THEOREM 4

(A and b is given)

Let A be an $m \times n$ matrix. Then the following statements are logically equivalent. That is, for a particular A, either they are all true statements or they are all false.

a. For each $\mathbf{b}$ in $\mathbb{R}^m$, the equation $A\mathbf{x}=\mathbf{b}$ has a solution. (Unique 한지와는 상관없이)

b. Each $\mathbf{b}$ in $\mathbb{R}^m$ is a linear combination of the columns of A. $b \in$ span{columns of A} $\Rightarrow \mathbb{R}^m \subset$ span{columns} span{columns of A} $= \mathbb{R}^m$

c. The columns of A span $\mathbb{R}^m$.

d. A has a pivot position in every row.

$A = [a_1 \dots a_n]$

Columns of A linearly independent

$\Leftrightarrow$ $x=0$ is only sol of $Ax=0$

$\Leftrightarrow$ $x = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$ is only weight of $x_1 a_1 + \cdots + x_n a_n = 0$

$\Leftrightarrow$ All columns in A are pivot columns (no free variable)

$\Leftrightarrow$ # pivots $= n$ ($=$ # of columns)

$\hookrightarrow$ 아니라면 linearly dependent

$Ax=b$ solution $=$ all row pivot $=$ b가 A에 대해 L.D

$=$ col span $\mathbb{R}^m$

col linearly independent $=$ all columns pivot $=$ $Ax=0$ trivial

THEOREM 7 ($v_j = c_1 v_1 + c_2 v_2 + \cdots + c_{j-1} v_{j-1}$ 인경우..)

Characterization of Linearly Dependent Sets

① An indexed set $S = \{v_1, \ldots, v_p\}$ of two or more vectors is linearly dependent if and only if at least one of the vectors in S is a linear combination of the others. In fact, if S is linearly dependent and $v_1 \neq 0$, ② then some $\mathbf{v}_j$ (with $j > 1$) is a linear combination of the preceding vectors, $\mathbf{v}_1, \ldots, \mathbf{v}_{j-1}$. (2번째 이상은)

THEOREM 8

$p > n$ 인경우 가정, $v_i \in \mathbb{R}^n$

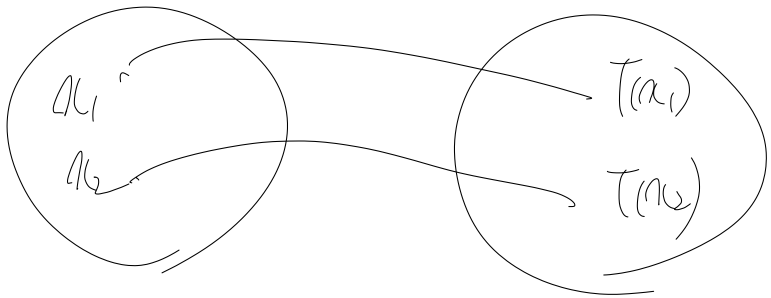
If a set contains more vectors than there are entries in each vector, then the set is linearly dependent. That is, any set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ is linearly dependent if $p > n$.

THEOREM 9

If a set $S = \{v_1, \ldots, v_p\}$ in $\mathbb{R}^n$ contains the zero vector, then the set is linearly dependent.

independent → No zero vector

For $x_1 \neq x_2 \rightarrow T(x_1) \neq T(x_2)$: One to One



For each $b \in$ range, b is the Image of only 1 vector in the domain

$Ax=b$ has a unique solution

$T(x)=Ax$: Onto $\Longleftrightarrow$ $Ax=b$ of solution 존재 (for arbitrary b)

One to One $\Longleftrightarrow$ Solution of $Ax=b$ unique (if it exist)

$\Leftrightarrow$ $x=0$ $Ax=0$의 유일해

$\Leftrightarrow$ No Free Var

$\Leftrightarrow$ All column = pivot column

$\Leftrightarrow$ ~~# pivots = # of Columns of A~~

~~= unknowns of A~~

$\Leftrightarrow$ One to One

$\Leftrightarrow$ range = co domain = span{columns}

$\Leftrightarrow$ $Ax=b$ has sol for each b

$\Leftrightarrow$ Each b is linear comb of columns of A

$\Leftrightarrow$ Every row pivot (in A)

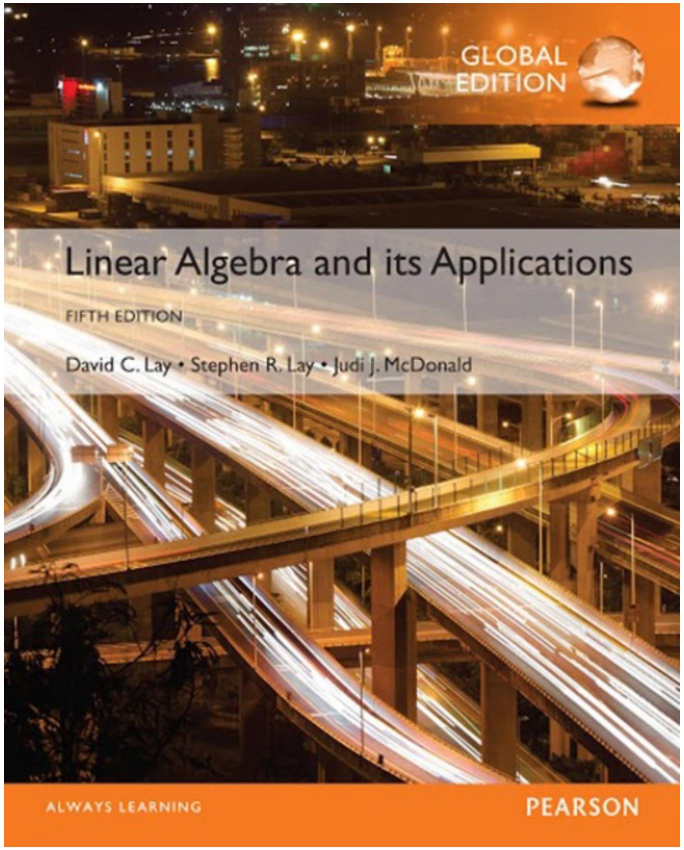
$\Leftrightarrow$ #pivots = # rows. (in A)

$\Leftrightarrow$ Onto

1 Linear Equations in Linear Algebra

1.4

THE MATRIX EQUATION $A\mathbf{x} = \mathbf{b}$



MATRIX EQUATION $A\mathbf{x} = \mathbf{b}$

- **Definition:** If A is an $m \times n$ matrix, with columns $\mathbf{a}_1, \ldots, \mathbf{a}_n$, and if $\mathbf{x}$ is in $\mathbb{R}^n$, then the **product of A and $\mathbf{x}$**, denoted by $A\mathbf{x}$, **is the linear combination of the columns of A using the corresponding entries in $\mathbf{x}$ as weights**; that is,

"weights"

$$A\mathbf{x} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \ldots + x_n\mathbf{a}_n$$

[illegible]

- Note that $A\mathbf{x}$ is defined only if the number of columns of A equals the number of entries in $\mathbf{x}$.

- **Example 2:** For $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ in $\mathbb{R}^m$, write the linear combination $3\mathbf{v}_1 - 5\mathbf{v}_2 + 7\mathbf{v}_3$ as a matrix times a vector.
- **Solution:** Place $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ into the columns of a matrix A and place the weights $3, -5$, and 7 into a vector $\mathbf{x}$.
- That is,

$$3\mathbf{v}_1 - 5\mathbf{v}_2 + 7\mathbf{v}_3 = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix} \begin{bmatrix} 3 \\ -5 \\ 7 \end{bmatrix} = A\mathbf{x}.$$

$A = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}$

MATRIX EQUATION $A\mathbf{x} = \mathbf{b}$

- Now, write the system of linear equations as a vector equation involving a linear combination of vectors.
- For example, the following system

$$\begin{aligned} x_1 + 2x_2 - x_3 &= 4 \\ -5x_2 + 3x_3 &= 1 \end{aligned} \qquad (1)$$

is equivalent to

$$x_1\begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2\begin{bmatrix} 2 \\ -5 \end{bmatrix} + x_3\begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}. \qquad (2)$$

MATRIX EQUATION $A\mathbf{x} = \mathbf{b}$

- As in the example, the linear combination on the left side is a matrix times a vector, so that (2) becomes

Column

unknowns

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}. \qquad (3)$$

- Equation (3) has the form $A\mathbf{x} = \mathbf{b}$. Such an equation is called a **matrix equation**, to distinguish it from a vector equation such as shown in (2).

Matrix ⇔ Vector

[illegible]

THEOREM 3

If A is an $m \times n$ matrix, with columns $\mathrm{a}_1, \ldots, \mathrm{a}_n$, and if b is in $\mathbb{R}^m$, then the matrix equation

$$A\mathbf{x} = \mathbf{b}$$

has the same solution set as the vector equation

$$x_1 a_1 + x_1 a_2 + \cdots + x_n a_n = b$$

which, in turn, has the same solution set as the system of linear equations whose augmented matrix is

$$[a_1 \quad a_2 \ \ldots \ a_n \quad \mathrm{b}]$$

행상이 거대한다.

Determine if $b \in \text{span}\{a_1, \dots, a_n\}$ — same procedure — b is linear comb of $a_1, \dots, a_n$

System of Eq
Vector Eq
Matrix E

$\rightarrow [a_1 \dots a_n \; b] \xrightarrow{\text{ERO}}$ echelon form $\rightarrow$ rightmost pivot column

Augment

Y $\rightarrow$ inconsistent

N $\rightarrow$ reduced echelon

free : many sol

No free : unique

EXISTENCE OF SOLUTIONS

- The equation $A\mathbf{x} = \mathbf{b}$ has a solution if and only if $\mathbf{b}$ is a linear combination of the columns of A.

(A and b is given) =

THEOREM 4

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c. The columns of A span $\mathbb{R}^m$.

d. A has a pivot position in every row.

→ Echelon form of A 에게 zero row 없다.

Only A given

b는 아무거나...

아래문장과는 다르다.

$Ax=b$ is consistent if and only if the right most column of $[A, b]$ is Not a pivot column

(A and b given)

$A: m\times n$ $A=[a_1\ a_2\ \dots\ a_n]$ $a_1, a_2, \dots a_n \in \mathbb{R}^m$

$c_1a_1+c_2a_2+\dots+c_na_n \in \mathbb{R}^m$ $c_1a_1+\dots+c_na_n$ 은 n entries vector

따라서. ↳ $\mathrm{span}\{a_1, a_2, \dots a_n\} \subset \mathbb{R}^m$

(a) $Ax=b$ has solution for each $b\in\mathbb{R}^m$

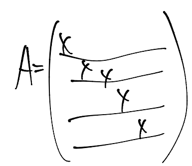
$$Ax=[a_1\ a_2 \dots a_n]\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1a_1+\dots+x_na_n=b \quad \text{for each } b\in\mathbb{R}^m$$

$\therefore$ (b) $b\in\mathbb{R}^m$ is linear comb of columns of A.

Each b in $\mathbb{R}^m \in$ Span{Colums of A}

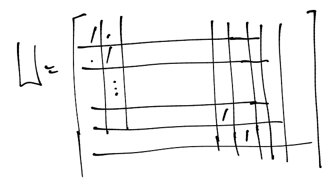
($\mathbb{R}^m$ 에 어떤걸 선택하던 span 이므로) $\therefore \mathbb{R}^m \subset$ Span{Colums of A}

$\mathbb{R}^m=$ Span{Colums of A} (c) 성립



$Ax=b$

A가 모든 row에 pivot



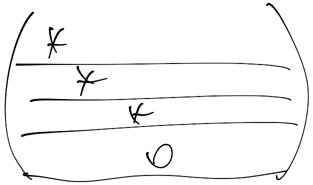
U가 A의 echelon form

(1) = (4)

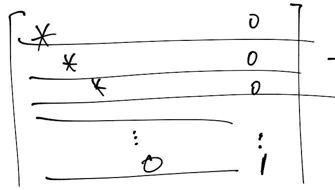
$[A\ b] \rightarrow [U\ b']$ 인데, 이러면 b'가 pivot이 있을수없다. U에 모든 pivot column이 있다.

$\Rightarrow Ax=b$는 solution 有

Some row에 pivot이 없다고 하면... U에 row of zeros 존재



이고, $[U \; b'] \Rightarrow$



→ no solution

(운이 좋아봐야 many solution)

$[U \; b']$ no sol → $[A \; b]$ no sol → $Ax = b$ no solution for some b

Pf.

① Suppose that A has a pivot in every row. $[A \; b] \to [U \; c]$

Then, the reduced echelon form U of A has a pivot in every row

In addition, pivot columns of U are $\begin{pmatrix} 1 \\ \vdots \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ \vdots \end{pmatrix} \cdots \begin{pmatrix} 0 \\ \vdots \\ 1 \end{pmatrix}$

Then $Ux = c$ is consistent means $Ax = b$ is consistent.

② Suppose that A does not have pivot in some row →

U doesnot have solution to some c $\left(ex = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}\right)$

Then, $[U, c] \to [A \; b]$ $Ax = b$ no solution.

COMPUTATION OF $A\mathbf{x}$

- **Example 4:** Compute $A\mathbf{x}$, where $A = \begin{bmatrix} 2 & 3 & 4 \\ -1 & 5 & -3 \\ 6 & -2 & 8 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$.
- **Solution:** From the definition,

$$\begin{bmatrix} 2 & 3 & 4 \\ -1 & 5 & -3 \\ 6 & -2 & 8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1\begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix} + x_2\begin{bmatrix} 3 \\ 5 \\ -2 \end{bmatrix} + x_3\begin{bmatrix} 4 \\ -3 \\ 8 \end{bmatrix}$$

$\rightarrow$

$+$

$+$

COMPUTATION OF $A\mathbf{x}$

$$= \begin{bmatrix} 2x_1 \\ -x_1 \\ 6x_1 \end{bmatrix} + \begin{bmatrix} 3x_2 \\ 5x_2 \\ -2x_2 \end{bmatrix} + \begin{bmatrix} 4x_3 \\ -3x_3 \\ 8x_3 \end{bmatrix} \quad (1)$$

$$= \begin{bmatrix} 2x_1 + 3x_2 + 4x_3 \\ -x_1 + 5x_2 - 3x_3 \\ 6x_1 - 2x_2 + 8x_3 \end{bmatrix}.$$

- The first entry in the product $A\mathbf{x}$ is a sum of products (sometimes called *a dot product*), using the first row of A and the entries in $\mathbf{x}$.

Scalar product

COMPUTATION OF *A***x**

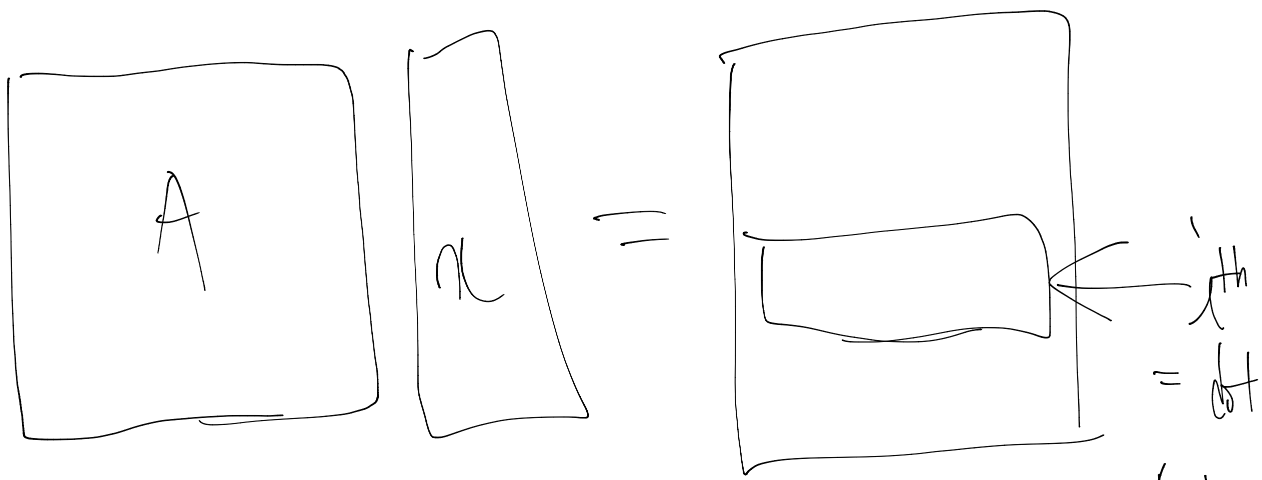
- That is, $\begin{bmatrix} 2 & 3 & 4 \\ & & \\ & & \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_1 + 3x_2 + 4x_3 \\ \\ \end{bmatrix}$.

