Unit - 4

Solutions of Algebraic & Transcedental equations

polynomial:- An expression of the form $f(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + a_3 x^{n-3} + \cdots + a_{n-1} x + a_n$ is called a polynomial in x of degree 'n'. where $a_0 \neq 0$, $a_1, a_2, \ldots a_n$ are constants & (real (or) complex). n is a +ve integer.

Eg:- 1) $f(x) = 5x^4 - 3x^2 + 4x + 1$ is a polynomial of degree 4.

2). $x^3 - 5x^2 + 9$ is a polynomial of degree 3.

Algebraic equation:-

An equation of the form $f(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_{n-1} x + a_n = 0$ is called an ~~polynomial~~ Algebraic eq. in x of degree 'n'. where $a_0 \neq 0$, $a_1, a_2, \ldots a_n$ are constants (real (or) complex). & n is +ve integer.

Eg:- 1) $x^4 - 3x^2 + 4x + 1 = 0$. 2) $x^2 - 5x + 9 = 0$.

Transcedental equation:-

If the equation $f(x) = 0$ involves the function of the form such as trignometric, logarithmic, exponential ... etc then the eq. ~~f~~ $f(x) = 0$ is called ~~a~~ Transcedental eq.

Eg:- 1) $x - \cot x = 0$ 2). $4\sin x = e^x$.

properties of polynomial equation:-

1. **fundamental theorem of Algebra:-** Every polynomial eq. of degree 'n' has exactly n - roots, real (or) complex.

This property is called fundamental theorem of Algebra.

(2). Every polynomial eq. $f(x)=0$ of degree $n \geq 1$. has atleast one root, real (or) Imaginary.

(3). Imaginary roots occurs in pairs. if $\alpha+i\beta$ is a root then $\alpha-i\beta$ is also a root of the equation and so, every equation of the odd degree has atleast one real root.

(4). Irrational roots occur in pairs. i.e if $a+\sqrt{b}$ is a root of an eq., $a-\sqrt{b}$ must be its root.

(5). Intermediate Value theorem :-

If $f(x)$ is continuous in the interval $[a,b]$ and $f(a)$ & $f(b)$ are of opp. signs. then the eq. $f(x)=0$ has atleast one root between $x=a$ & $x=b$.

this property is called Intermediate value theorem.

* $\boxed{(6)}$ Descrete's rule of signs:-

the no. of +ve roots of a polynomial eq $f(x)=0$ with real co-efficients can't exceeded the no. of changes of sign of the co-efficients in $f(x)$ & the no. of negative roots can't exceeded the no. of changes of sign of the co-efficients on $f(-x)$.

Root of an equation

A number 'α' (Real (or) Complex) is called a root (or) solution of an equation $f(x)=0$ if $f(\alpha)=0$.

* In this unit for Trignometric functions we use Radians in calculators

3. Algebaric and Transcendental Equations

3.0. Introduction

In Engineering Mathematics we often encounter problems of obtaining solutions of equations of the form $f(x) = 0$. In other words we have to find a number x_0 such that $f(x_0) = 0$. If $f(x)$ is a polynomial then the equation $f(x)$ is called an **algebraic equation.**

Equations which involve transcendental functions like $\sin x$, $\cos x$, $\tan x$, $\log x$, e^x etc. are called **transcendental** equations.

$$x^2 + 5x + 6 = 0; \qquad 2x^3 - x + 4 = 0; \qquad x^5 - x^3 + 3x + 3 = 0$$

are some examples of *algebraic equations.*

$2e^x+1 = 0$; $2x+\cos x-1 = 0$; $\log_{10} x-2x = 12$; $a+b\sin x+c\cos x+d\log x = 0$; $x^2 + \log_e x - 12 = 0$ are some examples of *transcendental equations.*

If $f(x) = 0$ is a quadratic equation $ax^2 + bx + c = 0$ we have simple formula $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ to find its roots. However if $f(x)$ is a polynomial of higher degree or an expression involving trancendental functions we have no simple formula to find the roots.

Due to limitations of analytical methods, formula giving exact numerical values of the solutions exist only in very simple cases. Hence we have to use *approximate* methods to get solutions with a good degree of accuracy. In this chapter we present the following methods for obtaining approximate solutions for algebraic and transcendental equations.

(i) Iteration method (Successive approximation method)
(ii) Bisection method (Bolzano method)
(iii) Regula-Falsi method (method of false position)
(iv) Newton-Raphson method (method of tangents)
(v) Horner's method.

To locate the root of an equation $f(x) = 0$ we use the following well known theorem in calculus.

If $f(x)$ is continuous in the interval $[a, b]$ and if $f(a)$ and $f(b)$ are of opposite signs then the equation $f(x) = 0$ has at least one root lying between a and b.

3.2. Bisection Method (Bolzano method).

Let $f(x)$ be a continuous function defined on $[a, b]$ such that $f(a)$ and $f(b)$ are of opposite signs. Hence one root of the equation $f(x) = 0$ lies between a and b. For definiteness we assume that $f(a) < 0$ and $f(b) > 0$. **Bisection method** is used to find the root between a and b to the desired approximation as follows.

(i) Let $x_1 = \dfrac{a+b}{2}$ be the first approximation of the required root (x_1 is the *midpoint* of a and b)

(ii) If $f(x_1) = 0$ then x_1 is a root of $f(x)$. If not the root lies between a and x_1 or x_1 and b depending on whether $f(x_1) > 0$ or $f(x_1) < 0$.

(iii) Bisect the interval in which the root lies and continue the process until the root is found to the desired accuracy.

Note. The bisection method is simple but the convergence is very slow. The interval within which the root lies is bisected each time until we get the root with desired accuracy.

Solved Problems

Problem 1. Find a real root of the equation $x^3 - 3x + 1 = 0$ lying between 1 and 2 correct to three places of decimal by using bisection method.

Solution. Let $f(x) = x^3 - 3x + 1$

Since $f(1) = -1$ and $f(2) = 3$, $f(x) = 0$ has one root lying between 1 and 2

Let $a = 1$ and $b = 2$.

The first approximation is

$$x_1 = \frac{a+b}{2} = 1.5.$$

Now $f(1.5) = -0.125$. Also $f(2)$ is positive. Hence the root lies between 1.5 and 2. Let $a = 1.5$ and $b = 2$.

$\therefore$ The second approximation is

$$x_2 = \frac{a+b}{2} = \frac{1.5+2}{2} = 1.75.$$

Now $f(1.75) = 1.1094$. Since $f(1.5)$ is negative and $f(1.75)$ is positive the root lies between 1.5 and 1.75. Let $a = 1.5$ and $b = 1.75$.

$\therefore$ The third approximation is

$$x_3 = \frac{1.5+1.75}{2} = 1.625.$$

Now $f(1.625) = 1.625^3 - 3 \times 1.625 + 1 = 4.2910 - 4.4875 + 1 = 0.4160$.

Since $f(1.5)$ is negative and $f(1.625)$ is positive the root lies between 1.5 and 1.625.

This process is repeated and the calculation of the successive approximations is given in the following table

i	a	b	$x_i = \frac{a+b}{2}$	$f(x_i)$
1	1	2	$x_1 = 1.5$	-0.125
2	1.5	2	$x_2 = 1.75$	1.1094
3	1.5	1.75	$x_3 = 1.625$	0.8035
4	1.5	1.1625	$x_4 = 1.5625$	0.1272
5	1.5	1.5625	$x_5 = 1.5313$	-0.0032
6	1.5313	1.5625	$x_6 = 1.5469$	0.0609
7	1.5313	1.5469	$x_7 = 1.5391$	0.0286
8	1.5313	1.5391	$x_8 = 1.5352$	0.0126
9	1.5313	1.5352	$x_9 = 1.5333$	0.0049
10	1.5313	1.5333	$x_{10} = 1.5323$	0.0009
11	1.5313	1.5323	$x_{11} = 1.5318$	–

We observe that $x_{10} = x_{11} = 1.532$ correct to 3 places decimals.

Hence the required root, correct to three places of decimals is 1.532.

Problem 2. Find a real root of the equation $x^3 - x - 11 = 0$ by using bisection method.

Solution. Let $f(x) = x^3 - x - 11$.

$$f(2) = -5 \text{ and } f(3) = 13.$$

$\therefore$ One root of $f(x) = 0$ lies between 2 and 3. Let $a = 2$ and $b = 3$.

The first approximation is $x_1 = \dfrac{a+b}{2} = 2.5$.

Now $f(2.5) = 2.125$ which is positive and $f(2)$ is negative.

Hence the root lies between 2 and 2.5.

Let $a = 2$ and $b = 2.5$

$\therefore$ The second approximation is $x_2 = \dfrac{2+2.5}{2} = 2.25$.

Now $f(2.25) = -1.8594$ and the root lies between 2.25 and 2.5.

This process is repeated and the calculation of the successive approximations is given in the following table.

i	a	b	$x_i = \frac{a+b}{2}$	$f(x_i)$
1	2	3	$x_1 = 2.5$	2.125
2	2	2.5	$x_2 = 2.25$	−1.18594
3	2.25	2.5	$x_3 = 2.375$	0.0215
4	2.25	2.375	$x_4 = 2.3125$	−0.9460
5	2.3125	2.375	$x_5 = 2.3438$	−0.4684
6	2.3438	2.375	$x_6 = 2.3594$	−0.2252
7	2.3594	2.375	$x_7 = 2.3672$	−0.1023
8	2.3672	2.375	$x_8 = 2.3711$	−0.0405
9	2.3711	2.375	$x_9 = 2.3731$	−0.0087
10	2.3731	2.375	$x_{10} = 2.3741$	0.0072
11	2.3731	2.3741	$x_{11} = 2.3736$	−0.0008
12	2.3736	2.3741	$x_{12} = 2.3739$	0.004
13	2.3736	2.3739	$x_{13} = 2.3738$	0.0024
14	2.3736	2.3738	$x_{14} = 2.3737$	0.0008
15	2.3736	2.3737	$x_{15} = 2.3737$	0.0008

From the table we see that $x_{10} = x_{11} = 2.374$, correct to three places of decimals and $x_{14} = x_{15} = 2.3737$. Hence the value of the root upto three places of decimals is 2.374 and upto four places of decimals is 2.3737.

Problem 3. Find the positive real root of $x \log_{10} x = 1.2$ using the bisection method in four iterations.

Solution. Let $f(x) = x \log_{10} x - 1.2$

$$
\begin{aligned}
f(2) &= 2\log_{10} 2 - 1.2 = 0.6020 - 1.2 = -0.598 \\
f(3) &= 3\log_{10} 3 - 1.2 = 3 \times 0.4771 - 1.2 = 0.2313.
\end{aligned}
$$

Hence the root lies between 2 and 3. Let $a = 2$ and $b = 3$.

The first approximation is $x_1 = \dfrac{a+b}{2} = \dfrac{2+3}{2} = 2.5.$

Now $f(2.5) = 2.5\log_{10} 2.5 - 1.2 = 2.5 \times 0.3979 - 1.2 = -0.2053$

Since $f(2.5) < 0$ and $f(3) > 0$ the root lies between 2.5 and 3. Let $a = 2.5$, $b = 3$.

$\therefore$ The second approximation is $x_2 = \frac{a+b}{2} = \frac{2.5+3}{2} = 2.75.$

Now $f(2.75) = 2.75 \log_{10} 2.75 - 1.2 = 2.75 \times 0.4393 - 1.2 = 0.008.$

Since $f(2.75) > 0$ and $f(2.5) < 0$ the root lies between 2.5 and 2.75. Let $a = 2.5$ and $b = 2.75$.

The third approximation $x_3 = \frac{a+b}{2} = \frac{2.5+2.75}{2} = 2.625.$

Now $f(2.625) = 2.625 \log_{10} 2.625 - 1.2 = 2.625 \times 0.4191 - 1.2 = -0.10.$

Since $f(2.625) < 0$ and $f(2.75) > 0$ the root lies between 2.625 and 2.75.

Take $a = 2.625$ and $b = 2.75$.

$\therefore$ The fourth approximation is $x_4 = \frac{2.625+2.75}{2} = 2.6875.$

$\therefore$ After fourth iteration the approximate value of the root of the given equation is 2.6875.

Exercises

1. Find a root of the equation $x^3 - 4x - 9 = 0$ (i) u g the bisection method in four stages (ii) perform five iterations of the bisection method to obtain the smallest positive root of $x^3 - 5x + 1 = 0$.

2. Find a root of the following equations using the bisection method

(i) $x^3 - x - 1 = 0$

(ii) $x^3 - 9x + 1 = 0$

(iii) $x \log_{10} x = 1.2$ lying between 2 and 3.

3. Find the negative root of $x^3 - 4x + 9 = 0$ by bisection method after five iterations,

4. Find the positive root of $x - \cos x = 0$ by bisection method.

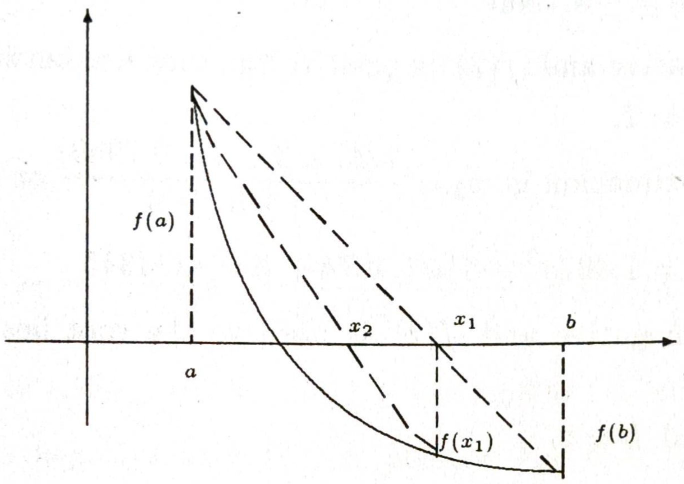
3.3. Regula Falsi Method (Method of false position).

Consider the equation $f(x) = 0$ where $f(x)$ is a continuous function. Choose two points a, b such that $f(a)$ and $f(b)$ are of opposite signs. Hence there exists a root lying between a and b. In this method we approximate the curve of the function $f(x)$ by a chord (Refer figure).

Equation of the chord joining the points $(a, f(a))$ and $(b, f(b))$ is given by

$$y - f(a) = \frac{f(b) - f(a)}{b - a}(x - a). \qquad \text{—— (1)}$$

The point of intersection of the chord with the x axis is taken as the first approximation x_1 to the root, which is obtained by putting $y = 0$ in (1).



$y = f(x)$

$$\therefore \frac{-f(a)(b-a)}{f(b)-f(a)} = x_1 - a$$

$$\therefore x_1 = a - \frac{f(a)(b-a)}{f(b)-f(a)}$$

$$\therefore x_1 = \frac{af(b) - bf(a)}{f(b) - f(a)}. \qquad \text{—— (2)}$$

Now if $f(a)$ and $f(x_1)$ are of opposite signs then the root lies between a and x_1. So we replace b by x_1 in (2) and get the next approximation x_2.

But if $f(a)$ and $f(x_1)$ are of same sign, then $f(x_1)$ and $f(b)$ will be of opposite signs. Hence the root lies between x_1 and b. We replace a by x_1 in (2).

The process is repeated until the root is found to the desired accuracy.

Solved Problems

Problem 1. Find the real root lying between 1 and 2 of the equation $x^3 - 3x + 1 = 0$ upto 3 places of decimals by using Regula-falsi method.

Solution. Let $f(x) = x^3 - 3x + 1$. Since $f(1) = -1$ and $f(2) = 3$ one root lies between 1 and 2.

