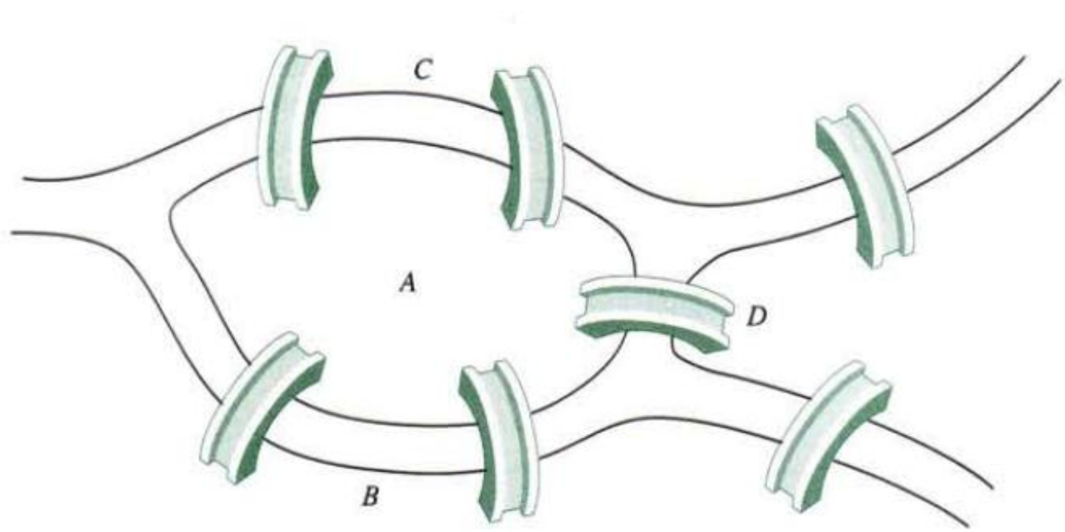
3.4

Euler and Hamilton Paths

Can we travel along the edges of a graph starting at a vertex and returning to it by traversing each edge exactly once? Similarly, can we travel along the edges of a graph starting at a vertex and returning to it by visiting each vertex of the graph exactly once? The first one gives the concept of **Euler circuit** (or **Eulerian circuit**) and the second one leads to the concept of **Hamilton circuit** (or **Hamiltonian circuit**). Although both questions have many practical applications in many different areas, both arose in old puzzles.

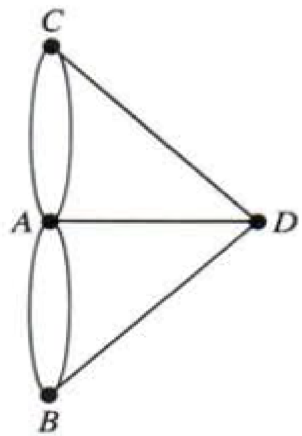
Konigsberg seven bridges problem: The town of Konigsberg, Prussia (now called Kaliningrad and part of Russian republic) was divided into four sections by the branches of the Pregel River. These four sections included the two regions on the banks of Pregel (marked as B, C), Kneiphof Island (marked as A), and the regions between the two branches of the Pregel (marked as D).

In the 18th century seven bridges connected these regions. The following figure depicts these regions and bridges.



The townspeople wondered whether it was possible to start at some location in the town, travel across all the bridges without crossing any bridge twice, and return to the starting point.

The Swiss mathematician **Leonard Euler** solved this problem, published his solution in *1736* and it was the first use of graph theory. *Euler* studied this problem and depicted the four regions as vertices A, B, C, D and the bridges as the edges, thus obtaining a multigraph as shown below:

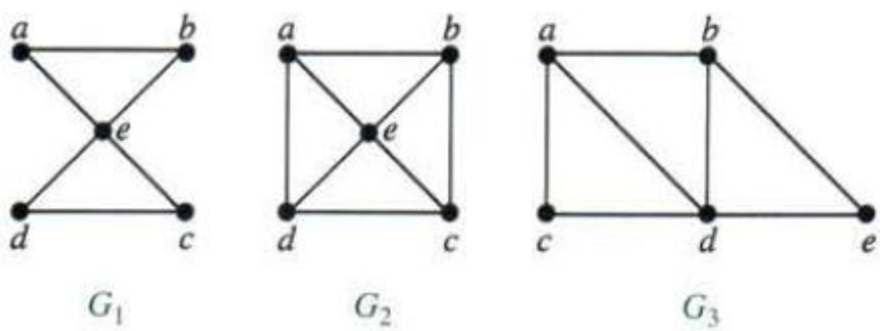


The question that the townspeople asked is:

Is there a simple circuit in this multigraph that contains every edge?

Euler Path and Euler Circuit: Let G be a graph.
An **Euler path** in G is a simple path containing every edge of G.
An **Euler circuit** in G is a simple circuit containing every edge of G.

Example 1: Which of the following undirected graphs have an Euler Circuit? Of those that do not, which have an Euler path?



Solution: The graph G_1 has an Euler circuit

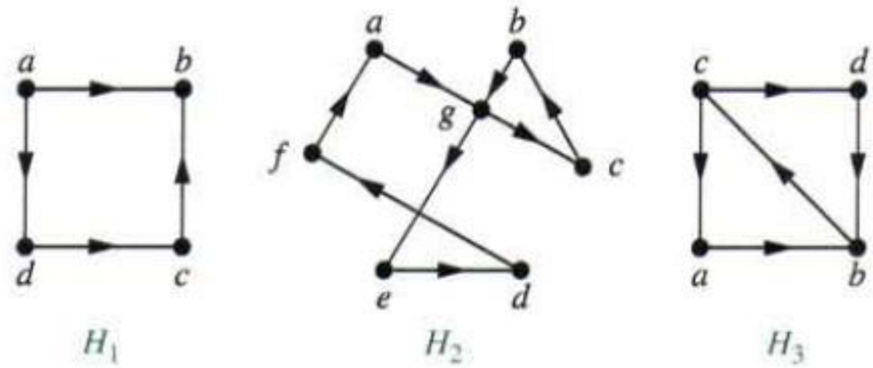
$$a, e, c, d, e, b, a$$

Neither G_2 nor G_3 has an Euler circuit (verify!). The graph G_3 has an Euler path

$$a, c, d, e, b, d, a, b$$

The graph G_3 does not have an Euler path (verify!)

