공업수학 II 목차

PART3. Vector Analysis

PART 3 Vector Analysis . Chapter 11. Vector Differential Calculus

Review.

* inner Product.
* Outer Product.
* Triple Product (Scalar 결과)

$(a\ b\ c) = a\cdot(b\times c)$

$= a_1\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$

$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$ = a, b, c가 이루는 육면체의 부피.

※ $(a\ b\ c) = a\cdot(b\times c) = (a\times b)\cdot c$

ex) $a=\langle 1\ 4\ -7\rangle\quad b=\langle 2\ -1\ 4\rangle\quad c=\langle 0\ -9\ 18\rangle$

~~세~~ 벡터가 한 평면에 존재함을 보여라.

sol) $(a\ b\ c) = \begin{vmatrix} 1 & 4 & -7 \\ 2 & -1 & 4 \\ 0 & -9 & 18 \end{vmatrix} = -18+36-4(36)-7(-18)$

$= 0 =$ volume of a, b, c

∴ a, b, c는 한 평면에 있다.

sol2) $\begin{pmatrix} 1 & 4 & -7 \\ 2 & -1 & 4 \\ 0 & -9 & 18 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ 의 근이 존재하면 a, b, c는 한 평면에 있게된다.

$\downarrow$

$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$

왜 자식의 근이 존재하면 한평면에 있게되는것인가?

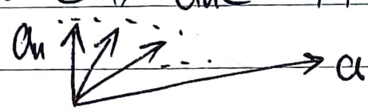
11.1 Vector Functions of One Variable

1. 개념.

* function
  - Scalar function : $y=f(x)$
  - vector function : $V(P)=\langle v_1(P), v_2(P), v_3(P)\rangle$

* Convergence (수렴)

① $\lim_{n\to\infty}|a_n-a|=\vec{0}$ 일때, a_n은 벡터 a로 수렴한다.
$\lim_{n\to\infty} a_n = a$

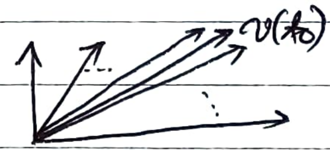


여러 벡터가 a되가는 과정.

② 이때 $a_n=\langle x_n, y_n, z_n\rangle$ $a=\langle x, y, z\rangle$ 이면,
$\lim_{n\to\infty} a_n=a \Rightarrow \lim_{n\to\infty}\langle x_n, y_n, z_n\rangle = \langle x, y, z\rangle$

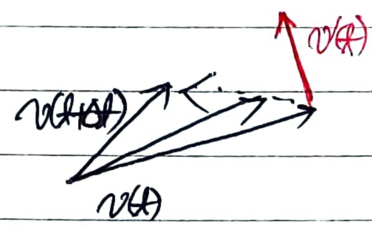
* Continuity (연속)

① $v(t_0)$가 정의되고, $\lim_{t\to t_0+0} v(t) = \lim_{t\to t_0-0} v(t)$, $\lim_{t\to t_0} v(t)=v(t_0)$ 이면,
$v(t)$는 $t=t_0$에서 연속이다.



* Derivative of Vector

① 벡터 $v(t)$의 미분은
$$v'(t)=\lim_{\Delta t\to 0}\frac{v(t+\Delta t)-v(t)}{\Delta t}$$ 이다.

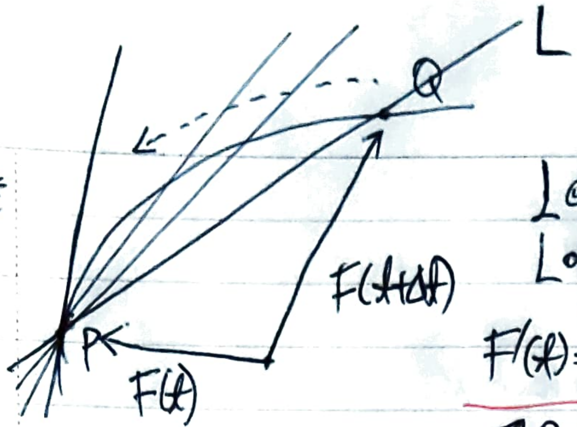


$v'(t)$는 $v(t)$가 그리는 Curve에 대해 tangent 해진다.

② $v(t)=\langle v_1(t), v_2(t), v_3(t)\rangle$ 을 미분하면 각 성분이 미분된다.
$v'(t)=\langle v_1'(t), v_2'(t), v_3'(t)\rangle$

③ Curve의 Tangent (접선)
vector function $F(t)$ 가 있다 하자.

F(t)는 Curve를 만든다.



L의 방향 : $F(t+\Delta t)-F(t)/\Delta t$

L이 tangent가 되므로,

$F'(t)=\lim_{\Delta t\to 0}\dfrac{F(t+\Delta t)-F(t)}{\Delta t}$ 은

점 P에서의 tangent vector가 된다.

이때 unit tangent vector는 $T(t)=\dfrac{1}{|F'(t)|}\cdot F'(t)$

tangent(직선)는 $g(\omega)=F(t)+F'(t)\cdot\omega$

ex) $\frac{x^2}{4}+y^2=1$ 의 vector function 표현 $F(t)=\langle 2\cos t,\sin t\rangle$

$t=\frac{\pi}{4}$의 접선은,

$F'(t)=\langle -2\sin t,\cos t\rangle$ $F'(\pi/4)=\langle -\sqrt{2},\frac{1}{\sqrt{2}}\rangle$

$g(\omega)=\langle \sqrt{2},\frac{1}{\sqrt{2}}\rangle+\omega\langle -\sqrt{2},\frac{1}{\sqrt{2}}\rangle=\langle -\sqrt{2}\omega+\sqrt{2},\frac{1}{\sqrt{2}}(\omega+1)\rangle$

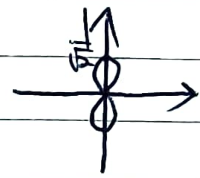
※ Curve, Arc length.

① 한 평면에 모든 점이 포함되면 plane curve

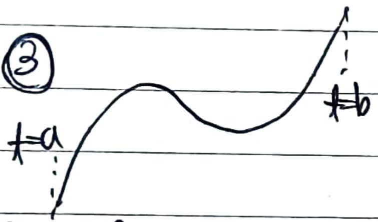
아니면 twisted curve

② 중복점이 없는 것은 simple curve

중복점 있는 커브 예시는 $F(t) = \langle \sin 2t, \cos t, 0 \rangle$



③



Curve of $F(t)$.

length of Curve는

$$\int_a^b |F'|\,dt = \int_a^b \sqrt{F' \cdot F'}\,dt$$

곡선거리길이

Arc는 Curve의 일부라는 뜻.

확장해서 Arc length of Curve는, $S(t) = \int_a^t |F'|\,dt = \int_a^t \sqrt{F' \cdot F'}\,dt$

④ $S(t) = \int_a^t |F'|\,dt$ 이용해 $t(s)$로 표현,

$F(t(s)) = G(s)$로 하면, $G'(s) = \frac{dF}{dt} \cdot \frac{dt}{ds} = \frac{1}{|F'|} F'$ = Unit tangent vector.

ex1) $F(t) = \langle \cos t, \sin t, t \rangle \quad 0 \le t \le 2\pi$.

$F'(t) = \langle -\sin t, \cos t, 1 \rangle$

곡선의 길이 $= \int_0^{2\pi} \sqrt{2}\,dt = 2\sqrt{2}\pi$.

arc length $\quad s(t) = \int_0^t \sqrt{2}\,dt = \sqrt{2}t$

ex2) $F(t) = \langle \cos t, \sin t, \frac{1}{3}t \rangle$

$F'(t) = \langle -\sin t, \cos t, \frac{1}{3} \rangle \Rightarrow s(t) = \int_{-4\pi}^t \frac{1}{3}\sqrt{10}\,d\xi = \frac{1}{3}\sqrt{10}(t + 4\pi)$

$\therefore t(s) = \frac{3}{\sqrt{10}}s - 4\pi$.

$\therefore G(s) = F(t(s)) = \cos\left(\frac{3}{\sqrt{10}}s\right)i + \sin\left(\frac{3}{\sqrt{10}}s\right)j + \left(\frac{1}{\sqrt{10}}s - \frac{4\pi}{3}\right)k$

$G'(s) = -\frac{3}{\sqrt{10}}\cos\left(\frac{3}{\sqrt{10}}s\right)i + \frac{3}{\sqrt{10}}\sin\left(\frac{3}{\sqrt{10}}s\right)j + \frac{1}{\sqrt{10}}k$: unit tangent vector to C

11.2 Velocity & Curvature.

[1] 속도, 가속도, unit tangent vector

curve를 표현하는 $F(t)$에 대해, $F(t)$가 변위를 의미한다면.

$\mathbf{V}(t) = F'(t)$: velocity

$v(t) = |\mathbf{V}(t)|$: speed.

$a(t) = F''(t) = \mathbf{V}'(t)$: acceleration.

$$T(t) = \frac{1}{|F'(t)|} \cdot F'(t) = \frac{1}{v(t)} \cdot \mathbf{V}(t)$$

[2] Curvature : arc length에 대한 unit tangent의 변화비. 굽은정도.

curve가 더 굽을수록, unit tangent vector가 더 크게 변한다.

여기서 s는 이동거리의 변수.

$$\kappa(s) = \left|\frac{dT}{ds}\right| = |T'(s)|$$

$$= \left|\frac{dT}{dt} \cdot \frac{dt}{ds}\right| = \frac{1}{|F'(s)|} \cdot |T'(t)|$$

보통 $\kappa(t)$로 이걸로 계산.

[3] Unit normal vector, acceleration의 분할.

① $N(s) = \frac{1}{\kappa(s)} \cdot T'(s)$ (unit normal vector)

F(t(s))를 s로 미분하면 unit tangent vector. 알잘 보기.

② $\mathbf{V}(t) = \frac{dF}{dt} = \frac{dF}{ds} \cdot \frac{ds}{dt} = v \cdot T$

$$a(t) = \frac{dv}{dt} T + vT' = \frac{dv}{dt} T + v \cdot \frac{ds}{dt} \cdot \frac{dT}{ds} = \frac{dv}{dt} T + v^2 \kappa(s) N(s)$$

$$= \frac{dv}{dt} \cdot T + v^2 \kappa N$$

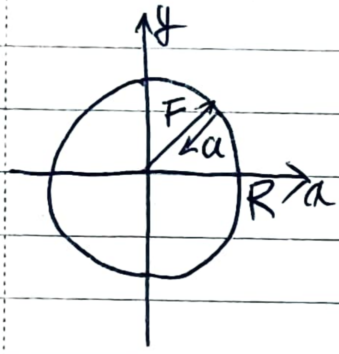
$$= a_T \cdot T + a_N N$$

$a_T = dv/dt$

$a_N = v^2 \kappa$

$$a^2 = a_T^2 + a_N^2$$

ex 1) Centripetal/Centrifugal Acceleration 구심/원심 가속도.



$r(t) = R\cos\omega t\, i + R\sin\omega t\, j$

$v(t) = -\omega R\sin\omega t\, i + \omega R\cos\omega t\, j$

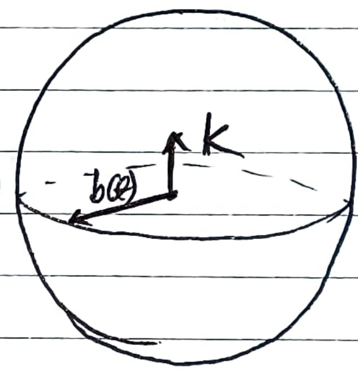
$v(t) = \omega R.$

$a(t) = r''(t) = -\omega^2 R\cos\omega t\, i - \omega^2 R\sin\omega t\, j$

$= -\omega^2 R$

a가 curve에 normal하고, 원점을 향한다.

ex2) Coriolis Acceleration.



$r(t) = R\cos\gamma t \cdot b(t) + R\sin\gamma t \cdot k$

$b(t) = \cos\omega t\, i + \sin\omega t\, j$ ($\omega > 0$, unit vector)

$b'(t) = -\omega\sin\omega t\, i + \omega\cos\omega t\, j$

$b''(t) = -\omega^2\cos\omega t\, i - \omega^2\sin\omega t\, j = -\omega^2 b(t)$

$v(t) = r'(t) = R\cos\gamma t\, b'(t) - \gamma R\sin\gamma t\, b(t) + \gamma R\cos\gamma t\, k$

$a(t) = R\cos\gamma t\, b''(t) - 2R\sin\gamma t\, b'(t) - \gamma^2 R\cos\gamma t\, b(t) - \gamma^2 R\sin\gamma t \cdot k$

$= \underbrace{R\cos\gamma t\, b''}_{\text{구심가속도(자전)}} \underbrace{- 2R\sin\gamma t\, b'}_{\text{코리올리 가속도}} \underbrace{- \gamma^2 R}_{\text{구심가속도(두 천체의 움직임에 의한)}}$

11.3 Vector Field & Streamline

1. vector function을 vector field 라고도 한다.
2. vector field F에 대해, F가 tangent vector인 curve를 streamline 이라 한다.
   ex) F가 유체의 속도면 그 stream line은 유체의 이동경로
3. stream line에 대해, 다음식이 성립한다.
   $F = fi + gj + hk$ 의 streamline은 다음이 성립.
   $$\frac{dx}{f} = \frac{dy}{g} = \frac{dz}{h}$$

ex) $F = \langle x^2, 2y, -1 \rangle$

$\frac{dx}{x^2} = \frac{dy}{2y} = \frac{dz}{-1}$. $-\frac{1}{x} = -z + C$, $\frac{1}{2}\ln|y| = -z + k$

$x = \frac{1}{z - C}$, $y = a \cdot e^{-2z}$

☆ Review of Calculus.

1. Chain Rule
   $w = f(x, y, z)$ 에서 x, y, z가 u, v의 함수면.
   $$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial u} + \frac{\partial w}{\partial z} \cdot \frac{\partial z}{\partial u}$$
   ex) $w = x^2 - y^2$, $x = r\cos\theta$, $y = r\sin\theta$.
   $\partial w / \partial r = 2x\cos\theta - 2y\sin\theta = 2r\cos 2\theta$.

2. 다변수함수에 대한 Mean Value Theorem.
   $f(x, y, z)$ 위의 점 $P_0(x_0, y_0, z_0)$, $P(x_0 + h, y_0 + k, z_0 + l)$ 에 대해
   $$f(x_0 + h, y_0 + k, z_0 + l) - f(x_0, y_0, z_0) = h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y} + l\frac{\partial f}{\partial z}$$
   을 만족시키는 지점이 적어도 하나 존재.
   = 좌변의 4차 volume을 만족시키는 (밑변 × 높이) 값이 존재

11.4 Gradient, Divergence, Curl

1 미분연산자 del (∇)

$$\nabla = \frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k$$

vector 처럼 다룬다.

2 Gradient, Divergence, Curl.

※ 세 연산 모두 좌표계에 독립이다.

① Gradient : scalar func → vector func. $\varphi(x, y, z)$

$$\operatorname{grad} \varphi = \nabla \varphi = \frac{\partial \varphi}{\partial x} i + \frac{\partial \varphi}{\partial y} j + \frac{\partial \varphi}{\partial z} k$$ (scalar product)

∴ irrotational.

② Divergence : vector func → scalar func. $F = \langle f, g, h \rangle$

$$\operatorname{div} \varphi = \nabla \cdot F = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}$$ (inner product)

③ Curl : vector func → vector func. $F = \langle f, g, h \rangle$

$$\operatorname{curl} F = \nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f & g & h \end{vmatrix}$$ (outer product)

④ Theorem. φ는 scalar function, F는 vector function

$\operatorname{curl} \operatorname{grad} \varphi = 0$. $\quad \nabla \times (\nabla \varphi) = 0$

$\operatorname{div} \operatorname{curl} F = 0 \quad \nabla \cdot (\nabla \times F) = 0$.

3 Gradient 와 방향미분 (directional derivative), 기하학적의미

① 특정 방향으로의 미분값 (rate of change)를 구할때.

$D_u \varphi = \nabla \varphi \cdot u$ 이때 u는 unit vector.

ex) $f(x,y,z) = 2x^2+3y^2+z^2$, $a=\langle 1,0,-2\rangle$ 으로의 $P(2,1,3)$에서 미분값.

$u = \frac{a}{|a|} = \langle \frac{1}{\sqrt{5}}, 0, -\frac{2}{\sqrt{5}} \rangle$

$D_u f = \nabla f \cdot u = \langle 4x, 6y, 2z \rangle \cdot \langle \frac{1}{\sqrt{5}}, 0, -\frac{2}{\sqrt{5}} \rangle$

$= \frac{4}{\sqrt{5}}x - \frac{4}{\sqrt{5}}z$

$= \frac{8}{\sqrt{5}} + \frac{12}{\sqrt{5}} = \frac{20}{\sqrt{5}} = 4\sqrt{5}$.

② Gradient는 level surface 에 대해 연직방향이다.
따라서 $\varphi(x,y,z)$는 $\nabla\varphi$ 방향으로 최대 변화 방향을 가지며,
그 최대 변화값은 $|\nabla\varphi|$ 이다.

④ Laplace Equation.

$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \Delta f$

$\nabla^2 = \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ (Laplace operator, Laplacian)

이때. $\nabla^2 f = \nabla \cdot (\nabla f) = \text{div grad } f$ 이다. (f는 scalar func)

flux 차원
양/시간·면적

$\boxed{4}$ Divergence의 기하학적의미.
total Flux per unit volume.
한 지점의 유속 변화율.
한 지점에서 벡터장이 퍼지는지 모이는지.
한 지점에서 나가고 들어오는 양의 차.

$\boxed{5}$ Curl의 기하학적의미

어떤 물체가 회전하고 있고, (각속도 ω로)
그걸 F가 나타내고 있다면
Curl은 그 물체에 직교하며
크기는 $|\text{Curl } F| = 2\omega$

12.1 Line Integrals

$\boxed{1}$ 세가지 표현방법

curve $C : r(t) = \langle x(t), y(t), z(t) \rangle \ (a \le t \le b)$를
path of integration으로 하여 어느 함수를 적분한다.
여기서 C는 유한개의 smooth curve로 구성 – piece wise smooth.

① f, g, h가 curve C에서 연속일때.

$\int_C f dx + g dy + h dz$ 를 line Integral 이라 한다.

$$= \int_a^b \left[f(x(t), y(t), z(t))\, x'(t) + g(x(t), y(t), z(t))\, y'(t) + h(x(t), y(t), z(t))\, z'(t) \right] dt$$

으로 계산한다. 즉, f, g, h에 C를 집어넣는다.
이때 $dx = x'(t)dt$, $dy = y'(t)dt$, $dz = z'(t)dt$.

② Vector Field를 선적분시.

$$\int_C F(r(t)) \cdot dr = \int_a^b F(r(t)) \cdot r'(t)\, dt \quad \because r' = dr/dt$$

$F(r) = \langle F_1\ F_2\ F_3 \rangle$ $dr = \langle dx\ dy\ dz \rangle$ 라 입하면

$$\int_C F_1 dx + F_2 dy + F_3 dz = \int_a^b (F_1 x' + F_2 y' + F_3 z')\, dt$$

로, ①과 같은 형태가 된다.

③ Scalar Field를 선적분시. $\varphi(x, y, z)$: scalar field.

$$\int_C \varphi(x, y, z)\, ds = \int_a^b \varphi(x(t), y(t), z(t)) \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\, dt$$

$\because ds = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\, dt$

순서에 따른 예시.

ex) $\int_C x\,dx - yz\,dy + e^z\,dz$

$C: x=t^3 \quad y=-t \quad z=t^2 \quad 1\le t\le 2$

sol) 선적분은, $\int_1^2 \{ t^3 \cdot 3t^2 - t^3(-1) + e^{t^2}\cdot 2t \}\,dt$

$= \frac{1}{2}t^6 + \frac{1}{4}t^4 + e^{t^2} \Big]_1^2 = \frac{1}{2}(64-1) + \frac{1}{4}(16-1) + e^4 - e$

$= \frac{111}{4} + e^4 - e$

ex2) $F(x,y) = \langle -y, -xy \rangle$

$C: x^2+y^2 \quad \Rightarrow x=\cos t, \cancel{y=}\ \sin t \quad 0\le t\le \frac{\pi}{2}$

sol) $F(r) = \langle -\sin t, -\cos t\sin t \rangle$

$\int_C F(r)\cdot dr = \int_0^{\frac{\pi}{2}} \langle -\sin t, -\cos t\sin t \rangle \cdot \langle -\sin t, \cos t \rangle\,dt$

$= \int_0^{\frac{\pi}{2}} \sin^2 t - \cos^2 t\sin t\,dt = \int_0^{\frac{\pi}{2}} \frac{1}{2}(1-\cos 2t) - \int_0^{\frac{\pi}{2}} \cos^2 t\sin t\,dt$

ex3) $\overset{\text{density}}{\delta}(x,y,z) = xy^2$, $\quad C: x=2\cos t,\ y=2\sin t,\ z=3.\ 0\le \cancel{\theta} \le \pi/2$

mass=?

sol) $m = \int_C \delta \cdot ds = \int_C 2\cos t \cdot 4\sin^2 t\sqrt{4\cdot 2}\,dt$

$= 16\int_0^{\pi/2} \cos t\sin^2 t\,dt = 16\cdot \frac{1}{3}(\sin^3 t)_0^{\pi/2} = \frac{16}{3}$

2 선적분의 특성.

① $\int_C kF\cdot dr = k\int_C F\,dr$

② $\int_C (F+G)\,dr = \int_C F\,dr + \int_C G\cdot dr$

③ $\int_C F\cdot dr = \int_{C_1} F\,dr + \int_{C_2} F\cdot dr$

④ $\int_{-C} F\cdot dr = -\int_C F\cdot dr$

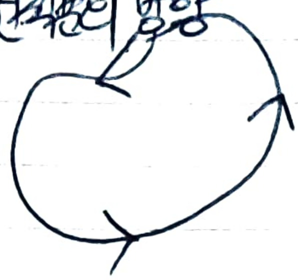
(반대 방향으로 선적분시.)

⑤ 닫힌곡선은 $\int_C = \oint_C$

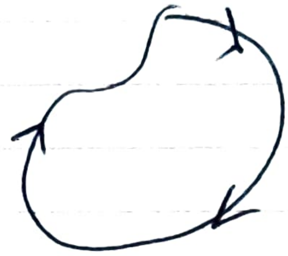
12.2 Green's Theorem.

선적분을 면적분으로.

① 선적분의 방향



Counter Clockwise
Positive Oriented



Clockwise
Negative Oriented

② Green's Theorem. (닫힌곡선용)

path of integration C가 simple closed, positive oriented

D는 C의 내부

$f, g, \partial f/\partial y, \partial g/\partial x$ 가 D에서 연속일때.

$$\oint_C f(x,y)\,dx + g(x,y)\,dy = \iint_D \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dA.$$

$$\oint_C F\cdot n\,ds = \iint_D \nabla\cdot F\,dA$$

으로도 볼수 있다.

