4.1 Review

- vector space = a set which closed under { addition / scalar multiplication } → vectors
- subspace = a subset of a vector space, which is a vector space.

▪ **Theorem 1:** If $\mathbf{v}_1,\ldots,\mathbf{v}_p$ are in a vector space V, then Span $\{\mathbf{v}_1,\ldots,\mathbf{v}_p\}$ is a subspace of V.

9.

9. Let H be the set of all vectors of the form $\begin{bmatrix} s \\ 3s \\ 2s \end{bmatrix}$. Find a ~~vector $\mathbf{v}$ in $\mathbb{R}^3$ such that $H = \text{Span}\{\mathbf{v}\}$~~. Why does this show that H is a subspace of $\mathbb{R}^3$?

+ find basis for H.

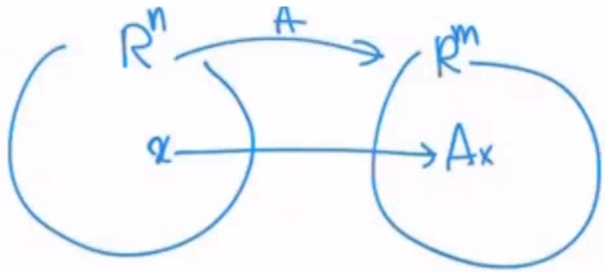
$$\begin{bmatrix} s \\ 3s \\ 2s \end{bmatrix} = s \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} \qquad v = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$$

상수곱과 덧셈에 closed

11.

11. Let W be the set of all vectors of the form $\begin{bmatrix} 5b + 2c \\ b \\ c \end{bmatrix}$, where b and c are arbitrary. ~~Find vectors $\mathbf{u}$ and $\mathbf{v}$ such that $W = \text{Span}\{\mathbf{u}, \mathbf{v}\}$~~. Why does this show that W is a subspace of $\mathbb{R}^3$?

$$\begin{bmatrix} 4b + 2c \\ 2b \\ c \end{bmatrix} = b \underbrace{\begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix}}_{= u} + c \underbrace{\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}}_{= v}$$



$A \xrightarrow{\text{row}} U \sim$ echelon form

$\text{Nul } A = \{x \mid Ax = 0\} = \{x \mid Ux = 0\} = \text{Nul } U$

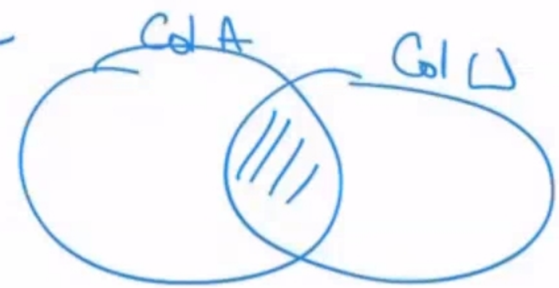
basis of Nul A = basis of Nul U

dim Nul A = dim Nul U

Col A = span{columns of A}

Col U = span{columns of U}

Col $A \neq$ Col U



basis of Col A = {pivot columns of A}

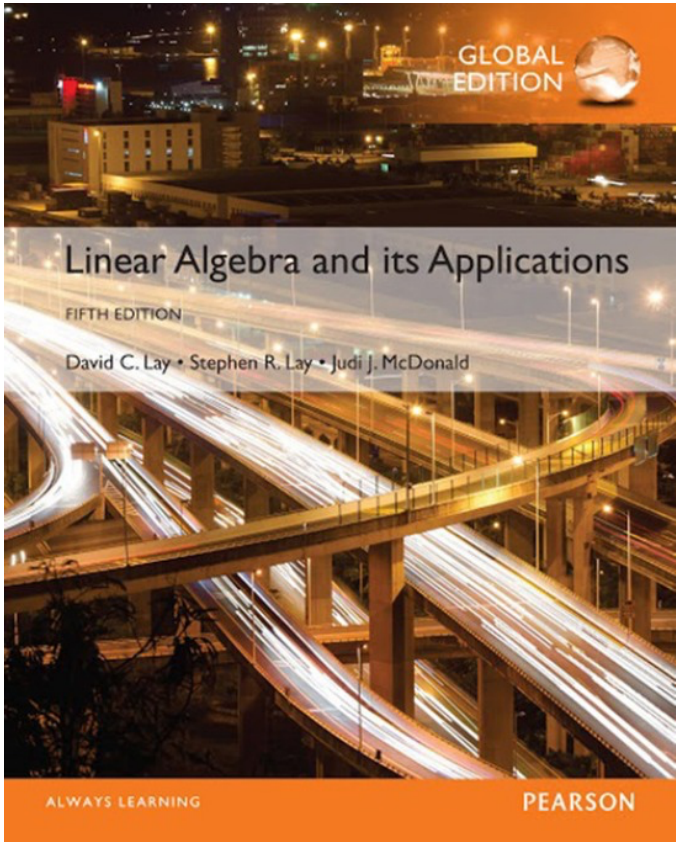
basis of Col U = {pivot columns of U} $\neq$

dim Col A = rank = # pivots = dim Col U

dim Col A + dim Nul A = # columns = n

4 Vector Spaces

4.1 VECTOR SPACES AND SUBSPACES



Vector Space 원소는 뭐든간에 전부
Vector라고 부른다.

- **Definition:** A **vector space** is a nonempty set V of objects, called *vectors*, on which are defined two operations, called *addition and multiplication by scalars* (real numbers), subject to the ten axioms (or rules) listed below. The axioms must hold for all vectors **u**, **v**, and **w** in V and for all scalars c and d.

  1. The sum of **u** and **v**, denoted by $\mathbf{u}+\mathbf{v}$, is in V.
  2. $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$. commutative
  3. $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$.
  4. There is a zero vector $\mathbf{0}$ in V such that

$$\mathbf{u}+\mathbf{0}=\mathbf{u}.$$

5. For each $\mathbf{u}$ in V, there is a vector $-\mathbf{u}$ in V such that $\mathbf{u}+(-\mathbf{u})=\mathbf{0}$.
6. The scalar multiple of $\mathbf{u}$ by c, denoted by $c\mathbf{u}$, is in V.
7. $c(\mathbf{u}+\mathbf{v})=c\mathbf{u}+c\mathbf{v}$.
8. $(c+d)\mathbf{u}=c\mathbf{u}+d\mathbf{u}$.
9. $c(d\mathbf{u})=(cd)\mathbf{u}$.
10. $1\mathbf{u}=\mathbf{u}$.

1, 6 → sum, multi also in V → 8 자동성립

모두 다 Vector Space 이다.

addition과 product에 단혀있기만하면 모두 괜찮

ex) $\mathbb{R}^2, \mathbb{R}^3, \mathbb{R}^n$

ex) $V = \{ 2\times 2 \text{ real matrices } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a,b,c,d \in \mathbb{R} \}$

전부 Vector라고 본다.

$$\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} + \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \in V$$

$$c \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \in V$$

ex) $V = \{ \text{continuous functions} \}$

$$f(x) + g(x) \in V$$

$$cf(x) \in V$$

$\sin(x)$

ex) $P_2 = \{ \text{polynomials of } \underline{\deg \le 2} \}$

$$= \{ a_2x^2 + a_1x + a_0 \mid a_0, a_1, a_2 \in \mathbb{R} \}$$

$$\begin{array}{r} x^2+1 \\ + \; -x^2-x-2 \\ \hline -x-1 \end{array} \in P_2$$

$3(x^2+1) \in P_2$

ex) $V = \{ \begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{R}^2 \mid ab \ge 0 \}$ $\quad \begin{pmatrix} 1 \\ 1 \end{pmatrix} \in V \quad \begin{pmatrix} -1 \\ -2 \end{pmatrix} \in V \cdots$

Is V a vector space?

No $\quad \begin{pmatrix} -1 \\ -3 \end{pmatrix}, \begin{pmatrix} 5 \\ 2 \end{pmatrix} \in V$

$$\begin{pmatrix} -1 \\ -3 \end{pmatrix} + \begin{pmatrix} 5 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ -1 \end{pmatrix} \notin V$$

반례

$P_n = \{ a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \mid a_0, a_1, \cdots, a_n \in \mathbb{R} \}$

$= \{$ polynomials of deg $\leq n \}$ ~ a vector space

Generalized Vector

<u>Note</u> that $Q_n = \{$ polynomials of degree $\underline{n} \}$ is NOT a vector space

ex) $x^2 + x + 1 \in Q_2$, $-x^2 + x \in Q_2$ $\}$ $(x^2 + x + 1) + (-x^2 + x) = \underline{2x + 1} \notin Q_2$

SUBSPACES

- **Definition:** A **subspace** of a vector space V is a subset H of V that has three properties:
  a. The zero vector of V is in H.
  b. H is closed under vector addition. That is, for each $\mathbf{u}$ and $\mathbf{v}$ in H, the sum $\mathbf{u} + \mathbf{v}$ is in H.
  c. H is closed under multiplication by scalars. That is, for each $\mathbf{u}$ in H and each scalar c, the vector $c\mathbf{u}$ is in H.

ex) for Subspace

ex) $V = \{$ continuous functions $\}$

$P_2 = \{$ polynomials of deg $\le 2\}$ is a subspace of V

ex) $V = \{\, 2\times 2$ real matrices $\}$

$H = \{\, 2\times 2$ upper triangular matrices $\}$ is a subspace of V.

$K = \{\, 2\times 2$ invertible matrices $\}$ ~ **NOT** a subspace. ← Subspace 아닌 경우

$$\underset{K}{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}} + \underset{K}{\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \notin K$$

SUBSPACES

- Properties (a), (b), and (c) guarantee that ~~a subspace H of V is itself a *vector space*~~, under the vector space operations already defined ~~in V~~.

- ~~Every subspace is a vector space.~~

A SUBSPACE SPANNED BY A SET

- The set consisting of only the zero vector in a vector space V is a subspace of V, called the **zero subspace** and written as $\{\mathbf{0}\}$.

A SUBSPACE SPANNED BY A SET

- **Theorem 1**: If $\mathbf{v}_1,\ldots,\mathbf{v}_p$ are in a vector space V, then Span $\{\mathbf{v}_1,\ldots,\mathbf{v}_p\}$ is a subspace of V.

자명하지 않나?

$u = c_1v_1 + \cdots + c_pv_p$

$v = c_1v_1 + \cdots + c_pv_p$

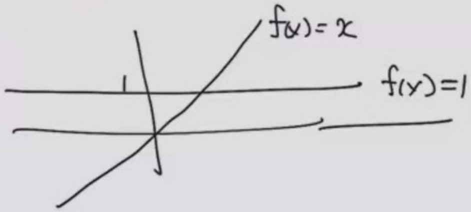
$\Rightarrow u+v = (c_1+d_1)v_1 + \cdots + (c_p+d_p)v_p$

$\in \text{span}\{v_1 \ldots v_p\}$

$cu = cc_1v_1 + \cdots + cc_pv_p$

$\in \text{span}\{v_1, \ldots v_p\}$

$P_2 \ni x$, 1



span$\{1, x\}$ is a subspace of P_2

$2x$, $x+1$,

$-x+3$

For the matrices in Exercises 17–20, (a) find k such that Nul A is a subspace of $\mathbb{R}^k$, and (b) find k such that Col A is a subspace of $\mathbb{R}^k$.

19. $A = \begin{bmatrix} 4 & 5 & -2 & 6 & 0 \\ 1 & 1 & 0 & 1 & 0 \end{bmatrix}$

20. $A = \begin{bmatrix} 1 & -3 & 9 & 0 & -5 \end{bmatrix}$

$$\begin{bmatrix} 4 & 5 & -2 & 6 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 5/4 & -1/2 & 3/2 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 5/4 & -1/2 & 3/2 & 0 & 0 \\ 0 & -1/4 & 1/2 & -1/2 & 0 & 0 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 5/4 & -1/2 & 3/2 & 0 & 0 \\ 0 & 1 & -2 & 2 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 & -1 & 0 & 0 \\ 0 & 1 & -2 & 2 & 0 & 0 \end{bmatrix}$$

$x_1 = -2x_3 + x_4$
$x_2 = 2x_3 - 2x_4$
x_3, x_4, x_5 free

$$X = \begin{bmatrix} -2x_3+x_4 \\ 2x_3-2x_4 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_3\begin{bmatrix} -2 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4\begin{bmatrix} 1 \\ -2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_5\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \qquad k=5$$

$\therefore$ dim Col $A \to 2$
dim Nul $A \to 5-2=3$

Col A: $\left\{\begin{bmatrix} 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 1 \end{bmatrix}\right\} \to k=2$

20.

$x_1 = 3x_2 - 9x_3 + 5x_5 \quad \therefore X = \begin{bmatrix} 3x_2-9x_3+5x_5 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$

$$= \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}x_2 + \begin{bmatrix} -9 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}x_3 + \begin{bmatrix} 5 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}x_5$$

dim Col $A = 1$ Col $A \subseteq R$
dim Nul $A = 5-1=4$ Nul $A \subseteq R$

24. Let $A = \begin{bmatrix} -8 & -2 & -9 \\ 6 & 4 & 8 \\ 4 & 0 & 4 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$. Determine if $\mathbf{w}$ is in Col A. Is $\mathbf{w}$ in Nul A?

$\text{Nul } A = \{x \mid Ax = 0\}$

$$\begin{bmatrix} -8 & -2 & -9 \\ 6 & 4 & 8 \\ 4 & 0 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore W \in \text{Nul} A$

$$\rightarrow \begin{bmatrix} 1 & 0 & 1 & -1/2 \\ 0 & 4 & 2 & 4 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

$\hookrightarrow$ inconsistent = no solution

$\therefore W \notin \text{Col} A$

32. Define a linear transformation $T : \mathbb{P}_2 \to \mathbb{R}^2$ by $T(\mathbf{p}) = \begin{bmatrix} \mathbf{p}(0) \\ \mathbf{p}(0) \end{bmatrix}$. Find polynomials $\mathbf{p}_1$ and $\mathbf{p}_2$ in $\mathbb{P}_2$ that span the kernel of T, and describe the range of T.

$P_2 = \{ a_2x^2 + a_1x + a_0 \mid a_0, a_1, a_2 \in \mathbb{R} \}$

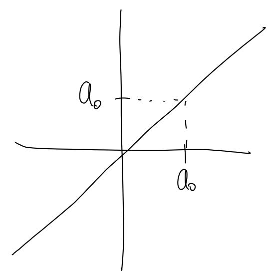
$\therefore$ kernel of $T = \{ p \mid T(p) = 0 \}$

$p(0) = 0 \rightarrow a_0 = 0 \rightarrow p(x) = a_2x^2 + a_1x \in \text{kernel}(T)$

$P_1 = x$

$P_2 = x^2$ $\rightarrow \text{kernel}(T) = \text{span}\{P_1, P_2\}$

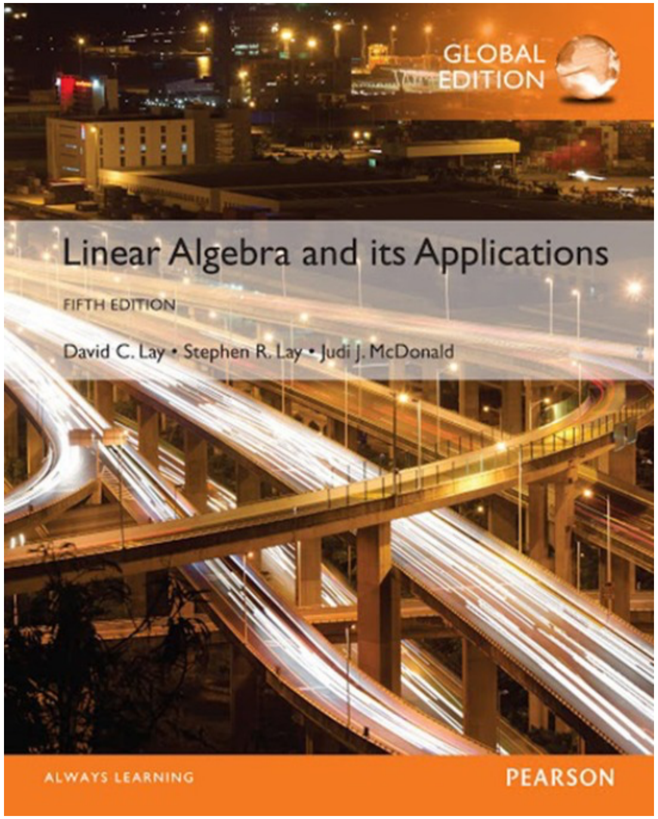
Range of $T = \left\{ \begin{pmatrix} p(0) \\ p(0) \end{pmatrix} \middle| \, p(x) \in P_2 \right\} = \left\{ \begin{pmatrix} a_0 \\ a_0 \end{pmatrix} \middle| \, a_0 \in R \right\}$



4 Vector Spaces

4.2

NULL SPACES, COLUMN SPACES, AND LINEAR TRANSFORMATIONS



PEARSON

NULL SPACE OF A MATRIX

- **Example 3:** Find a spanning set for the null space of the matrix

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$$

NULL SPACE OF A MATRIX

- **Solution:** The first step is to find the general solution of $A\mathrm{x} = 0$ in terms of free variables.

