1.4

Linear Dependence and Independence

In this module, we continue the development of vector space structure. We introduce concepts of dependence and independence of vectors. These will be useful tools in constructing "efficient" spanning sets for vector spaces–sets in which there are no redundant vectors.

Let us motivate the idea of dependence of vectors. Observe that the vector $(4,-1,0)$ is a linear combination of the vectors $(2,1,3)$ and $(0,1,2)$ since it can be written as $(4,-1,0) = 2(2,1,3) - 3(0,1,2)$.

The above equation can be rewritten in a number of ways. Each vector can be expressed in terms of the other vectors.

$(2,1,3) = (1/2)\,(4,-1,0) + (3/2)\,(0,1,2)$

$(0,1,2) = (2/3)\,(2,1,3) - (1/3)\,(4,-1,0)$.

Each of the three vectors is, in fact, dependent on the other two vectors. We express this by writing.

$(4,-1,0) - 2(2,1,3) + 3(0,1,2) = (0,0,0)$.

This concept of dependence of vectors is made precise with the following definition.

DEFINITION: (a) The set of vectors $\{ v_1, \dots, v_m \}$ in a vector space V is said to be linearly dependent if there exists scalars $c_1, \dots, c_m$ not all zero, such that $c_1 v_1 + \cdots + c_m v_m = 0$.

(b) The set of vectors $\{ v_1, \ldots, v_m \}$ is linearly independent if $c_1 v_1 + \cdots + c_m v_m = 0$ can only be satisfied when $c_1 = 0, \ldots., c_m = 0$.

We now present an important result that relates the concepts of linear dependence and linear combination.

THEOREM: A set consisting of two or more vectors in a vector space is linearly dependent if and only if it is possible to express one of the vectors as a linear combination of the other vectors.

Proof: Let the set $\{ v_1, v_2 \ldots, v_m \}$ be linearly dependent.

Therefore, there exists scalars $c_1, c_2, \ldots, c_m$ not all zero, such that

$c_1 v_1 + c_2 v_2 + \cdots, c_m v_m = 0$

Assume that $c_1 \neq 0$.

The above identity can be rewritten as

$$v_1 = \frac{-c_2}{c_1} v_2 + \cdots + \frac{-c_m}{c_m} v_m$$

Thus, v_1 is a linear combination of $v_2, \ldots, v_m$. Conversely, assume that v_1 is a linear combination of $v_2, \ldots, v_m$.

Therefore there exists scalars $d_2, \ldots, d_m$ such that

$v_1 = v_2 d_2, \ldots, d_m v_m$.

$\Rightarrow 1v_1 + (-d_2)v_2 + \cdots + (-d_m)v_m = 0$.

Thus the set $\{ v_1, v_2, \ldots, v_m \}$ is linearly dependent, completing the proof.

THEOREM: Let V be a vector space. Any set of vectors in V that contains the zero vector is linearly dependent.

Proof: Consider the set $\{\vec{0}, \vec{v}_2, \dots \vec{v}_m\}$, which contains the zero vector. Let us examine the identity.

$c_1 0 + c_2 v_2 + \cdots + c_n v_n = 0.$

We see that the identity is true for $c_1 = 1$, $c_2 = 0$, … , $c_m = 0$ (not all zero). Thus the set of vectors is linearly dependent, proving the theorem.

THEOREM: Let the set $\{v_1, \dots, v_m\}$ be linearly dependent in a vector space V. Any set of vectors in V that contains these vectors will also be linearly dependent.

Proof: Since the set $\{v_1, \dots, v_m\}$ is linearly dependent, there exists scalars $C_1, \dots C_m$, not all zero, such that $c_1 v_1 + \cdots + c_m v_m = 0$.

Consider the set of vectors, which contains the given vectors.

There are scalars, not all zero, namely $C_1, C_2, \dots C_m, 0, \dots 0$ such that

$C_1 v_1 + \cdots + C_m v_m + 0 v_m + 1 + \cdots .. + 0 v_n = 0.$

Thus the set$\{v_1, \dots, v_m, v_{m+1}, \dots v_n\}$ is linearly dependent.

