Chapter

27

Torsion of Circular Shafts

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27.1. Introduction

In workshops and factories, a turning force is always applied to transmit energy by rotation. This turning force is applied either to the rim of a pulley, keyed to the shaft or at any other suitable point at some distance from the axis of the shaft. The product of this turning force and the distance between the point of application of the force and the axis of the shaft is known as torque, turning moment or twisting moment. And the shaft is said to be subjected to torsion. Due to this torque, every cross-section of the shaft is subjected to some shear stress.

27.2. Assumptions for Shear Stress in a Circular Shaft Subjected to Torsion

Following assumptions are made, while

finding out shear stress in a circular shaft subjected to torsion:

1. The material of the shaft is uniform throughout.
2. The twist along the shaft is uniform.
3. Normal cross-sections of the shaft, which were plane and circular before the twist, remain plane and circular even after the twist.
4. All diameters of the normal cross-section, which were straight before the twist, remain straight with their magnitude unchanged, after the twist.

A little consideration will show that the above assumptions are justified, if the torque applied is small and the angle of twist is also small.

27.3. Torsional Stresses and Strains

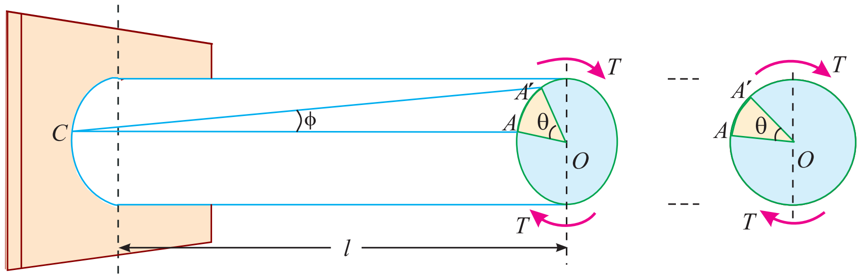

Fig. 27.1

Consider a circular shaft fixed at one end and subjected to a torque at the other end as shown in Fig. 27.1.

Let T = Torque in N-mm,

l = Length of the shaft in mm and

R = Radius of the circular shaft in mm.

As a result of this torque, every cross-section of the shaft will be subjected to shear stresses. Let the line CA on the surface of the shaft be deformed to CA' and OA to OA' as shown in Fig. 27.1.

Let $\angle ACA'$ = ϕ in degrees

$\angle AOA'$ = θ in radians

τ = Shear stress induced at the surface and

C = Modulus of rigidity, also known as torsional rigidity of the shaft material.

We know that shear strain = Deformation per unit length

$$= \frac{AA'}{l} = \tan\theta$$

$$= \phi \qquad \text{...(}\phi\text{ being very small, } \tan\phi = \phi\text{)}$$

We also know that the arc $AA' = R \cdot \theta$

$$\therefore \quad \phi = \frac{AA'}{l} = \frac{R \cdot \theta}{l} \qquad \text{...}(i)$$

If τ is the intensity of shear stress on the outermost layer and C the modulus of rigidity of the shaft, then

$$\phi = \frac{\tau}{C} \qquad \text{...}(ii)$$

From equations (*i*) and (*ii*), we find that

$$\frac{\tau}{C} = \frac{R \cdot \theta}{l} \qquad \text{or} \qquad \frac{\tau}{R} = \frac{C \cdot \theta}{l}$$

If τ_X be the intensity of shear stress, on any layer at a distance x from the centre of the shaft, then

$$\frac{\tau_X}{x} = \frac{\tau}{R} = \frac{C \cdot \theta}{l} \qquad \text{...}(iii)$$

27.4. Strength of a Solid Shaft

The term, strength of a shaft means the maximum torque or power, it can transmit. As a matter of fact, we are always interested in calculating the torque, a shaft can withstand or transmit.

Let R = Radius of the shaft in mm and

τ = Shear stress developed in the outermost layer of the shaft in N/mm^2

Consider a shaft subjected to a torque T as shown in Fig. 27.2. Now let us consider an element of area da of thickness dx at a distance x from the centre of the shaft as shown in Fig. 27.2.

$$\therefore \quad da = 2\pi x \cdot dx \qquad \text{...}(i)$$

and shear stress at this section,

$$\therefore \quad \tau_X = \tau \times \frac{x}{R} \qquad \text{...}(ii)$$

where τ = Maximum shear stress.

$$\therefore \quad \text{Turning force} = \text{Shear Stress} \times \text{Area}$$

$$= \tau_x \cdot dx$$

$$= \tau \times \frac{x}{R} \times da$$

$$= \tau \frac{x}{R} \times 2\pi x \cdot dx$$

$$= \frac{2\pi x}{R} \cdot x^2\, dx$$

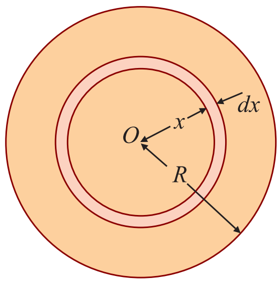

Fig. 27.2

We know that turning moment of this element,

$$dT = \text{Turning force} \times \text{Distance of element from axis of the shaft}$$

$$= \frac{2\pi\tau}{R} x^2\, dx \cdot x = \frac{2\pi\tau}{R} x^3 \cdot dx \qquad \text{...}(iii)$$

The total torque, which the shaft can withstand, may be found out by integrating the above equation between 0 and R *i.e.*,

$$T = \int_0^R \frac{2\pi\tau}{R} x^3 \cdot dx = \frac{2\pi\tau}{R} \int_0^R x^2 \cdot dx$$

$$= \frac{2\pi\tau}{R} \left[\frac{x^4}{4} \right]_0^R = \frac{\pi}{2} \tau \cdot R^3 = \frac{\pi}{16} \times \tau \times D^3 \quad \text{N-mm}$$

where D is the diameter of the shaft and is equal to $2R$.

EXAMPLE 27.1. *A circular shaft of 50 mm diameter is required to transmit torque from one shaft to another. Find the safe torque, which the shaft can transmit, if the shear stress is not to exceed 40 MPa.*

SOLUTION. Given: Diameter of shaft (D) = 50 mm and maximum shear stress (τ) = 40 MPa = 40 N/mm^2.

We know that the safe torque, which the shaft can transmit,

$$T = \frac{\pi}{16} \times \tau \times D^3 = \frac{\pi}{16} \times 40 \times (50)^3 \text{ N-mm}$$

$$= 0.982 \times 10^6 \text{ N-mm} = \mathbf{0.982 \text{ kN-m}} \qquad \textbf{Ans.}$$

EXAMPLE 27.2. *A solid steel shaft is to transmit a torque of 10 kN-m. If the shearing stress is not to exceed 45 MPa, find the minimum diameter of the shaft.*

SOLUTION. Given: Torque $(T) = 10$ kN-m $= 10 \times 10^6$ N-mm and maximum shearing stress $(\tau) = 45$ MPa $= 45$ N/mm^2.

Let D = Minimum diameter of the shaft in mm.

We know that torque transmitted by the shaft (T),

$$10 \times 10^6 = \frac{\pi}{16} \times \tau \times D^3 = \frac{\pi}{16} \times 45 \times D^3 = 8.836\, D^3$$

$$\therefore \quad D^3 = \frac{10 \times 10^6}{8.836} = 1.132 \times 10^6$$

or $$D = 1.04 \times 10^2 = \mathbf{104 \text{ mm}} \qquad \textbf{Ans.}$$

27.5. Strength of a Hollow Shaft

It means the maximum torque or power a hollow shaft can transmit from one pulley to another. Now consider a hollow circular shaft subjected to some torque.

Let R = Outer radius of the shaft in mm,

r = Inner radius of the shaft in mm, and

τ = Maximum shear stress developed in the outer most layer of the shaft material.

Now consider an elementary ring of thickness dx at a distance x from the centre as shown in Fig. 27.3.

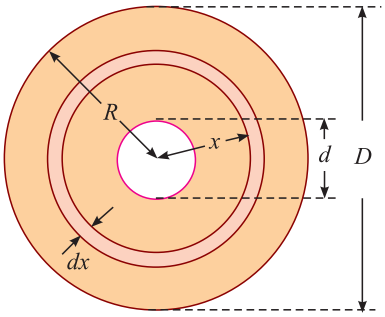

Fig. 27.3

We know that area of this ring,

$$da = 2\pi x \cdot dx \qquad ...(i)$$

and shear stress at this section,

$$\tau_X = \tau \times \frac{x}{R}$$

$$\therefore \quad \text{Turning force} = \text{Stress} \times \text{Area}$$

$$= \tau_X \cdot dx$$

$$...\left(\because \tau_X = \tau \times \frac{x}{R}\right)$$

$$= \tau \times \frac{x}{R} \times 2\pi\, x dx \qquad ...(\because da = 2\pi\, xdx)$$

$$= \frac{2\pi\tau}{R} x^2 \cdot dx \qquad ...(ii)$$

We know that turning moment of this element,

$$dT = \text{Turning force} \times \text{Distance of element from axis of the shaft}$$

$$= \frac{2\pi\tau}{R} x^2 \cdot dx \cdot x = \frac{2\pi\tau}{R} x^3 \cdot dx \qquad ...(iii)$$

The total torque, which the shaft can transmit, may be found out by integrating the above equation between r and R.

$$\therefore \qquad T = \int_r^R \frac{2\pi\tau}{R} x^3 \cdot dx = \frac{2\pi\tau}{R} \int_r^R x^3 \cdot dx$$

$$= \frac{2\pi\tau}{R}\left[\frac{x^4}{4}\right]_r^R = \frac{2\pi\tau}{R}\left(\frac{R^4 - r^4}{4}\right) = \frac{\pi}{16} \times \tau \times \left(\frac{D^4 - d^4}{D}\right) \text{ N-mm}$$

where D is the external diameter of the shaft and is equal to $2R$ and $2d$ is the internal diameter of the shaft and is equal to $2r$.

