Unit-5

Stochastic Processes

5.1

Stationarity of Stochastic Processes

In electrical systems voltage or current waveforms are used as signals for collecting, transmitting or processing information, as well as for controlling and providing power to a variety of devices. These signals (voltage or current waveforms) are functions of time and are of two classes: *deterministic* and *random*. Deterministic signals can be described by the usual mathematical functions with time t as the independent variable. But a random signal always has some element of uncertainty associated with it and hence it is not possible to determine its value exactly at any given point of time. However, we may be able to describe the random signal in terms of its average properties such as the average power in the random signal, its spectral distribution and the probability that the signal amplitude exceeds a given value. The probabilistic model used for characterising a random signal is called a **stochastic process** or **random process**.

A random variable (r.v) is a rule (or function) that assigns a real number to every outcome of a random experiment, while *a stochastic process is a rule* (or *function*) *that assigns a time function to every outcome of a random experiment*.
For example, consider the random experiment of throwing a die at $t = 0$ and observing the number on the top face. The sample space of this experiment consists of the outcomes $\{1,2,3, \ldots ,6\}$. For each outcome of the experiment, let us arbitrarily assign a function of time $t(0 \leq t < \infty)$ in the following manner:

Outcome:	1	2	3	4	5	6
Function of time:	$x_1(t)$	$x_2(t)$	$x_3(t)$	$x_4(t)$	$x_5(t)$	$x_6(t)$

The set of functions $\{x_1(t), x_2(t), \ldots , x_6(t)\}$ represents a stochastic process.

Definition: A **stochastic process** is a collection (or ensemble) of r.vs $\{X(s,t)\}$ that are functions of a real variable, namely time t where $s \in S$ (sample space) and $t \in T$ (parameter set or index set).

The set of possible values of any individual member of the stochastic process is called **state space**. Any individual member itself is called a ***sample function*** or a **realization of the process**.

Note:

1. If s and t are fixed, $\{X(s,t)\}$ is a number.
2. If t is fixed $\{X(s,t)\}$ is a r.v.
3. If s is fixed, $\{X(s,t)\}$ is a single time function
4. If s and t are variables, $\{X(s,t)\}$ is a collection of r.vs that are time functions.

Notation: As the dependence of a stochastic process on s is obvious, s will be omitted hereafter in the notation of a stochastic process.
If the parameter set T is discrete, the stochastic process will be noted by $\{X(n)\}$ or $\{X_n\}$.

If the parameter set T is continuous, the process will be denoted by $\{X(t)\}$.

Classification of stochastic Processes

Depending on the continuous or discrete nature of the state space S and parameter set T, a stochastic process can be classified into *four types*:

1. If both T and S are *discrete*, the stochastic process is called a discrete **stochastic sequence**.
   For example, if X_n represents the outcome of the n^{th} throw of a fair die, then $\{X_n, n \geq 1\}$ is a discrete sequence, since $T = \{1,2,3,\dots\}$ and $S = \{1,2,3,4,5,6\}$.
2. If T is discrete and S is continuous, the stochastic process is called a **continuous stochastic sequence.**
   For example, if X_n represents the temperature at the end of the n^{th} hour

of a day, then $\{X_n, 1 \le n \le 24\}$ is a continuous stochastic sequence, since temperature can take any value in an interval and hence continuous.

3. If T is continuous and S is discrete, the stochastic process is called a **discrete stochastic process.**
   For example, if $X(t)$ represents the number of telephone calls received in the interval $(0, t)$ then $\{X(t)\}$ is a discrete stochastic process, since $S = \{0,1,2,3, \dots\}$.
4. If both T and S are continuous, the stochastic process is called a **continuous stochastic process**.
   For example, if $X(t)$ represents the maximum temperature at a place in the interval $(0, t)$, $\{X(t)\}$ is a continuous stochastic process.
   In the names given above, the word *discrete* or *continuous* is used to refer to the nature of S and the word *sequence* or *process* is used to refer to the nature of T.

