Outline

Inviscid Stability of Nearly Parallel Flows: Rayleigh's Equation

When Re $\to \infty$, the Orr-Sommerfeld equation simplify to the **Rayleigh's equation**:

$$\frac{d^2\phi}{dy^2} - \left(\alpha^2 + \frac{1}{U-c}\frac{d^2U}{dy^2}\right)\phi = 0$$

The problem is simpified from a fourth-order ODE to a second-order ODE, and instead of requiring four B.C.s, we only require two. In fact, Rayleigh equation describes the behavior of the marginal stability curve $c_1(\alpha, \text{Re})$ as Re $\to \infty$.

Inviscid Stability of Nearly Parallel Flows: Some Theorems

There are two theorems based on inviscid stability that are useful to make predictions about stability.

Theorem (Rayleigh's Point-of-Inflection Theorem)

A necessary condition for inviscid instability is that the basic profile $U(y)$ has a point of inflection.

Theorem (Fjørtoft's Theorem)

If y_0 is the position of a point of inflection, then a necessary condition for inviscid instability is that $\frac{d^2U}{dy^2} < 0$ somewhere in the flow.

Inviscid Stability of Nearly Parallel Flows: Rayleigh Point-of-Inflection Theorem

We start with the Rayleigh Equation, multiplying it by the conjugate :

$$\int_{y_1}^{y_2} \left[\frac{d^2\phi}{dy^2} - \left(\alpha^2 + \frac{1}{U-c} \frac{d^2 U}{dy^2} \right) \phi = 0 \right] \phi^* dy$$

Consider the first term only, then:

$$\int_{y_1}^{y_2} \phi^* \frac{d^2\phi}{dy^2} dy = \underbrace{\left[\phi^* \frac{d\phi}{dy} \right]_{y_1}^{y_2}}_{=0} - \int_{y_1}^{y_2} \frac{d\phi^*}{dy} \frac{d\phi}{dy} dy = - \int_{y_1}^{y_2} \left| \frac{d\phi}{dy} \right|^2 dy$$

Hence, we obtain that:

$$\underbrace{\int_{y_1}^{y_2} \left(\left| \frac{d\phi}{dy} \right|^2 + \alpha^2 |\phi|^2 \right) dy}_{\text{Real Term}} + \underbrace{\int_{y_1}^{y_2} \frac{1}{U-c} \frac{d^2 U}{dy^2} |\phi|^2 dy}_{\text{Complex Term}} = 0$$

Inviscid Stability of Nearly Parallel Flows: Rayleigh Point-of-Inflection Theorem

The complex term can be decomposed into its real and imaginary part easily:

$$\int_{y_1}^{y_2} \frac{1}{U-c}\frac{d^2U}{dy^2}|\phi|^2 dy = \int_{y_1}^{y_2} \frac{U-c_R}{|U-c|^2}\frac{d^2U}{dy^2}|\phi|^2 dy + ic_I \int_{y_1}^{y_2} \frac{1}{|U-c|^2}\frac{d^2U}{dy^2}|\phi|^2 dy$$

This implies that:

$$ic_I \int_{y_1}^{y_2} \frac{1}{|U-c|^2}\frac{d^2U}{dy^2}|\phi|^2 dy = 0$$

which can only occur if $\frac{d^2U}{dy^2}$ changes sign at least once.

Theorem (Rayleigh's Point-of-Inflection Theorem)

A necessary condition for inviscid instability is that the basic profile $U(y)$ has a point of inflection.

Inviscid Stability of Nearly Parallel Flows: Fjørtoft Theorem

A much more stronger condition is that due to Fjørtoft. We work with the real part of:

$$\int_{y_1}^{y_2} \underbrace{\left(\left| \frac{d\phi}{dy} \right|^2 + \alpha^2 |\phi|^2 \right)}_{\text{Real Term}} dy + \int_{y_1}^{y_2} \underbrace{\frac{1}{U - c} \frac{d^2 U}{dy^2}}_{\text{Complex Term}} |\phi|^2 dy = 0$$