NOTE: We have already discussed in Art. 27.3 that the shear stress developed at a point is proportional to its distance from the centre of the shaft. It is thus obvious that in the central portion of a shaft, the shear stress induced is very small. In order to utilize the material to the fuller extent, hollow shafts are used.

EXAMPLE 27.3. *A hollow shaft of external and internal diameter of 80 mm and 50 mm is required to transmit torque from one end to the other. What is the safe torque it can transmit, if the allowable shear stress is 45 MPa ?*

SOLUTION. Given: External diameter (D) = 80 mm; Internal diameter (d) = 50 mm and allowable shear stress (τ) = 45 MPa = 45 N/mm^2.

We know that torque transmitted by the shaft,

$$T = \frac{\pi}{16} \times \tau \times \left[\frac{D^4 - d^4}{D}\right] = \frac{\pi}{16} \times 45 \times \left[\frac{(80)^4 - (50)^4}{80}\right] \text{ N-mm}$$

$$= 3.83 \times 10^6 \text{ N-mm} = \mathbf{3.83 \text{ kN-m}} \qquad \textbf{Ans.}$$

27.6. Power Transmitted by a Shaft

We have already discussed that the main purpose of a shaft is to transmit power from one shaft to another in factories and workshops. Now consider a rotating shaft, which transmits power from one of its ends to another.

Let N = No. of revolutions per minute and

T = Average torque in kN-m.

Work done per minute = Force × Distance = $T \times 2\pi N = 2\pi NT$

$$\text{Work done per second} = \frac{2\pi NT}{60} \text{ kN-m}$$

Power transmitted = Work done in kN-m per second

$$= \frac{2\pi NT}{60} \text{ kW}$$

NOTE: If the torque is in the N-m, then work done will also be in N-m and power will be in watt (W).

EXAMPLE 27.4. *A circular shaft of 60 mm diameter is running at 150 r.p.m. If the shear stress is not to exceed 50 MPa, find the power which can be transmitted by the shaft.*

Solution. Given: Diameter of the shaft (D) = 60 mm ; Speed of the shaft (N) = 150 r.p.m. and maximum shear stress (τ) = 50 MPa = 50 N/mm^2.

We know that torque transmitted by the shaft,

$$T = \frac{\pi}{16} \times \tau \times D^3 = \frac{\pi}{16} \times 50 \times (60)^3 \text{ N-mm}$$

$$= 2.12 \times 10^6 \text{ N-mm} = 2.12 \text{ kN-m}$$

and power which can be transmitted by the shaft,

$$P = \frac{2\pi NT}{60} = \frac{2\pi \times 150 \times 2.12}{60} = \mathbf{33.3\ kW} \qquad \textbf{Ans.}$$

EXAMPLE 27.5. *A hollow shaft of external and internal diameters as 100 mm and 40 mm is transmitting power at 120 r.p.m. Find the power the shaft can transmit, if the shearing stress is not to exceed 50 MPa.*

SOLUTION. Given: External diameter (D) = 100 mm; Internal diameter (d) = 40 mm ; Speed of the shaft (N) = 120 r.p.m. and allowable shear stress (τ) = 50 MPa = 50 N/mm^2.

We know that torque the shaft can transmit,

$$T = \frac{\pi}{16} \times \tau \times \left[\frac{D^4 - d^4}{D}\right] = \frac{\pi}{16} \times 50 \times \left[\frac{(100)^4 - (40)^4}{100}\right] \text{ N-mm}$$

$$= 9.56 \times 10^6 \text{ N-mm} = 9.56 \text{ kN-m}$$

and power the shaft can transmit,

$$P = \frac{2\pi NT}{60} = \frac{2\pi \times 120 \times 9.56}{60} = \mathbf{120\ kW} \qquad \textbf{Ans.}$$

EXAMPLE 27.6. *A solid circular shaft of 100 mm diameter is transmitting 120 kW at 150 r.p.m. Find the intensity of shear stress in the shaft.*

SOLUTION. Given : Diameter of the shaft (D) = 100 mm ; Power transmitted (P) = 120 kW and speed of the shaft (N) = 150 r.p.m.

Let T = Torque transmitted by the shaft, and

τ = Intensity of shear stress in the shaft.

We know that power transmitted by the shaft (P),

$$120 = \frac{2\pi NT}{60} = \frac{2\pi \times 150 \times T}{60} = 15.7\, T$$

$$\therefore \qquad T = \frac{120}{15.7} = 7.64 \text{ kN-m} = 7.64 \times 10^6 \text{ N-mm}$$

We also know that torque transmitted by the shaft (T),

$$7.64 \times 10^6 = \frac{\pi}{16} \times \tau \times D^3 = \frac{\pi}{16} \times \tau \times (100)^3 = 0.196 \times 10^6\, \tau$$

$$\tau = \frac{7.64}{0.196} = 39 \text{ N/mm}^2 = \mathbf{39\ MPa} \qquad \textbf{Ans.}$$

EXAMPLE 27.7. *A hollow shaft is to transmit 200 kW at 80 r.p.m. If the shear stress is not to exceed 60 MPa and internal diameter is 0.6 of the external diameter, find the diameters of the shaft.*

SOLUTION. Given : Power (P) = 200 kW ; Speed of shaft (N) = 80 r.p.m. ; Maximum shear stress (τ) = 60 MPa = 60 N/mm^2 and internal diameter of the shaft (d) = 0.6D (where D is the external diameter in mm).

We know that torque transmitted by the shaft,

$$T = \frac{\pi}{16} \times \tau \times \left[\frac{D^4 - d^4}{D}\right] = \frac{\pi}{16} \times 60 \times \left[\frac{D^4 - (0.6D)^4}{D}\right] \text{ N-mm}$$

$$= 10.3\,D^3 \text{ N-mm} = 10.3 \times 10^{-6}\,D^3 \text{ kN-m} \qquad ...(i)$$

We also know that power transmitted by the shaft (P),

$$200 = \frac{2\pi NT}{60} = \frac{2\pi \times 80 \times (10.3 \times 10^{-6} D^3)}{60} = 86.3 \times 10^{-6}\,D^3$$

$$\therefore \qquad D^3 = \frac{200}{(86.3 \times 10^{-6})} = 2.32 \times 10^6 \text{ mm}^3$$

or $\qquad D = 1.32 \times 10^2 = \mathbf{132\ mm}$ **Ans.**

and $\qquad d = 0.6\,D = 0.6 \times 132 = \mathbf{79.2\ mm}$ **Ans.**

EXAMPLE 27.8. *A solid steel shaft has to transmit 100 kW at 160 r.p.m. Taking allowable shear stress as 70 MPa, find the suitable diameter of the shaft. The maximum torque transmitted in each revolution exceeds the mean by 20%.*

SOLUTION. Given: Power (P) = 100 kW ; Speed of the shaft (N) = 160 r.p.m. ; Allowable shear stress (τ) = 70 MPa = 70 N/mm^2 and maximum torque (T_{max}) = 1.2 T (where T is the mean torque).

Let $\qquad D$ = Diameter of the shaft in mm.

We know that power transmitted by shaft (P),

$$100 = \frac{2\pi NT}{60} = \frac{2\pi \times 160 \times T}{60} = 16.8\,T$$

$$\therefore \qquad T = \frac{100}{16.8} = 5.95 \text{ kN-m} = 5.95 \times 10^6 \text{ N-mm}$$

and maximum torque, $\qquad T_{max} = 1.2T = 1.2 \times (5.95 \times 10^6) = 7.14 \times 10^6$ N-mm

We also know that maximum torque (T_{max}),

$$7.14 \times 10^6 = \frac{\pi}{16} \times \tau \times D^3 = \frac{\pi}{16} \times 70 \times D^3 = 13.7\,D^3$$

$$\therefore \qquad D^3 = \frac{7.14 \times 10^6}{13.7} = 0.521 \times 10^6$$

or $\qquad D = 0.8 \times 10^2 = \mathbf{80\ mm}$ **Ans.**

EXERCISE 27.1

1. A circular shaft of 80 mm diameter is required to transmit torque in a factory. Find the torque, which the shaft can transmit, if the allowable shear stress is 50 MPa. (**Ans.** 5.03 kN-mm)
2. A solid steel shaft is required to transmit a torque of 6.5 kN-m. What should be the minimum diameter of the shaft, if the maximum shear stress is 40 MPa? (**Ans.** 94 mm)
3. A solid shaft of 40 mm diameter is subjected to a torque of 0.8 kN-m. Find the maximum shear stress induced in the shaft. (**Ans.** 63.7 MPa)
4. A hollow shaft of external and internal diameters of 60 mm and 40 mm is transmitting torque. Find the torque it can transmit, if the shear stress is not to exceed 40 MPa. (**Ans.** 1.36 kN-m)
5. A circular shaft of 80 mm diameter is required to transmit power at 120 r.p.m. If the shear stress is not to exceed 40 MPa, find the power transmitted by the shaft. (**Ans.** 50.5 kW)

6. A hollow shaft of external and internal diameters as 80 mm and 50 mm respectively is transmitting power at 150 r.p.m. Determine the power, which the shaft can transmit, if the shearing stress is not to exceed 40 MPa. (**Ans.** 53.6 kW)

7. A hollow shaft has to transmit 53 kW at 160 r.p.m. If the maximum shear stress is 50 MPa and internal diameter is half of the external diameter, find the diameters of the shaft. (**Ans.** 70 mm; 35 mm)

27.7. Polar Moment of Inertia

The moment of inertia of a plane area, with respect to an axis perpendicular to the plane of the figure, is called polar moment of inertia with respect to the point, where the axis intersects the plane. In a circular plane, this point is always the centre of the circle. We know that

$$\frac{\tau}{R} = \frac{C \cdot \theta}{l} \qquad \text{...}(i) \qquad \text{... (from Art. 27.3)}$$

and

$$T = \frac{\pi}{16} \times \tau \times D^3 \qquad \text{...}(ii) \qquad \text{... (from Art. 27.3)}$$

or

$$\tau = \frac{16T}{\pi D^3}$$

Substituting the value of τ in equation (i),

$$\frac{16T}{\pi D^3 \times R} = \frac{C \cdot \theta}{l}$$

or

$$\frac{T}{\frac{\pi}{16} \times D^3 \times R} = \frac{C \cdot \theta}{l}$$

$$\frac{T}{\frac{\pi}{32} \times D^4} = \frac{C \cdot \theta}{l} \qquad \text{...}\left(\text{Radius, } R = \frac{D}{2}\right)$$

$$\frac{T}{J} = \frac{C \cdot \theta}{l} \qquad \text{...}(iii)$$

where $J = \frac{\pi}{32} \times D^4$. It is known as polar moment of inertia.

