UNIT-03 Discrete Fourier Transform (DFT)

$$x[n] \xleftrightarrow{\text{DTFT}} X[e^{j\omega}] = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$$

- $x[n]$ → not a finite sequence
- $\sum_{n=-\infty}^{\infty}$ → infinite summation
- $x[n]\,e^{-j\omega n}$ → i/p of infinite length

DFT : holds only for sequences of finite length

$x[n]$: 0 to $(N-1)$ → 'N' samples exists in the input

DTFT,

$$\underbrace{X[e^{j\omega}] = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}}_{\substack{\text{DTFT} \\ \text{infinite length}}} \xleftrightarrow{\text{DFT}} \underbrace{\sum_{n=0}^{N-1} x[n]\, e^{-j\omega n}}_{\substack{\text{DFT} \\ \text{finite length}}}$$

DFT,

$$X[e^{j\omega}] = \sum_{n=0}^{N-1} x[n]\, e^{-j\omega n}$$

$$X[e^{j\omega}] = x[0]\,e^{-j\omega 0} + x[1]\,e^{-j\omega 1} + x[2]\,e^{-j\omega 2} + \dots + x[N-1]\,e^{-j\omega(N-1)}$$

Polynomial in terms of $e^{-j\omega(\,)}$

* $X[e^{j\omega}]$ could be specified in terms of 'N' no. of coefficients i.e, $\{x[0], x[1], x[2], \dots x[N-1]\}$

Here, $X[e^{j\omega}]$ → continuous function of 'ω'.

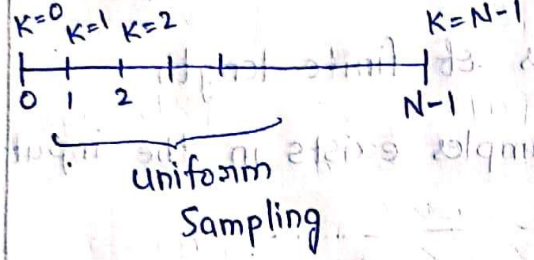
'N' no. of coefficients

- time domain: $\{x[0], x[1], x[2], \dots x[N-1]\}$
- frequency domain: $\{X[0], X[1], \dots X[N-1]\}$

UNIT - 03 Discrete Fourier Transform (DFT)

$X[e^{j\omega}] \longrightarrow$ Sampler $\longrightarrow \{X[0], X[1], \ldots X[N-1]\}$

$\underbrace{\text{Sampler}}_{\text{Samples at `N' no. of distinct points}}$

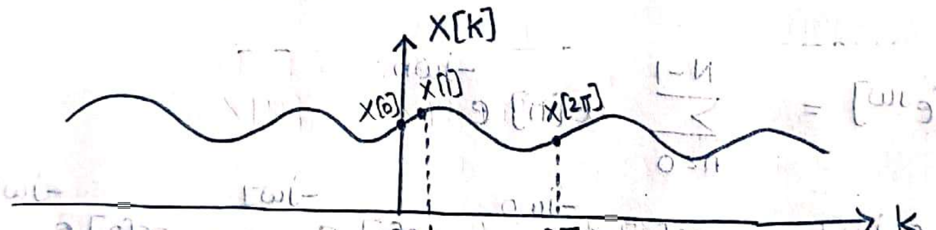


$X[e^{j\omega}] \longrightarrow$ Sampler $\longrightarrow X[e^{j\omega K}]$

$X[e^{j\omega}]$: Periodic with period 2π

$0 \leq \omega \leq 2\pi$

$K = 0, 1, 2, \ldots (N-1)$



$X[0] = \left(\frac{2\pi}{N}\right)0$

$X[1] = \left(\frac{2\pi}{N}\right)1$

$$\boxed{X[K] = \left(\frac{2\pi}{N}\right)K}$$

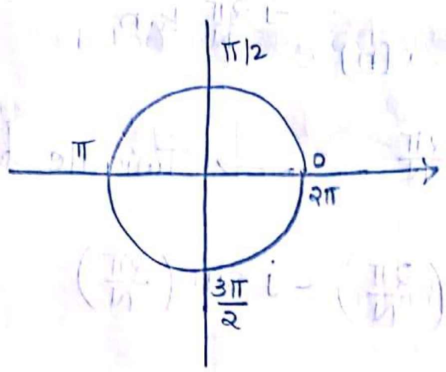
$X[2] = \left(\frac{2\pi}{N}\right)2$

$X[3] = \left(\frac{2\pi}{N}\right)3$

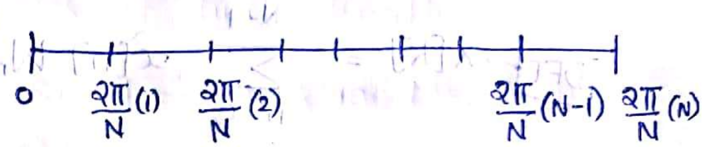
$X[e^{j\omega}] \longrightarrow$ Sampler $\longrightarrow X[e^{j\omega K}] = X[K]$

⋮

$X[N-1] = \left(\frac{2\pi}{N}\right)(N-1)$



Divide $0 \rightarrow 2\pi$ range into 'N' equal intervals.



$$\boxed{X[e^{j\omega k}] = X\left[e^{j\left(\frac{2\pi}{N}\right)\cdot k}\right] = X[K]}$$

$\left(\frac{2\pi}{N}\right)$ is the basic interval.

$$X[K] = \text{DFT of } x[n] = X\left[e^{j\frac{2\pi}{N}\cdot k}\right]$$

<u>DTFT</u> $\quad X[e^{j\omega}] = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$

<u>DFT</u> $\quad X\left[e^{j\frac{2\pi}{N}k}\right] = \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi}{N}\cdot k\cdot n}$

$$\boxed{X[K] = X\left[e^{j\frac{2\pi}{N}k}\right] = \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}\cdot k\cdot n}}$$ Analysis Equation

$K = 0, 1, 2, \ldots (N-1)$

$n = 0, 1, 2, \ldots (N-1)$

$$x[n] \xleftrightarrow{\text{DFT}} X[K], \quad x[n] \xleftrightarrow{\text{DTFT}} X[e^{j\omega}]$$

<u>Inverse DFT</u> : $\quad X[K] \xleftrightarrow{\text{IDTFT}} x[n]$

$$\boxed{x[n] = \frac{1}{N}\sum_{K=0}^{N-1} X[K]\, e^{j\frac{2\pi}{N}\cdot k\cdot n}}$$ Synthesis Equation.

$$X[K] = \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi}{N}\cdot k\cdot n}$$

$$W_n = e^{-j\frac{2\pi}{N}} \longrightarrow \text{Twiddle factor}$$

Scientist

Cookey Tukey

$$= \cos\left(\frac{2\pi}{N}\right) - j\sin\left(\frac{2\pi}{N}\right)$$

So,

DFT $\rightarrow$ $$X[K] = \sum_{n=0}^{N-1} x[n]\, W_n^{k\cdot n} \quad ; \; 0 \le K \le N-1$$

IDFT $\rightarrow$ $$x[n] = \frac{1}{N}\sum_{K=0}^{N-1} X[K]\, W_n^{-k\cdot n} \quad ; \; 0 \le n \le N-1$$

$$x[n] \xleftrightarrow[\text{DFT}]{\text{N-point}} X[K]$$

$$X[K] \xleftrightarrow[\text{IDFT}]{\text{N-point}} x[n]$$

$$X[K] = X[K+N]$$

LHS,

$$X[K] = \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}\cdot k\cdot n}$$

RHS,

$$X[K+N] = \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}[K+N]\cdot n}$$

$$= \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}\cdot kn} \cdot \underbrace{e^{-j\frac{2\pi}{N}\cdot N\cdot n}}_{= e^{-j2\pi n} = 1}$$

$$= \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}\cdot k\cdot n}$$

$$= \text{LHS}$$

DFT, $X[K] \Rightarrow$ Periodic with period N

DTFT, $X(e^{j\omega}) \Rightarrow$ Periodic with period 2π.

Problems:-

(1) $x[n] = \{\underset{\uparrow}{0}, 1, 2, 3\}$. Compute its 4-point DFT $X[K]$, $0 \le K \le 3$

Ans:- Given that, $N = 4$

By definition, the N-point DFT is given by

$$X[K] = \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N} \cdot K \cdot n}, \quad 0 \le K \le N-1$$

$$X[K] = \sum_{n=0}^{4-1} x[n]\, e^{-j\frac{2\pi}{4} \cdot K \cdot n}, \quad 0 \le K \le 4-1$$

$$0 \le K \le 3$$

$$= x[0]\, e^{-j\frac{\pi}{2} \cdot K \cdot 0} + x[1]\, e^{-j\frac{\pi}{2} K \cdot 1} + x[2]\, e^{-j\frac{\pi}{2} \cdot K \cdot 2} + x[3]\, e^{-j\frac{\pi}{2} \cdot K \cdot 3}$$

$$= \cancel{x[0]}^{\,0} + x[1]\, e^{-j\frac{\pi}{2}K} + x[2]\, e^{-j\pi} + x[3]\, e^{-j\frac{3\pi}{2}K}$$

$$X[K] = 1 \cdot e^{-j\frac{\pi}{2}K} + 2e^{-j\pi} + 3e^{-j\frac{3\pi}{2}K}$$

As $0 \le K \le 3$,

$K=0$, $X[0] = 1 + 2 + 3 = 6$

$K=1$, $X[1] = e^{-j\frac{\pi}{2}} + 2e^{-j\pi} + 3e^{-j\frac{3\pi}{2}} = -2 + 2j$

$K=2$, $X[2] = e^{-j\pi} + 2e^{-j2\pi} + 3e^{-j3\pi} = -2$

$K=3$, $X[3] = e^{-j\frac{3\pi}{2}} + 2e^{-j3\pi} + 3e^{-j\frac{9\pi}{2}} = -2 - 2j$

$$\therefore X[K] = \{\underset{\uparrow}{6}, -2+2j, -2, -2-2j\}$$

(2) Determine the 8-point DFT of the signal

$x[n] = \{1,1,1,1,1,1,0,0\}$. Find $X[K] = ?$

Ans:- Given that, $N = 8$

By definition, N-point DFT is given by

$$X[K] = \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}.K.n} \quad , \quad 0 \le K \le N-1$$

$$X[K] = \sum_{n=0}^{8-1} x[n]\, e^{-j\frac{2\pi}{8}.K.n} \quad , \quad 0 \le K \le 8-1$$

$$0 \le K \le 7$$

$$= x[0]\,e^{-j\frac{\pi}{4}0} + x[1]\,e^{-j\frac{\pi}{4}.K.1} + x[2]\,e^{-j\frac{\pi}{4}.K.2} +$$
$$x[3]\,e^{-j\frac{\pi}{4}.K.3} + x[4]\,e^{-j\frac{\pi}{4}.K.4} + x[5]\,e^{-j\frac{\pi}{4}.K.5} +$$
$$x[6]\,e^{-j\frac{\pi}{4}.K.6} + x[7]\,e^{-j\frac{\pi}{4}.K.7}$$

$$= x[0] + x[1]\,e^{-j\frac{\pi}{4}.K} + x[2]\,e^{-j\frac{\pi}{2}K} + x[3]\,e^{-j\frac{3\pi}{4}K} +$$
$$x[4]\,e^{-j\pi K} + x[5]\,e^{-j\frac{5\pi}{4}K} + x[6]\,e^{-j\frac{3\pi}{2}K} +$$
$$x[7]\,e^{-j\frac{7\pi}{4}K}$$

$$X[K] = 1 + e^{-j\frac{\pi}{4}K} + e^{-j\frac{\pi}{2}K} + e^{-j\frac{3\pi}{4}K} + e^{-j\pi K} + e^{-j\frac{5\pi}{4}K}$$

As $0 \le K \le 7$,

$K=0$, $X[0] = 1+1+1+1+1+1 = 6$

$K=1$, $X[1] = 1 + e^{-j\pi/4} + e^{-j\frac{\pi}{2}} + e^{-j\frac{3\pi}{4}} + e^{-j\pi} + e^{-j\frac{5\pi}{4}}$

$$= 1 + \left(\frac{1}{\sqrt{2}} - \frac{j}{\sqrt{2}}\right) + (0-j) + \left(\frac{-1}{\sqrt{2}} - \frac{j}{\sqrt{2}}\right) + (-1-0) + \left(\frac{-1}{\sqrt{2}} + \frac{j}{\sqrt{2}}\right)$$

$$= \frac{-1}{\sqrt{2}} - j\left(1 + \frac{1}{\sqrt{2}}\right)$$

$K=2$, $X[2] = 1 + e^{-j\frac{\pi}{2}} + e^{-j\pi} + e^{-j3\pi/2} + e^{-j2\pi} + e^{-j5\pi/2}$

$= 1 + (0-j) + (-1+0) + (0+j) + (1+0) + (0-j)$

$= 1-j$

$K=3$, $X[3] = 1 + e^{-j\frac{3\pi}{4}} + e^{-j\frac{3\pi}{2}} + e^{-j\frac{9\pi}{4}} + e^{-j3\pi} + e^{-j\frac{15\pi}{4}}$

$= 1 + \left(\frac{-1}{\sqrt{2}} - \frac{j}{\sqrt{2}}\right) + (0+j) + \left(\frac{1}{\sqrt{2}} - \frac{j}{\sqrt{2}}\right) + (-1+0) + \left(\frac{1}{\sqrt{2}} + \frac{j}{\sqrt{2}}\right)$

$= \frac{1}{\sqrt{2}} + j\left(1-\frac{1}{\sqrt{2}}\right)$

$K=4$, $X[4] = 1 + e^{-j\pi} + e^{-j2\pi} + e^{-j3\pi} + e^{-j4\pi} + e^{-j5\pi}$

$= 1 + (-1-0) + 1 + (-1) + 1 + (-1)$

$= 0$

$K=5$, $X[5] = 1 + e^{-j\frac{5\pi}{4}} + e^{-j\frac{5\pi}{2}} + e^{-j\frac{15\pi}{4}} + e^{-j5\pi} + e^{-j\frac{25\pi}{4}}$

$= 1 + \left(\frac{-1}{\sqrt{2}} + \frac{j}{\sqrt{2}}\right) + (0-j) + \left(\frac{1}{\sqrt{2}} + \frac{j}{\sqrt{2}}\right) + (-1+0) + \left(\frac{1}{\sqrt{2}} - \frac{j}{\sqrt{2}}\right)$

$= \frac{1}{\sqrt{2}} - j\left(1-\frac{1}{\sqrt{2}}\right)$

$K=6$, $X[6] = 1 + e^{-j\frac{3\pi}{2}} + e^{-j3\pi} + e^{-j\frac{9\pi}{2}} + e^{-j6\pi} + e^{-j\frac{15\pi}{2}}$

$= 1 + (0+j) + (-1) + (0-j) + (1+0) + (0+j)$

$= 1+j$

$K=7$, $X[7] = 1 + e^{-j\frac{7\pi}{4}} + e^{-j\frac{7\pi}{2}} + e^{-j\frac{21\pi}{4}} + e^{-j7\pi} + e^{-j\frac{35\pi}{4}}$

$= 1 + \left(\frac{1}{\sqrt{2}} + \frac{j}{\sqrt{2}}\right) + (0+j) + \left(\frac{-1}{\sqrt{2}} + \frac{j}{\sqrt{2}}\right) + (-1+0) + \left(\frac{-1}{\sqrt{2}} - \frac{j}{\sqrt{2}}\right)$

$= \frac{-1}{\sqrt{2}} + j\left(1+\frac{1}{\sqrt{2}}\right)$

$$\therefore X[K] = \left\{ \underset{\uparrow}{6}, \left(\frac{-1}{\sqrt{2}} - j\left(1+\frac{1}{\sqrt{2}}\right)\right), 1-j, \left(\frac{1}{\sqrt{2}} + j\left(1-\frac{1}{\sqrt{2}}\right)\right), 0, \left(\frac{1}{\sqrt{2}} - j\left(1-\frac{1}{\sqrt{2}}\right)\right), 1+j, \left(\frac{-1}{\sqrt{2}} + j\left(1+\frac{1}{\sqrt{2}}\right)\right) \right\}$$

(3) $X[K] = \{\underset{\uparrow}{1}, 0, 1, 0\}$. compute its 4-point IDTFT $x[n]$ $0 \le n \le 3$

Ans:- Given that, $N = 4$

By def, $x[n] = \frac{1}{N} \sum_{n=0}^{N-1} X[K] e^{j \cdot \frac{2\pi}{N} \cdot k \cdot n}$, $0 \le n \le N-1$

$$x[n] = \frac{1}{4} \sum_{K=0}^{3} X[K] e^{j \cdot \frac{2\pi}{4} \cdot k \cdot n}, \quad 0 \le n \le 3$$

$$= \frac{1}{4}\left[X[0] e^{j\frac{\pi}{2}n(0)} + X[1] e^{j\frac{\pi}{2}n(1)} + X[2] e^{j\frac{\pi}{2}n(2)} + X[3] e^{j\frac{\pi}{2}n(3)}\right]$$

$$= \frac{1}{4}\left[X[0] + \cancel{X[1] e^{j\frac{\pi}{2}n}}^{\,0} + X[2] e^{j\pi n} + \cancel{X[3] e^{j\frac{3\pi}{2}n}}^{\,0}\right]$$

$$= \frac{1}{4}\left[1 + 1 \cdot e^{jn\pi}\right]$$

$$x[n] = \frac{1}{4}\left(1 + e^{jn\pi}\right)$$

As, $0 \le n \le 3$,

$n=0$, $x[0] = \frac{1}{4}(1+e^{0}) = 1/2$

$n=1$, $x[1] = \frac{1}{4}(1+e^{j\pi}) = 0$

$n=2$, $x[2] = \frac{1}{4}(1+e^{j2\pi}) = 1/2$

$n=3$, $x[3] = \frac{1}{4}(1+e^{j3\pi}) = 0$

$$\therefore x[n] = \left\{\underset{\uparrow}{\frac{1}{2}}, 0, \frac{1}{2}, 0\right\}$$

$$X[K] \triangleq DFT[x[n]] = \sum_{n=0}^{N-1} x[n] \cdot e^{-j\frac{2\pi}{N} \cdot k \cdot n} \quad \{DFT\}$$

$\hookrightarrow K = 0$ to $N-1$

$$x[n] \triangleq IDFT[X[K]] = \frac{1}{N} \sum_{K=0}^{N-1} X[K]\, e^{j\frac{2\pi}{N} \cdot K \cdot n} \quad \{IDFT\}$$

<u>Twiddle factor:</u> $W_N = e^{-j\frac{2\pi}{N}} = \cos\left(\frac{2\pi}{N}\right) - j\sin\left(\frac{2\pi}{N}\right)$

$X[K] = X[K+N]$; $X(K)$ is periodic with period 'N'.

