3.1

Introduction to Vectors, Gradient and Directional Derivative

Introduction

The underlying elements in vector analysis are vectors and scalars. We use the notation $\mathbb{R}$ to denote the real line which is identified with the set of real numbers, $\mathbb{R}^2$ to denote the Cartesian plane, $\mathbb{R}^3$ to denote the ordinary 3-space. We denote vectors by bold face Roman letters.

Vectors

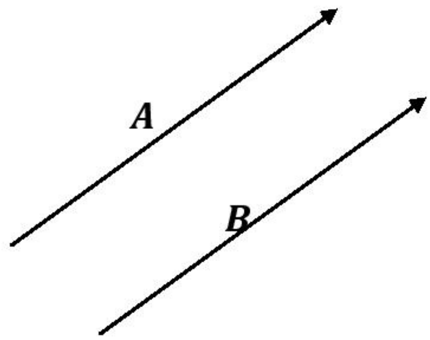
There are quantities in physics and science characterized by both magnitude and direction, such as displacement, velocity, force, and acceleration. To describe such quantities, we introduce the concept of a vector as a directed line segment $\overrightarrow{PQ}$ from one point P to another point Q. Here P is called the initial point or origin of $\overrightarrow{PQ}$, and Q is called the terminal point, end, or terminus of the vector.

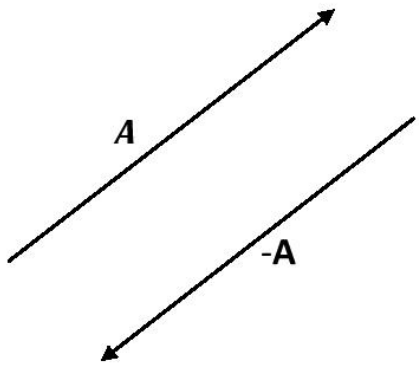
We will denote vectors by bold-faces letters or letters with an arrow over them. Thus the vector $\overrightarrow{PQ}$ may be denoted by $\boldsymbol{A}$ or $\vec{A}$ as in following figure. The magnitude or length of the vector is then denoted by $|\overrightarrow{PQ}|$, $|A|$, $|\vec{A}|$, or A.

The following comments apply.

(a) Two vectors $\boldsymbol{A}$ and $\boldsymbol{B}$ are equal if they have the same magnitude and direction regardless of their initial point. Thus $\boldsymbol{A} = \boldsymbol{B}$ in following 1st figure.

(b) A vector having direction opposite to that of a given vector $\boldsymbol{A}$ but having the same magnitude is denoted by $-\boldsymbol{A}$ [see 2nd following figure] and is called the negative of $\boldsymbol{A}$.





Scalars

Other quantities in physics and science are characterized by magnitude only, such as mass, length, and temperature. Such quantities are often called scalars to distinguish them from vectors. However, it must be emphasized that apart from units, such as feet, degrees, etc., scalars are nothing more than real numbers. Thus we can denote them, as usual, by ordinary letters. Also, the real numbers 0 and 1 are part of our set of scalars.

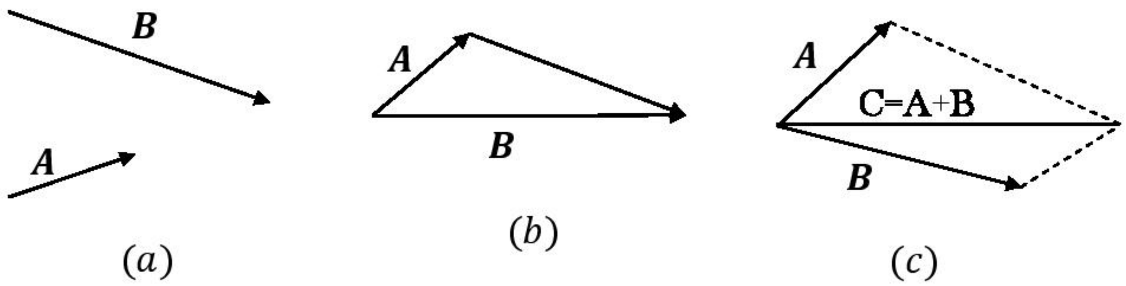
Vector Algebra

There are two basic operations with vectors:

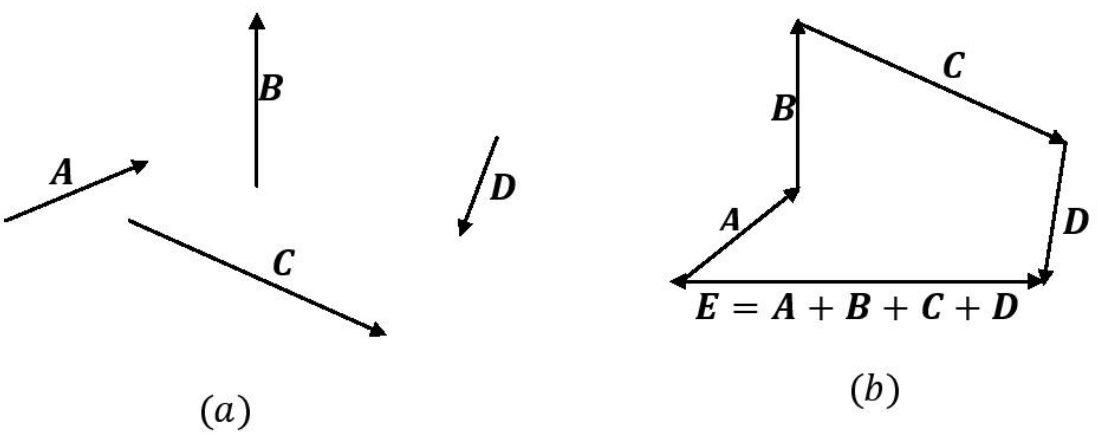
(a) Vector Addition; (b) Scalar Multiplication.

(a) Vector Addition

Consider vectors $\boldsymbol{A}$ and $\boldsymbol{B}$, pictured in figure (a). The sum or resultant of $\boldsymbol{A}$ and $\boldsymbol{B}$, is a vector $\boldsymbol{C}$ formed by placing the initial point of $\boldsymbol{B}$ on the terminal point of $\boldsymbol{A}$ and them joining the initial point of $\boldsymbol{A}$ to the terminal point of $\boldsymbol{B}$, pictured in figure (b). The sum $\boldsymbol{C}$ is written $\boldsymbol{C} = \boldsymbol{A} + \boldsymbol{B}$. This definition here is equivalent to the Parallelogram Law for vector addition, pictured in figure (c).



Extensions to sums of more than two vectors are immediate. Consider, for example, vectors $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D}$ in figure (a). Then figure (b) shows how to obtain the sum of resultant $\boldsymbol{E}$ of the vectors $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D}$, that is, by connecting the end of each vector to the beginning of the next vector.



The difference of vectors $\boldsymbol{A}$ and $\boldsymbol{B}$, denoted by $\boldsymbol{A}-\boldsymbol{B}$, is that vector $\boldsymbol{C}$, which added to $\boldsymbol{B}$, gives $\boldsymbol{A}$. Equivalently, $\boldsymbol{A}-\boldsymbol{B}$, may be defined as $\boldsymbol{A}+(-\boldsymbol{B})$.

