The Bernoulli equation in a moving reference frame

The Bernoulli equation in a moving reference frame

Carl E Mungan

Physics Department, US Naval Academy Annapolis, MD 21402-1363, USA

E-mail: mungan@usna.edu

Abstract
Unlike other standard equations in introductory classical mechanics, the Bernoulli equation is not Galilean invariant. The explanation is that, in a reference frame moving with respect to constrictions or obstacles, those surfaces do work on the fluid, constituting an extra term that needs to be included in the work–energy calculation. A quantitative example is presented here for a horizontal tapered pipe. A frame-independent expression for the pressure drop in the pipe is obtained. The concepts discussed in this paper are accessible to introductory undergraduate physics majors.

(Some figures in this article are in colour only in the electronic version)

1. Introduction

Suppose water is flowing from a horizontal pipe of large diameter to the one of small diameter. The speed of the water consequently increases, because the same volume of water passes through the larger pipe's cross section per second as through the smaller pipe's cross section. The Bernoulli equation then implies that the pressure decreases in the narrower pipe. Water is pushed from the higher pressure in the wide pipe towards the lower pressure in the narrow pipe and hence its speed increases. But now view the situation from a reference frame moving at the velocity of the water in the smaller diameter pipe. In that frame, the water in the narrow pipe is at rest, while the water in the wide pipe has a nonzero velocity (and is reversed in direction). However, pressure is a Galilean invariant and must therefore be the same in either frame of reference (since neither force nor area varies from one inertial frame to another). So we now have the paradoxical situation that the speed of the water is *larger* in the wide pipe where the pressure is *higher* [1]. How can that be?
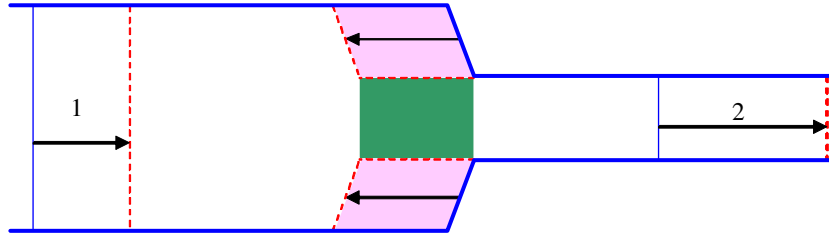
Figure 1. A pipe necks down from a large diameter in region 1 to a smaller diameter in region 2. In a frame at rest relative to the pipe, fluid flows from left to right.

2. Resolution of the paradox

Consider a straight cylindrical pipe (sketched with bold lines in figure 1) that has uniform cross-sectional area A_1 in region 1 and area A_2 in region 2. Between them there is a tapered neck connecting one diameter to the other.

In the pipe's frame of reference, an incompressible, inviscid fluid of density ρ flows steadily and irrotationally rightwards through the pipe. (Extensions to viscous, compressible, rotational and unsteady flows are possible [2, 3].) According to the equation of continuity, the speeds of the fluid in the two regions are related by

$$A_1 v_1 = A_2 v_2. \tag{1}$$

Now consider the situation from the point of view of a reference frame F moving to the right at speed v_F. In that frame, the fluid in region 1 has speed $v_1 - v_F$ to the right, the taper has speed v_F to the left, and the fluid in region 2 has speed $v_2 - v_F$ to the right, as indicated by the arrows in figure 1. Focus on the volume of the fluid between the two vertical solid lines. A time Δt later, the ends of that same volume have moved rightwards to the two vertical dashed lines and the taper has moved leftwards to the dashed positions. Work W done on the volume is equal to

$$W = P_1 A_1 (v_1 - v_F) \Delta t + P_{\mathrm{ave}} (A_1 - A_2) v_F \Delta t - P_2 A_2 (v_2 - v_F) \Delta t. \tag{2}$$

The first and third terms on the right-hand side are the work done by the fluid pressures P_1 and P_2 at the two ends marked 1 and 2 in figure 1. The middle term is the work done as the taper (of cross-sectional area $A_1 - A_2$) compresses the fluid [4]. In the appendix, it is shown that the static pressure [5] has an average value along the wall of the taper given by

$$P_{\mathrm{ave}} = \frac{P_1 A_1 + P_2 A_2}{A_1 + A_2} = \frac{P_1 v_2 + P_2 v_1}{v_1 + v_2} \tag{3}$$

using equation (1) to obtain the second equality. Note that $P_{\mathrm{ave}} \approx P_1$ if A_1 is much larger than A_2. (Already, P_2 is smaller than P_1 and weighting it by the areas makes its contribution even smaller.)

