Chapter 13

Bending Moment and Shear Force

Contents

13.1. Introduction

We see that whenever a horizontal beam is loaded with vertical loads, sometimes, it bends (*i.e.*, deflects) due to the action of the loads. The amount with which a beam bends, depends upon the amount and type of the loads, length of the beam, elasticity of the beam and type of the beam. The scientific way of studying the deflection or any other effect is to draw and analyse the shear force or bending moment diagrams of a beam. In general, the beams are classified as under:

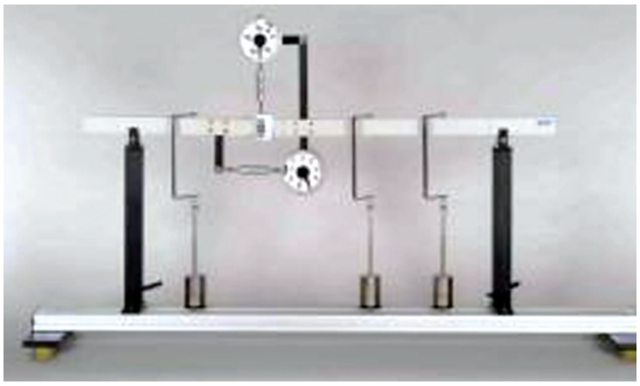
1. Cantilever beam,
2. Simply supported beam,
3. Overhanging beam,
4. Rigidly fixed or built-in-beam and
5. Continuous beam.

Note. In this chapter, we shall study the first three types of beams only.

13.2. Types of Loading

A beam may be subjected to either or in combination of the following types of loads:

1. Concentrated or point load,
2. Uniformly distributed load and
3. Uniformly varying load.



Shearing force

13.3. Shear Force

The shear force (briefly written as S.F.) at the cross-section of a beam may be defined as the unbalanced vertical force to the right or left of the section.

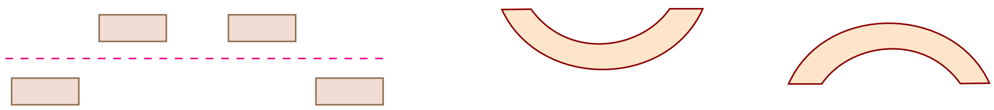
13.4. Bending Moment

The bending moment (briefly written as B.M.) at the cross-section of a beam may be defined as the algebraic sum of the moments of the forces, to the right or left of the section.

Note. While calculating the shear force or bending moment at a section, the end reactions must also be considered alongwith other external loads.

13.5. Sign Conventions

We find different sign conventions in different books, regarding shear force and bending moment at a section. But in this book the following sign conventions will be used, which are widely followed and internationally recognised.



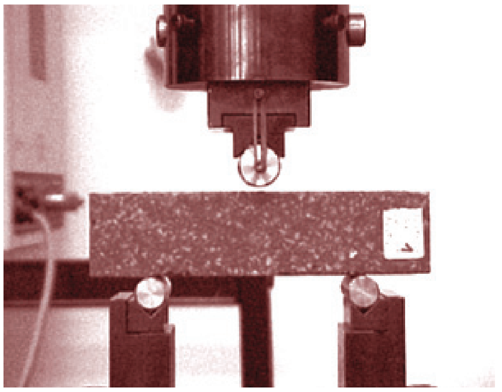
(*a*) Positive S.F. (*b*) Negative S.F. (*c*) Positive B.M. (*d*) Negative B.M.

Fig. 13.1

1. ***Shear Force.*** We know that as the shear force is the unbalanced vertical force, therefore it tends to slide one portion of the beam, upwards or downwards with respect to the other. The shear force is said to be positive, at a section, when the left hand portion tends to slide downwards or the right hand portion tends to slide upwards shown in Fig. 13.1 (*a*). Or in other words, all the downward forces to the left of the section cause positive shear and those acting upwards cause negative shear as shown in Fig. 13.1 (*a*).

Similarly, the shear force, is said to be negative at a section when the left hand portion tends to slide upwards or the right hand portion tends to slide downwards as shown in Fig. 13.1 (*b*). Or in other words, all the upward forces to the left of the section cause negative shear and those acting downwards cause positive shear as shown in Fig. 13.1 (*b*).

2. ***Bending Moment.*** At sections, where the bending moment, is such that it tends to bend the beam at that point to a curvature having concavity at the top, as shown in Fig. 13.1 (*c*) is taken as positive. On the other hand, where the bending moment is such that it tends to bend the beam at that point to a curvature having convexity at the top, as shown in Fig. 13.1 (*d*) is taken as negative. The positive bending moment is often called sagging moment and negative as hogging moment.



Bending test of resin concrete

A little consideration will show that the bending moment is said to be positive, at a section, when it is acting in an anticlockwise direction to the right and negative when acting in a clockwise direction. On the other hand, the bending moment is said to be negative when it is acting in a clockwise direction to the left and positive when it is acting in an anticlockwise direction.

NOTE. While calculating bending moment or shear force, at a section the beam will be assumed to be weightless.

13.6. Shear Force and Bending Moment Diagrams

The shear force and bending moment can be calculated numerically at any particular section. But sometimes, we are interested to know the manner, in which these values vary, along the length of the beam. This can be done by plotting the shear force or the bending moment as ordinate and the position of the cross as abscissa. These diagrams are very useful, as they give a clear picture of the distribution of shear force and bending moment all along the beam.

NOTE. While drawing the shear force or bending moment diagrams, all the positive values are plotted above the base line and negative values below it.

13.7. Relation between Loading, Shear Force and Bending Moment

The following relations between loading, shear force and bending moment at a point or between any two sections of a beam are important from the subject point of view:

1. If there is a point load at a section on the beam, then the shear force suddenly changes (*i.e.*, the shear force line is vertical). But the bending moment remains the same.
2. If there is no load between two points, then the shear force does not change (*i.e.*, shear force line is horizontal). But the bending moment changes linearly (*i.e.*, bending moment line is an inclined straight line).
3. If there is a uniformly distributed load between two points, then the shear force changes linearly (*i.e.*, shear force line is an inclined straight line). But the bending moment changes according to the parabolic law. (*i.e.*, bending moment line will be a parabola).
4. If there is a uniformly varying load between two points then the shear force changes according to the parabolic law (*i.e.*, shear force line will be a parabola). But the bending moment changes according to the cubic law.

13.8. Cantilever with a Point Load at its Free End

Consider a *cantilever AB of length l and carrying a point load W at its free end B as shown in Fig. 13.2 (*a*). We know that shear force at any section X, at a distance x from the free end, is equal to the total unbalanced vertical force. *i.e.*,

$$F_x = -W \qquad \text{...(Minus sign due to right downward)}$$

* It is a beam fixed at one end and free at the other.

and bending moment at this section,

$$M_x = -W \cdot x \qquad \text{...(Minus sign due to hogging)}$$

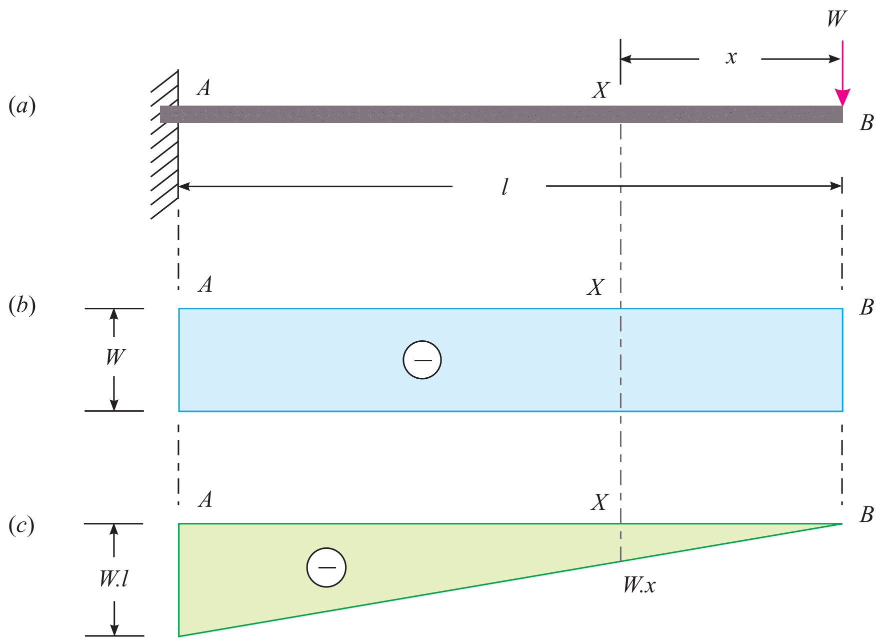


Fig. 13.2. Cantilever with a point load

Thus from the equation of shear force, we see that the shear force is constant and is equal to $-W$ at all sections between B and A. And from the bending moment equation, we see that the bending moment is zero at B (where $x = 0$) and increases by a straight line law to $-Wl$; . at (where $x = l$). Now draw the shear force and bending moment diagrams as shown in Fig. 13.2 (*b*) and 13.2 (*c*) respectively.

EXAMPLE 13.1. *Draw shear force and bending moment diagrams for a cantilever beam of span 1.5 m carrying point loads as shown in Fig. 13.3 (a).*

SOLUTION. Given : Span (l) = 1.5 m ; Point load at B (W_1) = 1.5 kN and point load at C (W_2) = 2 kN.

Shear force diagram

The shear force diagram is shown in Fig. 13.3 (*b*) and the values are tabulated here:

$$F_B = -W_1 = -1.5 \text{ kN}$$

$$F_C = -(1.5 + W_2) = -(1.5 + 2) = -3.5 \text{ kN}$$

$$F_A = -3.5 \text{ kN}$$

Bending moment diagram

The bending moment diagram is shown in Fig. 13.3 (*c*) and the values are tabulated here:

$$M_B = 0$$

$$M_C = -[1.5 \times 0.5] = -0.75 \text{ kN-m}$$

$$M_A = -[(1.5 \times 1.5) + (2 \times 1)] = -4.25 \text{ kN-m}$$

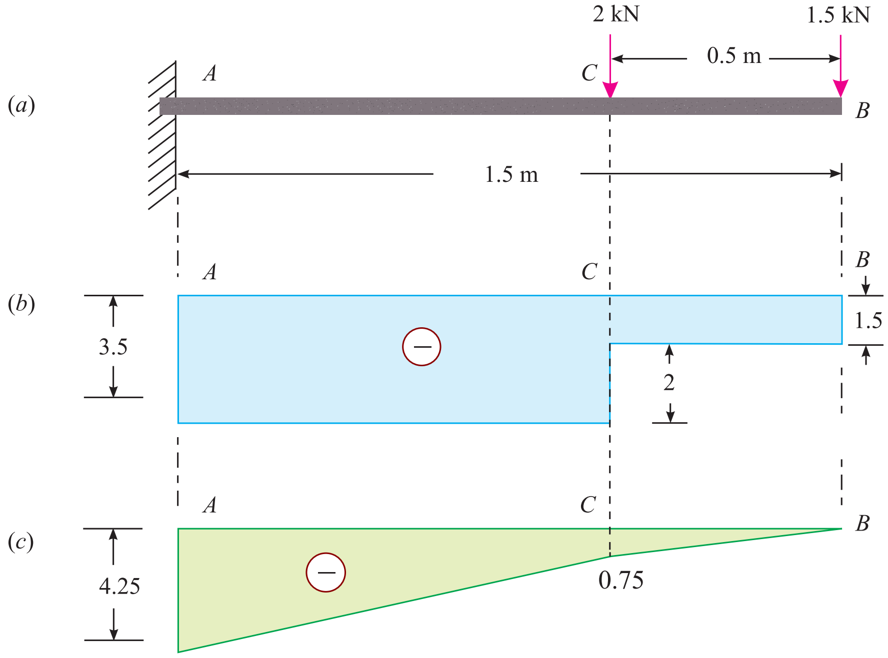


Fig. 13.3

13.9. Cantilever with a Uniformly Distributed Load

Consider a cantilever *AB* of length *l* and carrying a uniformly distributed load of *w* per unit length, over the entire length of the cantilever as shown in Fig. 13.4 (*a*).

We know that shear force at any section *X*, at a distance *x* from *B*,

$$F_x = -w \cdot x \qquad \text{... (Minus sign due to right downwards)}$$

Thus we see that shear force is zero at *B* (where $x = 0$) and increases by a straight line law to $-wl$ at *A* as shown in Fig. 13.4 (*b*).

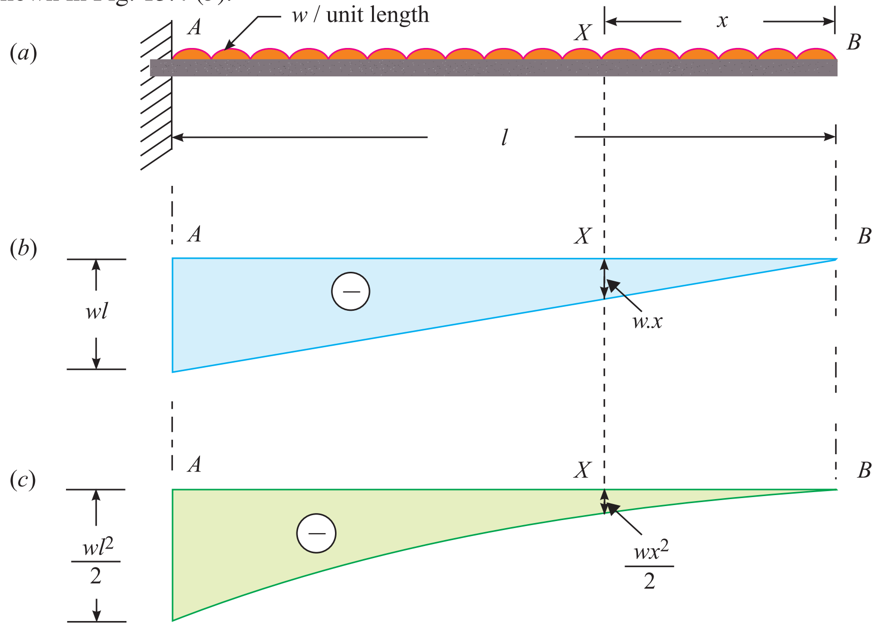


Fig. 13.4. Cantilever with a uniformly distributed load

We also know that bending moment at X,

$$M_x = -wx \cdot \frac{x}{2} = -\frac{wx^2}{2} \qquad \text{...(Minus sign due to hogging)}$$

Thus we also see that the bending moment is zero at B (where $x = 0$) and increases in the form of a parabolic curve to $-\frac{wl^2}{2}$ at B (where $x = 1$) as shown in Fig. 13.4 (c).

EXAMPLE 13.2. *A cantilever beam AB, 2 m long carries a uniformly distributed load of 1.5 kN/m over a length of 1.6 m from the free end. Draw shear force and bending moment diagrams for the beam.*

SOLUTION. Given : span (l) = 2 m ; Uniformly distributed load (w) = 1.5 kN/m and length of the cantilever CB carrying load (a) = 1.6 m.

Shear force diagram

The shear force diagram is shown in Fig. 13.5 (b) and the values are tabulated here:

$$F_B = 0$$

$$F_C = -w \cdot a = -1.5 \times 1.6 = -2.4 \text{ kN}$$

$$F_A = -2.4 \text{ kN}$$

Bending moment diagram

The bending moment diagram is shown in Fig. 13.5 (c) and the values are tabulated here:

$$M_B = 0$$

$$M_C = -\frac{wa^2}{2} = \frac{1.5 \times (1.6)^2}{2} = -1.92 \text{ kN-m}$$

$$M_A = -\left[(1.5 \times 1.6)\left(0.4 + \frac{1.6}{2}\right)\right] = -2.88 \text{ kN-m}$$

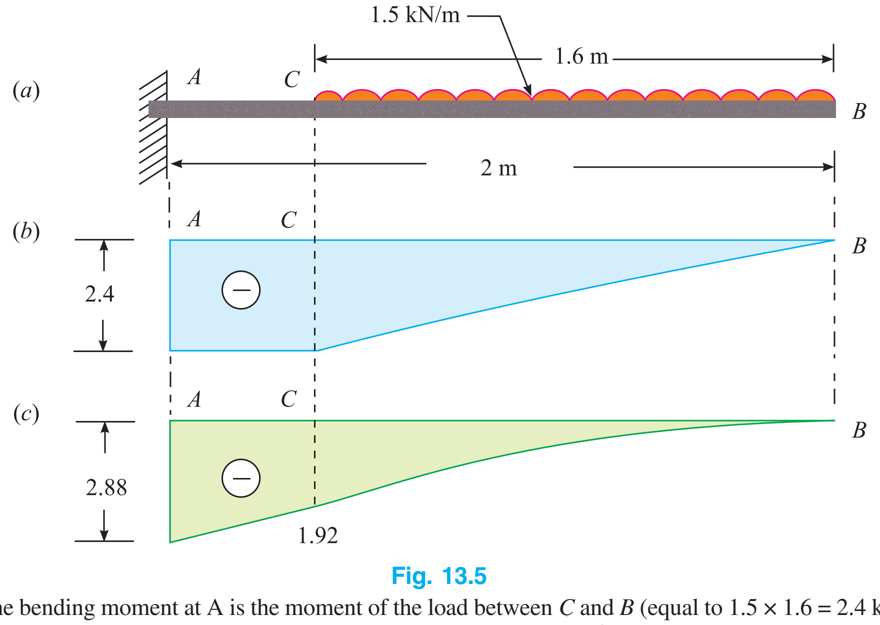


Fig. 13.5

NOTE. The bending moment at A is the moment of the load between C and B (equal to $1.5 \times 1.6 = 2.4$ kN) about A. The distance between the centre of the load and A is $0.4 + \frac{1.6}{2} = 1.2$ m.

Example 13.3. *A cantilever beam of 1.5 m span is loaded as shown in Fig. 13.6 (a). Draw the shear force and bending moment diagrams.*

Solution. Given : Span (l) = 1.5 m ; Point load at B (W) = 2 kN ; Uniformly distributed load (w) = 1 kN/m and length of the cantilever AC carrying the load (a) = 1 m.

Shear force diagram

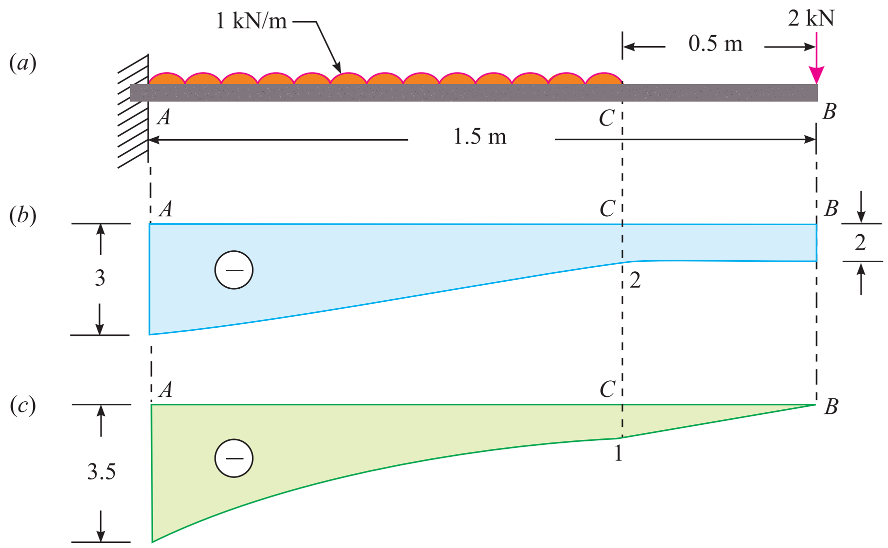


Fig. 13.6

The shear force diagram is shown in Fig. 13.6 (b) and the values are tabulated here:

$$F_B = -W = -2 \text{ kN}$$
$$F_C = -2 \text{ kN}$$
$$F_A = -[2 + (1 \times 1)] = -3 \text{ kN}$$

Bending moment diagram

The bending moment diagram is shown in Fig. 13.6 (c) and the values are tabulated here:

$$M_B = 0$$
$$M_C = -[2 \times 0.5] = -1 \text{ kN-m}$$
$$M_A = -\left[(2 \times 1.5) + (1 \times 1) \times \frac{1}{2}\right] = -3.5 \text{ kN-m}$$

13.10. Cantilever with a Gradually Varying Load

Consider a cantilever AB of length l, carrying a gradually varying load from zero at the free end to w per unit length at the fixed end, as shown in Fig. 13.7 (a).

We know that, the shear force at any section X, at a distance x from the free end B,

$$F_X = -\left(\frac{wx}{l} \cdot \frac{x}{2}\right) = -\frac{wx^2}{2l} \qquad ...(i) \quad \text{(Minus sign due to right downward)}$$

Thus, we see that the shear force is zero at the free end (where $x = 0$) and increases in the form of a parabolic curve [as given by equation (i) above] to $-\frac{wl^2}{2l} = -\frac{wl}{2}$ = at A (where $x = l$) as shown in Fig. 13.7 (b).

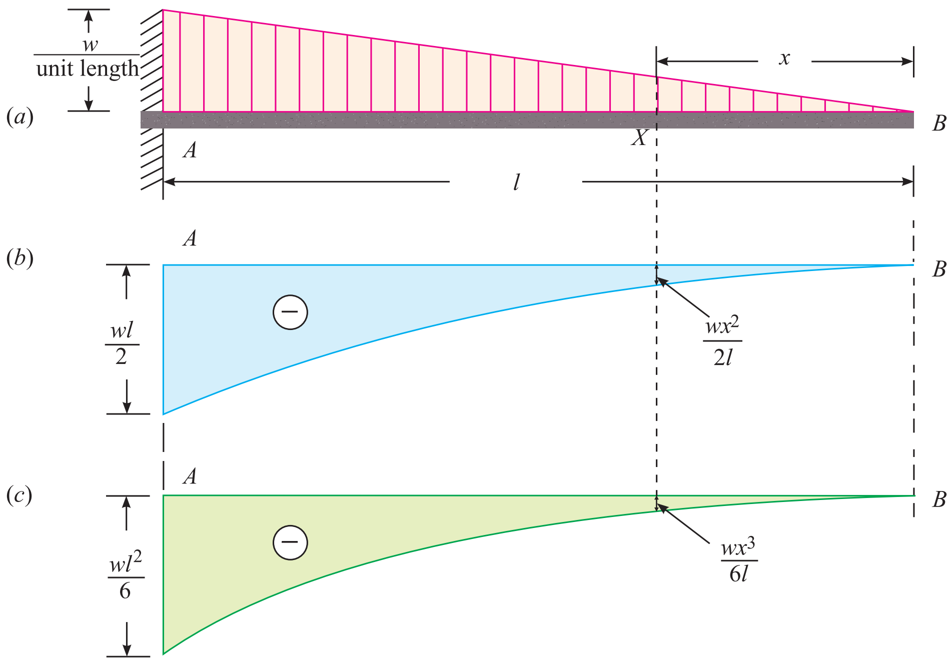


Fig. 13.7

We also know that the bending moment at X,

$$M_X = -\frac{wx^2}{2l} \times \frac{x}{3} = -\frac{wx^2}{6l} \qquad \text{...}(ii) \quad \text{(Minus sign due to hogging)}$$

Thus, we see that the bending moment is zero at the free end (where $x = 0$) and increases in the form of a *cubic parabolic curve* [as given by equation (ii) above] to $-\frac{wl^3}{6l} = -\frac{wl^2}{6}$ at A (where $x = l$) as shown in Fig. 13.7 (c).

EXAMPLE 13.4. *A cantilever beam 4 m long carries a gradually varying load, zero at the free end to 3 kN/m at the fixed end. Draw B.M. and S.F. diagrams for the beam.*

SOLUTION. Given : Span (l) = 4 m and gradually varying load at A (w) = 3 kN/m

The cantilever beam is shown in Fig. 13.8 (a).