If $\boldsymbol{A}=\boldsymbol{B}$, then $\boldsymbol{A}-\boldsymbol{B}$ is defined as the null or zero vector; it is represented by the symbol $\mathbf{0}$. It has zero magnitude and its direction is undefined. A vector that is not null is a proper vector. All vectors will be assumed to be proper unless otherwise stated.

(b) Scalar Multiplication

Multiplication of a vector $\boldsymbol{A}$ by a scalar m produces a vector $m\boldsymbol{A}$ with magnitude $|m|$ times the magnitude of $\boldsymbol{A}$ and the direction of $m\boldsymbol{A}$ is in the same or opposite of $\boldsymbol{A}$ according as m is positive or negative. If $m=0$, then $m\boldsymbol{A}=0$, the null vector.

Laws of Vector Algebra

The following theorem applies

Theorem: Suppose $\boldsymbol{A},\boldsymbol{B},\boldsymbol{C}$ are vectors and m and n are scalars. Then the following laws hold:

$[A_1]$ $(\boldsymbol{A}+\boldsymbol{B})+\boldsymbol{C}=(\boldsymbol{A}+\boldsymbol{B})+\boldsymbol{C}$ Associative Law for Addition

$[A_2]$ There exists a zero vector $\mathbf{0}$ such that, for every vector $\boldsymbol{A}$.

$$\boldsymbol{A}+\mathbf{0}=\mathbf{0}+\boldsymbol{A}=\boldsymbol{A} \qquad \text{Existence of Zero Element}$$

$[A_3]$ For every vector $\boldsymbol{A}$, there exists a vector $-\boldsymbol{A}$ such that

$\boldsymbol{A} + (-\boldsymbol{A}) = (-\boldsymbol{A}) + \boldsymbol{A} = \boldsymbol{0}$ Existence of Negatives

$[A_4]$ $\boldsymbol{A} + \boldsymbol{B} = \boldsymbol{B} + \boldsymbol{A}$ Commutative Law for Addition

$[M_1]$ $m(\boldsymbol{A} + \boldsymbol{B}) = m\boldsymbol{A} + m\boldsymbol{B}$ Distributive Law

$[M_2]$ $(m + n)\boldsymbol{A} = m\boldsymbol{A} + n\boldsymbol{A}$ Distributive Law

$[M_3]$ $m(n\boldsymbol{A}) = (mn)\boldsymbol{A}$ Associative Law

$[M_4]$ $1(\boldsymbol{A}) = \boldsymbol{A}$ Unit Multiplication

The above eight laws are the axioms that define an abstract structure called a Vector space.

The above laws split into two sets, as indicated by their labels. The first four laws refer to vector addition. One can then prove the following properties of vector addition.

a) Any sum $\boldsymbol{A}_1 + \boldsymbol{A}_2 + \cdots + \boldsymbol{A}_n$ of vectors requires no parentheses and does not depend on the order of the summands.
b) The zero vector $\boldsymbol{0}$ is unique and the negative $-\boldsymbol{A}$ of a vector $\boldsymbol{A}$ is unique
c) (Cancellation Law) If $\boldsymbol{A} + \boldsymbol{C} = \boldsymbol{B} + \boldsymbol{C}$, then $\boldsymbol{A} = \boldsymbol{B}$.
The remaining four laws refer to scalar multiplication. Using these additional laws, we can prove the following properties.

PROPOSITION:

a) For any scalar m and zero vector $\boldsymbol{0}$, we have $m\boldsymbol{0} = \boldsymbol{0}$.
b) For any vector $\boldsymbol{A}$ and scalar 0, we have $0\boldsymbol{A} = 0$.

c) If $m\boldsymbol{A} = 0$, then $m = 0$ or $\boldsymbol{A} = \boldsymbol{0}$.
d) For any vector $\boldsymbol{A}$ and scalar m, we have $(-m)\boldsymbol{A} = m(-\boldsymbol{A}) = -(m\boldsymbol{A})$.

Unit vectors

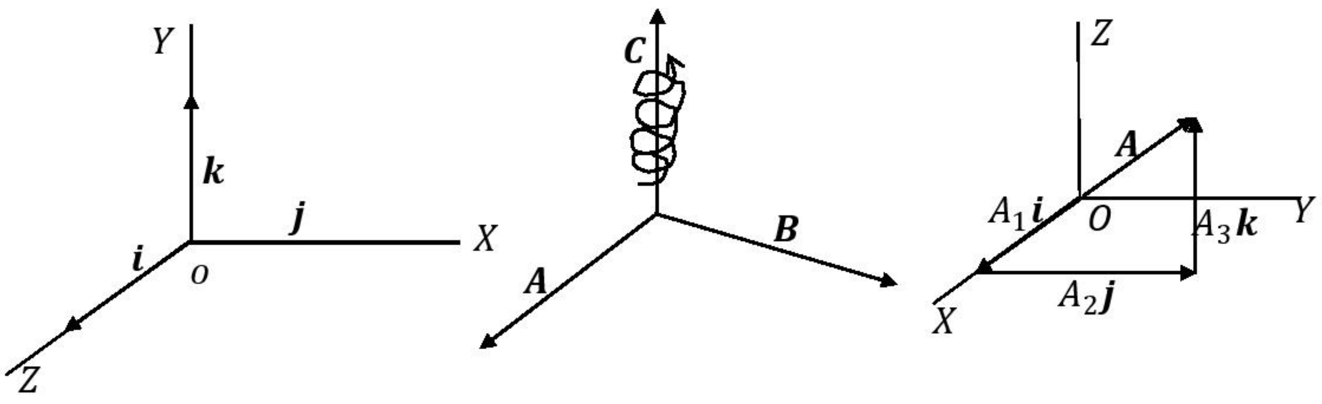
Unit vectors are vectors having unit length. Suppose $\boldsymbol{A}$ is any vector with length $|\boldsymbol{A}| > 0$. Then $\boldsymbol{A}/|\boldsymbol{A}|$ is a unit vector, denoted by $\boldsymbol{a}$, which has the same directions as $\boldsymbol{A}$. Also, every $\boldsymbol{A}$ may be represented by a unit vectors $\boldsymbol{a}$ in the direction of $\boldsymbol{A}$ multiplied by the magnitude of $\boldsymbol{A}$. That is, $\boldsymbol{A} = |\boldsymbol{A}|\boldsymbol{a}$.

Example: Suppose $|\boldsymbol{A}| = 3$. Then $\boldsymbol{a} = \boldsymbol{A}/3$ is a unit vector in the directions of $\boldsymbol{A}$. Also, $\boldsymbol{A} = 3\boldsymbol{a}$.

Rectangular Unit Vectors $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$

An important set of unit vectors, denoted by $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$, are those having the directions, respectively, of the positive X,Y,and Z axes of three-dimensional rectangular coordinate system. The coordinate system shown in figure, which we use unless otherwise stated, is called a right handed coordinate system. The system is characterized by the following property. If we curl the fingers of the right hand in the direction of a 90^0 rotation from the positive X –axis to the positive Y –axis, then the thumb will point in the direction of the positive Z –axis.

Generally speaking, suppose nonzero vectors $\boldsymbol{A}$,$\boldsymbol{B}$,$\boldsymbol{C}$ have the same initial point and are not coplanar. Then $\boldsymbol{A}$,$\boldsymbol{B}$,$\boldsymbol{C}$ are said to form a right-hand system or dextral system if a right-threaded screw rotated through an angle less than 180^0 from $\boldsymbol{A}$ to $\boldsymbol{B}$ will advance in the direction $\boldsymbol{C}$ as shown in figure.