Let $a = 1$ and $b = 2$. The first approximation is

$$x_1 = \frac{af(b) - bf(a)}{f(b) - f(a)}$$
$$= \frac{1 \times 3 - 2(-1)}{3 + 1} = \frac{5}{4} = 1.25.$$

Now $f(x_1) = f(1.25) = -0.7969$

Since $f(1.25)$ is negative and $f(2)$ is positive the root lies between 1.25 and 2.
Let $a = 1.25$ and $b = 2$.

The second approximation is $x_2 = \dfrac{1.25 \times 3 - 2(-0.7969)}{3 + 0.7969} = 1.4074.$

Now $\quad f(1.4074) = 1.4074^3 - 3 \times 1.4074 + 1 = -0.4345.$

Since $f(1.4074)$ is negative and $f(2)$ is positive the root lies between 1.4074 and 2.

Take $a = 1.4074$ and $b = 2$.

The third approximation is

$$x_3 = \frac{1.4074 \times 3 - 2 \times (-0.4345)}{3 + 0.4345} = \frac{4.2222 + 0.8690}{3.4345} = 1.4824.$$

Now $\quad f(1.4824) = 1.4824^3 - 3 \times 1.4824 + 1 = 3.2576 - 4.4472 + 1 = -0.1896.$

Since $f(1.4824)$ is negative and $f(2)$ is positive the root lies between 1.4824 and 2. Let $a = 1.4824$ and $b = 2$.

The fourth approximation is $x_4 = \dfrac{1.4824 \times 3 - 2 \times (-0.1896)}{3 + 0.1896} = 1.5132.$

Now $\quad f(1.5132) = 1.5132^3 - 3 \times 1.5132 + 1 = 3.4649 - 4.5396 + 1 = -0.0747.$

Since $f(1.5132)$ is negative and $f(2)$ is positive the root lies between 1.5132 and 2.

Let $a = 1.5132$ and $b = 2$.

The fifth approximation is

$$x_5 = \frac{1.5132 \times 3 - 2(-0.0747)}{3 + 0.0747} = 1.525.$$

Now $f(1.525) = 1.525^3 - 3 \times 1.525 + 1 = 3.5466 - 4.575 + 1 = -0.0284.$

Since $f(1.525)$ is negative and $f(2)$ is positive the root lies between 1.525 and 2. Take $a = 1.525$ and $b = 2$.

The sixth approximation is

$$x_6 = \frac{1.525 \times 3 - 2 \times (-0.0284)}{3 + 0.0284} = 1.5295.$$

Now $f(1.5295) = 1.5295^3 - 3 \times 1.5295 + 1 = 3.5781 - 4.5885 + 1 = -0.0104.$

Since $f(1.5295)$ is negative and $f(2)$ is positive the root lies between 1.5295 and 2. Take $a = 1.5295$ and $b = 2$.

$\therefore$ The seventh approximation is

$$x_7 = \frac{1.5295 \times 3 - 2 \times (-0.0104)}{3 + 0.0104} = 1.5311.$$

Now $f(1.5311) = 1.5311^3 - 3 \times 1.5311 + 1 = 3.5893 - 4.5933 + 1 = -0.004.$

Since $f(1.5311)$ is negative and $f(2)$ is positive the root lies between 1.5311 and 2. Take $a = 1.5311$ and $b = 2$.

The eighth approximation is

$$x_8 = \frac{1.5311 \times 3 - 2 \times (-0.0040)}{3 + 0.0040} = 1.5317.$$

Now $f(1.5317) = 1.5317^3 - 3 \times 1.5317 + 1 = 3.5935 - 4.5951 + 1 = -0.0016.$

Since $f(1.5317)$ is negative and $f(2)$ is positive the root lies between 1.5317 and 2. Take $a = 1.5317$ and $b = 2$.

The ninth approximation is

$$\begin{aligned} x_9 &= \frac{1.5317 \times 3 - 2(-0.0016)}{3 + 0.0016} = 1.5319 \\ &= 1.532 \text{ (corrected upto 3 places of decimals)} \\ \therefore x_8 = x_9 &= 1.532. \end{aligned}$$

$\therefore$ The required root is 1.532 (corrected to 3 places of decimals).

Problem 2. Find a root of the equation $x^3 - 3x - 5 = 0$ by the method of false position.

Solution. Let $f(x) = x^3 - 3x - 5$. Here $f(2) = -3$, $f(3) = 13$.

$\therefore$ The root lies between 2 and 3. Let $a = 2$ and $b = 3$.

The first approximation to the root is given by

$$x_1 = \frac{af(b) - bf(a)}{f(b) - f(a)} = \frac{2 \times 13 - 3 \times (-3)}{13 + 3} = 2.1875.$$

Now $f(x_1) = f(2.1875) = 10.4675 - 6.5625 - 5 = -1.095$.

Hence the root lies between 2.1875 and 3. Let $a = 2.1875$ and $b = 3$.

Hence the second approximation is given by

$$x_2 = \frac{2.1875 \times 13 - 3 \times (-1.095)}{13 + 1.095} = 2.2506.$$

Now $f(x_2) = f(2.2506) = 11.3997 - 6.7518 - 5 = -0.3521$.

Hence the root lies between 2.2506 and 3. Let $a = 2.2506$ and $b = 3$.

Hence the third approximation is given by

$$x_3 = \frac{2.2506 \times 13 - 3 \times (-0.3521)}{13 + 0.3521} = 2.2704.$$

Now $f(x_3) = f(2.2704) = 11.7033 - 6.8112 - 5 = -0.1079$.

The root lies between 2.2704 and 3. Let $a = 2.2704$ and $b = 3$.

$\therefore$ The fourth approximation is given by

$$x_4 = \frac{2.2704 \times 13 - 3 \times (-0.1079)}{13 + 0.1079} = 2.2764$$

Now, $f(x_4) = f(2.2764) = 11.7963 - 6.8292 - 5 = -0.0329$.

The root lies between 2.2764 and 3. Take $a = 2.2764$ and $b = 3$.

The fifth approximation is given by

$$x_5 = \frac{2.2764 \times 13 - 3 \times (-0.0329)}{13 + 0.0329} = 2.2782.$$

Now $f(x_5) = f(2.2782) = 11.8243 - 6.8346 - 5 = -0.0103$

Since $f(3)$ is positive and $f(2.2782)$ is negative the root lies between 2.2782 and 3. Take $a = 2.2782$ and $b = 3$.

$\therefore$ The sixth approximation is given by

$$x_6 = \frac{2.2782 \times 13 - 3 \times (-0.0103)}{13 + 0.0103} = 2.2788.$$

Now $f(2.2788) = 11.8336 - 6.8364 - 5 = -0.0028$

Since $f(x_6)$ is negative and $f(3)$ is positive, the root lies between 2.2788 and 3. Let $a = 2.2788,\ b = 3$

$$\therefore x_7 = \frac{2.2788 \times 13 - 3(-0.0028)}{13 + 0.0028} = 2.2790.$$

$\therefore\ x_6 \simeq x_7 = 2.279$ (corrected upto 3 places of decimals) .

Problem 3. Find the smallest positive root of $x^2 - \log_e x - 12 = 0$ by Regula falsi method.

Solution. Let

$$\begin{aligned} f(x) &= x^2 - \log_e x - 12 \\ f(3) &= 9 - 1.0986 - 12 = -4.0986 \\ f(4) &= 16 - 1.3863 - 12 = 2.6137. \end{aligned}$$

$\therefore$ The root lies between 3 and 4. Let $a = 3$ and $b = 4$.

$\therefore$ The first approximation to the root is given by

$$\begin{aligned} x_1 &= \frac{af(b) - bf(a)}{f(b) - f(a)} = \frac{3f(4) - 4f(3)}{f(4) - f(3)} \\ &= \frac{3 \times 2.6137 - 4 \times (-4.0986)}{2.6137 - (-4.0986)} = 3.6106. \end{aligned}$$

Now $f(x_1) = f(3.6106) = 13.0364 - 1.2839 - 12 = -0.2474.$

We note that $f(4)$ is positive and $f(3.6106)$ is negative.

$\therefore$ The root lies between 3.6106 and 4. Take $a = 3.6106$ and $b = 4$.

$\therefore$ The second approximation is

$$\therefore \qquad x_2 = \frac{3.6106 \times 2.6137 - 4 \times (-0.2474)}{2.6137 + 0.2474} = 3.6443.$$

Now $f(3.6443) = 13.2809 - 1.2932 - 12 = -0.0123$ the root lies between 4 and 3.6443. Hence take $a = 3.6443$ and $b = 4$.

$\therefore$ The third approximation is

$$x_3 = \frac{3.6443 \times 2.6137 - 4 \times (-0.0123)}{2.6137 - (-0.0123)} = 3.6460.$$

Now $f(3.6460) = 13.2933 - 1.2936 - 12 = -0.0003$ and the root lies between 4 and 3.646. Let $a = 3.646$ and $b = 4$.

The fourth approximation is

$$x_4 = \frac{3.646 \times 2.6137 - 4 \times (-0.0003)}{2.6137 - (-0.0003)} = 3.6461.$$

Hence the required root is 3.646 (corrected upto 3 decimals).

Problem 4. Find by Regula falsi method the positive root of $x^2 - \log_{10} x - 12 = 0$.

Solution. Let $f(x) = x^2 - \log_{10} x - 12$

$$\begin{aligned} f(3) &= 9 - 0.4771 - 12 = -3.4771 \\ f(4) &= 16 - 0.6021 - 12 = 3.3979 \end{aligned}$$

$\therefore$ The root lies between 3 and 4. Let $a = 3$ and $b = 4$.

$\therefore$ The first approximation to the root is given by

$$\begin{aligned} x_1 &= \frac{af(b) - bf(a)}{f(b) - f(a)} \\ &= \frac{3 \times 3.3979 - 4(-3.4771)}{3.3979 - 3.4771} = 3.5058. \end{aligned}$$

Now $f(x_1) = f(3.5058) = (3.5058)^2 - \log_{10} 3.5058 - 12 = -0.2542.$

$\therefore$ The root lies between 3.5058 and 4.

By taking $a = 3.5058$ and $b = 4$ we get the second approximation as

$$x_2 = \frac{3.5058 \times 3.3979 - 4(-0.2542)}{3.3979 - (-0.2542)} = 3.5402.$$

Now $f(x_2) = f(3.5402) = (3.5402)^2 - \log_{10} 3.5402 - 12 = -0.016.$

$\therefore$ The root lies between $a = 3.5402$ and $b = 4$.

Hence the third approximation is

$$x_3 = \frac{3.5402 \times 3.3979 - 4(-0.016)}{3.3979 - (-0.016)} = 3.542$$

Now $f(x_3) = -0.0007$. (Verify)

$\therefore$ The root lies between $a = 3.542$ and $b = 4$.

$\therefore$ The fourth approximation is

$$x_4 = \frac{3.5424 \times 3.3979 - 4(-0.0007)}{3.3979 - (-0.0007)} = 3.542.$$

Hence the required root is 3.542.

Exercises

Use Regula falsi method (method of false position) to solve the following equations.

1. Find the positive root of $x^3 - 2x - 50$
2. Find the root of $x^3 - 4x + 1 = 0$ which lies between 0 and 1
3. $xe^x - 2 = 0$
4. $x \log_{10} x = 1.2$
5. $e^x = 3x$
6. $2x - 3\sin x = 5$
7. $\cos x - 3x + 1 = 0$
8. $2x - \log_{10} x = 7$
9. Find the smallest positive root of $xe^x - \cos x = 0$.

3.4. Newton-Raphson method

Let x_0 be an approximate root of the equation $f(x) = 0$. Let $x_1 = x_0 + h$ be the exact root where h is very small, positive or negative.

$$\therefore f(x_1) = 0. \qquad \text{—— (1)}$$

By Taylor's series expansion, we have

$$f(x_1) = f(x_0 + h) = f(x_0) + \frac{h}{1!}f'(x_0) + \frac{h^2}{2!}f''(x_0) + \cdots.$$

Since $f(x_1) = 0$ and h is very small h^2 and higher powers of h can be neglected. Hence $f(x_0) + hf'(x_0) = 0$.

$$\therefore h = -\frac{f(x_0)}{f'(x_0)} \text{ if } f'(x_0) \neq 0.$$

Hence $x_1 = x - \dfrac{f(x_0)}{f'(x_0)}$ is a first approximation to the root.

Similarly starting with x_1 we get the next approximation to the root given by

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}.$$

In general, $$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, 2, \cdots}$$

and it is known as *Newton-Raphson's iteration formula* or simply Newton Raphson's formula.

Lemma. The order of convergence of the Newton-Raphson method is at least 2.

Proof. Newton-Raphson iteration formula is given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

On comparing this with the relation $x_{n+1} = \phi(x_n)$, we get $\phi(x) = x - \dfrac{f(x)}{f'(x)}$.

Hence $\phi'(x) = 1 - \dfrac{[f'(x)]^2 - f(x)f''(x)}{[f'(x)]^2} = \dfrac{f(x)f''(x)}{[f'(x)]^2}$.

If α is the desired root of the equation $f(x) = 0$, we have $\phi'(\alpha) = \dfrac{f(\alpha)f''(\alpha)}{[f'(\alpha)]^2} = 0$.

$$\begin{aligned}\therefore x_{n+1} &= \phi(\alpha) + \frac{\phi'(\alpha)}{1!}\epsilon_n + \frac{\phi''(\alpha)}{2!}\epsilon_n^2 + \cdots \\ &= \alpha + \frac{\phi''(\alpha)}{2!}\epsilon_n^2 + \cdots \\ \therefore \epsilon_{n+1} &= x_{n+1} - \alpha = \frac{\phi''(\alpha)}{2!}\epsilon_n^2 + \cdots.\end{aligned}$$

Hence the order of convergence is at least 2.

Geometrical interpretation of Newton-Raphson method

Let x_0 be a point near the root α of the equation $f(x) = 0$.

Let $A = (x_0, f(x_0))$.

Then the equation of the tangent to the curve $y = f(x)$ at the point A is given by

$$y - f(x_0) = f'(x_0)(x - x_0).$$

Let $(x_1, 0)$ be the point of intersection of the tangent with the x-axis.

Hence $f(x_0) = f'(x_0)(x_1 - x_0)$.

$$\therefore x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

which is the first approximation of the root. Similarly if the tangent to the curve $A_1 = (x_1, f(x_1))$ intersects the x-axis at $(x_2, 0)$ then x_2 gives the second approximation to the root.

Repeating this process we find the successive approximations to the root.

Note. Because of this geometrical interpretation Newton-Raphson method is also referred as the *method of tangents.*

Solved Problems

Problem 1. Find the first approximation of the root lying between 0 and 1 of the equation $x^3 + 3x - 1 = 0$ by Newton-Raphson formula.

Solution. Let $f(x) = x^3 + 3x - 1$. Hence $f'(x) = 3x^2 + 3$.
Given that the root lies between 0 and 1. Take $x_0 = 0$.

Newton's formula is $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$.

When $n = 0$, the first approximation is $x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)}$

Now
$$\begin{aligned} f(x_0) &= f(0) = -1 \\ f'(x_0) &= f'(0) = 3 \\ \therefore \quad x_1 &= 0 - \left(\frac{-1}{3}\right) = 0.3333. \end{aligned}$$

$\therefore$ The first approximation is 0.3333.

Problem 2. Write Newton-Raphson formula to obtain the cube root of N.

Solution. Let $x = \sqrt[3]{N}$.

$\therefore x^3 = N$. Hence $x^3 - N = 0$.

Let $f(x) = x^3 - N$ and hence $f'(x) = 3x^2$.

The Newton-Raphson formula is

$$\begin{aligned} x_{x+1} &= x_n - \frac{f(x_n)}{f'(x_n)}; \qquad n = 0, 1, 2 \cdots \\ &= x_n - \frac{(x_n^3 - N)}{3x_n^2} \end{aligned}$$

Problem 3. Find the real root of $x^3 - 3x + 1 = 0$ lying between 1 and 2 upto three decimal places by Newton Raphson method.

