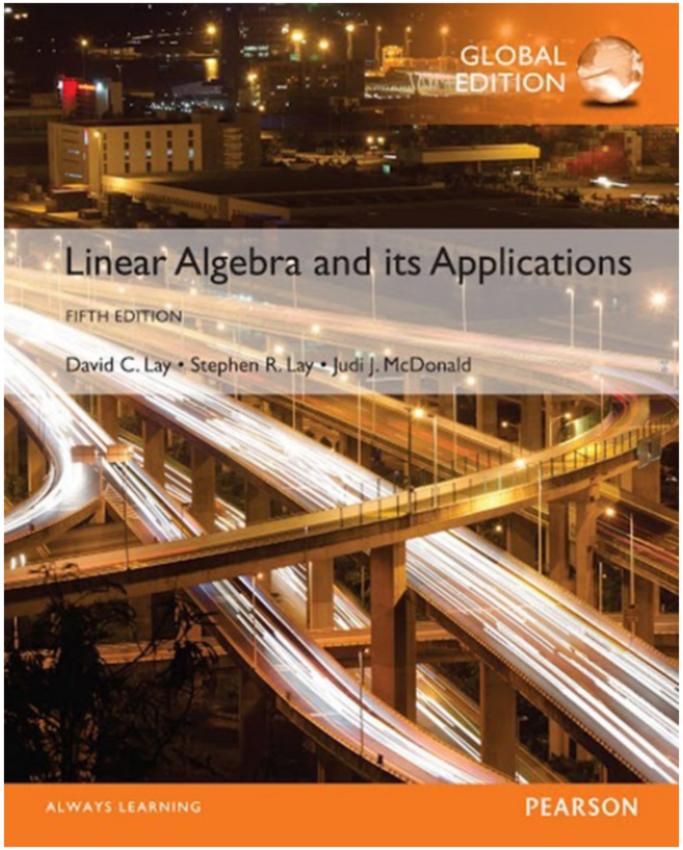
1 Linear Equations in Linear Algebra

1.1

SYSTEMS OF LINEAR EQUATIONS



LINEAR EQUATION

unknowns (variables)

- A **linear equation** in the variables $x_1, \ldots, x_n$ is an equation that can be written in the form

$$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = b$$

a, b ε number

where b and the coefficients $a_1, \ldots, a_n$ are real or complex numbers, usually known in advance.

linear Eq
$$\begin{cases} 3x = 5 \\ 3x + 4y = 5 \\ 2x_1 + x_2 - 3x_n = 7 \end{cases}$$

$$\left.\begin{array}{l} x^2 - x + 2 = 0 \\ \sin x + 7 = 0 \\ e^x - 2 \end{array}\right\}$$ not linear Eq

LINEAR EQUATION

- A **linear equation** in the variables $x_1, \ldots, x_n$ is an equation that can be written in the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

where b and the coefficients $a_1, \ldots, a_n$ are real or complex numbers, usually known in advance.

([illegible] Linear Equation)

- A **system of linear equations** (or a **linear system**) is a collection of one or more linear equations involving the same variables — say, $x_1, \ldots, x_n$.

$$\begin{cases} a+b=2 \\ a+b+h=0 \end{cases}$$

[illegible]

LINEAR EQUATION

- A **solution** of the system is a list $(s_1, s_2, \ldots, s_n)$ of numbers that makes each equation a true statement when the values $s_1, \ldots, s_n$ are substituted for $x_1, \ldots, x_n$, respectively.

$\begin{cases} x-y=2 \\ x+y=6 \end{cases}$ $x=4, y=2$ $(x,y)=(4,2)$

- The set of all possible solutions is called the **solution set** of the linear system.

[illegible] Solution [illegible]

- Two linear systems are called **equivalent** if they have the same solution set.

LINEAR EQUATION

2개를 딱 가지는 경우는 없다.
아예 무한

system is nonsingular
if the system has exactly one solution
otherwise, singular.

- A system of linear equations has
  1. no solution, or inconsistent singular
  2. exactly one solution, or consistent, non-singular
  3. infinitely many solutions. consistent, singular
- A system of linear equations is said to be **consistent** if it has either one solution or infinitely many solutions.
- A system is **inconsistent** if it has no solution.

MATRIX NOTATION

- The essential information of a linear system can be recorded compactly in a rectangular array called a **matrix**. Given the system,

$$
\begin{aligned}
x_1 - 2x_2 + x_3 &= 0 \\
2x_2 - 8x_3 &= 8 \\
-4x_1 + 5x_2 + 9x_3 &= -9,
\end{aligned}
$$

with the coefficients of each variable aligned in columns, the matrix

$$
\begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & -8 \\ -4 & 5 & 9 \end{bmatrix}
$$

is called the **coefficient matrix** (or matrix of **coefficients**) of the system

MATRIX NOTATION

- An **augmented matrix** of a system consists of the coefficient matrix with an added column containing the constants from the right sides of the equations.

- For the given system of equations

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ -4x_1 + 5x_2 + 9x_3 &= -9, \end{aligned}$$

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{bmatrix}$$

is called the **augmented matrix** of the system.

MATRIX SIZE

- The size of a matrix tells how many rows and columns it has. If m and n are positive integers, an $\boldsymbol{m \times n}$ **matrix** is a rectangular array of numbers with m rows and n columns. (The number of rows always comes first.)
- Ex) $\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{bmatrix}$ is a 3 x 4 matrix.

MATRIX SIZE

- The size of a matrix tells how many rows and columns it has. If m and n are positive integers, an $\boldsymbol{m \times n}$ **matrix** is a rectangular array of numbers with m rows and n columns. (The number of rows always comes first.)

- The basic strategy for solving a linear system is to *replace one system with an equivalent system (i.e., one with the same solution set) that is easier to solve.*

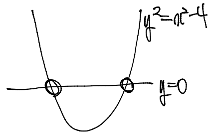
SOLVING SYSTEM OF EQUATIONS

- **Example 1:** Solve the given system of equations.

$x^2-4=0$

$(x-2)(x+2)=0$

$x=2,-2$



$x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$

공식으로 연계는 풀수있다.

→ 이런 공식 제시할 것.

$$x_1 - 2x_2 + x_3 = 0 \quad \text{----(1)}$$

$$2x_2 - 8x_3 = 8 \quad \text{----(2)}$$

$$-4x_1 + 5x_2 + 9x_3 = -9 \quad \text{----(3)}$$

- **Solution:** The elimination procedure is shown here with and without matrix notation, and the results are placed side by side for comparison.

$\begin{cases} x+y=0 \\ x-y=2 \end{cases} \overset{eq}{\longleftrightarrow} \begin{cases} x-y=2 \\ x+y=0 \end{cases}$ interchange

$\begin{cases} x+y=0 \\ 2x-2y=4 \end{cases} \overset{eq}{\longrightarrow} \begin{cases} x+y=0 \\ x-y=2 \end{cases} \longrightarrow \begin{cases} 2y=-2 \\ x-y=2 \end{cases}$ Constant Mult;

SOLVING SYSTEM OF EQUATIONS

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ -4x_1 + 5x_2 + 9x_3 &= -9 \end{aligned} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{bmatrix}$$

- Keep x_1 in the first equation and eliminate it from the other equations. To do so, add 4 times equation 1 to equation 3.

$$\begin{array}{r} 4x_1 - 8x_2 + 4x_3 = 0 \\ -4x_1 + 5x_2 + 9x_3 = -9 \\ \hline -3x_2 + 13x_3 = -9 \end{array}$$

행렬에 그대로 적용가능.

SOLVING SYSTEM OF EQUATIONS

- The result of this calculation is written in place of the original third equation:

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ \text{(New 3)} \quad -3x_2 + 13x_3 &= -9 \end{aligned}$$

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{bmatrix}$$

- Now, multiply equation 2 by $1/2$ in order to obtain 1 as the coefficient for x_2.

