UNIT-02 Discrete Time Fourier Transform (DTFT)

Signals
- → Continuous time signal
  - Periodic $x(t) = x(t+T)$
  - Aperiodic
- → Discrete time signal
  - Periodic $x[n] = x[n+N]$
  - Aperiodic

Fourier Series Family

Periodic signal
- CT - Continuous time fourier series (CTFS)
- DT - Discrete time fourier series (DTFS)

Aperiodic signal
- CT - continuous time fourier transform (CTFT)
- DT - Discrete time fourier transform (DTFT)

Discrete time fourier Series :-

Def :- Fourier series is a mathematical tool that allows the representation of any periodic signal as the sum of harmonically related complex exponential signals.

Periodic : $x[n] = x[n+mN]$ $m \rightarrow$ any integer

$N \rightarrow$ Smallest value (fundamental period)

Fundamental frequency : $\omega_0 = \frac{2\pi}{N}$ rad/samples

$$x[n] = \{e^{j0n}, e^{\pm j\omega_0 n}, e^{\pm 2j\omega_0 n}, \ldots\ldots, e^{\pm jk\omega_0 n}\}$$

↓

periodic signal with fund. frequency $= \omega_0$

$$x[n] = \sum_{K=K_0}^{K_0+N-1} X_K e^{jK\omega_0 n}$$ K_0 = arbitrary constant

$$x[n] = \sum_{\langle N \rangle} X_k e^{jk\omega_0 n}$$

Synthesis equation

$\langle N \rangle \rightarrow$ Summation over any range of consecutive k's exactly 'N' in length.

X_k = Fourier series coefficient / Spectral coefficients of $x[n]$

$$X_k = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-jk\omega_0 n}$$

Analysis equation

Q) Determine the Fourier series coefficient of the signal $x[n]$ and plot the magnitude and the phase spectra

$$x[n] = 1 + \sin\left(\frac{2\pi n}{N}\right) + 3\cos\left(\frac{2\pi n}{N}\right) + \cos\left(\frac{4\pi}{N} n + \frac{\pi}{2}\right)$$

Signal $\rightarrow$ periodic with period N

fundamental frequency : $\omega_0 = \frac{2\pi}{N}$

$$x[n] = 1 + \sin(\omega_0 n) + 3\cos(\omega_0 n) + \cos\left(2\omega_0 n + \frac{\pi}{2}\right)$$

$$x[n] = 1 + \frac{e^{j\omega_0 n} - e^{-j\omega_0 n}}{2j} + 3\,\frac{e^{j\omega_0 n} + e^{-j\omega_0 n}}{2} + \frac{e^{j(2\omega_0 n + \frac{\pi}{2})} - e^{-j(2\omega_0 n + \frac{\pi}{2})}}{2}$$

$$x[n] = 1 + e^{j\omega_0 n}\left[\frac{1}{2j} + \frac{3}{2}\right] + e^{-j\omega_0 n}\left(\frac{-1}{2j} + \frac{3}{2}\right] + e^{j2\omega_0 n}\left[\frac{1}{2} e^{j\pi/2}\right]$$

$$+ e^{-j2\omega_0 n}\left[\frac{1}{2} e^{-j\pi/2}\right]$$

$$x[n] = \sum_{\langle N \rangle} X_k e^{jk\omega_0 n}$$

$$x[n] = X_0 + X_1 e^{j\omega_0 n} + X_{-1} e^{-j\omega_0 n} + X_2 e^{j2\omega_0 n} + X_{-2} e^{-j2\omega_0 n}$$

$X_0 = 1$

$X_1 = \frac{3}{2} + \frac{1}{2j} = \frac{3}{2} - \frac{j}{2}$

$X_{-1} = \frac{3}{2} - \frac{1}{2j} = \frac{3}{2} + \frac{j}{2}$

$X_2 = \frac{1}{2} e^{j\pi/2} = \frac{j}{2}$

$X_{-2} = \frac{1}{2} e^{-j\pi/2} = \frac{-j}{2}$

Magnitude ~~of~~ fourier coefficients:

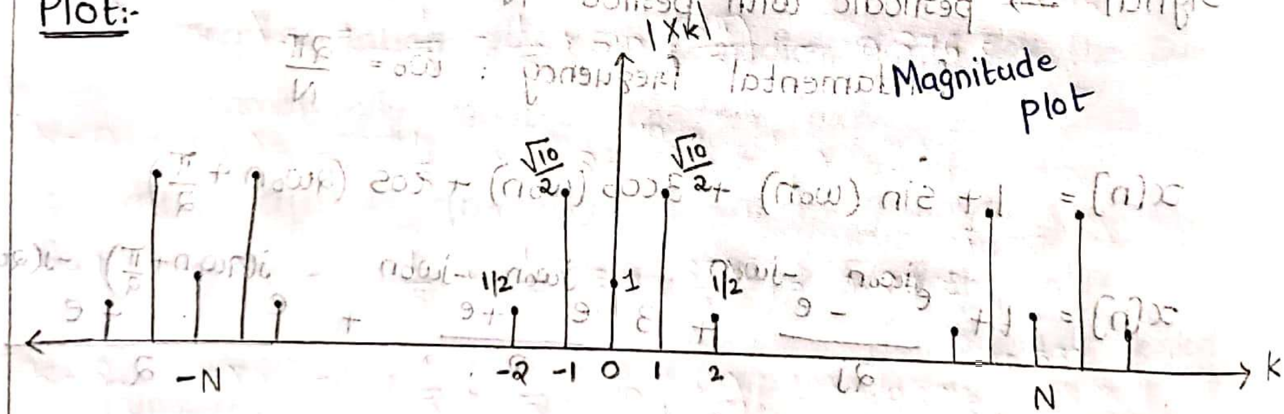
$|X_0| = 1$, $|X_1| = |X_{-1}| = \frac{\sqrt{10}}{2}$, $|X_2| = |X_{-2}| = \frac{1}{2}$

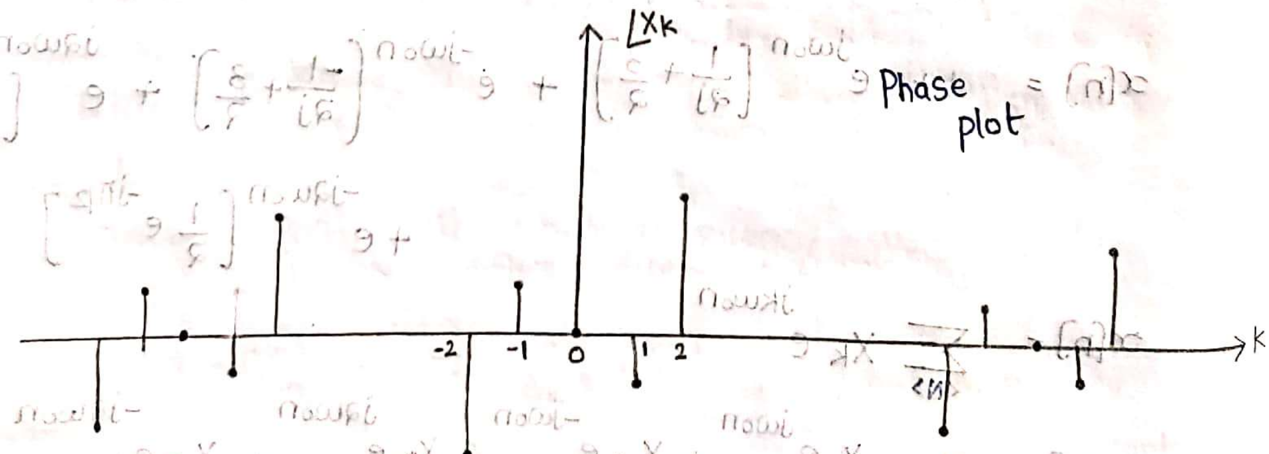
Phase ~~of~~ fourier coefficients,

$\angle X_0 = 0$, $\angle X_1 = -\tan^{-1}\left(\frac{1}{3}\right)$ $\frac{\text{img}}{\text{real}}$, $\angle X_{-1} = \tan^{-1}\left(\frac{1}{3}\right)$

$\angle X_2 = \frac{\pi}{2}$, $\angle X_{-2} = -\frac{\pi}{2}$

Plot:-





Note:- DTFS always converges because it is a finite sum defined entirely by the values of the signal over one period.

∴ It does not have "Gibbs phenomenon".

$\boxed{CT}$ $\quad x(t) = \int_{-\infty}^{\infty} C_k e^{j\omega_0 t}$

$\boxed{DT}$ $\quad x[n] = \sum X_k e^{jk\omega_0 n}$

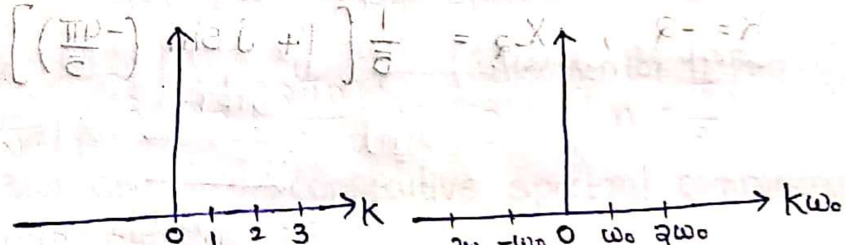
Note:- * Along the k scale/axis X_k repeats at intervals of 'N'

where $k \Rightarrow 0, 1, 2$

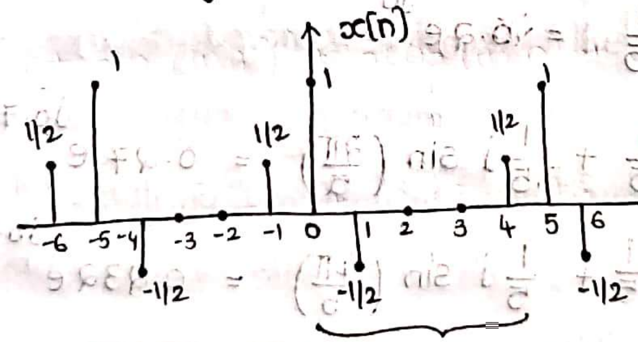
$\omega_0 k \Rightarrow 0\omega_0, 1\omega_0, 2\omega_0$

* Along the ω scale/axis X_k repeats for every 2π intervals

$k \Rightarrow$ N interval
$\omega_0 \rightarrow 2\pi$ interval



Q) Find the frequency domain representation of the signal.



$N = 5 \quad \{0, 1, 2, 3, 4\}$

Signal is periodic with period $N = 5$

$$\omega_0 = \frac{2\pi}{N} = \frac{2\pi}{5}$$

Odd symmetry, $n \Rightarrow -2$ to $+2$

$$X_k = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-jk\omega_0 n}$$

$$X_k = \frac{1}{5}\sum_{n=-2}^{2} x(n)e^{-jk\frac{2\pi}{5}n}$$

$$= \frac{1}{5}\left[x(-2)e^{jk\left(\frac{2\pi}{5}\right)(2)} + x(-1)e^{jk\frac{2\pi}{5}} + x(0) + x(1)e^{-jk\frac{2\pi}{5}} + x(2)e^{-jk\frac{4\pi}{5}}\right]$$

$$= \frac{1}{5}\left[0 + \frac{1}{2}e^{jk\frac{2\pi}{5}} + 1 - \frac{1}{2}e^{-jk\frac{2\pi}{5}} + 0\right]$$

$$= \frac{1}{5}\left[\frac{1}{2}\left(e^{jk\frac{2\pi}{5}} - e^{-jk\frac{2\pi}{5}}\right) + 1\right]$$

$$= \frac{1}{5}\left[1 + j\sin\left(\frac{2\pi}{5}k\right)\right]$$

$K = -2$, $X_{-2} = \frac{1}{5}\left[1 + j\sin\left(-\frac{4\pi}{5}\right)\right] = \frac{1}{5} - \frac{1}{5}j\sin\frac{4\pi}{5}$

$= 0.232\,e^{-j0.531}$

$K = -1$, $X_{-1} = \frac{1}{5} - \frac{1}{5}j\sin\left(\frac{2\pi}{5}\right) = 0.276\,e^{-j0.76}$

$K = 0$, $X_0 = \frac{1}{5} = 0.2\,e^{j0}$

$K = 1$, $X_1 = \frac{1}{5} + \frac{1}{5}j\sin\left(\frac{2\pi}{5}\right) = 0.27\,e^{j0.76}$

$K = 2$, $X_2 = \frac{1}{5} + \frac{1}{5}j\sin\left(\frac{4\pi}{5}\right) = 0.232\,e^{j0.531}$

Discrete time Fourier transform:-

→ Decompose any periodic signal with period N in terms of harmonically related complex exponentials of the form $e^{jk\omega_0 n}$

→ All such harmonics have the common period $N = \frac{2\pi}{\omega_0}$

→ Mathematical tool ⇒ Fourier transform [Aperiodic or periodic signals]

→ DTFT is used to find the spectral components in the DT signal

→ An aperiodic signal is viewed as periodic signal with an infinite period.

→ In fourier series representation of a periodic signal,
As period ↑ $\left[\downarrow\omega_0 = \frac{2\pi}{N\uparrow}\right]$, $\omega_0 \downarrow$ (fundamental frequency)

i.e, the spacing b/w any two consecutive spectral components decreases.

∴ Harmonically related components becomes closer in frequency.

∴ As the period becomes infinite, the frequency components form a continuous spectrum.

The resulting continuous spectrum of coefficients in this representation is called DTFT.

→ The synthesis integral that uses these coefficients to represent the discrete time signal as a linear combination of complex exponentials is called inverse DTFT.

→ DTFS representation of a periodic signal $x[n]$ with period N and frequency $\omega_0 = \frac{2\pi}{N}$

Periodic signal $\rightarrow x[n] = \sum_{K=\langle N\rangle} X_K e^{jK\omega_0 n}$

DTFT

Periodic or aperiodic $\rightarrow x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})\, e^{j\omega n}\, d\omega \rightarrow ①$

where, $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n} \rightarrow ②$

Equ ① refer to as the synthesis equation because it synthesis an arbitrary signal from its complex exponential components.

Equ ② refer to as the analysis equation bcoz it analyzes how much ~~of~~ each complex exponential signal is present in the original signal.

$$x[n] \xrightarrow{\text{DTFT}} X(e^{j\omega})$$
$$x[n] \xleftarrow{\text{IDTFT}} X(e^{j\omega})$$

$X[e^{j\omega}] = \text{DTFT}\{x[n]\} \Rightarrow \mathcal{F}\{x[n]\}$

$x[n] = \text{IDTFT}\{X(e^{j\omega})\} \Rightarrow \mathcal{F}^{-1}\{X(e^{j\omega})\}$

$x(n) \longleftrightarrow X[e^{j\omega}]$

Periodic or aperiodic signal (time domain 'n')

Periodic with period 2π : 0 to 2π, $-\pi$ to π

Complex function ~~of~~ 'ω'

$X[e^{j\omega}] \Rightarrow$ Complex function ~~of~~ Variable 'ω'.

$$X[e^{j\omega}] = \underbrace{X_R[e^{j\omega}]}_{\text{Real part}} + \underbrace{X_I[e^{j\omega}]}_{\text{Imaginary part}}$$

$$X_R[e^{j\omega}] = \frac{X[e^{j\omega}] + X^*[e^{j\omega}]}{2}$$

$$X_I[e^{j\omega}] = \frac{X[e^{j\omega}] - X^*[e^{j\omega}]}{2j}$$

$X^*[e^{j\omega}]$ denotes complex conjugate of $X[e^{j\omega}]$

⇒ <u>Polar form representation of DTFT</u> $X[e^{j\omega}]$ ↳ freq-domain representation of $x(n)$

$$X[e^{j\omega}] = \underbrace{|X[e^{j\omega}]|}_{\text{Magnitude Spectrum}} e^{j\theta(\omega)}$$

$\theta(\omega) = \angle X(e^{j\omega})$ ↳ Phase spectrum

→ $X_R[e^{j\omega}] = |X_R(e^{j\omega})| \cos(\theta(\omega))$

$X_I[e^{j\omega}] = |X_I(e^{j\omega})| \sin(\theta(\omega))$

$$|X[e^{j\omega}]| = \sqrt{X_R^2[e^{j\omega}] + X_I^2[e^{j\omega}]}$$

$$\theta(\omega) = \angle X(e^{j\omega}) = \tan^{-1}\left[\frac{X_I(e^{j\omega})}{X_R(e^{j\omega})}\right]$$

→ <u>For real signal,</u>

Magnitude: $|X(e^{-j\omega})| = \sqrt{X_R^2(e^{j\omega}) + X_I^2(e^{-j\omega})}$

$$|X(e^{j\omega})| = \sqrt{X_R^2(e^{j\omega}) + X_I^2(e^{j\omega})}$$

$$\boxed{\therefore\ |X(e^{j\omega})| = |X(e^{-j\omega})|}$$ Magnitude plot is same

Phase: $\angle X(e^{-j\omega}) = \tan^{-1}\left[\frac{X_I(e^{-j\omega})}{X_R(e^{-j\omega})}\right]$

$$= \tan^{-1}\left[\frac{-X_I(e^{j\omega})}{X_R(e^{j\omega})}\right]$$

$$= -\tan^{-1}\left[\frac{X_I(e^{j\omega})}{X_R(e^{j\omega})}\right]$$

$$\boxed{\angle X(e^{-j\omega}) = -\angle X(e^{+j\omega})}$$ Phase plot is -ve to each other.

* Magnitude spectrum $|X(e^{j\omega})|$ is an even function of 'ω'.
  Phase spectrum $\angle X(e^{j\omega})$ is an odd function of 'ω'.

Periodicity of DTFT $[X(e^{j\omega})]$:

$$X(e^{j\omega}) = X(e^{j(\omega+2\pi)})$$

The DTFT is a periodic function of 'ω' with a period of '2π'

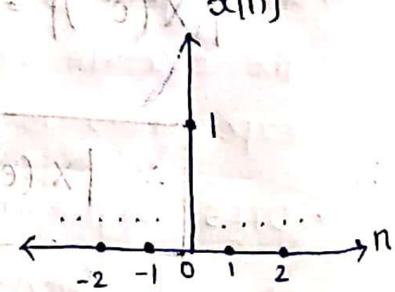
Range: $0 \rightarrow 2\pi$

$-\pi \rightarrow \pi$

Note:- The frequency range for any discrete time signal is unique over the frequency interval of $(-\pi, \pi)$ or $(0, 2\pi)$ and any frequency outside this interval is equivalent to a frequency with in this interval.

Q)

(1) $x[n] = \delta[n]$

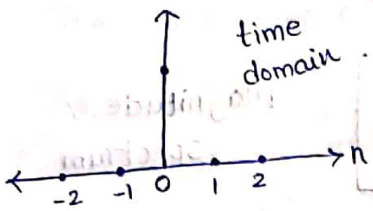


By def, $X[e^{j\omega}] = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$

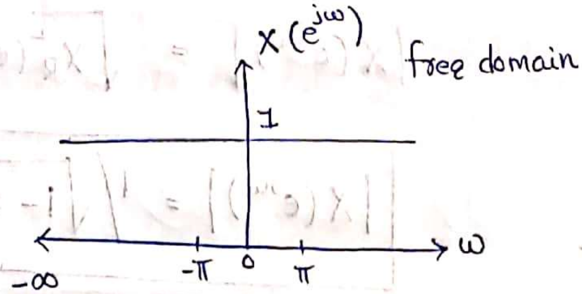
$$= \sum_{n=-\infty}^{\infty} \delta[n] e^{-j\omega n}$$

$$= e^{-j\omega n}\Big|_{n=0} = e^{0} = 1$$

$x[n] \quad \delta[n] \xrightarrow{DTFT} 1$



$\xrightarrow{DTFT}$



(2) $x[n] = a^n u[n] \qquad x[n] \xrightarrow{DTFT} X[e^{j\omega}]$

By def, $X[e^{j\omega}] = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$

$$= \sum_{n=-\infty}^{\infty} a^n u[n] e^{-j\omega n} \qquad u[n] = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$$

$$= \sum_{n=0}^{\infty} a^n e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} (a e^{-j\omega})^n = \frac{1}{1-ae^{-j\omega}} \; ; \; |ae^{-j\omega}| < 1$$

$$|a| < 1$$

$$\boxed{a^n u[n] \xrightarrow{DTFT} \frac{1}{1-ae^{-j\omega}} \; ; \; |a| < 1}$$

$$X[e^{j\omega}] = \frac{1}{1-ae^{-j\omega}} \times \frac{1-ae^{-j\omega}}{1-ae^{-j\omega}}$$

$$= \frac{1 - a[\cos\omega + j\sin\omega]}{1-ae^{j\omega} - ae^{-j\omega} + a^2 \cancel{e^{j\omega}} \cancel{e^{-j\omega}}}$$

$$= \frac{1 - a\cos\omega - aj\sin\omega}{1 - 2a\left[\frac{e^{j\omega} - e^{-j\omega}}{2}\right] + a^2} = \frac{1 - a\cos\omega - aj\sin\omega}{1 - 2a\cos\omega + a^2}$$

$$X[e^{j\omega}] = \underbrace{\frac{1 - a\cos\omega}{1 - 2a\cos\omega + a^2}}_{\text{Real part}} - \underbrace{\frac{j a\sin\omega}{1 - 2a\cos\omega + a^2}}_{\text{Imaginary part}}$$

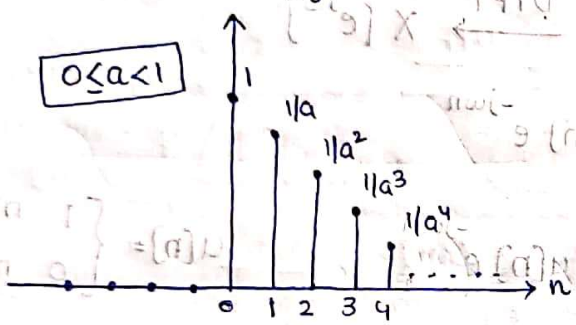
$$|X(e^{j\omega})| = \sqrt{X_R^2(e^{j\omega}) + X_I^2(e^{j\omega})}$$

$$\boxed{|X(e^{j\omega})| = 1/\sqrt{1-2a\cos\omega + a^2}}$$ Magnitude spectrum

<u>Case - 1</u>

$\boxed{0 \le a < 1}$

$x[n] = a^n u(n)$



$x[0] = a^0 = 1$

$x[1] = a$

$x[2] = a^2$

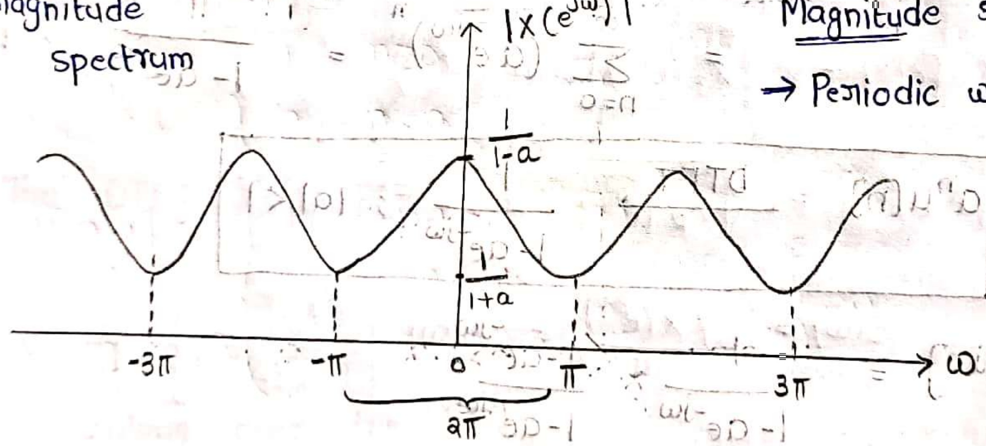
$x[3] = a^3$

$$|X(e^{j\omega})| = \frac{}{\sqrt{1-2a\cos\omega + a^2}}$$

Magnitude spectrum

<u>Magnitude spectrum</u>

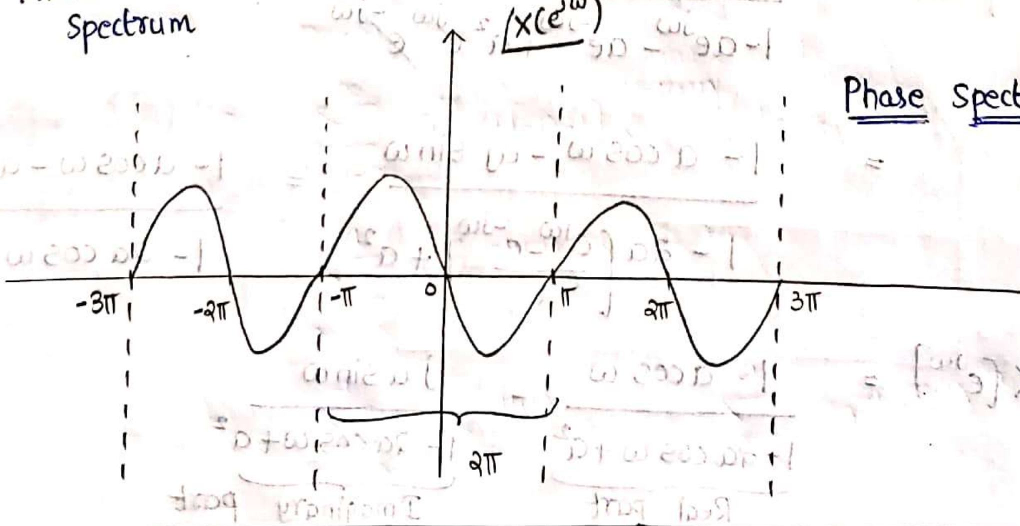
→ Periodic with period 2π



$$\underline{/X(e^{j\omega})} = \tan^{-1}\left(\frac{X_I(e^{j\omega})}{X_R(e^{j\omega})}\right) = \tan^{-1}\left[\frac{a\sin\omega}{1-a\cos\omega}\right]$$

Phase Spectrum

<u>Phase Spectrum</u>



⇒ Magnitude spectrum is even function of ω.