Methods of Description of a Stochastic Process

Since a stochastic process is an indexed set of r.vs, we can obviously use the *joint probability distribution functions* to describe a stochastic process.

For a specific $t, X(t)$ is a r.v as was observed earlier.

$F(x, t) = P\{X(t) \le x\}$ is called the ***first-order distribution*** of the process $\{X(t)\}$ and $f(x, t) = \frac{\partial}{\partial x}\left(F(x, t)\right)$ is called the ***first-order density*** of $\{X(t)\}$.

$F(x_1, x_2, t_1, t_2) = P\{X(t_1) \le x_1 ; X(t_2) \le x_2\}$ is the joint distribution of the r.vs $X(t_1)$ and $X(t_2)$ and is called the ***second-order distribution*** of the process $\{X(t)\}$ and $f(x_1, x_2, t_1, t_2) = \frac{\partial^2}{\partial x_1 \partial x_2} F(x_1, x_2, t_1, t_2)$ is called the ***second-order density*** of $\{X(t)\}$.

Similarly the n^{th} order distribution $\{X(t)\}$ is the joint distribution $F\{x_1, x_2, \dots, x_n ; t_1, t_2, \dots, t_n\}$ of the r.vs $X(t_1), X(t_2), \dots, X(t_n)$.

The first-order distribution function describes the instantaneous *amplitude* distribution of the process and the second-order distribution function tells us something about the structure of the signal in the time domain and hence the spectral content of the signal. Although the higher-order distributions describe the process in a more detailed manner, the first and second-order distribution functions are primarily used to describe the process.

Special Classes of Stochastic Processes

The important feature of a stochastic process is the relationship among the members of the family. Usually the nature of relationship is understood by the joint distribution function of the member r.vs.
A stochastic process is said to be **specified** only when the parameter set, the state space and the nature of dependence relationship existing among the members of the family are specified.

Based on the dependence relationship among the members of the process, stochastic processes are classified broadly into a few special types such as the ones explained below:

1. **Markov process**

If, for $t_1 < t_2 < t_3 < \cdots < t_n < t$,

$P\{X(t) \le x \mid X(t_1) = x_1, X(t_2) = x_2, \dots, X(t_n) = x_n\} = P\{X(t) \le x \mid X(t_n) = x_n\}$,
then the process $\{X(t)\}$ is called a **Markov process**.

In other words, if the future behavior of a process depends only on the present state, but not on the past, the process is a Markov process.

A discrete parameter Markov process is called a **Markov chain**.

2. **Process with independent increments**

If for all choices of $t_1, t_2, \dots, t_n$ such that $t_1 < t_2 < t_3 < \cdots < t_n$, the random variables $X(t_2) - X(t_1), X(t_3) - X(t_2), \dots, X(t_n) - X(t_{n-1})$ are independent, then the process $\{X(t)\}$ is said to be a **stochastic process with independent increments.**

Let $T = \{0,1,2, \dots\}$ be the parameter set for $\{X_n\}$. Then $\{Z_n\}$, where $Z_0 = X_0$ and $Z_n = X_n - X_{n-1}$, is a random sequence with independent increments if the r.vs $Z_0, Z_1, Z_2, \dots$, are independent.

Two processes with independent increments play an important role in the theory of random processes. One is the Poisson process that has Poisson distribution for the increments and the other is the Wiener process with a Gaussian distribution for the increments. We will take up the study of Poisson and Gaussian processes in Unit VI.

3. Stationary process

If certain probability distributions or averages do not depend on t, then stochastic process $\{X(t)\}$ is called **stationary**. A rigorous definition and detailed study of stationary processes will be discussed in this module.

Average Values of Stochastic Processes

As in the case of r.vs stochastic processes can be described in terms of averages or expected values, mostly derived from the first and second-order distributions of $\{X(t)\}$. Mean of the process $\{X(t)\}$ is the expected value of a typical member $X(t)$ of the process.