SOLVING SYSTEM OF EQUATIONS

$$
\begin{aligned}
x_1 - 2x_2 + x_3 &= 0 \\
x_2 - 4x_3 &= 4 \\
-3x_2 + 13x_3 &= -9
\end{aligned}
\qquad
\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{bmatrix}
$$

- Use the x_2 in equation 2 to eliminate the $-3x_2$ in equation 3.

$$
\begin{aligned}
3x_2 - 12x_3 &= 12 \\
\underline{-3x_2 + 13x_3} &\underline{= -9} \\
x_3 &= 3
\end{aligned}
$$

SOLVING SYSTEM OF EQUATIONS

- The new system has a *triangular* form.

$$
\begin{aligned}
x_1 - 2x_2 + x_3 &= 0 \\
x_2 - 4x_3 &= 4 \\
x_3 &= 3
\end{aligned}
\qquad
\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix}
\begin{array}{l} \leftarrow \text{3 unknowns} \\ \leftarrow \text{2 unknowns} \\ \leftarrow \text{1 unknown} \end{array}
$$

- You may find x_3, then x_2 and then x_1 with the system above.

SOLVING SYSTEM OF EQUATIONS

- The new system has a *triangular* form.

이번엔 이렇게
올려간다.

$$
\begin{aligned}
x_1 - 2x_2 + x_3 &= 0 \\
x_2 - 4x_3 &= 4 \\
x_3 &= 3
\end{aligned}
\qquad
\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix}
$$

밑에는 다 0으로

- Eventually, you want to eliminate the $-2x_2$ term from equation 1, but it is more efficient to use the x_3 term in equation 3 first to eliminate the $-4x_3$ and x_3 terms in equations 2 and 1.

나머지항은 두개씩. 1 unknown만 남게

SOLVING SYSTEM OF EQUATIONS

$$\begin{array}{r} 4x_3 = 12 \\ \underline{x_2 - 4x_3 = 4} \\ x_2 = 16 \end{array} \qquad \begin{array}{r} -x_3 = -3 \\ \underline{x_1 - 2x_2 + x_3 = 0} \\ x_1 - 2x_2 = -3 \end{array}$$

- Now, combine the results of these two operations.

$$\begin{array}{r} x_1 - 2x_2 = -3 \\ x_2 = 16 \\ x_3 = 3 \end{array} \qquad \begin{bmatrix} 1 & -2 & 0 & -3 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

SOLVING SYSTEM OF EQUATIONS

- Move back to the x_2 in equation 2, and use it to eliminate the $-2x_2$ above it. Because of the previous work with x_3, there is now no arithmetic involving x_3 terms. Add 2 times equation 2 to equation 1 and obtain the system:

$$\begin{aligned} x_1 &= 29 \\ x_2 &= 16 \\ x_3 &= 3 \end{aligned} \qquad \begin{bmatrix} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

SOLVING SYSTEM OF EQUATIONS

- Thus, the only solution of the original system is (29,16,3). To verify that (29,16,3) is a solution, substitute these values into the left side of the original system, and compute.

$$(29) - 2(16) + (3) = 29 - 32 + 3 = 0$$

$$2(16) - 8(3) = 32 - 24 = 8$$

$$-4(29) + 5(16) + 9(3) = -116 + 80 + 27 = -9$$

- The results agree with the right side of the original system, so (29,16,3) is a solution of the system.

$$\begin{cases} ax_1+bx_2+cx_3=d \\ ex_1+fx_2+gx_3=h \\ ix_1+jx_2+kx_3=l \end{cases} \rightsquigarrow \begin{cases} ax_1+bx_2+cx_3=d \\ f'x_2+g'x_3=h' \\ j'x_2+k'x_3=l' \end{cases} \rightsquigarrow \begin{cases} ax_1+bx_2+cx_3=d \\ f'x_2+g'x_3=h' \\ k''x_3=l'' \end{cases}$$

$$\begin{bmatrix} a & b & c & d \\ & f' & g' & h' \\ & & k'' & l'' \end{bmatrix}$$ triangular Sys

0 만들기

$$\rightsquigarrow \begin{bmatrix} a & b & 0 & d' \\ & f' & 0 & h'' \\ & & k'' & l'' \end{bmatrix} \rightarrow \begin{bmatrix} a & 0 & 0 & d'' \\ & f' & 0 & h'' \\ & & k'' & l'' \end{bmatrix}$$

0 만들기

ELEMENTARY ROW OPERATIONS

이들을 이용해서 간단히 변환 solution이 유지된다.

- Elementary row operations include the following:
  1. (Replacement) Replace one row by the sum of itself and a multiple of another row.
  2. (Interchange) Interchange two rows.
  3. (Scaling) Multiply all entries in a row by a nonzero constant.

- Two matrices are called **row equivalent** if there is a sequence of elementary row operations that transforms one matrix into the other.

ERO 써서 나온 행렬끼리는 항상 row equivalent

ELEMENTARY ROW OPERATIONS

- It is important to note that row operations are *reversible.*

ERO
$E1$, $E2$, $E3$, ⋮

$E2 \rightarrow E2 + c\cdot E1$ (replacement) E_i와 E_j에 대해서 $i \neq j$ 이어야. ($c=-1$인경우 [illegible])

$E2 \leftrightarrow E3$ (Interchange)

$E3 \times c$ (scaling) ($c \neq 0$)

모두 reversible

ex)

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & -1 & 0 \end{bmatrix} \xrightarrow{\text{1row} \times -1 \Rightarrow \text{2row}} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -3 \end{bmatrix} \xrightarrow{\times\left(-\frac{1}{3}\right)} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

ELEMENTARY ROW OPERATIONS

- It is important to note that row operations are *reversible*.
- If the augmented matrices of two linear systems are row equivalent, then the two systems have the same solution set.

ELEMENTARY ROW OPERATIONS

- It is important to note that row operations are *reversible*.
- If the augmented matrices of two linear systems are row equivalent, then the two systems have the same solution set.
- Two fundamental questions about a linear system are as follows:
  1. Is the system consistent; that is, does at least one solution *exist*?
  2. If a solution exists, is it the *only* one; that is, is the solution *unique*?

Consistent + { non-singular / singular

EXISTENCE AND UNIQUENESS OF SYSTEM OF EQUATIONS

- **Example 3:** Determine if the following system is consistent:

$$
\begin{aligned}
x_2 - 4x_3 &= 8 \\
2x_1 - 3x_2 + 2x_3 &= 1 \\
5x_1 - 8x_2 + 7x_3 &= 1
\end{aligned}
\qquad (5)
$$

- **Solution:** The augmented matrix is

$$
\begin{bmatrix} 0 & 1 & -4 & 8 \\ 2 & -3 & 2 & 1 \\ 5 & -8 & 7 & 1 \end{bmatrix}
$$

跳过去. ER03

EXISTENCE AND UNIQUENESS OF SYSTEM OF EQUATIONS

- To obtain an x_1 in in the first equation, interchange rows 1 and 2:

$$\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 5 & -8 & 7 & 1 \end{bmatrix} \times -5/2$$

- To eliminate the $5x_1$ term in the third equation, add $-5/2$ times row 1 to row 3.

$$\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & -1/2 & 2 & -3/2 \end{bmatrix} \times 1/2 \qquad (6)$$

EXISTENCE AND UNIQUENESS OF SYSTEM OF EQUATIONS

- Next, use the x_2 term in the second equation to eliminate the $-(1/2)x_2$ term from the third equation. Add $1/2$ times row 2 to row 3.

$$\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & 5/2 \end{bmatrix} \qquad (7)$$

no solution.