- Row reduce the augmented matrix $\begin{bmatrix} A & 0 \end{bmatrix}$ to *reduce* echelon form in order to write the basic variables in terms of the free variables:

$$\begin{bmatrix} 1 & -2 & 0 & -1 & 3 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{aligned} x_1 - 2x_2 - x_4 + 3x_5 &= 0 \\ x_3 + 2x_4 - 2x_5 &= 0 \\ 0 &= 0 \end{aligned}$$

NULL SPACE OF A MATRIX

- The general solution is $x_1 = 2x_2 + x_4 - 3x_5$, $x_3 = -2x_4 + 2x_5$, with x_2, x_4, and x_5 free.
- Next, decompose the vector giving the general solution into a linear combination of <u>*vectors where the weights are the free variables*</u>. That is,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2x_2 + x_4 - 3x_5 \\ x_2 \\ -2x_4 + 2x_5 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \underbrace{\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\mathbf{u}} + x_4 \underbrace{\begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}}_{\mathbf{v}} + x_5 \underbrace{\begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}}_{\mathbf{w}}$$

u, v, w 相互獨立

NULL SPACE OF A MATRIX

$$= x_2\mathbf{u} + x_4\mathbf{v} + x_5\mathbf{w} \qquad (3)$$

- Every linear combination of **u**, **v**, and **w** is an element of Nul A.
- Thus $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is a spanning set for Nul A.

basis of Nul A

COLUMN SPACE OF A MATRIX

- **Example 7:** Let $A = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} 3 \\ -2 \\ -1 \\ 0 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 3 \\ -1 \\ 3 \end{bmatrix}$.

$Au = 0$ 확인 No u는 R^4, Col $\in R^3$

a. Determine if **u** is in Nul A. Could **u** be in Col A?

b. Determine if **v** is in Col A. Could **v** be in Nul A?

$[A \ v]$ 칼 No. $\because R^3$ R^4

COLUMN SPACE OF A MATRIX

- **Solution:**
  a. An explicit description of Nul A is not needed here. Simply compute the product $A\mathbf{u}$.

$$A\mathrm{u} = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \\ 3 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

COLUMN SPACE OF A MATRIX

- $\mathbf{u}$ is *not* a solution of $A\mathrm{x} = 0$, so $\mathbf{u}$ is not in Nul$A$.
- Also, with four entries, $\mathbf{u}$ could not possibly be in Col A, since Col A is a subspace of $\mathbb{R}^3$.

  b. Reduce $[A \quad \mathrm{v}]$ to an echelon form.

$$[A \quad \mathrm{v}] = \begin{bmatrix} 2 & 4 & -2 & 1 & 3 \\ -2 & -5 & 7 & 3 & -1 \\ 3 & 7 & -8 & 6 & 3 \end{bmatrix} \sim \begin{bmatrix} 2 & 4 & -2 & 1 & 3 \\ 0 & 1 & -5 & -4 & -2 \\ 0 & 0 & 0 & 17 & 1 \end{bmatrix}$$

- The equation $A\mathrm{x} = \mathrm{v}$ is consistent, so $\mathbf{v}$ is in Col A.

KERNEL AND RANGE OF A LINEAR TRANSFORMATION

- With only three entries, $\mathbf{v}$ could not possibly be in Nul$A$, since Nul$A$ is a subspace of $\mathbb{R}^4$.

- standard basis for R^n $= \{e_1, \cdots, e_n\}$

- **Theorem 6:** The pivot columns of a matrix A form a basis for Col A.

not U

In Exercises 13 and 14, assume that A is row equivalent to B. Find bases for Nul A and Col A.

$$\left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} -5 \\ -5 \\ 0 \\ -5 \end{pmatrix}, \begin{pmatrix} -3 \\ 2 \\ 5 \\ -2 \end{pmatrix} \right\} \quad \text{NOT} \quad \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 5 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 5 \\ 8 \\ -9 \\ 0 \end{pmatrix} \right\}$$

14. $A = \begin{bmatrix} 1 & 2 & -5 & 11 & -3 \\ 2 & 4 & -5 & 15 & 2 \\ 1 & 2 & 0 & 4 & 5 \\ 3 & 6 & -5 & 19 & -2 \end{bmatrix}$,

$B = \begin{bmatrix} 1 & 2 & 0 & 4 & 5 \\ 0 & 0 & 5 & -7 & 8 \\ 0 & 0 & 0 & 0 & -9 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ $\Rightarrow$ $\left(\begin{array}{ccccc|c} 1 & 2 & 0 & 4 & 0 & 0 \\ 0 & 0 & 1 & -7/5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$

$x_1 + 2x_2 + 4x_4 = 0$

$x_3 - 7/5\, x_4 = 0$

$x_5 = 0$

$x_1 = -2x_2 - 4x_4$

$x_3 = 7/5\, x_4$

$x_5 = 0$

$Ax = 0$ or $Bx = 0$

$$x = x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -4 \\ 0 \\ 7/5 \\ 1 \\ 0 \end{pmatrix} \qquad \left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -4 \\ 0 \\ 7/5 \\ 1 \\ 0 \end{pmatrix} \right\}$$

Col A는 A에서만 가져와야하고, ($\because$ Col A $\neq$ Col U)

Nul A 에대해선 $Ax=0$, $Bx=0$ 둘다 해도 된다.

16. Exercises 15–18, find a basis for the space spanned by the given ctors, $\mathbf{v}_1, \ldots, \mathbf{v}_5$. → Identify the linearly independent columns

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ -1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 5 \\ -3 \\ 3 \\ -4 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ -1 \\ 1 \end{bmatrix}$$

$$\begin{pmatrix} 1 & -2 & 6 & 5 & 0 \\ 0 & 1 & -1 & -3 & 3 \\ 0 & -1 & 2 & 3 & -1 \\ 1 & 1 & -1 & -4 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & -2 & 6 & 5 & 0 \\ 0 & 1 & -1 & -3 & 3 \\ 0 & -1 & 2 & 3 & -1 \\ 0 & 3 & -7 & -9 & 1 \end{pmatrix} \to \begin{bmatrix} 1 & -2 & 6 & 5 & 0 \\ 0 & 1 & -1 & -3 & 3 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 6 \\ -1 \\ 2 \\ -1 \end{pmatrix} \right\}$$

Identify linearly Independent column vectors. make matrix and echelon it.

Consider the polynomials $\mathbf{p}_1(t) = 1 + t^2$ and $\mathbf{p}_2(t) = 1 - t^2$. Is $\{\mathbf{p}_1, \mathbf{p}_2\}$ a linearly independent set in $\mathbb{P}_3$? Why or why not?

Set $c_1 p_1 + c_2 p_2 = 0$

$$c_1(1+t^2) + c_2(1-t^2) = c_1 + c_2 + (c_1 - c_2)t^2 = 0$$

Then $c_1 + c_2 = 0$

+) $c_1 - c_2 = 0$

$2c_1 = 0$

$c_2 = 0$

$\boxed{c_1 = 0}$

get $c_1 p_1 + c_2 p_2 = 0$

$$c_1 + c_2 + (c_1 - c_2)t^2 = 0$$

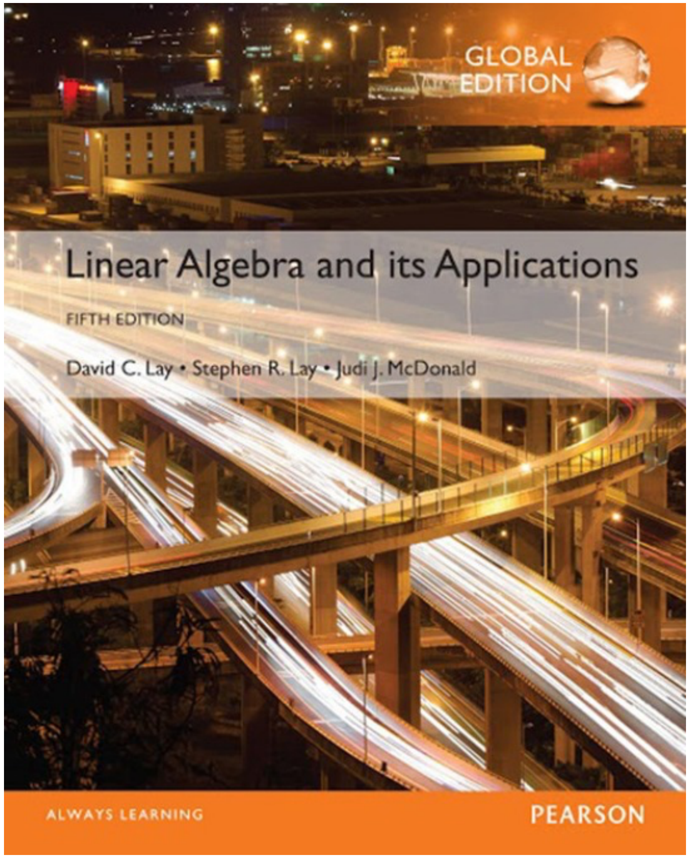
$\therefore$ $c_1 + c_2 = 0$, $c_1 - c_2 = 0$.

$\therefore$ $c_1 = c_2 = 0$

4 Vector Spaces

4.3

LINEARLY INDEPENDENT SETS; BASES



STANDARD BASIS

- Let $\mathbf{e}_1, \ldots, \mathbf{e}_n$ be the columns of the $n \times n$ matrix, I_n.
- That is,

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \ldots, e_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

- The set $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ is called the **standard basis** for $\mathbb{R}^n$.

BASIS FOR COL B

- **Example 8:** Find a basis for Col B, where

$$B = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_5 \end{bmatrix} = \begin{bmatrix} 1 & 4 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- **Solution:** Each nonpivot column of B is a linear combination of the pivot columns.
- In fact, $\mathbf{b}_2 = 4\mathbf{b}_1$ and $\mathbf{b}_4 = 2\mathbf{b}_1 - \mathbf{b}_3$.
- By the Spanning Set Theorem, we may discard $\mathbf{b}_2$ and $\mathbf{b}_4$, and $\{\mathbf{b}_1, \mathbf{b}_3, \mathbf{b}_5\}$ will still span Col B.

BASIS FOR COL *B*

- Let

$$S = \{b_1, b_3, b_5\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

- Since $b_1 \neq 0$ and no vector in S is a linear combination of the vectors that precede it, S is linearly independent. (Theorem 4).

- Thus S is a basis for Col B.

BASES FOR NUL *A* AND COL *A*

- **Theorem 6**: The pivot columns of a matrix A form a basis for Col A.

↳ not its echelon form

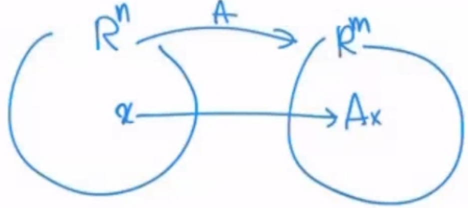
- **Proof:** Let B be the reduced echelon form of A.
- The set of pivot columns of B is linearly independent, for no vector in the set is a linear combination of the vectors that precede it.
- Since A is row equivalent to B, the pivot columns of A are linearly independent as well, because any linear dependence relation among the columns of A corresponds to a linear dependence relation among the columns of B.

BASES FOR NUL *A* AND COL *A*

- For this reason, every nonpivot column of A is a linear combination of the pivot columns of A.
- Thus the nonpivot columns of a may be discarded from the spanning set for Col A, by the Spanning Set Theorem.
- This leaves the pivot columns of A as a basis for Col A.

BASES FOR NUL *A* AND COL *A*

- **Warning:** The pivot columns of a matrix A are evident when A has been reduced only to echelon form.
- But, be careful to use the pivot columns of A itself for the basis of Col A.
- Row operations can change the column space of a matrix.
- The columns of an echelon form B of A are often not in the column space of A.



$A \xrightarrow{\text{row}} U$ ~ echelon form

$\mathrm{Nul}\,A = \{x \mid \underline{Ax} = 0\} = \{x \mid Ux = 0\} = \mathrm{Nul}\,U$

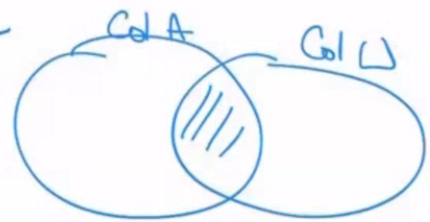
basis of Nul A = basis of Nul U

dim Nul A = dim Nul U

Col A = span{columns of A}

Col U = span{columns of U}

Col A $\neq$ Col U



basis of Col A = {pivot columns of A} $\neq$

basis of Col U = {pivot columns of U}

dim Col A = rank = # pivots = dim Col U

dim Col A + dim Nul A = # columns = n

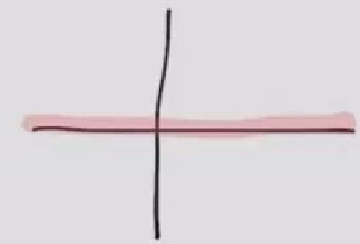
Given $v_1, \cdots, v_p$ in V, $v_1 \cdots, v_p$ are said to be linearly independent

if $c_1 v_1 + \cdots + c_p v_p = 0$ implies $c_1 = \cdots = c_p = 0$

(i.e. if $c_1 = \cdots = c_p = 0$ are only weights for $c_1 v_1 + \cdots + c_p v_p = 0$)

ex) Given P_2, $1, x, x^2 \in P_2$

$$\underbrace{c_1(1) + c_2(x) + c_3(x^2)}_{} = \underline{0}$$



$c_1 + c_2 x + c_3 x^2 = 0$ (zero function)

$$\left.\begin{array}{ll} x=1 & c_1 + c_2 + c_3 = 0 \\ x=0 & c_1 = 0 \\ x=-1 & c_1 - c_2 + c_3 = 0 \end{array}\right) \rightarrow c_1 = c_2 = c_3 = 0$$

$\vdots$

$\Rightarrow$ $1, x, x^2$ are linearly independent.

※ also $\{1, 1+x, 1+x+x^2\}$ is basis

+ Span P_2, $\{1, x, x^2\}$ $\rightarrow$ basis

$\dim P_2 = 3$

$V = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \ a, b, c, d \in \mathbb{R} \right\}$

basis dim $= 4$

$$\underbrace{c_1 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, + c_2 \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, + c_3 \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, + c_4}_{\begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix}} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$c_1 = c_2 = c_3 = c_4 = 0$ → linear independent

$$c_1 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + c_2 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + c_3 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + c_4 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} c_1 + c_3 & c_2 + c_4 \\ c_2 - c_4 & c_1 - c_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$C_1 = C_2 = C_3 = C_4$ → linearly independent

basis → dim $= 4$

- The coordinates of x relative to the basis B, coordinate vector of x (relative to B)
- coordinate mapping
- change-of-coordinate matrix in R^n

If $x = c_1 v_1 + \cdots + c_p v_p$, $\begin{pmatrix} c_1 \\ \vdots \\ c_p \end{pmatrix} = [x]_B$ where $B = \{v_1, \cdots, v_p\}$

$$\begin{aligned} V &\longrightarrow R^n \\ x &\longmapsto [x]_B \end{aligned}$$

$B = \{v_1, \cdots, v_n\}$ for R^n $\quad C = \{e_1, \cdots, e_n\}$ standard basis

$P = [v_1 \cdots v_n]$ satisfies

$$P[x]_B = x = [x]_C$$

$$A[x]_B = [x]_C$$

In Exercises 9 and 10, find the ~~change-of-coordinates matrix~~ from $\mathcal{B}$ to the standard basis in $\mathbb{R}^n$.

9. $\mathcal{B} = \left\{ \begin{bmatrix} 2 \\ -9 \end{bmatrix}, \begin{bmatrix} 1 \\ 8 \end{bmatrix} \right\}$

$P = \begin{pmatrix} 2 & 1 \\ -9 & 8 \end{pmatrix}$

10. $\mathcal{B} = \left\{ \begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -5 \end{bmatrix}, \begin{bmatrix} 8 \\ -2 \\ 7 \end{bmatrix} \right\}$

$P = \begin{pmatrix} 3 & 2 & 8 \\ -1 & 0 & -2 \\ 4 & -5 & 7 \end{pmatrix}$

21. Let $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ -4 \end{bmatrix}, \begin{bmatrix} -2 \\ 9 \end{bmatrix} \right\}$. Since the coordinate mapping determined by $\mathcal{B}$ is a linear transformation from $\mathbb{R}^2$ into $\mathbb{R}^2$, this mapping must be implemented by some 2×2 matrix A. Find it. [*Hint:* Multiplication by A should transform a vector $\mathbf{x}$ into its coordinate vector $[\mathbf{x}]_{\mathcal{B}}$.]

$\mathbb{R} \xrightarrow{T} \mathbb{R}^2$

$x \longmapsto [x_B]$

$Ax = [x]_B$

$A = \begin{bmatrix} T(e_1) & T(e_2) \end{bmatrix}$

$[e_1]_B = \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right]_B$ 구하려면 $\begin{bmatrix} 1 & -2 \\ -4 & 9 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ 구하라 $\to \begin{bmatrix} 1 & 0 & 9 \\ 0 & 1 & 4 \end{bmatrix}$ $\therefore c_1 = 9,\ c_2 = 4$ $\begin{bmatrix} 9 \\ 4 \end{bmatrix}$

$[e_2]_B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}_B$ $\begin{bmatrix} 1 & -2 \\ -4 & 9 \end{bmatrix} \begin{bmatrix} c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $\to \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix}$ $c_3 = 2,\ c_4 = 1$ $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$\therefore A = \begin{bmatrix} 9 & 2 \\ 4 & 1 \end{bmatrix}$

그냥 역함수해도.

그냥 $\begin{bmatrix} 1 & -2 \\ -4 & 9 \end{bmatrix}^{-1}$ 아닌가?

31.

There are 2 independent polynomials, which cannot span the 3-dimensional space.

