Example 11. *Using Runge 's Formula (third order), solve the differential equation* $\frac{dy}{dx} = x - y$ *subject to y = 1 when x = 1.*

Solution. $f(x, y) = x - y$

Here $h = 0.1,\ x_0 = 1, y_0 = 1$

$k_1 = hf(x_0, y_0) = 0.1\ (x - y) = 0.1\ (1 - 1) = 0$

$k_2 = hf(x_0 + h, y_0 + k_1) = 0.1f\ (1.1., 1 + 0) = 0.1(1.1-1) = 0.01$

$k_3 = h f(x_0 + h, y_0 + k_2) = 0.1f(1.1,1.01) = 0.1\ (1.1-1.01) = 0.009$

$$k_4 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = 0.1f\left(1.05, 1 + \frac{0}{2}\right) = 0.1(1.05 - 1) = 0.005$$

$$y_1 = y + \frac{1}{6}(k_1 + 4k_4 + k_3)$$

$$y(0.1) = 1 + \frac{1}{6}(0 + 0.02 + 0.009) = 1 + 0.004833 = 1.004833$$ **Ans.**

52.9 RUNGE-KUTTA FORMULA (FOURTH ORDER)

A fourth order Runge's-Kutta Formula for solving the differential equation is

$$y = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

where
$$k_1 = hf(x_0, y_0), \quad k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$k_3 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}), \quad k_4 = hf\left(x_0 + h, y_0 + k_3\right)$$

$$y = y_0 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$$

This is known as Runge-Kutta Formula. The error in this formula is of the order h^5. This method have greater accuracy. No deviatives are required to be tabulated.

It requires only functional values at some selected points on the sub interval.

Example 12. *Apply Runge-Kutta method to find an approximate value of y when x = 0.2, given that*

$$\frac{dy}{dx} = x + y,\ y = 1 \text{ when } x = 0$$

Solution. Let $h = 0.1$

Here $x_0 = 0,\ y_0 = 1, f(x, y) = x + y$

Now $k_1 = h f(x_0, y_0) = 0.1\ (0 + 1) = 0.1$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = 01\,f(0 + 0.05, 1 + 0.05) = 0.1[0.05 + 1.05] = 0.11$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = 01\,f(0 + 0.05, 1 + 0.055) = 0.1(0.05 + 1.055] = 0.1105$$

$k_4 = hf(x_0 + h, y_0 + k_3) = 0.1f(0 + 0.1,\ 1 + 0.1105)$

$$= 0.1f(0.1, 1.1105) = 0.1(0.1 + 1.1105) = 0.12105$$

According to Runge-Kutta (Fourth order) formula

$$y = y_0 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$$

$$y_{0.1} = 1 + \frac{1}{6}(0.1 + 0.22 + 0.221 + 0.12105) = 1 + \frac{1}{6}(0.66205) = 1.11034$$

For the second step

$$x_0 = 0.1, y_0 = 1.11034, h = 0.1$$

$$k_1 = hf(x_0, y_0) = 0.1(0.1 + 1.11034) = 0.121034$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = 0.1f(0.1 + 0.05, 1.11034 + 0.060517)$$

$$= 0.1\,(0.15 + 1.170857) = 0.1320857$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = 0.1f(0.1 + 0.05, 1.11034 + 0.0660428)$$

$$= 0.1\,(0.15 + 1.1763828) = 0.13263828$$

$$k_4 = hf(x_0 + h, y_0 + k_3) = 0.1\,(0.1 + 0.1, 1.11034 + 0.13263828)$$

$$= 0.1\,(0.2 + 1.24297828) = 0.144297828$$

$$y_1 = y_0 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$$

$$= 1.11034 + \frac{1}{6}[0.121034 + 2 \times 0.1320857 + 2 \times 0.13263828 + 0.144297828]$$

$$= 1.11034 + \frac{1}{6}[0.121034 + 0.2641714 + 0.26527656 + 0.144297828]$$

$$= 1.11034 + \frac{1}{6} \times 0.794779788 = 1.11034 + 0.132463298 = 1.242803298$$ **Ans.**

Example 13. *Apply Range-Kutta method of fourth order to solve :*

$$10\frac{dy}{dx} = x^2 + y^2; \quad y(0) = 1 \text{ for } x = 1.$$ *(R. G.P.V. Bhopal, III Semester, Dec. 2002)*

Solution. We have,

$$10\frac{dy}{dx} = x^2 + y^2 \quad \Rightarrow \quad \frac{dy}{dx} = \frac{x^2 + y^2}{10}$$

$$\Rightarrow \quad f(x, y) = \frac{x^2 + y^2}{10}$$

Here, let $h = 0.1$, $x_0 = 0$, $y_0 = 1$.