(a) (b) (b)

Components of a vector

Any vector $\boldsymbol{A}$ in three dimensions can be represented with an initial point at the origin $0 = (0,0,0)$ and its end point at some point, say, (A_1, A_2, A_3), then the vectors, $A_1\boldsymbol{i}$,$A_2\boldsymbol{j}$,$A_3\boldsymbol{k}$ are called the component vectors of $\boldsymbol{A}$ in the X,Y,Z directions, and the scalars A_1,A_2,A_3 are called the components of $\boldsymbol{A}$ in the X, Y, Z directions, respectively.

The sum of $A_1\boldsymbol{i}$, $A_2\boldsymbol{j}$, and $A_3\boldsymbol{k}$ is the vector $\boldsymbol{A}$, so we may write

$$\boldsymbol{A} = A_1\boldsymbol{i} + A_2\boldsymbol{j} + A_3\boldsymbol{k}$$

The magnitude of $\boldsymbol{A}$ follows

$$|\boldsymbol{A}| = \sqrt{{A_1}^2 + {A_2}^2 + {A_3}^2}$$

Consider a point $P(x, y, z)$ in space. The vector $\boldsymbol{r}$ form the origin O to the point P is called the position vector (or radius vector). Thus $\boldsymbol{r}$ may be written

$$\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$$

It has magnitude $|\boldsymbol{r}| = \sqrt{x^2 + y^2 + z^2}$.

The following proposition applies,

Proposition: Suppose $\boldsymbol{A} = A_1\boldsymbol{i} + A_2\boldsymbol{j} + A_3\boldsymbol{k}$ and $\boldsymbol{B} = B_1\boldsymbol{i} + B_2\boldsymbol{j} + B_3\boldsymbol{k}$. Then

i. $\boldsymbol{A} + \boldsymbol{B} = (A_1 + B_1)\boldsymbol{i} + (A_2 + B_2)\boldsymbol{j} + (A_3 + B_3)\boldsymbol{k}$
ii. $m\boldsymbol{A} = m(A_1\boldsymbol{i} + A_2\boldsymbol{j} + A_3\boldsymbol{k}) = (mA_1)\boldsymbol{i} + (mA_2)\boldsymbol{j} + (mA_3)\ \boldsymbol{k}$

Example: Suppose $\boldsymbol{A} = 3\boldsymbol{i} + 5\boldsymbol{j} - 2\boldsymbol{k}$ and $\boldsymbol{B} = 4\boldsymbol{i} - 8\boldsymbol{j} + 7\boldsymbol{k}$

a. To find $\boldsymbol{A} + \boldsymbol{B}$ and corresponding components, obtaining $\boldsymbol{A} + \boldsymbol{B} = 7\boldsymbol{i} - 3\boldsymbol{j} + 5\boldsymbol{k}$.
b. To find $3\boldsymbol{A} - 2\boldsymbol{B}$, first multiply by the scalars and then add:
$$\begin{aligned} 3\boldsymbol{A} - 2\boldsymbol{B} &= (9\boldsymbol{i} + 15\boldsymbol{j} - 6\boldsymbol{k}) + (-8\boldsymbol{i} + 16\boldsymbol{j} - 14\boldsymbol{k}) \\ &= \boldsymbol{i} + 31\boldsymbol{j} - 20\boldsymbol{k}. \end{aligned}$$
c. To find $|\boldsymbol{A}|$ and $|\boldsymbol{B}|$, take the square root of the sum of the square of the components:
$|\boldsymbol{A}| = \sqrt{9 + 25 + 4} = \sqrt{38}$ and $|\boldsymbol{B}| = \sqrt{16 + 64 + 49} = \sqrt{129}$.

The DOT and CROSS Product

Definition: Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be two vectors. The scalar product or dot product of $\boldsymbol{a}$ and $\boldsymbol{b}$ is defined to be $\boldsymbol{a}.\boldsymbol{b} = ab\cos\theta$ where θ is the angle between the two vectors when drawn from a common origin.

Note:

1. $\boldsymbol{a}.\boldsymbol{b} = \boldsymbol{b}.\boldsymbol{a}$.
2. $\boldsymbol{a}.\boldsymbol{a} = |\boldsymbol{a}|^2 = a^2$.
3. $\boldsymbol{a}.\boldsymbol{b} = 0$ if $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular vectors.
4. $\boldsymbol{a}.(\boldsymbol{b}+\boldsymbol{c}) = \boldsymbol{a}.\boldsymbol{b} + \boldsymbol{a}.\boldsymbol{c}$.
5. $\boldsymbol{i}.\boldsymbol{i} = \boldsymbol{j}.\boldsymbol{j} = \boldsymbol{k}.\boldsymbol{k} = 1$.
6. $\boldsymbol{i}.\boldsymbol{j} = \boldsymbol{j}.\boldsymbol{k} = \boldsymbol{k}.\boldsymbol{i} = 0$.
7. If $\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}$ and $\boldsymbol{b} = b_1\boldsymbol{i} + b_2\boldsymbol{j} + b_3\boldsymbol{k}$ then $\boldsymbol{a}.\boldsymbol{b} = a_1b_1 + a_2b_2 + a_3b_3$.

Definition: Let $\boldsymbol{a}, \boldsymbol{b}$ be two nonzero vectors. Then the vector product or cross product of $\boldsymbol{a}$ and $\boldsymbol{b}$ is a vector perpendicular to both $\boldsymbol{a}$ and $\boldsymbol{b}$ with magnitude $ab\sin\theta$ where $0 \le \theta \le \pi$ is the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$ and whose direction is along a unit vector $\boldsymbol{n}$ such that $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{n}$ from a right handed system. Thus $\boldsymbol{a} \times \boldsymbol{b} = ab\sin\theta\,\boldsymbol{n}$.

Note:

1. $|\boldsymbol{a} \times \boldsymbol{b}| =$ Area of the parallelogram with $\boldsymbol{a}$ and $\boldsymbol{b}$ as adjacent sides.
2. $\boldsymbol{a} \times \boldsymbol{b} = -(\boldsymbol{b} \times \boldsymbol{a})$.
3. $\boldsymbol{a} \times \boldsymbol{b} = 0$ if $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel.
4. $\boldsymbol{a} \times (\boldsymbol{b}+\boldsymbol{c}) = \boldsymbol{a} \times \boldsymbol{b} + \boldsymbol{a} \times \boldsymbol{c}$.