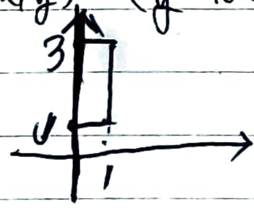
※ Vector function으로 주어지면, $F = \langle f, g\rangle$로 보고, $dr = \langle dx, dy\rangle$로.

$$\oint_C F(r)\cdot dr = \oint_C f\cdot dx + g\,dy = \iint_D \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dA$$

③ 예제

ex1) 계산 단순화

$F(x,y) = (y - x^2 e^x)\,i + (\cos 2y^2 - x)\,j$

$C=$ 

$$\oint_C F\cdot dR = \iint_D (-1-1)\,dA = -2\iint_D dA = -2\times 2 = -4$$

ex2) General Result 구하기.

모든 simple closed path C에서,

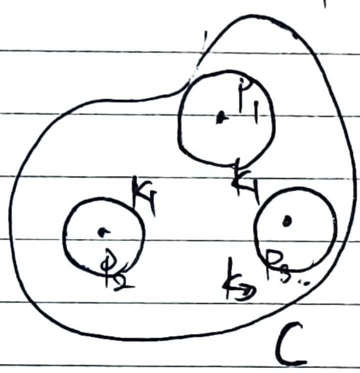
$$\oint_C 2x\cos 2y\,dx - 2x^2\sin 2y\,dy$$

$$= \iint_D (-4x\sin 2y + 4x\sin 2y)\,dA = 0.$$

∴ 모든 C에서 그 선적분이 0 이다..!

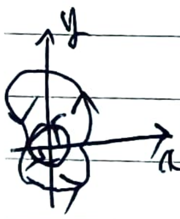
$\boxed{4}$ Green's Theorem의 확장

curve C 내부 점에서, $f, g, \partial g/\partial x, \partial f/\partial y$ 가 불연속하거나, 정의되지 않았을 때 사용.



불연속점 P_1, P_2, P_3 주위에 원 K_1, K_2, K_3를 그린다. 원래 C의 내부에서 K_1, K_2, K_3의 내부를 뺀 면적을 D^*라 하자.

$$\oint_C f\,dx + g\,dy = \sum_{j=1}^{n} \oint_{K_j} f\,dx + g\,dy + \iint_{D^*} \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dA$$



ex) $\oint_C \frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy$. $(x,y)=(0,0)$ 에서 불연속.

즉, 'C의 내부에 (0,0)이 있는가?' 문제

i) C의 내부에 (0,0)이 없다. → Green Theorem.

$$\oint_C f\,dx + g\,dy = \iint_D \left(\frac{y^2-x^2}{(x^2+y^2)^2} - \frac{y^2-x^2}{(x^2+y^2)^2}\right) dA = 0.$$

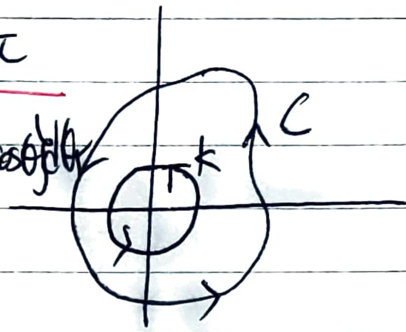
ii) C의 내부에 있다. → Green Theorem Expansion.

$$\oint_C f\,dx + g\,dy = \oint_K f\,dx + g\,dy + \cancel{\iint_D \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dA} \rightarrow 0$$

$$= \oint_K f\,dx + g\,dy.$$

이런식으로 임의의 원을 불연속 점에 만든다.

이때 $K : x = r\cos t \quad y = r\sin t. \quad 0 \leq \theta \leq 2\pi$



$$\oint_K f\,dx + g\,dy = \oint \left\{ \frac{-r\sin\theta}{r^2} \cdot (-r\sin\theta) + \frac{r\cos\theta}{r^2} r\cos\theta \right\} d\theta$$

$$= \oint_K d\theta = 2\pi.$$

$$\therefore \oint_C f\,dx + g\,dy = 2\pi.$$

12.4 Path Independence.

선적분의 값은 $\left\{ \begin{array}{l} F \\ \text{적분 종점, 시점} \\ \text{경로} \end{array} \right\}$ 에 따라 달라진다.

ex) $C_1 : r_1(t) = \langle t, t, 0 \rangle \quad 0 \le t \le 1$

$C_2 : r_2(t) = \langle t, t^2, 0 \rangle \quad 0 \le t \le 1$

이때 path Independence 한 선적분이 있다.

1 Vector field 가 conservative 하면
potential을 미분해서 Vector field가 나온다.

즉, $F = \nabla \varphi = \frac{\partial \varphi}{\partial x} i + \frac{\partial \varphi}{\partial y} j + \frac{\partial \varphi}{\partial z} k$ ※ φ는 scalar field.
이면 F는 conservative vector field 이다.

$$\Updownarrow$$

$$\int_{C_1} F \cdot dR = \int_{C_2} F \cdot dR \text{ (independence of path)}$$

$$\Updownarrow$$

$$\oint_C F \cdot dR = 0$$

2 Test for a Conservative Field in the Plane

① 범위(F가 정의된 곳) ② 함수 이 두개를 확인한다.

region D는 axis에 평행해야한다.

① F가 정의된 직사각 범위 D 가, domain이어야 한다.

domain 조건

1. D안의 한점 P에 대해, P를 감싸며 D의 점들만 감싸는 원이 적어도 하나 있어야 한다.
2. D의 어느 두점 사이에 D로만 구성된 path가 적어도 하나 있어야 한다.

② 적분 대상 F가 region D에서 $F = \langle f, g \rangle$ 일때
$\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}$ 이어야 한다.

※ 3차원으로 확장하면

① 원 → 구로 변경.

② region D에서 $F = \langle f, g, h \rangle$ 일때.

curl $F = \nabla \times F = 0$ 이어야한다.

즉, $\begin{cases} \frac{df}{dy} - \frac{dg}{dx} = 0 \\ \frac{dh}{dx} - \frac{df}{dz} = 0 \\ \frac{dh}{dy} - \frac{dg}{dz} = 0 \end{cases}$

ex1) $F = \langle 2x, 2y, 2z \rangle$의 선적분이 path independance 함을 보여라.

① $df/dx = 2x$, $f = x^2 + g(y,z)$.

$df/dy = dg/dy = 2y$, $\therefore g = y^2 + h(z)$. $\therefore f = x^2 + y^2 + h(z)$.

$df/dz = dh/dz = 2z$ $\therefore h = z^2$

$\therefore f = x^2 + y^2 + z^2$ 로 F의 potential이 존재.

② $df/dy = 0$

$dg/dx = 0$ $\therefore df/dy = dg/dx$.

domain 에 대해선 path independance 하다.

ex2) $\int_C (y+yz)dx + (x+3z^3+xz)dy + (9yz^2+xy-1)dz$

구하기. C의 시.종점은 $(1,1,1)$ $(2,1,4)$

sol) $\nabla \times F = \begin{vmatrix} i & j & k \\ \frac{d}{dx} & \frac{d}{dy} & \frac{d}{dz} \\ y+yz & \begin{matrix} x+ \\ 3z^3 \\ +xz \end{matrix} & \cdots \end{vmatrix} = (9z^2+x-(9z^2+x))i - (y-y)j + (1+z-(1+z))k = 0$

$\therefore$ 이 Region에서 위의 선적분은 path independance.

① $\partial f/\partial x = y+yz$, $f = xy+xyz+g(y,z)$

② $\partial f/\partial y = x+xz+\partial g/\partial y = x+3z^3+xz$ $\therefore g = 3yz^3+h(z)$.

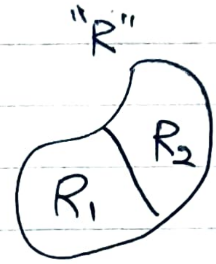
$\therefore f = xy+xyz+3yz^3+h(z)$.

③ $\partial f/\partial z = xy+9yz^2+h'(z) = 9yz^2+xy-1$. $\therefore h(z) = -z+C$.

$\therefore f = xy+xyz+3yz^3-z+C$.

$f(2,1,4) - f(1,1,1) = 194$.

12.5 Surface Integral.



선적분이 선을 경로로 한다면
면적분은 면을 경로로.

$$\iint_R f(x,y)\,dA = \iint_R f(x,y)\,dx\,dy$$ 이런식으로...

계산법: 중적분법을 먼저. Region이 plane에서 구불구불 surface 적분까지.

$\boxed{1}$ 특성

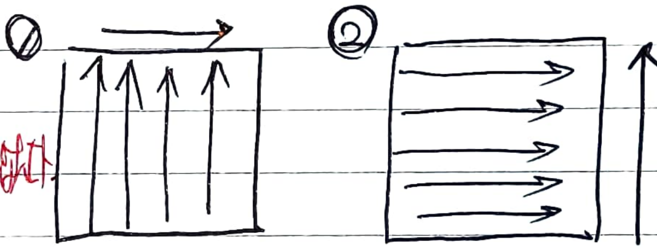
중적분과 유사

① $\iint_R kf\,dx\,dy = k\iint_R f\,dx\,dy$

② $\iint_R f\,dx\,dy = \iint_{R_1} f\,dx\,dy + \iint_{R_2} f\,dx\,dy$

$\boxed{2}$ 계산법

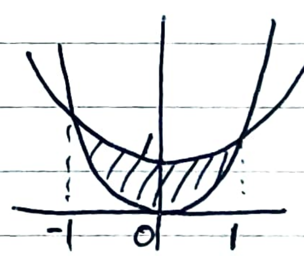
① y first
② x first.
차이없다

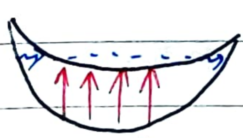


ex) $\int_0^3\int_1^2 x^2y\,dy\,dx = \int_0^3\left[\frac{1}{2}x^2y^2\right]_1^2 dx = \int_0^3 \frac{3}{2}\cdot x^2\,dx = \left.\frac{1}{2}x^3\right]_0^3$
$= \frac{27}{2}$

$\int_1^2\int_0^3 x^2y\,dx\,dy = \int_1^2\left(\frac{1}{3}x^3y\right)_0^3 dy = \int_1^2 9y\,dy = \left[\frac{9}{2}y^2\right]_1^2$
$= \frac{27}{2}$

ex2) $\iint_R (x+2y)\,dA$ region R은 $y=2x^2$, $y=1+x^2$ 사이





x먼저 적분하면
중간에 끊기니, y먼저하는게 유리하다.

$\int_{-1}^1\int_{2x^2}^{1+x^2}(x+2y)\,dy\,dx = \int_{-1}^1 x(1-x^2)+(1+2x^2-3x^4)\,dx$

$= \int_{-1}^1 1+\cancel{x}+2x^2-\cancel{x^3}-3x^4\,dx = 2\cdot\int_0^1 1+2x^2-3x^4\,dx$

$= 2\left[x+\frac{2}{3}x^3-\frac{3}{5}x^5\right] = 2\left(1+\frac{2}{3}-\frac{3}{5}\right) = 2\cdot\frac{15+10-9}{15} = \frac{32}{15}$

$\boxed{3}$ 중적분 물리적 현상 표현

면밀도 $\rho(x,y)$.

① mass $M = \iint_R \rho(x,y)\,dxdy$

② center of mass

$$\bar{x} = \frac{1}{M}\iint_R x\,\rho(x,y)\,dxdy$$

$$\bar{y} = \frac{1}{M}\iint_R y\,\rho(x,y)\,dxdy$$

③ moments of inertia.

$$I_x = \iint_R y^2\,\rho(x,y)\,dxdy$$

$$I_y = \iint_R x^2\,\rho(x,y)\,dxdy$$

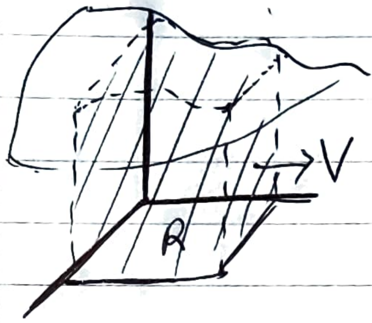
④ polar-moment of inertia

$$I_0 = I_x + I_y = \iint_R (x^2+y^2)\,\rho(x,y)\,dxdy.$$

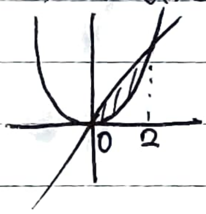
4 중적분의 응용

① $A = \iint_R dA = \iint_R dx\,dy$ (면적 of R)

② $V = \iint_R f(x,y)\,dA = \iint_R f(x,y)\,dx\,dy$ ($z=f(x,y)$의 region R 위의 체적)



ex) volume of $z=x^2+y^2$ above region $y=2x$, $y=x^2$



$$\int_0^2\int_{x^2}^{2x} x^2+y^2\,dy\,dx = \int_0^2 x^2(2x-x^2)+\frac{1}{3}(8x^3-x^6)\,dx$$

$$= \int_0^2 \frac{14}{3}x^3 - x^4 - \frac{1}{3}x^6\,dx = \left[\frac{7}{6}x^4 - \frac{1}{5}x^5 - \frac{1}{21}x^7\right]_0^2$$

region, 적분대상이 형태따라 region 읽는법을 바꾸는 것이다.

$\boxed{5}$ 적분변수 바꾸기: Change of variable - Jacobian.

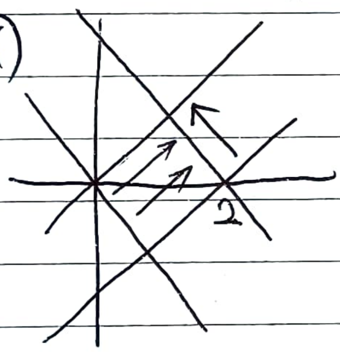
$$\iint_R f(x,y)\,dxdy = \iint_R f(x(u,v),\, y(u,v)) \underbrace{\left|\frac{\partial(x,y)}{\partial(u,v)}\right|}_{\text{Jacobian의 절대값.}} dudv$$

☆ Jacobian "J"

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\cdot\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\cdot\frac{\partial y}{\partial u}$$

Cartesian coordinate를 Polar coordinate로 바꿀때 사용.

ex)



$x+y=u,\ x-y=v\ \ (0 \le u \le 2,\ 0 \le v \le 2)$

이때 $x = \frac{1}{2}(u+v),\ y = \frac{1}{2}(u-v)$

$$J = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{4} - \frac{1}{4} = -\frac{1}{2}$$

이때 $\iint_R (x^2+y^2)\,dxdy = \int_0^2\int_0^2 \frac{1}{2}(u^2+v^2)\cdot\left|-\frac{1}{2}\right| dudv = \frac{1}{4}\int_0^2\int_0^2 u^2+v^2\, dudv$

$= \frac{8}{3}$

polar coordinate는,

$x = x(r,\theta) = r\cos\theta$

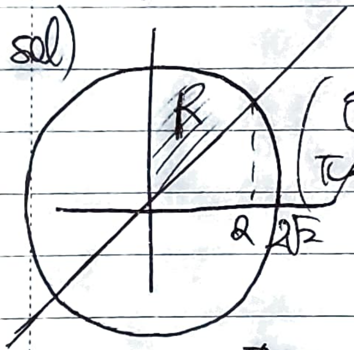
$y = y(r,\theta) = r\sin\theta$ 두고,

$$J = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$$

$$\therefore \iint_R f(x,y)\,dx\,dy = \iint_{R_p} f(r\cos\theta, r\sin\theta)\cdot r\cdot dr\,d\theta$$

ex) $\displaystyle\int_0^2\int_x^{\sqrt{8-x^2}} \frac{1}{5+x^2+y^2}\,dy\,dx$ $\quad\left(\begin{array}{l} x\le y\le\sqrt{8-x^2} \\ 0\le x\le 2\end{array}\right)$

sol)



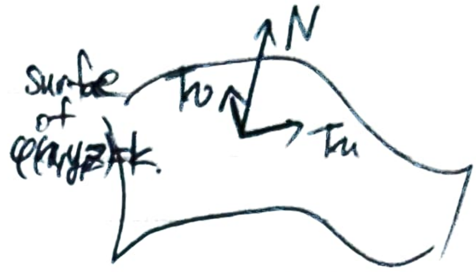
$\left(\begin{array}{l} 0\le r\le 2\sqrt{2} \\ \pi/4\le\theta\le\pi/2\end{array}\right)$

$$\therefore \text{원식} = \int_{\pi/4}^{\pi/2}\int_0^{2\sqrt{2}}\frac{r}{5+r^2}\,dr\,d\theta = \int_{\pi/4}^{\pi/2}\left[\frac{1}{2}\ln|5+r^2|\right]_0^{2\sqrt{2}}d\theta$$

$$= \frac{1}{2}\left(\ln\frac{13}{5}\right)\cdot\frac{\pi}{4} = \frac{\pi}{8}\ln\frac{13}{5}$$

적분경로 Surface가 평면이 아닌 구불구불할때...

$\boxed{6}$ Surface가 2개의 매개변수 u, v로 표현될 때,
(적분경로) $\varphi(x,y,z)=k$ 의 Normal Vector는.



u 방향, v 방향 tangent vector 각각 T_u, T_v라 한다.

$$N = T_u \times T_v = \begin{vmatrix} i & j & k \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix}$$

$$= \frac{\partial(y,z)}{\partial(u,v)} i - \frac{\partial(x,z)}{\partial(u,v)} j + \frac{\partial(x,y)}{\partial(u,v)} k$$

* $z = S(x,y)$ 형태는 $u=x, v=y$ 로 보고

$$N = -\frac{\partial z}{\partial x} i - \frac{\partial z}{\partial y} j + k$$

$\boxed{7}$ Normal Vector 통한 면적분.

(간단) { Curve Integral: tangent 적분
Surface Integral: Normal 적분

$$\text{area of } \Sigma = \iint_\Sigma |N(x,y)|\, dxdy$$

or

$$\iint_\Sigma |N(u,v)|\, dudv$$

$f(x,y,z)$ 이 적분 대상이면. ($d\sigma$는 면미분소) $d\sigma = |N(u,v)|\, dudv$

$$\iint_\Sigma f(x,y,z)\, d\sigma = \iint_\Sigma f(x(u,v), y(u,v), z(u,v))\, |N(u,v)|\, dudv$$

※ $z = S(x,y)$ 면

$$\text{area of } \Sigma = \iint_\Sigma \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dxdy$$

$$f(x,y,z) \text{적분} = \iint_\Sigma f(x,y,z)\, d\sigma = \iint_\Sigma f(x,y,z) \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dxdy$$

ex 1) $x+y+z=4$. $0 \le x \le 2$, $0 \le y \le 1$

$\iint_\Sigma z\, d\sigma = ?$

$d\sigma = N(x,y)\,dxdy = \sqrt{(i+j+k)^2}\,dxdy = \sqrt{3}\,dxdy$

풀이 $= \int_0^1 \int_0^2 (4-x-y)\sqrt{3}\,dxdy = \sqrt{3}\int_0^1 8-2-2y\,dy$

$= \sqrt{3}(6-1) = 5\sqrt{3}$.

$\boxed{\text{III}}$ Application of surface integrals.

① mass of Shell for $f(x,y,z)=k$. density $\delta(x,y,z)$.

Σ is shell

✓ mass of $\Sigma = m = \iint_\Sigma \delta(x,y,z)\,d\sigma$

✓ Center of mass

$\bar{x}=\frac{1}{m}\iint_\Sigma x\,\delta(x,y,z)\,d\sigma \quad \bar{y}=\frac{1}{m}\iint_\Sigma y\,\delta(x,y,z)\,d\sigma \quad \bar{z}=\frac{1}{m}\iint_\Sigma z\,\delta(x,y,z)\,d\sigma$

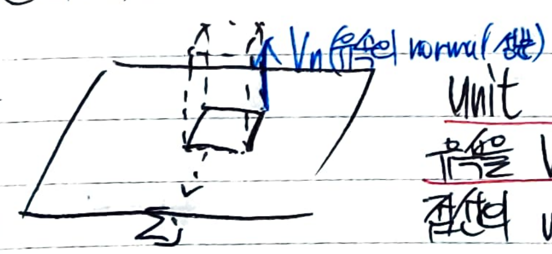
$d\sigma = N(x,y)\,dxdy = N(u,v)\,dudv$

※ $z=S(x,y)$ 이면

$m=\iint_D \delta(x,y,z)\sqrt{1+\left(\frac{\partial S}{\partial x}\right)^2+\left(\frac{\partial S}{\partial y}\right)^2}\,dydx.$

② Flux of Fluid across Surface ※ flux는 [유출량/시간·면적]

A_j = Area of Σ_j



unit normal vector $n(u,v,t)$
유속을 V라 할 때,
직선의 volume이 $V\cdot n\,\Delta t\,A_j$

이때 volume of fluid per unit time $= \dfrac{V\cdot n\,\Delta t\,A_j}{\Delta t} = V\cdot n\,A_j$

∴ 면적 A_j를 합산하면....
flux of V across Σ in the direction of n

$$= \iint_\Sigma V\cdot n\,d\sigma$$

ex) $F = \langle x, y, z \rangle$

plane $x^2+y^2+z^2=4$, 에서 $z=1,2$ 사이의 Flux 는?

$z=1$: $x^2+y^2=3$.

$z=2$: $x^2+y^2=0$. $\quad z \geq 1$

위쪽만...

