1 Linear Equations in Linear Algebra

1.1 SOLUTIONS

Notes: The key exercises are 7 (or 11 or 12), 19–22, and 25. For brevity, the symbols R1, R2,…, stand for row 1 (or equation 1), row 2 (or equation 2), and so on. Additional notes are at the end of the section.

1. $\begin{aligned} x_1 + 5x_2 &= 7 \\ -2x_1 - 7x_2 &= -5 \end{aligned}$ $\quad \begin{bmatrix} 1 & 5 & 7 \\ -2 & -7 & -5 \end{bmatrix}$

Replace R2 by R2 + (2)R1 and obtain: $\begin{aligned} x_1 + 5x_2 &= 7 \\ 3x_2 &= 9 \end{aligned}$ $\quad \begin{bmatrix} 1 & 5 & 7 \\ 0 & 3 & 9 \end{bmatrix}$

Scale R2 by 1/3: $\begin{aligned} x_1 + 5x_2 &= 7 \\ x_2 &= 3 \end{aligned}$ $\quad \begin{bmatrix} 1 & 5 & 7 \\ 0 & 1 & 3 \end{bmatrix}$

Replace R1 by R1 + (–5)R2: $\begin{aligned} x_1 \qquad &= -8 \\ x_2 &= 3 \end{aligned}$ $\quad \begin{bmatrix} 1 & 0 & -8 \\ 0 & 1 & 3 \end{bmatrix}$

The solution is $(x_1, x_2) = (-8, 3)$, or simply (–8, 3).

2. $\begin{aligned} 2x_1 + 4x_2 &= -4 \\ 5x_1 + 7x_2 &= 11 \end{aligned}$ $\quad \begin{bmatrix} 2 & 4 & -4 \\ 5 & 7 & 11 \end{bmatrix}$

Scale R1 by 1/2 and obtain: $\begin{aligned} x_1 + 2x_2 &= -2 \\ 5x_1 + 7x_2 &= 11 \end{aligned}$ $\quad \begin{bmatrix} 1 & 2 & -2 \\ 5 & 7 & 11 \end{bmatrix}$

Replace R2 by R2 + (–5)R1: $\begin{aligned} x_1 + 2x_2 &= -2 \\ -3x_2 &= 21 \end{aligned}$ $\quad \begin{bmatrix} 1 & 2 & -2 \\ 0 & -3 & 21 \end{bmatrix}$

Scale R2 by –1/3: $\begin{aligned} x_1 + 2x_2 &= -2 \\ x_2 &= -7 \end{aligned}$ $\quad \begin{bmatrix} 1 & 2 & -2 \\ 0 & 1 & -7 \end{bmatrix}$

Replace R1 by R1 + (–2)R2: $\begin{aligned} x_1 \qquad &= 12 \\ x_2 &= -7 \end{aligned}$ $\quad \begin{bmatrix} 1 & 0 & 12 \\ 0 & 1 & -7 \end{bmatrix}$

The solution is $(x_1, x_2) = (12, -7)$, or simply (12, –7).

3. The point of intersection satisfies the system of two linear equations:

$$\begin{aligned} x_1 + 5x_2 &= 7 \\ x_1 - 2x_2 &= -2 \end{aligned} \qquad \begin{bmatrix} 1 & 5 & 7 \\ 1 & -2 & -2 \end{bmatrix}$$

Replace R2 by R2 + (–1)R1 and obtain:
$$\begin{aligned} x_1 + 5x_2 &= 7 \\ -7x_2 &= -9 \end{aligned} \qquad \begin{bmatrix} 1 & 5 & 7 \\ 0 & -7 & -9 \end{bmatrix}$$

Scale R2 by –1/7:
$$\begin{aligned} x_1 + 5x_2 &= 7 \\ x_2 &= 9/7 \end{aligned} \qquad \begin{bmatrix} 1 & 5 & 7 \\ 0 & 1 & 9/7 \end{bmatrix}$$

Replace R1 by R1 + (–5)R2:
$$\begin{aligned} x_1 \qquad &= 4/7 \\ x_2 &= 9/7 \end{aligned} \qquad \begin{bmatrix} 1 & 0 & 4/7 \\ 0 & 1 & 9/7 \end{bmatrix}$$

The point of intersection is $(x_1, x_2) = (4/7, 9/7)$.

4. The point of intersection satisfies the system of two linear equations:

$$\begin{aligned} x_1 - 5x_2 &= 1 \\ 3x_1 - 7x_2 &= 5 \end{aligned} \qquad \begin{bmatrix} 1 & -5 & 1 \\ 3 & -7 & 5 \end{bmatrix}$$

Replace R2 by R2 + (–3)R1 and obtain:
$$\begin{aligned} x_1 - 5x_2 &= 1 \\ 8x_2 &= 2 \end{aligned} \qquad \begin{bmatrix} 1 & -5 & 1 \\ 0 & 8 & 2 \end{bmatrix}$$

Scale R2 by 1/8:
$$\begin{aligned} x_1 - 5x_2 &= 1 \\ x_2 &= 1/4 \end{aligned} \qquad \begin{bmatrix} 1 & -5 & 1 \\ 0 & 1 & 1/4 \end{bmatrix}$$

Replace R1 by R1 + (5)R2:
$$\begin{aligned} x_1 \qquad &= 9/4 \\ x_2 &= 1/4 \end{aligned} \qquad \begin{bmatrix} 1 & 0 & 9/4 \\ 0 & 1 & 1/4 \end{bmatrix}$$

The point of intersection is $(x_1, x_2) = (9/4, 1/4)$.

5. The system is already in "triangular" form. The fourth equation is $x_4 = -5$, and the other equations do not contain the variable x_4. The next two steps should be to use the variable x_3 in the third equation to eliminate that variable from the first two equations. In matrix notation, that means to replace R2 by its sum with 3 times R3, and then replace R1 by its sum with –5 times R3.

6. One more step will put the system in triangular form. Replace R4 by its sum with –3 times R3, which produces $\begin{bmatrix} 1 & -6 & 4 & 0 & -1 \\ 0 & 2 & -7 & 0 & 4 \\ 0 & 0 & 1 & 2 & -3 \\ 0 & 0 & 0 & -5 & 15 \end{bmatrix}$. After that, the next step is to scale the fourth row by –1/5.

7. Ordinarily, the next step would be to interchange R3 and R4, to put a 1 in the third row and third column. But in this case, the third row of the augmented matrix corresponds to the equation $0\,x_1 + 0\,x_2 + 0\,x_3 = 1$, or simply, 0 = 1. A system containing this condition has no solution. Further row operations are unnecessary once an equation such as 0 = 1 is evident.
The solution set is empty.

8. The standard row operations are:

$$\begin{bmatrix} 1 & -4 & 9 & 0 \\ 0 & 1 & 7 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -4 & 9 & 0 \\ 0 & 1 & 7 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -4 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

The solution set contains one solution: (0, 0, 0).

9. The system has already been reduced to triangular form. Begin by scaling the fourth row by 1/2 and then replacing R3 by R3 + (3)R4:

$$\begin{bmatrix} 1 & -1 & 0 & 0 & -4 \\ 0 & 1 & -3 & 0 & -7 \\ 0 & 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 2 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 & 0 & -4 \\ 0 & 1 & -3 & 0 & 7 \\ 0 & 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 & 0 & -4 \\ 0 & 1 & -3 & 0 & -7 \\ 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

Next, replace R2 by R2 + (3)R3. Finally, replace R1 by R1 + R2:

$$\sim \begin{bmatrix} 1 & -1 & 0 & 0 & -4 \\ 0 & 1 & 0 & 0 & 8 \\ 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 & 8 \\ 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

The solution set contains one solution: (4, 8, 5, 2).

10. The system has already been reduced to triangular form. Use the 1 in the fourth row to change the –4 and 3 above it to zeros. That is, replace R2 by R2 + (4)R4 and replace R1 by R1 + (–3)R4. For the final step, replace R1 by R1 + (2)R2.

$$\begin{bmatrix} 1 & -2 & 0 & 3 & -2 \\ 0 & 1 & 0 & -4 & 7 \\ 0 & 0 & 1 & 0 & 6 \\ 0 & 0 & 0 & 1 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 0 & 0 & 7 \\ 0 & 1 & 0 & 0 & -5 \\ 0 & 0 & 1 & 0 & 6 \\ 0 & 0 & 0 & 1 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & -3 \\ 0 & 1 & 0 & 0 & -5 \\ 0 & 0 & 1 & 0 & 6 \\ 0 & 0 & 0 & 1 & -3 \end{bmatrix}$$

The solution set contains one solution: (–3, –5, 6, –3).

11. First, swap R1 and R2. Then replace R3 by R3 + (–3)R1. Finally, replace R3 by R3 + (2)R2.

$$\begin{bmatrix} 0 & 1 & 4 & -5 \\ 1 & 3 & 5 & -2 \\ 3 & 7 & 7 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 5 & -2 \\ 0 & 1 & 4 & -5 \\ 3 & 7 & 7 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 5 & -2 \\ 0 & 1 & 4 & -5 \\ 0 & -2 & -8 & 12 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 5 & -2 \\ 0 & 1 & 4 & -5 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

The system is inconsistent, because the last row would require that 0 = 2 if there were a solution. The solution set is empty.

12. Replace R2 by R2 + (–3)R1 and replace R3 by R3 + (4)R1. Finally, replace R3 by R3 + (3)R2.

$$\begin{bmatrix} 1 & -3 & 4 & -4 \\ 3 & -7 & 7 & -8 \\ -4 & 6 & -1 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 4 & -4 \\ 0 & 2 & -5 & 4 \\ 0 & -6 & 15 & -9 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 4 & -4 \\ 0 & 2 & -5 & 4 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

The system is inconsistent, because the last row would require that 0 = 3 if there were a solution. The solution set is empty.

13. $\begin{bmatrix} 1 & 0 & -3 & 8 \\ 2 & 2 & 9 & 7 \\ 0 & 1 & 5 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -3 & 8 \\ 0 & 2 & 15 & -9 \\ 0 & 1 & 5 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -3 & 8 \\ 0 & 1 & 5 & -2 \\ 0 & 2 & 15 & -9 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -3 & 8 \\ 0 & 1 & 5 & -2 \\ 0 & 0 & 5 & -5 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 0 & -3 & 8 \\ 0 & 1 & 5 & -2 \\ 0 & 0 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{bmatrix}$. The solution is (5, 3, –1).

14. $\begin{bmatrix} 1 & -3 & 0 & 5 \\ -1 & 1 & 5 & 2 \\ 0 & 1 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 0 & 5 \\ 0 & -2 & 5 & 7 \\ 0 & 1 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 0 & 5 \\ 0 & 1 & 1 & 0 \\ 0 & -2 & 5 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 0 & 5 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 7 & 7 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & -3 & 0 & 5 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 0 & 5 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$. The solution is (2, –1, 1).

15. First, replace R4 by R4 + (–3)R1, then replace R3 by R3 + (2)R2, and finally replace R4 by R4 + (3)R3.

$$\begin{bmatrix} 1 & 0 & 3 & 0 & 2 \\ 0 & 1 & 0 & -3 & 3 \\ 0 & -2 & 3 & 2 & 1 \\ 3 & 0 & 0 & 7 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 0 & 2 \\ 0 & 1 & 0 & -3 & 3 \\ 0 & -2 & 3 & 2 & 1 \\ 0 & 0 & -9 & 7 & -11 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 3 & 0 & 2 \\ 0 & 1 & 0 & -3 & 3 \\ 0 & 0 & 3 & -4 & 7 \\ 0 & 0 & -9 & 7 & -11 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 0 & 2 \\ 0 & 1 & 0 & -3 & 3 \\ 0 & 0 & 3 & -4 & 7 \\ 0 & 0 & 0 & -5 & 10 \end{bmatrix}$$

The resulting triangular system indicates that a solution exists. In fact, using the argument from Example 2, one can see that the solution is unique.

16. First replace R4 by R4 + (2)R1 and replace R4 by R4 + (–3/2)R2. (One could also scale R2 before adding to R4, but the arithmetic is rather easy keeping R2 unchanged.) Finally, replace R4 by R4 + R3.

$$\begin{bmatrix} 1 & 0 & 0 & -2 & -3 \\ 0 & 2 & 2 & 0 & 0 \\ 0 & 0 & 1 & 3 & 1 \\ -2 & 3 & 2 & 1 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -2 & -3 \\ 0 & 2 & 2 & 0 & 0 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 3 & 2 & -3 & -1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & -2 & -3 \\ 0 & 2 & 2 & 0 & 0 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & -1 & -3 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -2 & -3 \\ 0 & 2 & 2 & 0 & 0 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The system is now in triangular form and has a solution. The next section discusses how to continue with this type of system.

17. Row reduce the augmented matrix corresponding to the given system of three equations:

$$\begin{bmatrix} 1 & -4 & 1 \\ 2 & -1 & -3 \\ -1 & -3 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & -4 & 1 \\ 0 & 7 & -5 \\ 0 & -7 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & -4 & 1 \\ 0 & 7 & -5 \\ 0 & 0 & 0 \end{bmatrix}$$

The system is consistent, and using the argument from Example 2, there is only one solution. So the three lines have only one point in common.

18. Row reduce the augmented matrix corresponding to the given system of three equations:

$$\begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & 1 & -1 & 1 \\ 1 & 3 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & -1 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & -5 \end{bmatrix}$$

The third equation, $0 = -5$, shows that the system is inconsistent, so the three planes have no point in common.

19. $\begin{bmatrix} 1 & h & 4 \\ 3 & 6 & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & h & 4 \\ 0 & 6-3h & -4 \end{bmatrix}$ Write c for $6 - 3h$. If $c = 0$, that is, if $h = 2$, then the system has no solution, because 0 cannot equal –4. Otherwise, when $h \neq 2$, the system has a solution.

20. $\begin{bmatrix} 1 & h & -3 \\ -2 & 4 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & h & -3 \\ 0 & 4+2h & 0 \end{bmatrix}$. Write c for $4 + 2h$. Then the second equation $cx_2 = 0$ has a solution for every value of c. So the system is consistent for all h.

21. $\begin{bmatrix} 1 & 3 & -2 \\ -4 & h & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & -2 \\ 0 & h+12 & 0 \end{bmatrix}$. Write c for $h + 12$. Then the second equation $cx_2 = 0$ has a solution for every value of c. So the system is consistent for all h.

22. $\begin{bmatrix} 2 & -3 & h \\ -6 & 9 & 5 \end{bmatrix} \sim \begin{bmatrix} 2 & -3 & h \\ 0 & 0 & 5+3h \end{bmatrix}$. The system is consistent if and only if $5 + 3h = 0$, that is, if and only if $h = -5/3$.

23. **a**. True. See the remarks following the box titled *Elementary Row Operations.*

b. False. A 5×6 matrix has five rows.

c. False. The description given applied to a single solution. The solution *set* consists of all possible solutions. Only in special cases does the solution set consist of exactly one solution. Mark a statement True only if the statement is *always* true.

d. True. See the box before Example 2.

24. **a**. True. See the box preceding the subsection titled *Existence and Uniqueness Questions.*

b. False. The definition of *row equivalent* requires that there exist a sequence of row operations that transforms one matrix into the other.

c. False. By definition, an inconsistent system has *no* solution.

d. True. This definition of *equivalent systems* is in the second paragraph after equation (2).

25. $\begin{bmatrix} 1 & -4 & 7 & g \\ 0 & 3 & -5 & h \\ -2 & 5 & -9 & k \end{bmatrix} \sim \begin{bmatrix} 1 & -4 & 7 & g \\ 0 & 3 & -5 & h \\ 0 & -3 & 5 & k+2g \end{bmatrix} \sim \begin{bmatrix} 1 & -4 & 7 & g \\ 0 & 3 & -5 & h \\ 0 & 0 & 0 & k+2g+h \end{bmatrix}$

Let b denote the number $k + 2g + h$. Then the third equation represented by the augmented matrix above is $0 = b$. This equation is possible if and only if b is zero. So the original system has a solution if and only if $k + 2g + h = 0$.

26. A basic principle of this section is that row operations do not affect the solution set of a linear system. Begin with a simple augmented matrix for which the solution is obviously (–2, 1, 0), and then perform any elementary row operations to produce other augmented matrices. Here are three examples. The fact that they are all row equivalent proves that they all have the solution set (–2, 1, 0).

$$\begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -2 \\ 2 & 1 & 0 & -3 \\ 0 & 0 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -2 \\ 2 & 1 & 0 & -3 \\ 2 & 0 & 1 & -4 \end{bmatrix}$$

27. Study the augmented matrix for the given system, replacing R2 by R2 + (–c)R1:

$$\begin{bmatrix} 1 & 3 & f \\ c & d & g \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & f \\ 0 & d-3c & g-cf \end{bmatrix}$$

This shows that shows $d - 3c$ must be nonzero, since f and g are arbitrary. Otherwise, for some choices of f and g the second row would correspond to an equation of the form $0 = b$, where b is nonzero. Thus $d \neq 3c$.

28. Row reduce the augmented matrix for the given system. Scale the first row by $1/a$, which is possible since a is nonzero. Then replace R2 by R2 + (–c)R1.

$$\begin{bmatrix} a & b & f \\ c & d & g \end{bmatrix} \sim \begin{bmatrix} 1 & b/a & f/a \\ c & d & g \end{bmatrix} \sim \begin{bmatrix} 1 & b/a & f/a \\ 0 & d-c(b/a) & g-c(f/a) \end{bmatrix}$$

The quantity $d - c(b/a)$ must be nonzero, in order for the system to be consistent when the quantity $g - c(f/a)$ is nonzero (which can certainly happen). The condition that $d - c(b/a) \neq 0$ can also be written as $ad - bc \neq 0$, or $ad \neq bc$.

29. Swap R1 and R2; swap R1 and R2.

30. Multiply R2 by –1/2; multiply R2 by –2.

31. Replace R3 by R3 + (–4)R1; replace R3 by R3 + (4)R1.

32. Replace R3 by R3 + (3)R2; replace R3 by R3 + (–3)R2.

33. The first equation was given. The others are:

$$T_2 = (T_1 + 20 + 40 + T_3)/4, \quad \text{or} \quad 4T_2 - T_1 - T_3 = 60$$
$$T_3 = (T_4 + T_2 + 40 + 30)/4, \quad \text{or} \quad 4T_3 - T_4 - T_2 = 70$$
$$T_4 = (10 + T_1 + T_3 + 30)/4, \quad \text{or} \quad 4T_4 - T_1 - T_3 = 40$$

Rearranging,

$$\begin{array}{rcrcrcrcl} 4T_1 & - & T_2 & & & - & T_4 & = & 30 \\ -T_1 & + & 4T_2 & - & T_3 & & & = & 60 \\ & & -T_2 & + & 4T_3 & - & T_4 & = & 70 \\ -T_1 & & & - & T_3 & + & 4T_4 & = & 40 \end{array}$$

34. Begin by interchanging R1 and R4, then create zeros in the first column:

$$\begin{bmatrix} 4 & -1 & 0 & -1 & 30 \\ -1 & 4 & -1 & 0 & 60 \\ 0 & -1 & 4 & -1 & 70 \\ -1 & 0 & -1 & 4 & 40 \end{bmatrix} \sim \begin{bmatrix} -1 & 0 & -1 & 4 & 40 \\ -1 & 4 & -1 & 0 & 60 \\ 0 & -1 & 4 & -1 & 70 \\ 4 & -1 & 0 & -1 & 30 \end{bmatrix} \sim \begin{bmatrix} -1 & 0 & -1 & 4 & 40 \\ 0 & 4 & 0 & -4 & 20 \\ 0 & -1 & 4 & -1 & 70 \\ 0 & -1 & -4 & 15 & 190 \end{bmatrix}$$

Scale R1 by –1 and R2 by 1/4, create zeros in the second column, and replace R4 by R4 + R3:

$$\sim \begin{bmatrix} 1 & 0 & 1 & -4 & -40 \\ 0 & 1 & 0 & -1 & 5 \\ 0 & -1 & 4 & -1 & 70 \\ 0 & -1 & -4 & 15 & 190 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & -4 & -40 \\ 0 & 1 & 0 & -1 & 5 \\ 0 & 0 & 4 & -2 & 75 \\ 0 & 0 & -4 & 14 & 195 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & -4 & -40 \\ 0 & 1 & 0 & -1 & 5 \\ 0 & 0 & 4 & -2 & 75 \\ 0 & 0 & 0 & 12 & 270 \end{bmatrix}$$

Scale R4 by 1/12, use R4 to create zeros in column 4, and then scale R3 by 1/4:

$$\sim \begin{bmatrix} 1 & 0 & 1 & -4 & -40 \\ 0 & 1 & 0 & -1 & 5 \\ 0 & 0 & 4 & -2 & 75 \\ 0 & 0 & 0 & 1 & 22.5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 0 & 50 \\ 0 & 1 & 0 & 0 & 27.5 \\ 0 & 0 & 4 & 0 & 120 \\ 0 & 0 & 0 & 1 & 22.5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 0 & 50 \\ 0 & 1 & 0 & 0 & 27.5 \\ 0 & 0 & 1 & 0 & 30 \\ 0 & 0 & 0 & 1 & 22.5 \end{bmatrix}$$

The last step is to replace R1 by R1 + (–1)R3:

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 & 20.0 \\ 0 & 1 & 0 & 0 & 27.5 \\ 0 & 0 & 1 & 0 & 30.0 \\ 0 & 0 & 0 & 1 & 22.5 \end{bmatrix}.$$ The solution is (20, 27.5, 30, 22.5).

Notes: The *Study Guide* includes a "Mathematical Note" about statements, "If … , then … ."

This early in the course, students typically use single row operations to reduce a matrix. As a result, even the small grid for Exercise 34 leads to about 25 multiplications or additions (not counting operations with zero). This exercise should give students an appreciation for matrix programs such as MATLAB. Exercise 14 in Section 1.10 returns to this problem and states the solution in case students have not already solved the system of equations. Exercise 31 in Section 2.5 uses this same type of problem in connection with an LU factorization.

For instructors who wish to use technology in the course, the *Study Guide* provides boxed MATLAB notes at the ends of many sections. Parallel notes for Maple, Mathematica, and the TI-83+/86/89 and HP-48G calculators appear in separate appendices at the end of the *Study Guide*. The MATLAB box for Section 1.1 describes how to access the data that is available for all numerical exercises in the text. This feature has the ability to save students time if they regularly have their matrix program at hand when studying linear algebra. The MATLAB box also explains the basic commands `replace`, `swap`, and `scale`. These commands are included in the text data sets, available from the text web site, www.laylinalgebra.com.

1.2 SOLUTIONS

Notes: The key exercises are 1–20 and 23–28. (Students should work at least four or five from Exercises 7–14, in preparation for Section 1.5.)

1. Reduced echelon form: a and b. Echelon form: d. Not echelon: c.

2. Reduced echelon form: a. Echelon form: b and d. Not echelon: c.

3. $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 5 & 6 & 7 \\ 6 & 7 & 8 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -6 & -9 \\ 0 & -5 & -10 & -15 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & -5 & -10 & -15 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Pivot cols 1 and 2. $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 5 & 6 & 7 \\ 6 & 7 & 8 & 9 \end{bmatrix}$

4. $\begin{bmatrix} 1 & 3 & 5 & 7 \\ 3 & 5 & 7 & 9 \\ 5 & 7 & 9 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 5 & 7 \\ 0 & -4 & -8 & -12 \\ 0 & -8 & -16 & -34 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 5 & 7 \\ 0 & 1 & 2 & 3 \\ 0 & -8 & -16 & -34 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 5 & 7 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & -10 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 3 & 5 & 7 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 5 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. Pivot cols 1, 2, and 4 $\begin{bmatrix} 1 & 3 & 5 & 7 \\ 3 & 5 & 7 & 9 \\ 5 & 7 & 9 & 1 \end{bmatrix}$

5. $\begin{bmatrix} \blacksquare & * \\ 0 & \blacksquare \end{bmatrix}, \begin{bmatrix} \blacksquare & * \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & \blacksquare \\ 0 & 0 \end{bmatrix}$

6. $\begin{bmatrix} \blacksquare & * \\ 0 & \blacksquare \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} \blacksquare & * \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & \blacksquare \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

7. $\begin{bmatrix} 1 & 3 & 4 & 7 \\ 3 & 9 & 7 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 4 & 7 \\ 0 & 0 & -5 & -15 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 4 & 7 \\ 0 & 0 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 0 & -5 \\ 0 & 0 & 1 & 3 \end{bmatrix}$

Corresponding system of equations: $\begin{aligned} x_1 + 3x_2 \qquad &= -5 \\ x_3 &= 3 \end{aligned}$

The basic variables (corresponding to the pivot positions) are x_1 and x_3. The remaining variable x_2 is free. Solve for the basic variables in terms of the free variable. The general solution is

$$\begin{cases} x_1 = -5 - 3x_2 \\ x_2 \text{ is free} \\ x_3 = 3 \end{cases}$$

Note: Exercise 7 is paired with Exercise 10.

8. $\begin{bmatrix} 1 & 4 & 0 & 7 \\ 2 & 7 & 0 & 10 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 0 & 7 \\ 0 & -1 & 0 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 0 & 7 \\ 0 & 1 & 0 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -9 \\ 0 & 1 & 0 & 4 \end{bmatrix}$

Corresponding system of equations: $\begin{aligned} x_1 \quad &= -9 \\ x_2 &= 4 \end{aligned}$

The basic variables (corresponding to the pivot positions) are x_1 and x_2. The remaining variable x_3 is free. Solve for the basic variables in terms of the free variable. In this particular problem, the basic variables do not depend on the value of the free variable.

General solution: $\begin{cases} x_1 = -9 \\ x_2 = 4 \\ x_3 \text{ is free} \end{cases}$

Note: A common error in Exercise 8 is to assume that x_3 is zero. To avoid this, identify the basic variables first. Any remaining variables are *free*. (This type of computation will arise in Chapter 5.)

9. $\begin{bmatrix} 0 & 1 & -6 & 5 \\ 1 & -2 & 7 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 7 & -6 \\ 0 & 1 & -6 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -5 & 4 \\ 0 & 1 & -6 & 5 \end{bmatrix}$

Corresponding system: $\begin{aligned} x_1 \quad - 5x_3 &= 4 \\ x_2 - 6x_3 &= 5 \end{aligned}$

Basic variables: x_1, x_2; free variable: x_3. General solution: $\begin{cases} x_1 = 4 + 5x_3 \\ x_2 = 5 + 6x_3 \\ x_3 \text{ is free} \end{cases}$

10. $\begin{bmatrix} 1 & -2 & -1 & 3 \\ 3 & -6 & -2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & -1 & 3 \\ 0 & 0 & 1 & -7 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 0 & -4 \\ 0 & 0 & 1 & -7 \end{bmatrix}$

Corresponding system: $\begin{aligned} x_1 - 2x_2 \quad &= -4 \\ x_3 &= -7 \end{aligned}$

Basic variables: x_1, x_3; free variable: x_2. General solution: $\begin{cases} x_1 = -4 + 2x_2 \\ x_2 \text{ is free} \\ x_3 = -7 \end{cases}$

11. $\begin{bmatrix} 3 & -4 & 2 & 0 \\ -9 & 12 & -6 & 0 \\ -6 & 8 & -4 & 0 \end{bmatrix} \sim \begin{bmatrix} 3 & -4 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -4/3 & 2/3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Corresponding system: $\begin{aligned} x_1 - \frac{4}{3}x_2 + \frac{2}{3}x_3 &= 0 \\ 0 &= 0 \\ 0 &= 0 \end{aligned}$

Basic variable: x_1; free variables x_2, x_3. General solution: $\begin{cases} x_1 = \frac{4}{3}x_2 - \frac{2}{3}x_3 \\ x_2 \text{ is free} \\ x_3 \text{ is free} \end{cases}$

12. $\begin{bmatrix} 1 & -7 & 0 & 6 & 5 \\ 0 & 0 & 1 & -2 & -3 \\ -1 & 7 & -4 & 2 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & -7 & 0 & 6 & 5 \\ 0 & 0 & 1 & -2 & -3 \\ 0 & 0 & -4 & 8 & 12 \end{bmatrix} \sim \begin{bmatrix} 1 & -7 & 0 & 6 & 5 \\ 0 & 0 & 1 & -2 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

Corresponding system:
$$\begin{aligned} x_1 - 7x_2 \quad + 6x_4 &= 5 \\ x_3 - 2x_4 &= -3 \\ 0 &= 0 \end{aligned}$$

Basic variables: x_1 and x_3; free variables: x_2, x_4. General solution: $\begin{cases} x_1 = 5 + 7x_2 - 6x_4 \\ x_2 \text{ is free} \\ x_3 = -3 + 2x_4 \\ x_4 \text{ is free} \end{cases}$

13. $\begin{bmatrix} 1 & -3 & 0 & -1 & 0 & -2 \\ 0 & 1 & 0 & 0 & -4 & 1 \\ 0 & 0 & 0 & 1 & 9 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 0 & 0 & 9 & 2 \\ 0 & 1 & 0 & 0 & -4 & 1 \\ 0 & 0 & 0 & 1 & 9 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & -3 & 5 \\ 0 & 1 & 0 & 0 & -4 & 1 \\ 0 & 0 & 0 & 1 & 9 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

Corresponding system:
$$\begin{aligned} x_1 \quad - 3x_5 &= 5 \\ x_2 \quad - 4x_5 &= 1 \\ x_4 + 9x_5 &= 4 \\ 0 &= 0 \end{aligned}$$

Basic variables: x_1, x_2, x_4; free variables: x_3, x_5. General solution: $\begin{cases} x_1 = 5 + 3x_5 \\ x_2 = 1 + 4x_5 \\ x_3 \text{ is free} \\ x_4 = 4 - 9x_5 \\ x_5 \text{ is free} \end{cases}$

Note: The *Study Guide* discusses the common mistake $x_3 = 0$.

14. $\begin{bmatrix} 1 & 2 & -5 & -6 & 0 & -5 \\ 0 & 1 & -6 & -3 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 7 & 0 & 0 & -9 \\ 0 & 1 & -6 & -3 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

Corresponding system:
$$\begin{array}{rcrcrcrcr} x_1 & & + & 7x_3 & & & & = & -9 \\ & x_2 & - & 6x_3 & - & 3x_4 & & = & 2 \\ & & & & & & x_5 & = & 0 \\ & & & & & & 0 & = & 0 \end{array}$$

Basic variables: x_1, x_2, x_5; free variables: x_3, x_4. General solution: $\begin{cases} x_1 = -9 - 7x_3 \\ x_2 = 2 + 6x_3 + 3x_4 \\ x_3 \text{ is free} \\ x_4 \text{ is free} \\ x_5 = 0 \end{cases}$

15. **a**. The system is consistent, with a unique solution.

b. The system is inconsistent. (The rightmost column of the augmented matrix is a pivot column).

16. **a**. The system is consistent, with a unique solution.

b. The system is consistent. There are many solutions because x_2 is a free variable.

17. $\begin{bmatrix} 2 & 3 & h \\ 4 & 6 & 7 \end{bmatrix} \sim \begin{bmatrix} 2 & 3 & h \\ 0 & 0 & 7-2h \end{bmatrix}$ The system has a solution only if $7 - 2h = 0$, that is, if $h = 7/2$.

18. $\begin{bmatrix} 1 & -3 & -2 \\ 5 & h & -7 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & -2 \\ 0 & h+15 & 3 \end{bmatrix}$ If $h + 15$ is zero, that is, if $h = -15$, then the system has no solution, because 0 cannot equal 3. Otherwise, when $h \neq -15$, the system has a solution.

19. $\begin{bmatrix} 1 & h & 2 \\ 4 & 8 & k \end{bmatrix} \sim \begin{bmatrix} 1 & h & 2 \\ 0 & 8-4h & k-8 \end{bmatrix}$

a. When $h = 2$ and $k \neq 8$, the augmented column is a pivot column, and the system is inconsistent.

b. When $h \neq 2$, the system is consistent and has a unique solution. There are no free variables.

c. When $h = 2$ and $k = 8$, the system is consistent and has many solutions.

20. $\begin{bmatrix} 1 & 3 & 2 \\ 3 & h & k \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 2 \\ 0 & h-9 & k-6 \end{bmatrix}$

a. When $h = 9$ and $k \neq 6$, the system is inconsistent, because the augmented column is a pivot column.

b. When $h \neq 9$, the system is consistent and has a unique solution. There are no free variables.

c. When $h = 9$ and $k = 6$, the system is consistent and has many solutions.

21. **a**. False. See Theorem 1.

b. False. See the second paragraph of the section.

c. True. Basic variables are defined after equation (4).

d. True. This statement is at the beginning of *Parametric Descriptions of Solution Sets*.

e. False. The row shown corresponds to the equation $5x_4 = 0$, which does not by itself lead to a contradiction. So the system might be consistent or it might be inconsistent.

22. **a**. False. See the statement preceding Theorem 1. Only the *reduced* echelon form is unique.

b. False. See the beginning of the subsection *Pivot Positions*. The pivot positions in a matrix are determined completely by the positions of the leading entries in the nonzero rows of any echelon form obtained from the matrix.

c. True. See the paragraph after Example 3.

d. False. The existence of at least one solution is not related to the presence or absence of free variables. If the system is inconsistent, the solution set is empty. See the solution of Practice Problem 2.

e. True. See the paragraph just before Example 4.

23. Yes. The system is consistent because with three pivots, there must be a pivot in the third (bottom) row of the coefficient matrix. The reduced echelon form cannot contain a row of the form $[0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 1]$.

24. The system is inconsistent because the pivot in column 5 means that there is a row of the form $[0\ \ 0\ \ 0\ \ 0\ \ 1]$. Since the matrix is the *augmented* matrix for a system, Theorem 2 shows that the system has no solution.

25. If the coefficient matrix has a pivot position in every row, then there is a pivot position in the bottom row, and there is no room for a pivot in the augmented column. So, the system is consistent, by Theorem 2.

26. Since there are three pivots (one in each row), the augmented matrix must reduce to the form

$$\begin{bmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \end{bmatrix} \quad \text{and so} \quad \begin{aligned} x_1 &= a \\ x_2 &= b \\ x_3 &= c \end{aligned}$$

No matter what the values of a, b, and c, the solution exists and is unique.

27. *"If a linear system is consistent, then the solution is unique if and only if every column in the coefficient matrix is a pivot column; otherwise there are infinitely many solutions."*

This statement is true because the free variables correspond to *nonpivot* columns of the coefficient matrix. The columns are all pivot columns if and only if there are no free variables. And there are no free variables if and only if the solution is unique, by Theorem 2.

28. Every column in the augmented matrix *except the rightmost column* is a pivot column, and the rightmost column is *not* a pivot column.

29. An underdetermined system always has more variables than equations. There cannot be more basic variables than there are equations, so there must be at least one free variable. Such a variable may be assigned infinitely many different values. If the system is consistent, each different value of a free variable will produce a different solution.

30. Example:
$$\begin{aligned} x_1 + x_2 + x_3 &= 4 \\ 2x_1 + 2x_2 + 2x_3 &= 5 \end{aligned}$$

31. Yes, a system of linear equations with more equations than unknowns can be consistent.

Example (in which $x_1 = x_2 = 1$):
$$\begin{aligned} x_1 + x_2 &= 2 \\ x_1 - x_2 &= 0 \\ 3x_1 + 2x_2 &= 5 \end{aligned}$$

32. According to the numerical note in Section 1.2, when $n = 30$ the reduction to echelon form takes about $2(30)^3/3 = 18{,}000$ flops, while further reduction to reduced echelon form needs at most $(30)^2 = 900$ flops. Of the total flops, the "backward phase" is about 900/18900 = .048 or about 5%.

When $n = 300$, the estimates are $2(300)^3/3 = 18{,}000{,}000$ phase for the reduction to echelon form and $(300)^2 = 90{,}000$ flops for the backward phase. The fraction associated with the backward phase is about $(9\times10^4)/(18\times10^6) = .005$, or about .5%.

33. For a quadratic polynomial $p(t) = a_0 + a_1t + a_2t^2$ to exactly fit the data (1, 12), (2, 15), and (3, 16), the coefficients a_0, a_1, a_2 must satisfy the systems of equations given in the text. Row reduce the augmented matrix:

$$\begin{bmatrix} 1 & 1 & 1 & 12 \\ 1 & 2 & 4 & 15 \\ 1 & 3 & 9 & 16 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 12 \\ 0 & 1 & 3 & 3 \\ 0 & 2 & 8 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 12 \\ 0 & 1 & 3 & 3 \\ 0 & 0 & 2 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 12 \\ 0 & 1 & 3 & 3 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 0 & 13 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

The polynomial is $p(t) = 7 + 6t - t^2$.

34. **[M]** The system of equations to be solved is:

$$\begin{array}{ccccccccccccc} a_0 & + & a_1\cdot 0 & + & a_2\cdot 0^2 & + & a_3\cdot 0^3 & + & a_4\cdot 0^4 & + & a_5\cdot 0^5 & = & 0 \\ a_0 & + & a_1\cdot 2 & + & a_2\cdot 2^2 & + & a_3\cdot 2^3 & + & a_4\cdot 2^4 & + & a_5\cdot 2^5 & = & 2.90 \\ a_0 & + & a_1\cdot 4 & + & a_2\cdot 4^2 & + & a_3\cdot 4^3 & + & a_4\cdot 4^4 & + & a_5\cdot 4^5 & = & 14.8 \\ a_0 & + & a_1\cdot 6 & + & a_2\cdot 6^2 & + & a_3\cdot 6^3 & + & a_4\cdot 6^4 & + & a_5\cdot 6^5 & = & 39.6 \\ a_0 & + & a_1\cdot 8 & + & a_2\cdot 8^2 & + & a_3\cdot 8^3 & + & a_4\cdot 8^4 & + & a_5\cdot 8^5 & = & 74.3 \\ a_0 & + & a_1\cdot 10 & + & a_2\cdot 10^2 & + & a_3\cdot 10^3 & + & a_4\cdot 10^4 & + & a_5\cdot 10^5 & = & 119 \end{array}$$

The unknowns are $a_0, a_1, \ldots, a_5$. Use technology to compute the reduced echelon of the augmented matrix:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 4 & 8 & 16 & 32 & 2.9 \\ 1 & 4 & 16 & 64 & 256 & 1024 & 14.8 \\ 1 & 6 & 36 & 216 & 1296 & 7776 & 39.6 \\ 1 & 8 & 64 & 512 & 4096 & 32768 & 74.3 \\ 1 & 10 & 10^2 & 10^3 & 10^4 & 10^5 & 119 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 4 & 8 & 16 & 32 & 2.9 \\ 0 & 0 & 8 & 48 & 224 & 960 & 9 \\ 0 & 0 & 24 & 192 & 1248 & 7680 & 30.9 \\ 0 & 0 & 48 & 480 & 4032 & 32640 & 62.7 \\ 0 & 0 & 80 & 960 & 9920 & 99840 & 104.5 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 4 & 8 & 16 & 32 & 2.9 \\ 0 & 0 & 8 & 48 & 224 & 960 & 9 \\ 0 & 0 & 0 & 48 & 576 & 4800 & 3.9 \\ 0 & 0 & 0 & 192 & 2688 & 26880 & 8.7 \\ 0 & 0 & 0 & 480 & 7680 & 90240 & 14.5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 4 & 8 & 16 & 32 & 2.9 \\ 0 & 0 & 8 & 48 & 224 & 960 & 9 \\ 0 & 0 & 0 & 48 & 576 & 4800 & 3.9 \\ 0 & 0 & 0 & 0 & 384 & 7680 & -6.9 \\ 0 & 0 & 0 & 0 & 1920 & 42240 & -24.5 \end{bmatrix}$$

$$\sim\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 4 & 8 & 16 & 32 & 2.9 \\ 0 & 0 & 8 & 48 & 224 & 960 & 9 \\ 0 & 0 & 0 & 48 & 576 & 4800 & 3.9 \\ 0 & 0 & 0 & 0 & 384 & 7680 & -6.9 \\ 0 & 0 & 0 & 0 & 0 & 3840 & 10 \end{bmatrix}\sim\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 4 & 8 & 16 & 32 & 2.9 \\ 0 & 0 & 8 & 48 & 224 & 960 & 9 \\ 0 & 0 & 0 & 48 & 576 & 4800 & 3.9 \\ 0 & 0 & 0 & 0 & 384 & 7680 & -6.9 \\ 0 & 0 & 0 & 0 & 0 & 1 & .0026 \end{bmatrix}$$

$$\sim\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 4 & 8 & 16 & 0 & 2.8167 \\ 0 & 0 & 8 & 48 & 224 & 0 & 6.5000 \\ 0 & 0 & 0 & 48 & 576 & 0 & -8.6000 \\ 0 & 0 & 0 & 0 & 384 & 0 & -26.900 \\ 0 & 0 & 0 & 0 & 0 & 1 & .002604 \end{bmatrix}\sim\cdots\sim\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1.7125 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1.1948 \\ 0 & 0 & 0 & 1 & 0 & 0 & .6615 \\ 0 & 0 & 0 & 0 & 1 & 0 & -.0701 \\ 0 & 0 & 0 & 0 & 0 & 1 & .0026 \end{bmatrix}$$

Thus $p(t) = 1.7125t - 1.1948t^2 + .6615t^3 - .0701t^4 + .0026t^5$, and $p(7.5) = 64.6$ hundred lb.

Notes: In Exercise 34, if the coefficients are retained to higher accuracy than shown here, then $p(7.5) = 64.8$. If a polynomial of lower degree is used, the resulting system of equations is overdetermined. The augmented matrix for such a system is the same as the one used to find p, except that at least column 6 is missing. When the augmented matrix is row reduced, the sixth row of the augmented matrix will be entirely zero except for a nonzero entry in the augmented column, indicating that no solution exists.

Exercise 34 requires 25 row operations. It should give students an appreciation for higher-level commands such as `gauss` and `bgauss`, discussed in Section 1.4 of the *Study Guide*. The command `ref` (reduced echelon form) is available, but I recommend postponing that command until Chapter 2.

The *Study Guide* includes a "Mathematical Note" about the phrase, "If and only if," used in Theorem 2.

1.3 SOLUTIONS

Notes: The key exercises are 11–14, 17–22, 25, and 26. A discussion of Exercise 25 will help students understand the notation $[\mathbf{a}_1\ \ \mathbf{a}_2\ \ \mathbf{a}_3]$, $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$, and Span$\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$.

1. $\mathbf{u}+\mathbf{v}=\begin{bmatrix}-1\\2\end{bmatrix}+\begin{bmatrix}-3\\-1\end{bmatrix}=\begin{bmatrix}-1+(-3)\\2+(-1)\end{bmatrix}=\begin{bmatrix}-4\\1\end{bmatrix}$.

Using the definitions carefully,

$$\mathbf{u}-2\mathbf{v}=\begin{bmatrix}-1\\2\end{bmatrix}+(-2)\begin{bmatrix}-3\\-1\end{bmatrix}=\begin{bmatrix}-1\\2\end{bmatrix}+\begin{bmatrix}(-2)(-3)\\(-2)(-1)\end{bmatrix}=\begin{bmatrix}-1+6\\2+2\end{bmatrix}=\begin{bmatrix}5\\4\end{bmatrix}$$, or, more quickly,

$$\mathbf{u}-2\mathbf{v}=\begin{bmatrix}-1\\2\end{bmatrix}-2\begin{bmatrix}-3\\-1\end{bmatrix}=\begin{bmatrix}-1+6\\2+2\end{bmatrix}=\begin{bmatrix}5\\4\end{bmatrix}$$. The intermediate step is often not written.

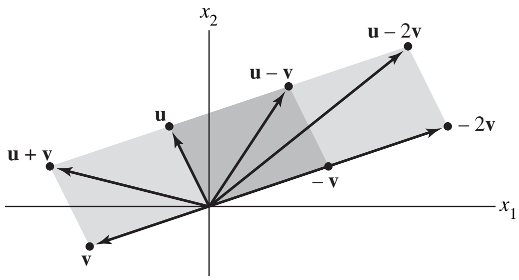
2. $\mathbf{u}+\mathbf{v}=\begin{bmatrix}3\\2\end{bmatrix}+\begin{bmatrix}2\\-1\end{bmatrix}=\begin{bmatrix}3+2\\2+(-1)\end{bmatrix}=\begin{bmatrix}5\\1\end{bmatrix}$.

Using the definitions carefully,

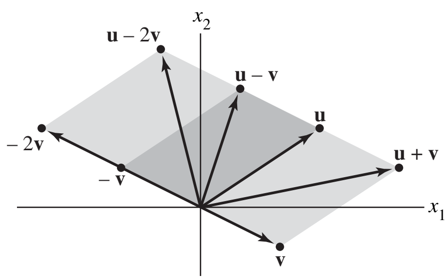
$$\mathbf{u}-2\mathbf{v}=\begin{bmatrix}3\\2\end{bmatrix}+(-2)\begin{bmatrix}2\\-1\end{bmatrix}=\begin{bmatrix}3\\2\end{bmatrix}+\begin{bmatrix}(-2)(2)\\(-2)(-1)\end{bmatrix}=\begin{bmatrix}3+(-4)\\2+2\end{bmatrix}=\begin{bmatrix}-1\\4\end{bmatrix}$$, or, more quickly,

$$\mathbf{u}-2\mathbf{v}=\begin{bmatrix}3\\2\end{bmatrix}-2\begin{bmatrix}2\\-1\end{bmatrix}=\begin{bmatrix}3-4\\2+2\end{bmatrix}=\begin{bmatrix}-1\\4\end{bmatrix}$$. The intermediate step is often not written.

3.



4.



5. $x_1\begin{bmatrix}6\\-1\\5\end{bmatrix}+x_2\begin{bmatrix}-3\\4\\0\end{bmatrix}=\begin{bmatrix}1\\-7\\-5\end{bmatrix}$, $\begin{bmatrix}6x_1\\-x_1\\5x_1\end{bmatrix}+\begin{bmatrix}-3x_2\\4x_2\\0\end{bmatrix}=\begin{bmatrix}1\\-7\\-5\end{bmatrix}$, $\begin{bmatrix}6x_1-3x_2\\-x_1+4x_2\\5x_1\end{bmatrix}=\begin{bmatrix}1\\-7\\-5\end{bmatrix}$

$$\begin{aligned}6x_1 &- 3x_2 &= 1\\ -x_1 &+ 4x_2 &= -7\\ 5x_1 & &= -5\end{aligned}$$

Usually the intermediate steps are not displayed.

6. $x_1\begin{bmatrix}-2\\3\end{bmatrix}+x_2\begin{bmatrix}8\\5\end{bmatrix}+x_3\begin{bmatrix}1\\-6\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$, $\begin{bmatrix}-2x_1\\3x_1\end{bmatrix}+\begin{bmatrix}8x_2\\5x_2\end{bmatrix}+\begin{bmatrix}x_3\\-6x_3\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$, $\begin{bmatrix}-2x_1+8x_2+x_3\\3x_1+5x_2-6x_3\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$

$$\begin{aligned}-2x_2 &+ 8x_2 &+ x_3 &= 0\\ 3x_1 &+ 5x_2 &- 6x_3 &= 0\end{aligned}$$

Usually the intermediate steps are not displayed.

7. See the figure below. Since the grid can be extended in every direction, the figure suggests that every vector in $\mathbf{R}^2$ can be written as a linear combination of **u** and **v**.

To write a vector **a** as a linear combination of **u** and **v**, imagine walking from the origin to **a** along the grid "streets" and keep track of how many "blocks" you travel in the **u**-direction and how many in the **v**-direction.

a. To reach **a** from the origin, you might travel 1 unit in the **u**-direction and –2 units in the **v**-direction (that is, 2 units in the negative **v**-direction). Hence **a** = **u** – 2**v**.

b. To reach **b** from the origin, travel 2 units in the **u**-direction and –2 units in the **v**-direction. So **b** = 2**u** – 2**v**. Or, use the fact that **b** is 1 unit in the **u**-direction from **a**, so that

$$\mathbf{b} = \mathbf{a} + \mathbf{u} = (\mathbf{u} - 2\mathbf{v}) + \mathbf{u} = 2\mathbf{u} - 2\mathbf{v}$$

c. The vector **c** is –1.5 units from **b** in the **v**-direction, so

$$\mathbf{c} = \mathbf{b} - 1.5\mathbf{v} = (2\mathbf{u} - 2\mathbf{v}) - 1.5\mathbf{v} = 2\mathbf{u} - 3.5\mathbf{v}$$

d. The "map" suggests that you can reach **d** if you travel 3 units in the **u**-direction and –4 units in the **v**-direction. If you prefer to stay on the paths displayed on the map, you might travel from the origin to –3**v**, then move 3 units in the **u**-direction, and finally move –1 unit in the **v**-direction. So

$$\mathbf{d} = -3\mathbf{v} + 3\mathbf{u} - \mathbf{v} = 3\mathbf{u} - 4\mathbf{v}$$

Another solution is

$$\mathbf{d} = \mathbf{b} - 2\mathbf{v} + \mathbf{u} = (2\mathbf{u} - 2\mathbf{v}) - 2\mathbf{v} + \mathbf{u} = 3\mathbf{u} - 4\mathbf{v}$$

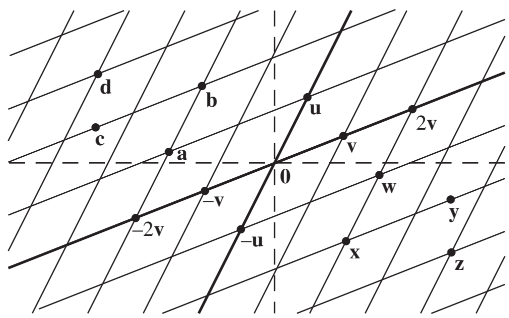


Figure for Exercises 7 and 8

8. See the figure above. Since the grid can be extended in every direction, the figure suggests that every vector in $\mathbf{R}^2$ can be written as a linear combination of **u** and **v**.

w. To reach **w** from the origin, travel –1 units in the **u**-direction (that is, 1 unit in the negative **u**-direction) and travel 2 units in the **v**-direction. Thus, **w** = (–1)**u** + 2**v**, or **w** = 2**v** – **u**.

x. To reach **x** from the origin, travel 2 units in the **v**-direction and –2 units in the **u**-direction. Thus, **x** = –2**u** + 2**v**. Or, use the fact that **x** is –1 units in the **u**-direction from **w**, so that

$$\mathbf{x} = \mathbf{w} - \mathbf{u} = (-\mathbf{u} + 2\mathbf{v}) - \mathbf{u} = -2\mathbf{u} + 2\mathbf{v}$$

y. The vector **y** is 1.5 units from **x** in the **v**-direction, so

$$\mathbf{y} = \mathbf{x} + 1.5\mathbf{v} = (-2\mathbf{u} + 2\mathbf{v}) + 1.5\mathbf{v} = -2\mathbf{u} + 3.5\mathbf{v}$$

z. The map suggests that you can reach **z** if you travel 4 units in the **v**-direction and –3 units in the **u**-direction. So **z** = 4**v** – 3**u** = –3**u** + 4**v**. If you prefer to stay on the paths displayed on the "map," you might travel from the origin to –2**u**, then 4 units in the **v**-direction, and finally move –1 unit in the **u**-direction. So

$$\mathbf{z} = -2\mathbf{u} + 4\mathbf{v} - \mathbf{u} = -3\mathbf{u} + 4\mathbf{v}$$

9. $$\begin{array}{rcrcrcl} & & x_2 & + & 5x_3 & = & 0 \\ 4x_1 & + & 6x_2 & - & x_3 & = & 0 \\ -x_1 & + & 3x_2 & - & 8x_3 & = & 0 \end{array}, \qquad \begin{bmatrix} x_2 + 5x_3 \\ 4x_1 + 6x_2 - x_3 \\ -x_1 + 3x_2 - 8x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 4x_1 \\ -x_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ 6x_2 \\ 3x_2 \end{bmatrix} + \begin{bmatrix} 5x_3 \\ -x_3 \\ -8x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \qquad x_1\begin{bmatrix} 0 \\ 4 \\ -1 \end{bmatrix} + x_2\begin{bmatrix} 1 \\ 6 \\ 3 \end{bmatrix} + x_3\begin{bmatrix} 5 \\ -1 \\ -8 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Usually, the intermediate calculations are not displayed.

Note: The *Study Guide* says, "Check with your instructor whether you need to "show work" on a problem such as Exercise 9."

10. $\begin{aligned} 4x_1 &+ x_2 + 3x_3 = 9 \\ x_1 &- 7x_2 - 2x_3 = 2 \\ 8x_1 &+ 6x_2 - 5x_3 = 15 \end{aligned}$, $\begin{bmatrix} 4x_1 + x_2 + 3x_3 \\ x_1 - 7x_2 - 2x_3 \\ 8x_1 + 6x_2 - 5x_3 \end{bmatrix} = \begin{bmatrix} 9 \\ 2 \\ 15 \end{bmatrix}$

$\begin{bmatrix} 4x_1 \\ x_1 \\ 8x_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ -7x_2 \\ 6x_2 \end{bmatrix} + \begin{bmatrix} 3x_3 \\ -2x_3 \\ -5x_3 \end{bmatrix} = \begin{bmatrix} 9 \\ 2 \\ 15 \end{bmatrix}$, $x_1 \begin{bmatrix} 4 \\ 1 \\ 8 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ -7 \\ 6 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ -2 \\ -5 \end{bmatrix} = \begin{bmatrix} 9 \\ 2 \\ 15 \end{bmatrix}$

Usually, the intermediate calculations are not displayed.

11. The question

Is **b** a linear combination of $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$?

is equivalent to the question

Does the vector equation $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3 = \mathbf{b}$ have a solution?

The equation

$$\underset{\mathbf{a}_1}{\underset{\uparrow}{x_1 \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}}} + \underset{\mathbf{a}_2}{\underset{\uparrow}{x_2 \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}}} + \underset{\mathbf{a}_3}{\underset{\uparrow}{x_3 \begin{bmatrix} 5 \\ -6 \\ 8 \end{bmatrix}}} = \underset{\mathbf{b}}{\underset{\uparrow}{\begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix}}} \qquad (*)$$

has the same solution set as the linear system whose augmented matrix is

$$M = \begin{bmatrix} 1 & 0 & 5 & 2 \\ -2 & 1 & -6 & -1 \\ 0 & 2 & 8 & 6 \end{bmatrix}$$

Row reduce M until the pivot positions are visible:

$$M \sim \begin{bmatrix} 1 & 0 & 5 & 2 \\ 0 & 1 & 4 & 3 \\ 0 & 2 & 8 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 5 & 2 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The linear system corresponding to M *has* a solution, so the vector equation (*) has a solution, and therefore **b** *is* a linear combination of $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$.

12. The equation

$$\underset{\mathbf{a}_1}{\underset{\uparrow}{x_1 \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}}} + \underset{\mathbf{a}_2}{\underset{\uparrow}{x_2 \begin{bmatrix} 0 \\ 5 \\ 5 \end{bmatrix}}} + \underset{\mathbf{a}_3}{\underset{\uparrow}{x_3 \begin{bmatrix} 2 \\ 0 \\ 8 \end{bmatrix}}} = \underset{\mathbf{b}}{\underset{\uparrow}{\begin{bmatrix} -5 \\ 11 \\ -7 \end{bmatrix}}} \qquad (*)$$

has the same solution set as the linear system whose augmented matrix is

$$M = \begin{bmatrix} 1 & 0 & 2 & -5 \\ -2 & 5 & 0 & 11 \\ 2 & 5 & 8 & -7 \end{bmatrix}$$

Row reduce M until the pivot positions are visible:

$$M \sim \begin{bmatrix} 1 & 0 & 2 & -5 \\ 0 & 5 & 4 & 1 \\ 0 & 5 & 4 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 & -5 \\ 0 & 5 & 4 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

The linear system corresponding to M has *no* solution, so the vector equation (*) has no solution, and therefore $\mathbf{b}$ is *not* a linear combination of $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$.

13. Denote the columns of A by $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$. To determine if $\mathbf{b}$ is a linear combination of these columns, use the boxed fact on page 34. Row reduced the augmented matrix until you reach echelon form:

$$\begin{bmatrix} 1 & -4 & 2 & 3 \\ 0 & 3 & 5 & -7 \\ -2 & 8 & -4 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & -4 & 2 & 3 \\ 0 & 3 & 5 & -7 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

The system for this augmented matrix is inconsistent, so $\mathbf{b}$ is *not* a linear combination of the columns of A.

14. $[\mathbf{a}_1 \;\; \mathbf{a}_2 \;\; \mathbf{a}_3 \;\; \mathbf{b}] = \begin{bmatrix} 1 & -2 & -6 & 11 \\ 0 & 3 & 7 & -5 \\ 1 & -2 & 5 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & -6 & 11 \\ 0 & 3 & 7 & -5 \\ 0 & 0 & 11 & -2 \end{bmatrix}$. The linear system corresponding to this matrix *has* a solution, so $\mathbf{b}$ is a linear combination of the columns of A.

15. Noninteger weights are acceptable, of course, but some simple choices are $0\cdot\mathbf{v}_1 + 0\cdot\mathbf{v}_2 = \mathbf{0}$, and

$$1\cdot\mathbf{v}_1 + 0\cdot\mathbf{v}_2 = \begin{bmatrix} 7 \\ 1 \\ -6 \end{bmatrix}, \quad 0\cdot\mathbf{v}_1 + 1\cdot\mathbf{v}_2 = \begin{bmatrix} -5 \\ 3 \\ 0 \end{bmatrix}$$

$$1\cdot\mathbf{v}_1 + 1\cdot\mathbf{v}_2 = \begin{bmatrix} 2 \\ 4 \\ -6 \end{bmatrix}, \quad 1\cdot\mathbf{v}_1 - 1\cdot\mathbf{v}_2 = \begin{bmatrix} 12 \\ -2 \\ -6 \end{bmatrix}$$

16. Some likely choices are $0\cdot\mathbf{v}_1 + 0\cdot\mathbf{v}_2 = \mathbf{0}$, and

$$1\cdot\mathbf{v}_1 + 0\cdot\mathbf{v}_2 = \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}, \quad 0\cdot\mathbf{v}_1 + 1\cdot\mathbf{v}_2 = \begin{bmatrix} -2 \\ 0 \\ 3 \end{bmatrix}$$

$$1\cdot\mathbf{v}_1 + 1\cdot\mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix}, \quad 1\cdot\mathbf{v}_1 - 1\cdot\mathbf{v}_2 = \begin{bmatrix} 5 \\ 0 \\ -1 \end{bmatrix}$$

17. $[\mathbf{a}_1\ \ \mathbf{a}_2\ \ \mathbf{b}] = \begin{bmatrix} 1 & -2 & 4 \\ 4 & -3 & 1 \\ -2 & 7 & h \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 4 \\ 0 & 5 & -15 \\ 0 & 3 & h+8 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 4 \\ 0 & 1 & -3 \\ 0 & 3 & h+8 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 4 \\ 0 & 1 & -3 \\ 0 & 0 & h+17 \end{bmatrix}$. The vector **b** is in Span$\{\mathbf{a}_1, \mathbf{a}_2\}$ when $h + 17$ is zero, that is, when $h = -17$.

18. $[\mathbf{v}_1\ \ \mathbf{v}_2\ \ \mathbf{y}] = \begin{bmatrix} 1 & -3 & h \\ 0 & 1 & -5 \\ -2 & 8 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & h \\ 0 & 1 & -5 \\ 0 & 2 & -3+2h \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & h \\ 0 & 1 & -5 \\ 0 & 0 & 7+2h \end{bmatrix}$. The vector **y** is in Span$\{\mathbf{v}_1, \mathbf{v}_2\}$ when $7 + 2h$ is zero, that is, when $h = -7/2$.

19. By inspection, $\mathbf{v}_2 = (3/2)\mathbf{v}_1$. Any linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$ is actually just a multiple of $\mathbf{v}_1$. For instance,

$$a\mathbf{v}_1 + b\mathbf{v}_2 = a\mathbf{v}_1 + b(3/2)\mathbf{v}_2 = (a + 3b/2)\mathbf{v}_1$$

So Span$\{\mathbf{v}_1, \mathbf{v}_2\}$ is the set of points on the line through $\mathbf{v}_1$ and **0**.

Note: Exercises 19 and 20 prepare the way for ideas in Sections 1.4 and 1.7.

20. Span$\{\mathbf{v}_1, \mathbf{v}_2\}$ is a plane in $\mathbf{R}^3$ through the origin, because the neither vector in this problem is a multiple of the other. Every vector in the set has 0 as its second entry and so lies in the *xz*-plane in ordinary 3-space. So Span$\{\mathbf{v}_1, \mathbf{v}_2\}$ *is* the *xz*-plane.

21. Let $\mathbf{y} = \begin{bmatrix} h \\ k \end{bmatrix}$. Then $[\mathbf{u}\ \ \mathbf{v}\ \ \mathbf{y}] = \begin{bmatrix} 2 & 2 & h \\ -1 & 1 & k \end{bmatrix} \sim \begin{bmatrix} 2 & 2 & h \\ 0 & 2 & k+h/2 \end{bmatrix}$. This augmented matrix corresponds to a consistent system for all h and k. So **y** is in Span$\{\mathbf{u}, \mathbf{v}\}$ for all h and k.

22. Construct any 3×4 matrix in echelon form that corresponds to an inconsistent system. Perform sufficient row operations on the matrix to eliminate all zero entries in the first three columns.

23. **a**. False. The alternative notation for a (column) vector is (–4, 3), using parentheses and commas.

b. False. Plot the points to verify this. Or, see the statement preceding Example 3. If $\begin{bmatrix} -5 \\ 2 \end{bmatrix}$ were on the line through $\begin{bmatrix} -2 \\ 5 \end{bmatrix}$ and the origin, then $\begin{bmatrix} -5 \\ 2 \end{bmatrix}$ would have to be a multiple of $\begin{bmatrix} -2 \\ 5 \end{bmatrix}$, which is not the case.

c. True. See the line displayed just before Example 4.

d. True. See the box that discusses the matrix in (5).

e. False. The statement is often true, but Span$\{\mathbf{u}, \mathbf{v}\}$ is not a plane when **v** is a multiple of **u**, or when **u** is the zero vector.

24. **a**. True. See the beginning of the subsection *Vectors in* $\mathbf{R}^n$.

b. True. Use Fig. 7 to draw the parallelogram determined by **u** – **v** and **v**.

c. False. See the first paragraph of the subsection *Linear Combinations*.

d. True. See the statement that refers to Fig. 11.

e. True. See the paragraph following the definition of Span$\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$.

25. **a**. There are only three vectors in the set $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$, and $\mathbf{b}$ is not one of them.

b. There are infinitely many vectors in $W = \text{Span}\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$. To determine if $\mathbf{b}$ is in W, use the method of Exercise 13.

$$\begin{array}{c}\begin{bmatrix} 1 & 0 & -4 & 4 \\ 0 & 3 & -2 & 1 \\ -2 & 6 & 3 & -4 \end{bmatrix} \\ \begin{matrix} \uparrow & \uparrow & \uparrow & \uparrow \\ \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{b} \end{matrix}\end{array} \sim \begin{bmatrix} 1 & 0 & -4 & 4 \\ 0 & 3 & -2 & 1 \\ 0 & 6 & -5 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -4 & 4 \\ 0 & 3 & -2 & 1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

The system for this augmented matrix is consistent, so $\mathbf{b}$ is in W.

c. $\mathbf{a}_1 = 1\mathbf{a}_1 + 0\mathbf{a}_2 + 0\mathbf{a}_3$. See the discussion in the text following the definition of $\text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$.

26. **a**. $[\mathbf{a}_1 \;\; \mathbf{a}_2 \;\; \mathbf{a}_3 \;\; \mathbf{b}] = \begin{bmatrix} 2 & 0 & 6 & 10 \\ -1 & 8 & 5 & 3 \\ 1 & -2 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 5 \\ -1 & 8 & 5 & 3 \\ 1 & -2 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 5 \\ 0 & 8 & 8 & 8 \\ 0 & -2 & -2 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 5 \\ 0 & 8 & 8 & 8 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Yes, $\mathbf{b}$ is a linear combination of the columns of A, that is, $\mathbf{b}$ is in W.

b. The third column of A is in W because $\mathbf{a}_3 = 0\cdot\mathbf{a}_1 + 0\cdot\mathbf{a}_2 + 1\cdot\mathbf{a}_3$.

27. **a**. $5\mathbf{v}_1$ is the output of 5 days' operation of mine #1.

b. The total output is $x_1\mathbf{v}_1 + x_2\mathbf{v}_2$, so x_1 and x_2 should satisfy $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 = \begin{bmatrix} 150 \\ 2825 \end{bmatrix}$.

c. **[M]** Reduce the augmented matrix $\begin{bmatrix} 20 & 30 & 150 \\ 550 & 500 & 2825 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1.5 \\ 0 & 1 & 4.0 \end{bmatrix}$.

Operate mine #1 for 1.5 days and mine #2 for 4 days. (This is the exact solution.)

28. **a**. The amount of heat produced when the steam plant burns x_1 tons of anthracite and x_2 tons of bituminous coal is $27.6x_1 + 30.2x_2$ million Btu.

b. The total output produced by x_1 tons of anthracite and x_2 tons of bituminous coal is given by the vector $x_1\begin{bmatrix} 27.6 \\ 3100 \\ 250 \end{bmatrix} + x_2\begin{bmatrix} 30.2 \\ 6400 \\ 360 \end{bmatrix}$.

c. **[M]** The appropriate values for x_1 and x_2 satisfy $x_1\begin{bmatrix} 27.6 \\ 3100 \\ 250 \end{bmatrix} + x_2\begin{bmatrix} 30.2 \\ 6400 \\ 360 \end{bmatrix} = \begin{bmatrix} 162 \\ 23{,}610 \\ 1{,}623 \end{bmatrix}$.

To solve, row reduce the augmented matrix:

$$\begin{bmatrix} 27.6 & 30.2 & 162 \\ 3100 & 6400 & 23610 \\ 250 & 360 & 1623 \end{bmatrix} \sim \begin{bmatrix} 1.000 & 0 & 3.900 \\ 0 & 1.000 & 1.800 \\ 0 & 0 & 0 \end{bmatrix}$$

The steam plant burned 3.9 tons of anthracite coal and 1.8 tons of bituminous coal.

29. The total mass is 2 + 5 + 2 + 1 = 10. So $\mathbf{v} = (2\mathbf{v}_1 + 5\mathbf{v}_2 + 2\mathbf{v}_3 + \mathbf{v}_4)/10$. That is,

$$\mathbf{v} = \frac{1}{10}\left(2\begin{bmatrix}5\\-4\\3\end{bmatrix} + 5\begin{bmatrix}4\\3\\-2\end{bmatrix} + 2\begin{bmatrix}-4\\-3\\-1\end{bmatrix} + \begin{bmatrix}-9\\8\\6\end{bmatrix}\right) = \frac{1}{10}\begin{bmatrix}10+20-8-9\\-8+15-6+8\\6-10-2+6\end{bmatrix} = \begin{bmatrix}1.3\\.9\\0\end{bmatrix}$$

30. Let m be the total mass of the system. By definition,

$$\mathbf{v} = \frac{1}{m}(m_1\mathbf{v}_1 + \cdots + m_k\mathbf{v}_k) = \frac{m_1}{m}\mathbf{v}_1 + \cdots + \frac{m_k}{m}\mathbf{v}_k$$

The second expression displays $\mathbf{v}$ as a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_k$, which shows that $\mathbf{v}$ is in Span$\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$.

31. **a**. The center of mass is $\frac{1}{3}\left(1\cdot\begin{bmatrix}0\\1\end{bmatrix} + 1\cdot\begin{bmatrix}8\\1\end{bmatrix} + 1\cdot\begin{bmatrix}2\\4\end{bmatrix}\right) = \begin{bmatrix}10/3\\2\end{bmatrix}$.

b. The total mass of the new system is 9 grams. The three masses added, w_1, w_2, and w_3, satisfy the equation

$$\frac{1}{9}\left((w_1+1)\cdot\begin{bmatrix}0\\1\end{bmatrix} + (w_2+1)\cdot\begin{bmatrix}8\\1\end{bmatrix} + (w_3+1)\cdot\begin{bmatrix}2\\4\end{bmatrix}\right) = \begin{bmatrix}2\\2\end{bmatrix}$$

which can be rearranged to

$$(w_1+1)\cdot\begin{bmatrix}0\\1\end{bmatrix} + (w_2+1)\cdot\begin{bmatrix}8\\1\end{bmatrix} + (w_3+1)\cdot\begin{bmatrix}2\\4\end{bmatrix} = \begin{bmatrix}18\\18\end{bmatrix}$$

and

$$w_1\cdot\begin{bmatrix}0\\1\end{bmatrix} + w_2\cdot\begin{bmatrix}8\\1\end{bmatrix} + w_3\cdot\begin{bmatrix}2\\4\end{bmatrix} = \begin{bmatrix}8\\12\end{bmatrix}$$

The condition $w_1 + w_2 + w_3 = 6$ and the vector equation above combine to produce a system of three equations whose augmented matrix is shown below, along with a sequence of row operations:

$$\begin{bmatrix}1&1&1&6\\0&8&2&8\\1&1&4&12\end{bmatrix} \sim \begin{bmatrix}1&1&1&6\\0&8&2&8\\0&0&3&6\end{bmatrix} \sim \begin{bmatrix}1&1&1&6\\0&8&2&8\\0&0&1&2\end{bmatrix}$$

$$\sim \begin{bmatrix}1&1&0&4\\0&8&0&4\\0&0&1&2\end{bmatrix} \sim \begin{bmatrix}1&0&0&3.5\\0&8&0&4\\0&0&1&2\end{bmatrix} \sim \begin{bmatrix}1&0&0&3.5\\0&1&0&.5\\0&0&1&2\end{bmatrix}$$

Answer: Add 3.5 g at (0, 1), add .5 g at (8, 1), and add 2 g at (2, 4).

Extra problem: Ignore the mass of the plate, and distribute 6 gm at the three vertices to make the center of mass at (2, 2). Answer: Place 3 g at (0, 1), 1 g at (8, 1), and 2 g at (2, 4).

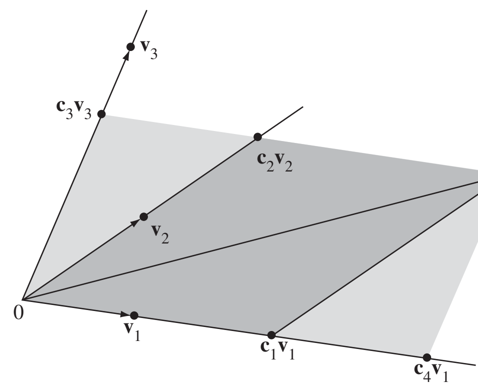
32. See the parallelograms drawn on Fig. 15 from the text. Here c_1, c_2, c_3, and c_4 are suitable scalars. The darker parallelogram shows that $\mathbf{b}$ is a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$, that is

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + 0\cdot\mathbf{v}_3 = \mathbf{b}$$

The larger parallelogram shows that **b** is a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_3$, that is,

$$c_4\mathbf{v}_1 + 0\cdot\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{b}$$

So the equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{b}$ has at least two solutions, not just one solution. (In fact, the equation has infinitely many solutions.)



33. **a**. For $j = 1,\ldots, n$, the jth entry of $(\mathbf{u} + \mathbf{v}) + \mathbf{w}$ is $(u_j + v_j) + w_j$. By associativity of addition in **R**, this entry equals $u_j + (v_j + w_j)$, which is the jth entry of $\mathbf{u} + (\mathbf{v} + \mathbf{w})$. By definition of equality of vectors, $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$.

b. For any scalar c, the jth entry of $c(\mathbf{u} + \mathbf{v})$ is $c(u_j + v_j)$, and the jth entry of $c\mathbf{u} + c\mathbf{v}$ is $cu_j + cv_j$ (by definition of scalar multiplication and vector addition). These entries are equal, by a distributive law in **R**. So $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$.

34. **a**. For $j = 1,\ldots, n$, $u_j + (-1)u_j = (-1)u_j + u_j = 0$, by properties of **R**. By vector equality,

$$\mathbf{u} + (-1)\mathbf{u} = (-1)\mathbf{u} + \mathbf{u} = \mathbf{0}.$$

b. For scalars c and d, the jth entries of $c(d\mathbf{u})$ and $(cd)\mathbf{u}$ are $c(du_j)$ and $(cd)u_j$, respectively. These entries in **R** are equal, so the vectors $c(d\mathbf{u})$ and $(cd)\mathbf{u}$ are equal.

Note: When an exercise in this section involves a vector equation, the corresponding technology data (in the data files on the web) is usually presented as a set of (column) vectors. To use MATLAB or other technology, a student must first construct an augmented matrix from these vectors. The MATLAB note in the *Study Guide* describes how to do this. The appendices in the *Study Guide* give corresponding information about Maple, Mathematica, and the TI and HP calculators.

1.4 SOLUTIONS

Notes: Key exercises are 1–20, 27, 28, 31 and 32. Exercises 29, 30, 33, and 34 are harder. Exercise 34 anticipates the Invertible Matrix Theorem but is not used in the proof of that theorem.

1. The matrix-vector product $A\mathbf{x}$ product is not defined because the number of columns (2) in the 3×2 matrix $\begin{bmatrix} -4 & 2 \\ 1 & 6 \\ 0 & 1 \end{bmatrix}$ does not match the number of entries (3) in the vector $\begin{bmatrix} 3 \\ -2 \\ 7 \end{bmatrix}$.

2. The matrix-vector product $A\mathbf{x}$ product is not defined because the number of columns (1) in the 3×1 matrix $\begin{bmatrix} 2 \\ 6 \\ -1 \end{bmatrix}$ does not match the number of entries (2) in the vector $\begin{bmatrix} 5 \\ -1 \end{bmatrix}$.

3. $A\mathbf{x} = \begin{bmatrix} 6 & 5 \\ -4 & -3 \\ 7 & 6 \end{bmatrix}\begin{bmatrix} 2 \\ -3 \end{bmatrix} = 2\begin{bmatrix} 6 \\ -4 \\ 7 \end{bmatrix} - 3\begin{bmatrix} 5 \\ -3 \\ 6 \end{bmatrix} = \begin{bmatrix} 12 \\ -8 \\ 14 \end{bmatrix} + \begin{bmatrix} -15 \\ 9 \\ -18 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ -4 \end{bmatrix}$, and

$$A\mathbf{x} = \begin{bmatrix} 6 & 5 \\ -4 & -3 \\ 7 & 6 \end{bmatrix}\begin{bmatrix} 2 \\ -3 \end{bmatrix} = \begin{bmatrix} 6\cdot 2 + 5\cdot(-3) \\ (-4)\cdot 2 + (-3)\cdot(-3) \\ 7\cdot 2 + 6\cdot(-3) \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ -4 \end{bmatrix}$$

4. $A\mathbf{x} = \begin{bmatrix} 8 & 3 & -4 \\ 5 & 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 1\cdot\begin{bmatrix} 8 \\ 5 \end{bmatrix} + 1\cdot\begin{bmatrix} 3 \\ 1 \end{bmatrix} + 1\cdot\begin{bmatrix} -4 \\ 2 \end{bmatrix} = \begin{bmatrix} 8+3-4 \\ 5+1+2 \end{bmatrix} = \begin{bmatrix} 7 \\ 8 \end{bmatrix}$, and

$$A\mathbf{x} = \begin{bmatrix} 8 & 3 & -4 \\ 5 & 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 8\cdot 1 + 3\cdot 1 + (-4)\cdot 1 \\ 5\cdot 1 + 1\cdot 1 + 2\cdot 1 \end{bmatrix} = \begin{bmatrix} 7 \\ 8 \end{bmatrix}$$

5. On the left side of the matrix equation, use the entries in the vector **x** as the weights in a linear combination of the columns of the matrix A:

$$5\cdot\begin{bmatrix} 5 \\ -2 \end{bmatrix} - 1\cdot\begin{bmatrix} 1 \\ -7 \end{bmatrix} + 3\cdot\begin{bmatrix} -8 \\ 3 \end{bmatrix} - 2\cdot\begin{bmatrix} 4 \\ -5 \end{bmatrix} = \begin{bmatrix} -8 \\ 16 \end{bmatrix}$$

6. On the left side of the matrix equation, use the entries in the vector **x** as the weights in a linear combination of the columns of the matrix A:

$$-2\cdot\begin{bmatrix} 7 \\ 2 \\ 9 \\ -3 \end{bmatrix} - 5\cdot\begin{bmatrix} -3 \\ 1 \\ -6 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ -9 \\ 12 \\ -4 \end{bmatrix}$$

7. The left side of the equation is a linear combination of three vectors. Write the matrix A whose columns are those three vectors, and create a variable vector **x** with three entries:

$A = \left[\begin{bmatrix} 4 \\ -1 \\ 7 \\ -4 \end{bmatrix} \begin{bmatrix} -5 \\ 3 \\ -5 \\ 1 \end{bmatrix} \begin{bmatrix} 7 \\ -8 \\ 0 \\ 2 \end{bmatrix}\right] = \begin{bmatrix} 4 & -5 & 7 \\ -1 & 3 & -8 \\ 7 & -5 & 0 \\ -4 & 1 & 2 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$. Thus the equation $A\mathbf{x} = \mathbf{b}$ is

$$\begin{bmatrix} 4 & -5 & 7 \\ -1 & 3 & -8 \\ 7 & -5 & 0 \\ -4 & 1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ -8 \\ 0 \\ -7 \end{bmatrix}$$

For your information: The unique solution of this equation is (5, 7, 3). Finding the solution by hand would be time-consuming.

Note: The skill of writing a vector equation as a matrix equation will be important for both theory and application throughout the text. See also Exercises 27 and 28.

8. The left side of the equation is a linear combination of four vectors. Write the matrix A whose columns are those four vectors, and create a variable vector with four entries:

$$A = \left[\begin{bmatrix} 4 \\ -2 \end{bmatrix} \begin{bmatrix} -4 \\ 5 \end{bmatrix} \begin{bmatrix} -5 \\ 4 \end{bmatrix} \begin{bmatrix} 3 \\ 0 \end{bmatrix}\right] = \begin{bmatrix} 4 & -4 & -5 & 3 \\ -2 & 5 & 4 & 0 \end{bmatrix}, \text{ and } \mathbf{z} = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix}. \text{ Then the equation } A\mathbf{z} = \mathbf{b}$$

is $\begin{bmatrix} 4 & -4 & -5 & 3 \\ -2 & 5 & 4 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} = \begin{bmatrix} 4 \\ 13 \end{bmatrix}$.

For your information: One solution is (7, 3, 3, 1). The general solution is $z_1 = 6 + .75z_3 - 1.25z_4$, $z_2 = 5 - .5z_3 - .5z_4$, with z_3 and z_4 free.