$x[n] = x[n+N]$; $x[n]$ is also periodic with period 'N'.

$$x[n] = \frac{1}{N} \sum_{K=0}^{N-1} X[K]\, e^{j\frac{2\pi}{N} kn}$$

RHS :

$$x[n+N] = \underbrace{\frac{1}{N} \sum_{k=0}^{N-1} X[k]\, e^{j\frac{2\pi}{N} kn}}_{= x[n]} \; \underbrace{e^{j\frac{2\pi}{\cancel{N}} k \cdot \cancel{N}}}_{=1}$$

$\therefore x[n+N] = x[n]$

* Show that if $0 \le n, m \le N-1$ then

$$\sum_{k=0}^{N-1} e^{j\frac{2\pi}{N}k(n-m)} = \sum_{k=0}^{N-1} W_N^{-k(n-m)} = N\,\delta[n-m]$$

$$= \begin{cases} N & \text{for } n=m \\ 0 & \text{otherwise} \end{cases}$$

Ans:- We know that, $W_N = e^{-j\frac{2\pi}{N}}$

$$\boxed{\sum_{k=0}^{N-1} a^k = \begin{cases} N & a=1 \\ \dfrac{1-a^N}{1-a} & a\neq 1 \end{cases}}$$

$$\sum_{k=0}^{N-1} 1^k = \underbrace{1^0+1^1+1^2+\cdots\cdots+1^{N-1}}_{N \text{ terms}} = N$$

Let $a = e^{j\frac{2\pi}{N}[n-m]}$

$$a^N = e^{j\frac{2\pi}{\cancel{N}}[n-m].\cancel{N}}$$

$$a^N = e^{j2\pi[n-m]}$$

$$a^N = e^{j2\pi(0)} \leftarrow [n=m] \Rightarrow \boxed{a^N = e^0 = 1 \text{ for } n=m}$$

$$\sum_{k=0}^{N-1} e^{j\frac{2\pi}{N}k[n-m]} = \sum_{k=0}^{N-1} e^{j\frac{2\pi}{N}k(0)} = \sum_{k=0}^{N-1} 1 = N$$

For $n\neq m$

$$\sum_{k=0}^{N-1} e^{j\frac{2\pi}{N}k(n-m)} = \frac{1-e^{j\frac{2\pi}{\cancel{N}}[n-m]\cdot\cancel{N}}}{1-e^{j\frac{2\pi}{N}[n-m]}} = \frac{1-1}{1-e^{j\frac{2\pi}{N}[n-m]}}$$

$$= 0$$

$$\therefore \sum_{k=0}^{N-1} e^{j\frac{2\pi}{N}k[n-m]} = \begin{cases} N \ ; \ n=m \\ 0 \ ; \ n \neq m \end{cases} = \underline{N\,\delta[n-m]}$$

$\Rightarrow$ **DFT AS A LINEAR TRANSFORMATION [Matrix form]**

$$X[k] = \sum_{n=0}^{N-1} x[n]\, W_N^{kn} \qquad 0 \le k \le N-1$$

Let $N = 4$ [4-point DFT]

$$X[k] = \sum_{n=0}^{3} x[n]\, W_4^{kn} \qquad 0 \le k \le 3$$

$$X[k] = x[0]W_4^0 + x[1]W_4^k + x[2]W_4^{2k} + x[3]W_4^{3k}$$

$k=0$, $X[0] = x[0] + x[1] + x[2] + x[3]$

$k=1$, $X[1] = x[0] + x[1]W_4^1 + x[2]W_4^2 + x[3]W_4^3$

$k=2$, $X[2] = x[0] + x[1]W_4^2 + x[2]W_4^4 + x[3]W_4^6$

$k=3$, $X[3] = x[0] + x[1]W_4^3 + x[2]W_4^6 + x[3]W_4^9$

Matrix Representation,

$$\underbrace{\begin{bmatrix} X[0] \\ X[1] \\ X[2] \\ X[3] \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & W_4^1 & W_4^2 & W_4^3 \\ 1 & W_4^2 & W_4^4 & W_4^6 \\ 1 & W_4^3 & W_4^6 & W_4^9 \end{bmatrix}}_{W_4} \underbrace{\begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \end{bmatrix}}_{x}$$

$X = W_4 \cdot x \Rightarrow$ 4-point DFT

$\boxed{X = W_N x} \Rightarrow$ N-point DFT

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$$\underbrace{\begin{bmatrix} X[0] \\ X[1] \\ \vdots \\ X[N-1] \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W_N^1 & W_N^2 & \cdots & W_N^{N-1} \\ 1 & W_N^2 & W_N^4 & \cdots & W_N^{2(N-1)} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & W_N^{N-1} & W_N^{2(N-1)} & \cdots & W_N^{(N-1)(N-1)} \end{bmatrix}}_{W_N} \underbrace{\begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ \vdots \\ x[N-1] \end{bmatrix}}_{x}$$

$$\boxed{X = W_N x}$$

<u>IDFT</u> : $\boxed{x = W_N^{-1} X}$

$$W_N^{-1} = \frac{1}{N} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W_N^{-1} & W_N^{-2} & \cdots & W_N^{-(N-1)} \\ 1 & W_N^{-2} & W_N^{-4} & \cdots & W_N^{-2(N-1)} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & W_N^{-(N-1)} & W_N^{-2(N-1)} & \cdots & W_N^{-(N-1)(N-1)} \end{bmatrix}$$

$$\boxed{W_N^{-1} = \frac{1}{N} W_N^*}$$

$$\boxed{I_N N = W_N \cdot W_N^*}$$ $I_N \Rightarrow N \times N$ Identity matrix.

→ Matrix W_N in the transformation is an orthogonal (unitary) matrix.

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$$\boxed{\begin{array}{lll} X = W_N x & [\text{DFT}] & W_N^{-1} = \frac{1}{N} \cdot W_N^* \\ x = W_N^{-1} X & [\text{IDFT}] & W_N \cdot W_N^* = N \cdot I_{N \times N} \end{array}}$$

Twiddle factor (W_N) and its Properties:-

$$W_N = e^{-j\frac{2\pi}{N}} = \cos\left(\frac{2\pi}{N}\right) - j\sin\left(\frac{2\pi}{N}\right)$$

$$|W_N| = \left|e^{-j\frac{2\pi}{N}}\right| = \left|\cos\left(\frac{2\pi}{N}\right) - j\sin\left(\frac{2\pi}{N}\right)\right| = 1$$

$$\angle W_N = \frac{-2\pi}{N}$$

$$W_N = e^{-j\frac{2\pi}{N}} \Rightarrow W_N^N = e^{-j\frac{2\pi}{\cancel{N}}\cdot\cancel{N}} = e^{-j2\pi} = 1$$

$$\boxed{W_N^N = 1} \longrightarrow N^{th} \text{ root of unity.}$$

Properties:

① $W_N^{N/4} = -j$

Proof:- We know that $W_N = e^{-j\frac{2\pi}{N}}$

$$W_N^{N/4} = e^{-j\frac{2\pi}{\cancel{N}}\cdot\frac{\cancel{N}}{4}} = e^{-j\frac{\pi}{2}} = \cos\frac{\pi}{2} - j\sin\frac{\pi}{2} = 0-j$$

$$\therefore \boxed{W_N^{N/4} = -j}$$

② $W_N^{N/2} = -1$

Proof:-

$$W_N = e^{-j\frac{2\pi}{N}}$$

$$W_N^{N/2} = e^{-j\frac{\cancel{2}\pi}{\cancel{N}}\cdot\frac{\cancel{N}}{\cancel{2}}} = e^{-j\pi} = \cos\pi - j\sin\pi = -1-0$$

$$= -1$$

$$\therefore \boxed{W_N^{N/2} = -1}$$

③ $W_N^{3N/4} = +j$

Proof:-

$$W_N = e^{-j\frac{2\pi}{N}} \Rightarrow W_N^{\frac{3N}{4}} = e^{-j\frac{2\pi}{\cancel{N}}\times\frac{3\cancel{N}}{4}} = e^{-j\frac{3\pi}{2}}$$

$$= \cos\frac{3\pi}{2} - j\sin\frac{3\pi}{2} = 0+j$$

$$= +j$$

$$\therefore \boxed{W_N^{3N/4} = +j}$$

④ $W_N^N = 1$

Proof:- $W_N = e^{-j\frac{2\pi}{N}}$

$$W_N^N = e^{-j\frac{2\pi}{\cancel{N}}\cdot\cancel{N}} = e^{-j2\pi} = \cos 2\pi - j\sin 2\pi$$

$$= 1 - 0 = 1$$

$$\therefore \boxed{W_N^N = 1}$$

⑤ Periodicity Property

$$W_N^{K+N} = W_N^K$$

Proof:- $W_N = e^{-j\frac{2\pi}{N}}$

$$W_N^{K+N} = e^{-j\frac{2\pi}{N}(K+N)} = e^{-j\frac{2\pi}{N}\cdot K} \cdot \underbrace{e^{-j\frac{2\pi}{\cancel{N}}\cancel{N}}}_{=1}$$

$$= e^{-j\frac{2\pi}{N}\cdot K}$$

$$= W_N^K$$

$$\therefore \boxed{W_N^{K+N} = W_N^K}$$

⑥ Symmetry Property

$$W_N^{K+N/2} = -W_N^K$$

Proof:- $W_N = e^{-j\frac{2\pi}{N}}$

$$W_N^{K+\frac{N}{2}} = e^{-j\frac{2\pi}{N}(K+\frac{N}{2})} = e^{-j\frac{2\pi}{N}\cdot K} \cdot \underbrace{e^{-j\frac{2\pi}{\cancel{N}}\cdot\frac{\cancel{N}}{2}}}_{=-1}$$

$$= -e^{-j\frac{2\pi}{N}\cdot K}$$

$$= -W_N^K$$

$$\therefore \boxed{W_N^{K+\frac{N}{2}} = -W_N^K}$$

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Ex:- For $N = 4$

$$W_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & W_4^1 & W_4^2 & W_4^3 \\ 1 & W_4^2 & W_4^4 & W_4^6 \\ 1 & W_4^3 & W_4^6 & W_4^9 \end{bmatrix}_{4\times4}$$

$$W_4^0 = e^{-j\frac{2\pi}{4}0} = e^0 = 1$$

$$W_4^4 = e^{-j\frac{2\pi}{4}4} = e^{-j2\pi} = 1$$

$$W_4^8 = e^{-j\frac{2\pi}{4}\cdot 8} = e^{-j4\pi} = 1$$

$$W_4^0 = W_4^4 = W_4^8 = 1$$ (multiples ~~of~~ base)

$$W_4^1 = e^{-j\frac{2\pi}{4}} = e^{-j\frac{\pi}{2}} = -j$$

$$W_4^5 = e^{-j\frac{2\pi}{4}\cdot 5} = e^{-j\frac{5\pi}{2}} = -j$$

$$W_4^9 = e^{-j\frac{2\pi}{4}\cdot 9} = e^{-j\frac{9\pi}{2}} = -j$$

$$W_4^1 = W_4^5 = W_4^9 = -j$$

$$W_4^2 = e^{-j\frac{2\pi}{4}\cdot 2} = e^{-j\pi} = -1$$

$$W_4^6 = e^{-j\frac{2\pi}{4}\cdot 6} = e^{-j3\pi} = -1$$

$$W_4^{10} = e^{-j\frac{2\pi}{4}\cdot 10} = e^{-j5\pi} = -1$$

$$W_4^2 = W_4^6 = W_4^{10} = -1$$

$$W_4^3 = e^{-j\frac{2\pi}{4}\cdot 3} = e^{-j\frac{3\pi}{2}} = +j$$

$$W_4^7 = e^{-j\frac{2\pi}{4}\cdot 7} = e^{-j\frac{7\pi}{2}} = +j$$

$$W_4^{11} = e^{-j\frac{2\pi}{4}\cdot 11} = e^{-j\frac{11\pi}{2}} = +j$$

$$W_4^3 = W_4^7 = W_4^{11} = +j$$

$$W_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ W_4^1 & W_4^1 & W_4^2 & W_4^3 \\ 1 & W_4^2 & W_4^4 = W_4^0 & W_4^6 = W_4^2 \\ 1 & W_4^3 & W_4^6 = W_4^2 & W_4^9 = W_4^1 \end{bmatrix}$$

$$W_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & W_4^1 & W_4^2 & W_4^3 \\ 1 & W_4^2 & W_4^0 & W_4^2 \\ 1 & W_4^3 & W_4^2 & W_4^1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$$

⑦ $W_N^{2K} = W_{N/2}^{K}$

Proof:- $W_N = e^{-j\frac{2\pi}{N}}$

$$W_N^{2K} = e^{-j\frac{2\pi}{N}\cdot 2K}$$

$$= e^{-j\frac{2\pi}{N/2}\cdot K}$$

$$= W_{N/2}^{K}$$

$$\therefore \boxed{W_N^{2K} = W_{N/2}^{K}}$$

⑧ $W_N^{*} = W_N^{-1}$

Proof:- $W_N = e^{-j\frac{2\pi}{N}}$

$$W_N^{*} = \left(e^{-j\frac{2\pi}{N}}\right)^{*} = e^{j\frac{2\pi}{N}} = W_N^{-1}$$

$$\therefore \boxed{W_N^{*} = W_N^{-1}}$$

Problems:-

(1) Compute the N-point DFT of the following sequences

(a) $x[n] = \delta[n]$

By definition,

$$DFT\{x[n]\} = X[K] = \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}.k.n}$$

$$= \sum_{n=0}^{N-1} \delta[n]\, e^{-j\frac{2\pi}{N}.k.n}$$

$$\left\{\begin{array}{ll} \delta[n] = 1 & , n = 0 \\ \quad = 0 & , n \neq 0 \end{array}\right\} \qquad = \delta[0]\, e^{0} = 1$$

$$X[K] = 1$$

$$\boxed{\delta[n] \xleftrightarrow{DFT} 1}$$

(b) $x[n] = \delta[n-n_0]$

By definition,

$$X[K] = \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}.k.n}$$

$$= \sum_{n=0}^{N-1} \delta[n-n_0]\, e^{-j\frac{2\pi}{N}.k.n} \qquad \left[\begin{array}{ll} \delta[n] = 1 & ; n = n_0 \\ \quad = 0 & ; n \neq n_0 \end{array}\right]$$

$$X[K] = 1 .\, e^{-j\frac{2\pi}{N}.k.n_0}$$

$$\boxed{\delta[n-n_0] \xleftrightarrow{DFT} e^{-j\frac{2\pi}{N}k.n_0}}$$

(c) $x[n] = a^n \qquad 0 \leq n \leq N-1$

By definition, $X[K] = \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}.k.n}$

$$= \sum_{n=0}^{N-1} a^n\, e^{-j\frac{2\pi}{N}.k.n}$$

$$X[k] = \sum_{n=0}^{N-1} \left(ae^{-j\frac{2\pi}{N}k}\right)^n$$

$$\left[\sum_{N_1}^{N_2} x^n = \frac{x^{N_1} - x^{N_2+1}}{1-x}, \quad x \neq 1\right]$$

$$X[k] = \frac{\left(ae^{-j\frac{2\pi}{N}k}\right)^0 - \left(ae^{-j\frac{2\pi}{N}k}\right)^N}{1 - ae^{-j\frac{2\pi}{N}k}}$$

$$X[k] = \frac{1 - a^N e^{-j2\pi k}}{1 - ae^{-j\frac{2\pi}{N}k}} \quad ; \; k = 0, 1, 2, \dots (N-1)$$

$$e^{-j2\pi k} = \cos 2\pi k - j\sin 2\pi k = 1 - 0 = 1$$

$$X[k] = \frac{1-a^N}{1-ae^{-j\frac{2\pi}{N}k}}$$

$$\boxed{a^n \xleftrightarrow{DFT} \frac{1-a^N}{1-ae^{-j\frac{2\pi}{N}k}}}$$

(d) $x(n) = \cos\left(\frac{2\pi m n}{N}\right) \quad ; \; 0 \le m \le (N-1)$

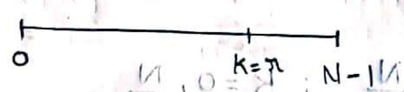
$$x[n] = \frac{e^{j\frac{2\pi m n}{N}} + e^{-j\frac{2\pi m n}{N}}}{2}$$

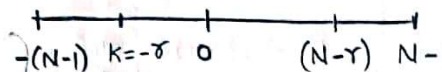
$$x[n] = \frac{1}{2}\left[W_N^{-mn} + W_N^{mn}\right]$$

$$X[k] = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi}{N}\cdot k\cdot n} = \sum_{n=0}^{N-1} x[n] W_N^{kn}$$

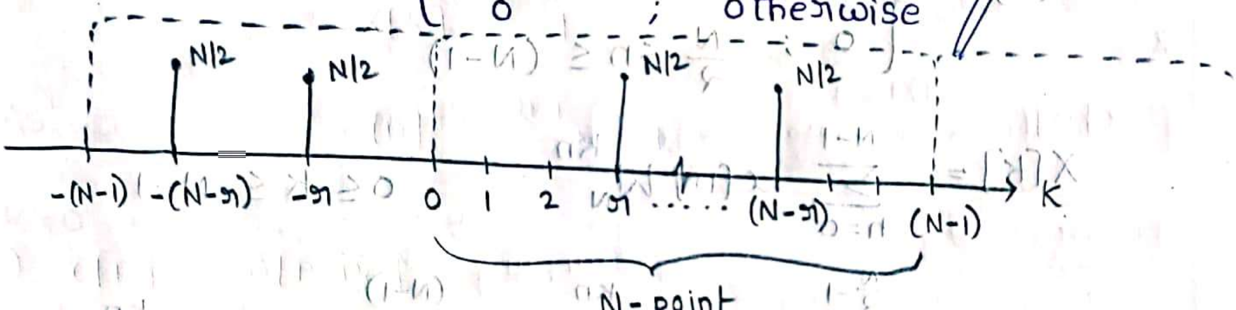
$$X[k] = \frac{1}{2}\left[\sum_{n=0}^{N-1} W_N^{-mn} \cdot W_N^{kn} + \sum_{n=0}^{N-1} W_N^{mn} \cdot W_N^{kn}\right]$$

$$X[k] = \frac{1}{2}\left[\underbrace{\sum_{n=0}^{N-1} W_N^{(k-r)n}}_{=\begin{cases} N, & k=r \\ 0, & \text{otherwise}\end{cases}} + \underbrace{\sum_{n=0}^{N-1} W_N^{(k+r)n}}_{=\begin{cases} N, & k=-r \text{ (or) } k=N-r \\ 0, & \text{otherwise}\end{cases}}\right]$$





$$\therefore \quad X[k] = \begin{cases} \frac{N}{2}; & k=r \\ \frac{N}{2}; & k=N-r \\ 0; & \text{otherwise} \end{cases}$$



(e) $x[n] = \begin{cases} 1 & n = \text{even} \quad 0 \le n \le (N-1) \\ 0 & n = \text{odd} \quad 0 \le n \le (N-1) \end{cases}$

$x[n] \longleftrightarrow X[k]$

$$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}k \cdot n} = \sum_{n=0}^{N-1} x[n]\, W_N^{kn}$$

$x[n]$ → Even $x[2n]$: $\{x[0], x[2], x[4]\}$; Odd $x[2n+1]$: $\{x[1], x[3], x[5]\}$

0 to $(N-1)$ → $\frac{N}{2}$; $\frac{N}{2}$

$$X[k] = \sum_{n=0}^{\frac{N}{2}-1} \underbrace{x[2n]}_{=1} W_N^{2kn} + \underbrace{\sum_{n=0}^{\frac{N}{2}-1} x[2n+1]\, W_N^{k[2n+1]}}_{=0}$$

$$X[k] = \sum_{n=0}^{\frac{N}{2}-1} e^{-j\frac{4\pi}{N}kn}$$

When $k=0$: $\sum_{n=0}^{\frac{N}{2}-1} e^{0} = \sum_{n=0}^{\frac{N}{2}-1} (1) = \frac{N}{2}$

When $k=\frac{N}{2}$: $\sum_{n=0}^{N/2-1} e^{-j\frac{4\pi}{N}\cdot\frac{N}{2}\cdot n} = \sum_{n=0}^{\frac{N}{2}-1} \underbrace{e^{-j2\pi n}}_{=1}$

$= \sum_{n=0}^{\frac{N}{2}-1} (1) = \underline{\underline{\frac{N}{2}}}$

$$X[k] = \begin{cases} \frac{N}{2} & ; k=0, \frac{N}{2} \\ 0 & \text{otherwise} \end{cases}$$

(f) $x[n] = \begin{cases} 1 & ; 0 \le n \le \left(\frac{N}{2}-1\right) \\ 0 & ; \frac{N}{2} \le n \le (N-1) \end{cases}$

★

$$X[k] = \sum_{n=0}^{N-1} x(n)\, W_N^{kn} \qquad 0 \le k \le N-1$$

$$= \sum_{n=0}^{\frac{N}{2}-1} \underbrace{x[n]}_{=1} W_N^{kn} + \sum_{n=\frac{N}{2}}^{(N-1)} \underbrace{x[n]}_{=0} W_N^{kn}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} W_N^{kn}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} e^{-j\frac{2\pi}{N}\cdot kn}$$

$$X[k] = \begin{cases} \frac{N}{2} & ; k=0 \\ \frac{1-e^{-j\pi k}}{1-e^{-j\frac{2\pi}{N}k}} & ; \underbrace{1 \le k \le N-1} \end{cases}$$

Even Odd

$$X[k] = \begin{cases} N/2 & ; k=0 \\ 2/1-e^{-j\frac{2\pi}{N}k} & ; k = \text{odd} \quad ; 1 \le k \le N-1 \\ 0 & ; k = \text{even} \quad ; 1 \le k \le N-1 \end{cases}$$

(g) $x[n] = \{1, 1, 1, 1\}$ $\Rightarrow$ 4-point Sequence
$\uparrow$

compute the 4-point DFT.

By definition,

$$X[K] = \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi}{N}kn} \qquad N = 4$$

$$= \sum_{n=0}^{3} x[n]\, e^{-j\frac{2\pi}{4}kn} \qquad 0 \le K \le 3$$

$$= x[0]\cdot e^{0} + x[1]\, e^{-j\frac{\pi}{2}\cdot 1\cdot K} + x[2]\, e^{-j\frac{\pi}{2}\cdot 2\cdot K} + x[3]\cdot e^{-j\frac{\pi}{2}\cdot 3\cdot K}$$

$$= 1 + e^{-j\frac{\pi}{2}K} + e^{-j\pi K} + e^{-j\frac{3\pi}{2}K}$$

As $0 \le K \le 3$,

$K=0 \Rightarrow X[0] = 1+1+1+1 = 4$

$K=1 \Rightarrow X[1] = 0$

$K=2 \Rightarrow X[2] = 0$

$K=3 \Rightarrow X[3] = 0$

$\therefore X[K] = \{4, 0, 0, 0\}$
$\uparrow$

Magnitude and phase of DFT i.e $X[K]$

$X[K] \rightarrow$ Complex function of real variable 'k'.