Solution. Let $f(x) = x^3 - 3x + 1$. Hence $f'(x) = 3x^2 - 3$

$f(1) = -1$ and $f(2) = 3$. Hence one root lies between 1 and 2.

Let the initial approximation be $x_0 = 1$. The Newton-Raphson formula is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}; \qquad n = 0, 1, 2 \cdots$$

$$\therefore \quad x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

Now $f(x_0) = f(1) = -1$; $f'(x_0) = f'(1) = 0$.

Since $f'(1) = 0$. Newton's approximation formula cannot be applied for the initial approximation as $x = 1$. Let us take $x_0 = 1.5$.

$$\begin{aligned} \text{Now} \quad f(x_0) &= f(1.5) = 3.375 - 4.5 + 1 = -0.125 \\ f'(x_0) &= f'(1.5) = 6.75 - 3 = 3.75 \end{aligned}$$

$$\begin{aligned} \therefore x_1 &= 1.5 - \left(\frac{-0.125}{3.75}\right) = 1.5 + 0.0333 \\ \therefore x_1 &= 1.5333 \\ \text{Now, } f(x_1) &= (1.5333)^3 - 3 \times 1.5333 + 1 = .0049 \\ \text{and } f'(x_1) &= 3 \times (1.5333)^2 - 3 = 4.053. \end{aligned}$$

$$\text{Hence } x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1.5333 - \left(\frac{0.0049}{4.053}\right) = 1.5321$$

Now, $f(x_2) = f(1.5321) = 3.5963 - 4.5963 + 1 = 0$

$\therefore$ x_2 is a root of $f(x) = 0$.

Hence the required root is 1.532 corrected upto 3 decimal places.

Problem 4. Find the real root of $xe^x - 2 = 0$ correct to three places of decimals using Newton-Raphson method.

Solution. Let $f(x) = xe^x - 2$

$$\therefore f'(x) = xe^x + e^x = e^x(x+1).$$

We note that $f(0) = -2$ and $f(1) = e - 2 = 2.7183 - 2 = 0.7183$.

$\therefore$ The root lies between 0 and 1

Since the numerical value of $f(1)$ is less than that of $f(0)$ we can take the initial approximation as $x_0 = 1$.

The Newton-Raphson formula is $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$; $\quad n = 0, 1, 2, \cdots$.

$$\therefore x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

Now, $f(x_0) = f(1) = 0.7183$; $f'(x_0) = f'(1) = 2.7183(1+1) = 5.4366$

$$\therefore x_1 = 1 - \frac{0.7183}{5.4366} = 1 - 0.1321 = 0.8679$$

Now, $f(x_1) = 0.0673$ and $f'(x_1) = 4.4492$ (Verify)

$$\therefore x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 0.8679 - \frac{0.0673}{4.4492} = 0.8679 - 0.0151 = 0.8528.$$

$$\begin{aligned} \text{Now,} \quad f(x_2) &= f(0.8528) = 0.8528 \times e^{0.8528} - 2 = .0008 \\ f'(x_2) &= e^{.8528} \times 1.8528 = 4.3471 \end{aligned}$$

$$\therefore \quad x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 0.8528 - \frac{0.0008}{4.3471} = 0.8526.$$

Since x_2 and x_3 are approximately equal, upto third decimals we can take the required root as 0.853.

The three iterations are given in the following table

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$	x_{n+1}
0	1	0.7183	5.4366	0.1321	0.8679	x_1
1	0.8679	0.0673	4.4492	0.0151	0.8528	x_2
2	0.8528	0.0008	4.3470	0.0002	0.8526	x_3

Problem 5. Evaluate $\sqrt{12}$ to four places of decimals by Newton-Raphson Method.

Solution. Let $x = \sqrt{12}$

$\therefore x^2 = 12$. Hence $x^2 - 12 = 0$.

Let $f(x) = x^2 - 12$. Hence $f'(x) = 2x$.

We note $f(3) = -3$ and $f(4) = 4$.

$\therefore$ The root lies between 3 and 4.

Since the numerical value of $f(3)$ is less than that of $f(4)$ we take the initial approximation as $x_0 = 3$.

The Newton-Raphson formula is $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$

$$\therefore x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Now $f(x_0) = f(3) = 9 - 12 = -3$; $f'(x_0) = f'(3) = 6$.

$$\therefore x_1 = 3 - \left(\frac{-3}{6}\right) = 3 + 0.5 = 3.5$$

$$\begin{aligned} \text{Now,} \quad f(x_1) &= f(3.5) = 12.25 - 12 = 0.25 \\ f'(x_1) &= f'(3.5) = 7. \end{aligned}$$

$$\therefore x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 3.5 - \left(\frac{0.25}{7}\right) = 3.4643$$

$$\begin{aligned} \text{Now,} \quad f(x_2) &= f(3.4643) = 12.0014 - 12 = .0014 \\ f'(x_2) &= f'(3.4643) = 2 \times 3.4643 = 6.9286 \end{aligned}$$

$$\therefore \ x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 3.4643 - \left(\frac{0.0014}{6.9286}\right) = 3.4641.$$

$\therefore$ The required root is 3.4641 to 4 places of decimals.

Problem 6. Find the *negative* root of $x^3 - 2x + 5 = 0$ correct to three places of decimals by the Newton-Raphson method.

Solution. Let $g(x) = x^3 - 2x + 5$. We find the positive root of $g(-x) = 0$. (i.e) $g(-x) = -x^3 + 2x + 5 = 0$.

$\therefore$ We find the positive root of $f(x) = x^3 - 2x - 5 = 0$

$$f(x) = x^3 - 2x - 5 \text{ and } f'(x) = 3x^2 - 2$$

Now, $f(2) = -1$; and $f(3) = 16$.

$\therefore$ One root lies between 2 and 3.

Since the numerical value of $f(2)$ is less than the numerical value of $f(3)$ we can take the initial approximation as $x_0 = 2$.

$$\text{Newton} - \text{Raphson formula is} \qquad x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$\therefore \qquad x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

Now $f(x_0) = f(2) = -1$ and $f'(x_0) = f'(2) = 10$

$$\therefore x_1 = 2 - \left(\frac{-1}{10}\right) = 2.1.$$

$$\begin{aligned}
\text{Now, } f(x_1) &= f(2.1) = 9.261 - 4.2 - 5 = 0.061 \\
f'(x_1) &= f'(2.1) = 3 \times 2.1^2 - 2 = 11.23 \\
\therefore x_2 &= x_1 - \frac{f(x_1)}{f'(x_1)} = 2.1 - \left(\frac{0.061}{11.23}\right) = 2.0946
\end{aligned}$$

$$\begin{aligned}
\text{Now, } f(x_2 &= 9.1897 - 4.1892 - 5 = 0.0005 \\
f'(x_2) &= 3 \times 4.3873 - 2 = 11.1619 \\
\therefore x_3 &= x_2 - \frac{f(x_2)}{f'(x_2)} = 2.0946 - \frac{0.0005}{11.1619} = 2.09456.
\end{aligned}$$

Since x_2 and x_3 are equal upto 3 places of decimals we can take the required root to be 2.095 correct to 3 places of decimals.

$\therefore$ The negative root of $x^3 - 2x + 5 = 0$ is -2.095

Problem 7. Using Newton-Raphson iterative method find the real root of $x \log_{10} x = 1.2$ correct to four decimal places.

Solution. Let $f(x) = x \log_{10} x - 1.2$

$$\begin{aligned} \therefore f'(x) &= x\left(\frac{\log_{10} e}{x}\right) + \log_{10} x = \log_{10} e + \log_{10} x \\ &= 0.4343 + \log_{10} x \end{aligned}$$

$$f(1) = -1.2; \qquad f(2) = 2 \times 0.3010 - 1.2 = 0.6020 - 1.2 = -0.598$$
$$f(3) = 3 \times 0.4771 - 1.2 = 1.4313 - 1.2 = 0.2313$$

$\therefore$ One root lies between 2 and 3. Let the initial approximation be $x_0 = 2$.

Newton-Raphson formula is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}; \qquad n = 0, 1, 2 \cdots.$$

When $n = 0$ the first approximation is $x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)}$.

$$\begin{aligned} \text{Now,} \quad f(x_0) &= f(2) = 0.6020 - 1.2 = -0.598 \\ f'(x_0) &= f'(2) = 0.4343 + 0.3010 = 0.7353 \\ \therefore x_1 &= 2 - \frac{(-0.598)}{(0.7353)} = 2 + 0.8133 = 2.8133. \end{aligned}$$

The second approximation is

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 2.8133 - \frac{f(2.8133)}{f'(2.8133)}$$

$$\begin{aligned} \text{Now,} \quad f(2.8133) &= 2.8133 \times 0.4492 - 1.2 = 1.2637 - 1.2 = 0.0637 \\ f'(2.8133) &= 0.4343 + 0.4492 = 0.8835 \\ \therefore x_2 &= 2.8133 - \left(\frac{0.0637}{0.8835}\right) = 2.8133 - 0.0721 = 2.7412 \end{aligned}$$

The third approximation is

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 2.7412 - \frac{f(2.7412)}{f'(2.7412)}$$

$$\text{Now,}\quad f(2.7412) = 2.7412 \times 0.4379 - 1.2$$
$$= 1.2004 - 1.2 = .0004$$
$$f'(2.7412) = 0.4343 + 0.4379 = 0.8722$$

$$\therefore\ x_3 = 2.7412 - \frac{.0004}{0.8722} = 2.7412 - 0.0005 = 2.7407.$$

$\therefore\quad x_2 \simeq x_3 = 2.741$ (correct to 3 decimal places).

$\therefore$ The required root is 2.741.

Problem 8. Find by Newton-Raphson method correct to 4 places of decimals the root between 0 and 1 of the equation $3x - \cos x - 1 = 0$.

Solution. Let $f(x) = 3x - \cos x - 1$. Hence $f'(x) = 3 + \sin x$.

Now $f(0) = -2$; $\qquad f(1) = 3 - 0.5403 - 1 = 1.4597$.

Since the numerical value of $f(1)$ is less than that of $f(0)$ we can take the initial approximation as $x_0 = 1$.

The Newton-Raphson formula is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)};\ n = 0, 1, 2, \cdots.$$

$\therefore$ The first approximation is

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$f(x_0) = f(1) = 1.4597; \qquad f'(x_0) = f'(1) = 3 + 0.8415 = 3.8415$$

$$\therefore\ x_1 = 1 - \frac{1.4597}{3.8415} = 0.62.$$

Now the second approximation is

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}.$$

$$f(x_1) = f(0.62) = 0.0461; \qquad f'(x_1) = f'(0.62) = 3.5810$$

$$\therefore x_2 = 0.62 - \frac{0.0461}{3.581} = 0.6071.$$

The third approximation is

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

Now, $f(x_2) = f(0.6071) = 0$; $f'(x_2) = f'(0.6071) = 3.5705$

$$\therefore\quad x_3 = 0.6071 - \left(\frac{0}{3.5705}\right) = 0.6071.$$

Since x_3 and x_4 are equal upto 4 decimal places the required root is 0.6071.

Exercises

Find a positive root of each of the following equations using Newton-Raphson method.

1. $x^3 + x - 1 = 0$
2. $x^3 - 5x - 6 = 0$
3. $2x^3 - 3x - 6 = 0$
4. $x^4 - x - 13 = 0$
5. $x^4 - 3x + 1 = 0$
6. $xe^x = \cos x$
7. $x \sin x + \cos x = 0$
8. $x \cos x = 0$
9. $4x - e^x = 0$
10. $\log x - x + 3 = 0$
11. $e^{-x} - \sin x = 0$
12. Find the approximate value of $\sqrt{5}$ using Newton-Raphson method
13. Find a positive real root of $\sqrt[3]{17}$ using Newton-Raphson method
14. Find the iterative formula for finding $\sqrt[3]{N}$ where N is a positive real number using Newton-Raphson method. Hence evaluate $\sqrt[5]{10}$ to four places of decimals
15. Find the negative root of $x^3 - \sin x + 1 = 0$
16. Find the negative root of $x^3 - 5x + 11 = 0$ correct to 2 places of decimals using Newton-Raphson method
17. Say true or false.

    (i) Finding a negative root of $2x^3 - 3x + 6 = 0$ is equivalent to finding a positive root of $2x^3 - 3x - 6 = 0$

    (ii) Finding a negative root of $x^3 - 6x - 4 = 0$ is equivalent to finding the positive root of $-x^3 + 6x + 4 = 0$

    (iii) Finding a negative root of $x^4 + x = 10$ is equivalent to finding the positive root of $x^4 + x - 10 = 0$
18. Using Newton-Raphson iterative method find the real root of $x \log_{10} x = 1.2$ correct to four decimal places.

Example 28.9. *Using Newton's iterative method, find the real root of $x \log_{10} x = 1.2$ correct to five decimal places.* *(V.T.U., 2005 ; Mumbai, 2004 ; Burdwan, 2003)*

Solution. Let $f(x) = x \log_{10} x - 1.2$

$f(1) = -1.2 = -\text{ve}, f(2) = 2 \log_{10} 2 - 1.2 = 0.59794 = -\text{ve}$

and $f(3) = 3 \log_{10} 3 - 1.2 = 1.4314 - 1.2 = 0.23136 = +\text{ve}$

So a root of $f(x) = 0$ lies between 2 and 3. Let us take $x_0 = 2$

Also $f'(x) = \log_{10} x + x \cdot \frac{1}{x} \log_{10} e = \log_{10} x + 0.43429$

$\therefore$ Newton's iteration formula gives

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = \frac{0.43429\, x_n + 1.2}{\log_{10} x_n + 0.43429} \quad \ldots(i)$$

Putting $n = 0$, the first approximation is

$$x_1 = \frac{0.43429 \times x_0 + 1.2}{\log_{10} x_0 + 0.43429} = \frac{0.43429 \times 2 + 1.2}{\log_{10} 2 + 0.43429} = \frac{0.86858 + 1.2}{0.30103 + 0.43429} = 2.81$$

Similarly putting $n = 1, 2, 3, 4$ in (i), we get

$$x_2 = \frac{0.43429 \times 2.81 + 1.2}{\log_{10} 2.81 + 0.43429} = 2.741$$

$$x_3 = \frac{0.43429 \times 2.741 + 1.2}{\log_{10} 2.81 + 0.43429} = 2.74064$$

$$x_4 = \frac{0.43429 \times 2.74064 + 1.2}{\log_{10} 2.74064 + 0.43429} = 2.74065$$

$$x_5 = \frac{0.43429 \times 2.74065 + 1.2}{\log_{10} 2.74065 + 0.43429} = 2.74065$$

Clearly $x_4 = x_5$.

Hence the required root is 2.74065 correct to five decimal places.