The above equation (iii) may also be written as :

$$\frac{\tau}{R} = \frac{T}{J} = \frac{C \cdot \theta}{l} \qquad \text{...}\left(\because \frac{\tau}{R} = \frac{C \cdot \theta}{l}\right)$$

Notes. 1. In a hollow circular shaft the polar moment of inertia,

$$J = \frac{\pi}{32}(D^4 - d^4)$$

where d is the internal diameter of the shaft.

2. The term $\frac{J}{R}$ is known as torsional section modulus or polar modulus. It is similar to section modulus which is equal to $\frac{I}{y}$.

3. Thus polar modulus for a solid shaft,

$$Zp = \frac{2\pi}{32D} \times D^4 = \frac{\pi}{16} D^3$$

and the polar modulus for a hollow shaft,

$$Zp = \frac{2\pi}{32D}(D^4 - d^4) = \frac{\pi}{16D}(D^4 - d^4)$$

EXAMPLE 27.9. *Calculate the maximum torque that a shaft of 125 mm diameter can transmit, if the maximum angle of twist is 1° in a length of 1.5 m. Take C = 70 GPa.*

SOLUTION. Given: Diameter of shaft $(D) = 125$ mm ; Angle of twist $(\theta) = 1° = \frac{\pi}{180}$ rad ; Length of the shaft $(l) = 1.5 \text{ m} = 1.5 \times 10^3$ mm and modulus of rigidity $(C) = 70 \text{ GPa} = 70 \times 10^3 \text{ N/mm}^2$.

Let T = Maximum torque the shaft can transmit.

We know that polar moment of inertia of a solid circular shaft,

$$J = \frac{\pi}{32} \times (D)^4 = \frac{\pi}{32}(125)^4 = 24.0 \times 10^6 \text{ mm}^4$$

and relation for torque transmitted by the shaft,

$$\frac{T}{J} = \frac{C \cdot \theta}{l}$$

$$\frac{T}{24.0 \times 10^6} = \frac{(70 \times 10^3)\pi/180}{1.5 \times 10^3} = 0.814$$

$$\therefore \quad T = 0.814 \times (24.0 \times 10^6) = 19.5 \times 10^6 \text{ N-mm}$$

$$= \mathbf{19.5 \text{ kN-m}} \quad \textbf{Ans.}$$

EXAMPLE 27.10. *Find the angle of twist per metre length of a hollow shaft of 100 mm external and 60 mm internal diameter, if the shear stress is not to exceed 35 MPa. Take C = 85 GPa.*

SOLUTION. Given: Length of the shaft $(l) = 1 \text{ m} = 1 \times 10^3$ mm ; External diameter $(D) = 100$ mm; Internal diameter $(d) = 60$ mm ; Maximum shear stress $(\tau) = 35 \text{ MPa} = 35 \text{ N/mm}^2$ and modulus of rigidity $(C) = 85 \text{ GPa} = 85 \times 10^3 \text{ N/mm}^2$.

Let θ = Angle of twist in the shaft.

We know that torque transmitted by the shaft,

$$T = \frac{\pi}{16} \times \tau \times \left[\frac{D^4 - d^4}{D}\right] = \frac{\pi}{16} \times 35 \times \left[\frac{(100)^4 - (60)^4}{100}\right] \text{ N-mm}$$

$$= 5.98 \times 10^6 \text{ N-mm}$$

We also know that polar moment of inertia of a hollow circular shaft,

$$J = \frac{\pi}{32}[D^4 - d^4] = \frac{\pi}{32}[(100)^4 - (60)^4] = 8.55 \times 10^6 \text{ mm}^4$$

and relation for the angle of twist,

$$\frac{T}{J} = \frac{C \cdot \theta}{l} \quad \text{or} \quad \frac{5.98 \times 10^6}{8.55 \times 10^6} = \frac{(85 \times 10^3)\,\theta}{1 \times 10^3} = 85.\theta$$

$$\therefore \quad \theta = \frac{5.98 \times 10^6}{(8.55 \times 10^6) \times 85} = 0.008 \text{ rad} = \mathbf{0.5°} \quad \textbf{Ans.}$$

EXAMPLE 27.11. *A solid shaft of 120 mm diameter is required to transmit 200 kW at 100 r.p.m. If the angle of twists not to exceed 2°, find the length of the shaft. Take modulus of rigidity for the shaft material as 90 GPa.*

SOLUTION. Given : Diameter of shaft $(D) = 120$ mm; Power $(P) = 200$ kW ; Speed of shaft $(N) = 100$ r.p.m. ; Angle of twist $(\theta) = 2° = \frac{2\pi}{180}$ rad. and modulus of rigidity $(C) = 90 \text{ GPa} = 90 \times 10^3 \text{ N/mm}^2$.

Let T = Torque transmitted by the shaft, and

l = Length of the shaft.

We know that power transmitted by the shaft (P),

$$200 = \frac{2\pi NT}{60} = \frac{2\pi \times 100 \times T}{60} = 10.5T$$

$$\therefore \quad T = \frac{200}{10.5} = 19 \text{ kN-m} = 19 \times 10^6 \text{ N-mm}$$

We also know that polar moment of inertia of a solid shaft,

$$J = \frac{\pi}{32} \times (D)^4 = \frac{\pi}{32} \times (120)^4 = 0.4 \times 10^6 \text{ mm}^4$$

and relation for the length of the shaft,

$$\frac{T}{J} = \frac{C \cdot \theta}{l} \quad \text{or} \quad \frac{19 \times 10^6}{20.4 \times 10^5} = \frac{(90 \times 10^3) \times (2\pi/180)}{l}$$

$$0.931 = \frac{3.14 \times 10^3}{l}$$

$$\therefore \quad l = \frac{(3.14 \times 10^3)}{0.931} = 3.37 \times 10^3 = \mathbf{3.37\ m} \quad \textbf{Ans.}$$

EXAMPLE 27.12. *Find the maximum torque, that can be safely applied to a shaft of 80 mm diameter. The permissible angle of twist is 1.5 degree in a length of 5 m and shear stress not to exceed 42 MPa. Take C = 84 GPa.*

SOLUTION. Given: Diameter of shaft (D) = 80 mm ; Angle of twist (θ) = 1.5° = $\frac{1.5\pi}{180}$ rad ; Length of shaft (l) = 5 m = 5×10^3 mm ; Maximum shear stress (τ) = 42 MPa = 42 N/mm^2 and Modulus of rigidity (C) = 84 GPa = 84×10^3 N/mm^2.

First of all, let us find out the values of torques based on shear stress and angle of twist.

1. ***Torque based on shear stress***

We know that the torque which can be applied to the shaft,

$$T_1 = \frac{\pi}{16} \times \tau \times D^3 = \frac{\pi}{16} \times 42 \times (80)^3 = 4.22 \times 10^6 \text{ N-mm} \quad ...(i)$$

2. ***Torque based on angle of twist***

We also know that polar moment of inertia of a solid circular shaft,

$$J = \frac{\pi}{32}(D)^4 = \frac{\pi}{32} \times (80)^4 = 4.02 \times 10^6 \text{ mm}^4$$

and relation for the torque that can be applied:

$$\frac{T_2}{J} = \frac{C \cdot \theta}{l} \quad \text{or} \quad \frac{T_2}{4.02 \times 10^6} = \frac{(84 \times 10^3) \times (1.5\pi/180)}{5 \times 10^3} = 0.44$$

$$\therefore \quad T_2 = 0.44 \times (4.02 \times 10^6) = 1.77 \times 10^6 \text{ N-mm} \quad ...(ii)$$

We shall apply a torque of 1.77×10^6 N-mm (*i.e.*, lesser of the two values). **Ans.**

EXAMPLE 27.13. *A solid shaft is subjected to a torque of 1.6 kN-m. Find the necessary diameter of the shaft, if the allowable shear stress is 60 MPa. The allowable twist is 1° for every 20 diameters length of the shaft. Take C = 80 GPa.*

SOLUTION. Given: Torque $(T) = 1.6$ kN-m $= 1.6 \times 10^6$ N-mm; Allowable shear stress $(\tau) = 60$ MPa $= 60$ N/mm^2; Angle of twist $(\theta) = 1° = \dfrac{\pi}{180}$ rad ; Length of shaft $(l) = 20\,D$ and modulus of rigidity $(C) = 80$ GPa $= 80 \times 10^3$ N/mm^2.

First of all, let us find out the value of diameter of the shaft for its strength and stiffness.