- The augmented matrix is now in triangular form. To interpret it correctly, go back to equation notation.

$$\begin{aligned} 2x_1 - 3x_2 + 2x_3 &= 1 \\ x_2 - 4x_3 &= 8 \\ 0 &= 5/2 \end{aligned} \qquad (8)$$

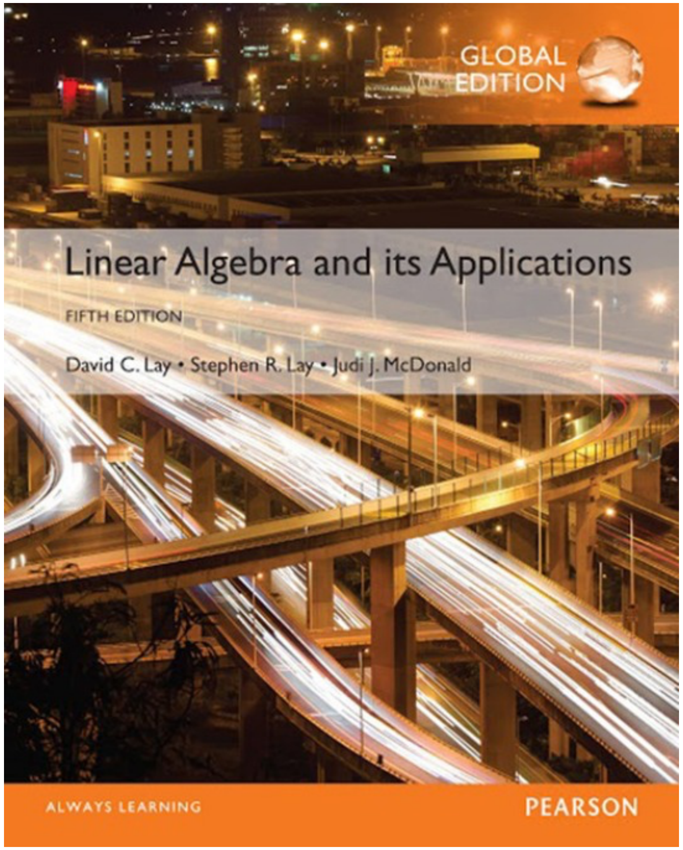
EXISTENCE AND UNIQUENESS OF SYSTEM OF EQUATIONS

- The equation $0 = 5/2$ is a short form of $0x_1 + 0x_2 + 0x_3 = 5/2$.
- There are no values of x_1, x_2, x_3 that satisfy (8) because the equation $0 = 5/2$ is never true.
- Since (8) and (5) have the same solution set, the original system is inconsistent (*i.e.*, has no solution).

1 Linear Equations in Linear Algebra

1.2

Row Reduction and Echelon Forms



ROW REDUCTION AND ECHELON FORMS

- In the definitions that follow, a *nonzero* row or column in a matrix means a row or column that contains at least one nonzero entry; a **leading entry** of a row refers to the *leftmost* nonzero entry (in a nonzero row).

$$\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & 5/2 \end{bmatrix}$$

non-zero row
$\rightarrow$ least 1 non-zero value

△: leading entry (좌측에서 최초 non zero)

non-zero column
least 1 non-zero value

ECHELON FORM

- A rectangular matrix is in **echelon form** (or **row echelon form**) if it has the following three properties:
  1. All nonzero rows are above any rows of all zeros. $\begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 4 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$
  2. Each leading entry of a row is in a column to the right of the leading entry of the row above it. $\begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \end{pmatrix}$
  3. All entries in a column below a leading entry are zeros.

자연스레
성립

Example 1 (revisited)

- Given

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ -4x_1 + 5x_2 + 9x_3 &= -9 \end{aligned} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{bmatrix}$$

the following *triangular* form is in **echelon form**

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ x_2 - 4x_3 &= 4 \\ x_3 &= 3 \end{aligned} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

ECHELON FORM

- If a matrix in echelon form satisfies the following additional conditions, then it is in **reduced echelon form** (or **reduced row echelon form**):

    4. The leading entry in each nonzero row is 1.

    5. Each leading 1 is the only nonzero entry in its column. 위아래 0

$$\begin{pmatrix} 2 & 4 & 6 \\ 0 & 3 & 6 \\ 0 & 0 & 4 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

reduced row echelon form.

Example 1 (revisited)

- Given

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ -4x_1 + 5x_2 + 9x_3 &= -9 \end{aligned} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{bmatrix}$$

the following form is in **reduced echelon form**

$$\begin{aligned} x_1 &= 29 \\ x_2 &= 16 \\ x_3 &= 3 \end{aligned} \qquad \begin{bmatrix} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

ECHELON FORM

- An **echelon matrix** (respectively, **reduced echelon matrix**) is one that is in echelon form (respectively, reduced echelon form.)

- Any nonzero matrix may be **row reduced** (i.e., transformed by elementary row operations) into more than one matrix in echelon form, using different sequences of row operations. However, the reduced echelon form one obtains from a matrix is unique.

ECHELON FORM

THEOREM 1

Uniqueness of the Reduced Echelon Form

Each matrix is row equivalent to one and only one reduced echelon matrix.

Reduced Echelon Form은 유일하다.

PIVOT POSITION

- If a matrix A is row equivalent to an echelon matrix U, we call U **an echelon form** (or row echelon form) **of A**; if U is in reduced echelon form, we call U **the reduced echelon form of A**.

  $A \to U$

- A **pivot position** in a matrix A is a location in A that corresponds to a leading 1 in the reduced echelon form of A. A **pivot column** is a column of A that contains a pivot position.

pivot column

pivot position

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix} \to \quad = U$$

PIVOT POSITION

- **Example 2:** Row reduce the matrix A below to echelon form, and locate the pivot columns of A.

$$\left\{\begin{matrix} 0 & 1 & \\ 0 & 2 & \cdots \\ 0 & 3 & \\ 0 & 4 & \end{matrix}\right\}$$

이경우 상관없나..

$$A = \begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}$$

교체

- **Solution:** The top of the leftmost nonzero column is the first pivot position. A nonzero entry, or *pivot*, must be placed in this position.

PIVOT POSITION

- Now, interchange rows 1 and 4.

Pivot

$$\begin{array}{c} \begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix} \\ \begin{array}{l} \uparrow \ \text{Pivot column} \end{array} \end{array}$$

- Create zeros below the pivot, 1, by adding multiples of the first row to the rows below, and obtain the next matrix.

PIVOT POSITION

- Choose the 2 in the second row as the next pivot.

pivot column

Pivot

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 5 & 10 & -15 & -15 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

$-5/2$

$3/2$

Next pivot column

- Add $-5/2$ times row 2 to row 3, and add $3/2$ times row 2 to row 4.

PIVOT POSITION

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 \end{bmatrix}$$

- There is no way to create a leading entry in column 3! However, if we interchange rows 3 and 4, we can produce a leading entry in column 4.

PIVOT POSITION

Δ: pivot position

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Pivot

Pivot columns

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 1 & 2 & -3 & -3 \\ 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- The matrix is in echelon form and thus reveals that columns 1, 2, and 4 of A are pivot columns.

pivot not fixed

pos, column of [illegible]

PIVOT POSITION

$$A = \begin{bmatrix} \boxed{0} & -3 & -6 & 4 & 9 \\ -1 & \boxed{-2} & -1 & 3 & 1 \\ -2 & -3 & 0 & \boxed{3} & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}$$

Pivot positions

Pivot columns

- The pivots in the example are 1, 2 and –5.

다음부터 ||| 이렇 (;|||)

ROW REDUCTION ALGORITHM

- **Example 3:** Apply elementary row operations to transform the following matrix first into echelon form and then into reduced echelon form:

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix}$$

- **Solution:**
- **STEP 1:** Begin with the leftmost nonzero column. This is a pivot column. The pivot position is at the top.

ROW REDUCTION ALGORITHM

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix}$$

Pivot column

- **STEP 2**: Select a nonzero entry in the pivot column as a pivot. If necessary, interchange rows to move this entry into the pivot position.