Shear force diagram

The shear force diagram is shown in Fig. 13.8 (b) and the values are tabulated here:

$$F_B = 0$$

$$F_A = -\frac{3 \times 4}{2} = -6 \text{ kN}$$

Bending moment diagram

The bending moment diagram is shown in Fig. 13.8 (c) and the values are tabulated here:

$$M_B = 0$$

$$M_A = -\frac{3 \times (4)^2}{6} = -8 \text{ kN-m}$$

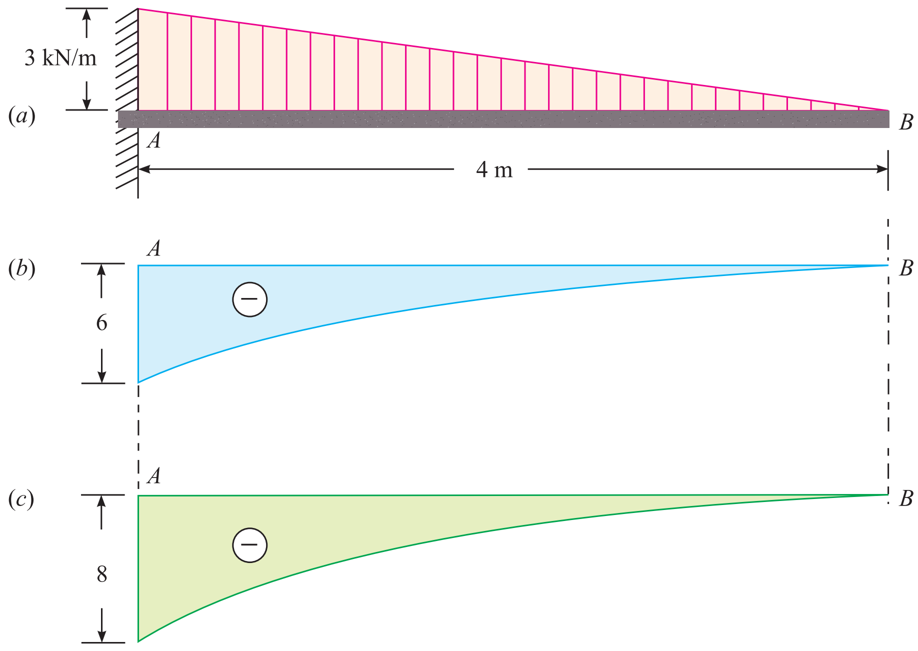


Fig. 13.8

EXAMPLE 13.5. *A cantilever beam of 2 m span is subjected to a gradually varying load from 2 kN/m to 5 kN/m as shown in Fig. 13.9.*

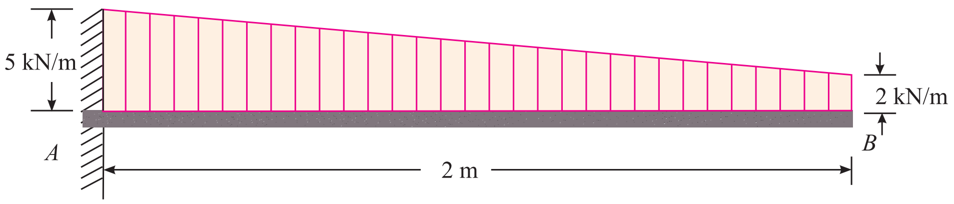


Fig. 13.9

Draw the shear force and bending moment diagrams for the beam.

SOLUTION. Given : Span (l) = 2 m ; Gradually varying load at A (w_A) = 5 kN/m and gradually varying load at B (w_B) = 2 kN/m.

The load may be assumed to be split up into (i) a uniformly distributed load (w_1) of 2 kN/m over the entire span and (ii) a gradually varying load (w_1) from zero at B to 3 kN/m at A as shown in Fig. 13.10 (a)

Shear force diagram

The shear force diagram is shown in Fig. 13.10 (b) and the values are tabulated here:

$$F_B = 0$$

$$F_A = -\left[(2 \times 2) + \left(\frac{3 \times 2}{2}\right)\right] = -7 \text{ kN}$$

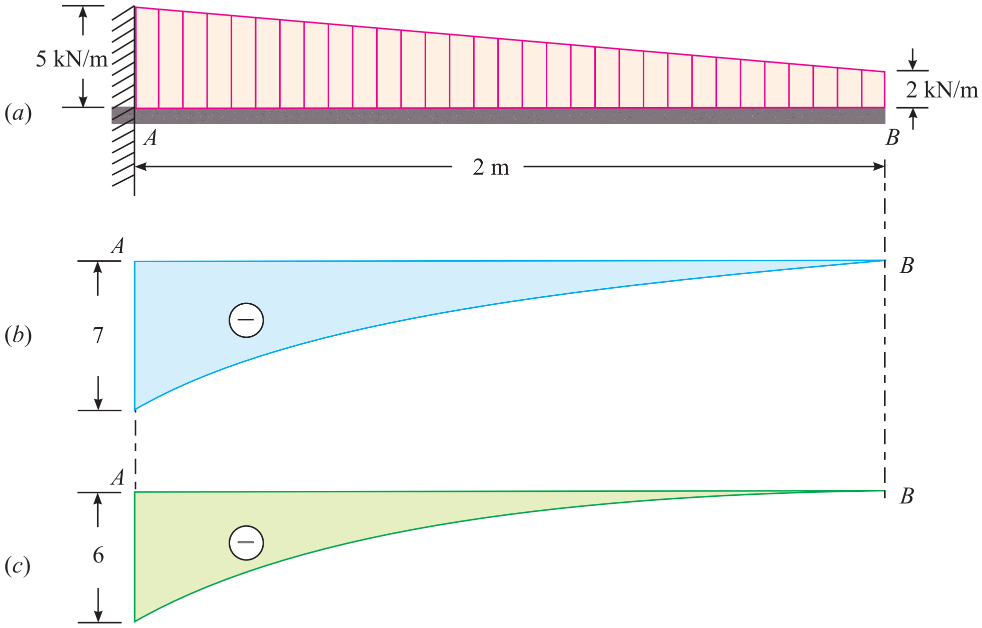


Fig. 13.10

Bending moment diagram

The bending moment diagram is shown in Fig. 13.10 (c) and the values are tabulated here:

$$M_B = 0$$

$$M_A = -\left[\left(\frac{2\times(2)^2}{2}\right)+\left(\frac{3(2)^2}{6}\right)\right] = -6 \text{ kN-m}$$

EXERCISE 13.1

1. A cantilever beam 2 m long carries a point load of 1.8 kN at its free end. Draw shear force and bending moment diagrams for the cantilever. [**Ans.** $F_{max} = -1.8$ kN ; $M_{max} = -3.6$ kN-m]
2. A cantilever beam 1.5 m long carries point loads of 1 kN, 2 kN and 3 kN at 0.5 m, 1.0 m and 1.5 m from the fixed end respectively. Draw the shear force and bending moment diagrams for the beam. [**Ans.** $F_{max} = -6$ kN ; $M_{max} = -7$ kN-m]
3. A cantilever beam of 1.4 m length carries a uniformly distributed load of 1.5 kN/m over its entire length. Draw S.F. and B.M. diagrams for the cantilever. [**Ans.** $F_{max} = -2.1$ kN ; $M_{max} = -1.47$ kN-m]
4. A cantilever AB 1.8 m long carries a point load of 2.5 kN at its free end and a uniformly distributed load of 1 kN/m from A to B. Draw the shear force the bending moment diagrams for the beam. [**Ans.** $F_{max} = -4.3$ kN ; $M_{max} = -6.12$ kN-m]
5. A cantilever 1.5 m long is loaded with a uniformly distributed load of 2 kN/m and a point load of 3 kN as shown in Fig. 13.11

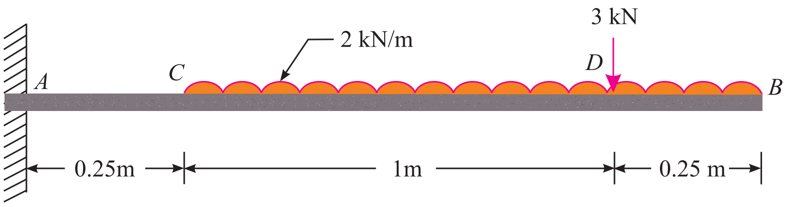


Fig. 13.11

Draw the shear force and bending moment diagrams for the cantilever.

[**Ans.** $F_{max} = -5.5$ kN ; $M_{max} = -5.94$ kN-m]

6. A cantilever beam 2 m long is subjected to a gradually varying load from zero at the free end to 2 kN/m at the fixed end. Find the values of maximum shear force and bending moment and draw the shear force and bending moment diagrams. [**Ans.** $F_{max} = -2$ kN ; $M_{max} = -1.33$ kN-m]

13.11. Simply Supported Beam with a Point Load at its Mid-point

Consider a *simply supported beam AB of span l and carrying a point load W at its mid-point C as shown in Fig. 13.12 (a). Since the load is at the mid-point of the beam, therefore the reaction at the support A,

$$R_A = R_B = 0.5\ W$$

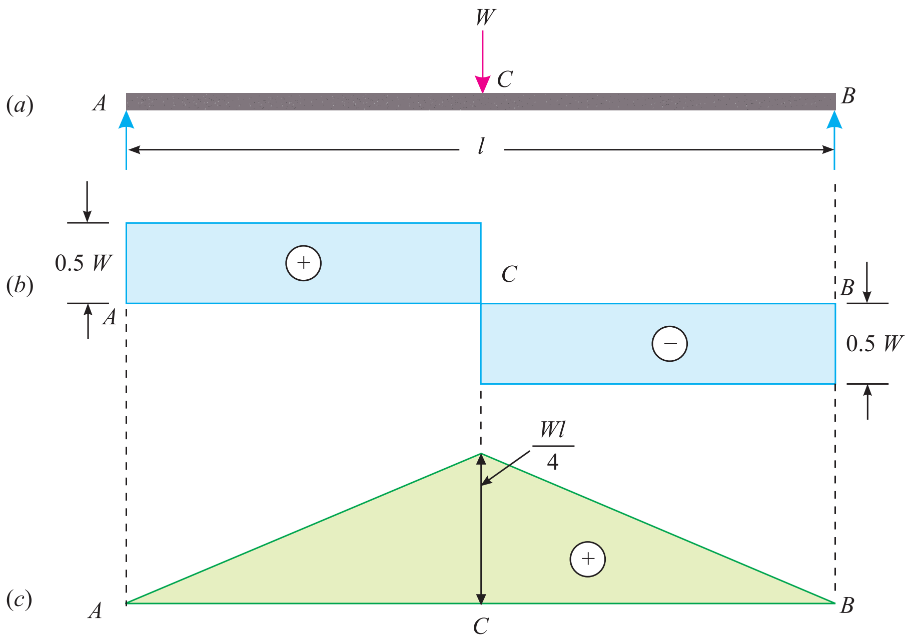


Fig. 13.12. Simply supported beam with a point load

Thus we see that the shear force at any section between A and C (*i.e.*, up to the point just before the load W) is constant and is equal to the unbalanced vertical force, *i.e.*, + 0.5 W. Shear force at any section between C and B (*i.e.*, just after the load W) is also constant and is equal to the unbalanced vertical force, *i.e.*, – 0.5 W as shown in Fig. 13.12 (b).

We also see that the bending moment at A and B is zero. It increases by a straight line law and is maximum at centre of beam, where shear force changes sign as shown in Fig. 13.12 (c).

* It is beam supported or resting freely on the walls or columns on both ends.

Therefore bending moment at C,

$$M_C = \frac{W}{2} \times \frac{1}{2} = \frac{Wl}{4} \qquad \text{...(Plus sign due to sagging)}$$

Note. If the point load does not act at the mid-point of the beam, then the two reactions are obtained and the diagrams are drawn as usual.

Example 13.6. *A simply supported beam AB of span 2.5 m is carrying two point loads as shown in Fig. 13.13.*

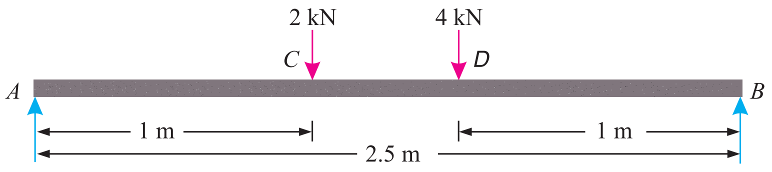


Fig. 13.13

Draw the shear force and bending moment diagrams for the beam.

Solution. Given : Span $(l) = 2.5$ m ; Point load at C $(W_1) = 2$ kN and point load at B $(W_2) = 4$ kN.

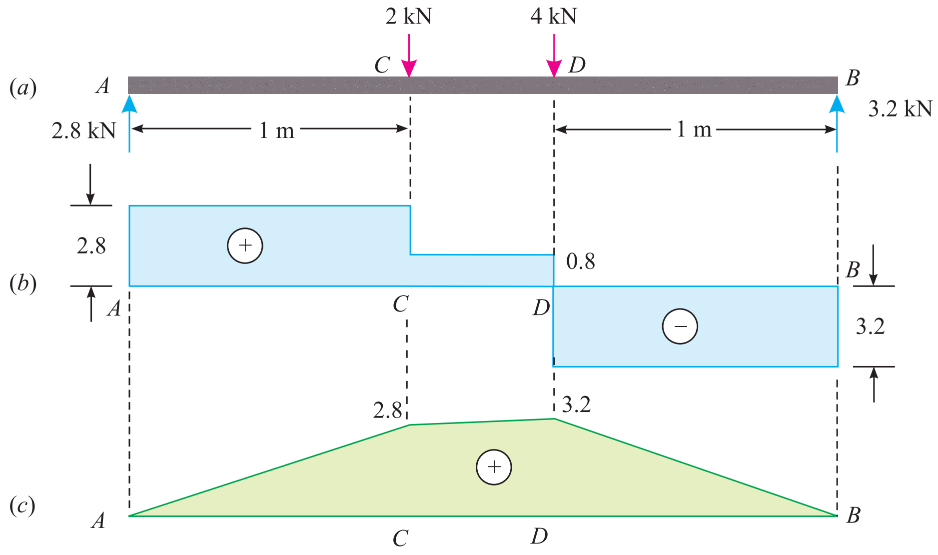


Fig. 13.14

First of all let us find out the reactions R_A and R_B. Taking moments about A and equating the same,

$$R_B \times 2.5 = (2 \times 1) + (4 \times 1.5) = 8$$

$$R_B = 8/2.5 = 3.2 \text{ kN}$$

and

$$R_A = (2 + 4) - 3.2 = 2.8 \text{ kN}$$

Shear force diagram

The shear force diagram is shown in Fig. 13.14 (*b*) and the values are tabulated here:

$$F_A = +R_A = 2.8 \text{ kN}$$

$$F_C = +2.8 - 2 = 0.8 \text{ kN}$$

$$F_D = 0.8 - 4 = -3.2 \text{ kN}$$

$$F_B = -3.2 \text{ kN}$$

Bending moment diagram

The bending moment diagram is shown in Fig. 13.14 (*c*) and the values are tabulated here:

$$M_A = 0$$
$$M_C = 2.8 \times 1 = 2.8 \text{ kN-m}$$
$$M_D = 3.2 \times 1 = 3.2 \text{ kN-m}$$
$$M_B = 0$$

Note. The value of M_D may also be found and from the reaction R_A. *i.e.*,

$$M_D = (2.8 \times 1.5) - (2 \times 0.5) = 4.2 - 1.0 = 3.2 \text{ kN-m}$$

13.12. Simply Supported Beam with a Uniformly Distributed Load

Consider a simply supported beam *AB* of length *l* and carrying a uniformly distributed load of *w* per unit length as shown in Fig. 13.15. Since the load is uniformly distributed over the entire length of the beam, therefore the reactions at the supports *A*,

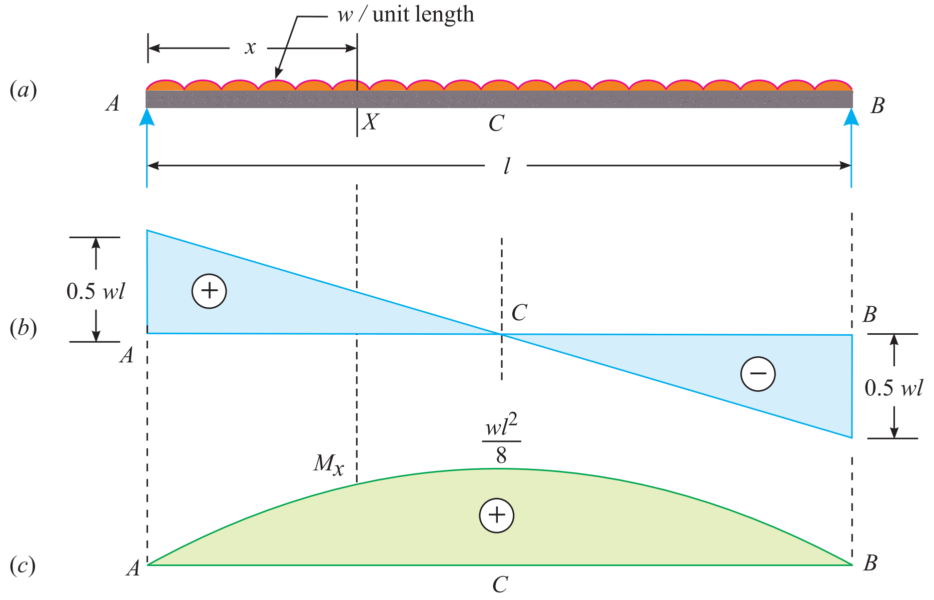


Fig. 13.15. Simply supported beam with a uniformly distributed load

$$R_A = R_B = \frac{wl}{2} = 0.5\, wl$$

We know that shear force at any section *X* at a distance *x* from *A*,

$$F_x = R_A - wx = 0.5\, wl - wx$$

We see that the shear force at *A* is equal to $R_A = 0.5\, wl$, where $x = 0$ and decreases uniformly by a straight line law, to zero at the mid-point of the beam ; beyond which it continues to decrease uniformly to $-0.5\, wl$ at *B* *i.e.*, R_B as shown in Fig. 13.15 (*b*). We also know that bending moment at any section at a distance *x* from *A*,

$$M_x = R_A \,.\, x - \frac{wx^2}{2} = \frac{wl}{2}x - \frac{wx^2}{2}$$

We also see that the bending moment is zero at A and B (where $x = 0$ and $x = l$) and increases in the form of a parabolic curve at C, *i.e.*, mid-point of the beam where shear force changes sign as shown in Fig. 13.15 (c). Thus bending moment at C,

$$M_C = \frac{wl}{2}\left(\frac{l}{2}\right) - \frac{w}{2}\left(\frac{l}{2}\right)^2 = \frac{wl^2}{4} - \frac{wl^2}{8} = \frac{wl^2}{8}$$

EXAMPLE 13.7. *A simply supported beam 6 m long is carrying a uniformly distributed load of 5 kN/m over a length of 3 m from the right end. Draw the S.F. and B.M. diagrams for the beam and also calculate the maximum B.M. on the section.*

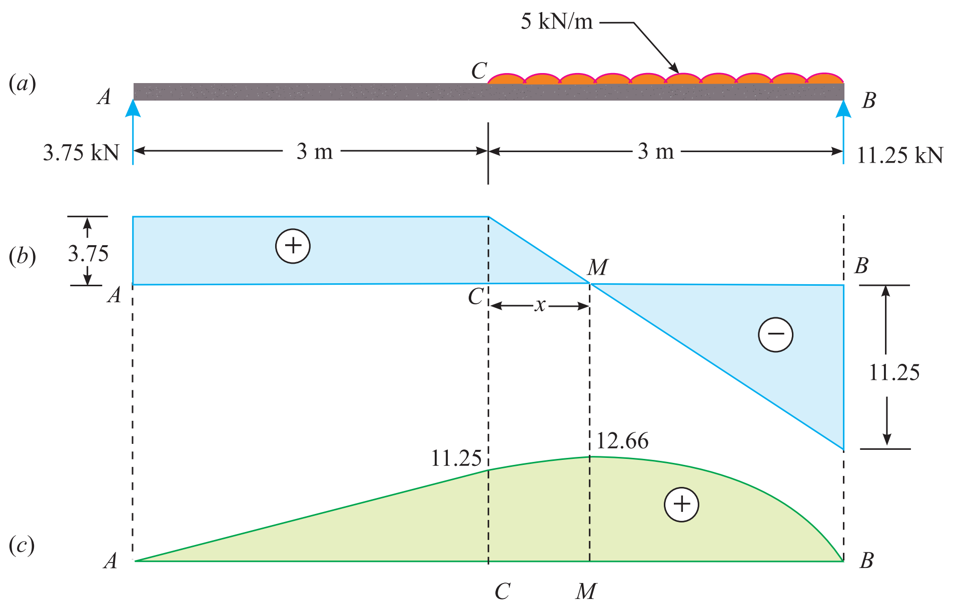


Fig. 13.16

SOLUTION. Given : Span (l) = 6 m ; Uniformly distributed load (w) = 5 kN/m and length of the beam CB carrying load (a) = 3 m.

First of all, let us find out the reactions R_A and R_B. Taking moments about A and equating the same,

$$R_B \times 6 = (5 \times 3) \times 4.5 = 67.5$$

$$\therefore \quad R_B = \frac{67.5}{6} = 11.25 \text{ kN}$$

$$\text{and} \quad R_A = (5 \times 3) - 11.25 = 3.75 \text{ kN}$$

Shear force diagram

The shear force diagram is shown in Fig. 13.16 (b) and the values are tabulated here:

$$F_A = + R_A = + 3.75 \text{ kN}$$

$$F_C = + 3.75 \text{ kN}$$

$$F_B = + 3.75 - (5 \times 3) = - 11.25 \text{ kN}$$

Bending moment diagram

The bending moment is shown in Fig. 13.16 (*c*) and the values are tabulated here:

$$M_A = 0$$
$$M_C = 3.75 \times 3 = 11.25 \text{ kN}$$
$$M_B = 0$$

We know that the maximum bending moment will occur at *M*, where the shear force changes sign. Let *x* be the distance between *C* and *M*. From the geometry of the figure between *C* and *B*, we find that

$$\frac{x}{3.75} = \frac{3-x}{11.25} \quad \text{or} \quad 11.25\,x = 11.25 - 3.75\,x$$

$$15\,x = 11.25 \quad \text{or} \quad x = 11.25/15 = 0.75 \text{ m}$$

$$\therefore \quad M_M = 3.75 \times (3 + 0.75) - 5 \times \frac{0.75}{2} = 12.66 \text{ kN-m}$$

EXAMPLE 13.8. *A simply supported beam 5 m long is loaded with a uniformly distributed load of 10 kN/m over a length of 2 m as shown in Fig. 13.17.*

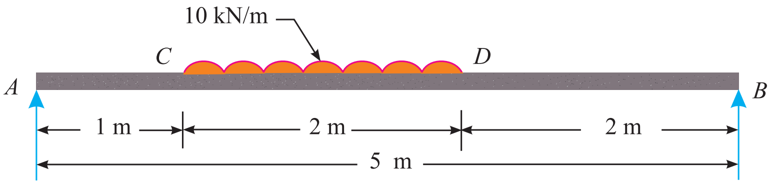


Fig. 13.17

Draw shear force and bending moment diagrams for the beam indicating the value of maximum bending moment.

SOLUTION. Given : Span (l) = 5 m ; Uniformly distributed load (w) = 10 kN/m and length of the beam *CD* carrying load (a) = 2 m.