- Similarly, the second entry in $A\mathbf{x}$ can be calculated by multiplying the entries in the second row of A by the corresponding entries in $\mathbf{x}$ and then summing the resulting products.

$$\begin{bmatrix} & & \\ -1 & 5 & -3 \\ & & \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \\ -x_1 + 5x_2 - 3x_3 \\ \\ \end{bmatrix}$$

ROW-VECTOR RULE FOR COMPUTING $A\mathbf{x}$

- Likewise, the third entry in $A\mathbf{x}$ can be calculated from the third row of A and the entries in $\mathbf{x}$.
- If the product $A\mathbf{x}$ is defined, then the ith entry in $A\mathbf{x}$ is the sum of the products of corresponding entries from row i of A and from the vector $\mathbf{x}$.



= dot product between
A ith row • x

ROW-VECTOR RULE FOR COMPUTING $A\mathbf{x}$

- The matrix with 1's on the diagonal and 0's elsewhere is called an **identity matrix** and is denoted by I.
- For example, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is an identity matrix.

PROPERTIES OF THE MATRIX-VECTOR PRODUCT $A\mathbf{x}$

THEOREM 5

If A is an $m \times n$ matrix, u and v are vectors in $\mathbb{R}^n$, and c is a scalar, then

a. $A(\mathbf{u}+\mathbf{v}) = A\mathbf{u} + A\mathbf{v}$;

b. $A(c\mathbf{u}) = c(A\mathbf{u})$.

- **Proof:** For simplicity, take $n = 3$, $A = [\mathrm{a}_1 \ \ \mathrm{a}_2 \ \ \mathrm{a}_3]$, and **u**, **v** in $\mathbb{R}^3$.
- For $i = 1, 2, 3$, let u_i and v_i be the ith entries in **u** and **v**, respectively.

PROPERTIES OF THE MATRIX-VECTOR PRODUCT *A*x

- To prove statement (a), compute $A(\mathbf{u}+\mathbf{v})$ as a linear combination of the columns of A using the entries in $\mathbf{u}+\mathbf{v}$ as weights.

$$A(\mathbf{u}+\mathbf{v}) = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \end{bmatrix} \begin{bmatrix} u_1+v_1 \\ u_2+v_2 \\ u_3+v_3 \end{bmatrix}$$

Entries in $\mathbf{u}+\mathbf{v}$

$$= (u_1+v_1)\mathbf{a}_1 + (u_2+v_2)\mathbf{a}_2 + (u_3+v_3)\mathbf{a}_3$$

Columns of A

$$= (u_1\mathbf{a}_1 + u_2\mathbf{a}_2 + u_3\mathbf{a}_3) + (v_1\mathbf{a}_1 + v_2\mathbf{a}_2 + v_3\mathbf{a}_3)$$

$$= A\mathbf{u} + A\mathbf{v}$$

PROPERTIES OF THE MATRIX-VECTOR PRODUCT $A\mathbf{x}$

- To prove statement (b), compute $A(c\mathbf{u})$ as a linear combination of the columns of A using the entries in $c\mathbf{u}$ as weights.

$$A(c\mathrm{u}) = \begin{bmatrix} \mathrm{a}_1 & \mathrm{a}_2 & \mathrm{a}_3 \end{bmatrix} \begin{bmatrix} cu_1 \\ cu_2 \\ cu_3 \end{bmatrix} = (cu_1)\mathrm{a}_1 + (cu_2)\mathrm{a}_2 + (cu_3)\mathrm{a}_3$$

$$= c(u_1\mathrm{a}_1) + c(u_2\mathrm{a}_2) + c(u_3\mathrm{a}_3)$$

$$= c(u_1\mathrm{a}_1 + u_2\mathrm{a}_2 + u_3\mathrm{a}_3)$$

$$= c(A\mathrm{u})$$

Matrix x (Vector Sum) = Sum (Matrix x Vector)

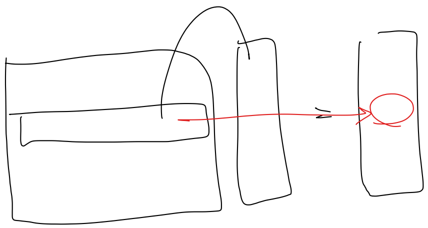
Matrix x (Vector Scalar Mult) = Scalar Mult (Matrix x Vector)

Product of matrix and Vector

$$Ax = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1 a_1 + \cdots + x_n a_n$$

dot Product

dot Product



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which, in turn, has the same solution set as the system of linear equations whose augmented matrix is

$$[a_1 \quad a_2 \ \ldots \ a_n \quad b]$$

항상이거다

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d. A has a pivot position in every row.

Echelon form of A에게 zero row 없다.

Only A given

b는 아무거나...

아래문장과는 다르다.

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b. $A(c\mathbf{u}) = c(A\mathbf{u})$.

1. $\begin{bmatrix} -4 & 2 \\ 1 & 6 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \\ 7 \end{bmatrix} = 3\begin{bmatrix} -4 \\ 1 \\ 0 \end{bmatrix} - 2\begin{bmatrix} 2 \\ 6 \\ 1 \end{bmatrix} + 7\left(? \right)$ Not defined

(3×2) (3×1)

not same.

multi the matrix & vector → # of columns in matrix = vector entry

2. $\begin{bmatrix} 2 \\ 6 \\ -1 \end{bmatrix} \begin{bmatrix} 4 \\ -1 \end{bmatrix}$ 이것도.

$\left(\begin{array}{l} \text{column} = \text{entry of vector} \\ \#\text{of} \ldots \end{array} \right)$

9.

$$x_1\begin{pmatrix}7\\0\end{pmatrix}+x_2\begin{pmatrix}1\\1\end{pmatrix}+x_3\begin{pmatrix}-5\\4\end{pmatrix}=\begin{pmatrix}9\\0\end{pmatrix}$$

$$\begin{bmatrix}7 & 1 & -5\\0 & 1 & 4\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}9\\0\end{bmatrix}$$

11.

$$\begin{bmatrix}1 & 2 & 4 & -2\\0 & 1 & 5 & 2\\-2 & -4 & -3 & 9\end{bmatrix}\Rightarrow\begin{bmatrix}1 & 2 & 4 & -2\\0 & 1 & 5 & 2\\0 & 0 & 1 & 1\end{bmatrix}\Rightarrow\begin{bmatrix}1 & 0 & 0 & 0\\ & 1 & 0 & -3\\ & & 1 & 1\end{bmatrix}$$

write sol as vector

$x_1=0,\ x_2=-3,\ x_3=1$

$$\therefore\quad\begin{bmatrix}1\\0\\-2\end{bmatrix}0+\begin{bmatrix}2\\1\\-4\end{bmatrix}(-3)+\begin{bmatrix}4\\5\\-3\end{bmatrix}\cdot 1=\begin{bmatrix}-2\\2\\9\end{bmatrix}$$

$$u = \begin{bmatrix} 2 \\ -3 \\ 2 \end{bmatrix} \qquad A = \begin{bmatrix} 5 & 8 & 7 \\ 0 & 1 & -1 \\ 1 & 3 & 0 \end{bmatrix}$$

is u subset of $\mathbb{R}^3$ spanned by column of A ?

$Ax = u$

$$\left[\begin{array}{ccc|c} 5 & 8 & 7 & 2 \\ 0 & 1 & -1 & -3 \\ 1 & 3 & 0 & 2 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 3 & 0 & 2 \\ 0 & 1 & -1 & -3 \\ 5 & 8 & 7 & 2 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 3 & 0 & 2 \\ 0 & 1 & -1 & -3 \\ 0 & -7 & 7 & -8 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 3 & 0 & 2 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 0 & -29 \end{array}\right]$$

no solution $\therefore u \notin \text{span}\{\text{columns of } A\}$

17. $\begin{bmatrix} 1 & 3 & 0 & 3 \\ -1 & -1 & -1 & 1 \\ 0 & -4 & 2 & -8 \\ 2 & 0 & 3 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & 0 & 3 \\ 0 & 2 & -1 & 4 \\ 0 & -4 & 2 & -8 \\ 0 & -6 & 3 & -7 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & 0 & 3 \\ 0 & 2 & -1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix}$

inconsistent

3 row

row ecns pivt

18. $\mathbb{R}^4$ vector가 A의 column의 linear combination인가?

A의 column이 $\mathbb{R}^4$를 span하나?

No. No.

↳ 법칙(4)

A = coefficient Matrix

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

일반적. (4)에 대해

$$= [v_1 \ v_2 \ v_3]$$

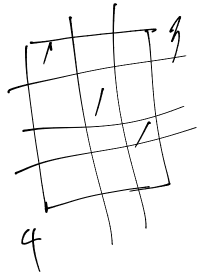
$\rightarrow$ # pivots $\le 3 < 4$ $\leftarrow$ ERO 불필요

There is (최대) at most one pivot in each row/column. $A \sim m \times n$ (# pivots) $\le \min\{m, n\}$

$\mathbb{R}^3$은 가능하지만 $\mathbb{R}^4$ ×

Ans) No.

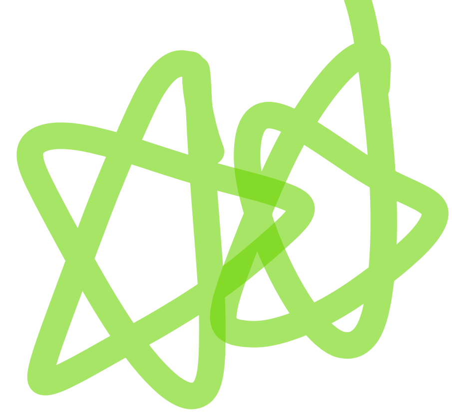
$A \sim 4 \times 3$ $b \sim 4 \times 1$ $Ax = b$ unique.



$[A \ b]$ (rightmost o ×)

No free variable
3 pivot columns

$A \sim 4\times3$. $b \in \mathbb{R}^4$ vector, $Ax=b$ Unique.



A의 reduced echelon form 에 대한 설명

1		
	1	
		1

$[A \ b] \rightarrow [U \ b']$

(right most zero X)

+

No free Variable

+

3 pivot columns

Thm 4번의 차이에 주의

Thm 4는 A만 유효함

이문제는 풀다가 끝

Augment

Matrix

$$\begin{pmatrix} 1 & - & \cdot & / \\ 0 & 1 & \cdot & / \\ 0 & 0 & 1 & / \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

이런 형태로 가능

3. $\begin{bmatrix} 6 & 5 \\ -4 & -3 \\ 7 & 6 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \end{bmatrix} = \begin{bmatrix} 12 \\ -8 \\ 14 \end{bmatrix} + \begin{bmatrix} -15 \\ 9 \\ -18 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ 4 \end{bmatrix}$

4. $\begin{bmatrix} 8 & 3 & -4 \\ 5 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 8+3-4 \\ 5+1+2 \end{bmatrix} = \begin{bmatrix} 7 \\ 8 \end{bmatrix}$

10. $\begin{bmatrix} 8x_1 \\ 5x_1 \\ x_1 \end{bmatrix} + \begin{bmatrix} -x_2 \\ 4x_2 \\ -3x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$

$x_1 \begin{bmatrix} 8 \\ 5 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 4 \\ -3 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$

$\begin{bmatrix} 8 & -1 \\ 5 & 4 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$

12.

$$A=\begin{bmatrix}1&2&1\\-3&-1&2\\0&5&3\end{bmatrix}\quad b=\begin{bmatrix}0\\1\\-1\end{bmatrix}\qquad Ax=b \text{ 인 } x=?$$

$$\left[\begin{array}{ccc|c}1&2&1&0\\-3&-1&2&1\\0&5&3&-1\end{array}\right]\to\left[\begin{array}{ccc|c}1&0&0&3/5\\0&1&0&-4/5\\0&0&1&1\end{array}\right]\quad\therefore\ \begin{aligned}x_1&=3/5\\x_2&=-4/5\\x_3&=1\end{aligned}$$

15.

$$A=\begin{bmatrix}2&-1\\-6&3\end{bmatrix}\quad b=\begin{bmatrix}b_1\\b_2\end{bmatrix}$$

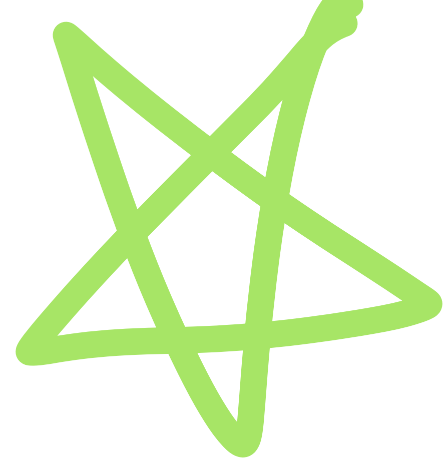
모든 b에 sol x 인경우에도 sol b를 구해라.

$$\begin{bmatrix}2&-1&b_1\\-6&3&b_2\end{bmatrix}\to\begin{bmatrix}2&-1&b_1\\0&0&3b_1+b_2\end{bmatrix}$$

$3b_1+b_2=0$ 이어야 consistent $\to$ $b=b_1\begin{pmatrix}1\\-3\end{pmatrix}$ 이어야 한다.

A만 봐도 안되는걸 ~~알 수~~ 알 수 있다.

$$B=\begin{bmatrix}1 & 3 & -2 & 2\\ 0 & 1 & 1 & -5\\ 1 & 2 & -3 & 7\\ -2 & -8 & 2 & -1\end{bmatrix} \to \begin{bmatrix}1 & 3 & -2 & 2\\ 0 & 1 & 1 & -5\\ 0 & -1 & -1 & 5\\ 0 & -2 & -2 & 3\end{bmatrix} \to \begin{bmatrix}1 & 3 & -2 & 2\\ 0 & 1 & 1 & -5\\ 0 & 0 & 0 & -7\\ 0 & 0 & 0 & 0\end{bmatrix} \leftarrow \text{zero row}$$



18. span $\mathbb{R}^4$? $\to$ No

$Bx = y$ solution for $y \in \mathbb{R}^4$ $\to$ No

20. every vector is L.C. by columns? $\to$ No

(every row of A should have pivot)

[illegible] $\to$ No. columns are all $\in \mathbb{R}^4$

~~This spans for $\mathbb{R}^3$~~

22. $v_1 = \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix}$ $v_2 = \begin{bmatrix} 0 \\ -3 \\ 8 \end{bmatrix}$ $v_3 = \begin{bmatrix} 4 \\ -1 \\ -5 \end{bmatrix}$ $\Rightarrow \begin{bmatrix} 0 & 0 & 4 \\ 0 & -3 & -1 \\ -2 & 8 & -5 \end{bmatrix} \rightarrow \begin{bmatrix} -2 & 8 & -5 \\ & -3 & -1 \\ & & 4 \end{bmatrix} \Rightarrow$ Every row pivot.

$a_1 a_2 a_3$ span $\mathbb{R}^3$

$Ax=b$ solution for any $b \in \mathbb{R}^3$

$b \in \mathbb{R}^3$ is linear comb by $a_1 a_2 a_3$

암기

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24. 3×3 $Ax=b$ Unique Sol $\rightarrow$ No free Var $\Rightarrow$ All columns in A is pivot columns.