$i.e.,$ $$\mu(t) = E\{X(t)\}$$

Autocorrelation of the process $\{\boldsymbol{X}(\boldsymbol{t})\}$, denoted by $R_{xx}(t_1, t_2)$ or $R_x(t_1, t_2)$ or $R(t_1, t_2)$, is the expected value of the product of any two members $X(t_1)$ and $X(t_2)$ of the process.

$i.e.,$ $$R(t_1, t_2) = E\{X(t_1) \cdot X(t_2)\}$$

Autocovariance of the process $\{\boldsymbol{X}(\boldsymbol{t})\}$**,** denoted by $C_{xx}(t_1, t_2)$ or $C_x(t_1, t_2)$ or $C(t_1, t_2)$, is defined as

$$C(t_1, t_2) = E\big[\big(X(t_1) - \mu(t_1)\big)\big(X(t_2) - \mu(t_2)\big)\big]$$

$$= R(t_1, t_2) - \mu(t_1)\mu(t_2)$$

Correlation co-efficient of the process $\{\boldsymbol{X(t)}\}$, denoted by $\rho_{xx}(t_1, t_2)$ or $\rho(t_1, t_2)$, is defined as

$$\rho(t_1, t_2) = \frac{C(t_1, t_2)}{\sqrt{C(t_1, t_1).C(t_2, t_2)}}$$

where $C(t_1, t_1)$ is the variance of $X(t_1)$.

When we deal with two or more stochastic processes, we can use joint distribution functions or averages to describe the relationship between them.

Cross-correlation of two processes $\{X(t)\}$ and $\{Y(t)\}$ is defined as

$$R_{xy}(t_1, t_2) = E\{X(t_1) \cdot Y(t_2)\}$$

Cross-covariance of two processes $\{X(t)\}$ and $\{Y(t)\}$ is defined as

$$C_{xy}(t_1, t_2) = R_{xy}(t_1, t_2) - \mu_x(t_1)\mu_y(t_2)$$

Cross correlation co-efficient of two processes $\{X(t)\}$ and $\{Y(t)\}$ is defined as

$$\rho_{xy}(t_1, t_2) = \frac{C_{xy}(t_1, t_2)}{\sqrt{C_{xx}(t_1, t_1)C_{yy}(t_2, t_2)}}$$

Stationarity

A stochastic process is called a **strongly stationary process** or **strict sense stationary process** (abbreviated as **SSS** process), if all its finite dimensional distributions are invariant under translation of time parameter.That is, if the joint distribution (and hence the joint density) of $X(t_1), X(t_2), \dots, X(t_n)$ is the same as that of $X(t_1 + h), X(t_2 + h), \dots, X(t_n + h)$ for all $t_1, t_2, \dots, t_n$ and $h > 0$ and for all $n \geq 1$, then the stochastic process $\{X(t)\}$ is called a **SSS process**. If the definition given above holds good for $n = 1,2, \dots, k$ only and not for $n > k$, then the process is called $\boldsymbol{k}^{\text{th}}$ **order stationary**.

Note: If a stochastic process is a SSS process, as per the definition, its first-order densities must be invariant under translation of time, $i.e.$, the densities of $X(t)$ and $X(t + h)$ are the same, $i.e.$, $f(x, t) = f(x, t + h)$.This is possible only if

$f(x,t)$ is independent of t.Therefore, *first-order densities (and hence distribution functions) of a SSS process are independent of time*.

As a consequence, $\mu_r' = E\{X^r(t)\}$, $r \geq 1$ *is also independent of time* t. *From this it follows that* $\mu = E\{X(t)\}$ *and variance=* $V\{X(t)\}$ *are independent of time* t

Also the second-order densities must be invariant under translation of time, $i.e.$, the joint p.d.f of $\{X(t_1), X(t_2)\}$ is the same as that of $\{X(t_1 + h), X(t_2 + h)\}$.

$$i.e., f(x_1, x_2, t_1, t_2) = f(x_1, x_2, t_1 + h, t_2 + h).$$

This is possible only if $f(x_1, x_2, t_1, t_2)$ is function of $t = t_1 - t_2$.