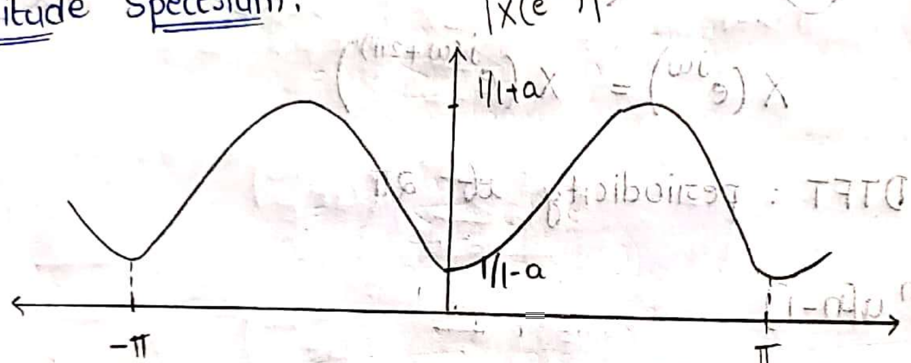
The phase spectrum is odd function of ω.

Case - 2 $\boxed{-1 < a \leq 0}$

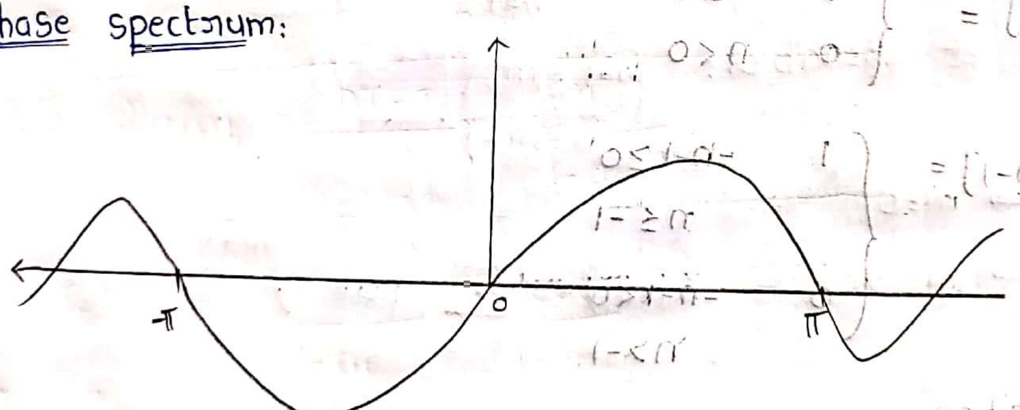
$$|X(e^{j\omega})| = \frac{1}{\sqrt{1-2a\cos\omega + a^2}}$$

$$\angle X(e^{j\omega}) = \tan^{-1}\left(\frac{X_I(e^{j\omega})}{X_R(e^{j\omega})}\right) = \tan^{-1}\left[\frac{a\sin\omega}{1-a\cos\omega}\right]$$

Magnitude Spectrum:



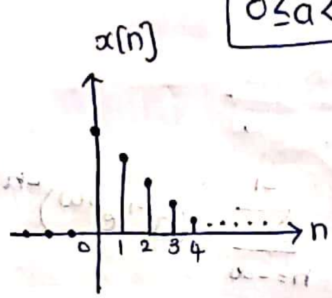
Phase spectrum:

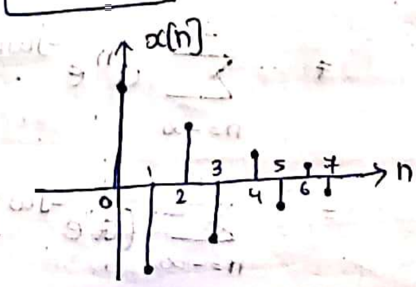


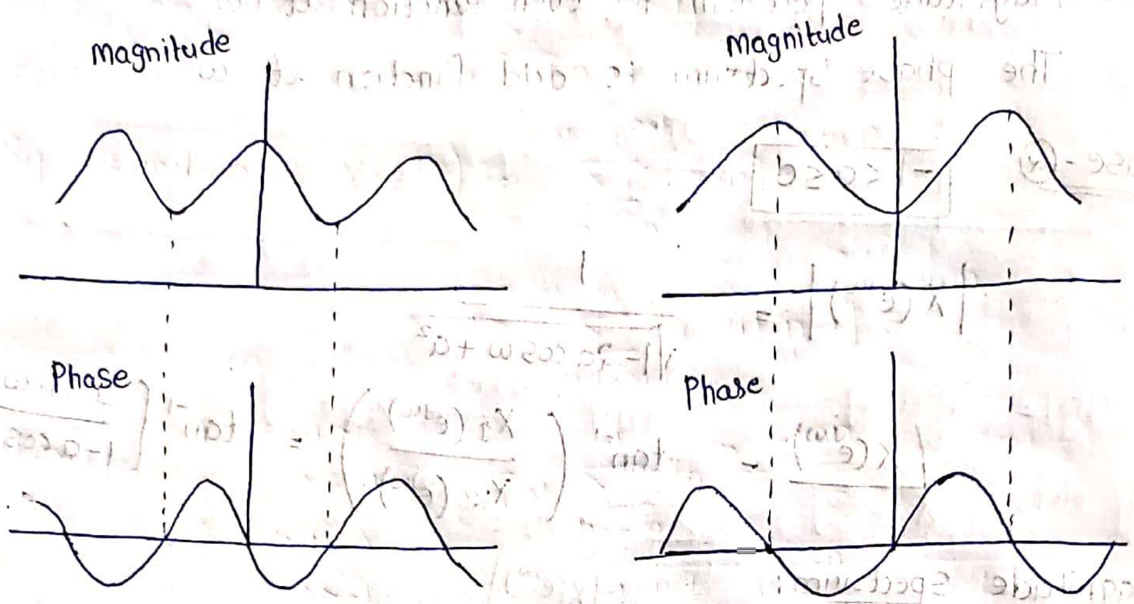
⇒ Both magnitude & phase spectrum are in odd symmetry.

$$x(n) = a^n u(n)$$

$\boxed{0 \leq a < 1}$ $\boxed{-1 < a \leq 0}$







$$X(e^{j\omega}) = X(e^{j(\omega+2\pi)})$$

DTFT : periodicity of 2π

(3) $x[n] = a^n u[n-1]$

$$u[n] = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$$

$$u[-n-1] = \begin{cases} 1 & -n-1 \geq 0 \\ & n \leq -1 \\ 0 & -n-1 < 0 \\ & n > -1 \end{cases}$$

By definition,

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} a^n u[-n-1]\, e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{-1} a^n e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{-1} (a e^{-j\omega})^n = \sum_{n=-\infty}^{-1} (a^{-1} e^{j\omega})^{-n}$$

$$X(e^{j\omega}) = \sum_{n=\infty}^{1} (a^{-1}e^{j\omega})^n$$

$$= \sum_{n=1}^{\infty} \left(\frac{1}{a}e^{j\omega}\right)^n$$

A change of variables is performed by letting $n-1 = m$

$n = m+1$

As $n=1 \Rightarrow m=0$

$n=\infty \Rightarrow m=\infty$

$$= \sum_{m=0}^{\infty} \left(\frac{1}{a}e^{j\omega}\right)^{m+1}$$

$$= \frac{1}{a}e^{j\omega} \sum_{m=0}^{\infty} \left(\frac{1}{a}e^{j\omega}\right)^m$$

$$= \left(\frac{1}{a}e^{j\omega}\right)\left(\frac{1}{1-\frac{1}{a}e^{j\omega}}\right)$$

$$= \frac{\left(\frac{1}{a}e^{j\omega}\right)}{\left(1-\frac{1}{a}e^{j\omega}\right)}, \quad |a|>1$$

$$\left[\begin{array}{c} \left|\frac{1}{a}e^{j\omega}\right| < 1 \\ \hookrightarrow =1 \\ \left|\frac{1}{a}\right| < 1 \\ |a|>1 \end{array}\right]$$

$$= \frac{e^{j\omega}}{a-e^{j\omega}} = \frac{e^{j\omega}}{e^{j\omega}(ae^{-j\omega}-1)} = \frac{1}{ae^{-j\omega}-1}$$

$$X(e^{j\omega}) = \frac{1}{ae^{-j\omega}-1} \; ; \quad |a|>1$$

$$\boxed{X(e^{j\omega}) = \frac{-1}{1-ae^{-j\omega}} \; ; \quad |a|>1}$$

* $a^n u[n] \xrightarrow{DTFT} \dfrac{1}{1-ae^{-j\omega}} \; ; \quad |a|<1$

$a^n u[-n-1] \xrightarrow{DTFT} \dfrac{-1}{1-ae^{-j\omega}} \; ; \quad |a|>1$

(4) Find the discrete time fourier transform of an anticausal sequence. $x[n] = a^{-n}u[-n-1]$; $|a| < 1$

By definition, $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$

$$= \sum_{n=-\infty}^{\infty} a^{-n}u[-n-1]\, e^{-j\omega n}$$

$$u[-n-1] = \begin{cases} 1 & -n-1 \geq 0 \\ & n \leq -1 \\ 0 & -n-1 < 0 \\ & n > -1 \end{cases} \qquad = \sum_{n=-\infty}^{-1} a^{-n} e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{-1} (ae^{j\omega})^{-n}$$

$$= \sum_{n=\infty}^{1} (ae^{j\omega})^{n}$$

$$= \sum_{n=1}^{\infty} (ae^{j\omega})^{n}$$

Change of variables: $n-1 = m$
$n = m+1$

$n = 1 \Rightarrow m = 0$

$n = \infty \Rightarrow m = \infty$ $\quad X(e^{j\omega}) = \sum_{m=0}^{\infty} (ae^{j\omega})^{m+1}$

$$= ae^{j\omega} \sum_{m=0}^{\infty} (ae^{j\omega})^{m}$$

$$= ae^{j\omega} \left(\frac{1}{1-ae^{j\omega}} \right) \qquad |ae^{j\omega}| < 1$$

$|a| < 1$

$$= \frac{ae^{j\omega}}{1-ae^{j\omega}}$$

$$\boxed{X(e^{j\omega}) = \frac{ae^{j\omega}}{1-ae^{j\omega}} ; \; |a|<1}$$

(5) $x[n] = a^{|n|}$; $|a| < 1$ $\qquad x[n] = \begin{cases} a^{n} & n \geq 0 \\ a^{-n} & n \leq -1 \end{cases}$

So, $x[n] = a^{n}u[n] + a^{-n}u[-n-1]$

$$\mathcal{F}\{x[n]\} = \mathcal{F}\{a^{n}u(n)\} + \mathcal{F}\{a^{-n}u[-n-1]\}$$

$$= \frac{1}{1-ae^{-j\omega}} + \frac{ae^{j\omega}}{1-ae^{j\omega}}$$

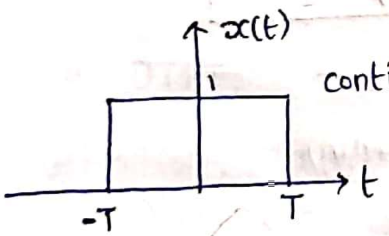
$$\Rightarrow \frac{1-\cancel{ae^{j\omega}} + \cancel{ae^{j\omega}} - a^{2}\cancel{e^{j\omega}}\,\cancel{e^{-j\omega}}}{(1-ae^{-j\omega})(1-ae^{j\omega})}$$

$$X(e^{j\omega}) = \frac{1-a^{2}}{1-2a\cos\omega + a^{2}} \quad ; \quad |a| < 1$$

$$\therefore \boxed{a^{|n|} \xrightarrow{DTFT} \frac{1-a^{2}}{1-2a\cos\omega + a^{2}} \;, \; |a|<1}$$

Imp

(6) $x[n] = \begin{cases} 1 & |n| \leq N \\ 0 & |n| > N \end{cases} = \begin{cases} 1 & -N \leq n \leq N \\ 0 & \text{otherwise} \end{cases}$



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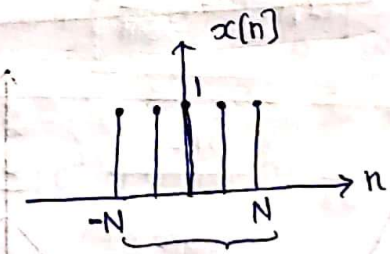


$\longleftrightarrow$ 2N+1 Samples

By definition, $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$

$$= \sum_{n=-N}^{N} 1 \cdot e^{-j\omega n}$$

By change of variables, $m = n+N$

$n = m - N$

$n = -N \Rightarrow m = 0$

$n = N \Rightarrow m = 2N$

So, $$X(e^{j\omega}) = \sum_{m=0}^{2N} \left(e^{-j\omega}\right)^{m-N}$$

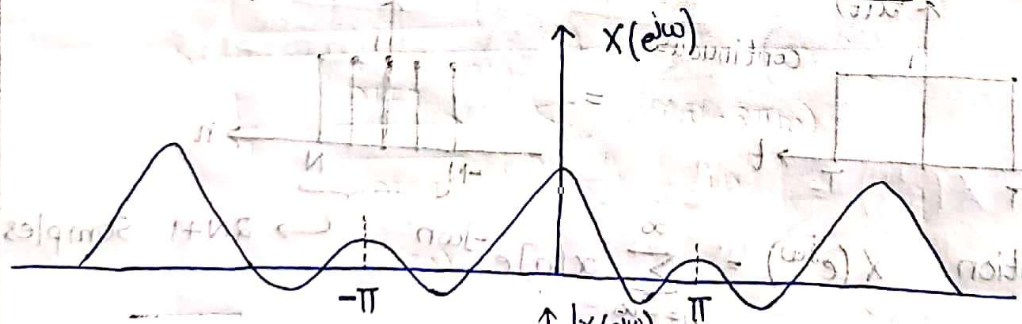
$$= e^{j\omega N} \sum_{m=0}^{2N} e^{-j\omega m}$$

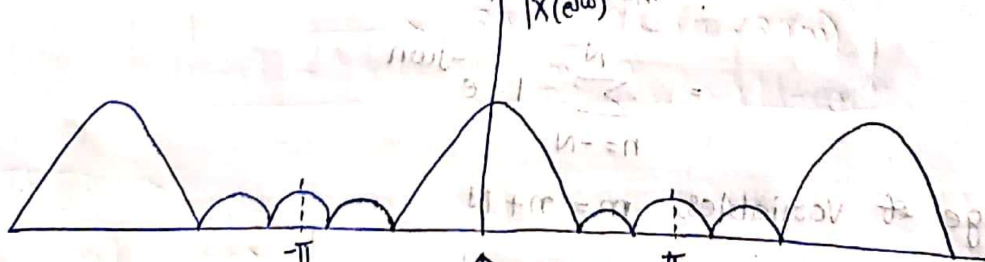
$$= e^{j\omega N} \left[\frac{1 - e^{-j\omega(2N+1)}}{1 - e^{-j\omega}} \right]$$

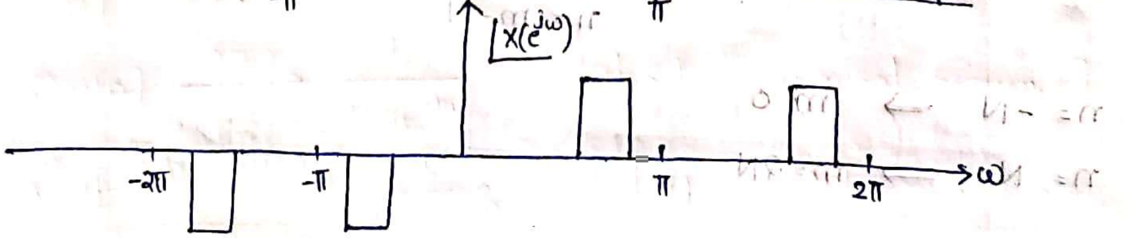
$$= e^{j\omega N} \left[\frac{\cancel{e^{-j\omega/2}} \left[e^{j\omega/2} - e^{-j\omega\left(2N+\frac{1}{2}\right)} \right]}{\cancel{e^{-j\omega/2}} \left[e^{j\omega/2} - e^{-j\omega/2} \right]} \right]$$

$$= \frac{\left[e^{j\omega\left(N+\frac{1}{2}\right)} - e^{-j\omega\left(N+\frac{1}{2}\right)} \right] \left(\frac{2j}{2j}\right)}{\left[e^{j\omega/2} - e^{-j\omega/2} \right] \left(\frac{2j}{2j}\right)}$$

$$\boxed{X(e^{j\omega}) = \frac{\sin\left[\omega\left(N+\frac{1}{2}\right)\right]}{\sin\left(\frac{\omega}{2}\right)}}$$

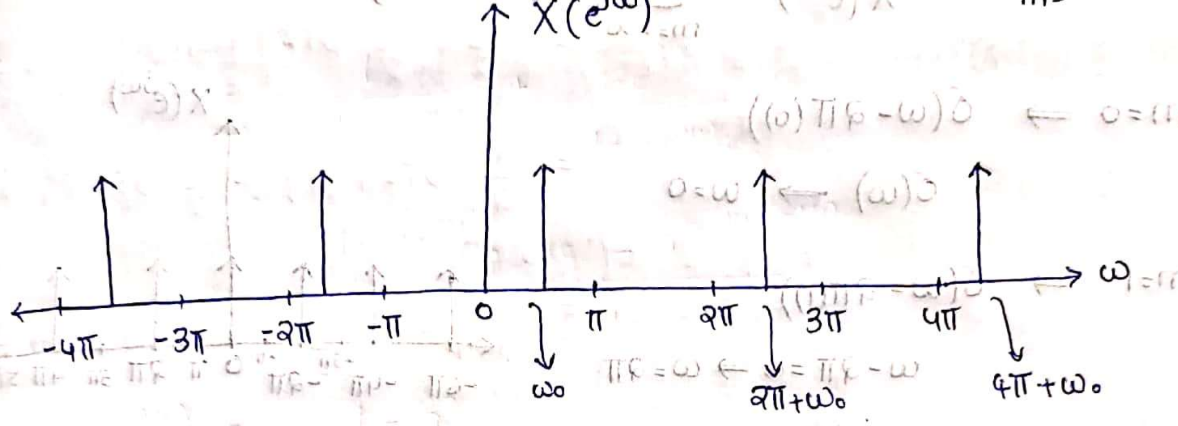






$$\angle X(e^{j\omega}) = \begin{cases} 0 & X(e^{j\omega}) > 0 \\ \pi & (X(e^{j\omega}) < 0 \end{cases}$$

(7) Find the inverse discrete time FT of $X(e^{j\omega}) = \sum_{m=-\infty}^{\infty} 2\pi(\delta(\omega-\omega_0-2\pi))$



For $m=0$, $2\pi\delta(\omega-\omega_0-0) = 2\pi\delta(\omega-\omega_0)$

$\omega-\omega_0 = 0 \rightarrow \omega = \omega_0$

Impulse located at $\omega=\omega_0$ with amplitude '2π'

$m=1$, $2\pi\delta(\omega-\omega_0-2\pi) \Rightarrow \omega-\omega_0-2\pi = 0$

$\omega = 2\pi+\omega_0$

$X(e^{j\omega}) = 2\pi\delta(\omega-\omega_0)$ for ω lies b/w 0 to 2π $\boxed{0 \le \omega \le 2\pi.}$

Inverse DTFT,

By definition, $x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega$

$$= \frac{1}{2\pi}\int_0^{2\pi} 2\pi\underbrace{\delta(\omega-\omega_0)}_{}e^{j\omega n}d\omega$$

$\hookrightarrow$ Exists at $\omega=\omega_0$

$$= \frac{1}{\cancel{2\pi}} \times \cancel{2\pi} \; e^{j\omega n}\Big|_{\omega=\omega_0}$$

$$\boxed{x[n] = e^{j\omega_0 n}}$$

$$* \boxed{e^{j\omega_0 n} \underset{IDTFT}{\overset{DTFT}{\rightleftharpoons}} \sum_{m=-\infty}^{\infty} 2\pi\delta(\omega-\omega_0-2\pi m)} *$$

(8) Find the inverse discrete time fourier transform of

$$X(e^{j\omega}) = \delta(\omega) \quad ; \quad -\pi \le \omega \le \pi$$

(or)

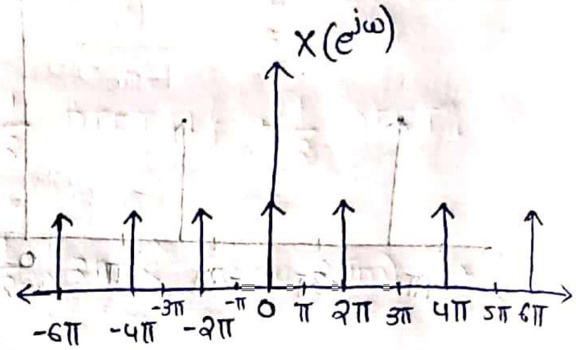
$$X(e^{j\omega}) = \sum_{m=-\infty}^{\infty} \delta(\omega - 2\pi m)$$

$m=0 \Rightarrow \delta(\omega - 2\pi(0))$

$\delta(\omega) \rightarrow \omega = 0$

$m=1 \Rightarrow \delta(\omega - 2\pi(1))$

$\omega - 2\pi = 0 \Rightarrow \omega = 2\pi$



By definition, $x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega}) e^{j\omega n} d\omega$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \underbrace{\delta(\omega)}_{\text{exists at } \omega=0} e^{j\omega n} d\omega$$

$$= \frac{1}{2\pi} e^{j\omega n} \Big|_{\omega=0} = \frac{1}{2\pi} e^{0} = \frac{1}{2\pi} \;//$$

$$\boxed{x[n] = \frac{1}{2\pi}}$$

$$\boxed{\begin{array}{c} \frac{1}{2\pi} \underset{\text{IDTFT}}{\overset{\text{DTFT}}{\rightleftarrows}} \sum_{m=-\infty}^{\infty} \delta(\omega - 2\pi m) \\ \text{(or)} \\ 1 \longleftrightarrow 2\pi \sum_{m=-\infty}^{\infty} \delta(\omega - 2\pi m) \end{array}}$$

<u>DTFT</u>:-

(1) $\delta[n] \overset{\text{DTFT}}{\longleftrightarrow} 1$

(2) $a^n u[n] \overset{\text{DTFT}}{\longleftrightarrow} \frac{1}{1 - ae^{-j\omega}}$, $|a| < 1$ [causal sequence]