1. *Diameter for strength*

We know that torque transmitted by the shaft (T),

$$1.6 \times 10^6 = \frac{\pi}{16} \times \tau \times D_1^3 = \frac{\pi}{16} \times 60 \times D_1^3 = 11.78\, D_1^{\,3}$$

$$\therefore \quad D_1^{\,3} = \frac{1.6 \times 10^6}{11.78} = 0.136 \times 10^6 \text{ mm}^3$$

$$\text{or} \quad D_1 = 0.514 \times 10^2 = 51.4 \text{ mm} \qquad ...(i)$$

2. *Diameter for stiffness*

We know that polar moment of inertia of a solid circular shaft,

$$J = \frac{\pi}{32} \times (D_2)^4 = 0.098\, D_2^4$$

and relation for the diameter,

$$\frac{T}{J} = \frac{C \cdot \theta}{l} \quad \text{or} \quad \frac{1.6 \times 10^6}{0.098\, D_2^4} = \frac{(80 \times 10^3) \times (\pi/180)}{20 D_2}$$

$$\therefore \quad D_2^3 = \frac{(1.6 \times 10^6) \times 20}{0.098 \times (80 \times 10^3) \times (\pi/180)} = 234 \times 10^3 \text{ mm}^3$$

$$\text{or} \quad D_2 = 6.16 \times 10^1 = 61.6 \text{ mm} \qquad ...(ii)$$

We shall provide a shaft of diameter of 61.6 mm (*i.e.*, greater of the two values). **Ans.**

EXAMPLE 27.14. *A solid shaft of 200 mm diameter has the same cross-sectional area as a hollow shaft of the same material with inside diameter of 150 mm. Find the ratio of*

(*a*) *powers transmitted by both the shafts at the same angular velocity.*

(*b*) *angles of twist in equal lengths of these shafts, when stressed to the same intensity.*

SOLUTION. Given: Diameter of solid shaft $(D_1) = 200$ mm and inside diameter of hollow shaft $(d) = 150$ mm.

(*a*) *Ratio of powers transmitted by both the shafts*

We know that cross-sectional area of the solid shaft,

$$A_1 = \frac{\pi}{4} \times D_1^2 = \frac{\pi}{4} \times (200)^2 = 10\,000\,\pi \text{ mm}^2$$

and cross-sectional area of hollow shaft,

$$A_2 = \frac{\pi}{4} \times (D^2 - d^2) = \frac{\pi}{4} \times [D^2 - (150)^2] = \frac{\pi}{4}(D^2 - 22\,500)$$

Since the cross-sectional areas of both the shafts are same, therefore equating A_1 and A_2,

$$\frac{\pi}{4}(200)^2 = \frac{\pi}{4}(D^2 - 22\,500)$$

$$\therefore \quad 40\,000 = D^2 - 22\,500$$

$$D^2 = 40\,000 + 22\,500 = 62\,500 \text{ mm}^2$$

$$\text{or} \quad D = 250 \text{ mm}$$

We also know that torque transmitted by the solid shaft,

$$T_1 = \frac{\pi}{16} \times \tau \times D_1^3 = \frac{\pi}{16} \times \tau \times (200)^3 = 500 \times 10^3 \, \pi \, \tau \text{ N-mm} \qquad ...(i)$$

Similarly, torque transmitted by the hollow shaft,

$$T_2 = \frac{\pi}{16} \times \tau \times \left[\frac{D^4 - d^4}{D}\right] = \frac{\pi}{16} \times \tau \times \left[\frac{(250)^4 - (150)^4}{250}\right] \text{ N-mm}$$

$$= 850 \times 10^3 \, \pi \, \tau \text{ N-mm}$$

$$\therefore \quad \frac{\text{Power transmitted by hollow shaft}}{\text{Power transmitted by solid shaft}}$$

$$= \frac{T_2}{T_1} = \frac{50 \times 10^3 \, \pi \, \tau}{500 \times 10^3 \, \pi \, \tau} = \mathbf{1.7} \qquad \textbf{Ans.}$$

(b) Ratio of angles of twist in both the shafts

We know that relation for angle of twist for a shaft,

$$\frac{\tau}{R} = \frac{C \cdot \theta}{l} \quad \text{or} \quad \theta = \frac{\tau l}{RC}$$

∴ Angle of twist for the solid shaft,

$$\theta_1 = \frac{\tau l}{RC} = \frac{\tau l}{100C} \qquad ...\left(\text{where } R = \frac{D_1}{2} = \frac{200}{2} = 100 \text{ mm}\right)$$

Similarly angle of twist for the hollow shaft,

$$\theta_2 = \frac{\tau l}{RC} = \frac{\tau l}{125C} \qquad ...\left(\text{where } R = \frac{D_1}{2} = \frac{250}{2} = 125 \text{ mm}\right)$$

$$\therefore \quad \frac{\text{Angle of twist of hollow shaft}}{\text{Angle of twist of solid shaft}} = \frac{\theta_2}{\theta_1} = \frac{\dfrac{\tau l}{125C}}{\dfrac{\tau l}{100C}} = \frac{100}{125} = \mathbf{0.8} \qquad \textbf{Ans.}$$

27.8. Replacing a Shaft

Sometimes, we are required to replace a solid shaft by a hollow one, or vice versa. In such cases, the torque transmitted by the new shaft should be equal to that by the replaced shaft. But sometimes, there are certain other conditions which have also to be considered while designing the new shaft.

EXAMPLE 27.15. *A solid steel shaft of 60 mm diameter is to be replaced by a hollow steel shaft of the same material with internal diameter equal to half of the external diameter. Find the diametres of the hollow shaft and saving in material, if the maximum allowable shear stress is same for both shafts.*

SOLUTION. Given: Diameter of solid shaft (D) = 60 mm.

Diameter of the hollow shaft

Let D = External diameter of the hollow shaft,

d = Internal diameter of the hollow shaft (equal to $D_1/2$) and

τ = Shear stress developed in both the shafts.

We know that torque transmitted by the solid shaft,

$$T = \frac{\pi}{16} \times \tau \times D^3 = \frac{\pi}{16} \times \tau \times (60)^3 \qquad ...(i)$$

and torque transmitted by the hollow shaft,

$$T_1 = \frac{\pi}{16} \times \tau \times \left[\frac{D^4 - d^4}{D_1}\right] = \frac{\pi}{16} \times \tau \times \left[\frac{D_1^4 - (0.5D_1)^4}{D_1}\right]$$

$$= \frac{\pi}{16} \times \tau \times 0.9375\, D_1^3 \qquad ...(ii)$$

Since the torque transmitted and allowable shear stress in both the cases are same, therefore equating the equations (*i*) and (*ii*),

$$\frac{\pi}{16} \times \tau \times (60)^3 = \frac{\pi}{16} \times \tau \times 0.9375\, D_1^3$$

$$\therefore \qquad D_1^3 = \frac{(60)^3}{0.9375} = 230400 \text{ mm}^3$$

or $\qquad D_1 = \mathbf{61.3\ mm} \qquad$ **Ans.**

and $\qquad d = \frac{61.3}{2} = \mathbf{30.65\ mm} \qquad$ **Ans.**

Saving in material

We know that saving in material

$$= \frac{\left[\frac{\pi}{4}(60)^2\right] - \left[\frac{\pi}{4}((61.3)^2 - (30.65)^2)\right]}{\frac{\pi}{4}(60)^2} = \frac{3600 - 2819}{3600}$$

$$= 0.217 = \mathbf{21.7\%} \qquad \textbf{Ans.}$$

EXAMPLE 27.16. *A solid shaft of 80 mm diameter is to be replaced by a hollow shaft of external diameter 100 mm. Determine the internal diameter of the hollow shaft if the same power is to be transmitted by both the shafts at the same angular velocity and shear stress.*

SOLUTION. Given: Diameter of solid shaft (D) = 80 mm and external diameter of hollow shaft (D_1) = 100 mm.

Let $\qquad d$ = Internal diameter of the hollow shaft, and

$\qquad \tau$ = Shear stress developed in both the shafts.

We know that torque transmitted by the solid shaft,

$$T = \frac{\pi}{16} \times \tau \times D^3 = \frac{\pi}{16} \times \tau \times (80)^3 \qquad ...(i)$$

and torque transmitted by the hollow shaft,

$$T_1 = \frac{\pi}{16} \times \tau \times \left[\frac{D^4 - d^4}{D}\right] = \frac{\pi}{16} \times \tau \times \left[\frac{(100)^4 - d^4}{100}\right] \qquad ...(ii)$$

Since both the torques are equal, therefore equating the equations (*i*) and (*ii*),

$$\frac{\pi}{16} \times \tau \times (80)^2 = \frac{\pi}{16} \times \tau \times \left[\frac{(100)^4 - d^4}{100}\right]$$

$$(80)^3 = \frac{(100)^4 - d^4}{100} = (100)^3 - \frac{d^4}{100}$$

$$\frac{d^4}{100} = (100)^3 - (80)^3 = 488 \times 10^3$$

$$d^4 = (488 \times 10^3) \times 100 = 488 \times 10^5 = 4880 \times 10^4$$

$$\therefore \qquad d = 8.36 \times 10 = \mathbf{83.6\ mm} \qquad \textbf{Ans.}$$

EXAMPLE 27.17. *A solid aluminium shaft 1 m long and of 50 mm diameter is to be replaced by a hollow shaft of the same length and same outside diameter, so that the hollow shaft could carry the same torque and has the same angle of twist. What must be the inner diameter of the hollow shaft ?*

Take modulus of rigidity for the aluminium as 28 GPa and that for steel as 85 GPa.