Example 2: Which of the following digraphs have an Euler circuit? Of those that do not, which have an Euler path?



Solution: The digraph H_2 has an Euler circuit

$$a, g, c, b, g, e, d, f, a$$

Neither H_1 nor H_3 has an Euler circuit (verify!).The digraph H_3 has an Euler path

$$c, a, b, c, d, b$$

The digraph H_1 does not have an Euler path (verify!).

Necessary and sufficient conditions for Euler circuits and paths in an undirected graph

Let G be an undirected connected graph with at least two vertices.

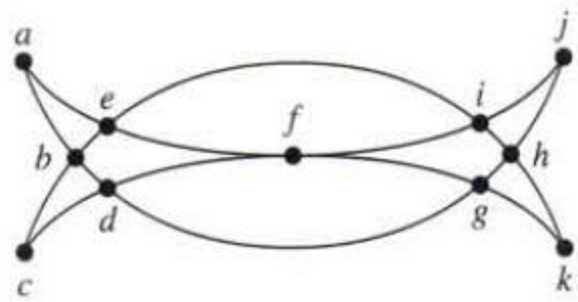
Theorem 1: The graph G has an Euler circuit if and only if each of its vertices has even degree.

Theorem 2: The graph G has an Euler path if and only if it has exactly two vertices of odd degree.

Note: The Euler path of the above theorem is from one of the two vertices of odd degree to the other.

Many puzzles ask you to draw a picture in a continuous motion without lifting the pencil so that no part of the picture is retraced. We can solve such puzzles using Euler paths and circuits.

Example 3: Can the following graph (called Mohammed's Scimitars) be drawn in the above manner, where the drawing begins and ends at the same point?



G

Solution: Note that the graph is connected and the degree sequence of the graph is 4,4,4,4,4,4,4,4,2,2,2,2. Observe that each of its vertices has even degree. By Theorem 1 , it has an Euler circuit. We will now construct an Euler circuit in the following way:

First we form a simple circuit starting at a:

$$a, b, d, g, h, j, i, f, e, a$$

Now, delete the edges in this circuit and obtain the subgraph H (some vertices may become isolated, in this case the vertices a, j become isolated).

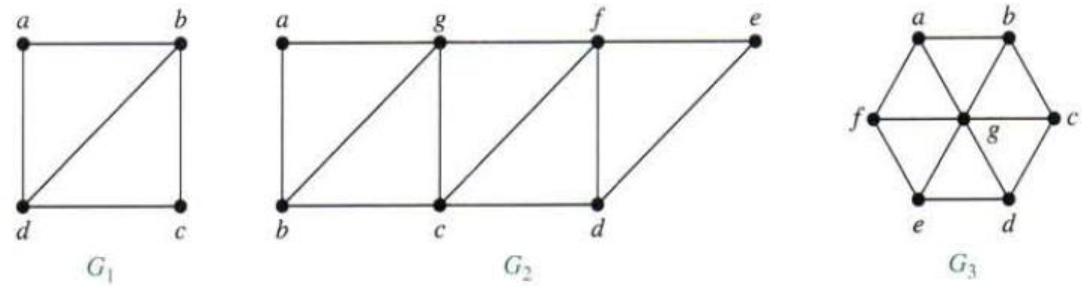
Now, form a simple circuit starting at a vertex which is common to the above circuit and H, say b. Form a simple circuit starting at b with the edges of H.

$$b, e, i, h, k, g, f, d, c, b$$

Observe that we have used all edges in the given graph. Join this new circuit in the first circuit at b we get the following Euler circuit:

$$a, b, e, i, h, k, g, f, d, c, b, d, g, h, j, i, f, e, a$$

Example 4: Which of the following graphs have an Euler path?



Solution: First note that all graphs G_1 , G_2 and G_3 are connected.

(i) The degree sequence of G_1 is 3,3,2,2. It contains exactly two vertices b, d of odd degree. By Theorem 2, G_1 has an Euler path (and it must have b and d as its end points). First start at one of the vertices of odd degree and reach the other tracing the edges exactly once. The following is a simple path starting at b and ending at d.

$$b, a, d$$

Now delete the edges in this path and obtain the sub graph H. Now every vertex in H has even degree. In H form a simple circuit with the edges of H starting at d:

$$d, b, c, d$$

Now splice(join) the two, we get an Euler path:

$$b, a, d, b, c, d$$

(ii) The degree sequence of G_2 is 4,4,4,3,3,2,2. It contains exactly two vertices b, d of odd order. By Theorem 2 G_2 has an Euler path from b to d or from d to b. We will construct the Euler path as in (i).

The first simple path is : $\quad b, c, d$

Simple circuit at c $\quad$: $\quad c, f, g, c$

(not to traverse the edges which are already included in the earlier path)

Join : b, c, f, g, c, d

Simple circuit at g : g, b, a, g

Join : $b, c, f, g, b, a, g, c, d$

Simple circuit at f : f, e, d, f

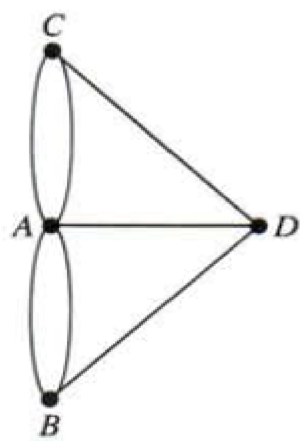
Join : $b, c, f, e, d, f, g, b, a, g, c, d$

We have traversed each edge exactly once and so the above is an Euler path.