ROW REDUCTION ALGORITHM

- Interchange rows 1 and 3. (We could have interchanged rows 1 and 2 instead)

Pivot

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix} \quad \times -1$$

- **STEP 3**: Use row replacement operations to create zeros in all positions below the pivot.

ROW REDUCTION ALGORITHM

- We could have divided the top row by the pivot, 3, but with two 3s in column 1, it is just as easy to add -1 times row 1 to row 2.

Pivot

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix} \begin{array}{l} \\ \times\ -3/2 \\ \\ \end{array}$$

- **STEP 4**: Cover (or ignore) the row containing the pivot position and cover all rows, if any, above it. Apply steps 1–3 to the submatrix that remains. Repeat the process until there are no more nonzero rows to modify.

ROW REDUCTION ALGORITHM

- With row 1 covered, step 1 shows that column 2 is the next pivot column; for step 2, select as a pivot the "top" entry in that column.

$$\begin{bmatrix} \boxed{3 & -9 & 12 & -9 & 6 & 15} \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

Pivot

New pivot column

- For step 3, we could insert an optional step of dividing the "top" row of the submatrix by the pivot, 2. Instead, we add $-3/2$ times the "top" row to the row below.

ROW REDUCTION ALGORITHM

- This produces the following matrix.

echelon form

$$\begin{bmatrix} \boxed{3 \quad -9 \quad 12 \quad -9 \quad 6 \quad 15} \\ 0 \quad 2 \quad -4 \quad 4 \quad 2 \quad -6 \\ 0 \quad 0 \quad 0 \quad 0 \quad 1 \quad 4 \end{bmatrix}$$

(1) (2)

- When we cover the row containing the second pivot position for step 4, we are left with a new submatrix that has only one row.

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

ROW REDUCTION ALGORITHM

- Steps 1–3 require no work for this submatrix, and we have reached an echelon form of the full matrix. We perform one more step to obtain the reduced echelon form.
- **STEP 5**: Beginning with the rightmost pivot and working upward and to the left, create zeros above each pivot. If a pivot is not 1, make it 1 by a scaling operation.
- The rightmost pivot is in row 3. Create zeros above it, adding suitable multiples of row 3 to rows 2 and 1.

ROW REDUCTION ALGORITHM reduced echelon form

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 0 & -9 \\ 0 & 2 & -4 & 4 & 0 & -14 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \begin{matrix} \leftarrow \text{Row } 1 + (-6) \times \text{row } 3 \\ \leftarrow \text{Row } 2 + (-2) \times \text{row } 3 \\ \end{matrix}$$

- The next pivot is in row 2. Scale this row, dividing by the pivot.

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 0 & -9 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \begin{matrix} \\ \leftarrow \text{Row scaled by } \dfrac{1}{2} \\ \end{matrix}$$

ROW REDUCTION ALGORITHM

- Create a zero in column 2 by adding 9 times row 2 to row 1.

$$\begin{bmatrix} 3 & 0 & -6 & 9 & 0 & -72 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \leftarrow \text{Row } 1 + (9) \times \text{row } 2$$

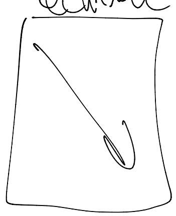
- Finally, scale row 1, dividing by the pivot, 3.

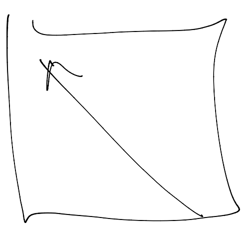
ROW REDUCTION ALGORITHM

$$\begin{bmatrix} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \leftarrow \text{Row scaled by } \frac{1}{3}$$

- This is the reduced echelon form of the original matrix.

echelon





reduced echelon

ROW REDUCTION ALGORITHM

- The combination of steps 1–4 is called the **forward phase** of the row reduction algorithm.
  I.e. the steps from a rectangular matrix

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix}$$

  to an echelon form

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

ROW REDUCTION ALGORITHM

- Step 5, which produces the unique reduced echelon form, is called the **backward phase**.
  I.e. from an echelon form

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

  to the reduced echelon form

$$\begin{bmatrix} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

SOLUTIONS OF LINEAR SYSTEMS

- The row reduction algorithm leads to an explicit description of the solution set of a linear system when the algorithm is applied to the augmented matrix of the system.
- Suppose, for example, that the augmented matrix of a linear system has been changed into the equivalent *reduced* echelon form.

$$\begin{bmatrix} 1 & 0 & -5 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

이어서

SOLUTIONS OF LINEAR SYSTEMS

- There are three variables because the augmented matrix has four columns. The associated system of equations is

$$\begin{aligned} x_1 - 5x_3 &= 1 \\ x_2 + x_3 &= 4 \\ 0 &= 0 \end{aligned} \qquad (4)$$

$x_1 = 1 + 5x_3$

$x_2 = 4 - x_3$

- The variables x_1 and x_2 corresponding to pivot columns in the matrix are called **basic variables**. The other variable, x_3, is called a **free variable**.

SOLUTIONS OF LINEAR SYSTEMS

- Whenever a system is consistent, as in

$$\begin{aligned} x_1 - 5x_3 &= 1 \\ x_2 + x_3 &= 4 \\ 0 &= 0 \end{aligned} \qquad (4),$$

the solution set can be described explicitly by solving the *reduced* system of equations for the basic variables in terms of the free variables.

- This operation is possible because the reduced echelon form places each basic variable in one and only one equation.

SOLUTIONS OF LINEAR SYSTEMS

- Whenever a system is consistent, as in

$$\begin{aligned} x_1 - 5x_3 &= 1 \\ x_2 + x_3 &= 4 \\ 0 &= 0 \end{aligned} \qquad (4),$$

  the solution set can be described explicitly by solving the *reduced* system of equations for the basic variables in terms of the free variables.

- In (4), solve the first and second equations for x_1 and the second for x_2. (Ignore the third equation; it offers no restriction on the variables.)

SOLUTIONS OF LINEAR SYSTEMS

$$\begin{aligned} x_1 &= 1 + 5x_3 \\ x_2 &= 4 - x_3 \qquad\qquad (5) \\ x_3 &\text{ is free} \end{aligned}$$

- The statement "x_3 is free" means that you are free to choose any value for x_3. Once that is done, the formulas in (5) determine the values for x_1 and x_2. For instance, when $x_3 = 0$, the solution is (1,4,0); when $x_3 = 1$, the solution is (6,3,1).

- *Each different choice of x_3 determines a (different) solution of the system, and every solution of the system is determined by a choice of x_3.*

Consistent, singular

PARAMETRIC DESCRIPTIONS OF SOLUTION SETS

- The descriptions in (5)

$$\begin{aligned} x_1 &= 1 + 5x_3 \\ x_2 &= 4 - x_3 \\ x_3 &\text{ is free} \end{aligned}$$

매개 parameter.

are *parametric descriptions* of solutions sets in which the free variables act as parameters.

- *Solving a system* amounts to finding a parametric description of the solution set or determining that the solution set is empty.

PARAMETRIC DESCRIPTIONS OF SOLUTION SETS

- Whenever a system is consistent and has free variables, the solution set has many parametric descriptions.

- For instance, in system (4),

$$
\begin{aligned}
x_1 - 5x_3 &= 1 \\
x_2 + x_3 &= 4 \\
0 &= 0
\end{aligned}
$$

we may add 5 times equation 2 to equation 1 and obtain the following equivalent system

$$\cancel{\begin{cases} x_1 = 21 - 5x_2 \\ x_3 = -x_2 + 4 \end{cases}}$$

x_2: free Var

$$
\begin{aligned}
x_1 + 5x_2 &= 21 \\
x_2 + x_3 &= 4
\end{aligned}
$$

pivot column의 basic Var를

pivot column이 없는 free Var로 설정하자.

PARAMETRIC DESCRIPTIONS OF SOLUTION SETS

- Then in the equivalent system

$$x_1 + 5x_2 = 21$$

$$x_2 + x_3 = 4$$

we could treat x_2 as a parameter and solve for x_1 and x_3 in terms of x_2, and we would have an accurate description of the solution set.