5. $\boldsymbol{i}\times\boldsymbol{i}=\boldsymbol{j}\times\boldsymbol{j}=\boldsymbol{k}\times\boldsymbol{k}=0$.
6. $\boldsymbol{i}\times\boldsymbol{j}=\boldsymbol{k}=-\boldsymbol{j}\times\boldsymbol{i}, \boldsymbol{j}\times\boldsymbol{k}=\boldsymbol{i}=-\boldsymbol{k}\times\boldsymbol{j}, \boldsymbol{k}\times\boldsymbol{i}=\boldsymbol{j}=-\boldsymbol{i}\times\boldsymbol{k}$.
7. If $\boldsymbol{a}=a_1\boldsymbol{i}+a_2\boldsymbol{j}+a_3\boldsymbol{k}$ and $\boldsymbol{b}=b_1\boldsymbol{i}+b_2\boldsymbol{j}+b_3\boldsymbol{k}$ then $\boldsymbol{a}\times\boldsymbol{b}=(a_2b_3-a_3b_2)\boldsymbol{i}+(a_3b_1-a_1b_3)\boldsymbol{j}+(a_1b_2-a_2b_1)\boldsymbol{k}$.
$$=\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}.$$

Definition: The scalar triple product or box product of three vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ is defined to be the scalar $\boldsymbol{a}.(\boldsymbol{b}\times\boldsymbol{c})$. It is sometimes denoted by $[\boldsymbol{a}\ \boldsymbol{b}\ \boldsymbol{c}]$.

$$\boldsymbol{a}.(\boldsymbol{b}\times\boldsymbol{c})=[\boldsymbol{a}\ \boldsymbol{b}\ \boldsymbol{c}]=\begin{vmatrix}a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix}.$$

Note:

1. $\boldsymbol{a}.(\boldsymbol{b}\times\boldsymbol{c})$ =Volume of the parallelepiped turned by the coterminous edges $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$.
2. $[\boldsymbol{a}\ \boldsymbol{b}\ \boldsymbol{c}]=[\boldsymbol{b}\ \boldsymbol{c}\ \boldsymbol{a}]=[\boldsymbol{c}\ \boldsymbol{a}\ \boldsymbol{b}]$.
3. $[\boldsymbol{a}\ \boldsymbol{b}\ \boldsymbol{c}]=-[\boldsymbol{b}\ \boldsymbol{a}\ \boldsymbol{c}]=-[\boldsymbol{c}\ \boldsymbol{b}\ \boldsymbol{a}]=-[\boldsymbol{a}\ \boldsymbol{c}\ \boldsymbol{b}]$.
4. The vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ are coplanar if and only if $[\boldsymbol{a}\ \boldsymbol{b}\ \boldsymbol{c}]=0$.
5. $\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c})=(\boldsymbol{a}.\boldsymbol{c})\boldsymbol{b}-(\boldsymbol{a}.\boldsymbol{b})\boldsymbol{c}$.
6. $(\boldsymbol{a}\times\boldsymbol{b})\times\boldsymbol{c}=(\boldsymbol{a}.\boldsymbol{c})\boldsymbol{b}-(\boldsymbol{b}.\boldsymbol{c})\boldsymbol{a}$.
7. $(\boldsymbol{a}\times\boldsymbol{b}).(\boldsymbol{c}\times\boldsymbol{d})=\begin{vmatrix}\boldsymbol{a}.\boldsymbol{c} & \boldsymbol{a}.\boldsymbol{d}\\ \boldsymbol{b}.\boldsymbol{c} & \boldsymbol{b}.\boldsymbol{d}\end{vmatrix}$.
8. $(\boldsymbol{a}\times\boldsymbol{b})\times(\boldsymbol{c}\times\boldsymbol{d})=[\boldsymbol{a}\ \boldsymbol{b}\ \boldsymbol{d}]\boldsymbol{c}-[\boldsymbol{a}\ \boldsymbol{b}\ \boldsymbol{c}]\boldsymbol{d}$.

Scalar and vector fields:

Scalar filed: A scalar function $F(x,y,z)$ defined over some region of space D is a function that assigns to each point P_0

in D with coordinates (x_0, y_0, z_0) the number $F(P_0) = F(x_0, y_0, z_0)$. The set of all numbers $F(P)$ for all points P in D are said to form a **scalar field** over D.

Example:

The temperature at any point within or on the earth's surface at a certain time defines a scalar field.

The scalar function of position $F(x, y, z) = xyz^2$ for (x, y, z) inside the unit sphere $x^2 + y^2 + z^2 = 1$ defines a scalar field throughout the unit sphere.

Vector field: More general than a scalar field $F(x, y, z)$ is a vector field defined by a vector function $\boldsymbol{F}(x, y, z)$ over some region of space D that assigns to each point P_0 in D with coordinates (x_0, y_0, z_0) the vector $\boldsymbol{F}(P_0) = \boldsymbol{F}(x_0, y_0, z_0)$ with its tail at P_0. Functions of this type are called either **vector functions** or **vector-valued functions**.

Example:

The vector-valued function $\boldsymbol{F}(x, y, z) = (x - y)\boldsymbol{i} - (y - z)\boldsymbol{j} + (xyz - 2)\boldsymbol{k}$, for (x, y, z) inside the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, defines a vector field throughout the ellipsoid.

Suppose the velocity at any point within a moving fluid is known at a certain time defines a vector field.

Limits and continuity of vector functions of a single real variable

A vector function of a single real variable $\boldsymbol{F}(t) = f_1(t)\boldsymbol{i} + f_2(t)\boldsymbol{j} + f_3(t)\boldsymbol{k}$ is said to have $\boldsymbol{l}$ as its **limit** at t_0, written

$\lim_{t \to t_0} \boldsymbol{F}(t) = \boldsymbol{l}$, where $\boldsymbol{l} = l_1\boldsymbol{i} + l_2\boldsymbol{j} + l_3\boldsymbol{k}$. If $\lim_{t \to t_0} f_1(t) = l_1$, $\lim_{t \to t_0} f_2(t) = l_2$, and $\lim_{t \to t_0} f_3(t) = l_3$. If, in addition, the vector function is defined at t_0 and $\lim_{t \to t_0} \boldsymbol{F}(t) = \boldsymbol{F}(t_0)$, then $\boldsymbol{F}(t)$ is said to be **continuous** at t_0. A vector function $\boldsymbol{F}(t)$ that is continuous for each t in the interval $a \le t \le b$ is said to be continuous over the interval. A vector function of a single real variable that is not continuous at a point t_0 is said to be **discontinuous** at t_0.

Differentiability and the derivative of a vector function of a single real variable

The vector function of a single real variable $\boldsymbol{F}(t) = f_1(t)\boldsymbol{i} + f_2(t)\boldsymbol{j} + f_3(t)\boldsymbol{k}$ defined over the interval $a \le t \le b$ and $t_0 \epsilon [a\ b]$. Then $\lim_{t \to t_0} \frac{\boldsymbol{F}(t) - \boldsymbol{F}(t_0)}{t - t_0}$ it exists, is called the **derivative** of $\boldsymbol{F}(t)$ at t_0 and is denoted by $\boldsymbol{F}'(t_0)$ or $\left(\frac{d\boldsymbol{F}}{dt}\right)$ at $t = t_0$. We also say that $\boldsymbol{F}(t)$ is **differentiable** at a point t_0 in the interval if its components are differentiable at t_0. It is said to be differentiable over the interval if it is differentiable at each point of the interval, and when $\boldsymbol{F}(t)$ is differentiable its derivative with respect to t is

$$\frac{d\boldsymbol{F}}{dt} = \frac{df_1}{dt}\boldsymbol{i} + \frac{df_2}{dt}\boldsymbol{j} + \frac{df_3}{dt}\boldsymbol{k}.$$

Note: Every differentiable function is continuous.

When $\frac{d\boldsymbol{F}}{dt}$ is differentiable, the second order derivative $d^2\boldsymbol{F}/dt^2$ is defined as $\frac{d^2\boldsymbol{F}}{dt^2} = \frac{d}{dt}\left(\frac{d\boldsymbol{F}}{dt}\right)$ and, in general, provided the derivates exist,

$$\frac{d^n\boldsymbol{F}}{dt^n} = \frac{d}{dt}\left(\frac{d^{n-1}\boldsymbol{F}}{dt^{n-1}}\right), \text{ for } n \geq 2.$$

Properties:

Let $\boldsymbol{u}(t)$ and $\boldsymbol{v}(t)$ be differentiable function of t over some interval $a \leq t \leq b$.