$A = \begin{bmatrix} * & & \\ & * & \\ & & * \end{bmatrix} \Rightarrow U_A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

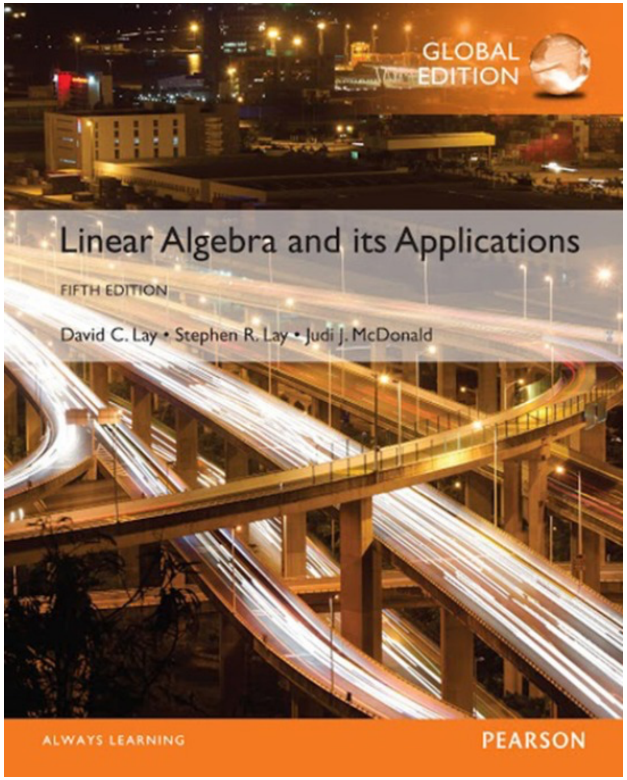
$\Rightarrow$ A is 3×3 $\therefore$ each row has pivot

$\Rightarrow$ columns span $\mathbb{R}^3$

1 Linear Equations in Linear Algebra

1.5

SOLUTION SETS OF LINEAR SYSTEMS



HOMOGENEOUS LINEAR SYSTEMS

- A system of linear equations is said to be **homogeneous** if it can be written in the form $A\mathbf{x} = \mathbf{0}$, where A is an $m \times n$ matrix and $\mathbf{0}$ is the zero vector in $\mathbb{R}^m$.
- Such a system $A\mathbf{x} = \mathbf{0}$ *always* has at least one solution, namely, $\mathbf{x} = \mathbf{0}$ (the zero vector in $\mathbb{R}^n$).
- This zero solution is usually called the **trivial solution**.
- The homogenous equation $A\mathbf{x} = \mathbf{0}$, the important question is whether there exists a **nontrivial solution**, that is, a nonzero vector $\mathbf{x}$ that satisfies $A\mathbf{x} = \mathbf{0}$.

HOMOGENEOUS LINEAR SYSTEMS

- **Example 1:** Determine if the following homogeneous system has a nontrivial solution. Then describe the solution set.

$$\begin{aligned} 3x_1 + 5x_2 - 4x_3 &= 0 \\ -3x_1 - 2x_2 + 4x_3 &= 0 \\ 6x_1 + x_2 - 8x_3 &= 0 \end{aligned}$$

- **Solution:** Let A be the matrix of coefficients of the system and row reduce the augmented matrix $\begin{bmatrix} A & 0 \end{bmatrix}$ to echelon form:

HOMOGENEOUS LINEAR SYSTEMS

$$\begin{bmatrix} 3 & 5 & -4 & 0 \\ -3 & -2 & 4 & 0 \\ 6 & 1 & -8 & 0 \end{bmatrix} \sim \begin{bmatrix} 3 & 5 & -4 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & -9 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 3 & 5 & -4 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

- Since x_3 is a free variable, $A\mathbf{x} = 0$ has nontrivial solutions (one for each choice of x_3.)
- Continue the row reduction of $\begin{bmatrix} A & 0 \end{bmatrix}$ to reduced echelon form:

$$\begin{bmatrix} 1 & 0 & -\frac{4}{3} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{aligned} x_1 - \frac{4}{3}x_3 &= 0 \\ x_2 &= 0 \\ 0 &= 0 \end{aligned}$$

x_3 free

HOMOGENEOUS LINEAR SYSTEMS

- Solve for the basic variables x_1 and x_2 to obtain $x_1 = \frac{4}{3}x_3$, $x_2 = 0$, with x_3 free.
- As a vector, the general solution of $A\mathbf{x} = \mathbf{0}$ has the form given below.

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{4}{3}x_3 \\ 0 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} \frac{4}{3} \\ 0 \\ 1 \end{bmatrix} = x_3\mathbf{v}, \text{ where } \mathbf{v} = \begin{bmatrix} \frac{4}{3} \\ 0 \\ 1 \end{bmatrix}$$

x는 어떤값이와도 가능

HOMOGENEOUS LINEAR SYSTEMS

- Here x_3 is factored out of the expression for the general solution vector.
- This shows that every solution of $A\mathbf{x} = \mathbf{0}$ in this case is a scalar multiple of $\mathbf{v}$.
- The trivial solution is obtained by choosing $x_3 = 0$.

PARAMETRIC VECTOR FORM

Parameter Vector

- The equation of the form $\mathbf{x} = s\mathbf{u} + t\mathbf{v}$ (s, t in $\mathbb{R}$) is called a **parametric vector equation** of the plane.

Parameter

- In Example 1, the equation $\mathbf{x} = x_3\mathbf{v}$ (with x_3 free), or $\mathbf{x} = t\mathbf{v}$ (with t in $\mathbb{R}$), is a parametric vector equation of a line.

- Whenever a solution set is described explicitly with vectors as in Example 1, we say that the solution is in **parametric vector form**.

SOLUTIONS OF NONHOMOGENEOUS SYSTEMS

$Ax=b$. $(b \neq 0)$ 인경우

- When a nonhomogeneous linear system has many solutions, the general solution can be written in parametric vector form as one vector plus an arbitrary linear combination of vectors that satisfy the corresponding homogeneous system.

p. one of the solution

이전에 예시 $\Rightarrow$ $x = p + tv$

general solution

$=$ one solution of $Ax=b$ $+$ general solution of $Ax=0$

SOLUTIONS OF NONHOMOGENEOUS SYSTEMS

- **Example 3 :** Describe all solutions of $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 3 & 5 & -4 \\ -3 & -2 & 4 \\ 6 & 1 & -8 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} 7 \\ -1 \\ -4 \end{bmatrix}.$$

Note: A is matrix in Ex1 $\quad v = \begin{bmatrix} 4/3 \\ 0 \\ 1 \end{bmatrix} (?)$

SOLUTIONS OF NONHOMOGENEOUS SYSTEMS

- **Solution:** Row operations on $\begin{bmatrix} A & \mathbf{0} \end{bmatrix}$ produce

Augmented

$$\begin{bmatrix} 3 & 5 & -4 & 7 \\ -3 & -2 & 4 & -1 \\ 6 & 1 & -8 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -\frac{4}{3} & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{aligned} x_1 - \frac{4}{3}x_3 &= -1 \\ x_2 &= 2 \\ 0 &= 0 \end{aligned}$$

이뿐 변함

- Thus $x_1 = -1 + \frac{4}{3}x_3$, $x_2 = 2$, and x_3 is free.

SOLUTIONS OF NONHOMOGENEOUS SYSTEMS

- As a vector, the general solution of $A\mathbf{x} = \mathbf{b}$ has the form

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 + \frac{4}{3}x_3 \\ 2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{4}{3}x_3 \\ 0 \\ x_3 \end{bmatrix} = \underbrace{\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}}_{\mathbf{p}} + x_3 \underbrace{\begin{bmatrix} \frac{4}{3} \\ 0 \\ 1 \end{bmatrix}}_{\mathbf{v}}$$

New by $\vec{b}$

homogeneous

$x = p + x_3 v$

$b = A(p + x_3 v) = Ap + Ax_3 v = Ap$

$\therefore b = Ap$

SOLUTIONS OF NONHOMOGENEOUS SYSTEMS

- The equation $\mathbf{x} = \mathbf{p} + x_3\mathbf{v}$, or, writing t as a general parameter,

$$\mathbf{x} = \mathbf{p} + t\mathbf{v} \quad (t \text{ in } \mathbb{R}) \qquad (3)$$

describes the solution set of $A\mathbf{x} = \mathbf{b}$ in parametric vector form.

- The solution set of $A\mathbf{x} = \mathbf{0}$ has the parametric vector equation

해내의 부분해

$$\mathbf{x} = t\mathbf{v} \quad (t \text{ in } \mathbb{R}) \qquad (4)$$

[with the same **v** that appears in (3)].

- Thus the solutions of $A\mathbf{x} = \mathbf{b}$ are obtained by adding the vector **p** to the solutions of $A\mathbf{x} = \mathbf{0}$.

$Ax=b$ 에서

One solution p is known

Then $Ax=b$

$Ap=b$

$A(x-p)=0$

$\therefore x-p = x_3 v$ (general)

$\therefore x = \underbrace{p}_{\text{particular solution}} + \underbrace{x_3 v}_{\text{homogeneous solution}}$

homogeneous System $Ax=0$

trivial solution: $x=0$

non-trivial solution: $x \neq 0$. non-zero

parametric solution: $c_1 v_1 + c_2 v_2 + \cdots + c_p v_p$

non homogeneous solution: $x = x_p + x_h$

particular solution of $Ax=b$

General solution of $Ax=b$. parametric.

- Determine if the system has a nontrivial solution.

2.
$$\begin{aligned} x_1 - 3x_2 + 7x_3 &= 0 \\ -2x_1 + x_2 - 4x_3 &= 0 \\ x_1 + 2x_2 + 9x_3 &= 0 \end{aligned}$$

$$\begin{bmatrix} 1 & -3 & 7 & 0 \\ & -5 & 10 & 0 \\ & & 12 & 0 \end{bmatrix}$$

→ No free

→ $x=0$ is unique

→ No nontrivial sol

4.
$$\begin{aligned} -5x_1 + 7x_2 + 9x_3 &= 0 \\ x_1 - 2x_2 + 6x_3 &= 0 \end{aligned}$$

→ max pivot = 2 → 1 var should be free.

→ many sol.

→ nontrivial

Write the solution set in parametric form.

6.
$$\begin{aligned} x_1 + 3x_2 - 5x_3 &= 0 \\ x_1 + 4x_2 - 8x_3 &= 0 \\ -3x_1 - 7x_2 + 9x_3 &= 0 \end{aligned}$$

$$\to \begin{bmatrix} 1 & 0 & -4 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore x_1 = -4x_3$

$x_2 = 3x_3$

x_3 ~ free

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -4x_3 \\ 3x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -4 \\ 3 \\ 1 \end{bmatrix}$$

Describe all solutions of $Ax = 0$ in parametric vector form, where A is row equivalent to the given matrix.

8. $\begin{bmatrix} 1 & -2 & -9 & 5 \\ 0 & 1 & 2 & -6 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -5 & -7 & 0 \\ 0 & 1 & 2 & -6 & 0 \end{bmatrix}$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 5x_3 \\ -2x_3 \\ x_3 \\ 0 \end{bmatrix} + \begin{bmatrix} 7x_4 \\ 6x_4 \\ 0 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 5 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 7 \\ 6 \\ 0 \\ 1 \end{bmatrix}$$

10. $\begin{bmatrix} 1 & 3 & 0 & -4 \\ 2 & 6 & 0 & -8 \end{bmatrix}$

$$\rightarrow \begin{bmatrix} 1 & 3 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore x_1 + 3x_2 - 4x_4 = 0$

$$X = \begin{bmatrix} -3x_2 + 4x_4 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$Ax=0$ parametric

$$A = \begin{bmatrix} \boxed{1} & 5 & 2 & -8 & 9 & 0 \\ & & \boxed{1} & -7 & 4 & -8 \\ & & & & & \boxed{1} \\ & \cdots & & & & 0 \end{bmatrix} \rightarrow \begin{bmatrix} \boxed{1} & 5 & 0 & 8 & 1 & 0 & 0 \\ & & \boxed{1} & -7 & 4 & 0 & 0 \\ & & & & & \boxed{1} & 0 \\ & \cdots & & & & 0 & 0 \end{bmatrix}$$

$$\begin{cases} x_1 + 5x_2 + 8x_4 + x_5 = 0 \\ x_3 - 7x_4 + 4x_5 = 0 \\ x_6 = 0 \end{cases}$$

$x_1 = -5x_2 - 8x_4 - x_5$

$x_3 = 7x_4 \; 4x_5$

$x_6 = 0$

x_1, x_3, x_6 는 basic

나머지는 free.

16.

$$\begin{bmatrix} 1 & 3 & -5 & 4 \\ 1 & 4 & -8 & 7 \\ -3 & -7 & 9 & -6 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & -5 & 4 \\ 0 & 1 & -3 & 3 \\ 0 & 2 & -6 & 6 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & -5 & 4 \\ 0 & 1 & -3 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 4 & -5 \\ 0 & 1 & -3 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_1 = -4x_3 - 5$

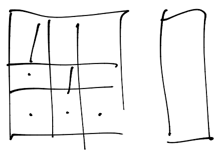
$x_2 = 3x_3 + 3$

$$\rightarrow x_3 \begin{bmatrix} -4 \\ 3 \\ 1 \end{bmatrix} + \begin{bmatrix} -5 \\ 3 \\ 0 \end{bmatrix}$$

General Sol　　Particular → not only particular sol.

1B. (3x3) $Ax=y$ no solution $\rightarrow$ not every row in A has pivot

최대 2 pivot



$Ax=z$ has 2 basic var at least 1 free var.

No Unique Solution.

$\rightarrow$ ($Ax=z$ has many sol

or

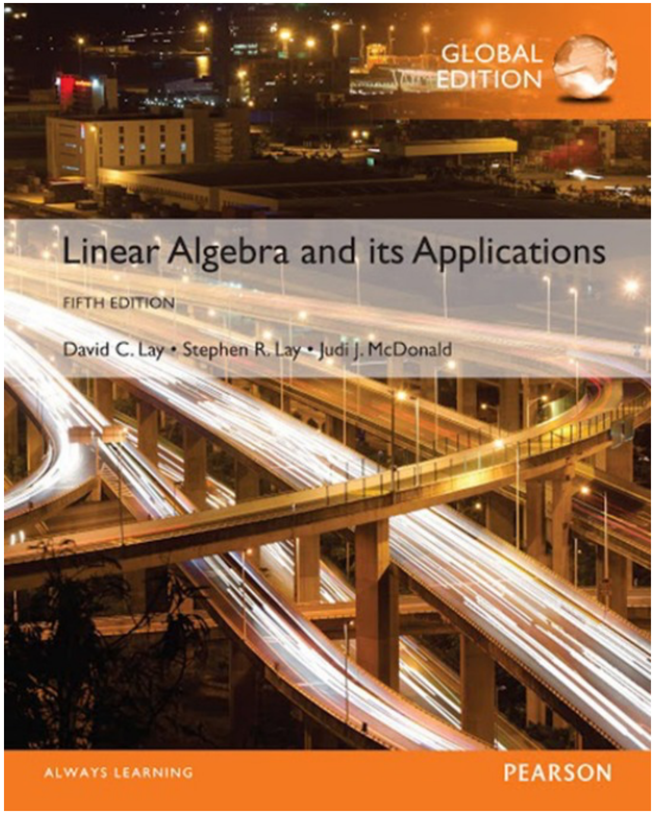
No sol)

1 Linear Equations in Linear Algebra

1.7

LINEAR INDEPENDENCE

벡터사이 관계 → 곧 된다
연관.



PEARSON

$A = [a_1 \ldots a_n]$

Columns of A linearly independent

$\Leftrightarrow$ $x = 0$ is only sol of $Ax = 0$

$\Leftrightarrow$ $x = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$ is only weight of $x_1 a_1 + \cdots + x_n a_n = 0$

$\Leftrightarrow$ All columns in A are pivot columns (no free variable)

$\Leftrightarrow$ # pivots $= n$ ($=$ # of columns)

$\hookrightarrow$ 아니라면 linearly dependent

LINEAR INDEPENDENCE

- **Definition:** An indexed set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ is said to be **linearly independent** if the vector equation

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \ldots + x_p\mathbf{v}_p = 0$$

$x_1 = x_2 \cdots = x_p = 0$ (trivial solution)

has only the trivial solution. The set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is said to be **linearly dependent** if there exist weights $c_1, \ldots, c_p$, not all zero, such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_p\mathbf{v}_p = 0 \qquad (2)$$

LINEAR INDEPENDENCE

- Equation (2)

$$c_1\mathrm{v}_1 + c_2\mathrm{v}_2 + \ldots + c_p\mathrm{v}_p = 0$$

  is called a **linear dependence relation** among $\mathbf{v}_1, \ldots, \mathbf{v}_p$ when the weights are not all zero.

- An indexed set is linearly dependent if and only if it is not linearly independent.

LINEAR INDEPENDENCE

- **Example 1:** Let $\mathrm{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \mathrm{v}_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$, and $\mathrm{v}_3 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$.

a. Determine if the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent. $c_1 = c_2 = c_3 = 0$ 일때 $[v_1\ v_2\ v_3]\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = 0$ 의 해 $\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

b. If possible, find a linear dependence relation among $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$.