(3) $a^n u[-n-1] \overset{\text{DTFT}}{\longleftrightarrow} \frac{-1}{1 - ae^{-j\omega}}$, $|a| > 1$ [Anti-causal sequence]

(4) $a^{-n}u[-n-1] \xleftrightarrow{DTFT} \dfrac{ae^{j\omega}}{1-ae^{j\omega}}$, $|a|<1$

(5) $a^{|n|} \xleftrightarrow{DTFT} \dfrac{1-a^2}{1-2a\cos(\omega)+a^2}$; $|a|<1$

(6) $x[n] = \begin{cases} 1 & |n| \le N \\ 0 & |n| > N \end{cases} = \begin{cases} 1 & -N \le n \le N \\ 0 & \text{otherwise} \end{cases} \longrightarrow (2N+1)$ samples

$\xrightarrow{DTFT} \dfrac{\sin\left[\omega\left(N+\frac{1}{2}\right)\right]}{\sin\left(\frac{\omega}{2}\right)}$

(7) $e^{j\omega_0 n} \longleftrightarrow \sum_{m=-\infty}^{\infty} 2\pi\,\delta(\omega-\omega_0-2\pi m)$

(8) $\longleftrightarrow \delta(\omega) \quad -\pi < \omega \le \pi$

$\dfrac{1}{2\pi} \longleftrightarrow \sum_{m=-\infty}^{\infty} \delta(\omega-2\pi m)$

$1 \longleftrightarrow 2\pi \sum_{m=-\infty}^{\infty} \delta(\omega-2\pi m)$

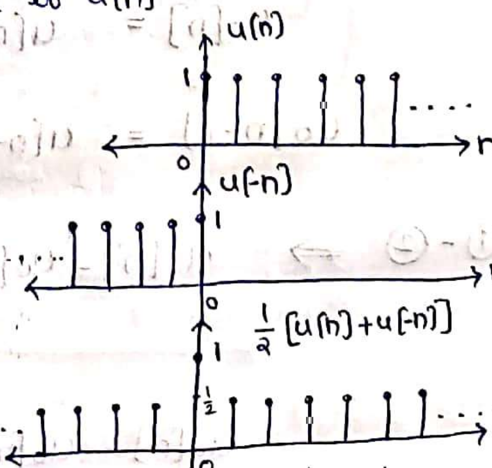
(9) Find the DTFT ~~of~~ the unit step signal $u[n]$

$u[n] \longleftrightarrow ?$

$u[n] = u_e[n] + u_o[n]$

$u_e[n]$ → even part ~~of~~ $u[n]$

$u_o[n]$ → odd part ~~of~~ $u[n]$



Even part : $u_e[n] = \dfrac{u[n]+u[-n]}{2}$

$= \dfrac{1}{2}\left[1+\delta[n]\right]$

$u_e[n] = \dfrac{1}{2} + \dfrac{1}{2}\delta[n]$

$= \dfrac{1}{2} + \dfrac{1}{2}\delta(n)$

By taking DTFT for $\underbrace{\left[\frac{1}{2}+\frac{1}{2}\delta[n]\right]}$

$$DTFT\left\{\frac{1}{2}+\frac{1}{2}\delta[n]\right\} = DTFT\left\{\frac{1}{2}\right\}+\frac{1}{2}DTFT\{\delta[n]\}$$

$$= \cancel{\frac{1}{2}} + \cancel{\frac{1}{2}}\, \cancel{DTFT\{\delta[n]\}}$$

$$= \frac{1}{2}DTFT\{1\}+\frac{1}{2}DTFT\{\delta[n]\}$$

$$= \frac{1}{2}\left\{2\pi\sum_{m=-\infty}^{\infty}\delta(\omega-2\pi m)\right\}+\frac{1}{2}\times 1$$

$$\therefore u_e[n] = \underbrace{\frac{1}{2}+\frac{1}{2}\delta[n]}_{x[n]=u_e[n]} \xleftrightarrow{DTFT} \underbrace{\frac{1}{2}\left\{2\pi\sum_{m=-\infty}^{\infty}\delta(\omega-2\pi m)\right\}+\frac{1}{2}}_{X(e^{j\omega}) = U_e(e^{j\omega})}$$

→ Now consider the odd part of $u[n]$

$$u_o[n] = \frac{u[n]-u[-n]}{2}$$

$$u[n] = u_e[n]+u_o[n]$$

$$\Rightarrow u_o[n] = u[n]-u_e[n]$$

$$= u[n]-\left[\frac{1}{2}+\frac{1}{2}\delta[n]\right]$$

$$u_o[n] = u[n]-\frac{1}{2}-\frac{1}{2}\delta[n] \longrightarrow ①$$

$$u_o[n-1] = u[n-1]-\frac{1}{2}-\frac{1}{2}\delta(n-1) \longrightarrow ②$$

$$① - ② \Rightarrow u_o[n]-u_o[n-1] = \underbrace{\{u[n]-u[n-1]\}}_{=\delta[n]} - \cancel{\frac{1}{2}}+\cancel{\frac{1}{2}}-\frac{1}{2}\{\delta[n]-\delta[n-1]\}$$

$$u_o[n]-u_o[n-1] = \underbrace{1\delta[n]-\frac{1}{2}\delta[n]}_{=\frac{1}{2}\delta[n]}+\frac{1}{2}\delta[n-1]$$

$$u_o[n] - u_o[n-1] = \frac{1}{2}\delta(n) + \frac{1}{2}\delta(n-1) \longrightarrow (*)$$

DTFT

$$u_o(n) \xleftrightarrow{\text{DTFT}} U_o[e^{j\omega}]$$

$$x[n] \xleftrightarrow{\text{DTFT}} X(e^{j\omega})$$

$$U_o[e^{j\omega}] - e^{-j\omega \cdot 1} U_o[e^{j\omega}] = \frac{1}{2}(1) + e^{-j\omega \cdot 1} \cdot \frac{1}{2}(1) = \frac{1}{2}[1 + e^{-j\omega}]$$

$$U_o[e^{j\omega}] = \frac{\frac{1}{2}[1 + e^{-j\omega}]}{1 - e^{-j\omega}}$$

$$u[n] = u_e[n] + u_o[n] \longleftrightarrow U_e[e^{j\omega}] + U_o[e^{j\omega}]$$

$$= \pi \sum_{m=-\infty}^{\infty} \delta[\omega - 2\pi m] + \frac{1}{2} + \frac{\frac{1}{2}[1 + e^{-j\omega}]}{1 - e^{-j\omega}}$$

$$= \pi \sum_{m=-\infty}^{\infty} \delta[\omega - 2\pi m] + \frac{1}{2}\left[1 + \frac{1 + e^{-j\omega}}{1 - e^{-j\omega}}\right]$$

$$= \pi \sum_{m=-\infty}^{\infty} \delta[\omega - 2\pi m] + \frac{1}{2}\left[\frac{1 - \cancel{e^{-j\omega}} + 1 + \cancel{e^{-j\omega}}}{1 - e^{-j\omega}}\right]$$

$$= \pi \sum_{m=-\infty}^{\infty} \delta(\omega - 2\pi m) + \frac{1}{1 - e^{-j\omega}}$$

$**$

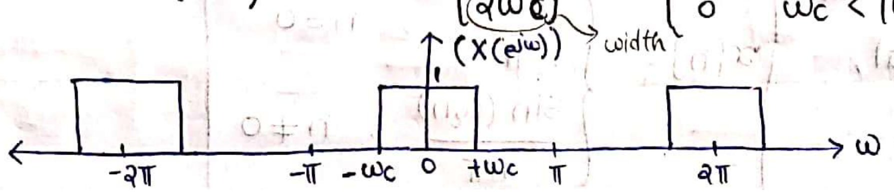
$$\therefore \boxed{u[n] \longleftrightarrow \frac{1}{1 - e^{-j\omega}} + \pi \sum_{m=-\infty}^{\infty} \delta[\omega - 2\pi m]}$$

(10) Find the inverse DTFT of the rectangular pulse spectrum defined only for $-\pi \le \omega \le \pi$.

$$X(e^{j\omega}) = \text{rect}\left[\frac{\omega}{2\omega_c}\right] = \begin{cases} 1 & |\omega| \le \omega_0 \\ 0 & \omega_c < |\omega| < \pi \end{cases}$$

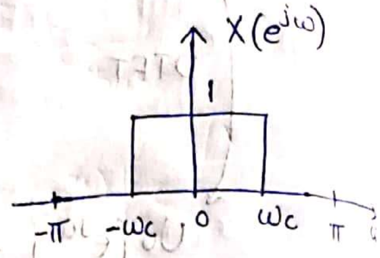
width

(Infinite)



$\rightarrow$ Periodic with 2π

$$X(e^{j\omega}) = rect\left[\frac{\omega - 2\pi m}{2\omega_c}\right] \longrightarrow \text{for infinite sequence.}$$



for one period.

The inverse DTFT of $X(e^{j\omega})$

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega}) \cdot e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\omega_c}^{+\omega_c} 1 \cdot e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi} \left. \frac{e^{j\omega n}}{jn} \right|_{-\omega_c}^{\omega_c}$$

$$= \frac{1}{2\pi}\left(\frac{e^{j\omega_c n} - e^{-j\omega_c n}}{jn}\right) = \frac{\sin(\omega_c n)}{n\pi}, \quad n \neq 0$$

*
$$\boxed{x[n] = \frac{\sin(\omega_c n)}{n\pi}, \quad n \neq 0}$$

For $n=0$ the IDTFT expression reduces to,

$$x(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})\, e^{j\omega \times 0}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} 1\, d\omega$$

$$x[0] = \frac{1}{2\pi}(\omega_c - (-\omega_c)) = \frac{\cancel{2}\omega_c}{\cancel{2}\pi} = \frac{\omega_c}{\pi} \Rightarrow \boxed{x[0] = \frac{\omega_c}{\pi}}$$

*
In general, $\boxed{x[n] = \begin{cases} \dfrac{\omega_c}{\pi}; & n = 0 \\ \dfrac{\sin(\omega_c n)}{n\pi}; & n \neq 0 \end{cases}}$

Compact form,

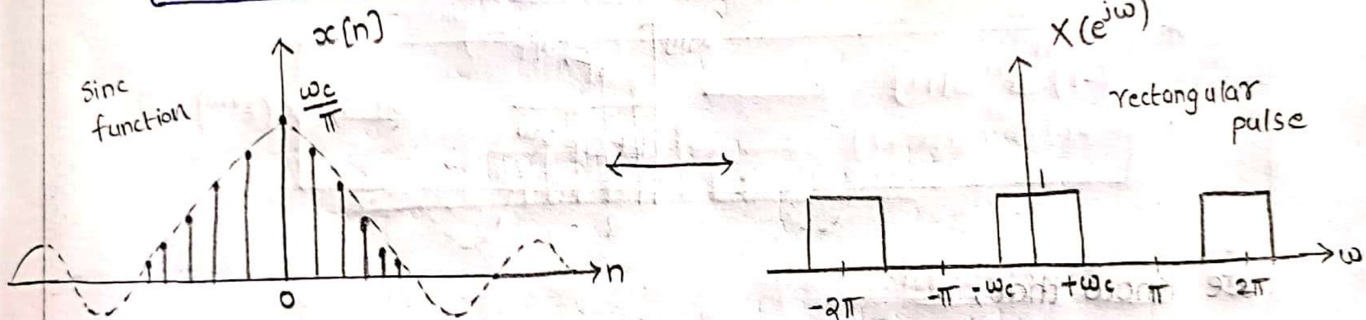
$$\boxed{x[n] = \frac{\sin(\omega_c n)}{n\pi}}, \quad -\infty < n < \infty$$

In the form of sinc func, $x[n] = \frac{\omega_c \cancel{n}}{\cancel{n}\pi} \frac{\sin(\omega_c n)}{\omega_c n}$

$$= \frac{\omega_c}{\pi} \operatorname{sinc}\left(\frac{\omega_c n}{\pi}\right)$$

$$* \quad \frac{\omega_c}{\pi} \operatorname{sinc}\left[\frac{\omega_c n}{\pi}\right] \longleftrightarrow \sum_{m=-\infty}^{\infty} \operatorname{rect}\left[\frac{\omega - 2\pi m}{2\omega_c}\right]$$

$$* \quad \frac{\omega_c}{\pi} \frac{\sin(\omega_c n)}{\omega_c n} \longleftrightarrow \sum_{m=-\infty}^{\infty} \operatorname{rect}\left[\frac{\omega - 2\pi m}{2\omega_c}\right]$$



Properties of DTFT :- $\longrightarrow$ Reduce the complexity of evaluation

$$x[n] \longleftrightarrow X(e^{j\omega})$$

① Linearity: If $x_1[n] \longleftrightarrow X_1(e^{j\omega})$

$$x_2[n] \longleftrightarrow X_2(e^{j\omega})$$

then $\boxed{a x_1[n] + b x_2[n] \longleftrightarrow a X_1(e^{j\omega}) + b X_2(e^{j\omega})}$

② Time shifting: If $x[n] \longleftrightarrow X(e^{j\omega})$

then $\boxed{x[n-n_0] \longleftrightarrow X(e^{j\omega}) \cdot e^{j\omega[-n_0]}}$

Note:- when a signal is shifted in time, the magnitude of its

Fourier transform remains unaltered.

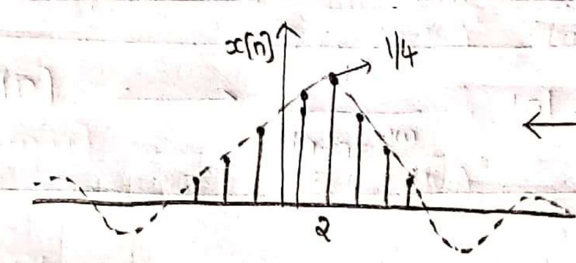
so, $\mathcal{F}\{x[n]\} = X(e^{j\omega}) = \underbrace{|X(e^{j\omega})|}_{\text{Magnitude}} \underbrace{e^{j\angle X(e^{j\omega})}}_{\text{Phase}}$

$$\mathcal{F}\{x[n-n_0]\} = X(e^{j\omega})\, e^{-j\omega n_0} = |X(e^{j\omega})|\, e^{j\angle X(e^{j\omega}) - \omega n_0}$$

<u>Note</u>:- The effect of a time shift on a signal is to introduce into its Fourier transform, a phase shift ~~of~~ "ωn_0" which is a linear function of 'ω'.

Q) Find the DTFT ~~of~~ the given sequence

$$\boxed{x[n] = \frac{1}{4}\,\text{sinc}\left[\frac{1}{4}(n-2)\right] \longleftrightarrow X(e^{j\omega}) = ?}$$



$\longleftrightarrow X(e^{j\omega}) = ?$

we know that,

$$\frac{\omega_c}{\pi}\,\text{sinc}\left[\frac{\omega_c n}{\pi}\right] \longleftrightarrow \sum_{m=-\infty}^{\infty} \text{rect}\left[\frac{\omega - 2\pi m}{2\omega_c}\right]$$

Substituting, $\omega_c = \pi/4$

$$\frac{\pi/4}{\pi}\,\text{sinc}\left(\frac{n\pi/4}{\pi}\right) \longleftrightarrow \sum_{m=-\infty}^{\infty} \text{rect}\left[\frac{\omega - 2\pi m}{2(\pi/4)}\right]$$

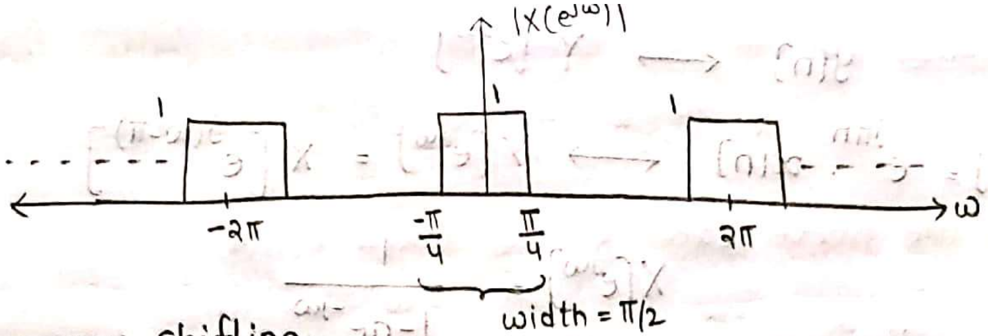
$$\frac{1}{4}\,\text{sinc}\left[\frac{1}{4}n\right] \longleftrightarrow \sum_{m=-\infty}^{\infty} \text{rect}\left[\frac{\omega - 2\pi m}{\pi/2}\right]$$

Using time shifting property,

$\Rightarrow$ shifting by a factor ~~of~~ 2 is $n \rightarrow n-2$

$$\frac{1}{4}\,\text{sinc}\left[\frac{1}{4}(n-2)\right] \longleftrightarrow \sum_{m=-\infty}^{\infty} \text{rect}\left[\frac{\omega - 2\pi m}{\pi/2}\right] \cdot e^{-j2\omega}$$

$\pi/2 \rightarrow$ width



③ Frequency shifting:-

$$\text{If} \quad x[n] \longleftrightarrow X[e^{j\omega}]$$

$$\text{then} \quad x[n] \cdot e^{j\omega_0 n} \longleftrightarrow X[e^{j(\omega-\omega_0)}]$$

Note:- Frequency shift corresponds to multiplication in time domain by a complex sinusoid i.e $e^{j\omega_0 n}$ whose frequency is equal to the frequency shift i.e ω_0.

Q) Find DTFT of an exponential sinusoid sequence with alternating signs $\quad y[n] = (-1)^n a^n u[n] \quad , \quad |a| < 1$

Q) Find the DTFT of an exponential sinusoid sequence with alternating sign. $y[n] = (-1)^n a^n u[n] \quad |a| < 1$

We know that, $(-1)^n = (e^{j\pi})^n$

$$y[n] = (-1)^n a^n u[n] = e^{j\pi n} \cdot \underbrace{a^n u[n]}_{x[n]}$$

$$y[n] = e^{j\pi n} \cdot x[n]$$

$$x[n] \longleftrightarrow X(e^{j\omega})$$

$$a^n u[n] \longleftrightarrow \frac{1}{1-ae^{-j\omega}}$$

$$y[n] \longleftrightarrow Y[e^{j\omega}]$$

$$y[n] = e^{j\pi n} \cdot x[n] \longleftrightarrow Y[e^{j\omega}] = X[e^{j(\omega-\pi)}]$$

$$X[e^{j\omega}] = \frac{1}{1-ae^{-j\omega}}$$

$$X[e^{j(\omega-\pi)}] = \frac{1}{1-ae^{-j(\omega-\pi)}}$$

$$= \frac{1}{1-a\underbrace{e^{j\pi}}_{=+1} \cdot e^{-j\omega}}$$

$$X[e^{j(\omega-\pi)}] = \frac{1}{1+ae^{-j\omega}}$$

*
$$\boxed{(-1)^n a^n u[n] \longleftrightarrow \frac{1}{1+ae^{-j\omega}}, \quad |a|<1}$$

4 Time Reversal:-

If $x[n] \longleftrightarrow X(e^{j\omega})$

then, $y[n] = x[-n] \longleftrightarrow Y(e^{j\omega}) = X[e^{-j\omega}]$

Note:- ① If $x[n]$ is even, then its Fourier transform is also even.

if $x[n] = x[-n]$ then $X[e^{j\omega}] = X[e^{-j\omega}]$

② If $x[n]$ is odd, then its Fourier transform is also odd.

if $x[n] = -x[-n]$ then $X[e^{j\omega}] = -X[e^{-j\omega}]$

5 Differentiation in time Domain:-

First order difference operation, $y[n] = x[n] - x[n-1]$

↳ maximum shift of '1'.

If $x[n] \longleftrightarrow X[e^{j\omega}]$

then $y[n] = x[n] - x[n-1] \longleftrightarrow Y[e^{j\omega}] = [1-e^{-j\omega}] \cdot X[e^{j\omega}]$

Proof:- Given that $x[n] \longleftrightarrow X[e^{j\omega}]$

using time shifting property,

$$x[n-1] \longleftrightarrow X[e^{j\omega}] \cdot e^{-j\omega}$$

$$x[n] - x[n-1] \longleftrightarrow X[e^{j\omega}] - X[e^{j\omega}] \cdot e^{-j\omega}$$

$$x[n] - x[n-1] \longleftrightarrow [1-e^{-j\omega}] X[e^{j\omega}]$$

⑥ Differentiation in frequency domain:-

If $x[n] \longleftrightarrow X[e^{j\omega}]$

then $-jn \cdot x[n] \longleftrightarrow \frac{d}{d\omega} X[e^{j\omega}]$

(or)

$$n \cdot x[n] \longleftrightarrow j \frac{d}{d\omega} X[e^{j\omega}]$$

$\because j^2 = -1$

$-j \leftrightarrow +1$

$j \leftrightarrow (-1)$

$\cancel{j} \leftrightarrow j^2$

Q) Determine the DTFT of the sequence $y[n] = n a^n u[n]$, $|a| < 1$

Let $x[n] = a^n \cdot u[n]$; $|a| < 1$

$$x[n] \longleftrightarrow X[e^{j\omega}]$$

$$a^n u[n] \longleftrightarrow \frac{1}{1-ae^{-j\omega}}$$

$\therefore y[n] = n \cdot a^n u[n] = n x[n]$

$$y[n] \longleftrightarrow Y[e^{j\omega}]$$

$$y[n] = n x[n] \longleftrightarrow Y[e^{j\omega}] = j \frac{d}{d\omega} X(e^{j\omega})$$

$$= j \frac{d}{d\omega}\left[\frac{1}{1-ae^{-j\omega}}\right]$$

$j \frac{d}{d\omega}\left(\frac{1}{(1-ae^{-j\omega})}\right) = j \cdot -1 (1-ae^{-j\omega})^{-2} \cdot (-jae^{-j\omega})$

$= \frac{-ae^{-j\omega}}{(1-ae^{-j\omega})^2}$

$$= \frac{ae^{-j\omega}}{(1-ae^{-j\omega})^2}$$

$$n a^n u[n] \longleftrightarrow \frac{ae^{-j\omega}}{(1-ae^{-j\omega})^2}.$$

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Q) Determine the inverse DTFT of

$$X[e^{j\omega}] = \frac{1}{(1-ae^{-j\omega})^2}$$

From the above example, i.e from equ ⊛

$$na^n u[n] \longleftrightarrow \frac{ae^{-j\omega}}{(1-ae^{-j\omega})^2}, \quad |a|<1$$

By using time shifting property, shift factor of '1' unit

$$n \longrightarrow n+1$$

$$(n+1)a^{n+1}u[n+1] \longleftrightarrow \frac{a\cancel{e^{-j\omega}}}{(1-ae^{-j\omega})^2} \cdot \cancel{e^{(+j\omega)}}$$

$$(n+1)\frac{a^{\cancel{n+1}}}{\cancel{a}}u[n+1] \longleftrightarrow \frac{1}{(1-ae^{-j\omega})^2}$$

$$(n+1)a^n u[n+1] \longleftrightarrow \frac{1}{(1-ae^{-j\omega})^2}$$

⑦ Convolution Property:-

If $x_1[n] \longleftrightarrow X_1[e^{j\omega}]$ & $x_2[n] \longleftrightarrow X_2[e^{j\omega}]$

then $x_1[n] * x_2[n] \longleftrightarrow X_1[e^{j\omega}] \cdot X_2[e^{j\omega}]$

Note:- The convolution property states that the convolution in the time domain corresponds to multiplication in the frequency domain.