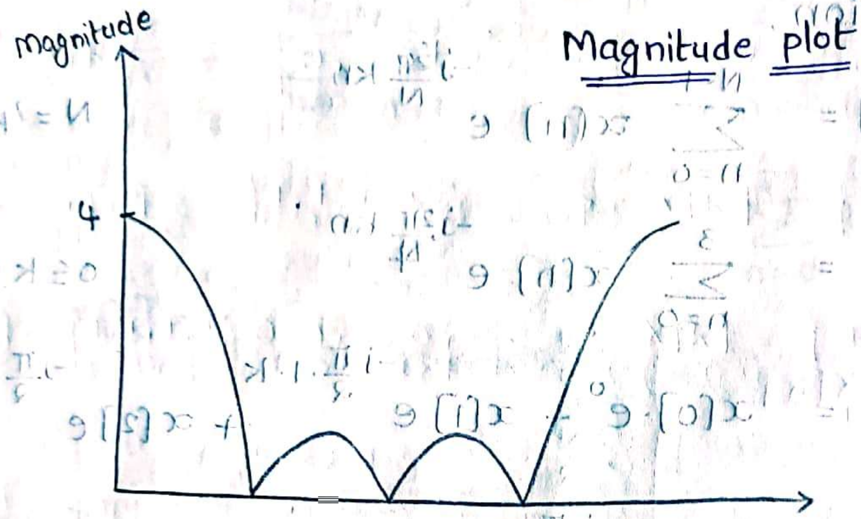
Rectangular form : $X[K] = X_R[K] + j X_I[K]$

Polar form :- $X[K] = |X[K]| \cdot e^{j\angle X[K]}$

$$|X[K]| = \sqrt{X_R^2[K] + X_I^2[K]}$$

$$\angle X[K] = \tan^{-1}\left[\frac{X_I(K)}{X_R(K)}\right]$$

$$x[n] = \{\underset{\uparrow}{1}, 1, 1, 1\} \longleftrightarrow X[K] = \{\underset{\uparrow}{4}, 0, 0, 0\}$$



$\frac{2\pi}{N} k = \frac{2\pi}{4} K$

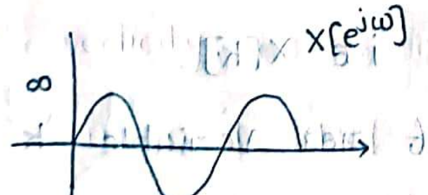
$$x[n] \xleftrightarrow{\text{DTFT}} X(e^{j\omega}) \rightarrow \text{continuous}$$

$\updownarrow$ Sampling

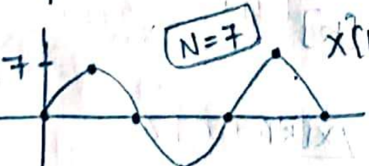
$X[K] \longrightarrow$ DFT of $x[n]$

As $N=4$, we can represent the DFT of $x[n]$ in only 4-points.

Ex:-

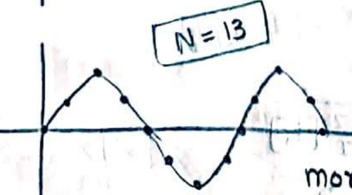


So, in our problem to get rid of loss of information we append zeroes at the end.



$x[n] = \{1, 1, 1, 1, 0, 0, 0, 0\}$ $N=4$, $m=8$

$\Rightarrow (m-N)$ zeroes should be appended



more denser than $N=7$

$x[n] = \{1, 1, 1, 1, 0, 0, 0, 0\}$

$$X[K] = \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}Kn} \qquad 0 \le K \le N-1$$

$$X[K] = \sum_{n=0}^{7} x[n]\, e^{-j\frac{\pi}{4}Kn}$$

$$= x[0]\cdot e^{-j\frac{\pi}{4}\cdot K\cdot 0} + x[1]\, e^{+j\frac{\pi}{4}\cdot K\cdot 1} + x[2]\, e^{-j\frac{\pi}{4}\cdot K\cdot 2} + \cdots\cdots x[7]\, e^{-j\frac{\pi}{4}\cdot K\cdot 7}$$

$$X[K] = 1 + e^{-j\frac{\pi}{4}K} + e^{-j\frac{\pi}{2}K} + e^{-j\frac{3\pi}{4}K}$$

AS $0 \le K \le 7$,

$K=0 \Rightarrow X[0] = 1+1+1+1 = 4$

$K=1 \Rightarrow X[1] = 1+e^{-j\frac{\pi}{4}} + e^{-j\frac{\pi}{2}} + e^{-j\frac{3\pi}{4}}$

$= 1 + [0.707 - j0.707] - j + [-0.707 - j0.707]$

$= 1 - j2.414 = 2.613\, e^{-j67.5^\circ}$

$K=2 \Rightarrow X[2] = 1 + e^{-j\frac{\pi}{2}} + e^{-j\pi} + e^{-j\frac{3\pi}{2}} = 0$

$K=3 \Rightarrow X[3] = 1 + e^{-j\frac{3\pi}{4}} + e^{-j\frac{3\pi}{2}} + e^{-j\frac{9\pi}{4}}$

$= 1 + (-0.707 - j0.707) + j + (0.707 - j0.707)$

$= 1 - j0.414 = 1.082\, e^{-j22.5^\circ}$

$K=4 \Rightarrow X[4] = 1 + e^{-j\pi K} + e^{-j2\pi} + e^{-j3\pi} = 0$

$K=5 \Rightarrow X[5] = 1 + j0.414 = 1.082\, e^{j22.5^\circ}$

$K=6 \Rightarrow X[6] = 0$

$K=7 \Rightarrow X[7] = 1 + j2.414 = 2.613\, e^{j67.5^\circ}$

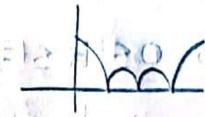
$$|X[k]| = \{\underset{\uparrow}{4}, 2.613, 0, 1.082, 0, 1.082, 0, 2.613\}$$

$$\angle X[k] = \{0, -67.5^\circ, 0, -22.5^\circ, 0, 22.5^\circ, 0, 67.5^\circ\}$$

For, $x[n] = \{1,1,1,1\} \longleftrightarrow X[k] = \{4,0,0,0\}$

$x[n] = \{1,1,1,1,0,0,0,0\} \longleftrightarrow X[k] = \{4, 2.613, 0, 1.082, 0, 1.082, 0, 2.613\}$

By adding zeroes to $x[n]$, we will get more dense and clear spectrum than previous one.



Properties of DFT:-

① Linearity:

If $x_1[n] \overset{DFT}{\underset{N}{\longleftrightarrow}} X_1[k]$

$x_2[n] \overset{DFT}{\underset{N}{\longleftrightarrow}} X_2[k]$

then $a x_1[n] + b x_2[n] \overset{DFT}{\underset{N}{\longleftrightarrow}} a X_1[k] + b X_2[k]$

② Periodicity:

If $x[n]$ and $X[k]$ are N-point DFT pair

$$x[n] \overset{DFT}{\underset{N}{\longleftrightarrow}} X[k]$$

then,

$X[k+N] = X[k]$ for all 'k'

$x[n+N] = x[n]$ for all 'n'.

Proof:

$$X[k] = \sum_{n=0}^{N-1} x[n]\, W_N^{kn}$$

$$X[k+N] = \sum_{n=0}^{N-1} x[n]\, W_N^{(k+N)n}$$

$$= \sum_{n=0}^{N-1} x[n]\, W_N^{kn} \cdot \underbrace{W_N^{Nn}}_{=1}$$

$$= \sum_{n=0}^{N-1} x[n]\, W_N^{kn} = \underline{\underline{X[k]}}$$

$$x[n+N] = x[n]$$

By definition of IDFT,

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k]\, W_N^{-kn}$$

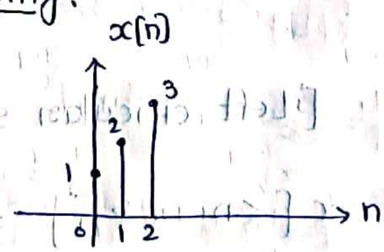
$$x[n+N] = \frac{1}{N} \sum_{k=0}^{N-1} X[k]\, W_N^{-kn} \cdot \underbrace{W_N^{-kN}}_{=1}$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} X[k]\, W_N^{-kn} = \underline{\underline{x[n]}}$$

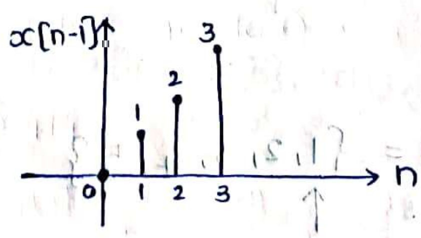
③ Circular time shifting:-

Time Shifting

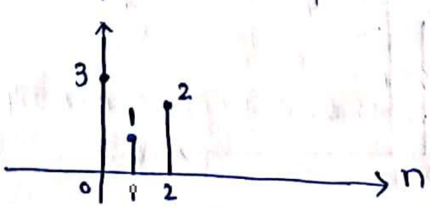
$x[n] = \{1, 2, 3\}$
$\uparrow$



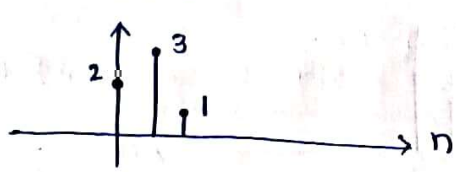
$x[n-1] = \{0, 1, 2, 3\}$
$\uparrow$



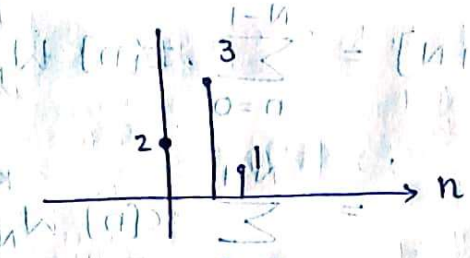
$x_c[n-1]$ $\longleftrightarrow$



$x_c[n-2]$ $\longleftrightarrow$

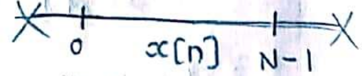


$x_c[n+1] \longleftrightarrow$



Consider a length 'N' sequence $x[n]$ defined $0 \le n \le (N-1)$

$x[n] = 0$ for $n < 0$ and $n \ge N$



$x[n-n_0] \longrightarrow x_c[n] = x(\underbrace{<n-n_0>}_{\text{argument}}{}_N)$

$\Rightarrow$ If argument $(n-n_0)$ is in b/w 0 to N-1, then leave it as it is.

otherwise, add/subtract multiples of 'N' from $(n-n_0)$ until the result is in between 0 to N-1

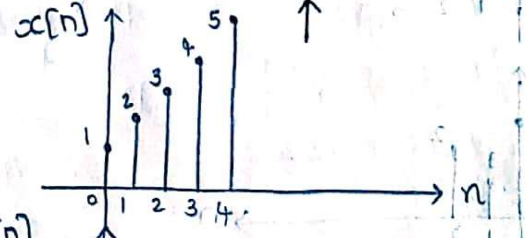
$\longrightarrow$ $n_0 > 0$ [right circular shift]

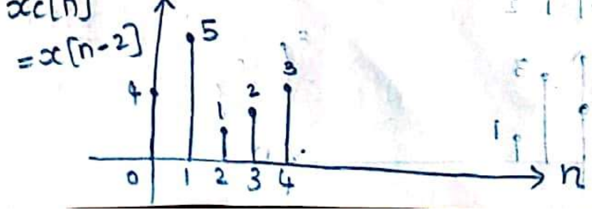
$$x_c[n] = x[<n-n_0>_N] = \begin{cases} x[n-n_0] \;; & n_0 \le n \le (N-1) \\ x[n-n_0+N] \;; & 0 \le n \le n_0-1 \end{cases}$$

$\longrightarrow$ $n_0 < 0$ [Left circular shift]

$$x_c[n] = x[<n-n_0>_N] = \begin{cases} x[n+n_0] \;; & 0 \le n \le N-n_0-1 \\ x[n+n_0-N] \;; & N-n_0 \le n \le N-1 \end{cases}$$

<u>Ex:-</u> $x[n] = \{1, 2, 3, 4, 5\}$, $N = 5$, $n_0 = 2$
$\uparrow$





Right circular shift : $x_c[n] = x[\langle n-n_0\rangle_5] \quad 0 \le n \le N-1$

$= x[\langle n-2\rangle_5] \quad 0 \le n \le 4$

$x_c[0] = x[\langle 0-2\rangle_5]$

$= x[\langle -2+5\rangle] = x[\langle +3\rangle]$

$= x[3]$

$x_c[1] = x[\langle 1-2\rangle_5]$

$= x[\langle -1\rangle_5] = x[\langle -1+5\rangle]$

$= x[\langle 4\rangle] = x[4]$

$x_c[2] = x[\langle 2-2\rangle_5]$

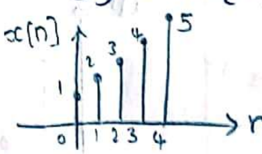
$= x[\langle 0\rangle_5] = x[0]$

$x_c[3] = x[\langle 3-2\rangle_5]$

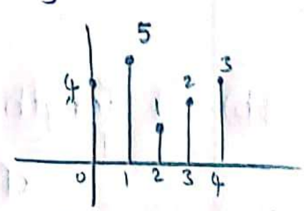
$= x[\langle 1\rangle_5] = x[1]$

$x_c[4] = x[\langle 4-2\rangle_5] = x[\langle 2\rangle_5] = x[2]$

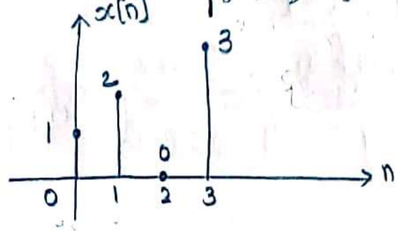
$\therefore x[n] = \{\underset{\uparrow}{\overset{1}{x[0]}}, \overset{2}{x[1]}, \overset{3}{x[2]}, \overset{4}{x[3]}, \overset{5}{x[4]}\}$

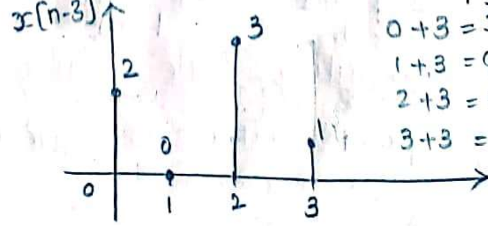


$x_c[n] = x[n-2] = \{\underset{\uparrow 4}{x[3]}, \underset{5}{x[4]}, \underset{1}{x[0]}, \underset{2}{x[1]}, \underset{3}{x[2]}\}$



Ex:- (1) Let $x[n] = \{\underset{\uparrow 0}{1}, \underset{1}{2}, \underset{2}{0}, \underset{3}{3}\}$ find $x_1[n] = x[\langle n-3\rangle_4]$





$0+3=3 \quad = 3 = 1$

$1+3=4 \quad -4=0 = 2$

$2+3=5 \quad -4=1 = 0$

$3+3=6 \quad -4=2 = 3$

$\{\underset{\uparrow}{2}, 0, 3, 1\}$

$$\boxed{\begin{array}{l} \text{If } x[n] \xleftrightarrow[N]{DFT} X[k] \\ \text{then } x[\langle n-n_0\rangle_N] \xleftrightarrow[N]{DFT} X[k]\, e^{-j\frac{2\pi}{N}kn_0} = X[k]\, W_N^{kn_0} \\ \text{then } x[\langle n+n_0\rangle_N] \xleftrightarrow[N]{DFT} X[k]\, e^{+j\frac{2\pi}{N}kn_0} = X[k]\, W_N^{-kn_0} \end{array}}$$

Ans:- Right circular shift: $x_c[n] = x[\langle n-n_0\rangle_N] = x[\langle n-3\rangle_4]$

$0 \le n \le 3$

$x_c[0] = x[\langle 0-3\rangle_4] = x[\langle -3+4\rangle]$

$= x[\langle +1\rangle] = x[1]$

$x_c[1] = x[\langle 1-3\rangle_4] = x[\langle -2+4\rangle] = x[2]$

$x_c[2] = x[\langle 2-3\rangle_4] = x[\langle -1+4\rangle] = x[3]$

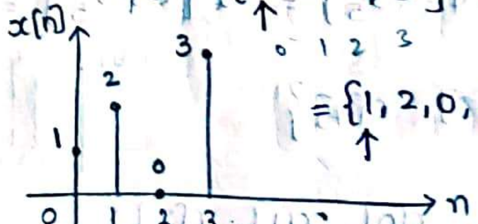
$x_c[3] = x[\langle 3-3\rangle_4] = x[\langle 0\rangle_4] = x[0]$

$\therefore x[n] = \{\overset{1}{\underset{\uparrow}{x[0]}}, \overset{2}{x[1]}, \overset{0}{x[2]}, \overset{3}{x[3]}\} = \{\underset{\uparrow}{1}, 2, 0, 3\}$

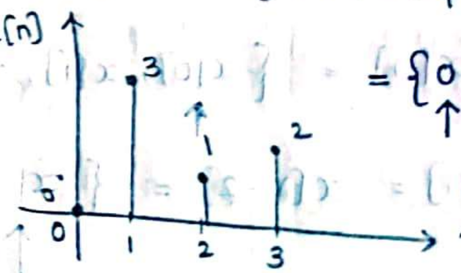
$x[n-3] = \{\overset{2}{\underset{\uparrow}{x[1]}}, \overset{0}{x[2]}, \overset{3}{x[3]}, \overset{1}{x[0]}\} = \{\underset{\uparrow}{2}, 0, 3, 1\}$

Ex2:- Let $x[n] = \{\underset{\uparrow}{1}, 2, 0, 3\}$ find $x_2[n] = x[\langle n+2\rangle_4]$

0 to 4



$= \{\underset{\uparrow}{1}, 2, 0, 3\}$



$= \{\underset{\uparrow}{0}, 3, 1, 2\}$

$0 - 2 = -2 + 4 = 2 \quad x[2] \rightarrow 1$

$1 - 2 = -1 + 4 = 3 \quad x[3] \rightarrow 2$

$2 - 2 = 0 = 0 \quad x[0] \rightarrow 0$

$3 - 2 = 1 = 1 \quad x[1] \rightarrow 3$

$x_2[n] = \{\underset{\uparrow}{0}, 3, 1, 2\}$

(Pb) Consider a finite length sequence $x[n] = \delta[n] + 2\delta[n-5]$

(a) Find the 10-point DFT ~~of~~ $x[n]$

By definition,

$$X[k] = \sum_{n=0}^{N-1} x[n]\, W_N^{kn} \qquad N=10$$

$$X[k] = \sum_{n=0}^{9} \left[\delta[n] + 2\delta[n-5]\right] W_{10}^{kn}$$

$$= \sum_{n=0}^{9} \left[\delta[n] W_{10}^{kn} + 2\delta[n-5] W_{10}^{kn}\right]$$

$$= W_{10}^{0} + 2 W_{10}^{5k}$$

$$= 1 + 2e^{-j\frac{2\pi}{10}\cdot 5\cdot k} = 1 + 2e^{-j\pi k} = \underline{1 + 2(-1)^k}$$

(b) Find the sequence that has a DFT

$$Y[k] = X[k]\cdot e^{-j\frac{2\pi}{10}\cdot 3k} \qquad n=3,\ N=10$$

By using right circular shift property,

$$x[\langle n-3\rangle_{10}] \xleftrightarrow[10]{DFT} X[k]\, e^{-j\frac{2\pi}{10}\cdot 3k}$$

~~(c) Find the sequence that has a DFT $Y[k] = X[k] e^{-j2\pi}$~~

$$\therefore\ y[n] = x[\langle n-3\rangle_{10}] = \delta[n-3] + 2\delta[n-3-5]$$

$$= \delta[n-3] + 2\delta[n-8]$$

(c) $Y[k] = X[k]\cdot e^{-j\frac{2\pi}{10}\cdot 7k}$ $\quad n=3,\ N=10$

Using right circular shift property,

$$x[\langle n-7\rangle_{10}] \xleftrightarrow[10]{DFT} X[k]\, e^{-j\frac{2\pi}{10}\cdot 7k}$$

$$y[n] = x[\langle n-7\rangle_{10}] = \delta[n-7] + 2\delta[n-7-5]$$

$$= \delta[n-7] + 2\delta[n-12+10]$$

$$= \delta[n-7] + 2\delta[n-2]$$

ⓓ $Y[K] = X[K] \cdot e^{j\frac{2\pi}{10} \cdot 3K}$ $\to n=3$, $\to N=10$

$$x[\langle n+3\rangle_{10}] \xleftrightarrow[10]{DFT} X[K]e^{j\frac{2\pi}{10}\cdot 3K}$$

$$y[n] = x[\langle n+3\rangle_{10}]$$

$$= \underbrace{\delta[n+3]}_{n=-3\ \times} + 2\underbrace{\delta[n+3-5]}_{n=2\ \checkmark}$$

0 to N-1
0 to 9

$$= \delta[n+3-10] + 2\delta[n-2]$$

$$y[n] = \delta[n-7] + 2\delta[n-2]$$

(Pb) If $X[K] = \{\underset{\uparrow}{4}, -j2, 0, j2\}$ is the 4-point DFT of $x[n]$, Determine the DFT of $y[n] = x[\langle n-2\rangle_4]$

By using right circular shift property,

$$x[\langle n-2\rangle_4] \xleftrightarrow[4]{DFT} X[K]e^{-j\frac{2\pi}{4}\cdot 2k}$$

$$Y[K] = X[K]\cdot e^{-j\frac{2\pi}{4}\cdot 2k} \Rightarrow X[K]\cdot e^{-j\pi k}$$

$$Y[K] = (-1)^k X[K]$$

$$= \{\underset{\uparrow}{4}, -(-j2), 0, -j2\}$$

$$= \{4, j2, 0, -j2\}$$

(Pb) A length 8-sequence is given by

$$x[n] = \{\underset{\uparrow}{-4}, 5, 2, -3, 0, -2, 3, 4\} \quad 0 \le n \le 7$$

with an 8-point DFT given by $X[K]$.
Without computing the IDFT, determine the sequence $y[n]$, whose 8-point DFT is given by $Y[K] = W_4^{3K} X[K]$.