$z=\sqrt{4-x^2-y^2}$. $0 \le x^2+y^2 \le 3$.

$$N = -\frac{\partial z}{\partial x} i - \frac{\partial z}{\partial y} j + k$$

$$= \frac{x}{\sqrt{4-x^2-y^2}} i + \frac{y}{\sqrt{4-x^2-y^2}} j + k = \frac{x}{z} i + \frac{y}{z} j + k.$$

$$n = \frac{1}{|N|}\cdot N = \frac{1}{\frac{\sqrt{4}}{z}} * \left(\frac{x}{z} i + \frac{y}{z} j + k\right)$$

$$= \frac{1}{2}(x i + y j + z k)$$

$$\iint_\Sigma F\cdot n \, d\sigma = \iint_D \frac{1}{2}(x^2+y^2+z^2)\sqrt{\frac{x^2+y^2+z^2}{z^2}}\, dxdy$$

$$= 4\iint_D \frac{1}{z}\, dxdy = 4\iint \frac{1}{\sqrt{4-x^2-y^2}}\, dxdy$$

이때 $x=r\cos\theta$, $y=r\sin\theta$ $0\le r\le\sqrt{3}$, $0\le\theta\le 2\pi$.

$$= 4\cdot\iint -\frac{1}{2}\frac{-2r}{\sqrt{4-r^2}}\, dr d\theta =$$

$$= -2\times 2\int_0^{2\pi} d\theta \times \left[\sqrt{4-r^2}\right]_0^{\sqrt{3}}$$

$$= -4\cdot 2\pi\cdot(1-2) = 8\pi.$$

12.7 Green Theorem R^2 to R^3.

$\boxed{1}$ R^2에서 Green Theorem은,

$$\oint f(x,y)dx + g(x,y)dy = \iint_D \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dA.$$

이때 $F = g(x,y)i - f(x,y)j$ 로 두면 $\nabla \cdot F = \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}$

곡선을 $C: x(s), y(s).\ 0 \le s \le L$ 로 두고

unit tangent vector $T(s) = x'(s)i + y'(s)j$

unit normal vector $n(s) = y'(s)i - x'(s)j$

이때 $F \cdot n\, ds = \left[g(x,y)\frac{dy}{ds} + f(x,y)\frac{dx}{ds}\right] ds$

$\therefore$ 원래 Green Theorem은 $\oint_C F \cdot n\, ds = \iint_D \nabla \cdot F\, dA$

$\boxed{2}$ 곡선 C를 plane으로 확장. Σ closed surface로 Green Theorem을 적용하면 M은 Σ 내부의 solid.

$$\iint_\Sigma F \cdot n\, d\sigma = \iiint_M \nabla \cdot F\, dV$$ (Gauss Divergence Theorem)

3 Stokes's Theorem.

$$\int_C F \cdot T\, ds = \iint_\Sigma (\nabla \times F) \cdot n\, d\sigma$$

어떻게 구하나? $F(x,y,z) = f i + g j + 0 k$

$$\nabla \times F = \left(\frac{dg}{dx} - \frac{df}{dy}\right) k$$

이때 unit tangent vector $T = x'(s) i + y'(s) j$

$$F \cdot T\, ds = \left[f(x,y)\frac{dx}{ds} + g(x,y)\frac{dy}{ds}\right] ds$$

Green Theorem에 따라 $\int_C F \cdot T ds = \iint_D (\nabla \times F) k\, dA$

이를 일반화한 것.

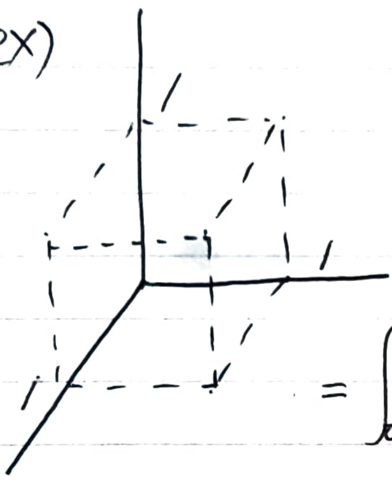
12.8 Gauss Divergence Theorem.

$$\iint_{\Sigma} F\cdot n \, d\sigma = \iiint_{M} \nabla\cdot F \, dV$$

이걸, 질량보존법칙으로 볼 수 있다.

"표면으로 나가는 flux 양" = "부피에서 손실량"

ex)



$F = \langle x^2, y^2, z^2 \rangle$ 에 대해 flux 양을 구하려한다.

각 면에 대해 $\iint_{\Sigma} F\cdot n \, d\sigma$ 하기 어렵다.

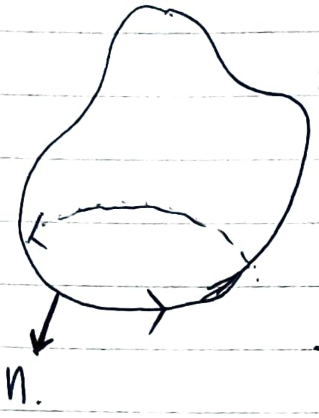
이때

$$\iint_{\Sigma} F\cdot n \, d\sigma = \iiint_{M} \nabla\cdot F \, dV$$

$$= \int_0^1\int_0^1\int_0^1 2x+2y+2z \, dxdydz = 3.$$

12.9 Stokes's Theorem. 선적분 ⟷ 면적분

*



n을 머리라고 생각하고 curve 따라 걸을때
region이 왼쪽에 있을때.
그 curve가 oriented coherently.

Green Theorem

$\oint_C F\cdot n\, ds$

$\parallel$

$\iint_\Sigma \nabla\cdot F\, dA$

1 Stokes's Theorem.

surface Σ가 C로 bounded 되어 있을 때.

$$\oint_C F\cdot dR = \iint_\Sigma (\nabla\times F)\cdot n\, d\sigma$$

선적분 ⟺ 면적분

ex 12.27) Stokes's Theorem

$F(x,y,z) = \langle -y, x, -xyz\rangle$

$\Sigma:\ z=\sqrt{x^2+y^2}\quad 0\le x^2+y^2\le 9$ 에서

bounding curve는 $x^2+y^2=9,\ z=3$ ⤳ $x=3\cos\theta,\ y=3\sin\theta,\ z=3.$

unit normal curve $n=\frac{1}{\sqrt2}\left(-\frac{x}{z}i-\frac{y}{z}j+k\right)$

$$\oint_C F\cdot dR = \int_C \langle -3\sin\theta, 3\cos\theta, -27\sin\theta\cos\theta\rangle\cdot\langle -3\sin\theta, 3\cos\theta, 0\rangle\, d\theta$$

$= 18\pi.$

$\nabla\times F = \langle -xz, yz, 2\rangle$

$$\therefore \iint_\Sigma \langle -xz, yz, 2\rangle\cdot\left\langle -\frac{1}{\sqrt2}\frac{x}{z}, -\frac{1}{\sqrt2}\frac{y}{z}, \frac{1}{\sqrt2}\right\rangle d\sigma$$

$$=\iint_\Sigma\left(\frac{1}{\sqrt2}x^2-\frac{1}{\sqrt2}y^2+\sqrt2\right)\sqrt2\, dx\,dy$$

$$=\iint_\Sigma (x^2-y^2+2)\, dx\,dy = \int_0^3\int_0^{2\pi}(r^3\cos2\theta+2r)\,d\theta\, dr$$

$$\int_0^3 \left(\frac{r^2}{2}\cancel{\sin 2\theta} + 2r\theta\right)_0^{2\pi} dr = \int_0^3 4\pi r \, dr$$

$$= 2\pi r^2 \Big)_0^3 = 18\pi.$$

같은값이 나왔다..ㅎ

$\boxed{2}$ Potential Theory in 3-space.

$\nabla \times F = 0$ in D $\Longleftrightarrow$ conservative F in D

(D: simply connected region in 3-space)

proof)

① $\nabla \times F = \nabla \times (\nabla \varphi) = 0.$

② if $\nabla \times F = 0 \ldots$

$$\varphi(x,y,z) = \int_{P_0}^{(x,y,z)} F \cdot dR$$

$$\oint F \cdot dR = \int_C F \cdot dR = \int_D F \cdot dR$$

$$= \iint_\Sigma (\nabla \times F) \cdot n \, d\sigma = 0 \quad \because \nabla \times F = 0.$$

$$\therefore \int_C F \cdot dR = \int_D F \cdot dR \quad \text{(conservative)}$$

12.10 Curvilinear Coordinates

[1] coordinates를 새로운 변수 $q_1 q_2 q_3$ 로 표현한다면

$$\begin{cases} x = x(q_1 q_2 q_3) \\ y = y(q_1 q_2 q_3) \\ z = z(q_1 q_2 q_3) \end{cases} \rightsquigarrow \begin{cases} q_1 = q_1(x\ y\ z) \\ q_2 = q_2(x\ y\ z) \\ q_3 = q_3(x\ y\ z) \end{cases}$$

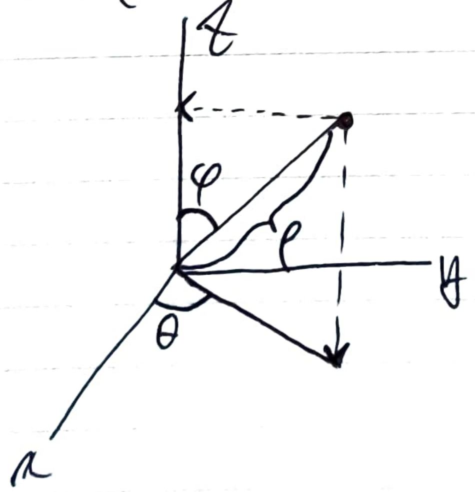
이렇게 표시가능.

① Cylinderical Coordinate

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases} \rightarrow \begin{cases} r = \sqrt{x^2+y^2} \\ \theta = \tan^{-1}(y/x) \\ z = z \end{cases}$$

② spherical coordinate

$$\begin{cases} x = \rho\cos\theta\sin\varphi \\ y = \rho\sin\theta\sin\varphi \\ z = \rho\cos\varphi \end{cases} \rightarrow \begin{cases} \rho = \sqrt{x^2+y^2+z^2} \\ \theta = \arcsin(y/\sqrt{x^2+y^2+z^2}) \\ \varphi = \arccos(y/\sqrt{x^2+y^2+z^2}) \end{cases}$$



② 서로 독립변수인가?

orthogonal curvilinear coordinate는 서로의 coordinate surface가 orthogonal 하다.

$$\begin{cases} r=\sqrt{x^2+y^2} \\ \theta=\arctan(y/x) \\ z=z \end{cases} \rightarrow \begin{matrix} \nabla r=\langle -2x/\sqrt{x^2+y^2}, -2y/\sqrt{x^2+y^2}, 0\rangle \\ \nabla\theta=\langle -y/x^2+y^2, x/x^2+y^2, 0\rangle \\ \nabla z=\langle 0,0,1\rangle \end{matrix}$$

여기서, $\nabla r\cdot\nabla\theta=\nabla r\cdot\nabla z=\nabla\theta\cdot\nabla z=0$.
위 coordinates는 orthogonal curvilinear coordinates.
즉, 서로 독립변수다.

③ curvilinear coordinate의 미소

cartesian에서 arc length는

$$(ds)^2=(dx)^2+(dy)^2+(dz)^2 \xleftarrow[\text{하면}]{\text{대입}} \begin{cases} dx=\frac{dx}{dq_1}dq_1+\frac{dx}{dq_2}dq_2+\frac{dx}{dq_3}dq_3 \\ dy=\frac{dy}{dq_1}dq_1+\frac{dy}{dq_2}dq_2+\frac{dy}{dq_3}dq_3 \\ dz=\frac{dz}{dq_1}dq_1+\frac{dz}{dq_2}dq_2+\frac{dz}{dq_3}dq_3 \end{cases} \text{나타냄}$$

그러므로 coordinate에서 arc length는

$$(ds)^2=\sum_{i=1}^{3}\sum_{j=1}^{3}h_{ij}^2\,dq_i\,dq_j \quad (h_{ij}=\text{scale factor})$$

만약 coordinates $q_1 q_2 q_3$가 orthogonal curvilinear coordinate면,

$$(ds)^2=(h_1dq_1)^2+(h_2dq_2)^2+(h_3dq_3)^2$$

$$\left.\begin{matrix} dx=ds_1=h_1dq_1 \\ dy=ds_2=h_2dq_2 \\ dz=ds_3=h_3dq_3 \end{matrix}\right\} \text{이다.}$$

$$h_1=\sqrt{\left(\frac{dx}{dq_1}\right)^2+\left(\frac{dy}{dq_1}\right)^2+\left(\frac{dz}{dq_1}\right)^2}$$
$$h_2=\sqrt{\left(\frac{dx}{dq_2}\right)^2+\left(\frac{dy}{dq_2}\right)^2+\left(\frac{dz}{dq_2}\right)^2}$$
$$h_3=\sqrt{\left(\frac{dx}{dq_3}\right)^2+\left(\frac{dy}{dq_3}\right)^2+\left(\frac{dz}{dq_3}\right)^2}$$

* Cylindrical Coordinates 에서는, $q_r = r$, $q_\theta = \theta$, $q_z = z$

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases} \qquad \begin{aligned} h_r &= \sqrt{\cos^2\theta + \sin^2\theta + 0^2} = 1 \\ h_\theta &= r \\ h_z &= 1 \end{aligned}$$

$$\therefore\ dx\,dy = ds_1\,ds_2 = h_r h_\theta\,dr\,d\theta = r\,dr\,d\theta$$

이는 $\left|\dfrac{\partial(x,y)}{\partial(r,\theta)}\right| = r$ 인 것과 일맥상통.

$$dx\,dy\,dz = r\,dr\,d\theta\,dz = \left|\frac{\partial(x,y,z)}{\partial(r,\theta,z)}\right| dr\,d\theta\,dz$$

* Spherical Coordinates.

$$\begin{cases} x = \rho\sin\varphi\cos\theta \\ y = \rho\sin\varphi\sin\theta \\ z = \rho\cos\varphi \end{cases} \qquad \begin{aligned} h_\rho &= 1 \\ h_\varphi &= \rho \\ h_\theta &= \rho\sin\varphi \end{aligned}$$

$$\therefore\ dx\,dy\,dz = \rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta = \left|\frac{\partial(x,y,z)}{\partial(\rho,\theta,\varphi)}\right| d\rho\,d\varphi\,d\theta$$

$$(ds)^2 = (d\rho)^2 + (\rho\sin\varphi)^2(d\theta)^2 + \rho^2(d\varphi)^2$$

4 Curvilinear Coordinate에서 Gradient / Laplacian / Divergence / Curl

① Gradient

scalar field ψ 에서 $\nabla\psi$의 성분중 q_1 방향은 $\frac{\partial\psi}{\partial s_1} = \frac{1}{h_1}\cdot\frac{\partial\psi}{\partial q_1}$

$$\therefore \nabla\psi(q_1 q_2 q_3) = \frac{1}{h_1}\frac{\partial\psi}{\partial q_1}u_1 + \frac{1}{h_2}\frac{\partial\psi}{\partial q_2}u_2 + \frac{1}{h_3}\frac{\partial\psi}{\partial q_3}u_3$$

② Divergence $F = \langle F_1, F_2, F_3 \rangle$

$$\nabla\cdot F(q_1 q_2 q_3) = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial q_1}(F_1 h_2 h_3) + \frac{\partial}{\partial q_2}(F_2 h_1 h_3) + \frac{\partial}{\partial q_3}(F_3 h_1 h_2)\right]$$

③ Curl

$$\nabla\times F = \begin{vmatrix} h_1 u_1 & h_2 u_2 & h_3 u_3 \\ \frac{\partial}{\partial q_1} & \frac{\partial}{\partial q_2} & \frac{\partial}{\partial q_3} \\ F_1 h_1 & F_2 h_2 & F_3 h_3 \end{vmatrix}$$

④ Laplacian (scalar field φ)

$$\Delta\varphi = \nabla^2\varphi = \nabla\cdot\nabla\varphi = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial q_1}\left(\frac{h_2 h_3}{h_1}\cdot\frac{\partial\varphi}{\partial q_1}\right) + \frac{\partial}{\partial q_2}\left(\frac{h_1 h_3}{h_2}\frac{\partial\varphi}{\partial q_2}\right) + \frac{\partial}{\partial q_3}\left(\frac{h_1 h_2}{h_3}\frac{\partial\varphi}{\partial q_3}\right)\right]$$

CH 13. Fourier Series

기본배경을 먼저 논한다.

1 Function Orthogonality

벡터처럼 함수에도 내적을 정의한다.

$[a,b]$ 에서 함수의 내적은

$$\langle f, g \rangle = \int_a^b w(x)\, f(x) \cdot g(x)\, dx$$

이때 weight function $w(x) = 1$로 두면

$\int_a^b f(x) \cdot g(x)\, dx$ 이 f와 g의 내적이다.

이 값이 0이면, f, g는 $[a,b]$에서 직교한다.

ex) $[-\pi, \pi]$ 에서 $\sin x, \sin 2x, \ldots$ 직교하나?

($m \neq n$ 가정)

sol) $$\int_{-\pi}^{\pi} \sin mx \sin nx\, dx = \int_{-\pi}^{\pi} \frac{1}{2}\left[\cos(m-n)x - \cos(m+n)x\right] dx$$

$$= \frac{1}{2}\left[\sin(m-n)x - \sin(m+n)x\right]_{-\pi}^{\pi} = 0.$$

※ $m = n$인 경우

$$\int_{-\pi}^{\pi} \sin^2 nx\, dx = \int_{-\pi}^{\pi} \frac{1}{2} - \frac{1}{2}\cos 2nx\, dx = \left[\frac{1}{2}x - \frac{1}{4n}\sin 2nx\right]_{-\pi}^{\pi}$$

$$= \pi$$

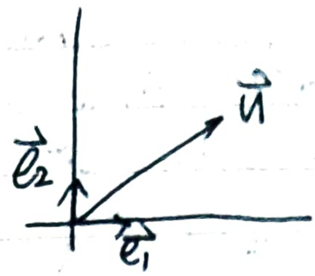
ex) $[-\pi, \pi]$ 에서 $\cos x, \cos 2x, \cos 3x \ldots$ 인 경우에도,

$m \neq n$: sol) $$\int_{-\pi}^{\pi} \cos mx \cos nx\, dx = \int_{-\pi}^{\pi} \frac{1}{2}\left[\cos(m+n)x + \cos(m-n)x\right] dx$$

$$= \frac{1}{2}\left[\frac{1}{m+n}\sin(m+n)x + \frac{1}{m-n}\sin(m-n)x\right]_{-\pi}^{\pi} = 0.$$

$m = n$: sol) $$\int_{-\pi}^{\pi} \cos^2 nx\, dx = \int_{-\pi}^{\pi} \frac{1}{2} + \frac{1}{2}\cos 2nx\, dx = \left.\frac{1}{2}x + \frac{1}{4n}\sin 2nx\right]_{-\pi}^{\pi} = \pi.$$

[2] Orthogonal Expansion of Function



* 벡터의 경우 그 직교전개는 ...

$\vec{u} = c_1\vec{e_1} + c_2\vec{e_2}$ 에서,

$\vec{e_1}\cdot\vec{e_2} = 0$ 을 이용해, $\vec{e_1}$을 양변 내적.

$\therefore \vec{u}\cdot\vec{e_1} = c_1\vec{e_1}\cdot\vec{e_1} \quad \therefore c_1 = \dfrac{\vec{u}\cdot\vec{e_1}}{\vec{e_1}\cdot\vec{e_1}}$

마찬가지로 $\quad c_2 = \dfrac{\vec{u}\cdot\vec{e_2}}{\vec{e_2}\cdot\vec{e_2}}$

$$\therefore \vec{u} = \frac{\vec{u}\cdot\vec{e_1}}{\vec{e_1}\cdot\vec{e_1}}\vec{e_1} + \frac{\vec{u}\cdot\vec{e_2}}{\vec{e_2}\cdot\vec{e_2}}\vec{e_2}$$

* 함수의 직교전개도 같은 motive로.

서로 orthogonal한 $y_0, y_1, y_2 \ldots$

$f(x) = a_0y_0(x) + a_1y_1(x) + a_2y_2(x) \cdots$ 로 나타낸다면.

$$a_0 = \frac{\langle f, y_0\rangle}{\langle y_0, y_0\rangle} \quad a_1 = \frac{\langle f, y_1\rangle}{\langle y_1, y_1\rangle} \quad a_2 = \frac{\langle f, y_2\rangle}{\langle y_2, y_2\rangle} \quad \cdots$$

즉, y_0, y_1, y_2 이 기저함수가 되는 것이다.

13.2 The Fourier Series of a Function.

$[-\pi,\pi]$에서,

$$\langle \sin mx, \sin nx \rangle = \int_{-\pi}^{\pi} \sin mx \sin nx \, dx = \begin{cases} 0 & (m \neq n) \\ \pi & (m = n) \end{cases}$$

$$\langle \cos mx, \cos nx \rangle = \int_{-\pi}^{\pi} \cos mx \cos nx \, dx = \begin{cases} 0 & (m \neq n) \\ \pi & (m = n) \end{cases}$$

인걸 염두한다.

$\boxed{1}$ 기본적인 접근. ($[-\pi,\pi]$의 주기함수 f 가정)

$\left.\begin{matrix} \cos x, \cos 2x, \cos 3x, \dots \\ \sin x, \sin 2x, \sin 3x, \dots \end{matrix}\right\}$ 서로 직교이므로, 직교기저를 사용한다.

$$f(x)\text{의 Fourier Series} = a_0 + \begin{matrix} a_1 \cos x + a_2 \cos 2x + \cdots \\ b_1 \sin x + b_2 \sin 2x + \cdots \end{matrix}$$

$$= a_0 + \sum_{n=1}^{\infty} \left[a_n \cos nx + b_n \sin nx \right]$$

여기서

$$\begin{cases} a_0 = \dfrac{\langle f, 1 \rangle}{\langle 1, 1 \rangle} = \dfrac{\int_{-\pi}^{\pi} f(x) \cdot 1 \, dx}{2\pi} = \dfrac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \, dx \\ a_n = \dfrac{\langle f, \cos nx \rangle}{\langle \cos nx, \cos nx \rangle} = \dfrac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx \\ b_n = \dfrac{\langle f, \sin nx \rangle}{\langle \sin nx, \sin nx \rangle} = \dfrac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx \end{cases}$$

이 Fourier Series Coefficient는 각 직교성분의 비중으로 본다.