First of all, let us find out the reactions R_A and R_B. Taking moments about *A* and equating the same,

$$R_B \times 5 = (10 \times 2) \times 2 = 40$$
$$\therefore \quad R_B = 40/5 = 8 \text{ kN}$$
$$\text{and} \quad R_A = (10 \times 2) - 8 = 12 \text{ kN}$$

Shear force diagram

The shear force diagram is shown in Fig. 13.18 (*b*) and the values are tabulated here:

$$F_A = +R_A = +12 \text{ kN}$$
$$F_C = +12 \text{ kN}$$
$$F_D = +12 - (10 \times 2) = -8 \text{ kN}$$
$$F_B = -8 \text{ kN}$$

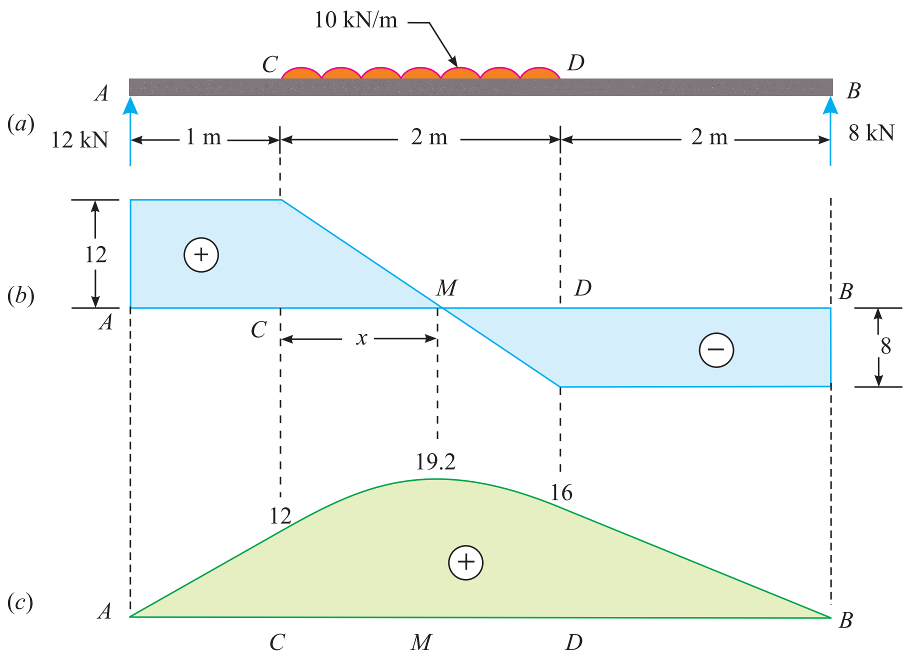


Fig. 13.18

Bending moment diagram

The bending moment diagram is shown in Fig. 13.18 (c) and the values are tabulated here:

$$M_A = 0$$

$$M_C = 12 \times 1 = 12 \text{ kN-m}$$

$$M_D = 8 \times 2 = 16 \text{ kN-m}$$

We know that maximum bending moment will occur at M, where the shear force changes sign. Let x be the distance between C and M. From the geometry of the figure between C and D, we find that

$$\frac{x}{12} = \frac{2-x}{8} \quad \text{or} \quad 8\,x = 24 - 12\,x$$

$$20\,x = 24 \quad \text{or} \quad x = 24/20 = 1.2 \text{ m}$$

$$M_M = 12\,(1 + 1.2) - 10 \times 1.2 \times \frac{1.2}{2} = 19.2 \text{ kN-m}$$

EXAMPLE 13.9. *A simply supported beam of 4 m span is carrying loads as shown in Fig. 13.19.*

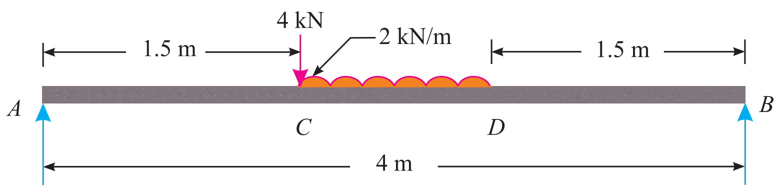


Fig. 13.19

Draw shear force and bending moment diagrams for the beam.

SOLUTION. Given : Span (l) = 4 m ; Point load at C (W) = 4 kN and uniformly distributed load between C and D (w) = 2 kN/m.

First of all, let us find out the reactions R_A and R_B. Taking moments about A and equating the same,

$$R_B \times 4 = (4 \times 1.5) + (2 \times 1) \times 2 = 10$$

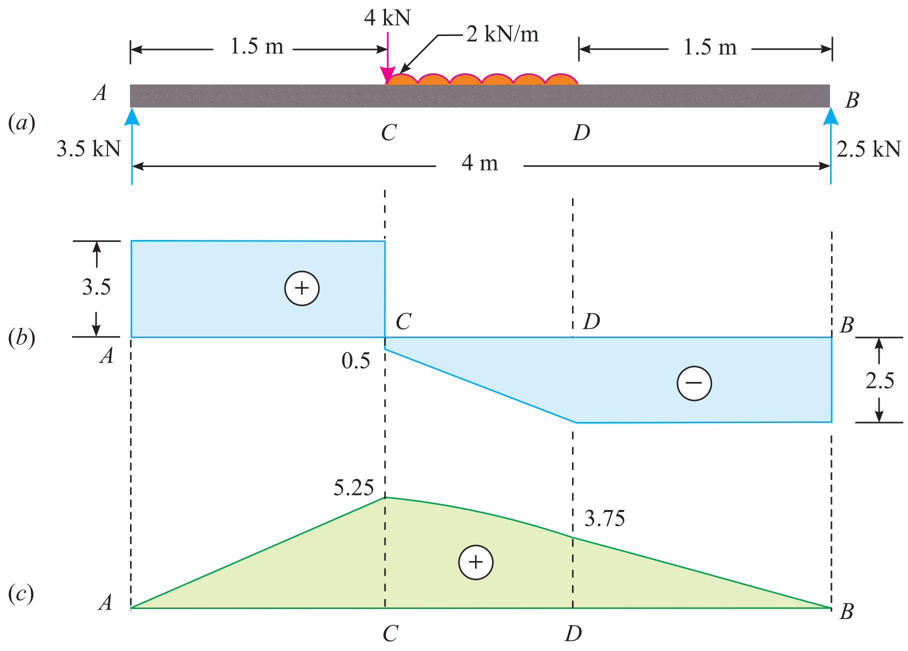


Fig. 13.20

$$R_B = 10/4 = 2.5 \text{ kN}$$

and $$R_A = 4 + (2 \times 1) - 2.5 = 3.5 \text{ kN}$$

Shear force diagram

The shear force diagram is shown in Fig. 13.20 (*b*) and the values are tabulated here:

$$F_A = +R_A = +3.5 \text{ kN}$$

$$F_C = +3.5 - 4 = -0.5 \text{ kN}$$

$$F_D = -0.5 - (2 \times 1) = -2.5 \text{ kN}$$

$$F_B = -2.5 \text{ kN}$$

Bending moment diagram

The bending moment diagram is shown in Fig. 13.20 (*c*) and the values are tabulated here:

$$M_A = 0$$

$$M_C = 3.5 \times 1.5 = 5.25 \text{ kN-m}$$

$$M_D = 2.5 \times 1.5 = 3.75 \text{ kN-m}$$

$$M_B = 0$$

We know that the maximum bending moment will occur at C, where the shear force changes sign, *i.e.*, at C as shown in the figure.

EXAMPLE 13.10. *A simply supported beam AB, 6 m long is loaded as shown in Fig. 13.21.*

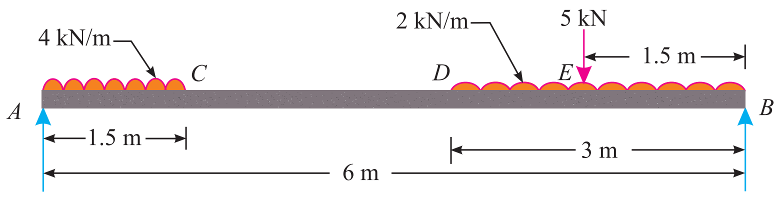


Fig. 13.21

Construct the shear force and bending moment diagrams for the beam and find the position and value of maximum bending moment.

SOLUTION. Given : Span (l) = 6 m ; Point load at E (W) = 5 kN ; Uniformly distributed load between A and C (w_1) = 4 kN/m and uniformly distributed load between D and B = 2 kN/m.

First of all, let us find out the reactions R_A and R_B. Taking moments about A and equating the same,

$$R_B \times 6 = (4 \times 1.5 \times 0.75) + (2 \times 3 \times 4.5) + (5 \times 4.5) = 54$$

$$R_B = 54/6 = 9 \text{ kN}$$

and

$$R_A = (4 \times 1.5) + (2 \times 3) + 5 - 9 = 8 \text{ kN}$$

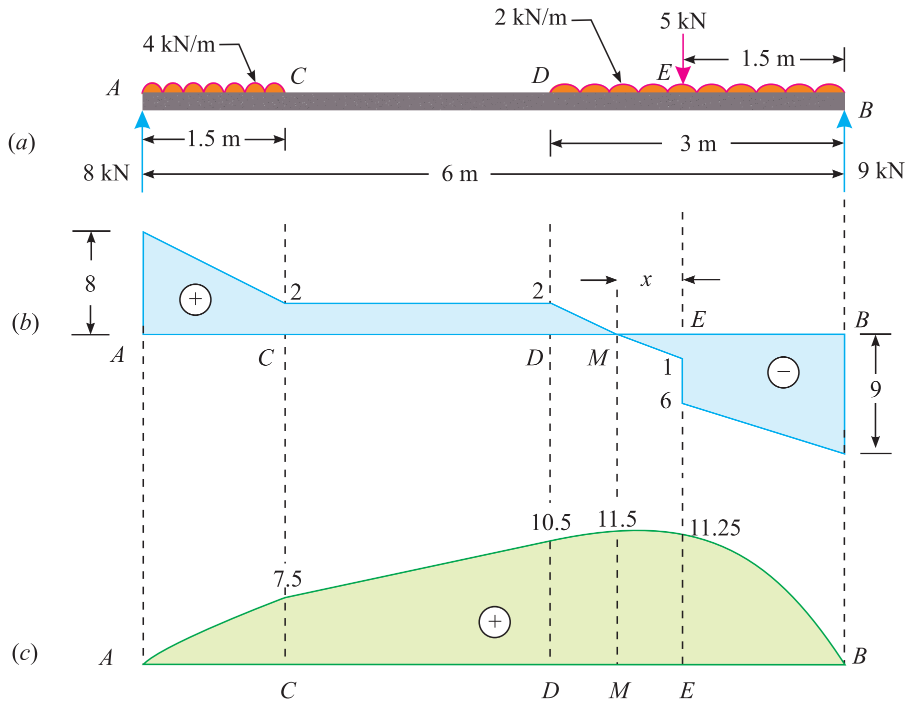


Fig. 13.22

Shear force diagram

The shear force diagram is shown in Fig. 13.22 (*b*) and the values are tabulated here:

$$F_A = +R_A = +8 \text{ kN}$$
$$F_C = 8 - (4 \times 1.5) = 2 \text{ kN}$$
$$F_D = 2 \text{ kN}$$
$$F_E = 2 - (2 \times 1.5) - 5 = -6 \text{ kN}$$
$$F_B = -6 - (2 \times 1.5) = -9 \text{ kN}$$

Bending moment diagram

The bending moment diagram is shown in Fig. 13.22 (c) and the values are tabulated here:

$$M_A = 0$$
$$M_C = (8 \times 1.5) - (4 \times 1.5 \times 0.75) = 7.5 \text{ kN-m}$$
$$M_D = (8 \times 3) - (4 \times 1.5 \times 2.25) = 10.5 \text{ kN-m}$$
$$M_E = (9 \times 1.5) - (2 \times 1.5 \times 0.75) = 11.25 \text{ kN-m}$$
$$M_B = 0$$

We know that maximum bending moment will occur at M, where the shear force changes sign. Let x be the distance between E and M. From the geometry of the figure between D and E, we find that

$$\frac{x}{1} = \frac{1.5 - x}{2} \quad \text{or} \quad 2\,x = 1.5 - x$$

$$3\,x = 1.5 \quad \text{or} \quad x = 1.5/3 = 0.5 \text{ m}$$

$$\therefore \quad M_M = 9\,(1.5 + 0.5) - (2 \times 2 \times 1) - (5 \times 0.5) = 11.5 \text{ kN-m}$$

13.13. Simply Supported Beam with a Triangle Load, Varying Gradually from Zero at Both Ends to *w* per unit length at the Centre

Consider a simply supported beam AB of span l and carrying a triangular load, varying gradually from zero at both the ends to w per unit length, at the centre as shown in Fig. 13.23 (*a*). Since the load is symmetrical, therefore the reactions R_A and R_B will be equal.

or
$$R_A = R_B = \frac{1}{2} \times \text{w} \times \frac{1}{2} = \frac{wl}{4}$$

$$= \frac{W}{2} \qquad \ldots\left(\text{where } \; W = \text{Total load} = \frac{wl}{2}\right)$$

The shear force at any section X at a distance x from B,

$$F_X = -R_B + \frac{wx^2}{l} = \frac{wx^2}{l} - \frac{wl}{4} = \frac{wx^2}{l} - \frac{W}{2} \qquad \ldots(i)$$

Thus we see that shear force is equal to $-\dfrac{W}{2}$ at B, where $x = 0$ and increases in the form of a *parabolic curve* [as given by equation (*i*) above] to zero at C, *i.e.*, mid-point of the span ; beyond which it continues to increase to $+\dfrac{W}{2}$ at A where $x = l$ as shown in Fig. 13.23 (*b*). The bending moment at any section X at a distance x from B,

$$M_X = R_B \,.\, x - \frac{wx}{\frac{l}{2}} \times \frac{x}{2} \times \frac{x}{3} = \frac{wlx}{4} - \frac{wx^3}{3l} \qquad \ldots(ii)$$

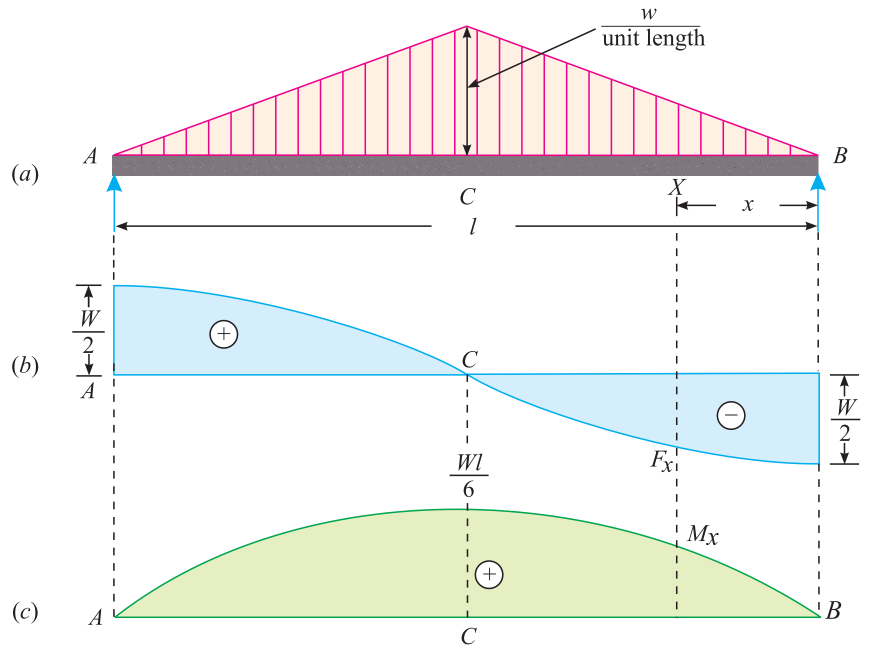


Fig. 13.23

Thus we see that the bending moment at A and B is zero and increases in the form of a cubic curve [as given by the equation (*ii*) above] at C, *i.e.*, mid-point of the beam, where bending moment will be maximum because shear force changes sign.

$$\therefore \qquad M_M = \frac{wl}{4}\left(\frac{l}{2}\right) - \frac{w}{3l}\left(\frac{l}{2}\right)^3 = \frac{wl^2}{12}$$

$$= \frac{Wl}{6} \qquad \ldots\left(\text{where} \quad W = \text{Total load} = \frac{wl}{2}\right)$$

EXAMPLE 13.11. *A simply supported beam of 5 m span carries a triangular load of 30 kN. Draw S.F. and B.M. diagrams for the beam.*

SOLUTION. Given : Span (l) = 5 m and total triangular load (W) = 30 kN

By symmetry,
$$R_A = R_B = \frac{30}{2} = 15 \text{ kN}$$

Shear force diagram

The shear force diagram is shown in Fig. 13.24 (*b*) and the values are tabulated here:

$$F_A = +R_A = +15 \text{ kN}$$
$$F_B = -R_B = -15 \text{ kN}$$

Bending moment diagram

The bending moment diagram is shown in Fig. 13.24 (*c*). It is zero at A and B and the maximum bending moment will occur at the centre *i.e.*, at M, where the shear force changes sign. We know that maximum bending moment,

$$M_M = \frac{Wl}{6} = \frac{30 \times 5}{6} = 25 \text{ kN-m}$$

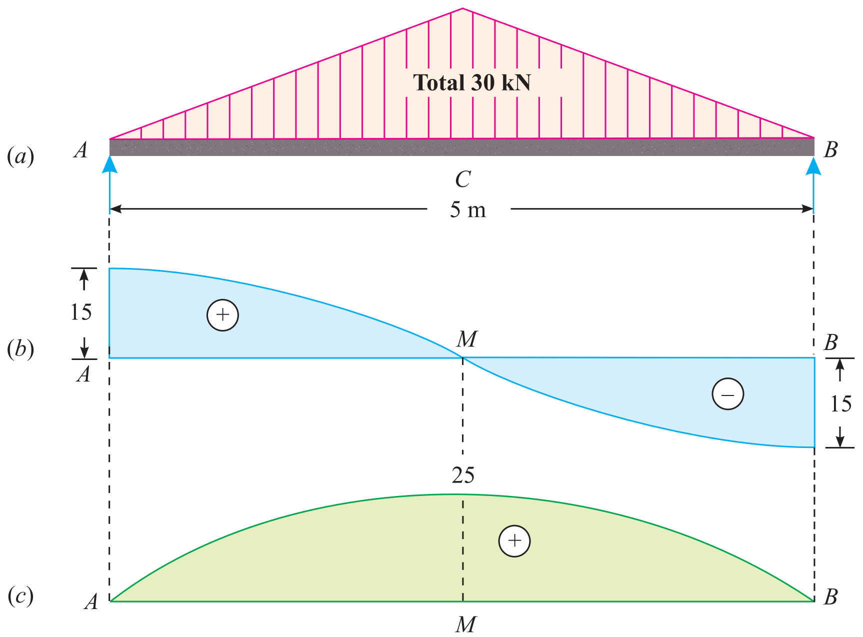


Fig. 13.24

13.14. Simply Supported Beam with a Gradually Varying Load from Zero at One End to *w* per unit length at the Other End

Consider a simply supported beam *AB* of length *l* and carrying a gradually varying load zero at one end and *w* per unit length at the other as shown in Fig. 13.25 (*a*). Since the load is varying gradually from zero at one end to *w* per unit length at the other, therefore both the reactions at *A* and *B* will have to be first calculated.

Taking moments about *A*,

$$R_B \times 1 = \left(\frac{0+w}{2}\right) \times l \times \frac{l}{3} = \frac{wl^2}{6}$$

$$\therefore \qquad R_B = \frac{wl^2}{6} \times \frac{1}{l} = \frac{wl}{6}$$

$$= \frac{W}{3} \qquad \ldots \left(\text{where } W = \text{Total load} = \frac{wl}{2}\right)$$

and

$$R_A = \frac{wl}{2} - \frac{wl}{6} = \frac{wl}{3}$$

$$= \frac{2W}{3} \qquad \ldots \left(\text{where } W = \frac{wl}{2}\right)$$

We know that the shear force at any section *X* at a distance *x* from *B*,

$$F_X = -R_B + \frac{wx^2}{2l} = \frac{wx^2}{2l} - \frac{W}{3} \qquad \ldots(i)$$

Thus we see that the shear force is equal to $-\frac{W}{3}$ at *B* (where $x = 0$) and increases in the form of a *parabolic curve* [as is given by the equation (*i*) above] to zero at *M* ; beyond which it continues to increase to $+\frac{2W}{3}$ at *A* (where $x = l$) as shown in Fig. 13.25 (*b*). The bending moment at any section *X* at a distance *x* from *B*,

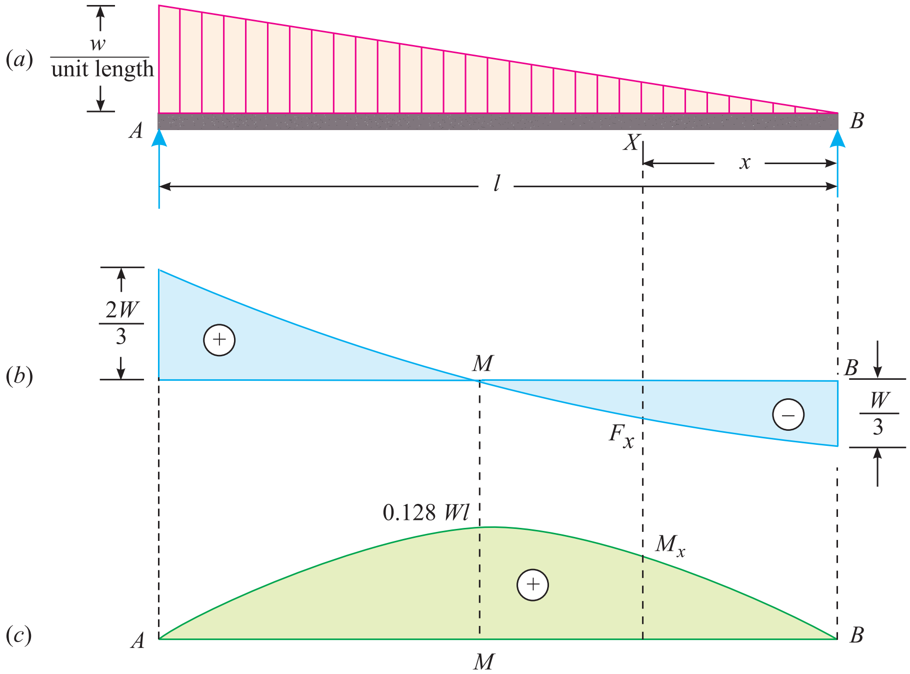


Fig. 13.25

$$M_X = R_B \cdot x - \frac{wx}{l} \times \frac{x}{2} \times \frac{x}{3}$$

$$= \frac{wlx}{6} - \frac{wx^3}{6l} \qquad \text{...(ii)}$$

Thus bending moment at *A* and *B* is zero and it increases in the form of a cubic curve [as given by the equation (*ii*) above] at *M*, where the shear force changes sign. To find out the position *M*, let us equate the equation (*i*) to zero, *i.e.*,

$$\frac{wx^2}{2l} - \frac{wl}{6} = 0 \quad \text{or} \quad \frac{wx^2}{2l} - \frac{wl}{6}$$

$$\therefore \qquad x^2 = \frac{l^2}{3} \quad \text{or} \quad x = \frac{l}{\sqrt{3}} = 0.577\, l$$

$$\therefore \qquad M_M = \frac{wl}{6}\left(\frac{l}{\sqrt{3}}\right) - \frac{w}{6l}\left(\frac{l}{\sqrt{3}}\right)^3 = \frac{wl^2}{9\sqrt{3}}$$

$$= \frac{2\,Wl}{9\sqrt{3}} = 0.128\ Wl \qquad \text{...}\left(\text{where } W = \frac{wl}{2}\right)$$

Note. In such cases the different values of shear force and bending moment should be calculated at intervals of 0.5 m or 1 m [as per equations (*i*) and (*ii*) above] and then the diagrams should be drawn.