(iii) The degree sequence of G_3 is 6,3,3,3,3,3,3. Notice that the degree of each vertex is not even and it does not have exactly two vertices of odd degree. Therefore, by Theorem 1, G_3 has no Euler circuit and by Theorem 2, G_3 has no Euler path. Thus, G_3 has neither an Euler circuit nor an Euler path.

Solution of Konigsberg Seven bridges problem:

The question can now be rephrased as: **Is there is an Euler circuit in this multigraph?**



The degree sequence of this graph is 5,3,3,3. Notice that the degree of each vertex is not even. Therefore, by Theorem 1 there is no Euler circuit in this graph.

Thus, Euler answered that it was not possible reaching the starting point by crossing each bridge exactly once.

Note that this graph does not have exactly two vertices of odd degree. By Theorem 2, there is no Euler path in this graph.

Necessary and sufficient conditions for Euler circuits and paths in a digraph

Theorem 3: **A digraph possesses an Euler circuit if and only if it is weakly connected and the in-degree of every vertex is equal to its out-degree.**

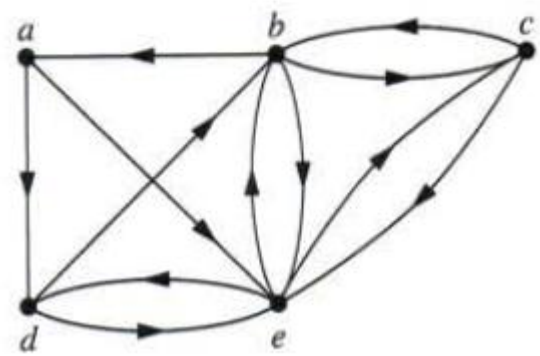
Theorem 4: **A digraph possesses an Euler path if and only if it is weakly connected and in-degree of every vertex is equal to its out-degree with the exception of two vertices $\boldsymbol{a}$, $\boldsymbol{b}$ with**

$$\boldsymbol{deg^-(a) = deg^+(a) + 1}$$

$$\boldsymbol{deg^-(b) = deg^+(b) - 1}$$

Note: The Euler path will be from b to a.

Example 5: **Determine whether the following digraph has an Euler circuit. Construct an Euler circuit if one exists. If no Euler circuit exists, determine whether the digraph has an Euler path, construct an Euler path if one exists.**



Solution: The following is the in-degree and out-degree table for vertices:

Vertex x	$\deg^-(x)$	$\deg^+(x)$
a	1	2
b	3	3
c	2	2
d	2	2

e	4	3

Notice that the given graph is weakly connected and the in-degree of every vertex is equal to its out-degree with the exception of two vertices a, e with

$$deg^{-}(e) = deg^{+}(e) + 1$$

$$deg^{-}(a) = deg^{+}(a) - 1$$

Therefore by Theorem 4, the given digraph has an Euler path (from a to e). Further, by Theorem 3 the digraph has no Euler circuit.

Construction of an Euler path:

Step 1: Take a simple path from a to e

a, d, e

Step 2: Circuit at d : d, b, e, d

(not to traverse the edges which are already included in the earlier path)

Step 3: Join : a, d, b, e, d, e

Step 4: Circuit at b b, a, e, b

Step 5: Join : $a, d, b, a, e, b, e, d, e$

Step 6: Circuit at b : b, c, b

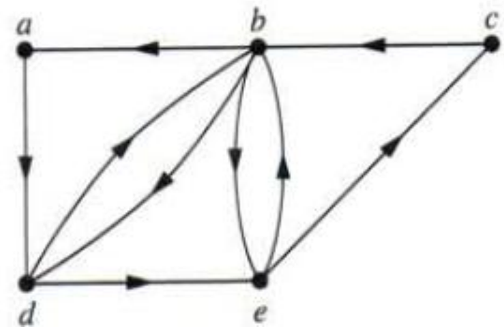
Step 7: Join : $a, d, b, c, b, a, e, b, e, d, e$

Step 8: Circuit at c : c, e, c

Step 9: Join : $a, d, b, c, e, c, b, a, e, b, e, d, e$

We have traversed each directed edge exactly once and so the above is an Euler path.

Example 6: **Determine whether the following digraph has an Euler circuit. Construct an Euler circuit if one exists. If no Euler circuit exists, determine whether the digraph has an Euler path, construct an Euler path if one exists.**



Solution: The given digraph is weakly connected. The in-degree and out-degree of each vertex is given in the following table:

Vertex x	$\deg^{-}(x)$	$\deg^{+}(x)$
a	1	1
b	3	3
c	1	1
d	2	2
e	2	2

Notice that in-degree of every vertex is equal to its out-degree. By Theorem 3 this graph has an Euler circuit. By Theorem 4 this graph has no Euler path.