SOLUTION. Given: Length of aluminium shaft (l_A) = 1 m = 1×10^3 mm ; Diameter of aluminium shaft (D_A) = 50 mm ; Length of steel shaft (l_S) = 1 m = 1×10^3 mm ; Outside diameter of steel shaft (D_S) = 50 mm; Modulus of rigidity for aluminium (C_A) = 28 GPa = 28×10^3 N/mm^2 and modulus of rigidity for steel = 85 GPa = 85×10^3 N/mm^2.

Let d_S = Inner diameter of steel shaft in mm.

We know that polar moment of inertia of the solid aluminium shaft,

$$J_A = \frac{\pi}{32} \times D^4 = \frac{\pi}{32} \times (50)^4 \text{ mm}^4$$

We also know that relation for angle of twist

$$\frac{T}{J} = \frac{C \cdot \theta}{l} \quad \text{or} \quad \theta = \frac{T \cdot l}{JC}$$

$$\therefore \quad \theta_A = \frac{T_A \cdot l_A}{\frac{\pi}{32} \times (50)^4 \times 28 \times 10^3} \text{ rad.}$$

and

$$\theta_S = \frac{T_S \cdot l_S}{\frac{\pi}{32} \times [(50)^4 - (d)^4] \times 85 \times 10^3} \text{ rad.}$$

Since both the angles of twists (*i.e.*, θ_A and θ_B) are same, therefore equating these values,

$$\frac{T_A \cdot l_A}{\frac{\pi}{32} \times (50)^4 \times 28 \times 10^3} = \frac{T_S \cdot l_S}{\frac{\pi}{32} \times [(50)^4 - (d)^4] \times 85 \times 10^3}$$

Substituting $T_A = T_S$ and $l_A = l_S$ in the above equation,

$$(50)^4 \times 28 = [(50)^4 - d^4] \times 85$$

$$175 \times 10^6 = (531.25 \times 10^6) - 85\, d^4$$

$$85\, d^4 = (531.25 \times 10^6) - (175 \times 10^6) = 356.25 \times 10^6$$

$$d^4 = \frac{356.25 \times 10^6}{85} = 4.191 \times 10^6 \text{ mm}^4$$

$$\therefore \quad d = \mathbf{45.25 \text{ mm}} \quad \textbf{Ans.}$$

EXAMPLE 27.18. *A hollow steel shaft of 300 mm external diameter and 200 mm internal diameter has to be replaced by a solid alloy shaft. Assuming the same values of polar modulus for both, calculate the diameter of the latter and work out the ratio of their torsional rigidities. Take C for steel as 2.4 C for alloy.*

SOLUTION. Given: External diameter of steel shaft (D) = 300 mm ; Internal diameter of steel shaft (d) = 200 mm and modulus of rigidity for steel (C_S) = 2.4 (where C_A is the modulus of rigidity for the alloy).

Diameter of the solid alloy shaft

Let D_1 = Diameter of the solid alloy shaft.

We know that polar modulus of hollow steel shaft,

$$Z_S = \frac{\pi}{16D}(D^4 - d^4) = \frac{\pi}{16 \times 300}[(300)^4 - (200)^4] \text{ mm}^3$$

$$= \frac{8.125 \times 10^6 \, \pi}{6} \text{ mm}^3 \qquad \ldots(i)$$

Similarly, polar modulus of solid alloy shaft,

$$Z_A = \frac{\pi}{16} D_1^3 \text{ mm}^2 \qquad \ldots(ii)$$

Since the polar modulus for both the shafts are the same, therefore equating (*i*) and (*ii*),

$$\frac{8.125 \times 10^6 \, \pi}{6} = \frac{\pi}{16} D_1^3$$

or $$D_1^3 = \frac{8.125 \times 10^6 \times 16}{6} = 21.67 \times 10^6$$

$\therefore$ $$D_1 = \mathbf{278.8 \text{ mm}} \qquad \textbf{Ans.}$$

Ratio of torsional rigidities

We know that the torsional rigidity of hollow steel shaft

$$= C_S \times J_S = 2.4 \, C_A \times \frac{\pi}{32} [(300)^4 - (200)^4] \qquad \ldots(iii)$$

Similarly, torsional rigidity for solid alloy shaft

$$= C_A \times J_A = C_A \times \frac{\pi}{32} \times D^4 = C_A \times \frac{\pi}{32} \times (278.8)^4 \qquad \ldots(iv)$$

$$\therefore \frac{\text{Torsional rigidity of hollow steel shaft}}{\text{Torsional rigidity of solid alloy shaft}}$$

$$= \frac{2.4 \, C_A \times \frac{\pi}{32} [(300)^4 - (200)^4]}{C_A \times \frac{\pi}{32} (278.8)^4} = \mathbf{2.58} \qquad \textbf{Ans.}$$

EXERCISE 27.2

1. Find the torque a solid shaft of 100 mm diameter can transmit, if the maximum angle of twist is 1.5° in a length of 2 m. Take C = 70 GPa. (**Ans.** 9.0 kN-m)
2. A hollow shaft of external and internal diameters as 80 mm and 40 mm is required to transmit torque from one pulley to another. What is the value of torque transmitted, if angle of twist is not to exceed 1° in a length of 2 m. Take C = 80 GPa (**Ans.** 2.63 kN-m)
3. A solid shaft and a hollow circular shaft, whose inside diameter is 3/4 of the outside diameter are of equal lengths and are required to transmit a given torque. Compare the weights of these two shafts, if maximum shear stress developed in both the shaft is also equal. (**Ans.** 1.76)
4. A solid shaft of 150 mm diameter is to be replaced by a hollow shaft of the same material with internal diameter equal to 60% of the external diameter. Find the saving in material, if maximum allowable shear stress is the same for both the shafts. (**Ans.** 30.9%)
5. A shaft is transmitting 100 kW at 180 r.p.m. If the allowable shear stress in the shaft material is 60 MPa, determine the suitable diameter for the shaft. The shaft is not to twist more than 1° in a length of 3 metres. Take C = 80 GPa. (**Ans.** 103.8 mm)

27.9. Shaft of Varying Sections

Sometimes a shaft, made up of different lengths having different cross-sectional areas, is required to transmit some torque (or horse power) from one pulley to another.

A little consideration will show that for such a shaft, the torque transmitted by individual sections have to be calculated first and the minimum value of these torques will be the strength of such a shaft. The angle of twist for such a shaft may be found out as usual.

Example 27.19. *The stepped steel shaft shown in Fig. 27.4 is subjected to a torque (T) at the free end, and a torque (2T) in the opposite direction at the junction of the two sizes.*

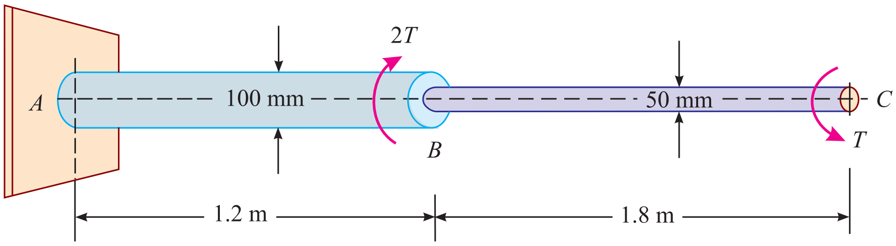

Fig. 27.4

What is the total angle of twist at the free end, if maximum shear stress in the shaft is limited to 70 MPa? Assume the modulus of rigidity to be 84 GPa.

Solution. Given: Torque at $C = T$ (anticlockwise); Torque at $B = 2T$ (clockwise) ; Diameter of shaft AB $(D_{AB}) = 100$ mm ; Diameter of shaft BC $(D_{BC}) = 50$ mm ; Maximum shear stress $(\tau) = 70$ MPa $= 70$ N/mm^2 and modulus of rigidity $(C) = 84$ GPa $= 84 \times 10^3$ N/mm^2.

Since the torques at B and C are in opposite directions, therefore the effect of these two torques will be studied first independently, sum of the two twists (one in clockwise direction and the other in anticlockwise direction).

First of all, let us first find out the value of torque T at C. It may be noted that if the value of torque is obtained for the portion AB, it will induce more stress in the portion BC (because the portion BC is of less diameter). Therefore we shall calculate the torque for the portion BC (because it will not induce stress more than the permissible in the portion AB).

We know that the torque at C,

$$T = \frac{\pi}{16} \times \tau \times (D_{BC})^3 = \frac{\pi}{16} \times 70 \times (50)^3 = 1.718 \times 10^6 \text{ N-mm}$$

We also know that polar moment of inertia of the solid circular shaft AB,

$$J_{AB} = \frac{\pi}{32} \times (D_{AB})^4 = \frac{\pi}{32} \times (100)^4 = 9.82 \times 10^6 \text{ mm}^4$$

Similarly,

$$J_{BC} = \frac{\pi}{32} \times (D_{BC})^4 = \frac{\pi}{32} \times (50)^4 = 0.614 \times 10^6 \text{ mm}^4$$

$\therefore$ Angle of twist at C due to torque (T) at C,

$$\theta = \frac{T \cdot l}{J \cdot C} = \frac{T}{C}\left(\frac{l_{AB}}{J_{AB}} + \frac{l_{BC}}{J_{BC}}\right)$$

$$= \frac{1.718 \times 10^6}{84 \times 10^3}\left(\frac{1.2 \times 10^3}{9.82 \times 10^6} + \frac{1.8 \times 10^3}{0.614 \times 10^6}\right) \text{rad}$$

$$= 20.45 \times (30.54 \times 10^{-4}) = 0.0624 \text{ rad} \qquad \ldots(i)$$

Similarly, angle of twist at C due to torque $(2T)$ at B,

$$\theta = \frac{2T}{C} \times \frac{l_{AB}}{J_{AB}} = \frac{2 \times (1.718 \times 10^6)}{84 \times 10^3} \times \frac{1.2 \times 10^3}{9.82 \times 10^6} \text{ rad}$$

$$= 40.9 \times (1.222 \times 10^{-4}) = 0.005 \text{ rad} \quad ...(ii)$$

From the geometry of the shaft, we find that the twist at B (due to torque of $2T$ at B) will continue at C also. Since the directions of both the twists are opposite to each other, therefore net angle of twist at C

$$= 0.0624 - 0.005 = 0.0574 \text{ rad} = \mathbf{3.29°} \quad \textbf{Ans.}$$

EXAMPLE 27.20. *A shaft ABC of 500 mm length and 40 mm external diameter is bored, for a part of its length AB to a 20 mm diameter and for the remaining length BC to a 30 mm diameter bore as shown in Fig. 27.5. If the shear stress is not to exceed 80 MPa, find the maximum power, the shaft can transmit at a speed of 200 r.p.m.*

If the angle of twist in the length of 20 mm diameter bore is equal to that in the 30 mm diameter bore, find the length of the shaft that has been bored to 20 mm and 30 mm diameter.