$\boxed{2}$ 주기가 바뀌면...

f의 주기를 $2L$로 두면, 정의범위가 $[-L, L]$이 된다. 이때,

$$f의\ Fourier\ Series = a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right]$$

$$\begin{cases} a_0 = \dfrac{1}{2L}\displaystyle\int_{-L}^{L} f(x)\,dx \\ a_n = \dfrac{1}{L}\displaystyle\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx \\ b_n = \dfrac{1}{L}\displaystyle\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx \end{cases}$$

f가 $[-L, L]$에서 piecewise smooth

$-L < x < L$: $\frac{1}{2}[f(x+0) + f(x-0)]$

양끝점 : $\frac{1}{2}[f(L-0) + f(L+0)]$

∴ f가 연속이면

푸리에 급수 $= f(x)$ 성립

$\boxed{3}$ Even & Odd.

$$\begin{cases} even: f(-x) = f(x) & \displaystyle\int_{-L}^{L} f(x)\,dx = 2\int_{0}^{L} f(x)\,dx \\ odd: f(-x) = -f(x) & \displaystyle\int_{-L}^{L} f(x)\,dx = 0 \end{cases}$$

$$\begin{cases} even \cdot even = even \\ even \cdot odd = odd \\ odd \cdot odd = even \end{cases}$$

4 Fourier Cosine/Sine Series.

① Cosine Series

f는 $0 \le x \le L$, even function (e로 가정도 ok)

$\Rightarrow b_n = \int_{-L}^{L} f(x) \cdot \sin\frac{n\pi}{L}x\,dx = 0$. ($\because$ even · odd = odd)

f의 Fourier Cosine Series = $a_0 + \sum_{n=1}^{\infty} a_n \cdot \cos\frac{n\pi x}{L}$

$$\begin{cases} a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx \\ a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx = \frac{2}{L}\int_{0}^{L} f(x)\cos\frac{n\pi x}{L}\,dx \end{cases}$$

수렴성

$$\left[\begin{array}{l} 0<x<L : \frac{1}{2}[f(x+)+f(x-)] \\ x=0 : f(0+) \\ x=L : f(L-) \end{array}\right.$$

② Sine Series

f는 $0 \le x \le L$, odd function (e로 가정도 ok)

마찬가지로 $a_n = \int_{-L}^{L} f(x) \cdot \cos\frac{n\pi x}{L}\,dx = 0$ ($\because$ odd · even = odd), $a_0 = 0$

f의 Fourier Sine Series = $\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx = \frac{2}{L}\int_{0}^{L} f(x)\sin\frac{n\pi x}{L}\,dx$$

수렴성

$$\left[\begin{array}{l} 0<x<L = \frac{1}{2}[f(x+0)+f(x-0)] \\ \text{양끝} = 0 \end{array}\right.$$

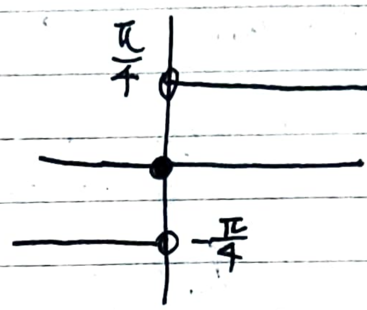
5 푸리에 급수 수렴정리와 Gibbs Phenomenon.

f : $[-L, L]$ continuous
f' : $[-L, L]$ piecewise continuous
$f(L) = f(-L)$

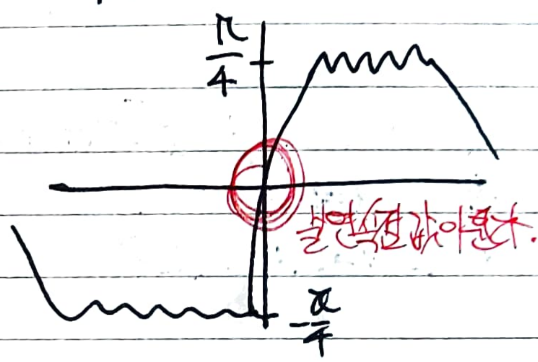
$\Bigg\} \rightarrow$ $[-L, L]$ $f(x) =$ Fourier Series of $f(x)$

이렇지 않은 경우, 불연속점, 불연속점에서 Fourier Series 값이 된다.

Ex)
$$f(x) = \begin{cases} -\pi/4 & (x<0) \\ 0 & (x=0) \\ \pi/4 & (x>0) \end{cases}$$



$$\text{Fourier Series} = \sum_{n=1}^{\infty} \frac{1}{2n-1} \sin(2n-1)x$$



Ex1) $f(x)=\begin{cases} 0 & (-2<x<-1) \\ k & (-1<x<1) \\ 0 & (1<x<2) \end{cases}$ 의 Fourier Series.

sol) $L=2$.

$a_0=\frac{1}{2\cdot 2}\int_{-2}^{2} f(x)dx = \frac{1}{4}\int_{-1}^{1} f(x)dx = \frac{k}{2}$

$a_n=\frac{1}{2}\int_{-1}^{1} k\cos\frac{n\pi}{2}x\,dx = \frac{2k}{n\pi}$

$b_n=\frac{1}{2}\int_{-1}^{1} k\sin\frac{n\pi x}{2}dx=0.$

$\therefore f(x)=\frac{k}{2}+\frac{2k}{\pi}\cos\frac{\pi}{2}x+\frac{k}{\pi}\cos\pi x+\frac{2k}{3\pi}\cos\frac{3\pi}{2}x+\cdots$

Ex2) $f(x)=e^{2x}$ $(0\le x\le 1)$ 에 대해 cosine series는?

sol) $a_0=\frac{1}{2}\cdot\int_{-1}^{1}e^{2x}dx=\frac{2}{2}\cdot\int_{0}^{1}e^{2x}dx=\frac{1}{2}(e^2-1)$ ($\because$ even 가정)

$a_n=2\cdot\int_0^1 e^{2x}\cos n\pi x\,dx=\frac{4}{4+(n\pi)^2}(e^2(-1)^n-1)$

$\therefore$ Fourier Series $=\frac{1}{2}(e^2-1)+\sum_{n=1}\frac{4}{4+(n\pi)^2}(e^2(-1)^n-1)\cos n\pi x$

Ex3) $f(x)=e^{2x}$ $(0\le x\le 1)$ 의 sine Series.

sol) $b_n=2\int_0^1 e^{2x}\sin n\pi x\,dx=\frac{2n\pi}{4+(n\pi)^2}(e^2(-1)^n-1)$

$\therefore$ Fourier Series $=\sum_{n=1}\frac{2n\pi}{4+(n\pi)^2}(e^2(-1)^n-1)\sin n\pi x$

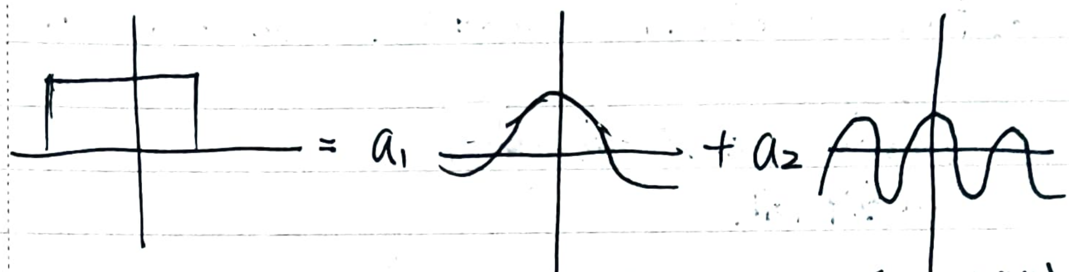
CH14. The Fourier Integral & Transforms.

14.1 The Fourier Integral

$\boxed{1}$ 기본 생각.

$L=\pi$ 에서 Fourier Series는

$$f(x) = a_0 + \begin{matrix} a_1\cos x \\ b_1 \sin x \end{matrix} + \begin{matrix} a_2\cos 2x \\ b_2 \sin 2x \end{matrix} + \cdots$$



〈W=1 인 성분〉 〈W=2 인 성분〉

여기서 W=0.5인 성분을 보려면 L=2π로 둔다.

$$f(x) = a_0 + \begin{matrix} a_1\cos 0.5x \\ b_1 \sin 0.5x \end{matrix} + \begin{matrix} a_2\cos x \\ b_2 \sin x \end{matrix} + \cdots$$

즉, $L\to\infty$ 하면 모든 주파수의 무게를 볼수있다.

$$f_L(x) = a_0 + \sum_{n=1}^{\infty}\left[\left(\frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx\right)\cos\frac{n\pi x}{L} + \left(\frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\right)\sin\frac{n\pi x}{L}\right]$$

에서 $W_n = \frac{n\pi}{L}$, $\Delta W = \frac{\pi}{L}$, $L\to\infty$, $\Delta W\to 0$ 이고, $\frac{1}{L} = \frac{\Delta W}{\pi}$ 대입하면

Fourier Integral of $f(x) = \int_0^{\infty}(A(\omega)\cos\omega x + B(\omega)\sin\omega x)\,d\omega$

$$A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(\varepsilon)\cos\omega\varepsilon\, d\varepsilon$$

$$B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(\varepsilon)\sin\omega\varepsilon\, d\varepsilon$$

※ f가 연속 → Fourier Integral 이 $f(x)$로 수렴.

일반적: $\int_{-\infty}^{\infty}|f(x)|dx$ 수렴. f가 piecewise smooth → $\frac{1}{2}[f(x+)+f(x-)]$ 수렴.

※ Fourier Integral은 $\frac{1}{\pi}\int_0^\infty \int_{-\infty}^{\infty} f(\varepsilon)\cos(\omega(\varepsilon - x))\,d\varepsilon\,d\omega$
로 표현 가능.

$\boxed{2}$ 푸리에 Cosine/Sine Integral. ($f(x)$ on $x \geq 0$에 대해)

① Fourier cosine integral
even 함수 가정.

Fourier Cosine Integral of $f(x) = \int_0^\infty A_\omega \cos \omega x\, d\omega$

$$A_\omega = \frac{2}{\pi}\int_0^\infty f(\varepsilon)\cos(\omega\varepsilon)\,d\varepsilon$$

※ $x=0$ 에서 $f(0+)$ 로 수렴
$x>0$ 에서 $[f(x+)+f(x-)]/2$

② Fourier sine integral
odd 가정

Fourier sine Integral of $f(x) = \int_0^\infty B_\omega \sin \omega x\, d\omega$

$$B_\omega = \frac{2}{\pi}\int_0^\infty f(\varepsilon)\sin(\omega\varepsilon)\,d\varepsilon$$

Ex) $f(x) = \begin{cases} 1 & |x| \le 1 \\ 0 & \end{cases}$ 를 푸리에 적분 표현하라.

sol) $A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\cos\omega\varepsilon\, d\varepsilon = \frac{1}{\pi}\int_{-1}^{1} \cos\omega\varepsilon\, d\varepsilon = \frac{2\sin\omega}{\omega\pi}$

$B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} \sin\omega\varepsilon\, d\varepsilon = 0.$

$\therefore f(x) = \int_{0}^{\infty} \frac{2\sin\omega}{\omega\pi}\cos\omega x\, d\omega$

여기에서, $A(\omega)$에 $\omega = 0.5$ 대입하면, $\cos 0.5x$의 성분의 비중이 나올 것이다.

14.3 The Fourier Transform

$\boxed{1}$ 기본 공식.

오일러공식 $\left.\begin{aligned} e^{i\theta} &= \cos\theta + i\sin\theta \\ e^{-i\theta} &= \cos\theta - i\sin\theta \end{aligned}\right\} \rightarrow \begin{aligned} \cos\theta &= \frac{1}{2}(e^{i\theta}+e^{-i\theta}) \\ \sin\theta &= \frac{1}{2i}(e^{i\theta}-e^{-i\theta}) \end{aligned}$

푸리에적분에 대입하면

$$f(x)=\int_{-\infty}^{\infty} C(\omega)e^{i\omega x}\,d\omega,\quad C(\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{-i\omega x}\,dx$$

로 본다..

이걸 "변환"으로 보고, 시간 t 영역에서 주파수 ω 영역으로 보낸다 하면,
$C(\omega)$를 $F(\omega)$로 보면, (혹은 $\hat{f}(\omega) = \mathcal{F}[f](\omega)$)

$$f(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{f}(\omega)e^{i\omega t}\,d\omega,\quad \hat{f}(\omega)=\int_{-\infty}^{\infty} f(t)e^{-i\omega t}\,dt$$

<Complex Fourier Integral representation><Fourier Transform>

푸리에 변환은 한 주파수의 비중인 "계수"에만 치중한 것이다.

$\boxed{2}$ 푸리에 변환 / 역 푸리에 변환

$$\mathcal{F}[f](\omega)=\hat{f}(\omega)=\int_{-\infty}^{\infty} f(t)e^{-i\omega t}\,dt$$

$$\mathcal{F}^{-1}[\hat{f}](\omega)=f(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{f}(\omega)e^{i\omega t}\,d\omega$$

3 Cosine Transform / Sine Transform

① $\hat{f}_c(\omega) = \int_0^{\infty} f(t)\cos\omega t \, dt$

$\hat{f}_c(\omega) = \frac{\pi}{2} a_\omega$ (a_ω는 푸리에 적분에서 $\cos\omega t$의 계수)

$f(t) = \frac{2}{\pi}\int_0^{\infty} \hat{f}_c(\omega)\cos\omega t \, d\omega$

② $\hat{f}_s(\omega) = \int_0^{\infty} f(t)\sin\omega t \, dt$

$\hat{f}_s(\omega) = \frac{\pi}{2} b_\omega$ (b_ω는 푸리에 적분에서 $\sin\omega t$의 계수)

$f(t) = \frac{2}{\pi}\int_0^{\infty} \hat{f}_s(\omega)\sin\omega t \, d\omega$

4 Cosine/Sine Transform의 이계도함수 변환

① $\mathcal{F}_c[f''(t)](\omega) = -\omega^2 \hat{f}_c(\omega) - f'(0)$

② $\mathcal{F}_s[f''(t)](\omega) = -\omega^2 \hat{f}_s(\omega) + \omega f(0)$

5 Dirac Impulse Function (Delta Function)

① $\mathcal{F}[\delta(t)](\omega) = 1$

② $\delta * f = f * \delta = f$ (Fourier Convolution)

6 Properties of Fourier Transform.

라플라스 변환과 비슷하다.

① $\mathcal{F}[f^{(n)}(t)](\omega) = (i\omega)^n \hat{f}(\omega)$

$d^n\hat{f}(\omega)/d\omega^n = i^n \mathcal{F}[t^n f(t)](\omega)$

↓ $\mathcal{F}[f'(t)] = i\omega \hat{f}(\omega)$

② $\mathcal{F}[\int_0^t f(t)dt](\omega) = \hat{f}(\omega)/i\omega$

③ Convolution

↓ time convolution

$\mathcal{F}[f*g](\omega) = \hat{f}(\omega)\hat{g}(\omega)$

$\mathcal{F}^{-1}[\hat{f}(\omega)\hat{g}(\omega)](t) = (f*g)(t)$

2 frequency convolution

$\mathcal{F}[f\cdot g](\omega) = \frac{1}{2\pi}(\hat{f}*\hat{g})(\omega)$

④ Shifting

① time shifting

$\mathcal{F}[f(t-t_0)](\omega) = e^{-i\omega t_0}\hat{f}(\omega)$

$\mathcal{F}^{-1}[e^{-i\omega t_0}\hat{f}(\omega)](t) = f(t-t_0)$

② frequency shifting

$\mathcal{F}[e^{i\omega_0 t}f(t)] = \hat{f}(\omega-\omega_0)$

$\mathcal{F}^{-1}[\hat{f}(\omega-\omega_0)](t) = e^{i\omega_0 t}f(t)$

⑤ 기타...

$$\begin{cases} \mathcal{F}[f(ct)](\omega) = \frac{1}{|c|}\hat{f}\left(\frac{\omega}{c}\right) \\ \mathcal{F}^{-1}\left[\hat{f}\left(\frac{\omega}{c}\right)\right](t) = |c|\, f(ct) \end{cases}$$

$$\begin{cases} \mathcal{F}[f(-t)](\omega) = \hat{f}(-\omega) \\ \mathcal{F}[\hat{f}(t)](\omega) = 2\pi f(-\omega) \end{cases}$$

$$\begin{cases} \mathcal{F}[f(t)\cos\omega_0 t](\omega) = \frac{1}{2}\left(\hat{f}(\omega+\omega_0) + \hat{f}(\omega-\omega_0)\right) \\ \mathcal{F}[f(t)\sin\omega_0 t](\omega) = \frac{i}{2}\left(\hat{f}(\omega+\omega_0) - \hat{f}(\omega-\omega_0)\right) \end{cases}$$

Ex1) $f(x) = \begin{cases} \frac{1}{2} & (-1 \le x \le 1) \\ 0 & (|x|>1) \end{cases}$ 의 푸리에 변환.

$$\hat{f}(\omega) = \int_{-1}^{1} \frac{1}{2} e^{-i\omega x} dx = -\frac{1}{2i\omega}\left[e^{-i\omega x}\right]_{-1}^{1} = \frac{1}{2\omega i}(e^{i\omega} - e^{-i\omega})$$

$$= \frac{\sin\omega}{\omega} \quad (\because \sin\omega = \frac{1}{2i}(e^{i\omega} - e^{-i\omega}))$$

Ex2) $f(x) = \begin{cases} x & (0<x<a) \\ 0 & (\text{그 외}) \end{cases}$

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx = \int_{0}^{a} x e^{-i\omega x} dx$$

$$= \frac{(1+ia\omega)e^{-ia\omega} - 1}{\omega^2}$$

Ex3) Cosine Transform.

$f(t) = \begin{cases} 1 & (0 \le t \le K) \\ 0 & (t>K) \end{cases}$

$$\hat{f}_c(\omega) = \int_{0}^{\infty} f(t)\cos\omega t\, dt = \int_{0}^{K} \cos\omega t\, dt = \frac{\sin\omega K}{\omega}$$

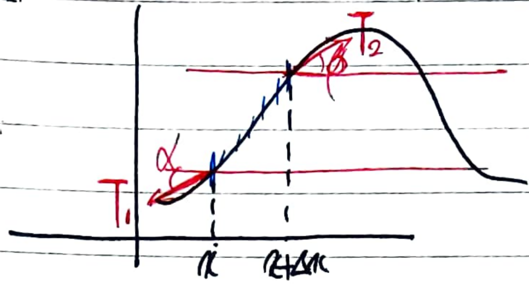
Ex4) Sine Transform

$f(t) = \begin{cases} 1 & (0 \le t \le K) \\ 0 & (t>K) \end{cases}$

$$\hat{f}_s(\omega) = \int_{0}^{K} \sin\omega t\, dt = \frac{1-\cos\omega K}{\omega}$$

16.1 Derivation of Wave Equation.

$\boxed{1}$ 유도. $u_{tt} = c^2 u_{xx}$ 구하기.



wave는 x방향 motion x.
장력 T_1, T_2 존재.
ρ는 선밀도.

ΣF_x : $T_1\cos\alpha - T_2\cos\beta = 0$. → $T = T_1\cos\alpha = T_2\cos\beta$. …①

ΣF_y : $T_2\sin\beta - T_1\sin\alpha = \rho \cdot \Delta x \dfrac{\partial^2 u}{\partial t^2}$ …②

①로 ②를 나눈다.

$$\frac{\tan\beta - \tan\alpha}{\Delta x} = \frac{\rho}{T} \cdot \frac{\partial^2 u}{\partial t^2}$$ 여기서 $\tan\alpha = \left.\dfrac{\partial u}{\partial x}\right)_x$, $\tan\beta = \left.\dfrac{\partial u}{\partial x}\right)_{x+\Delta x}$

불식은 $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\rho}{T}\dfrac{\partial^2 u}{\partial t^2}$ → $c^2 = \dfrac{\rho}{T}$ → $u_{tt} = c^2 u_{xx}$

$\boxed{2}$ 기타 종류.

2차원에서는 $u_{tt} = c^2(u_{xx} + u_{yy})$

polar coordinate 에서 $u_{tt} = c^2\left(u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}\right)$

Forcing term 추가시 $u_{tt} = c^2 u_{xx} + F(x,t)$

16.2 Wave Motion on an Interval

Ⅲ 풀이법...

$$u_{tt} = c^2 u_{xx}$$

(시간) 초기조건 initial condition 2개 : $u(x,0) = f(x)$; $\overset{\text{속도}}{u_t}(x,0) = g(x)$

(경계) 경계조건 boundary condition 2개 : $u(0,t) = 0$, $u(L,t) = 0$

sol)

Fourier Method (method of separation of variables)

$u(x,t) = F(x) \cdot G(t)$ 가정. 대입하면...

$\frac{1}{c^2 G} \cdot G_{tt} = \frac{1}{F} F_{xx} = -k$ 로 정리하고, 새 상수 k를 도입.

$$\begin{cases} F_{xx} + kF = 0 \quad \cdots ① \\ G_{tt} + c^2 k G = 0 \quad \cdots ② \end{cases}$$

이 방정식 상미분방정식 풀이와 다를게 없다.

① 식에서...

i) $k < 0$ 이면 $F = c_1 e^{\sqrt{-k}x} + c_2 e^{-\sqrt{-k}x}$

$F(0) = c_1 + c_2 = 0$

$F(L) = c_1 e^{\sqrt{-k}L} + c_2 e^{-\sqrt{-k}L} = c_1 (e^{\sqrt{-k}L} - e^{-\sqrt{-k}L}) = 0$.

$c_1 = c_2 = 0$: trivial solution $\rightarrow k \nless 0$

ii) $k = 0$. $F = (c_1 + c_2 x)e^0 = c_1 + c_2 x$

$F(0) = c_1 = 0$

$F(L) = c_2 L = 0$. $\therefore c_1 = c_2 = 0$: $k \neq 0$.

iii) $k > 0$ $F = c_1 \cos\sqrt{k}x + c_2 \sin\sqrt{k}x \xrightarrow{k = p^2} c_1 \cos px + c_2 \sin px$

$F(0) = c_1 = 0$

$F(L) = c_2 \sin pL = 0$.

$c_2 \neq 0$ 이어야 nontrivial $\therefore pL = n\pi$, $p = \frac{n\pi}{L}$.

근이 여러 n에서 나오므로, 근이 여러개고, linear 한 equation 이므로 그 합도 근이다. $F_n(x) = \sin \frac{n\pi}{L} x$.

② 식에서 $k=-p^2$ 대입시...

$$\frac{d^2G}{dt^2} + c^2p^2G = 0.$$

① 식에서와 같은 방법으로.

$$G_n(t) = A_n\cos cpt + B_n\sin cpt$$

$$= A_n\cos\frac{cn\pi}{L}t + B_n\sin\frac{cn\pi}{L}t$$

$$\therefore u_n(x,t) = \left(A_n\cos\frac{cn\pi}{L}t + B_n\sin\frac{cn\pi}{L}t\right)\sin\frac{n\pi}{L}x$$

① $$\therefore u(x,t) = \sum_{n=1}^{\infty}\left(A_n\cos\frac{cn\pi}{L}t + B_n\sin\frac{cn\pi}{L}t\right)\sin\frac{n\pi}{L}x$$

여기서 initial condition에 의해.

$$\left.\begin{aligned} u(x,0) &= \sum_{n=1} A_n\sin\frac{n\pi}{L}x = f(x) \\ u_t(x,0) &= \sum_{n=1}\frac{cn\pi}{L}B_n\sin\frac{n\pi}{L}x = g(x) \end{aligned}\right\}$$

이걸 푸리에 사인 급수라고 놓으면...