PARAMETRIC DESCRIPTIONS OF SOLUTION SETS

- Whenever a system is inconsistent, the solution set is empty, even when the system has free variables. In this case, the solution set has *no* parametric representation.

$$\begin{pmatrix} x_2 - 4x_3 = 8 \\ 2x_1 - 3x_2 + 2x_3 = 1 \\ 5x_1 - 8x_2 + 7x_3 = 1 \end{pmatrix} \sim \begin{array}{c} \text{not pivot column} \\ \begin{pmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & 5/2 \end{pmatrix} \end{array}$$

Existence of free variable does not mean the system is consistent

$\hookrightarrow$ Just No Answer

EXISTENCE AND UNIQUENESS THEOREM

rightmost column = pivot column
Y → No Solution − inconsistent → singular
N → free variable
Y → Many sol → singular → consistent
N → Unique sol → non-singular → consistent

THEOREM 2

Existence and Uniqueness Theorem

A linear system is consistent if and only if the rightmost column of the augmented matrix is *not* a pivot column—i.e., if and only if an echelon form of the augmented matrix has *no* row of the form

$$[0 \ \dots \ 0 \quad b] \text{ with } b \text{ nonzero.}$$ (이런 경우 해는 없다.)

이런게 없는 경우,

If a linear system is consistent, then the solution set contains either (i) a unique solution, when there are no free variables, or (ii) infinitely many solutions, when there is at least on free variable.

rightmost column = pivot column

Y → No Solution – inconsistent → singular ——— inconsistent

N → free variable

Y → Many sol → singular ——— consistent

N → Unique sol → non-singular ——— consistent

Using Row Reduction to Solve a Linear System

1. Write the augmented matrix of the system.
2. Use the row reduction algorithm to obtain an equivalent augmented matrix in echelon form. Decide whether the system is consistent. If there is no solution, stop; otherwise, go to the next step.

   ie : Check if the right-most column is a pivot column
   : Yes → 모순. 해X
3. Continue row reduction to obtain the reduced echelon form.
4. Write the system of equations corresponding to the matrix obtained in step 3.

ROW REDUCTION TO SOLVE A LINEAR SYSTEM

5. Rewrite each nonzero equation from step 4 so that its one basic variable is expressed in terms of any free variables appearing in the equation.

Consistent

non singular

ERO — replacement
scaling
Interchange

echelon form

reduced echelon form

pivot columns

free vs basic variables

equivalent → same solution set

consistent → at least 1 solution

coefficient / Augmented Matrix
$$\begin{cases} x + 2y + 3z = 4 \\ -x - y + 2z = 1 \end{cases}$$

$$\begin{pmatrix} 1 & 2 & 3 \\ -1 & -1 & 2 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 & 3 & 4 \\ -1 & -1 & 2 & 1 \end{pmatrix}$$

ERO

Replacement : $E_i \to E_i + c(E_j)$

Interchange : $E_i \leftrightarrow E_j$

Scaling : $E_i \to c \times E_i$

row equivalent : ERO로 구해지는 다른 Matrix.

ERO is reversible, you can undo

$$\begin{cases} x_2 + 4x_3 = -5 \\ x_1 + 3x_2 + 5x_3 = -2 \\ 3x_1 + 7x_2 + 7x_3 = 6 \end{cases}$$

[illegible]

$$\begin{bmatrix} 0 & 1 & 4 & -5 \\ 1 & 3 & 5 & -2 \\ 3 & 7 & 7 & 6 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & 5 & -2 \\ 0 & 1 & 4 & -5 \\ 3 & 7 & 7 & 6 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & 5 & -2 \\ 0 & 1 & 4 & -5 \\ 0 & -2 & -8 & 12 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & 5 & -2 \\ 0 & 1 & 4 & -5 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

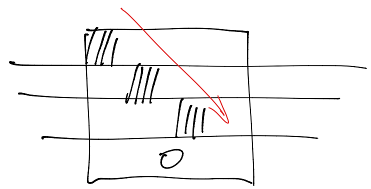
$\Leftarrow$ No solution (inconsistence)

$$\begin{cases} x_1 - 3x_3 = 8 \\ 2x_1 + 2x_2 + 9x_3 = 7 \\ x_2 + 5x_3 = -2 \end{cases} \rightsquigarrow \begin{bmatrix} 1 & 0 & -3 & 8 \\ 2 & 2 & 9 & 7 \\ 0 & 1 & 5 & -2 \end{bmatrix} \xrightarrow{(-2)} \begin{bmatrix} 1 & 0 & -3 & 8 \\ 0 & 2 & 15 & -9 \\ 0 & 1 & 5 & -2 \end{bmatrix} \xrightarrow{-1/2} \begin{bmatrix} 1 & 0 & -3 & 8 \\ 0 & 2 & 15 & -9 \\ 0 & 0 & -5/2 & 5/2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -3 & 8 \\ 0 & 1 & 15/2 & -9/2 \\ 0 & 0 & 1 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{bmatrix} \rightarrow \begin{pmatrix} x_1 = 5 \\ x_2 = 3 \\ x_3 = -1 \end{pmatrix}$$

Leading entry : leftmost nonzero entry

Echelon Form: bottom zero row

right going pivot



Reduced Echelon Form : Echelon Form + (all pivot = 1) + (pivots are only nonzero on the column)

모든 행렬은 하나의 Reduced Echelon Form과 row equivalent

Pivot position, pivot column (pivot : leading entries in the reduced echelon form)

Basic Var : pivot column에 해당하는 Variable.

free Var : pivot column이 아닌 Variable

Augmented Matrix에서 rightmost column이 pivot이 아니어야 consistent

- unique
- many infinitely

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 5 & 6 & 7 \\ 6 & 7 & 8 & 9 \end{bmatrix} (\times -4) \rightarrow \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -6 & -9 \\ 6 & 7 & 8 & 9 \end{bmatrix} (\times -6) \rightarrow \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -6 & -9 \\ 0 & -5 & -10 & -15 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -6 & -9 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

pivot

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \therefore$$

$$x_1 - x_3 = -2$$

$$x_2 + 2x_3 = 3$$

$$\begin{bmatrix} 1 & 3 & 5 & 7 \\ 3 & 5 & 7 & 9 \\ 5 & 7 & 9 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & 5 & 7 \\ 0 & -4 & -8 & -12 \\ 0 & -8 & -16 & -35 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & 5 & 7 \\ 0 & -4 & -8 & -12 \\ 0 & 0 & 0 & -11 \end{bmatrix}$$

<u>no sol</u>

pivot

augmented

$$\begin{bmatrix} 1 & -2 & -1 & 3 \\ 3 & -6 & -2 & 2 \end{bmatrix} \xrightarrow{\times -3} \begin{bmatrix} 1 & -2 & -1 & 3 \\ 0 & 0 & 1 & -7 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 & 0 & -4 \\ 0 & 0 & 1 & -7 \end{bmatrix}$$

pivot

$$x_1 - 2x_2 = -4$$
$$x_3 = -7$$

x_2: free

x_1, x_3 : basic

$$\begin{bmatrix} 1 & -3 & 0 & -1 & 0 & -2 \\ 0 & 1 & 0 & 0 & -4 & 1 \\ 0 & 0 & 0 & 1 & 9 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -3 & 0 & 0 & 9 & 2 \\ 0 & 1 & 0 & 0 & -4 & 1 \\ & & & 1 & 9 & 4 \\ & & & & & \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & -3 & 5 \\ & 1 & 0 & 0 & -4 & 1 \\ & & & 1 & 9 & 4 \\ & & & & & \end{bmatrix}$$

$$\begin{cases} x_1 - 3x_5 = 5 \\ x_2 - 4x_5 = 1 \\ x_4 - 9x_5 = 4 \end{cases}$$

x_1, x_2, x_4 : basic

x_5 : free

x_3 : free too.

