First let $k = 1$. Expand along the last row to obtain

$$\det A_1 = \det\begin{bmatrix} A & O \\ O & 1 \end{bmatrix} = (-1)^{(n+1)+(n+1)} \cdot 1 \cdot \det A = \det A.$$

Now let $1 < k \le n$ and assume that $\det A_{k-1} = \det A$. Expand along the last row of A_k to obtain

$$\det A_k = \det\begin{bmatrix} A & O \\ O & I_k \end{bmatrix} = (-1)^{(n+k)+(n+k)} \cdot 1 \cdot \det A_{k-1} = \det A_{k-1} = \det A.$$ Thus we have proven the result, and the determinant of the matrix in question is det A.

b. Consider the matrix $A_k = \begin{bmatrix} I_k & O \\ C_k & D \end{bmatrix}$, where $1 \le k \le n$, C_k is an $n \times k$ matrix and O is an appropriately sized zero matrix. We will show that $\det A_k = \det D$ for all $1 \le k \le n$ by mathematical induction.

First let $k = 1$. Expand along the first row to obtain

$$\det A_1 = \det\begin{bmatrix} 1 & O \\ C_1 & D \end{bmatrix} = (-1)^{1+1} \cdot 1 \cdot \det D = \det D.$$

Now let $1 < k \le n$ and assume that $\det A_{k-1} = \det D$. Expand along the first row of A_k to obtain

$$\det A_k = \det\begin{bmatrix} I_k & O \\ C_k & D \end{bmatrix} = (-1)^{1+1} \cdot 1 \cdot \det A_{k-1} = \det A_{k-1} = \det D.$$ Thus we have proven the result, and the determinant of the matrix in question is det D.

c. By combining parts a. and b., we have shown that

$$\det\begin{bmatrix} A & O \\ C & D \end{bmatrix} = \left(\det\begin{bmatrix} A & O \\ O & I \end{bmatrix}\right)\left(\det\begin{bmatrix} I & O \\ C & D \end{bmatrix}\right) = (\det A)(\det D).$$

From this result and Theorem 5, we have

$$\det\begin{bmatrix} A & B \\ O & D \end{bmatrix} = \det\begin{bmatrix} A & B \\ O & D \end{bmatrix}^T = \det\begin{bmatrix} A^T & O \\ B^T & D^T \end{bmatrix} = (\det A^T)(\det D^T) = (\det A)(\det D).$$

15. **a**. Compute the right side of the equation:

$$\begin{bmatrix} I & O \\ X & I \end{bmatrix}\begin{bmatrix} A & B \\ O & Y \end{bmatrix} = \begin{bmatrix} A & B \\ XA & XB+Y \end{bmatrix}$$

Set this equal to the left side of the equation:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A & B \\ XA & XB+Y \end{bmatrix} \text{so that } \quad XA = C \qquad XB + Y = D$$

Since $XA = C$ and A is invertible, $X = CA^{-1}$. Since $XB + Y = D$, $Y = D - XB = D - CA^{-1}B$. Thus by Exercise 14(c),

$$\det\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \det\begin{bmatrix} I & O \\ CA^{-1} & I \end{bmatrix}\det\begin{bmatrix} A & B \\ O & D - CA^{-1}B \end{bmatrix}$$

$$= (\det A)(\det(D - CA^{-1}B))$$

b. From part a.,

$$\det\begin{bmatrix} A & B \\ C & D \end{bmatrix} = (\det A)(\det(D - CA^{-1}B)) = \det[A(D - CA^{-1}B)]$$

$$= \det[AD - ACA^{-1}B] = \det[AD - CAA^{-1}B]$$

$$= \det[AD - CB]$$

16. **a**. Doing the given operations does not change the determinant of A since the given operations are all row replacement operations. The resulting matrix is

$$\begin{bmatrix} a-b & -a+b & 0 & \dots & 0 \\ 0 & a-b & -a+b & \dots & 0 \\ 0 & 0 & a-b & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b & b & b & \dots & a \end{bmatrix}$$

b. Since column replacement operations are equivalent to row operations on A^T and $\det A^T = \det A$, the given operations do not change the determinant of the matrix. The resulting matrix is

$$\begin{bmatrix} a-b & 0 & 0 & \dots & 0 \\ 0 & a-b & 0 & \dots & 0 \\ 0 & 0 & a-b & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b & 2b & 3b & \dots & a+(n-1)b \end{bmatrix}$$

c. Since the preceding matrix is a triangular matrix with the same determinant as A,

$$\det A = (a-b)^{n-1}(a+(n-1)b).$$

17. First consider the case $n = 2$. In this case

$$\det B = \begin{vmatrix} a-b & b \\ 0 & a \end{vmatrix} = a(a-b), \det C = \begin{vmatrix} b & b \\ b & a \end{vmatrix} = ab-b^2,$$

so $\det A = \det B + \det C = a(a-b) + ab - b^2 = a^2 - b^2 = (a-b)(a+b) = (a-b)^{2-1}(a+(2-1)b)$, and the formula holds for $n = 2$.

Now assume that the formula holds for all $(k-1) \times (k-1)$ matrices, and let A, B, and C be $k \times k$ matrices. By a cofactor expansion along the first column,

$$\det B = (a-b)\begin{vmatrix} a & b & \dots & b \\ b & a & \dots & b \\ \vdots & \vdots & \ddots & \vdots \\ b & b & \dots & a \end{vmatrix} = (a-b)(a-b)^{k-2}(a+(k-2)b) = (a-b)^{k-1}(a+(k-2)b)$$

since the matrix in the above formula is a $(k-1) \times (k-1)$ matrix. We can perform a series of row operations on C to "zero out" below the first pivot, and produce the following matrix whose determinant is $\det C$:

$$\begin{bmatrix} b & b & \dots & b \\ 0 & a-b & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a-b \end{bmatrix}.$$

Since this is a triangular matrix, we have found that $\det C = b(a-b)^{k-1}$. Thus

$$\det A = \det B + \det C = (a-b)^{k-1}(a+(k-2)b) + b(a-b)^{k-1} = (a-b)^{k-1}(a+(k-1)b),$$

which is what was to be shown. Thus the formula has been proven by mathematical induction.

18. **[M]** Since the first matrix has $a = 3$, $b = 8$, and $n = 4$, its determinant is $(3-8)^{4-1}(3+(4-1)8) = (-5)^3(3+24) = (-125)(27) = -3375$. Since the second matrix has $a = 8$, $b = 3$, and $n = 5$, its determinant is $(8-3)^{5-1}(8+(5-1)3) = (5)^4(8+12) = (625)(20) = 12{,}500$.

19. **[M]** We find that

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{vmatrix} = 1, \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 \\ 1 & 2 & 3 & 4 \end{vmatrix} = 1, \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 & 3 \\ 1 & 2 & 3 & 4 & 4 \\ 1 & 2 & 3 & 4 & 5 \end{vmatrix} = 1.$$

Our conjecture then is that

$$\begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & 2 & 2 & \dots & 2 \\ 1 & 2 & 3 & \dots & 3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 2 & 3 & \dots & n \end{vmatrix} = 1.$$

To show this, consider using row replacement operations to "zero out" below the first pivot. The resulting matrix is

$$\begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 0 & 1 & 1 & \dots & 1 \\ 0 & 1 & 2 & \dots & 2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 1 & 2 & \dots & n-1 \end{bmatrix}.$$

Now use row replacement operations to "zero out" below the second pivot, and so on. The final matrix which results from this process is

$$\begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 0 & 1 & 1 & \dots & 1 \\ 0 & 0 & 1 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix},$$

which is an upper triangular matrix with determinant 1.

20. **[M]** We find that

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 3 & 3 \\ 1 & 3 & 6 \end{vmatrix} = 6, \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & 3 & 3 \\ 1 & 3 & 6 & 6 \\ 1 & 3 & 6 & 9 \end{vmatrix} = 18, \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 3 & 3 & 3 & 3 \\ 1 & 3 & 6 & 6 & 6 \\ 1 & 3 & 6 & 9 & 9 \\ 1 & 3 & 6 & 9 & 12 \end{vmatrix} = 54.$$

Our conjecture then is that

$$\begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & 3 & 3 & \dots & 3 \\ 1 & 3 & 6 & \dots & 6 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 3 & 6 & \dots & 3(n-1) \end{vmatrix} = 2 \cdot 3^{n-2}.$$

To show this, consider using row replacement operations to "zero out" below the first pivot. The resulting matrix is

$$\begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 0 & 2 & 2 & \dots & 2 \\ 0 & 2 & 5 & \dots & 5 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 2 & 5 & \dots & 3(n-1)-1 \end{bmatrix}.$$

Now use row replacement operations to "zero out" below the second pivot. The matrix which results from this process is

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & \dots & 1 \\ 0 & 2 & 2 & 2 & 2 & 2 & \dots & 2 \\ 0 & 0 & 3 & 3 & 3 & 3 & \dots & 3 \\ 0 & 0 & 3 & 6 & 6 & 6 & \dots & 6 \\ 0 & 0 & 3 & 6 & 9 & 9 & \dots & 9 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 3 & 6 & 9 & 12 & \dots & 3(n-2) \end{bmatrix}.$$

This matrix has the same determinant as the original matrix, and is recognizable as a block matrix of the form

$$\begin{bmatrix} A & B \\ O & D \end{bmatrix},$$

where

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \text{and } D = \begin{bmatrix} 3 & 3 & 3 & 3 & \dots & 3 \\ 3 & 6 & 6 & 6 & \dots & 6 \\ 3 & 6 & 9 & 9 & \dots & 9 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 3 & 6 & 9 & 12 & \dots & 3(n-2) \end{bmatrix} = 3\begin{bmatrix} 1 & 1 & 1 & 1 & \dots & 1 \\ 1 & 2 & 2 & 2 & \dots & 2 \\ 1 & 2 & 3 & 3 & \dots & 3 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 2 & 3 & 4 & \dots & n-2 \end{bmatrix}.$$

As in Exercise 14(c), the determinant of the matrix $\begin{bmatrix} A & B \\ O & D \end{bmatrix}$ is (det A)(det D) = 2 det D.

Since D is an $(n-2) \times (n-2)$ matrix,

$$\det D = 3^{n-2}\begin{vmatrix} 1 & 1 & 1 & 1 & \dots & 1 \\ 1 & 2 & 2 & 2 & \dots & 2 \\ 1 & 2 & 3 & 3 & \dots & 3 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 2 & 3 & 4 & \dots & n-2 \end{vmatrix} = 3^{n-2}(1) = 3^{n-2}$$

by Exercise 19. Thus the determinant of the matrix $\begin{bmatrix} A & B \\ O & D \end{bmatrix}$ is $2\det D = 2 \cdot 3^{n-2}$.

4 Vector Spaces

4.1 SOLUTIONS

Notes: This section is designed to avoid the standard exercises in which a student is asked to check ten axioms on an array of sets. Theorem 1 provides the main homework tool in this section for showing that a set is a subspace. Students should be taught how to check the closure axioms. The exercises in this section (and the next few sections) emphasize $\mathbb{R}^n$, to give students time to absorb the abstract concepts. Other vectors do appear later in the chapter: the space $\mathbb{S}$ of signals is used in Section 4.8, and the spaces $\mathbb{P}_n$ of polynomials are used in many sections of Chapters 4 and 6.

1. **a**. If **u** and **v** are in V, then their entries are nonnegative. Since a sum of nonnegative numbers is nonnegative, the vector **u** + **v** has nonnegative entries. Thus **u** + **v** is in V.

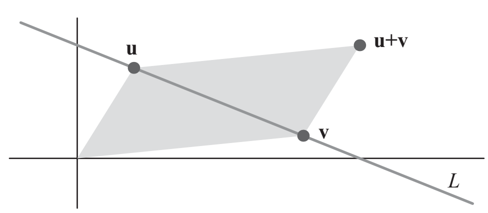
b. *Example:* If $\mathbf{u} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ and $c = -1$, then **u** is in V but c**u** is not in V.

2. **a**. If $\mathbf{u} = \begin{bmatrix} x \\ y \end{bmatrix}$ is in W, then the vector $c\mathbf{u} = c\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} cx \\ cy \end{bmatrix}$ is in W because $(cx)(cy) = c^2(xy) \geq 0$ since $xy \geq 0$.

b. *Example:* If $\mathbf{u} = \begin{bmatrix} -1 \\ -7 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, then **u** and **v** are in W but **u** + **v** is not in W.

3. *Example:* If $\mathbf{u} = \begin{bmatrix} .5 \\ .5 \end{bmatrix}$ and $c = 4$, then **u** is in H but c**u** is not in H. Since H is not closed under scalar multiplication, H is not a subspace of $\mathbb{R}^2$.

4. Note that **u** and **v** are on the line L, but **u** + **v** is not.



5. Yes. Since the set is $\text{Span}\{t^2\}$, the set is a subspace by Theorem 1.

6. No. The zero vector is not in the set.

7. No. The set is not closed under multiplication by scalars which are not integers.

8. Yes. The zero vector is in the set H. If **p** and **q** are in H, then $(\mathbf{p} + \mathbf{q})(0) = \mathbf{p}(0) + \mathbf{q}(0) = 0 + 0 = 0$, so $\mathbf{p} + \mathbf{q}$ is in H. For any scalar c, $(c\mathbf{p})(0) = c \cdot \mathbf{p}(0) = c \cdot 0 = 0$, so $c\mathbf{p}$ is in H. Thus H is a subspace by Theorem 1.

9. The set $H = \text{Span}\{\mathbf{v}\}$, where $\mathbf{v} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$. Thus H is a subspace of $\mathbb{R}^3$ by Theorem 1.

10. The set $H = \text{Span}\{\mathbf{v}\}$, where $\mathbf{v} = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$. Thus H is a subspace of $\mathbb{R}^3$ by Theorem 1.

11. The set $W = \text{Span}\{\mathbf{u}, \mathbf{v}\}$, where $\mathbf{u} = \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$. Thus W is a subspace of $\mathbb{R}^3$ by Theorem 1.

12. The set $W = \text{Span}\{\mathbf{u}, \mathbf{v}\}$, where $\mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 3 \\ -1 \\ -1 \\ 4 \end{bmatrix}$. Thus W is a subspace of $\mathbb{R}^4$ by Theorem 1.

13. **a**. The vector **w** is not in the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. There are 3 vectors in the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

b. The set $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ contains infinitely many vectors.

c. The vector **w** is in the subspace spanned by $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ if and only if the equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{w}$ has a solution. Row reducing the augmented matrix for this system of linear equations gives

$$\begin{bmatrix} 1 & 2 & 4 & 3 \\ 0 & 1 & 2 & 1 \\ -1 & 3 & 6 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

so the equation has a solution and **w** is in the subspace spanned by $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

14. The augmented matrix is found as in Exercise 13c. Since

$$\begin{bmatrix} 1 & 2 & 4 & 8 \\ 0 & 1 & 2 & 4 \\ -1 & 3 & 6 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

the equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{w}$ has no solution, and **w** is not in the subspace spanned by $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

15. Since the zero vector is not in W, W is not a vector space.

16. Since the zero vector is not in W, W is not a vector space.

17. Since a vector **w** in W may be written as

$$\mathbf{w} = a\begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} + b\begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix} + c\begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}$$

$$S = \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} \right\}$$

is a set that spans W.

18. Since a vector **w** in W may be written as

$$\mathbf{w} = a\begin{bmatrix} 4 \\ 0 \\ 1 \\ -2 \end{bmatrix} + b\begin{bmatrix} 3 \\ 0 \\ 1 \\ 0 \end{bmatrix} + c\begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

$$S = \left\{ \begin{bmatrix} 4 \\ 0 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

is a set that spans W.

19. Let H be the set of all functions described by $y(t) = c_1 \cos \omega t + c_2 \sin \omega t$. Then H is a subset of the vector space V of all real-valued functions, and may be written as $H = \text{Span}\{\cos \omega t, \sin \omega t\}$. By Theorem 1, H is a subspace of V and is hence a vector space.

20. **a**. The following facts about continuous functions must be shown.

1. The constant function $\mathbf{f}(t) = 0$ is continuous.

2. The sum of two continuous functions is continuous.

3. A constant multiple of a continuous function is continuous.

b. Let $H = \{\mathbf{f} \text{ in } C[a, b]: \mathbf{f}(a) = \mathbf{f}(b)\}$.

1. Let $\mathbf{g}(t) = 0$ for all t in $[a, b]$. Then $\mathbf{g}(a) = \mathbf{g}(b) = 0$, so **g** is in H.

2. Let **g** and **h** be in H. Then $\mathbf{g}(a) = \mathbf{g}(b)$ and $\mathbf{h}(a) = \mathbf{h}(b)$, and $(\mathbf{g} + \mathbf{h})(a) = \mathbf{g}(a) + \mathbf{h}(a) = \mathbf{g}(b) + \mathbf{h}(b) = (\mathbf{g} + \mathbf{h})(b)$, so $\mathbf{g} + \mathbf{h}$ is in H.

3. Let **g** be in H. Then $\mathbf{g}(a) = \mathbf{g}(b)$, and $(c\mathbf{g})(a) = c\mathbf{g}(a) = c\mathbf{g}(b) = (c\mathbf{g})(b)$, so $c\mathbf{g}$ is in H.

Thus H is a subspace of $C[a, b]$.

21. The set H is a subspace of $M_{2\times 2}$. The zero matrix is in H, the sum of two upper triangular matrices is upper triangular, and a scalar multiple of an upper triangular matrix is upper triangular.

22. The set H is a subspace of $M_{2\times 4}$. The 2×4 zero matrix 0 is in H because $F0 = 0$. If A and B are matrices in H, then $F(A + B) = FA + FB = 0 + 0 = 0$, so $A + B$ is in H. If A is in H and c is a scalar, then $F(cA) = c(FA) = c0 = 0$, so cA is in H.

23. **a**. False. The zero vector in V is the function $\mathbf{f}$ whose values $\mathbf{f}(t)$ are zero *for all t* in $\mathbb{R}$.

b. False. An arrow in three-dimensional space is an example of a vector, but not every arrow is a vector.

c. False. See Exercises 1, 2, and 3 for examples of subsets which contain the zero vector but are not subspaces.

d. True. See the paragraph before Example 6.

e. False. Digital signals are used. See Example 3.

24. **a**. True. See the definition of a vector space.

b. True. See statement (3) in the box before Example 1.

c. True. See the paragraph before Example 6.

d. False. See Example 8.

e. False. The second and third parts of the conditions are stated incorrectly. For example, part (ii) does not state that $\mathbf{u}$ and $\mathbf{v}$ represent all possible elements of H.

25. 2, 4

26. **a**. 3

b. 5

c. 4

27. **a**. 8

b. 3

c. 5

d. 4

28. **a**. 4

b. 7

c. 3

d. 5

e. 4

29. Consider $\mathbf{u} + (-1)\mathbf{u}$. By Axiom 10, $\mathbf{u} + (-1)\mathbf{u} = 1\mathbf{u} + (-1)\mathbf{u}$. By Axiom 8, $1\mathbf{u} + (-1)\mathbf{u} = (1 + (-1))\mathbf{u} = 0\mathbf{u}$. By Exercise 27, $0\mathbf{u} = \mathbf{0}$. Thus $\mathbf{u} + (-1)\mathbf{u} = \mathbf{0}$, and by Exercise 26 $(-1)\mathbf{u} = -\mathbf{u}$.

30. By Axiom 10 $\mathbf{u} = 1\mathbf{u}$. Since c is nonzero, $c^{-1}c = 1$, and $\mathbf{u} = (c^{-1}c)\mathbf{u}$. By Axiom 9, $(c^{-1}c)\mathbf{u} = c^{-1}(c\mathbf{u}) = c^{-1}\mathbf{0}$ since $c\mathbf{u} = \mathbf{0}$. Thus $\mathbf{u} = c^{-1}\mathbf{0} = \mathbf{0}$ by Property (2), proven in Exercise 28.

31. Any subspace H that contains $\mathbf{u}$ and $\mathbf{v}$ must also contain all scalar multiples of $\mathbf{u}$ and $\mathbf{v}$, and hence must also contain all sums of scalar multiples of $\mathbf{u}$ and $\mathbf{v}$. Thus H must contain all linear combinations of $\mathbf{u}$ and $\mathbf{v}$, or Span$\{\mathbf{u}, \mathbf{v}\}$.

Note: Exercises 32–34 provide good practice for mathematics majors because these arguments involve simple symbol manipulation typical of mathematical proofs. Most students outside mathematics might profit more from other types of exercises.

32. Both H and K contain the zero vector of V because they are subspaces of V. Thus the zero vector of V is in $H \cap K$. Let $\mathbf{u}$ and $\mathbf{v}$ be in $H \cap K$. Then $\mathbf{u}$ and $\mathbf{v}$ are in H. Since H is a subspace $\mathbf{u} + \mathbf{v}$ is in H. Likewise $\mathbf{u}$ and $\mathbf{v}$ are in K. Since K is a subspace $\mathbf{u} + \mathbf{v}$ is in K. Thus $\mathbf{u} + \mathbf{v}$ is in $H \cap K$. Let $\mathbf{u}$ be in $H \cap K$. Then $\mathbf{u}$ is in H. Since H is a subspace $c\mathbf{u}$ is in H. Likewise $\mathbf{v}$ is in K. Since K is a subspace $c\mathbf{u}$ is in K. Thus $c\mathbf{u}$ is in $H \cap K$ for any scalar c, and $H \cap K$ is a subspace of V.

The union of two subspaces is not in general a subspace. For an example in $\mathbb{R}^2$ let H be the x-axis and let K be the y-axis. Then both H and K are subspaces of $\mathbb{R}^2$, but $H \cup K$ is not closed under vector addition. The subset $H \cup K$ is thus not a subspace of $\mathbb{R}^2$.

33. **a**. Given subspaces H and K of a vector space V, the zero vector of V belongs to $H + K$, because $\mathbf{0}$ is in both H and K (since they are subspaces) and $\mathbf{0} = \mathbf{0} + \mathbf{0}$. Next, take two vectors in $H + K$, say $\mathbf{w}_1 = \mathbf{u}_1 + \mathbf{v}_1$ and $\mathbf{w}_2 = \mathbf{u}_2 + \mathbf{v}_2$ where $\mathbf{u}_1$ and $\mathbf{u}_2$ are in H, and $\mathbf{v}_1$ and $\mathbf{v}_2$ are in K. Then

$$\mathbf{w}_1 + \mathbf{w}_2 = \mathbf{u}_1 + \mathbf{v}_1 + \mathbf{u}_2 + \mathbf{v}_2 = (\mathbf{u}_1 + \mathbf{u}_2) + (\mathbf{v}_1 + \mathbf{v}_2)$$

because vector addition in V is commutative and associative. Now $\mathbf{u}_1 + \mathbf{u}_2$ is in H and $\mathbf{v}_1 + \mathbf{v}_2$ is in K because H and K are subspaces. This shows that $\mathbf{w}_1 + \mathbf{w}_2$ is in $H + K$. Thus $H + K$ is closed under addition of vectors. Finally, for any scalar c,

$$c\mathbf{w}_1 = c(\mathbf{u}_1 + \mathbf{v}_1) = c\mathbf{u}_1 + c\mathbf{v}_1$$

The vector $c\mathbf{u}_1$ belongs to H and $c\mathbf{v}_1$ belongs to K, because H and K are subspaces. Thus, $c\mathbf{w}_1$ belongs to $H + K$, so $H + K$ is closed under multiplication by scalars. These arguments show that $H + K$ satisfies all three conditions necessary to be a subspace of V.

b. Certainly H is a subset of $H + K$ because every vector $\mathbf{u}$ in H may be written as $\mathbf{u} + \mathbf{0}$, where the zero vector $\mathbf{0}$ is in K (and also in H, of course). Since H contains the zero vector of $H + K$, and H is closed under vector addition and multiplication by scalars (because H is a subspace of V), H is a subspace of $H + K$. The same argument applies when H is replaced by K, so K is also a subspace of $H + K$.

34. A proof that $H + K = \text{Span}\{\mathbf{u}_1, \ldots, \mathbf{u}_p, \mathbf{v}_1, \ldots, \mathbf{v}_q\}$ has two parts. First, one must show that $H + K$ is a subset of $\text{Span}\{\mathbf{u}_1, \ldots, \mathbf{u}_p, \mathbf{v}_1, \ldots, \mathbf{v}_q\}$. Second, one must show that $\text{Span}\{\mathbf{u}_1, \ldots, \mathbf{u}_p, \mathbf{v}_1, \ldots, \mathbf{v}_q\}$ is a subset of $H + K$.

(1) A typical vector H has the form $c_1\mathbf{u}_1 + \ldots + c_p\mathbf{u}_p$ and a typical vector in K has the form $d_1\mathbf{v}_1 + \ldots + d_q\mathbf{v}_q$. The sum of these two vectors is a linear combination of $\mathbf{u}_1, \ldots, \mathbf{u}_p, \mathbf{v}_1, \ldots, \mathbf{v}_q$ and so belongs to $\text{Span}\{\mathbf{u}_1, \ldots, \mathbf{u}_p, \mathbf{v}_1, \ldots, \mathbf{v}_q\}$. Thus $H + K$ is a subset of $\text{Span}\{\mathbf{u}_1, \ldots, \mathbf{u}_p, \mathbf{v}_1, \ldots, \mathbf{v}_q\}$.

(2) Each of the vectors $\mathbf{u}_1, \ldots, \mathbf{u}_p, \mathbf{v}_1, \ldots, \mathbf{v}_q$ belongs to $H + K$, by Exercise 33(b), and so any linear combination of these vectors belongs to $H + K$, since $H + K$ is a subspace, by Exercise 33(a). Thus, $\text{Span}\{\mathbf{u}_1, \ldots, \mathbf{u}_p, \mathbf{v}_1, \ldots, \mathbf{v}_q\}$ is a subset of $H + K$.

35. **[M]** Since

$$\begin{bmatrix} 7 & -4 & -9 & -9 \\ -4 & 5 & 4 & 7 \\ -2 & -1 & 4 & 4 \\ 9 & -7 & -7 & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 15/2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 11/2 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

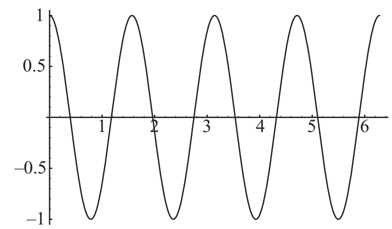
$\mathbf{w}$ is in the subspace spanned by $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

36. **[M]** Since

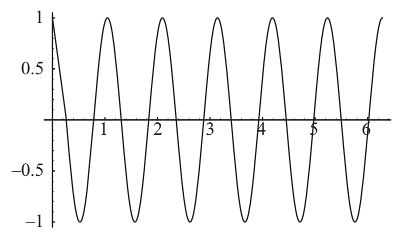
$$[A \quad \mathbf{y}] = \begin{bmatrix} 5 & -5 & -9 & 6 \\ 8 & 8 & -6 & 7 \\ -5 & -9 & 3 & 1 \\ 3 & -2 & -7 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 11/2 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 7/2 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$\mathbf{y}$ is in the subspace spanned by the columns of A.

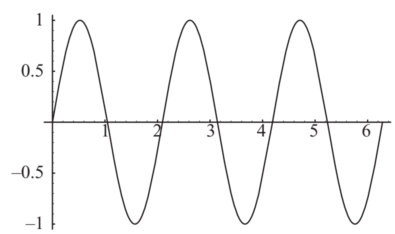
37. **[M]** The graph of $\mathbf{f}(t)$ is given below. A conjecture is that $\mathbf{f}(t) = \cos 4t$.



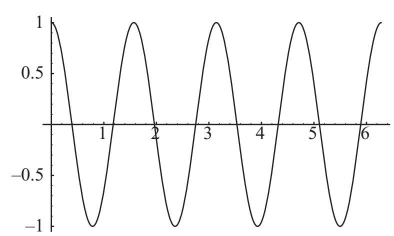
The graph of $\mathbf{g}(t)$ is given below. A conjecture is that $\mathbf{g}(t) = \cos 6t$.



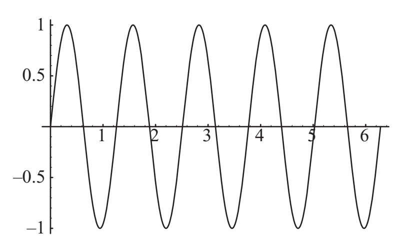
38. **[M]** The graph of $\mathbf{f}(t)$ is given below. A conjecture is that $\mathbf{f}(t) = \sin 3t$.



The graph of $\mathbf{g}(t)$ is given below. A conjecture is that $\mathbf{g}(t) = \cos 4t$.



The graph of $\mathbf{h}(t)$ is given below. A conjecture is that $\mathbf{h}(t) = \sin 5t$.



4.2 SOLUTIONS

Notes: This section provides a review of Chapter 1 using the new terminology. Linear tranformations are introduced quickly since students are already comfortable with the idea from $\mathbb{R}^n$. The key exercises are 17–26, which are straightforward but help to solidify the notions of null spaces and column spaces. Exercises 30–36 deal with the kernel and range of a linear transformation and are progressively more advanced theoretically. The idea in Exercises 7–14 is for the student to use Theorems 1, 2, or 3 to determine whether a given set is a subspace.

1. One calculates that

$$A\mathbf{w} = \begin{bmatrix} 3 & -5 & -3 \\ 6 & -2 & 0 \\ -8 & 4 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ -4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

so $\mathbf{w}$ is in Nul A.

2. One calculates that

$$A\mathbf{w} = \begin{bmatrix} 5 & 21 & 19 \\ 13 & 23 & 2 \\ 8 & 14 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ -3 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

so $\mathbf{w}$ is in Nul A.

3. First find the general solution of $A\mathbf{x} = \mathbf{0}$ in terms of the free variables. Since

$$\begin{bmatrix} A & \mathbf{0} \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -7 & 6 & 0 \\ 0 & 1 & 4 & -2 & 0 \end{bmatrix},$$

the general solution is $x_1 = 7x_3 - 6x_4$, $x_2 = -4x_3 + 2x_4$, with x_3 and x_4 free. So

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 7 \\ -4 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -6 \\ 2 \\ 0 \\ 1 \end{bmatrix},$$

and a spanning set for Nul A is

$$\left\{ \begin{bmatrix} 7 \\ -4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -6 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

4. First find the general solution of $A\mathbf{x} = \mathbf{0}$ in terms of the free variables. Since

$$\begin{bmatrix} A & \mathbf{0} \end{bmatrix} \sim \begin{bmatrix} 1 & -6 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix},$$

the general solution is $x_1 = 6x_2$, $x_3 = 0$, with x_2 and x_4 free. So

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} 6 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix},$$

and a spanning set for Nul A is

$$\left\{\begin{bmatrix}6\\1\\0\\0\end{bmatrix},\begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right\}.$$

5. First find the general solution of $A\mathbf{x} = \mathbf{0}$ in terms of the free variables. Since

$$\begin{bmatrix}A & \mathbf{0}\end{bmatrix} \sim \begin{bmatrix}1 & -2 & 0 & 4 & 0 & 0\\0 & 0 & 1 & -9 & 0 & 0\\0 & 0 & 0 & 0 & 1 & 0\end{bmatrix},$$

the general solution is $x_1 = 2x_2 - 4x_4$, $x_3 = 9x_4$, $x_5 = 0$, with x_2 and x_4 free. So

$$\mathbf{x} = \begin{bmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{bmatrix} = x_2\begin{bmatrix}2\\1\\0\\0\\0\end{bmatrix} + x_4\begin{bmatrix}-4\\0\\9\\1\\0\end{bmatrix},$$

and a spanning set for Nul A is

$$\left\{\begin{bmatrix}2\\1\\0\\0\\0\end{bmatrix},\begin{bmatrix}-4\\0\\9\\1\\0\end{bmatrix}\right\}.$$

6. First find the general solution of $A\mathbf{x} = \mathbf{0}$ in terms of the free variables. Since

$$\begin{bmatrix}A & \mathbf{0}\end{bmatrix} \sim \begin{bmatrix}1 & 0 & 6 & -8 & 1 & 0\\0 & 1 & -2 & 1 & 0 & 0\\0 & 0 & 0 & 0 & 0 & 0\end{bmatrix},$$

the general solution is $x_1 = -6x_3 + 8x_4 - x_5$, $x_2 = 2x_3 - x_4$, with x_3, x_4, and x_5 free. So

$$\mathbf{x} = \begin{bmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{bmatrix} = x_3\begin{bmatrix}-6\\2\\1\\0\\0\end{bmatrix} + x_4\begin{bmatrix}8\\-1\\0\\1\\0\end{bmatrix} + x_5\begin{bmatrix}-1\\0\\0\\0\\1\end{bmatrix},$$

and a spanning set for Nul A is

$$\left\{\begin{bmatrix}-6\\2\\1\\0\\0\end{bmatrix},\begin{bmatrix}8\\-1\\0\\1\\0\end{bmatrix},\begin{bmatrix}-1\\0\\0\\0\\1\end{bmatrix}\right\}.$$

7. The set W is a subset of $\mathbb{R}^3$. If W were a vector space (under the standard operations in $\mathbb{R}^3$), then it would be a subspace of $\mathbb{R}^3$. But W is not a subspace of $\mathbb{R}^3$ since the zero vector is not in W. Thus W is not a vector space.

8. The set W is a subset of $\mathbb{R}^3$. If W were a vector space (under the standard operations in $\mathbb{R}^3$), then it would be a subspace of $\mathbb{R}^3$. But W is not a subspace of $\mathbb{R}^3$ since the zero vector is not in W. Thus W is not a vector space.

9. The set W is the set of all solutions to the homogeneous system of equations $a - 2b - 4c = 0$, $2a - c - 3d = 0$. Thus $W = \text{Nul } A$, where $A = \begin{bmatrix} 1 & -2 & -4 & 0 \\ 2 & 0 & -1 & -3 \end{bmatrix}$. Thus W is a subspace of $\mathbb{R}^4$ by Theorem 2, and is a vector space.

10. The set W is the set of all solutions to the homogeneous system of equations $a + 3b - c = 0$, $a + b + c - d = 0$. Thus $W = \text{Nul } A$, where $A = \begin{bmatrix} 1 & 3 & -1 & 0 \\ 1 & 1 & 1 & -1 \end{bmatrix}$. Thus W is a subspace of $\mathbb{R}^4$ by Theorem 2, and is a vector space.

11. The set W is a subset of $\mathbb{R}^4$. If W were a vector space (under the standard operations in $\mathbb{R}^4$), then it would be a subspace of $\mathbb{R}^4$. But W is not a subspace of $\mathbb{R}^4$ since the zero vector is not in W. Thus W is not a vector space.

12. The set W is a subset of $\mathbb{R}^4$. If W were a vector space (under the standard operations in $\mathbb{R}^4$), then it would be a subspace of $\mathbb{R}^4$. But W is not a subspace of $\mathbb{R}^4$ since the zero vector is not in W. Thus W is not a vector space.

13. An element $\mathbf{w}$ on W may be written as

$$\mathbf{w} = c\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + d\begin{bmatrix} -6 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & -6 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} c \\ d \end{bmatrix}$$

where c and d are any real numbers. So $W = \text{Col } A$ where $A = \begin{bmatrix} 1 & -6 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$. Thus W is a subspace of $\mathbb{R}^3$ by Theorem 3, and is a vector space.

14. An element $\mathbf{w}$ on W may be written as

$$\mathbf{w} = a\begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix} + b\begin{bmatrix} 2 \\ -2 \\ -6 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 1 & -2 \\ 3 & -6 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix}$$

where a and b are any real numbers. So $W = \text{Col } A$ where $A = \begin{bmatrix} -1 & 2 \\ 1 & -2 \\ 3 & -6 \end{bmatrix}$. Thus W is a subspace of $\mathbb{R}^3$ by Theorem 3, and is a vector space.

15. An element in this set may be written as

$$r\begin{bmatrix}0\\1\\4\\3\end{bmatrix}+s\begin{bmatrix}2\\1\\1\\-1\end{bmatrix}+t\begin{bmatrix}3\\-2\\0\\-1\end{bmatrix}=\begin{bmatrix}0&2&3\\1&1&-2\\4&1&0\\3&-1&-1\end{bmatrix}\begin{bmatrix}r\\s\\t\end{bmatrix}$$

where r, s and t are any real numbers. So the set is Col A where $A=\begin{bmatrix}0&2&3\\1&1&-2\\4&1&0\\3&-1&-1\end{bmatrix}$.

16. An element in this set may be written as

$$b\begin{bmatrix}1\\2\\0\\0\end{bmatrix}+c\begin{bmatrix}-1\\1\\5\\0\end{bmatrix}+d\begin{bmatrix}0\\1\\-4\\1\end{bmatrix}=\begin{bmatrix}1&-1&0\\2&1&1\\0&5&-4\\0&0&1\end{bmatrix}\begin{bmatrix}b\\c\\d\end{bmatrix}$$

where b, c and d are any real numbers. So the set is Col A where $A=\begin{bmatrix}1&-1&0\\2&1&1\\0&5&-4\\0&0&1\end{bmatrix}$.

17. The matrix A is a 4×2 matrix. Thus
(a) Nul A is a subspace of $\mathbb{R}^2$, and
(b) Col A is a subspace of $\mathbb{R}^4$.

18. The matrix A is a 4×3 matrix. Thus
(a) Nul A is a subspace of $\mathbb{R}^3$, and
(b) Col A is a subspace of $\mathbb{R}^4$.

19. The matrix A is a 2×5 matrix. Thus
(a) Nul A is a subspace of $\mathbb{R}^5$, and
(b) Col A is a subspace of $\mathbb{R}^2$.

20. The matrix A is a 1×5 matrix. Thus
(a) Nul A is a subspace of $\mathbb{R}^5$, and
(b) Col A is a subspace of $\mathbb{R}^1=\mathbb{R}$.

21. Either column of A is a nonzero vector in Col A. To find a nonzero vector in Nul A, find the general solution of $A\mathbf{x}=\mathbf{0}$ in terms of the free variables. Since

$$\begin{bmatrix}A & \mathbf{0}\end{bmatrix}\sim\begin{bmatrix}1&-3&0\\0&0&0\\0&0&0\\0&0&0\end{bmatrix},$$

the general solution is $x_1 = 3x_2$, with x_2 free. Letting x_2 be a nonzero value (say $x_2 = 1$) gives the nonzero vector

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

which is in Nul A.

22. Any column of A is a nonzero vector in Col A. To find a nonzero vector in Nul A, find the general solution of $A\mathbf{x} = \mathbf{0}$ in terms of the free variables. Since

$$\begin{bmatrix} A & \mathbf{0} \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -7 & 6 & 0 \\ 0 & 1 & 4 & -2 & 0 \end{bmatrix},$$

the general solution is $x_1 = 7x_3 - 6x_4$, $x_2 = -4x_3 + 2x_4$, with x_3 and x_4 free. Letting x_3 and x_4 be nonzero values (say $x_3 = x_4 = 1$) gives the nonzero vector

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \\ 1 \end{bmatrix}$$

which is in Nul A.

23. Consider the system with augmented matrix $\begin{bmatrix} A & \mathbf{w} \end{bmatrix}$. Since

$$\begin{bmatrix} A & \mathbf{w} \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & -1/3 \\ 0 & 0 & 0 \end{bmatrix},$$

the system is consistent and $\mathbf{w}$ is in Col A. Also, since

$$A\mathbf{w} = \begin{bmatrix} -6 & 12 \\ -3 & 6 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\mathbf{w}$ is in Nul A.

24. Consider the system with augmented matrix $\begin{bmatrix} A & \mathbf{w} \end{bmatrix}$. Since

$$\begin{bmatrix} A & \mathbf{w} \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & -1/2 \\ 0 & 1 & 1/2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

the system is consistent and $\mathbf{w}$ is in Col A. Also, since

$$A\mathbf{w} = \begin{bmatrix} -8 & -2 & -9 \\ 6 & 4 & 8 \\ 4 & 0 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\mathbf{w}$ is in Nul A.

25. **a**. True. See the definition before Example 1.

b. False. See Theorem 2.

c. True. See the remark just before Example 4.

d. False. The equation $A\mathbf{x} = \mathbf{b}$ must be consistent *for every* $\mathbf{b}$. See #7 in the table on page 226.

e. True. See Figure 2.

f. True. See the remark after Theorem 3.

26. **a**. True. See Theorem 2.
b. True. See Theorem 3.
c. False. See the box after Theorem 3.
d. True. See the paragraph after the definition of a linear transformation.
e. True. See Figure 2.
f. True. See the paragraph before Example 8.

27. Let A be the coefficient matrix of the given homogeneous system of equations. Since $A\mathbf{x} = \mathbf{0}$ for $\mathbf{x} = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$, $\mathbf{x}$ is in Nul$A$. Since Nul$A$ is a subspace of $\mathbb{R}^3$, it is closed under scalar multiplication. Thus $10\mathbf{x} = \begin{bmatrix} 30 \\ 20 \\ -10 \end{bmatrix}$ is also in Nul$A$, and $x_1 = 30$, $x_2 = 20$, $x_3 = -10$ is also a solution to the system of equations.

28. Let A be the coefficient matrix of the given systems of equations. Since the first system has a solution, the constant vector $\mathbf{b} = \begin{bmatrix} 0 \\ 1 \\ 9 \end{bmatrix}$ is in Col$A$. Since Col A is a subspace of $\mathbb{R}^3$, it is closed under scalar multiplication. Thus $5\mathbf{b} = \begin{bmatrix} 0 \\ 5 \\ 45 \end{bmatrix}$ is also in Col A, and the second system of equations must thus have a solution.

29. **a**. Since $A\mathbf{0} = \mathbf{0}$, the zero vector is in Col A.
b. Since $A\mathbf{x} + A\mathbf{w} = A(\mathbf{x} + \mathbf{w}), A\mathbf{x} + A\mathbf{w}$ is in Col A.
c. Since $c(A\mathbf{x}) = A(c\mathbf{x}), cA\mathbf{x}$ is in Col A.

30. Since $T(\mathbf{0}_V) = \mathbf{0}_W$, the zero vector $\mathbf{0}_W$ of W is in the range of T. Let $T(\mathbf{x})$ and $T(\mathbf{w})$ be typical elements in the range of T. Then since $T(\mathbf{x}) + T(\mathbf{w}) = T(\mathbf{x} + \mathbf{w}), T(\mathbf{x}) + T(\mathbf{w})$ is in the range of T and the range of T is closed under vector addition. Let c be any scalar. Then since $cT(\mathbf{x}) = T(c\mathbf{x}), cT(\mathbf{x})$ is in the range of T and the range of T is closed under scalar multiplication. Hence the range of T is a subspace of W.

31. **a**. Let $\mathbf{p}$ and $\mathbf{q}$ be arbitary polynomials in $\mathbb{P}_2$, and let c be any scalar. Then

$$T(\mathbf{p} + \mathbf{q}) = \begin{bmatrix} (\mathbf{p} + \mathbf{q})(0) \\ (\mathbf{p} + \mathbf{q})(1) \end{bmatrix} = \begin{bmatrix} \mathbf{p}(0) + \mathbf{q}(0) \\ \mathbf{p}(1) + \mathbf{q}(1) \end{bmatrix} = \begin{bmatrix} \mathbf{p}(0) \\ \mathbf{p}(1) \end{bmatrix} + \begin{bmatrix} \mathbf{q}(0) \\ \mathbf{q}(1) \end{bmatrix} = T(\mathbf{p}) + T(\mathbf{q})$$

and

$$T(c\mathbf{p}) = \begin{bmatrix} (c\mathbf{p})(0) \\ (c\mathbf{p})(1) \end{bmatrix} = c\begin{bmatrix} \mathbf{p}(0) \\ \mathbf{p}(1) \end{bmatrix} = cT(\mathbf{p})$$

so T is a linear transformation.

b. Any quadratic polynomial **q** for which $\mathbf{q}(0)=0$ and $\mathbf{q}(1)=0$ will be in the kernel of *T*. The polynomial **q** must then be a multiple of $\mathbf{p}(t)=t(t-1)$. Given any vector $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ in $\mathbb{R}^2$, the polynomial $\mathbf{p}=x_1+(x_2-x_1)t$ has $\mathbf{p}(0)=x_1$ and $\mathbf{p}(1)=x_2$. Thus the range of *T* is all of $\mathbb{R}^2$.

32. Any quadratic polynomial **q** for which $\mathbf{q}(0)=0$ will be in the kernel of *T*. The polynomial **q** must then be $\mathbf{q}=at+bt^2$. Thus the polynomials $\mathbf{p}_1(t)=t$ and $\mathbf{p}_2(t)=t^2$ span the kernel of *T*. If a vector is in the range of *T*, it must be of the form $\begin{bmatrix} a \\ a \end{bmatrix}$. If a vector is of this form, it is the image of the polynomial $\mathbf{p}(t)=a$ in $\mathbb{P}_2$. Thus the range of *T* is $\left\{ \begin{bmatrix} a \\ a \end{bmatrix} : a \text{ real} \right\}$.

33. **a**. For any *A* and *B* in $M_{2\times 2}$ and for any scalar *c*,

$$T(A+B)=(A+B)+(A+B)^T=A+B+A^T+B^T=(A+A^T)+(B+B^T)=T(A)+T(B)$$

and

$$T(cA)=(cA)^T=c(A^T)=cT(A)$$

so *T* is a linear transformation.

b. Let *B* be an element of $M_{2\times 2}$ with $B^T=B$, and let $A=\frac{1}{2}B$. Then

$$T(A)=A+A^T=\frac{1}{2}B+(\frac{1}{2}B)^T=\frac{1}{2}B+\frac{1}{2}B^T=\frac{1}{2}B+\frac{1}{2}B=B$$

c. Part b. showed that the range of *T* contains the set of all *B* in $M_{2\times 2}$ with $B^T=B$. It must also be shown that any *B* in the range of *T* has this property. Let *B* be in the range of *T*. Then *B* = *T*(*A*) for some *A* in $M_{2\times 2}$. Then $B=A+A^T$, and

$$B^T=(A+A^T)^T=A^T+(A^T)^T=A^T+A=A+A^T=B$$

so *B* has the property that $B^T=B$.

d. Let $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be in the kernel of *T*. Then $T(A)=A+A^T=0$, so

$$A+A^T=\begin{bmatrix} a & b \\ c & d \end{bmatrix}+\begin{bmatrix} a & c \\ b & d \end{bmatrix}=\begin{bmatrix} 2a & c+b \\ b+c & 2d \end{bmatrix}=\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Solving it is found that $a=d=0$ and $c=-b$. Thus the kernel of *T* is $\left\{ \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix} : b \text{ real} \right\}$.

34. Let **f** and **g** be any elements in *C*[0, 1] and let *c* be any scalar. Then *T*(**f**) is the antiderivative **F** of **f** with **F**(0) = 0 and *T*(**g**) is the antiderivative **G** of **g** with **G**(0) = 0. By the rules for antidifferentiation $\mathbf{F}+\mathbf{G}$ will be an antiderivative of $\mathbf{f}+\mathbf{g}$, and $(\mathbf{F}+\mathbf{G})(0)=\mathbf{F}(0)+\mathbf{G}(0)=0+0=0$. Thus $T(\mathbf{f}+\mathbf{g})=T(\mathbf{f})+T(\mathbf{g})$. Likewise *c***F** will be an antiderivative of *c***f**, and $(c\mathbf{F})(0)=c\mathbf{F}(0)=c0=0$. Thus $T(c\mathbf{f})=cT(\mathbf{f})$, and *T* is a linear transformation. To find the kernel of *T*, we must find all functions *f* in *C*[0,1] with antiderivative equal to the zero function. The only function with this property is the zero function **0**, so the kernel of *T* is $\{\mathbf{0}\}$.

35. Since U is a subspace of V, $\mathbf{0}_V$ is in U. Since T is linear, $T(\mathbf{0}_V)=\mathbf{0}_W$. So $\mathbf{0}_W$ is in $T(U)$. Let $T(\mathbf{x})$ and $T(\mathbf{y})$ be typical elements in $T(U)$. Then $\mathbf{x}$ and $\mathbf{y}$ are in U, and since U is a subspace of V, $\mathbf{x}+\mathbf{y}$ is also in U. Since T is linear, $T(\mathbf{x})+T(\mathbf{y})=T(\mathbf{x}+\mathbf{y})$. So $T(\mathbf{x})+T(\mathbf{y})$ is in $T(U)$, and $T(U)$ is closed under vector addition. Let c be any scalar. Then since $\mathbf{x}$ is in U and U is a subspace of V, $c\mathbf{x}$ is in U. Since T is linear, $T(c\mathbf{x})=cT(\mathbf{x})$ and $cT(\mathbf{x})$ is in $T(U)$. Thus $T(U)$ is closed under scalar multiplication, and $T(U)$ is a subspace of W.

36. Since Z is a subspace of W, $\mathbf{0}_W$ is in Z. Since T is linear, $T(\mathbf{0}_V)=\mathbf{0}_W$. So $\mathbf{0}_V$ is in U. Let $\mathbf{x}$ and $\mathbf{y}$ be typical elements in U. Then $T(\mathbf{x})$ and $T(\mathbf{y})$ are in Z, and since Z is a subspace of W, $T(\mathbf{x})+T(\mathbf{y})$ is also in Z. Since T is linear, $T(\mathbf{x})+T(\mathbf{y})=T(\mathbf{x}+\mathbf{y})$. So $T(\mathbf{x}+\mathbf{y})$ is in Z, and $\mathbf{x}+\mathbf{y}$ is in U. Thus U is closed under vector addition. Let c be any scalar. Then since $\mathbf{x}$ is in U, $T(\mathbf{x})$ is in Z. Since Z is a subspace of W, $cT(\mathbf{x})$ is also in Z. Since T is linear, $cT(\mathbf{x})=T(c\mathbf{x})$ and $T(c\mathbf{x})$ is in $T(U)$. Thus $c\mathbf{x}$ is in U and U is closed under scalar multiplication. Hence U is a subspace of V.

37. **[M]** Consider the system with augmented matrix $\begin{bmatrix} A & \mathbf{w}\end{bmatrix}$. Since

$$\begin{bmatrix} A & \mathbf{w}\end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -1/95 & 1/95 \\ 0 & 1 & 0 & 39/19 & -20/19 \\ 0 & 0 & 1 & 267/95 & -172/95 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

the system is consistent and $\mathbf{w}$ is in Col$A$. Also, since

$$A\mathbf{w} = \begin{bmatrix} 7 & 6 & -4 & 1 \\ -5 & -1 & 0 & -2 \\ 9 & -11 & 7 & -3 \\ 19 & -9 & 7 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ -1 \\ -3 \end{bmatrix} = \begin{bmatrix} 14 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$\mathbf{w}$ is not in Nul$A$.

38. **[M]** Consider the system with augmented matrix $\begin{bmatrix} A & \mathbf{w}\end{bmatrix}$. Since

$$\begin{bmatrix} A & \mathbf{w}\end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & 0 & -2 \\ 0 & 1 & -2 & 0 & -3 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

the system is consistent and $\mathbf{w}$ is in Col$A$. Also, since

$$A\mathbf{w} = \begin{bmatrix} -8 & 5 & -2 & 0 \\ -5 & 2 & 1 & -2 \\ 10 & -8 & 6 & -3 \\ 3 & -2 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$\mathbf{w}$ is in Nul$A$.

39. **[M]**

a. To show that $\mathbf{a}_3$ and $\mathbf{a}_5$ are in the column space of *B*, we can row reduce the matrices $[B \quad \mathbf{a}_3]$ and $[B \quad \mathbf{a}_3]$:

$$[B \quad \mathbf{a}_3] \sim \begin{bmatrix} 1 & 0 & 0 & 1/3 \\ 0 & 1 & 0 & 1/3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$[B \quad \mathbf{a}_5] \sim \begin{bmatrix} 1 & 0 & 0 & 10/3 \\ 0 & 1 & 0 & -26/3 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since both these systems are consistent, $\mathbf{a}_3$ and $\mathbf{a}_5$ are in the column space of *B*. Notice that the same conclusions can be drawn by observing the reduced row echelon form for *A*:

$$A \sim \begin{bmatrix} 1 & 0 & 1/3 & 0 & 10/3 \\ 0 & 1 & 1/3 & 0 & -26/3 \\ 0 & 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

b. We find the general solution of $A\mathbf{x} = \mathbf{0}$ in terms of the free variables by using the reduced row echelon form of *A* given above: $x_1 = (-1/3)x_3 - (10/3)x_5$, $x_2 = (-1/3)x_3 + (26/3)x_5$, $x_4 = 4x_5$ with x_3 and x_5 free. So

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} -1/3 \\ -1/3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -10/3 \\ 26/3 \\ 0 \\ 4 \\ 1 \end{bmatrix},$$

and a spanning set for Nul *A* is

$$\left\{ \begin{bmatrix} -1/3 \\ -1/3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -10/3 \\ 26/3 \\ 0 \\ 4 \\ 1 \end{bmatrix} \right\}.$$

c. The reduced row echelon form of *A* shows that the columns of *A* are linearly dependent and do not span $\mathbb{R}^4$. Thus by Theorem 12 in Section 1.9, *T* is neither one-to-one nor onto.

40. **[M]** Since the line lies both in $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$ and in $K = \text{Span}\{\mathbf{v}_3, \mathbf{v}_4\}$, **w** can be written both as $c_1\mathbf{v}_1 + c_2\mathbf{v}_2$ and $c_3\mathbf{v}_3 + c_4\mathbf{v}_4$. To find ***w*** we must find the c_j's which solve $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 - c_3\mathbf{v}_3 - c_4\mathbf{v}_4 = \mathbf{0}$. Row reduction of $[\mathbf{v}_1 \quad \mathbf{v}_2 \quad -\mathbf{v}_3 \quad -\mathbf{v}_4 \quad \mathbf{0}]$ yields

$$\begin{bmatrix} 5 & 1 & -2 & 0 & 0 \\ 3 & 3 & 1 & 12 & 0 \\ 8 & 4 & -5 & 28 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -10/3 & 0 \\ 0 & 1 & 0 & 26/3 & 0 \\ 0 & 0 & 1 & -4 & 0 \end{bmatrix},$$

so the vector of c_j's must be a multiple of (10/3, –26/3, 4, 1). One simple choice is (10, –26, 12, 3), which gives $\mathbf{w} = 10\mathbf{v}_1 - 26\mathbf{v}_2 = 12\mathbf{v}_3 + 3\mathbf{v}_4 = (24, -48, -24)$. Another choice for **w** is (1, –2, –1).

4.3 SOLUTIONS

Notes: The definition for basis is given initially for subspaces because this emphasizes that the basis elements must be in the subspace. Students often overlook this point when the definition is given for a vector space (see Exercise 25). The subsection on bases for Nul A and Col A is essential for Sections 4.5 and 4.6. The subsection on "Two Views of a Basis" is also fundamental to understanding the interplay between linearly independent sets, spanning sets, and bases. Key exercises in this section are Exercises 21–25, which help to deepen students' understanding of these different subsets of a vector space.

1. Consider the matrix whose columns are the given set of vectors. This 3×3 matrix is in echelon form, and has 3 pivot positions. Thus by the Invertible Matrix Theorem, its columns are linearly independent and span $\mathbb{R}^3$. So the given set of vectors is a basis for $\mathbb{R}^3$.

2. Since the zero vector is a member of the given set of vectors, the set cannot be linearly independent and thus cannot be a basis for $\mathbb{R}^3$. Now consider the matrix whose columns are the given set of vectors. This 3×3 matrix has only 2 pivot positions. Thus by the Invertible Matrix Theorem, its columns do not span $\mathbb{R}^3$.

3. Consider the matrix whose columns are the given set of vectors. The reduced echelon form of this matrix is

$$\begin{bmatrix} 1 & 3 & -3 \\ 0 & 2 & -5 \\ -2 & -4 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 9/2 \\ 0 & 1 & -5/2 \\ 0 & 0 & 0 \end{bmatrix}$$

so the matrix has only two pivot positions. Thus its columns do not form a basis for $\mathbb{R}^3$; the set of vectors is neither linearly independent nor does it span $\mathbb{R}^3$.

4. Consider the matrix whose columns are the given set of vectors. The reduced echelon form of this matrix is

$$\begin{bmatrix} 2 & 1 & -7 \\ -2 & -3 & 5 \\ 1 & 2 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

so the matrix has three pivot positions. Thus its columns form a basis for $\mathbb{R}^3$.

5. Since the zero vector is a member of the given set of vectors, the set cannot be linearly independent and thus cannot be a basis for $\mathbb{R}^3$. Now consider the matrix whose columns are the given set of vectors. The reduced echelon form of this matrix is

$$\begin{bmatrix} 1 & -2 & 0 & 0 \\ -3 & 9 & 0 & -3 \\ 0 & 0 & 0 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

so the matrix has a pivot in each row. Thus the given set of vectors spans $\mathbb{R}^3$.

6. Consider the matrix whose columns are the given set of vectors. Since the matrix cannot have a pivot in each row, its columns cannot span $\mathbb{R}^3$; thus the given set of vectors is not a basis for $\mathbb{R}^3$. The reduced echelon form of the matrix is

$$\begin{bmatrix} 1 & -4 \\ 2 & -5 \\ -3 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

so the matrix has a pivot in each column. Thus the given set of vectors is linearly independent.

7. Consider the matrix whose columns are the given set of vectors. Since the matrix cannot have a pivot in each row, its columns cannot span $\mathbb{R}^3$; thus the given set of vectors is not a basis for $\mathbb{R}^3$. The reduced echelon form of the matrix is

$$\begin{bmatrix} -2 & 6 \\ 3 & -1 \\ 0 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

so the matrix has a pivot in each column. Thus the given set of vectors is linearly independent.

8. Consider the matrix whose columns are the given set of vectors. Since the matrix cannot have a pivot in each column, the set cannot be linearly independent and thus cannot be a basis for $\mathbb{R}^3$. The reduced echelon form of this matrix is

$$\begin{bmatrix} 1 & 0 & 3 & 0 \\ -4 & 3 & -5 & 2 \\ 3 & -1 & 4 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -3/2 \\ 0 & 1 & 0 & -1/2 \\ 0 & 0 & 1 & 1/2 \end{bmatrix}$$

so the matrix has a pivot in each row. Thus the given set of vectors spans $\mathbb{R}^3$.

9. We find the general solution of $A\mathbf{x} = \mathbf{0}$ in terms of the free variables by using the reduced echelon form of A:

$$\begin{bmatrix} 1 & 0 & -3 & 2 \\ 0 & 1 & -5 & 4 \\ 3 & -2 & 1 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -3 & 2 \\ 0 & 1 & -5 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

So $x_1 = 3x_3 - 2x_4$, $x_2 = 5x_3 - 4x_4$, with x_3 and x_4 free. So

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 3 \\ 5 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ -4 \\ 0 \\ 1 \end{bmatrix},$$

and a basis for Nul A is

$$\left\{ \begin{bmatrix} 3 \\ 5 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ -4 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

10. We find the general solution of $A\mathbf{x} = \mathbf{0}$ in terms of the free variables by using the reduced echelon form of A:

$$\begin{bmatrix} 1 & 0 & -5 & 1 & 4 \\ -2 & 1 & 6 & -2 & -2 \\ 0 & 2 & -8 & 1 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -5 & 0 & 7 \\ 0 & 1 & -4 & 0 & 6 \\ 0 & 0 & 0 & 1 & -3 \end{bmatrix}.$$

So $x_1 = 5x_3 - 7x_5$, $x_2 = 4x_3 - 6x_5$, $x_4 = 3x_5$, with x_3 and x_5 free. So

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} 5 \\ 4 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -7 \\ -6 \\ 0 \\ 3 \\ 1 \end{bmatrix},$$

and a basis for Nul A is

$$\left\{ \begin{bmatrix} 5 \\ 4 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -7 \\ -6 \\ 0 \\ 3 \\ 1 \end{bmatrix} \right\}.$$

11. Let $A = \begin{bmatrix} 1 & 2 & 1 \end{bmatrix}$. Then we wish to find a basis for Nul A. We find the general solution of $A\mathbf{x} = \mathbf{0}$ in terms of the free variables: $x = -2y - z$ with y and z free. So

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = y \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix},$$

and a basis for Nul A is

$$\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

12. We want to find a basis for the set of vectors in $\mathbb{R}^2$ in the line $5x - y = 0$. Let $A = \begin{bmatrix} 5 & -1 \end{bmatrix}$. Then we wish to find a basis for Nul A. We find the general solution of $A\mathbf{x} = \mathbf{0}$ in terms of the free variables: $y = 5x$ with x free. So

$$\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix} = x \begin{bmatrix} 1 \\ 5 \end{bmatrix},$$

and a basis for Nul A is

$$\left\{ \begin{bmatrix} 1 \\ 5 \end{bmatrix} \right\}.$$

13. Since B is a row echelon form of A, we see that the first and second columns of A are its pivot columns. Thus a basis for Col A is

$$\left\{\begin{bmatrix}-2\\2\\-3\end{bmatrix},\begin{bmatrix}4\\-6\\8\end{bmatrix}\right\}.$$

To find a basis for Nul A, we find the general solution of $A\mathbf{x} = \mathbf{0}$ in terms of the free variables: $x_1 = -6x_3 - 5x_4$, $x_2 = (-5/2)x_3 - (3/2)x_4$, with x_3 and x_4 free. So

$$\mathbf{x} = \begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = x_3\begin{bmatrix}-6\\-5/2\\1\\0\end{bmatrix} + x_4\begin{bmatrix}-5\\-3/2\\0\\1\end{bmatrix},$$

and a basis for Nul A is

$$\left\{\begin{bmatrix}-6\\-5/2\\1\\0\end{bmatrix},\begin{bmatrix}-5\\-3/2\\0\\1\end{bmatrix}\right\}.$$

14. Since B is a row echelon form of A, we see that the first, third, and fifth columns of A are its pivot columns. Thus a basis for Col A is

$$\left\{\begin{bmatrix}1\\2\\1\\3\end{bmatrix},\begin{bmatrix}-5\\-5\\0\\-5\end{bmatrix},\begin{bmatrix}-3\\2\\5\\-2\end{bmatrix}\right\}.$$

To find a basis for Nul A, we find the general solution of $A\mathbf{x} = \mathbf{0}$ in terms of the free variables, mentally completing the row reduction of B to get: $x_1 = -2x_2 - 4x_4$, $x_3 = (7/5)x_4$, $x_5 = 0$, with x_2 and x_4 free. So

$$\mathbf{x} = \begin{bmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{bmatrix} = x_2\begin{bmatrix}-2\\1\\0\\0\\0\end{bmatrix} + x_4\begin{bmatrix}-4\\0\\7/5\\1\\0\end{bmatrix},$$

and a basis for Nul A is

$$\left\{\begin{bmatrix}-2\\1\\0\\0\\0\end{bmatrix},\begin{bmatrix}-4\\0\\7/5\\1\\0\end{bmatrix}\right\}.$$

15. This problem is equivalent to finding a basis for Col A, where $A = [\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3 \quad \mathbf{v}_4 \quad \mathbf{v}_5]$. Since the reduced echelon form of A is

$$\begin{bmatrix} 1 & 0 & -3 & 1 & 2 \\ 0 & 1 & -4 & -3 & 1 \\ -3 & 2 & 1 & -8 & -6 \\ 2 & -3 & 6 & 7 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -3 & 0 & 4 \\ 0 & 1 & -4 & 0 & -5 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

we see that the first, second, and fourth columns of A are its pivot columns. Thus a basis for the space spanned by the given vectors is

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ -8 \\ 7 \end{bmatrix} \right\}.$$

16. This problem is equivalent to finding a basis for Col A, where $A = [\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3 \quad \mathbf{v}_4 \quad \mathbf{v}_5]$. Since the reduced echelon form of A is

$$\begin{bmatrix} 1 & -2 & 6 & 5 & 0 \\ 0 & 1 & -1 & -3 & 3 \\ 0 & -1 & 2 & 3 & -1 \\ 1 & 1 & -1 & -4 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -1 & -2 \\ 0 & 1 & 0 & -3 & 5 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

we see that the first, second, and third columns of A are its pivot columns. Thus a basis for the space spanned by the given vectors is

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ -1 \\ 2 \\ -1 \end{bmatrix} \right\}.$$

17. **[M]** This problem is equivalent to finding a basis for Col A, where $A = [\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3 \quad \mathbf{v}_4 \quad \mathbf{v}_5]$. Since the reduced echelon form of A is

$$\begin{bmatrix} 8 & 4 & -1 & 6 & -1 \\ 9 & 5 & -4 & 8 & 4 \\ -3 & 1 & -9 & 4 & 11 \\ -6 & -4 & 6 & -7 & -8 \\ 0 & 4 & -7 & 10 & -7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -1/2 & 3 \\ 0 & 1 & 0 & 5/2 & -7 \\ 0 & 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

we see that the first, second, and third columns of A are its pivot columns. Thus a basis for the space spanned by the given vectors is

$$\left\{ \begin{bmatrix} 8 \\ 9 \\ -3 \\ -6 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 1 \\ -4 \\ 4 \end{bmatrix}, \begin{bmatrix} -1 \\ -4 \\ -9 \\ 6 \\ -7 \end{bmatrix} \right\}.$$

18. **[M]** This problem is equivalent to finding a basis for Col A, where $A=[\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3 \quad \mathbf{v}_4 \quad \mathbf{v}_5]$. Since the reduced echelon form of A is

$$\begin{bmatrix} -8 & 8 & -8 & 1 & -9 \\ 7 & -7 & 7 & 4 & 3 \\ 6 & -9 & 4 & 9 & -4 \\ 5 & -5 & 5 & 6 & -1 \\ -7 & 7 & -7 & -7 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 5/3 & 0 & 4/3 \\ 0 & 1 & 2/3 & 0 & 1/3 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

we see that the first, second, and fourth columns of A are its pivot columns. Thus a basis for the space spanned by the given vectors is

$$\left\{ \begin{bmatrix} -8 \\ 7 \\ 6 \\ 5 \\ -7 \end{bmatrix}, \begin{bmatrix} 8 \\ -7 \\ -9 \\ -5 \\ 7 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 9 \\ 6 \\ -7 \end{bmatrix} \right\}.$$

19. Since $4\mathbf{v}_1 + 5\mathbf{v}_2 - 3\mathbf{v}_3 = \mathbf{0}$, we see that each of the vectors is a linear combination of the others. Thus the sets $\{\mathbf{v}_1, \mathbf{v}_2\}$, $\{\mathbf{v}_1, \mathbf{v}_3\}$, and $\{\mathbf{v}_2, \mathbf{v}_3\}$ all span H. Since we may confirm that none of the three vectors is a multiple of any of the others, the sets $\{\mathbf{v}_1, \mathbf{v}_2\}$, $\{\mathbf{v}_1, \mathbf{v}_3\}$, and $\{\mathbf{v}_2, \mathbf{v}_3\}$ are linearly independent and thus each forms a basis for H.

20. Since $\mathbf{v}_1 - 3\mathbf{v}_2 + 5\mathbf{v}_3 = \mathbf{0}$, we see that each of the vectors is a linear combination of the others. Thus the sets $\{\mathbf{v}_1, \mathbf{v}_2\}$, $\{\mathbf{v}_1, \mathbf{v}_3\}$, and $\{\mathbf{v}_2, \mathbf{v}_3\}$ all span H. Since we may confirm that none of the three vectors is a multiple of any of the others, the sets $\{\mathbf{v}_1, \mathbf{v}_2\}$, $\{\mathbf{v}_1, \mathbf{v}_3\}$, and $\{\mathbf{v}_2, \mathbf{v}_3\}$ are linearly independent and thus each forms a basis for H.

21. **a**. False. The zero vector by itself is linearly dependent. See the paragraph preceding Theorem 4.

b. False. The set $\{\mathbf{b}_1, \ldots, \mathbf{b}_p\}$ must also be linearly independent. See the definition of a basis.

c. True. See Example 3.

d. False. See the subsection "Two Views of a Basis."

e. False. See the box before Example 9.

22. **a**. False. The subspace spanned by the set must also coincide with H. See the definition of a basis.

b. True. Apply the Spanning Set Theorem to V instead of H. The space V is nonzero because the spanning set uses nonzero vectors.

c. True. See the subsection "Two Views of a Basis."

d. False. See the two paragraphs before Example 8.

e. False. See the warning after Theorem 6.

23. Let $A=[\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3 \quad \mathbf{v}_4]$. Then A is square and its columns span $\mathbb{R}^4$ since $\mathbb{R}^4 = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$. So its columns are linearly independent by the Invertible Matrix Theorem, and $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is a basis for $\mathbb{R}^4$.

24. Let $A=[\mathbf{v}_1 \quad \ldots \quad \mathbf{v}_n]$. Then A is square and its columns are linearly independent, so its columns span $\mathbb{R}^n$ by the Invertible Matrix Theorem. Thus $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a basis for $\mathbb{R}^n$.

25. In order for the set to be a basis for H, $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ must be a spanning set for H; that is, $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. The exercise shows that H is a subset of $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. but there are vectors in $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ which are not in H ($\mathbf{v}_1$ and $\mathbf{v}_3$, for example). So $H \neq \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, and $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is not a basis for H.

26. Since $\sin t \cos t = (1/2) \sin 2t$, the set $\{\sin t, \sin 2t\}$ spans the subspace. By inspection we note that this set is linearly independent, so $\{\sin t, \sin 2t\}$ is a basis for the subspace.

27. The set $\{\cos \omega t, \sin \omega t\}$ spans the subspace. By inspection we note that this set is linearly independent, so $\{\cos \omega t, \sin \omega t\}$ is a basis for the subspace.

28. The set $\{e^{-bt}, te^{-bt}\}$ spans the subspace. By inspection we note that this set is linearly independent, so $\{e^{-bt}, te^{-bt}\}$ is a basis for the subspace.

29. Let A be the $n \times k$ matrix $[\mathbf{v}_1 \quad \dots \quad \mathbf{v}_k]$. Since A has fewer columns than rows, there cannot be a pivot position in each row of A. By Theorem 4 in Section 1.4, the columns of A do not span $\mathbb{R}^n$ and thus are not a basis for $\mathbb{R}^n$.

30. Let A be the $n \times k$ matrix $[\mathbf{v}_1 \quad \dots \quad \mathbf{v}_k]$. Since A has fewer rows than columns rows, there cannot be a pivot position in each column of A. By Theorem 8 in Section 1.6, the columns of A are not linearly independent and thus are not a basis for $\mathbb{R}^n$.

31. Suppose that $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is linearly dependent. Then there exist scalars $c_1, \dots, c_p$ not all zero with

$$c_1\mathbf{v}_1 + \dots + c_p\mathbf{v}_p = \mathbf{0}.$$

Since T is linear,

$$T(c_1\mathbf{v}_1 + \dots + c_p\mathbf{v}_p) = c_1T(\mathbf{v}_1) + \dots + c_pT(\mathbf{v}_p)$$

and

$$T(c_1\mathbf{v}_1 + \dots + c_p\mathbf{v}_p) = T(\mathbf{0}) = \mathbf{0}.$$

Thus

$$c_1T(\mathbf{v}_1) + \dots + c_pT(\mathbf{v}_p) = \mathbf{0}$$

and since not all of the c_i are zero, $\{T(\mathbf{v}_1), \dots, T(\mathbf{v}_p)\}$ is linearly dependent.

32. Suppose that $\{T(\mathbf{v}_1), \dots, T(\mathbf{v}_p)\}$ is linearly dependent. Then there exist scalars $c_1, \dots, c_p$ not all zero with

$$c_1T(\mathbf{v}_1) + \dots + c_pT(\mathbf{v}_p) = \mathbf{0}.$$

Since T is linear,

$$T(c_1\mathbf{v}_1 + \dots + c_p\mathbf{v}_p) = c_1T(\mathbf{v}_1) + \dots + c_pT(\mathbf{v}_p) = \mathbf{0} = T(\mathbf{0})$$

Since T is one-to-one

$$T(c_1\mathbf{v}_1 + \dots + c_p\mathbf{v}_p) = T(\mathbf{0})$$

implies that

$$c_1\mathbf{v}_1 + \dots + c_p\mathbf{v}_p = \mathbf{0}.$$

Since not all of the c_i are zero, $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is linearly dependent.

33. Neither polynomial is a multiple of the other polynomial. So $\{\mathbf{p}_1, \mathbf{p}_2\}$ is a linearly independent set in $\mathbb{P}_3$. Note: $\{\mathbf{p}_1, \mathbf{p}_2\}$ is also a linearly independent set in $\mathbb{P}_2$ since $\mathbf{p}_1$ and $\mathbf{p}_2$ both happen to be in $\mathbb{P}_2$.

34. By inspection, $\mathbf{p}_3 = \mathbf{p}_1 + \mathbf{p}_2$, or $\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}_3 = \mathbf{0}$. By the Spanning Set Theorem, $\text{Span}\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\} = \text{Span}\{\mathbf{p}_1, \mathbf{p}_2\}$. Since neither $\mathbf{p}_1$ nor $\mathbf{p}_2$ is a multiple of the other, they are linearly independent and hence $\{\mathbf{p}_1, \mathbf{p}_2\}$ is a basis for $\text{Span}\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$.

35. Let $\{\mathbf{v}_1, \mathbf{v}_3\}$ be any linearly independent set in a vector space *V*, and let $\mathbf{v}_2$ and $\mathbf{v}_4$ each be linear combinations of $\mathbf{v}_1$ and $\mathbf{v}_3$. For instance, let $\mathbf{v}_2 = 5\mathbf{v}_1$ and $\mathbf{v}_4 = \mathbf{v}_1 + \mathbf{v}_3$. Then $\{\mathbf{v}_1, \mathbf{v}_3\}$ is a basis for $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$.

36. **[M]** Row reduce the following matrices to identify their pivot columns:

$$[\mathbf{u}_1 \quad \mathbf{u}_2 \quad \mathbf{u}_3] = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 2 & 2 \\ 3 & -1 & 7 \\ -1 & 1 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \text{ so } \{\mathbf{u}_1, \mathbf{u}_2\} \text{ is a basis for } H.$$

$$[\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3] = \begin{bmatrix} 1 & 2 & -1 \\ 0 & -2 & 4 \\ 8 & 9 & 6 \\ -4 & -5 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \text{ so } \{\mathbf{v}_1, \mathbf{v}_2\} \text{ is a basis for } K.$$

$$[\mathbf{u}_1 \quad \mathbf{u}_2 \quad \mathbf{u}_3 \quad \mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3] = \begin{bmatrix} 1 & 0 & 2 & 1 & 2 & -1 \\ 2 & 2 & 2 & 0 & -2 & 4 \\ 3 & -1 & 7 & 8 & 9 & 6 \\ -1 & 1 & -3 & -4 & -5 & -2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 2 & 0 & 2 & -4 \\ 0 & 1 & -1 & 0 & -3 & 6 \\ 0 & 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \text{ so } \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{v}_1\} \text{ is a basis for } H + K.$$

37. **[M]** For example, writing

$$c_1 \cdot t + c_2 \cdot \sin t + c_3 \cos 2t + c_4 \sin t \cos t = 0$$

with $t = 0, .1, .2, .3$ gives the following coefficent matrix *A* for the homogeneous system $A\mathbf{c} = \mathbf{0}$ (to four decimal places):

$$A = \begin{bmatrix} 0 & \sin 0 & \cos 0 & \sin 0 \cos 0 \\ .1 & \sin .1 & \cos .2 & \sin .1 \cos .1 \\ .2 & \sin .2 & \cos .4 & \sin .2 \cos .2 \\ .3 & \sin .3 & \cos .6 & \sin .3 \cos .3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ .1 & .0998 & .9801 & .0993 \\ .2 & .1987 & .9211 & .1947 \\ .3 & .2955 & .8253 & .2823 \end{bmatrix}.$$

This matrix is invertible, so the system $A\mathbf{c} = \mathbf{0}$ has only the trivial solution and $\{t, \sin t, \cos 2t, \sin t \cos t\}$ is a linearly independent set of functions.