28.3 USEFUL DEDUCTIONS FROM THE NEWTON-RAPHSON FORMULA

(1) *Iterative formula to find $1/N$ is* $\quad \boldsymbol{x_{n+1} = x_n (2 - Nx_n)}$

(2) *Iterative formula to find $\sqrt{N}$ is* $\quad \boldsymbol{x_{n+1} = \frac{1}{2}(x_n + N/x_n)}$

(3) *Iterative formula to find $1/\sqrt{N}$ is* $\quad \boldsymbol{x_{n+1} = \frac{1}{2}(x_n + 1/Nx_n)}$

(4) *Iterative formula to find $\sqrt[k]{N}$ is* $\quad \boldsymbol{x_{n+1} = \frac{1}{k}[(k-1)x_n + N/x_n^{k-1})]}$

Proofs. (1) Let $x = 1/N$ or $1/x - N = 0$

Taking $f(x) = 1/x - N$, we have $f'(x) = -x^{-2}$

Then Newton's formula gives

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{(1/x_n - N)}{-x_n^{-2}} = x_n + \left(\frac{1}{x^n} - N\right) x_n^2 = x_n + x_n - Nx_n^2 = x_n(2 - Nx_n)$$ *(Anna, 2013)*

(2) Let $x = \sqrt{N}$ or $x^2 - N = 0$

Taking $f(x) = x^2 - N$, we have $f'(x) = 2x$

Then Newton's formula gives

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2 - N}{2x_n} = \frac{1}{2}(x_n + N/x_n)$$ (Madras, 2006)

(3) Let $x = \frac{1}{\sqrt{N}}$ or $x^2 - \frac{1}{N} = 0$

Taking $f(x) = x^2 - 1/N$, we have $f'(x) = 2x$

Then Newton's formula gives

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2 - 1/N}{2x_n} = \frac{1}{2}\left(x_n + \frac{1}{Nx_n}\right)$$

(4) Let $x = \sqrt[k]{N}$ or $x^k - N = 0$

Taking $f(x) = x^k - N$, we have $f'(x) = kx^{k-1}$

Then Newton's formula gives

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^k - N}{kx_n^{k-1}} = \frac{1}{k}\left[(k-1)x_n + \frac{N}{x_n^{k-1}}\right].$$

Example 28.10. *Evaluate the following (correct to four decimal places) by Newton's iteration method :*

(*i*) $1/31$ (*ii*) $\sqrt{5}$ (Anna, 2012)

(*iii*) $1/\sqrt{14}$ (*iv*) $\sqrt[3]{24}$ (Madras, 2003) (*v*) $(30)^{-1/5}$.

Solution. (*i*) Taking $N = 31$, the above formula (1) becomes

$$x_{n+1} = x_n(2 - 31x_n)$$

Since an approximate value of $1/31 = 0.03$, we take $x_0 = 0.03$

Then $x_1 = x_0(2 - 31x_0) = 0.03\,(2 - 31 \times 0.03) = 0.0321$

$x_2 = x_1(2 - 31x_1) = 0.0321(2 - 31 \times 0.0321) = 0.032257$

$x_3 = x_2\,(2 - 31x_2) = 0.032257\,(2 - 31 \times 0.032257) = 0.03226$

Since $x_2 = x_3$ upto 4 decimal places, we have $1/31 = 0.0323$.

(*ii*) Taking $N = 5$, the above formula (2), becomes $x_{n+1} = \frac{1}{2}(x_n + 5/x_n)$

Since an approximate value of $\sqrt{5} = 2$, we take $x_0 = 2$

Then $x_1 = \frac{1}{2}(x_0 + 5/x_0) = \frac{1}{2}(2 + 5/2) = 2.25$

$x_2 = \frac{1}{2}(x_1 + 5/x_1) = 2.2361$

$x_3 = \frac{1}{2}(x_2 + 5/x_2) = 2.2361$

Since $x_2 = x_3$ upto 4 decimal places, we have $\sqrt{5} = 2.2361$.

(*iii*) Taking $N = 14$, the above formula (3), becomes $x_{n+1} = \frac{1}{2}[x_n + 1/(14x_n)]$

Since an approximate value of $1/\sqrt{14} = 1/\sqrt{16} = \frac{1}{4} = 0.25$, we take $x_0 = 0.25$

Then $x_1 = \frac{1}{2}[x_0 + (14x_0)^{-1}] = \frac{1}{2}[0.25 + (14 \times 0.25)^{-1}] = 0.26785$

$x_2 = \frac{1}{2}[x_1 + (14x_1)^{-1}] = \frac{1}{2}[0.26785 + (14 \times 0.26785)^{-1}] = 0.2672618$

$x_3 = \frac{1}{2}[x_2 + (14x_2)^{-1}] = \frac{1}{2}[0.2672618 + (14 \times 0.2672618)^{-1}] = 0.2672612$

Since $x_2 = x_3$ upto 4 decimal places, we take $1/\sqrt{14} = 0.2673$.

(*iv*) Taking $N = 24$ and $k = 3$, the above formula (4) becomes $x_{n+1} = \frac{1}{3}[2x_n + 24/x_n^2]$

Since an approximate value of $(24)^{1/3} = (27)^{1/3} = 3$, we take $x_0 = 3$.

Then
$$x_1 = \frac{1}{3}(2x_0 + 24/x_0^2) = \frac{1}{3}(6 + 24/9) = 2.88889$$
$$x_2 = \frac{1}{3}(2x_1 + 24/x_1^2) = \frac{1}{3}[(2 \times 2.88889) + 24/(2.88889)^2] = 2.88451$$
$$x_3 = \frac{1}{3}(2x_2 + 24/x_2^2) = \frac{1}{3}[2 \times 2.88451 + 24/(2.88451)^2] = 2.8845$$

Since $x_2 = x_3$ upto 4 decimal places, we take $(24)^{1/3} = 2.8845$

(*v*) Taking $N = 30$ and $k = -5$, the above formula (4) becomes
$$x_{n+1} = \frac{1}{-5}(-6x_n + 30/x_n^{-6}) = \frac{x_n}{5}(6 - 30x_n^5)$$

Since an approximate value of $(30)^{-1/5} = (32)^{-1/5} = 1/2$, we take $x_0 = 1/2$

Then
$$x_1 = \frac{x_0}{5}(6 - 30\,x_0^5) = \frac{1}{10}(6 - 30/2^5) = 0.50625$$
$$x_2 = \frac{x_1}{5}(6 - 30\,x_1^5) = \frac{0.50625}{5}[6 - 30(0.50625)^5] = 0.506495$$
$$x_3 = \frac{x_2}{5}(6 - 30\,x_2^5) = \frac{0.506495}{5}[6 - 30(0.506495)^5] = 0.506496.$$

Since $x_2 = x_3$ upto 4 decimal places, we take $(30)^{-1/5} = 0.5065$.

Example 28.1. *(a) Find a root of the equation $x^3 - 4x - 9 = 0$, using the bisection method correct to three decimal places.* *(Mumbai, 2003)*

(b) Using bisection method, find the negative root of the equation $x^2 - 4x + 9 = 0$. *(J.N.T.U., 2009)*

Solution. (*a*) Let $f(x) = x^3 - 4x - 9$

Since $f(2)$ is –ve and $f(3)$ is +ve, a root lies between 2 and 3

$\therefore$ first approximate to the root is

$$x_1 = \frac{1}{2}(2 + 3) = 2.5$$

Thus $f(x_1) = (2.5)^3 - 4(2.5) - 9 = -3.375$ *i.e.*, – ve

$\therefore$ the root lies between x_1 and 3. Thus the second approximation to the root is

$$x_2 = \frac{1}{2}(x_1 + 3) = 2.75$$

Then $f(x_2) = (2.75)^3 - 4(2.75) - 9 = 0.7969$ *i.e.*, + ve

$\therefore$ the root lies between x_1 and x_2. Thus the third approximation to the root is

$$x_3 = \frac{1}{2}(x_1 + x_2) = 2.625$$

Then $f(x_3) = (2.625)^3 - 4(2.625) - 9 = -1.4121$ *i.e.*, – ve

$\therefore$ the root lies between x_2 and x_3. Thus the fourth approximation to the root is

$$x_4 = \frac{1}{2}(x_2 + x_3) = 2.6875$$

Repeating this process, the successive approximations are

$$x_5 = 2.71875, \quad x_6 = 2.70313, \quad x_7 = 2.71094$$
$$x_8 = 2.70703, \quad x_9 = 2.70508, \quad x_{10} = 2.70605$$
$$x_{11} = 2.70654, \quad x_{12} = 2.70642$$

Hence the root is 2.7064

(*b*) If α, β, γ are the roots of the given equation, then $-\alpha, -\beta, -\gamma$ are the roots of $(-x)^3 - 4(-x) + 9 = 0$

$\therefore$ the negative root of the given equation is the positive root of $x^3 - 4x - 9 = 0$ which we have found above to be 2.7064.

Hence the negative root for the given equation is – 2.7064.

6. Finite Differences

6.0. Introduction

In this chapter we introduce the idea of finite differences and associated concepts, which have important applications in numerical analysis.

For example Interpolation formulae are based on finite differences. Through finite differences we study the relations that exist between the values that are assumed by the functions whenever the independent variables change by finite jumps.

In this chapter we study the variations when the independent variables change by equal intervals.

6.1. Difference Operators

In this section we introduce three difference operators namely forward, backward and central difference operators. Consider the function $y = f(x)$. Suppose we are given a table of values of the function at the points

$$x_0, x_1 = x_0 + h, \quad x_2 = x_0 + 2h, \cdots, x_n = x_0 + nh.$$

Let $f(x_0) = y_0, \; f(x_1) = y_1, \cdots, f(x_n) = y_n$.

We define

$$\boxed{\Delta[f(x)] = f(x+h) - f(x).}$$

Thus $\Delta y_0 = f(x_0 + h) - f(x_0) = f(x_1) - f(x_0) = y_1 - y_0$.

Similararly
$$\begin{aligned} \Delta y_1 &= y_2 - y_1 \\ \cdots &\quad \cdots \quad \cdots \\ \Delta y_{n-1} &= y_n - y_{n-1}. \end{aligned}$$

Δ is called the **forward difference operator** and $\Delta y_0, \Delta y_1, \cdots, \Delta y_{n-1}$ are called the **first forward differences** of the functions $y = f(x)$.

The **second order differences** of the function are defined by

$$\begin{aligned} \Delta^2 y_0 &= \Delta y_1 - \Delta y_0 \\ \Delta^2 y_1 &= \Delta y_2 - \Delta y_1 \\ \cdots &\quad \cdots \quad \cdots \\ \Delta^2 y_{n-1} &= \Delta y_n - \Delta y_{n-1}. \end{aligned}$$

In a similar manner higher order differences can be defined. In general the n th order differences are defined by the equations

$$\boxed{\Delta^n y_i = \Delta^{n-1} y_{i+1} - \Delta^{n-1} y_i.}$$

These differences of the function $y = f(x)$ can be systematically represented in the form of a table called **forward difference table**. We can construct the difference table for any number of arguments and a sample difference table is given for six consecutive arguments.

Forward differences table

x	$y=f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
x_0	y_0					
		Δy_0				
$x_1 = x_0 + h$	y_1		$\Delta^2 y_0$			
		Δy_1		$\Delta^3 y_0$		
$x_2 = x_0 + 2h$	y_2		$\Delta^2 y_1$		$\Delta^4 y_0$	
		Δy_2		$\Delta^3 y_1$		$\Delta^5 y_0$
$x_3 = x_0 + 3h$	y_3		$\Delta^2 y_2$		$\Delta^4 y_1$	
		Δy_3		$\Delta^3 y_2$		
$x_4 = x_0 + 4h$	y_4		$\Delta^2 y_3$			
		Δy_4				
$x_5 = x_0 + 5h$	y_5					

Note. In this table y_0 is known as the **first entry** and $\Delta y_0, \Delta^2 y_0, \cdots, \Delta^5 y_0$ are called **leading differences**.

Remark. Since each higher order difference is defined in terms of the previous lower differences by continuous substitution each higher order difference can be expressed in terms of the values of the function.

$$\begin{aligned}
\text{Thus,}\quad \Delta^2 y_0 &= \Delta y_1 - \Delta y_0 \\
&= (y_2 - y_1) - (y_1 - y_0) = y_2 - 2y_1 + y_0. \\
\Delta^3 y_0 &= \Delta^2 y_1 - \Delta^2 y_0 \\
&= (y_3 - 2y_2 + y_1) - (y_2 - 2y_1 + y_0) = y_3 - 3y_2 + 3y_1 - y_0. \\
\Delta^4 y_0 &= \Delta^3 y_1 - \Delta^3 y_0 \\
&= (y_4 - 3y_3 + 3y_2 - y_1) - (y_3 - 3y_2 + 3y_1 - y_0) \\
&= y_4 - 4y_3 + 6y_2 - 4y_1 + y_0.
\end{aligned}$$

We observe that the coefficients occurring in the RHS are simply the binomial coefficients in $(1-x)^n$. Hence in general we have

$$\Delta^n y_0 = y_n - n_{c_1} y_{n-1} + n_{c_2} y_{n-2} + \cdots + (-1)^n y_0$$

Properties of the operator Δ

1. Δ is linear. i.e. $\Delta[af(x) + bg(x)] = a\Delta[f(x)] + b\Delta[g(x)]$ where a, b are constants.

Proof.
$$\begin{aligned}\Delta[af(x)+bg(x)] &= [af(x+h)+bg(x+h)]-[af(x)+bg(x)]\\ &= a[f(x+h)-f(x)]+b[g(x+h)-g(x)]\\ &= a\Delta[f(x)]+b\Delta[g(x)]\end{aligned}$$

2. $\Delta^m\Delta^n[f(x)] = \Delta^{m+n}[f(x)]$

Proof.
$$\begin{aligned}\Delta^m\Delta^n[f(x)] &= (\Delta\Delta\cdots m \text{ times})\ (\Delta\Delta\cdots n\text{times})f(x)\\ &= [\Delta\Delta\cdots(m+n)\text{times}]f(x)\\ &= \Delta^{m+n}[f(x)].\end{aligned}$$

3. $\Delta[f(x)g(x)] = f(x+h)\Delta[g(x)]+g(x)\Delta[f(x)]$

Proof.
$$\begin{aligned}\Delta[f(x)g(x)] &= f(x+h)g(x+h)-f(x)g(x)\\ &= [f(x+h)g(x+h)-f(x+h)g(x)]\\ &\qquad +[f(x+h)g(x)-f(x)g(x)]\\ &= f(x+h)[g(x+h)-g(x)]+g(x)[f(x+h)-f(x)]\\ &= f(x+h)\Delta[g(x)]+g(x)\Delta[f(x)]\end{aligned}$$

4. $\Delta\left[\dfrac{f(x)}{g(x)}\right] = \dfrac{g(x)\Delta[f(x)]-f(x)\Delta[g(x)]}{g(x+h)g(x)}$

Proof.
$$\begin{aligned}\Delta\left[\frac{f(x)}{g(x)}\right] &= \frac{f(x+h)}{g(x+h)}-\frac{f(x)}{g(x)}\\ &= \frac{f(x+h)g(x)-f(x)g(x+h)}{g(x+h)g(x)}\\ &= \frac{f(x+h)g(x)-f(x)g(x)+f(x)g(x)-f(x)g(x+h)}{g(x+h)g(x)}\\ &= \frac{g(x)[f(x+h)-f(x)]-f(x)[g(x+h)-g(x)]}{g(x+h)g(x)}\\ &= \frac{g(x)\Delta[f(x)]-f(x)\Delta[g(x)]}{g(x+h)g(x)}\end{aligned}$$

Backward differences

Consider the function $y=f(x)$. Suppose we are given a table of values of the function at the points

$$x_0, x_1 = x_0+h, x_2 = x_0+2h, \cdots, x_n = x_0+nh$$

Let $f(x_0) = y_0, f(x_1){=}y_1, \cdots, f(x_n) = y_n$.
we define

$$\boxed{\nabla[f(x)] = f(x) - f(x-h)}$$

Thus,
$$\begin{aligned}
\nabla y_1 &= y_1 - y_0 \\
\nabla y_2 &= y_2 - y_1 \\
\cdots & \quad \cdots \quad \cdots \\
\nabla y_n &= y_n - y_{n-1}.
\end{aligned}$$

∇ is called the **backward difference operator** and $\nabla y_1, \nabla y_2, \cdots, \nabla y_n$ are called the **first order backward differences** of the function $y = f(x)$.

