- Inverse Given A, $CA = I = AC$ then $C = A^{-1}$
- Inverse of $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ $(ad-bc \neq 0)$

- ▪ **Theorem 5:** If A is an invertible $n \times n$ matrix, then for each **b** in $\mathbb{R}^n$, the equation $A\mathbf{x} = \mathbf{b}$ has the unique solution $\mathbf{x} = A^{-1}\mathbf{b}$.

- Thm 6. $(A^{-1})^{-1} = A$, $(AB)^{-1} = B^{-1}A^{-1}$, $(A^T)^{-1} = (A^{-1})^T$

elementary Matrix

$$A \xrightarrow{\text{row op}} A_1 = EA$$
$$I \xrightarrow{\text{row op}} E$$

$$A \xrightarrow{\text{row op 1}} A_1 \xrightarrow{\text{row op}_2} A_2$$
$$I \xrightarrow{\text{op 1}} E_1 \qquad A_1 = E_1 A \qquad A_2 = E_2 A_1 = E_2 E_1 A$$
$$I \xrightarrow{\text{op 2}} E_2$$

Theorem 7: An $n \times n$ matrix A is invertible if and only if A is row equivalent to I_n, and in this case, any sequence of elementary row operations that reduces A to I_n also transforms I_n into A^{-1}.

A ~ invertible

$E_p \cdots E_1 A = I$

$$\boxed{E_p \cdots E_1 A = I \quad \text{then} \quad E_p \cdots E_1 I = A^{-1}} \Rightarrow [A \; I] \to [I \; A^{-1}]$$

If Inverse Exist

Span of columns of A

= Codomain = Range

= each b is linear Combination of Columns of A

= Each row have pivot

(Onto) $X = A^{-1}B$

$Ax=0$ unique $x=0$

= columns linearly independent

= No free Var

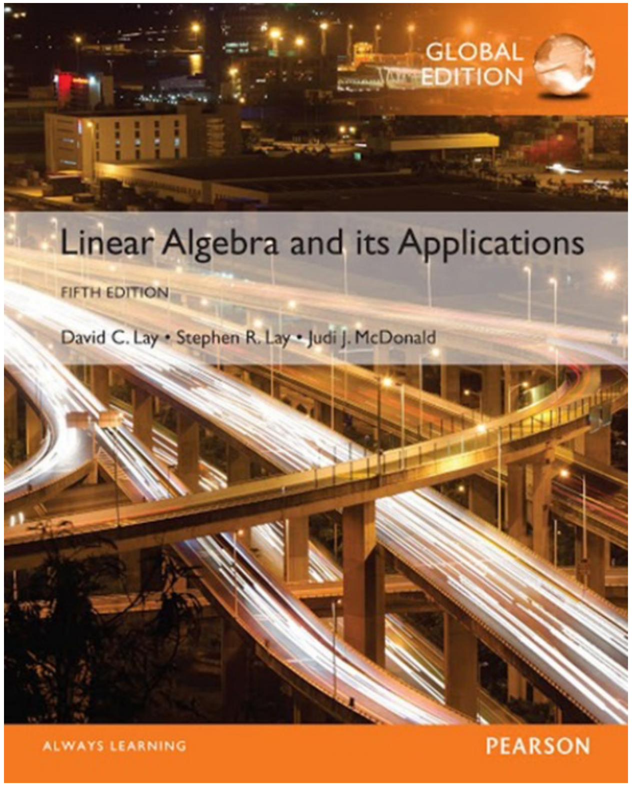
= pivots = columns

(One to One) Unique

2 Matrix Algebra

2.2

THE INVERSE OF A MATRIX



Let A be an $n \times n$ matrix. $A = [a_1 \cdots a_n]$ Uniqueness of Soln

Suppose some $n \times n$ C exists satisfying $CA = I$

Then $Ax = 0$ implies $C(Ax) = C0$ so that $x = 0$

So, If you consider a homogeneous system, $x = 0$ is the only solution

$$Ax = 0$$

$$\boxed{x_1 a_1 + \cdots + x_n a_n = 0}$$

If a linear combination of the columns of A is zero, then the weights are all 0

i.e. $a_1 \cdots a_n$ are linearly independent → Columns are pivot columns

$\Downarrow$

The mapping defined by $x \mapsto Ax$ is 1-1

If C exists such that $\boxed{AC = I}$ ← existence of a solution

then for any $b \in \mathbb{R}^n$, $A(Cb) = (AC)b = b$

i.e. for any $b \in \mathbb{R}^n$, we can find a solution of $Ax = b$
$(x = Cb)$

$\Downarrow$

$x \mapsto Ax$ is onto and range $= \mathbb{R}^n$

$\Downarrow$

There is a pivot in every row

If A is invertible, C exists such that $CA = I$ ←
$AC = I$ ←

$\Downarrow$

For any b,
$Ax = b$ has a unique solution

If $AC = I$, $\quad C = A^{-1}$

$C = [x_1 \cdots x_n]$

$$\left\{\begin{array}{l} Ax_1 = e_1 \\ \vdots \\ Ax_n = e_n \end{array}\right. \longrightarrow [A \; e_1 \cdots e_n] \longrightarrow [I \; \overbrace{\boxed{\;/////\;}}^{A^{-1}}]$$

MATRIX OPERATIONS

- An $n \times n$ matrix A is said to be **invertible** if there is an $n \times n$ matrix C such that

x=0 is only solution of Ax=0

columns of A are linearly independent

T(x)=Ax is 1-1

$$CA = I \quad \text{and} \quad AC = I$$

For each b in R^n, Ax=b solution

each b linear comb to col of A

range = span {col of A} = R^n

Each row of A has a pivot

A onto

where $I = I_n$, the $n \times n$ identity matrix.

- In this case, C is an **inverse** of A.
- In fact, C is uniquely determined by A, because if B were another inverse of A, then

$$B = BI = B(AC) = (BA)C = IC = C.$$

- This unique inverse is denoted by A^{-1}, so that

$$A^{-1}A = I \text{ and } AA^{-1} = I.$$

If $CA = I$, then the solution of $Ax = b$ is unique
(if it exists)

If $AC = I$ then there is a solution (always)

If some C satisfies both $CA = I$ and $AC = I$,
then $Ax = b$ has a ~~sol~~ unique solution

Inverse → sol Exist & Unique 보장

MATRIX OPERATIONS

- **Theorem 4:** Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $ad - bc \neq 0$, then

=0이면 not invertable

  A is invertible and

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

  If $ad - bc = 0$, then A is not invertible.

$$A\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} ad-bc & -ab+ab \\ cd-cd & -bc+ad \end{bmatrix} = \begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix} = (ad-bc)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

MATRIX OPERATIONS

$$\therefore A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

- **Theorem 4:** Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $ad - bc \neq 0$, then

  A is invertible and

  $$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

  If $ad - bc = 0$, then A is not invertible.
- The quantity $ad - bc$ is called the **determinant** of A, and we write $\det A = ad - bc$
- This theorem says that a 2×2 matrix A is invertible if and only if $\det A \neq 0$

MATRIX OPERATIONS

- **Theorem 5:** If A is an invertible $n \times n$ matrix, then for each **b** in $\mathbb{R}^n$, the equation $A\mathbf{x} = \mathbf{b}$ has the unique solution $\mathbf{x} = A^{-1}\mathbf{b}$.

existance

Onto, One to One

range = Codomain = span of A

- **Proof:** Take any **b** in $\mathbb{R}^n$.
- A solution exists because if $A^{-1}\mathbf{b}$ is substituted for **x**, then $A\mathbf{x} = A(A^{-1}\mathbf{b}) = (AA^{-1})\mathbf{b} = I\mathbf{b} = \mathbf{b}$.
- So $A^{-1}\mathbf{b}$ is a solution.
- To prove that the solution is unique, show that if **u** is any solution, then **u** must be $A^{-1}\mathbf{b}$.
- If $A\mathbf{u} = \mathbf{b}$, we can multiply both sides by A^{-1} and obtain $A^{-1}A\mathbf{u} = A^{-1}\mathbf{b}$, $I\mathbf{u} = A^{-1}\mathbf{b}$, and $\mathbf{u} = A^{-1}\mathbf{b}$.

MATRIX OPERATIONS

- **Theorem 6:**

a. If A is an invertible matrix, then A^{-1} is invertible and

$AA^{-1}=I \therefore A=(A^{-1})^{-1}$

$$(A^{-1})^{-1} = A$$

b. If A and B are $n \times n$ invertible matrices, then so is AB, and the inverse of AB is the product of the inverses of A and B in the reverse order. That is,

$AB(B^{-1}A^{-1})=I$

$$(AB)^{-1} = B^{-1}A^{-1}$$

c. If A is an invertible matrix, then so is A^T, and the inverse of A^T is the transpose of A^{-1}. That is,

$I=I^T (A^{-1}A)^T = A^T (A^{-1})^T$

$$(A^T)^{-1} = (A^{-1})^T$$

MATRIX OPERATIONS

- **Proof:** To verify statement (a), find a matrix C such that

$$A^{-1}C = I \quad \text{and} \quad CA^{-1} = I$$

- These equations are satisfied with A in place of C. Hence A^{-1} is invertible, and A is its inverse.
- Next, to prove statement (b), compute:

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I$$

- A similar calculation shows that $(B^{-1}A^{-1})(AB) = I$.

MATRIX OPERATIONS

- For statement (c), use Theorem 3(d), read from right to left, $(A^{-1})^T A^T = (AA^{-1})^T = I^T = I$.
- Similarly, $A^T (A^{-1})^T = I^T = I$.
- Hence A^T is invertible, and its inverse is $(A^{-1})^T$.

$(A^{-1})^T A^T = (AA^{-1})^T = I^T$

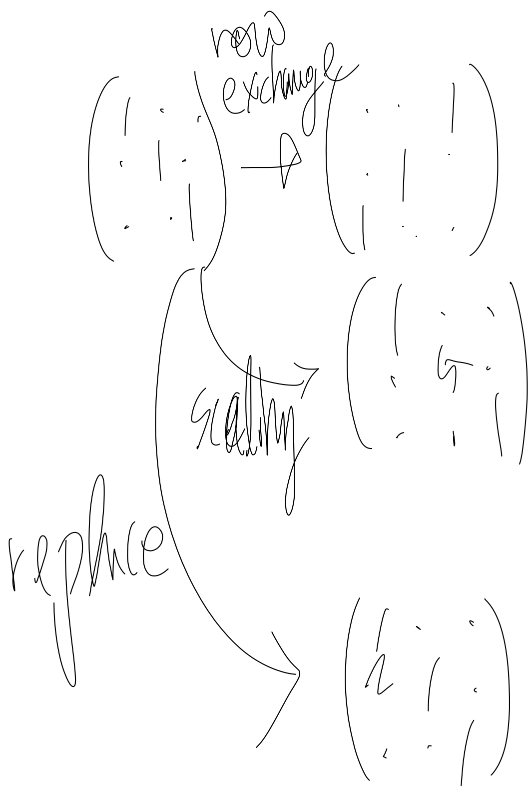
ELEMENTARY MATRICES

- The generalization of Theorem 6(b) is as follows:

  The product of $n \times n$ invertible matrices is invertible, and the inverse is the product of their inverses in the reverse order.

ELEMENTARY MATRICES

- An **elementary matrix** is one that is obtained by performing a single elementary row operation on an identity matrix.



When an elementary matrix is multiplied to A, the result is the same as the case when a row operation to obtain the elementary matrix from the identity matrix is applied to ~~the~~ A.

★

$$I \xrightarrow{\text{row op}} E$$

$$A \xrightarrow{\text{row op}} U = EA$$

ELEMENTARY MATRICES

- **Example 5:** Let $E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}$, $E_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$,

  $E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}$, $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$

  Compute E_1A, E_2A, and E_3A, and describe how these products can be obtained by elementary row operations on A.

ELEMENTARY MATRICES

- **Solution:** Verify that

$$E_1A = \begin{bmatrix} a & b & c \\ d & e & f \\ g-4a & h-4b & i-4c \end{bmatrix}, \quad E_2A = \begin{bmatrix} d & e & f \\ a & b & c \\ g & h & i \end{bmatrix},$$

$$E_3A = \begin{bmatrix} a & b & c \\ d & e & f \\ 5g & 5h & 5i \end{bmatrix}.$$

기본 ERO 시행

$I \xrightarrow{ERO} E$

EA: 기본 A에 같은 ERO 시행한 것.

- Addition of -4 times row 1 of A to row 3 produces E_1A.

ELEMENTARY MATRICES

- An interchange of rows 1 and 2 of A produces E_2A, and multiplication of row 3 of A by 5 produces E_3A.
- Left-multiplication (that is, multiplication on the left) by E_1 in Example 1 has the same effect on any $n \times n$ matrix.
- Since $E_1 \cdot I = E_1$, we see that E_1 *itself* is produced by this same row operation on the identity.

ELEMENTARY MATRICES

- Example 5 illustrates the following general fact about elementary matrices.

- If an elementary row operation is performed on an $m \times n$ matrix A, the resulting matrix can be written as EA, where the $m \times m$ matrix E is created by performing the same row operation on I_m.

- Each elementary matrix E is invertible. The inverse of E is the elementary matrix of the same type that transforms E back into I.

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ 3 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 3 & 0 & 0 \end{pmatrix} = EA \to \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & -3 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = E$$

$$\hookrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix} = F$$

E_1: 3rd row of (-4) 1st + 3rd equal

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{pmatrix} \quad E_1^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{pmatrix}$$

E_1^{-1}: $4 \times$ 1st + 3rd eq.

$E_2 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ Interchange 1st & 2nd

$\downarrow$

E_2^{-1}: Inter 1st & 2nd

E_3: $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{pmatrix}$ $5 \times$ 3rd

E_3^{-1}: $1/5 \times$ 3rd

replacement

$$E = \begin{array}{c} \\ i \rightarrow \end{array}\begin{array}{c} \quad j \\ \begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & \downarrow & \ddots & \\ & c & & 1 \end{bmatrix}\end{array}$$

replaces ith row by i^{th} row $+ c$ (j^{th} row)

$$E^{-1} = \begin{array}{c} \\ i \rightarrow \end{array}\begin{array}{c} \quad j \\ \begin{bmatrix} 1 & & & \\ & 1 & & \\ & \downarrow & \ddots & \\ & -c & & 1 \end{bmatrix}\end{array}$$

Interchange

$$E_2 = \begin{bmatrix} 1 & & & & 0 \\ & \ddots & & & \\ & & \boxed{0} & \boxed{1} & \\ & & \boxed{1} & \boxed{0} & \\ 0 & & & \ddots & 1 \end{bmatrix} = E_2^{-1}$$

Scaling

$$E_n = \begin{bmatrix} 1 & \ddots & 0 \\ & \boxed{c} & \\ 0 & & 1 \end{bmatrix} \qquad E_3^{-1} = \begin{bmatrix} 1 & & 0 \\ & \ddots \ \boxed{1/c} & \\ 0 & & \ddots \ 1 \end{bmatrix}$$

$CA = I \rightarrow$ col all pivot $\rightarrow$ $Ax=0$ trivial

$AC = I \rightarrow$ row all pivot $\rightarrow$ $Ax=b$ sol

ELEMENTARY MATRICES

- **Theorem 7:** An $n \times n$ matrix A is invertible if and only if A is row equivalent to I_n, and in this case, any sequence of elementary row operations that reduces A to I_n also transforms I_n into A^{-1}.
- **Proof:** Suppose that A is invertible.
- Then, since the equation $A\mathbf{x} = \mathbf{b}$ has a solution for each $\mathbf{b}$ (Theorem 5), A has a pivot position in every row.
- Because A is square, the n pivot positions must be on the diagonal, which implies that the reduced echelon form of A is I_n. That is, $A \sim I_n$.

ELEMENTARY MATRICES

- Now suppose, conversely, that $A \sim I_n$.
- Then, since each step of the row reduction of A corresponds to left-multiplication by an elementary matrix, there exist elementary matrices $E_1, \ldots, E_p$ such that $A \sim E_1 A \sim E_2(E_1 A) \sim \ldots \sim E_p(E_{p-1} \ldots E_1 A) = I_n$
- That is,

$$(1) \qquad E_p \ldots E_1 A = I_n$$

- Since the product $E_p \ldots E_1$ of invertible matrices is invertible, (1) leads to

$$(E_p \ldots E_1)^{-1}(E_p \ldots E_1)A = (E_p \ldots E_1)^{-1} I_n$$

$$A = (E_p \ldots E_1)^{-1} = E_1^{-1} E_2^{-1} \ldots E_p^{-1}$$

ALGORITHM FOR FINDING A^{-1}

- Thus A is invertible, as it is the inverse of an invertible matrix (Theorem 6). Also,

$$A^{-1} = \left[(E_p \ldots E_1)^{-1} \right]^{-1} = E_p \ldots E_1$$

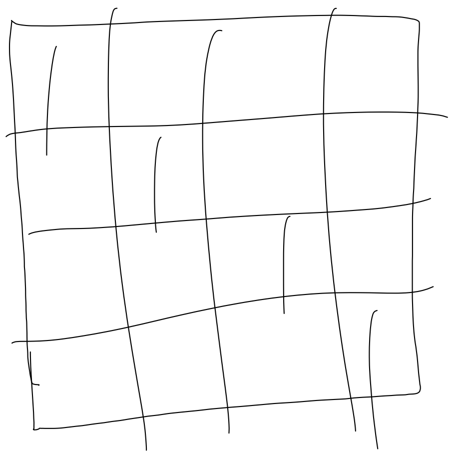
- Then $A^{-1} = E_p \ldots E_1 \cdot I_n$, which says that A^{-1} results from applying E_1, ..., E_p successively to I_n.