$\frac{d\boldsymbol{C}}{dt} = 0$ ($\boldsymbol{C}$ is constant vector).

$\frac{d}{dt}(\lambda\boldsymbol{u}) = \lambda\frac{d\boldsymbol{u}}{dt}$ (λ an arbitrary constant scalar).

$\frac{d}{dt}(\boldsymbol{u} \pm \boldsymbol{v}) = \frac{d\boldsymbol{u}}{dt} \pm \frac{d\boldsymbol{v}}{dt}$.

$\frac{d}{dt}(\boldsymbol{u}.\boldsymbol{v}) = \frac{d\boldsymbol{u}}{dt}.\boldsymbol{v} + \boldsymbol{u}.\frac{d\boldsymbol{v}}{dt}$.

$\frac{d}{dt}(\boldsymbol{u} \times \boldsymbol{v}) = \frac{d\boldsymbol{u}}{ddt} \times \boldsymbol{v} + \boldsymbol{u} \times \frac{d\boldsymbol{v}}{dt}$.

It $\boldsymbol{u}(t)$ is a differentiable function of t and $t = t(s)$ is a differentiable function of s, then $\frac{d\boldsymbol{u}}{ds} = \frac{d\boldsymbol{u}}{dt}.\frac{dt}{ds}$.

Partial derivatives of a vector function of real variables

Let $\boldsymbol{F}$ be a vector function of scalar variable p,q, t. Then we write $\boldsymbol{F} = \boldsymbol{F}(p, q, t)$. Treating t as a variable and p, q as constants, we define $\lim_{\delta t \to 0} \frac{\boldsymbol{F}(p,q,t+\delta t) - \boldsymbol{F}(p,q,t)}{\delta t}$ if exists, is called **partial derivative** of $\boldsymbol{F}$ with respect to t and is donated by $\frac{\partial \boldsymbol{F}}{\partial t}$.

Similarly, we can define $\frac{\partial \boldsymbol{F}}{\partial p}, \frac{\partial \boldsymbol{F}}{\partial q}$ also.

If $\frac{\partial \boldsymbol{F}}{\partial t}$ exists $\frac{\partial^2 \boldsymbol{F}}{\partial t^2}$ and $\frac{\partial^2 \boldsymbol{F}}{\partial p \partial t}$ are defined as

$$\frac{\partial^2 \boldsymbol{F}}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{\partial \boldsymbol{F}}{\partial t}\right), \frac{\partial^2 \boldsymbol{F}}{\partial p \partial t} = \frac{\partial}{\partial p}\left(\frac{\partial \boldsymbol{F}}{\partial t}\right) \text{ etc.}$$

Properties:

1. $\frac{\partial}{\partial t}\ (\phi \boldsymbol{u} = \frac{\partial \phi}{\partial t}\ \boldsymbol{u} + \phi\ \frac{\partial \boldsymbol{u}}{\partial t}$ (ϕ is a scalar differential function).
2. $\frac{\partial}{\partial t}\ (\lambda \boldsymbol{u}) = \lambda\ \frac{\partial \boldsymbol{u}}{\partial t}$.
3. $\frac{\partial}{\partial t}\ (\boldsymbol{u} \pm \boldsymbol{v}) = \frac{\partial \boldsymbol{u}}{\partial t} \pm \frac{\partial \boldsymbol{v}}{\partial t}$.
4. $\frac{\partial}{\partial t}\ (\boldsymbol{u}.\boldsymbol{v}) = \frac{\partial \boldsymbol{u}}{\partial t}.\boldsymbol{v} + \boldsymbol{u}.\frac{\partial \boldsymbol{v}}{\partial t}$.
5. $\frac{\partial}{\partial t}\ (\boldsymbol{u} \times \boldsymbol{v}) = \frac{\partial \boldsymbol{u}}{\partial t} \times \boldsymbol{v} + \boldsymbol{u} \times \frac{\partial \boldsymbol{v}}{\partial t}$.
6. Let $\boldsymbol{F} = F_1\boldsymbol{i} + F_2\boldsymbol{j} + F_3\boldsymbol{k}$, where F_1, F_2, F_3 are differential scalar functions of more than one variable, then
   $\frac{\partial \boldsymbol{F}}{\partial t} = \frac{\partial F_1}{\partial t}\ \boldsymbol{i} + \frac{\partial F_2}{\partial t}\ \boldsymbol{j} + \frac{\partial F_3}{\partial t}\mathbf{k}$

Level surfaces:

Let $f(x,y,z)$ be a single valued continuous scalar function defined at every point $p \in D$. Then $f(x,y,z) = c$ (constant), defines the equation of a surface and is called a **level surface** of the function. For different values of c we obtain different surface, no two of which intersect. For example, if $f(x,y,z)$ represent temperature in a medium, then $f(x,y,z) = c$ represents a surface on which the temperature is a constant c. Such surfaces are called **isothermal** surfaces.

Example:

Find the level surface of the scalar fields in space, defined by $f(x, y, z) = z - \sqrt{x^2 + y^2}$.

Solution: We find that $f(x, y, z) = c$

$$\Rightarrow z - \sqrt{x^2 + y^2} = c \Rightarrow x^2 + y^2 = (z - c)^2$$

The level surfaces are cones.

Directional Derivatives and the Gradient Operator

Consider a scalar function $w = f(x, y, z)$ with continuous first order partial derivatives with respect to $x, y,$ and z that is defined in same region D of space, and let a space curve Γ in D have the parametric equations

$x = x(t)$, $y = y(t)$, and $z = z(t)$. Then from the chain rule

$$\frac{dw}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$$

$$= \left(\frac{\partial f}{\partial x}\,\boldsymbol{i} + \frac{\partial f}{\partial y}\boldsymbol{j} + \frac{\partial f}{\partial z}\,\boldsymbol{k}\right).\left(\frac{dx}{dt}\,\boldsymbol{i} + \frac{dy}{dt}\,\boldsymbol{j} + \frac{dz}{dt}\,\boldsymbol{k}\right).$$

The first vector, denoted by $\text{grad} f = \frac{\partial f}{\partial x}\,\boldsymbol{i} + \frac{\partial f}{\partial y}\,\boldsymbol{j} + \frac{\partial f}{\partial z}\,\boldsymbol{k}$ is called the **gradient** of the scalar function f expressed in terms of Cartesian coordinates, the second vector $\frac{d\boldsymbol{r}}{dt} = \frac{dx}{dt}\boldsymbol{i} + \frac{dy}{dt}\boldsymbol{j} + \frac{dz}{dt}\,\boldsymbol{k}$ is seen to be a vector that is tangent to the space curve Γ, consequently $\frac{dw}{dt}$ is the scalar product of $\text{grad} f$ and

$\frac{d\boldsymbol{r}}{dt}$ at the point $x = x(t)$, $y = y(t)$, and $z = z(t)$ for any given value of t.