The following is the construction of Euler circuit:

Step 1: Take a circuit (at any vertex): a, d, b, a

Step 2: Circuit at d : d, e, b, d

(not to traverse the edges which are already included in the earlier path)

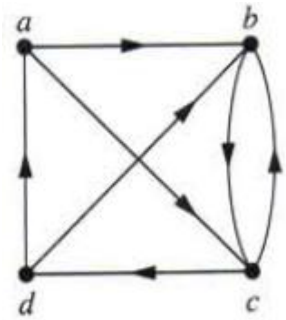
Step 3: Join : a, d, e, b, d, b, a

Step 4: Circuit at b : b, e, c, b

Step 5: Join : $a, d, e, b, d, b, e, c, b, a$

An Euler circuit is $a, d, e, b, d, b, e, c, b, a$

Example 7: **Determine whether the following digraph has an Euler circuit. Construct an Euler circuit if one exists. If no Euler circuit exists, determine whether the digraph has an Euler path, construct an Euler path if one exists.**



Solution: The given digraph is weakly connected and the in-degree and out-degree of each vertex is given in the following table.

Vertex x	$\deg^-(x)$	$\deg^+(x)$
a	1	2
b	3	1
c	2	2
d	1	2

From the table it is clear that the in-degree and out-degree of vertices are satisfying neither the conditions of Theorem 3 nor the conditions of Theorem 4. Therefore, neither an Euler circuit nor an Euler path exists in the given digraph.

Hamilton Paths and circuits

Let G be a graph.

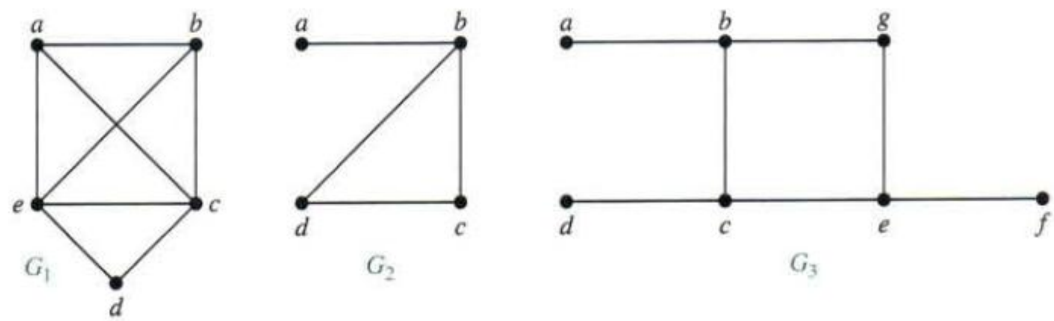
A simple path in G that passes through every vertex of G exactly once is called a **Hamilton path**.

A simple circuit in G that passes through every vertex of G exactly once is called a **Hamilton circuit**.

Note: If a graph has a Hamilton circuit, then it has a Hamilton path. This Hamilton path can be obtained from the Hamilton circuit by dropping an edge in it. However, the existence of a Hamilton path does not guarantee a Hamilton circuit.

These concepts and terminology comes from a game called **Icosian puzzle** invented in 1857 by the Irish mathematician ***Sir William Rowan Hamilton***.

Example 8: Which of the following simple graphs have a Hamilton circuit or, if not, a Hamilton path?



Solution:

(i) The graph G_1, has a Hamilton circuit a, b, c, d, e, a

(ii) The graph G_2 has no Hamilton circuit (It may be seen that every circuit containing every vertex must contain b twice), but it has a Hamilton path: a, b, c, d.

(iii) The graph G_3 has neither a Hamilton path nor a Hamilton circuit. (Note that every path containing all vertices must visit one of the vertices b, c, e more than once).

There are no known simple necessary and sufficient criteria for the existence of Hamilton paths/circuits. However, many theorems are known that give sufficient conditions for the existence of Hamilton circuits. Also certain properties can be used to show that a graph has no Hamilton circuit.

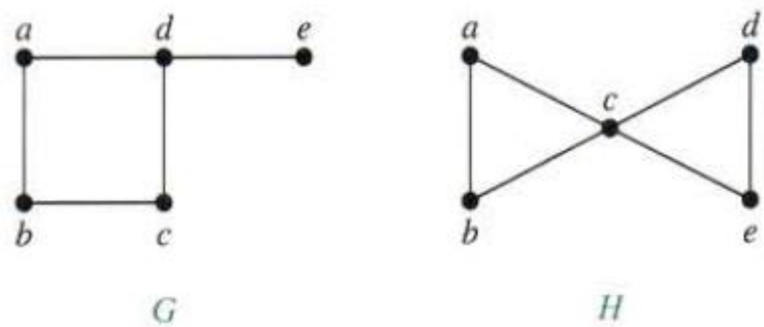
Note:

(i) A graph with a vertex of degree 1 cannot have a Hamilton circuit (because each vertex incident with two edges in a Hamilton circuit).

(ii) If a vertex in a graph has degree two, then both the edges that incident with this vertex must be part of any Hamilton circuit.

(iii) A Hamilton circuit cannot contain a smaller circuit within it.

Example 9: Show that neither graph given below has a Hamilton circuit.



Solution: The graph G has no Hamilton circuit because G has a vertex e of degree one.

Notice that the degrees of the vertices a, b, d and e in H are all two. Therefore, every edge incident with these vertices must be a part of any Hamilton circuit. Now any Hamilton circuit must contain the four edges incident with c and this is not possible. Therefore, H has no Hamilton circuit.

Example 10: Every complete graph K_n has a Hamilton circuit whenever $n \geq 3$.

Solution: A Hamilton circuit in K_n can be formed by starting at any vertex and visiting vertices in any order. This is possible because every pair of vertices are adjacent in K_n.

Although no useful necessary and sufficient conditions for the existence of Hamilton circuits are known, quite a few sufficient conditions have been found.

Note: Adding edges, but not vertices, to a graph with a Hamilton circuit produces a graph with the same Hamilton circuit.