Therefore, second-order densities (and hence distribution functions) of a SSS process are functions of $t = t_1 - t_2$.

As a consequence, $R(t_1, t_2) = E\{X(t_1)X(t_2)\}$ is also a function of $t = t_1 - t_2$.It is pointed out that if $E\{X(t)\}$ is a constant and $R(t_1, t_2)$ is a function of $(t_1 - t_2)$, the stochastic process $\{X(t)\}$ need not be a SSS process.

The definition of strict sense Stationarity can be extended as follows:

Two real-valued stochastic processes $\{X(t)\}$ and $\{Y(t)\}$ are said to be **jointly stationary** in the strict sense, if the joint distribution of $X(t)$ and $Y(t)$ are invariant under translation of time.

The complex stochastic process $\{Z(t)\}$, where $Z(t) = X(t) + iY(t)$, is said to be a SSS process if $\{X(t)\}$ and $\{Y(t)\}$ are jointly stationary in the strict sense.

Wide-sense Stationarity

A stochastic process $\{X(t)\}$ with finite first-and second-order moments is called a **weakly stationary process** or **covariance stationary process** or **wide-sense stationary process** (abbreviated as **WSS process**), if its mean is a constant and the autocorrelation depends only on the time difference.

$$i.e., \text{ if } E\{X(t)\} = \mu \text{ and } E\{X(t)X(t-\tau)\} = R(\tau)$$

Note: From the definition given above, it is clear that a SSS process with finite first-and second-order moments is a WSS process, while a WSS process need not be a SSS process.

A stochastic process that is not stationary in any sense is called an ***evolutionary process.***

Two stochastic processes $\{X(t)\}$ and $\{Y(t)\}$ are said to be jointly stationary in the wide sense, if each process is individually a WSS process and $R_{xy}(t_1, t_2)$ is a function of $(t_1 - t_2)$ only.

Example of an SSS Process

Let X_n denote the presence or absence of a pulse at the n^{th} time instant in a digital communication system or digital data processing system.

If $P\{X_n = 1\} = p$ and $P\{X_n = 0\} = 1 - p = q$, then the stochastic processes (sequence) $\{X_n, n \geq 1\}$, called the **Bernoulli's process**, is a SSS process, for its first-order distribution is given by

$X_n = r$	1	0
$P(X_n = r)$	p	q

This distribution is the same for any X_n, $i.e.$, for X_m and X_{m+p}.

Consider the second-order distribution of the process, $i.e.$, the joint distribution of X_r and X_s.

X_s \ X_r	1	0
1	p^2	pq
0	pq	q^2

This joint distribution is the same for the pair of members X_r and X_s and for the pair X_{r+p} and X_{s+p} of the process.

Consider the third-order distribution of the process, $i.e.$, the joint distribution of X_r , X_s and X_t that is given below:

$$P\{X_r = 0, X_s = 0, X_t = 0\} = q^3$$

$$P\{X_r = 0, X_s = 0, X_t = 1\} = pq^2$$

$$P\{X_r = 0, X_s = 1, X_t = 0\} = pq^2$$

$$P\{X_r = 0, X_s = 1, X_t = 1\} = p^2q$$

$$P\{X_r = 1, X_s = 0, X_t = 0\} = pq^2$$

$$P\{X_r = 1, X_s = 0, X_t = 1\} = p^2q$$

$$P\{X_r = 1, X_s = 1, X_t = 0\} = p^2q$$

$$P\{X_r = 1, X_s = 1, X_t = 1\} = p^3$$

This joint distribution is the same for the triple of members X_r, X_s, X_t and for $X_{r+p}, X_{s+p}, X_{t+p}$ of the process and so on, $i.e.$ distributions of all orders are invariant under translation of time.

Note: If $Y_n = \sum_{n=1}^{n} X_n =$ the total number of pulses from time instant 1 through n, then the stochastic processes $\{Y_n , n \geq 1\}$, called the **Binomial process**, is not a SSS process, for $P\{Y_n = i\} = {}^nC_i \, p^i \, q^{n-i}$ $(i = 0,1,2, \dots , n)$ depends on n, $i.e.$, the distributions of Y_m and Y_{m+p} are not the same).