결론

- This is the same sequence in (1) that reduced A to I_n.
- Row reduce the augmented matrix $\begin{bmatrix} A & I \end{bmatrix}$. If A is row equivalent to I, then $\begin{bmatrix} A & I \end{bmatrix}$ is row equivalent to $\begin{bmatrix} I & A^{-1} \end{bmatrix}$. Otherwise, A does not have an inverse.

A is invertible $\Leftrightarrow$ C exist that $\begin{cases} CA = I \rightarrow \text{Col are independent} \rightarrow \text{each col pivot (n pivot)} \\ AC = I \rightarrow \text{Each row pivot (m pivot)} \end{cases}$

m x n



main diagonal 에서 [illegible].

Given A

suppose $E_p \cdots E_1 A = I$

Then $C = E_p \cdots E_1$ is inverse of A.

$A = E_1^{-1} \cdots E_p^{-1} I$

$A^{-1} = \underbrace{E_p \cdots E_1}_{\text{Row Operations Which change } A \text{ to } I} I$

↳ If the same row operation are performed on I, you obtain A^{-1}

ALGORITHM FOR FINDING A^{-1}

- **Example 2:** Find the inverse of the matrix

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix}, \text{ if it exists.}$$

A I

- **Solution:**

$$\begin{bmatrix} A & I \end{bmatrix} = \begin{bmatrix} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{bmatrix}$$

ALGORITHM FOR FINDING A^{-1}

$$\sim \begin{bmatrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & -3 & -4 & 0 & -4 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 3 & -4 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3/2 & -2 & 1/2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & -9/2 & 7 & -3/2 \\ 0 & 1 & 0 & -2 & 4 & -1 \\ 0 & 0 & 1 & 3/2 & -2 & 1/2 \end{bmatrix}$$

A^{-1}

ALGORITHM FOR FINDING A^{-1}

- Theorem 7 shows, since $A \sim I$, that A is invertible, and

$$A^{-1} = \begin{bmatrix} -9/2 & 7 & -3/2 \\ -2 & 4 & -1 \\ 3/2 & -2 & 1/2 \end{bmatrix}$$

- Now, check the final answer.

$$AA^{-1} = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix} \begin{bmatrix} -9/2 & 7 & -3/2 \\ -2 & 4 & -1 \\ 3/2 & -2 & 1/2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

ANOTHER VIEW OF MATRIX INVERSION

- It is not necessary to check that $A^{-1}A = I$ since A is invertible.

- Denote the columns of I_n by $\mathbf{e}_1, \ldots, \mathbf{e}_n$.

- Then row reduction of $\begin{bmatrix} A & I \end{bmatrix}$ to $\begin{bmatrix} I & A^{-1} \end{bmatrix}$ can be viewed as the simultaneous solution of the n systems

$$A\mathrm{x} = \mathrm{e}_1,\ A\mathrm{x} = \mathrm{e}_2,\ \ldots,\ A\mathrm{x} = \mathrm{e}_n \qquad (2)$$

where the "augmented columns" of these systems have all been placed next to A to form

$$\begin{bmatrix} A & \mathrm{e}_1 & \mathrm{e}_2 & \cdots & \mathrm{e}_n \end{bmatrix} = \begin{bmatrix} A & I \end{bmatrix}.$$

ANOTHER VIEW OF MATRIX INVERSION

- The equation $AA^{-1} = I$ and the definition of matrix multiplication show that <u>the columns of A^{-1} are precisely the solutions of the systems in (2)</u>.

$Ax = b_1 \quad [A\ b_1] \xrightarrow{\text{row op}} [I\ c_1] \quad c_1 = A^{-1}b_1$

$Ax = b_2 \quad [A\ b_2] \rightarrow [I\ c_2] \quad c_2 = A^{-1}b_2$

$[A\ b_1 b_2] \rightarrow [I\ c_1 c_2]$:

$Ax = b_n \quad [A\ b_1 b_2 b_3] \rightarrow [I\ c_1 c_2 c_3]$

$AC = I \rightarrow C = A^{-1}$

↳ $A[a_1 a_2 \cdots a_n] = [e_1 \cdots e_n] \rightarrow$

↳ $[Ax_1\ Ax_2 \cdots Ax_n] = [e_1 \cdots e_n]$

$[A\ e_1 e_2 \cdots e_n]$

$\rightarrow [I\ \ c_1\ c_2 \cdots c_n]$, $c_1 = x_1,\ c_2 = x_2,\ \ldots,\ c_n = x_n$

Finding A^{-1} is equivalent to finding $x_1 \ldots x_n$

$Ax_1 = e_1$

$\vdots$

$Ax_n = e_n$

A^{-1} 찾기위해 $[A\ I]$에서 $[I\ C]$로 변환.

fail to change → A is not invertible

- Compute the product AB

6. $A = \begin{bmatrix} 4 & -2 \\ -3 & 0 \\ 3 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}$

$AB = \begin{bmatrix} Ab_1 & Ab_2 \end{bmatrix} = \begin{bmatrix} 0 & 14 \\ -3 & -9 \\ 13 & 4 \end{bmatrix}$

$Ab_1 = \begin{bmatrix} 0 \\ -3 \\ 13 \end{bmatrix}$

$Ab_2 = \begin{bmatrix} 14 \\ -9 \\ 4 \end{bmatrix}$

or dot product

$AB = \begin{bmatrix} 4-4 & 12+2 \\ -3+0 & -9+0 \\ 3+10 & 9-5 \end{bmatrix}$

23. Suppose $CA = I_n$ (the $n \times n$ identity matrix). Show that the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution. Explain why A cannot have more columns than rows.

24. Suppose $AD = I_m$ (the $m \times m$ identity matrix). Show that for any $\mathbf{b}$ in $\mathbb{R}^m$, the equation $A\mathbf{x} = \mathbf{b}$ has a solution. [*Hint:* Think about the equation $AD\mathbf{b} = \mathbf{b}$.] Explain why A cannot have more rows than columns.

23. $CA = I_n$ 일 때 $C(Ax) = C0 = 0$
$(CA)x = x$ // 이된다. 즉, 항상 $x=0$이 성립한다.

→ Columns of A are linearly independent
→ 근이 Unique

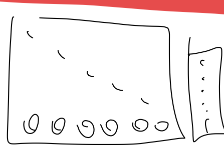
If A has more columns than rows,
A has at least one non-pivot column
→ free variable → many solution.

24. $AD = I_m$ 일 때 $Ax=b$ 일관성이다 (any $b \in \mathbb{R}^m$)

$A(Db) = (AD)b = Ib = b$, Db satisfies the equation.

any b에 대하여 성립 → The range of A is $\mathbb{R}^m$ → Each row has pivot

No more rows than columns. why?



근이 안생길수있다.

$\#$pivots $\le$ $\#$columns $<$ $\#$rows

some row cannot have pivot → range $\neq$ codomain

Find the inverse

1. $\begin{bmatrix} 8 & 6 \\ 5 & 4 \end{bmatrix}$

①

$\frac{1}{8 \times 4 - 30} \begin{pmatrix} 4 & -6 \\ -5 & 8 \end{pmatrix} =$

No

②

$\left(\begin{array}{cc|cc} 8 & 6 & 1 & 0 \\ 5 & 4 & 0 & 1 \end{array}\right) \rightarrow$

Yes

15. Suppose A, B, and C are invertible n x n matrices. Show that ~~ABC is also invertible~~

$(ABC)(C^{-1}B^{-1}A^{-1}) = I$ 이므로,

$(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$ 이다.

31.

Find the inverse

31. $\begin{bmatrix} 1 & 0 & -2 \\ -3 & 1 & 4 \\ 2 & -3 & 4 \end{bmatrix}$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ -3 & 1 & 4 & 0 & 1 & 0 \\ 2 & -3 & 4 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & & & 8 & 3 & 1 \\ & 1 & & 10 & 4 & 1 \\ & & 1 & 7/2 & 9/2 & 1/2 \end{array}\right]$$

- Find the inverse

2. $\begin{bmatrix} 3 & 2 \\ 7 & 4 \end{bmatrix}$

16. Suppose A and B are n x n, B is invertible, and AB is invertible. Show that A is invertible

Let $C = AB$

Then $CB^{-1} = (AB)B^{-1} = A$

A = product of two invertible matrices.

$C = AB$, The $CB^{-1} = (AB)B^{-1} = A$ $\therefore$ A is product of two invertable Matrices.

$\to$ invertable.

inverti

Find the inverse

32. $\begin{bmatrix} 1 & -2 & 1 \\ 4 & -7 & 3 \\ -2 & 6 & -4 \end{bmatrix}$

$$\left[\begin{array}{ccc|ccc} 1 & -2 & 1 & 1 & 0 & 0 \\ 4 & -7 & 3 & 0 & 1 & 0 \\ -2 & 6 & -4 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & -2 & 1 & & & \\ & 1 & -1 & & & \\ & & 0 & & & \end{array}\right]$$

Theorem 8: Let A be a square $n \times n$ matrix. Then the following statements are equivalent. That is, for a given A, the statements are either all true or all false

a. A is an invertible matrix.

b. A is row equivalent to the $n \times n$ identity matrix.

c. A has n pivot positions.

d. The equation $Ax = 0$ has only the trivial solution.

e. The columns of A form a linearly independent set.

f. The linear transformation $x \mapsto Ax$ is one-to-one.

g. The equation $Ax = b$ has at least one solution for each b in $\mathbb{R}^n$. $\Longleftrightarrow$ range = codomain = span{columns of A}

h. The columns of A span $\mathbb{R}^n$.

i. The linear transformation $x \mapsto Ax$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$.

All columns are pivot columns
$\Updownarrow$
no free variable

j. There is an $n \times n$ matrix C such that $CA = I$. $\to$ uniqueness

k. There is an $n \times n$ matrix D such that $AD = I$. $\to$ existence $Ax=b$

l. A^T is an invertible matrix.

nonsingular etc.

A ~ invertible $\left(\begin{matrix} CA = I \\ AC = I \end{matrix}\right) \to CA = I \to$ uniqueness

$x = 0$ is the only solution of $Ax = 0$

$\Rightarrow x = 0$ is the only weight of $x_1a_1 + \cdots + x_na_n = 0$

$\Rightarrow a_1 \cdots a_n$ are linearly independent

$\Rightarrow x \mapsto Ax$ is 1-1

$\to$ Every column has a pivot

(Every column is a pivot column)

$\Downarrow$

$AC = I$ (existence)

$Ax = b$ has a solution for any $b \in \mathbb{R}^n$

$x_1a_1 + \cdots + x_na_n$

$\Downarrow$

Each b is a linear combination of the columns of A

$\Downarrow$

$\mathbb{R}^n$ = range = span$\{a_1, \cdots, a_n\}$

$\Downarrow$

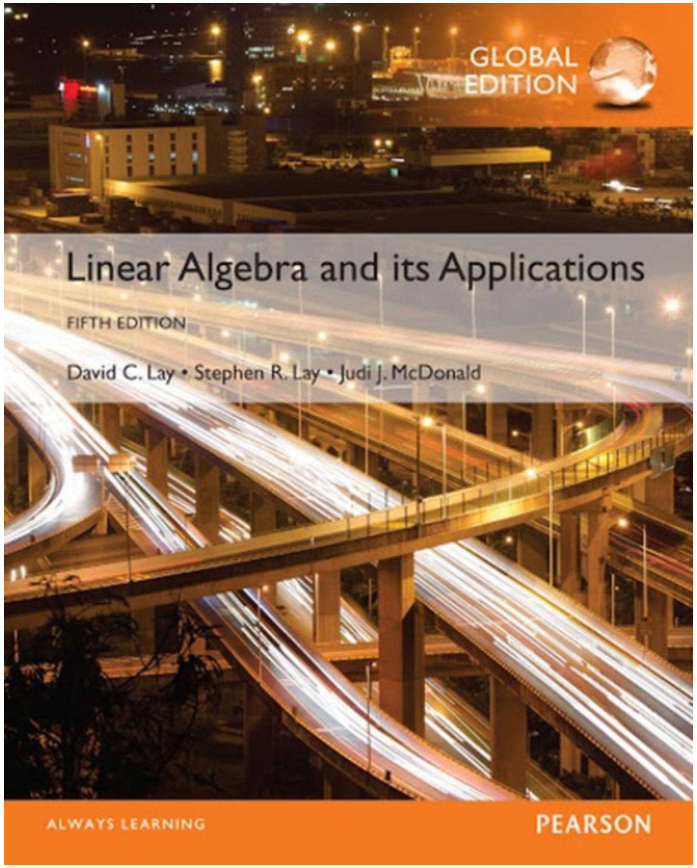
Every row has a pivot

2

Matrix Algebra

2.3

CHARACTERIZATIONS OF INVERTIBLE MATRICES



PEARSON

THE INVERTIBLE MATRIX THEOREM

- **Theorem 8**: Let A be a square $n \times n$ matrix. Then the following statements are equivalent. That is, for a given A, the statements are either all true or all false.
  a. A is an invertible matrix.
  $CA = I$ (j) ⇒ (d) ⇒ (c) ⇒ (b) ⇒ (a)
  $AC = I$ (k) ⇒ (g) ⇒ (h) ⇒ (I) ⇒ (a)
  b. A is row equivalent to the $n \times n$ identity matrix.
  $E_p \cdots E_1 A = I$
  c. A has n pivot positions. → $n \times n$ → pivot to all col/row (main diagonal)
  d. The equation $Ax = 0$ has only the trivial solution.
  ||
  e. The columns of A form a linearly independent set.

THE INVERTIBLE MATRIX THEOREM

f. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.

g. The equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b}$ in $\mathbb{R}^n$.

h. The columns of A span $\mathbb{R}^n$.

i. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$.

j. There is an $n \times n$ matrix C such that $CA = I$.

k. There is an $n \times n$ matrix D such that $AD = I$.

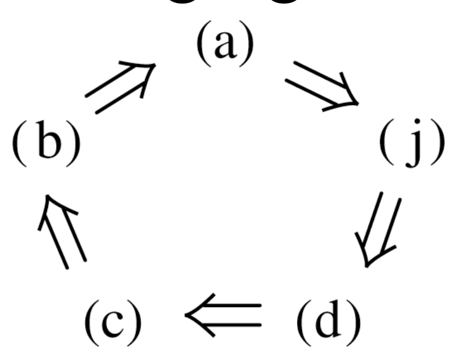
l. A^T is an invertible matrix.

$A[x_1 \cdots x_n] = [e_1 \cdots e_n] = I$

$\therefore$ Invertible)

THE INVERTIBLE MATRIX THEOREM

- First, we need some notation.
- If the truth of statement (a) always implies that statement (j) is true, we say that (a) *implies* (j) and write (a) $\Rightarrow$ (j) .
- The proof will establish the “circle” of implications as shown in the following figure.



- If any one of these five statements is true, then so are the others.

THE INVERTIBLE MATRIX THEOREM

- Finally, the proof will link the remaining statements of the theorem to the statements in this circle.
- **Proof:** If statement (a) is true, then A^{-1} works for C in (j), so (a) $\Rightarrow$ (j).
- Next, (j) $\Rightarrow$ (d).
- Also, (d) $\Rightarrow$ (c).
- If A is square and has n pivot positions, then the pivots must lie on the main diagonal, in which case the reduced echelon form of A is I_n.
- Thus (c) $\Rightarrow$ (b).
- Also, (b) $\Rightarrow$ (a).

THE INVERTIBLE MATRIX THEOREM

- This completes the circle in the previous figure.
- Next, $(a) \Rightarrow (k)$ because A^{-1} works for D.
- Also, $(k) \Rightarrow (g)$ and $(g) \Rightarrow (a)$.
- So (k) and (g) are linked to the circle.
- Further, (g), (h), and (i) are equivalent for any matrix.
- Thus, (h) and (i) are linked through (g) to the circle.
- Since (d) is linked to the circle, so are (e) and (f), because (d), (e), and (f) are all equivalent for *any* matrix A.
- Finally, $(a) \Rightarrow (l)$ and $(l) \Rightarrow (a)$.
- This completes the proof.

THE INVERTIBLE MATRIX THEOREM

- Theorem 8 could also be written as "The equation $A\mathbf{x} = \mathbf{b}$ has a *unique* solution for each **b** in $\mathbb{R}^n$."
- This statement implies (b) and hence implies that A is invertible.
- The following fact follows from Theorem 8.

  Let A and B be square matrices. If $AB = I$, then A and B are both invertible, with $B = A^{-1}$ and $A = B^{-1}$.
- The Invertible Matrix Theorem divides the set of all $n \times n$ matrices into two disjoint classes: the invertible (nonsingular) matrices, and the noninvertible (singular) matrices.

THE INVERTIBLE MATRIX THEOREM

- Each statement in the theorem describes a property of every $n \times n$ invertible matrix.

위 statement 의 부정 → No sol or Many sol

- The *negation* of a statement in the theorem describes a property of every $n \times n$ singular matrix.