Example 13.12. *The intensity of loading on a simply supported beam of 6 m span increases gradually from 800 N/m run at one end to 2000 N/m run at the other as shown in Fig. 13.26.*

Find the position and amount of maximum bending moment. Also draw the shear force and bending moment diagrams.

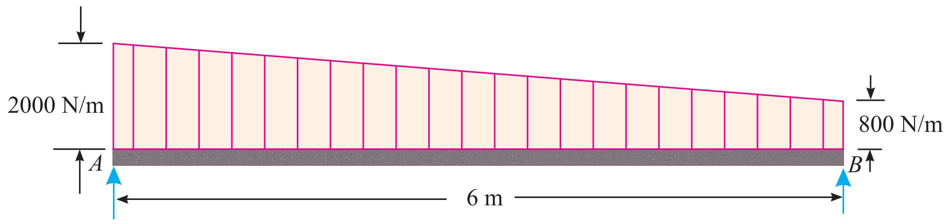


Fig. 13.26

SOLUTION. Given : Span (l) = 6 m ; Gradually varying load at A (w_A) = 2000 N/m and gradually varying load at B (w_B) = 800 N/m.

The weight may be assumed to be split up with

(*i*) a uniformly distributed load of 800 N/m over the entire span and

(*ii*) a gradually varying load of zero at B to 1200 N/m at A.

∴ Total uniformly distributed load,

$$W_1 = 800 \times 6 = 4800 \text{ N}$$

∴
$$R_B = \frac{4800}{2} + \frac{3600}{3} = 3600 \text{ N}$$

and
$$R_A = \frac{4800}{2} + \frac{2 \times 3600}{3} = 4800 \text{ N.}$$

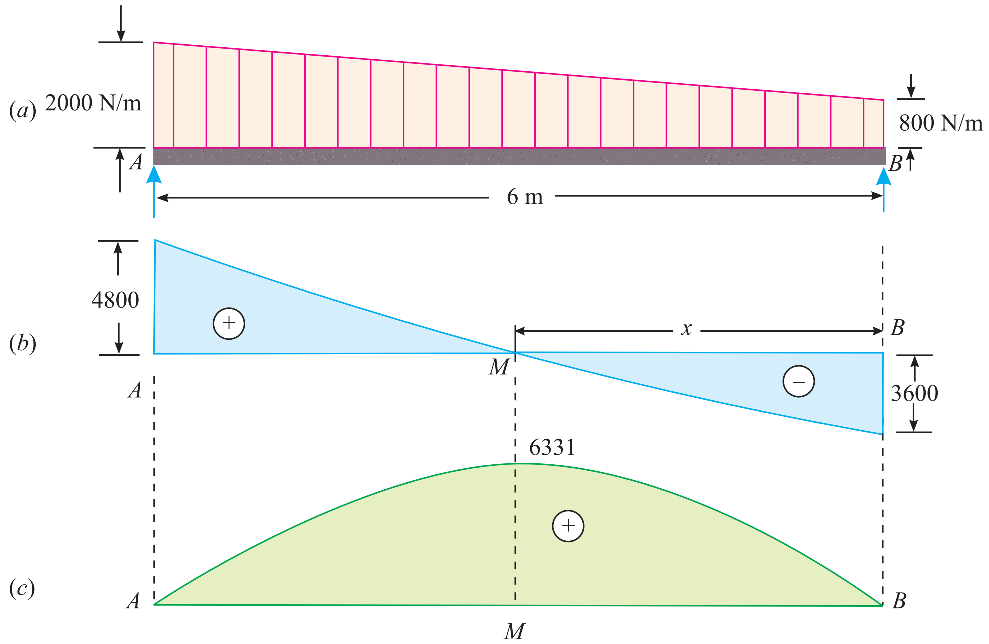


Fig. 13.27

Shear force diagram

The shear force diagram is shown in Fig. 13.27 (*b*), and the values are tabulated here:

$$F_A = +R_A = 4800 \text{ N}$$
$$F_B = -R_B = -3600 \text{ N}$$

Bending moment diagram

The bending moment diagram is shown in Fig. 13.27 (*c*). It is zero at A and B and the maximum bending moment will occur at M, where the shear force changes sign.

Maximum bending moment

We know that maximum bending moment will occur at a point (M), where shear force changes sign. Let x be the distance between B and M. We also know that shear force at a distance x from M,

$$= -3600 + 800\,x + \frac{1}{2} \times 1200\,x \times \frac{x}{6}$$
$$= -3600 + 800\,x + 100\,x^2$$
$$= 100\,x^2 + 800\,x - 3600$$

Now to find the position of M (*i.e.*, the point where shear force changes sign), let us equate the above equation to zero. *i.e.*,

$$100\,x^2 + 800\,x - 3600 = 0$$

or $$x^2 + 8\,x - 36 = 0$$

This is a quadratic equation. Therefore

$$\therefore \quad x = \frac{-8 \pm \sqrt{(8)^2 + (4 \times 36)}}{2} = 3.21 \text{ m}$$

and bending moment at M,

$$M_M = 3600\,x - \left(800 \times \frac{x^2}{2}\right) - \left(\frac{1}{2} \times 1200\,x \times \frac{x}{6} \times \frac{x}{3}\right)$$
$$= 3600\,x - 400\,x^2 - \frac{100}{3}x^3$$
$$= (3600 \times 3.21) - 400 \times (3.21)^2 - \frac{100}{3} \times (3.21)^3 \text{ N-m}$$
$$= 11556 - 4122 - 1102 = 6332 \text{ N-m}$$

EXERCISE 13.2

1. A simply supported beam of 3 m span carries two loads of 5 kN each at 1 m and 2 m from the left hand support. Draw the shear force and bending moment diagrams for the beam. [**Ans.** M_{max} = 5 kN-m]
2. A simply supported beam of span 4.5 m carries a uniformly distributed load of 3.6 kN/m over a length of 2 m from the left end A. Draw the shear force and bending moment diagrams for the beam. [**Ans.** M_{max} = 4.36 kN-m at 1.56 m from A]
3. A simply supported beam $ABCD$ is of 5 m span, such that AB = 2 m, BC = 1 m and CD = 2 m. It is loaded with 5 kN/m over AB and 2 kN/m over CD. Draw shear force and bending moment diagrams for the beam. [**Ans.** M_{max} = 7.74 kN-m at 1.76 m from A]
4. Draw shear force and bending moment diagrams for a simply supported beam, loaded as shown in Fig. 13.28.

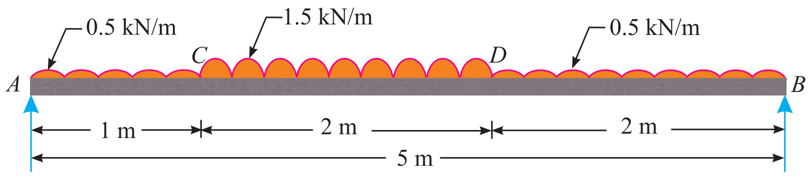


Fig. 13.28

Find the position and value of the maximum bending moment that will occur in the beam. [**Ans.** 3.47 kN-m at 1.3 m from C]

5. A simply supported beam AB, 6 m long is loaded as shown in Fig. 13.29.

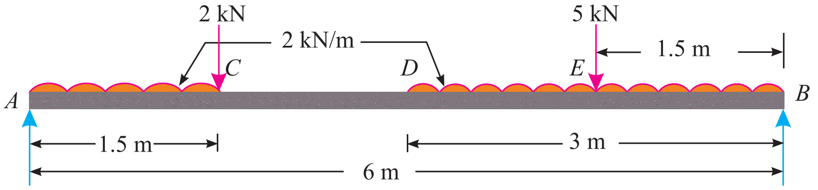


Fig. 13.29

Draw the shear force and bending moment diagrams for the beam.

[*Ans.* M_{max} = 11.75 kN-m at 0.56 m from E]

6. A simply supported beam 3 m long carries a triangular load of 12 kN. Draw the S.F and B.M. diagrams for the beam. [**Ans.** M_{max} = 6 kN-m]

13.15. Overhanging Beam

It is a simply supported beam which overhangs (*i.e.*, extends in the form of a cantilever) from its support.

For the purposes of shear force and bending moment diagrams, the overhanging beam is analysed as a combination of a simply supported beam and a cantilever. An overhanging beam may overhang on one side only or on both sides of the supports.

13.16. Point of Contraflexure

We have already discussed in the previous article that an overhanging beam is analysed as a combination of simply supported beam and a cantilever. In the previous examples, we have seen that the bending moment in a cantilever is negative, whereas that in a simply supported beam is positive. It is thus obvious that in an overhanging beam, there will be a point, where the bending moment will change sign from negative to positive or *vice versa*. Such a point, where the bending moment changes sign, is known as a point of contraflexure.

Example 13.13. *An overhanging beam ABC is loaded as shown in Fig. 13.30.*

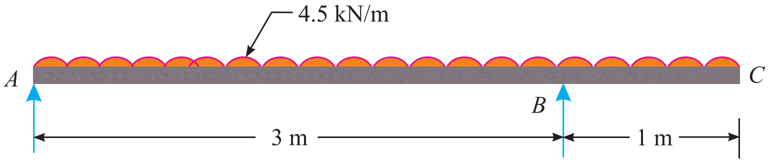


Fig. 13.30

Draw the shear force and bending moment diagrams and find the point of contraflexure, if any.

Solution. Given : Span (l) = 4 m ; Uniformly distributed load (w) = 4.5 kN/m and overhanging length (c) = 1 m.

First of all, let us find out the reactions R_A and R_B. Taking moment about A and equating the same,

$$R_B \times 3 = (4.5 \times 4) \times 2 = 36$$

$$\therefore \quad R_B = 36/3 = 12 \text{ kN}$$

$$\text{and} \quad R_A = (4.5 \times 4) - 12 = 6 \text{ kN}$$

Shear force diagram

The shear force diagram is shown in Fig. 13.31 (b) and the values are tabulated here:

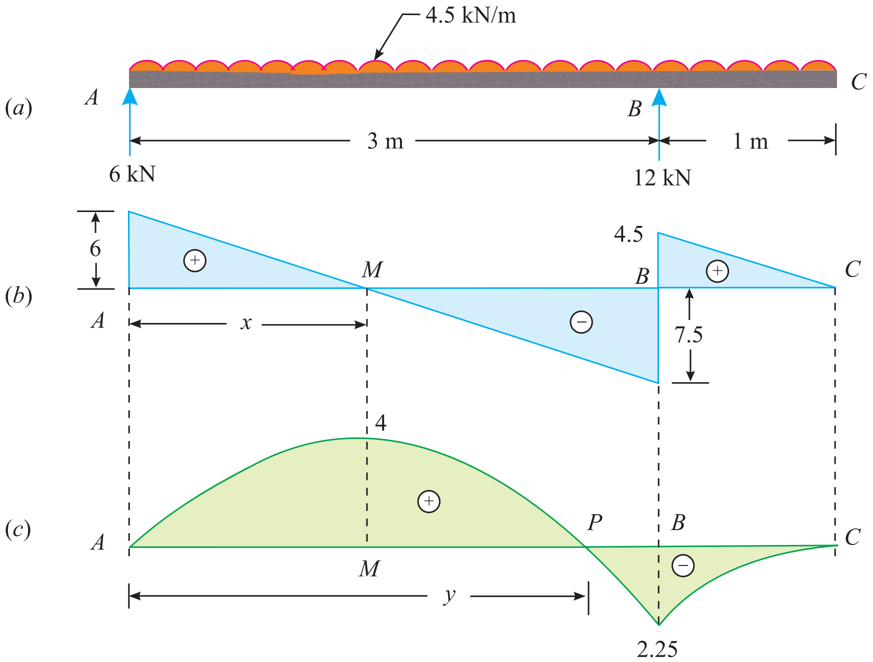


Fig. 13.31

$$F_A = +R_A = +6 \text{ kN}$$
$$F_B = +6 - (4.5 \times 3) + 12 = 4.5 \text{ kN}$$
$$F_C = +4.5 - (4.5 \times 1) = 0$$

Bending moment diagram

The bending moment diagram is shown in Fig. 13.31 (*c*) and the values are tabulated here:

$$M_A = 0$$

$$M_B = -\left(4.5 \times 1 \times \frac{1}{2}\right) = -2.25 \text{ kN-m}$$

$$M_C = 0$$

We know that maximum bending moment will occur at *M*, where the shear force changes sign. Let *x* be the distance between *A* and *M*. From the geometry of the figure between *A* and *B*, we find that

$$\frac{x}{6} = \frac{3-x}{7.5} \quad \text{or} \quad 7.5\,x = 18 - 6\,x$$

$$13.5\,x = 18 \quad \text{or} \quad x = 18/13.5 = 1.33 \text{ m}$$

$$\therefore \quad M_M = (6 \times 1.33) - 4.5 \times 1.33 \times \frac{1.33}{2} = 4 \text{ kN-m}$$

Point of contraflexure

Let *P* be the point of contraflexure at a distance *y* from the support *A*. We know that bending moment at *P*.

$$M_P = 6 \times y - 4.5 \times y \times \frac{y}{2} = 0$$

$$2.25\,y^2 - 6\,y = 0 \quad \text{or} \quad 2.25\,y = 6$$

$$\therefore \quad y = 6/2.25 = 2.67 \text{ m} \quad \textbf{Ans.}$$

EXAMPLE 13.14. *A beam ABCD, 4 m long is overhanging by 1 m and carries load as shown in Fig. 13.32.*

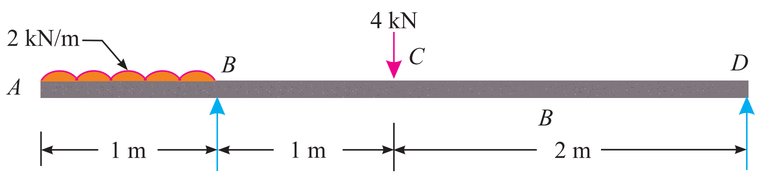


Fig. 13.32

Draw the shear force and bending moment diagrams for the beam and locate the point of contraflexure.

SOLUTION. Given : Span (l) = 4 m ; Uniformly distributed load over AB (w) = 2 kN/m and point load at C (W) = 4 kN.

First of all, let us find out the reactions R_B and R_D. Taking moments about B and equating the same,

$$R_D \times 3 = (4 \times 1) - (2 \times 1) \times \frac{1}{2} = 3$$

$\therefore$ $$R_D = 3/3 = 1 \text{ kN}$$

and $$R_B = (2 \times 1) + 4 - 1 = 5 \text{ kN}$$

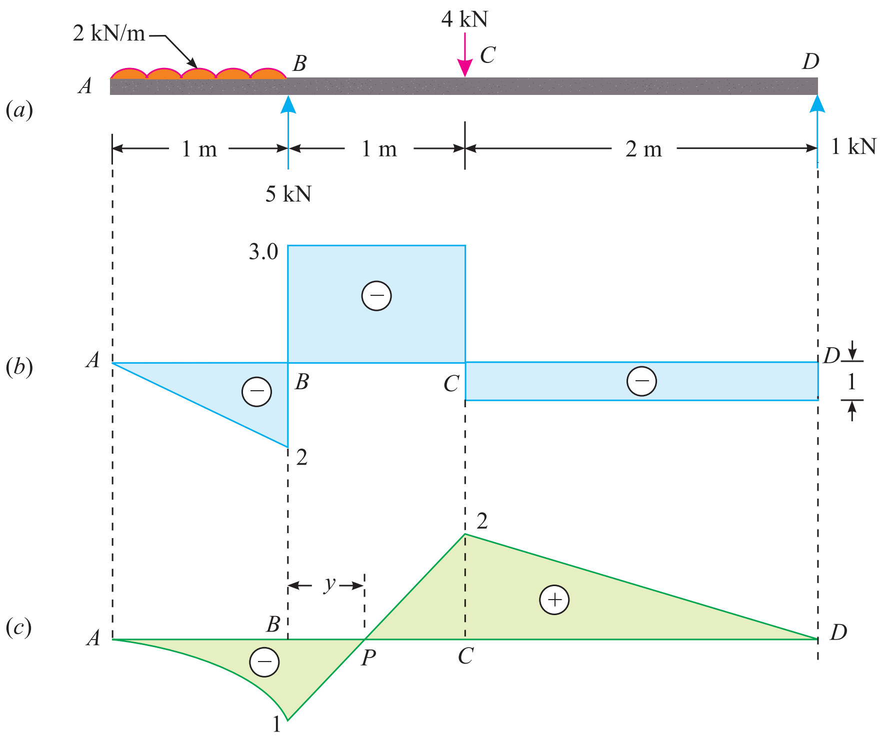


Fig. 13.33

Shear force diagram

The shear force diagram is shown in Fig. 13.33 (*b*) and the values are tabulated here:

$$F_A = 0$$

$$F_B = 0 - (2 \times 1) + 5 = +3 \text{ kN}$$

$$F_C = +3 - 4 = -1 \text{ kN}$$

$$F_D = 1 \text{ kN}$$

Bending moment diagram

The bending moment diagram is shown in Fig. 13.33 (*c*) and the values are tabulated here:

$$M_A = 0$$

$$M_B = -(2 \times 1)\, 0.5 = -1 \text{ kN-m}$$

$$M_C = 1 \times 2 = +2 \text{ kN}$$

$$M_D = 0$$

We know that maximum bending moment occurs either at *B* or *C*, where the shear force changes sign. From the geometry of the bending moment diagram, we find that maximum negative bending moment occurs at *B* and maximum positive bending moment occurs at *C*.

Point of contraflexure

Let *P* be the point of contraflexure at a distance *y* from the support *B*. From the geometry of the figure between *B* and *C*, we find that

$$\frac{y}{1.0} = \frac{1-y}{2.0}$$

$$2y = 1 - y \qquad \text{or} \qquad 3y = 1$$

or $$y = 1/3 = 0.33 \text{ m} \qquad \textbf{Ans.}$$

EXAMPLE 13.15. *Draw shear force and bending moment diagrams for the beam shown in Fig. 13.34. Indicate the numerical values at all important sections.*

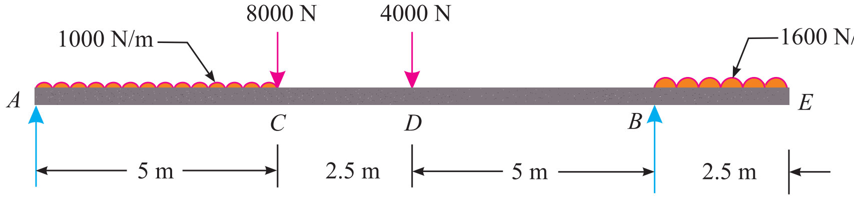


Fig. 13.34

SOLUTION. Given : Span (l) = 15 m ; Uniformly distributed load between *A* and *B* (w_1) = 1000 N/m ; Point load at *C* (W_1) = 8000 N ; Point load at *D* (W_2) = 4000 N and uniformly distributed load between *B* and *E* (w_2) = 1600 N/m.

First of all, let us find out the reactions R_A and R_B.

Taking moments about *A* and equating the same,

$$R_B \times 12.5 = (1600 \times 2.5) \times 13.75 + (4000 \times 7.5) + (8000 \times 5) + (1000 \times 5) \times 2.5$$

$$= 137500$$

$$R_B = \frac{137500}{12.6} = 110000 \text{ N}$$

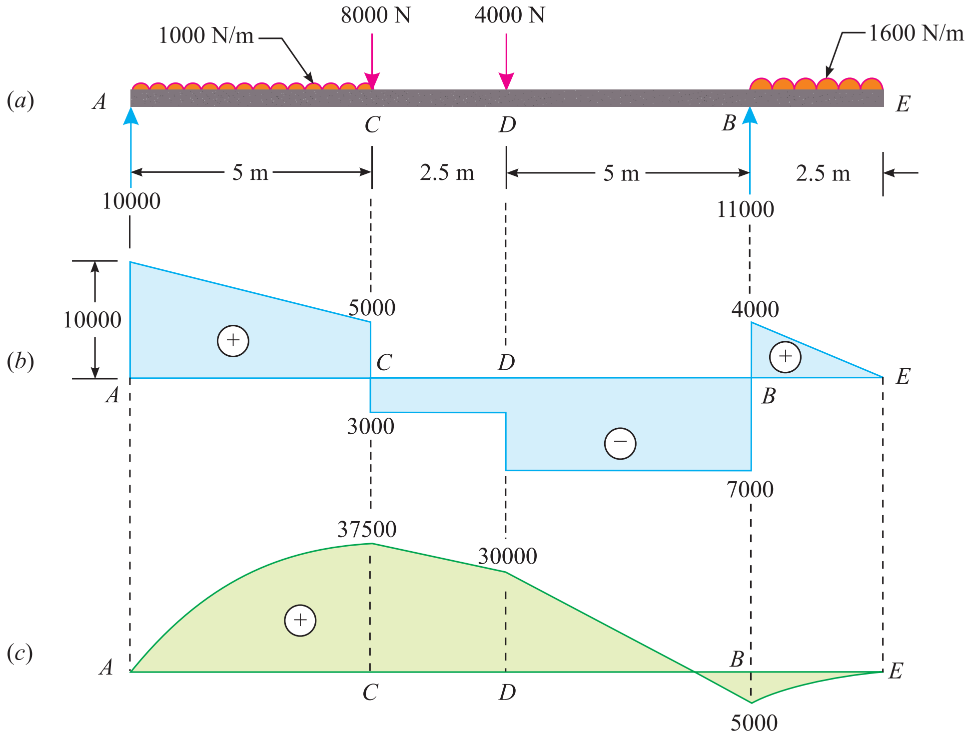


Fig. 13.35

and $\quad R_A = (1000 \times 5 + 8000 + 4000 + 1600 \times 2.5) - 11000$ N

$\quad = 10000$ N

Shear force

The shear force diagram is shown in Fig. 13.35 (*b*) and the values are tabulated here:

$$F_A = +\,10000 \text{ N}$$

$$F_C = +\,10000 - (1000 \times 5) - 800 = -\,3000 \text{ N}$$

$$F_D = -\,3000 - 4000 = -\,7000 \text{ N}$$

$$F_B = -\,7000 + 11000 = +\,4000 \text{ N}$$

$$F_E = +\,4000 - 1600 \times 2.5 = 0$$

Bending moment

The bending moment diagram is shown in Fig. 13.35 (*c*), and the values are tabulated here:

$$M_A = 0$$

$$M_C = (10000 \times 5) - (1000) \times (5 \times 2.5) = 37500 \text{ N-m}$$

$$M_D = (10000 \times 7.5) - (1000 \times 5 \times 5) - (8000 \times 2.5) \text{ N-m}$$

$$= 30000 \text{ N-m}$$

$$M_B = -\,1600 \times 2.5 \times \frac{2.5}{2} = -\,5000 \text{ N-m}$$

Maximum bending moment

The maximum bending moment, positive or negative will occur at C or at B because the shear force changes sign at both these points. But from the bending moment diagram, we see that the maximum positive bending moment occurs at C and the maximum negative bending moment occurs at B.