Another notation for grad$f$ is $\nabla f = \frac{\partial f}{\partial x}\boldsymbol{i} + \frac{\partial f}{\partial y}\boldsymbol{j} + \frac{\partial f}{\partial z}\boldsymbol{k}$, where the symbol ∇f is either read "del$f$" or "grad$f$". In this notation, the vector operator $\nabla = \boldsymbol{i}\frac{\partial}{\partial x} + \boldsymbol{j}\frac{\partial}{\partial y} + \boldsymbol{k}\frac{\partial}{\partial z}$ is the gradient operator expressed in terms of Cartesian coordinates, and if ϕ is a suitably differentiable scalar function of x, y and z.

$$\therefore\ \nabla\phi = \frac{\partial\phi}{\partial x}\boldsymbol{i} + \frac{\partial\phi}{\partial y}\boldsymbol{j} + \frac{\partial\phi}{\partial z}\boldsymbol{k}.$$

Geometrical representation of the gradient:

Let $\phi(P) = \phi(x, y, z)$ be a differentiable scalar field. Let $\phi(x, y, z) = c$ be a level surface and $P_0\ (x_0, y_0, z_0)$ be a point on it. There are infinite numbers of smooth curves on the surface passing through the point P_0. Each of these curves has a tangent at P_0 the totality of all these tangent lines from a tangent plane to the surface at the point P_0. A vector normal to this plane at P_0 is called the normal vector to the surface at this point.

Consider now a smooth curve C on the surface passing through a point P on the surface. Let $x = x(t),\ y = y(t), z = z(t)$ be the parametric representation of the curve C. Any point P on C has the position vector $\boldsymbol{r} = x(t)\boldsymbol{i} + y(t)\,\boldsymbol{j} + z(t)\boldsymbol{k}$. Since the curve lies on the surface, we have

$$f(x(t), y(t), z(t)) = k$$

Then $\frac{d}{dt} f(x(t), y(t), z(t)) = 0$

By chain rule, we have $\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} = 0$

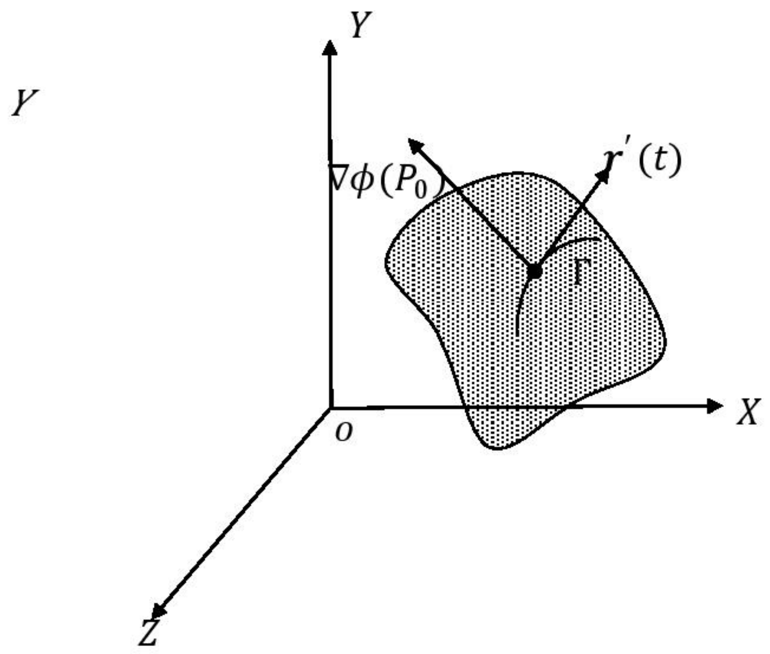
or

$$(\boldsymbol{i}\frac{\partial f}{\partial x} + \boldsymbol{j}\frac{\partial f}{\partial y} + \boldsymbol{k}\frac{\partial f}{\partial z}).(\boldsymbol{i}\frac{dx}{dt} + \boldsymbol{j}\frac{dy}{dt} + \boldsymbol{k}\frac{dz}{dt}) = 0$$

or

$$\nabla f.\boldsymbol{r}'(t) = 0.$$

Let $\nabla f(P) \neq 0$ and $\boldsymbol{r}'(t) \neq 0$. Now, $\boldsymbol{r}'(t)$ is a tangent vector to C at the point P and lies in the tangent plane to the surface at P. Hence $\nabla f(P)$ is orthogonal to every tangent vector at P. Therefore, $\nabla f(P)$ is the vector normal to the surface $f(x, y, z) = k$ at the point P.



The unit normal vector is $\boldsymbol{n} = \frac{\text{grad } \phi}{|\text{grad } \phi|}$.

Example: Find a unit normal vector to the surface $xy^2 - 2xyz = 3$ at the point $(1, 4, 3)$.

Solution: Let $\phi(x, y, z) = xy^2 - 2xyz = 3$

$\Rightarrow \frac{\partial \phi}{\partial x} = y^2 - 2yz,\ \frac{\partial \phi}{\partial y} = 2xy - 2xz$ and $\frac{\partial \phi}{\partial z} = -2xy$.

$$\therefore\ \nabla\phi = \frac{\partial \emptyset}{\partial x}\boldsymbol{i} + \frac{\partial \emptyset}{\partial y}\boldsymbol{j} + \frac{\partial \emptyset}{\partial z}\boldsymbol{k}$$

$$= (y^2 - 2yz)\boldsymbol{i} + (2xy - 2xz)\boldsymbol{j} - 2xy\boldsymbol{k}$$

$$\nabla\phi(1,4,3) = -8\boldsymbol{i} + 2\boldsymbol{j} - 8\boldsymbol{k}.$$

The unit normal vector at (1,4,3) is

$$\widehat{\boldsymbol{n}} = \frac{-8\boldsymbol{i}+2\boldsymbol{j}-8\boldsymbol{k}}{\sqrt{64+4+64}} = \frac{2(-4\boldsymbol{i}+\boldsymbol{j}-4\boldsymbol{k})}{\sqrt{132}}.$$

Note: The angle between two surfaces $\phi_1\ (x, y, z)$ and $\phi_{\ 2}(x, y, z)$ at a point is the angle between their normal's at that point.

i.e. $\cos\theta = \left|\frac{\widehat{\boldsymbol{n}}_1.\widehat{\boldsymbol{n}}_2}{|\widehat{\boldsymbol{n}}_1|.|\widehat{\boldsymbol{n}}_2|}\right|$, where $\widehat{\boldsymbol{n}}_1 = \frac{\nabla\phi_1}{|\nabla\phi_1|}$, $\widehat{\boldsymbol{n}}_2 = \frac{\nabla\phi_2}{|\nabla\phi_2|}$.

Properties:

Let f and g be any two differentiable scalar fields. Then

$\nabla(f \pm g) = \nabla f \pm \nabla g$.

$\nabla(c_1 f + c_2 g) = c_1\nabla f + c_2\nabla g$, where c_1,c_2 are arbitrary constants.