Analytical Representation of a Stochastic Process

Deterministic signals are usually expressed in simple analytical forms such as $X(t) = e^{-t^2}$ and $Y(t) = 20 \sin 10t$. It is sometimes possible to express a stochastic process in an analytical form using one or more r.vs. For example, consider an FM station that is broadcasting a *tone*, $X(t) = 100 \cos(10^8 t)$, to a large number of receivers distributed randomly in a metropolitan area. The amplitude and phase of the waveform received by any receiver will depend on the distance between the transmitter and the receiver. Since there are a large number of receivers distributed randomly over an area, the distance can be considered as a continuous r.v. Since the amplitude and the phase are functions of distance, they are also r.vs. So we can represent the ensemble (collection) of received waveforms by a stochastic process $\{X(t)\}$ of the form

$$X(t) = A\cos(10^8 t + \theta)$$

where A and θ are r.vs representing the amplitude and phase of the received waveforms.

Such representation of a stochastic process in terms of one or more r.vs whose probability law is known is used in several applications in communication systems.

Example 1: Examine whether Poisson process $\{X(t)\}$. Given by the probability law $P\{X(t) = r\} = \frac{e^{-\lambda t}(\lambda t)^r}{r!}, r = 0, 1, 2, \ldots$, is covariance stationary.

Solution: The probability distribution of $X(t)$ is a Poisson distribution with parameter λt.

$\therefore$ $$E\{X(t)\} = \lambda t \neq a \text{ constant.}$$

Therefore, the Poisson process is not covariance stationary.

Example 2:The process $\{X(t)\}$ whose probability distribution is given by

$$\mathbf{P}\{X(t) = n\} = \frac{(at)^{n-1}}{(1+at)^{n+1}}, n = 1, 2, \ldots$$

$$= \frac{at}{1+at}, n = 0$$

Show that it is not stationary.

Solution: The probability distribution of $X(t)$ is

$X(t) = n$	0	1	2	3	...
P_n	$\frac{at}{1+at}$	$\frac{1}{(1+at)^2}$	$\frac{at}{(1+at)^3}$	$\frac{(at)^2}{(1+at)^4}$	...

$$E\{X(t)\} = \sum_{n=0}^{\infty} np_n = \frac{1}{(1+at)^2} + \frac{2at}{(1+at)^3} + \frac{3(at)^2}{(1+at)^4} + \cdots$$

$$= \frac{1}{(1+at)^2}\{1 + 2\alpha + 3\alpha^2 + \cdots\}, \text{ where } \alpha = \frac{at}{1+at}$$

$$= \frac{1}{(1+at)^2}(1-\alpha)^{-2} = \frac{1}{(1+at)^2}(1+at)^2 = 1$$

$$E\{X^2(t)\} = \sum_{n=0}^{\infty} n^2 p_n = \sum_{n=1}^{\infty} n^2 \frac{(at)^{n-1}}{(1+at)^{n+1}}$$

$$= \frac{1}{(1+at)^2}\left[\sum_{n=1}^{\infty} n(n+1)\left(\frac{at}{1+at}\right)^{n-1} - \sum_{n=1}^{\infty} n\left(\frac{at}{1+at}\right)^{n-1}\right]$$

$$= \frac{1}{(1+at)^2}\left[\frac{2}{\left(1-\frac{at}{1+at}\right)^3} - \frac{1}{\left(1-\frac{at}{1+at}\right)^2}\right]$$

$$= 1 + 2at$$

$$\therefore Var\{X(t)\} = E\{X^2(t)\} - (E\{X(t)\})^2 = 2at$$

Since $Var\,(X(t))$ is a function of t, the given process is not stationary.