9. The system has the same solution set as the vector equation

$$x_1\begin{bmatrix} 3 \\ 0 \end{bmatrix} + x_2\begin{bmatrix} 1 \\ 1 \end{bmatrix} + x_3\begin{bmatrix} -5 \\ 4 \end{bmatrix} = \begin{bmatrix} 9 \\ 0 \end{bmatrix}$$

and this equation has the same solution set as the matrix equation

$$\begin{bmatrix} 3 & 1 & -5 \\ 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 9 \\ 0 \end{bmatrix}$$

10. The system has the same solution set as the vector equation

$$x_1\begin{bmatrix} 8 \\ 5 \\ 1 \end{bmatrix} + x_2\begin{bmatrix} -1 \\ 4 \\ -3 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$$

and this equation has the same solution set as the matrix equation

$$\begin{bmatrix} 8 & -1 \\ 5 & 4 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$$

11. To solve $A\mathbf{x} = \mathbf{b}$, row reduce the augmented matrix $[\mathbf{a}_1 \;\; \mathbf{a}_2 \;\; \mathbf{a}_3 \;\; \mathbf{b}]$ for the corresponding linear system:

$$\begin{bmatrix} 1 & 2 & 4 & -2 \\ 0 & 1 & 5 & 2 \\ -2 & -4 & -3 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 4 & -2 \\ 0 & 1 & 5 & 2 \\ 0 & 0 & 5 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 4 & -2 \\ 0 & 1 & 5 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & -6 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

The solution is $\begin{cases} x_1 &= & 0 \\ x_2 &= & -3 \\ x_3 &= & 1 \end{cases}$. As a vector, the solution is $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \\ 1 \end{bmatrix}$.

12. To solve $A\mathbf{x} = \mathbf{b}$, row reduce the augmented matrix $[\mathbf{a}_1 \ \ \mathbf{a}_2 \ \ \mathbf{a}_3 \ \ \mathbf{b}]$ for the corresponding linear system:

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ -3 & -1 & 2 & 1 \\ 0 & 5 & 3 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 5 & 5 & 1 \\ 0 & 5 & 3 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 5 & 5 & 1 \\ 0 & 0 & -2 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 5 & 5 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 5 & 0 & -4 \\ 0 & 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 1 & 0 & -4/5 \\ 0 & 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 3/5 \\ 0 & 1 & 0 & -4/5 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

The solution is $\begin{cases} x_1 &= & 3/5 \\ x_2 &= & -4/5 \\ x_3 &= & 1 \end{cases}$. As a vector, the solution is $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3/5 \\ -4/5 \\ 1 \end{bmatrix}$.

13. The vector $\mathbf{u}$ is in the plane spanned by the columns of A if and only if $\mathbf{u}$ is a linear combination of the columns of A. This happens if and only if the equation $A\mathbf{x} = \mathbf{u}$ has a solution. (See the box preceding Example 3 in Section 1.4.) To study this equation, reduce the augmented matrix $[A \ \ \mathbf{u}]$

$$\begin{bmatrix} 3 & -5 & 0 \\ -2 & 6 & 4 \\ 1 & 1 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 4 \\ -2 & 6 & 4 \\ 3 & -5 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 4 \\ 0 & 8 & 12 \\ 0 & -8 & -12 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 4 \\ 0 & 8 & 12 \\ 0 & 0 & 0 \end{bmatrix}$$

The equation $A\mathbf{x} = \mathbf{u}$ has a solution, so $\mathbf{u}$ is in the plane spanned by the columns of A.

For your information: The unique solution of $A\mathbf{x} = \mathbf{u}$ is (5/2, 3/2).

14. Reduce the augmented matrix $[A \ \ \mathbf{u}]$ to echelon form:

$$\begin{bmatrix} 5 & 8 & 7 & 2 \\ 0 & 1 & -1 & -3 \\ 1 & 3 & 0 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 0 & 2 \\ 0 & 1 & -1 & -3 \\ 5 & 8 & 7 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 0 & 2 \\ 0 & 1 & -1 & -3 \\ 0 & -7 & 7 & -8 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 0 & 2 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 0 & -29 \end{bmatrix}$$

The equation $A\mathbf{x} = \mathbf{u}$ has no solution, so $\mathbf{u}$ is not in the subset spanned by the columns of A.

15. The augmented matrix for $A\mathbf{x} = \mathbf{b}$ is $\begin{bmatrix} 2 & -1 & b_1 \\ -6 & 3 & b_2 \end{bmatrix}$, which is row equivalent to $\begin{bmatrix} 2 & -1 & b_1 \\ 0 & 0 & b_2 + 3b_1 \end{bmatrix}$.

This shows that the equation $A\mathbf{x} = \mathbf{b}$ is not consistent when $3b_1 + b_2$ is nonzero. The set of $\mathbf{b}$ for which the equation *is* consistent is a line through the origin–the set of all points (b_1, b_2) satisfying $b_2 = -3b_1$.

16. Row reduce the augmented matrix $[A \ \ \mathbf{b}]$: $A = \begin{bmatrix} 1 & -3 & -4 \\ -3 & 2 & 6 \\ 5 & -1 & -8 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$.

$$\begin{bmatrix} 1 & -3 & -4 & b_1 \\ -3 & 2 & 6 & b_2 \\ 5 & -1 & -8 & b_3 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & -4 & b_1 \\ 0 & -7 & -6 & b_2 + 3b_1 \\ 0 & 14 & 12 & b_3 - 5b_1 \end{bmatrix}$$

$$\sim\begin{bmatrix} 1 & -3 & -4 & b_1 \\ 0 & -7 & -6 & b_2+3b_1 \\ 0 & 0 & 0 & b_3-5b_1+2(b_2+3b_1) \end{bmatrix} = \begin{bmatrix} 1 & -3 & -4 & b_1 \\ 0 & -7 & -6 & b_2+3b_1 \\ 0 & 0 & 0 & b_1+2b_2+b_3 \end{bmatrix}$$

The equation $A\mathbf{x} = \mathbf{b}$ is consistent if and only if $b_1 + 2b_2 + b_3 = 0$. The set of such $\mathbf{b}$ is a plane through the origin in $\mathbf{R}^3$.

17. Row reduction shows that only three rows of A contain a pivot position:

$$A = \begin{bmatrix} 1 & 3 & 0 & 3 \\ -1 & -1 & -1 & 1 \\ 0 & -4 & 2 & -8 \\ 2 & 0 & 3 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 0 & 3 \\ 0 & 2 & -1 & 4 \\ 0 & -4 & 2 & -8 \\ 0 & -6 & 3 & -7 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 0 & 3 \\ 0 & 2 & -1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 0 & 3 \\ 0 & 2 & -1 & 4 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Because not every row of A contains a pivot position, Theorem 4 in Section 1.4 shows that the equation $A\mathbf{x} = \mathbf{b}$ does *not* have a solution for each $\mathbf{b}$ in $\mathbf{R}^4$.

18. Row reduction shows that only three rows of B contain a pivot position:

$$B = \begin{bmatrix} 1 & 3 & -2 & 2 \\ 0 & 1 & 1 & -5 \\ 1 & 2 & -3 & 7 \\ -2 & -8 & 2 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & -2 & 2 \\ 0 & 1 & 1 & -5 \\ 0 & -1 & -1 & 5 \\ 0 & -2 & -2 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & -2 & 2 \\ 0 & 1 & 1 & -5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -7 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & -2 & 2 \\ 0 & 1 & 1 & -5 \\ 0 & 0 & 0 & -7 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Because not every row of B contains a pivot position, Theorem 4 in Section 1.4 shows that the equation $B\mathbf{x} = \mathbf{y}$ does *not* have a solution for each $\mathbf{y}$ in $\mathbf{R}^4$.

19. The work in Exercise 17 shows that statement (d) in Theorem 4 is false. So all four statements in Theorem 4 are false. Thus, not all vectors in $\mathbf{R}^4$ can be written as a linear combination of the columns of A. Also, the columns of A do *not* span $\mathbf{R}^4$.

20. The work in Exercise 18 shows that statement (d) in Theorem 4 is false. So all four statements in Theorem 4 are false. Thus, not all vectors in $\mathbf{R}^4$ can be written as a linear combination of the columns of B. The columns of B certainly do *not* span $\mathbf{R}^3$, because each column of B is in $\mathbf{R}^4$, not $\mathbf{R}^3$. (This question was asked to alert students to a fairly common misconception among students who are just learning about spanning.)

21. Row reduce the matrix $[\mathbf{v}_1\ \ \mathbf{v}_2\ \ \mathbf{v}_3]$ to determine whether it has a pivot in each row.

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

The matrix $[\mathbf{v}_1\ \ \mathbf{v}_2\ \ \mathbf{v}_3]$ does not have a pivot in each row, so the columns of the matrix do not span $\mathbf{R}^4$, by Theorem 4. That is, $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ does not span $\mathbf{R}^4$.

Note: Some students may realize that row operations are not needed, and thereby discover the principle covered in Exercises 31 and 32.

22. Row reduce the matrix $[\mathbf{v}_1\ \ \mathbf{v}_2\ \ \mathbf{v}_3]$ to determine whether it has a pivot in each row.

$$\begin{bmatrix} 0 & 0 & 4 \\ 0 & -3 & -1 \\ -2 & 8 & -5 \end{bmatrix} \sim \begin{bmatrix} -2 & 8 & -5 \\ 0 & -3 & -1 \\ 0 & 0 & 4 \end{bmatrix}$$

The matrix $[\mathbf{v}_1\ \ \mathbf{v}_2\ \ \mathbf{v}_3]$ has a pivot in each row, so the columns of the matrix span $\mathbf{R}^4$, by Theorem 4. That is, $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ spans $\mathbf{R}^4$.

23. **a**. False. See the paragraph following equation (3). The text calls $A\mathbf{x} = \mathbf{b}$ a *matrix equation*.

b. True. See the box before Example 3.

c. False. See the warning following Theorem 4.

d. True. See Example 4.

e. True. See parts (c) and (a) in Theorem 4.

f. True. In Theorem 4, statement (a) is false if and only if statement (d) is also false.

24. **a**. True. This statement is in Theorem 3. However, the statement is true without any "proof" because, by definition, $A\mathbf{x}$ is simply a notation for $x_1\mathbf{a}_1 + \cdots + x_n\mathbf{a}_n$, where $\mathbf{a}_1, \ldots, \mathbf{a}_n$ are the columns of A.

b. True. See Example 2.

c. True, by Theorem 3.

d. True. See the box before Example 2. Saying that $\mathbf{b}$ is not in the set spanned by the columns of A is the same a saying that $\mathbf{b}$ is not a linear combination of the columns of A.

e. False. See the warning that follows Theorem 4.

f. True. In Theorem 4, statement (c) is false if and only if statement (a) is also false.

25. By definition, the matrix-vector product on the left is a linear combination of the columns of the matrix, in this case using weights –3, –1, and 2. So $c_1 = -3$, $c_2 = -1$, and $c_3 = 2$.

26. The equation in x_1 and x_2 involves the vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$, and it may be viewed as

$\begin{bmatrix} \mathbf{u} & \mathbf{v} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{w}$. By definition of a matrix-vector product, $x_1\mathbf{u} + x_2\mathbf{v} = \mathbf{w}$. The stated fact that

$3\mathbf{u} - 5\mathbf{v} - \mathbf{w} = \mathbf{0}$ can be rewritten as $3\mathbf{u} - 5\mathbf{v} = \mathbf{w}$. So, a solution is $x_1 = 3$, $x_2 = -5$.

27. Place the vectors $\mathbf{q}_1$, $\mathbf{q}_2$, and $\mathbf{q}_3$ into the columns of a matrix, say, Q and place the weights x_1, x_2, and x_3 into a vector, say, $\mathbf{x}$. Then the vector equation becomes

$$Q\mathbf{x} = \mathbf{v}, \text{ where } Q = [\mathbf{q}_1\ \ \mathbf{q}_2\ \ \mathbf{q}_3] \text{ and } \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Note: If your answer is the equation $A\mathbf{x} = \mathbf{b}$, you need to specify what A and $\mathbf{b}$ are.

28. The matrix equation can be written as $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + c_4\mathbf{v}_4 + c_5\mathbf{v}_5 = \mathbf{v}_6$, where

$c_1 = -3$, $c_2 = 2$, $c_3 = 4$, $c_4 = -1$, $c_5 = 2$, and

$$\mathbf{v}_1 = \begin{bmatrix} -3 \\ 5 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 5 \\ 8 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} -4 \\ 1 \end{bmatrix}, \mathbf{v}_4 = \begin{bmatrix} 9 \\ -2 \end{bmatrix}, \mathbf{v}_5 = \begin{bmatrix} 7 \\ -4 \end{bmatrix}, \mathbf{v}_6 = \begin{bmatrix} 8 \\ -1 \end{bmatrix}$$

29. Start with any 3×3 matrix B in echelon form that has three pivot positions. Perform a row operation (a row interchange or a row replacement) that creates a matrix A that is *not* in echelon form. Then A has the desired property. The justification is given by row reducing A to B, in order to display the pivot positions. Since A has a pivot position in every row, the columns of A span $\mathbf{R}^3$, by Theorem 4.

30. Start with any nonzero 3×3 matrix B in echelon form that has fewer than three pivot positions. Perform a row operation that creates a matrix A that is *not* in echelon form. Then A has the desired property. Since A does not have a pivot position in every row, the columns of A do not span $\mathbf{R}^3$, by Theorem 4.

31. A 3×2 matrix has three rows and two columns. With only two columns, A can have at most two pivot columns, and so A has at most two pivot positions, which is not enough to fill all three rows. By Theorem 4, the equation $A\mathbf{x} = \mathbf{b}$ cannot be consistent for all $\mathbf{b}$ in $\mathbf{R}^3$. Generally, if A is an $m\times n$ matrix with $m > n$, then A can have at most n pivot positions, which is not enough to fill all m rows. Thus, the equation $A\mathbf{x} = \mathbf{b}$ cannot be consistent for all $\mathbf{b}$ in $\mathbf{R}^3$.

32. A set of three vectors in cannot span $\mathbf{R}^4$. Reason: the matrix A whose columns are these three vectors has four rows. To have a pivot in each row, A would have to have at least four columns (one for each pivot), which is not the case. Since A does not have a pivot in every row, its columns do not span $\mathbf{R}^4$, by Theorem 4. In general, a set of n vectors in $\mathbf{R}^m$ cannot span $\mathbf{R}^m$ when n is less than m.

33. If the equation $A\mathbf{x} = \mathbf{b}$ has a unique solution, then the associated system of equations does not have any free variables. If every variable is a basic variable, then each column of A is a pivot column. So the reduced echelon form of A must be $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

Note: Exercises 33 and 34 are difficult in the context of this section because the focus in Section 1.4 is on existence of solutions, not uniqueness. However, these exercises serve to review ideas from Section 1.2, and they anticipate ideas that will come later.

34. If the equation $A\mathbf{x} = \mathbf{b}$ has a unique solution, then the associated system of equations does not have any free variables. If every variable is a basic variable, then each column of A is a pivot column. So the reduced echelon form of A must be $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Now it is clear that A has a pivot position in each *row*.

By Theorem 4, the columns of A span $\mathbf{R}^3$.

35. Given $A\mathbf{x}_1 = \mathbf{y}_1$ and $A\mathbf{x}_2 = \mathbf{y}_2$, you are asked to show that the equation $A\mathbf{x} = \mathbf{w}$ has a solution, where $\mathbf{w} = \mathbf{y}_1 + \mathbf{y}_2$. Observe that $\mathbf{w} = A\mathbf{x}_1 + A\mathbf{x}_2$ and use Theorem 5(a) with $\mathbf{x}_1$ and $\mathbf{x}_2$ in place of $\mathbf{u}$ and $\mathbf{v}$, respectively. That is, $\mathbf{w} = A\mathbf{x}_1 + A\mathbf{x}_2 = A(\mathbf{x}_1 + \mathbf{x}_2)$. So the vector $\mathbf{x} = \mathbf{x}_1 + \mathbf{x}_2$ is a solution of $\mathbf{w} = A\mathbf{x}$.

36. Suppose that y and $\mathbf{z}$ satisfy $A\mathbf{y} = \mathbf{z}$. Then $4\mathbf{z} = 4A\mathbf{y}$. By Theorem 5(b), $4A\mathbf{y} = A(4\mathbf{y})$. So $4\mathbf{z} = A(4\mathbf{y})$, which shows that $4\mathbf{y}$ is a solution of $A\mathbf{x} = 4\mathbf{z}$. Thus, the equation $A\mathbf{x} = 4\mathbf{z}$ is consistent.

37. **[M]** $$\begin{bmatrix} 7 & 2 & -5 & 8 \\ -5 & -3 & 4 & -9 \\ 6 & 10 & -2 & 7 \\ -7 & 9 & 2 & 15 \end{bmatrix} \sim \begin{bmatrix} 7 & 2 & -5 & 8 \\ 0 & -11/7 & 3/7 & -23/7 \\ 0 & 58/7 & 16/7 & 1/7 \\ 0 & 11 & -3 & 23 \end{bmatrix} \sim \begin{bmatrix} 7 & 2 & -5 & 8 \\ 0 & -11/7 & 3/7 & -23/7 \\ 0 & 0 & 50/11 & -189/11 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

or, approximately $\begin{bmatrix} 7 & 2 & -5 & 8 \\ 0 & -1.57 & .429 & -3.29 \\ 0 & 0 & 4.55 & -17.2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, to three significant figures. The original matrix does not have a pivot in every row, so its columns do not span $\mathbf{R}^4$, by Theorem 4.

38. **[M]** $\begin{bmatrix} 5 & -7 & -4 & 9 \\ 6 & -8 & -7 & 5 \\ 4 & -4 & -9 & -9 \\ -9 & 11 & 16 & 7 \end{bmatrix} \sim \begin{bmatrix} 5 & -7 & -4 & 9 \\ 0 & 2/5 & -11/5 & -29/5 \\ 0 & 8/5 & -29/5 & -81/5 \\ 0 & -8/5 & 44/5 & 116/5 \end{bmatrix} \sim \begin{bmatrix} 5 & -7 & -4 & 9 \\ 0 & 2/5 & -11/5 & -29/5 \\ 0 & 0 & 3 & 7 \\ 0 & 0 & * & * \end{bmatrix}$

MATLAB shows starred entries for numbers that are essentially zero (to many decimal places). So, with pivots only in the first three rows, the original matrix has columns that do not span $\mathbf{R}^4$, by Theorem 4.

39. **[M]** $\begin{bmatrix} 12 & -7 & 11 & -9 & 5 \\ -9 & 4 & -8 & 7 & -3 \\ -6 & 11 & -7 & 3 & -9 \\ 4 & -6 & 10 & -5 & 12 \end{bmatrix} \sim \begin{bmatrix} 12 & -7 & 11 & -9 & 5 \\ 0 & -5/4 & 1/4 & 1/4 & 3/4 \\ 0 & 15/2 & -3/2 & -3/2 & -13/2 \\ 0 & -11/3 & 19/3 & -2 & 31/3 \end{bmatrix}$

$$\sim \begin{bmatrix} 12 & -7 & 11 & -9 & 5 \\ 0 & -5/4 & 1/4 & 1/4 & 3/4 \\ 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 28/5 & -41/15 & 122/15 \end{bmatrix} \sim \begin{bmatrix} 12 & -7 & 11 & -9 & 5 \\ 0 & -5/4 & 1/4 & 1/4 & 3/4 \\ 0 & 0 & 28/5 & -41/15 & 122/15 \\ 0 & 0 & 0 & 0 & -2 \end{bmatrix}$$

The original matrix has a pivot in every row, so its columns span $\mathbf{R}^4$, by Theorem 4.

40. **[M]** $\begin{bmatrix} 8 & 11 & -6 & -7 & 13 \\ -7 & -8 & 5 & 6 & -9 \\ 11 & 7 & -7 & -9 & -6 \\ -3 & 4 & 1 & 8 & 7 \end{bmatrix} \sim \begin{bmatrix} 8 & 11 & -6 & -7 & 13 \\ 0 & 13/8 & -1/4 & -1/8 & 19/8 \\ 0 & -65/8 & 5/4 & 5/8 & -191/8 \\ 0 & 65/8 & -5/4 & 43/8 & 95/8 \end{bmatrix}$

$$\sim \begin{bmatrix} 8 & 11 & -6 & -7 & 13 \\ 0 & 13/8 & -1/4 & -1/8 & 19/8 \\ 0 & 0 & 0 & 0 & -12 \\ 0 & 0 & 0 & 6 & 0 \end{bmatrix} \sim \begin{bmatrix} 8 & 11 & -6 & -7 & 13 \\ 0 & 13/8 & -1/4 & -1/8 & 19/8 \\ 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & -12 \end{bmatrix}$$

The original matrix has a pivot in every row, so its columns span $\mathbf{R}^4$, by Theorem 4.

41. **[M]** Examine the calculations in Exercise 39. Notice that the fourth column of the original matrix, say A, is not a pivot column. Let A^{o} be the matrix formed by deleting column 4 of A, let B be the echelon form obtained from A, and let B^{o} be the matrix obtained by deleting column 4 of B. The sequence of row operations that reduces A to B also reduces A^{o} to B^{o}. Since B^{o} is in echelon form, it shows that A^{o} has a pivot position in each row. Therefore, the columns of A^{o} span $\mathbf{R}^4$.

It is possible to delete column 3 of A instead of column 4. In this case, the fourth column of A becomes a pivot column of A^{o}, as you can see by looking at what happens when column 3 of B is deleted. For later work, it is desirable to delete a nonpivot column.

Note: Exercises 41 and 42 help to prepare for later work on the column space of a matrix. (See Section 2.9 or 4.6.) The *Study Guide* points out that these exercises depend on the following idea, not explicitly mentioned in the text: when a row operation is performed on a matrix A, the calculations for each new entry depend only on the other entries in the *same column*. If a column of A is removed, forming a new matrix, the absence of this column has no affect on any row-operation calculations for entries in the other columns of A. (The absence of a column might affect the particular *choice* of row operations performed for some purpose, but that is not being considered here.)

42. **[M]** Examine the calculations in Exercise 40. The third column of the original matrix, say A, is not a pivot column. Let A^{o} be the matrix formed by deleting column 3 of A, let B be the echelon form obtained from A, and let B^{o} be the matrix obtained by deleting column 3 of B. The sequence of row operations that reduces A to B also reduces A^{o} to B^{o}. Since B^{o} is in echelon form, it shows that A^{o} has a pivot position in each row. Therefore, the columns of A^{o} span $\mathbf{R}^4$.

    It is possible to delete column 2 of A instead of column 3. (See the remark for Exercise 41.) However, only *one* column can be deleted. If two or more columns were deleted from A, the resulting matrix would have fewer than four columns, so it would have fewer than four pivot positions. In such a case, not every row could contain a pivot position, and the columns of the matrix would not span $\mathbf{R}^4$, by Theorem 4.

Notes: At the end of Section 1.4, the *Study Guide* gives students a method for learning and mastering linear algebra concepts. Specific directions are given for constructing a review sheet that connects the basic definition of “span” with related ideas: equivalent descriptions, theorems, geometric interpretations, special cases, algorithms, and typical computations. I require my students to prepare such a sheet that reflects their choices of material connected with “span”, and I make comments on their sheets to help them refine their review. Later, the students use these sheets when studying for exams.

The MATLAB box for Section 1.4 introduces two useful commands **`gauss`** and **`bgauss`** that allow a student to speed up row reduction while still visualizing all the steps involved. The command **`B = gauss(A,1)`** causes MATLAB to find the left-most nonzero entry in row 1 of matrix A, and use that entry as a pivot to create zeros in the entries below, using row replacement operations. The result is a matrix that a student might write next to A as the first stage of row reduction, since there is no need to write a new matrix after each separate row replacement. I use the **`gauss`** command frequently in lectures to obtain an echelon form that provides data for solving various problems. For instance, if a matrix has 5 rows, and if row swaps are not needed, the following commands produce an echelon form of A:

```
B = gauss(A,1),  B = gauss(B,2),  B = gauss(B,3),  B = gauss(B,4)
```

If an interchange is required, I can insert a command such as **`B = swap(B,2,5)`** . The command **`bgauss`** uses the left-most nonzero entry in a row to produce zeros *above* that entry. This command, together with **`scale`**, can change an echelon form into reduced echelon form.

The use of **`gauss`** and **`bgauss`** creates an environment in which students use their computer program the same way they work a problem by hand on an exam. Unless you are able to conduct your exams in a computer laboratory, it may be unwise to give students too early the power to obtain reduced echelon forms with one command—they may have difficulty performing row reduction by hand during an exam. Instructors whose students use a graphic calculator in class each day do not face this problem. In such a case, you may wish to introduce **`rref`** earlier in the course than Chapter 4 (or Section 2.8), which is where I finally allow students to use that command.

1.5 SOLUTIONS

Notes: The geometry helps students understand Span$\{\mathbf{u}, \mathbf{v}\}$, in preparation for later discussions of subspaces. The parametric vector form of a solution set will be used throughout the text. Figure 6 will appear again in Sections 2.9 and 4.8.

For solving homogeneous systems, the text recommends working with the augmented matrix, although no calculations take place in the augmented column. See the *Study Guide* comments on Exercise 7 that illustrate two common student errors.

All students need the practice of Exercises 1–14. (Assign all odd, all even, or a mixture. If you do not assign Exercise 7, be sure to assign both 8 and 10.) Otherwise, a few students may be unable later to find a basis for a null space or an eigenspace. Exercises 29–34 are important. Exercises 33 and 34 help students later understand how solutions of $A\mathbf{x} = \mathbf{0}$ encode linear dependence relations among the columns of A. Exercises 35–38 are more challenging. Exercise 37 will help students avoid the standard mistake of forgetting that Theorem 6 applies only to a *consistent* equation $A\mathbf{x} = \mathbf{b}$.

1. Reduce the augmented matrix to echelon form and circle the pivot positions. If a column of the *coefficient* matrix is not a pivot column, the corresponding variable is free and the system of equations has a nontrivial solution. Otherwise, the system has *only* the trivial solution.

$$\begin{bmatrix} 2 & -5 & 8 & 0 \\ -2 & -7 & 1 & 0 \\ 4 & 2 & 7 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & -5 & 8 & 0 \\ 0 & -12 & 9 & 0 \\ 0 & 12 & -9 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & -5 & 8 & 0 \\ 0 & -12 & 9 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The variable x_3 is free, so the system has a nontrivial solution.

2. $$\begin{bmatrix} 1 & -3 & 7 & 0 \\ -2 & 1 & -4 & 0 \\ 1 & 2 & 9 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 7 & 0 \\ 0 & -5 & 10 & 0 \\ 0 & 5 & 2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 7 & 0 \\ 0 & -5 & 10 & 0 \\ 0 & 0 & 12 & 0 \end{bmatrix}$$

There is no free variable; the system has only the trivial solution.

3. $\begin{bmatrix} -3 & 5 & -7 & 0 \\ -6 & 7 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} -3 & 5 & -7 & 0 \\ 0 & -3 & 15 & 0 \end{bmatrix}$. The variable x_3 is free; the system has nontrivial solutions.

An alert student will realize that row operations are unnecessary. With only two equations, there can be at most two basic variables. One variable *must* be free. Refer to Exercise 31 in Section 1.2.

4. $\begin{bmatrix} -5 & 7 & 9 & 0 \\ 1 & -2 & 6 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 6 & 0 \\ -5 & 7 & 9 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 6 & 0 \\ 0 & -3 & 39 & 0 \end{bmatrix}$. x_3 is a free variable; the system has nontrivial solutions. As in Exercise 3, row operations are unnecessary.

5. $$\begin{bmatrix} 1 & 3 & 1 & 0 \\ -4 & -9 & 2 & 0 \\ 0 & -3 & -6 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 1 & 0 \\ 0 & 3 & 6 & 0 \\ 0 & -3 & -6 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -5 & 0 \\ 0 & 3 & 6 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -5 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} x_1 \quad - 5x_3 &= 0 \\ x_2 + 2x_3 &= 0 \\ 0 &= 0 \end{aligned}$$

The variable x_3 is free, $x_1 = 5x_3$, and $x_2 = -2x_3$.

In parametric vector form, the general solution is $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5x_3 \\ -2x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}$.

6. $\begin{bmatrix} 1 & 3 & -5 & 0 \\ 1 & 4 & -8 & 0 \\ -3 & -7 & 9 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & -5 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 2 & -6 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & -5 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 4 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$$\begin{aligned} x_1 \quad + 4x_3 &= 0 \\ x_2 - 3x_3 &= 0 \\ 0 &= 0 \end{aligned}$$

. The variable x_3 is free, $x_1 = -4x_3$, and $x_2 = 3x_3$.

In parametric vector form, the general solution is $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -4x_3 \\ 3x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -4 \\ 3 \\ 1 \end{bmatrix}$.

7. $\begin{bmatrix} 1 & 3 & -3 & 7 & 0 \\ 0 & 1 & -4 & 5 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 9 & -8 & 0 \\ 0 & 1 & -4 & 5 & 0 \end{bmatrix}$. $\quad \begin{aligned} x_1 \quad + 9x_3 - 8x_4 &= 0 \\ x_2 - 4x_3 + 5x_4 &= 0 \end{aligned}$

The basic variables are x_1 and x_2, with x_3 and x_4 free. Next, $x_1 = -9x_3 + 8x_4$, and $x_2 = 4x_3 - 5x_4$. The general solution is

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -9x_3 + 8x_4 \\ 4x_3 - 5x_4 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -9x_3 \\ 4x_3 \\ x_3 \\ 0 \end{bmatrix} + \begin{bmatrix} 8x_4 \\ -5x_4 \\ 0 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -9 \\ 4 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 8 \\ -5 \\ 0 \\ 1 \end{bmatrix}$$

8. $\begin{bmatrix} 1 & -2 & -9 & 5 & 0 \\ 0 & 1 & 2 & -6 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -5 & -7 & 0 \\ 0 & 1 & 2 & -6 & 0 \end{bmatrix}$. $\quad \begin{aligned} x_1 \quad - 5x_3 - 7x_4 &= 0 \\ x_2 + 2x_3 - 6x_4 &= 0 \end{aligned}$

The basic variables are x_1 and x_2, with x_3 and x_4 free. Next, $x_1 = 5x_3 + 7x_4$ and $x_2 = -2x_3 + 6x_4$. The general solution in parametric vector form is

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 5x_3 + 7x_4 \\ -2x_3 + 6x_4 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 5x_3 \\ -2x_3 \\ x_3 \\ 0 \end{bmatrix} + \begin{bmatrix} 7x_4 \\ 6x_4 \\ 0 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 5 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 7 \\ 6 \\ 0 \\ 1 \end{bmatrix}$$

9. $\begin{bmatrix} 3 & -9 & 6 & 0 \\ -1 & 3 & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 2 & 0 \\ 3 & -9 & 6 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{aligned} x_1 - 3x_2 + 2x_3 &= 0 \\ 0 &= 0 \end{aligned}$.

The solution is $x_1 = 3x_2 - 2x_3$, with x_2 and x_3 free. In parametric vector form,

$$\mathbf{x} = \begin{bmatrix} 3x_2 - 2x_3 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3x_2 \\ x_2 \\ 0 \end{bmatrix} + \begin{bmatrix} -2x_3 \\ 0 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}.$$

10. $\begin{bmatrix} 1 & 3 & 0 & -4 & 0 \\ 2 & 6 & 0 & -8 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{aligned} x_1 - 3x_2 \quad - 4x_4 &= 0 \\ 0 &= 0 \end{aligned}$.

The only basic variable is x_1, so x_2, x_3, and x_4 are free. (Note that x_3 is not zero.) Also, $x_1 = 3x_2 + 4x_4$. The general solution is

$$\mathbf{x}=\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=\begin{bmatrix}3x_2+4x_4\\x_2\\x_3\\x_4\end{bmatrix}=\begin{bmatrix}3x_2\\x_2\\0\\0\end{bmatrix}+\begin{bmatrix}0\\0\\x_3\\0\end{bmatrix}+\begin{bmatrix}4x_4\\0\\0\\x_4\end{bmatrix}=x_2\begin{bmatrix}3\\1\\0\\0\end{bmatrix}+x_3\begin{bmatrix}0\\0\\1\\0\end{bmatrix}+x_4\begin{bmatrix}4\\0\\0\\1\end{bmatrix}$$

11. $$\begin{bmatrix}1&-4&-2&0&3&-5&0\\0&0&1&0&0&-1&0\\0&0&0&0&1&-4&0\\0&0&0&0&0&0&0\end{bmatrix}\sim\begin{bmatrix}1&-4&-2&0&0&7&0\\0&0&1&0&0&-1&0\\0&0&0&0&1&-4&0\\0&0&0&0&0&0&0\end{bmatrix}\sim\begin{bmatrix}1&-4&0&0&0&5&0\\0&0&1&0&0&-1&0\\0&0&0&0&1&-4&0\\0&0&0&0&0&0&0\end{bmatrix}$$

$$\begin{aligned}x_1-4x_2\qquad\qquad+5x_6&=0\\x_3\qquad-x_6&=0\\x_5-4x_6&=0\\0&=0\end{aligned}$$

. The basic variables are x_1, x_3, and x_5. The remaining variables are free.

In particular, x_4 is free (and not zero as some may assume). The solution is $x_1 = 4x_2 - 5x_6$, $x_3 = x_6$, $x_5 = 4x_6$, with x_2, x_4, and x_6 free. In parametric vector form,

$$\mathbf{x}=\begin{bmatrix}x_1\\x_2\\x_3\\x_4\\x_5\\x_6\end{bmatrix}=\begin{bmatrix}4x_2-5x_6\\x_2\\x_6\\x_4\\4x_6\\x_6\end{bmatrix}=\begin{bmatrix}4x_2\\x_2\\0\\0\\0\\0\end{bmatrix}+\begin{bmatrix}0\\0\\0\\x_4\\0\\0\end{bmatrix}+\begin{bmatrix}-5x_6\\0\\x_6\\0\\4x_6\\x_6\end{bmatrix}=x_2\underbrace{\begin{bmatrix}4\\1\\0\\0\\0\\0\end{bmatrix}}_{\uparrow\ \mathbf{u}}+x_4\underbrace{\begin{bmatrix}0\\0\\0\\1\\0\\0\end{bmatrix}}_{\uparrow\ \mathbf{v}}+x_6\underbrace{\begin{bmatrix}-5\\0\\1\\0\\4\\1\end{bmatrix}}_{\uparrow\ \mathbf{w}}$$

Note: The *Study Guide* discusses two mistakes that students often make on this type of problem.

12. $$\begin{bmatrix}1&5&2&-6&9&0&0\\0&0&1&-7&4&-8&0\\0&0&0&0&0&1&0\\0&0&0&0&0&0&0\end{bmatrix}\sim\begin{bmatrix}1&5&2&-6&9&0&0\\0&0&1&-7&4&0&0\\0&0&0&0&0&1&0\\0&0&0&0&0&0&0\end{bmatrix}\sim\begin{bmatrix}1&5&0&8&1&0&0\\0&0&1&-7&4&0&0\\0&0&0&0&0&1&0\\0&0&0&0&0&0&0\end{bmatrix}$$

$$\begin{aligned}x_1+5x_2\qquad+8x_4+x_5\qquad&=0\\x_3-7x_4+4x_5\qquad&=0\\x_6&=0\\0&=0\end{aligned}.$$

The basic variables are x_1, x_3, and x_6; the free variables are x_2, x_4, and x_5. The general solution is $x_1 = -5x_2 - 8x_4 - x_5$, $x_3 = 7x_4 - 4x_5$, and $x_6 = 0$. In parametric vector form, the solution is

$$\mathbf{x}=\begin{bmatrix}x_1\\x_2\\x_3\\x_4\\x_5\\x_6\end{bmatrix}=\begin{bmatrix}-5x_2-8x_4-x_5\\x_2\\7x_4-4x_5\\x_4\\x_5\\0\end{bmatrix}=\begin{bmatrix}-5x_2\\x_2\\0\\0\\0\\0\end{bmatrix}+\begin{bmatrix}-8x_4\\0\\7x_4\\x_4\\0\\0\end{bmatrix}+\begin{bmatrix}-x_5\\0\\-4x_5\\0\\x_5\\0\end{bmatrix}=x_2\begin{bmatrix}-5\\1\\0\\0\\0\\0\end{bmatrix}+x_4\begin{bmatrix}-8\\0\\7\\1\\0\\0\end{bmatrix}+x_5\begin{bmatrix}-1\\0\\-4\\0\\1\\0\end{bmatrix}$$

13. To write the general solution in parametric vector form, pull out the constant terms that do not involve the free variable:

$$\mathbf{x}=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}5+4x_3\\-2-7x_3\\x_3\end{bmatrix}=\begin{bmatrix}5\\-2\\0\end{bmatrix}+\begin{bmatrix}4x_3\\-7x_3\\x_3\end{bmatrix}=\underset{\substack{\uparrow\\\mathbf{p}}}{\begin{bmatrix}5\\-2\\0\end{bmatrix}}+x_3\underset{\substack{\uparrow\\\mathbf{q}}}{\begin{bmatrix}4\\-7\\1\end{bmatrix}}=\mathbf{p}+x_3\mathbf{q}.$$

Geometrically, the solution set is the line through $\begin{bmatrix}5\\-2\\0\end{bmatrix}$ in the direction of $\begin{bmatrix}4\\-7\\1\end{bmatrix}$.

14. To write the general solution in parametric vector form, pull out the constant terms that do not involve the free variable:

$$\mathbf{x}=\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=\begin{bmatrix}3x_4\\8+x_4\\2-5x_4\\x_4\end{bmatrix}=\begin{bmatrix}0\\8\\2\\0\end{bmatrix}+\begin{bmatrix}3x_4\\x_4\\-5x_4\\x_4\end{bmatrix}=\underset{\substack{\uparrow\\\mathbf{p}}}{\begin{bmatrix}0\\8\\2\\0\end{bmatrix}}+x_4\underset{\substack{\uparrow\\\mathbf{q}}}{\begin{bmatrix}3\\1\\-5\\1\end{bmatrix}}=\mathbf{p}+x_4\mathbf{q}$$

The solution set is the line through **p** in the direction of **q**.

15. Row reduce the augmented matrix for the system:

$$\begin{bmatrix}1&3&1&1\\-4&-9&2&-1\\0&-3&-6&-3\end{bmatrix}\sim\begin{bmatrix}1&3&1&1\\0&3&6&3\\0&-3&-6&-3\end{bmatrix}\sim\begin{bmatrix}1&3&1&1\\0&3&6&3\\0&0&0&0\end{bmatrix}$$

$$\sim\begin{bmatrix}1&3&1&1\\0&1&2&1\\0&0&0&0\end{bmatrix}\sim\begin{bmatrix}1&0&-5&-2\\0&1&2&1\\0&0&0&0\end{bmatrix}.\qquad\begin{aligned}x_1\quad-5x_3&=-2\\x_2+2x_3&=1.\\0&=0\end{aligned}$$

Thus $x_1=-2+5x_3$, $x_2=1-2x_3$, and x_3 is free. In parametric vector form,

$$\mathbf{x}=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}-2+5x_3\\1-2x_3\\x_3\end{bmatrix}=\begin{bmatrix}-2\\1\\0\end{bmatrix}+\begin{bmatrix}5x_3\\-2x_3\\x_3\end{bmatrix}=\begin{bmatrix}-2\\1\\0\end{bmatrix}+x_3\begin{bmatrix}5\\-2\\1\end{bmatrix}$$

The solution set is the line through $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$, parallel to the line that is the solution set of the homogeneous system in Exercise 5.

16. Row reduce the augmented matrix for the system:

$$\begin{bmatrix} 1 & 3 & -5 & 4 \\ 1 & 4 & -8 & 7 \\ -3 & -7 & 9 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & -5 & 4 \\ 0 & 1 & -3 & 3 \\ 0 & 2 & -6 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & -5 & 4 \\ 0 & 1 & -3 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 4 & -5 \\ 0 & 1 & -3 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} x_1 \quad + 4x_3 &= -5 \\ x_2 - 3x_3 &= 3 \\ 0 &= 0 \end{aligned}$$

Thus $x_1 = -5 - 4x_3$, $x_2 = 3 + 3x_3$, and x_3 is free. In parametric vector form,

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -5-4x_3 \\ 3+3x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} -5 \\ 3 \\ 0 \end{bmatrix} + \begin{bmatrix} -4x_3 \\ 3x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} -5 \\ 3 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -4 \\ 3 \\ 1 \end{bmatrix}$$

The solution set is the line through $\begin{bmatrix} -5 \\ 3 \\ 0 \end{bmatrix}$, parallel to the line that is the solution set of the homogeneous system in Exercise 6.

17. Solve $x_1 + 9x_2 - 4x_3 = -2$ for the basic variable: $x_1 = -2 - 9x_2 + 4x_3$, with x_2 and x_3 free. In vector form, the solution is

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2-9x_2+4x_3 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -9x_2 \\ x_2 \\ 0 \end{bmatrix} + \begin{bmatrix} 4x_3 \\ 0 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -9 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix}$$

The solution of $x_1 + 9x_2 - 4x_3 = 0$ is $x_1 = -9x_2 + 4x_3$, with x_2 and x_3 free. In vector form,

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -9x_2+4x_3 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -9x_2 \\ x_2 \\ 0 \end{bmatrix} + \begin{bmatrix} 4x_3 \\ 0 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -9 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix} = x_2\mathbf{u} + x_3\mathbf{v}$$

The solution set of the homogeneous equation is the plane through the origin in $\mathbf{R}^3$ spanned by **u** and **v**. The solution set of the nonhomogeneous equation is parallel to this plane and passes through the point $\mathbf{p} = \begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix}$.

18. Solve $x_1 - 3x_2 + 5x_3 = 4$ for the basic variable: $x_1 = 4 + 3x_2 - 5x_3$, with x_2 and x_3 free. In vector form, the solution is

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4+3x_2-5x_3 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 3x_2 \\ x_2 \\ 0 \end{bmatrix} + \begin{bmatrix} -5x_3 \\ 0 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix}$$

The solution of $x_1 - 3x_2 + 5x_3 = 0$ is $x_1 = 3x_2 - 5x_3$, with x_2 and x_3 free. In vector form,

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3x_2 - 5x_3 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3x_2 \\ x_2 \\ 0 \end{bmatrix} + \begin{bmatrix} -5x_3 \\ 0 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix} = x_2\mathbf{u} + x_3\mathbf{v}$$

The solution set of the homogeneous equation is the plane through the origin in $\mathbf{R}^3$ spanned by **u** and **v**. The solution set of the nonhomogeneous equation is parallel to this plane and passes through the point $\mathbf{p} = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix}$.

19. The line through **a** parallel to **b** can be written as $\mathbf{x} = \mathbf{a} + t\,\mathbf{b}$, where t represents a parameter:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \end{bmatrix} + t\begin{bmatrix} -5 \\ 3 \end{bmatrix}, \text{ or } \begin{cases} x_1 = -2 - 5t \\ x_2 = 3t \end{cases}$$

20. The line through **a** parallel to **b** can be written as $\mathbf{x} = \mathbf{a} + t\mathbf{b}$, where t represents a parameter:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ -4 \end{bmatrix} + t\begin{bmatrix} -7 \\ 8 \end{bmatrix}, \text{ or } \begin{cases} x_1 = 3 - 7t \\ x_2 = -4 + 8t \end{cases}$$

21. The line through **p** and **q** is parallel to $\mathbf{q} - \mathbf{p}$. So, given $\mathbf{p} = \begin{bmatrix} 2 \\ -5 \end{bmatrix}$ and $\mathbf{q} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$, form

$$\mathbf{q} - \mathbf{p} = \begin{bmatrix} -3 - 2 \\ 1 - (-5) \end{bmatrix} = \begin{bmatrix} -5 \\ 6 \end{bmatrix}, \text{ and write the line as } \mathbf{x} = \mathbf{p} + t(\mathbf{q} - \mathbf{p}) = \begin{bmatrix} 2 \\ -5 \end{bmatrix} + t\begin{bmatrix} -5 \\ 6 \end{bmatrix}.$$

22. The line through **p** and **q** is parallel to $\mathbf{q} - \mathbf{p}$. So, given $\mathbf{p} = \begin{bmatrix} -6 \\ 3 \end{bmatrix}$ and $\mathbf{q} = \begin{bmatrix} 0 \\ -4 \end{bmatrix}$, form

$$\mathbf{q} - \mathbf{p} = \begin{bmatrix} 0 - (-6) \\ -4 - 3 \end{bmatrix} = \begin{bmatrix} 6 \\ -7 \end{bmatrix}, \text{ and write the line as } \mathbf{x} = \mathbf{p} + t(\mathbf{q} - \mathbf{p}) = \begin{bmatrix} -6 \\ 3 \end{bmatrix} + t\begin{bmatrix} 6 \\ -7 \end{bmatrix}$$

Note: Exercises 21 and 22 prepare for Exercise 27 in Section 1.8.

23. **a**. True. See the first paragraph of the subsection titled *Homogeneous Linear Systems*.

b. False. The equation $A\mathbf{x} = \mathbf{0}$ gives an *implicit* description of its solution set. See the subsection entitled *Parametric Vector Form*.

c. False. The equation $A\mathbf{x} = \mathbf{0}$ *always* has the trivial solution. The box before Example 1 uses the word *nontrivial* instead of *trivial*.

d. False. The line goes through **p** parallel to **v**. See the paragraph that precedes Fig. 5.

e. False. The solution set could be *empty*! The statement (from Theorem 6) is true only when there exists a vector **p** such that $A\mathbf{p} = \mathbf{b}$.

24. **a**. False. A nontrivial solution of $A\mathbf{x} = \mathbf{0}$ is any nonzero **x** that satisfies the equation. See the sentence before Example 2.

b. True. See Example 2 and the paragraph following it.

c. True. If the zero vector is a solution, then $\mathbf{b} = A\mathbf{x} = A\mathbf{0} = \mathbf{0}$.

d. True. See the paragraph following Example 3.

e. False. The statement is true only when the solution set of $A\mathbf{x} = \mathbf{0}$ is nonempty. Theorem 6 applies only to a consistent system.

25. Suppose $\mathbf{p}$ satisfies $A\mathbf{x} = \mathbf{b}$. Then $A\mathbf{p} = \mathbf{b}$. Theorem 6 says that the solution set of $A\mathbf{x} = \mathbf{b}$ equals the set $S = \{\mathbf{w} : \mathbf{w} = \mathbf{p} + \mathbf{v}_h \text{ for some } \mathbf{v}_h \text{ such that } A\mathbf{v}_h = \mathbf{0}\}$. There are two things to prove: (a) every vector in S satisfies $A\mathbf{x} = \mathbf{b}$, (b) every vector that satisfies $A\mathbf{x} = \mathbf{b}$ is in S.

a. Let $\mathbf{w}$ have the form $\mathbf{w} = \mathbf{p} + \mathbf{v}_h$, where $A\mathbf{v}_h = \mathbf{0}$. Then

$$A\mathbf{w} = A(\mathbf{p} + \mathbf{v}_h) = A\mathbf{p} + A\mathbf{v}_h. \text{ By Theorem 5(a) in section 1.4}$$
$$= \mathbf{b} + \mathbf{0} = \mathbf{b}$$

So every vector of the form $\mathbf{p} + \mathbf{v}_h$ satisfies $A\mathbf{x} = \mathbf{b}$.

b. Now let $\mathbf{w}$ be any solution of $A\mathbf{x} = \mathbf{b}$, and set $\mathbf{v}_h = \mathbf{w} - \mathbf{p}$. Then

$$A\mathbf{v}_h = A(\mathbf{w} - \mathbf{p}) = A\mathbf{w} - A\mathbf{p} = \mathbf{b} - \mathbf{b} = \mathbf{0}$$

So $\mathbf{v}_h$ satisfies $A\mathbf{x} = \mathbf{0}$. Thus every solution of $A\mathbf{x} = \mathbf{b}$ has the form $\mathbf{w} = \mathbf{p} + \mathbf{v}_h$.

26. (*Geometric argument using Theorem* 6.) Since $A\mathbf{x} = \mathbf{b}$ is consistent, its solution set is obtained by translating the solution set of $A\mathbf{x} = \mathbf{0}$, by Theorem 6. So the solution set of $A\mathbf{x} = \mathbf{b}$ is a single vector if and only if the solution set of $A\mathbf{x} = \mathbf{0}$ is a single vector, and that happens if and only if $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

(*Proof using free variables.*) If $A\mathbf{x} = \mathbf{b}$ has a solution, then the solution is unique if and only if there are no free variables in the corresponding system of equations, that is, if and only if every column of A is a pivot column. This happens if and only if the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

27. When A is the 3×3 zero matrix, *every* $\mathbf{x}$ in $\mathbf{R}^3$ satisfies $A\mathbf{x} = \mathbf{0}$. So the solution set is all vectors in $\mathbf{R}^3$.

28. No. If the solution set of $A\mathbf{x} = \mathbf{b}$ contained the origin, then $\mathbf{0}$ would satisfy $A\mathbf{0} = \mathbf{b}$, which is not true since $\mathbf{b}$ is not the zero vector.

29. **a**. When A is a 3×3 matrix with three pivot positions, the equation $A\mathbf{x} = \mathbf{0}$ has no free variables and hence has no nontrivial solution.

b. With three pivot positions, A has a pivot position in each of its three rows. By Theorem 4 in Section 1.4, the equation $A\mathbf{x} = \mathbf{b}$ has a solution for every possible $\mathbf{b}$. The term "possible" in the exercise means that the only vectors considered in this case are those in $\mathbf{R}^3$, because A has three rows.

30. **a**. When A is a 3×3 matrix with two pivot positions, the equation $A\mathbf{x} = \mathbf{0}$ has two basic variables and one free variable. So $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution.

b. With only two pivot positions, A cannot have a pivot in every row, so by Theorem 4 in Section 1.4, the equation $A\mathbf{x} = \mathbf{b}$ cannot have a solution for every possible $\mathbf{b}$ (in $\mathbf{R}^3$).

31. **a**. When A is a 3×2 matrix with two pivot positions, each column is a pivot column. So the equation $A\mathbf{x} = \mathbf{0}$ has no free variables and hence no nontrivial solution.

b. With two pivot positions and three rows, A cannot have a pivot in every row. So the equation $A\mathbf{x} = \mathbf{b}$ cannot have a solution for every possible $\mathbf{b}$ (in $\mathbf{R}^3$), by Theorem 4 in Section 1.4.

32. **a**. When A is a 2×4 matrix with two pivot positions, the equation $A\mathbf{x} = \mathbf{0}$ has two basic variables and two free variables. So $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution.

b. With two pivot positions and only two rows, A has a pivot position in every row. By Theorem 4 in Section 1.4, the equation $A\mathbf{x} = \mathbf{b}$ has a solution for every possible $\mathbf{b}$ (in $\mathbf{R}^2$).

33. Look at $x_1\begin{bmatrix}-2\\7\\-3\end{bmatrix}+x_2\begin{bmatrix}-6\\21\\-9\end{bmatrix}$ and notice that the second column is 3 times the first. So suitable values for x_1 and x_2 would be 3 and –1 respectively. (Another pair would be 6 and –2, etc.) Thus $\mathbf{x}=\begin{bmatrix}3\\-1\end{bmatrix}$ satisfies $A\mathbf{x} = \mathbf{0}$.

34. Inspect how the columns $\mathbf{a}_1$ and $\mathbf{a}_2$ of A are related. The second column is –3/2 times the first. Put another way, $3\mathbf{a}_1 + 2\mathbf{a}_2 = \mathbf{0}$. Thus $\begin{bmatrix}3\\2\end{bmatrix}$ satisfies $A\mathbf{x} = \mathbf{0}$.

Note: Exercises 33 and 34 set the stage for the concept of linear dependence.

35. Look for $A = [\mathbf{a}_1\ \ \mathbf{a}_2\ \ \mathbf{a}_3]$ such that $1\cdot\mathbf{a}_1 + 1\mathbf{a}_2 + 1\cdot\mathbf{a}_3 = \mathbf{0}$. That is, construct A so that each row sum (the sum of the entries in a row) is zero.

36. Look for $A = [\mathbf{a}_1\ \ \mathbf{a}_2\ \ \mathbf{a}_3]$ such that $1\cdot\mathbf{a}_1 - 2\cdot\mathbf{a}_2 + 1\cdot\mathbf{a}_3 = \mathbf{0}$. That is, construct A so that the sum of the first and third columns is twice the second column.

37. Since the solution set of $A\mathbf{x} = \mathbf{0}$ contains the point (4,1), the vector $\mathbf{x} = (4,1)$ satisfies $A\mathbf{x} = \mathbf{0}$. Write this equation as a vector equation, using $\mathbf{a}_1$ and $\mathbf{a}_2$ for the columns of A:

$$4\cdot\mathbf{a}_1 + 1\cdot\mathbf{a}_2 = \mathbf{0}$$

Then $\mathbf{a}_2 = -4\mathbf{a}_1$. So choose any nonzero vector for the first column of A and multiply that column by – 4 to get the second column of A. For example, set $A=\begin{bmatrix}1 & -4\\1 & -4\end{bmatrix}$.

Finally, the only way the solution set of $A\mathbf{x} = \mathbf{b}$ could *not* be parallel to the line through (1,4) and the origin is for the solution set of $A\mathbf{x} = \mathbf{b}$ to be *empty*. This does not contradict Theorem 6, because that theorem applies only to the case when the equation $A\mathbf{x} = \mathbf{b}$ has a nonempty solution set. For $\mathbf{b}$, take any vector that is *not* a multiple of the columns of A.

Note: In the *Study Guide*, a "Checkpoint" for Section 1.5 will help students with Exercise 37.

38. No. If $A\mathbf{x} = \mathbf{y}$ has no solution, then A cannot have a pivot in each row. Since A is 3×3, it has at most two pivot positions. So the equation $A\mathbf{x} = \mathbf{z}$ for any $\mathbf{z}$ has at most two basic variables and at least one free variable. Thus, the solution set for $A\mathbf{x} = \mathbf{z}$ is either empty or has infinitely many elements.

39. If $\mathbf{u}$ satisfies $A\mathbf{x} = \mathbf{0}$, then $A\mathbf{u} = \mathbf{0}$. For any scalar c, Theorem 5(b) in Section 1.4 shows that $A(c\mathbf{u}) = cA\mathbf{u} = c\cdot\mathbf{0} = \mathbf{0}$.

40. Suppose $A\mathbf{u} = \mathbf{0}$ and $A\mathbf{v} = \mathbf{0}$. Then, since $A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v}$ by Theorem 5(a) in Section 1.4,

$$A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v} = \mathbf{0} + \mathbf{0} = \mathbf{0}.$$

Now, let c and d be scalars. Using both parts of Theorem 5,

$$A(c\mathbf{u} + d\mathbf{v}) = A(c\mathbf{u}) + A(d\mathbf{v}) = cA\mathbf{u} + dA\mathbf{v} = c\mathbf{0} + d\mathbf{0} = \mathbf{0}.$$

Note: The MATLAB box in the *Study Guide* introduces the `zeros` command, in order to augment a matrix with a column of zeros.

1.6 SOLUTIONS

1. Fill in the exchange table one column at a time. The entries in a column describe where a sector's output goes. The decimal fractions in each column sum to 1.

	Distribution of Output From:			
	Goods	Services		Purchased by:
output	↓	↓	input	
	.2	.7	→	Goods
	.8	.3	→	Services

Denote the total annual output (in dollars) of the sectors by p_G and p_S. From the first row, the total input to the Goods sector is $.2\,p_G + .7\,p_S$. The Goods sector must pay for that. So the equilibrium prices must satisfy

$$\begin{array}{ccc} \text{income} & & \text{expenses} \\ p_G & = & .2p_G + .7p_S \end{array}$$

From the second row, the input (that is, the expense) of the Services sector is $.8\,p_G + .3\,p_S$. The equilibrium equation for the Services sector is

$$\begin{array}{ccc} \text{income} & & \text{expenses} \\ p_S & = & .8p_G + .3p_S \end{array}$$

Move all variables to the left side and combine like terms:

$$\begin{array}{rcrcl} .8p_G & - & .7p_S & = & 0 \\ -.8p_G & + & .7p_S & = & 0 \end{array}$$

Row reduce the augmented matrix:

$$\begin{bmatrix} .8 & -.7 & 0 \\ -.8 & .7 & 0 \end{bmatrix} \sim \begin{bmatrix} .8 & -.7 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -.875 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The general solution is $p_G = .875\,p_S$, with p_S free. One equilibrium solution is $p_S = 1000$ and $p_G = 875$. If one uses fractions instead of decimals in the calculations, the general solution would be written $p_G = (7/8)\,p_S$, and a natural choice of prices might be $p_S = 80$ and $p_G = 70$. Only the *ratio* of the prices is important: $p_G = .875\,p_S$. The economic equilibrium is unaffected by a proportional change in prices.

2. Take some other value for p_S, say 200 million dollars. The other equilibrium prices are then $p_C = 188$ million, $p_E = 170$ million. Any constant nonnegative multiple of these prices is a set of equilibrium prices, because the solution set of the system of equations consists of all multiples of one vector. Changing the unit of measurement to, say, European euros has the same effect as multiplying all equilibrium prices by a constant. The *ratios* of the prices remain the same, no matter what currency is used.

3. **a.** Fill in the exchange table one column at a time. The entries in a column describe where a sector's output goes. The decimal fractions in each column sum to 1.

	Distribution of Output From:				Purchased
	Chemicals	Fuels	Machinery		by:
output	↓	↓	↓	input	
	.2	.8	.4	⟶	Chemicals
	.3	.1	.4	⟶	Fuels
	.5	.1	.2	⟶	Machinery

b. Denote the total annual output (in dollars) of the sectors by p_C, p_F, and p_M. From the first row of the table, the total input to the Chemical & Metals sector is .2 p_C + .8 p_F + .4 p_M. So the equillibrium prices must satisfy

$$\begin{array}{ccc} \text{income} & & \text{expenses} \\ p_C & = & .2p_C + .8p_F + .4p_M \end{array}$$

From the second and third rows of the table, the income/expense requirements for the Fuels & Power sector and the Machinery sector are, respectively,

$$p_F = .3p_C + .1p_F + .4p_M$$
$$p_M = .5p_C + .1p_F + .2p_M$$

Move all variables to the left side and combine like terms:

$$.8p_C - .8p_F - .4p_M = 0$$
$$-.3p_C + .9p_F - .4p_M = 0$$
$$-.5p_C - .1p_F + .8p_M = 0$$

c. **[M]** You can obtain the reduced echelon form with a matrix program. Actually, hand calculations are not too messy. To simplify the calculations, first scale each row of the augmented matrix by 10, then continue as usual.

$$\begin{bmatrix} 8 & -8 & -4 & 0 \\ -3 & 9 & -4 & 0 \\ -5 & -1 & 8 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -.5 & 0 \\ -3 & 9 & -4 & 0 \\ -5 & -1 & 8 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -.5 & 0 \\ 0 & 6 & -5.5 & 0 \\ 0 & -6 & 5.5 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 & -.5 & 0 \\ 0 & 1 & -.917 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1.417 & 0 \\ 0 & 1 & -.917 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The number of decimal places displayed is somewhat arbitrary.

The general solution is p_C = 1.417 p_M, p_F = .917 p_M, with p_M free. If p_M is assigned the value 100, then p_C = 141.7 and p_F = 91.7. Note that only the *ratios* of the prices are determined. This makes sense, for if the were converted from, say, dollars to yen or Euros, the inputs and outputs of each sector would still balance. The economic equilibrium is not affected by a proportional change in prices.

4. a. Fill in the exchange table one column at a time. The entries in each column must sum to 1.

	Distribution of Output From:					Purchased by:
	Agric.	Energy	Manuf.	Transp.		
output	↓	↓	↓	↓	input	
	.65	.30	.30	.20	→	Agric.
	.10	.10	.15	.10	→	Energy
	.25	.35	.15	.30	→	Manuf.
	0	.25	.40	.40	→	Transp.

b. Denote the total annual output of the sectors by p_A, p_E, p_M, and p_T, respectively. From the first row of the table, the total input to Agriculture is $.65p_A + .30p_E + .30p_M + .20\,p_T$. So the equilibrium prices must satisfy

income expenses

$$p_A = .65p_A + .30p_E + .30p_M + .20p_T$$

From the second, third, and fourth rows of the table, the equilibrium equations are

$$\begin{aligned} p_E &= .10p_A + .10p_E + .15p_M + .10p_T \\ p_M &= .25p_A + .35p_E + .15p_M + .30p_T \\ p_T &= \qquad\quad\;\; .25p_E + .40p_M + .40p_T \end{aligned}$$

Move all variables to the left side and combine like terms:

$$\begin{aligned} .35p_A - .30p_E - .30p_M - .20p_T &= 0 \\ -.10p_A + .90p_E - .15p_M - .10p_T &= 0 \\ -.25p_A - .35p_E + .85p_M - .30p_T &= 0 \\ -.25p_E - .40p_M + .60p_T &= 0 \end{aligned}$$

Use gauss, bgauss, and scale operations to reduce the augmented matrix to reduced echelon form

$$\begin{bmatrix} .35 & -.3 & -.3 & -.2 & 0 \\ 0 & .81 & -.24 & -.16 & 0 \\ 0 & 0 & 1.0 & -1.17 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} .35 & -.3 & 0 & -.55 & 0 \\ 0 & .81 & 0 & -.43 & 0 \\ 0 & 0 & 1 & -1.17 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 35 & 0 & 0 & -.71 & 0 \\ 0 & 1 & 0 & -.53 & 0 \\ 0 & 0 & 1 & -1.17 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Scale the first row and solve for the basic variables in terms of the free variable p_T, and obtain $p_A = 2.03p_T$, $p_E = .53p_T$, and $p_M = 1.17p_T$. The data probably justifies at most two significant figures, so take $p_T = 100$ and round off the other prices to $p_A = 200$, $p_E = 53$, and $p_M = 120$.

5. The following vectors list the numbers of atoms of boron (B), sulfur (S), hydrogen (H), and oxygen (O):

$$B_2S_3: \begin{bmatrix} 2 \\ 3 \\ 0 \\ 0 \end{bmatrix}, \quad H_2O: \begin{bmatrix} 0 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \quad H_3BO_3: \begin{bmatrix} 1 \\ 0 \\ 3 \\ 3 \end{bmatrix}, \quad H_2S: \begin{bmatrix} 0 \\ 1 \\ 2 \\ 0 \end{bmatrix} \begin{array}{l} \text{boron} \\ \text{sulfur} \\ \text{hydrogen} \\ \text{oxygen} \end{array}$$

The coefficients in the equation $x_1 \cdot B_2S_3 + x_2 \cdot H_20 \;\rightarrow\; x_3 \cdot H_3BO_3 + x_4 \cdot H_2S$ satisfy

$$x_1\begin{bmatrix}2\\3\\0\\0\end{bmatrix}+x_2\begin{bmatrix}0\\0\\2\\1\end{bmatrix}=x_3\begin{bmatrix}1\\0\\3\\3\end{bmatrix}+x_4\begin{bmatrix}0\\1\\2\\0\end{bmatrix}$$

Move the right terms to the left side (changing the sign of each entry in the third and fourth vectors) and row reduce the augmented matrix of the homogeneous system:

$$\begin{bmatrix}2&0&-1&0&0\\3&0&0&-1&0\\0&2&-3&-2&0\\0&1&-3&0&0\end{bmatrix}\sim\begin{bmatrix}2&0&-1&0&0\\0&0&3/2&-1&0\\0&2&-3&-2&0\\0&1&-3&0&0\end{bmatrix}\sim\begin{bmatrix}2&0&-1&0&0\\0&1&-3&0&0\\0&0&3/2&-1&0\\0&2&-3&-2&0\end{bmatrix}\sim\begin{bmatrix}2&0&-1&0&0\\0&1&-3&0&0\\0&0&3/2&-1&0\\0&0&3&-2&0\end{bmatrix}$$

$$\sim\begin{bmatrix}2&0&-1&0&0\\0&1&-3&0&0\\0&0&1&-2/3&0\\0&0&3&-2&0\end{bmatrix}\sim\begin{bmatrix}2&0&0&-2/3&0\\0&1&0&-2&0\\0&0&1&-2/3&0\\0&0&0&0&0\end{bmatrix}\sim\begin{bmatrix}1&0&0&-1/3&0\\0&1&0&-2&0\\0&0&1&-2/3&0\\0&0&0&0&0\end{bmatrix}$$

The general solution is $x_1 = (1/3)\,x_4$, $x_2 = 2x_4$, $x_3 = (2/3)\,x_4$, with x_4 free. Take $x_4 = 3$. Then $x_1 = 1$, $x_2 = 6$, and $x_3 = 2$. The balanced equation is

$$B_2S_3 + 6H_20 \rightarrow 2H_3BO_3 + 3H_2S$$

6. The following vectors list the numbers of atoms of sodium (Na), phosphorus (P), oxygen (O), barium (Ba), and nitrogen(N):

$$Na_3PO_4:\begin{bmatrix}3\\1\\4\\0\\0\end{bmatrix},\quad Ba(NO_3)_2:\begin{bmatrix}0\\0\\6\\1\\2\end{bmatrix},\quad Ba_3(PO_4)_2:\begin{bmatrix}0\\2\\8\\3\\0\end{bmatrix},\quad NaNO_3:\begin{bmatrix}1\\0\\3\\0\\1\end{bmatrix}\begin{matrix}\text{sodium}\\\text{phosphorus}\\\text{oxygen}\\\text{barium}\\\text{nitrogen}\end{matrix}$$

The coefficients in the equation $x_1{\cdot}Na_3PO_4 + x_2{\cdot}Ba(NO_3)_2 \rightarrow x_3{\cdot}Ba_3(PO_4)_2 + x_4{\cdot}NaNO_3$ satisfy

$$x_1\begin{bmatrix}3\\1\\4\\0\\0\end{bmatrix}+x_2\begin{bmatrix}0\\0\\6\\1\\2\end{bmatrix}=x_3\begin{bmatrix}0\\2\\8\\3\\0\end{bmatrix}+x_4\begin{bmatrix}1\\0\\3\\0\\1\end{bmatrix}$$

Move the right terms to the left side (changing the sign of each entry in the third and fourth vectors) and row reduce the augmented matrix of the homogeneous system:

$$\begin{bmatrix}3&0&0&-1&0\\1&0&-2&0&0\\4&6&-8&-3&0\\0&1&-3&0&0\\0&2&0&-1&0\end{bmatrix}\sim\begin{bmatrix}1&0&-2&0&0\\3&0&0&-1&0\\4&6&-8&-3&0\\0&1&-3&0&0\\0&2&0&-1&0\end{bmatrix}\sim\begin{bmatrix}1&0&-2&0&0\\0&0&6&-1&0\\0&6&0&-3&0\\0&1&-3&0&0\\0&2&0&-1&0\end{bmatrix}\sim\begin{bmatrix}1&0&-2&0&0\\0&1&-3&0&0\\0&6&0&-3&0\\0&0&6&-1&0\\0&2&0&-1&0\end{bmatrix}$$

$$\sim\begin{bmatrix}1&0&-2&0&0\\0&1&-3&0&0\\0&0&18&-3&0\\0&0&6&-1&0\\0&0&6&-1&0\end{bmatrix}\sim\begin{bmatrix}1&0&-2&0&0\\0&1&-3&0&0\\0&0&1&-1/6&0\\0&0&0&0&0\\0&0&0&0&0\end{bmatrix}\sim\begin{bmatrix}1&0&0&-1/3&0\\0&1&0&-1/2&0\\0&0&1&-1/6&0\\0&0&0&0&0\\0&0&0&0&0\end{bmatrix}$$

The general solution is $x_1 = (1/3)x_4$, $x_2 = (1/2)x_4$, $x_3 = (1/6)x_4$, with x_4 free. Take $x_4 = 6$. Then $x_1 = 2$, $x_2 = 3$, and $x_3 = 1$. The balanced equation is

$$2Na_3PO_4 + 3Ba(NO_3)_2 \rightarrow Ba_3(PO_4)_2 + 6NaNO_3$$

7. The following vectors list the numbers of atoms of sodium (Na), hydrogen (H), carbon (C), and oxygen (O):

$$NaHCO_3:\begin{bmatrix}1\\1\\1\\3\end{bmatrix},\ H_3C_6H_5O_7:\begin{bmatrix}0\\8\\6\\7\end{bmatrix},\ Na_3C_6H_5O_7:\begin{bmatrix}3\\5\\6\\7\end{bmatrix},\ H_2O:\begin{bmatrix}0\\2\\0\\1\end{bmatrix},\ CO_2:\begin{bmatrix}0\\0\\1\\2\end{bmatrix}\begin{matrix}\text{sodium}\\\text{hydrogen}\\\text{carbon}\\\text{oxygen}\end{matrix}$$

The order of the various atoms is not important. The list here was selected by writing the elements in the order in which they first appear in the chemical equation, reading left to right:

$$x_1\cdot NaHCO_3 + x_2\cdot H_3C_6H_5O_7 \rightarrow x_3\cdot Na_3C_6H_5O_7 + x_4\cdot H_2O + x_5\cdot CO_2.$$

The coefficients $x_1, \ldots, x_5$ satisfy the vector equation

$$x_1\begin{bmatrix}1\\1\\1\\3\end{bmatrix}+x_2\begin{bmatrix}0\\8\\6\\7\end{bmatrix}=x_3\begin{bmatrix}3\\5\\6\\7\end{bmatrix}+x_4\begin{bmatrix}0\\2\\0\\1\end{bmatrix}+x_5\begin{bmatrix}0\\0\\1\\2\end{bmatrix}$$

Move all the terms to the left side (changing the sign of each entry in the third, fourth, and fifth vectors) and reduce the augmented matrix:

$$\begin{bmatrix}1&0&-3&0&0&0\\1&8&-5&-2&0&0\\1&6&-6&0&-1&0\\3&7&-7&-1&-2&0\end{bmatrix}\sim\cdots\sim\begin{bmatrix}1&0&0&0&-1&0\\0&1&0&0&-1/3&0\\0&0&1&0&-1/3&0\\0&0&0&1&-1&0\end{bmatrix}$$

The general solution is $x_1 = x_5$, $x_2 = (1/3)x_5$, $x_3 = (1/3)x_5$, $x_4 = x_5$, and x_5 is free. Take $x_5 = 3$. Then $x_1 = x_4 = 3$, and $x_2 = x_3 = 1$. The balanced equation is

$$3NaHCO_3 + H_3C_6H_5O_7 \rightarrow Na_3C_6H_5O_7 + 3H_2O + 3CO_2$$

8. The following vectors list the numbers of atoms of potassium (K), manganese (Mn), oxygen (O), sulfur (S), and hydrogen (H):

$$KMnO_4:\begin{bmatrix}1\\1\\4\\0\\0\end{bmatrix},\ MnSO_4:\begin{bmatrix}0\\1\\4\\1\\0\end{bmatrix},\ H_2O:\begin{bmatrix}0\\0\\1\\0\\2\end{bmatrix},\ MnO_2:\begin{bmatrix}0\\1\\2\\0\\0\end{bmatrix},\ K_2SO_4:\begin{bmatrix}2\\0\\4\\1\\0\end{bmatrix},\ H_2SO_4:\begin{bmatrix}0\\0\\4\\1\\2\end{bmatrix}\begin{matrix}\text{potassium}\\\text{manganese}\\\text{oxygen}\\\text{sulfur}\\\text{hydrogen}\end{matrix}$$

The coefficients in the chemical equation

$$x_1 \cdot KMnO_4 + x_2 \cdot MnSO_4 + x_3 \cdot H_2O \rightarrow x_4 \cdot MnO_2 + x_5 \cdot K_2SO_4 + x_6 \cdot H_2SO_4$$

satisfy the vector equation

$$x_1\begin{bmatrix}1\\1\\4\\0\\0\end{bmatrix}+x_2\begin{bmatrix}0\\1\\4\\1\\0\end{bmatrix}+x_3\begin{bmatrix}0\\0\\1\\0\\2\end{bmatrix}=x_4\begin{bmatrix}0\\1\\2\\0\\0\end{bmatrix}+x_5\begin{bmatrix}2\\0\\4\\1\\0\end{bmatrix}+x_6\begin{bmatrix}0\\0\\4\\1\\2\end{bmatrix}$$

Move the terms to the left side (changing the sign of each entry in the last three vectors) and reduce the augmented matrix:

$$\begin{bmatrix}1&0&0&0&-2&0&0\\1&1&0&-1&0&0&0\\4&4&1&-2&-4&-4&0\\0&1&0&0&-1&-1&0\\0&0&2&0&0&-2&0\end{bmatrix}\sim\begin{bmatrix}1&0&0&0&0&-1.0&0\\0&1&0&0&0&-1.5&0\\0&0&1&0&0&-1.0&0\\0&0&0&1&0&-2.5&0\\0&0&0&0&1&-.5&0\end{bmatrix}$$

The general solution is $x_1 = x_6$, $x_2 = (1.5)x_6$, $x_3 = x_6$, $x_4 = (2.5)x_6$, $x_5 = .5x_6$, and x_6 is free. Take $x_6 = 2$. Then $x_1 = x_3 = 2$, and $x_2 = 3$, $x_4 = 5$, and $x_5 = 1$. The balanced equation is

$$2KMnO_4 + 3MnSO_4 + 2H_2O \rightarrow 5MnO_2 + K_2SO_4 + 2H_2SO_4$$

9. **[M]** Set up vectors that list the atoms per molecule. Using the order lead (Pb), nitrogen (N), chromium (Cr), manganese (Mn), and oxygen (O), the vector equation to be solved is

$$x_1\begin{bmatrix}1\\6\\0\\0\\0\end{bmatrix}+x_2\begin{bmatrix}0\\0\\1\\2\\8\end{bmatrix}=x_3\begin{bmatrix}3\\0\\0\\0\\4\end{bmatrix}+x_4\begin{bmatrix}0\\0\\2\\0\\3\end{bmatrix}+x_5\begin{bmatrix}0\\0\\0\\1\\2\end{bmatrix}+x_6\begin{bmatrix}0\\1\\0\\0\\1\end{bmatrix}\begin{array}{l}\text{lead}\\\text{nitrogen}\\\text{chromium}\\\text{manganese}\\\text{oxygen}\end{array}$$

The general solution is $x_1 = (1/6)x_6$, $x_2 = (22/45)x_6$, $x_3 = (1/18)x_6$, $x_4 = (11/45)x_6$, $x_5 = (44/45)x_6$, and x_6 is free. Take $x_6 = 90$. Then $x_1 = 15$, $x_2 = 44$, $x_3 = 5$, $x_4 = 22$, and $x_5 = 88$. The balanced equation is

$$15PbN_6 + 44CrMn_2O_8 \rightarrow 5Pb_3O_4 + 22Cr_2O_3 + 88MnO_2 + 90NO$$

10. **[M]** Set up vectors that list the atoms per molecule. Using the order manganese (Mn), sulfur (S), arsenic (As), chromium (Cr), oxygen (O), and hydrogen (H), the vector equation to be solved is

$$x_1\begin{bmatrix}1\\1\\0\\0\\0\\0\end{bmatrix}+x_2\begin{bmatrix}0\\0\\2\\10\\35\\0\end{bmatrix}+x_3\begin{bmatrix}0\\1\\0\\0\\4\\2\end{bmatrix}=x_4\begin{bmatrix}1\\0\\0\\0\\4\\1\end{bmatrix}+x_5\begin{bmatrix}0\\0\\1\\0\\0\\3\end{bmatrix}+x_6\begin{bmatrix}0\\3\\0\\1\\12\\0\end{bmatrix}+x_7\begin{bmatrix}0\\0\\0\\0\\1\\2\end{bmatrix}\begin{array}{l}\text{manganese}\\\text{sulfur}\\\text{arsenic}\\\text{chromium}\\\text{oxygen}\\\text{hydrogen}\end{array}$$

In rational format, the general solution is $x_1 = (16/327)x_7$, $x_2 = (13/327)x_7$, $x_3 = (374/327)x_7$, $x_4 = (16/327)x_7$, $x_5 = (26/327)x_7$, $x_6 = (130/327)x_7$, and x_7 is free. Take $x_7 = 327$ to make the other variables whole numbers. The balanced equation is

$$16MnS + 13As_2Cr_{10}O_{35} + 374H_2SO_4 \rightarrow 16HMnO_4 + 26AsH_3 + 130CrS_3O_{12} + 327H_2O$$

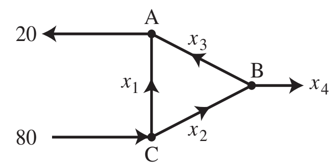
Note that some students may use decimal calculation and simply "round off" the fractions that relate x_1, ..., x_6 to x_7. The equations they construct may balance most of the elements but miss an atom or two. Here is a solution submitted by two of my students:

$5MnS + 4As_2Cr_{10}O_{35} + 115H_2SO_4 \rightarrow 5HMnO_4 + 8AsH_3 + 40CrS_3O_{12} + 100H_2O$

Everything balances except the hydrogen. The right side is short 8 hydrogen atoms. Perhaps the students thought that the $4H_2$ (hydrogen gas) escaped!

11. Write the equations for each node:

Node	Flow in		Flow out
A	$x_1 + x_3$	=	20
B	x_2	=	$x_3 + x_4$
C	80	=	$x_1 + x_2$
Total flow:	80	=	$x_4 + 20$



Rearrange the equations:

$$\begin{aligned} x_1 \quad\quad\quad + x_3 \quad\quad\quad &= 20 \\ x_2 - x_3 - x_4 &= 0 \\ x_1 + x_2 \quad\quad\quad\quad\quad &= 80 \\ x_4 &= 60 \end{aligned}$$

Reduce the augmented matrix:

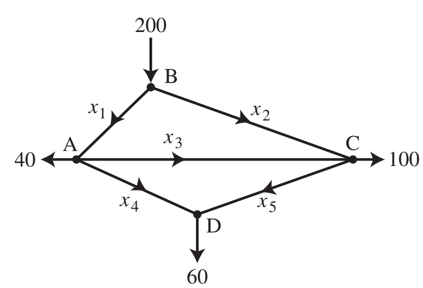
$$\begin{bmatrix} 1 & 0 & 1 & 0 & 20 \\ 0 & 1 & -1 & -1 & 0 \\ 1 & 1 & 0 & 0 & 80 \\ 0 & 0 & 0 & 1 & 60 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} 1 & 0 & 1 & 0 & 20 \\ 0 & 1 & -1 & 0 & 60 \\ 0 & 0 & 0 & 1 & 60 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

For this type of problem, the best description of the general solution uses the style of Section 1.2 rather than parametric vector form:

$\begin{cases} x_1 = 20 - x_3 \\ x_2 = 60 + x_3 \\ x_3 \text{ is free} \\ x_4 = 60 \end{cases}$. Since x_1 cannot be negative, the largest value of x_3 is 20.