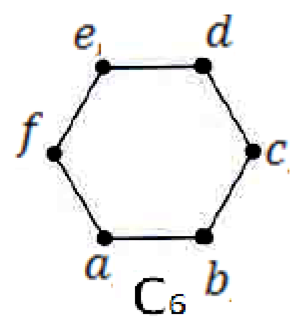
The following are the most important sufficient conditions for the existence of a Hamilton circuit.

Theorem 5 (Dirac's Theorem): If G is a simple graph with n vertices, $n \geq 3$ such that the degree of every vertex is at least $\frac{n}{2}$, then G has a Hamilton circuit.

Theorem 6(Ore's Theorem): If G is a simple graph with n vertices, $n \geq 3$ such that deg(u) + deg(v) $\geq n$ for every pair of nonadjacent vertices u and v in G, then G has a Hamilton circuit.

Both Ore's Theorem and Dirac's Theorem provide sufficient conditions for a connected simple graph to have a Hamilton circuit. However, these theorems do not provide necessary conditions for the existence of a Hamiltonian circuit.

Counter example: Consider the cycle graph C_6.



Note that C_6 has a Hamilton cycle: a, b, c, d, e, f, a. In this graph, the degree of every vertex x is 2, and thus $\deg(x)$ is atleast $\frac{n}{2} = \frac{6}{2} = 3$ is not satisfied but, C_6 has a Hamilton circuit. Further, $\deg(u) + \deg(v) = 4 \ngeq n = 6$ for every pair of non adjacent vertices u and v of C_6 , but C_6 has a Hamiltonian circuit. That is, the graph C_6 has a Hamilton circuit but does not satisfy the hypothesis of Ore's Theorem and Dirac's Theorem.

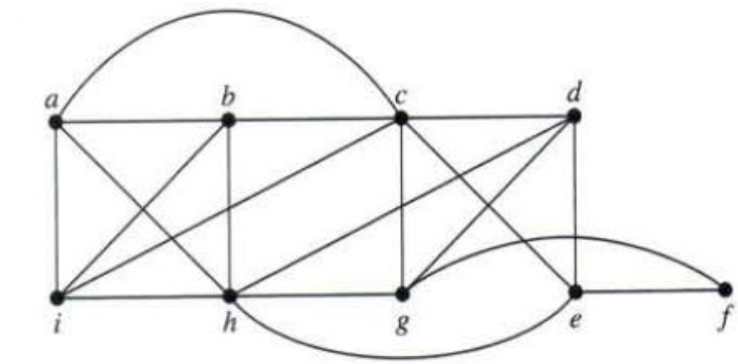
Applications:

Hamilton paths and circuits can be used to solve practical problems. For example, many applications ask for a path or circuit that visits *each road intersection in a city, each place pipelines intersection in a utility grid*, or *each node in a* *communications network exactly once*. Finding a Hamilton path or circuit in the appropriate graph model can solve such problems.

The famous **Travelling Salesman Problem** asks for the shortest route a travelling salesman should take to visit a set of cities. This problem reduces to finding a Hamilton circuit in a complete graph such that the total weight of its edges is as small as possible.

P1:

Determine whether the given graph has an Euler circuit. Construct such a circuit when one exists. If no Euler circuit exists, determine whether the graph has an Euler path and construct such a path if one exists.



Solution:

The degrees of the vertices of the graph G are

vertex x	a	b	c	d	e	f	g	h	i
$\deg(x)$	4	4	6	4	4	2	4	6	4

Observe that the degree of each vertex of G is even. By Theorem 1, G has an Euler circuit.

Now Construction of an Euler circuit:

Take a circuit (at any vertex) : a, c, e, h, a

Simple circuit at e : e, f, g, d, e

(not to traverse the edges which are already included in the earlier path)

Join : $a, c, e, f, g, d, e, h, a$

Simple circuit at c : c, d, h, b, c

Join : $a, c, d, h, b, c, e, f, g, d, e, h, a$

Simple circuit at b : b, i, a, b

Join : $a, c, d, h, b, i, a, b, c, e, f, g, d, e, h, a$

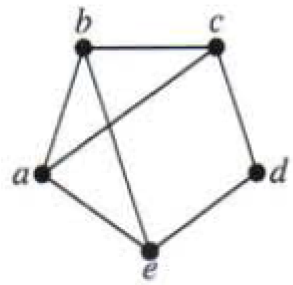
Simple circuit at i : i, c, g, h, i

Join : $a, c, d, h, b, i, c, g, h, i, a, b, c, e, f, g, d, e, h, a$

The above is the Euler circuit in G.

P3:

Determine whether the given graph has an Euler circuit. Construct such a circuit when one exists. If no Euler circuit exists, determine whether the graph has an Euler path and construct such a path if one exists.

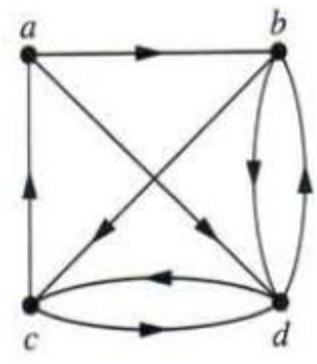


Solution:

The degree sequence of the given graph G is 3,3,3,3,2. Notice that the degree of each vertex is not even and it does not have exactly two vertices of odd degree. Therefore, by Theorem 1, G has no Euler circuit and by Theorem 2, G has no Euler path. Thus, G has neither an Euler circuit nor an Euler path.

P4:

Determine whether the given digraph has an Euler circuit. Construct such a circuit when one exists. If no Euler circuit exists, determine whether the graph has an Euler path and construct such a path if one exists.



Solution:

The given digraph G is weakly connected. The in-degree and out-degree of each vertex is given in the following table.