Q) By using the convolution property, determine the convolution $x[n] = x_1[n] * x_2[n]$ of the sequence, $x_1[n] = x_2[n] = \{1, \underset{\uparrow}{1}, 1\}$

$$x_1[n] \longleftrightarrow X_1[e^{j\omega}]$$

$$x_2[n] \longleftrightarrow X_2[e^{j\omega}]$$

$$x_1[n] * x_2[n] \longleftrightarrow X_1(e^{j\omega}) \cdot X_2(e^{j\omega})$$

$$x_1[n] * x_2[n] \longleftrightarrow X_1[e^{j\omega}] \cdot X_2[e^{j\omega}]$$

⑧ Accumulation property:-

If $x[n] \longleftrightarrow X[e^{j\omega}]$

then $$\underbrace{\sum_{k=-\infty}^{n} x[k]} \longleftrightarrow \underbrace{\frac{1}{1-e^{-j\omega}} X[e^{j\omega}] + \pi X[e^{j\omega}] \sum_{m=-\infty}^{\infty} \delta[\omega-2\pi m]}$$

Proof:- convolving $x[n]$ and $u[n] \Rightarrow$ unit step signal.

$$x[n] * u[n] = \sum_{k=-\infty}^{\infty} x[k]\, u[n-k]$$

$$u[n-k] = \begin{cases} 1 & n-k \geq 0 \Rightarrow k \leq n \\ 0 & n-k < 0 \Rightarrow k > n \end{cases}$$

$$x[n] * u[n] = \sum_{k=-\infty}^{n} x[k] \cdot 1 = \sum_{k=-\infty}^{n} x[k] \Rightarrow \text{Accumulation}$$

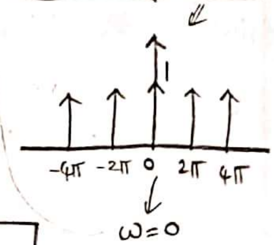
$$\mathcal{F}\left\{\sum_{k=-\infty}^{n} x[k]\right\} = \mathcal{F}\{x[n] * u[n]\}$$

By convolution property, $= \mathcal{F}\{x[n]\} \cdot \mathcal{F}\{u[n]\} = X[e^{j\omega}] \cdot U[e^{j\omega}]$

$$= X[e^{j\omega}]\left[\frac{1}{1-e^{-j\omega}} + \pi \sum_{m=-\infty}^{\infty} \delta[\omega - 2\pi m]\right]$$

$$= X[e^{j\omega}] \cdot \frac{1}{1-e^{-j\omega}} + X[e^{j\omega}] \cdot \pi \sum_{m=-\infty}^{\infty} \delta(\omega - 2\pi m)$$

$$= X[e^{j\omega}] \cdot \frac{1}{1-e^{-j\omega}} + \pi X[e^{j0}] \underbrace{\sum_{m=-\infty}^{\infty} \delta(\omega - 2\pi m)}$$



$$\boxed{\sum_{k=-\infty}^{\infty} x(k) \longleftrightarrow \frac{1}{1-e^{-j\omega}} X[e^{j\omega}] + \pi X[e^{j0}] \sum_{m=-\infty}^{\infty} \delta(\omega - 2\pi m)}$$

(9) Multiplication / Modulation property:-

If $x_1(n) \longleftrightarrow X_1[e^{j\omega}]$ & $x_2(n) \longleftrightarrow X_2[e^{j\omega}]$

then $$x_1[n] \cdot x_2[n] \longleftrightarrow \frac{1}{2\pi}\left[X_1[e^{j\omega}] \circledast X_2[e^{j\omega}]\right]$$

↳ Circular convolution

Discrete

$$x_1[n] * x_2[n] = \sum_{k=-\infty}^{\infty} x_1[k]\, x_2[n-k]$$

Continuous

$$X_1[e^{j\omega}] * X_2[e^{j\omega}] = \int_{2\pi} X_1[e^{j\theta}] \cdot X_2[e^{j(\omega-\theta)}]\, d\theta$$

$$\boxed{\begin{aligned} x_1[n] \cdot x_2[n] &\longleftrightarrow \frac{1}{2\pi}\left[X_1[e^{j\omega}] \circledast X_2[e^{j\omega}]\right] \\ &\longleftrightarrow \frac{1}{2\pi}\int_{2\pi} X_1[e^{j\theta}] \cdot X_2[e^{j(\omega-\theta)}]\, d\theta \end{aligned}}$$

(10) Conjugation and Conjugate Symmetry:-

If $x[n] \longleftrightarrow X[e^{j\omega}]$

then $x^*[n] \longleftrightarrow X^*[e^{-j\omega}]$

Proof:- By definition, $\mathcal{F}\{x[n]\} = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$

$$\mathcal{F}\{x^*[n]\} = \sum_{n=-\infty}^{\infty} x^*[n]\, e^{-j\omega n}$$

$$= \left[\sum_{n=-\infty}^{\infty} x[n]\, e^{+j\omega n}\right]^*$$

$\sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} \Rightarrow X[e^{j\omega}]$

$\sum_{n=-\infty}^{\infty} x[n] e^{-j(-\omega)n} \Rightarrow X[e^{-j(-\omega)}]$

$$= \left[\sum_{n=-\infty}^{\infty} x[n] \cdot e^{-j(-\omega)n}\right]^*$$

$$= \left[X[e^{-j\omega}]\right]^* = X^*[e^{-j\omega}]$$

$$\boxed{\therefore\ \mathcal{F}\{x^*[n]\} = X^*[e^{-j\omega}]}$$

Case-1:- If $x[n]$ is a real signal i.e $x[n] = x^*[n]$

then $\mathcal{F}\{x[n]\} = \mathcal{F}\{x^*[n]\}$

$$\updownarrow \qquad\qquad \updownarrow$$

$$X[e^{j\omega}] = X^*[e^{-j\omega}]$$

Apply conjugate on both sides,

$$\boxed{X^*[e^{j\omega}] = X[e^{-j\omega}]} \Rightarrow \text{Conjugate Symmetry}$$

→ The DTFT of a real signal is conjugate symmetry.

Case-2:- If $x[n]$ is real and even

i.e $x[n] = x^*[n] = x[-n]$

Real

Even

then $\mathcal{F}\{x[n]\} = \mathcal{F}\{x^*[n]\} = \mathcal{F}\{x[-n]\}$

$$X[e^{j\omega}] = X^*[e^{-j\omega}] = X[e^{-j\omega}]$$

Real

Even

⇒ The DTFT of a real and even signal is real & even i.e if $x[n]$ is real & even then its DTFT $X[e^{j\omega}]$ is also real and even.

Case-3:- If $x[n]$ is real and odd signal i.e,

$$\underbrace{x[n] = x^*[n]}_{\text{Real}} = -x[-n] \quad (\text{odd})$$

then $$\mathcal{F}\{x[n]\} = \mathcal{F}\{x^*[n]\} = \mathcal{F}\{-x[-n]\}$$

$$X[e^{j\omega}] = X^*[e^{-j\omega}] = -X[e^{-j\omega}]$$

Real Imaginary   odd

⇒ If $x[n]$ is real and odd signal, then its DTFT $X[e^{j\omega}]$ is Imaginary and odd.

Case-4:- If $x[n]$ is imaginary i.e $x^*[n] = -x[n]$

then $$\mathcal{F}\{x^*[n]\} = \mathcal{F}\{-x[n]\}$$

$$X^*[e^{-j\omega}] = -X[e^{j\omega}] \Rightarrow \text{Conjugate antisymmetry.}$$

Case-5:- If $x[n]$ is imaginary & even signal

i.e $$x^*[n] = -x[n] = x[-n]$$

Img   Even

then $$\mathcal{F}\{x^*[n]\} = \mathcal{F}\{-x[n]\} = \mathcal{F}\{x[-n]\}$$

$$X^*[e^{-j\omega}] = -X[e^{j\omega}] = X[e^{-j\omega}]$$

img

Case-6:- If $x[n]$ is imaginary and odd

that is $x[n] = \underbrace{-x^*[n]}_{\text{img}} = \underbrace{-x[-n]}_{\text{odd}}$

$$\mathcal{F}\{x[n]\} = \mathcal{F}\{-x^*[n]\} = \mathcal{F}\{-x[-n]\}$$

$$X[e^{j\omega}] = \underbrace{-X^*[e^{-j\omega}]}_{\text{Real}} = \underbrace{-X[e^{-j\omega}]}_{\text{odd}}$$

⇒ If $x[n]$ is imaginary and odd then its DTFT $X[e^{j\omega}]$ is real and odd.

Case-7:- If $x[n] \longleftrightarrow X[e^{j\omega}]$

then ① The DTFT ~~of~~ the even part ~~of~~ $x[n]$ is the real value ~~of~~ $X[e^{j\omega}]$ i.e,

$$x_e[n] = E\{x[n]\} \longleftrightarrow Re\{X[e^{j\omega}]\} = X_R[e^{j\omega}]$$

② The DTFT ~~of~~ the odd part ~~of~~ $x[n]$ is $jX_{img}[e^{j\omega}]$

⑪ Parsevals Theorem / Relation:-

Let $x[n]$ be an energy signal and if

$$x[n] \longleftrightarrow X[e^{j\omega}]$$

then $$E_x = \sum_{n=-\infty}^{\infty} |x(n)|^2 = \frac{1}{2\pi}\int_{2\pi} |X[e^{j\omega}]|^2 d\omega$$

Proof:- Consider the LHS ~~of~~ above equation,

$$E_x = \sum_{n=-\infty}^{\infty} |x(n)|^2 = \sum_{n=-\infty}^{\infty} x[n].\, x^*[n]$$

$$= \sum_{n=-\infty}^{\infty} x[n]\left[\frac{1}{2\pi}\int_{2\pi} X[e^{j\omega}]\, e^{j\omega n} d\omega\right]^*$$

$$= \sum_{n=-\infty}^{\infty} x[n].\, \frac{1}{2\pi}\int_{2\pi} X^*[e^{j\omega}].\, e^{-j\omega n} d\omega$$

$$= \frac{1}{2\pi}\int_{2\pi} X^*[e^{j\omega}]\,\underbrace{\left[\sum_{n=-\infty}^{\infty} x[n]\,e^{-j\omega n}\right]}_{=X[e^{j\omega}]} d\omega$$

$$= \frac{1}{2\pi}\int_{2\pi} X^*[e^{j\omega}]\cdot X[e^{j\omega}]\, d\omega$$

$$= \frac{1}{2\pi}\int_{2\pi} |X(e^{j\omega})|^2\, d\omega \Rightarrow \text{RHS}$$

**

$$\boxed{E_x = \sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{2\pi} |X[e^{j\omega}]|^2\, d\omega}$$

Q) Find the Energy of the sequence $x[n] = \text{sinc}\left[\frac{\omega_c n}{\pi}\right]$ assuming $\omega_c < \pi$.

We know that,

$$\frac{\omega_c}{\pi}\,\text{sinc}\left[\frac{\omega_c n}{\pi}\right] \longleftrightarrow \sum_{m=-\infty}^{\infty} \text{rect}\left[\frac{\omega-2\pi m}{2\omega_c}\right]$$

$$x[n] = \text{sinc}\left[\frac{\omega_c n}{\pi}\right] \longleftrightarrow X[e^{j\omega}]$$

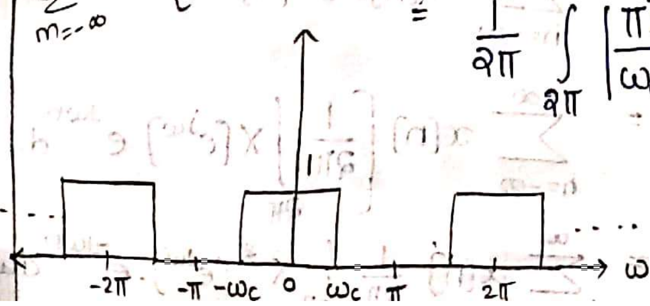
$$\therefore \text{sinc}\left[\frac{\omega_c n}{\pi}\right] \longleftrightarrow \frac{\pi}{\omega_c}\sum_{m=-\infty}^{\infty} \text{rect}\left[\frac{\omega-2\pi m}{2\omega_c}\right]$$

$$\text{Energy} = \sum_{n=-\infty}^{\infty} |x[n]|^2 \longleftrightarrow \frac{1}{2\pi}\int_{2\pi} |X(e^{j\omega})|^2\, d\omega$$

$$\text{Energy of } x[n] = \frac{1}{2\pi}\int_{2\pi} |X(e^{j\omega})|^2\, d\omega$$

$$= \frac{1}{2\pi}\int_{2\pi}\left|\frac{\pi^2}{\omega_c^2}\sum_{m=-\infty}^{\infty} \text{rect}\left[\frac{\omega-2\pi m}{2\omega_c}\right]\right|^2 d\omega$$

$$\sum_{m=-\infty}^{\infty} \text{rect}\left[\frac{\omega-2\pi m}{2\omega_c}\right]$$



$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \frac{\pi}{\omega_c} \, rect\left[\frac{\omega}{2\omega_c}\right] \right|^2 d\omega$$

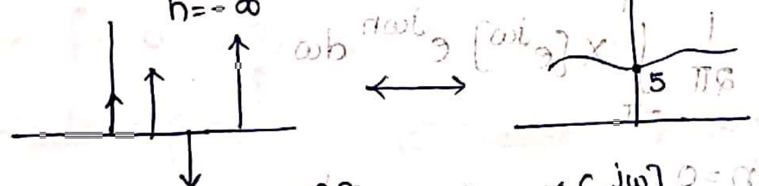
$$rect\left[\frac{\omega}{2\omega_c}\right] = \begin{cases} 1 & -\omega_c \le \omega < \omega_c \Rightarrow |\omega| < \omega_c \\ 0 & \omega_c < |\omega| < \pi \end{cases}$$

$$= \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} \frac{\pi^2}{\omega_c^2} \left| rect\left(\frac{\omega}{2\omega_c}\right) \right|^2 d\omega$$

$$= \frac{1}{2\pi} \frac{\pi^2}{\omega_c^2} \int_{-\omega_c}^{\omega_c} 1 \, d\omega = \frac{\pi^{\cancel{2}}}{\cancel{2}\cancel{\pi} \cdot \omega_c^{\cancel{2}}} \cdot \cancel{2}\cancel{\omega_c} = \frac{\pi}{\omega_c}$$

Eneary of $x(n) \Rightarrow$ $\boxed{E_x = \frac{\pi}{\omega_c} \, J}$

① $$X\left[e^{j0}\right] = \sum_{n=-\infty}^{\infty} x(n)$$



$$x(n) \longleftrightarrow X\left[e^{j\omega}\right]$$

$1+2-1+3 = 5 \Rightarrow$ sum of all values in discrete is equal to value of $X\left[e^{j\omega}\right]$ at $\omega = 0$.

Proof:- By definition,

$$X\left[e^{j\omega}\right] = \sum_{n=-\infty}^{\infty} x(n) \, e^{-j\omega n}$$

Substituting $\omega = 0$ in above equ,

$$X\left[e^{j\omega}\right]\Big|_{\omega=0} = X\left[e^{j0}\right] = \sum_{n=-\infty}^{\infty} x(n) e^{-j0n}$$

$$\boxed{\sum_{n=-\infty}^{\infty} x(n) = X\left[e^{j0}\right]}$$

② $X[e^{j\pi}] = \sum_{n=-\infty}^{\infty} (-1)^n x(n)$

Proof:- By definition,

$$X[e^{j\omega}] = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

Substituting $\omega = \pi$,

$$X[e^{j\omega}]\Big|_{\omega=\pi} = X[e^{j\pi}] = \sum_{n=-\infty}^{\infty} x[n] e^{-j\pi n} \quad (\because e^{-j\pi} = (-1))$$

$$= \sum_{n=-\infty}^{\infty} x(n)(-1)^n$$

$$\boxed{X[e^{j\pi}] = \sum_{n=-\infty}^{\infty} (-1)^n x(n)}$$

③ $\int_{-\pi}^{\pi} X[e^{j\omega}]\, d\omega = 2\pi x(0)$

By definition of IDTFT,

$$x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X[e^{j\omega}] e^{j\omega n}\, d\omega$$

Substituting $n=0$,

$$x(0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X[e^{j\omega}] \cancel{e^{j\omega(0)}}\, d\omega = \frac{1}{2\pi} \int_{-\pi}^{\pi} X[e^{j\omega}]\, d\omega$$

$$\boxed{\int_{-\pi}^{\pi} X[e^{j\omega}]\, d\omega = 2\pi x(0)}$$

④ $\int_{-\pi}^{\pi} |X[e^{j\omega}]|^2\, d\omega = 2\pi \sum_{n=-\infty}^{\infty} |x(n)|^2$

By parseval's Relationship,

⑤ $$\int_{-\pi}^{\pi} \left| \frac{dX[e^{j\omega}]}{d\omega} \right|^2 d\omega = 2\pi \sum_{n=-\infty}^{\infty} |n\,x[n]|^2$$

Proof:- By using differentiation in freq domain,

2) Let $X[e^{j\omega}]$ is the DTFT ~~of~~ the signal $x[n]$

$$x[n] \longleftrightarrow X[e^{j\omega}]$$

Perform the following calculations without explicitly evaluating $X[e^{j\omega}]$.

$$x[n] = \{-1, 0, 1, \underset{\uparrow}{2}, 1, 0, 1, 2, 1, 0, -1\}$$

(a) Evaluate $X[e^{j0}]$

(b) Find $\angle X[e^{j\omega}]$

(c) Evaluate $\int_{-\pi}^{\pi} X[e^{j\omega}]\, d\omega$

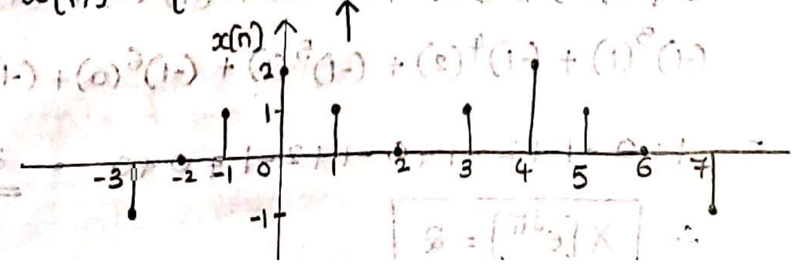
(d) Find $X[e^{j\pi}]$

(e) Determine and sketch the signal whose fourier transform is $Re\{X[e^{j\omega}]\}$

(f) Evaluate $\int_{-\pi}^{\pi} |X[e^{j\omega}]|^2 d\omega$

(g) Evaluate $\int_{-\pi}^{\pi} \left| \frac{dX(e^{j\omega})}{d\omega} \right|^2 d\omega$

Sol:- Given, $x[n] = \{-1, 0, 1, \underset{\uparrow}{2}, 1, 0, 1, 2, 1, 0, -1\}$



(a) We know that,

$$X[e^{j0}] = \sum_{n=-\infty}^{\infty} x[n]$$

$$= \sum_{n=-3}^{7} x[n]$$

$$= -1+0+1+2+1+0+1+2+1+0+-1$$

$$\boxed{X[e^{j0}] = 6}$$

(b)

(c) We know that,

$$\int_{-\pi}^{\pi} X[e^{j\omega}]\, d\omega = 2\pi\, x[0]$$

$$= 2\pi(2) = \underline{\underline{4\pi}}$$

(d) $$X[e^{j\pi}] = \sum_{n=-\infty}^{\infty} (-1)^n x[n]$$

$$= \sum_{n=-3}^{7} (-1)^n x[n]$$

$$= (-1)^{-3}(-1) + (-1)^{2}(0) + (-1)^{-1}(1) + (-1)^0(2) + (-1)^1(1) + (-1)^2(0) +$$
$$(-1)^3(1) + (-1)^4(2) + (-1)^5(1) + (-1)^6(0) + (-1)^7(-1)$$

$$= 1+0-1+2-1+0-1+2-1+0+0 = \underline{\underline{2}}$$

$$\therefore \boxed{X[e^{j\pi}] = 2}$$

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(e) Given, DTFT of $x[n]$ is $Re\{X[e^{j\omega}]\}$

we know that DTFT of even part of $x[n]$ is the real value of $X[e^{j\omega}]$ i.e,

$$E\{x[n]\} \longleftrightarrow Re\{X[e^{j\omega}]\}$$

We know that, $E\{x[n]\} = \dfrac{x[n] + x[-n]}{2}$

$$x[n] = \{-1,0,1,2,\underset{\uparrow}{1},0,1,2,1,0,-1\}$$

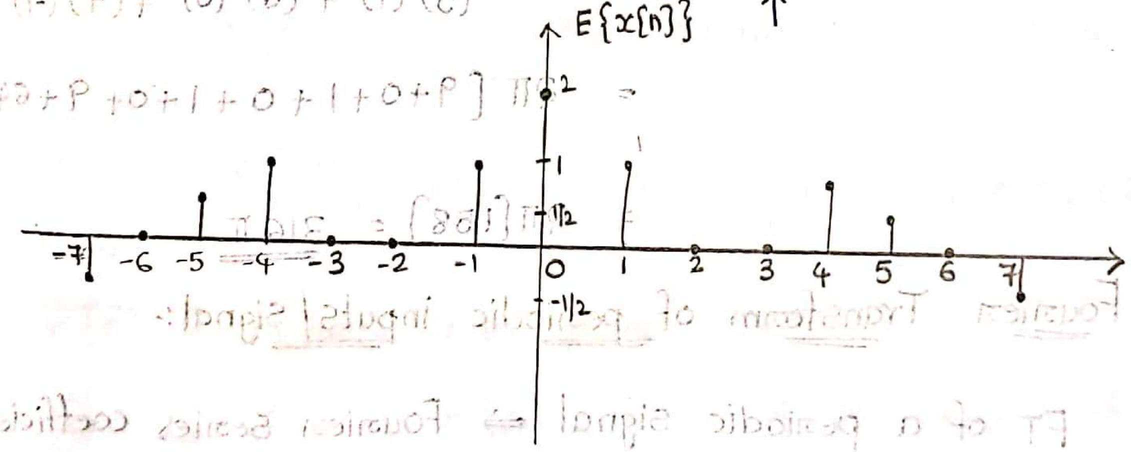
$-3 \; -2 \; -1 \; 0 \; 1 \; 2 \; 3 \; 4 \; 5 \; 6 \; 7$

$$x[-n] = \{-1,0,1,2,1,0,1,2,\underset{\uparrow}{1},0,-1\}$$

$-7 \; -6 \; -5 \; -4 \; -3 \; -2 \; -1 \; 0 \; 1 \; 2 \; 3$

$$E\{x[n]\} = \left\{\frac{-1}{2},\frac{0}{2},\frac{1}{2},\frac{2}{2},\frac{0}{2},\frac{0}{2},\frac{2}{2},\underset{\uparrow}{\frac{4}{2}},\frac{2}{2},\frac{0}{2},\frac{0}{2},\frac{2}{2},\frac{1}{2},\frac{0}{2},\frac{-1}{2}\right\}$$

$$E\{x[n]\} = \{-0.5,0,0.5,1,0,0,1,\underset{\uparrow}{2},1,0,0,1,0.5,0,-0.5\}$$



(f) We know that,

$$\int_{-\pi}^{\pi} |X[e^{j\omega}]|^2 \, d\omega = 2\pi \sum_{n=-\infty}^{\infty} |x[n]|^2$$

$$= 2\pi \sum_{n=-\infty}^{\infty} x[n] \cdot x^*[n]$$

$$x[n] = \{-1,0,1,\underset{\uparrow}{2},1,0,1,2,1,0,-1\}$$

$$x^*[n] = \{-1,0,1,\underset{\uparrow}{2},1,0,1,2,1,0,-1\}$$

$$|x[n]|^2 = x(n) \cdot x^*[n] = \{1,0,1,4,1,0,1,4,1,0,1\}$$
$$\uparrow$$

$$\int_{-\pi}^{\pi} |X[e^{j\omega}]|^2 d\omega = 2\pi \sum_{n=-\infty}^{\infty} |x[n]|^2$$

$$= 2\pi [1+0+1+4+1+0+1+4+1+0+1]$$

$$= 2\pi [14] = \underline{28\pi}$$

(g) $$\int_{-\pi}^{\pi} \left| \frac{dX[e^{j\omega}]}{d\omega} \right|^2 d\omega = 2\pi \sum_{n=-\infty}^{\infty} |n x[n]|^2$$

$$= 2\pi \sum_{n=-\infty}^{\infty} n^2 |x[n]|^2$$

$$= 2\pi \left[(-3)^2(-1)^2 + (-2)^2(0)^2 + (1)^2(1)^2 + (0)^2(2)^2 + (1)^2(1)^2 + (2)^2(0)^2 + (3)^2(1)^2 + (4)^2(2)^2 + (5)^2(1)^2 + (6)^2(0)^2 + (7)^2(-1)^2 \right]$$

$$= 2\pi [9+0+1+0+1+0+9+64+25+0+49]$$

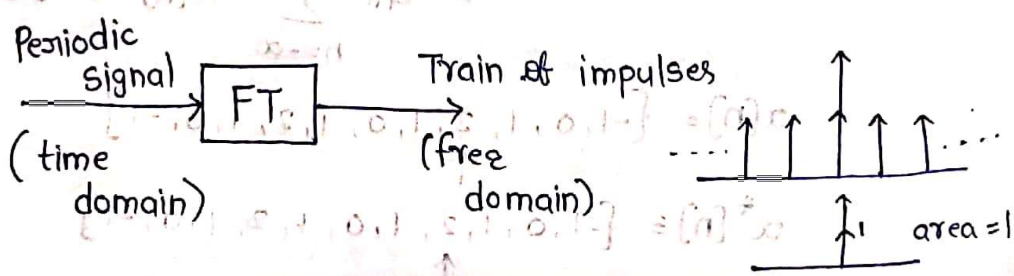
$$= 2\pi[158] = \underline{316\pi}$$

Fourier Transform of periodic inputs/ signal:-

FT of a periodic signal $\Rightarrow$ Fourier Series coefficients

$\Rightarrow$ The resulting transform consists of a train of impulses in the frequency domain.