The **second order differences** of the function are defined by

$$\begin{aligned}
\nabla^2 y_2 &= \nabla y_2 - \nabla y_1 \\
\nabla^2 y_3 &= \nabla y_3 - \nabla y_2 \\
\cdots & \quad \cdots \quad \cdots \\
\nabla^2 y_n &= \nabla y_n - \nabla y_{n-1}.
\end{aligned}$$

In a similar manner higher order differences can be defined. In general the nth order differences are defined by

$$\boxed{\nabla^n y_i = \nabla^{n-1} y_i - \nabla^{n-1} y_{i-1}.}$$

These differences of the function $y = f(x)$ can be systematically represented in the form of a table called **backward difference table.**

Backward difference table (for 6 arguments)

x	$y=f(x)$	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$	$\nabla^5 y$
x_0	y_0					
		∇y_1				
$x_1 = x_0 + h$	y_1		$\nabla^2 y_2$			
		∇y_2		$\nabla^3 y_3$		
$x_2 = x_0 + 2h$	y_2		$\nabla^2 y_3$		$\nabla^4 y_4$	
		∇y_3		$\nabla^3 y_4$		$\nabla^5 y_5$
$x_3 = x_0 + 3h$	y_3		$\nabla^2 y_4$		$\nabla y^4 y_5$	
		∇y_4		$\nabla^3 y_5$		
$x_4 = x_0 + 4h$	y_4		$\nabla^2 y_5$			
		∇y_5				
$x_5 = x_0 + 5h$	y_5					

Remark 1. The relation between the two difference operators is given by $\nabla[f(x+h)] = \Delta f(x)$.

For, $\nabla[f(x+h)] = f(x+h) - f(x) = \Delta f(x)$

Similarly
$$\begin{aligned}\nabla^2[f(x+2h)] &= \nabla[f(x+2h) - f(x+h)] \\ &= \nabla f(x+2h) - \nabla f(x+h) \\ &= \Delta f(x+h) - \Delta f(x) \\ &= \Delta[f(x+h) - f(x)] \\ &= \Delta^2 f(x)\end{aligned}$$

In general $\boxed{\nabla[f(x+nh)] = \Delta^n f(x).}$

Hence from the forward difference table of the function $f(x)$ we can obtain backward differences of all orders.

Central difference Operator

Sometimes it is convenient to employ another system of differences known as central differences. We define **central difference operator** δ as

$$\boxed{\delta f(x) = f\left(x + \frac{h}{2}\right) - f\left(x - \frac{h}{2}\right).}$$

Thus if $f(x_i) = y_i$ then we have

$$\begin{aligned}\delta y_{\frac{1}{2}} &= y_1 - y_0 \\ \delta y_{\frac{3}{2}} &= y_2 - y_1 \\ &\vdots \\ \delta y_{\frac{n-1}{2}} &= y_n - y_{n-1}\end{aligned}$$

Here the subscript of δy is the average of the subscripts of the two members of the difference. The higher order differences can be defined similar to forward and backward differences.

$$\begin{aligned}\delta^2 y_1 &= \delta y_{\frac{3}{2}} - \delta y_{\frac{1}{2}} \\ \delta^2 y_2 &= \delta y_{\frac{5}{2}} - \delta y_{\frac{3}{2}} \\ \delta^3 y_{\frac{3}{2}} &= \delta^2 y_2 - \delta^2 y_1 \quad \text{etc.}\end{aligned}$$

These differences of the function $y = f(x)$ can be systematically represented in the form of a table called *central difference* table

Central difference table

x	$y = f(x)$	δy	$\delta^2 y$	$\delta^3 y$	$\delta^4 y$
x_0	y_0				
		$\delta y_{\frac{1}{2}}$			
$x_1 = x_0 + h$	y_1		$\delta^2 y_1$		
		$\delta y_{\frac{3}{2}}$		$\delta^3 y_{\frac{3}{2}}$	
$x_2 = x_0 + 2h$	y_2		$\delta^2 y_2$		$\delta^4 y_2$
		$\delta y_{\frac{5}{2}}$		$\delta^3 y_{\frac{5}{2}}$	
$x_3 = x_0 + 3h$	y_3		$\delta^2 y_3$		
		$\delta y_{\frac{7}{2}}$			
$x_4 = x_0 + 4h$	y_4				

Theorem 6.1 (Fundamental theorem for finite differences). Let $f(x)$ be a polynomial of degree n. Then the nth difference of $f(x)$ is a constant and all higher order differences are zero.

$$(i.e.)\Delta^r[f(x)] = \begin{cases} \text{constant} & \text{if } r = n \\ 0 & \text{if } r > n. \end{cases}$$

Proof. Let $f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n$ where $a_0 \neq 0$ be a polynomial of degree n. (a_0 is called the leading coefficient of $f(x)$).

$$\begin{aligned} \Delta f(x) &= f(x+h) - f(x) \\ &= [a_0(x+h)^n + a_1(x+h)^{n-1} + \cdots + a_{n-1}(x+h) + a_n] \\ &\qquad -[a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n] \\ &= [a_0(x^n + n_{c_1}x^{n-1}h + \cdots + h^n) \\ &\qquad +a_1(x^{n-1} + (n-1)_{c_1}x^{n-2}h + \cdots + h^{n-1}) + \cdots + a_n] \\ &\qquad -[a_0x^n + a_1x^{n-1} + \cdots + a_n]. \end{aligned}$$

$= a_0nhx^{n-1} +$ terms involving lower powers of x.

Thus $\Delta f(x)$ is a polynomial of degree $n-1$ with leading coefficient nha_0.

Similarly $\Delta^2 f(x)$ is a polynomial of degree $n-2$ with leading coefficient $n(n-1)h^2a_0$.

Continuing this process we get $\Delta^n f(x)$ is a polynomial of degree zero with leading coefficient $n(n-1)\cdots 1.h^na_0 = n!h^na_0$.

Since $\Delta^n f(x)$ is a constant polynomial it follows that $\Delta^n f(x) = n!h^na_0$.

$\therefore$ The nth difference of $f(x)$ is a constant and all the higher order differences of order greater than n are zero.

Definition A product of the form $x(x-h)(x-2h)\cdots[x-(n-1)h]$ is called a **factorial function** and is denoted by $x^{(n)}$

$$\text{i.e. } x^{(n)} = x(x-h)(x-2h)\cdots[x-(n-1)h].$$

If we take $h = 1$, then $x^{(1)} = x$, $x^{(2)} = x(x-1)$ and $x^{(3)} = x(x-1)(x-2)$.

We observe that $x^{(n)}$ is a polynomial of degree n with leading coefficient 1.

The following theorem shows that the formula for the first difference of $x^{(n)}$ is obtained by the simple rule of differentiation.

Theorem 6.2. $\Delta x^{(n)} = nhx^{(n-1)}$.

In particular when $h = 1, \Delta x^{(n)} = nx^{(n-1)}$.

Proof.

$$\begin{aligned}
\Delta x^{(n)} &= (x+h)^{(n)} - x^{(n)} \\
&= (x+h)x(x-h)\cdots[x-(n-2)h] \\
&\qquad -x(x-h)(x-2h)\cdots[x-(n-1)h] \\
&= x(x-h)(x-2h)\cdots[x-(n-2)h]\{(x+h)-[x-(n-1)h]\} \\
&= x^{(n-1)}nh. \\
\therefore\ \Delta x^{(n)} &= nhx^{(n-1)}.
\end{aligned}$$

When $h = 1$, $\Delta x^{(n)} = nx^{(n-1)}$. —— (I)

Remark 1. From (I) we get the formula for first order difference which is obtained by the simple differentiation rule.

For example
$$\begin{aligned}
\Delta^2 x^{(n)} &= \Delta[nhx^{(n-1)}] \\
&= nh\Delta x^{(n-1)} \\
&= n(n-1)h^2 x^{(n-2)}
\end{aligned}$$

Proceeding like this we get,

$$\Delta^n x^{(n)} = n(n-1)\cdots 1h^n x^{(0)} = n!h^n$$

Remark 2. Any polynomial $f(x)$ of degree n can be expressed in the form

$f(x) = c_0 x^{(n)} + c_1 x^{(n-1)} + \cdots + c_{n-1}x^{(1)} + c_n$. If $f(x)$ is divided succesively by $x-0$, $x-1$, $x-2, \cdots, x-(n-1)$, then the remainders give the coefficients $c_n, c_{n-1}, \cdots, c_0$. Using this expresion and Theorem 6.2, $\Delta f(x)$ can be computed.

Definition. The **reciprocal factorial** function $x^{(-n)}$ is defined as

$$\boxed{x^{(-n)} = \frac{1}{(x+h)(x+2h)\cdots(x+nh)}}$$

where n is a positive integer.

As in the case of factorial function the formula for first order difference of $x^{(-n)}$ is similar to differentiation rule when $h = 1$.

Theorem 6.3. $\Delta x^{(-n)} = (-n)hx^{[-(n+1)]}$.

In particular when $h = 1, \Delta x^{(-n)} = -nx^{[-(n+1)]}$

Proof.

$$\begin{aligned}
\Delta x^{(-n)} &= (x+h)^{(-n)} - x^{(-n)} \\
&= \frac{1}{(x+2h)(x+3h)\cdots[x+(n+1)h]} - \frac{1}{(x+h)(x+2h)\cdots(x+nh)} \\
&= \frac{1}{(x+h)(x+2h)\cdots[x+(n+1)h]}\{x+h-[x+(n+1)h]\} \\
&= \frac{-nh}{(x+h)(x+2h)\cdots[x+(n+1)h]} \\
\therefore \Delta x^{(-n)} &= -nhx^{[-(n+1)]}.
\end{aligned}$$

When $h = 1$, $\Delta x^{(-n)} = -nx^{[-(n+1)]}$

Remark.

$$\begin{aligned}
\Delta^2 x^{(-n)} = \Delta[-nhx^{[-(n+1)]} &= -nh[-(n+1)h]x^{[-(n+2)]} \\
&= (-1)^2h^2n(n+1)x^{[-(n+2)]}.
\end{aligned}$$

In general $\Delta^r x^{(-n)} = (-1)^r h^r n(n+1)\cdots(n+r-1)x^{[-(n+r)]}$.

Example 1. Let $f(x) = \dfrac{1}{(x+1)(x+2)(x+3)} = x^{(-3)}$.

$\Delta f(x) = -3x^{(-4)}$; $\Delta^2 f(x) = 3\times 4x^{(-5)}$; $\Delta^3 f(x) = -3\times 4\times 5x^{(-6)}$ etc.

Example 2. Let

$$\begin{aligned}
y &= \frac{1}{x(x+1)(x+2)} \\
&= \frac{1}{[(x-1)+1][(x-1)+2][(x-1)+3]} \\
&= (x-1)^{(-3)} \\
\Delta y &= -3(x-1)^{(-4)} \\
\Delta^2 y &= 3\times 4(x-1)^{(-5)} \\
\Delta^3 y &= -3\times 4\times 5(x-1)^{(-6)} \text{ etc.}
\end{aligned}$$

Solved problems

Problem 1. Form the forward difference table for the following data.

$x:$	0	1	2	3	4
$y:$	8	11	9	15	6

Solution.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0	8				
		3			
1	11		-5		
		-2		13	
2	9		8		-36
		6		-23	
3	15		-15		
		-9			
4	6				

Problem 2. Find $\Delta(2^x)$

Solution.

$$\begin{aligned} \Delta 2^x &= 2^{x+h} - 2^x \\ &= 2^x(2^h - 1) \end{aligned}$$

Problem 3. Find the *nth* difference of e^x.

Solution.

$$\begin{aligned} \Delta e^x &= e^{x+h} - e^x \\ &= e^x(e^h - 1) \\ \Delta^2 e^x &= \Delta(\Delta e^x) \\ &= \Delta[e^x(e^h - 1)] \\ &= (e^h - 1)\Delta e^x \\ &= (e^h - 1)e^x(e^h - 1) \\ &= e^x(e^h - 1)^2. \end{aligned}$$

Similarly $\Delta^3 e^x = e^x(e^h - 1)^3$.

Proceeding like this we get $\Delta^n e^x = e^x(e^h - 1)^n$.

Problem 4. Prove that $\Delta \log f(x) = \log\left[1 + \dfrac{\Delta f(x)}{f(x)}\right]$

Solution.

$$\begin{aligned}
\Delta \log f(x) &= \log f(x+h) - \log f(x) \\
&= \log\left[\frac{f(x+h)}{f(x)}\right] \\
&= \log\left[\frac{f(x)+f(x+h)-f(x)}{f(x)}\right] \\
&= \log\left[\frac{f(x)+\Delta f(x)}{f(x)}\right] \\
&= \log\left[1+\frac{\Delta f(x)}{f(x)}\right]
\end{aligned}$$

Problem 5. Prove that $\Delta\left[\tan^{-1}\left(\frac{n-1}{n}\right)\right] = \tan^{-1}\left(\frac{1}{2n^2}\right)$.

Solution. Without loss of generality we take the interval of differencing as $h = 1$.

$$\begin{aligned}
\Delta \tan^{-1}\left(\frac{n-1}{n}\right) &= \tan^{-1}\left(\frac{(n+1)-1}{n+1}\right) - \tan^{-1}\left(\frac{n-1}{n}\right) \\
&= \tan^{-1}\left(\frac{n}{n+1}\right) - \tan^{-1}\left(\frac{n-1}{n}\right) \\
&= \tan^{-1}\left(\frac{\frac{n}{n+1} - \frac{n-1}{n}}{1+\frac{n}{n+1}\times\frac{n-1}{n}}\right) \\
&= \tan^{-1}\left[\frac{n^2-(n^2-1)}{n(n+1)+n(n-1)}\right] \\
&= \tan^{-1}\left(\frac{1}{2n^2}\right).
\end{aligned}$$

Problem 6. If $f(x) = \dfrac{x}{x^2+7x+12}$ find $\Delta f(x)$ taking the interval of differencing as unity.

Solution. $f(x) = \dfrac{x}{x^2+7x+12} = \dfrac{4}{x+4} - \dfrac{3}{x+3}$ (by partial fraction)

$$\begin{aligned}
\therefore \Delta f(x) &= \left[\frac{4}{(x+1)+4} - \frac{3}{(x+1)+3}\right] - \left[\frac{4}{x+4} - \frac{3}{x+3}\right] \\
&= \frac{4}{x+5} - \frac{3}{x+4} - \frac{4}{x+4} + \frac{3}{x+3} \\
&= \frac{4}{x+5} - \frac{7}{x+4} + \frac{3}{x+3}.
\end{aligned}$$

Problem 7. Find the first and second order differences for $f(x) = ab^{cx}$

Solution.

$$\begin{aligned}
\Delta f(x) &= f(x+h) - f(x) \\
&= ab^{c(x+h)} - ab^{cx} \\
&= ab^{cx}b^{ch} - ab^{cx} \\
&= ab^{cx}(b^{ch} - 1) \\
&= (b^{ch} - 1)ab^{cx} \\
\Delta^2 f(x) &= (b^{ch} - 1)\Delta(ab^{cx}) \\
&= (b^{ch} - 1)[(b^{ch} - 1)ab^{cx}] \\
&= (b^{ch} - 1)^2 ab^{cx}.
\end{aligned}$$

Problem 8. Evaluate $(\Delta - \nabla)x^2$ taking the interval of differencing as h.

Solution.

$$\begin{aligned}
(\Delta - \nabla)x^2 &= \Delta x^2 - \nabla x^2 \\
&= [(x+h)^2 - x^2] - [x^2 - (x-h)^2] \\
&= (2xh + h^2) - (2xh - h^2) \\
&= 2h^2.
\end{aligned}$$

Problem 9. Find $\Delta^n \sin x$ taking $h = 1$.