The net work is equal to the change in the kinetic energy (KE) of the fluid within the volume. The fluid gains the KE K_{f} equal to

$$K_{\mathrm{f}} = \tfrac{1}{2} \rho A_2 (v_2 - v_F) \Delta t (v_2 - v_F)^2 + \tfrac{1}{2} \rho A_2 v_F \Delta t (v_2 - v_F)^2 \tag{4}$$

where the first term on the right-hand side is the KE of the fluid inside the region marked 2 in figure 1, and the second term is the final KE of the fluid inside the dark-coloured cylinder near the taper. Similarly, the fluid lost the KE K_{i}:

$$K_{\mathrm{i}} = \tfrac{1}{2} \rho A_1 (v_1 - v_F) \Delta t (v_1 - v_F)^2 + \tfrac{1}{2} \rho A_1 v_F \Delta t (v_1 - v_F)^2 \tag{5}$$

where the first term on the right-hand side is the KE of the fluid inside the region marked 1, and the second term is the initial KE of the fluid both inside the dark-coloured cylinder and inside the light-coloured parallelogram-shaped annular region next to the taper. Here, it has been assumed that the taper is steep enough and Δt is long enough that we can neglect the small volume of the fluid within the cross section of the taper itself (which has a speed intermediate between v_1 and v_2). Then, dividing $W = K_f - K_i$ by $A_1 v_1 \Delta t = A_2 v_2 \Delta t$ and simplifying the result gives

$$(P_1 - P_2)\left[1 - \frac{2v_F}{v_1 + v_2}\right] = \frac{\rho}{2}\left(v_2^2 - v_1^2\right)\left[1 - \frac{2v_F}{v_1 + v_2}\right]. \tag{6}$$

Cancelling the term in square brackets on both sides of this equation, one recovers the standard Bernoulli equation [6–8] for horizontal flow, which thus only holds for fluid speeds measured in the rest frame of the pipe. In particular, defining $\tilde{v}_1 \equiv v_1 - v_F$ and $\tilde{v}_2 \equiv v_2 - v_F$, one cannot write

$$P_1 + \tfrac{1}{2}\rho\tilde{v}_1^2 = P_2 + \tfrac{1}{2}\rho\tilde{v}_2^2 \tag{7}$$

if $v_F \neq 0$. The pressure drop $\Delta P \equiv P_1 - P_2$ is given not by $\rho(\tilde{v}_2^2 - \tilde{v}_1^2)/2$ but instead by

$$\Delta P = \frac{\rho}{2}(\tilde{v}_2 - \tilde{v}_1)^2 \frac{A_1 + A_2}{A_1 - A_2}. \tag{8}$$

Note that $\tilde{v}_2 - \tilde{v}_1 = v_2 - v_1$ so that equation (8) is properly frame independent, unlike (7). But if $v_F = 0$, equation (1) can be used to show that (8) can be rewritten as (7).

3. Summary

In a frame in which a tapered pipe moves, the taper does work on the fluid moving through it. In the standard energy derivation of the Bernoulli equation, the pressure term represents work per unit volume on the fluid. In a frame moving with respect to obstacles or constrictions that the fluid passes over or through, the work they do must be included in the calculation of the net work to which the change in KE is equated. Quantitative agreement then results, as demonstrated here, and a Galilean-invariant expression for the pressure drop along a horizontal tapered pipe can be obtained.