- For instance, an $n \times n$ singular matrix is *not* row equivalent to I_n, does *not* have n pivot position, and has linearly *dependent* columns.

THE INVERTIBLE MATRIX THEOREM

- **Example 1:** Use the Invertible Matrix Theorem to decide if A is invertible:

$$A = \begin{bmatrix} 1 & 0 & -2 \\ 3 & 1 & -2 \\ -5 & -1 & 9 \end{bmatrix}$$

- **Solution:**

$$A \sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 4 \\ 0 & -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 4 \\ 0 & 0 & 3 \end{bmatrix}$$

THE INVERTIBLE MATRIX THEOREM

- So A has t~~hree pivot positions and hence is invertible,~~ by the Invertible Matrix Theorem, statement (c).

THE INVERTIBLE MATRIX THEOREM

- The Invertible Matrix Theorem *applies only to square matrices*.

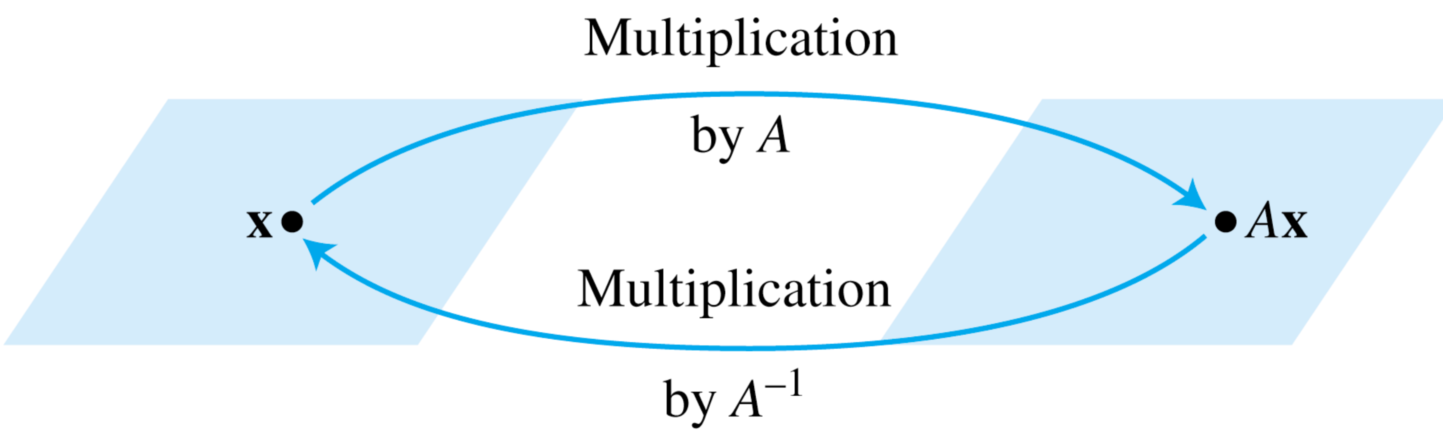
- For example, if the columns of a 4×3 matrix are linearly independent, we cannot use the Invertible Matrix Theorem to conclude anything about the existence or nonexistence of solutions of equation of the form $A\mathbf{x} = \mathbf{b}$.

INVERTIBLE LINEAR TRANSFORMATIONS

- Matrix multiplication corresponds to composition of linear transformations.
- When a matrix A is invertible, the equation $A^{-1}A\mathbf{x} = \mathbf{x}$ can be viewed as a statement about linear transformations. See the following figure.

I

$A^{-1}(A\mathbf{x})$



A^{-1} transforms $A\mathbf{x}$ back to $\mathbf{x}$.

INVERTIBLE LINEAR TRANSFORMATIONS

- A linear transformation $T: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is said to be **invertible** if there exists a function $S: \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that

$$S(T(\mathrm{x})) = \mathrm{x} \quad \text{for all } \mathbf{x} \text{ in } \mathbb{R}^n \qquad (1)$$

$$T(S(\mathrm{x})) = \mathrm{x} \quad \text{for all } \mathbf{x} \text{ in } \mathbb{R}^n \qquad (2)$$

- **Theorem 9**: Let $T: \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a linear transformation and let A be the standard matrix for T. Then T is invertible if and only if A is an invertible matrix. In that case, the linear transformation S given by $S(\mathrm{x}) = A^{-1}\mathrm{x}$ is the unique function satisfying equation (1) and (2).

INVERTIBLE LINEAR TRANSFORMATIONS

→ S exist such that
$T(S(x)) = x$, $S(T(x)) = x$

- **Proof:** Suppose that T is invertible.
- Then (2) shows that T is onto $\mathbb{R}^n$, for if **b** is in $\mathbb{R}^n$ and $\mathbf{x} = S(\mathbf{b})$, then $T(\mathbf{x}) = T(S(\mathbf{b})) = \mathbf{b}$, so each **b** is in the range of T.

$S(x) = Bx$ $T(S(x)) = A(Bx) = ABx$, $AB = I$

- Thus A is invertible, by the Invertible Matrix Theorem, statement (i).
- Conversely, suppose that A is invertible, and let $S(\mathbf{x}) = A^{-1}\mathbf{x}$. Then, S is a linear transformation, and S satisfies (1) and (2).
- For instance, $S(T(\mathbf{x})) = S(A\mathbf{x}) = A^{-1}(A\mathbf{x}) = \mathbf{x}$.
- Thus, T is invertible.

$\therefore$ A is invertible

15. Can a square matrix with two identical columns be invertible? Why or why not?

16. Is it possible for a 5×5 matrix to be invertible when its columns do not span $\mathbb{R}^5$? Why or why not?

17. If A is invertible, then the columns of A^{-1} are linearly independent. Explain why.

15 linearly dependent $\Rightarrow$ not invertible

16 No. means not all row have pivot $\Rightarrow$ No sol can happen. Should span $\mathbb{R}^5$

17 Yes. A^{-1} is invertible $\Rightarrow$ col linearly independent

21. If the equation $G\mathbf{x} = \mathbf{y}$ has more than one solution for some $\mathbf{y}$ in $\mathbb{R}^n$, can the columns of G span $\mathbb{R}^n$? Why or why not?

22. If the equation $H\mathbf{x} = \mathbf{c}$ is inconsistent for some $\mathbf{c}$ in $\mathbb{R}^n$, what can you say about the equation $H\mathbf{x} = \mathbf{0}$? Why?

23. If an $n \times n$ matrix K cannot be row reduced to I_n, what can you say about the columns of K? Why?

21 free var $\to$ not independent $\to$ not invertable $\to$ no span $\mathbb{R}^n$

22 range $\neq \mathbb{R}^n$ $\to$ not invertable $\to$ free var $\to$ many solutions

23 K is Not Invertable $\Rightarrow$ Col are dependent

18. If C is 6×6 and the equation $C\mathbf{x} = \mathbf{v}$ is consistent for every $\mathbf{v}$ in $\mathbb{R}^6$, is it possible that for some $\mathbf{v}$, the equation $C\mathbf{x} = \mathbf{v}$ has more than one solution? Why or why not? Solution is unique

NO

19. If the columns of a 7×7 matrix D are linearly independent, what can you say about solutions of $D\mathbf{x} = \mathbf{b}$? Why?

20. If $n \times n$ matrices E and F have the property that $EF = I$, then E and F commute. Explain why.

Use Th8

18. C is invertable $\rightarrow$ No Uniqueness.

19. Invertable $\rightarrow$ Unique solution, solution for each $b \in \mathbb{R}^7$

20. $EF = I$, $F = E^{-1}$, $EE^{-1} = E^{-1}E = I$ 이니까

24. If L is $n \times n$ and the equation $L\mathbf{x} = \mathbf{0}$ has the trivial solution, do the columns of L span $\mathbb{R}^n$? Why?

$\Downarrow$

No special information.

No special Info (trivial이 유일해가 아니므로서) $\rightarrow$ We do not know.

26. Explain why the columns of A^2 span $\mathbb{R}^n$ whenever the columns of A are linearly independent.

27. Show that if AB is invertible, so is A. You cannot use Theorem 6(b), because you cannot *assume* that A and B are invertible.

26. A is invertable $\Rightarrow$ A^2 = product of invertable matrices $\Rightarrow$ A^2 is invertible $\Rightarrow$ columns of A^2 span $\mathbb{R}^n$

27. If $\begin{pmatrix} AB \\ B \end{pmatrix}$ are invertable $\Rightarrow$ $A = AB \cdot B^{-1}$ ~~[illegible]~~ 이 불가능하다...

Let C be the inverse of AB

$$A(BC) = (AB)C = I$$

$\therefore$

$\hookrightarrow$ AD로 보면 A는 invertable

28. Show that if AB is invertible, so is B.

29. If A is an $n \times n$ matrix and the equation $A\mathbf{x} = \mathbf{b}$ has more than one solution for some $\mathbf{b}$, then the transformation $\mathbf{x} \mapsto A\mathbf{x}$ is not one-to-one. What else can you say about this transformation? Justify your answer.

28. Let C inverse of A $\to$ $I = C(AB) = (C\cancel{A})B$

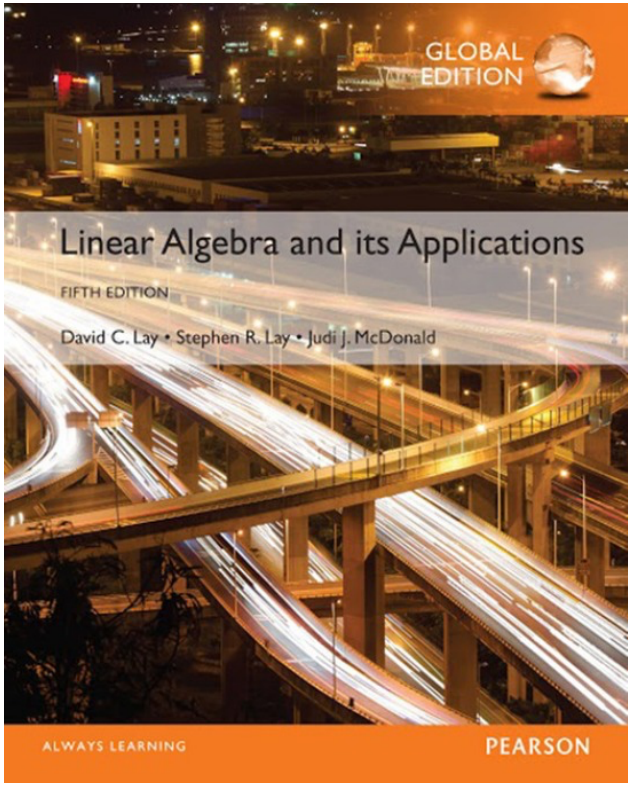
29. $A \sim$ Not invertable $\to$ It is not Onto either

The transformation defined by $x \mapsto Ax$ is not invertable.

2 Matrix Algebra

2.4

PARTITIONED MATRICES



PARTITIONED MATRICES

- A key feature of our work with matrices has been the ability to regard matrix A as a list of column vectors rather than just a rectangular array of numbers.
- This point of view has been so useful that we wish to consider other **partitions** of A, indicated by horizontal and vertical dividing rules, as in Example 1 on the next slide.

PARTITIONED MATRICES

- **Example 1** The matrix

$$A = \left[\begin{array}{ccc|cc|c} 3 & 0 & -1 & 5 & 9 & -2 \\ -5 & 2 & 4 & 0 & -3 & 1 \\ \hline -8 & -6 & 3 & 1 & 7 & -4 \end{array}\right]$$

can also be written as the 2×3 **partitioned** (or **block**) **matrix**

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \end{bmatrix}$$

whose entries are the *blocks* (or *submatrices*)

$$A_{11} = \begin{bmatrix} 3 & 0 & -1 \\ -5 & 2 & 4 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} 5 & 9 \\ 0 & -3 \end{bmatrix}, \quad A_{13} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$A_{21} = \begin{bmatrix} -8 & -6 & 3 \end{bmatrix}, \quad A_{22} = \begin{bmatrix} 1 & 7 \end{bmatrix}, \quad A_{23} = \begin{bmatrix} -4 \end{bmatrix}$$

ADDITION AND SCALAR MULTIPLICATION

- If matrices *A* and *B* are the same size and are partitioned in exactly the same way, then it is natural to make the same partition of the ordinary matrix sum *A* + *B*.
- In this case, each block of *A* + *B* is the (matrix) sum of the corresponding blocks of *A* and *B*.
- Multiplication of a partitioned matrix by a scalar is also computed block by block.

$A = \left[\begin{array}{cc|c} 1 & 2 & 3 \\ 4 & 5 & 6 \end{array}\right] \to [A_1 \;\; A_2]$

$B = \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 2 & 1 & 0 \end{array}\right] \to [B_1 \;\; B_2]$

$A_1 + B_1 = \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 6 & 6 \end{bmatrix}$

$A_2 + B_2 = \begin{bmatrix} 3 \\ 6 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$

$A + B = \begin{bmatrix} 1 & 3 & 3 \\ 6 & 6 & 6 \end{bmatrix}$

$2A = [2A_1 \;\; 2A_2] = \begin{bmatrix} 2 & 4 & 6 \\ 8 & 10 & 12 \end{bmatrix}$

MULTIPLICATION OF PARTITIONED MATRICES

- Partitioned matrices can be multiplied by the usual row—column rule as if the block entries were scalars, provided that for a product AB, the column partition of A matches the row partition of B. $A_{11}B_1$, $A_{21}B_1$, $A_{12}B_2$, $A_{22}B_2$ well defined
- **Example 3** Let

$$A = \left[\begin{array}{ccc|cc} 2 & -3 & 1 & 0 & -4 \\ 1 & 5 & -2 & 3 & -1 \\ \hline 0 & -4 & -2 & 7 & -1 \end{array}\right] = \begin{array}{cc} 2\times 3 & 2\times 2 \\ \left[\begin{array}{c} A_{11} \\ A_{21} \end{array}\right. & \left.\begin{array}{c} A_{12} \\ A_{22} \end{array}\right] \\ 1\times 3 & 1\times 2 \end{array}, \quad B = \left[\begin{array}{cc} 6 & 4 \\ -2 & 1 \\ -3 & 7 \\ \hline -1 & 3 \\ 5 & 2 \end{array}\right] = \begin{array}{c} 3\times 2 \\ \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \\ 2\times 2 \end{array}$$

The 5 columns of A are partitioned into a set of 3 columns and then a set of 2 columns. The 5 rows of B are partitioned in the same way—into a set of 3 rows and then a set of 2 rows.

$$\begin{array}{c} 2\times 2 \\ 1\times 2 \end{array} \begin{bmatrix} A_{11}B_1 + A_{12}B_2 \\ A_{21}B_1 + A_{22}B_2 \end{bmatrix} \text{ 가능}$$

MULTIPLICATION OF PARTITIONED MATRICES

- We say that the partitions of A and B are conformable for block multiplication. It can be shown that the ordinary product AB can be written as

$$AB = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} A_{11}B_1 + A_{12}B_2 \\ A_{21}B_1 + A_{22}B_2 \end{bmatrix} = \left[\begin{array}{cc} -5 & 4 \\ -6 & 2 \\ \hline 2 & 1 \end{array}\right]$$

- It is important for each smaller product in the expression for AB to be written with the submatrix from A on the left, since matrix multiplication is not commutative.

$$\left(\begin{array}{cc|c} 1 & 2 & 3 \\ 1 & 2 & 0 \end{array}\right) \left(\begin{array}{cc} 1 & 0 \\ \hline 1 & 1 \\ 1 & 0 \end{array}\right) \rightarrow$$

AB는 되지만

But $(A_1 \; A_2)\begin{pmatrix} B_1 \\ B_2 \end{pmatrix}$는 안된다. → Submatrix로 잘게 해야.

MULTIPLICATION OF PARTITIONED MATRICES

- For instance,

$$A_{11}B_1 = \begin{bmatrix} 2 & -3 & 1 \\ 1 & 5 & -2 \end{bmatrix} \begin{bmatrix} 6 & 4 \\ -2 & 1 \\ -3 & 7 \end{bmatrix} = \begin{bmatrix} 15 & 12 \\ 2 & -5 \end{bmatrix}$$

$$A_{12}B_2 = \begin{bmatrix} 0 & -4 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} -1 & 3 \\ 5 & 2 \end{bmatrix} = \begin{bmatrix} -20 & -8 \\ -8 & 7 \end{bmatrix}$$

- Hence the top block in AB is

$$A_{11}B_1 + A_{12}B_2 = \begin{bmatrix} 15 & 12 \\ 2 & -5 \end{bmatrix} + \begin{bmatrix} -20 & -8 \\ -8 & 7 \end{bmatrix} = \begin{bmatrix} -5 & 4 \\ -6 & 2 \end{bmatrix}$$

MULTIPLICATION OF PARTITIONED MATRICES

- **Theorem 10**: Column—Row Expansion of AB

If A is $m \times n$ and B is $n \times p$, then AB can be defined

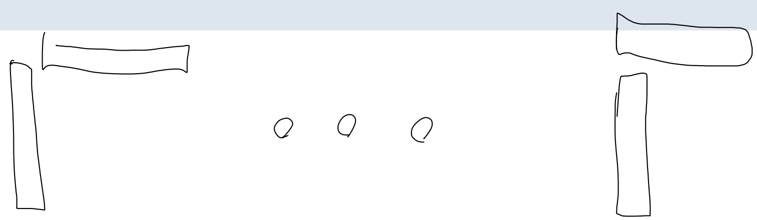
1xp

mx1

$$AB = \begin{bmatrix} \text{col}_1(A) & \text{col}_2(A) & \cdots & \text{col}_n(A) \end{bmatrix} \begin{bmatrix} \text{row}_1(B) \\ \text{row}_2(B) \\ \vdots \\ \text{row}_n(B) \end{bmatrix} \qquad (1)$$

dot product

$$= \text{col}_1(A)\,\text{row}_1(B) + \cdots + \text{col}_n(A)\,\text{row}_n(B)$$



INVERSES OF PARTIONED MATRICES

- The next example illustrates calculations involving inverses and partitioned matrices.
- **Example 5** A matrix of the form

$$A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$$

is said to be *block upper triangular*.