Example 13.16. *Draw the complete shear force diagram for the overhanging beam shown in Fig. 13.36.*

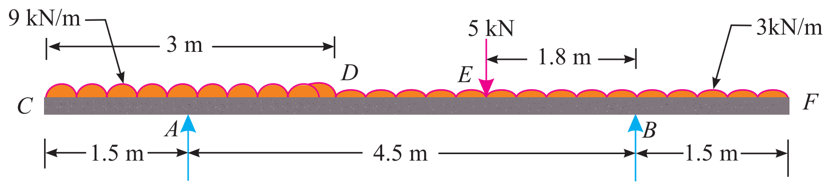


Fig. 13.36

Hence, determine the position in the central bay, at which the positive bending moment occurs. Find also magnitude of the maximum positive and negative bending moment.

Solution. Given : Span (l) = 7.5 m ; Uniformly distributed load between C and D = 9 kN/m ; Point load at E (W) = 5 kN ; Uniformly distributed load between D and F (w_2) = 3 kN/m and overhanging on both sides = 1.5 m.

Taking moments about A,

$$R_B \times 4.5 = (3 \times 4.5) \times 3{,}75 + (5 \times 2.7) = 64.125$$

($\because$ U.D.L. of 9 kN/m will have zero moment about A)

$$\therefore \quad R_B = \frac{64.125}{4.5} = 14.25 \text{ kN}$$

$$\text{and} \quad R_A = (9 \times 3 + 5 + 3 \times 4.5) - 14.25 = 31.25 \text{ kN}$$

Shear force diagram

The shear force diagram is shown in Fig. 13.37 (b) and the values are tabulated here:

$$F_C = 0$$

$$F_A = 0 - 9 \times 1.5 + 31.25 = +\ 17.75 \text{ kN}$$

$$F_D = +\ 17.75 - 9 \times 1.5 = +\ 4.25 \text{ kN}$$

$$F_E = +\ 4.25 - 3 \times 1.2 - 5.0 = -\ 4.35 \text{ kN}$$

$$F_B = -\ 4.35 - 3 \times 1.8 + 14.25 = +\ 4.5 \text{ kN}$$

$$F_F = +\ 4.5 - 3 \times 1.5 = 0$$

Bending moment diagram

The bending moment diagram is shown in Fig. 13.37 (c) and the values are tabulated here:

$$M_C = 0$$

$$M_A = -\frac{9 \times (1.5)^2}{2} = -\ 10.125 \text{ kN-m}$$

$$M_D = -\frac{9 \times (3)^2}{2} + 31.25 \times 1.5 = 6.375 \text{ kN-m}$$

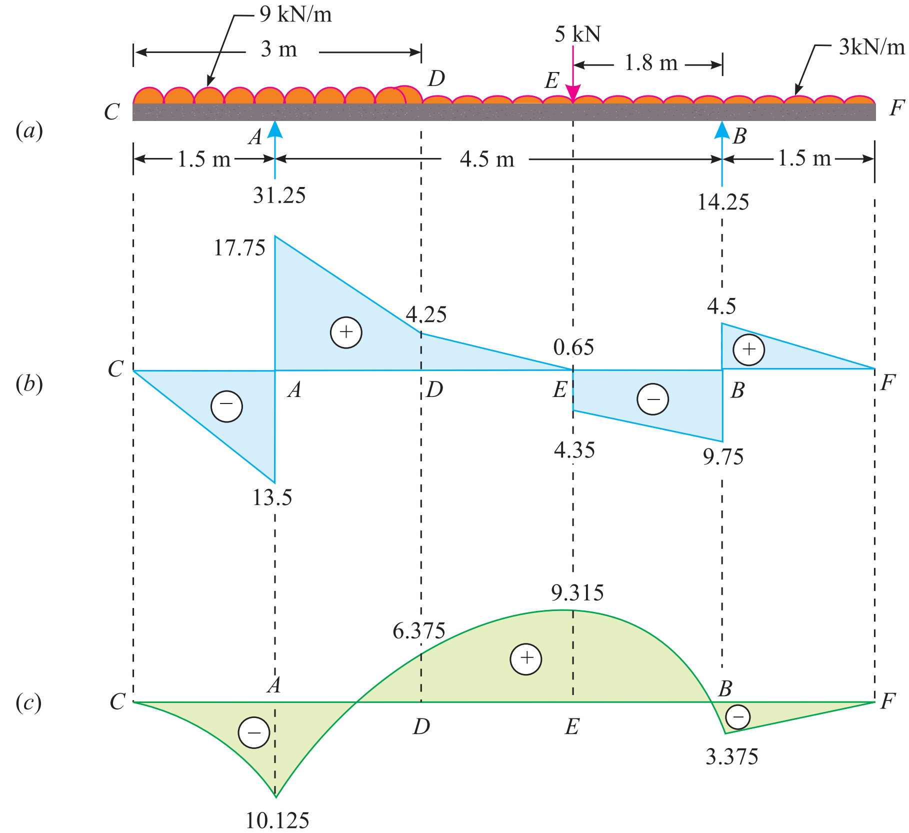


Fig. 13.37

$$M_B = -\frac{3 \times (1.5)^2}{2} = -3.375 \text{ kN-m}$$

$$M_E = -\frac{3 \times (3.3)^2}{2} + 14.25 \times 1.8 = 9.315 \text{ kN-m}$$

$$M_F = 0$$

Maximum bending moment

The maximum bending moment, positive or negative will occur at *A*, *E* or *B*, because the shear force changes sign at all these three points. But from the bending moment diagram, we see that the maximum negative bending moment occurs at *A* and the maximum positive bending moment occurs at *E*.

Example 13.17. *A simply supported beam with over-hanging ends carries transverse loads as shown in Fig. 13.38.*

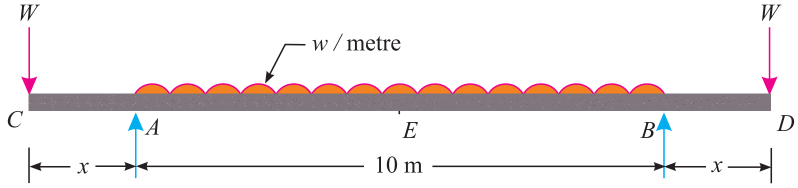


Fig. 13.38

If W = 10 w, what is the overhanging length on each side, such that the bending moment at the middle of the beam, is zero? Sketch the shear force and bending moment diagrams.

SOLUTION. Given : Span (l) = 10 m ; Point loads at C and $D = W$ and uniformly distributed load between A and $B = w$/metre.

Since the beam is symmetrically loaded, therefore, the two reactions (*i.e.*, R_A and R_B) will be equal. From the geometry of the figure, we find that the reaction at A,

$$R_A = R_B = \frac{1}{2}(W + 10\,w + W) = W + 5\,w$$

$$= 10\,w + 5\,w = 15\,w \qquad (\because W = 10\,w)$$

Overhanging length of the beam on each side

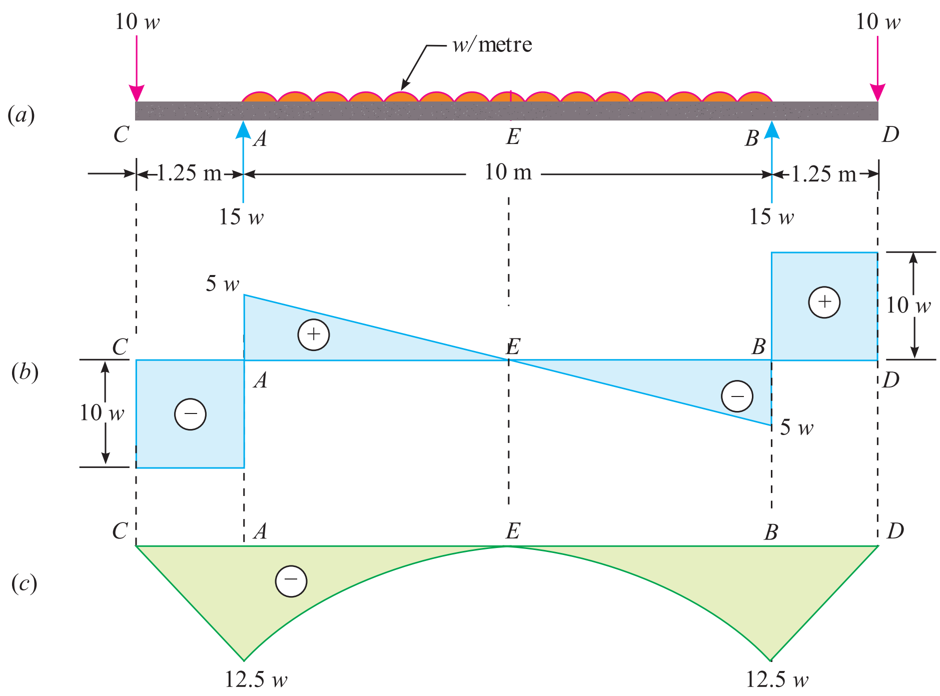


Fig. 13.39

We know that the bending moment at the middle of the beam A,

$$M_E = W(5 + x) + w \times 5 \times \frac{5}{2} - 15\,w \times 5$$

$$= 10\,w\,(5 + x) + 12.5\,w - 75\,w \qquad (\because W = 10\,w)$$

$$= 50\,w + 10\,wx - 62.5\,w$$

$$= 10\,wx - 12.5\,w \qquad ...(i)$$

Since the bending moment at the middle of the beam is zero, therefore equating the above equation to zero,

$$10\,wx - 12.5\,w = 0$$

$$\therefore \qquad x = \frac{12.5}{10} = 1.25 \text{ m} \qquad \textbf{Ans.}$$

Shear force

The shear force diagram is shown in Fig. 13.39 (*b*), and the values are tabulated here:

$$F_C = -10\,w$$
$$F_A = -10\,w + 15\,w = +5\,w$$
$$F_B = +5\,w - 10\,w + 15\,w = +10\,w$$
$$F_D = +10\,w$$

Bending moment

The bending moment diagram is shown in Fig. 13.39 (*c*) and the values are tabulated here:

$$M_C = 0$$
$$M_A = -10\,w \times 1.25 = -12.5\,w$$
$$^{*}M_E = 0 \qquad \text{...(given)}$$
$$M_B = -10\,w \times 1.25 = -12.5\,w$$
$$M_D = 0$$

EXAMPLE 13.18. *A beam of length l carries a uniformly distributed load of w per unit length. The beam is supported on two supports at equal distances from the two ends. Determine the position of the supports, if the B.M., to which the beam is subjected to, is as small as possible. Draw the B.M. and S.F. diagrams for the beam.*

SOLUTION. Given : Total span $= l$; Uniformly distributed load $= w$/unit length and overhanging on both sides $= a$

Let a be the distance of the supports from the ends. The bending moment will be minimum, only if the positive bending moment is equal to the negative bending moment. Since the beam is carrying a uniformly distributed load and the two supports are equally spaced from the ends, therefore the two reactions are equal.

or
$$R_A = R_B = \frac{wl}{2}$$

From the geometry of the figure, we find that the maximum negative bending moment will be at the two supports, whereas the maximum positive bending moment will be at the middle of the beam. Now bending moment at A,

$$M_A = -wa \times \frac{a}{2} = -\frac{wa^2}{2} \qquad \text{...(}i\text{)}$$

and bending moment at the middle of the beam,

$$M_M = R_A\left(\frac{l}{2} - a\right) - \left(\frac{wl}{2} \times \frac{l}{4}\right)$$
$$= \frac{wl}{2}\left(\frac{l}{2} - a\right) - \frac{wl^2}{8} \qquad \text{...(}ii\text{)}$$

Equating (*i*) and (*ii*) and ignoring the nature of M_A,

$$\frac{wa^2}{2} = \frac{wl}{2}\left(\frac{l}{2} - a\right) - \frac{wl^2}{8}$$
$$a^2 = \frac{l^2}{2} - la - \frac{l^2}{4} = \frac{l^2}{4} - la$$

* The moment at E (*i.e.*, M_E) may also be found out as discussed below:
$M_E = (10\,w \times 6.25) + (5\,w \times 2.5) - (15\,w \times 5) = 0$

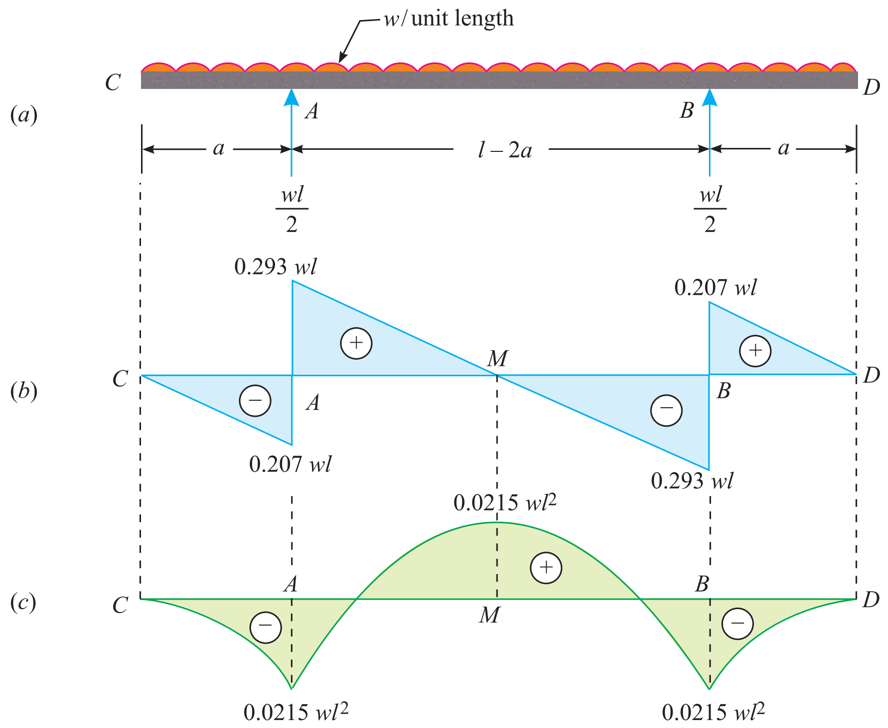


Fig. 13.40

$$\text{or} \qquad a^2 + la - \frac{l^2}{4} = 0$$

Solving it as a quadratic equation for a,

$$a = \frac{-l \pm \sqrt{l^2 + \dfrac{4 \times l^2}{4}}}{2} = \frac{-l \pm \sqrt{2\, l^2}}{2}$$

$$= 0.\,5\,1 + 0.707\, l = 0.207\, l \qquad \text{(Taking + sign)}$$

Shear force

The shear force diagram is shown in Fig. 13.40 (b), and values are tabulated here:

$$F_C = 0$$

$$F_A = 0 - w \times 0.207\, l + 0.5\, wl = +\, 0.293\, wl$$

$$F_M = +\, 0.293\, wl - w \times 0.293\, l = 0$$

$$F_B = 0 - w \times 0.293\, l + 0.5\, wl = +\, 0.207\, wl$$

$$F_D = +\, 0.207\, wl - w \times 0.207\, l = 0$$

Bending moment

The bending moment diagram is shown in Fig. 13.40 (c) and the values are tabulated here:

$$M_C = 0$$

$$M_A = M_B = -\frac{wa^2}{2} = -\frac{w}{2}\,(0.207\, l)^2 = -\, 0.0215\, wl^2$$

$$M_M = -\frac{wl}{2} \times \frac{l}{4} + \frac{wl}{2}\left(\frac{l}{2} - a\right) = -\frac{wl^2}{8} + \frac{wl}{2}\,(0.5\, l - 0.207\, l)$$

$$= -\frac{wl^2}{8} + \frac{wl}{2} \times 0.293\, l = 0.021\,5\, wl^2$$

EXAMPLE 13.19. *A horizontal beam 10 m long is carrying a uniformly distributed load of 1 kN/m. The beam is supported on two supports 6 m apart. Find the position of the supports, so that bending moment on the beam is as small as possible. Also draw the shear force and bending moment diagrams.*

SOLUTION. Given : Total length of beam = 10 m ; Uniformly distributed load (w) = 1 kN/m and span (l) = 6 m

Let a be the distance between the support A and the left end of the beam as shown in Fig. 13.41 (a).

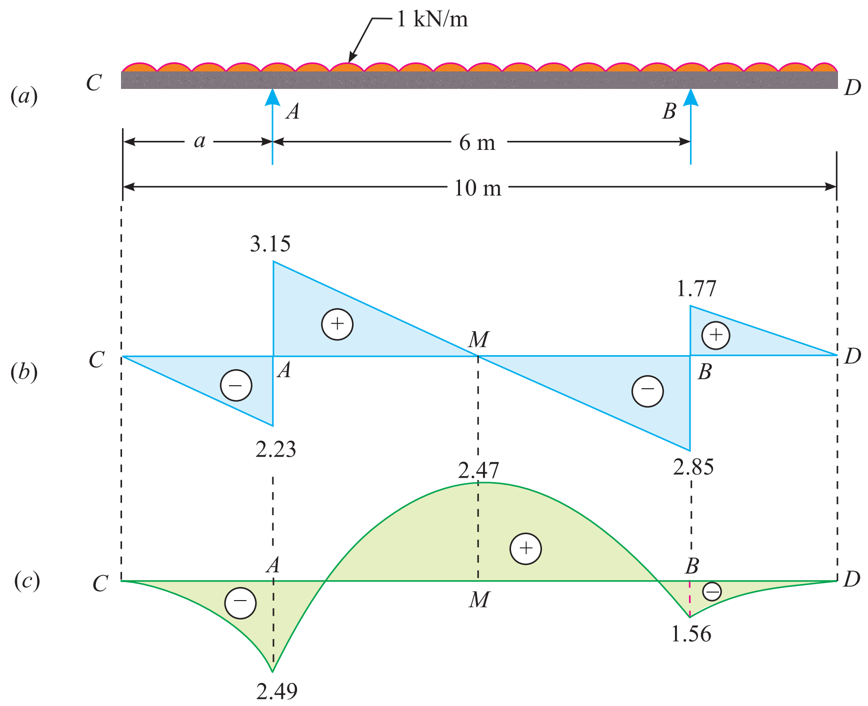


Fig. 13.41

Taking moments about ,

$$R_B \times 6 = 1 \times 10\,(5 - a) = 10\,(5 - a)$$

$$\therefore \quad R_B = \frac{10\,(5-a)}{6} = \frac{5}{3}(5 - a)$$

$$\text{and} \quad R_A = 10 - \frac{5}{3}(5 - a) = \frac{5}{3}(1 + a)$$

From the geometry of the figure, we find that the maximum negative bending moment will be at either of the two supports and the maximum positive bending moment will be in the span AB. Let the maximum positive bending moment be at M at a distance of x from C.

Since the shear force at M is zero, therefore

$$1 \times x - R_A = 0$$

$$\therefore \quad x = R_A = \frac{5}{3}\,(1 + a)$$

We know that the bending moment at A,

$$M_A = -1 \times a \times \frac{a}{2} = -\frac{x^2}{2} \qquad ...(i)$$

and bending moment, where shear force is zero (*i.e.*, at a distance of x from C),

$$M_M = 1 \times x \times \frac{x}{2} + R_A (x - a) = R_A (x - a) - \frac{x^2}{2}$$

$$= \frac{5}{3}(1+a)\left[\frac{5}{3}(1+a) - a\right] - \frac{1}{2}\left[\frac{5}{3}(1+a)\right]^2$$

...($\because$ Substituting the values of R_A and x)

$$= \frac{5}{3}(1+a)\left[\frac{5}{3} + \frac{5a}{3} - a\right] - \frac{25}{18}(1+a)^2$$

$$= \frac{5}{3}(1+a)\frac{5}{3}\left[1 + a - \frac{3a}{5}\right] - \frac{25}{18}(1+a)^2$$

$$= \frac{25}{9}(1+a)\left[1 + \frac{2a}{5}\right] - \frac{25}{18}(1+a)^2$$

$$= \frac{25}{9}(1+a)\left[\left(\frac{5+2a)}{5}\right) - \frac{1}{2}(1+a)\right]$$

$$= \frac{25}{9}(1+a)\left[\frac{10 + 4a - 5 - 5a}{10}\right]$$

$$= \frac{25}{9}(1+a)\left(\frac{5-a}{10}\right)$$

$$= \frac{5}{9}(1+a)\left[\frac{5-a}{2}\right] = \frac{5}{18}(1+a)(5-a)$$

$$= \frac{5}{18}(5 - a + 5a - a^2)$$

$$= \frac{5}{18}(5 + 4a - a^2) \qquad ...(ii)$$

Equating (i) and (ii) and ignoring the nature of M_A,

$$\frac{a^2}{2} = \frac{5}{18}(5 + 4a - a^2) = \frac{25}{18} + \frac{20a}{18} - \frac{5a^2}{18}$$

$\therefore$ $$a^2 = \frac{25}{9} + \frac{20a}{9} - \frac{5a^2}{9}$$

or $$14a^2 - 20a - 25 = 0$$

Solving it as a quadratic equation for a,

$$a = \frac{20 \pm \sqrt{(20)^2 + (4 \times 14 \times 25)}}{2 \times 14} = 2.23 \text{ m}$$

$\therefore$ $$x = \frac{5}{3}(1+a) = \frac{5}{3}(1 + 2.23) = 5.38 \text{ m}$$

Now reaction at B,

$$R_B = \frac{5}{3}(5-a) = \frac{5}{3}(5 - 2.23) = 4.62 \text{ kN}$$

and $$R_A = \frac{5}{3}(1+a) = \frac{5}{3}(1 + 2.23) = 5.38 \text{ kN}$$

Shear force diagram

The shear force diagram is shown in Fig. 13.41 (*b*), and the values are tabulated here:

$$F_C = 0$$
$$F_A = 0 - 1 \times 2.23 + 5.38 = +3.15 \text{ kN}$$
$$F_B = +3.15 - 1 \times 6 + 4.62 = +1.77 \text{ kN}$$
$$F_D = +1.77 - 1.77 = 0$$

Bending moment diagram

The bending moment diagram is drawn in Fig. 13.41 (*c*), and the values are tabulated here:

$$M_C = 0$$
$$M_D = 0$$
$$M_A = -1 \times 2.23 \times \frac{2.23}{2} = -2.49 \text{ kN-m}$$
$$M_M = -1 \times 5.38 \times \frac{5.38}{2} + 5.38 \times 3.15 = 2.47 \text{ kN-m}$$
$$M_B = 1 \times 1.77 \times \frac{1.77}{2} = 1.56 \text{ kN-m}$$

13.17. Load and Bending Moment Diagrams from a Shear Force Diagram

Sometimes, instead of load diagram, a shear force diagram for a beam is given. In such cases, we first draw the actual load diagram and then the bending moment diagram. The load diagram for the beam may be easily drawn by keeping the following points in view:

1. If there is a sudden increase or decrease (*i.e.*, vertical line of the shear force diagram), it indicates that there is either a point load or reaction (*i.e.*, support) at that point.
2. If there is no increase or decrease in shear force diagram between any two points (*i.e.*, the shear force line is horizontal and consists of rectangle), it indicates that there is no loading between the two points.
3. If the shear force line is an inclined straight line between any two points, it indicates that there is a uniformly distributed load between the two points.
4. If the shear force line is a parabolic curve between any two points, it indicates that there is a uniformly varying load between the two points.

After drawing the load diagram, for the beam the bending moment diagram may be drawn as usual.