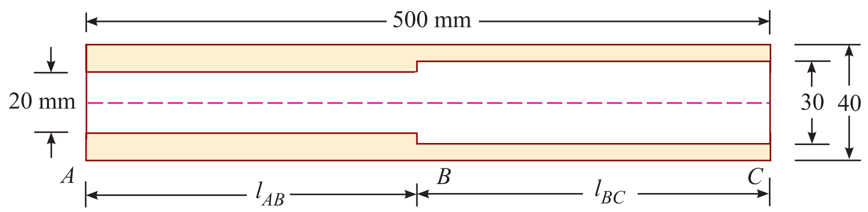

Fig. 27.5

SOLUTION. Given: Total length of the shaft (l) = 500 mm; External diameter of the shaft (D) = 40 mm ; Internal diameter of shaft AB (d_{AB}) = 20 mm ; Internal diameter of shaft BC (d_{BC}) = 30 mm ; Maximum shear stress (τ) = 80 MPa = 80 N/mm^2 and speed of the shaft (N) = 200 r.p.m.

Maximum power the shaft can transmit

We know that torque transmitted by the shaft AB,

$$T_{AB} = \frac{\pi}{16} \times \tau \times \left(\frac{D^4 - d_{AB}^4}{D} \right) = \frac{\pi}{16} \times 80 \times \left[\frac{(40)^4 - (20)^4}{40} \right] \text{ N-mm}$$

$$= 942.5 \times 10^3 \text{ N-mm} \quad ...(i)$$

Similarly,

$$T_{BC} = \frac{\pi}{16} \times \tau \times \left(\frac{D^4 - d_{BC}^4}{D} \right) = \frac{\pi}{16} \times 80 \times \left[\frac{(40)^4 - (30)^4}{40} \right] \text{ N-mm}$$

$$= 687.3 \times 10^3 \text{ N-mm} \quad ...(ii)$$

From the above two values, we see that the safe torque transmitted by the shaft is minimum of the two, *i.e.*, 687.3×10^3 N-mm = 687.3 N-m. Therefore maximum power the shaft can transmit,

$$P = \frac{2\pi NT}{60} = \frac{2 \times \pi \times 200 \times (687.3)}{60} = 14\ 394 \text{ W}$$

$$= \mathbf{14.39 \text{ kW}} \quad \textbf{Ans.}$$

Length of the shaft, that has been bored to 20 mm diameter

Let l_{AB} = Length of the shaft AB (*i.e.*, 20 mm diameter bore) and

l_{BC} = Length of the shaft BC (*i.e.*, 30 mm diameter bore) equal to $(500 - l_{AB})$ mm.

We know that polar moment of inertia for the shaft AB,

$$J_{AB} = \frac{\pi}{32} \times (D^4 - d_{AB}^4) = \frac{\pi}{32} \times [(40)^4 - (20)^4] \text{ mm}^4$$

Similarly,
$$J_{BC} = \frac{\pi}{32} \times (D^4 - d_{BC}^4) = \frac{\pi}{32} \times [(40)^4 - (30)^4] \text{ mm}^4$$

We know that relation for the angle of twist:

$$\frac{T}{J} = \frac{C\,\theta}{l} \quad \text{or} \quad \theta = \frac{T \cdot l}{JC}$$

$\therefore$
$$\theta_{AB} = \frac{T \cdot l_{AB}}{J_{AB} \cdot C} \quad \text{and} \quad \theta_{BC} = \frac{T \cdot l_{BC}}{J_{BC} \cdot C}$$

Since $\theta_{AB} = \theta_{AC}$ and T as well as C is equal in both these cases, therefore

$$\frac{l_{AB}}{J_{AB}} = \frac{l_{BC}}{J_{BC}} \quad \text{or} \quad \frac{l_{AB}}{\frac{\pi}{32} \times [(40)^4 - (20)^4]} = \frac{l_{BC}}{\frac{\pi}{32} \times [(40)^4 - (30)^4]}$$

or
$$\frac{l_{AB}}{l_{BC}} = \frac{(40)^4 - (20)^4}{(40)^4 - (30)^4} = \frac{2400000}{1750000} = 1.37$$

$\therefore$
$$l_{AB} = 1.37\, l_{BC}$$

$$1.37\, l_{BC} + l_{BC} = 500 \qquad \ldots(\because l_{AB} + l_{BC} = 500)$$

$\therefore$
$$l_{BC} = \frac{500}{2.37} = \mathbf{211 \text{ mm}} \qquad \textbf{Ans.}$$

and
$$l_{AB} = 500 - 211 = \mathbf{289 \text{ mm}} \qquad \textbf{Ans.}$$

27.10. Composite Shaft

Sometimes, a shaft is made up of composite section *i.e.*, one type of shaft rigidly sleeved over another type of shaft. At the time of sleeving, the two shafts are joined together in such a way, that the composite shaft behaves like a single shaft. The total torque transmitted by the composite shaft is shared by the two shafts, depending upon their diameters and elastic properties.

EXAMPLE 27.21. *A composite shaft consists of copper rod of 30 mm diameter enclosed in a steel tube of external diameter 40 mm and 5 mm thick. The shaft is required to transmit a torque of 0.5 kN-m. Determine the shearing stresses developed in the copper and steel, if both the shafts have equal lengths and welded to a plate at each end, so that their twists are equal. Take C_C = 40 GPa and C_S = 80 GPa.*

SOLUTION. Given: Diameter of copper rod (D_C) = 30 mm ; External diameter of steel tube (D_S) = 40 mm ; Thickness of steel tube = 5 mm ; Therefore internal diameter of steel tube (d_S) = 40 – (2 × 5) = 30 mm; Total torque to be transmitted (T) = 0.5 kN-m = 0.5×10^6 N-mm ; Modulus of rigidity for copper (C_C) = 40 GPa = 40×10^3 N/mm² and modulus of rigidity for steel (C_S) = 80 GPa = 80×10^3 N/mm².

Let
T_C = Torque shared by copper rod,
τ_C = Shear stress developed in the copper rod and
T_S, τ_S = Corresponding values for steel tube.

$\therefore$ Total torque (T)

$$T_C + T_S = 0.5 \times 10^6 \text{ N-mm} \qquad \ldots(i)$$

We know that polar moment of inertia of copper rod,

$$J_C = \frac{\pi}{32} \times (D_C^4) = \frac{\pi}{32} \times (30)^4 = \frac{0.81 \times 10^6\, \pi}{32} \text{ mm}^4$$

and polar moment of inertia of steel tube,

$$J_S = \frac{\pi}{32}\times(D_S^4 - d_S^4) = \frac{\pi}{32}[(40)^4 - (30)^4] = \frac{1.75\times10^6\ \pi}{32}\ \text{mm}^4$$

We also know that relation for angle of twist:

$$\frac{T}{J} = \frac{C\,\theta}{l} \qquad \text{or} \qquad \theta = \frac{T\cdot l}{JC}$$

$$\therefore \qquad \theta_C = \frac{T_C\cdot l}{J_C\cdot C_C} = \frac{T_C\cdot l}{\frac{0.81\times10^6\ \pi}{32}\times(40\times10^3)} = \frac{T_C\cdot l}{1012.5\times10^6\ \pi}\ \text{rad.}$$

Similarly,

$$\theta_S = \frac{T_S\cdot l}{J_S\cdot C_S} = \frac{T_S\cdot l}{\frac{1.75\times10^6\ \pi}{32}\times(80\times10^3)} = \frac{T_S\cdot l}{4375\times10^6\ \pi}\ \text{rad.}$$

Since θ_C is equal to θ_S, therefore equating these values,

$$\frac{T_C\cdot l}{1012.5\times10^6\ \pi} = \frac{T_S\cdot l}{4375\times10^6\ \pi} \qquad \text{or} \qquad T_C = \frac{81\,T_S}{350}$$

Substituting this value of T_C in equation (*i*),

$$\frac{81\,T_S}{350} + T_S = 0.5\times10^6 \qquad \text{or} \qquad \frac{431\,T_S}{350} = 0.5\times10^6$$

$$\therefore \qquad T_S = \frac{(0.5\times10^6)\times350}{431} = 0.406\times10^6\ \text{N-mm}$$

and

$$T_C = \frac{81\,T_S}{350} = \frac{81\times(0.406\times10^6)}{350}$$

$$= \mathbf{0.094\times10^6\ N\text{-}mm} \qquad \textbf{Ans.}$$

We know that torque transmitted by copper rod (T_C),

$$0.094\times10^6 = \frac{\pi}{16}\times\tau_C\times D_C^3 = \frac{\pi}{16}\times\tau_C\times(30)^3 = 5301\ \tau_C$$

$$\therefore \qquad \tau_C = \frac{0.094\times10^6}{5301} = 17.7\ \text{N/mm}^2 = \mathbf{17.7\ MPa} \qquad \textbf{Ans.}$$

Similarly, torque transmitted by steel tube (T_S),

$$0.406\times10^6 = \frac{\pi}{16}\times\tau_S\times\left(\frac{D_S^4 - d_S^4}{D_S}\right) = \frac{\pi}{16}\times\tau_S\times\left(\frac{(40)^4-(30)^4}{40}\right) = 8590\ \tau_S$$

$$\therefore \qquad \tau_S = \frac{0.406\times10^6}{8590} = 47.3\ \text{N/mm}^2 = \mathbf{47.3\ MPa} \qquad \textbf{Ans.}$$

EXAMPLE 27.22. *A composite shaft consists of a steel rod of 60 mm diameter surrounded by a closely fitting tube of brass. Find the outside diameter of the brass tube, when a torque of 1 kN-m is applied on the composite shaft and shared equally by the two materials. Take C for steel as 84 GPa and C for brass as 42 GPa.*

Also determine the common angle of twist in a length of 4 metres.