Assume that A_{11} is $p \times p$, A_{22} is $q \times q$, and A is invertible. Find a formula for A^{-1}.

INVERSES OF PARTIONED MATRICES

- **Solution** Denote A^{-1} by B and partition B so that

$$\begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} = \begin{bmatrix} I_p & 0 \\ 0 & I_q \end{bmatrix} \quad (2)$$

$$\begin{pmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{22}B_{21} & A_{22}B_{22} \end{pmatrix}$$

- This matrix equation provides four equations that will lead to the unknown blocks $B_{11}, \ldots, B_{22}$. Compute the product on the left side of equation (2), and equate each entry with the corresponding block in the identity matrix on the right.

INVERSES OF PARTIONED MATRICES

- That is, set

[illegible]

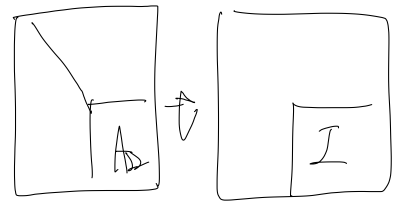
$$\begin{cases} A_{11}B_{11} + A_{12}B_{21} = I_p & (3) \\ A_{11}B_{12} + A_{12}B_{22} = 0 & (4) \\ A_{22}B_{21} = 0 & (5) \\ A_{22}B_{22} = I_q & (6) \end{cases}$$

- By itself, equation (6) does not show that A_{22} is invertible. However, since A_{22} is square, the Invertible Matrix Theorem and (6) together show that A_{22} is invertible and $B_{22} = A_{22}^{-1}$.

Claim:

Is A_{22} invertible? $A \sim I_{n\times n}$

Yes.



$\Rightarrow B_{22} = A_{22}^{-1}$, $B_{21} = 0$ $\longrightarrow$ $A_{11}B_{12} + A_{12}A_{22}^{-1} = 0$, $\underline{B_{12} = -A_{11}^{-1}A_{12}A_{22}^{-1}}$

$\Rightarrow B_{11} = A_{11}^{-1}$

$$\therefore B = \begin{pmatrix} A_{11}^{-1} & -A_{11}^{-1}A_{12}A_{22}^{-1} \\ 0 & A_{22}^{-1} \end{pmatrix}$$

INVERSES OF PARTIONED MATRICES

- Next, left-multiply both sides of (5) by A_{22}^{-1} and obtain

$$B_{21} = A_{22}^{-1}0 = 0$$

- So that (3) simplifies to

$$A_{11}B_{11} + 0 = I_p$$

- Since A_{11} is square, this shows that A_{11} is invertible and $B_{11} = A_{11}^{-1}$. Finally, use these results with (4) to find that

$$A_{11}B_{12} = -A_{12}B_{22} = -A_{12}A_{22}^{-1} \quad \text{and} \quad B_{12} = -A_{11}^{-1}A_{12}A_{22}^{-1}$$

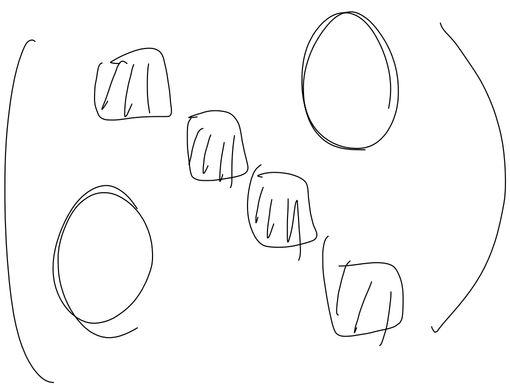
INVERSES OF PARTIONED MATRICES

- Thus

$$A^{-1} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}^{-1} = \begin{bmatrix} A_{11}^{-1} & -A_{11}^{-1} A_{12} A_{22}^{-1} \\ 0 & A_{22}^{-1} \end{bmatrix}$$

INVERSES OF PARTIONED MATRICES

- A **block diagonal matrix** is a partitioned matrix with zero blocks off the main diagonal (of blocks). Such a matrix is invertible if and only if each block on the diagonal is invertible.



ex)

$$\begin{pmatrix} 1 & & & & & \\ & 1 & 2 & & & 0 \\ & 0 & 1 & & & \\ & & & 2 & 3 & \\ 0 & & & 0 & 2 & \\ & & & & & 1 \end{pmatrix}$$

$$\begin{pmatrix} A & B \\ C & 0 \end{pmatrix}\begin{pmatrix} I & 0 \\ X & Y \end{pmatrix} = \begin{pmatrix} 0 & I \\ Z & 0 \end{pmatrix}$$ for $X\,Y\,Z$

$$\begin{pmatrix} AI+BX & A0+BY \\ CI+0X & C0+0Y \end{pmatrix} = \begin{pmatrix} A+BX & BY \\ C & 0 \end{pmatrix}$$

$\therefore\ A+BX=0 \quad BY=I \quad C=Z$

$\Rightarrow\ BX=-A \quad X=-B^{-1}A \quad Y=B^{-1}$

$\therefore\ C=Z$

$$\begin{bmatrix} X & 0 \\ Y & Z \end{bmatrix}\begin{bmatrix} A & 0 \\ B & C \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

$$\begin{bmatrix} XA+0B & X0+0C \\ YA+ZB & Y0+ZC \end{bmatrix} \quad \begin{pmatrix} XA=I \\ YA+ZB=0 \\ ZC=I \end{pmatrix} \quad \begin{Bmatrix} X=A^{-1} \\ Y=-C^{-1}BA^{-1} \\ Z=C^{-1} \end{Bmatrix}$$

- LU factorization

$$A = \underset{L}{(\text{unit lower triangular})}\ \underset{U}{(\underline{\underline{\text{echelon}}})}$$

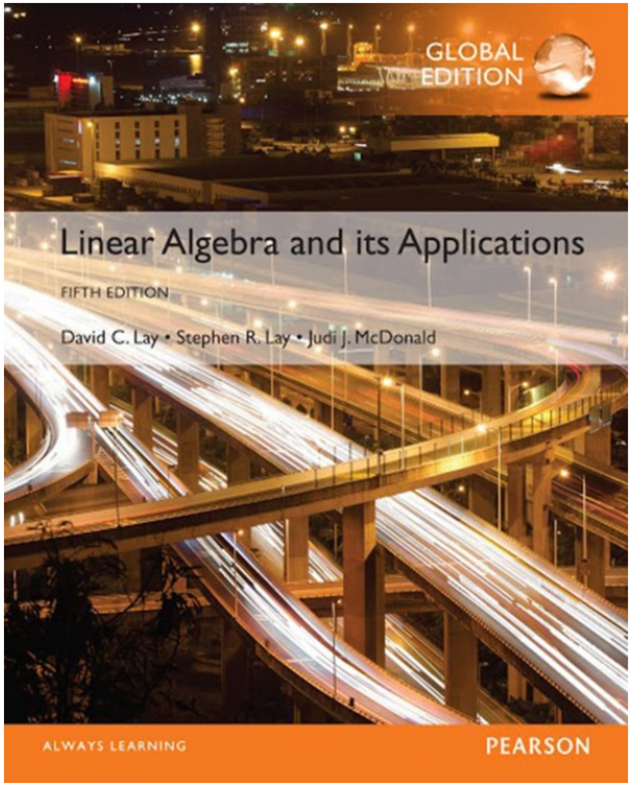
For i^{th} pivot column, $\begin{pmatrix} \text{pivot} \\ \text{values below} \end{pmatrix} \div \text{pivot} = \begin{pmatrix} 1 \\ \text{values}/\text{pivot} \end{pmatrix} \rightarrow i^{th}$ column of L

i

2 Matrix Algebra

2.5

MATRIX FACTORIZATIONS



MATRIX FACTORIZATIONS

- A *factorization* of a matrix A is an equation that expresses A as a product of two or more matrices.
- Whereas matrix multiplication involves a *synthesis* of data (combining the effects of two or more linear transformations into a single matrix), matrix factorization is an *analysis* of data.

THE LU FACTORIZATION

- The LU factorization, described on the next few slides, is motivated by the fairly common industrial and business problem of solving a sequence of equations, all with the same coefficient matrix:

$$Ax = b_1, \quad Ax = b_2, \ldots, \quad Ax = b_p \qquad (1)$$

- When A is invertible, one could compute A^{-1} and then compute $A^{-1}b_1$, $A^{-1}b_2$, and so on.
- However, it is more efficient to solve the first equation in the sequence (1) by row reduction and obtain the LU factorization of A at the same time. Thereafter, the remaining equations in sequence (1) are solved with the LU factorization.

THE LU FACTORIZATION

ERO { replacing, Interchange, Scaling }

The numbers in pivot positions during ERO are always non-zero

- At first, assume that A is an $m \times n$ matrix that can be row reduced to echelon form, *without row interchanges*.
- Then A can be written in the form $A = LU$, were L is an $m \times m$ lower triangular matrix with 1's on the diagonal and U is an $m \times n$ echelon form of A.

unit lower triangular matrix.

- For instance, see Fig. 1 below. Such a factorization is called an **LU factorization** of A. The matrix L is invertible and is called a unit lower triangular matrix.

L^{-1} also a unit lower triangular matrix

$$A=\underbrace{\begin{bmatrix}1&0&0&0\\ *&1&0&0\\ *&*&1&0\\ *&*&*&1\end{bmatrix}}_{L}\underbrace{\begin{bmatrix}\blacksquare&*&*&*&*\\ 0&\blacksquare&*&*&*\\ 0&0&0&\blacksquare&*\\ 0&0&0&0&0\end{bmatrix}}_{U}$$

$*$: any value

$\square$: pivot

THE LU FACTORIZATION

- Before studying how to construct L and U, we should look at why they are so useful. When $A = LU$, the equation $Ax = b$ ~~can be written as~~ $L(Ux) = b$.
- Writing y for Ux, we can find x by solving the pair of equations

$$\begin{aligned} Ly &= b \quad \text{for } y \\ Ux &= y \quad \text{for } x \end{aligned}$$

- First solve $Ly = b$ for y, and then solve $Ux = y$ for x. See Fig. 2 on the next slide. Each equation is easy to solve because L and U are triangular.

$$A = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}}_{L} \underbrace{\begin{pmatrix} 1 & 2 & n & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 1 \end{pmatrix}}_{U} \qquad b = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

$$Ly = b \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \quad \therefore \quad \begin{aligned} y_1 &= 1 \\ y_2 &= 0 \\ y_3 &= 2 \end{aligned}$$

$$\begin{pmatrix} 1 & 2 & n & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \qquad x = \begin{bmatrix} \cdots \\ -1 - \frac{1}{2}x_4 \\ 1 - \frac{1}{2}x_4 \\ x_4 \end{bmatrix}$$

THE LU FACTORIZATION

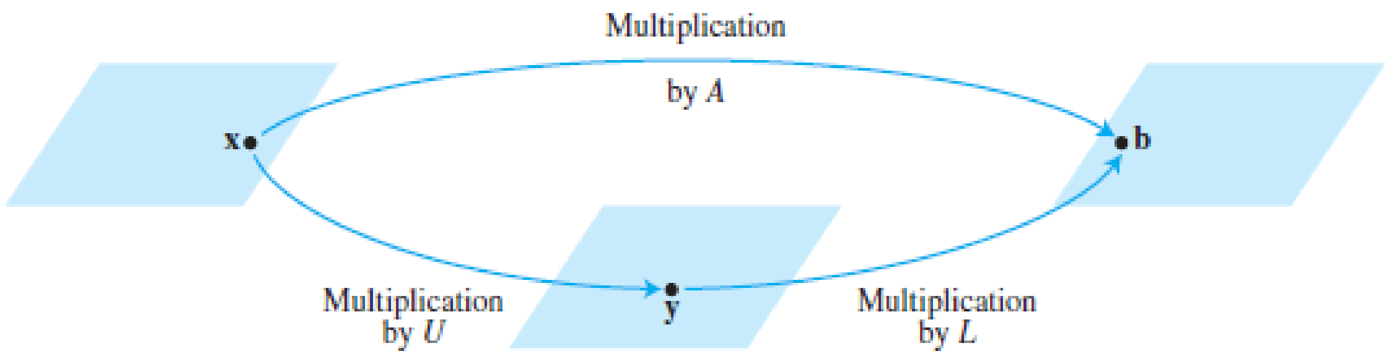


FIGURE 2 Factorization of the mapping $\mathbf{x} \mapsto A\mathbf{x}$.

- **Example 1** It can be verified that

$$A = \begin{bmatrix} 3 & -7 & -2 & 2 \\ -3 & 5 & 1 & 0 \\ 6 & -4 & 0 & -5 \\ -9 & 5 & -5 & 12 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & -5 & 1 & 0 \\ -3 & 8 & 3 & 1 \end{bmatrix} \begin{bmatrix} 3 & -7 & -2 & 2 \\ 0 & -2 & -1 & 2 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix} = LU$$

- Use this factorization of A to solve $Ax=b$, where $\mathrm{b} = \begin{bmatrix} -9 \\ 5 \\ 7 \\ 11 \end{bmatrix}$

THE LU FACTORIZATION

- **Solution** The solution of $Ly = b$ needs only 6 multiplications and 6 additions, because the arithmetic takes place only in column 5.

$$[L \quad \mathbf{b}] = \begin{bmatrix} 1 & 0 & 0 & 0 & -9 \\ -1 & 1 & 0 & 0 & 5 \\ 2 & -5 & 1 & 0 & 7 \\ -3 & 8 & 3 & 1 & 11 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & -9 \\ 0 & 1 & 0 & 0 & -4 \\ 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} = [I \quad \mathbf{y}]$$

- Then, for $Ux = y$, the "backward" phase of row reduction requires 4 divisions, 6 multiplications, and 6 additions.

THE LU FACTORIZATION

- For instance, creating the zeros in column 4 of $[U \quad y]$ requires 1 division in row 4 and 3 multiplication-addition pairs to add multiples of row 4 to the rows above.

$$[U \quad \mathbf{y}] = \begin{bmatrix} 3 & -7 & -2 & 2 & -9 \\ 0 & -2 & -1 & 2 & -4 \\ 0 & 0 & -1 & 1 & 5 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 & -6 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} 3 \\ 4 \\ -6 \\ -1 \end{bmatrix}$$

- To find x requires 28 arithmetic operations, or “flops” (floating point operations), excluding the cost of finding L and U. In contrast, row reduction of $[A \quad b]$ to $[I \quad x]$ takes 62 operations. (계산 줄어짐)

AN LU FACTORIZATION ALGORITHM

How To LU?

- Suppose A can be reduced to an echelon form U using only row replacements that add a multiple of one row to another below it.
- In this case, there exist unit lower triangular elementary matrices $E_1, \ldots, E_p$ such that

echelon form

$$E_p \ldots E_1 A = U$$

※ $E_i = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & c & & \ddots & \\ & & & & 1 \end{pmatrix}$

$E_i^{-1} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & -c & & \ddots & \\ & & & & 1 \end{pmatrix}$

마찬가지로 Unit Lower Triangular

- Then

Unit Lower Triangular

$$A = (E_p \ldots E_1)^{-1} U = LU \tag{3}$$

- where

$$L = (E_p \cdots E_1)^{-1} \tag{4}$$

- It can be shown that products and inverses of unit lower triangular matrices are also unit lower triangular. Thus L is unit lower triangular.

AN LU FACTORIZATION ALGORITHM

- Note that row operations in equation (3), which reduce A to U, also reduce the L in equation (4) to I, because $E_p \ldots E_1 L = (E_p \ldots E_1)(E_p \cdots E_1)^{-1} = I$. This observation is the key to *constructing* L.

Algorithm for an LU Factorization

1. Reduce A to an echelon form U by a sequence of row replacement operations, if possible.
2. Place entries in L such that the *same sequence of row operations* reduces L to I.

AN LU FACTORIZATION ALGORITHM

- Step 1 is not always possible, but when it is, the argument above shows that an LU factorization exists.
- Example 2 on the followings slides will show how to implement step 2. By construction, L will satisfy

$$(E_p \ldots E_1)L = I$$

- using the same $E_p, \ldots, E_1$ as in equation (3). Thus L will be invertible, by the Invertible Matrix Theorem, with $(E_p \ldots E_1) = L^{-1}$. From (3), $L^{-1}A = U$, and $A = LU$. So step 2 will produce an acceptable L.