$vertex$ x	$\deg^-(x)$	$\deg^+(x)$
a	1	2
b	2	2
c	2	2
d	3	2

Notice that in-degree of every vertex is equal to its out-degree with exception of two vertices a, d with

$$\deg^-(d) = \deg^+(d) + 1$$

$$\deg^-(a) = \deg^+(a) - 1$$

Therefore, by Theorem 4, the digraph G has an Euler path from a to d.

Take a simple path from a to d : a, b, d

Simple Circuit at b : b, c, a, d, b

(not to traverse the edges which are already included in the earlier path)

Join : a, b, c, a, d, b, d

Simple circuit at d : d, c, d

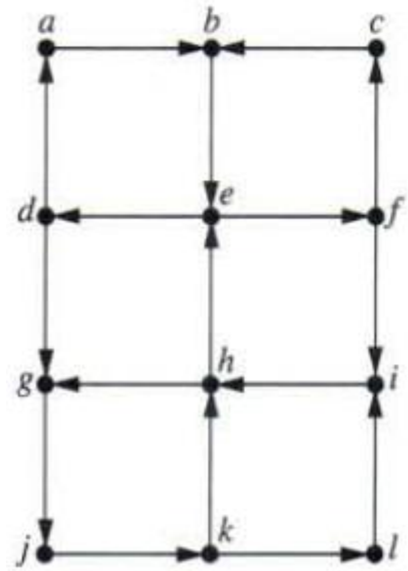
Join : $a, b, c, a, d, c, d, b, d$

An Euler path in the digraph is

$$a, b, c, a, d, c, d, b, d$$

P5:

Determine whether the given digraph has an Euler circuit. Construct such a circuit when one exists. If no Euler circuit exists, determine whether the graph has an Euler path and construct such a path if one exists.

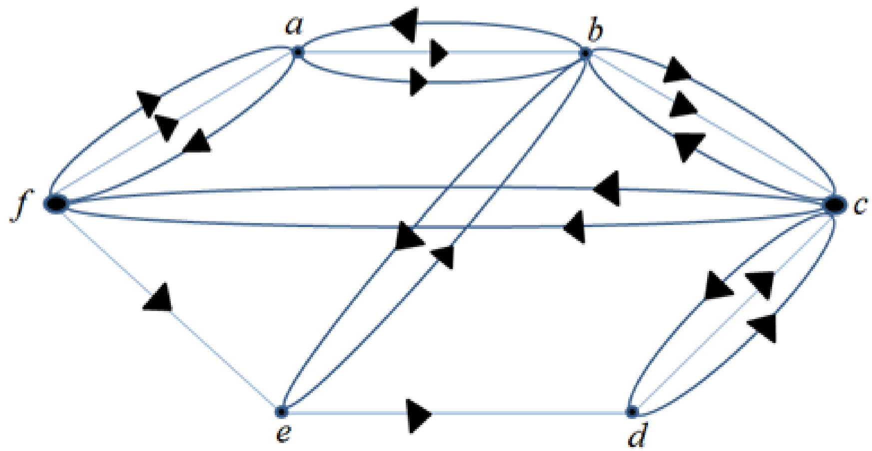


Solution:

Notice that there are 6 vertices b, d, f, g, i and k each of degree 3 and 3 is to be divided between its in-degree and out-degree. Thus this digraph has 6 vertices with unequal in-degree and out-degree. This shows that the condition of Theorem 4 and Theorem 5 are not satisfied. Therefore, neither an Euler circuit nor an Euler path exists in this digraph.

P6:

Determine whether the given digraph has an Euler circuit. Construct such a circuit when one exists. If no Euler circuit exists, determine whether the graph has an Euler path and construct such a path if one exists.



Solution:

The given digraph G is weakly connected. The in-degree and out-degree of each vertex is given in the following table.

$vertex$ x	$\deg^-(x)$	$\deg^+(x)$
a	3	3
b	3	3
c	4	4
d	2	2
e	2	2
f	3	3

Notice that the in-degree of each vertex is equal to its out-degree. By Theorem 3, this digraph has an Euler circuit.

Construction of an Euler circuit

(1) Take a circuit at any vertex : a, b, c, d, c, f, a

(2) Circuit at d : d, c, f, e, d

(not to traverse the edges which are already included in the earlier path)

(3) Join : $a, b, c, d, c, f, e, d, c, f, a$

(4) Circuit at c : c, b, a, f, a, b, c

(5) Join : $a, b, c, b, a, f, a, b, c, d, c, f, e, d, c, f, a$

(6) Circuit at b : b, e, b

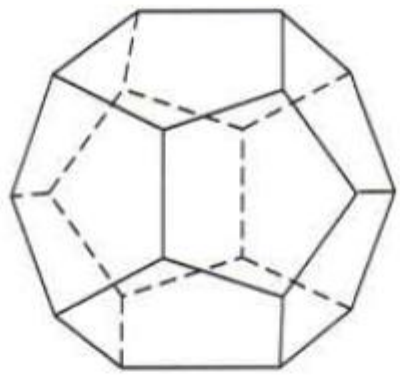
(7) Join : $a, b, e, b, c, b, a, f, a, b, c, d, c, f, e, d, c, f, a$

The path given in (7) is an Euler circuit.

P7:

A puzzle called **Icosian puzzle** (“***A voyage round the world” puzzle***) was posed by the Irish mathematician **Sir William Rowan Hamilton** in 1859 and this puzzle led to the concepts of Hamilton circuit and Hamilton path.

Hamilton devised a toy consisting of a wooden regular dodecahedron
(a polyhedron with 20 vertices, 30 edges and 12 regular pentagons as faces) and sold it to a toy manufacturer in Dublin.