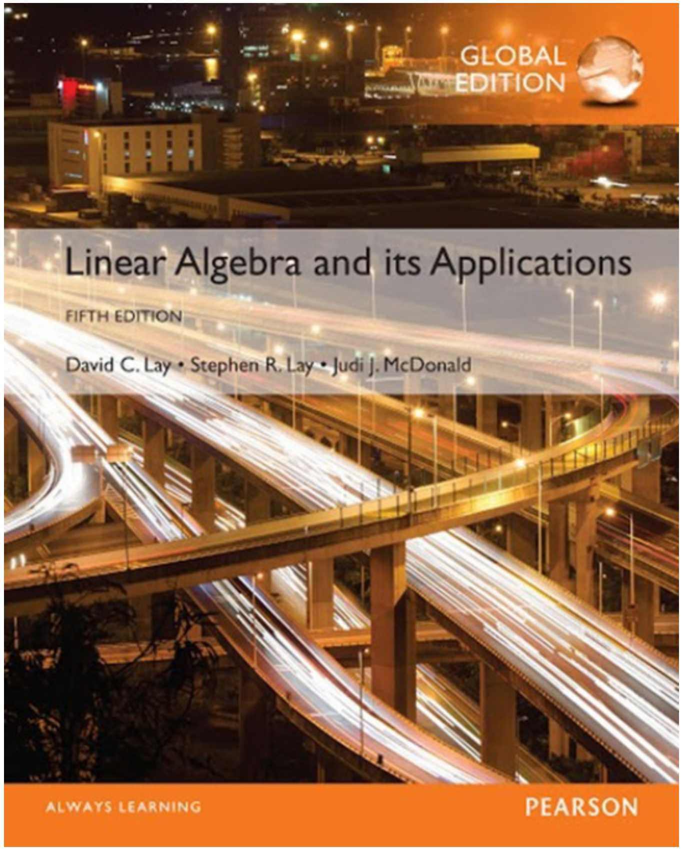
No x_3.

1

Linear Equations in Linear Algebra

1.3

VECTOR EQUATIONS



VECTOR EQUATIONS

Vectors in $\mathbb{R}^2$

- A matrix with only one column is called a **column vector**, or simply a **vector**.
- An example of a vector with two entries is

$$\mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix},$$

where w_1 and w_2 are any real numbers.

- The set of all vectors with two entries is denoted by $\mathbb{R}^2$ (read "r-two").

VECTOR EQUATIONS

- The $\mathbb{R}$ stands for the real numbers that appear as entries in the vector, and the exponent 2 indicates that each vector contains two entries.
- Two vectors in $\mathbb{R}^2$ are **equal** if and only if their corresponding entries are equal.

ex) $\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} b \\ 3 \end{bmatrix} \Leftrightarrow a=b,\ b=3$ [illegible]

VECTOR EQUATIONS

- The $\mathbb{R}$ stands for the real numbers that appear as entries in the vector, and the exponent 2 indicates that each vector contains two entries.
- Two vectors in $\mathbb{R}^2$ are **equal** if and only if their corresponding entries are equal.
- Given two vectors **u** and **v** in $\mathbb{R}^2$, their **sum** is the vector $\mathbf{u}+\mathbf{v}$ obtained by adding corresponding entries of **u** and **v**.

$$u = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad v = \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \quad u+v = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$$

VECTOR EQUATIONS

- The $\mathbb{R}$ stands for the real numbers that appear as entries in the vector, and the exponent 2 indicates that each vector contains two entries.
- Two vectors in $\mathbb{R}^2$ are **equal** if and only if their corresponding entries are equal.
- Given two vectors **u** and **v** in $\mathbb{R}^2$, their **sum** is the vector $\mathbf{u} + \mathbf{v}$ obtained by adding corresponding entries of **u** and **v**.
- Given a vector **u** and a real number c, the **scalar multiple** of **u** by c is the vector c**u** obtained by multiplying each entry in **u** by c.

$$3\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

VECTOR EQUATIONS

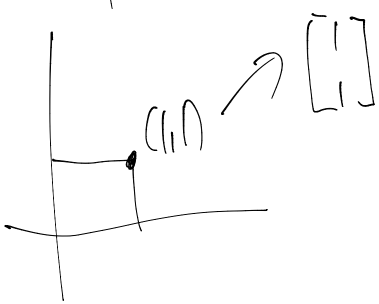
Example 1: Given $\mathbf{u} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ -5 \end{bmatrix}$, find $4\mathbf{u}, (-3)\mathbf{v}$, and $4\mathbf{u} + (-3)\mathbf{v}$.

Solution: $4\mathbf{u} = \begin{bmatrix} 4 \\ -8 \end{bmatrix}$, $(-3)\mathbf{v} = \begin{bmatrix} -6 \\ 15 \end{bmatrix}$ and

$$4\mathbf{u} + (-3)\mathbf{v} = \begin{bmatrix} 4 \\ -8 \end{bmatrix} + \begin{bmatrix} -6 \\ 15 \end{bmatrix} = \begin{bmatrix} -2 \\ 7 \end{bmatrix}$$

GEOMETRIC DESCRIPTIONS OF $\mathbb{R}^2$

- Consider a rectangular coordinate system in the plane. Because each point in the plane is determined by an ordered pair of numbers, *we can identify a geometric point (a, b) with the column vector* $\begin{bmatrix} a \\ b \end{bmatrix}$.



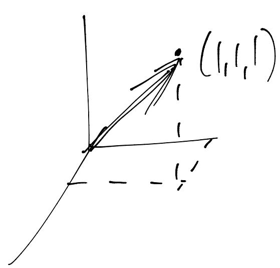
GEOMETRIC DESCRIPTIONS OF $\mathbb{R}^2$

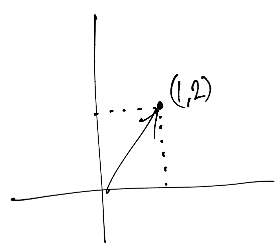
- Consider a rectangular coordinate system in the plane. Because each point in the plane is determined by an ordered pair of numbers, *we can identify a geometric point (a, b) with the column vector* $\begin{bmatrix} a \\ b \end{bmatrix}$.

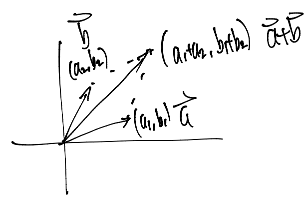
- So we may regard $\mathbb{R}^2$ as the set of all points in the plane.

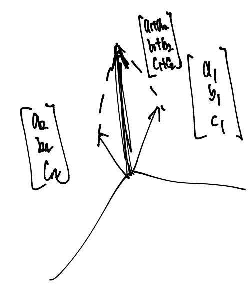
VECTORS IN $\mathbb{R}^3$ and $\mathbb{R}^n$

- Vectors in $\mathbb{R}^3$ are 3×1 column matrices with three entries.
- They are represented geometrically by points in a three-dimensional coordinate space, with arrows from the origin sometimes included for visual clarity.









VECTORS IN $\mathbb{R}^3$ and $\mathbb{R}^n$

- Vectors in $\mathbb{R}^3$ are 3×1 column matrices with three entries.
- They are represented geometrically by points in a three-dimensional coordinate space, with arrows from the origin sometimes included for visual clarity.
- If n is a positive integer, $\mathbb{R}^n$ (read "r-n") denotes the collection of all lists (or *ordered n-tuples*) of n real numbers, usually written as $n \times 1$ column matrices, such as

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}.$$

记住!!

$\mathbb{R}^2 = \{ \begin{pmatrix} a \\ b \end{pmatrix} \mid a, b \in \mathbb{R} \}$

$\mathbb{R}^3 = \{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \mid a, b, c \in \mathbb{R} \}$

ALGEBRAIC PROPERTIES OF $\mathbb{R}^n$

- The vector whose entries are all zero is called the **zero vector** and is denoted by **0**.