31. Use coordinate vectors to test ~~whether the following sets of polynomials span~~ $\mathbb{P}_2$. Justify your conclusions.

a. $\boxed{1-3t+5t^2, -3+5t-7t^2}, -4+5t-6t^2, 1-t^2$

b. $5t+t^2, 1-8t-2t^2, -3+4t+2t^2, 2-3t$

$$\begin{pmatrix} 1 & -3 & -4 & 1 \\ -3 & 5 & 5 & 0 \\ 5 & -7 & -6 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -3 & -4 & 1 \\ 0 & -4 & -7 & 3 \\ 0 & 8 & 14 & -6 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -3 & -4 & 1 \\ 0 & -4 & -7 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

rank = 2

$$\begin{pmatrix} 0 & 1 & -3 & 2 \\ 5 & -8 & 4 & -3 \\ 1 & -2 & 2 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -2 & 2 & 0 \\ 5 & -8 & 4 & -3 \\ 0 & 1 & -3 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -2 & 2 & 0 \\ 0 & 2 & -6 & -3 \\ 0 & 1 & -3 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -2 & 2 & 0 \\ 0 & 1 & -3 & 2 \\ 0 & 2 & -6 & -3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -2 & 2 & 0 \\ 0 & 1 & -3 & 2 \\ 0 & 0 & 0 & -7 \end{pmatrix} \Rightarrow$$

There are 3 independent polynomials

$\Rightarrow$ They form a basis for $\mathbb{P}_2$.

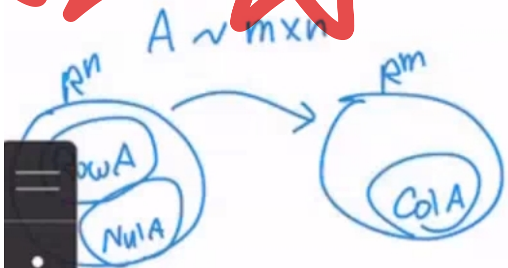
a. $1, t, t^2$ 의 계수를 Column에 놓는다.

→ rank = 2

2 column 만 독립.

cannot span 3 dimensional space

b. 같은 방법으로.



$A \xrightarrow{\text{row}} B$

basis for Col A = pivot columns of A

Col A $\neq$ Col B

dim Col A = dim Col B = # pivot columns = rank

$\underline{\text{Nul } A} = \underline{\text{Nul } B}$

(dim Nul A = dim Nul B = # free = n − rank

basis for Nul A = basis for Nul B)

$\underline{\text{Row } A}$ = $\underline{\text{Row } B}$

(Col A^T) (Col B^T)

dim Row A = dim Row B = # nonzero rows = # pivot columns = rank

(basis for Row A = basis for Row B)

$\hookrightarrow$ nonzero rows in echelon form of A

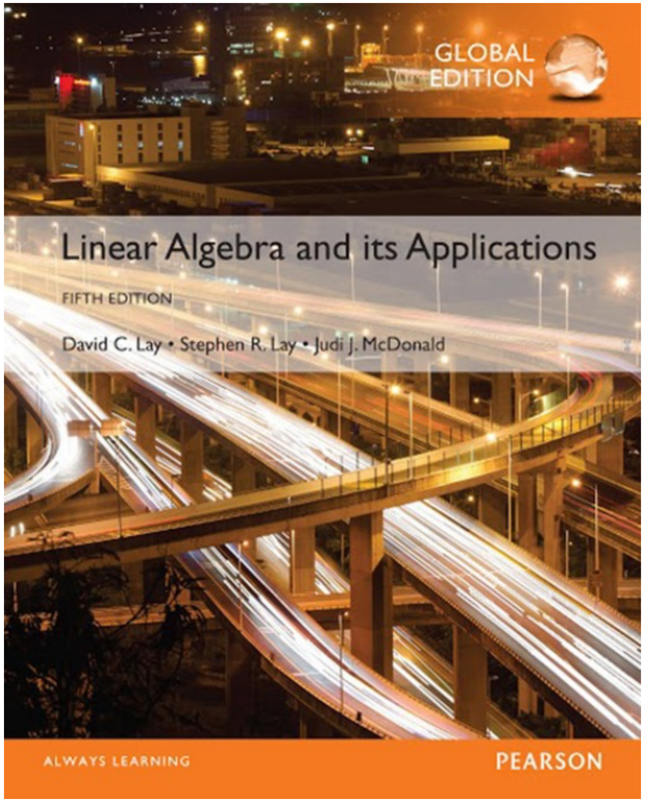
---

dim Row A = dim Col A but $\begin{cases} \text{Col } A \subseteq \underline{\underline{\mathbb{R}^m}} \\ \text{Row } A = \text{Col } A^T \subseteq \mathbb{R}^n \end{cases}$

4 Vector Spaces

4.4

COORDINATE SYSTEMS



THE UNIQUE REPRESENTATION THEOREM

- **Definition:** Suppose $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ is a basis for V and $\mathbf{x}$ is in V. **The coordinates of x relative to the basis $\mathcal{B}$** (or the **$\mathcal{B}$-coordinate of x**) are the weights $c_1, \ldots, c_n$ such that $\mathbf{x} = c_1\mathbf{b}_1 + \ldots + c_n\mathbf{b}_n$.

In P_2, $B=\{1, x, x^2\}$

$x^2-x = 0\cdot 1+(-1)x+1\cdot x^2$ $\qquad [x^2-x]_B=\begin{pmatrix}0\\-1\\1\end{pmatrix}$

$-x+3 = 3\cdot 1+(-1)\cdot x+0\cdot x^2$ $\qquad [-x+3]_B=\begin{pmatrix}3\\-1\\0\end{pmatrix}$

$C=\{1, 1+x, 1+x+x^2\}$

$x^2-x = 1\cdot 1+(-2)(1+x)+1\,(1+x+x^2)$ $\qquad [x^2-x]_C=\begin{pmatrix}1\\-2\\1\end{pmatrix}$

$-x+3 = 4\cdot 1+(-1)(1+x)+0\,(1+x+x^2)$ $\qquad [-x+3]_C=\begin{pmatrix}4\\-1\\0\end{pmatrix}$

THE UNIQUE REPRESENTATION THEOREM

- If $c_1, \ldots, c_n$ are the **B-coordinates of x**, then the vector in $\mathbb{R}^n$

$$[\mathrm{x}]_{\mathrm{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

is the **coordinate vector of x (relative to B)**, or the **B-coordinatevector ofx**.

$[x]_c$

일반적인식 x를 Vector로 바꾼것

- The mapping $\mathrm{x} \mapsto [\mathrm{x}]_{\mathrm{B}}$ is the **coordinate mapping (determined by B)**.

Coordinate Mapping is Onto $\mathbb{R}^n$ and 1-1

COORDINATES IN $\mathbb{R}^n$

- **Example 1:** Let $\mathbf{b}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$, and

$\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$. Find the coordinate vector $[\mathbf{x}]_{\mathcal{B}}$ of $\mathbf{x}$ relative to $\mathcal{B}$.

- **Solution:** The $\mathcal{B}$-coordinate c_1, c_2 of $\mathbf{x}$ satisfy

$$\underbrace{c_1\begin{bmatrix} 2 \\ 1 \end{bmatrix}}_{\mathbf{b}_1} + \underbrace{c_2\begin{bmatrix} -1 \\ 1 \end{bmatrix}}_{\mathbf{b}_2} = \underbrace{\begin{bmatrix} 4 \\ 5 \end{bmatrix}}_{\mathbf{x}}$$

$\begin{bmatrix}2 & -1\\ 1 & 1\end{bmatrix}[x]_B = [x]_C$

or

$$\begin{bmatrix}2 & -1\\ 1 & 1\end{bmatrix}\begin{bmatrix}c_1\\ c_2\end{bmatrix} = \begin{bmatrix}4\\ 5\end{bmatrix} = 4\begin{bmatrix}1\\ 0\end{bmatrix} + 5\begin{bmatrix}0\\ 1\end{bmatrix} \quad (3)$$

$\mathbf{b}_1$ $\mathbf{b}_2$ $\mathbf{x}$

$\begin{bmatrix}2 & -1\\ 1 & 1\end{bmatrix}[x]_B = [x]_C$

$C = \{e_1, e_2\}$

$C = \left\{\begin{bmatrix}1\\ 0\end{bmatrix}, \begin{bmatrix}0\\ 1\end{bmatrix}\right\}$

Transforms $b \to c$

- This equation can be solved by row operations on an augmented matrix or by using the inverse of the matrix on the left.
- In any case, the solution is $c_1 = 3, c_2 = 2$.
- Thus $\mathbf{x} = 3\mathbf{b}_1 + 2\mathbf{b}_2$ and

$$[\mathbf{x}]_B = \begin{bmatrix}c_1\\ c_2\end{bmatrix} = \begin{bmatrix}3\\ 2\end{bmatrix}$$

COORDINATES IN $\mathbb{R}^n$

- The matrix in (3) changes the $\mathcal{B}$-coordinates of a vector **x** into the standard coordinates for **x**.

COORDINATES IN $\mathbb{R}^n$

- An analogous change of coordinates can be carried out in $\mathbb{R}^n$ for a basis $B = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$.

이렇게 만든다.

- Let $P_B = [\mathbf{b}_1 \quad \mathbf{b}_2 \quad \cdots \quad \mathbf{b}_n]$

in R^n

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

In R^n, $B = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}$ $\quad C = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

$$\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}_B = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \longleftrightarrow \left[\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}\right]_C = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$$

$$P_B \underset{[x]_B}{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}} = \underset{[x]_C}{\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}}$$

$$P_B = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

Converting 하는 함수

COORDINATES IN $\mathbb{R}^n$

- Then the vector equation

$$\mathbf{x} = c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + \ldots + c_n\mathbf{b}_n$$

is equivalent to

(4) $[x]_C = $ $$\mathbf{x} = P_{\mathcal{B}}[\mathbf{x}]_{\mathcal{B}}$$

$C = \{e_1 \ldots e_n\}$

- $P_{\mathcal{B}}$ is called the **change-of-coordinates matrix** from $\mathcal{B}$ to the standard basis in $\mathbb{R}^n$.

- Left-multiplication by $P_{\mathcal{B}}$ transforms the coordinate vector $[\mathbf{x}]_{\mathcal{B}}$ into $\mathbf{x}$.

$\cdot x$: $\mathcal{B}$ 는 basis

- Since the columns of $P_{\mathcal{B}}$ form a basis for $\mathbb{R}^n$, $P_{\mathcal{B}}$ is invertible (by the Invertible Matrix Theorem).

COORDINATES IN $\mathbb{R}^n$

- Left-multiplication by $P_{\mathcal{B}}^{-1}$ converts **x** into its B-coordinate vector:

$$P_{\mathcal{B}}^{-1}\mathbf{x} = [\mathbf{x}]_{\mathcal{B}}$$

$[x_B] = P^{-1}[x]_C$
$= P^{-1}x$

standard basis to
B basis

- The correspondence $\mathbf{x} \mapsto [\mathbf{x}]_{\mathcal{B}}$, produced by $P_{\mathcal{B}}^{-1}$, is the coordinate mapping.

- Since $P_{\mathcal{B}}^{-1}$ is an invertible matrix, the coordinate mapping is a one-to-one linear transformation from $\mathbb{R}^n$ onto $\mathbb{R}^n$, by the Invertible Matrix Theorem.

THE COORDINATE MAPPING

- **Example 7:** Let $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \ \mathbf{x} = \begin{bmatrix} 3 \\ 12 \\ 7 \end{bmatrix},$

  and $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2\}$. Then $\mathcal{B}$ is a basis for $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$. Determine if **x** is in H, and if it is, find the coordinate vector of **x** relative to $\mathcal{B}$.

THE COORDINATE MAPPING

- **Solution:** If $\mathbf{x}$ is in H, then the following vector equation is consistent:

$$c_1\begin{bmatrix}3\\6\\2\end{bmatrix}+c_2\begin{bmatrix}-1\\0\\1\end{bmatrix}=\begin{bmatrix}3\\12\\7\end{bmatrix}$$

- The scalars c_1 and c_2, if they exist, are the B-coordinates of $\mathbf{x}$.

THE COORDINATE MAPPING

- Using row operations, we obtain

$$\begin{bmatrix} 3 & -1 & 3 \\ 6 & 0 & 12 \\ 2 & 1 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

- Thus $c_1 = 2$, $c_2 = 3$ and $[\mathbf{x}]_B = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

Nonlinear Problem $\longrightarrow$ Linear Problem $\quad f(x) = x^3 + x + 1$

$B = \{x^3, x^2, x, 1\}$

일반적인 문제를

- Matrix problem

$$\begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \in \mathbb{R}^4$$

Matrix 문제로 만들수 있다.

$V \sim$ n-dimensional vector sp.

$$V \xrightarrow{T} V$$

$\updownarrow$ coordinate mapping

$$\mathbb{R}^3 \xrightarrow{A} \mathbb{R}^3$$

$$\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

$$A \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}$$

$$0 \cdot x^2 + 2 \cdot x + (-1) \cdot 1 = \underline{2x - 1}$$

ex) $V = P_2$

$p(x) = x^2 - x + 1$

$\frac{d}{dx} p(x) = 2x - 1$

미분하는 Matrix 만들기.

$B = \{x^2, x, 1\}$

$$[p]_B = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \in \mathbb{R}^3$$

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

- **Theorem 9**: If a vector space V has a basis $B = \{b_1, \ldots, b_n\}$, then any set in V containing more than n vectors must be linearly dependent.

- **Theorem 10**: If a vector space V has a basis of n vectors, then every basis of V must consist of exactly n vectors.

- **Theorem 11**: Let H be a subspace of a finite-dimensional vector space V. Any linearly independent set in H can be expanded, if necessary, to a basis for H. Also, H is finite-dimensional and

$$\dim H \leq \dim V$$

Determine the dimensions of Nul A and Col A for the matrices shown in Exercises 13–18.

13. $A = \begin{bmatrix} 1 & -6 & 9 & 0 & -2 \\ 0 & 1 & 2 & -4 & 5 \\ 0 & 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

$\uparrow \quad \uparrow \quad \uparrow$

$\underbrace{\text{pivot columns}}$

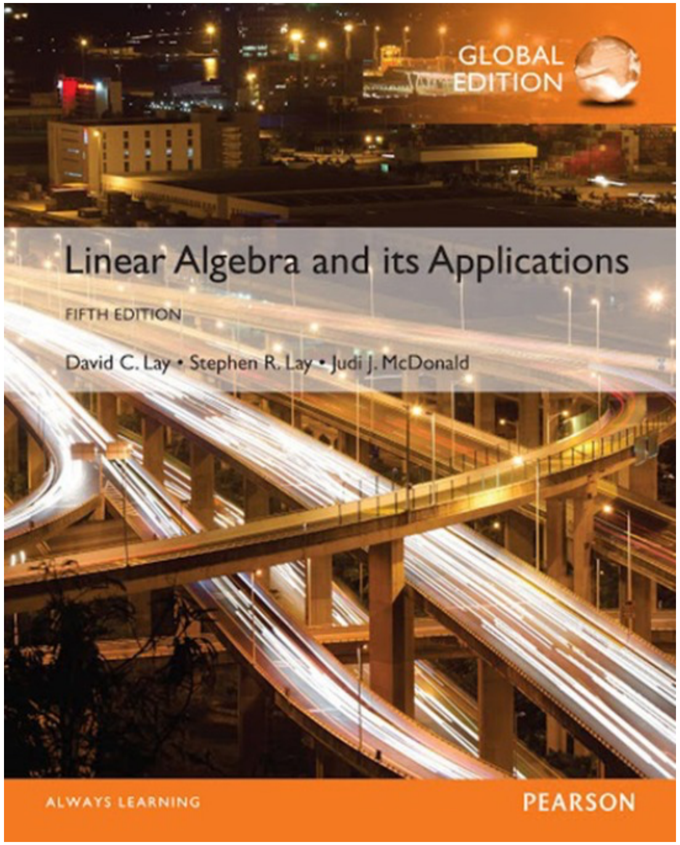
dim Col A = 3

dim Nul A = 5 − 3 = 2

4 Vector Spaces

4.5

THE DIMENSION OF A VECTOR SPACE



$P_2 : \{1, x, x^2\}$ basis → $\{1, 1+x, -x^2-x, x-x^2-1\}$ → dependent

DIMENSION OF A VECTOR SPACE

- **Theorem 9:** If a vector space V has a basis $B = \{b_1, \dots, b_n\}$, then any set in V containing more than n vectors must be linearly dependent.

- **Proof:** Let $\{\mathbf{u}_1, \dots, \mathbf{u}_p\}$ be a set in V with more than n vectors. $p > n$

$p > n$ 이므로.

- The coordinate vectors $[\mathbf{u}_1]_B, \dots, [\mathbf{u}_p]_B$ form a linearly dependent set in $\mathbb{R}^n$, because there are more vectors (p) than entries (n) in each vector.

DIMENSION OF A VECTOR SPACE

- So there exist scalars $c_1, \ldots, c_p$, not all zero, such that

$$c_1[\mathbf{u}_1]_B + \ldots + c_p[\mathbf{u}_p]_B = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

Linear Dependency 가정

The zero vector in $\mathbb{R}^n$

- Since the coordinate mapping is a linear transformation,

$$[c_1\mathbf{u}_1 + \ldots + c_p\mathbf{u}_p]_B = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

DIMENSION OF A VECTOR SPACE

- The zero vector on the right displays the n weights needed to build the vector $c_1\mathbf{u}_1 + \ldots + c_p\mathbf{u}_p$ from the basis vectors in B.

- That is, $c_1\mathbf{u}_1 + \ldots + c_p\mathbf{u}_p = 0 \cdot \mathbf{b}_1 + \ldots + 0 \cdot \mathbf{b}_n = \mathbf{0}$.

- Since the c_i are not all zero, $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ is linearly dependent.

* Coordinate Mapping → One to One & Onto

DIMENSION OF A VECTOR SPACE

- Theorem 9 implies that if a vector space V has a basis $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$, then each linearly independent set in V has no more than n vectors.

DIMENSION OF A VECTOR SPACE

- **Theorem 10:** If a vector space V has a basis of n vectors, then every basis of V must consist of exactly n vectors.