12. Write the equations for each intersection:

Intersection	Flow in		Flow out
A	x_1	=	$x_3 + x_4 + 40$
B	200	=	$x_1 + x_2$
C	$x_2 + x_3$	=	$x_5 + 100$
D	$x_4 + x_5$	=	60
Total flow:	200	=	200



Rearrange the equations:

$$\begin{array}{ccccccccccc}
x_1 & & & - & x_3 & - & x_4 & & & = & 40 \\
x_1 & + & x_2 & & & & & & & = & 200 \\
 & & x_2 & + & x_3 & & & - & x_5 & = & 100 \\
 & & & & & & x_4 & + & x_5 & = & 60
\end{array}$$

Reduce the augmented matrix:

$$\begin{bmatrix} 1 & 0 & -1 & -1 & 0 & 40 \\ 1 & 1 & 0 & 0 & 0 & 200 \\ 0 & 1 & 1 & 0 & -1 & 100 \\ 0 & 0 & 0 & 1 & 1 & 60 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & 0 & 1 & 100 \\ 0 & 1 & 1 & 0 & -1 & 100 \\ 0 & 0 & 0 & 1 & 1 & 60 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The general solution (written in the style of Section 1.2) is

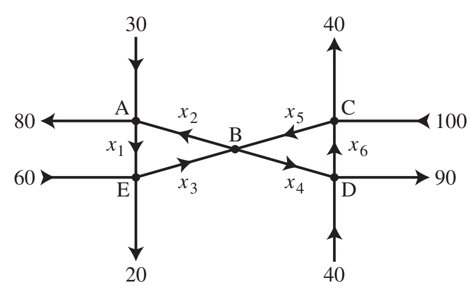
$$\begin{cases} x_1 = 100 + x_3 - x_5 \\ x_2 = 100 - x_3 + x_5 \\ x_3 \text{ is free} \\ x_4 = 60 - x_5 \\ x_5 \text{ is free} \end{cases}$$

b. When $x_4 = 0$, x_5 must be 60, and $\begin{cases} x_1 = 40 + x_3 \\ x_2 = 160 - x_3 \\ x_3 \text{ is free} \\ x_4 = 0 \\ x_5 = 60 \end{cases}$

c. The minimum value of x_1 is 40 cars/minute, because x_3 cannot be negative.

13. Write the equations for each intersection:

Intersection	Flow in		Flow out
A	$x_2 + 30$	=	$x_1 + 80$
B	$x_3 + x_5$	=	$x_2 + x_4$
C	$x_6 + 100$	=	$x_5 + 40$
D	$x_4 + 40$	=	$x_6 + 90$
E	$x_1 + 60$	=	$x_3 + 20$
Total flow:	230	=	230



Rearrange the equations:

$$\begin{array}{ccccccccccccc}
x_1 & - & x_2 & & & & & & & & & = & -50 \\
 & & x_2 & - & x_3 & + & x_4 & - & x_5 & & & = & 0 \\
 & & & & & & & & x_5 & - & x_6 & = & 60 \\
 & & & & & & x_4 & & & - & x_6 & = & 50 \\
x_1 & & & - & x_3 & & & & & & & & -40
\end{array}$$

Reduce the augmented matrix:

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & -50 \\ 0 & 1 & -1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 60 \\ 0 & 0 & 0 & 1 & 0 & -1 & 50 \\ 1 & 0 & -1 & 0 & 0 & 0 & -40 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & -50 \\ 0 & 1 & -1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & 50 \\ 0 & 0 & 0 & 0 & 1 & -1 & 60 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

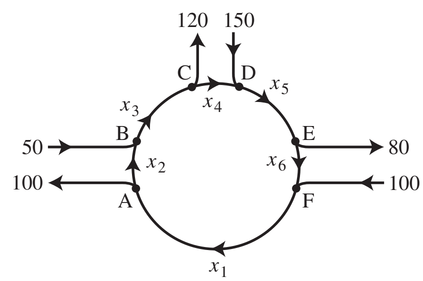
$$\sim\cdots\sim\begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 0 & -40 \\ 0 & 1 & -1 & 0 & 0 & 0 & 10 \\ 0 & 0 & 0 & 1 & 0 & -1 & 50 \\ 0 & 0 & 0 & 0 & 1 & -1 & 60 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

a. The general solution is $\begin{cases} x_1 = x_3 - 40 \\ x_2 = x_3 + 10 \\ x_3 \text{ is free} \\ x_4 = x_6 + 50 \\ x_5 = x_6 + 60 \\ x_6 \text{ is free} \end{cases}$

b. To find minimum flows, note that since x_1 cannot be negative, $x_3 \geq 40$. This implies that $x_2 \geq 50$. Also, since x_6 cannot be negative, $x_4 \geq 50$ and $x_5 \geq 60$. The minimum flows are $x_2 = 50$, $x_3 = 40$, $x_4 = 50$, $x_5 = 60$ (when $x_1 = 0$ and $x_6 = 0$).

14. Write the equations for each intersection.

Intersection	Flow in		Flow out
A	x_1	=	$x_2 + 100$
B	$x_2 + 50$	=	x_3
C	x_3	=	$x_4 + 120$
D	$x_4 + 150$	=	x_5
E	x_5	=	$x_6 + 80$
F	$x_6 + 100$	=	x_1



Rearrange the equations:

$$\begin{aligned} x_1 - x_2 &= 100 \\ x_2 - x_3 &= -50 \\ x_3 - x_4 &= 120 \\ x_4 - x_5 &= -150 \\ x_5 - x_6 &= 80 \\ -x_1 + x_6 &= -100 \end{aligned}$$

Reduce the augmented matrix:

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 100 \\ 0 & 1 & -1 & 0 & 0 & 0 & -50 \\ 0 & 0 & 1 & -1 & 0 & 0 & 120 \\ 0 & 0 & 0 & 1 & -1 & 0 & -150 \\ 0 & 0 & 0 & 0 & 1 & -1 & 80 \\ -1 & 0 & 0 & 0 & 0 & 1 & -100 \end{bmatrix} \sim\cdots\sim \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 100 \\ 0 & 1 & -1 & 0 & 0 & 0 & -50 \\ 0 & 0 & 1 & -1 & 0 & 0 & 120 \\ 0 & 0 & 0 & 1 & -1 & 0 & -150 \\ 0 & 0 & 0 & 0 & 1 & -1 & 80 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim\cdots\sim\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -1 & 100 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 50 \\ 0 & 0 & 0 & 1 & 0 & -1 & -70 \\ 0 & 0 & 0 & 0 & 1 & -1 & 80 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \text{ The general solution is } \begin{cases} x_1 = 100 + x_6 \\ x_2 = x_6 \\ x_3 = 50 + x_6 \\ x_4 = -70 + x_6 \\ x_5 = 80 + x_6 \\ x_6 \text{ is free} \end{cases}.$$

Since x_4 cannot be negative, the minimum value of x_6 is 70.

Note: The MATLAB box in the *Study Guide* discusses rational calculations, needed for balancing the chemical equations in Exercises 9 and 10. As usual, the appendices cover this material for Maple, Mathematica, and the TI and HP graphic calculators.

1.7 SOLUTIONS

Note: Key exercises are 9–20 and 23–30. Exercise 30 states a result that could be a theorem in the text. There is a danger, however, that students will memorize the result without understanding the proof, and then later mix up the words row and column. Exercises 37 and 38 anticipate the discussion in Section 1.9 of one-to-one transformations. Exercise 44 is fairly difficult for my students.

1. Use an augmented matrix to study the solution set of $x_1\mathbf{u} + x_2\mathbf{v} + x_3\mathbf{w} = \mathbf{0}$ (*), where **u**, **v**, and **w** are the three given vectors. Since $\begin{bmatrix} 5 & 7 & 9 & 0 \\ 0 & 2 & 4 & 0 \\ 0 & -6 & -8 & 0 \end{bmatrix} \sim \begin{bmatrix} 5 & 7 & 9 & 0 \\ 0 & 2 & 4 & 0 \\ 0 & 0 & 4 & 0 \end{bmatrix}$, there are no free variables. So the homogeneous equation (*) has only the trivial solution. The vectors are linearly independent.

2. Use an augmented matrix to study the solution set of $x_1\mathbf{u} + x_2\mathbf{v} + x_3\mathbf{w} = \mathbf{0}$ (*), where **u**, **v**, and **w** are the three given vectors. Since $\begin{bmatrix} 0 & 0 & -3 & 0 \\ 0 & 5 & 4 & 0 \\ 2 & -8 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & -8 & 1 & 0 \\ 0 & 5 & 4 & 0 \\ 0 & 0 & -3 & 0 \end{bmatrix}$, there are no free variables. So the homogeneous equation (*) has only the trivial solution. The vectors are linearly independent.

3. Use the method of Example 3 (or the box following the example). By comparing entries of the vectors, one sees that the second vector is –3 times the first vector. Thus, the two vectors are linearly dependent.

4. From the first entries in the vectors, it seems that the second vector of the pair $\begin{bmatrix} -1 \\ 4 \end{bmatrix}, \begin{bmatrix} -2 \\ -8 \end{bmatrix}$ may be 2 times the first vector. But there is a sign problem with the second entries. So neither of the vectors is a multiple of the other. The vectors are linearly independent.

5. Use the method of Example 2. Row reduce the augmented matrix for $A\mathbf{x} = \mathbf{0}$:

$$\begin{bmatrix} 0 & -8 & 5 & 0 \\ 3 & -7 & 4 & 0 \\ -1 & 5 & -4 & 0 \\ 1 & -3 & 2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 2 & 0 \\ 3 & -7 & 4 & 0 \\ -1 & 5 & -4 & 0 \\ 0 & -8 & 5 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 2 & 0 \\ 0 & 2 & -2 & 0 \\ 0 & 2 & -2 & 0 \\ 0 & -8 & 5 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 2 & 0 \\ 0 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 2 & 0 \\ 0 & 2 & -2 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

There are no free variables. The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution and so the columns of A are linearly independent.

6. Use the method of Example 2. Row reduce the augmented matrix for $A\mathbf{x} = \mathbf{0}$:

$$\begin{bmatrix} -4 & -3 & 0 & 0 \\ 0 & -1 & 4 & 0 \\ 1 & 0 & 3 & 0 \\ 5 & 4 & 6 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & -1 & 4 & 0 \\ -4 & -3 & 0 & 0 \\ 5 & 4 & 6 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & -1 & 4 & 0 \\ 0 & -3 & 12 & 0 \\ 0 & 4 & -9 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & -1 & 4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 7 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & -1 & 4 & 0 \\ 0 & 0 & 7 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

There are no free variables. The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution and so the columns of A are linearly independent.

7. Study the equation $A\mathbf{x} = \mathbf{0}$. Some people may start with the method of Example 2:

$$\begin{bmatrix} 1 & 4 & -3 & 0 & 0 \\ -2 & -7 & 5 & 1 & 0 \\ -4 & -5 & 7 & 5 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & -3 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 11 & -5 & 5 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & -3 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 6 & -6 & 0 \end{bmatrix}$$

But this is a waste of time. There are only 3 rows, so there are at most three pivot positions. Hence, at least one of the four variables must be free. So the equation $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution and the columns of A are linearly dependent.

8. Same situation as with Exercise 7. The (unnecessary) row operations are

$$\begin{bmatrix} 1 & -3 & 3 & -2 & 0 \\ -3 & 7 & -1 & 2 & 0 \\ 0 & 1 & -4 & 3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 3 & -2 & 0 \\ 0 & -2 & 8 & -4 & 0 \\ 0 & 1 & -4 & 3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 3 & -2 & 0 \\ 0 & -2 & 8 & -4 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Again, because there are at most three pivot positions yet there are four variables, the equation $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution and the columns of A are linearly dependent.

9. **a**. The vector $\mathbf{v}_3$ is in Span$\{\mathbf{v}_1, \mathbf{v}_2\}$ if and only if the equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 = \mathbf{v}_3$ has a solution. To find out, row reduce $[\mathbf{v}_1\ \ \mathbf{v}_2\ \ \mathbf{v}_3]$, considered as an augmented matrix:

$$\begin{bmatrix} 1 & -3 & 5 \\ -3 & 9 & -7 \\ 2 & -6 & h \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 5 \\ 0 & 0 & 8 \\ 0 & 0 & h-10 \end{bmatrix}$$

At this point, the equation $0 = 8$ shows that the original vector equation has no solution. So $\mathbf{v}_3$ is in Span$\{\mathbf{v}_1, \mathbf{v}_2\}$ for *no* value of h.

b. For $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ to be linearly independent, the equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{0}$ must have only the trivial solution. Row reduce the augmented matrix $[\mathbf{v}_1\ \ \mathbf{v}_2\ \ \mathbf{v}_3\ \ \mathbf{0}]$

$$\begin{bmatrix} 1 & -3 & 5 & 0 \\ -3 & 9 & -7 & 0 \\ 2 & -6 & h & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 5 & 0 \\ 0 & 0 & 8 & 0 \\ 0 & 0 & h-10 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 5 & 0 \\ 0 & 0 & 8 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

For every value of h, x_2 is a free variable, and so the homogeneous equation has a nontrivial solution. Thus $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a linearly dependent set for all h.

10. **a**. The vector $\mathbf{v}_3$ is in Span$\{\mathbf{v}_1, \mathbf{v}_2\}$ if and only if the equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 = \mathbf{v}_3$ has a solution. To find out, row reduce $[\mathbf{v}_1 \ \ \mathbf{v}_2 \ \ \mathbf{v}_3]$, considered as an augmented matrix:

$$\begin{bmatrix} 1 & -2 & 2 \\ -5 & 10 & -9 \\ -3 & 6 & h \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & h+6 \end{bmatrix}$$

At this point, the equation 0 = 1 shows that the original vector equation has no solution. So $\mathbf{v}_3$ is in Span$\{\mathbf{v}_1, \mathbf{v}_2\}$ for *no* value of h.

b. For $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ to be linearly independent, the equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{0}$ must have only the trivial solution. Row reduce the augmented matrix $[\mathbf{v}_1 \ \ \mathbf{v}_2 \ \ \mathbf{v}_3 \ \ \mathbf{0}]$

$$\begin{bmatrix} 1 & -2 & 2 & 0 \\ -5 & 10 & -9 & 0 \\ -3 & 6 & h & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & h+6 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

For every value of h, x_2 is a free variable, and so the homogeneous equation has a nontrivial solution. Thus $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a linearly dependent set for all h.

11. To study the linear dependence of three vectors, say $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$, row reduce the augmented matrix $[\mathbf{v}_1 \ \ \mathbf{v}_2 \ \ \mathbf{v}_3 \ \ \mathbf{0}]$:

$$\begin{bmatrix} 1 & 3 & -1 & 0 \\ -1 & -5 & 5 & 0 \\ 4 & 7 & h & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & -1 & 0 \\ 0 & -2 & 4 & 0 \\ 0 & -5 & h+4 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & -1 & 0 \\ 0 & -2 & 4 & 0 \\ 0 & 0 & h-6 & 0 \end{bmatrix}$$

The equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{0}$ has a nontrivial solution if and only if $h - 6 = 0$ (which corresponds to x_3 being a free variable). Thus, the vectors are linearly dependent if and only if $h = 6$.

12. To study the linear dependence of three vectors, say $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$, row reduce the augmented matrix $[\mathbf{v}_1 \ \ \mathbf{v}_2 \ \ \mathbf{v}_3 \ \ \mathbf{0}]$:

$$\begin{bmatrix} 2 & -6 & 8 & 0 \\ -4 & 7 & h & 0 \\ 1 & -3 & 4 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & -6 & 8 & 0 \\ 0 & -5 & h+16 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{0}$ has a free variable and hence a nontrivial solution no matter what the value of h. So the vectors are linearly dependent for all values of h.

13. To study the linear dependence of three vectors, say $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$, row reduce the augmented matrix $[\mathbf{v}_1 \ \ \mathbf{v}_2 \ \ \mathbf{v}_3 \ \ \mathbf{0}]$:

$$\begin{bmatrix} 1 & -2 & 3 & 0 \\ 5 & -9 & h & 0 \\ -3 & 6 & -9 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 3 & 0 \\ 0 & 1 & h-15 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{0}$ has a free variable and hence a nontrivial solution no matter what the value of h. So the vectors are linearly dependent for all values of h.

14. To study the linear dependence of three vectors, say $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, row reduce the augmented matrix $[\mathbf{v}_1\ \ \mathbf{v}_2\ \ \mathbf{v}_3\ \ \mathbf{0}]$:

$$\begin{bmatrix} 1 & -5 & 1 & 0 \\ -1 & 7 & 1 & 0 \\ -3 & 8 & h & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -5 & 1 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & -7 & h+3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -5 & 1 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & h+10 & 0 \end{bmatrix}$$

The equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{0}$ has a nontrivial solution if and only if $h + 10 = 0$ (which corresponds to x_3 being a free variable). Thus, the vectors are linearly dependent if and only if $h = -10$.

15. The set is linearly dependent, by Theorem 8, because there are four vectors in the set but only two entries in each vector.

16. The set is linearly dependent because the second vector is 3/2 times the first vector.

17. The set is linearly dependent, by Theorem 9, because the list of vectors contains a zero vector.

18. The set is linearly dependent, by Theorem 8, because there are four vectors in the set but only two entries in each vector.

19. The set is linearly independent because neither vector is a multiple of the other vector. [Two of the entries in the first vector are – 4 times the corresponding entry in the second vector. But this multiple does not work for the third entries.]

20. The set is linearly dependent, by Theorem 9, because the list of vectors contains a zero vector.

21. **a**. False. A homogeneous system *always* has the trivial solution. See the box before Example 2.

b. False. See the warning after Theorem 7.

c. True. See Fig. 3, after Theorem 8.

d. True. See the remark following Example 4.

22. **a**. True. See Fig. 1.

b. False. For instance, the set consisting of $\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ -4 \\ 6 \end{bmatrix}$ is linearly dependent. See the warning after Theorem 8.

c. True. See the remark following Example 4.

d. False. See Example 3(a).

23. $\begin{bmatrix} \blacksquare & * & * \\ 0 & \blacksquare & * \\ 0 & 0 & \blacksquare \end{bmatrix}$

24. $\begin{bmatrix} \blacksquare & * \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & \blacksquare \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

25. $\begin{bmatrix} \blacksquare & * \\ 0 & \blacksquare \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & \blacksquare \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

26. $\begin{bmatrix} \blacksquare & * & * \\ 0 & \blacksquare & * \\ 0 & 0 & \blacksquare \\ 0 & 0 & 0 \end{bmatrix}$. The columns must linearly independent, by Theorem 7, because the first column is not zero, the second column is not a multiple of the first, and the third column is not a linear combination of the preceding two columns (because $\mathbf{a}_3$ is not in Span$\{\mathbf{a}_1, \mathbf{a}_2\}$).

27. All five columns of the 7×5 matrix A must be pivot columns. Otherwise, the equation $A\mathbf{x} = \mathbf{0}$ would have a free variable, in which case the columns of A would be linearly dependent.

28. If the columns of a 5×7 matrix A span $\mathbf{R}^5$, then A has a pivot in each row, by Theorem 4. Since each pivot position is in a different column, A has five pivot columns.

29. A: any 3×2 matrix with two nonzero columns such that neither column is a multiple of the other. In this case the columns are linearly independent and so the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
B: any 3×2 matrix with one column a multiple of the other.

30. **a**. n

b. The columns of A are linearly independent if and only if the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution. This happens if and only if $A\mathbf{x} = \mathbf{0}$ has no free variables, which in turn happens if and only if every variable is a basic variable, that is, if and only if every column of A is a pivot column.

31. Think of $A = [\mathbf{a}_1 \ \ \mathbf{a}_2 \ \ \mathbf{a}_3]$. The text points out that $\mathbf{a}_3 = \mathbf{a}_1 + \mathbf{a}_2$. Rewrite this as $\mathbf{a}_1 + \mathbf{a}_2 - \mathbf{a}_3 = \mathbf{0}$. As a matrix equation, $A\mathbf{x} = \mathbf{0}$ for $\mathbf{x} = (1, 1, -1)$.

32. Think of $A = [\mathbf{a}_1 \ \ \mathbf{a}_2 \ \ \mathbf{a}_3]$. The text points out that $\mathbf{a}_1 + 2\mathbf{a}_2 = \mathbf{a}_3$. Rewrite this as $\mathbf{a}_1 + 2\mathbf{a}_2 - \mathbf{a}_3 = \mathbf{0}$. As a matrix equation, $A\mathbf{x} = \mathbf{0}$ for $\mathbf{x} = (1, 2, -1)$.

33. True, by Theorem 7. (The *Study Guide* adds another justification.)

34. True, by Theorem 9.

35. False. The vector $\mathbf{v}_1$ could be the zero vector.

36. False. Counterexample: Take $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_4$ all to be multiples of one vector. Take $\mathbf{v}_3$ to be *not* a multiple of that vector. For example,

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \mathbf{v}_4 = \begin{bmatrix} 4 \\ 4 \\ 4 \\ 4 \end{bmatrix}$$

37. True. A linear dependence relation among $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ may be extended to a linear dependence relation among $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, $\mathbf{v}_4$ by placing a zero weight on $\mathbf{v}_4$.

38. True. If the equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{0}$ had a nontrivial solution (with at least one of x_1, x_2, x_3 nonzero), then so would the equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 + 0{\cdot}\mathbf{v}_4 = \mathbf{0}$. But that cannot happen because $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is linearly independent. So $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ must be linearly independent. This problem can also be solved using Exercise 37, if you know that the statement there is true.

39. If for all **b** the equation $A\mathbf{x} = \mathbf{b}$ has at most one solution, then take $\mathbf{b} = \mathbf{0}$, and conclude that the equation $A\mathbf{x} = \mathbf{0}$ has at most one solution. Then the trivial solution is the only solution, and so the columns of A are linearly independent.

40. An $m \times n$ matrix with n pivot columns has a pivot in each column. So the equation $A\mathbf{x} = \mathbf{b}$ has no free variables. If there is a solution, it must be unique.

41. **[M]** $A = \begin{bmatrix} 8 & -3 & 0 & -7 & 2 \\ -9 & 4 & 5 & 11 & -7 \\ 6 & -2 & 2 & -4 & 4 \\ 5 & -1 & 7 & 0 & 10 \end{bmatrix} \sim \begin{bmatrix} 8 & -3 & 0 & -7 & 2 \\ 0 & 5/8 & 5 & 25/8 & -19/4 \\ 0 & 1/4 & 2 & 5/4 & 5/2 \\ 0 & 7/8 & 7 & 35/8 & 35/4 \end{bmatrix}$

$$\sim \begin{bmatrix} 8 & -3 & 0 & -7 & 2 \\ 0 & 5/8 & 5 & 25/8 & -19/4 \\ 0 & 0 & 0 & 0 & 22/5 \\ 0 & 0 & 0 & 0 & 77/5 \end{bmatrix} \sim \begin{bmatrix} 8 & -3 & 0 & -7 & 2 \\ 0 & 5/8 & 5 & 25/8 & -19/4 \\ 0 & 0 & 0 & 0 & 22/5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The pivot columns of A are 1, 2, and 5. Use them to form $B = \begin{bmatrix} 8 & -3 & 2 \\ -9 & 4 & -7 \\ 6 & -2 & 4 \\ 5 & -1 & 10 \end{bmatrix}$.

Other likely choices use columns 3 or 4 of A instead of 2: $\begin{bmatrix} 8 & 0 & 2 \\ -9 & 5 & -7 \\ 6 & 2 & 4 \\ 5 & 7 & 10 \end{bmatrix}, \begin{bmatrix} 8 & -7 & 2 \\ -9 & 11 & -7 \\ 6 & -4 & 4 \\ 5 & 0 & 10 \end{bmatrix}$.

Actually, any set of three columns of A that includes column 5 will work for B, but the concepts needed to prove that are not available now. (Column 5 is not in the two-dimensional subspace spanned by the first four columns.)

42. [M]

$$\begin{bmatrix} 12 & 10 & -6 & -3 & 7 & 10 \\ -7 & -6 & 4 & 7 & -9 & 5 \\ 9 & 9 & -9 & -5 & 5 & -1 \\ -4 & -3 & 1 & 6 & -8 & 9 \\ 8 & 7 & -5 & -9 & 11 & -8 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} 12 & 10 & -6 & -3 & 7 & 10 \\ 0 & -1/6 & 1/2 & 21/4 & -59/12 & 65/6 \\ 0 & 0 & 0 & 89/2 & -89/2 & 89 \\ 0 & 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The pivot columns of A are 1, 2, 4, and 6. Use them to form $B = \begin{bmatrix} 12 & 10 & -3 & 10 \\ -7 & -6 & 7 & 5 \\ 9 & 9 & -5 & -1 \\ -4 & -3 & 6 & 9 \\ 8 & 7 & -9 & -8 \end{bmatrix}$.

Other likely choices might use column 3 of A instead of 2, and/or use column 5 instead of 4.

43. **[M]** Make **v** any one of the columns of A that is not in B and row reduce the augmented matrix $[B \quad \mathbf{v}]$. The calculations will show that the equation $B\mathbf{x} = \mathbf{v}$ is consistent, which means that **v** is a linear combination of the columns of B. Thus, each column of A that is not a column of B is in the set spanned by the columns of B.

44. **[M]** Calculations made as for Exercise 43 will show that each column of A that is not a column of B is in the set spanned by the columns of B. *Reason*: The original matrix A has only four pivot columns. If one or more columns of A are removed, the resulting matrix will have at most four pivot columns. (Use exactly the same row operations on the new matrix that were used to reduce A to echelon form.) If **v** is a column of A that is not in B, then row reduction of the augmented matrix $[B \quad \mathbf{v}]$ will display at most four pivot columns. Since B itself was constructed to have four pivot columns, adjoining **v** cannot produce a fifth pivot column. Thus the first four columns of $[B \quad \mathbf{v}]$ are the pivot columns. This implies that the equation $B\mathbf{x} = \mathbf{v}$ has a solution.

Note: At the end of Section 1.7, the *Study Guide* has another note to students about "Mastering Linear Algebra Concepts." The note describes how to organize a review sheet that will help students form a mental image of linear independence. The note also lists typical misuses of terminology, in which an adjective is applied to an inappropriate noun. (This is a major problem for my students.) I require my students to prepare a review sheet as described in the *Study Guide*, and I try to make helpful comments on their sheets. I am convinced, through personal observation and student surveys, that the students who prepare many of these review sheets consistently perform better than other students. Hopefully, these students will remember important concepts for some time beyond the final exam.

1.8 SOLUTIONS

Notes: The key exercises are 17–20, 25 and 31. Exercise 20 is worth assigning even if you normally assign only odd exercises. Exercise 25 (and 27) can be used to make a few comments about computer graphics, even if you do not plan to cover Section 2.6. For Exercise 31, the *Study Guide* encourages students *not* to look at the proof before trying hard to construct it. Then the *Guide* explains how to create the proof.

Exercises 19 and 20 provide a natural segue into Section 1.9. I arrange to discuss the homework on these exercises when I am ready to begin Section 1.9. The definition of the standard matrix in Section 1.9 follows naturally from the homework, and so I've covered the first page of Section 1.9 before students realize we are working on new material.

The text does not provide much practice determining whether a transformation is linear, because the time needed to develop this skill would have to be taken away from some other topic. If you want your students to be able to do this, you may need to supplement Exercises 29, 30, 32 and 33.

If you skip the concepts of one-to-one and "onto" in Section 1.9, you can use the result of Exercise 31 to show that the coordinate mapping from a vector space onto $\mathbf{R}^n$ (in Section 4.4) preserves linear independence and dependence of sets of vectors. (See Example 6 in Section 4.4.)

1. $T(\mathbf{u}) = A\mathbf{u} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 2 \\ -6 \end{bmatrix}$, $T(\mathbf{v}) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 2a \\ 2b \end{bmatrix}$

2. $T(\mathbf{u}) = A\mathbf{u} = \begin{bmatrix} .5 & 0 & 0 \\ 0 & .5 & 0 \\ 0 & 0 & .5 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ -4 \end{bmatrix} = \begin{bmatrix} .5 \\ 0 \\ -2 \end{bmatrix}$, $T(\mathbf{v}) = \begin{bmatrix} .5 & 0 & 0 \\ 0 & .5 & 0 \\ 0 & 0 & .5 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} .5a \\ .5b \\ .5c \end{bmatrix}$

3. $\begin{bmatrix} A & \mathbf{b} \end{bmatrix} = \begin{bmatrix} 1 & 0 & -2 & -1 \\ -2 & 1 & 6 & 7 \\ 3 & -2 & -5 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 & -1 \\ 0 & 1 & 2 & 5 \\ 0 & -2 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 & -1 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 5 & 10 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 0 & -2 & -1 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \quad \mathbf{x} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$, unique solution

4. $\begin{bmatrix} A & \mathbf{b} \end{bmatrix} = \begin{bmatrix} 1 & -3 & 2 & 6 \\ 0 & 1 & -4 & -7 \\ 3 & -5 & -9 & -9 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 2 & 6 \\ 0 & 1 & -4 & -7 \\ 0 & 4 & -15 & -27 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 2 & 6 \\ 0 & 1 & -4 & -7 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & -3 & 0 & 4 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -5 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 1 \end{bmatrix} \quad \mathbf{x} = \begin{bmatrix} -5 \\ -3 \\ 1 \end{bmatrix}$, unique solution

5. $\begin{bmatrix} A & \mathbf{b} \end{bmatrix} = \begin{bmatrix} 1 & -5 & -7 & -2 \\ -3 & 7 & 5 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & -5 & -7 & -2 \\ 0 & 1 & 2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 3 \\ 0 & 1 & 2 & 1 \end{bmatrix}$

Note that a solution is *not* $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$. To avoid this common error, write the equations:

$\begin{array}{ccccc} x_1 & & + & 3x_3 & = & 3 \\ & x_2 & + & 2x_3 & = & 1 \end{array}$ and solve for the basic variables: $\begin{cases} x_1 = 3 - 3x_3 \\ x_2 = 1 - 2x_3 \\ x_3 \text{ is free} \end{cases}$

General solution $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 - 3x_3 \\ 1 - 2x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ -2 \\ 1 \end{bmatrix}$. For a particular solution, one might choose

$x_3 = 0$ and $\mathbf{x} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$.

6. $\begin{bmatrix} A & \mathbf{b} \end{bmatrix} = \begin{bmatrix} 1 & -2 & 1 & 1 \\ 3 & -4 & 5 & 9 \\ 0 & 1 & 1 & 3 \\ -3 & 5 & -4 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 1 & 1 \\ 0 & 2 & 2 & 6 \\ 0 & 1 & 1 & 3 \\ 0 & -1 & -1 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 1 & 1 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 7 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\begin{array}{ccccc} x_1 & & + & 3x_3 & = & 7 \\ & x_2 & + & x_3 & = & 3 \end{array}$. $\begin{cases} x_1 = 7 - 3x_3 \\ x_2 = 3 - x_3 \\ x_3 \text{ is free} \end{cases}$

General solution: $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 - 3x_3 \\ 3 - x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ -1 \\ 1 \end{bmatrix}$, one choice: $\begin{bmatrix} 7 \\ 3 \\ 0 \end{bmatrix}$.

7. $a = 5$; the domain of T is $\mathbf{R}^5$, because a 6×5 matrix has 5 columns and for $A\mathbf{x}$ to be defined, $\mathbf{x}$ must be in $\mathbf{R}^5$. $b = 6$; the codomain of T is $\mathbf{R}^6$, because $A\mathbf{x}$ is a linear combination of the columns of A, and each column of A is in $\mathbf{R}^6$.

8. A must have 5 rows and 4 columns. For the domain of T to be $\mathbf{R}^4$, A must have four columns so that $A\mathbf{x}$ is defined for $\mathbf{x}$ in $\mathbf{R}^4$. For the codomain of T to be $\mathbf{R}^5$, the columns of A must have five entries (in which case A must have five rows), because $A\mathbf{x}$ is a linear combination of the columns of A.

9. Solve $A\mathbf{x} = \mathbf{0}$. $\begin{bmatrix} 1 & -4 & 7 & -5 & 0 \\ 0 & 1 & -4 & 3 & 0 \\ 2 & -6 & 6 & -4 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -4 & 7 & -5 & 0 \\ 0 & 1 & -4 & 3 & 0 \\ 0 & 2 & -8 & 6 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -4 & 7 & -5 & 0 \\ 0 & 1 & -4 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & 0 & -9 & 7 & 0 \\ 0 & 1 & -4 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{aligned} x_1 \quad - 9x_3 + 7x_4 &= 0 \\ x_2 - 4x_3 + 3x_4 &= 0, \\ 0 &= 0 \end{aligned} \quad \begin{cases} x_1 = 9x_3 - 7x_4 \\ x_2 = 4x_3 - 3x_4 \\ x_3 \text{ is free} \\ x_4 \text{ is free} \end{cases}$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 9x_3 - 7x_4 \\ 4x_3 - 3x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 9 \\ 4 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -7 \\ -3 \\ 0 \\ 1 \end{bmatrix}$$

10. Solve $A\mathbf{x} = \mathbf{0}$. $\begin{bmatrix} 1 & 3 & 9 & 2 & 0 \\ 1 & 0 & 3 & -4 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ -2 & 3 & 0 & 5 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 9 & 2 & 0 \\ 0 & -3 & -6 & -6 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ 0 & 9 & 18 & 9 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 9 & 2 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ 0 & -3 & -6 & -6 & 0 \\ 0 & 9 & 18 & 9 & 0 \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & 3 & 9 & 2 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & -18 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 9 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} x_1 \quad + 3x_3 \quad &= 0 \\ x_2 + 2x_3 \quad &= 0 \\ x_4 &= 0 \end{aligned} \quad \begin{cases} x_1 = -3x_3 \\ x_2 = -2x_3 \\ x_3 \text{ is free} \\ x_4 = 0 \end{cases} \quad \mathbf{x} = \begin{bmatrix} -3x_3 \\ -2x_3 \\ x_3 \\ 0 \end{bmatrix} = x_3 \begin{bmatrix} -3 \\ -2 \\ 1 \\ 0 \end{bmatrix}$$

11. Is the system represented by $[A \;\; \mathbf{b}]$ consistent? Yes, as the following calculation shows.

$$\begin{bmatrix} 1 & -4 & 7 & -5 & -1 \\ 0 & 1 & -4 & 3 & 1 \\ 2 & -6 & 6 & -4 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -4 & 7 & -5 & -1 \\ 0 & 1 & -4 & 3 & 1 \\ 0 & 2 & -8 & 6 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -4 & 7 & -5 & -1 \\ 0 & 1 & -4 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The system is consistent, so $\mathbf{b}$ is in the range of the transformation $\mathbf{x} \mapsto A\mathbf{x}$.

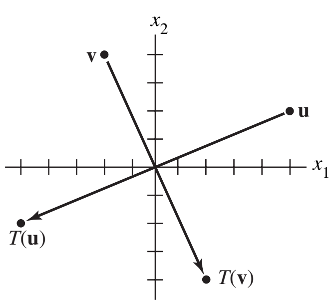
12. Is the system represented by $[A \quad \mathbf{b}]$ consistent?

$$\begin{bmatrix} 1 & 3 & 9 & 2 & -1 \\ 1 & 0 & 3 & -4 & 3 \\ 0 & 1 & 2 & 3 & -1 \\ -2 & 3 & 0 & 5 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 9 & 2 & -1 \\ 0 & -3 & -6 & -6 & 4 \\ 0 & 1 & 2 & 3 & -1 \\ 0 & 9 & 18 & 9 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 9 & 2 & -1 \\ 0 & 1 & 2 & 3 & -1 \\ 0 & -3 & -6 & -6 & 4 \\ 0 & 9 & 18 & 9 & 2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 3 & 9 & 2 & -1 \\ 0 & 1 & 2 & 3 & -1 \\ 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & -18 & 11 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 9 & 2 & -1 \\ 0 & 1 & 2 & 3 & -1 \\ 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 17 \end{bmatrix}$$

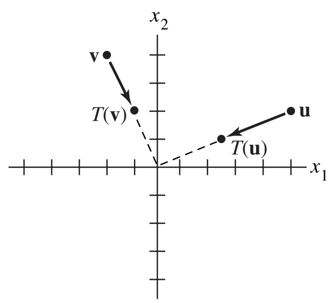
The system is inconsistent, so **b** is not in the range of the transformation $\mathbf{x} \mapsto A\mathbf{x}$.

13.



A reflection through the origin.

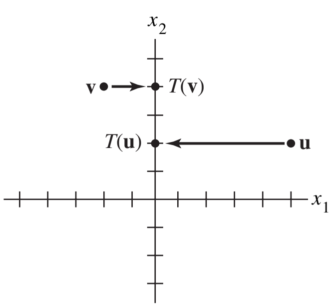
14.



A contraction by the factor .5.

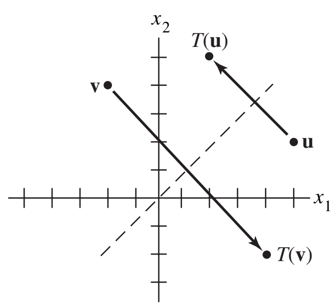
The transformation in Exercise 13 may also be described as a rotation of π radians about the origin or a rotation of –π radians about the origin.

15.



A projection onto the x_2-axis

16.

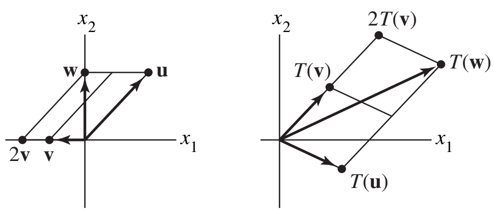


A reflection through the line $x_2 = x_1$.

17. $T(3\mathbf{u}) = 3T(\mathbf{u}) = 3\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix}$, $T(2\mathbf{v}) = 2T(\mathbf{v}) = 2\begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ 6 \end{bmatrix}$, and

$$T(3\mathbf{u} + 2\mathbf{v}) = 3T(\mathbf{u}) = 2T(\mathbf{v}) = \begin{bmatrix} 6 \\ 3 \end{bmatrix} + \begin{bmatrix} -2 \\ 6 \end{bmatrix} = \begin{bmatrix} 4 \\ 9 \end{bmatrix}.$$

18. Draw a line through **w** parallel to **v**, and draw a line through **w** parallel to **u**. See the left part of the figure below. From this, estimate that **w** = **u** + 2**v**. Since T is linear, $T(\mathbf{w})$ = T(**u**) + 2T(**v**). Locate T(**u**) and 2T(**v**) as in the right part of the figure and form the associated parallelogram to locate T(**w**).



19. All we know are the images of $\mathbf{e}_1$ and $\mathbf{e}_2$ and the fact that T is linear. The key idea is to write

$\mathbf{x} = \begin{bmatrix} 5 \\ -3 \end{bmatrix} = 5\begin{bmatrix} 1 \\ 0 \end{bmatrix} - 3\begin{bmatrix} 0 \\ 1 \end{bmatrix} = 5\mathbf{e}_1 - 3\mathbf{e}_2$. Then, from the linearity of T, write

$$T(\mathbf{x}) = T(5\mathbf{e}_1 - 3\mathbf{e}_2) = 5T(\mathbf{e}_1) - 3T(\mathbf{e}_2) = 5\mathbf{y}_1 - 3\mathbf{y}_2 = 5\begin{bmatrix} 2 \\ 5 \end{bmatrix} - 3\begin{bmatrix} -1 \\ 6 \end{bmatrix} = \begin{bmatrix} 13 \\ 7 \end{bmatrix}.$$

To find the image of $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, observe that $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1\begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2\begin{bmatrix} 0 \\ 1 \end{bmatrix} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2$. Then

$$T(\mathbf{x}) = T(x_1\mathbf{e}_1 + x_2\mathbf{e}_2) = x_1T(\mathbf{e}_1) + x_2T(\mathbf{e}_2) = x_1\begin{bmatrix} 2 \\ 5 \end{bmatrix} + x_2\begin{bmatrix} -1 \\ 6 \end{bmatrix} = \begin{bmatrix} 2x_1 - x_2 \\ 5x_1 + 6x_2 \end{bmatrix}$$

20. Use the basic definition of $A\mathbf{x}$ to construct A. Write

$$T(\mathbf{x}) = x_1\mathbf{v}_1 + x_2\mathbf{v}_2 = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2 & 7 \\ 5 & -3 \end{bmatrix}\mathbf{x}, \quad A = \begin{bmatrix} -2 & 7 \\ 5 & -3 \end{bmatrix}$$

21. **a**. True. Functions from $\mathbf{R}^n$ to $\mathbf{R}^m$ are defined before Fig. 2. A linear transformation is a function with certain properties.

b. False. The domain is $\mathbf{R}^5$. See the paragraph before Example 1.

c. False. The range is the set of all linear combinations of the columns of A. See the paragraph before Example 1.

d. False. See the paragraph after the definition of a linear transformation.

e. True. See the paragraph following the box that contains equation (4).

22. **a**. True. See the paragraph following the definition of a linear transformation.

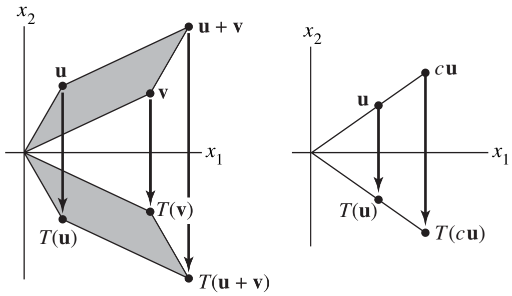
b. False. If A is an $m \times n$ matrix, the codomain is $\mathbf{R}^m$. See the paragraph before Example 1.

c. False. The question is an existence question. See the remark about Example 1(d), following the solution of Example 1.

d. True. See the discussion following the definition of a linear transformation.

e. True. See the paragraph following equation (5).

23.



24. Given any $\mathbf{x}$ in $\mathbf{R}^n$, there are constants $c_1, \ldots, c_p$ such that $\mathbf{x} = c_1\mathbf{v}_1 + \cdots c_p\mathbf{v}_p$, because $\mathbf{v}_1, \ldots, \mathbf{v}_p$ span $\mathbf{R}^n$. Then, from property (5) of a linear transformation,

$$T(\mathbf{x}) = c_1T(\mathbf{v}_1) + \cdots + c_pT(\mathbf{v}_p) = c_1\mathbf{0} + \cdots + c_p\mathbf{0} = \mathbf{0}$$

25. Any point $\mathbf{x}$ on the line through $\mathbf{p}$ in the direction of $\mathbf{v}$ satisfies the parametric equation $\mathbf{x} = \mathbf{p} + t\mathbf{v}$ for some value of t. By linearity, the image $T(\mathbf{x})$ satisfies the parametric equation

$$T(\mathbf{x}) = T(\mathbf{p} + t\mathbf{v}) = T(\mathbf{p}) + tT(\mathbf{v}) \qquad (*)$$

If T($\mathbf{v}$) = $\mathbf{0}$, then $T(\mathbf{x}) = T(\mathbf{p})$ for all values of t, and the image of the original line is just a single point. Otherwise, (*) is the parametric equation of a line through $T(\mathbf{p})$ in the direction of $T(\mathbf{v})$.

26. Any point $\mathbf{x}$ on the plane P satisfies the parametric equation $\mathbf{x} = s\mathbf{u} + t\mathbf{v}$ for some values of s and t. By linearity, the image $T(\mathbf{x})$ satisfies the parametric equation

$$T(\mathbf{x}) = sT(\mathbf{u}) + tT(\mathbf{v}) \qquad (s, t \text{ in } \mathbf{R}) \qquad (*)$$

The set of images is just Span$\{T(\mathbf{u}), T(\mathbf{v})\}$. If $T(\mathbf{u})$ and $T(\mathbf{v})$ are linearly independent, Span$\{T(\mathbf{u}), T(\mathbf{v})\}$ is a plane through $T(\mathbf{u})$, $T(\mathbf{v})$, and $\mathbf{0}$. If $T(\mathbf{u})$ and $T(\mathbf{v})$ are linearly dependent and not both zero, then Span$\{T(\mathbf{u}), T(\mathbf{v})\}$ is a line through $\mathbf{0}$. If $T(\mathbf{u}) = T(\mathbf{v}) = \mathbf{0}$, then Span$\{T(\mathbf{u}), T(\mathbf{v})\}$ is $\{\mathbf{0}\}$.

27. **a**. From Fig. 7 in the exercises for Section 1.5, the line through $T(\mathbf{p})$ and $T(\mathbf{q})$ is in the direction of $\mathbf{q} - \mathbf{p}$, and so the equation of the line is $\mathbf{x} = \mathbf{p} + t(\mathbf{q} - \mathbf{p}) = \mathbf{p} + t\mathbf{q} - t\mathbf{p} = (1 - t)\mathbf{p} + t\mathbf{q}$.

b. Consider $\mathbf{x} = (1 - t)\mathbf{p} + t\mathbf{q}$ for t such that $0 \le t \le 1$. Then, by linearity of T,

$$T(\mathbf{x}) = T((1 - t)\mathbf{p} + t\mathbf{q}) = (1 - t)T(\mathbf{p}) + tT(\mathbf{q}) \qquad 0 \le t \le 1 \qquad (*)$$

If $T(\mathbf{p})$ and $T(\mathbf{q})$ are distinct, then (*) is the equation for the line segment between $T(\mathbf{p})$ and $T(\mathbf{q})$, as shown in part (a) Otherwise, the set of images is just the single point $T(\mathbf{p})$, because

$$(1 - t)T(\mathbf{p}) + tT(\mathbf{q}) =(1 - t)T(\mathbf{p}) + tT(\mathbf{p}) = T(\mathbf{p})$$

28. Consider a point $\mathbf{x}$ in the parallelogram determined by $\mathbf{u}$ and $\mathbf{v}$, say $\mathbf{x} = a\mathbf{u} + b\mathbf{v}$ for $0 \le a \le 1$, $0 \le b \le 1$. By linearity of T, the image of $\mathbf{x}$ is

$$T(\mathbf{x}) = T(a\mathbf{u} + b\mathbf{v}) = aT(\mathbf{u}) + bT(\mathbf{v}), \text{ for } 0 \le a \le 1,\ 0 \le b \le 1 \qquad (*)$$

This image point lies in the parallelogram determined by $T(\mathbf{u})$ and $T(\mathbf{v})$.

Special "degenerate" cases arise when $T(\mathbf{u})$ and $T(\mathbf{v})$ are linearly dependent. If one of the images is not zero, then the "parallelogram" is actually the line segment from $\mathbf{0}$ to $T(\mathbf{u}) + T(\mathbf{v})$. If both $T(\mathbf{u})$ and $T(\mathbf{v})$ are zero, then the parallelogram is just $\{\mathbf{0}\}$. Another possibility is that even $\mathbf{u}$ and $\mathbf{v}$ are linearly dependent, in which case the original parallelogram is degenerate (either a line segment or the zero vector). In this case, the set of images must be degenerate, too.

29. **a**. When $b = 0$, $f(x) = mx$. In this case, for all x,y in $\mathbf{R}$ and all scalars c and d,

$$f(cx + dy) = m(cx + dy) = mcx + mdy = c(mx) + d(my) = c{\cdot}f(x) + d{\cdot}f(y)$$

This shows that f is linear.

b. When $f(x) = mx + b$, with b nonzero, $f(0) = m(0) = b = b \neq 0$. This shows that f is not linear, because every linear transformation maps the zero vector in its domain into the zero vector in the codomain. (In this case, both zero vectors are just the number 0.) Another argument, for instance, would be to calculate $f(2x) = m(2x) + b$ and $2f(x) = 2mx + 2b$. If b is nonzero, then $f(2x)$ is not equal to $2f(x)$ and so f is not a linear transformation.

c. In calculus, f is called a "linear function" because the graph of f is a line.

30. Let $T(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$ for $\mathbf{x}$ in $\mathbf{R}^n$. If $\mathbf{b}$ is not zero, $T(\mathbf{0}) = A\mathbf{0} + \mathbf{b} = \mathbf{b} \neq \mathbf{0}$. Actually, T fails both properties of a linear transformation. For instance, $T(2\mathbf{x}) = A(2\mathbf{x}) + \mathbf{b} = 2A\mathbf{x} + \mathbf{b}$, which is not the same as $2T(\mathbf{x}) = 2(A\mathbf{x} + \mathbf{b}) = 2A\mathbf{x} + 2\mathbf{b}$. Also,

$$T(\mathbf{x} + \mathbf{y}) = A(\mathbf{x} + \mathbf{y}) + \mathbf{b} = A\mathbf{x} + A\mathbf{y} + \mathbf{b}$$

which is not the same as

$$T(\mathbf{x}) + T(\mathbf{y}) = A\mathbf{x} + \mathbf{b} + A\mathbf{y} + \mathbf{b}$$

31. (The *Study Guide* has a more detailed discussion of the proof.) Suppose that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent. Then there exist scalars c_1, c_2, c_3, not all zero, such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}$$

Then $T(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3) = T(\mathbf{0}) = \mathbf{0}$. Since T is linear,

$$c_1T(\mathbf{v}_1) + c_2T(\mathbf{v}_2) + c_3T(\mathbf{v}_3) = \mathbf{0}$$

Since not all the weights are zero, $\{T(\mathbf{v}_1), T(\mathbf{v}_2), T(\mathbf{v}_3)\}$ is a linearly dependent set.

32. Take any vector (x_1, x_2) with $x_2 \neq 0$, and use a negative scalar. For instance, $T(0, 1) = (-2, 3)$, but $T(-1\cdot(0, 1)) = T(0, -1) = (2, 3) \neq (-1)\cdot T(0, 1)$.

33. One possibility is to show that T does not map the zero vector into the zero vector, something that every linear transformation *does* do. $T(0, 0) = (0, 4, 0)$.

34. Suppose that $\{\mathbf{u}, \mathbf{v}\}$ is a linearly independent set in $\mathbf{R}^n$ and yet $T(\mathbf{u})$ and $T(\mathbf{v})$ are linearly dependent. Then there exist weights c_1, c_2, not both zero, such that

$$c_1T(\mathbf{u}) + c_2T(\mathbf{v}) = \mathbf{0}$$

Because T is linear, $T(c_1\mathbf{u} + c_2\mathbf{v}) = \mathbf{0}$. That is, the vector $\mathbf{x} = c_1\mathbf{u} + c_2\mathbf{v}$ satisfies $T(\mathbf{x}) = \mathbf{0}$. Furthermore, $\mathbf{x}$ cannot be the zero vector, since that would mean that a nontrivial linear combination of $\mathbf{u}$ and $\mathbf{v}$ is zero, which is impossible because $\mathbf{u}$ and $\mathbf{v}$ are linearly independent. Thus, the equation $T(\mathbf{x}) = \mathbf{0}$ has a nontrivial solution.

35. Take $\mathbf{u}$ and $\mathbf{v}$ in $\mathbf{R}^3$ and let c and d be scalars. Then

$c\mathbf{u} + d\mathbf{v} = (cu_1 + dv_1, cu_2 + dv_2, cu_3 + dv_3)$. The transformation T is linear because

$$\begin{aligned} T(c\mathbf{u} + d\mathbf{v}) &= (cu_1 + dv_1, cu_2 + dv_2, -(cu_3 + dv_3)) = (cu_1 + dv_1, cu_2 + dv_2,\ cu_3\ \ dv_3) \\ &= (cu_1, cu_2,\ \ cu_3) + (dv_1, dv_2,\ dv_3) = c(u_1, u_2,\ \ \ u_3) + d(v_1, v_2,\ \ v_3) \\ &= cT(\mathbf{u}) + dT(\mathbf{v}) \end{aligned}$$

36. Take $\mathbf{u}$ and $\mathbf{v}$ in $\mathbf{R}^3$ and let c and d be scalars. Then

$c\mathbf{u} + d\mathbf{v} = (cu_1 + dv_1, cu_2 + dv_2, cu_3 + dv_3)$. The transformation T is linear because

$$\begin{aligned} T(c\mathbf{u} + d\mathbf{v}) &= (cu_1 + dv_1, 0, cu_3 + dv_3) = (cu_1, 0, cu_3) + (dv_1, 0, dv_3) \\ &= c(u_1, 0, u_3) + d(v_1, 0, v_3) \\ &= cT(\mathbf{u}) + dT(\mathbf{v}) \end{aligned}$$

37. [M] $\begin{bmatrix} 4 & -2 & 5 & -5 & 0 \\ -9 & 7 & -8 & 0 & 0 \\ -6 & 4 & 5 & 3 & 0 \\ 5 & -3 & 8 & -4 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -7/2 & 0 \\ 0 & 1 & 0 & -9/2 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, $\begin{cases} x_1 = (7/2)x_4 \\ x_2 = (9/2)x_4 \\ x_3 = 0 \\ x_4 \text{ is free} \end{cases}$ $\mathbf{x} = x_4 \begin{bmatrix} 7/2 \\ 9/2 \\ 0 \\ 1 \end{bmatrix}$

38. [M] $\begin{bmatrix} -9 & -4 & -9 & 4 & 0 \\ 5 & -8 & -7 & 6 & 0 \\ 7 & 11 & 16 & -9 & 0 \\ 9 & -7 & -4 & 5 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 3/4 & 0 \\ 0 & 1 & 0 & 5/4 & 0 \\ 0 & 0 & 1 & -7/4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, $\begin{cases} x_1 = -(3/4)x_4 \\ x_2 = -(5/4)x_4 \\ x_3 = (7/4)x_4 \\ x_4 \text{ is free} \end{cases}$ $\mathbf{x} = x_4 \begin{bmatrix} -3/4 \\ -5/4 \\ 7/4 \\ 1 \end{bmatrix}$

39. [M] $\begin{bmatrix} 4 & -2 & 5 & -5 & 7 \\ -9 & 7 & -8 & 0 & 5 \\ -6 & 4 & 5 & 3 & 9 \\ 5 & -3 & 8 & -4 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -7/2 & 4 \\ 0 & 1 & 0 & -9/2 & 7 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, yes, **b** is in the range of the transformation, because the augmented matrix shows a consistent system. In fact,

the general solution is $\begin{cases} x_1 = 4 + (7/2)x_4 \\ x_2 = 7 + (9/2)x_4 \\ x_3 = 1 \\ x_4 \text{ is free} \end{cases}$; when $x_4 = 0$ a solution is $\mathbf{x} = \begin{bmatrix} 4 \\ 7 \\ 1 \\ 0 \end{bmatrix}$.

40. [M] $\begin{bmatrix} -9 & -4 & -9 & 4 & -7 \\ 5 & -8 & -7 & 6 & -7 \\ 7 & 11 & 16 & -9 & 13 \\ 9 & -7 & -4 & 5 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 3/4 & -5/4 \\ 0 & 1 & 0 & 5/4 & -11/4 \\ 0 & 0 & 1 & -7/4 & 13/4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, yes, **b** is in the range of the transformation, because the augmented matrix shows a consistent system. In fact,

the general solution is $\begin{cases} x_1 = -5/4 - (3/4)x_4 \\ x_2 = -11/4 - (5/4)x_4 \\ x_3 = 13/4 + (7/4)x_4 \\ x_4 \text{ is free} \end{cases}$; when $x_4 = 1$ a solution is $\mathbf{x} = \begin{bmatrix} -2 \\ -4 \\ 5 \\ 1 \end{bmatrix}$.

Notes: At the end of Section 1.8, the *Study Guide* provides a list of equations, figures, examples, and connections with concepts that will strengthen a student's understanding of linear transformations. I encourage my students to continue the construction of review sheets similar to those for "span" and "linear independence," but I refrain from collecting these sheets. At some point the students have to assume the responsibility for mastering this material.

If your students are using MATLAB or another matrix program, you might insert the definition of matrix multiplication after this section, and then assign a project that uses random matrices to explore properties of matrix multiplication. See Exercises 34–36 in Section 2.1. Meanwhile, in class you can continue with your plans for finishing Chapter 1. When you get to Section 2.1, you won't have much to do. The *Study Guide*'s MATLAB note for Section 2.1 contains the matrix notation students will need for a project on matrix multiplication. The appendices in the *Study Guide* have the corresponding material for Mathematica, Maple, and the T-83+/86/89 and HP-48G graphic calculators.

1.9 SOLUTIONS

Notes: This section is optional if you plan to treat linear transformations only lightly, but many instructors will want to cover at least Theorem 10 and a few geometric examples. Exercises 15 and 16 illustrate a fast way to solve Exercises 17–22 without explicitly computing the images of the standard basis.

The purpose of introducing *one-to-one* and *onto* is to prepare for the term *isomorphism* (in Section 4.4) and to acquaint math majors with these terms. Mastery of these concepts would require a substantial digression, and some instructors prefer to omit these topics (and Exercises 25–40). In this case, you can use the result of Exercise 31 in Section 1.8 to show that the coordinate mapping from a vector space onto $\mathbf{R}^n$ (in Section 4.4) preserves linear independence and dependence of sets of vectors. (See Example 6 in Section 4.4.) The notions of one-to-one and onto appear in the Invertible Matrix Theorem (Section 2.3), but can be omitted there if desired

Exercises 25–28 and 31–36 offer fairly easy writing practice. Exercises 31, 32, and 35 provide important links to earlier material.

1. $A = [T(\mathbf{e}_1) \quad T(\mathbf{e}_2)] = \begin{bmatrix} 3 & -5 \\ 1 & 2 \\ 3 & 0 \\ 1 & 0 \end{bmatrix}$

2. $A = [T(\mathbf{e}_1) \quad T(\mathbf{e}_2) \quad T(\mathbf{e}_3)] = \begin{bmatrix} 1 & 4 & -5 \\ 3 & -7 & 4 \end{bmatrix}$

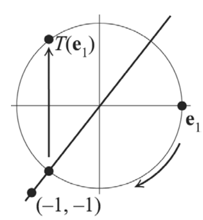
3. $T(\mathbf{e}_1) = -\mathbf{e}_2$, $T(\mathbf{e}_2) = \mathbf{e}_1$. $A = \begin{bmatrix} -\mathbf{e}_2 & \mathbf{e}_1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

4. $T(\mathbf{e}_1) = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$, $T(\mathbf{e}_2) = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$, $A = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$

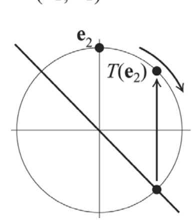
5. $T(\mathbf{e}_1) = \mathbf{e}_1 - 2\mathbf{e}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$, $T(\mathbf{e}_2) = \mathbf{e}_2$, $A = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$

6. $T(\mathbf{e}_1) = \mathbf{e}_1$, $T(\mathbf{e}_2) = \mathbf{e}_2 + 3\mathbf{e}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, $A = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$

7. Follow what happens to $\mathbf{e}_1$ and $\mathbf{e}_2$. Since $\mathbf{e}_1$ is on the unit circle in the plane, it rotates through $-3\pi/4$ radians into a point on the unit circle that lies in the third quadrant and on the line $x_2 = x_1$ (that is, $y = x$ in more familiar notation). The point (–1,–1) is on the ine $x_2 = x_1$, but its distance from the origin is $\sqrt{2}$. So the rotational image of $\mathbf{e}_1$ is $(-1/\sqrt{2}, -1/\sqrt{2})$. Then this image reflects in the horizontal axis to $(-1/\sqrt{2}, 1/\sqrt{2})$.



Similarly, $\mathbf{e}_2$ rotates into a point on the unit circle that lies in the second quadrant and on the line $x_2 = x_1$, namely,



$(-1/\sqrt{2},-1/\sqrt{2})$. Then this image reflects in the horizontal axis to $(-1/\sqrt{2},1/\sqrt{2})$.

When the two calculations described above are written in vertical vector notation, the transformation's standard matrix [T($\mathbf{e}_1$) T($\mathbf{e}_2$)] is easily seen:

$$\mathbf{e}_1 \to \begin{bmatrix} -1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} \to \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix},\ \mathbf{e}_2 \to \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} \to \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix},\ A = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

8. $\mathbf{e}_1 \to \mathbf{e}_1 \to \mathbf{e}_2$ and $\mathbf{e}_2 \to -\mathbf{e}_2 \to -\mathbf{e}_1$, so $A = [\mathbf{e}_2 \quad -\mathbf{e}_1] = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

9. The horizontal shear maps $\mathbf{e}_1$ into $\mathbf{e}_1$, and then the reflection in the line $x_2 = -x_1$ maps $\mathbf{e}_1$ into $-\mathbf{e}_2$. (See Table 1.) The horizontal shear maps $\mathbf{e}_2$ into $\mathbf{e}_2$ into $\mathbf{e}_2 - 2\mathbf{e}_1$. To find the image of $\mathbf{e}_2 - 2\mathbf{e}_1$ when it is reflected in the line $x_2 = -x_1$, use the fact that such a reflection is a linear transformation. So, the image of $\mathbf{e}_2 - 2\mathbf{e}_1$ is the same linear combination of the images of $\mathbf{e}_2$ and $\mathbf{e}_1$, namely, $-\mathbf{e}_1 - 2(-\mathbf{e}_2) = -\mathbf{e}_1 + 2\mathbf{e}_2$.
To summarize,

$$\mathbf{e}_1 \to \mathbf{e}_1 \to -\mathbf{e}_2 \text{ and } \mathbf{e}_2 \to \mathbf{e}_2 - 2\mathbf{e}_1 \to -\mathbf{e}_1 + 2\mathbf{e}_2, \text{ so } A = \begin{bmatrix} 0 & -1 \\ -1 & 2 \end{bmatrix}$$

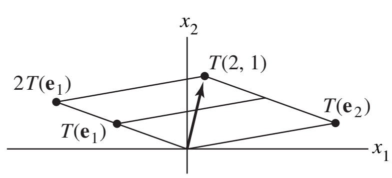
To find the image of $\mathbf{e}_2 - 2\mathbf{e}_1$ when it is reflected through the vertical axis use the fact that such a reflection is a linear transformation. So, the image of $\mathbf{e}_2 - 2\mathbf{e}_1$ is the same linear combination of the images of $\mathbf{e}_2$ and $\mathbf{e}_1$, namely, $\mathbf{e}_2 + 2\mathbf{e}_1$.

10. $\mathbf{e}_1 \to -\mathbf{e}_1 \to -\mathbf{e}_2$ and $\mathbf{e}_2 \to \mathbf{e}_2 \to -\mathbf{e}_1$, so $A = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

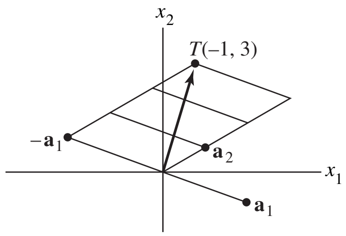
11. The transformation T described maps $\mathbf{e}_1 \to \mathbf{e}_1 \to -\mathbf{e}_1$ and maps $\mathbf{e}_2 \to -\mathbf{e}_2 \to -\mathbf{e}_2$. A rotation through π radians also maps $\mathbf{e}_1$ into $-\mathbf{e}_1$ and maps $\mathbf{e}_2$ into $-\mathbf{e}_2$. Since a linear transformation is completely determined by what it does to the columns of the identity matrix, the rotation transformation has the same effect as T on every vector in $\mathbb{R}^2$.

12. The transformation T in Exercise 8 maps $\mathbf{e}_1 \to \mathbf{e}_1 \to \mathbf{e}_2$ and maps $\mathbf{e}_2 \to -\mathbf{e}_2 \to -\mathbf{e}_1$. A rotation about the origin through $\pi/2$ radians also maps $\mathbf{e}_1$ into $\mathbf{e}_2$ and maps $\mathbf{e}_2$ into $-\mathbf{e}_1$. Since a linear transformation is completely determined by what it does to the columns of the identity matrix, the rotation transformation has the same effect as T on every vector in $\mathbb{R}^2$.

13. Since (2, 1) = 2$\mathbf{e}_1$ + $\mathbf{e}_2$, the image of (2, 1) under T is 2T($\mathbf{e}_1$) + T($\mathbf{e}_2$), by linearity of T. On the figure in the exercise, locate 2T($\mathbf{e}_1$) and use it with T($\mathbf{e}_2$) to form the parallelogram shown below.



14. Since $T(\mathbf{x}) = A\mathbf{x} = [\mathbf{a}_1 \;\; \mathbf{a}_2]\mathbf{x} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 = -\mathbf{a}_1 + 3\mathbf{a}_2$, when $\mathbf{x} = (-1, 3)$, the image of $\mathbf{x}$ is located by forming the parallelogram shown below.



15. By inspection, $\begin{bmatrix} 3 & 0 & -2 \\ 4 & 0 & 0 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3x_1 - 2x_3 \\ 4x_1 \\ x_1 - x_2 + x_3 \end{bmatrix}$

16. By inspection, $\begin{bmatrix} 1 & -1 \\ -2 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 - x_2 \\ -2x_1 + x_2 \\ x_1 \end{bmatrix}$

17. To express $T(\mathbf{x})$ as $A\mathbf{x}$, write $T(\mathbf{x})$ and $\mathbf{x}$ as column vectors, and then fill in the entries in A by inspection, as done in Exercises 15 and 16. Note that since $T(\mathbf{x})$ and $\mathbf{x}$ have four entries, A must be a 4×4 matrix.

$$T(\mathbf{x}) = \begin{bmatrix} 0 \\ x_1 + x_2 \\ x_2 + x_3 \\ x_3 + x_4 \end{bmatrix} = \begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

18. As in Exercise 17, write $T(\mathbf{x})$ and $\mathbf{x}$ as column vectors. Since $\mathbf{x}$ has 2 entries, A has 2 columns. Since $T(\mathbf{x})$ has 4 entries, A has 4 rows.

$$\begin{bmatrix} 2x_2 - 3x_1 \\ x_1 - 4x_2 \\ 0 \\ x_2 \end{bmatrix} = \begin{bmatrix} & \\ A & \\ & \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ 1 & -4 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

19. Since $T(\mathbf{x})$ has 2 entries, A has 2 rows. Since $\mathbf{x}$ has 3 entries, A has 3 columns.

$$\begin{bmatrix} x_1 - 5x_2 + 4x_3 \\ x_2 - 6x_3 \end{bmatrix} = \begin{bmatrix} & A & \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & -5 & 4 \\ 0 & 1 & -6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

20. Since $T(\mathbf{x})$ has 1 entry, A has 1 row. Since $\mathbf{x}$ has 4 entries, A has 4 columns.

$$[2x_1 + 3x_3 - 4x_4] = [\quad A \quad]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = [2 \quad 0 \quad 3 \quad -4]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

21. $T(\mathbf{x}) = \begin{bmatrix} x_1 + x_2 \\ 4x_1 + 5x_2 \end{bmatrix} = \begin{bmatrix} & A & \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. To solve $T(\mathbf{x}) = \begin{bmatrix} 3 \\ 8 \end{bmatrix}$, row reduce the augmented matrix:

$$\begin{bmatrix} 1 & 1 & 3 \\ 4 & 5 & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 7 \\ 0 & 1 & -4 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} 7 \\ -4 \end{bmatrix}.$$

22. $T(\mathbf{x}) = \begin{bmatrix} x_1 - 2x_2 \\ -x_1 + 3x_2 \\ 3x_1 - 2x_2 \end{bmatrix} = \begin{bmatrix} & A & \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ -1 & 3 \\ 3 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. To solve $T(\mathbf{x}) = \begin{bmatrix} -1 \\ 4 \\ 9 \end{bmatrix}$, row reduce the augmented matrix:

$$\begin{bmatrix} 1 & -2 & -1 \\ -1 & 3 & 4 \\ 3 & -2 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & -1 \\ 0 & 1 & 3 \\ 0 & 4 & 12 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & -1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}.$$

23. **a.** True. See Theorem 10.
b. True. See Example 3.
c. False. See the paragraph before Table 1.
d. False. See the definition of *onto*. *Any* function from $\mathbf{R}^n$ to $\mathbf{R}^m$ maps each vector onto another vector.
e. False. See Example 5.

24. **a.** False. See the paragraph preceding Example 2.
b. True. See Theorem 10.
c. True. See Table 1.
d. False. See the definition of one-to-one. Any *function* from $\mathbf{R}^n$ to $\mathbf{R}^m$ maps a vector onto a single (unique) vector.
e. True. See the solution of Example 5.

25. Three row interchanges on the standard matrix A of the transformation T in Exercise 17 produce $\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. This matrix shows that A has only three pivot positions, so the equation $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution. By Theorem 11, the transformation T is *not* one-to-one. Also, since A does not have a pivot in each row, the columns of A do not span $\mathbf{R}^4$. By Theorem 12, T does *not* map $\mathbf{R}^4$ onto $\mathbf{R}^4$.

26. The standard matrix A of the transformation T in Exercise 2 is 2×3. Its columns are linearly dependent because A has more columns than rows. So T is *not* one-to-one, by Theorem 12. Also, A is row equivalent to $\begin{bmatrix} 1 & 4 & -5 \\ 0 & -19 & 19 \end{bmatrix}$, which shows that the rows of A span $\mathbf{R}^2$. By Theorem 12, T maps $\mathbf{R}^3$ onto $\mathbf{R}^2$.

27. The standard matrix A of the transformation T in Exercise 19 is $\begin{bmatrix} 1 & -5 & 4 \\ 0 & 1 & -6 \end{bmatrix}$. The columns of A are linearly dependent because A has more columns than rows. So T is *not* one-to-one, by Theorem 12. Also, A has a pivot in each row, so the rows of A span $\mathbf{R}^2$. By Theorem 12, T maps $\mathbf{R}^3$ onto $\mathbf{R}^2$.

28. The standard matrix A of the transformation T in Exercise 14 has linearly independent columns, because the figure in that exercise shows that $\mathbf{a}_1$ and $\mathbf{a}_2$ are not multiples. So T is one-to-one, by Theorem 12. Also, A must have a pivot in each column because the equation $A\mathbf{x} = \mathbf{0}$ has no free variables. Thus, the echelon form of A is $\begin{bmatrix} \blacksquare & * \\ 0 & \blacksquare \end{bmatrix}$. Since A has a pivot in each row, the columns of A span $\mathbf{R}^2$. So T maps $\mathbf{R}^2$ onto $\mathbf{R}^2$. An alternate argument for the second part is to observe directly from the figure in Exercise 14 that $\mathbf{a}_1$ and $\mathbf{a}_2$ span $\mathbf{R}^2$. This is more or less evident, based on experience with grids such as those in Figure 8 and Exercise 7 of Section 1.3.

29. By Theorem 12, the columns of the standard matrix A must be linearly independent and hence the equation $A\boldsymbol{x} = \mathbf{0}$ has no free variables. So each column of A must be a pivot column: $A \sim \begin{bmatrix} \blacksquare & * & * \\ 0 & \blacksquare & * \\ 0 & 0 & \blacksquare \\ 0 & 0 & 0 \end{bmatrix}$.

Note that T cannot be onto because of the shape of A.

30. By Theorem 12, the columns of the standard matrix A must span $\mathbb{R}^3$. By Theorem 4, the matrix must have a pivot in each row. There are four possibilities for the echelon form:

$$\begin{bmatrix} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & \blacksquare & * \end{bmatrix}, \begin{bmatrix} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & 0 & \blacksquare \end{bmatrix}, \begin{bmatrix} \blacksquare & * & * & * \\ 0 & 0 & \blacksquare & * \\ 0 & 0 & 0 & \blacksquare \end{bmatrix}, \begin{bmatrix} 0 & \blacksquare & * & * \\ 0 & 0 & \blacksquare & * \\ 0 & 0 & 0 & \blacksquare \end{bmatrix}$$

Note that T cannot be one-to-one because of the shape of A.

31. "T is one-to-one if and only if A has n pivot columns." By Theorem 12(b), T is one-to-one if and only if the columns of A are linearly independent. And from the statement in Exercise 30 in Section 1.7, the columns of A are linearly independent if and only if A has n pivot columns.

32. The transformation T maps $\mathbf{R}^n$ onto $\mathbf{R}^m$ if and only if the columns of A span $\mathbf{R}^m$, by Theorem 12. This happens if and only if A has a pivot position in each row, by Theorem 4 in Section 1.4. Since A has m rows, this happens if and only if A has m pivot columns. Thus, "T maps $\mathbf{R}^n$ onto $\mathbf{R}^m$ if and only A has m pivot columns."

33. Define $T:\mathbb{R}^n \to \mathbb{R}^m$ by $T(\mathbf{x}) = B\mathbf{x}$ for some $m \times n$ matrix B, and let A be the standard matrix for T. By definition, $A = [T(\mathbf{e}_1) \ \cdots \ T(\mathbf{e}_n)]$, where $\mathbf{e}_j$ is the jth column of I_n. However, by matrix-vector multiplication, $T(\mathbf{e}_j) = B\mathbf{e}_j = \mathbf{b}_j$, the jth column of B. So $A = [\mathbf{b}_1 \ \cdots \ \mathbf{b}_n] = B$.

34. The transformation T maps $\mathbf{R}^n$ *onto* $\mathbf{R}^m$ if and only if for each $\mathbf{y}$ in $\mathbf{R}^m$ *there exists* an $\mathbf{x}$ in $\mathbf{R}^n$ such that $\mathbf{y} = T(\mathbf{x})$.

35. If $T:\mathbb{R}^n \to \mathbb{R}^m$ maps $\mathbb{R}^n$ onto $\mathbb{R}^m$, then its standard matrix A has a pivot in each row, by Theorem 12 and by Theorem 4 in Section 1.4. So A must have at least as many columns as rows. That is, $m \le n$. When T is one-to-one, A must have a pivot in each column, by Theorem 12, so $m \ge n$.

36. Take $\mathbf{u}$ and $\mathbf{v}$ in $\mathbf{R}^p$ and let c and d be scalars. Then

$$\begin{aligned} T(S(c\mathbf{u} + d\mathbf{v})) &= T(c\cdot S(\mathbf{u}) + d\cdot S(\mathbf{v})) && \text{because } S \text{ is linear} \\ &= c\cdot T(S(\mathbf{u})) + d\cdot T(S(\mathbf{v})) && \text{because } T \text{ is linear} \end{aligned}$$

This calculation shows that the mapping $\mathbf{x} \to T(S(\mathbf{x}))$ is linear. See equation (4) in Section 1.8.

37. **[M]** $\begin{bmatrix} -5 & 10 & -5 & 4 \\ 8 & 3 & -4 & 7 \\ 4 & -9 & 5 & -3 \\ -3 & -2 & 5 & 4 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} 1 & 0 & 0 & 44/35 \\ 0 & 1 & 0 & 79/35 \\ 0 & 0 & 1 & 86/35 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1.2571 \\ 0 & 1 & 0 & 2.2571 \\ 0 & 0 & 1 & 2.4571 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. There is no pivot in the fourth column of the standard matrix A, so the equation $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution. By Theorem 11, the transformation T is *not* one-to-one. (For a shorter argument, use the result of Exercise 31.)

38. **[M]** $\begin{bmatrix} 7 & 5 & 4 & -9 \\ 10 & 6 & 16 & -4 \\ 12 & 8 & 12 & 7 \\ -8 & -6 & -2 & 5 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} 1 & 0 & 7 & 0 \\ 0 & 1 & -9 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. No. There is no pivot in the third column of the standard matrix A, so the equation $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution. By Theorem 11, the transformation T is *not* one-to-one. (For a shorter argument, use the result of Exercise 31.)

39. **[M]** $\begin{bmatrix} 4 & -7 & 3 & 7 & 5 \\ 6 & -8 & 5 & 12 & -8 \\ -7 & 10 & -8 & -9 & 14 \\ 3 & -5 & 4 & 2 & -6 \\ -5 & 6 & -6 & -7 & 3 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} 1 & 0 & 0 & 5 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. There is not a pivot in every row, so the columns of the standard matrix do not span $\mathbf{R}^5$. By Theorem 12, the transformation T does *not* map $\mathbf{R}^5$ onto $\mathbf{R}^5$.

40. **[M]** $\begin{bmatrix} 9 & 13 & 5 & 6 & -1 \\ 14 & 15 & -7 & -6 & 4 \\ -8 & -9 & 12 & -5 & -9 \\ -5 & -6 & -8 & 9 & 8 \\ 13 & 14 & 15 & 2 & 11 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 5 \\ 0 & 1 & 0 & 0 & -4 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. There is not a pivot in every row, so the columns of the standard matrix do not span $\mathbf{R}^5$. By Theorem 12, the transformation T does *not* map $\mathbf{R}^5$ onto $\mathbf{R}^5$.

1.10 SOLUTIONS

1. **a**. If x_1 is the number of servings of Cheerios and x_2 is the number of servings of 100% Natural Cereal, then x_1 and x_2 should satisfy

$$x_1 \begin{bmatrix} \text{nutrients} \\ \text{per serving} \\ \text{of Cheerios} \end{bmatrix} + x_2 \begin{bmatrix} \text{nutrients} \\ \text{per serving of} \\ \text{100\% Natural} \end{bmatrix} = \begin{bmatrix} \text{quantities} \\ \text{of nutrients} \\ \text{required} \end{bmatrix}$$

That is,

$$x_1 \begin{bmatrix} 110 \\ 4 \\ 20 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 130 \\ 3 \\ 18 \\ 5 \end{bmatrix} = \begin{bmatrix} 295 \\ 9 \\ 48 \\ 8 \end{bmatrix}$$

b. The equivalent matrix equation is $\begin{bmatrix} 110 & 130 \\ 4 & 3 \\ 20 & 18 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 295 \\ 9 \\ 48 \\ 8 \end{bmatrix}$. To solve this, row reduce the augmented matrix for this equation.

$$\begin{bmatrix} 110 & 130 & 295 \\ 4 & 3 & 9 \\ 20 & 18 & 48 \\ 2 & 5 & 8 \end{bmatrix} \sim \begin{bmatrix} 2 & 5 & 8 \\ 4 & 3 & 9 \\ 20 & 18 & 48 \\ 110 & 130 & 295 \end{bmatrix} \sim \begin{bmatrix} 1 & 2.5 & 4 \\ 4 & 3 & 9 \\ 10 & 9 & 24 \\ 110 & 130 & 295 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2.5 & 4 \\ 0 & -7 & -7 \\ 0 & -16 & -16 \\ 0 & -145 & -145 \end{bmatrix} \sim \begin{bmatrix} 1 & 2.5 & 4 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1.5 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The desired nutrients are provided by 1.5 servings of Cheerios together with 1 serving of 100% Natural Cereal.

2. Set up nutrient vectors for one serving of Kellogg's Cracklin' Oat Bran (COB) and Kellogg's Crispix (Crp):

$$\begin{array}{lcc} \text{Nutrients:} & \text{COB} & \text{Crp} \\ \begin{array}{l} \text{calories} \\ \text{protein} \\ \text{carbohydrate} \\ \text{fat} \end{array} & \begin{bmatrix} 110 \\ 3 \\ 21 \\ 3 \end{bmatrix} & \begin{bmatrix} 110 \\ 2 \\ 25 \\ .4 \end{bmatrix}. \end{array}$$

a. Let $B = [\text{COB} \quad \text{Crp}] = \begin{bmatrix} 110 & 110 \\ 3 & 2 \\ 21 & 25 \\ 3 & .4 \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$.

Then $B\mathbf{u}$ lists the amounts of calories, protein, carbohydrate, and fat in a mixture of three servings of Cracklin' Oat Bran and two servings of Crispix.

b. Let u_1 and u_2 be the number of servings of Cracklin' Oat Bran and Crispix, respectively. Can these numbers satisfy the equation $B\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 110 \\ 2.25 \\ 24 \\ 1 \end{bmatrix}$? To find out, row reduce the augmented matrix

$$\begin{bmatrix} 110 & 110 & 110 \\ 3 & 2 & 2.25 \\ 21 & 25 & 24 \\ 3 & .4 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 3 & 2 & 2.25 \\ 21 & 25 & 24 \\ 3 & .4 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -.75 \\ 0 & 4 & 3 \\ 0 & -2.6 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -.75 \\ 0 & 0 & 0 \\ 0 & 0 & -.05 \end{bmatrix}$$

The last row identifies an inconsistent system, because 0 = –.05 is impossible. So, technically, there is no mixture of the two cereals that will supply *exactly* the desired list of nutrients. However, one could tentatively ignore the final equation and see what the other equations prescribe. They reduce to $u_1 = .25$ and $u_2 = .75$. What does the corresponding mixture provide?

$$.25\cdot\text{COB} + .75\cdot\text{Crp} = .25\begin{bmatrix}110\\3\\21\\3\end{bmatrix}+.75\begin{bmatrix}110\\2\\25\\.4\end{bmatrix}=\begin{bmatrix}110\\2.25\\24\\1.05\end{bmatrix}$$

The error of 5% for fat might be acceptable for practical purposes. Actually, the data in COB and Crp are certainly not precise and may have some errors even greater than 5%.