LINEAR INDEPENDENCE

- **Solution:** We must determine if there is a nontrivial solution of the equation :

$$x_1\begin{pmatrix}1\\2\\3\end{pmatrix}+x_2\begin{pmatrix}4\\5\\6\end{pmatrix}+x_3\begin{pmatrix}2\\1\\0\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix} \qquad (1)$$

LINEAR INDEPENDENCE

- Row operations on the associated augmented matrix show that

$$\begin{bmatrix} 1 & 4 & 2 & 0 \\ 2 & 5 & 1 & 0 \\ 3 & 6 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 2 & 0 \\ 0 & -3 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

echelon form

free variable

↓

many sol

- x_1 and x_2 are basic variables, and x_3 is free.
- Each nonzero value of x_3 determines a nontrivial solution of (1).
- Hence, $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly dependent.

LINEAR INDEPENDENCE

b. To find a linear dependence relation among $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$, completely row reduce the augmented matrix and write the new system:

$$\begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{aligned} x_1 - 2x_3 &= 0 \\ x_2 + x_3 &= 0 \\ 0 &= 0 \end{aligned}$$

- Thus, $x_1 = 2x_3$, $x_2 = -x_3$, and x_3 is free.
- Choose any nonzero value for x_3—say, $x_3 = 5$.
- Then $x_1 = 10$ and $x_2 = -5$.

LINEAR INDEPENDENCE

- Substitute these values into equation (1) and obtain the equation below.

$$10\mathbf{v}_1 - 5\mathbf{v}_2 + 5\mathbf{v}_3 = \mathbf{0}$$

- This is one (out of infinitely many) possible linear dependence relations among $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$.

LINEAR INDEPENDENCE OF MATRIX COLUMNS

- Suppose that we begin with a matrix $A = [\mathbf{a}_1 \quad \cdots \quad \mathbf{a}_n]$ instead of a set of vectors.

- The matrix equation $A\mathbf{x} = \mathbf{0}$ can be written as

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \ldots + x_n\mathbf{a}_n = \mathbf{0}.$$

$x_1, x_2 \ldots x_n \neq 0$ 일경우

- *Each linear dependence relation among the columns of A corresponds* ~~*to*~~ *a nontrivial solution of* $A\mathbf{x} = \mathbf{0}$

- $\boxed{\text{The columns of matrix } A \text{ are linearly independent if and only if the equation } A\mathbf{x} = \mathbf{0} \text{ has } \textit{only} \text{ the trivial solution.}}$

SETS OF ONE OR TWO VECTORS

- A set containing only one vector – say, **v** – is linearly independent if and only if **v** is not the zero vector.
- This is because the vector equation $x_1\mathbf{v} = \mathbf{0}$ has only the trivial solution when $\mathbf{v} \neq \mathbf{0}$.
- The zero vector is linearly dependent because $x_1\mathbf{0} = \mathbf{0}$ has many nontrivial solutions.

SETS OF ONE OR TWO VECTORS

- A set of two vectors $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly dependent if at least one of the vectors is a multiple of the other.

$v_1 = c v_2$ 인 경우

- The set is linearly independent if and only if neither of the vectors is a multiple of the other.

$c_1 v_1 + c_2 v_2 = 0 \quad \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \therefore \quad v_2 = -\frac{c_1}{c_2} v_1$

$\{v_1, v_2\}$ ~ dependent only if one is multiple of the other

SETS OF TWO OR MORE VECTORS

THEOREM 7

($v_j = c_1 v_1 + c_2 v_2 + \cdots + c_{j-1} v_{j-1}$ 인 형태..)

Characterization of Linearly Dependent Sets

An indexed set $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ of two or more vectors is linearly dependent if and only if at least one of the vectors in S is a linear combination of the others. In fact, if S is linearly dependent and $\mathbf{v}_1 \neq \mathbf{0}$, then some $\mathbf{v}_j$ (with $j > 1$) is a linear combination of the preceding vectors, $\mathbf{v}_1, \ldots, \mathbf{v}_{j-1}$. (그 뒤에 이렇게)

SETS OF TWO OR MORE VECTORS

- **Proof:** If some $\mathbf{v}_j$ in S equals a linear combination of the other vectors, then $\mathbf{v}_j$ can be subtracted from both sides of the equation, producing a linear dependence relation with a nonzero weight (-1) on $\mathbf{v}_j$.

①

- [For instance, if $\mathrm{v}_1 = c_2\mathrm{v}_2 + c_3\mathrm{v}_3$, then
$0 = (-1)\mathrm{v}_1 + c_2\mathrm{v}_2 + c_3\mathrm{v}_3 + 0\mathrm{v}_4 + \ldots + 0\mathrm{v}_p$.] 0이 아닌 계수로 linearly dependent
- Thus S is linearly dependent.

②

- Conversely, suppose S is linearly dependent.
- If $\mathbf{v}_1$ is zero, then it is a (trivial) linear combination of the other vectors in S.

$V_1 = 0V_2 + 0V_3 + \cdots + 0V_p$ 로 표현가능

SETS OF TWO OR MORE VECTORS

- Otherwise, $\mathbf{v}_1 \neq \mathbf{0}$, and there exist weights $c_1, \ldots, c_p$, not all zero, such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_p\mathbf{v}_p = \mathbf{0}.$$

exclude by $(c_{j+1} = \cdots = c_p = 0)$

- Let j be the largest subscript for which $c_j \neq 0$.

j가 가장 큰거

i)

- If $j = 1$, then $c_1\mathbf{v}_1 = \mathbf{0}$, which is impossible because $\mathbf{v}_1 \neq \mathbf{0}$.

SETS OF TWO OR MORE VECTORS

- ii) So $j > 1$, and

$$c_1 v_1 + \cdots + c_j v_j + 0 v_{j+1} + \cdots + 0 v_p = 0$$

$$c_j \mathrm{v}_j = -c_1 \mathrm{v}_1 - \ldots - c_{j-1} \mathrm{v}_{j-1}$$

$$\mathrm{v}_j = \left(-\frac{c_1}{c_j}\right)\mathrm{v}_1 + \ldots + \left(-\frac{c_{j-1}}{c_j}\right)\mathrm{v}_{j-1}.$$

linearly dependent

SETS OF TWO OR MORE VECTORS

Some!

- Theorem 7 does *not* say that *every* vector in a linearly dependent set is a linear combination of the preceding vectors.
- A vector in a linearly dependent set may fail to be a linear combination of the other vectors.

Redundant Info를 제거할 때 중요

SETS OF TWO OR MORE VECTORS

- **Example 4:** Let $\mathbf{u} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ 6 \\ 0 \end{bmatrix}$. Describe the set spanned by **u** and **v**, and explain why a vector **w** is in Span {**u**, **v**} if and only if {**u**, **v**, **w**} is linearly dependent.

u, v ~ independent, $u \neq 0$
$u \neq c \cdot v$

SETS OF TWO OR MORE VECTORS

- **Solution:** The vectors **u** and **v** are linearly independent because neither vector is a multiple of the other, and so they span a plane in $\mathbb{R}^3$.
- Span $\{\mathbf{u}, \mathbf{v}\}$ is the x_1x_2-plane (with $x_3 = 0$). (필수 X)
- If **w** is a linear combination of **u** and **v**, then $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly dependent, by Theorem 7.
- Conversely, suppose that $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly dependent. 참고 $\therefore W = C_1U_1 + C_2V_2$
- By theorem 7, some vector in $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is a linear combination of the preceding vectors (since $\mathbf{u} \neq \mathbf{0}$).
- That vector must be **w**, since **v** is not a multiple of **u**.

SETS OF TWO OR MORE VECTORS

- So **w** is in Span $\{\mathbf{u}, \mathbf{v}\}$.
- Example 4 generalizes to any set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ in $\mathbb{R}^3$ with **u** and **v** linearly independent.
- The set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ will be linearly dependent if and only if **w** is in the plane spanned by **u** and **v**.

$w \in \text{Span}\{u, v\} \Leftrightarrow \{u, v, w\}$ dependent

[illegible]

$\Rightarrow$ $w = c_1u + c_2v$ then $c_1u + c_2v - w = 0$

$\Leftarrow$ Suppose that $\{u\ v\ w\}$ ~dependent. $v \neq cu$. Thm 7 $\Rightarrow$ $\therefore w = c_1u + c_2v$

$u \neq 0$

SETS OF TWO OR MORE VECTORS

THEOREM 8

$p > n$인 경우 가정, $v_i \in \mathbb{R}^n$

If a set contains more vectors than there are entries in each vector, then the set is linearly dependent. That is, any set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ is linearly dependent if $p > n$.

- **Proof:** Let $A = \begin{bmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_p \end{bmatrix}$.
- Then A is $n \times p$, and the equation $A\mathbf{x} = \mathbf{0}$ corresponds to a system of n equations in p unknowns.
- If $p > n$, there are more variables than equations, so there must be a free variable.

at most n pivot $\rightarrow$ A has $p-n$ free var $\rightarrow$ many sol $\rightarrow$ dependent

SETS OF TWO OR MORE VECTORS

- Hence $A\mathbf{x} = 0$ has a nontrivial solution, and the columns of A are linearly dependent.
- See the figure below for a matrix version of this theorem.

$$\begin{array}{cc} & p \\ n & \begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{bmatrix} \end{array}$$

If $p > n$, the columns are linearly dependent.

ex)

$\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ p \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} \right\}$ linearly dependent

$\left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ n \end{pmatrix}, \begin{pmatrix} a \\ b \end{pmatrix} \right\}$ linearly dependent

↳ free variable 발생

- Theorem 8 says nothing about the case in which the number of vectors in the set does *not* exceed the number of entries in each vector. 넘지 않는경우는 알아서계산.

SETS OF TWO OR MORE VECTORS

THEOREM 9

If a set $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ contains the zero vector, then the set is linearly dependent.

independent → No zero vector

- **Proof:** By renumbering the vectors, we may suppose $\mathbf{v}_1 = \mathbf{0}$.
- Then the equation $1\mathbf{v}_1 + 0\mathbf{v}_2 + \ldots + 0\mathbf{v}_p = \mathbf{0}$ shows that S in linearly dependent.

1. $\begin{bmatrix}5\\0\\0\end{bmatrix} \begin{bmatrix}7\\2\\-6\end{bmatrix} \begin{bmatrix}9\\4\\-8\end{bmatrix}$ $\quad x_1v_1+x_2v_2+x_3v_3=0 \quad \begin{bmatrix}v_1\,v_2\,v_3 & \begin{matrix}0\\0\\0\end{matrix}\end{bmatrix} \rightarrow \begin{bmatrix}5 & 7 & 9 & 0\\0 & 2 & 4 & 0\\0 & 0 & 4 & 0\end{bmatrix}$

3 pivots.
No free Var
↳ Linearly independent

2. $\begin{bmatrix}1\\-3\end{bmatrix} \begin{bmatrix}-3\\9\end{bmatrix} = -3 \times \begin{bmatrix}1\\-3\end{bmatrix}$ dependent

5. $\begin{bmatrix}0 & -8 & 5\\3 & -7 & 4\\-1 & 5 & -4\\1 & -3 & 2\end{bmatrix} \rightarrow \begin{bmatrix}0 & -8 & 5 & 0\\3 & -7 & 4 & 0\\-1 & 5 & -4 & 0\\1 & -3 & 2 & 0\end{bmatrix} \rightarrow \begin{bmatrix}1 & -3 & 2 & 0\\3 & -7 & 4 & 0\\-1 & 5 & -4 & 0\\0 & -8 & 5 & 0\end{bmatrix} \rightarrow \begin{bmatrix}1 & -3 & 2 & 0\\0 & 2 & -2 & 0\\0 & 2 & -2 & 0\\0 & -8 & 5 & 0\end{bmatrix} \rightarrow \begin{bmatrix}1 & -3 & 2 & 0\\0 & 2 & -2 & 0\\0 & 0 & -3 & 0\\0 & 0 & 0 & 0\end{bmatrix}$

3 pivot

No free Var → Independent $\quad Ax=0$ Trivial

7. $\begin{bmatrix}1 & 4 & -3 & 0\\-2 & -7 & 5 & 1\\-4 & -5 & 7 & 5\end{bmatrix}$

3x4 3 pivot Max ⇒ linearly dependent (Thm 8)

13. $v_1 \sim v_4 \in \mathbb{R}^4$ & $v_3 = 2v_1 + v_2 \Rightarrow \{v_1\ v_2\ v_3\ v_4\}$ 는 linearly dependent

Yes

14. $v_1 \sim v_4 \in \mathbb{R}^4$ & $v_3 = 0 \Rightarrow \{v_1\ v_2\ v_3\ v_4\}$ 는 linearly dependent

Yes. v_3가 어디에나 포함된다.

15. $v_1, v_2 \in \mathbb{R}^4$ $v_2 \neq c v_1 \Rightarrow v_1, v_2$ 는 linearly independent

$\Rightarrow$ v_1 may be a zero vector $\Rightarrow$ False.

THEOREM 7

($v_j = c_1 v_1 + c_2 v_2 + \cdots + c_{j-1} v_{j-1}$ 인 경우..)

Characterization of Linearly Dependent Sets

① An indexed set $S = \{v_1, \ldots, v_p\}$ of two or more vectors is linearly dependent if and only if at least one of the vectors in S is a linear combination of the others. In fact, if S is linearly dependent and $v_1 \neq 0$, ② then some $\mathbf{v}_j$ (with $j > 1$) is a linear combination of the preceding vectors, $\mathbf{v}_1, \ldots, v_{j-1}$. (2번째에 아닌)

THEOREM 8

$p > n$ 인 경우 가정, $v_i \in \mathbb{R}^n$

If a set contains more vectors than there are entries in each vector, then the set is linearly dependent. That is, any set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ is linearly dependent if $p > n$.

THEOREM 9

If a set $S = \{v_1, \ldots, v_p\}$ in $\mathbb{R}^n$ contains the zero vector, then the set is linearly dependent.

independent → No zero vector

1. $\begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} \begin{bmatrix} 0 \\ 5 \\ -8 \end{bmatrix} \begin{bmatrix} -3 \\ 4 \\ 1 \end{bmatrix}$ independent? $\begin{bmatrix} 0 & 0 & -3 \\ 0 & 5 & 4 \\ 2 & -8 & 1 \end{bmatrix} \sim \begin{bmatrix} 2 & -8 & 1 \\ \cdot & 5 & 4 \\ \cdot & \cdot & -3 \end{bmatrix}$ 3 pivot, No free var in $Ax=0$ → Linearly independent

2. $\begin{bmatrix} -1 \\ 4 \end{bmatrix} \begin{bmatrix} -2 \\ 8 \end{bmatrix}$ $V_1 \neq V_2 \times C$, $V_1 \neq 0$
V_1 V_2 ∴ Linearly Independent

6. $\begin{bmatrix} -4 & -3 & 0 \\ 0 & -1 & 4 \\ 1 & 0 & 3 \\ 5 & 4 & 6 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 3 & 0 \\ \cdot & -1 & 4 & 0 \\ \cdot & \cdot & 7 & 0 \\ \cdot & \cdot & \cdot & 0 \end{bmatrix}$ 4 pivot, No free
→ Independent

8. $\begin{bmatrix} -1 & -3 & 3 & -2 \\ -3 & 7 & -1 & 2 \\ 0 & 1 & -4 & 3 \end{bmatrix}$ # columns > # entries
→ dependent

16. $v_1 \dots v_4 \in \mathbb{R}^4$ v_3 not linear comb $v_1 \dots v_4$, $\{v_1 \dots v_4\}$ independent. No

$v_1 = v_2 = v_4 = 0$ $v_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ 인 경우가 있다.