38. **[M]** For example, writing

$$c_1 \cdot 1 + c_2 \cdot \cos t + c_3 \cdot \cos^2 t + c_4 \cdot \cos^3 t + c_5 \cdot \cos^4 t + c_6 \cdot \cos^5 t + c_7 \cdot \cos^6 t = 0$$

with $t = 0, .1, .2, .3, .4, .5, .6$ gives the following coefficent matrix A for the homogeneous system $A\mathbf{c} = \mathbf{0}$ (to four decimal places):

$$A = \begin{bmatrix} 1 & \cos 0 & \cos^2 0 & \cos^3 0 & \cos^4 0 & \cos^5 0 & \cos^6 0 \\ 1 & \cos .1 & \cos^2 .1 & \cos^3 .1 & \cos^4 .1 & \cos^5 .1 & \cos^6 .1 \\ 1 & \cos .2 & \cos^2 .2 & \cos^3 .2 & \cos^4 .2 & \cos^5 .2 & \cos^6 .2 \\ 1 & \cos .3 & \cos^2 .3 & \cos^3 .3 & \cos^4 .3 & \cos^5 .3 & \cos^6 .3 \\ 1 & \cos .4 & \cos^2 .4 & \cos^3 .4 & \cos^4 .4 & \cos^5 .4 & \cos^6 .4 \\ 1 & \cos .5 & \cos^2 .5 & \cos^3 .5 & \cos^4 .5 & \cos^5 .5 & \cos^6 .5 \\ 1 & \cos .6 & \cos^2 .6 & \cos^3 .6 & \cos^4 .6 & \cos^5 .6 & \cos^6 .6 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & .9950 & .9900 & .9851 & .9802 & .9753 & .9704 \\ 1 & .9801 & .9605 & .9414 & .9226 & .9042 & .8862 \\ 1 & .9553 & .9127 & .8719 & .8330 & .7958 & .7602 \\ 1 & .9211 & .8484 & .7814 & .7197 & .6629 & .6106 \\ 1 & .8776 & .7702 & .6759 & .5931 & .5205 & .4568 \\ 1 & .8253 & .6812 & .5622 & .4640 & .3830 & .3161 \end{bmatrix}$$

This matrix is invertible, so the system Ac = 0 has only the trivial solution and $\{1, \cos t, \cos^2 t, \cos^3 t, \cos^4 t, \cos^5 t, \cos^6 t\}$ is a linearly independent set of functions.

4.4 SOLUTIONS

Notes: Section 4.7 depends heavily on this section, as does Section 5.4. It is possible to cover the $\mathbb{R}^n$ parts of the two later sections, however, if the first half of Section 4.4 (and perhaps Example 7) is covered. The linearity of the coordinate mapping is used in Section 5.4 to find the matrix of a transformation relative to two bases. The change-of-coordinates matrix appears in Section 5.4, Theorem 8 and Exercise 27. The concept of an isomorphism is needed in the proof of Theorem 17 in Section 4.8. Exercise 25 is used in Section 4.7 to show that the change-of-coordinates matrix is invertible.

1. We calculate that

$$\mathbf{x} = 5\begin{bmatrix} 3 \\ -5 \end{bmatrix} + 3\begin{bmatrix} -4 \\ 6 \end{bmatrix} = \begin{bmatrix} 3 \\ -7 \end{bmatrix}.$$

2. We calculate that

$$\mathbf{x} = 8\begin{bmatrix} 4 \\ 5 \end{bmatrix} + (-5)\begin{bmatrix} 6 \\ 7 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}.$$

3. We calculate that

$$\mathbf{x} = 3\begin{bmatrix} 1 \\ -4 \\ 3 \end{bmatrix} + 0\begin{bmatrix} 5 \\ 2 \\ -2 \end{bmatrix} + (-1)\begin{bmatrix} 4 \\ -7 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ -5 \\ 9 \end{bmatrix}.$$

4. We calculate that

$$\mathbf{x} = (-4)\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + 8\begin{bmatrix} 3 \\ -5 \\ 2 \end{bmatrix} + (-7)\begin{bmatrix} 4 \\ -7 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -5 \end{bmatrix}.$$

5. The matrix $\begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{x} \end{bmatrix}$ row reduces to $\begin{bmatrix} 1 & 0 & 8 \\ 0 & 1 & -5 \end{bmatrix}$, so $[\mathbf{x}]_B = \begin{bmatrix} 8 \\ -5 \end{bmatrix}$.

6. The matrix $\begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{x} \end{bmatrix}$ row reduces to $\begin{bmatrix} 1 & 0 & -6 \\ 0 & 1 & 2 \end{bmatrix}$, so $[\mathbf{x}]_B = \begin{bmatrix} -6 \\ 2 \end{bmatrix}$.

7. The matrix $\begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 & \mathbf{x} \end{bmatrix}$ row reduces to $\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{bmatrix}$, so $[\mathbf{x}]_B = \begin{bmatrix} -1 \\ -1 \\ 3 \end{bmatrix}$.

8. The matrix $\begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 & \mathbf{x} \end{bmatrix}$ row reduces to $\begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 5 \end{bmatrix}$, so $[\mathbf{x}]_B = \begin{bmatrix} -2 \\ 0 \\ 5 \end{bmatrix}$.

9. The change-of-coordinates matrix from B to the standard basis in $\mathbb{R}^2$ is

$$P_B = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -9 & 8 \end{bmatrix}.$$

10. The change-of-coordinates matrix from B to the standard basis in $\mathbb{R}^3$ is

$$P_B = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 8 \\ -1 & 0 & -2 \\ 4 & -5 & 7 \end{bmatrix}.$$

11. Since P_B^{-1} converts $\mathbf{x}$ into its B-coordinate vector, we find that

$$[\mathbf{x}]_B = P_B^{-1}\mathbf{x} = \begin{bmatrix} 3 & -4 \\ -5 & 6 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ -6 \end{bmatrix} = \begin{bmatrix} -3 & -2 \\ -5/2 & -3/2 \end{bmatrix}\begin{bmatrix} 2 \\ -6 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}.$$

12. Since P_B^{-1} converts $\mathbf{x}$ into its B-coordinate vector, we find that

$$[\mathbf{x}]_B = P_B^{-1}\mathbf{x} = \begin{bmatrix} 4 & 6 \\ 5 & 7 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} -7/2 & 3 \\ 5/2 & -2 \end{bmatrix}\begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} -7 \\ 5 \end{bmatrix}.$$

13. We must find c_1, c_2, and c_3 such that

$$c_1(1+t^2) + c_2(t+t^2) + c_3(1+2t+t^2) = \mathbf{p}(t) = 1 + 4t + 7t^2.$$

Equating the coefficients of the two polynomials produces the system of equations

$$\begin{aligned} c_1 \quad + \quad\quad\; c_3 &= 1 \\ c_2 + 2c_3 &= 4 \\ c_1 + c_2 + c_3 &= 7 \end{aligned}$$

We row reduce the augmented matrix for the system of equations to find

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 2 & 4 \\ 1 & 1 & 1 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & -1 \end{bmatrix}, \text{so } [\mathbf{p}]_B = \begin{bmatrix} 2 \\ 6 \\ -1 \end{bmatrix}.$$

One may also solve this problem using the coordinate vectors of the given polynomials relative to the standard basis $\{1, t, t^2\}$; the same system of linear equations results.

14. We must find c_1, c_2, and c_3 such that

$$c_1(1-t^2) + c_2(t-t^2) + c_3(2-2t+t^2) = \mathbf{p}(t) = 3 + t - 6t^2.$$

Equating the coefficients of the two polynomials produces the system of equations

$$\begin{array}{rcrcrcr} c_1 & + & & & 2c_3 & = & 3 \\ & & c_2 & - & 2c_3 & = & 1 \\ -c_1 & - & c_2 & + & c_3 & = & -6 \end{array}$$

We row reduce the augmented matrix for the system of equations to find

$$\begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & -2 & 1 \\ -1 & -1 & 1 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & -2 \end{bmatrix}, \text{so } [\mathbf{p}]_B = \begin{bmatrix} 7 \\ -3 \\ -2 \end{bmatrix}.$$

One may also solve this problem using the coordinate vectors of the given polynomials relative to the standard basis $\{1, t, t^2\}$; the same system of linear equations results.

15. **a**. True. See the definition of the *B*-coordinate vector.

b. False. See Equation (4).

c. False. $\mathbb{P}_3$ is isomorphic to $\mathbb{R}^4$. See Example 5.

16. **a**. True. See Example 2.

b. False. By definition, the coordinate mapping goes in the opposite direction.

c. True. If the plane passes through the origin, as in Example 7, the plane is isomorphic to $\mathbb{R}^2$.

17. We must solve the vector equation $x_1\begin{bmatrix} 1 \\ -3 \end{bmatrix} + x_2\begin{bmatrix} 2 \\ -8 \end{bmatrix} + x_3\begin{bmatrix} -3 \\ 7 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. We row reduce the augmented matrix for the system of equations to find

$$\begin{bmatrix} 1 & 2 & -3 & 1 \\ -3 & -8 & 7 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -5 & 5 \\ 0 & 1 & 1 & -2 \end{bmatrix}.$$

Thus we can let $x_1 = 5 + 5x_3$ and $x_2 = -2 - x_3$, where x_3 can be any real number. Letting $x_3 = 0$ and $x_3 = 1$ produces two different ways to express $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ as a linear combination of the other vectors: $5\mathbf{v}_1 - 2\mathbf{v}_2$ and $10\mathbf{v}_1 - 3\mathbf{v}_2 + \mathbf{v}_3$. There are infintely many correct answers to this problem.

18. For each *k*, $\mathbf{b}_k = 0 \cdot \mathbf{b}_1 + \cdots + 1 \cdot \mathbf{b}_k + \cdots + 0 \cdot \mathbf{b}_n$, so $[\mathbf{b}_k]_B = (0,\ldots,1,\ldots,0) = \mathbf{e}_k$.

19. The set *S* spans *V* because every **x** in *V* has a representation as a (unique) linear combination of elements in *S*. To show linear independence, suppose that $S = \{\mathbf{v}_1,\ldots,\mathbf{v}_n\}$ and that $c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n = \mathbf{0}$ for some scalars $c_1, \ldots, c_n$. The case when $c_1 = \cdots = c_n = 0$ is one possibility. By hypothesis, this is the unique (and thus the only) possible representation of the zero vector as a linear combination of the elements in S. So S is linearly independent and is thus a basis for V.

20. For $\boldsymbol{w}$ in V there exist scalars k_1, k_2, k_3, and k_4 such that

$$\mathbf{w} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 + k_4\mathbf{v}_4 \tag{1}$$

because $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ spans V. Because the set is linearly dependent, there exist scalars c_1, c_2, c_3, and c_4 not all zero, such that

$$\mathbf{0} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + c_4\mathbf{v}_4 \tag{2}$$

Adding (1) and (2) gives

$$\mathbf{w} = \mathbf{w} + \mathbf{0} = (k_1 + c_1)\mathbf{v}_1 + (k_2 + c_2)\mathbf{v}_2 + (k_3 + c_3)\mathbf{v}_3 + (k_4 + c_4)\mathbf{v}_4 \tag{3}$$

At least one of the weights in (3) differs from the corresponding weight in (1) because at least one of the c_i is nonzero. So $\mathbf{w}$ is expressed in more than one way as a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, and $\mathbf{v}_4$.

21. The matrix of the transformation will be $P_B^{-1} = \begin{bmatrix} 1 & -2 \\ -4 & 9 \end{bmatrix}^{-1} = \begin{bmatrix} 9 & 2 \\ 4 & 1 \end{bmatrix}$.

22. The matrix of the transformation will be $P_B^{-1} = \begin{bmatrix} \mathbf{b}_1 & \cdots & \mathbf{b}_n \end{bmatrix}^{-1}$.

23. Suppose that

$$[\mathbf{u}]_B = [\mathbf{w}]_B = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}.$$

By definition of coordinate vectors,

$$\mathbf{u} = \mathbf{w} = c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n.$$

Since $\mathbf{u}$ and $\mathbf{w}$ were arbitrary elements of V, the coordinate mapping is one-to-one.

24. Given $\mathbf{y} = (y_1, \ldots, y_n)$ in $\mathbb{R}^n$, let $\mathbf{u} = y_1\mathbf{b}_1 + \cdots + y_n\mathbf{b}_n$. Then, by definition, $[\mathbf{u}]_B = \mathbf{y}$. Since $\mathbf{y}$ was arbitrary, the coordinate mapping is onto $\mathbb{R}^n$.

25. Since the coordinate mapping is one-to-one, the following equations have the same solutions $c_1, \ldots, c_p$:

$$c_1\mathbf{u}_1 + \cdots + c_p\mathbf{u}_p = \mathbf{0} \qquad \text{(the zero vector in } V\text{)} \tag{4}$$

$$\left[c_1\mathbf{u}_1 + \cdots + c_p\mathbf{u}_p\right]_B = [\mathbf{0}]_B \qquad \text{(the zero vector in } \mathbb{R}^n\text{)} \tag{5}$$

Since the coordinate mapping is linear, (5) is equivalent to

$$c_1[\mathbf{u}_1]_B + \cdots + c_p[\mathbf{u}_p]_B = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \tag{6}$$

Thus (4) has only the trivial solution if and only if (6) has only the trivial solution. It follows that $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ is linearly independent if and only if $\{[\mathbf{u}_1]_B, \ldots, [\mathbf{u}_p]_B\}$ is linearly independent. This result also follows directly from Exercises 31 and 32 in Section 4.3.

26. By definition, **w** is a linear combination of $\mathbf{u}_1,\ldots,\mathbf{u}_p$ if and only if there exist scalars $c_1,\ldots,c_p$ such that

$$\mathbf{w} = c_1\mathbf{u}_1 + \cdots + c_p\mathbf{u}_p \qquad (7)$$

Since the coordinate mapping is linear,

$$[\mathbf{w}]_B = c_1[\mathbf{u}_1]_B + \cdots + c_p[\mathbf{u}_p]_B \qquad (8)$$

Conversely, (8) implies (7) because the coordinate mapping is one-to-one. Thus **w** is a linear combination of $\mathbf{u}_1,\ldots,\mathbf{u}_p$ if and only if $[\mathbf{w}]_B$ is a linear combination of $[\mathbf{u}]_1,\ldots,[\mathbf{u}]_p$.

Note: Students need to be urged to *write* not just to compute in Exercises 27–34. The language in the *Study Guide* solution of Exercise 31 provides a model for the students. In Exercise 32, students may have difficulty distinguishing between the two isomorphic vector spaces, sometimes giving a vector in $\mathbb{R}^3$ as an answer for part (b).

27. The coordinate mapping produces the coordinate vectors (1, 0, 0, 1), (3, 1, –2, 0), and (0, –1, 3, –1) respectively. We test for linear independence of these vectors by writing them as columns of a matrix and row reducing:

$$\begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & -1 \\ 0 & -2 & 3 \\ 1 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Since the matrix has a pivot in each column, its columns (and thus the given polynomials) are linearly independent.

28. The coordinate mapping produces the coordinate vectors (1, 0, –2, –3), (0, 1, 0, 1), and (1, 3, –2, 0) respectively. We test for linear independence of these vectors by writing them as columns of a matrix and row reducing:

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 3 \\ -2 & 0 & -2 \\ -3 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Since the matrix does not have a pivot in each column, its columns (and thus the given polynomials) are linearly dependent.

29. The coordinate mapping produces the coordinate vectors (1, –2, 1, 0), (–2, 0, 0, 1), and (–8, 12, –6, 1) respectively. We test for linear independence of these vectors by writing them as columns of a matrix and row reducing:

$$\begin{bmatrix} 1 & -2 & -8 \\ -2 & 0 & 12 \\ 1 & 0 & -6 \\ 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -6 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Since the matrix does not have a pivot in each column, its columns (and thus the given polynomials) are linearly dependent.

30. The coordinate mapping produces the coordinate vectors (1, –3, 3, –1), (4, –12, 9, 0), and (0, 0, 3, –4) respectively. We test for linear independence of these vectors by writing them as columns of a matrix and row reducing:

$$\begin{bmatrix} 1 & 4 & 0 \\ -3 & -12 & 0 \\ 3 & 9 & 3 \\ -1 & 0 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Since the matrix does not have a pivot in each column, its columns (and thus the given polynomials) are linearly dependent.

31. In each part, place the coordinate vectors of the polynomials into the columns of a matrix and reduce the matrix to echelon form.

a. $\begin{bmatrix} 1 & -3 & -4 & 1 \\ -3 & 5 & 5 & 0 \\ 5 & -7 & -6 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & -4 & 1 \\ 0 & -4 & -7 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Since there is not a pivot in each row, the original four column vectors do not span $\mathbb{R}^3$. By the isomorphism between $\mathbb{R}^3$ and $\mathbb{P}_2$, the given set of polynomials does not span $\mathbb{P}_2$.

b. $\begin{bmatrix} 0 & 1 & -3 & 2 \\ 5 & -8 & 4 & -3 \\ 1 & -2 & 2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 2 & 0 \\ 0 & 2 & -6 & -3 \\ 0 & 0 & 0 & 7/2 \end{bmatrix}$

Since there is a pivot in each row, the original four column vectors span $\mathbb{R}^3$. By the isomorphism between $\mathbb{R}^3$ and $\mathbb{P}_2$, the given set of polynomials spans $\mathbb{P}_2$.

32. **a**. Place the coordinate vectors of the polynomials into the columns of a matrix and reduce the matrix to echelon form: $\begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 2 \\ 1 & 3 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & -3 \end{bmatrix}$

The resulting matrix is invertible since it row equivalent to I_3. The original three column vectors form a basis for $\mathbb{R}^3$ by the Invertible Matrix Theorem. By the isomorphism between $\mathbb{R}^3$ and $\mathbb{P}_2$, the corresponding polynomials form a basis for $\mathbb{P}_2$.

b. Since $[\mathbf{q}]_B = (-3, 1, 2)$, $\mathbf{q} = -3\mathbf{p}_1 + \mathbf{p}_2 + 2\mathbf{p}_3$. One might do the algebra in $\mathbb{P}_2$ or choose to compute $\begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 2 \\ 1 & 3 & -4 \end{bmatrix}\begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ -8 \end{bmatrix}$. This combination of the columns of the matrix corresponds to the same combination of $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$. So $\mathbf{q}(t) = 1 + 3t - 8t^2$.

33. The coordinate mapping produces the coordinate vectors (3, 7, 0, 0), (5, 1, 0, –2), (0, 1, –2, 0) and (1, 16, –6, 2) respectively. To determine whether the set of polynomials is a basis for $\mathbb{P}_3$, we investigate whether the coordinate vectors form a basis for $\mathbb{R}^4$. Writing the vectors as the columns of a matrix and row reducing

$$\begin{bmatrix} 3 & 5 & 0 & 1 \\ 7 & 1 & 1 & 16 \\ 0 & 0 & -2 & -6 \\ 0 & -2 & 0 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

we find that the matrix is not row equivalent to I_4. Thus the coordinate vectors do not form a basis for $\mathbb{R}^4$. By the isomorphism between $\mathbb{R}^4$ and $\mathbb{P}_3$, the given set of polynomials does not form a basis for $\mathbb{P}_3$.

34. The coordinate mapping produces the coordinate vectors (5, –3, 4, 2), (9, 1, 8, –6), (6, –2, 5, 0), and (0, 0, 0, 1) respectively. To determine whether the set of polynomials is a basis for $\mathbb{P}_3$, we investigate whether the coordinate vectors form a basis for $\mathbb{R}^4$. Writing the vectors as the columns of a matrix, and row reducing

$$\begin{bmatrix} 5 & 9 & 6 & 0 \\ -3 & 1 & -2 & 0 \\ 4 & 8 & 5 & 0 \\ 2 & -6 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3/4 & 0 \\ 0 & 1 & 1/4 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

we find that the matrix is not row equivalent to I_4. Thus the coordinate vectors do not form a basis for $\mathbb{R}^4$. By the isomorphism between $\mathbb{R}^4$ and $\mathbb{P}_3$, the given set of polynomials does not form a basis for $\mathbb{P}_3$.

35. To show that **x** is in $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$, we must show that the vector equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 = \mathbf{x}$ has a solution. The augmented matrix $[\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{x}]$ may be row reduced to show

$$\begin{bmatrix} 11 & 14 & 19 \\ -5 & -8 & -13 \\ 10 & 13 & 18 \\ 7 & 10 & 15 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -5/3 \\ 0 & 1 & 8/3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Since this system has a solution, $\boldsymbol{x}$ is in *H*. The solution allows us to find the *B*-coordinate vector for $\boldsymbol{x}$: since $\mathbf{x} = x_1\mathbf{v}_1 + x_2\mathbf{v}_2 = (-5/3)\mathbf{v}_1 + (8/3)\mathbf{v}_2$, $[\mathbf{x}]_B = \begin{bmatrix} -5/3 \\ 8/3 \end{bmatrix}$.

36. To show that **x** is in $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, we must show that the vector equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{x}$ has a solution. The augmented matrix $[\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3 \quad \mathbf{x}]$ may be row reduced to show

$$\begin{bmatrix} -6 & 8 & -9 & 4 \\ 4 & -3 & 5 & 7 \\ -9 & 7 & -8 & -8 \\ 4 & -3 & 3 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The first three columns show that *B* is a basis for *H*. Moreover, since this system has a solution, **x** is in *H*. The solution allows us to find the *B*-coordinate vector for **x**: since

$$\mathbf{x} = x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = 3\mathbf{v}_1 + 5\mathbf{v}_2 + 2\mathbf{v}_3,\ [\mathbf{x}]_B = \begin{bmatrix} 3 \\ 5 \\ 2 \end{bmatrix}.$$

37. We are given that $[\mathbf{x}]_B = \begin{bmatrix} 1/2 \\ 1/4 \\ 1/6 \end{bmatrix}$, where $B = \left\{ \begin{bmatrix} 2.6 \\ -1.5 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 4.8 \end{bmatrix} \right\}$. To find the coordinates of $\boldsymbol{x}$ relative to the standard basis in $\mathbb{R}^3$, we must find **x**. We compute that

$$\mathbf{x} = P_B[\mathbf{x}]_B = \begin{bmatrix} 2.6 & 0 & 0 \\ -1.5 & 3 & 0 \\ 0 & 0 & 4.8 \end{bmatrix} \begin{bmatrix} 1/2 \\ 1/4 \\ 1/6 \end{bmatrix} = \begin{bmatrix} 1.3 \\ 0 \\ 0.8 \end{bmatrix}.$$

38. We are given that $[\mathbf{x}]_B = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/3 \end{bmatrix}$, where $B = \left\{ \begin{bmatrix} 2.6 \\ -1.5 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 4.8 \end{bmatrix} \right\}$. To find the coordinates of **x** relative to the standard basis in $\mathbb{R}^3$, we must find **x**. We compute that

$$\mathbf{x} = P_B[\mathbf{x}]_B = \begin{bmatrix} 2.6 & 0 & 0 \\ -1.5 & 3 & 0 \\ 0 & 0 & 4.8 \end{bmatrix} \begin{bmatrix} 1/2 \\ 1/2 \\ 1/3 \end{bmatrix} = \begin{bmatrix} 1.3 \\ 0.75 \\ 1.6 \end{bmatrix}.$$

4.5 SOLUTIONS

Notes: Theorem 9 is true because a vector space isomorphic to $\mathbb{R}^n$ has the same algebraic properties as $\mathbb{R}^n$; a proof of this result may not be needed to convince the class. The proof of Theorem 9 relies upon the fact that the coordinate mapping is a linear transformation (which is Theorem 8 in Section 4.4). If you have skipped this result, you can prove Theorem 9 as is done in *Introduction to Linear Algebra* by Serge Lang (Springer-Verlag, New York, 1986). There are two separate groups of true-false questions in this section; the second batch is more theoretical in nature. Example 4 is useful to get students to visualize subspaces of different dimensions, and to see the relationships between subspaces of different dimensions. Exercises 31 and 32 investigate the relationship between the dimensions of the domain and the range of a linear transformation; Exercise 32 is mentioned in the proof of Theorem 17 in Section 4.8.

1. This subspace is $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$, where $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix}$. Since $\mathbf{v}_1$ and $\mathbf{v}_2$ are not multiples of each other, $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly independent and is thus a basis for *H*. Hence the dimension of *H* is 2.

2. This subspace is $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$, where $\mathbf{v}_1 = \begin{bmatrix} 4 \\ -3 \\ 0 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$. Since $\mathbf{v}_1$ and $\mathbf{v}_2$ are not multiples of each other, $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly independent and is thus a basis for *H*. Hence the dimension of *H* is 2.

3. This subspace is $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, where $\mathbf{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 0 \\ -1 \\ 1 \\ 2 \end{bmatrix}$, and $\mathbf{v}_3 = \begin{bmatrix} 2 \\ 0 \\ -3 \\ 0 \end{bmatrix}$. Theorem 4 in Section 4.3 can be used to show that this set is linearly independent: $\mathbf{v}_1 \neq \mathbf{0}$, $\mathbf{v}_2$ is not a multiple of $\mathbf{v}_1$, and (since its first entry is not zero) $\mathbf{v}_3$ is not a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$. Thus $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent and is thus a basis for *H*. Alternatively, one can show that this set is linearly independent by row reducing the matrix $[\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3 \quad \mathbf{0}]$. Hence the dimension of the subspace is 3.

4. This subspace is $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$, where $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ -1 \end{bmatrix}$. Since $\mathbf{v}_1$ and $\mathbf{v}_2$ are not multiples of each other, $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly independent and is thus a basis for *H*. Hence the dimension of *H* is 2.

5. This subspace is $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, where $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \\ -3 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -4 \\ 5 \\ 0 \\ 7 \end{bmatrix}$, and $\mathbf{v}_3 = \begin{bmatrix} -2 \\ -4 \\ 2 \\ 6 \end{bmatrix}$. Since $\mathbf{v}_3 = -2\mathbf{v}_1$, $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent. By the Spanning Set Theorem, $\mathbf{v}_3$ may be removed from the set with no change in the span of the set, so $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$. Since $\mathbf{v}_1$ and $\mathbf{v}_2$ are not multiples of each other, $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly independent and is thus a basis for H. Hence the dimension of H is 2.

6. This subspace is $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, where $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 6 \\ -9 \\ -3 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 6 \\ -2 \\ 5 \\ 1 \end{bmatrix}$, and $\mathbf{v}_3 = \begin{bmatrix} -1 \\ -2 \\ 3 \\ 1 \end{bmatrix}$. Since $\mathbf{v}_3 = -(1/3)\mathbf{v}_1$, $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent. By the Spanning Set Theorem, $\mathbf{v}_3$ may be removed from the set with no change in the span of the set, so $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$. Since $\mathbf{v}_1$ and $\mathbf{v}_2$ are not multiples of each other, $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly independent and is thus a basis for H. Hence the dimension of H is 2.

7. This subspace is H = Nul A, where $A = \begin{bmatrix} 1 & -3 & 1 \\ 0 & 1 & -2 \\ 0 & 2 & -1 \end{bmatrix}$. Since $\begin{bmatrix} A & \mathbf{0} \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$, the homogeneous system has only the trivial solution. Thus H = Nul A = $\{\mathbf{0}\}$, and the dimension of H is 0.

8. From the equation $a - 3b + c = 0$, it is seen that $(a, b, c, d) = b(3, 1, 0, 0) + c(-1, 0, 1, 0) + d(0, 0, 0, 1)$. Thus the subspace is $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, where $\mathbf{v}_1 = (3,1,0,0)$, $\mathbf{v}_2 = (-1,0,1,0)$, and $\mathbf{v}_3 = (0,0,0,1)$. It is easily checked that this set of vectors is linearly independent, either by appealing to Theorem 4 in Section 4.3, or by row reducing $\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{0} \end{bmatrix}$. Hence the dimension of the subspace is 3.

9. This subspace is $H = \left\{ \begin{bmatrix} a \\ b \\ a \end{bmatrix} : a, b \text{ in } \mathbb{R} \right\} = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$, where $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. Since $\mathbf{v}_1$ and $\mathbf{v}_2$ are not multiples of each other, $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly independent and is thus a basis for H. Hence the dimension of H is 2.

10. The matrix A with these vectors as its columns row reduces to

$$\begin{bmatrix} 2 & -4 & -3 \\ -5 & 10 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

There are two pivot columns, so the dimension of Col A (which is the dimension of H) is 2.

11. The matrix A with these vectors as its columns row reduces to

$$\begin{bmatrix} 1 & 3 & 9 & -7 \\ 0 & 1 & 4 & -3 \\ 2 & 1 & -2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -3 & 2 \\ 0 & 1 & 4 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

There are two pivot columns, so the dimension of Col A (which is the dimension of the subspace spanned by the vectors) is 2.

12. The matrix A with these vectors as its columns row reduces to

$$\begin{bmatrix} 1 & -3 & -8 & -3 \\ -2 & 4 & 6 & 0 \\ 0 & 1 & 5 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 7 & 0 \\ 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

There are three pivot columns, so the dimension of Col A (which is the dimension of the subspace spanned by the vectors) is 3.

13. The matrix A is in echelon form. There are three pivot columns, so the dimension of Col A is 3. There are two columns without pivots, so the equation $A\mathbf{x} = \mathbf{0}$ has two free variables. Thus the dimension of Nul A is 2.

14. The matrix A is in echelon form. There are three pivot columns, so the dimension of Col A is 3. There are three columns without pivots, so the equation $A\mathbf{x} = \mathbf{0}$ has three free variables. Thus the dimension of Nul A is 3.

15. The matrix A is in echelon form. There are two pivot columns, so the dimension of Col A is 2. There are two columns without pivots, so the equation $A\mathbf{x} = \mathbf{0}$ has two free variables. Thus the dimension of Nul A is 2.

16. The matrix A row reduces to

$$\begin{bmatrix} 3 & 4 \\ -6 & 10 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

There are two pivot columns, so the dimension of Col A is 2. There are no columns without pivots, so the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution $\mathbf{0}$. Thus Nul $A = \{\mathbf{0}\}$, and the dimension of Nul A is 0.

17. The matrix A is in echelon form. There are three pivot columns, so the dimension of Col A is 3. There are no columns without pivots, so the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution $\mathbf{0}$. Thus Nul $A = \{\mathbf{0}\}$, and the dimension of Nul A is 0.

18. The matrix A is in echelon form. There are two pivot columns, so the dimension of Col A is 2. There is one column without a pivot, so the equation $A\mathbf{x} = \mathbf{0}$ has one free variable. Thus the dimension of Nul A is 1.

19. **a**. True. See the box before Example 5.

b. False. The plane must pass through the origin; see Example 4.

c. False. The dimension of $\mathbb{P}_n$ is $n + 1$; see Example 1.

d. False. The set S must also have n elements; see Theorem 12.

e. True. See Theorem 9.

20. **a**. False. The set $\mathbb{R}^2$ is not even a sub**set** of $\mathbb{R}^3$.

b. False. The number of **free** variables is equal to the dimension of Nul A; see the box before Example 5.

c. False. A basis could still have only finitely many elements, which would make the vector space finite-dimensional.

d. False. The set S must also have n elements; see Theorem 12.

e. True. See Example 4.

21. The matrix whose columns are the coordinate vectors of the Hermite polynomials relative to the standard basis $\{1, t, t^2, t^3\}$ of $\mathbb{P}_3$ is

$$A = \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 2 & 0 & -12 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 8 \end{bmatrix}.$$

This matrix has 4 pivots, so its columns are linearly independent. Since their coordinate vectors form a linearly independent set, the Hermite polynomials themselves are linearly independent in $\mathbb{P}_3$. Since there are four Hermite polynomials and dim $\mathbb{P}_3 = 4$, the Basis Theorem states that the Hermite polynomials form a basis for $\mathbb{P}_3$.

22. The matrix whose columns are the coordinate vectors of the Laguerre polynomials relative to the standard basis $\{1, t, t^2, t^3\}$ of $\mathbb{P}_3$ is

$$A = \begin{bmatrix} 1 & 1 & 2 & 6 \\ 0 & -1 & -4 & -18 \\ 0 & 0 & 1 & 9 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

This matrix has 4 pivots, so its columns are linearly independent. Since their coordinate vectors form a linearly independent set, the Laguerre polynomials themselves are linearly independent in $\mathbb{P}_3$. Since there are four Laguerre polynomials and dim $\mathbb{P}_3 = 4$, the Basis Theorem states that the Laguerre polynomials form a basis for $\mathbb{P}_3$.

23. The coordinates of $\mathbf{p}(t) = 7 - 12t - 8t^2 + 12t^3$ with respect to B satisfy

$$c_1(1) + c_2(2t) + c_3(-2 + 4t^2) + c_4(-12t + 8t^3) = 7 - 12t - 8t^2 + 12t^3$$

Equating coefficients of like powers of t produces the system of equations

$$\begin{array}{rcrcrcrcr} c_1 & & & - & 2c_3 & & & = & 7 \\ & & 2c_2 & & & - & 12c_4 & = & -12 \\ & & & & 4c_3 & & & = & -8 \\ & & & & & & 8c_4 & = & 12 \end{array}$$

Solving this system gives $c_1 = 3,\ c_2 = 3,\ c_3 = -2,\ c_4 = 3/2,$ and $[\mathbf{p}]_B = \begin{bmatrix} 3 \\ 3 \\ -2 \\ 3/2 \end{bmatrix}$.

24. The coordinates of $\mathbf{p}(t) = 7 - 8t + 3t^2$ with respect to B satisfy

$$c_1(1) + c_2(1 - t) + c_3(2 - 4t + t^2) = 7 - 8t + 3t^2$$

Equating coefficients of like powers of t produces the system of equations

$$\begin{array}{rcrcrcr} c_1 & + & c_2 & + & 2c_3 & = & 7 \\ & & -c_2 & - & 4c_3 & = & -8 \\ & & & & c_3 & = & 3 \end{array}$$

Solving this system gives $c_1 = 5,\ c_2 = -4,\ c_3 = 3,$ and $[\mathbf{p}]_B = \begin{bmatrix} 5 \\ -4 \\ 3 \end{bmatrix}$.

25. Note first that $n \geq 1$ since S cannot have fewer than 1 vector. Since $n \geq 1$, $V \neq \mathbf{0}$. Suppose that S spans V and that S contains fewer than n vectors. By the Spanning Set Theorem, some subset S' of S is a basis for V. Since S contains fewer than n vectors, and S' is a subset of S, S' also contains fewer than n vectors. Thus there is a basis S' for V with fewer than n vectors, but this is impossible by Theorem 10 since $\dim V = n$. Thus S cannot span V.

26. If $\dim V = \dim H = 0$, then $V = \{\mathbf{0}\}$ and $H = \{\mathbf{0}\}$, so $H = V$. Suppose that $\dim V = \dim H > 0$. Then H contains a basis S consisting of n vectors. But applying the Basis Theorem to V, S is also a basis for V. Thus $H = V = \text{Span}S$.

27. Suppose that $\dim \mathbb{P} = k < \infty$. Now $\mathbb{P}_n$ is a subspace of $\mathbb{P}$ for all n, and $\dim \mathbb{P}_{k-1} = k$, so $\dim \mathbb{P}_{k-1} = \dim \mathbb{P}$. This would imply that $\mathbb{P}_{k-1} = \mathbb{P}$, which is clearly untrue: for example $\mathbf{p}(t) = t^k$ is in $\mathbb{P}$ but not in $\mathbb{P}_{k-1}$. Thus the dimension of $\mathbb{P}$ cannot be finite.

28. The space $C(\mathbb{R})$ contains $\mathbb{P}$ as a subspace. If $C(\mathbb{R})$ were finite-dimensional, then $\mathbb{P}$ would also be finite-dimensional by Theorem 11. But $\mathbb{P}$ is infinite-dimensional by Exercise 27, so $C(\mathbb{R})$ must also be infinite-dimensional.

29. **a**. True. Apply the Spanning Set Theorem to the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ and produce a basis for V. This basis will not have more than p elements in it, so $\dim V \leq p$.

b. True. By Theorem 11, $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ can be expanded to find a basis for V. This basis will have at least p elements in it, so $\dim V \geq p$.

c. True. Take any basis (which will contain p vectors) for V and adjoin the zero vector to it.

30. **a**. False. For a counterexample, let $\mathbf{v}$ be a non-zero vector in $\mathbb{R}^3$, and consider the set $\{\mathbf{v}, 2\mathbf{v}\}$. This is a linearly dependent set in $\mathbb{R}^3$, but $\dim \mathbb{R}^3 = 3 > 2$.

b. True. If $\dim V \leq p$, there is a basis for V with p or fewer vectors. This basis would be a spanning set for V with p or fewer vectors, which contradicts the assumption.

c. False. For a counterexample, let $\mathbf{v}$ be a non-zero vector in $\mathbb{R}^3$, and consider the set $\{\mathbf{v}, 2\mathbf{v}\}$. This is a linearly dependent set in $\mathbb{R}^3$ with $3 - 1 = 2$ vectors, and $\dim \mathbb{R}^3 = 3$.

31. Since H is a nonzero subspace of a finite-dimensional vector space V, H is finite-dimensional and has a basis. Let $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ be a basis for H. We show that the set $\{T(\mathbf{u}_1), \ldots, T(\mathbf{u}_p)\}$ spans $T(H)$. Let $\mathbf{y}$ be in $T(H)$. Then there is a vector $\mathbf{x}$ in H with $T(\mathbf{x}) = \mathbf{y}$. Since $\mathbf{x}$ is in H and $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ is a basis for H, $\mathbf{x}$ may be written as $\mathbf{x} = c_1\mathbf{u}_1 + \ldots + c_p\mathbf{u}_p$ for some scalars $c_1, \ldots, c_p$. Since the transformation T is linear,

$$\mathbf{y} = T(\mathbf{x}) = T(c_1\mathbf{u}_1 + \ldots + c_p\mathbf{u}_p) = c_1T(\mathbf{u}_1) + \ldots + c_pT(\mathbf{u}_p)$$

Thus $\mathbf{y}$ is a linear combination of $T(\mathbf{u}_1), \ldots, T(\mathbf{u}_p)$, and $\{T(\mathbf{u}_1), \ldots, T(\mathbf{u}_p)\}$ spans $T(H)$. By the Spanning Set Theorem, this set contains a basis for $T(H)$. This basis then has not more than p vectors, and $\dim T(H) \leq p = \dim H$.

32. Since H is a nonzero subspace of a finite-dimensional vector space V, H is finite-dimensional and has a basis. Let $\{\mathbf{u}_1, \ldots \mathbf{u}_p\}$ be a basis for H. In Exercise 31 above it was shown that $\{T(\mathbf{u}_1), \ldots, T(\mathbf{u}_p)\}$ spans $T(H)$. In Exercise 32 in Section 4.3, it was shown that $\{T(\mathbf{u}_1), \ldots, T(\mathbf{u}_p)\}$ is linearly independent. Thus $\{T(\mathbf{u}_1), \ldots, T(\mathbf{u}_p)\}$ is a basis for $T(H)$, and $\dim T(H) = p = \dim H$.

33. **[M]**

a. To find a basis for $\mathbb{R}^5$ which contains the given vectors, we row reduce

$$\begin{bmatrix} -9 & 9 & 6 & 1 & 0 & 0 & 0 & 0 \\ -7 & 4 & 7 & 0 & 1 & 0 & 0 & 0 \\ 8 & 1 & -8 & 0 & 0 & 1 & 0 & 0 \\ -5 & 6 & 5 & 0 & 0 & 0 & 1 & 0 \\ 7 & -7 & -7 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -1/3 & 0 & 0 & 1 & 3/7 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 5/7 \\ 0 & 0 & 1 & -1/3 & 0 & 0 & 0 & -3/7 \\ 0 & 0 & 0 & 0 & 1 & 0 & 3 & 22/7 \\ 0 & 0 & 0 & 0 & 0 & 1 & -9 & -53/7 \end{bmatrix}.$$

The first, second, third, fifth, and sixth columns are pivot columns, so these columns of the original matrix ($\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{e}_2, \mathbf{e}_3\}$) form a basis for $\mathbb{R}^5$:

b. The original vectors are the first *k* columns of *A*. Since the set of original vectors is assumed to be linearly independent, these columns of *A* will be pivot columns and the original set of vectors will be included in the basis. Since the columns of *A* include all the columns of the identity matrix, Col $A = \mathbb{R}^n$.

34. **[M]**

a. The *B*-coordinate vectors of the vectors in *C* are the columns of the matrix

$$P = \begin{bmatrix} 1 & 0 & -1 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & -3 & 0 & 5 & 0 \\ 0 & 0 & 2 & 0 & -8 & 0 & 18 \\ 0 & 0 & 0 & 4 & 0 & -20 & 0 \\ 0 & 0 & 0 & 0 & 8 & 0 & -48 \\ 0 & 0 & 0 & 0 & 0 & 16 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 32 \end{bmatrix}.$$

The matrix *P* is invertible because it is triangular with nonzero entries along its main diagonal. Thus its columns are linearly independent. Since the coordinate mapping is an isomorphism, this shows that the vectors in *C* are linearly independent.

b. We know that dim $H = 7$ because *B* is a basis for *H*. Now *C* is a linearly independent set, and the vectors in *C* lie in *H* by the trigonometric identities. Thus by the Basis Theorem, *C* is a basis for *H*.

4.6 SOLUTIONS

Notes: This section puts together most of the ideas from Chapter 4. The Rank Theorem is the main result in this section. Many students have difficulty with the difference in finding bases for the row space and the column space of a matrix. The first process uses the nonzero rows of an echelon form of the matrix. The second process uses the pivots columns of the original matrix, which are usually found through row reduction. Students may also have problems with the varied effects of row operations on the linear dependence relations among the rows and columns of a matrix. Problems of the type found in Exercises 19–26 make excellent test questions. Figure 1 and Example 4 prepare the way for Theorem 3 in Section 6.1; Exercises 27–29 anticipate Example 6 in Section 7.4.

1. The matrix B is in echelon form. There are two pivot columns, so the dimension of Col A is 2. There are two pivot rows, so the dimension of Row A is 2. There are two columns without pivots, so the equation $A\mathbf{x} = \mathbf{0}$ has two free variables. Thus the dimension of Nul A is 2. A basis for Col A is the pivot columns of A:

$$\left\{\begin{bmatrix}1\\-1\\5\end{bmatrix},\begin{bmatrix}-4\\2\\-6\end{bmatrix}\right\}.$$

A basis for Row A is the pivot rows of B: $\{(1,0,-1,5),(0,-2,5,-6)\}$. To find a basis for Nul A row reduce to reduced echelon form:

$$A \sim \begin{bmatrix}1 & 0 & -1 & 5\\0 & 1 & -5/2 & 3\end{bmatrix}.$$

The solution to $A\mathbf{x} = \mathbf{0}$ in terms of free variables is $x_1 = x_3 - 5x_4$, $x_2 = (5/2)x_3 - 3x_4$ with x_3 and x_4 free. Thus a basis for Nul A is

$$\left\{\begin{bmatrix}1\\5/2\\1\\0\end{bmatrix},\begin{bmatrix}-5\\-3\\0\\1\end{bmatrix}\right\}.$$

2. The matrix B is in echelon form. There are three pivot columns, so the dimension of Col A is 3. There are three pivot rows, so the dimension of Row A is 3. There are two columns without pivots, so the equation $A\mathbf{x} = \mathbf{0}$ has two free variables. Thus the dimension of Nul A is 2. A basis for Col A is the pivot columns of A:

$$\left\{\begin{bmatrix}1\\-2\\-3\\3\end{bmatrix},\begin{bmatrix}4\\-6\\-6\\4\end{bmatrix},\begin{bmatrix}9\\-10\\-3\\0\end{bmatrix}\right\}.$$

A basis for Row A is the pivot rows of B: $\{(1,-3,0,5,-7),(0,0,2,-3,8),(0,0,0,0,5)\}$. To find a basis for Nul A row reduce to reduced echelon form:

$$A \sim \begin{bmatrix}1 & -3 & 0 & 5 & 0\\0 & 0 & 1 & -3/2 & 0\\0 & 0 & 0 & 0 & 1\\0 & 0 & 0 & 0 & 0\end{bmatrix}.$$

The solution to $A\mathbf{x} = \mathbf{0}$ in terms of free variables is $x_1 = 3x_2 - 5x_4$, $x_3 = (3/2)x_4$, $x_5 = 0$, with x_2 and x_4 free. Thus a basis for Nul A is

$$\left\{\begin{bmatrix}3\\1\\0\\0\\0\end{bmatrix},\begin{bmatrix}-5\\0\\3/2\\1\\0\end{bmatrix}\right\}.$$

3. The matrix B is in echelon form. There are three pivot columns, so the dimension of Col A is 3. There are three pivot rows, so the dimension of Row A is 3. There are two columns without pivots, so the equation $A\mathbf{x}=\mathbf{0}$ has two free variables. Thus the dimension of Nul A is 2. A basis for Col A is the pivot columns of A:

$$\left\{\begin{bmatrix}2\\-2\\4\\-2\end{bmatrix},\begin{bmatrix}6\\-3\\9\\3\end{bmatrix},\begin{bmatrix}2\\-3\\5\\-4\end{bmatrix}\right\}.$$

A basis for Row A is the pivot rows of B: $\{(2,-3,6,2,5),(0,0,3,-1,1),(0,0,0,1,3)\}$. To find a basis for Nul A row reduce to reduced echelon form:

$$A\sim\begin{bmatrix}1 & -3/2 & 0 & 0 & -9/2\\0 & 0 & 1 & 0 & 4/3\\0 & 0 & 0 & 1 & 3\\0 & 0 & 0 & 0 & 0\end{bmatrix}.$$

The solution to $A\mathbf{x}=\mathbf{0}$ in terms of free variables is $x_1=(3/2)x_2+(9/2)x_5$, $x_3=-(4/3)x_5$, $x_4=-3x_5$, with x_2 and x_5 free. Thus a basis for Nul A is

$$\left\{\begin{bmatrix}3/2\\1\\0\\0\\0\end{bmatrix},\begin{bmatrix}9/2\\0\\-4/3\\-3\\1\end{bmatrix}\right\}.$$

4. The matrix B is in echelon form. There are three pivot columns, so the dimension of Col A is 3. There are three pivot rows, so the dimension of Row A is 3. There are three columns without pivots, so the equation $A\mathbf{x}=\mathbf{0}$ has three free variables. Thus the dimension of Nul A is 3. A basis for Col A is the pivot columns of A:

$$\left\{\begin{bmatrix}1\\1\\1\\1\\1\end{bmatrix},\begin{bmatrix}1\\2\\-1\\-3\\-2\end{bmatrix},\begin{bmatrix}7\\10\\1\\-5\\0\end{bmatrix}\right\}.$$

A basis for Row A is the pivot rows of B:

$$\{(1,1,-3,7,9,-9),(0,1,-1,3,4,-3),(0,0,0,1,-1,-2)\}.$$

To find a basis for Nul A row reduce to reduced echelon form:

$$A\sim\begin{bmatrix}1 & 0 & -2 & 0 & 9 & 2\\0 & 1 & -1 & 0 & 7 & 3\\0 & 0 & 0 & 1 & -1 & -2\\0 & 0 & 0 & 0 & 0 & 0\\0 & 0 & 0 & 0 & 0 & 0\end{bmatrix}.$$

The solution to $A\mathbf{x}=\mathbf{0}$ in terms of free variables is $x_1 = 2x_3 - 9x_5 - 2x_6$, $x_2 = x_3 - 7x_5 - 3x_6$, $x_4 = x_5 + 2x_6$, with x_3, x_5, and x_6 free. Thus a basis for Nul A is

$$\left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -9 \\ -7 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

5. By the Rank Theorem, $\dim \text{Nul } A = 8 - \text{rank } A = 8 - 3 = 5$. Since $\dim \text{Row } A = \text{rank } A$, $\dim \text{Row } A = 3$. Since $\text{rank } A^T = \dim \text{Col } A^T = \dim \text{Row } A$, $\text{rank} A^T = 3$.

6. By the Rank Theorem, $\dim \text{Nul } A = 3 - \text{rank } A = 3 - 3 = 0$. Since $\dim \text{Row } A = \text{rank } A$, $\dim \text{Row } A = 3$. Since $\text{rank } A^T = \dim \text{Col } A^T = \dim \text{Row } A$, $\text{rank } A^T = 3$.

7. Yes, Col $A = \mathbb{R}^4$. Since A has four pivot columns, $\dim \text{Col } A = 4$. Thus Col A is a four-dimensional subspace of $\mathbb{R}^4$, and Col $A = \mathbb{R}^4$.
No, Nul $A \neq \mathbb{R}^3$. It is true that $\dim \text{Nul } A = 3$, but Nul A is a subspace of $\mathbb{R}^7$.

8. Since A has four pivot columns, $\text{rank } A = 4$, and $\dim \text{Nul } A = 6 - \text{rank } A = 6 - 4 = 2$.
No. Col $A \neq \mathbb{R}^4$. It is true that $\dim \text{Col } A = \text{rank } A = 4$, but Col A is a subspace of $\mathbb{R}^5$.

9. Since $\dim \text{Nul } A = 4$, $\text{rank } A = 6 - \dim \text{Nul } A = 6 - 4 = 2$. So $\dim \text{Col } A = \text{rank } A = 2$.

10. Since $\dim \text{Nul } A = 5$, $\text{rank } A = 6 - \dim \text{Nul } A = 6 - 5 = 1$. So $\dim \text{Col } A = \text{rank } A = 1$.

11. Since $\dim \text{Nul } A = 2$, $\text{rank } A = 5 - \dim \text{Nul } A = 5 - 2 = 3$. So $\dim \text{Row } A = \dim \text{Col } A = \text{rank } A = 3$.

12. Since $\dim \text{Nul } A = 4$, $\text{rank } A = 6 - \dim \text{Nul } A = 6 - 4 = 2$. So $\dim \text{Row } A = \dim \text{Col } A = \text{rank } A = 2$.

13. The rank of a matrix A equals the number of pivot positions which the matrix has. If A is either a 7×5 matrix or a 5×7 matrix, the largest number of pivot positions that A could have is 5. Thus the largest possible value for rank A is 5.

14. The dimension of the row space of a matrix A is equal to rank A, which equals the number of pivot positions which the matrix has. If A is either a 4×3 matrix or a 3×4 matrix, the largest number of pivot positions that A could have is 3. Thus the largest possible value for dimRow A is 3.

15. Since the rank of A equals the number of pivot positions which the matrix has, and A could have at most 6 pivot positions, $\text{rank } A \le 6$. Thus $\dim \text{Nul } A = 8 - \text{rank } A \ge 8 - 6 = 2$.

16. Since the rank of A equals the number of pivot positions which the matrix has, and A could have at most 4 pivot positions, $\text{rank } A \le 4$. Thus $\dim \text{Nul } A = 4 - \text{rank } A \ge 4 - 4 = 0$.

17. **a**. True. The rows of A are identified with the columns of A^T. See the paragraph before Example 1.
b. False. See the warning after Example 2.
c. True. See the Rank Theorem.
d. False. See the Rank Theorem.
e. True. See the Numerical Note before the Practice Problem.

18. **a**. False. Review the warning after Theorem 6 in Section 4.3.

b. False. See the warning after Example 2.

c. True. See the remark in the proof of the Rank Theorem.

d. True. This fact was noted in the paragraph before Example 4. It also follows from the fact that the rows of A^T are the columns of $(A^T)^T = A$.

e. True. See Theorem 13.

19. Yes. Consider the system as $A\mathbf{x} = \mathbf{0}$, where A is a 5×6 matrix. The problem states that $\dim \text{Nul} A = 1$. By the Rank Theorem, $\text{rank } A = 6 - \dim\text{Nul } A = 5$. Thus $\dim \text{Col } A = \text{rank } A = 5$, and since Col A is a subspace of $\mathbb{R}^5$, Col $A = \mathbb{R}^5$ So every vector **b** in $\mathbb{R}^5$ is also in Col A, and $A\mathbf{x} = \mathbf{b}$, has a solution for all **b**.

20. No. Consider the system as $A\mathbf{x} = \mathbf{b}$, where A is a 6×8 matrix. The problem states that $\dim\text{Nul } A = 2$. By the Rank Theorem, $\text{rank } A = 8 - \dim\text{Nul } A = 6$. Thus $\dim\text{Col } A = \text{rank } A = 6$, and since Col A is a subspace of $\mathbb{R}^6$, Col $A = \mathbb{R}^6$ So every vector **b** in $\mathbb{R}^6$ is also in Col A, and $A\mathbf{x} = \mathbf{b}$ has a solution for all **b**. Thus it is impossible to change the entries in **b** to make $A\mathbf{x} = \mathbf{b}$ into an inconsistent system.

21. No. Consider the system as $A\mathbf{x} = \mathbf{b}$, where A is a 9×10 matrix. Since the system has a solution for all **b** in $\mathbb{R}^9$, A must have a pivot in each row, and so $\text{rank} A = 9$. By the Rank Theorem, $\dim\text{Nul} A = 10 - 9 = 1$. Thus it is impossible to find two linearly independent vectors in Nul A.

22. No. Consider the system as $A\mathbf{x} = \mathbf{0}$, where A is a 10×12 matrix. Since A has at most 10 pivot positions, $\text{rank} A \le 10$. By the Rank Theorem, $\dim\text{Nul} A = 12 - \text{rank} A \ge 2$. Thus it is impossible to find a single vector in Nul A which spans Nul A.

23. Yes, six equations are sufficient. Consider the system as $A\mathbf{x} = \mathbf{0}$, where A is a 12×8 matrix. The problem states that $\dim\text{Nul } A = 2$. By the Rank Theorem, $\text{rank } A = 8 - \dim\text{Nul } A = 6$. Thus $\dim\text{Col } A = \text{rank } A = 6$. So the system $A\mathbf{x} = \mathbf{0}$ is equivalent to the system $B\mathbf{x} = \mathbf{0}$, where B is an echelon form of A with 6 nonzero rows. So the six equations in this system are sufficient to describe the solution set of $A\mathbf{x} = \mathbf{0}$.

24. Yes, No. Consider the system as $A\mathbf{x} = \mathbf{b}$, where A is a 7×6 matrix. Since A has at most 6 pivot positions, $\text{rank } A \le 6$. By the Rank Theorem, $\dim\text{Nul } A = 6 - \text{rank } A \ge 0$. If $\dim\text{Nul } A = 0$, then the system $A\mathbf{x} = \mathbf{b}$ will have no free variables. The solution to $A\mathbf{x} = \mathbf{b}$, if it exists, would thus have to be unique. Since $\text{rank } A \le 6$, Col A will be a proper subspace of $\mathbb{R}^7$. Thus there exists a **b** in $\mathbb{R}^7$ for which the system $A\mathbf{x} = \mathbf{b}$ is inconsistent, and the system $A\mathbf{x} = \mathbf{b}$ cannot have a unique solution for all **b**.

25. No. Consider the system as $A\mathbf{x} = \mathbf{b}$, where A is a 10×12 matrix. The problem states that $\dim\text{Nul} A = 3$. By the Rank Theorem, $\dim\text{Col } A = \text{rank } A = 12 - \dim\text{Nul } A = 9$. Thus Col A will be a proper subspace of $\mathbb{R}^{10}$ Thus there exists a **b** in $\mathbb{R}^{10}$ for which the system $A\mathbf{x} = \mathbf{b}$ is inconsistent, and the system $A\mathbf{x} = \mathbf{b}$ cannot have a solution for all **b**.

26. Consider the system $A\mathbf{x} = \mathbf{0}$, where A is a $m\times n$ matrix with $m > n$. Since the rank of A is the number of pivot positions that A has and A is assumed to have full rank, $\text{rank } A = n$. By the Rank Theorem, $\dim\text{Nul} A = n - \text{rank} A = 0$. So $\text{Nul} A = \{\mathbf{0}\}$, and the system $A\mathbf{x} = \mathbf{0}$ has only the trivial solution. This happens if and only if the columns of A are linearly independent.

27. Since A is an $m \times n$ matrix, Row A is a subspace of $\mathbb{R}^n$, Col A is a subspace of $\mathbb{R}^m$, and Nul A is a subspace of $\mathbb{R}^n$. Likewise since A^T is an $n \times m$ matrix, Row A^T is a subspace of $\mathbb{R}^m$, Col A^T is a subspace of $\mathbb{R}^n$, and $\text{Nul}\,A^T$ is a subspace of $\mathbb{R}^m$. Since $\text{Row}\,A = \text{Col}\,A^T$ and $\text{Col}\,A = \text{Row}\,A^T$, there are four dinstict subspaces in the list: Row A, Col A, Nul A, and $\text{Nul}\,A^T$.

28. **a**. Since A is an $m \times n$ matrix and dimRow A = rank A,

$$\text{dimRow}\,A + \text{dimNul}\,A = \text{rank}\,A + \text{dimNul}\,A = n.$$

b. Since A^T is an $n \times m$ matrix and $\text{dimCol}\,A = \text{dimRow}\,A = \text{dimCol}\,A^T = \text{rank}\,A^T$,

$$\text{dimCol}\,A + \text{dimNul}\,A^T = \text{rank}\,A^T + \text{dimNul}\,A^T = m.$$

29. Let A be an $m \times n$ matrix. The system $A\mathbf{x} = \mathbf{b}$ will have a solution for all $\mathbf{b}$ in $\mathbb{R}^m$ if and only if A has a pivot position in each row, which happens if and only if dimCol $A = m$. By Exercise 28 b., dimCol $A = m$ if and only if $\text{dimNul}\,A^T = m - m = 0$, or $\text{Nul}\,A^T = \{\mathbf{0}\}$. Finally, $\text{Nul}\,A^T = \{\mathbf{0}\}$ if and only if the equation $A^T\mathbf{x} = \mathbf{0}$ has only the trivial solution.

30. The equation $A\mathbf{x} = \mathbf{b}$ is consistent if and only if $\text{rank}\begin{bmatrix} A & \mathbf{b} \end{bmatrix} = \text{rank}\,A$ because the two ranks will be equal if and only if $\mathbf{b}$ is not a pivot column of $\begin{bmatrix} A & \mathbf{b} \end{bmatrix}$. The result then follows from Theorem 2 in Section 1.2.

31. Compute that $\mathbf{u}\mathbf{v}^T = \begin{bmatrix} 2 \\ -3 \\ 5 \end{bmatrix}\begin{bmatrix} a & b & c \end{bmatrix} = \begin{bmatrix} 2a & 2b & 2c \\ -3a & -3b & -3c \\ 5a & 5b & 5c \end{bmatrix}$. Each column of $\mathbf{u}\mathbf{v}^T$ is a multiple of $\mathbf{u}$, so $\text{dimCol}\,\mathbf{u}\mathbf{v}^T = 1$, unless $a = b = c = 0$, in which case $\mathbf{u}\mathbf{v}^T$ is the 3×3 zero matrix and $\text{dimCol}\,\mathbf{u}\mathbf{v}^T = 0$. In any case, $\text{rank}\,\mathbf{u}\mathbf{v}^T = \text{dimCol}\,\mathbf{u}\mathbf{v}^T \le 1$

32. Note that the second row of the matrix is twice the first row. Thus if $\mathbf{v} = (1, -3, 4)$, which is the first row of the matrix,

$$\mathbf{u}\mathbf{v}^T = \begin{bmatrix} 1 \\ 2 \end{bmatrix}\begin{bmatrix} 1 & -3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 4 \\ 2 & -6 & 8 \end{bmatrix}.$$

33. Let $A = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 \end{bmatrix}$, and assume that rank $A = 1$. Suppose that $\mathbf{u}_1 \ne \mathbf{0}$. Then $\{\mathbf{u}_1\}$ is basis for Col A, since Col A is assumed to be one-dimensional. Thus there are scalars x and y with $\mathbf{u}_2 = x\mathbf{u}_1$ and $\mathbf{u}_3 = y\mathbf{u}_1$, and $A = \mathbf{u}_1\mathbf{v}^T$, where $\mathbf{v} = \begin{bmatrix} 1 \\ x \\ y \end{bmatrix}$.

If $\mathbf{u}_1 = \mathbf{0}$ but $\mathbf{u}_2 \ne \mathbf{0}$, then similarly $\{\mathbf{u}_2\}$ is basis for Col A, since Col A is assumed to be one-dimensional. Thus there is a scalar x with $\mathbf{u}_3 = x\mathbf{u}_2$, and $A = \mathbf{u}_2\mathbf{v}^T$, where $\mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ x \end{bmatrix}$.

If $\mathbf{u}_1 = \mathbf{u}_2 = \mathbf{0}$ but $\mathbf{u}_3 \ne \mathbf{0}$, then $A = \mathbf{u}_3\mathbf{v}^T$, where $\mathbf{v} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

34. Let A be an $m \times n$ matrix with of rank $r > 0$, and let U be an echelon form of A. Since A can be reduced to U by row operations, there exist invertible elementary matrices $E_1, \ldots, E_p$ with $(E_p \cdots E_1)A = U$. Thus

$A = (E_p \cdots E_1)^{-1}U$, since the product of invertible matrices is invertible. Let $E = (E_p \cdots E_1)^{-1}$; then $A = EU$. Let the columns of E be denoted by $\mathbf{c}_1, \ldots, \mathbf{c}_m$. Since the rank of A is r, U has r nonzero rows, which can be denoted $\mathbf{d}_1^T, \ldots, \mathbf{d}_r^T$. By the column-row expansion of A (Theorem 10 in Section 2.4):

$$A = EU = \begin{bmatrix} \mathbf{c}_1 & \ldots & \mathbf{c}_m \end{bmatrix} \begin{bmatrix} \mathbf{d}_1^T \\ \vdots \\ \mathbf{d}_r^T \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{bmatrix} = \mathbf{c}_1\mathbf{d}_1^T + \ldots + \mathbf{c}_r\mathbf{d}_r^T,$$

which is the sum of r rank 1 matrices.

35. [M]

a. Begin by reducing A to reduced echelon form:

$$A \sim \begin{bmatrix} 1 & 0 & 13/2 & 0 & 5 & 0 & -3 \\ 0 & 1 & 11/2 & 0 & 1/2 & 0 & 2 \\ 0 & 0 & 0 & 1 & -11/2 & 0 & 7 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

A basis for Col A is the pivot columns of A, so matrix C contains these columns:

$$C = \begin{bmatrix} 7 & -9 & 5 & -3 \\ -4 & 6 & -2 & -5 \\ 5 & -7 & 5 & 2 \\ -3 & 5 & -1 & -4 \\ 6 & -8 & 4 & 9 \end{bmatrix}.$$

A basis for Row A is the pivot rows of the reduced echelon form of A, so matrix R contains these rows:

$$R = \begin{bmatrix} 1 & 0 & 13/2 & 0 & 5 & 0 & -3 \\ 0 & 1 & 11/2 & 0 & 1/2 & 0 & 2 \\ 0 & 0 & 0 & 1 & -11/2 & 0 & 7 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

To find a basis for Nul A row reduce to reduced echelon form, note that the solution to $A\mathbf{x} = \mathbf{0}$ in terms of free variables is $x_1 = -(13/2)x_3 - 5x_5 + 3x_7$, $x_2 = -(11/2)x_3 - (1/2)x_5 - 2x_7$, $x_4 = (11/2)x_5 - 7x_7$, $x_6 = -x_7$, with x_3, x_5, and x_7 free. Thus matrix N is

$$N = \begin{bmatrix} -13/2 & -5 & 3 \\ -11/2 & -1/2 & -2 \\ 1 & 0 & 0 \\ 0 & 11/2 & -7 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix}.$$

b. The reduced echelon form of A^T is

$$A^T \sim \begin{bmatrix} 1 & 0 & 0 & 0 & -2/11 \\ 0 & 1 & 0 & 0 & -41/11 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 28/11 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

so the solution to $A^T\mathbf{x} = \mathbf{0}$ in terms of free variables is $x_1 = (2/11)x_5$, $x_2 = (41/11)x_5$, $x_3 = 0$, $x_4 = -(28/11)x_5$, with x_5 free. Thus matrix *M* is

$$M = \begin{bmatrix} 2/11 \\ 41/11 \\ 0 \\ -28/11 \\ 1 \end{bmatrix}.$$

The matrix $S = \begin{bmatrix} R^T & N \end{bmatrix}$ is 7 × 7 because the columns of R^T and *N* are in $\mathbb{R}^7$ and dimRow *A* + dimNul *A* = 7. The matrix $T = \begin{bmatrix} C & M \end{bmatrix}$ is 5 × 5 because the columns of *C* and *M* are in $\mathbb{R}^5$ and dimCol $A +$ dimNul $A^T = 5$. Both *S* and *T* are invertible because their columns are linearly independent. This fact will be proven in general in Theorem 3 of Section 6.1.

36. **[M]** Answers will vary, but in most cases *C* will be 6 × 4, and will be constructed from the first 4 columns of *A*. In most cases *R* will be 4 × 7, *N* will be 7 × 3, and *M* will be 6 × 2.

37. **[M]** The *C* and *R* from Exercise 35 work here, and *A* = *CR*.

38. **[M]** If *A* is nonzero, then *A* = *CR*. Note that $CR = \begin{bmatrix} C\mathbf{r}_1 & C\mathbf{r}_2 & \dots & C\mathbf{r}_n \end{bmatrix}$, where $\mathbf{r}_1, \dots, \mathbf{r}_n$ are the columns of *R*. The columns of *R* are either pivot columns of *R* or are not pivot columns of *R*.

Consider first the pivot columns of *R*. The i^{th} pivot column of *R* is $\mathbf{e}_i$, the i^{th} column in the identity matrix, so $C\mathbf{e}_i$ is the i^{th} pivot column of *A*. Since *A* and *R* have pivot columns in the same locations, when *C* multiplies a pivot column of *R*, the result is the corresponding pivot column of *A* in its proper location.

Suppose $\mathbf{r}_j$ is a nonpivot column of *R*. Then $\mathbf{r}_j$ contains the weights needed to construct the j^{th} column of *A* from the pivot columns of *A*, as is discussed in Example 9 of Section 4.3 and in the paragraph preceding that example. Thus $\mathbf{r}_j$ contains the weights needed to construct the j^{th} column of *A* from the columns of *C*, and $C\mathbf{r}_j = \mathbf{a}_j$.

4.7 SOLUTIONS

Notes: This section depends heavily on the coordinate systems introduced in Section 4.4. The row reduction algorithm that produces $\underset{c\leftarrow B}{P}$ can also be deduced from Exercise 12 in Section 2.2, by row reducing $\left[P_C \mid P_B\right]$. to $\left[I \mid P_C^{-1}P_B\right]$. The change-of-coordinates matrix here is interpreted in Section 5.4 as the matrix of the identity transformation relative to two bases.

1. **a**. Since $\mathbf{b}_1 = 6\mathbf{c}_1 - 2\mathbf{c}_2$ and $\mathbf{b}_2 = 9\mathbf{c}_1 - 4\mathbf{c}_2$, $[\mathbf{b}_1]_C = \begin{bmatrix} 6 \\ -2 \end{bmatrix}$, $[\mathbf{b}_2]_C = \begin{bmatrix} 9 \\ -4 \end{bmatrix}$, and $\underset{C\leftarrow B}{P} = \begin{bmatrix} 6 & 9 \\ -2 & -4 \end{bmatrix}$.

b. Since $\mathbf{x} = -3\mathbf{b}_1 + 2\mathbf{b}_2$, $[\mathbf{x}]_B = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$ and

$$[\mathbf{x}]_C = \underset{C\leftarrow B}{P}[x]_B = \begin{bmatrix} 6 & 9 \\ -2 & -4 \end{bmatrix}\begin{bmatrix} -3 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$$

2. **a**. Since $\mathbf{b}_1 = -\mathbf{c}_1 + 4\mathbf{c}_2$ and $\mathbf{b}_2 = 5\mathbf{c}_1 - 3\mathbf{c}_2$, $[\mathbf{b}_1]_C = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$, $[\mathbf{b}_2]_C = \begin{bmatrix} 5 \\ -3 \end{bmatrix}$, and $\underset{C\leftarrow B}{P} = \begin{bmatrix} -1 & 5 \\ 4 & -3 \end{bmatrix}$.

b. Since $\mathbf{x} = 5\mathbf{b}_1 + 3\mathbf{b}_2$, $[\mathbf{x}]_B = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$ and

$$[\mathbf{x}]_C = \underset{C\leftarrow B}{P}[\mathbf{x}]_B = \begin{bmatrix} -1 & 5 \\ 4 & -3 \end{bmatrix}\begin{bmatrix} 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 10 \\ 11 \end{bmatrix}$$

3. Equation (ii) is satisfied by P for all $\mathbf{x}$ in V.

4. Equation (i) is satisfied by P for all $\mathbf{x}$ in V.

5. **a**. Since $\mathbf{a}_1 = 4\mathbf{b}_1 - \mathbf{b}_2$, $\mathbf{a}_2 = -\mathbf{b}_1 + \mathbf{b}_2 + \mathbf{b}_3$, and $\mathbf{a}_3 = \mathbf{b}_2 - 2\mathbf{b}_3$, $[\mathbf{a}_1]_B = \begin{bmatrix} 4 \\ -1 \\ 0 \end{bmatrix}$, $[\mathbf{a}_2]_B = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$,

$[\mathbf{a}_3]_B = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}$, and $\underset{B\leftarrow A}{P} = \begin{bmatrix} 4 & -1 & 0 \\ -1 & 1 & 1 \\ 0 & 1 & -2 \end{bmatrix}$.

b. Since $\mathbf{x} = 3\mathbf{a}_1 + 4\mathbf{a}_2 + \mathbf{a}_3$, $[\mathbf{x}]_A = \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix}$ and

$$[\mathbf{x}]_B = \underset{B\leftarrow A}{P} = \begin{bmatrix} 4 & -1 & 0 \\ -1 & 1 & 1 \\ 0 & 1 & -2 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 2 \\ 2 \end{bmatrix}$$

6. **a**. Since $\mathbf{f}_1 = 2\mathbf{d}_1 - \mathbf{d}_2 + \mathbf{d}_3$, $\mathbf{f}_2 = 3\mathbf{d}_2 + \mathbf{d}_3$, and $\mathbf{f}_3 = -3\mathbf{d}_1 + 2\mathbf{d}_3$, $[\mathbf{f}_1]_D = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$, $[\mathbf{f}_2]_D = \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}$, $[\mathbf{f}_3]_D = \begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix}$,

and $\underset{D\leftarrow F}{P} = \begin{bmatrix} 2 & 0 & -3 \\ -1 & 3 & 0 \\ 1 & 1 & 2 \end{bmatrix}$.

b. Since $\mathbf{x} = \mathbf{f}_1 - 2\mathbf{f}_2 + 2\mathbf{f}_3$, $[\mathbf{x}]_F = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$ and

$$[\mathbf{x}]_D = \underset{D\leftarrow F}{P}[\mathbf{x}]_F = \begin{bmatrix} 2 & 0 & -3 \\ -1 & 3 & 0 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} = \begin{bmatrix} -4 \\ -7 \\ 3 \end{bmatrix}$$

7. To find $\underset{C\leftarrow B}{P}$, row reduce the matrix $\begin{bmatrix} \mathbf{c}_1 & \mathbf{c}_2 & \mathbf{b}_1 & \mathbf{b}_2 \end{bmatrix}$:

$$\begin{bmatrix} \mathbf{c}_1 & \mathbf{c}_2 & \mathbf{b}_1 & \mathbf{b}_2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -3 & 1 \\ 0 & 1 & -5 & 2 \end{bmatrix}.$$

Thus $\underset{C\leftarrow B}{P} = \begin{bmatrix} -3 & 1 \\ -5 & 2 \end{bmatrix}$, and $\underset{B\leftarrow C}{P} = \underset{C\leftarrow B}{P}^{-1} = \begin{bmatrix} -2 & 1 \\ -5 & 3 \end{bmatrix}$.

8. To find $\underset{C\leftarrow B}{P}$, row reduce the matrix $\begin{bmatrix} \mathbf{c}_1 & \mathbf{c}_2 & \mathbf{b}_1 & \mathbf{b}_2 \end{bmatrix}$:

$$\begin{bmatrix} \mathbf{c}_1 & \mathbf{c}_2 & \mathbf{b}_1 & \mathbf{b}_2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & -2 \\ 0 & 1 & -4 & 3 \end{bmatrix}.$$

Thus $\underset{C\leftarrow B}{P} = \begin{bmatrix} 3 & -2 \\ -4 & 3 \end{bmatrix}$, and $\underset{B\leftarrow C}{P} = \underset{C\leftarrow B}{P}^{-1} = \begin{bmatrix} 3 & 2 \\ 4 & 3 \end{bmatrix}$.

9. To find $\underset{C\leftarrow B}{P}$, row reduce the matrix $\begin{bmatrix} \mathbf{c}_1 & \mathbf{c}_2 & \mathbf{b}_1 & \mathbf{b}_2 \end{bmatrix}$:

$$\begin{bmatrix} \mathbf{c}_1 & \mathbf{c}_2 & \mathbf{b}_1 & \mathbf{b}_2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 9 & -2 \\ 0 & 1 & -4 & 1 \end{bmatrix}.$$

Thus $\underset{C\leftarrow B}{P} = \begin{bmatrix} 9 & -2 \\ -4 & 1 \end{bmatrix}$, and $\underset{B\leftarrow C}{P} = \underset{C\leftarrow B}{P}^{-1} = \begin{bmatrix} 1 & 2 \\ 4 & 9 \end{bmatrix}$.

10. To find $\underset{C\leftarrow B}{P}$, row reduce the matrix $\begin{bmatrix} \mathbf{c}_1 & \mathbf{c}_2 & \mathbf{b}_1 & \mathbf{b}_2 \end{bmatrix}$:

$$\begin{bmatrix} \mathbf{c}_1 & \mathbf{c}_2 & \mathbf{b}_1 & \mathbf{b}_2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 8 & 3 \\ 0 & 1 & -5 & -2 \end{bmatrix}.$$

Thus $\underset{C\leftarrow B}{P} = \begin{bmatrix} 8 & 3 \\ -5 & -2 \end{bmatrix}$, and $\underset{B\leftarrow C}{P} = \underset{C\leftarrow B}{P}^{-1} = \begin{bmatrix} 2 & 3 \\ -5 & -8 \end{bmatrix}$.

11. **a**. False. See Theorem 15.

b. True. See the first paragraph in the subsection "Change of Basis in $\mathbb{R}^n$."