3. Here are the data, assembled from Table 1 and Exercise 3:

Nutrient	Mg of Nutrients/Unit: milk	soy flour	whey	soy prot.	Nutrients Required (milligrams)
protein	36	51	13	80	33
carboh.	52	34	74	0	45
fat	0	7	1.1	3.4	3
calcium	1.26	.19	.8	.18	.8

a. Let x_1, x_2, x_3, x_4 represent the number of units of nonfat milk, soy flour, whey, and isolated soy protein, respectively. These amounts must satisfy the following matrix equation

$$\begin{bmatrix}36&51&13&80\\52&34&74&0\\0&7&1.1&3.4\\1.26&.19&.8&.18\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=\begin{bmatrix}33\\45\\3\\.8\end{bmatrix}$$

b. **[M]** $$\begin{bmatrix}36&51&13&80&33\\52&34&74&0&45\\0&7&1.1&3.4&3\\1.26&.19&.8&.18&.8\end{bmatrix}\sim\cdots\sim\begin{bmatrix}1&0&0&0&.64\\0&1&0&0&.54\\0&0&1&0&-.09\\0&0&0&1&-.21\end{bmatrix}$$

The "solution" is $x_1 = .64$, $x_2 = .54$, $x_3 = -.09$, $x_4 = -.21$. This solution is not feasible, because the mixture cannot include negative amounts of whey and isolated soy protein. Although the coefficients of these two ingredients are fairly small, they cannot be ignored. The mixture of .64 units of nonfat milk and .54 units of soy flour provide 50.6 g of protein, 51.6 g of carbohydrate, 3.8 g of fat, and .9 g of calcium. Some of these nutrients are nowhere close to the desired amounts.

4. Let x_1, x_2, and x_3 be the number of units of foods 1, 2, and 3, respectively, needed for a meal. The values of x_1, x_2, and x_3 should satisfy

$$x_1\begin{bmatrix}\text{nutrients}\\\text{(in mg)}\\\text{per unit}\\\text{of Food 1}\end{bmatrix}+x_2\begin{bmatrix}\text{nutrients}\\\text{(in mg)}\\\text{per unit}\\\text{of Food 2}\end{bmatrix}+x_3\begin{bmatrix}\text{nutrients}\\\text{(in mg)}\\\text{per unit}\\\text{of Food 3}\end{bmatrix}=\begin{bmatrix}\text{milligrams}\\\text{of nutrients}\\\text{required}\end{bmatrix}$$

From the given data,

$$x_1\begin{bmatrix}10\\50\\30\end{bmatrix}+x_2\begin{bmatrix}20\\40\\10\end{bmatrix}+x_3\begin{bmatrix}20\\10\\40\end{bmatrix}=\begin{bmatrix}100\\300\\200\end{bmatrix}$$

To solve, row reduce the corresponding augmented matrix:

$$\begin{bmatrix}10&20&20&100\\50&40&10&300\\30&10&40&200\end{bmatrix}\sim\begin{bmatrix}10&20&20&100\\0&-60&-90&-200\\0&-50&-20&-100\end{bmatrix}\sim\begin{bmatrix}1&2&2&10\\0&1&3/2&10/3\\0&5&2&10\end{bmatrix}$$

$$\sim\begin{bmatrix}1&2&2&10\\0&1&3/2&10/3\\0&0&1&40/33\end{bmatrix}\sim\begin{bmatrix}1&2&0&250/33\\0&1&0&50/33\\0&0&1&40/33\end{bmatrix}\sim\begin{bmatrix}1&0&0&50/11\\0&1&0&50/33\\0&0&1&40/33\end{bmatrix}$$

$$\mathbf{x}=\begin{bmatrix}50/11\\50/33\\40/33\end{bmatrix}\doteq\begin{bmatrix}4.55\\1.52\\1.21\end{bmatrix}=\begin{bmatrix}\text{units of Food 1}\\\text{units of Food 2}\\\text{units of Food 3}\end{bmatrix}$$

5. Loop 1: The resistance vector is

$$\mathbf{r}_1=\begin{bmatrix}5\\-2\\0\\0\end{bmatrix}\begin{array}{l}\text{Total of four RI voltage drops for current } I_1\\\text{Voltage drop for } I_2 \text{ is negative; } I_2 \text{ flows in opposite direction}\\\text{Current } I_3 \text{ does not flow in loop 1}\\\text{Current } I_4 \text{ does not flow in loop 1}\end{array}$$

Loop 2: The resistance vector is

$$\mathbf{r}_2=\begin{bmatrix}-2\\11\\-3\\0\end{bmatrix}\begin{array}{l}\text{Voltage drop for } I_1 \text{ is negative; } I_1 \text{ flows in opposite direction}\\\text{Total of four RI voltage drops for current } I_2\\\text{Voltage drop for } I_3 \text{ is negative; } I_3 \text{ flows in opposite direction}\\\text{Current } I_4 \text{ does not flow in loop 2}\end{array}$$

Also, $\mathbf{r}_3=\begin{bmatrix}0\\-3\\17\\-4\end{bmatrix}$, $\mathbf{r}_4=\begin{bmatrix}0\\0\\-4\\25\end{bmatrix}$, and $R=[\mathbf{r}_1\ \ \mathbf{r}_2\ \ \mathbf{r}_3\ \ \mathbf{r}_4]=\begin{bmatrix}5&-2&0&0\\-2&11&-3&0\\0&-3&17&-4\\0&0&-4&25\end{bmatrix}$.

Notice that each off-diagonal entry of R is negative (or zero). This happens because the loop current directions are all chosen in the same direction on the figure. (For each loop j, this choice forces the currents in other loops adjacent to loop j to flow in the direction opposite to current I_j.)

Next, set $\mathbf{v}=\begin{bmatrix}40\\-30\\20\\-10\end{bmatrix}$. The voltages in loops 2 and 4 are negative because the battery orientation in each loop is opposite to the direction chosen for positive current flow. Thus, the equation $R\mathbf{i}=\mathbf{v}$ becomes

$$\begin{bmatrix} 5 & -2 & 0 & 0 \\ -2 & 11 & -3 & 0 \\ 0 & -3 & 17 & -4 \\ 0 & 0 & -4 & 25 \end{bmatrix}\begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \end{bmatrix} = \begin{bmatrix} 40 \\ -30 \\ 20 \\ -10 \end{bmatrix}. \quad \textbf{[M]}\text{: The solution is } \mathbf{i} = \begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \end{bmatrix} = \begin{bmatrix} 7.56 \\ -1.10 \\ .93 \\ -.25 \end{bmatrix}.$$

6. Loop 1: The resistance vector is

$$\mathbf{r}_1 = \begin{bmatrix} 4 \\ -1 \\ 0 \\ 0 \end{bmatrix} \begin{array}{l} \text{Total of four RI voltage drops for current } I_1 \\ \text{Voltage drop for } I_2 \text{ is negative; } I_2 \text{ flows in opposite direction} \\ \text{Current } I_3 \text{ does not flow in loop 1} \\ \text{Current } I_4 \text{ does not flow in loop 1} \end{array}$$

Loop 2: The resistance vector is

$$\mathbf{r}_2 = \begin{bmatrix} -1 \\ 6 \\ -2 \\ 0 \end{bmatrix} \begin{array}{l} \text{Voltage drop for } I_1 \text{ is negative; } I_1 \text{ flows in opposite direction} \\ \text{Total of four RI voltage drops for current } I_2 \\ \text{Voltage drop for } I_3 \text{ is negative; } I_3 \text{ flows in opposite direction} \\ \text{Current } I_4 \text{ does not flow in loop 2} \end{array}$$

Also, $\mathbf{r}_3 = \begin{bmatrix} 0 \\ -2 \\ 10 \\ -3 \end{bmatrix}$, $\mathbf{r}_4 = \begin{bmatrix} 0 \\ 0 \\ -3 \\ 12 \end{bmatrix}$, and $R = [\mathbf{r}_1 \ \ \mathbf{r}_2 \ \ \mathbf{r}_3 \ \ \mathbf{r}_4]$. Set $\mathbf{v} = \begin{bmatrix} 40 \\ 30 \\ 20 \\ 10 \end{bmatrix}$. Then $R\mathbf{i} = \mathbf{v}$ becomes

$$\begin{bmatrix} 4 & -1 & 0 & 0 \\ -1 & 6 & -2 & 0 \\ 0 & -2 & 10 & -3 \\ 0 & 0 & -3 & 12 \end{bmatrix}\begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \end{bmatrix} = \begin{bmatrix} 40 \\ 30 \\ 20 \\ 10 \end{bmatrix}. \quad \textbf{[M]}\text{: The solution is } \mathbf{i} = \begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \end{bmatrix} = \begin{bmatrix} 12.11 \\ 8.44 \\ 4.26 \\ 1.90 \end{bmatrix}.$$

7. Loop 1: The resistance vector is

$$\mathbf{r}_1 = \begin{bmatrix} 12 \\ -7 \\ 0 \\ -4 \end{bmatrix} \begin{array}{l} \text{Total of three RI voltage drops for current } I_1 \\ \text{Voltage drop for } I_2 \text{ is negative; } I_2 \text{ flows in opposite direction} \\ \text{Current } I_3 \text{ does not flow in loop 1} \\ \text{Voltage drop for } I_4 \text{ is negative; } I_4 \text{ flows in opposite direction} \end{array}$$

Loop 2: The resistance vector is

$$\mathbf{r}_2 = \begin{bmatrix} -7 \\ 15 \\ -6 \\ 0 \end{bmatrix} \begin{array}{l} \text{Voltage drop for } I_1 \text{ is negative; } I_1 \text{ flows in opposite direction} \\ \text{Total of three RI voltage drops for current } I_2 \\ \text{Voltage drop for } I_3 \text{ is negative; } I_3 \text{ flows in opposite direction} \\ \text{Current } I_4 \text{ does not flow in loop 2} \end{array}$$

Also, $\mathbf{r}_3 = \begin{bmatrix} 0 \\ -6 \\ 14 \\ -5 \end{bmatrix}$, $\mathbf{r}_4 = \begin{bmatrix} -4 \\ 0 \\ -5 \\ 13 \end{bmatrix}$, and $R = [\mathbf{r}_1 \quad \mathbf{r}_2 \quad \mathbf{r}_3 \quad \mathbf{r}_4] = \begin{bmatrix} 12 & -7 & 0 & -4 \\ -7 & 15 & -6 & 0 \\ 0 & -6 & 14 & -5 \\ -4 & 0 & -5 & 13 \end{bmatrix}$.

Notice that each off-diagonal entry of R is negative (or zero). This happens because the loop current directions are all chosen in the same direction on the figure. (For each loop j, this choice forces the currents in other loops adjacent to loop j to flow in the direction opposite to current I_j.)

Next, set $\mathbf{v} = \begin{bmatrix} 40 \\ 30 \\ 20 \\ -10 \end{bmatrix}$. Note the negative voltage in loop 4. The current direction chosen in loop 4 is opposed by the orientation of the voltage source in that loop. Thus $R\mathbf{i} = \mathbf{v}$ becomes

$\begin{bmatrix} 12 & -7 & 0 & -4 \\ -7 & 15 & -6 & 0 \\ 0 & -6 & 14 & -5 \\ -4 & 0 & -5 & 13 \end{bmatrix}\begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \end{bmatrix} = \begin{bmatrix} 40 \\ 30 \\ 20 \\ -10 \end{bmatrix}$. **[M]**: The solution is $\mathbf{i} = \begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \end{bmatrix} = \begin{bmatrix} 11.43 \\ 10.55 \\ 8.04 \\ 5.84 \end{bmatrix}$.

8. Loop 1: The resistance vector is

$$\mathbf{r}_1 = \begin{bmatrix} 15 \\ -5 \\ 0 \\ -5 \\ -1 \end{bmatrix} \begin{array}{l} \text{Total of four RI voltage drops for current } I_1 \\ \text{Voltage drop for } I_2 \text{ is negative; } I_2 \text{ flows in opposite direction} \\ \text{Current } I_3 \text{ does not flow in loop 1} \\ \text{Voltage drop for } I_4 \text{ is negative; } I_4 \text{ flows in opposite direction} \\ \text{Voltage drop for } I_5 \text{ is negative; } I_5 \text{ flows in opposite direction} \end{array}$$

Loop 2: The resistance vector is

$$\mathbf{r}_2 = \begin{bmatrix} -5 \\ 15 \\ -5 \\ 0 \\ -2 \end{bmatrix} \begin{array}{l} \text{Voltage drop for } I_1 \text{ is negative; } I_1 \text{ flows in opposite direction} \\ \text{Total of four RI voltage drops for current } I_2 \\ \text{Voltage drop for } I_3 \text{ is negative; } I_3 \text{ flows in opposite direction} \\ \text{Current } I_4 \text{ does not flow in loop 2} \\ \text{Voltage drop for } I_5 \text{ is negative; } I_5 \text{ flows in opposite direction} \end{array}$$

Also, $\mathbf{r}_3 = \begin{bmatrix} 0 \\ -5 \\ 15 \\ -5 \\ -3 \end{bmatrix}$, $\mathbf{r}_4 = \begin{bmatrix} -5 \\ 0 \\ -5 \\ 15 \\ -4 \end{bmatrix}$, $\mathbf{r}_5 = \begin{bmatrix} -1 \\ -2 \\ -3 \\ -4 \\ 10 \end{bmatrix}$, and $R = \begin{bmatrix} 15 & -5 & 0 & -5 & -1 \\ -5 & 15 & -5 & 0 & -2 \\ 0 & -5 & 15 & -5 & -3 \\ -5 & 0 & -5 & 15 & -4 \\ -1 & -2 & -3 & -4 & 10 \end{bmatrix}$. Set $\mathbf{v} = \begin{bmatrix} 40 \\ -30 \\ 20 \\ -10 \\ 0 \end{bmatrix}$. Note the negative voltages for loops where the chosen current direction is opposed by the orientation of the voltage source in that loop. Thus $R\mathbf{i} = \mathbf{v}$ becomes:

$$\begin{bmatrix} 15 & -5 & 0 & -5 & -1 \\ -5 & 15 & -5 & 0 & -2 \\ 0 & -5 & 15 & -5 & -3 \\ -5 & 0 & -5 & 15 & -4 \\ -1 & -2 & -3 & -4 & 10 \end{bmatrix}\begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \\ I_5 \end{bmatrix} = \begin{bmatrix} 40 \\ -30 \\ 20 \\ -10 \\ 0 \end{bmatrix}.$$ **[M]** The solution is $\begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \\ I_5 \end{bmatrix} = \begin{bmatrix} 3.37 \\ .11 \\ 2.27 \\ 1.67 \\ 1.70 \end{bmatrix}$.

9. The population movement problems in this section assume that the total population is constant, with no migration or immigration. The statement that "about 5% of the city's population moves to the suburbs" means also that the rest of the city's population (95%) remain in the city. This determines the entries in the first column of the migration matrix (which concerns movement *from* the city).

$$\begin{array}{cc} \text{From:} & \\ \begin{array}{cc}\text{City} & \text{Suburbs}\end{array} & \text{To:} \\ \begin{bmatrix} .95 & \\ .05 & \end{bmatrix} & \begin{array}{l}\text{City} \\ \text{Suburbs}\end{array} \end{array}$$

Likewise, if 4% of the suburban population moves to the city, then the other 96% remain in the suburbs. This determines the second column of the migration matrix:, $M = \begin{bmatrix} .95 & .04 \\ .05 & .96 \end{bmatrix}$. The difference equation is $\mathbf{x}_{k+1} = M\mathbf{x}_k$ for $k = 0, 1, 2, \ldots$. Also, $\mathbf{x}_0 = \begin{bmatrix} 600{,}000 \\ 400{,}000 \end{bmatrix}$

The population in 2001 (when $k = 1$) is $\mathbf{x}_1 = M\mathbf{x}_0 = \begin{bmatrix} .95 & .04 \\ .05 & .96 \end{bmatrix}\begin{bmatrix} 600{,}000 \\ 400{,}000 \end{bmatrix} = \begin{bmatrix} 586{,}000 \\ 414{,}000 \end{bmatrix}$

The population in 2002 (when $k = 2$) is $\mathbf{x}_2 = M\mathbf{x}_1 = \begin{bmatrix} .95 & .04 \\ .05 & .96 \end{bmatrix}\begin{bmatrix} 586{,}000 \\ 414{,}000 \end{bmatrix} = \begin{bmatrix} 573{,}260 \\ 426{,}740 \end{bmatrix}$

10. The data in the first sentence implies that the migration matrix has the form:

$$\begin{array}{cc} \text{From:} & \\ \begin{array}{cc}\text{City} & \text{Suburbs}\end{array} & \text{To:} \\ \begin{bmatrix} & .03 \\ .07 & \end{bmatrix} & \begin{array}{l}\text{City} \\ \text{Suburbs}\end{array} \end{array}$$

The remaining entries are determined by the fact that the numbers in each column must sum to 1. (For instance, if 7% of the city people move to the suburbs, then the rest, or 93%, remain in the city.) So the migration matrix is $M = \begin{bmatrix} .93 & .03 \\ .07 & .97 \end{bmatrix}$. The initial population is $\mathbf{x}_0 = \begin{bmatrix} 800{,}000 \\ 500{,}000 \end{bmatrix}$.

The population in 2001 (when $k = 1$) is $\mathbf{x}_1 = M\mathbf{x}_0 = \begin{bmatrix} .93 & .03 \\ .07 & .97 \end{bmatrix}\begin{bmatrix} 800{,}000 \\ 500{,}000 \end{bmatrix} = \begin{bmatrix} 759{,}000 \\ 541{,}000 \end{bmatrix}$

The population in 2002 (when $k = 2$) is $\mathbf{x}_2 = M\mathbf{x}_1 = \begin{bmatrix} .93 & .03 \\ .07 & .97 \end{bmatrix}\begin{bmatrix} 759{,}000 \\ 541{,}000 \end{bmatrix} = \begin{bmatrix} 722{,}100 \\ 577{,}900 \end{bmatrix}$

11. The problem concerns two groups of people–those living in California and those living outside California (and in the United States). It is reasonable, but not essential, to consider the people living inside

California first. That is, the first entry in a column or row of a vector will concern the people living in California. With this choice, the migration matrix has the form:

$$\begin{array}{ccl} \multicolumn{2}{c}{\text{From:}} & \\ \text{Calif.} & \text{Outside} & \text{To:} \\ \multicolumn{2}{c}{\left[\begin{matrix} & \\ & \end{matrix}\right]} & \begin{matrix}\text{Calif.} \\ \text{Outside}\end{matrix} \end{array}$$

a. For the first column of the migration matrix M, compute

$$\frac{\left\{\begin{matrix}\text{Calif. persons} \\ \text{who moved}\end{matrix}\right\}}{\{\text{Total Calif. pop.}\}} = \frac{509{,}500}{29{,}726{,}000} = .017146$$

The other entry in the first column is 1 – .017146 = .982854. The exercise requests that 5 decimal places be used. So this number should be rounded to .98285. Whatever number of decimal places is used, it is important that the two entries sum to 1. So, for the first fraction, use .01715.

For the second column of M, compute $\dfrac{\left\{\begin{matrix}\text{outside persons} \\ \text{who moved}\end{matrix}\right\}}{\{\text{Total outside pop.}\}} = \dfrac{564{,}100}{218{,}994{,}000} = .00258$. The other entry is 1 – .00258 = .99742. Thus, the migration matrix is

$$\begin{array}{ccl} \multicolumn{2}{c}{\text{From:}} & \\ \text{Calif.} & \text{Outside} & \text{To:} \\ \multicolumn{2}{c}{\begin{bmatrix} .98285 & .00258 \\ .01715 & .99742 \end{bmatrix}} & \begin{matrix}\text{Calif.} \\ \text{Outside}\end{matrix} \end{array}$$

b. **[M]** The initial vector is $\mathbf{x}_0 = (29.716, 218.994)$, with data in millions of persons. Since $\mathbf{x}_0$ describes the population in 1990, and $\mathbf{x}_1$ describes the population in 1991, the vector $\mathbf{x}_{10}$ describes the projected population for the year 2000, assuming that the migration rates remain constant and there are no deaths, births, or migration. Here are some of the vectors in the calculation, with only the first 4 or 5 figures displayed. Numbers are in millions of persons:

$$\begin{bmatrix} 29.7 \\ 219.0 \end{bmatrix}, \begin{bmatrix} 29.8 \\ 218.9 \end{bmatrix}, \begin{bmatrix} 29.8 \\ 218.9 \end{bmatrix}, \ldots, \begin{bmatrix} 30.1 \\ 218.6 \end{bmatrix}, \begin{bmatrix} 30.18 \\ 218.53 \end{bmatrix}, \begin{bmatrix} 30.223 \\ 218.487 \end{bmatrix} = \mathbf{x}_{10}.$$

12. Set $M = \begin{bmatrix} .97 & .05 & .10 \\ .00 & .90 & .05 \\ .03 & .05 & .85 \end{bmatrix}$ and $\mathbf{x}_0 = \begin{bmatrix} 305 \\ 48 \\ 98 \end{bmatrix}$. Then $\mathbf{x}_1 = \begin{bmatrix} .97 & .05 & .10 \\ .00 & .90 & .05 \\ .03 & .05 & .85 \end{bmatrix}\begin{bmatrix} 305 \\ 48 \\ 98 \end{bmatrix} \approx \begin{bmatrix} 308 \\ 48 \\ 95 \end{bmatrix}$, and

$\mathbf{x}_2 = \begin{bmatrix} .97 & .05 & .10 \\ .00 & .90 & .05 \\ .03 & .05 & .85 \end{bmatrix}\begin{bmatrix} 308 \\ 48 \\ 95 \end{bmatrix} \approx \begin{bmatrix} 311 \\ 48 \\ 92 \end{bmatrix}$. The entries in $\mathbf{x}_2$ give the approximate distribution of cars on Wednesday, two days after Monday.

13. **[M]** The order of entries in a column of a migration matrix must match the order of the columns. For instance, if the first column concerns the population in the city, then the first entry in *each* column must be the fraction of the population that moves to (or remains in) the city. In this case, the data in the exercise leads to $M = \begin{bmatrix} .95 & .03 \\ .05 & .97 \end{bmatrix}$ and $\mathbf{x}_0 = \begin{bmatrix} 600{,}000 \\ 400{,}000 \end{bmatrix}$

a. Some of the population vectors are

$$\mathbf{x}_5 = \begin{bmatrix} 523{,}293 \\ 476{,}707 \end{bmatrix},\ \mathbf{x}_{10} = \begin{bmatrix} 472{,}737 \\ 527{,}263 \end{bmatrix},\ \mathbf{x}_{15} = \begin{bmatrix} 439{,}417 \\ 560{,}583 \end{bmatrix},\ \mathbf{x}_{20} = \begin{bmatrix} 417{,}456 \\ 582{,}544 \end{bmatrix}$$

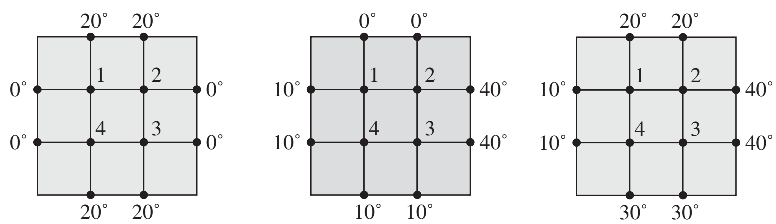
The data here shows that the city population is declining and the suburban population is increasing, but the changes in population each year seem to grow smaller.

b. When $\mathbf{x}_0 = \begin{bmatrix} 350{,}000 \\ 650{,}000 \end{bmatrix}$, the situation is different. Now

$$\mathbf{x}_5 = \begin{bmatrix} 358{,}523 \\ 641{,}477 \end{bmatrix},\ \mathbf{x}_{10} = \begin{bmatrix} 364{,}140 \\ 635{,}860 \end{bmatrix},\ \mathbf{x}_{15} = \begin{bmatrix} 367{,}843 \\ 632{,}157 \end{bmatrix},\ \mathbf{x}_{20} = \begin{bmatrix} 370{,}283 \\ 629{,}717 \end{bmatrix}$$

The city population is increasing slowly and the suburban population is decreasing. No other conclusions are expected. (This example will be analyzed in greater detail later in the text.)

14. Here are Figs. (a) and (b) for Exercise 13, followed by the figure for Exercise 34 in Section 1.1:



(a) (b) Section 1.1

For Fig. (a), the equations are:

$$4T_1 = 0 + 20 + T_2 + T_4$$
$$4T_2 = T_1 + 20 + 0 + T_3$$
$$4T_3 = T_4 + T_2 + 0 + 20$$
$$4T_4 = 0 + T_1 + T_3 + 20$$

To solve the system, rearrange the equations and row reduce the augmented matrix. Interchanging rows 1 and 4 speeds up the calculations. The first five steps are shown in detail.

$$\begin{bmatrix} 4 & -1 & 0 & -1 & 20 \\ -1 & 4 & -1 & 0 & 20 \\ 0 & -1 & 4 & -1 & 20 \\ -1 & 0 & -1 & 4 & 20 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & -4 & -20 \\ -1 & 4 & -1 & 0 & 20 \\ 0 & -1 & 4 & -1 & 20 \\ 4 & -1 & 0 & -1 & 20 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & -4 & -20 \\ 0 & 4 & 0 & -4 & 0 \\ 0 & -1 & 4 & -1 & 20 \\ 0 & -1 & -4 & 15 & 100 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & -4 & -20 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & -1 & 4 & -1 & 20 \\ 0 & -1 & -4 & 15 & 100 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 1 & -4 & -20 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 4 & -2 & 20 \\ 0 & 0 & -4 & 14 & 100 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & -4 & -20 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 4 & -2 & 20 \\ 0 & 0 & 0 & 12 & 120 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 10 \\ 0 & 1 & 0 & 0 & 10 \\ 0 & 0 & 1 & 0 & 10 \\ 0 & 0 & 0 & 1 & 10 \end{bmatrix}$$

For Fig (b), the equations are

$$4T_1 = 10 + 0 + T_2 + T_4$$
$$4T_2 = T_1 + 0 + 40 + T_3$$
$$4T_3 = T_4 + T_2 + 40 + 10$$
$$4T_4 = 10 + T_1 + T_3 + 10$$

Rearrange the equations and row reduce the augmented matrix:

$$\begin{bmatrix} 4 & -1 & 0 & -1 & 10 \\ -1 & 4 & -1 & 0 & 40 \\ 0 & -1 & 4 & -1 & 50 \\ -1 & 0 & -1 & 4 & 20 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 10 \\ 0 & 1 & 0 & 0 & 17.5 \\ 0 & 0 & 1 & 0 & 20 \\ 0 & 0 & 0 & 1 & 12.5 \end{bmatrix}$$

a. Here are the solution temperatures for the three problems studied:

Fig. (a) in Exercise 14 of Section 1.10:	(10, 10, 10, 10)
Fig. (b) in Exercise 14 of Section 1.10:	(10, 17.5, 20, 12.5)
Figure for Exercises 34 in Section 1.1	(20, 27.5, 30, 22.5)

When the solutions are arranged this way, it is evident that the third solution is the sum of the first two solutions. What might not be so evident is that list of boundary temperatures of the third problem is the sum of the lists of boundary temperatures of the first two problems. (The temperatures are listed clockwise, starting at the left of T_1.)

Fig. (a):	(0, 20, 20, 0, 0, 20, 20, 0)
Fig. (b):	(10, 0, 0, 40, 40, 10, 10, 10)
Fig. from Section 1.1:	(10, 20, 20, 40, 40, 30, 30, 10)

b. When the boundary temperatures in Fig. (a) are multiplied by 3, the new interior temperatures are also multiplied by 3.

c. The correspondence from the list of eight boundary temperatures to the list of four interior temperatures is a linear transformation. A verification of this statement is not expected. However, it can be shown that the solutions of the steady-state temperature problem here satisfy a superposition principle. The system of equations that approximate the interior temperatures can be written in the form $A\mathbf{x} = \mathbf{b}$, where A is determined by the arrangement of the four interior points on the plate and $\mathbf{b}$ is a vector in $\mathbf{R}^4$ determined by the boundary temperatures.

Note: The MATLAB box in the *Study Guide* for Section 1.10 discusses scientific notation and shows how to generate a matrix whose columns list the vectors $\mathbf{x}_0$, $\mathbf{x}_1$, $\mathbf{x}_2$, …, determined by an equation $\mathbf{x}_{k+1} = M\mathbf{x}_k$ for $k = 0, 1, \ldots$.

Chapter 1 SUPPLEMENTARY EXERCISES

1. **a**. False. (The word “reduced” is missing.) Counterexample:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix},\ B = \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix},\ C = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

The matrix A is row equivalent to matrices B and C, both in echelon form.

b. False. Counterexample: Let A be any $n \times n$ matrix with fewer than n pivot columns. Then the equation $A\mathbf{x} = \mathbf{0}$ has infinitely many solutions. (Theorem 2 in Section 1.2 says that a system has either zero, one, or infinitely many solutions, but it does not say that a system with infinitely many solutions exists. Some counterexample is needed.)

c. True. If a linear system has more than one solution, it is a consistent system and has a free variable. By the Existence and Uniqueness Theorem in Section 1.2, the system has infinitely many solutions.

d. False. Counterexample: The following system has no free variables and no solution:

$$\begin{aligned} x_1 + x_2 &= 1 \\ x_2 &= 5 \\ x_1 + x_2 &= 2 \end{aligned}$$

e. True. See the box after the definition of elementary row operations, in Section 1.1. If $[A \;\; \mathbf{b}]$ is transformed into $[C \;\; \mathbf{d}]$ by elementary row operations, then the two augmented matrices are row equivalent.

f. True. Theorem 6 in Section 1.5 essentially says that when $A\mathbf{x} = \mathbf{b}$ is consistent, the solution sets of the nonhomogeneous equation and the homogeneous equation are translates of each other. In this case, the two equations have the same number of solutions.

g. False. For the columns of A to span $\mathbf{R}^m$, the equation $A\mathbf{x} = \mathbf{b}$ must be consistent for *all* $\mathbf{b}$ in $\mathbf{R}^{\mathrm{m}}$, not for just one vector $\mathbf{b}$ in $\mathbf{R}^{\mathrm{m}}$.

h. False. *Any* matrix can be transformed by elementary row operations into reduced echelon form, but not every matrix equation $A\mathbf{x} = \mathbf{b}$ is consistent.

i. True. If A is row equivalent to B, then A can be transformed by elementary row operations first into B and then further transformed into the reduced echelon form U of B. Since the reduced echelon form of A is unique, it must be U.

j. False. Every equation $A\mathbf{x} = \mathbf{0}$ has the trivial solution whether or not some variables are free.

k. True, by Theorem 4 in Section 1.4. If the equation $A\mathbf{x} = \mathbf{b}$ is consistent for every $\mathbf{b}$ in $\mathbf{R}^{\mathrm{m}}$, then A must have a position in every one of its m rows. If A has m pivot positions, then A has m pivot columns, each containing one pivot position.

l. False. The word "unique" should be deleted. Let A be any matrix with m pivot columns but more than m columns altogether. Then the equation $A\mathbf{x} = \mathbf{b}$ is consistent and has m basic variables and at least one free variable. Thus the equation does not does not have a unique solution.

m. True. If A has n pivot positions, it has a pivot in each of its n columns and in each of its n rows. The reduced echelon form has a 1 in each pivot position, so the reduced echelon form is the $n \times n$ identity matrix.

n. True. Both matrices A and B can be row reduced to the 3×3 identity matrix, as discussed in the previous question. Since the row operations that transform B into I_3 are reversible, A can be transformed first into I_3 and then into B.

o. True. The reason is essentially the same as that given for question f.

p. True. If the columns of A span $\mathbf{R}^m$, then the reduced echelon form of A is a matrix U with a pivot in each row, by Theorem 4 in Section 1.4. Since B is row equivalent to A, B can be transformed by row operations first into A and then further transformed into U. Since U has a pivot in each row, so does B. By Theorem 4, the columns of B span $\mathbf{R}^m$.

q. False. See Example 5 in Section 1.6.

r. True. Any set of three vectors in $\mathbf{R}^2$ would have to be linearly dependent, by Theorem 8 in Section 1.6.

s. False. If a set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ were to span $\mathbf{R}^5$, then the matrix $A = [\mathbf{v}_1 \ \ \mathbf{v}_2 \ \ \mathbf{v}_3 \ \ \mathbf{v}_4]$ would have a pivot position in each of its five rows, which is impossible since A has only four columns.

t. True. The vector $-\mathbf{u}$ is a linear combination of $\mathbf{u}$ and $\mathbf{v}$, namely, $-\mathbf{u} = (-1)\mathbf{u} + 0\mathbf{v}$.

u. False. If $\mathbf{u}$ and $\mathbf{v}$ are multiples, then Span$\{\mathbf{u}, \mathbf{v}\}$ is a line, and $\mathbf{w}$ need not be on that line.

v. False. Let $\mathbf{u}$ and $\mathbf{v}$ be any linearly independent pair of vectors and let $\mathbf{w} = 2\mathbf{v}$. Then $\mathbf{w} = 0\mathbf{u} + 2\mathbf{v}$, so $\mathbf{w}$ is a linear combination of $\mathbf{u}$ and $\mathbf{v}$. However, $\mathbf{u}$ cannot be a linear combination of $\mathbf{v}$ and $\mathbf{w}$ because if it were, $\mathbf{u}$ would be a multiple of $\mathbf{v}$. That is not possible since $\{\mathbf{u}, \mathbf{v}\}$ is linearly independent.

w. False. The statement would be true if the condition $\mathbf{v}_1$ is not zero were present. See Theorem 7 in Section 1.7. However, if $\mathbf{v}_1 = \mathbf{0}$, then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent, no matter what else might be true about $\mathbf{v}_2$ and $\mathbf{v}_3$.

x. True. "Function" is another word used for "transformation" (as mentioned in the definition of "transformation" in Section 1.8), and a linear transformation is a special type of transformation.

y. True. For the transformation $\mathbf{x} \mapsto A\mathbf{x}$ to map $\mathbf{R}^5$ onto $\mathbf{R}^6$, the matrix A would have to have a pivot in every row and hence have six pivot columns. This is impossible because A has only five columns.

z. False. For the transformation $\mathbf{x} \mapsto A\mathbf{x}$ to be one-to-one, A must have a pivot in each column. Since A has n columns and m pivots, m might be less than n.

2. If $a \neq 0$, then $x = b/a$; the solution is unique. If $a = 0$, and $b \neq 0$, the solution set is empty, because $0x = 0 \neq b$. If $a = 0$ and $b = 0$, the equation $0x = 0$ has infinitely many solutions.

3. **a**. Any consistent linear system whose echelon form is

$$\begin{bmatrix} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ or } \begin{bmatrix} \blacksquare & * & * & * \\ 0 & 0 & \blacksquare & * \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ or } \begin{bmatrix} 0 & \blacksquare & * & * \\ 0 & 0 & \blacksquare & * \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

b. Any consistent linear system whose coefficient matrix has reduced echelon form I_3.

c. Any inconsistent linear system of three equations in three variables.

4. Since there are three pivots (one in each row), the augmented matrix must reduce to the form

$\begin{bmatrix} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & \blacksquare & * \end{bmatrix}$. A solution of $A\mathbf{x} = \mathbf{b}$ exists for all $\mathbf{b}$ because there is a pivot in each row of A. Each solution is unique because there are no free variables.

5. **a**. $\begin{bmatrix} 1 & 3 & k \\ 4 & h & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & k \\ 0 & h-12 & 8-4k \end{bmatrix}$. If $h = 12$ and $k \neq 2$, the second row of the augmented matrix indicates an inconsistent system of the form $0x_2 = b$, with b nonzero. If $h = 12$, and $k = 2$, there is only one nonzero equation, and the system has infinitely many solutions. Finally, if $h \neq 12$, the coefficient matrix has two pivots and the system has a unique solution.

b. $\begin{bmatrix} -2 & h & 1 \\ 6 & k & -2 \end{bmatrix} \sim \begin{bmatrix} -2 & h & 1 \\ 0 & k+3h & 1 \end{bmatrix}$. If $k + 3h = 0$, the system is inconsistent. Otherwise, the coefficient matrix has two pivots and the system has a unique solution.

6. **a**. Set $\mathbf{v}_1 = \begin{bmatrix} 4 \\ 8 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -2 \\ -3 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 7 \\ 10 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} -5 \\ -3 \end{bmatrix}$. "Determine if **b** is a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$." Or, "Determine if **b** is in Span$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$." To do this, compute $\begin{bmatrix} 4 & -2 & 7 & -5 \\ 8 & -3 & 10 & -3 \end{bmatrix} \sim \begin{bmatrix} 4 & -2 & 7 & -5 \\ 0 & 1 & -4 & 7 \end{bmatrix}$. The system is consistent, so **b** *is* in Span$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

b. Set $A = \begin{bmatrix} 4 & -2 & 7 \\ 8 & -3 & 10 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} -5 \\ -3 \end{bmatrix}$. "Determine if **b** is a linear combination of the columns of A."

c. Define $T(\mathbf{x}) = A\mathbf{x}$. "Determine if **b** is in the range of T."

7. **a**. Set $\mathbf{v}_1 = \begin{bmatrix} 2 \\ -5 \\ 7 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -4 \\ 1 \\ -5 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$. "Determine if $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ span $\mathbf{R}^3$." To do this, row reduce $[\mathbf{v}_1 \;\; \mathbf{v}_2 \;\; \mathbf{v}_3]$:

$\begin{bmatrix} 2 & -4 & -2 \\ -5 & 1 & 1 \\ 7 & -5 & -3 \end{bmatrix} \sim \begin{bmatrix} 2 & -4 & -2 \\ 0 & -9 & -4 \\ 0 & 9 & 4 \end{bmatrix} \sim \begin{bmatrix} 2 & -4 & -2 \\ 0 & -9 & -4 \\ 0 & 0 & 0 \end{bmatrix}$. The matrix does not have a pivot in each row, so its columns do not span $\mathbf{R}^3$, by Theorem 4 in Section 1.4.

b. Set $A = \begin{bmatrix} 2 & -4 & -2 \\ -5 & 1 & 1 \\ 7 & -5 & -3 \end{bmatrix}$. "Determine if the columns of A span $\mathbf{R}^3$."

c. Define $T(\mathbf{x}) = A\mathbf{x}$. "Determine if T maps $\mathbf{R}^3$ onto $\mathbf{R}^3$."

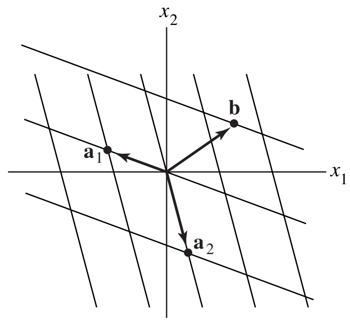
8. **a**. $\begin{bmatrix} \blacksquare & * & * \\ 0 & \blacksquare & * \end{bmatrix}, \begin{bmatrix} \blacksquare & * & * \\ 0 & 0 & \blacksquare \end{bmatrix}, \begin{bmatrix} 0 & \blacksquare & * \\ 0 & 0 & \blacksquare \end{bmatrix}$ **b**. $\begin{bmatrix} \blacksquare & * & * \\ 0 & \blacksquare & * \\ 0 & 0 & \blacksquare \end{bmatrix}$

9. The first line is the line spanned by $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$. The second line is spanned by $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$. So the problem is to write $\begin{bmatrix} 5 \\ 6 \end{bmatrix}$ as the sum of a multiple of $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and a multiple of $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$. That is, find x_1 and x_2 such that $x_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$. Reduce the augmented matrix for this equation:

$$\begin{bmatrix} 2 & 1 & 5 \\ 1 & 2 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 6 \\ 2 & 1 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 6 \\ 0 & -3 & -7 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 6 \\ 0 & 1 & 7/3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 4/3 \\ 0 & 1 & 7/3 \end{bmatrix}$$

Thus, $\begin{bmatrix} 5 \\ 6 \end{bmatrix} = \frac{4}{3} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \frac{7}{3} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ or $\begin{bmatrix} 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 8/3 \\ 4/3 \end{bmatrix} + \begin{bmatrix} 7/3 \\ 14/3 \end{bmatrix}$.

10. The line through $\mathbf{a}_1$ and the origin and the line through $\mathbf{a}_2$ and the origin determine a "grid" on the x_1x_2-plane as shown below. Every point in $\mathbf{R}^2$ can be described uniquely in terms of this grid. Thus, **b** can be reached from the origin by traveling a certain number of units in the $\mathbf{a}_1$-direction and a certain number of units in the $\mathbf{a}_2$-direction.



11. A solution set is a line when the system has one free variable. If the coefficient matrix is 2×3, then two of the columns should be pivot columns. For instance, take $\begin{bmatrix} 1 & 2 & * \\ 0 & 3 & * \end{bmatrix}$. Put anything in column 3. The resulting matrix will be in echelon form. Make one row replacement operation on the second row to create a matrix *not* in echelon form, such as $\begin{bmatrix} 1 & 2 & 1 \\ 0 & 3 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 \\ 1 & 5 & 2 \end{bmatrix}$

12. A solution set is a plane where there are two free variables. If the coefficient matrix is 2×3, then only one column can be a pivot column. The echelon form will have all zeros in the second row. Use a row replacement to create a matrix not in echelon form. For instance, let A $= \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$.

13. The reduced echelon form of A looks like $E = \begin{bmatrix} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & 0 \end{bmatrix}$. Since E is row equivalent to A, the equation $E\mathbf{x} = \mathbf{0}$ has the same solutions as $A\mathbf{x} = \mathbf{0}$. Thus $\begin{bmatrix} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

By inspection, $E = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$.

14. Row reduce the augmented matrix for $x_1 \begin{bmatrix} 1 \\ a \end{bmatrix} + x_2 \begin{bmatrix} a \\ a+2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ (*).

$$\begin{bmatrix} 1 & a & 0 \\ a & a+2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & a & 0 \\ 0 & a+2-a^2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & a & 0 \\ 0 & (2-a)(1+a) & 0 \end{bmatrix}$$

The equation (*) has a nontrivial solution only when $(2 - a)(1 + a) = 0$. So the vectors are linearly independent for all a except $a = 2$ and $a = -1$.

15. **a**. If the three vectors are linearly independent, then a, c, and f must all be nonzero. (The converse is true, too.) Let A be the matrix whose columns are the three linearly independent vectors. Then

A must have three pivot columns. (See Exercise 30 in Section 1.7, or realize that the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution and so there can be no free variables in the system of equations.) Since A is 3×3, the pivot positions are exactly where a, c, and f are located.

b. The numbers $a, \ldots, f$ can have any values. Here's why. Denote the columns by $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$. Observe that v_1 is not the zero vector. Next, $\mathbf{v}_2$ is not a multiple of $\mathbf{v}_1$ because the third entry of $\mathbf{v}_2$ is nonzero. Finally, $\mathbf{v}_3$ is not a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$ because the fourth entry of $\mathbf{v}_3$ is nonzero. By Theorem 7 in Section 1.7, $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent.

16. Denote the columns from right to left by $\mathbf{v}_1, \ldots, \mathbf{v}_4$. The "first" vector $\mathbf{v}_1$ is nonzero, $\mathbf{v}_2$ is not a multiple of $\mathbf{v}_1$ (because the third entry of $\mathbf{v}_2$ is nonzero), and $\mathbf{v}_3$ is not a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$ (because the second entry of $\mathbf{v}_3$ is nonzero). Finally, by looking at first entries in the vectors, $\mathbf{v}_4$ cannot be a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$. By Theorem 7 in Section 1.7, the columns are linearly independent.

17. Here are two arguments. The first is a "direct" proof. The second is called a "proof by contradiction."

i. Since $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a linearly independent set, $\mathbf{v}_1 \neq \mathbf{0}$. Also, Theorem 7 shows that $\mathbf{v}_2$ cannot be a multiple of $\mathbf{v}_1$, and $\mathbf{v}_3$ cannot be a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$. By hypothesis, $\mathbf{v}_4$ is not a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$. Thus, by Theorem 7, $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ cannot be a linearly dependent set and so must be linearly independent.

ii. Suppose that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is linearly dependent. Then by Theorem 7, one of the vectors in the set is a linear combination of the preceding vectors. This vector cannot be $\mathbf{v}_4$ because $\mathbf{v}_4$ is *not* in Span$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. Also, none of the vectors in $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a linear combinations of the preceding vectors, by Theorem 7. So the linear dependence of $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is impossible. Thus $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is linearly independent.

18. Suppose that c_1 and c_2 are constants such that

$$c_1\mathbf{v}_1 + c_2(\mathbf{v}_1 + \mathbf{v}_2) = \mathbf{0} \qquad (*)$$

Then $(c_1 + c_2)\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{0}$. Since $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent, both $c_1 + c_2 = 0$ and $c_2 = 0$. It follows that both c_1 and c_2 in (*) must be zero, which shows that $\{\mathbf{v}_1, \mathbf{v}_1 + \mathbf{v}_2\}$ is linearly independent.

19. Let M be the line through the origin that is parallel to the line through $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$. Then $\mathbf{v}_2 - \mathbf{v}_1$ and $\mathbf{v}_3 - \mathbf{v}_1$ are both on M. So one of these two vectors is a multiple of the other, say $\mathbf{v}_2 - \mathbf{v}_1 = k(\mathbf{v}_3 - \mathbf{v}_1)$. This equation produces a linear dependence relation $(k - 1)\mathbf{v}_1 + \mathbf{v}_2 - k\mathbf{v}_3 = \mathbf{0}$.

A second solution: A parametric equation of the line is $\mathbf{x} = \mathbf{v}_1 + t(\mathbf{v}_2 - \mathbf{v}_1)$. Since $\mathbf{v}_3$ is on the line, there is some t_0 such that $\mathbf{v}_3 = \mathbf{v}_1 + t_0(\mathbf{v}_2 - \mathbf{v}_1) = (1 - t_0)\mathbf{v}_1 + t_0\mathbf{v}_2$. So $\mathbf{v}_3$ is a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$, and $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent.

20. If $T(\mathbf{u}) = \mathbf{v}$, then since T is linear,

$$T(-\mathbf{u}) = T((-1)\mathbf{u}) = (-1)T(\mathbf{u}) = -\mathbf{v}.$$

21. Either compute $T(\mathbf{e}_1)$, $T(\mathbf{e}_2)$, and $T(\mathbf{e}_3)$ to make the columns of A, or write the vectors vertically in the definition of T and fill in the entries of A by inspection:

$$A\mathbf{x} = \begin{bmatrix} ? & ? & ? \\ ? & A & ? \\ ? & ? & ? \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ -x_2 \\ x_3 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

22. By Theorem 12 in Section 1.9, the columns of A span $\mathbf{R}^3$. By Theorem 4 in Section 1.4, A has a pivot in each of its three rows. Since A has three columns, each column must be a pivot column. So the equation

$A\mathbf{x} = \mathbf{0}$ has no free variables, and the columns of A are linearly independent. By Theorem 12 in Section 1.9, the transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.

23. $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}\begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$ implies that $\begin{array}{rcrcr} 4a & - & 3b & = & 5 \\ 3a & + & 4b & = & 0 \end{array}$. Solve:

$$\begin{bmatrix} 4 & -3 & 5 \\ 3 & 4 & 0 \end{bmatrix} \sim \begin{bmatrix} 4 & -3 & 5 \\ 0 & 25/4 & -15/4 \end{bmatrix} \sim \begin{bmatrix} 4 & -3 & 5 \\ 0 & 1 & -3/5 \end{bmatrix} \sim \begin{bmatrix} 4 & 0 & 16/5 \\ 0 & 1 & -3/5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 4/5 \\ 0 & 1 & -3/5 \end{bmatrix}$$

Thus $a = 4/5$ and $b = -3/5$.

24. The matrix equation displayed gives the information $2a - 4b = 2\sqrt{5}$ and $4a + 2b = 0$. Solve for a and b:

$$\begin{bmatrix} 2 & -4 & 2\sqrt{5} \\ 4 & 2 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & -4 & 2\sqrt{5} \\ 0 & 10 & -4\sqrt{5} \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & \sqrt{5} \\ 0 & 1 & -2/\sqrt{5} \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1/\sqrt{5} \\ 0 & 1 & -2/\sqrt{5} \end{bmatrix}$$

So $a = 1/\sqrt{5},\ b = -2/\sqrt{5}$.

25. a. The vector lists the number of three-, two-, and one-bedroom apartments provided when x_1 floors of plan A are constructed.

b. $x_1\begin{bmatrix} 3 \\ 7 \\ 8 \end{bmatrix} + x_2\begin{bmatrix} 4 \\ 4 \\ 8 \end{bmatrix} + x_3\begin{bmatrix} 5 \\ 3 \\ 9 \end{bmatrix}$

c. [M] Solve $x_1\begin{bmatrix} 3 \\ 7 \\ 8 \end{bmatrix} + x_2\begin{bmatrix} 4 \\ 4 \\ 8 \end{bmatrix} + x_3\begin{bmatrix} 5 \\ 3 \\ 9 \end{bmatrix} = \begin{bmatrix} 66 \\ 74 \\ 136 \end{bmatrix}$

$$\begin{bmatrix} 3 & 4 & 5 & 66 \\ 7 & 4 & 3 & 74 \\ 8 & 8 & 9 & 136 \end{bmatrix} \sim \cdots \begin{bmatrix} 1 & 0 & -1/2 & 2 \\ 0 & 1 & 13/8 & 15 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{array}{rcrcrcr} x_1 & & & - & (1/2)x_3 & = & 2 \\ & & x_2 & + & (13/8)x_3 & = & 15 \\ & & & & 0 & = & 0 \end{array}$$

The general solution is

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 + (1/2)x_3 \\ 15 - (13/8)x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 15 \\ 0 \end{bmatrix} + x_3\begin{bmatrix} 1/2 \\ -13/8 \\ 1 \end{bmatrix}$$

However, the only feasible solutions must have whole numbers of floors for each plan. Thus, x_3 must be a multiple of 8, to avoid fractions. One solution, for $x_3 = 0$, is to use 2 floors of plan A and 15 floors of plan B. Another solution, for $x_3 = 8$, is to use 6 floors of plan A , 2 floors of plan B, and 8 floors of plan C. These are the only feasible solutions. A larger positive multiple of 8 for x_3 makes x_2 negative. A negative value for x_3, of course, is not feasible either.

2 Matrix Algebra

2.1 SOLUTIONS

Notes: The definition here of a matrix product AB gives the proper view of AB for nearly all matrix calculations. (The dual fact about the rows of A and the rows of AB is seldom needed, mainly because vectors here are usually written as columns.) I assign Exercise 13 and most of Exercises 17–22 to reinforce the definition of AB.

Exercises 23 and 24 are used in the proof of the Invertible Matrix Theorem, in Section 2.3. Exercises 23–25 are mentioned in a footnote in Section 2.2. A class discussion of the solutions of Exercises 23–25 can provide a transition to Section 2.2. Or, these exercises could be assigned after starting Section 2.2.

Exercises 27 and 28 are optional, but they are mentioned in Example 4 of Section 2.4. Outer products also appear in Exercises 31–34 of Section 4.6 and in the spectral decomposition of a symmetric matrix, in Section 7.1. Exercises 29–33 provide good training for mathematics majors.

1. $-2A=(-2)\begin{bmatrix}2&0&-1\\4&-5&2\end{bmatrix}=\begin{bmatrix}-4&0&2\\-8&10&-4\end{bmatrix}$. Next, use $B-2A=B+(-2A)$:

$$B-2A=\begin{bmatrix}7&-5&1\\1&-4&-3\end{bmatrix}+\begin{bmatrix}-4&0&2\\-8&10&-4\end{bmatrix}=\begin{bmatrix}3&-5&3\\-7&6&-7\end{bmatrix}$$

The product AC is not defined because the number of columns of A does not match the number of rows of C. $CD=\begin{bmatrix}1&2\\-2&1\end{bmatrix}\begin{bmatrix}3&5\\-1&4\end{bmatrix}=\begin{bmatrix}1\cdot3+2(-1)&1\cdot5+2\cdot4\\-2\cdot3+1(-1)&-2\cdot5+1\cdot4\end{bmatrix}=\begin{bmatrix}1&13\\-7&-6\end{bmatrix}$. For mental computation, the row-column rule is probably easier to use than the definition.

2. $A+2B=\begin{bmatrix}2&0&-1\\4&-5&2\end{bmatrix}+2\begin{bmatrix}7&-5&1\\1&-4&-3\end{bmatrix}=\begin{bmatrix}2+14&0-10&-1+2\\4+2&-5-8&2-6\end{bmatrix}=\begin{bmatrix}16&-10&1\\6&-13&-4\end{bmatrix}$

The expression $3C-E$ is not defined because $3C$ has 2 columns and $-E$ has only 1 column.

$$CB=\begin{bmatrix}1&2\\-2&1\end{bmatrix}\begin{bmatrix}7&-5&1\\1&-4&-3\end{bmatrix}=\begin{bmatrix}1\cdot7+2\cdot1&1(-5)+2(-4)&1\cdot1+2(-3)\\-2\cdot7+1\cdot1&-2(-5)+1(-4)&-2\cdot1+1(-3)\end{bmatrix}=\begin{bmatrix}9&-13&-5\\-13&6&-5\end{bmatrix}$$

The product EB is not defined because the number of columns of E does not match the number of rows of R.

3. $3I_2 - A = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} - \begin{bmatrix} 4 & -1 \\ 5 & -2 \end{bmatrix} = \begin{bmatrix} 3-4 & 0-(-1) \\ 0-5 & 3-(-2) \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -5 & 5 \end{bmatrix}$

$(3I_2)A = 3(I_2 A) = 3\begin{bmatrix} 4 & -1 \\ 5 & -2 \end{bmatrix} = \begin{bmatrix} 12 & -3 \\ 15 & -6 \end{bmatrix}$, or

$(3I_2)A = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 4 & -1 \\ 5 & -2 \end{bmatrix} = \begin{bmatrix} 3\cdot 4+0 & 3(-1)+0 \\ 0+3\cdot 5 & 0+3(-2) \end{bmatrix} = \begin{bmatrix} 12 & -3 \\ 15 & -6 \end{bmatrix}$

4. $A - 5I_3 = \begin{bmatrix} 9 & -1 & 3 \\ -8 & 7 & -6 \\ -4 & 1 & 8 \end{bmatrix} - \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 4 & -1 & 3 \\ -8 & 2 & -6 \\ -4 & 1 & 3 \end{bmatrix}$

$(5I_3)A = 5(I_3 A) = 5A = 5\begin{bmatrix} 9 & -1 & 3 \\ -8 & 7 & -6 \\ -4 & 1 & 8 \end{bmatrix} = \begin{bmatrix} 45 & -5 & 15 \\ -40 & 35 & -30 \\ -20 & 5 & 40 \end{bmatrix}$, or

$$(5I_3)A = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}\begin{bmatrix} 9 & -1 & 3 \\ -8 & 7 & -6 \\ -4 & 1 & 8 \end{bmatrix}$$

$$= \begin{bmatrix} 5\cdot 9+0+0 & 5(-1)+0+0 & 5\cdot 3+0+0 \\ 0+5(-8)+0 & 0+5\cdot 7+0 & 0+5(-6)+0 \\ 0+0+5(-4) & 0+0+5\cdot 1 & 0+0+5\cdot 8 \end{bmatrix} = \begin{bmatrix} 45 & -5 & 15 \\ -45 & 35 & -30 \\ -20 & 5 & 40 \end{bmatrix}$$

5. **a**. $A\mathbf{b}_1 = \begin{bmatrix} -1 & 2 \\ 5 & 4 \\ 2 & -3 \end{bmatrix}\begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} -7 \\ 7 \\ 12 \end{bmatrix}$, $A\mathbf{b}_2 = \begin{bmatrix} -1 & 2 \\ 5 & 4 \\ 2 & -3 \end{bmatrix}\begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ -6 \\ -7 \end{bmatrix}$

$AB = \begin{bmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 \end{bmatrix} = \begin{bmatrix} -7 & 4 \\ 7 & -6 \\ 12 & -7 \end{bmatrix}$

b. $\begin{bmatrix} -1 & 2 \\ 5 & 4 \\ 2 & -3 \end{bmatrix}\begin{bmatrix} 3 & -2 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} -1\cdot 3+2(-2) & -1(-2)+2\cdot 1 \\ 5\cdot 3+4(-2) & 5(-2)+4\cdot 1 \\ 2\cdot 3-3(-2) & 2(-2)-3\cdot 1 \end{bmatrix} = \begin{bmatrix} -7 & 4 \\ 7 & -6 \\ 12 & -7 \end{bmatrix}$

6. **a**. $A\mathbf{b}_1 = \begin{bmatrix} 4 & -2 \\ -3 & 0 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \\ 13 \end{bmatrix}$, $A\mathbf{b}_2 = \begin{bmatrix} 4 & -2 \\ -3 & 0 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 14 \\ -9 \\ 4 \end{bmatrix}$

$AB = \begin{bmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 \end{bmatrix} = \begin{bmatrix} 0 & 14 \\ -3 & -9 \\ 13 & 4 \end{bmatrix}$

b. $\begin{bmatrix} 4 & -2 \\ -3 & 0 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 4\cdot 1-2\cdot 2 & 4\cdot 3-2(-1) \\ -3\cdot 1+0\cdot 2 & -3\cdot 3+0(-1) \\ 3\cdot 1+5\cdot 2 & 3\cdot 3+5(-1) \end{bmatrix} = \begin{bmatrix} 0 & 14 \\ -3 & -9 \\ 13 & 4 \end{bmatrix}$

7. Since A has 3 columns, B must match with 3 rows. Otherwise, AB is undefined. Since AB has 7 columns, so does B. Thus, B is 3×7.

8. The number of rows of B matches the number of rows of BC, so B has 3 rows.

9. $AB = \begin{bmatrix} 2 & 5 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 4 & -5 \\ 3 & k \end{bmatrix} = \begin{bmatrix} 23 & -10+5k \\ -9 & 15+k \end{bmatrix}$, while $BA = \begin{bmatrix} 4 & -5 \\ 3 & k \end{bmatrix}\begin{bmatrix} 2 & 5 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} 23 & 15 \\ 6-3k & 15+k \end{bmatrix}$.

Then $AB = BA$ if and only if $-10 + 5k = 15$ and $-9 = 6 - 3k$, which happens if and only if $k = 5$.

10. $AB = \begin{bmatrix} 2 & -3 \\ -4 & 6 \end{bmatrix}\begin{bmatrix} 8 & 4 \\ 5 & 5 \end{bmatrix} = \begin{bmatrix} 1 & -7 \\ -2 & 14 \end{bmatrix}$, $AC = \begin{bmatrix} 2 & -3 \\ -4 & 6 \end{bmatrix}\begin{bmatrix} 5 & -2 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -7 \\ -2 & 14 \end{bmatrix}$

11. $AD = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 5 \end{bmatrix}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 5 \\ 2 & 6 & 15 \\ 2 & 12 & 25 \end{bmatrix}$

$DA = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 5 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 2 \\ 3 & 6 & 9 \\ 5 & 20 & 25 \end{bmatrix}$

Right-multiplication (that is, multiplication on the right) by the diagonal matrix D multiplies each *column* of A by the corresponding diagonal entry of D. Left-multiplication by D multiplies each *row* of A by the corresponding diagonal entry of D. To make $AB = BA$, one can take B to be a multiple of I_3. For instance, if $B = 4I_3$, then AB and BA are both the same as $4A$.

12. Consider $B = [\mathbf{b}_1 \;\; \mathbf{b}_2]$. To make $AB = 0$, one needs $A\mathbf{b}_1 = \mathbf{0}$ and $A\mathbf{b}_2 = \mathbf{0}$. By inspection of A, a suitable $\mathbf{b}_1$ is $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$, or any multiple of $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Example: $B = \begin{bmatrix} 2 & 6 \\ 1 & 3 \end{bmatrix}$.

13. Use the definition of AB written in reverse order: $[A\mathbf{b}_1 \;\cdots\; A\mathbf{b}_p] = A[\mathbf{b}_1 \;\cdots\; \mathbf{b}_p]$. Thus

$$[Q\mathbf{r}_1 \;\cdots\; Q\mathbf{r}_p] = QR, \text{ when } R = [\mathbf{r}_1 \;\cdots\; \mathbf{r}_p].$$

14. By definition, $UQ = U[\mathbf{q}_1 \;\cdots\; \mathbf{q}_4] = [U\mathbf{q}_1 \;\cdots\; U\mathbf{q}_4]$. From Example 6 of Section 1.8, the vector $U\mathbf{q}_1$ lists the total costs (material, labor, and overhead) corresponding to the amounts of products B and C specified in the vector $\mathbf{q}_1$. That is, the first column of UQ lists the total costs for materials, labor, and overhead used to manufacture products B and C during the first quarter of the year. Columns 2, 3, and 4 of UQ list the total amounts spent to manufacture B and C during the 2nd, 3rd, and 4th quarters, respectively.

15. **a**. False. See the definition of AB.

b. False. The roles of A and B should be reversed in the second half of the statement. See the box after Example 3.

c. True. See Theorem 2(b), read right to left.

d. True. See Theorem 3(b), read right to left.

e. False. The phrase "in the same order" should be "in the reverse order." See the box after Theorem 3.

16. **a**. False. AB must be a 3×3 matrix, but the formula for AB implies that it is 3×1. The plus signs should be just spaces (between columns). This is a common mistake.

b. True. See the box after Example 6.

c. False. The left-to-right order of B and C cannot be changed, in general.

d. False. See Theorem 3(d).

e. True. This general statement follows from Theorem 3(b).

17. Since $\begin{bmatrix} -1 & 2 & -1 \\ 6 & -9 & 3 \end{bmatrix} = AB = \begin{bmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 & A\mathbf{b}_3 \end{bmatrix}$, the first column of B satisfies the equation $A\mathbf{x} = \begin{bmatrix} -1 \\ 6 \end{bmatrix}$. Row reduction: $\begin{bmatrix} A & A\mathbf{b}_1 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & -1 \\ -2 & 5 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 7 \\ 0 & 1 & 4 \end{bmatrix}$. So $\mathbf{b}_1 = \begin{bmatrix} 7 \\ 4 \end{bmatrix}$. Similarly, $\begin{bmatrix} A & A\mathbf{b}_2 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 2 \\ -2 & 5 & -9 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -8 \\ 0 & 1 & -5 \end{bmatrix}$ and $\mathbf{b}_2 = \begin{bmatrix} -8 \\ -5 \end{bmatrix}$.

Note: An alternative solution of Exercise 17 is to row reduce $[A \;\; A\mathbf{b}_1 \;\; A\mathbf{b}_2]$ with one sequence of row operations. This observation can prepare the way for the inversion algorithm in Section 2.2.

18. The first two columns of AB are $A\mathbf{b}_1$ and $A\mathbf{b}_2$. They are equal since $\mathbf{b}_1$ and $\mathbf{b}_2$ are equal.

19. (A solution is in the text). Write $B = [\mathbf{b}_1 \;\; \mathbf{b}_2 \;\; \mathbf{b}_3]$. By definition, the third column of AB is $A\mathbf{b}_3$. By hypothesis, $\mathbf{b}_3 = \mathbf{b}_1 + \mathbf{b}_2$. So $A\mathbf{b}_3 = A(\mathbf{b}_1 + \mathbf{b}_2) = A\mathbf{b}_1 + A\mathbf{b}_2$, by a property of matrix-vector multiplication. Thus, the third column of AB is the sum of the first two columns of AB.

20. The second column of AB is also all zeros because $A\mathbf{b}_2 = A\mathbf{0} = \mathbf{0}$.

21. Let $\mathbf{b}_p$ be the last column of B. By hypothesis, the last column of AB is zero. Thus, $A\mathbf{b}_p = \mathbf{0}$. However, $\mathbf{b}_p$ is not the zero vector, because B has no column of zeros. Thus, the equation $A\mathbf{b}_p = \mathbf{0}$ is a linear dependence relation among the columns of A, and so the columns of A are linearly dependent.

Note: The text answer for Exercise 21 is, "The columns of A are linearly dependent. Why?" The *Study Guide* supplies the argument above, in case a student needs help.

22. If the columns of B are linearly dependent, then there exists a nonzero vector $\mathbf{x}$ such that $B\mathbf{x} = \mathbf{0}$. From this, $A(B\mathbf{x}) = A\mathbf{0}$ and $(AB)\mathbf{x} = \mathbf{0}$ (by associativity). Since $\mathbf{x}$ is nonzero, the columns of AB must be linearly dependent.

23. If $\mathbf{x}$ satisfies $A\mathbf{x} = \mathbf{0}$, then $CA\mathbf{x} = C\mathbf{0} = \mathbf{0}$ and so $I_n\mathbf{x} = \mathbf{0}$ and $\mathbf{x} = \mathbf{0}$. This shows that the equation $A\mathbf{x} = \mathbf{0}$ has no free variables. So every variable is a basic variable and every column of A is a pivot column. (A variation of this argument could be made using linear independence and Exercise 30 in Section 1.7.) Since each pivot is in a different row, A must have at least as many rows as columns.

24. Take any $\mathbf{b}$ in $\mathbf{R}^m$. By hypothesis, $AD\mathbf{b} = I_m\mathbf{b} = \mathbf{b}$. Rewrite this equation as $A(D\mathbf{b}) = \mathbf{b}$. Thus, the vector $\mathbf{x} = D\mathbf{b}$ satisfies $A\mathbf{x} = \mathbf{b}$. This proves that the equation $A\mathbf{x} = \mathbf{b}$ has a solution for each $\mathbf{b}$ in $\mathbf{R}^m$. By Theorem 4 in Section 1.4, A has a pivot position in each row. Since each pivot is in a different column, A must have at least as many columns as rows.

25. By Exercise 23, the equation $CA = I_n$ implies that (number of rows in A) $\geq$ (number of columns), that is, $m \geq n$. By Exercise 24, the equation $AD = I_m$ implies that (number of rows in A) $\leq$ (number of columns), that is, $m \leq n$. Thus $m = n$. To prove the second statement, observe that $DAC = (DA)C = I_nC = C$, and also $DAC = D(AC) = DI_m = D$. Thus $C = D$. A shorter calculation is

$$C = I_nC = (DA)C = D(AC) = DI_n = D$$

26. Write $I_3 = [\mathbf{e}_1 \;\; \mathbf{e}_2 \;\; \mathbf{e}_3]$ and $D = [\mathbf{d}_1 \;\; \mathbf{d}_2 \;\; \mathbf{d}_3]$. By definition of AD, the equation $AD = I_3$ is equivalent |to the three equations $A\mathbf{d}_1 = \mathbf{e}_1$, $A\mathbf{d}_2 = \mathbf{e}_2$, and $A\mathbf{d}_3 = \mathbf{e}_3$. Each of these equations has at least one solution because the columns of A span $\mathbf{R}^3$. (See Theorem 4 in Section 1.4.) Select one solution of each equation and use them for the columns of D. Then $AD = I_3$.

27. The product $\mathbf{u}^T\mathbf{v}$ is a 1×1 matrix, which usually is identified with a real number and is written without the matrix brackets.

$$\mathbf{u}^T\mathbf{v} = \begin{bmatrix} -2 & 3 & -4 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = -2a+3b-4c\,,\ \mathbf{v}^T\mathbf{u} = \begin{bmatrix} a & b & c \end{bmatrix}\begin{bmatrix} -2 \\ 3 \\ -4 \end{bmatrix} = -2a+3b-4c$$

$$\mathbf{u}\mathbf{v}^T = \begin{bmatrix} -2 \\ 3 \\ -4 \end{bmatrix}\begin{bmatrix} a & b & c \end{bmatrix} = \begin{bmatrix} -2a & -2b & -2c \\ 3a & 3b & 3c \\ -4a & -4b & -4c \end{bmatrix}$$

$$\mathbf{v}\mathbf{u}^T = \begin{bmatrix} a \\ b \\ c \end{bmatrix}\begin{bmatrix} -2 & 3 & -4 \end{bmatrix} = \begin{bmatrix} -2a & 3a & -4a \\ -2b & 3b & -4b \\ -2c & 3c & -4c \end{bmatrix}$$

28. Since the inner product $\mathbf{u}^T\mathbf{v}$ is a real number, it equals its transpose. That is, $\mathbf{u}^T\mathbf{v} = (\mathbf{u}^T\mathbf{v})^T = \mathbf{v}^T(\mathbf{u}^T)^T = \mathbf{v}^T\mathbf{u}$, by Theorem 3(d) regarding the transpose of a product of matrices and by Theorem 3(a). The outer product $\mathbf{u}\mathbf{v}^T$ is an $n\times n$ matrix. By Theorem 3, $(\mathbf{u}\mathbf{v}^T)^T = (\mathbf{v}^T)^T\mathbf{u}^T = \mathbf{v}\mathbf{u}^T$.

29. The (i, j)-entry of $A(B + C)$ equals the (i, j)-entry of $AB + AC$, because

$$\sum_{k=1}^{n} a_{ik}(b_{kj}+c_{kj}) = \sum_{k=1}^{n} a_{ik}b_{kj} + \sum_{k=1}^{n} a_{ik}c_{kj}$$

The (i, j)-entry of $(B + C)A$ equals the (i, j)-entry of $BA + CA$, because

$$\sum_{k=1}^{n} (b_{ik}+c_{ik})a_{kj} = \sum_{k=1}^{n} b_{ik}a_{kj} + \sum_{k=1}^{n} c_{ik}a_{kj}$$

30. The (i, j))-entries of $r(AB)$, $(rA)B$, and $A(rB)$ are all equal, because

$$r\sum_{k=1}^{n} a_{ik}b_{kj} = \sum_{k=1}^{n} (ra_{ik})b_{kj} = \sum_{k=1}^{n} a_{ik}(rb_{kj})$$

31. Use the definition of the product I_mA and the fact that $I_m\mathbf{x} = \mathbf{x}$ for $\mathbf{x}$ in $\mathbf{R}^m$.

$$I_mA = I_m[\mathbf{a}_1 \ \cdots \ \mathbf{a}_n] = [I_m\mathbf{a}_1 \ \cdots \ I_m\mathbf{a}_n] = [\mathbf{a}_1 \ \cdots \ \mathbf{a}_n] = A$$

32. Let $\mathbf{e}_j$ and $\mathbf{a}_j$ denote the jth columns of I_n and A, respectively. By definition, the jth column of AI_n *is* $A\mathbf{e}_j$, which is simply $\mathbf{a}_j$ because $\mathbf{e}_j$ has 1 in the jth position and zeros elsewhere. Thus corresponding columns of AI_n and A are equal. Hence $AI_n = A$.

33. The (i, j)-entry of $(AB)^T$ is the (j, i)-entry of AB, which is

$$a_{j1}b_{1i} + \cdots + a_{jn}b_{ni}$$

The entries in row i of B^T are $b_{1i}, \dots , b_{ni}$, because they come from column i of B. Likewise, the entries in column j of A^T are $a_{j1}, \dots, a_{jn}$, because they come from row j of A. Thus the (i, j)-entry in B^TA^T is $a_{j1}b_{1i} + \cdots + a_{jn}b_{ni}$, as above.

34. Use Theorem 3(d), treating $\mathbf{x}$ as an $n\times 1$ matrix: $(AB\mathbf{x})^T = \mathbf{x}^T(AB)^T = \mathbf{x}^TB^TA^T$.

35. [**M**] The answer here depends on the choice of matrix program. For MATLAB, use the **`help`** command to read about **`zeros`**, **`ones`**, **`eye`**, and **`diag`**. For other programs see the appendices in the *Study Guide*. (The TI calculators have fewer single commands that produce special matrices.)

36. [**M**] The answer depends on the choice of matrix program. In MATLAB, the command **`rand(6,4)`** creates a 6×4 matrix with random entries uniformly distributed between 0 and 1. The command

`round(19*(rand(6,4)-.5))`

creates a random 6×4 matrix with integer entries between –9 and 9. The same result is produced by the command **`randomint`** in the Laydata Toolbox on text website. For other matrix programs see the appendices in the *Study Guide*.

37. [**M**] $(A + I)(A - I) - (A^2 - I) = 0$ for all 4×4 matrices. However, $(A + B)(A - B) - A^2 - B^2$ is the zero matrix only in the special cases when $AB = BA$. In general,

$$(A + B)(A - B) = A(A - B) + B(A - B) = AA - AB + BA - BB.$$

38. [**M**] The equality $(AB)^T = A^TB^T$ is very likely to be false for 4×4 matrices selected at random.

39. [**M**] The matrix S "shifts" the entries in a vector (a, b, c, d, e) to yield $(b, c, d, e, 0)$. The entries in S^2 result from applying S to the columns of S, and similarly for S^3, and so on. This explains the patterns of entries in the powers of S:

$$S^2 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, S^3 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, S^4 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

S^5 is the 5×5 zero matrix. S^6 is also the 5×5 zero matrix.

40. [**M**] $A^5 = \begin{bmatrix} .3318 & .3346 & .3336 \\ .3346 & .3323 & .3331 \\ .3336 & .3331 & .3333 \end{bmatrix}, A^{10} = \begin{bmatrix} .333337 & .333330 & .333333 \\ .333330 & .333336 & .333334 \\ .333333 & .333334 & .333333 \end{bmatrix}$

The entries in A^{20} all agree with .3333333333 to 9 or 10 decimal places. The entries in A^{30} all agree with .33333333333333 to at least 14 decimal places. The matrices appear to approach the matrix $\begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}$. Further exploration of this behavior appears in Sections 4.9 and 5.2.

Note: The MATLAB box in the *Study Guide* introduces basic matrix notation and operations, including the commands that create special matrices needed in Exercises 35, 36 and elsewhere. The *Study Guide* appendices treat the corresponding information for the other matrix programs.

2.2 SOLUTIONS

Notes: The text includes the matrix inversion algorithm at the end of the section because this topic is popular. Students like it because it is a simple mechanical procedure. However, I no longer cover it in my classes because technology is readily available to invert a matrix whenever needed, and class time is better spent on more useful topics such as partitioned matrices. The final subsection is independent of the inversion algorithm and is needed for Exercises 35 and 36.

Key Exercises: 8, 11–24, 35. (Actually, Exercise 8 is only helpful for some exercises in this section. Section 2.3 has a stronger result.) Exercises 23 and 24 are used in the proof of the Invertible Matrix Theorem (IMT) in Section 2.3, along with Exercises 23 and 24 in Section 2.1. I recommend letting students work on two or more of these four exercises before proceeding to Section 2.3. In this way students *participate* in the proof of the IMT rather than simply watch an instructor carry out the proof. Also, this activity will help students understand *why* the theorem is true.

1. $\begin{bmatrix} 8 & 6 \\ 5 & 4 \end{bmatrix}^{-1} = \dfrac{1}{32-30}\begin{bmatrix} 4 & -6 \\ -5 & 8 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ -5/2 & 4 \end{bmatrix}$

2. $\begin{bmatrix} 3 & 2 \\ 7 & 4 \end{bmatrix}^{-1} = \dfrac{1}{12-14}\begin{bmatrix} 4 & -2 \\ -7 & 3 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 7/2 & -3/2 \end{bmatrix}$

3. $\begin{bmatrix} 8 & 5 \\ -7 & -5 \end{bmatrix}^{-1} = \dfrac{1}{-40-(-35)}\begin{bmatrix} -5 & -5 \\ 7 & 8 \end{bmatrix} = -\dfrac{1}{5}\begin{bmatrix} -5 & -5 \\ 7 & 8 \end{bmatrix}$ or $\begin{bmatrix} 1 & 1 \\ -1.4 & -1.6 \end{bmatrix}$

4. $\begin{bmatrix} 3 & -4 \\ 7 & -8 \end{bmatrix}^{-1} = \dfrac{1}{-24-(-28)}\begin{bmatrix} -8 & 4 \\ -7 & 3 \end{bmatrix} = \dfrac{1}{4}\begin{bmatrix} -8 & 4 \\ -7 & 3 \end{bmatrix}$ or $\begin{bmatrix} -2 & 1 \\ -7/4 & 3/4 \end{bmatrix}$

5. The system is equivalent to $A\mathbf{x} = \mathbf{b}$, where $A = \begin{bmatrix} 8 & 6 \\ 5 & 4 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, and the solution is

$\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} 2 & -3 \\ -5/2 & 4 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 7 \\ -9 \end{bmatrix}$. Thus $x_1 = 7$ and $x_2 = -9$.

6. The system is equivalent to $A\mathbf{x} = \mathbf{b}$, where $A = \begin{bmatrix} 8 & 5 \\ -7 & -5 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 9 \\ 11 \end{bmatrix}$, and the solution is $\mathbf{x} = A^{-1}\mathbf{b}$. To compute this by hand, the arithmetic is simplified by keeping the fraction 1/det(A) in front of the matrix for A^{-1}. (The *Study Guide* comments on this in its discussion of Exercise 7.) From Exercise 3,

$\mathbf{x} = A^{-1}\mathbf{b} = -\dfrac{1}{5}\begin{bmatrix} -5 & -5 \\ 7 & 8 \end{bmatrix}\begin{bmatrix} -9 \\ 11 \end{bmatrix} = -\dfrac{1}{5}\begin{bmatrix} -10 \\ 25 \end{bmatrix} = \begin{bmatrix} 2 \\ -5 \end{bmatrix}$. Thus $x_1 = 2$ and $x_2 = -5$.

7. **a**. $\begin{bmatrix} 1 & 2 \\ 5 & 12 \end{bmatrix}^{-1} = \dfrac{1}{1\cdot 12 - 2\cdot 5}\begin{bmatrix} 12 & -2 \\ -5 & 1 \end{bmatrix} = \dfrac{1}{2}\begin{bmatrix} 12 & -2 \\ -5 & 1 \end{bmatrix}$ or $\begin{bmatrix} 6 & -1 \\ -2.5 & .5 \end{bmatrix}$

$\mathbf{x} = A^{-1}\mathbf{b}_1 = \dfrac{1}{2}\begin{bmatrix} 12 & -2 \\ -5 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 3 \end{bmatrix} = \dfrac{1}{2}\begin{bmatrix} -18 \\ 8 \end{bmatrix} = \begin{bmatrix} -9 \\ 4 \end{bmatrix}$. Similar calculations give

$A^{-1}\mathbf{b}_2 = \begin{bmatrix} 11 \\ -5 \end{bmatrix}, A^{-1}\mathbf{b}_3 = \begin{bmatrix} 6 \\ -2 \end{bmatrix}, A^{-1}\mathbf{b}_4 = \begin{bmatrix} 13 \\ -5 \end{bmatrix}$.

b. $[A \;\; \mathbf{b}_1 \;\; \mathbf{b}_2 \;\; \mathbf{b}_3 \;\; \mathbf{b}_4] = \begin{bmatrix} 1 & 2 & -1 & 1 & 2 & 3 \\ 5 & 12 & 3 & -5 & 6 & 5 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 2 & -1 & 1 & 2 & 3 \\ 0 & 2 & 8 & -10 & -4 & -10 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 & 1 & 2 & 3 \\ 0 & 1 & 4 & -5 & -2 & -5 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 0 & -9 & 11 & 6 & 13 \\ 0 & 1 & 4 & -5 & -2 & -5 \end{bmatrix}$

The solutions are $\begin{bmatrix} -9 \\ 4 \end{bmatrix}, \begin{bmatrix} 11 \\ -5 \end{bmatrix}, \begin{bmatrix} 6 \\ -2 \end{bmatrix}$, and $\begin{bmatrix} 13 \\ -5 \end{bmatrix}$, the same as in part (a).