Given that,

$$Y[K] = W_4^{3k} X[K] = e^{-j\frac{2\pi}{4}\cdot 3k} \cdot X[K]$$

$$= e^{-j\frac{2\pi}{8}\cdot 6k} X[K]$$

Using circular time shifting property,

$$y[n] = x[\langle n-6\rangle_8]$$

$n=0 \Rightarrow y[0] = x[\langle 0-6\rangle_8] = x[\langle -6+8\rangle_8] = x[\langle 2\rangle_8]$

$= x[2] = 2$

$n=1 \Rightarrow y[1] = x[\langle 1-6\rangle_8] = x[\langle -5+8\rangle_8] = x[\langle 3\rangle_8]$

$= x[3] = -3$

$n=2 \Rightarrow y[2] = x[\langle 2-6\rangle_8] = x[\langle -4+8\rangle_8] = x[\langle 4\rangle_8]$

$= x[4] = 0$

$n=3 \Rightarrow y[3] = x[\langle 3-6\rangle_8] = x[\langle -3+8\rangle_8] = x[\langle 5\rangle_8]$

$= x[5] = -2$

$n=4 \Rightarrow y[4] = x[\langle 4-6\rangle_8] = x[\langle -2+8\rangle_8] = x[\langle 6\rangle_8]$

$= x[6] = 3$

$n=5 \Rightarrow y[5] = x[\langle 5-6\rangle_8] = x[\langle -1+8\rangle_8] = x[\langle 7\rangle_8]$

$= x[7] = 4$

$n=6 \Rightarrow y[6] = x[\langle 6-6\rangle_8] = x[\langle 0\rangle_8] = x[0]$

$= -4$

$n=7 \Rightarrow y[7] = x[\langle 7-6\rangle_8] = x[\langle 1\rangle_8] = x[1]$

$= 5$

$$\therefore y[n] = \{\underset{\uparrow}{2}, -3, 0, -2, 3, 4, -4, 5\}$$

(4) Circular Frequency Shifting:

If $x[n] \overset{DFT}{\underset{N}{\longleftrightarrow}} X[k]$

then, $e^{j\frac{2\pi}{N}k_0 n} x[n] = W_N^{-k_0 n} \cdot x[n] \overset{DFT}{\underset{N}{\longleftrightarrow}} X[\langle k-k_0 \rangle_N]$

$$e^{j\frac{2\pi}{N}(-k_0)n} \cdot x[n] = W_N^{+k_0 n} \cdot x[n] \overset{DFT}{\underset{N}{\longleftrightarrow}} X[\langle k+k_0 \rangle_N]$$

(Pb) If $x[n]$ has an N-point DFT $X[k]$, find the N-point DFT ~~of~~ $y[n] = \cos\left(\frac{2\pi}{N}n\right) \cdot x[n]$

Given that,

$$y[n] = \cos\left[\frac{2\pi}{N}n\right] x[n]$$

$$= \frac{1}{2}\left[e^{j\frac{2\pi}{N}n} + e^{-j\frac{2\pi}{N}n}\right] \cdot x[n]$$

$$= \frac{1}{2}\left[e^{j\frac{2\pi}{N}n} \cdot x[n] + e^{-j\frac{2\pi}{N}n} x[n]\right]$$

Using Circular frequency shifting property,

$$e^{j\frac{2\pi}{N}n} \cdot x[n] \overset{DFT}{\underset{N}{\longleftrightarrow}} X[\langle k-1 \rangle_N]$$

$$e^{-j\frac{2\pi}{N}n} \cdot x[n] \overset{DFT}{\underset{N}{\longleftrightarrow}} X[\langle k+1 \rangle_N]$$

$$\therefore Y[k] = \frac{1}{2}\left[X[\langle k-1 \rangle_N] + X[\langle k+1 \rangle_N]\right]$$

(Pb) Determine the N-point IDFT of

$$X[k] = \delta[k] \quad ; \quad 0 \le k \le N-1$$

and hence for $0 \le k_0 \le N-1$, Determine the DFT of

ⓐ $x_1[n] = e^{j\frac{2\pi}{N}k_0 n}$ ⓒ $x_3[n] = \cos\left[\frac{2\pi}{N}k_0 n\right]$

ⓑ $x_2[n] = e^{-j\frac{2\pi}{N}k_0 n}$ ⓓ $x_4[n] = \sin\left[\frac{2\pi}{N}k_0 n\right]$

Ans:- By definition,

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]\, e^{j\frac{2\pi}{N}kn}$$

$$= \frac{1}{N}\sum_{k=0}^{N-1} \delta[k]\, e^{j\frac{2\pi}{N}kn} = \frac{1}{N} e^{j\frac{2\pi}{N}kn}\Big|_{k=0}$$

$$= \frac{1}{N}$$

$$\boxed{\frac{1}{N} \underset{IDFT}{\overset{DFT}{\longleftrightarrow}} \delta[k]} \equiv \boxed{1 \underset{IDFT}{\overset{DFT}{\longleftrightarrow}} N\cdot\delta[k]}$$

ⓐ $x_1[n] = e^{j\frac{2\pi}{N}k_0 n} \cdot 1$

Using circular frequency shifting property,

$$\underbrace{e^{j\frac{2\pi}{N}k_0 n} \cdot 1}_{x[n]} \underset{N}{\overset{DFT}{\longleftrightarrow}} N\,\delta[k-k_0]$$

$$N\,\delta[\langle k-k_0\rangle_N]$$

ⓑ $x_2[n] = e^{-j\frac{2\pi}{N}k_0 n} \cdot 1$

Using circular frequency shifting property,

$$e^{-j\frac{2\pi}{N}k_0 n} \cdot 1 \underset{N}{\overset{DFT}{\longleftrightarrow}} N\cdot\delta[\langle k+k_0\rangle_N]$$

ⓒ $x_3[n] = \cos\left[\frac{2\pi}{N}k_0 n\right] = \frac{1}{2}\left[e^{j\frac{2\pi}{N}k_0 n} + e^{-j\frac{2\pi}{N}k_0 n}\right]$

$$X_3[k] = \frac{1}{2}\left[N\,\delta[\langle k-k_0\rangle_N] + N\cdot\delta[\langle k+k_0\rangle_N]\right]$$

$$\therefore \cos\left[\frac{2\pi}{N}k_0 n\right] \underset{N}{\overset{DFT}{\longleftrightarrow}} \frac{1}{2}\left[N\,\delta[\langle k-k_0\rangle_N] + N\cdot\delta[\langle k+k_0\rangle_N]\right]$$

(d) $x_4[n] = \sin\left[\frac{2\pi}{N} k_0 n\right]$

$$= \frac{1}{2j}\left[e^{j\frac{2\pi}{N}k_0 n} - e^{-j\frac{2\pi}{N}k_0 n}\right]$$

$$X_4[k] = \frac{1}{2j}\left[N\,\delta[\langle k-k_0\rangle_N] - N\,\delta[\langle k+k_0\rangle_N]\right]$$

$$\therefore \sin\left[\frac{2\pi}{N}k_0 n\right] \xleftrightarrow[N]{DFT} \frac{1}{2j}\left[N\,\delta[\langle k-k_0\rangle_N] - N\,\delta[\langle k+k_0\rangle_N]\right]$$

(5) Circular Time Reversal:

$$\boxed{x_c[n] = x[\langle -n\rangle_N]}$$

$$x_c[n] = \begin{cases} x[0] & n=0 \\ x[-n+N] & 1 \le n \le N-1 \end{cases}$$

$x(t) = x(-t)$

$x[n] = x[-n]$

$x[n] = x_c[n]$

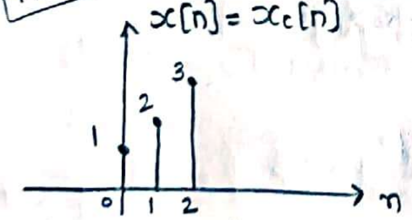
$= x[\langle -n\rangle_N]$

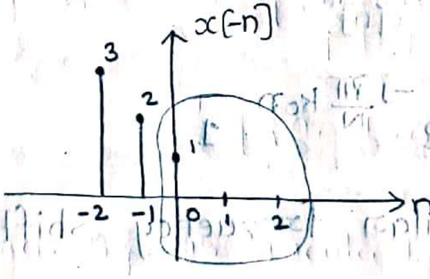
$= x[-n+N]$

$1 \le n \le (N-1)$

Ex:-

$\boxed{N=3}$





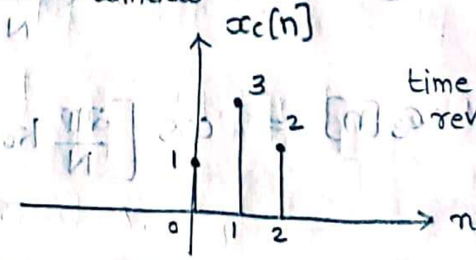
Sample at zero remains at zero only

$x[n] = \{\underset{\uparrow}{1}, 2, 3\}$

Samples out of the window

$n=1 \Rightarrow x[-1+3] = x[2]$

$n=2 \Rightarrow x[-2+3] = x[1]$

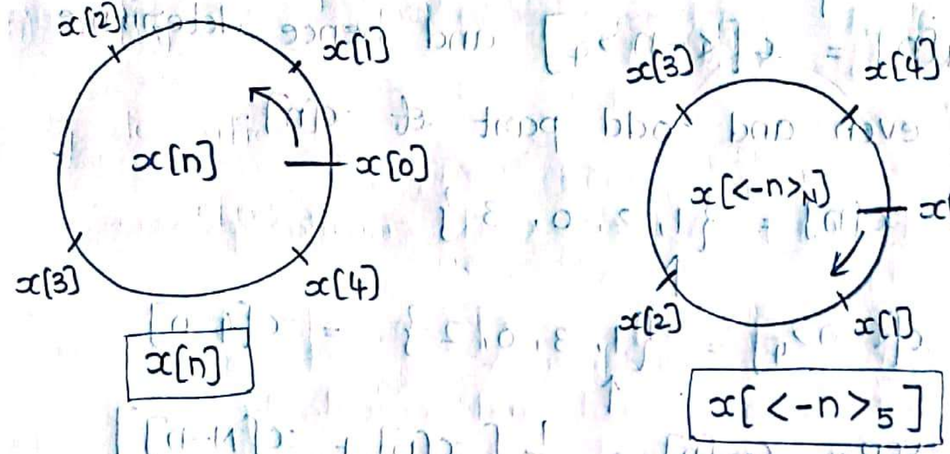


time reversal

⇒ The time reversal of an N-point sequence is attained by reversing its sample about the point zero on the circle.

⇒ Plotting $x[n]$ in a clockwise direction on a circle.

$$x[n] = \{x[0], x[1], x[2], x[3], x[4]\}$$



(a) Circularly even sequence [periodic Even]

An N-point sequence $x[n]$ is called circularly even if it is symmetric about the point "zero" on the circle.

$$x(n) = x[\langle -n \rangle_N] = x[N-n] \qquad 1 \le n \le N-1$$

(b) Circularly Odd sequence [periodic Odd]

An N-point sequence $x(n)$ is called circularly odd, if it is anti-symmetric about the point "zero" on the circle.

$$x(n) = -x[\langle -n \rangle_N] = -x[N-n] \qquad 1 \le n \le N-1$$

$$x(n) = \underbrace{x_{ce}(n)}_{\text{circularly even part}} + \underbrace{x_{co}(n)}_{\text{circularly odd part}}$$

$$x_{ce}(n) = \frac{1}{2}\left[x(n) + x[\langle -n \rangle_N]\right]$$

$$= \frac{1}{2}\left[x(n) + x[N-n]\right] \qquad 0 \le n \le N-1$$

$$x_{co}(n) = \frac{1}{2}\left[x(n) - x[\langle -n \rangle_N]\right]$$

$$= \frac{1}{2}\left[x(n) - x[N-n]\right] \qquad 0 \le n \le N-1$$

(Pb) Let $x[n] = \{\underset{\uparrow}{1}, 2, 0, 3\}$. Find the circularly folded signal $x_1[n] = x[\langle -n \rangle_4]$ and hence determine the circularly even and odd part ~~of~~ of $x[n]$.

Ans:- Given, $x[n] = \{\underset{\uparrow}{1}, 2, 0, 3\}$

$$x[\langle -n \rangle_4] = \{\underset{\uparrow}{1}, 3, 0, 2\} = x[4-n]$$

Circularly even, $x_{ce}[n] = \frac{1}{2}\left[x[n] + x[N-n]\right]$

$$= \frac{1}{2}\left[\{\underset{\uparrow}{1}, 2, 0, 3\} + \{\underset{\uparrow}{1}, 3, 0, 2\}\right]$$

$$= \frac{1}{2}\left[\{\underset{\uparrow}{2}, 5, 0, 5\}\right]$$

$$= \underline{\underline{\{1, \tfrac{5}{2}, 0, \tfrac{5}{2}\}}}$$

Circularly odd, $x_{co}[n] = \frac{1}{2}\left[x[n] + x[N-n]\right]$

$$= \frac{1}{2}\left[\{\underset{\uparrow}{1}, 2, 0, 3\} - \{\underset{\uparrow}{1}, 3, 0, 2\}\right]$$

$$= \frac{1}{2}\left[\{0, -1, 0, +1\}\right] = \underline{\underline{\{0, -\tfrac{1}{2}, 0, \tfrac{1}{2}\}}}$$

(Pb) If $x[n] = \{\underset{\uparrow}{3}, 2, A, 0, B\}$ is circularly even. Find A & B.

Ans:- Given, $x[n]$ is circularly even.

$$x[n] = x[\langle -n \rangle_5] = x[5-n] \qquad 1 \le n \le 5-1$$

$$1 \le n \le 4$$

$$\{\underset{\uparrow}{3}, 2, A, 0, B\} = \{\underset{\uparrow}{3}, B, 0, A, 2\}$$

By comparing samples on either sides,

We get $\boxed{B=2}$ $\boxed{A=0}$

Note:- Circular time reversal Property

⇒ If $x[n]$ is circularly even then its DFT is also circularly even.

$$\text{if} \quad x[n] = x[\langle -n \rangle_N]$$

$$\text{then} \quad X[k] = X[\langle -k \rangle_N] = X[N-k]$$

⇒ If $x[n]$ is circularly odd, then its DFT is also circularly odd,

$$\text{if} \quad x[n] = -x[\langle -n \rangle_N]$$

$$\text{then} \quad X[k] = -X[\langle -k \rangle_N] = -X[N-k]$$

(Pb) Consider the DFT pair $x[n] \xleftrightarrow[N=4]{DFT} X[k] = \{4, -j2, 0, j2\}$ with $N=4$. Find the DFT of $x[\langle -n \rangle_N]$

$$DFT[x[\langle -n \rangle_4]] = X[\langle -k \rangle_4] = \{\underset{\uparrow}{4}, j2, 0, -j2\}$$

$k=0 \Rightarrow X[\langle 0 \rangle_4] = X[0] = 4$

$k=1 \Rightarrow X[\langle -1 \rangle_4] = X[\langle -1+4 \rangle] = X[\langle +3 \rangle] = X[3] = j2$

$k=2 \Rightarrow X[\langle -2 \rangle_4] = X[\langle -2+4 \rangle] = X[\langle +2 \rangle] = X[2] = 0$

$k=3 \Rightarrow X[\langle -3 \rangle_4] = X[\langle -3+4 \rangle] = X[\langle +1 \rangle] = X[1] = -j2$

⑥ Conjugate and Conjugate Symmetry [Symmetry Property]:-

$$\text{If} \quad x[n] \xleftrightarrow[N]{DFT} X[k]$$

$$\text{then} \quad x^*[n] \xleftrightarrow[N]{DFT} X^*[\langle -k \rangle_N] = X^*[N-k]$$

$$\text{and} \quad x^*[N-n] = x^*[\langle -n \rangle_N] \xleftrightarrow[N]{DFT} X^*[k]$$

case-(i): If $x[n]$ is real, $x[n] = x^*[n]$

$$DFT\{x[n]\} = DFT\{x^*[n]\}$$

$$X[k] = X^*[\langle -k \rangle_N] = X^*[N-k]$$

$$X[K] = X^*[\langle -K \rangle_N] = X^*[N-K]$$

$$X^*[K] = X[N-K]$$

(Conjugate Symmetric)

<u>Case-2</u>: If $x[n]$ is real and circularly even

$$x[n] = x^*[n] = x[\langle -n \rangle_N]$$

then its DFT is also real and circularly even.

<u>Case-3</u>: If $x[n]$ is real and circularly odd

$$x[n] = x^*[n] = -x[\langle -n \rangle_N]$$

then its DFT is also imaginary and circularly odd.

<u>Case-4</u>: If $x[n]$ is imaginary

$$x[n] = -x^*[n]$$

then its DFT is conjugate Anti-symmetric.

$$DFT\{x[n]\} = DFT\{-x^*[n]\}$$

$$X[K] = -X^*[\langle -K \rangle_N] = -X^*[N-K]$$

$$X^*[K] = -X[N-K]$$ (Conjugate Anti-symmetric)

<u>Case-5</u>: If $x[n]$ is imaginary and circularly even

$$x[n] = -x^*[n] = x[\langle -n \rangle_N]$$

then its DFT is imaginary and circularly even.

<u>Case-6</u>: If $x[n]$ is imaginary and circularly odd

$$x[n] = -x^*[n] = -x[\langle -n \rangle_N]$$

then its DFT is real and circularly odd.

Case-7: Circularly even and odd decomposition of real slg.

Let $x[n]$ be a real signal

$$x[n] \xleftrightarrow[N]{DFT} X[K]$$

① The DFT of the circularly even part of $x[n]$ is the real value of $X[K]$.

$$x_{ce}[n] \xleftrightarrow[N]{DFT} X_R[K] = Re\{X[K]\}$$

② The DFT of the circularly odd part of $x[n]$ is the imaginary value of $X[K]$.

$$x_{co}[n] \xleftrightarrow[N]{DFT} j\,X_I[K] = j\,Img\{X[K]\}$$

(Pb) The even samples of the 11-point DFT of a length-11 real sequence are given by

$X[0] = 4$, $X[2] = -1+j3$, $X[4] = 2+j5$

$X[6] = 9-j6$, $X[8] = -5-j8$, $X[10] = \sqrt{3}-j2$

Determine the missing odd samples of the DFT.

Ans:- $\because$ $x[n]$ is real sequence,

then its DFT $X[K] = X^*[\langle -K\rangle_N] = X^*[N-K]$

Here, $N = 11$ $0,1,2,3,\ldots,10$

$$X[K] = X^*[N-K]$$

$K=1 \Rightarrow X[1] = X^*[11-1] = X^*[10] = (\sqrt{3}-j2)^* = \sqrt{3}+j2$

$K=3 \Rightarrow X[3] = X^*[11-3] = X^*[8] = (-5-j8)^* = -5+j8$

$K=5 \Rightarrow X[5] = X^*[11-5] = X^*[6] = (9-j6)^* = 9+j6$

$K=7 \Rightarrow X[7] = X^*[11-7] = X^*[4] = (2+j5)^* = 2-j5$

$K=9 \Rightarrow X[9] = X^*[11-9] = X^*[2] = (-1+j3)^* = -1-j3$

(Pb) A length-10 sequence $x[n]$ has a real valued 10-point DFT $X[K]$. The first six samples of $x[n]$ are given by

$x[0] = 2.5$ $\qquad$ $x[3] = -2+j5$

$x[1] = 7-j3$ $\qquad$ $x[4] = 7+j$

$x[2] = -3.2+j1.3$ $\qquad$ $x[5] = 5$

Find the remaining 4-samples of $x[n]$.

Ans:- $\because$ $x[n]$ has real valued 10-point DFT $X[K]$ i.e,

$$X[K] = X^*[K]$$

$$IDFT\{X[K]\} = IDFT\{X^*[K]\}$$

$$x[n] = x^*[\langle -n \rangle_N] = x^*[N-n]$$

Here, $N = 10$

$n=6 \implies x[6] = x^*[10-6] = x^*[4] = (7+j)^* = 7-j$

$n=7 \implies x[7] = x^*[10-7] = x^*[3] = (-2+j5)^* = -2-j5$

$n=8 \implies x[8] = x^*[10-8] = x^*[2] = -3.2-j1.3$

$n=9 \implies x[9] = x^*[10-9] = x^*[1] = (7-j3)^* = 7+j3$

⑦ <u>Duality</u> :-

$$\text{If} \quad x[n] \xleftrightarrow[N]{DFT} X[K]$$

$$\text{then} \quad X[n] \xleftrightarrow[N]{DFT} N \cdot x[\langle -K \rangle_N]$$

where 'n' is the time index

'k' is the frequency index.

$X[n]$ indicates that $X[K]$ is evaluated as a function of the time index 'n'.