AN LU FACTORIZATION ALGORITHM

- **Example 2** Find an LU factorization of

$$A = \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ -4 & -5 & 3 & -8 & 1 \\ 2 & -5 & -4 & 1 & 8 \\ -6 & 0 & 7 & -3 & 1 \end{bmatrix}$$

- **Solution** Since A has four rows, L should be 4×4. ~~The first column of L is the first column of A divided by the top pivot entry:~~

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & & 1 & 0 \\ -3 & & & 1 \end{bmatrix}$$

AN LU FACTORIZATION ALGORITHM

- Compare the first columns of A and L. *The row operations that create zeros in the first column of A will also create zeros in the first column of L.*
- To make this same correspondence of row operations on A hold for the rest of L, watch a row reduction of A to an echelon form U. That is, *highlight the entries* in each matrix that are used to determine the sequence of row operations that transform A onto U.

$$A = \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ -4 & -5 & 3 & -8 & 1 \\ 2 & -5 & -4 & 1 & 8 \\ -6 & 0 & 7 & -3 & 1 \end{bmatrix} \sim \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ 0 & 3 & 1 & 2 & -3 \\ 0 & -9 & -3 & -4 & 10 \\ 0 & 12 & 4 & 12 & -5 \end{bmatrix} = A_1$$

$$\sim A_2 = \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ 0 & 3 & 1 & 2 & -3 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 4 & 7 \end{bmatrix} \sim \boxed{\begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ 0 & 3 & 1 & 2 & -3 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix}} = U \qquad (5)$$

$$L = \begin{bmatrix} 1 & \cdot & \cdot & \cdot \\ -2 & 1 & \cdot & \cdot \\ 1 & -3 & 1 & \cdot \\ -3 & 4 & 2 & 1 \end{bmatrix}$$

AN LU FACTORIZATION ALGORITHM

$\left(\begin{pmatrix} 1 \end{pmatrix} \begin{pmatrix} 2 \\ 2 \end{pmatrix} \begin{pmatrix} 3 \\ 3 \end{pmatrix} \cdots \right)$

- The highlighted entries above determine the row reduction of A to U. At each pivot column, divide the highlighted entries by the pivot and place the result onto L:

$$\begin{bmatrix} 2 \\ -4 \\ 2 \\ -6 \end{bmatrix} \quad \begin{bmatrix} 3 \\ -9 \\ 12 \end{bmatrix} \quad \begin{bmatrix} 2 \\ 4 \end{bmatrix} \quad \begin{bmatrix} 5 \end{bmatrix}$$

$$\div 2 \qquad \div 3 \qquad \div 2 \qquad \div 5$$

$$\downarrow \qquad \downarrow \qquad \downarrow \qquad \downarrow$$

$$\begin{bmatrix} 1 & & & \\ -2 & 1 & & \\ 1 & -3 & 1 & \\ -3 & 4 & 2 & 1 \end{bmatrix}, \quad \text{and} \quad L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & -3 & 1 & 0 \\ -3 & 4 & 2 & 1 \end{bmatrix}$$

trick

pivot으로 나눠서

- An easy calculation verifies that this L and U satisfy $LU = A$.

$$A=\begin{pmatrix} 1 & -1 & -2 & -1 \\ -1 & 5 & 1 & 5 \\ 6 & -4 & 0 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & -2 & -1 \\ 0 & -2 & -1 & -2 \\ 0 & 16 & 4 & 16 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & -2 & -1 \\ 0 & -2 & -1 & -2 \\ 0 & 0 & -1 & 6 \end{pmatrix}$$

$$L=\begin{array}{ccc} 1 & 2 & 3 \\ \hline \end{array}$$

$$L=\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & -5 & 1 \end{pmatrix} \qquad A=LU$$

$$\begin{pmatrix} \boxed{\begin{matrix} 1 \\ -1 \\ 4 \\ -2 \end{matrix}} & \begin{matrix} 1 \\ -5 \\ 2 \\ -4 \end{matrix} & \begin{matrix} -5 \\ 8 \\ -5 \\ 17 \end{matrix} & \begin{matrix} -3 \\ 4 \\ -1 \\ 5 \end{matrix} \end{pmatrix} \to \begin{pmatrix} 1 & 3 & -4 & -3 \\ 0 & -2 & 3 & 1 \\ 0 & -10 & 15 & 5 \\ 0 & 2 & -3 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & -5 & -3 \\ 0 & -2 & 3 & 1 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} \to \text{No mor pivot}$$

$$L=\begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ -1 & 1 & \cdot & \cdot \\ 4 & 5 & 1 & 0 \\ -2 & -1 & 0 & 1 \end{pmatrix}$$

Identity matrix 2 和 4 一

- Find the LU factorization of A and solve $Ax = b$

2. $A = \begin{bmatrix} 4 & 3 & -5 \\ -4 & -5 & 7 \\ 8 & 6 & -8 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 2 \\ -4 \\ 6 \end{bmatrix}$

2.

$\begin{bmatrix} 4 & 3 & -5 \\ 0 & -2 & 2 \\ 0 & 0 & 2 \end{bmatrix}$ 아래 echelon.

$L = \begin{bmatrix} 1 & & \\ -1 & 1 & \\ 2 & 0 & 1 \end{bmatrix}$ 부호.

$Ly = b$

$\begin{bmatrix} 1 & 0 & 0 & 2 \\ -1 & 1 & 0 & -4 \\ 2 & 0 & 1 & 6 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 2 \end{bmatrix}$

$\therefore y = \begin{bmatrix} 2 \\ -2 \\ 2 \end{bmatrix}$

$Ux = y$

$\begin{bmatrix} 4 & 3 & -5 & 2 \\ 0 & -2 & 2 & -2 \\ 0 & 0 & 2 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 4 & 3 & -5 & 2 \\ 0 & -2 & 2 & -2 \\ 0 & 0 & 1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 4 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 1/4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ $x = \begin{bmatrix} 1/4 \\ 2 \\ 1 \end{bmatrix}$

- Find the LU factorization of A and solve $Ax = b$

4. $A = \begin{bmatrix} 2 & -2 & 4 \\ 1 & -3 & 1 \\ 3 & 7 & 5 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 0 \\ -5 \\ 7 \end{bmatrix}$

4.

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 1/3 & 5 & 1 \end{bmatrix}$$

$$A' \Rightarrow \begin{bmatrix} 2 & -2 & 4 \\ 0 & -2 & -1 \\ 0 & 10 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & -2 & 4 \\ 0 & -2 & -1 \\ 0 & 0 & -6 \end{bmatrix} = U$$

$$\left.\begin{matrix} Ly = b \\ Ux = y \end{matrix}\right\} \text{구하기}$$

- Find the LU factorization of A

14. $\begin{bmatrix} 1 & 4 & -1 & 5 \\ 3 & 7 & -2 & 9 \\ -2 & -3 & 1 & -4 \\ -1 & 6 & -1 & 7 \end{bmatrix}$

16.

$$L = \begin{bmatrix} 1 & \cdot & \cdot & \cdot \\ 3 & 1 & \cdot & \cdot \\ -2 & -1 & 1 & \cdot \\ -1 & -2 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 4 & -1 & 5 \\ 0 & -5 & 1 & -6 \\ 0 & 5 & -1 & 6 \\ 0 & 10 & -2 & 12 \end{bmatrix} \to \begin{bmatrix} 1 & 4 & -1 & 5 \\ 0 & -5 & 1 & -6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = U$$

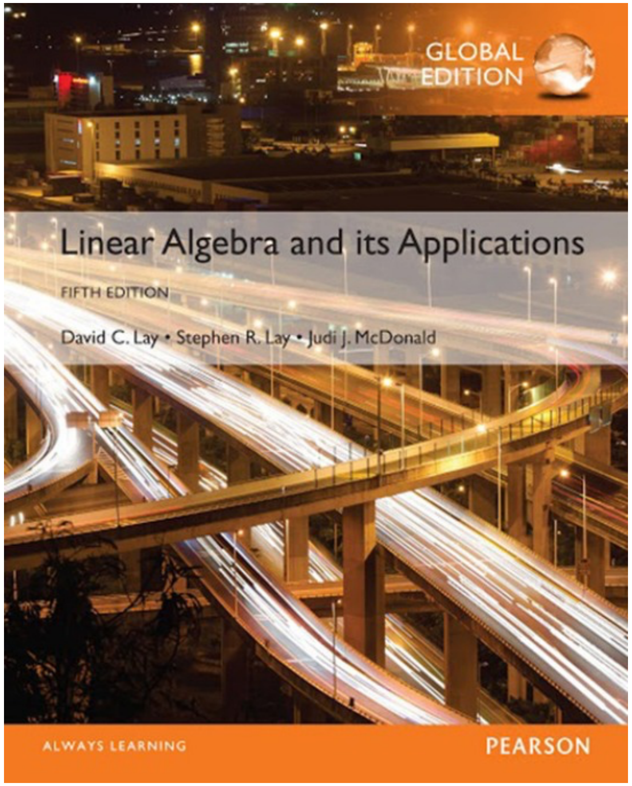
없다.

$$\left.\begin{matrix} \\ \\ \end{matrix}\right\} \quad \begin{matrix} Ly = b \\ Ux = y \end{matrix}$$

2 Matrix Algebra

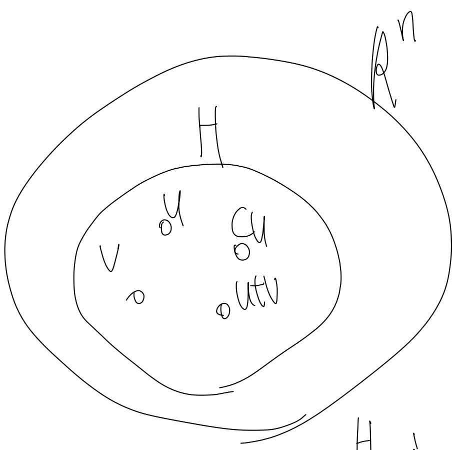
2.8

SUBSPACES OF $\mathbb{R}^n$



SUBSPACES OF $\mathbb{R}^n$

- **Definition**: A **subspace** of $\mathbb{R}^n$ is any set H in $\mathbb{R}^n$ that has three properties:
  a) The zero vector is in H.
  b) For each $\boldsymbol{u}$ and $\boldsymbol{v}$ in H, the sum $\mathbf{u}+\mathbf{v}$ is in H.
  c) For each $\mathbf{u}$ in H and each scalar c, the vector $c\mathbf{u}$ is in H.



H: subspace

H is a subset of $\mathbb{R}^n$ which is closed under vector addition and scalar

SUBSPACES OF $\mathbb{R}^n$

- A plane through the origin is the standard way to visualize the subspace in Example 1
  See Fig. 1 below:

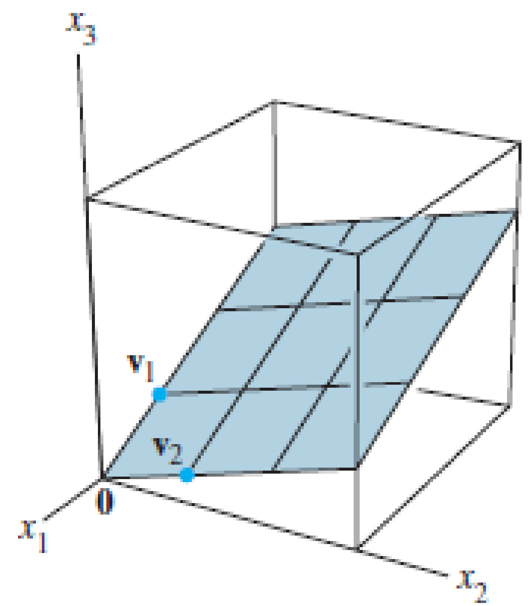


FIGURE 1
Span $\{\mathbf{v}_1, \mathbf{v}_2\}$ as a plane through the origin.

Example 1 shows that

SUBSPACES OF $\mathbb{R}^n$

span $\{v_1, \cdots, v_p\}$ is a subspace of $\mathbb{R}^n$ given $v_1, \cdots, v_p \in \mathbb{R}^n$

- **Example 1** If v_1 and v_2 are in $\mathbb{R}^n$ and $H = \text{Span}\{v_1, v_2\}$, then H is a subspace of $\mathbb{R}^n$. To verify this statement, note that the zero vector is in H (because $0v_1 + 0v_2$ is a linear combination of v_1 and v_2).
- Now take two arbitrary vectors in H, say,

$$u = s_1 v_1 + s_2 v_2 \quad \text{and} \quad v = t_1 v_1 + t_2 v_2$$

- Then

$$u + v = (s_1 + t_1)v_1 + (s_2 + t_2)v_2$$

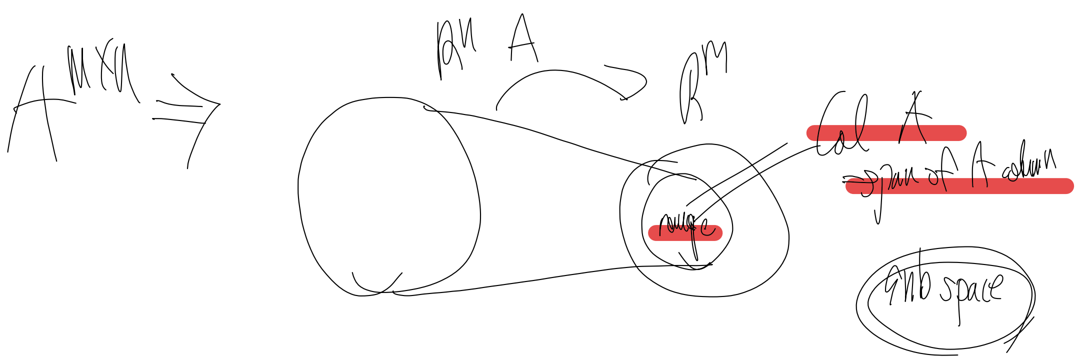
- which shows that $u + v$ is a linear combination of v_1 and v_2 and hence is in H. Also, for any scalar c, the vector cu is in H, because $cu = c(s_1 v_1 + s_2 v_2) = cs_1(v_1) + cs_2(v_2)$.

COLUMN SPACE AND NULL SPACE OF A MATRIX

- **Definition:** The **column space** of a matrix A is the set Col A of all linear combinations of the columns of A.

= span {column of A} = range

- If $A = [a_1 \; . \; . \; . \; a_n]$ with the columns of $\mathbb{R}^n$, then Col A is the same as Span$\{a_1 \; . \; . \; . \; a_n\}$. Example 4 shows that the column space of an $m \times n$ matrix is a subspace of $\mathbb{R}^m$.



COLUMN SPACE AND NULL SPACE OF A MATRIX

- **Example 4** Let $A = \begin{bmatrix} 1 & -3 & -4 \\ -4 & 6 & -2 \\ -3 & 7 & 6 \end{bmatrix}$ and $b = \begin{bmatrix} 3 \\ 3 \\ -4 \end{bmatrix}$.

~~Determine whether b is in the column space of *A*.~~

$\begin{bmatrix} A & b \end{bmatrix}$

$b \in$ span {column of A}

b is range

b is $Ax = b$ consistent

b is linear combi of the columns of A

Image of some x

COLUMN SPACE AND NULL SPACE OF A MATRIX

- **Solution:** The vector **b** is a linear combination of the columns of A if and only if **b** can be written as A**x** for some x, that is, if and only if the equation $A\mathrm{x} = \mathrm{b}$ has a solution.

- Row reducing the ~~augmented matrix [A **b**],~~

$$\begin{bmatrix} 1 & -3 & -4 & 3 \\ -4 & 6 & -2 & 3 \\ -3 & 7 & 6 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & -4 & 3 \\ 0 & -6 & -18 & 15 \\ 0 & -2 & -6 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & -4 & 3 \\ 0 & -6 & -18 & 15 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

- We conclude that $A\mathrm{x} = \mathrm{b}$ is consistent and **b** is in Col A.

COLUMN SPACE AND NULL SPACE OF A MATRIX

- **Definition:** The **null space** of a matrix A is the set Nul$A$ of all solutions of the homogenous equation $A\mathbf{x} = 0$.

collection of x, that $Ax = 0$

$u, v \in \text{Nul } A \Leftrightarrow Au = 0,\ Av = 0$

$A(u+v) = Au + Av = 0 \qquad u+v \in \text{Nul } A$

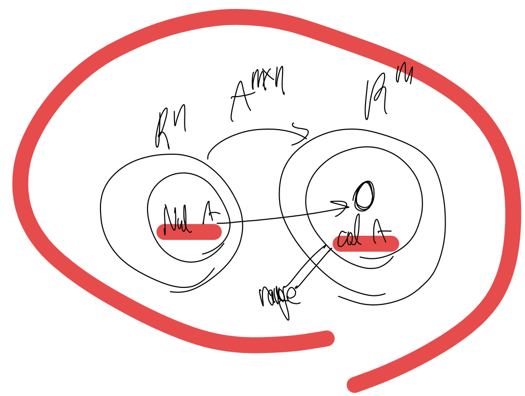
$A(cu) = cAu = 0 \qquad cu \in \text{Nul } A$

$\text{Nul } A = \{x \mid Ax = 0\}$

$= \{x \mid (\text{image of } x) = 0\}$

$= \{\text{preimages of } \vec{0}\}$

Nul A는 Subspace이다.