17. $\{v_1 v_2 v_3\}$ dependent → $\{v_1 v_2 v_3 v_4\}$ dependent. Yes. for some $\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ $c_1v_1 + c_2v_2 + c_3v_3 = 0$

$\Rightarrow c_1v_1 + c_2v_2 + c_3v_3 + 0 \cdot v_4 = 0$

성립

18. $c_1v_1 + c_2v_2 + c_3v_3 = 0$

$\rightarrow c_1v_1 + c_2v_2 + c_3v_3 + 0\,v_4 = 0$

since $\{v_1 \sim v_4\}$ independent, $c_1 = c_2 = c_3 = 0$

$v_1 \sim v_4$ independent,

$v_1 \sim v_3$ independent

Yes

10. Ax가 zero vector 되는 x는?

$$\begin{pmatrix} 1 & 2 & 9 & 2 & 0 \\ 1 & 0 & 3 & -4 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ -2 & 3 & 0 & 5 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 9 & 2 & 0 \\ 0 & 1 & 2 & 2 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \_ & \_ & \_ & \_ & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 3 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \_ & \_ & \_ & \_ & 0 \end{pmatrix}$$

$$x = \begin{pmatrix} -3x_3 \\ -2x_3 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -3 \\ -2 \\ 1 \\ 0 \end{pmatrix} x_3 +$$

12. $b \in T(x)$ 인가? → $Ax=b = \begin{bmatrix} -1 \\ 3 \\ -1 \\ 4 \end{bmatrix}$ 해?

$$\begin{bmatrix} 1 & 2 & 9 & -2 & -1 \\ 1 & 0 & 3 & -4 & 3 \\ 0 & 1 & 2 & 3 & -1 \\ -2 & 3 & 0 & 5 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 3 & 0 & -1 \\ 0 & 1 & 2 & 0 & -4/3 \\ \_ & \_ & \_ & 1 & 1/3 \\ \_ & \_ & \_ & \_ & 1 \end{bmatrix}$$

In Consistent

- Not in Range

20. $T(x) = x_1 v_1 + x_2 v_2 = [v_1 \; v_2]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2 & 7 \\ 5 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

25. $v \neq 0$, $p \in \mathbb{R}^n$, $x = p + tv$ (line)

$T(x) = T(p) + tT(v)$,

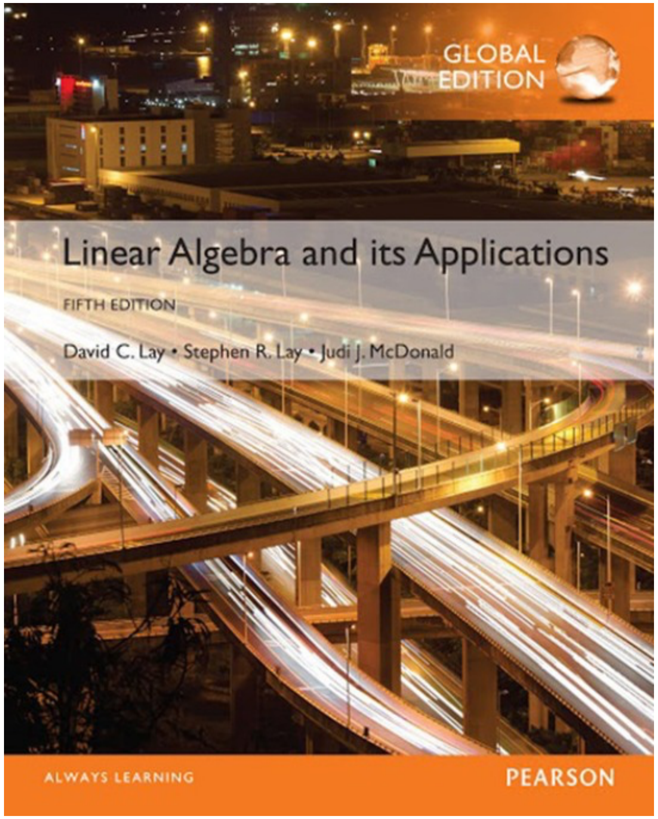
If $T(v) = 0 \Rightarrow T(x) = T(p) \Rightarrow$ single point

$T(v) \neq 0 \rightarrow T(p) + tT(v) \Rightarrow$ line

1 Linear Equations in Linear Algebra

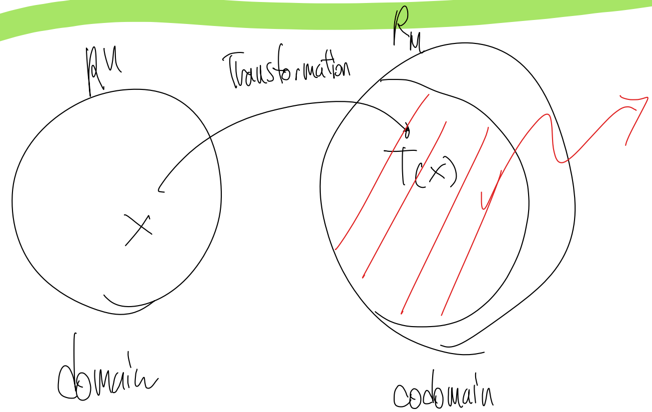
1.8

INTRODUCTION TO LINEAR TRANSFORMATIONS



LINEAR TRANSFORMATIONS

- A **transformation** (or **function** or **mapping**) T from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a rule that assigns to each vector $\mathbf{x}$ in $\mathbb{R}^n$ a vector $T(\mathbf{x})$ in $\mathbb{R}^m$.
- The set $\mathbb{R}^n$ is called **domain** of T, and $\mathbb{R}^m$ is called the **codomain** of T.
- The notation $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ indicates that the domain of T is $\mathbb{R}^n$ and the codomain is $\mathbb{R}^m$.
- For $\mathbf{x}$ in $\mathbb{R}^n$, the vector $T(\mathbf{x})$ in $\mathbb{R}^m$ is called the **image** of $\mathbf{x}$ (under the action of T).
- The set of all images $T(\mathbf{x})$ is called the **range** of T. See Fig. 2 on the next slide



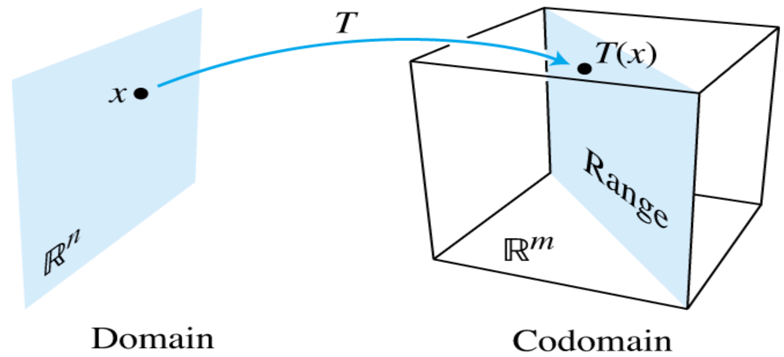
Range = {images $T(x)$ | $x \in \mathbb{R}^n$}

x : pre image of $T(x)$

$T(x)$: the Image of x

$T(x) = Ax$ 로 본다.

MATRIX TRANSFORMATIONS



Domain, codomain, and range of $T : \mathbb{R}^n \to \mathbb{R}^m$.

- For each $\mathbf{x}$ in $\mathbb{R}^n$, $T(\mathbf{x})$ is computed as $A\mathbf{x}$, where A is an $m \times n$ matrix. A는 m×n 이어야한다.
- Such a mapping is called *matrix transformation* and is denoted by $\mathbf{x} \mapsto A\mathbf{x}$.
- Observe that the domain of T is $\mathbb{R}^n$ when A has n columns and the codomain of T is $\mathbb{R}^m$ when each column of A has m entries.

MATRIX TRANSFORMATIONS

- The range of T is the set of all linear combinations of the columns of A, because each image $T(\mathbf{x})$ is of the form $A\mathbf{x}$.

range $= \{T(x) \mid x \in \mathbb{R}^n\}$

$= \{Ax \mid x \in \mathbb{R}^n\}$ $A = [a_1 \ a_2 \cdots a_n]$ fw 이걸 행렬로 놓다

$= \{a_1x_1 + a_2x_2 + \cdots + a_nx_n \mid x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n\}$

$=$ {linear Combination of columns of A} $=$ Span{columns of A}

MATRIX TRANSFORMATIONS

- **Example 1:**

Let $A = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix}$, $\mathrm{u} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, $b = \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix}$, $\mathrm{c} = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$.

and define a transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ by $T(\mathrm{x}) = A\mathrm{x}$, so that

$$T(\mathrm{x}) = A\mathrm{x} = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 - 3x_2 \\ 3x_1 + 5x_2 \\ -x_1 + 7x_2 \end{bmatrix}.$$

MATRIX TRANSFORMATIONS

a. Find $T(\mathbf{u})$, the image of $\mathbf{u}$ under the transformation T.

b. Find an $\mathbf{x}$ in $\mathbb{R}^2$ whose image under T is $\mathbf{b}$.

Solve $Ax=b$

c. Is there more than one $\mathbf{x}$ whose image under T is $\mathbf{b}$?

Unique? free var?

No

d. Determine if $\mathbf{c}$ is in the range of the transformation T.

$c \in$ range $\Leftrightarrow$ $Ax=c$ 의 x가 존재? $\Leftrightarrow$ c is an image of some x

$[A\ c]$ 의 끝열이 pivot column이 없나?

MATRIX TRANSFORMATIONS

Solution:

a. Compute

$$T(\mathbf{u}) = A\mathbf{u} = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ -9 \end{bmatrix}.$$

b. Solve $T(\mathbf{x}) = \mathbf{b}$ for **x**. That is, solve $A\mathbf{x} = \mathbf{b}$, or

$$\begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix} \qquad (1)$$

MATRIX TRANSFORMATIONS

- Row reduce the augmented matrix:

$$\begin{bmatrix} 1 & -3 & 3 \\ 3 & 5 & 2 \\ -1 & 7 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 3 \\ 0 & 14 & -7 \\ 0 & 4 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 3 \\ 0 & 1 & -.5 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1.5 \\ 0 & 1 & -.5 \\ 0 & 0 & 0 \end{bmatrix} \quad (2)$$

pre-image

- Hence $x_1 = 1.5$, $x_2 = -.5$, and $\mathbf{x} = \begin{bmatrix} 1.5 \\ -.5 \end{bmatrix}$.

- The image of this **x** under T is the given vector **b**.

MATRIX TRANSFORMATIONS

c. Any **x** whose image under T is **b** must satisfy equation (1).
   - From (2), it is clear that equation (1) has a unique solution.
   - So there is exactly one **x** whose image is **b**.

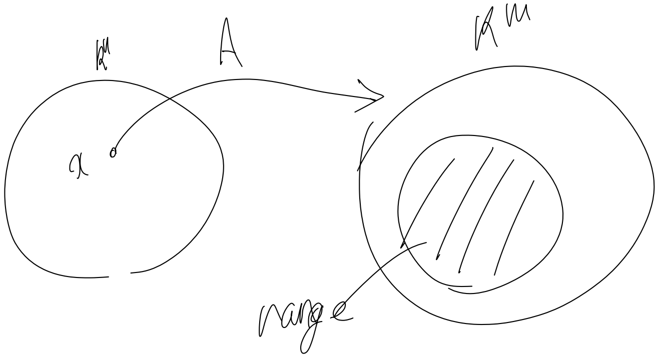
d. The vector **c** is in the range of T if **c** is the image of some **x** in $\mathbb{R}^2$, that is, if $\mathbf{c} = T(\mathbf{x})$ for some **x**.
   - This is another way of asking if the system $A\mathbf{x} = \mathbf{c}$ is consistent.

MATRIX TRANSFORMATIONS

- To find the answer, row reduce the augmented matrix:

$$\begin{bmatrix} 1 & -3 & 3 \\ 3 & 5 & 2 \\ -1 & 7 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 3 \\ 0 & 14 & -7 \\ 0 & 4 & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 3 \\ 0 & 1 & 2 \\ 0 & 14 & -7 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & \boxed{3} \\ 0 & 1 & \boxed{2} \\ 0 & 0 & \boxed{-35} \end{bmatrix}$$

- The third equation, $0 = -35$, shows that the system is inconsistent.
- So **c** is *not* in the range of T.



$b \in$ range $\Leftrightarrow$ b is an image of some $x \in \mathbb{R}^n$

$\Leftrightarrow$ $Ax=b$ consistent (has solution)

$\Leftrightarrow$ right-most of $[A\,b]$ not a pivot column

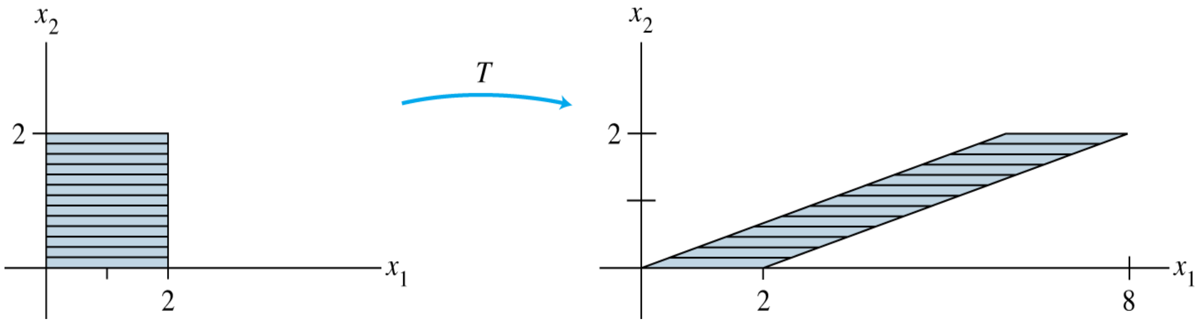
Ae column of Independent $\Leftrightarrow$ Codomain = Range

SHEAR TRANSFORMATION

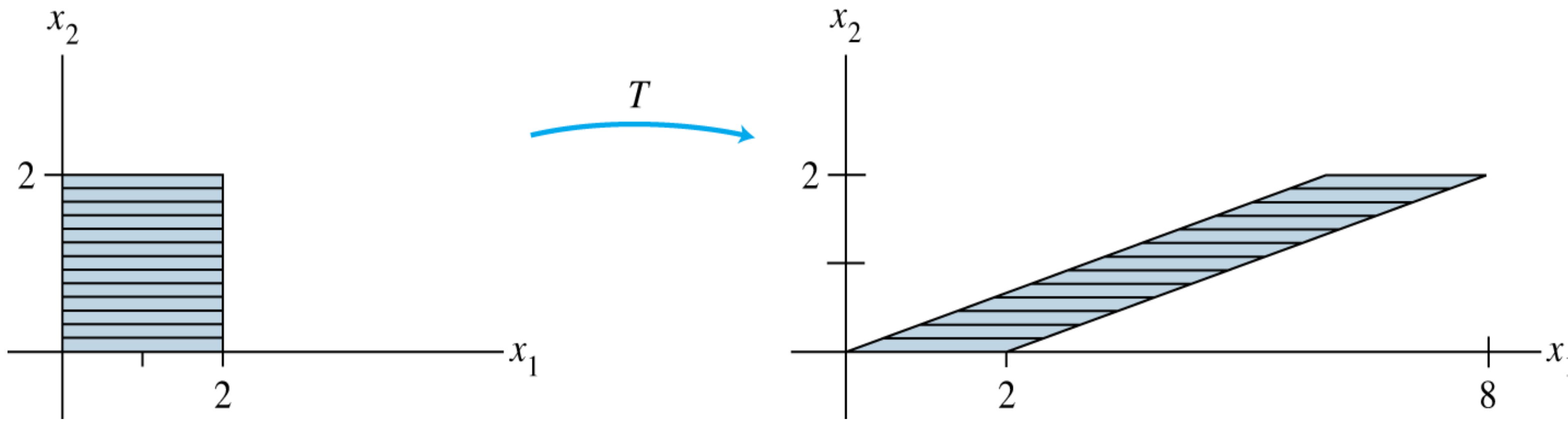
- **Example 3:** Let $A = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$. The transformation

$T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined by $T(\mathrm{x}) = A\mathrm{x}$ is called a **shear transformation**.

- It can be shown that if T acts on each point in the 2×2 square shown in the figure below, then the set of images forms the shaded parallelogram.



SHEAR TRANSFORMATION



- The key idea is to show that T maps line segments onto line segments and then to check that the corners of the square map onto the vertices of the parallelogram.
- For instance, the image of the point $\mathbf{u} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$ is

$$T(\mathbf{u}) = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \end{bmatrix},$$

LINEAR TRANSFORMATIONS

and the image of $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$ is $\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 8 \\ 2 \end{bmatrix}$.

- T deforms the square as if the top of the square were pushed to the right while the base is held fixed.

LINEAR TRANSFORMATIONS

Definition: A transformation (or mapping) T is **linear** if:

i. $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all **u**, **v** in the domain of T;

ii. $T(c\mathbf{u}) = cT(\mathbf{u})$ for all scalars c and all **u** in the domain of T.

$\boxed{A(u+v) = Au + Av \quad A(cu) = c(Au)}$ linearity

matrix multi $\subset$ linear transformation

From Calculus, $(f(x)+g(x))' = f'(x) + g'(x)$

$(cf(x))' = cf'(x)$

$\therefore$ Linear transformation

LINEAR TRANSFORMATIONS

- Linear transformations *preserve the operations of vector addition and scalar multiplication*.
- Property (i) says that the result $T(\mathrm{u}+\mathrm{v})$ of first adding $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$ and then applying T is the same as first applying T to $\mathbf{u}$ and $\mathbf{v}$ and then adding $T(\mathbf{u})$ and $T(\mathbf{v})$ in $\mathbb{R}^m$.