12. **a**. True. The columns of $\underset{C \leftarrow B}{P}$ are coordinate vectors of the linearly independent set B. See the second paragraph after Theorem 15.

b. False. The row reduction is discussed after Example 2. The matrix P obtained there satisfies $[\mathbf{x}]_C = P[\mathbf{x}]_B$

13. Let $B = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\} = \{1 - 2t + t^2, 3 - 5t + 4t^2, 2t + 3t^2\}$ and let $C = \{\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3\} = \{1, t, t^2\}$. The C-coordinate vectors of $\mathbf{b}_1$, $\mathbf{b}_2$, and $\mathbf{b}_3$ are

$$[\mathbf{b}_1]_C = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, [\mathbf{b}_2]_C = \begin{bmatrix} 3 \\ -5 \\ 4 \end{bmatrix}, [\mathbf{b}_3]_C = \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}$$

So

$$\underset{C \leftarrow B}{P} = \begin{bmatrix} 1 & 3 & 0 \\ -2 & -5 & 2 \\ 1 & 4 & 3 \end{bmatrix}$$

Let $\mathbf{x} = -1 + 2t$. Then the coordinate vector $[\mathbf{x}]_B$ satisfies

$$\underset{C \leftarrow B}{P}[\mathbf{x}]_B = [\mathbf{x}]_C = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$$

This system may be solved by row reducing its augmented matrix:

$$\begin{bmatrix} 1 & 3 & 0 & -1 \\ -2 & -5 & 2 & 2 \\ 1 & 4 & 3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \text{so } [\mathbf{x}]_B = \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}$$

14. Let $B = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\} = \{1 - 3t^2, 2 + t - 5t^2, 1 + 2t\}$ and let $C = \{\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3\} = \{1, t, t^2\}$. The C-coordinate vectors of $\mathbf{b}_1$, $\mathbf{b}_2$, and $\mathbf{b}_3$ are

$$[\mathbf{b}_1]_C = \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, [\mathbf{b}_2]_C = \begin{bmatrix} 2 \\ 1 \\ -5 \end{bmatrix}, [\mathbf{b}_3]_C = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

So

$$\underset{C \leftarrow B}{P} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ -3 & -5 & 0 \end{bmatrix}$$

Let $\mathbf{x} = t^2$. Then the coordinate vector $[\mathbf{x}]_B$ satisfies

$$\underset{C \leftarrow B}{P}[\mathbf{x}]_B = [\mathbf{x}]_C = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

This system may be solved by row reducing its augmented matrix:

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ -3 & -5 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \text{ so } [\mathbf{x}]_B = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$$

and $t^2 = 3(1-3t^2) - 2(2+t-5t^2) + (1+2t)$.

15. (a) B is a basis for V

(b) the coordinate mapping is a linear transformation

(c) of the product of a matrix and a vector

(d) the coordinate vector of $\mathbf{v}$ relative to B

16. (a) $[\mathbf{b}_1]_C = Q[\mathbf{b}_1]_B = Q\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = Q\mathbf{e}_1$

(b) $[\mathbf{b}_k]_C$

(c) $[\mathbf{b}_k]_C = Q[\mathbf{b}_k]_B = Q\mathbf{e}_k$

17. **[M]**

a. Since we found P in Exercise 34 of Section 4.5, we can calculate that

$$P^{-1} = \frac{1}{32}\begin{bmatrix} 32 & 0 & 16 & 0 & 12 & 0 & 10 \\ 0 & 32 & 0 & 24 & 0 & 20 & 0 \\ 0 & 0 & 16 & 0 & 16 & 0 & 15 \\ 0 & 0 & 0 & 8 & 0 & 10 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 & 6 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

b. Since P is the change-of-coordinates matrix from C to B, P^{-1} will be the change-of-coordinates matrix from B to C. By Theorem 15, the columns of P^{-1} will be the C-coordinate vectors of the basis vectors in B. Thus

$$\cos^2 t = \frac{1}{2}(1+\cos 2t)$$

$$\cos^3 t = \frac{1}{4}(3\cos t + \cos 3t)$$

$$\cos^4 t = \frac{1}{8}(3 + 4\cos 2t + \cos 4t)$$

$$\cos^5 t = \frac{1}{16}(10\cos t + 5\cos 3t + \cos 5t)$$

$$\cos^6 t = \frac{1}{32}(10 + 15\cos 2t + 6\cos 4t + \cos 6t)$$

18. **[M]** The C-coordinate vector of the integrand is (0, 0, 0, 5, –6, 5, –12). Using P^{-1} from the previous exercise, the B- coordinate vector of the integrand will be

$$P^{-1}(0, 0, 0, 5, -6, 5, -12) = (-6, 55/8, -69/8, 45/16, -3, 5/16, -3/8)$$

Thus the integral may be rewritten as

$$\int -6+\frac{55}{8}\cos t-\frac{69}{8}\cos 2t+\frac{45}{16}\cos 3t-3\cos 4t+\frac{5}{16}\cos 5t-\frac{3}{8}\cos 6t\,dt,$$

which equals

$$-6t+\frac{55}{8}\sin t-\frac{69}{16}\sin 2t+\frac{15}{16}\sin 3t-\frac{3}{4}\sin 4t+\frac{1}{16}\sin 5t-\frac{1}{16}\sin 6t+C.$$

19. **[M]**

a. If C is the basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, then the columns of P are $[\mathbf{u}_1]_C$, $[\mathbf{u}_2]_C$, and $[\mathbf{u}_3]_C$. So $\mathbf{u}_j = [\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3][\mathbf{u}_1]_C$, and $[\mathbf{u}_1 \quad \mathbf{u}_2 \quad \mathbf{u}_3] = [\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3]P$. In the current exercise,

$$[\mathbf{u}_1 \quad \mathbf{u}_2 \quad \mathbf{u}_3] = \begin{bmatrix} -2 & -8 & -7 \\ 2 & 5 & 2 \\ 3 & 2 & 6 \end{bmatrix}\begin{bmatrix} 1 & 2 & -1 \\ -3 & -5 & 0 \\ 4 & 6 & 1 \end{bmatrix} = \begin{bmatrix} -6 & -6 & -5 \\ -5 & -9 & 0 \\ 21 & 32 & 3 \end{bmatrix}.$$

b. Analogously to part a., $[\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3] = [\mathbf{w}_1 \quad \mathbf{w}_2 \quad \mathbf{w}_3]P$, so $[\mathbf{w}_1 \quad \mathbf{w}_2 \quad \mathbf{w}_3] =$ $[\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3]P^{-1}$. In the current exercise,

$$[\mathbf{w}_1 \quad \mathbf{w}_2 \quad \mathbf{w}_3] = \begin{bmatrix} -2 & -8 & -7 \\ 2 & 5 & 2 \\ 3 & 2 & 6 \end{bmatrix}\begin{bmatrix} 1 & 2 & -1 \\ -3 & -5 & 0 \\ 4 & 6 & 1 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} -2 & -8 & -7 \\ 2 & 5 & 2 \\ 3 & 2 & 6 \end{bmatrix}\begin{bmatrix} 5 & 8 & 5 \\ -3 & -5 & -3 \\ -2 & -2 & -1 \end{bmatrix} = \begin{bmatrix} 28 & 38 & 21 \\ -9 & -13 & -7 \\ -3 & 2 & 3 \end{bmatrix}.$$

20. **a**. $\underset{D\leftarrow B}{P} = \underset{D\leftarrow C}{P}\,\underset{C\leftarrow B}{P}$

Let $\mathbf{x}$ be any vector in the two-dimensional vector space. Since $\underset{C\leftarrow B}{P}$ is the change-of-coordinates matrix from B to C and $\underset{D\leftarrow C}{P}$ is the change-of-coordinates matrix from C to D,

$$[\mathbf{x}]_C = \underset{C\leftarrow B}{P}[\mathbf{x}]_B \text{ and } [\mathbf{x}]_D = \underset{D\leftarrow C}{P}[\mathbf{x}]_C = \underset{D\leftarrow C}{P}\,\underset{C\leftarrow B}{P}[\mathbf{x}]_B$$

But since $\underset{D\leftarrow B}{P}$ is the change-of-coordinates matrix from B to D,

$$[\mathbf{x}]_D = \underset{D\leftarrow B}{P}[\mathbf{x}]_B$$

Thus

$$\underset{D\leftarrow B}{P}[\mathbf{x}]_B = \underset{D\leftarrow C}{P}\,\underset{C\leftarrow B}{P}[\mathbf{x}]_B$$

for any vector $[\mathbf{x}]_B$ in $\mathbb{R}^2$, and

$$\underset{D\leftarrow B}{P} = \underset{D\leftarrow C}{P}\,\underset{C\leftarrow B}{P}$$

b. **[M]** For example, let $B=\left\{\begin{bmatrix}7\\5\end{bmatrix},\begin{bmatrix}-3\\-1\end{bmatrix}\right\}$, $C=\left\{\begin{bmatrix}1\\-5\end{bmatrix},\begin{bmatrix}-2\\2\end{bmatrix}\right\}$, and $D=\left\{\begin{bmatrix}-1\\8\end{bmatrix},\begin{bmatrix}1\\-5\end{bmatrix}\right\}$. Then we can calculate the change-of-coordinates matrices:

$$\begin{bmatrix}1&-2&7&-3\\-5&2&5&-1\end{bmatrix}\sim\begin{bmatrix}1&0&-3&1\\0&1&-5&2\end{bmatrix}\Rightarrow \underset{C\leftarrow B}{P}=\begin{bmatrix}-3&1\\-5&2\end{bmatrix}$$

$$\begin{bmatrix}-1&1&1&-2\\8&-5&-5&2\end{bmatrix}\sim\begin{bmatrix}1&0&0&-8/3\\0&1&1&-14/3\end{bmatrix}\Rightarrow \underset{D\leftarrow C}{P}=\begin{bmatrix}0&-8/3\\1&-14/3\end{bmatrix}$$

$$\begin{bmatrix}-1&1&7&-3\\8&-5&5&-1\end{bmatrix}\sim\begin{bmatrix}1&0&40/3&-16/3\\0&1&61/3&-25/3\end{bmatrix}\Rightarrow \underset{D\leftarrow B}{P}=\begin{bmatrix}40/3&-16/3\\61/3&-25/3\end{bmatrix}$$

One confirms easily that

$$\underset{D\leftarrow B}{P}=\begin{bmatrix}40/3&-16/3\\61/3&-25/3\end{bmatrix}=\begin{bmatrix}0&-8/3\\1&-14/3\end{bmatrix}\begin{bmatrix}-3&1\\-5&2\end{bmatrix}=\underset{D\leftarrow C}{P}\,\underset{C\leftarrow B}{P}$$

4.8 SOLUTIONS

Notes: This is an important section for engineering students and worth extra class time. To spend only one lecture on this section, you could cover through Example 5, but assign the somewhat lengthy Example 3 for reading. Finding a spanning set for the solution space of a difference equation uses the Basis Theorem (Section 4.5) and Theorem 17 in this section, and demonstrates the power of the theory of Chapter 4 in helping to solve applied problems. This section anticipates Section 5.7 on differential equations. The reduction of an n^{th} order difference equation to a linear system of first order difference equations was introduced in Section 1.10, and is revisited in Sections 4.9 and 5.6. Example 3 is the background for Exercise 26 in Section 6.5.

1. Let $y_k=2^k$. Then

$$\begin{aligned}y_{k+2}+2y_{k+1}-8y_k&=2^{k+2}+2(2^{k+1})-8(2^k)\\&=2^k(2^2+2^2-8)\\&=2^k(0)=0\text{ for all }k\end{aligned}$$

Since the difference equation holds for all k, 2^k is a solution.

Let $y_k=(-4)^k$. Then

$$\begin{aligned}y_{k+2}+2y_{k+1}-8y_k&=(-4)^{k+2}+2(-4)^{k+1}-8(-4)^k\\&=(-4)^k((-4)^2+2(-4)-8)\\&=(-4)^k(0)=0\text{ for all }k\end{aligned}$$

Since the difference equation holds for all k, $(-4)^k$ is a solution.

2. Let $y_k=3^k$. Then

$$\begin{aligned}y_{k+2}-9y_k&=3^{k+2}-9(3^k)\\&=3^k(3^2-9)\\&=3^k(0)=0\text{ for all }k\end{aligned}$$

Since the difference equation holds for all k, 3^k is a solution.

Let $y_k = (-3)^k$. Then

$$\begin{aligned} y_{k+2} - 9y_k &= (-3)^{k+2} - 9(-3)^k \\ &= (-3)^k((-3)^2 - 9) \\ &= (-3)^k(0) = 0 \text{ for all } k \end{aligned}$$

Since the difference equation holds for all k, $(-3)^k$ is a solution.

3. The signals 2^k and $(-4)^k$ are linearly independent because neither is a multiple of the other; that is, there is no scalar c such that $2^k = c(-4)^k$ for all k. By Theorem 17, the solution set H of the difference equation $y_{k+2} + 2y_{k+1} - 8y_k = 0$ is two-dimensional. By the Basis Theorem, the two linearly independent signals 2^k and $(-4)^k$ form a basis for H.

4. The signals 3^k and $(-3)^k$ are linearly independent because neither is a multiple of the other; that is, there is no scalar c such that $3^k = c(-3)^k$ for all k. By Theorem 17, the solution set H of the difference equation $y_{k+2} - 9y_k = 0$ is two-dimensional. By the Basis Theorem, the two linearly independent signals 3^k and $(-3)^k$ form a basis for H.

5. Let $y_k = (-3)^k$. Then

$$\begin{aligned} y_{k+2} + 6y_{k+1} + 9y_k &= (-3)^{k+2} + 6(-3)^{k+1} + 9(-3)^k \\ &= (-3)^k((-3)^2 + 6(-3) + 9) \\ &= (-3)^k(0) = 0 \text{ for all } k \end{aligned}$$

Since the difference equation holds for all k, $(-3)^k$ is in the solution set H.

Let $y_k = k(-3)^k$. Then

$$\begin{aligned} y_{k+2} + 6y_{k+1} + 9y_k &= (k+2)(-3)^{k+2} + 6(k+1)(-3)^{k+1} + 9k(-3)^k \\ &= (-3)^k((k+2)(-3)^2 + 6(k+1)(-3) + 9k) \\ &= (-3)^k(9k + 18 - 18k - 18 + 9k) \\ &= (-3)^k(0) = 0 \text{ for all } k \end{aligned}$$

Since the difference equation holds for all k, $k(-3)^k$ is in the solution set H.

The signals $(-3)^k$ and $k(-3)^k$ are linearly independent because neither is a multiple of the other; that is, there is no scalar c such that $(-3)^k = ck(-3)^k$ for all k and there is no scalar c such that $c(-3)^k = k(-3)^k$ for all k. By Theorem 17, dim H = 2, so the two linearly independent signals 3^k and $(-3)^k$ form a basis for H by the Basis Theorem.

6. Let $y_k = 5^k \cos\frac{k\pi}{2}$. Then

$$\begin{aligned} y_{k+2} + 25y_k &= 5^{k+2}\cos\frac{(k+2)\pi}{2} + 25\left(5^k \cos\frac{k\pi}{2}\right) \\ &= 5^k\left(5^2\cos\frac{(k+2)\pi}{2} + 25\cos\frac{k\pi}{2}\right) \\ &= 25\cdot 5^k\left(\cos\left(\frac{k\pi}{2}+\pi\right) + \cos\frac{k\pi}{2}\right) \\ &= 25\cdot 5^k(0) = 0 \text{ for all } k \end{aligned}$$

since $\cos(t + \pi) = -\cos t$ for all t. Since the difference equation holds for all k, $5^k\cos\frac{k\pi}{2}$ is in the solution set H.

Let $y_k = 5^k \sin\frac{k\pi}{2}$. Then

$$\begin{aligned} y_{k+2} + 25y_k &= 5^{k+2}\sin\frac{(k+2)\pi}{2} + 25\left(5^k \sin\frac{k\pi}{2}\right) \\ &= 5^k\left(5^2\sin\frac{(k+2)\pi}{2} + 25\sin\frac{k\pi}{2}\right) \\ &= 25\cdot 5^k\left(\sin\left(\frac{k\pi}{2}+\pi\right) + \sin\frac{k\pi}{2}\right) \\ &= 25\cdot 5^k(0) = 0 \text{ for all } k \end{aligned}$$

since $\sin(t + \pi) = -\sin t$ for all t. Since the difference equation holds for all k, $5^k\sin\frac{k\pi}{2}$ is in the solution set H.

The signals $5^k\cos\frac{k\pi}{2}$ and $5^k\sin\frac{k\pi}{2}$ are linearly independent because neither is a multiple of the other. By Theorem 17, dim $H = 2$, so the two linearly independent signals $5^k\cos\frac{k\pi}{2}$ and $5^k\sin\frac{k\pi}{2}$ form a basis for H by the Basis Theorem.

7. Compute and row reduce the Casorati matrix for the signals 1^k, 2^k, and $(-2)^k$, setting $k = 0$ for convenience:

$$\begin{bmatrix} 1^0 & 2^0 & (-2)^0 \\ 1^1 & 2^1 & (-2)^1 \\ 1^2 & 2^2 & (-2)^2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

This Casorati matrix is row equivalent to the identity matrix, thus is invertible by the IMT. Hence the set of signals $\{1^k, 2^k, (-2)^k\}$ is linearly independent in $\mathbb{S}$. The exercise states that these signals are in the solution set H of a third-order difference equation. By Theorem 17, dim $H = 3$, so the three linearly independent signals 1^k, 2^k, $(-2)^k$ form a basis for H by the Basis Theorem.

8. Compute and row reduce the Casorati matrix for the signals 2^k, 4^k, and $(-5)^k$, setting $k=0$ for convenience:

$$\begin{bmatrix} 2^0 & 4^0 & (-5)^0 \\ 2^1 & 4^1 & (-5)^1 \\ 2^2 & 4^2 & (-5)^2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

This Casorati matrix is row equivalent to the identity matrix, thus is invertible by the IMT. Hence the set of signals $\{2^k, 4^k, (-5)^k\}$ is linearly independent in $\mathbb{S}$. The exercise states that these signals are in the solution set *H* of a third-order difference equation. By Theorem 17, dim $H=3$, so the three linearly independent signals 2^k, 4^k, $(-5)^k$ form a basis for *H* by the Basis Theorem.

9. Compute and row reduce the Casorati matrix for the signals 1^k, $3^k \cos\frac{k\pi}{2}$, and $3^k \sin\frac{k\pi}{2}$, setting $k=0$ for convenience:

$$\begin{bmatrix} 1^0 & 3^0\cos 0 & 3^0 \sin 0 \\ 1^1 & 3^1\cos\frac{\pi}{2} & 3^1\sin\frac{\pi}{2} \\ 1^2 & 3^2\cos\pi & 3^2\sin\pi \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

This Casorati matrix is row equivalent to the identity matrix, thus is invertible by the IMT. Hence the set of signals $\{1^k, 3^k\cos\frac{k\pi}{2}, 3^k\sin\frac{k\pi}{2}\}$ is linearly independent in $\mathbb{S}$. The exercise states that these signals are in the solution set *H* of a third-order difference equation. By Theorem 17, dim $H=3$, so the three linearly independent signals 1^k, $3^k\cos\frac{k\pi}{2}$, and $3^k\sin\frac{k\pi}{2}$, form a basis for *H* by the Basis Theorem.

10. Compute and row reduce the Casorati matrix for the signals $(-1)^k$, $k(-1)^k$, and 5^k, setting $k=0$ for convenience:

$$\begin{bmatrix} (-1)^0 & 0(-1)^0 & 5^0 \\ (-1)^1 & 1(-1)^1 & 5^1 \\ (-1)^2 & 2(-1)^2 & 5^2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

This Casorati matrix is row equivalent to the identity matrix, thus is invertible by the IMT. Hence the set of signals $\{(-1)^k, k(-1)^k, 5^k\}$ is linearly independent in $\mathbb{S}$. The exercise states that these signals are in the solution set *H* of a third-order difference equation. By Theorem 17, dim $H=3$, so the three linearly independent signals $(-1)^k$, $k(-1)^k$, and 5^k form a basis for *H* by the Basis Theorem.

11. The solution set *H* of this third-order difference equation has dim $H=3$ by Theorem 17. The two signals $(-1)^k$ and 3^k cannot possibly span a three-dimensional space, and so cannot be a basis for *H*.

12. The solution set *H* of this fourth-order difference equation has dim $H=4$ by Theorem 17. The two signals 1^k and $(-1)^k$ cannot possibly span a four-dimensional space, and so cannot be a basis for *H*.

13. The auxiliary equation for this difference equation is $r^2 - r + 2/9 = 0$. By the quadratic formula (or factoring), $r = 2/3$ or $r = 1/3$, so two solutions of the difference equation are $(2/3)^k$ and $(1/3)^k$. The signals $(2/3)^k$ and $(1/3)^k$ are linearly independent because neither is a multiple of the other.

By Theorem 17, the solution space is two-dimensional, so the two linearly independent signals $(2/3)^k$ and $(1/3)^k$ form a basis for the solution space by the Basis Theorem.

14. The auxiliary equation for this difference equation is $r^2-7r+12=0$. By the quadratic formula (or factoring), $r = 3$ or $r = 4$, so two solutions of the difference equation are 3^k and 4^k. The signals 3^k and 4^k are linearly independent because neither is a multiple of the other. By Theorem 17, the solution space is two-dimensional, so the two linearly independent signals 3^k and 4^k form a basis for the solution space by the Basis Theorem.

15. The auxiliary equation for this difference equation is $r^2-25=0$. By the quadratic formula (or factoring), $r = 5$ or $r = -5$, so two solutions of the difference equation are 5^k and $(-5)^k$. The signals 5^k and $(-5)^k$ are linearly independent because neither is a multiple of the other. By Theorem 17, the solution space is two-dimensional, so the two linearly independent signals 5^k and $(-5)^k$ form a basis for the solution space by the Basis Theorem.

16. The auxiliary equation for this difference equation is $16r^2+8r-3=0$. By the quadratic formula (or factoring), $r = 1/4$ or $r = -3/4$, so two solutions of the difference equation are $(1/4)^k$ and $(-3/4)^k$. The signals $(1/4)^k$ and $(-3/4)^k$ are linearly independent because neither is a multiple of the other. By Theorem 17, the solution space is two-dimensional, so the two linearly independent signals $(1/4)^k$ and $(-3/4)^k$ form a basis for the solution space by the Basis Theorem.

17. Letting $a = .9$ and $b = 4/9$ gives the difference equation $Y_{k+2}-1.3Y_{k+1}+.4Y_k=1$. First we find a particular solution $Y_k=T$ of this equation, where T is a constant. The solution of the equation $T-1.3T+.4T=1$ is $T = 10$, so 10 is a particular solution to $Y_{k+2}-1.3Y_{k+1}+.4Y_k=1$. Next we solve the homogeneous difference equation $Y_{k+2}-1.3Y_{k+1}+.4Y_k=0$. The auxiliary equation for this difference equation is $r^2-1.3r+.4=0$. By the quadratic formula (or factoring), $r = .8$ or $r = .5$, so two solutions of the homogeneous difference equation are $.8^k$ and $.5^k$. The signals $(.8)^k$ and $(.5)^k$ are linearly independent because neither is a multiple of the other. By Theorem 17, the solution space is two-dimensional, so the two linearly independent signals $(.8)^k$ and $(.5)^k$ form a basis for the solution space of the homogeneous difference equation by the Basis Theorem. Translating the solution space of the homogeneous difference equation by the particular solution 10 of the nonhomogeneous difference equation gives us the general solution of $Y_{k+2}-1.3Y_{k+1}+.4Y_k=1$: $Y_k=c_1(.8)^k+c_2(.5)^k+10$. As k increases the first two terms in the solution approach 0, so Y_k approaches 10.

18. Letting $a = .9$ and $b = .5$ gives the difference equation $Y_{k+2}-1.35Y_{k+1}+.45Y_k=1$. First we find a particular solution $Y_k=T$ of this equation, where T is a constant. The solution of the equation $T-1.35T+.45T=1$ is $T = 10$, so 10 is a particular solution to $Y_{k+2}-1.3Y_{k+1}+.4Y_k=1$. Next we solve the homogeneous difference equation $Y_{k+2}-1.35Y_{k+1}+.45Y_k=0$. The auxiliary equation for this difference equation is $r^2-1.35r+.45=0$. By the quadratic formula (or factoring), $r = .6$ or $r = .75$, so two solutions of the homogeneous difference equation are $.6^k$ and $.75^k$. The signals $(.6)^k$ and $(.75)^k$ are linearly independent because neither is a multiple of the other. By Theorem 17, the solution space is two-dimensional, so the two linearly independent signals $(.6)^k$ and $(.75)^k$ form a basis for the solution space of the homogeneous difference equation by the Basis Theorem. Translating the solution space of the homogeneous difference equation by the particular solution 10 of the nonhomogeneous difference equation gives us the general solution of $Y_{k+2} - 1.35Y_{k+1} + .45Y_k = 1$: $Y_k = c_1(.6)^k + c_2(.75)^k + 10$.

19. The auxiliary equation for this difference equation is $r^2 + 4r + 1 = 0$. By the quadratic formula, $r = -2 + \sqrt{3}$ or $r = -2 - \sqrt{3}$, so two solutions of the difference equation are $(-2+\sqrt{3})^k$ and $(-2-\sqrt{3})^k$. The signals $(-2+\sqrt{3})^k$ and $(-2-\sqrt{3})^k$ are linearly independent because neither is a multiple of the other. By Theorem 17, the solution space is two-dimensional, so the two linearly independent signals $(-2+\sqrt{3})^k$ and $(-2-\sqrt{3})^k$ form a basis for the solution space by the Basis Theorem. Thus a general solution to this difference equation is $y_k = c_1(-2+\sqrt{3})^k + c_2(-2-\sqrt{3})^k$.

20. Let $a = -2 + \sqrt{3}$ and $b = -2 - \sqrt{3}$. Using the solution from the previous exercise, we find that $y_1 = c_1a + c_2b = 5000$ and $y_N = c_1a^N + c_2b^N = 0$. This is a system of linear equations with variables c_1 and c_2 whose augmented matrix may be row reduced:

$$\begin{bmatrix} a & b & 5000 \\ a^N & b^N & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & \dfrac{5000b^N}{b^Na - a^Nb} \\ 0 & 1 & \dfrac{5000a^N}{b^Na - a^Nb} \end{bmatrix}$$

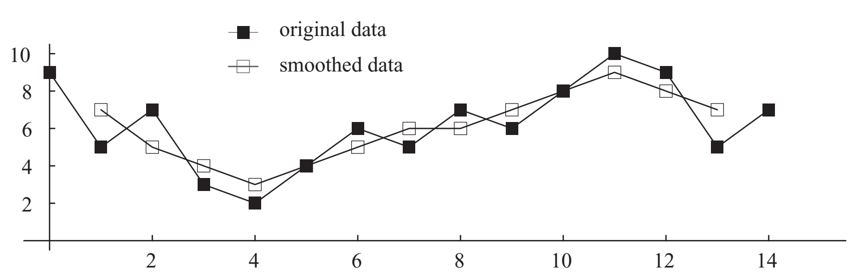
so

$$c_1 = \frac{5000b^N}{b^Na - a^Nb}, c_2 = \frac{5000a^N}{b^Na - a^Nb}$$

(Alternatively, Cramer's Rule may be applied to get the same solution). Thus

$$\begin{aligned} y_k &= c_1a^k + c_2b^k \\ &= \frac{5000(a^kb^N - a^Nb^k)}{b^Na - a^Nb} \end{aligned}$$

21. The smoothed signal z_k has the following values: $z_1 = (9+5+7)/3 = 7$, $z_2 = (5+7+3)/3 = 5$, $z_3 = (7+3+2)/3 = 4$, $z_4 = (3+2+4)/3 = 3$, $z_5 = (2+4+6)/3 = 4$, $z_6 = (4+6+5)/3 = 5$, $z_7 = (6+5+7)/3 = 6$, $z_8 = (5+7+6)/3 = 6$, $z_9 = (7+6+8)/3 = 7$, $z_{10} = (6+8+10)/3 = 8$, $z_{11} = (8+10+9)/3 = 9$, $z_{12} = (10+9+5)/3 = 8$, $z_{13} = (9+5+7)/3 = 7$.



22. **a**. The smoothed signal z_k has the following values:

$$\begin{aligned} z_0 &= .35y_2 + .5y_1 + .35y_0 = .35(0) + .5(.7) + .35(3) = 1.4, \\ z_1 &= .35y_3 + .5y_2 + .35y_1 = .35(-.7) + .5(0) + .35(.7) = 0, \\ z_2 &= .35y_4 + .5y_3 + .35y_2 = .35(-.3) + .5(-.7) + .35(0) = -1.4, \\ z_3 &= .35y_5 + .5y_4 + .35y_3 = .35(-.7) + .5(-.3) + .35(-.7) = -2, \end{aligned}$$

$$z_4 = .35y_6 + .5y_5 + .35y_4 = .35(0) + .5(-.7) + .35(-.3) = -1.4,$$
$$z_5 = .35y_7 + .5y_6 + .35y_5 = .35(.7) + .5(0) + .35(-.7) = 0,$$
$$z_6 = .35y_8 + .5y_7 + .35y_6 = .35(3) + .5(.7) + .35(0) = 1.4,$$
$$z_7 = .35y_9 + .5y_8 + .35y_7 = .35(.7) + .5(3) + .35(.7) = 2,$$
$$z_8 = .35y_{10} + .5y_9 + .35y_8 = .35(0) + .5(.7) + .35(3) = 1.4,\ldots$$

b. This signal is two times the signal output by the filter when the input (in Example 3) was $y = \cos(\pi t/4)$. This is expected because the filter is linear. The output from the input $2\cos(\pi t/4) + \cos(3\pi t/4)$ should be two times the output from $\cos(\pi t/4)$ plus the output from $\cos(3\pi t/4)$ (which is zero).

23. **a**. $y_{k+1} - 1.01y_k = -450,\ y_0 = 10,000.$

b. **[M]** MATLAB code to create the table:

```
pay=450, y=10000, m=0, table=[0;y]
while y>450
    y=1.01*y-pay
    m=m+1
    table=[table [m;y]]
end
m,y
```

Mathematica code to create the table:

```
pay = 450; y = 10000; m = 0; balancetable = {{0, y}};
While[y > 450, {y = 1.01*y - pay; m = m + 1,
    AppendTo[balancetable, {m, y}]}];
m
y
```

c. **[M]** At month 26, the last payment is \$114.88. The total paid by the borrower is \$11,364.88.

24. **a**. $y_{k+1} - 1.005y_k = 200,\ y_0 = 1,000.$

b. **[M]** MATLAB code to create the table:

```
pay=200, y=1000, m=0, table=[0;y]
for m=1: 60
    y=1.005*y+pay
    table=[table [m;y]]
end
interest=y-60*pay-1000
```

Mathematica code to create the table:

```
pay = 200; y = 1000; amounttable = {{0, y}};
Do[{y = 1.005*y + pay;
    AppendTo[amounttable, {m, y}]},{m,1,60}];
interest=y-60*pay-1000
```

c. **[M]** The total is \$6213.55 at $k = 24$, \$12,090.06 at $k = 48$, and \$15,302.86 at $k = 60$. When $k = 60$, the interest earned is \$2302.86.

25. To show that $y_k = k^2$ is a solution of $y_{k+2} + 3_{k+1} - 4y_k = 10k + 7$, substitute $y_k = k^2$, $y_{k+1} = (k+1)^2$, and $y_{k+2} = (k+2)^2$:

$$\begin{aligned} y_{k+2} + 3_{k+1} - 4y_k &= (k+2)^2 + 3(k+1)^2 - 4k^2 \\ &= (k^2 + 4k + 4) + 3(k^2 + 2k + 1) - 4k^2 \\ &= k^2 + 4k + 4 + 3k^2 + 6k + 3 - 4k^2 \\ &= 10k + 7 \text{ for all } k \end{aligned}$$

The auxiliary equation for the homogeneous difference equation $y_{k+2} + 3y_{k+1} - 4y_k = 0$ is $r^2 + 3r - 4 = 0$. By the quadratic formula (or factoring), $r = -4$ or $r = 1$, so two solutions of the difference equation are $(-4)^k$ and 1^k. The signals $(-4)^k$ and 1^k are linearly independent because neither is a multiple of the other. By Theorem 17, the solution space is two-dimensional, so the two linearly independent signals $(-4)^k$ and 1^k form a basis for the solution space of the homogeneous difference equation by the Basis Theorem. The general solution to the homogeneous difference equation is thus $c_1(-4)^k + c_2 \cdot 1^k = c_1(-4)^k + c_2$. Adding the particular solution k^2 of the nonhomogeneous difference equation, we find that the general solution of the difference equation $y_{k+2} + 3y_{k+1} - 4y_k = 10k + 7$ is $y_k = k^2 + c_1(-4)^k + c_2$.

26. To show that $y_k = 1 + k$ is a solution of $y_{k+2} - 8y_{k+1} + 15y_k = 8k + 2$, substitute $y_k = 1 + k$, $y_{k+1} = 1 + (k+1) = 2 + k$, and $y_{k+2} = 1 + (k+2) = 3 + k$:

$$\begin{aligned} y_{k+2} - 8y_{k+1} + 15y_k &= (3+k) - 8(2+k) + 15(1+k) \\ &= 3 + k - 16 - 8k + 15 + 15k \\ &= 8k + 2 \text{ for all } k \end{aligned}$$

The auxiliary equation for the homogeneous difference equation $y_{k+2} - 8y_{k+1} + 15y_k = 0$ is $r^2 - 8r + 15 = 0$. By the quadratic formula (or factoring), $r = 5$ or $r = 3$, so two solutions of the difference equation are 5^k and 3^k. The signals 5^k and 3^k are linearly independent because neither is a multiple of the other. By Theorem 17, the solution space is two-dimensional, so the two linearly independent signals 5^k and 3^k form a basis for the solution space of the homogeneous difference equation by the Basis Theorem. The general solution to the homogeneous difference equation is thus $c_1 \cdot 5^k + c_2 \cdot 3^k$. Adding the particular solution $1 + k$ of the nonhomogeneous difference equation, we find that the general solution of the difference equation $y_{k+2} - 8y_{k+1} + 15y_k = 8k + 2$ is $y_k = 1 + k + c_1 \cdot 5^k + c_2 \cdot 3^k$.

27. To show that $y_k = 2 - 2k$ is a solution of $y_{k+2} - (9/2)y_{k+1} + 2y_k = 3k + 2$, substitute $y_k = 2 - 2k$, $y_{k+1} = 2 - 2(k+1) = -2k$, and $y_{k+2} = 2 - 2(k+2) = -2 - 2k$:

$$\begin{aligned} y_{k+2} - (9/2)y_{k+1} + 2y_k &= (-2 - 2k) - (9/2)(-2k) + 2(2 - 2k) \\ &= -2 - 2k + 9k + 4 - 4k \\ &= 3k + 2 \text{ for all } k \end{aligned}$$

The auxiliary equation for the homogeneous difference equation $y_{k+2} - (9/2)y_{k+1} + 2y_k = 0$ is $r^2 - (9/2)r + 2 = 0$. By the quadratic formula (or factoring), $r = 4$ or $r = 1/2$, so two solutions of the difference equation are 4^k and $(1/2)^k$. The signals 4^k and $(1/2)^k$ are linearly independent because neither is a multiple of the other. By Theorem 17, the solution space is two-dimensional, so the two linearly independent signals 4^k and $(1/2)^k$ form a basis for the solution space of the homogeneous difference equation by the Basis Theorem. The general solution to the homogeneous difference equation is thus $c_1 \cdot 4^k + c_2 \cdot (1/2)^k = c_1 \cdot 4^k + c_2 \cdot 2^{-k}$. Adding the particular solution $2 - 2k$ of the nonhomogeneous difference equation, we find that the general solution of the difference equation $y_{k+2} - (9/2)y_{k+1} + 2y_k = 3k + 2$ is $y_k = 2 - 2k + c_1 \cdot 4^k + c_2 \cdot 2^{-k}$.

28. To show that $y_k = 2k - 4$ is a solution of $y_{k+2} + (3/2)y_{k+1} - y_k = 1 + 3k$, substitute $y_k = 2k - 4$, $y_{k+1} = 2(k+1) - 4 = 2k - 2$, and $y_{k+2} = 2(k+2) - 4 = 2k$:

$$\begin{aligned} y_{k+2} + (3/2)y_{k+1} - y_k &= 2k + (3/2)(2k-2) - (2k-4) \\ &= 2k + 3k - 3 - 2k + 4 \\ &= 1 + 3k \text{ for all } k \end{aligned}$$

The auxiliary equation for the homogeneous difference equation $y_{k+2} + (3/2)y_{k+1} - y_k = 0$ is $r^2 + (3/2)r - 1 = 0$. By the quadratic formula (or factoring), $r = -2$ or $r = 1/2$, so two solutions of the difference equation are $(-2)^k$ and $(1/2)^k$. The signals $(-2)^k$ and $(1/2)^k$ are linearly independent because neither is a multiple of the other. By Theorem 17, the solution space is two-dimensional, so the two linearly independent signals $(-2)^k$ and $(1/2)^k$ form a basis for the solution space of the homogeneous difference equation by the Basis Theorem. The general solution to the homogeneous difference equation is thus $c_1 \cdot (-2)^k + c_2 \cdot (1/2)^k = c_1 \cdot (-2)^k + c_2 \cdot 2^{-k}$. Adding the particular solution $2k - 4$ of the nonhomogeneous difference equation, we find that the general solution of the difference equation $y_{k+2} + (3/2)y_{k+1} - y_k = 1 + 3k$ is $y_k = 2k - 4 + c_1 \cdot (-2)^k + c_2 \cdot 2^{-k}$.

29. Let $\mathbf{x}_k = \begin{bmatrix} y_k \\ y_{k+1} \\ y_{k+2} \\ y_{k+3} \end{bmatrix}$. Then $\mathbf{x}_{k+1} = \begin{bmatrix} y_{k+1} \\ y_{k+2} \\ y_{k+3} \\ y_{k+4} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 9 & -6 & -8 & 6 \end{bmatrix} \begin{bmatrix} y_k \\ y_{k+1} \\ y_{k+2} \\ y_{k+3} \end{bmatrix} = A\mathbf{x}_k$.

30. Let $\mathbf{x}_k = \begin{bmatrix} y_k \\ y_{k+1} \\ y_{k+2} \end{bmatrix}$. Then $\mathbf{x}_{k+1} = \begin{bmatrix} y_{k+1} \\ y_{k+2} \\ y_{k+3} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1/16 & 0 & 3/4 \end{bmatrix} \begin{bmatrix} y_k \\ y_{k+1} \\ y_{k+2} \end{bmatrix} = A\mathbf{x}_k$.

31. The difference equation is of order 2. Since the equation $y_{k+3} + 5y_{k+2} + 6y_{k+1} = 0$ holds for all k, it holds if k is replaced by $k - 1$. Performing this replacement transforms the equation into $y_{k+2} + 5y_{k+1} + 6y_k = 0$, which is also true for all k. The transformed equation has order 2.

32. The order of the difference equation depends on the values of a_1, a_2, and a_3. If $a_3 \neq 0$, then the order is 3. If $a_3 = 0$ and $a_2 \neq 0$, then the order is 2. If $a_3 = a_2 = 0$ and $a_1 \neq 0$, then the order is 1. If $a_3 = a_2 = a_1 = 0$, then the order is 0, and the equation has only the zero signal for a solution.

33. The Casorati matrix $C(k)$ is

$$C(k) = \begin{bmatrix} y_k & z_k \\ y_{k+1} & z_{k+1} \end{bmatrix} = \begin{bmatrix} k^2 & 2k\,|k| \\ (k+1)^2 & 2(k+1)\,|k+1| \end{bmatrix}$$

In particular,

$$C(0)=\begin{bmatrix}0 & 0\\ 1 & 2\end{bmatrix}, C(-1)=\begin{bmatrix}1 & -2\\ 0 & 0\end{bmatrix}, \text{and } C(-2)=\begin{bmatrix}4 & -8\\ 1 & -2\end{bmatrix}$$

none of which are invertible. In fact, $C(k)$ is not invertible for all k, since

$$\det C(k) = 2k^2(k+1)|k+1| - 2(k+1)^2 k|k| = 2k(k+1)\big(k|k+1| - (k+1)|k|\big)$$

If $k = 0$ or $k = -1$, det $C(k) = 0$. If $k > 0$, then $k + 1 > 0$ and $k|k + 1| - (k + 1)|k| = k(k + 1) - (k + 1)k = 0$, so det $C(k) = 0$. If $k < -1$, then $k + 1 < 0$ and $k|k + 1| - (k + 1)|k| = -k(k + 1) + (k + 1)k = 0$, so det $C(k) = 0$. Thus det$C(k)$=0 for all k, and $C(k)$ is not invertible for all k. Since $C(k)$ is not invertible for all k, it provides no information about whether the signals $\{y_k\}$ and $\{z_k\}$ are linearly dependent or linearly independent. In fact, neither signal is a multiple of the other, so the signals $\{y_k\}$ and $\{z_k\}$ are linearly independent.

34. No, the signals could be linearly dependent, since the vector space V of functions considered on the entire real line is not the vector space $\mathbb{S}$ of signals. For example, consider the functions $f(t) = \sin\pi t$, $g(t) = \sin 2\pi t$, and $h(t) = \sin 3\pi t$. The functions f, g, and h are linearly independent in V since they have different periods and thus no function could be a linear combination of the other two. However, sampling the functions at any integer n gives $f(n) = g(n) = h(n) = 0$, so the signals are linearly dependent in $\mathbb{S}$.

35. Let $\{y_k\}$ and $\{z_k\}$ be in $\mathbb{S}$, and let r be any scalar. The k^{th} term of $\{y_k\}+\{z_k\}$ is y_k+z_k, while the k^{th} term of $r\{y_k\}$ is ry_k. Thus

$$\begin{aligned}T(\{y_k\}+\{z_k\}) &= T\{y_k+z_k\}\\ &= (y_{k+2}+z_{k+2})+a(y_{k+1}+z_{k+1})+b(y_k+z_k)\\ &= (y_{k+2}+ay_{k+1}+by_k)+(z_{k+2}+az_{k+1}+bz_k)\\ &= T\{y_k\}+T\{z_k\}, \text{and}\\ T(r\{y_k\}) &= T\{ry_k\}\\ &= ry_{k+2}+a(ry_{k+1})+b(ry_k)\\ &= r(y_{k+2}+ay_{k+1}+by_k)\\ &= rT\{y_k\}\end{aligned}$$

so T has the two properties that define a linear transformation.

36. Let $\mathbf{z}$ be in V, and suppose that $\mathbf{x}_p$ in V satisfies $T(\mathbf{x}_p)=\mathbf{z}$. Let $\mathbf{u}$ be in the kernel of T; then $T(\mathbf{u}) = \mathbf{0}$. Since T is a linear transformation, $T(\mathbf{u}+\mathbf{x}_p)=T(\mathbf{u})+T(\mathbf{x}_p)=\mathbf{0}+\mathbf{z}=\mathbf{z}$, so the vector $\mathbf{x}=\mathbf{u}+\mathbf{x}_p$ satisfies the nonhomogeneous equation $T(\mathbf{x}) = \mathbf{z}$.

37. We compute that

$$(TD)(y_0,y_1,y_2,\ldots)=T(D(y_0,y_1,y_2,\ldots))=T(0,y_0,y_1,y_2,\ldots)=(y_0,y_1,y_2,\ldots)$$

while

$$(DT)(y_0,y_1,y_2,\ldots)=D(T(y_0,y_1,y_2,\ldots))=D(y_1,y_2,y_3,\ldots)=(0,y_1,y_2,y_3,\ldots)$$

Thus $TD = I$ (the identity transformation on $\mathbb{S}_0$), while $DT \neq I$.

4.9 SOLUTIONS

Notes: This section builds on the population movement example in Section 1.10. The migration matrix is examined again in Section 5.2, where an eigenvector decomposition shows explicitly why the sequence of state vectors $\mathbf{x}_k$ tends to a steady state vector. The discussion in Section 5.2 does not depend on prior knowledge of this section.

1. **a**. Let N stand for "News" and M stand for "Music." Then the listeners' behavior is given by the table

From: N	From: M	To:
.7	.6	N
.3	.4	M

so the stochastic matrix is $P = \begin{bmatrix} .7 & .6 \\ .3 & .4 \end{bmatrix}$.

b. Since 100% of the listeners are listening to news at 8: 15, the initial state vector is $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

c. There are two breaks between 8: 15 and 9: 25, so we calculate $\mathbf{x}_2$:

$$\mathbf{x}_1 = P\mathbf{x}_0 = \begin{bmatrix} .7 & .6 \\ .3 & .4 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} .7 \\ .3 \end{bmatrix}$$

$$\mathbf{x}_2 = P\mathbf{x}_1 = \begin{bmatrix} .7 & .6 \\ .3 & .4 \end{bmatrix}\begin{bmatrix} .7 \\ .3 \end{bmatrix} = \begin{bmatrix} .67 \\ .33 \end{bmatrix}$$

Thus 33% of the listeners are listening to news at 9: 25.

2. **a**. Let the foods be labelled "1," "2," and "3." Then the animals' behavior is given by the table

From: 1	From: 2	From: 3	To:
.5	.25	.25	1
.25	.5	.25	2
.25	.25	.5	3

so the stochastic matrix is $P = \begin{bmatrix} .5 & .25 & .25 \\ .25 & .5 & .25 \\ .25 & .25 & .5 \end{bmatrix}$.

b. There are two trials after the initial trial, so we calculate $\mathbf{x}_2$. The initial state vector is $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

$$\mathbf{x}_1 = P\mathbf{x}_0 = \begin{bmatrix} .5 & .25 & .25 \\ .25 & .5 & .25 \\ .25 & .25 & .5 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} .5 \\ .25 \\ .25 \end{bmatrix}$$

$$\mathbf{x}_2 = P\mathbf{x}_1 = \begin{bmatrix} .5 & .25 & .25 \\ .25 & .5 & .25 \\ .25 & .25 & .5 \end{bmatrix}\begin{bmatrix} .5 \\ .25 \\ .25 \end{bmatrix} = \begin{bmatrix} .375 \\ .3125 \\ .3125 \end{bmatrix}$$

Thus the probability that the animal will choose food #2 is .3125.

3. **a**. Let H stand for "Healthy" and I stand for "Ill." Then the students' conditions are given by the table

From: H	I	To:
.95	.45	H
.05	.55	I

so the stochastic matrix is $P = \begin{bmatrix} .95 & .45 \\ .05 & .55 \end{bmatrix}$.

b. Since 20% of the students are ill on Monday, the initial state vector is $\mathbf{x}_0 = \begin{bmatrix} .8 \\ .2 \end{bmatrix}$. For Tuesday's percentages, we calculate $\mathbf{x}_1$; for Wednesday's percentages, we calculate $\mathbf{x}_2$:

$$\mathbf{x}_1 = P\mathbf{x}_0 = \begin{bmatrix} .95 & .45 \\ .05 & .55 \end{bmatrix}\begin{bmatrix} .8 \\ .2 \end{bmatrix} = \begin{bmatrix} .85 \\ .15 \end{bmatrix}$$

$$\mathbf{x}_2 = P\mathbf{x}_1 = \begin{bmatrix} .95 & .45 \\ .05 & .55 \end{bmatrix}\begin{bmatrix} .85 \\ .15 \end{bmatrix} = \begin{bmatrix} .875 \\ .125 \end{bmatrix}$$

Thus 15% of the students are ill on Tuesday, and 12.5% are ill on Wednesday.

c. Since the student is well today, the initial state vector is $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. We calculate $\mathbf{x}_2$:

$$\mathbf{x}_1 = P\mathbf{x}_0 = \begin{bmatrix} .95 & .45 \\ .05 & .55 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} .95 \\ .05 \end{bmatrix}$$

$$\mathbf{x}_2 = P\mathbf{x}_1 = \begin{bmatrix} .95 & .45 \\ .05 & .55 \end{bmatrix}\begin{bmatrix} .95 \\ .05 \end{bmatrix} = \begin{bmatrix} .925 \\ .075 \end{bmatrix}$$

Thus the probability that the student is well two days from now is .925.

4. **a**. Let G stand for good weather, I for indifferent weather, and B for bad weather. Then the change in the weather is given by the table

From: G	I	B	To:
.6	.4	.4	G
.3	.3	.5	I
.1	.3	.1	B

so the stochastic matrix is $P = \begin{bmatrix} .6 & .4 & .4 \\ .3 & .3 & .5 \\ .1 & .3 & .1 \end{bmatrix}$.

b. The initial state vector is $\begin{bmatrix} .5 \\ .5 \\ 0 \end{bmatrix}$. We calculate $\mathbf{x}_1$:

$$\mathbf{x}_1 = P\mathbf{x}_0 = \begin{bmatrix} .6 & .4 & .4 \\ .3 & .3 & .5 \\ .1 & .3 & .1 \end{bmatrix}\begin{bmatrix} .5 \\ .5 \\ 0 \end{bmatrix} = \begin{bmatrix} .5 \\ .3 \\ .2 \end{bmatrix}$$

Thus the chance of bad weather tomorrow is 20%.

c. The initial state vector is $\mathbf{x}_0 = \begin{bmatrix} 0 \\ .4 \\ .6 \end{bmatrix}$. We calculate $\mathbf{x}_2$:

$$\mathbf{x}_1 = P\mathbf{x}_0 = \begin{bmatrix} .6 & .4 & .4 \\ .3 & .3 & .5 \\ .1 & .3 & .1 \end{bmatrix}\begin{bmatrix} 0 \\ .4 \\ .6 \end{bmatrix} = \begin{bmatrix} .4 \\ .42 \\ .18 \end{bmatrix}$$

$$\mathbf{x}_2 = P\mathbf{x}_1 = \begin{bmatrix} .6 & .4 & .4 \\ .3 & .3 & .5 \\ .1 & .3 & .1 \end{bmatrix}\begin{bmatrix} .4 \\ .42 \\ .18 \end{bmatrix} = \begin{bmatrix} .48 \\ .336 \\ .184 \end{bmatrix}$$

Thus the chance of good weather on Wednesday is 48%.

5. We solve $P\mathbf{x} = \mathbf{x}$ by rewriting the equation as $(P - I)\mathbf{x} = \mathbf{0}$, where $P - I = \begin{bmatrix} -.9 & .6 \\ .9 & -.6 \end{bmatrix}$. Row reducing the augmented matrix for the homogeneous system $(P - I)\mathbf{x} = \mathbf{0}$ gives

$$\begin{bmatrix} -.9 & .6 & 0 \\ .9 & -.6 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -2/3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2\begin{bmatrix} 2/3 \\ 1 \end{bmatrix}$, and one solution is $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$. Since the entries in $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ sum to 5, multiply by 1/5 to obtain the steady-state vector $\mathbf{q} = \begin{bmatrix} 2/5 \\ 3/5 \end{bmatrix} = \begin{bmatrix} .4 \\ .6 \end{bmatrix}$.

6. We solve $P\mathbf{x} = \mathbf{x}$ by rewriting the equation as $(P - I)\mathbf{x} = \mathbf{0}$, where $P - I = \begin{bmatrix} -.2 & .5 \\ .2 & -.5 \end{bmatrix}$. Row reducing the augmented matrix for the homogeneous system $(P - I)\mathbf{x} = \mathbf{0}$ gives

$$\begin{bmatrix} -.2 & .5 & 0 \\ .2 & -.5 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -5/2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2\begin{bmatrix} 5/2 \\ 1 \end{bmatrix}$, and one solution is $\begin{bmatrix} 5 \\ 2 \end{bmatrix}$. Since the entries in $\begin{bmatrix} 5 \\ 2 \end{bmatrix}$ sum to 7, multiply by 1/7 to obtain the steady-state vector $\mathbf{q} = \begin{bmatrix} 5/7 \\ 2/7 \end{bmatrix} \approx \begin{bmatrix} .714 \\ .286 \end{bmatrix}$.

7. We solve $P\mathbf{x} = \mathbf{x}$ by rewriting the equation as $(P - I)\mathbf{x} = \mathbf{0}$, where $P - I = \begin{bmatrix} -.3 & .1 & .1 \\ .2 & -.2 & .2 \\ .1 & .1 & -.3 \end{bmatrix}$. Row reducing the augmented matrix for the homogeneous system $(P - I)\mathbf{x} = \mathbf{0}$ gives

$$\begin{bmatrix} -.3 & .1 & .1 & 0 \\ .2 & -.2 & .2 & 0 \\ .1 & .1 & -.3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$, and one solution is $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$. Since the entries in $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ sum to 4, multiply by 1/4 to obtain the steady-state vector $\mathbf{q} = \begin{bmatrix} 1/4 \\ 1/2 \\ 1/4 \end{bmatrix} = \begin{bmatrix} .25 \\ .5 \\ .25 \end{bmatrix}$.

8. We solve $P\mathbf{x} = \mathbf{x}$ by rewriting the equation as $(P - I)\mathbf{x} = \mathbf{0}$, where $P - I = \begin{bmatrix} -.3 & .2 & .2 \\ 0 & -.8 & .4 \\ .3 & .6 & -.6 \end{bmatrix}$. Row reducing the augmented matrix for the homogeneous system $(P - I)\mathbf{x} = \mathbf{0}$ gives

$$\begin{bmatrix} -.3 & .2 & .2 & 0 \\ 0 & -.8 & .4 & 0 \\ .3 & .6 & -.6 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1/2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 1/2 \\ 1 \end{bmatrix}$, and one solution is $\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$. Since the entries in $\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$ sum to 5, multiply by 1/5 to obtain the steady-state vector $\mathbf{q} = \begin{bmatrix} 2/5 \\ 1/5 \\ 2/5 \end{bmatrix} = \begin{bmatrix} .4 \\ .2 \\ .4 \end{bmatrix}$.

9. Since $P^2 = \begin{bmatrix} .84 & .2 \\ .16 & .8 \end{bmatrix}$ has all positive entries, P is a regular stochastic matrix.

10. Since $P^k = \begin{bmatrix} 1 & 1-.8^k \\ 0 & .8^k \end{bmatrix}$ will have a zero as its (2,1) entry for all k, so P is not a regular stochastic matrix.

11. From Exercise 1, $P = \begin{bmatrix} .7 & .6 \\ .3 & .4 \end{bmatrix}$, so $P - I = \begin{bmatrix} -.3 & .6 \\ .3 & -.6 \end{bmatrix}$. Solving $(P - I)\mathbf{x} = \mathbf{0}$ by row reducing the augmented matrix gives

$$\begin{bmatrix} -.3 & .6 & 0 \\ .3 & -.6 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, and one solution is $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Since the entries in $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ sum to 3, multiply by 1/3 to obtain the steady-state vector $\mathbf{q} = \begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix} \approx \begin{bmatrix} .667 \\ .333 \end{bmatrix}$.

12. From Exercise 2, $P=\begin{bmatrix} .5 & .25 & .25 \\ .25 & .5 & .25 \\ .25 & .25 & .5 \end{bmatrix}$, so $P-I=\begin{bmatrix} -.5 & .25 & .25 \\ .25 & -.5 & .25 \\ .25 & .25 & -.5 \end{bmatrix}$. Solving $(P-I)\mathbf{x}=\mathbf{0}$ by row reducing the augmented matrix gives

$$\begin{bmatrix} -.5 & .25 & .25 & 0 \\ .25 & -.5 & .25 & 0 \\ .25 & .25 & -.5 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus $\mathbf{x}=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=x_3\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, and one solution is $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. Since the entries in $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ sum to 3, multiply by 1/3 to obtain the steady-state vector $\mathbf{q}=\begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix} \approx \begin{bmatrix} .333 \\ .333 \\ .333 \end{bmatrix}$. Thus in the long run each food will be preferred equally.

13. a. From Exercise 3, $P=\begin{bmatrix} .95 & .45 \\ .05 & .55 \end{bmatrix}$, so $P-I=\begin{bmatrix} -.05 & .45 \\ .05 & -.45 \end{bmatrix}$. Solving $(P-I)\mathbf{x}=\mathbf{0}$ by row reducing the augmented matrix gives

$$\begin{bmatrix} -.05 & .45 & 0 \\ .05 & -.45 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -9 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus $\mathbf{x}=\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=x_2\begin{bmatrix} 9 \\ 1 \end{bmatrix}$, and one solution is $\begin{bmatrix} 9 \\ 1 \end{bmatrix}$. Since the entries in $\begin{bmatrix} 9 \\ 1 \end{bmatrix}$ sum to 10, multiply by 1/10 to obtain the steady-state vector $\mathbf{q}=\begin{bmatrix} 9/10 \\ 1/10 \end{bmatrix}=\begin{bmatrix} .9 \\ .1 \end{bmatrix}$.

b. After many days, a specific student is ill with probability .1, and it does not matter whether that student is ill today or not.

14. From Exercise 4, $P=\begin{bmatrix} .6 & .4 & .4 \\ .3 & .3 & .5 \\ .1 & .3 & .1 \end{bmatrix}$, so $P-I=\begin{bmatrix} -.4 & .4 & .4 \\ .3 & -.7 & .5 \\ .1 & .3 & -.9 \end{bmatrix}$. Solving $(P-I)\mathbf{x}=\mathbf{0}$ by row reducing the augmented matrix gives

$$\begin{bmatrix} -.4 & .4 & .4 & 0 \\ .3 & -.7 & .5 & 0 \\ .1 & .3 & -.9 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus $\mathbf{x}=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=x_3\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$, and one solution is $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$. Since the entries in $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$ sum to 6, multiply by 1/6 to obtain the steady-state vector $\mathbf{q}=\begin{bmatrix} 1/2 \\ 1/3 \\ 1/6 \end{bmatrix} \approx \begin{bmatrix} .5 \\ .333 \\ .167 \end{bmatrix}$. Thus in the long run the chance that a day has good weather is 50%.

15. **[M]** Let $P=\begin{bmatrix} .9821 & .0029 \\ .0179 & .9971 \end{bmatrix}$, so $P-I=\begin{bmatrix} -.0179 & .0029 \\ .0179 & -.0029 \end{bmatrix}$. Solving $(P-I)\mathbf{x}=\mathbf{0}$ by row reducing the augmented matrix gives

$$\begin{bmatrix} -.0179 & .0029 & 0 \\ .0179 & -.0029 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -.162011 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus $\mathbf{x}=\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=x_2\begin{bmatrix} .162011 \\ 1 \end{bmatrix}$, and one solution is $\begin{bmatrix} .162011 \\ 1 \end{bmatrix}$. Since the entries in $\begin{bmatrix} .162011 \\ 1 \end{bmatrix}$ sum to 1.162011, multiply by 1/1.162011 to obtain the steady-state vector $\mathbf{q}=\begin{bmatrix} .139423 \\ .860577 \end{bmatrix}$. Thus about 13.9% of the total U.S. population would eventually live in California.

16. **[M]** Let $P=\begin{bmatrix} .90 & .01 & .09 \\ .01 & .90 & .01 \\ .09 & .09 & .90 \end{bmatrix}$, so $P-I=\begin{bmatrix} -.10 & .01 & .09 \\ .01 & -.10 & .01 \\ .09 & .09 & -.1 \end{bmatrix}$. Solving $(P-I)\mathbf{x}=\mathbf{0}$ by row reducing the augmented matrix gives

$$\begin{bmatrix} -.10 & .01 & .09 & 0 \\ .01 & -.10 & .01 & 0 \\ .09 & .09 & -.1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -.919192 & 0 \\ 0 & 1 & -.191919 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus $\mathbf{x}=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=x_3\begin{bmatrix} .919192 \\ .191919 \\ 1 \end{bmatrix}$, and one solution is $\begin{bmatrix} .919192 \\ .191919 \\ 1 \end{bmatrix}$. Since the entries in $\begin{bmatrix} .919192 \\ .191919 \\ 1 \end{bmatrix}$ sum to 2.111111, multiply by 1/2.111111 to obtain the steady-state vector $\mathbf{q}=\begin{bmatrix} .435407 \\ .090909 \\ .473684 \end{bmatrix}$. Thus on a typical day, about (.090909)(2000) = 182 cars will be rented or available from the downtown location.

17. **a**. The entries in each column of P sum to 1. Each column in the matrix $P-I$ has the same entries as in P except one of the entries is decreased by 1. Thus the entries in each column of $P-I$ sum to 0, and adding all of the other rows of $P-I$ to its bottom row produces a row of zeros.

b. By part a., the bottom row of $P-I$ is the negative of the sum of the other rows, so the rows of $P-I$ are linearly dependent.

c. By part b. and the Spanning Set Theorem, the bottom row of $P-I$ can be removed and the remaining $(n-1)$ rows will still span the row space of $P-I$. Thus the dimension of the row space of $P-I$ is less than n. Alternatively, let A be the matrix obtained from $P-I$ by adding to the bottom row all the other rows. These row operations did not change the row space, so the row space of $P-I$ is spanned by the nonzero rows of A. By part a., the bottom row of A is a zero row, so the row space of $P-I$ is spanned by the first $(n-1)$ rows of A.

d. By part c., the rank of $P-I$ is less than n, so the Rank Theorem may be used to show that $\dim\text{Nul}(P-I)=n-\text{rank}(P-I)>0$. Alternatively the Invertible Martix Theorem may be used since $P-I$ is a square matrix.

18. If $\alpha = \beta = 0$ then $P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Notice that $P\mathbf{x} = \mathbf{x}$ for any vector $\mathbf{x}$ in $\mathbb{R}^2$, and that $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are two linearly independent steady-state vectors in this case.

If $\alpha \neq 0$ or $\beta \neq 0$, we solve $(P - I)\mathbf{x} = \mathbf{0}$ where $P - I = \begin{bmatrix} -\alpha & \beta \\ \alpha & -\beta \end{bmatrix}$. Row reducing the augmented matrix gives

$$\begin{bmatrix} -\alpha & \beta & 0 \\ \alpha & -\beta & 0 \end{bmatrix} \sim \begin{bmatrix} \alpha & -\beta & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

So $\alpha x_1 = \beta x_2$, and one possible solution is to let $x_1 = \beta$, $x_2 = \alpha$. Thus $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \beta \\ \alpha \end{bmatrix}$. Since the entries in $\begin{bmatrix} \beta \\ \alpha \end{bmatrix}$ sum to $\alpha + \beta$, multiply by $1/(\alpha + \beta)$ to obtain the steady-state vector $\mathbf{q} = \dfrac{1}{\alpha + \beta} \begin{bmatrix} \beta \\ \alpha \end{bmatrix}$.

19. **a**. The product $S\mathbf{x}$ equals the sum of the entries in $\mathbf{x}$. Thus $\mathbf{x}$ is a probability vector if and only if its entries are nonnegative and $S\mathbf{x} = 1$.

b. Let $P = [\mathbf{p}_1 \quad \mathbf{p}_2 \quad \cdots \quad \mathbf{p}_n]$, where $\mathbf{p}_1$, $\mathbf{p}_2$, …, $\mathbf{p}_n$ are probability vectors. By part a.,

$$SP = [S\mathbf{p}_1 \quad S\mathbf{p}_2 \quad \cdots \quad S\mathbf{p}_n] = [1 \quad 1 \quad \cdots \quad 1] = S$$

c. By part b., $S(P\mathbf{x}) = (SP)\mathbf{x} = S\mathbf{x} = 1$. The entries in $P\mathbf{x}$ are nonnegative since P and $\mathbf{x}$ have only nonnegative entries. By part a., the condition $S(P\mathbf{x}) = 1$ shows that $P\mathbf{x}$ is a probability vector.

20. Let $P = [\mathbf{p}_1 \quad \mathbf{p}_2 \quad \cdots \quad \mathbf{p}_n]$, so $P^2 = PP = [P\mathbf{p}_1 \quad P\mathbf{p}_2 \quad \cdots \quad P\mathbf{p}_n]$. By Exercise 19c., the columns of P^2 are probability vectors, so P^2 is a stochastic matrix.

Alternatively, $SP = S$ by Exercise 19b., since P is a stochastic matrix. Right multiplication by P gives $SP^2 = SP$, so $SP = S$ implies that $SP^2 = S$. Since the entries in P are nonnegative, so are the entries in P^2, and P^2 is stochastic matrix.

21. **[M]**

a. To four decimal places,

$$P^2 = \begin{bmatrix} .2779 & .2780 & .2803 & .2941 \\ .3368 & .3355 & .3357 & .3335 \\ .1847 & .1861 & .1833 & .1697 \\ .2005 & .2004 & .2007 & .2027 \end{bmatrix}, P^3 = \begin{bmatrix} .2817 & .2817 & .2817 & .2814 \\ .3356 & .3356 & .3355 & .3352 \\ .1817 & .1817 & .1819 & .1825 \\ .2010 & .2010 & .2010 & .2009 \end{bmatrix},$$

$$P^4 = P^5 = \begin{bmatrix} .2816 & .2816 & .2816 & .2816 \\ .3355 & .3355 & .3355 & .3355 \\ .1819 & .1819 & .1819 & .1819 \\ .2009 & .2009 & .2009 & .2009 \end{bmatrix}$$

The columns of P^k are converging to a common vector as k increases. The steady state vector $\mathbf{q}$ for P is $\mathbf{q} = \begin{bmatrix} .2816 \\ .3355 \\ .1819 \\ .2009 \end{bmatrix}$, which is the vector to which the columns of P^k are converging.

b. To four decimal places,

$$Q^{10}=\begin{bmatrix} .8222 & .4044 & .5385 \\ .0324 & .3966 & .1666 \\ .1453 & .1990 & .2949 \end{bmatrix}, Q^{20}=\begin{bmatrix} .7674 & .6000 & .6690 \\ .0637 & .2036 & .1326 \\ .1688 & .1964 & .1984 \end{bmatrix},$$

$$Q^{30}=\begin{bmatrix} .7477 & .6815 & .7105 \\ .0783 & .1329 & .1074 \\ .1740 & .1856 & .1821 \end{bmatrix}, Q^{40}=\begin{bmatrix} .7401 & .7140 & .7257 \\ .0843 & .1057 & .0960 \\ .1756 & .1802 & .1783 \end{bmatrix},$$

$$Q^{50}=\begin{bmatrix} .7372 & .7269 & .7315 \\ .0867 & .0951 & .0913 \\ .1761 & .1780 & .1772 \end{bmatrix}, Q^{60}=\begin{bmatrix} .7360 & .7320 & .7338 \\ .0876 & .0909 & .0894 \\ .1763 & .1771 & .1767 \end{bmatrix},$$

$$Q^{70}=\begin{bmatrix} .7356 & .7340 & .7347 \\ .0880 & .0893 & .0887 \\ .1764 & .1767 & .1766 \end{bmatrix}, Q^{80}=\begin{bmatrix} .7354 & .7348 & .7351 \\ .0881 & .0887 & .0884 \\ .1764 & .1766 & .1765 \end{bmatrix},$$

$$Q^{116}=Q^{117}=\begin{bmatrix} .7353 & .7353 & .7353 \\ .0882 & .0882 & .0882 \\ .1765 & .1765 & .1765 \end{bmatrix}$$

The steady state vector **q** for Q is $\mathbf{q}=\begin{bmatrix} .7353 \\ .0882 \\ .1765 \end{bmatrix}$ Conjecture: the columns of P^k, where P is a regular stochastic matrix, converge to the steady state vector for P as k increases.

c. Let P be an $n\times n$ regular stochastic matrix, **q** the steady state vector of P, and $\mathbf{e}_j$ the j^{th} column of the $n\times n$ identity matrix. Consider the Markov chain $\{\mathbf{x}_k\}$ where $\mathbf{x}_{k+1}=P\mathbf{x}_k$ and $\mathbf{x}_0=e_j$. By Theorem 18, $\mathbf{x}_k=P^k\mathbf{x}_0$ converges to **q** as $k\to\infty$. But $P^k\mathbf{x}_0=P^k\mathbf{e}_j$, which is the j^{th} column of P^k. Thus the j^{th} column of P^k converges to **q** as $k\to\infty$; that is, $P^k\to[\mathbf{q}\quad\mathbf{q}\quad\ldots\quad\mathbf{q}]$.

22. **[M]** Answers will vary.

MATLAB Student Version 4.0 code for Method (1):

```
A=randstoc(32); flops(0);
tic, x=nulbasis(A-eye(32));
q=x/sum(x); toc, flops
```

MATLAB Student Version 4.0 code for Method (2):

```
A=randstoc(32); flops(0);
tic, B=A^100; q=B(: ,1); toc, flops
```

Chapter 4 SUPPLEMENTARY EXERCISES

1. **a.** True. This set is $\text{Span}\{\mathbf{v}_1, \dots \mathbf{v}_p\}$, and every subspace is itself a vector space.

b. True. Any linear combination of $\mathbf{v}_1, \dots, \mathbf{v}_{p-1}$ is also a linear combination of $\mathbf{v}_1, \dots, \mathbf{v}_{p-1}, \mathbf{v}_p$ using the zero weight on $\mathbf{v}_p$.

c. False. Counterexample: Take $\mathbf{v}_p = 2\mathbf{v}_1$. Then $\{\mathbf{v}_1, \dots \mathbf{v}_p\}$ is linearly dependent.

d. False. Counterexample: Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be the standard basis for $\mathbb{R}^3$. Then $\{\mathbf{e}_1, \mathbf{e}_2\}$ is a linearly independent set but is not a basis for $\mathbb{R}^3$.

e. True. See the Spanning Set Theorem (Section 4.3).

f. True. By the Basis Theorem, S is a basis for V because S spans V and has exactly p elements. So S must be linearly independent.

g. False. The plane must pass through the origin to be a subspace.

h. False. Counterexample: $\begin{bmatrix} 2 & 5 & -2 & 0 \\ 0 & 0 & 7 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

i. True. This statement appears before Theorem 13 in Section 4.6.

j. False. Row operations on A do not change the solutions of $A\mathbf{x} = \mathbf{0}$.

k. False. Counterexample: $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$; A has two nonzero rows but the rank of A is 1.

l. False. If U has k nonzero rows, then rank $A = k$ and dimNul $A = n - k$ by the Rank Theorem.

m. True. Row equivalent matrices have the same number of pivot columns.

n. False. The nonzero rows of A span Row A but they may not be linearly independent.

o. True. The nonzero rows of the reduced echelon form E form a basis for the row space of each matrix that is row equivalent to E.

p. True. If H is the zero subspace, let A be the 3×3 zero matrix. If dim $H = 1$, let $\{\mathbf{v}\}$ be a basis for H and set $A = [\mathbf{v} \quad \mathbf{v} \quad \mathbf{v}]$. If dim $H = 2$, let $\{\mathbf{u},\mathbf{v}\}$ be a basis for H and set $A = [\mathbf{u} \quad \mathbf{v} \quad \mathbf{v}]$, for example. If dim $H = 3$, then $H = \mathbb{R}^3$, so A can be any 3×3 invertible matrix. Or, let $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ be a basis for H and set $A = [\mathbf{u} \quad \mathbf{v} \quad \mathbf{w}]$.

q. False. Counterexample: $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$. If rank $A = n$ (the number of *columns* in A), then the transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.

r. True. If $\mathbf{x} \mapsto A\mathbf{x}$ is onto, then Col $A = \mathbb{R}^m$ and rank $A = m$. See Theorem 12(a) in Section 1.9.

s. True. See the second paragraph after Theorem 15 in Section 4.7.

t. False. The j^{th} column of $\underset{C \leftarrow B}{P}$ is $[\mathbf{b}_j]_C$.

2. The set is Span$S$, where $S = \left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} -2 \\ 5 \\ -4 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ -8 \\ 7 \\ 1 \end{bmatrix} \right\}$. Note that S is a linearly dependent set, but each pair of vectors in S forms a linearly independent set. Thus any two of the three vectors $\begin{bmatrix} 1 \\ 2 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} -2 \\ 5 \\ -4 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ -8 \\ 7 \\ 1 \end{bmatrix}$ will be a basis for Span$S$.

3. The vector $\mathbf{b}$ will be in $W = \text{Span}\{\mathbf{u}_1, \mathbf{u}_2\}$ if and only if there exist constants c_1 and c_2 with $c_1\mathbf{u}_1 + c_2\mathbf{u}_2 = \mathbf{b}$. Row reducing the augmented matrix gives

$$\begin{bmatrix} -2 & 1 & b_1 \\ 4 & 2 & b_2 \\ -6 & -5 & b_3 \end{bmatrix} \sim \begin{bmatrix} -2 & 1 & b_1 \\ 0 & 4 & 2b_1 + b_2 \\ 0 & 0 & b_1 + 2b_2 + b_3 \end{bmatrix}$$

so $W = \text{Span}\{\mathbf{u}_1, \mathbf{u}_2\}$ is the set of all (b_1, b_2, b_3) satisfying $b_1 + 2b_2 + b_3 = 0$.

4. The vector $\mathbf{g}$ is not a scalar multiple of the vector $\mathbf{f}$, and $\mathbf{f}$ is not a scalar multiple of $\mathbf{g}$, so the set $\{\mathbf{f}, \mathbf{g}\}$ is linearly independent. Even though the *number* $\mathbf{g}(t)$ is a scalar multiple of $\mathbf{f}(t)$ for each t, the scalar depends on t.

5. The vector $\mathbf{p}_1$ is not zero, and $\mathbf{p}_2$ is not a multiple of $\mathbf{p}_1$. However, $\mathbf{p}_3$ is $2\mathbf{p}_1 + 2\mathbf{p}_2$, so $\mathbf{p}_3$ is discarded. The vector $\mathbf{p}_4$ cannot be a linear combination of $\mathbf{p}_1$ and $\mathbf{p}_2$ since $\mathbf{p}_4$ involves t^2 but $\mathbf{p}_1$ and $\mathbf{p}_2$ do not involve t^2. The vector $\mathbf{p}_5$ is $(3/2)\mathbf{p}_1 - (1/2)\mathbf{p}_2 + \mathbf{p}_4$ (which may not be so easy to see at first.) Thus $\mathbf{p}_5$ is a linear combination of $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_4$, so $\mathbf{p}_5$ is discarded. So the resulting basis is $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_4\}$.

6. Find two polynomials from the set $\{\mathbf{p}_1, \ldots, \mathbf{p}_4\}$ that are not multiples of one another. This is easy, because one compares only two polynomials at a time. Since these two polynomials form a linearly independent set in a two-dimensional space, they form a basis for H by the Basis Theorem.

7. You would have to know that the solution set of the homogeneous system is spanned by two solutions. In this case, the null space of the 18×20 coefficient matrix A is at most two-dimensional. By the Rank Theorem, dimCol $A = 20 -$ dimNul $A \geq 20 - 2 = 18$. Since Col A is a subspace of $\mathbb{R}^{18}$, Col $A = \mathbb{R}^{18}$. Thus $A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b}$ in $\mathbb{R}^{18}$.

8. If $n = 0$, then H and V are both the zero subspace, and $H = V$. If $n > 0$, then a basis for H consists of n linearly independent vectors $\mathbf{u}_1, \ldots, \mathbf{u}_n$. These vectors are also linearly independent as elements of V. But since dim$V = n$, any set of n linearly independent vectors in V must be a basis for V by the Basis Theorem. So $\mathbf{u}_1, \ldots, \mathbf{u}_n$ span V, and $H = \text{Span}\{\mathbf{u}_1, \ldots, \mathbf{u}_n\} = V$.

9. Let T: $\mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a linear transformation, and let A be the $m \times n$ standard matrix of T.

a. If T is one-to-one, then the columns of A are linearly independent by Theoerm 12 in Section 1.9, so dimNul $A = 0$. By the Rank Theorem, dimCol $A = n - 0 = n$, which is the number of columns of A. As noted in Section 4.2, the range of T is Col A, so the dimension of the range of T is n.

b. If T maps $\mathbb{R}^n$ onto $\mathbb{R}^m$, then the columns of A span $\mathbb{R}^m$ by Theoerm 12 in Section 1.9, so dimCol $A = m$. By the Rank Theorem, dimNul $A = n - m$. As noted in Section 4.2, the kernel of T is Nul A, so the dimension of the kernel of T is $n - m$. Note that $n - m$ must be nonnegative in this case: since A must have a pivot in each row, $n \geq m$.