Acknowledgments

Appendix. Derivation of equation (3)

Consider an annular area element $\mathrm{d}A$ of the tapered region located at radius r relative to the centre of the pipe as sketched in figure A1.

The differential bit of work done by this area element as it moves horizontally leftwards a distance $\Delta x = v_F \Delta t$ is

$$\mathrm{d}W = P(r)\,\mathrm{d}A\,\Delta x \cos\theta \tag{A.1}$$

where $\mathrm{d}A$ is the left-hand surface area of the small rectangle in figure A1 revolved around the axis of the pipe. But $\mathrm{d}A \cos\theta$ is just the vertical projection of that area and is thus equal to $2\pi r\,\mathrm{d}r$. Equation (A.1) now becomes

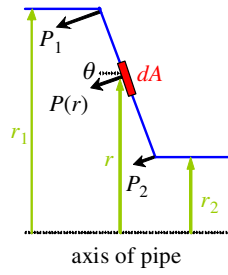$$\mathrm{d}W = P(r)\,2\pi r\,\mathrm{d}r\,v_F \Delta t. \tag{A.2}$$

Figure A1. Close-up view of a small bit of area on the taper of the pipe and of the pressures at different radii along the taper.

In the pipe's rest frame, the Bernoulli equation is

$$P(r) + \tfrac{1}{2}\rho v^2(r) = P_1 + \tfrac{1}{2}\rho v_1^2 \tag{A.3}$$

which at $r = r_2$ can be rearranged as

$$\frac{1}{2}\rho = \frac{P_1 - P_2}{v_2^2 - v_1^2}. \tag{A.4}$$

Substituting equation (A.4) into (A.3) leads to

$$P(r) = P_1 + \frac{P_1 - P_2}{v_2^2 - v_1^2}\left[v_1^2 - v^2(r)\right]. \tag{A.5}$$

But the equation of continuity implies that

$$A_1 v_1 = \pi r^2 v(r). \tag{A.6}$$

Solving equation (A.6) for $v(r)$ and substituting it into (A.5) and then, that equation into (A.2) gives

$$\mathrm{d}W = \left[P_1 + \frac{P_1 - P_2}{A_1^2/A_2^2 - 1}\left(1 - \frac{A_1^2}{\pi^2 r^4}\right)\right] 2\pi r\, \mathrm{d}r\, v_F \Delta t \tag{A.7}$$

where equation (1) was used to eliminate v_2 in favour of v_1. The right-hand side of equation (A.7) is integrated from r_2 to r_1 to get the work W_{taper} done on the fluid by the taper. But that work defines the middle term in equation (2) so that

$$P_{\mathrm{ave}} = \int_{r_2}^{r_1} \left[P_1 + \frac{P_1 - P_2}{A_1^2/A_2^2 - 1}\left(1 - \frac{A_1^2}{\pi^2 r^4}\right)\right] \frac{2\pi r\, \mathrm{d}r}{A_1 - A_2}. \tag{A.8}$$

Performing the integral and simplifying the result gives equation (3), as desired.

References

[1] Bedford D and Lindsay R 1977 A misinterpretation of Bernoulli's theorem *Phys. Educ.* **12** 311–2
[2] Synolakis C E and Badeer H S 1989 On combining the Bernoulli and Poiseuille equation—a plea to authors of college physics texts *Am. J. Phys.* **57** 1013–9
[3] Bauman R P and Schwaneberg R 1994 Interpretation of Bernoulli's equation *Phys. Teach.* **32** 478–88
[4] Sleator W W 1949 Check and proofs of the Bernoulli equation *Am. J. Phys.* **17** 110–3
[5] Kamela M 2007 Thinking about Bernoulli *Phys. Teach.* **45** 379–81
[6] Van Lear G A 1934 Pressure energy—a misconception *Am. J. Phys.* **2** 99–100
[7] Bottom V E 1941 The Bernoulli equation *Am. J. Phys.* **9** 190
[8] Bauman R P 2000 An alternative derivation of Bernoulli's principle *Am. J. Phys.* **68** 288–9