EXAMPLE 13.20. *The diagram shown in Fig. 13.42 is the shear force diagram in metric units, for a beam, which rests on two supports, one being at the left hand end.*

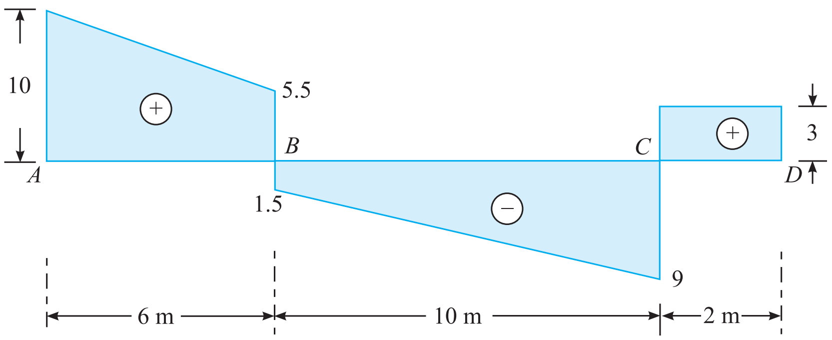


Fig. 13.42

Deduce directly from the shear force diagram, (a) loading on the beam, (b) bending moment at 2 m intervals along the beam and (c) position of the second support. Also draw bending moment diagram for the beam and indicate the position and magnitude of the maximum value on it.

Solution. Given : Total length (l) = 18 m ; Shear force at A = 10 kN and shear force at D = 3 kN.

First of all, let us analyse the shear force diagram as discussed below:

1. *At A*

We see that the shear force increases suddenly from 0 to 10 kN. Therefore there is a support reaction of 10 kN at A.

2. *Between A and B*

We see that the shear force diagram has an inclined straight line between A and B. Therefore the beam is carrying a uniformly distributed load between A and B. We also see that there is a decrease of 10 – 5.5 = 4.5 kN shear force in 6 m length of beam. Therefore the beam carries a uniformly distributed load of 4.5/6 = 3/4 kN/m.

3. *At B*

We see that the shear force has a sudden decrease of 5.5 + 1.5 = 7 kN. Thus there is a point load of 7 kN at B.

4. *Between B and C*

We see that the shear force diagram has an inclined straight line between B and C. Therefore the beam is carrying a uniformly distributed load between B and C. We also see that there is a decrease of 9 – 1.5 = 7.5 kN shear force in 10 m length of beam. Therefore the beam carries a uniformly distributed load of 7.5/10 = 3/4 kN/m.

5. *At C*

We see that the shear force has a sudden increase of 9 + 3 = 12 kN. Thus there is a support reaction of 12 kN at C.

6. *Between C and D*

We see that the shear force diagram has a straight horizontal line between C and D. Therefore there is no load between C and D.

7. *At D*

We see that the shear force decreases suddenly from + 3 kN to 0. Therefore there is a point load of 3 kN at D.

The load diagram is shown in Fig. 13.43 (b).

Bending Moment

Let us calculate bending moments at 2 meters interval along the beam.

$$M_0 = 0$$

$$M_2 = 10 \times 2 - \frac{3}{4} \times 2 \times 1 = 18.5 \text{ kN-m}$$

$$M_4 = 10 \times 4 - \frac{3}{4} \times 4 \times 2 = 34 \text{ kN-m}$$

$$M_6 = 10 \times 6 - \frac{3}{4} \times 6 \times 3 = 46.5 \text{ kN-m}$$

$$M_8 = 10 \times 8 - \frac{3}{4} \times 8 \times 4 - 7 \times 2 = 42 \text{ kN-m}$$

$$M_{10} = 10 \times 10 - \frac{3}{4} \times 10 \times 5 - 7 \times 4 = 34.5 \text{ kN-m}$$

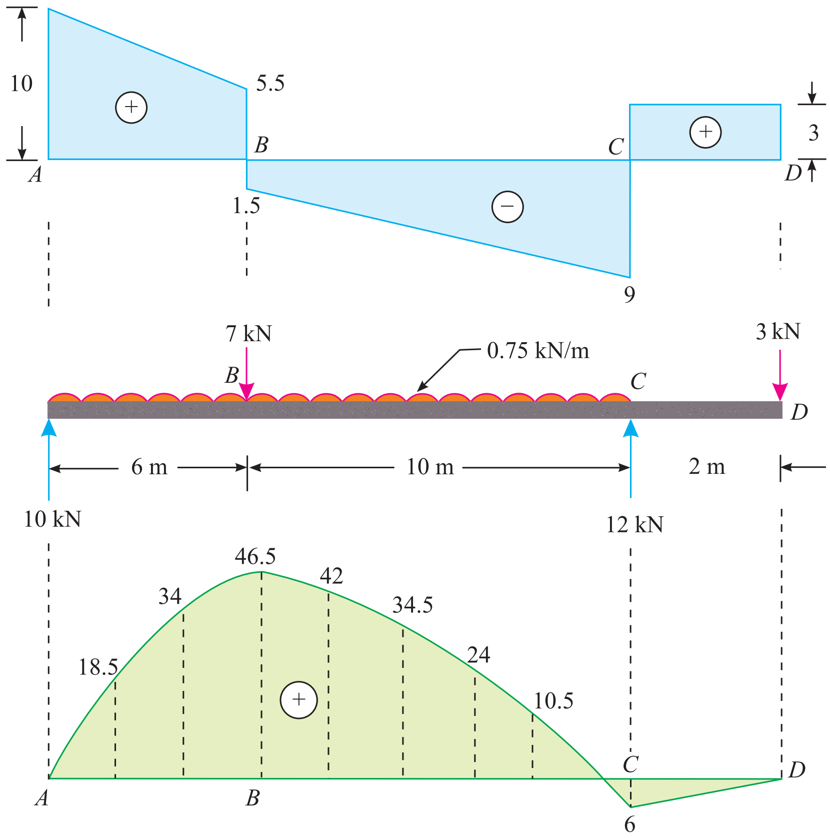


Fig. 13.43

$$M_{12} = 10 \times 12 - \frac{3}{2} \times 12 \times 6 - 7 \times 6 = 24 \text{ kN-m}$$

$$M_{14} = 10 \times 14 - \frac{3}{2} \times 14 \times 7 - 7 \times 8 = 10.5 \text{ kN-m}$$

$$M_{16} = 10 \times 16 - \frac{3}{2} \times 16 \times 8 - 7 \times 10 = -6 \text{ kN-m}$$

$$M_{18} = 0$$

Maximum bending moment

The maximum bending moment, positive or negative will occur at B (*i.e.*, 6 m from A) and C (*i.e.*, 16 m from A) because the shear force changes sign at both the points. But from the bending moment diagram, we see that maximum positive bending moment occurs at B and the maximum negative bending moment at C. Now complete the diagram as shown in Fig. 13.43 (c).

EXAMPLE 13.21. *Figure 13.44 shows the shear force diagram of a loaded beam.*

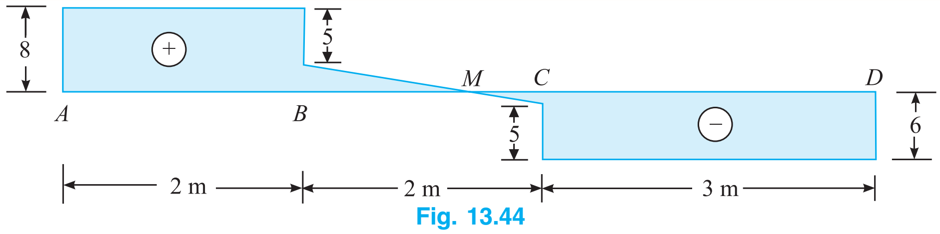


Fig. 13.44

Find the loading on the beam and draw the bending moment diagram.

SOLUTION. Given : Total length (L) = 7 m ; Shear force at A = 8 kN and shear force at D = 6 kN

First of all, let us analyse the shear force diagram as discussed below:

1. *At A*

We see that the shear force increase suddenly from 0 to 8 kN. Therefore there is a support reaction of 8 kN at A.

2. *Between A and B*

We see that shear force diagram has a straight horizontal line between A and B. Therefore there is no load between A and B.

3. *At B*

We see that the shear force has a sudden decrease of $8 - 3 = 5$ kN. Therefore there is a point load of 5 kN at B.

4. *Between B and C*

We see that the shear force diagram has an inclined straight line between B and C. Therefore the beam is carrying a uniformly distributed load between B and C. We also see that there is a decrease of $3 + 1 = 4$ kN in 2 m length of the beam. Therefore the beam is carrying a uniformly distributed load of $4/2 = 2$ kN/m.

5. *At C*

We see that the shear force has sudden decrease of $6 - 1 = 5$ kN. Therefore there is a point load of 5 kN at C.

6. *Between C and D*

We see that the shear force has a straight horizontal line between C and D. Therefore there is no load between C and D.

7. *At D*

We see that the shear force suddenly decreases from – 6 kN to 0. Therefore there is a section of 6 kN at D.

The load diagram is shown in Fig. 13.45.

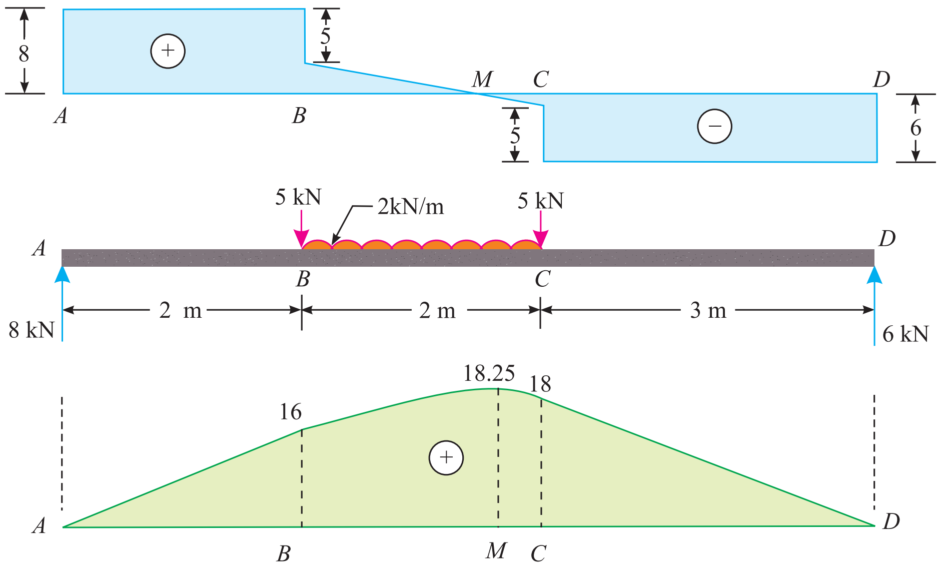


Fig. 13.45

Bending moment diagram

The bending moment diagram is shown in Fig. 13.45 and the values are tabulated here:

$$M_A = 0$$
$$M_B = 8 \times 2 = 16 \text{ kN-m}$$
$$M_C = 6 \times 3 = 18 \text{ kN-m}$$
$$M_D = 0$$

We know that the maximum bending moment will occur at M, where the shear force changes sign. Let x be the distance between B and M. From the geometry of the figure between B and M,

$$\frac{x}{3} = \frac{2-x}{1} \quad \text{or} \quad x = 6 - 3x$$
$$4x = 6 \quad \text{or} \quad x = 1.5 \text{ m}$$

$$\therefore \quad M_M = (8 \times 3.5) - (5 \times 1.5) - (2 \times 1.5 \times \frac{1.5}{2})$$
$$= 18.25 \text{ kN-m}$$

EXAMPLE 13.22. *Shear force diagram for a loaded beam is shown in Fig. 13.46.*

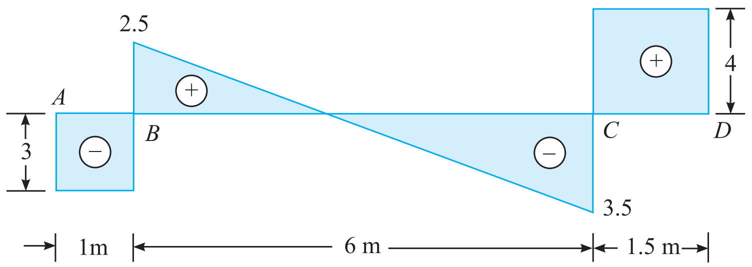


Fig. 13.46

Determine the loading on the beam and bence draw bending moment diagram. Locate the point of contraflexure, if any. All the values are in kilonewtons.

SOLUTION. Given : Total span (L) = 8.5 m ; Shear force at A = – 3 kN and shear force at D = + 4 kN

First of all, let us analyse the shear force diagram discussed below:

1. *At A*

We see that the shear force decreases suddenly from 0 to 3 kN at A. Therefore there is a point load of 3 kN at A.

2. *Between A and B*

We see that the shear force diagram is a straight horizontal line between A and B. Therefore there is no load between A and B.

3. *At B*

We see that the shear force diagram has a sudden increase of 3 + 2.5 = 5.5 kN at B. Thus there is a support reaction of 5.5 kN at B.

4. *Between B and C*

We see that the shear force diagram is an inclined straight line between B and C. Therefore the beam is carrying a uniformly distributed load between B and C. We also see that there is a decrease of 2.5 + 3.5 = 6 kN shear force in 6 m length of beam. Therefore the beam carries a uniformly distributed load of 6/6 = 1 kN/m.

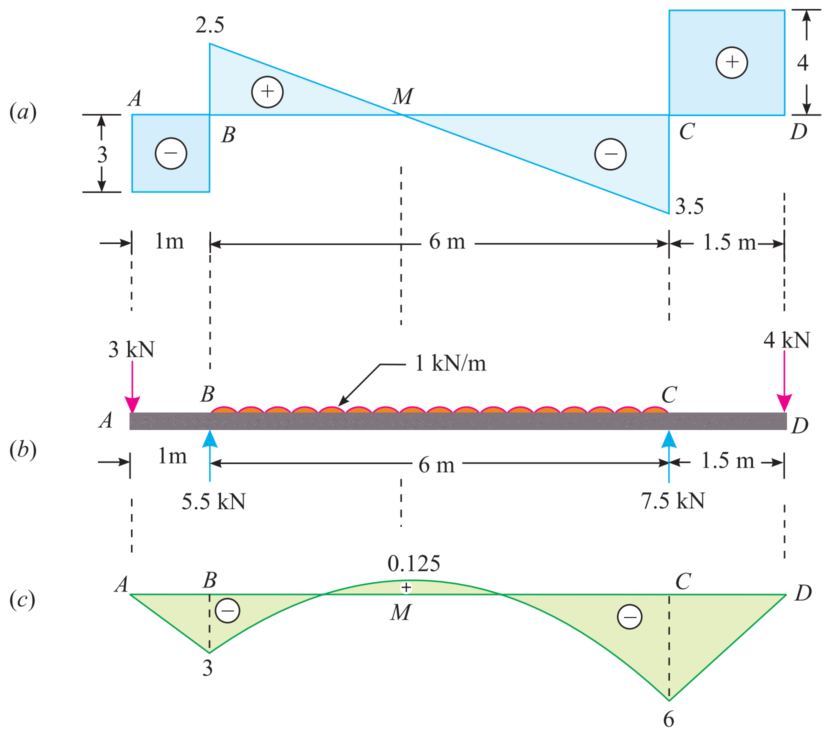


Fig. 13.47

5. *At C*

We see that the shear force diagram has a sudden increase of 3.5 + 4 = 7.5 kN. Thus there is a support reaction of 7.5 kN at *C*.

6. *Between C and D*

We see that the shear force diagram is a straight horizontal line between *C* and *D*. Therefore there is no load between *C* and *D*.

7. *At D*

We see that the shear force decreases suddenly from + 4 kN to 0. Therefore there is a point load of 4 kN at *D*.

The load diagram is shown in Fig. 13.47 (*b*).

Bending moment diagram

The bending moment diagram is shown in Fig. 13.47 (*c*) and the values are tabulated here:

$$M_A = 0$$
$$M_B = -3 \times 1 = -3 \text{ kN-m}$$
$$M_C = -4 \times 1.5 = -6 \text{ kN-m}$$
$$M_D = 0$$

Maximum bending moment

The maximum bending moment, positive or negative will occur at *B*, *M* or *C* because shear force changes sign at all three points. Let *x* be the distance between *B* and *M*. From the geometry of the figure between *B* and *C*,

$$\frac{x}{2.5} = \frac{6-x}{3.5}$$

$$3.5\,x = 15 - 2.5\,x$$

or $$x = 2.5 \text{ m}$$

$\therefore$ $$M_M = -(3 \times 3.5) + (5.5 \times 2.5) - \left(1 \times 2.5 \times \frac{2.5}{2}\right) = 0.125 \text{ kN-m}$$

Thus we see that the maximum positive bending moment occurs at M and maximum negative bending moment occurs at C.

Points of Contraflexures

Let the point of contraflexure be at a distance of x metres from B (it will be between B and C as is seen in the bending moment diagram). We know that bending moment at any section X at a distance of x from B,

$$M_X = -(x+1) + 5.5\,x - 1 \times x \times \frac{x}{2}$$

$$= -3\,x - 3 + 5.5\,x - \frac{x^2}{2} = -\frac{x^2}{2} + 2.5\,x - 3$$

Equating the above equation to zero, we get

$$-\frac{x^2}{2} + 2.5\,x - 3 = 0$$

or $$x^2 - 5\,x + 6 = 0$$

$$x = \frac{5 \pm \sqrt{(5)^2 - 4 \times 6}}{2} = \frac{5 \pm 1}{2} = 2 \text{ m and } 3 \text{ m}$$ **Ans.**

EXERCISE 13.3

1. A beam 6 m long rests on two supports 5 m apart. The right end is overhanging by 1 m. The beam carries a uniformly distributed load of 1.5 kN/m over the entire length of the beam. Draw S.F. and B.M. diagram and find the amount and position of maximum bending moment.
[**Ans.** 4.32 kN-m at 2.4 m from left end]

2. Draw the shear force and bending moment diagrams, for the overhanging beam carrying loads as shown in Fig. 13.48.

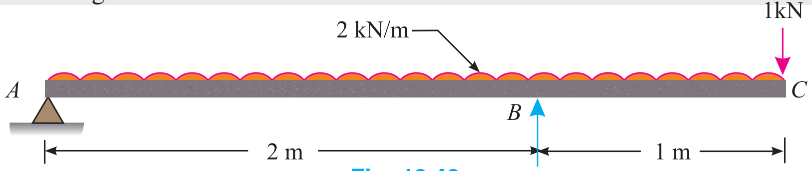


Fig. 13.48

Mark the values of the principal ordinates and locate the point of contraflexure, if any.
[**Ans.** 1 m from A]

3. A beam 10 m long carries load as shown in Fig. 13.49.

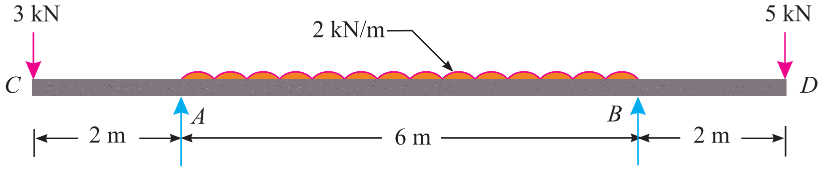


Fig. 13.49

Draw shear force and bending moment diagrams for the beam and determine the points of contraflexures, if any. [**Ans.** 3.62 m and 5.72 m from C]

4. A beam AB 20 metres long, carries a uniformly distributed load 0.6 kN/m together with concentrated loads of 3 kN at left hand end A and 5 kN at right hand-end B as shown in Fig. 13.50.

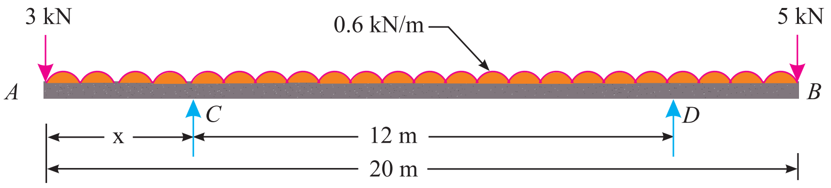


Fig. 13.50

The props are so located that the reaction is the same at each support. Determine the position of the props and draw bending moment and shear force diagrams. Mark the values of the maximum bending moment and maximum shear force. [**Ans.** 5 m ; 17 m]

13.18. Beams Subjected to a Moment

Sometimes, a beam is subjected to a clockwise or anticlockwise moment (or couple) at a section. In such a case, the magnitude of the moment is considered while calculating the reactions. The bending moment at the section of the couple changes suddenly in magnitude equal to that of the couple. This may also be found out by calculating the bending moment separately with the help of both the reactions. Since the bending moment does not involve any load, therefore the shear force does not change at the section of couple.

Notes: **1.** A clockwise moment (called positive moment) causes negative shear force over the beam and positive bending moment at the section. Similarly, an anticlockwise moment (called negative moment) causes positive shear force over the beam and negative bending moment at the section.

2. The bending moment will suddenly increase due to clockwise moment and decrease due to anticlockwise moment at the point of its application when we move from left to right along the beam.

Example 13.23. *A simply supported beam of 5 m span is subjected to a clockwise moment of 15 kN-m at a distance of 2 m from the left end as shown in Fig. 13.51.*

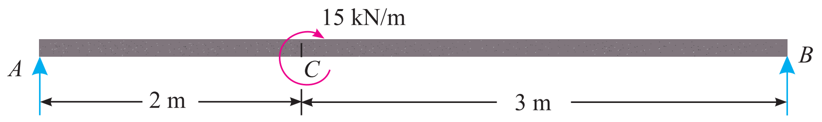


Fig. 13.51

Draw the shear force and bending moment diagrams for the beam.

Solution. Given : Span (l) = 5 m and couple at C (μ) = 15 kN-m

We know that the tendency of the moment is to uplift the beam from its support A and to depress it at its support B. It is thus obvious that the reaction at A will be downwards and that at B will be upwards as shown in Fig. 13.52 (a).

Taking moments about ,

$$R_B \times 5 = 15 \qquad \text{...(Since the beam is subjected to moment only)}$$

$$\therefore \qquad R_B = \frac{15}{5} = 3 \text{ kN} \qquad \text{(upwards)}$$

Since there is no external loading on the beam, therefore the reaction at A will be of the same magnitude but in opposite direction. Therefore reaction at A,

$$R_A = 3 \text{ kN} \qquad \text{(downwards)}$$

Shear force diagram

We know the shear force is constant from A to B and is equal to – 3 kN (because of downward reaction at A or upward reaction at B) as shown in Fig. 13.52 (b).

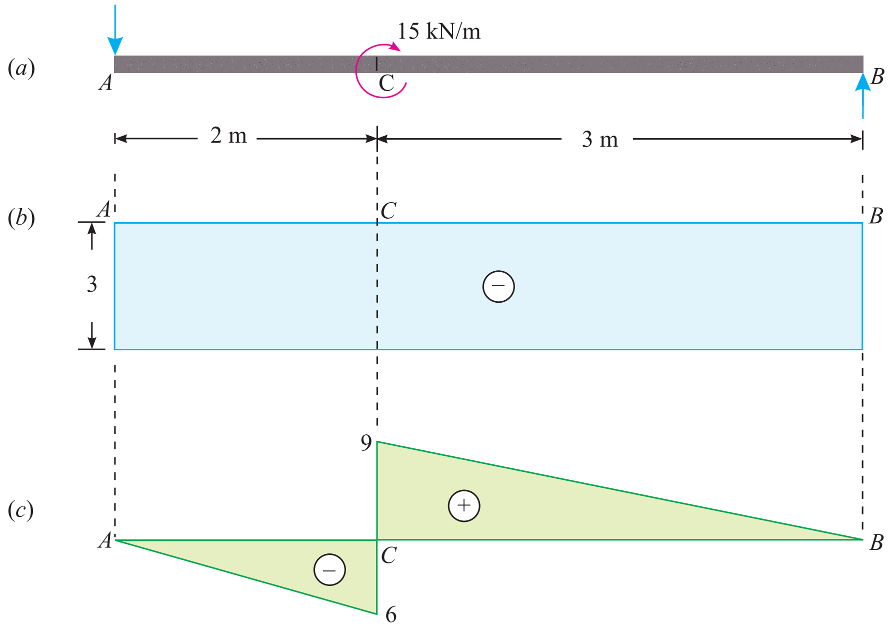


Fig. 13.52

Bending moment diagram

The bending moment diagram is shown in Fig. 13.52 (c) and the values are tabulated here:

$$M_A = 0$$

$$M_B = 0$$

Bending moment just on the left side of C,

$$= R_A \times 2 = -3 \times 2 = -6 \text{ kN-m}$$

and bending moment just on the right side of C*

$$= -6 + 15 = +9 \text{ kN-m}$$

EXAMPLE 13.24. *A simply supported beam of span 2.5 m is subjected to a uniformly distributed load and a clockwise couple as shown in Fig. 13.53.*

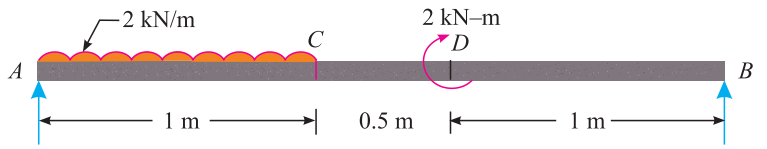


Fig. 13.53

Draw the shear force and bending moment diagrams for the beam.