10. Let $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$. If S were linearly independent and not a basis for V, then S would not span V. In this case, there would be a vector $\mathbf{v}_{p+1}$ in V that is not in Span$\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$. Let $S' = \{\mathbf{v}_1, \ldots, \mathbf{v}_p, \mathbf{v}_{p+1}\}$. Then S' is linearly independent since none of the vectors in S' is a linear combination of vectors that precede it. Since S' has more elements than S, this would contradict the maximality of S. Hence S must be a basis for V.

11. If S is a finite spanning set for V, then a subset of S is a basis for V. Denote this subset of S by S'. Since S' is a basis for V, S' must span V. Since S is a minimal spanning set, S' cannot be a proper subset of S. Thus $S' = S$, and S is a basis for V.

12. **a**. Let $\mathbf{y}$ be in Col AB. Then $\mathbf{y} = AB\mathbf{x}$ for some $\mathbf{x}$. But $AB\mathbf{x} = A(B\mathbf{x})$, so $\mathbf{y} = A(B\mathbf{x})$, and $\mathbf{y}$ is in Col A. Thus Col AB is a subspace of Col A, so rank AB = dimCol $AB \leq$ dimCol A = rank A by Theorem 11 in Section 4.5.

b. By the Rank Theorem and part a.:

$$\text{rank } AB = \text{rank}(AB)^T = \text{rank } B^T A^T \leq \text{rank } B^T = \text{rank } B$$

13. By Exercise 12, rank $PA \leq$ rank A, and rank $A = \text{rank}(P^{-1}P)A = \text{rank } P^{-1}(PA) \leq \text{rank } PA$, so rank PA = rank A.

14. Note that $(AQ)^T = Q^T A^T$. Since Q^T is invertible, we can use Exercise 13 to conclude that $\text{rank}(AQ)^T = \text{rank } Q^T A^T = \text{rank } A^T$. Since the ranks of a matrix and its transpose are equal (by the Rank Theorem), rank AQ = rank A.

15. The equation $AB = O$ shows that each column of B is in Nul A. Since Nul A is a subspace of $\mathbb{R}^n$, all linear combinations of the columns of B are in Nul A. That is, Col B is a subspace of Nul A. By Theorem 11 in Section 4.5, rank B = dimCol $B \leq$ dimNul A. By this inequality and the Rank Theorem applied to A,

$$n = \text{rank } A + \text{dimNul } A \geq \text{rank } A + \text{rank } B$$

16. Suppose that rank $A = r_1$ and rank $B = r_2$. Then there are rank factorizations $A = C_1R_1$ and $B = C_2R_2$ of A and B, where C_1 is $m \times r_1$ with rank r_1, C_2 is $m \times r_2$ with rank r_2, R_1 is $r_1 \times n$ with rank r_1, and R_2 is $r_2 \times n$ with rank r_2. Create an $m \times (r_1 + r_2)$ matrix $C = \begin{bmatrix} C_1 & C_2 \end{bmatrix}$ and an $(r_1 + r_2) \times n$ matrix R by stacking R_1 over R_2. Then

$$A + B = C_1R_1 + C_2R_2 = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} R_1 \\ R_2 \end{bmatrix} = CR$$

Since the matrix CR is a product, its rank cannot exceed the rank of either of its factors by Exercise 12. Since C has $r_1 + r_2$ columns, the rank of C cannot exceed $r_1 + r_2$. Likewise R has $r_1 + r_2$ rows, so the rank of R cannot exceed $r_1 + r_2$. Thus the rank of $A + B$ cannot exceed $r_1 + r_2 = \text{rank } A + \text{rank } B$, or rank $(A + B) \leq$ rank A + rank B.

17. Let A be an $m \times n$ matrix with rank r.

(a) Let A_1 consist of the r pivot columns of A. The columns of A_1 are linearly independent, so A_1 is an $m \times r$ matrix with rank r.

(b) By the Rank Theorem applied to A_1, the dimension of RowA_1 is r, so A_1 has r linearly independent rows. Let A_2 consist of the r linearly independent rows of A_1. Then A_2 is an $r \times r$ matrix with linearly independent rows. By the Invertible Matrix Theorem, A_2 is invertible.

18. Let A be a 4×4 matrix and B be a 4×2 matrix, and let $\mathbf{u}_0, \ldots, \mathbf{u}_3$ be a sequence of input vectors in $\mathbb{R}^2$.

a. Use the equation $\mathbf{x}_{k+1} = A\mathbf{x}_k + B\mathbf{u}_k$ for $k = 0, \ldots, 4$, $k = 0, \ldots, 4$, with $\mathbf{x}_0 = \mathbf{0}$.

$$\mathbf{x}_1 = A\mathbf{x}_0 + B\mathbf{u}_0 = B\mathbf{u}_0$$

$$\mathbf{x}_2 = A\mathbf{x}_1 + B\mathbf{u}_1 = AB\mathbf{u}_0 + B\mathbf{u}_1$$

$$\mathbf{x}_3 = A\mathbf{x}_2 + B\mathbf{u}_2 = A(AB\mathbf{u}_0 + B\mathbf{u}_1) + B\mathbf{u}_2 = A^2B\mathbf{u}_0 + AB\mathbf{u}_1 + B\mathbf{u}_2$$

$$\begin{aligned}\mathbf{x}_4 &= A\mathbf{x}_3 + B\mathbf{u}_3 = A(A^2B\mathbf{u}_0 + AB\mathbf{u}_1 + B\mathbf{u}_2) + B\mathbf{u}_3 \\ &= A^3B\mathbf{u}_0 + A^2B\mathbf{u}_1 + AB\mathbf{u}_2 + B\mathbf{u}_3 \\ &= \begin{bmatrix} B & AB & A^2B & A^3B \end{bmatrix} \begin{bmatrix} \mathbf{u}_3 \\ \mathbf{u}_2 \\ \mathbf{u}_1 \\ \mathbf{u}_0 \end{bmatrix} = M\,\mathbf{u}\end{aligned}$$

Note that M has 4 rows because B does, and that M has 8 columns because B and each of the matrices A^kB have 2 columns. The vector $\mathbf{u}$ in the final equation is in $\mathbb{R}^8$, because each $\mathbf{u}_k$ is in $\mathbb{R}^2$.

b. If (A, B) is controllable, then the controlability matrix has rank 4, with a pivot in each row, and the columns of M span $\mathbb{R}^4$. Therefore, for any vector $\mathbf{v}$ in $\mathbb{R}^4$, there is a vector $\mathbf{u}$ in $\mathbb{R}^8$ such that $\mathbf{v} = M\mathbf{u}$. However, from part a. we know that $\mathbf{x}_4 = M\,\mathbf{u}$ when $\mathbf{u}$ is partitioned into a control sequence $\mathbf{u}_0, \ldots, \mathbf{u}_3$. This particular control sequence makes $\mathbf{x}_4 = \mathbf{v}$.

19. To determine if the matrix pair (A, B) is controllable, we compute the rank of the matrix $\begin{bmatrix} B & AB & A^2B \end{bmatrix}$. To find the rank, we row reduce:

$$\begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -.9 & .81 \\ 1 & .5 & .25 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The rank of the matrix is 3, and the pair (A, B) is controllable.

20. To determine if the matrix pair (A, B) is controllable, we compute the rank of the matrix $\begin{bmatrix} B & AB & A^2B \end{bmatrix}$. To find the rank, we note that :

$$\begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} 1 & .5 & .19 \\ 1 & .7 & .45 \\ 0 & 0 & 0 \end{bmatrix}.$$

The rank of the matrix must be less than 3, and the pair (A, B) is not controllable.

21. **[M]** To determine if the matrix pair (A, B) is controllable, we compute the rank of the matrix $\begin{bmatrix} B & AB & A^2B & A^3B \end{bmatrix}$. To find the rank, we row reduce:

$$\begin{bmatrix} B & AB & A^2B & A^3B \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & -1 & 1.6 \\ 0 & -1 & 1.6 & -.96 \\ -1 & 1.6 & -.96 & -.024 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1.6 \\ 0 & 0 & 1 & -1.6 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The rank of the matrix is 3, and the pair (A, B) is not controllable.

22. **[M]** To determine if the matrix pair (A, B) is controllable, we compute the rank of the matrix $\begin{bmatrix} B & AB & A^2B & A^3B \end{bmatrix}$. To find the rank, we row reduce:

$$\begin{bmatrix} B & AB & A^2B & A^3B \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & -1 & .5 \\ 0 & -1 & .5 & 11.45 \\ -1 & .5 & 11.45 & -10.275 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The rank of the matrix is 4, and the pair (A, B) is controllable.

5 Eigenvalues and Eigenvectors

5.1 SOLUTIONS

Notes: Exercises 1–6 reinforce the definitions of eigenvalues and eigenvectors. The subsection on eigenvectors and difference equations, along with Exercises 33 and 34, refers to the chapter introductory example and anticipates discussions of dynamical systems in Sections 5.2 and 5.6.

1. The number 2 is an eigenvalue of A if and only if the equation $A\mathbf{x} = 2\mathbf{x}$ has a nontrivial solution. This equation is equivalent to $(A - 2I)\mathbf{x} = \mathbf{0}$. Compute

$$A - 2I = \begin{bmatrix} 3 & 2 \\ 3 & 8 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$$

The columns of A are obviously linearly dependent, so $(A - 2I)\mathbf{x} = \mathbf{0}$ has a nontrivial solution, and so 2 is an eigenvalue of A.

2. The number -2 is an eigenvalue of A if and only if the equation $A\mathbf{x} = -2\mathbf{x}$ has a nontrivial solution. This equation is equivalent to $(A + 2I)\mathbf{x} = \mathbf{0}$. Compute

$$A + 2I = \begin{bmatrix} 7 & 3 \\ 3 & -1 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 9 & 3 \\ 3 & 1 \end{bmatrix}$$

The columns of A are obviously linearly dependent, so $(A + 2I)\mathbf{x} = \mathbf{0}$ has a nontrivial solution, and so -2 is an eigenvalue of A.

3. Is $A\mathbf{x}$ a multiple of $\mathbf{x}$? Compute $\begin{bmatrix} -3 & 1 \\ -3 & 8 \end{bmatrix}\begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 29 \end{bmatrix} \neq \lambda \begin{bmatrix} 1 \\ 4 \end{bmatrix}$. So $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$ is *not* an eigenvector of A.

4. Is $A\mathbf{x}$ a multiple of $\mathbf{x}$? Compute $\begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} -1+\sqrt{2} \\ 1 \end{bmatrix} = \begin{bmatrix} -1+2\sqrt{2} \\ 3+\sqrt{2} \end{bmatrix}$ The second entries of $\mathbf{x}$ and $A\mathbf{x}$ shows that if $A\mathbf{x}$ is a multiple of $\mathbf{x}$, then that multiple must be $3+\sqrt{2}$. Check $3+\sqrt{2}$ times the first entry of $\mathbf{x}$:

$$(3+\sqrt{2})(-1+\sqrt{2}) = -3 + \left(\sqrt{2}\right)^2 + 2\sqrt{2} = -1 + 2\sqrt{2}$$

This matches the first entry of $A\mathbf{x}$, so $\begin{bmatrix} -1+\sqrt{2} \\ 1 \end{bmatrix}$ is an eigenvector of A, and the corresponding eigenvalue is $3+\sqrt{2}$.

5. Is $A\mathbf{x}$ a multiple of $\mathbf{x}$? Compute $\begin{bmatrix} 3 & 7 & 9 \\ -4 & -5 & 1 \\ 2 & 4 & 4 \end{bmatrix}\begin{bmatrix} 4 \\ -3 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. So $\begin{bmatrix} 4 \\ -3 \\ 1 \end{bmatrix}$ is an eigenvector of A for the eigenvalue 0.

6. Is $A\mathbf{x}$ a multiple of $\mathbf{x}$? Compute $\begin{bmatrix} 3 & 6 & 7 \\ 3 & 3 & 7 \\ 5 & 6 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \\ -2 \end{bmatrix} = (-2)\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ So $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ is an eigenvector of A for the eigenvalue -2.

7. To determine if 4 is an eigenvalue of A, decide if the matrix $A-4I$ is invertible.

$$A-4I = \begin{bmatrix} 3 & 0 & -1 \\ 2 & 3 & 1 \\ -3 & 4 & 5 \end{bmatrix} - \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} -1 & 0 & -1 \\ 2 & -1 & 1 \\ -3 & 4 & 1 \end{bmatrix}$$

Invertibility can be checked in several ways, but since an eigenvector is needed in the event that one exists, the best strategy is to row reduce the augmented matrix for $(A-4I)\mathbf{x} = \mathbf{0}$:

$$\begin{bmatrix} -1 & 0 & -1 & 0 \\ 2 & -1 & 1 & 0 \\ -3 & 4 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} -1 & 0 & -1 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 4 & 4 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The equation $(A-4I)\mathbf{x} = \mathbf{0}$ has a nontrivial solution, so 4 is an eigenvalue. Any nonzero solution of $(A-4I)\mathbf{x} = \mathbf{0}$ is a corresponding eigenvector. The entries in a solution satisfy $x_1 + x_3 = 0$ and $-x_2 - x_3 = 0$, with x_3 free. The general solution is *not* requested, so to save time, simply take any nonzero value for x_3 to produce an eigenvector. If $x_3 = 1$, then $\mathbf{x} = (-1, -1, 1)$.

Note: The answer in the text is $(1, 1, -1)$, written in this form to make the students wonder whether the more common answer given above is also correct. This may initiate a class discussion of what answers are "correct."

8. To determine if 3 is an eigenvalue of A, decide if the matrix $A-3I$ is invertible.

$$A-3I = \begin{bmatrix} 1 & 2 & 2 \\ 3 & -2 & 1 \\ 0 & 1 & 1 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} -2 & 2 & 2 \\ 3 & -5 & 1 \\ 0 & 1 & -2 \end{bmatrix}$$

Row reducing the augmented matrix $[(\mathrm{A}-3I) \quad \mathbf{0}]$ yields:

$$\begin{bmatrix} -2 & 2 & 2 & 0 \\ 3 & -5 & 1 & 0 \\ 0 & 1 & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & -2 & 4 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The equation $(A-3I)\mathbf{x} = \mathbf{0}$ has a nontrivial solution, so 3 is an eigenvalue. Any nonzero solution of $(A-3I)\mathbf{x} = \mathbf{0}$ is a corresponding eigenvector. The entries in a solution satisfy $x_1 - 3x_3 = 0$ and $x_2 - 2x_3 = 0$, with x_3 free. The general solution is *not* requested, so to save time, simply take any nonzero value for x_3 to produce an eigenvector. If $x_3 = 1$, then $\mathbf{x} = (3, 2, 1)$.

9. For $\lambda = 1$: $A - 1I = \begin{bmatrix} 5 & 0 \\ 2 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 2 & 0 \end{bmatrix}$

The augmented matrix for $(A - I)\mathbf{x} = \mathbf{0}$ is $\begin{bmatrix} 4 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix}$. Thus $x_1 = 0$ and x_2 is free. The general solution of $(A - I)\mathbf{x} = \mathbf{0}$ is $x_2\mathbf{e}_2$, where $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and so $\mathbf{e}_2$ is a basis for the eigenspace corresponding to the eigenvalue 1.

For $\lambda = 5$: $A - 5I = \begin{bmatrix} 5 & 0 \\ 2 & 1 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 2 & -4 \end{bmatrix}$

The equation $(A - 5I)\mathbf{x} = \mathbf{0}$ leads to $2x_1 - 4x_2 = 0$, so that $x_1 = 2x_2$ and x_2 is free. The general solution is $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2x_2 \\ x_2 \end{bmatrix} = x_2\begin{bmatrix} 2 \\ 1 \end{bmatrix}$. So $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is a basis for the eigenspace.

10. For $\lambda = 4$: $A - 4I = \begin{bmatrix} 10 & -9 \\ 4 & -2 \end{bmatrix} - \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 6 & -9 \\ 4 & -6 \end{bmatrix}$.

The augmented matrix for $(A - 4I)\mathbf{x} = \mathbf{0}$ is $\begin{bmatrix} 6 & -9 & 0 \\ 4 & -6 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -9/6 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Thus $x_1 = (3/2)x_2$ and x_2 is free. The general solution is $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} (3/2)x_2 \\ x_2 \end{bmatrix} = x_2\begin{bmatrix} 3/2 \\ 1 \end{bmatrix}$. A basis for the eigenspace corresponding to 4 is $\begin{bmatrix} 3/2 \\ 1 \end{bmatrix}$. Another choice is $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$.

11. $A - 10I = \begin{bmatrix} 4 & -2 \\ -3 & 9 \end{bmatrix} - \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix} = \begin{bmatrix} -6 & -2 \\ -3 & -1 \end{bmatrix}$

The augmented matrix for $(A - 10I)\mathbf{x} = \mathbf{0}$ is $\begin{bmatrix} -6 & -2 & 0 \\ -3 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1/3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Thus $x_1 = (-1/3)x_2$ and x_2 is free. The general solution is $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -(1/3)x_2 \\ x_2 \end{bmatrix} = x_2\begin{bmatrix} -1/3 \\ 1 \end{bmatrix}$. A basis for the eigenspace corresponding to 10 is $\begin{bmatrix} -1/3 \\ 1 \end{bmatrix}$. Another choice is $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$.

12. For $\lambda = 1$: $A - I = \begin{bmatrix} 7 & 4 \\ -3 & -1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 4 \\ -3 & -2 \end{bmatrix}$

The augmented matrix for $(A - I)\mathbf{x} = \mathbf{0}$ is $\begin{bmatrix} 6 & 4 & 0 \\ -3 & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2/3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Thus $x_1 = (-2/3)x_2$ and x_2 is free. A basis for the eigenspace corresponding to 1 is $\begin{bmatrix} -2/3 \\ 1 \end{bmatrix}$. Another choice is $\begin{bmatrix} -2 \\ 3 \end{bmatrix}$.

For $\lambda = 5$: $A - 5I = \begin{bmatrix} 7 & 4 \\ -3 & -1 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ -3 & -6 \end{bmatrix}$.

The augmented matrix for $(A-5I)\mathbf{x}=\mathbf{0}$ is $\begin{bmatrix} 2 & 4 & 0 \\ -3 & -6 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Thus $x_1 = 2x_2$ and x_2 is free. The general solution is $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix}$. A basis for the eigenspace is $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

13. <u>For $\lambda = 1$</u>:

$$A - 1I = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 1 \\ -2 & 0 & 0 \\ -2 & 0 & 0 \end{bmatrix}$$

The equations for $(A-I)\mathbf{x}=\mathbf{0}$ are easy to solve: $\left\{\begin{aligned} 3x_1 + x_3 &= 0 \\ -2x_1 \quad\;\; &= 0 \end{aligned}\right\}$

Row operations hardly seem necessary. Obviously x_1 is zero, and hence x_3 is also zero. There are three-variables, so x_2 is free. The general solution of $(A-I)\mathbf{x}=\mathbf{0}$ is $x_2\mathbf{e}_2$, where $\mathbf{e}_2 = (0,1,0)$, and so $\mathbf{e}_2$ provides a basis for the eigenspace.

<u>For $\lambda = 2$</u>:

$$A - 2I = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 1 \\ -2 & -1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

$$[(A-2I)\;\;\mathbf{0}] = \begin{bmatrix} 2 & 0 & 1 & 0 \\ -2 & -1 & 0 & 0 \\ -2 & 0 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1/2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So $x_1 = -(1/2)x_3, x_2 = x_3,$ with x_3 free. The general solution of $(A-2I)\mathbf{x}=\mathbf{0}$ is $x_3\begin{bmatrix} -1/2 \\ 1 \\ 1 \end{bmatrix}$. A nice basis vector for the eigenspace is $\begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$.

<u>For $\lambda = 3$</u>:

$$A - 3I = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ -2 & -2 & 0 \\ -2 & 0 & -2 \end{bmatrix}$$

$$[(A-3I)\;\;\mathbf{0}] = \begin{bmatrix} 1 & 0 & 1 & 0 \\ -2 & -2 & 0 & 0 \\ -2 & 0 & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & -2 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So $x_1 = -x_3, x_2 = x_3,$ with x_3 free. A basis vector for the eigenspace is $\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$.

14. For $\lambda=-2$: $A-(-2I)=A+2I=\begin{bmatrix}1&0&-1\\1&-3&0\\4&-13&1\end{bmatrix}+\begin{bmatrix}2&0&0\\0&2&0\\0&0&2\end{bmatrix}=\begin{bmatrix}3&0&-1\\1&-1&0\\4&-13&3\end{bmatrix}$.

The augmented matrix for $[A-(-2)I]\mathbf{x}=\mathbf{0}$, or $(A+2I)\mathbf{x}=\mathbf{0}$, is

$$[(A+2I)\quad \mathbf{0}]=\begin{bmatrix}3&0&-1&0\\1&-1&0&0\\4&-13&3&0\end{bmatrix}\sim\begin{bmatrix}1&0&-1/3&0\\0&1&-1/3&0\\0&-13&13/3&0\end{bmatrix}\sim\begin{bmatrix}1&0&-1/3&0\\0&1&-1/3&0\\0&0&0&0\end{bmatrix}$$

Thus $x_1=(1/3)x_3, x_2=(1/3)x_3$, with x_3 free. The general solution of $(A+2I)\mathbf{x}=\mathbf{0}$ is $x_3\begin{bmatrix}1/3\\1/3\\1\end{bmatrix}$.

A basis for the eigenspace corresponding to -2 is $\begin{bmatrix}1/3\\1/3\\1\end{bmatrix}$; another is $\begin{bmatrix}1\\1\\3\end{bmatrix}$.

15. For $\lambda=3$: $[(A-3I)\quad \mathbf{0}]=\begin{bmatrix}1&2&3&0\\-1&-2&-3&0\\2&4&6&0\end{bmatrix}\sim\begin{bmatrix}1&2&3&0\\0&0&0&0\\0&0&0&0\end{bmatrix}$. Thus $x_1+2x_2+3x_3=0,$ with x_2 and x_3 free. The general solution of $(A-3I)\mathbf{x}=\mathbf{0}$, is

$$\mathbf{x}=\begin{bmatrix}-2x_2-3x_3\\x_2\\x_3\end{bmatrix}=x_2\begin{bmatrix}-2\\1\\0\end{bmatrix}+x_3\begin{bmatrix}-3\\0\\1\end{bmatrix}.\text{ Basis for the eigenspace}:\left\{\begin{bmatrix}-2\\1\\0\end{bmatrix},\begin{bmatrix}-3\\0\\1\end{bmatrix}\right\}$$

Note: For simplicity, the text answer omits the set brackets. I permit my students to list a basis without the set brackets. Some instructors may prefer to include brackets.

16. For $\lambda=4$: $A-4I=\begin{bmatrix}3&0&2&0\\1&3&1&0\\0&1&1&0\\0&0&0&4\end{bmatrix}-\begin{bmatrix}4&0&0&0\\0&4&0&0\\0&0&4&0\\0&0&0&4\end{bmatrix}=\begin{bmatrix}-1&0&2&0\\1&-1&1&0\\0&1&-3&0\\0&0&0&0\end{bmatrix}$.

$[(A-4I)\quad \mathbf{0}]=\begin{bmatrix}-1&0&2&0&0\\1&-1&1&0&0\\0&1&-3&0&0\\0&0&0&0&0\end{bmatrix}\sim\begin{bmatrix}1&0&-2&0&0\\0&1&-3&0&0\\0&0&0&0&0\\0&0&0&0&0\end{bmatrix}$. So $x_1=2x_3, x_2=3x_3$, with x_3 and x_4 free variables. The general solution of $(A-4I)\mathbf{x}=\mathbf{0}$ is

$$\mathbf{x}=\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=\begin{bmatrix}2x_3\\3x_3\\x_3\\x_4\end{bmatrix}=x_3\begin{bmatrix}2\\3\\1\\0\end{bmatrix}+x_4\begin{bmatrix}0\\0\\0\\1\end{bmatrix}.\text{ Basis for the eigenspace}:\left\{\begin{bmatrix}2\\3\\1\\0\end{bmatrix},\begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right\}$$

Note: I urge my students always to include the extra column of zeros when solving a homogeneous system. Exercise 16 provides a situation in which *failing* to add the column is likely to create problems for a student, because the matrix $A-4I$ itself has a column of zeros.

17. The eigenvalues of $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 5 \\ 0 & 0 & -1 \end{bmatrix}$ are 0, 2, and -1, on the main diagonal, by Theorem 1.

18. The eigenvalues of $\begin{bmatrix} 4 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & -3 \end{bmatrix}$ are 4, 0, and -3, on the main diagonal, by Theorem 1.

19. The matrix $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$ is not invertible because its columns are linearly dependent. So the number 0 is an eigenvalue of the matrix. See the discussion following Example 5.

20. The matrix $A = \begin{bmatrix} 5 & 5 & 5 \\ 5 & 5 & 5 \\ 5 & 5 & 5 \end{bmatrix}$ is not invertible because its columns are linearly dependent. So the number 0 is an eigenvalue of A. Eigenvectors for the eigenvalue 0 are solutions of $A\mathbf{x} = \mathbf{0}$ and therefore have entries that produce a linear dependence relation among the columns of A. Any nonzero vector (in $\mathbf{R}^3$) whose entries sum to 0 will work. Find any two such vectors that are not multiples; for instance, $(1, 1, -2)$ and $(1, -1, 0)$.

21. **a**. False. The equation $A\mathbf{x} = \lambda\mathbf{x}$ must have a *nontrivial* solution.
b. True. See the paragraph after Example 5.
c. True. See the discussion of equation (3).
d. True. See Example 2 and the paragraph preceding it. Also, see the Numerical Note.
e. False. See the warning after Example 3.

22. **a**. False. The vector $\mathbf{x}$ in $A\mathbf{x} = \lambda\mathbf{x}$ must be *nonzero*.
b. False. See Example 4 for a two-dimensional eigenspace, which contains two linearly independent eigenvectors corresponding to the same eigenvalue. The statement given is not at all the same as Theorem 2. In fact, it is the *converse* of Theorem 2 (for the case $r = 2$).
c. True. See the paragraph after Example 1.
d. False. Theorem 1 concerns a *triangular* matrix. See Examples 3 and 4 for counterexamples.
e. True. See the paragraph following Example 3. The eigenspace of A corresponding to λ is the null space of the matrix $A - \lambda I$.

23. If a 2×2 matrix A were to have three distinct eigenvalues, then by Theorem 2 there would correspond three linearly independent eigenvectors (one for each eigenvalue). This is impossible because the vectors all belong to a two-dimensional vector space, in which any set of three vectors is linearly dependent. See Theorem 8 in Section 1.7. In general, if an $n\times n$ matrix has p distinct eigenvalues, then by Theorem 2 there would be a linearly independent set of p eigenvectors (one for each eigenvalue). Since these vectors belong to an n-dimensional vector space, p cannot exceed n.

24. A simple example of a 2×2 matrix with only one distinct eigenvalue is a triangular matrix with the same number on the diagonal. By experimentation, one finds that if such a matrix is actually a diagonal matrix then the eigenspace is two dimensional, and otherwise the eigenspace is only one dimensional.

Examples: $\begin{bmatrix} 4 & 1 \\ 0 & 4 \end{bmatrix}$ and $\begin{bmatrix} 4 & 5 \\ 0 & 4 \end{bmatrix}$.

25. If λ is an eigenvalue of A, then there is a nonzero vector $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$. Since A is invertible, $A^{-1}A\mathbf{x} = A^{-1}(\lambda\mathbf{x})$, and so $\mathbf{x} = \lambda(A^{-1}\mathbf{x})$. Since $\mathbf{x} \neq \mathbf{0}$ (and since A is invertible), λ cannot be zero. Then $\lambda^{-1}\mathbf{x} = A^{-1}\mathbf{x}$, which shows that λ^{-1} is an eigenvalue of A^{-1}.

Note: The *Study Guide* points out here that the relation between the eigenvalues of A and A^{-1} is important in the so-called *inverse power method* for estimating an eigenvalue of a matrix. See Section 5.8.

26. Suppose that A^2 is the zero matrix. If $A\mathbf{x} = \lambda\mathbf{x}$ for some $\mathbf{x} \neq \mathbf{0}$, then $A^2\mathbf{x} = A(A\mathbf{x}) = A(\lambda\mathbf{x}) = \lambda A\mathbf{x} = \lambda^2\mathbf{x}$. Since $\mathbf{x}$ is nonzero, λ must be nonzero. Thus each eigenvalue of A is zero.

27. Use the *Hint* in the text to write, for any $\lambda, (A-\lambda I)^T = A^T - (\lambda I)^T = A^T - \lambda I$. Since $(A-\lambda I)^T$ is invertible if and only if $A-\lambda I$ is invertible (by Theorem 6(c) in Section 2.2), it follows that $A^T - \lambda I$ is *not* invertible if and only if $A-\lambda I$ is *not* invertible. That is, λ is an eigenvalue of A^T if and only if λ is an eigenvalue of A.

Note: If you discuss Exercise 27, you might ask students on a test to show that A and A^T have the same characteristic polynomial (discussed in Section 5.2). Since det $A = \det A^T$, for any square matrix A,

$$\det(A-\lambda I) = \det(A-\lambda I)^T = \det(A^T - (\lambda I)^T) = \det(A-\lambda I).$$

28. If A is lower triangular, then A^T is upper triangular and has the same diagonal entries as A. Hence, by the part of Theorem 1 already proved in the text, these diagonal entries are eigenvalues of A^T. By Exercise 27, they are also eigenvalues of A.

29. Let $\mathbf{v}$ be the vector in $\mathbf{R}^n$ whose entries are all ones. Then $A\mathbf{v} = s\mathbf{v}$.

30. Suppose the column sums of an $n \times n$ matrix A all equal the same number s. By Exercise 29 applied to A^T in place of A, the number s is an eigenvalue of A^T. By Exercise 27, s is an eigenvalue of A.

31. Suppose T reflects points across (or through) a line that passes through the origin. That line consists of all multiples of some nonzero vector $\mathbf{v}$. The points on this line do not move under the action of A. So $T(\mathbf{v}) = \mathbf{v}$. If A is the standard matrix of T, then $A\mathbf{v} = \mathbf{v}$. Thus $\mathbf{v}$ is an eigenvector of A corresponding to the eigenvalue 1. The eigenspace is Span $\{\mathbf{v}\}$. Another eigenspace is generated by any nonzero vector $\mathbf{u}$ that is perpendicular to the given line. (Perpendicularity in $\mathbf{R}^2$ should be a familiar concept even though orthogonality in $\mathbf{R}^n$ has not been discussed yet.) Each vector $\mathbf{x}$ on the line through $\mathbf{u}$ is transformed into the vector $-\mathbf{x}$. The eigenvalue is -1.

33. (The solution is given in the text.)

a. Replace k by $k+1$ in the definition of $\mathbf{x}_k$, and obtain $\mathbf{x}_{k+1} = c_1\lambda^{k+1}\mathbf{u} + c_2\mu^{k+1}\mathbf{v}$.

b.
$$\begin{aligned} A\mathbf{x}_k &= A(c_1\lambda^k\mathbf{u} + c_2\mu^k\mathbf{v}) \\ &= c_1\lambda^k A\mathbf{u} + c_2\mu^k A\mathbf{v} \quad \text{by linearity} \\ &= c_1\lambda^k \lambda\mathbf{u} + c_2\mu^k \mu\mathbf{v} \quad \text{since } \mathbf{u} \text{ and } \mathbf{v} \text{ are eigenvectors} \\ &= \mathbf{x}_{k+1} \end{aligned}$$

34. You could try to write $\mathbf{x}_0$ as linear combination of eigenvectors, $\mathbf{v}_1, ..., \mathbf{v}_p$. If $\lambda_1, ..., \lambda_p$ are corresponding eigenvalues, and if $\mathbf{x}_0 = c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p$, then you could *define*

$$\mathbf{x}_k = c_1\lambda_1^k\mathbf{v}_1 + \cdots + c_p\lambda_p^k\mathbf{v}_p$$

In this case, for $k = 0, 1, 2, ...,$

$$\begin{aligned} A\mathbf{x}_k &= A(c_1\lambda_1^k\mathbf{v}_1 + \cdots + c_p\lambda_p^k\mathbf{v}_p) \\ &= c_1\lambda_1^k A\mathbf{v}_1 + \cdots + c_p\lambda_p^k A\mathbf{v}_p \quad \text{Linearity} \\ &= c_1\lambda_1^{k+1}\mathbf{v}_1 + \cdots + c_p\lambda_p^{k+1}\mathbf{v}_p \quad \text{The } \mathbf{v}_i \text{ are eigenvectors.} \\ &= \mathbf{x}_{k+1} \end{aligned}$$

35. Using the figure in the exercise, plot $T(\mathbf{u})$ as $2\mathbf{u}$, because **u** is an eigenvector for the eigenvalue 2 of the standard matrix A. Likewise, plot $T(\mathbf{v})$ as $3\mathbf{v}$, because **v** is an eigenvector for the eigenvalue 3. Since T is linear, the image of **w** is $T(\mathbf{w}) = T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$.

36. As in Exercise 35, $T(\mathbf{u}) = -\mathbf{u}$ and $T(\mathbf{v}) = 3\mathbf{v}$ because **u** and **v** are eigenvectors for the eigenvalues -1 and 3, respectively, of the standard matrix A. Since T is linear, the image of **w** is $T(\mathbf{w}) = T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$.

Note: The matrix programs supported by this text all have an eigenvalue command. In some cases, such as MATLAB, the command can be structured so it provides eigenvectors as well as a list of the eigenvalues. At this point in the course, students should *not* use the extra power that produces eigenvectors. Students need to be reminded frequently that eigenvectors of A are null vectors of a translate of A. That is why the instructions for Exercises 35–38 tell students to use the method of Example 4.

It is my experience that nearly all students need manual practice finding eigenvectors by the method of Example 4, at least in this section if not also in Sections 5.2 and 5.3. However, [M] exercises do create a burden if eigenvectors must be found manually. For this reason, the data files for the text include a special command, `nulbasis` for each matrix program (MATLAB, Maple, etc.). The output of `nulbasis (A)` is a matrix whose columns provide a basis for the null space of A, and these columns are identical to the ones a student would find by row reducing the augmented matrix $[A \;\; \mathbf{0}]$. With nulbasis, student answers will be the same (up to multiples) as those in the text. I encourage my students to use technology to speed up all numerical homework here, not just the $[\mathbf{M}]$ exercises,

37. **[M]** Let A be the given matrix. Use the MATLAB commands `eig` and `nulbasis` (or equivalent commands). The command `ev = eig(A)` computes the three eigenvalues of A and stores them in a vector `ev`. In this exercise, `ev` $= (3, 13, 13)$. The eigenspace for the eigenvalue 3 is the null space of $A - 3I$. Use `nulbasis` to produce a basis for each null space. If the format is set for rational display, the result is

$$\texttt{nulbasis(A-ev(1)*eye(3))} = \begin{bmatrix} 5/9 \\ -2/9 \\ 1 \end{bmatrix}.$$

For simplicity, scale the entries by 9. A basis for the eigenspace for $\lambda = 3$: $\begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix}$

For the next eigenvalue, 13, compute `nulbasis(A-ev(2)*eye(3))` $=\begin{bmatrix} -2 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Basis for eigenspace for $\lambda = 13$: $\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$

There is no need to use `ev(3)` because it is the same as `ev(2)`.

38. **[M]** `ev = eig(A)` $= (13, -12, -12, 13)$. For $\lambda = 13$:

`nulbasis (A-ev(1)*eye(4))` $= \begin{bmatrix} -1/2 & 1/3 \\ 0 & -4/3 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$. Basis for eigenspace: $\left\{ \begin{bmatrix} -1 \\ 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -4 \\ 0 \\ 3 \end{bmatrix} \right\}$

For $\lambda = -12$: `nulbasis(A-ev(2)*eye(4))` $= \begin{bmatrix} 2/7 & 0 \\ 1 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$. Basis: $\left\{ \begin{bmatrix} 2 \\ 7 \\ 7 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$

39. **[M]** For $\lambda = 5$, basis: $\left\{ \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$. For $\lambda = -2$, basis: $\left\{ \begin{bmatrix} -2 \\ 7 \\ -5 \\ 5 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 7 \\ -5 \\ 0 \\ 5 \end{bmatrix} \right\}$

40. **[M]** `ev = eig(A)` $= (21.68984106239549, -16.68984106239549, 3, 2, 2)$. The first two eigenvalues are the roots of $\lambda^2 - 5\lambda - 362 = 0$.

Basis for $\lambda = \text{ev}(1)$: $\begin{bmatrix} -0.33333333333333 \\ 2.39082008853296 \\ 0.33333333333333 \\ 0.58333333333333 \\ 1.00000000000000 \end{bmatrix}$, for $\lambda = \text{ev}(2)$: $\begin{bmatrix} -0.33333333333333 \\ -0.80748675519962 \\ 0.33333333333333 \\ 0.58333333333333 \\ 1.00000000000000 \end{bmatrix}$.

For the eigenvalues 3 and 2, the eigenbases are $\begin{bmatrix} 0 \\ -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, and $\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -.5 \\ .5 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$, respectively.

Note: Since so many eigenvalues in text problems are small integers, it is easy for students to form a habit of entering a value for λ in `nulbasis (A-λI)` based on a *visual examination* of the eigenvalues produced by `eig(A)` when only a few decimal places for λ are displayed. Exercise 40 may help your students discover the dangers of this approach.

5.2 SOLUTIONS

Notes: Exercises 9–14 can be omitted, unless you want your students to have some facility with determinants of 3×3 matrices. In later sections, the text will provide eigenvalues when they are needed for matrices larger than 2×2. If you discussed partitioned matrices in Section 2.4, you might wish to bring in Supplementary Exercises 12–14 in Chapter 5. (Also, see Exercise 14 of Section 2.4.)

Exercises 25 and 27 support the subsection on dynamical systems. The calculations in these exercises and Example 5 prepare for the discussion in Section 5.6 about eigenvector decompositions.

1. $A=\begin{bmatrix}2&7\\7&2\end{bmatrix}$, $A-\lambda I=\begin{bmatrix}2&7\\7&2\end{bmatrix}-\begin{bmatrix}\lambda&0\\0&\lambda\end{bmatrix}=\begin{bmatrix}2-\lambda&7\\7&2-\lambda\end{bmatrix}$. The characteristic polynomial is

$$\det(A-\lambda I)=(2-\lambda)^2-7^2=4-4\lambda+\lambda^2-49=\lambda^2-4\lambda-45$$

In factored form, the characteristic equation is $(\lambda-9)(\lambda+5)=0$, so the eigenvalues of A are 9 and -5.

2. $A=\begin{bmatrix}5&3\\3&5\end{bmatrix}$, $A-\lambda I=\begin{bmatrix}5-\lambda&3\\3&5-\lambda\end{bmatrix}$. The characteristic polynomial is

$$\det(A-\lambda I)=(5-\lambda)(5-\lambda)-3\cdot3=\lambda^2-10\lambda+16$$

Since $\lambda^2-10\lambda+16=(\lambda-8)(\lambda-2)$, the eigenvalues of A are 8 and 2.

3. $A=\begin{bmatrix}3&-2\\1&-1\end{bmatrix}$, $A-\lambda I=\begin{bmatrix}3-\lambda&-2\\1&-1-\lambda\end{bmatrix}$. The characteristic polynomial is

$$\det(A-\lambda I)=(3-\lambda)(-1-\lambda)-(-2)(1)=\lambda^2-2\lambda-1$$

Use the quadratic formula to solve the characteristic equation and find the eigenvalues:

$$\lambda=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{2\pm\sqrt{4+4}}{2}=1\pm\sqrt{2}$$

4. $A=\begin{bmatrix}5&-3\\-4&3\end{bmatrix}$, $A-\lambda I=\begin{bmatrix}5-\lambda&-3\\-4&3-\lambda\end{bmatrix}$. The characteristic polynomial of A is

$$\det(A-\lambda I)=(5-\lambda)(3-\lambda)-(-3)(-4)=\lambda^2-8\lambda+3$$

Use the quadratic formula to solve the characteristic equation and find the eigenvalues:

$$\lambda=\frac{8\pm\sqrt{64-4(3)}}{2}=\frac{8\pm2\sqrt{13}}{2}=4\pm\sqrt{13}$$

5. $A=\begin{bmatrix}2&1\\-1&4\end{bmatrix}$, $A-\lambda I=\begin{bmatrix}2-\lambda&1\\-1&4-\lambda\end{bmatrix}$. The characteristic polynomial of A is

$$\det(A-\lambda I)=(2-\lambda)(4-\lambda)-(1)(-1)=\lambda^2-6\lambda+9=(\lambda-3)^2$$

Thus, A has only one eigenvalue 3, with multiplicity 2.

6. $A=\begin{bmatrix}3&-4\\4&8\end{bmatrix}$, $A-\lambda I=\begin{bmatrix}3-\lambda&-4\\4&8-\lambda\end{bmatrix}$. The characteristic polynomial is

$$\det(A-\lambda I)=(3-\lambda)(8-\lambda)-(-4)(4)=\lambda^2-11\lambda+40$$

Use the quadratic formula to solve det $(A-\lambda I)=0$:

$$\lambda=\frac{-11\pm\sqrt{121-4(40)}}{2}=\frac{-11\pm\sqrt{-39}}{2}$$

These values are complex numbers, not real numbers, so A has no real eigenvalues. There is no nonzero vector $\mathbf{x}$ in $\mathbf{R}^2$ such that $A\mathbf{x}=\lambda\mathbf{x}$, because a real vector $A\mathbf{x}$ cannot equal a complex multiple of $\mathbf{x}$.

7. $A=\begin{bmatrix}5 & 3\\ -4 & 4\end{bmatrix}, A-\lambda I=\begin{bmatrix}5-\lambda & 3\\ -4 & 4-\lambda\end{bmatrix}$. The characteristic polynomial is

$$\det(A-\lambda I)=(5-\lambda)(4-\lambda)-(3)(-4)=\lambda^2-9\lambda+32$$

Use the quadratic formula to solve det $(A-\lambda I)=0$:

$$\lambda=\frac{9\pm\sqrt{81-4(32)}}{2}=\frac{9\pm\sqrt{-47}}{2}$$

These values are complex numbers, not real numbers, so A has no real eigenvalues. There is no nonzero vector $\mathbf{x}$ in $\mathbf{R}^2$ such that $A\mathbf{x}=\lambda\mathbf{x}$, because a real vector $A\mathbf{x}$ cannot equal a complex multiple of $\mathbf{x}$.

8. $A=\begin{bmatrix}7 & -2\\ 2 & 3\end{bmatrix}, A-\lambda I=\begin{bmatrix}7-\lambda & -2\\ 2 & 3-\lambda\end{bmatrix}$. The characteristic polynomial is

$$\det(A-\lambda I)=(7-\lambda)(3-\lambda)-(-2)(2)=\lambda^2-10\lambda+25$$

Since $\lambda^2-10\lambda+25=(\lambda-5)^2$, the only eigenvalue is 5, with multiplicity 2.

9. $\det(A-\lambda I)=\det\begin{bmatrix}1-\lambda & 0 & -1\\ 2 & 3-\lambda & -1\\ 0 & 6 & 0-\lambda\end{bmatrix}$. From the special formula for 3×3 determinants, the characteristic polynomial is

$$\begin{aligned}\det(A-\lambda I)&=(1-\lambda)(3-\lambda)(-\lambda)+0+(-1)(2)(6)-0-(6)(-1)(1-\lambda)-0\\&=(\lambda^2-4\lambda+3)(-\lambda)-12+6(1-\lambda)\\&=-\lambda^3+4\lambda^2-3\lambda-12+6-6\lambda\\&=-\lambda^3+4\lambda^2-9\lambda-6\end{aligned}$$

(This polynomial has one irrational zero and two imaginary zeros.) Another way to evaluate the determinant is to interchange rows 1 and 2 (which reverses the sign of the determinant) and then make one row replacement:

$$\det\begin{bmatrix}1-\lambda & 0 & -1\\ 2 & 3-\lambda & -1\\ 0 & 6 & 0-\lambda\end{bmatrix}=-\det\begin{bmatrix}2 & 3-\lambda & -1\\ 1-\lambda & 0 & -1\\ 0 & 6 & 0-\lambda\end{bmatrix}$$

$$=-\det\begin{bmatrix}2 & 3-\lambda & -1\\ 0 & 0+(.5\lambda-.5)(3-\lambda) & -1+(.5\lambda-.5)(-1)\\ 0 & 6 & 0-\lambda\end{bmatrix}$$

Next, expand by cofactors down the first column. The quantity above equals

$$-2\det\begin{bmatrix}(.5\lambda-.5)(3-\lambda) & -.5-.5\lambda\\ 6 & -\lambda\end{bmatrix}=-2[(.5\lambda-.5)(3-\lambda)(-\lambda)-(-.5-.5\lambda)(6)]$$

$$=(1-\lambda)(3-\lambda)(-\lambda)-(1+\lambda)(6)=(\lambda^2-4\lambda+3)(-\lambda)-6-6\lambda=-\lambda^3+4\lambda^2-9\lambda-6$$

10. $\det(A-\lambda I)=\det\begin{bmatrix}0-\lambda & 3 & 1\\ 3 & 0-\lambda & 2\\ 1 & 2 & 0-\lambda\end{bmatrix}$. From the special formula for 3×3 determinants, the characteristic polynomial is

$$\det(A-\lambda I)=(-\lambda)(-\lambda)(-\lambda)+3\cdot 2\cdot 1+1\cdot 3\cdot 2-1\cdot(-\lambda)\cdot 1-2\cdot 2\cdot(-\lambda)-(-\lambda)\cdot 3\cdot 3$$
$$=-\lambda^3+6+6+\lambda+4\lambda+9\lambda=-\lambda^3+14\lambda+12$$

11. The special arrangements of zeros in A makes a cofactor expansion along the first row highly effective.

$$\det(A-\lambda I)=\det\begin{bmatrix}4-\lambda & 0 & 0\\ 5 & 3-\lambda & 2\\ -2 & 0 & 2-\lambda\end{bmatrix}=(4-\lambda)\det\begin{bmatrix}3-\lambda & 2\\ 0 & 2-\lambda\end{bmatrix}$$
$$=(4-\lambda)(3-\lambda)(2-\lambda)=(4-\lambda)(\lambda^2-5\lambda+6)=-\lambda^3+9\lambda^2-26\lambda+24$$

If only the eigenvalues were required, there would be no need here to write the characteristic polynomial in expanded form.

12. Make a cofactor expansion along the third row:

$$\det(A-\lambda I)=\det\begin{bmatrix}-1-\lambda & 0 & 1\\ -3 & 4-\lambda & 1\\ 0 & 0 & 2-\lambda\end{bmatrix}=(2-\lambda)\cdot\det\begin{bmatrix}-1-\lambda & 0\\ -3 & 4-\lambda\end{bmatrix}$$
$$=(2-\lambda)(-1-\lambda)(4-\lambda)=-\lambda^3+5\lambda^2-2\lambda-8$$

13. Make a cofactor expansion down the third column:

$$\det(A-\lambda I)=\det\begin{bmatrix}6-\lambda & -2 & 0\\ -2 & 9-\lambda & 0\\ 5 & 8 & 3-\lambda\end{bmatrix}=(3-\lambda)\cdot\det\begin{bmatrix}6-\lambda & -2\\ -2 & 9-\lambda\end{bmatrix}$$
$$=(3-\lambda)[(6-\lambda)(9-\lambda)-(-2)(-2)]=(3-\lambda)(\lambda^2-15\lambda+50)$$
$$=-\lambda^3+18\lambda^2-95\lambda+150 \text{ or } (3-\lambda)(\lambda-5)(\lambda-10)$$

14. Make a cofactor expansion along the second row:

$$\det(A-\lambda I)=\det\begin{bmatrix}5-\lambda & -2 & 3\\ 0 & 1-\lambda & 0\\ 6 & 7 & -2-\lambda\end{bmatrix}=(1-\lambda)\cdot\det\begin{bmatrix}5-\lambda & 3\\ 6 & -2-\lambda\end{bmatrix}$$
$$=(1-\lambda)\cdot[(5-\lambda)(-2-\lambda)-3\cdot 6]=(1-\lambda)(\lambda^2-3\lambda-28)$$
$$=-\lambda^3+4\lambda^2+25\lambda-28 \text{ or } (1-\lambda)(\lambda-7)(\lambda+4)$$

15. Use the fact that the determinant of a triangular matrix is the product of the diagonal entries:

$$\det(A-\lambda I)=\det\begin{bmatrix}4-\lambda & -7 & 0 & 2\\ 0 & 3-\lambda & -4 & 6\\ 0 & 0 & 3-\lambda & -8\\ 0 & 0 & 0 & 1-\lambda\end{bmatrix}=(4-\lambda)(3-\lambda)^2(1-\lambda)$$

The eigenvalues are 4, 3, 3, and 1.

16. The determinant of a triangular matrix is the product of its diagonal entries:

$$\det(A-\lambda I)=\det\begin{bmatrix}5-\lambda & 0 & 0 & 0\\ 8 & -4-\lambda & 0 & 0\\ 0 & 7 & 1-\lambda & 0\\ 1 & -5 & 2 & 1-\lambda\end{bmatrix}=(5-\lambda)(-4-\lambda)(1-\lambda)^2$$

The eigenvalues are 5, 1, 1, and -4.

17. The determinant of a triangular matrix is the product of its diagonal entries:

$$\begin{bmatrix}3-\lambda & 0 & 0 & 0 & 0\\ -5 & 1-\lambda & 0 & 0 & 0\\ 3 & 8 & 0-\lambda & 0 & 0\\ 0 & -7 & 2 & 1-\lambda & 0\\ -4 & 1 & 9 & -2 & 3-\lambda\end{bmatrix}=(3-\lambda)^2(1-\lambda)^2(-\lambda)$$

The eigenvalues are 3, 3, 1, 1, and 0.

18. Row reduce the augmented matrix for the equation $(A-5I)\mathbf{x}=\mathbf{0}$:

$$\begin{bmatrix}0 & -2 & 6 & -1 & 0\\ 0 & -2 & h & 0 & 0\\ 0 & 0 & 0 & 4 & 0\\ 0 & 0 & 0 & -4 & 0\end{bmatrix}\sim\begin{bmatrix}0 & -2 & 6 & -1 & 0\\ 0 & 0 & h-6 & 1 & 0\\ 0 & 0 & 0 & 4 & 0\\ 0 & 0 & 0 & 4 & 0\end{bmatrix}\sim\begin{bmatrix}0 & 1 & -3 & 0 & 0\\ 0 & 0 & h-6 & 0 & 0\\ 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0\end{bmatrix}$$

For a two-dimensional eigenspace, the system above needs two free variables. This happens if and only if $h=6$.

19. Since the equation $\det(A-\lambda I)=(\lambda_1-\lambda)(\lambda_2-\lambda)\cdots(\lambda_n-\lambda)$ holds for all λ, set $\lambda=0$ and conclude that $\det A=\lambda_1\lambda_2\cdots\lambda_n$.

20. $\det(A^T-\lambda I)=\det(A^T-\lambda I^T)$

$$\begin{aligned}&=\det(A-\lambda I)^T && \text{Transpose property}\\ &=\det(A-\lambda I) && \text{Theorem 3(c)}\end{aligned}$$

21. **a**. False. See Example 1.
b. False. See Theorem 3.
c. True. See Theorem 3.
d. False. See the solution of Example 4.

22. **a**. False. See the paragraph before Theorem 3.
b. False. See Theorem 3.
c. True. See the paragraph before Example 4.
d. False. See the warning after Theorem 4.

23. If $A=QR$, with Q invertible, and if $A_1=RQ$, then write $A_1=Q^{-1}QRQ=Q^{-1}AQ$, which shows that A_1 is similar to A.

24. First, observe that if P is invertible, then Theorem 3(b) shows that

$$1 = \det I = \det(PP^{-1}) = (\det P)(\det P^{-1})$$

Use Theorem 3(b) again when $A = PBP^{-1}$,

$$\det A = \det(PBP^{-1}) = (\det P)(\det B)(\det P^{-1}) = (\det B)(\det P)(\det P^{-1}) = \det B$$

25. Example 5 of Section 4.9 showed that $A\mathbf{v}_1 = \mathbf{v}_1$, which means that $\mathbf{v}_1$ is an eigenvector of A corresponding to the eigenvalue 1.

a. Since A is a 2×2 matrix, the eigenvalues are easy to find, and factoring the characteristic polynomial is easy when one of the two factors is known.

$$\det\begin{bmatrix} .6-\lambda & .3 \\ .4 & .7-\lambda \end{bmatrix} = (.6-\lambda)(.7-\lambda)-(.3)(.4) = \lambda^2 - 1.3\lambda + .3 = (\lambda - 1)(\lambda - .3)$$

The eigenvalues are 1 and .3. For the eigenvalue .3, solve $(A - .3I)\mathbf{x} = \mathbf{0}$:

$$\begin{bmatrix} .6-.3 & .3 & 0 \\ .4 & .7-.3 & 0 \end{bmatrix} = \begin{bmatrix} .3 & .3 & 0 \\ .4 & .4 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Here $x_1 - x_2 = 0$, with x_2 free. The general solution is not needed. Set $x_2 = 1$ to find an eigenvector $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. A suitable basis for $\mathbf{R}^2$ is $\{\mathbf{v}_1, \mathbf{v}_2\}$.

b. Write $\mathbf{x}_0 = \mathbf{v}_1 + c\mathbf{v}_2$: $\begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} = \begin{bmatrix} 3/7 \\ 4/7 \end{bmatrix} + c\begin{bmatrix} -1 \\ 1 \end{bmatrix}$. By inspection, c is $-1/14$. (The value of c depends on how $\mathbf{v}_2$ is scaled.)

c. For $k = 1, 2, \ldots,$ define $\mathbf{x}_k = A^k\mathbf{x}_0$. Then $\mathbf{x}_1 = A(\mathbf{v}_1 + c\mathbf{v}_2) = A\mathbf{v}_1 + cA\mathbf{v}_2 = \mathbf{v}_1 + c(.3)\mathbf{v}_2$, because $\mathbf{v}_1$ and $\mathbf{v}_2$ are eigenvectors. Again

$$\mathbf{x}_2 = A\mathbf{x}_1 = A(\mathbf{v}_1 + c(.3)\mathbf{v}_2) = A\mathbf{v}_1 + c(.3)A\mathbf{v}_2 = \mathbf{v}_1 + c(.3)(.3)\mathbf{v}_2.$$

Continuing, the general pattern is $\mathbf{x}_k = \mathbf{v}_1 + c(.3)^k\mathbf{v}_2$. As k increases, the second term tends to $\mathbf{0}$ and so $\mathbf{x}_k$ tends to $\mathbf{v}_1$.

26. If $a \neq 0$, then $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \sim \begin{bmatrix} a & b \\ 0 & d - ca^{-1}b \end{bmatrix} = U$, and $\det A = (a)(d - ca^{-1}b) = ad - bc$. If $a = 0$, then $A = \begin{bmatrix} 0 & b \\ c & d \end{bmatrix} \sim \begin{bmatrix} c & d \\ 0 & b \end{bmatrix} = U$ (with one interchange), so $\det A = (-1)^1(cb) = 0 - bc = ad - bc$.

27. **a**. $A\mathbf{v}_1 = \mathbf{v}_1$, $A\mathbf{v}_2 = .5\mathbf{v}_2$, $A\mathbf{v}_3 = .2\mathbf{v}_3$.

b. The set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent because the eigenvectors correspond to different eigenvalues (Theorem 2). Since there are three vectors in the set, the set is a basis for $\mathbb{R}^3$. So there exist unique constants such that $\mathbf{x}_0 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$, and $\mathbf{w}^T\mathbf{x}_0 = c_1\mathbf{w}^T\mathbf{v}_1 + c_2\mathbf{w}^T\mathbf{v}_2 + c_3\mathbf{w}^T\mathbf{v}_3$. Since $\mathbf{x}_0$ and $\mathbf{v}_1$ are probability vectors and since the entries in $\mathbf{v}_2$ and $\mathbf{v}_3$ sum to 0, the above equation shows that $c_1 = 1$.

c. By (b), $\mathbf{x}_0 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$. Using (a),

$$\mathbf{x}_k = A^k\mathbf{x}_0 = c_1A^k\mathbf{v}_1 + c_2A^k\mathbf{v}_2 + c_3A^k\mathbf{v}_3 = \mathbf{v}_1 + c_2(.5)^k\mathbf{v}_2 + c_3(.2)^k\mathbf{v}_3 \to \mathbf{v}_1 \text{ as } k \to \infty$$

28. [M]

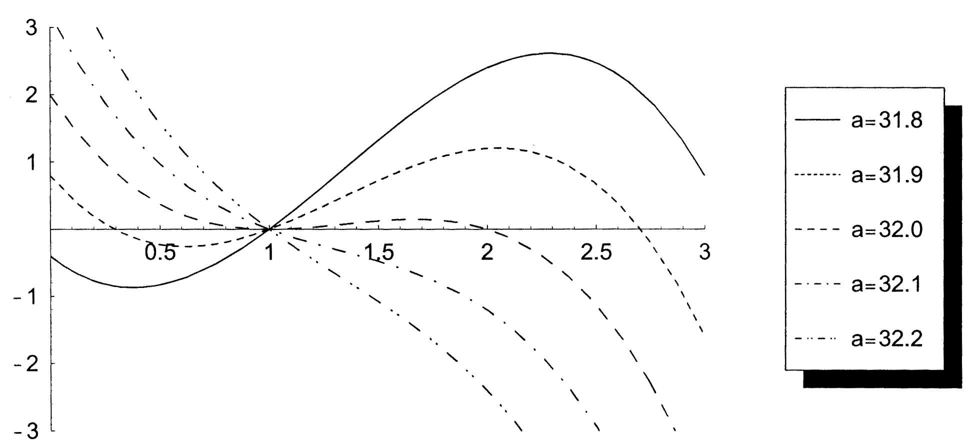
Answers will vary, but should show that the eigenvectors of A are not the same as the eigenvectors of A^T, unless, of course, $A^T = A$.

29. [M] Answers will vary. The product of the eigenvalues of A should equal det A.

30. [M] The characteristic polynomials and the eigenvalues for the various values of a are given in the following table:

a	Characteristic Polynomial	Eigenvalues
31.8	$-.4-2.6t+4t^2-t^3$	$3.1279, 1, -.1279$
31.9	$.8-3.8t+4t^2-t^3$	2.7042, 1, .2958
32.0	$2-5t+4t^2-t^3$	2, 1, 1
32.1	$3.2-6.2t+4t^2-t^3$	$1.5\pm.9747i, 1$
32.2	$4.4-7.4t+4t^2-t^3$	$1.5\pm1.4663i, 1$

The graphs of the characteristic polynomials are:



Notes: An appendix in Section 5.3 of the *Study Guide* gives an example of factoring a cubic polynomial with integer coefficients, in case you want your students to find integer eigenvalues of simple 3×3 or perhaps 4×4 matrices.

The MATLAB box for Section 5.3 introduces the command `poly (A)`, which lists the coefficients of the characteristic polynomial of the matrix A, and it gives MATLAB code that will produce a graph of the characteristic polynomial. (This is needed for Exercise 30.) The Maple and Mathematica appendices have corresponding information. The appendices for the TI and HP calculators contain only the commands that list the coefficients of the characteristic polynomial.

5.3 SOLUTIONS

1. $P=\begin{bmatrix}5&7\\2&3\end{bmatrix}, D=\begin{bmatrix}2&0\\0&1\end{bmatrix}, A=PDP^{-1}$, and $A^4=PD^4P^{-1}$. We compute $P^{-1}=\begin{bmatrix}3&-7\\-2&5\end{bmatrix}, D^4=\begin{bmatrix}16&0\\0&1\end{bmatrix}$, and $A^4=\begin{bmatrix}5&7\\2&3\end{bmatrix}\begin{bmatrix}16&0\\0&1\end{bmatrix}\begin{bmatrix}3&-7\\-2&5\end{bmatrix}=\begin{bmatrix}226&-525\\90&-209\end{bmatrix}$

2. $P=\begin{bmatrix}2&-3\\-3&5\end{bmatrix}, D=\begin{bmatrix}1&0\\0&1/2\end{bmatrix}, A=PDP^{-1}$, and $A^4=PD^4P^{-1}$. We compute $P^{-1}=\begin{bmatrix}5&3\\3&2\end{bmatrix}, D^4=\begin{bmatrix}1&0\\0&1/16\end{bmatrix}$, and $A^4=\begin{bmatrix}2&-3\\-3&5\end{bmatrix}\begin{bmatrix}1&0\\0&1/16\end{bmatrix}\begin{bmatrix}5&3\\3&2\end{bmatrix}=\frac{1}{16}\begin{bmatrix}151&90\\-225&-134\end{bmatrix}$

3. $A^k=PD^kP^{-1}=\begin{bmatrix}1&0\\3&1\end{bmatrix}\begin{bmatrix}a^k&0\\0&b^k\end{bmatrix}\begin{bmatrix}1&0\\-3&1\end{bmatrix}=\begin{bmatrix}a^k&0\\3a^k-3b^k&b^k\end{bmatrix}$.

4. $A^k=PD^kP^{-1}=\begin{bmatrix}3&4\\1&1\end{bmatrix}\begin{bmatrix}2^k&0\\0&1^k\end{bmatrix}\begin{bmatrix}-1&4\\1&-3\end{bmatrix}=\begin{bmatrix}4-3\cdot 2^k&12\cdot 2^k-12\\1-2^k&4\cdot 2^k-3\end{bmatrix}$.

5. By the Diagonalization Theorem, eigenvectors form the columns of the left factor, and they correspond respectively to the eigenvalues on the diagonal of the middle factor.

$$\lambda=5:\begin{bmatrix}1\\1\\1\end{bmatrix};\lambda=1:\begin{bmatrix}1\\0\\-1\end{bmatrix},\begin{bmatrix}2\\-1\\0\end{bmatrix}$$

6. As in Exercise 5, inspection of the factorization gives:

$$\lambda=4:\begin{bmatrix}-1\\2\\0\end{bmatrix};\lambda=5:\begin{bmatrix}-2\\0\\1\end{bmatrix},\begin{bmatrix}0\\1\\0\end{bmatrix}$$

7. Since A is triangular, its eigenvalues are obviously ± 1.

For $\lambda = 1$: $A-1I=\begin{bmatrix}0&0\\6&-2\end{bmatrix}$. The equation $(A-1I)\mathbf{x}=\mathbf{0}$ amounts to $6x_1-2x_2=0$, so $x_1=(1/3)x_2$ with x_2 free. The general solution is $x_2\begin{bmatrix}1/3\\1\end{bmatrix}$, and a nice basis vector for the eigenspace is $\mathbf{v}_1=\begin{bmatrix}1\\3\end{bmatrix}$.

For $\lambda = -1$: $A+1I=\begin{bmatrix}2&0\\6&0\end{bmatrix}$. The equation $(A+1I)\mathbf{x}=\mathbf{0}$ amounts to $2x_1=0$, so $x_1=0$ with x_2 free. The general solution is $x_2\begin{bmatrix}0\\1\end{bmatrix}$, and a basis vector for the eigenspace is $\mathbf{v}_2=\begin{bmatrix}0\\1\end{bmatrix}$.

From $\mathbf{v}_1$ and $\mathbf{v}_2$ construct $P=\begin{bmatrix}\mathbf{v}_1&\mathbf{v}_2\end{bmatrix}=\begin{bmatrix}1&0\\3&1\end{bmatrix}$. Then set $D=\begin{bmatrix}1&0\\0&-1\end{bmatrix}$, where the eigenvalues in D correspond to $\mathbf{v}_1$ and $\mathbf{v}_2$ respectively.

8. Since A is triangular, its only eigenvalue is obviously 5.

For $\lambda = 5$: $A-5I=\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}$. The equation $(A-5I)\mathbf{x}=\mathbf{0}$ amounts to $x_2=0$, so $x_2=0$ with x_1 free. The general solution is $x_1\begin{bmatrix}1\\ 0\end{bmatrix}$. Since we cannot generate an eigenvector basis for $\mathbb{R}^2$, A is not diagonalizable.

9. To find the eigenvalues of A, compute its characteristic polynomial:

$$\det(A-\lambda I)=\det\begin{bmatrix}3-\lambda & -1\\ 1 & 5-\lambda\end{bmatrix}=(3-\lambda)(5-\lambda)-(-1)(1)=\lambda^2-8\lambda+16=(\lambda-4)^2$$

Thus the only eigenvalue of A is 4.

For $\lambda = 4$: $A-4I=\begin{bmatrix}-1 & -1\\ 1 & 1\end{bmatrix}$. The equation $(A-4I)\mathbf{x}=\mathbf{0}$ amounts to $x_1+x_2=0$, so $x_1=-x_2$ with x_2 free. The general solution is $x_2\begin{bmatrix}-1\\ 1\end{bmatrix}$. Since we cannot generate an eigenvector basis for $\mathbb{R}^2$, A is not diagonalizable.

10. To find the eigenvalues of A, compute its characteristic polynomial:

$$\det(A-\lambda I)=\det\begin{bmatrix}2-\lambda & 3\\ 4 & 1-\lambda\end{bmatrix}=(2-\lambda)(1-\lambda)-(3)(4)=\lambda^2-3\lambda-10=(\lambda-5)(\lambda+2)$$

Thus the eigenvalues of A are 5 and -2.

For $\lambda = 5$: $A-5I=\begin{bmatrix}-3 & 3\\ 4 & -4\end{bmatrix}$. The equation $(A-5I)\mathbf{x}=\mathbf{0}$ amounts to $x_1-x_2=0$, so $x_1=x_2$ with x_2 free. The general solution is $x_2\begin{bmatrix}1\\ 1\end{bmatrix}$, and a basis vector for the eigenspace is $\mathbf{v}_1=\begin{bmatrix}1\\ 1\end{bmatrix}$.

For $\lambda = -2$: $A+2I=\begin{bmatrix}4 & 3\\ 4 & 3\end{bmatrix}$. The equation $(A+1I)\mathbf{x}=\mathbf{0}$ amounts to $4x_1+3x_2=0$, so $x_1=(-3/4)x_2$ with x_2 free. The general solution is $x_2\begin{bmatrix}-3/4\\ 1\end{bmatrix}$, and a nice basis vector for the eigenspace is $\mathbf{v}_2=\begin{bmatrix}-3\\ 4\end{bmatrix}$.

From $\mathbf{v}_1$ and $\mathbf{v}_2$ construct $P=\begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2\end{bmatrix}=\begin{bmatrix}1 & -3\\ 1 & 4\end{bmatrix}$. Then set $D=\begin{bmatrix}5 & 0\\ 0 & -2\end{bmatrix}$, where the eigenvalues in D correspond to $\mathbf{v}_1$ and $\mathbf{v}_2$ respectively.

11. The eigenvalues of A are given to be 1, 2, and 3.

For $\lambda = 3$: $A-3I=\begin{bmatrix}-4 & 4 & -2\\ -3 & 1 & 0\\ -3 & 1 & 0\end{bmatrix}$, and row reducing $\begin{bmatrix}A-3I & \mathbf{0}\end{bmatrix}$ yields $\begin{bmatrix}1 & 0 & -1/4 & 0\\ 0 & 1 & -3/4 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$. The general solution is $x_3\begin{bmatrix}1/4\\ 3/4\\ 1\end{bmatrix}$, and a nice basis vector for the eigenspace is $\mathbf{v}_1=\begin{bmatrix}1\\ 3\\ 4\end{bmatrix}$.

For $\lambda = 2$: $A - 2I = \begin{bmatrix} -3 & 4 & -2 \\ -3 & 2 & 0 \\ -3 & 1 & 1 \end{bmatrix}$, and row reducing $\begin{bmatrix} A - 2I & \mathbf{0} \end{bmatrix}$ yields $\begin{bmatrix} 1 & 0 & -2/3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. The general solution is $x_3 \begin{bmatrix} 2/3 \\ 1 \\ 1 \end{bmatrix}$, and a nice basis vector for the eigenspace is $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix}$.

For $\lambda = 1$: $A - I = \begin{bmatrix} -2 & 4 & -2 \\ -3 & 3 & 0 \\ -3 & 1 & 2 \end{bmatrix}$, and row reducing $\begin{bmatrix} A - 1I & \mathbf{0} \end{bmatrix}$ yields $\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. The general solution is $x_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, and a basis vector for the eigenspace is $\mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

From $\mathbf{v}_1, \mathbf{v}_2$ and $\mathbf{v}_3$ construct $P = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 3 & 1 \\ 4 & 3 & 1 \end{bmatrix}$. Then set $D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, where the eigenvalues in D correspond to $\mathbf{v}_1, \mathbf{v}_2$ and $\mathbf{v}_3$ respectively.

12. The eigenvalues of A are given to be 2 and 8.

For $\lambda = 8$: $A - 8I = \begin{bmatrix} -4 & 2 & 2 \\ 2 & -4 & 2 \\ 2 & 2 & -4 \end{bmatrix}$, and row reducing $\begin{bmatrix} A - 8I & \mathbf{0} \end{bmatrix}$ yields $\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. The general solution is $x_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, and a basis vector for the eigenspace is $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

For $\lambda = 2$: $A - 2I = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix}$, and row reducing $\begin{bmatrix} A - 2I & \mathbf{0} \end{bmatrix}$ yields $\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. The general solution is $x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, and a basis for the eigenspace is $\{\mathbf{v}_2, \mathbf{v}_3\} = \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$.

From $\mathbf{v}_1, \mathbf{v}_2$ and $\mathbf{v}_3$ construct $P = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$. Then set $D = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$, where the eigenvalues in D correspond to $\mathbf{v}_1, \mathbf{v}_2$ and $\mathbf{v}_3$ respectively.

13. The eigenvalues of A are given to be 5 and 1.

For $\lambda = 5$: $A-5I=\begin{bmatrix}-3 & 2 & -1\\ 1 & -2 & -1\\ -1 & -2 & -3\end{bmatrix}$, and row reducing $\begin{bmatrix}A-5I & \mathbf{0}\end{bmatrix}$ yields $\begin{bmatrix}1 & 0 & 1 & 0\\ 0 & 1 & 1 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$. The general solution is $x_3\begin{bmatrix}-1\\ -1\\ 1\end{bmatrix}$, and a basis for the eigenspace is $\mathbf{v}_1=\begin{bmatrix}-1\\ -1\\ 1\end{bmatrix}$.

For $\lambda = 1$: $A-1I=\begin{bmatrix}1 & 2 & -1\\ 1 & 2 & -1\\ -1 & -2 & 1\end{bmatrix}$, and row reducing $\begin{bmatrix}A-I & \mathbf{0}\end{bmatrix}$ yields $\begin{bmatrix}1 & 2 & -1 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$. The general solution is $x_2\begin{bmatrix}-2\\ 1\\ 0\end{bmatrix}+x_3\begin{bmatrix}1\\ 0\\ 1\end{bmatrix}$, and a basis for the eigenspace is $\{\mathbf{v}_2,\mathbf{v}_3\}=\left\{\begin{bmatrix}-2\\ 1\\ 0\end{bmatrix},\begin{bmatrix}1\\ 0\\ 1\end{bmatrix}\right\}$.

From $\mathbf{v}_1,\mathbf{v}_2$ and $\mathbf{v}_3$ construct $P=\begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3\end{bmatrix}=\begin{bmatrix}-1 & -2 & 1\\ -1 & 1 & 0\\ 1 & 0 & 1\end{bmatrix}$. Then set $D=\begin{bmatrix}5 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$, where the eigenvalues in D correspond to $\mathbf{v}_1,\mathbf{v}_2$ and $\mathbf{v}_3$ respectively.

14. The eigenvalues of A are given to be 5 and 4.

For $\lambda = 5$: $A-5I=\begin{bmatrix}-1 & 0 & -2\\ 2 & 0 & 4\\ 0 & 0 & 0\end{bmatrix}$, and row reducing $\begin{bmatrix}A-5I & \mathbf{0}\end{bmatrix}$ yields $\begin{bmatrix}1 & 0 & 2 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$. The general solution is $x_2\begin{bmatrix}0\\ 1\\ 0\end{bmatrix}+x_3\begin{bmatrix}-2\\ 0\\ 1\end{bmatrix}$, and a basis for the eigenspace is $\{\mathbf{v}_1,\mathbf{v}_2\}=\left\{\begin{bmatrix}-2\\ 0\\ 1\end{bmatrix},\begin{bmatrix}0\\ 1\\ 0\end{bmatrix}\right\}$.

For $\lambda = 4$: $A-4I=\begin{bmatrix}0 & 0 & -2\\ 2 & 1 & 4\\ 0 & 0 & 1\end{bmatrix}$, and row reducing $\begin{bmatrix}A-4I & \mathbf{0}\end{bmatrix}$ yields $\begin{bmatrix}1 & 1/2 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$. The general solution is $x_3\begin{bmatrix}-1/2\\ 1\\ 0\end{bmatrix}$, and a nice basis vector for the eigenspace is $\mathbf{v}_3=\begin{bmatrix}-1\\ 2\\ 0\end{bmatrix}$.

From $\mathbf{v}_1,\mathbf{v}_2$ and $\mathbf{v}_3$ construct $P=\begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3\end{bmatrix}=\begin{bmatrix}-2 & 0 & -1\\ 0 & 1 & 2\\ 1 & 0 & 0\end{bmatrix}$. Then set $D=\begin{bmatrix}5 & 0 & 0\\ 0 & 5 & 0\\ 0 & 0 & 4\end{bmatrix}$, where the eigenvalues in D correspond to $\mathbf{v}_1,\mathbf{v}_2$ and $\mathbf{v}_3$ respectively.

15. The eigenvalues of A are given to be 3 and 1.

For $\lambda = 3$: $A-3I=\begin{bmatrix} 4 & 4 & 16 \\ 2 & 2 & 8 \\ -2 & -2 & -8 \end{bmatrix}$, and row reducing $\begin{bmatrix} A-3I & \mathbf{0} \end{bmatrix}$ yields $\begin{bmatrix} 1 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. The general solution is $x_2\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}+x_3\begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix}$, and a basis for the eigenspace is $\{\mathbf{v}_1,\mathbf{v}_2\}=\left\{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix},\begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix}\right\}$

For $\lambda = 1$: $A-I=\begin{bmatrix} 6 & 4 & 16 \\ 2 & 4 & 8 \\ -2 & -2 & -6 \end{bmatrix}$, and row reducing $\begin{bmatrix} A-I & \mathbf{0} \end{bmatrix}$ yields $\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. The general solution is $x_3\begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}$, and a basis for the eigenspace is $\mathbf{v}_3=\begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}$.

From $\mathbf{v}_1,\mathbf{v}_2$ and $\mathbf{v}_3$ construct $P=\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix}=\begin{bmatrix} -1 & -4 & -2 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}$. Then set $D=\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, where the eigenvalues in D correspond to $\mathbf{v}_1,\mathbf{v}_2$ and $\mathbf{v}_3$ respectively.

16. The eigenvalues of A are given to be 2 and 1.

For $\lambda = 2$: $A-2I=\begin{bmatrix} -2 & -4 & -6 \\ -1 & -2 & -3 \\ 1 & 2 & 3 \end{bmatrix}$, and row reducing $\begin{bmatrix} A-2I & \mathbf{0} \end{bmatrix}$ yields $\begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. The general solution is $x_2\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}+x_3\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$, and a basis for the eigenspace is $\{\mathbf{v}_1,\mathbf{v}_2\}=\left\{\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix},\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}\right\}$.