Proof: By definition,

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] \cdot e^{j\frac{2\pi}{N}kn}$$

$$N \cdot x[n] = \sum_{k=0}^{N-1} X[k] \cdot e^{j\frac{2\pi}{N}kn}$$

Interchanging the variables 'n' and 'k',

$$N \cdot x[k] = \sum_{n=0}^{N-1} X[n] \cdot e^{j\frac{2\pi}{N}kn}$$

Now, replacing 'k' by '-k' in the above equation,

$$N \cdot x[\langle -k \rangle_N] = \sum_{n=0}^{N-1} X[n] \cdot e^{-j\frac{2\pi}{N}kn}$$

$$N \cdot x[\langle -k \rangle_N] = \text{DFT}\{X[n]\}$$

$$X[n] \xleftrightarrow[N]{\text{DFT}} N \cdot x[\langle -k \rangle_N]$$

$$\boxed{X[n] \xleftrightarrow[N]{\text{DFT}} N \cdot x[N-k]}$$

Interpretation of duality property:

Let $X[k]$ denote the N-point DFT of an N-point sequence $x[n]$.

$$x[n] \xleftrightarrow[N]{\text{DFT}} X[k]$$

The DFT $X[k]$ itself is an N-point sequence. If the DFT of $X[k]$ is computed to obtain another N-point sequence.

$$\text{DFT}\{X[k]\} = N \cdot x[\langle -n \rangle_N]$$

$$\text{DFT}\{\text{DFT}\{x[n]\}\} = N \cdot x[\langle -n \rangle_N]$$

Proof: By definition,

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] \cdot e^{j\frac{2\pi}{N}kn}$$

$$N \cdot x[n] = \sum_{k=0}^{N-1} X[k] \cdot e^{j\frac{2\pi}{N}kn}$$

Replacing 'n' by '-n' in the above equation,

$$N \cdot x[\langle -n \rangle_N] = \sum_{k=0}^{N-1} X[k]\, e^{-j\frac{2\pi}{N}kn}$$

$$= \mathrm{DFT}\{X[k]\}$$

$$\left[\sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}kn} = \mathrm{DFT}\{x[n]\}\right]$$

$$N \cdot x[\langle -n \rangle_N] = \mathrm{DFT}\{X[k]\}$$

$$* \quad N \cdot x[\langle -n \rangle_N] = \mathrm{DFT}\{\mathrm{DFT}\{x[n]\}\}$$

<u>Note</u>:- Important points

$\Rightarrow$ $DFT\{X[K]\} = DFT\{DFT\{x[n]\}\} = N \cdot x[\langle -n \rangle_N] = x_1[n]$

$\Rightarrow$ $DFT[DFT[DFT[DFT[x[n]]]]] = DFT[DFT[x_1[n]]]$

$= N \cdot x_1[\langle -n \rangle_N]$

$= N \cdot N \cdot x[n]$

$= N^2 x[n]$

2 - times DFT $\Rightarrow$ $N \cdot x[n]$

4 - times DFT $\Rightarrow$ $N^2 x[n]$

6 - times DFT $\Rightarrow$ $N^3 x[\langle -n \rangle_N]$

8 - times DFT $\Rightarrow$ $N^4 x[n]$

(Pb) Determine the N-point DFT of the sequence

$$g[n] = 1 \quad , \quad 0 \le n \le N-1$$

Ans:- Define $X[k] = 1$

Replacing 'n' with 'k' in the expression $g[n]$,

$$\delta[n] \underset{N}{\overset{DFT}{\longleftrightarrow}} 1$$

$$x[n] \underset{N}{\overset{DFT}{\longleftrightarrow}} X[k]$$

$$x[n] = \delta[n] \quad \text{and} \quad X[k] = 1$$

By using duality property,

$$X[n] \underset{N}{\overset{DFT}{\longleftrightarrow}} N\, x[\langle -k \rangle_N]$$

$$1 \underset{N}{\overset{DFT}{\longleftrightarrow}} N\, \delta[\langle -k \rangle_N] = N\, \delta[k]$$

$$\boxed{\begin{array}{c} 1 \underset{N}{\overset{DFT}{\longleftrightarrow}} N\cdot\delta[k] \\ \frac{1}{N} \underset{N}{\overset{DFT}{\longleftrightarrow}} \delta[k] \end{array}}$$

(Pb) Find the 10-point inverse DFT of

$$X[k] = 1 + 2\delta[k] = \begin{cases} 3 & k=0 \\ 1 & 1 \le k \le 9 \end{cases}$$

Ans:- Given that, $X[k] = 1 + 2\delta[k] \quad 0 \le k \le 9$

Here, $N = 10$

We know that the inverse DFT of a constant is a unit impulse.

$$\delta[n] \overset{DFT}{\longleftrightarrow} 1$$

Similarly, the DFT of a constant is an impulse

$$1 \underset{N}{\overset{DFT}{\longleftrightarrow}} N\cdot\delta[k]$$

$$1 \underset{10}{\overset{DFT}{\longleftrightarrow}} 10\cdot\delta[k]$$

$$\frac{1}{5} \xleftrightarrow[10]{DFT} 2 \cdot \delta[K]$$

$$\therefore \quad x[n] = \delta[n] + \frac{1}{5} \qquad 0 \le n \le 9$$

⑧ Circular Convolution [Multiplication of two DFT's]

If $x_1[n] \xleftrightarrow[N]{DFT} X_1[K]$

$x_2[n] \xleftrightarrow[N]{DFT} X_2[K]$

$$x_1[n] \, ⓝ \, x_2[n] = x_1[n] \circledast x_2[n] \xleftrightarrow[N]{DFT} X_1[K] \cdot X_2[K] = X_3[K]$$

Here, Ⓝ / $\circledast$ denotes N-point circular convolution.

$$x_3[n] = x_1[n] \circledast x_2[n] = \sum_{m=0}^{N-1} x_1[m] \cdot x_2[\langle n-m \rangle_N]$$

(Circular convolution)

$$x_1[n] * x_2[n] = \sum_{m=-\infty}^{\infty} x_1[m] \cdot x_2[n-m]$$

(Linear convolution)

Note: → Linear convolution is a commutative operation

$$x_1[n] * x_2[n] = x_2[n] * x_1[n]$$

→ Circular convolution is also a commutative operation.

⇒ Methods to perform circular convolution.

① Analytical method

② Circulant matrix method

③ Graphical method

④ Frequency domain method.

(Pb) Perform the 4-point Circular convolution of the following two sequences.

$x_1[n] = \{\underset{\uparrow}{1}, 2, 2\}$ $\quad x_2[n] = \{\underset{\uparrow}{1}, 2, 3, 4\}$

Ans:- <u>Analytical method</u>: Both sequences should be of same length.

$x_1[n]$ = length of 3 $\quad x_1[n] = \{1, 2, 2, 0\}$

$x_2[n]$ = length of 4 $\quad x_2[n] = \{1, 2, 3, 4\}$

The 4-point circular convolution is given by,

$$N=4 \Longrightarrow x_3[n] = \sum_{m=0}^{N-1} x_1[m] \cdot x_2[\langle n-m\rangle_N]$$

$$= \sum_{m=0}^{3} x_1[m] \cdot x_2[\langle n-m\rangle_4] \quad , 0 \le n \le 3$$

As $0 \le n \le 3$,

For $n=0 \Rightarrow x_3[0] = \sum_{m=0}^{3} x_1[m] \cdot x_2[\langle -m\rangle_4]$

$$= x_1[0]\, x_2[0] + x_1[1]\, x_2[\langle -1\rangle_4] + x_1[2]\, x_2[\langle -2\rangle_4] + x_1[3]\, x_2[\langle -3\rangle_4]$$

$$= x_1[0] \cdot x_2[0] + x_1[1] \cdot x_2[-1+4] + x_1[2] \cdot x_2[-2+4] + x_1[3] \cdot x_2[4-3]$$

$$= \underset{1 \times 1}{x_1[0] \cdot x_2[0]} + \underset{2\times 4}{x_1[1] \cdot x_2[3]} + \underset{2 \times 3}{x_1[2] \cdot x_2[2]} + \underset{0 \times 2}{x_1[3] \cdot x_2[1]}$$

$$x_3[0] = 1 + 8 + 6 + 0 = \underline{\underline{15}}$$

For $n=1 \Rightarrow x_3[1] = \sum_{m=0}^{3} x_1[m] \cdot x_2[\langle 1-m\rangle_4]$

$$= x_1[0] \cdot x_2[1] + x_1[1] \cdot x_2[0] + x_1[2]\, x_2[\langle -1\rangle_4] + x_1[3]\, x_2[\langle -2\rangle_4]$$

$$= \underset{1\times 2}{x_1[0] \cdot x_2[1]} + \underset{2\times 1}{x_1[1]\, x_2[0]} + \overset{2\times 4}{x_1[2]\, x_2[3]} + x_1[3]\, x_2[2] \; 0\times 3$$

$x_3[1] = 2 + 2 + 8 + 0 = 12$

For $n=2 \Rightarrow x_3[2] = \sum_{m=0}^{3} x_1[m] \cdot x_2[\langle 2-m \rangle_4]$

$$= x_1[0] \cdot x_2[2] + x_1[1] \cdot x_2[1] + x_1[2] \cdot x_2[0] + x_1[3] \cdot x_2[3]$$

$$= 3 + 4 + 2 + 0 = 9$$

For $n=3 \Rightarrow x_3[3] = \sum_{m=0}^{3} x_1[m] \cdot x_2[\langle 3-m \rangle_4]$

$$= x_1[0] \cdot x_2[3] + x_1[1] \cdot x_2[2] + x_1[2]\, x_2[1] + x_1[3] \cdot x_2[0]$$

$$= 4 + 6 + 4 + 0 = 14$$

$\therefore x_3[n] = \{ \underset{\uparrow}{15}, 12, 9, 14 \}$

Circulant matrix method:-

$$x_3[n] = x_1[n] \circledast x_2[n]$$

Circulant matrix $N \times N$ whose columns equal to one of sequence and its cyclically shifted version.

For example, $x_2[n]$

$$C = \begin{bmatrix} x_2[0] & x_2[N-1] & \cdots & x_2[1] \\ x_2[1] & x_2[0] & \cdots & x_2[2] \\ x_2[2] & x_2[1] & \cdots & x_2[3] \\ x_2[3] & x_2[2] & \cdots & x_2[4] \\ \vdots & \vdots & & \vdots \\ x_2[N-1] & x_2[N-2] & & x_2[0] \end{bmatrix}_{N \times N} \Rightarrow \text{diagonal elements same } x_2[0].$$

$x_1[n] = \{ \underset{\uparrow}{1}, 2, 2, 0 \}$ Length of $x_1[n]$ & $x_2[n]$ should be same.

$x_2[n] = \{ \underset{\uparrow}{1}, 2, 3, 4 \}$

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Writing circulant matrix,

$$\begin{bmatrix} x_3(0) \\ x_3(1) \\ x_3(2) \\ x_3(3) \end{bmatrix} = \underset{4\times4}{\begin{bmatrix} x_2[0] & x_2[3] & x_2[2] & x_2[1] \\ x_2[1] & x_2[0] & x_2[3] & x_2[2] \\ x_2[2] & x_2[1] & x_2[0] & x_2[3] \\ x_2[3] & x_2[2] & x_2[1] & x_2[0] \end{bmatrix}} \times \underset{4\times1}{\begin{bmatrix} x_1[0] \\ x_1[1] \\ x_1[2] \\ x_1[3] \end{bmatrix}}$$

$$x_3[n] = x_1[n] \circledast x_2[n] = x_2[n] \circledast x_1[n]$$

$$\begin{bmatrix} x_3(0) \\ x_3(1) \\ x_3(2) \\ x_3[3] \end{bmatrix} = \begin{bmatrix} 1 & 4 & 3 & 2 \\ 2 & 1 & 4 & 3 \\ 3 & 2 & 1 & 4 \\ 4 & 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 2 \\ 0 \end{bmatrix} \begin{matrix} 1+8+6+0 \\ 2+2+8+0 \\ 3+4+2+0 \\ 4+6+4+0 \end{matrix} = \begin{bmatrix} 15 \\ 12 \\ 9 \\ 14 \end{bmatrix}$$

$$\therefore x_3[n] = \{\underline{\underline{15, 12, 9, 14}}\}$$

<u>Frequency</u> <u>domain</u> <u>method</u> (using DFT & IDFT):

$$x_3[n] = x_1(n) \circledast x_2(n)$$

① Take N-point DFT of $x_1[n]$ and $x_2[n]$ to obtain

$$X_1[K] = \text{DFT}\{x_1[n]\}$$

$$X_2[K] = \text{DFT}\{x_2[n]\}$$

② Determine the product of $X_1[K]$ & $X_2[K]$

$$X_3[K] = X_1(K) \cdot X_2[K]$$

③ Take IDFT of $X_3[K]$ to obtain $x_3[n] = \text{IDFT}\{X_3[K]\}$

$\longrightarrow$ $x_1[n] = \{\underset{\uparrow}{1}, 2, 2\}$ $\quad$ $x_2[n] = \{\underset{\uparrow}{1}, 2, 3, 4\}$

The 4-point DFT is given by

$$X_1[K] = \sum_{n=0}^{3} x_1[n] \cdot W_4^{kn} \qquad 0 \le k \le 3$$

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$$X_1[K] = x_1[0]W_4^0 + x_1[1]W_4^K + x_1[2]W_4^{2K} + x_1[3]W_4^{3k}$$

$$X_1[K] = 1 + 2W_4^K + 2W_4^{2K} + 0$$

As $0 \le k \le 3$,

$K=0 \Rightarrow X_1[0] = 1 + 2 + 2 = 5$

$K=1 \Rightarrow X_1[1] = 1 + 2W_4^2 + 2W_4^1 = -1-j2$

$K=2 \Rightarrow X_1[2] = 1 + 2W_4^2 + 2W_4^4 = +1$

$K=3 \Rightarrow X_1[3] = 1 + 2W_4^3 + 2W_4^6 = -1+j2$

$\therefore X_1[K] = \{5, -1-j2, +1, -1+j2\}$

Now, $X_2[K] = \sum_{n=0}^{3} x_2[n] W_4^{kn} = \sum_{n=0}^{3} x_2[n] e^{-j\frac{2\pi}{4}kn}$

$$X_2[K] = x_2[0]W_4^0 + x_2[1]W_4^K + x_2[2]W_4^{2K} + x_2[3]W_4^{3k}$$

$$X_2[K] = 1 + 2W_4^K + 3W_4^{2K} + 4W_4^{3K}$$

$$1 + 2e^{-j\frac{\pi}{2}K} + 3e^{-j\frac{\pi}{2}\cdot 2K} + 4e^{-j\frac{\pi}{2}\cdot 3k}$$

As $0 \le k \le 3$,

$K=0 \Rightarrow X_2[0] = 1 + 2 + 3 + 4 = 10$

$K=1 \Rightarrow X_2[1] = 1 + 2e^{-j\frac{\pi}{2}} + 3e^{-j\pi} + 4e^{-j\frac{3\pi}{2}} = 1 + 2(-j) + 3(-1) + 4(j)$

$= -2 + j2$

$K=2 \Rightarrow X_2[2] = 1 + 2e^{-j\pi} + 3e^{-j2\pi} + 4e^{-j3\pi} = 1 - 2 + 3 - 4$

$= -2$

$K=3 \Rightarrow X_2[3] = 1 + 2e^{-j\frac{3\pi}{2}} + 3e^{-j3\pi} + 4e^{-j\frac{9\pi}{2}}$

$= 1 + 2(j) + 3(-1) + 4(-j) = -2-j2$

$$\therefore \; X_2[K] = \{10, -2+j2, -2, -2-j2\}$$

$$X_3[K] = X_1[K] \cdot X_2[K] = \{X_1[0]\cdot X_2[0],\; X_1[1]\cdot X_2[1],\; X_1[2]\, X_2[2],\; X_1[3]\cdot X_2[3]\}$$
$$\uparrow$$

$$= \{5\cdot 10,\; (-1-j2)(-2+j2),\; 1\cdot -2,\; (-1+j2)(-2-j2)\}$$
$$\uparrow$$

$$X_3[K] = \{50,\; 6+j2,\; -2,\; 6-j2\}$$
$$\uparrow$$

Finding $x_3(n)$, $\quad x_3[n] = \text{IDFT}\{X_3[K]\} = \frac{1}{4}\sum_{K=0}^{3} X_3[K]\, W_4^{-kn}$

$$= \frac{1}{4}\left[X_3[0]\, W_4^{0} + X_3[1]\, W_4^{-n} + X_3[2]\, W_4^{-2n} + X_3[3]\, W_4^{-3n}\right]$$

$$x_3(n) = \frac{1}{4}\left[50 + (6+j2)\, e^{j\frac{2\pi}{4}n} + (-2)\, e^{j\frac{2\pi}{4}(2n)} + (6-j2)\, e^{j\frac{2\pi}{4}(3n)}\right]$$

As $0 \le n \le 3$,

$n=0 \Rightarrow x_3[0] = \frac{1}{4}[50 + 6+j2 - 2 + 6-j2] = \frac{60}{4} = 15$

$n=1 \Rightarrow x_3[1] = \frac{1}{4}\left[50 + (6+j2)\, e^{j\frac{\pi}{2}} + (-2)\, e^{j\pi} + (6-j2)\, e^{j\frac{3\pi}{2}}\right]$

$$= \frac{1}{4}\left[50 + (6+j2)(j) + (-2)(-1) + (6-j2)(-j)\right]$$

$$= \frac{1}{4}\left[50 + \cancel{6j} - \cancel{2} + \cancel{2} - \cancel{6j} - 2\right] = \frac{48}{4} = 12$$

$n=2 \Rightarrow x_3[2] = \frac{1}{4}\left[50 + (6+j2)\, e^{j\pi} + (-2)\, e^{j2\pi} + (6-j2)\, e^{j3\pi}\right]$

$$= \frac{1}{4}\left[50 + (6+j2)(-1) + (-2)(1) + (6-j2)(-1)\right]$$

$$= \frac{1}{4}\left[50 - 6 - \cancel{j2} - 2 - 6 + \cancel{j2}\right] = \frac{36}{4} = 9$$

$n=3 \Rightarrow x_3[3] = \frac{1}{4}\left[50 + (6+j2)\, e^{j\frac{3\pi}{2}} + (-2)\, e^{j3\pi} + (6-j2)\, e^{j\frac{9\pi}{2}}\right]$

$$= \frac{1}{4}\left[50 + (6+j2)(-j) + (-2)(-1) + (6-j2)(j)\right]$$

$$x_3(3) = \frac{1}{4}\left[50 - \cancel{6j} + 2 + 2 + \cancel{6j} + 2\right] = \frac{56}{4} = 14$$

$$\therefore x_3(n) = \{ \underset{\uparrow}{x_3(0)}, x_3(1), x_3(2), x_3(3) \}$$

$$x_3(n) = \{ \underset{\uparrow}{15}, 12, 9, 14 \}$$

(Pb) Let $x_1(n) = \{\underset{\uparrow}{1}, 2, 2, 1\}$, $x_2(n) = \{\underset{\uparrow}{1}, 2, 3, 4\}$

Compute the 4-point circular convolution by

① Analytical method

② Matrix method

③ Frequency domain method.