Solution.

$$\begin{aligned}
\Delta \sin x &= \sin(x+1) - \sin x \\
&= 2\cos\left(x + \frac{1}{2}\right)\sin\left(\frac{1}{2}\right) \\
&= 2\sin\left(\frac{1}{2}\right)\sin\left(\frac{\pi}{2} + x + \frac{1}{2}\right) \\
\Delta^2 \sin x &= 2\sin\left(\frac{1}{2}\right)\Delta\sin\left(\frac{\pi}{2} + x + \frac{1}{2}\right) \\
&= 2\sin\left(\frac{1}{2}\right)\left[\sin\left(\frac{\pi}{2} + x + 1 + \frac{1}{2}\right) - \sin\left(\frac{\pi}{2} + x + \frac{1}{2}\right)\right] \\
&= 2\sin\left(\frac{1}{2}\right)\left[2\cos\left(\frac{\pi}{2} + x + 1\right).\sin\left(\frac{1}{2}\right)\right] \\
&= \left[2\sin(\frac{1}{2})\right]^2 \sin\left[\frac{\pi}{2} + \left(\frac{\pi}{2} + x + 1\right)\right] \\
&= \left[2\sin\left(\frac{1}{2}\right)\right]^2 \sin\left[2\left(\frac{\pi}{2} + \frac{1}{2}\right) + x\right].
\end{aligned}$$

Proceeding like this we get,

$$\Delta^n \sin x = \left[2\sin\left(\frac{1}{2}\right)\right]^n \sin\left[n\left(\frac{\pi}{2}+\frac{1}{2}\right)+x\right].$$

Problem 10. Evaluate $\Delta^{10}[(1-x)(1-2x^2)(1-3x^3)(1-4x^4)]$.

Solution. Let $y=(1-x)(1-2x^2)(1-3x^3)(1-4x^4)$

This is a polynomial of degree 10 with leading coefficient 24.

Since $\Delta^n y = \begin{cases} 0 & \text{if } n>10 \\ \text{constant} & \text{if } n=10 \end{cases}$ we have

$$\begin{aligned} \Delta^{10} y &= \Delta^{10}(24x^{10}) \\ &= 24\Delta^{10}(x^{10}) \\ &= 24 \times 10!. \end{aligned}$$

Problem 11. Find the sixth term of the sequence $2, 6, 12, 20, 30, \cdots$

Solution. Let a be the sixth term.

The difference table for the given data is as follows:

x	y	Δy	$\Delta^2 y$	Δ^3	$\Delta^4 y$	$\Delta^5 y$
0	2					
		4				
1	6		2			
		6		0		
2	12		2		0	
		8		0		$a-42$
3	20		2		$a-42$	
		10		$a-42$		
4	30		$a-40$			
		$a-30$				
5	a					

Since five values are given $\Delta^5 y = 0$.

Hence $a-42=0$. Thus $a=42$.

$\therefore$ The sixth term of the given sequence is 42.

Problem 12. If $u_1=1$, $u_3=17$, $u_4=43$, $u_5=89$ find u_2.

Solution. Let $u_2=a$.

We have the following difference table.

x	u	Δu	$\Delta^2 u$	$\Delta^3 u$	$\Delta^4 u$
1	1				
		$a-1$			
2	a		$18-2a$		
		$17-a$		$-9+3a$	
3	17		$9+a$		$20-4a$
		26		$11-a$	
4	43		20		
		46			
5	89				

Since four values are given $\Delta^4 u = 0$.

$\therefore \qquad 20 - 4a = 0$. Hence $a = 5$.

Hence $u_2 = 5$.

Problem 13. Express $2x^3 - 3x^2 + 4x - 8$ as a factorial polynomial.

Solution. Let the factorial polynomial be $Ax^{(3)} + Bx^{(2)} + Cx^{(1)} + D$.

$$\therefore \qquad 2x^3 - 3x^2 + 4x - 8 = Ax^{(3)} + Bx^{(2)} + Cx^{(1)} + D$$

where A, B, C, D are to be determined by synthetic division.

$$\begin{array}{c|cccc}
0 & 2 & -3 & 4 & -8 \\
 & & 0 & 0 & 0 \\
\hline
1 & 2 & -3 & 4 & \boxed{-8} \\
 & & 2 & -1 & \\
\hline
2 & 2 & -1 & \boxed{3} & \\
 & & 4 & & \\
\hline
 & 2 & \boxed{3} & &
\end{array}$$

$\therefore$ The factorial polynomial is $2x^{(3)} + 3x^{(2)} + 3x^{(1)} - 8$.

Problem 14. Show that

$$\Delta(5x^4 + 6x^3 + x^2 - x + 7) = 20x^{(3)} + 108x^{(2)} + 108x^{(1)} + 11.$$

Solution. Let $y = 5x^4 + 6x^3 + x^2 - x + 7$.

Let $Ax^{(4)} + Bx^{(3)} + Cx^{(2)} + Dx^{(1)} + E$ be the factorial polynomial of y.

$$\begin{array}{c|ccccc}
0 & 5 & 6 & 1 & -1 & 7 \\
 & & 0 & 0 & 0 & 0 \\
\hline
1 & 5 & 6 & 1 & -1 & \boxed{7} \\
 & & 5 & 11 & 12 & \\
\hline
2 & 5 & 11 & 12 & \boxed{11} & \\
 & & 10 & 42 & & \\
\hline
3 & 5 & 21 & \boxed{54} & & \\
 & & 15 & & & \\
\hline
 & 5 & \boxed{36} & & &
\end{array}$$

$$\begin{aligned}
\therefore \quad y &= 5x^{(4)} + 36x^{(3)} + 54x^{(2)} + 11x^{(1)} + 7 \\
\therefore \quad \Delta y &= 20x^{(3)} + 108x^{(2)} + 108x^{(1)} + 11.
\end{aligned}$$

Problem 15. Find the second difference of the polynomial

$$f(x) = x^4 - 12x^3 + 42x^2 - 30x + 9 \text{ with } h = 2.$$

Solution. First we shall express the given polynomial $f(x)$ in terms of factorial polynomial by synthetic division with $h = 2$.

$$\begin{array}{c|ccccc}
0 & 1 & -12 & 42 & -30 & 9 \\
 & & 0 & 0 & 0 & 0 \\
\hline
2 & 1 & -12 & 42 & -30 & \boxed{9} \\
 & & 2 & -20 & 44 & \\
\hline
4 & 1 & -10 & 22 & \boxed{14} & \\
 & & 4 & -24 & & \\
\hline
6 & 1 & -6 & \boxed{-2} & & \\
 & & 6 & & & \\
\hline
 & 1 & \boxed{0} & & &
\end{array}$$

$$\begin{aligned}
\therefore f(x) &= x^{(4)} - 2x^{(2)} + 14x^{(1)} + 9. \\
\therefore \Delta f(x) &= 8x^{(3)} - 8x^{(1)} + 28 \qquad (\because\ h = 2) \\
\Delta^2 f(x) &= 48x^{(2)} - 16 \\
&= 48x(x-2) - 16 \\
&= 48x^2 - 96x - 16.
\end{aligned}$$

Problem 16. Find the function whose first difference is $x^3 + 3x^2 + 5x + 12$.

Solution. Given $\Delta y = x^3 + 3x^2 + 5x + 12$.

We express this in terms of factorial polynomial.

$$
\begin{array}{c|cccc}
 & 1 & 3 & 5 & 12 \\
 & & 0 & 0 & 0 \\
\hline
1 & 1 & 3 & 5 & \boxed{12} \\
 & & 1 & 4 & \\
\hline
2 & 1 & 4 & \boxed{9} & \\
 & & 2 & & \\
\hline
 & 1 & \boxed{6} & &
\end{array}
$$

$$
\begin{aligned}
\therefore \Delta y &= x^{(3)} + 6x^{(2)} + 9x^{(1)} + 12 \\
\therefore \ y &= \Delta^{-1}[x^{(3)} + 6x^{(2)} + 9x^{(1)} + 12] \\
&= \frac{x^{(4)}}{4} + 2x^{(3)} + \frac{9x^{(2)}}{2} + 12x^{(1)} + c \\
&= \frac{1}{4}[x(x-1)(x-2)(x-3)] + 2[x(x-1)(x-2)] \\
&\qquad + \frac{9}{2}x(x-1) + 12x + c
\end{aligned}
$$

6.2. Other Difference Operators

In this section we introduce the shift operator E and averaging operator μ.

Definition. The shift operator E is defined by

$$\boxed{Ef(x) = f(x+h)}$$

Hence $E^2 f(x) = Ef(x+h) = f(x+2h)$.

In general for any *positive integer* n

$$\boxed{E^n f(x) = f(x+nh)}$$

In particular we have
$$\begin{aligned} Ey_0 &= y_1, \\ E^2 y_0 &= y_2 \\ \cdots & \quad \cdots \\ E^n y_0 &= y_n. \end{aligned}$$

The inverse operator E^{-1} is defined as

$$\boxed{E^{-1}f(x) = f(x-h)}$$

For any *real number* n we have

$$\boxed{E^n f(x) = f(x+nh)}$$

Note. $E^m E^n f(x) = E^{m+n} f(x)$.

Definition. The averaging operator μ is defined by

$$\boxed{\mu f(x) = \frac{f\left(x+\frac{h}{2}\right) + f\left(x-\frac{h}{2}\right)}{2}.}$$

There are several relations connecting the operators $\Delta, \nabla, \delta, E, \mu$ and the differentiation operator D. These results are presented in the following theorems.

Theorem 6.4. $E = 1 + \Delta$

Proof.

$$\begin{aligned} \Delta f(x) &= f(x+h) - f(x) \\ &= Ef(x) - f(x) \\ &= (E-1)f(x) \\ \text{Hence} \quad \Delta &= E - 1 \\ \therefore E &= 1 + \Delta. \end{aligned}$$

Theorem 6.5. $\nabla = 1 - E^{-1}$.

Proof.

$$\begin{aligned} \nabla f(x) &= f(x) - f(x-h) \\ &= f(x) - E^{-1}f(x) \\ &= (1 - E^{-1})f(x) \\ \text{Hence} \quad \nabla &= 1 - E^{-1}. \end{aligned}$$

Theorem 6.6. $\delta = E^{\frac{1}{2}} - E^{-\frac{1}{2}}$.

Proof.

$$\begin{aligned} \delta f(x) &= f\left(x+\frac{h}{2}\right) - f\left(x-\frac{h}{2}\right) \\ &= E^{\frac{1}{2}} f(x) - E^{-\frac{1}{2}} f(x) \\ &= \left(E^{\frac{1}{2}} - E^{-\frac{1}{2}}\right) f(x) \\ \therefore \delta &= E^{\frac{1}{2}} - E^{-\frac{1}{2}}. \end{aligned}$$

Theorem 6.7. $\mu = \dfrac{E^{\frac{1}{2}} + E^{-\frac{1}{2}}}{2}$

Proof.

$$\begin{aligned}
\mu f(x) &= \frac{f(x+\frac{h}{2}) + f(x-\frac{h}{2})}{2} \\
&= \frac{E^{\frac{1}{2}} f(x) + E^{-\frac{1}{2}} f(x)}{2} \\
&= \left(\frac{E^{\frac{1}{2}} + E^{-\frac{1}{2}}}{2}\right) f(x) \\
\therefore \mu &= \frac{E^{\frac{1}{2}} + E^{-\frac{1}{2}}}{2}
\end{aligned}$$

Theorem 6.8. $\delta = E^{\frac{1}{2}} \nabla$

Proof.

$$\begin{aligned}
\delta f(x) &= (E^{\frac{1}{2}} - E^{-\frac{1}{2}}) f(x) \\
&= E^{\frac{1}{2}} (1 - E^{-1}) f(x) \\
&= E^{\frac{1}{2}} \nabla f(x) \quad \text{(using theorem 6.5)} \\
\therefore \delta &= E^{\frac{1}{2}} \nabla
\end{aligned}$$

Theorem 6.9. $E = e^{hD}$.

Proof. The Taylor's series expansion of $y = f(x)$ is given by

$$\begin{aligned}
f(x+h) &= f(x) + hf'(x) + \frac{h^2 f''(x)}{2!} + \cdots + \frac{h^n f^{(n)}(x)}{n!} + \cdots \\
\therefore Ef(x) &= f(x) + hD[f(x)] + \frac{h^2}{2!} D^2[f(x)] + \cdots + \frac{h^n}{n!} D^n[f(x)] + \cdots \\
&= [1 + hD + \frac{h^2}{2!} D^2 + \cdots + \frac{h^n}{n!} D^n + \cdots] f(x) \\
\text{Hence } E &= 1 + hD + \frac{h^2 D^2}{2!} + \cdots + \frac{h^n D^n}{n!} + \cdots = e^{hD}.
\end{aligned}$$

Thus $E = e^{hD}$.

Theorem 6.10. $D = \dfrac{1}{h}\left[\Delta - \dfrac{\Delta^2}{2} + \dfrac{\Delta^3}{3} - \cdots\right]$.

Proof.

$$E = e^{hD} \text{ (by Theorem 6.9)}$$

$$\therefore hD = \log E = \log(1+\Delta)$$
$$= \Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} - \dots$$
$$\therefore D = \frac{1}{h}\left[\Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} - \dots\right].$$

Note. Taking E as the fundamental operator we have expressed the other operators $\Delta, \nabla, \delta, \mu, D$ in terms of E as

1. $\Delta = E - 1$
2. $\nabla = 1 - E^{-1}$
3. $\delta = E^{\frac{1}{2}} - E^{-\frac{1}{2}}$
4. $\mu = \frac{E^{\frac{1}{2}} + E^{-\frac{1}{2}}}{2}$
5. $D = \frac{1}{h}\log E.$

Solved Problems

Problem 1. Prove that $E\nabla = \nabla E = \Delta$

Solution. $E\nabla = E(1 - E^{-1}) = E - 1 = \Delta.$

Also $\nabla E = (1 - E^{-1})E = E - 1 = \Delta.$

Problem 2. Prove that $(E^{\frac{1}{2}} + E^{-\frac{1}{2}})(1+\Delta)^{\frac{1}{2}} = 2 + \Delta.$

Solution.

$$(E^{\frac{1}{2}} + E^{-\frac{1}{2}})(1+\Delta)^{\frac{1}{2}} = (E^{\frac{1}{2}} + E^{-\frac{1}{2}})E^{\frac{1}{2}}$$
$$= E + 1$$
$$= (1+\Delta) + 1$$
$$= 2 + \Delta.$$

Problem 3. Prove $\nabla\Delta = \Delta - \nabla = \delta^2.$

Solution.

$$\nabla\Delta = (1 - E^{-1})(E - 1)$$
$$= E - 1 - 1 + E^{-1}$$
$$= E + E^{-1} - 2$$
$$= (E^{\frac{1}{2}} - E^{-\frac{1}{2}})^2$$
$$= \delta^2.$$

Also
$$
\begin{aligned}
\Delta - \nabla &= (E-1)-(1-E^{-1}) \\
&= E-1-1+E^{-1} \\
&= E+E^{-1}-2 \\
&= (E^{\frac{1}{2}}-E^{-\frac{1}{2}})^2 \\
&= \delta^2.
\end{aligned}
$$

Problem 4. Prove that $E^{\frac{1}{2}} = \mu + \frac{1}{2}\delta$.