$\nabla(fg) = f\nabla g + g\nabla f$.

$$\nabla\left(\frac{f}{g}\right) = \frac{g\nabla f - f\nabla g}{g^2},\ g \neq 0.$$

Directional derivative:

Consider a scalar function $\phi(x, y, z)$ with continuous first order partial derivatives with respect to $x, y,$ and z that is defined in some region of space. The partial derivatives $\frac{\partial \emptyset}{\partial x}, \frac{\partial \emptyset}{\partial y}$ and $\frac{\partial \emptyset}{\partial z}$ can be interpreted as the slope (or rate of change) of $\phi(x, y, z)$ along $\boldsymbol{i}$,$\boldsymbol{j}$ and $\boldsymbol{k}$ directions, respectively. We can also evaluate derivatives along intermediate between $\boldsymbol{i}$,$\boldsymbol{j}$ and $\boldsymbol{k}$. The result is a directional derivative. The **directional derivative** of ϕ at a point $P(x, y, z)$ in the direction of a unit vector $\boldsymbol{u}$ is denoted by $D_{\boldsymbol{u}}(\phi)$. The directional derivative is the slope (or rate of change) of $\phi(x, y, z)$ in the direction of $\boldsymbol{u}$.

$$D_{\boldsymbol{u}}(\phi) = \nabla\phi.\boldsymbol{u}.$$

Example: Find the directional derivative of $\phi(x, y, z) = x^2 + 3y^2 + 2y^2$ in the direction of the vector $2\boldsymbol{i} - \boldsymbol{j} - 2\boldsymbol{k}$ and determine its value at the point $(1, -3, 2)$.

Solution: $\nabla\phi = 2x\boldsymbol{i} + 6y\boldsymbol{j} + 4z\boldsymbol{k}$ and the unit vector is

$$\boldsymbol{u} = \frac{2\boldsymbol{i}-\boldsymbol{j}-2\boldsymbol{k}}{\sqrt{4+1+4}} = \frac{2\boldsymbol{i}-\boldsymbol{j}-2\boldsymbol{k}}{\sqrt{9}} = \frac{1}{3}(2\boldsymbol{i} - \boldsymbol{j} - 2\boldsymbol{k}).$$

Directional derivative of ϕ in the direction $\boldsymbol{u}$

$$D_{\boldsymbol{u}}(\phi) = \nabla\emptyset.\boldsymbol{u}$$

$$= (2x\boldsymbol{i} + 6y\boldsymbol{j} + 4z\boldsymbol{k}).\frac{1}{3}\,(2\boldsymbol{i} - \boldsymbol{j} - 2\boldsymbol{k})$$

$$= \frac{4}{3}\,x - 2y - \frac{8}{3}\,z.$$

Directional derivative $D_{\boldsymbol{u}}(\phi)$ at the point $(1,-3,2)$ is

$$D_{\boldsymbol{u}}(\phi)_{(1,-3,2)} = \frac{4}{3} + 6 - \frac{16}{3} = 2.$$

Properties:

The most rapid increase of a differentiable function $\phi(x, y, z)$ at a point P in space occurs in the direction of the vector $\boldsymbol{u}_P = \text{grad}\phi(P)$. The directional derivative at P is

$$D_{\boldsymbol{u}}\phi(P) = |\text{grad}\phi(P)| = \left(\left(\frac{\partial \phi}{\partial x}\right)_P^2 + \left(\frac{\partial \phi}{\partial y}\right)_P^2 + \left(\frac{\partial \phi}{\partial z}\right)_P^2\right)^{\frac{1}{2}}.$$

The most rapid decrease of a differentiable function $\phi(x, y, z)$ at a point P in space occurs when the vector $\boldsymbol{u}_P$ just defined and $\text{grad}\phi$ are oppositely directed, so that $\boldsymbol{u}_P = -\text{grad}\phi(P)$ the directional derivative at P is

$$D_{\boldsymbol{u}}\phi(P) = -|\text{grad}\phi(P)| = -\left(\left(\frac{\partial \phi}{\partial x}\right)_P^2 + \left(\frac{\partial \phi}{\partial y}\right)_P^2 + \left(\frac{\partial \phi}{\partial z}\right)_P^2\right)^{\frac{1}{2}}.$$

There is a zero local rate of change of a differentiable function $\phi(x, y, z)$ at a point P in space in the direction of any vector $\boldsymbol{u}_P$ that is orthogonal to $\text{grad}\phi$ at P, so that $\boldsymbol{u}_P.\text{grad}\ \phi(P) = 0$.

1. Find $\nabla\emptyset$ at $(1, -1, 2)$ if $\emptyset(x, y, z) = x^2 - y^2 - z^2 - 2$.

Solution:

$$\nabla\emptyset = \left(\boldsymbol{i}\frac{\partial}{\partial x} + \boldsymbol{j}\frac{\partial}{\partial y} + \boldsymbol{k}\frac{\partial}{\partial z}\right)(x^2 - y^2 - z^2 - 2)$$

$$= \boldsymbol{i}(2x) + \boldsymbol{j}(-2y) + \boldsymbol{k}(-2z)$$

$$= 2x\boldsymbol{i} - 2y\boldsymbol{j} - 2z\boldsymbol{k}$$

$\therefore \nabla\emptyset$ at $(1, -1, 2) = 2\boldsymbol{i} + 2\boldsymbol{j} - 4\boldsymbol{k}$.

2. Find the unit normal to the surface $x^2 + y^2 + z^2 = 3$ at $(1, 1, 1)$.

Solution:

Let $\emptyset(x, y, z) = x^2 + y^2 + z^2 - 3$

Let **n** denote the unit normal to the surface.

$$\therefore \boldsymbol{n} = \frac{\nabla\emptyset}{|\nabla\emptyset|}.$$

$$\text{Now } \nabla\emptyset = 2x\boldsymbol{i} + 2y\boldsymbol{j} + 2z\boldsymbol{k}$$

$$\therefore \nabla\emptyset \text{ at } (1, 1, 1) = 2\boldsymbol{i} + 2\boldsymbol{j} + 2\boldsymbol{k}$$

$$\therefore \boldsymbol{n} = \frac{2\boldsymbol{i}+2\boldsymbol{j}+2\boldsymbol{k}}{\sqrt{2^2+2^2+2^2}} = \frac{2(\boldsymbol{i}+\boldsymbol{j}+\boldsymbol{k})}{\sqrt{12}} = \frac{\boldsymbol{i}+\boldsymbol{j}+\boldsymbol{k}}{\sqrt{3}}.$$

3. Find a unit normal vector to the surface $x = t, y = t^2, z = t^3$ at $t = 1$.

Solution:

The given surface can be written in Cartesian form as $z = xy$

Let $\emptyset(x, y, z) = z - xy$.

Let $\boldsymbol{n}$ be the unit normal to this surface

$$\therefore \boldsymbol{n} = \frac{\nabla\emptyset}{|\nabla\emptyset|}.$$

$$\text{Now } \nabla\emptyset = -y\boldsymbol{i} - x\boldsymbol{j} + \boldsymbol{k}$$

$$\therefore \nabla\emptyset \text{ at } (1, 1, 1) = -\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}$$

$$\therefore \boldsymbol{n} = \frac{-\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}}{\sqrt{(-1)^2 + (-1)^2 + 1^2}} = \frac{-\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}}{\sqrt{3}}.$$

4. Find the directional derivative of the function $xy^2 + yz^2 + zx^2$ along the tangent to the curve $x = t, y = t^2, z = t^3$ at the point $(1, 1, 1)$.

Solution:

Let $\emptyset(x, y, z) = xy^2 + yz^2 + zx^2$

Now $\nabla\emptyset = (y^2 + 2xz)\boldsymbol{i} + (z^2 + 2xy)\boldsymbol{j} + (x^2 + 2yz)\boldsymbol{k}$.

$$\nabla\emptyset(1,1,1) = 3\boldsymbol{i} + 3\boldsymbol{j} + 3\boldsymbol{k}.$$

Let $\boldsymbol{r}$ be the position vector of any point on the curve $x = t, y = t^2, z = t^3$

Then $\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k} = t\boldsymbol{i} + t^2\boldsymbol{j} + t^3\boldsymbol{k}$.