Ans:- Analytical method:-

$x_1(n) =$ length of 4 $\quad x_1(n) = \{\underset{\uparrow}{1}, 2, 2, 1\}$

$x_2(n) =$ length of 4 $\quad x_2(n) = \{\underset{\uparrow}{1}, 2, 3, 4\}$

The 4-point circular convolution is given by,

$$N=4 \Rightarrow x_3(n) = \sum_{m=0}^{N-1} x_1(m) \cdot x_2[\langle n-m\rangle_N]$$

$$= \sum_{m=0}^{3} x_1(m) \cdot x_2[\langle n-m\rangle_4], \quad 0 \le n \le 3$$

As $0 \le n \le 3$,

For $n=0 \Rightarrow x_3(0) = \sum_{m=0}^{3} x_1(m) \cdot x_2[\langle -m\rangle_4]$

$$= x_1(0) \cdot x_2(0) + x_1(1) \cdot x_2(3) + x_1(2) \cdot x_2(2) + x_1(3) \cdot x_2(1)$$

$$= 1 + 8 + 6 + 2 = 17$$

For $n=1 \Rightarrow x_3(1) = \sum_{m=0}^{3} x_1(m) \cdot x_2[\langle 1-m\rangle_4]$

$$x_3[1] = \underset{1\times 2}{x_1[0]\cdot x_2[1]} + \underset{2\times 1}{x_1[1]\cdot x_2[0]} + \underset{2\times 4}{x_1[2]\cdot x_2[3]} + \underset{1\times 3}{x_1[3]\cdot x_2[2]}$$

$$= 2 + 2 + 8 + 3 = 15$$

For $n = 2 \Rightarrow x_3[2] = \sum_{m=0}^{3} x_3[m]\cdot x_2[\langle 2-m\rangle_4]$

$$= \underset{1\times 3}{x_1[0]\cdot x_2[2]} + \underset{2\times 2}{x_1[1]\cdot x_2[1]} + \underset{2\times 1}{x_1[2]\cdot x_2[0]} + \underset{1\times 4}{x_1[3]\cdot x_2[3]}$$

$$= 3 + 4 + 2 + 4$$

$$= 13$$

For $n=3 \Rightarrow x_3[3] = \sum_{m=0}^{3} x_1[m]\cdot x_2[\langle 3-m\rangle_4]$

$$= \underset{1\times 4}{x_1[0]\cdot x_2[3]} + \underset{2\times 3}{x_1[1]\cdot x_2[2]} + \underset{2\times 2}{x_1[2]\cdot x_2[1]} + \underset{1\times 1}{x_1[3]\cdot x_2[0]}$$

$$= 4 + 6 + 4 + 1 = 15$$

$$\therefore x_3[n] = \{\underset{\uparrow}{17}, 15, 13, 15\}$$

Circulant matrix method:-

$x_3(n) = x_1(n) \circledast x_2(n)$

$x_1(n) = \{\underset{\uparrow}{1}, 2, 2, 1\}$

$x_2(n) = \{\underset{\uparrow}{1}, 2, 3, 4\}$

Writing circulant matrix, $x_2(n)$

$$\begin{bmatrix} x_3[0] \\ x_3[1] \\ x_3[2] \\ x_3[3] \end{bmatrix} = \begin{bmatrix} 1 & 4 & 3 & 2 \\ 2 & 1 & 4 & 3 \\ 3 & 2 & 1 & 4 \\ 4 & 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1+8+6+2 \\ 2+2+8+3 \\ 3+4+2+4 \\ 4+6+4+1 \end{bmatrix}$$

$$= \begin{bmatrix} 17 \\ 15 \\ 13 \\ 15 \end{bmatrix}$$

$$\therefore x_3[n] = \{\underset{\uparrow}{17}, 15, 13, 15\}$$

Frequency domain method:-

$x_1[n] = \{1, 2, 2, 1\}$
$\uparrow$

$$x_3[n] = x_1[n] \circledast x_2[n]$$

$x_2[n] = \{1, 2, 3, 4\}$
$\uparrow$

The 4-point DFT is given by,

$$X_1[K] = \sum_{n=0}^{3} x_1[n]\, W_4^{kn} \qquad 0 \le k \le 3$$

$$X_1[K] = x_1[0] W_4^0 + x_1[1] W_4^K + x_1[2] W_4^{2K} + x_1[3] W_4^{3K}$$

$$X_1[K] = 1 + 2W_4^K + 2W_4^{2K} + 1 \cdot W_4^{3K}$$

$$X_1[K] = 1 + 2e^{-j\frac{\pi}{2}K} + 2e^{-j\frac{\pi}{2}\cdot 2K} + 1 \cdot e^{-j\frac{\pi}{2}\cdot 3K}$$

As $0 \le K \le 3$,

$K=0 \Rightarrow X_1[0] = 1 + 2 + 2 + 1 = 6$

$K=1 \Rightarrow X_1[1] = 1 + 2e^{-j\frac{\pi}{2}} + 2e^{-j\pi} + e^{-j\frac{3\pi}{2}} = 1 - 2j - 2 + j$

$= -1 - j$

$K=2 \Rightarrow X_1[2] = 1 + 2e^{-j\pi} + 2e^{-j2\pi} + e^{-j3\pi} = 1 - 2 + 2 - 1 = 0$

$K=3 \Rightarrow X_1[3] = 1 + 2e^{-j\frac{3\pi}{2}} + 2e^{-j3\pi} + e^{-j\frac{9\pi}{2}}$

$= 1 + 2j - 2 - j = -1 + j$

$\therefore X_1[K] = \{6, -1-j, 0, -1+j\}$
$\uparrow$

Now, The 4-point DFT is given by

$$X_2[K] = \sum_{n=0}^{3} x_2[n]\, W_4^{kn} \qquad 0 \le K \le 3$$

$$X_2[K] = x_2[0] W_4^0 + x_2[1] W_4^K + x_2[2] W_4^{2K} + x_2[3] W_4^{3K}$$

$$X_2[K] = 1 + 2e^{-j\frac{\pi}{2}K} + 3e^{-j\frac{\pi}{2}\cdot 2K} + 4e^{-j\frac{\pi}{2}\cdot 3K}$$

As $0 \le K \le 3$,

$K=0 \Rightarrow X_2[0] = 1+2+3+4 = 10$

$K=1 \Rightarrow X_2[1] = 1+2e^{-j\frac{\pi}{2}} + 3e^{-j\pi} + 4e^{-j\frac{3\pi}{2}} = -2+j2$

$K=2 \Rightarrow X_2[2] = 1+2e^{-j\pi} + 3e^{-j2\pi} + 4e^{-j3\pi} = -2$

$K=3 \Rightarrow X_2[3] = 1+2e^{-j\frac{3\pi}{2}} + 3e^{-j3\pi} + 4e^{-j\frac{9\pi}{2}} = -2-j2$

$$\therefore X_2[K] = \{\underset{\uparrow}{10}, -2+j2, -2, -2-j2\}$$

Now, $X_3[K] = X_1[K] \cdot X_2[K]$

$$= \{X_1[0]\cdot X_2[0], X_1[1]\cdot X_2[1], X_1[2]\cdot X_2[2], X_1[3]\cdot X_2[3]\}$$

$$= \{(6)(10), (-1-j)(-2+j2), (0)(-2), (1+j)(-2-j2)\}$$

$$X_3[K] = \{\underset{\uparrow}{60}, 4, 0, 4\}$$

Finding $x_3(n)$, $x_3(n) = \text{IDFT}\{X_3[K]\} = \frac{1}{4}\sum_{K=0}^{3} X_3[K]\, W_4^{-Kn}$

$$= \frac{1}{4}\left[X_3[0]W_4^0 + X_3[1]W_4^{-n} + \cancel{X_3[2]}\, W_4^{-2n} + X_3[3]W_4^{-3n}\right]$$

$$x_3[n] = \frac{1}{4}\left[60 + 4e^{+j\frac{\pi}{2}n} + 4e^{j\frac{\pi}{2}\cdot 3n}\right]$$

As $0 \le n \le 3$,

$n=0 \Rightarrow x_3[0] = \frac{1}{4}[60+4+4] = \frac{68}{4} = 17$

$n=1 \Rightarrow x_3[1] = \frac{1}{4}\left[60+4e^{j\frac{\pi}{2}} + 4e^{j\frac{3\pi}{2}}\right] = \frac{60}{4} = 15$

$n=2 \Rightarrow x_3[2] = \frac{1}{4}\left[60+4e^{j\pi} + 4e^{j3\pi}\right] = \frac{52}{4} = 13$

$n=3 \Rightarrow x_3[3] = \frac{1}{4}\left[60+4e^{j\frac{3\pi}{2}} + 4e^{j\frac{9\pi}{2}}\right] = \frac{60}{4} = 15$

$$\therefore x_3[n] = \{\underset{\uparrow}{17}, 15, 13, 15\}$$

⑨ Multiplication property [Modulation Property]

If $x_1[n] \xleftrightarrow[N]{DFT} X_1[K]$

$x_2[n] \xleftrightarrow[N]{DFT} X_2[K]$

then $x_1[n] \cdot x_2[n] \xleftrightarrow[N]{DFT} \frac{1}{N}\left[X_1[K] \circledast X_2[K]\right]$

Proof:-

$$DFT\{x_1[n] \cdot x_2[n]\} = \sum_{n=0}^{N-1} x_1[n] \cdot x_2[n] W_N^{kn}$$

$$= \sum_{n=0}^{N-1}\left[\frac{1}{N}\sum_{m=0}^{N-1} X_1[m] W_N^{-mn}\right] \cdot x_2[n] W_N^{kn}$$

$$= \frac{1}{N}\sum_{m=0}^{N-1} X_1[m] \underbrace{\left[\sum_{n=0}^{N-1} x_2[n] W_N^{(k-m)n}\right]}_{= X_2[\langle k-m\rangle_N]}$$

$$= \frac{1}{N} \underbrace{\sum_{m=0}^{N-1} X_1[m] \cdot X_2[\langle k-m\rangle_N]}_{= X_1[K] \circledast X_2[K]}$$

$$= \frac{1}{N} \cdot X_1[K] \circledast X_2[K].$$

$$\therefore \boxed{DFT\{x_1[n] \cdot x_2[n]\} = \frac{1}{N} X_1[K] \circledast X_2[K]}$$

⑩ Parsevals Relation:

For complex valued $x[n]$ and $y[n]$

$x[n] \xleftrightarrow[N]{DFT} X[K]$ $\quad y[n] \xleftrightarrow[N]{DFT} Y[K]$

then
$$\boxed{\sum_{n=0}^{N-1} x[n]\, y^*[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[K]\, Y^*[K]}$$

Proof:- Consider LHS

$$\sum_{n=0}^{N-1} x[n]\, y^*[n] = \sum_{n=0}^{N-1} x[n] \cdot [y[n]]^*$$

$$= \sum_{n=0}^{N-1} x[n] \left[\frac{1}{N} \sum_{k=0}^{N-1} Y[k]\, e^{j\frac{2\pi}{N}kn} \right]^*$$

$$= \sum_{n=0}^{N-1} x[n] \left[\frac{1}{N} \sum_{k=0}^{N-1} Y^*[k]\, e^{-j\frac{2\pi}{N}kn} \right]$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} Y^*[k] \underbrace{\left[\sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}kn} \right]}_{DFT\{x[n]\} = X[k]}$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} X[k]\, Y^*[k] = RHS$$

If $y[n] = x[n]$,

$$\sum_{n=0}^{N-1} x[n]\, y^*[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k]\, Y^*[k]$$

$$\boxed{\sum_{n=0}^{N-1} |x[n]|^2 = \frac{1}{N} \sum_{k=0}^{N-1} |X[k]|^2}$$

Important Results:-

Let $X[k]$ denotes the N-point DFT of $x[n]$

① $X[0] = \sum_{n=0}^{N-1} x[n]$

Proof:- By definition, $X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}kn}$

$$X[k]\Big|_{k=0} = \sum_{n=0}^{N-1} x[n]\, e^{0} = \sum_{n=0}^{N-1} x[n]$$

$$X[0] = \sum_{n=0}^{N-1} x[n]$$

② $X\left[\frac{N}{2}\right] = \sum_{n=0}^{N-1} e^{j\pi n} x[n] = \sum_{n=0}^{N-1} (-1)^n x[n]$

Proof:- By definition,

$$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}kn}$$ (Pb)

$$X[k]\Big|_{k=\frac{N}{2}} = \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{\cancel{N}}\cdot\frac{\cancel{N}}{2}\cdot n}$$

$$= \sum_{n=0}^{N-1} x[n]\, e^{-j\pi n} = \sum_{n=0}^{N-1} (-1)^n x[n]$$

③ $x[0] = \frac{1}{N}\sum_{k=0}^{N-1} X[k] \equiv \sum_{k=0}^{N-1} X[k] = N\cdot x[0]$ Ans:

Proof:- By definition,

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]\, e^{j\frac{2\pi}{N}kn}$$

$$x[n]\Big|_{n=0} = x[0] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]\, e^{0}$$

$$x[0] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]$$

④ $x\left[\frac{N}{2}\right] = \frac{1}{N}\sum_{k=0}^{N-1} e^{j\pi k} X[k] = \frac{1}{N}\sum_{k=0}^{N-1} (-1)^k X[k]$

Proof:- By definition,

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]\, e^{j\frac{2\pi}{N}kn}$$

$$x[n]\Big|_{n=\frac{N}{2}} = \frac{1}{N}\sum_{k=0}^{N-1} X[k]\, e^{j\frac{2\pi}{\cancel{N}}\cdot\frac{\cancel{N}}{2}\cdot k}$$

$$x\left[\frac{N}{2}\right] = \frac{1}{N}\sum_{k=0}^{N-1} e^{j\pi k}\cdot X[k] = \frac{1}{N}\sum_{k=0}^{N-1} (-1)^k X[k]$$

⑤ $\sum_{K=0}^{N-1} |X[K]|^2 = N \cdot \sum_{n=0}^{N-1} |x[n]|^2$ $\longrightarrow$ Parsevals Property.

(Pb) Consider the length-12 sequence defined for $0 \le n \le 11$

$x[n] = \{ \underset{\uparrow}{3}, -1, 2, 4, -3, -2, 0, 1, -4, 6, 2, 5 \}$ with a

12-point DFT given by $X[K]$, $0 \le K \le 11$. Evaluate the following functions of $X[K]$ without computing the DFT.

(a) $X[0]$ (b) $X[6]$ (c) $\sum_{K=0}^{11} X[K]$

(d) $\sum_{K=0}^{11} X[K] e^{-j\frac{4\pi}{6}K}$ (e) $\sum_{K=0}^{11} |X[K]|^2$

Ans:- Given that, $N = 12$

(a) $X[0] = \sum_{n=0}^{11} x[n] = 3-1+2+4-3-2+0+1-4+6+2+5$

$$X[0] = 13$$

(b) $X[6] = X\left[\frac{12}{2}\right] = \sum_{n=0}^{11} (-1)^n x[n]$

$$= (-1)^0 x[0] + (-1)^1 x[1] + (-1)^2 x[2] + \cdots + (-1)^{11} x[11]$$

$$= 3+1+2+4-3+2+0-1-4-6+2-5 = -13$$

(c) $\sum_{K=0}^{11} X[K] = N \cdot x[0] = 12(3) = 36$

(d) $\sum_{K=0}^{11} X[K] e^{-j\frac{2\pi}{3}K} = \sum_{K=0}^{11} X[K] e^{-j\frac{4\pi}{6}\times\frac{2}{2}K} = \sum_{K=0}^{11} X[K] e^{-j\frac{2\pi}{12}\cdot 4k}$

$e^{-j\frac{2\pi}{N}\cdot kn}$

By definition of IDFT $[N=12]$

$$\frac{1}{N} \sum_{K=0}^{N-1} X[K] e^{j\frac{2\pi}{N}kn} = x[n]$$

$$\frac{1}{12} \sum_{K=0}^{N-1} X[K] e^{j\frac{2\pi}{12}\cdot nk} = x[n]$$

$n = -4$ ✓

$$\sum_{K=0}^{11} X[K]\, e^{-j\frac{2\pi}{12}\cdot 4K} = 12 \cdot x[\langle -4 \rangle_{12}]$$

$$= 12 \cdot x[12-4] = 12 \cdot x[8] = 12 \times -4$$

$$= \underline{\underline{-48}}$$

(e) $$\sum_{K=0}^{11} |X[K]|^2 = 12 \sum_{n=0}^{11} |x[n]|^2$$

$$= 12\,[9+1+4+16+9+4+0+1+16+36+4+25]$$

$$= \underline{\underline{1500}}$$

(Pb) Let $X[K]$ be a 14-point DFT of a length-14 real sequence $x[n]$. The first eight samples are given by

$X[0] = 12$, $X[1] = -1+j3$, $X[2] = 3+j4$, $X[3] = 1-j5$

$X[4] = -2+j2$, $X[5] = 6+j3$, $X[6] = -2-j3$, $X[7] = 10$

Determine the remaining samples of $X[K]$. Evaluate the following functions of $x[n]$ without computing the IDFT of $X[K]$.

(a) $x[0]$ (b) $x[7]$ (c) $\sum_{n=0}^{13} x[n]$ (d) $\sum_{n=0}^{13} x[n]\, e^{j\frac{4\pi}{7}n}$

(e) $\sum_{n=0}^{13} |x[n]|^2$

Ans:- $\because x[n]$ is real sequence,

then its DFT $X[K] = X^*[\langle -K \rangle_N] = X^*[N-K]$

Here, $N = 14$, $0, 1, 2, 3, \ldots, 13$

$X[K] = X^*[N-K]$

$K=8 \Rightarrow X[8] = X^*[14-8] = X^*[6] = (-2-j3)^* = -2+j3$

$K=9 \Rightarrow X[9] = X^*[14-9] = X^*[5] = (6+j3)^* = 6-j3$

$K=10 \Rightarrow X[10] = X^*[14-10] = X^*[4] = (-2+j2)^* = -2-j2$

$K=11 \Rightarrow X[11] = X^*[14-11] = X^*[3] = (1-j5)^* = 1+j5$

$K=12 \Rightarrow X[12] = X^*[14-12] = X^*[2] = (3+j4)^* = 3-j4$

$K=13 \Rightarrow X[13] = X^*[14-13] = X^*[1] = (-1+j3)^* = -1-j3$

(a) $x[0] = \frac{1}{N} \sum_{K=0}^{13} X[K]$

$= \frac{1}{14}\left[X[0] + X[1] + X[2] + \cdots + X[13]\right]$

$= \frac{1}{14}\left[12 - 1 + \cancel{j3} + 3 + \cancel{j4} + 1 - \cancel{j5} - 2 + \cancel{j2} + 6 + \cancel{j3} - 2 - \cancel{j3} + 10 - 2 + \cancel{j3} + 6 - \cancel{j3} - 2 - \cancel{j2} + 1 + \cancel{j5} + 3 - \cancel{j4} - 1 - \cancel{j3}\right]$

$= \frac{32}{14} = \frac{16}{7}$

(b) $x[7] = x\left[\frac{14}{2}\right] = \frac{1}{14} \sum_{K=0}^{13} (-1)^K X[K]$

$= \frac{1}{14}\left[(-1)^0 X[0] + (-1)^1 X[1] + (-1)^2 X[2] + \cdots + (-1)^{13} X[13]\right]$

$= \frac{1}{14}\left[12 + 1 - \cancel{j3} + 3 + \cancel{j4} - 1 + \cancel{j5} - 2 + \cancel{j2} - 6 - \cancel{j3} - 2 - j3 - 10 - 2 + \cancel{j3} - 6 + \cancel{j3} - 2 - \cancel{j2} - 1 - \cancel{j5} + 3 - \cancel{j4} + 1 + \cancel{j3}\right]$

$= \frac{-12}{14} = -\frac{6}{7}$

(c) $\sum_{n=0}^{13} x[n] = X[0] = 12$

(d) $\sum_{n=0}^{13} x(n)\, e^{j\frac{4\pi}{7}n} = \sum_{n=0}^{13} x[n]\, e^{j\frac{2\pi}{14}\cdot 4n}$

Using definition of DFT $(N=14)$

$$\sum_{n=0}^{13} x[n]\, e^{-j\frac{2\pi}{14}kn} = X[k]$$

$$k=-4 \checkmark$$

$$\sum_{n=0}^{13} x[n]\, e^{j\frac{2\pi}{14}4n} = X[\langle -4 \rangle_{14}] = X[14-4]$$

$$= X[10] = -2-j2$$

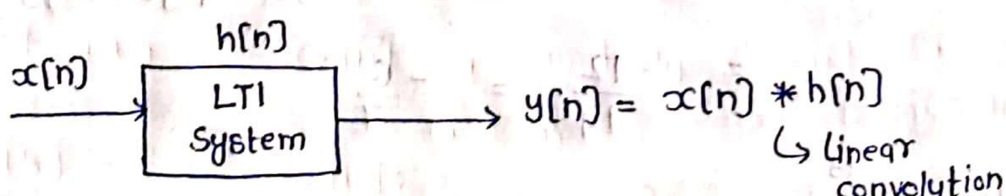
(e) $$\sum_{n=0}^{13} |x[n]|^2 = \frac{1}{14}\sum_{k=0}^{13} |X[k]|^2$$

$$= \frac{1}{14}\Big[(12)^2 + (1+j3)^2 + (3+j4)^2 + (1-j5)^2 + (-2+j2)^2 + (6+j3)^2 + (-2-j3)^2 + (10)^2 + (-2+j3)^2 + (6-j3)^2 + (-2-j2)^2 + (1+j5)^2 + (3-j4)^2 + (-1-j3)^2\Big]$$

$$= \frac{1}{14}\Big[144 + (1-9-\cancel{6j}) + (9-16+\cancel{24j}) + (1-25-\cancel{10j}) + (4-4-\cancel{8j}) + (36-9+\cancel{36j}) + (4-9+\cancel{12j}) + 100 + (4-9-\cancel{12j}) + (36-9-\cancel{36j}) + (4-4+\cancel{8j}) + (1-25+\cancel{10j}) + (9-16-\cancel{24j}) + (1-9+\cancel{6j})\Big]$$

$$= \frac{210}{14} = \underline{\underline{15}}$$

Linear Convolution using Circular Convolution :-



Filtering a sequence such as speech signal, radar signal, computing correlation of signals (it tells the degree of equality b/w two signals).

DFT ⇒ approach for implementing linear system operations in frequency domain.

DFT ⇒ Circular Convolution

$X[K] \cdot H[K] \Rightarrow$ Circular convolution

IDFT ⇒ $x[n] \circledast h[n]$

Let $x[n]$ length → N_x Let $N_x > N_h$

$h[n]$ length → N_h

Result of c.c $x[n] \circledast h[n]$ ≠ Result of L.c $x[n] * h[n]$

↓ Length of the result $N = N_x$

↓ Length of the Result $N = N_x + N_h - 1$

⇒ In order to get result of linear convolution same as circular convolution, we append 0's for $x[n]$ & $h[n]$

N_x & N_h ⇒ $N = N_x + N_h - 1$

$N = 3 + 2 - 1 = 4$

$$\begin{array}{cc} 3 & 2 \\ +1 & +2 \\ \hline 4 & 4 \end{array}$$

(Appending zeroes)

(Pb) Let $x[n] = \{2, 5, 0, 4\}$ $h[n] = \{4, 1, 3\}$

(a) Perform linear convolution using circular convolution.

(b) DFT and IDFT.