Solution. We know that

$$
\begin{aligned}
\mu &= \frac{1}{2}(E^{\frac{1}{2}}+E^{-\frac{1}{2}}) \text{ and } \delta = E^{\frac{1}{2}}-E^{-\frac{1}{2}} \\
\therefore \quad \mu + \frac{1}{2}\delta &= \frac{1}{2}(E^{\frac{1}{2}}+E^{-\frac{1}{2}}) + \frac{1}{2}(E^{\frac{1}{2}}-E^{-\frac{1}{2}}) = E^{\frac{1}{2}}.
\end{aligned}
$$

Problem 5. Prove that $\mu\delta = \dfrac{\Delta}{2} + \dfrac{\Delta E^{-1}}{2}$

Solution.
$$
\begin{aligned}
\frac{\Delta}{2} + \frac{\Delta E^{-1}}{2} &= \frac{\Delta}{2}(1+E^{-1}) \\
&= \frac{1}{2}(E-1)(1+E^{-1}) (\because \Delta = E-1) \\
&= \frac{1}{2}(E+EE^{-1}-1-E^{-1}) \\
&= \frac{1}{2}(E-E^{-1}) \\
&= \left(\frac{E^{\frac{1}{2}}+E^{-\frac{1}{2}}}{2}\right)(E^{\frac{1}{2}}-E^{-\frac{1}{2}}) \\
&= \mu\delta \text{ (by definition of } \mu \text{ and } \delta).
\end{aligned}
$$

Problem 6. Prove that $1 - e^{-hD} = \nabla$.

Solution. We know that
$$
\begin{aligned}
D &= \frac{1}{h}\log E \\
\therefore \quad hD &= \log E \\
\therefore \quad e^{hD} &= E \\
\therefore \quad \frac{1}{e^{hD}} &= \frac{1}{E} \\
\therefore \quad e^{-hD} &= E^{-1} = 1-\nabla \\
\therefore \quad \nabla &= 1-e^{-hD}.
\end{aligned}
$$

Problem 7. Prove that $\Delta + \nabla = \frac{\Delta}{\nabla} - \frac{\nabla}{\Delta}$.

Solution. We know that $\Delta = E - 1$ and $\nabla = 1 - E^{-1}$.

$$\begin{aligned}
\therefore \quad \frac{\Delta}{\nabla} - \frac{\nabla}{\Delta} &= \frac{E-1}{1-E^{-1}} - \frac{1-E^{-1}}{E-1} \\
&= \frac{(E-1)^2 - (1-E^{-1})^2}{(1-E^{-1})(E-1)} \\
&= \frac{(E-1-1+E^{-1})(E-1+1-E^{-1})}{(E-1-1+E^{-1})} \\
&= E - 1 + 1 - E^{-1} \\
&= \Delta + \nabla
\end{aligned}$$

Problem 8. Prove that $\delta = \Delta E^{-\frac{1}{2}}$ and hence prove that $E = \left(\frac{\Delta}{\delta}\right)^2$.

Solution.

$$\begin{aligned}
\Delta E^{-\frac{1}{2}} f(x) &= \Delta f\left(x - \frac{h}{2}\right) \\
&= f\left(x - \frac{h}{2} + h\right) - f\left(x - \frac{h}{2}\right) \\
&= f\left(x + \frac{h}{2}\right) - f\left(x - \frac{h}{2}\right) \\
&= \delta f(x) \\
\therefore \Delta E^{\frac{-1}{2}} &= \delta \\
\therefore E^{-\frac{1}{2}} &= \frac{\delta}{\Delta} \\
\therefore E^{\frac{1}{2}} &= \frac{\Delta}{\delta} \\
\therefore E &= \left(\frac{\Delta}{\delta}\right)^2.
\end{aligned}$$

Problem 9. Prove that

$$hD = \log(1 + \Delta) = -\log(1 - \nabla) = \sinh^{-1}(\mu\delta).$$

Solution. We know that

$$\begin{aligned}
E &= e^{hD} \text{ (by Theorem 6.9)} \\
\therefore e^{hD} &= 1 + \Delta.
\end{aligned}$$

Taking logarithm on both sides we have $hD = \log(1+\Delta)$.

Also $\nabla = 1 - E^{-1}$.

$$\begin{aligned} \therefore\ E^{-1} &= 1-\nabla \\ (e^{hD})^{-1} &= 1-\nabla \\ \text{i.e.,}\quad e^{-hD} &= 1-\nabla. \end{aligned}$$

Taking logarithm on both sides we have

$$\begin{aligned} -hD &= \log(1-\nabla) \\ \therefore\ hD &= -\log(1-\nabla). \end{aligned}$$

$$\begin{aligned} \text{Now, } \sinh(hD) &= \frac{e^{hD}-e^{-hD}}{2} \quad \text{(by definition of hyperbolic function)} \\ &= \frac{E-E^{-1}}{2} \\ &= \frac{1}{2}\left[(E^{\frac{1}{2}}+E^{-\frac{1}{2}})(E^{\frac{1}{2}}-E^{-\frac{1}{2}})\right] \\ &= \left(\frac{E^{\frac{1}{2}}+E^{-\frac{1}{2}}}{2}\right)(E^{\frac{1}{2}}-E^{-\frac{1}{2}}) \\ &= \mu\delta \\ \therefore\ hD &= \sinh^{(-1)}(\mu\delta). \end{aligned}$$

Problem 10. Prove that $\frac{1}{2}\delta^2 + \delta\sqrt{1+\frac{\delta^2}{4}} = \Delta$.

Solution.

$$\begin{aligned} \frac{1}{2}\delta^2 + \delta\sqrt{1+\frac{\delta^2}{4}} &= \frac{1}{2}\delta\left[\delta + 2\sqrt{1+\frac{\delta^2}{4}}\right] \\ &= \frac{1}{2}\delta\left[\delta + \sqrt{4+\delta^2}\right] \\ &= \frac{1}{2}\delta\left[(E^{\frac{1}{2}}-E^{-\frac{1}{2}}) + \sqrt{4+(E^{\frac{1}{2}}-E^{-\frac{1}{2}})^2}\right] \\ &= \frac{1}{2}\delta\left[(E^{\frac{1}{2}}-E^{-\frac{1}{2}}) + \sqrt{(E^{\frac{1}{2}}+E^{-\frac{1}{2}})^2}\right] \\ &= \frac{1}{2}\delta\left[(E^{\frac{1}{2}}-E^{-\frac{1}{2}}) + (E^{\frac{1}{2}}+E^{-\frac{1}{2}})\right] \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\delta[2E^{\frac{1}{2}}] \\
&= \delta E^{\frac{1}{2}} \\
&= (E^{\frac{1}{2}} - E^{-\frac{1}{2}})E^{\frac{1}{2}} \\
&= E - 1 \\
&= \Delta.
\end{aligned}
$$

Problem 11. Prove $\nabla^r f(x) = \Delta^r f(x-r)$ for any positive integer r.

Solution. We know that $\nabla = 1 - E^{-1}$ and $\Delta = E - 1$.

We prove the required result by induction on r. When $r = 1$,

$$
\begin{aligned}
\nabla f(x) &= (1 - E^{-1})f(x) \\
&= f(x) - f(x-1) \\
&= \Delta f(x-1)
\end{aligned}
$$

$\therefore$ The result is true for $r = 1$.

Let us assume that the result is true for $r = k$.

$$
\begin{aligned}
\therefore \nabla^k f(x) &= \Delta^k f(x-k) \\
\text{Now, } \nabla^{k+1} f(x) &= \nabla(\nabla^k f(x)) \\
&= \nabla(\Delta^k f(x-k)) \\
&= (1 - E^{-1})\Delta^k f(x-k) \\
&= \Delta^k f(x-k) - \Delta^k f(x-k-1) \\
&= \Delta^k [f(x-k) - f(x-(k+1))] \\
&= \Delta^k [\Delta f(x-(k+1))] \\
&= \Delta^{k+1} f(x-(k+1)).
\end{aligned}
$$

Hence the result is true for $r = k+1$.

$\therefore$ $\Delta^r f(x) = \Delta^r f(x-r)$ for all natural numbers r.

Problem 12. Taking $h = 1$, find $(\Delta + \nabla)^2 f(x)$ where $f(x) = x^2 + x$.

Solution.

$$
\begin{aligned}
(\Delta + \nabla)^2 f(x) &= (E - 1 + 1 - E^{-1})^2(x^2 + x) \\
&= (E - E^{-1})^2(x^2 + x) \\
&= (E^2 + E^{-2} - 2)(x^2 + x) \\
&= \left[(x+2)^2 + (x+2)\right] + \left[(x-2)^2 + (x-2)\right] - 2(x^2 + x) \\
&= 8.
\end{aligned}
$$

Problem 13. Prove that $y_4 = y_3 + \Delta y_2 + \Delta^2 y_1 + \Delta^3 y_1$.

Solution.

$$\begin{aligned}
y_3 + \Delta y_2 + \Delta^2 y_1 + \Delta^3 y_1 &= y_3 + (E-1)y_2 + (E-1)^2 y_1 + (E-1)^3 y_1 \\
&= y_3 + y_3 - y_2 + (E^2 - 2E + 1)y_1 \\
&\qquad + (E^3 - 3E^2 + 3E - 1)y_1 \\
&= 2y_3 - y_2 + y_3 - 2y_2 + y_1 + y_4 - 3y_3 + 3y_2 - y_1 \\
&= y_4.
\end{aligned}$$

Problem 14. Prove that $\Delta^2 y_2 = \nabla^2 y_4$.

Solution.

$$\begin{aligned}
\Delta^2 y_2 &= (E-1)^2 y_2 \\
&= (E^2 - 2E + 1)y_2 \\
&= y_4 - 2y_3 + y_2 \qquad \text{——} \quad (1) \\
\text{Also} \quad \nabla^2 y_4 &= (1 - E^{-1})^2 y_4 \\
&= (1 - 2E^{-1} + E^{-2})y_4 \\
&= y_4 - 2y_3 + y_2. \qquad \text{——} \quad (2)
\end{aligned}$$

From (1) and (2), $\Delta^2 y_2 = \nabla^2 y_4$.

Problem 15. Given $u_0 = 2, u_1 = 11, u_2 = 80, u_3 = 200, u_4 = 100, u_5 = 8$ find $\nabla^5 u_5$ **(i)** without constructing the difference table

(ii) by constructing the difference table.

Solution. (i) We know that $\nabla = 1 - E^{-1}$

$$\begin{aligned}
\therefore \nabla^5 u_5 &= (1 - E^{-1})^5 u_5 \\
&= (1 - 5E^{-1} + 10E^{-2} - 10E^{-3} + 5E^{-4} - E^{-5})u_5 \\
&= u_5 - 5E^{-1}(u_5) + 10E^{-2}(u_5) - 10E^{-3}(u_5) + 5E^{-4}(u_5) - E^{-5}(u_5) \\
&= u_5 - 5u_4 + 10u_3 - 10u_2 + 5u_1 - u_0 \\
&= 8 - 500 + 2000 - 800 + 55 - 2 \\
&= 761.
\end{aligned}$$

(iii) We have $\nabla^n u(x) = \Delta^n u(x-n)$. Hence $\nabla^5 u_5 = \Delta^5 u_0$. We construct the forward difference table.

x	u	Δu	$\Delta^2 u$	$\Delta^3 u$	$\Delta^4 u$	$\Delta^5 u$
0	2					
		9				
1	11		60			
		69		-9		
2	80		51		-262	
		120		-271		761
3	200		-220			
		-100			499	
4	100		8	228		
		-92				
5	8					

$$\therefore \nabla^5 u_5 = \Delta^5 u_0 = 761.$$

Problem 16. If $u_0 = 1, u_1 = 5, u_2 = 8, u_3 = 3, u_4 = 7, u_5 = 0$ find $\Delta^5 u_0$.

Solution.

$$\begin{aligned}
\Delta^5 u_0 &= (E-1)^5 u_0 \\
&= (E^5 - 5E^4 + 10E^3 - 10E^2 + 5E - 1)u_0 \\
&= (E^5 u_0 - 5E^4 u_0 + 10E^3 u_0 - 10E^2 u_0 + 5E u_0 - u_0 \\
&= u_5 - 5u_4 + 10u_3 - 10u_2 + 5u_1 - u_0 \\
&= -35 + 30 - 80 + 25 - 1 \\
&= -61.
\end{aligned}$$

Problem 17. Estimate the missing term in the following table.

x	0	1	2	3	4
$u(x)$	1	3	9	–	81

Explain why the resulting value differs from 3^3.

Solution. Let the missing term in $u(x)$ be a.

$$\begin{aligned}
\text{Consider} \quad \Delta^4 u_0 &= 0 \qquad \text{(since 4 values are given)} \\
\therefore (E-1)^4 u_0 &= 0 \\
\therefore (E^4 - 4E^3 + 6E^2 - 4E + 1)u_0 &= 0 \\
\therefore u_4 - 4u_3 + 6u_2 - 4u_1 + u_0 &= 0
\end{aligned}$$

$$
\begin{aligned}
\therefore 81 - 4a + 54 - 12 + 1 &= 0\\
\therefore 124 - 4a &= 0\\
\therefore a &= 31.
\end{aligned}
$$

We understand from the data that $u(x)$ satisfies the relation $u(x) = 3^x$. While estimating for $u(3)$ the basic assumption is that $u(x)$ is a polynomial of degree 3. But 3^x is not a polynomial but an exponential function. Hence the assumption is violated in this case and so we are not getting $u(3) = 3^3 = 27$.

Problem 18. Give an estimate of the population in 1971 from the following table.

Year	1941	1951	1961	1971	1981	1991
Population in lakhs	363	391	421	?	467	501

Solution. Let the population in 1971 be a. Let $u_0 = 363$; $u_1 = 391$; $u_2 = 421$; $u_3 = a$; $u_4 = 467$ and $u_5 = 501$.

Since five values are given $\Delta^5 u_0 = 0$

$\therefore \quad (E-1)^5 u_0 = 0.$

$$
\begin{aligned}
\therefore (E^5 - 5E^4 + 10E^3 - 10E^2 + 5E - 1)u_0 &= 0\\
\therefore u_5 - 5u_4 + 10u_3 - 10u_2 + 5u_1 - u_0 &= 0\\
\therefore 501 - 2335 + 10a - 4210 + 1955 - 363 &= 0\\
\therefore 10a - 4452 &= 0\\
\therefore \qquad a &= 445.2 \text{ lakhs}.
\end{aligned}
$$

Hence the estimated population in 1971 is 445.2 lakhs.

Problem 19. Given that $u_0+u_8 = 80$; $u_1+u_7 = 10$; $u_2+u_6 = 5$; $u_3+u_5 = 10$, find u_4.

Solution. Since 4 values are given $\Delta^n u(x) = 0$ for all $n \geq 4$.

In particular $\Delta^8 u_0 = 0$. Hence $(E-1)^8 u_0 = 0$

$$
\begin{aligned}
u_8 - 8u_7 + 28u_6 - 56u_5 + 70u_4 - 56u_3 + 28u_2 - 8u_1 + u_0 &= 0\\
(u_0 + u_8) - 8(u_1 + u_7) + 28(u_2 + u_6) - 56(u_3 + u_5) + 70u_4 &= 0\\
\therefore 80 - 80 + 140 - 560 + 70u_4 &= 0\\
\therefore 70u_4 &= 420\\
\therefore u_4 &= 6.
\end{aligned}
$$

Interpolation

Augment & Entry :- Let $y = f(x)$ be a function. $f(x_1), f(x_2), f(x_3), \ldots f(x_n)$ are the values of y corresponding to values of $x = x_1, x_2, x_3, \ldots x_n$. Here each value of x is called Arguement and each value of y is called 'Entry'.

Interpolation:-

The process of computing the value of a function inside a given range is called Interpolation.

If the point x lies outside the domain $[x_0, x_n]$ then the process of computing the value of a function is called Extrapolation.

To determine the values of $f(x), f'(x)$ for some intermediate value of x. The following three types of operators are used.

1). forward difference operator [Δ - delta]
2). Backward difference operator [∇ - Nabla]
3). Central difference operator [δ - delta].

Interpolation with equal Intervals :-

Let the function $y = f(x)$ takes the values $y_0, y_1, y_2 \cdots y_n$ Corresponding to the values $x_0, x_1, x_2, \cdots x_n$. If the values of x are at equal intervals we will estimate the function value by using some interpolation methods.