Note: The *Study Guide* also discusses the number of arithmetic calculations for this Exercise 7, stating that when A is large, the method used in (b) is much faster than using A^{-1}.

8. Left-multiply each side of the equation $AD = I$ by A^{-1} to obtain

$$A^{-1}AD = A^{-1}I,\ ID = A^{-1}, \text{ and } D = A^{-1}.$$

Parentheses are routinely suppressed because of the associative property of matrix multiplication.

9. **a**. True, by definition of *invertible.* **b**. False. See Theorem 6(b).

c. False. If $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, then $ab - cd = 1 - 0 \neq 0$, but Theorem 4 shows that this matrix is not invertible, because $ad - bc = 0$.

d. True. This follows from Theorem 5, which also says that the solution of $A\mathbf{x} = \mathbf{b}$ is unique, for each $\mathbf{b}$.

e. True, by the box just before Example 6.

10. **a**. False. The product matrix is invertible, but the product of inverses should be in the *reverse* order. See Theorem 6(b).

b. True, by Theorem 6(a). **c**. True, by Theorem 4.

d. True, by Theorem 7. **e**. False. The last part of Theorem 7 is misstated here.

11. (The proof can be modeled after the proof of Theorem 5.) The $n \times p$ matrix B is given (but is arbitrary). Since A is invertible, the matrix $A^{-1}B$ satisfies $AX = B$, because $A(A^{-1}B) = A\,A^{-1}B = IB = B$. To show this solution is unique, let X be any solution of $AX = B$. Then, left-multiplication of each side by A^{-1} shows that X must be $A^{-1}B$:

$$A^{-1}(AX) = A^{-1}B, \quad IX = A^{-1}B, \quad \text{and} \quad X = A^{-1}B.$$

12. If you assign this exercise, consider giving the following *Hint*: Use elementary matrices and imitate the proof of Theorem 7. The solution in the Instructor's Edition follows this hint. Here is another solution, based on the idea at the end of Section 2.2.

Write $B = [\mathbf{b}_1 \ \cdots \ \mathbf{b}_p]$ and $X = [\mathbf{u}_1 \ \cdots \ \mathbf{u}_p]$. By definition of matrix multiplication, $AX = [A\mathbf{u}_1 \ \cdots \ A\mathbf{u}_p]$. Thus, the equation $AX = B$ is equivalent to the p systems:

$$A\mathbf{u}_1 = \mathbf{b}_1, \ \ldots \ A\mathbf{u}_p = \mathbf{b}_p$$

Since A is the coefficient matrix in each system, these systems may be solved simultaneously, placing the augmented columns of these systems next to A to form $[A \ \ \mathbf{b}_1 \ \cdots \ \mathbf{b}_p] = [A \ \ B]$. Since A is invertible, the solutions $\mathbf{u}_1, \ldots, \mathbf{u}_p$ are uniquely determined, and $[A \ \ \mathbf{b}_1 \ \cdots \ \mathbf{b}_p]$ must row reduce to $[I \ \ \mathbf{u}_1 \ \cdots \ \mathbf{u}_p] = [I \ \ X]$. By Exercise 11, X is the unique solution $A^{-1}B$ of $AX = B$.

13. Left-multiply each side of the equation $AB = AC$ by A^{-1} to obtain

$$A^{-1}AB = A^{-1}AC, \quad IB = IC, \quad \text{and} \quad B = C.$$

This conclusion does not always follow when A is singular. Exercise 10 of Section 2.1 provides a counterexample.

14. Right-multiply each side of the equation $(B - C)D = 0$ by D^{-1} to obtain

$$(B - C)DD^{-1} = 0D^{-1}, \quad (B - C)I = 0, \quad B - C = 0, \quad \text{and} \quad B = C.$$

15. The box following Theorem 6 suggests what the inverse of ABC should be, namely, $C^{-1}B^{-1}A^{-1}$. To verify that this is correct, compute:

$$(ABC)\, C^{-1}B^{-1}A^{-1} = ABCC^{-1}B^{-1}A^{-1} = ABIB^{-1}A^{-1} = ABB^{-1}A^{-1} = AIA^{-1} = AA^{-1} = I$$

and

$$C^{-1}B^{-1}A^{-1}\,(ABC) = C^{-1}B^{-1}A^{-1}ABC = C^{-1}B^{-1}IBC = C^{-1}B^{-1}BC = C^{-1}IC = C^{-1}C = I$$

16. Let $C = AB$. Then $CB^{-1} = ABB^{-1}$, so $CB^{-1} = AI = A$. This shows that A is the product of invertible matrices and hence is invertible, by Theorem 6.

Note: The *Study Guide* warns against using the formula $(AB)^{-1} = B^{-1}A^{-1}$ here, because this formula can be used only when both A and B are already known to be invertible.

17. Right-multiply each side of $AB = BC$ by B^{-1}:

$$ABB^{-1} = BCB^{-1}, \quad AI = BCB^{-1}, \quad A = BCB^{-1}.$$

18. Left-multiply each side of $A = PBP^{-1}$ by P^{-1}:

$$P^{-1}A = P^{-1}PBP^{-1}, \quad P^{-1}A = IBP^{-1}, \quad P^{-1}A = BP^{-1}$$

Then right-multiply each side of the result by P:

$$P^{-1}AP = BP^{-1}P, \quad P^{-1}AP = BI, \quad P^{-1}AP = B$$

19. Unlike Exercise 17, this exercise asks two things, "Does a solution exist and what is it?" First, find what the solution must be, if it exists. That is, suppose X satisfies the equation $C^{-1}(A + X)B^{-1} = I$. Left-multiply each side by C, and then right-multiply each side by B:

$$CC^{-1}(A + X)B^{-1} = CI, \quad I(A + X)B^{-1} = C, \quad (A + X)B^{-1}B = CB, \quad (A + X)I = CB$$

Expand the left side and then subtract A from both sides:

$$AI + XI = CB, \quad A + X = CB, \quad X = CB - A$$

If a solution exists, it must be $CB - A$. To show that $CB - A$ really *is* a solution, substitute it for X:

$$C^{-1}[A + (CB - A)]B^{-1} = C^{-1}[CB]B^{-1} = C^{-1}CBB^{-1} = II = I.$$

Note: The *Study Guide* suggests that students ask their instructor about how many details to include in their proofs. After some practice with algebra, an expression such as $CC^{-1}(A + X)B^{-1}$ could be simplified directly to $(A + X)B^{-1}$ without first replacing CC^{-1} by I. However, you may wish this detail to be included in the homework for this section.

20. **a**. Left-multiply both sides of $(A - AX)^{-1} = X^{-1}B$ by X to see that B is invertible because it is the product of invertible matrices.

b. Invert both sides of the original equation and use Theorem 6 about the inverse of a product (which applies because X^{-1} and B are invertible):

$$A - AX = (X^{-1}B)^{-1} = B^{-1}(X^{-1})^{-1} = B^{-1}X$$

Then $A = AX + B^{-1}X = (A + B^{-1})X$. The product $(A + B^{-1})X$ is invertible because A is invertible. Since X is known to be invertible, so is the other factor, $A + B^{-1}$, by Exercise 16 or by an argument similar to part (a). Finally,

$$(A + B^{-1})^{-1}A = (A + B^{-1})^{-1}(A + B^{-1})X = X$$

Note: This exercise is difficult. The algebra is not trivial, and at this point in the course, most students will not recognize the need to verify that a matrix is invertible.

21. Suppose A is invertible. By Theorem 5, the equation $A\mathbf{x} = \mathbf{0}$ has only one solution, namely, the zero solution. This means that the columns of A are linearly independent, by a remark in Section 1.7.

22. Suppose A is invertible. By Theorem 5, the equation $A\mathbf{x} = \mathbf{b}$ has a solution (in fact, a unique solution) for each $\mathbf{b}$. By Theorem 4 in Section 1.4, the columns of A span $\mathbf{R}^n$.

23. Suppose A is $n\times n$ and the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution. Then there are no free variables in this equation, and so A has n pivot columns. Since A is *square* and the n pivot positions must be in different rows, the pivots in an echelon form of A must be on the main diagonal. Hence A is row equivalent to the $n\times n$ identity matrix.

24. If the equation $A\mathbf{x} = \mathbf{b}$ has a solution for each $\mathbf{b}$ in $\mathbf{R}^n$, then A has a pivot position in each row, by Theorem 4 in Section 1.4. Since A is square, the pivots must be on the diagonal of A. It follows that A is row equivalent to I_n. By Theorem 7, A is invertible.

25. Suppose $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $ad - bc = 0$. If $a = b = 0$, then examine $\begin{bmatrix} 0 & 0 \\ c & d \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ This has the solution $\mathbf{x}_1 = \begin{bmatrix} d \\ -c \end{bmatrix}$. This solution is nonzero, except when $a = b = c = d$. In that case, however, A is the zero matrix, and $A\mathbf{x} = \mathbf{0}$ for *every* vector $\mathbf{x}$. Finally, if a and b are not both zero, set $\mathbf{x}_2 = \begin{bmatrix} -b \\ a \end{bmatrix}$. Then $A\mathbf{x}_2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} -b \\ a \end{bmatrix} = \begin{bmatrix} -ab+ba \\ -cb+da \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, because $-cb + da = 0$. Thus, $\mathbf{x}_2$ is a nontrivial solution of $A\mathbf{x} = \mathbf{0}$. So, in all cases, the equation $A\mathbf{x} = \mathbf{0}$ has more than one solution. This is impossible when A is invertible (by Theorem 5), so A is *not* invertible.

26. $\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} da-bc & 0 \\ 0 & -cb+ad \end{bmatrix}$. Divide both sides by $ad - bc$ to get $CA = I$.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} ad-bc & 0 \\ 0 & -cb+da \end{bmatrix}.$$

Divide both sides by $ad - bc$. The right side is I. The left side is AC, because

$$\frac{1}{ad-bc}\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = AC$$

27. **a**. Interchange A and B in equation (1) after Example 6 in Section 2.1: $\text{row}_i(BA) = \text{row}_i(B)\cdot A$. Then replace B by the identity matrix: $\text{row}_i(A) = \text{row}_i(IA) = \text{row}_i(I)\cdot A$.

b. Using part (a), when rows 1 and 2 of A are interchanged, write the result as

$$\begin{bmatrix} \text{row}_2(A) \\ \text{row}_1(A) \\ \text{row}_3(A) \end{bmatrix} = \begin{bmatrix} \text{row}_2(I)\cdot A \\ \text{row}_1(I)\cdot A \\ \text{row}_3(I)\cdot A \end{bmatrix} = \begin{bmatrix} \text{row}_2(I) \\ \text{row}_1(I) \\ \text{row}_3(I) \end{bmatrix} A = EA \qquad (*)$$

Here, E is obtained by interchanging rows 1 and 2 of I. The second equality in (*) is a consequence of the fact that $\text{row}_i(EA) = \text{row}_i(E)\cdot A$.

c. Using part (a), when row 3 of A is multiplied by 5, write the result as

$$\begin{bmatrix} \text{row}_1(A) \\ \text{row}_2(A) \\ 5\cdot\text{row}_3(A) \end{bmatrix} = \begin{bmatrix} \text{row}_1(I)\cdot A \\ \text{row}_2(I)\cdot A \\ 5\cdot\text{row}_3(I)\cdot A \end{bmatrix} = \begin{bmatrix} \text{row}_1(I) \\ \text{row}_2(I) \\ 5\cdot\text{row}_3(I) \end{bmatrix} A = EA$$

Here, E is obtained by multiplying row 3 of I by 5.

28. When row 3 of A is replaced by $\text{row}_3(A) - 4\cdot\text{row}_1(A)$, write the result as

$$\begin{bmatrix} \text{row}_1(A) \\ \text{row}_2(A) \\ \text{row}_3(A) - 4\cdot\text{row}_1(A) \end{bmatrix} = \begin{bmatrix} \text{row}_1(I)\cdot A \\ \text{row}_2(I)\cdot A \\ \text{row}_3(I)\cdot A - 4\cdot\text{row}_1(I)\cdot A \end{bmatrix}$$

$$= \begin{bmatrix} \text{row}_1(I)\cdot A \\ \text{row}_2(I)\cdot A \\ [\text{row}_3(I) - 4\cdot\text{row}_1(I)]\cdot A \end{bmatrix} = \begin{bmatrix} \text{row}_1(I) \\ \text{row}_2(I) \\ \text{row}_3(I) - 4\cdot\text{row}_1(I) \end{bmatrix} A = EA$$

Here, E is obtained by replacing $\text{row}_3(I)$ by $\text{row}_3(I) - 4\cdot\text{row}_1(I)$.

29. $[A \;\; I] = \begin{bmatrix} 1 & 2 & 1 & 0 \\ 4 & 7 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & -1 & -4 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 4 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -7 & 2 \\ 0 & 1 & 4 & -1 \end{bmatrix}$

$$A^{-1} = \begin{bmatrix} -7 & 2 \\ 4 & -1 \end{bmatrix}$$

30. $[A \quad I] = \begin{bmatrix} 5 & 10 & 1 & 0 \\ 4 & 7 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1/5 & 0 \\ 4 & 7 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1/5 & 0 \\ 0 & -1 & -4/5 & 1 \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & 2 & 1/5 & 0 \\ 0 & 1 & 4/5 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -7/5 & 2 \\ 0 & 1 & 4/5 & -1 \end{bmatrix}. \quad A^{-1} = \begin{bmatrix} -7/5 & 2 \\ 4/5 & -1 \end{bmatrix}$$

31. $[A \quad I] = \begin{bmatrix} 1 & 0 & -2 & 1 & 0 & 0 \\ -3 & 1 & 4 & 0 & 1 & 0 \\ 2 & -3 & 4 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 3 & 1 & 0 \\ 0 & -3 & 8 & -2 & 0 & 1 \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 3 & 1 & 0 \\ 0 & 0 & 2 & 7 & 3 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 8 & 3 & 1 \\ 0 & 1 & 0 & 10 & 4 & 1 \\ 0 & 0 & 2 & 7 & 3 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 8 & 3 & 1 \\ 0 & 1 & 0 & 10 & 4 & 1 \\ 0 & 0 & 1 & 7/2 & 3/2 & 1/2 \end{bmatrix}. \quad A^{-1} = \begin{bmatrix} 8 & 3 & 1 \\ 10 & 4 & 1 \\ 7/2 & 3/2 & 1/2 \end{bmatrix}$$

32. $[A \quad I] = \begin{bmatrix} 1 & -2 & 1 & 1 & 0 & 0 \\ 4 & -7 & 3 & 0 & 1 & 0 \\ -2 & 6 & -4 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -4 & 1 & 0 \\ 0 & 2 & -2 & 2 & 0 & 1 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & -2 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -4 & 1 & 0 \\ 0 & 0 & 0 & 10 & -2 & 1 \end{bmatrix}$. The matrix A is not invertible.

33. Let $B = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ -1 & 1 & 0 & & 0 \\ 0 & -1 & 1 & & \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & -1 & 1 \end{bmatrix}$, and for $j = 1, \ldots, n$, let $\mathbf{a}_j$, $\mathbf{b}_j$, and $\mathbf{e}_j$ denote the jth columns of A, B, and I, respectively. Note that for $j = 1, \ldots, n - 1$, $\mathbf{a}_j - \mathbf{a}_{j+1} = \mathbf{e}_j$ (because $\mathbf{a}_j$ and $\mathbf{a}_{j+1}$ have the same entries except for the jth row), $\mathbf{b}_j = \mathbf{e}_j - \mathbf{e}_{j+1}$ and $\mathbf{a}_n = \mathbf{b}_n = \mathbf{e}_n$.

To show that $AB = I$, it suffices to show that $A\mathbf{b}_j = \mathbf{e}_j$ for each j. For $j = 1, \ldots, n - 1$,

$$A\mathbf{b}_j = A(\mathbf{e}_j - \mathbf{e}_{j+1}) = A\mathbf{e}_j - A\mathbf{e}_{j+1} = \mathbf{a}_j - \mathbf{a}_{j+1} = \mathbf{e}_j$$

and $A\mathbf{b}_n = A\mathbf{e}_n = \mathbf{a}_n = \mathbf{e}_n$. Next, observe that $\mathbf{a}_j = \mathbf{e}_j + \cdots + \mathbf{e}_n$ for each j. Thus,

$$\begin{aligned} B\mathbf{a}_j &= B(\mathbf{e}_j + \cdots + \mathbf{e}_n) = \mathbf{b}_j + \cdots + \mathbf{b}_n \\ &= (\mathbf{e}_j - \mathbf{e}_{j+1}) + (\mathbf{e}_{j+1} - \mathbf{e}_{j+2}) + \cdots + (\mathbf{e}_{n-1} - \mathbf{e}_n) + \mathbf{e}_n = \mathbf{e}_j \end{aligned}$$

This proves that $BA = I$. Combined with the first part, this proves that $B = A^{-1}$.

Note: Students who do this problem and then do the corresponding exercise in Section 2.4 will appreciate the Invertible Matrix Theorem, partitioned matrix notation, and the power of a proof by induction.

34. Let

$$A = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 2 & 0 & & 0 \\ 1 & 2 & 3 & & 0 \\ \vdots & & & \ddots & \vdots \\ 1 & 2 & 3 & \cdots & n \end{bmatrix}, \text{ and } B = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ -1/2 & 1/2 & 0 & & \\ 0 & -1/3 & 1/3 & & \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & 0 & & -1/n & 1/n \end{bmatrix}$$

and for $j = 1, \ldots, n$, let $\mathbf{a}_j$, $\mathbf{b}_j$, and $\mathbf{e}_j$ denote the jth columns of A, B, and I, respectively. Note that for $j = 1, \ldots, n-1$, $\mathbf{a}_j = j(\mathbf{e}_j + \cdots + \mathbf{e}_n)$, $\mathbf{b}_j = \frac{1}{j}\mathbf{e}_j - \frac{1}{j+1}\mathbf{e}_{j+1}$, and $\mathbf{b}_n = \frac{1}{n}\mathbf{e}_n$.

To show that $AB = I$, it suffices to show that $A\mathbf{b}_j = \mathbf{e}_j$ for each j. For $j = 1, \ldots, n-1$,

$$\begin{aligned} A\mathbf{b}_j &= A\left(\frac{1}{j}\mathbf{e}_j - \frac{1}{j+1}\mathbf{e}_{j+1}\right) = \frac{1}{j}\mathbf{a}_j - \frac{1}{j+1}\mathbf{a}_{j+1} \\ &= (\mathbf{e}_j + \cdots + \mathbf{e}_n) - (\mathbf{e}_{j+1} + \cdots + \mathbf{e}_n) = \mathbf{e}_j \end{aligned}$$

Also, $A\mathbf{b}_n = A\left(\frac{1}{n}\mathbf{e}_n\right) = \frac{1}{n}\mathbf{a}_n = \mathbf{e}_n$. Finally, for $j = 1, \ldots, n$, the sum $\mathbf{b}_j + \cdots + \mathbf{b}_n$ is a "telescoping sum" whose value is $\frac{1}{j}\mathbf{e}_j$. Thus,

$$B\mathbf{a}_j = j(B\mathbf{e}_j + \cdots + B\mathbf{e}_n) = j(\mathbf{b}_j + \cdots + \mathbf{b}_n) = j\left(\frac{1}{j}\mathbf{e}_j\right) = \mathbf{e}_j$$

which proves that $BA = I$. Combined with the first part, this proves that $B = A^{-1}$.

Note: If you assign Exercise 34, you may wish to supply a hint using the notation from Exercise 33: Express each column of A in terms of the columns $\mathbf{e}_1, \ldots, \mathbf{e}_n$ of the identity matrix. Do the same for B.

35. Row reduce $[A \;\; \mathbf{e}_3]$:

$$\begin{bmatrix} -2 & -7 & -9 & 0 \\ 2 & 5 & 6 & 0 \\ 1 & 3 & 4 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 4 & 1 \\ 2 & 5 & 6 & 0 \\ -2 & -7 & -9 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 4 & 1 \\ 0 & -1 & -2 & -2 \\ 0 & -1 & -1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 4 & 1 \\ 0 & -1 & -2 & -2 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 3 & 0 & -15 \\ 0 & -1 & 0 & 6 \\ 0 & 0 & 1 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 0 & -15 \\ 0 & 1 & 0 & -6 \\ 0 & 0 & 1 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -6 \\ 0 & 0 & 1 & 4 \end{bmatrix}.$$

Answer: The third column of A^{-1} is $\begin{bmatrix} 3 \\ -6 \\ 4 \end{bmatrix}$.

36. **[M]** Write $B = [A \quad F]$, where F consists of the last two columns of I_3, and row reduce:

$$B = \begin{bmatrix} -25 & -9 & -27 & 0 & 0 \\ 546 & 180 & 537 & 1 & 0 \\ 154 & 50 & 149 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 3/2 & -9/2 \\ 0 & 1 & 0 & -433/6 & 439/2 \\ 0 & 0 & 1 & 68/3 & -69 \end{bmatrix}$$

The last two columns of A^{-1} are $\begin{bmatrix} 1.5000 & -4.5000 \\ -72.1667 & 219.5000 \\ 22.6667 & -69.0000 \end{bmatrix}$

37. There are many possibilities for C, but $C = \begin{bmatrix} 1 & 1 & -1 \\ -1 & 1 & 0 \end{bmatrix}$ is the only one whose entries are 1, –1, and 0.

With only three possibilities for each entry, the construction of C can be done by trial and error. This is probably faster than setting up a system of 4 equations in 6 unknowns. The fact that A cannot be invertible follows from Exercise 25 in Section 2.1, because A is not square.

38. Write $AD = A[\mathbf{d}_1 \quad \mathbf{d}_2] = [A\mathbf{d}_1 \quad A\mathbf{d}_2]$. The structure of A shows that $D = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$.

[There are 25 possibilities for D if entries of D are allowed to be 1, –1, and 0.] There is *no* 4×2 matrix C such that $CA = I_4$. If this were true, then $CA\mathbf{x}$ would equal $\mathbf{x}$ for all $\mathbf{x}$ in $\mathbf{R}^4$. This cannot happen because the columns of A are linearly dependent and so $A\mathbf{x} = \mathbf{0}$ for some nonzero vector $\mathbf{x}$. For such an $\mathbf{x}$, $CA\mathbf{x} = C(\mathbf{0}) = \mathbf{0}$. An alternate justification would be to cite Exercise 23 or 25 in Section 2.1.

39. $\mathbf{y} = D\mathbf{f} = \begin{bmatrix} .005 & .002 & .001 \\ .002 & .004 & .002 \\ .001 & .002 & .005 \end{bmatrix}\begin{bmatrix} 30 \\ 50 \\ 20 \end{bmatrix} = \begin{bmatrix} .27 \\ .30 \\ .23 \end{bmatrix}$. The deflections are .27 in., .30 in., and .23 in. at points 1, 2, and 3, respectively.

40. **[M]** The *stiffness matrix* is D^{-1}. Use an "inverse" command to produce

$$D^{-1} = 125\begin{bmatrix} 2 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

To find the forces (in pounds) required to produce a deflection of .04 cm at point 3, most students will use technology to solve $D\mathbf{f} = (0, 0, .04)$ and obtain (0, –5, 10).

Here is another method, based on the idea suggested in Exercise 42. The first column of D^{-1} lists the forces required to produce a deflection of 1 in. at point 1 (with zero deflection at the other points). Since the transformation $\mathbf{y} \mapsto D^{-1}\mathbf{y}$ is linear, the forces required to produce a deflection of .04 cm at point 3 is given by .04 times the third column of D^{-1}, namely (.04)(125) times (0, –1, 2), or (0, –5, 10) pounds.

41. To determine the forces that produce a deflections of .08, .12, .16, and .12 cm at the four points on the beam, use technology to solve $D\mathbf{f} = \mathbf{y}$, where $\mathbf{y} = (.08, .12, .16, .12)$. The forces at the four points are 12, 1.5, 21.5, and 12 newtons, respectively.

42. **[M]** To determine the forces that produce a deflection of .240 cm at the second point on the beam, use technology to solve $D\mathbf{f} = \mathbf{y}$, where $\mathbf{y} = (0, .24, 0, 0)$. The forces at the four points are –104, 167, –113, and 56.0 newtons, respectively (to three significant digits). These forces are .24 times the entries in the second column of D^{-1}. *Reason*: The transformation $\mathbf{y} \mapsto D^{-1}\mathbf{y}$ is linear, so the forces required to produce a deflection of .24 cm at the second point are .24 times the forces required to produce a deflection of 1 cm at the second point. These forces are listed in the second column of D^{-1}.

Another possible discussion: The solution of $D\mathbf{x} = (0, 1, 0, 0)$ is the second column of D^{-1}. Multiply both sides of this equation by .24 to obtain $D(.24\mathbf{x}) = (0, .24, 0, 0)$. So $.24\mathbf{x}$ is the solution of $D\mathbf{f} = (0, .24, 0, 0)$. (The argument uses linearity, but students may not mention this.)

Note: The *Study Guide* suggests using **`gauss`**, **`swap`**, **`bgauss`**, and **`scale`** to reduce $[A \quad I]$, because I prefer to postpone the use of **`ref`** (or **`rref`**) until later. If you wish to introduce **`ref`** now, see the *Study Guide*'s technology notes for Sections 2.8 or 4.3. (Recall that Sections 2.8 and 2.9 are only covered when an instructor plans to skip Chapter 4 and get quickly to eigenvalues.)

2.3 SOLUTIONS

Notes: This section ties together most of the concepts studied thus far. With strong encouragement from an instructor, most students can use this opportunity to review and reflect upon what they have learned, and form a solid foundation for future work. Students who fail to do this now usually struggle throughout the rest of the course. Section 2.3 can be used in at least three different ways.

(1) Stop after Example 1 and assign exercises only from among the Practice Problems and Exercises 1 to 28. I do this when teaching “Course 3” described in the text's “Notes to the Instructor. ” If you did not cover Theorem 12 in Section 1.9, omit statements (f) and (i) from the Invertible Matrix Theorem.

(2) Include the subsection “Invertible Linear Transformations” in Section 2.3, if you covered Section 1.9. I do this when teaching “Course 1” because our mathematics and computer science majors take this class. Exercises 29–40 support this material.

(3) Skip the linear transformation material here, but discusses the **condition number** and the Numerical Notes. Assign exercises from among 1–28 and 41–45, and perhaps add a computer project on the condition number. (See the projects on our web site.) I do this when teaching “Course 2” for our engineers.

The abbreviation IMT (here and in the *Study Guide*) denotes the Invertible Matrix Theorem (Theorem 8).

1. The columns of the matrix $\begin{bmatrix} 5 & 7 \\ -3 & -6 \end{bmatrix}$ are not multiples, so they are linearly independent. By (e) in the IMT, the matrix is invertible. Also, the matrix is invertible by Theorem 4 in Section 2.2 because the determinant is nonzero.

2. The fact that the columns of $\begin{bmatrix} -4 & 6 \\ 6 & -9 \end{bmatrix}$ are multiples is not so obvious. The fastest check in this case may be the determinant, which is easily seen to be zero. By Theorem 4 in Section 2.2, the matrix is not invertible.

3. Row reduction to echelon form is trivial because there is really no need for arithmetic calculations:
$\begin{bmatrix} 5 & 0 & 0 \\ -3 & -7 & 0 \\ 8 & 5 & -1 \end{bmatrix} \sim \begin{bmatrix} 5 & 0 & 0 \\ 0 & -7 & 0 \\ 0 & 5 & -1 \end{bmatrix} \sim \begin{bmatrix} 5 & 0 & 0 \\ 0 & -7 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ The 3×3 matrix has 3 pivot positions and hence is invertible, by (c) of the IMT. [Another explanation could be given using the transposed matrix. But see the note below that follows the solution of Exercise 14.]

4. The matrix $\begin{bmatrix} -7 & 0 & 4 \\ 3 & 0 & -1 \\ 2 & 0 & 9 \end{bmatrix}$ obviously has linearly dependent columns (because one column is zero), and so the matrix is not invertible (or singular) by (e) in the IMT.

5. $\begin{bmatrix} 0 & 3 & -5 \\ 1 & 0 & 2 \\ -4 & -9 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 3 & -5 \\ -4 & -9 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 3 & -5 \\ 0 & -9 & 15 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 3 & -5 \\ 0 & 0 & 0 \end{bmatrix}$

The matrix is not invertible because it is not row equivalent to the identity matrix.

6. $\begin{bmatrix} 1 & -5 & -4 \\ 0 & 3 & 4 \\ -3 & 6 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -5 & -4 \\ 0 & 3 & 4 \\ 0 & -9 & -12 \end{bmatrix} \sim \begin{bmatrix} 1 & -5 & -4 \\ 0 & 3 & 4 \\ 0 & 0 & 0 \end{bmatrix}$

The matrix is not invertible because it is not row equivalent to the identity matrix.

7. $\begin{bmatrix} -1 & -3 & 0 & 1 \\ 3 & 5 & 8 & -3 \\ -2 & -6 & 3 & 2 \\ 0 & -1 & 2 & 1 \end{bmatrix} \sim \begin{bmatrix} -1 & -3 & 0 & 1 \\ 0 & -4 & 8 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & -1 & 2 & 1 \end{bmatrix} \sim \begin{bmatrix} -1 & -3 & 0 & 1 \\ 0 & -4 & 8 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

The 4×4 matrix has four pivot positions and so is invertible by (c) of the IMT.

8. The 4×4 matrix $\begin{bmatrix} 1 & 3 & 7 & 4 \\ 0 & 5 & 9 & 6 \\ 0 & 0 & 2 & 8 \\ 0 & 0 & 0 & 10 \end{bmatrix}$ is invertible because it has four pivot positions, by (c) of the IMT.

9. **[M]** $\begin{bmatrix} 4 & 0 & -7 & -7 \\ -6 & 1 & 11 & 9 \\ 7 & -5 & 10 & 19 \\ -1 & 2 & 3 & -1 \end{bmatrix} \sim \begin{bmatrix} -1 & 2 & 3 & -1 \\ -6 & 1 & 11 & 9 \\ 7 & -5 & 10 & 19 \\ 4 & 0 & -7 & -7 \end{bmatrix} \sim \begin{bmatrix} -1 & 2 & 3 & -1 \\ 0 & -11 & -7 & 15 \\ 0 & 9 & 31 & 12 \\ 0 & 8 & 5 & -11 \end{bmatrix}$

$\sim \begin{bmatrix} -1 & 2 & 3 & -1 \\ 0 & 8 & 5 & -11 \\ 0 & 9 & 31 & 12 \\ 0 & -11 & -7 & 15 \end{bmatrix} \sim \begin{bmatrix} -1 & 2 & 3 & -1 \\ 0 & 8 & 5 & -11 \\ 0 & 0 & 25.375 & 24.375 \\ 0 & 0 & -.1250 & -.1250 \end{bmatrix} \sim \begin{bmatrix} -1 & 2 & 3 & -1 \\ 0 & 8 & 5 & -11 \\ 0 & 0 & 25.375 & 24.375 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

$\sim \begin{bmatrix} -1 & 2 & 3 & -1 \\ 0 & 8 & 5 & -11 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 25.375 & 24.375 \end{bmatrix} \sim \begin{bmatrix} -1 & 2 & 3 & -1 \\ 0 & 8 & 5 & -11 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}$

The 4×4 matrix is invertible because it has four pivot positions, by (c) of the IMT.

10. **[M]** $\begin{bmatrix} 5 & 3 & 1 & 7 & 9 \\ 6 & 4 & 2 & 8 & -8 \\ 7 & 5 & 3 & 10 & 9 \\ 9 & 6 & 4 & -9 & -5 \\ 8 & 5 & 2 & 11 & 4 \end{bmatrix} \sim \begin{bmatrix} 5 & 3 & 1 & 7 & 9 \\ 0 & .4 & .8 & -.4 & -18.8 \\ 0 & .8 & 1.6 & .2 & -3.6 \\ 0 & .6 & 2.2 & -21.6 & -21.2 \\ 0 & .2 & .4 & -.2 & -10.4 \end{bmatrix}$

$$\sim \begin{bmatrix} 5 & 3 & 1 & 7 & 9 \\ 0 & .4 & .8 & -.4 & -18.8 \\ 0 & 0 & 0 & 1 & 34 \\ 0 & 0 & 1 & -21 & 7 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 5 & 3 & 1 & 7 & 9 \\ 0 & .4 & .8 & -.4 & -18.8 \\ 0 & 0 & 1 & -21 & 7 \\ 0 & 0 & 0 & 1 & 34 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

The 5×5 matrix is invertible because it has five pivot positions, by (c) of the IMT.

11. **a**. True, by the IMT. If statement (d) of the IMT is true, then so is statement (b).
b. True. If statement (h) of the IMT is true, then so is statement (e).
c. False. Statement (g) of the IMT is true only for invertible matrices.
d. True, by the IMT. If the equation $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution, then statement (d) of the IMT is false. In this case, all the lettered statements in the IMT are false, including statement (c), which means that A must have fewer than n pivot positions.
e. True, by the IMT. If A^T is not invertible, then statement (1) of the IMT is false, and hence statement (a) must also be false.

12. **a**. True. If statement (k) of the IMT is true, then so is statement (j).
b. True. If statement (e) of the IMT is true, then so is statement (h).
c. True. See the remark immediately following the proof of the IMT.
d. False. The first part of the statement is not part (i) of the IMT. In fact, if A is any $n\times n$ matrix, the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ into $\mathbb{R}^n$, yet not every such matrix has n pivot positions.
e. True, by the IMT. If there is a $\mathbf{b}$ in $\mathbb{R}^n$ such that the equation $A\mathbf{x} = \mathbf{b}$ is inconsistent, then statement (g) of the IMT is false, and hence statement (f) is also false. That is, the transformation $\mathbf{x} \mapsto A\mathbf{x}$ cannot be one-to-one.

Note: The solutions below for Exercises 13–30 refer mostly to the IMT. In many cases, however, part or all of an acceptable solution could also be based on various results that were used to establish the IMT.

13. If a square upper triangular $n\times n$ matrix has nonzero diagonal entries, then because it is already in echelon form, the matrix is row equivalent to I_n and hence is invertible, by the IMT. Conversely, if the matrix is invertible, it has n pivots on the diagonal and hence the diagonal entries are nonzero.

14. If A is lower triangular with nonzero entries on the diagonal, then these n diagonal entries can be used as pivots to produce zeros below the diagonal. Thus A has n pivots and so is invertible, by the IMT. If one of the diagonal entries in A is zero, A will have fewer than n pivots and hence be singular.

Notes: For Exercise 14, another correct analysis of the case when A has nonzero diagonal entries is to apply the IMT (or Exercise 13) to A^T. Then use Theorem 6 in Section 2.2 to conclude that since A^T is invertible so is its transpose, A. You might mention this idea in class, but I recommend that you not spend much time discussing A^T and problems related to it, in order to keep from making this section too lengthy. (The transpose is treated infrequently in the text until Chapter 6.)

If you do plan to ask a test question that involves A^T and the IMT, then you should give the students some extra homework that develops skill using A^T. For instance, in Exercise 14 replace "columns" by "rows."

Also, you could ask students to explain why an $n \times n$ matrix with linearly independent columns must also have linearly independent rows.

15. If A has two identical columns then its columns are linearly dependent. Part (e) of the IMT shows that A cannot be invertible.

16. Part (h) of the IMT shows that a 5×5 matrix cannot be invertible when its columns do not span $\mathbf{R}^5$.

17. If A is invertible, so is A^{-1}, by Theorem 6 in Section 2.2. By (e) of the IMT applied to A^{-1}, the columns of A^{-1} are linearly independent.

18. By (g) of the IMT, C is invertible. Hence, each equation $C\mathbf{x} = \mathbf{v}$ has a unique solution, by Theorem 5 in Section 2.2. This fact was pointed out in the paragraph following the proof of the IMT.

19. By (e) of the IMT, D is invertible. Thus the equation $D\mathbf{x} = \mathbf{b}$ has a solution for each $\mathbf{b}$ in $\mathbf{R}^7$, by (g) of the IMT. Even better, the equation $D\mathbf{x} = \mathbf{b}$ has a *unique* solution for each $\mathbf{b}$ in $\mathbf{R}^7$, by Theorem 5 in Section 2.2. (See the paragraph following the proof of the IMT.)

20. By the box following the IMT, E and F are invertible and are inverses. So $FE = I = EF$, and so E and F commute.

21. The matrix G cannot be invertible, by Theorem 5 in Section 2.2 or by the box following the IMT. So (h) of the IMT is false and the columns of G do not span $\mathbf{R}^n$.

22. Statement (g) of the IMT is false for H, so statement (d) is false, too. That is, the equation $H\mathbf{x} = \mathbf{0}$ has a nontrivial solution.

23. Statement (b) of the IMT is false for K, so statements (e) and (h) are also false. That is, the columns of K are linearly *de*pendent and the columns do *not* span $\mathbf{R}^n$.

24. No conclusion about the columns of L may be drawn, because no information about L has been given. The equation $L\mathbf{x} = \mathbf{0}$ *always* has the trivial solution.

25. Suppose that A is square and $AB = I$. Then A is invertible, by the (k) of the IMT. Left-multiplying each side of the equation $AB = I$ by A^{-1}, one has

$$A^{-1}AB = A^{-1}I, \quad IB = A^{-1}, \quad \text{and } B = A^{-1}.$$

By Theorem 6 in Section 2.2, the matrix B (which is A^{-1}) is invertible, and its inverse is $(A^{-1})^{-1}$, which is A.

26. If the columns of A are linearly independent, then since A is square, A is invertible, by the IMT. So A^2, which is the product of invertible matrices, is invertible. By the IMT, the columns of A^2 span $\mathbf{R}^n$.

27. Let W be the inverse of AB. Then $ABW = I$ and $A(BW) = I$. Since A is square, A is invertible, by (k) of the IMT.

Note: The *Study Guide* for Exercise 27 emphasizes here that the equation $A(BW) = I$, *by itself*, does not show that A is invertible. Students are referred to Exercise 38 in Section 2.2 for a counterexample. Although there is an overall assumption that matrices in this section are square, I insist that my students mention this fact when using the IMT. Even so, at the end of the course, I still sometimes find a student who thinks that an equation $AB = I$ implies that A is invertible.

28. Let W be the inverse of AB. Then $WAB = I$ and $(WA)B = I$. By (j) of the IMT applied to B in place of A, the matrix B is invertible.

29. Since the transformation $\mathbf{x} \mapsto A\mathbf{x}$ is not one-to-one, statement (f) of the IMT is false. Then (i) is also false and the transformation $\mathbf{x} \mapsto A\mathbf{x}$ does not map $\mathbf{R}^n$ onto $\mathbf{R}^n$. Also, A is not invertible, which implies that the transformation $\mathbf{x} \mapsto A\mathbf{x}$ is not invertible, by Theorem 9.

30. Since the transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one, statement (f) of the IMT is true. Then (i) is also true and the transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbf{R}^n$ onto $\mathbf{R}^n$. Also, A is invertible, which implies that the transformation $\mathbf{x} \mapsto A\mathbf{x}$ is invertible, by Theorem 9.

31. Since the equation $A\mathbf{x} = \mathbf{b}$ has a solution for each $\mathbf{b}$, the matrix A has a pivot in each row (Theorem 4 in Section 1.4). Since A is square, A has a pivot in each column, and so there are no free variables in the equation $A\mathbf{x} = \mathbf{b}$, which shows that the solution is unique.

Note: The preceding argument shows that the (square) shape of A plays a crucial role. A less revealing proof is to use the "pivot in each row" and the IMT to conclude that A is invertible. Then Theorem 5 in Section 2.2 shows that the solution of $A\mathbf{x} = \mathbf{b}$ is unique.

32. If $A\mathbf{x} = \mathbf{0}$ has only the trivial solution, then A must have a pivot in each of its n columns. Since A is square (and this is the key point), there must be a pivot in each *row* of A. By Theorem 4 in Section 1.4, the equation $A\mathbf{x} = \mathbf{b}$ has a solution for each $\mathbf{b}$ in $\mathbf{R}^n$.

Another argument: Statement (d) of the IMT is true, so A is invertible. By Theorem 5 in Section 2.2, the equation $A\mathbf{x} = \mathbf{b}$ has a (unique) solution for each $\mathbf{b}$ in $\mathbf{R}^n$.

33. (Solution in *Study Guide*) The standard matrix of T is $A = \begin{bmatrix} -5 & 9 \\ 4 & -7 \end{bmatrix}$, which is invertible because det $A \neq 0$. By Theorem 9, the transformation T is invertible and the standard matrix of T^{-1} is A^{-1}. From the formula for a 2×2 inverse, $A^{-1} = \begin{bmatrix} 7 & 9 \\ 4 & 5 \end{bmatrix}$. So

$$T^{-1}(x_1, x_2) = \begin{bmatrix} 7 & 9 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = (7x_1 + 9x_2, 4x_1 + 5x_2)$$

34. The standard matrix of T is $A = \begin{bmatrix} 6 & -8 \\ -5 & 7 \end{bmatrix}$, which is invertible because det $A = 2 \neq 0$. By Theorem 9, T is invertible, and $T^{-1}(\mathbf{x}) = B\mathbf{x}$, where $B = A^{-1} = \frac{1}{2}\begin{bmatrix} 7 & 8 \\ 5 & 6 \end{bmatrix}$. Thus

$$T^{-1}(x_1, x_2) = \frac{1}{2}\begin{bmatrix} 7 & 8 \\ 5 & 6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \left(\frac{7}{2}x_1 + 4x_2, \frac{5}{2}x_1 + 3x_2\right)$$

35. (Solution in *Study Guide*) To show that T is one-to-one, suppose that $T(\mathbf{u}) = T(\mathbf{v})$ for some vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbf{R}^n$. Then $S(T(\mathbf{u})) = S(T(\mathbf{v}))$, where S is the inverse of T. By Equation (1), $\mathbf{u} = S(T(\mathbf{u}))$ and $S(T(\mathbf{v})) = \mathbf{v}$, so $\mathbf{u} = \mathbf{v}$. Thus T is one-to-one. To show that T is onto, suppose $\mathbf{y}$ represents an arbitrary vector in $\mathbf{R}^n$ and define $\mathbf{x} = S(\mathbf{y})$. Then, using Equation (2), $T(\mathbf{x}) = T(S(\mathbf{y})) = \mathbf{y}$, which shows that T maps $\mathbf{R}^n$ onto $\mathbf{R}^n$.

Second proof: By Theorem 9, the standard matrix A of T is invertible. By the IMT, the columns of A are linearly independent and span $\mathbf{R}^n$. By Theorem 12 in Section 1.9, T is one-to-one and maps $\mathbf{R}^n$ onto $\mathbf{R}^n$.

36. If T maps $\mathbf{R}^n$ onto $\mathbf{R}^n$, then the columns of its standard matrix A span $\mathbf{R}^n$, by Theorem 12 in Section 1.9. By the IMT, A is invertible. Hence, by Theorem 9 in Section 2.3, T is invertible, and A^{-1} is the standard matrix of T^{-1}. Since A^{-1} is also invertible, by the IMT, its columns are linearly independent and span $\mathbf{R}^n$. Applying Theorem 12 in Section 1.9 to the transformation T^{-1}, we conclude that T^{-1} is a one-to-one mapping of $\mathbf{R}^n$ onto $\mathbf{R}^n$.

37. Let A and B be the standard matrices of T and U, respectively. Then AB is the standard matrix of the mapping $\mathbf{x} \mapsto T(U(\mathbf{x}))$, because of the way matrix multiplication is defined (in Section 2.1). By hypothesis, this mapping is the identity mapping, so $AB = I$. Since A and B are square, they are invertible, by the IMT, and $B = A^{-1}$. Thus, $BA = I$. This means that the mapping $\mathbf{x} \mapsto U(T(\mathbf{x}))$ is the identity mapping, i.e., $U(T(\mathbf{x})) = \mathbf{x}$ for all $\mathbf{x}$ in $\mathbf{R}^n$.

38. Let A be the standard matrix of T. By hypothesis, T is not a one-to-one mapping. So, by Theorem 12 in Section 1.9, the standard matrix A of T has linearly dependent columns. Since A is square, the columns of A do not span $\mathbf{R}^n$. By Theorem 12, again, T cannot map $\mathbf{R}^n$ onto $\mathbf{R}^n$.

39. Given any $\mathbf{v}$ in $\mathbf{R}^n$, we may write $\mathbf{v} = T(\mathbf{x})$ for some $\mathbf{x}$, because T is an onto mapping. Then, the assumed properties of S and U show that $S(\mathbf{v}) = S(T(\mathbf{x})) = \mathbf{x}$ and $U(\mathbf{v}) = U(T(\mathbf{x})) = \mathbf{x}$. So $S(\mathbf{v})$ and $U(\mathbf{v})$ are equal for each $\mathbf{v}$. That is, S and U are the same function from $\mathbf{R}^n$ into $\mathbf{R}^n$.

40. Given $\mathbf{u}$, $\mathbf{v}$ in $\mathbb{R}^n$, let $\mathbf{x} = S(\mathbf{u})$ and $\mathbf{y} = S(\mathbf{v})$. Then $T(\mathbf{x}) = T(S(\mathbf{u})) = \mathbf{u}$ and $T(\mathbf{y}) = T(S(\mathbf{v})) = \mathbf{v}$, by equation (2). Hence

$$\begin{aligned} S(\mathbf{u}+\mathbf{v}) &= S(T(\mathbf{x})+T(\mathbf{y})) \\ &= S(T(\mathbf{x}+\mathbf{y})) && \text{Because } T \text{ is linear} \\ &= \mathbf{x}+\mathbf{y} && \text{By equation (1)} \\ &= S(\mathbf{u})+S(\mathbf{v}) \end{aligned}$$

So, S preserves sums. For any scalar r,

$$\begin{aligned} S(r\mathbf{u}) &= S(rT(\mathbf{x})) = S(T(r\mathbf{x})) && \text{Because } T \text{ is linear} \\ &= r\mathbf{x} && \text{By equation (1)} \\ &= rS(\mathbf{u}) \end{aligned}$$

So S preserves scalar multiples. Thus S ia a linear transformation.

41. **[M]** **a**. The exact solution of (3) is $x_1 = 3.94$ and $x_2 = .49$. The exact solution of (4) is $x_1 = 2.90$ and $x_2 = 2.00$.

b. When the solution of (4) is used as an approximation for the solution in (3) , the error in using the value of 2.90 for x_1 is about 26%, and the error in using 2.0 for x_2 is about 308%.

c. The condition number of the coefficient matrix is 3363. The percentage change in the solution from (3) to (4) is about 7700 times the percentage change in the right side of the equation. This is the same order of magnitude as the condition number. The condition number gives a rough measure of how sensitive the solution of $A\mathbf{x} = \mathbf{b}$ can be to changes in $\mathbf{b}$. Further information about the condition number is given at the end of Chapter 6 and in Chapter 7.

Note: See the *Study Guide*'s MATLAB box, or a technology appendix, for information on condition number. Only the TI-83+ and TI-89 lack a command for this.

42. **[M]** MATLAB gives cond(A) = 23683, which is approximately 10^4. If you make several trials with MATLAB, which records 16 digits accurately, you should find that $\mathbf{x}$ and $\mathbf{x}_1$ agree to at least 12 or 13 significant digits. So about 4 significant digits are lost. Here is the result of one experiment. The vectors were all computed to the maximum 16 decimal places but are here displayed with only four decimal places:

$$\mathbf{x} = \text{rand}(4,1) = \begin{bmatrix} .9501 \\ .21311 \\ .6068 \\ .4860 \end{bmatrix}, \ \mathbf{b} = A\mathbf{x} = \begin{bmatrix} -3.8493 \\ 5.5795 \\ 20.7973 \\ .8467 \end{bmatrix}. \text{ The MATLAB solution is } \mathbf{x}_1 = A\backslash\mathbf{b} = \begin{bmatrix} .9501 \\ .2311 \\ .6068 \\ .4860 \end{bmatrix}.$$

However, $\mathbf{x} - \mathbf{x}_1 = \begin{bmatrix} .0171 \\ .4858 \\ -.2360 \\ .2456 \end{bmatrix} \times 10^{-12}$. The computed solution $\mathbf{x}_1$ is accurate to about 12 decimal places.

43. **[M]** MATLAB gives cond(A) = 68,622. Since this has magnitude between 10^4 and 10^5, the estimated accuracy of a solution of $A\mathbf{x} = \mathbf{b}$ should be to about four or five decimal places *less* than the 16 decimal places that MATLAB usually computes accurately. That is, one should expect the solution to be accurate to only about 11 or 12 decimal places. Here is the result of one experiment. The vectors were all computed to the maximum 16 decimal places but are here displayed with only four decimal places:

$$\mathbf{x} = \text{rand}(5,1) = \begin{bmatrix} .2190 \\ .0470 \\ .6789 \\ .6793 \\ .9347 \end{bmatrix}, \mathbf{b} = A\mathbf{x} = \begin{bmatrix} 15.0821 \\ .8165 \\ 19.0097 \\ -5.8188 \\ 14.5557 \end{bmatrix}. \text{ The MATLAB solution is } \mathbf{x}_1 = A\backslash\mathbf{b} = \begin{bmatrix} .2190 \\ .0470 \\ .6789 \\ .6793 \\ .9347 \end{bmatrix}.$$

However, $\mathbf{x} - \mathbf{x}_1 = \begin{bmatrix} .3165 \\ -.6743 \\ .3343 \\ .0158 \\ -.0005 \end{bmatrix} \times 10^{-11}$. The computed solution $\mathbf{x}_1$ is accurate to about 11 decimal places.

44. **[M]** Solve $A\mathbf{x} = (0, 0, 0, 0, 1)$. MATLAB shows that $\text{cond}(A) \approx 4.8 \times 10^5$. Since MATLAB computes numbers accurately to 16 decimal places, the entries in the computed value of $\mathbf{x}$ should be accurate to at least 11 digits. The exact solution is (630, –12600, 56700, –88200, 44100).

45. **[M]** Some versions of MATLAB issue a warning when asked to invert a Hilbert matrix of order 12 or larger using floating-point arithmetic. The product AA^{-1} should have several off-diagonal entries that are far from being zero. If not, try a larger matrix.

Note: All matrix programs supported by the *Study Guide* have data for Exercise 45, but only MATLAB and Maple have a single command to create a Hilbert matrix. The HP-48G data for Exercise 45 contain a program that can be edited to create other Hilbert matrices.

Notes: The *Study Guide* for Section 2.3 organizes the statements of the Invertible Matrix Theorem in a table that imbeds these ideas in a broader discussion of rectangular matrices. The statements are arranged in three columns: statements that are logically equivalent for any $m \times n$ matrix and are related to existence concepts, those that are equivalent only for any $n \times n$ matrix, and those that are equivalent for any $n \times p$ matrix and are related to uniqueness concepts. Four statements are included that are not in the text's official list of statements, to give more symmetry to the three columns. You may or may not wish to comment on them.

I believe that students cannot fully understand the concepts in the IMT if they do not know the correct wording of each statement. (Of course, this knowledge is not sufficient for understanding.) The *Study Guide*'s Section 2.3 has an example of the type of question I often put on an exam at this point in the course. The section concludes with a discussion of reviewing and reflecting, as important steps to a mastery of linear algebra.

2.4 SOLUTIONS

Notes: Partitioned matrices arise in theoretical discussions in essentially every field that makes use of matrices. The *Study Guide* mentions some examples (with references).

Every student should be exposed to some of the ideas in this section. If time is short, you might omit Example 4 and Theorem 10, and replace Example 5 by a problem similar to one in Exercises 1–10. (A sample replacement is given at the end of these solutions.) Then select homework from Exercises 1–13, 15, and 21–24.

The exercises just mentioned provide a good environment for practicing matrix manipulation. Also, students will be reminded that an equation of the form $AB = I$ does not by itself make A or B invertible. (The matrices must be square and the IMT is required.)

1. Apply the row-column rule as if the matrix entries were numbers, but for each product always write the entry of the left block-matrix on the *left.*

$$\begin{bmatrix} I & 0 \\ E & I \end{bmatrix}\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} IA+0C & IB+0D \\ EA+IC & EB+ID \end{bmatrix} = \begin{bmatrix} A & B \\ EA+C & EB+D \end{bmatrix}$$

2. Apply the row-column rule as if the matrix entries were numbers, but for each product always write the entry of the left block-matrix on the *left.*

$$\begin{bmatrix} E & 0 \\ 0 & F \end{bmatrix}\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} EA+0C & EB+0D \\ 0A+FC & 0B+FD \end{bmatrix} = \begin{bmatrix} EA & EB \\ FC & FD \end{bmatrix}$$

3. Apply the row-column rule as if the matrix entries were numbers, but for each product always write the entry of the left block-matrix on the *left.*

$$\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}\begin{bmatrix} W & X \\ Y & Z \end{bmatrix} = \begin{bmatrix} 0W+IY & 0X+IZ \\ IW+0Y & IX+0Z \end{bmatrix} = \begin{bmatrix} Y & Z \\ W & X \end{bmatrix}$$

4. Apply the row-column rule as if the matrix entries were numbers, but for each product always write the entry of the left block-matrix on the *left.*

$$\begin{bmatrix} I & 0 \\ -X & I \end{bmatrix}\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} IA+0C & IB+0D \\ -XA+IC & -XB+ID \end{bmatrix} = \begin{bmatrix} A & B \\ -XA+C & -XB+D \end{bmatrix}$$

5. Compute the left side of the equation:

$$\begin{bmatrix} A & B \\ C & 0 \end{bmatrix}\begin{bmatrix} I & 0 \\ X & Y \end{bmatrix} = \begin{bmatrix} AI+BX & A0+BY \\ CI+0X & C0+0Y \end{bmatrix}$$

Set this equal to the right side of the equation:

$$\begin{bmatrix} A+BX & BY \\ C & 0 \end{bmatrix} = \begin{bmatrix} 0 & I \\ Z & 0 \end{bmatrix} \quad \text{so that} \quad \begin{matrix} A+BX=0 & BY=I \\ C=Z & 0=0 \end{matrix}$$

Since the (2, 1) blocks are equal, $Z = C$. Since the (1, 2) blocks are equal, $BY = I$. To proceed further, assume that B and Y are square. Then the equation $BY = I$ implies that B is invertible, by the IMT, and $Y = B^{-1}$. (See the boxed remark that follows the IMT.) Finally, from the equality of the (1, 1) blocks,

$$BX = -A, \quad B^{-1}BX = B^{-1}(-A), \quad \text{and} \quad X = -B^{-1}A.$$

The order of the factors for X is crucial.

Note: For simplicity, statements (j) and (k) in the Invertible Matrix Theorem involve square matrices C and D. Actually, if A is $n\times n$ and if C is any matrix such that AC is the $n\times n$ identity matrix, then C must be $n\times n$, too. (For AC to be defined, C must have n rows, and the equation $AC = I$ implies that C has n columns.) Similarly, $DA = I$ implies that D is $n\times n$. Rather than discuss this in class, I expect that in Exercises 5–8, when students see an equation such as $BY = I$, they will decide that *both* B and Y should be square in order to use the IMT.

6. Compute the left side of the equation:

$$\begin{bmatrix} X & 0 \\ Y & Z \end{bmatrix}\begin{bmatrix} A & 0 \\ B & C \end{bmatrix} = \begin{bmatrix} XA+0B & X0+0C \\ YA+ZB & Y0+ZC \end{bmatrix} = \begin{bmatrix} XA & 0 \\ YA+ZB & ZC \end{bmatrix}$$

Set this equal to the right side of the equation:

$$\begin{bmatrix} XA & 0 \\ YA+ZB & ZC \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \quad \text{so that} \quad \begin{matrix} XA = I & 0 = 0 \\ YA+ZB = 0 & ZC = I \end{matrix}$$

To use the equality of the (1, 1) blocks, assume that A and X are square. By the IMT, the equation $XA = I$ implies that A is invertible and $X = A^{-1}$. (See the boxed remark that follows the IMT.) Similarly, if C and Z are assumed to be square, then the equation $ZC = I$ implies that C is invertible, by the IMT, and $Z = C^{-1}$. Finally, use the (2, 1) blocks and right-multiplication by A^{-1}:

$$YA = -ZB = -C^{-1}B, \quad YAA^{-1} = (-C^{-1}B)A^{-1}, \quad \text{and} \quad Y = -C^{-1}BA^{-1}$$

The order of the factors for Y is crucial.

7. Compute the left side of the equation:

$$\begin{bmatrix} X & 0 & 0 \\ Y & 0 & I \end{bmatrix}\begin{bmatrix} A & Z \\ 0 & 0 \\ B & I \end{bmatrix} = \begin{bmatrix} XA+0+0B & XZ+0+0I \\ YA+0+IB & YZ+0+II \end{bmatrix}$$

Set this equal to the right side of the equation:

$$\begin{bmatrix} XA & XZ \\ YA+B & YZ+I \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \quad \text{so that} \quad \begin{matrix} XA = I & XZ = 0 \\ YA+B = 0 & YZ+I = I \end{matrix}$$

To use the equality of the (1, 1) blocks, assume that A and X are square. By the IMT, the equation $XA = I$ implies that A is invertible and $X = A^{-1}$. (See the boxed remark that follows the IMT) Also, X is invertible. Since $XZ = 0$, $X^{-1}XZ = X^{-1}0 = 0$, so Z must be 0. Finally, from the equality of the (2, 1) blocks, $YA = -B$. Right-multiplication by A^{-1} shows that $YAA^{-1} = -BA^{-1}$ and $Y = -BA^{-1}$. The order of the factors for Y is crucial.

8. Compute the left side of the equation:

$$\begin{bmatrix} A & B \\ 0 & I \end{bmatrix}\begin{bmatrix} X & Y & Z \\ 0 & 0 & I \end{bmatrix} = \begin{bmatrix} AX+B0 & AY+B0 & AZ+BI \\ 0X+I0 & 0Y+I0 & 0Z+II \end{bmatrix}$$

Set this equal to the right side of the equation:

$$\begin{bmatrix} AX & AY & AZ+B \\ 0 & 0 & I \end{bmatrix} = \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & I \end{bmatrix}$$

To use the equality of the (1, 1) blocks, assume that A and X are square. By the IMT, the equation $XA = I$ implies that A is invertible and $X = A^{-1}$. (See the boxed remark that follows the IMT. Since $AY = 0$, from the equality of the (1, 2) blocks, left-multiplication by A^{-1} gives $A^{-1}AY = A^{-1}0 = 0$, so $Y = 0$. Finally, from the (1, 3) blocks, $AZ = -B$. Left-multiplication by A^{-1} gives $A^{-1}AZ = A^{-1}(-B)$, and $Z = -A^{-1}B$. The order of the factors for Z is crucial.

Note: The *Study Guide* tells students, "Problems such as 5–10 make good exam questions. Remember to mention the IMT when appropriate, and remember that matrix multiplication is generally not commutative." When a problem statement includes a condition that a matrix is square, I expect my students to mention this fact when they apply the IMT.

9. Compute the left side of the equation:

$$\begin{bmatrix} I & 0 & 0 \\ X & I & 0 \\ Y & 0 & I \end{bmatrix}\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \\ A_{31} & A_{32} \end{bmatrix} = \begin{bmatrix} IA_{11}+0A_{21}+0A_{31} & IA_{12}+0A_{22}+0A_{32} \\ XA_{11}+IA_{21}+0A_{31} & XA_{12}+IA_{22}+0A_{32} \\ YA_{11}+0A_{21}+IA_{31} & YA_{12}+0A_{22}+IA_{32} \end{bmatrix}$$

Set this equal to the right side of the equation:

$$\begin{bmatrix} A_{11} & A_{12} \\ XA_{11}+A_{21} & XA_{12}+A_{22} \\ YA_{11}+A_{31} & YA_{12}+A_{32} \end{bmatrix} = \begin{bmatrix} B_{11} & B_{12} \\ 0 & B_{22} \\ 0 & B_{32} \end{bmatrix}$$

$$\begin{aligned} & A_{11} = B_{11} && A_{12} = B_{12} \\ \text{so that } & XA_{11}+A_{21} = 0 && XA_{12}+A_{22} = B_{22} \\ & YA_{11}+A_{31} = 0 && YA_{12}+A_{32} = B_{32} \end{aligned}$$

Since the (2,1) blocks are equal, $XA_{11}+A_{21}=0$ and $XA_{11}=-A_{21}$. Since A_{11} is invertible, right multiplication by A_{11}^{-1} gives $X=-A_{21}A_{11}^{-1}$. Likewise since the (3,1) blocks are equal, $YA_{11}+A_{31}=0$ and $YA_{11}=-A_{31}$. Since A_{11} is invertible, right multiplication by A_{11}^{-1} gives $Y=-A_{31}A_{11}^{-1}$. Finally, from the (2,2) entries, $XA_{12}+A_{22}=B_{22}$. Since $X=-A_{21}A_{11}^{-1}, B_{22}=-A_{21}A_{11}^{-1}A_{12}+A_{22}$.

10. Since the two matrices are inverses,

$$\begin{bmatrix} I & 0 & 0 \\ C & I & 0 \\ A & B & I \end{bmatrix}\begin{bmatrix} T & 0 & 0 \\ Z & I & 0 \\ X & Y & I \end{bmatrix} = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}$$

Compute the left side of the equation:

$$\begin{bmatrix} I & 0 & 0 \\ C & I & 0 \\ A & B & I \end{bmatrix}\begin{bmatrix} I & 0 & 0 \\ Z & I & 0 \\ X & Y & I \end{bmatrix} = \begin{bmatrix} II+0Z+0X & I0+0I+0Y & I0+00+0I \\ CI+IZ+0X & C0+II+0Y & C0+I0+0I \\ AI+BZ+IX & A0+BI+IY & A0+B0+II \end{bmatrix}$$

Set this equal to the right side of the equation:

$$\begin{bmatrix} I & 0 & 0 \\ C+Z & I & 0 \\ A+BZ+X & B+Y & I \end{bmatrix} = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}$$

$$\begin{aligned} & I = I && 0=0 && 0=0 \\ \text{so that } & C+Z=0 && I=I && 0=0 \\ & A+BZ+X=0 && B+Y=0 && I=I \end{aligned}$$

Since the (2,1) blocks are equal, $C+Z=0$ and $Z=-C$. Likewise since the (3, 2) blocks are equal, $B+Y=0$ and $Y=-B$. Finally, from the (3,1) entries, $A+BZ+X=0$ and $X=-A-BZ$. Since $Z=-C,\ X=-A-B(-C)=-A+BC$.

11. **a**. True. See the subsection *Addition and Scalar Multiplication*.
b. False. See the paragraph before Example 3.

12. **a**. True. See the paragraph before Example 4.
b. False. See the paragraph before Example 3.

13. You are asked to establish an *if and only if* statement. First, supose that A is invertible, and let $A^{-1} = \begin{bmatrix} D & E \\ F & G \end{bmatrix}$. Then

$$\begin{bmatrix} B & 0 \\ 0 & C \end{bmatrix}\begin{bmatrix} D & E \\ F & G \end{bmatrix} = \begin{bmatrix} BD & BE \\ CF & CG \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

Since B is square, the equation $BD = I$ implies that B is invertible, by the IMT. Similarly, $CG = I$ implies that C is invertible. Also, the equation $BE = 0$ imples that $E = B^{-1}0 = 0$. Similarly $F = 0$. Thus

$$A^{-1} = \begin{bmatrix} B & 0 \\ 0 & C \end{bmatrix}^{-1} = \begin{bmatrix} D & E \\ E & G \end{bmatrix} = \begin{bmatrix} B^{-1} & 0 \\ 0 & C^{-1} \end{bmatrix} \qquad (*)$$

This proves that A is invertible *only if* B and C are invertible. For the "*if*" part of the statement, suppose that B and C are invertible. Then (*) provides a likely candidate for A^{-1} which can be used to show that A is invertible. Compute:

$$\begin{bmatrix} B & 0 \\ 0 & C \end{bmatrix}\begin{bmatrix} B^{-1} & 0 \\ 0 & C^{-1} \end{bmatrix} = \begin{bmatrix} BB^{-1} & 0 \\ 0 & CC^{-1} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

Since A is square, this calculation and the IMT imply that A is invertible. (Don't forget this final sentence. Without it, the argument is incomplete.) Instead of that sentence, you could add the equation:

$$\begin{bmatrix} B^{-1} & 0 \\ 0 & C^{-1} \end{bmatrix}\begin{bmatrix} B & 0 \\ 0 & C \end{bmatrix} = \begin{bmatrix} B^{-1}B & 0 \\ 0 & C^{-1}C \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

14. You are asked to establish an *if and only if* statement. First suppose that A is invertible. Example 5 shows that A_{11} and A_{22} are invertible. This proves that A is invertible *only if* A_{11} A_{22} are invertible. For the *if* part of this statement, suppose that A_{11} and A_{22} are invertible. Then the formula in Example 5 provides a likely candidate for A^{-1} which can be used to show that A is invertible . Compute:

$$\begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}\begin{bmatrix} A_{11}^{-1} & -A_{11}^{-1}A_{12}A_{22}^{-1} \\ 0 & A_{22}^{-1} \end{bmatrix} = \begin{bmatrix} A_{11}A_{11}^{-1} + A_{12}0 & A_{11}(-A_{11}^{-1})A_{12}A_{22}^{-1} + A_{12}A_{22}^{-1} \\ 0A_{11}^{-1} + A_{22}0 & 0(-A_{11}^{-1})A_{12}A_{22}^{-1} + A_{22}A_{22}^{-1} \end{bmatrix}$$

$$= \begin{bmatrix} I & -(A_{11}A_{11}^{-1})A_{12}A_{22}^{-1} + A_{12}A_{22}^{-1} \\ 0 & I \end{bmatrix}$$

$$= \begin{bmatrix} I & -A_{12}A_{22}^{-1} + A_{12}A_{22}^{-1} \\ 0 & I \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

Since A is square, this calculation and the IMT imply that A is invertible.

15. Compute the right side of the equation:

$$\begin{bmatrix} I & 0 \\ X & I \end{bmatrix}\begin{bmatrix} A_{11} & 0 \\ 0 & S \end{bmatrix}\begin{bmatrix} I & Y \\ 0 & I \end{bmatrix} = \begin{bmatrix} A_{11} & 0 \\ XA_{11} & S \end{bmatrix}\begin{bmatrix} I & Y \\ 0 & I \end{bmatrix} = \begin{bmatrix} A_{11} & A_{11}Y \\ XA_{11} & XA_{11}Y + S \end{bmatrix}$$

Set this equal to the left side of the equation:

$$\begin{bmatrix} A_{11} & A_{11}Y \\ XA_{11} & XA_{11}Y + S \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \text{ so that } \begin{matrix} A_{11} = A_{11} & A_{11}Y = A_{12} \\ XA_{11} = A_{21} & XA_{11}Y + S = A_{22} \end{matrix}$$

Since the (1, 2) blocks are equal, $A_{11}Y = A_{12}$. Since A_{11} is invertible, left multiplication by A_{11}^{-1} gives $Y = A_{11}^{-1}A_{12}$. Likewise since the (2,1) blocks are equal, $XA_{11} = A_{21}$. Since A_{11} is invertible, right multiplication by A_{11}^{-1} gives that $X = A_{21}A_{11}^{-1}$. One can check that the matrix S as given in the exercise satisfies the equation $XA_{11}Y + S = A_{22}$ with the calculated values of X and Y given above.

16. Suppose that A and A_{11} are invertible. First note that

$$\begin{bmatrix} I & 0 \\ X & I \end{bmatrix}\begin{bmatrix} I & 0 \\ -X & I \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

and

$$\begin{bmatrix} I & Y \\ 0 & I \end{bmatrix}\begin{bmatrix} I & -Y \\ 0 & I \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

Since the matrices $\begin{bmatrix} I & 0 \\ X & I \end{bmatrix}$ and $\begin{bmatrix} I & Y \\ 0 & I \end{bmatrix}$

are square, they are both invertible by the IMT. Equation (7) may be left multipled by $\begin{bmatrix} I & 0 \\ X & I \end{bmatrix}^{-1}$ and right multipled by $\begin{bmatrix} I & Y \\ 0 & I \end{bmatrix}^{-1}$ to find

$$\begin{bmatrix} A_{11} & 0 \\ 0 & S \end{bmatrix} = \begin{bmatrix} I & 0 \\ X & I \end{bmatrix}^{-1} A \begin{bmatrix} I & Y \\ 0 & I \end{bmatrix}^{-1}$$

Thus by Theorem 6, the matrix $\begin{bmatrix} A_{11} & 0 \\ 0 & S \end{bmatrix}$ is invertible as the product of invertible matrices. Finally, Exercise 13 above may be used to show that S is invertible.

17. The column-row expansions of G_k and G_{k+1} are:

$$\begin{aligned} G_k &= X_k X_k^T \\ &= \text{col}_1(X_k)\text{row}_1(X_k^T) + \cdots + \text{col}_k(X_k)\text{row}_k(X_k^T) \end{aligned}$$

and

$$\begin{aligned} G_{k+1} &= X_{k+1}X_{k+1}^T \\ &= \text{col}_1(X_{k+1})\text{row}_1(X_{k+1}^T) + \cdots + \text{col}_k(X_{k+1})\text{row}_k(X_{k+1}^T) + \text{col}_{k+1}(X_{k+1})\text{row}_{k+1}(X_{k+1}^T) \\ &= \text{col}_1(X_k)\text{row}_1(X_k^T) + \cdots + \text{col}_k(X_k)\text{row}_k(X_k^T) + \text{col}_{k+1}(X_{k+1})\text{row}_{k+1}(X_k^T) \\ &= G_k + \text{col}_{k+1}(X_{k+1})\text{row}_{k+1}(X_k^T) \end{aligned}$$

since the first k columns of X_{k+1} are identical to the first k columns of X_k. Thus to update G_k to produce G_{k+1}, the number $\text{col}_{k+1}(X_{k+1})\ \text{row}_{k+1}\ (X_k^T)$ should be added to G_k.

18. Since $W = [X\ \mathbf{x}_0]$,

$$W^TW = \begin{bmatrix} X^T \\ \mathbf{x}_0^T \end{bmatrix}[X\ \ \mathbf{x}_0] = \begin{bmatrix} X^TX & X^T\mathbf{x}_0 \\ \mathbf{x}_0^TX & \mathbf{x}_0^T\mathbf{x}_0 \end{bmatrix}$$

By applying the formula for S from Exercise 15, S may be computed:

$$\begin{aligned} S &= \mathbf{x}_0^T\mathbf{x}_0 - \mathbf{x}_0^TX(X^TX)^{-1}X^T\mathbf{x}_0 \\ &= \mathbf{x}_0^T(I_m - X(X^TX)^{-1}X^T)\mathbf{x}_0 \\ &= \mathbf{x}_0^TM\mathbf{x}_0 \end{aligned}$$

19. The matrix equation (8) in the text is equivalent to

$$(A - sI_n)\mathbf{x} + B\mathbf{u} = 0 \text{ and } C\mathbf{x} + \mathbf{u} = \mathbf{y}$$

Rewrite the first equation as $(A - sI_n)\mathbf{x} = -B\mathbf{u}$. When $A - sI_n$ is invertible,

$$\mathbf{x} = (A - sI_n)^{-1}(-B\mathbf{u}) = -(A - sI_n)^{-1}B\mathbf{u}$$

Substitute this formula for **x** into the second equation above:

$$C(-(A - sI_n)^{-1}B\mathbf{u}) + \mathbf{u} = \mathbf{y}, \text{ so that } I_m\mathbf{u} - C(A - sI_n)^{-1}B\mathbf{u} = \mathbf{y}$$

Thus $\mathbf{y} = (I_m - C(A - sI_n)^{-1}B)\mathbf{u}$. If $W(s) = I_m - C(A - sI_n)^{-1}B$, then $\mathbf{y} = W(s)\mathbf{u}$. The matrix W(s) is the Schur complement of the matrix $A - sI_n$ in the system matrix in equation (8)

20. The matrix in question is

$$\begin{bmatrix} A - BC - sI_n & B \\ -C & I_m \end{bmatrix}$$

By applying the formula for S from Exercise 15, S may be computed:

$$\begin{aligned} S &= I_m - (-C)(A - BC - sI_m)^{-1}B \\ &= I_m + C(A - BC - sI_m)^{-1}B \end{aligned}$$

21. **a**. $A^2 = \begin{bmatrix} 1 & 0 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} 1+0 & 0+0 \\ 3-3 & 0+(-1)^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

b. $M^2 = \begin{bmatrix} A & 0 \\ I & -A \end{bmatrix}\begin{bmatrix} A & 0 \\ I & -A \end{bmatrix} = \begin{bmatrix} A^2+0 & 0+0 \\ A-A & 0+(-A)^2 \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$

22. Let C be any nonzero 2×3 matrix. Define $A = \begin{bmatrix} I_3 & 0 \\ C & -I_2 \end{bmatrix}$. Then

$$A^2 = \begin{bmatrix} I_3 & 0 \\ C & -I_2 \end{bmatrix}\begin{bmatrix} I_3 & 0 \\ C & -I_2 \end{bmatrix} = \begin{bmatrix} I_3+0 & 0+0 \\ CI_3 - I_2C & 0+(-I_2)^2 \end{bmatrix} = \begin{bmatrix} I_3 & 0 \\ 0 & I_2 \end{bmatrix}$$

23. The product of two 1×1 "lower triangular" matrices is "lower triangular." Suppose that for $n = k$, the product of two $k\times k$ lower triangular matrices is lower triangular, and consider any $(k+1)\times(k+1)$ matrices A_1 and B_1. Partition these matrices as

$$A_1 = \begin{bmatrix} a & \mathbf{0}^T \\ \mathbf{v} & A \end{bmatrix}, \quad B_1 = \begin{bmatrix} b & \mathbf{0}^T \\ \mathbf{w} & B \end{bmatrix}$$

where A and B are $k\times k$ matrices, **v** and **w** are in $\mathbf{R}^k$, and a and b are scalars. Since A_1 and B_1 are lower triangular, so are A and B. Then

$$A_1B_1 = \begin{bmatrix} a & \mathbf{0}^T \\ \mathbf{v} & A \end{bmatrix}\begin{bmatrix} b & \mathbf{0}^T \\ \mathbf{w} & B \end{bmatrix} = \begin{bmatrix} ab + \mathbf{0}^T\mathbf{w} & a\mathbf{0}^T + \mathbf{0}^TB \\ \mathbf{v}b + A\mathbf{w} & \mathbf{v0}^T + AB \end{bmatrix} = \begin{bmatrix} ab & \mathbf{0}^T \\ b\mathbf{v} + A\mathbf{w} & AB \end{bmatrix}$$

Since A and B are $k\times k$, AB is lower triangular. The form of A_1B_1 shows that it, too, is lower triangular. Thus the statement about lower triangular matrices is true for $n = k+1$ if it is true for $n = k$. By the principle of induction, the statement is true for all $n \geq 1$.

Note: Exercise 23 is good for mathematics and computer science students. The solution of Exercise 23 in the *Study Guide* shows students how to use the principle of induction. The *Study Guide* also has an appendix on "The Principle of Induction," at the end of Section 2.4. The text presents more applications of induction in Section 3.2 and in the Supplementary Exercises for Chapter 3.

24. Let $A_n = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & & 0 \\ 1 & 1 & 1 & & 0 \\ \vdots & & & \ddots & \\ 1 & 1 & 1 & \cdots & 1 \end{bmatrix}$, $B_n = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ -1 & 1 & 0 & & 0 \\ 0 & -1 & 1 & & 0 \\ \vdots & & \ddots & \ddots & \\ 0 & & \cdots & -1 & 1 \end{bmatrix}$.

By direct computation $A_2B_2 = I_2$. Assume that for $n = k$, the matrix A_kB_k is I_k, and write

$$A_{k+1} = \begin{bmatrix} 1 & \mathbf{0}^T \\ \mathbf{v} & A_k \end{bmatrix} \quad \text{and} \quad B_{k+1} = \begin{bmatrix} 1 & \mathbf{0}^T \\ \mathbf{w} & B_k \end{bmatrix}$$

where $\mathbf{v}$ and $\mathbf{w}$ are in $\mathbf{R}^k$, $\mathbf{v}^T = [1 \;\; 1 \;\; \cdots \;\; 1]$, and $\mathbf{w}^T = [-1 \;\; 0 \;\; \cdots \;\; 0]$. Then

$$A_{k+1}B_{k+1} = \begin{bmatrix} 1 & \mathbf{0}^T \\ \mathbf{v} & A_k \end{bmatrix}\begin{bmatrix} 1 & \mathbf{0}^T \\ \mathbf{w} & B_k \end{bmatrix} = \begin{bmatrix} 1+\mathbf{0}^T\mathbf{w} & \mathbf{0}^T + \mathbf{0}^T B_k \\ \mathbf{v} + A_k\mathbf{w} & \mathbf{v}\mathbf{0}^T + A_kB_k \end{bmatrix} = \begin{bmatrix} 1 & \mathbf{0}^T \\ \mathbf{0} & I_k \end{bmatrix} = I_{k+1}$$

The (2,1)-entry is $\mathbf{0}$ because $\mathbf{v}$ equals the first column of A_k., and $A_k\mathbf{w}$ is –1 times the first column of A_k. By the principle of induction, $A_nB_n = I_n$ for all $n \geq 2$. Since A_n and B_n are square, the IMT shows that these matrices are invertible, and $B_n = A_n^{-1}$.

Note: An induction proof can also be given using partitions with the form shown below. The details are slightly more complicated.

$$A_{k+1} = \begin{bmatrix} A_k & \mathbf{0} \\ \mathbf{v}^T & 1 \end{bmatrix} \quad \text{and} \quad B_{k+1} = \begin{bmatrix} B_k & \mathbf{0} \\ \mathbf{w}^T & 1 \end{bmatrix}$$

$$A_{k+1}B_{k+1} = \begin{bmatrix} A_k & \mathbf{0} \\ \mathbf{v}^T & 1 \end{bmatrix}\begin{bmatrix} B_k & \mathbf{0} \\ \mathbf{w}^T & 1 \end{bmatrix} = \begin{bmatrix} A_kB_k + \mathbf{0}\mathbf{w}^T & A^k\mathbf{0} + \mathbf{0} \\ \mathbf{v}^T B_k + \mathbf{w}^T & \mathbf{v}^T 0 + 1 \end{bmatrix} = \begin{bmatrix} I_k & \mathbf{0} \\ \mathbf{0}^{\mathbf{T}} & 1 \end{bmatrix} = I_{k+1}$$

The (2,1)-entry is $\mathbf{0}^T$ because $\mathbf{v}^T$ times a column of B_k equals the sum of the entries in the column, and all of such sums are zero except the last, which is 1. So $\mathbf{v}^TB_k$ is the negative of $\mathbf{w}^T$. By the principle of induction, $A_nB_n = I_n$ for all $n \geq 2$. Since A_n and B_n are square, the IMT shows that these matrices are invertible, and $B_n = A_n^{-1}$.