The area of these impulses proportional to the fourier series coefficients.



→ Consider a periodic signal $x[n]$ with period of 'N' and fundamental frequency $\omega_0 = \frac{2\pi}{N}$.

→ The signal $x[n]$ has the DTFS representation

$$x[n] = \sum_{k=\langle N\rangle} X_k e^{jk\omega_0 n}$$

Discrete time periodic signal

Taking the F.T on both sides,

$$\mathcal{F}\{x[n]\} = \mathcal{F}\Big\{\sum_{k=\langle N\rangle} X_k e^{jk\omega_0 n}\Big\}$$

$$X[e^{j\omega}] = \sum_{k=\langle N\rangle} X_k \mathcal{F}\{e^{jk\omega_0 n}\}$$

$$\because \mathcal{F}\{e^{jk\omega_0 n}\} = 2\pi\,\delta[\omega - k\omega_0] \quad ; \; 0 \le \omega < 2\pi$$

$$X[e^{j\omega}] = \sum_{k=\langle N\rangle} X_k\, 2\pi\,\delta[\omega - k\omega_0] \quad ; \; 0 \le \omega < 2\pi$$

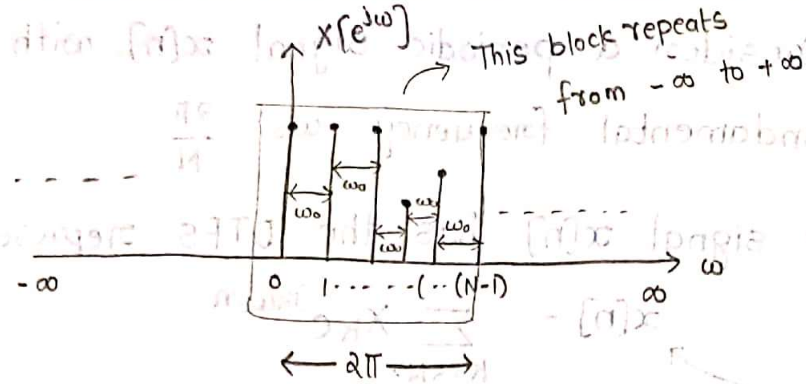
$$X[e^{j\omega}] = 2\pi \sum_{k=\langle N\rangle} X_k\,\delta[\omega - k\omega_0] \quad ; \; 0 \le \omega < 2\pi$$

∵ DTFT is periodic with period '2π'

$X[e^{j\omega}]$ consists a set of 'N' impulses of strength '$2\pi X_k$' where $k = 0, 1, 2 \ldots, (N-1)$ repeated at intervals of $N\omega_0 = 2\pi$

$$\boxed{X[e^{j\omega}] = 2\pi \sum_{k=-\infty}^{\infty} X_k\,\delta[\omega - k\omega_0]}$$

The Fourier transform of a periodic signal is an impulse train with impulses located at $\omega = k\omega_0$, each of which has a strength of $2\pi X_k$ and all impulses are seperated from each other by ω_0.



As $N \uparrow \Rightarrow$ spacing b/w any two spectral components $\downarrow$.

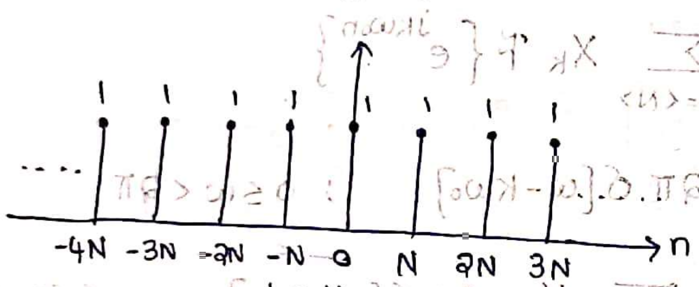
Q) Find and sketch the F.T of the discrete time impulse train $x[n] = \sum_{m=-\infty}^{\infty} \delta[n-mN]$

$m=0 \Rightarrow \delta[n] \rightarrow n=0$

$m=1 \Rightarrow \delta[n-N] \rightarrow n=N$

$m=2 \Rightarrow \delta[n-2N] \rightarrow n=2N$

$\vdots$



The given signal is periodic with period 'N'

Fundamental freq : $\omega_0 = \frac{2\pi}{N}$

$$x[n] = \delta[n] = \begin{cases} 1 & n=0 \\ 0 & 1 \le n \le N-1 \end{cases}$$

The Fourier Series coefficient,

$$X_K = \frac{1}{N}\sum_{n=0}^{N-1} x[n]\, e^{-jK\omega_0 n}$$

F.S representation

$$\boxed{x[n] = \sum_{K=\langle N\rangle} X_K e^{jK\omega_0 n}}$$

$\hookrightarrow$ FS coefficient.

$$= \frac{1}{N}\sum_{n=0}^{N-1} \delta[n]\, e^{-jK\frac{2\pi}{N}n}$$

$$= \frac{1}{N} \left. e^{-jK\frac{2\pi}{N}n} \right|_{n=0} = \frac{1}{N} e^0 = \frac{1}{N}.$$

$$\boxed{\therefore X_K = \frac{1}{N}}$$

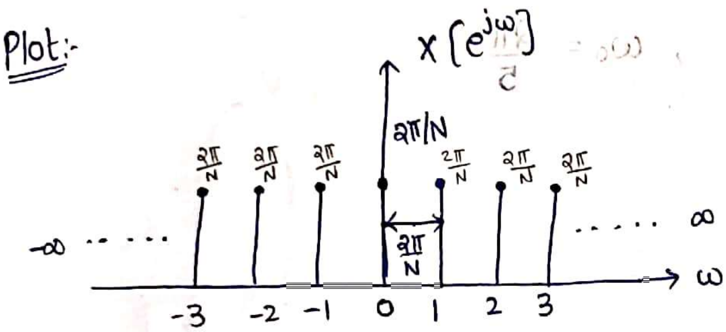
The DTFT of this periodic signal $x(n)$ is given as

$$X(e^{j\omega}) = 2\pi \sum_{k=-\infty}^{\infty} X_k \, \delta(\omega - k\omega_0)$$

$$X(e^{j\omega}) = 2\pi \sum_{k=-\infty}^{\infty} \frac{1}{N} \delta(\omega - k\omega_0)$$

$$\boxed{X(e^{j\omega}) = \frac{2\pi}{N} \sum_{k=-\infty}^{\infty} \delta\left(\omega - k \cdot \frac{2\pi}{N}\right)}$$

Plot:-



(2) Compute the FT of the periodic signal $x(n) = \cos(\omega_0 n)$ with $\omega_0 = \frac{2\pi}{5}$

Given, $x(n) = \cos(\omega_0 n) = \frac{1}{2} e^{j\omega_0 n} + \frac{1}{2} e^{-j\omega_0 n}$

We know that, $e^{j\omega_0 n} \longleftrightarrow \sum_{m=-\infty}^{\infty} 2\pi \, \delta(\omega - \omega_0 - 2\pi m)$

$e^{-j\omega_0 n} \longleftrightarrow \sum_{m=-\infty}^{\infty} 2\pi \, \delta(\omega + \omega_0 - 2\pi m)$

Substitute in above equ, we get

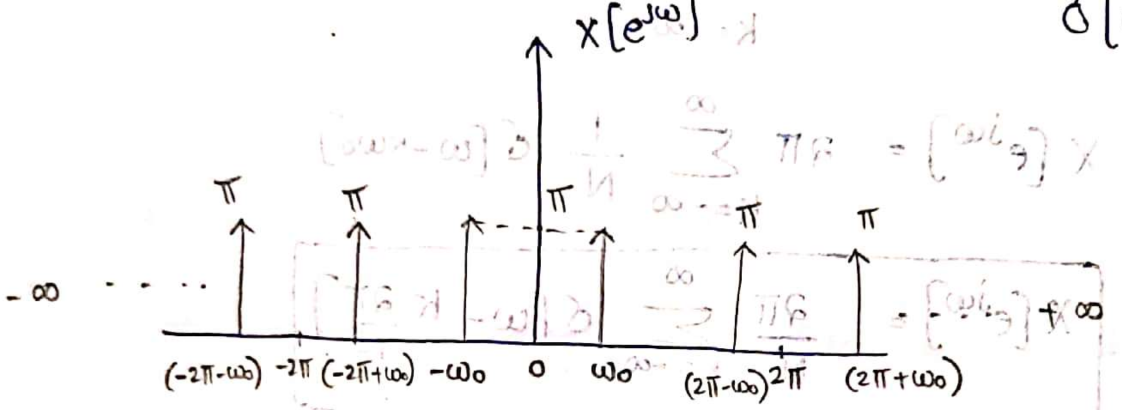
$$x(n) \longleftrightarrow X(e^{j\omega})$$

$$\mathcal{F}\{x[n]\} = \frac{1}{2} \mathcal{F}\{e^{j\omega_0 n}\} + \frac{1}{2} \mathcal{F}\{e^{-j\omega_0 n}\}$$

$$X(e^{j\omega}) = \frac{1}{2} \sum_{m=-\infty}^{\infty} 2\pi \, \delta(\omega - \omega_0 - 2\pi m) + \frac{1}{2} \sum_{m=-\infty}^{\infty} 2\pi \, \delta(\omega + \omega_0 - 2\pi m)$$

We know that, $\omega_0 = \frac{2\pi}{5}$

$$X[e^{j\omega}] = \frac{1}{2}\sum_{m=-\infty}^{\infty} 2\pi\,\delta\left[\omega - \frac{2\pi}{5} - 2\pi m\right] + \frac{1}{2}\sum_{m=-\infty}^{\infty} 2\pi\,\delta[\omega + 2\pi - 2\pi m]$$



③ $x[n] = \sin(\omega_0 n)$; $\omega_0 = \dfrac{2\pi}{5}$

Signal Transmission through Linear time Invariant System (LTI)

$x[n] \Rightarrow$ input for the LTI System

$y(n) \Rightarrow$ output " " " "

$h[n] \Rightarrow$ impulse response " "

LTI System

$$\xrightarrow[X[e^{j\omega}]]{x[n]} \boxed{h[n]}_{H[e^{j\omega}]} \xrightarrow{} \quad y[n] = x(n) * h[n], \quad Y[e^{j\omega}] = X[e^{j\omega}] \cdot H[e^{j\omega}]$$

$$\mathcal{F}\{y[n]\} = \mathcal{F}\{x[n] * h[n]\}$$

$$Y[e^{j\omega}] = X[e^{j\omega}] \cdot H[e^{j\omega}]$$

$$\boxed{H[e^{j\omega}] = \frac{Y[e^{j\omega}]}{X[e^{j\omega}]}}$$

In polar form,

$$|Y[e^{j\omega}]| e^{j\angle Y(e^{j\omega})} = \left[|X[e^{j\omega}]| e^{j\angle X[e^{j\omega}]}\right] \cdot \left[|H[e^{j\omega}]| \cdot e^{j\angle H(e^{j\omega})}\right]$$

$$= |X[e^{j\omega})| \cdot |H(e^{j\omega})| e^{j\left(\angle X(e^{j\omega}) + \angle H(e^{j\omega})\right)}$$

magnitude Spectrum: $|Y(e^{j\omega})| = |X(e^{j\omega})| \cdot |H(e^{j\omega})|$

Phase Spectrum: $\angle Y(e^{j\omega}) = \angle X(e^{j\omega}) + \angle H(e^{j\omega})$

$\Rightarrow$ During transmission, the input signal magnitude spectrum $|X(e^{j\omega})|$ is changed to $|X(e^{j\omega})| \cdot |H(e^{j\omega})|$

$\Rightarrow$ Similarly, the input signal phase spectrum $\angle X(e^{j\omega})$ is changed to $\angle X(e^{j\omega}) + \angle H(e^{j\omega})$

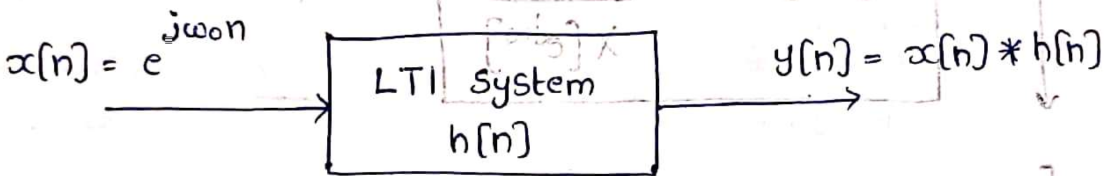
⇒ An input signal spectral component of frequency 'ω' is modified in amplitude by a factor of $|H(e^{j\omega})|$ and is shifted by an angle $\angle H(e^{j\omega})$.

$|H(e^{j\omega})| \rightarrow$ Magnitude response

$\angle H(e^{j\omega}) \rightarrow$ Phase response of the system

$H(e^{j\omega}) \rightarrow$ Frequency response of the system

Response of an LTI system to complex exponential signals :-



$$y[n] = x[n] * h[n]$$

$$= h[n] * x[n]$$

$x[n] = e^{j\omega_0 n}$

$x[n-m] = e^{j\omega_0 (n-m)}$

$$y[n] = \sum_{m=-\infty}^{\infty} h[m]\, x[n-m]$$

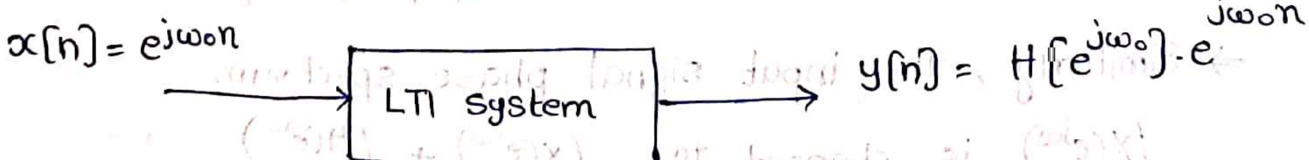
$$= \sum_{m=-\infty}^{\infty} h[m]\, e^{j\omega_0 (n-m)}$$

$$= \underbrace{\sum_{m=-\infty}^{\infty} h[m] . e^{-j\omega_0 m}}_{\Rightarrow H[e^{j\omega_0}]} . e^{j\omega_0 n}$$

$x[n] \longleftrightarrow X[e^{j\omega}]$

$X[e^{j\omega}] = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$

$$\boxed{y[n] = H[e^{j\omega_0}]\, e^{j\omega_0 n}}$$



Polar form: $H[e^{j\omega_0}] = |H(e^{j\omega_0})| \cdot e^{j\angle H(e^{j\omega_0})}$

$$y[n] = |H(e^{j\omega_0 n})| \, e^{j(\omega_0 n + \angle H(e^{j\omega_0}))}$$

$$e^{j\omega_0 n} \longrightarrow H[e^{j\omega_0}] \cdot e^{j\omega_0 n}$$

$$1 \cdot e^{j\omega_0 n} \longrightarrow |H(e^{j\omega_0})| \cdot e^{j[\omega_0 n + \angle H(e^{j\omega_0})]}$$

→ Amplitude of the i/p is modified by a factor of $|H(e^{j\omega_0})|$ and phase by a factor of $\angle H(e^{j\omega_0})$

i/p		multiplication factor		o/p
$e^{j\omega_0 n}$	$\longrightarrow$	$\lvert H(e^{j\omega_0})\rvert$		$e^{j\omega_0 n}$
1	$\longrightarrow$	2×1	$=$	2
1	$\longrightarrow$	0.5×1	$=$	0.5
1	$\longrightarrow$	5×1	$=$	5
1	$\longrightarrow$	0.1×1	$=$	0.1

* Multiplication factor $|H(e^{j\omega_0})| < 1 \rightarrow$ Attenuate system (Attenuator)

$|H(e^{j\omega_0})| > 1 \rightarrow$ Amplifies system (Amplifier)

i/p		system		i/p
$e^{j\omega_0 n}$	$\longrightarrow$	$\angle H(e^{j\omega_0})$	$+$	$e^{j\omega_0 n}$

→ Phase of each component in the o/p also changes accordingly with frequency.

$$\underbrace{x(n)=e^{j\omega_0 n}}_{\text{Eigen function}} \longrightarrow \boxed{\text{LTI System}} \longrightarrow \underbrace{H[e^{j\omega_0}]}_{\text{Eigen Value}}\, e^{j\omega_0 n}$$

* **Eigen function**: An Eigen function of a system is an input signal that produces an output that differs from the input by a constant multiplicative factor.

$x(n) = e^{j\omega_0 n} \Rightarrow$ Eigen function

$H[e^{j\omega_0}] \Rightarrow$ Eigen value.

$$x(n) = \sum_i A_i e^{j\omega_i n} \longrightarrow \boxed{\begin{matrix}\text{LTI System} \\ H[e^{j\omega_i}]\end{matrix}} \longrightarrow y(n) = \sum_i A_i H[e^{j\omega_i}] e^{j\omega_i n}$$

Response to sinusoidal signals:-

$$x(n) = A\cos\omega_0 n \longrightarrow \boxed{\text{LTI}} \longrightarrow y(n)$$

$-\infty < n < \infty$

$$x(n) = A\cos(\omega_0 n) \qquad -\infty < n < \infty$$

$$= A\left[\frac{e^{j\omega_0 n} + e^{-j\omega_0 n}}{2}\right]$$

$$= \frac{A}{2}e^{j\omega_0 n} + \frac{A}{2}e^{-j\omega_0 n}$$

$$A\cos(\omega_0 n) \longrightarrow \boxed{H[e^{j\omega}]} \longrightarrow A\left|H(e^{j\omega_0})\right| \cdot \cos[\omega_0 n + \theta(\omega_0)]$$

→ * Input sinusoidal signals of different frequencies will be affected differently, by the system, because of the dependence on the frequency response $H[e^{j\omega}]$.

Ex:- If $H[e^{j\omega}] = 0$ at some frequencies, then the sinusoidal signals at these frequencies may be completely supressed by the system. And other sinusoidal signals may receive no attenuation by the system.

Ex:- $H[e^{j\omega}] = 0$ for $\left.\begin{matrix} 10\text{ Hz} \\ 20\text{ Hz} \end{matrix}\right\}$ then for that freq o/p will be zero.

$$x(t) = \underbrace{2\cos \cancel{20\pi t}^{\,0}}_{f = 10\text{ Hz}} + \underbrace{\cos \cancel{40\pi t}^{\,0}}_{f = 20\text{ Hz}} + 5\sin 60\pi t$$

$$y(t) = |H(e^{j\omega})|\, 5\sin(60\pi t + \theta(60\pi))$$

⇒ The LTI system is functioning as a filter to sinusoidal signals of different frequencies, passing some of the frequency components to the output and suppressing other frequency components.

$$\sum_{i=1}^{L} A_i \cos(\omega_i n + \phi_i) \longrightarrow \boxed{H[e^{j\omega}]} \longrightarrow \sum_{i=1}^{L} A_i \left| H(e^{j\omega_i}) \right| . \cos(\omega_i n + \phi_i + \theta(\omega_i))$$

Phase of i^{th} sinusoidal signal

Phase incorporated by system at ω_i freq

) Consider a causal LTI system that is characterised by the difference equation

$$y[n] - \frac{3}{4} y[n-1] + \frac{1}{8} y[n-2] = 2x[n]$$

(a) Find the frequency response $H[e^{j\omega}]$ and the impulse-response $h[n]$ of the system.

Given, $y[n] - \frac{3}{4}y[n-1] + \frac{1}{8}y[n-2] = 2x[n]$

Taking the DTFT of this equation,

$$Y[e^{j\omega}] - \frac{3}{4}Y[e^{j\omega}].e^{-j\omega} + \frac{1}{8}Y(e^{j\omega}).e^{-2j\omega} = 2X[e^{j\omega}]$$

$$Y[e^{j\omega}]\left[1 - \frac{3}{4}e^{-j\omega} + \frac{1}{8}e^{-2j\omega}\right] = 2X[e^{j\omega}].$$

By the definition of frequency response of the system,

$$H[e^{j\omega}] = \frac{Y[e^{j\omega}]}{X[e^{j\omega}]}$$

$$= \frac{2}{\left[1 - \frac{3}{4}e^{-j\omega} + \frac{1}{8}e^{-2j\omega}\right]}$$

$$= \frac{2}{\left[1 - \frac{1}{2}e^{-j\omega}\right]\left[1 - \frac{1}{4}e^{-j\omega}\right]}$$

$-\frac{1}{2} \times -\frac{1}{4} = +\frac{1}{8}$

$-\frac{1}{2} - \frac{1}{4} = -\frac{3}{4}$

Assuming $e^{-j\omega} = P$, $e^{j\omega} = \frac{1}{P}$

$$H\left[\frac{1}{P}\right] = \frac{2}{(1-\frac{1}{2}P)(1-\frac{1}{4}P)}$$

Using the partial fraction expansion,

$$H\left[\frac{1}{P}\right] = \frac{A_1}{(1-\frac{1}{2}P)} + \frac{A_2}{(1-\frac{1}{4}P)}$$

$A_1(1-\frac{1}{4}P) + A_2(1-\frac{1}{2}P) = 2$, comparing coefficients.