SOLUTION. Given: Diameter of steel rod $(D_S) = 60$ mm ; Inner diameter of brass tube $(d_S) = 60$ mm ; Total torque $(T) = 1$ kN-m $= 1 \times 10^6$ N-mm ; Torque shared by steel rod $(T_S) = \frac{1}{2} \times (1 \times 10^6) = 0.5 \times 10^6$ N-mm ; Torque shared by brass tube $(T_B) = (1 \times 10^6) - (0.5 \times 10^6) = 0.5 \times 10^6$ N-mm ; Modulus of rigidity for steel $(C_S) = 84$ GPa $= 84 \times 10^3$ N/mm^2 ; Modulus of rigidity for brass $(\sigma_B) = 42$ GPa $= 42 \times 10^3$ N/mm^2 and length of shaft $(l) = 4$ m $= 4 \times 10^3$ mm.

Outside diameter of the brass tube

Let D_B = Outside diameter of the brass tube in mm.

We know that polar moment of inertia of steel rod,

$$J_S = \frac{\pi}{32} \times (D_S)^4 = \frac{\pi}{32} \times (60)^4 = \frac{12.96 \times 10^6\ \pi}{32}\ \text{mm}^4$$

and polar moment of inertia of brass tube,

$$J_B = \frac{\pi}{32} \times (D_B^4 - d_B^4) = \frac{\pi}{32} \times [D_B^4 - (60)^4]\ \text{mm}^4$$

We also know that the relation for angle of twist:

$$\frac{T}{J} = \frac{C\,\theta}{l} \qquad \text{or} \qquad \theta = \frac{T \cdot l}{JC}$$

$$\therefore \qquad \theta_S = \frac{T \cdot l}{J_S \cdot C_S} = \frac{T \cdot l}{\frac{\pi}{32} \times (60)^4 \times (84 \times 10^3)} \qquad ...(i)$$

Similarly,

$$\theta_B = \frac{T \cdot l}{J_B \cdot C_B} = \frac{T \cdot l}{\frac{\pi}{32} \times [D_B^4 - (60)^4] \times 42 \times 10^3} \qquad ...(ii)$$

Since θ_S is equal to θ_B, therefore equating these two values,

$$\frac{T \cdot l}{\frac{\pi}{32} \times (60)^4 \times (84 \times 10^3)} = \frac{T \cdot l}{\frac{\pi}{32} \times [D_B^4 - (60)^4] \times 42 \times 10^3}$$

$$D_B^4 - 12.96 \times 10^6 = 2 \times (12.96 \times 10^6) = 25.92 \times 10^6$$

or

$$D_B^4 = (25.92 \times 10^6) + (12.96 \times 10^6) = 38.88 \times 10^6\ \text{mm}^4$$

$$\therefore \qquad D_B = \mathbf{79\ mm} \qquad \textbf{Ans.}$$

Common angle of twist

Let θ = Common angle of twist.

Substituting the values of T (equal to 0.5×10^6 N-mm) and l (equal to 4×10^3 mm) in equation (*i*),

$$\theta = \frac{(0.5 \times 10^6) \times (4 \times 10^3)}{\frac{\pi}{32} \times (60)^4 \times (84 \times 10^3)} = 0.0187\ \text{rad} = \mathbf{1.07°} \qquad \textbf{Ans.}$$

27.11. Strain Energy due to Torsion

We have already discussed in Chapter of Strain Energy that when a body is subjected to a shear stress, the strain energy stored is,

$$U = \frac{\tau^2}{2C} \times V \qquad ...\text{(See Art. 8.10)}$$

where τ = Shear stress,

C = Modulus of rigidity or shear modulus and

V = Volume of the body.

But in the case of a solid circular shaft, the torsional stress varies from zero at the central axis to a maximum τ at the surface. Now, consider a circular shaft of diameter D, subjected to shear stress (τ) at its surface. Now, let us consider an elementary ring of thickness dx and a distance x from the axis of the shaft.

$\therefore$ Area of the ring, $da = 2\pi x \, dx$

and its volume $V = l \cdot 2\pi x \, dx$

We know that shear stress at this section,

$$\theta = \tau \times \frac{x}{R}$$

$\therefore$ Strain energy stored in this ring

$$= \frac{q^2}{2C} \times l \cdot 2\pi x \, dx = \frac{\tau^2}{2C} \times \frac{x^2}{R^2} \cdot l \cdot 2\pi x \, dx = \frac{\tau^2}{2CR^2} \cdot 2\pi l \cdot x^3 \, dx$$

The total strain energy stored in the shaft may be found out by integrating the above equation from zero to R.

$$\therefore \quad U = \int_0^R \frac{\tau^2}{2\,CR^2} \cdot 2\pi l \cdot x^3 \, dx$$

$$= \frac{\tau^2\,\pi l}{CR^2} \int_0^R x^3 \, dx = \frac{\tau^2\,\pi l}{CR^2} \left[\frac{x^4}{4} \right]_0^R$$

$$= \frac{\tau^2\,\pi l\, R^2}{4C} = \frac{\tau^2}{4C} \times \pi l \times \left(\frac{D}{2} \right)^2 \qquad \ldots \left(\because R = \frac{D}{2} \right)$$

$$= \frac{\tau^2}{4C} \times V \qquad \ldots \left(\because V = \frac{\pi}{4} \times D^2 \times l \right)$$

$$\boxed{\therefore \text{ For Solid Circular Shaft, } U = \frac{\tau^2}{4C} \times V}$$

Note. If the shaft is a hollow one, then by integrating the equation between r and R instead of integrating it from zero to R, we get the strain energy stored,

$$\boxed{\therefore \text{ For Hollow Circular Shaft, } U = \frac{\tau^2}{4C} \left(\frac{D^2 + d^2}{D^2} \right) \times V}$$

Example 27.23. *A solid steel shaft 120 mm diameter and 1.5 m long is used to transmit power from one pulley to another. Determine the maximum strain energy that can be stored in the shaft, if maximum allowable shear stress is 50 MPa. Take shear modulus as 80 GPa.*

Solution. Given: Diameter of shaft (D) = 120 mm ; Length of shaft (l) = 1.5 m = 1.5×10^3 mm; Allowable shear stress (τ) = 50 MPa = 50 N/mm^2 and shear modulus (C) = 80 GPa = 80×10^3 N/mm^2.

We know that volume of the shaft,

$$V = \frac{\pi}{4} \times (120)^2 \times (1.5 \times 10^3) = 16.96 \times 10^6 \text{ mm}^3$$

and strain energy stored in the shaft,

$$U = \frac{\tau^2}{4C} \times V = \frac{(50)^2}{4 \times (80 \times 10^3)} \times 16.96 \times 10^6 \text{ N-mm}$$

$$= \mathbf{132.5 \times 10^3 \text{ N-mm}} \qquad \textbf{Ans.}$$

27.12. Shaft Couplings

Sometimes, due to the non-availability of a single shaft of the required length, it becomes necessary to connect two shafts together. This is usually done by means of flanged couplings as shown in Fig. 27.6 (*a*) and (*b*).

The flanges of the two shafts are joined together by bolts and nuts or rivets and the torque is then transferred from one shaft to another through the couplings. A little consideration will show that as the torque is transferred through the bolts, it is thus obvious that the bolts are subjected to shear stress. As the diameter of bolts is small, as compared to the diameter of the flanges, therefore shear stress is as assumed to be uniform in the bolts. The design of a shaft coupling means (*a*) design of bolts and (*b*) design of keys.

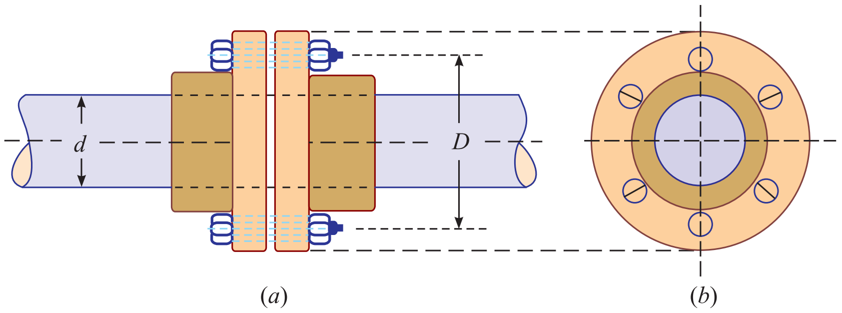

Fig. 27.6

Now we shall discuss the above two designs one by one.

27.13. Design of Bolts

Consider a shaft coupling, transmitting torque from one shaft to another.

Let τ_S = Shear stress in the shaft,

d = Diameter of the shaft,

D = Diameter of the bolt pitch circle,

(*i.e.*, the circle on which the bolts are arranged)

d_b = Diameter of the bolts,

n = No. of bolts and

τ_b = Shear stress in the bolts.