Bold

Number와
Vector zero를 구분

- For all **u**, **v**, **w** in $\mathbb{R}^n$ and all scalars c and d:

(i) $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$

(ii) $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$

(iii) $\mathbf{u}+\mathbf{0}=\mathbf{0}+\mathbf{u}=\mathbf{u}$

(iv) $\mathbf{u}+(-\mathbf{u})=-\mathbf{u}+\mathbf{u}=\mathbf{0}$,

where $-\mathbf{u}$ denotes $(-1)\mathbf{u}$

(v) $c(\mathbf{u}+\mathbf{v})=c\mathbf{u}+c\mathbf{v}$

(vi) $(c+d)\mathbf{u}=c\mathbf{u}+d\mathbf{u}$

LINEAR COMBINATIONS

- (vii) $c(d\mathbf{u})=(cd)(\mathbf{u})$

  (viii) $1\mathbf{u} = \mathbf{u}$

$$u+v = \begin{pmatrix} u_1+v_1 \\ u_2+v_2 \end{pmatrix} = \begin{pmatrix} v_1+u_1 \\ v_2+u_2 \end{pmatrix} = v+u$$

$$(u+v)+w = \begin{pmatrix} u_1+v_1 \\ u_2+v_2 \end{pmatrix} + \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} u_1+v_1+w_1 \\ u_2+v_2+w_2 \end{pmatrix}$$

$$u+(v+w) = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} v_1+w_1 \\ v_2+w_2 \end{pmatrix}$$

$$c(u+v) = \begin{pmatrix} c(u_1+v_1) \\ c(u_2+v_2) \end{pmatrix} = \begin{pmatrix} cu_1+cv_1 \\ cu_2+cv_2 \end{pmatrix}$$

$$= cu+cv$$

$$u+(-u) = \begin{pmatrix} u_1-u_1 \\ u_2-u_2 \end{pmatrix} = 0$$

$$c(d\vec{u}) = c\begin{pmatrix} du_1 \\ du_2 \end{pmatrix} = \begin{pmatrix} cdu_1 \\ cdu_2 \end{pmatrix} = cd\vec{u}$$

$$1\cdot\vec{u} = \begin{pmatrix} 1\times u_1 \\ 1\times u_2 \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \vec{u}$$

LINEAR COMBINATIONS

$$\left.\begin{cases} v_1 + v_2 + v_3 \\ v_2 + v_3 \\ v_1 - v_2 \end{cases}\right\}$$ 是 linear combination

- Given vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$ in $\mathbb{R}^n$ and given scalars c_1, $c_2, \ldots, c_p$, the vector $\mathbf{y}$ defined by

$$\mathbf{y} = c_1\mathbf{v}_1 + \ldots + c_p\mathbf{v}_p$$

  is called a **linear combination** of $\mathbf{v}_1, \ldots, \mathbf{v}_p$ with **weights** $c_1, \ldots, c_p$.

- The weights in a linear combination can be any real numbers, including zero.

LINEAR COMBINATIONS

- **Example 5:** Let $\mathbf{a}_1 = \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix}$, $\mathbf{a}_2 = \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}$.

  Determine whether **b** can be generated (or written) as a linear combination of $\mathbf{a}_1$ and $\mathbf{a}_2$. That is, determine whether weights x_1 and x_2 exist such that

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 = \mathbf{b} \qquad (1)$$

  If vector equation (1) has a solution, find it.

LINEAR COMBINATIONS

Solution: Use the definitions of scalar multiplication and vector addition to rewrite the vector equation

$$x_1 \underset{\uparrow \atop \mathbf{a}_1}{\begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix}} + x_2 \underset{\uparrow \atop \mathbf{a}_2}{\begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix}} = \underset{\uparrow \atop \mathbf{b}}{\begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}},$$

which is same as $$\begin{bmatrix} x_1 \\ -2x_1 \\ -5x_1 \end{bmatrix} + \begin{bmatrix} 2x_2 \\ 5x_2 \\ 6x_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}$$

LINEAR COMBINATIONS

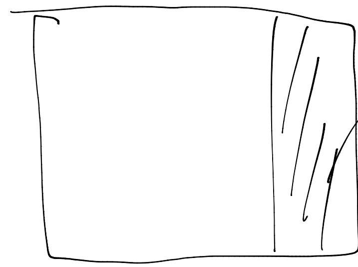
and $$\begin{bmatrix} x_1 + 2x_2 \\ -2x_1 + 5x_2 \\ -5x_1 + 6x_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}. \qquad (2)$$

$$\begin{bmatrix} 1 & 2 \\ -2 & 5 \\ -5 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

- The vectors on the left and right sides of (2) are equal if and only if their corresponding entries are both equal. That is, x_1 and x_2 make the vector equation (1) true if and only if x_1 and x_2 satisfy the following system.

$$\begin{aligned} x_1 + 2x_2 &= 7 \\ -2x_1 + 5x_2 &= 4 \\ -5x_1 + 6x_2 &= -3 \end{aligned} \qquad (3)$$

b가 a_1, a_2 의 linear combination이 되는가? = Matrix 가 근이 있나? consistant 한가?

→ augmented matrix  → 여기에 pivot 아니면, x_1, x_2 모두 근이 존재.

LINEAR COMBINATIONS

- To solve this system, row reduce the augmented matrix of the system as follows:

$$\begin{bmatrix} 1 & 2 & 7 \\ -2 & 5 & 4 \\ -5 & 6 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 7 \\ 0 & 9 & 18 \\ 0 & 16 & 32 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 7 \\ 0 & 1 & 2 \\ 0 & 16 & 32 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

- The solution of (3) is $x_1 = 3$ and $x_2 = 2$. Hence **b** is a linear combination of $\mathbf{a}_1$ and $\mathbf{a}_2$, with weights $x_1 = 3$ and $x_2 = 2$. That is,

$$3\begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix} + 2\begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}.$$

LINEAR COMBINATIONS

- Now, observe that the original vectors $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{b}$ are the columns of the augmented matrix that we row reduced:

$$\begin{array}{ccc} \begin{bmatrix} 1 & 2 & 7 \\ -2 & 5 & 4 \\ -5 & 6 & -3 \end{bmatrix} \\ \begin{array}{ccc} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{b} \end{array} \end{array}$$

- Write this matrix in a way that identifies its columns.

$$\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{b} \end{bmatrix} \qquad (4)$$

LINEAR COMBINATIONS

- A vector equation

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \dots + x_n\mathbf{a}_n = \mathbf{b}$$

has the same solution set as the linear system whose augmented matrix is

$$\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n & \mathbf{b} \end{bmatrix} \qquad (5)$$

- In particular, **b** can be generated by a linear combination of $\mathbf{a}_1, \dots, \mathbf{a}_n$ if and only if there exists a solution to the linear system corresponding to the matrix (5).

LINEAR COMBINATIONS

- **Definition:** If $\mathbf{v}_1, \ldots, \mathbf{v}_p$ are in $\mathbb{R}^n$, then the set of all linear combinations of $\mathbf{v}_1, \ldots, \mathbf{v}_p$ is denoted by Span $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ and is called the **subset of $\mathbb{R}^n$ spanned (or generated) by $\mathbf{v}_1, \ldots, \mathbf{v}_p$**. That is, Span $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is the collection of all vectors that can be written in the form

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_p\mathbf{v}_p$$

with $c_1, \ldots, c_p$ scalars.