LINEAR TRANSFORMATIONS

- These two properties lead to the following useful facts.
- If T is a linear transformation, then

$$T(\mathbf{0}) = \mathbf{0} \qquad (3)$$

Origin not change

and

$$T(c\mathbf{u} + d\mathbf{v}) = cT(\mathbf{u}) + dT(\mathbf{v}) \qquad (4)$$

for all vectors **u**, **v** in the domain of T and all scalars c, d.

- Property (3) follows from condition (ii) in the definition, because $T(\mathbf{0}) = T(0\mathbf{u}) = 0T(\mathbf{u}) = \mathbf{0}$.

- Property (4) requires both (i) and (ii):

$$T(c\mathbf{u} + d\mathbf{v}) = T(c\mathbf{u}) + T(d\mathbf{v}) = cT(\mathbf{u}) + dT(\mathbf{v})$$

LINEAR TRANSFORMATIONS

If a transformation satisfies (4)

$$T(c\mathrm{u} + d\mathrm{v}) = cT(\mathrm{u}) + dT(\mathrm{v})$$

for all **u**, **v** and c, d, *it must be linear.*

- (Set $c = d = 1$ for preservation of addition, and set for $d = 0$ preservation of scalar multiplication.)

linear $\Leftrightarrow$ $T(u+v) = T(u) + T(v)$ $\Leftrightarrow$ $T(cu + dv) = cT(u) + dT(v)$

$T(cu) = cT(u)$

LINEAR TRANSFORMATIONS

- Repeated application of (4) produces a useful generalization:

$$T(c_1\mathrm{v}_1 + \ldots + c_p\mathrm{v}_p) = c_1T(\mathrm{v}_1) + \ldots + c_pT(\mathrm{v}_p) \qquad (5)$$

Calculus

$y = mx + n$

linear function

Linear Alg

T Linear $\rightarrow T(0) = 0$

다르다.

$A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ $T(x) = Ax$ $T(u) = Au = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 2 \\ -6 \end{bmatrix}$

$u = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$ $v = \begin{bmatrix} a \\ b \end{bmatrix}$ 의 이미지는 $T(v) = Av = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 2a \\ 2b \end{bmatrix}$

$Ax=b$ 인 T에서 b는 unique 한가?

7, 4 $\Rightarrow Ax=b \Rightarrow$ Augmented

$\begin{bmatrix} 1 & 0 & -2 & -1 \\ -2 & 1 & 6 & 7 \\ 3 & -2 & -5 & -3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -2 & -1 \\ 0 & 1 & 2 & 5 \\ 0 & -2 & 1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -2 & -1 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 5 & 10 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$ $x_1 = 3$, $x_2 = 1$, $x_3 = 2$

No free Var, Unique

Unique

$\begin{bmatrix} 1 & -3 & 2 & 6 \\ 0 & 1 & -4 & -7 \\ 7 & -5 & -9 & -9 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -3 & 2 & 6 \\ 0 & 1 & -4 & -7 \\ 0 & 4 & -15 & -27 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -3 & 2 & 6 \\ 0 & 1 & -4 & -7 \\ 0 & 0 & 1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -3 & 0 & 4 \\ \cdot & 1 & 0 & -3 \\ \cdot & \cdot & 1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & -5 \\ \cdot & 1 & 0 & -3 \\ \cdot & \cdot & 1 & 1 \end{bmatrix}$ $x_1 = -5$, $x_2 = -3$, $x_3 = 1$

No free Var, Unique.

Unique

9. $Ax=0$ 인 x 찾기 = homogeneous general sol

$$A=\begin{bmatrix}1&-4&7&-5\\0&1&-4&3\\2&-6&6&-4\end{bmatrix}\quad \begin{bmatrix}1&-4&7&-5&0\\0&1&-4&3&0\\2&-6&6&-4&0\end{bmatrix}\to\begin{bmatrix}1&-4&7&-5&0\\0&1&-4&3&0\\0&2&-8&6&0\end{bmatrix}\to\begin{bmatrix}1&-4&7&-5&0\\&1&-4&3&0\\\hline\end{bmatrix}\to\begin{bmatrix}1&0&-9&7&0\\0&1&-4&3&0\\\hline\end{bmatrix}$$

$$x_1=9x_3-7x_4$$
$$x_2=4x_3-3x_4$$

$$x=\begin{pmatrix}9x_3\\4x_3\\x_3\\0\end{pmatrix}+\begin{pmatrix}-7x_4\\-3x_4\\0\\x_4\end{pmatrix}=\begin{bmatrix}9\\4\\1\\0\end{bmatrix}x_3+\begin{bmatrix}-7\\-3\\0\\1\end{bmatrix}x_4$$

11. $b=\begin{bmatrix}-1\\1\\0\end{bmatrix}$ in range of $x\to Ax$?

$\Leftrightarrow$ $Ax=b$ has sol (= consistent)

$\Leftrightarrow$ b is a linear comb of colums of A?

$\Leftrightarrow$ $b\in$ span{colums of A}

$\Rightarrow$ $[A,b]$ the rightmost column pivot column(X).

$$\begin{bmatrix}1&0&-9&7&7\\0&1&-4&3&1\\\hline&&&&0\end{bmatrix}$$

has sol $\therefore$ $Ax=b$ True

$b\in$ range

$e_1 = \begin{bmatrix}1\\0\end{bmatrix}$ $e_2 = \begin{bmatrix}0\\1\end{bmatrix}$ $y_1 = \begin{bmatrix}2\\5\end{bmatrix}$ $y_2 = \begin{bmatrix}-1\\6\end{bmatrix}$ / $T\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}2\\5\end{bmatrix}$ $T\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}-1\\6\end{bmatrix}$

$$T\begin{pmatrix}5\\-3\end{pmatrix} = T\left(5\begin{bmatrix}1\\0\end{bmatrix} + (-3)\begin{bmatrix}0\\1\end{bmatrix}\right)$$

$$= 5T\begin{bmatrix}1\\0\end{bmatrix} + (-3)T\begin{bmatrix}0\\1\end{bmatrix} = 5\times\begin{bmatrix}2\\5\end{bmatrix} + (-3)\times\begin{bmatrix}-1\\6\end{bmatrix} = \begin{pmatrix}10\\25\end{pmatrix} + \begin{pmatrix}3\\-18\end{pmatrix} = \begin{pmatrix}13\\7\end{pmatrix}$$

$$x_1\begin{bmatrix}1\\0\end{bmatrix} + x_2\begin{bmatrix}0\\1\end{bmatrix} \Rightarrow T\left(\begin{pmatrix}x_1\\x_2\end{pmatrix}\right) = x_1 T\begin{pmatrix}1\\0\end{pmatrix} + x_2 T\begin{pmatrix}0\\1\end{pmatrix} = x_1\begin{bmatrix}2\\5\end{bmatrix} + x_2\begin{bmatrix}-1\\6\end{bmatrix} = \begin{bmatrix}2x_1 - x_2\\5x_1 + 6x_2\end{bmatrix}$$

Vector $v_1 \cdots v_p$ span $\mathbb{R}^n$, $T: \mathbb{R}^n \to \mathbb{R}^n$, $T(v_i)=0$.

모든 x 에게 $T(x)=0$

$T(v_i)=0$ 이고, zero transformation 가정

$v \in \mathbb{R}^n$, $c_1v_1 + c_2v_2 + \cdots + c_pv_p = v$ 존재하는

$\therefore T(v) = c_1T(v_1) + c_2T(v_2) + \cdots + c_pT(v_p) = 0 + 0 + \cdots + 0 = 0$

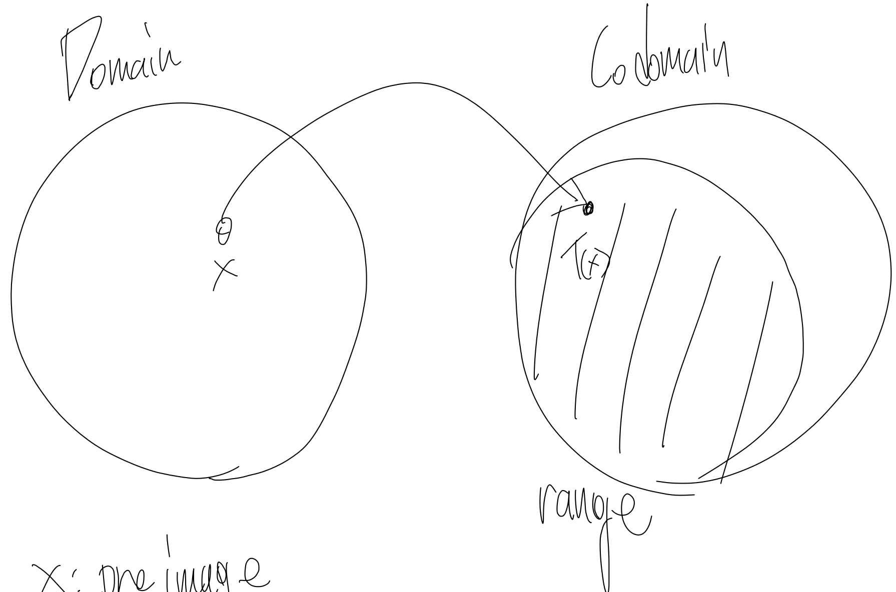
(did not mention $c_1, \ldots c_p$ unique.)

transformation (function, mapping)

domain, codomain, range

image, preimage

matrix transformation



x: preimage

T(x): image

$$T(u) = \begin{pmatrix} 1/2 & & \\ & 1/2 & \\ & & 1/2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ -4 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 0 \\ -2 \end{pmatrix}$$

$$T(v) = \begin{pmatrix} 1/2 & & \\ & 1/2 & \\ & & 1/2 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 1/2a \\ 1/2b \\ 1/2c \end{pmatrix}$$

---

5. $$\begin{bmatrix} 1 & -5 & -4 & -2 \\ -3 & 4 & 5 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -5 & -4 & -2 \\ 0 & 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 3 & 3 \\ 0 & 1 & 2 & 1 \end{bmatrix}$$

$x_1 = 3 - 3x_3$
$x_2 = 1 - 2x_3$
x_3 free

Free Variable
Not Unique

6. $$\begin{bmatrix} 1 & -2 & 1 & 1 \\ 3 & -4 & 5 & 9 \\ 0 & 1 & 1 & 3 \\ -3 & 5 & -4 & -6 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 3 & 7 \\ 0 & 1 & 1 & 3 \\ & & & \\ & & & \end{bmatrix}$$

Free Variable
Not Unique

10. $Ax=0$ 이 되게 하는 x는?

$$A=\begin{bmatrix}1&3&9&2\\1&0&3&-4\\0&1&2&3\\-2&3&0&5\end{bmatrix} \rightarrow \begin{bmatrix}1&3&9&2&0\\1&0&3&-4&0\\0&1&2&3&0\\-2&3&0&5&0\end{bmatrix} \rightarrow \begin{bmatrix}1&0&3&0&0\\ \cdot&1&2&0&0\\ \cdot&\cdot&\cdot&1&0\\ \cdot&\cdot&\cdot&\cdot&0\end{bmatrix}$$

$$\begin{cases} x_1+3x_3 \\ x_2+2x_3 \\ x_4=0 \\ x_3 \text{ free} \end{cases} \qquad x = x_3\begin{bmatrix}-3\\-2\\1\\0\end{bmatrix}$$

11. $b=\begin{bmatrix}-1\\3\\-1\\4\end{bmatrix}$ 의 경우는?

$$\rightarrow \begin{bmatrix}1&0&3&0&-1\\-1&2&0&-4/3\\ & & &1&1/3\\ & & & &1\end{bmatrix}$$

$Ax=b$ inconsistent

$\longrightarrow$ $b \notin$ range.

20. $T(x)=x_1V_1+x_2V_2=\begin{bmatrix}V_1 & V_2\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}$ 이다.

25. $v \neq 0$ 와 $p \in \mathbb{R}^n$ 에서 선이 $x = p + tv$ 로 표현된다. $T: \mathbb{R}^n \to \mathbb{R}^n$ 이 점/선으로 가는것을 밝혀라

$$T(x) = T(p + tv) = T(p) + tT(v)$$

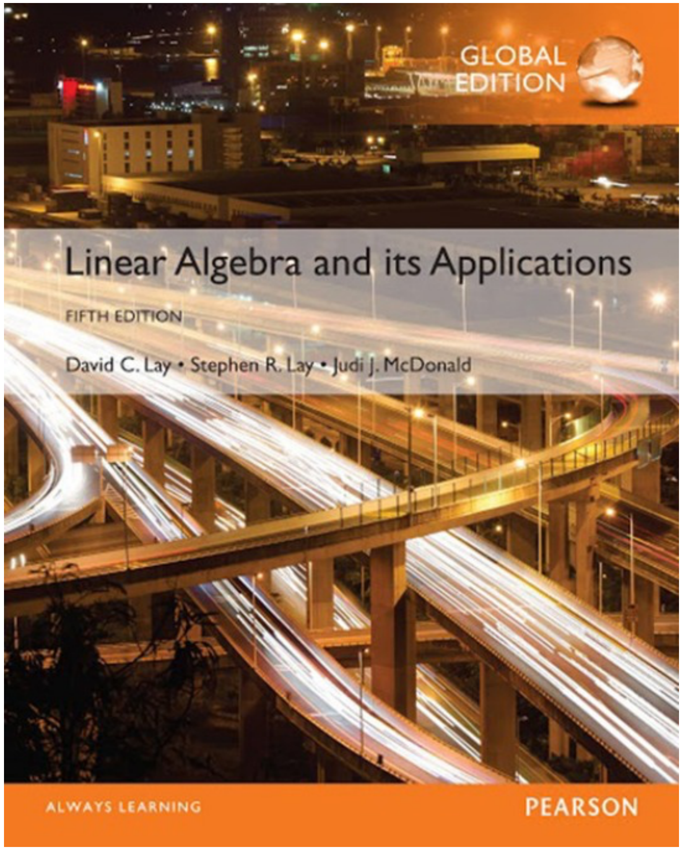
i) $T(v) = 0$ $\therefore T(x) = T(p)$ $\rightarrow$ Image = single point

ii) $T(v) \neq 0$ $T(x) = T(p) + tT(v)$ $\rightarrow$ Image = line

1 Linear Equations in Linear Algebra

1.9

THE MATRIX OF A LINEAR TRANSFORMATION



THE MATRIX OF A LINEAR TRANSFORMATION

Linear Transformation은: $T(x)=Ax$ 로 표현됨 A는 m×n 이고, $T: \mathbb{R}^n \to \mathbb{R}^m$

- **Theorem 10**: Let T: $\mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then there exists a unique matrix A such that

$$T(x) = Ax \text{ for all x in } \mathbb{R}^n$$

- In fact, A is the m×n matrix whose j th column is the vector $T(e_j)$, where e_j is the j th column of the identity matrix in $\mathbb{R}^n$

$$A = [T(e_1) \ \cdots \ T(e_n)] \qquad (3)$$

THE MATRIX OF A LINEAR TRANSFORMATION

- **Proof**: Write $x = I_n x = [e_1 \cdots e_n]x = x_1 e_1 + \cdots + x_n e_n$, and use the linearity of T to compute

Linear Transformation

$$T(x) = T(x_1 e_1 + \cdots + x_n e_n) = x_1 T(e_1) + \cdots + x_n T(e_n)$$

$$= [T(e_1) \quad \ldots \quad T(e_1)] \begin{bmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} = Ax$$

THE MATRIX OF A LINEAR TRANSFORMATION

- The matrix A in (3)

$$A = [T(e_1)\ \cdots\ T(e_n)]$$

is called the **standard matrix for the linear transformation** T.

- We know now that every linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ can be viewed as a matrix transformation, and vice versa. The term *linear transformation* focuses on a property of a mapping, while *matrix transformation* describes how such a mapping is implemented, as the example on the next slide illustrates.