Problem 1: Show that the set $\{(1,2,3), (-2,1,1), (8,6,10)\}$ is linearly dependent in $\mathbb{R}^3$.

Solution: Let us examine the identity $C_1\,(1,2,3) + C_2\,(-2,1,1) + C_3\,(8,6,10) = \vec{0}$. We want to show that at least one of the C's can be nonzero. We get

$$(C_1 - 2C_2 + 8C_3, 2C_1 + C_2 + 6C_3, 3C_1 + C_2 + 10C_3) = (0,0,0).$$

Equating each component of this vector to zero gives the system of equations.

$$C_1 - 2C_2 + 8C_3 = 0$$

$$2C_1 + C_2 + 6C_3 = 0$$

$$3C_1 + C_2 + 10C_3 = 0$$

This system has the solution $C_1 = 4$, $C_2 = -2$, $C_3 = -1$. Since at least one of the C's is nonzero, the set of vectors is linearly dependent.

The Linear dependence is expressed by the equation

$$4(1,2,3) - 2(-2,1,1) - (8,6,10) = \vec{0}.$$

Problem 2: Show that the set $\{(3,-2,2),(3,-1,4),(1,0,5)\}$ is linearly dependent in $\mathbb{R}^3$.

Solution: We examine the identity $C_1(3,-2,2)+C_2(3,-1,4)+C_3(1,0,5)=\vec{0}$.

We want to show that this identity can only hold if C_1, C_2 and C_3 are all zero. We get

$(3C_1+3C_2+C_3,-2C_1-\ C_2,2C_1+4C_2+5C_3)\ =\ \vec{0}.$

Equating the components to zero gives.

$3\ C_1+3\ C_2+\ C_3\ =\ 0$

$-2\ C_1-\ C_2\ =\ 0$

$2\ C_1+4\ C_2+5\ C_3\ =\ 0$

This system has the solution $C_1=0,\ C_2=0,\ C_3=\ 0$. Since the set of vectors is linearly independent.

Problem 3: Let the set $\{v_1, v_2\}$ be linearly independent. Prove that $\{v_1 + v_2, v_1 - v_2\}$ is also linearly independent.

Solution: Let us examine the identity.

$$a(v_1 + v_2) + b(v_1 - v_2) \_\_\_\_\_\_(1)$$

If we can show that this identity implies $a = 0$ and $b = 0$, then $\{v_1 + +v_2, v_1 - v_2\}$ will be linearly independent,

we get

$av_1 + av_2 + bv_1 - bv_2 = 0$

$$\Rightarrow (a + b)v_1 + (a - b)v_2 = 0$$

Since $\{v_1, v_2\}$ is linearly independent $a + b = 0, a - b = 0$

This system has the unique solution $a = 0, b = 0$.

EXCERCISE

1. Find values of 't' for which the following sets are linearly dependent.

   (a) $\{(-1, 2), (t, -4)\}$

   (b) $\{(2, -t), (2t + 6, 4t)\}$.

2. Let the set $\{v_1, v_2, v_3\}$ be linearly dependent in a vector space V. Let 'c' be a non-zero scalar. Prove that the following sets are also linearly dependent.

   (a) $\{v_1, v_1 + v_2, v_3\}$

   (b) $\{v_1, cv_2, v_3\}$

   (c) $\{v_1, v_1 + cv_2, v_3\}$.

3. Same question as above replacing linearly dependent with linearly independent.
4. Let a set 'S' be linearly independent in a vector space V . Prove that every subset of S is also linearly independent. Let P be linearly dependent. Is every subset of P linearly dependent?
5. Let $\{v_1, v_2\}$ be linearly independent in a vector space V. Show that if a vector v_3 is not of the form $av_1 + bv_2$, then the set $\{v_1, v_2, v_3\}$ is linearly independent.
6. Prove that a set of two or more vectors in a vector space is linearly independent if no vector in the set can be expressed as a linear combination of the other vectors.

Answers

1. (a) 2
   (b) -7