(span$\{v_1, \ldots v_p\} = \{c_1v_1 + \cdots c_pv_p \mid c_1, \cdots, c_p \in \mathbb{R}\}$ = linear combination의 [illegible])

9.

$$\begin{cases} x_2 + 5x_3 = 0 \\ 4x_1 + 6x_2 - x_3 = 0 \\ -x_1 + 3x_2 - 8x_3 = 0 \end{cases}, \quad \begin{bmatrix} 0 & 1 & 5 \\ 4 & 6 & -1 \\ -1 & 3 & -8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 4 \\ -1 \end{bmatrix}x_1 + \begin{bmatrix} 1 \\ 6 \\ 3 \end{bmatrix}x_2 + \begin{bmatrix} 5 \\ -1 \\ -8 \end{bmatrix}x_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Leftarrow \begin{bmatrix} 0x_1 + x_2 + 5x_3 \\ 4x_1 + 6x_2 - x_3 \\ -x_1 + 3x_2 - 8x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$[a_1\ a_2\ a_3\ b] = \begin{bmatrix} 1 & 0 & 5 & 2 \\ -2 & 1 & -6 & -1 \\ 0 & 2 & 8 & 6 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 5 & 2 \\ 0 & 1 & 4 & 3 \\ 0 & 2 & 8 & 6 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 5 & 2 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore x_1 = 2 - 5x_3$, $x_2 = 3 - 4x_3$ (x_3 free)

$\therefore$ linearly dependent

$x_3 = 0 \to 2a_1 + 3a_2 + 0a_3 = b$

$x_3 = 1 \to -3a_1 - a_2 + a_3 = b$

vector parallel $\nparallel$ not same

(not parallel, not meet in space) 이경우를 포함못한다:

rightmost column is a pivot column $\Longleftrightarrow$ system is inconsistent

$\Longleftrightarrow$ no solution

$$A = \begin{bmatrix} 1 & -4 & 2 \\ 0 & 3 & 5 \\ -2 & 8 & -4 \end{bmatrix}, \quad b = \begin{bmatrix} 3 \\ -7 \\ -3 \end{bmatrix}$$

$$\begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} x_1 + \begin{pmatrix} -4 \\ 3 \\ 8 \end{pmatrix} x_2 + \begin{pmatrix} 2 \\ 5 \\ -4 \end{pmatrix} x_3 = \begin{pmatrix} 3 \\ -7 \\ -3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -4 & 2 & 3 \\ 0 & 3 & 5 & -7 \\ -2 & 8 & -4 & -3 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & -4 & 2 & 3 \\ 0 & 3 & 5 & -7 \\ 0 & 0 & 0 & 3 \end{pmatrix} \Leftarrow \text{no solution}$$

→ independent!!

Span → b가 a_1, a_2로 생성되어야 한다 ⇔ augment의 조건!!

$$\begin{pmatrix} 1 & -2 & 4 \\ 4 & -3 & 1 \\ -2 & 7 & h \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -2 & 4 \\ 0 & 5 & -15 \\ 0 & 3 & h+8 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -2 & 4 \\ 0 & 1 & -3 \\ 0 & 3 & h+8 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -2 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & h+17 \end{pmatrix}$$

h = -17 이어야 한다.
right most가 pivot이 되면
안되니께 ...

$$A = \begin{bmatrix} 1 & 0 & -4 \\ 0 & 3 & -2 \\ -2 & 6 & 3 \end{bmatrix} \quad b = \begin{bmatrix} 4 \\ 1 \\ -4 \end{bmatrix}$$

a. no, 3 vectors

b. b가 span에 속함?

$$[a_1, a_2, a_3, b] = \begin{pmatrix} 1 & 0 & -4 & 4 \\ 0 & 3 & -2 & 1 \\ -2 & 6 & 3 & -4 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -4 & 4 \\ 0 & 3 & -2 & 1 \\ 0 & 6 & -5 & 4 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -4 & 4 \\ 0 & 3 & -2 & 1 \\ 0 & 0 & -1 & 2 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 0 & 0 & -8 \\ 0 & 3 & 0 & -3 \\ 0 & 0 & 1 & -2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & -8 \\ \cdot & 1 & 0 & -1 \\ \cdot & \cdot & 1 & -2 \end{pmatrix}$$

consistent $b \in W$

infinite elements

c. $1 \cdot a_1 + 0a_2 + 0 \cdot a_3 \in W$

그저 linear combination.

(그냥 linear의 결합이니까...)

$$\begin{pmatrix} 1 & 0 & -4 & 1 \\ 0 & 3 & -2 & 0 \\ -2 & 6 & 3 & -2 \end{pmatrix}$$

Review

① column vector $n \times 1$ matrix

② Two vector in $\mathbb{R}^2$ is equal when two components are equal

③ linear Combination is $c_1v_1 + c_2v_2 + c_3v_3 + \cdots + c_pv_p$

④ $\text{Span}\{v_1 \cdots v_p\} = \{c_1v_1 + \cdots c_pv_p \mid c_1 \cdots c_p \in \mathbb{R}\}$

$\begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} \begin{pmatrix} 0 \\ 5 \\ 5 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \\ 8 \end{pmatrix} \rightarrow \begin{pmatrix} -5 \\ 11 \\ -7 \end{pmatrix}$ 이 셋의 LC 되는지

$$[a_1\ a_2\ a_3\ b] = \begin{bmatrix} 1 & 0 & 2 & -5 \\ -2 & 5 & 0 & 11 \\ 2 & 5 & 8 & -7 \end{bmatrix} \xrightarrow{\times 2} \begin{bmatrix} 1 & 0 & 2 & -5 \\ 0 & 5 & 4 & 1 \\ 0 & 5 & 4 & 3 \end{bmatrix} \xrightarrow{\times -1} \begin{bmatrix} 1 & 0 & 2 & -5 \\ 0 & 5 & 4 & 3 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

$0 \cdot x_3 = 2$

No sol

$\therefore$ b는 linearly independent

$$A = \begin{bmatrix} 1 & -2 & -6 \\ 0 & 3 & 7 \\ 1 & -2 & 5 \end{bmatrix} \quad b = \begin{bmatrix} 11 \\ -5 \\ 9 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & -2 & -6 & 11 \\ 0 & 3 & 7 & -5 \\ 1 & -2 & 5 & 9 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 & -6 & 11 \\ 0 & 3 & 7 & -5 \\ 0 & 0 & 11 & -2 \end{bmatrix} \rightarrow$$ b is linear comb of colums of A

$$\begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} \quad \begin{pmatrix} -3 \\ 1 \\ 8 \end{pmatrix} \quad y = \begin{pmatrix} h \\ -5 \\ -3 \end{pmatrix}$$ y가 span에 들도록.

$$\begin{pmatrix} 1 & -3 & h \\ 0 & 1 & -5 \\ -2 & 8 & -3 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & -3 & h \\ 0 & 1 & -5 \\ 0 & 2 & 2h-3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -3 & h \\ 0 & 1 & -5 \\ 0 & 0 & 2h+7 \end{pmatrix}$$

$\therefore$ $2h+7=0$ 이어야.

$h=-7/2$

$2h+7 \neq 0$ 이면, 근이 없다. $\therefore$ $2h+7=0$ 이어야 한다.

$$A = \begin{bmatrix} 2 & 0 & 6 \\ -1 & 8 & 5 \\ 1 & 2 & 1 \end{bmatrix} \quad b = \begin{bmatrix} 10 \\ 3 \\ 3 \end{bmatrix}$$

W는 A의 Column Vector의 Span

a. Is $b \in W$?

b. Show A의 third column $\in W$

(a)

$$\left[\begin{array}{ccc|c} 2 & 0 & 6 & 10 \\ -1 & 8 & 5 & 3 \\ 1 & -2 & 1 & 3 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 3 & 5 \\ -1 & 8 & 5 & 3 \\ 1 & -2 & 1 & 3 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 3 & 5 \\ 0 & 8 & 8 & 8 \\ 0 & -2 & -2 & -2 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & 3 & 5 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

pivot

$\therefore$ $x_1 = 5 - 3x_3$, $x_2 = 1 - x_3$ (x_3: free)

(b) $a_3 = 0a_1 + 0a_2 + 1 \cdot a_3 \in W$