Example 3: Given a r.v Y with characteristic function

$$\emptyset(\boldsymbol{w}) = \boldsymbol{E}\left(\boldsymbol{e}^{i\omega Y}\right) = \boldsymbol{E}(\mathbf{cos}\,\boldsymbol{\omega Y} + \boldsymbol{i}\,\mathbf{sin}\,\boldsymbol{\omega Y})$$

and a stochastic process defined by $X(t) = \cos(\lambda t + Y)$, Show that $\{X(t)\}$ is stationary in the wide sense if $\emptyset(1) = \emptyset(2) = 0$

Solution: We have $E\{X(t)\} = E\{\cos(\lambda t + Y)\} = E\{cos\lambda t\; cosY - sin\lambda t\; sinY\}$

$$= \cos\lambda t\, E(\cos Y) - \sin\lambda t\, E(\sin Y) \quad \text{... (1)}$$

Given $\emptyset(1) = 0$ $i.e.$, $E\{\cos Y + i\sin Y\} = 0$

Therefore, $E(\cos Y) = 0 = E(\sin Y)$ ... (2)

Using (2) in (1), we get $E\{X(t)\} = 0$ ... (3)

$$E\{X(t_1)X(t_2)\} = E\{\cos(\lambda t_1 + Y)\cos(\lambda t_2 + Y)\}$$

$$= \cos\lambda\, t_1 \cos\lambda\, t_2 E(cos^2 Y) + \sin\lambda\, t_1 \sin\lambda\, t_2 E(sin^2 Y) - \sin\lambda\,(t_1 + t_2)E(\sin Y \cos Y)$$

$$= \cos\lambda\, t_1 \cos\lambda\, t_2 E\left(\frac{1}{2} + \frac{1}{2}\cos 2\,Y\right) + \sin\lambda\, t_1 \sin\lambda\, t_2\, E\left(\frac{1}{2} - \frac{1}{2}\cos 2\,Y\right)$$

$$-\tfrac{1}{2}\sin\lambda\,(t_1 + t_2)E(\sin 2\,Y) \qquad \ldots (4)$$

Given: $\emptyset(2) = 0$, $i.e.$, $E\{\cos 2\,Y + i\sin 2\,Y\} = 0$

Therefore, $E(\cos 2\,Y) = 0 = E(\sin 2\,Y)$ ... (5)

Using (5) in (4), we get

$$R(t_1, t_2) = E\{X(t_1)X(t_2)\} = \tfrac{1}{2}\{\cos\lambda\, t_1 \cos\lambda\, t_2 + \sin\lambda\, t_1 \sin\lambda\, t_2\}$$

$$= \tfrac{1}{2}\cos\lambda\,(t_1 - t_2)$$

From (3) and (6), it follows that $\{X(t)\}$ is a WSS process.

Example 4: In the fair coin experiment, we define the process $\{X(t)\}$ as follows.

$X(t) = \sin\pi\, t$, if head shows, and $2t$, if tail shows.

(a) Find $E\{X(t)\}$ and (b) find $F(x, t)$ for $t = 0.25$

Solution:

(a). The probability distribution of $\{X(t)\}$ is given by

$P\{X(t) = \sin\pi t\} = \frac{1}{2}$ and $P\{X(t) = 2t\} = \frac{1}{2}$

Therefore, $E\{X(t)\} = \frac{1}{2}\sin\pi t + \frac{1}{2}.\,2t = \frac{1}{2}sin\,\pi t + t$

(b) When $t = 0.25$, $P\left\{X(t) = \frac{1}{\sqrt{2}}\right\} = \frac{1}{2}$ and $P\left\{X(t) = \frac{1}{2}\right\} = \frac{1}{2}$

$\therefore F(x, 0.25)$ is given by

$$F(x, 0.25) = \begin{cases} 0 & , \text{if } x < \frac{1}{2} \\ \frac{1}{2} & , \text{if } \frac{1}{2} \leq x < \frac{1}{\sqrt{2}} \\ 1 & , \text{if } \frac{1}{\sqrt{2}} \leq x \end{cases}$$

Example 5: If $\{X(t)\}$ is a wide – sense stationary process with autocorrelation $R(\tau) = Ae^{-\alpha|\tau|}$, determine the second order moment of the r.v $X(8) - X(5)$