Nul A is subspace of domain $\mathbb{R}^n$
Col A is subspace of codomain $\mathbb{R}^m$

COLUMN SPACE AND NULL SPACE OF A MATRIX

- **Theorem 12**: ~~The null space of an m x n matrix A is a subspace of $\mathbb{R}^n$~~. Equivalently, ~~the set of all solutions of a system Ax = 0 of m homogenous linear equations~~ in n unknowns is a subspace of $\mathbb{R}^n$.

- **Proof:** The zero vector is in Nul$A$ (because $A0 = 0$). To show that Nul$A$ satisfies that other two properties required for a subspace, take any **u** and **v** in Nul$A$.

COLUMN SPACE AND NULL SPACE OF A MATRIX

- That is, suppose $A\mathbf{u} = 0$ and $A\mathbf{v} = 0$. Then, by a property of matrix multiplication,

$$A(u+v) = Au + Av = 0 + 0 = 0$$

- Thus $\mathbf{u} + \mathbf{v}$ satisfies $A = 0$, and so $\mathbf{u} + \mathbf{v}$ is in Nul$A$. Also, for any scalar c, $A(c\mathbf{u}) = c(A\mathbf{u}) = c(0) = 0$, which shows that $c\mathbf{u}$ is in Nul$A$.

BASIS FOR A SUBSPACE

- **Definition**: A **basis** for a subspace H of $\mathbb{R}^n$ is a linearly independent set in H that spans H.

① ②

H를 만드는 최소개수의 벡터 집합

$\beta = \{v_1 \sim v_p\}$ is a basis of H if
- $v_1 \sim v_p$ are linearly independent
- $\text{span}\{v_1 \sim v_p\} = H$ $\quad (\text{span}\{v_1 \dots v_p\} \supsetneq H)$ ↳ not basis.

ex)

$\text{span}\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right\} = \mathbb{R}^2$

$H = \text{span}\left\{\begin{pmatrix}1\\1\end{pmatrix}\right\}$

Claim: $\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right\}$ is a basis of H $\quad$ False

$\text{span}\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right\} = \mathbb{R}^2 \supsetneq H$

Note. A basis of a subspace H consists of the vectors in H only

ex) $H = \text{span}\left\{\begin{pmatrix}1\\1\end{pmatrix}\right\}$

$\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}$은 basis가 될수 없다.

BASIS FOR A SUBSPACE

- **Example 5** The columns of an invertible $n \times n$ matrix form a basis for all of $\mathbb{R}^n$ because they are linearly independent and span $\mathbb{R}^n$, by the Invertible Matrix Theorem.

BASIS FOR A SUBSPACE

- One such matrix is the $n \times n$ identity matrix. Its columns are denoted by $e_1, \ldots, e_n$:

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix}, \quad \ldots, \quad e_n = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{bmatrix},$$

- The set $\{e_1, \ldots, e_n\}$ is called the **standard basis** for $\mathbb{R}^n$.

BASIS FOR A SUBSPACE

- Ex. The set $\{e_1, e_2, e_3\}$ is called the **standard basis** for $\mathbb{R}^3$.

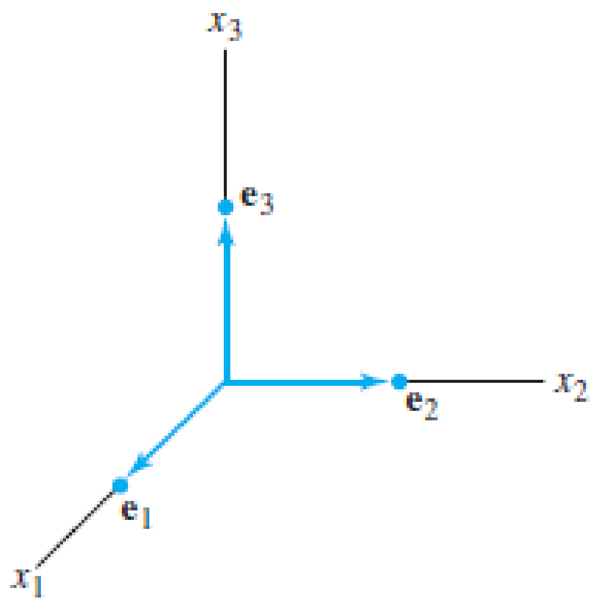


FIGURE 3
The standard basis for $\mathbb{R}^3$.

BASIS FOR A SUBSPACE

어떤 column이 다른 col의 linear combination 인가?

- **Theorem 13**: The pivot columns of a matrix A form a basis for the column space of A.

※ not an echelon form U. Only original A

= linear comb of cols of A

$= \{c_1 a_1 + c_2 a_2 + \cdots + c_n a_n \mid \}$

1) Ax=0 iff Ux=0

corresponding columns

2) Columns of A are independent iff columns of U are independent

(Pf. Let columns of A be independent. Suppose linear combination of columns of U is 0. Then, weight is 0 and columns of U are independent. Let columns of A be dependent. Then linear dependence relation for A exists. So does U and the columns of U are dependent.)

3) Some rows of U may be zero but the corresponding rows of A may not be.

※ pivot Columns of U forms basis for the column space of U

The pivots columns of U may not be a basis of column space of A

ex)

$$A=\begin{array}{c}\begin{array}{ccc} a_1 & & \end{array}\\ \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{pmatrix}\end{array} \qquad U=\begin{array}{c}\begin{array}{ccc} u_1 & & \end{array}\\ \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\end{array}$$

$\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right\}$ is NOT a basis of Col A

$\left\{\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}\right\}$ ~ basis of Col A

$$U = \begin{pmatrix} 1 & 1 & 2 & 1 \\ 0 & 2 & 4 & 2 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\uparrow v_1 \quad \uparrow v_2 \quad v_3 \quad \uparrow v_4$

Note

- The pivot columns in U are linearly independent

  $(v_1 \neq 0, \quad v_2 \neq c v_1 \quad v_4 \neq c v_1 + d v_2)$

- Any non-pivot column of U can be written as a linear combination of preceeding pivot columns

  (i.e. $v_3 = c_1 v_1 + c_2 v_2$)

$$U = \begin{array}{cccccc} v_1 & v_2 & v_3 & v_4 & & \\ \end{array}$$
$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 0 & 1 & 2 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$v_3 = c_1 v_1 + c_2 v_2$ $\quad \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

$v_4 = d_1 v_1 + d_2 v_2$ $\quad \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} \begin{pmatrix} 4 \\ 1 \end{pmatrix}$

$A \longrightarrow U$

$[a_1\, a_2\, a_3\, a_4]$

Claim 1 Pivot columns of A are independent.

Set $c_1 a_1 + c_2 a_2 + c_4 a_4 = 0$ i.e. $A \begin{pmatrix} c_1 \\ c_2 \\ 0 \\ c_4 \end{pmatrix} = 0$

Then $U \begin{pmatrix} c_1 \\ c_2 \\ 0 \\ c_4 \end{pmatrix} = 0$ $\quad c_1 v_1 + c_2 v_2 + c_4 v_4 = 0$

$\Downarrow$

$c_1 = c_2 = c_4 = 0$

Claim 2 Any non pivot column of A can be written as a linear combination of preceeding pivot columns of A.

$U \begin{pmatrix} c_1 \\ c_2 \\ -1 \\ 0 \end{pmatrix} = 0$ $\quad A \begin{pmatrix} c_1 \\ c_2 \\ -1 \\ 0 \end{pmatrix} = 0$ $\quad c_1 a_1 + c_2 a_2 - a_3 = 0$

$a_3 = c_1 a_1 + c_2 a_2$

ex) In the previous example, $a_3 = c_1 a_1 + c_2 a_2$ for some c_1, c_2

Any linear combination of the columns of A

$$d_1 a_1 + d_2 a_2 + d_3 a_3 + d_4 a_4$$

$$= d_1 a_1 + d_2 a_2 + d_3 (c_1 a_1 + c_2 a_2) + d_4 a_4$$

$= \underline{\text{a linear combination}}$ of pivot columns of A

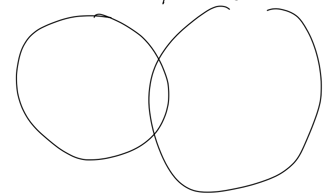
$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 2 & 6 & 6 \end{bmatrix} \rightarrow U = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{basis of } \operatorname{Col} U = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

동형인가

같은가.

$$\text{basis of } \operatorname{Col} A = \left\{ \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \right\}$$

$\operatorname{Col} A \neq \operatorname{Col} U$



$\because \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \notin \operatorname{Col} A, \quad \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \notin \operatorname{Col} U$

[illegible] 중요

$\operatorname{Nul} A = \{ x \mid Ax = 0 \} = \{ x \mid Ux = 0 \} = \operatorname{Nul} U$

$\therefore$ basis of $\operatorname{Nul} A$ = basis of $\operatorname{Nul} U$

5. Let $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 3 \\ -5 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -4 \\ -5 \\ 8 \end{bmatrix}$, and $\mathbf{w} = \begin{bmatrix} 8 \\ 2 \\ -9 \end{bmatrix}$. Determine if $\mathbf{w}$ is in the subspace of $\mathbb{R}^3$ generated by $\mathbf{v}_1$ and $\mathbf{v}_2$.

Sol.

$$\begin{bmatrix} V_1 & V_2 & W \end{bmatrix} = \begin{bmatrix} 2 & -4 & 8 \\ 3 & -5 & 2 \\ -5 & 8 & -9 \end{bmatrix} \sim \begin{bmatrix} 2 & -4 & 8 \\ \cdot & 1 & -10 \\ \cdot & -2 & 11 \end{bmatrix} \sim \begin{bmatrix} 2 & 4 & 8 \\ & 1 & -10 \\ & & -9 \end{bmatrix}$$

$$W \notin \operatorname{span}\{V_1\ V_2\}$$

7. Let $\mathbf{v}_1 = \begin{bmatrix} 2 \\ -8 \\ 6 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -3 \\ 8 \\ -7 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} -4 \\ 6 \\ -7 \end{bmatrix},$

$\mathbf{p} = \begin{bmatrix} 6 \\ -10 \\ 11 \end{bmatrix}$, and $A = [\mathbf{v}_1 \ \mathbf{v}_2 \ \mathbf{v}_3]$.

a. How many vectors are in $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$? 3

b. How many vectors are in Col A? ∞

c. Is $\mathbf{p}$ in Col A? Why or why not?

9. With A and $\mathbf{p}$ as in Exercise 7, determine if $\mathbf{p}$ is in Nul A.

a. 3

b. ∞

c. $\begin{bmatrix} 2 & -3 & -4 & 6 \\ -8 & 8 & 6 & -10 \\ 6 & -7 & -7 & 11 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & -3 & -4 & -6 \\ 0 & -4 & -10 & 14 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

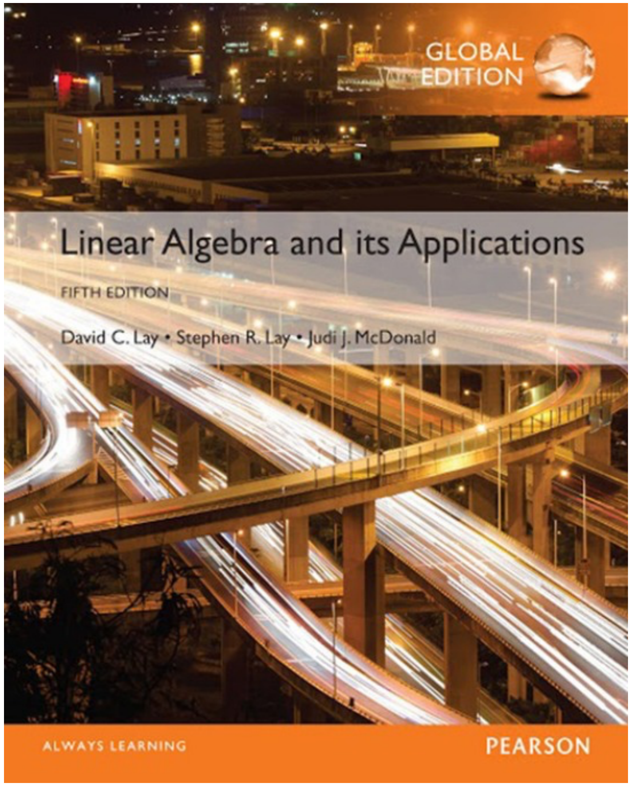
$p \in \mathrm{Col}\,A$ [illegible]

9. $Ap = 0$? $\begin{bmatrix} 2 & -3 & -4 \\ -8 & 8 & 6 \\ 6 & -7 & 7 \end{bmatrix} \begin{bmatrix} 6 \\ -10 \\ 11 \end{bmatrix} = \begin{bmatrix} -2 \\ ? \\ ? \end{bmatrix} \rightarrow$ No.

2 Matrix Algebra

2.9

DIMENSION AND RANK



PEARSON

COORDINATE SYSTEMS

- Suppose $\beta = \{b_1, \ldots, b_p\}$ is a basis for H, and suppose a vector **x** in H can be generated in two ways, say,

$$x = c_1 b_1 + \cdots + c_p b_p \text{ and } x = d_1 b_1 + \cdots + d_p b_p \quad (1)$$

- Then, subtracting gives

$$0 = x - x = (c_1 - d_1)b_1 + \cdots + (c_p - d_p)b_p \qquad (2)$$

- Since β is linearly independent, the weights in (2) must all be zero. That is, $c_j = d_j$ for $1 \le j \le p$, which shows that the two representations in (1) are actually the same.

representation is unique

COORDINATE SYSTEMS

- **Definition**: Suppose the set $\beta = \{b_1, \ldots, b_p\}$ is a basis for a subspace H. For each x in H, the **coordinates of x relative to the basis** β are the weights $c_1, \ldots, c_p$ such that $x = c_1 b_1 + \cdots + c_p b_p$, and the vector in $\mathbb{R}^p$

$$[x]_\beta = \begin{bmatrix} c_1 \\ \cdot \\ \cdot \\ \cdot \\ c_p \end{bmatrix}$$

- is called the **coordinate vector of x (relative to** β**)** or the β**-coordinate vector of x.**

$$B = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\} \qquad C = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$$

$$\begin{pmatrix} 3 \\ 5 \end{pmatrix} = 3 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 5 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 3 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + 2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\left[\begin{pmatrix} 3 \\ 5 \end{pmatrix} \right]_B = \underline{\underline{\begin{pmatrix} 3 \\ 5 \end{pmatrix}}} \qquad \left[\begin{pmatrix} 3 \\ 5 \end{pmatrix} \right]_C = \underline{\underline{\begin{pmatrix} 3 \\ 2 \end{pmatrix}}}$$

COORDINATE SYSTEMS

- **Example 1** Let $v_1 = \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix}$, $v_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, $x = \begin{bmatrix} 3 \\ 12 \\ 7 \end{bmatrix}$, and $\beta = \{v_1, v_2\}$. Then β is a basis for H = Span $\{v_1, v_2\}$ because v_1 and v_2 are linearly independent. Determine if x is in *H*, and if it is, find the coordinate vector of x relative to β.

dim H = 2

- **Solution** If x is in *H*, then the following vector equation is consistent:

eq Augmented Matrix

$$c_1 \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 12 \\ 7 \end{bmatrix}$$

COORDINATE SYSTEMS

- The scalars c_1 and c_2, if they exist, are the β-coordinates of $\mathbf{x}$. Row operations show that

$$\begin{bmatrix} 3 & -1 & 3 \\ 6 & 0 & 12 \\ 2 & 1 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

- Thus $c_1 = 2$, $c_2 = 3$ and $[x]_\beta = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$. The basis β determines a "coordinate system" on H, which can be visualized by the grid shown in Fig. 1 on the next slide.

COORDINATE SYSTEMS

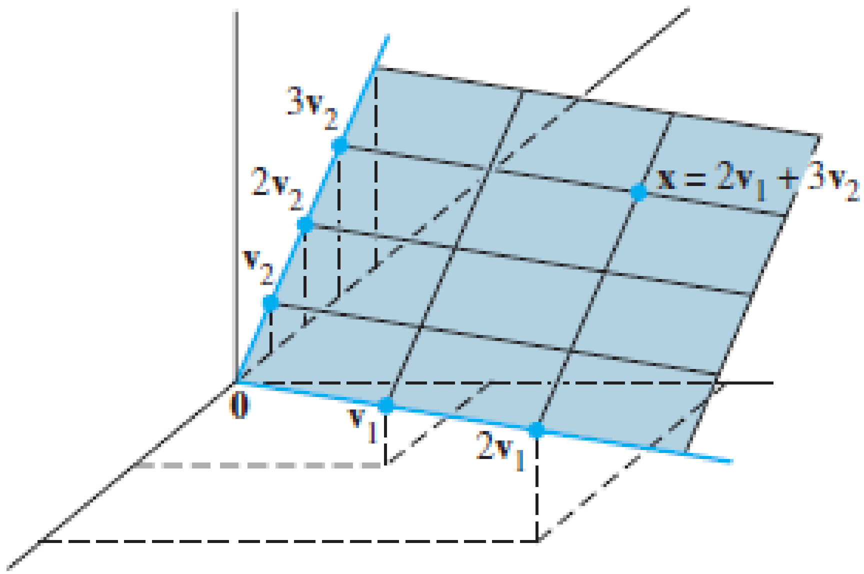


FIGURE 1 A coordinate system on a plane H in $\mathbb{R}^3$.