Now, $k_1 = h\,f(x_0, y_0) = (0.1)\,f(0,1) = (0.1)\left(\frac{0+1}{10}\right) = 0.01$

$$k_2 = h\,f\left(x_0 + \frac{h}{2}, \; y_0 + \frac{1}{2}k_1\right) = (0.1)f\left(0 + \frac{0.1}{2}, \; 1 + \frac{0.01}{2}\right)$$

$$= (0.1)f(0.05, \; 1.005) = (0.1)\left[\frac{(0.05)^2 + (1.005)^2}{10}\right] = 0.01012525$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, \; y_0 + \frac{k_2}{2}\right) = (0.1)f\left(0.05, \; 1 + \frac{0.01012525}{2}\right)$$

$$= (0.1)f(0.05, \; 1.00506263) = (0.1)\left[\frac{(0.05)^2 + (1.00506263)^2}{10}\right] = 0.01012651$$

$$k_4 = hf(x_0 + h,\ \ y_0 + k_3) = (0.1)\,f(0.1,\ \ 1 + 0.01012651)$$

$$= (0.1)\,f(0.1,\ \ 1.01012651)\ \ = (0.1)\left[\frac{(0.1)^2 + (1.01012651)^2}{10}\right] = 0.010303556$$

$$y_{0.1} = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= 1 + \frac{1}{6}[0.01 + 2(0.01012525) + 2(0.01012651) + 0.010303556]$$

$$= 1 + 0.01013451 = 1.01013451$$

Hence, y at $x = 0.1$ is 1.01013451. **Ans.**

Example 14. *Apply Runge-Kutta method (fourth order), to find an approximate value of y when x = 0.2, given that* $\frac{dy}{dx} = x + y^2$ *and y = 1 when x = 0.*

(RGPV., Bhopal, III Sem. Dec. 2004, AMIETE, Dec. 2010)

Solution. Let $h = 0.1$,

Here $\quad x_0 = 0, y_0 = 1,\ f(x, y) = x + y^2$

Now $\quad k_1 = hf(x_0, y_0) = 0.1\ (0 + 1) = 0.1$

$$k_2 = hf\left(x_0 + \frac{h}{2},\ y_0 + \frac{k_1}{2}\right) = 0.1 f(0 + 0.05,\ 1 + 0.05) = 0.1[0.05 + (1.05)^2] = 0.11525$$

$$k_3 = hf\left(x_0 + \frac{h}{2},\ y_0 + \frac{k_2}{2}\right) = 0.1 f(0 + 0.05,\ 1 + 0.057625)$$

$$= 0.1[0.05 + (1.057625)^2] = 0.11685$$

$$k_4 = hf(x_0 + h, y_0 + k_3) = 0.1f(0 + 0.1,\ 1 + 0.11685) = 0.1\ [0.1 + (0.11685)^2] = 0.13474$$

According to Runge-Kutta (fourth order) formula

$$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$y_{0.1} = 1 + \frac{1}{6}[0.1 + 2(.11525) + 2(0.11685) + 0.13474]$$

$$y_{0.1} = 1 + 0.1165 = 1.1165$$

For the second step $\quad x_0 = 0.1, y_0 = 1.1165$

$$k_1 = 0.1\ (0.1 + 1.2466) = 0.1347$$

$$k_2 = 0.1\ (0.15 + 1.4014) = 0.1551$$

$$k_3 = 0.1\ (0.15 + 1.4259) = 0.1576$$

$$k_4 = 0.1\ (0.2 + 1.6233) = 0.1823$$

$$y_{0.2} = y_{0.1} + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= 1.1165 + \frac{1}{6}[0.1347 + 2(0.1551) + 2(0.1576) + 0.1823]$$

$$= 1.1165 + 0.1571 = 1.2736$$

Ans.