② $$A_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi}{L}x\,dx = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi}{L}x\,dx$$

③ $$B_n = \frac{L}{cn\pi}\times\frac{1}{L}\int_{-L}^{L} g(x)\sin\frac{n\pi}{L}x\,dx = \frac{L}{cn\pi}\times\frac{2}{L}\int_0^L g(x)\sin\frac{n\pi}{L}x\,dx$$

즉, $f(x)=0$: $u(x,t) = \sum_{n=1} B_n\sin\frac{cn\pi}{L}t\,\sin\frac{n\pi}{L}x$

$g(x)=0$: $u(x,t) = \sum_{n=1} A_n\cos\frac{cn\pi}{L}t\,\sin\frac{n\pi}{L}x$

Ex 1)

- $f(x) = \begin{cases} x & (0 \le x \le L/2) \\ L-x & (L/2 \le x \le L) \end{cases}$

- $g(x) = x(1+\cos(\pi x/L))$

- $u_{tt} = c^2 u_{xx}$

$u(0,t) = u(L,t) = 0$

$u(x,0) = f(x)$, $u_t(x,0) = g(x)$, $L = \pi$, $c = 1$.

$$u(x,t) = \sum_{n=1} \left(A_n \cos\frac{cn\pi}{L}t + B_n \sin\frac{cn\pi}{L}t \right) \sin\frac{n\pi}{L}x$$

$$A_n = \frac{2}{L}\int_0^L f(\varepsilon)\sin\frac{n\pi}{L}\varepsilon \, d\varepsilon \Rightarrow \frac{4\sin(n\pi/2)}{\pi n^2}$$

$$B_1 = \frac{3}{2}, \quad B_n = \frac{2(-1)^n}{n^2(n^2-1)} \quad (n \neq 2)$$

Ex2) Forcing Term..

$$\begin{cases} y_{tt} = y_{xx} + Ax \\ y(0,t) = y(L,t) = 0 \\ y(x,0) = 0,\ y_t(x,0) = 1 \end{cases}$$

sol)

$y(x,t) = Y(x,t) + \psi(x)$ 로 두고 대입하면

$Y_{tt} = Y_{xx} + \psi''(x) + Ax$

이걸 $\psi''(x) + Ax = 0$ 인 $\psi(x)$를 지정. Y에 대한 initial-boundary problem으로 바꾼다.

$\psi'(x) = -\frac{1}{2}Ax^2 + B$, $\psi(x) = -\frac{1}{6}Ax^3 + Bx + C$

① boundary condition.

$$\left.\begin{aligned} y(0,t) &= Y(0,t) + \psi(0) = 0 \\ y(L,t) &= Y(L,t) + \psi(L) = 0 \end{aligned}\right\}$$ $Y(0,t) = Y(L,t) = 0$ 인 ψ를 고른다.

$\psi(0) = 0 = C$

$\psi(L) = 0$

$\therefore \psi(x) = \frac{1}{6}Ax(L^2 - x^2)$

② Initial condition.

$$\left.\begin{aligned} y(x,0) &= Y(x,0) + \psi(x) = 0 \\ y_t(x,0) &= Y_t(x,0) = 1 \end{aligned}\right\} \Rightarrow \begin{aligned} Y(x,0) &= -\psi(x) = \tfrac{1}{6}Ax(x^2 - L^2) \\ Y_t(x,0) &= 1 \end{aligned}$$

③ 새로운 문제

$$\left.\begin{aligned} & Y_{tt} = Y_{xx} \\ & Y(0,t) = Y(L,t) = 0 \\ & Y(x,0) = \tfrac{1}{6}Ax(x^2 - L^2) \quad Y_t(x,0) = 1 \end{aligned}\right\} \Rightarrow$$ 새로운 Initial-Boundary problem.

$Y(x,t) = \sum_{n=1}^{\infty}\left(A_n \cos\frac{n\pi}{L}t + B_n \sin\frac{n\pi}{L}t\right)\sin\frac{n\pi}{L}x$

$A_n = \frac{1}{L}\int_{-L}^{L} \frac{1}{6}Ax(x^2 - L^2)\cos\frac{n\pi}{L}x\,dx$

$B_n = \frac{1}{L}\int_{-L}^{L} 1 \cdot \sin\frac{n\pi}{L}x\,dx$

$y(x,t) = Y(x,t) + \frac{1}{6}Ax(L^2 - x^2)$

$= Y(x,t) + \psi(x)$

17.1 Heat Equation.

1 Initial. Boundary condition.

$$\begin{cases} u_t = k\,u_{xx} \quad \text{(열전도 방정식)} \\ \text{B.C}: \; u(0,t) = u(L,t) = 0 \\ \text{I.C}: \; u(x,0) = f(x) \end{cases}$$

이때 두 끝단이 모두 0인 조건을 제차방향.
아닌 조건을 비제차방향이라 한다.
제차방향을 먼저 정리하는 게 간단하다.

2 solution 유도. - 양끝단 점은 0 상태.

$u(x,t) = F(x)\,G(t)$ 가정.

$$\frac{1}{F}F_{xx} = \frac{1}{kG}G_{tt} = -\lambda = -p^2 \quad \therefore \begin{cases} F'' + \lambda F = 0 & \cdots ① \\ G' + k\lambda G = 0 & \cdots ② \end{cases}$$

①식에서

$F(x) = A\cos px + B\sin px$

$F(0) = 0 = A$

$F(L) = 0 = B\sin pL \quad \therefore \; p = n\pi/L$

$\Rightarrow F_n(x) = \sin\frac{n\pi}{L}x$

②식에서 linear 방식으로 풀면

$G_n(t) = Ce^{-k\lambda^2 t} = Ce^{-k\left(\frac{n\pi}{L}\right)^2 t}$ (C는 상수)

$$\therefore\ u(x,t) = \sum_{n=1} C_n e^{-k\left(\frac{n\pi}{L}\right)^2 t} \sin\frac{n\pi}{L}x$$

$$C_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi}{L}x\,dx$$

$$= \frac{2}{L}\int_{0}^{L} f(x)\sin\frac{n\pi}{L}x\,dx$$

$\because$ $u(x,0) = \sum C_n \sin\frac{n\pi}{L}x = f(x)$ 푸리에 급수로 볼 수 있으므로

$$C_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi}{L}x\,dx$$

[3] solution 유도 - Insulated Ends.

$$\begin{cases} u_t = k u_{xx} \\ u_x(0,t) = u_x(L,t) = 0 \\ u(x,0) = f(x) \end{cases}$$

$u(x,t) = F(x)G(t)$ 대입

$$\begin{cases} F'' + \lambda F = 0 \\ G' + k\lambda G = 0 \end{cases} \qquad (\lambda = p^2)$$

이때 $u_x = F'(x)G(t)$.

$\therefore F'(0) = 0,\ F'(L) = 0.$

이걸 $F(x) = c_1 \cos px + c_2 \sin px$ 에 대입하면,

$$F_n(x) = \cos \frac{n\pi}{L} x$$

$G(t)$는 전과 같은 방법으로 $G_n(t) = e^{-k\left(\frac{n\pi}{L}\right)^2 t}$

$\therefore u_n(x,t) = F_n G_n = e^{-k\left(\frac{n\pi}{L}\right)^2 t} \cos \frac{n\pi}{L} x$

$$u(x,t) = c_0 + \sum_{n=1}^{\infty} c_n e^{-k\left(\frac{n\pi}{L}\right)^2 t} \cos \frac{n\pi}{L} x.$$

여기서 $c_0 = \frac{1}{2} \cdot \frac{1}{L} \int_{-L}^{L} f(x)\,dx = \frac{1}{L} \int_0^L f(x)\,dx$

$$c_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi}{L} x\,dx = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi}{L} x\,dx$$

Steady Temperature

Ex 1) $\begin{cases} u_t = k u_{xx} \\ u(0,t) = 0 = u(L,t) \\ u(x,0) = f(x) = A \end{cases}$

$$C_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi}{L}x\,dx = \frac{2A}{n\pi}\{1-(-1)^n\}.$$

$$u(x,t) = \sum C_n e^{-k\left(\frac{n\pi}{L}\right)^2 t}\sin\frac{n\pi}{L}x$$

Insulation

Ex2) $u_t = k u_{xx}$

$u_x(0,t) = u_x(L,t) = 0$

$u(x,0) = f(x) = \begin{cases} A & 0 \le x \le L/2 \\ 0 & L/2 \le x \le L \end{cases}$

$$C_0 = \frac{1}{2}\cdot\frac{2}{L}\int_0^{\frac{L}{2}} A\,d\xi = \frac{A}{2}$$

$$C_n = \frac{2}{L}\int_0^{\frac{L}{2}} A\cos\frac{n\pi}{L}\xi\,d\xi = \frac{2A}{n\pi}\sin\frac{n\pi}{2}$$

$$\therefore u(x,t) = \frac{A}{2} + \sum \frac{2A}{n\pi}\sin\frac{n\pi}{2}e^{-k\left(\frac{n\pi}{L}\right)^2 t}\cos\frac{n\pi}{L}x$$

CH18. Laplace Equation.

1 개요.

Laplace Equation = Potential Equation

$$\nabla^2 u = u_{xx} + u_{yy} = 0$$

시간에 대한 조건없이
한 시점에서의 현황만 볼수 있다.

※ polar coordinate 에서는 $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ x^2+y^2=r^2 \end{cases}$ 대입

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0 \quad (\because \text{chain rule에 의해 } u_x = u_r r_x + u_\theta \theta_x \cdots)$$

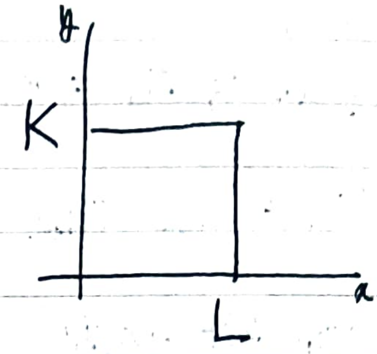
$u = R(r)\,\Theta(\theta)$ 를 대입하,

$$\frac{r^2}{R}R'' + \frac{r}{R}R' = -\frac{1}{\Theta}\Theta'' = -\lambda$$

$\left.\begin{cases} r^2R'' + rR' + \lambda R = 0 \\ \Theta'' - \lambda\Theta = 0 \end{cases}\right\}$ 로 놓고 푼다.

$\boxed{2}$ 직사각형 region 에서의 solution

$$\begin{cases} u_{xx}+u_{yy}=0 \\ u(0,y)=0 \quad u(L,y)=0 \\ u(x,0)=0 \quad u(x,K)=f(x) \end{cases}$$



$u(x,y)=F(x)G(y)$ 가정. 대입

$$\frac{1}{F}F''=-\frac{1}{G}G''=-\lambda=-p^2$$

$$\begin{cases} F''+p^2F=0 & \cdots ① \\ G''-p^2G=0 & \cdots ② \end{cases}$$

①식에서는 그전과 같이

$p=\frac{n\pi}{L}$, $F_n(x)=\sin\frac{n\pi}{L}x$

②식에서는

$G(y)=C_1e^{py}+C_2e^{-py}$

$G(0)=C_1+C_2=0$ $\therefore$ $G(y)=C_1(e^{py}+e^{-py})$

$=2C_1\sinh py=2C_1\sinh\frac{n\pi}{L}y$

$\therefore u_n=\sin\frac{n\pi}{L}x\sinh\frac{n\pi}{L}y$.

$u(x,y)=\sum C_n\sin\frac{n\pi}{L}x\sinh\frac{n\pi}{L}y$ $\quad$ $u(x,K)=f(x)$에 대입하고, 푸리에 사인급수로 보면.

$$C_n=\frac{1}{L\cdot\sinh\frac{n\pi K}{L}}\int_{-L}^{L}f(x)\sin\frac{n\pi}{L}x\,dx$$

$$=\frac{2}{L\cdot\sinh\frac{n\pi K}{L}}\int_{0}^{L}f(x)\sin\frac{n\pi}{L}x\,dx \quad (f(x)\text{ odd 가정})$$

Ex1)

$$\begin{cases} u_{xx} + u_{yy} = 0 \\ u(x,0) = 0 \quad u(x,\pi) = f(x) \\ u(0,y) = 0, \ u(\pi,y) = 0 \end{cases}$$

$L = K = \pi$, $f(x) = x(\pi - x)$ 풀기.

sol)

$$u(x,y) = \sum C_n \sin nx \sinh ny$$

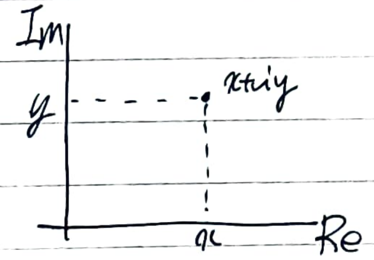
$$C_n = \frac{2}{\pi \sinh n\pi} \int_0^L f(\varepsilon) \sin n\varepsilon \, d\varepsilon$$

CH19. Complex Numbers & Functions

19.1 Complex Numbers – Geometry & Arithmetic.

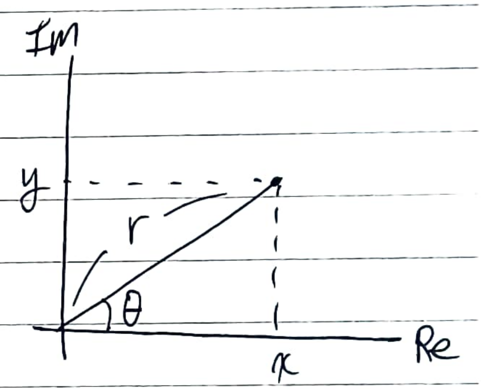
1 복소수 표현

① $z = x + iy$ $\begin{cases} \text{실수부}(Re) : x \\ \text{허수부}(Im) : y \end{cases}$



② 복소수 극형식. polar form.

$$z = x + iy = r(\cos\theta + i\sin\theta) = re^{i\theta}$$



여기서,

r : 절대값 $= |z| = \sqrt{x^2+y^2} = r > 0.$

θ : 편각, argument. $= \arg(z) = \theta$

$-\pi < \theta < \pi$ 로 한정하면 $\mathrm{Arg}(z)$로 표현한다.

2 복소수 연산 법칙.

$\begin{cases} z_1 = r_1(\cos\theta_1 + i\sin\theta_1) \\ z_2 = r_2(\cos\theta_2 + i\sin\theta_2) \end{cases}$

① 곱.

$$z_1 z_2 = r_1 r_2[\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)]$$
$$= r_1 r_2(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2))$$
$$= r_1 r_2 e^{i(\theta_1+\theta_2)}$$

② 나눗셈

$$z_1/z_2 = \frac{r_1(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 - i\sin\theta_2)}{r_2(\cos\theta_2 + i\sin\theta_2)(\cos\theta_2 - i\sin\theta_2)}$$
$$= \frac{r_1}{r_2}[\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)] = \frac{r_1}{r_2}e^{i(\theta_1-\theta_2)}$$

③ $z^n = \overbrace{z \cdot z \cdots z}^{n} = r^n(\cos n\theta + i\sin n\theta)$

ex1) $x^8=1$ 일때, x는 복소수다. x를 구하라.

1을 polar form으로 표현.

$$1 = 1\cdot(\cos 2\pi n + i\sin 2\pi n)$$
$$= 1 + i\cdot 0$$

$x = r\cdot(\cos\theta + i\sin\theta)$로 두면, $x^8 = r^8(\cos 8\theta + i\sin 8\theta)$

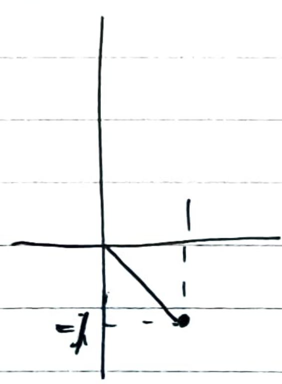
$\therefore\ r^8 = 1 \rightarrow r = 1.$

$\theta = 2\pi n/8 = \pi n/4.$

해가 8개이므로, $n=0, 1, \ldots 7$ 대입해서,

$\begin{cases} r=1 \\ \theta=0 \end{cases}$ $\begin{cases} r=1 \\ \theta=\frac{\pi}{4} \end{cases}$ $\cdots$ $\begin{cases} r=1 \\ \theta=\frac{7\pi}{4} \end{cases}$

ex2) $1-i = x^3$ 에서 x는?



$$1-i = \sqrt{2}\left(\cos\left(-\frac{\pi}{4}+2\pi n\right) + i\sin\left(-\frac{\pi}{4}+2\pi n\right)\right)$$
$$x^3 = r^3(\cos 3\theta + i\sin 3\theta)$$

$r^3 = \sqrt{2}$, $\therefore\ r = \sqrt[6]{2}$

$\theta = -\frac{\pi}{12} + \frac{2\pi n}{3}$

$n=0, 1, 2$ 대입. 근이 세개니까.

$\begin{cases} r=\sqrt[6]{2} \\ \theta=-\frac{\pi}{12} \end{cases}$ $\begin{cases} r=\sqrt[6]{2} \\ \theta=\frac{7\pi}{12} \end{cases}$ $\begin{cases} \sqrt[6]{2} \\ \frac{15\pi}{12} \end{cases}$

ex3) $\sqrt{i} = x$ 의 x를 구하라.

sol) $i = x^2$ 에서,

$i = 1 \cdot (\cos(\frac{\pi}{2}+2n\pi) + i\sin(\frac{\pi}{2}+2n\pi))$

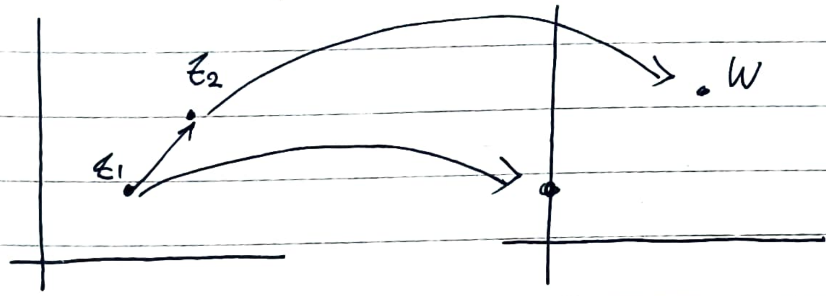
$x^2 = r^2 e^{i2\theta} = r^2(\cos 2\theta + i\sin 2\theta)$

$\therefore r = 1, \ \theta = \frac{\pi}{4}+n\pi$

$(r, \theta) = (1, \frac{\pi}{4}), (1, \frac{5}{4}\pi)$

19.2 Complex Functions.

$\boxed{1}$ 복소함수의 대응관계.

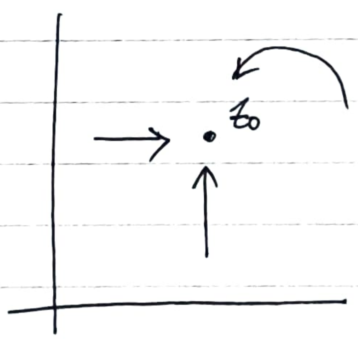


이렇게 두서없이... Mapping 된다.

$f(z) = z^2 = (x+iy)^2 = x^2 - y^2 + i \cdot 2xy$ 에서.

실수부 $U = x^2 - y^2$, 허수부 $V = 2xy$ 가 나온다.

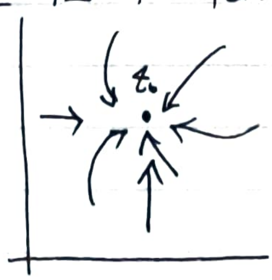
$\boxed{2}$ 복소함수의 연속.



어떤방향으로든, $f(z)$가 같은 값으로 수렴하면, 연속이다.

$$\lim_{z \to z_0} f(z) = L.$$

$\boxed{3}$ 복소함수 미분.



모든 방향으로, 미분값이 같은 값으로 수렴하면 그 함수는 그 지점에서 미분가능하다.

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

$$= \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\Delta u + i \Delta v}{\Delta x + i \Delta y}$$

$(f(z) = w = u + iv,\ z = x + iy)$

$\boxed{4}$ Cauchy-Rieman Equation.

미분값이 존재함을 보이려면, 무수히 많은 경로를 확인해야 한다.
But, Cauchy-Rieman Equation을 통해 수직/수평 두방향만 확인해서 미분가능을 보일 수 있다.

1) 수평 $\rightarrow z_0 \leftarrow$, $\Delta y=0$.

$$\lim_{\Delta x\to 0}\frac{\Delta u+i\Delta v}{\Delta x}=u_x+iv_x.$$

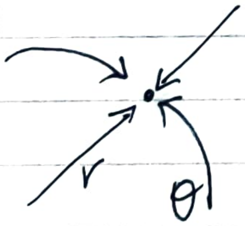
2) 수직 z_0 , $\Delta x=0$

$$\lim_{\Delta y\to 0}\frac{\Delta u+i\Delta v}{i\Delta y}=v_y+i(-u_y)$$

종합하면, $f(z)$가 미분가능하려면,

$$\begin{cases} u_x=v_y \\ v_x=-u_y \end{cases}$$ 가 성립해야 한다.

※ 극형식의 경우, $\Delta z=\Delta r e^{i\theta}+ire^{i\theta}\Delta\theta$ 로 두면,



1) r 방향 : $\lim_{\Delta r\to 0}\frac{\Delta u+i\Delta v}{\Delta r e^{i\theta}}=e^{-i\theta}(u_r+iv_r)$

2) θ방향 : $\lim_{\Delta\theta\to 0}\frac{\Delta u+i\Delta v}{ire^{i\theta}\Delta\theta}=\frac{1}{r}e^{-i\theta}(v_\theta+i(-u_\theta))$

$$\begin{cases} u_r=\frac{1}{r}v_\theta \\ v_r=-\frac{1}{r}u_\theta \end{cases}$$

$\boxed{5}$. $f(z) = u + iv$ 해석적 (미분가능) $\Leftrightarrow$ u, v 가 harmonic.

$$u_{xx} + u_{yy} = 0$$
$$v_{xx} + v_{yy} = 0$$

을 만족하는 두함수 u, v를 조화함수라고 한다.

이때 u, v를 서로 공액조화함수(Harmonic Conjugate)라고 부른다.

위 정리에 의해 조화함수 u를 알면 Cauchy-Rieman 정리로 공액조화함수 v를 구할수 있다.

ex1) $f(z)=\frac{1}{z}$ 는 미분가능한가?

sol) $f(z)=\frac{1}{x+iy}=\frac{x}{x^2+y^2}-i\frac{y}{x^2+y^2}$

$= u+iv$

$u_x=\frac{x^2+y^2-2x^2}{(x^2+y^2)^2}=\frac{-x^2+y^2}{(x^2+y^2)^2}$

$v_y=-\frac{x^2+y^2-2y^2}{(x^2+y^2)^2}=\frac{-x^2+y^2}{(x^2+y^2)^2}$

$\therefore u_x=v_y$.

$v_x=-\frac{-2xy}{(x^2+y^2)^2}=\frac{2xy}{(x^2+y^2)^2}$

$-u_y=-\frac{-2yx}{(x^2+y^2)^2}=\frac{2xy}{(x^2+y^2)^2}$

$\therefore v_x=-u_y$

$\therefore f(z)=\frac{1}{z}$ 는 미분가능하다.

ex2) $f(z) = \bar{z} = x - iy$는 미분가능?

sol) $u_x = 1$, $v_y = -1$.
미분 불가.

ex3) 극좌표의 Cauchy-Riemann.

$f(z) = \ln z = \ln|z| + i\,Arg(z)$ 는 미분가능한가?
$= \ln r + i\theta$

$$\begin{cases} u_r = \frac{1}{r} \\ \frac{1}{r}v_\theta = \frac{1}{r}\cdot 1 = \frac{1}{r}. \end{cases}$$

$$\begin{cases} v_r = 0 \\ -\frac{1}{r}u_\theta = 0 \end{cases}$$

∴ $f(z)$는 해석적이다.

ex4) 공액조화함수 찾기.
$u = x^2 - y^2 - y$의 공액조화 함수는?