25. First, visualize a partition of A as a 2×2 block–diagonal matrix, as below, and then visualize the (2,2)-block itself as a block-diagonal matrix. That is,

$$A = \left[\begin{array}{cc|ccc} 1 & 2 & 0 & 0 & 0 \\ 3 & 5 & 0 & 0 & 0 \\ \hline 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 7 & 8 \\ 0 & 0 & 0 & 5 & 6 \end{array}\right] = \left[\begin{array}{c|c} A_{11} & 0 \\ \hline 0 & A_{22} \end{array}\right], \text{ where } A_{22} = \left[\begin{array}{c|cc} 2 & 0 & 0 \\ \hline 0 & 7 & 8 \\ 0 & 5 & 6 \end{array}\right] = \begin{bmatrix} 2 & 0 \\ 0 & B \end{bmatrix}$$

Observe that B is invertible and $B^{-1} = \begin{bmatrix} 3 & -4 \\ -2.5 & 3.5 \end{bmatrix}$. By Exercise 13, the block diagonal matrix A_{22} is invertible, and

$$A_{22}^{-1} = \left[\begin{array}{c|cc} .5 & \multicolumn{2}{c}{0} \\ \hline 0 & 3 & -4 \\ & -2.5 & 3.5 \end{array}\right] = \begin{bmatrix} .5 & 0 & 0 \\ 0 & 3 & -4 \\ 0 & -2.5 & 3.5 \end{bmatrix}$$

Next, observe that A_{11} is also invertible, with inverse $\begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix}$. By Exercise 13, A itself is invertible, and its inverse is block diagonal:

$$A^{-1} = \begin{bmatrix} A_{11}^{-1} & 0 \\ 0 & A_{22}^{-1} \end{bmatrix} = \left[\begin{array}{cc|ccc} -5 & 2 & & 0 & \\ 3 & -1 & & & \\ \hline & & .5 & 0 & 0 \\ & 0 & 0 & 3 & -4 \\ & & 0 & -2.5 & 3.5 \end{array}\right] = \begin{bmatrix} -5 & 2 & 0 & 0 & 0 \\ 3 & -1 & 0 & 0 & 0 \\ 0 & 0 & .5 & 0 & 0 \\ 0 & 0 & 0 & 3 & -4 \\ 0 & 0 & 0 & -2.5 & 3.5 \end{bmatrix}$$

26. **[M]** This exercise and the next, which involve large matrices, are more appropriate for MATLAB, Maple, and Mathematica, than for the graphic calculators.

a. Display the submatrix of A obtained from rows 15 to 20 and columns 5 to 10.

MATLAB: `A(15:20, 5:10)`

Maple: `submatrix(A, 15..20, 5..10)`

Mathematica: `Take[ A, {15,20}, {5,10} ]`

b. Insert a 5×10 matrix B into rows 10 to 14 and columns 20 to 29 of matrix A:

MATLAB: `A(10:14, 20:29) = B ;` The semicolon suppresses output display.

Maple: `copyinto(B, A, 10, 20):` The colon suppresses output display.

Mathematica:
```
For [ i=10, i<=14, i++,
  For [ j=20, j<=29, j++,
    A [[ i,j ]] = B [[ i-9, j-19 ]] ] ];
```
Colon suppresses output.

c. To create $B = \begin{bmatrix} A & 0 \\ 0 & A^T \end{bmatrix}$ with MATLAB, build B out of four blocks:

```
B = [A zeros(30,20); zeros(20,30) A'];
```

Another method: first enter `B = A ;` and then enlarge B with the command

```
B(21:50, 31:50) = A';
```

This places A^T in the (2, 2) block of the larger B and fills in the (1, 2) and (2, 1) blocks with zeros.

For Maple:

```
B := matrix(50,50,0):
copyinto(A, B, 1, 1):
copyinto( transpose(A), B, 21, 31):
```

For Mathematica:

```
B = BlockMatrix[ {{A, ZeroMatrix[30,20]}, ZeroMatrix[20,30],
Transpose[A]}} ]
```

27. **a**. **[M]** Construct A from four blocks, say C_{11}, C_{12}, C_{21}, and C_{22}, for example with C_{11} a 30×30 matrix and C_{22} a 20×20 matrix.

MATLAB:

```
C11 = A(1:30, 1:30) + B(1:30, 1:30)
C12 = A(1:30, 31:50) + B(1:30, 31:50)
C21 = A(31:50, 1:30)+ B(31:50, 1:30)
C22 = A(31:50, 31:50) + B(31:50, 31:50)
C = [C11 C12; C21 C22]
```

The commands in Maple and Mathematica are analogous, but with different syntax. The first commands are:

Maple: `C11 := submatrix(A, 1..30, 1..30) + submatrix(B, 1..30, 1..30)`

Mathematica: `c11 := Take[ A, {1,30), {1,30} ] + Take[B, {1,30), {1,30} ]`

b. The algebra needed comes from block matrix multiplication:

$$AB = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} = \begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{bmatrix}$$

Partition both A and B, for example with 30×30 (1, 1) blocks and 20×20 (2, 2) blocks. The four necessary submatrix computations use syntax analogous to that shown for (a).

c. The algebra needed comes from the block matrix equation $\begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix}\begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \end{bmatrix}$, where $\mathbf{x}_1$ and $\mathbf{b}_1$ are in $\mathbf{R}^{30}$ and $\mathbf{x}_2$ and $\mathbf{b}_2$ are in $\mathbf{R}^{20}$. Then $A_{1\,1}\mathbf{x}_1 = \mathbf{b}_1$, which can be solved to produce $\mathbf{x}_1$. Once $\mathbf{x}_1$ is found, rewrite the equation $A_{21}\mathbf{x}_1 + A_{22}\mathbf{x}_2 = \mathbf{b}_2$ as $A_{22}\mathbf{x}_2 = \mathbf{c}$, where $\mathbf{c} = \mathbf{b}_2 - A_{21}\mathbf{x}_1$, and solve $A_{22}\mathbf{x}_2 = \mathbf{c}$ for $\mathbf{x}_2$.

Notes: The following may be used in place of Example 5:

Example 5: Use equation (*) to find formulas for X, Y, and Z in terms of A, B, and C. Mention any assumptions you make in order to produce the formulas.

$$\begin{bmatrix} X & 0 \\ Y & Z \end{bmatrix}\begin{bmatrix} I & 0 \\ A & B \end{bmatrix} = \begin{bmatrix} I & 0 \\ C & I \end{bmatrix} \qquad (*)$$

Solution:

This matrix equation provides four equations that can be used to find X, Y, and Z:

$X + 0 = I,$ $\qquad 0 = 0$

$YI + ZA = C,$ $\qquad Y0 + ZB = I$ $\qquad$ (Note the order of the factors.)

The first equation says that $X = I$. To solve the fourth equation, $ZB = I$, assume that B and Z are square. In this case, the equation $ZB = I$ implies that B and Z are invertible, by the IMT. (Actually, it suffices to assume either that B is square or that Z is square.) Then, right-multiply each side of $ZB = I$ to get $ZBB^{-1} = IB^{-1}$ and $Z = B^{-1}$. Finally, the third equation is $Y + ZA = C$. So, $Y + B^{-1}A = C$, and $Y = C - B^{-1}A$.

The following counterexample shows that Z need not be square for the equation (*) above to be true.

$$\left[\begin{array}{cc|ccc} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \hline 1 & 2 & 1 & 3 & 1 \\ 3 & 4 & 1 & 0 & -1 \end{array}\right]\left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 1 & 1 & 2 & 5 \\ 1 & 1 & -1 & -3 \\ 1 & -1 & 2 & 4 \end{array}\right] = \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 6 & 5 & 1 & 0 \\ 3 & 6 & 0 & 1 \end{array}\right]$$

Note that Z is not determined by A, B, and C, when B is not square. For instance, another Z that works in this counterexample is $Z = \begin{bmatrix} 3 & 5 & 0 \\ -1 & -2 & 0 \end{bmatrix}$.

2.5 SOLUTIONS

Notes: Modern algorithms in numerical linear algebra are often described using matrix factorizations. For practical work, this section is more important than Sections 4.7 and 5.4, even though matrix factorizations are explained nicely in terms of change of bases. Computational exercises in this section emphasize the use of the LU factorization to solve linear systems. The LU factorization is performed using the algorithm explained in the paragraphs before Example 2, and performed in Example 2. The text discusses how to build L when no interchanges are needed to reduce the given matrix to U. An appendix in the *Study Guide* discusses how to build L in permuted unit lower triangular form when row interchanges are needed. Other factorizations are introduced in Exercises 22–26.

1. $L = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & -5 & 1 \end{bmatrix}, U = \begin{bmatrix} 3 & -7 & -2 \\ 0 & -2 & -1 \\ 0 & 0 & -1 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} -7 \\ 5 \\ 2 \end{bmatrix}$. First, solve $L\mathbf{y} = \mathbf{b}$.

$$[L \;\; \mathbf{b}] = \begin{bmatrix} 1 & 0 & 0 & -7 \\ -1 & 1 & 0 & 5 \\ 2 & -5 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -7 \\ 0 & 1 & 0 & -2 \\ 0 & -5 & 1 & 16 \end{bmatrix} \text{ The only arithmetic is in column 4}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & -7 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 6 \end{bmatrix}, \text{ so } \mathbf{y} = \begin{bmatrix} -7 \\ -2 \\ 6 \end{bmatrix}.$$

Next, solve $U\mathbf{x} = \mathbf{y}$, using back-substitution (with matrix notation).

$$[U \;\; \mathbf{y}] = \begin{bmatrix} 3 & -7 & -2 & -7 \\ 0 & -2 & -1 & -2 \\ 0 & 0 & -1 & 6 \end{bmatrix} \sim \begin{bmatrix} 3 & -7 & -2 & -7 \\ 0 & -2 & -1 & -2 \\ 0 & 0 & 1 & -6 \end{bmatrix} \sim \begin{bmatrix} 3 & -7 & 0 & -19 \\ 0 & -2 & 0 & -8 \\ 0 & 0 & 1 & -6 \end{bmatrix}$$

$$\sim \begin{bmatrix} 3 & -7 & 0 & -19 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -6 \end{bmatrix} \sim \begin{bmatrix} 3 & 0 & 0 & 9 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -6 \end{bmatrix}$$

So $\mathbf{x} = (3, 4, -6)$.

To confirm this result, row reduce the matrix $[A \;\; \mathbf{b}]$:

$$[A \;\; \mathbf{b}] = \begin{bmatrix} 3 & -7 & -2 & -7 \\ -3 & 5 & 1 & 5 \\ 6 & -4 & 0 & 2 \end{bmatrix} \sim \begin{bmatrix} 3 & -7 & -2 & -7 \\ 0 & -2 & -1 & -2 \\ 0 & 10 & 4 & 16 \end{bmatrix} \sim \begin{bmatrix} 3 & -7 & -2 & -7 \\ 0 & -2 & -1 & -2 \\ 0 & 0 & -1 & 6 \end{bmatrix}$$

From this point the row reduction follows that of $[U \;\; \mathbf{y}]$ above, yielding the same result.

2. $L=\begin{bmatrix}1&0&0\\-1&1&0\\2&0&1\end{bmatrix}, U=\begin{bmatrix}4&3&-5\\0&-2&2\\0&0&2\end{bmatrix}, \mathbf{b}=\begin{bmatrix}2\\-4\\6\end{bmatrix}$. First, solve $L\mathbf{y}=\mathbf{b}$:

$$[L\ \ \mathbf{b}]=\begin{bmatrix}1&0&0&2\\-1&1&0&-4\\2&0&1&6\end{bmatrix}\sim\begin{bmatrix}1&0&0&2\\0&1&0&-2\\0&0&1&2\end{bmatrix},$$

so $\mathbf{y}=\begin{bmatrix}2\\-2\\2\end{bmatrix}$.

Next solve $U\mathbf{x}=\mathbf{y}$, using back-substitution (with matrix notation):

$$[U\ \ \mathbf{y}]=\begin{bmatrix}4&3&-5&2\\0&-2&2&-2\\0&0&2&2\end{bmatrix}\sim\begin{bmatrix}4&3&-5&2\\0&-2&2&-2\\0&0&1&1\end{bmatrix}\sim\begin{bmatrix}4&3&0&7\\0&-2&0&-4\\0&0&1&1\end{bmatrix}$$

$$\sim\begin{bmatrix}4&3&0&7\\0&1&0&2\\0&0&1&1\end{bmatrix}\sim\begin{bmatrix}4&0&0&1\\0&1&0&2\\0&0&1&1\end{bmatrix}\sim\begin{bmatrix}1&0&0&1/4\\0&1&0&2\\0&0&1&1\end{bmatrix},$$

so $\mathbf{x}=(1/4,2,1)$. To confirm this result, row reduce the matrix $[A\ \ \mathbf{b}]$:

$$[A\ \ \mathbf{b}]=\begin{bmatrix}4&3&-5&2\\-4&-5&7&-4\\8&6&-8&6\end{bmatrix}\sim\begin{bmatrix}4&3&-5&2\\0&-2&2&-2\\0&0&2&2\end{bmatrix}$$

From this point the row reduction follows that of $[U\ \ \mathbf{y}]$ above, yielding the same result.

3. $L=\begin{bmatrix}1&0&0\\-3&1&0\\4&-1&1\end{bmatrix}, U=\begin{bmatrix}2&-1&2\\0&-3&4\\0&0&1\end{bmatrix}, \mathbf{b}=\begin{bmatrix}1\\0\\4\end{bmatrix}$. First, solve $L\mathbf{y}=\mathbf{b}$:

$$[L\ \ \mathbf{b}]=\begin{bmatrix}1&0&0&1\\-3&1&0&0\\4&-1&1&4\end{bmatrix}\sim\begin{bmatrix}1&0&0&1\\0&1&0&3\\0&-1&1&0\end{bmatrix}\sim\begin{bmatrix}1&0&0&1\\0&1&0&3\\0&0&1&0\end{bmatrix},$$

so $\mathbf{y}=\begin{bmatrix}1\\3\\3\end{bmatrix}$.

Next solve $U\mathbf{x}=\mathbf{y}$, using back-substitution (with matrix notation):

$$[U\ \ \mathbf{y}]=\begin{bmatrix}2&-1&2&1\\0&-3&4&3\\0&0&1&3\end{bmatrix}\sim\begin{bmatrix}2&-1&0&-5\\0&-3&0&-9\\0&0&1&3\end{bmatrix}\sim\begin{bmatrix}2&-1&0&-5\\0&1&0&3\\0&0&1&3\end{bmatrix}$$

$$\sim\begin{bmatrix}2&0&0&-2\\0&1&0&3\\0&0&1&3\end{bmatrix},$$

so $\mathbf{x}=(-1, 3, 3)$.

4. $L = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 3/2 & -5 & 1 \end{bmatrix}, U = \begin{bmatrix} 2 & -2 & 4 \\ 0 & -2 & -1 \\ 0 & 0 & -6 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 0 \\ -5 \\ 7 \end{bmatrix}$. First, solve $L\mathbf{y} = \mathbf{b}$:

$$[L \ \ \mathbf{b}] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 1 & 0 & -5 \\ 3/2 & -5 & 1 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -5 \\ 0 & -5 & 1 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & -18 \end{bmatrix},$$

so $\mathbf{y} = \begin{bmatrix} 0 \\ -5 \\ -18 \end{bmatrix}$.

Next solve $U\mathbf{x} = \mathbf{y}$, using back-substitution (with matrix notation):

$$[U \ \ \mathbf{y}] = \begin{bmatrix} 2 & -2 & 4 & 0 \\ 0 & -2 & -1 & -5 \\ 0 & 0 & -6 & -18 \end{bmatrix} \sim \begin{bmatrix} 2 & -2 & 4 & 0 \\ 0 & -2 & -1 & -5 \\ 0 & 0 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 2 & -2 & 0 & -12 \\ 0 & -2 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 2 & -2 & 0 & -12 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 2 & 0 & 0 & -10 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -5 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 \end{bmatrix},$$

so $\mathbf{x} = (-5, 1, 3)$.

5. $L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -4 & 3 & -5 & 1 \end{bmatrix}, U = \begin{bmatrix} 1 & -2 & -4 & -3 \\ 0 & -3 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \ \mathbf{b} = \begin{bmatrix} 1 \\ 7 \\ 0 \\ 3 \end{bmatrix}$. First solve $L\mathbf{y} = \mathbf{b}$:

$$[L \ \ \mathbf{b}] = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 2 & 1 & 0 & 0 & 7 \\ -1 & 0 & 1 & 0 & 0 \\ -4 & 3 & -5 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 5 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 3 & -5 & 1 & 7 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 5 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & -5 & 1 & -8 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 5 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -3 \end{bmatrix},$$

so $\mathbf{y} = \begin{bmatrix} 1 \\ 5 \\ 1 \\ -3 \end{bmatrix}$.

Next solve $U\mathbf{x} = \mathbf{y}$, using back-substitution (with matrix notation):

$$[U \ \ \mathbf{y}] = \begin{bmatrix} 1 & -2 & -4 & -3 & 1 \\ 0 & -3 & 1 & 0 & 5 \\ 0 & 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & -4 & 0 & -8 \\ 0 & -3 & 1 & 0 & 5 \\ 0 & 0 & 2 & 0 & 4 \\ 0 & 0 & 0 & 1 & -3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -2 & -4 & 0 & -8 \\ 0 & -3 & 1 & 0 & 5 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & -3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & -3 \end{bmatrix},$$

so $\mathbf{x} = (-2, -1, 2, -3)$.

6. $L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 1 & 0 & 0 \\ 3 & -2 & 1 & 0 \\ -5 & 4 & -1 & 1 \end{bmatrix}, U = \begin{bmatrix} 1 & 3 & 4 & 0 \\ 0 & 3 & 5 & 2 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 1 \\ -2 \\ -1 \\ 2 \end{bmatrix}$. First, solve $L\mathbf{y} = \mathbf{b}$:

$$[L \;\; \mathbf{b}] = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ -3 & 1 & 0 & 0 & -2 \\ 3 & -2 & 1 & 0 & -1 \\ -5 & 4 & -1 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & -2 & 1 & 0 & -4 \\ 0 & 4 & -1 & 1 & 7 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & -1 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix},$$

so $\mathbf{y} = \begin{bmatrix} 1 \\ 1 \\ -2 \\ 1 \end{bmatrix}$.

Next solve $U\mathbf{x} = \mathbf{y}$, using back-substitution (with matrix notation):

$$[U \;\; \mathbf{y}] = \begin{bmatrix} 1 & 3 & 4 & 0 & 1 \\ 0 & 3 & 5 & 2 & 1 \\ 0 & 0 & -2 & 0 & -2 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 4 & 0 & 1 \\ 0 & 3 & 5 & 0 & -1 \\ 0 & 0 & -2 & 0 & -2 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 3 & 4 & 0 & 1 \\ 0 & 3 & 5 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 0 & 0 & -3 \\ 0 & 3 & 0 & 0 & -6 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 3 & 0 & 0 & -3 \\ 0 & 1 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix},$$

so $\mathbf{x} = (3, -2, 1, 1)$.

7. Place the first pivot column of $\begin{bmatrix}2 & 5\\ -3 & -4\end{bmatrix}$ into L, after dividing the column by 2 (the pivot), then add 3/2 times row 1 to row 2, yielding U.

$$A=\begin{bmatrix}2 & 5\\ -3 & -4\end{bmatrix}\sim\begin{bmatrix}2 & 5\\ 0 & 7/2\end{bmatrix}=U$$

$$\begin{array}{cc}\begin{bmatrix}2\\ -3\end{bmatrix} & [7/2]\\ \div 2 & \div 7/2\end{array}$$

$$\begin{bmatrix}1 & \\ -3/2 & 1\end{bmatrix},\ L=\begin{bmatrix}1 & 0\\ -3/2 & 1\end{bmatrix}$$

8. Row reduce A to echelon form using only row replacement operations. Then follow the algorithm in Example 2 to find L.

$$A=\begin{bmatrix}6 & 9\\ 4 & 5\end{bmatrix}\sim\begin{bmatrix}6 & 9\\ 0 & -1\end{bmatrix}=U$$

$$\begin{array}{cc}\begin{bmatrix}6\\ 4\end{bmatrix} & [-1]\\ \div 6 & \div -1\end{array}$$

$$\begin{bmatrix}1 & \\ 2/3 & 1\end{bmatrix},\ L=\begin{bmatrix}1 & 0\\ 2/3 & 1\end{bmatrix}$$

9. $$A=\begin{bmatrix}3 & -1 & 2\\ -3 & -2 & 10\\ 9 & -5 & 6\end{bmatrix}\sim\begin{bmatrix}3 & -1 & 2\\ 0 & -3 & 12\\ 0 & -2 & 0\end{bmatrix}\sim\begin{bmatrix}3 & -1 & 2\\ 0 & -3 & 12\\ 0 & 0 & -8\end{bmatrix}=U$$

$$\begin{array}{ccc}\begin{bmatrix}3\\ -3\\ 9\end{bmatrix} & \begin{bmatrix}-3\\ -2\end{bmatrix} & [-8]\\ \div 3 & \div -3 & \div -8\end{array}$$

$$\begin{bmatrix}1 & & \\ -1 & 1 & \\ 3 & 2/3 & 1\end{bmatrix},\ L=\begin{bmatrix}1 & 0 & 0\\ -1 & 1 & 0\\ 3 & 2/3 & 1\end{bmatrix}$$

10. $$A = \begin{bmatrix} -5 & 3 & 4 \\ 10 & -8 & -9 \\ 15 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} -5 & 3 & 4 \\ 0 & -2 & -1 \\ 0 & 10 & 14 \end{bmatrix} \sim \begin{bmatrix} -5 & 3 & 4 \\ 0 & -2 & -1 \\ 0 & 0 & 9 \end{bmatrix} = U$$

$$\begin{bmatrix} -5 \\ 10 \\ 15 \end{bmatrix} \quad \begin{bmatrix} -2 \\ 10 \end{bmatrix} \quad \begin{bmatrix} 9 \end{bmatrix}$$

$$\div -5 \quad \div -2 \quad \div 9$$

$$\downarrow \quad \downarrow \quad \downarrow$$

$$\begin{bmatrix} 1 & & \\ -2 & 1 & \\ -3 & -5 & 1 \end{bmatrix}, \quad L = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & -5 & 1 \end{bmatrix}$$

11. $$A = \begin{bmatrix} 3 & -6 & 3 \\ 6 & -7 & 2 \\ -1 & 7 & 0 \end{bmatrix} \sim \begin{bmatrix} 3 & -6 & 3 \\ 0 & 5 & -4 \\ 0 & 5 & 1 \end{bmatrix} \sim \begin{bmatrix} 3 & -6 & 3 \\ 0 & 5 & -4 \\ 0 & 0 & 5 \end{bmatrix} = U$$

$$\begin{bmatrix} 3 \\ 6 \\ -1 \end{bmatrix} \quad \begin{bmatrix} 5 \\ 5 \end{bmatrix} \quad \begin{bmatrix} 5 \end{bmatrix}$$

$$\div 3 \quad \div 5 \quad \div 5$$

$$\downarrow \quad \downarrow \quad \downarrow$$

$$\begin{bmatrix} 1 & & \\ 2 & 1 & \\ -1/3 & 1 & 1 \end{bmatrix}, \quad L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1/3 & 1 & 1 \end{bmatrix}$$

12. Row reduce A to echelon form using only row replacement operations. Then follow the algorithm in Example 2 to find L. Use the last column of I_3 to make L unit lower triangular.

$$A = \begin{bmatrix} 2 & -4 & 2 \\ 1 & 5 & -4 \\ -6 & -2 & 4 \end{bmatrix} \sim \begin{bmatrix} 2 & -4 & 2 \\ 0 & 7 & -5 \\ 0 & -14 & 10 \end{bmatrix} \sim \begin{bmatrix} 2 & -4 & 2 \\ 0 & 7 & -5 \\ 0 & 0 & 0 \end{bmatrix} = U$$

$$\begin{bmatrix} 2 \\ 1 \\ -6 \end{bmatrix} \quad \begin{bmatrix} 7 \\ -14 \end{bmatrix}$$

$$\div 2 \quad \div 7$$

$$\downarrow \quad \downarrow$$

$$\begin{bmatrix} 1 & & \\ 1/2 & 1 & \\ -3 & -2 & 1 \end{bmatrix}, \quad L = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ -3 & -2 & 1 \end{bmatrix}$$

13. $\begin{bmatrix} 1 & 3 & -5 & -3 \\ -1 & -5 & 8 & 4 \\ 4 & 2 & -5 & -7 \\ -2 & -4 & 7 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & -5 & -3 \\ 0 & -2 & 3 & 1 \\ 0 & -10 & 15 & 5 \\ 0 & 2 & -3 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & -5 & -3 \\ 0 & -2 & 3 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = U$ No more pivots!

$$\begin{bmatrix} 1 \\ -1 \\ 4 \\ -2 \end{bmatrix} \begin{bmatrix} -2 \\ -10 \\ 2 \end{bmatrix}$$ Use the last two columns of I_4 to make L unit lower triangular.

$$\begin{array}{cc} \div 1 & \div -2 \end{array}$$

$$\begin{bmatrix} 1 & & & \\ -1 & 1 & & \\ 4 & 5 & 1 & \\ -2 & -1 & 0 & 1 \end{bmatrix}, \quad L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 4 & 5 & 1 & 0 \\ -2 & -1 & 0 & 1 \end{bmatrix}$$

14. $A = \begin{bmatrix} 1 & 4 & -1 & 5 \\ 3 & 7 & -2 & 9 \\ -2 & -3 & 1 & -4 \\ -1 & 6 & -1 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & -1 & 5 \\ 0 & -5 & 1 & -6 \\ 0 & 5 & -1 & 6 \\ 0 & 10 & -2 & 12 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & -1 & 5 \\ 0 & -5 & 1 & -6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = U$

$$\begin{bmatrix} 1 \\ 3 \\ -2 \\ -1 \end{bmatrix} \begin{bmatrix} -5 \\ 5 \\ 10 \end{bmatrix}$$ Use the last two columns of I_4 to make L unit lower triangular.

$$\begin{array}{cc} \div 1 & \div -5 \end{array}$$

$$\begin{bmatrix} 1 & & & \\ 3 & 1 & & \\ -2 & -1 & 1 & \\ -1 & -2 & 0 & 1 \end{bmatrix}, \quad L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ -2 & -1 & 1 & 0 \\ -1 & -2 & 0 & 1 \end{bmatrix}$$

15. $A=\begin{bmatrix} 2 & -4 & 4 & -2 \\ 6 & -9 & 7 & -3 \\ -1 & -4 & 8 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & -4 & 4 & -2 \\ 0 & 3 & -5 & 3 \\ 0 & -6 & 10 & -1 \end{bmatrix} \sim \begin{bmatrix} 2 & -4 & 4 & -2 \\ 0 & 3 & -5 & 3 \\ 0 & 0 & 0 & 5 \end{bmatrix} = U$

$$\begin{bmatrix} 2 \\ 6 \\ -1 \end{bmatrix} \quad \begin{bmatrix} 3 \\ -6 \end{bmatrix} \quad [5]$$

$$\div 2 \quad \div 3 \quad \div 5$$

$$\begin{bmatrix} 1 & & \\ 3 & 1 & \\ -1/2 & -2 & 1 \end{bmatrix}, \quad L=\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -1/2 & -2 & 1 \end{bmatrix}$$

16. $A=\begin{bmatrix} 2 & -6 & 6 \\ -4 & 5 & -7 \\ 3 & 5 & -1 \\ -6 & 4 & -8 \\ 8 & -3 & 9 \end{bmatrix} \sim \begin{bmatrix} 2 & -6 & 6 \\ 0 & -7 & 5 \\ 0 & 14 & -10 \\ 0 & -14 & 10 \\ 0 & 21 & -15 \end{bmatrix} \sim \begin{bmatrix} 2 & -6 & 6 \\ 0 & -7 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = U$

$$\begin{bmatrix} 2 \\ -4 \\ 3 \\ -6 \\ 8 \end{bmatrix} \quad \begin{bmatrix} -7 \\ 14 \\ -14 \\ 21 \end{bmatrix}$$

Use the last three columns of I_5 to make L unit lower triangular.

$$\div 2 \quad \div -7$$

$$\begin{bmatrix} 1 & & & & \\ -2 & 1 & & & \\ 3/2 & -2 & 1 & & \\ -3 & 2 & 0 & 1 & \\ 4 & -3 & 0 & 0 & 1 \end{bmatrix}, \quad L=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 & 0 \\ 3/2 & -2 & 1 & 0 & 0 \\ -3 & 2 & 0 & 1 & 0 \\ 4 & -3 & 0 & 0 & 1 \end{bmatrix}$$

17. $L=\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$, $U=\begin{bmatrix} 4 & 3 & -5 \\ 0 & -2 & 2 \\ 0 & 0 & 2 \end{bmatrix}$ To find L^{-1}, use the method of Section 2.2; that is, row reduce $[L \quad I]$:

$$[L \quad I]=\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & 0 \\ 2 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -2 & 0 & 1 \end{bmatrix} = [I \quad L^{-1}],$$

so $L^{-1}=\begin{bmatrix}1&0&0\\1&1&0\\-2&0&1\end{bmatrix}$. Likewise to find U^{-1}, row reduce $[U\ I]$:

$$[U\quad I]=\begin{bmatrix}4&3&-5&1&0&0\\0&-2&2&0&1&0\\0&0&2&0&0&1\end{bmatrix}\sim\begin{bmatrix}4&3&0&1&0&5/2\\0&-2&0&0&1&-1\\0&0&2&0&0&1\end{bmatrix}$$

$$\sim\begin{bmatrix}4&0&0&1&3/2&1\\0&-2&0&0&1&-1\\0&0&2&0&0&1\end{bmatrix}\sim\begin{bmatrix}1&0&0&1/4&3/8&1/4\\0&1&0&0&-1/2&1/2\\0&0&1&0&0&1/2\end{bmatrix}=[I\quad U^{-1}],$$

so $U^{-1}=\begin{bmatrix}1/4&3/8&1/4\\0&-1/2&1/2\\0&0&1/2\end{bmatrix}$. Thus

$$A^{-1}=U^{-1}L^{-1}=\begin{bmatrix}1/4&3/8&1/4\\0&-1/2&1/2\\0&0&1/2\end{bmatrix}\begin{bmatrix}1&0&0\\1&1&0\\-2&0&1\end{bmatrix}=\begin{bmatrix}1/8&3/8&1/4\\-3/2&-1/2&1/2\\-1&0&1/2\end{bmatrix}$$

18. $L=\begin{bmatrix}1&0&0\\-3&1&0\\4&-1&1\end{bmatrix}$, $U=\begin{bmatrix}2&-1&2\\0&-3&4\\0&0&1\end{bmatrix}$ To find L^{-1}, row reduce $[L\quad I]$:

$$[L\quad I]=\begin{bmatrix}1&0&0&1&0&0\\-3&1&0&0&1&0\\4&-1&1&0&0&1\end{bmatrix}\sim\begin{bmatrix}1&0&0&1&0&0\\0&1&0&3&1&0\\0&-1&1&-4&0&1\end{bmatrix}$$

$$\sim\begin{bmatrix}1&0&0&1&0&0\\0&1&0&3&1&0\\0&0&1&-1&1&1\end{bmatrix}=[I\quad L^{-1}],$$

so $L^{-1}=\begin{bmatrix}1&0&0\\3&1&0\\-1&1&1\end{bmatrix}$. Likewise to find U^{-1}, row reduce $[U\quad I]$:

$$[U\quad I]=\begin{bmatrix}2&-1&2&1&0&0\\0&-3&4&0&1&0\\0&0&1&0&0&1\end{bmatrix}\sim\begin{bmatrix}2&-1&0&1&0&-2\\0&-3&0&0&1&-4\\0&0&1&0&0&1\end{bmatrix}$$

$$\sim\begin{bmatrix}2&-1&0&1&0&-2\\0&1&0&0&-1/3&4/3\\0&0&1&0&0&1\end{bmatrix}\sim\begin{bmatrix}2&0&0&1&-1/3&-2/3\\0&1&0&0&-1/3&4/3\\0&0&1&0&0&1\end{bmatrix}$$

$$\sim\begin{bmatrix}1&0&0&1/2&-1/6&-1/3\\0&1&0&0&-1/3&4/3\\0&0&1&0&0&1\end{bmatrix}=[I\quad U^{-1}],$$

so $U^{-1} = \begin{bmatrix} 1/2 & -1/6 & -1/3 \\ 0 & -1/3 & 4/3 \\ 0 & 0 & 1 \end{bmatrix}$. Thus

$$A^{-1} = U^{-1}L^{-1} = \begin{bmatrix} 1/2 & -1/6 & -1/3 \\ 0 & -1/3 & 4/3 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1/3 & -1/2 & -1/3 \\ -7/3 & 1 & 4/3 \\ -1 & 1 & 1 \end{bmatrix}$$

19. Let A be a lower-triangular $n \times n$ matrix with nonzero entries on the diagonal, and consider the augmented matrix $[A \;\; I]$.

a. The (1, 1)-entry can be scaled to 1 and the entries below it can be changed to 0 by adding multiples of row 1 to the rows below. This affects only the first column of A and the first column of I. So the (2, 2)-entry in the new matrix is still nonzero and now is the only nonzero entry of row 2 in the first n columns (because A was lower triangular).

The (2, 2)-entry can be scaled to 1, the entries below it can be changed to 0 by adding multiples of row 2 to the rows below. This affects only columns 2 and $n + 2$ of the augmented matrix. Now the (3, 3) entry in A is the only nonzero entry of the third row in the first n columns, so it can be scaled to 1 and then used as a pivot to zero out entries below it. Continuing in this way, A is eventually reduced to I, by scaling each row with a pivot and then using only row operations that add multiples of the pivot row to rows below.

b. The row operations just described only add rows to rows below, so the I on the right in $[A \; I]$ changes into a lower triangular matrix. By Theorem 7 in Section 2.2, that matrix is A^{-1}.

20. Let $A = LU$ be an LU factorization for A. Since L is unit lower triangular, it is invertible by Exercise 19. Thus by the Invertible Matrix Theroem, L may be row reduced to I. But L is unit lower triangular, so it can be row reduced to I by adding suitable multiples of a row to the rows below it, beginning with the top row. Note that all of the described row operations done to L are row-replacement operations. If elementary matrices E_1, E_2, … E_p implement these row-replacement operations, then

$$E_p \ldots E_2E_1A = (E_p \ldots E_2E_1)LU = IU = U$$

This shows that A may be row reduced to U using only row-replacement operations.

21. (Solution in *Study Guide*.) Suppose $A = BC$, with B invertible. Then there exist elementary matrices $E_1, \ldots, E_p$ corresponding to row operations that reduce B to I, in the sense that $E_p \ldots E_1B = I$. Applying the same sequence of row operations to A amounts to left-multiplying A by the product $E_p \ldots E_1$. By associativity of matrix multiplication.

$$E_p \ldots E_1A = E_p \ldots E_1BC = IC = C$$

so the same sequence of row operations reduces A to C.

22. First find an LU factorization for A. Row reduce A to echelon form using only row replacement operations:

$$A = \begin{bmatrix} 2 & -4 & -2 & 3 \\ 6 & -9 & -5 & 8 \\ 2 & -7 & -3 & 9 \\ 4 & -2 & -2 & -1 \\ -6 & 3 & 3 & 4 \end{bmatrix} \sim \begin{bmatrix} 2 & -4 & -2 & 3 \\ 0 & 3 & 1 & -1 \\ 0 & -3 & -1 & 6 \\ 0 & 6 & 2 & -7 \\ 0 & -9 & -3 & 13 \end{bmatrix} \sim \begin{bmatrix} 2 & -4 & -2 & 3 \\ 0 & 3 & 1 & -1 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & -5 \\ 0 & 0 & 0 & 10 \end{bmatrix}$$

$$\sim \begin{bmatrix} 2 & -4 & -2 & 3 \\ 0 & 3 & 1 & -1 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = U$$

then follow the algorithm in Example 2 to find L. Use the last two columns of I_5 to make L unit lower triangular.

$$\begin{bmatrix} 2 \\ 6 \\ 2 \\ 4 \\ -6 \end{bmatrix} \begin{bmatrix} 3 \\ -3 \\ 6 \\ -9 \end{bmatrix} \begin{bmatrix} 5 \\ -5 \\ 10 \end{bmatrix}$$

$$\div 2 \quad \div 3 \quad \div 5$$

$$\downarrow \quad \downarrow \quad \downarrow$$

$$\begin{bmatrix} 1 & & & & \\ 3 & 1 & & & \\ 1 & -1 & 1 & & \\ 2 & 2 & -1 & 1 & \\ -3 & -3 & 2 & 0 & 1 \end{bmatrix}, \ L = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 \\ 2 & 2 & -1 & 1 & 0 \\ -3 & 3 & 2 & 0 & 1 \end{bmatrix}$$

Now notice that the bottom two rows of U contain only zeros. If one uses the row-column method to find LU, the entries in the final two columns of L will not be used, since these entries will be multiplied zeros from the bottom two rows of U. So let B be the first three columns of L and let C be the top three rows of U. That is,

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 1 & -1 & 1 \\ 2 & 2 & -1 \\ -3 & 3 & 2 \end{bmatrix}, \ C = \begin{bmatrix} 2 & -4 & -2 & 3 \\ 0 & 3 & 1 & -1 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

Then B and C have the desired sizes and $BC = LU = A$. We can generalize this process to the case where A in $m \times n$, $A = LU$, and U has only three non-zero rows: let B be the first three columns of L and let C be the top three rows of U.

23. **a**. Express each row of D as the transpose of a column vector. Then use the multiplication rule for partitioned matrices to write

$$A = CD = \begin{bmatrix} \mathbf{c}_1 & \mathbf{c}_2 & \mathbf{c}_3 & \mathbf{c}_4 \end{bmatrix} \begin{bmatrix} \mathbf{d}_1^T \\ \mathbf{d}_2^T \\ \mathbf{d}_3^T \\ \mathbf{d}_4^T \end{bmatrix} = \mathbf{c}_1\mathbf{d}_1^T + \mathbf{c}_2\mathbf{d}_2^T + \mathbf{c}_3\mathbf{d}_3^T + \mathbf{c}_4\mathbf{d}_4^T$$

which is the sum of four outer products.

b. Since A has $400 \times 100 = 40000$ entries, C has $400 \times 4 = 1600$ entries and D has $4 \times 100 = 400$ entries, to store C and D together requires only 2000 entries, which is 5% of the amount of entries needed to store A directly.

24. Since Q is square and $Q^TQ = I$, Q is invertible by the Invertible Matrix Theorem and $Q^{-1} = Q^T$. Thus A is the product of invertible matrices and hence is invertible. Thus by Theorem 5, the equation $A\mathbf{x} = \mathbf{b}$ has a unique solution for all $\mathbf{b}$. From $A\mathbf{x} = \mathbf{b}$, we have $QR\mathbf{x} = \mathbf{b}$, $Q^TQR\mathbf{x} = Q^T\mathbf{b}$, $R\mathbf{x} = Q^T\mathbf{b}$, and finally $\mathbf{x} = R^{-1}Q^T\mathbf{b}$. A good algorithm for finding $\mathbf{x}$ is to compute $Q^T\mathbf{b}$ and then row reduce the matrix $[\ R\ \ Q^T\mathbf{b}\ ]$. See Exercise 11 in Section 2.2 for details on why this process works. The reduction is fast in this case because R is a triangular matrix.

25. $A = UDV^T$. Since U and V^T are square, the equations $U^TU = I$ and $V^TV = I$ imply that U and V^T are invertible, by the IMT, and hence $U^{-1} = U^T$ and $(V^T)^{-1} = V$. Since the diagonal entries $\sigma_1,\ldots,\sigma_n$ in D are nonzero, D is invertible, with the inverse of D being the diagonal matrix with $\sigma_1^{-1},\ldots,\sigma_n^{-1}$ on the diagonal. Thus A is a product of invertible matrices. By Theorem 6, A is invertible and $A^{-1} = (UDV^T)^{-1} = (V^T)^{-1}D^{-1}U^{-1} = VD^{-1}U^T$.

26. If $A = PDP^{-1}$, where P is an invertible 3 × 3 matrix and D is the diagonal matrix

$$D=\begin{bmatrix}1 & 0 & 0\\ 0 & 1/2 & 0\\ 0 & 0 & 1/3\end{bmatrix}$$

then

$$A^2=(PDP^{-1})(PDP^{-1})=PD(P^{-1}P)DP^{-1}=PDIDP^{-1}=PD^2P^{-1}$$

and since

$$D^2=\begin{bmatrix}1 & 0 & 0\\ 0 & 1/2 & 0\\ 0 & 0 & 1/3\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & 1/2 & 0\\ 0 & 0 & 1/3\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 0 & 1/2^2 & 0\\ 0 & 0 & 1/3^2\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 0 & 1/4 & 0\\ 0 & 0 & 1/9\end{bmatrix}$$

$$A^2=P\begin{bmatrix}1 & 0 & 0\\ 0 & 1/4 & 0\\ 0 & 0 & 1/9\end{bmatrix}P^{-1}$$

Likewise, $A^3 = PD^3P^{-1}$, so

$$A^3=P\begin{bmatrix}1 & 0 & 0\\ 0 & 1/2^3 & 0\\ 0 & 0 & 1/3^3\end{bmatrix}P^{-1}=P\begin{bmatrix}1 & 0 & 0\\ 0 & 1/8 & 0\\ 0 & 0 & 1/27\end{bmatrix}P^{-1}$$

In general, $A^k = PD^kP^{-1}$, so

$$A^k=P\begin{bmatrix}1 & 0 & 0\\ 0 & 1/2^k & 0\\ 0 & 0 & 1/3^k\end{bmatrix}P^{-1}$$

27. First consider using a series circuit with resistance R_1 followed by a shunt circuit with resistance R_2 for the network. The transfer matrix for this network is

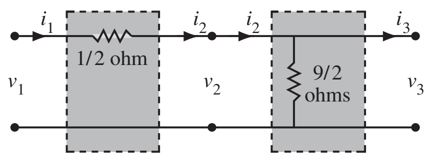
$$\begin{bmatrix}1 & 0\\ -1/R_2 & 1\end{bmatrix}\begin{bmatrix}1 & -R_1\\ 0 & 1\end{bmatrix}=\begin{bmatrix}1 & -R_1\\ -1/R_2 & (R_1+R_2)/R_2\end{bmatrix}$$

For an input of 12 volts and 6 amps to produce an output of 9 volts and 4 amps, the transfer matrix must satisfy

$$\begin{bmatrix} 1 & -R_1 \\ -1/R_2 & (R_1+R_2)/R_2 \end{bmatrix}\begin{bmatrix} 12 \\ 6 \end{bmatrix} = \begin{bmatrix} 12-6R_1 \\ (-12+6R_1+6R_2)/R_2 \end{bmatrix} = \begin{bmatrix} 9 \\ 4 \end{bmatrix}$$

Equate the top entries and obtain $R_1 = \frac{1}{2}$ ohm. Substitute this value in the bottom entry and solve to obtain $R_2 = \frac{9}{2}$ ohms. The ladder network is

a.



Next consider using a shunt circuit with resistance R_1 followed by a series circuit with resistance R_2 for the network. The transfer matrix for this network is

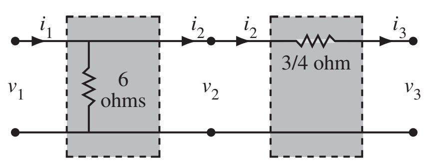
$$\begin{bmatrix} 1 & -R_2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1/R_1 & 1 \end{bmatrix} = \begin{bmatrix} (R_1+R_2)/R_1 & -R_2 \\ -1/R_1 & 1 \end{bmatrix}$$

For an input of 12 volts and 6 amps to produce an output of 9 volts and 4 amps, the transfer matrix must satisfy

$$\begin{bmatrix} (R_1+R_2)/R_1 & -R_2 \\ -1/R_1 & 1 \end{bmatrix}\begin{bmatrix} 12 \\ 6 \end{bmatrix} = \begin{bmatrix} (12R_1+12R_2)/R_1 - 6R_2 \\ -12/R_1+6 \end{bmatrix} = \begin{bmatrix} 9 \\ 4 \end{bmatrix}$$

Equate the bottom entries and obtain $R_1 = 6$ ohms. Substitute this value in the top entry and solve to obtain $R_2 = \frac{3}{4}$ ohms. The ladder network is

b.



28. The three shunt circuits have transfer matrices

$$\begin{bmatrix} 1 & 0 \\ -1/R_1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ -1/R_2 & 1 \end{bmatrix}, \text{ and } \begin{bmatrix} 1 & 0 \\ -1/R_3 & 1 \end{bmatrix}$$

respectively. To find the transfer matrix for the series of circuits, multiply these matrices

$$\begin{bmatrix} 1 & 0 \\ -1/R_3 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ -1/R_2 & 1 \end{bmatrix}, \text{ and } \begin{bmatrix} 1 & 0 \\ -1/R_1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -(1/R_1+1/R_2+1/R_3) & 1 \end{bmatrix}$$

Thus the resulting network is itself a shunt circuit with resistance $1/R_1 + 1/R_2 + 1/R_3$.

29. **a**. The first circuit is a shunt circuit with resistance R_1 ohms, so its transfer matrix is $\begin{bmatrix} 1 & 0 \\ -1/R_1 & 1 \end{bmatrix}$.

The second circuit is a series circuit with resistance R_2 ohms, so its transfer matrix is $\begin{bmatrix} 1 & -R_2 \\ 0 & 1 \end{bmatrix}$.

The third circuit is a shunt circuit with resistance R_3 ohms so its transfer matrix is $\begin{bmatrix} 1 & 0 \\ -1/R_3 & 1 \end{bmatrix}$.

The transfer matrix of the network is the product of these matrices, in *right-to-left* order:

$$\begin{bmatrix} 1 & 0 \\ -1/R_3 & 1 \end{bmatrix}\begin{bmatrix} 1 & -R_2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1/R_1 & 1 \end{bmatrix} = \begin{bmatrix} (R_1+R_2)/R_1 & -R_2 \\ -(R_1+R_2+R_3)/R_3 & (R_2+R_3)/R_3 \end{bmatrix}$$

b. To find a ladder network with a structure like that in part (a) and with the given transfer matrix A, we must find resistances R_1, R_2, and R_3 such that

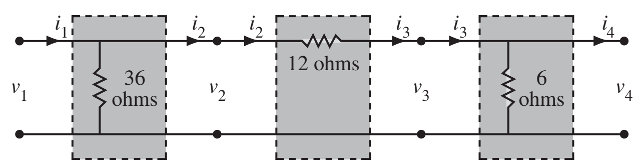
$$A = \begin{bmatrix} 4/3 & -12 \\ -1/4 & 3 \end{bmatrix} = \begin{bmatrix} (R_1+R_2)/R_1 & -R_2 \\ -(R_1+R_2+R_3)/R_3 & (R_2+R_3)/R_3 \end{bmatrix}$$

From the (1, 2) entries, $R_2 = 12$ ohms. The (1, 1) entries now give $(R_1 + 12)/R_1 = 4/3$, which may be solved to obtain $R_1 = 36$ ohms. Likewise the (2, 2) entries give $(R_3 + 12)/R_3 = 3$, which also may be solved to obtain $R_3 = 6$ ohms. Thus the matrix A may be factored as

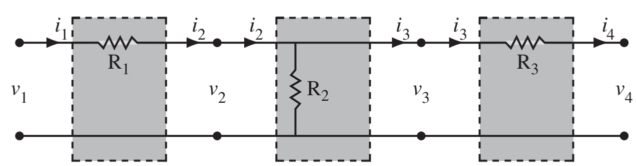
$$A = \begin{bmatrix} 1 & 0 \\ -1/R_3 & 1 \end{bmatrix}\begin{bmatrix} 1 & -R_2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1/R_1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ -1/6 & 1 \end{bmatrix}\begin{bmatrix} 1 & -12 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1/36 & 1 \end{bmatrix}$$

The ladder network is



30. Answers may vary. The network below interchanges the series and shunt circuits.



The transfer matrix of this network is the product of the individual transfer matrices, in *right-to-left* order.

$$\begin{bmatrix} 1 & -R_3 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1/R_2 & 1 \end{bmatrix}\begin{bmatrix} 1 & -R_1 \\ 0 & 1 \end{bmatrix} =$$

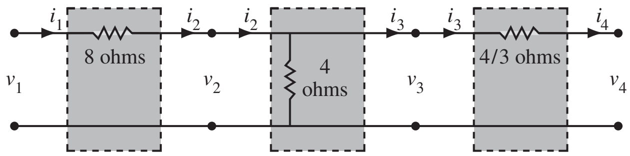
$$\begin{bmatrix} (R_2+R_3)/R_2 & -R_3 - R_1(R_2+R_3)/R_2 \\ -1/R_2 & (R_1+R_2)/R_2 \end{bmatrix}$$

By setting the matrix A from the previous exercise equal to this matrix, one may find that

$$\begin{bmatrix} (R_2+R_3)/R_2 & -R_3 - R_1(R_2+R_3)/R_2 \\ -1/R_2 & (R_1+R_2)/R_2 \end{bmatrix} = \begin{bmatrix} 4/3 & -12 \\ -1/4 & 3 \end{bmatrix}$$

Set the (2, 1) entries equal and obtain $R_2 = 4$ ohms. Substitute this value for R_2, equating the (2, 2) entries and solving gives $R_1 = 8$ ohms. Likewise equating the (1, 1) entries gives $R_3 = 4/3$ ohms.

The ladder network is



Note: The *Study Guide's* MATLAB box for Section 2.5 suggests that for most LU factorizations in this section, students can use the **gauss** command repeatedly to produce U, and use paper and mental arithmetic to write down the columns of L as the row reduction to U proceeds. This is because for Exercises 7–16 the pivots are integers and other entries are simple fractions. However, for Exercises 31 and 32 this is not reasonable, and students are expected to solve an elementary programming problem. (The *Study Guide* provides no hints.)

31. **[M]** Store the matrix A in a temporary matrix B and create L initially as the 8×8 identity matrix. The following sequence of MATLAB commands fills in the entries of L below the diagonal, one column at a time, until the first seven columns are filled. (The eighth column is the final column of the identity matrix.)

```
L(2:8, 1) = B(2:8, 1)/B(1, 1)
B = gauss(B, 1)
L(3:8, 2) = B(3:8, 2)/B(2, 2)
B = gauss(B, 2)
⋮
L(8:8, 7) = B(8:8, 7)/B(7, 7)
U = gauss(B,7)
```

Of course, some students may realize that a loop will speed up the process. The **for..end** syntax is illustrated in the MATLAB box for Section 5.6. Here is a MATLAB program that includes the initial setup of B and L:

```
B = A
L = eye(8)
for j=1:7
  L(j+1:8, j) = B(j+1:8, j)/B(j, j)
  B = gauss(B, j)
end
U = B
```

a. To four decimal places, the results of the LU decomposition are

$$L=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -.25 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -.25 & -.0667 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -.2667 & -.2857 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -.2679 & -.0833 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -.2917 & -.2921 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -.2697 & -.0861 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -.2948 & -.2931 & 1 \end{bmatrix}$$

$$U = \begin{bmatrix} 4 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3.75 & -.25 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3.7333 & -1.0667 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3.4286 & -.2857 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3.7083 & -1.0833 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3.3919 & -.2921 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3.7052 & -1.0861 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3.3868 \end{bmatrix}$$

b. The result of solving $L\mathbf{y} = \mathbf{b}$ and then $U\mathbf{x} = \mathbf{y}$ is

$\mathbf{x} = (3.9569, 6.5885, 4.2392, 7.3971, 5.6029, 8.7608, 9.4115, 12.0431)$

c. $$A^{-1} = \begin{bmatrix} .2953 & .0866 & .0945 & .0509 & .0318 & .0227 & .0010 & .0082 \\ .0866 & .2953 & .0509 & .0945 & .0227 & .0318 & .0082 & .0100 \\ .0945 & .0509 & .3271 & .1093 & .1045 & .0591 & .0318 & .0227 \\ .0509 & .0945 & .1093 & .3271 & .0591 & .1045 & .0227 & .0318 \\ .0318 & .0227 & .1045 & .0591 & .3271 & .1093 & .0945 & .0509 \\ .0227 & .0318 & .0591 & .1045 & .1093 & .3271 & .0509 & .0945 \\ .0010 & .0082 & .0318 & .0227 & .0945 & .0509 & .2953 & .0866 \\ .0082 & .0100 & .0227 & .0318 & .0509 & .0945 & .0866 & .2953 \end{bmatrix}$$

32. **[M]** $A = \begin{bmatrix} 3 & -1 & 0 & 0 & 0 \\ -1 & 3 & -1 & 0 & 0 \\ 0 & -1 & 3 & -1 & 0 \\ 0 & 0 & -1 & 3 & -1 \\ 0 & 0 & 0 & -1 & 3 \end{bmatrix}$. The commands shown for Exercise 31, but modified for 5×5 matrices, produce

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -\frac{1}{3} & 1 & 0 & 0 & 0 \\ 0 & -\frac{3}{8} & 1 & 0 & 0 \\ 0 & 0 & -\frac{8}{21} & 1 & 0 \\ 0 & 0 & 0 & -\frac{21}{55} & 1 \end{bmatrix}$$

$$U = \begin{bmatrix} 3 & -1 & 0 & 0 & 0 \\ 0 & \frac{8}{3} & -1 & 0 & 0 \\ 0 & 0 & \frac{21}{8} & -1 & 0 \\ 0 & 0 & 0 & \frac{55}{21} & -1 \\ 0 & 0 & 0 & 0 & \frac{144}{55} \end{bmatrix}$$

b. Let $\mathbf{s}_{k+1}$ be the solution of $L\mathbf{s}_{k+1} = \mathbf{t}_k$ for $k = 0, 1, 2, \ldots$. Then $\mathbf{t}_{k+1}$ is the solution of $U\mathbf{t}_{k+1} = \mathbf{s}_{k+1}$ for $k = 0, 1, 2, \ldots$. The results are

$$\mathbf{s}_1 = \begin{bmatrix} 10.0000 \\ 15.3333 \\ 17.7500 \\ 18.7619 \\ 17.1636 \end{bmatrix}, \mathbf{t}_1 = \begin{bmatrix} 6.5556 \\ 9.6667 \\ 10.4444 \\ 9.6667 \\ 6.5556 \end{bmatrix}, \mathbf{s}_2 = \begin{bmatrix} 6.5556 \\ 11.8519 \\ 14.8889 \\ 15.3386 \\ 12.4121 \end{bmatrix}, \mathbf{t}_2 = \begin{bmatrix} 4.7407 \\ 7.6667 \\ 8.5926 \\ 7.6667 \\ 4.7407 \end{bmatrix},$$

$$\mathbf{s}_3 = \begin{bmatrix} 4.7407 \\ 9.2469 \\ 12.0602 \\ 12.2610 \\ 9.4222 \end{bmatrix}, \mathbf{t}_3 = \begin{bmatrix} 3.5988 \\ 6.0556 \\ 6.9012 \\ 6.0556 \\ 3.5988 \end{bmatrix}, \mathbf{s}_4 = \begin{bmatrix} 3.5988 \\ 7.2551 \\ 9.6219 \\ 9.7210 \\ 7.3104 \end{bmatrix}, \mathbf{t}_4 = \begin{bmatrix} 2.7922 \\ 4.7778 \\ 5.4856 \\ 4.7778 \\ 2.7922 \end{bmatrix}.$$

2.6 SOLUTIONS

Notes: This section is independent of Section 1.10. The material here makes a good backdrop for the series expansion of $(I–C)^{-1}$ because this formula is actually used in some practical economic work. Exercise 8 gives an interpretation to entries of an inverse matrix that could be stated without the economic context.

1. The answer to this exercise will depend upon the order in which the student chooses to list the sectors. The important fact to remember is that each column is the unit consumption vector for the appropriate sector. If we order the sectors manufacturing, agriculture, and services, then the consumption matrix is

$$C = \begin{bmatrix} .10 & .60 & .60 \\ .30 & .20 & 0 \\ .30 & .10 & .10 \end{bmatrix}$$

The intermediate demands created by the production vector $\mathbf{x}$ are given by $C\mathbf{x}$. Thus in this case the intermediate demand is

$$C\mathbf{x} = \begin{bmatrix} .10 & .60 & .60 \\ .30 & .20 & .00 \\ .30 & .10 & .10 \end{bmatrix}\begin{bmatrix} 0 \\ 100 \\ 0 \end{bmatrix} = \begin{bmatrix} 60 \\ 20 \\ 10 \end{bmatrix}$$

2. Solve the equation $\mathbf{x} = C\mathbf{x} + \mathbf{d}$ for $\mathbf{d}$:

$$\mathbf{d} = \mathbf{x} - C\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} - \begin{bmatrix} .10 & .60 & .60 \\ .30 & .20 & .00 \\ .30 & .10 & .10 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} .9x_1 & -.6x_2 & -.6x_3 \\ -.3x_1 & +.8x_2 & \\ -.3x_1 & -.1x_2 & +.9x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 18 \\ 0 \end{bmatrix}$$

This system of equations has the augmented matrix

$$\begin{bmatrix} -.90 & -.60 & -.60 & 0 \\ -.30 & .80 & .00 & 18 \\ -.30 & -.10 & .90 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 33.33 \\ 0 & 1 & 0 & 35.00 \\ 0 & 0 & 1 & 15.00 \end{bmatrix}$$

so $\mathbf{x} = (33.33, 35.00, 15.00)$.

3. Solving as in Exercise 2:

$$\mathbf{d}=\mathbf{x}-C\mathbf{x}=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}-\begin{bmatrix}.10&.60&.60\\.30&.20&.00\\.30&.10&.10\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}.9x_1&-.6x_2&-.6x_3\\-.3x_1&+.8x_2&\\-.3x_1&-.1x_2&+.9x_3\end{bmatrix}=\begin{bmatrix}18\\0\\0\end{bmatrix}$$

This system of equations has the augmented matrix

$$\begin{bmatrix}.90&-.60&-.60&18\\-.30&.80&.00&0\\-.30&-.10&.90&0\end{bmatrix}\sim\begin{bmatrix}1&0&0&40.00\\0&1&0&15.00\\0&0&1&15.00\end{bmatrix}$$

so $\mathbf{x}$ = (40.00, 15.00, 15.00).

4. Solving as in Exercise 2:

$$\mathbf{d}=\mathbf{x}-C\mathbf{x}=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}-\begin{bmatrix}.10&.60&.60\\.30&.20&.00\\.30&.10&.10\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}.9x_1&-.6x_2&-.6x_3\\-.3x_1&+.8x_2&\\-.3x_1&-.1x_2&+.9x_3\end{bmatrix}=\begin{bmatrix}18\\18\\0\end{bmatrix}$$

This system of equations has the augmented matrix

$$\begin{bmatrix}-.90&-.60&-.60&18\\-.30&.80&.00&18\\-.30&-.10&.90&0\end{bmatrix}\sim\begin{bmatrix}1&0&0&73.33\\0&1&0&50.00\\0&0&1&30.00\end{bmatrix}$$

so $\mathbf{x}$ = (73.33, 50.00, 30.00).

Note: Exercises 2–4 may be used by students to discover the linearity of the Leontief model.

5. $\mathbf{x}=(I-C)^{-1}\mathbf{d}=\begin{bmatrix}1&-.5\\-.6&.8\end{bmatrix}^{-1}\begin{bmatrix}50\\20\end{bmatrix}=\begin{bmatrix}1.6&1\\1.2&2\end{bmatrix}\begin{bmatrix}50\\20\end{bmatrix}=\begin{bmatrix}110\\120\end{bmatrix}$

6. $\mathbf{x}=(I-C)^{-1}\mathbf{d}=\begin{bmatrix}.9&-.6\\-.5&.8\end{bmatrix}^{-1}\begin{bmatrix}18\\11\end{bmatrix}=\begin{bmatrix}40/21&30/21\\25/21&45/21\end{bmatrix}\begin{bmatrix}18\\11\end{bmatrix}=\begin{bmatrix}50\\45\end{bmatrix}$

7. **a**. From Exercise 5,

$$(I-C)^{-1}=\begin{bmatrix}1.6&1\\1.2&2\end{bmatrix}$$

so

$$\mathbf{x}_1=(I-C)^{-1}\mathbf{d}_1=\begin{bmatrix}1.6&1\\1.2&2\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}1.6\\1.2\end{bmatrix}$$

which is the first column of $(I-C)^{-1}$.

b. $$\mathbf{x}_2=(I-C)^{-1}\mathbf{d}_2=\begin{bmatrix}1.6&1\\1.2&2\end{bmatrix}\begin{bmatrix}51\\30\end{bmatrix}=\begin{bmatrix}111.6\\121.2\end{bmatrix}$$

c. From Exercise 5, the production **x** corressponding to $\mathbf{d}=\begin{bmatrix}50\\20\end{bmatrix}$ is $\mathbf{x}=\begin{bmatrix}110\\120\end{bmatrix}$.

Note that $\mathbf{d}_2=\mathbf{d}+\mathbf{d}_1$. Thus

$$\begin{aligned}\mathbf{x}_2&=(I-C)^{-1}\mathbf{d}_2\\&=(I-C)^{-1}(\mathbf{d}+\mathbf{d}_1)\\&=(I-C)^{-1}\mathbf{d}+(I-C)^{-1}\mathbf{d}_1\\&=\mathbf{x}+\mathbf{x}_1\end{aligned}$$

8. **a**. Given $(I-C)\mathbf{x}=\mathbf{d}$ and $(I-C)\Delta\mathbf{x}=\Delta\mathbf{d}$,

$$(I-C)(\mathbf{x}+\Delta\mathbf{x})=(I-C)\mathbf{x}+(I-C)\Delta\,\mathbf{x}=\mathbf{d}+\Delta\mathbf{d}$$

Thus $\mathbf{x}+\Delta\,\mathbf{x}$ is the production level corresponding to a demand of $\mathbf{d}+\Delta\mathbf{d}$.

b. Since $\Delta\mathbf{x}=(I-C)^{-1}\Delta\mathbf{d}$ and $\Delta\mathbf{d}$ is the first column of *I*, $\Delta\mathbf{x}$ will be the first column of $(I-C)^{-1}$.

9. In this case

$$I-C=\begin{bmatrix}.8&-.2&.0\\-.3&.9&-.3\\-.1&.0&.8\end{bmatrix}$$

Row reduce $[\,I-C\;\;\mathbf{d}\,]$ to find

$$\begin{bmatrix}.8&-.2&.0&40.0\\-.3&.9&-.3&60.0\\-.1&.0&.8&80.0\end{bmatrix}\sim\begin{bmatrix}1&0&0&82.8\\0&1&0&131.0\\0&0&1&110.3\end{bmatrix}$$

So **x** = (82.8, 131.0, 110.3).

10. From Exercise 8, the (*i, j*) entry in $(I-C)^{-1}$ corresponds to the effect on production of sector *i* when the final demand for the output of sector *j* increases by one unit. Since these entries are all positive, an increase in the final demand for any sector will cause the production of all sectors to increase. Thus an increase in the demand for any sector will lead to an increase in the demand for all sectors.

11. (Solution in *study Guide*) Following the hint in the text, compute $\mathbf{p}^T\mathbf{x}$ in two ways. First, take the transpose of both sides of the price equation, $\mathbf{p}=C^T\mathbf{p}+\mathbf{v}$, to obtain

$$\mathbf{p}^T=(C^T\mathbf{p}+\mathrm{v})^T=(C^T\mathbf{p})^T+\mathbf{v}^T=\mathbf{p}^TC+\mathbf{v}^T$$

and right-multiply by **x** to get

$$\mathbf{p}^T\mathbf{x}=(\mathbf{p}^TC+\mathbf{v}^T)\mathbf{x}=\mathbf{p}^TC\mathbf{x}+\mathbf{v}^T\mathbf{x}$$

Another way to compute $\mathbf{p}^T\mathbf{x}$ starts with the production equation $\mathbf{x}=C\mathbf{x}+\mathbf{d}$. Left multiply by $\mathbf{p}^T$ to get

$$\mathbf{p}^T\mathbf{x}=\mathbf{p}^T(C\mathbf{x}+\mathbf{d})=\mathbf{p}^TC\,\mathbf{x}+\mathbf{p}^T\mathbf{d}$$

The two expression for $\mathbf{p}^T\mathbf{x}$ show that

$$\mathbf{p}^TC\,\mathbf{x}+\mathbf{v}^T\,\mathbf{x}=\mathbf{p}^TC\,\mathbf{x}+\mathbf{p}^T\mathbf{d}$$

so $\mathbf{v}^T\mathbf{x}=\mathbf{p}^T\mathbf{d}$. The *Study Guide* also provides a slightly different solution.

12. Since

$$D_{m+1}=I+C+C^2+\ldots+C^{m+1}=I+C(I+C+\ldots+C^m)=I+CD_m$$

D_{m+1} may be found iteratively by $D_{m+1}=I+CD_m$.

13. **[M]** The matrix $I-C$ is

$$\begin{bmatrix} 0.8412 & -0.0064 & -0.0025 & -0.0304 & -0.0014 & -0.0083 & -0.1594 \\ -0.0057 & 0.7355 & -0.0436 & -0.0099 & -0.0083 & -0.0201 & -0.3413 \\ -0.0264 & -0.1506 & 0.6443 & -0.0139 & -0.0142 & -0.0070 & -0.0236 \\ -0.3299 & -0.0565 & -0.0495 & 0.6364 & -0.0204 & -0.0483 & -0.0649 \\ -0.0089 & -0.0081 & -0.0333 & -0.0295 & 0.6588 & -0.0237 & -0.0020 \\ -0.1190 & -0.0901 & -0.0996 & -0.1260 & -0.1722 & 0.7632 & -0.3369 \\ -0.0063 & -0.0126 & -0.0196 & -0.0098 & -0.0064 & -0.0132 & 0.9988 \end{bmatrix}$$

so the augmented matrix $[\,I-C \;\; \mathbf{d}\,]$ may be row reduced to find

$$\begin{bmatrix} 0.8412 & -0.0064 & -0.0025 & -0.0304 & -0.0014 & -0.0083 & -0.1594 & 74000 \\ -0.0057 & 0.7355 & -0.0436 & -0.0099 & -0.0083 & -0.0201 & -0.3413 & 56000 \\ -0.0264 & -0.1506 & 0.6443 & -0.0139 & -0.0142 & -0.0070 & -0.0236 & 10500 \\ -0.3299 & -0.0565 & -0.0495 & 0.6364 & -0.0204 & -0.0483 & -0.0649 & 25000 \\ -0.0089 & -0.0081 & -0.0333 & -0.0295 & 0.6588 & -0.0237 & -0.0020 & 17500 \\ -0.1190 & -0.0901 & -0.0996 & -0.1260 & -0.1722 & 0.7632 & -0.3369 & 196000 \\ -0.0063 & -0.0126 & -0.0196 & -0.0098 & -0.0064 & -0.0132 & 0.9988 & 5000 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 99576 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 97703 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 51231 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 131570 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 49488 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 329554 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 13835 \end{bmatrix}$$

so $\mathbf{x}$ = (99576, 97703, 51321, 131570, 49488, 329554, 13835). Since the entries in $\mathbf{d}$ seem to be accurate to the nearest thousand, a more realistic answer would be $\mathbf{x}$ = (100000, 98000, 51000, 132000, 49000, 330000, 14000).

14. **[M]** The augmented matrix $[\,I-C \;\; \mathbf{d}\,]$ in this case may be row reduced to find

$$\begin{bmatrix} 0.8412 & -0.0064 & -0.0025 & -0.0304 & -0.0014 & -0.0083 & -0.1594 & 99640 \\ -0.0057 & 0.7355 & -0.0436 & -0.0099 & -0.0083 & -0.0201 & -0.3413 & 75548 \\ -0.0264 & -0.1506 & 0.6443 & -0.0139 & -0.0142 & -0.0070 & -0.0236 & 14444 \\ -0.3299 & -0.0565 & -0.0495 & 0.6364 & -0.0204 & -0.0483 & -0.0649 & 33501 \\ -0.0089 & -0.0081 & -0.0333 & -0.0295 & 0.6588 & -0.0237 & -0.0020 & 23527 \\ -0.1190 & -0.0901 & -0.0996 & -0.1260 & -0.1722 & 0.7632 & -0.3369 & 263985 \\ -0.0063 & -0.0126 & -0.0196 & -0.0098 & -0.0064 & -0.0132 & 0.9988 & 6526 \end{bmatrix}$$

$$\sim\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 134034 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 131687 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 69472 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 176912 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 66596 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 443773 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 18431 \end{bmatrix}$$

so $\mathbf{x}$ = (134034, 131687, 69472, 176912, 66596, 443773, 18431). To the nearest thousand, $\mathbf{x}$ = (134000, 132000, 69000, 177000, 67000, 444000, 18000).

15. **[M]** Here are the iterations rounded to the nearest tenth:

$$\mathbf{x}^{(0)} = (74000.0, 56000.0, 10500.0, 25000.0, 17500.0, 196000.0, 5000.0)$$
$$\mathbf{x}^{(1)} = (89344.2, 77730.5, 26708.1, 72334.7, 30325.6, 265158.2, 9327.8)$$
$$\mathbf{x}^{(2)} = (94681.2, 87714.5, 37577.3, 100520.5, 38598.0, 296563.8, 11480.0)$$
$$\mathbf{x}^{(3)} = (97091.9, 92573.1, 43867.8, 115457.0, 43491.0, 312319.0, 12598.8)$$
$$\mathbf{x}^{(4)} = (98291.6, 95033.2, 47314.5, 123202.5, 46247.0, 320502.4, 13185.5)$$
$$\mathbf{x}^{(5)} = (98907.2, 96305.3, 49160.6, 127213.7, 47756.4, 324796.1, 13493.8)$$
$$\mathbf{x}^{(6)} = (99226.6, 96969.6, 50139.6, 129296.7, 48569.3, 327053.8, 13655.9)$$
$$\mathbf{x}^{(7)} = (99393.1, 97317.8, 50656.4, 130381.6, 49002.8, 328240.9, 13741.1)$$
$$\mathbf{x}^{(8)} = (99480.0, 97500.7, 50928.7, 130948.0, 49232.5, 328864.7, 13785.9)$$
$$\mathbf{x}^{(9)} = (99525.5, 97596.8, 51071.9, 131244.1, 49353.8, 329192.3, 13809.4)$$
$$\mathbf{x}^{(10)} = (99549.4, 97647.2, 51147.2, 131399.2, 49417.7, 329364.4, 13821.7)$$
$$\mathbf{x}^{(11)} = (99561.9, 97673.7, 51186.8, 131480.4, 49451.3, 329454.7, 13828.2)$$
$$\mathbf{x}^{(12)} = (99568.4, 97687.6, 51207.5, 131523.0, 49469.0, 329502.1, 13831.6)$$

so $\mathbf{x}^{(12)}$ is the first vector whose entries are accurate to the nearest thousand. The calculation of $\mathbf{x}^{(12)}$ takes about 1260 flops, while the row reduction above takes about 550 flops. If C is larger than 20×20, then fewer flops are required to compute $\mathbf{x}^{(12)}$ by iteration than by row reduction. The advantage of the iterative method increases with the size of C. The matrix C also becomes more sparse for larger models, so fewer iterations are needed for good accuracy.

2.7 SOLUTIONS

Notes: The content of this section seems to have universal appeal with students. It also provides practice with composition of linear transformations. The case study for Chapter 2 concerns computer graphics – see this case study (available as a project on the website) for more examples of computer graphics in action. The *Study Guide* encourages the student to examine the book by Foley referenced in the text. This section could form the beginning of an independent study on computer graphics with an interested student.

1. Refer to Example 5. The representation in homogenous coordinates can be written as a partitioned matrix of the form $\begin{bmatrix} A & \mathbf{0} \\ \mathbf{0}^T & 1 \end{bmatrix}$, where A is the matrix of the linear transformation. Since in this case $A = \begin{bmatrix} 1 & .25 \\ 0 & 1 \end{bmatrix}$, the representation of the transformation with respect to homogenous coordinates is

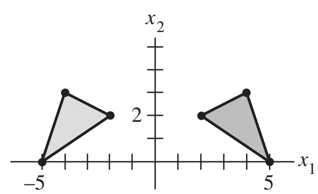
$$\begin{bmatrix} 1 & .25 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Note: The *Study Guide* shows the student why the action of $\begin{bmatrix} A & \mathbf{0} \\ \mathbf{0}^T & 1 \end{bmatrix}$ on the vector $\begin{bmatrix} \mathrm{x} \\ 1 \end{bmatrix}$ corresponds to the action of A on $\mathbf{x}$.

2. The matrix of the transformation is $A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$, so the transformed data matrix is

$$AD = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 5 & 2 & 4 \\ 0 & 2 & 3 \end{bmatrix} = \begin{bmatrix} -5 & -2 & -4 \\ 0 & 2 & 3 \end{bmatrix}$$

Both the original triangle and the transformed triangle are shown in the following sketch.



3. Following Examples 4–6,

$$\begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 & 0 \\ \sqrt{2}/2 & \sqrt{2}/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 & \sqrt{2} \\ \sqrt{2}/2 & \sqrt{2}/2 & 2\sqrt{2} \\ 0 & 0 & 1 \end{bmatrix}$$

4. $$\begin{bmatrix} .8 & 0 & 0 \\ 0 & 1.2 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} .8 & 0 & -1.6 \\ 0 & 1.2 & 3.6 \\ 0 & 0 & 1 \end{bmatrix}$$

5. $$\begin{bmatrix} \sqrt{3}/2 & -1/2 & 0 \\ 1/2 & \sqrt{3}/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \sqrt{3}/2 & 1/2 & 0 \\ 1/2 & -\sqrt{3}/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

6. $$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \sqrt{3}/2 & -1/2 & 0 \\ 1/2 & \sqrt{3}/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \sqrt{3}/2 & -1/2 & 0 \\ -1/2 & -\sqrt{3}/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

7. A 60° rotation about the origin is given in homogeneous coordinates by the matrix $\begin{bmatrix} 1/2 & -\sqrt{3}/2 & 0 \\ \sqrt{3}/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. To rotate about the point (6, 8), first translate by (–6, –8), then rotate about the origin, then translate back by (6, 8) (see the Practice Problem in this section). A 60° rotation about (6, 8) is thus is given in homogeneous coordinates by the matrix

$$\begin{bmatrix} 1 & 0 & 6 \\ 0 & 1 & 8 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1/2 & -\sqrt{3}/2 & 0 \\ \sqrt{3}/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & -6 \\ 0 & 1 & -8 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/2 & -\sqrt{3}/2 & 3+4\sqrt{3} \\ \sqrt{3}/2 & 1/2 & 4-3\sqrt{3} \\ 0 & 0 & 1 \end{bmatrix}$$

8. A 45° rotation about the origin is given in homogeneous coordinates by the matrix $\begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 & 0 \\ \sqrt{2}/2 & \sqrt{2}/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. To rotate about the point (3, 7), first translate by (–3, –7), then rotate about the origin, then translate back by (3, 7) (see the Practice Problem in this section). A 45° rotation about (3, 7) is thus is given in homogeneous coordinates by the matrix

$$\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 7 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 & 0 \\ \sqrt{2}/2 & \sqrt{2}/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & -7 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 & 3+2\sqrt{2} \\ \sqrt{2}/2 & \sqrt{2}/2 & 7-5\sqrt{2} \\ 0 & 0 & 1 \end{bmatrix}$$

9. To produce each entry in *BD* two multiplications are necessary. Since *BD* is a 2×200 matrix, it will take $2\times 2\times 200 = 800$ multiplications to compute *BD*. By the same reasoning it will take $2\times 2\times 200 =$ 800 multiplications to compute *A*(*BD*). Thus to compute *A*(*BD*) from the beginning will take 800 + 800 = 1600 multiplications.

To compute the 2×2 matrix *AB* it will take $2\times 2\times 2 = 8$ multiplications, and to compute (*AB*)*D* it will take $2\times 2\times 200 = 800$ multiplications. Thus to compute (*AB*)*D* from the beginning will take 8 + 800 = 808 multiplications.

For computer graphics calculations that require applying multiple transformations to data matrices, it is thus more efficient to compute the product of the transformation matrices before applying the result to the data matrix.

10. Let the transformation matrices in homogeneous coordinates for the dilation, rotation, and translation be called respectively *D*, and *R*, and *T*. Then for some value of *s*, φ, *h*, and *k*,

$$D = \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & 1 \end{bmatrix}, R = \begin{bmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}, T = \begin{bmatrix} 1 & 0 & h \\ 0 & 1 & k \\ 0 & 0 & 1 \end{bmatrix}$$

Compute the products of these matrices:

$$DR = \begin{bmatrix} s\cos\varphi & -s\sin\varphi & 0 \\ s\sin\varphi & s\cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}, RD = \begin{bmatrix} s\cos\varphi & -s\sin\varphi & 0 \\ s\sin\varphi & s\cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$DT = \begin{bmatrix} s & 0 & sh \\ 0 & s & sk \\ 0 & 0 & 1 \end{bmatrix}, TD = \begin{bmatrix} s & 0 & h \\ 0 & s & k \\ 0 & 0 & 1 \end{bmatrix}$$

$$RT = \begin{bmatrix} \cos\varphi & -\sin\varphi & h\cos\varphi - k\sin\varphi \\ \sin\varphi & \cos\varphi & h\sin\varphi + k\cos\varphi \\ 0 & 0 & 1 \end{bmatrix}, TR = \begin{bmatrix} \cos\varphi & -\sin\varphi & h \\ \sin\varphi & \cos\varphi & k \\ 0 & 0 & 1 \end{bmatrix}$$

Since $DR = RD$, $DT \neq TD$ and $RT \neq TR$, D and R commute, D and T do not commute and R and T do not commute.

11. To simplify A_2A_1 completely, the following trigonometric identities will be needed:

1. $-\tan\varphi\cos\varphi = -\frac{\sin\varphi}{\cos\varphi}\cos\varphi = -\sin\varphi$

2. $\sec\varphi - \tan\varphi\sin\varphi = \frac{1}{\cos\varphi} - \frac{\sin\varphi}{\cos\varphi}\sin\varphi = \frac{1-\sin^2\varphi}{\cos\varphi} = \frac{\cos^2\varphi}{\cos\varphi} = \cos\varphi$

Using these identities,

$$A_2A_1 = \begin{bmatrix} \sec\varphi & -\tan\varphi & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \sec\varphi - \tan\varphi\sin\varphi & -\tan\varphi\cos\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

which is the transformation matrix in homogeneous coordinates for a rotation in $\mathbb{R}^2$.