To obtain the following inequality:

$$\Rightarrow \int_{y_1}^{y_2} \frac{U - c_R}{|U - c|^2} \frac{d^2 U}{dy^2} |\phi|^2 dy = - \int_{y_1}^{y_2} \underbrace{\left(\left| \frac{d\phi}{dy} \right|^2 + \alpha^2 |\phi|^2 \right)}_{\text{Real Term}} dy < 0$$

From the previous theorem, we had that:

$$ic_I \int_{y_1}^{y_2} \frac{1}{|U - c|^2} \frac{d^2 U}{dy^2} |\phi|^2 dy = \text{constant} \int_{y_1}^{y_2} \frac{1}{|U - c|^2} \frac{d^2 U}{dy^2} |\phi|^2 dy = 0$$

Inviscid Stability of Nearly Parallel Flows: Fjørtoft Theorem

We add the two previous equations to have:

$$\int_{y_1}^{y_2} \frac{U - c_R + \text{constant}}{|U-c|^2} \frac{d^2U}{dy^2} |\phi|^2 dy < 0$$

If we choose the constant to be $c_\mathrm{R} - U_1$:

$$\int_{y_1}^{y_2} \frac{U - U_1}{|U-c|^2} \frac{d^2U}{dy^2} |\phi|^2 dy < 0$$

This implies atleast that $\frac{d^2U}{dy^2} < 0$ $((U - U_1)\frac{d^2U}{dy^2} < 0)$ somewhere in the flow.

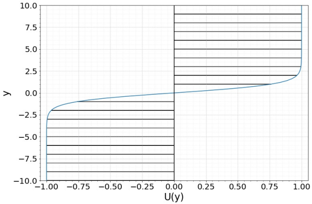
Theorem (Fjørtoft's Theorem)

If y_0 is the position of a point of inflection, then a necessary condition for inviscid instability is that $\frac{d^2U}{dy^2} < 0$ $((U - U_1)\frac{d^2U}{dy^2} < 0)$ somewhere in the flow.

The Inviscid Shear Layer

Although in class we derived the Orr-Sommerfield equation for the case of channel flow, for learning purposes it proves more convenient to start with the shear layer flow. Therefore, the velocity profile will be:

$$U = U_0 \tanh \frac{y}{H}$$



where U_0 is the free-stream speed and H is a measure of the thickness of the layer.

The Inviscid Shear Layer

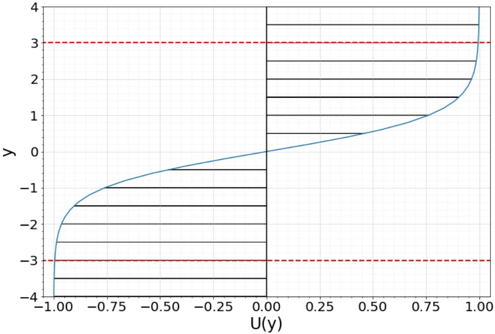
From the previous discussion, we had the boundary conditions:

- Duct Flows: $v(\pm h) = v'(\pm h) = 0$
- Boundary Layers: $v(0) = v'(0) = 0 \qquad v(\infty) = v'(\infty) = 0$
- Free-Shear Layers: $v(\pm\infty) = v'(\pm\infty) = 0$

We have to drop two boundary conditions due to only having a second order ODE. Perhaps it's worth noting now that the boundary conditions are located infinitely far away, thus perhaps leading to the possibility of doing a coordinate transform. Instead, we can truncate the domain from $(-\infty, +\infty)$ to $[-L, L]$, where L is an appropriate finite positive number.

The Inviscid Shear Layer

Perhaps an appropriate truncation length L would be $L = 3$. It can be noted that for $|y| > 3$, the velocity is close to constant, and it could be of use to define appropriate initial guess for the eigenvalue problem.