- **Proof:** Let B_1 be a basis of n vectors and B_2 be any other basis (of V). $\#$ in $B_2 \leq \#$ in B_1 (아니면 linearly dependent)
- Since B_1 is a basis and B_2 is linearly independent, B_2 has no more than n vectors, by Theorem 9.
- Also, since B_2 is a basis and B_1 is linearly independent, B_2 has at least n vectors. $\#$ in $B_1 \leq \#$ in B_2 (왜냐 마찬가지)
- Thus B_2 consists of exactly n vectors. $\therefore \#$ in $B_1 = \#$ in B_2

Example of Dimension

$$\dim P_2 = 3$$

$$\dim P_n = n+1$$

$$\dim \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \ a, b, c, d \in \mathbb{R} \right\} = 4$$

$$\dim \{ m \times n \text{ real matrices} \} = mn$$

DIMENSION OF A VECTOR SPACE

- **Definition:** If V is spanned by a finite set, then V is said to be **finite-dimensional**, and the **dimension** of V, written as dim V, is the number of vectors in a basis for V. The dimension of the zero vector space $\{\mathbf{0}\}$ is defined to be zero. If V is not spanned by a finite set, then V is said to be **infinite-dimensional**.

- **Example 3:** Find the dimension of the subspace

$\mathbb{R}^2 \to 2$

$\mathbb{R}^3 \to 3$

$\mathbb{R}^n \to n$ dimensional

$$H = \left\{ \begin{bmatrix} a-3b+6c \\ 5a+4d \\ b-2c-d \\ 5d \end{bmatrix} : a,b,c,d \text{ in } \mathbb{R} \right\}$$

$$= a\begin{pmatrix}1\\5\\0\\0\end{pmatrix} + b\begin{pmatrix}-3\\0\\1\\0\end{pmatrix} + c\begin{pmatrix}6\\0\\-2\\0\end{pmatrix} + d\begin{pmatrix}0\\4\\-1\\5\end{pmatrix} \to$$

linearly independent 的 spanning set

→ basis

$$= \left\{ \begin{pmatrix}1\\5\\0\\0\end{pmatrix} \begin{pmatrix}-3\\0\\1\\0\end{pmatrix} \cancel{\begin{pmatrix}6\\0\\-2\\0\end{pmatrix}} \begin{pmatrix}0\\4\\-1\\5\end{pmatrix} \right\}$$

DIMENSION OF A VECTOR SPACE

- H is the set of all linear combinations of the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 5 \\ 0 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 6 \\ 0 \\ -2 \\ 0 \end{bmatrix}, \mathbf{v}_4 = \begin{bmatrix} 0 \\ 4 \\ -1 \\ 5 \end{bmatrix}$$

- Clearly, $\mathbf{v}_1 \neq 0$, $\mathbf{v}_2$ is not a multiple of $\mathbf{v}_1$, but $\mathbf{v}_3$ is a multiple of $\mathbf{v}_2$.
- By the Spanning Set Theorem, we may discard $\mathbf{v}_3$ and still have a set that spans H.

SUBSPACES OF A FINITE-DIMENSIONAL SPACE

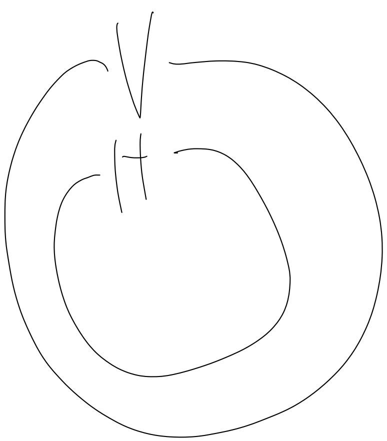
- Finally, $\mathbf{v}_4$ is not a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$.
- So $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4\}$ is linearly independent and hence is a basis for H.
- Thus $\dim H = 3$.

$$\begin{vmatrix} 1 & -3 & 6 & 0 \\ 5 & 0 & 0 & 4 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & 5 \end{vmatrix} \quad \begin{vmatrix} 1 & -3 & 6 & 0 \\ 0 & 15 & -30 & 4 \\ & 1 & -2 & -1 \\ & & & 5 \end{vmatrix} \rightarrow \begin{vmatrix} 1 & -3 & 6 & 0 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & 19 \\ 0 & 0 & 0 & 5 \end{vmatrix} \leftarrow \text{column 3 dependent}$$

THE BASIS THEOREM

- **Theorem 11**: Let H be a subspace of a finite-dimensional vector space V. Any linearly independent set in H can be expanded, if necessary, to a basis for H. Also, H is finite-dimensional and

$$\dim H \leq \dim V$$



basis vectors for H

basis vectors for V

DIMENSIONS OF NUL *A* AND COL *A*

- **Example 5:** Find the dimensions of the null space and the column space of $\cancel{A\mathbf{x} = \mathbf{0}}$

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$$

DIMENSIONS OF NUL A AND COL A

- **Solution:** Row reduce the augmented matrix $\begin{bmatrix} A & 0 \end{bmatrix}$ to echelon form:

$$\begin{bmatrix} 1 & -2 & 2 & 3 & -1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- There are three free variable—x_2, x_4 and x_5.
- Hence the dimension of Nul A is 3.
- Also $\dim \text{Col } A = 2$ because A has two pivot columns.

$n = 5$

invertible $\text{Col}\,A = 3$ $\dim \text{Nul}\,A = 0$

Ex) Suppose $A \sim 3\times 3$ Invertible matrix $\xrightarrow{\text{row}} I$

Define $B = \left(\begin{array}{cc} A & A \\ \hline A & A \end{array}\right) \xrightarrow{\text{row } A} \begin{pmatrix} I & I \\ I & I \end{pmatrix} \to \underbrace{\begin{pmatrix} \boxed{I} & I \\ 0\,0\,0 & 0\,0\,0 \\ 0\,0\,0 & 0\,0\,0 \\ 0\,0\,0 & 0\,0\,0 \end{pmatrix}}_{6}$

$\dim \text{Col}\, B = ?$ 3

$\dim \text{Nul}\, B = ?$ $6-3=3$

$\dim \text{Col}\, B = 3$

$\dim \text{Nul}\, B = 3$

$B = \begin{pmatrix} A & A \\ A & A \end{pmatrix}$

$\to \begin{pmatrix} I & I \\ I & I \end{pmatrix} \to \begin{pmatrix} 1 & & & 1 & & \\ & 1 & & & 1 & \\ 0 & 0 & 1 & & & 1 \\ & 0 & 0 & 0 & & \\ & & \ddots & & \ddots & \end{pmatrix}$ (ERO를 이용해서 pivot 아래 제거하면...)

- Row space = {linear combinations of rows of A} = span {rows of A}

- **Theorem 13:** If two matrices A and B are row equivalent, then their row spaces are the same. If B is in echelon form, the nonzero rows of B form a basis for the row space of A as well as for that of B.

→ Each row of A is a linear combination of rows of B.

ex) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

row exchange 때문에 echelon form에서 고려하라.

- **Theorem 14:** The dimensions of the column space and the row space of an $m \times n$ matrix A are equal. This common dimension, the rank of A, also equals the number of pivot positions in A and satisfies the equation

$$\text{rank } A + \underbrace{\dim \text{Nul } A}_{\text{nullity}} = n$$

There are n pivots of echelon form (U) $\Leftarrow A \sim I \Leftarrow A$ is invertible (nonsingular)

Theorem: Let A be an $n \times n$ matrix. Then the following statements are each equivalent to the statement that A is an invertible matrix.

m. The columns of A form a basis of $\mathbb{R}^n$.

n. Col $A = \mathbb{R}^n$

o. Dim Col $A = n$

p. Rank$A = n$

q. Nul $A = \{0\}$

r. Dim Nul $A = 0$

$CA = I$ $\Downarrow$ uniqueness $\Downarrow$

$x = 0$ is the only soln of $Ax = 0$

$\Downarrow$

Columns of A are linearly independent

$\Downarrow$

Columns of A form a basis of $\mathbb{R}^n$

$\Rightarrow$ nullity $= \dim \text{Nul} A = 0$

$\Downarrow$

$\text{Nul} A = \{0\}$

$AC = I$

$\Downarrow$

existence

$\Downarrow$

$Ax = b$ has a solution for each b

$\Downarrow$

Each b is a linear combination of the columns of A

$\Downarrow$

range $=$ span{columns} $= \mathbb{R}^n$

$|A| \neq 0$

pivots $=$ rank $= \boxed{\dim \text{Col} A} = n$

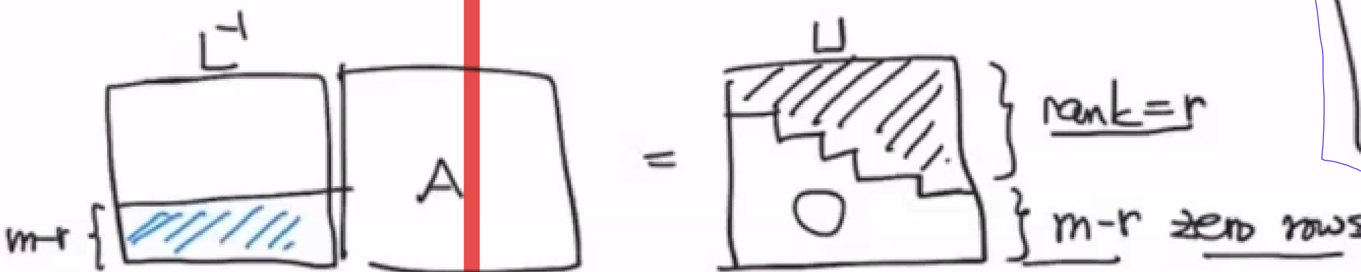
m-r rows at the bottom of L^{-1}

- $\mathrm{Nul}(A^T)$ and its basis

$\{y \mid A^T y = 0\} = \{y \mid y^T A = 0\}$

Suppose $A = LU$

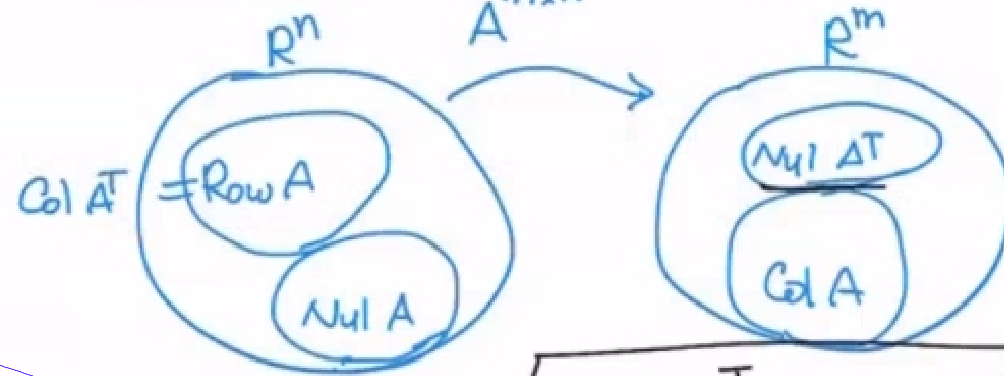
Since L is invertible, $L^{-1}A = U$



(m-r rows at the bottom of L^{-1}) $\times A$ = (zero rows)

↳ independent because they are rows of invertible matrix (L^{-1})

fundamental subspaces of $A_{m \times n}$



rank A^T + nullity A^T

= # columns in A^T

r + nullity A^T = m

||

vectors in a basis for Nul (A^T)

$\boxed{m-r = \dim \mathrm{Nul}\,(A^T)}$

Find bases for Col(A), Row(A), Nul(A), Nul(A^T)

Rows of U ~ a basis for Row (A)

$$A = \underbrace{\begin{bmatrix} 3 & -7 & -2 & 2 \\ -3 & 5 & 1 & 0 \\ 6 & -4 & 0 & -5 \\ -9 & 5 & -5 & 12 \end{bmatrix}}_{} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & -5 & 1 & 0 \\ -3 & 8 & 3 & 1 \end{bmatrix} \begin{bmatrix} 3 & -7 & -2 & 2 \\ 0 & -2 & -1 & 2 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix} = \underline{LU}$$

Columns of A ~ a basis for Col(A)

$Ax = 0 \Rightarrow \underline{x = 0}$

Nul $A \Rightarrow \{\ \}$ + 모두 pivot이 있어서 parametric form 없다.

Nul $A^T \Rightarrow \{\ \}$ + zero row가 없으므로, parametric form 없다.

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & -3 & 1 & 0 \\ -3 & 4 & 2 & 1 \end{pmatrix}$$

- Find bases for Col(A), Row(A), Nul(A), Nul(A^T)

$$A = \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ -4 & -5 & 3 & -8 & 1 \\ 2 & -5 & -4 & 1 & 8 \\ -6 & 0 & 7 & -3 & 1 \end{bmatrix} \rightarrow \begin{pmatrix} 2 & 4 & -1 & 5 & -2 \\ 0 & 3 & 1 & 2 & -3 \\ 0 & -9 & -3 & -4 & 10 \\ 0 & 12 & 4 & 12 & -5 \end{pmatrix} \rightarrow \begin{pmatrix} 2 & 4 & -1 & 5 & -2 \\ 0 & 3 & 1 & 2 & -3 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 5 \end{pmatrix} = LU$$

Col(A)

Rows of U ~ Row A

$[U \quad 0]$ 풀면

Nul $A^T = \{0\}$

No zero row.

basis Nul $A^T = \{\ \}$

$x_4 = x_5 = 0 \qquad 3x_2 + x_3 = 0 \qquad x_2 = -\frac{1}{3}x_3$

$2x_1 + 4(-\frac{1}{3}x_3) - x_3 = 0 \qquad 2x_1 = \frac{7}{3}x_3$

$$x = x_3 \begin{pmatrix} 7/6 \\ -1/3 \\ 1 \\ 0 \\ 0 \end{pmatrix} \sim \text{Nul } A$$

A is row equivalent to $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$

Then, the first two row of A form a basis for Row A

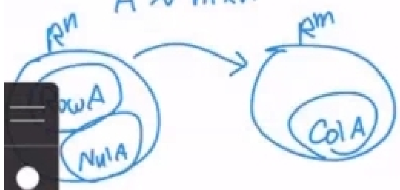
~~True~~ / False ?

ex) $A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 2 \end{pmatrix}$

0 벡터는 basis가 될수 없으므로

→ linearly independent 깨짐

$A \sim m \times n$



$A \xrightarrow{\text{row}} B$

basis for Col A
= pivot columns of A

Col A $\neq$ Col B

dim Col A = dim Col B = # pivot columns
= rank

Nul A = Nul B

(dim Nul A = dim Nul B = # free = n − # rank
basis for Nul A = basis for Nul B)

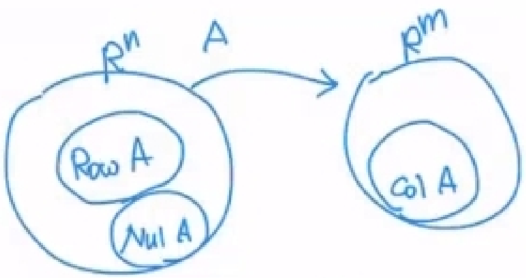
$\underset{(\mathrm{Col}\ A^T)}{\text{Row } A} = \underset{(\mathrm{Col}\ B^T)}{\text{Row } B}$

dim Row A = dim Row B = # nonzero rows = # pivot columns
= rank

(basis for Row A = basis for Row B)

↳ nonzero rows in echelon form of A

dim Row A = dim Col A but $\begin{cases} \mathrm{Col}\ A \subseteq \mathbb{R}^m \\ \mathrm{Row}\ A = \mathrm{Col}\ A^T \subseteq \mathbb{R}^n \end{cases}$



$\text{Row } A = \text{Col } A^T$

ex)

$$\begin{array}{ccc} A & & B & & C \\ \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 2 & 3 & 4 \end{pmatrix} & \rightarrow & \begin{pmatrix} 1 & 1 & 1 \\ \boxed{0 \; 1 \; 2} \\ \boxed{0 \; 1 \; 2} \end{pmatrix} & \rightarrow & \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \end{array}$$

2^{nd} row B = 2^{nd} row of A + (−1) 1^{st} row of A ←

→ 2^{nd} row of A = 2^{nd} row of B + 1^{st} row of A
= 2^{nd} row of B + 2^{nd} row of B

3^{rd} row B = 3^{rd} row of A + (−2) 1^{st} row of A ←

→ 3^{rd} row of A = 3^{rd} row of B + 2 (1^{st} row of A)
= 3^{rd} row of B + 2 (1^{st} row of B)

A의 row vector의 조합

B의 row vector의 조합

a linear combination of rows of A
= a linear combination of rows of B

Row A = Row B = Row C

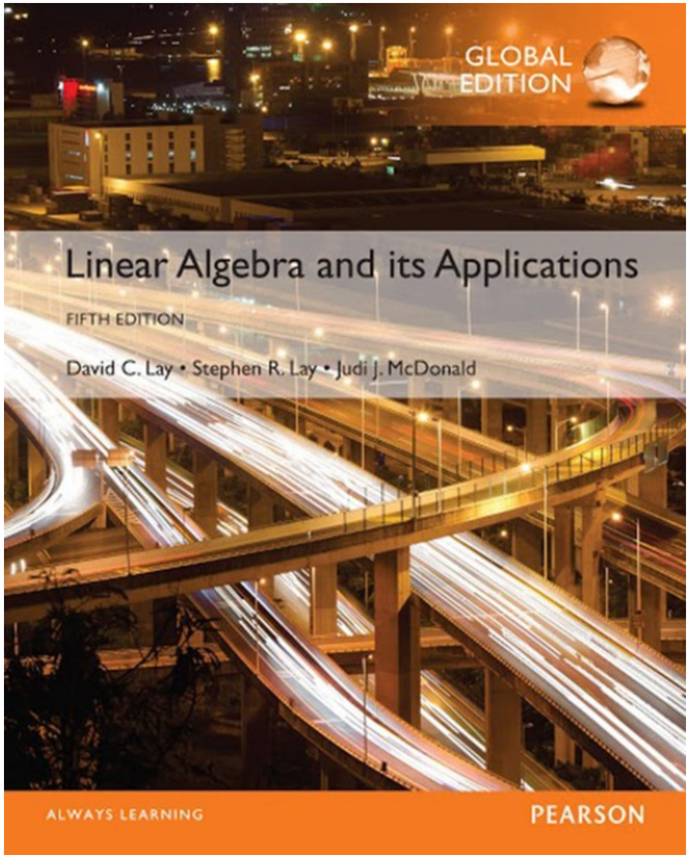
$$\text{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, \cancel{\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}} \right\}$$

basis

4 Vector Spaces

4.6

RANK



PEARSON

THE ROW SPACE

- If A is an $m \times n$ matrix, each row of A has n entries and thus can be identified with a vector in $\mathbb{R}^n$.
- The set of all linear combinations of the row vectors is called the **row space** of A and is denoted by Row A.