12. To simplify this product completely, the following trigonometric identity will be needed:

$$\tan\varphi/2 = \frac{1-\cos\varphi}{\sin\varphi} = \frac{\sin\varphi}{1+\cos\varphi}$$

This identity has two important consequences:

$$1-(\tan\varphi/2)(\sin\varphi) = 1 - \frac{1-\cos\varphi}{\sin\varphi}\sin\varphi = \cos\varphi$$

$$(\cos\varphi)(-\tan\varphi/2) - \tan\varphi/2 = -(\cos\varphi+1)\tan\varphi/2 = -(\cos\varphi+1)\frac{\sin\varphi}{1+\cos\varphi} = -\sin\varphi$$

The product may be computed and simplified using these results:

$$\begin{bmatrix} 1 & -\tan\varphi/2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ \sin\varphi & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -\tan\varphi/2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1-(\tan\varphi/2)(\sin\varphi) & -\tan\varphi/2 & 0 \\ \sin\varphi & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -\tan\varphi/2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\varphi & -\tan\varphi/2 & 0 \\ \sin\varphi & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -\tan\varphi/2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\varphi & (\cos\varphi)(-\tan\varphi/2) - \tan\varphi/2 & 0 \\ \sin\varphi & -(\sin\varphi)(\tan\varphi/2) + 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

which is the transformation matrix in homogeneous coordinates for a rotation in $\mathbb{R}^2$.

13. Consider first applying the linear transformation on $\mathbb{R}^2$ whose matrix is A, then applying a translation by the vector $\mathbf{p}$ to the result. The matrix representation in homogeneous coordinates of the linear transformation is $\begin{bmatrix} A & \mathbf{0} \\ \mathbf{0}^T & 1 \end{bmatrix}$, while the matrix representation in homogeneous coordinates of the translation is $\begin{bmatrix} I & \mathbf{p} \\ \mathbf{0}^T & 1 \end{bmatrix}$. Applying these transformations in order leads to a transformation whose matrix representation in homogeneous coordinates is

$$\begin{bmatrix} I & \mathbf{p} \\ \mathbf{0}^T & 1 \end{bmatrix} \begin{bmatrix} A & \mathbf{0} \\ \mathbf{0}^T & 1 \end{bmatrix} = \begin{bmatrix} A & \mathbf{p} \\ \mathbf{0}^T & 1 \end{bmatrix}$$

which is the desired matrix.

14. The matrix for the transformation in Exercise 7 was found to be

$$\begin{bmatrix} 1/2 & -\sqrt{3}/2 & 3+4\sqrt{3} \\ \sqrt{3}/2 & 1/2 & 4-3\sqrt{3} \\ 0 & 0 & 1 \end{bmatrix}$$

This matrix is of the form $\begin{bmatrix} A & \mathbf{p} \\ \mathbf{0}^T & 1 \end{bmatrix}$, where

$$A = \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix}, \mathbf{p} = \begin{bmatrix} 3+4\sqrt{3} \\ 4-3\sqrt{3} \end{bmatrix}$$

By Exercise 13, this matrix may be written as

$$\begin{bmatrix} I & \mathbf{p} \\ \mathbf{0}^T & 1 \end{bmatrix} \begin{bmatrix} A & \mathbf{0} \\ \mathbf{0}^T & 1 \end{bmatrix}$$

that is, the composition of a linear transformation on $\mathbb{R}^2$ and a translation. The matrix A is the matrix of a rotation about the origin in $\mathbb{R}^2$. Thus the transformation in Exercise 7 is the composition of a rotation about the origin and a translation by $\mathbf{p} = \begin{bmatrix} 3+4\sqrt{3} \\ 4-3\sqrt{3} \end{bmatrix}$.

15. Since $(X, Y, Z, H) = (\frac{1}{2}, -\frac{1}{4}, \frac{1}{8}, \frac{1}{24})$, the corresponding point in $\mathbb{R}^3$ has coordinates

$$(x, y, z) = \left(\frac{X}{H}, \frac{Y}{H}, \frac{Z}{H}\right) = \left(\frac{\frac{1}{2}}{\frac{1}{24}}, \frac{-\frac{1}{4}}{\frac{1}{24}}, \frac{\frac{1}{8}}{\frac{1}{24}}\right) = (12, -6, 3)$$

16. The homogeneous coordinates (1, –2, 3, 4) represent the point

$$(1/4, -2/4, 3/4) = (1/4, -1/2, 3/4)$$

while the homogeneous coordinates (10, –20, 30, 40) represent the point

$$(10/40, -20/40, 30/40) = (1/4, -1/2, 3/4)$$

so the two sets of homogeneous coordinates represent the same point in $\mathbb{R}^3$.

17. Follow Example 7a by first constructing that 3×3 matrix for this rotation. The vector $\mathbf{e}_1$ is not changed by this rotation. The vector $\mathbf{e}_2$ is rotated 60° toward the positive z-axis, ending up at the point (0, cos 60°, sin 60°) = $(0, 1/2, \sqrt{3}/2)$. The vector $\mathbf{e}_3$ is rotated 60° toward the negative y-axis, stopping at the point (0, cos 150°, sin 150°) = $(0, -\sqrt{3}/2, 1/2)$. The matrix A for this rotation is thus

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & -\sqrt{3}/2 \\ 0 & \sqrt{3}/2 & 1/2 \end{bmatrix}$$

so in homogeneous coordinates the transformation is represented by the matrix

$$\begin{bmatrix} A & \mathbf{0} \\ \mathbf{0}^T & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/2 & -\sqrt{3}/2 & 0 \\ 0 & \sqrt{3}/2 & 1/2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

18. First construct the 3×3 matrix for the rotation. The vector $\mathbf{e}_1$ is rotated 30° toward the negative y-axis, ending up at the point (cos(–30)°, sin (–30)°, 0) = $(\sqrt{3}/2, -1/2, 0)$. The vector $\mathbf{e}_2$ is rotated 60° toward the positive x-axis, ending up at the point (cos 60°, sin 60°, 0) $= (1/2, \sqrt{3}/2, 0)$. The vector $\mathbf{e}_3$ is not changed by the rotation. The matrix A for the rotation is thus

$$A = \begin{bmatrix} \sqrt{3}/2 & 1/2 & 0 \\ -1/2 & \sqrt{3}/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

so in homogeneous coordinates the rotation is represented by the matrix

$$\begin{bmatrix} A & \mathbf{0} \\ \mathbf{0}^T & 1 \end{bmatrix} = \begin{bmatrix} \sqrt{3}/2 & 1/2 & 0 & 0 \\ -1/2 & \sqrt{3}/2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Following Example 7b, in homogeneous coordinates the translation by the vector (5, –2, 1) is represented by the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Thus the complete transformation is represented in homogeneous coordinates by the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \sqrt{3}/2 & 1/2 & 0 & 0 \\ -1/2 & \sqrt{3}/2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \sqrt{3}/2 & 1/2 & 0 & 5 \\ -1/2 & \sqrt{3}/2 & 0 & -2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

19. Referring to the material preceding Example 8 in the text, we find that the matrix P that performs a perspective projection with center of projection (0, 0, 10) is

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -.1 & 1 \end{bmatrix}$$

The homogeneous coordinates of the vertices of the triangle may be written as (4.2, 1.2, 4, 1), (6, 4, 2, 1), and (2, 2, 6, 1), so the data matrix for S is

$$\begin{bmatrix} 4.2 & 6 & 2 \\ 1.2 & 4 & 2 \\ 4 & 2 & 6 \\ 1 & 1 & 1 \end{bmatrix}$$

and the data matrix for the transformed triangle is

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -.1 & 1 \end{bmatrix}\begin{bmatrix} 4.2 & 6 & 2 \\ 1.2 & 4 & 2 \\ 4 & 2 & 6 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 4.2 & 6 & 2 \\ 1.2 & 4 & 2 \\ 0 & 0 & 0 \\ .6 & .8 & .4 \end{bmatrix}$$

Finally, the columns of this matrix may be converted from homogeneous coordinates by dividing by the final coordinate:

$$\begin{aligned} &(4.2, 1.2, 0, .6) \to (4.2/.6, 1.2/.6, 0/.6) = (7, 2, 0) \\ &(6, 4, 0, .8) \quad\ \to (6/.8, 2/.8, 0/.8) = (7.5, 5, 0) \\ &(2, 2, 0, .4) \quad\ \to (2/.4, 2/.4, 0/.4) = (5, 5, 0) \end{aligned}$$

So the coordinates of the vertices of the transformed triangle are (7, 2, 0), (7.5, 5, 0), and (5, 5, 0).

20. As in the previous exercise, the matrix P that performs the perspective projection is

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -.1 & 1 \end{bmatrix}$$

The homogeneous coordinates of the vertices of the triangle may be written as (9, 3, –5, 1), (12, 8, 2, 1), and (1.8, 2.7, 1, 1), so the data matrix for S is

$$\begin{bmatrix} 9 & 12 & 1.8 \\ 3 & 8 & 2.7 \\ -5 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

and the data matrix for the transformed triangle is

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -.1 & 1 \end{bmatrix}\begin{bmatrix} 9 & 12 & 1.8 \\ 3 & 8 & 2.7 \\ -5 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 9 & 12 & 1.8 \\ 3 & 8 & 2.7 \\ 0 & 0 & 0 \\ 1.5 & .8 & .9 \end{bmatrix}$$

Finally, the columns of this matrix may be converted from homogeneous coordinates by dividing by the final coordinate:

$$\begin{aligned} &(9, 3, 0, 1.5) &&\to (9/1.5, 3/1.5, 0/1.5) = (6, 2, 0) \\ &(12, 8, 0, .8) &&\to (12/.8, 8/.8, 0/.8) = (15, 10, 0) \\ &(1.8, 2.7, 0, .9) &&\to (1.8/.9, 2.7/.9, 0/.9) = (2, 3, 0) \end{aligned}$$

So the coordinates of the vertices of the transformed triangle are (6, 2, 0), (15, 10, 0), and (2, 3, 0).

21. **[M]** Solve the given equation for the vector (R, G, B), giving

$$\begin{bmatrix} R \\ G \\ B \end{bmatrix} = \begin{bmatrix} .61 & .29 & .15 \\ .35 & .59 & .063 \\ .04 & .12 & .787 \end{bmatrix}^{-1}\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} 2.2586 & -1.0395 & -.3473 \\ -1.3495 & 2.3441 & .0696 \\ .0910 & -.3046 & 1.2777 \end{bmatrix}\begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

22. **[M]** Solve the given equation for the vector (R, G, B), giving

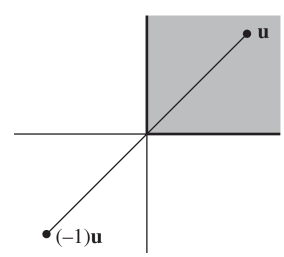
$$\begin{bmatrix} R \\ G \\ B \end{bmatrix} = \begin{bmatrix} .299 & .587 & .114 \\ .596 & -.275 & -.321 \\ .212 & -.528 & .311 \end{bmatrix}^{-1}\begin{bmatrix} Y \\ I \\ Q \end{bmatrix} = \begin{bmatrix} 1.0031 & .9548 & .6179 \\ .9968 & -.2707 & -.6448 \\ 1.0085 & -1.1105 & 1.6996 \end{bmatrix}\begin{bmatrix} Y \\ I \\ Q \end{bmatrix}$$

2.8 SOLUTIONS

Notes: Cover this section only if you plan to skip most or all of Chapter 4. This section and the next cover everything you need from Sections 4.1–4.6 to discuss the topics in Section 4.9 and Chapters 5–7 (except for the general inner product spaces in Sections 6.7 and 6.8). Students may use Section 4.2 for review, particularly the Table near the end of the section. (The final subsection on linear transformations should be omitted.) Example 6 and the associated exercises are critical for work with eigenspaces in Chapters 5 and 7. Exercises 31–36 review the Invertible Matrix Theorem. New statements will be added to this theorem in Section 2.9.

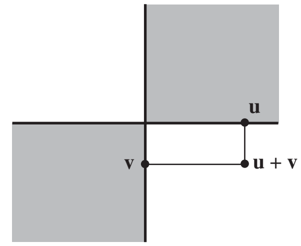
Key Exercises: 5–20 and 23–26.

1. The set is closed under sums but not under multiplication by a negative scalar. A counterexample to the subspace condition is shown at the right.

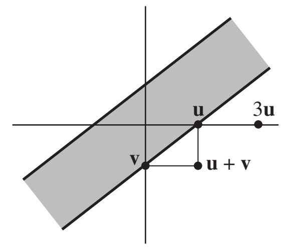


Note: Most students prefer to give a geometric counterexample, but some may choose an algebraic calculation. The four exercises here should help students develop an understanding of subspaces, but they may be insufficient if you want students to be able to analyze an unfamiliar set on an exam. Developing that skill seems more appropriate for classes covering Sections 4.1–4.6.

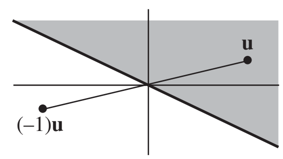
2. The set is closed under scalar multiples but not sums. For example, the sum of the vectors **u** and **v** shown here is not in H.



3. No. The set is not closed under sums or scalar multiples. The subset consisting of the points on the line $x_2 = x_1$ is a subspace, so any "counterexample" must use at least one point not on this line. Here are two counterexamples to the subspace conditions:



4. No. The set is closed under sums, but not under multiplication by a negative scalar.



5. The vector **w** is in the subspace generated by $\mathbf{v}_1$ and $\mathbf{v}_2$ if and only if the vector equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 = \mathbf{w}$ is consistent. The row operations below show that **w** is *not* in the subspace generated by $\mathbf{v}_1$ and $\mathbf{v}_2$.

$$[\mathbf{v}_1\ \ \mathbf{v}_2\ \ \mathbf{w}] \sim \begin{bmatrix} 2 & -4 & 8 \\ 3 & -5 & 2 \\ -5 & 8 & -9 \end{bmatrix} \sim \begin{bmatrix} 2 & -4 & 8 \\ 0 & 1 & -10 \\ 0 & -2 & 11 \end{bmatrix} \sim \begin{bmatrix} 2 & -4 & 8 \\ 0 & 1 & -10 \\ 0 & 0 & -9 \end{bmatrix}$$

6. The vector **u** is in the subspace generated by $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ if and only if the vector equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{u}$ is consistent. The row operations below show that **u** is *not* in the subspace generated by $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

$$[\mathbf{v}_1\ \ \mathbf{v}_2\ \ \mathbf{v}_3\ \ \mathbf{u}] \sim \begin{bmatrix} 1 & 4 & 5 & -4 \\ -2 & -7 & -8 & 10 \\ 4 & 9 & 6 & -7 \\ 3 & 7 & 5 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 5 & -4 \\ 0 & 1 & 2 & 2 \\ 0 & -7 & -14 & 9 \\ 0 & -5 & -10 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 5 & -4 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 23 \\ 0 & 0 & 0 & 17 \end{bmatrix}$$

Note: For a quiz, you could use $\mathbf{w} = (1, -3, 11, 8)$, which is *in* Span$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

7. **a**. There are three vectors: $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ in the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

b. There are infinitely many vectors in Span$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ = Col A.

c. Deciding whether **p** is in Col A requires calculation:

$$[A\ \ \mathbf{p}] \sim \begin{bmatrix} 2 & -3 & -4 & 6 \\ -8 & 8 & 6 & -10 \\ 6 & -7 & -7 & 11 \end{bmatrix} \sim \begin{bmatrix} 2 & -3 & -4 & 6 \\ 0 & -4 & -10 & 14 \\ 0 & 2 & 5 & -7 \end{bmatrix} \sim \begin{bmatrix} 2 & -3 & -4 & 6 \\ 0 & -4 & -10 & 14 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The equation $A\mathbf{x} = \mathbf{p}$ has a solution, so **p** is in Col A.

8. $[A \quad \mathbf{p}] = \begin{bmatrix} -3 & -2 & 0 & 1 \\ 0 & 2 & -6 & 14 \\ 6 & 3 & 3 & -9 \end{bmatrix} \sim \begin{bmatrix} -3 & -2 & 0 & 1 \\ 0 & 2 & -6 & 14 \\ 0 & -1 & 3 & -7 \end{bmatrix} \sim \begin{bmatrix} -3 & -2 & 0 & 1 \\ 0 & 2 & -6 & 14 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Yes, the augmented matrix $[A \quad \mathbf{p}]$ corresponds to a consistent system, so $\mathbf{p}$ is in Col A.

9. To determine whether $\mathbf{p}$ is in Nul A, simply compute $A\mathbf{p}$. Using A and $\mathbf{p}$ as in Exercise 7,

$A\mathbf{p} = \begin{bmatrix} 2 & -3 & -4 \\ -8 & 8 & 6 \\ 6 & -7 & -7 \end{bmatrix}\begin{bmatrix} 6 \\ -10 \\ 11 \end{bmatrix} = \begin{bmatrix} -2 \\ -62 \\ 29 \end{bmatrix}$. Since $A\mathbf{p} \neq \mathbf{0}$, $\mathbf{p}$ is *not* in Nul A.

10. To determine whether $\mathbf{u}$ is in Nul A, simply compute $A\mathbf{u}$. Using A as in Exercise 7 and $\mathbf{u} = (-2, 3, 1)$,

$A\mathbf{u} = \begin{bmatrix} -3 & -2 & 0 \\ 0 & 2 & -6 \\ 6 & 3 & 3 \end{bmatrix}\begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. Yes, $\mathbf{u}$ is in Nul A.

11. $p = 4$ and $q = 3$. Nul A is a subspace of $\mathbf{R}^4$ because solutions of $A\mathbf{x} = \mathbf{0}$ must have 4 entries, to match the columns of A. Col A is a subspace of $\mathbf{R}^3$ because each column vector has 3 entries.

12. $p = 3$ and $q = 4$. Nul A is a subspace of $\mathbf{R}^3$ because solutions of $A\mathbf{x} = \mathbf{0}$ must have 3 entries, to match the columns of A. Col A is a subspace of $\mathbf{R}^4$ because each column vector has 4 entries.

13. To produce a vector in Col A, select any column of A. For Nul A, solve the equation $A\mathbf{x} = \mathbf{0}$. (Include an augmented column of zeros, to avoid errors.)

$$\begin{bmatrix} 3 & 2 & 1 & -5 & 0 \\ -9 & -4 & 1 & 7 & 0 \\ 9 & 2 & -5 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 3 & 2 & 1 & -5 & 0 \\ 0 & 2 & 4 & -8 & 0 \\ 0 & -4 & -8 & 16 & 0 \end{bmatrix} \sim \begin{bmatrix} 3 & 2 & 1 & -5 & 0 \\ 0 & 2 & 4 & -8 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 3 & 2 & 1 & -5 & 0 \\ 0 & 1 & 2 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 2 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{aligned} x_1 \quad - x_3 + x_4 &= 0 \\ x_2 + 2x_3 - 4x_4 &= 0 \\ 0 &= 0 \end{aligned}$$

The general solution is $x_1 = x_3 - x_4$, and $x_2 = -2x_3 + 4x_4$, with x_3 and x_4 free. The general solution in parametric vector form is not needed. All that is required here is one nonzero vector. So choose any values for x_3 and x_4 (not both zero). For instance, set $x_3 = 1$ and $x_4 = 0$ to obtain the vector $(1, -2, 1, 0)$ in Nul A.

Note: Section 2.8 of *Study Guide* introduces the **`ref`** command (or **`rref`**, depending on the technology), which produces the reduced echelon form of a matrix. This will greatly speed up homework for students who have a matrix program available.

14. To produce a vector in Col A, select any column of A. For Nul A, solve the equation $A\mathbf{x} = \mathbf{0}$:

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 4 & 5 & 7 & 0 \\ -5 & -1 & 0 & 0 \\ 2 & 7 & 11 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -3 & -5 & 0 \\ 0 & 9 & 15 & 0 \\ 0 & 3 & 5 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 1 & 5/3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1/3 & 0 \\ 0 & 1 & 5/3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The general solution is $x_1 = (1/3)x_3$ and $x_2 = (-5/3)\,x_3$, with x_3 free. The general solution in parametric vector form is not needed. All that is required here is one nonzero vector. So choose any values of x_3 and x_4 (not both zero). For instance, set $x_3 = 3$ to obtain the vector $(1, -5, 3)$ in Nul A.

15. Yes. Let A be the matrix whose columns are the vectors given. Then A is invertible because its determinant is nonzero, and so its columns form a basis for $\mathbf{R}^2$, by the Invertible Matrix Theorem (or by Example 5). (Other reasons for the invertibility of A could be given.)

16. No. One vector is a multiple of the other, so they are linearly dependent and hence cannot be a basis for any subspace.

17. No. Place the three vectors into a 3×3 matrix A and determine whether A is invertible:

$$A=\begin{bmatrix} 0 & 5 & 6 \\ 1 & -7 & 3 \\ -2 & 4 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & -7 & 3 \\ 0 & 5 & 6 \\ -2 & 4 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & -7 & 3 \\ 0 & 5 & 6 \\ 0 & -10 & 11 \end{bmatrix} \sim \begin{bmatrix} 1 & -7 & 3 \\ 0 & 5 & 6 \\ 0 & 0 & 23 \end{bmatrix}$$

The matrix A has three pivots, so A is invertible by the IMT and its columns form a basis for $\mathbf{R}^3$ (as pointed out in Example 5).

18. Yes. Place the three vectors into a 3×3 matrix A and determine whether A is invertible:

$$A=\begin{bmatrix} 1 & -5 & 7 \\ 1 & -1 & 0 \\ -2 & 2 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & -5 & 7 \\ 0 & 4 & -7 \\ 0 & -8 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & -5 & 7 \\ 0 & 4 & -7 \\ 0 & 0 & -5 \end{bmatrix}$$

The matrix A has three pivots, so A is invertible by the IMT and its columns form a basis for $\mathbf{R}^3$ (as pointed out in Example 5).

19. No. The vectors cannot be a basis for $\mathbf{R}^3$ because they only span a plan in $\mathbf{R}^3$. Or, point out that the columns of the matrix $\begin{bmatrix} 1 & -5 \\ 1 & -1 \\ -2 & 2 \end{bmatrix}$ cannot possibly span $\mathbf{R}^3$ because the matrix cannot have a pivot in every row. So the columns are not a basis for $\mathbf{R}^3$.

Note: The *Study Guide* warns students not to say that the two vectors here are a basis for $\mathbf{R}^2$.

20. No. The vectors are linearly dependent because there are more vectors in the set than entries in each vector. (Theorem 8 in Section 1.7.) So the vectors cannot be a basis for any subspace.

21. **a**. False. See the definition at the beginning of the section. The critical phrases “for each” are missing.
b. True. See the paragraph before Example 4.
c. False. See Theorem 12. The null space is a subspace of $\mathbf{R}^n$, not $\mathbf{R}^m$.
d. True. See Example 5.
e. True. See the first part of the solution of Example 8.

22. **a**. False. See the definition at the beginning of the section. The condition about the zero vector is only one of the conditions for a subspace.
b. True. See Example 3.
c. True. See Theorem 12.
d. False. See the paragraph after Example 4.
e. False. See the Warning that follows Theorem 13.

23. (Solution in *Study Guide*) $A=\begin{bmatrix}4&5&9&-2\\6&5&1&12\\3&4&8&-3\end{bmatrix}\sim\begin{bmatrix}1&2&6&-5\\0&1&5&-6\\0&0&0&0\end{bmatrix}$. The echelon form identifies columns 1 and 2 as the pivot columns. A basis for Col A uses columns 1 and 2 of A: $\begin{bmatrix}4\\6\\3\end{bmatrix},\begin{bmatrix}5\\5\\4\end{bmatrix}$. This is not the only choice, but it is the "standard" choice. A *wrong* choice is to select columns 1 and 2 of the echelon form. These columns have zero in the third entry and could not possibly generate the columns displayed in A.

24. For Nul A, obtain the reduced (and augmented) echelon form for $A\mathbf{x}=\mathbf{0}$:

$$\begin{bmatrix}1&0&-4&7&0\\0&1&5&-6&0\\0&0&0&0&0\end{bmatrix}. \text{ This corresponds to: } \begin{aligned}x_1 \quad - 4x_3 + 7x_4 &= 0\\ x_2 + 5x_3 - 6x_4 &= 0.\\ 0 &= 0\end{aligned}$$

Solve for the basic variables and write the solution of $A\mathbf{x}=\mathbf{0}$ in parametric vector form:

$$\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=\begin{bmatrix}4x_3-7x_4\\-5x_3+6x_4\\x_3\\x_4\end{bmatrix}=x_3\begin{bmatrix}4\\-5\\1\\0\end{bmatrix}+x_4\begin{bmatrix}-7\\6\\0\\1\end{bmatrix}. \text{ Basis for Nul } A\text{: } \begin{bmatrix}4\\-5\\1\\0\end{bmatrix},\begin{bmatrix}-7\\6\\0\\1\end{bmatrix}$$

Notes: (1) A basis is a *set* of vectors. For simplicity, the answers here and in the text list the vectors without enclosing the list inside set brackets. This style is also easier for students. I am careful, however, to distinguish between a matrix and the set or list whose elements are the columns of the matrix.

(2) Recall from Chapter 1 that students are encouraged to use the augmented matrix when solving $A\mathbf{x}=\mathbf{0}$, to avoid the common error of misinterpreting the reduced echelon form of A as itself the augmented matrix for a nonhomogeneous system.

(3) Because the concept of a basis is just being introduced, I insist that my students write the parametric vector form of the solution of $A\mathbf{x}=\mathbf{0}$. They see how the basis vectors span the solution space and are obviously linearly independent. A shortcut, which some instructors might introduce later in the course, is only to solve for the basic variables and to produce each basis vector one at a time. Namely, set all free variables equal to zero except for one free variable, and set that variable equal to a suitable nonzero number.

24. $A=\begin{bmatrix}-3&9&-2&-7\\2&-6&4&8\\3&-9&-2&2\end{bmatrix}\sim\begin{bmatrix}1&-3&6&9\\0&0&4&5\\0&0&0&0\end{bmatrix}$. Basis for Col A: $\begin{bmatrix}-3\\2\\3\end{bmatrix},\begin{bmatrix}-2\\4\\-2\end{bmatrix}$.

For Nul A, obtain the reduced (and augmented) echelon form for $A\mathbf{x}=\mathbf{0}$:

$$\begin{bmatrix}1&-3&0&1.50&0\\0&0&1&1.25&0\\0&0&0&0&0\end{bmatrix}. \text{ This corresponds to: } \begin{aligned}x_1 - 3x_2 \quad + 1.50x_4 &= 0\\ x_3 + 1.25x_4 &= 0.\\ 0 &= 0\end{aligned}$$

Solve for the basic variables and write the solution of $A\mathbf{x}=\mathbf{0}$ in parametric vector form:

$$\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=\begin{bmatrix}3x_2-1.5x_4\\x_2\\-1.25x_4\\x_4\end{bmatrix}=x_2\begin{bmatrix}3\\1\\0\\0\end{bmatrix}+x_4\begin{bmatrix}-1.5\\0\\-1.25\\1\end{bmatrix}. \text{ Basis for Nul } A\text{: } \begin{bmatrix}3\\1\\0\\0\end{bmatrix},\begin{bmatrix}-1.5\\0\\-1.25\\1\end{bmatrix}.$$

25. $A = \begin{bmatrix} 1 & 4 & 8 & -3 & -7 \\ -1 & 2 & 7 & 3 & 4 \\ -2 & 2 & 9 & 5 & 5 \\ 3 & 6 & 9 & -5 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 8 & 0 & 5 \\ 0 & 2 & 5 & 0 & -1 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. Basis for Col A: $\begin{bmatrix} 1 \\ -1 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 2 \\ 6 \end{bmatrix}, \begin{bmatrix} -3 \\ 3 \\ 5 \\ -5 \end{bmatrix}$.

For Nul A, obtain the reduced (and augmented) echelon form for $A\mathbf{x} = \mathbf{0}$:

$$[A \quad \mathbf{0}] \sim \begin{bmatrix} 1 & 0 & -2 & 0 & 7 & 0 \\ 0 & 1 & 2.5 & 0 & -.5 & 0 \\ 0 & 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \qquad \begin{array}{rcl} x_1 - 2x_3 + 7x_5 & = & 0 \\ x_2 + 2.5x_3 - .5x_5 & = & 0 \\ x_4 + 4x_5 & = & 0 \\ 0 & = & 0 \end{array}.$$

The solution of $A\mathbf{x} = 0$ in parametric vector form: $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2x_3 - 7x_5 \\ -2.5x_3 + .5x_5 \\ x_3 \\ -4x_5 \\ x_5 \end{bmatrix} = x_3 \underbrace{\begin{bmatrix} 2 \\ -2.5 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{\mathbf{u}} + x_5 \underbrace{\begin{bmatrix} -7 \\ .5 \\ 0 \\ -4 \\ 1 \end{bmatrix}}_{\mathbf{v}}$.

Basis for Nul A: $\{\mathbf{u}, \mathbf{v}\}$.

Note: The solution above illustrates how students could write a solution on an exam, when time is precious, namely, describe the basis by giving names to appropriate vectors found in the calculations.

26. $A = \begin{bmatrix} 3 & -1 & 7 & 3 & 9 \\ -2 & 2 & -2 & 7 & 5 \\ -5 & 9 & 3 & 3 & 4 \\ -2 & 6 & 6 & 3 & 7 \end{bmatrix} \sim \begin{bmatrix} 3 & -1 & 7 & 0 & 6 \\ 0 & 2 & 4 & 0 & 3 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. Basis for Col A: $\begin{bmatrix} 3 \\ -2 \\ -5 \\ -2 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 9 \\ 6 \end{bmatrix}, \begin{bmatrix} 3 \\ 7 \\ 3 \\ 3 \end{bmatrix}$.

For Nul A,

$$[A \quad \mathbf{0}] \sim \begin{bmatrix} 1 & 0 & 3 & 0 & 2.5 & 0 \\ 0 & 1 & 2 & 0 & 1.5 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \qquad \begin{array}{rcl} x_1 + 3x + 2.5x_5 & = & 0 \\ x_2 + 2x_3 + 1.5x_5 & = & 0 \\ x_4 + x_5 & = & 0 \\ 0 & = & 0 \end{array}$$

The solution of $A\mathbf{x} = \mathbf{0}$ in parametric vector form:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -3x_3 - 2.5x_5 \\ -2x_3 - 1.5x_5 \\ x_3 \\ -x_5 \\ x_5 \end{bmatrix} = x_3 \underbrace{\begin{bmatrix} -3 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{\mathbf{u}} + x_5 \underbrace{\begin{bmatrix} -2.5 \\ -1.5 \\ 0 \\ -1 \\ 1 \end{bmatrix}}_{\mathbf{v}}.$$ Basis for Nul A: $\{\mathbf{u}, \mathbf{v}\}$.

27. Construct a nonzero 3×3 matrix A and construct $\mathbf{b}$ to be almost any convenient linear combination of the columns of A.

28. The easiest construction is to write a 3×3 matrix in echelon form that has only 2 pivots, and let **b** be any vector in $\mathbf{R}^3$ whose third entry is nonzero.

29. (Solution in *Study Guide*) A simple construction is to write any nonzero 3×3 matrix whose columns are obviously linearly dependent, and then make **b** a vector of weights from a linear dependence relation among the columns. For instance, if the first two columns of A are equal, then **b** could be (1, –1, 0).

30. Since Col A is the set of all linear combinations of $\mathbf{a}_1, \ldots, \mathbf{a}_p$, the set $\{\mathbf{a}_1, \ldots, \mathbf{a}_p\}$ spans Col A. Because $\{\mathbf{a}_1, \ldots, \mathbf{a}_p\}$ is also linearly independent, it is a basis for Col A. (There is no need to discuss pivot columns and Theorem 13, though a proof could be given using this information.)

31. If Col $F \neq \mathbf{R}^5$, then the columns of F do not span $\mathbf{R}^5$. Since F is square, the IMT shows that F is not invertible and the equation $F\mathbf{x} = \mathbf{0}$ has a nontrivial solution. That is, Nul F contains a nonzero vector. Another way to describe this is to write Nul $F \neq \{\mathbf{0}\}$.

32. If Nul R contains nonzero vectors, then the equation $R\mathbf{x} = \mathbf{0}$ has nontrivial solutions. Since R is square, the IMT shows that R is not invertible and the columns of R do not span $\mathbf{R}^6$. So Col R is a subspace of $\mathbf{R}^6$, but Col $R \neq \mathbf{R}^6$.

33. If Col $Q = \mathbf{R}^4$, then the columns of Q span $\mathbf{R}^4$. Since Q is square, the IMT shows that Q is invertible and the equation $Q\mathbf{x} = \mathbf{b}$ has a solution for each **b** in $\mathbf{R}^4$. Also, each solution is unique, by Theorem 5 in Section 2.2.

34. If Nul $P = \{\mathbf{0}\}$, then the equation $P\mathbf{x} = \mathbf{0}$ has only the trivial solution. Since P is square, the IMT shows that P is invertible and the equation $P\mathbf{x} = \mathbf{b}$ has a solution for each **b** in $\mathbf{R}^5$. Also, each solution is unique, by Theorem 5 in Section 2.2.

35. If the columns of B are linearly independent, then the equation $B\mathbf{x} = \mathbf{0}$ has only the trivial (zero) solution. That is, Nul $B = \{\mathbf{0}\}$.

36. If the columns of A form a basis, they are linearly independent. This means that A cannot have more columns than rows. Since the columns also span $\mathbf{R}^m$, A must have a pivot in each row, which means that A cannot have more rows than columns. As a result, A must be a square matrix.

37. **[M]** Use the command that produces the reduced echelon form in one step (`ref` or `rref` depending on the program). See the Section 2.8 in the *Study Guide* for details. By Theorem 13, the pivot columns of A form a basis for Col A.

$$A = \begin{bmatrix} 3 & -5 & 0 & -1 & 3 \\ -7 & 9 & -4 & 9 & -11 \\ -5 & 7 & -2 & 5 & -7 \\ 3 & -7 & -3 & 4 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2.5 & -4.5 & 3.5 \\ 0 & 1 & 1.5 & -2.5 & 1.5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{Basis for Col } A\text{: } \begin{bmatrix} 3 \\ -7 \\ -5 \\ 3 \end{bmatrix}, \begin{bmatrix} -5 \\ 9 \\ 7 \\ -7 \end{bmatrix}$$

For Nul A, obtain the solution of $A\mathbf{x} = \mathbf{0}$ in parametric vector form:

$$\begin{aligned} x_1 \quad + 2.5x_3 - 4.5x_4 + 3.5x_5 &= 0 \\ x_2 + 1.5x_3 - 2.5x_4 + 1.5x_5 &= 0 \end{aligned}$$

$$\text{Solution: } \begin{cases} x_1 = -2.5x_3 + 4.5x_4 - 3.5x_5 \\ x_2 = -1.5x_3 + 2.5x_4 - 1.5x_5 \\ x_3, x_4, \text{ and } x_5 \text{ are free} \end{cases}$$

$$\mathbf{x}=\begin{bmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{bmatrix}=\begin{bmatrix}-2.5x_3+4.5x_4-3.5x_5\\-1.5x_3+2.5x_4-1.5x_5\\x_3\\x_4\\x_5\end{bmatrix}=x_3\begin{bmatrix}-2.5\\-1.5\\1\\0\\0\end{bmatrix}+x_4\begin{bmatrix}4.5\\2.5\\0\\1\\0\end{bmatrix}+x_5\begin{bmatrix}-3.5\\-1.5\\0\\0\\1\end{bmatrix}=x_3\mathbf{u}+x_4\mathbf{v}+x_5\mathbf{w}$$

By the argument in Example 6, a basis for Nul A is $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$.

38. **[M]** $A=\begin{bmatrix}5&2&0&-8&-8\\4&1&2&-8&-9\\5&1&3&5&19\\-8&-5&6&8&5\end{bmatrix}\sim\begin{bmatrix}1&0&0&60&122\\0&1&0&-154&-309\\0&0&1&-47&-94\\0&0&0&0&0\end{bmatrix}.$

The pivot columns of A form a basis for Col A: $\begin{bmatrix}5\\4\\5\\-8\end{bmatrix},\begin{bmatrix}2\\1\\1\\-5\end{bmatrix},\begin{bmatrix}0\\2\\3\\6\end{bmatrix}.$

For Nul A, solve $A\mathbf{x}=\mathbf{0}$:
$$\begin{aligned}x_1\qquad\quad + 60x_4 + 122x_5 &= 0\\ x_2\quad - 154x_4 - 309x_5 &= 0\\ x_3 - 47x_4 - 94x_5 &= 0\end{aligned}$$

Solution: $\begin{cases}x_1 = -60x_4 - 122x_5\\ x_2 = 154x_4 + 309x_5\\ x_3 = 47x_4 + 94x_5\\ x_4 \text{ and } x_5 \text{ are free}\end{cases}$

$$\mathbf{x}=\begin{bmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{bmatrix}=\begin{bmatrix}-60x_4-122x_5\\154x_4+309x_5\\47x_4+94x_5\\x_4\\x_5\end{bmatrix}=x_4\begin{bmatrix}-60\\154\\47\\1\\0\end{bmatrix}+x_5\begin{bmatrix}-122\\309\\94\\0\\1\end{bmatrix}=x_4\mathbf{u}+x_5\mathbf{v}$$

By the method of Example 6, a basis for Nul A is $\{\mathbf{u}, \mathbf{v}\}$

Note: The *Study Guide* for Section 2.8 gives directions for students to construct a review sheet for the concept of a subspace and the two main types of subspaces, Col A and Nul A, and a review sheet for the concept of a basis. I encourage you to consider making this an assignment for your class.

2.9 SOLUTIONS

Notes: This section contains the ideas from Sections 4.4–4.6 that are needed for later work in Chapters 5–7. If you have time, you can enrich the geometric content of "coordinate systems" by discussing crystal lattices (Example 3 and Exercises 35 and 36 in Section 4.4.) Some students might profit from reading Examples 1–3 from Section 4.4 and Examples 2, 4, and 5 from Section 4.6. Section 4.5 is probably *not* a good reference for students who have not considered general vector spaces.

Coordinate vectors are important mainly to give an intuitive and geometric feeling for the isomorphism between a k-dimensional subspace and $\mathbf{R}^k$. If you plan to omit Sections 5.4, 5.6, 5.7 and 7.2, you can safely omit Exercises 1–8 here.

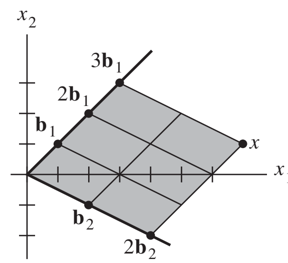
Exercises 1–16 may be assigned after students have read as far as Example 2. Exercises 19 and 20 use the Rank Theorem, but they can also be assigned before the Rank Theorem is discussed.

The Rank Theorem in this section omits the nontrivial fact about Row A which is included in the Rank Theorem of Section 4.6, but that is used only in Section 7.4. The row space itself can be introduced in Section 6.2, for use in Chapter 6 and Section 7.4.

Exercises 9–16 include important review of techniques taught in Section 2.8 (and in Sections 1.2 and 2.5). They make good test questions because they require little arithmetic. My students need the practice here. Nearly every time I teach the course and start Chapter 5, I find that at least one or two students cannot find a basis for a two-dimensional eigenspace!

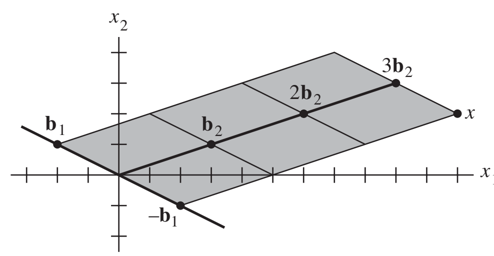
1. If $[\mathbf{x}]_B = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$, then $\mathbf{x}$ is formed from $\mathbf{b}_1$ and $\mathbf{b}_2$ using weights 3 and 2:

$$\mathbf{x} = 3\mathbf{b}_1 + 2\mathbf{b}_2 = 3\begin{bmatrix} 1 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 7 \\ 1 \end{bmatrix}$$



2. If $[\mathbf{x}]_B = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$, then $\mathbf{x}$ is formed from $\mathbf{b}_1$ and $\mathbf{b}_2$ using weights –1 and 3:

$$\mathbf{x} = (-1)\mathbf{b}_1 + 3\mathbf{b}_2 = (-1)\begin{bmatrix} -2 \\ 1 \end{bmatrix} + 3\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 11 \\ 2 \end{bmatrix}$$



3. To find c_1 and c_2 that satisfy $\mathbf{x} = c_1\mathbf{b}_1 + c_2\mathbf{b}_2$, row reduce the augmented matrix:

$[\mathbf{b}_1\ \ \mathbf{b}_2\ \ \mathbf{x}] = \begin{bmatrix} 1 & -2 & -3 \\ -4 & 7 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & -3 \\ 0 & -1 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 7 \\ 0 & 1 & 5 \end{bmatrix}$. Or, one can write a matrix equation as suggested by Exercise 7 and solve using the matrix inverse. In either case,

$[\mathbf{x}]_B = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 5 \end{bmatrix}$.

4. As in Exercise 3, $[\mathbf{b}_1\ \ \mathbf{b}_2\ \ \mathbf{x}] = \begin{bmatrix} 1 & -3 & -7 \\ -3 & 5 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & -7 \\ 0 & -4 & -16 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 4 \end{bmatrix}$, and

$[\mathbf{x}]_B = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$.

5. $[\mathbf{b}_1\ \ \mathbf{b}_2\ \ \mathbf{x}] = \begin{bmatrix} 1 & -3 & 4 \\ 5 & -7 & 10 \\ -3 & 5 & -7 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 4 \\ 0 & 8 & -10 \\ 0 & -4 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1/4 \\ 0 & 1 & -5/4 \\ 0 & 0 & 0 \end{bmatrix}$. $[\mathbf{x}]_B = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1/4 \\ -5/4 \end{bmatrix}$.

6. $[\mathbf{b}_1 \quad \mathbf{b}_2 \quad \mathbf{x}] = \begin{bmatrix} -3 & 7 & 11 \\ 1 & 5 & 0 \\ -4 & -6 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 5 & 0 \\ 0 & 22 & 11 \\ 0 & 14 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -5/2 \\ 0 & 1 & 1/2 \\ 0 & 0 & 0 \end{bmatrix}$

$$[\mathbf{x}]_B = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -5/2 \\ 1/2 \end{bmatrix}.$$

7. Fig. 1 suggests that $\mathbf{w} = 2\mathbf{b}_1 - \mathbf{b}_2$ and $\mathbf{x} = 1.5\mathbf{b}_1 + .5\mathbf{b}_2$, in which case,

$[\mathbf{w}]_B = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ and $[\mathbf{x}]_B = \begin{bmatrix} 1.5 \\ .5 \end{bmatrix}$. To confirm $[\mathbf{x}]_B$, compute

$$1.5\mathbf{b}_1 + .5\mathbf{b}_2 = 1.5\begin{bmatrix} 3 \\ 0 \end{bmatrix} + .5\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix} = \mathbf{x}$$

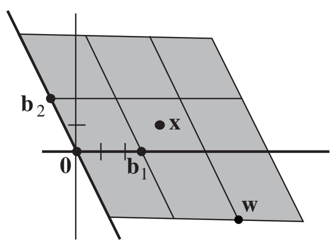


Figure 1

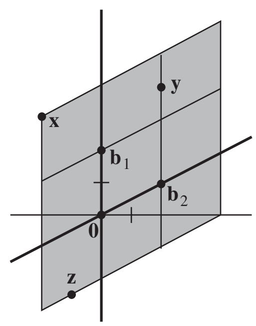


Figure 2

Note: Figures 1 and 2 display what Section 4.4 calls *B*-graph paper.

8. Fig. 2 suggests that $\mathbf{x} = 2\mathbf{b}_1 - \mathbf{b}_2$, $\mathbf{y} = 1.5\mathbf{b}_1 + \mathbf{b}_2$, and $\mathbf{z} = -\mathbf{b}_1 - .5\mathbf{b}_2$. If so, then

$[\mathbf{x}]_B = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, $[\mathbf{y}]_B = \begin{bmatrix} 1.5 \\ 1.0 \end{bmatrix}$, and $[\mathbf{z}]_B = \begin{bmatrix} -1 \\ -.5 \end{bmatrix}$. To confirm $[\mathbf{y}]_B$ and $[\mathbf{z}]_B$, compute

$$1.5\mathbf{b}_1 + \mathbf{b}_2 = 1.5\begin{bmatrix} 0 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \mathbf{y} \text{ and } -\mathbf{b}_1 - .5\mathbf{b}_2 = -1\begin{bmatrix} 0 \\ 2 \end{bmatrix} - .5\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -2.5 \end{bmatrix} = \mathbf{z}.$$

9. The information $A = \begin{bmatrix} 1 & -3 & 2 & -4 \\ -3 & 9 & -1 & 5 \\ 2 & -6 & 4 & -3 \\ -4 & 12 & 2 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 2 & -4 \\ 0 & 0 & 5 & -7 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ is enough to see that columns 1, 3, and 4 of

A form a basis for Col *A*: $\begin{bmatrix} 1 \\ -3 \\ 2 \\ -4 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 4 \\ 2 \end{bmatrix}, \begin{bmatrix} -4 \\ 5 \\ -3 \\ 7 \end{bmatrix}.$

Columns 1, 2 and 4, of the echelon form certain cannot span Col *A* since those vectors all have zero in their fourth entries. For Nul *A*, use the reduced echelon form, augmented with a zero column to insure that the equation $A\mathbf{x} = \mathbf{0}$ is kept in mind:

$$\begin{bmatrix} 1 & -3 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \quad \begin{aligned} x_1 - 3x_2 &= 0 \\ x_3 &= 0 \\ x_4 &= 0 \\ x_2 \text{ is the free variable} \end{aligned}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3x_2 \\ x_2 \\ 0 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}. \text{ So } \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} \text{ is}$$

a basis for Nul A. From this information, dim Col A = 3 (because A has three pivot columns) and dim Nul A = 1 (because the equation $A\mathbf{x} = \mathbf{0}$ has only one free variable).

10. The information $A = \begin{bmatrix} 1 & -2 & 9 & 5 & 4 \\ 1 & -1 & 6 & 5 & -3 \\ -2 & 0 & -6 & 1 & -2 \\ 4 & 1 & 9 & 1 & -9 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 9 & 5 & 4 \\ 0 & 1 & -3 & 0 & -7 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ shows that columns 1, 2, and 4 of A form a basis for Col A: $\begin{bmatrix} 1 \\ 1 \\ -2 \\ 4 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 5 \\ 1 \\ 1 \end{bmatrix}$. For Nul A,

$$[A \quad \mathbf{0}] \sim \begin{bmatrix} 1 & 0 & 3 & 0 & 0 & 0 \\ 0 & 1 & -3 & 0 & -7 & 0 \\ 0 & 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \quad \begin{aligned} x_1 + 3x_3 &= 0 \\ x_2 - 3x_3 - 7x_5 &= 0 \\ x_4 - 2x_5 &= 0 \\ x_3 \text{ and } x_5 \text{ are free variables} \end{aligned}$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -3x_3 \\ 3x_3 + 7x_5 \\ x_3 \\ 2x_5 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} -3 \\ 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 0 \\ 7 \\ 0 \\ 2 \\ 1 \end{bmatrix}. \text{ Basis for Nul } A\text{: } \begin{bmatrix} -3 \\ 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 7 \\ 0 \\ 2 \\ 1 \end{bmatrix}.$$

From this, dim Col A = 3 and dim Nul A = 2.

11. The information $A = \begin{bmatrix} 1 & 2 & -5 & 0 & -1 \\ 2 & 5 & -8 & 4 & 3 \\ -3 & -9 & 9 & -7 & -2 \\ 3 & 10 & -7 & 11 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -5 & 0 & -1 \\ 0 & 1 & 2 & 4 & 5 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ shows that columns 1, 2, and 4 of A form a basis for Col A: $\begin{bmatrix} 1 \\ 2 \\ -3 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ -9 \\ 10 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \\ -7 \\ 11 \end{bmatrix}$. For Nul A,

$$[A \quad \mathbf{0}] \sim \begin{bmatrix} 1 & 0 & -9 & 0 & 5 & 0 \\ 0 & 1 & 2 & 0 & -3 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \quad \begin{aligned} x_1 - 9x_3 + 5x_5 &= 0 \\ x_2 + 2x_3 - 3x_5 &= 0 \\ x_4 + 2x_5 &= 0 \\ x_3 \text{ and } x_5 \text{ are free variables} \end{aligned}$$

$$\mathbf{x}=\begin{bmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{bmatrix}=\begin{bmatrix}9x_3-5x_5\\-2x_3+3x_5\\x_3\\-2x_5\\x_5\end{bmatrix}=x_3\begin{bmatrix}9\\-2\\1\\0\\0\end{bmatrix}+x_5\begin{bmatrix}-5\\3\\0\\-2\\1\end{bmatrix}.\text{ Basis for Nul }A\text{: }\begin{bmatrix}9\\-2\\1\\0\\0\end{bmatrix},\begin{bmatrix}-5\\3\\0\\-2\\1\end{bmatrix}.$$

From this, dim Col A = 3 and dim Nul A = 2.

12. The information $A=\begin{bmatrix}1&2&-4&3&3\\5&10&-9&-7&8\\4&8&-9&-2&7\\-2&-4&5&0&-6\end{bmatrix}\sim\begin{bmatrix}1&2&-4&3&3\\0&0&1&-2&0\\0&0&0&0&-5\\0&0&0&0&0\end{bmatrix}$ shows that columns 1, 3, and 5 of A form a basis for Col A: $\begin{bmatrix}1\\5\\4\\-2\end{bmatrix},\begin{bmatrix}-4\\-9\\-9\\5\end{bmatrix},\begin{bmatrix}3\\8\\7\\-6\end{bmatrix}$. For Nul A

$$[A\quad \mathbf{0}]\sim\begin{bmatrix}1&2&0&-5&0&0\\0&0&1&-2&0&0\\0&0&0&0&1&0\\0&0&0&0&0&0\end{bmatrix}.\qquad\begin{aligned}x_1+2x_2-5x_4&=0\\x_3-2x_4&=0\\x_5&=0\end{aligned}$$

x_2 and x_4 are free variables

$$\mathbf{x}=\begin{bmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{bmatrix}=\begin{bmatrix}-2x_2+5x_4\\x_2\\2x_4\\x_4\\0\end{bmatrix}=x_2\begin{bmatrix}-2\\1\\0\\0\\0\end{bmatrix}+x_4\begin{bmatrix}5\\0\\2\\1\\0\end{bmatrix}.\text{ Basis for Nul }A\text{: }\begin{bmatrix}-2\\1\\0\\0\\0\end{bmatrix},\begin{bmatrix}5\\0\\2\\1\\0\end{bmatrix}.$$

From this, dim Col A = 3 and dim Nul A = 2.

13. The four vectors span the column space H of a matrix that can be reduced to echelon form:

$$\begin{bmatrix}1&-3&2&-4\\-3&9&-1&5\\2&-6&4&-3\\-4&12&2&7\end{bmatrix}\sim\begin{bmatrix}1&-3&2&-4\\0&0&5&-7\\0&0&0&5\\0&0&10&-9\end{bmatrix}\sim\begin{bmatrix}1&-3&2&-4\\0&0&5&-7\\0&0&0&5\\0&0&0&5\end{bmatrix}\sim\begin{bmatrix}1&-3&2&-4\\0&0&5&-7\\0&0&0&5\\0&0&0&0\end{bmatrix}$$

Columns 1, 3, and 4 of the original matrix form a basis for H, so dim H = 3.

Note: Either Exercise 13 or 14 should be assigned because there are always one or two students who confuse Col A with Nul A. Or, they wrongly connect "set of linear combinations" with "parametric vector form" (of the general solution of $A\mathbf{x} = \mathbf{0}$).

14. The five vectors span the column space H of a matrix that can be reduced to echelon form:

$$\begin{bmatrix}1&2&0&-1&3\\-1&-3&2&4&-8\\-2&-1&-6&-7&9\\5&6&8&7&-5\end{bmatrix}\sim\begin{bmatrix}1&2&0&-1&3\\0&-1&2&3&-5\\0&3&-6&-9&15\\0&-4&8&12&-20\end{bmatrix}\sim\begin{bmatrix}1&2&0&-1&3\\0&-1&2&3&-5\\0&0&0&0&0\\0&0&0&0&0\end{bmatrix}$$

Columns 1 and 2 of the original matrix form a basis for H, so dim H = 2.

15. Col $A = \mathbf{R}^3$, because A has a pivot in each row and so the columns of A span $\mathbf{R}^3$. Nul A *cannot* equal $\mathbf{R}^2$, because Nul A is a subspace of $\mathbf{R}^5$. It is true, however, that Nul A is two-dimensional. Reason: the equation $A\mathbf{x} = \mathbf{0}$ has two free variables, because A has five columns and only three of them are pivot columns.

16. Col A *cannot* be $\mathbf{R}^3$ because the columns of A have four entries. (In fact, Col A is a 3-dimensional subspace of $\mathbf{R}^4$, because the 3 pivot columns of A form a basis for Col A.) Since A has 7 columns and 3 pivot columns, the equation $A\mathbf{x} = \mathbf{0}$ has 4 free variables. So, dim Nul $A = 4$.

17. **a**. True. This is the definition of a B-coordinate vector.

b. False. Dimension is defined only for a subspace. A line must be through the origin in $\mathbf{R}^n$ to be a subspace of $\mathbf{R}^n$.

c. True. The sentence before Example 1 concludes that the number of pivot columns of A is the rank of A, which is the dimension of Col A by definition.

d. True. This is equivalent to the Rank Theorem because rank A *is* the dimension of Col A.

e. True, by the Basis Theorem. In this case, the spanning set is automatically a linearly independent set.

18. **a**. True. This fact is justified in the second paragraph of this section.

b. True. See the second paragraph after Fig. 1.

c. False. The dimension of Nul A is the number of *free* variables in the equation $A\mathbf{x} = \mathbf{0}$. See Example 2.

d. True, by the definition of *rank*.

e. True, by the Basis Theorem. In this case, the linearly independent set is automatically a spanning set.

19. The fact that the solution space of $A\mathbf{x} = \mathbf{0}$ has a basis of three vectors means that dim Nul $A = 3$. Since a 5×7 matrix A has 7 columns, the Rank Theorem shows that rank $A = 7 -$ dim Nul $A = 4$.

Note: One can solve Exercises 19–22 without explicit reference to the Rank Theorem. For instance, in Exercise 19, if the null space of a matrix A is three-dimensional, then the equation $A\mathbf{x} = \mathbf{0}$ has three free variables, and three of the columns of A are nonpivot columns. Since a 5×7 matrix has seven columns, A must have four pivot columns (which form a basis of Col A). So rank $A =$ dim Col $A = 4$.

20. A 4×5 matrix A has 5 columns. By the Rank Theorem, rank $A = 5 -$ dim Nul A. Since the null space is three-dimensional, rank $A = 2$.

21. A 7×6 matrix has 6 columns. By the Rank Theorem, dim Nul $A = 6 -$ rank A. Since the rank is four, dim Nul $A = 2$. That is, the dimension of the solution space of $A\mathbf{x} = \mathbf{0}$ is two.

22. The wording of this problem was poor in the first printing, because the phrase "it spans a four-dimensional subspace" was never defined. Here is a revision that I will put in later printings of the third edition:

Show that a set $\{\mathbf{v}_1, \ldots, \mathbf{v}_5\}$ in $\mathbf{R}^n$ is linearly dependent if dim Span$\{\mathbf{v}_1, \ldots, \mathbf{v}_5\} = 4$.

Solution: Suppose that the subspace $H = \text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_5\}$ is four-dimensional. If $\{\mathbf{v}_1, \ldots, \mathbf{v}_5\}$ were linearly independent, it would be a basis for H. This is impossible, by the statement just before the definition of *dimension* in Section 2.9, which essentially says that *every* basis of a p-dimensional subspace consists of p vectors. Thus, $\{\mathbf{v}_1, \ldots, \mathbf{v}_5\}$ must be linearly dependent.

23. A 3×4 matrix A with a two-dimensional column space has two pivot columns. The remaining two columns will correspond to free variables in the equation $A\mathbf{x} = \mathbf{0}$. So the desired construction is possible.

There are six possible locations for the two pivot columns, one of which is $\begin{bmatrix} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 \end{bmatrix}$. A simple construction is to take two vectors in $\mathbf{R}^3$ that are obviously not linearly dependent, and put two copies of these two vectors in any order. The resulting matrix will obviously have a two-dimensional column space. There is no need to worry about whether Nul A has the correct dimension, since this is guaranteed by the Rank Theorem: dim Nul $A = 4 -$ rank A.

24. A rank 1 matrix has a one-dimensional column space. Every column is a multiple of some fixed vector. To construct a 4×3 matrix, choose any nonzero vector in $\mathbf{R}^4$, and use it for one column. Choose any multiples of the vector for the other two columns.

25. The p columns of A span Col A by definition. If dim Col $A = p$, then the spanning set of p columns is automatically a basis for Col A, by the Basis Theorem. In particular, the columns are linearly independent.

26. If columns $\mathbf{a}_1$, $\mathbf{a}_3$, $\mathbf{a}_5$, and $\mathbf{a}_6$ of A are linearly independent and if dim Col $A = 4$, then $\{\mathbf{a}_1, \mathbf{a}_3, \mathbf{a}_5, \mathbf{a}_6\}$ is a linearly independent set in a 4-dimensional column space. By the Basis Theorem, this set of four vectors is a basis for the column space.

27. **a**. Start with $B = [\mathbf{b}_1 \ \cdots \ \mathbf{b}_p]$ and $A = [\mathbf{a}_1 \ \cdots \ \mathbf{a}_q]$, where $q > p$. For $j = 1, \ldots, q$, the vector $\mathbf{a}_j$ is in W. Since the columns of B span W, the vector $\mathbf{a}_j$ is in the column space of B. That is, $\mathbf{a}_j = B\mathbf{c}_j$ for some vector $\mathbf{c}_j$ of weights. Note that $\mathbf{c}_j$ is in $\mathbf{R}^p$ because B has p columns.

b. Let $C = [\mathbf{c}_1 \ \cdots \ \mathbf{c}_q]$. Then C is a $p\times q$ matrix because each of the q columns is in $\mathbf{R}^p$. By hypothesis, q is larger than p, so C has more columns than rows. By a theorem, the columns of C are linearly dependent and there exists a nonzero vector $\mathbf{u}$ in $\mathbf{R}^q$ such that $C\mathbf{u} = \mathbf{0}$.

c. From part (a) and the definition of matrix multiplication

$$A = [\mathbf{a}_1 \ \cdots \ \mathbf{a}_q] = [B\mathbf{c}_1 \ \cdots \ B\mathbf{c}_q] = BC$$

From part (b), $A\mathbf{u} = (BC)\mathbf{u} = B(C\mathbf{u}) = B\mathbf{0} = \mathbf{0}$. Since $\mathbf{u}$ is nonzero, the columns of A are linearly dependent.

28. If A contained more vectors than B, then A would be linearly dependent, by Exercise 27, because B spans W. Repeat the argument with B and A interchanged to conclude that B cannot contain more vectors than A.

29. **[M]** Apply the matrix command **ref** or **rref** to the matrix $[\mathbf{v}_1 \ \mathbf{v}_2 \ \mathbf{x}]$:

$$\begin{bmatrix} 11 & 14 & 19 \\ -5 & -8 & -13 \\ 10 & 13 & 18 \\ 7 & 10 & 15 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1.667 \\ 0 & 1 & 2.667 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The equation $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{x}$ is consistent, so $\mathbf{x}$ is in the subspace H. The decimal approximations suggest $c_1 = -5/3$ and $c_2 = 8/3$, and it can be checked that these values are precise. Thus, the B-coordinate of $\mathbf{x}$ is $(-5/3, 8/3)$.

30. **[M]** Apply the matrix command **ref** or **rref** to the matrix $[\mathbf{v}_1 \ \mathbf{v}_2 \ \mathbf{v}_3 \ \mathbf{x}]$:

$$\begin{bmatrix} -6 & 8 & -9 & 4 \\ 4 & -3 & 5 & 7 \\ -9 & 7 & -8 & -8 \\ 4 & -3 & 3 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The first three columns of $[\mathbf{v}_1\ \ \mathbf{v}_2\ \ \mathbf{v}_3\ \ \mathbf{x}]$ are pivot columns, so $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ are linearly independent. Thus $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ form a basis B for the subspace H which they span. View $[\mathbf{v}_1\ \ \mathbf{v}_2\ \ \mathbf{v}_3\ \ \mathbf{x}]$ as an augmented matrix for $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{x}$. The reduced echelon form shows that $\mathbf{x}$ is in H and

$$[\mathbf{x}]_B = \begin{bmatrix} 3 \\ 5 \\ 2 \end{bmatrix}.$$

Notes: The *Study Guide* for Section 2.9 contains a complete list of the statements in the Invertible Matrix Theorem that have been given so far. The format is the same as that used in Section 2.3, with three columns: statements that are logically equivalent for any $m\times n$ matrix and are related to existence concepts, those that are equivalent only for any $n\times n$ matrix, and those that are equivalent for any $n\times p$ matrix and are related to uniqueness concepts. Four statements are included that are not in the text's official list of statements, to give more symmetry to the three columns.

The *Study Guide* section also contains directions for making a review sheet for "dimension" and "rank."

Chapter 2 SUPPLEMENTARY EXERCISES

1. a. True. If A and B are $m\times n$ matrices, then B^T has as many rows as A has columns, so AB^T is defined. Also, A^TB is defined because A^T has m columns and B has m rows.

b. False. B must have 2 columns. A has as many columns as B has rows.

c. True. The ith row of A has the form $(0, \ldots, d_i, \ldots, 0)$. So the ith row of AB is $(0, \ldots, d_i, \ldots, 0)B$, which is d_i times the ith row of B.

d. False. Take the zero matrix for B. Or, construct a matrix B such that the equation $B\mathbf{x} = \mathbf{0}$ has nontrivial solutions, and construct C and D so that $C \neq D$ and the columns of $C - D$ satisfy the equation $B\mathbf{x} = \mathbf{0}$. Then $B(C - D) = 0$ and $BC = BD$.

e. False. Counterexample: $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $C = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

f. False. $(A + B)(A - B) = A^2 - AB + BA - B^2$. This equals $A^2 - B^2$ if and only if A commutes with B.

g. True. An $n\times n$ replacement matrix has $n + 1$ nonzero entries. The $n\times n$ scale and interchange matrices have n nonzero entries.

h. True. The transpose of an elementary matrix is an elementary matrix of the same type.

i. True. An $n\times n$ elementary matrix is obtained by a row operation on I_n.

j. False. Elementary matrices are invertible, so a product of such matrices is invertible. But not every square matrix is invertible.

k. True. If A is 3×3 with three pivot positions, then A is row equivalent to I_3.

l. False. A must be square in order to conclude from the equation $AB = I$ that A is invertible.

m. False. AB is invertible, but $(AB)^{-1} = B^{-1}A^{-1}$, and this product is not always equal to $A^{-1}B^{-1}$.

n. True. Given $AB = BA$, left-multiply by A^{-1} to get $B = A^{-1}BA$, and then right-multiply by A^{-1} to obtain $BA^{-1} = A^{-1}B$.

o. False. The correct equation is $(rA)^{-1} = r^{-1}A^{-1}$, because

$(rA)(r^{-1}A^{-1}) = (rr^{-1})(AA^{-1}) = 1\cdot I = I.$

p. True. If the equation $A\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ has a unique solution, then there are no free variables in this equation, which means that A must have three pivot positions (since A is 3×3). By the Invertible Matrix Theorem, A is invertible.

2. $C = (C^{-1})^{-1} = \dfrac{1}{-2}\begin{bmatrix} 7 & -5 \\ -6 & 4 \end{bmatrix} = \begin{bmatrix} -7/2 & 5/2 \\ 3 & -2 \end{bmatrix}$

3. $A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad A^2 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

$$A^3 = A \cdot A^2 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Next, $(I - A)(I + A + A^2) = I + A + A^2 - A(I + A + A^2) = I + A + A^2 - A - A^2 - A^3 = I - A^3$.

Since $A^3 = 0$, $(I - A)(I + A + A^2) = I$.

4. From Exercise 3, the inverse of $I - A$ is probably $I + A + A^2 + \cdots + A^{n-1}$. To verify this, compute

$$(I - A)(I + A + \cdots + A^{n-1}) = I + A + \cdots + A^{n-1} - A(I + A + \cdots + A^{n-1}) = I - AA^{n-1} = I - A^n$$

If $A^n = 0$, then the matrix $B = I + A + A^2 + \cdots + A^{n-1}$ satisfies $(I - A)B = I$. Since $I - A$ and B are square, they are invertible by the Invertible Matrix Theorem, and B is the inverse of $I - A$.

5. $A^2 = 2A - I$. Multiply by A: $A^3 = 2A^2 - A$. Substitute $A^2 = 2A - I$: $A^3 = 2(2A - I) - A = 3A - 2I$.
Multiply by A again: $A^4 = A(3A - 2I) = 3A^2 - 2A$. Substitute the identity $A^2 = 2A - I$ again:
Finally, $A^4 = 3(2A - I) - 2A = 4A - 3I$.

6. Let $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. By direct computation, $A^2 = I$, $B^2 = I$, and $AB = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = -BA$.

7. (Partial answer in *Study Guide*) Since $A^{-1}B$ is the solution of $AX = B$, row reduction of $[A \quad B]$ to $[I \quad X]$ will produce $X = A^{-1}B$. See Exercise 12 in Section 2.2.

$$\begin{bmatrix} A & B \end{bmatrix} = \left[\begin{array}{ccc|cc} 1 & 3 & 8 & -3 & 5 \\ 2 & 4 & 11 & 1 & 5 \\ 1 & 2 & 5 & 3 & 4 \end{array}\right] \sim \begin{bmatrix} 1 & 3 & 8 & -3 & 5 \\ 0 & -2 & -5 & 7 & -5 \\ 0 & -1 & -3 & 6 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 8 & -3 & 5 \\ 0 & 1 & 3 & -6 & 1 \\ 0 & -2 & -5 & 7 & -5 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 3 & 8 & -3 & 5 \\ 0 & 1 & 3 & -6 & 1 \\ 0 & 0 & 1 & -5 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 0 & 37 & 29 \\ 0 & 1 & 0 & 9 & 10 \\ 0 & 0 & 1 & -5 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 10 & -1 \\ 0 & 1 & 0 & 9 & 10 \\ 0 & 0 & 1 & -5 & -3 \end{bmatrix}$$

Thus, $A^{-1}B = \begin{bmatrix} 10 & -1 \\ 9 & 10 \\ -5 & -3 \end{bmatrix}$.

8. By definition of matrix multiplication, the matrix A satisfies

$$A\begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix}$$

Right-multiply both sides by the inverse of $\begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix}$. The left side becomes A. Thus,

$$A = \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 7 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 4 & -1 \end{bmatrix}$$

9. Given $AB = \begin{bmatrix} 5 & 4 \\ -2 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 7 & 3 \\ 2 & 1 \end{bmatrix}$, notice that $ABB^{-1} = A$. Since $\det B = 7 - 6 = 1$,

$$B^{-1} = \begin{bmatrix} 1 & -3 \\ -2 & 7 \end{bmatrix} \text{ and } A = (AB)B^{-1} = \begin{bmatrix} 5 & 4 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 1 & -3 \\ -2 & 7 \end{bmatrix} = \begin{bmatrix} -3 & 13 \\ -8 & 27 \end{bmatrix}$$

Note: Variants of this question make simple exam questions.

10. Since A is invertible, so is A^T, by the Invertible Matrix Theorem. Then A^TA is the product of invertible matrices and so is invertible. Thus, the formula $(A^TA)^{-1}A^T$ makes sense. By Theorem 6 in Section 2.2,

$$(A^TA)^{-1} \cdot A^T = A^{-1}(A^T)^{-1}A^T = A^{-1}I = A^{-1}$$

An alternative calculation: $(A^TA)^{-1}A^T \cdot A = (A^TA)^{-1}(A^TA) = I$. Since A is invertible, this equation shows that its inverse is $(A^TA)^{-1}A^T$.

11. **a**. For $i = 1,\dots, n$, $p(x_i) = c_0 + c_1x_i + \cdots + c_{n-1}x_i^{n-1} = \text{row}_i(V) \cdot \begin{bmatrix} c_0 \\ \vdots \\ c_{n-1} \end{bmatrix} = \text{row}_i(V)\mathbf{c}$.

By a property of matrix multiplication, shown after Example 6 in Section 2.1, and the fact that **c** was chosen to satisfy $V\mathbf{c} = \mathbf{y}$,

$$\text{row}_i(V)\mathbf{c} = \text{row}_i(V\mathbf{c}) = \text{row}_i(\mathbf{y}) = y_i$$

Thus, $p(x_i) = y_i$. To summarize, the entries in $V\mathbf{c}$ are the values of the polynomial $p(x)$ at $x_1, \dots, x_n$.

b. Suppose $x_1, \dots, x_n$ are distinct, and suppose $V\mathbf{c} = \mathbf{0}$ for some vector **c**. Then the entries in **c** are the coefficients of a polynomial whose value is zero at the distinct points $x_1, \dots, x_n$. However, a nonzero polynomial of degree $n - 1$ cannot have n zeros, so the polynomial must be identically zero. That is, the entries in **c** must all be zero. This shows that the columns of V are linearly independent.

c. (Solution in *Study Guide*) When $x_1, \dots, x_n$ are distinct, the columns of V are linearly independent, by (b). By the Invertible Matrix Theorem, V is invertible and its columns span $\mathbf{R}^n$. So, for every $\mathbf{y} = (y_1, \dots, y_n)$ in $\mathbf{R}^n$, there is a vector **c** such that $V\mathbf{c} = \mathbf{y}$. Let p be the polynomial whose coefficients are listed in **c**. Then, by (a), p is an interpolating polynomial for $(x_1, y_1), \dots, (x_n, y_n)$.

12. If $A = LU$, then $\text{col}_1(A) = L\cdot\text{col}_1(U)$. Since $\text{col}_1(U)$ has a zero in every entry except possibly the first, $L\cdot\text{col}_1(U)$ is a linear combination of the columns of L in which all weights except possibly the first are zero. So $\text{col}_1(A)$ is a multiple of $\text{col}_1(L)$.

Similarly, $\text{col}_2(A) = L\cdot\text{col}_2(U)$, which is a linear combination of the columns of L using the first two entries in $\text{col}_2(U)$ as weights, because the other entries in $\text{col}_2(U)$ are zero. Thus $\text{col}_2(A)$ is a linear combination of the first two columns of L.

13. **a**. $P^2 = (\mathbf{u}\mathbf{u}^T)(\mathbf{u}\mathbf{u}^T) = \mathbf{u}(\mathbf{u}^T\mathbf{u})\mathbf{u}^T = \mathbf{u}(1)\mathbf{u}^T = P$, because **u** satisfies $\mathbf{u}^T\mathbf{u} = 1$.

b. $P^T = (\mathbf{u}\mathbf{u}^T)^T = \mathbf{u}^{TT}\mathbf{u}^T = \mathbf{u}\mathbf{u}^T = P$

c. $Q^2 = (I - 2P)(I - 2P) = I - I(2P) - 2PI + 2P(2P)$

$= I - 4P + 4P^2 = I$, because of part (a).

14. Given $\mathbf{u}=\begin{bmatrix}0\\0\\1\end{bmatrix}$, define P and Q as in Exercise 13 by

$$P=\mathbf{u}\mathbf{u}^T=\begin{bmatrix}0\\0\\1\end{bmatrix}\begin{bmatrix}0 & 0 & 1\end{bmatrix}=\begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix},\quad Q=I-2P=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}-2\begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&-1\end{bmatrix}$$

If $\mathbf{x}=\begin{bmatrix}1\\5\\3\end{bmatrix}$, then $P\mathbf{x}=\begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}\begin{bmatrix}1\\5\\3\end{bmatrix}=\begin{bmatrix}0\\0\\3\end{bmatrix}$ and $Q\mathbf{x}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&-1\end{bmatrix}\begin{bmatrix}1\\5\\3\end{bmatrix}=\begin{bmatrix}1\\5\\-3\end{bmatrix}$.

15. Left-multiplication by an elementary matrix produces an elementary row operation:

$$B\sim E_1B\sim E_2E_1B\sim E_3E_2E_1B=C$$

so B is row equivalent to C. Since row operations are reversible, C is row equivalent to B. (Alternatively, show C being changed into B by row operations using the inverse of the E_i .)

16. Since A is not invertible, there is a nonzero vector $\mathbf{v}$ in $\mathbf{R}^n$ such that $A\mathbf{v}=\mathbf{0}$. Place n copies of $\mathbf{v}$ into an $n\times n$ matrix B. Then $AB = A[\mathbf{v}\ \cdots\ \mathbf{v}] = [A\mathbf{v}\ \cdots\ A\mathbf{v}] = 0$.

17. Let A be a 6×4 matrix and B a 4×6 matrix. Since B has more columns than rows, its six columns are linearly dependent and there is a nonzero $\mathbf{x}$ such that $B\mathbf{x}=\mathbf{0}$. Thus $AB\mathbf{x} = A\mathbf{0} = \mathbf{0}$. This shows that the matrix AB is not invertible, by the IMT. (Basically the same argument was used to solve Exercise 22 in Section 2.1.)

Note: (In the *Study Guide*) It is possible that BA is invertible. For example, let C be an invertible 4×4 matrix and construct $A=\begin{bmatrix}C\\0\end{bmatrix}$ and $B=[C^{-1}\ \ 0]$. Then $BA = I_4$, which is invertible.

18. By hypothesis, A is 5×3, C is 3×5, and $AC = I_3$. Suppose $\mathbf{x}$ satisfies $A\mathbf{x}=\mathbf{b}$. Then $CA\mathbf{x} = C\mathbf{b}$. Since $CA = I$, $\mathbf{x}$ must be $C\mathbf{b}$. This shows that $C\mathbf{b}$ is the only solution of $A\mathbf{x}=\mathbf{b}$.

19. **[M]** Let $A=\begin{bmatrix}.4&.2&.3\\.3&.6&.3\\.3&.2&.4\end{bmatrix}$. Then $A^2=\begin{bmatrix}.31&.26&.30\\.39&.48&.39\\.30&.26&.31\end{bmatrix}$. Instead of computing A^3 next, speed up the calculations by computing

$$A^4=A^2A^2=\begin{bmatrix}.2875&.2834&.2874\\.4251&.4332&.4251\\.2874&.2834&.2875\end{bmatrix},\quad A^8=A^4A^4=\begin{bmatrix}.2857&.2857&.2857\\.4285&.4286&.4285\\.2857&.2857&.2857\end{bmatrix}$$

To four decimal places, as k increases,

$$A^k\to\begin{bmatrix}.2857&.2857&.2857\\.4286&.4286&.4286\\.2857&.2857&.2857\end{bmatrix},\text{ or, in rational format, }A^k\to\begin{bmatrix}2/7&2/7&2/7\\3/7&3/7&3/7\\2/7&2/7&2/7\end{bmatrix}.$$

If $B = \begin{bmatrix} 0 & .2 & .3 \\ .1 & .6 & .3 \\ .9 & .2 & .4 \end{bmatrix}$, then $B^2 = \begin{bmatrix} .29 & .18 & .18 \\ .33 & .44 & .33 \\ .38 & .38 & .49 \end{bmatrix}$,

$$B^4 = \begin{bmatrix} .2119 & .1998 & .1998 \\ .3663 & .3764 & .3663 \\ .4218 & .4218 & .4339 \end{bmatrix}, \quad B^8 = \begin{bmatrix} .2024 & .2022 & .2022 \\ .3707 & .3709 & .3707 \\ .4269 & .4269 & .4271 \end{bmatrix}$$

To four decimal places, as k increases,

$$B^k \to \begin{bmatrix} .2022 & .2022 & .2022 \\ .3708 & .3708 & .3708 \\ .4270 & .4270 & .4270 \end{bmatrix}, \text{ or, in rational format, } B^k \to \begin{bmatrix} 18/89 & 18/89 & 18/89 \\ 33/89 & 33/89 & 33/89 \\ 38/89 & 38/89 & 38/89 \end{bmatrix}.$$

20. **[M]** The 4×4 matrix A_4 is the 4×4 matrix of ones, minus the 4×4 identity matrix. The MATLAB command is `A4 = ones(4) - eye(4)`. For the inverse, use `inv(A4)`.

$$A_4 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}, \quad A_4^{-1} = \begin{bmatrix} -2/3 & 1/3 & 1/3 & 1/3 \\ 1/3 & -2/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & -2/3 & 1/3 \\ 1/3 & 1/3 & 1/3 & -2/3 \end{bmatrix}$$

$$A_5 = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix}, \quad A_5^{-1} = \begin{bmatrix} -3/4 & 1/4 & 1/4 & 1/4 & 1/4 \\ 1/4 & -3/4 & 1/4 & 1/4 & 1/4 \\ 1/4 & 1/4 & -3/4 & 1/4 & 1/4 \\ 1/4 & 1/4 & 1/4 & -3/4 & 1/4 \\ 1/4 & 1/4 & 1/4 & 1/4 & -3/4 \end{bmatrix}$$

$$A_6 = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}, \quad A_6^{-1} = \begin{bmatrix} -4/5 & 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \\ 1/5 & -4/5 & 1/5 & 1/5 & 1/5 & 1/5 \\ 1/5 & 1/5 & -4/5 & 1/5 & 1/5 & 1/5 \\ 1/5 & 1/5 & 1/5 & -4/5 & 1/5 & 1/5 \\ 1/5 & 1/5 & 1/5 & 1/5 & -4/5 & 1/5 \\ 1/5 & 1/5 & 1/5 & 1/5 & 1/5 & -4/5 \end{bmatrix}$$

The construction of A_6 and the appearance of its inverse suggest that the inverse is related to I_6. In fact, $A_6^{-1} + I_6$ is 1/5 times the 6×6 matrix of ones. Let J denotes the $n \times n$ matrix of ones. The conjecture is:

$$A_n = J - I_n \quad \text{and} \quad A_n^{-1} = \frac{1}{n-1} \cdot J - I_n$$

Proof: (Not required) Observe that $J^2 = nJ$ and $A_n J = (J - I)J = J^2 - J = (n - 1)J$. Now compute

$$A_n((n-1)^{-1}J - I) = (n-1)^{-1}A_n J - A_n = J - (J - I) = I$$

Since A_n is square, A_n is invertible and its inverse is $(n - 1)^{-1}J - I$.

3 Determinants

3.1 SOLUTIONS

Notes: Some exercises in this section provide practice in computing determinants, while others allow the student to discover the properties of determinants which will be studied in the next section. Determinants are developed through the cofactor expansion, which is given in Theorem 1. Exercises 33–36 in this section provide the first step in the inductive proof of Theorem 3 in the next section.