For $\lambda = 1$: $A-I=\begin{bmatrix} -1 & -4 & -6 \\ -1 & -1 & -3 \\ 1 & 2 & 4 \end{bmatrix}$, and row reducing $\begin{bmatrix} A-I & \mathbf{0} \end{bmatrix}$ yields $\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. The general solution is $x_3\begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}$, and a basis for the eigenspace is $\mathbf{v}_3=\begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}$.

From $\mathbf{v}_1,\mathbf{v}_2$ and $\mathbf{v}_3$ construct $P=\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix}=\begin{bmatrix} -2 & -3 & -2 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}$. Then set $D=\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, where the eigenvalues in D correspond to $\mathbf{v}_1,\mathbf{v}_2$ and $\mathbf{v}_3$ respectively.

17. Since A is triangular, its eigenvalues are obviously 4 and 5.

For $\lambda = 4$: $A-4I=\begin{bmatrix}0&0&0\\1&0&0\\0&0&1\end{bmatrix}$, and row reducing $\begin{bmatrix}A-4I & \mathbf{0}\end{bmatrix}$ yields $\begin{bmatrix}1&0&0&0\\0&0&1&0\\0&0&0&0\end{bmatrix}$. The general solution is $x_2\begin{bmatrix}0\\1\\0\end{bmatrix}$, and a basis for the eigenspace is $\mathbf{v}_1=\begin{bmatrix}0\\1\\0\end{bmatrix}$.

Since $\lambda = 5$ must have only a one-dimensional eigenspace, we can find at most 2 linearly independent eigenvectors for A, so A is not diagonalizable.

18. An eigenvalue of A is given to be 5; an eigenvector $\mathbf{v}_1=\begin{bmatrix}-2\\1\\2\end{bmatrix}$ is also given. To find the eigenvalue corresponding to $\mathbf{v}_1$, compute $A\mathbf{v}_1=\begin{bmatrix}-7&-16&4\\6&13&-2\\12&16&1\end{bmatrix}\begin{bmatrix}-2\\1\\2\end{bmatrix}=\begin{bmatrix}6\\-3\\6\end{bmatrix}=-3\mathbf{v}_1$. Thus the eigenvalue in question is -3.

For $\lambda = 5$: $A-5I=\begin{bmatrix}-12&-16&4\\6&8&-2\\12&16&-4\end{bmatrix}$, and row reducing $\begin{bmatrix}A-5I & \mathbf{0}\end{bmatrix}$ yields $\begin{bmatrix}1&4/3&-1/3&0\\0&0&0&0\\0&0&0&0\end{bmatrix}$.

The general solution is $x_2\begin{bmatrix}-4/3\\1\\0\end{bmatrix}+x_3\begin{bmatrix}1/3\\0\\1\end{bmatrix}$, and a nice basis for the eigenspace is

$\{\mathbf{v}_2,\mathbf{v}_3\}=\left\{\begin{bmatrix}-4\\3\\0\end{bmatrix},\begin{bmatrix}1\\0\\3\end{bmatrix}\right\}$.

From $\mathbf{v}_1,\mathbf{v}_2$ and $\mathbf{v}_3$ construct $P=\begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3\end{bmatrix}=\begin{bmatrix}-2&-4&1\\1&3&0\\2&0&3\end{bmatrix}$. Then set $D=\begin{bmatrix}-3&0&0\\0&5&0\\0&0&5\end{bmatrix}$, where the eigenvalues in D correspond to $\mathbf{v}_1,\mathbf{v}_2$ and $\mathbf{v}_3$ respectively. Note that this answer differs from the text. There, $P=\begin{bmatrix}\mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_1\end{bmatrix}$ and the entries in D are rearranged to match the new order of the eigenvectors. According to the Diagonalization Theorem, both answers are correct.

19. Since A is triangular, its eigenvalues are obviously 2, 3, and 5.

For $\lambda = 2$: $A-2I=\begin{bmatrix}3&-3&0&9\\0&1&1&-2\\0&0&0&0\\0&0&0&0\end{bmatrix}$, and row reducing $\begin{bmatrix}A-2I & \mathbf{0}\end{bmatrix}$ yields $\begin{bmatrix}1&0&1&1&0\\0&1&1&-2&0\\0&0&0&0&0\\0&0&0&0&0\end{bmatrix}$. The general solution is $x_3\begin{bmatrix}-1\\-1\\1\\0\end{bmatrix}+x_4\begin{bmatrix}-1\\2\\0\\1\end{bmatrix}$, and a nice basis for the eigenspace is $\{\mathbf{v}_1,\mathbf{v}_2\}=\left\{\begin{bmatrix}-1\\-1\\1\\0\end{bmatrix},\begin{bmatrix}-1\\2\\0\\1\end{bmatrix}\right\}$.

<u>For $\lambda = 3$</u>: $A-3I=\begin{bmatrix}2&-3&0&9\\0&0&1&-2\\0&0&-1&0\\0&0&0&-1\end{bmatrix}$, and row reducing $[A-3I\ \ \mathbf{0}]$ yields $\begin{bmatrix}1&-3/2&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&0\end{bmatrix}$.

The general solution is $x_2\begin{bmatrix}3/2\\1\\0\\0\end{bmatrix}$, and a nice basis for the eigenspace is $\mathbf{v}_3=\begin{bmatrix}3\\2\\0\\0\end{bmatrix}$.

<u>For $\lambda = 5$</u>: $A-5I=\begin{bmatrix}0&-3&0&9\\0&-2&1&-2\\0&0&-3&0\\0&0&0&-3\end{bmatrix}$, and row reducing $[A-5I\ \ \mathbf{0}]$ yields $\begin{bmatrix}0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&0\end{bmatrix}$. The general solution is $x_1\begin{bmatrix}1\\0\\0\\0\end{bmatrix}$, and a basis for the eigenspace is $\mathbf{v}_4=\begin{bmatrix}1\\0\\0\\0\end{bmatrix}$.

From $\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3$ and $\mathbf{v}_4$ construct $P=[\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3\ \mathbf{v}_4]=\begin{bmatrix}-1&-1&3&1\\-1&2&2&0\\1&0&0&0\\0&1&0&0\end{bmatrix}$. Then set $D=\begin{bmatrix}2&0&0&0\\0&2&0&0\\0&0&3&0\\0&0&0&5\end{bmatrix}$, where the eigenvalues in D correspond to $\mathbf{v}_1,\mathbf{v}_2$ and $\mathbf{v}_3$ respectively. Note that this answer differs from the text. There, $P=[\mathbf{v}_4\ \mathbf{v}_3\ \mathbf{v}_1\ \mathbf{v}_2]$ and the entries in D are rearranged to match the new order of the eigenvectors. According to the Diagonalization Theorem, both answers are correct.

20. Since A is triangular, its eigenvalues are obviously 4 and 2.

<u>For $\lambda = 4$</u>: $A-4I=\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&-2&0\\1&0&0&-2\end{bmatrix}$, and row reducing $[A-4I\ \ \mathbf{0}]$ yields $\begin{bmatrix}1&0&0&-2&0\\0&0&1&0&0\\0&0&0&0&0\\0&0&0&0&0\end{bmatrix}$. The general solution is $x_2\begin{bmatrix}0\\1\\0\\0\end{bmatrix}+x_4\begin{bmatrix}2\\0\\0\\1\end{bmatrix}$, and a basis for the eigenspace is $\{\mathbf{v}_1,\mathbf{v}_2\}=\left\{\begin{bmatrix}0\\1\\0\\0\end{bmatrix},\begin{bmatrix}2\\0\\0\\1\end{bmatrix}\right\}$.

<u>For $\lambda = 2$</u>: $A-2I=\begin{bmatrix}2&0&0&0\\0&2&0&0\\0&0&0&0\\1&0&0&0\end{bmatrix}$, and row reducing $[A-2I\ \ \mathbf{0}]$ yields $\begin{bmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{bmatrix}$. The general solution is $x_3\begin{bmatrix}0\\0\\1\\0\end{bmatrix}+x_4\begin{bmatrix}0\\0\\0\\1\end{bmatrix}$, and a basis for the eigenspace is $\{\mathbf{v}_3,\mathbf{v}_4\}=\left\{\begin{bmatrix}0\\0\\1\\0\end{bmatrix},\begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right\}$.

From $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ and $\mathbf{v}_4$ construct $P = [\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3\ \mathbf{v}_4] = \begin{bmatrix} 0 & 2 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$. Then set $D = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$, where the eigenvalues in D correspond to $\mathbf{v}_1, \mathbf{v}_2$ and $\mathbf{v}_3$ respectively.

21. **a**. False. The symbol D does not automatically denote a diagonal matrix.

b. True. See the remark after the statement of the Diagonalization Theorem.

c. False. The 3×3 matrix in Example 4 has 3 eigenvalues, counting multiplicities, but it is not diagonalizable.

d. False. Invertibility depends on 0 not being an eigenvalue. (See the Invertible Matrix Theorem.) A diagonalizable matrix may or may not have 0 as an eigenvalue. See Examples 3 and 5 for both possibilities.

22. **a**. False. The *n* eigenvectors must be linearly independent. See the Diagonalization Theorem.

b. False. The matrix in Example 3 is diagonalizable, but it has only 2 distinct eigenvalues. (The statement given is the *converse* of Theorem 6.)

c. True. This follows from $AP = PD$ and formulas (1) and (2) in the proof of the Diagonalization Theorem.

d. False. See Example 4. The matrix there is invertible because 0 is not an eigenvalue, but the matrix is not diagonalizable.

23. A is diagonalizable because you know that five linearly independent eigenvectors exist: three in the three-dimensional eigenspace and two in the two-dimensional eigenspace. Theorem 7 guarantees that the set of all five eigenvectors is linearly independent.

24. No, by Theorem 7(b). Here is an explanation that does not appeal to Theorem 7: Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be eigenvectors that span the two one-dimensional eigenspaces. If **v** is any other eigenvector, then it belongs to one of the eigenspaces and hence is a multiple of either $\mathbf{v}_1$ or $\mathbf{v}_2$. So there cannot exist three linearly independent eigenvectors. By the Diagonalization Theorem, A cannot be diagonalizable.

25. Let $\{\mathbf{v}_1\}$ be a basis for the one-dimensional eigenspace, let $\mathbf{v}_2$ and $\mathbf{v}_3$ form a basis for the two-dimensional eigenspace, and let $\mathbf{v}_4$ be any eigenvector in the remaining eigenspace. By Theorem 7, $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is linearly independent. Since A is 4×4, the Diagonalization Theorem shows that A is diagonalizable.

26. Yes, if the third eigenspace is only one-dimensional. In this case, the sum of the dimensions of the eigenspaces will be six, whereas the matrix is 7×7. See Theorem 7(b). An argument similar to that for Exercise 24 can also be given.

27. If A is diagonalizable, then $A = PDP^{-1}$ for some invertible P and diagonal D. Since A is invertible, 0 is not an eigenvalue of A. So the diagonal entries in D (which are eigenvalues of A) are not zero, and D is invertible. By the theorem on the inverse of a product,

$$A^{-1} = (PDP^{-1})^{-1} = (P^{-1})^{-1}D^{-1}P^{-1} = PD^{-1}P^{-1}$$

Since D^{-1} is obviously diagonal, A^{-1} is diagonalizable.

28. If A has n linearly independent eigenvectors, then by the Diagonalization Theorem, $A = PDP^{-1}$ for some invertible P and diagonal D. Using properties of transposes,

$$\begin{aligned} A^T &= (PDP^{-1})^T = (P^{-1})^T D^T P^T \\ &= (P^T)^{-1} DP^T = QDQ^{-1} \end{aligned}$$

where $Q = (P^T)^{-1}$. Thus A^T is diagonalizable. By the Diagonalization Theorem, the columns of Q are n linearly independent eigenvectors of A^T.

29. The diagonal entries in D_1 are reversed from those in D. So interchange the (eigenvector) columns of P to make them correspond properly to the eigenvalues in D_1. In this case,

$$P_1 = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix} \text{ and } D_1 = \begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix}$$

Although the first column of P must be an eigenvector corresponding to the eigenvalue 3, there is nothing to prevent us from selecting some multiple of $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$, say $\begin{bmatrix} -3 \\ 6 \end{bmatrix}$, and letting $P_2 = \begin{bmatrix} -3 & 1 \\ 6 & -1 \end{bmatrix}$. We now have three different factorizations or “diagonalizations” of A:

$$A = PDP^{-1} = P_1 D_1 P_1^{-1} = P_2 D_1 P_2^{-1}$$

30. A nonzero multiple of an eigenvector is another eigenvector. To produce P_2, simply multiply one or both columns of P by a nonzero scalar unequal to 1.

31. For a 2×2 matrix A to be invertible, its eigenvalues must be nonzero. A first attempt at a construction might be something such as $\begin{bmatrix} 2 & 3 \\ 0 & 4 \end{bmatrix}$, whose eigenvalues are 2 and 4. Unfortunately, a 2×2 matrix with two distinct eigenvalues is diagonalizable (Theorem 6). So, adjust the construction to $\begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}$, which works. In fact, any matrix of the form $\begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$ has the desired properties when a and b are nonzero. The eigenspace for the eigenvalue a is one-dimensional, as a simple calculation shows, and there is no other eigenvalue to produce a second eigenvector.

32. Any 2×2 matrix with two distinct eigenvalues is diagonalizable, by Theorem 6. If one of those eigenvalues is zero, then the matrix will not be invertible. Any matrix of the form $\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$ has the desired properties when a and b are nonzero. The number a must be nonzero to make the matrix diagonalizable; b must be nonzero to make the matrix not diagonal. Other solutions are $\begin{bmatrix} 0 & 0 \\ a & b \end{bmatrix}$ and $\begin{bmatrix} 0 & a \\ 0 & b \end{bmatrix}$.

33. $A=\begin{bmatrix}-6 & 4 & 0 & 9\\ -3 & 0 & 1 & 6\\ -1 & -2 & 1 & 0\\ -4 & 4 & 0 & 7\end{bmatrix}$,

`ev = eig(A)=(5,1,-2,-2)`

`nulbasis(A-ev(1)*eye(4))` $=\begin{bmatrix}1.0000\\ 0.5000\\ -0.5000\\ 1.0000\end{bmatrix}$

A basis for the eigenspace of $\lambda=5$ is $\begin{bmatrix}2\\ 1\\ -1\\ 2\end{bmatrix}$.

`nulbasis(A-ev(2)*eye(4))` $=\begin{bmatrix}1.0000\\ -0.5000\\ -3.5000\\ 1.0000\end{bmatrix}$

A basis for the eigenspace of $\lambda=1$ is $\begin{bmatrix}2\\ -1\\ -7\\ 2\end{bmatrix}$.

`nulbasis(A-ev(3)*eye(4))` $=\begin{bmatrix}1.0000\\ 1.0000\\ 1.0000\\ 0\end{bmatrix},\begin{bmatrix}1.5000\\ -0.7500\\ 0\\ 1.0000\end{bmatrix}$

A basis for the eigenspace of $\lambda=-2$ is $\begin{bmatrix}1\\ 1\\ 1\\ 0\end{bmatrix},\begin{bmatrix}6\\ -3\\ 0\\ 4\end{bmatrix}$.

Thus we construct $P=\begin{bmatrix}2 & 2 & 1 & 6\\ 1 & -1 & 1 & -3\\ -1 & -7 & 1 & 0\\ 2 & 2 & 0 & 4\end{bmatrix}$ and $D=\begin{bmatrix}5 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & -2 & 0\\ 0 & 0 & 0 & -2\end{bmatrix}$.

34. $A=\begin{bmatrix}0 & 13 & 8 & 4\\ 4 & 9 & 8 & 4\\ 8 & 6 & 12 & 8\\ 0 & 5 & 0 & -4\end{bmatrix}$,

`ev = eig(A) = (-4,24,1,-4)`

`nulbasis(A-ev(1)*eye(4))` $= \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

A basis for the eigenspace of $\lambda = -4$ is $\begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$.

`nulbasis(A-ev(2)*eye(4))` $= \begin{bmatrix} 5.6000 \\ 5.6000 \\ 7.2000 \\ 1.0000 \end{bmatrix}$

A basis for the eigenspace of $\lambda = 24$ is $\begin{bmatrix} 28 \\ 28 \\ 36 \\ 5 \end{bmatrix}$.

`nulbasis(A-ev(3)*eye(4))` $= \begin{bmatrix} 1.0000 \\ 1.0000 \\ -2.0000 \\ 1.0000 \end{bmatrix}$

A basis for the eigenspace of $\lambda = 1$ is $\begin{bmatrix} 1 \\ 1 \\ -2 \\ 1 \end{bmatrix}$.

Thus we construct $P = \begin{bmatrix} -2 & -1 & 28 & 1 \\ 0 & 0 & 28 & 1 \\ 1 & 0 & 36 & -2 \\ 0 & 1 & 5 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} -4 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & 0 & 24 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

35. $A = \begin{bmatrix} 11 & -6 & 4 & -10 & -4 \\ -3 & 5 & -2 & 4 & 1 \\ -8 & 12 & -3 & 12 & 4 \\ 1 & 6 & -2 & 3 & -1 \\ 8 & -18 & 8 & -14 & -1 \end{bmatrix}$,

`ev = eig(A)=(5,1,3,5,1)`

`nulbasis(A-ev(1)*eye(5))` $= \begin{bmatrix} 2.0000 \\ -0.3333 \\ -1.0000 \\ 1.0000 \\ 0 \end{bmatrix}, \begin{bmatrix} 1.0000 \\ -0.3333 \\ -1.0000 \\ 0 \\ 1.0000 \end{bmatrix}$

A basis for the eigenspace of $\lambda = 5$ is $\begin{bmatrix} 6 \\ -1 \\ -3 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ -3 \\ 0 \\ 3 \end{bmatrix}$.

`nulbasis(A-ev(2)*eye(5))` $= \begin{bmatrix} 0.8000 \\ -0.6000 \\ -0.4000 \\ 1.0000 \\ 0 \end{bmatrix}, \begin{bmatrix} 0.6000 \\ -0.2000 \\ -0.8000 \\ 0 \\ 1.0000 \end{bmatrix}$

A basis for the eigenspace of $\lambda = 1$ is $\begin{bmatrix} 4 \\ -3 \\ -2 \\ 5 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ -4 \\ 0 \\ 5 \end{bmatrix}$.

`nulbasis(A-ev(3)*eye(5))` $= \begin{bmatrix} 0.5000 \\ -0.2500 \\ -1.0000 \\ -0.2500 \\ 1.0000 \end{bmatrix}$

A basis for the eigenspace of $\lambda = 3$ is $\begin{bmatrix} 2 \\ -1 \\ -4 \\ -1 \\ 4 \end{bmatrix}$.

Thus we construct $P = \begin{bmatrix} 6 & 3 & 4 & 3 & 2 \\ -1 & -1 & -3 & -1 & -1 \\ -3 & -3 & -2 & -4 & -4 \\ 3 & 0 & 5 & 0 & -1 \\ 0 & 3 & 0 & 5 & 4 \end{bmatrix}$ and $D = \begin{bmatrix} 5 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}$.

36. $A = \begin{bmatrix} 4 & 4 & 2 & 3 & -2 \\ 0 & 1 & -2 & -2 & 2 \\ 6 & 12 & 11 & 2 & -4 \\ 9 & 20 & 10 & 10 & -6 \\ 15 & 28 & 14 & 5 & -3 \end{bmatrix}$,

`ev = eig(A) = (3,5,7,5,3)`

`nulbasis(A-ev(1)*eye(5))` $= \begin{bmatrix} 2.0000 \\ -1.5000 \\ 0.5000 \\ 1.0000 \\ 0 \end{bmatrix}, \begin{bmatrix} -1.0000 \\ 0.5000 \\ 0.5000 \\ 0 \\ 1.0000 \end{bmatrix}$

A basis for the eigenspace of $\lambda = 3$ is $\begin{bmatrix} 4 \\ -3 \\ 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \\ 2 \end{bmatrix}$.

`nulbasis(A-ev(2)*eye(5))` $= \begin{bmatrix} 0 \\ -0.5000 \\ 1.0000 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1.0000 \\ 1.0000 \\ 0 \\ -1.0000 \\ 1.0000 \end{bmatrix}$

A basis for the eigenspace of $\lambda = 5$ is $\begin{bmatrix} 0 \\ -1 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$.

`nulbasis(A-ev(3)*eye(5))` $= \begin{bmatrix} 0.3333 \\ 0.0000 \\ 0.0000 \\ 1.0000 \\ 1.0000 \end{bmatrix}$

A basis for the eigenspace of $\lambda = 7$ is $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 3 \\ 3 \end{bmatrix}$.

Thus we construct $P = \begin{bmatrix} 4 & -2 & 0 & -1 & 1 \\ -3 & 1 & -1 & 1 & 0 \\ 1 & 1 & 2 & 0 & 0 \\ 2 & 0 & 0 & -1 & 3 \\ 0 & 2 & 0 & 1 & 3 \end{bmatrix}$ and $D = \begin{bmatrix} 3 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 7 \end{bmatrix}$.

Notes: For your use, here is another matrix with five distinct real eigenvalues. To four decimal places, they are 11.0654, 9.8785, 3.8238, -3.7332, and -6.0345.

$$\begin{bmatrix} 6 & -8 & 5 & -3 & 0 \\ -7 & 3 & -5 & 3 & 0 \\ -3 & -7 & 5 & -3 & 5 \\ 0 & -4 & 1 & -7 & 5 \\ -5 & -3 & -2 & 0 & 8 \end{bmatrix}$$

The MATLAB box in the *Study Guide* encourages students to use `eig (A)` and `nulbasis` to practice the diagonalization procedure in this section. It also remarks that in later work, a student may automate the process, using the command `[P D]= eig (A)`. You may wish to permit students to use the full power of `eig` in some problems in Sections 5.5 and 5.7.

5.4 SOLUTIONS

1. Since $T(\mathbf{b}_1) = 3\mathbf{d}_1 - 5\mathbf{d}_2, [T(\mathbf{b}_1)]_D = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$. Likewise $T(\mathbf{b}_2) = -\mathbf{d}_1 + 6\mathbf{d}_2$ implies that $[T(\mathbf{b}_2)]_D = \begin{bmatrix} -1 \\ 6 \end{bmatrix}$ and $T(\mathbf{b}_3) = 4\mathbf{d}_2$ implies that $[T(\mathbf{b}_3)]_D = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$. Thus the matrix for T relative to B and D is $\left[[T(\mathbf{b}_1)]_D [T(\mathbf{b}_2)]_D [T(\mathbf{b}_3)]_D\right] = \begin{bmatrix} 3 & -1 & 0 \\ -5 & 6 & 4 \end{bmatrix}$.

2. Since $T(\mathbf{d}_1) = 2\mathbf{b}_1 - 3\mathbf{b}_2, [T(\mathbf{d}_1)]_B = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$. Likewise $T(\mathbf{d}_2) = -4\mathbf{b}_1 + 5\mathbf{b}_2$ implies that $[T(\mathbf{d}_2)]_B = \begin{bmatrix} -4 \\ 5 \end{bmatrix}$. Thus the matrix for T relative to D and B is $\left[[T(\mathbf{d}_1)]_B [T(\mathbf{d}_2)]_B\right] = \begin{bmatrix} 2 & -4 \\ -3 & 5 \end{bmatrix}$.

3. **a**. $T(\mathbf{e}_1) = 0\mathbf{b}_1 - 1\mathbf{b}_2 + \mathbf{b}_3, T(\mathbf{e}_2) = -1\mathbf{b}_1 - 0\mathbf{b}_2 - 1\mathbf{b}_3, T(\mathbf{e}_3) = 1\mathbf{b}_1 - 1\mathbf{b}_2 + 0\mathbf{b}_3$

b. $[T(\mathbf{e}_1)]_B = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, [T(\mathbf{e}_2)]_B = \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix}, [T(\mathbf{e}_3)]_B = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

c. The matrix for T relative to $\mathcal{E}$ and B is $[\ [T(\mathbf{e}_1)]_B \quad [T(\mathbf{e}_2)]_B \quad [T(\mathbf{e}_3)]_B] = \begin{bmatrix} 0 & -1 & 1 \\ -1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix}$.

4. Let $\mathcal{E} = \{\mathbf{e}_1, \mathbf{e}_2\}$ be the standard basis for $\mathbb{R}^2$. Since $[T(\mathbf{b}_1)]_{\mathcal{E}} = T(\mathbf{b}_1) = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, [T(\mathbf{b}_2)]_{\mathcal{E}} = T(\mathbf{b}_2) = \begin{bmatrix} -4 \\ -1 \end{bmatrix}$, and $[T(\mathbf{b}_3)]_{\mathcal{E}} = T(\mathbf{b}_3) = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$, the matrix for T relative to B and $\mathcal{E}$ is $[[T(\mathbf{b}_1)]_{\mathcal{E}} \quad [T(\mathbf{b}_2)]_{\mathcal{E}} \quad [T(\mathbf{b}_3)]_{\mathcal{E}}] = \begin{bmatrix} 2 & -4 & 5 \\ 0 & -1 & 3 \end{bmatrix}$.

5. **a**. $T(\mathbf{p}) = (t+5)(2-t+t^2) = 10-3t+4t^2+t^3$

b. Let $\mathbf{p}$ and $\mathbf{q}$ be polynomials in $\mathbb{P}_2$, and let c be any scalar. Then

$$\begin{aligned} T(\mathbf{p}(t)+\mathbf{q}(t)) &= (t+5)[\mathbf{p}(t)+\mathbf{q}(t)] = (t+5)\mathbf{p}(t)+(t+5)\mathbf{q}(t) \\ &= T(\mathbf{p}(t))+T(\mathbf{q}(t)) \\ T(c\cdot\mathbf{p}(t)) &= (t+5)[c\cdot\mathbf{p}(t)] = c\cdot(t+5)\mathbf{p}(t) \\ &= c\cdot T[\mathbf{p}(t)] \end{aligned}$$

and T is a linear transformation.

c. Let $B = \{1, t, t^2\}$ and $C = \{1, t, t^2, t^3\}$. Since $T(\mathbf{b}_1) = T(1) = (t+5)(1) = t+5$, $[T(\mathbf{b}_1)]_C = \begin{bmatrix} 5 \\ 1 \\ 0 \\ 0 \end{bmatrix}$. Likewise since $T(\mathbf{b}_2) = T(t) = (t+5)(t) = t^2+5t$, $[T(\mathbf{b}_2)]_C = \begin{bmatrix} 0 \\ 5 \\ 1 \\ 0 \end{bmatrix}$, and since $T(\mathbf{b}_3) = T(t^2) = (t+5)(t^2) = t^3+5t^2$, $[T(\mathbf{b}_3)]_C = \begin{bmatrix} 0 \\ 0 \\ 5 \\ 1 \end{bmatrix}$. Thus the matrix for T relative to B and C is $[\ [T(\mathbf{b}_1)]_C \quad [T(\mathbf{b}_2)]_C \quad [T(\mathbf{b}_3)]_C\ ] = \begin{bmatrix} 5 & 0 & 0 \\ 1 & 5 & 0 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix}$.

6. **a**. $T(\mathbf{p}) = (2-t+t^2)+t^2(2-t+t^2) = 2-t+3t^2-t^3+t^4$

b. Let $\mathbf{p}$ and $\mathbf{q}$ be polynomials in $\mathbb{P}_2$, and let c be any scalar. Then

$$\begin{aligned} T(\mathbf{p}(t)+\mathbf{q}(t)) &= [\mathbf{p}(t)+\mathbf{q}(t)]+t^2[\mathbf{p}(t)+\mathbf{q}(t)] \\ &= [\mathbf{p}(t)+t^2\mathbf{p}(t)]+[\mathbf{q}(t)+t^2\mathbf{q}(t)] \\ &= T(\mathbf{p}(t))+T(\mathbf{q}(t)) \\ T(c\cdot\mathbf{p}(t)) &= [c\cdot\mathbf{p}(t)]+t^2[c\cdot\mathbf{p}(t)] \\ &= c\cdot[\mathbf{p}(t)+t^2\mathbf{p}(t)] \\ &= c\cdot T[\mathbf{p}(t)] \end{aligned}$$

and T is a linear transformation.

c. Let $B=\{1,t,t^2\}$ and $C=\{1,t,t^2,t^3,t^4\}$. Since $T(\mathbf{b}_1)=T(1)=1+t^2(1)=t^2+1, [T(\mathbf{b}_1)]_C=\begin{bmatrix}1\\0\\1\\0\\0\end{bmatrix}$.

Likewise since $T(\mathbf{b}_2)=T(t)=t+(t^2)(t)=t^3+t, [T(\mathbf{b}_2)]_C=\begin{bmatrix}0\\1\\0\\1\\0\end{bmatrix}$, and

since $T(\mathbf{b}_3)=T(t^2)=t^2+(t^2)(t^2)=t^4+t^2, [T(\mathbf{b}_3)]_C=\begin{bmatrix}0\\0\\1\\0\\1\end{bmatrix}$. Thus the matrix for T relative to

B and C is $[\ [T(\mathbf{b}_1)]_C \quad [T(\mathbf{b}_2)]_C \quad [T(\mathbf{b}_3)]_C\,]=\begin{bmatrix}1&0&0\\0&1&0\\1&0&1\\0&1&0\\0&0&1\end{bmatrix}$.

7. Since $T(\mathbf{b}_1)=T(1)=3+5t, [T(\mathbf{b}_1)]_B=\begin{bmatrix}3\\5\\0\end{bmatrix}$. Likewise since $T(\mathbf{b}_2)=T(t)=-2t+4t^2, [T(\mathbf{b}_2)]_B=\begin{bmatrix}0\\-2\\4\end{bmatrix}$,

and since $T(\mathbf{b}_3)=T(t^2)=t^2, [T(\mathbf{b}_3)]_B=\begin{bmatrix}0\\0\\1\end{bmatrix}$. Thus the matrix representation of T relative to the basis

B is $\big[[T(\mathbf{b}_1)]_B \quad [T(\mathbf{b}_2)]_B \quad [T(\mathbf{b}_3)]_B\big]=\begin{bmatrix}3&0&0\\5&-2&0\\0&4&1\end{bmatrix}$. Perhaps a faster way is to realize that the

information given provides the general form of $T(\mathbf{p})$ as shown in the figure below:

$$\begin{tikzcd}
a_0+a_1t+a_2t^2 \arrow[r, "T"] \arrow[d, "\text{coordinate mapping}"'] & 3a_0+(5a_0-2a_1)t+(4a_1+a_2)t^2 \arrow[d, "\text{coordinate mapping}"] \\
\begin{bmatrix}a_0\\a_1\\a_2\end{bmatrix} \arrow[r, "\text{multiplication}", "\text{by}[T]_B"'] & \begin{bmatrix}3a_0\\5a_0-2a_1\\4a_1+a_2\end{bmatrix}
\end{tikzcd}$$

The matrix that implements the multiplication along the bottom of the figure is easily filled in by inspection:

$$\begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix}\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 3a_0 \\ 5a_0 - 2a_1 \\ 4a_1 + a_2 \end{bmatrix} \text{ implies that } [T]_B = \begin{bmatrix} 3 & 0 & 0 \\ 5 & -2 & 0 \\ 0 & 4 & 1 \end{bmatrix}$$

8. Since $[3\mathbf{b}_1 - 4\mathbf{b}_2]_B = \begin{bmatrix} 3 \\ -4 \\ 0 \end{bmatrix}$, $[T(3\mathbf{b}_1 - 4\mathbf{b}_2)]_B = [T]_B[3\mathbf{b}_1 - 4\mathbf{b}_2]_B = \begin{bmatrix} 0 & -6 & 1 \\ 0 & 5 & -1 \\ 1 & -2 & 7 \end{bmatrix}\begin{bmatrix} 3 \\ -4 \\ 0 \end{bmatrix} = \begin{bmatrix} 24 \\ -20 \\ 11 \end{bmatrix}$

and $T(3\mathbf{b}_1 - 4\mathbf{b}_2) = 24\mathbf{b}_1 - 20\mathbf{b}_2 + 11\mathbf{b}_3$.

9. **a**. $T(\mathbf{p}) = \begin{bmatrix} 5+3(-1) \\ 5+3(0) \\ 5+3(1) \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix}$

b. Let $\mathbf{p}$ and $\mathbf{q}$ be polynomials in $\mathbb{P}_2$, and let c be any scalar. Then

$$T(\mathbf{p}+\mathbf{q}) = \begin{bmatrix} (\mathbf{p}+\mathbf{q})(-1) \\ (\mathbf{p}+\mathbf{q})(0) \\ (\mathbf{p}+\mathbf{q})(1) \end{bmatrix} = \begin{bmatrix} \mathbf{p}(-1)+\mathbf{q}(-1) \\ \mathbf{p}(0)+\mathbf{q}(0) \\ \mathbf{p}(1)+\mathbf{q}(1) \end{bmatrix} = \begin{bmatrix} \mathbf{p}(-1) \\ \mathbf{p}(0) \\ \mathbf{p}(1) \end{bmatrix} + \begin{bmatrix} \mathbf{q}(-1) \\ \mathbf{q}(0) \\ \mathbf{q}(1) \end{bmatrix} = T(\mathbf{p}) + T(\mathbf{q})$$

$$T(c\cdot\mathbf{p}) = \begin{bmatrix} (c\cdot\mathbf{p})(-1) \\ (c\cdot\mathbf{p})(0) \\ (c\cdot\mathbf{p})(1) \end{bmatrix} = \begin{bmatrix} c\cdot(\mathbf{p}(-1)) \\ c\cdot(\mathbf{p}(0)) \\ c\cdot(\mathbf{p}(1)) \end{bmatrix} = c\cdot\begin{bmatrix} \mathbf{p}(-1) \\ \mathbf{p}(0) \\ \mathbf{p}(1) \end{bmatrix} = c\cdot T(\mathbf{p})$$

and T is a linear transformation.

c. Let $B = \{1, t, t^2\}$ and $\mathcal{E} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be the standard basis for $\mathbb{R}^3$. Since

$[T(\mathbf{b}_1)]_{\mathcal{E}} = T(\mathbf{b}_1) = T(1) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $[T(\mathbf{b}_2)]_{\mathcal{E}} = T(\mathbf{b}_2) = T(t) = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, and $[T(\mathbf{b}_3)]_{\mathcal{E}} = T(\mathbf{b}_3) = T(t^2) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$,

the matrix for T relative to B and $\mathcal{E}$ is $\begin{bmatrix} [T(\mathbf{b}_1)]_{\mathcal{E}} & [T(\mathbf{b}_2)]_{\mathcal{E}} & [T(\mathbf{b}_3)]_{\mathcal{E}} \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$.

10. **a**. Let $\mathbf{p}$ and $\mathbf{q}$ be polynomials in $\mathbb{P}_3$, and let c be any scalar. Then

$$T(\mathbf{p}+\mathbf{q}) = \begin{bmatrix} (\mathbf{p}+\mathbf{q})(-3) \\ (\mathbf{p}+\mathbf{q})(-1) \\ (\mathbf{p}+\mathbf{q})(1) \\ (\mathbf{p}+\mathbf{q})(3) \end{bmatrix} = \begin{bmatrix} \mathbf{p}(-3)+\mathbf{q}(-3) \\ \mathbf{p}(-1)+\mathbf{q}(-1) \\ \mathbf{p}(1)+\mathbf{q}(1) \\ \mathbf{p}(3)+\mathbf{q}(3) \end{bmatrix} = \begin{bmatrix} \mathbf{p}(-3) \\ \mathbf{p}(-1) \\ \mathbf{p}(1) \\ \mathbf{p}(3) \end{bmatrix} + \begin{bmatrix} \mathbf{q}(-3) \\ \mathbf{q}(-1) \\ \mathbf{q}(1) \\ \mathbf{q}(3) \end{bmatrix} = T(\mathbf{p}) + T(\mathbf{q})$$

$$T(c\cdot\mathbf{p}) = \begin{bmatrix} (c\cdot\mathbf{p})(-3) \\ (c\cdot\mathbf{p})(-1) \\ (c\cdot\mathbf{p})(1) \\ (c\cdot\mathbf{p})(3) \end{bmatrix} = \begin{bmatrix} c\cdot(\mathbf{p}(-3)) \\ c\cdot(\mathbf{p}(-1)) \\ c\cdot(\mathbf{p}(1)) \\ c\cdot(\mathbf{p}(3)) \end{bmatrix} = c\cdot\begin{bmatrix} \mathbf{p}(-3) \\ \mathbf{p}(-1) \\ \mathbf{p}(1) \\ \mathbf{p}(3) \end{bmatrix} = c\cdot T(\mathbf{p})$$

and T is a linear transformation.

b. Let $B=\{1,t,t^2,t^3\}$ and $\mathcal{E}=\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3,\mathbf{e}_4\}$ be the standard basis for $\mathbb{R}^3$. Since

$$[T(\mathbf{b}_1)]_{\mathcal{E}}=T(\mathbf{b}_1)=T(1)=\begin{bmatrix}1\\1\\1\\1\end{bmatrix},\ [T(\mathbf{b}_2)]_{\mathcal{E}}=T(\mathbf{b}_2)=T(t)=\begin{bmatrix}-3\\-1\\1\\3\end{bmatrix},\ [T(\mathbf{b}_3)]_{\mathcal{E}}=T(\mathbf{b}_3)=T(t^2)=\begin{bmatrix}9\\1\\1\\9\end{bmatrix},\text{ and}$$

$$[T(\mathbf{b}_4)]_{\mathcal{E}}=T(\mathbf{b}_4)=T(t^3)=\begin{bmatrix}-27\\-1\\1\\27\end{bmatrix},$$ the matrix for T relative to B and $\mathcal{E}$ is

$$\begin{bmatrix}[T(\mathbf{b}_1)]_{\mathcal{E}} & [T(\mathbf{b}_2)]_{\mathcal{E}} & [T(\mathbf{b}_3)]_{\mathcal{E}} & [T(\mathbf{b}_4)]_{\mathcal{E}}\end{bmatrix}=\begin{bmatrix}1&-3&9&-27\\1&-1&1&-1\\1&1&1&1\\1&3&9&27\end{bmatrix}.$$

11. Following Example 4, if $P=\begin{bmatrix}\mathbf{b}_1 & \mathbf{b}_2\end{bmatrix}=\begin{bmatrix}2&1\\-1&2\end{bmatrix}$, then the B-matrix is

$$P^{-1}AP=\frac{1}{5}\begin{bmatrix}2&-1\\1&2\end{bmatrix}\begin{bmatrix}3&4\\-1&-1\end{bmatrix}\begin{bmatrix}2&1\\-1&2\end{bmatrix}=\begin{bmatrix}1&5\\0&1\end{bmatrix}$$

12. Following Example 4, if $P=\begin{bmatrix}\mathbf{b}_1 & \mathbf{b}_2\end{bmatrix}=\begin{bmatrix}3&-1\\2&1\end{bmatrix}$, then the B-matrix is

$$P^{-1}AP=\frac{1}{5}\begin{bmatrix}1&1\\-2&3\end{bmatrix}\begin{bmatrix}-1&4\\-2&3\end{bmatrix}\begin{bmatrix}3&-1\\2&1\end{bmatrix}=\begin{bmatrix}1&2\\-2&1\end{bmatrix}$$

13. Start by diagonalizing A. The characteristic polynomial is $\lambda^2-4\lambda+3=(\lambda-1)(\lambda-3)$, so the eigenvalues of A are 1 and 3.

<u>For $\lambda = 1$:</u> $A-I=\begin{bmatrix}-1&1\\-3&3\end{bmatrix}$. The equation $(A-I)\mathbf{x}=\mathbf{0}$ amounts to $-x_1+x_2=0$, so $x_1=x_2$ with x_2 free. A basis vector for the eigenspace is thus $\mathbf{v}_1=\begin{bmatrix}1\\1\end{bmatrix}$.

<u>For $\lambda = 3$:</u> $A-3I=\begin{bmatrix}-3&1\\-3&1\end{bmatrix}$. The equation $(A-3I)\mathbf{x}=\mathbf{0}$ amounts to $-3x_1+x_2=0$, so $x_1=(1/3)x_2$ with x_2 free. A nice basis vector for the eigenspace is thus $\mathbf{v}_2=\begin{bmatrix}1\\3\end{bmatrix}$.

From $\mathbf{v}_1$ and $\mathbf{v}_2$ we may construct $P=\begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2\end{bmatrix}=\begin{bmatrix}1&1\\1&3\end{bmatrix}$ which diagonalizes A. By Theorem 8, the basis $B=\{\mathbf{v}_1,\mathbf{v}_2\}$ has the property that the B-matrix of the transformation $\mathbf{x}\mapsto A\mathbf{x}$ is a diagonal matrix.

14. Start by diagonalizing A. The characteristic polynomial is $\lambda^2 - 6\lambda - 16 = (\lambda - 8)(\lambda + 2)$, so the eigenvalues of A are 8 and -2.

For $\lambda = 8$: $A - 8I = \begin{bmatrix} -3 & -3 \\ -7 & -7 \end{bmatrix}$. The equation $(A - 8I)\mathbf{x} = \mathbf{0}$ amounts to $x_1 + x_2 = 0$, so $x_1 = -x_2$ with x_2 free. A basis vector for the eigenspace is thus $\mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

For $\lambda = 2$: $A + 2I = \begin{bmatrix} 7 & -3 \\ -7 & 3 \end{bmatrix}$. The equation $(A - 2I)\mathbf{x} = \mathbf{0}$ amounts to $7x_1 - 3x_2 = 0$, so $x_1 = (3/7)x_2$ with x_2 free. A nice basis vector for the eigenspace is thus $\mathbf{v}_2 = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$.

From $\mathbf{v}_1$ and $\mathbf{v}_2$ we may construct $P = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} -1 & 3 \\ 1 & 7 \end{bmatrix}$ which diagonalizes A. By Theorem 8, the basis $B = \{\mathbf{v}_1, \mathbf{v}_2\}$ has the property that the B-matrix of the transformation $\mathbf{x} \mapsto A\mathbf{x}$ is a diagonal matrix.

15. Start by diagonalizing A. The characteristic polynomial is $\lambda^2 - 7\lambda + 10 = (\lambda - 5)(\lambda - 2)$, so the eigenvalues of A are 5 and 2.

For $\lambda = 5$: $A - 5I = \begin{bmatrix} -1 & -2 \\ -1 & -2 \end{bmatrix}$. The equation $(A - 5I)\mathbf{x} = \mathbf{0}$ amounts to $x_1 + 2x_2 = 0$, so $x_1 = -2x_2$ with x_2 free. A basis vector for the eigenspace is thus $\mathbf{v}_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

For $\lambda = 2$: $A - 2I = \begin{bmatrix} 2 & -2 \\ -1 & 1 \end{bmatrix}$. The equation $(A - 2I)\mathbf{x} = \mathbf{0}$ amounts to $x_1 - x_2 = 0$, so $x_1 = x_2$ with x_2 free. A basis vector for the eigenspace is thus $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

From $\mathbf{v}_1$ and $\mathbf{v}_2$ we may construct $P = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix}$ which diagonalizes A. By Theorem 8, the basis $B = \{\mathbf{v}_1, \mathbf{v}_2\}$ has the property that the B-matrix of the transformation $\mathbf{x} \mapsto A\mathbf{x}$ is a diagonal matrix.

16. Start by diagonalizing A. The characteristic polynomial is $\lambda^2 - 5\lambda = \lambda(\lambda - 5)$, so the eigenvalues of A are 5 and 0.

For $\lambda = 5$: $A - 5I = \begin{bmatrix} -3 & -6 \\ -1 & -2 \end{bmatrix}$. The equation $(A - 5I)\mathbf{x} = \mathbf{0}$ amounts to $x_1 + 2x_2 = 0$, so $x_1 = -2x_2$ with x_2 free. A basis vector for the eigenspace is thus $\mathbf{v}_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

For $\lambda = 0$: $A - 0I = \begin{bmatrix} 2 & -6 \\ -1 & 3 \end{bmatrix}$. The equation $(A - 0I)\mathbf{x} = \mathbf{0}$ amounts to $x_1 - 3x_2 = 0$, so $x_1 = 3x_2$ with x_2 free. A basis vector for the eigenspace is thus $\mathbf{v}_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

From $\mathbf{v}_1$ and $\mathbf{v}_2$ we may construct $P = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} -2 & 3 \\ 1 & 1 \end{bmatrix}$ which diagonalizes A. By Theorem 8, the basis $B = \{\mathbf{v}_1, \mathbf{v}_2\}$ has the property that the B-matrix of the transformation $\mathbf{x} \mapsto A\mathbf{x}$ is a diagonal matrix.

17. **a**. We compute that

$$A\mathbf{b}_1 = \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} = 2\mathbf{b}_1$$

so $\mathbf{b}_1$ is an eigenvector of A corresponding to the eigenvalue 2. The characteristic polynomial of A is $\lambda^2 - 4\lambda + 4 = (\lambda - 2)^2$, so 2 is the only eigenvalue for A. Now $A - 2I = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}$, which implies that the eigenspace corresponding to the eigenvalue 2 is one-dimensional. Thus the matrix A is not diagonalizable.

b. Following Example 4, if $P = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 \end{bmatrix}$, then the B-matrix for T is

$$P^{-1}AP = \begin{bmatrix} -4 & 5 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 5 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 0 & 2 \end{bmatrix}$$

18. If there is a basis B such that $[T]_B$ is diagonal, then A is similar to a diagonal matrix, by the second paragraph following Example 3. In this case, A would have three linearly independent eigenvectors. However, this is not necessarily the case, because A has only two distinct eigenvalues.

19. If A is similar to B, then there exists an invertible matrix P such that $P^{-1}AP = B.$ Thus B is invertible because it is the product of invertible matrices. By a theorem about inverses of products, $B^{-1} = P^{-1}A^{-1}(P^{-1})^{-1} = P^{-1}A^{-1}P,$ which shows that A^{-1} is similar to B^{-1}.

20. If $A = PBP^{-1}$, then $A^2 = (PBP^{-1})(PBP^{-1}) = PB(P^{-1}P)BP^{-1} = PB \cdot I \cdot BP^{-1} = PB^2P^{-1}$. So A^2 is similar to B^2.

21. By hypothesis, there exist invertible P and Q such that $P^{-1}BP = A$ and $Q^{-1}CQ = A$. Then $P^{-1}BP = Q^{-1}CQ.$ Left-multiply by Q and right-multiply by Q^{-1} to obtain $QP^{-1}BPQ^{-1} = QQ^{-1}CQQ^{-1}$. So $C = QP^{-1}BPQ^{-1} = (PQ^{-1})^{-1}B(PQ^{-1}),$ which shows that B is similar to C.

22. If A is diagonalizable, then $A = PDP^{-1}$ for some P. Also, if B is similar to A, then $B = QAQ^{-1}$ for some Q. Then $B = Q(PDP^{-1})Q^{-1} = (QP)D(P^{-1}Q^{-1}) = (QP)D(QP)^{-1}$
So B is diagonalizable.

23. If $A\mathbf{x} = \lambda\mathbf{x}, \mathbf{x} \neq 0,$ then $P^{-1}A\mathbf{x} = \lambda P^{-1}\mathbf{x}.$ If $B = P^{-1}AP,$ then

$$B(P^{-1}\mathbf{x}) = P^{-1}AP(P^{-1}\mathbf{x}) = P^{-1}A\mathbf{x} = \lambda P^{-1}\mathbf{x} \qquad (*)$$

by the first calculation. Note that $P^{-1}\mathbf{x} \neq 0,$ because $\mathbf{x} \neq 0$ and P^{-1} is invertible. Hence (*) shows that $P^{-1}\mathbf{x}$ is an eigenvector of B corresponding to λ. (Of course, λ is an eigenvalue of both A and B because the matrices are similar, by Theorem 4 in Section 5.2.)

24. If $A = PBP^{-1}$, then $\text{rank}\, A = \text{rank}\, P(BP^{-1}) = \text{rank}\, BP^{-1}$, by Supplementary Exercise 13 in Chapter 4. Also, $\text{rank}\, BP^{-1} = \text{rank}\, B,$ by Supplementary Exercise 14 in Chapter 4, since P^{-1} is invertible. Thus $\text{rank}\, A = \text{rank}\, B.$

25. If $A = PBP^{-1}$, then

$$\begin{aligned} \text{tr}(A) &= \text{tr}((PB)P^{-1}) = \text{tr}(P^{-1}(PB)) \qquad \text{By the trace property} \\ &= \text{tr}(P^{-1}PB) = \text{tr}(IB) = \text{tr}(B) \end{aligned}$$

If B is diagonal, then the diagonal entries of B must be the eigenvalues of A, by the Diagonalization Theorem (Theorem 5 in Section 5.3). So $\text{tr } A = \text{tr } B = \{\text{sum of the eigenvalues of } A\}$.

26. If $A = PDP^{-1}$ for some P, then the general trace property from Exercise 25 shows that $\text{tr } A = \text{tr }[(PD)P^{-1}] = \text{tr }[P^{-1}PD] = \text{tr } D.$ (Or, one can use the result of Exercise 25 that since A is similar to D, $\text{tr } A = \text{tr } D$.) Since the eigenvalues of A are on the main diagonal of D, tr D is the sum of the eigenvalues of A.

27. For each $j, I(\mathbf{b}_j) = \mathbf{b}_j$. Since the standard coordinate vector of any vector in $\mathbb{R}^n$ is just the vector itself, $[I(\mathbf{b}_j)]_{\mathcal{E}} = \mathbf{b}_j$. Thus the matrix for I relative to B and the standard basis $\mathcal{E}$ is simply $\begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \dots & \mathbf{b}_n \end{bmatrix}$. This matrix is precisely the *change-of-coordinates* matrix P_B defined in Section 4.4.

28. For each $j, I(\mathbf{b}_j) = \mathbf{b}_j$, and $[I(\mathbf{b}_j)]_C = [\mathbf{b}_j]_C$. By formula (4), the matrix for I relative to the bases B and C is

$$M = \begin{bmatrix} [\mathbf{b}_1]_C & [\mathbf{b}_2]_C & \dots & [\mathbf{b}_n]_C \end{bmatrix}$$

In Theorem 15 of Section 4.7, this matrix was denoted by $\underset{C\leftarrow B}{P}$ and was called the *change-of-coordinates matrix from* B *to* C.

29. If $B = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$, then the B-coordinate vector of $\mathbf{b}_j$ is $\mathbf{e}_j$, the standard basis vector for $\mathbb{R}^n$. For instance,

$$\mathbf{b}_1 = 1 \cdot \mathbf{b}_1 + 0 \cdot \mathbf{b}_2 + \dots + 0 \cdot \mathbf{b}_n$$

Thus $[I(\mathbf{b}_\text{j})]_B = [\mathbf{b}_\text{j}]_B = \mathbf{e}_\text{j}$, and

$$[I]_B = \begin{bmatrix} [I(\mathbf{b}_1)]_B \cdots [I(\mathbf{b}_n)]_B \end{bmatrix} = [\mathbf{e}_1 \cdots \mathbf{e}_n] = I$$

30. **[M]** If P is the matrix whose columns come from B, then the B-matrix of the transformation $\mathbf{x} \mapsto A\mathbf{x}$ is $D = P^{-1}AP$. From the data in the text,

$$A = \begin{bmatrix} -14 & 4 & -14 \\ -33 & 9 & -31 \\ 11 & -4 & 11 \end{bmatrix}, P = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 \end{bmatrix} = \begin{bmatrix} -1 & -1 & -1 \\ -2 & -1 & -2 \\ 1 & 1 & 0 \end{bmatrix},$$

$$D = \begin{bmatrix} 2 & -1 & 1 \\ -2 & 1 & 0 \\ -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} -14 & 4 & -14 \\ -33 & 9 & -31 \\ 11 & -4 & 11 \end{bmatrix} \begin{bmatrix} -1 & -1 & -1 \\ -2 & -1 & -2 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 8 & 3 & -6 \\ 0 & 1 & 3 \\ 0 & 0 & -3 \end{bmatrix}$$

31. **[M]** If P is the matrix whose columns come from B, then the B-matrix of the transformation $\mathbf{x} \mapsto A\mathbf{x}$ is $D = P^{-1}AP$. From the data in the text,

$$A = \begin{bmatrix} -7 & -48 & -16 \\ 1 & 14 & 6 \\ -3 & -45 & -19 \end{bmatrix}, P = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 \end{bmatrix} = \begin{bmatrix} -3 & -2 & 3 \\ 1 & 1 & -1 \\ -3 & -3 & 0 \end{bmatrix},$$

$$D = \begin{bmatrix} -1 & -3 & -1/3 \\ 1 & 3 & 0 \\ 0 & -1 & -1/3 \end{bmatrix} \begin{bmatrix} -7 & -48 & -16 \\ 1 & 14 & 6 \\ -3 & -45 & -19 \end{bmatrix} \begin{bmatrix} -3 & -2 & 3 \\ 1 & 1 & -1 \\ -3 & -3 & 0 \end{bmatrix} = \begin{bmatrix} -7 & -2 & -6 \\ 0 & -4 & -6 \\ 0 & 0 & -1 \end{bmatrix}$$

32. **[M]** $A = \begin{bmatrix} 15 & -66 & -44 & -33 \\ 0 & 13 & 21 & -15 \\ 1 & -15 & -21 & 12 \\ 2 & -18 & -22 & 8 \end{bmatrix},$

`ev=eig(A)=(2, 4, 4, 5)`

`nulbasis(A-ev(1)*eye(4))` $= \begin{bmatrix} 0.0000 \\ -1.5000 \\ 1.5000 \\ 1.0000 \end{bmatrix}$

A basis for the eigenspace of $\lambda = 2$ is $\mathbf{b}_1 = \begin{bmatrix} 0 \\ -3 \\ 3 \\ 2 \end{bmatrix}.$

`nulbasis(A-ev(2)*eye(4))` $= \begin{bmatrix} -10.0000 \\ -2.3333 \\ 1.0000 \\ 0 \end{bmatrix}, \begin{bmatrix} 13.0000 \\ 1.6667 \\ 0 \\ 1.0000 \end{bmatrix}$

A basis for the eigenspace of $\lambda = 4$ is $\{\mathbf{b}_2, \mathbf{b}_3\} = \left\{ \begin{bmatrix} -30 \\ -7 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 39 \\ 5 \\ 0 \\ 3 \end{bmatrix} \right\}.$

`nulbasis(A-ev(4)*eye(4))` $= \begin{bmatrix} 2.7500 \\ -0.7500 \\ 1.0000 \\ 1.0000 \end{bmatrix}$

A basis for the eigenspace of $\lambda = 5$ is $\mathbf{b}_4 = \begin{bmatrix} 11 \\ -3 \\ 4 \\ 4 \end{bmatrix}.$

The basis $B = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \mathbf{b}_4\}$ is a basis for $\mathbb{R}^4$ with the property that $[T]_B$ is diagonal.

Note: The *Study Guide* comments on Exercise 25 and tells students that the trace of *any* square matrix A equals the sum of the eigenvalues of A, counted according to multiplicities. This provides a quick check on the accuracy of an eigenvalue calculation. You could also refer students to the property of the determinant described in Exercise 19 of Section 5.2.

5.5 SOLUTIONS

1. $A=\begin{bmatrix}1 & -2\\ 1 & 3\end{bmatrix}, A-\lambda I=\begin{bmatrix}1-\lambda & -2\\ 1 & 3-\lambda\end{bmatrix}$

$$\det(A-\lambda I)=(1-\lambda)(3-\lambda)-(-2)=\lambda^2-4\lambda+5$$

Use the quadratic formula to find the eigenvalues: $\lambda=\dfrac{4\pm\sqrt{16-20}}{2}=2\pm i.$ Example 2 gives a shortcut for finding one eigenvector, and Example 5 shows how to write the other eigenvector with no effort.

<u>For $\lambda = 2 + i$:</u> $A-(2+i)I=\begin{bmatrix}-1-i & -2\\ 1 & 1-i\end{bmatrix}$. The equation $(A-\lambda I)\mathbf{x}=\mathbf{0}$ gives

$$\begin{aligned}(-1-i)x_1-2x_2&=0\\ x_1+(1-i)x_2&=0\end{aligned}$$

As in Example 2, the two equations are equivalent—each determines the same relation between x_1 and x_2. So use the second equation to obtain $x_1=-(1-i)x_2$, with x_2 free. The general solution is $x_2\begin{bmatrix}-1+i\\ 1\end{bmatrix}$, and the vector $\mathbf{v}_1=\begin{bmatrix}-1+i\\ 1\end{bmatrix}$ provides a basis for the eigenspace.

<u>For ~$\lambda = 2 - i$:</u> Let $\mathbf{v}_2=\overline{\mathbf{v}}_1=\begin{bmatrix}-1-i\\ 1\end{bmatrix}$. The remark prior to Example 5 shows that $\mathbf{v}_2$ is automatically an eigenvector for $\overline{2+i}$. In fact, calculations similar to those above would show that $\{\mathbf{v}_2\}$ is a basis for the eigenspace. (In general, for a real matrix A, it can be shown that the set of complex conjugates of the vectors in a basis of the eigenspace for λ is a basis of the eigenspace for $\overline{\lambda}$.)

2. $A=\begin{bmatrix}5 & -5\\ 1 & 1\end{bmatrix}$. The characteristic polynomial is $\lambda^2-6\lambda+10$, so the eigenvalues of A are $\lambda=\dfrac{6\pm\sqrt{36-40}}{2}=3\pm i.$

<u>For $\lambda = 3 + i$:</u> $A-(3+i)I=\begin{bmatrix}2-i & -5\\ 1 & -2-i\end{bmatrix}$. The equation $(A-(3+i)I)\mathbf{x}=\mathbf{0}$ amounts to $x_1+(-2-i)x_2=0$, so $x_1=(2+i)x_2$ with x_2 free. A basis vector for the eigenspace is thus $\mathbf{v}_1=\begin{bmatrix}2+i\\ 1\end{bmatrix}$.

<u>For $\lambda = 3 - i$:</u> A basis vector for the eigenspace is $\mathbf{v}_2=\overline{\mathbf{v}}_1=\begin{bmatrix}2-i\\ 1\end{bmatrix}$.

3. $A=\begin{bmatrix}1 & 5\\ -2 & 3\end{bmatrix}$. The characteristic polynomial is $\lambda^2-4\lambda+13$, so the eigenvalues of A are $\lambda=\dfrac{4\pm\sqrt{-36}}{2}=2\pm 3i.$

<u>For λ = 2 + 3*i*:</u> $A-(2+3i)I=\begin{bmatrix}-1-3i & 5\\ -2 & 1-3i\end{bmatrix}$. The equation $(A-(2+3i)I)\mathbf{x}=\mathbf{0}$ amounts to $-2x_1+(1-3i)x_2=0$, so $x_1=\dfrac{1-3i}{2}x_2$ with x_2 free. A nice basis vector for the eigenspace is thus $\mathbf{v}_1=\begin{bmatrix}1-3i\\ 2\end{bmatrix}$.

<u>For λ = 2 – 3*i*:</u> A basis vector for the eigenspace is $\mathbf{v}_2=\overline{\mathbf{v}}_1=\begin{bmatrix}1+3i\\ 2\end{bmatrix}$.

4. $A=\begin{bmatrix}5 & -2\\ 1 & 3\end{bmatrix}$. The characteristic polynomial is $\lambda^2-8\lambda+17$, so the eigenvalues of A are $\lambda=\dfrac{8\pm\sqrt{-4}}{2}=4\pm i.$

<u>For λ = 4 + *i*:</u> $A-(4+i)I=\begin{bmatrix}1-i & -2\\ 1 & -1-i\end{bmatrix}$. The equation $(A-(4+i)I)\mathbf{x}=\mathbf{0}$ amounts to $x_1+(-1-i)x_2=0$, so $x_1=(1+i)x_2$ with x_2 free. A basis vector for the eigenspace is thus $\mathbf{v}_1=\begin{bmatrix}1+i\\ 1\end{bmatrix}$.

<u>For λ = 4 – *i*:</u> A basis vector for the eigenspace is $\mathbf{v}_2=\overline{\mathbf{v}}_1=\begin{bmatrix}1-i\\ 1\end{bmatrix}$.

5. $A=\begin{bmatrix}0 & 1\\ -8 & 4\end{bmatrix}$. The characteristic polynomial is $\lambda^2-4\lambda+8$, so the eigenvalues of A are $\lambda=\dfrac{4\pm\sqrt{-16}}{2}=2\pm 2i.$

<u>For λ = 2 + 2*i*:</u> $A-(2+2i)I=\begin{bmatrix}-2-2i & 1\\ -8 & 2-2i\end{bmatrix}$. The equation $(A-(2+2i)I)\mathbf{x}=\mathbf{0}$ amounts to $(-2-2i)x_1+x_2=0$, so $x_2=(2+2i)x_1$ with x_1 free. A basis vector for the eigenspace is thus $\mathbf{v}_1=\begin{bmatrix}1\\ 2+2i\end{bmatrix}$.

<u>For λ = 2 – 2*i*:</u> A basis vector for the eigenspace is $\mathbf{v}_2=\overline{\mathbf{v}}_1=\begin{bmatrix}1\\ 2-2i\end{bmatrix}$.

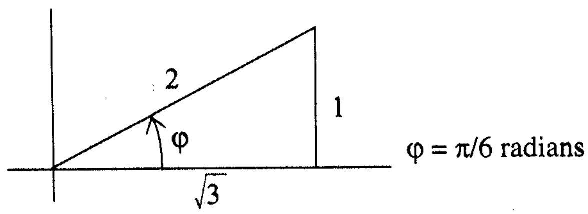
6. $A=\begin{bmatrix}4 & 3\\ -3 & 4\end{bmatrix}$. The characteristic polynomial is $\lambda^2-8\lambda+25$, so the eigenvalues of A are $\lambda=\dfrac{8\pm\sqrt{-36}}{2}=4\pm 3i.$

For $\lambda = 4 + 3i$: $A-(4+3i)I=\begin{bmatrix} -3i & 3 \\ -3 & -3i \end{bmatrix}$. The equation $(A-(4+3i)I)\mathbf{x}=\mathbf{0}$ amounts to $x_1+ix_2=0$, so $x_1=-ix_2$ with x_2 free. A basis vector for the eigenspace is thus $\mathbf{v}_1=\begin{bmatrix} -i \\ 1 \end{bmatrix}$.

For $\lambda = 4 - 3i$: A basis vector for the eigenspace is $\mathbf{v}_2=\overline{\mathbf{v}}_1=\begin{bmatrix} i \\ 1 \end{bmatrix}$.

7. $A=\begin{bmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{bmatrix}$. From Example 6, the eigenvalues are $\sqrt{3}\pm i$. The scale factor for the transformation $\mathbf{x}\mapsto A\mathbf{x}$ is $r=|\lambda|=\sqrt{(\sqrt{3})^2+1^2}=2$. For the angle of rotation, plot the point $(a,b)=(\sqrt{3},1)$ in the xy-plane and use trigonometry:

$$\varphi = \arctan(b/a) = \arctan(1/\sqrt{3}) = \pi/6 \text{ radians.}$$



Note: Your students will want to know whether you permit them on an exam to omit calculations for a matrix of the form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ and simply write the eigenvalues $a\pm bi$. A similar question may arise about the corresponding eigenvectors, $\begin{bmatrix} 1 \\ -i \end{bmatrix}$ and $\begin{bmatrix} 1 \\ i \end{bmatrix}$, which are announced in the Practice Problem. Students may have trouble keeping track of the correspondence between eigenvalues and eigenvectors.

8. $A=\begin{bmatrix} \sqrt{3} & 3 \\ -3 & \sqrt{3} \end{bmatrix}$. From Example 6, the eigenvalues are $\sqrt{3}\pm 3i$. The scale factor for the transformation $\mathbf{x}\mapsto A\mathbf{x}$ is $r=|\lambda|=\sqrt{(\sqrt{3})^2+3^2}=2\sqrt{3}$. From trigonometry, the angle of rotation φ is $\arctan(b/a)=$ $\arctan(-3/\sqrt{3})=-\pi/3$ radians.

9. $A=\begin{bmatrix} -\sqrt{3}/2 & 1/2 \\ -1/2 & -\sqrt{3}/2 \end{bmatrix}$. From Example 6, the eigenvalues are $-\sqrt{3}/2\pm(1/2)i$. The scale factor for the transformation $\mathbf{x}\mapsto A\mathbf{x}$ is $r=|\lambda|=\sqrt{(-\sqrt{3}/2)^2+(1/2)^2}=1$. From trigonometry, the angle of rotation φ is $\arctan(b/a)=$ $\arctan((-1/2)/(-\sqrt{3}/2))=-5\pi/6$ radians.

10. $A=\begin{bmatrix} -5 & -5 \\ 5 & -5 \end{bmatrix}$. From Example 6, the eigenvalues are $-5\pm 5i$. The scale factor for the transformation $\mathbf{x}\mapsto A\mathbf{x}$ is $r=|\lambda|=\sqrt{(-5)^2+5^2}=5\sqrt{2}$. From trigonometry, the angle of rotation φ is $\arctan(b/a)=\arctan(5/(-5))=3\pi/4$ radians.

11. $A=\begin{bmatrix} .1 & .1 \\ -.1 & .1 \end{bmatrix}$. From Example 6, the eigenvalues are $.1\pm .1i$. The scale factor for the transformation $\mathbf{x}\mapsto A\mathbf{x}$ is $r=|\lambda|=\sqrt{(.1)^2+(.1)^2}=\sqrt{2}/10$. From trigonometry, the angle of rotation φ is $\arctan(b/a)=\arctan(-.1/.1)=-\pi/4$ radians.

12. $A=\begin{bmatrix} 0 & .3 \\ -.3 & 0 \end{bmatrix}$. From Example 6, the eigenvalues are $0\pm .3i$. The scale factor for the transformation $\mathbf{x}\mapsto A\mathbf{x}$ is $r=|\lambda|=\sqrt{0^2+(.3)^2}=.3$. From trigonometry, the angle of rotation φ is $\arctan(b/a)=$ arctan $(-\infty)=-\pi/2$ radians.

13. From Exercise 1, $\lambda=2\pm i$, and the eigenvector $\mathbf{v}=\begin{bmatrix} -1-i \\ 1 \end{bmatrix}$ corresponds to $\lambda=2-i$. Since Re $\mathbf{v}=\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and Im $\mathbf{v}=\begin{bmatrix} -1 \\ 0 \end{bmatrix}$, take $P=\begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}$. Then compute

$$C=P^{-1}AP=\begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}=\begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} -3 & -1 \\ 2 & -1 \end{bmatrix}=\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$$

Actually, Theorem 9 gives the formula for *C*. Note that the eigenvector **v** corresponds to $a-bi$ instead of $a+bi$. If, for instance, you use the eigenvector for $2+i$, your *C* will be $\begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$.

Notes: The *Study Guide* points out that the matrix *C* is described in Theorem 9 and the first column of *C* is the real part of the eigenvector corresponding to $a-bi$, not $a+bi$, as one might expect. Since students may forget this, they are encouraged to compute *C* from the formula $C=P^{-1}AP$, as in the solution above.

The *Study Guide* also comments that because there are two possibilities for *C* in the factorization of a 2×2 matrix as in Exercise 13, the measure of rotation of the angle associated with the transformation $\mathbf{x}\mapsto A\mathbf{x}$ is determined only up to a change of sign. The "orientation" of the angle is determined by the change of variable $\mathbf{x}=P\mathbf{u}$. See Figure 4 in the text.

14. $A=\begin{bmatrix} 5 & -5 \\ 1 & 1 \end{bmatrix}$. From Exercise 2, the eigenvalues of *A* are $\lambda=3\pm i$, and the eigenvector $\mathbf{v}=\begin{bmatrix} 2-i \\ 1 \end{bmatrix}$ corresponds to $\lambda=3-i$. By Theorem 9, $P=[\text{Re }\mathbf{v}\ \ \text{Im }\mathbf{v}]=\begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}$ and

$$C=P^{-1}AP=\begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 5 & -5 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}=\begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix}$$

15. $A=\begin{bmatrix}1 & 5\\ -2 & 3\end{bmatrix}$. From Exercise 3, the eigenvalues of A are $\lambda = 2\pm 3i$, and the eigenvector $\mathbf{v}=\begin{bmatrix}1+3i\\ 2\end{bmatrix}$ corresponds to $\lambda = 2-3i$. By Theorem 9, $P=[\text{Re }\mathbf{v}\ \ \text{Im }\mathbf{v}]=\begin{bmatrix}1 & 3\\ 2 & 0\end{bmatrix}$ and

$$C=P^{-1}AP=\frac{1}{6}\begin{bmatrix}0 & -3\\ -2 & 1\end{bmatrix}\begin{bmatrix}1 & 5\\ -2 & 3\end{bmatrix}\begin{bmatrix}1 & 3\\ 2 & 0\end{bmatrix}=\begin{bmatrix}2 & -3\\ 3 & 2\end{bmatrix}$$

16. $A=\begin{bmatrix}5 & -2\\ 1 & 3\end{bmatrix}$. From Exercise 4, the eigenvalues of A are $\lambda = 4\pm i$, and the eigenvector $\mathbf{v}=\begin{bmatrix}1-i\\ 1\end{bmatrix}$ corresponds to $\lambda = 4-i$. By Theorem 9, $P=\begin{bmatrix}\text{Re }\mathbf{v} & \text{Im }\mathbf{v}\end{bmatrix}=\begin{bmatrix}1 & -1\\ 1 & 0\end{bmatrix}$ and

$$C=P^{-1}AP=\begin{bmatrix}0 & 1\\ -1 & 1\end{bmatrix}\begin{bmatrix}5 & -2\\ 1 & 3\end{bmatrix}\begin{bmatrix}1 & -1\\ 1 & 0\end{bmatrix}=\begin{bmatrix}4 & -1\\ 1 & 4\end{bmatrix}$$

17. $A=\begin{bmatrix}1 & -.8\\ 4 & -2.2\end{bmatrix}$. The characteristic polynomial is $\lambda^2+1.2\lambda+1$, so the eigenvalues of A are $\lambda=-.6\pm .8i$. To find an eigenvector corresponding to $-.6-.8i$, we compute

$$A-(-.6-.8i)I=\begin{bmatrix}1.6+.8i & -.8\\ 4 & -1.6+.8i\end{bmatrix}$$

The equation $(A-(-.6-.8i)I)\mathbf{x}=\mathbf{0}$ amounts to $4x_1+(-1.6+.8i)x_2=0$, so $x_1=((2-i)/5)x_2$ with x_2 free. A nice eigenvector corresponding to $-.6-.8i$ is thus $\mathbf{v}=\begin{bmatrix}2-i\\ 5\end{bmatrix}$. By Theorem 9,

$$P=\begin{bmatrix}\text{Re }\mathbf{v} & \text{Im }\mathbf{v}\end{bmatrix}=\begin{bmatrix}2 & -1\\ 5 & 0\end{bmatrix} \text{ and } C=P^{-1}AP=\frac{1}{5}\begin{bmatrix}0 & 1\\ -5 & 1\end{bmatrix}\begin{bmatrix}1 & -.8\\ 4 & -2.2\end{bmatrix}\begin{bmatrix}2 & -1\\ 5 & 0\end{bmatrix}=\begin{bmatrix}-.6 & -.8\\ .8 & -.6\end{bmatrix}$$

18. $A=\begin{bmatrix}1 & -1\\ .4 & .6\end{bmatrix}$. The characteristic polynomial is $\lambda^2-1.6\lambda+1$, so the eigenvalues of A are $\lambda=.8\pm .6i$. To find an eigenvector corresponding to $.8-.6i$, we compute

$$A-(.8-.6i)I=\begin{bmatrix}.2+.6i & -1\\ .4 & -.2+.6i\end{bmatrix}$$

The equation $(A-(.8-.6i)I)\mathbf{x}=\mathbf{0}$ amounts to $.4x_1+(-.2+.6i)x_2=0$, so $x_1=((1-3i)/2)x_2$ with x_2 free. A nice eigenvector corresponding to $.8-.6i$ is thus $\mathbf{v}=\begin{bmatrix}1-3i\\ 2\end{bmatrix}$. By Theorem 9,

$$P=\begin{bmatrix}\text{Re }\mathbf{v} & \text{Im }\mathbf{v}\end{bmatrix}=\begin{bmatrix}1 & -3\\ 2 & 0\end{bmatrix} \text{ and } C=P^{-1}AP=\frac{1}{6}\begin{bmatrix}0 & 3\\ -2 & 1\end{bmatrix}\begin{bmatrix}1 & -1\\ .4 & .6\end{bmatrix}\begin{bmatrix}1 & -3\\ 2 & 0\end{bmatrix}=\begin{bmatrix}.8 & -.6\\ .6 & .8\end{bmatrix}$$

19. $A=\begin{bmatrix}1.52 & -.7\\ .56 & .4\end{bmatrix}$. The characteristic polynomial is $\lambda^2-1.92\lambda+1$, so the eigenvalues of A are $\lambda=.96\pm.28i$. To find an eigenvector corresponding to $.96-.28i$, we compute

$$A-(.96-.28i)I=\begin{bmatrix}.56+.28i & -.7\\ .56 & -.56+.28i\end{bmatrix}$$

The equation $(A-(.96-.28i)I)\mathbf{x}=\mathbf{0}$ amounts to $.56x_1+(-.56+.28i)x_2=0$, so $x_1=((2-i)/2)x_2$ with x_2 free. A nice eigenvector corresponding to $.96-.28i$ is thus $\mathbf{v}=\begin{bmatrix}2-i\\ 2\end{bmatrix}$. By Theorem 9,

$$P=\begin{bmatrix}\operatorname{Re}\mathbf{v} & \operatorname{Im}\mathbf{v}\end{bmatrix}=\begin{bmatrix}2 & -1\\ 2 & 0\end{bmatrix} \text{ and } C=P^{-1}AP=\frac{1}{2}\begin{bmatrix}0 & 1\\ -2 & 2\end{bmatrix}\begin{bmatrix}1.52 & -.7\\ .56 & .4\end{bmatrix}\begin{bmatrix}2 & -1\\ 2 & 0\end{bmatrix}=\begin{bmatrix}.96 & -.28\\ .28 & .96\end{bmatrix}$$

20. $A=\begin{bmatrix}-1.64 & -2.4\\ 1.92 & 2.2\end{bmatrix}$. The characteristic polynomial is $\lambda^2-.56\lambda+1$, so the eigenvalues of A are $\lambda=.28\pm.96i$. To find an eigenvector corresponding to $.28-.96i$, we compute

$$A-(.28-.96i)I=\begin{bmatrix}-1.92+.96i & -2.4\\ 1.92 & 1.92+.96i\end{bmatrix}$$

The equation $(A-(.28-.96i)I)\mathbf{x}=\mathbf{0}$ amounts to $1.92x_1+(1.92+.96i)x_2=0$, so $x_1=((-2-i)/2)x_2$ with x_2 free. A nice eigenvector corresponding to $.28-.96i$ is thus $\mathbf{v}=\begin{bmatrix}-2-i\\ 2\end{bmatrix}$. By Theorem 9,

$$P=\begin{bmatrix}\operatorname{Re}\mathbf{v} & \operatorname{Im}\mathbf{v}\end{bmatrix}=\begin{bmatrix}-2 & -1\\ 2 & 0\end{bmatrix} \text{ and } C=P^{-1}AP=\frac{1}{2}\begin{bmatrix}0 & 1\\ -2 & -2\end{bmatrix}\begin{bmatrix}-1.64 & -2.4\\ 1.92 & 2.2\end{bmatrix}\begin{bmatrix}-2 & -1\\ 2 & 0\end{bmatrix}=\begin{bmatrix}.28 & -.96\\ .96 & .28\end{bmatrix}$$

21. The first equation in (2) is $(-.3+.6i)x_1-.6x_2=0$. We solve this for x_2 to find that $x_2=((-.3+.6i)/.6)x_1=((-1+2i)/2)x_1$. Letting $x_1=2$, we find that $\mathbf{y}=\begin{bmatrix}2\\ -1+2i\end{bmatrix}$ is an eigenvector for the matrix A. Since $\mathbf{y}=\begin{bmatrix}2\\ -1+2i\end{bmatrix}=\frac{-1+2i}{5}\begin{bmatrix}-2-4i\\ 5\end{bmatrix}=\frac{-1+2i}{5}\mathbf{v}_1$ the vector $\mathbf{y}$ is a complex multiple of the vector $\mathbf{v}_1$ used in Example 2.