  = span {rows of A}
- Each row has n entries, so Row A is a subspace of $\mathbb{R}^n$.
- Since the rows of A are identified with the columns of A^T, we could also write Col A^T in place of Row A.

B is result of a sequence of row operations from A.

THE ROW SPACE

- **Theorem 13:** If two matrices A and B are row equivalent, then their row spaces are the same. If B is in echelon form, the nonzero rows of B form a basis for the row space of A as well as for that of B.

- **Proof:** If B is obtained from A by row operations, the rows of B are linear combinations of the rows of A.

- It follows that any linear combination of the rows of B is automatically a linear combination of the rows of A.

THE ROW SPACE

- Thus the row space of B is contained in the row space of A.
- Since row operations are reversible, the same argument shows that the row space of A is a subset of the row space of B.
- So the two row spaces are the same.

THE ROW SPACE

- If B is in echelon form, its nonzero rows are linearly independent because no nonzero row is a linear combination of the nonzero rows below it. (Apply Theorem 4 to the nonzero rows of B in reverse order, with the first row last).

- Thus the nonzero rows of B form a basis of the (common) row space of B and A.

THE ROW SPACE

- **Example 2:** Find bases for the row space, the column space, and the null space of the matrix

$$\begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & 5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix}$$

- **Solution:** To find bases for the row space and the column space, row reduce A to an echelon form:

THE ROW SPACE

$$A \sim B = \begin{bmatrix} 1 & 3 & -5 & 1 & 5 \\ 0 & 1 & -2 & 2 & -7 \\ 0 & 0 & 0 & -4 & 20 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- By Theorem 13, the first three rows of B form a basis for the row space of A (as well as for the row space of B).
- Thus　Row和 Col [illegible] 比较

  Basis for Row A: $\{(1,3,-5,1,5),(0,1,-2,2,-7),(0,0,0,-4,20)\}$

THE ROW SPACE

- For the column space, observe from B that the pivots are in columns 1, 2, and 4.
- Hence columns 1~~, 2, and 4 of A (not B) form a basis~~ ~~for Col A:~~

$$\text{Basis for Col } A: \left\{ \begin{bmatrix} -2 \\ 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -5 \\ 3 \\ 11 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 7 \\ 5 \end{bmatrix} \right\}$$

- Notice that any echelon form of A provides (in its nonzero rows) a basis for Row A and also identifies the pivot columns of A for Col A.

THE ROW SPACE

- However, for Nul A, we need the *reduced echelon form*.

- Further row operations on B yield

$$A \sim B \sim C = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & -2 & 0 & 3 \\ 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

THE ROW SPACE

- The equation $A\mathrm{x} = 0$ is equivalent to $C\mathrm{x} = 0$, that is,

$$
\begin{aligned}
x_1 + x_3 + x_5 &= 0 \\
x_2 - 2x_3 + 3x_5 &= 0 \\
x_4 - 5x_5 &= 0
\end{aligned}
$$

- So $x_1 = -x_3 - x_5$, $x_2 = 2x_3 - 3x_5$, $x_4 = 5x_5$, with x_3 and x_5 free variables.

THE ROW SPACE

- The calculations show that

$$\text{Basis for Nul } A: \left\{ \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 0 \\ 5 \\ 1 \end{bmatrix} \right\}$$

- Observe that, unlike the basis for Col A, the bases for Row A and Nul A have no simple connection with the entries in A itself.

THE RANK THEOREM

\# of pivot column

\# nonzero rows in echelon form

- **Definition:** The **rank** of A is the dimension of the column space of A.
- Since Row A is the same as Col A^T, the dimension of the row space of A is the rank of A^T.

\# columns

- The dimension of the null space is sometimes called the **nullity** of A.

rank of A + nullity of A = n

- **Theorem 14:** The dimensions of the column space and the row space of an $m \times n$ matrix A are equal. This common dimension, the rank of A, also equals the number of pivot positions in A and satisfies the equation

$$\text{rank } A + \dim \text{Nul } A = n$$

THE RANK THEOREM

- **Example 3:**
  a. If A is a 7×9 matrix with a two-dimensional null space, what is the rank of A?
  b. Could a 6×9 matrix have a two-dimensional null space?

every row pivot

$\rightarrow Ax=b$ solution each $b \in \mathbb{R}^7$

- **Solution:**
  a. Since A has 9 columns, $(\text{rank } A) + 2 = 9$, and hence rank $A = 7$.
  b. No. If a 6×9 matrix, call it B, has a two-dimensional null space, it would have to have rank 7, by the Rank Theorem.

Column pivot은 6이
7개

THE INVERTIBLE MATRIX THEOREM (CONTINUED)

- But the columns of B are vectors in $\mathbb{R}^6$, and so the dimension of Col B cannot exceed 6; that is, rank B cannot exceed 6.

THE INVERTIBLE MATRIX THEOREM (CONTINUED)

- **Theorem:** Let A be an $n \times n$ matrix. Then the following statements are each equivalent to the statement that A is an invertible matrix.

  m. The columns of A form a basis of $\mathbb{R}^n$.

  n. Col $A = \mathbb{R}^n$

  o. Dim Col $A = n$ $\Leftrightarrow$ Dim Row $A = n$

  p. Rank$A = n$

  q. Nul $A = \{0\}$

  r. Dim Nul $A = 0$

$\Updownarrow$

Rows of A form a basis of R^n

$\Updownarrow$

Row $A = R^n$

RANK AND THE INVERTIBLE MATRIX THEOREM

- **Proof:** Statement (m) is logically equivalent to statements (e) and (h) regarding linear independence and spanning.

- The other five statements are linked to the earlier ones of the theorem by the following chain of almost trivial implications:

$$(g) \Rightarrow (n) \Rightarrow (o) \Rightarrow (p) \Rightarrow (r) \Rightarrow (q) \Rightarrow (d)$$

RANK AND THE INVERTIBLE MATRIX THEOREM

- Statement (g), which says that the equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each **b** in $\mathbb{R}^n$, implies (n), because Col A is precisely the set of all **b** such that the equation $A\mathbf{x} = \mathbf{b}$ is consistent.

- The implications $(\mathrm{n}) \Rightarrow (\mathrm{o}) \Rightarrow (\mathrm{p})$ follow from the definitions of dimension and rank.

- If the rank of A is n, the number of columns of A, then $\dim \mathrm{Nul}\, A = 0$, by the Rank Theorem, and so $\mathrm{Nul}\, A = \{\mathbf{0}\}$.

RANK AND THE INVERTIBLE MATRIX THEOREM

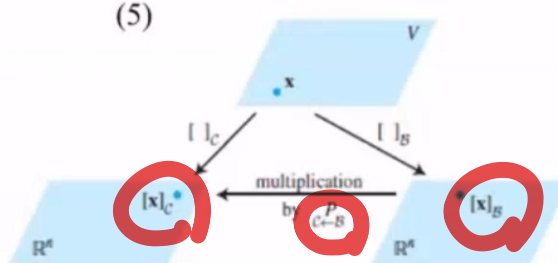
- Thus (p) $\Rightarrow$ (r) $\Rightarrow$ (q).
- Also, (q) implies that the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution, which is statement (d).
- Since statements (d) and (g) are already known to be equivalent to the statement that A is invertible, the proof is complete.

- **Theorem 15**: Let $\beta = \{b_1, \ldots, b_n\}$ and $C = \{c_1, \ldots, c_2\}$ for a vector space V. Then there is a unique $n \times n$ matrix $\overset{P}{c \leftarrow \beta}$ such that

$$[x]_C = \overset{P}{c \leftarrow \beta}\,[x]_\beta \qquad (4)$$

- The columns of $\overset{P}{c \leftarrow \beta}$ are the C-coordinate vectors of the vectors in the basis β. That is,

$$\overset{P}{c \leftarrow \beta} = [\ [b_1]_C [b_2]_C \quad \cdots \quad [b_n]_C] \qquad (5)$$



- Find the change-of-coordinates matrix from B to C and the change-of-coordinates matrix from C to B.

9. $\mathbf{b}_1 = \begin{bmatrix} -6 \\ -1 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$, $\mathbf{c}_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, $\mathbf{c}_2 = \begin{bmatrix} 6 \\ -2 \end{bmatrix}$ basis

$B \to C$

$C \to B$

$B \to C$: $\left[[b_1]_C \ [b_2]_C \right]$

$[b_1]_C = \begin{pmatrix} d_1 \\ \beta_1 \end{pmatrix}$ $\quad d_1 c_1 + \beta_1 c_2 = b_1 \quad [c_1 c_2 b_1] = \begin{bmatrix} 2 & 6 & -6 \\ -1 & -2 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 9 \\ 0 & 1 & -4 \end{bmatrix}$

$[b_2]_C = \begin{pmatrix} d_2 \\ \beta_2 \end{pmatrix}$ $\quad d_2 c_1 + \beta_2 c_2 = b_2 \quad [c_1 c_2 b_2] = \begin{bmatrix} 2 & 6 & 2 \\ -1 & -2 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix}$

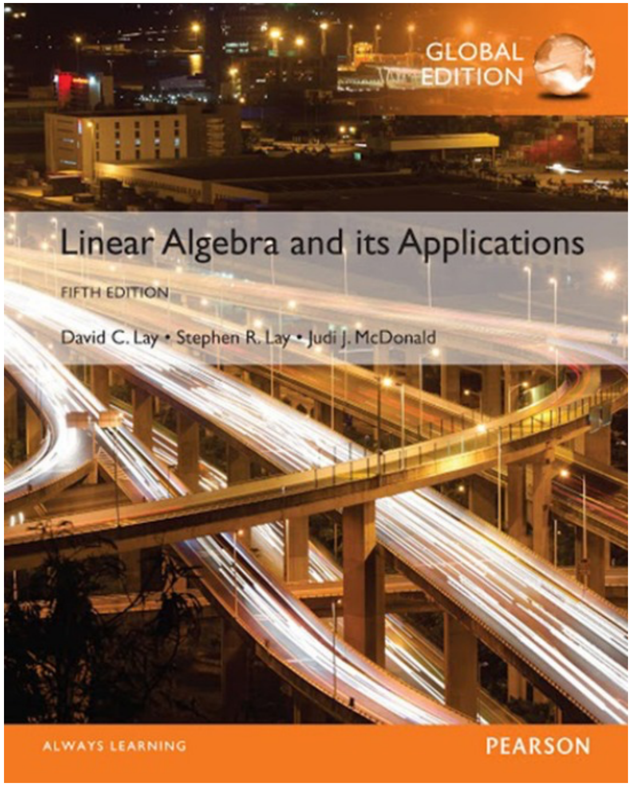
$\therefore \ P_{B \to C} = \begin{bmatrix} 9 & -2 \\ -4 & 1 \end{bmatrix}$

14. In $\mathbb{P}_2$, find the change-of-coordinates matrix from the basis $\mathcal{B} = \{1 - 3t^2, 2 + t - 5t^2, 1 + 2t\}$ to the standard basis. Then write t^2 as a linear combination of the polynomials in $\mathcal{B}$. $\longrightarrow$ Skip

4 Determinants

4.7

CHANGE OF BASIS



CHANGE OF BASIS

- **Example 1** Consider two bases $\beta = \{b_1, b_2\}$ and $C = \{c_1, c_2\}$ for a vector space V, such that

$$b_1 = 4c_1 + c_2 \quad \text{and} \quad b_2 = -6c_1 + c_2 \tag{1}$$

- Suppose

$$x = 3b_1 + b_2 \tag{2}$$

- That is, suppose $[x]_\beta = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$. Find $[x]_C$.

CHANGE OF BASIS

- **Solution** Apply the coordinate mapping determined by C to **x** in (2). Since the coordinate mapping is a linear transformation,

$$
\begin{aligned}
[x]_C &= [3\mathrm{b}_1 + \mathrm{b}_2]_C \\
&= [3\mathrm{b}_1]_C + [\mathrm{b}_2]_C
\end{aligned}
$$

- We can write the vector equation as a matrix equation, using the vectors in the linear combination as the columns of a matrix:

Standard의 좌표

$$[x]_{\varepsilon} = [b_1 \;\; b_2][x]_B$$

$$[x]_C = \underbrace{([\mathrm{b}_1]_C[\mathrm{b}_2]_C)}_{P} \underbrace{\begin{bmatrix} 3 \\ 1 \end{bmatrix}}_{[x]_B} \qquad (3)$$

CHANGE OF BASIS

- This formula gives $[x]_C$, once we know the columns of the matrix. From (1),

$$[\mathrm{b}_1]_C = \begin{bmatrix} 4 \\ 1 \end{bmatrix} \text{and } [\mathrm{b}_2]_C = \begin{bmatrix} -6 \\ 1 \end{bmatrix}$$

- Thus, (3) provides the solution:

C가 standard basis가 아닌 any basis일수 있다.

$$[x]_C = \begin{bmatrix} 4 & -6 \\ 1 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} 3 \\ 1 \end{bmatrix}}_{[x]_B} = \underbrace{\begin{bmatrix} 6 \\ 4 \end{bmatrix}}_{[x]_C}$$

$B \to C$ basis 한다.

CHANGE OF BASIS

old new c_n

- **Theorem 15**: Let $\beta = \{b_1, \ldots, b_n\}$ and $C = \{c_1, \ldots, \cancel{c_2}\}$ for a vector space V. Then there is a unique $n \times n$ matrix $\underset{C\leftarrow\beta}{P}$ such that

$$[x]_C = \underset{C\leftarrow\beta}{P}\,[x]_\beta \qquad (4)$$

- The columns of $\underset{C\leftarrow\beta}{P}$ are the C-coordinate vectors of the vectors in the basis β. That is,

$$\underset{C\leftarrow\beta}{P} = \left[\,[b_1]_C\,[b_2]_C \quad \cdots \quad [b_n]_C\right] \qquad (5)$$

$\left[\,[\text{old}]_{\text{new}} \quad \cdots \quad\right]$

CHANGE OF BASIS

- The matrix $\underset{C\leftarrow\beta}{P}$ in Theorem 15 is called the **change-of-coordinates matrix from β to C**. Multiplication by $\underset{C\leftarrow\beta}{P}$ converts β-coordinates into C-coordinates.
- Figure 2 below illustrates the change-of-coordinates equation (4).

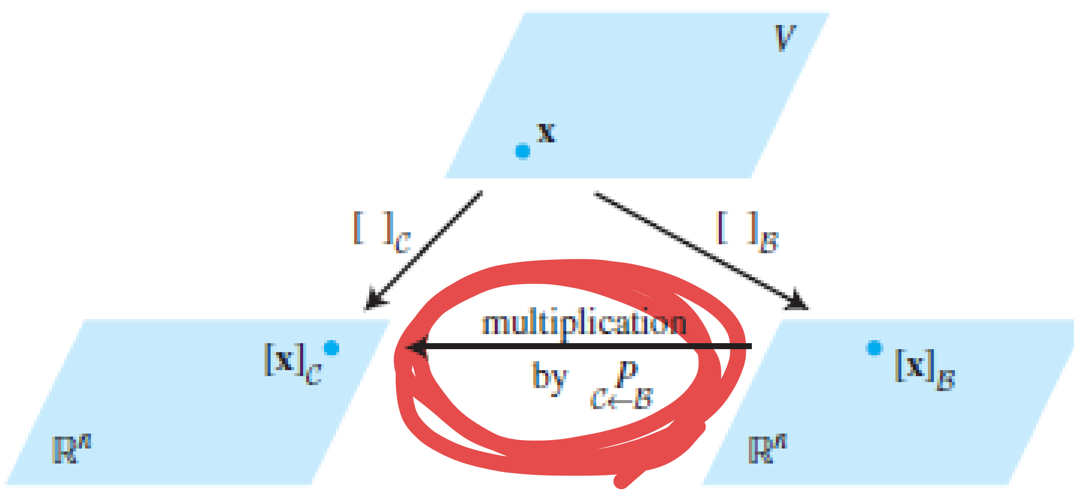


FIGURE 2 Two coordinate systems for V.

CHANGE OF BASIS

- The columns of $\overset{P}{c\leftarrow\beta}$ are linearly independent because they are the coordinate vectors of the linearly independent set β.

- Since $\overset{P}{c\leftarrow\beta}$ is square, it must be invertible, by the Invertible Matrix Theorem. Left-multiplying both sides of equation (4) by $(\overset{P}{c\leftarrow\beta})^{-1}$ yields

$$(\overset{P}{c\leftarrow\beta})^{-1}[x]_C = [x]_\beta$$

- Thus $(\overset{P}{c\leftarrow\beta})^{-1}$ is the matrix that converts C-coordinates into β–coordinates. That is,

$$= \left([b_1]_C \cdots [b_n]_C\right)^{-1}$$

$$(\overset{P}{c\leftarrow\beta})^{-1} = \overset{P}{\beta\leftarrow C} \qquad (6)$$

$$= \left([c_1]_\beta \cdots [c_n]_\beta\right)$$

CHANGE OF BASIS IN $\mathbb{R}^n$

- If $\beta = \{b_1, \ldots, b_n\}$ and $\mathcal{E}$ is the standard basis $\{e_1, \ldots, e_n\}$ in $\mathbb{R}^n$, then $[b_1]_{\mathcal{E}} = b_1$, and likewise for the other vectors in β. In this case, $\underset{\mathcal{E} \leftarrow \beta}{P}$ is thesame as the change-of-coordinates matrix P_β introduced in Section 4.4, namely,

$$P_\beta = [\, b_1 \quad b_2 \ldots b_n \,]$$

$\beta \to$ standard

- To change coordinates between two nonstandard bases in $\mathbb{R}^n$,we need Theorem 15. The theorem shows that to solve the change-of-basis problem, we need the coordinate vectors of the old basis relative to the new basis.