THE DIMESION OF A SUBSPACE

- **Definition**: The **dimension** of a nonzero subspace *H*, denoted by dim *H*, is the number of vectors in any basis for *H*. The dimension of the zero subspace {0} is defined to be zero.

이번에는 했다가 안됐어...

- **Definition:** The **rank** of a matrix *A*, denoted by rank *A*, is the dimension of the column space of *A*. # pivot columns of A

- The pivot columns of A are linearly independent
- A linear combination of the columns of A can be a linear combination of the pivot columns of A

→ The pivot columns of A form a basis

THE DIMESION OF A SUBSPACE

- **Example 3** Determine the rank of the matrix

$$A \sim \begin{bmatrix} 2 & 5 & -3 & -4 & 8 \\ 0 & -3 & 2 & 5 & -7 \\ 0 & -6 & 4 & 14 & -20 \\ 0 & -9 & 6 & 5 & -6 \end{bmatrix}$$

- **Solution** Reduce A to echelon form:

$$A \sim \begin{bmatrix} 2 & 5 & -3 & -4 & 8 \\ 0 & -3 & 2 & 5 & -7 \\ 0 & -6 & 4 & 14 & -20 \\ 0 & -9 & 6 & 5 & -6 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} 2 & 5 & -3 & -4 & 8 \\ 0 & -3 & 2 & 5 & -7 \\ 0 & 0 & 0 & 4 & -6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Pivot columns

- The matrix A has 3 pivot columns, so rank $A = 3$.

THE DIMESION OF A SUBSPACE

- **Theorem 14** If a matrix A has n columns, then rank A + dim Nul$A = n$.

basis of H는 p개의 linearly Independent vector로 구성되어있다

dim H = p = # vectors in a basis

- **Theorem 15** Let H be a p-dimensional subspace of $\mathbb{R}^n$. Any linearly independent set of exactly p elements in H is automatically a basis for H. Also, any set of p elements of H that spans H is automatically a basis for H.

If H is known to have the dimension p,

then { any p linearly independent vectors will span H
any p vectors which span H will be linearly independent }

Suppose $A \sim m \times n$

$\underline{Ax=0} \rightarrow$ # basic variables $=$ # pivot columns $=$ rank $(=\dim \mathrm{Col}\, A)$

$\underline{\dim \mathrm{Nul}\, A} =$ # free variables $= \underline{\text{\# unknowns}} -$ # basic variables

$=$ # columns $-$ rank

$=$ n $-$ rank

pivot column

→ basic Var 이 된다.

non-pivot column

→ free Var 이 된다.

$$\{x \mid \underline{Ax=0}\} = \{\text{parametric vector form with one vector for each free variable}\}$$

$$\left\{\begin{aligned} \underline{\underline{x_1}} \quad & - x_3 + x_4 = 0 \\ \underline{\underline{x_2}} \quad & - 2x_3 \quad\;\; = 0 \end{aligned}\right.$$

$x_1 = x_3 - x_4$

$x_2 = 2x_3$

$\left.\begin{matrix} x_3 \\ x_4 \end{matrix}\right\} \sim$ free.

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_3 \\ 2x_3 \\ x_3 \\ 0 \end{pmatrix} + \begin{pmatrix} -x_4 \\ 0 \\ 0 \\ x_4 \end{pmatrix} = \underline{x_3} \begin{pmatrix} 1 \\ 2 \\ \boxed{1} \\ \boxed{0} \end{pmatrix} + \underline{x_4} \begin{pmatrix} -1 \\ 0 \\ \boxed{0} \\ \boxed{1} \end{pmatrix}$$

independent

이 form에 basis의 개수 = free Var 개수

RANK AND THE INVERTIBLE MATRIX THEOREM

- **The Invertible Theorem (continued)** Let A be an $n \times n$ matrix. Then the following statements are each equivalent to the statement that A is an invertible matrix.

  m. The columns of A form a basis of $\mathbb{R}^n$. solution uniquely exist!

  n. $\text{Col } A = \mathbb{R}^n$ = Range → all pivot cols...

  o. $\dim \text{Col } A = n$

  p. $\text{rank } A = n$ rank + dim Nul A = n

  Onto

  q. $\text{Nul} A = \{0\}$

  r. $\dim \text{Nul} A = 0$

  One to One

  { x = 0 is the only solution of Ax = 0 }

$A \sim$ invertible (nonsingular).

$\Uparrow$

$AC = I$

$\Downarrow$

$Ax = b$ has a solution for each $b \in \mathbb{R}^n$

$\Downarrow$

Each b is an image of some x

a linear combination of the columns of A

$\Downarrow$

Range $= \mathbb{R}^n = \text{Col } A = \text{span}\{\text{columns}\}$

$\Downarrow \quad \Downarrow$

rank $= \#$ pivots $= n$

$\Downarrow$

$\dim \text{Col } A = n$

$\Downarrow$

All the columns form a basis of $\text{Col}(A) = \mathbb{R}^n$

$CA = I \rightarrow Ax = b$ has a unique solution (if it exists)

$\Downarrow$

$x = 0$ is the only solution of $Ax = 0$

$\Downarrow$

$\text{Nul } A = \{0\}$

$\Downarrow$

$\dim \text{Nul } A = 0$

$\Downarrow$

Columns are independent

$\Downarrow$

1–1

$\Downarrow$

RANK AND THE INVERTIBLE MATRIX THEOREM

- **Proof** Statement (m) is logically equivalent to statements (e) and (h) regarding linear independence and spanning. The other five statements are linked to the earlier ones of the theorem by the following chain of almost trivial implications:

$$(g) \Longrightarrow (n) \Rightarrow (o) \Rightarrow (p) \Rightarrow (r) \Rightarrow (q) \Rightarrow (d)$$

- Statement (g), which says that the equation Ax = b has at least one solution for each **b** in $\mathbb{R}^n$, implies statement (n), because Col A is precisely the set of all **b** such that the equation Ax = b is consistent.

RANK AND THE INVERTIBLE MATRIX THEOREM

- The implications $(n) \Rightarrow (o) \Rightarrow (p)$ follow from the definitions of *dimension* and *rank*.
- If the rank of A is n, the number of columns of A, then dim Nul$A$ = 0, by the Rank Theorem, and so Nul$A$ $\{0\}$. Thus $(p) \Rightarrow (r) \Rightarrow (q)$.
- Also, statement (q) implies that the equation $A\mathbf{x} = 0$ has only the trivial solution, which is statement (d).
- Since statements (d) and (g) are already known to be equivalent to the statement that A is invertible, the proof is complete.

- Find the B-coordinate of x

3. $\mathbf{b}_1 = \begin{bmatrix} 1 \\ -4 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} -2 \\ 7 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} -3 \\ 7 \end{bmatrix}$

4. $\mathbf{b}_1 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} -3 \\ 5 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} -7 \\ 5 \end{bmatrix}$

$$\begin{bmatrix} 1 & -2 & -3 \\ -4 & 7 & 7 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & -3 \\ 0 & -1 & -5 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 7 \\ 0 & 1 & 5 \end{bmatrix} \quad \begin{matrix} c_1 = 7 \\ c_2 = 5 \end{matrix}$$

$$\begin{bmatrix} 1 & -3 & -7 \\ -3 & 5 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & -3 & -7 \\ 0 & -4 & -16 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 4 \end{bmatrix} \quad \begin{matrix} c_1 = 5 \\ c_2 = 4 \end{matrix}$$

Find bases for Col A and Nul A, and then state the dimensions of these subspaces.

9. $A = \begin{bmatrix} 1 & -3 & 2 & -4 \\ -3 & 9 & -1 & 5 \\ 2 & -6 & 4 & -3 \\ -4 & 12 & 2 & 7 \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & -3 & 2 & -4 \\ \cdot & \cdot & 5 & -7 \\ \cdot & \cdot & \cdot & 5 \\ \cdot & \cdot & 10 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 2 & -4 \\ 0 & 0 & 5 & -7 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 23 \end{bmatrix}$$

linearly independent

$\therefore$ basis for Col A $\to \left\{ \begin{pmatrix} 1 \\ -3 \\ 2 \\ -4 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 4 \\ 2 \end{pmatrix}, \begin{pmatrix} -4 \\ 5 \\ -3 \\ 7 \end{pmatrix} \right\}$

dim Col A $= 3$

$Ax = 0 \iff Ux = 0$

$$\left[\begin{array}{cccc|c} 1 & -3 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{cases} x_1 = 3x_2 \\ x_3 = x_4 = 0 \end{cases} \quad X = \begin{pmatrix} 3 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

basis of Nul A $= \left\{ \begin{pmatrix} 3 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\}$

dim Nul A $= 1$

- Fnd a basis for the subspace spanned by the given vectors. What is the dimension of the subspace?

13. $\begin{bmatrix} 1 \\ -3 \\ 2 \\ -4 \end{bmatrix}, \begin{bmatrix} -3 \\ 9 \\ -6 \\ 12 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 4 \\ 2 \end{bmatrix}, \begin{bmatrix} -4 \\ 5 \\ -3 \\ 7 \end{bmatrix}$

13. Identify the independent vectors → Identify the pivot columns.

$$\sim \begin{vmatrix} 1 & -3 & 2 & -4 \\ . & . & 5 & -7 \\ . & . & . & 5 \\ . & . & 10 & -9 \end{vmatrix} \sim \begin{vmatrix} 1 & -3 & 2 & -4 \\ . & . & 5 & -7 \\ . & . & . & 5 \\ . & . & . & 5 \end{vmatrix} \sim \begin{vmatrix} 1 & -3 & 2 & -4 \\ . & . & 5 & -7 \\ . & . & . & 5 \\ . & . & . & 0 \end{vmatrix}$$

$\dim = 3$

15. Suppose a 3×5 matrix A has three pivot columns. Is Col $A = \mathbb{R}^3$? Is Nul $A = \mathbb{R}^2$? Explain your answers.

15. 3×5. 3 pivot (= rank = 3)

$$\begin{matrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{matrix}$$

Col $A = \mathbb{R}^3$. Yes

Nul $A = \mathbb{R}^2$. No $\because$ Nul $A \leq \mathbb{R}^5$

$\dim(\text{Nul } A) + \text{rank} = 5$

$\therefore$ Nul $A = 2$

Nul A is the 2-dimensional subspace of $\mathbb{R}^5$

19. If the subspace of all solutions of $A\mathbf{x} = \mathbf{0}$ has a basis consisting of three vectors and if A is a 5×7 matrix, what is the rank of A?

19.

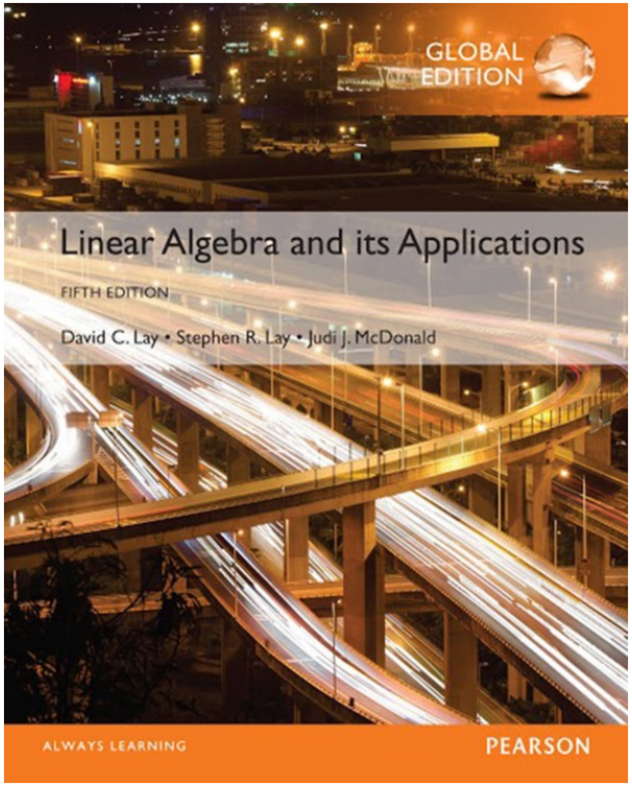
$\dim \operatorname{Nul} A = 3$

rank of $A = 7 - 3 = 4$.

3 Determinants

3.1

INTRODUCTION TO DETERMINANTS



이게 정의

INTRODUCTION TO DETERMINANTS

- **Definition**: For $n \geq 2$, ~~the **determinant** of an $n \times n$~~ matrix $A = [a_{ij}]$ is the sum of n terms of the form $\pm\, a_{1j} \det A_{1j}$, with plus and minus signs alternating, where the entries $a_{11}, a_{12}, \ldots, a_{1n}$ are from the first row of A. In symbols,

$$\det A = a_{11} \det A_{11} - a_{12} \det A_{12} + \cdots + (-1)^{1+n} a_{1n} \det A_{1n}$$

$$= \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det A_{1j}$$

For $A(a)$ $\det A = a$ (not $|a|$)

For $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $\det A = ad - bc$

A_{ij} is the $(n-1) \times (n-1)$ matrix is obtained from A by deleting the ith row and jth column

※ $C_{ij} = (-1)^{i+j} \det A_{ij}$ = Co factor

INTRODUCTION TO DETERMINANTS

- **Example 1** Compute the determinant of

$$A = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$$

- **Solution** Compute $\det A = a_{11}\det A_{11} - a_{12}\det A_{12} + a_{13}\det A_{13}$:

$$detA = 1 \cdot det\begin{bmatrix} 4 & -1 \\ -2 & 0 \end{bmatrix} - 5 \cdot det\begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix} + 0 \cdot det\begin{bmatrix} 2 & 4 \\ 0 & -2 \end{bmatrix}$$

$$= 1(0 - 2) - 5(0 - 0) + 0(-4 - 0) = -2$$

INTRODUCTION TO DETERMINANTS

- Another common notation for the determinant of a matrix uses a pair of vertical lines in place of brackets.
- Thus the calculation in Example 1 can be written as

$$|A| = det A = 1 \cdot \begin{vmatrix} 4 & -1 \\ -2 & 0 \end{vmatrix} - 5 \begin{vmatrix} 2 & -1 \\ 0 & 0 \end{vmatrix} + 0 \begin{vmatrix} 2 & 4 \\ 0 & -2 \end{vmatrix} = \cdots = -2$$

INTRODUCTION TO DETERMINANTS

- To state the next theorem, it is convenient to write the definition of det A in a slightly different form. Given $A = [a_{ij}]$, the **(*i*, *j*)-cofactor** of A is the number C_{ij} given by

$$C_{ij} = (-1)^{i+j} \det A_{ij} \qquad (4)$$

- Then

$$\det A = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}$$

This formula is called a **cofactor expansion across the first row** of A.

INTRODUCTION TO DETERMINANTS

nxn Matrix A

- **Theorem 1**: The determinant of an $\cancel{n\times}$ $a_{1n}C_{1n}$ A can be computed by a cofactor across any row or down any column. The expansion across the i th row using the cofactors in (4) is

$$\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$

- The cofactor expansion down the j th column is

$$\det A = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$$

INTRODUCTION TO DETERMINANTS

- **Example 2** Use a cofactor expansion across the third row to compute det A, where

$$A = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$$

여기가 제일 편하다

- **Solution** Compute

$$detA = a_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33}$$

$$= (-1)^{3+1}a_{31}detA_{31} + (-1)^{3+2}a_{32}\det A_{32} + (-1)^{3+3}\, a_{33}\det A_{33}$$

$$= \cancel{\begin{vmatrix} 5 & 0 \\ 4 & -1 \end{vmatrix}} - (-2)\begin{vmatrix} 1 & 0 \\ 2 & -1 \end{vmatrix} + \cancel{0\begin{vmatrix} 1 & 5 \\ 2 & 4 \end{vmatrix}}$$

$$= 0 + 2(-1) + 0 = -2$$

INTRODUCTION TO DETERMINANTS

- **Theorem 2**: If A is a triangular matrix, then det A is the product of the entries on the main diagonal of A.

$$\left|\begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix}\right| = a\begin{vmatrix} d & e \\ 0 & f \end{vmatrix} + \cancel{0(\ldots) + 0(\ldots)}$$

$$= a(df - 0)$$

$$= adf$$

$$\begin{vmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & f \end{vmatrix} = a\begin{vmatrix} c & 0 \\ e & f \end{vmatrix} + 0 = \underline{acf}$$

- Compute the determinant

1. $\begin{vmatrix} 3 & 0 & 4 \\ 2 & 3 & 2 \\ 0 & 5 & -1 \end{vmatrix}$ **2.** $\begin{vmatrix} 0 & 4 & 1 \\ 5 & -3 & 0 \\ 2 & 3 & 1 \end{vmatrix}$

1. $3\begin{vmatrix} 3 & 2 \\ 5 & -1 \end{vmatrix} + 0 + 4\begin{vmatrix} 2 & 3 \\ 0 & 5 \end{vmatrix}$

2. $-4\begin{vmatrix} 5 & 0 \\ 2 & 1 \end{vmatrix} + 1\begin{vmatrix} 5 & -3 \\ 2 & 3 \end{vmatrix}$