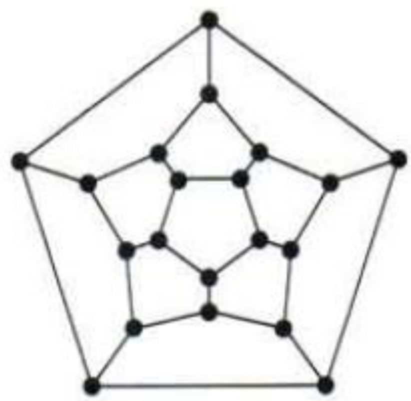


The 20 vertices were labeled with different cities in the world and 30 edges represent routes connecting the cities.

The aim of the puzzle was to start at a city, say a, and travel along the edges, visiting each of the other cities exactly once and coming back to the first city a.

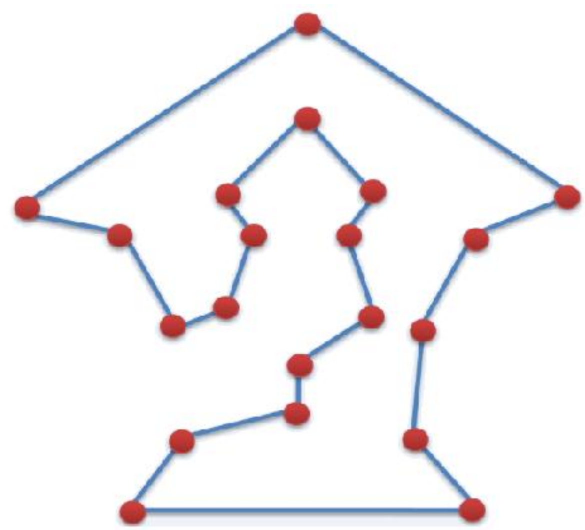
The following is an equivalent question:

Is there a circuit in the graph shown below that passes through each vertex exactly once?



This solves the puzzle because this graph is isomorphic to the graph consisting of vertices and edges of the dodecahedron.

A solution of Hamilton's puzzle is shown below



P8:

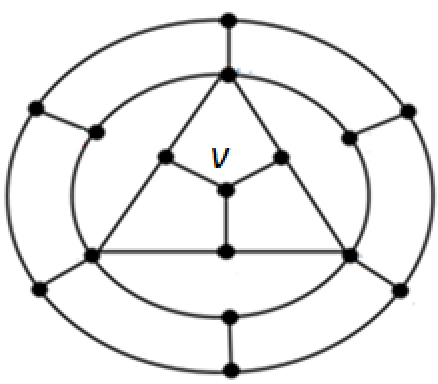
Labeling technique for the existence/nonexistence of a Hamilton circuit

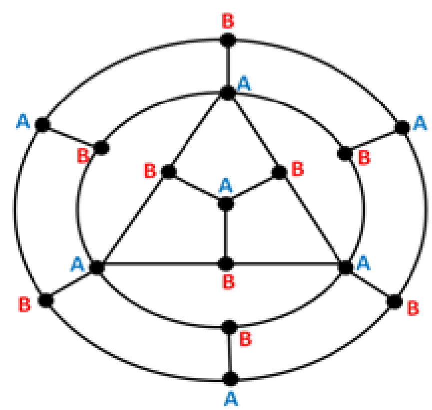
Solution:

we assign a label say A to some vertex v in G. All vertices adjacent to v with label A are labeled with B. All the vertices adjacent to vertices with label B are assigned the label A. The process is continued until all the vertices are labeled.

If there is a Hamilton circuit/path it must pass through the vertices with label A and vertices with label B alternately. In this case **the number of vertices with label A and the number of vertices with label B must differ by at most 1**. Otherwise the given graph has no Hamilton circuit/path.

Example: Determine whether the following graph has a Hamilton circuit/path.





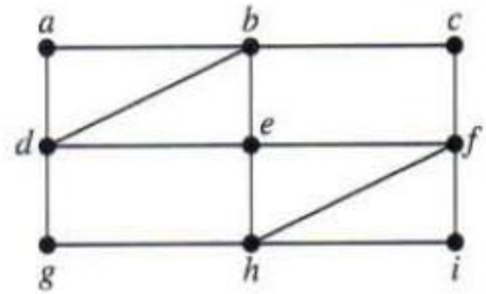
We use labeling technique. Assign label A to the center v of the graph. Assign label B to all the vertices adjacent to v. Continue the process till all the vertices are labeled. Notice that we have 7 vertices with label A and 9 vertices with label B. If there is a Hamilton circuit or path it must pass through the vertices with label A and the vertices with label B alternately and this is not possible. Thus the given graph has no Hamilton circuit. Further, there is no Hamilton path also.

3.4. Euler and Hamilton paths

Exercise:

1. Determine whether the given graph has an Euler circuit. Construct such circuit when one exists. If no Euler circuit exists. If no Euler circuit exists, determine whether the graph has an Euler path and construct such a path if one exists**.**

i



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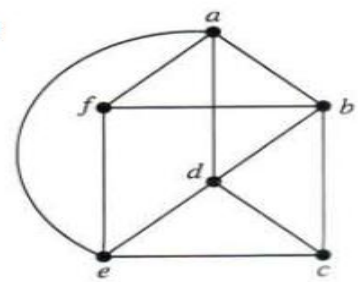

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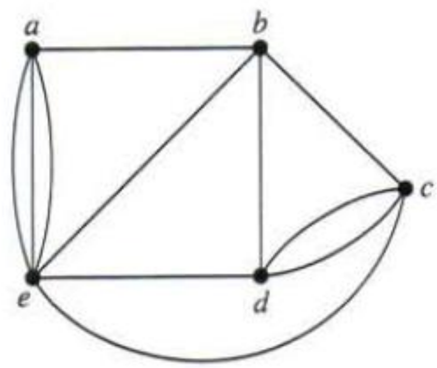