1). Newton's forward Interpolation formula
2). Newton's backward " "
3). Gauss forward Interpolation formula
4). Gauss backward " "

1). Newton forward difference formula (or). Newton forward Interpolation formula (or). Newton gregory forward Interpolation formula :-

Let the function $y = f(x)$ satisfies the values $y_0, y_1, y_2, \cdots y_n$ corresponding to the values $x_0, x_1, x_2, \cdots x_n$ of the Independent variable 'x'.

Let the value of x be ~~equipped~~ equiplaced such that $x_i = x_0 + ih$. where $i = 1, 2, 3, 4, \cdots$

Suppose it is required to estimate the values of y ~~at~~ some point x. where $x_0 < x < x_n$.

$$\therefore y = f(x) = y_0 + p \cdot \Delta y_0 + \frac{p(p-1)}{2!} \Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!} \Delta^3 y_0 + \cdots + \frac{p(p-1)(p-2)\cdots(p-(n-1))}{n!} \Delta^n y_0 .$$

this is called Newtons gregory forward Interpolation formula. where $p = \frac{x - x_0}{h}$.

Where, x = where you want estimate the Value of y.

x_0 = 1st Arguement.

h = Interval difference of x

i.e $h = x_1 - x_0 = x_2 - x_1 = \dots\dots$

Note :- Newton's forward Interpolation is Applicable only for Equal Intervals.

Forward difference Table :-

Value of x	Value of y	First difference Δy	Second difference $\Delta^2 y$	third difference $\Delta^3 y$	fourth difference $\Delta^4 y$
x_0	y_0				
		$\Delta y_0 = y_1 - y_0$			
x_1	y_1		$\Delta^2 y_0 = \Delta y_1 - \Delta y_0$		
		$\Delta y_1 = y_2 - y_1$		$\Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_0$	$\Delta^4 y_0 = \Delta^3 y_1 - \Delta^3 y_0$
x_2	y_2		$\Delta^2 y_1 = \Delta y_2 - \Delta y_1$		
		$\Delta y_2 = y_3 - y_2$		$\Delta^3 y_1 = \Delta^2 y_2 - \Delta^2 y_1$	
x_3	y_3		$\Delta^2 y_2 = \Delta y_3 - \Delta y_2$		
		$\Delta y_3 = y_4 - y_3$			
x_4	y_4				

Eg: Construct the F. D. Table.

x	1	2	3	4	5
y	4	13	34	73	136

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1	4				
		$13-4=9$			
2	13		$21-9=12$		
		$34-13=21$		$18-12=6$	
3	34		$39-21=18$		$6-6=0$
		$73-34=39$		$24-18=6$	
4	73		$63-39=24$		
		$136-73=63$			
5	136				

(i) $\Delta^3(1) = 6$, (ii) $\Delta^2(3) = 24$.

Newtons Backward difference formula :- (or)

Newtons backword Interpolation (or) Newtons gregory backward Interpolation formula :-

Let the function $y = f(x)$ satisfies the values $y_0, y_1, y_2, \ldots y_n$ corresponding to the values $x_0, x_1, x_2, \ldots x_n$ of the Independent variable 'x'. Let the values of x be equiplaced such that $x_i = x_0 + ih$ $\forall i = 1, 2, 3, \ldots$

Suppose it is required to estimate the value of y at some point x, where $x_0 < x < x_n$.

$$\therefore y = f(x) = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \cdots + \frac{p(p+1)(p+2)\cdots(p+(n-1))}{n!}\nabla^n y_n.$$

Where $p = \dfrac{x - x_n}{h}$.

Where x = where you want to estimate the value of y.

x_n = Last arguement.

h = Interval differences of x.

i.e $h = x_2 - x_1 = x_3 - x_2 = \cdots$

The Backward difference table :-

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$
x_0	y_0			
		$\nabla y_1 = y_1 - y_0$		
x_1	y_1		$\nabla^2 y_2 = \nabla y_2 - \nabla y_1$	$\nabla^3 y_3 = \nabla^2 y_3 - \nabla^2 y_2$
		$\nabla y_2 = y_2 - y_1$		
x_2	y_2		$\nabla^2 y_3 = \nabla y_3 - \nabla y_2$	
		$\nabla y_3 = y_3 - y_2$		
x_3	y_3			

eg:-

x	1	3	5	7	9
y	8	12	21	36	62

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
1	8				
		4			
3	12		5		
		9		1	
5	21		6		4.
		15		5	
7	36		11		
		26			
9	62				

Remark. Newton's forward interpolation formula is used to interpolate the values of y *near the beginning* of the set of tabulated values or for extrapolating values of y to the *left* of the beginning. Newton's backward interpolation formula is used to interpolate the values of y *near the end* of the set of tabulated values or for extrapolating values of y to the *right* of the last tabulated value y_n.

Solved Problems

Problem 1. If $y(75) = 246, y(80) = 202, y(85) = 118, y(90) = 40$ find $y(79)$.

Solution. Here $x_0 = 75, h = 5$ and we have to find the value of y at $x = 79$.

Newton's forward interpolation formula is

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \cdots + \frac{p(p-1)\cdots(p-(n-1))}{n!}\Delta^n y_0 \quad \text{——} \quad (1)$$

$$\text{where} \quad p = \frac{x - x_0}{h}$$

$$\therefore p = \frac{79 - 75}{5} = 0.8.$$

We now form the forward difference table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
75	246			
		−44		
80	202		**−40**	
		−84		**46**
85	118		6	
		−78		
90	40			

Using the values of $\Delta y_0, \Delta^2 y_0, \Delta^3 y_0$ in (1) we get,

$$y_{0.8} = 246 + 0.8(-44) + \frac{(0.8)(0.8-1)}{2!}(-40) + \frac{0.8(0.8-1)(0.8-2)}{3!}(46)$$
$$= 215.472.$$

Problem 2. Find a cubic polynomial which takes the following values

x	0	1	2	3
$f(x)$	1	2	1	10

Solution. Let us form the difference table first

x	$f(x)$	Δf	$\Delta^2 f$	$\Delta^3 f$
0	1			
		1		
1	2		-2	
		-1		12
2	1		10	
		9		
3	10			

By Newton-Gregory formula

$$f(x) = f(x_0) + (x - x_0)\frac{\Delta f(x_0)}{1!h} + (x - x_0)(x - x_0 - h)\frac{\Delta^2 f(x_0)}{2!h^2} + \cdots$$

Here $x_0 = 0$ and $h = 1$.

$$\begin{aligned}
\therefore f(x) &= 1 + (x-0)\frac{1}{1!} + (x-0)(x-1)\frac{(-2)}{2!} + (x-0)(x-1)(x-2)\frac{12}{3!} \\
&= 1 + x - x(x-1) + 2x(x-1)(x-2) \\
&= 1 + x - x^2 + x + 2x^3 - 6x^2 + 4x \\
&= 2x^3 - 7x^2 + 6x + 1
\end{aligned}$$

Problem 3. A function $y = f(x)$ is given by the following table. Find $f(0.2)$ by a suitable formula.

x	0	1	2	3	4	5	6
$y = f(x)$	176	185	194	203	212	220	229

Solution. Since the value $x = 0.2$ is near the beginning of the table we use Newton's forward interpolation formula.

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \cdots + \frac{p(p-1)\cdots p(p-(n-1))}{n!}\Delta^n y_0 \quad \text{— (1)}$$

where $p = \dfrac{x - x_0}{h}$.

Here $x_0 = 0, h = 1$ and we want to find the value of $f(x)$ at $x = 0.2$

$$\therefore p = \frac{0.2 - 0}{1} = 0.2.$$

Now we form the forward difference table

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$	$\Delta^5 f(x)$	$\Delta^6 f(x)$
0	176						
		9					
1	185		**0**				
		9		**0**			
2	194		0		**0**		
		9		0		**−1**	
3	203		0		−1		**5**
		9		−1		4	
4	212		−1		3		
		8		2			
5	220		1				
		9					
6	229						

$\therefore$ (1) becomes,

$$\begin{aligned} y_{0.2} &= 176 + 0.2 \times 9 + \frac{(0.2)(0.2-1)(0.2-2)(0.2-3)(0.2-4)}{5!}(-1) \\ &\quad + \frac{(0.2)(0.2-1)(0.2-2)(0.2-3)(0.2-4)(0.2-5)}{6!}(5) \\ &= 177.67232. \end{aligned}$$

Hence $f(0.2) = 177.67$.

Problem 4. Construct Newton's forward interpolation polynomial for the following data

x	4	6	8	10
y	1	3	8	16

Use it to find the value of y for $x = 5$.

Solution. Here $x_0 = 4$ and $h = 2$. The Newton's forward interpolation formula is

$$y = y_0 + \frac{(x-x_0)\Delta y_0}{1!h} + (x-x_0)(x-x_0-h)\frac{\Delta^2 y_0}{2!h^2} + \cdots. \qquad \text{——— (1)}$$

We form the difference table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
4	1			
		2		
6	3		3	
		5		0
8	8		3	
		8		
10	16			

$$\therefore\ y = 1 + \frac{(x-4)\times 2}{1!2} + \frac{(x-4)(x-6)\times 3}{2!2^2} + \frac{(x-4)(x-6)(x-8)\times 0}{3!2^3}$$

$$= 1 + (x-4) + \frac{3(x-4)(x-6)}{8}.$$

$\therefore$ $y = 1+(x-4)+\frac{3}{8}(x^2-10x+24)$ is the required interpolating polynomial

When $x = 5$, $y_5 = 1+1+\frac{3}{8}(5^2-50+24) = 1.625$.

Problem 5. Find the value of y from the following data at $x = 2.65$

x	-1	0	1	2	3
y	-21	6	15	12	3

Solution. Since the value of $x(= 2.65)$ is near the end of the table, we use Newton's backward interpolation formula. The formula is

$$y_p = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \cdots$$

where $p = \dfrac{x - x_n}{h}$.

Here $x = 2.65$, $x_n = 3$ and $h = 1$.

$$\therefore\ p = \frac{2.65-3}{1} = -0.35$$

To find $\nabla y_n, \nabla^2 y_n$ etc. we form the backward difference table.

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
-1	-21				
		27			
0	6		-18		
		9		6	
1	15		-12		**0**
		-3		**6**	
2	12		**-6**		
		-9			
3	**3**				

$$\begin{aligned}\therefore y_{-0.35} &= 3 + (-0.35)(-9) + \frac{(-0.35)(-0.35+1)}{2!}(-6) \\ &\quad + \frac{(-0.35)(-0.35+1)(-0.35+2)}{3!} \times 6 \\ &= 6.4571.\end{aligned}$$

Problem 6. The following data gives the melting point of an alloy of zinc and lead, θ is the temperature and x is the percentage of lead. Using Newton's interpolation formula find **(i)** θ when $x = 48$ **(ii)** θ when $x = 84$.

x	40	50	60	70	80	90
θ	184	204	226	250	276	304

Solution. **(i)** Since $x = 48$ is near the beginning of the table we use Newton's forward interpolation formula. The formula can be written as

$$\theta_p = \theta_0 + p\Delta\theta_0 + \frac{p(p-1)}{2!}\Delta^2\theta_0 + \cdots$$

where $p = \dfrac{x - x_0}{h} = \dfrac{48 - 40}{10} = 0.8.$

To find $\Delta\theta_0, \Delta^2\theta_0$ etc. we form the forward difference table.

x	θ	$\Delta\theta$	$\Delta^2\theta$	$\Delta^3\theta$	$\Delta^4\theta$	$\Delta^5\theta$
40	**184**					
		20				
50	204		**2**			
		22		**0**		
60	226		2		**0**	
		24		0		**0**
70	250		2		0	
		26		0		
80	276		2			
		28				
90	304					

$$\begin{aligned}\theta_p &= 184 + 0.8 \times 20 + \frac{0.8(0.8-1)}{2!} \times 2 \\ &= 199.84 \simeq 200.\end{aligned}$$

(ii) Since $x = 84$ is nearer to the end of the table we use Newton's backward interpolation formula.

$$\theta_p = \theta_n + p\nabla\theta_n + \frac{p(p+1)}{2!}\nabla^2\theta_n + \cdots$$

where $p = \dfrac{x - x_n}{h} = \dfrac{84 - 90}{10} = -0.6.$

The values of $\theta_n, \nabla\theta_n, \nabla^2\theta_n$ etc. can be obtained from the forward difference table by sloping backwards with respect to the increasing direction of x.

$$\therefore \theta_n = 304, \nabla\theta_n = 28, \nabla^2\theta_n = 2, \nabla^3\theta_n = \nabla^4\theta_n = \nabla^5\theta_n = 0.$$

Hence we have

$$\begin{aligned}\theta_{-0.6} &= 304 + (-0.6) \times 28 + \frac{(-0.6)(-0.6+1)}{2!}(2) \\ &= 286.96 \simeq 287.\end{aligned}$$

Problem 7. From the data given below, find the number of students whose weight is between 60 and 70.

Weight	0 – 40	40 – 60	60 – 80	80 – 100	100 – 120
No. of Students	250	120	100	70	50

Solution. The less than cumulative frequency table of the given data is as shown below:

Weight less than x	40	60	80	100	120
No of students	250	370	470	540	590

We now form the difference table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
40	**250**				
		120			
60	370		**−20**		
		100		**−10**	
80	470		−30		**20**
		70		10	
100	540		−20		
		50			
120	590				

Number of students whose weight is between 60 and 70 is got from $y_{70} - y_{60}$.

We have $y_{60} = 370$.

Now we shall find y_{70} by Newton's forward interpolation formula.

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \cdots + \frac{p(p-1)\cdots(p-(n-1))}{n!}\Delta^n y_0$$

where $p = \dfrac{x - x_0}{h}$.

Here $x_0 = 40, h = 20$ and $x = 70$. Hence $p = \dfrac{70-40}{20} = 1.5$.

$$\begin{aligned} \text{Now} \quad y_{70} &= y_{1.5} = 250 + 1.5 \times 120 + \frac{(1.5)(0.5)}{2!}(-20) \\ &\quad + \frac{(1.5)(0.5)(-0.5)}{3!}(-10) + \frac{(1.5)(0.5)(-0.5)(-1.5)}{4!}(20) \\ &= 423.59375 \\ \therefore \quad y_{70} &= 424 \text{ (approximately)}. \end{aligned}$$

$\therefore$ Number of persons whose weight is between 60 and 70 is $424 - 370 = 54$.

Exercises

1. For the following data find $f(9)$ using Newton's forward interpolation formula

x	8	10	12	14	16
$f(x)$	1000	1900	3250	5400	8950

2. Find the value of y at $x = 21$ from the following data

x	20	23	26	29
y	0.3420	0.3907	0.4384	0.4848

3. Find $f(2.5)$ using Newton's forward difference formula for the given data

x	1	2	3	4	5	6
y	0	1	8	27	64	125

4. Using Newton's formula find the area of a circle of diameter 98 from the given table of diameter and area of a circle

Diameter	80	85	90	95	100
Area	5026	5674	6362	7088	7854

5. From the following table find the value of tan $45°15'$

$x°$	45	46	47	48	49	50
$\tan x°$	1.00000	1.03553	1.07237	1.11061	1.15037	1.19175

6. Population was recorded as follows in a village

Year	1941	1951	1961	1971	1981	1991
Population	2500	2800	3200	3700	4350	5225

Estimate the population for the year 1945.

7. From the table given below find $\sin 52°$ by using Newton's forward interpolation formula

x	45	50	55	60
$\sin x$	0.7071	0.7660	0.8192	0.8660

8. The following temperature readings were taken on a day

Time t	2am	6am	10am	12noon
Temperature θ	40.2°	42.4°	51°	72.4°

Find the temperature at 3am and 4am.