22. Since $A(\mu\mathbf{x})=\mu(A\mathbf{x})=\mu(\lambda\mathbf{x})=\lambda(\mu\mathbf{x}), \mu\mathbf{x}$ is an eigenvector of A.

23. (a) properties of conjugates and the fact that $\overline{\mathbf{x}}^T=\overline{\mathbf{x}^T}$
(b) $\overline{A\mathbf{x}}=A\overline{\mathbf{x}}$ and A is real
(c) $\mathbf{x}^TA\overline{\mathbf{x}}$ is a scalar and hence may be viewed as a 1×1 matrix
(d) properties of transposes
(e) $A^T=A$ and the definition of q

24. $\overline{\mathbf{x}}^TA\mathbf{x}=\overline{\mathbf{x}}^T(\lambda\mathbf{x})=\lambda\cdot\overline{\mathbf{x}}^T\mathbf{x}$ because $\mathbf{x}$ is an eigenvector. It is easy to see that $\overline{\mathbf{x}}^T\mathbf{x}$ is real (and positive) because $\bar{z}z$ is nonnegative for every complex number z. Since $\overline{\mathbf{x}}^TA\mathbf{x}$ is real, by Exercise 23, so is λ. Next, write $\mathbf{x}=\mathbf{u}+i\mathbf{v}$, where $\mathbf{u}$ and $\mathbf{v}$ are real vectors. Then

$A\mathbf{x}=A(\mathbf{u}+i\mathbf{v})=A\mathbf{u}+iA\mathbf{v}$ and $\lambda\mathbf{x}=\lambda\mathbf{u}+i\lambda\mathbf{v}$

The real part of $A\mathbf{x}$ is $A\mathbf{u}$ because the entries in A, $\mathbf{u}$, and $\mathbf{v}$ are all real. The real part of $\lambda\mathbf{x}$ is $\lambda\mathbf{u}$ because λ and the entries in $\mathbf{u}$ and $\mathbf{v}$ are real. Since $A\mathbf{x}$ and $\lambda\mathbf{x}$ are equal, their real parts are equal, too. (Apply the corresponding statement about complex numbers to each entry of $A\mathbf{x}$.) Thus $A\mathbf{u} = \lambda\mathbf{u}$, which shows that the real part of $\mathbf{x}$ is an eigenvector of A.

25. Write $\mathbf{x} = \text{Re }\mathbf{x} + i(\text{Im }\mathbf{x})$, so that $A\mathbf{x} = A(\text{Re }\mathbf{x}) + iA(\text{Im }\mathbf{x})$. Since A is real, so are $A(\text{Re }\mathbf{x})$ and $A(\text{Im }\mathbf{x})$. Thus $A(\text{Re }\mathbf{x})$ is the real part of $A\mathbf{x}$ and $A(\text{Im }\mathbf{x})$ is the imaginary part of $A\mathbf{x}$.

26. **a**. If $\lambda = a - bi$, then

$$\begin{aligned} A\mathbf{v} &= \lambda\mathbf{v} = (a-bi)(\text{Re }\mathbf{v} + i \text{ Im }\mathbf{v}) \\ &= \underbrace{(a \text{ Re }\mathbf{v} + b \text{ Im }\mathbf{v})}_{\text{Re } Av} + i\underbrace{(a \text{ Im }\mathbf{v} - b \text{ Re }\mathbf{v})}_{\text{Im } Av} \end{aligned}$$

By Exercise 25,

$$A(\text{Re }\mathbf{v}) = \text{Re } A\mathbf{v} = a \text{ Re }\mathbf{v} + b \text{ Im }\mathbf{v}$$
$$A(\text{Im }\mathbf{v}) = \text{Im } A\mathbf{v} = -b \text{ Re }\mathbf{v} + a \text{ Im }\mathbf{v}$$

b. Let $P = \begin{bmatrix} \text{Re }\mathbf{v} & \text{Im }\mathbf{v} \end{bmatrix}$. By (a),

$$A(\text{Re }\mathbf{v}) = P\begin{bmatrix} a \\ b \end{bmatrix}, A(\text{Im }\mathbf{v}) = P\begin{bmatrix} -b \\ a \end{bmatrix}$$

So

$$\begin{aligned} AP &= \begin{bmatrix} A(\text{Re }\mathbf{v}) & A(\text{Im }\mathbf{v}) \end{bmatrix} \\ &= \begin{bmatrix} P\begin{bmatrix} a \\ b \end{bmatrix} & P\begin{bmatrix} -b \\ a \end{bmatrix} \end{bmatrix} = P\begin{bmatrix} a & -b \\ b & a \end{bmatrix} = PC \end{aligned}$$

27. **[M]** $A = \begin{bmatrix} .7 & 1.1 & 2.0 & 1.7 \\ -2.0 & -4.0 & -8.6 & -7.4 \\ 0 & -.5 & -1.0 & -1.0 \\ 1.0 & 2.8 & 6.0 & 5.3 \end{bmatrix}$

```
ev=eig(A)=(.2+.5i,.2-.5i,.3+.1i,.3-.1i)
```

For $\lambda = .2 - .5i$, an eigenvector is

```
nulbasis(A-ev(2)*eye(4)) =
 0.5000 - 0.5000i
-2.0000 + 0.0000i
 0.0000 - 0.0000i
 1.0000
```

so that $\mathbf{v}_1 = \begin{bmatrix} .5 - .5i \\ -2 \\ 0 \\ 1 \end{bmatrix}$

For $\lambda = .3 - .1i$, an eigenvector is

```
nulbasis(A-ev(4)*eye(4))=
-0.5000 - 0.0000i
 0.0000 + 0.5000i
```

```
-0.7500 - 0.2500i
 1.0000
```

so that $\mathbf{v}_2 = \begin{bmatrix} -.5 \\ .5i \\ -.75-.25i \\ 1 \end{bmatrix}$

Hence by Theorem 9, $P = \begin{bmatrix} \operatorname{Re}\mathbf{v}_1 & \operatorname{Im}\mathbf{v}_1 & \operatorname{Re}\mathbf{v}_2 & \operatorname{Im}\mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} .5 & -.5 & -.5 & 0 \\ -2 & 0 & 0 & .5 \\ 0 & 0 & -.75 & -.25 \\ 1 & 0 & 1 & 0 \end{bmatrix}$ and

$C = \begin{bmatrix} .2 & -.5 & 0 & 0 \\ .5 & .2 & 0 & 0 \\ 0 & 0 & .3 & -.1 \\ 0 & 0 & .1 & .3 \end{bmatrix}$. Other choices are possible, but C must equal $P^{-1}AP$.

28. [**M**] $A = \begin{bmatrix} -1.4 & -2.0 & -2.0 & -2.0 \\ -1.3 & -.8 & -.1 & -.6 \\ .3 & -1.9 & -1.6 & -1.4 \\ 2.0 & 3.3 & 2.3 & 2.6 \end{bmatrix}$

```
ev=eig(A)=(-.4+i,-.4-i,-.2+.5i,-.2-.5i)
```

For $\lambda = -.4 - i$, an eigenvector is

```
nulbasis(A-ev(2)*eye(4))=
-1.0000 - 1.0000i
-1.0000 + 1.0000i
 1.0000 - 1.0000i
 1.0000
```

so that $\mathbf{v}_1 = \begin{bmatrix} -1-i \\ -1+i \\ 1-i \\ 1 \end{bmatrix}$

For $\lambda = -.2 - .5i$, an eigenvector is

```
nulbasis(A-ev(4)*eye(4))=
 0.0000 - 0.0000i
-0.5000 - 0.5000i
-0.5000 + 0.5000i
 1.0000
```

so that $\mathbf{v}_2 = \begin{bmatrix} 0 \\ -1-i \\ -1+i \\ 2 \end{bmatrix}$

Hence by Theorem 9, $P=\begin{bmatrix}\text{Re }\mathbf{v}_1 & \text{Im }\mathbf{v}_1 & \text{Re }\mathbf{v}_2 & \text{Im }\mathbf{v}_2\end{bmatrix}=\begin{bmatrix}-1 & -1 & 0 & 0\\ -1 & 1 & -1 & -1\\ 1 & -1 & -1 & 1\\ 1 & 0 & 2 & 0\end{bmatrix}$ and

$C=\begin{bmatrix}-.4 & -1 & 0 & 0\\ 1 & -.4 & 0 & 0\\ 0 & 0 & -.2 & -.5\\ 0 & 0 & .5 & -.2\end{bmatrix}$. Other choices are possible, but C must equal $P^{-1}AP$.

5.6 SOLUTIONS

1. The exercise does not specify the matrix A, but only lists the eigenvalues 3 and 1/3, and the corresponding eigenvectors $\mathbf{v}_1=\begin{bmatrix}1\\1\end{bmatrix}$ and $\mathbf{v}_2=\begin{bmatrix}-1\\1\end{bmatrix}$. Also, $\mathbf{x}_0=\begin{bmatrix}9\\1\end{bmatrix}$.

a. To find the action of A on $\mathbf{x}_0$, express $\mathbf{x}_0$ in terms of $\mathbf{v}_1$ and $\mathbf{v}_2$. That is, find c_1 and c_2 such that $\mathbf{x}_0=c_1\mathbf{v}_1+c_2\mathbf{v}_2$. This is certainly possible because the eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent (by inspection and also because they correspond to distinct eigenvalues) and hence form a basis for $\mathbf{R}^2$. (Two linearly independent vectors in $\mathbf{R}^2$ automatically span $\mathbf{R}^2$.) The row reduction $\begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \mathbf{x}_0\end{bmatrix}=\begin{bmatrix}1 & -1 & 9\\ 1 & 1 & 1\end{bmatrix}\sim\begin{bmatrix}1 & 0 & 5\\ 0 & 1 & -4\end{bmatrix}$ shows that $\mathbf{x}_0=5\mathbf{v}_1-4\mathbf{v}_2$. Since $\mathbf{v}_1$ and $\mathbf{v}_2$ are eigenvectors (for the eigenvalues 3 and 1/3):

$$\mathbf{x}_1=A\mathbf{x}_0=5A\mathbf{v}_1-4A\mathbf{v}_2=5\cdot 3\mathbf{v}_1-4\cdot(1/3)\mathbf{v}_2=\begin{bmatrix}15\\15\end{bmatrix}-\begin{bmatrix}-4/3\\4/3\end{bmatrix}=\begin{bmatrix}49/3\\41/3\end{bmatrix}$$

b. Each time A acts on a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$, the $\mathbf{v}_1$ term is multiplied by the eigenvalue 3 and the $\mathbf{v}_2$ term is multiplied by the eigenvalue 1/3:

$$\mathbf{x}_2=A\mathbf{x}_1=A[5\cdot 3\mathbf{v}_1-4(1/3)\mathbf{v}_2]=5(3)^2\mathbf{v}_1-4(1/3)^2\mathbf{v}_2$$

In general, $\mathbf{x}_k=5(3)^k\mathbf{v}_1-4(1/3)^k\mathbf{v}_2$, for $k\ge 0$.

2. The vectors $\mathbf{v}_1=\begin{bmatrix}1\\0\\-3\end{bmatrix},\mathbf{v}_2=\begin{bmatrix}2\\1\\-5\end{bmatrix},\mathbf{v}_3=\begin{bmatrix}-3\\-3\\7\end{bmatrix}$ are eigenvectors of a 3×3 matrix A, corresponding to eigenvalues 3, 4/5, and 3/5, respectively. Also, $\mathbf{x}_0=\begin{bmatrix}-2\\-5\\3\end{bmatrix}$. To describe the solution of the equation $\mathbf{x}_{k+1}=A\mathbf{x}_k\,(k=1,2,...)$, first write $\mathbf{x}_0$ in terms of the eigenvectors.

$$\begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{x}_0\end{bmatrix}=\begin{bmatrix}1 & 2 & -3 & -2\\ 0 & 1 & -3 & -5\\ -3 & -5 & 7 & 3\end{bmatrix}\sim\begin{bmatrix}1 & 0 & 0 & 2\\ 0 & 1 & 0 & 1\\ 0 & 0 & 0 & 2\end{bmatrix}\Rightarrow\mathbf{x}_0=2\mathbf{v}_1+\mathbf{v}_2+2\mathbf{v}_3$$

Then, $\mathbf{x}_1 = A(2\mathbf{v}_1 + \mathbf{v}_2 + 2\mathbf{v}_3) = 2A\mathbf{v}_1 + A\mathbf{v}_2 + 2A\mathbf{v}_3 = 2\cdot 3\mathbf{v}_1 + (4/5)\mathbf{v}_2 + 2\cdot(3/5)\mathbf{v}_3$. In general, $\mathbf{x}_k = 2\cdot 3^k\,\mathbf{v}_1 + (4/5)^k\,\mathbf{v}_2 + 2\cdot(3/5)^k\,\mathbf{v}_3$. For all k sufficiently large,

$$\mathbf{x}_k \approx 2\cdot 3^k\,\mathbf{v}_1 = 2\cdot 3^k\begin{bmatrix}1\\0\\-3\end{bmatrix}$$

3. $A=\begin{bmatrix}.5 & .4\\-.2 & 1.1\end{bmatrix}$, $\det(A-\lambda I)=(.5-\lambda)(1.1-\lambda)+.08=\lambda^2-1.6\lambda+.63$. This characteristic polynomial factors as $(\lambda-.9)(\lambda-.7)$, so the eigenvalues are .9 and .7. If $\mathbf{v}_1$ and $\mathbf{v}_2$ denote corresponding eigenvectors, and if $\mathbf{x}_0 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2$, then

$$\mathbf{x}_1 = A(c_1\mathbf{v}_1 + c_2\mathbf{v}_2) = c_1A\mathbf{v}_1 + c_2A\mathbf{v}_2 = c_1(.9)\mathbf{v}_1 + c_2(.7)\mathbf{v}_2$$

and for $k \ge 1$,

$$\mathbf{x}_k = c_1(.9)^k\,\mathbf{v}_1 + c_2(.7)^k\,\mathbf{v}_2$$

For any choices of c_1 and c_2, both the owl and wood rat populations decline over time.

4. $A=\begin{bmatrix}.5 & .4\\-.125 & 1.1\end{bmatrix}$, $\det(A-\lambda I)=(.5-\lambda)(1.1-\lambda)-(.4)(.125)=\lambda^2-1.6\lambda+.6$. This characteristic polynomial factors as $(\lambda-1)(\lambda-.6)$, so the eigenvalues are 1 and .6. For the eigenvalue 1, solve $(A-I)\mathbf{x}=0:\begin{bmatrix}-.5 & .4 & 0\\-.125 & .1 & 0\end{bmatrix}\sim\begin{bmatrix}-5 & 4 & 0\\0 & 0 & 0\end{bmatrix}$. A basis for the eigenspace is $\mathbf{v}_1=\begin{bmatrix}4\\5\end{bmatrix}$. Let $\mathbf{v}_2$ be an eigenvector for the eigenvalue .6. (The entries in $\mathbf{v}_2$ are not important for the long-term behavior of the system.) If $\mathbf{x}_0 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2$, then $\mathbf{x}_1 = c_1A\mathbf{v}_1 + c_2A\mathbf{v}_2 = c_1\mathbf{v}_1 + c_2(.6)\mathbf{v}_2$, and for k sufficiently large,

$$\mathbf{x}_k = c_1\begin{bmatrix}4\\5\end{bmatrix} + c_2(.6)^k\,\mathbf{v}_2 \approx c_1\begin{bmatrix}4\\5\end{bmatrix}$$

Provided that $c_1 \ne 0$, the owl and wood rat populations each stabilize in size, and eventually the populations are in the ratio of 4 owls for each 5 thousand rats. If some aspect of the model were to change slightly, the characteristic equation would change slightly and the perturbed matrix A might not have 1 as an eigenvalue. If the eigenvalue becomes slightly large than 1, the two populations will grow; if the eigenvalue becomes slightly less than 1, both populations will decline.

5. $A=\begin{bmatrix}.4 & .3\\-.325 & 1.2\end{bmatrix}$, $\det(A-\lambda I)=\lambda^2-1.6\lambda+.5775$. The quadratic formula provides the roots of the characteristic equation:

$$\lambda=\frac{1.6\pm\sqrt{1.6^2-4(.5775)}}{2}=\frac{1.6\pm\sqrt{.25}}{2}=1.05 \text{ and } .55$$

Because one eigenvalue is larger than one, both populations grow in size. Their relative sizes are determined eventually by the entries in the eigenvector corresponding to 1.05. Solve $(A-1.05I)\mathbf{x}=\mathbf{0}$:

$$\begin{bmatrix}-.65 & .3 & 0\\-.325 & .15 & 0\end{bmatrix}\sim\begin{bmatrix}-13 & 6 & 0\\0 & 0 & 0\end{bmatrix}. \text{ An eigenvector is } \mathbf{v}_1=\begin{bmatrix}6\\13\end{bmatrix}.$$

Eventually, there will be about 6 spotted owls for every 13 (thousand) flying squirrels.

6. When $p=.5$, $A=\begin{bmatrix} .4 & .3 \\ -.5 & 1.2 \end{bmatrix}$, and $\det(A-\lambda I)=\lambda^2-1.6\lambda+.63=(\lambda-.9)(\lambda-.7)$.

The eigenvalues of A are .9 and .7, both less than 1 in magnitude. The origin is an attractor for the dynamical system and each trajectory tends toward **0**. So both populations of owls and squirrels eventually perish.
The calculations in Exercise 4 (as well as those in Exercises 35 and 27 in Section 5.1) show that if the largest eigenvalue of A is 1, then in most cases the population vector $\mathbf{x}_k$ will tend toward a multiple of the eigenvector corresponding to the eigenvalue 1. [If $\mathbf{v}_1$ and $\mathbf{v}_2$ are eigenvectors, with $\mathbf{v}_1$ corresponding to $\lambda=1$, and if $\mathbf{x}_0=c_1\mathbf{v}_1+c_2\mathbf{v}_2$, then $\mathbf{x}_k$ tends toward $c_1\mathbf{v}_1$, provided c_1 is not zero.] So the problem here is to determine the value of the predation parameter p such that the largest eigenvalue of A is 1. Compute the characteristic polynomial:

$$\det\begin{bmatrix} .4-\lambda & .3 \\ -p & 1.2-\lambda \end{bmatrix}=(.4-\lambda)(1.2-\lambda)+.3p=\lambda^2-1.6\lambda+(.48+.3p)$$

By the quadratic formula,

$$\lambda=\frac{1.6\pm\sqrt{1.6^2-4(.48+.3p)}}{2}$$

The larger eigenvalue is 1 when

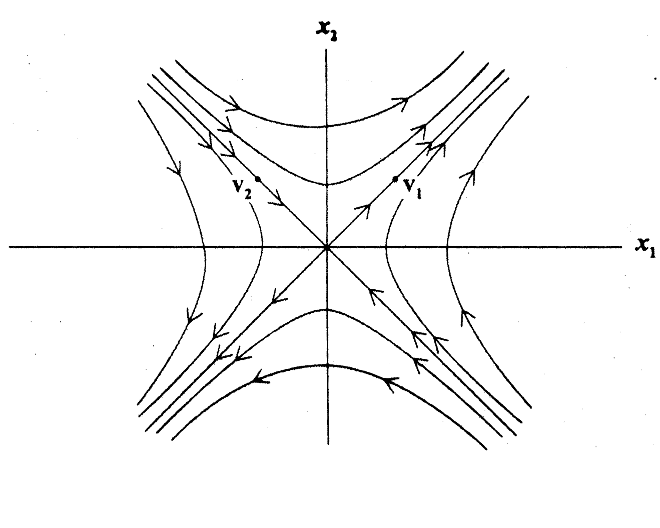
$$1.6+\sqrt{1.6^2-4(.48+.3p)}=2 \text{ and } \sqrt{2.56-1.92-1.2p}=.4$$

In this case, $.64-1.2p=.16$, and $p=.4$.

7. **a**. The matrix A in Exercise 1 has eigenvalues 3 and 1/3. Since $|3|>1$ and $|1/3|<1$, the origin is a saddle point.

b. The direction of greatest attraction is determined by $\mathbf{v}_2=\begin{bmatrix} -1 \\ 1 \end{bmatrix}$, the eigenvector corresponding to the eigenvalue with absolute value less than 1. The direction of greatest repulsion is determined by $\mathbf{v}_1=\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, the eigenvector corresponding to the eigenvalue greater than 1.

c. The drawing below shows: (1) lines through the eigenvectors and the origin, (2) arrows toward the origin (showing attraction) on the line through $\mathbf{v}_2$ and arrows away from the origin (showing repulsion) on the line through $\mathbf{v}_1$, (3) several typical trajectories (with arrows) that show the general flow of points. No specific points other than $\mathbf{v}_1$ and $\mathbf{v}_2$ were computed. This type of drawing is about all that one can make without using a computer to plot points.



Note: If you wish your class to sketch trajectories for anything except saddle points, you will need to go beyond the discussion in the text. The following remarks from the *Study Guide* are relevant.

Sketching trajectories for a dynamical system in which the origin is an attractor or a repellor is more difficult than the sketch in Exercise 7. There has been no discussion of the direction in which the trajectories "bend" as they move toward or away from the origin. For instance, if you rotate Figure 1 of Section 5.6 through a quarter-turn and relabel the axes so that x_1 is on the horizontal axis, then the new figure corresponds to the matrix A with the diagonal entries .8 and .64 interchanged. In general, if A is a diagonal matrix, with positive diagonal entries a and d, unequal to 1, then the trajectories lie on the axes or on curves whose equations have the form $x_2 = r(x_1)^s$, where $s = (\ln d)/(\ln a)$ and r depends on the initial point $\mathbf{x}_0$. (See *Encounters with Chaos*, by Denny Gulick, New York: McGraw-Hill, 1992, pp. 147–150.)

8. The matrix from Exercise 2 has eigenvalues 3, 4/5, and 3/5. Since one eigenvalue is greater than 1 and the others are less than one in magnitude, the origin is a saddle point. The direction of greatest repulsion is the line through the origin and the eigenvector $(1, 0, -3)$ for the eigenvalue 3. The direction of greatest attraction is the line through the origin and the eigenvector $(-3, -3, 7)$ for the smallest eigenvalue 3/5.

9. $A = \begin{bmatrix} 1.7 & -.3 \\ -1.2 & .8 \end{bmatrix}$, $\det(A - \lambda I) = \lambda^2 - 2.5\lambda + 1 = 0$

$$\lambda = \frac{2.5 \pm \sqrt{2.5^2 - 4(1)}}{2} = \frac{2.5 \pm \sqrt{2.25}}{2} = \frac{2.5 \pm 1.5}{2} = 2 \text{ and } .5$$

The origin is a saddle point because one eigenvalue is greater than 1 and the other eigenvalue is less than 1 in magnitude. The direction of greatest repulsion is through the origin and the eigenvector $\mathbf{v}_1$ found below. Solve $(A - 2I)\mathbf{x} = \mathbf{0}$: $\begin{bmatrix} -.3 & -.3 & 0 \\ -1.2 & -1.2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, so $x_1 = -x_2$, and x_2 is free. Take $\mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

The direction of greatest attraction is through the origin and the eigenvector $\mathbf{v}_2$ found below. Solve $(A - .5I)\mathbf{x} = \mathbf{0}$: $\begin{bmatrix} 1.2 & -.3 & 0 \\ -1.2 & .3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -.25 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, so $x_1 = -.25x_2$, and x_2 is free. Take $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$.

10. $A = \begin{bmatrix} .3 & .4 \\ -.3 & 1.1 \end{bmatrix}$, $\det(A - \lambda I) = \lambda^2 - 1.4\lambda + .45 = 0$

$$\lambda = \frac{1.4 \pm \sqrt{1.4^2 - 4(.45)}}{2} = \frac{1.4 \pm \sqrt{.16}}{2} = \frac{1.4 \pm .4}{2} = .5 \text{ and } .9$$

The origin is an attractor because both eigenvalues are less than 1 in magnitude. The direction of greatest attraction is through the origin and the eigenvector $\mathbf{v}_1$ found below. Solve $(A - .5I)\mathbf{x} = \mathbf{0}$: $\begin{bmatrix} -.2 & .4 & 0 \\ -.3 & .6 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, so $x_1 = 2x_2$, and x_2 is free. Take $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

11. $A = \begin{bmatrix} .4 & .5 \\ -.4 & 1.3 \end{bmatrix}$, $\det(A - \lambda I) = \lambda^2 - 1.7\lambda + .72 = 0$

$$\lambda = \frac{1.7 \pm \sqrt{1.7^2 - 4(.72)}}{2} = \frac{1.7 \pm \sqrt{.01}}{2} = \frac{1.7 \pm .1}{2} = .8 \text{ and } .9$$

The origin is an attractor because both eigenvalues are less than 1 in magnitude. The direction of greatest attraction is through the origin and the eigenvector $\mathbf{v}_1$ found below. Solve $(A - .8I)\mathbf{x} = \mathbf{0}$: $\begin{bmatrix} -.4 & .5 & 0 \\ -.4 & .5 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1.25 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, so $x_1 = 1.25x_2$, and x_2 is free. Take $\mathbf{v}_1 = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$.

12. $A=\begin{bmatrix} .5 & .6 \\ -.3 & 1.4 \end{bmatrix}$, $\det(A-\lambda I)=\lambda^2-1.9\lambda+.88=0$

$$\lambda=\frac{1.9\pm\sqrt{1.9^2-4(.88)}}{2}=\frac{1.9\pm\sqrt{.09}}{2}=\frac{1.9\pm .3}{2}=.8 \text{ and } 1.1$$

The origin is a saddle point because one eigenvalue is greater than 1 and the other eigenvalue is less than 1 in magnitude. The direction of greatest repulsion is through the origin and the eigenvector $\mathbf{v}_1$ found below. Solve $(A-1.1I)\mathbf{x}=\mathbf{0}$: $\begin{bmatrix} -.6 & .6 & 0 \\ -.3 & .3 & 0 \end{bmatrix}\sim\begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, so $x_1=x_2$, and x_2 is free. Take $\mathbf{v}_1=\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

The direction of greatest attraction is through the origin and the eigenvector $\mathbf{v}_2$ found below. Solve $(A-.8I)\mathbf{x}=\mathbf{0}$: $\begin{bmatrix} -.3 & .6 & 0 \\ -.3 & .6 & 0 \end{bmatrix}\sim\begin{bmatrix} 1 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, so $x_1=2x_2$, and x_2 is free. Take $\mathbf{v}_2=\begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

13. $A=\begin{bmatrix} .8 & .3 \\ -.4 & 1.5 \end{bmatrix}$, $\det(A-\lambda I)=\lambda^2-2.3\lambda+1.32=0$

$$\lambda=\frac{2.3\pm\sqrt{2.3^2-4(1.32)}}{2}=\frac{2.3\pm\sqrt{.01}}{2}=\frac{2.3\pm .1}{2}=1.1 \text{ and } 1.2$$

The origin is a repellor because both eigenvalues are greater than 1 in magnitude. The direction of greatest repulsion is through the origin and the eigenvector $\mathbf{v}_1$ found below. Solve $(A-1.2I)\mathbf{x}=\mathbf{0}$: $\begin{bmatrix} -.4 & .3 & 0 \\ -.4 & .3 & 0 \end{bmatrix}\sim\begin{bmatrix} 1 & -.75 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, so $x_1=.75x_2$, and x_2 is free. Take $\mathbf{v}_1=\begin{bmatrix} 3 \\ 4 \end{bmatrix}$.

14. $A=\begin{bmatrix} 1.7 & .6 \\ -.4 & .7 \end{bmatrix}$, $\det(A-\lambda I)=\lambda^2-2.4\lambda+1.43=0$

$$\lambda=\frac{2.4\pm\sqrt{2.4^2-4(1.43)}}{2}=\frac{2.4\pm\sqrt{.04}}{2}=\frac{2.4\pm .2}{2}=1.1 \text{ and } 1.3$$

The origin is a repellor because both eigenvalues are greater than 1 in magnitude. The direction of greatest repulsion is through the origin and the eigenvector $\mathbf{v}_1$ found below. Solve $(A-1.3I)\mathbf{x}=\mathbf{0}$: $\begin{bmatrix} .4 & .6 & 0 \\ -.4 & -.6 & 0 \end{bmatrix}\sim\begin{bmatrix} 1 & 1.5 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, so $x_1=-1.5x_2$, and x_2 is free. Take $\mathbf{v}_1=\begin{bmatrix} -3 \\ 2 \end{bmatrix}$.

15. $A=\begin{bmatrix} .4 & 0 & .2 \\ .3 & .8 & .3 \\ .3 & .2 & .5 \end{bmatrix}$. Given eigenvector $\mathbf{v}_1=\begin{bmatrix} .1 \\ .6 \\ .3 \end{bmatrix}$ and eigenvalues .5 and .2. To find the eigenvalue for $\mathbf{v}_1$, compute

$$A\mathbf{v}_1=\begin{bmatrix} .4 & 0 & .2 \\ .3 & .8 & .3 \\ .3 & .2 & .5 \end{bmatrix}\begin{bmatrix} .1 \\ .6 \\ .3 \end{bmatrix}=\begin{bmatrix} .1 \\ .6 \\ .3 \end{bmatrix}=1\cdot\mathbf{v}_1 \quad \text{Thus } \mathbf{v}_1 \text{ is an eigenvector for } \lambda=1.$$

For $\lambda=.5$: $\begin{bmatrix} -.1 & 0 & .2 & 0 \\ .3 & .3 & .3 & 0 \\ .3 & .2 & 0 & 0 \end{bmatrix}\sim\begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, $\begin{matrix} x_1=2x_3 \\ x_2=-3x_3 \\ x_3 \text{ is free} \end{matrix}$. Set $\mathbf{v}_2=\begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$.

For $\lambda = .2$: $\begin{bmatrix} .2 & 0 & .2 & 0 \\ .3 & .6 & .3 & 0 \\ .3 & .2 & .3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, $\begin{matrix} x_1 = -x_3 \\ x_2 = 0 \\ x_3 \text{ is free} \end{matrix}$. Set $\mathbf{v}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

Given $\mathbf{x}_0 = (0, .3, .7)$, find weights such that $\mathbf{x}_0 = c_1\mathbf{v}_1 + c\mathbf{v}_2 + c_3\mathbf{v}_3$.

$$\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{x}_0 \end{bmatrix} = \begin{bmatrix} .1 & 2 & -1 & 0 \\ .6 & -3 & 0 & .3 \\ .3 & 1 & 1 & .7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & .1 \\ 0 & 0 & 0 & .3 \end{bmatrix}.$$

$\mathbf{x}_0 = \mathbf{v}_1 + .1\mathbf{v}_2 + .3\mathbf{v}_3$

$\mathbf{x}_1 = A\mathbf{v}_1 + .1A\mathbf{v}_2 + .3A\mathbf{v}_3 = \mathbf{v}_1 + .1(.5)\mathbf{v}_2 + .3(.2)\mathbf{v}_3$, and

$\mathbf{x}_k = \mathbf{v}_1 + .1(.5)^k\mathbf{v}_2 + .3(.2)^k\mathbf{v}_3$. As k increases, $\mathbf{x}_k$ approaches $\mathbf{v}_1$.

16. **[M]**

$$A = \begin{bmatrix} .90 & .01 & .09 \\ .01 & .90 & .01 \\ .09 & .09 & .90 \end{bmatrix} \cdot \texttt{ev = eig(A)} = \begin{bmatrix} 1.0000 \\ 0.8900 \\ .8100 \end{bmatrix}. \text{ To four decimal places,}$$

$$\texttt{v}_1 = \texttt{nulbasis(A-eye(3))} = \begin{bmatrix} 0.9192 \\ 0.1919 \\ 1.0000 \end{bmatrix}. \text{ Exact: } \begin{bmatrix} 91/99 \\ 19/99 \\ 1 \end{bmatrix}$$

$$\texttt{v}_2 = \texttt{nulbasis(A-ev(2)*eye(3))} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$$\texttt{v}_3 = \texttt{nulbasis(A-ev(3)*eye(3))} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

The general solution of the dynamical system is $\mathbf{x}_k = c_1\mathbf{v}_1 + c_2(.89)^k\mathbf{v}_2 + c_3(.81)^k\mathbf{v}_3$.

Note: When working with stochastic matrices and starting with a probability vector (having nonnegative entries whose sum is 1), it helps to scale $\mathbf{v}_1$ to make its entries sum to 1. If $\mathbf{v}_1 = (91/209, 19/209, 99/209)$, or $(.435, .091, .474)$ to three decimal places, then the weight c_1 above turns out to be 1. See the text's discussion of Exercise 27 in Section 5.2.

17. **a**. $A = \begin{bmatrix} 0 & 1.6 \\ .3 & .8 \end{bmatrix}$

b. $\det\begin{bmatrix} -\lambda & 1.6 \\ .3 & .8-\lambda \end{bmatrix} = \lambda^2 - .8\lambda - .48 = 0$. The eigenvalues of A are given by

$$\lambda = \frac{.8 \pm \sqrt{(-.8)^2 - 4(-.48)}}{2} = \frac{.8 \pm \sqrt{2.56}}{2} = \frac{.8 \pm 1.6}{2} = 1.2 \text{ and } -.4$$

The numbers of juveniles and adults are increasing because the largest eigenvalue is greater than 1. The eventual growth rate of each age class is 1.2, which is 20% per year.

To find the eventual relative population sizes, solve $(A-1.2I)\mathbf{x}=\mathbf{0}$:

$$\begin{bmatrix} -1.2 & 1.6 & 0 \\ .3 & -.4 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -4/3 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \quad \begin{matrix} x_1 = (4/3)x_2 \\ x_2 \text{ is free} \end{matrix}. \quad \text{Set } \mathbf{v}_1 = \begin{bmatrix} 4 \\ 3 \end{bmatrix}.$$

Eventually, there will be about 4 juveniles for every 3 adults.

c. **[M]** Suppose that the initial populations are given by $\mathbf{x}_0 = (15, 10)$. The *Study Guide* describes how to generate the trajectory for as many years as desired and then to plot the values for each population. Let $\mathbf{x}_k = (\mathrm{j}_k, \mathrm{a}_k)$. Then we need to plot the sequences $\{\mathrm{j}_k\}, \{\mathrm{a}_k\}, \{\mathrm{j}_k + \mathrm{a}_k\}$, and $\{\mathrm{j}_k/\mathrm{a}_k\}$. Adjacent points in a sequence can be connected with a line segment. When a sequence is plotted, the resulting graph can be captured on the screen and printed (if done on a computer) or copied by hand onto paper (if working with a graphics calculator).

18. **a.** $A = \begin{bmatrix} 0 & 0 & .42 \\ .6 & 0 & 0 \\ 0 & .75 & .95 \end{bmatrix}$

b. `ev = eig(A) =` $\begin{bmatrix} 0.0774 + 0.4063i \\ 0.0774 - 0.4063i \\ 1.1048 \end{bmatrix}$

The long-term growth rate is 1.105, about 10.5 % per year.

`v = nulbasis(A - ev(3)*eye(3)) =` $\begin{bmatrix} 0.3801 \\ 0.2064 \\ 1.0000 \end{bmatrix}$

For each 100 adults, there will be approximately 38 calves and 21 yearlings.

Note: The MATLAB box in the *Study Guide* and the various technology appendices all give directions for generating the sequence of points in a trajectory of a dynamical system. Details for producing a graphical representation of a trajectory are also given, with several options available in MATLAB, Maple, and Mathematica.

5.7 SOLUTIONS

1. From the "eigendata" (eigenvalues and corresponding eigenvectors) given, the eigenfunctions for the differential equation $\mathbf{x}' = A\mathbf{x}$ are $\mathbf{v}_1 e^{4t}$ and $\mathbf{v}_2 e^{2t}$. The general solution of $\mathbf{x}' = A\mathbf{x}$ has the form

$$c_1 \begin{bmatrix} -3 \\ 1 \end{bmatrix} e^{4t} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{2t}$$

The initial condition $\mathbf{x}(0) = \begin{bmatrix} -6 \\ 1 \end{bmatrix}$ determines c_1 and c_2:

$$c_1 \begin{bmatrix} -3 \\ 1 \end{bmatrix} e^{4(0)} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{2(0)} = \begin{bmatrix} -6 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} -3 & -1 & -6 \\ 1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 5/2 \\ 0 & 1 & -3/2 \end{bmatrix}$$

Thus $c_1 = 5/2, c_2 = -3/2$, and $\mathbf{x}(t) = \frac{5}{2}\begin{bmatrix} -3 \\ 1 \end{bmatrix} e^{4t} - \frac{3}{2}\begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{2t}$.

2. From the eigendata given, the eigenfunctions for the differential equation $\mathbf{x}' = A\mathbf{x}$ are $\mathbf{v}_1 e^{-3t}$ and $\mathbf{v}_2 e^{-1t}$. The general solution of $\mathbf{x}' = A\mathbf{x}$ has the form

$$c_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-3t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-1t}$$

The initial condition $\mathbf{x}(0) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ determines c_1 and c_2:

$$c_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-3(0)} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-1(0)} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 & 2 \\ 1 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 5/2 \end{bmatrix}$$

Thus $c_1 = 1/2, c_2 = 5/2,$ and $\mathbf{x}(t) = \frac{1}{2}\begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-3t} + \frac{5}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t}.$

3. $A = \begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix}$, $\det(A - \lambda I) = \lambda^2 - 1 = (\lambda - 1)(\lambda + 1) = 0.$ Eigenvalues: 1 and -1.

<u>For $\lambda = 1$:</u> $\begin{bmatrix} 1 & 3 & 0 \\ -1 & -3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, so $x_1 = -3x_2$ with x_2 free. Take $x_2 = 1$ and $\mathbf{v}_1 = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$.

<u>For $\lambda = -1$:</u> $\begin{bmatrix} 3 & 3 & 0 \\ -1 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, so $x_1 = -x_2$ with x_2 free. Take $x_2 = 1$ and $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

For the initial condition $\mathbf{x}(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$, find c_1 and c_2 such that $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{x}(0)$:

$$\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{x}(0) \end{bmatrix} = \begin{bmatrix} -3 & -1 & 3 \\ 1 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -5/2 \\ 0 & 1 & 9/2 \end{bmatrix}$$

Thus $c_1 = -5/2, c_2 = 9/2,$ and $\mathbf{x}(t) = -\frac{5}{2}\begin{bmatrix} -3 \\ 1 \end{bmatrix} e^{t} + \frac{9}{2}\begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-t}.$

Since one eigenvalue is positive and the other is negative, the origin is a saddle point of the dynamical system described by $\mathbf{x}' = A\mathbf{x}$. The direction of greatest attraction is the line through $\mathbf{v}_2$ and the origin. The direction of greatest repulsion is the line through $\mathbf{v}_1$ and the origin.

4. $A = \begin{bmatrix} -2 & -5 \\ 1 & 4 \end{bmatrix}$, $\det(A - \lambda I) = \lambda^2 - 2\lambda - 3 = (\lambda + 1)(\lambda - 3) = 0.$ Eigenvalues: -1 and 3.

<u>For $\lambda = 3$:</u> $\begin{bmatrix} -5 & -5 & 0 \\ 1 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, so $x_1 = -x_2$ with x_2 free. Take $x_2 = 1$ and $\mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

<u>For $\lambda = -1$:</u> $\begin{bmatrix} -1 & -5 & 0 \\ 1 & 5 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 5 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, so $x_1 = -5x_2$ with x_2 free. Take $x_2 = 1$ and $\mathbf{v}_2 = \begin{bmatrix} -5 \\ 1 \end{bmatrix}$.

For the initial condition $\mathbf{x}(0)=\begin{bmatrix}3\\2\end{bmatrix}$, find c_1 and c_2 such that $c_1\mathbf{v}_1+c_2\mathbf{v}_2=\mathbf{x}(0)$:

$$\begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \mathbf{x}(0)\end{bmatrix}=\begin{bmatrix}-1 & -5 & 3\\ 1 & 1 & 2\end{bmatrix}\sim\begin{bmatrix}1 & 0 & 13/4\\ 0 & 1 & -5/4\end{bmatrix}$$

Thus $c_1=13/4, c_2=-5/4$, and $\mathbf{x}(t)=\frac{13}{4}\begin{bmatrix}-1\\1\end{bmatrix}e^{3t}-\frac{5}{4}\begin{bmatrix}-5\\1\end{bmatrix}e^{-t}$.

Since one eigenvalue is positive and the other is negative, the origin is a saddle point of the dynamical system described by $\mathbf{x}'=A\mathbf{x}$. The direction of greatest attraction is the line through $\mathbf{v}_2$ and the origin. The direction of greatest repulsion is the line through $\mathbf{v}_1$ and the origin.

5. $A=\begin{bmatrix}7 & -1\\ 3 & 3\end{bmatrix}$, det $(A-\lambda I)=\lambda^2-10\lambda+24=(\lambda-4)(\lambda-6)=0$. Eigenvalues: 4 and 6.

<u>For $\lambda = 4$:</u> $\begin{bmatrix}3 & -1 & 0\\ 3 & -1 & 0\end{bmatrix}\sim\begin{bmatrix}1 & -1/3 & 0\\ 0 & 0 & 0\end{bmatrix}$, so $x_1=(1/3)x_2$ with x_2 free. Take $x_2=3$ and $\mathbf{v}_1=\begin{bmatrix}1\\3\end{bmatrix}$.

<u>For $\lambda = 6$:</u> $\begin{bmatrix}1 & -1 & 0\\ 3 & -3 & 0\end{bmatrix}\sim\begin{bmatrix}1 & -1 & 0\\ 0 & 0 & 0\end{bmatrix}$, so $x_1=x_2$ with x_2 free. Take $x_2=1$ and $\mathbf{v}_2=\begin{bmatrix}1\\1\end{bmatrix}$.

For the initial condition $\mathbf{x}(0)=\begin{bmatrix}3\\2\end{bmatrix}$, find c_1 and c_2 such that $c_1\mathbf{v}_1+c_2\mathbf{v}_2=\mathbf{x}(0)$:

$$\begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \mathbf{x}(0)\end{bmatrix}=\begin{bmatrix}1 & 1 & 3\\ 3 & 1 & 2\end{bmatrix}\sim\begin{bmatrix}1 & 0 & -1/2\\ 0 & 1 & 7/2\end{bmatrix}$$

Thus $c_1=-1/2, c_2=7/2$, and $\mathbf{x}(t)=-\frac{1}{2}\begin{bmatrix}1\\3\end{bmatrix}e^{4t}+\frac{7}{2}\begin{bmatrix}1\\1\end{bmatrix}e^{6t}$.

Since both eigenvalues are positive, the origin is a repellor of the dynamical system described by $\mathbf{x}'=A\mathbf{x}$. The direction of greatest repulsion is the line through $\mathbf{v}_2$ and the origin.

6. $A=\begin{bmatrix}1 & -2\\ 3 & -4\end{bmatrix}$, det $(A-\lambda I)=\lambda^2+3\lambda+2=(\lambda+1)(\lambda+2)=0$. Eigenvalues: -1 and -2.

<u>For $\lambda = -2$:</u> $\begin{bmatrix}3 & -2 & 0\\ 3 & -2 & 0\end{bmatrix}\sim\begin{bmatrix}1 & -2/3 & 0\\ 0 & 0 & 0\end{bmatrix}$, so $x_1=(2/3)x_2$ with x_2 free. Take $x_2=3$ and $\mathbf{v}_1=\begin{bmatrix}2\\3\end{bmatrix}$.

<u>For $\lambda = -1$:</u> $\begin{bmatrix}2 & -2 & 0\\ 3 & -3 & 0\end{bmatrix}\sim\begin{bmatrix}1 & -1 & 0\\ 0 & 0 & 0\end{bmatrix}$, so $x_1=x_2$ with x_2 free. Take $x_2=1$ and $\mathbf{v}_2=\begin{bmatrix}1\\1\end{bmatrix}$.

For the initial condition $\mathbf{x}(0)=\begin{bmatrix}3\\2\end{bmatrix}$, find c_1 and c_2 such that $c_1\mathbf{v}_1+c_2\mathbf{v}_2=\mathbf{x}(0)$:

$$[\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{x}(0)]=\begin{bmatrix}2 & 1 & 3\\ 3 & 1 & 2\end{bmatrix}\sim\begin{bmatrix}1 & 0 & -1\\ 0 & 1 & 5\end{bmatrix}$$

Thus $c_1=-1, c_2=5$, and $\mathbf{x}(t)=-\begin{bmatrix}2\\3\end{bmatrix}e^{-2t}+5\begin{bmatrix}1\\1\end{bmatrix}e^{-t}$.

Since both eigenvalues are negative, the origin is an attractor of the dynamical system described by $\mathbf{x}'=A\mathbf{x}$. The direction of greatest attraction is the line through $\mathbf{v}_1$ and the origin.

7. From Exercise 5, $A=\begin{bmatrix}7 & -1\\ 3 & 3\end{bmatrix}$, with eigenvectors $\mathbf{v}_1=\begin{bmatrix}1\\3\end{bmatrix}$ and $\mathbf{v}_2=\begin{bmatrix}1\\1\end{bmatrix}$ corresponding to eigenvalues 4 and 6 respectively. To decouple the equation $\mathbf{x}'=A\mathbf{x}$, set $P=[\mathbf{v}_1\ \ \mathbf{v}_2]=\begin{bmatrix}1 & 1\\ 3 & 1\end{bmatrix}$ and let $D=\begin{bmatrix}4 & 0\\ 0 & 6\end{bmatrix}$, so that $A=PDP^{-1}$ and $D=P^{-1}AP$. Substituting $\mathbf{x}(t)=P\mathbf{y}(t)$ into $\mathbf{x}'=A\mathbf{x}$ we have

$$\frac{d}{dt}(P\mathbf{y})=A(P\mathbf{y})=PDP^{-1}(P\mathbf{y})=PD\mathbf{y}$$

Since P has constant entries, $\frac{d}{dt}(P\mathbf{y})=P(\frac{d}{dt}(\mathbf{y}))$, so that left-multiplying the equality $P(\frac{d}{dt}(\mathbf{y}))=PD\mathbf{y}$ by P^{-1} yields $\mathbf{y}'=D\mathbf{y}$, or

$$\begin{bmatrix}y_1'(t)\\ y_2'(t)\end{bmatrix}=\begin{bmatrix}4 & 0\\ 0 & 6\end{bmatrix}\begin{bmatrix}y_1(t)\\ y_2(t)\end{bmatrix}$$

8. From Exercise 6, $A=\begin{bmatrix}1 & -2\\ 3 & -4\end{bmatrix}$, with eigenvectors $\mathbf{v}_1=\begin{bmatrix}2\\3\end{bmatrix}$ and $\mathbf{v}_2=\begin{bmatrix}1\\1\end{bmatrix}$ corresponding to eigenvalues -2 and -1 respectively. To decouple the equation $\mathbf{x}'=A\mathbf{x}$, set $P=\begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2\end{bmatrix}=\begin{bmatrix}2 & 1\\ 3 & 1\end{bmatrix}$ and let $D=\begin{bmatrix}-2 & 0\\ 0 & -1\end{bmatrix}$, so that $A=PDP^{-1}$ and $D=P^{-1}AP$. Substituting $\mathbf{x}(t)=P\mathbf{y}(t)$ into $\mathbf{x}'=A\mathbf{x}$ we have

$$\frac{d}{dt}(P\mathbf{y})=A(P\mathbf{y})=PDP^{-1}(P\mathbf{y})=PD\mathbf{y}$$

Since P has constant entries, $\frac{d}{dt}(P\mathbf{y})=P\left(\frac{d}{dt}(\mathbf{y})\right)$, so that left-multiplying the equality $P\left(\frac{d}{dt}(\mathbf{y})\right)=PD\mathbf{y}$ by P^{-1} yields $\mathbf{y}'=D\mathbf{y}$, or

$$\begin{bmatrix}y_1'(t)\\ y_2'(t)\end{bmatrix}=\begin{bmatrix}-2 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}y_1(t)\\ y_2(t)\end{bmatrix}$$

9. $A=\begin{bmatrix}-3 & 2\\ -1 & -1\end{bmatrix}$. An eigenvalue of A is $-2+i$ with corresponding eigenvector $\mathbf{v}=\begin{bmatrix}1-i\\ 1\end{bmatrix}$. The complex eigenfunctions $\mathbf{v}e^{\lambda t}$ and $\overline{\mathbf{v}}e^{\overline{\lambda}t}$ form a basis for the set of all complex solutions to $\mathbf{x}'=A\mathbf{x}$. The general complex solution is

$$c_1\begin{bmatrix}1-i\\ 1\end{bmatrix}e^{(-2+i)t}+c_2\begin{bmatrix}1+i\\ 1\end{bmatrix}e^{(-2-i)t}$$

where c_1 and c_2 are arbitrary complex numbers. To build the general real solution, rewrite $\mathbf{v}e^{(-2+i)t}$ as:

$$\begin{aligned}\mathbf{v}e^{(-2+i)t}&=\begin{bmatrix}1-i\\ 1\end{bmatrix}e^{-2t}e^{it}=\begin{bmatrix}1-i\\ 1\end{bmatrix}e^{-2t}(\cos t+i\sin t)\\ &=\begin{bmatrix}\cos t-i\cos t+i\sin t-i^2\sin t\\ \cos t+i\sin t\end{bmatrix}e^{-2t}\\ &=\begin{bmatrix}\cos t+\sin t\\ \cos t\end{bmatrix}e^{-2t}+i\begin{bmatrix}\sin t-\cos t\\ \sin t\end{bmatrix}e^{-2t}\end{aligned}$$

The general real solution has the form

$$c_1\begin{bmatrix}\cos t+\sin t\\ \cos t\end{bmatrix}e^{-2t}+c_2\begin{bmatrix}\sin t-\cos t\\ \sin t\end{bmatrix}e^{-2t}$$

where c_1 and c_2 now are real numbers. The trajectories are spirals because the eigenvalues are complex. The spirals tend toward the origin because the real parts of the eigenvalues are negative.

10. $A=\begin{bmatrix}3 & 1\\ -2 & 1\end{bmatrix}$. An eigenvalue of A is $2+i$ with corresponding eigenvector $\mathbf{v}=\begin{bmatrix}1+i\\ -2\end{bmatrix}$. The complex eigenfunctions $\mathbf{v}e^{\lambda t}$ and $\overline{\mathbf{v}}e^{\bar{\lambda}t}$ form a basis for the set of all complex solutions to $\mathbf{x}'=A\mathbf{x}$. The general complex solution is

$$c_1\begin{bmatrix}1+i\\ -2\end{bmatrix}e^{(2+i)t}+c_2\begin{bmatrix}1-i\\ -2\end{bmatrix}e^{(2-i)t}$$

where c_1 and c_2 are arbitrary complex numbers. To build the general real solution, rewrite $\mathbf{v}e^{(2+i)t}$ as:

$$\begin{aligned}\mathbf{v}e^{(2+i)t}&=\begin{bmatrix}1+i\\ -2\end{bmatrix}e^{2t}e^{it}=\begin{bmatrix}1+i\\ -2\end{bmatrix}e^{2t}(\cos t+i\sin t)\\ &=\begin{bmatrix}\cos t+i\cos t+i\sin t+i^2\sin t\\ -2\cos t-2i\sin t\end{bmatrix}e^{2t}\\ &=\begin{bmatrix}\cos t-\sin t\\ -2\cos t\end{bmatrix}e^{2t}+i\begin{bmatrix}\sin t+\cos t\\ -2\sin t\end{bmatrix}e^{2t}\end{aligned}$$

The general real solution has the form

$$c_1\begin{bmatrix}\cos t-\sin t\\ -2\cos t\end{bmatrix}e^{2t}+c_2\begin{bmatrix}\sin t+\cos t\\ -2\sin t\end{bmatrix}e^{2t}$$

where c_1 and c_2 now are real numbers. The trajectories are spirals because the eigenvalues are complex. The spirals tend away from the origin because the real parts of the eigenvalues are positive.

11. $A=\begin{bmatrix}-3 & -9\\ 2 & 3\end{bmatrix}$. An eigenvalue of A is $3i$ with corresponding eigenvector $\mathbf{v}=\begin{bmatrix}-3+3i\\ 2\end{bmatrix}$. The complex eigenfunctions $\mathbf{v}e^{\lambda t}$ and $\overline{\mathbf{v}}e^{\bar{\lambda}t}$ form a basis for the set of all complex solutions to $\mathbf{x}'=A\mathbf{x}$. The general complex solution is

$$c_1\begin{bmatrix}-3+3i\\ 2\end{bmatrix}e^{(3i)t}+c_2\begin{bmatrix}-3-3i\\ 2\end{bmatrix}e^{(-3i)t}$$

where c_1 and c_2 are arbitrary complex numbers. To build the general real solution, rewrite $\mathbf{v}e^{(3i)t}$ as:

$$\begin{aligned}\mathbf{v}e^{(3i)t}&=\begin{bmatrix}-3+3i\\ 2\end{bmatrix}(\cos 3t+i\sin 3t)\\ &=\begin{bmatrix}-3\cos 3t-3\sin 3t\\ 2\cos 3t\end{bmatrix}+i\begin{bmatrix}-3\sin 3t+3\cos 3t\\ 2\sin 3t\end{bmatrix}\end{aligned}$$

The general real solution has the form

$$c_1\begin{bmatrix}-3\cos 3t-3\sin 3t\\ 2\cos 3t\end{bmatrix}+c_2\begin{bmatrix}-3\sin 3t+3\cos 3t\\ 2\sin 3t\end{bmatrix}$$

where c_1 and c_2 now are real numbers. The trajectories are ellipses about the origin because the real parts of the eigenvalues are zero.

12. $A=\begin{bmatrix}-7 & 10\\ -4 & 5\end{bmatrix}$. An eigenvalue of A is $-1+2i$ with corresponding eigenvector $\mathbf{v}=\begin{bmatrix}3-i\\ 2\end{bmatrix}$. The complex eigenfunctions $\mathbf{v}e^{\lambda t}$ and $\overline{\mathbf{v}}e^{\bar{\lambda}t}$ form a basis for the set of all complex solutions to $\mathbf{x}'=A\mathbf{x}$. The general complex solution is

$$c_1\begin{bmatrix}3-i\\ 2\end{bmatrix}e^{(-1+2i)t}+c_2\begin{bmatrix}3+i\\ 1\end{bmatrix}e^{(-1-2i)t}$$

where c_1 and c_2 are arbitrary complex numbers. To build the general real solution, rewrite $\mathbf{v}e^{(-1+2i)t}$ as:

$$\begin{aligned}\mathbf{v}e^{(-1+2i)t}&=\begin{bmatrix}3-i\\ 2\end{bmatrix}e^{-t}(\cos 2t+i\sin 2t)\\ &=\begin{bmatrix}3\cos 2t+\sin 2t\\ 2\cos 2t\end{bmatrix}e^{-t}+i\begin{bmatrix}3\sin 2t-\cos 2t\\ 2\sin 2t\end{bmatrix}e^{-t}\end{aligned}$$

The general real solution has the form

$$c_1\begin{bmatrix}3\cos 2t+\sin 2t\\ 2\cos 2t\end{bmatrix}e^{-t}+c_2\begin{bmatrix}3\sin 2t-\cos 2t\\ 2\sin 2t\end{bmatrix}e^{-t}$$

where c_1 and c_2 now are real numbers. The trajectories are spirals because the eigenvalues are complex. The spirals tend toward the origin because the real parts of the eigenvalues are negative.

13. $A=\begin{bmatrix}4 & -3\\ 6 & -2\end{bmatrix}$. An eigenvalue of A is $1+3i$ with corresponding eigenvector $\mathbf{v}=\begin{bmatrix}1+i\\ 2\end{bmatrix}$. The complex eigenfunctions $\mathbf{v}e^{\lambda t}$ and $\overline{\mathbf{v}}e^{\bar{\lambda}t}$ form a basis for the set of all complex solutions to $\mathbf{x}'=A\mathbf{x}$. The general complex solution is

$$c_1\begin{bmatrix}1+i\\ 2\end{bmatrix}e^{(1+3i)t}+c_2\begin{bmatrix}1-i\\ 1\end{bmatrix}e^{(1-3i)t}$$

where c_1 and c_2 are arbitrary complex numbers. To build the general real solution, rewrite $\mathbf{v}e^{(1+3i)t}$ as:

$$\begin{aligned}\mathbf{v}e^{(1+3i)t}&=\begin{bmatrix}1+i\\ 2\end{bmatrix}e^{t}(\cos 3t+i\sin 3t)\\ &=\begin{bmatrix}\cos 3t-\sin 3t\\ 2\cos 3t\end{bmatrix}e^{t}+i\begin{bmatrix}\sin 3t+\cos 3t\\ 2\sin 3t\end{bmatrix}e^{t}\end{aligned}$$

The general real solution has the form

$$c_1\begin{bmatrix}\cos 3t-\sin 3t\\ 2\cos 3t\end{bmatrix}e^{t}+c_2\begin{bmatrix}\sin 3t+\cos 3t\\ 2\sin 3t\end{bmatrix}e^{t}$$

where c_1 and c_2 now are real numbers. The trajectories are spirals because the eigenvalues are complex. The spirals tend away from the origin because the real parts of the eigenvalues are positive.

14. $A=\begin{bmatrix}-2 & 1\\ -8 & 2\end{bmatrix}$. An eigenvalue of A is $2i$ with corresponding eigenvector $\mathbf{v}=\begin{bmatrix}1-i\\ 4\end{bmatrix}$. The complex eigenfunctions $\mathbf{v}e^{\lambda t}$ and $\overline{\mathbf{v}}e^{\bar{\lambda}t}$ form a basis for the set of all complex solutions to $\mathbf{x}'=A\mathbf{x}$. The general complex solution is

$$c_1\begin{bmatrix}1-i\\ 4\end{bmatrix}e^{(2i)t}+c_2\begin{bmatrix}1+i\\ 4\end{bmatrix}e^{(-2i)t}$$

where c_1 and c_2 are arbitrary complex numbers. To build the general real solution, rewrite $\mathbf{v}e^{(2i)t}$ as:

$$\mathbf{v}e^{(2i)t} = \begin{bmatrix} 1-i \\ 4 \end{bmatrix}(\cos 2t + i\sin 2t)$$
$$= \begin{bmatrix} \cos 2t + \sin 2t \\ 4\cos 2t \end{bmatrix} + i\begin{bmatrix} \sin 2t - \cos 2t \\ 4\sin 2t \end{bmatrix}$$

The general real solution has the form

$$c_1\begin{bmatrix} \cos 2t + \sin 2t \\ 4\cos 2t \end{bmatrix} + c_2\begin{bmatrix} \sin 2t - \cos 2t \\ 4\sin 2t \end{bmatrix}$$

where c_1 and c_2 now are real numbers. The trajectories are ellipses about the origin because the real parts of the eigenvalues are zero.

15. **[M]** $A = \begin{bmatrix} -8 & -12 & -6 \\ 2 & 1 & 2 \\ 7 & 12 & 5 \end{bmatrix}$. The eigenvalues of A are:

```
ev = eig(A)=
 1.0000
-1.0000
-2.0000
nulbasis(A-ev(1)*eye(3)) =
-1.0000
 0.2500
 1.0000
```

so that $\mathbf{v}_1 = \begin{bmatrix} -4 \\ 1 \\ 4 \end{bmatrix}$

```
nulbasis(A-ev(2)*eye(3)) =
-1.2000
 0.2000
 1.0000
```

so that $\mathbf{v}_2 = \begin{bmatrix} -6 \\ 1 \\ 5 \end{bmatrix}$

```
nulbasis (A-ev(3)*eye(3)) =
-1.0000
 0.0000
 1.0000
```

so that $\mathbf{v}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

Hence the general solution is $\mathbf{x}(t)=c_1\begin{bmatrix}-4\\1\\4\end{bmatrix}e^{t}+c_2\begin{bmatrix}-6\\1\\5\end{bmatrix}e^{-t}+c_3\begin{bmatrix}-1\\0\\1\end{bmatrix}e^{-2t}$. The origin is a saddle point.

A solution with $c_1=0$ is attracted to the origin while a solution with $c_2=c_3=0$ is repelled.

16. **[M]** $A=\begin{bmatrix}-6&-11&16\\2&5&-4\\-4&-5&10\end{bmatrix}$. The eigenvalues of A are:

```
ev = eig(A)=
4.0000
3.0000
2.0000
nulbasis(A-ev(1)*eye(3)) =
 2.3333
-0.6667
 1.0000
```

so that $\mathbf{v}_1=\begin{bmatrix}7\\-2\\3\end{bmatrix}$

```
nulbasis(A-ev(2)*eye(3)) =
 3.0000
-1.0000
 1.0000
```

so that $\mathbf{v}_2=\begin{bmatrix}3\\-1\\1\end{bmatrix}$

```
nulbasis(A-ev(3)*eye(3)) =
2.0000
0.0000
1.0000
```

so that $\mathbf{v}_3=\begin{bmatrix}2\\0\\1\end{bmatrix}$

Hence the general solution is $\mathbf{x}(t)=c_1\begin{bmatrix}7\\-2\\3\end{bmatrix}e^{4t}+c_2\begin{bmatrix}3\\-1\\1\end{bmatrix}e^{3t}+c_3\begin{bmatrix}2\\0\\1\end{bmatrix}e^{2t}$. The origin is a repellor, because all eigenvalues are positive. All trajectories tend away from the origin.

17. [M] $A=\begin{bmatrix} 30 & 64 & 23 \\ -11 & -23 & -9 \\ 6 & 15 & 4 \end{bmatrix}$. The eigenvalues of A are:

```
ev = eig(A) =
5.0000 + 2.0000i
5.0000 - 2.0000i
1.0000
nulbasis(A-ev(1)*eye(3)) =
7.6667 - 11.3333i
-3.0000 + 4.6667i
1.0000
```

so that $\mathbf{v}_1=\begin{bmatrix} 23-34i \\ -9+14i \\ 3 \end{bmatrix}$

```
nulbasis (A-ev(2)*eye(3)) =
 7.6667 + 11.3333i
-3.0000 - 4.6667i
 1.0000
```

so that $\mathbf{v}_2=\begin{bmatrix} 23+34i \\ -9-14i \\ 3 \end{bmatrix}$

```
nulbasis (A-ev(3)*eye(3)) =
-3.0000
 1.0000
 1.0000
```

so that $\mathbf{v}_3=\begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix}$

Hence the general complex solution is

$$\mathbf{x}(t)=c_1\begin{bmatrix} 23-34i \\ -9+14i \\ 3 \end{bmatrix}e^{(5+2i)t}+c_2\begin{bmatrix} 23+34i \\ -9-14i \\ 3 \end{bmatrix}e^{(5-2i)t}+c_3\begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix}e^{t}$$

Rewriting the first eigenfunction yields

$$\begin{bmatrix} 23-34i \\ -9+14i \\ 3 \end{bmatrix}e^{5t}(\cos 2t+i\sin 2t)=\begin{bmatrix} 23\cos 2t+34\sin 2t \\ -9\cos 2t-14\sin 2t \\ 3\cos 2t \end{bmatrix}e^{5t}+i\begin{bmatrix} 23\sin 2t-34\cos 2t \\ -9\sin 2t+14\cos 2t \\ 3\sin 2t \end{bmatrix}e^{5t}$$

Hence the general real solution is

$$\mathbf{x}(t) = c_1 \begin{bmatrix} 23\cos 2t + 34\sin 2t \\ -9\cos 2t - 14\sin 2t \\ 3\cos 2t \end{bmatrix} e^{5t} + c_2 \begin{bmatrix} 23\sin 2t - 34\cos 2t \\ -9\sin 2t + 14\cos 2t \\ 3\sin 2t \end{bmatrix} e^{5t} + c_3 \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix} e^{t}$$

where $c_1, c_2,$ and c_3 are real. The origin is a repellor, because the real parts of all eigenvalues are positive. All trajectories spiral away from the origin.

18. **[M]** $A = \begin{bmatrix} 53 & -30 & -2 \\ 90 & -52 & -3 \\ 20 & -10 & 2 \end{bmatrix}$. The eigenvalues of A are:

```
ev = eig(A)=
-7.0000
 5.0000  +  1.0000i
 5.0000  -  1.0000i
nulbasis(A-ev(1)*eye(3)) =
0.5000
1.0000
0.0000
```

so that $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$

```
nulbasis(A-ev(2)*eye(3)) =
0.6000  +  0.2000i
0.9000  +  0.3000i
1.0000
```

so that $\mathbf{v}_2 = \begin{bmatrix} 6+2i \\ 9+3i \\ 10 \end{bmatrix}$

```
nulbasis(A-ev(3)*eye(3)) =
0.6000  -  0.20000
0.9000  -  0.3000i
1.0000
```

so that $\mathbf{v}_3 = \begin{bmatrix} 6-2i \\ 9-3i \\ 10 \end{bmatrix}$

Hence the general complex solution is

$$\mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} e^{-7t} + c_2 \begin{bmatrix} 6+2i \\ 9+3i \\ 10 \end{bmatrix} e^{(5+i)t} + c_3 \begin{bmatrix} 6-2i \\ 9-3i \\ 10 \end{bmatrix} e^{(5-i)t}$$

Rewriting the second eigenfunction yields

$$\begin{bmatrix} 6+2i \\ 9+3i \\ 10 \end{bmatrix} e^{5t}(\cos t + i\sin t) = \begin{bmatrix} 6\cos t - 2\sin t \\ 9\cos t - 3\sin t \\ 10\cos t \end{bmatrix} e^{5t} + i \begin{bmatrix} 6\sin t + 2\cos t \\ 9\sin t + 3\cos t \\ 10\sin t \end{bmatrix} e^{5t}$$

Hence the general real solution is

$$\mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} e^{-7t} + c_2 \begin{bmatrix} 6\cos t - 2\sin t \\ 9\cos t - 3\sin t \\ 10\cos t \end{bmatrix} e^{5t} + c_3 \begin{bmatrix} 6\sin t + 2\cos t \\ 9\sin t + 3\cos t \\ 10\sin t \end{bmatrix} e^{5t}$$

where $c_1, c_2,$ and c_3 are real. When $c_2 = c_3 = 0$ the trajectories tend toward the origin, and in other cases the trajectories spiral away from the origin.

19. **[M]** Substitute $R_1 = 1/5, R_2 = 1/3, C_1 = 4,$ and $C_2 = 3$ into the formula for A given in Example 1, and use a matrix program to find the eigenvalues and eigenvectors:

$$A = \begin{bmatrix} -2 & 3/4 \\ 1 & -1 \end{bmatrix}, \quad \lambda_1 = -.5 : \mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \lambda_2 = -2.5 : \mathbf{v}_1 = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$$

The general solution is thus $\mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-.5t} + c_2 \begin{bmatrix} -3 \\ 2 \end{bmatrix} e^{-2.5t}$. The condition $\mathbf{x}(0) = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$ implies

that $\begin{bmatrix} 1 & -3 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$. By a matrix program, $c_1 = 5/2$ and $c_2 = -1/2,$ so that

$$\begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix} = \mathbf{x}(t) = \frac{5}{2} \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-.5t} - \frac{1}{2} \begin{bmatrix} -3 \\ 2 \end{bmatrix} e^{-2.5t}$$

20. **[M]** Substitute $R_1 = 1/15, R_2 = 1/3, C_1 = 4,$ and $C_2 = 2$ into the formula for A given in Example 1, and use a matrix program to find the eigenvalues and eigenvectors:

$$A = \begin{bmatrix} -2 & 1/3 \\ 3/2 & -3/2 \end{bmatrix}, \quad \lambda_1 = -1 : \mathbf{v}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad \lambda_2 = -2.5 : \mathbf{v}_2 = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$$

The general solution is thus $\mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} -2 \\ 3 \end{bmatrix} e^{-2.5t}$. The condition $\mathbf{x}(0) = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$ implies

that $\begin{bmatrix} 1 & -2 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$. By a matrix program, $c_1 = 5/3$ and $c_2 = -2/3,$ so that

$$\begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix} = \mathbf{x}(t) = \frac{5}{3} \begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{-t} - \frac{2}{3} \begin{bmatrix} -2 \\ 3 \end{bmatrix} e^{-2.5t}$$

21. **[M]** $A = \begin{bmatrix} -1 & -8 \\ 5 & -5 \end{bmatrix}$. Using a matrix program we find that an eigenvalue of A is $-3+6i$ with

corresponding eigenvector $\mathbf{v} = \begin{bmatrix} 2+6i \\ 5 \end{bmatrix}$. The conjugates of these form the second

eigenvalue-eigenvector pair. The general complex solution is

$$\mathbf{x}(t) = c_1 \begin{bmatrix} 2+6i \\ 5 \end{bmatrix} e^{(-3+6i)t} + c_2 \begin{bmatrix} 2-6i \\ 5 \end{bmatrix} e^{(-3-6i)t}$$

where c_1 and c_2 are arbitrary complex numbers. Rewriting the first eigenfunction and taking its real and imaginary parts, we have

$$\begin{aligned}\mathbf{v}e^{(-3+6i)t} &= \begin{bmatrix}2+6i\\5\end{bmatrix}e^{-3t}(\cos 6t + i\sin 6t)\\ &= \begin{bmatrix}2\cos 6t - 6\sin 6t\\5\cos 6t\end{bmatrix}e^{-3t} + i\begin{bmatrix}2\sin 6t + 6\cos 6t\\5\sin 6t\end{bmatrix}e^{-3t}\end{aligned}$$

The general real solution has the form

$$\mathbf{x}(t) = c_1\begin{bmatrix}2\cos 6t - 6\sin 6t\\5\cos 6t\end{bmatrix}e^{-3t} + c_2\begin{bmatrix}2\sin 6t + 6\cos 6t\\5\sin 6t\end{bmatrix}e^{-3t}$$

where c_1 and c_2 now are real numbers. To satisfy the initial condition $\mathbf{x}(0) = \begin{bmatrix}0\\15\end{bmatrix}$, we solve

$c_1\begin{bmatrix}2\\5\end{bmatrix} + c_2\begin{bmatrix}6\\0\end{bmatrix} = \begin{bmatrix}0\\15\end{bmatrix}$ to get $c_1 = 3, c_2 = -1$. We now have

$$\begin{bmatrix}i_L(t)\\v_C(t)\end{bmatrix} = \mathbf{x}(t) = 3\begin{bmatrix}2\cos 6t - 6\sin 6t\\5\cos 6t\end{bmatrix}e^{-3t} - \begin{bmatrix}2\sin 6t + 6\cos 6t\\5\sin 6t\end{bmatrix}e^{-3t} = \begin{bmatrix}-20\sin 6t\\15\cos 6t - 5\sin 6t\end{bmatrix}e^{-3t}$$

22. **[M]** $A = \begin{bmatrix}0 & 2\\-.4 & -.8\end{bmatrix}$. Using a matrix program we find that an eigenvalue of A is $-.4+.8i$ with corresponding eigenvector $\mathbf{v} = \begin{bmatrix}-1-2i\\1\end{bmatrix}$. The conjugates of these form the second eigenvalue-eigenvector pair. The general complex solution is

$$\mathbf{x}(t) = c_1\begin{bmatrix}-1-2i\\1\end{bmatrix}e^{(-.4+.8i)t} + c_2\begin{bmatrix}-1+2i\\1\end{bmatrix}e^{(-.4-.8i)t}$$

where c_1 and c_2 are arbitrary complex numbers. Rewriting the first eigenfunction and taking its real and imaginary parts, we have

$$\begin{aligned}\mathbf{v}e^{(-.4+.8i)t} &= \begin{bmatrix}-1-2i\\1\end{bmatrix}e^{-.4t}(\cos .8t + i\sin .8t)\\ &= \begin{bmatrix}-\cos .8t + 2\sin .8t\\\cos .8t\end{bmatrix}e^{-.4t} + i\begin{bmatrix}-\sin .8t - 2\cos .8t\\\sin .8t\end{bmatrix}e^{-.4t}\end{aligned}$$

The general real solution has the form

$$\mathbf{x}(t) = c_1\begin{bmatrix}-\cos .8t + 2\sin .8t\\\cos .8t\end{bmatrix}e^{-.4t} + c_2\begin{bmatrix}-\sin .8t - 2\cos .8t\\\sin .8t\end{bmatrix}e^{-.4t}$$

where c_1 and c_2 now are real numbers. To satisfy the initial condition $\mathbf{x}(0) = \begin{bmatrix}0\\12\end{bmatrix}$, we solve

$c_1\begin{bmatrix}-1\\1\end{bmatrix} + c_2\begin{bmatrix}-2\\0\end{bmatrix} = \begin{bmatrix}0\\12\end{bmatrix}$ to get $c_1 = 12, c_2 = -6$. We now have

$$\begin{bmatrix}i_L(t)\\v_C(t)\end{bmatrix} = \mathbf{x}(t) = 12\begin{bmatrix}-\cos .8t + 2\sin .8t\\\cos .8t\end{bmatrix}e^{-.4t} - 6\begin{bmatrix}-\sin .8t - 2\cos .8t\\\sin .8t\end{bmatrix}e^{-.4t} = \begin{bmatrix}30\sin .8t\\12\cos .8t - 6\sin .8t\end{bmatrix}e^{-.4t}$$

5.8 SOLUTIONS

1. The vectors in the given sequence approach an eigenvector $\mathbf{v}_1$. The last vector in the sequence, $\mathbf{x}_4 = \begin{bmatrix} 1 \\ .3326 \end{bmatrix}$, is probably the best estimate for $\mathbf{v}_1$. To compute an estimate for λ_1, examine $A\mathbf{x}_4 = \begin{bmatrix} 4.9978 \\ 1.6652 \end{bmatrix}$. This vector is approximately $\lambda_1\mathbf{v}_1$. From the first entry in this vector, an estimate of λ_1 is 4.9978.

2. The vectors in the given sequence approach an eigenvector $\mathbf{v}_1$. The last vector in the sequence, $\mathbf{x}_4 = \begin{bmatrix} -.2520 \\ 1 \end{bmatrix}$, is probably the best estimate for $\mathbf{v}_1$. To compute an estimate for λ_1, examine $A\mathbf{x}_4 = \begin{bmatrix} -1.2536 \\ 5.0064 \end{bmatrix}$. This vector is approximately $\lambda_1\mathbf{v}_1$. From the second entry in this vector, an estimate of λ_1 is 5.0064.

3. The vectors in the given sequence approach an eigenvector $\mathbf{v}_1$. The last vector in the sequence, $\mathbf{x}_4 = \begin{bmatrix} .5188 \\ 1 \end{bmatrix}$, is probably the best estimate for $\mathbf{v}_1$. To compute an estimate for λ_1, examine $A\mathbf{x}_4 = \begin{bmatrix} .4594 \\ .9075 \end{bmatrix}$. This vector is approximately $\lambda_1\mathbf{v}_1$. From the second entry in this vector, an estimate of λ_1 is .9075.