$A_1 + A_2 = 2$, $-\frac{A_1}{4}P - \frac{A_2}{2}P = 0$

$-2A_2 + A_2 = 2$

$\boxed{A_2 = -2}$, $\boxed{A_1 = 4}$ $\quad \frac{-A_1P - 2A_2P}{4} = 0 \Rightarrow -A_1 - 2A_2 = 0$

$A_1 = -2A_2$

Then, $$H\left[\frac{1}{p}\right] = \frac{4}{\left(1-\frac{1}{2}p\right)} + \frac{(-2)}{\left(1-\frac{1}{4}p\right)}$$

$$H\left[e^{j\omega}\right] = \frac{4}{\left(1-\frac{1}{2}e^{-j\omega}\right)} - \frac{2}{\left(1-\frac{1}{4}e^{-j\omega}\right)}$$

By taking inverse DTFT ~~of~~ the above equ, We get

$$\boxed{h[n] = 4\left(\frac{1}{2}\right)^n u(n) - 2\left(\frac{1}{4}\right)^n u[n]} \qquad a^n u(n) \leftrightarrow \frac{1}{1-ae^{-j\omega}}$$

(b) Find the response $y(n)$, if the input to this system is

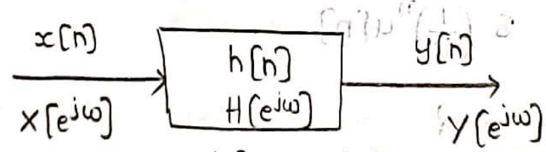
$$x[n] = \left(\frac{1}{4}\right)^n u(n)$$

The DTFT for $x[n] = \left(\frac{1}{4}\right)^n u[n]$

$$x[n] \xrightarrow{\text{DTFT}} X\left[e^{j\omega}\right]$$

$$\left(\frac{1}{4}\right)^n u[n] \longrightarrow \frac{1}{1-\left(\frac{1}{4}\right)e^{-j\omega}}$$

We know that, $Y\left[e^{j\omega}\right] = X\left[e^{j\omega}\right] \cdot H\left[e^{j\omega}\right]$



$$Y\left[e^{j\omega}\right] = X\left[e^{j\omega}\right] \cdot H\left[e^{j\omega}\right]$$

$$Y\left[e^{j\omega}\right] = \left[\frac{1}{1-\frac{1}{4}e^{-j\omega}}\right] \cdot \frac{2}{\left[1-\frac{1}{2}e^{-j\omega}\right]\left[1-\frac{1}{4}e^{-j\omega}\right]}$$

$$Y\left[e^{j\omega}\right] = \frac{2}{\left[1-\frac{1}{2}e^{-j\omega}\right]\left[1-\frac{1}{4}e^{-j\omega}\right]^2}$$

Using partial fraction expansion,

$$Y[e^{j\omega}] = \frac{-4}{\left(1-\frac{1}{4}e^{-j\omega}\right)} - \frac{2}{\left(1-\frac{1}{4}e^{-j\omega}\right)^2} + \frac{8}{\left(1-\frac{1}{2}e^{-j\omega}\right)}$$

$$Y[e^{j\omega}] = Y_1[e^{j\omega}] + Y_2[e^{j\omega}] + Y_3[e^{j\omega}]$$

IDTFT

$$y[n] = -y_1[n] - y_2[n] + y_3[n]$$

$$y_1[n] = 4\left(\frac{1}{4}\right)^n u[n]$$

$$y_3[n] = 8\left(\frac{1}{2}\right)^n u[n]$$

Here, $Y_2[e^{j\omega}] = \frac{-2}{\left(1-\frac{1}{4}e^{-j\omega}\right)^2}$

$$x[n] \longleftrightarrow X[e^{j\omega}]$$

$$a^n u[n] \longleftrightarrow \frac{1}{1-ae^{-j\omega}} \quad |a|<1$$

$$na^n u[n] \longleftrightarrow j\frac{d}{d\omega}X[e^{j\omega}] = j\frac{d}{d\omega}\left[\frac{1}{1-ae^{-j\omega}}\right]$$

$$= \frac{ae^{-j\omega}}{(1-ae^{-j\omega})^2}$$

$$na^n u[n] \longleftrightarrow \frac{ae^{-j\omega}}{(1-ae^{-j\omega})^2}$$

By using time shifting property, $n \longrightarrow n+1$

$$(n+1)\, a^{n+1} u[n+1] \longleftrightarrow \frac{a\cancel{e^{-j\omega}}}{(1-ae^{-j\omega})^2}\, \cancel{e^{j\omega}}$$

$$(n+1)\, a^n u[n+1] \longleftrightarrow \frac{1}{(1-ae^{-j\omega})^2}$$

$$(n+1)\, a^n u[n] \longleftrightarrow \frac{1}{(1-ae^{-j\omega})^2}$$

From,

$$(n+1)a^n u[n] \longleftrightarrow \frac{1}{(1-ae^{-j\omega})^2}$$

$$(n+1)\left(\frac{1}{4}\right)^n u[n] \longleftrightarrow \frac{1}{\left(1-\frac{1}{4}e^{-j\omega}\right)^2}$$

$$2(n+1)\left(\frac{1}{4}\right)^n u[n] \longleftrightarrow \frac{2}{\left(1-\frac{1}{4}e^{-j\omega}\right)^2}$$

$$\therefore\ y_2[n] = 2(n+1)\left(\frac{1}{4}\right)^n u[n]$$

Here, $y[n] = -y_1[n] - y_2[n] + y_3[n]$

$$\boxed{y[n] = -4\left(\frac{1}{4}\right)^n u[n] - 2(n+1)\left(\frac{1}{4}\right)^n u[n] + 8\left(\frac{1}{2}\right)^n u[n]}$$

Assignment

(1) Suppose that we want to design a discrete time LTI System that has the property that if the input is

$$x[n] = \left(\frac{1}{2}\right)^n u[n] - \frac{1}{4}\left(\frac{1}{2}\right)^{n-1} u[n-1]$$

$$y[n] = \left(\frac{1}{3}\right)^n u[n]$$

(a) Find the impulse response $h[n]$ and frequency response $H[e^{j\omega}]$ of the discrete time LTI system that has the foregoing property.

We know that, $x[n] \longleftrightarrow X[e^{j\omega}]$

$$y[n] \longleftrightarrow Y[e^{j\omega}]$$

$$y[n] = \left(\frac{1}{3}\right)^n u[n] \longleftrightarrow Y[e^{j\omega}] = \frac{1}{1-\frac{1}{3}e^{-j\omega}}$$

$$a^n u[n] \longleftrightarrow \frac{1}{1-ae^{-j\omega}}$$

By time shifting property , $n \to n-1$

$$a^{n-1} u[n-1] \longleftrightarrow \frac{e^{-j\omega}}{1-ae^{-j\omega}}$$

$$\left(\frac{1}{2}\right)^{n-1} u[n-1] \longleftrightarrow \frac{e^{-j\omega}}{1-\frac{1}{2}e^{-j\omega}}$$

$$\left(\frac{1}{4}\right)\left(\frac{1}{2}\right)^{n-1} u[n-1] \longleftrightarrow \frac{\left(\frac{1}{4}\right)e^{-j\omega}}{1-\frac{1}{2}e^{-j\omega}}$$

$$x[n] = \left(\frac{1}{2}\right)^n u[n] - \left(\frac{1}{4}\right)\left(\frac{1}{2}\right)^{n-1} u[n-1] \longleftrightarrow \frac{1}{1-\frac{1}{2}e^{-j\omega}} - \frac{\left(\frac{1}{4}\right)e^{-j\omega}}{1-\frac{1}{2}e^{-j\omega}}$$

$$\therefore \quad X[e^{j\omega}] = \frac{1-\left(\frac{1}{4}\right)e^{-j\omega}}{1-\left(\frac{1}{2}\right)e^{-j\omega}}$$

$$Y[e^{j\omega}] = \frac{1}{1-\left(\frac{1}{3}\right)e^{-j\omega}}$$

By the definition of frequency response of the system,

$$H[e^{j\omega}] = \frac{Y[e^{j\omega}]}{X[e^{j\omega}]}$$

$$= \frac{1/1-\left(\frac{1}{3}\right)e^{-j\omega}}{\dfrac{1-\left(\frac{1}{4}\right)e^{-j\omega}}{1-\left(\frac{1}{2}\right)e^{-j\omega}}}$$

$$= \frac{1-\left(\frac{1}{2}\right)e^{-j\omega}}{\left(1-\frac{1}{3}e^{-j\omega}\right)\left(1-\frac{1}{4}e^{-j\omega}\right)}$$

Assuming $e^{-j\omega} = P$, $e^{j\omega} = \frac{1}{P}$

$$H\left[\frac{1}{P}\right] = \frac{1-\frac{1}{2}P}{\left(1-\frac{1}{3}P\right)\left(1-\frac{1}{4}P\right)}$$

Using partial fractions expansion,

$$\frac{1-\left(\frac{1}{2}\right)P}{\left(1-\frac{1}{3}P\right)\left(1-\frac{1}{4}P\right)} = \frac{A}{\left(1-\frac{1}{3}P\right)} + \frac{B}{\left(1-\frac{1}{4}P\right)}$$

$$A\left(1-\frac{1}{4}P\right) + B\left(1-\frac{1}{3}P\right) = 1-\frac{1}{2}P$$

Comparing coefficients $\Rightarrow$ $A+B = 1$ $\qquad -\frac{A}{4} - \frac{B}{3} = -\frac{1}{2}$

$A = 1-B$

$\frac{-3A-4B}{\cancel{12}\,6} = -\frac{1}{\cancel{2}}$

$A = 1-3$

$-3A-4B = -6$

$\boxed{A = -2}$

$-3(1-B) - 4B = -6$

$-3+3B-4B = -6$

$-B = -3$

$\boxed{B = 3}$

$$H\left[\frac{1}{P}\right] = \frac{-2}{1-\frac{1}{3}P} + \frac{3}{1-\frac{1}{4}P}$$

$$\boxed{H[e^{j\omega}] = \frac{-2}{1-\frac{1}{3}e^{-j\omega}} + \frac{3}{1-\frac{1}{4}e^{-j\omega}}}$$

By taking IDTFT ~~of~~ above equ, We get

$$\boxed{h[n] = -2\left(\frac{1}{3}\right)^n u[n] + 3\left(\frac{1}{4}\right)^n u[n]}$$

(b) Find the difference equation relating $x[n]$ and $y[n]$ that characterizes the system.

From (a), $h[n] = -2\left(\frac{1}{3}\right)^n u[n] + 3\left(\frac{1}{4}\right)^n u[n]$

$$\frac{y[n]}{x[n]} = -2\left(\frac{1}{3}\right)^n u[n] + 3\left(\frac{1}{4}\right)^n u[n]$$

Difference equ: $y[n] = \left[-2\left(\frac{1}{3}\right)^n u[n] + 3\left(\frac{1}{4}\right)^n u[n]\right] x[n]$

(2) Suppose that a system has the response $\left(\frac{1}{4}\right)^n u[n]$ to the input $(n+2)\left(\frac{1}{2}\right)^n u[n]$. If the output of this system is $\delta[n] - \left(\frac{-1}{2}\right)^n u[n]$, what is the input?

Given, $x[n] = (n+2)\left(\frac{1}{2}\right)^n u[n]$

$y[n] = \left(\frac{1}{4}\right)^n u[n]$

$$Y[e^{j\omega}] = \frac{1}{1-\frac{1}{4}e^{-j\omega}}$$

We know that, $na^n u[n] \longleftrightarrow \dfrac{ae^{-j\omega}}{(1-ae^{-j\omega})^2}$

$$(n+2)a^{n+2}\boxed{u[n+2]} \longleftrightarrow \frac{ae^{-j\omega}\cdot e^{2j\omega}}{(1-ae^{-j\omega})^2}$$

$$(n+2)a^n u[n+2] \longleftrightarrow \frac{ae^{j\omega}}{a^2(1-ae^{-j\omega})^2}$$

$$(n+2)a^n u[n] \longleftrightarrow \frac{\cancel{a}\, e^{j\omega}}{a(1-ae^{-j\omega})^2}$$

$$x[n] = (n+2)\left(\frac{1}{2}\right)^n u[n] \longleftrightarrow \frac{e^{j\omega}}{\left(\frac{1}{2}\right)\left(1-\frac{1}{2}e^{-j\omega}\right)^2} = \frac{2e^{j\omega}}{\left(1-\frac{1}{2}e^{-j\omega}\right)^2}$$

$$= X[e^{j\omega}]$$

$$H[e^{j\omega}] = \frac{Y[e^{j\omega}]}{X[e^{j\omega}]} = \frac{1/1-\frac{1}{4}e^{-j\omega}}{\frac{2e^{j\omega}}{\left(1-\frac{1}{2}e^{-j\omega}\right)^2}} = \frac{\left(1-\frac{1}{2}e^{-j\omega}\right)^2}{2e^{j\omega}\left(1-\frac{1}{4}e^{-j\omega}\right)}$$

If $y[n] = \delta[n] - \left(\frac{-1}{2}\right)^n u[n]$. then $x[n] = ?$

$$Y[e^{j\omega}] = 1 - \frac{1}{1+\frac{1}{2}e^{-j\omega}} = \frac{\cancel{1} + \frac{1}{2}e^{-j\omega} - \cancel{1}}{1+\frac{1}{2}e^{-j\omega}}$$

$$Y[e^{j\omega}] = \frac{e^{-j\omega}}{2\left(1+\frac{1}{2}e^{-j\omega}\right)}$$

$$H[e^{j\omega}] = \frac{Y(e^{j\omega})}{X(e^{j\omega})}$$

$$H[e^{j\omega}]\, X[e^{j\omega}] = \frac{e^{-j\omega}}{2\left(1+\frac{1}{2}e^{-j\omega}\right)}$$

$$X[e^{j\omega}] = \frac{e^{-j\omega}}{\cancel{2}\left(1+\frac{1}{2}e^{-j\omega}\right)} \times \frac{\cancel{2}e^{j\omega}\left(1-\frac{1}{4}e^{-j\omega}\right)}{\left(1-\frac{1}{2}e^{-j\omega}\right)^2}$$

$$X[e^{j\omega}] = \frac{\left(1-\frac{1}{4}e^{-j\omega}\right)}{\left(1+\frac{1}{2}e^{-j\omega}\right)\left(1-\frac{1}{2}e^{-j\omega}\right)^2}$$

$$X\left[\frac{1}{P}\right] = \frac{\left(1-\frac{1}{4}P\right)}{\left(1+\frac{1}{2}P\right)\left(1-\frac{1}{2}P\right)^2}$$

$$\frac{3/8}{1+\frac{1}{2}e^{-j\omega}} + \frac{3/8}{1-\frac{1}{2}e^{-j\omega}} + \frac{1/4}{\left(1-\frac{1}{2}e^{-j\omega}\right)^2}$$

$$= \frac{A}{1+\frac{1}{2}P} + \frac{B}{1-\frac{1}{2}P} + \frac{C}{\left(1-\frac{1}{2}P\right)^2}$$

Comparing coefficients; $A\left(1-\frac{1}{2}P\right)^2 + B\left(1+\frac{1}{2}P\right)\left(1-\frac{1}{2}P\right) + C\left(1+\frac{1}{2}P\right) = 1-\frac{1}{4}P$

$$A\left(1+\frac{1}{4}P^2 - \cancel{2}\cdot\frac{1}{\cancel{2}}P\right) + B\left(1+\frac{1}{\cancel{2}}P - \frac{1}{\cancel{2}}P - \frac{1}{4}P^2\right) + C\left(1+\frac{1}{2}P\right) = 1-\frac{1}{4}P$$

$A+B+C = 1$, $-A + \frac{C}{2} = -\frac{1}{4}$, $\frac{A}{4} - \frac{B}{4} = 0$

$2A + C = 1$ $\qquad -A + \frac{1-2A}{2} = -\frac{1}{4}$ $\qquad A = B$

$C = 1-2A$ $\qquad -2A + 1 - 2A = -\frac{1}{2}$ $\qquad \boxed{B = \frac{3}{8}}$

$C = 1 - \cancel{2}\times\frac{3}{\cancel{8}4} \Rightarrow \boxed{C = 1/4}$ $\qquad -4A = -\frac{3}{2} \Rightarrow \boxed{A = 3/8}$

$$X\left[\frac{1}{P}\right] = \frac{3/8}{1+\frac{1}{2}P} + \frac{3/8}{1-\frac{1}{2}P} + \frac{1/4}{\left(1-\frac{1}{2}P\right)^2}$$

$$X\left[e^{j\omega}\right] = \frac{3/8}{1+\frac{1}{2}e^{-j\omega}} + \frac{3/8}{1-\frac{1}{2}e^{-j\omega}} + \frac{1/4}{\left(1-\frac{1}{2}e^{-j\omega}\right)^2}$$

By taking IDTFT, We get,

$$\left[a^n (n+1) u[n+1] = \frac{1}{(1-ae^{-j\omega})^2}\right]$$

$$\boxed{x[n] = \frac{3}{8}\left(\frac{-1}{2}\right)^n u(n) + \frac{3}{8}\left(\frac{1}{2}\right)^n u[n] + (n+1)\left(\frac{1}{4}\right)^n u[n+1]}$$

(3) Determine the response of the system with impulse response $h[n] = \left(\frac{1}{2}\right)^n u[n]$.
when the input is the complex exponential sequence

$$x[n] = Ae^{j\frac{\pi}{2}n} \qquad -\infty < n < \infty$$

Given, $h[n] = \left(\frac{1}{2}\right)^n u[n]$, $x[n] = Ae^{j\frac{\pi}{2}n}$

$$H[e^{j\omega}] = \frac{1}{1-\frac{1}{2}e^{-j\omega}}$$

At $\omega_0 = \frac{\pi}{2} \Rightarrow H\left[e^{j\pi/2}\right] = \frac{1}{1-\frac{1}{2}e^{-j\pi/2}} = \frac{1}{1+\frac{j}{2}}$

we know that, $H(e^{j\omega_0}) = |H(e^{j\omega_0})| \, e^{j\angle H(e^{j\omega_0})}$

$$= \frac{1}{\sqrt{1+\left(\frac{1}{2}\right)^2}} \, e^{j(-26.56^\circ)}$$

$$= \frac{1}{\sqrt{5/4}} \, e^{-j26.56} = \frac{2}{\sqrt{5}} \, e^{-j26.56^\circ}$$

$$y[n] = H[e^{j\omega_0}] \, e^{j\omega_0 n} = \frac{2}{\sqrt{5}} \, e^{-j26.56^\circ} \left(Ae^{j\frac{\pi}{2}n}\right)$$

$$\therefore\ y[n] = \frac{A2}{\sqrt{5}} e^{j\left(\frac{\pi n}{2} - 26.56^\circ\right)}$$

(4) Determine the response of the system with impulse response $h[n] = \left(\frac{1}{2}\right)^n u[n]$ to the input signal

$$x[n] = 5 - 5\sin\left(\frac{\pi}{2}\right)n + 10\cos(\pi n) \quad ; -\infty < n < \infty.$$

$$h[n] = \left(\frac{1}{2}\right)^n u[n]$$

$$H[e^{j\omega}] = \frac{1}{1-\frac{1}{2}e^{-j\omega}}$$

For, $x[n] = 5\cos 0$

$$\omega_0 = 0 \Rightarrow H[e^{j0}] = \frac{1}{1-\frac{1}{2}e^{-0}}$$

$$= \frac{1}{1-\frac{1}{2}}$$

$$= \frac{1}{1/2} = 2$$

$$|H(e^{j\omega_0})| = 2$$

For, $5\cos(0) = 5(2).\cos(0+0)$

$$= 10$$

For $x[n] = 5\sin\left(\frac{\pi}{2}\right)n$

$$H[e^{j\omega_0}] = \frac{1}{1-\frac{1}{2}e^{-j\pi/2}}$$

$$= \frac{1}{1+\frac{j}{2}}$$

$$|H(e^{j\omega_0})| = \frac{1}{\sqrt{1+\left(\frac{1}{2}\right)^2}} = \frac{2}{\sqrt{5}}$$

$$\underline{/H(e^{j\omega_0})} = \tan^{-1}(0) - \tan^{-1}\left(\frac{1/2}{1}\right)$$

$$= -26.6^\circ$$

$$5\sin\left(\frac{\pi}{2}n\right) \rightarrow 5\left(\frac{2}{\sqrt{5}}\right)\overset{\sin}{\cancel{\cos}}\left(\frac{\pi}{2}n - 26.6\right)$$

$$\boxed{\left(\because A\cos\omega_0 n = A\left|H(e^{j\omega_0})\right|\cos(\omega_0 n + \theta(\omega_0))\right)}$$

For $x(n) = 10\cos\pi n$

$\omega_0 = \pi$

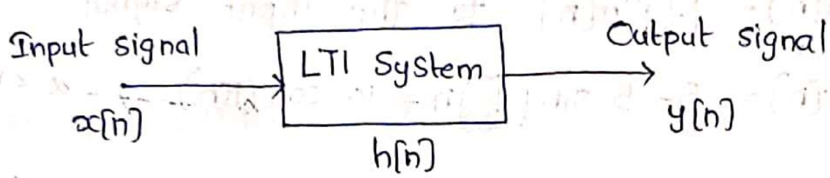
$$H(e^{j\omega_0}) = \frac{1}{1-\frac{1}{2}e^{-j\pi}} = \frac{1}{1-\frac{1}{2}(-1-0)} = \frac{1}{3/2} = \frac{2}{3}$$

$$|H(e^{j\omega_0})| = \frac{2}{3} \quad , \quad \underline{/H(e^{j\omega_0})} = 0$$

$$10\cos(\pi n) \rightarrow 10\left(\frac{2}{3}\right)\cos(\pi n + 0) = \frac{20}{3}\cos(\pi n)$$

$$\boxed{y[n] = 10 - 2\sqrt{5}\,\sin\left(\frac{\pi}{2}n - 26.56^\circ\right) + \frac{20}{3}\cos(\pi n)}$$

Linear and Non-linear phase :-



$$y[n] = x[n] * h[n]$$

DTFT

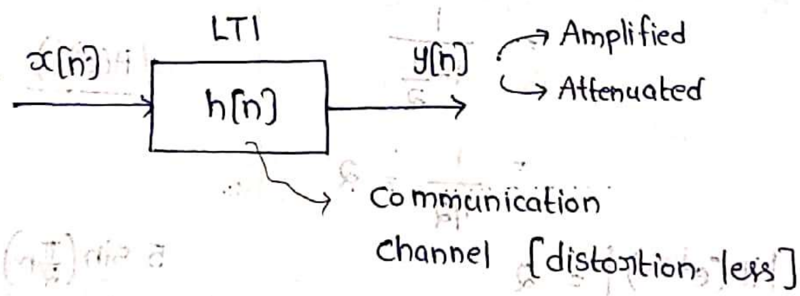
$$\underbrace{Y[e^{j\omega}]}_{o/p} = \underbrace{X[e^{j\omega}]}_{i/p} \cdot \underbrace{H[e^{j\omega}]}_{\text{multiplicative factor}}$$

output is input times the multiplicative factor

o/p should be exact replica of the i/p.

$>1 \Rightarrow$ o/p $\rightarrow$ Amplified version of i/p

$<1 \Rightarrow$ o/p $\rightarrow$ Attenuated version of i/p



$\Rightarrow$ The transmission is said to be distortionless if the i/p and o/p have identical wave-shapes within a multiplicative factor / constant and a constant time delay.

$$\boxed{y[n] = G \cdot x[n-n_d]}$$

G $\rightarrow$ multiplicative constant

n_d $\rightarrow$ time delay

DTFT

$$\boxed{Y[e^{j\omega}] = G \cdot X[e^{j\omega}] \cdot e^{-jn_d\omega}}$$

$x[n] \leftrightarrow X[e^{j\omega}]$

$x[n-n_d] \leftrightarrow X[e^{j\omega}] \cdot e^{-j\omega n_d}$

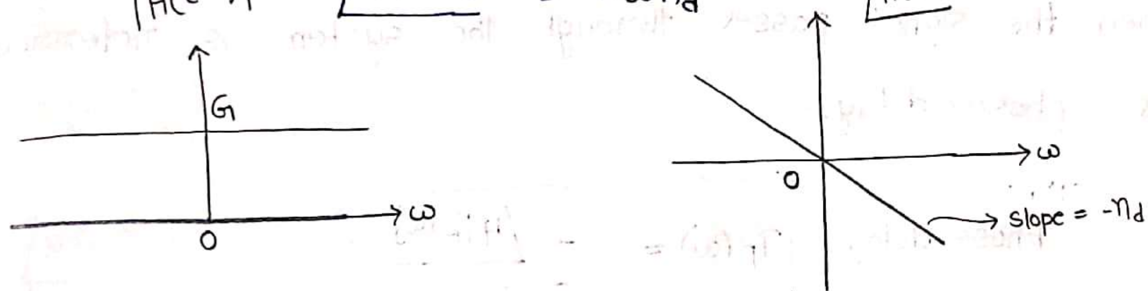
For distortionless transmission

① The Magnitude Response $|H(e^{j\omega})|$ must be constant i.e.,

$$|H(e^{j\omega})| = G \qquad G: \text{Constant}$$

② The phase Response $\angle H(e^{j\omega})$ must be a linear function of 'ω' with slope "$-n_d$" and intercept is "zero".

$$|H(e^{j\omega})| \qquad \angle H(e^{j\omega}) = -\omega n_d \qquad \angle H(e^{j\omega})$$



System for distortionless transmission

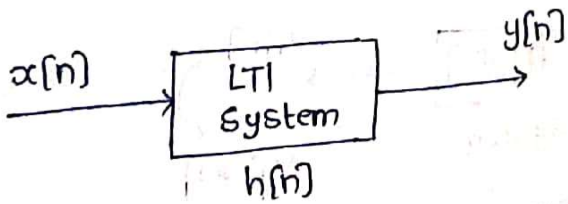
$$|H(e^{j\omega})| = G \text{ [constant]} \longrightarrow \text{Gain}$$

①Every spectral component is multiplied by a constant 'G'.