CHANGE OF BASIS IN $\mathbb{R}^n$

- **Example 2** Let $\mathbf{b}_1 = \begin{bmatrix} -9 \\ 1 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} -5 \\ -1 \end{bmatrix}$, $\mathbf{c}_1 = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$, $\mathbf{c}_2 = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$ and consider the bases for $\mathbb{R}^n$ given by $\beta = \{\mathbf{b}_1, \mathbf{b}_2\}$ and $C = \{\mathbf{c}_1, \mathbf{c}_2\}$. Find the change-of-coordinates matrix from β to C.

- **Solution** The matrix $\underset{\beta \leftarrow C}{P}$ involves the C-coordinate vectors of $\mathbf{b}_1$ and $\mathbf{b}_2$. Let $[\mathbf{b}_1]_C = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $[\mathbf{b}_2]_C = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$. Then, by definition,

$$[c_1 \quad c_2]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{b}_1 \quad \text{and} \quad [c_1 \quad c_2]\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \mathbf{b}_2$$

CHANGE OF BASIS IN $\mathbb{R}^n$

- To solve both systems simultaneously, augment the coefficient matrix with $\mathbf{b}_1$ and $\mathbf{b}_2$, and row reduce:

$$[c_1 \;\; c_2 \;\vdots\; b_1 \;\; b_2] = \left[\begin{array}{cc:cc} 1 & 3 & -9 & -5 \\ -4 & -5 & 1 & -1 \end{array}\right] \sim \left[\begin{array}{cc:cc} 1 & 0 & 6 & 4 \\ 0 & 1 & -5 & -3 \end{array}\right] \quad (7)$$

- Thus

$$[\mathbf{b}_1]_C = \begin{bmatrix} 6 \\ -5 \end{bmatrix} \text{and } [\mathbf{b}_2]_C = \begin{bmatrix} 4 \\ -3 \end{bmatrix}$$

- The desired change-of-coordinates matrix is therefore

$\mathcal{B} \to C$

$$\underset{C\leftarrow\beta}{P} = [[b_1]_c \;\; [b_2]_c] = \begin{bmatrix} 6 & 4 \\ -5 & -3 \end{bmatrix}$$

$A \sim n \times n$

If $Ax = \lambda x$ for some $x \neq 0$,

λ is an eigenvalue (corresponding to x)

x is an eigenvector

$\lambda x = \lambda I x$ gives $Ax = \lambda I x$ or $(A - \lambda I) x = 0$

Since $x \neq 0$, $A - \lambda I$ is a singular matrix.

so that $|A - \lambda I| = 0$

characteristic equation

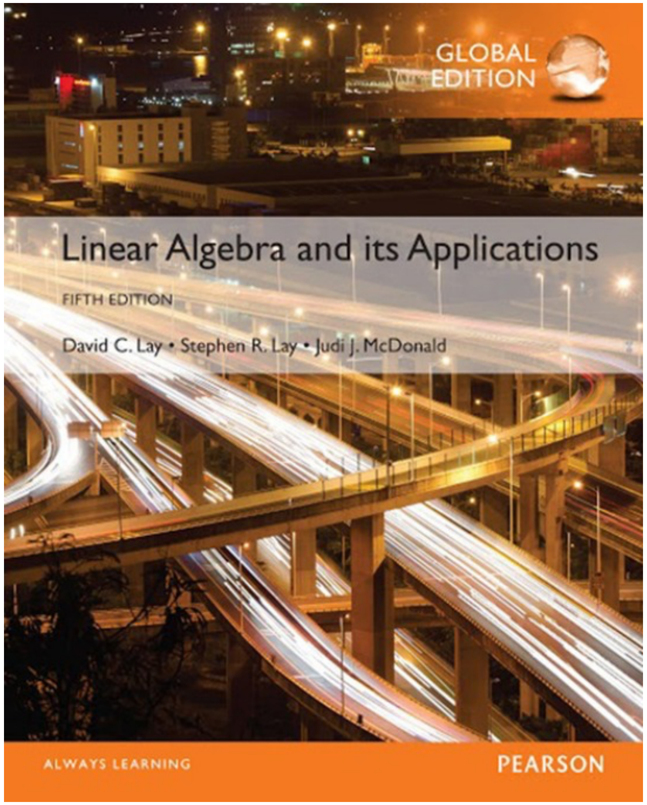
λ is the only unknown

5 Eigenvalues and Eigenvectors

5.1

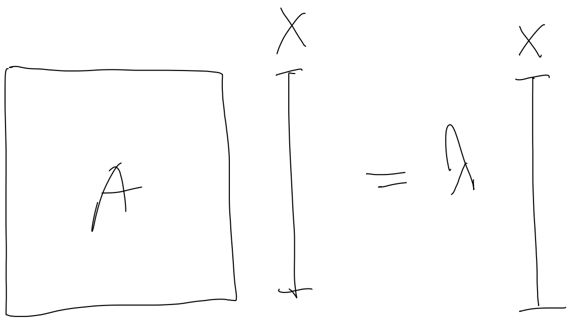
EIGENVECTORS AND EIGENVALUES

Use Determinants



EIGENVECTORS AND EIGENVALUES

- **Definition:** An **eigenvector** of an $n \times n$ matrix A is a nonzero vector $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$ for some scalar λ. A scalar λ is called an **eigenvalue** of A if there is a nontrivial solution $\mathbf{x}$ of $A\mathbf{x} = \lambda\mathbf{x}$; such an $\mathbf{x}$ is called an *eigenvector corresponding to* λ.



$\lambda, x \to$ unknown

$x = 0 \to$ trivial $\to$ 제외. $\to$ $x \neq 0$ 만 따지자.

x is an eigenvector of A

$\begin{cases} Ax = \lambda x \text{ for some } \lambda \\ x \neq 0 \end{cases}$

EIGENVECTORS AND EIGENVALUES

- **Definition:** An **eigenvector** of an $n \times n$ matrix A is a nonzero vector **x** such that $A\mathrm{x} = \lambda\mathrm{x}$ for some scalar λ. A scalar λ is called an **eigenvalue** of A if there is a nontrivial solution **x** of $A\mathrm{x} = \lambda\mathrm{x}$; such an **x** is called an *eigenvector corresponding to λ*.
- λ is an eigenvalue of an $n \times n$ matrix A if and only if the equation

$$\underline{(A - \lambda I)\mathrm{x} = 0} \qquad (3)$$

has a nontrivial solution.

x is an eigenvector of A corresponding to λ $\Longleftrightarrow$ x is a nontrivial solution of a homogeneous system $(A-\lambda I)x = 0$

↳ $A - \lambda I = 0$ 성립

EIGENVECTORS AND EIGENVALUES

- **Definition:** An **eigenvector** of an $n \times n$ matrix A is a nonzero vector **x** such that $A\mathrm{x} = \lambda\mathrm{x}$ for some scalar λ. A scalar λ is called an **eigenvalue** of A if there is a nontrivial solution **x** of $A\mathrm{x} = \lambda\mathrm{x}$; such an **x** is called an *eigenvector corresponding to λ*.
- λ is an eigenvalue of an $n \times n$ matrix A if and only if the equation

$$(A - \lambda I)\mathrm{x} = 0 \qquad (3)$$

has a nontrivial solution. (nonzero 여야.)

- The set of *all* solutions of (3) is just the null space of the matrix $A - \lambda I$.

Each Vector in Nul of $(A-\lambda I)$ is an eigenvector of A? ← False. zero vector가 있으니까.

EIGENVECTORS AND EIGENVALUES

- So this set is a *subspace* of $\mathbb{R}^n$ and is called the **eigenspace** of A corresponding to λ. null space of $A-\lambda I$

- The eigenspace consists of the zero vector and all the eigenvectors corresponding to λ.

$x=0$ 이면 뻔하다. $x \neq 0$ 인 nontrivial solution을 원한다.

EIGENVECTORS AND EIGENVALUES

- **Example 3:** Show that 7 is an eigenvalue of matrix

$A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$ and find the corresponding eigenvectors.

EIGENVECTORS AND EIGENVALUES

- **Solution:** The scalar 7 is an eigenvalue of A if and only if the equation

$$A\mathbf{x} = 7\mathbf{x} \qquad (1)$$

has a nontrivial solution.

- But (1) is equivalent to $A\mathbf{x} - 7\mathbf{x} = \mathbf{0}$, or

$$(A - 7I)\mathbf{x} = \mathbf{0} \qquad (2)$$

- To solve this homogeneous equation, form the matrix

$$A - 7I = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} - \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix} = \begin{bmatrix} -6 & 6 \\ 5 & -5 \end{bmatrix}$$

EIGENVECTORS AND EIGENVALUES

- The columns of $A-7I$ are obviously linearly dependent, so (2) has nontrivial solutions.
- To find the corresponding eigenvectors, use row operations:

$$\begin{bmatrix} -6 & 6 & 0 \\ 5 & -5 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

($x_2 \neq 0$ 이어야)

- The general solution has the form $x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.
- Each vector of this form with $x_2 \neq 0$ is an eigenvector corresponding to $\lambda = 7$.

$= \lambda$

EIGENVECTORS AND EIGENVALUES

- **Example 4:** Let $A = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$. An eigenvalue of A is 2. Find a basis for the corresponding eigenspace.

EIGENVECTORS AND EIGENVALUES

- **Example 4:** Let $A = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$. An eigenvalue of A is 2. Find a basis for the corresponding eigenspace.

- **Solution:** Form

$$A - 2I = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 6 \\ 2 & -1 & 6 \\ 2 & -1 & 6 \end{bmatrix}$$

and row reduce the augmented matrix for $(A - 2I)\mathrm{x} = 0$.

EIGENVECTORS AND EIGENVALUES

$$\begin{bmatrix} 2 & -1 & 6 & 0 \\ 2 & -1 & 6 & 0 \\ 2 & -1 & 6 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & -1 & 6 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

- At this point, it is clear that 2 is indeed an eigenvalue of A because the equation $(A-2I)\mathbf{x}=0$ has free variables.
- The general solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}, \; x_2 \text{ and } x_3 \text{ free.}$$

$E_1, E_2 = \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$

EIGENVECTORS AND EIGENVALUES

- **Theorem 1**: The eigenvalues of a triangular matrix are the entries on its main diagonal.

Use Determinant

$$A = \begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix}$$

$$A - aI = \begin{pmatrix} 0 & b & c \\ 0 & d-a & e \\ 0 & 0 & f-a \end{pmatrix} \quad |A - aI| = 0 \cdot (d-a)(f-a) = 0$$

$$A - dI = \begin{pmatrix} a-d & b & c \\ 0 & 0 & e \\ 0 & 0 & f-d \end{pmatrix} \quad |A - dI| = (a-d)(f-d)0 = 0$$

$$A - fI = \begin{pmatrix} a-f & b & c \\ 0 & d-f & e \\ 0 & 0 & 0 \end{pmatrix}$$

$$|A - fI| = (a-f)(d-f)0 = 0$$

$\Rightarrow$ a, d, f is eigenvalue

x is an eigenvector of A corresponding to eigenvalue λ

$\Leftrightarrow$ $Ax = \lambda x$ for some $x \neq 0$

$\Leftrightarrow$ $(A - \lambda I)x = 0$ has a non-trivial solution x

$\Leftrightarrow$ $A - \lambda I$ is singular (NOT invertible)

$\Leftrightarrow$ $|A - \lambda I| = 0$.

EIGENVECTORS AND EIGENVALUES

- **Theorem 1**: The eigenvalues of a triangular matrix are the entries on its main diagonal.
- **Proof:** For simplicity, consider the 3×3 case.

EIGENVECTORS AND EIGENVALUES

- **Theorem 1**: The eigenvalues of a triangular matrix are the entries on its main diagonal.
- **Proof:** For simplicity, consider the 3×3 case.
- If A is upper triangular, the $A - \lambda I$ has the form

$$A - \lambda I = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} - \lambda & a_{12} & a_{13} \\ 0 & a_{22} - \lambda & a_{23} \\ 0 & 0 & a_{33} - \lambda \end{bmatrix}$$

EIGENVECTORS AND EIGENVALUES

- The scalar λ is an eigenvalue of A if and only if the equation $(A-\lambda I)\mathrm{x}=0$ has a nontrivial solution, that is, if and only if the equation has a free variable.

- Because of the zero entries in $A-\lambda I$, it is easy to see that $(A-\lambda I)\mathrm{x}=0$ has a free variable if and only if at least one of the entries on the diagonal of $A-\lambda I$ is zero.

- This happens if and only if λ equals one of the entries a_{11}, a_{22}, a_{33} in A.

EIGENVECTORS AND EIGENVALUES

- **Theorem 2:** If $\mathbf{v}_1, \ldots, \mathbf{v}_r$ are eigenvectors that correspond to distinct eigenvalues $\lambda_1, \ldots, \lambda_r$ of an $n \times n$ matrix A, then the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ is linearly independent.

$Av_1 = \lambda_1 v_1$ $\quad \lambda_1 \neq \lambda_2 \neq \lambda_3 \neq \lambda_1$

$Av_2 = \lambda_2 v_2$

$Av_3 = \lambda_3 v_3$

Set $\underline{c_1 v_1 + c_2 v_2 + c_3 v_3 = 0}$

$\lambda_1 (c_1 v_1 + c_2 v_2 + c_3 v_3) = \lambda_1 0 = 0$

$\boxed{c_1 \lambda_1 v_1 + c_2 \lambda_2 v_2 + c_3 \lambda_3 v_3 = 0}$

$\rightarrow c_1 \lambda_1 v_1 + c_2 \lambda_1 v_2 + c_3 \lambda_1 v_3 = 0$

$c_2(\lambda_2 - \lambda_1) v_2 + c_3(\lambda_3 - \lambda_1) v_3 = 0$

$A \times$ $\quad c_2 \cancel{(\lambda_2 - \lambda_1)} \lambda_2 v_2 + c_3(\lambda_3 - \lambda_1) \lambda_3 v_3 = 0$

$\lambda_2 \times$ $\quad \rightarrow c_2 \cancel{(\lambda_2 - \lambda_1)} \lambda_2 v_2 + c_3(\lambda_3 - \lambda_1) \lambda_2 v_3 = 0$

$\underset{=0}{c_3} \underset{\neq 0}{(\lambda_3 - \lambda_1)} \underset{\neq 0}{(\lambda_3 - \lambda_2)} \underset{\neq 0}{v_3} = 0$

→ 같은 방식으로 $c_1 = c_2 = c_n = 0$

→ linearly independent

Note The converse may not be true.

If $Av_1 = \lambda_1 v_1$, $v_i \neq 0$, $\{v_1, \ldots, v_p\}$ ~~independent~~

$Av_2 = \lambda_2 v_2$

$\vdots$

$Av_p = \lambda_p v_p$ $\qquad$ False.

then eigenvalues are all distinct

※ $Ix = x = 1 \cdot x$ for any $x \neq 0$

→ $x \neq 0$ is an eigenvector of I corresponding to 1

$\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ are linearly independent eigenvector of 1

1이 유일한 eigen value → 둘다 same eigenvalue.

- **Theorem 2:** If $\mathbf{v}_1, \ldots, \mathbf{v}_r$ are eigenvectors that correspond to distinct eigenvalues $\lambda_1, \ldots, \lambda_r$ of an $n \times n$ matrix A, then the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ is linearly independent.
- **Proof:** Suppose $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ is linearly dependent.
- Since $\mathbf{v}_1$ is nonzero, Theorem 7 in Section 1.7 says that one of the vectors in the set is a linear combination of the preceding vectors.
- Let p be the least index such that v_{p+1} is a linear combination of the preceding (linearly independent) vectors.

EIGENVECTORS AND EIGENVALUES

- Then there exist scalars $c_1, \ldots, c_p$ such that

(5) $$c_1 \mathrm{v}_1 + \cdots + c_p \mathrm{v}_p = \mathrm{v}_{p+1}$$

- Multiplying both sides of (5) by A and using the fact that $Av_k = \lambda_k v_k$ for each k, we obtain

$$c_1 A\mathrm{v}_1 + \cdots + c_p A\mathrm{v}_p = A\mathrm{v}_{p+1}$$

$$c_1 \lambda_1 \mathrm{v}_1 + \cdots + c_p \lambda_p \mathrm{v}_p = \lambda_{p+1} \mathrm{v}_{p+1} \qquad (6)$$

- Multiplying both sides of (5) by λ_{p+1} and subtracting the result from (6), we have

$$c_1(\lambda_1 - \lambda_{p+1})\mathrm{v}_1 + \cdots + c_p(\lambda_p - \lambda_{p+1})\mathrm{v}_p = 0 \qquad (7)$$

EIGENVECTORS AND EIGENVALUES

- Since $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is linearly independent, the weights in (7) are all zero.
- But none of the factors $\lambda_i - \lambda_{p+1}$ are zero, because the eigenvalues are distinct.
- Hence $c_i = 0$ for $i = 1, \ldots, p$.
- But then (5) says that $\mathrm{v}_{p+1} = 0$, which is impossible.

EIGENVECTORS AND DIFFERENCE EQUATIONS

- Hence $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ cannot be linearly dependent and therefore must be linearly independent.