$u_x = 2x = v_y$, ∴ $v = 2xy + g(x)$

$v_x = 2y + g'(x) = -u_y = -(-2y-1) = 2y+1$

∴ $g'(x) = 1$ $\quad g(x) = x + C$.

∴ $v = 2xy + x + C$.

ex4) $u = ax^3 + bxy$ 에서 조화함수가 되는 a, b를 찾고

공액조화함수를 찾고,

u를 실수부로 두는 복소함수(미분가능한)를 찾아라.

sol) $u_{xx} + u_{yy} = 6ax = 0$. $\therefore a = 0,\ b = \text{constant}$

$\therefore u = bxy$.

$u_x = by = v_y, \quad v = \frac{1}{2}by^2 + g(x)$.

$v_x = g'(x) = -u_y = -bx$.

$\therefore g(x) = -\frac{1}{2}bx^2 + C$.

$\therefore v = \frac{1}{2}b(y^2 - x^2) + C$

$f(z) = u + iv$

$= bxy + i\left(\frac{1}{2}b(y^2 - x^2) + C\right)$ 는 미분가능하다.

대표적인 함수들의 특징과 미분법을 알아보자.

1 지수함수

$z = x+iy = re^{i\theta}$ 일때...

$e^{z} = e^{x+iy} = e^{x}(\cos y + i\sin y)$

① $e^{z_1+z_2} = e^{z_1}e^{z_2}$
② $e^{z} = e^{z+i2n\pi}$
③ $\overline{e^{it}} = e^{-it}$
④ $(e^{z})' = e^{z}$

※ $z = u+iv$의 미분은
$u_x + iv_x = v_y + i(-u_y)$ 이다.
polar coordinate 에서,
$e^{-i\theta}(u_r + iv_r) = \frac{1}{r}e^{-i\theta}(v_\theta + i(-u_\theta))$

[2] 삼각함수, 쌍곡선함수

오일러공식 $e^{ix} = \cos x + i\sin x$, $e^{-ix} = \cos x - i\sin x$ 풀면

$$\begin{cases} \cos z = \dfrac{e^{iz}+e^{-iz}}{2} \\[2ex] \sin z = \dfrac{e^{iz}-e^{-iz}}{2i} \\[2ex] \tan z = \dfrac{\sin z}{\cos z} \\[2ex] \cosh z = \dfrac{e^{z}+e^{-z}}{2} \\[2ex] \sinh z = \dfrac{e^{z}-e^{-z}}{2} \end{cases} \qquad \begin{cases} (\cos z)' = -\sin z \\[2ex] (\sin z)' = \cos z \\[2ex] (\tan z)' = \sec^2 z \\[2ex] (\cosh z)' = \sinh z \\[2ex] (\sinh z)' = \cosh z \end{cases}$$

* 성질

① $\begin{cases} \cos(iz) = \cosh(z) \\ \sin(iz) = i\sinh(z) \end{cases}$

② $\begin{cases} \cosh(iz) = \cos(z) \\ \sinh(iz) = i\sin(z) \end{cases}$

③ $\cos z = \cos(x+iy)$
$\quad = \cos x\cosh y - i\sin x\sinh y$

$\sin z = \sin(x+iy)$
$\quad = \sin x\cosh y + i\cos x\sinh y$

④ $\cos^2 z + \sin^2 z = 1$

⑤ $\cosh^2 z = 1 + \sinh^2 z$

$\boxed{3}$ 로그 함수.

$z = re^{i\theta}$로 두면, $r = |z| = \sqrt{x^2+y^2}$

$\theta = \arg(z) = \frac{y}{x}$

$\ln(z) = \ln(re^{i\theta})$

$= \underline{\ln|z| + i\arg(z)}$

① $(\ln z)' = \frac{1}{z}$

② 음수에 로그 취할 수 있다. 복소함수면,

ex) $3e^{(2n+1)\pi i} = -3$ 은,

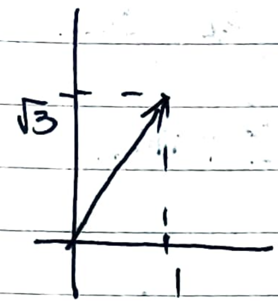
$\ln(-3) = \ln 3 + (2n+1)\pi i$

$\boxed{4}$ 거듭제곱

$z^c = e^{c\ln z}$ 이다.

ex) $i^i = e^{i\ln i} = e^{i \cdot i(\frac{\pi}{2}+2n\pi)} = e^{-\frac{\pi}{2}-2n\pi}$

Ex1) $e^z = 1+\sqrt{3}i$ 를 풀라.



$1+\sqrt{3}i = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

$e^z = e^x(\cos y + i\sin y)$.

$\therefore x = \ln 2, \quad y = \frac{\pi}{3} + 2n\pi$.

Ex2) $|\cos z|^2 = \cos^2 x\cosh^2 y + \sin^2 x\sinh^2 y$

$= \cos^2 x + \sinh^2 y$

if $y \to \infty$: $\sinh^2 y \to \infty$ $\quad \therefore |\cos z|^2 \to \infty$

즉 $\cos z$는 크기 제한이 없다. not $|\cos z| \le 1$

(단, z는 복소수)

Ex3) $\cos z = 5$ 에서 z는?

sol) $\dfrac{e^{iz}+e^{-iz}}{2} = 5 \quad e^{iz} = t$로 두면,

$t^2 - 10t + 1 = 0$.

$\therefore t = e^{iz} = 5 \pm \sqrt{24} \fallingdotseq 9.9, 0.1$

$e^{-y+ix} = e^{-y}e^{ix} = 9.9, 0.1$

$\begin{cases} e^{-y} = 9.9, & x = 2n\pi \\ e^{-y} = 0.1, & x = 2n\pi \ldots \end{cases}$

Ex4) $\cos z = 0$ 에서 $z = ?$

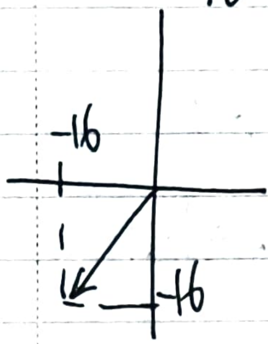
$\frac{e^{iz}+e^{-iz}}{2} = 0$. $\therefore e^{2iz} = -1 = e^{-2y} e^{i\cdot 2x}$

$\therefore y = 0$, $2x = \pi + 2n\pi$, $\therefore x = \frac{\pi}{2} + n\pi \ldots$

Ex5) $(2-2i)^{3/5} = z$ 구하기

sol) $(2-2i)^3 = z^5$

$-16-16i = r^5 e^{i\cdot 5\theta}$



$r^5 = 16\sqrt{2} = 2^{\frac{11}{2}}$ $\therefore r = 2^{\frac{11}{10}}$

$5\theta = \frac{5}{4}\pi + 2n\pi$, $\theta = \frac{\pi}{4} + \frac{2n}{5}\pi$.

Ex6) $(1-i)^{1+i} = z$ 풀기.

sol) $e^{(1+i)\ln(1-i)}$ 에서,

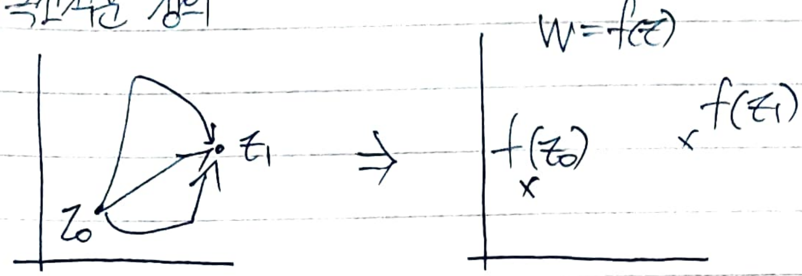
$\ln(1-i) = \ln\sqrt{2} + i(-\frac{\pi}{4} + 2n\pi)$

$\therefore e^{(1+i)\ln(1-i)} = e^{(1+i)(\ln\sqrt{2} + i(-\frac{\pi}{4}+2n\pi))}$

$= \sqrt{2}\, e^{\frac{\pi}{4}-2n\pi} \cdot e^{i(\ln\sqrt{2} - \frac{\pi}{4} + 2n\pi)}$

20.1 The Integral of a Complex Function

① 복소적분 정의

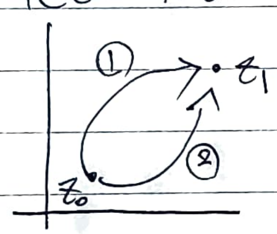


적분경로 C를 정의해서 적분한다.

$$\int_C f(z)\,dz = \int_{t_0}^{t_1} f(z(t))\frac{dz}{dt}\,dt$$

$$= \int_{t_0}^{t_1} f(z(t))\left(\frac{dx}{dt}+i\frac{dy}{dt}\right)dt$$

② 적분경로 의존성.



※ 미분가능 : 점 미분가능
해석적 : 특정영역 미분가능
완전함수 : 전영역 미분가능

경로 ①, ②가 f(z)에 따라 적분값이 다르다.

f(z) analytic ⇒ 경로 무관 ⇒ potential 구하자

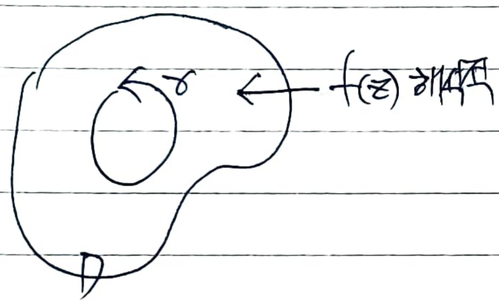
20.3 Cauchy 적분 정리

1 Cauchy Integral Theorem.

Simply connected region D에서 f(z)가 analytic 하면,
D 안의 simply connected closed curve γ에 대해,

$$\oint_\gamma f(z)\,dz = 0.$$

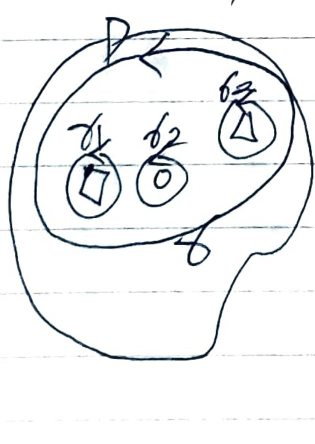
즉,



$$\rightarrow \oint_\gamma f(z)\,dz = 0.$$

2 Cauchy Integral Theorem 2.

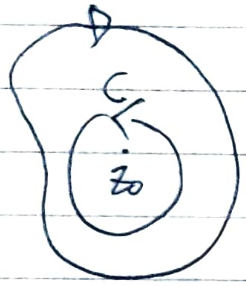
Multiply connected region D에서, f(z)가 analytic 하고,
D 안의 simply connected closed curve γ가 ~~구멍들을~~ 감싸면,



$$\oint_\gamma f(z)\,dz = \sum_{i=1}^{N} \oint_{\gamma_i} f(z)\,dz.$$

③ Cauchy Integral Theorem 3.

Simply Connected Region D 에서 analytic f(z) 가,
$z=z_0$ 에서 비해석적이다.
점 $z=z_0$ 를 둘러싼 폐곡선 C에 대해,



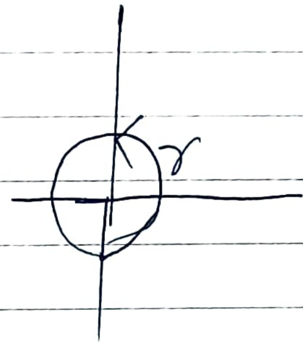
$$\oint_C \frac{f(z)}{z-z_0}\,dz = 2\pi i\, f(z_0)$$

※ 같은 조건으로

$$n!\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz = 2\pi i\, f^{(n)}(z_0)$$

Ex1) Ex 에서, $f(z) = \frac{1}{z}$ 이고, 적분경로 $\gamma = \cos t + i\sin t$.

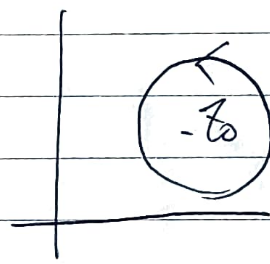
$0 \le t \le 2\pi$: $dz/dt = -\sin t + i\cos t$



$$\int_\gamma f(z)dz = \int_0^{2\pi} \frac{-\sin t + i\cos t}{\cos t + i\sin t} dt$$
$$= \int_0^{2\pi} i \, dt = 2\pi i$$

$z=0$ 에서 비해석적인 $f(z)$로서 이런 결과다.

Ex2) $f(z) = (z-z_0)^n$ 에서, $\gamma = z_0 + \rho e^{it}$ $(0 \le t \le 2\pi)$
선적분은?



i) $\int_0^{2\pi} (\rho\cos t + i\rho\sin t)^n (-\rho\sin t + i\rho\cos t) dt$

$= \int \rho^n e^{nit} \cdot \rho i e^{it} dt$

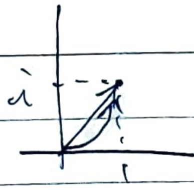
$= \int_0^{2\pi} \rho^{n+1} e^{(n+1)it} dt = \frac{\rho^{n+1}}{n+1} (\underbrace{e^{2(n+1)\pi i}}_{=1} - 1) = 0$ (단, $n \ne -1$)

ii) $n=-1$ 일때 $f(z)$는 비해석적.

$$\int_0^{2\pi} \frac{\rho(-\sin t + i\cos t)}{\rho(\cos t + i\sin t)} dt = \int_0^{2\pi} \frac{i e^{it}}{e^{it}} dt = \int_0^{2\pi} i \, dt = 2\pi i$$

ii)의 결과를 Cauchy Integral Theorem 3의 결과로 간단히 구할 수 있다.

Ex3) $C_1 : z = t + it \ (0 \le t \le 1)$
$C_2 : z = t + it^2 \ (0 \le t \le 1)$



$f(z) = z$ 로 둘때,
$f(z) = z = x + iy = U + iV$ 에서 Cauchi-Rieman 미분가능 Test에 의해,

$$\left.\begin{aligned} U_x = V_y = 1 \\ U_y = -V_y = 0 \end{aligned}\right\} \rightarrow f(z)\text{는 해석적.}$$

∴ C_1, C_2 에서 선적분 값은 같을 것이다.

$$\int_{C_1} f(z)\,dz = \int_0^1 t(1+i)(1+i)\,dt = i$$

$$\int_{C_2} f(z)\,dz = \int_0^1 t(1+it)(1+i \cdot 2t)\,dt = i$$

※ potential function을 이용하면,

$$\int_C f(z)\,dz = \frac{1}{2}z^2\Big]_0^{1+i} = \frac{1}{2}(2i) = i$$

Ex4) $f(z)=\bar{z}$ not analytic 한것이,

$u_x=1,\ v_y=-1$ 이라...

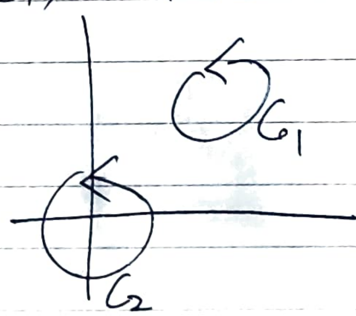
$\therefore \int_{C_1} f(z)dz = \int_0^1 t(1-i)(1+i)\,dt = 1$

$\int_{C_2} f(z)dz = \int_0^1 (t-it^2)(1+i\cdot 2t)\,dt = 1+\frac{1}{3}i$

Ex5) Cauchy Integral Theorem 에서

① $\oint e^z dz = 0$ $\because$ $f(z)=e^z$는 전평면해석.

② $f(z)=\frac{1}{z}$는 특이점을 C_1, C_2 에따라...



$\int_{C_1} \frac{1}{z}dz = 0$

$\int_{C_2} \frac{1}{z}dz = 2\pi i \neq 0.$

Ex 6) $C: z = z_0 + e^{it}$
$dz/dt = ie^{it}$

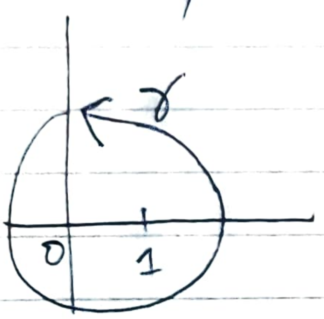
에서

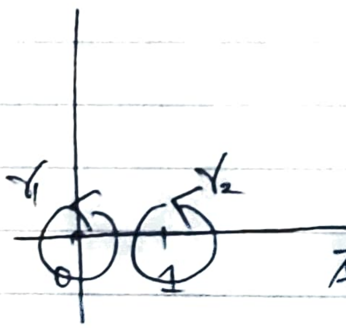
$$\oint_C \frac{1}{(z-z_0)^n} dz = \int_0^{2\pi} \frac{1}{e^{itn}} i e^{it} dt = \int_0^{2\pi} i e^{it(1-n)} dt$$

$$= \begin{cases} n=1 : & 2\pi i \\ n \neq 1 : & \left[\frac{1}{1-n} e^{it(1-n)} \right]_0^{2\pi} = 0. \end{cases}$$

* 비해석적이어도 0일 수 있다.
* Cauchy Integral Theorem으로 해석할 수 있다.

Ex 7) Cauchy Integral Theorem 2.





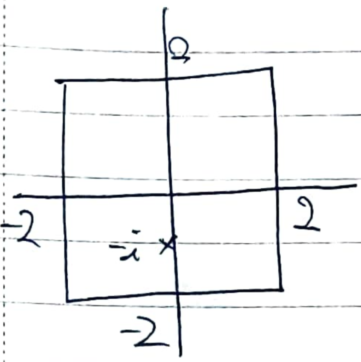
작은원들로 쪼갠다..!

$$\oint_\gamma \frac{1}{z(z-1)} dz = \oint_\gamma \frac{1}{z-1} - \frac{1}{z} dz$$

$$= \underbrace{\oint_{\gamma_1} \frac{1}{z-1} dz}_{=0} + \oint_{\gamma_1} \frac{1}{z} dz + \oint_{\gamma_2} \frac{1}{z-1} dz + \underbrace{\oint_{\gamma_2} \frac{1}{z} dz}_{=0}$$

$$= 2\pi i + 2\pi i = 4\pi i$$

Ex2)



에서 $\oint_C \frac{1}{z+i}\,dz$ 를 구하라.

방법 1) 선적분.

$\begin{cases} x=2 \\ y=t \end{cases}$ $\begin{cases} x=t \\ y=2 \end{cases}$ $\begin{cases} x=-2 \\ y=t \end{cases}$ $\begin{cases} x=t \\ y=-2 \end{cases}$ $(-2 \le t \le 2)$.

$z=2+it$ $\quad z=-t+i2$ $\quad z=-2-it$ $\quad z=t-2i$

$dz/dt=i$ $\quad dz/dt=-1$ $\quad dz/dt=-i$ $\quad dz/dt=1$.

$$\int_{-2}^{2}\frac{1}{it+i+2}\,i\,dt+\int_{-2}^{2}\frac{-1}{-t+3i}\,dt+\int_{-2}^{2}\frac{1}{-it+i-2}(-i)\,dt$$

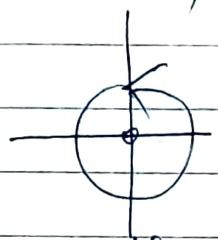
$$+\int_{-2}^{2}\frac{1}{t-i}\,dt = \dots$$ 복잡하다.

방법2) Cauchy Integral Theorem 2.

$\oint_C \frac{1}{z+i}\,dz \Rightarrow$ z에 적분경로 $z=-i+e^{it}$ 를 정의.
$dz/dt = ie^{it}$

그대로 적분하면, Cauchy Integral Theorem 2를 쓰면,
답은 $2\pi i$

Ex) Cauchy Integral Theorem 3.



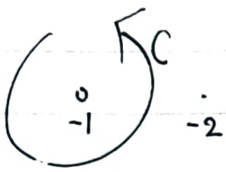
γ : 원.

여기서 $\oint_C \frac{\cos z}{z} dz$ 는, $f(z) = \cos z$ 로 보고,

$\oint_C \frac{\cos z}{z} dz = 2\pi i f(0) = 2\pi i$

경로 위에 singular point 가 있는 경우는?

ex) Cauchy 적분정리 3.



$$\oint_C \frac{\cos z}{(z+1)(z+2)}\,dz = \oint_C \frac{\cos z}{z+1} - \cancel{\frac{\cos z}{z+2}}\,dz \quad \cancel{} \to 0$$

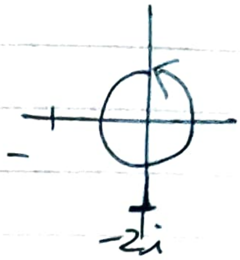
$$= \oint_C \frac{\cos z}{z+1}\,dz = 2\pi i \cos(-1)$$

ex) Cauchy 도함수 적분정리

$$\oint_C \frac{1}{(z-z_0)^2}\,dz = \frac{2\pi i}{1!}\cdot 0 = 0$$

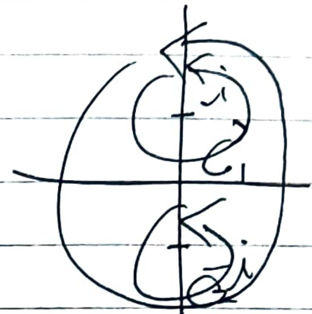
$$\oint_C \frac{z^4-3z^2+6}{(z+i)^3}\,dz = 2\pi i/2! \cdot (12i^2-6)$$

ex) $\oint_C \frac{1}{z^3+2iz^2}\,dz = \oint_C \frac{1}{z^2(z+2i)}\,dz.$



$$= \oint_C \frac{1}{4z} - \frac{i}{2z^2} - \frac{1}{4(z+2i)}\,dz$$

$$= \frac{2\pi i}{4} - 0 - 0 = \frac{\pi}{2} i$$



ex) $\oint_C \frac{e^{\pi z}}{z^2+1}\,dz = \oint_C \frac{e^{\pi z}}{(z+i)(z-i)}\,dz$

$$= \frac{1}{2i}\oint_C \frac{e^{\pi z}}{z-i} - \frac{e^{\pi z}}{z+i}\,dz = \cancel{\frac{1}{2i}\cdot 2\pi i\left(e^{\pi i} - e^{-\pi i}\right)}$$

$$= \frac{1}{2i}\int_{C_1} \frac{e^{\pi z}}{z-i} - \frac{e^{\pi z}}{z+i}\,dz + \frac{1}{2i}\int_{C_2} \frac{e^{\pi z}}{z-i} - \frac{e^{\pi z}}{z+i}\,dz$$

$$= \frac{1}{2i}\left(e^{\pi i} - e^{-\pi i}\right) = 0$$

복소함수 적분정리.