We know that the torque transmitted by the shaft,

$$T = \frac{\pi}{16} \times \tau_S \times d^3 \qquad ...(i)$$

and torque resisted by one bolt

= Area × Stress × Radius of bolt circle

$$= \frac{\pi}{4} \times d_b^2 \times \tau_b \times R = \frac{\pi}{4} \times d_b^2 \times \tau_b \times \frac{D}{2} = \frac{\pi\, d_b^2 \times \tau_b \cdot D}{8}$$

∴ Total torque resisted by the bolts

$$= n \times \frac{\pi \times d_b^2 \times \tau_b\, D}{8} \qquad ...(ii)$$

Since the torque resisted by the bolts should be equal to the torque transmitted by the shaft, therefore equating (*i*) and (*ii*),

$$\frac{\pi}{16}\tau_S\, d^3 = \frac{n\times\pi\, d_b^2\cdot\tau_b\, D}{8}$$

This is the required equation for the number of bolts or the diameter of bolts.

27.14. Design of Keys

As a matter of fact, a flange is attached to a shaft by means of a key. A rectangular notch is cut on the circumference of the shaft and a similar notch is cut on inner side of the flange. The flange is then placed over the shaft in such a way, that the two notches form a rectangular hole. A rectangular key is then inserted into the hole, and the flange is said to be keyed to the shaft is shown in Fig. 27.7.

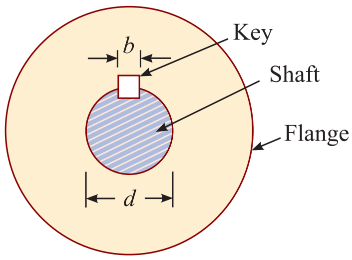

Fig. 27.7

A little consideration will show that the torque is transmitted by the shaft to the flange through the key. It is thus obvious that the key is also subjected to the shear stress.

Let l = Length of the key,

b = Width of the key and

τ_K = Shear stress in the key.

We know that the torque resisted by the key,

$$T = \text{Area}\times\text{Stress}\times\text{Radius}$$

$$= l\cdot b\times\tau_K\times r = \frac{l\cdot b\cdot\tau_K\cdot d}{2} \quad \text{...(i)}$$

We also know that the torque transmitted by the shaft,

$$T = \frac{\pi}{16}\times\tau_S\times d^3 \quad \text{...}(ii)$$

Since the torque resisted by the key should be equal to the torque transmitted by the shaft, therefore equating (*i*) and (*ii*),

$$\frac{\pi}{16}\times\tau_S\times d^3 = \frac{l\cdot b\cdot\tau_K\cdot d}{2}$$

This is the required equation for the length or width of the key.

Note. Sometimes the torque is not transmitted by flange coupling and key. But it is transmitted through gears. In such a case, the gear ratio should also be taken into account, for calculating the revolutions of the shaft.

Example 27.24. *A 80 mm shaft transmits power at maximum shear stress of 60 MPa when the stress in key and coupling bolts is 50 MPa and 40 MPa respectively. The coupling has 4 bolts arranged symmetrically along a circle of 200 mm diameter. Determine the diameter of bolts. If the key is 20 mm wide, determine its length.*

Solution. Given: Diameter of shaft (d) = 80 mm ; Shear stress in shaft (τ_S) = 60 MPa = 60 N/mm^2; Stress in key (τ_K) = 50 MPa = 50 N/mm^2 ; Stress in bolts (τ_b) = 40 MPa = 40 N/mm^2 ; No. of bolts (n) = 4 ; Bolt circle diameter (D) = 200 mm and width of the key (b) = 20 mm.

Diameter of bolts

Let d_b = Diameter of the bolts.

We know that relation for the diameter of bolts:

$$\frac{\pi}{16} \times \tau_S \times d^3 = n \times \frac{\pi\, d_b^2 \cdot \tau_b \cdot D}{8}$$

$$\frac{\pi}{16} \times 60 \times (80)^3 = 4 \times \frac{\pi\, d_b^2 \times 40 \times 200}{8}$$

$$1\,920\,000 = 4000\, d_b^2$$

$$\therefore \quad d_b^2 = \frac{1\,920\,000}{4000} = 480$$

or $\quad d_b =$ **21.9 mm say 22 mm** **Ans.**

Length of the key

Let $\quad l =$ Length of the key in mm.

We know that relation for the length of the key:

$$\frac{\pi}{16} \times \tau_S \times d^3 = \frac{l \cdot b \cdot \tau_K \cdot d}{2}$$

$$\frac{\pi}{16} \times 60 \times (80)^3 = \frac{l \times 20 \times 50 \times 80}{2}$$

$$1\,920\,000\,\pi = 40\,000\, l$$

$$\therefore \quad l = \frac{1\,920\,000\,\pi}{40\,000} = \textbf{150.8 mm} \quad \textbf{Ans.}$$

EXAMPLE 27.25. *A motor shaft consists of a steel tube 30 mm external diameter and 3 mm thick. The engine develops 10 kW at 2000 r.p.m. What will be the maximum stress in the tube, when the power is transmitted through 4 : 1 gearing?*

SOLUTION. Given: External diameter of shaft (D) = 30 mm ; Thickness = 3 mm or internal diameter (d) = 30 – (2 × 3) = 24 mm ; Power (P) = 10 kW; Engine speed = 2000 r.p.m. and gearing = 4 : 1.

Let $\quad \tau =$ Torque transmitted by the shaft and

$\quad \tau =$ Maximum shear stress in the shaft.

Since the power is transmitted through 4 : 1 gearing, therefore speed of the shaft,

$$N = \frac{2000}{4} = 500 \text{ r.p.m.}$$

We know that power transmitted by the shaft (P)

$$10 = \frac{2\pi NT}{60} = \frac{2\pi \times 500 \times T}{60} = 52.36\, T$$

$$\therefore \quad T = \frac{10}{52.36} = 0.19 \text{ kN-m} = 0.19 \times 10^6 \text{ N-mm}$$

We also know that torque transmitted by the shaft (T)

$$0.19 \times 10^6 = \frac{\pi}{16} \times \tau \times \left[\frac{D^4 - d^4}{D}\right] = \frac{\pi}{16} \times \tau \times \left[\frac{(30)^4 - (24)^4}{30}\right] = 3130\, \tau \text{ N-mm}$$

$$= 3130\, \tau$$

$$\therefore \quad \tau = \frac{0.19 \times 10^6}{3130} = 60.7 \text{ N/mm}^2 = \textbf{60.7 MPa} \quad \textbf{Ans.}$$

EXERCISE 27.3

1. A steel shaft *ABC* is subjected to two equal and opposite torques as shown in Fig. 27.8.

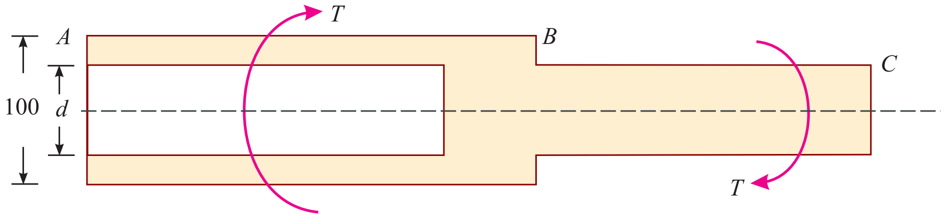

Fig. 27.8

If the shearing stresses developed in *AB* and *BC* are equal, then find the value of internal diameter of the hollow shaft. (**Ans.** 72 mm)

2. A solid steel shaft *AB* of 30 mm diameter has enlarged ends at *A* and *B*. On this enlarged portion is held a steel tube of internal diameter 6 cm and 2 mm thick. If a torque of is applied on the composite shaft, determine torque shared by the shaft and sleeve. (**Ans.** 66 kN-m ; 14 kN-m)
3. A solid shaft 1 m long and 30 mm diameter is transmitting power from one pulley to another. Find the strain energy that can be stored in the shaft, when the shaft is subjected to a shear stress of 40 MPa. Take *C* as 80 GPa. (**Ans.** 3.53×10^3 N-mm)
4. Two 100 mm diameter shafts are to be connected by means of flanges with 20 mm diameter bolts equally spaced in a circle of diameter 200 mm. If the maximum shear stress in the shaft is not to exceed 75 MPa and the average shear stress in the bolts is not to exceed 60 MPa, determine the number of rivets. (**Ans.** 8)

QUESTIONS

1. Define the term 'torque'.
2. Write the assumptions for finding out the shear stress in a circular shaft, subjected to torsion.
3. Prove $\frac{\tau}{R} = \frac{C\theta}{l}$ in case of torsion of a circular shaft.
4. Obtain a relation for the torque and power, a solid shaft can transmit.
5. Explain the term 'polar modulus'.
6. Derive an expression for the angle of twist in the case of a member of circular cross-section subjected to torsional moment.

OBJECTIVE TYPE QUESTIONS

1. Torque transmitted by a solid shaft of diameter (*D*), when subjected to a shear stress (τ) is equal to

   (*a*) $\frac{\pi}{16} \times \tau \times D^2$ (*b*) $\frac{\pi}{16} \times \tau \times D^3$ (*c*) $\frac{\pi}{32} \times \tau \times D^2$ (*d*) $\frac{\pi}{32} \times \tau \times D^3$

2. A shaft revolving at r.p.m. transmits torque (*T*) in kg-m. The power developed is

   (*a*) $2\pi NT$ kW (*b*) $\frac{2\pi NT}{30}$ kW (*c*) $\frac{2\pi NT}{60}$ kW (*d*) $\frac{2\pi NT}{120}$ kW