4. The vectors in the given sequence approach an eigenvector $\mathbf{v}_1$. The last vector in the sequence, $\mathbf{x}_4 = \begin{bmatrix} 1 \\ .7502 \end{bmatrix}$, is probably the best estimate for $\mathbf{v}_1$. To compute an estimate for λ_1, examine $A\mathbf{x}_4 = \begin{bmatrix} -.4012 \\ -.3009 \end{bmatrix}$. This vector is approximately $\lambda_1\mathbf{v}_1$. From the first entry in this vector, an estimate of λ_1 is $-.4012$.

5. Since $A^5\mathbf{x} = \begin{bmatrix} 24991 \\ -31241 \end{bmatrix}$ is an estimate for an eigenvector, the vector $\mathbf{v} = -\frac{1}{31241}\begin{bmatrix} 24991 \\ -31241 \end{bmatrix} = \begin{bmatrix} -.7999 \\ 1 \end{bmatrix}$ is a vector with a 1 in its second entry that is close to an eigenvector of A. To estimate the dominant eigenvalue λ_1 of A, compute $A\mathbf{v} = \begin{bmatrix} 4.0015 \\ -5.0020 \end{bmatrix}$. From the second entry in this vector, an estimate of λ_1 is -5.0020.

6. Since $A^5\mathbf{x} = \begin{bmatrix} -2045 \\ 4093 \end{bmatrix}$ is an estimate for an eigenvector, the vector $\mathbf{v} = \frac{1}{4093}\begin{bmatrix} -2045 \\ 4093 \end{bmatrix} = \begin{bmatrix} -.4996 \\ 1 \end{bmatrix}$ is a vector with a 1 in its second entry that is close to an eigenvector of A. To estimate the dominant eigenvalue λ_1 of A, compute $A\mathbf{v} = \begin{bmatrix} -2.0008 \\ 4.0024 \end{bmatrix}$. From the second entry in this vector, an estimate of λ_1 is 4.0024.

7. **[M]** $A = \begin{bmatrix} 6 & 7 \\ 8 & 5 \end{bmatrix}$, $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. The data in the table below was calculated using Mathematica, which carried more digits than shown here.

k	0	1	2	3	4	5
$\mathbf{x}_k$	$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$	$\begin{bmatrix} .75 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 1 \\ .9565 \end{bmatrix}$	$\begin{bmatrix} .9932 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 1 \\ .9990 \end{bmatrix}$	$\begin{bmatrix} .9998 \\ 1 \end{bmatrix}$
$A\mathbf{x}_k$	$\begin{bmatrix} 6 \\ 8 \end{bmatrix}$	$\begin{bmatrix} 11.5 \\ 11.0 \end{bmatrix}$	$\begin{bmatrix} 12.6957 \\ 12.7826 \end{bmatrix}$	$\begin{bmatrix} 12.9592 \\ 12.9456 \end{bmatrix}$	$\begin{bmatrix} 12.9927 \\ 12.9948 \end{bmatrix}$	$\begin{bmatrix} 12.9990 \\ 12.9987 \end{bmatrix}$
μ_k	8	11.5	12.7826	12.9592	12.9948	12.9990

The actual eigenvalue is 13.

8. **[M]** $A = \begin{bmatrix} 2 & 1 \\ 4 & 5 \end{bmatrix}$, $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. The data in the table below was calculated using Mathematica, which carried more digits than shown here.

k	0	1	2	3	4	5
$\mathbf{x}_k$	$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$	$\begin{bmatrix} .5 \\ 1 \end{bmatrix}$	$\begin{bmatrix} .2857 \\ 1 \end{bmatrix}$	$\begin{bmatrix} .2558 \\ 1 \end{bmatrix}$	$\begin{bmatrix} .2510 \\ 1 \end{bmatrix}$	$\begin{bmatrix} .2502 \\ 1 \end{bmatrix}$
$A\mathbf{x}_k$	$\begin{bmatrix} 2 \\ 4 \end{bmatrix}$	$\begin{bmatrix} 2 \\ 7 \end{bmatrix}$	$\begin{bmatrix} 1.5714 \\ 6.1429 \end{bmatrix}$	$\begin{bmatrix} 1.5116 \\ 6.0233 \end{bmatrix}$	$\begin{bmatrix} 1.5019 \\ 6.0039 \end{bmatrix}$	$\begin{bmatrix} 1.5003 \\ 6.0006 \end{bmatrix}$
μ_k	4	7	6.1429	6.0233	6.0039	6.0006

The actual eigenvalue is 6.

9. **[M]** $A = \begin{bmatrix} 8 & 0 & 12 \\ 1 & -2 & 1 \\ 0 & 3 & 0 \end{bmatrix}$, $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. The data in the table below was calculated using Mathematica, which carried more digits than shown here.

k	0	1	2	3	4	5	6
$\mathbf{x}_k$	$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 1 \\ .125 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 1 \\ .0938 \\ .0469 \end{bmatrix}$	$\begin{bmatrix} 1 \\ .1004 \\ .0328 \end{bmatrix}$	$\begin{bmatrix} 1 \\ .0991 \\ .0359 \end{bmatrix}$	$\begin{bmatrix} 1 \\ .0994 \\ .0353 \end{bmatrix}$	$\begin{bmatrix} 1 \\ .0993 \\ .0354 \end{bmatrix}$
$A\mathbf{x}_k$	$\begin{bmatrix} 8 \\ 1 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 8 \\ .75 \\ .375 \end{bmatrix}$	$\begin{bmatrix} 8.5625 \\ .8594 \\ .2812 \end{bmatrix}$	$\begin{bmatrix} 8.3942 \\ .8321 \\ .3011 \end{bmatrix}$	$\begin{bmatrix} 8.4304 \\ .8376 \\ .2974 \end{bmatrix}$	$\begin{bmatrix} 8.4233 \\ .8366 \\ .2981 \end{bmatrix}$	$\begin{bmatrix} 8.4246 \\ .8368 \\ .2979 \end{bmatrix}$
μ_k	8	8	8.5625	8.3942	8.4304	8.4233	8.4246

Thus $\mu_5 = 8.4233$ and $\mu_6 = 8.4246$. The actual eigenvalue is $(7+\sqrt{97})/2$, or 8.42443 to five decimal places.

10. **[M]** $A = \begin{bmatrix} 1 & 2 & -2 \\ 1 & 1 & 9 \\ 0 & 1 & 9 \end{bmatrix}$, $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. The data in the table below was calculated using Mathematica, which carried more digits than shown here.

k	0	1	2	3	4	5	6
$\mathbf{x}_k$	$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 1 \\ .6667 \\ .3333 \end{bmatrix}$	$\begin{bmatrix} .3571 \\ 1 \\ .7857 \end{bmatrix}$	$\begin{bmatrix} .0932 \\ 1 \\ .9576 \end{bmatrix}$	$\begin{bmatrix} .0183 \\ 1 \\ .9904 \end{bmatrix}$	$\begin{bmatrix} .0038 \\ 1 \\ .9982 \end{bmatrix}$
$A\mathbf{x}_k$	$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 1.6667 \\ 4.6667 \\ 3.6667 \end{bmatrix}$	$\begin{bmatrix} .7857 \\ 8.4286 \\ 8.0714 \end{bmatrix}$	$\begin{bmatrix} .1780 \\ 9.7119 \\ 9.6186 \end{bmatrix}$	$\begin{bmatrix} .0375 \\ 9.9319 \\ 9.9136 \end{bmatrix}$	$\begin{bmatrix} .0075 \\ 9.9872 \\ 9.9834 \end{bmatrix}$
μ_k	1	3	4.6667	8.4286	9.7119	9.9319	9.9872

Thus $\mu_5 = 9.9319$ and $\mu_6 = 9.9872$. The actual eigenvalue is 10.

11. **[M]** $A = \begin{bmatrix} 5 & 2 \\ 2 & 2 \end{bmatrix}$, $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. The data in the table below was calculated using Mathematica, which carried more digits than shown here.

k	0	1	2	3	4
$\mathbf{x}_k$	$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 1 \\ .4 \end{bmatrix}$	$\begin{bmatrix} 1 \\ .4828 \end{bmatrix}$	$\begin{bmatrix} 1 \\ .4971 \end{bmatrix}$	$\begin{bmatrix} 1 \\ .4995 \end{bmatrix}$
$A\mathbf{x}_k$	$\begin{bmatrix} 5 \\ 2 \end{bmatrix}$	$\begin{bmatrix} 5.8 \\ 2.8 \end{bmatrix}$	$\begin{bmatrix} 5.9655 \\ 2.9655 \end{bmatrix}$	$\begin{bmatrix} 5.9942 \\ 2.9942 \end{bmatrix}$	$\begin{bmatrix} 5.9990 \\ 2.9990 \end{bmatrix}$
μ_k	5	5.8	5.9655	5.9942	5.9990
$R(\mathbf{x}_k)$	5	5.9655	5.9990	5.99997	5.9999993

The actual eigenvalue is 6. The bottom two columns of the table show that $R(\mathbf{x}_k)$ estimates the eigenvalue more accurately than μ_k.

12. **[M]** $A = \begin{bmatrix} -3 & 2 \\ 2 & 2 \end{bmatrix}$, $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. The data in the table below was calculated using Mathematica, which carried more digits than shown here.

k	0	1	2	3	4
$\mathbf{x}_k$	$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$	$\begin{bmatrix} -1 \\ .6667 \end{bmatrix}$	$\begin{bmatrix} 1 \\ -.4615 \end{bmatrix}$	$\begin{bmatrix} -1 \\ .5098 \end{bmatrix}$	$\begin{bmatrix} 1 \\ -.4976 \end{bmatrix}$
$A\mathbf{x}_k$	$\begin{bmatrix} -3 \\ 2 \end{bmatrix}$	$\begin{bmatrix} 4.3333 \\ -2.0000 \end{bmatrix}$	$\begin{bmatrix} -3.9231 \\ 2.0000 \end{bmatrix}$	$\begin{bmatrix} 4.0196 \\ -2.0000 \end{bmatrix}$	$\begin{bmatrix} -3.9951 \\ 2.0000 \end{bmatrix}$
μ_k	–3	–4.3333	–3.9231	–4.0196	–3.9951
$R(\mathbf{x}_k)$	–3	–3.9231	–3.9951	–3.9997	–3.99998

The actual eigenvalue is –4. The bottom two columns of the table show that $R(\mathbf{x}_k)$ estimates the eigenvalue more accurately than μ_k.

13. If the eigenvalues close to 4 and -4 have different absolute values, then one of these is a strictly dominant eigenvalue, so the power method will work. But the power method depends on powers of the quotients λ_2/λ_1 and λ_3/λ_1 going to zero. If $|\lambda_2/\lambda_1|$ is close to 1, its powers will go to zero slowly, and the power method will converge slowly.

14. If the eigenvalues close to 4 and -4 have the same absolute value, then neither of these is a strictly dominant eigenvalue, so the power method will not work. However, the inverse power method may still be used. If the initial estimate is chosen near the eigenvalue close to 4, then the inverse power method should produce a sequence that estimates the eigenvalue close to 4.

15. Suppose $A\mathbf{x} = \lambda\mathbf{x}$, with $\mathbf{x} \neq 0$. For any $\alpha, A\mathbf{x} - \alpha I\mathbf{x} = (\lambda - \alpha)\mathbf{x}$. If α is *not* an eigenvalue of A, then $A - \alpha I$ is invertible and $\lambda - \alpha$ is not 0; hence

$\mathbf{x} = (A - \alpha I)^{-1}(\lambda - \alpha)\mathbf{x}$ and $(\lambda - \alpha)^{-1}\mathbf{x} = (A - \alpha I)^{-1}\mathbf{x}$

This last equation shows that **x** is an eigenvector of $(A - \alpha I)^{-1}$ corresponding to the eigenvalue $(\lambda - \alpha)^{-1}$.

16. Suppose that μ is an eigenvalue of $(A - \alpha I)^{-1}$ with corresponding eigenvector **x**. Since $(A - \alpha I)^{-1}\mathbf{x} = \mu\mathbf{x}$,

$$\mathbf{x} = (A - \alpha I)(\mu\mathbf{x}) = A(\mu\mathbf{x}) - (\alpha I)(\mu\mathbf{x}) = \mu(A\mathbf{x}) - \alpha\mu\mathbf{x}$$

Solving this equation for A**x**, we find that

$$A\mathbf{x} = \left(\frac{1}{\mu}\right)(\alpha\mu\mathbf{x} + \mathbf{x}) = \left(\alpha + \frac{1}{\mu}\right)\mathbf{x}$$

Thus $\lambda = \alpha + (1/\mu)$ is an eigenvalue of A with corresponding eigenvector **x**.

17. **[M]** $A = \begin{bmatrix} 10 & -8 & -4 \\ -8 & 13 & 4 \\ -4 & 5 & 4 \end{bmatrix}, \mathbf{x}_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \alpha = 3.3.$ The data in the table below was calculated using Mathematica, which carried more digits than shown here.

k	0	1	2
$\mathbf{x}_k$	$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 1 \\ .7873 \\ .0908 \end{bmatrix}$	$\begin{bmatrix} 1 \\ .7870 \\ .0957 \end{bmatrix}$
$\mathbf{y}_k$	$\begin{bmatrix} 26.0552 \\ 20.5128 \\ 2.3669 \end{bmatrix}$	$\begin{bmatrix} 47.1975 \\ 37.1436 \\ 4.5187 \end{bmatrix}$	$\begin{bmatrix} 47.1233 \\ 37.0866 \\ 4.5083 \end{bmatrix}$
μ_k	26.0552	47.1975	47.1233
ν_k	3.3384	3.32119	3.3212209

Thus an estimate for the eigenvalue to four decimal places is 3.3212. The actual eigenvalue is $(25 - \sqrt{337})/2$, or 3.3212201 to seven decimal places.

18. **[M]** $A=\begin{bmatrix}8&0&12\\1&-2&1\\0&3&0\end{bmatrix}, \mathbf{x}_0=\begin{bmatrix}1\\0\\0\end{bmatrix}, \alpha=-1.4.$ The data in the table below was calculated using Mathematica, which carried more digits than shown here.

k	0	1	2	3	4
$\mathbf{x}_k$	$\begin{bmatrix}1\\0\\0\end{bmatrix}$	$\begin{bmatrix}1\\.3646\\-.7813\end{bmatrix}$	$\begin{bmatrix}1\\.3734\\-.7854\end{bmatrix}$	$\begin{bmatrix}1\\.3729\\-.7854\end{bmatrix}$	$\begin{bmatrix}1\\.3729\\-.7854\end{bmatrix}$
$\mathbf{y}_k$	$\begin{bmatrix}40\\14.5833\\-31.25\end{bmatrix}$	$\begin{bmatrix}-38.125\\-14.2361\\29.9479\end{bmatrix}$	$\begin{bmatrix}-41.1134\\-15.3300\\32.2888\end{bmatrix}$	$\begin{bmatrix}-40.9243\\-15.2608\\32.1407\end{bmatrix}$	$\begin{bmatrix}-40.9358\\-15.2650\\32.1497\end{bmatrix}$
μ_k	40	−38.125	−41.1134	−40.9243	−40.9358
ν_k	−1.375	−1.42623	−1.42432	−1.42444	−1.42443

Thus an estimate for the eigenvalue to four decimal places is -1.4244. The actual eigenvalue is $(7-\sqrt{97})/2$, or -1.424429 to six decimal places.

19. **[M]** $A=\begin{bmatrix}10&7&8&7\\7&5&6&5\\8&6&10&9\\7&5&9&10\end{bmatrix}, \mathbf{x}_0=\begin{bmatrix}1\\0\\0\\0\end{bmatrix}.$

(a) The data in the table below was calculated using Mathematica, which carried more digits than shown here.

k	0	1	2	3
$\mathbf{x}_k$	$\begin{bmatrix}1\\0\\0\\0\end{bmatrix}$	$\begin{bmatrix}1\\.7\\.8\\.7\end{bmatrix}$	$\begin{bmatrix}.988679\\.709434\\1\\.932075\end{bmatrix}$	$\begin{bmatrix}.961467\\.691491\\1\\.942201\end{bmatrix}$
$A\mathbf{x}_k$	$\begin{bmatrix}10\\7\\8\\7\end{bmatrix}$	$\begin{bmatrix}26.2\\18.8\\26.5\\24.7\end{bmatrix}$	$\begin{bmatrix}29.3774\\21.1283\\30.5547\\28.7887\end{bmatrix}$	$\begin{bmatrix}29.0505\\20.8987\\30.3205\\28.6097\end{bmatrix}$
μ_k	10	26.5	30.5547	30.3205

k	4	5	6	7
$\mathbf{x}_k$	$\begin{bmatrix} .958115 \\ .689261 \\ 1 \\ .943578 \end{bmatrix}$	$\begin{bmatrix} .957691 \\ .688978 \\ 1 \\ .943755 \end{bmatrix}$	$\begin{bmatrix} .957637 \\ .688942 \\ 1 \\ .943778 \end{bmatrix}$	$\begin{bmatrix} .957630 \\ .688938 \\ 1 \\ .943781 \end{bmatrix}$
$A\mathbf{x}_k$	$\begin{bmatrix} 29.0110 \\ 20.8710 \\ 30.2927 \\ 28.5889 \end{bmatrix}$	$\begin{bmatrix} 29.0060 \\ 20.8675 \\ 30.2892 \\ 28.5863 \end{bmatrix}$	$\begin{bmatrix} 29.0054 \\ 20.8671 \\ 30.2887 \\ 28.5859 \end{bmatrix}$	$\begin{bmatrix} 29.0053 \\ 20.8670 \\ 30.2887 \\ 28.5859 \end{bmatrix}$
μ_k	30.2927	30.2892	30.2887	30.2887

Thus an estimate for the eigenvalue to four decimal places is 30.2887. The actual eigenvalue is 30.2886853 to seven decimal places. An estimate for the corresponding eigenvector is $\begin{bmatrix} .957630 \\ .688938 \\ 1 \\ .943781 \end{bmatrix}$.

(b) The data in the table below was calculated using Mathematica, which carried more digits than shown here.

k	0	1	2	3	4
$\mathbf{x}_k$	$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} -.609756 \\ 1 \\ -.243902 \\ .146341 \end{bmatrix}$	$\begin{bmatrix} -.604007 \\ 1 \\ -.251051 \\ .148899 \end{bmatrix}$	$\begin{bmatrix} -.603973 \\ 1 \\ -.251134 \\ .148953 \end{bmatrix}$	$\begin{bmatrix} -.603972 \\ 1 \\ -.251135 \\ .148953 \end{bmatrix}$
$\mathbf{y}_k$	$\begin{bmatrix} 25 \\ -41 \\ 10 \\ -6 \end{bmatrix}$	$\begin{bmatrix} -59.5610 \\ 98.6098 \\ -24.7561 \\ 14.6829 \end{bmatrix}$	$\begin{bmatrix} -59.5041 \\ 98.5211 \\ -24.7420 \\ 14.6750 \end{bmatrix}$	$\begin{bmatrix} -59.5044 \\ 98.5217 \\ -24.7423 \\ 14.6751 \end{bmatrix}$	$\begin{bmatrix} -59.5044 \\ 98.5217 \\ -24.7423 \\ 14.6751 \end{bmatrix}$
μ_k	−41	98.6098	98.5211	98.5217	98.5217
ν_k	−.0243902	.0101410	.0101501	.0101500	.0101500

Thus an estimate for the eigenvalue to five decimal places is .01015. The actual eigenvalue is .01015005 to eight decimal places. An estimate for the corresponding eigenvector is $\begin{bmatrix} -.603972 \\ 1 \\ -.251135 \\ .148953 \end{bmatrix}$.

20. [M] $A = \begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 12 & 13 & 11 \\ -2 & 3 & 0 & 2 \\ 4 & 5 & 7 & 2 \end{bmatrix}$, $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$.

(a) The data in the table below was calculated using Mathematica, which carried more digits than shown here.

k	0	1	2	3	4
$\mathbf{x}_k$	$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} .25 \\ .5 \\ -.5 \\ 1 \end{bmatrix}$	$\begin{bmatrix} .159091 \\ 1 \\ .272727 \\ .181818 \end{bmatrix}$	$\begin{bmatrix} .187023 \\ 1 \\ .170483 \\ .442748 \end{bmatrix}$	$\begin{bmatrix} .184166 \\ 1 \\ .180439 \\ .402197 \end{bmatrix}$
$A\mathbf{x}_k$	$\begin{bmatrix} 1 \\ 2 \\ -2 \\ 4 \end{bmatrix}$	$\begin{bmatrix} 1.75 \\ 11 \\ 3 \\ 2 \end{bmatrix}$	$\begin{bmatrix} 3.34091 \\ 17.8636 \\ 3.04545 \\ 7.90909 \end{bmatrix}$	$\begin{bmatrix} 3.58397 \\ 19.4606 \\ 3.51145 \\ 7.82697 \end{bmatrix}$	$\begin{bmatrix} 3.52988 \\ 19.1382 \\ 3.43606 \\ 7.80413 \end{bmatrix}$
μ_k	4	11	17.8636	19.4606	19.1382

k	5	6	7	8	9
$\mathbf{x}_k$	$\begin{bmatrix} .184441 \\ 1 \\ .179539 \\ .407778 \end{bmatrix}$	$\begin{bmatrix} .184414 \\ 1 \\ .179622 \\ .407021 \end{bmatrix}$	$\begin{bmatrix} .184417 \\ 1 \\ .179615 \\ .407121 \end{bmatrix}$	$\begin{bmatrix} .184416 \\ 1 \\ .179615 \\ .407108 \end{bmatrix}$	$\begin{bmatrix} .184416 \\ 1 \\ .179615 \\ .407110 \end{bmatrix}$
$A\mathbf{x}_k$	$\begin{bmatrix} 3.53861 \\ 19.1884 \\ 3.44667 \\ 7.81010 \end{bmatrix}$	$\begin{bmatrix} 3.53732 \\ 19.1811 \\ 3.44521 \\ 7.80905 \end{bmatrix}$	$\begin{bmatrix} 3.53750 \\ 19.1822 \\ 3.44541 \\ 7.80921 \end{bmatrix}$	$\begin{bmatrix} 3.53748 \\ 19.1820 \\ 3.44538 \\ 7.80919 \end{bmatrix}$	$\begin{bmatrix} 3.53748 \\ 19.1811 \\ 3.44539 \\ 7.80919 \end{bmatrix}$
μ_k	19.1884	19.1811	19.1822	19.1820	19.1820

Thus an estimate for the eigenvalue to four decimal places is 19.1820. The actual eigenvalue is 19.1820368 to seven decimal places. An estimate for the corresponding eigenvector is $\begin{bmatrix} .184416 \\ 1 \\ .179615 \\ .407110 \end{bmatrix}$.

(b) The data in the table below was calculated using Mathematica, which carried more digits than shown here.

k	0	1	2
$\mathbf{x}_k$	$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 1 \\ .226087 \\ -.921739 \\ .660870 \end{bmatrix}$	$\begin{bmatrix} 1 \\ .222577 \\ -.917970 \\ .660496 \end{bmatrix}$
$\mathbf{y}_k$	$\begin{bmatrix} 115 \\ 26 \\ -106 \\ 76 \end{bmatrix}$	$\begin{bmatrix} 81.7304 \\ 18.1913 \\ -75.0261 \\ 53.9826 \end{bmatrix}$	$\begin{bmatrix} 81.9314 \\ 18.2387 \\ -75.2125 \\ 54.1143 \end{bmatrix}$
μ_k	115	81.7304	81.9314
ν_k	.00869565	.0122353	.0122053

Thus an estimate for the eigenvalue to four decimal places is .0122. The actual eigenvalue is .01220556 to eight decimal places. An estimate for the corresponding eigenvector is $\begin{bmatrix} 1 \\ .222577 \\ -.917970 \\ .660496 \end{bmatrix}$.

21. **a**. $A = \begin{bmatrix} .8 & 0 \\ 0 & .2 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} .5 \\ .5 \end{bmatrix}$. Here is the sequence $A^k\mathbf{x}$ for $k = 1, \dots 5$:

$$\begin{bmatrix} .4 \\ .1 \end{bmatrix}, \begin{bmatrix} .32 \\ .02 \end{bmatrix}, \begin{bmatrix} .256 \\ .004 \end{bmatrix}, \begin{bmatrix} .2048 \\ .0008 \end{bmatrix}, \begin{bmatrix} .16384 \\ .00016 \end{bmatrix}$$

Notice that $A^5\mathbf{x}$ is approximately $.8(A^4\mathbf{x})$.

Conclusion: If the eigenvalues of A are all less than 1 in magnitude, and if $\mathbf{x} \neq 0$, then $A^k\mathbf{x}$ is approximately an eigenvector for large k.

b. $A = \begin{bmatrix} 1 & 0 \\ 0 & .8 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} .5 \\ .5 \end{bmatrix}$. Here is the sequence $A^k\mathbf{x}$ for $k = 1, \dots 5$:

$$\begin{bmatrix} .5 \\ .4 \end{bmatrix}, \begin{bmatrix} .5 \\ .32 \end{bmatrix}, \begin{bmatrix} .5 \\ .256 \end{bmatrix}, \begin{bmatrix} .5 \\ .2048 \end{bmatrix}, \begin{bmatrix} .5 \\ .16384 \end{bmatrix}$$

Notice that $A^k\mathbf{x}$ seems to be converging to $\begin{bmatrix} .5 \\ 0 \end{bmatrix}$.

Conclusion: If the strictly dominant eigenvalue of A is 1, and if $\mathbf{x}$ has a component in the direction of the corresponding eigenvector, then $\{A^k\mathbf{x}\}$ will converge to a multiple of that eigenvector.

c. $A = \begin{bmatrix} 8 & 0 \\ 0 & 2 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} .5 \\ .5 \end{bmatrix}$. Here is the sequence $A^k\mathbf{x}$ for $k = 1, \dots 5$:

$$\begin{bmatrix} 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 32 \\ 2 \end{bmatrix}, \begin{bmatrix} 256 \\ 4 \end{bmatrix}, \begin{bmatrix} 2048 \\ 8 \end{bmatrix}, \begin{bmatrix} 16384 \\ 16 \end{bmatrix}$$

Notice that the distance of $A^k\mathbf{x}$ from either eigenvector of A is increasing rapidly as k increases.

Conclusion: If the eigenvalues of A are all greater than 1 in magnitude, and if $\mathbf{x}$ is not an eigenvector, then the distance from $A^k\mathbf{x}$ to the nearest eigenvector will *increase* as $k \to \infty$.

Chapter 5 SUPPLEMENTARY EXERCISES

1. **a**. True. If A is invertible and if $A\mathbf{x} = 1 \cdot \mathbf{x}$ for some nonzero $\mathbf{x}$, then left-multiply by A^{-1} to obtain $\mathbf{x} = A^{-1}\mathbf{x}$, which may be rewritten as $A^{-1}\mathbf{x} = 1 \cdot \mathbf{x}$. Since $\mathbf{x}$ is nonzero, this shows 1 is an eigenvalue of A^{-1}.

b. False. If A is row equivalent to the identity matrix, then A is invertible. The matrix in Example 4 of Section 5.3 shows that an invertible matrix need not be diagonalizable. Also, see Exercise 31 in Section 5.3.

c. True. If A contains a row or column of zeros, then A is not row equivalent to the identity matrix and thus is not invertible. By the Invertible Matrix Theorem (as stated in Section 5.2), 0 is an eigenvalue of A.

d. False. Consider a diagonal matrix D whose eigenvalues are 1 and 3, that is, its diagonal entries are 1 and 3. Then D^2 is a diagonal matrix whose eigenvalues (diagonal entries) are 1 and 9. In general, the eigenvalues of A^2 are the *squares* of the eigenvalues of A.

e. True. Suppose a nonzero vector $\mathbf{x}$ satisfies $A\mathbf{x} = \lambda\mathbf{x}$, then

$$A^2\mathbf{x} = A(A\mathbf{x}) = A(\lambda\mathbf{x}) = \lambda A\mathbf{x} = \lambda^2\mathbf{x}$$

This shows that $\mathbf{x}$ is also an eigenvector for A^2

f. True. Suppose a nonzero vector $\mathbf{x}$ satisfies $A\mathbf{x} = \lambda\mathbf{x}$, then left-multiply by A^{-1} to obtain $\mathbf{x} = A^{-1}(\lambda\mathbf{x}) = \lambda A^{-1}\mathbf{x}$. Since A is invertible, the eigenvalue λ is not zero. So $\lambda^{-1}\mathbf{x} = A^{-1}\mathbf{x}$, which shows that $\mathbf{x}$ is also an eigenvector of A^{-1}.

g. False. Zero is an eigenvalue of each singular square matrix.

h. True. By definition, an eigenvector must be nonzero.

i. False. Let $\mathbf{v}$ be an eigenvector for A. Then $\mathbf{v}$ and $2\mathbf{v}$ are distinct eigenvectors for the same eigenvalue (because the eigenspace is a subspace), but $\mathbf{v}$ and $2\mathbf{v}$ are linearly dependent.

j. True. This follows from Theorem 4 in Section 5.2

k. False. Let A be the 3×3 matrix in Example 3 of Section 5.3. Then A is similar to a diagonal matrix D. The eigenvectors of D are the columns of I_3, but the eigenvectors of A are entirely different.

l. False. Let $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$. Then $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are eigenvectors of A, but $\mathbf{e}_1 + \mathbf{e}_2$ is not.

(Actually, it can be shown that if two eigenvectors of A correspond to distinct eigenvalues, then their sum cannot be an eigenvector.)

m. False. *All* the diagonal entries of an upper triangular matrix are the eigenvalues of the matrix (Theorem 1 in Section 5.1). A diagonal entry may be zero.

n. True. Matrices A and A^T have the same characteristic polynomial, because $\det(A^T - \lambda I) = \det(A - \lambda I)^T = \det(A - \lambda I)$, by the determinant transpose property.

o. False. Counterexample: Let A be the 5×5 identity matrix.

p. True. For example, let A be the matrix that rotates vectors through $\pi/2$ radians about the origin. Then $A\mathbf{x}$ is not a multiple of $\mathbf{x}$ when $\mathbf{x}$ is nonzero.

q. False. If A is a diagonal matrix with 0 on the diagonal, then the columns of A are not linearly independent.

r. True. If $A\mathbf{x} = \lambda_1\mathbf{x}$ and $A\mathbf{x} = \lambda_2\mathbf{x}$, then $\lambda_1\mathbf{x} = \lambda_2\mathbf{x}$ and $(\lambda_1 - \lambda_2)\mathbf{x} = \mathbf{0}$. If $\mathbf{x} \neq \mathbf{0}$, then λ_1 must equal λ_2.

s. False. Let A be a singular matrix that is diagonalizable. (For instance, let A be a diagonal matrix with 0 on the diagonal.) Then, by Theorem 8 in Section 5.4, the transformation $\mathbf{x} \mapsto A\mathbf{x}$ is represented by a diagonal matrix relative to a coordinate system determined by eigenvectors of A.

t. True. By definition of matrix multiplication,

$$A = AI = A[\mathbf{e}_1 \quad \mathbf{e}_2 \quad \cdots \quad \mathbf{e}_n] = [A\mathbf{e}_1 \quad A\mathbf{e}_2 \quad \cdots \quad A\mathbf{e}_n]$$

If $A\mathbf{e}_j = d_j\mathbf{e}_j$ for $j = 1, \ldots, n$, then A is a diagonal matrix with diagonal entries $d_1, \ldots, d_n$.

u. True. If $B = PDP^{-1}$, where D is a diagonal matrix, and if $A = QBQ^{-1}$, then $A = Q(PDP^{-1})Q^{-1} = (QP)D(PQ)^{-1}$, which shows that A is diagonalizable.

v. True. Since B is invertible, AB is similar to $B(AB)B^{-1}$, which equals BA.

w. False. Having n linearly independent eigenvectors makes an $n \times n$ matrix diagonalizable (by the Diagonalization Theorem 5 in Section 5.3), but not necessarily invertible. One of the eigenvalues of the matrix could be zero.

x. True. If A is diagonalizable, then by the Diagonalization Theorem, A has n linearly independent eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ in $\mathbf{R}^n$. By the Basis Theorem, $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ spans $\mathbf{R}^n$. This means that each vector in $\mathbf{R}^n$ can be written as a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_n$.

2. Suppose $B\mathbf{x} \neq \mathbf{0}$ and $AB\mathbf{x} = \lambda\mathbf{x}$ for some λ. Then $A(B\mathbf{x}) = \lambda\mathbf{x}$. Left-multiply each side by B and obtain $BA(B\mathbf{x}) = B(\lambda\mathbf{x}) = \lambda(B\mathbf{x})$. This equation says that $B\mathbf{x}$ is an eigenvector of BA, because $B\mathbf{x} \neq \mathbf{0}$.

3. **a**. Suppose $A\mathbf{x} = \lambda\mathbf{x}$, with $\mathbf{x} \neq \mathbf{0}$. Then $(5I - A)\mathbf{x} = 5\mathbf{x} - A\mathbf{x} = 5\mathbf{x} - \lambda\mathbf{x} = (5 - \lambda)\mathbf{x}$. The eigenvalue is $5 - \lambda$.

b. $(5I - 3A + A^2)\mathbf{x} = 5\mathbf{x} - 3A\mathbf{x} + A(A\mathbf{x}) = 5\mathbf{x} - 3(\lambda\mathbf{x}) + \lambda^2\mathbf{x} = (5 - 3\lambda + \lambda^2)\mathbf{x}$. The eigenvalue is $5 - 3\lambda + \lambda^2$.

4. Assume that $A\mathbf{x} = \lambda\mathbf{x}$ for some nonzero vector $\mathbf{x}$. The desired statement is true for $m = 1$, by the assumption about λ. Suppose that for some $k \geq 1$, the statement holds when $m = k$. That is, suppose that $A^k\mathbf{x} = \lambda^k\mathbf{x}$. Then $A^{k+1}\mathbf{x} = A(A^k\mathbf{x}) = A(\lambda^k\mathbf{x})$ by the induction hypothesis. Continuing, $A^{k+1}\mathbf{x} = \lambda^k A\mathbf{x} = \lambda^{k+1}\mathbf{x}$, because $\mathbf{x}$ is an eigenvector of A corresponding to A. Since $\mathbf{x}$ is nonzero, this equation shows that λ^{k+1} is an eigenvalue of A^{k+1}, with corresponding eigenvector $\mathbf{x}$. Thus the desired statement is true when $m = k + 1$. By the principle of induction, the statement is true for each positive integer m.

5. Suppose $A\mathbf{x} = \lambda\mathbf{x}$, with $\mathbf{x} \neq \mathbf{0}$. Then

$$\begin{aligned} p(A)\mathbf{x} &= (c_0I + c_1A + c_2A^2 + \ldots + c_nA^n)\mathbf{x} \\ &= c_0\mathbf{x} + c_1A\mathbf{x} + c_2A^2\mathbf{x} + \ldots + c_nA^n\mathbf{x} \\ &= c_0\mathbf{x} + c_1\lambda\mathbf{x} + c_2\lambda^2\mathbf{x} + \ldots + c_n\lambda^n\mathbf{x} = p(\lambda)\mathbf{x} \end{aligned}$$

So $p(\lambda)$ is an eigenvalue of $p(A)$.

6. **a**. If $A = PDP^{-1}$, then $A^k = PD^kP^{-1}$, and

$$B = 5I - 3A + A^2 = 5PIP^{-1} - 3PDP^{-1} + PD^2P^{-1}$$
$$= P(5I - 3D + D^2)P^{-1}$$

Since D is diagonal, so is $5I - 3D + D^2$. Thus B is similar to a diagonal matrix.

b. $p(A) = c_0I + c_1PDP^{-1} + c_2PD^2P^{-1} + \cdots + c_nPD^nP^{-1}$

$$= P(c_0I + c_1D + c_2D^2 + \cdots + c_nD^n)P^{-1}$$
$$= Pp(D)P^{-1}$$

This shows that $p(A)$ is diagonalizable, because $p(D)$ is a linear combination of diagonal matrices and hence is diagonal. In fact, because D is diagonal, it is easy to see that

$$p(D) = \begin{bmatrix} p(2) & 0 \\ 0 & p(7) \end{bmatrix}$$

7. If $A = PDP^{-1}$, then $p(A) = Pp(D)P^{-1}$, as shown in Exercise 6. If the (j, j) entry in D is λ, then the (j, j) entry in D^k is λ^k, and so the (j, j) entry in $p(D)$ is $p(\lambda)$. If p is the characteristic polynomial of A, then $p(\lambda) = 0$ for each diagonal entry of D, because these entries in D are the eigenvalues of A. Thus $p(D)$ is the zero matrix. Thus $p(A) = P \cdot 0 \cdot P^{-1} = 0$.

8. **a**. If λ is an eigenvalue of an $n \times n$ diagonalizable matrix A, then $A = PDP^{-1}$ for an invertible matrix P and an $n \times n$ diagonal matrix D whose diagonal entries are the eigenvalues of A. If the multiplicity of λ is n, then λ must appear in every diagonal entry of D. That is, $D = \lambda I$. In this case, $A = P(\lambda I)P^{-1} = \lambda PIP^{-1} = \lambda PP^{-1} = \lambda I$.

b. Since the matrix $A = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$ is triangular, its eigenvalues are on the diagonal. Thus 3 is an eigenvalue with multiplicity 2. If the 2×2 matrix A were diagonalizable, then A would be $3I$, by part (a). This is not the case, so A is not diagonalizable.

9. If $I - A$ were not invertible, then the equation $(I - A)\mathbf{x} = \mathbf{0}$. would have a nontrivial solution $\mathbf{x}$. Then $\mathbf{x} - A\mathbf{x} = \mathbf{0}$ and $A\mathbf{x} = 1 \cdot \mathbf{x}$, which shows that A would have 1 as an eigenvalue. This cannot happen if all the eigenvalues are less than 1 in magnitude. So $I - A$ must be invertible.

10. To show that A^k tends to the zero matrix, it suffices to show that each column of A^k can be made as close to the zero vector as desired by taking k sufficiently large. The jth column of A is $A\mathbf{e}_j$, where $\mathbf{e}_j$ is the jth column of the identity matrix. Since A is diagonalizable, there is a basis for $\mathbb{R}^n$ consisting of eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$, corresponding to eigenvalues $\lambda_1, \ldots, \lambda_n$. So there exist scalars $c_1, \ldots, c_n$, such that

$\mathbf{e}_j = c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n$ (an eigenvector decomposition of $\mathbf{e}_j$)

Then, for $k = 1, 2, \ldots,$

$$A^k\mathbf{e}_j = c_1(\lambda_1)^k\mathbf{v}_1 + \cdots + c_n(\lambda_n)^k\mathbf{v}_n \qquad (*)$$

If the eigenvalues are all less than 1 in absolute value, then their kth powers all tend to zero. So $(*)$ shows that $A^k\mathbf{e}_j$ tends to the zero vector, as desired.

11. **a**. Take $\mathbf{x}$ in H. Then $\mathbf{x} = c\mathbf{u}$ for some scalar c. So $A\mathbf{x} = A(c\mathbf{u}) = c(A\mathbf{u}) = c(\lambda\mathbf{u}) = (c\lambda)\mathbf{u}$, which shows that $A\mathbf{x}$ is in H.

b. Let $\mathbf{x}$ be a nonzero vector in K. Since K is one-dimensional, K must be the set of all scalar multiples of $\mathbf{x}$. If K is invariant under A, then $A\mathbf{x}$ is in K and hence $A\mathbf{x}$ is a multiple of $\mathbf{x}$. Thus $\mathbf{x}$ is an eigenvector of A.

12. Let U and V be echelon forms of A and B, obtained with r and s row interchanges, respectively, and no scaling. Then $\det A = (-1)^r \det U$ and $\det B = (-1)^s \det V$

Using first the row operations that reduce A to U, we can reduce G to a matrix of the form $G' = \begin{bmatrix} U & Y \\ 0 & B \end{bmatrix}$.

Then, using the row operations that reduce B to V, we can further reduce G' to $G'' = \begin{bmatrix} U & Y \\ 0 & V \end{bmatrix}$. There will be $r+s$ row interchanges, and so $\det G = \det \begin{bmatrix} A & X \\ 0 & B \end{bmatrix} = (-1)^{r+s} \det \begin{bmatrix} U & Y \\ 0 & V \end{bmatrix}$ Since $\begin{bmatrix} U & Y \\ 0 & V \end{bmatrix}$ is upper triangular, its determinant equals the product of the diagonal entries, and since U and V are upper triangular, this product also equals (det U) (det V). Thus

$\det G = (-1)^{r+s}(\det U)(\det V) = (\det A)(\det B)$

For any scalar λ, the matrix $G - \lambda I$ has the same partitioned form as G, with $A - \lambda I$ and $B - \lambda I$ as its diagonal blocks. (Here I represents various identity matrices of appropriate sizes.) Hence the result about det G shows that $\det(G - \lambda I) = \det(A - \lambda I) \cdot \det(B - \lambda I)$

13. By Exercise 12, the eigenvalues of A are the eigenvalues of the matrix $[3]$ together with the eigenvalues of $\begin{bmatrix} 5 & -2 \\ -4 & 3 \end{bmatrix}$. The only eigenvalue of $[3]$ is 3, while the eigenvalues of $\begin{bmatrix} 5 & -2 \\ -4 & 3 \end{bmatrix}$ are 1 and 7. Thus the eigenvalues of A are 1, 3, and 7.

14. By Exercise 12, the eigenvalues of A are the eigenvalues of the matrix $\begin{bmatrix} 1 & 5 \\ 2 & 4 \end{bmatrix}$ together with the eigenvalues of $\begin{bmatrix} -7 & -4 \\ 3 & 1 \end{bmatrix}$. The eigenvalues of $\begin{bmatrix} 1 & 5 \\ 2 & 4 \end{bmatrix}$ are -1 and 6, while the eigenvalues of $\begin{bmatrix} -7 & -4 \\ 3 & 1 \end{bmatrix}$ are -5 and -1. Thus the eigenvalues of A are $-1, -5,$ and 6, and the eigenvalue -1 has multiplicity 2.

15. Replace A by $A - \lambda$ in the determinant formula from Exercise 16 in Chapter 3 Supplementary Exercises.

$\det(A - \lambda I) = (a - b - \lambda)^{n-1}[a - \lambda + (n-1)b]$

This determinant is zero only if $a - b - \lambda = 0$ or $a - \lambda + (n-1)b = 0$. Thus λ is an eigenvalue of A if and only if $\lambda = a - b$ or $\lambda = a + (n-1)$. From the formula for $\det(A - \lambda I)$ above, the algebraic multiplicity is $n-1$ for $a - b$ and 1 for $a + (n-1)b$.

16. The 3×3 matrix has eigenvalues $1-2$ and $1+(2)(2)$, that is, -1 and 5. The eigenvalues of the 5×5 matrix are $7-3$ and $7+(4)(3)$, that is 4 and 19.

17. Note that $\det(A-\lambda I)=(a_{11}-\lambda)(a_{22}-\lambda)-a_{12}a_{21}=\lambda^2-(a_{11}+a_{22})\lambda+(a_{11}a_{22}-a_{12}a_{21})$ $=\lambda^2-(\text{tr }A)\lambda+\det A,$ and use the quadratic formula to solve the characteristic equation:

$$\lambda=\frac{\text{tr }A\pm\sqrt{(\text{tr }A)^2-4\det A}}{2}$$

The eigenvalues are both real if and only if the discriminant is nonnegative, that is, $(\text{tr }A)^2-4\det A\geq 0.$

This inequality simplifies to $(\text{tr }A)^2\geq 4\det A$ and $\left(\frac{trA}{2}\right)^2\geq \det A.$

18. The eigenvalues of A are 1 and .6. Use this to factor A and A^k.

$$A=\begin{bmatrix}-1&-3\\2&2\end{bmatrix}\begin{bmatrix}1&0\\0&.6\end{bmatrix}\cdot\frac{1}{4}\begin{bmatrix}2&3\\-2&-1\end{bmatrix}$$

$$A^k=\begin{bmatrix}-1&-3\\2&2\end{bmatrix}\begin{bmatrix}1^k&0\\0&.6^k\end{bmatrix}\cdot\frac{1}{4}\begin{bmatrix}2&3\\-2&-1\end{bmatrix}$$

$$=\frac{1}{4}\begin{bmatrix}-1&-3\\2&2\end{bmatrix}\begin{bmatrix}2&3\\-2\cdot(.6)^k&-(.6)^k\end{bmatrix}$$

$$=\frac{1}{4}\begin{bmatrix}-2+6(.6)^k&-3+3(.6)^k\\4-4(.6)^k&6-2(.6)^k\end{bmatrix}$$

$$\to\frac{1}{4}\begin{bmatrix}-2&-3\\4&6\end{bmatrix}\text{ as } k\to\infty$$

19. $C_p=\begin{bmatrix}0&1\\-6&5\end{bmatrix};\ \det(C_p-\lambda I)=6-5\lambda+\lambda^2=p(\lambda)$

20. $C_p=\begin{bmatrix}0&1&0\\0&0&1\\24&-26&9\end{bmatrix};$

$\det(C_p-\lambda I)=24-26\lambda+9\lambda^2-\lambda^3=p(\lambda)$

21. If p is a polynomial of order 2, then a calculation such as in Exercise 19 shows that the characteristic polynomial of C_p is $p(\lambda)=(-1)^2p(\lambda),$ so the result is true for $n=2.$ Suppose the result is true for $n=k$ for some $k\geq 2,$ and consider a polynomial p of degree $k+1.$ Then expanding $\det(C_p-\lambda I)$ by cofactors down the first column, the determinant of $C_p-\lambda I$ equals

$$(-\lambda)\det\begin{bmatrix}-\lambda&1&\cdots&0\\\vdots&&&\vdots\\0&&&1\\-a_1&-a_2&\cdots&-a_k-\lambda\end{bmatrix}+(-1)^{k+1}a_0$$

The $k \times k$ matrix shown is $C_q - \lambda I$, where $q(t) = a_1 + a_2 t + \cdots + a_k t^{k-1} + t^k$. By the induction assumption, the determinant of $C_q - \lambda I$ is $(-1)^k q(\lambda)$. Thus

$$\begin{aligned}\det(C_p - \lambda I) &= (-1)^{k+1} a_0 + (-\lambda)(-1)^k q(\lambda) \\ &= (-1)^{k+1}[a_0 + \lambda(a_1 + \cdots + a_k \lambda^{k-1} + \lambda^k)] \\ &= (-1)^{k+1} p(\lambda)\end{aligned}$$

So the formula holds for $n = k+1$ when it holds for $n = k$. By the principle of induction, the formula for $\det(C_p - \lambda I)$ is true for all $n \ge 2$.

22. **a**. $C_p = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix}$

b. Since λ is a zero of p, $a_0 + a_1\lambda + a_2\lambda^2 + \lambda^3 = 0$ and $-a_0 - a_1\lambda - a_2\lambda^2 = \lambda^3$. Thus

$$C_p \begin{bmatrix} 1 \\ \lambda \\ \lambda^2 \end{bmatrix} = \begin{bmatrix} \lambda \\ \lambda^2 \\ -a_0 - a_1\lambda - a_2\lambda^2 \end{bmatrix} = \begin{bmatrix} \lambda \\ \lambda^2 \\ \lambda^3 \end{bmatrix}$$

That is, $C_p(1, \lambda, \lambda^2) = \lambda(1, \lambda, \lambda^2)$, which shows that $(1, \lambda, \lambda^2)$ is an eigenvector of C_p corresponding to the eigenvalue λ.

23. From Exercise 22, the columns of the Vandermonde matrix *V* are eigenvectors of C_p, corresponding to the eigenvalues $\lambda_1, \lambda_2, \lambda_3$ (the roots of the polynomial *p*). Since these eigenvalues are distinct, the eigenvectors from a linearly independent set, by Theorem 2 in Section 5.1. Thus *V* has linearly independent columns and hence is invertible, by the Invertible Matrix Theorem. Finally, since the columns of *V* are eigenvectors of C_p, the Diagonalization Theorem (Theorem 5 in Section 5.3) shows that $V^{-1}C_pV$ is diagonal.

24. **[M]** The MATLAB command `roots (p)` requires as input a row vector *p* whose entries are the coefficients of a polynomial, with the highest order coefficient listed first. MATLAB constructs a companion matrix C_p whose characteristic polynomial is *p*, so the roots of *p* are the eigenvalues of C_p. The numerical values of the eigenvalues (roots) are found by the same QR algorithm used by the command `eig(A)`.

25. **[M]** The MATLAB command `[P D] = eig(A)` produces a matrix *P*, whose condition number is 1.6×10^8, and a diagonal matrix *D*, whose entries are *almost* 2, 2, 1. However, the exact eigenvalues of *A* are 2, 2, 1, and *A* is not diagonalizable.

26. **[M]** This matrix may cause the same sort of trouble as the matrix in Exercise 25. A matrix program that computes eigenvalues by an interative process may indicate that *A* has four distinct eigenvalues, all close to zero. However, the only eigenvalue is 0, with multiplicity 4, because $A^4 = 0$.

6 Orthogonality and Least Squares

6.1 SOLUTIONS

Notes: The first half of this section is computational and is easily learned. The second half concerns the concepts of orthogonality and orthogonal complements, which are essential for later work. Theorem 3 is an important general fact, but is needed only for Supplementary Exercise 13 at the end of the chapter and in Section 7.4. The optional material on angles is not used later. Exercises 27–31 concern facts used later.

1. Since $\mathbf{u}=\begin{bmatrix}-1\\2\end{bmatrix}$ and $\mathbf{v}=\begin{bmatrix}4\\6\end{bmatrix}$, $\mathbf{u}\cdot\mathbf{u}=(-1)^2+2^2=5$, $\mathbf{v}\cdot\mathbf{u}=4(-1)+6(2)=8$, and $\dfrac{\mathbf{v}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}=\dfrac{8}{5}$.

2. Since $\mathbf{w}=\begin{bmatrix}3\\-1\\-5\end{bmatrix}$ and $\mathbf{x}=\begin{bmatrix}6\\-2\\3\end{bmatrix}$, $\mathbf{w}\cdot\mathbf{w}=3^2+(-1)^2+(-5)^2=35$, $\mathbf{x}\cdot\mathbf{w}=6(3)+(-2)(-1)+3(-5)=5$, and

$\dfrac{\mathbf{x}\cdot\mathbf{w}}{\mathbf{w}\cdot\mathbf{w}}=\dfrac{5}{35}=\dfrac{1}{7}$.

3. Since $\mathbf{w}=\begin{bmatrix}3\\-1\\-5\end{bmatrix}$, $\mathbf{w}\cdot\mathbf{w}=3^2+(-1)^2+(-5)^2=35$, and $\dfrac{1}{\mathbf{w}\cdot\mathbf{w}}\mathbf{w}=\begin{bmatrix}3/35\\-1/35\\-1/7\end{bmatrix}$.

4. Since $\mathbf{u}=\begin{bmatrix}-1\\2\end{bmatrix}$, $\mathbf{u}\cdot\mathbf{u}=(-1)^2+2^2=5$ and $\dfrac{1}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u}=\begin{bmatrix}-1/5\\2/5\end{bmatrix}$.

5. Since $\mathbf{u}=\begin{bmatrix}-1\\2\end{bmatrix}$ and $\mathbf{v}=\begin{bmatrix}4\\6\end{bmatrix}$, $\mathbf{u}\cdot\mathbf{v}=(-1)(4)+2(6)=8$, $\mathbf{v}\cdot\mathbf{v}=4^2+6^2=52$, and

$\left(\dfrac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v}=\dfrac{2}{13}\begin{bmatrix}4\\6\end{bmatrix}=\begin{bmatrix}8/13\\12/13\end{bmatrix}$.

6. Since $\mathbf{x}=\begin{bmatrix}6\\-2\\3\end{bmatrix}$ and $\mathbf{w}=\begin{bmatrix}3\\-1\\-5\end{bmatrix}$, $\mathbf{x}\cdot\mathbf{w}=6(3)+(-2)(-1)+3(-5)=5$, $\mathbf{x}\cdot\mathbf{x}=6^2+(-2)^2+3^2=49$, and

$\left(\dfrac{\mathbf{x}\cdot\mathbf{w}}{\mathbf{x}\cdot\mathbf{x}}\right)\mathbf{x}=\dfrac{5}{49}\begin{bmatrix}6\\-2\\3\end{bmatrix}=\begin{bmatrix}30/49\\-10/49\\15/49\end{bmatrix}$.

7. Since $\mathbf{w}=\begin{bmatrix}3\\-1\\-5\end{bmatrix}$, $\|\mathbf{w}\|=\sqrt{\mathbf{w}\cdot\mathbf{w}}=\sqrt{3^2+(-1)^2+(-5)^2}=\sqrt{35}.$

8. Since $\mathbf{x}=\begin{bmatrix}6\\-2\\3\end{bmatrix}$, $\|\mathbf{x}\|=\sqrt{\mathbf{x}\cdot\mathbf{x}}=\sqrt{6^2+(-2)^2+3^2}=\sqrt{49}=7.$

9. A unit vector in the direction of the given vector is

$$\frac{1}{\sqrt{(-30)^2+40^2}}\begin{bmatrix}-30\\40\end{bmatrix}=\frac{1}{50}\begin{bmatrix}-30\\40\end{bmatrix}=\begin{bmatrix}-3/5\\4/5\end{bmatrix}$$

10. A unit vector in the direction of the given vector is

$$\frac{1}{\sqrt{(-6)^2+4^2+(-3)^2}}\begin{bmatrix}-6\\4\\-3\end{bmatrix}=\frac{1}{\sqrt{61}}\begin{bmatrix}-6\\4\\-3\end{bmatrix}=\begin{bmatrix}-6/\sqrt{61}\\4/\sqrt{61}\\-3\sqrt{61}\end{bmatrix}$$

11. A unit vector in the direction of the given vector is

$$\frac{1}{\sqrt{(7/4)^2+(1/2)^2+1^2}}\begin{bmatrix}7/4\\1/2\\1\end{bmatrix}=\frac{1}{\sqrt{69/16}}\begin{bmatrix}7/4\\1/2\\1\end{bmatrix}=\begin{bmatrix}7/\sqrt{69}\\2/\sqrt{69}\\4/\sqrt{69}\end{bmatrix}$$

12. A unit vector in the direction of the given vector is

$$\frac{1}{\sqrt{(8/3)^2+2^2}}\begin{bmatrix}8/3\\2\end{bmatrix}=\frac{1}{\sqrt{100/9}}\begin{bmatrix}8/3\\2\end{bmatrix}=\begin{bmatrix}4/5\\3/5\end{bmatrix}$$

13. Since $\mathbf{x}=\begin{bmatrix}10\\-3\end{bmatrix}$ and $\mathbf{y}=\begin{bmatrix}-1\\-5\end{bmatrix}$, $\|\mathbf{x}-\mathbf{y}\|^2=[10-(-1)]^2+[-3-(-5)]^2=125$ and $\text{dist}\,(\mathbf{x},\mathbf{y})=\sqrt{125}=5\sqrt{5}.$

14. Since $\mathbf{u}=\begin{bmatrix}0\\-5\\2\end{bmatrix}$ and $\mathbf{z}=\begin{bmatrix}-4\\-1\\8\end{bmatrix}$, $\|\mathbf{u}-\mathbf{z}\|^2=[0-(-4)]^2+[-5-(-1)]^2+[2-8]^2=68$ and

$\text{dist}\,(\mathbf{u},\mathbf{z})=\sqrt{68}=2\sqrt{17}.$

15. Since $\mathbf{a}\cdot\mathbf{b}=8(-2)+(-5)(-3)=-1\neq 0$, **a** and **b** are not orthogonal.

16. Since $\mathbf{u}\cdot\mathbf{v}=12(2)+(3)(-3)+(-5)(3)=0$, **u** and **v** are orthogonal.

17. Since $\mathbf{u}\cdot\mathbf{v}=3(-4)+2(1)+(-5)(-2)+0(6)=0$, **u** and **v** are orthogonal.

18. Since $\mathbf{y}\cdot\mathbf{z}=(-3)(1)+7(-8)+4(15)+0(-7)=1\neq 0$, **y** and **z** are not orthogonal.

19. **a**. True. See the definition of $\|\mathbf{v}\|$.
b. True. See Theorem 1(c).
c. True. See the discussion of Figure 5.

d. False. Counterexample: $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$.

e. True. See the box following Example 6.

20. **a**. True. See Example 1 and Theorem 1(a).

b. False. The absolute value sign is missing. See the box before Example 2.

c. True. See the defintion of orthogonal complement.

d. True. See the Pythagorean Theorem.

e. True. See Theorem 3.

21. Theorem 1(b):

$$(\mathbf{u}+\mathbf{v})\cdot\mathbf{w} = (\mathbf{u}+\mathbf{v})^T\mathbf{w} = (\mathbf{u}^T+\mathbf{v}^T)\mathbf{w} = \mathbf{u}^T\mathbf{w}+\mathbf{v}^T\mathbf{w} = \mathbf{u}\cdot\mathbf{w}+\mathbf{v}\cdot\mathbf{w}$$

The second and third equalities used Theorems 3(b) and 2(c), respectively, from Section 2.1.

Theorem 1(c):

$$(c\mathbf{u})\cdot\mathbf{v} = (c\mathbf{u})^T\mathbf{v} = c(\mathbf{u}^T\mathbf{v}) = c(\mathbf{u}\cdot\mathbf{v})$$

The second and third equalities used Theorems 3(c) and 2(d), respectively, from Section 2.1.

22. Since $\mathbf{u}\cdot\mathbf{u}$ is the sum of the squares of the entries in $\mathbf{u}$, $\mathbf{u}\cdot\mathbf{u} \geq 0$. The sum of squares of numbers is zero if and only if all the numbers are themselves zero.

23. One computes that $\mathbf{u}\cdot\mathbf{v} = 2(-7) + (-5)(-4) + (-1)6 = 0$, $\|\mathbf{u}\|^2 = \mathbf{u}\cdot\mathbf{u} = 2^2+(-5)^2+(-1)^2 = 30$, $\|\mathbf{v}\|^2 = \mathbf{v}\cdot\mathbf{v} = (-7)^2+(-4)^2+6^2 = 101$, and $\|\mathbf{u}+\mathbf{v}\|^2 = (\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v}) =$ $(2+(-7))^2+(-5+(-4))^2+(-1+6)^2 = 131$.

24. One computes that

$$\|\mathbf{u}+\mathbf{v}\|^2 = (\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v}) = \mathbf{u}\cdot\mathbf{u}+2\mathbf{u}\cdot\mathbf{v}+\mathbf{v}\cdot\mathbf{v} = \|\mathbf{u}\|^2+2\mathbf{u}\cdot\mathbf{v}+\|\mathbf{v}\|^2$$

and

$$\|\mathbf{u}-\mathbf{v}\|^2 = (\mathbf{u}-\mathbf{v})\cdot(\mathbf{u}-\mathbf{v}) = \mathbf{u}\cdot\mathbf{u}-2\mathbf{u}\cdot\mathbf{v}+\mathbf{v}\cdot\mathbf{v} = \|\mathbf{u}\|^2-2\mathbf{u}\cdot\mathbf{v}+\|\mathbf{v}\|^2$$

so

$$\|\mathbf{u}+\mathbf{v}\|^2+\|\mathbf{u}-\mathbf{v}\|^2 = \|\mathbf{u}\|^2+2\mathbf{u}\cdot\mathbf{v}+\|\mathbf{v}\|^2+\|\mathbf{u}\|^2-2\mathbf{u}\cdot\mathbf{v}+\|\mathbf{v}\|^2 = 2\|\mathbf{u}\|^2+2\|\mathbf{v}\|^2$$

25. When $\mathbf{v} = \begin{bmatrix} a \\ b \end{bmatrix}$, the set H of all vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ that are orthogonal to $\boldsymbol{v}$ is the subspace of vectors whose entries satisfy $ax + by = 0$. If $a \neq 0$, then $x = -(b/a)y$ with y a free variable, and H is a line through the origin. A natural choice for a basis for H in this case is $\left\{\begin{bmatrix} -b \\ a \end{bmatrix}\right\}$. If $a = 0$ and $b \neq 0$, then $by = 0$. Since $b \neq 0$, $y = 0$ and x is a free variable. The subspace H is again a line through the origin. A natural choice for a basis for H in this case is $\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$, but $\left\{\begin{bmatrix} -b \\ a \end{bmatrix}\right\}$ is still a basis for H since $a = 0$ and $b \neq 0$. If $a = 0$ and $b = 0$, then $H = \mathbb{R}^2$ since the equation $0x + 0y = 0$ places no restrictions on x or y.

26. Theorem 2 in Chapter 4 may be used to show that W is a subspace of $\mathbb{R}^3$, because W is the null space of the 1×3 matrix $\mathbf{u}^T$. Geometrically, W is a plane through the origin.

27. If $\mathbf{y}$ is orthogonal to $\mathbf{u}$ and $\mathbf{v}$, then $\mathbf{y}\cdot\mathbf{u} = \mathbf{y}\cdot\mathbf{v} = 0$, and hence by a property of the inner product, $\mathbf{y}\cdot(\mathbf{u}+\mathbf{v}) = \mathbf{y}\cdot\mathbf{u} + \mathbf{y}\cdot\mathbf{v} = 0 + 0 = 0$. Thus $\mathbf{y}$ is orthogonal to $\mathbf{u}+\mathbf{v}$.

28. An arbitrary $\mathbf{w}$ in Span{$\mathbf{u}$, $\mathbf{v}$} has the form $\mathbf{w} = c_1\mathbf{u} + c_2\mathbf{v}$. If $\mathbf{y}$ is orthogonal to $\mathbf{u}$ and $\mathbf{v}$, then $\mathbf{u}\cdot\mathbf{y} = \mathbf{v}\cdot\mathbf{y} = 0$. By Theorem 1(b) and 1(c),

$$\mathbf{w}\cdot\mathbf{y} = (c_1\mathbf{u} + c_2\mathbf{v})\cdot\mathbf{y} = c_1(\mathbf{u}\cdot\mathbf{y}) + c_2(\mathbf{v}\cdot\mathbf{y}) = 0 + 0 = 0$$

29. A typical vector in W has the form $\mathbf{w} = c_1\mathbf{v}_1 + \ldots + c_p\mathbf{v}_p$. If $\mathbf{x}$ is orthogonal to each $\mathbf{v}_j$, then by Theorems 1(b) and 1(c),

$$\mathbf{w}\cdot\mathbf{x} = (c_1\mathbf{v}_1 + \ldots + c_p\mathbf{v}_p)\cdot\mathbf{y} = c_1(\mathbf{v}_1\cdot\mathbf{x}) + \ldots + c_p(\mathbf{v}_p\cdot\mathbf{x}) = 0$$

So $\mathbf{x}$ is orthogonal to each $\mathbf{w}$ in W.

30. **a**. If $\mathbf{z}$ is in $W^\perp$, $\mathbf{u}$ is in W, and c is any scalar, then $(c\mathbf{z})\cdot\mathbf{u} = c(\mathbf{z}\cdot\mathbf{u}) - c0 = 0$. Since $\mathbf{u}$ is any element of W, $c\mathbf{z}$ is in $W^\perp$.

b. Let $\mathbf{z}_1$ and $\mathbf{z}_2$ be in $W^\perp$. Then for any $\mathbf{u}$ in W, $(\mathbf{z}_1 + \mathbf{z}_2)\cdot\mathbf{u} = \mathbf{z}_1\cdot\mathbf{u} + \mathbf{z}_2\cdot\mathbf{u} = 0 + 0 = 0$. Thus $\mathbf{z}_1 + \mathbf{z}_2$ is in $W^\perp$.

c. Since $\mathbf{0}$ is orthogonal to every vector, $\mathbf{0}$ is in $W^\perp$. Thus $W^\perp$ is a subspace.

31. Suppose that $\mathbf{x}$ is in W and $W^\perp$. Since $\mathbf{x}$ is in $W^\perp$, $\mathbf{x}$ is orthogonal to every vector in W, including $\mathbf{x}$ itself. So $\mathbf{x}\cdot\mathbf{x} = 0$, which happens only when $\mathbf{x} = \mathbf{0}$.

32. **[M]**

a. One computes that $\|\mathbf{a}_1\| = \|\mathbf{a}_2\| = \|\mathbf{a}_3\| = \|\mathbf{a}_4\| = 1$ and that $\mathbf{a}_i\cdot\mathbf{a}_j = 0$ for $i \neq j$.

b. Answers will vary, but it should be that $\|A\mathbf{u}\| = \|\mathbf{u}\|$ and $\|A\mathbf{v}\| = \|\mathbf{v}\|$.

c. Answers will again vary, but the cosines should be equal.

d. A conjecture is that multiplying by A does not change the lengths of vectors or the angles between vectors.

33. **[M]** Answers to the calculations will vary, but will demonstrate that the mapping $\mathbf{x} \mapsto T(\mathbf{x}) = \left(\dfrac{\mathbf{x}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v}$ (for $\mathbf{v} \neq \mathbf{0}$) is a linear transformation. To confirm this, let $\mathbf{x}$ and $\mathbf{y}$ be in $\mathbb{R}^n$, and let c be any scalar. Then

$$T(\mathbf{x}+\mathbf{y}) = \left(\frac{(\mathbf{x}+\mathbf{y})\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v} = \left(\frac{(\mathbf{x}\cdot\mathbf{v}) + (\mathbf{y}\cdot\mathbf{v})}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v} = \left(\frac{\mathbf{x}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v} + \left(\frac{\mathbf{y}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v} = T(\mathbf{x}) + T(\mathbf{y})$$

and

$$T(c\mathbf{x}) = \left(\frac{(c\mathbf{x})\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v} = \left(\frac{c(\mathbf{x}\cdot\mathbf{v})}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v} = c\left(\frac{\mathbf{x}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v} = cT(\mathbf{x})$$

34. **[M]** One finds that

$$N = \begin{bmatrix} -5 & 1 \\ -1 & 4 \\ 1 & 0 \\ 0 & -1 \\ 0 & 3 \end{bmatrix}, R = \begin{bmatrix} 1 & 0 & 5 & 0 & -1/3 \\ 0 & 1 & 1 & 0 & -4/3 \\ 0 & 0 & 0 & 1 & 1/3 \end{bmatrix}$$

The row-column rule for computing RN produces the 3×2 zero matrix, which shows that the rows of R are orthogonal to the columns of N. This is expected by Theorem 3 since each row of R is in Row A and each column of N is in Nul A.

6.2 SOLUTIONS

Notes: The nonsquare matrices in Theorems 6 and 7 are needed for the QR factorizarion in Section 6.4. It is important to emphasize that the term *orthogonal matrix* applies only to certain *square* matrices. The subsection on orthogonal projections not only sets the stage for the general case in Section 6.3, it also provides what is needed for the orthogonal diagonalization exercises in Section 7.1, because none of the eigenspaces there have dimension greater than 2. For this reason, the Gram-Schmidt process (Section 6.4) is not really needed in Chapter 7. Exercises 13 and 14 prepare for Section 6.3.

1. Since $\begin{bmatrix} -1 \\ 4 \\ -3 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ -4 \\ -7 \end{bmatrix} = 2 \neq 0,$ the set is not orthogonal.

2. Since $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -5 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} -5 \\ -2 \\ 1 \end{bmatrix} = 0,$ the set is orthogonal.

3. Since $\begin{bmatrix} -6 \\ -3 \\ 9 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} = -30 \neq 0,$ the set is not orthogonal.

4. Since $\begin{bmatrix} 2 \\ -5 \\ -3 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -5 \\ -3 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ -2 \\ 6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ -2 \\ 6 \end{bmatrix} = 0,$ the set is orthogonal.

5. Since $\begin{bmatrix} 3 \\ -2 \\ 1 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 3 \\ -3 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ 1 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 8 \\ 7 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ -3 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 8 \\ 7 \\ 0 \end{bmatrix} = 0,$ the set is orthogonal.

6. Since $\begin{bmatrix} -4 \\ 1 \\ -3 \\ 8 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 3 \\ 5 \\ -1 \end{bmatrix} = -32 \neq 0,$ the set is not orthogonal.

7. Since $\mathbf{u}_1 \cdot \mathbf{u}_2 = 12 - 12 = 0,$ $\{\mathbf{u}_1, \mathbf{u}_2\}$ is an orthogonal set. Since the vectors are non-zero, $\mathbf{u}_1$ and $\mathbf{u}_2$ are linearly independent by Theorem 4. Two such vectors in $\mathbb{R}^2$ automatically form a basis for $\mathbb{R}^2$. So $\{\mathbf{u}_1, \mathbf{u}_2\}$ is an orthogonal basis for $\mathbb{R}^2$. By Theorem 5,

$$\mathbf{x} = \frac{\mathbf{x} \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1}\mathbf{u}_1 + \frac{\mathbf{x} \cdot \mathbf{u}_2}{\mathbf{u}_2 \cdot \mathbf{u}_2}\mathbf{u}_2 = 3\mathbf{u}_1 + \frac{1}{2}\mathbf{u}_2$$

8. Since $\mathbf{u}_1 \cdot \mathbf{u}_2 = -6 + 6 = 0$, $\{\mathbf{u}_1, \mathbf{u}_2\}$ is an orthogonal set. Since the vectors are non-zero, $\mathbf{u}_1$ and $\mathbf{u}_2$ are linearly independent by Theorem 4. Two such vectors in $\mathbb{R}^2$ automatically form a basis for $\mathbb{R}^2$. So $\{\mathbf{u}_1, \mathbf{u}_2\}$ is an orthogonal basis for $\mathbb{R}^2$. By Theorem 5,

$$\mathbf{x} = \frac{\mathbf{x} \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1}\mathbf{u}_1 + \frac{\mathbf{x} \cdot \mathbf{u}_2}{\mathbf{u}_2 \cdot \mathbf{u}_2} = -\frac{3}{2}\mathbf{u}_1 + \frac{3}{4}\mathbf{u}_2$$

9. Since $\mathbf{u}_1 \cdot \mathbf{u}_2 = \mathbf{u}_1 \cdot \mathbf{u}_3 = \mathbf{u}_2 \cdot \mathbf{u}_3 = 0$, $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthogonal set. Since the vectors are non-zero, $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$ are linearly independent by Theorem 4. Three such vectors in $\mathbb{R}^3$ automatically form a basis for $\mathbb{R}^3$. So $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthogonal basis for $\mathbb{R}^3$. By Theorem 5,

$$\mathbf{x} = \frac{\mathbf{x} \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1}\mathbf{u}_1 + \frac{\mathbf{x} \cdot \mathbf{u}_2}{\mathbf{u}_2 \cdot \mathbf{u}_2} + \frac{\mathbf{x} \cdot \mathbf{u}_3}{\mathbf{u}_3 \cdot \mathbf{u}_3}\mathbf{u}_3 = \frac{5}{2}\mathbf{u}_1 - \frac{3}{2}\mathbf{u}_2 + 2\mathbf{u}_3$$

10. Since $\mathbf{u}_1 \cdot \mathbf{u}_2 = \mathbf{u}_1 \cdot \mathbf{u}_3 = \mathbf{u}_2 \cdot \mathbf{u}_3 = 0$, $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthogonal set. Since the vectors are non-zero, $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$ are linearly independent by Theorem 4. Three such vectors in $\mathbb{R}^3$ automatically form a basis for $\mathbb{R}^3$. So $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthogonal basis for $\mathbb{R}^3$. By Theorem 5,

$$\mathbf{x} = \frac{\mathbf{x} \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1}\mathbf{u}_1 + \frac{\mathbf{x} \cdot \mathbf{u}_2}{\mathbf{u}_2 \cdot \mathbf{u}_2} + \frac{\mathbf{x} \cdot \mathbf{u}_3}{\mathbf{u}_3 \cdot \mathbf{u}_3}\mathbf{u}_3 = \frac{4}{3}\mathbf{u}_1 + \frac{1}{3}\mathbf{u}_2 + \frac{1}{3}\mathbf{u}_3$$

11. Let $\mathbf{y} = \begin{bmatrix} 1 \\ 7 \end{bmatrix}$ and $\mathbf{u} = \begin{bmatrix} -4 \\ 2 \end{bmatrix}$. The orthogonal projection of **y** onto the line through **u** and the origin is the orthogonal projection of **y** onto **u**, and this vector is

$$\hat{\mathbf{y}} = \frac{\mathbf{y} \cdot \mathbf{u}}{\mathbf{u} \cdot \mathbf{u}}\mathbf{u} = \frac{1}{2}\mathbf{u} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

12. Let $\mathbf{y} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\mathbf{u} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$. The orthogonal projection of **y** onto the line through **u** and the origin is the orthogonal projection of **y** onto **u**, and this vector is

$$\hat{\mathbf{y}} = \frac{\mathbf{y} \cdot \mathbf{u}}{\mathbf{u} \cdot \mathbf{u}}\mathbf{u} = -\frac{2}{5}\mathbf{u} = \begin{bmatrix} 2/5 \\ -6/5 \end{bmatrix}$$

13. The orthogonal projection of **y** onto **u** is

$$\hat{\mathbf{y}} = \frac{\mathbf{y} \cdot \mathbf{u}}{\mathbf{u} \cdot \mathbf{u}}\mathbf{u} = -\frac{13}{65}\mathbf{u} = \begin{bmatrix} -4/5 \\ 7/5 \end{bmatrix}$$

The component of **y** orthogonal to **u** is

$$\mathbf{y} - \hat{\mathbf{y}} = \begin{bmatrix} 14/5 \\ 8/5 \end{bmatrix}$$

Thus $\mathbf{y} = \hat{\mathbf{y}} + (\mathbf{y} - \hat{\mathbf{y}}) = \begin{bmatrix} -4/5 \\ 7/5 \end{bmatrix} + \begin{bmatrix} 14/5 \\ 8/5 \end{bmatrix}$.

14. The orthogonal projection of **y** onto **u** is

$$\hat{\mathbf{y}} = \frac{\mathbf{y} \cdot \mathbf{u}}{\mathbf{u} \cdot \mathbf{u}}\mathbf{u} = \frac{2}{5}\mathbf{u} = \begin{bmatrix} 14/5 \\ 2/5 \end{bmatrix}$$

The component of **y** orthogonal to **u** is

$$\mathbf{y}-\hat{\mathbf{y}}=\begin{bmatrix}-4/5\\28/5\end{bmatrix}$$

Thus $\mathbf{y}=\hat{\mathbf{y}}+(\mathbf{y}-\hat{\mathbf{y}})=\begin{bmatrix}14/5\\2/5\end{bmatrix}+\begin{bmatrix}-4/5\\28/5\end{bmatrix}$.

15. The distance from **y** to the line through **u** and the origin is $\|\mathbf{y}-\hat{\mathbf{y}}\|$. One computes that

$$\mathbf{y}-\hat{\mathbf{y}}=\mathbf{y}-\frac{\mathbf{y}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u}=\begin{bmatrix}3\\1\end{bmatrix}-\frac{3}{10}\begin{bmatrix}8\\6\end{bmatrix}=\begin{bmatrix}3/5\\-4/5\end{bmatrix}$$

so $\|\mathbf{y}-\hat{\mathbf{y}}\|=\sqrt{9/25+16/25}=1$ is the desired distance.

16. The distance from **y** to the line through **u** and the origin is $\|\mathbf{y}-\hat{\mathbf{y}}\|$. One computes that

$$\mathbf{y}-\hat{\mathbf{y}}=\mathbf{y}-\frac{\mathbf{y}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u}=\begin{bmatrix}-3\\9\end{bmatrix}-3\begin{bmatrix}1\\2\end{bmatrix}=\begin{bmatrix}-6\\3\end{bmatrix}$$

so $\|\mathbf{y}-\hat{\mathbf{y}}\|=\sqrt{36+9}=3\sqrt{5}$ is the desired distance.