1)

내부는 모두 해석적

$$\oint_C f(z)\,dz = 0$$

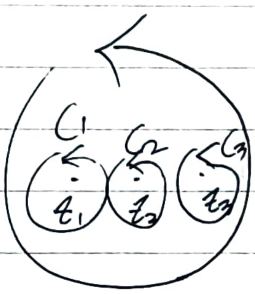
2)

C내부의 특이점 z_0

① $\displaystyle\oint_C \frac{f(z)}{z-z_0}\,dz = 2\pi i f(z_0)$

② $\displaystyle\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz = \frac{2\pi i}{n!} f^{(n)}(z_0).$

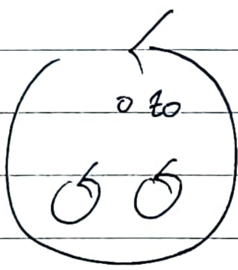
3)



Multiply connected region 의 내부해석적

$$\oint_C f(z)\,dz = \sum_{i=1} \oint_{C_i} f(z)\,dz$$

4)



singular point + Multiply connected region.

$$\oint_C \frac{f(z)}{z-z_0}\,dz = 2\pi i f(z_0) + \sum_{i=1} \int_{C_i} f(z)\,dz$$

21.1 Series Presentation of Complex Function.

1 Taylor Series.

- $f(z) = a_n(z-z_0)^n = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots$

- $a_n = \dfrac{f^{(n)}(z_0)}{n!} = \dfrac{1}{2\pi i}\oint \dfrac{f(z)}{(z-z_0)^{n+1}}\,dz$

단, f는 $|z-z_0|<R$ 에서 analytic

✓ $z_0=0$ 이면 Maclaurin Series

✓ 과정을 이용할 곳 근처에서 근사하는 것이 좋다.

2 자주 쓰는 Taylor Series. ($z_0=0$)

① $\dfrac{1}{1-z} = 1 + z + z^2 + \cdots \quad (|z|<1)$

② $\dfrac{1}{1+z} = 1 - z + z^2 - \cdots \quad (|z|<1)$

③ $\cos z = 1 - \dfrac{1}{2!}z^2 + \dfrac{1}{4!}z^4 - \cdots \quad (|z|<\infty)$

④ $\sin z = z - \dfrac{1}{3!}z^3 + \dfrac{1}{5!}z^5 - \cdots \quad (|z|<\infty)$

⑤ $e^z = 1 + z + \dfrac{1}{2!}z^2 + \cdots \quad (|z|<\infty)$

3 Isolated Zero

$f(z)=0$ 인 z는 zero 라고 한다. 이 zero가 단 하나만 고립되어 있으면 isolated zero 라고 한다.

① 미분가능한 복소함수는 isolated zero 만 가져야한다.
② Taylor Series 계수중 0이 아닌 최초의 계수의 변수를 order
   즉, $C_m \neq 0$ 인 최초 m.
③ h, k 함수에서 모두 z 에서 order m, n을 가지면
   $h(z)k(z)$는 order $m+n$ at z
   $h(z)/k(z)$는 order $m-n$ at z
④ zero of order을 알아보기.
   함수 f를 미분하고 z_0를 대입해서
   0이 안나오는 최초의 값(m)을 구한다.

ex) $\frac{1}{3-z} = \frac{1}{3(1-\frac{z}{3})} = 1/3 \cdot \left(1 + \frac{1}{3}z + \left(\frac{1}{3}z\right)^2 + \cdots\right)$

수렴반경 $\left|\frac{z}{3}\right| < 1$, $|z| < 3$

ex) $\frac{5}{4+8z} = \frac{5}{4} \frac{1}{1+2x} = \frac{5}{4}\left(1 - 2x + (2x)^2 - (2x)^3 \cdots\right)$

$= \frac{5}{4} \sum_{n=0} (-1)^n \cdot (2x)^n$

ex) Taylor Series 이용하기

$\frac{1}{(3-z)^2}$ 전개하라.

sol) $\left.\frac{1}{3-z}\right)' = (3-z)^{-2}$

$\sum_{n=0}^{\infty} \left(\frac{1}{3^{n+1}} z^n\right)' = \sum_{n=1} \frac{n}{3^{n+1}} z^{n-1}$

ex) $\frac{1}{(z-1)^2}$ 의 전개

$\left.\frac{1}{z-1}\right)' = -\frac{1}{(z-1)^2}$

$\left.-\sum_{n=0} z^n\right)' = -\sum_{n=1} n z^{n-1} = -\sum_{n=0} (n+1) z^n$

$\therefore \frac{1}{(z-1)^2} = \sum_{n=0} (n+1) z^n$

ex) $f(z) = \cos^3(z)$ $z_0 \frac{\pi}{2}$.

$f'(z) = 3\cos^2 z \sin z$ $f'(z_0) = 0$

$f''(z) = -6\cos z \sin^2 z - 3\cos^3 z$ $f''(z_0) = 0$

$f'''(z) = \ldots$ $f'''(z_0) = 6 \neq 0$.

$\therefore$ $f(z)$의 $\frac{\pi}{2}$에서 zero of order = 3.

21.2 Laurant Series

[1] Definition

- $f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n = \cdots a_{-2}\frac{1}{(z-z_0)^2} + a_{-1}\frac{1}{(z-z_0)} + a_0 + a_1(z-z_0) \cdots$

- $a_n = \frac{1}{2\pi i} \oint \frac{f(z)}{(z-z_0)^{n+1}} dz$

단, f는 annulus $r < |z-z_0| < R$ 에서 analytic.
보통의 경우 기존의 급수식을 변형하여 구한다.
singularity point에서 급수식을 전개하기 위해 사용한다.

[2] 3 kinds of Singularity. (미분불가점)

① Removable
모든 $n<0$ 인 $C_n = 0$. 사실상 Taylor Series.

② pole of order m.
$C_{-m} \neq 0$ 이고 $C_{-m-1} = C_{-m-2} = \cdots = 0$.
인 경우.
$m=1$: simple pole
$m=2$: double pole.

③ Essential singularity
$k>0$ 에서 $C_{-k} \neq 0$ 가 무한히 존재.

$\boxed{3}$ 찾기 방법.

① pole of order m 찾기.

$\lim_{z \to z_0} (z-z_0)^m f(z) \neq 0$ 으로 수렴시.

z_0에서 pole of order $= m$.

② $f(z) = \dfrac{h(z)}{g(z)}$

$h(z_0) \neq 0$ 이고 $g(z)$가 z_0에서 zero order m이면
f는 pole of order $= m$ at z_0.

③ $f(z) = \dfrac{h(z)}{g(z)}$

h : zero of order k at z_0
g : 〃 m at z_0.

f : pole of order $m-k$ at z_0

ex) $e^{1/z}$ 전개.

$$e^{\frac{1}{z}} = 1 + \frac{1}{z} + \left(\frac{1}{z}\right)^2 + \cdots$$

수렴반경은 $0 < |z| < \infty$

ex) $f(z) = \frac{2z-1}{z^3 - z^2}$ 의 Laurant Series. $z_0 = 0$.

i) $f(z) = \frac{1}{z^2} \frac{2z-1}{z-1} = \frac{1}{z^2}\left(2 - \frac{1}{1-z}\right)$

$= \frac{1}{z^2}\left(1 - z - z^2 - \cdots\right) = \frac{1}{z^2} - \frac{1}{z} - 1 - z - z^2$

$\cdots = \frac{1}{z^2} - \sum_{n=0}^{\infty} z^{n-1}$

단, $0 < |z| < 1$.

ii) 이경우, $|z| > 1 \Rightarrow \left|\frac{1}{z}\right| < 1$.

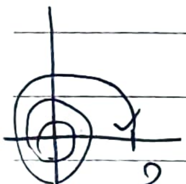
$f(z) = \frac{2z-1}{z^2(z-1)} = \frac{z}{z-1} \cdot \frac{1}{z^3}(2z-1)$

$= \frac{1}{1 - \frac{1}{z}} \cdot \left(\frac{2}{z^2} - \frac{1}{z^3}\right)$

$= \left(1 + \frac{1}{z} + \frac{1}{z^2} \cdots\right)\left(\frac{2}{z^2} - \frac{1}{z^3}\right)$

$= \sum_{n=0} \frac{2}{z^{n+2}} - \sum_{n=0} \frac{1}{z^{n+3}} = \sum_{n=3}^{\infty} \frac{1}{z^n} + \frac{2}{z^2}$.

ex) $f(z) = \dfrac{1}{z(z-2)}$ 의 로랑급수의 annulus에 따른 전개.



여기에
문제가 있는
경우...

i) $0 < |z-2| < 2 \Rightarrow \dfrac{|z-2|}{2} < 1.$

$f(z) = \dfrac{1}{z-2} \cdot \dfrac{1}{z}$ 에서

$$\frac{1}{z} = \frac{1}{2+z-2} = \frac{1}{2} \cdot \frac{1}{1+\frac{z-2}{2}} = \frac{1}{2}\left(1 - \frac{z-2}{2} + \left(\frac{z-2}{2}\right)^2 \cdots\right)$$

$$\therefore f(z) = \frac{1}{2}\left(\frac{1}{z-2} - \frac{1}{2} + \frac{z-2}{2^2} \cdots\right)$$

---

ii) $|z-2| > 2 : \quad \left|\dfrac{2}{z-2}\right| < 1$

$$\frac{1}{z} = \frac{1}{z-2+2} = \frac{1}{z-2} \cdot \frac{1}{1+\frac{2}{z-2}}$$

$$= \frac{1}{z-2} \cdot \left(1 - \frac{2}{z-2} + \left(\frac{2}{z-2}\right)^2 \cdots\right)$$

$$\therefore f(z) = \frac{1}{(z-2)^2} - \frac{2}{(z-2)^3} + \frac{2^2}{(z-2)^4} \cdots$$

---

iii) $0 < |z-3| < 1$

$$f(z) = \frac{1}{2} \cdot \frac{1}{z-2} - \frac{1}{2} \cdot \frac{1}{z}$$

$$= \frac{1}{2} \cdot \frac{1}{1+(z-3)} - \frac{1}{2} \cdot \frac{1}{3} \cdot \frac{1}{1+\frac{z-3}{3}}$$

$$= \frac{1}{2} \sum_{n=0} (-1)^n (z-3)^n - \frac{1}{6} \sum_{n=0} (-1)^n \left(\frac{z-3}{3}\right)^n$$

$$= \sum_{n=0} (-1)^n \frac{3^{n+1}-1}{2 \cdot 3^{n+1}} \cdot (z-3)^n$$

⋮

ex) $f(z) = \frac{\cos z}{z^5}$ 의 Laurent

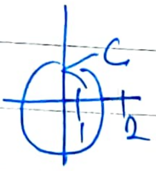
$$\cos z = 1 - \frac{1}{2!}z^2 + \frac{1}{4!}z^4 \cdots$$

$$f(z) = \frac{1}{z^5} - \frac{1}{2!}z^3 + \frac{1}{4!}\frac{1}{z} \cdots$$

22.2 Residue Theorem

I 배경

Cauchy Theorem $\oint_C f(z)\,dz = 0$



Cauchy Integral Formula $\oint_C \frac{f(z)}{(z-z_0)^{n+1}}dz = \frac{2\pi i}{n!} f^{(n)}(z_0)$

$f(z)=\frac{1}{(z-1)(z-2)^5}$ 에서 $\oint_C f(z)dz$ 처럼 부분분수 만들기 어려울 때.

이것말고 다른 적분 방법.. $\Rightarrow \oint_C \frac{1}{(z-z_0)^n}dz = \begin{cases} 2\pi i & (n=1) \\ 0 & (n\neq 1) \end{cases}$

① Residue 정의.

Laurent Series 의 $(z-z_0)^{-1}$ 의 계수.

$f(z) = \cdots \frac{a_{-2}}{(z-z_0)^2} + \frac{a_{-1}}{z-z_0} + a_0 \cdots$ 에서의 $a_{-1} = Res(f(z), z_0)$

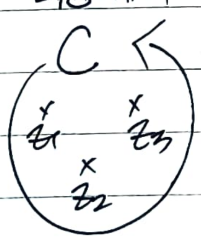
② Residue Theorem (유수정리)

$$\oint_C f(z)dz = \oint_C \cdots \underbrace{\frac{a_{-2}}{(z-z_0)^2}}_{\to 0} + \frac{a_{-1}}{z-z_0} + \underbrace{a_0}_{\to 0} + \underbrace{a_1(z-z_0)}_{\to 0} + \cdots dz$$

$$= a_{-1}\oint_C \frac{1}{z-z_0}dz = 2\pi i\, a_{-1}$$

$$= 2\pi i\, Res(f(z), z_0).$$

확장해서



z_n = pole 들)

$$\oint_C f(z)dz = 2\pi i \sum_{n=1}^{N} Res(f(z), z_n)$$

2 Residue 계산법 $f(z) = \frac{a_{-1}}{z-z_0} + \sum_{n=0}^{\infty} a_n (z-z_0)^n$ 에서.

① 단순극, simple pole, $m=1$

$$\lim_{z \to z_0} (z-z_0) f(z_0) = a_{-1} = \text{Residue}(f, z_0)$$

② m중극, multi pole,

$$\lim_{z \to z_0} \frac{1}{(m-1)!} \cdot \frac{d^{m-1}}{dz^{m-1}} \left((z-z_0)^m f(z_0) \right) = \text{Residue}(f, z_0)$$

③ 분수가 다항식이 아니거나 위 2개가 어려울때.

$f(z) = \frac{h(z)}{g(z)}$, $h(z_0) \neq 0$, $g(z)$ 가 simple zero

$$\frac{h(z_0)}{g'(z_0)} = \text{Residue}(f, z_0)$$

ex) 가분유수구하기

$f=\cos\frac{1}{z}$ 에서 $\mathrm{Res}(\cos\frac{1}{z}, 0)$

$$\cos\frac{1}{z} = 1 - \frac{1}{2!z^2} + \frac{1}{4!z^4} + \cdots$$

$$\therefore\ a_{-1} = 0 = \mathrm{Res}(f, 0)$$

ex) $f(z) = \frac{1}{(z-1)^3(z-2)^2}$ 에서 $\mathrm{Res}(f, 1)$

$$\frac{1}{z-2} = \frac{-1}{1-(z-2)} = -\sum_{n=0}(z-1)^n$$

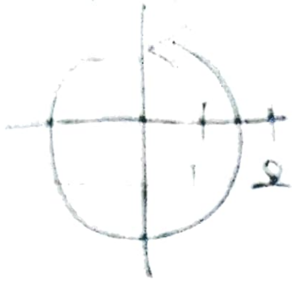
$$-\frac{1}{(z-2)^2} = -\sum_{n=1} n(z-1)^{n-1}$$

$$\frac{1}{(z-2)^2} = \sum_{n=0}(n+1)(z-1)^n$$

$$\therefore\ f(z) = \sum_{n=0}(n+1)(z-1)^{n-3}$$

$n=2$ 에서 $a_{-1} = 3 = \mathrm{Res}(f, 1)$.

유수 계산의 이용



$$\oint_C \frac{1}{(z-1)^3(z-2)^2}\,dz$$

여기서 $\mathrm{Res}(f,1) = \frac{1}{2!}\frac{d^2}{dz^2}\left((z-1)^3\cdot\frac{1}{(z-1)^3(z-2)^2}\right)$

$= \frac{1}{2}\cdot\frac{d^2}{dz^2}\left(\frac{1}{(z-2)^2}\right) = \frac{1}{2}\cdot\frac{d}{dz}\left(-\frac{2}{(z-2)^3}\right)$

$= \frac{1}{2}\cdot\frac{6}{(z-2)^4}$ 에 $z_0=1$ 대입.

$\mathrm{Res}(f,1) = 3\cdot 1 = 3.$

$\therefore \oint_C \frac{1}{(z-1)^3(z-2)^2}\,dz = 2\pi i\cdot 3$
$= 6\pi i$

ex) $f(z) = \frac{\sin z}{z^2}$

$\sin(0)=0 \quad \therefore$ order of pole $= 2-1 = 1.$

$\therefore \lim_{z\to 0} z f(z) = \lim_{z\to 0}\frac{\sin z}{z} = 1 = \mathrm{Res}(f,0)$

$\therefore \oint_C \frac{\sin z}{z^2}\,dz = 2\pi i$

ex) $f(z) = \frac{4iz-1}{\sin z}.$

$\mathrm{Res}(f,\pi) = \frac{4i\pi-1}{\cos\pi} = 1-4\pi i$

22.3 Evaluation of Real Integrals.

1 개요.

실함수의 적분을 복소평면으로 가져와 행한다.

① $\int_0^{2\pi} f(\cos\theta, \sin\theta)\, d\theta$

② $\int_{-\infty}^{\infty} f(x)\, dx$

③ $\int_{-\infty}^{\infty} f(x) \cos\alpha x\, dx$

$\int_{-\infty}^{\infty} f(x) \sin\alpha x\, dx$

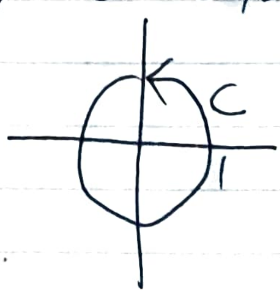
2 Case 1.

$$\int_0^{2\pi} f(\cos\theta, \sin\theta)\, d\theta \xrightarrow{z=e^{i\theta}} = \oint_C g(z)\, dz.$$

$z = e^{i\theta}$ (unit circle)로 두고 치환한다.

이때 $dz = iz\,d\theta$, $\therefore d\theta = dz/iz$

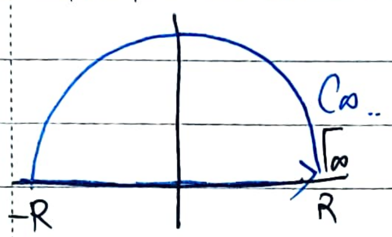
경로 C는 단위원



3 case 2

$$\int_{-\infty}^{\infty} f(x)\,dx = \oint_{C_\infty} f(z)\,dz$$

여기서 C_∞는 무한히 큰 반원이다.



proof)

$$\int_{-R}^{R} f(x)\,dx + \int_{\Gamma} f(z)\,dz = \oint_{C} f(z)\,dz$$

$R\to\infty$ 하면

$$\int_{-\infty}^{\infty} f(x)\,dx + \int_{\Gamma_\infty} f(z)\,dz = \oint_{C_\infty} f(z)\,dz$$

(Jordan's Path)

Jordan's lemma 에 의해

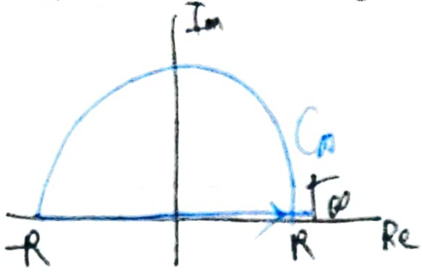
$$\lim_{|z|\to\infty} |z f(z)| = 0 \Rightarrow \int_{\Gamma_\infty} f(z)\,dz = 0$$

$$\therefore \int_{-\infty}^{\infty} f(x)\,dx = \oint_{C_\infty} f(z)\,dz.$$

Residue Theorem 이용

case 3

Case 2와 같이 경로 C_∞는 무한히 큰 반원



여기서

$$\int_{-R}^{R} f(x)e^{i\alpha x}dx + \int_{\Gamma} f(z)e^{i\alpha z}dz = \oint_{C} f(z)e^{i\alpha z}dz$$

R→∞ 하면

$$\int_{-\infty}^{\infty} f(x)e^{i\alpha x}dx + \underbrace{\int_{\Gamma_\infty} f(z)e^{i\alpha z}dz}_{\text{(Jordan's Path)}} = \oint_{C_\infty} f(z)e^{i\alpha z}dz$$

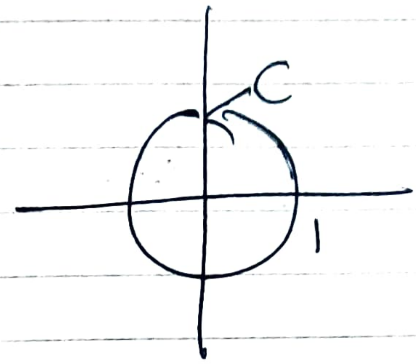
Jordan's lemma에 의해

$$\lim_{|z|\to\infty}|f(z)| = 0 \Rightarrow \int_{\Gamma_\infty} f(z)e^{i\alpha z}dz = 0$$

$$\therefore \int_{-\infty}^{\infty} f(x)e^{i\alpha x}dx = \oint_{C_\infty} f(z)e^{i\alpha z}dz$$

$$\underbrace{\int_{-\infty}^{\infty} f(x)\cos(\alpha x)dx}_{\text{실수부}} + i\underbrace{\int_{-\infty}^{\infty} f(x)\sin(\alpha x)dx}_{\text{허수부}} = \underbrace{\oint_{C_\infty} f(z)e^{i\alpha z}dz}_{\text{Residue Theorem 이용}}$$

ex) Case ① 의 이용.



$$\int_0^{2\pi} \frac{1}{5/4+\sin\theta}\, d\theta.$$

$z = e^{i\theta}$ 로 두면 $\frac{dz}{iz} = d\theta.$

$z = \cos\theta + i\sin\theta$

$z^{-1} = \cos\theta - i\sin\theta$ $\quad$ ∴ $\sin\theta = \frac{1}{2i}(z - z^{-1})$

원식 $= \oint_C \frac{1}{5/4+(z-z^{-1})/2i}\cdot\frac{1}{iz}\, dz$

$= \oint_C \frac{4}{2z^2+5zi-2}\, dz = \oint_C \frac{4}{(2z+i)(z+2i)}\, dz$

$= 2\pi i\, \mathrm{Res}\left(f, -\frac{i}{2}\right)$

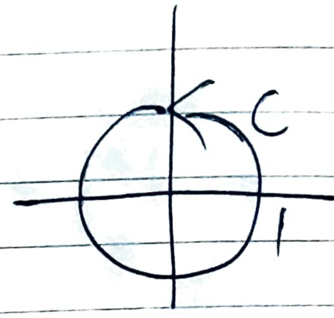
이때 $\mathrm{Res}\left(f, -\frac{i}{2}\right) = \lim_{z\to -\frac{i}{2}} \frac{4}{z+2i} = 4\cdot\frac{2}{3i}$

$= -\frac{8}{3}i$

∴ 원식 $= \frac{16\pi}{3}$

ex) case 1

$$\int_0^{2\pi} \frac{1}{1+\frac{1}{2}\cos\theta} d\theta$$



$z = e^{i\theta}$, $\frac{dz}{iz} = d\theta$

$\cos\theta = \frac{1}{2}(z+z^{-1})$

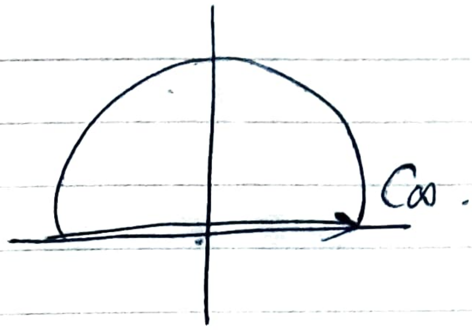
원식 $= \oint_C \frac{1}{1+\frac{1}{4}(z+\frac{1}{z})} \cdot \frac{1}{iz} dz$