Ans:- (a) Given that, $x[n] = \{\underset{\uparrow}{2}, 5, 0, 4\}$ $N_x = 4$

$h[n] = \{\underset{\uparrow}{4}, 1, 3\}$ $N_h = 3$

→ We know that the linear convolution of $x[n]$ & $h[n]$ is having a length $N = N_x + N_h - 1 = 4 + 3 - 1 = 6$

→ We have to append zeroes to $x[n]$ & $h[n]$

$x[n] \rightarrow N - N_x = 6 - 4 = 2$, $h[n] \Rightarrow N - N_h = 6 - 3 = 3$

Scanned by CamScanner

$$x_z[n] = \{\underset{\uparrow}{2}, 5, 0, 4, 0, 0\}$$

$$h_z[n] = \{\underset{\uparrow}{4}, 1, 3, 0, 0, 0\}$$

$$y_c[n] = x_z[n] \circledast h_z[n]$$

By using circulant matrix method,

$$\begin{bmatrix} y_c[0] \\ y_c[1] \\ y_c[2] \\ y_c[3] \\ y_c[4] \\ y_c[5] \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 & 0 & 3 & 1 \\ 1 & 4 & 0 & 0 & 0 & 3 \\ 3 & 1 & 4 & 0 & 0 & 0 \\ 0 & 3 & 1 & 4 & 0 & 0 \\ 0 & 0 & 3 & 1 & 4 & 0 \\ 0 & 0 & 0 & 3 & 1 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \\ 0 \\ 4 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 8 \\ 22 \\ 11 \\ 31 \\ 4 \\ 12 \end{bmatrix}$$

Checking:-

$x[n]$ \ $h[n]$	4	1	3
2	8	2	6
5	20	5	15
0	0	0	0
4	16	4	12

$\Rightarrow \{8, 22, 11, 31, 4, 12\}$

(b) $x_z[n] = \{\overset{0}{\underset{\uparrow}{2}}, \overset{1}{5}, \overset{2}{0}, \overset{3}{4}, \overset{4}{0}, \overset{5}{0}\}$ $\quad h_z[n] = \{\overset{0}{\underset{\uparrow}{4}}, \overset{1}{1}, \overset{2}{3}, \overset{3}{0}, \overset{4}{0}, \overset{5}{0}\}$

$$N = N_x + N_h - 1 = 4 + 3 - 1 = 6$$

Size of DFT $\Rightarrow$ Atleast six

6-point DFT of $x_z[n]$ & similarly 6-point DFT of $h_z[n]$

$$X_z[k] = \sum_{n=0}^{5} x_z[n]\, e^{-j\frac{2\pi}{6}kn} \qquad 0 \le k \le 5$$

$$X_z[k] = 2 + 5e^{-j\frac{2\pi}{6}k\cdot 1} + 4e^{-j\frac{2\pi}{6}k\cdot 3} \qquad 0 \le k \le 5$$

$k=0 \Rightarrow X_z[0] = 2+5+4 = 11$

$k=1 \Rightarrow X_z[1] = 2 + 5e^{-j\frac{\pi}{3}} + 4e^{-j\frac{\pi}{\cancel{3}}\cdot\cancel{3}} = 0.5 - j4.3301$

$k=2 \Rightarrow X_z[2] = 2 + 5e^{-j\frac{\pi}{3}\cdot 2} + 4e^{-j2\pi} = 3.5 - j4.3301$

$k=3 \Rightarrow X_z[3] = 2 + 5e^{-j\frac{\pi}{\cancel{3}}\cdot\cancel{3}} + 4e^{-j3\pi} = -7$

$k=4 \Rightarrow X_z[4] = 2 + 5e^{-j\frac{\pi}{3}\cdot 4} + 4e^{-j4\pi} = 3.5 + j4.3301$

$k=5 \Rightarrow X_z[5] = 2 + 5e^{-j\frac{\pi}{3}\cdot 5} + 4e^{-j5\pi} = 0.5 + j4.3301$

Similarly, 6-point DFT ~~of~~ $h_z(n)$ is given by,

$$H_z[k] = \sum_{n=0}^{5} h_z[n] e^{-j\frac{2\pi}{6}kn} \quad , 0 \le k \le 5$$

$$H_z[k] = 4 + 1\cdot e^{-j\frac{\pi}{3}k} + 3e^{-j\frac{\pi}{3}\cdot 2k}$$

$k=0 \Rightarrow H_z[0] = 4+1+3 = 8$

$k=1 \Rightarrow H_z[1] = 4 + e^{-j\frac{\pi}{3}} + 3e^{-j\frac{2\pi}{3}} = 3 - j3.464$

$k=2 \Rightarrow H_z[2] = 4 + e^{-j\frac{2\pi}{3}} + 3e^{-j\frac{4\pi}{3}} = 2 + j1.732$

$k=3 \Rightarrow H_z[3] = 4 + e^{-j\frac{3\pi}{3}} + 3e^{-j\frac{6\pi}{3}} = 6$

$k=4 \Rightarrow H_z[4] = 4 + e^{-j\frac{4\pi}{3}} + 3e^{-j\frac{8\pi}{3}} = 2 - j1.732$

$k=5 \Rightarrow H_z[5] = 4 + e^{-j\frac{5\pi}{3}} + 3e^{-j\frac{10\pi}{3}} = 3 + j3.464$

$$\therefore Y_c[k] = X_z[k] \cdot H_z[k]$$

$$= \{\underset{\uparrow}{X_z[0]\cdot H_z[0]}, X_z[1]H_z[1], \ldots\ldots, X_z[5]H_z[5]\}$$

$$= \{\underset{\uparrow}{88}, -13.5 - j14.72, 14.5 - j2.59, -42, 14.5 + j2.59, -13.5 + j14.72\}$$

Now, finding IDFT

6-point IDFT : $y_c[n] = \frac{1}{6}\sum_{k=0}^{5} Y_c[k] e^{j\frac{2\pi}{6}kn} \qquad 0 \le n \le 5$

$$y_c[n] = \frac{1}{6}\left[Y_c[0]e^{0} + Y_c[1]e^{j\frac{\pi}{3}n} + Y_c[2]e^{j\frac{\pi}{3}2n} + Y_c[3]e^{j\frac{\pi}{3}3n} + Y_c[4]e^{j\frac{\pi}{3}4n} + Y_c[5]e^{j\frac{\pi}{3}5n} \right]$$

As $0 \le n \le 5$,

$$n=0 \Rightarrow y_c[0] = \frac{1}{6}\left[Y_c[0] + Y_c[1] + Y_c[2] + Y_c[3] + Y_c[4] + Y_c[5]\right]$$

$$= \frac{1}{6}\left[88 - 13.5 - \cancel{j14.72} + 14.5 - \cancel{j2.59} - 42 + 14.5 + \cancel{j2.59} - 13.5 + \cancel{j14.72}\right]$$

$$= \frac{1}{6}\left[88 - 42 - 2(13.5) + 2(14.5)\right]$$

$$= \frac{48}{6} = 8$$

$$n=1 \rightarrow y_c[1] = \frac{1}{6}\left[Y_c[0] + Y_c[1]e^{j\frac{\pi}{3}} + Y_c[2]e^{j\frac{2\pi}{3}} + Y_c[3]e^{j\pi} + Y_c[4]e^{j\frac{4\pi}{3}} + Y_c[5]e^{j\frac{5\pi}{3}}\right]$$

$$= \frac{1}{6}\Big[88 + (-13.5 - j14.72)(0.5 + 0.866) + (14.5 - j2.59)(-0.5 + 0.866) + (-42)(-1) + (14.5 + j2.59)(-0.5 - 0.866) + (-13.5 + j14.72)(0.5$$

$$= \frac{1}{6}\Big[88 + (-18.441 - j20.10) + (5.307 - j0.947794) + 42 + (-19.807 - j3.53794) + ($$

Assignment

(1) Let $x[n]$ $0 \le n \le N-1$, be an even length sequence with an N-point DFT $X[K]$, $0 \le k \le N-1$.

Determine the N-point DFT's of the following N-length sequences in terms of $X[K]$.

ⓐ $g[n] = x[n] - x[n - \frac{N}{2}]$

$$x[n] \xleftrightarrow{DFT} X[K]$$

By using circular time shifting property,

$$x[n - \frac{N}{2}] \xleftrightarrow{DFT} X[K]\, e^{-j \frac{2\pi}{\cancel{N}} \cdot \frac{\cancel{N}}{\cancel{2}} \cdot k}$$

$$x[n - \frac{N}{2}] \xleftrightarrow{DFT} X[K]\, e^{-j\pi k} = (-1)^k X[K]$$

$$\therefore g[n] = x[n] - x[n - \tfrac{N}{2}] \xleftrightarrow{DFT} G[K] = X[K] - (-1)^k X[K]$$

$$\boxed{G[K] = X[K]\left(1 - (-1)^k\right)}$$

ⓑ $h[n] = x[n] + x[n - \frac{N}{2}]$

$$x[n - \frac{N}{2}] \xleftrightarrow{DFT} (-1)^k X[K]$$

$$\therefore h[n] = x[n] + x[n - \tfrac{N}{2}] \xleftrightarrow{DFT} H[K] = X[K] + (-1)^k X[K]$$

$$\boxed{H[K] = X[K]\left(1 + (-1)^k\right)}$$

ⓒ $y[n] = (-1)^n x[n]$

$$y[n] = e^{-j\pi n} \cdot x[n] = e^{-j \frac{2\pi}{N} \cdot \frac{N}{2} \cdot n} \cdot x[n]$$

By using circular frequency shifting property,

$$e^{j \frac{2\pi}{N} (-k_0) n} x[n] \xleftrightarrow{DFT} X[\langle k + k_0 \rangle_N]$$

By comparing, Here $k_0 = \frac{N}{2}$

$$\therefore \boxed{(-1)^n x[n] \xleftrightarrow{DFT} X[\langle k + \tfrac{N}{2} \rangle_N]}$$

(2) Let $X[e^{j\omega}]$ denote the DTFT of the sequence $x[n] = \left(\frac{1}{2}\right)^n u[n]$. Let $y[n]$ denote a finite duration sequence of length $N = 10$, that is $y[n] = 0$; $n < 0$ and $y[n] = 0$, $n \geq 10$. The 10- point DFT of $y[n]$ denoted by $Y[k]$, corresponds to 10 equally spaced samples of $X[e^{j\omega}]$ that is $Y[k] = X[e^{j\frac{2\pi}{10}k}]$. Determine $y[n]$.

Ans:- Given, $x[n] = \left(\frac{1}{2}\right)^n u[n]$

We know that, $a^n u[n] \xleftrightarrow{DTFT} \dfrac{1}{1-ae^{-j\omega}}$

$$\left(\frac{1}{2}\right)^n u[n] \xleftrightarrow{DTFT} \frac{1}{1-\frac{1}{2}e^{-j\omega}} \quad , |a|<1$$

$$\therefore X[e^{j\omega}] = \frac{1}{1-\frac{1}{2}e^{-j\omega}} ,$$

$$Y[k] = X[e^{j\frac{2\pi}{10}k}] = \frac{1}{1-\frac{1}{2}e^{-j\frac{2\pi}{10}k}}$$

By definition of IDFT,

$$y[n] = \frac{1}{N}\sum_{k=0}^{N-1} Y[k]\, W_N^{-kn} \qquad \text{Here, } N=10$$

$$y[n] = \frac{1}{10}\sum_{k=0}^{9} \frac{1}{1-\frac{1}{2}e^{-j\frac{2\pi}{10}k}} \cdot e^{j\frac{2\pi}{10}kn}$$

We know that,

$$a^n \xleftrightarrow[N]{DFT} \begin{cases} \dfrac{1-a^N}{1-ae^{-j\frac{2\pi}{N}k}} & , 0 \leq n \leq N-1 \\ 0 & , \text{otherwise} \end{cases}$$

$$\left(\frac{1}{2}\right)^n \xleftrightarrow[10]{DFT} \frac{1-(1/2)^{10}}{1-\left(\frac{1}{2}\right)e^{-j\frac{2\pi}{10}k}}$$

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$$\underbrace{\frac{\left(\frac{1}{2}\right)^n}{1-\left(\frac{1}{2}\right)^{10}}}_{y(n)} \xleftrightarrow[10]{DFT} \underbrace{\frac{1}{1-\frac{1}{2}e^{-j\frac{2\pi}{10}k}}}_{Y(k)}$$

$$\boxed{\therefore \quad y(n) = \begin{cases} \dfrac{(1/2)^n}{1-\left(\frac{1}{2}\right)^{10}} & ; 0 \le n \le 9 \\ 0 & ; \text{otherwise} \end{cases}}$$

(3) If $X[K]$ is the 10-point DFT of the sequence $x[n] = \delta[n-1] + 2\delta[n-4] - \delta[n-7]$. What sequence $y[n]$ has a 10-point DFT.

$$Y[K] = 2X[K] \cdot \cos\left[\frac{6\pi}{10}K\right]$$

Ans:- Given, $x[n] = \delta[n-1] + 2\delta[n-4] - \delta[n-7]$ $N=10$

$$Y[K] = 2X[K] \cdot \cos\left[\frac{6\pi}{10}K\right]$$

$$= \cancel{2}X[K]\left[\frac{e^{j\frac{6\pi}{10}K} + e^{-j\frac{6\pi}{10}K}}{\cancel{2}}\right]$$

$$= X[K]\left[e^{j\frac{2\pi}{10}\cdot 3K} + e^{-j\frac{2\pi}{10}\cdot 3K}\right] \rightarrow n_0 = 3$$

$$Y[K] = X[K]e^{j\frac{2\pi}{10}\cdot 3k} + X[K]e^{-j\frac{2\pi}{10}\cdot 3k} \rightarrow N=10$$

By using left circular shift property,

$$x[\langle n+3\rangle_{10}] \xleftrightarrow{DFT} X[K]e^{j\frac{2\pi}{10}\cdot 3k}$$

By using Right circular shift property,

$$x[\langle n-3\rangle_{10}] \xleftrightarrow{DFT} X[K]e^{-j\frac{2\pi}{10}\cdot 3k}$$

$$\therefore \quad y[n] = x[\langle n+3\rangle_{10}] + x[\langle n-3\rangle_{10}]$$

$$y[n] = \delta[n+3-1] + 2\delta[n+3-4] - \delta[n+3-7] + \delta[n-3-1] + 2\delta[n-3-4] - \delta[n-3-7]$$

$$y[n] = \delta[n+2] + 2\delta[n-1] - \cancel{\delta[n-4]} + \cancel{\delta[n-4]} + 2\delta[n-7] - \delta[n-10]$$

$$y[n] = \delta[n+2] + 2\delta[n-1] + 2\delta[n-7] - \delta[n-10]$$

$\delta[n+2]$ ↓ $n=-2$ Not in the range

$2\delta[n-1]$ ↓ $n=+1$

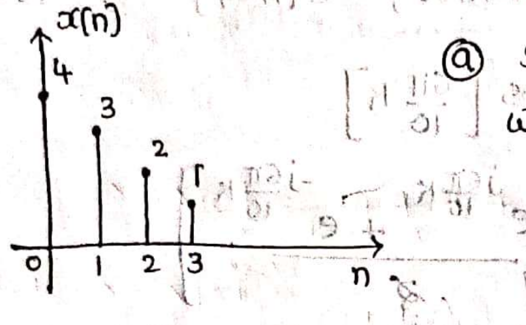
$2\delta[n-7]$ ↓ $n=7$

$\delta[n-10]$ ↓ $n=10$ Not in the range

$$y(n) = \delta[n+2-10] + 2\delta(n-1) + 2\delta(n-7) - \delta[n-10+10]$$

$$\therefore \boxed{y(n) = \delta[n-8] + 2\delta[n-1] + 2\delta[n-7] - \delta[n]}$$

(4) Consider the real finite length signal $x(n)$ as shown.



(a) Sketch the finite length signal $y(n)$ whose 6-point DFT is

$$Y[K] = X[K] \cdot W_6^{4K} \qquad 0 \le K \le 5$$

Ans:- $x(n)$ is a 6-point sequence $x(n) = \{\underset{\uparrow}{4}, 3, 2, 1, 0, 0\}$

$N = 6$

Given that, $Y[K] = X[K]\, W_6^{4K} \qquad 0 \le K \le 5$

$$Y[K] = X[K]\, e^{-j\frac{2\pi}{6}\cdot 4K} \qquad n_0 = 4, \; N = 6$$

By using circular time shifting property,

$$x[\langle n-4\rangle_6] \xleftrightarrow{DFT} X[K]\, e^{-j\frac{2\pi}{6}4K}$$

$$\therefore y(n) = x[\langle n-4\rangle_6] \qquad 0 \le n \le 5$$

$y[0] = x[\langle 0-4\rangle_6] = x[6-4] = x[2] = 2$

$y[1] = x[\langle 1-4\rangle_6] = x[6-3] = x[3] = 1$

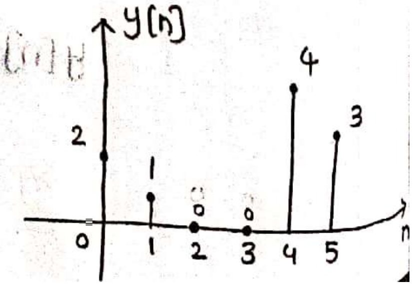
$y[2] = x[\langle 2-4\rangle_6] = x[6-2] = x[4] = 0$

$y[3] = x[\langle 3-4\rangle_6] = x[6-1] = x[5] = 0$

$y[4] = x[\langle 4-4\rangle_6] = x[0] = 4$

$y[5] = x[\langle 5-4\rangle_6] = x[1] = 3$

$$\therefore y(n) = \{\underset{\uparrow}{2}, 1, 0, 0, 4, 3\}$$



(b) Sketch the finite length signal $g[n]$ whose 6-point DFT is $G[K] = Re\{X[K]\}$

Given that, $x[n]$ is a real finite length signal.

By using Conjugate property,

⇒ The DFT of circularly even part of $x[n]$ is the real value of $X[K]$.

$$x_{ce}[n] \xleftrightarrow[N]{DFT} X_R[K] = Re\{X[K]\}$$

$$\therefore g[n] = x_{ce}[n]$$

$$g[n] = \frac{1}{2}\left[x[n] + x[N-n]\right] \qquad N=6,\quad 0 \le n \le 5$$

$$g[0] = \frac{1}{2}\left[x[0] + x[0]\right] = \frac{1}{2}[4+4] = \frac{8}{2} = 4$$

$$g[1] = \frac{1}{2}\left[x[1] + x[6-1]\right] = \frac{1}{2}[3+0] = \frac{3}{2} = 1.5$$

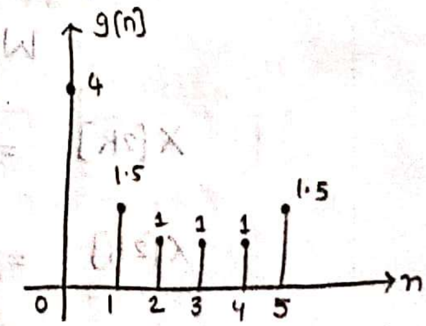
$$g[2] = \frac{1}{2}\left[x[2] + x[6-2]\right] = \frac{1}{2}[2+0] = \frac{2}{2} = 1$$

$$g[3] = \frac{1}{2}\left[x[3] + x[6-3]\right] = \frac{1}{2}[1+1] = \frac{2}{2} = 1$$

$$g[4] = \frac{1}{2}\left[x[4] + x[6-4]\right] = \frac{1}{2}[0+2] = \frac{2}{2} = 1$$

$$g[5] = \frac{1}{2}\left[x[5] + x[6-5]\right] = \frac{1}{2}[0+3] = \frac{3}{2} = 1.5$$

$$\therefore g[n] = \{\underset{\uparrow}{4}, 1.5, 1, 1, 1, 1.5\}$$



(c) Sketch the finite length signal $h[n]$ whose 6-point DFT is $H[K] = X[2K] \qquad K = 0, 1, 2, \ldots$

We know that,
$$X[K] = \sum_{n=0}^{5} x[n]\, e^{-j\frac{2\pi}{6}Kn}$$

$$X[2K] = \sum_{n=0}^{5} x[n]\, e^{-j\frac{2\pi}{6}(2K)n}$$

Ans:- Given, $H[K] = X[2K] \qquad K = 0, 1, 2$

$$X[K] = 4 + 3W_6^{K} + 2W_6^{2K} + W_6^{3K}$$

$$X[2K] = 4 + 3W_6^{2K} + 2W_6^{2(2K)} + W_6^{3(2K)}$$

By the property of twiddle factor,

$$W_N^{2K} = W_{N/2}^{K}$$

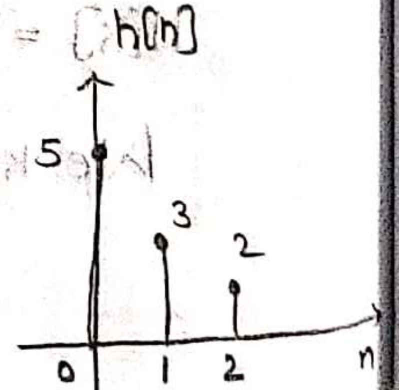
$$X[2K] = 4 + 3W_3^{K} + 2W_3^{2K} + W_3^{3K}$$

$$X[2K] = 4 + 3W_3^{k} + 2W_3^{2k} + 1 \qquad (\because W_N^{N} = 1)$$

$$H[K] = X[2K] = 5 + 3e^{-j\frac{2\pi}{3}K} + 2e^{-j\frac{2\pi}{3}\cdot 2k}$$

$$\delta[n-n_0] \xleftrightarrow{DFT} e^{-j\frac{2\pi}{N}k\cdot n_0}$$

$$\boxed{h[n] = 5\delta[n] + 3\delta[n-1] + 2\delta[n-2]}$$



(5) Consider the following two 4-point sequences

$$x[n] = \cos\left(\frac{\pi}{2}n\right) \text{ and } h[n] = 2^n$$

ⓐ Calculate the 4-point DFT $X[k]$

$$x[n] = \frac{1}{2}\left[e^{j\frac{\pi}{2}n} + e^{-j\frac{\pi}{2}n}\right] = \frac{1}{2}\left[e^{j\frac{2\pi}{4}n} + e^{-j\frac{2\pi}{4}n}\right]$$

We know that,

$$e^{j\frac{2\pi}{N}k_0 n} \xleftrightarrow[N]{DFT} N.\,\delta[\langle k-k_0\rangle_N]$$

Circular freq Shifting

$$e^{-j\frac{2\pi}{N}k_0 n} \xleftrightarrow[N]{DFT} N.\,\delta[\langle k+k_0\rangle_N]$$

$$\frac{1}{2}\left[e^{j\frac{2\pi}{4}n} + e^{-j\frac{2\pi}{4}n}\right] \xleftrightarrow[4]{DFT} \frac{4}{2}\left[\delta[\langle k-1\rangle_4] + \delta[\langle k+1\rangle_4]\right]$$

$$\cos\left(\frac{\pi}{2}n\right) \xleftrightarrow{DFT} 2\left[\delta[\langle k-1\rangle_4] + \delta[\langle k+1\rangle_4]\right]$$

ⓑ Calculate the 4-point DFT $H[k]$

We know that, $h[n] = 2^n$

$$H[k] = \sum_{n=0}^{N-1} h[n]\, W_N^{kn}$$

$$N=4, \quad H[k] = \sum_{n=0}^{3} h[n]\, e^{-j\frac{2\pi}{4}kn}$$

$$= \sum_{n=0}^{3} 2^n \cdot e^{-j\frac{2\pi}{4}kn} = \sum_{n=0}^{3}\left[2e^{-j\frac{2\pi}{4}k}\right]^n$$

$$H[k] = \frac{1 - 2^4 \cdot e^{-j2\pi k}}{1 - 2e^{-j\frac{2\pi}{4}k}} \qquad \left(\because a = 2e^{-j\frac{2\pi}{4}k}\right)$$

$k = 0, 1, 2, 3$

$$H[0] = \frac{1-2^4}{1-2} = 15$$

$$H[2] = \frac{1-2^4}{1-2e^{-j\pi}} = -5$$

$$H[1] = \frac{1-2^4}{1-2e^{-j\frac{\pi}{2}}} = \frac{-15}{1-2e^{-j\pi/2}} = \frac{-15}{1+2j}$$

$$H[3] = \frac{1-2^4}{1-2e^{-j\frac{2\pi}{4}\cdot 3}} = \frac{-15}{1-2j}$$

(c) Calculate the 4-point circular convolution of $x[n]$ with $h[n]$ by multiplying $X[K]$ and $H[K]$ and performing an inverse DFT.