17. Let $\mathbf{u}=\begin{bmatrix}1/3\\1/3\\1/3\end{bmatrix}$, $\mathbf{v}=\begin{bmatrix}-1/2\\0\\1/2\end{bmatrix}$. Since $\mathbf{u}\cdot\mathbf{v}=0$, $\{\mathbf{u},\mathbf{v}\}$ is an orthogonal set. However, $\|\mathbf{u}\|^2=\mathbf{u}\cdot\mathbf{u}=1/3$ and $\|\mathbf{v}\|^2=\mathbf{v}\cdot\mathbf{v}=1/2$, so $\{\mathbf{u},\mathbf{v}\}$ is not an orthonormal set. The vectors **u** and **v** may be normalized to form the orthonormal set

$$\left\{\frac{\mathbf{u}}{\|\mathbf{u}\|},\frac{\mathbf{v}}{\|\mathbf{v}\|}\right\}=\left\{\begin{bmatrix}\sqrt{3}/3\\\sqrt{3}/3\\\sqrt{3}/3\end{bmatrix},\begin{bmatrix}-\sqrt{2}/2\\0\\\sqrt{2}/2\end{bmatrix}\right\}$$

18. Let $\mathbf{u}=\begin{bmatrix}0\\1\\0\end{bmatrix}$, $\mathbf{v}=\begin{bmatrix}0\\-1\\0\end{bmatrix}$. Since $\mathbf{u}\cdot\mathbf{v}=-1\neq 0$, $\{\mathbf{u},\mathbf{v}\}$ is not an orthogonal set.

19. Let $\mathbf{u}=\begin{bmatrix}-.6\\.8\end{bmatrix}$, $\mathbf{v}=\begin{bmatrix}.8\\.6\end{bmatrix}$. Since $\mathbf{u}\cdot\mathbf{v}=0$, $\{\mathbf{u},\mathbf{v}\}$ is an orthogonal set. Also, $\|\mathbf{u}\|^2=\mathbf{u}\cdot\mathbf{u}=1$ and $\|\mathbf{v}\|^2=\mathbf{v}\cdot\mathbf{v}=1$, so $\{\mathbf{u},\mathbf{v}\}$ is an orthonormal set.

20. Let $\mathbf{u}=\begin{bmatrix}-2/3\\1/3\\2/3\end{bmatrix}$, $\mathbf{v}=\begin{bmatrix}1/3\\2/3\\0\end{bmatrix}$. Since $\mathbf{u}\cdot\mathbf{v}=0$, $\{\mathbf{u},\mathbf{v}\}$ is an orthogonal set. However, $\|\mathbf{u}\|^2=\mathbf{u}\cdot\mathbf{u}=1$ and $\|\mathbf{v}\|^2=\mathbf{v}\cdot\mathbf{v}=5/9$, so $\{\mathbf{u},\mathbf{v}\}$ is not an orthonormal set. The vectors **u** and **v** may be normalized to form the orthonormal set

$$\left\{\frac{\mathbf{u}}{\|\mathbf{u}\|},\frac{\mathbf{v}}{\|\mathbf{v}\|}\right\}=\left\{\begin{bmatrix}-2/3\\1/3\\2/3\end{bmatrix},\begin{bmatrix}1/\sqrt{5}\\2/\sqrt{5}\\0\end{bmatrix}\right\}$$

21. Let $\mathbf{u}=\begin{bmatrix}1/\sqrt{10}\\3/\sqrt{20}\\3/\sqrt{20}\end{bmatrix}$, $\mathbf{v}=\begin{bmatrix}3/\sqrt{10}\\-1/\sqrt{20}\\-1/\sqrt{20}\end{bmatrix}$, and $\mathbf{w}=\begin{bmatrix}0\\-1/\sqrt{2}\\1/\sqrt{2}\end{bmatrix}$. Since $\mathbf{u}\cdot\mathbf{v}=\mathbf{u}\cdot\mathbf{w}=\mathbf{v}\cdot\mathbf{w}=0$, {**u**, **v**, **w**} is an orthogonal set. Also, $\|\mathbf{u}\|^2=\mathbf{u}\cdot\mathbf{u}=1$, $\|\mathbf{v}\|^2=\mathbf{v}\cdot\mathbf{v}=1$, and $\|\mathbf{w}\|^2=\mathbf{w}\cdot\mathbf{w}=1$, so {**u**, **v**, **w**} is an orthonormal set.

22. Let $\mathbf{u}=\begin{bmatrix}1/\sqrt{18}\\4/\sqrt{18}\\1/\sqrt{18}\end{bmatrix}$, $\mathbf{v}=\begin{bmatrix}1/\sqrt{2}\\0\\-1/\sqrt{2}\end{bmatrix}$, and $\mathbf{w}=\begin{bmatrix}-2/3\\1/3\\-2/3\end{bmatrix}$. Since $\mathbf{u}\cdot\mathbf{v}=\mathbf{u}\cdot\mathbf{w}=\mathbf{v}\cdot\mathbf{w}=0$, {**u**, **v**, **w**} is an orthogonal set. Also, $\|\mathbf{u}\|^2=\mathbf{u}\cdot\mathbf{u}=1$, $\|\mathbf{v}\|^2=\mathbf{v}\cdot\mathbf{v}=1$, and $\|\mathbf{w}\|^2=\mathbf{w}\cdot\mathbf{w}=1$, so {**u**, **v**, **w**} is an orthonormal set.

23. **a**. True. For example, the vectors **u** and **y** in Example 3 are linearly independent but not orthogonal.
b. True. The formulas for the weights are given in Theorem 5.
c. False. See the paragraph following Example 5.
d. False. The matrix must also be square. See the paragraph before Example 7.
e. False. See Example 4. The distance is $\|\mathbf{y}-\hat{\mathbf{y}}\|$.

24. **a**. True. But every orthogonal set of *nonzero vectors* is linearly independent. See Theorem 4.
b. False. To be orthonormal, the vectors is S must be unit vectors as well as being orthogonal to each other.
c. True. See Theorem 7(a).
d. True. See the paragraph before Example 3.
e. True. See the paragraph before Example 7.

25. To prove part (b), note that

$$(U\mathbf{x})\cdot(U\mathbf{y})=(U\mathbf{x})^T(U\mathbf{y})=\mathbf{x}^TU^TU\mathbf{y}=\mathbf{x}^T\mathbf{y}=\mathbf{x}\cdot\mathbf{y}$$

because $U^TU=I$. If $\mathbf{y}=\mathbf{x}$ in part (b), $(U\mathbf{x})\cdot(U\mathbf{x})=\mathbf{x}\cdot\mathbf{x}$, which implies part (a). Part (c) of the Theorem follows immediately fom part (b).

26. A set of n nonzero orthogonal vectors must be linearly independent by Theorem 4, so if such a set spans W it is a basis for W. Thus W is an n-dimensional subspace of $\mathbb{R}^n$, and $W=\mathbb{R}^n$.

27. If U has orthonormal columns, then $U^TU=I$ by Theorem 6. If U is also a square matrix, then the equation $U^TU=I$ implies that U is invertible by the Invertible Matrix Theorem.

28. If U is an $n\times n$ orthogonal matrix, then $I=UU^{-1}=UU^T$. Since U is the transpose of U^T, Theorem 6 applied to U^T says that U^T has orthogonal columns. In particular, the columns of U^T are linearly independent and hence form a basis for $\mathbb{R}^n$ by the Invertible Matrix Theorem. That is, the rows of U form a basis (an orthonormal basis) for $\mathbb{R}^n$.

29. Since U and V are orthogonal, each is invertible. By Theorem 6 in Section 2.2, UV is invertible and $(UV)^{-1}=V^{-1}U^{-1}=V^TU^T=(UV)^T$, where the final equality holds by Theorem 3 in Section 2.1. Thus UV is an orthogonal matrix.

30. If U is an orthogonal matrix, its columns are orthonormal. Interchanging the columns does not change their orthonormality, so the new matrix – say, V – still has orthonormal columns. By Theorem 6, $V^TV = I$. Since V is square, $V^T = V^{-1}$ by the Invertible Matrix Theorem.

31. Suppose that $\hat{\mathbf{y}} = \dfrac{\mathbf{y}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u}$. Replacing **u** by c**u** with $c \neq 0$ gives

$$\frac{\mathbf{y}\cdot(c\mathbf{u})}{(c\mathbf{u})\cdot(c\mathbf{u})}(c\mathbf{u}) = \frac{c(\mathbf{y}\cdot\mathbf{u})}{c^2(\mathbf{u}\cdot\mathbf{u})}(c)\mathbf{u} = \frac{c^2(\mathbf{y}\cdot\mathbf{u})}{c^2(\mathbf{u}\cdot\mathbf{u})}\mathbf{u} = \frac{\mathbf{y}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u} = \hat{\mathbf{y}}$$

So $\hat{\mathbf{y}}$ does not depend on the choice of a nonzero **u** in the line L used in the formula.

32. If $\mathbf{v}_1\cdot\mathbf{v}_2 = 0$, then by Theorem 1(c) in Section 6.1,

$$(c_1\mathbf{v}_1)\cdot(c_2\mathbf{v}_2) = c_1[\mathbf{v}_1\cdot(c_2\mathbf{v}_2)] = c_1c_2(\mathbf{v}_1\cdot\mathbf{v}_2) = c_1c_2 0 = 0$$

33. Let L = Span{**u**}, where **u** is nonzero, and let $T(\mathbf{x}) = \dfrac{\mathbf{x}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u}$. For any vectors **x** and **y** in $\mathbb{R}^n$ and any scalars c and d, the properties of the inner product (Theorem 1) show that

$$\begin{aligned} T(c\mathbf{x}+d\mathbf{y}) &= \frac{(c\mathbf{x}+d\mathbf{y})\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u} \\ &= \frac{c\mathbf{x}\cdot\mathbf{u}+d\mathbf{y}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u} \\ &= \frac{c\mathbf{x}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u} + \frac{d\mathbf{y}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u} \\ &= cT(\mathbf{x}) + dT(\mathbf{y}) \end{aligned}$$

Thus T is a linear transformation. Another approach is to view T as the composition of the following three linear mappings: $\mathbf{x} \mapsto a = \mathbf{x}\cdot\mathbf{v}$, $a \mapsto b = a / \mathbf{v}\cdot\mathbf{v}$, and $b \mapsto b\mathbf{v}$.

34. Let L = Span{**u**}, where **u** is nonzero, and let $T(\mathbf{x}) = \text{refl}_L\mathbf{y} = 2\text{proj}_L\mathbf{y} - \mathbf{y}$. By Exercise 33, the mapping $\mathbf{y} \mapsto \text{proj}_L\mathbf{y}$ is linear. Thus for any vectors **y** and **z** in $\mathbb{R}^n$ and any scalars c and d,

$$\begin{aligned} T(c\mathbf{y}+d\mathbf{z}) &= 2\ \text{proj}_L(c\mathbf{y}+d\mathbf{z}) - (c\mathbf{y}+d\mathbf{z}) \\ &= 2(c\ \text{proj}_L\mathbf{y} + d\ \text{proj}_L\mathbf{z}) - c\mathbf{y} - d\mathbf{z} \\ &= 2c\ \text{proj}_L\mathbf{y} - c\mathbf{y} + 2d\ \text{proj}_L\mathbf{z} - d\mathbf{z} \\ &= c(2\ \text{proj}_L\mathbf{y} - \mathbf{y}) + d(2\ \text{proj}_L\mathbf{z} - \mathbf{z}) \\ &= cT(\mathbf{y}) + dT(\mathbf{z}) \end{aligned}$$

Thus T is a linear transformation.

35. **[M]** One can compute that $A^TA = 100I_4$. Since the off-diagonal entries in A^TA are zero, the columns of A are orthogonal.

36. **[M]**

a. One computes that $U^T U = I_4$, while

$$UU^T = \left(\frac{1}{100}\right)\begin{bmatrix} 82 & 0 & -20 & 8 & 6 & 20 & 24 & 0 \\ 0 & 42 & 24 & 0 & -20 & 6 & 20 & -32 \\ -20 & 24 & 58 & 20 & 0 & 32 & 0 & 6 \\ 8 & 0 & 20 & 82 & 24 & -20 & 6 & 0 \\ 6 & -20 & 0 & 24 & 18 & 0 & -8 & 20 \\ 20 & 6 & 32 & -20 & 0 & 58 & 0 & 24 \\ 24 & 20 & 0 & 6 & -8 & 0 & 18 & -20 \\ 0 & -32 & 6 & 0 & 20 & 24 & -20 & 42 \end{bmatrix}$$

The matrices $U^T U$ and UU^T are of different sizes and look nothing like each other.

b. Answers will vary. The vector $\mathbf{p} = UU^T\mathbf{y}$ is in Col U because $\mathbf{p} = U(U^T\mathbf{y})$. Since the columns of U are simply scaled versions of the columns of A, Col U = Col A. Thus each **p** is in Col A.

c. One computes that $U^T\mathbf{z} = \mathbf{0}$.

d. From (c), **z** is orthogonal to each column of A. By Exercise 29 in Section 6.1, **z** must be orthogonal to every vector in Col A; that is, **z** is in $(\text{Col } A)^\perp$.

6.3 SOLUTIONS

Notes: Example 1 seems to help students understand Theorem 8. Theorem 8 is needed for the Gram-Schmidt process (but only for a subspace that itself has an orthogonal basis). Theorems 8 and 9 are needed for the discussions of least squares in Sections 6.5 and 6.6. Theorem 10 is used with the QR factorization to provide a good numerical method for solving least squares problems, in Section 6.5. Exercises 19 and 20 lead naturally into consideration of the Gram-Schmidt process.

1. The vector in $\text{Span}\{\mathbf{u}_4\}$ is

$$\frac{\mathbf{x}\cdot\mathbf{u}_4}{\mathbf{u}_4\cdot\mathbf{u}_4}\mathbf{u}_4 = \frac{72}{36}\mathbf{u}_4 = 2\mathbf{u}_4 = \begin{bmatrix} 10 \\ -6 \\ -2 \\ 2 \end{bmatrix}$$

Since $\mathbf{x} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3 + \dfrac{\mathbf{x}\cdot\mathbf{u}_4}{\mathbf{u}_4\cdot\mathbf{u}_4}\mathbf{u}_4$, the vector

$$\mathbf{x} - \frac{\mathbf{x}\cdot\mathbf{u}_4}{\mathbf{u}_4\cdot\mathbf{u}_4}\mathbf{u}_4 = \begin{bmatrix} 10 \\ -8 \\ 2 \\ 0 \end{bmatrix} - \begin{bmatrix} 10 \\ -6 \\ -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ 4 \\ -2 \end{bmatrix}$$

is in $\text{Span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$.

2. The vector in $\text{Span}\{\mathbf{u}_1\}$ is

$$\frac{\mathbf{v}\cdot\mathbf{u}_1}{\mathbf{u}_1\cdot\mathbf{u}_1}\mathbf{u}_1=\frac{14}{7}\mathbf{u}_1=2\mathbf{u}_1=\begin{bmatrix}2\\4\\2\\2\end{bmatrix}$$

Since $\mathbf{x}=\frac{\mathbf{v}\cdot\mathbf{u}_1}{\mathbf{u}_1\cdot\mathbf{u}_1}\mathbf{u}_1+c_2\mathbf{u}_2+c_3\mathbf{u}_3+c_4\mathbf{u}_4$, the vector

$$\mathbf{v}-\frac{\mathbf{v}\cdot\mathbf{u}_1}{\mathbf{u}_1\cdot\mathbf{u}_1}\mathbf{u}_1=\begin{bmatrix}4\\5\\-3\\3\end{bmatrix}-\begin{bmatrix}2\\4\\2\\2\end{bmatrix}=\begin{bmatrix}2\\1\\-5\\1\end{bmatrix}$$

is in $\text{Span}\{\mathbf{u}_2,\mathbf{u}_3,\mathbf{u}_4\}$.

3. Since $\mathbf{u}_1\cdot\mathbf{u}_2=-1+1+0=0$, $\{\mathbf{u}_1,\mathbf{u}_2\}$ is an orthogonal set. The orthogonal projection of $\mathbf{y}$ onto $\text{Span}\{\mathbf{u}_1,\mathbf{u}_2\}$ is

$$\hat{\mathbf{y}}=\frac{\mathbf{y}\cdot\mathbf{u}_1}{\mathbf{u}_1\cdot\mathbf{u}_1}\mathbf{u}_1+\frac{\mathbf{y}\cdot\mathbf{u}_2}{\mathbf{u}_2\cdot\mathbf{u}_2}\mathbf{u}_2=\frac{3}{2}\mathbf{u}_1+\frac{5}{2}\mathbf{u}_2=\frac{3}{2}\begin{bmatrix}1\\1\\0\end{bmatrix}+\frac{5}{2}\begin{bmatrix}-1\\1\\0\end{bmatrix}=\begin{bmatrix}-1\\4\\0\end{bmatrix}$$

4. Since $\mathbf{u}_1\cdot\mathbf{u}_2=-12+12+0=0$, $\{\mathbf{u}_1,\mathbf{u}_2\}$ is an orthogonal set. The orthogonal projection of $\mathbf{y}$ onto $\text{Span}\{\mathbf{u}_1,\mathbf{u}_2\}$ is

$$\hat{\mathbf{y}}=\frac{\mathbf{y}\cdot\mathbf{u}_1}{\mathbf{u}_1\cdot\mathbf{u}_1}\mathbf{u}_1+\frac{\mathbf{y}\cdot\mathbf{u}_2}{\mathbf{u}_2\cdot\mathbf{u}_2}\mathbf{u}_2=\frac{30}{25}\mathbf{u}_1-\frac{15}{25}\mathbf{u}_2=\frac{6}{5}\begin{bmatrix}3\\4\\0\end{bmatrix}-\frac{3}{5}\begin{bmatrix}-4\\3\\0\end{bmatrix}=\begin{bmatrix}6\\3\\0\end{bmatrix}$$

5. Since $\mathbf{u}_1\cdot\mathbf{u}_2=3+1-4=0$, $\{\mathbf{u}_1,\mathbf{u}_2\}$ is an orthogonal set. The orthogonal projection of $\mathbf{y}$ onto $\text{Span}\{\mathbf{u}_1,\mathbf{u}_2\}$ is

$$\hat{\mathbf{y}}=\frac{\mathbf{y}\cdot\mathbf{u}_1}{\mathbf{u}_1\cdot\mathbf{u}_1}\mathbf{u}_1+\frac{\mathbf{y}\cdot\mathbf{u}_2}{\mathbf{u}_2\cdot\mathbf{u}_2}\mathbf{u}_2=\frac{7}{14}\mathbf{u}_1-\frac{15}{6}\mathbf{u}_2=\frac{1}{2}\begin{bmatrix}3\\-1\\2\end{bmatrix}-\frac{5}{2}\begin{bmatrix}1\\-1\\2\end{bmatrix}=\begin{bmatrix}-1\\2\\6\end{bmatrix}$$

6. Since $\mathbf{u}_1\cdot\mathbf{u}_2=0-1+1=0$, $\{\mathbf{u}_1,\mathbf{u}_2\}$ is an orthogonal set. The orthogonal projection of $\mathbf{y}$ onto $\text{Span}\{\mathbf{u}_1,\mathbf{u}_2\}$ is

$$\hat{\mathbf{y}}=\frac{\mathbf{y}\cdot\mathbf{u}_1}{\mathbf{u}_1\cdot\mathbf{u}_1}\mathbf{u}_1+\frac{\mathbf{y}\cdot\mathbf{u}_2}{\mathbf{u}_2\cdot\mathbf{u}_2}\mathbf{u}_2=-\frac{27}{18}\mathbf{u}_1+\frac{5}{2}\mathbf{u}_2=-\frac{3}{2}\begin{bmatrix}-4\\-1\\1\end{bmatrix}+\frac{5}{2}\begin{bmatrix}0\\1\\1\end{bmatrix}=\begin{bmatrix}6\\4\\1\end{bmatrix}$$

7. Since $\mathbf{u}_1\cdot\mathbf{u}_2=5+3-8=0$, $\{\mathbf{u}_1,\mathbf{u}_2\}$ is an orthogonal set. By the Orthogonal Decomposition Theorem,

$$\hat{\mathbf{y}}=\frac{\mathbf{y}\cdot\mathbf{u}_1}{\mathbf{u}_1\cdot\mathbf{u}_1}\mathbf{u}_1+\frac{\mathbf{y}\cdot\mathbf{u}_2}{\mathbf{u}_2\cdot\mathbf{u}_2}\mathbf{u}_2=0\mathbf{u}_1+\frac{2}{3}\mathbf{u}_2=\begin{bmatrix}10/3\\2/3\\8/3\end{bmatrix},\mathbf{z}=\mathbf{y}-\hat{\mathbf{y}}=\begin{bmatrix}-7/3\\7/3\\7/3\end{bmatrix}$$

and $\mathbf{y}=\hat{\mathbf{y}}+\mathbf{z}$, where $\hat{\mathbf{y}}$ is in W and $\mathbf{z}$ is in $W^{\perp}$.

8. Since $\mathbf{u}_1 \cdot \mathbf{u}_2 = -1 + 3 - 2 = 0$, $\{\mathbf{u}_1, \mathbf{u}_2\}$ is an orthogonal set. By the Orthogonal Decomposition Theorem,

$$\hat{\mathbf{y}} = \frac{\mathbf{y} \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1}\mathbf{u}_1 + \frac{\mathbf{y} \cdot \mathbf{u}_2}{\mathbf{u}_2 \cdot \mathbf{u}_2}\mathbf{u}_2 = 2\mathbf{u}_1 + \frac{1}{2}\mathbf{u}_2 = \begin{bmatrix} 3/2 \\ 7/2 \\ 1 \end{bmatrix}, \mathbf{z} = \mathbf{y} - \hat{\mathbf{y}} = \begin{bmatrix} -5/2 \\ 1/2 \\ 2 \end{bmatrix}$$

and $\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z}$, where $\hat{\mathbf{y}}$ is in W and $\mathbf{z}$ is in $W^{\perp}$.

9. Since $\mathbf{u}_1 \cdot \mathbf{u}_2 = \mathbf{u}_1 \cdot \mathbf{u}_3 = \mathbf{u}_2 \cdot \mathbf{u}_3 = 0$, $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthogonal set. By the Orthogonal Decomposition Theorem,

$$\hat{\mathbf{y}} = \frac{\mathbf{y} \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1}\mathbf{u}_1 + \frac{\mathbf{y} \cdot \mathbf{u}_2}{\mathbf{u}_2 \cdot \mathbf{u}_2}\mathbf{u}_2 + \frac{\mathbf{y} \cdot \mathbf{u}_3}{\mathbf{u}_3 \cdot \mathbf{u}_3}\mathbf{u}_3 = 2\mathbf{u}_1 + \frac{2}{3}\mathbf{u}_2 - \frac{2}{3}\mathbf{u}_3 = \begin{bmatrix} 2 \\ 4 \\ 0 \\ 0 \end{bmatrix}, \mathbf{z} = \mathbf{y} - \hat{\mathbf{y}} = \begin{bmatrix} 2 \\ -1 \\ 3 \\ -1 \end{bmatrix}$$

and $\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z}$, where $\hat{\mathbf{y}}$ is in W and $\mathbf{z}$ is in $W^{\perp}$.

10. Since $\mathbf{u}_1 \cdot \mathbf{u}_2 = \mathbf{u}_1 \cdot \mathbf{u}_3 = \mathbf{u}_2 \cdot \mathbf{u}_3 = 0$, $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthogonal set. By the Orthogonal Decomposition Theorem,

$$\hat{\mathbf{y}} = \frac{\mathbf{y} \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1}\mathbf{u}_1 + \frac{\mathbf{y} \cdot \mathbf{u}_2}{\mathbf{u}_2 \cdot \mathbf{u}_2}\mathbf{u}_2 + \frac{\mathbf{y} \cdot \mathbf{u}_3}{\mathbf{u}_3 \cdot \mathbf{u}_3}\mathbf{u}_3 = \frac{1}{3}\mathbf{u}_1 + \frac{14}{3}\mathbf{u}_2 - \frac{5}{3}\mathbf{u}_3 = \begin{bmatrix} 5 \\ 2 \\ 3 \\ 6 \end{bmatrix}, \mathbf{z} = \mathbf{y} - \hat{\mathbf{y}} = \begin{bmatrix} -2 \\ 2 \\ 2 \\ 0 \end{bmatrix}$$

and $\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z}$, where $\hat{\mathbf{y}}$ is in W and $\mathbf{z}$ is in $W^{\perp}$.

11. Note that $\mathbf{v}_1$ and $\mathbf{v}_2$ are orthogonal. The Best Approximation Theorem says that $\hat{\mathbf{y}}$, which is the orthogonal projection of $\mathbf{y}$ onto $W = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$, is the closest point to $\mathbf{y}$ in W. This vector is

$$\hat{\mathbf{y}} = \frac{\mathbf{y} \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 + \frac{\mathbf{y} \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2 = \frac{1}{2}\mathbf{v}_1 + \frac{3}{2}\mathbf{v}_2 = \begin{bmatrix} 3 \\ -1 \\ 1 \\ -1 \end{bmatrix}$$

12. Note that $\mathbf{v}_1$ and $\mathbf{v}_2$ are orthogonal. The Best Approximation Theorem says that $\hat{\mathbf{y}}$, which is the orthogonal projection of y onto $W = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$, is the closest point to $\mathbf{y}$ in W. This vector is

$$\hat{\mathbf{y}} = \frac{\mathbf{y} \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 + \frac{\mathbf{y} \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2 = 3\mathbf{v}_1 + 1\mathbf{v}_2 = \begin{bmatrix} -1 \\ -5 \\ -3 \\ 9 \end{bmatrix}$$

13. Note that $\mathbf{v}_1$ and $\mathbf{v}_2$ are orthogonal. By the Best Approximation Theorem, the closest point in $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$ to $\mathbf{z}$ is

$$\hat{\mathbf{z}} = \frac{\mathbf{z} \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 + \frac{\mathbf{z} \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2 = \frac{2}{3}\mathbf{v}_1 - \frac{7}{3}\mathbf{v}_2 = \begin{bmatrix} -1 \\ -3 \\ -2 \\ 3 \end{bmatrix}$$

14. Note that $\mathbf{v}_1$ and $\mathbf{v}_2$ are orthogonal. By the Best Approximation Theorem, the closest point in $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$ to **z** is

$$\hat{\mathbf{z}} = \frac{\mathbf{z}\cdot\mathbf{v}_1}{\mathbf{v}_1\cdot\mathbf{v}_1}\mathbf{v}_1 + \frac{\mathbf{z}\cdot\mathbf{v}_2}{\mathbf{v}_2\cdot\mathbf{v}_2}\mathbf{v}_2 = \frac{1}{2}\mathbf{v}_1 + 0\mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ -1/2 \\ -3/2 \end{bmatrix}$$

15. The distance from the point **y** in $\mathbb{R}^3$ to a subspace *W* is defined as the distance from **y** to the closest point in *W*. Since the closest point in *W* to **y** is $\hat{\mathbf{y}} = \text{proj}_W \mathbf{y}$, the desired distance is $\|\mathbf{y} - \hat{\mathbf{y}}\|$. One computes that $\hat{\mathbf{y}} = \begin{bmatrix} 3 \\ -9 \\ -1 \end{bmatrix}, \mathbf{y} - \hat{\mathbf{y}} = \begin{bmatrix} 2 \\ 0 \\ 6 \end{bmatrix}$, and $\|\mathbf{y} - \hat{\mathbf{y}}\| = \sqrt{40} = 2\sqrt{10}.$

16. The distance from the point **y** in $\mathbb{R}^4$ to a subspace *W* is defined as the distance from **y** to the closest point in *W*. Since the closest point in *W* to **y** is $\hat{\mathbf{y}} = \text{proj}_W \mathbf{y}$, the desired distance is $\|\mathbf{y} - \hat{\mathbf{y}}\|$. One computes that $\hat{\mathbf{y}} = \begin{bmatrix} -1 \\ -5 \\ -3 \\ 9 \end{bmatrix}, \mathbf{y} - \hat{\mathbf{y}} = \begin{bmatrix} 4 \\ 4 \\ 4 \\ 4 \end{bmatrix}$, and $\|\mathbf{y} - \hat{\mathbf{y}}\| = 8.$

17. **a**. $U^TU = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, UU^T = \begin{bmatrix} 8/9 & -2/9 & 2/9 \\ -2/9 & 5/9 & 4/9 \\ 2/9 & 4/9 & 5/9 \end{bmatrix}$

b. Since $U^TU = I_2$, the columns of *U* form an orthonormal basis for *W*, and by Theorem 10

$$\text{proj}_W \mathbf{y} = UU^T\mathbf{y} = \begin{bmatrix} 8/9 & -2/9 & 2/9 \\ -2/9 & 5/9 & 4/9 \\ 2/9 & 4/9 & 5/9 \end{bmatrix}\begin{bmatrix} 4 \\ 8 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix}.$$

18. **a**. $U^TU = [1] = 1, UU^T = \begin{bmatrix} 1/10 & -3/10 \\ -3/10 & 9/10 \end{bmatrix}$

b. Since $U^TU = 1$, $\{\mathbf{u}_1\}$ forms an orthonormal basis for *W*, and by Theorem 10

$$\text{proj}_W \mathbf{y} = UU^T\mathbf{y} = \begin{bmatrix} 1/10 & -3/10 \\ -3/10 & 9/10 \end{bmatrix}\begin{bmatrix} 7 \\ 9 \end{bmatrix} = \begin{bmatrix} -2 \\ 6 \end{bmatrix}.$$

19. By the Orthogonal Decomposition Theorem, $\mathbf{u}_3$ is the sum of a vector in $W = \text{Span}\{\mathbf{u}_1, \mathbf{u}_2\}$ and a vector **v** orthogonal to *W*. This exercise asks for the vector **v**:

$$\mathbf{v} = \mathbf{u}_3 - \text{proj}_W \mathbf{u}_3 = \mathbf{u}_3 - \left(-\frac{1}{3}\mathbf{u}_1 + \frac{1}{15}\mathbf{u}_2\right) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ -2/5 \\ 4/5 \end{bmatrix} = \begin{bmatrix} 0 \\ 2/5 \\ 1/5 \end{bmatrix}$$

Any multiple of the vector **v** will also be in $W^{\perp}$.

20. By the Orthogonal Decomposition Theorem, $\mathbf{u}_4$ is the sum of a vector in $W = \text{Span}\{\mathbf{u}_1, \mathbf{u}_2\}$ and a vector $\mathbf{v}$ orthogonal to W. This exercise asks for the vector $\mathbf{v}$:

$$\mathbf{v} = \mathbf{u}_4 - \text{proj}_W \mathbf{u}_4 = \mathbf{u}_4 - \left(\frac{1}{6}\mathbf{u}_1 - \frac{1}{30}\mathbf{u}_2\right) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1/5 \\ -2/5 \end{bmatrix} = \begin{bmatrix} 0 \\ 4/5 \\ 2/5 \end{bmatrix}$$

Any multiple of the vector $\mathbf{v}$ will also be in $W^\perp$.

21. **a**. True. See the calculations for $\mathbf{z}_2$ in Example 1 or the box after Example 6 in Section 6.1.

b. True. See the Orthogonal Decomposition Theorem.

c. False. See the last paragraph in the proof of Theorem 8, or see the second paragraph after the statement of Theorem 9.

d. True. See the box before the Best Approximation Theorem.

e. True. Theorem 10 applies to the column space W of U because the columns of U are linearly independent and hence form a basis for W.

22. **a**. True. See the proof of the Orthogonal Decomposition Theorem.

b. True. See the subsection "A Geometric Interpretation of the Orthogonal Projection."

c. True. The orthgonal decomposition in Theorem 8 is unique.

d. False. The Best Approximation Theorem says that the best approximation to $\mathbf{y}$ is $\text{proj}_W \mathbf{y}$.

e. False. This statement is only true if $\mathbf{x}$ is in the column space of U. If $n > p$, then the column space of U will not be all of $\mathbb{R}^n$, so the statement cannot be true for all $\mathbf{x}$ in $\mathbb{R}^n$.

23. By the Orthogonal Decomposition Theorem, each $\mathbf{x}$ in $\mathbb{R}^n$ can be written uniquely as $\mathbf{x} = \mathbf{p} + \mathbf{u}$, with $\mathbf{p}$ in Row A and $\mathbf{u}$ in $(\text{Row } A)^\perp$. By Theorem 3 in Section 6.1, $(\text{Row } A)^\perp = \text{Nul } A$, so $\mathbf{u}$ is in Nul$A$.

Next, suppose $A\mathbf{x} = \mathbf{b}$ is consistent. Let $\mathbf{x}$ be a solution and write $\mathbf{x} = \mathbf{p} + \mathbf{u}$ as above. Then $A\mathbf{p} = A(\mathbf{x} - \mathbf{u}) = A\mathbf{x} - A\mathbf{u} = \mathbf{b} - \mathbf{0} = \mathbf{b}$, so the equation $A\mathbf{x} = \mathbf{b}$ has at least one solution $\mathbf{p}$ in Row A.

Finally, suppose that $\mathbf{p}$ and $\mathbf{p}_1$ are both in Row$A$ and both satisfy $A\mathbf{x} = \mathbf{b}$. Then $\mathbf{p} - \mathbf{p}_1$ is in $\text{Nul } A = (\text{Row } A)^\perp$, since $A(\mathbf{p} - \mathbf{p}_1) = A\mathbf{p} - A\mathbf{p}_1 = \mathbf{b} - \mathbf{b} = \mathbf{0}$. The equations $\mathbf{p} = \mathbf{p}_1 + (\mathbf{p} - \mathbf{p}_1)$ and $\mathbf{p} = \mathbf{p} + \mathbf{0}$ both then decompose $\mathbf{p}$ as the sum of a vector in Row$A$ and a vector in $(\text{Row } A)^\perp$. By the uniqueness of the orthogonal decomposition (Theorem 8), $\mathbf{p} = \mathbf{p}_1$, and $\mathbf{p}$ is unique.

24. **a**. By hypothesis, the vectors $\mathbf{w}_1, \ldots, \mathbf{w}_p$ are pairwise orthogonal, and the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_q$ are pairwise orthogonal. Since $\mathbf{w}_i$ is in W for any i and $\mathbf{v}_j$ is in $W^\perp$ for any j, $\mathbf{w}_i \cdot \mathbf{v}_j = 0$ for any i and j. Thus $\{\mathbf{w}_1, \ldots, \mathbf{w}_p, \mathbf{v}_1, \ldots, \mathbf{v}_q\}$ forms an orthogonal set.

b. For any $\mathbf{y}$ in $\mathbb{R}^n$, write $\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z}$ as in the Orthogonal Decomposition Theorem, with $\hat{\mathbf{y}}$ in W and $\mathbf{z}$ in $W^\perp$. Then there exist scalars $c_1, \ldots, c_p$ and $d_1, \ldots, d_q$ such that $\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z} = c_1\mathbf{w}_1 + \ldots + c_p\mathbf{w}_p + d_1\mathbf{v}_1 + \ldots + d_q\mathbf{v}_q$. Thus the set $\{\mathbf{w}_1, \ldots, \mathbf{w}_p, \mathbf{v}_1, \ldots, \mathbf{v}_q\}$ spans $\mathbb{R}^n$.

c. The set $\{\mathbf{w}_1, \ldots, \mathbf{w}_p, \mathbf{v}_1, \ldots, \mathbf{v}_q\}$ is linearly independent by (a) and spans $\mathbb{R}^n$ by (b), and is thus a basis for $\mathbb{R}^n$. Hence $\dim W + \dim W^\perp = p + q = \dim \mathbb{R}^n$.

25. **[M]** Since $U^TU = I_4$, U has orthonormal columns by Theorem 6 in Section 6.2. The closest point to **y** in Col U is the orthogonal projection $\hat{\mathbf{y}}$ of **y** onto Col U. From Theorem 10,

$$\hat{\mathbf{y}} = UU^{\mathrm{T}}\mathbf{y} = \begin{bmatrix} 1.2 \\ .4 \\ 1.2 \\ 1.2 \\ .4 \\ 1.2 \\ .4 \\ .4 \end{bmatrix}$$

26. **[M]** The distance from **b** to Col U is $\|\mathbf{b} - \hat{\mathbf{b}}\|$, where $\hat{\mathbf{b}} = UU^{\mathrm{T}}\mathbf{b}$. One computes that

$$\hat{\mathbf{b}} = UU^{\mathrm{T}}\mathbf{b} = \begin{bmatrix} .2 \\ .92 \\ .44 \\ 1 \\ -.2 \\ -.44 \\ .6 \\ -.92 \end{bmatrix}, \mathbf{b} - \hat{\mathbf{b}} = \begin{bmatrix} .8 \\ .08 \\ .56 \\ 0 \\ -.8 \\ -.56 \\ -1.6 \\ -.08 \end{bmatrix}, \|\mathbf{b} - \hat{\mathbf{b}}\| = \frac{\sqrt{112}}{5}$$

which is 2.1166 to four decimal places.

6.4 SOLUTIONS

Notes: The QR factorization encapsulates the essential outcome of the Gram-Schmidt process, just as the LU factorization describes the result of a row reduction process. For practical use of linear algebra, the factorizations are more important than the algorithms that produce them. In fact, the Gram-Schmidt process is *not* the appropriate way to compute the QR factorization. For that reason, one should consider deemphasizing the hand calculation of the Gram-Schmidt process, even though it provides easy exam questions.

The Gram-Schmidt process is used in Sections 6.7 and 6.8, in connection with various sets of orthogonal polynomials. The process is mentioned in Sections 7.1 and 7.4, but the one-dimensional projection constructed in Section 6.2 will suffice. The QR factorization is used in an optional subsection of Section 6.5, and it is needed in Supplementary Exercise 7 of Chapter 7 to produce the Cholesky factorization of a positive definite matrix.

1. Set $\mathbf{v}_1 = \mathbf{x}_1$ and compute that $\mathbf{v}_2 = \mathbf{x}_2 - \dfrac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 = \mathbf{x}_2 - 3\mathbf{v}_1 = \begin{bmatrix} -1 \\ 5 \\ -3 \end{bmatrix}$. Thus an orthogonal basis for W is

$$\left\{ \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 5 \\ -3 \end{bmatrix} \right\}.$$

2. Set $\mathbf{v}_1 = \mathbf{x}_1$ and compute that $\mathbf{v}_2 = \mathbf{x}_2 - \dfrac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 = \mathbf{x}_2 - \dfrac{1}{2}\mathbf{v}_1 = \begin{bmatrix} 5 \\ 4 \\ -8 \end{bmatrix}$. Thus an orthogonal basis for W is

$$\left\{ \begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix}, \begin{bmatrix} 5 \\ 4 \\ -8 \end{bmatrix} \right\}.$$

3. Set $\mathbf{v}_1 = \mathbf{x}_1$ and compute that $\mathbf{v}_2 = \mathbf{x}_2 - \dfrac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 = \mathbf{x}_2 - \dfrac{1}{2}\mathbf{v}_1 = \begin{bmatrix} 3 \\ 3/2 \\ 3/2 \end{bmatrix}$. Thus an orthogonal basis for W is

$$\left\{ \begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 3/2 \\ 3/2 \end{bmatrix} \right\}.$$

4. Set $\mathbf{v}_1 = \mathbf{x}_1$ and compute that $\mathbf{v}_2 = \mathbf{x}_2 - \dfrac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 = \mathbf{x}_2 - (-2)\mathbf{v}_1 = \begin{bmatrix} 3 \\ 6 \\ 3 \end{bmatrix}$. Thus an orthogonal basis for W is

$$\left\{ \begin{bmatrix} 3 \\ -4 \\ 5 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ 3 \end{bmatrix} \right\}.$$

5. Set $\mathbf{v}_1 = \mathbf{x}_1$ and compute that $\mathbf{v}_2 = \mathbf{x}_2 - \dfrac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 = \mathbf{x}_2 - 2\mathbf{v}_1 = \begin{bmatrix} 5 \\ 1 \\ -4 \\ -1 \end{bmatrix}$. Thus an orthogonal basis for W is

$$\left\{ \begin{bmatrix} 1 \\ -4 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 1 \\ -4 \\ -1 \end{bmatrix} \right\}.$$

6. Set $\mathbf{v}_1 = \mathbf{x}_1$ and compute that $\mathbf{v}_2 = \mathbf{x}_2 - \dfrac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 = \mathbf{x}_2 - (-3)\mathbf{v}_1 = \begin{bmatrix} 4 \\ 6 \\ -3 \\ 0 \end{bmatrix}$. Thus an orthogonal basis for W is

$$\left\{ \begin{bmatrix} 3 \\ -1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 4 \\ 6 \\ -3 \\ 0 \end{bmatrix} \right\}.$$

7. Since $\|\mathbf{v}_1\| = \sqrt{30}$ and $\|\mathbf{v}_2\| = \sqrt{27/2} = 3\sqrt{6}/2$, an orthonormal basis for W is

$$\left\{\frac{\mathbf{v}_1}{\|\mathbf{v}_1\|}, \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|}\right\} = \left\{\begin{bmatrix} 2/\sqrt{30} \\ -5/\sqrt{30} \\ 1/\sqrt{30} \end{bmatrix}, \begin{bmatrix} 2/\sqrt{6} \\ 1/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}\right\}.$$

8. Since $\|\mathbf{v}_1\| = \sqrt{50}$ and $\|\mathbf{v}_2\| = \sqrt{54} = 3\sqrt{6}$, an orthonormal basis for W is

$$\left\{\frac{\mathbf{v}_1}{\|\mathbf{v}_1\|}, \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|}\right\} = \left\{\begin{bmatrix} 3/\sqrt{50} \\ -4/\sqrt{50} \\ 5/\sqrt{50} \end{bmatrix}, \begin{bmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}\right\}.$$

9. Call the columns of the matrix $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_3$ and perform the Gram-Schmidt process on these vectors:

$$\mathbf{v}_1 = \mathbf{x}_1$$

$$\mathbf{v}_2 = \mathbf{x}_2 - \frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 = \mathbf{x}_2 - (-2)\mathbf{v}_1 = \begin{bmatrix} 1 \\ 3 \\ 3 \\ -1 \end{bmatrix}$$

$$\mathbf{v}_3 = \mathbf{x}_3 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2 = \mathbf{x}_3 - \frac{3}{2}\mathbf{v}_1 - \left(-\frac{1}{2}\right)\mathbf{v}_2 = \begin{bmatrix} -3 \\ 1 \\ 1 \\ 3 \end{bmatrix}$$

Thus an orthogonal basis for W is $\left\{\begin{bmatrix} 3 \\ 1 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 1 \\ 3 \end{bmatrix}\right\}$.

10. Call the columns of the matrix $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_3$ and perform the Gram-Schmidt process on these vectors:

$$\mathbf{v}_1 = \mathbf{x}_1$$

$$\mathbf{v}_2 = \mathbf{x}_2 - \frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 = \mathbf{x}_2 - (-3)\mathbf{v}_1 = \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

$$\mathbf{v}_3 = \mathbf{x}_3 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2 = \mathbf{x}_3 - \frac{1}{2}\mathbf{v}_1 - \frac{5}{2}\mathbf{v}_2 = \begin{bmatrix} -1 \\ -1 \\ 3 \\ -1 \end{bmatrix}$$

Thus an orthogonal basis for W is $\left\{\begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 3 \\ -1 \end{bmatrix}\right\}$.

11. Call the columns of the matrix $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_3$ and perform the Gram-Schmidt process on these vectors:

$$\mathbf{v}_1 = \mathbf{x}_1$$

$$\mathbf{v}_2 = \mathbf{x}_2 - \frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 = \mathbf{x}_2 - (-1)\mathbf{v}_1 = \begin{bmatrix} 3 \\ 0 \\ 3 \\ -3 \\ 3 \end{bmatrix}$$

$$\mathbf{v}_3 = \mathbf{x}_3 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2 = \mathbf{x}_3 - 4\mathbf{v}_1 - \left(-\frac{1}{3}\right)\mathbf{v}_2 = \begin{bmatrix} 2 \\ 0 \\ 2 \\ 2 \\ -2 \end{bmatrix}$$

Thus an orthogonal basis for W is $\left\{ \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 3 \\ -3 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 2 \\ 2 \\ -2 \end{bmatrix} \right\}$.

12. Call the columns of the matrix $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_3$ and perform the Gram-Schmidt process on these vectors:

$$\mathbf{v}_1 = \mathbf{x}_1$$

$$\mathbf{v}_2 = \mathbf{x}_2 - \frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 = \mathbf{x}_2 - 4\mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}$$

$$\mathbf{v}_3 = \mathbf{x}_3 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2 = \mathbf{x}_3 - \frac{7}{2}\mathbf{v}_1 - \frac{3}{2}\mathbf{v}_2 = \begin{bmatrix} 3 \\ 3 \\ 0 \\ -3 \\ -3 \end{bmatrix}$$

Thus an orthogonal basis for W is $\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 0 \\ -3 \\ -3 \end{bmatrix} \right\}$.

13. Since A and Q are given,

$$R = Q^T A = \begin{bmatrix} 5/6 & 1/6 & -3/6 & 1/6 \\ -1/6 & 5/6 & 1/6 & 3/6 \end{bmatrix} \begin{bmatrix} 5 & 9 \\ 1 & 7 \\ -3 & -5 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} 6 & 12 \\ 0 & 6 \end{bmatrix}$$

14. Since A and Q are given,

$$R = Q^T A = \begin{bmatrix} -2/7 & 5/7 & 2/7 & 4/7 \\ 5/7 & 2/7 & -4/7 & 2/7 \end{bmatrix} \begin{bmatrix} -2 & 3 \\ 5 & 7 \\ 2 & -2 \\ 4 & 6 \end{bmatrix} = \begin{bmatrix} 7 & 7 \\ 0 & 7 \end{bmatrix}$$

15. The columns of Q will be normalized versions of the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ found in Exercise 11. Thus

$$Q = \begin{bmatrix} 1/\sqrt{5} & 1/2 & 1/2 \\ -1/\sqrt{5} & 0 & 0 \\ -1/\sqrt{5} & 1/2 & 1/2 \\ 1/\sqrt{5} & -1/2 & 1/2 \\ 1/\sqrt{5} & 1/2 & -1/2 \end{bmatrix}, R = Q^T A = \begin{bmatrix} \sqrt{5} & -\sqrt{5} & 4\sqrt{5} \\ 0 & 6 & -2 \\ 0 & 0 & 4 \end{bmatrix}$$

16. The columns of Q will be normalized versions of the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ found in Exercise 12. Thus

$$Q = \begin{bmatrix} 1/2 & -1/2\sqrt{2} & 1/2 \\ -1/2 & 1/2\sqrt{2} & 1/2 \\ 0 & 1/\sqrt{2} & 0 \\ 1/2 & 1/2\sqrt{2} & -1/2 \\ 1/2 & 1/2\sqrt{2} & 1/2 \end{bmatrix}, R = Q^T A = \begin{bmatrix} 2 & 8 & 7 \\ 0 & 2\sqrt{2} & 3\sqrt{2} \\ 0 & 0 & 6 \end{bmatrix}$$

17. **a**. False. Scaling was used in Example 2, but the scale factor was nonzero.

b. True. See (1) in the statement of Theorem 11.

c. True. See the solution of Example 4.

18. **a**. False. The three orthogonal vectors must be *nonzero* to be a basis for a three-dimensional subspace. (This was the case in Step 3 of the solution of Example 2.)

b. True. If $\mathbf{x}$ is not in a subspace $\mathbf{w}$, then $\mathbf{x}$ cannot equal $\text{proj}_W \mathbf{x}$, because $\text{proj}_W \mathbf{x}$ is in W. This idea was used for $\mathbf{v}_{k+1}$ in the proof of Theorem 11.

c. True. See Theorem 12.

19. Suppose that $\mathbf{x}$ satisfies $R\mathbf{x} = \mathbf{0}$; then $Q\,R\mathbf{x} = Q\mathbf{0} = \mathbf{0}$, and $A\mathbf{x} = \mathbf{0}$. Since the columns of A are linearly independent, $\mathbf{x}$ must be $\mathbf{0}$. This fact, in turn, shows that the columns of R are linearly indepedent. Since R is square, it is invertible by the Invertible Matrix Theorem.

20. If $\mathbf{y}$ is in Col$A$, then $\mathbf{y} = A\mathbf{x}$ for some $\mathbf{x}$. Then $\mathbf{y} = QR\mathbf{x} = Q(R\mathbf{x})$, which shows that $\mathbf{y}$ is a linear combination of the columns of Q using the entries in $R\mathbf{x}$ as weights. Conversly, suppose that $\mathbf{y} = Q\mathbf{x}$ for some $\mathbf{x}$. Since R is invertible, the equation $A = QR$ implies that $Q = AR^{-1}$. So $\mathbf{y} = AR^{-1}\mathbf{x} = A(R^{-1}\mathbf{x})$, which shows that $\mathbf{y}$ is in Col A.

21. Denote the columns of Q by $\{\mathbf{q}_1,\ldots,\mathbf{q}_n\}$. Note that $n \le m$, because A is $m \times n$ and has linearly independent columns. The columns of Q can be extended to an orthonormal basis for $\mathbb{R}^m$ as follows. Let $\mathbf{f}_1$ be the first vector in the standard basis for $\mathbb{R}^m$ that is *not* in $W_n = \text{Span}\{\mathbf{q}_1,\ldots,\mathbf{q}_n\}$, let $\mathbf{u}_1 = \mathbf{f}_1 - \text{proj}_{W_n}\mathbf{f}_1$, and let $\mathbf{q}_{n+1} = \mathbf{u}_1 / \|\mathbf{u}_1\|$. Then $\{\mathbf{q}_1,\ldots,\mathbf{q}_n,\mathbf{q}_{n+1}\}$ is an orthonormal basis for $W_{n+1} = \text{Span}\{\mathbf{q}_1,\ldots,\mathbf{q}_n,\mathbf{q}_{n+1}\}$. Next let $\mathbf{f}_2$ be the first vector in the standard basis for $\mathbb{R}^m$ that is *not* in W_{n+1}, let $\mathbf{u}_2 = \mathbf{f}_2 - \text{proj}_{W_{n+1}}\mathbf{f}_2$, and let $\mathbf{q}_{n+2} = \mathbf{u}_2 / \|\mathbf{u}_2\|$. Then $\{\mathbf{q}_1,\ldots,\mathbf{q}_n,\mathbf{q}_{n+1},\mathbf{q}_{n+2}\}$ is an orthogonal basis for $W_{n+2} = \text{Span}\{\mathbf{q}_1,\ldots,\mathbf{q}_n,\mathbf{q}_{n+1},\mathbf{q}_{n+2}\}$. This process will continue until $m - n$ vectors have been added to the original n vectors, and $\{\mathbf{q}_1,\ldots,\mathbf{q}_n,\mathbf{q}_{n+1},\ldots,\mathbf{q}_m\}$ is an orthonormal basis for $\mathbb{R}^m$. Let $Q_0 = \begin{bmatrix}\mathbf{q}_{n+1} & \cdots & \mathbf{q}_m\end{bmatrix}$ and $Q_1 = \begin{bmatrix}Q & Q_0\end{bmatrix}$. Then, using partitioned matrix multiplication, $Q_1\begin{bmatrix}R \\ O\end{bmatrix} = QR = A$.

22. We may assume that $\{\mathbf{u}_1,\ldots,\mathbf{u}_p\}$ is an orthonormal basis for W, by normalizing the vectors in the original basis given for W, if necessary. Let U be the matrix whose columns are $\mathbf{u}_1,\ldots,\mathbf{u}_p$. Then, by Theorem 10 in Section 6.3, $T(\mathbf{x}) = \text{proj}_W\mathbf{x} = (UU^T)\mathbf{x}$ for $\mathbf{x}$ in $\mathbb{R}^n$. Thus T is a matrix transformation and hence is a linear transformation, as was shown in Section 1.8.

23. Given $A = QR$, partition $A = \begin{bmatrix}A_1 & A_2\end{bmatrix}$, where A_1 has p columns. Partition Q as $Q = \begin{bmatrix}Q_1 & Q_2\end{bmatrix}$ where Q_1 has p columns, and partition R as $R = \begin{bmatrix}R_{11} & R_{12} \\ O & R_{22}\end{bmatrix}$, where R_{11} is a $p \times p$ matrix. Then

$$A = \begin{bmatrix}A_1 & A_2\end{bmatrix} = QR = \begin{bmatrix}Q_1 & Q_2\end{bmatrix}\begin{bmatrix}R_{11} & R_{12} \\ O & R_{22}\end{bmatrix} = \begin{bmatrix}Q_1R_{11} & Q_1R_{12} + Q_2R_{22}\end{bmatrix}$$

Thus $A_1 = Q_1R_{11}$. The matrix Q_1 has orthonormal columns because its columns come from Q. The matrix R_{11} is square and upper triangular due to its position within the upper triangular matrix R. The diagonal entries of R_{11} are positive because they are diagonal entries of R. Thus Q_1R_{11} is a QR factorization of A_1.

24. **[M]** Call the columns of the matrix $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$, and $\mathbf{x}_4$ and perform the Gram-Schmidt process on these vectors:

$$\mathbf{v}_1 = \mathbf{x}_1$$

$$\mathbf{v}_2 = \mathbf{x}_2 - \frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 = \mathbf{x}_2 - (-1)\mathbf{v}_1 = \begin{bmatrix}3 \\ 3 \\ -3 \\ 0 \\ 3\end{bmatrix}$$

$$\mathbf{v}_3 = \mathbf{x}_3 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2 = \mathbf{x}_3 - \left(-\frac{1}{2}\right)\mathbf{v}_1 - \left(-\frac{4}{3}\right)\mathbf{v}_2 = \begin{bmatrix} 6 \\ 0 \\ 6 \\ 6 \\ 0 \end{bmatrix}$$

$$\mathbf{v}_4 = \mathbf{x}_4 - \frac{\mathbf{x}_4 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 - \frac{\mathbf{x}_4 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2 - \frac{\mathbf{x}_4 \cdot \mathbf{v}_3}{\mathbf{v}_3 \cdot \mathbf{v}_3}\mathbf{v}_3 = \mathbf{x}_4 - \frac{1}{2}\mathbf{v}_1 - (-1)\mathbf{v}_2 - \left(-\frac{1}{2}\right)\mathbf{v}_3 = \begin{bmatrix} 0 \\ 5 \\ 0 \\ 0 \\ -5 \end{bmatrix}$$

Thus an orthogonal basis for W is $\left\{ \begin{bmatrix} -10 \\ 2 \\ -6 \\ 16 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ -3 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 6 \\ 0 \\ 6 \\ 6 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 5 \\ 0 \\ 0 \\ -5 \end{bmatrix} \right\}$.

25. **[M]** The columns of Q will be normalized versions of the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ found in Exercise 24. Thus

$$Q = \begin{bmatrix} -1/2 & 1/2 & 1/\sqrt{3} & 0 \\ 1/10 & 1/2 & 0 & 1/\sqrt{2} \\ -3/10 & -1/2 & 1/\sqrt{3} & 0 \\ 4/5 & 0 & 1/\sqrt{3} & 0 \\ 1/10 & 1/2 & 0 & -1/\sqrt{2} \end{bmatrix}, R = Q^T A = \begin{bmatrix} 20 & -20 & -10 & 10 \\ 0 & 6 & -8 & -6 \\ 0 & 0 & 6\sqrt{3} & -3\sqrt{3} \\ 0 & 0 & 0 & 5\sqrt{2} \end{bmatrix}$$

26. **[M]** In MATLAB, when A has n columns, suitable commands are

```
Q = A(:,1)/norm(A(:,1))
    %  The first column of Q
    for j=2: n
       v=A(:,j) – Q*(Q'*A(:,j))
       Q(:,j)=v/norm(v)
       % Add a new column to Q
    end
```

6.5 SOLUTIONS

Notes: This is a core section – the basic geometric principles in this section provide the foundation for all the applications in Sections 6.6–6.8. Yet this section need not take a full day. Each example provides a stopping place. Theorem 13 and Example 1 are all that is needed for Section 6.6. Theorem 15, however, gives an illustration of why the QR factorization is important. Example 4 is related to Exercise 17 in Section 6.6.

1. To find the normal equations and to find $\hat{\mathbf{x}}$, compute

$$A^TA = \begin{bmatrix} -1 & 2 & -1 \\ 2 & -3 & 3 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 2 & -3 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 6 & -11 \\ -11 & 22 \end{bmatrix}$$

$$A^T\mathbf{b} = \begin{bmatrix} -1 & 2 & -1 \\ 2 & -3 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -4 \\ 11 \end{bmatrix}$$

a. The normal equations are $(A^TA)\mathbf{x} = A^T\mathbf{b}$: $\begin{bmatrix} 6 & -11 \\ -11 & 22 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -4 \\ 11 \end{bmatrix}$.

b. Compute

$$\hat{\mathrm{x}} = (A^TA)^{-1}A^T\mathbf{b} = \begin{bmatrix} 6 & -11 \\ -11 & 22 \end{bmatrix}^{-1} \begin{bmatrix} -4 \\ 11 \end{bmatrix} = \frac{1}{11} \begin{bmatrix} 22 & 11 \\ 11 & 6 \end{bmatrix} \begin{bmatrix} -4 \\ 11 \end{bmatrix}$$

$$= \frac{1}{11} \begin{bmatrix} 33 \\ 22 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

2. To find the normal equations and to find $\hat{\mathbf{x}}$, compute

$$A^TA = \begin{bmatrix} 2 & -2 & 2 \\ 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -2 & 0 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 12 & 8 \\ 8 & 10 \end{bmatrix}$$

$$A^T\mathbf{b} = \begin{bmatrix} 2 & -2 & 2 \\ 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} -5 \\ 8 \\ 1 \end{bmatrix} = \begin{bmatrix} -24 \\ -2 \end{bmatrix}$$

a. The normal equations are $(A^TA)\mathbf{x} = A^T\mathbf{b}$: $\begin{bmatrix} 12 & 8 \\ 8 & 10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -24 \\ -2 \end{bmatrix}$.

b. Compute

$$\hat{\mathrm{x}} = (A^TA)^{-1}A^T\mathbf{b} = \begin{bmatrix} 12 & 8 \\ 8 & 10 \end{bmatrix}^{-1} \begin{bmatrix} -24 \\ -2 \end{bmatrix} = \frac{1}{56} \begin{bmatrix} 10 & -8 \\ -8 & 12 \end{bmatrix} \begin{bmatrix} -24 \\ -2 \end{bmatrix}$$

$$= \frac{1}{56} \begin{bmatrix} 224 \\ 168 \end{bmatrix} = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$$

3. To find the normal equations and to find $\hat{\mathbf{x}}$, compute

$$A^TA = \begin{bmatrix} 1 & -1 & 0 & 2 \\ -2 & 2 & 3 & 5 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ -1 & 2 \\ 0 & 3 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 6 & 6 \\ 6 & 42 \end{bmatrix}$$

$$A^T\mathbf{b} = \begin{bmatrix} 1 & -1 & 0 & 2 \\ -2 & 2 & 3 & 5 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ -4 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ -6 \end{bmatrix}$$

a. The normal equations are $(A^TA)\mathbf{x}=A^T\mathbf{b}$: $\begin{bmatrix}6 & 6\\ 6 & 42\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}6\\ -6\end{bmatrix}$

b. Compute

$$\hat{\mathbf{x}}=(\mathrm{A}^T\mathrm{A})^{-1}\mathrm{A}^T\mathbf{b}=\begin{bmatrix}6 & 6\\ 6 & 42\end{bmatrix}^{-1}\begin{bmatrix}6\\ -6\end{bmatrix}=\frac{1}{216}\begin{bmatrix}42 & -6\\ -6 & 6\end{bmatrix}\begin{bmatrix}6\\ -6\end{bmatrix}$$
$$=\frac{1}{216}\begin{bmatrix}288\\ -72\end{bmatrix}=\begin{bmatrix}4/3\\ -1/3\end{bmatrix}$$

4. To find the normal equations and to find $\hat{\mathbf{x}}$, compute

$$A^TA=\begin{bmatrix}1 & 1 & 1\\ 3 & -1 & 1\end{bmatrix}\begin{bmatrix}1 & 3\\ 1 & -1\\ 1 & 1\end{bmatrix}=\begin{bmatrix}3 & 3\\ 3 & 11\end{bmatrix}$$

$$A^T\mathbf{b}=\begin{bmatrix}1 & 1 & 1\\ 3 & -1 & 1\end{bmatrix}\begin{bmatrix}5\\ 1\\ 0\end{bmatrix}=\begin{bmatrix}6\\ 14\end{bmatrix}$$

a. The normal equations are $(A^TA)\mathbf{x}=A^T\mathbf{b}$: $\begin{bmatrix}3 & 3\\ 3 & 11\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}6\\ 14\end{bmatrix}$

b. Compute

$$\hat{\mathbf{x}}=(\mathrm{A}^T\mathrm{A})^{-1}\mathrm{A}^T\mathbf{b}=\begin{bmatrix}3 & 3\\ 3 & 11\end{bmatrix}^{-1}\begin{bmatrix}6\\ 14\end{bmatrix}=\frac{1}{24}\begin{bmatrix}11 & -3\\ -3 & 3\end{bmatrix}\begin{bmatrix}6\\ 14\end{bmatrix}$$
$$=\frac{1}{24}\begin{bmatrix}24\\ 24\end{bmatrix}=\begin{bmatrix}1\\ 1\end{bmatrix}$$

5. To find the least squares solutions to $A\mathbf{x}=\mathbf{b}$, compute and row reduce the augmented matrix for the system $A^TA\mathbf{x}=A^T\mathbf{b}$:

$$\begin{bmatrix}A^TA & A^T\mathbf{b}\end{bmatrix}=\begin{bmatrix}4 & 2 & 2 & 14\\ 2 & 2 & 0 & 4\\ 2 & 0 & 2 & 10\end{bmatrix}\sim\begin{bmatrix}1 & 0 & 1 & 5\\ 0 & 1 & -1 & -3\\ 0 & 0 & 0 & 0\end{bmatrix}$$

so all vectors of the form $\hat{\mathbf{x}}=\begin{bmatrix}5\\ -3\\ 0\end{bmatrix}+x_3\begin{bmatrix}-1\\ 1\\ 1\end{bmatrix}$ are the least-squares solutions of $A\mathbf{x}=\mathbf{b}$.

6. To find the least squares solutions to $A\mathbf{x}=\mathbf{b}$, compute and row reduce the augmented matrix for the system $A^TA\mathbf{x}=A^T\mathbf{b}$:

$$\begin{bmatrix}A^TA & A^T\mathbf{b}\end{bmatrix}=\begin{bmatrix}6 & 3 & 3 & 27\\ 3 & 3 & 0 & 12\\ 3 & 0 & 3 & 15\end{bmatrix}\sim\begin{bmatrix}1 & 0 & 1 & 5\\ 0 & 1 & -1 & -1\\ 0 & 0 & 0 & 0\end{bmatrix}$$

so all vectors of the form $\hat{\mathbf{x}}=\begin{bmatrix}5\\ -1\\ 0\end{bmatrix}+x_3\begin{bmatrix}-1\\ 1\\ 1\end{bmatrix}$ are the least-squares solutions of $A\mathbf{x}=\mathbf{b}$.

7. From Exercise 3, $A=\begin{bmatrix}1 & -2\\ -1 & 2\\ 0 & 3\\ 2 & 5\end{bmatrix}$, $\mathbf{b}=\begin{bmatrix}3\\ 1\\ -4\\ 2\end{bmatrix}$, and $\hat{\mathbf{x}}=\begin{bmatrix}4/3\\ -1/3\end{bmatrix}$. Since

$$A\hat{\mathbf{x}}-\mathbf{b}=\begin{bmatrix}1 & -2\\ -1 & 2\\ 0 & 3\\ 2 & 5\end{bmatrix}\begin{bmatrix}4/3\\ -1/3\end{bmatrix}-\begin{bmatrix}3\\ 1\\ -4\\ 2\end{bmatrix}=\begin{bmatrix}2\\ -2\\ -1\\ 1\end{bmatrix}-\begin{bmatrix}3\\ 1\\ -4\\ 2\end{bmatrix}=\begin{bmatrix}-1\\ -3\\ 3\\ -1\end{bmatrix}$$

the least squares error is $\|A\hat{\mathbf{x}}-\mathbf{b}\|=\sqrt{20}=2\sqrt{5}.$

8. From Exercise 4, $A=\begin{bmatrix}1 & 3\\ 1 & -1\\ 1 & 1\end{bmatrix}$, $\mathbf{b}=\begin{bmatrix}5\\ 1\\ 0\end{bmatrix}$, and $\hat{\mathbf{x}}=\begin{bmatrix}1\\ 1\end{bmatrix}$. Since

$$A\hat{\mathbf{x}}-\mathbf{b}=\begin{bmatrix}1 & 3\\ 1 & -1\\ 1 & 1\end{bmatrix}\begin{bmatrix}1\\ 1\end{bmatrix}-\begin{bmatrix}5\\ 1\\ 0\end{bmatrix}=\begin{bmatrix}4\\ 0\\ 2\end{bmatrix}-\begin{bmatrix}5\\ 1\\ 0\end{bmatrix}=\begin{bmatrix}-1\\ -1\\ 2\end{bmatrix}$$

the least squares error is $\|A\hat{\mathbf{x}}-\mathbf{b}\|=\sqrt{6}.$

9. (a) Because the columns $\mathbf{a}_1$ and $\mathbf{a}_2$ of A are orthogonal, the method of Example 4 may be used to find $\hat{\mathbf{b}}$, the orthogonal projection of $\mathbf{b}$ onto Col A:

$$\hat{\mathbf{b}}=\frac{\mathbf{b}\cdot\mathbf{a}_1}{\mathbf{a}_1\cdot\mathbf{a}_1}\mathbf{a}_1+\frac{\mathbf{b}\cdot\mathbf{a}_2}{\mathbf{a}_2\cdot\mathbf{a}_2}\mathbf{a}_2=\frac{2}{7}\mathbf{a}_1+\frac{1}{7}\mathbf{a}_2=\frac{2}{7}\begin{bmatrix}1\\ 3\\ -2\end{bmatrix}+\frac{1}{7}\begin{bmatrix}5\\ 1\\ 4\end{bmatrix}=\begin{bmatrix}1\\ 1\\ 0\end{bmatrix}$$

(b) The vector $\hat{\mathbf{x}}$ contains the weights which must be placed on $\mathbf{a}_1$ and $\mathbf{a}_2$ to produce $\hat{\mathbf{b}}$. These weights are easily read from the above equation, so $\hat{\mathbf{x}}=\begin{bmatrix}2/7\\ 1/7\end{bmatrix}$.

10. (a) Because the columns $\mathbf{a}_1$ and $\mathbf{a}_2$ of A are orthogonal, the method of Example 4 may be used to find $\hat{\mathbf{b}}$, the orthogonal projection of $\mathbf{b}$ onto Col A:

$$\hat{\mathbf{b}}=\frac{\mathbf{b}\cdot\mathbf{a}_1}{\mathbf{a}_1\cdot\mathbf{a}_1}\mathbf{a}_1+\frac{\mathbf{b}\cdot\mathbf{a}_2}{\mathbf{a}_2\cdot\mathbf{a}_2}\mathbf{a}_2=3\mathbf{a}_1+\frac{1}{2}\mathbf{a}_2=3\begin{bmatrix}1\\ -1\\ 1\end{bmatrix}+\frac{1}{2}\begin{bmatrix}2\\ 4\\ 2\end{bmatrix}=\begin{bmatrix}4\\ -1\\ 4\end{bmatrix}$$

(b) The vector $\hat{\mathbf{x}}$ contains the weights which must be placed on $\mathbf{a}_1$ and $\mathbf{a}_2$ to produce $\hat{\mathbf{b}}$. These weights are easily read from the above equation, so $\hat{\mathbf{x}}=\begin{bmatrix}3\\ 1/2\end{bmatrix}$.

11. (a) Because the columns $\mathbf{a}_1$, $\mathbf{a}_2$ and $\mathbf{a}_3$ of A are orthogonal, the method of Example 4 may be used to find $\hat{\mathbf{b}}$, the orthogonal projection of **b** onto Col A:

$$\hat{\mathbf{b}} = \frac{\mathbf{b}\cdot\mathbf{a}_1}{\mathbf{a}_1\cdot\mathbf{a}_1}\mathbf{a}_1 + \frac{\mathbf{b}\cdot\mathbf{a}_2}{\mathbf{a}_2\cdot\mathbf{a}_2}\mathbf{a}_2 + \frac{\mathbf{b}\cdot\mathbf{a}_3}{\mathbf{a}_3\cdot\mathbf{a}_3}\mathbf{a}_3 = \frac{2}{3}\mathbf{a}_1 + 0\mathbf{a}_2 + \frac{1}{3}\mathbf{a}_3$$

$$= \frac{2}{3}\begin{bmatrix}4\\1\\6\\1\end{bmatrix} + 0\begin{bmatrix}0\\-5\\1\\-1\end{bmatrix} + \frac{1}{3}\begin{bmatrix}1\\1\\0\\-5\end{bmatrix} = \begin{bmatrix}3\\1\\4\\-1\end{bmatrix}$$

(b) The vector $\hat{\mathbf{x}}$ contains the weights which must be placed on $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$ to produce $\hat{\mathbf{b}}$. These weights are easily read from the above equation, so $\hat{\mathbf{x}} = \begin{bmatrix}2/3\\0\\1/3\end{bmatrix}$.

12. (a) Because the columns $\mathbf{a}_1$, $\mathbf{a}_2$ and $\mathbf{a}_3$ of A are orthogonal, the method of Example 4 may be used to find $\hat{\mathbf{b}}$, the orthogonal projection of **b** onto Col A:

$$\hat{\mathbf{b}} = \frac{\mathbf{b}\cdot\mathbf{a}_1}{\mathbf{a}_1\cdot\mathbf{a}_1}\mathbf{a}_1 + \frac{\mathbf{b}\cdot\mathbf{a}_2}{\mathbf{a}_2\cdot\mathbf{a}_2}\mathbf{a}_2 + \frac{\mathbf{b}\cdot\mathbf{a}_3}{\mathbf{a}_3\cdot\mathbf{a}_3}\mathbf{a}_3 = \frac{1}{3}\mathbf{a}_1 + \frac{14}{3}\mathbf{a}_2 + \left(-\frac{5}{3}\right)\mathbf{a}_3$$

$$= \frac{1}{3}\begin{bmatrix}1\\1\\0\\-1\end{bmatrix} + \frac{14}{3}\begin{bmatrix}1\\0\\1\\1\end{bmatrix} - \frac{5}{3}\begin{bmatrix}0\\-1\\1\\-1\end{bmatrix} = \begin{bmatrix}5\\2\\3\\6\end{bmatrix}$$

(b) The vector $\hat{\mathbf{x}}$ contains the weights which must be placed on $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$ to produce $\hat{\mathbf{b}}$. These weights are easily read from the above equation, so $\hat{\mathbf{x}} = \begin{bmatrix}1/3\\14/3\\-5/3\end{bmatrix}$.

13. One computes that

$$A\mathbf{u} = \begin{bmatrix}11\\-11\\11\end{bmatrix}, \mathbf{b} - A\mathbf{u} = \begin{bmatrix}0\\2\\-6\end{bmatrix}, \|\mathbf{b} - A\mathbf{u}\| = \sqrt{40}$$

$$A\mathbf{v} = \begin{bmatrix}7\\-12\\7\end{bmatrix}, \mathbf{b} - A\mathbf{v} = \begin{bmatrix}4\\3\\-2\end{bmatrix}, \|\mathbf{b} - A\mathbf{v}\| = \sqrt{29}$$

Since $A\mathbf{v}$ is closer to **b** than $A\mathbf{u}$ is, $A\mathbf{u}$ is not the closest point in Col A to **b**. Thus **u** cannot be a least-squares solution of $A\mathbf{x} = \mathbf{b}$.

14. One computes that

$$A\mathbf{u} = \begin{bmatrix} 3 \\ 8 \\ 2 \end{bmatrix}, \mathbf{b} - A\mathbf{u} = \begin{bmatrix} 2 \\ -4 \\ 2 \end{bmatrix}, \|\mathbf{b} - A\mathbf{u}\| = \sqrt{24}$$

$$A\mathbf{v} = \begin{bmatrix} 7 \\ 2 \\ 8 \end{bmatrix}, \mathbf{b} - A\mathbf{v} = \begin{bmatrix} -2 \\ 2 \\ -4 \end{bmatrix}, \|\mathbf{b} - A\mathbf{v}\| = \sqrt{24}$$

Since $A\mathbf{u}$ and $A\mathbf{u}$ are equally close to $\mathbf{b}$, and the orthogonal projection is the *unique* closest point in Col A to $\mathbf{b}$, neither $A\mathbf{u}$ nor $A\mathbf{v}$ can be the closest point in Col A to $\mathbf{b}$. Thus neither $\mathbf{u}$ nor $\mathbf{v}$ can be a least-squares solution of $A\mathbf{x} = \mathbf{b}$.

15. The least squares solution satisfies $R\hat{\mathbf{x}} = Q^T\mathbf{b}$. Since $R = \begin{bmatrix} 3 & 5 \\ 0 & 1 \end{bmatrix}$ and $Q^T\mathbf{b} = \begin{bmatrix} 7 \\ -1 \end{bmatrix}$, the augmented matrix for the system may be row reduced to find

$$\begin{bmatrix} R & Q^T\mathbf{b} \end{bmatrix} = \begin{bmatrix} 3 & 5 & 7 \\ 0 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & -1 \end{bmatrix}$$

and so $\hat{\mathbf{x}} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$ is the least squares solution of $A\mathbf{x} = \mathbf{b}$.

16. The least squares solution satisfies $R\hat{\mathbf{x}} = Q^T\mathbf{b}$. Since $R = \begin{bmatrix} 2 & 3 \\ 0 & 5 \end{bmatrix}$ and $Q^T\mathbf{b} = \begin{bmatrix} 17/2 \\ 9/2 \end{bmatrix}$, the augmented matrix for the system may be row reduced to find

$$\begin{bmatrix} R & Q^T\mathbf{b} \end{bmatrix} = \begin{bmatrix} 2 & 3 & 17/2 \\ 0 & 5 & 9/2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2.9 \\ 0 & 1 & .9 \end{bmatrix}$$

and so $\hat{\mathbf{x}} = \begin{bmatrix} 2.9 \\ .9 \end{bmatrix}$ is the least squares solution of $A\mathbf{x} = \mathbf{b}$.

17. **a**. True. See the beginning of the section. The distance from $A\mathbf{x}$ to $\mathbf{b}$ is $\| A\mathbf{x} - \mathbf{b} \|$.
b. True. See the comments about equation (1).
c. False. The inequality points in the wrong direction. See the definition of a least-squares solution.
d. True. See Theorem 13.
e. True. See Theorem 14.

18. **a**. True. See the paragraph following the definition of a least-squares solution.
b. False. If $\hat{\mathbf{x}}$ is the least-squares solution, then $A\hat{\mathbf{x}}$ is the point in the column space of A closest to $\mathbf{b}$. See Figure 1 and the paragraph preceding it.
c. True. See the discussion following equation (1).
d. False. The formula applies only when the columns of A are linearly independent. See Theorem 14.
e. False. See the comments after Example 4.
f. False. See the Numerical Note.