$= \oint_C \frac{4}{i(4z+z^2+1)} dz = \frac{4}{i}\oint_C \frac{1}{(z+2-\sqrt{3})(z+2+\sqrt{3})} dz$

$= \frac{4}{i} \cdot 2\pi i \ \mathrm{Res}(f, -2+\sqrt{3})$

$= 8\pi \cdot \frac{1}{2\sqrt{3}} = \frac{4}{\sqrt{3}}\pi$

ex) Case ②



$$\int_{-\infty}^{\infty} \frac{1}{(1+x^2)^3}\, dx$$

이때 $\displaystyle\lim_{|z|\to\infty} \frac{z}{(1+z^2)^3} = 0$

$$\therefore \int_{-\infty}^{\infty} \frac{1}{(1+x^2)^3}\, dx = \oint_{C_\infty} \frac{1}{(1+z^2)^3}\, dz$$

$$= \oint_{C_\infty} \frac{1}{(z+i)^3 (z-i)^3}\, dz$$

$$= 2\pi i \cdot \mathrm{Res}(f, i)$$

$$\mathrm{Res}(f, i) = \frac{1}{2!} \frac{d^2}{dz^2}\left(\frac{1}{(z+i)^3}\right) = -\frac{3}{16} i$$

$\therefore$ 원식 $= \frac{3}{8}\pi$.

ex) Case ②

$$\int_{-\infty}^{\infty} \frac{1}{(x^2+2x+2)^2}\,dx$$

$$\lim_{|z|\to\infty} \frac{z}{(z^2+2z+2)^2}\,dz = 0$$

$$\therefore \text{준식} = \oint_{C_\infty} \frac{1}{(z^2+2z+2)^2}\,dz = \oint_{C_\infty} \frac{1}{(z+1+i)^2(z+1-i)^2}\,dz$$

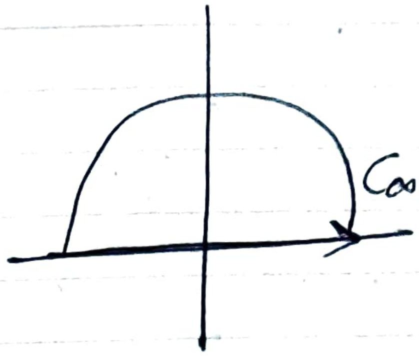
$$= 2\pi i \cdot \mathrm{Res}(f, -1+i)$$

$$\mathrm{Res}(f, -1+i) = \frac{1}{1!}\frac{d}{dz}\left(\frac{1}{(z+1+i)^2}\right) = -\frac{i}{4}$$

$$\therefore \text{준식} = \frac{\pi}{2}$$

ex) Case ③

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2+1}\,dx$$



$$\lim_{|z|\to\infty} \left| \frac{1}{z^2+1} \right| = 0 \;\therefore$$

원식 $\Rightarrow \oint_{C_\infty} \frac{e^{iz}}{(z+i)(z-i)}\,dz$ 에서 본다.

$$= 2\pi i\ \mathrm{Res}(f, i)$$

$$= 2\pi i\, \frac{e^{-1}}{2i} = \pi/e = \int_{-\infty}^{\infty} \frac{1}{x^2+1}\cdot\cos x\,dx$$

$$+ i\int_{-\infty}^{\infty} \frac{1}{x^2+1}\sin x\,dx$$

$$\therefore \int_{-\infty}^{\infty} \frac{\cos x}{x^2+1}\,dx = \frac{\pi}{e}$$

ex) Case ②

실수축 위에 pole이 있을때... $\int_{-\infty}^{\infty}\frac{\sin x}{x}dx = ?$

path를 수정하자.

$$\oint_{C_\infty}\frac{e^{iz}}{z}dz = \int_{-\infty}^{-\varepsilon}\frac{e^{ix}}{x}dx + \int_{\Gamma'}\frac{e^{iz}}{z}dz + \int_{\varepsilon}^{\infty}\frac{e^{ix}}{x}dx + \int_{\Gamma_\infty}\frac{e^{iz}}{z}dz$$

여기서 $\varepsilon \to 0$

$$\oint_{C_\infty}\frac{e^{iz}}{z}dz = \int_{-\infty}^{\infty}\frac{e^{ix}}{x}dx + \int_{C_1}\frac{e^{iz}}{z}dz$$

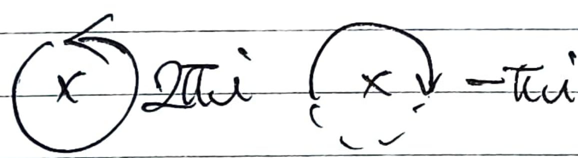
이때 극이 안에 없으므로

$$\oint_{C_\infty}\frac{e^{iz}}{z}dz = 0 = \int_{-\infty}^{\infty}\frac{e^{ix}}{x}dx + \int_{C_1}\frac{e^{iz}}{z}dz$$

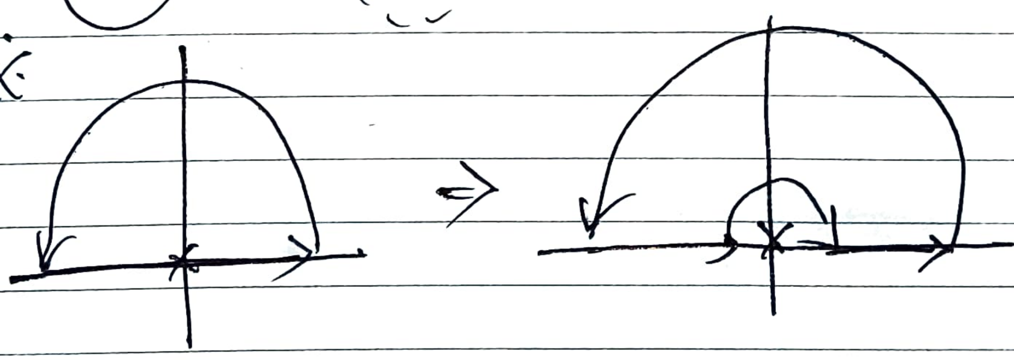
$$\therefore \int_{-\infty}^{\infty}\frac{e^{ix}}{x}dx = -\int_{C_1}\frac{e^{iz}}{z}dz$$

$$= -\pi i \cdot \mathrm{Res}(f, \text{pole})$$

$\because$



$\because$



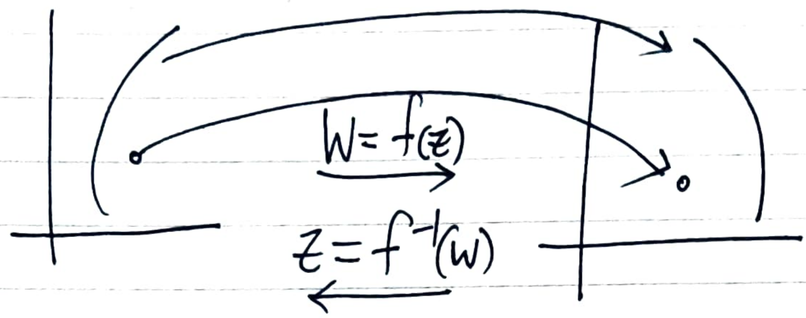
$$\therefore \int_{-\infty}^{\infty} \frac{e^{ix}}{x}\,dx = -\pi i \cdot Res(f, 0)$$

$$= -\pi i \cdot 1 = -\pi i$$

23.1. Conformal Mapping

1 개요

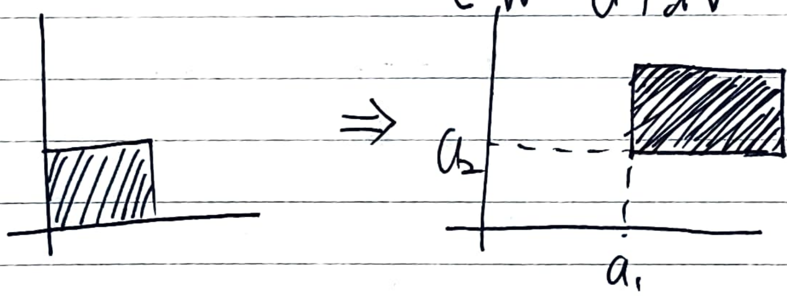
복소함수를 다른 공간으로의 mapping으로 보는 것이다.



2 Mapping 종류.

① Translation (평행이동)

$$W = Z + a \qquad \begin{cases} Z = x+iy \\ a = a_1+ia_2 \\ W = U+iV \end{cases}$$

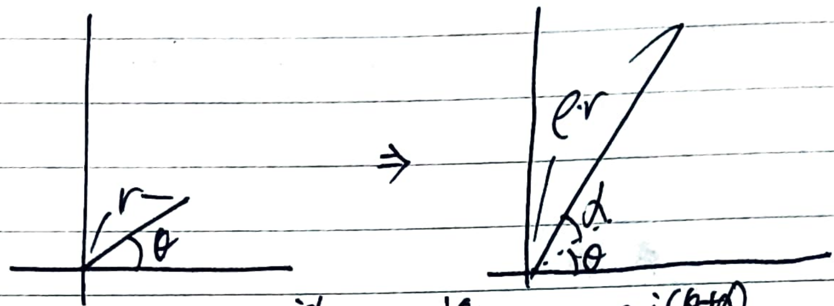


※ 선대에선

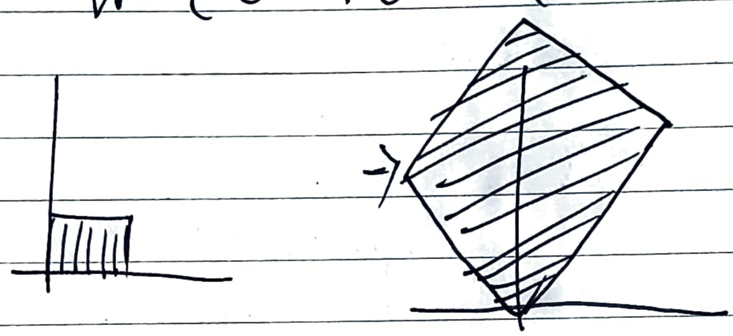
$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

② Rotation / Scale (회전 + 확대축소)

$$W = aZ \qquad \begin{cases} z = re^{i\theta} \\ a = \rho e^{i\alpha} \\ w = Re^{i\phi} \end{cases}$$



$$W = \rho e^{i\alpha} \cdot r e^{i\theta} = \rho r e^{i(\theta+\alpha)}$$

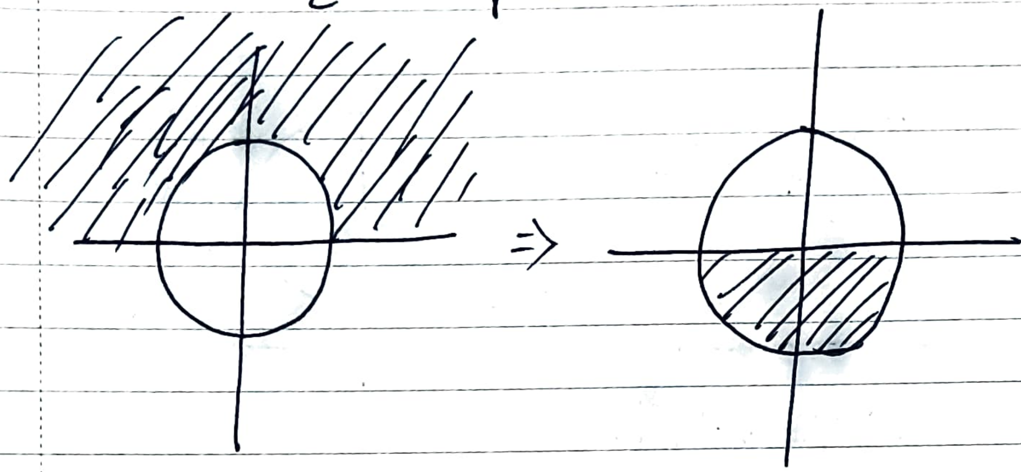


※ 선개

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \rho & 0 \\ 0 & \rho \end{pmatrix} \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

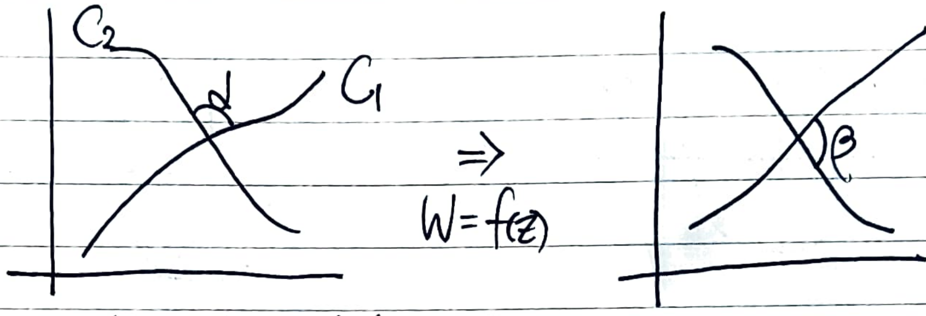
③ Inversion (반전)

$$W = \frac{1}{z} = \frac{1}{r} e^{i(-\theta)}$$



[3] Conformal Mapping (등각사상)

orientation, angle이 유지되는 function이 conformal mapping



$\alpha=\beta$인 것이다.

☆ 영역 D에서 해석적함수 $f(z)$는
$\frac{df}{dz}=0$을 제외한 모든 점에서 Conformal 하다.

ex) $W=z+b$
$\frac{dw}{dz}=1\neq0$ ∴ 항상 Conformal

$\boxed{4}$ Möbius 변환

$$w = \frac{az+b}{cz+d} \quad (ad-bc \neq 0)$$

*주요 등각사상. Translation, Rotation, Scale, Inversion 모두 포함.

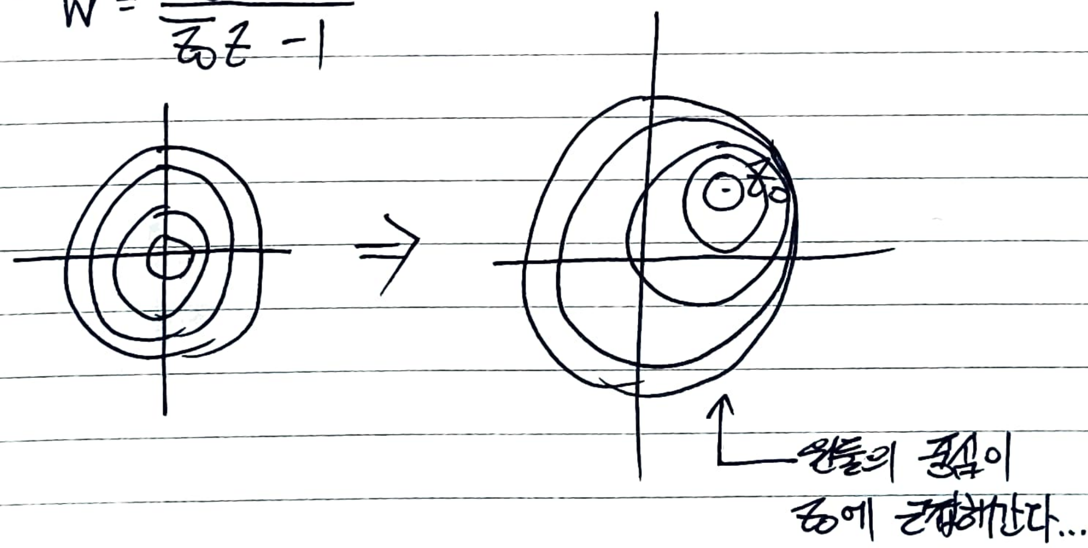
* circle은 line/circle이 결과물로
  line은 line/circle이 결과물로.

* 세 지점이 주어졌을때 Möbius 변환 구하기.

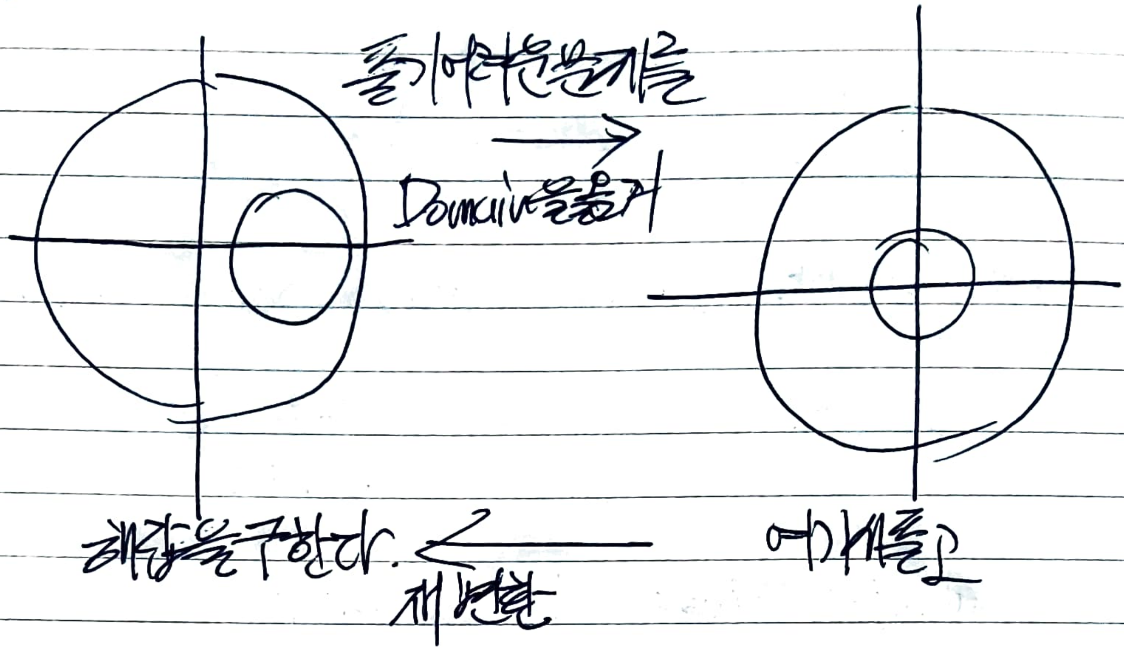
$$\left.\begin{matrix} z_1 \to w_1 \\ z_2 \to w_2 \\ z_3 \to w_3 \end{matrix}\right\} \quad \frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$\boxed{5}$ 단위원의 중심이동 변환

$$w = \frac{z - z_0}{\bar{z_0} z - 1}$$



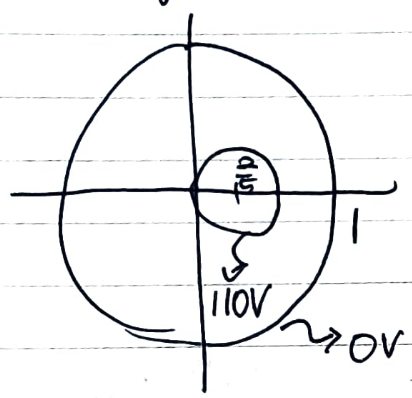
6 등각사상을 통한 문제풀이



ex) Möbius 변환 구하기.

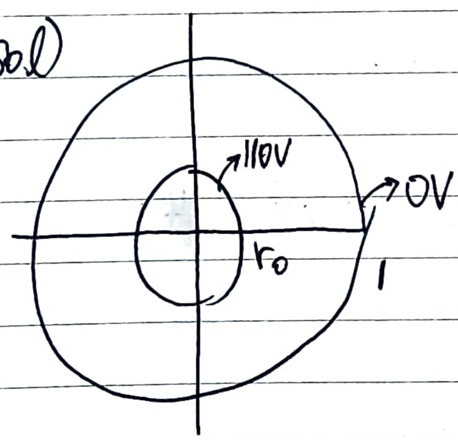
$$\left.\begin{aligned} -1 &\to 1 \\ i &\to 0 \\ 1 &\to -1 \end{aligned}\right\} \Rightarrow \frac{(w-1)(0+1)}{(w+1)(0-1)} = \frac{(z+1)(i-1)}{(z-1)(i+1)}$$

ex) Mapping을 통한 문제해결



이때 내부의 전압 분포는?
이때 전압 u는 Laplace Eq 만족.

sol)



여기서 풀어본다.

극좌표의 Laplace Eq.

$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$. θ에 따른 변화가 없으므로

$u_{rr} + \frac{1}{r}u_r = 0$, $-\frac{1}{r} = \frac{u_{rr}}{u_r}$

$-\ln r + C = \ln|u_r|$ $\therefore u_r = \frac{1}{r}e^c$

$\therefore u = A\ln r + B$

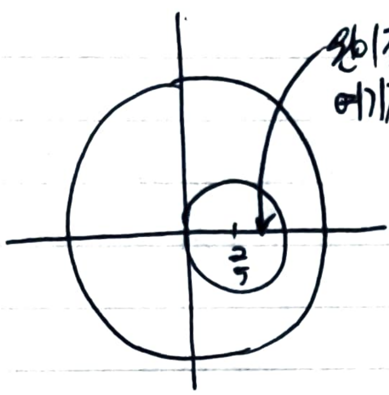
여기서

$r=r_0$: $u = 110 = A\ln r_0 + B$

$r=1$: $u = 0 = B$

$\therefore A = \dfrac{110}{\ln r_0}$, $u = \dfrac{110}{\ln r_0}\ln r = \dfrac{110}{\ln r_0}\ln|z|$

z_0를 구하자...



원이 점점 작아져 중심이 여기까지와야하므로, $z = \frac{2}{5}$

여기서 $r_0 \to 0$으로, $-r_0 \to \frac{4}{5}$ 으로가는 변환이므로...

$$0 = \frac{r_0 - z_0}{\bar{z}_0 r_0 - 1} \quad \therefore r_0 = z_0$$

$$\frac{4}{5} = \frac{-r_0 - z_0}{-\bar{z}_0 r_0 - 1} = \frac{-2z_0}{-\bar{z}_0 z_0 - 1}, \quad -4\bar{z}_0 z_0 - 4 = -10z_0$$

$z_0 = x+iy$ 하면

$-4(x^2+y^2) - 4 = -10(x+iy)$

$4(x^2+y^2) + 4 = 10x + i\cdot 10y$

$y=0$, $4x^2 - 10x + 4 = 0$

$x = \frac{1}{2}$.

$\therefore z_0 = \frac{1}{2}$, $\quad w = \dfrac{z-\frac{1}{2}}{\frac{1}{2}z - 1} = \dfrac{2z-1}{z-2}$

$z = \dfrac{2w-1}{w-2}$

$\therefore u = \dfrac{110}{\ln r_0} \ln\left|\dfrac{2w-1}{w-2}\right|$ 이다..