$\therefore \frac{d\boldsymbol{r}}{dt} = \boldsymbol{i} + 2t\boldsymbol{j} + 3t^2\boldsymbol{k} \Rightarrow \frac{d\boldsymbol{r}}{dt}$ at $(1,1,1) = \boldsymbol{i} + 2\boldsymbol{j} + 3\boldsymbol{k}$.

We know that $\frac{d\boldsymbol{r}}{dt}$ is the vector along the tangent to the curve.

$\therefore$ unit vector along the tangent is $\frac{\boldsymbol{i}+2\boldsymbol{j}+3\boldsymbol{k}}{\sqrt{1^2+2^2+3^2}} = \frac{\boldsymbol{i}+2\boldsymbol{j}+3\boldsymbol{k}}{\sqrt{14}} = \boldsymbol{u}$(say).

$\therefore$ Directional derivative along the tangent at the point $(1,1,1)$ is $D_{\boldsymbol{u}}\emptyset = \boldsymbol{u}.\nabla\emptyset$.

$$= \frac{1}{\sqrt{14}}(\boldsymbol{i} + 2\boldsymbol{j} + 3\boldsymbol{k}).(3(\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}) = \frac{3}{\sqrt{14}}(1 + 2 + 3)$$

$$= \frac{18}{\sqrt{14}}.$$

5. Find the angle of intersection of the spheres $x^2 + y^2 + z^2 = 29$ and $x^2 + y^2 + z^2 + 4x - 6y - 8z - 27 = 0$ at the point $(4, -3,2)$.

Solution:

Let $\emptyset_1(x, y, z) = x^2 + y^2 + z^2 - 29 = 0$ and $\emptyset_2(x, y, z) = x^2 + y^2 + z^2 + 4x - 6y - 8z - 27 = 0$

Now, $\nabla\emptyset_1 = 2x\boldsymbol{i} + 2y\boldsymbol{j} + 2z\boldsymbol{k}$ and $\nabla\emptyset_2 = (2x + 4)\boldsymbol{i} + (2y - 6)\boldsymbol{j} + (2z - 8)\boldsymbol{k}$.

The angle between two surfaces at a point is the angle between the normals to the surface at that point.

Let $\boldsymbol{n}_1 = \nabla\emptyset_1$ at $(4, -3,2) = 8\boldsymbol{i} - 6\boldsymbol{j} + 4\boldsymbol{k}$

and $\boldsymbol{n}_2 = \nabla\emptyset_1$at $(4, -3,2) = 12\boldsymbol{i} - 12\boldsymbol{j} - 4\boldsymbol{k}$.

The vectors $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$ are along the normals to the two surfaces at $(4, -3,2)$.

Let θ be the angle between the surfaces.

Then $\cos\theta = \frac{\boldsymbol{n}_1.\boldsymbol{n}_2}{|\boldsymbol{n}_1||\boldsymbol{n}_2|} = \frac{152}{\sqrt{116}\sqrt{304}}$

$$\therefore \theta = \cos^{-1}(\sqrt{\frac{19}{29}}).$$

6. Find the greatest value of the directional derivative of the function $\emptyset(x, y, z) = x^2yz^2$ at $(2,1,1)$.

Solution:

$$\nabla\emptyset = \boldsymbol{i}\frac{\partial\emptyset}{\partial x} + \boldsymbol{j}\frac{\partial\emptyset}{\partial y} + \boldsymbol{k}\frac{\partial\emptyset}{\partial z}$$

$$= 2xyz^2\boldsymbol{i} + x^2z^2\boldsymbol{j} + 2x^2yz\boldsymbol{k}.$$

$$\nabla\emptyset \text{ at } (2,1,1) = 4\boldsymbol{i} + 4\boldsymbol{j} + 8\boldsymbol{k} = 4(\boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k})$$

$\therefore$ greatest value of the directional derivative of $\emptyset(x, y, z) = |\nabla\emptyset| = \sqrt{16 + 16 + 64} = 4\sqrt{6}$.

Exercise

1. Find grad $\emptyset$ Where
   a. $\emptyset(x, y, z) = x^3 + y^3 + 3xyz$
   b. $\emptyset(x, y, z) = x^2y + y^2x + z^2$
2. If $\emptyset(x, y, z) = 2xz^4 - x^2y$, find $|\nabla\emptyset|$ at the point $(2, -2, -1)$.
3. Find a unit normal to the surface $x^3 + y^3 + 3xyz = 3$ at the point $(1, 2, -1)$.
4. Find the directional derivative of
   a. $\emptyset(x, y, z) = x^2 - 2y^2 + 4z^2$ at $(1, 1, -1)$ in the direction of $2\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k}$.
   b. Find the directional derivative of the scalar point function $\emptyset(x, y, z) = 4xy^2 + 2x^2yz$ at the point $P(1, 2, 3)$ in the direction of the line PQ where $Q = (5, 0, 4)$.
   c. Find the directional derivative of $xyz^2 + xz$ at $(1, 1, 1)$ in a direction of the normal to the surface $3xy^2 + y = z$ at $(0, 1, 1)$.
5. Find the maximum value of the directional derivative of $\emptyset(x, y, z) = x^2yz$ at $(1, 4, 1)$.
6. Find the values of a and b so that the surfaces $ax^2 - byz = (a + 2)x$ and $4x^2y + z^3 = 4$ may intersect orthogonally at the point $(1, -1, 2)$.
7. Find the angle between the following surfaces.
   a. $x^2 + y^2 + z^2 = 9$ and $z = x^2 + y^2 - 3$ at the point $(2, -1, 2)$.
   b. $xy^2z = 3x + z^2$ and $3x^2 - y^2 + 2z = 1$ at the point $(1, -2, 1)$.

8. If $\emptyset(x, y) = x^2 - xy - y + y^2$, find all points where the directional derivative in the direction $\boldsymbol{u} = \frac{\boldsymbol{i}+\sqrt{3}\boldsymbol{j}}{2}$ is zero.
9. Find the directional derivative of $\emptyset(x, y) = x^2y^3 + xy$ at $(2, 1)$, in the direction of a unit vector which makes an angle of $\frac{\pi}{6}$ with $X -$ axis.

Answers

1.
   a. $(3x^2 + 3yz)\boldsymbol{i} + (3y^2 + 3xy)\boldsymbol{j} + 3xy\boldsymbol{k}$.
   b. $(2xy + y^2)\boldsymbol{i} + (x^2 + 2xy)\boldsymbol{j} + 2z\boldsymbol{k}$.
2. $2\sqrt{93}$.
3. $\frac{1}{\sqrt{14}}(-\boldsymbol{i} + 3\boldsymbol{j} + 2\boldsymbol{k})$.
4.
   a. $\frac{8}{\sqrt{6}}$,
   b. $\frac{120}{\sqrt{21}}$,
   c. $\frac{4}{\sqrt{11}}$.
5. 9.
6. $a = \frac{5}{2}$ and $b = 1$.
7.
   a. $\cos^{-1}\left(\frac{8}{3\sqrt{21}}\right)$,
   b. $\cos^{-1}\left(\frac{3}{7\sqrt{6}}\right)$.
8. All points on the line $\left(2 - \sqrt{3}\right)x + \left(2\sqrt{3} - 1\right) = \sqrt{3}$.
9. $\frac{1}{2}(14 + 5\sqrt{3})$.