Compute the determinant

9. $\begin{vmatrix} 4 & 0 & 0 & 5 \\ 1 & 7 & 2 & -5 \\ 3 & 0 & 0 & 0 \\ 8 & 3 & 1 & 7 \end{vmatrix}$ **10.** $\begin{vmatrix} 1 & -2 & 5 & 2 \\ 0 & 0 & 3 & 0 \\ 2 & -4 & -3 & 5 \\ 2 & 0 & 3 & 5 \end{vmatrix}$

9.

$$(-1)^{3+1} 3 \begin{vmatrix} 0 & 0 & 5 \\ 7 & 2 & -5 \\ 3 & 1 & 7 \end{vmatrix} = 3 \times (-1)^{1+3} \times 5 \begin{vmatrix} 7 & 2 \\ 3 & 1 \end{vmatrix}$$

10.

$$(-1)^{2+3} \times 3 \begin{vmatrix} 1 & -2 & 2 \\ 2 & -4 & 5 \\ 2 & 0 & 5 \end{vmatrix} = -3 \times \left[(-1)^{3+1} \cdot 2 \begin{vmatrix} -2 & 2 \\ -4 & 5 \end{vmatrix} + (-1)^{3+3} \cdot 5 \begin{vmatrix} 1 & -2 \\ 2 & -4 \end{vmatrix} \right]$$

$$\begin{bmatrix} 2 & -2 & 3 \\ 3 & 1 & 2 \\ 1 & 3 & -1 \end{bmatrix} \rightarrow 2\begin{vmatrix} 1 & 2 \\ 3 & -1 \end{vmatrix} - (-2)\begin{vmatrix} 3 & 2 \\ 1 & -1 \end{vmatrix} + 3\begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix}$$

$$\begin{bmatrix} 1 & 2 & 4 \\ 3 & 1 & 1 \\ 2 & 4 & 2 \end{bmatrix} = 1 \cdot \begin{vmatrix} 1 & 1 \\ 4 & 2 \end{vmatrix} + 2\begin{vmatrix} 3 & 1 \\ 2 & 2 \end{vmatrix} + 4\begin{vmatrix} 3 & 1 \\ 2 & 4 \end{vmatrix}$$

$$\begin{bmatrix} 3 & 5 & -6 & 4 \\ 0 & -2 & 3 & -3 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 3 \end{bmatrix} = 3 \times (-2) \times 1 \times 3$$

$$\begin{bmatrix} 3 & 0 & 0 & 0 \\ 3 & -2 & 0 & 0 \\ 2 & 6 & 3 & 0 \\ 3 & -8 & 4 & -3 \end{bmatrix} = 3 \times (-2) \times 3 \times (-3)$$

(a) $E_1 \sim$ elementary matrix for a replacement

$$B = E_1 A$$

$$E_1 = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ c & & & 1 \end{pmatrix}$$

(1) $|A| = |B| = |E_1 A| \Rightarrow \underline{|E_1||A| = |E_1 A|}$

$\underline{|E_1| = 1}$

(b) $E_2 \sim$ elementary matrix for an interchange

$$B = E_2 A$$

$$E_2 = \begin{pmatrix} 1 & & & & \\ & 0 & \cdots & 1 & \\ & \vdots & 1 & \vdots & \\ & 1 & \cdots & 0 & \\ & & & & 1 \end{pmatrix}$$

$|B| = (-1)|A| = |E_2||A|$

$|B| = |E_2 A|$

Check $|E_2| = -1$

(c) $E_3 \sim$ elementary matrix for a scaling

$$B = E_3 A$$

$$E_3 = \begin{pmatrix} 1 & & \\ & k & \\ & & 1 \end{pmatrix}$$

$|E_3 A| = |B| = k|A| = |E_3||A|$

$|E_3| = k$

A is invertible, $E_p \cdots E_1 A = I$

$|E_p \cdots E_1 A| = |I| = 1$

$|E_p \cdots E_1 A| = |E_p||E_{p-1}| \cdots |E_1||A|$

product of $1, -1, k$

$\neq 0$

$\boxed{|A| \neq 0}$

0이 될 수 없다.

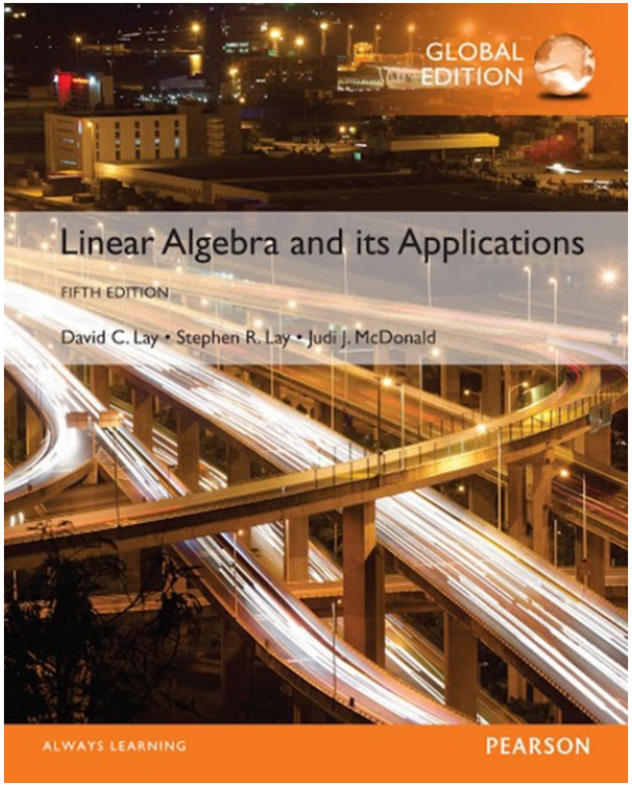
$A \sim$ invertible $\Longleftrightarrow |A| \neq 0$

- A matrix with a $\left\{\begin{array}{l}\text{row}\\\text{column}\end{array}\right\}$ of zeros has determinant 0
- A matrix with two identical $\left\{\begin{array}{l}\text{rows}\\\text{columns}\end{array}\right\}$

3 Determinants

3.2

PROPERTIES OF DETERMINANTS



PROPERTIES OF DETERMINANTS

- **Theorem 3**: Let A be a square matrix
  a) If a multiple of one row of A is added to another row to produce a matrix B, then $\det B = \det A$.
  b) If two rows of A are interchanged to produce B, then $\det B = - \det A$.
  c) If one row of A is multiplied by k to produce B, then $\det B = k \det A$

replacement

interchange

scaling

$k \neq 0$.

$E_p \cdots E_1 A = I$, 이때 $|I| = 1$

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \qquad B = \begin{pmatrix} a & b & c \\ d+2a & e+2b & f+2c \\ g & h & i \end{pmatrix}$$

$$|A| = d\,C_{21}^A + e\,C_{22}^A + f\,C_{23}^A \qquad |B| = (d+2a)\,C_{21}^B + (e+2b)\,C_{22}^B + (f+2c)\,C_{23}^B$$

$$C_{21}^A = \begin{pmatrix} b & c \\ h & i \end{pmatrix} = C_{21}^B, \qquad = d\,C_{21}^A + e\,C_{22}^A + f\,C_{23}^A + \underbrace{2(a\,C_{21}^A + b\,C_{22}^A + c\,C_{23}^A)}$$

$$|D| = \begin{vmatrix} a & b & c \\ a & b & c \\ g & h & i \end{vmatrix}$$

Thm 1: 증명
+
α

<u>Note</u> 1^{st} row expansion of $D = 2^{nd}$ row expansion

<u>Note</u> $\boxed{a\begin{vmatrix} b & c \\ h & i \end{vmatrix} - b\begin{vmatrix} a & c \\ g & i \end{vmatrix} + c\begin{vmatrix} a & b \\ g & h \end{vmatrix}}$

$$= \boxed{-a\begin{vmatrix} b & c \\ h & i \end{vmatrix} + b\begin{vmatrix} a & c \\ g & i \end{vmatrix} - c\begin{vmatrix} a & b \\ g & h \end{vmatrix}}$$

$$|D| = -|D|$$

$$|D| = 0$$

2 identical row → determinant 0

$$A=\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \qquad B=\begin{pmatrix} d & e & f \\ a & b & c \\ g & h & i \end{pmatrix}$$

$$-a\begin{vmatrix} e & f \\ h & i \end{vmatrix} + b\begin{vmatrix} d & f \\ g & i \end{vmatrix} - c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

[illegible] 증명

$$|A| = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

$$C=\begin{pmatrix} 3a & 3b & 3c \\ d & e & f \\ g & h & i \end{pmatrix}$$

$$|C| = 3a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - 3b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + 3c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

ex) $|I| = 1$

$|2I| = 2^3$ ← 3개의 row에 ×2 하므로.

PROPERTIES OF DETERMINANTS

- **Example 1** Compute det A, where $A = \begin{bmatrix} 1 & -4 & 2 \\ -2 & 8 & -9 \\ -1 & 7 & 0 \end{bmatrix}$
- **Solution** The strategy is to reduce A to echelon form and then to use the fact that the determinant of a triangular matrix is the product of the diagonal entries. The first two row replacements in column 1 do not change the determinant:

$$detA = \begin{vmatrix} 1 & -4 & 2 \\ -2 & 8 & -9 \\ -1 & 7 & 0 \end{vmatrix} = \begin{vmatrix} 1 & -4 & 2 \\ 0 & 0 & -5 \\ -1 & 7 & 0 \end{vmatrix} = \begin{vmatrix} 1 & -4 & 2 \\ 0 & 0 & -5 \\ 0 & 3 & 2 \end{vmatrix}$$

still same.

PROPERTIES OF DETERMINANTS

- An interchange of rows 2 and 3 reverses the sign of the determinant, so

$$\det A = -\begin{vmatrix} 1 & -4 & 2 \\ 0 & 3 & 2 \\ 0 & 0 & -5 \end{vmatrix} = -(1)(3)(-5) = 15$$

sign change

PROPERTIES OF DETERMINANTS

- **Theorem 4**: A square matrix A is invertible if and only if $\det A \neq 0$.

$E_p \cdots E_1 A = I$ → ERO 들은 determinant에 $(-1)^k$ 변화만

- **Example 3** Compute det A, where $A = \begin{bmatrix} 3 & -1 & 2 & -5 \\ 0 & 5 & -3 & -6 \\ -6 & 7 & -7 & 4 \\ -5 & -8 & 0 & 9 \end{bmatrix}$

- **Solution** Add 2 times row 1 to row 3 to obtain

$$detA = det\begin{bmatrix} 3 & -1 & 2 & -5 \\ 0 & 5 & -3 & -6 \\ -6 & 7 & -7 & 4 \\ -5 & -8 & 0 & 9 \end{bmatrix} = 0$$

(identical → 0)

0 4 -3 -6

because the second and third rows of the second matrix are equal.

singular → not invertable

COLUMN OPERATIONS

- **Theorem 5:** If A is a $n \times n$ matrix, then $\det A^T = \det A$.
- **Proof**: The theorem is obvious for $n = 1$. Suppose the theorem is true for $k \times k$ determinants and let $n = k + 1$.
- Then the cofactor of a_{1j} in A equals the cofactor of a_{j1} in A^T, because the cofactors involve $k \times k$ determinants.
- Hence the cofactor expansion of det A along the first row equals the cofactor expansion of det A^T down the first column. That is, A and A^T have equal determinants.
- Thus the theorem is true for $n = 1$, and the truth of the theorem for one value of n implies its truth for the next value of n. By the principle of induction, the theorem is true for all $n \geq 1$.

DETERMINANTS AND MATRIX PRODUCTS

- **Theorem 6:** If A and B are $n \times n$ matrices, then $\det AB = (\det A)(\det B)$.

- **Example 5** Verify Theorem 6 for $A = \begin{bmatrix} 6 & 1 \\ 3 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}$.
- Solution

$$AB = \begin{bmatrix} 6 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 25 & 20 \\ 14 & 13 \end{bmatrix}$$

- and

$$\det AB = 25 \cdot 13 - 20 \cdot 14 = 325 - 280 = 45$$

Since det A = 9 and det B = 5,

$$(\det A)(\det B) = 9 \cdot 5 = 45 = \det AB$$

~~A or B is Not Invertible $\Longleftrightarrow$ AB is not Invertible~~

$$|A| = 0 \text{ or } |B| = 0 \quad \Longleftrightarrow \quad |AB| = 0$$

If both A and B are invertible

then $A = E_1 E_2 \cdots E_p$ for some elementary matrices $E_1 \cdots E_p$

$$
\begin{aligned}
|AB| &= |E_1 \cdots E_p B| \\
&= |E_1|\,|E_2 \cdots E_p B| \quad \text{by } (*) \\
&= |E_1|\,|E_2|\,|E_3 \cdots E_p B| \\
&\;\;\vdots \\
&= \underbrace{|E_1|\,|E_2| \cdots |E_p|}\,|B| \\
&\;\;\vdots \\
&= |E_1 \cdots E_p|\,|B| = |A|\,|B|
\end{aligned}
$$

ex) $AA^{-1} = I$

$$\left.\begin{aligned} &|AA^{-1}| = |I| = 1 \\ &\;\;\;\| \\ &|A|\,|A^{-1}| \end{aligned}\right\} \quad |A^{-1}| = \frac{1}{|A|}$$

ex) For an $n \times n$ A, if $A = LU$ and A is inve

then $|A| = |LU| = |L|\,|U| = |U| =$ ~~product of pivots~~

↳ A가 invertable 는 pivot 모두 1. ($A_{reduced} = I$)

5. $\begin{vmatrix} 1 & 5 & -4 \\ -1 & -4 & 5 \\ -2 & -8 & 7 \end{vmatrix}$

7. $\begin{vmatrix} 1 & 3 & 0 & 2 \\ -2 & -5 & 7 & 4 \\ 3 & 5 & 2 & 1 \\ 1 & -1 & 2 & -3 \end{vmatrix}$

5.

$$\begin{vmatrix} 1 & 5 & -4 \\ 0 & 1 & 1 \\ 0 & 2 & -1 \end{vmatrix} \sim \begin{vmatrix} 1 & 5 & -4 \\ 0 & 1 & 1 \\ 0 & 0 & -3 \end{vmatrix} = (1)(1)(-3) = -3$$

7

[illegible]

Use determinants to find out if the matrix is invertible.
the matrix is invertible.

21. $\begin{bmatrix} 2 & 6 & 0 \\ 1 & 3 & 2 \\ 3 & 9 & 2 \end{bmatrix}$

21. $\begin{vmatrix} 2 & 6 & 0 \\ 1 & 3 & 2 \\ 3 & 9 & 2 \end{vmatrix} = -\begin{vmatrix} 1 & 3 & 2 \\ 2 & 6 & 0 \\ 3 & 9 & 2 \end{vmatrix} = -\begin{vmatrix} 1 & 3 & 2 \\ 0 & 0 & -4 \\ 0 & 0 & -4 \end{vmatrix} = 0$

Not Invertable

• Use determinants to decide if the set of vectors is linearly independent.

24. $\begin{bmatrix} 4 \\ 6 \\ 2 \end{bmatrix}, \begin{bmatrix} -7 \\ 0 \\ 7 \end{bmatrix}, \begin{bmatrix} -3 \\ -5 \\ -2 \end{bmatrix}$

24

$$\begin{vmatrix} 4 & -7 & -3 \\ 6 & 0 & -5 \\ 2 & 7 & -2 \end{vmatrix} = -\begin{vmatrix} 2 & 7 & -2 \\ 6 & 0 & -5 \\ 4 & -7 & -3 \end{vmatrix} = -\begin{vmatrix} 2 & 7 & -2 \\ 0 & -21 & 1 \\ 0 & -21 & 1 \end{vmatrix} = 0 \rightarrow \text{Not Invertable} \rightarrow \text{Columns are not Linearly Independent}$$

ERO

b. $3\begin{vmatrix} 1 & 1 & -1 \\ 3 & 4 & -4 \\ 2 & -3 & -5 \end{vmatrix} = 3\begin{vmatrix} 1 & 1 & -1 \\ 0 & 1 & -1 \\ 0 & -5 & -3 \end{vmatrix} = 3\begin{vmatrix} 1 & 1 & -1 \\ & 1 & -1 \\ & & -8 \end{vmatrix} = 3 \times 1 \times 1(-8)$

p. $\begin{vmatrix} 1 & 3 & 2 & -4 \\ 0 & 1 & 2 & -5 \\ 0 & 1 & 2 & 5 \\ 0 & -1 & -1 & -10 \end{vmatrix} = -\begin{vmatrix} 1 & 3 & 2 & -4 \\ & 1 & 2 & -5 \\ & & 1 & -5 \\ & & & 10 \end{vmatrix} = -10$

22. $\begin{vmatrix} 5 & 1 & -1 \\ 1 & -3 & -2 \\ 0 & 5 & 3 \end{vmatrix} = 5\begin{vmatrix} -3 & -2 \\ 5 & 3 \end{vmatrix} - 1\begin{vmatrix} 1 & -1 \\ 5 & 3 \end{vmatrix} \rightarrow$ Invertable

$= -3 \neq 0$

25. $\begin{vmatrix} 7 & -8 & 7 \\ -4 & 5 & 0 \\ -6 & 7 & -5 \end{vmatrix} = 7\begin{vmatrix} -4 & 5 \\ -6 & 7 \end{vmatrix} - 5\begin{vmatrix} 7 & -8 \\ -4 & 5 \end{vmatrix}$

$= -1 \neq 0$

Quiz 2.2 ~ 3.2