$|\lambda I - A|$ is called the characteristic polynomial.

ex) $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ $\quad A - \lambda I = \begin{pmatrix} 1-\lambda & 2 \\ 3 & 4-\lambda \end{pmatrix}$

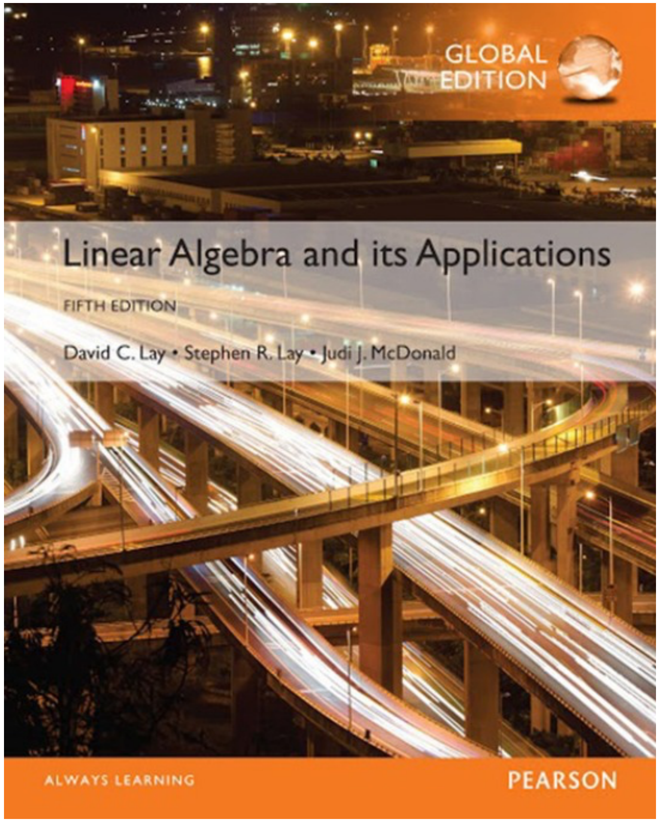
$$|A - \lambda I| = (1-\lambda)(4-\lambda) - 2\cdot 3 = \lambda^2 - 5\lambda + 4 - 6 = \lambda^2 - 5\lambda - 2$$

$$\boxed{\lambda^2 - 5\lambda - 2 = 0} \longrightarrow \lambda = \frac{5 \pm \sqrt{25 - 4\cdot(-2)}}{2} \Rightarrow (A - \lambda I)x = 0$$

5 Eigenvalues and Eigenvectors

5.2

THE CHARACTERISTIC EQUATION



THE CHARACTERISTIC EQUATION

- The scalar equation $\det(A - \lambda I) = 0$ is called the **characteristic equation** of A.

x is an eigenvector of A corresponding to λ

$\Leftrightarrow$ $Ax = \lambda x$ for some $x \neq 0$

$\Leftrightarrow$ $(A - \lambda I)x = 0$ (x, λ unknown)

$\Leftrightarrow$ $x \neq 0$ nontrivial solution $\rightarrow$ singular

$|A - \lambda I| = 0$, $A - \lambda I$ is not invertible (λ the only unknown)

Find Eigenvalues from $|A - \lambda I| = 0$

Then eigenvector from $(A - \lambda I)x = 0$

$\downarrow$

이젠 known

THE CHARACTERISTIC EQUATION

- The scalar equation $\det(A - \lambda I) = 0$ is called the **characteristic equation** of A.

- A scalar λ is an eigenvalue of an $n \times n$ matrix A if and only if λ satisfies the characteristic equation

$$\det(A - \lambda I) = 0$$

THE CHARACTERISTIC EQUATION

- **Example 3:** Find the characteristic equation of

$$A = \begin{bmatrix} 5 & -2 & 6 & -1 \\ 0 & 3 & -8 & 0 \\ 0 & 0 & 5 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$|A - \lambda I| = (5-\lambda)^2(3-\lambda)(1-\lambda) \Rightarrow$

THE CHARACTERISTIC EQUATION

- **Example 3:** Find the characteristic equation of

$$A = \begin{bmatrix} 5 & -2 & 6 & -1 \\ 0 & 3 & -8 & 0 \\ 0 & 0 & 5 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- **Solution:** Form $A - \lambda I$, and use Theorem 3(d):

THE CHARACTERISTIC EQUATION

$$\det(A-\lambda I)=\det\begin{bmatrix} 5-\lambda & -2 & 6 & -1 \\ 0 & 3-\lambda & -8 & 0 \\ 0 & 0 & 5-\lambda & 4 \\ 0 & 0 & 0 & 1-\lambda \end{bmatrix}$$

$$=(5-\lambda)(3-\lambda)(5-\lambda)(1-\lambda)$$

- The characteristic equation is

$$(5-\lambda)^2(3-\lambda)(1-\lambda)=0$$

or

$$(\lambda-5)^2(\lambda-3)(\lambda-1)=0$$

THE CHARACTERISTIC EQUATION

- Expanding the product, we can also write

$$\lambda^4 - 14\lambda^3 + 68\lambda^2 - 130\lambda + 75 = 0$$

THE CHARACTERISTIC EQUATION

- Expanding the product, we can also write

$$\lambda^4 - 14\lambda^3 + 68\lambda^2 - 130\lambda + 75 = 0$$

- If A is an $n \times n$ matrix, then $\det(A - \lambda I)$ is a polynomial of degree n called the **characteristic polynomial** of A.

THE CHARACTERISTIC EQUATION

- The eigenvalue 5 in Example 3 is said to have *multiplicity* 2 because $(\lambda - 5)$ occurs two times as a factor of the characteristic polynomial

$$(\lambda - 5)^2(\lambda - 3)(\lambda - 1) = 0$$

multiplicity 2 1 1

- In general, the **(algebraic) multiplicity** of an eigenvalue λ is its multiplicity as a root of the characteristic equation.

SIMILARITY

- If A and B are $n \times n$ matrices, then A **is similar to** B if there is an invertible matrix P such that $P^{-1}AP = B$, or, equivalently, $A = PBP^{-1}$.

$$A = \boxed{P}\, B\, \boxed{\text{inv } P}$$

To find eigenvalues and eigenvectors of A,

1) Solve $\underbrace{|A - \lambda I| = 0}_{\text{There is only one unknown } \lambda}$ for λ

2) Solve $\underbrace{(A - \lambda I)x = 0}$ for x

one unknown x

because λ is already obtained from 1)

SIMILARITY

- If A and B are $n \times n$ matrices, then A **is similar to** B if there is an invertible matrix P such that $P^{-1}AP = B$, or, equivalently, $A = PBP^{-1}$.

- Writing Q for P^{-1}, we have $Q^{-1}BQ = A$.

  So B is also similar to A, and we say simply that A and B **are similar**.

SIMILARITY

- If A and B are $n \times n$ matrices, then A **is similar to** B if there is an invertible matrix P such that $P^{-1}AP = B$, or, equivalently, $A = PBP^{-1}$.

- Writing Q for P^{-1}, we have $Q^{-1}BQ = A$.

  So B is also similar to A, and we say simply that A and B **are similar**.

- Changing A into $P^{-1}AP$ is called a **similarity transformation**.

SIMILARITY

- **Theorem 4**: If $n \times n$ matrices A and B are similar, then they have the same characteristic polynomial and hence the same eigenvalues (with the same multiplicities).

not mentioned bout eigne vector.

SIMILARITY

- **Theorem 4:** If $n \times n$ matrices A and B are similar, then they have the same characteristic polynomial and hence the same eigenvalues (with the same multiplicities).

not mentioned but eigenvector.

- **Proof:** If $B = P^{-1}AP$ then,

$$B - \lambda I = P^{-1}AP - \lambda P^{-1}P = P^{-1}(AP - \lambda P) = P^{-1}(A - \lambda I)P$$

$P^{-1}AP - P^{-1}(\lambda I)P$

- Using the multiplicative property (b) in Theorem (3), we compute

$$\det(B - \lambda I) = \det\left[P^{-1}(A - \lambda I)P\right]$$

(nxn Mat이라서)

$$= \det(P^{-1}) \cdot \det(A - \lambda I) \cdot \det(P) \qquad (2)$$

$= \det(P^{-1}P) \cdot \det(A - \lambda I) = |A - \lambda I|$

If A and B are similar, the eigenvectors of A are the same as the eigenvectors of B False

If $B = P^{-1}AP$ and $Bx = \lambda x$,

then

$$PP^{-1}AP\,x = P\lambda x \qquad A(Px) = P(\lambda x) = \lambda(Px)$$

$\Rightarrow$ Px is an eigenvector of A corresponding to ~~B~~ λ

$\hookrightarrow$ eigenvector 바뀜

SIMILARITY

- Since $\det(P^{-1})\cdot\det(P) = \underline{\det(P^{-1}P) = \det I = 1}$, we see from equation (1) that $\underline{\det(B-\lambda I) = \det(A-\lambda I)}$.

SIMILARITY

- **Warnings:**
  1. The matrices

$$\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

are not similar even though they have the same eigenvalues.

triangluar matrix

→ 대각성분 eigenvalue

SIMILARITY

2. Similarity is not the same as row equivalence. (If A is row equivalent to B, then $B = EA$ for some invertible matrix E). Row operations on a matrix usually change its eigenvalues.

Similarity는 eigenvalue 그대로

ex) $\begin{bmatrix} 2 & 4 \\ 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

Find the characteristic polynomial and the eigenvalues of the matrices in Exercises 1–8.

1. $\begin{bmatrix} 2 & 7 \\ 7 & 2 \end{bmatrix}$ **2.** $\begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}$

$\begin{vmatrix} 2-\lambda & 7 \\ 7 & 2-\lambda \end{vmatrix} = (2-\lambda)^2 - 7^2 = \lambda^2 - 4\lambda - 45$

$= (\lambda+5)(\lambda-9) = 0$

$\lambda = -5, 9$

$\lambda = -5$

$(A-\lambda I)x = \begin{pmatrix} 7 & 7 \\ 7 & 7 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$x_1 + x_2 = 0 \quad x_2 = -x_1$

$x = \begin{pmatrix} x_1 \\ -x_1 \end{pmatrix} = x_1\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\lambda = 9$

$(A-\lambda I)x = \begin{pmatrix} -7 & 7 \\ 7 & -7 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$x_1 = x_2$

$x = x_1\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\begin{vmatrix} 5-\lambda & 3 \\ 3 & 5-\lambda \end{vmatrix} = (5-\lambda)^2 - 9 = \lambda^2 - 10\lambda + 16$

$= (\lambda-2)(\lambda-8) = 0$

$\lambda = 2$

$\begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$x = c_1\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\lambda = 8$

$\begin{pmatrix} -3 & 3 \\ 3 & -3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad x_1 = x_2$

$x = x_1\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

9. $\begin{bmatrix} 1 & 0 & -1 \\ 2 & 3 & -1 \\ 0 & 6 & 0 \end{bmatrix}$

10. $\begin{bmatrix} 0 & 3 & 1 \\ 3 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix}$

$$\begin{vmatrix} 1-\lambda & 0 & -1 \\ 2 & 3-\lambda & -1 \\ 0 & 6 & -\lambda \end{vmatrix} = (1-\lambda)\begin{vmatrix} 3-\lambda & -1 \\ 6 & -\lambda \end{vmatrix} - 1\begin{vmatrix} 2 & 3-\lambda \\ 0 & 6 \end{vmatrix} = (1-\lambda)\{(3-\lambda)(-\lambda)+6\} - (12)$$

$$= (1-\lambda)(\lambda^2-3\lambda+6)-12$$

$$= \lambda^2-3\lambda+6-\lambda^3+3\lambda^2-6\lambda-12$$

$$= -\lambda^3+4\lambda^2-9\lambda-6=0$$

$$\lambda^3-4\lambda^2+9\lambda+6=0$$

$$\begin{array}{r|rrrr} -3 & 1 & -4 & 9 & 6 \\ & & +2 & 12 & \\ \hline & 1 & -6 & & \end{array}$$

$$|A-\lambda I| = \begin{vmatrix} -\lambda & 3 & 1 \\ 3 & -\lambda & 2 \\ 1 & 2 & -\lambda \end{vmatrix} = (-\lambda)\begin{vmatrix} -\lambda & 2 \\ 2 & -\lambda \end{vmatrix} - 3\begin{vmatrix} 3 & 2 \\ 1 & -\lambda \end{vmatrix} + \begin{vmatrix} 3 & -\lambda \\ 1 & 2 \end{vmatrix}$$

$$= (-\lambda)(\lambda^2-4)-3(-3\lambda-2)+6+\lambda$$

- Diagonalize the matrix:

5. $\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$ **6.** $\begin{bmatrix} 4 & 0 & -2 \\ 2 & 5 & 4 \\ 0 & 0 & 5 \end{bmatrix}$

$$|A-\lambda I| = \begin{vmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{vmatrix} = (2-\lambda)\begin{vmatrix} 3-\lambda & 1 \\ 2 & 2-\lambda \end{vmatrix} - 2\begin{vmatrix} 1 & 1 \\ 1 & 2-\lambda \end{vmatrix} + 1\begin{vmatrix} 1 & 3-\lambda \\ 1 & 2 \end{vmatrix}$$

$$= -(\lambda-1)^2(\lambda-5) \qquad \lambda = 1, 5$$

$$\lambda = 1 \rightarrow \begin{pmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{pmatrix} x = 0 \qquad x = x_2\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + x_3\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$$\lambda = 5 \rightarrow \begin{pmatrix} -3 & 2 & 1 \\ 1 & -2 & 1 \\ 1 & 2 & -3 \end{pmatrix} x = 0 \rightarrow \left|\begin{matrix} 1 & -2 & 1 & 0 \\ 0 & -4 & 4 & 0 \\ 0 & 4 & -4 & 0 \end{matrix}\right| \rightarrow x = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

don't find P^{-1} 그냥 Eigenvector로

P와 D 만 찾다.

$$P = \begin{bmatrix} -2 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \rightarrow D = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 5 \end{pmatrix}$$

or

$$P = \begin{pmatrix} 1 & -2 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \rightarrow D = \begin{pmatrix} 5 & & \\ & 1 & \\ & & 1 \end{pmatrix}$$

- Diagonalize the matrix:

11. $\begin{bmatrix} -1 & 4 & -2 \\ -3 & 4 & 0 \\ -3 & 1 & 3 \end{bmatrix}$

e matrix:

$$\begin{vmatrix} -1-\lambda & 4 & -2 \\ -3 & 4-\lambda & 0 \\ -3 & 1 & 3-\lambda \end{vmatrix} = 0 \rightarrow \lambda = 1, 2, 3$$

$$\lambda = 3 \rightarrow x = x_3 \begin{pmatrix} 1/4 \\ 3/4 \\ 1 \end{pmatrix}$$

$$\lambda = 1 \rightarrow x = x_3 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad \lambda = 2 \rightarrow x = x_3 \begin{pmatrix} 2/3 \\ 1 \\ 1 \end{pmatrix}$$

$$P = \begin{pmatrix} 1 & 2/3 & 1/4 \\ 1 & 1 & 3/4 \\ 1 & 1 & 1 \end{pmatrix} \qquad D = \begin{pmatrix} 1 & & \\ & 2 & \\ & & 3 \end{pmatrix}$$

$$P = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 3 & 3 \\ 1 & 3 & 4 \end{pmatrix}$$

24. A is a 3×3 matrix with two eigenvalues. Each eigenspace is one-dimensional. Is A diagonalizable? Why?

No.

There is only one linearly independent eigenvector for each eigenvalue.

$\Downarrow$

At most 2 linearly independent eigenvectors

25. A is a 4×4 matrix with three eigenvalues. One eigenspace is one-dimensional, and one of the other eigenspaces is two-dimensional. Is it possible that A is *not* diagonalizable? Justify your answer.

No

We can find at least one (linearly independent) eigenvector for the 3rd eigenvalue

$\Downarrow$

dim eigenspace $= 1$

$|A - \lambda I| = 0$

$(A - \lambda I)x = 0$

$[A - \lambda I \;\; 0]$

26. A is a 7×7 matrix with three eigenvalues. One eigenspace is two-dimensional, and one of the other eigenspaces is three-dimensional. Is it possible that A is *not* diagonalizable? Justify your answer.

Yes

$1 \le \dim(3^{rd}\text{ eigenspace}) \overset{?}{\le} 2$

이건 확정 아니죠..

$\rightarrow$ eigenvector가 6개일 수 있다.

- diagonalizable if $A = \boxed{PDP^{-1}}$ for some invertible P, diagonal D
- **Theorem 5:** An $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors.
- $\boxed{AP = PD}$ (for any $n \times n$ matrix A) $\Rightarrow$ $A[v_1 \cdots v_n] = [v_1 \cdots v_n]\begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$ $\Rightarrow$ $Av_i = \lambda_i v_i$

  If $v_i \neq 0$, v_i is an eigenvector of A corresponding to λ_i
- **Theorem 6:** An $n \times n$ matrix with n distinct eigenvalues is diagonalizable.

- Diagonalize the following:

14. $\begin{bmatrix} 4 & 0 & -2 \\ 2 & 5 & 4 \\ 0 & 0 & 5 \end{bmatrix}$ **16.** $\begin{bmatrix} 0 & -4 & -6 \\ -1 & 0 & -3 \\ 1 & 2 & 5 \end{bmatrix}$

여기에서는 λ 때문에 ERO가 적용이 안된다.