THE MATRIX OF A LINEAR TRANSFORMATION

- **Example 2**: Find the standard matrix A for the dilation transformation T(x)= 3x, for x in $\mathbb{R}^2$.
- **Solution**: Write

$A\begin{bmatrix}2\\3\end{bmatrix}=\begin{bmatrix}3&0\\0&3\end{bmatrix}=\begin{bmatrix}6\\9\end{bmatrix}$

$T\begin{bmatrix}2\\3\end{bmatrix}$ 의 결과

$$T(e_1) = 3e_1 = \begin{bmatrix}3\\0\end{bmatrix} \textit{ and } T(e_2) = 3e_2 = \begin{pmatrix}0\\3\end{pmatrix}$$

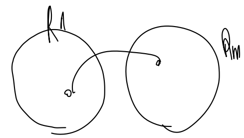
$$A = \begin{bmatrix}3 & 0\\0 & 3\end{bmatrix}$$

EXISTENCE AND UNIQUENESS QUESTIONS

Transformation 결과가 $\mathbb{R}^m$ 전체를 span 하는 것 = $Ax=b$ 에서 모든 $b \in \mathbb{R}^m$ 에 mapping 되는 것

- **Definition**: A mapping T: $\mathbb{R}^n \rightarrow \mathbb{R}^m$ is said to be **onto** $\mathbb{R}^n$ if each **b** in $\mathbb{R}^m$ is the image of *at least one* **x** in $\mathbb{R}^n$.

$\Leftrightarrow$ if each $b \in$ Range
$\Leftrightarrow$ if $\mathbb{R}^m$ = Range = span {columns of A}



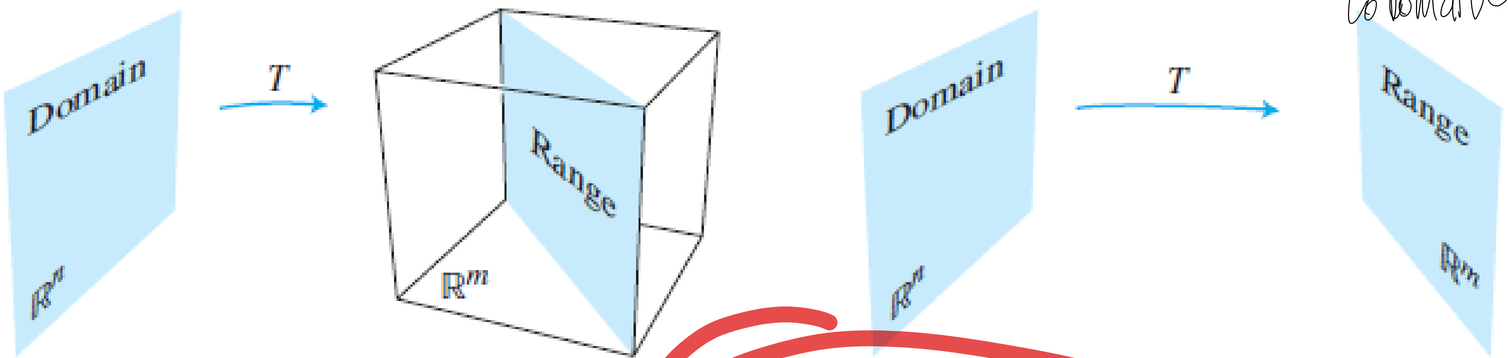
최소 하나

- Equivalently, T is onto $\mathbb{R}^m$ when the range of *T* is all of the codomain $\mathbb{R}^m$. That is, *T* maps $\mathbb{R}^n$ onto $\mathbb{R}^m$ if, for each **b** in the codomain $\mathbb{R}^m$, there exists at least one solution of T(x)= b . “Does T map $\mathbb{R}^n$ onto $\mathbb{R}^m$?” is an existence question. The mapping *T* is not onto when there is some **b** in $\mathbb{R}^m$ for which the equation T(x)=b has no solution. See the figure on the next slide.

EXISTENCE AND UNIQUENESS QUESTIONS

①

Codomain



T is *not* onto $\mathbb{R}^m$ | T *is* onto $\mathbb{R}^m$

FIGURE 3 Is the range of T all of $\mathbb{R}^m$?

$T: \mathbb{R}^n \to \mathbb{R}^m$ onto

$Ax=b$ has solution for any $b \in \mathbb{R}^m$

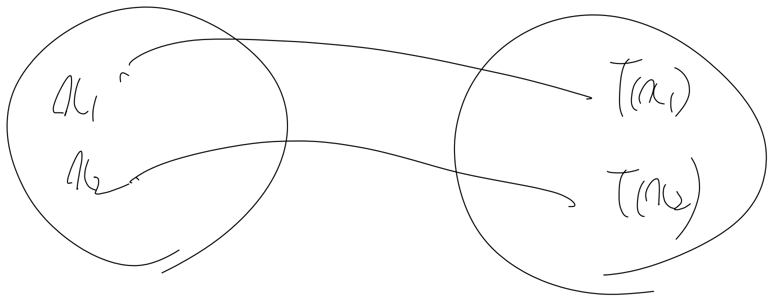
every row of A has a pivot

②

- **Definition**: A mapping T: $\mathbb{R}^n \rightarrow \mathbb{R}^m$ is said to be **one-to-one** if each **b** in $\mathbb{R}^m$ is the image of *at most one* **x** in $\mathbb{R}^n$.

최대

for $x_1 \neq x_2 \rightarrow T(x_1) \neq T(x_2)$: One to One



For each $b \in$ range, b is the Image of only 1 vector in the domain

$Ax=b$ has a unique solution

$T(x)=Ax$: Onto $\Longleftrightarrow$ $Ax=b$ of solution 존재 (for arbitrary b)

One to One $\Longleftrightarrow$ Solution of $Ax=b$ unique (if it exist)

EXISTENCE AND UNIQUENESS QUESTIONS

- **Example 4**: Let T be the linear transformation whose standard matrix is

$$A=\begin{bmatrix} 1 & -4 & 8 & 1 \\ 0 & 2 & -1 & 3 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

每 row pivot

$Ax=b$ solution for all $b \in \mathbb{R}^3$

Any $b \in \mathbb{R}^3 \in$ span{columns of A}

free var

- Does T map $\mathbb{R}^4$ onto $\mathbb{R}^3$? Is T a one-to-one mapping?

Yes

No

free Var $\Rightarrow$ No one to one solution, not unique.

EXISTENCE AND UNIQUENESS QUESTIONS

- **Solution**: Since A happens to be in echelon form, we can see at once that A has a pivot position in each row. By Theorem 4 in Section 1.4, for each $\mathbf{b}$ in $\mathbb{R}^3$, the equation Ax=b is consistent. In other words, the linear transformation T maps $\mathbb{R}^4$(its domain) onto $\mathbb{R}^3$.
- However, since the equation Ax=b has a free variable (because there are four variables and only three basic variables), each $\mathbf{b}$ is the image of more than one $\mathbf{x}$. This is, T is *not* one-to-one.

EXISTENCE AND UNIQUENESS QUESTIONS

- **Theorem 11**: Let T: $\mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear transformation. Then T is one-to-one if and only if the equation T(x)=0 has only the trivial solution.
- **Proof**: Since T is linear, T(0) =0. If T is one-to-one, then the equation T(x)=0 has at most one solution and hence only the trivial solution.
- 아정리-반대로

  If T is not one-to-one, then there is a b that is the image of at least two different vectors in $\mathbb{R}^n$, say **u** and **v**. That is T(u)=b and T(v)=b. But then, since T is linear,

$$T(u-v) = T(u) - T(v) = b - b = 0$$

EXISTENCE AND UNIQUENESS QUESTIONS

- The vector **u** – **v** is not zero, since u ≠ v. Hence the equation T(x)=0 has more than one solution. So, either the two conditions in the theorem are both true or they are both false.

일대일함수에서

$x_1a_1+\cdots+x_na_n$

T is 1-1 ⟺ x=0 가 유일한해 Ax=0 (해가 없거나, two many sol은 dependent)

⟺ $\{a_1, \ldots, a_n\}$ 이 linearly independent 이렇다.

EXISTENCE AND UNIQUENESS QUESTIONS

- **Theorem 12**: Let T: $\mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation and let A be the standard matrix for T. Then:

a) *T* maps $\mathbb{R}^n$ onto $\mathbb{R}^m$ if and only if the columns of *A* span $\mathbb{R}^m$;

b) *T* is one-to-one if and only if the columns of *A* are linearly independent.

⟺ x=0 Ax=0의 유일해
⟺ No Free Var
⟺ All column = pivot column
⟺ # pivots = # of Columns of A
= unknowns of A
⟺ One to One

⟺ range = co domain = span {columns}
⟺ Ax=b has sol for each b
⟺ Each b is linear comb of Columns of A
⟺ Every row pivot (in A)
⟺ #pivots = # rows. (in A)
⟺ Onto

EXISTENCE AND UNIQUENESS QUESTIONS

- **Proof:**

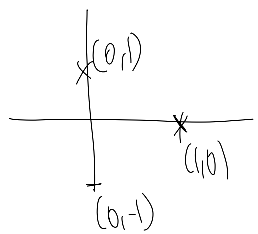
a) By Theorem 4 in Section 1.4, the columns of A span $\mathbb{R}^m$ if and only if for each b in $\mathbb{R}^m$ the equation Ax=b is consistent—in other words, if and only if for every **b**, the equation T(x)=b has at least one solution. This is true if and only if T maps $\mathbb{R}^n$ onto $\mathbb{R}^m$.

b) The equations T(x)=0 and Ax=0 are the same except for notation. So, by Theorem 11, T is one-to-one if and only if Ax=0 has only the trivial solution. This happens if and only if the columns of A are linearly independent.

1. Find standard Matrix A for T

$$A = [T(e_1)\ T(e_2)] = \begin{bmatrix} 3 & -5 \\ 1 & 2 \\ 3 & 0 \\ 1 & 0 \end{bmatrix}$$

2. Standard Matrix



$T(e_1) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$ $T(e_2) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\therefore A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

$\therefore T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

17, 18

One To One or Onto

$$T(x_1, x_2, x_3, x_4) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

No pivot, Not Onto.

Not one-to-one. (pivot ≤ 3)

$$\begin{bmatrix} -3 & 2 \\ 1 & -4 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

pivot ≤ 2

not onto

Independent → Each Column pivot → One to One

Q1. $Ax = \begin{bmatrix}1 & 1\\ 4 & 5\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$

$T(e_1) = \begin{bmatrix}1\\ 0\end{bmatrix} = \begin{bmatrix}1+0\\ 4+5\times0\end{bmatrix}$

$T(e_2) = \begin{bmatrix}0\\ 1\end{bmatrix} = \begin{bmatrix}0+1\\ 4\times0+5\times1\end{bmatrix}$

$Ax = \begin{bmatrix}7\\ 8\end{bmatrix}$ 인경우 $\begin{bmatrix}1 & 1 & 7\\ 4 & 5 & 8\end{bmatrix} \rightarrow \begin{bmatrix}1 & 0 & 7\\ 0 & 1 & -4\end{bmatrix}$

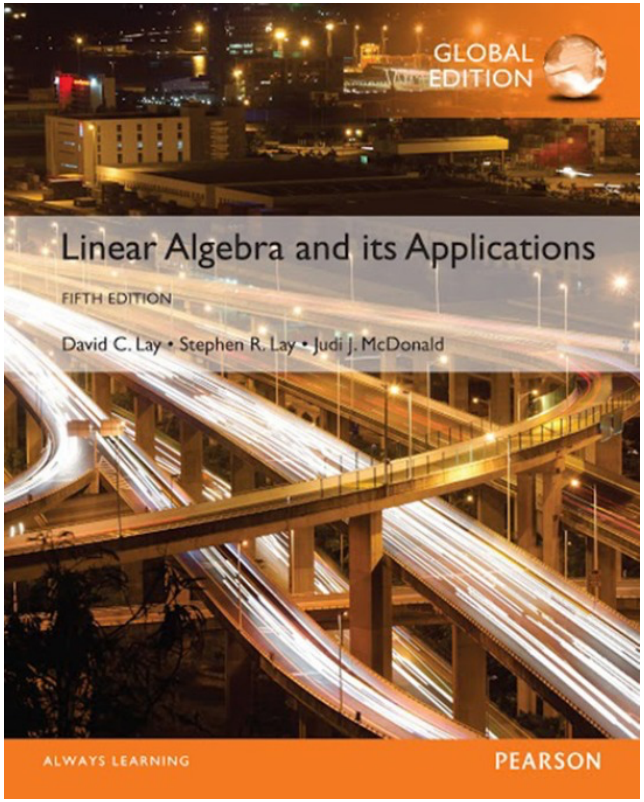
$\therefore x_1 = 7$

$x_2 = -4$

2 Matrix Algebra

2.1

MATRIX OPERATIONS



MATRIX OPERATIONS

- If A is an $m \times n$ matrix—that is, a matrix with m rows and n columns—then the scalar entry in the ith row and jth column of A is denoted by a_{ij} and is called the (i, j)-entry of A. See the Fig. 1 below.
- Each column of A is a list of m real numbers, which identifies a vector in $\mathbb{R}^m$.

$$\begin{array}{cc}
 & \begin{array}{ccccc} \phantom{a_{11}} & & \text{Column} & & \phantom{a_{1n}} \\ & & j & & \end{array} \\
\text{Row } i & \begin{bmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{m1} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix} = A \\
 & \begin{array}{ccccc} \uparrow & & \uparrow & & \uparrow \\ \mathbf{a}_1 & & \mathbf{a}_j & & \mathbf{a}_n \end{array}
\end{array}$$

Matrix notation.

MATRIX OPERATIONS

- The columns are denoted by $\mathbf{a}_1, \ldots, \mathbf{a}_n$, and the matrix A is written as

$$A = [a_1 \quad a_2 \quad \ldots \quad a_n]$$

- The number a_{ij} is the ith entry (from the top) of the jth column vector $\mathbf{a}_j$.
- The **diagonal entries** in an $m \times n$ matrix $A = \left[a_{ij} \right]$ are $a_{11}, a_{22}, a_{33}, \ldots,$ and they form the **main diagonal** of A.
- A **diagonal matrix** is a square $n \times n$ matrix whose nondiagonal entries are zero. $n \times n$ 중에 $a_{ij}=0$ $(i \neq j)$ 이면 diagonal Matrix
- An example is the $n \times n$ identity matrix, I_n.

diagonal

SUMS AND SCALAR MULTIPLES

- An $m \times n$ matrix whose entries are all zero is a **zero matrix** and is written as 0.

- The two matrices are **equal** if they have the same size (*i.e.*, the same number of rows and the same number of columns) and if their corresponding columns are equal, which amounts to saying that their corresponding entries are equal.

- If A and B are $m \times n$ matrices, then the **sum** $A + B$ is the $m \times n$ matrix whose columns are the sums of the corresponding columns in A and B.

SUMS AND SCALAR MULTIPLES

- Since vector addition of the columns is done entrywise, each entry in $A+B$ is the sum of the corresponding entries in A and B.

- The sum $A+B$ is defined only when A and B are the same size.

SUMS AND SCALAR MULTIPLES

- **Example 1:** Let $A = \begin{bmatrix} 4 & 0 & 5 \\ -1 & 3 & 2 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 5 & 7 \end{bmatrix},$

  and $C = \begin{bmatrix} 2 & -3 \\ 0 & 1 \end{bmatrix}$. Find $A+B$ and $A+C$

- **Solution:** $A+B = \begin{bmatrix} 5 & 1 & 6 \\ 2 & 8 & 9 \end{bmatrix}$ but $A+C$ is not defined because A and C have different sizes.

SUMS AND SCALAR MULTIPLES

r: real number

- If r is a scalar and A is a matrix, then the **scalar multiple** rA is the matrix whose columns are r times the corresponding columns in A.

multiply r to each entry to A

- **Theorem 1:** Let A, B, and C be matrices of the same size, and let r and s be scalars.