Multiplying the two DFT'S,

$$Y[K] = X[K] \cdot H[K]$$

$$Y[K] = 2H[K]\left[\underbrace{\delta[\langle K-1\rangle_4]}_{\text{at } K=1} + \underbrace{\delta[\langle K+1\rangle_4]}_{\text{at } K=-1 \text{ or } 3}\right]$$

Taking 4-point inverse DFT, $0 \le n \le 3$

$$y[n] = \frac{1}{2} H[1] e^{j\frac{2\pi}{4}n} + \frac{1}{2} H[3] e^{-j\frac{2\pi}{4}n}$$

$$= \frac{-15}{2}\left[\frac{e^{j\frac{\pi}{2}n}}{1+2j} + \frac{e^{-j\frac{\pi}{2}n}}{1-2j}\right]$$

$$= -\frac{15}{2}\left[\frac{e^{j\frac{\pi}{2}n}(1-2j)}{(1+2j)(1-2j)} + \frac{e^{-j\frac{\pi}{2}n}(1+2j)}{(1-2j)(1+2j)}\right]$$

$$= -\frac{15}{2}\left[\frac{e^{j\frac{\pi}{2}n} - 2je^{j\frac{\pi}{2}n}}{(1)^2-(2j)^2} + \frac{e^{-j\frac{\pi}{2}n} + 2je^{-j\frac{\pi}{2}n}}{1-(2j)^2}\right]$$

$$= \frac{-15}{2\times5}\left[\underbrace{\left(e^{j\frac{\pi}{2}n} + e^{-j\frac{\pi}{2}n}\right)}_{\Rightarrow \text{Cosine func}} + \underbrace{\left(2je^{-j\frac{\pi}{2}n} - 2je^{j\frac{\pi}{2}n}\right)}_{\Rightarrow \text{Sine func}}\right]$$

$$y[n] = -\frac{15}{10}\left[2\cos\left(\frac{\pi}{2}n\right) + 4\sin\left(\frac{\pi}{2}n\right)\right] \quad 0 \le n \le 3$$

$$y[n] = -3\left[\cos\frac{\pi}{2}n + 2\sin\frac{\pi}{2}n\right] \ ; \ 0 \le n \le 3$$

$n=0 \Rightarrow y[0] = -3$

$n=1 \Rightarrow y[1] = -6$

$n=2 \Rightarrow y[2] = 3$

$n=3 \Rightarrow y[3] = 6$

$$\therefore y[n] = \{\underset{\uparrow}{-3}, -6, 3, 6\}$$

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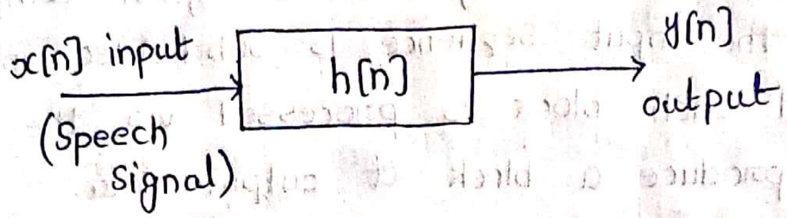
Filtering of long sequences using DFT:



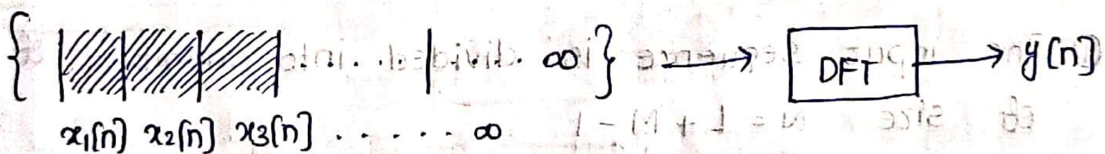
↳ infinite length sequence

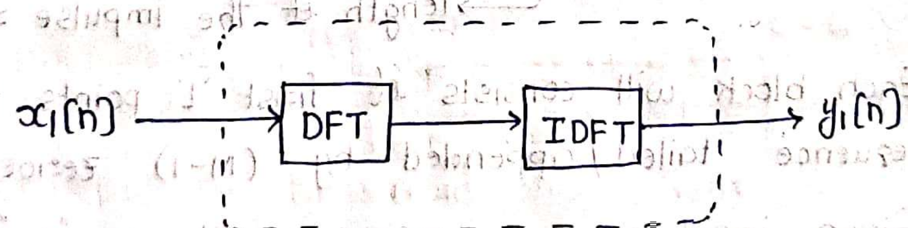
$x[n] \Rightarrow$ long sequence

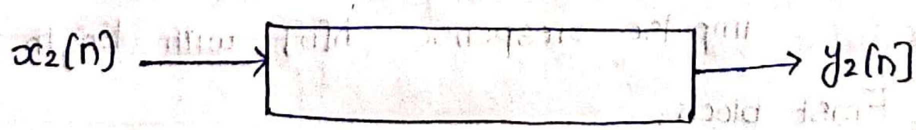
$DFT\{x[n]\} \Rightarrow$ impractical

$y[n] \Rightarrow$ o/p cannot be computed / processed

DFT of long sequences in two methods ↗ Overlap add method ↘ Overlap Save method







and so on

$y[n] = \{y_1[n], y_2[n], \dots\dots\dots y_L[n]\}$

① For the filtering of long sequence data and due to limitation on the memory size of digital computer/processor the input sequence is subdivided into smaller blocks and each block is processed via the DFT and IDFT to produce a block of output data.

② The output blocks are fitted together to yield the desired output sequence, which is identical to the sequence obtained by linear convolution.

③ This procedure is called as block convolution/processing.

④ Two methods used for filtering the blocks and combining the results are

ⓐ Overlap add Method

ⓑ Overlap save method.

Overlap Add Method:-

① The input sequence is divided into number of blocks of size $N = L + \underbrace{M}_{\text{length of the impulse response}} - 1$

② Each block will consists of first 'L' points from input sequence tailed / appended by $(M-1)$ zeroes.

③ Given input sequence $x[n]$ with length "L_s"

impulse response $h[n]$ with length "M".

④ First block:

$$x_1[n] = \{\underbrace{x[0], x[1], x[2], \dots x[L-1]}_{\text{first 'L' data points from input sequence}}, \underbrace{0, 0, 0, \dots 0}_{(M-1) \text{ zeros}}\}$$

Second block:

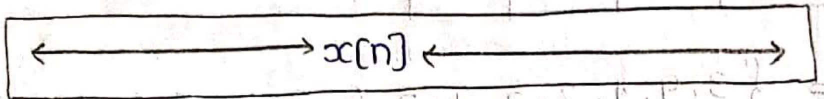
$$x_2[n] = \{\underbrace{x[L], x[L+1], x[L+2], \dots x[2L-1]}_{\text{next 'L' data points from input sequence}}, \underbrace{0, 0, \dots 0}_{(m-1) \text{ zeros}}\}$$

$x_3[n]$

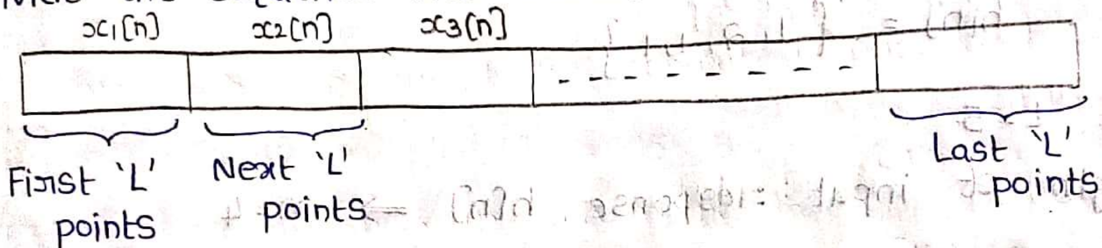
$x_4[n]$ ..... and so forth

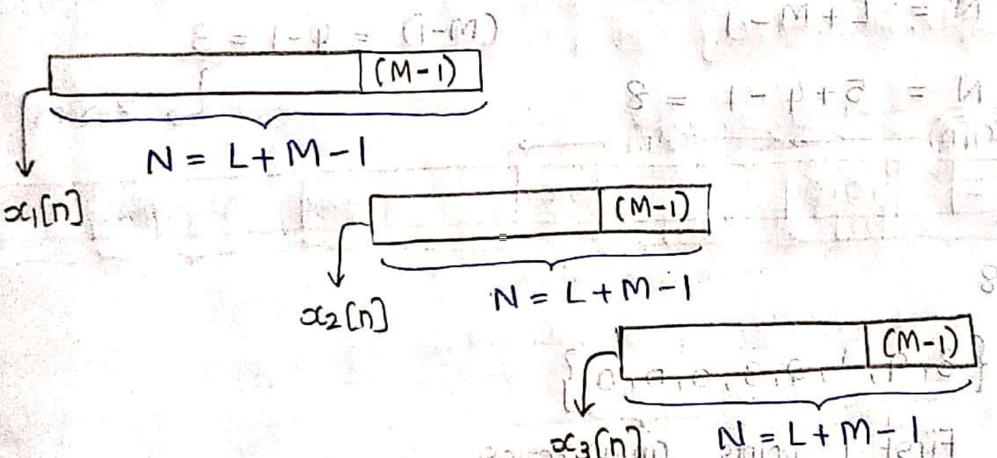
By using these segmented blocks of input the respective output blocks are computed using DFT and IDFT. Then these output blocks are fitted together to form the complete output.

input sequence $x[n]$

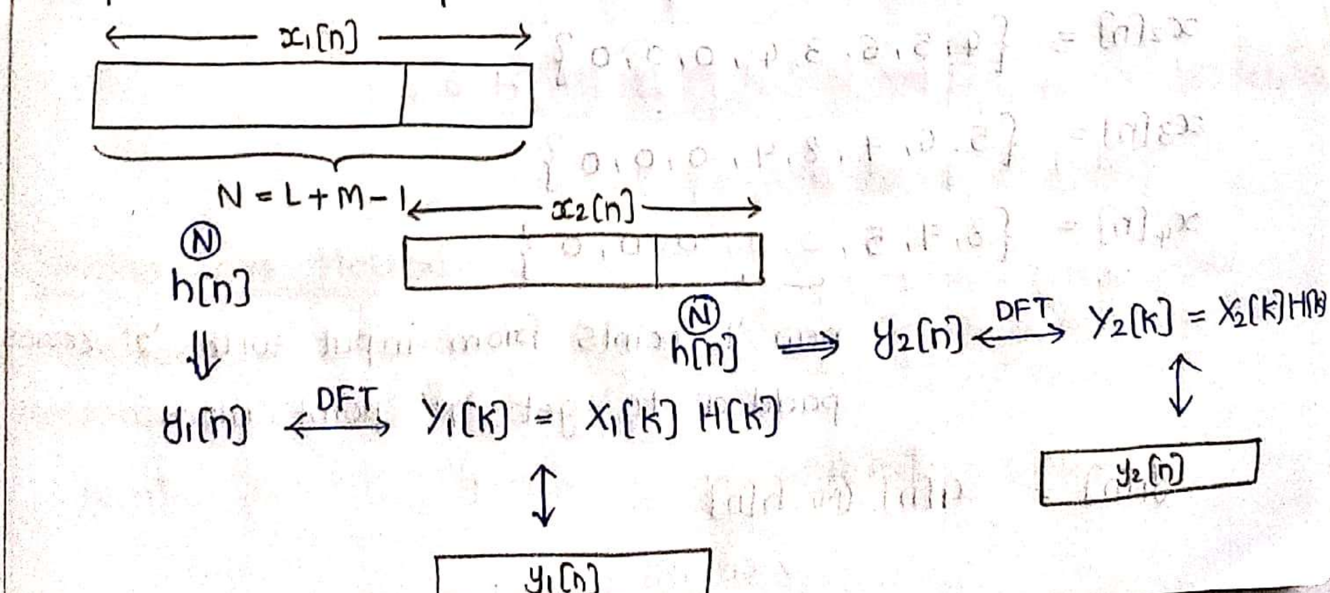


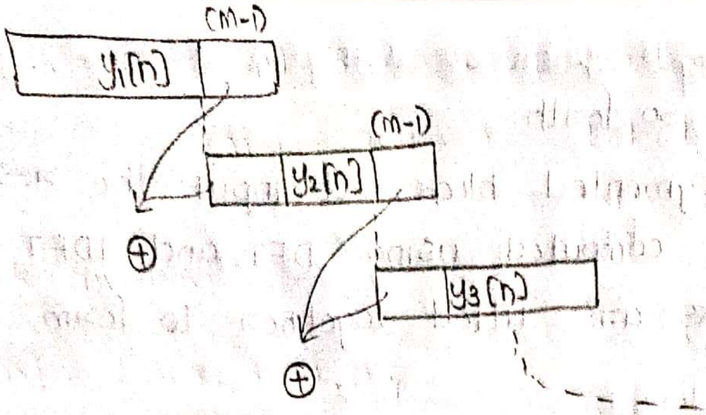
Divide the sequence into blocks,

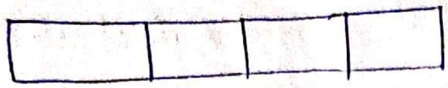




Computation of output block:





$y[n] =$ 

Ex:- $x(n) = \{3, 9, 1, 2, 3, 4, 5, 6, 3, 4, 5, 6, 7, 8, 9, 8, 7, 5\}$

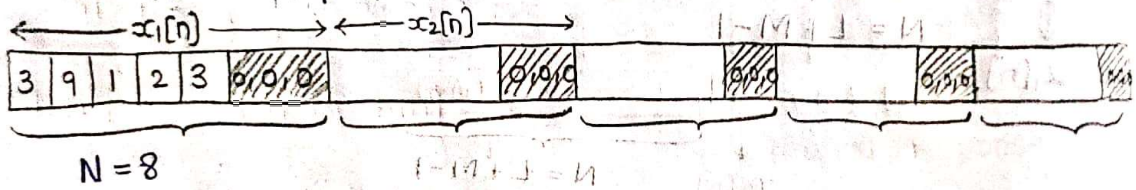
$h[n] = \{1, 2, 1, 1\}$

Let $L = 5$

Length of input response $h[n] \Rightarrow M = 4$

$N = L + M - 1$ $\qquad (M-1) = 4 - 1 = 3$ ↳ zeroes.

$N = 5 + 4 - 1 = 8$



$x_1[n] = \{\underbrace{3, 9, 1, 2, 3}_{\text{First 'L' points in input}}, \underbrace{0, 0, 0}_{(M-1)\text{ zeros}}\}$

$x_2[n] = \{4, 5, 6, 3, 4, 0, 0, 0\}$

$x_3[n] = \{5, 6, 7, 8, 9, 0, 0, 0\}$

$x_4[n] = \{8, 7, 5, 0, 0, 0, 0, 0\}$

new 'L' points from input with '2' zeros padded to get 'N' point block.

$y_1[n] = x_1[n] \; (N) \; h[n]$

↳ circular convolution

$y_1[n] = \{3, 9, 1, 2, 3, 0, 0, 0\} \; Ⓝ \; \{1, 2, 1, 1\}$

By using circulant matrix method,

$$y_1[n] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 2 \\ 2 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 9 \\ 1 \\ 2 \\ 3 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 15 \\ 22 \\ 16 \\ 17 \\ 9 \\ 5 \\ 3 \end{bmatrix}$$

$y_1[n] \Rightarrow$ 3 15 22 16 17 9 5 3

$y_2[n] \Rightarrow$ 4 13 20 24 21 17 7 4

$y_3[n] \Rightarrow$ 5 6 24 33 38

$y_1[n]$: 3 15 22 16 17 9 5 3

$y_2[n]$: 4 13 20 24 21 17 7 4

$y_3[n]$: 5 6 24 33 38 33 17 9

$y_4[n]$: 8 23 27 25

12, 5 0 0,

$\therefore y[n] = \{3, 15, 22, 16, 17, 13, 18, 23, 24, 21, 22, 13, 28, 33, 38, 41, 40,$

$36, 25, 12, 5, 0, 0\}$

Overlap Save Method:-

Exam question

Consider a sequence $x(n)$ of length '10' and an impulse response $h[n]$ of length '3'.

$x(n) = \{3, -1, 0, 1, 3, 2, 0, 1, 2, 1\}$ $\quad h[n] = \{1, 1, 1\}$

$L_s = 10$, $M = 3$ $\quad N = L + M - 1$

$(M-1) = 3 - 1 = 2$ $\quad$ Let $L = 3$

$N = 3 + 3 - 1 = 5$

Block division:

$x_1[n] = \{0, 0, 3, -1, 0\}$

$x_2[n] = \{-1, 0, 1, 3, 2\}$

$x_3[n] = \{3, 2, 0, 1, 2\}$

$x_4[n] = \{1, 2, 1, 0, 0\}$

$x_1[n] = \{\underbrace{\quad\quad}_{(M-1)\text{ zeroes}} \underbrace{\quad\quad}_{L \text{ of i/p}}\}$

$x_2[n] = \{\underbrace{\quad\quad}_{(M-1)} \underbrace{\quad\quad}_{L \text{ of i/p}}\}$

Last samples of previous block

Convolution:

① $y_1[n] = x_1[n] \text{ Ⓝ } h[n]$

$y_1[n] = \{0, 0, 3, -1, 0\} \text{ Ⓝ } \{1, 1, 1, 0, 0\}$

$y_1[n] = \{-1, 0, 3, 2, 2\}$

② $y_2[n] = x_2[n] \text{ Ⓝ } h[n]$

$= \{-1, 0, 1, 3, 2\} \text{ Ⓝ } \{1, 1, 1, 0, 0\}$

$y_2[n] = \{4, 1, 0, 4, 6\}$

③ $y_3[n] = x_3[n] \text{ Ⓝ } h[n]$

$= \{3, 2, 0, 1, 2\} \text{ Ⓝ } \{1, 1, 1, 0, 0\}$

$y_3[n] = \{6, 7, 5, 3, 3\}$

④ $y_4[n] = x_4[n] \text{ Ⓝ } h[n]$

$= \{1, 2, 1, 0, 0\} \text{ Ⓝ } \{1, 1, 1, 0, 0\}$

$y_4[n] = \{1, 3, 4, 3, 1\}$

Arranging:

$$y_1(n) = \{-\underline{1,0},3,2,2\}$$

$$y_2(n) = \{\cancel{4,1,0},\ \underline{4,1},0,4,6\}$$

$$y_3(n) = \{\qquad \underline{6,7},5,3,3\}$$

$$y_4(n) = \{\qquad \underline{1,3},4,3,1\}$$

$$\therefore y(n) = \{3,2,2,0,4,6,5,3,3,4,3,1\}$$

Discard first two samples in every sequence