SOLUTION. Given : Span (l) = 2.5 m ; Uniformly distributed load between A and C (w)=2 kN/m and couple at D (μ) = 2 kN-m

* At C, the bending moment will suddenly increase due to clockwise moment at C. The bending moment just on the right side C may also be found out from the reaction R_B, *i.e.*, = $R_B \times 3 = +3.0 \times 3 = +9.0$ kN-m

First of all, let us find out the reactions R_A and R_B. Taking moments about A and equating the same,

$$R_B \times 2.5 = \left(2 \times 1 \times \frac{1}{2}\right) + 2 = 3 \qquad \text{...(+ 3 due to clockwise moment)}$$

$\therefore$ $R_B = 3/2.5 = 1.2$ kN

and RA $= (2 \times 1) - 1.2 = 0.8$ kN

Shear force diagram

The shear force diagram is shown in Fig. 13.54 (*b*) and the values are tabulated here:

$$F_A = +R_A = +8 \text{ kN}$$
$$F_C = +0.8 - (2 \times 1) = -1.2 \text{ kN}$$
$$F_B = -1.2 \text{ kN}$$

Bending moment diagram

The bending moment diagram is shown in Fig. 13.54 (*c*) and the values are tabulated here:

$$M_A = 0$$
$$M_C = (0.8 \times 1) - (2 \times 1 \times 0.5) = -0.2 \text{ kN-m}$$
$$M_D = (0.8 \times 1.5) - (2 \times 1 \times 1) = -0.8 \text{ kN-m} \qquad \text{...(With the help of } R_A)$$
$$= 1.2 \times 1 = 1.2 \text{ kN-m} \qquad \text{...(With the help of } R_B)$$

We know that maximum bending moment will occur either at E where shear force changes sign or at D due to couple. Let x be the distance between A and E. From the geometry of the figure between A and C, we find that

$$\frac{x}{0.8} = \frac{1-x}{1.2}$$

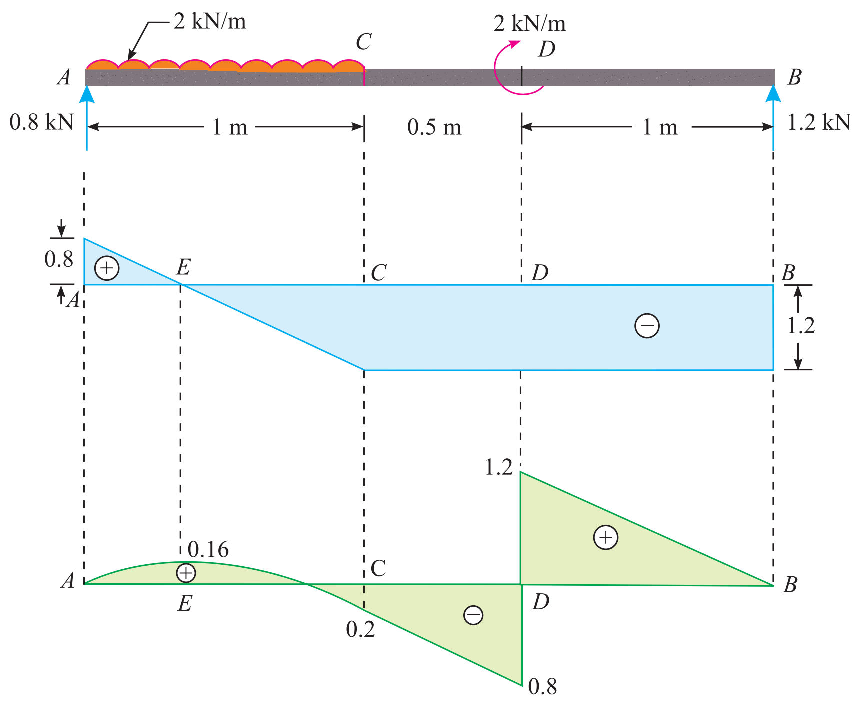


Fig. 13.54

or $$1.2\,x = 0.8 - 0.8\,x$$

$$2\,x = 0.8 \qquad \text{or} \qquad x = \frac{0.8}{2} = 0.4 \text{ m}$$

$$\therefore \qquad M_E = (0.8 \times 0.4) - \left(2 \times 0.4 \times \frac{0.4}{2}\right) = +\,0.16 \text{ kN-m}$$

From the above two values of M_D, we find that it will suddenly increase from – 0.8 kN-m to + 1.2 kN-m due to the clockwise moment of 2 kN-m,

$$M_B = 0$$

EXAMPLE 13.25. *A simply supported beam 5 metres long carries a load of 10 kN on a bracket welded to the beam as shown in Fig. 13.55.*

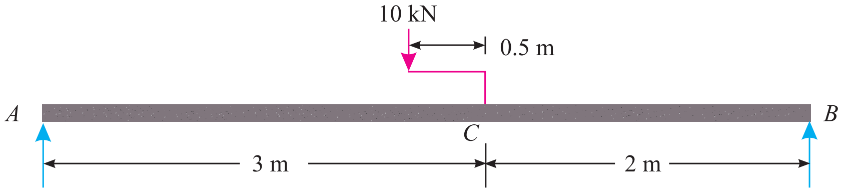


Fig. 13.55

Draw the shear force and bending moment diagrams for the beam.

SOLUTION. Given : Span (l) = 5 m and load on the bracket at C = 10 kN.

It will be interesting to know that the 10 kN load, applied on the bracket will have the following two effects:

1. Vertical load of 10 kN at C,
2. An anticlockwise couple of moment equal to 10 × 0.5 = 5 kN-m at C.

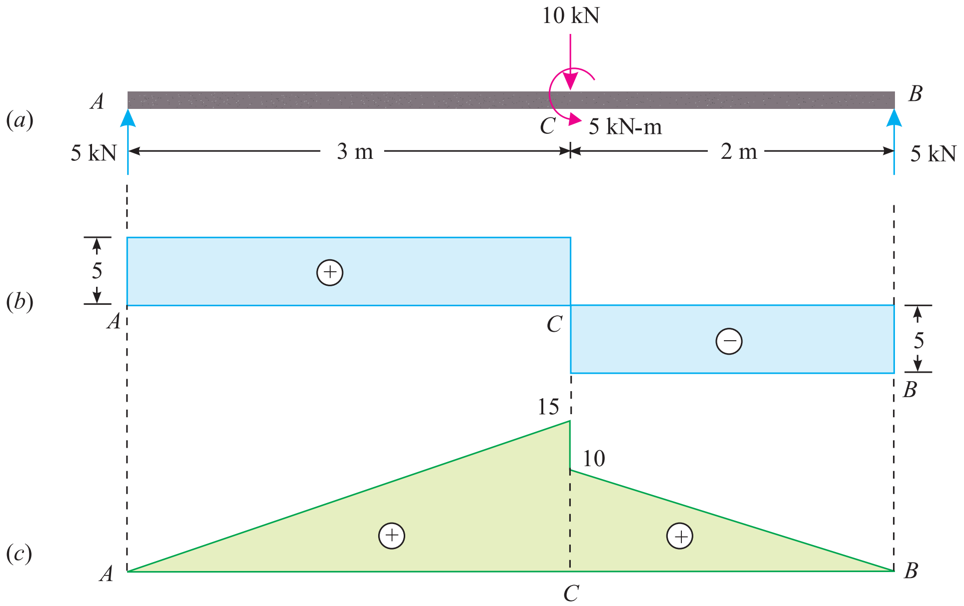


Fig. 13.56

Now the shear force and bending moment diagrams should be drawn by combining the above two mentioned effects as shown in Fig. 13.56 (*a*). First of all, let us find out the reactions R_A and R_B. Taking moments about A and equating the same,

$$R_B \times 5 = (10 \times 3) - 5 = 25 \quad \text{...(– 5 due to anticlockwise moment)}$$

$\therefore$ $$R_B = 25/5 = 5 \text{ kN}$$

and $$R_A = 10 - 5 = 5 \text{ kN}$$

Shear force diagram

The shear force diagram is shown in Fig. 13.56 (*b*) and the values are tabulated here:

$$F_A = +R_A = +5 \text{ kN}$$
$$F_C = +5 - 10 = -5 \text{ kN}$$
$$F_B = -5 \text{ kN}$$

Bending moment diagram

The bending moment diagram is shown in Fig. 13.56 (*c*) and the values are tabulated here:

$$M_A = 0$$
$$M_C = 5 \times 3 = 15 \text{ kN-m} \quad \text{...(With the help of } R_A\text{)}$$
$$= 5 \times 2 = 10 \text{ kN-m} \quad \text{...(With the help of } R_B\text{)}$$
$$M_B = 0$$

From the above two values of M_C we find that it will suddenly decrease from 15 kN-m to 10 kN-m due to the anticlockwise moment of 5 kN-m.

EXAMPLE 13.26. *A beam is loaded as shown in Fig. 13.57.*

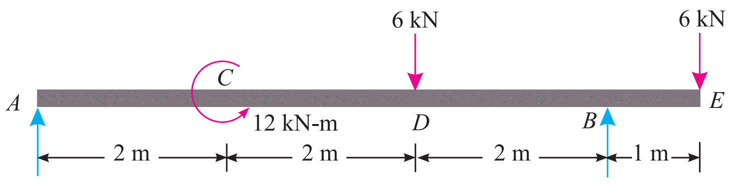


Fig. 13.57

Construct the shear force and bending moment diagrams for the beam and mark the values of the important ordinates.

SOLUTION. Given : Span (l) = 7 m ; Couple at C (μ) = 12 kN-m ; Point load at D (W_1) = 6 kN and point load at E (W_2) = 6 kN

Taking moments about A,

$$R_B \times 6 = (6 \times 4) + (6 \times 7) - 12 = 54$$
...(– 12 due to anticlockwise moment)

$$R_B = \frac{54}{6} = 9 \text{ kN}$$

$\therefore$ $$R_A = (6 + 6) - 9 = 3 \text{ kN}$$

Shear force diagram

The shear force diagram is shown in Fig. 13.58 (*b*) and the values are tabulated here:

$$F_A = +3 \text{ kN}$$

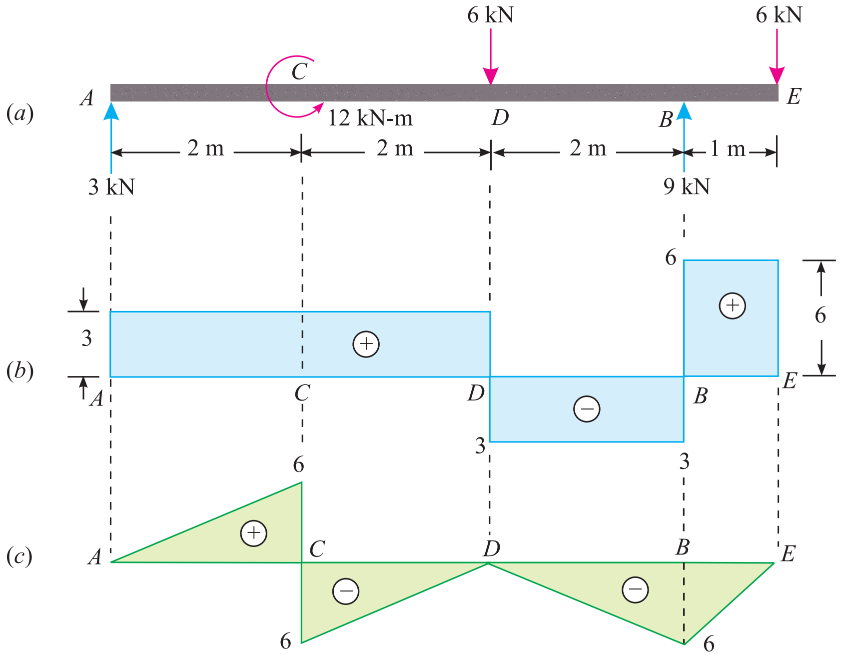


Fig. 13.58

$$F_D = +3 - 6 = -3 \text{ kN} \qquad \text{...(With the help of } R_A\text{)}$$
$$F_B = -3 + 9 = +6 \text{ kN}$$
$$F_E = +6 \text{ kN}$$

Bending moment diagram

The bending moment diagram is shown in Fig. 13.58 (*c*) and the values are tabulated here:

$$M_A = 0$$
$$M_C = 3 \times 2 = 6 \text{ kN-m}$$
$$M_D = 3 \times 4 - 12 = 0$$
$$M_B = -6 \times 1 = -6 \text{ kN-m}$$
$$M_E = 0$$

At *C*, the bending moment will suddenly decrease from 6 kN-m to 6 – 12 = – 6 kN-m because of anticlockwise couple as shown in Fig. 13.58 (*c*).

13.19. Beams Subjected to Inclined Loads

In the previous articles, we have been discussing the cases, when the load used to act at right angles to the axis of the beam. But in actual practice, there may be cases when a beam is subjected to inclined loads. These inclined loads are resolved at right angles and along the axis of the beam. A little consideration will show that the transverse components (*i.e.*, components, which are resolved at right angles to the axis of the beam) will cause shear force and bending moments. The axial components (*i.e.*, components, which are resolved along the axis of the beam) will cause thrust *i.e.*, pulls or pushes in the beam, depending upon its end position.

In such cases, one end of the beam is hinged, whereas the other is simply supported or supported on rollers. The hinged end will be subjected to horizontal thrust equal to the unbalanced horizontal force of the axial components of the inclined loads. In such cases, like shear force and bending moment diagrams, an axial force diagram is drawn, which represents the horizontal thrust. The general practice, to draw the axial force diagram is that the tensile force is taken as positive, whereas the compressive force as negative.

EXAMPLE 13.27. *Analyse the beam shown in Fig. 13.59 and draw the bending moment and shear force diagrams.*

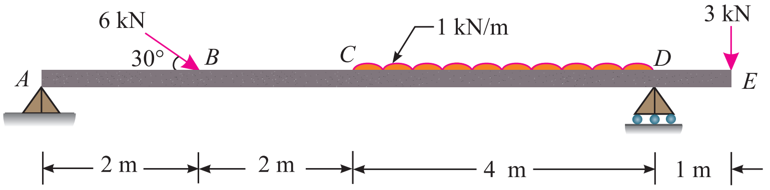


Fig. 13.59

Locate the points of contraflexure, if any.

SOLUTION. Given : Span $l = 9$ m ; Inclined load at $B = 6$ N ; Uniformly distributed load between C and D $(w) = 1$ kN/m and point load at $E = 3$ kN.

Resolving vertically the force of 6 kN at B

$$= 6 \sin 30° = 6 \times 0.5 = 3 \text{ kN}$$

and now resolving horizontally the force of 6 kN at B

$$= 6 \cos 30° = 6 \times 0.866 = 5.196 \text{ kN}$$

*Taking moments about A, $R_D \times 8 = (3 \times 9) + (1 \times 4 \times 6) + (3 \times 2) = 57$

$$\therefore \quad R_D = \frac{57}{8} = 7.125 \text{ kN}$$

and $\quad R_A = (3 + 4 + 3) - 7.125 = 2.875$ kN

The load diagram and reactions are shown in Fig. 13.60 (*a*).

Shear force diagram

The shear force diagram is shown in Fig. 13.60 (*b*) and the values are tabulated here:

$$F_A = +2.875 \text{ kN}$$
$$F_B = +2.875 - 3 = -0.125 \text{ kN}$$
$$F_C = -0.125 \text{ kN}$$
$$F_D = -0.125 - (1 \times 4) + 7.125 = +3 \text{ kN}$$
$$F_E = +3 \text{ kN}$$

Bending moment diagram

The bending moment diagram is shown in Fig. 13.60 (*c*) and the values are tabulated here:

$$M_A = 0$$
$$M_B = 2.875 \times 2 = 5.75 \text{ kN-m}$$
$$M_C = (2.875 \times 4) - (3 \times 2) = 5.5 \text{ kN-m}$$
$$M_D = -3 \times 1 = -3 \text{ kN-m}$$

* The moment of axial component *i.e.*, horizontal component of the 6 kN force will have no moment about A.

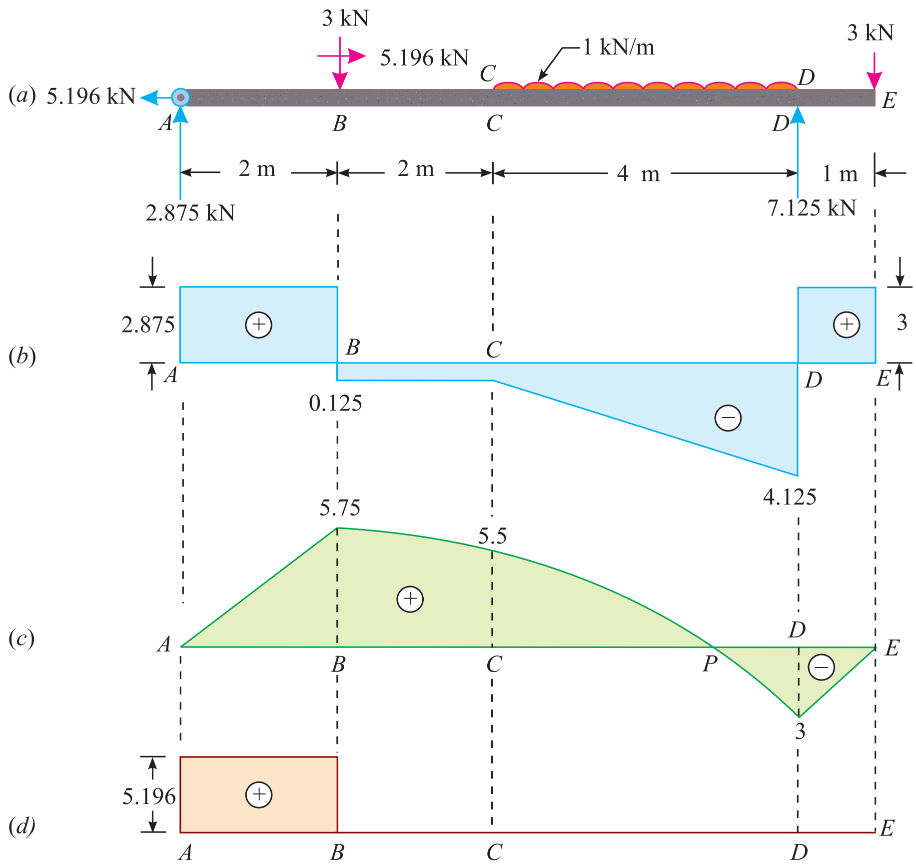


Fig. 13.60

Point of contraflexure

Let the point of contraflexure (*P*) be at a distance of *x* from *D* (It will be between *C* and *D* as is seen in the bending moment diagram). We know that the bending moment at any section *X* in *CD* at a distance *x* from *D*,

$$M_X = 3\,(x+1) + \left(1 \times x \times \frac{x}{2}\right) - 7.125\,x$$

Equating the above equation to zero,

$$3\,(x+1) + \frac{x^2}{2} - 7.125\,x = 0$$

$$3\,x + 3 + \frac{x^2}{2} - 7.125\,x = 0$$

$$\frac{x^2}{2} - 4.125\,x + 3 = 0$$

$$x^2 - 8.25\,x + 6 = 0$$

Solving it as a quadratic equation for *x*,

$$x = \frac{8.25 \pm \sqrt{(8.25)^2 - (4 \times 6)}}{2} = 0.8 \text{ m} \qquad \textbf{Ans.}$$

Axial force diagram

From the load diagram, we see that horizontal reaction at A (being a hinged end) is equal to 5.196 kN (←). Therefore the section AB of the beam is subjected to an axial tensile force (A_{AB}) of 5.196 kN. The beam from B to E is not subjected to any axial force. The axial force diagram is drawn in Fig. 13.60 (d).

EXAMPLE 13.28. *A horizontal beam AB 6 m long is hinged at A and freely supported at B. The beam is loaded as shown in Fig. 13.61.*

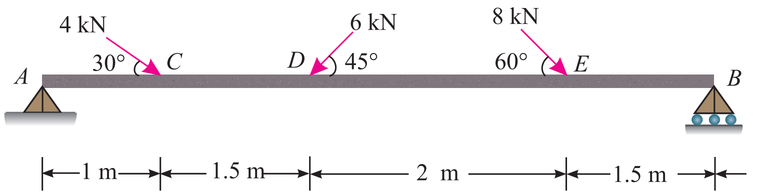


Fig. 13.61

Draw the shear force, bending moment and thrust diagrams for the beam.

SOLUTION. Given : Span (l) = 6 m ; Inclined load of C = 4 kN ; Inclined load of D = 6 kN and inclined load of E = 8 kN.

Resolving vertically the force of 4 kN at C

$$= 4 \sin 30° = 4 \times 0.5 = 2 \text{ kN}$$

and now resolving horizontally the force of 4 kN at C

$$= 4 \cos 30° = 4 \times 0.866 = 3.464 \text{ kN}$$

Similarly, resolving vertically the force of 6 kN at D

$$= 6 \sin 45° = 6 \times 0.707 = 4.242 \text{ kN}$$

and now resolving horizontally the force of 6 kN at D

$$= 6 \cos 45° = 6 \times 0.707 = 4.242 \text{ kN}$$

Similarly, resolving vertically the force of 8 kN at E

$$= 8 \sin 60° = 8 \times 0.866 = 6.928 \text{ kN}$$

and now resolving horizontally the force of 8 kN at E

$$= 8 \cos 60° = 8 \times 0.5 = 4 \text{ kN } (\rightarrow)$$

Taking moments about A,

$$R_B \times 6 = (2 \times 1) + (4.242 \times 2.5) + 6.928 \times 4.5 = 43.78$$

$$\therefore \quad R_B = \frac{43.78}{6} = 7.3 \text{ kN}$$

and

$$R_A = 2 + 4.242 + 6.928 - 7.3 = 5.87 \text{ kN}$$

The load diagram and reactions are shown in Fig. 13.62 (a).