14. $\begin{vmatrix} 4-\lambda & 0 & -2 \\ 5 & 5-\lambda & 4 \\ 0 & 0 & 5-\lambda \end{vmatrix} = (4-\lambda)(5-\lambda)^2$

$\lambda=5 \to \left[\begin{array}{ccc|c} -1 & 0 & -2 & 0 \\ & & & 0 \\ & & & 0 \end{array}\right] \to \begin{array}{l} x_1 = -2x_3 \\ x_2 \text{ free} \end{array} \to x = x_2\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_3\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$

$\lambda=4 \to \begin{bmatrix} 2 & 1 & 4 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{array}{l} 2x_1+x_2=0 \\ x_3=0 \end{array} \to x = x_2\begin{bmatrix} -1/2 \\ 1 \\ 0 \end{bmatrix} \Rightarrow x_2\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$ $P = \begin{bmatrix} 0 & -2 & -1 \\ 1 & 0 & 2 \\ 0 & 1 & 0 \end{bmatrix}$ $D = \begin{bmatrix} 5 & & \\ & 5 & \\ & & 4 \end{bmatrix}$

16. $\begin{vmatrix} -\lambda & -4 & -6 \\ -1 & -\lambda & -3 \\ 1 & 2 & 5-\lambda \end{vmatrix} = -\lambda\begin{vmatrix} -\lambda & -3 \\ 2 & 5-\lambda \end{vmatrix} + 4\begin{vmatrix} -1 & -3 \\ 1 & 5-\lambda \end{vmatrix} - 6\begin{vmatrix} -1 & -\lambda \\ 1 & 2 \end{vmatrix}$

$= (\lambda-2)\{-\lambda^2+3\lambda-2\} = -(\lambda-2)^2(\lambda-1)$

$\lambda=2 \to \left[\begin{array}{ccc|c} -2 & -4 & -6 & 0 \\ -1 & -2 & -3 & 0 \\ 1 & 2 & 3 & 0 \end{array}\right] \to \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $x = \begin{bmatrix} -2x_2-3x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + x_3\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$

$\lambda=1 \to \begin{bmatrix} -1 & -4 & -6 \\ -1 & -1 & -3 \\ 1 & 2 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 4 \\ -1 & -3 & -3 \\ -3 & -4 & -6 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \to$

$x = \begin{bmatrix} -2x_3 \\ -x_3 \\ x_3 \end{bmatrix} = x_3\begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}$ $P = \begin{bmatrix} -2 & -3 & -2 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}$ $D = \begin{bmatrix} 2 & & \\ & 2 & \\ & & 3 \end{bmatrix}$

$|A-\lambda I| = \begin{vmatrix} -\lambda & -4 & -6 \\ -1 & -\lambda & -3 \\ 1 & 2 & 5-\lambda \end{vmatrix} = (-\lambda)\underbrace{\begin{vmatrix} -\lambda & -3 \\ 2 & 5-\lambda \end{vmatrix}}_{(n-1)\ \lambda's} - (-4)\begin{vmatrix} -1 & -3 \\ 1 & 5-\lambda \end{vmatrix} + (-6)\underbrace{\begin{vmatrix} -1 & -\lambda \\ 1 & 2 \end{vmatrix}}_{\text{contains only } (n-2)\ \lambda s}$

n λs

polynomials of deg $\leq (n-2)$

$\|$

n^{th} order polynomial in λ

$\|$

$(-1)^n\lambda^n + \boxed{a_{n-1}}\lambda^{n-1} + \cdots$

a_{n-1}는 앞의 determinant만 보면된다.

$$\begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ \vdots & a_{22}-\lambda & & \\ & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{vmatrix} = (a_{11}-\lambda)\begin{vmatrix} a_{22}-\lambda & \cdots & \\ & \ddots & \\ & & a_{nn}-\lambda \end{vmatrix} + \text{the other terms}$$

$$\| \qquad\qquad \|$$

$$(a_{11}-\lambda)\{(a_{22}-\lambda)| \quad | + \cdots\} + \boxed{\cdots}$$

$$(-1)^n\lambda^n + a_{n-1}\lambda^{n-1} + \cdots$$

$$= (a_{11}-\lambda)(a_{22}-\lambda)\cdots(a_{nn}-\lambda) + \text{other terms}$$

$$= (-1)^n\lambda^n + (-1)^{n-1}(a_{11}+a_{22}+\cdots+a_{nn})\lambda^{n-1} + \cdots$$

$$\begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ \vdots & a_{22}-\lambda & & \\ & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{vmatrix} = \boxed{a_n\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0}$$

$a_n = (-1)^n$

$a_{n-1} = (-1)^{n-1}(a_{11} + a_{22} + \cdots + a_{nn})$

Given

$\boxed{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0}$

there are n solutions : $x_1, x_2, \cdots, x_n$

Then

$$\begin{cases} x_1 + x_2 + \cdots + x_n = -\dfrac{a_{n-1}}{a_n} \\ x_1 \cdot x_2 \cdots x_n = (-1)^n \dfrac{a_0}{a_n} \end{cases}$$

$a_n(x-x_1)(x-x_2)\cdots(x-x_n)=0$

$\Rightarrow$

Let $\lambda_1, \lambda_2, \cdots, \lambda_n$ be n eigenvalues.

Then

$\lambda_1 + \lambda_2 + \cdots + \lambda_n = a_{11} + a_{22} + \cdots + a_{nn}$

$=$ trace of A

(대각원소의 합)

$\boxed{\lambda_1 \cdot \lambda_2 \cdots \lambda_n} = a_0 =$ the value of $|A-\lambda I|$ when $\lambda=0$

$= |A|$

Given $A = \begin{pmatrix} a_{11} & & a_{1n} \\ & \ddots & \\ a_{n1} & & a_{nn} \end{pmatrix}$,

- $a_{11} + a_{22} + \cdots + a_{nn} = \lambda_1 + \lambda_2 + \cdots + \lambda_n$
- $|A| = \lambda_1 \cdot \lambda_2 \cdots \lambda_n$

대각원소 합 = sum of $\lambda_1 \cdots \lambda_n$

$|A|$ = product of $\lambda_1 \cdots \lambda_n$

A is invertible $\Leftrightarrow |A| \neq 0$

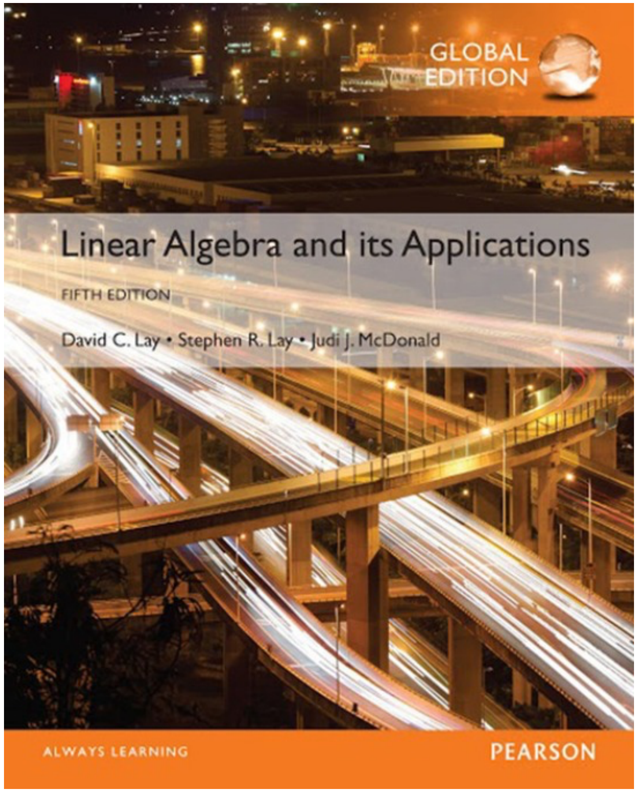
$\Leftrightarrow$ None of the eigenvalues is zero.

A invertible $\rightarrow |A| \neq 0$

$\rightarrow$ None of Eigenvalues is zero

5 Eigenvalues and Eigenvectors

5.3 DIAGONALIZATION



DIAGONALIZATION

- A square matrix A is said to be **diagonalizable** if A is similar to a diagonal matrix, that is, if $A = PDP^{-1}$ for some invertible matrix P and some diagonal matrix D.

ex) $A = I = \overset{P}{I}\, I\, \overset{P^{-1}}{I}$

혹은 $A = Q^{-1} D Q$

$(Q = P^{-1})$

혹은 $D = P^{-1} A P$

THE DIAGONALIZATION THEOREM

- **Theorem 5:** An $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors.

※ eigen value는 n개 아니어도
eigen vector는 n개 일수있다
올가지 확인할 것.

THE DIAGONALIZATION THEOREM

- **Theorem 5:** An $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors.

  In fact, $A = PDP^{-1}$, with D a diagonal matrix, if and only if the columns of P ~~and~~ are n linearly independent eigenvectors of A. In this case, the diagonal entries of D are eigenvalues of A that correspond, respectively, to the eigenvectors in P.

$$AP = PD$$

$$A[v_1 \cdots v_n] = [v_1 \cdots v_n]\begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix} = [\lambda_1 v_1 \ \ \lambda_2 v_2 \cdots \lambda_n v_n]$$

$$= [Av_1 \ \cdots \ Av_n]$$

$$Av_1 = \lambda_1 v_1$$
$$Av_2 = \lambda_2 v_2$$
$$\vdots$$
$$Av_n = \lambda_n v_n$$

P는 eigenvector를 column으로

λ는 그에 맞는 eigenvalue

THE DIAGONALIZATION THEOREM

- **Theorem 5:** An $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors.

In fact, $A = PDP^{-1}$, with D a diagonal matrix, if and only if the columns of P and n linearly independent eigenvectors of A. In this case, the diagonal entries of D are eigenvalues of A that correspond, respectively, to the eigenvectors in P.

P는 invertible → linearly independent

In other words, A is diagonalizable if and only if there are enough eigenvectors to form a basis of $\mathbb{R}^n$. We call such a basis an **eigenvector basis** of $\mathbb{R}^n$.

= Column 벡터들 eigenvector 인 경우

THE DIAGONALIZATION THEOREM

- **Proof:** First, observe that if P is any $n \times n$ matrix with columns $\mathbf{v}_1, \ldots, \mathbf{v}_n$, and if D is any diagonal matrix with diagonal entries $\lambda_1, \ldots, \lambda_n$, then

$$AP = A\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{bmatrix} = \begin{bmatrix} A\mathbf{v}_1 & A\mathbf{v}_2 & \cdots & A\mathbf{v}_n \end{bmatrix} \quad (1)$$

while

$$PD = P\begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} = \begin{bmatrix} \lambda_1\mathbf{v}_1 & \lambda_2\mathbf{v}_2 & \cdots & \lambda_n\mathbf{v}_n \end{bmatrix} \quad (2)$$

THE DIAGONALIZATION THEOREM

- Now suppose A is diagonalizable and $A = PDP^{-1}$. Then right-multiplying this relation by P, we have $AP = PD$.
- In this case, equations (1) and (2) imply that

$$\begin{bmatrix} A\mathbf{v}_1 & A\mathbf{v}_2 & \cdots & A\mathbf{v}_n \end{bmatrix} = \begin{bmatrix} \lambda_1\mathbf{v}_1 & \lambda_2\mathbf{v}_2 & \cdots & \lambda_n\mathbf{v}_n \end{bmatrix} \quad (3)$$

- Equating columns, we find that 아직은 v 가 eigenvector 인지는 모름

$$A\mathbf{v}_1 = \lambda_1\mathbf{v}_1, A\mathbf{v}_2 = \lambda_2\mathbf{v}_2, \ldots, A\mathbf{v}_n = \lambda_n\mathbf{v}_n \quad (4)$$

(nonzero 인지 몰라서)

- Since P is invertible, its columns $\mathbf{v}_1, \ldots, \mathbf{v}_n$ must be linearly independent. → These columns are non zero vectors.

THE DIAGONALIZATION THEOREM

- Also, since these columns are nonzero, the equations in (4) show that $\lambda_1, \ldots, \lambda_n$ are eigenvalues and $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are corresponding eigenvectors.

- This argument proves the "only if" parts of the first and second statements, along with the third statement, of the theorem.

- Finally, given any n eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$, use them to construct the columns of P and use corresponding eigenvalues $\lambda_1, \ldots, \lambda_n$ to construct D.

THE DIAGONALIZATION THEOREM

$Av_1 = \lambda_1 v_1$

$\vdots$

$Av_n = \lambda_n v_n$

$[Av_1 \cdots Av_n] = [\lambda_1 v_1 \cdots \lambda_n v_n]$

$AP = PD$

- By equations (1)–(3), $AP = PD$.

- This is true without any condition on the eigenvectors.

- If, in fact, the eigenvectors are linearly independent, then P is invertible (by the Invertible Matrix Theorem), and $AP = PD$ implies that $A = PDP^{-1}$.

n degree polynomial → n개의 eigenvalue → n 개의 eigenvector를 가져

If $\boxed{\begin{matrix} Av_1 = \lambda_1 v_1 \\ \vdots \\ Av_n = \lambda_n v_n \end{matrix}}$, then

$$A[v_1 \cdots v_n] = [v_1 \cdots v_n]\begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

$$A \quad P \quad = P \quad D \text{ for some diagonal } D$$

i.e. If A is $n \times n$, then A always satisfies $\boxed{AP = PD}$ for some diagonal D

That is, $\boxed{AP=PD}$ can be derived in any situtation.

Note that the previous theorem tells:

A is diagonalizable $\Longleftrightarrow$ $A = (\text{inv})(\text{diag})(\text{inv})^{-1}$ $\Longleftrightarrow$ there are n linearly independent eigenvectors

(i.e. $A = PDP^{-1}$)

↳ implies the invertibility of P

DIAGONALIZING MATRICES

- **Example 3:** Diagonalize the following matrix, if possible.

$$A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$$

That is, find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$.

DIAGONALIZING MATRICES

- **Example 3:** Diagonalize the following matrix, if possible.

$$A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$$

  That is, find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$.

- **Solution:** There are four steps to implement the description in Theorem 5.
- *Step 1.* ***Find the <u>eigenvalues</u> of A.***
- Here, the characteristic equation turns out to involve a cubic polynomial that can be factored:

DIAGONALIZING MATRICES

$$0 = \det(A - \lambda I) = -\lambda^3 - 3\lambda^2 + 4$$
$$= -(\lambda - 1)(\lambda + 2)^2$$

- The eigenvalues are $\underline{\lambda = 1 \text{ and } \lambda = -2.}$
- *Step 2.* ***Find three linearly independent eigenvectors of A.***
- *Three* vectors are needed because A is a 3×3 matrix.
- This is a critical step.
- If it fails, then Theorem 5 says that A cannot be diagonalized.

$\lambda=1$

$$(A-\lambda I)x = \begin{pmatrix} 0 & 3 & 3 \\ -3 & -6 & -3 \\ 3 & 3 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 3 & 3 & 0 \\ -3 & -6 & -3 & 0 \\ 3 & 3 & 0 & 0 \end{pmatrix} \updownarrow \begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & -2 & -1 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\downarrow$$

$$\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\cancel{x_1 + x_2 =}$ $x_1 = x_3$

$x_2 = -x_3$

$$x = x_3\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

$\lambda=-2$

$$(A-\lambda I)x = \begin{pmatrix} 3 & 3 & 3 \\ -3 & -3 & -3 \\ 3 & 3 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$x_1 + x_2 + x_3 = 0$

$$x = \begin{pmatrix} -x_2 - x_3 \\ x_2 \\ x_3 \end{pmatrix} = x_2\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + x_3\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

DIAGONALIZING MATRICES 풀어서 나온 결과

- Basis for $\lambda = 1$: $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

- Basis for $\lambda = -2$: $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{v}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

- You can check that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a linearly independent set.

DIAGONALIZING MATRICES

$P = [v_2 \; v_3 \; v_1]$ 도 가능 → $D = \begin{bmatrix} \lambda_2 & & \\ & \lambda_3 & \\ & & \lambda_1 \end{bmatrix}$

- *Step 3.* ***Construct P from the vectors in step 2.***
- The order of the vectors is unimportant.
- Using the order chosen in step 2, form

$$P = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

다른 eigvalue이므로 linearly independent

- *Step 4.* ***Construct D from the corresponding eigenvalues.***
- In this step, it is essential that the order of the eigenvalues matches the order chosen for the columns of P.

DIAGONALIZING MATRICES

- Use the eigenvalue $\lambda = -2$ twice, once for each of the eigenvectors corresponding to $\lambda = -2$:

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

- To avoid computing P^{-1}, simply verify that $AD = PD$.
- Compute

$$AP = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 2 \\ -1 & -2 & 0 \\ 1 & 0 & -2 \end{bmatrix}$$

DIAGONALIZING MATRICES

$$PD = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 2 \\ -1 & -2 & 0 \\ 1 & 0 & -2 \end{bmatrix}$$

DIAGONALIZING MATRICES

→ distinct eigenvalue → linearly independent

- **Theorem 6:** An $n \times n$ matrix with n distinct eigenvalues is diagonalizable.

Note. The converse of Thm 6 is not true in general

ex) I

The eigenvalue of I is 1 (w/ multiplicity n)

DIAGONALIZING MATRICES

- **Theorem 6:** An $n \times n$ matrix with n distinct eigenvalues is diagonalizable.

- **Proof:** Let $\mathbf{v}_1, \ldots, \mathbf{v}_n$ be eigenvectors corresponding to the n distinct eigenvalues of a matrix A.
- Then $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is linearly independent, by Theorem 2 in Section 5.1.
- Hence A is diagonalizable, by Theorem 5.

MATRICES WHOSE EIGENVALUES ARE NOT DISTINCT

- It is not *necessary* for an $n \times n$ matrix to have n distinct eigenvalues in order to be diagonalizable.
- The 3 x 3 matrix in Example 3 is diagonalizable even though it has only two distinct eigenvalues.

eigenvector 만 n개만큼 있으면된다.

MATRICES WHOSE EIGENVALUES ARE NOT DISTINCT

- If an $n \times n$ matrix A has n distinct eigenvalues, with corresponding eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$, and if $P = \begin{bmatrix} \mathrm{v}_1 & \cdots & \mathrm{v}_2 \end{bmatrix}$, then P is automatically invertible because its columns are linearly independent, by Theorem 2.

Quize → 5.3

- Same prob : Lecture Note / Ex

- 90 → A^+

  80 → B^+

  70 →