Solution: Second moment of $X(8) - X(5)$ is given by

$$E[\{X(8) - X(5)\}^2] = E\{X^2(8)\} + E\{X^2(5)\} - 2E\{X(8)X(5)\} \quad \text{... (1)}$$

Given: $R(\tau) = Ae^{-\alpha|\tau|}$

i.e., $R(t_1, t_2) = Ae^{-\alpha|t_1 - t_2|}$

$\therefore \ E\{X^2(t)\} = R(t, t) = A$

$\therefore \ E\{X^2(8)\} = E\{X^2(5)\} = A$ ... (2)

Also $E\{X(8)X(5)\} = R(8, 5) = Ae^{-3\alpha}$ ... (3)

Using (2) and (3) in (1), we get

$$E[\{X(8) - X(5)\}^2] = 2A(1 - e^{-3\alpha})$$

Example 6: Show that the process $X(t) = A\cos\lambda t + B\sin\lambda t$ (where A and B are r.vs) is wide – sense stationary, if

(i) $E(A) = E(B) = 0$

(ii) $E(A^2) = E(B^2)$ **and** $E(AB) = 0$

Solution: $E\{X(t)\} = \cos\lambda t E(A) + \sin\lambda t \times E(B)$ ... (1)

If $\{X(t)\}$ is to be a WSS process, $E\{X(t)\}$ must be a constant ($i.e.$,independent of t)

In (1), if $E(A)$ and $E(B)$ are any constants other than zero, $E\{X(t)\}$ will be a function of t.Therefore,

$$E(A) = E(B) = 0$$

$$R(t_1, t_2) = E\{X(t_1)X(t_2)\}$$

$$= E\{(A\cos\lambda\, t_1 + B\sin\lambda t_2)(A\cos\lambda\, t_2 + B\sin\lambda\, t_2)\}$$

$$= E(A^2)\cos\lambda\, t_1 \cos\lambda\, t_2 + E(B^2)\sin\lambda\, t_2 \sin\lambda\, t_2 +$$

$$E(AB)\sin\lambda\,(t_1 + t_2) \qquad \text{...(2)}$$

If $\{X(t)\}$ is to be a WSS process, $R(t_1, t_2)$ must be a function of $(t_1 - t_2)$.

Therefore, In (2),if $E(AB) = 0$ and $E(A^2) = E(B^2) = k$, then,

$$R(t_1, t_2) = k\cos\lambda(t_1 - t_2)$$

Example 7: If the $2n$ r.vs A_r and B_r are uncorrelated with zero mean and $E(A_r^2) = E(B_r^2) = \sigma_r^2$,show that the process $X(t) = \sum_{r=1}^{n}(A_r\cos\omega_r t + B_r\sin\omega_r t)$ is wide – sense stationary. What are the mean and autocorrelation of $X(t)$?

Solution: The mean of $X(t)$ $= E\{X(t)\} = \sum_{r=1}^{n} E(A_r)\cos\omega_r t + E(B_r)\sin\omega_r t = 0$

$$E\{X(t_1)X(t_2)\} = E\left\{\sum_{r=1}^{n}\sum_{s=1}^{n}(A_r\cos\omega_r t_1 + B_r\sin\omega_r t_1)(A_s\cos\omega_s t_2 + B_s\sin\omega_s t_2)\right\}$$

Since $E\{A_r\, A_s\}, E\{B_r B_s\}, E\{A_r B_r\}$ and $E\{A_s B_s\}$ are all zero, for $r \neq s$, we have

$$R(t_1, t_2) = E\{X(t_1)X(t_2)\} = \sum_{r=1}^{n} E(A_r^2)\cos\omega_r t_1\cos\omega_r t_2 + E(B_r^2)\sin\omega_r t_1\sin\omega_r t_2$$

$$= \sum_{r=1}^{n}\sigma_r^2\cos\omega_r(t_1 - t_2)$$

Therefore, $\{X(t)\}$ is a WSS process.