The Inviscid Shear Layer

The original Rayleigh equation:

$$\frac{d^2\phi}{dy^2} - \left(\alpha^2 + \frac{1}{U-c}\frac{d^2U}{dy^2}\right)\phi = 0$$

is better rewrote as:

$$\phi = \int \phi' \, dy$$

$$\phi' = \int \phi'' \, dy = \int \left(\alpha^2 + \frac{1}{U-c}\frac{d^2U}{dy^2}\right)\phi dy$$

for the numerical solution of the problem. For the integration of the problem we use the classical fourth-order Runge-Kutta with constant step-size.

```
q[0,0] = 1.; q[1,0] = α
for jj in range(N-1):
    q1 = dy * rhs(U0[jj], U0pp[jj], α, c, A, q[:,jj])
    q2 = dy * rhs(U0[jj], U0pp[jj], α, c, A, q[:,jj] + q1/2 )
    q3 = dy * rhs(U0[jj], U0pp[jj], α, c, A, q[:,jj] + q2/2 )
    q4 = dy * rhs(U0[jj], U0pp[jj], α, c, A, q[:,jj] + q3 )
    q[:, jj+1] = q[:, jj] + (1/6) * ( q1 + 2*q2 + 2*q3 + q4 )
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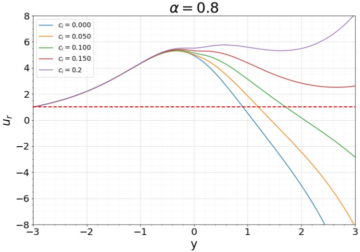
The Inviscid Shear Layer

Lets consider first the numerical integration of the Rayleigh equation for $\alpha = 0.8$, where the only free-parameter is c_i:



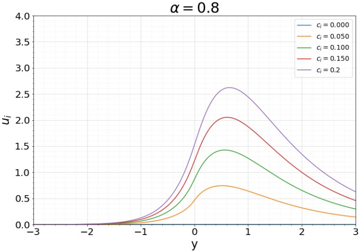


Figure: Solutions obtained for $\alpha = 0.8$

From the left-side figure, we can be lead to think that the solution c_i must lie in the region $(0.100, 0.150)$, which is actually true.

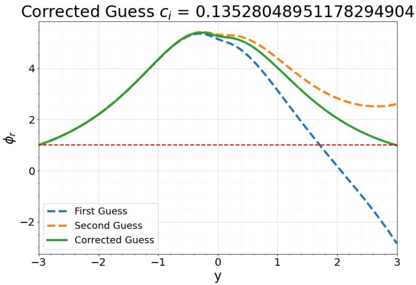
The Inviscid Shear Layer

If two guesses for c_i, i.e., $c_{i,1} = 0.1$ and $c_{i,2} = 0.15$, were to be used, two different solutions ϕ_1 and ϕ_2 could be obtained from the numerical integration. The parameter m defined as:

$$m = \frac{c_{i,1} - c_{i,2}}{\phi_1(L) - \phi_2(L)}$$

Then a new guess $c_{i,3}$ can be obtained from:

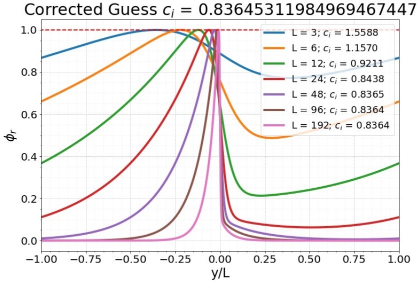
$$c_{i,3} = c_{i,2} + m(\phi(L) - \phi_2(L))$$



This process can be iterated as:

The Inviscid Shear Layer

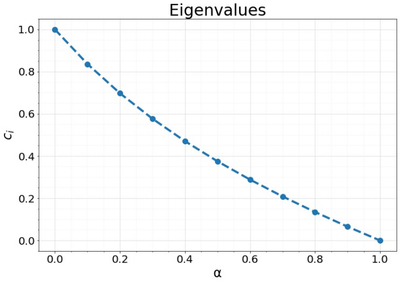
One last thing before finishing the presentation. The truncation length of the domain L was on the previous problem chosen to be $L = 3.0$, but say that for a very small wavenumber α, to find it's c_i that satisfies the eigenvalue problem.



The results above are after 8 iterations of search, and normalized with the maximum magnitude of the numerical solution.

The Inviscid Shear Layer

Finally, the eigenvalues of the problem are:



Note: the eigenvalue corresponding to $\alpha = 0$ wasn't obtained from the program, but from the reference.