Example 15. *Use the fourth order Runge-Kutta method to find u (0, 2), of the initial value problem* $u' = -2\,t\,u^2$, $u\,(0) = 1$, *using* $h = 0.2$. *(U.P. III Sem., Dec. 2009)*

Solution. $h = 0.2$

Here $t = 0$, $u = 1$, $f(t, u) = -2\,tu^2$

$$k_1 = hf(t_0, u_0) = 0.2\,(-2\,tu^2) = 0.2\,(0) = 0$$

$$k_2 = hf\left(t_0 + \frac{h}{2}, u_0 + \frac{k_1}{2}\right)$$

$$= 0.2 f(0.1, 1 + 0) = 0.2\, f(0.1, 1) = 0.2\,(-2 \times 0.1 \times 1^2) = -0.04$$

$$k_3 = hf\left(t_0 + \frac{h}{2}, u_0 + \frac{k_2}{2}\right) = 0.2 f(0 + 0.1, 1 - 0.02) = 0.2 f(0.1, 0.98)$$

$$= 0.2 f[-2 \times 0.1 \times (0.98)^2] = -0.2[0.2 \times 0.9604] = -\,0.038416$$

$$k_4 = hf(t_0 + h, u_0 + k_3) \quad = 0.2\, f(0.2, 1 - 0.038416) = 0.2(-2) \times (0.2) \times (0.961584)^2$$

$$= -\,0.08 \times 0.9246 = -\,0.073971503$$

$$u = u_0 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4] = 1 + \frac{1}{6}[0 + 2(-0.04) + 2(-0.038416) + (-0.073968)]$$

$$= 1 - \frac{1}{6}|0.08 + 0.076832 + 0.07391503|$$

$$= 1 - \frac{1}{6}(0.2308) = 1 - 0.03847 = 0.961532749 \qquad \textbf{Ans.}$$

EXERCISE 52.6

1. The initial value problem $y' = x(y + x) - 2$, $y(1) = 2$ is given. Find the value of $y(1.2)$ with $h = 0.2$ using the Runge-Kutta method of fourth order. **Ans.** $y(1.2) = 2.3\,1\,38$
2. Use the Runge-Kutta method of fourth order to find $y(0.8)$ with $h = 0.2$ for the initial value problem.

$$\frac{dy}{dx} = \sqrt{x + y},\ y(0.4) = 0.41 \qquad \textbf{Ans. } 0.8489912$$

3. Find $y(0.2)$ for the equation

$$\frac{dy}{dx} = -xy, \quad y(0) = 1, \text{ using Runge-Kutta method.}$$

4. Apply the Runge-Kutta method to obtain $y(1.1)$ from the differential equation

$$\frac{dy}{dx} = xy^{1/3}, \quad y(1) = 1, \text{ taking } h = 0.1.$$

5. Apply Runge-Kutta (fourth order) formula to find an approximate value of y when $x = 1.1$, given that

$$\frac{dy}{dx} = x - y \text{ and } y = 1 \text{ at } x = 1. \qquad \textbf{Ans. } 1.004837$$

6. Using Ringe-Kutta method of fourth order solve $\frac{dy}{dx} = \frac{y^2 - x^2}{y^2 + x^2}$ with y (0) = 1 at x = 0.2 and 0.4.

(*RGPV, Bhopal III Sem. June 2008, 2004*) **Ans.** $y_{1.2} = 1.19600$, $y_{1.4} = 1.37527$

52.10 HIGHER ORDER DIFFERENTIAL EQUATIONS

Let $\frac{dy}{dx} = f(x, y, z),\ \frac{dz}{dx} = g(x, y, z),\ y(x_0) = y_0,\ z(x_0) = z_0$

Formulae for the application of Runge-Kutta method are as follows :

$$k_1 = hf(x_n, y_n, z_n),\ m_1 = hg(x_n, y_n, z_n)$$