②Every spectral component is delayed by 'n_d' time units.

For distortionless Transmission
- → Linear phase characteristics
- → phase not only a linear func of 'ω' but should also pass through the origin. ($\omega = 0$)

Phase delay and Group delay:-



$\downarrow$

$H[e^{j\omega}] \Rightarrow$ Frequency response of the system.

* For a linear phase spectrum [whose phase varies linearly with frequency] ⇒ both the phase delay and group delay are constant.

Phase Delay:-

The time delay experienced by a single frequency signal, when the signal passes through the system is referred to as phase delay.

$$\text{Phase delay: } \boxed{\tau_p(\omega) = \frac{-\angle H[e^{j\omega}]}{\omega}}$$

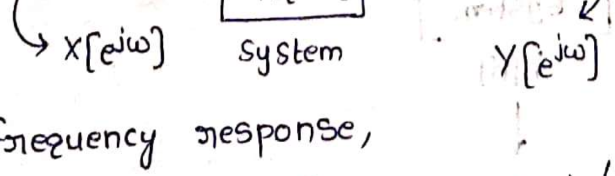
$$\tau_p(\omega) = \frac{-\theta(\omega)}{\omega}$$

Let us assume that the i/p signal is single frequency signal.

$$x[n] = A \cos(\omega_0 n + \theta)$$

$$x[n] = A\left[\frac{e^{j(\omega_0 n+\theta)} + e^{-j(\omega_0 n+\theta)}}{2}\right]$$

$$x[n] = \frac{A}{2}e^{j(\omega_0 n+\theta)} + \frac{A}{2}e^{-j(\omega_0 n+\theta)}$$



System frequency response,

$$H[e^{j\omega}] = \underbrace{\left|H[e^{j\omega}]\right|}_{\text{Magnitude}} \cdot \underbrace{e^{j\angle H(e^{j\omega})}}_{\text{Phase}}$$

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System o/p signal is given by,

$$y[n] = \underbrace{\frac{A}{2}\left|H[e^{j\omega}]\right|}_{\text{Magnitude}} \cdot e^{\underbrace{j(\omega_0 n + \theta + \angle H[e^{j\omega_0}])}_{\text{Phase}}} +$$

$$\underbrace{\frac{A}{2}\left|H(e^{-j\omega})\right|}_{\text{Magnitude}} e^{\underbrace{-j(\omega_0 n + \theta + \angle H(e^{-j\omega_0}))}_{\text{Phase}}}$$

Since,

$$\left|H[e^{j\omega_0}]\right| = \left|H[e^{-j\omega_0}]\right| \rightarrow \text{Magnitude is even function of } \omega_0$$

$$\angle H[e^{j\omega_0}] = -\angle H[e^{-j\omega_0}] \rightarrow \text{Phase is odd function of } \omega_0$$

$$y[n] = \frac{A}{2}\left|H[e^{j\omega_0}]\right| \cdot e^{j(\omega_0 n + \theta + \angle H(e^{j\omega_0}))} +$$

$$\frac{A}{2}\left|H(e^{j\omega_0})\right| \cdot e^{-j(\omega_0 n + \theta + \angle H(e^{j\omega_0}))}$$

$$y[n] = A\left|H[e^{j\omega_0}]\right| \cos\left[\omega_0 n + \theta + \angle H[e^{j\omega_0}]\right]$$

$$y[n] = A\left|H[e^{j\omega_0}]\right| \cos\left[\omega_0\left(n + \frac{\angle H[e^{j\omega_0}]}{\omega_0}\right) + \theta\right]$$

$$\frac{\angle H[e^{j\omega_0}]}{\omega_0} = -\tau_p(\omega_0)$$

$$\boxed{y[n] = A\left|H[e^{j\omega_0}]\right| \cos\left[\omega_0\left(n - \tau_p(\omega_0)\right) + \theta\right]}$$

where, $$\tau_p(\omega)\Big|_{\omega=\omega_0} = \tau_p(\omega_0) = \frac{-\angle H(e^{j\omega_0})}{\omega_0} = \frac{-\theta(\omega_0)}{\omega_0}$$

$\downarrow$

Time delay experienced by the single frequency signal with frequency 'ω_0' when it is passed through the system.

⇒ A negative system phase response at positive frequencies indicates that a signal is delayed in time when it passes through the system.

⇒ A positive system phase response at positive frequencies indicates that a signal is advanced in time when it passes through the system.

Group delay:-

When the input signal contains many different signals [many sinusoids] with different frequencies that are not harmonically related, each sinusoid / signal / component will go through different phase delays when passed through the system.

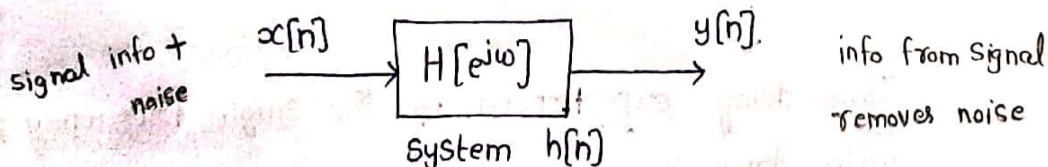
→ Group delay : $\tau_g(\omega) = \dfrac{-d}{d\omega} \angle H[e^{j\omega}] = -\dfrac{d\theta(\omega)}{d\omega}$

group delay / Envelop delay

⇒ The group delay at each frequency equals to the negative of the slope of the phase at that frequency.

If $\tau_g(\omega)$ = constant then all the sinusoids / components are delayed by the same interval.

Ideal and Practical Filters:-



$$y[n] = x[n] * h[n] \xleftrightarrow{DTFT} Y[e^{j\omega}] = X[e^{j\omega}] \cdot H[e^{j\omega}]$$

$H[e^{j\omega}]$ ⇒ weighted function / spectral shaping function to different frequency components in the i/p signal.

$$Y[e^{j\omega}] = X[e^{j\omega}] \cdot H[e^{j\omega}]$$

$H[e^{j\omega}] \rightarrow$ Filter

Filter

$$x[n] \longrightarrow \boxed{H[e^{j\omega}]} \longrightarrow y[n]$$

$\omega = 0 \implies H[e^{j\omega}] = 0$ → Blocks sig at $\omega = 0$

$\omega = 1 \implies H[e^{j\omega}] = 1/2$ → Allows half sig at $\omega = 1$

$\omega = 2 \implies H[e^{j\omega}] = 1$ → Allows completely at $\omega = 2$

* Ideal filters allows distortionless transmission of certain band of frequencies and completely suppress the remaining frequencies.

* The pass band of a filter is the band of frequencies that are passed through the system.

* The stop band of a filter is the band of frequencies that are attenuated by the system.

$$|H[e^{j\omega}]| = 1 \implies \text{Pass band}$$

$$|H[e^{j\omega}]| = 0 \implies \text{Stop band}$$

Types of filters:-

① Ideal low pass filter (LPF)

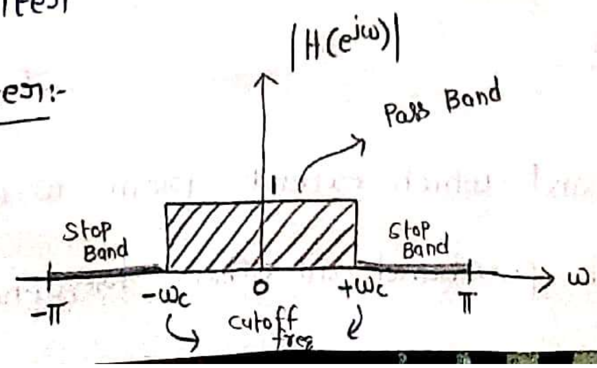
② Ideal high pass filter (HPF)

③ Ideal Band pass filter (BPF)

④ Ideal Band stop filter (BSF)

⑤ All pass filter

1. Low Pass Filter:-



Ex:- $x[n] \Rightarrow 1\ KHz,\ 2\ KHz,\ 1.5\ KHz$

$x[n]$ (1 KHz, 1.5 KHz, 2 KHz) $\longrightarrow$ LPF $\longrightarrow$ $y[n]$ (1. KHz)

$\omega_c = 1.4\ KHz$

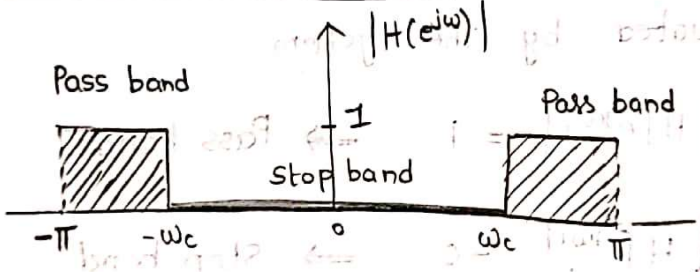
Allows signals which has frequency less than $\omega_c = 1.4\ KHz$

Low Pass Filter (LPF) $\Longrightarrow$ Pass band : $\omega = 0$ to $\omega_c \Rightarrow$ passes

Stop band : $\omega = \omega_c$ to $\pi \Rightarrow$ attenuated

* A Low pass filter allows only low frequency signals and attenuates the high frequency signals.

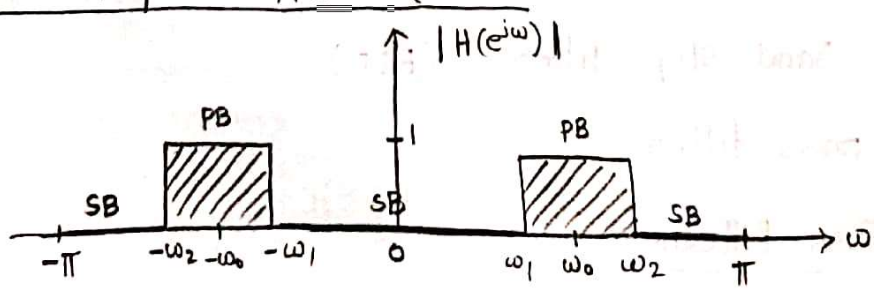
2. Ideal high pass filter (HPF):



$\Rightarrow$ High pass filter allows high frequency signals and attenuates low frequency signals.

HPF — Stopband which extends from $\omega = 0$ to ω_c (low freq)
— Pass band which extends from $\omega = \omega_c$ to π (high freq)

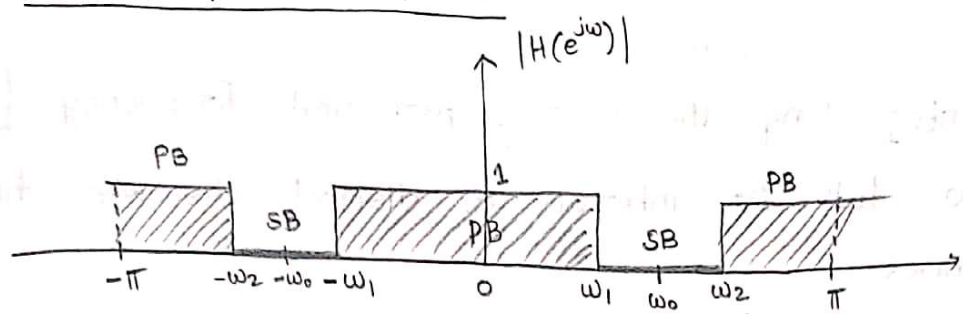
3. Ideal Bandpass filter (BPF):



BPF $\rightarrow$ Pass band which extends from $\omega = \omega_1$ to ω_2 (Passed through the system).

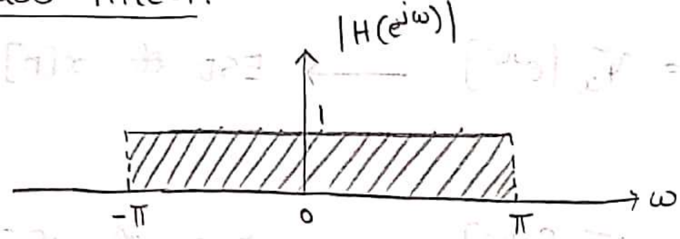
$\hookrightarrow$ Except passband all other frequencies get attenuated.

4. Band stop filter (BSF):



BSF → stop frequencies extending from $\omega = \omega_1$ to ω_2 and passes other frequencies

5. All pass filter:



⇒ All pass filter is characterized by a magnitude response that is constant for all frequencies.

⇒ The characteristics of an all pass filter are completely determined by its phase shift characteristics.

⇒ An all pass filter has a non-constant group delay i.e., different frequencies in the input are delayed by different amounts.

Energy spectral Density (ESD):-

Parseval's theorem relates the total signal energy in a signal $x[n]$

$$E_x = \sum_{n=-\infty}^{\infty} |x[n]|^2 \quad \text{DTFT}$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} |X[e^{j\omega}]|^2 \, d\omega$$

⇒ Parseval's theorem states that the Energy E_x may be determined either by computing the energy per unit time

$[|x[n]|^2]$ and summing over all the time.

(or)

$\Rightarrow$ By integrating the energy per unit frequency $\frac{|X(e^{j\omega})|^2}{2\pi}$ over a full '2π' interval of distinct discrete time frequencies.

$$\underbrace{|X[e^{j\omega}]|^2}_{} \Rightarrow \text{Energy per unit bandwidth } \left[\begin{matrix}-\pi \text{ to } \pi \\ = 2\pi\end{matrix}\right]$$

$\hookrightarrow$ Energy spectral density (ESD) of the signal $x[n]$

$$|X[e^{j\omega}]|^2 = \Psi_x[e^{j\omega}] \longrightarrow \text{ESD of } x[n]$$

Similarly,

$$|Y[e^{j\omega}]|^2 = \Psi_y[e^{j\omega}] \longrightarrow \text{ESD of } y[n]$$

$$X[e^{j\omega}] \longrightarrow \boxed{H[e^{j\omega}]} \longrightarrow Y[e^{j\omega}]$$

$$Y[e^{j\omega}] = H[e^{j\omega}] \cdot X[e^{j\omega}]$$

$$|Y[e^{j\omega}]|^2 = |H[e^{j\omega}]|^2 \cdot |X[e^{j\omega}]|^2$$

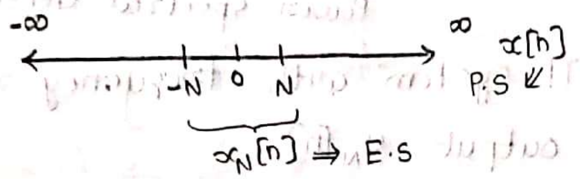
Relation b/w i/p & o/p energy spectral densities $\Leftarrow$ $\boxed{\underbrace{\Psi_y[e^{j\omega}]}_{\text{o/p energy spectral density}} = |H[e^{j\omega}]|^2 \cdot \underbrace{\Psi_x[e^{j\omega}]}_{\text{i/p energy spectral density}}}$

<u>Power Spectral Density (PSD)</u>:-

Energy signals $\Rightarrow$ Energy spectral Density (ESD)

Power signals $\Rightarrow$ Power spectral Density (PSD)

PSD $\Rightarrow$ Obtained from power signals by assuming power signals as limiting case of an Energy signals.



Consider a power signal of infinite duration.

$\because$ Total Energy of a power signal cannot be determined, we consider truncated version of $x[n]$ as $x_N[n]$

$x[n]$ ↓ Power s/g

$x_N[n]$ ↓ Small part (or) truncated version of $x[n]$

$$x_N[n] = \begin{cases} x[n] & ; -N \le n \le N \\ 0 & ; \text{otherwise} \end{cases}$$

↓

finite duration $\Rightarrow$ So, Energy signal

$$x_N[n] \xleftrightarrow{DTFT} X_N[e^{j\omega}]$$

then by using Parseval's theorem,

$$E_{x_N} = \sum_{n=-\infty}^{\infty} |x_N[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |x_N[e^{j\omega}]|^2 d\omega$$

$$\sum_{n=-N}^{N} |x_N[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X_N[e^{j\omega}]|^2 d\omega$$

$$\underset{N\to\infty}{Lt} \frac{1}{2N+1} \sum_{n=-N}^{N} |x_N[n]|^2 = \underset{N\to\infty}{Lt} \frac{1}{2N+1} \frac{1}{2\pi}\int_{-\pi}^{\pi} |X_N[e^{j\omega}]|^2 d\omega$$

$$\underbrace{\underset{N\to\infty}{Lt} \frac{1}{2N+1} \sum_{n=-N}^{N} |x_N[n]|^2}_{\text{Average power } P_x \text{ of signal } x[n]} = \frac{1}{2\pi}\int_{+\pi}^{\pi} \underset{N\to\infty}{Lt} \frac{|X_N[e^{j\omega}]|^2}{2N+1} d\omega$$

$$P_x = \frac{1}{2\pi}\int_{-\pi}^{\pi} \underset{N\to\infty}{Lt} \frac{|X_N[e^{j\omega}]|^2}{2N+1} d\omega$$

$$P_x = \frac{1}{2\pi}\int_{-\pi}^{\pi} \underbrace{G_x[e^{j\omega}]}_{\text{Power spectral density (PSD).}} d\omega$$

→ Consider an LTI system with frequency response $H[e^{j\omega}]$, input $x_N[n]$, output $y_N[n]$.

→ If $x[n]$ and $y[n]$ are power signals, then their PSD are $G_x[e^{j\omega}]$ and $G_y[e^{j\omega}]$

$$G_x[e^{j\omega}] = \underset{N\to\infty}{Lt} \frac{|X_N[e^{j\omega}]|^2}{2N+1} \quad ; \quad G_y[e^{j\omega}] = \underset{N\to\infty}{Lt} \frac{|Y_N[e^{j\omega}]|^2}{2N+1}$$

then,
$$Y_N[e^{j\omega}] = H[e^{j\omega}] \cdot X_N[e^{j\omega}]$$

$$|Y_N[e^{j\omega}]|^2 = |H[e^{j\omega}] \cdot X_N[e^{j\omega}]|^2$$

$$\underbrace{\underset{N\to\infty}{Lt} \frac{|Y_N[e^{j\omega}]|^2}{2N+1}}_{G_y[e^{j\omega}] \to \text{PSD of output}} = \underbrace{|H[e^{j\omega}]|^2}_{\text{system function}} \cdot \underbrace{\underset{N\to\infty}{Lt} \frac{|X_N[e^{j\omega}]|^2}{2N+1}}_{G_x[e^{j\omega}] \to \text{PSD of input}}$$

* $$\boxed{G_y[e^{j\omega}] = |H[e^{j\omega}]|^2 \, G_x[e^{j\omega}]}$$

Problems:-

(1) Given that $x[n]$ has fourier transform $X[e^{j\omega}]$, express the Fourier transform of the following signals in terms of $X[e^{j\omega}]$

$$x[n] \xleftrightarrow{DTFT} X[e^{j\omega}]$$

(a) $x_1[n] = x[1-n] + x[-1-n]$

We know that, $x[n] \xleftrightarrow{DTFT} X[e^{j\omega}]$

By using time shifting property,

$$x[n+1] \longleftrightarrow X[e^{j\omega}] \cdot e^{j\omega}$$

By applying time reversal property,

$$x[-n+1] \longleftrightarrow X[e^{-j\omega}] \cdot e^{-j\omega}$$

Similarly, $x[n-1] \longleftrightarrow X[e^{j\omega}] \cdot e^{-j\omega}$ (shifting)

$\downarrow$

$x[-n-1] \longleftrightarrow X[e^{-j\omega}] \cdot e^{j\omega}$ (Reversal)

Given, $x_1[n] = x[1-n] + x[-1-n]$

$$\mathcal{F}\{x_1[n]\} = \mathcal{F}\{x[1-n]\} + \mathcal{F}\{x[-1-n]\}$$

$$X_1[e^{j\omega}] = X[e^{-j\omega}] \cdot e^{-j\omega} + X[e^{-j\omega}] \cdot e^{j\omega}$$

$$= X[e^{-j\omega}]\left(e^{-j\omega} + e^{j\omega}\right)$$

$$= X[e^{-j\omega}]\,(2\cos\omega)$$

$$\boxed{X_1[e^{j\omega}] = 2X[e^{-j\omega}] \cdot \cos\omega}$$

(b) $x_2[n] = x[2n+1]$

$x[n] \longleftrightarrow X[e^{j\omega}]$

$x[n-1] \longrightarrow X[e^{j\omega}]\, e^{j\omega}$

$x[2n+1] \longrightarrow \frac{1}{2} X[e^{j\frac{\omega}{2}}]\, e^{\frac{j\omega}{2}}$

Using time shifting property

$$x[n+1] \longleftrightarrow X[e^{j\omega}] \cdot e^{j\omega}$$

Using time scaling property

$$x[2n+1] \longleftrightarrow X[e^{\frac{j\omega}{2}}] \cdot e^{\frac{j\omega}{2}}$$

(c) $x_3[n] = \dfrac{x^*[n] + x[n]}{2}$

$$x[n] \longleftrightarrow X[e^{j\omega}]$$

Using conjugation property,

$$x^*[n] \longleftrightarrow X^*[e^{-j\omega}]$$

Using time reversal property,

$$x^*[-n] \longleftrightarrow X^*[e^{j\omega}]$$

$$\mathcal{F}\{x_3[n]\} = \mathcal{F}\left[\frac{x[n] + x^*(n)}{2}\right]$$

$$= \frac{\mathcal{F}\{x[n]\}}{2} + \frac{\mathcal{F}\{x^*[n]\}}{2}$$

$$X_3[e^{j\omega}] = \frac{X[e^{j\omega}] + X^*[e^{-j\omega}]}{2}$$

$$X_3[e^{j\omega}] = \frac{X[e^{j\omega}] + X^*[e^{j\omega}]}{2} = Re\{X[e^{j\omega}]\}$$

(d) $x_4[n] = (n-1)^2 x[n]$

$$x[n] \longleftrightarrow X[e^{j\omega}]$$

$$x_4[n] = n^2 x[n] - 2n\, x[n] + 1 \cdot x[n]$$

By differentiation in frequency domain property,

$$n \cdot x[n] \longleftrightarrow j \frac{dX[e^{j\omega}]}{d\omega}$$

By differentiation in freq domain property,

$$n[n x[n]] \longleftrightarrow j \cdot \frac{d}{d\omega}\left[\frac{d}{d\omega} X[e^{j\omega}]\right]$$

$$n^2 x[n] \longleftrightarrow j^2 \frac{d^2}{d\omega^2} X[e^{j\omega}]$$

$$x_4[n] = n^2 x[n] - 2n\, x[n] + x[n]$$

$$\mathcal{F}\{x_4[n]\} = \mathcal{F}\{n^2 x[n]\} - 2\,\mathcal{F}\{n x[n]\} + \mathcal{F}\{x[n]\}$$

$$X_4[e^{j\omega}] = (-1)\frac{d^2}{d\omega^2} X[e^{j\omega}] - 2j \frac{d}{d\omega} X[e^{j\omega}] + X[e^{j\omega}]$$

(e) $x_5[n] = e^{\frac{j\pi n}{2}} \cdot x[n+2]$

$x[n] \longleftrightarrow x[n+2] \longleftrightarrow e^{j\frac{\pi}{2}n} x[n+2]$

$X[e^{j\omega}] \longleftrightarrow X[e^{j\omega}] \, e^{j2\omega}$

$X[e^{j(\omega-\frac{\pi}{2})}] \, e^{j2(\omega-\frac{\pi}{2})}$

Using the ~~time~~ freq shifting property,

$$e^{\frac{j\pi}{2}n} x[n+2] \longleftrightarrow X\left[e^{j(\omega-\frac{\pi}{2})}\right] \cdot e^{j2(\omega-\frac{\pi}{2})}$$

(2) Find the DTFT ~~of~~

$x_1[n] = \cos \omega_0 n \; u[n]$

$x_2[n] = \sin \omega_0 n \; u[n]$

$X_1[e^{j\omega}] = \pi + \pi^2 \left[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)\right]$

$X_2[e^{j\omega}] = -\pi \cot\left[\frac{\omega_0}{2}\right] + \pi^2 \left[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)\right]$

$$\cos \omega_0 n = \frac{1}{2} e^{j\omega_0 n} + \frac{1}{2} e^{-j\omega_0 n}$$

$$e^{j\omega_0 n} \leftrightarrow \sum_{m=-\infty}^{\infty} 2\pi \, \delta(\omega-\omega_0-2\pi m) = \sum_{m=0}^{\infty} 2\pi \, \delta(\omega-\omega_0$$

$$e^{-j\omega_0 n} \leftrightarrow \sum_{m=-\infty}^{\infty} 2\pi \, \delta[\omega+\omega_0-2\pi m]$$