Shear force diagram

The shear force diagram is shown in Fig. 13.62 (b) and the values are tabulated here:

$$F_A = +5.87 \text{ kN}$$
$$F_C = +5.87 - 2 = +3.87 \text{ kN}$$
$$F_D = +3.87 - 4.242 = -0.372 \text{ kN}$$
$$F_E = -0.372 - 6.928 = -7.3 \text{ kN}$$
$$F_B = -7.3 + 7.3 = 0$$

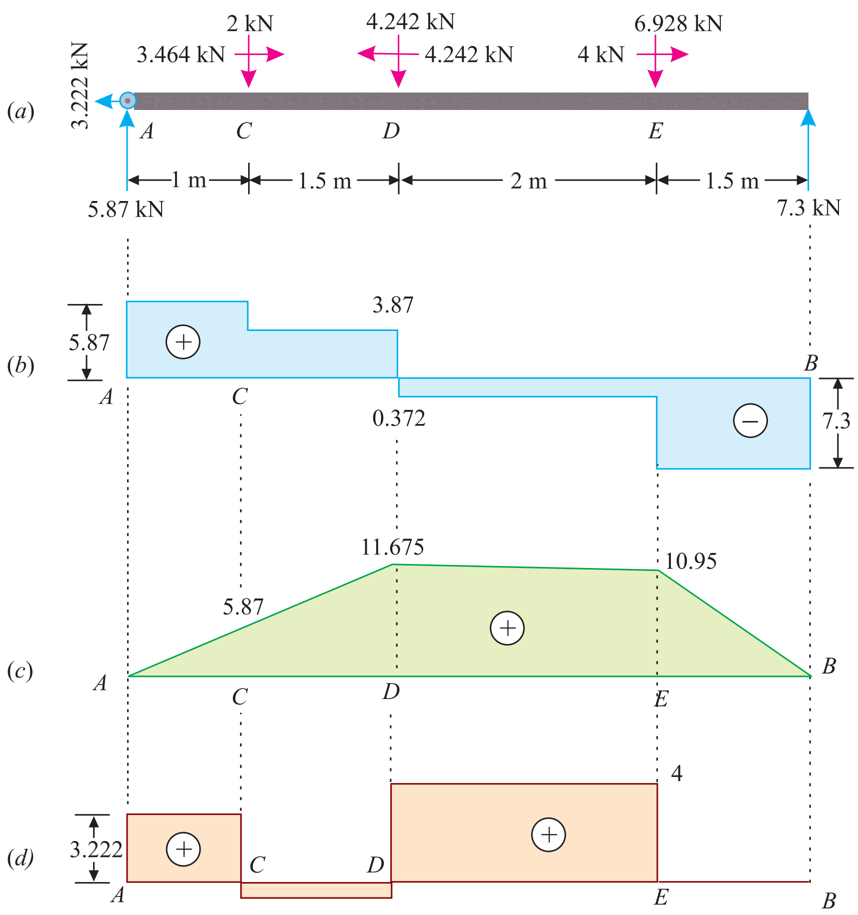


Fig. 13.62

Bending moment diagram

The bending moment diagram is shown in Fig. 13.62 (*c*) and the values are tabulated here:

$$M_A = 0$$
$$M_C = 5.87 \times 1 = 5.87 \text{ kN-m}$$
$$M_D = 5.87 \times 2.5 - 2 \times 1.5 = 11.675 \text{ kN-m}$$
$$M_E = 7.3 \times 1.5 = 10.95 \text{ kN-m}$$
$$M_B = 0$$

Maximum bending moment

It will occur at *D*, where shear force changes sign. Thus we see that maximum bending moment occurs at *D*.

Axial force diagram

From the load diagram, we see that the horizontal reaction at *A* (being a hinged end) is $\overleftarrow{3.464} + \overleftarrow{4.0} - \overrightarrow{4.242} = 3.222$ kN (⟵) The axial force diagram is shown in Fig. 13.62 (*d*) and the values are tabulated here:

$$A_{AC} = 3.222 \text{ kN (Tensile)}$$

$$A_{CD} = 3.464 - 3.222 = 0.242 \text{ kN (Compressive)}$$
$$A_{DE} = 4.242 - 0.242 = 4 \text{ kN (Tensile)}$$
$$A_{EB} = 0$$

13.20. Shear Force and Bending Moment Diagrams for Inclined Beams

In the previous articles, we have discussed the cases of horizontal beams, subjected to various types of loadings. But sometimes, we come across inclined beams or members (such as ladders etc.) and carrying vertical loads. In such cases, the given loads are resolved at right angles and along the axis of the beam. The beam is further analysed in the same manner as a beam is subjected to inclined loads. The horizontal and vertical reactions at the two supports of the inclined beam are found out from the simple laws of statics.

EXAMPLE 13.29. *A ladder AB 5 m long, weighing 500 N/m, rests against a smooth wall and on a rough floor as shown in Fig. 13.63.*

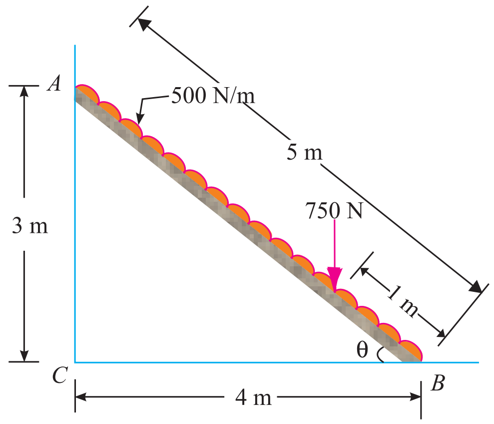


Fig. 13.63

Find the reactions at A and B and construct the shear force, bending moment and axial thrust diagrams for the ladder.

SOLUTION. Given : Span (l) = 5 m ; Uniformly distributed load (w) = 500 N/m and point load at D = 750 N.

From the geometry of the figure, we find that

$$\tan \theta = \frac{3}{4} = 0.75$$

$$\therefore \quad \sin \theta = \frac{3}{5} = 0.6 \quad \text{and} \quad \cos \theta = \frac{4}{5} = 0.8$$

R_A and R_B = Normal reactions at the wall and floor,

R_f = *Frictional resistance at the floor.

Equating the vertical and horizontal forces,

$$R_B = (500 \times 5) + 750 = 3250 \text{ N}$$

and $$R_A = R_f$$

Taking moments about B,

$$R_A \times 3 = (500 \times 5 \times 2) + (750 \times 0.8) = 5600$$

* Since the wall is smooth, therefore there is no frictional resistance at the wall.

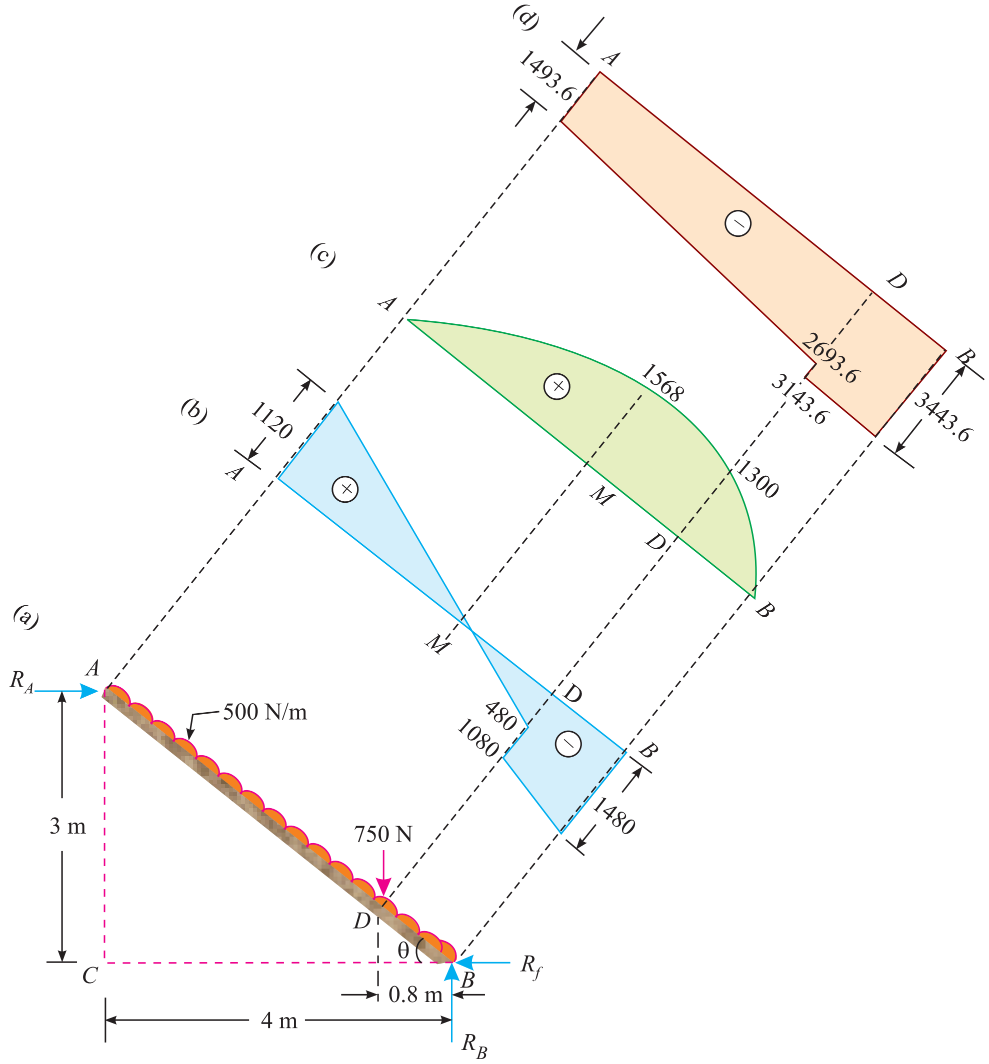


Fig. 13.64

$$\therefore \qquad R_A = R_f = \frac{5600}{3} = 1867 \text{ N}$$

Resolving the reaction R_A at A along the beam

$$= R_A \cos\theta = 1867 \times 0.8 = 1493.6 \text{ N}$$

and now resolving the reaction R_A at right angles to the beam

$$= R_A \sin\theta = 1867 \times 0.6 = 1120 \text{ N}$$

Similarly, resolving the reactions R_B and R_f at B along the beam

$$= R_B \sin\theta + R_f \cos\theta$$

$$= 3250 \times 0.6 + 1867 \times 0.8 = 3443.6 \text{ N}$$

and now resolving the reactions R_B and R_f at right angles to the beam

$$= R_B \cos\theta - R_f \sin\theta$$

$$= 3250 \times 0.8 - 1867 \times 0.6 = 1480 \text{ N}$$

Resolving the force 750 N at D along the beam

$$= 750 \sin \theta = 750 \times 0.6 = 450 \text{ N}$$

and now resolving this force 750 N at right angle to the beam

$$= 750 \cos \theta = 750 \times 0.8 = 600 \text{ N}$$

Resolving the weight of ladder 500 N/m along the beam

$$= 500 \sin \theta = 500 \times 0.6 = 300 \text{ N/m}$$

and now resolving this weight of 500 N/m at right angles to the beam

$$= 500 \cos \theta = 500 \times 0.8 = 400 \text{ N/m}$$

Shear force

The shear force diagram is shown in Fig. 13.64 (*b*) and the values are tabulated here:

$$F_A = +1120 \text{ N}$$

$$F_D = +1120 - (400 \times 4) - 600 = -1080 \text{ N}$$

$$F_B = -1080 - (400 \times 1) + 1480 = 0$$

Bending moment

The bending moment diagram is shown in Fig. 13.64 (*c*) and the values are tabulated here:

$$M_A = 0$$

$$M_D = 3250 \times 0.8 - 1867 \times 0.6 - 400 \times 1 \times 0.5 \text{ N}$$

$$= 1279.8 \text{ N}$$

$$M_B = 0$$

Maximum bending moment

It will occur at M, where shear force changes sign. Let x be the distance between D and M. From the geometry of the figure, distance between A and D, we find that

$$\frac{x}{1120} = \frac{4-x}{480}$$

or $$480\,x = 4480 - 1120\,x$$

$$x = 2.8$$

$\therefore$ $$M_M = 1120 \times 2.8 - 400 \times 2.8 \times \frac{2.8}{2} = 1568 \text{ N}$$

Axial force diagram

The axial force diagram as shown in Fig. 13.64 (*d*) and the values are tabulated here:

$$P_A = -1493.6 \text{ N}$$

$$P_D = -1493.6 - (300 \times 4) - 450 \text{ N}$$

$$= -3143.6 \text{ N}$$

$$P_B = -3143.6 - (300 \times 1) = -3443.6 \text{ N}$$

EXERCISE 13.4

1. A simply supported beam AB of 4 m span is subjected to a clockwise moment of 20 kN-m at its centre. Draw the S.F. and B.M. diagrams. [**Ans.** $R_A = R_B = 5$ kN ; M = 10 kN-m]
2. A simply supported beam 7.5 m long is subjected to a couple of 30 kN-m in an anticlockwise direction at a distance of 5.5 m from the left support. Draw the S.F. and B.M. diagrams for the beam. [**Ans.** $R_A = R_B = 4$ kN ; M = – 22 kN-m ; + 8 kN-m]
3. Analyse the beam subjected to the moment and uniformly distributed load as shown in Fig. 13.65.

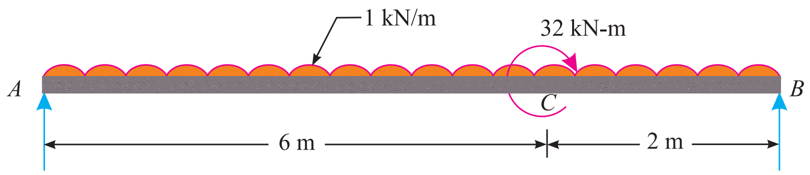


Fig. 13.65

Draw the moment and bending diagrams. [**Ans.** M_{max} = – 18.0 kN.m at C]

4. Calculate the reactions at A and B for the beam shown in Fig. 13.66 and draw the bending moment diagram and shear force diagram. $\left[\textbf{Ans. } \frac{4W}{3} ; \frac{2W}{2}\right]$

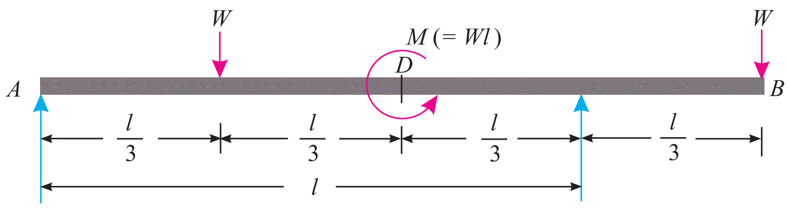


Fig. 13.66

5. Analyse the beam shown in Fig. 13.67.

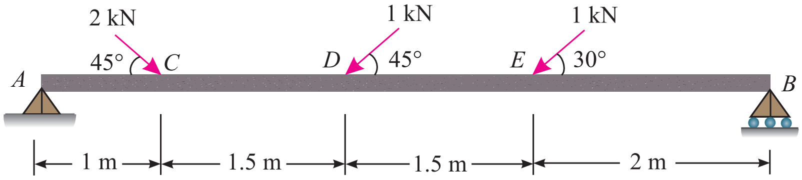


Fig. 13.67

Draw the shear force, bending moment and thrust diagrams.

[**Ans.** = 2.09 kN ; $R_B = 1.53$ kN ; $M_C = 2.09$ kN-m ; $M_D = 3.11$ kN-m ; $M_A = 3.06$ kN-m ; $P_A = -1.893$ kN ; $P_C = -3.307$ kN ; $P_D = 2.6$ kN ; $P_E = -2.6$ kN]

QUESTIONS

1. Define the terms shear force and bending moment.
2. Discuss the utility of shear force and bending moment diagrams.
3. Explain briefly the relationship between shear force and bending moment at a section.

4. How will you determine the maximum bending moment in a simply supported beam?
5. What do you understand by the term, ‘point of contraflexture’?
6. Describe the effect of a couple on the S.F. and B.M. diagram of a beam.
7. Explain the procedure adopted for analysing simply supported beam subjected to inclined loads.

OBJECTIVE TYPE QUESTIONS

1. If a cantilever beam is subjected to a point load at its free end, then the shear force under the point load is
   (*a*) zero (*b*) less than the load
   (*c*) equal to the load (*d*) more than the load.
2. The bending moment at the free end of a cantilever beam carrying any type of load is
   (*a*) zero (*b*) minimum (*c*) maximum (*d*) equal to the load.
3. The B.M. at the centre of a simply supported beam carrying a uniformly distributed load is
   (*a*) $w \cdot l$ (*b*) $\frac{wl}{2}$ (*c*) $\frac{wl^2}{4}$ (*d*) $\frac{wl^2}{8}$
   When w = Uniformly distributed load and
   l = Span of the beam.
4. When shear force at a point is zero, then bending moment at that point will be
   (*a*) zero (*b*) minimum (*c*) maximum (*d*) infinity.
5. The point of contraflexure is a point where
   (*a*) shear force changes sign (*b*) bending moment changes sign
   (*c*) shear force is maximum (*d*) bending moment is maximum.

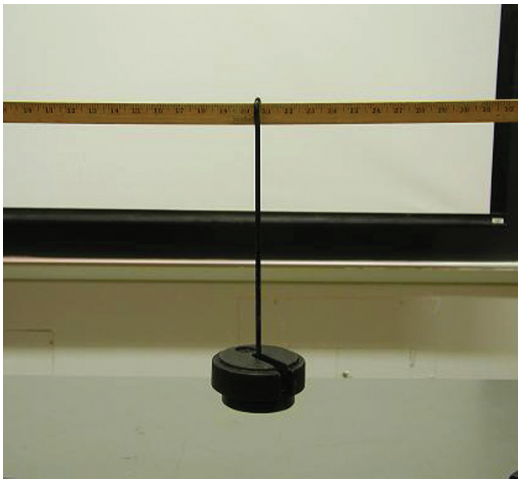
ANSWERS

1. (*c*) **2.** (*a*) **3.** (*d*) **4.** (*c*) **5.** (*b*)

Chapter 14

Bending Stresses in Simple Beams

Contents



14.1. Introduction

We have already discussed in Chapter 13 that the bending moments and shearing forces are set up at all sections of a beam, when it is loaded with some external loads. We have also discussed the methods of estimating the bending moments and shear forces at various sections of the beams and cantilevers.

As a matter of fact, the bending moment at a section tends to bend or deflect the beam and the internal stresses resist its bending. The process of bending stops, when every cross-section sets up full resistance to the bending moment. The *resistance, offered by the internal stresses, to the

* The resistance offered by the internal stresses to the shear force is called shearing stresses. It will be discussed in the next chapter.

bending, is called bending stress, and the relevant theory is called the theory of simple bending.

14.2. Assumptions in the Theory of Simple Bending

The following assumptions are made in the theory of simple bending:

1. The material of the beam is perfectly homogeneous (*i.e.*, of the same kind throughout) and isotropic (*i.e.*, of equal elastic properties in all directions).
2. The beam material is stressed within its elastic limit and thus, obeys Hooke's law.
3. The transverse sections, which were plane before bending, remains plane after bending also.
4. Each layer of the beam is free to expand or contract, independently, of the layer above or below it.
5. The value of E (Young's modulus of elasticity) is the same in tension and compression.
6. The beam is in equilibrium *i.e.*, there is no resultant pull or push in the beam section.

14.3. Theory of Simple Bending

Consider a small length of a simply supported beam subjected to a bending moment as shown in Fig. 14.1 (*a*). Now consider two sections AB and CD, which are normal to the axis of the beam RS. Due to action of the bending moment, the beam as a whole will bend as shown in Fig. 14.1 (*b*).

Since we are considering a small length of dx of the beam, therefore the curvature of the beam in this length, is taken to be circular. A little consideration will show that all the layers of the beam, which were originally of the same length do not remain of the same length any more. The top layer of the beam has suffered compression and reduced to $A'C'$. As we proceed towards the lower layers of the beam, we find that the layers have no doubt suffered compression, but to lesser degree; until we come across the layer RS, which has suffered no change in its length, though bent into $R'S'$. If we further proceed towards the lower layers, we find the layers have suffered tension, as a result of which the layers are stretched. The amount of extension increases as we proceed lower, until we come across the lowermost layer BD which has been stretched to $B'D'$.

(*a*) Before bending (*b*) After bending

Fig. 14.1. Simple bending

Now we see that the layers above have been compressed and those below RS have been stretched. The amount, by which layer is compressed or stretched, depends upon the position of the layer with reference to RS. This layer RS, which is neither compressed nor stretched, is known as neutral plane or neutral layer. This theory of bending is called theory of simple bending.

14.4. Bending Stress

Consider a small length dx of a beam subjected to a bending moment as shown in Fig. 14.2 (*a*). As a result of this moment, let this small length of beam bend into an arc of a circle with O as centre as shown in Fig. 14.2 (*b*).

Let M = Moment acting at the beam,

θ = Angle subtended at the centre by the arc and

R = Radius of curvature of the beam.

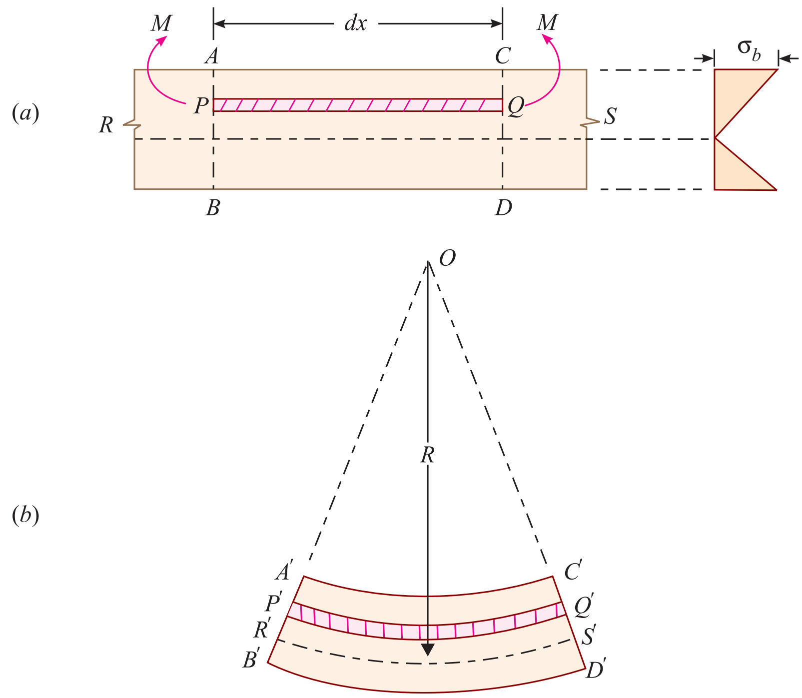


Fig. 14.2. Bending stress

Now consider a layer PQ at a distance y from RS the neutral axis of the beam. Let this layer be compressed to $P'Q'$ after bending as shown in Fig. 14.2 (*b*).

We know that decrease in length of this layer,

$$\delta l = PQ - P'Q'$$

$$\therefore \quad \text{Strain } \varepsilon = \frac{\delta l}{\text{Original length}} = \frac{PQ - P'Q'}{PQ}$$

Now from the geometry of the curved beam, we find that the two sections $OP'Q'$ and $OR'S'$ are similar.

$$\therefore \quad \frac{P'Q'}{R'S'} = \frac{R-y}{R}$$

$$\text{or} \quad 1 - \frac{P'Q'}{R'S'} = 1 - \frac{R-y}{R}$$

$$\text{or} \quad \frac{R'S' - P'Q'}{PQ} = \frac{y}{R}$$

$$\frac{PQ - P'Q'}{PQ} = \frac{y}{R} \qquad \ldots(PQ = R'S' = \text{Neutral axis})$$

$$\varepsilon = \frac{y}{R} \qquad \ldots\left(\because \varepsilon = \frac{PQ - P'Q'}{PQ}\right)$$

It is thus obvious, that the strain (ε) of a layer is proportional to its distance from the neutral axis. We also know that the bending stress,

$$\sigma_b = \text{Strain} \times \text{Elasticity} = \varepsilon \times E$$