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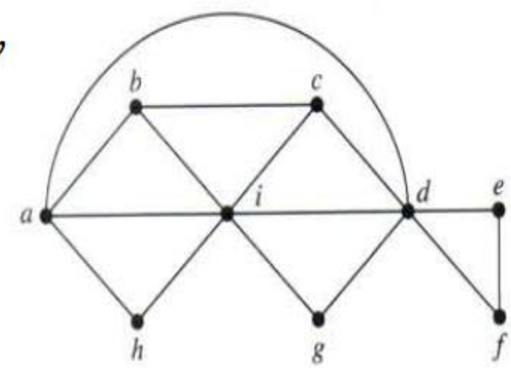
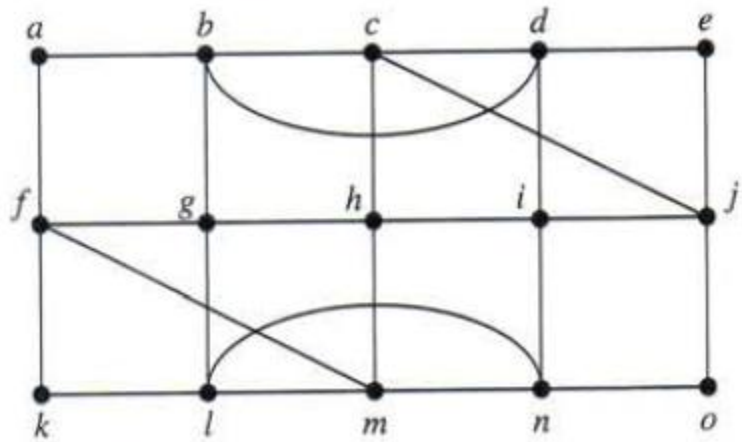
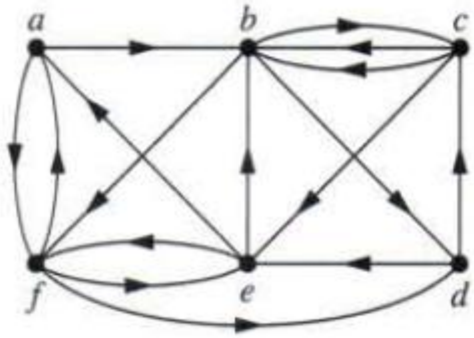


v

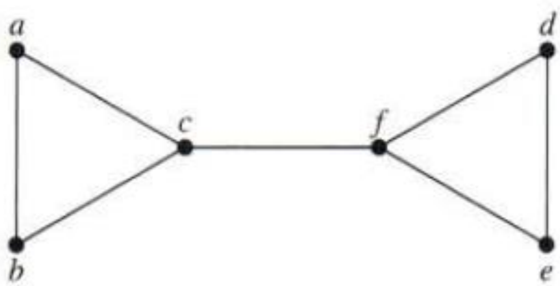


2. Determine whether the directed graph shown has an Euler circuit. Construct an Euler circuit if one exists. If no Euler circuit exists,determine whether the directed graph has an Euler path. Construct an Euler path if one exists.



3. Determine whether the given graph has a Hamilton circuit. If it does,find such a circuit. If it does not, give an argument to show that an argument to show why no such circuit exists.

i



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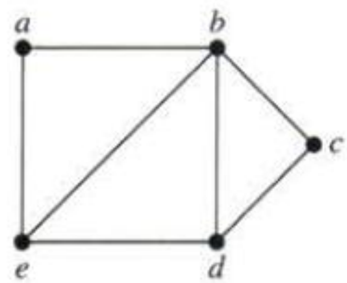
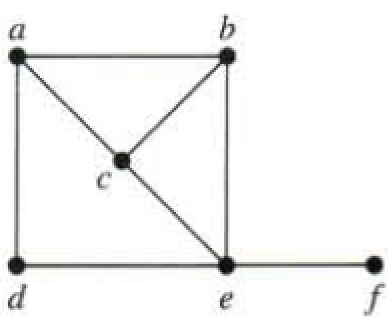


iii



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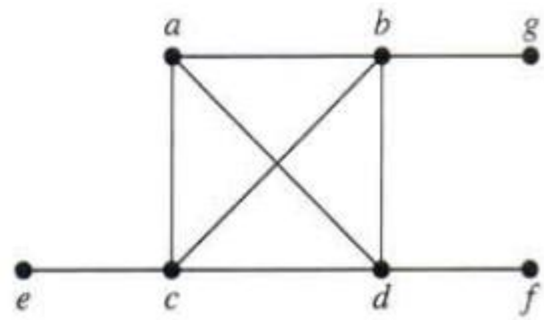

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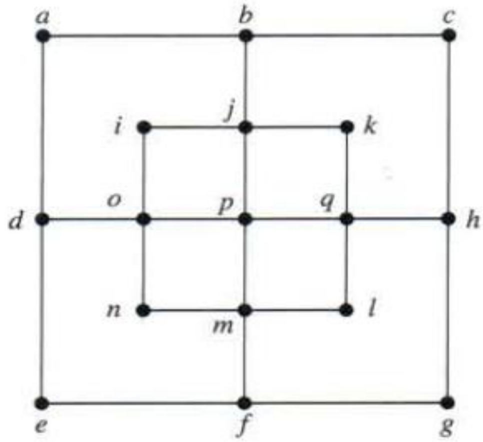
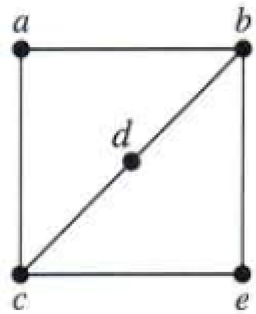


vi



vii