1. Expanding along the first row:

$$\begin{vmatrix} 3 & 0 & 4 \\ 2 & 3 & 2 \\ 0 & 5 & -1 \end{vmatrix} = 3\begin{vmatrix} 3 & 2 \\ 5 & -1 \end{vmatrix} - 0\begin{vmatrix} 2 & 2 \\ 0 & -1 \end{vmatrix} + 4\begin{vmatrix} 2 & 3 \\ 0 & 5 \end{vmatrix} = 3(-13) + 4(10) = 1$$

Expanding along the second column:

$$\begin{vmatrix} 3 & 0 & 4 \\ 2 & 3 & 2 \\ 0 & 5 & -1 \end{vmatrix} = (-1)^{1+2} \cdot 0\begin{vmatrix} 2 & 2 \\ 0 & -1 \end{vmatrix} + (-1)^{2+2} \cdot 3\begin{vmatrix} 3 & 4 \\ 0 & -1 \end{vmatrix} + (-1)^{3+2} \cdot 5\begin{vmatrix} 3 & 4 \\ 2 & 2 \end{vmatrix} = 3(-3) - 5(-2) = 1$$

2. Expanding along the first row:

$$\begin{vmatrix} 0 & 5 & 1 \\ 4 & -3 & 0 \\ 2 & 4 & 1 \end{vmatrix} = 0\begin{vmatrix} -3 & 0 \\ 4 & 1 \end{vmatrix} - 5\begin{vmatrix} 4 & 0 \\ 2 & 1 \end{vmatrix} + 1\begin{vmatrix} 4 & -3 \\ 2 & 4 \end{vmatrix} = -5(4) + 1(22) = 2$$

Expanding along the second column:

$$\begin{vmatrix} 0 & 5 & 1 \\ 4 & -3 & 0 \\ 2 & 4 & 1 \end{vmatrix} = (-1)^{1+2} \cdot 5\begin{vmatrix} 4 & 0 \\ 2 & 1 \end{vmatrix} + (-1)^{2+2} \cdot (-3)\begin{vmatrix} 0 & 1 \\ 2 & 1 \end{vmatrix} + (-1)^{3+2} \cdot 4\begin{vmatrix} 0 & 1 \\ 4 & 0 \end{vmatrix} = -5(4) - 3(-2) - 4(-4) = 2$$

3. Expanding along the first row:

$$\begin{vmatrix} 2 & -4 & 3 \\ 3 & 1 & 2 \\ 1 & 4 & -1 \end{vmatrix} = 2\begin{vmatrix} 1 & 2 \\ 4 & -1 \end{vmatrix} - (-4)\begin{vmatrix} 3 & 2 \\ 1 & -1 \end{vmatrix} + 3\begin{vmatrix} 3 & 1 \\ 1 & 4 \end{vmatrix} = 2(-9) + 4(-5) + (3)(11) = -5$$

Expanding along the second column:

$$\begin{vmatrix} 2 & -4 & 3 \\ 3 & 1 & 2 \\ 1 & 4 & -1 \end{vmatrix} = (-1)^{1+2}\cdot(-4)\begin{vmatrix} 3 & 2 \\ 1 & -1 \end{vmatrix} + (-1)^{2+2}\cdot 1\begin{vmatrix} 2 & 3 \\ 1 & -1 \end{vmatrix} + (-1)^{3+2}\cdot 4\begin{vmatrix} 2 & 3 \\ 3 & 2 \end{vmatrix} = 4(-5)+1(-5)-4(-5) = -5$$

4. Expanding along the first row:

$$\begin{vmatrix} 1 & 3 & 5 \\ 2 & 1 & 1 \\ 3 & 4 & 2 \end{vmatrix} = 1\begin{vmatrix} 1 & 1 \\ 4 & 2 \end{vmatrix} - 3\begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} + 5\begin{vmatrix} 2 & 1 \\ 3 & 4 \end{vmatrix} = 1(-2)-3(1)+5(5) = 20$$

Expanding along the second column:

$$\begin{vmatrix} 1 & 3 & 5 \\ 2 & 1 & 1 \\ 3 & 4 & 2 \end{vmatrix} = (-1)^{1+2}\cdot 3\begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} + (-1)^{2+2}\cdot 1\begin{vmatrix} 1 & 5 \\ 3 & 2 \end{vmatrix} + (-1)^{3+2}\cdot 4\begin{vmatrix} 1 & 5 \\ 2 & 1 \end{vmatrix} = -3(1)+1(-13)-4(-9) = 20$$

5. Expanding along the first row:

$$\begin{vmatrix} 2 & 3 & -4 \\ 4 & 0 & 5 \\ 5 & 1 & 6 \end{vmatrix} = 2\begin{vmatrix} 0 & 5 \\ 1 & 6 \end{vmatrix} - 3\begin{vmatrix} 4 & 5 \\ 5 & 6 \end{vmatrix} + (-4)\begin{vmatrix} 4 & 0 \\ 5 & 1 \end{vmatrix} = 2(-5)-3(-1)-4(4) = -23$$

6. Expanding along the first row:

$$\begin{vmatrix} 5 & -2 & 4 \\ 0 & 3 & -5 \\ 2 & -4 & 7 \end{vmatrix} = 5\begin{vmatrix} 3 & -5 \\ -4 & 7 \end{vmatrix} - (-2)\begin{vmatrix} 0 & -5 \\ 2 & 7 \end{vmatrix} + 4\begin{vmatrix} 0 & 3 \\ 2 & -4 \end{vmatrix} = 5(1)+2(10)+4(-6) = 1$$

7. Expanding along the first row:

$$\begin{vmatrix} 4 & 3 & 0 \\ 6 & 5 & 2 \\ 9 & 7 & 3 \end{vmatrix} = 4\begin{vmatrix} 5 & 2 \\ 7 & 3 \end{vmatrix} - 3\begin{vmatrix} 6 & 2 \\ 9 & 3 \end{vmatrix} + 0\begin{vmatrix} 6 & 5 \\ 9 & 7 \end{vmatrix} = 4(1)-3(0) = 4$$

8. Expanding along the first row:

$$\begin{vmatrix} 8 & 1 & 6 \\ 4 & 0 & 3 \\ 3 & -2 & 5 \end{vmatrix} = 8\begin{vmatrix} 0 & 3 \\ -2 & 5 \end{vmatrix} - 1\begin{vmatrix} 4 & 3 \\ 3 & 5 \end{vmatrix} + 6\begin{vmatrix} 4 & 0 \\ 3 & -2 \end{vmatrix} = 8(6)-1(11)+6(-8) = -11$$

9. First expand along the third row, then expand along the first row of the remaining matrix:

$$\begin{vmatrix} 6 & 0 & 0 & 5 \\ 1 & 7 & 2 & -5 \\ 2 & 0 & 0 & 0 \\ 8 & 3 & 1 & 8 \end{vmatrix} = (-1)^{3+1}\cdot 2\begin{vmatrix} 0 & 0 & 5 \\ 7 & 2 & -5 \\ 3 & 1 & 8 \end{vmatrix} = 2\cdot(-1)^{1+3}\cdot 5\begin{vmatrix} 7 & 2 \\ 3 & 1 \end{vmatrix} = 10(1) = 10$$

10. First expand along the second row, then expand along either the third row or the second column of the remaining matrix.

$$\begin{vmatrix} 1 & -2 & 5 & 2 \\ 0 & 0 & 3 & 0 \\ 2 & -6 & -7 & 5 \\ 5 & 0 & 4 & 4 \end{vmatrix} = (-1)^{2+3} \cdot 3 \begin{vmatrix} 1 & -2 & 2 \\ 2 & -6 & 5 \\ 5 & 0 & 4 \end{vmatrix}$$

$$= (-3)\left((-1)^{3+1} \cdot 5 \begin{vmatrix} -2 & 2 \\ -6 & 5 \end{vmatrix} + (-1)^{3+3} \cdot 4 \begin{vmatrix} 1 & -2 \\ 2 & -6 \end{vmatrix} \right) = (-3)(5(2) + 4(-2)) = -6$$

or

$$\begin{vmatrix} 1 & -2 & 5 & 2 \\ 0 & 0 & 3 & 0 \\ 2 & -6 & -7 & 5 \\ 5 & 0 & 4 & 4 \end{vmatrix} = (-1)^{2+3} \cdot 3 \begin{vmatrix} 1 & -2 & 2 \\ 2 & -6 & 5 \\ 5 & 0 & 4 \end{vmatrix}$$

$$= (-3)\left((-1)^{1+2} \cdot (-2) \begin{vmatrix} 2 & 5 \\ 5 & 4 \end{vmatrix} + (-1)^{2+2} \cdot (-6) \begin{vmatrix} 1 & 2 \\ 5 & 4 \end{vmatrix} \right) = (-3)\big(2(-17) - 6(-6)\big) = -6$$

11. There are many ways to do this determinant efficiently. One strategy is to always expand along the first column of each matrix:

$$\begin{vmatrix} 3 & 5 & -8 & 4 \\ 0 & -2 & 3 & -7 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 2 \end{vmatrix} = (-1)^{1+1} \cdot 3 \begin{vmatrix} -2 & 3 & -7 \\ 0 & 1 & 5 \\ 0 & 0 & 2 \end{vmatrix} = 3 \cdot (-1)^{1+1} \cdot (-2) \begin{vmatrix} 1 & 5 \\ 0 & 2 \end{vmatrix} = 3(-2)(2) = -12$$

12. There are many ways to do this determinant efficiently. One strategy is to always expand along the first row of each matrix:

$$\begin{vmatrix} 4 & 0 & 0 & 0 \\ 7 & -1 & 0 & 0 \\ 2 & 6 & 3 & 0 \\ 5 & -8 & 4 & -3 \end{vmatrix} = (-1)^{1+1} \cdot 4 \begin{vmatrix} -1 & 0 & 0 \\ 6 & 3 & 0 \\ -8 & 4 & -3 \end{vmatrix} = 4 \cdot (-1)^{1+1} \cdot (-1) \begin{vmatrix} 3 & 0 \\ 4 & -3 \end{vmatrix} = 4(-1)(-9) = 36$$

13. First expand along either the second row or the second column. Using the second row,

$$\begin{vmatrix} 4 & 0 & -7 & 3 & -5 \\ 0 & 0 & 2 & 0 & 0 \\ 7 & 3 & -6 & 4 & -8 \\ 5 & 0 & 5 & 2 & -3 \\ 0 & 0 & 9 & -1 & 2 \end{vmatrix} = (-1)^{2+3} \cdot 2 \begin{vmatrix} 4 & 0 & 3 & -5 \\ 7 & 3 & 4 & -8 \\ 5 & 0 & 2 & -3 \\ 0 & 0 & -1 & 2 \end{vmatrix}$$

Now expand along the second column to find:

$$(-1)^{2+3} \cdot 2 \begin{vmatrix} 4 & 0 & 3 & -5 \\ 7 & 3 & 4 & -8 \\ 5 & 0 & 2 & -3 \\ 0 & 0 & -1 & 2 \end{vmatrix} = -2\left((-1)^{2+2} \cdot 3 \begin{vmatrix} 4 & 3 & -5 \\ 5 & 2 & -3 \\ 0 & -1 & 2 \end{vmatrix} \right)$$

Now expand along either the first column or third row. The first column is used below.

$$-2\left((-1)^{2+2}\cdot 3\begin{vmatrix}4 & 3 & -5\\ 5 & 2 & -3\\ 0 & -1 & 2\end{vmatrix}\right)=-6\left((-1)^{1+1}\cdot 4\begin{vmatrix}2 & -3\\ -1 & 2\end{vmatrix}+(-1)^{2+1}\cdot 5\begin{vmatrix}3 & -5\\ -1 & 2\end{vmatrix}\right)=(-6)(4(1)-5(1))=6$$

14. First expand along either the fourth row or the fifth column. Using the fifth column,

$$\begin{vmatrix}6 & 3 & 2 & 4 & 0\\ 9 & 0 & -4 & 1 & 0\\ 8 & -5 & 6 & 7 & 1\\ 3 & 0 & 0 & 0 & 0\\ 4 & 2 & 3 & 2 & 0\end{vmatrix}=(-1)^{3+5}\cdot 1\begin{vmatrix}6 & 3 & 2 & 4\\ 9 & 0 & -4 & 1\\ 3 & 0 & 0 & 0\\ 4 & 2 & 3 & 2\end{vmatrix}$$

Now expand along the third row to find:

$$(-1)^{3+5}\cdot 1\begin{vmatrix}6 & 3 & 2 & 4\\ 9 & 0 & -4 & 1\\ 3 & 0 & 0 & 0\\ 4 & 2 & 3 & 2\end{vmatrix}=1\left((-1)^{3+1}\cdot 3\begin{vmatrix}3 & 2 & 4\\ 0 & -4 & 1\\ 2 & 3 & 2\end{vmatrix}\right)$$

Now expand along either the first column or second row. The first column is used below.

$$1\left((-1)^{3+1}\cdot 3\begin{vmatrix}3 & 2 & 4\\ 0 & -4 & 1\\ 2 & 3 & 2\end{vmatrix}\right)=3\left((-1)^{1+1}\cdot 3\begin{vmatrix}-4 & 1\\ 3 & 2\end{vmatrix}+(-1)^{3+1}\cdot 2\begin{vmatrix}2 & 4\\ -4 & 1\end{vmatrix}\right)=(3)(3(-11)+2(18))=9$$

15. $\begin{vmatrix}3 & 0 & 4\\ 2 & 3 & 2\\ 0 & 5 & -1\end{vmatrix}=$ (3)(3)(–1) + (0)(2)(0) + (4)(2)(5) – (0)(3)(4) – (5)(2)(3) – (–1)(2)(0) =

–9 + 0 + 40 – 0 – 30 –0 = 1

16. $\begin{vmatrix}0 & 5 & 1\\ 4 & -3 & 0\\ 2 & 4 & 1\end{vmatrix}=$ (0)(–3)(1) + (5)(0)(2) + (1)(4)(4) – (2)(–3)(1) – (4)(0)(0) – (1)(4)(5) =

0 + 0 + 16 – (–6) – 0 – 20 = 2

17. $\begin{vmatrix}2 & -4 & 3\\ 3 & 1 & 2\\ 1 & 4 & -1\end{vmatrix}=$ (2)(1)(–1) + (–4)(2)(1) + (3)(3)(4) – (1)(1)(3) – (4)(2)(2) – (–1)(3)(–4) =

–2 + (–8) + 36 – 3 – 16 – 12 = –5

18. $\begin{vmatrix}1 & 3 & 5\\ 2 & 1 & 1\\ 3 & 4 & 2\end{vmatrix}=$ (1)(1)(2) + (3)(1)(3) + (5)(2)(4) – (3)(1)(5) – (4)(1)(1) – (2)(2)(3) =

2 + 9 + 40 – 15 – 4 – 12 = 20

19. $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc, \begin{vmatrix} c & d \\ a & b \end{vmatrix} = cb - da = -(ad - bc)$

The row operation swaps rows 1 and 2 of the matrix, and the sign of the determinant is reversed.

20. $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc, \begin{vmatrix} a & b \\ kc & kd \end{vmatrix} = a(kd) - (kc)b = kad - kbc = k(ad - bc)$

The row operation scales row 2 by k, and the determinant is multiplied by k.

21. $\begin{vmatrix} 3 & 4 \\ 5 & 6 \end{vmatrix} = 18 - 20 = -2, \begin{vmatrix} 3 & 4 \\ 5+3k & 6+4k \end{vmatrix} = 3(6+4k) - (5+3k)4 = -2$

The row operation replaces row 2 with k times row 1 plus row 2, and the determinant is unchanged.

22. $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc, \begin{vmatrix} a+kc & b+kd \\ c & d \end{vmatrix} = (a+kc)d - c(b+kd) = ad + kcd - bc - kcd = ad - bc$

The row operation replaces row 1 with k times row 2 plus row 1, and the determinant is unchanged.

23. $\begin{vmatrix} 1 & 1 & 1 \\ -3 & 8 & -4 \\ 2 & -3 & 2 \end{vmatrix} = 1(4) - 1(2) + 1(-7) = -5, \begin{vmatrix} k & k & k \\ -3 & 8 & -4 \\ 2 & -3 & 2 \end{vmatrix} = k(4) - k(2) + k(-7) = -5k$

The row operation scales row 1 by k, and the determinant is multiplied by k.

24. $\begin{vmatrix} a & b & c \\ 3 & 2 & 2 \\ 6 & 5 & 6 \end{vmatrix} = a(2) - b(6) + c(3) = 2a - 6b + 3c,$

$\begin{vmatrix} 3 & 2 & 2 \\ a & b & c \\ 6 & 5 & 6 \end{vmatrix} = 3(6b - 5c) - 2(6a - 6c) + 2(5a - 6b) = -2a + 6b - 3c$

The row operation swaps rows 1 and 2 of the matrix, and the sign of the determinant is reversed.

25. Since the matrix is triangular, by Theorem 2 the determinant is the product of the diagonal entries:

$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & k & 1 \end{vmatrix} = (1)(1)(1) = 1$

26. Since the matrix is triangular, by Theorem 2 the determinant is the product of the diagonal entries:

$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ k & 0 & 1 \end{vmatrix} = (1)(1)(1) = 1$

27. Since the matrix is triangular, by Theorem 2 the determinant is the product of the diagonal entries:

$\begin{vmatrix} k & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = (k)(1)(1) = k$

28. Since the matrix is triangular, by Theorem 2 the determinant is the product of the diagonal entries:

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & 1 \end{vmatrix} = (1)(k)(1) = k$$

29. A cofactor expansion along row 1 gives

$$\begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} = -1\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = -1$$

30. A cofactor expansion along row 1 gives

$$\begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix} = 1\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1$$

31. A 3×3 elementary row replacement matrix looks like one of the six matrices

$$\begin{bmatrix} 1 & 0 & 0 \\ k & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ k & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & k & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \mathrm{k} \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & k \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & k & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

In each of these cases, the matrix is triangular and its determinant is the product of its diagonal entries, which is 1. Thus the determinant of a 3×3 elementary row replacement matrix is 1.

32. A 3×3 elementary scaling matrix with k on the diagonal looks like one of the three matrices

$$\begin{bmatrix} k & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & k \end{bmatrix}$$

In each of these cases, the matrix is triangular and its determinant is the product of its diagonal entries, which is k. Thus the determinant of a 3×3 elementary scaling matrix with k on the diagonal is k.

33. $E = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, EA = \begin{bmatrix} c & d \\ a & b \end{bmatrix}$

det $E = -1$, det $A = ad - bc$,
det $EA = cb - da = -1(ad - bc) =$ (det E)(det A)

34. $E = \begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}, A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, EA = \begin{bmatrix} a & b \\ kc & kd \end{bmatrix}$

det $E = k$, det $A = ad - bc$,
det $EA = a(kd) - (kc)b = k(ad - bc) =$ (det E)(det A)

35. $E = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}, A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, EA = \begin{bmatrix} a + kc & b + kd \\ c & d \end{bmatrix}$

det $E = 1$, det $A = ad - bc$,
det $EA = (a + kc)d - c(b + kd) = ad + kcd - bc - kcd = 1(ad - bc) =$ (det E)(det A)

36. $E = \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$, $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $EA = \begin{bmatrix} a & b \\ ka+c & kb+d \end{bmatrix}$

det E = 1, det $A = ad - bc$,
det $EA = a(kb + d) - (ka + c)b = kab + ad - kab - bc = 1(ad - bc)$ = (det E)(det A)

37. $A = \begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix}$, $5A = \begin{bmatrix} 15 & 5 \\ 20 & 10 \end{bmatrix}$, det A = 2, det $5A = 50 \neq 5\det A$

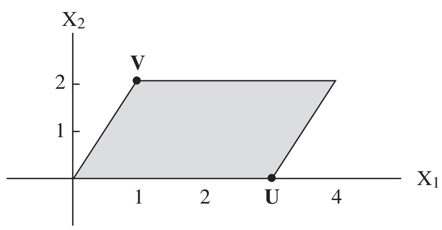
38. $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $kA = \begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix}$, det $A = ad - bc$,

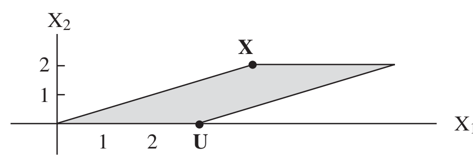
$\det kA = (ka)(kd) - (kb)(kc) = k^2(ad - bc) = k^2 \det A$

39. **a**. True. See the paragraph preceding the definition of the determinant.
b. False. See the definition of cofactor, which precedes Theorem 1.

40. **a**. False. See Theorem 1.
b. False. See Theorem 2.

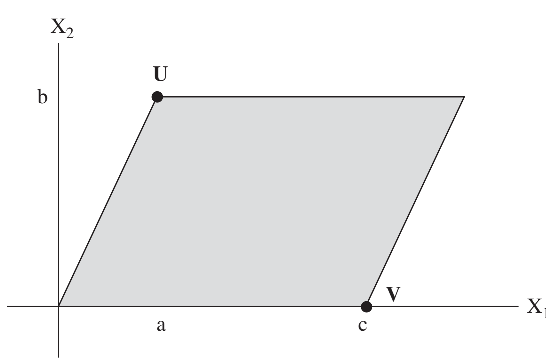
41. The area of the parallelogram determined by $\mathbf{u} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, **u** + **v**, and **0** is 6, since the base of the parallelogram has length 3 and the height of the parallelogram is 2. By the same reasoning, the area of the parallelogram determined by $\mathbf{u} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} x \\ 2 \end{bmatrix}$, **u** + **x**, and **0** is also 6.





Also note that $\det\begin{bmatrix}\mathbf{u} & \mathbf{v}\end{bmatrix} = \det\begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} = 6,$ and $\det\begin{bmatrix}\mathbf{u} & \mathbf{x}\end{bmatrix} = \det\begin{bmatrix} 3 & x \\ 0 & 2 \end{bmatrix} = 6.$ The determinant of the matrix whose columns are those vectors which define the sides of the parallelogram adjacent to **0** is equal to the area of the parallelogram.

42. The area of the parallelogram determined by $\mathbf{u} = \begin{bmatrix} a \\ b \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} c \\ 0 \end{bmatrix}$, **u** + **v**, and **0** is cb, since the base of the parallelogram has length c and the height of the parallelogram is b.



Also note that $\det[\mathbf{u} \quad \mathbf{v}] = \det\begin{bmatrix} a & c \\ b & 0 \end{bmatrix} = -cb$, and $\det[\mathbf{v} \quad \mathbf{u}] = \det\begin{bmatrix} c & a \\ 0 & b \end{bmatrix} = cb.$ The determinant of the matrix whose columns are those vectors which define the sides of the parallelogram adjacent to **0** either is equal to the area of the parallelogram or is equal to the negative of the area of the parallelogram.

43. **[M]** Answers will vary. The conclusion should be that $\det(A + B) \neq \det A + \det B$.

44. **[M]** Answers will vary. The conclusion should be that $\det(AB) = (\det A)(\det B)$.

45. **[M]** Answers will vary. For 4×4 matrices, the conclusions should be that $\det A^T = \det A$, $\det(-A) = \det A$, $\det(2A) = 16\det A$, and $\det(10A) = 10^4 \det A$. For 5×5 matrices, the conclusions should be that $\det A^T = \det A$, $\det(-A) = -\det A$, $\det(2A) = 32\det A$, and $\det(10A) = 10^5 \det A$. For 6×6 matrices, the conclusions should be that $\det A^T = \det A$, $\det(-A) = \det A$, $\det(2A) = 64\det A$, and $\det(10A) = 10^6 \det A$.

46. **[M]** Answers will vary. The conclusion should be that $\det A^{-1} = 1/\det A$.

3.2 SOLUTIONS

Notes: This section presents the main properties of the determinant, including the effects of row operations on the determinant of a matrix. These properties are first studied by examples in Exercises 1–20. The properties are treated in a more theoretical manner in later exercises. An efficient method for computing the determinant using row reduction and selective cofactor expansion is presented in this section and used in Exercises 11–14. Theorems 4 and 6 are used extensively in Chapter 5. The linearity property of the determinant studied in the text is optional, but is used in more advanced courses.

1. Rows 1 and 2 are interchanged, so the determinant changes sign (Theorem 3b.).

2. The constant 2 may be factored out of the Row 1 (Theorem 3c.).

3. The row replacement operation does not change the determinant (Theorem 3a.).

4. The row replacement operation does not change the determinant (Theorem 3a.).

5. $\begin{vmatrix} 1 & 5 & -6 \\ -1 & -4 & 4 \\ -2 & -7 & 9 \end{vmatrix} = \begin{vmatrix} 1 & 5 & -6 \\ 0 & 1 & -2 \\ 0 & 3 & -3 \end{vmatrix} = \begin{vmatrix} 1 & 5 & -6 \\ 0 & 1 & -2 \\ 0 & 0 & 3 \end{vmatrix} = 3$

6. $\begin{vmatrix} 1 & 5 & -3 \\ 3 & -3 & 3 \\ 2 & 13 & -7 \end{vmatrix} = \begin{vmatrix} 1 & 5 & -3 \\ 0 & -18 & 12 \\ 0 & 3 & -1 \end{vmatrix} = 6\begin{vmatrix} 1 & 5 & -3 \\ 0 & -3 & 2 \\ 0 & 3 & -1 \end{vmatrix} = 6\begin{vmatrix} 1 & 5 & -3 \\ 0 & -3 & 2 \\ 0 & 0 & 1 \end{vmatrix} = (6)(-3) = -18$

7. $\begin{vmatrix} 1 & 3 & 0 & 2 \\ -2 & -5 & 7 & 4 \\ 3 & 5 & 2 & 1 \\ 1 & -1 & 2 & -3 \end{vmatrix} = \begin{vmatrix} 1 & 3 & 0 & 2 \\ 0 & 1 & 7 & 8 \\ 0 & -4 & 2 & -5 \\ 0 & -4 & 2 & -5 \end{vmatrix} = \begin{vmatrix} 1 & 3 & 0 & 2 \\ 0 & 1 & 7 & 8 \\ 0 & 0 & 30 & 27 \\ 0 & 0 & 30 & 27 \end{vmatrix} = \begin{vmatrix} 1 & 3 & 0 & 2 \\ 0 & 1 & 7 & 8 \\ 0 & 0 & 30 & 27 \\ 0 & 0 & 0 & 0 \end{vmatrix} = 0$

8. $\begin{vmatrix} 1 & 3 & 3 & -4 \\ 0 & 1 & 2 & -5 \\ 2 & 5 & 4 & -3 \\ -3 & -7 & -5 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 3 & 3 & -4 \\ 0 & 1 & 2 & -5 \\ 0 & -1 & -2 & 5 \\ 0 & 2 & 4 & -10 \end{vmatrix} = \begin{vmatrix} 1 & 3 & 3 & -4 \\ 0 & 1 & 2 & -5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix} = 0$

9. $\begin{vmatrix} 1 & -1 & -3 & 0 \\ 0 & 1 & 5 & 4 \\ -1 & 2 & 8 & 5 \\ 3 & -1 & -2 & 3 \end{vmatrix} = \begin{vmatrix} 1 & -1 & -3 & 0 \\ 0 & 1 & 5 & 4 \\ 0 & 1 & 5 & 5 \\ 0 & 2 & 7 & 3 \end{vmatrix} = \begin{vmatrix} 1 & -1 & -3 & 0 \\ 0 & 1 & 5 & 4 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -3 & -5 \end{vmatrix} = -\begin{vmatrix} 1 & -1 & -3 & 0 \\ 0 & 1 & 5 & 4 \\ 0 & 0 & -3 & -5 \\ 0 & 0 & 0 & 1 \end{vmatrix} = -(-3) = 3$

10. $\begin{vmatrix} 1 & 3 & -1 & 0 & -2 \\ 0 & 2 & -4 & -1 & -6 \\ -2 & -6 & 2 & 3 & 9 \\ 3 & 7 & -3 & 8 & -7 \\ 3 & 5 & 5 & 2 & 7 \end{vmatrix} = \begin{vmatrix} 1 & 3 & -1 & 0 & -2 \\ 0 & 2 & -4 & -1 & -6 \\ 0 & 0 & 0 & 3 & 5 \\ 0 & -2 & 0 & 8 & -1 \\ 0 & -4 & 8 & 2 & 13 \end{vmatrix} = \begin{vmatrix} 1 & 3 & -1 & 0 & -2 \\ 0 & 2 & -4 & -1 & -6 \\ 0 & 0 & 0 & 3 & 5 \\ 0 & 0 & -4 & 7 & -7 \\ 0 & 0 & 0 & 0 & 1 \end{vmatrix} =$

$-\begin{vmatrix} 1 & 3 & -1 & 0 & -2 \\ 0 & 2 & -4 & -1 & -6 \\ 0 & 0 & -4 & 7 & -7 \\ 0 & 0 & 0 & 3 & 5 \\ 0 & 0 & 0 & 0 & 1 \end{vmatrix} = -(-24) = 24$

11. First use a row replacement to create zeros in the second column, and then expand down the second column:

$\begin{vmatrix} 2 & 5 & -3 & -1 \\ 3 & 0 & 1 & -3 \\ -6 & 0 & -4 & 9 \\ 4 & 10 & -4 & -1 \end{vmatrix} = \begin{vmatrix} 2 & 5 & -3 & -1 \\ 3 & 0 & 1 & -3 \\ -6 & 0 & -4 & 9 \\ 0 & 0 & 2 & 1 \end{vmatrix} = -5\begin{vmatrix} 3 & 1 & -3 \\ -6 & -4 & 9 \\ 0 & 2 & 1 \end{vmatrix}$

Now use a row replacement to create zeros in the first column, and then expand down the first column:

$-5\begin{vmatrix} 3 & 1 & -3 \\ -6 & -4 & 9 \\ 0 & 2 & 1 \end{vmatrix} = -5\begin{vmatrix} 3 & 1 & -3 \\ 0 & -2 & 3 \\ 0 & 2 & 1 \end{vmatrix} = (-5)(3)\begin{vmatrix} -2 & 3 \\ 2 & 1 \end{vmatrix} = (-5)(3)(-8) = 120$

12. First use a row replacement to create zeros in the fourth column, and then expand down the fourth column:

$\begin{vmatrix} -1 & 2 & 3 & 0 \\ 3 & 4 & 3 & 0 \\ 5 & 4 & 6 & 6 \\ 4 & 2 & 4 & 3 \end{vmatrix} = \begin{vmatrix} -1 & 2 & 3 & 0 \\ 3 & 4 & 3 & 0 \\ -3 & 0 & -2 & 0 \\ 4 & 2 & 4 & 3 \end{vmatrix} = 3\begin{vmatrix} -1 & 2 & 3 \\ 3 & 4 & 3 \\ -3 & 0 & -2 \end{vmatrix}$

Now use a row replacement to create zeros in the first column, and then expand down the first column:

$3\begin{vmatrix} -1 & 2 & 3 \\ 3 & 4 & 3 \\ -3 & 0 & -2 \end{vmatrix} = 3\begin{vmatrix} -1 & 2 & 3 \\ 0 & 10 & 12 \\ 0 & -6 & -11 \end{vmatrix} = 3(-1)\begin{vmatrix} 10 & 12 \\ -6 & -11 \end{vmatrix} = 3(-1)(-38) = 114$

13. First use a row replacement to create zeros in the fourth column, and then expand down the fourth column:

$$\begin{vmatrix} 2 & 5 & 4 & 1 \\ 4 & 7 & 6 & 2 \\ 6 & -2 & -4 & 0 \\ -6 & 7 & 7 & 0 \end{vmatrix} = \begin{vmatrix} 2 & 5 & 4 & 1 \\ 0 & -3 & -2 & 0 \\ 6 & -2 & -4 & 0 \\ -6 & 7 & 7 & 0 \end{vmatrix} = -1\begin{vmatrix} 0 & -3 & -2 \\ 6 & -2 & -4 \\ -6 & 7 & 7 \end{vmatrix}$$

Now use a row replacement to create zeros in the first column, and then expand down the first column:

$$-1\begin{vmatrix} 0 & -3 & -2 \\ 6 & -2 & -4 \\ -6 & 7 & 7 \end{vmatrix} = -1\begin{vmatrix} 0 & -3 & -2 \\ 6 & -2 & -4 \\ 0 & 5 & 3 \end{vmatrix} = (-1)(-6)\begin{vmatrix} -3 & -2 \\ 5 & 3 \end{vmatrix} = (-1)(-6)(1) = 6$$

14. First use a row replacement to create zeros in the third column, and then expand down the third column:

$$\begin{vmatrix} -3 & -2 & 1 & -4 \\ 1 & 3 & 0 & -3 \\ -3 & 4 & -2 & 8 \\ 3 & -4 & 0 & 4 \end{vmatrix} = \begin{vmatrix} -3 & -2 & 1 & -4 \\ 1 & 3 & 0 & -3 \\ -9 & 0 & 0 & 0 \\ 3 & -4 & 0 & 4 \end{vmatrix} = 1\begin{vmatrix} 1 & 3 & -3 \\ -9 & 0 & 0 \\ 3 & -4 & 4 \end{vmatrix}$$

Now expand along the second row:

$$1\begin{vmatrix} 1 & 3 & -3 \\ -9 & 0 & 0 \\ 3 & -4 & 4 \end{vmatrix} = 1(-(-9))\begin{vmatrix} 3 & -3 \\ -4 & 4 \end{vmatrix} = (1)(9)(0) = 0$$

15. $\begin{vmatrix} a & b & c \\ d & e & f \\ 5g & 5h & 5i \end{vmatrix} = 5\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 5(7) = 35$

16. $\begin{vmatrix} a & b & c \\ 3d & 3e & 3f \\ g & h & i \end{vmatrix} = 3\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 3(7) = 21$

17. $\begin{vmatrix} a & b & c \\ g & h & i \\ d & e & f \end{vmatrix} = -\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = -7$

18. $\begin{vmatrix} g & h & i \\ a & b & c \\ d & e & f \end{vmatrix} = -\begin{vmatrix} a & b & c \\ g & h & i \\ d & e & f \end{vmatrix} = -\left(-\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}\right) = -(-7) = 7$

19. $\begin{vmatrix} a & b & c \\ 2d+a & 2e+b & 2f+c \\ g & h & i \end{vmatrix} = \begin{vmatrix} a & b & c \\ 2d & 2e & 2f \\ g & h & i \end{vmatrix} = 2\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 2(7) = 14$

20. $\begin{vmatrix} a+d & b+e & c+f \\ d & e & f \\ g & h & i \end{vmatrix} = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 7$

21. Since $\begin{vmatrix} 2 & 3 & 0 \\ 1 & 3 & 4 \\ 1 & 2 & 1 \end{vmatrix} = -1 \neq 0$, the matrix is invertible.

22. Since $\begin{vmatrix} 5 & 0 & -1 \\ 1 & -3 & -2 \\ 0 & 5 & 3 \end{vmatrix} = 0$, the matrix is not invertible.

23. Since $\begin{vmatrix} 2 & 0 & 0 & 8 \\ 1 & -7 & -5 & 0 \\ 3 & 8 & 6 & 0 \\ 0 & 7 & 5 & 4 \end{vmatrix} = 0$, the matrix is not invertible.

24. Since $\begin{vmatrix} 4 & -7 & -3 \\ 6 & 0 & -5 \\ -7 & 2 & 6 \end{vmatrix} = 11 \neq 0$, the columns of the matrix form a linearly independent set.

25. Since $\begin{vmatrix} 7 & -8 & 7 \\ -4 & 5 & 0 \\ -6 & 7 & -5 \end{vmatrix} = -1 \neq 0$, the columns of the matrix form a linearly independent set.

26. Since $\begin{vmatrix} 3 & 2 & -2 & 0 \\ 5 & -6 & -1 & 0 \\ -6 & 0 & 3 & 0 \\ 4 & 7 & 0 & -3 \end{vmatrix} = 0$, the columns of the matrix form a linearly dependent set.

27. **a**. True. See Theorem 3.
b. True. See the paragraph following Example 2.
c. True. See the paragraph following Theorem 4.
d. False. See the warning following Example 5.

28. **a**. True. See Theorem 3.
b. False. See the paragraphs following Example 2.
c. False. See Example 3.
d. False. See Theorem 5.

29. By Theorem 6, $\det B^5 = (\det B)^5 = (-2)^5 = -32$.

30. Suppose the two rows of a square matrix A are equal. By swapping these two rows, the matrix A is not changed so its determinant should not change. But since swapping rows changes the sign of the determinant, det A = – det A. This is only possible if det A = 0. The same may be proven true for columns by applying the above result to A^T and using Theorem 5.

31. By Theorem 6, $(\det A)(\det A^{-1}) = \det I = 1$, so $\det A^{-1} = 1/\det A$.

32. By factoring an r out of each of the n rows, $\det(rA) = r^n \det A$.

33. By Theorem 6, $\det AB = (\det A)(\det B) = (\det B)(\det A) = \det BA$.

34. By Theorem 6 and Exercise 31,

$$\begin{aligned}\det(PAP^{-1}) &= (\det P)(\det A)(\det P^{-1}) = (\det P)(\det P^{-1})(\det A)\\ &= (\det P)\left(\frac{1}{\det P}\right)(\det A) = 1 \det A\\ &= \det A\end{aligned}$$

35. By Theorem 6 and Theorem 5, $\det U^T U = (\det U^T)(\det U) = (\det U)^2$. Since $U^T U = I$, $\det U^T U = \det I = 1$, so $(\det U)^2 = 1$. Thus $\det U = \pm 1$.

36. By Theorem 6 $\det A^4 = (\det A)^4$. Since $\det A^4 = 0$, then $(\det A)^4 = 0$. Thus $\det A = 0$, and A is not invertible by Theorem 4.

37. One may compute using Theorem 2 that $\det A = 3$ and $\det B = 8$, while $AB = \begin{bmatrix} 6 & 0 \\ 17 & 4 \end{bmatrix}$. Thus $\det AB = 24 = 3 \times 8 = (\det A)(\det B)$.

38. One may compute that $\det A = 0$ and $\det B = -2$, while $AB = \begin{bmatrix} 6 & 0 \\ -2 & 0 \end{bmatrix}$. Thus $\det AB = 0 = 0 \times -2 = (\det A)(\det B)$.

39. **a**. By Theorem 6, $\det AB = (\det A)(\det B) = 4 \times -3 = -12$.

b. By Exercise 32, $\det 5A = 5^3 \det A = 125 \times 4 = 500$.

c. By Theorem 5, $\det B^T = \det B = -3$.

d. By Exercise 31, $\det A^{-1} = 1/\det A = 1/4$.

e. By Theorem 6, $\det A^3 = (\det A)^3 = 4^3 = 64$.

40. **a**. By Theorem 6, $\det AB = (\det A)(\det B) = -1 \times 2 = -2$.

b. By Theorem 6, $\det B^5 = (\det B)^5 = 2^5 = 32$.

c. By Exercise 32, $\det 2A = 2^4 \det A = 16 \times -1 = -16$.

d. By Theorems 5 and 6, $\det A^T A = (\det A^T)(\det A) = (\det A)(\det A) = -1 \times -1 = 1$.

e. By Theorem 6 and Exercise 31,
$\det B^{-1}AB = (\det B^{-1})(\det A)(\det B) = (1/\det B)(\det A)(\det B) = \det A = -1$.

41. $\det A = (a + e)d - c(b + f) = ad + ed - bc - cf = (ad - bc) + (ed - cf) = \det B + \det C$.

42. $\det(A+B) = \begin{vmatrix} 1+a & b \\ c & 1+d \end{vmatrix} = (1+a)(1+d) - cb = 1 + a + d + ad - cb = \det A + a + d + \det B$, so $\det(A + B) = \det A + \det B$ if and only if $a + d = 0$.

43. Compute det A by using a cofactor expansion down the third column:

$$\det A = (u_1 + v_1)\det A_{13} - (u_2 + v_2)\det A_{23} + (u_3 + v_3)\det A_{33}$$
$$= u_1 \det A_{13} - u_2 \det A_{23} + u_3 \det A_{33} + v_1 \det A_{13} - v_2 \det A_{23} + v_3 \det A_{33}$$
$$= \det B + \det C$$

44. By Theorem 5, $\det AE = \det (AE)^T$. Since $(AE)^T = E^T A^T$, $\det AE = \det(E^T A^T)$. Now E^T is itself an elementary matrix, so by the proof of Theorem 3, $\det(E^T A^T) = (\det E^T)(\det A^T)$. Thus it is true that $\det AE = (\det E^T)(\det A^T)$, and by applying Theorem 5, det AE = (det E)(det A).

45. **[M]** Answers will vary, but will show that $\det A^T A$ always equals 0 while $\det AA^T$ should seldom be zero. To see why $A^T A$ should not be invertible (and thus $\det A^T A = 0$), let A be a matrix with more columns than rows. Then the columns of A must be linearly dependent, so the equation $A\mathbf{x} = \mathbf{0}$ must have a non-trivial solution $\mathbf{x}$. Thus $(A^T A)\mathbf{x} = A^T(A\mathbf{x}) = A^T\mathbf{0} = \mathbf{0}$, and the equation $(A^T A)\mathbf{x} = \mathbf{0}$ has a non-trivial solution. Since $A^T A$ is a square matrix, the Invertible Matrix Theorem now says that $A^T A$ is not invertible. Notice that the same argument will not work in general for AA^T, since A^T has more rows than columns, so its columns are not automatically linearly dependent.

46. **[M]** One may compute for this matrix that det A = 1 and cond $A \approx 23683$. Note that this is the ℓ_2 condition number, which is used in Section 2.3. Since det $A \neq 0$, it is invertible and

$$A^{-1} = \begin{bmatrix} -19 & -14 & 0 & 7 \\ -549 & -401 & -2 & 196 \\ 267 & 195 & 1 & -95 \\ -278 & -203 & -1 & 99 \end{bmatrix}$$

The determinant is very sensitive to scaling, as $\det 10A = 10^4 \det A = 10{,}000$ and $\det 0.1A = (0.1)^4 \det A = 0.0001$. The condition number is not changed at all by scaling: cond($10A$) = cond($0.1A$) = cond$A \approx 23683$.

When $A = I_4$, det A=1 and cond A = 1. As before the determinant is sensitive to scaling: $\det 10A = 10^4 \det A = 10{,}000$ and $\det 0.1A = (0.1)^4 \det A = 0.0001$. Yet the condition number is not changed by scaling: cond($10A$) = cond($0.1A$) = cond A = 1.

3.3 SOLUTIONS

Notes: This section features several independent topics from which to choose. The geometric interpretation of the determinant (Theorem 10) provides the key to changes of variables in multiple integrals. Students of economics and engineering are likely to need Cramer's Rule in later courses. Exercises 1–10 concern Cramer's Rule, exercises 11–18 deal with the adjugate, and exercises 19–32 cover the geometric interpretation of the determinant. In particular, Exercise 25 examines students' understanding of linear independence and requires a careful explanation, which is discussed in the *Study Guide*. The *Study Guide* also contains a heuristic proof of Theorem 9 for 2×2 matrices.

1. The system is equivalent to $A\mathbf{x} = \mathbf{b}$, where $A = \begin{bmatrix} 5 & 7 \\ 2 & 4 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$. We compute

$$A_1(\mathbf{b}) = \begin{bmatrix} 3 & 7 \\ 1 & 4 \end{bmatrix}, A_2(\mathbf{b}) = \begin{bmatrix} 5 & 3 \\ 2 & 1 \end{bmatrix}, \det A = 6, \det A_1(\mathbf{b}) = 5, \det A_2(\mathbf{b}) = -1,$$

$$x_1 = \frac{\det A_1(\mathbf{b})}{\det A} = \frac{5}{6}, x_2 = \frac{\det A_2(\mathbf{b})}{\det A} = -\frac{1}{6}.$$

2. The system is equivalent to $A\mathbf{x} = \mathbf{b}$, where $A = \begin{bmatrix} 4 & 1 \\ 5 & 2 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 6 \\ 7 \end{bmatrix}$. We compute

$$A_1(\mathbf{b}) = \begin{bmatrix} 6 & 1 \\ 7 & 2 \end{bmatrix}, A_2(\mathbf{b}) = \begin{bmatrix} 4 & 6 \\ 5 & 7 \end{bmatrix}, \det A = 3, \det A_1(\mathbf{b}) = 5, \det A_2(\mathbf{b}) = -2,$$

$$x_1 = \frac{\det A_1(\mathbf{b})}{\det A} = \frac{5}{3}, x_2 = \frac{\det A_2(\mathbf{b})}{\det A} = -\frac{2}{3}.$$

3. The system is equivalent to $A\mathbf{x} = \mathbf{b}$, where $A = \begin{bmatrix} 3 & -2 \\ -5 & 6 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 7 \\ -5 \end{bmatrix}$. We compute

$$A_1(\mathbf{b}) = \begin{bmatrix} 7 & -2 \\ -5 & 6 \end{bmatrix}, A_2(\mathbf{b}) = \begin{bmatrix} 3 & 7 \\ -5 & -5 \end{bmatrix}, \det A = 8, \det A_1(\mathbf{b}) = 32, \det A_2(\mathbf{b}) = 20,$$

$$x_1 = \frac{\det A_1(\mathbf{b})}{\det A} = \frac{32}{8} = 4, x_2 = \frac{\det A_2(\mathbf{b})}{\det A} = \frac{20}{8} = \frac{5}{2}.$$

4. The system is equivalent to $A\mathbf{x} = \mathbf{b}$, where $A = \begin{bmatrix} -5 & 3 \\ 3 & -1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 9 \\ -5 \end{bmatrix}$. We compute

$$A_1(\mathbf{b}) = \begin{bmatrix} 9 & 3 \\ -5 & -1 \end{bmatrix}, A_2(\mathbf{b}) = \begin{bmatrix} -5 & 9 \\ 3 & -5 \end{bmatrix}, \det A = -4, \det A_1(\mathbf{b}) = 6, \det A_2(\mathbf{b}) = -2,$$

$$x_1 = \frac{\det A_1(\mathbf{b})}{\det A} = \frac{6}{-4} = -\frac{3}{2}, x_2 = \frac{\det A_2(\mathbf{b})}{\det A} = \frac{-2}{-4} = \frac{1}{2}.$$

5. The system is equivalent to $A\mathbf{x} = \mathbf{b}$, where $A = \begin{bmatrix} 2 & 1 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 7 \\ -8 \\ -3 \end{bmatrix}$. We compute

$$A_1(\mathbf{b}) = \begin{bmatrix} 7 & 1 & 0 \\ -8 & 0 & 1 \\ -3 & 1 & 2 \end{bmatrix}, A_2(\mathbf{b}) = \begin{bmatrix} 2 & 7 & 0 \\ -3 & -8 & 1 \\ 0 & -3 & 2 \end{bmatrix}, A_3(\mathbf{b}) = \begin{bmatrix} 2 & 1 & 7 \\ -3 & 0 & -8 \\ 0 & 1 & -3 \end{bmatrix},$$

$$\det A = 4, \det A_1(\mathbf{b}) = 6, \det A_2(\mathbf{b}) = 16, \det A_3(\mathbf{b}) = -14,$$

$$x_1 = \frac{\det A_1(\mathbf{b})}{\det A} = \frac{6}{4} = \frac{3}{2}, x_2 = \frac{\det A_2(\mathbf{b})}{\det A} = \frac{16}{4} = 4, x_3 = \frac{\det A_3(\mathbf{b})}{\det A} = \frac{-14}{4} = -\frac{7}{2}.$$

6. The system is equivalent to $A\mathbf{x} = \mathbf{b}$, where $A = \begin{bmatrix} 2 & 1 & 1 \\ -1 & 0 & 2 \\ 3 & 1 & 3 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 4 \\ 2 \\ -2 \end{bmatrix}$. We compute

$$A_1(\mathbf{b}) = \begin{bmatrix} 4 & 1 & 1 \\ 2 & 0 & 2 \\ -2 & 1 & 3 \end{bmatrix}, A_2(\mathbf{b}) = \begin{bmatrix} 2 & 4 & 1 \\ -1 & 2 & 2 \\ 3 & -2 & 3 \end{bmatrix}, A_3(\mathbf{b}) = \begin{bmatrix} 2 & 1 & 4 \\ -1 & 0 & 2 \\ 3 & 1 & -2 \end{bmatrix},$$

$$\det A = 4, \det A_1(\mathbf{b}) = -16, \det A_2(\mathbf{b}) = 52, \det A_3(\mathbf{b}) = -4,$$

$$x_1 = \frac{\det A_1(\mathbf{b})}{\det A} = \frac{-16}{4} = -4, x_2 = \frac{\det A_2(\mathbf{b})}{\det A} = \frac{52}{4} = 13, x_3 = \frac{\det A_3(\mathbf{b})}{\det A} = \frac{-4}{4} = -1.$$

7. The system is equivalent to $A\mathbf{x} = \mathbf{b}$, where $A = \begin{bmatrix} 6s & 4 \\ 9 & 2s \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$. We compute

$$A_1(\mathbf{b}) = \begin{bmatrix} 5 & 4 \\ -2 & 2s \end{bmatrix}, A_2(\mathbf{b}) = \begin{bmatrix} 6s & 5 \\ 9 & -2 \end{bmatrix}, \det A_1(\mathbf{b}) = 10s + 8, \det A_2(\mathbf{b}) = -12s - 45.$$

Since $\det A = 12s^2 - 36 = 12(s^2 - 3) \neq 0$ for $s \neq \pm\sqrt{3}$, the system will have a unique solution when $s \neq \pm\sqrt{3}$. For such a system, the solution will be

$$x_1 = \frac{\det A_1(\mathbf{b})}{\det A} = \frac{10s+8}{12(s^2-3)} = \frac{5s+4}{6(s^2-3)}, x_2 = \frac{\det A_2(\mathbf{b})}{\det A} = \frac{-12s-45}{12(s^2-3)} = \frac{-4s-15}{4(s^2-3)}.$$

8. The system is equivalent to $A\mathbf{x} = \mathbf{b}$, where $A = \begin{bmatrix} 3s & -5 \\ 9 & 5s \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$. We compute

$$A_1(\mathbf{b}) = \begin{bmatrix} 3 & -5 \\ 2 & 5s \end{bmatrix}, A_2(\mathbf{b}) = \begin{bmatrix} 3s & 3 \\ 9 & 2 \end{bmatrix}, \det A_1(\mathbf{b}) = 15s + 10, \det A_2(\mathbf{b}) = 6s - 27.$$

Since $\det A = 15s^2 + 45 = 15(s^2 + 3) \neq 0$ for all values of s, the system will have a unique solution for all values of s. For such a system, the solution will be

$$x_1 = \frac{\det A_1(\mathbf{b})}{\det A} = \frac{15s+10}{15(s^2+3)} = \frac{3s+2}{3(s^2+3)}, x_2 = \frac{\det A_2(\mathbf{b})}{\det A} = \frac{6s-27}{15(s^2+3)} = \frac{2s-9}{5(s^2+3)}.$$

9. The system is equivalent to $A\mathbf{x} = \mathbf{b}$, where $A = \begin{bmatrix} s & -2s \\ 3 & 6s \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$. We compute

$$A_1(\mathbf{b}) = \begin{bmatrix} -1 & -2s \\ 4 & 6s \end{bmatrix}, A_2(\mathbf{b}) = \begin{bmatrix} s & -1 \\ 3 & 4 \end{bmatrix}, \det A_1(\mathbf{b}) = 2s, \det A_2(\mathbf{b}) = 4s + 3.$$

Since $\det A = 6s^2 + 6s = 6s(s+1) = 0$ for $s = 0, -1$, the system will have a unique solution when $s \neq 0, -1$. For such a system, the solution will be

$$x_1 = \frac{\det A_1(\mathbf{b})}{\det A} = \frac{2s}{6s(s+1)} = \frac{1}{3(s+1)}, x_2 = \frac{\det A_2(\mathbf{b})}{\det A} = \frac{4s+3}{6s(s+1)}.$$

10. The system is equivalent to $A\mathbf{x} = \mathbf{b}$, where $A = \begin{bmatrix} 2s & 1 \\ 3s & 6s \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. We compute

$$A_1(\mathbf{b}) = \begin{bmatrix} 1 & 1 \\ 2 & 6s \end{bmatrix}, A_2(\mathbf{b}) = \begin{bmatrix} 2s & 1 \\ 3s & 2 \end{bmatrix}, \det A_1(\mathbf{b}) = 6s - 2, \det A_2(\mathbf{b}) = s.$$

Since $\det A = 12s^2 - 3s = 3s(4s-1) = 0$ for $s = 0, 1/4$, the system will have a unique solution when $s \neq 0, 1/4$. For such a system, the solution will be

$$x_1 = \frac{\det A_1(\mathbf{b})}{\det A} = \frac{6s-2}{3s(4s-1)},\ x_2 = \frac{\det A_2(\mathbf{b})}{\det A} = \frac{s}{3s(4s-1)} = \frac{1}{3(4s-1)}.$$

11. Since det A = 3 and the cofactors of the given matrix are

$$C_{11} = \begin{vmatrix} 0 & 0 \\ 1 & 1 \end{vmatrix} = 0, \quad C_{12} = -\begin{vmatrix} 3 & 0 \\ -1 & 1 \end{vmatrix} = -3, \quad C_{13} = \begin{vmatrix} 3 & 0 \\ -1 & 1 \end{vmatrix} = 3,$$

$$C_{21} = -\begin{vmatrix} -2 & -1 \\ 1 & 1 \end{vmatrix} = 1, \quad C_{22} = \begin{vmatrix} 0 & -1 \\ -1 & 1 \end{vmatrix} = -1, \quad C_{23} = -\begin{vmatrix} 0 & -2 \\ -1 & 1 \end{vmatrix} = 2,$$

$$C_{31} = \begin{vmatrix} -2 & -1 \\ 0 & 0 \end{vmatrix} = 0, \quad C_{32} = -\begin{vmatrix} 0 & -1 \\ 3 & 0 \end{vmatrix} = -3, \quad C_{33} = \begin{vmatrix} 0 & -2 \\ 3 & 0 \end{vmatrix} = 6,$$

$$\operatorname{adj} A = \begin{bmatrix} 0 & 1 & 0 \\ -3 & -1 & -3 \\ 3 & 2 & 6 \end{bmatrix} \text{ and } A^{-1} = \frac{1}{\det A}\operatorname{adj} A = \begin{bmatrix} 0 & 1/3 & 0 \\ -1 & -1/3 & -1 \\ 1 & 2/3 & 2 \end{bmatrix}.$$

12. Since det A = 5 and the cofactors of the given matrix are

$$C_{11} = \begin{vmatrix} -2 & 1 \\ 1 & 0 \end{vmatrix} = -1, \quad C_{12} = -\begin{vmatrix} 2 & 1 \\ 0 & 0 \end{vmatrix} = 0, \quad C_{13} = \begin{vmatrix} 2 & -2 \\ 0 & 1 \end{vmatrix} = 2,$$

$$C_{21} = -\begin{vmatrix} 1 & 3 \\ 1 & 0 \end{vmatrix} = 3, \quad C_{22} = \begin{vmatrix} 1 & 3 \\ 0 & 0 \end{vmatrix} = 0, \quad C_{23} = -\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = -1,$$

$$C_{31} = \begin{vmatrix} 1 & 3 \\ -2 & 1 \end{vmatrix} = 7, \quad C_{32} = -\begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} = 5, \quad C_{33} = \begin{vmatrix} 1 & 1 \\ 2 & -2 \end{vmatrix} = -4,$$

$$\operatorname{adj} A = \begin{bmatrix} -1 & 3 & 7 \\ 0 & 0 & 5 \\ 2 & -1 & -4 \end{bmatrix} \text{ and } A^{-1} = \frac{1}{\det A}\operatorname{adj} A = \begin{bmatrix} -1/5 & 3/5 & 7/5 \\ 0 & 0 & 1 \\ 2/5 & -1/5 & -4/5 \end{bmatrix}.$$

13. Since det A = 6 and the cofactors of the given matrix are

$$C_{11} = \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} = -1, \quad C_{12} = -\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} = 1, \quad C_{13} = \begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} = 1,$$

$$C_{21} = -\begin{vmatrix} 5 & 4 \\ 1 & 1 \end{vmatrix} = -1, \quad C_{22} = \begin{vmatrix} 3 & 4 \\ 2 & 1 \end{vmatrix} = -5, \quad C_{23} = -\begin{vmatrix} 3 & 5 \\ 2 & 1 \end{vmatrix} = 7,$$

$$C_{31} = \begin{vmatrix} 5 & 4 \\ 0 & 1 \end{vmatrix} = 5, \quad C_{32} = -\begin{vmatrix} 3 & 4 \\ 1 & 1 \end{vmatrix} = 1, \quad C_{33} = \begin{vmatrix} 3 & 5 \\ 1 & 0 \end{vmatrix} = -5,$$

$$\operatorname{adj} A = \begin{bmatrix} -1 & -1 & 5 \\ 1 & -5 & 1 \\ 1 & 7 & -5 \end{bmatrix} \text{ and } A^{-1} = \frac{1}{\det A}\operatorname{adj} A = \begin{bmatrix} -1/6 & -1/6 & 5/6 \\ 1/6 & -5/6 & 1/6 \\ 1/6 & 7/6 & -5/6 \end{bmatrix}.$$

14. Since det $A = -1$ and the cofactors of the given matrix are

$$C_{11}=\begin{vmatrix}2 & 1\\ 3 & 4\end{vmatrix}=5,\qquad C_{12}=-\begin{vmatrix}0 & 1\\ 2 & 4\end{vmatrix}=2,\qquad C_{13}=\begin{vmatrix}0 & 2\\ 2 & 3\end{vmatrix}=-4,$$

$$C_{21}=-\begin{vmatrix}6 & 7\\ 3 & 3\end{vmatrix}=-3,\quad C_{22}=\begin{vmatrix}3 & 7\\ 2 & 4\end{vmatrix}=-2,\qquad C_{23}=-\begin{vmatrix}3 & 6\\ 2 & 3\end{vmatrix}=3,$$

$$C_{31}=\begin{vmatrix}6 & 7\\ 2 & 1\end{vmatrix}=-8,\qquad C_{32}=-\begin{vmatrix}3 & 7\\ 0 & 1\end{vmatrix}=-3,\quad C_{33}=\begin{vmatrix}3 & 6\\ 0 & 2\end{vmatrix}=6,$$

$$\operatorname{adj} A=\begin{bmatrix}5 & -3 & -8\\ 2 & -2 & -3\\ -4 & 3 & 6\end{bmatrix} \text{ and } A^{-1}=\frac{1}{\det A}\operatorname{adj} A=\begin{bmatrix}-5 & 3 & 8\\ -2 & 2 & 3\\ 4 & -3 & -6\end{bmatrix}.$$

15. Since det $A = 6$ and the cofactors of the given matrix are

$$C_{11}=\begin{vmatrix}1 & 0\\ 3 & 2\end{vmatrix}=2,\qquad C_{12}=-\begin{vmatrix}-1 & 0\\ -2 & 2\end{vmatrix}=2,\qquad C_{13}=\begin{vmatrix}-1 & 1\\ -2 & 3\end{vmatrix}=-1,$$

$$C_{21}=-\begin{vmatrix}0 & 0\\ 3 & 2\end{vmatrix}=0,\quad C_{22}=\begin{vmatrix}3 & 0\\ -2 & 2\end{vmatrix}=6,\qquad C_{23}=-\begin{vmatrix}3 & 0\\ -2 & 3\end{vmatrix}=-9,$$

$$C_{31}=\begin{vmatrix}0 & 0\\ 1 & 0\end{vmatrix}=0,\qquad C_{31}=\begin{vmatrix}0 & 0\\ 1 & 0\end{vmatrix}=0,\qquad C_{33}=\begin{vmatrix}3 & 0\\ -1 & 1\end{vmatrix}=3,$$

$$\operatorname{adj} A=\begin{bmatrix}2 & 0 & 0\\ 2 & 6 & 0\\ -1 & -9 & 3\end{bmatrix} \text{ and } A^{-1}=\frac{1}{\det A}\operatorname{adj} A=\begin{bmatrix}1/3 & 0 & 0\\ 1/3 & 1 & 0\\ -1/6 & -3/2 & 1/2\end{bmatrix}.$$

16. Since det $A = -9$ and the cofactors of the given matrix are

$$C_{11}=\begin{vmatrix}-3 & 1\\ 0 & 3\end{vmatrix}=-9,\qquad C_{12}=-\begin{vmatrix}0 & 1\\ 0 & 3\end{vmatrix}=0,\qquad C_{13}=\begin{vmatrix}0 & -3\\ 0 & 0\end{vmatrix}=0,$$

$$C_{21}=-\begin{vmatrix}2 & 4\\ 0 & 3\end{vmatrix}=-6,\quad C_{22}=\begin{vmatrix}1 & 4\\ 0 & 3\end{vmatrix}=3,\qquad C_{23}=-\begin{vmatrix}1 & 2\\ 0 & 0\end{vmatrix}=0,$$

$$C_{31}=\begin{vmatrix}2 & 4\\ -3 & 1\end{vmatrix}=14,\qquad C_{32}=-\begin{vmatrix}1 & 4\\ 0 & 1\end{vmatrix}=-1,\qquad C_{33}=\begin{vmatrix}1 & 2\\ 0 & -3\end{vmatrix}=-3,$$

$$\operatorname{adj} A=\begin{bmatrix}-9 & -6 & 14\\ 0 & 3 & -1\\ 0 & 0 & -3\end{bmatrix} \text{ and } A^{-1}=\frac{1}{\det A}\operatorname{adj} A=\begin{bmatrix}1 & 2/3 & -14/9\\ 0 & -1/3 & 1/9\\ 0 & 0 & 1/3\end{bmatrix}.$$

17. Let $A=\begin{bmatrix}a & b\\ c & d\end{bmatrix}$. Then the cofactors of A are $C_{11}=|d|=d$, $C_{12}=-|c|=-c$, $C_{21}=-|b|=-b$, and $C_{22}=|a|=a$. Thus $\operatorname{adj} A=\begin{bmatrix}d & -b\\ -c & a\end{bmatrix}$. Since det $A = ad - bc$, Theorem 8 gives that $A^{-1}=\frac{1}{\det A}\operatorname{adj} A=\frac{1}{ad-bc}\begin{bmatrix}d & -b\\ -c & a\end{bmatrix}$. This result is identical to that of Theorem 4 in Section 2.2.

18. Each cofactor of A is an integer since it is a sum of products of entries in A. Hence all entries in adj A will be integers. Since det A = 1, the inverse formula in Theorem 8 shows that all the entries in A^{-1} will be integers.

19. The parallelogram is determined by the columns of $A = \begin{bmatrix} 5 & 6 \\ 2 & 4 \end{bmatrix}$, so the area of the parallelogram is $|\det A| = |8| = 8$.

20. The parallelogram is determined by the columns of $A = \begin{bmatrix} -1 & 4 \\ 3 & -5 \end{bmatrix}$, so the area of the parallelogram is $|\det A| = |-7| = 7$.

21. First translate one vertex to the origin. For example, subtract (–1, 0) from each vertex to get a new parallelogram with vertices (0, 0),(1, 5),(2, –4), and (3, 1). This parallelogram has the same area as the original, and is determined by the columns of $A = \begin{bmatrix} 1 & 2 \\ 5 & -4 \end{bmatrix}$, so the area of the parallelogram is $|\det A| = |-14| = 14$.

22. First translate one vertex to the origin. For example, subtract (0, –2) from each vertex to get a new parallelogram with vertices (0, 0),(6, 1),(–3, 3), and (3, 4). This parallelogram has the same area as the original, and is determined by the columns of $A = \begin{bmatrix} 6 & -3 \\ 1 & 3 \end{bmatrix}$, so the area of the parallelogram is $|\det A| = |21| = 21$.

23. The parallelepiped is determined by the columns of $A = \begin{bmatrix} 1 & 1 & 7 \\ 0 & 2 & 1 \\ -2 & 4 & 0 \end{bmatrix}$, so the volume of the parallelepiped is $|\det A| = |22| = 22$.

24. The parallelepiped is determined by the columns of $A = \begin{bmatrix} 1 & -2 & -1 \\ 4 & -5 & 2 \\ 0 & 2 & -1 \end{bmatrix}$, so the volume of the parallelepiped is $|\det A| = |-15| = 15$.

25. The Invertible Matrix Theorem says that a 3×3 matrix A is not invertible if and only if its columns are linearly dependent. This will happen if and only if one of the columns is a linear combination of the others; that is, if one of the vectors is in the plane spanned by the other two vectors. This is equivalent to the condition that the parallelepiped determined by the three vectors has zero volume, which is in turn equivalent to the condition that det A = 0.

26. By definition, $\mathbf{p} + S$ is the set of all vectors of the form $\mathbf{p} + \mathbf{v}$, where $\mathbf{v}$ is in S. Applying T to a typical vector in $\mathbf{p} + S$, we have $T(\mathbf{p} + \mathbf{v}) = T(\mathbf{p}) + T(\mathbf{v})$. This vector is in the set denoted by $T(\mathbf{p}) + T(S)$. This proves that T maps the set $\mathbf{p} + S$ into the set $T(\mathbf{p}) + T(S)$.

Conversely, any vector in $T(\mathbf{p}) + T(S)$ has the form $T(\mathbf{p}) + T(\mathbf{v})$ for some $\mathbf{v}$ in S. This vector may be written as $T(\mathbf{p} + \mathbf{v})$. This shows that every vector in $T(\mathbf{p}) + T(S)$ is the image under T of some point $\mathbf{p} + \mathbf{v}$ in $\mathbf{p} + S$.

27. Since the parallelogram S is determined by the columns of $\begin{bmatrix} -2 & -2 \\ 3 & 5 \end{bmatrix}$, the area of S is

$\left|\det\begin{bmatrix} -2 & -2 \\ 3 & 5 \end{bmatrix}\right| = |-4| = 4.$ The matrix A has $\det A = \begin{vmatrix} 6 & -2 \\ -3 & 2 \end{vmatrix} = 6$. By Theorem 10, the area of $T(S)$ is $|\det A|\{\text{area of } S\} = 6 \cdot 4 = 24$.

Alternatively, one may compute the vectors that determine the image, namely, the columns of

$$A[\mathbf{b}_1 \quad \mathbf{b}_2] = \begin{bmatrix} 6 & -2 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} -2 & -2 \\ 3 & 5 \end{bmatrix} = \begin{bmatrix} -18 & -22 \\ 12 & 16 \end{bmatrix}$$

The determinant of this matrix is –24, so the area of the image is 24.

28. Since the parallelogram S is determined by the columns of $\begin{bmatrix} 4 & 0 \\ -7 & 1 \end{bmatrix}$, the area of S is

$\left|\det\begin{bmatrix} 4 & 0 \\ -7 & 1 \end{bmatrix}\right| = |4| = 4$. The matrix A has $\det A = \begin{vmatrix} 7 & 2 \\ 1 & 1 \end{vmatrix} = 5$. By Theorem 10, the area of $T(S)$ is $|\det A|\{\text{area of } S\} = 5 \cdot 4 = 20$.

Alternatively, one may compute the vectors that determine the image, namely, the columns of

$$A[\mathbf{b}_1 \quad \mathbf{b}_2] = \begin{bmatrix} 7 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 4 & 0 \\ -7 & 1 \end{bmatrix} = \begin{bmatrix} 14 & 2 \\ -3 & 1 \end{bmatrix}$$

The determinant of this matrix is 20, so the area of the image is 20.

29. The area of the triangle will be one half of the area of the parallelogram determined by $\mathbf{v}_1$ and $\mathbf{v}_2$. By Theorem 9, the area of the triangle will be (1/2)|det A|, where $A = [\mathbf{v}_1 \quad \mathbf{v}_2]$.

30. Translate R to a new triangle of equal area by subtracting (x_3, y_3) from each vertex. The new triangle has vertices (0, 0), $(x_1 - x_3, y_1 - y_3)$, and $(x_2 - x_3, y_2 - y_3)$. By Exercise 29, the area of the triangle will be

$$\frac{1}{2}\left|\det\begin{bmatrix} x_1 - x_3 & x_2 - x_3 \\ y_1 - y_3 & y_2 - y_3 \end{bmatrix}\right|.$$

Now consider using row operations and a cofactor expansion to compute the determinant in the formula:

$$\det\begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix} = \det\begin{bmatrix} x_1 - x_3 & y_1 - y_3 & 0 \\ x_2 - x_3 & y_2 - y_3 & 0 \\ x_3 & y_3 & 1 \end{bmatrix} = \det\begin{bmatrix} x_1 - x_3 & y_1 - y_3 \\ x_2 - x_3 & y_2 - y_3 \end{bmatrix}$$

By Theorem 5,

$$\det\begin{bmatrix} x_1 - x_3 & y_1 - y_3 \\ x_2 - x_3 & y_2 - y_3 \end{bmatrix} = \det\begin{bmatrix} x_1 - x_3 & x_2 - x_3 \\ y_1 - y_3 & y_2 - y_3 \end{bmatrix}$$

So the above observation allows us to state that the area of the triangle will be

$$\frac{1}{2}\left|\det\begin{bmatrix} x_1 - x_3 & x_2 - x_3 \\ y_1 - y_3 & y_2 - y_3 \end{bmatrix}\right| = \frac{1}{2}\left|\det\begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix}\right|$$

31. **a**. To show that $T(S)$ is bounded by the ellipsoid with equation $\frac{x_1^2}{a^2}+\frac{x_2^2}{b^2}+\frac{x_3^2}{c^2}=1$, let $\mathbf{u}=\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$ and let $\mathbf{x}=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=A\mathbf{u}$. Then $u_1=x_1/a$, $u_2=x_2/b$, and $u_3=x_3/c$, and $\mathbf{u}$ lies inside S (or $u_1^2+u_2^2+u_3^2\le 1$) if and only if $\mathbf{x}$ lies inside $T(S)$ (or $\frac{x_1^2}{a^2}+\frac{x_2^2}{b^2}+\frac{x_3^2}{c^2}\le 1$).

b. By the generalization of Theorem 10,

$$\{\text{volume of ellipsoid}\}=\{\text{volume of } T(S)\}$$
$$=|\det A|\cdot\{\text{volume of } S\}=abc\frac{4\pi}{3}=\frac{4\pi abc}{3}$$

32. **a**. A linear transformation T that maps S onto S' will map $\mathbf{e}_1$ to $\mathbf{v}_1$, $\mathbf{e}_2$ to $\mathbf{v}_2$, and $\mathbf{e}_3$ to $\mathbf{v}_3$; that is, $T(\mathbf{e}_1)=\mathbf{v}_1$, $T(\mathbf{e}_2)=\mathbf{v}_2$, and $T(\mathbf{e}_3)=\mathbf{v}_3$. The standard matrix for this transformation will be $A=\begin{bmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) & T(\mathbf{e}_3)\end{bmatrix}=\begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3\end{bmatrix}$.

b. The area of the base of S is (1/2)(1)(1) = 1/2, so the volume of S is (1/3)(1/2)(1) = 1/6. By part a. $T(S) = S'$, so the generalization of Theorem 10 gives that the volume of S' is $|\det A|$\{volume of S\} = $(1/6)|\det A|$.

33. **[M]** Answers will vary. In MATLAB, entries in B – inv(A) are approximately 10^{-15} or smaller.

34. **[M]** Answers will vary, as will the commands which produce the second entry of **x**. For example, the MATLAB command is `x2 = det([A(:,1) b A(:,3:4)])/det(A)` while the Mathematica command is `x2 = Det[{Transpose[A][[1]],b,Transpose[A][[3]], Transpose[A][[4]]}]/Det[A]`.

35. **[M]** MATLAB Student Version 4.0 uses 57,771 flops for inv A and 14,269,045 flops for the inverse formula. The `inv(A)` command requires only about 0.4% of the operations for the inverse formula.

Chapter 3 SUPPLEMENTARY EXERCISES

1. **a**. True. The columns of A are linearly dependent.

b. True. See Exercise 30 in Section 3.2.

c. False. See Theorem 3(c); in this case $\det 5A=5^3\det A$.

d. False. Consider $A=\begin{bmatrix} 2 & 0 \\ 0 & 1\end{bmatrix}$, $B=\begin{bmatrix} 1 & 0 \\ 0 & 3\end{bmatrix}$, and $A+B=\begin{bmatrix} 3 & 0 \\ 0 & 4\end{bmatrix}$.

e. False. By Theorem 6, $\det A^3=2^3$.

f. False. See Theorem 3(b).

g. True. See Theorem 3(c).

h. True. See Theorem 3(a).

i. False. See Theorem 5.

j. False. See Theorem 3(c); this statement is false for $n\times n$ invertible matrices with n an even integer.

k. True. See Theorems 6 and 5; $\det A^TA=(\det A)^2$.

l. False. The coefficient matrix must be invertible.

m. False. The area of the **triangle** is 5.

n. True. See Theorem 6; $\det A^3 = (\det A)^3$.

o. False. See Exercise 31 in Section 3.2.

p. True. See Theorem 6.

2. $$\begin{vmatrix} 12 & 13 & 14 \\ 15 & 16 & 17 \\ 18 & 19 & 20 \end{vmatrix} = \begin{vmatrix} 12 & 13 & 14 \\ 3 & 3 & 3 \\ 6 & 6 & 6 \end{vmatrix} = 0$$

3. $$\begin{vmatrix} 1 & a & b+c \\ 1 & b & a+c \\ 1 & c & a+b \end{vmatrix} = \begin{vmatrix} 1 & a & b+c \\ 0 & b-a & a-b \\ 0 & c-a & a-c \end{vmatrix} = (b-a)(c-a)\begin{vmatrix} 1 & a & b+c \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{vmatrix} = 0$$

4. $$\begin{vmatrix} a & b & c \\ a+x & b+x & c+x \\ a+y & b+y & c+y \end{vmatrix} = \begin{vmatrix} a & b & c \\ x & x & x \\ y & y & y \end{vmatrix} = xy\begin{vmatrix} a & b & c \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 0$$

5. $$\begin{vmatrix} 9 & 1 & 9 & 9 & 9 \\ 9 & 0 & 9 & 9 & 2 \\ 4 & 0 & 0 & 5 & 0 \\ 9 & 0 & 3 & 9 & 0 \\ 6 & 0 & 0 & 7 & 0 \end{vmatrix} = (-1)\begin{vmatrix} 9 & 9 & 9 & 2 \\ 4 & 0 & 5 & 0 \\ 9 & 3 & 9 & 0 \\ 6 & 0 & 7 & 0 \end{vmatrix} = (-1)(-2)\begin{vmatrix} 4 & 0 & 5 \\ 9 & 3 & 9 \\ 6 & 0 & 7 \end{vmatrix}$$

$$= (-1)(-2)(3)\begin{vmatrix} 4 & 5 \\ 6 & 7 \end{vmatrix} = (-1)(-2)(3)(-2) = -12$$

6. $$\begin{vmatrix} 4 & 8 & 8 & 8 & 5 \\ 0 & 1 & 0 & 0 & 0 \\ 6 & 8 & 8 & 8 & 7 \\ 0 & 8 & 8 & 3 & 0 \\ 0 & 8 & 2 & 0 & 0 \end{vmatrix} = (1)\begin{vmatrix} 4 & 8 & 8 & 5 \\ 6 & 8 & 8 & 7 \\ 0 & 8 & 3 & 0 \\ 0 & 2 & 0 & 0 \end{vmatrix} = (1)(2)\begin{vmatrix} 4 & 8 & 5 \\ 6 & 8 & 7 \\ 0 & 3 & 0 \end{vmatrix} = (1)(2)(-3)\begin{vmatrix} 4 & 5 \\ 6 & 7 \end{vmatrix} = (1)(2)(-3)(-2) = 12$$

7. Expand along the first row to obtain

$$\begin{vmatrix} 1 & x & y \\ 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \end{vmatrix} = 1\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} - x\begin{vmatrix} 1 & y_1 \\ 1 & y_2 \end{vmatrix} + y\begin{vmatrix} 1 & x_1 \\ 1 & x_2 \end{vmatrix} = 0.$$ This is an equation of the form $ax + by + c = 0$, and since the points (x_1, y_1) and (x_2, y_2) are distinct, at least one of a and b is not zero. Thus the equation
is the equation of a line. The points (x_1, y_1) and (x_2, y_2) are on the line, because when the coordinates of one of the points are substituted for x and y, two rows of the matrix are equal and so the determinant is zero.

8. Expand along the first row to obtain

$\begin{vmatrix} 1 & x & y \\ 1 & x_1 & y_1 \\ 0 & 1 & m \end{vmatrix} = 1\begin{vmatrix} x_1 & y_1 \\ 1 & m \end{vmatrix} - x\begin{vmatrix} 1 & y_1 \\ 0 & m \end{vmatrix} + y\begin{vmatrix} 1 & x_1 \\ 0 & 1 \end{vmatrix} = 1(mx_1 - y_1) - x(m) + y(1) = 0.$ This equation may be rewritten as $mx_1 - y_1 - mx + y = 0,$ or $y - y_1 = m(x - x_1).$

9. $\det T = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & (b-a)(b+a) \\ 0 & c-a & (c-a)(c+a) \end{vmatrix}$

$$= (b-a)(c-a)\begin{vmatrix} 1 & a & a^2 \\ 0 & 1 & b+a \\ 0 & 1 & c+a \end{vmatrix} = (b-a)(c-a)\begin{vmatrix} 1 & a & a^2 \\ 0 & 1 & b+a \\ 0 & 0 & c-b \end{vmatrix} = (b-a)(c-a)(c-b)$$

10. Expanding along the first row will show that $f(t) = \det V = c_0 + c_1 t + c_2 t^2 + c_3 t^3$. By Exercise 9,

$$c_3 = \begin{vmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{vmatrix} = (x_2 - x_1)(x_3 - x_1)(x_3 - x_2) \neq 0$$

since x_1, x_2, and x_3 are distinct. Thus $f(t)$ is a cubic polynomial. The points $(x_1, 0)$, $(x_2, 0)$, and $(x_3, 0)$ are on the graph of f, since when any of x_1, x_2 or x_3 are substituted for t, the matrix has two equal rows and thus its determinant (which is $f(t)$) is zero. Thus $f(x_i) = 0$ for i = 1, 2, 3.

11. To tell if a quadrilateral determined by four points is a parallelogram, first translate one of the vertices to the origin. If we label the vertices of this new quadrilateral as **0**, $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$, then they will be the vertices of a parallelogram if one of $\mathbf{v}_1$, $\mathbf{v}_2$, or $\mathbf{v}_3$ is the sum of the other two. In this example, subtract (1, 4) from each vertex to get a new parallelogram with vertices **0** = (0, 0), $\mathbf{v}_1 = (-2, 1)$, $\mathbf{v}_2 = (2, 5)$, and $\mathbf{v}_3 = (4, 4)$. Since $\mathbf{v}_2 = \mathbf{v}_3 + \mathbf{v}_1$, the quadrilateral is a parallelogram as stated. The translated parallelogram has the same area as the original, and is determined by the columns of $A = [\mathbf{v}_1 \quad \mathbf{v}_3] = \begin{bmatrix} -2 & 4 \\ 1 & 4 \end{bmatrix}$, so the area of the parallelogram is $|\det A| = |-12| = 12$.

12. A 2×2 matrix A is invertible if and only if the parallelogram determined by the columns of A has nonzero area.

13. By Theorem 8, $(\text{adj } A) \cdot \dfrac{1}{\det A} A = A^{-1}A = I$. By the Invertible Matrix Theorem, adj A is invertible and

$$(\text{adj } A)^{-1} = \frac{1}{\det A} A.$$

14. **a**. Consider the matrix $A_k = \begin{bmatrix} A & O \\ O & I_k \end{bmatrix}$, where $1 \le k \le n$ and O is an appropriately sized zero matrix. We will show that $\det A_k = \det A$ for all $1 \le k \le n$ by mathematical induction.