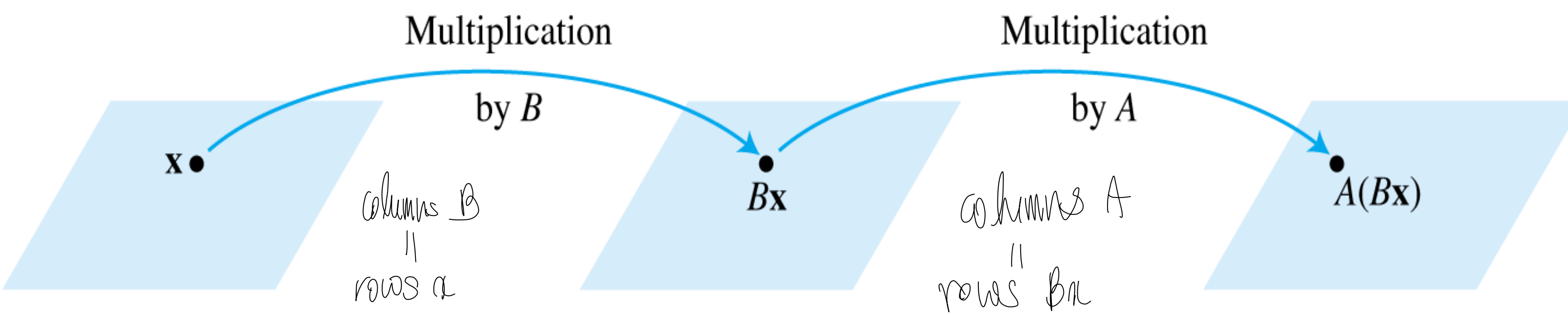
  a. $A + B = B + A$ Commutative
  b. $(A + B) + C = A + (B + C)$ Associative
  c. $A + 0 = A$ 영행렬
  d. $r(A + B) = rA + rB$ ~> $r(a_{ij}+b_{ij}) = ra_{ij}+rb_{ij}$
  e. $(r + s)A = rA + sA$
  f. $r(sA) = (rs)A$

SUMS AND SCALAR MULTIPLES

- Each quantity in Theorem 1 is verified by showing that the matrix on the left side has the same size as the matrix on the right and that corresponding columns are equal.

MATRIX MULTIPLICATION

- When a matrix B multiplies a vector $\mathbf{x}$, it transforms $\mathbf{x}$ into the vector $B\mathbf{x}$.
- If this vector is then multiplied in turn by a matrix A, the resulting vector is $A(B\mathbf{x})$. See the Fig. 2 below.

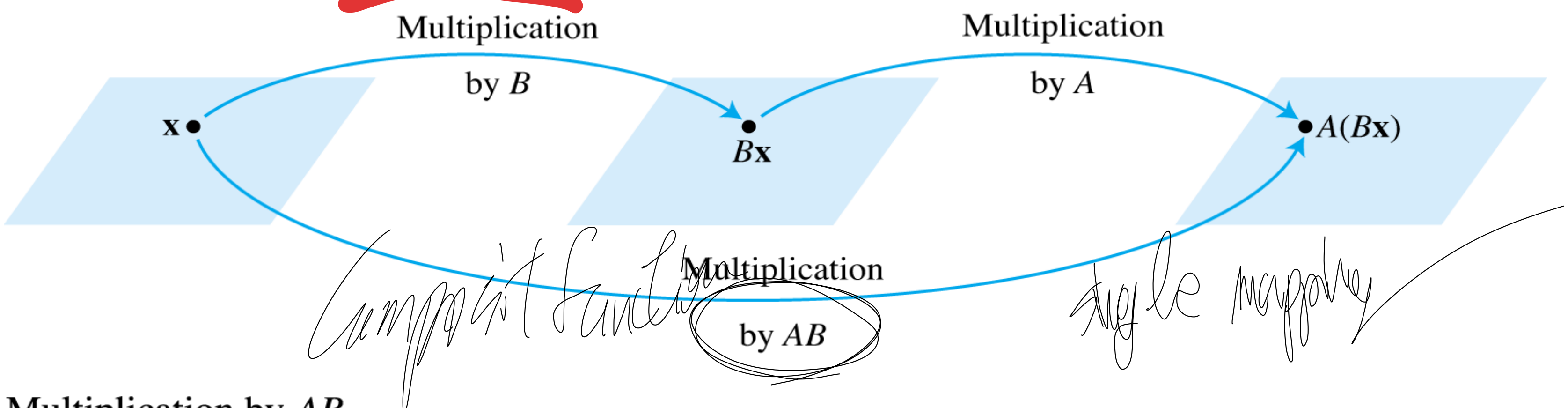


Multiplication by B and then A.

- Thus $A\,(B\mathbf{x})$ is produced from x by a *composition of mappings*—the linear transformations.

MATRIX MULTIPLICATION

- Our goal is to represent this composite mapping as multiplication by a single matrix, denoted by AB, so that $A(B\mathrm{x})=(AB)\mathrm{x}$. See Fig. 3 below



Multiplication by AB.

- If A is $m \times n$, B is $n \times p$, and $\mathbf{x}$ is in $\mathbb{R}^p$, denote the columns of B by $\mathbf{b}_1, \ldots, \mathbf{b}_p$ and the entries in $\mathbf{x}$ by $x_1, \ldots, x_p$.

MATRIX MULTIPLICATION

- Then

$$Bx = x_1b_1 + \ldots + x_pb_p$$

- By the linearity of multiplication by A,

$$A(Bx) = A(x_1b_1) + \ldots + A(x_pb_p)$$

$$= x_1Ab_1 + \ldots + x_pAb_p$$

- The vector $A(B\mathbf{x})$ is a linear combination of the vectors $A\mathbf{b}_1, \ldots, A\mathbf{b}_p$, using the entries in $\mathbf{x}$ as weights.
- In matrix notation, this linear combination is written as

$$A(Bx) = \begin{bmatrix} Ab_1 & Ab_2 & \cdots & Ab_p \end{bmatrix} x$$

MATRIX MULTIPLICATION

- Thus multiplication by $\begin{bmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 & \cdots & A\mathbf{b}_p \end{bmatrix}$ transforms $\mathbf{x}$ into $A(B\mathbf{x})$.

- **Definition:** If A is an $m \times n$ matrix, and if B is an $n \times p$ matrix with columns $\mathbf{b}_1, \ldots, \mathbf{b}_p$, then the product AB is the $m \times p$ matrix whose columns are $A\mathbf{b}_1, \ldots, A\mathbf{b}_p$.

- That is,

$$AB = A\begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_p \end{bmatrix} = \begin{bmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 & \cdots & A\mathbf{b}_p \end{bmatrix}$$

- *Multiplication of matrices corresponds to composition of linear transformations.*

MATRIX MULTIPLICATION

- **Example 3:** Compute AB, where $A = \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 3 & 6 \\ 1 & -2 & 3 \end{bmatrix}$.

- **Solution:** Write $B = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 \end{bmatrix}$, and compute:

MATRIX MULTIPLICATION

$$Ab_1 = \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix}\begin{bmatrix} 4 \\ 1 \end{bmatrix}, \quad Ab_2 = \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix}\begin{bmatrix} 3 \\ -2 \end{bmatrix}, \quad Ab_3 = \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix}\begin{bmatrix} 6 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 11 \\ -1 \end{bmatrix} \qquad = \begin{bmatrix} 0 \\ 13 \end{bmatrix} \qquad = \begin{bmatrix} 21 \\ -9 \end{bmatrix}$$

- Then

$$AB = A\begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix} = \begin{bmatrix} 11 & 0 & 21 \\ -1 & 13 & -9 \end{bmatrix}$$

합친다

$$\begin{array}{ccc} Ab_1 & Ab_2 & Ab_3 \end{array}$$

MATRIX MULTIPLICATION

- Each column of AB is a linear combination of the columns of A using weights from the corresponding column of B.

MATRIX MULTIPLICATION

- Each column of AB is a linear combination of the columns of A using weights from the corresponding column of B.

Row—column rule for computing AB

- If a product AB is defined, then the entry in row i and column j of AB is the sum of the products of corresponding entries from row i of A and column j of B. If $(AB)_{ij}$ denotes the (i, j)-entry in AB, and if A is an $m \times n$ matrix, then

= dot product of Row i of A and Column j of B

$$(AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$$

PROPERTIES OF MATRIX MULTIPLICATION

- **Theorem 2**: Let A be an $m \times n$ matrix, and let B and C have sizes for which the indicated sums and products are defined.
  a. $A(BC) = (AB)C$ (associative law of multiplication) $m \times n$, $n \times p$, $n \times q$
  b. $A(B + C) = AB + AC$ (left distributive law)
  c. $(B + C)A = BA + CA$ (right distributive law)
  d. $r(AB) = (rA)B = A(rB)$ for any scalar r
  e. $I_m A = A = AI_n$ (identity for matrix multiplication)

PROPERTIES OF MATRIX MULTIPLICATION

- **Proof:** Property (a) follows from the fact that matrix multiplication corresponds to composition of linear transformations (which are functions), and it is known that the composition of functions is associative. Let

$$C = \begin{bmatrix} \mathbf{c}_1 & \cdots & \mathbf{c}_p \end{bmatrix}$$

- By the definition of matrix multiplication,

$$BC = \begin{bmatrix} B\mathbf{c}_1 & \cdots & B\mathbf{c}_p \end{bmatrix}$$

$$A(BC) = \begin{bmatrix} A(B\mathbf{c}_1) & \cdots & A(B\mathbf{c}_p) \end{bmatrix}$$

PROPERTIES OF MATRIX MULTIPLICATION

- The definition of AB makes $A(B\mathbf{x}) = (AB)\mathbf{x}$ for all $\mathbf{x}$, so

$$A(BC) = \begin{bmatrix} (AB)\mathbf{c}_1 & \cdots & (AB)\mathbf{c}_p \end{bmatrix} = (AB)C$$

PROPERTIES OF MATRIX MULTIPLICATION

- The left-to-right order in products is critical because AB and BA are usually not the same.
- Because the columns of AB are linear combinations of the columns of A, whereas the columns of BA are constructed from the columns of B.
- The position of the factors in the product AB is emphasized by saying that A is *right-multiplied* by B or that B is *left-multiplied* by A.
- If $AB = BA$, we say that A and B **commute** with one another.

PROPERTIES OF MATRIX MULTIPLICATION

$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ $C = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ $AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = AC$

- **Warnings:**
  1. In general, $AB \neq BA$.
  2. The cancellation laws do *not* hold for matrix multiplication. That is, if $AB = AC$, then it is *not* true in general that $B = C$.
  3. If a product AB is the zero matrix, you *cannot* conclude in general that either $A = 0$ or $B = 0$.

POWERS OF A MATRIX

- If A is an $n \times n$ matrix and if k is a positive integer, then A^k denotes the product of k copies of A:

$$A^k = \underbrace{A \cdots A}_{k}$$

- If A is nonzero and if $\mathbf{x}$ is in $\mathbb{R}^n$, then $A^k\mathbf{x}$ is the result of left-multiplying $\mathbf{x}$ by A repeatedly k times.

$A^0 x = I x$

- If $k = 0$, then $A^0\mathbf{x}$ should be $\mathbf{x}$ itself.

- Thus A^0 is interpreted as the identity matrix.

I

THE TRANSPOSE OF A MATRIX

- Given an $m \times n$ matrix A, the **transpose** of A is the $n \times m$ matrix, denoted by A^T, whose columns are formed from the corresponding rows of A.

$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ $A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$

(i, j) entry $\rightarrow$ (j, i) entry

THE TRANSPOSE OF A MATRIX

- Given an $m \times n$ matrix A, the **transpose** of A is the $n \times m$ matrix, denoted by A^T, whose columns are formed from the corresponding rows of A.

Theorem 3: Let A and B denote matrices whose sizes are appropriate for the following sums and products.

a. $(A^T)^T = A$

b. $(A+B)^T = A^T + B^T$

c. For any scalar r, $(rA)^T = rA^T$

d. $(AB)^T = B^T A^T$

THE TRANSPOSE OF A MATRIX

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Theorem 3: Let A and B denote matrices whose sizes are appropriate for the following sums and products.

a. $(A^T)^T = A$ $\quad$ (i,j) of $(A^T)^T = (j,i)$ of $A^T = (i,j)$ of A

b. $(A+B)^T = A^T + B^T$

c. For any scalar r, $(rA)^T = rA^T$

d. $(AB)^T = B^T A^T$

- The transpose of a product of matrices equals the product of their transposes in the *reverse* order.

J: 증명

$$(AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$$

$$((AB)^T)_{ij} = (AB)_{ji} = a_{j1}b_{1i} + \cdots + a_{jn}b_{ni}$$

$$(B^TA^T)_{ij} = (B^T)_{i1}(A^T)_{1j} + (B^T)_{i2}(A^T)_{2j} + \cdots + (B^T)_{in}(A^T)_{nj}$$

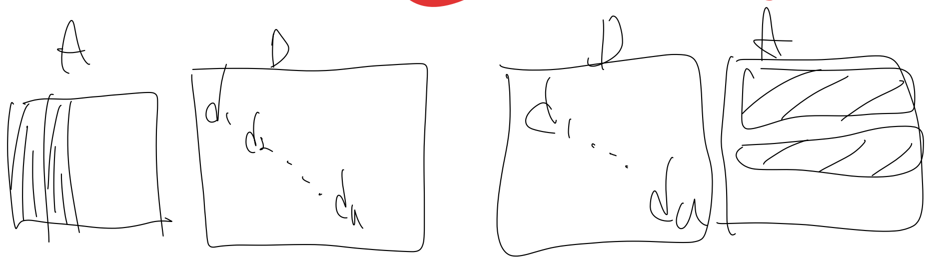
5.

$$AB = [Ab_1 \;\; Ab_2] = \left[\, 3\begin{pmatrix}-1\\5\\2\end{pmatrix} - 2\begin{pmatrix}2\\4\\-3\end{pmatrix} \quad -2\begin{pmatrix}-1\\5\\2\end{pmatrix} + \begin{pmatrix}2\\4\\-3\end{pmatrix} \,\right] = \begin{bmatrix}-7 & 4\\ 7 & -6\\ 12 & -7\end{bmatrix}$$

11.

$$A = \begin{bmatrix}1&1&1\\1&2&3\\1&4&5\end{bmatrix} \quad D = \begin{bmatrix}2&0&0\\0&3&0\\0&0&5\end{bmatrix} \quad AD = \begin{bmatrix}2&3&5\\2&6&15\\2&12&25\end{bmatrix}$$

$$DA = \begin{bmatrix}2&0&0\\0&3&0\\0&0&5\end{bmatrix}\begin{bmatrix}1&1&1\\1&2&3\\1&4&5\end{bmatrix} = \begin{bmatrix}2&2&2\\3&6&9\\5&20&25\end{bmatrix}$$



21. $AB = [Ab_1 \; Ab_2 \; \cdots \; Ab_n]$ 마지막 column은 0   ② 왜 zero row 없다.

①

$Ab_n = b_{1n}a_1 + b_{2n}a_2 + \cdots + b_{mn}a_n = 0$ 이므로,

$\therefore \{a_1 \cdots a_n\} \to$ linearly dependent 인 것이다.

22. B가 linearly dependent 할 때, AB는 linearly dependent 이다.

$Bx = 0$ for some $x \neq 0 \Rightarrow ABx = 0$ $\therefore$ AB는 linearly dependent

(= linearly dependent)

성립

같은 방법으로

정리

$v_1 \dots v_p$

$c_1v_1 + c_2v_2 + \cdots + c_pv_p$: linear combination of $v_1 \sim v_p$
weights $c_1 \cdots c_p$

ex)

$$0\begin{pmatrix}1\\0\\8\end{pmatrix} + \begin{pmatrix}1\\2\\0\end{pmatrix} - \begin{pmatrix}1\\1\\1\end{pmatrix} \qquad n\begin{pmatrix}1\\9\end{pmatrix} + \begin{pmatrix}1\\2\\3\end{pmatrix} - 2\begin{pmatrix}1\\1\\1\end{pmatrix}$$

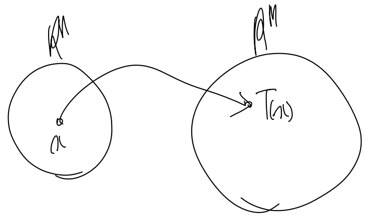
~~$\text{span}\{v_1 \dots v_p\}$ = 모든 linear combination = $\{c_1v_1 + \cdots + c_pv_p \mid c_1\ c_2 \cdots c_p \in \mathbb{R}\}$~~

$A = [a_1 \dots a_n]$ $\quad a_1 + a_2 + \cdots + a_n$, $\quad 2a_1 + 0a_2 + \cdots + a_n$ — linear combination

$\text{span}\{a_1 \dots a_n\} = \{c_1a_1 + c_2a_2 + \cdots + c_na_n \mid c_1, c_2, \dots c_n \in \mathbb{R}\}$

= set of all linear combinations of the columns of A

Given $A=[a_1 \dots a_n] \rightarrow$ span $W = \text{span}\{a_1 \dots a_n\} = \{c_1a_1 \cdots c_na_n \mid c_i \in \mathbb{R}\}$



$T(x)=Ax$ 로 표현하고 이를 Matrix Transformation 이라고 한다.

range = the collection of all images $\{Ax \mid x \in \mathbb{R}^n\}$

$= \{x_1a_1+\cdots+x_na_n \mid x_1 \dots x_n \in \mathbb{R}\}$

= span{columns of A}