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Q) Find the DTFT of the following

(a) $x_1[n] = \{\underset{\uparrow 0}{1}, \underset{1}{-1}, \underset{2}{2}, \underset{3}{2}\}$

$$x_1[n] \xleftrightarrow{\text{DTFT}} X_1[e^{j\omega}]$$

By definition, $X_1[e^{j\omega}] = \sum_{n=-\infty}^{\infty} x_1[n] \cdot e^{-j\omega n}$

$$= \sum_{n=0}^{3} x_1[n] \cdot e^{-j\omega n}$$

$$= x_1[0] \cdot e^{-j\omega 0} + x_1[1] e^{-j\omega(1)} + x_1[2] e^{-j\omega 2} + x_1[3] e^{-j\omega 3}$$

$$= (1)(1) + (-1) e^{-j\omega} + (2) e^{-j\omega 2} + (2) e^{-j\omega 3}$$

$$X_1[e^{j\omega}] = 1 - e^{-j\omega} + 2e^{-j\omega 2} + 2e^{-j\omega 3}$$

In Trignometric form $= 2j e^{-j\frac{\omega}{2}} \left[\frac{e^{j\frac{\omega}{2}} - e^{-j\frac{\omega}{2}}}{2j}\right] + 4e^{-j2.5\omega} \left[\frac{e^{j\frac{\omega}{2}} + e^{-j\frac{\omega}{2}}}{2}\right]$

$$X_1[e^{j\omega}] = 2j \sin\left(\frac{\omega}{2}\right) e^{-j\frac{\omega}{2}} + 4 \cos\left(\frac{\omega}{2}\right) e^{-j2.5\omega}$$

(b) $x_2[n] = \delta[n-1] + \delta[n+1]$

$$\mathcal{F}\{x_2[n]\} = \mathcal{F}\{\delta[n-1]\} + \mathcal{F}\{\delta[n+1]\}$$

$$X_2[e^{j\omega}] = e^{-j\omega} + e^{j\omega} = \underline{\underline{2\cos(\omega)}}$$

(c) $x_3[n] = (0.5)^n u[n] + 2^n u[-n-1]$

$$a^n u[n] \longleftrightarrow \frac{1}{1-ae^{-j\omega}} \qquad |a|<1$$

$$a^n u[-n-1] \longleftrightarrow \frac{1}{ae^{-j\omega}-1} \qquad |a|>1$$

$$a^{-n} u[-n-1] \longleftrightarrow \frac{-ae^{+j\omega}}{1-ae^{+j\omega}} \qquad |a|<1$$

$$x_3[n] = (0.5)^n u[n] + 2^n u[-n-1]$$

$$\mathcal{F}\{x_3[n]\} = \mathcal{F}\{(0.5)^n u[n]\} + \mathcal{F}\{2^n u[-n-1]\}$$

$$X_3[e^{j\omega}] = \frac{1}{1-0.5e^{-j\omega}} + \mathcal{F}\left\{\left(\tfrac{1}{2}\right)^{-n} u[-n-1]\right\}$$

$$X_3[e^{j\omega}] = \frac{1}{1-0.5e^{-j\omega}} + \frac{0.5e^{+j\omega}}{1-0.5e^{j\omega}}$$

Q) Determine the signal $x[n]$ for the following given discrete time Fourier transform

(a) $X[e^{j\omega}] = e^{-j\alpha\omega}$ for $-\omega_c \le \omega \le \omega_c$

(b) $X[e^{j\omega}] = e^{-j\omega}[1+\cos\omega]$

Ans:- The inverse DTFT of $X[e^{j\omega}]$ is given by

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})\, e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{-j\alpha\omega} \cdot e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega(n-\alpha)}\, d\omega$$

$$= \frac{1}{2\pi}\left[\frac{e^{j\omega(n-\alpha)}}{j(n-\alpha)}\right]_{-\omega_c}^{\omega_c}$$

$$= \frac{1}{2\pi} \cdot \frac{1}{j(n-\alpha)}\left[e^{j\omega_c(n-\alpha)} - e^{-j\omega_c(n-\alpha)}\right]$$

$$= \frac{1}{\pi(n-\alpha)}\left[\frac{e^{j\omega_c(n-\alpha)} - e^{-j\omega_c(n-\alpha)}}{2j}\right] = \frac{1}{\pi(n-\alpha)}\sin\omega_c(n-\alpha)$$

(b) $x[n]$

Given, $X[e^{j\omega}] = e^{-j\omega}[1+\cos\omega]$

$$= e^{-j\omega}\left[1 + \frac{e^{j\omega}+e^{-j\omega}}{2}\right]$$

$$= e^{-j\omega} + \frac{1}{2} + \frac{1}{2}e^{-2j\omega}$$

$$= \frac{1}{2}e^{j0\omega} + 1\cdot e^{-1\cdot j\omega} + \frac{1}{2}e^{-2j\omega}$$

$$X[e^{j\omega}] = \sum_{n=-\infty}^{\infty} x[n]\,e^{-j\omega n}$$

$$X[e^{j\omega}] = \frac{1}{2}e^{j0\omega} + 1\cdot e^{-1j\omega} + \frac{1}{2}e^{-2j\omega}$$

I.D.T.F.T

$$\boxed{x[n] = 0.5\,\delta[n] + 1\cdot\delta[n-1] + 0.5\,\delta[n-2]}$$

So, $x[n] = \left\{\underset{\uparrow}{\frac{1}{2}}, 1, \frac{1}{2}\right\}$

$x[0] = 1/2$

$x[1] = 1$

$x[2] = 1/2$

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(Pb) A causal system is described by the difference equation

$$y[n] - a\,y[n-1] = b\,x[n] + x[n-1]$$

where 'a' is real and less than '1' in magnitude. Find the value of 'b' such that the freq response of the system satisfies $|H[e^{j\omega}]| = 1$ for all 'ω'.

Ans:- Consider the given difference Equation,

$$y[n] - a y[n-1] = b x[n] + x[n-1]$$

Taking the DTFT of the above equation,

$$Y[e^{j\omega}] - a e^{-j\omega} Y[e^{j\omega}] = b X[e^{j\omega}] + e^{-j\omega} X[e^{j\omega}]$$

$$\frac{Y[e^{j\omega}]}{X[e^{j\omega}]} = H[e^{j\omega}] \longrightarrow \text{System response}$$

$$X[e^{j\omega}] \longrightarrow \boxed{H[e^{j\omega}]} \longrightarrow Y[e^{j\omega}]$$

$$H[e^{j\omega}] = \frac{b + e^{-j\omega}}{1 - a e^{-j\omega}}$$

$$\frac{\sqrt{b^2+1}}{\sqrt{1+a^2}} = 1$$

$$\sqrt{b^2+1} = \sqrt{1+a^2}$$

$$b^2 + \cancel{1} = \cancel{1} + a^2 \Rightarrow a^2 = b^2$$

$$b = \pm a$$

$(\because a<1)$ $\boxed{b=-a}$

$$|H[e^{j\omega}]| = \frac{|b + e^{-j\omega}|}{|1 - a e^{-j\omega}|} = \frac{|b + \cos\omega - j\sin\omega|}{|1 - a\cos\omega + ja\sin\omega|}$$

$$|H[e^{j\omega}]| = \frac{\sqrt{(b+\cos\omega)^2 + \sin^2\omega}}{\sqrt{(1+a\cos\omega)^2 + a^2\sin^2\omega}}$$

$$\cancel{1} + a^2 - 2a\cos\omega = \cancel{1} + b^2 + 2b\cos\omega$$

$$a^2 - 2a\cos\omega = b^2 + 2b\cos\omega$$

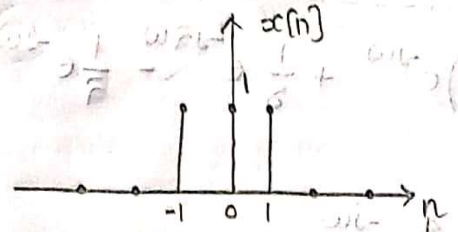
$$a(a - 2\cos\omega) = b(b + 2\cos\omega)$$

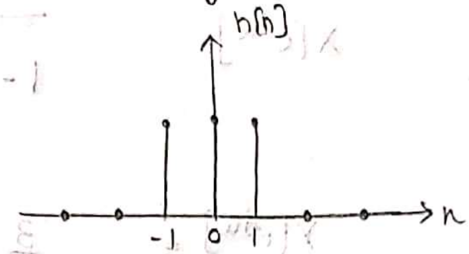
$$1 = \frac{\sqrt{b^2 + \cancel{\cos^2}\omega + 2b\cos\omega + \cancel{\sin^2}\omega}}{\sqrt{1 + a^2\cancel{\cos^2}\omega - 2a\cos\omega + a^2\cancel{\sin^2}\omega}}$$

$$1 = \frac{\sqrt{b^2 + 1 + 2b\cos\omega}}{\sqrt{a^2 + 1 - 2a\cos\omega}} \Rightarrow 1 + a^2 - 2a\cos\omega = 1 + b^2 + 2b\cos\omega$$

$$\boxed{b = -a}$$

(Pb) Given i/p, $x[n]$ and impulse response $h[n]$ are shown in the figure. Evaluate $y[n] = x[n] * h[n]$ using DTFT.





$$X[e^{j\omega}] = e^{-j\omega} + 1 + e^{j\omega}$$
$$= 1 + e^{j\omega} + e^{-j\omega}$$

$$H[e^{j\omega}] = e^{-j\omega} + 1 + e^{j\omega}$$
$$= 1 + e^{j\omega} + e^{-j\omega}$$

DTFT

$$y[n] = x[n] * h[n]$$
$$Y[e^{j\omega}] = X[e^{j\omega}] \cdot H[e^{j\omega}]$$
$$= (1 + e^{j\omega} + e^{-j\omega})(1 + e^{j\omega} + e^{-j\omega})$$

$$Y[e^{j\omega}] = 1 + e^{j\omega} + e^{-j\omega} + e^{j\omega} + e^{j2\omega} + 1 + e^{-j\omega} + 1 + e^{-j2\omega}$$

$$Y[e^{j\omega}] = e^{j2\omega} + 2e^{j\omega} + 3 + 2e^{-j\omega} + e^{-j2\omega}$$

IDTFT

$$y[n] = \delta[n+2] + 2\delta[n+1] + 3\delta[n] + 2\delta[n-1] + \delta[n-2]$$

(Pb) Consider a discrete time LTI system with unit impulse response $h[n] = \left(\frac{1}{2}\right)^n u[n] + \left(\frac{1}{2}\right)\left(\frac{1}{4}\right)^n u[n]$. Determine a linear constant coefficient difference Equation relating input and output of the system.

Ans:- Consider the given impulse response

$$h[n] = \left(\frac{1}{2}\right)^n u[n] + \left(\frac{1}{2}\right)\left(\frac{1}{4}\right)^n u[n]$$

Taking the DTFT for the above equa,

$$H[e^{j\omega}] = \frac{1}{1 - \frac{1}{2}e^{-j\omega}} + \frac{(1/2)\cdot 1}{1 - \left(\frac{1}{4}\right)e^{-j\omega}}$$

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$$H[e^{j\omega}] = \frac{Y[e^{j\omega}]}{X[e^{j\omega}]} = \frac{1-\left(\frac{1}{4}\right)e^{-j\omega} + \left(\frac{1}{2}\right)\left(1-\frac{1}{2}e^{-j\omega}\right)}{1-\left(\frac{1}{4}\right)e^{-j\omega} + \frac{1}{8}e^{-j2\omega} - \frac{1}{2}e^{-j\omega}}$$

$$\frac{Y[e^{j\omega}]}{X[e^{j\omega}]} = \frac{\frac{3}{2} - \frac{1}{2}e^{-j\omega}}{1-\frac{3}{4}e^{-j\omega} + \frac{1}{8}e^{-j2\omega}}$$

$$Y[e^{j\omega}] - \frac{3}{4}Y[e^{j\omega}]e^{-j\omega} + \frac{1}{8}Y[e^{j\omega}]e^{-j2\omega} = \frac{3}{2}X[e^{j\omega}] - \frac{1}{2}X[e^{j\omega}]e^{-j\omega}$$

IDTFT

$$\boxed{y[n] - \frac{3}{4}y[n-1] + \frac{1}{8}y[n-2] = \frac{3}{2}x[n] - \frac{1}{2}x[n-1]}$$

Assignment :

(1) The following four facts are given about a particular signal $x[n]$ with Fourier transform $X[e^{j\omega}]$.

① $x[n] = 0$ for $n > 0$

② $x[0] > 0$

③ $\text{Img}\{X[e^{j\omega}]\} = X_I[e^{j\omega}] = \sin(\omega) - \sin(2\omega)$

④ $\frac{1}{2\pi}\int_{-\pi}^{\pi} |X[e^{j\omega}]|^2 \, d\omega = 3$

Determine $x[n]$.

Ans:-

$$x[n] \xleftrightarrow{DTFT} X[e^{j\omega}]$$

From the odd and even properties of $x[n]$,

$$O\{x[n]\} \xleftrightarrow{DTFT} j\,\text{Im}\{X[e^{j\omega}]\}$$

$$E\{x[n]\} \xleftrightarrow{DTFT} \text{Re}\{X[e^{j\omega}]\}$$

Given, $\text{Img}\{X[e^{j\omega}]\} = \sin(\omega) - \sin(2\omega)$

$j\,\text{Img}\{X[e^{j\omega}]\} = j\sin\omega - j\sin(2\omega)$

The IDTFT of $j\,\text{Img}\{X[e^{j\omega}]\} \xleftrightarrow{\text{IDTFT}} O\{x[n]\}$

Then, $j\,\text{Img}\{X[e^{j\omega}]\} = j\sin\omega - j\sin(2\omega)$

$$= j\left[\frac{e^{j\omega} - e^{-j\omega}}{2j} - \frac{e^{j2\omega} - e^{-j2\omega}}{2j}\right]$$

$$j\,\text{Img}\{X[e^{j\omega}]\} = \frac{1}{2}e^{j\omega} - \frac{1}{2}e^{-j\omega} - \frac{1}{2}e^{2j\omega} + \frac{1}{2}e^{-2j\omega}$$

IDTFT

$$O\{x[n]\} = \frac{1}{2}\delta[n+1] - \frac{1}{2}\delta[n-1] - \frac{1}{2}\delta[n+2] + \frac{1}{2}\delta[n-2]$$

From the fact ① $\Rightarrow x[n] = 0$ for $n > 0$

$$O\{x[n]\} = \underset{n=-1}{\frac{1}{2}\delta[n+1]} - \underset{n=+1}{\frac{1}{2}\cancel{\delta[n-1]}^{\,0}} - \underset{n=-2}{\frac{1}{2}\delta[n+2]} + \underset{n=+2}{\frac{1}{2}\cancel{\delta[n-2]}^{\,0}}$$

$$O\{x[n]\} = \frac{1}{2}\left[\delta[n+1] - \delta[n+2]\right] \longrightarrow ①$$

Given, $\frac{1}{2\pi}\int_{-\pi}^{\pi} |X[e^{j\omega}]|^2 d\omega = 3$

By using Parseval's Relation,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |X[e^{j\omega}]|^2 d\omega = \sum_{n=-\infty}^{\infty} |x[n]|^2$$

$$3 = \sum_{n=-\infty}^{\infty} |x[n]|^2$$

$$3 = \sum_{n=-\infty}^{-1} |x[n]|^2 + |x[0]|^2 + \sum_{n=1}^{\infty} |\cancel{x[n]}^{\,0}|^2$$

$$3 = |x[0]|^2 + \sum_{n=-\infty}^{-1} |x[n]|^2$$

$$3 = |x[0]|^2 + [\underbrace{\ldots\ldots}_{=0} + |x[-2]|^2 + |x[-1]|^2]$$

$$3 = |x[0]|^2 + |x[-2]|^2 + |x[-1]|^2$$

$$3 = |x[0]|^2 + (1)^2 + (1)^2$$

$$|x[0]|^2 = 3-2 = 1$$

$$|x[0]| = \pm 1$$

From fact ② $\Rightarrow$ $x[0] > 0$

$$\boxed{\therefore |x[0]| = 1}$$

From equ ①,

$$O\{x[n]\} = \frac{1}{2}\delta[n+1] - \frac{1}{2}\delta[n+2]$$

$$\frac{x[n]-x[-n]}{\cancel{2}} = \frac{1}{\cancel{2}}[\delta[n+1] - \delta[n+2]]$$

$$x[n] = \delta[n+1] - \delta[n+2]$$

$$x[n] = x[0] + \delta[n+1] - \delta[n+2]$$

$$x[n] = 1 + \delta[n+1] - \delta[n+2]$$

$$\therefore \boxed{x[n] = \delta[n] + \delta[n+1] - \delta[n+2]}$$

(2) A linear time invariant system is described by the following difference equation

$$y[n] = a\,y[n-1] + b\,x[n] \quad ; \quad 0<a<1$$

(a) Determine the magnitude response $|H[e^{j\omega}]|$ and phase response $\angle H(e^{j\omega})$ of the system.

Ans:- Given, $y[n] = a\,y[n-1] + b\,x[n] \quad ; 0<a<1$

Taking the DTFT of the above equation,

$$Y[e^{j\omega}] = ae^{-j\omega}\,Y[e^{j\omega}] + b\,X[e^{j\omega}]$$

$$Y[e^{j\omega}]\left[1-ae^{-j\omega}\right] = b\,X[e^{j\omega}]$$

$$\frac{Y[e^{j\omega}]}{X[e^{j\omega}]} = \frac{b}{1-ae^{-j\omega}}$$

$$H[e^{j\omega}] = \frac{b}{1-ae^{-j\omega}} = \frac{b}{1-a(\cos\omega - j\sin\omega)}$$

By calculating magnitude of the above equ, We get

$$|H[e^{j\omega}]| = \frac{|b|}{\sqrt{(1-a\cos\omega)^2 + \sin^2\omega\cdot a^2}}$$

$$|H[e^{j\omega}]| = \frac{|b|}{\sqrt{1+a^2\cos^2\omega - 2a\cos\omega + a^2\sin^2\omega}}$$

$$|H[e^{j\omega}]| = \frac{|b|}{\sqrt{1-2a\cos\omega + a^2}}$$

$$\angle H[e^{j\omega}] = \frac{\angle b}{\angle(1-a\cos\omega)+ja\sin\omega}$$

$$= \tan^{-1}\left(\frac{0}{b}\right) - \tan^{-1}\left(\frac{a\sin\omega}{1-a\cos\omega}\right)$$

$$\angle H[e^{j\omega}] = -\tan^{-1}\left(\frac{a\sin\omega}{1-a\cos\omega}\right)$$

(b) choose the parameter 'b' such that the maximum value of $|H[e^{j\omega}]|$ is unity.

From (a) $\Rightarrow$ $|H[e^{j\omega}]| = \dfrac{|b|}{\sqrt{1-2a\cos\omega + a^2}}$; $0<a<1$

Given, $\max(|H(e^{j\omega})|) = 1$

$\because$ At $\omega=0$, denominator becomes minimum then $|H(e^{j\omega})|$ becomes maximum.

$$|H(e^{j0})| = \frac{|b|}{\sqrt{1-2a\cos 0 + a^2}}$$

$$1 = \frac{|b|}{\sqrt{(1-a)^2}} = \frac{|b|}{1-a}$$

$|b| = 1-a$

$\boxed{\therefore b = 1-a}$

(c) Determine the output of the system to the input $x[n] = 5 + 12\sin\left(\frac{\pi}{2}n\right) + 20\cos\left(\pi n + \frac{\pi}{4}\right)$ for $a=0.9$

Given, $a = 0.9 \Rightarrow b = 1-a$

$b = 1-0.9 \Rightarrow \boxed{b = 0.1}$

$$x[n] = 5 + 12\sin\left(\frac{\pi}{2}n\right) + 20\cos\left(\pi n + \frac{\pi}{4}\right)$$

$$x[n] = \underbrace{5\cos 0}_{\omega_0 = 0} + \underbrace{12\sin\left(\frac{\pi}{2}n\right)}_{\omega_0 = \frac{\pi}{2}} + \underbrace{20\cos\left(\pi n + \frac{\pi}{4}\right)}_{\omega_0 = \pi}$$

From (a), $H[e^{j\omega}] = \dfrac{b}{1-ae^{-j\omega}}$

$$H[e^{j\omega}] = \frac{0.1}{1-0.9e^{-j\omega}}$$

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$\boxed{\omega_0 = 0}$

$$H[e^{j0}] = \frac{0.1}{1-0.9e^{-j0}}$$

$$H[e^{j0}] = 1$$

$$\angle H(e^{j\omega_0}) = \tan^{-1}\left(\frac{0}{1}\right) = 0$$

$\boxed{\omega_0 = \pi/2}$

$$H[e^{j\frac{\pi}{2}}] = \frac{0.1}{1-0.9e^{-j\frac{\pi}{2}}}$$

$$= \frac{0.1}{1+0.9j}$$

$$|H(e^{j\pi/2})| = \frac{0.1}{\sqrt{1+(0.9)^2}}$$

$$= \underline{\underline{0.074}}$$

$$\angle H(e^{j\omega_0}) = \tan^{-1}\left(\frac{0.1}{0.1}\right) - \tan^{-1}\left(\frac{0.9}{1}\right)$$

$$= \underline{\underline{-42^\circ}}$$

$\boxed{\omega_0 = \pi}$

$$H[e^{j\pi}] = \frac{0.1}{1-0.9e^{-j\pi}}$$

$$H[e^{j\pi}] = \frac{0.1}{1+0.9} = \underline{\underline{0.053}}$$

$$\angle H(e^{j\omega_0}) = \tan^{-1}\left(\frac{0}{0.053}\right) = \underline{\underline{0}}$$

$\therefore$ By using response to sinusoidal inputs,

$$A\cos\omega_0 n = A\,|H(e^{j\omega_0})|\cos(\omega_0 n + \theta(\omega_0))$$

$$5\cos 0 = 5(1)\cos(0+0) = 5$$

$$12\sin\left(\frac{\pi}{2}n\right) = 12(0.074)\sin\left(\frac{\pi}{2}n - 42^\circ\right) = 0.888\sin\left(\frac{\pi}{2}n - 42\right)$$

$$20\cos\left(\pi n + \frac{\pi}{4}\right) = 20(0.053)\cos\left(\pi n + \frac{\pi}{4}\right) = 1.06\cos\left(\pi n + \frac{\pi}{4}\right)$$

$$\boxed{\therefore\ y[n] = 5 + 0.888\sin\left(\frac{\pi}{2}n - 42\right) + 1.06\cos\left(\pi n + \frac{\pi}{4}\right)}$$

(3) Suppose we are given the following facts about an LTI system with impulse response $h[n]$ and frequency response $H[e^{j\omega}]$.

① $\left(\frac{1}{4}\right)^n u[n] \longrightarrow g[n]$ where $g[n] = 0$ for $n \geq 2$ and $n<0$

② $H[e^{j\pi/2}] = 1$

③ $H[e^{j\omega}] = H[e^{j(\omega-\pi)}]$

Determine $h[n]$.

Ans:- Given, $x[n] = \left(\frac{1}{4}\right)^n u[n]$

$$x[n] \xrightarrow{DTFT} X[e^{j\omega}] = \frac{1}{1-\frac{1}{4}e^{-j\omega}}$$

$$y[n] = g[n] \xleftrightarrow{DTFT} G[e^{j\omega}]$$

exists only at $n=0,1$

$g[n] = 0$ for all other values.

$$X[e^{j\omega}] = \frac{1}{1-\left(\frac{1}{4}\right)e^{-j\omega}} \quad ; \quad G[e^{j\omega}] = g[0] + g[1]e^{-j\omega}$$

We know that,

$$H[e^{j\omega}] = \frac{G[e^{j\omega}]}{X[e^{j\omega}]} = \frac{g[0]+g[1]e^{-j\omega}}{1/1-\left(\frac{1}{4}\right)e^{-j\omega}}$$

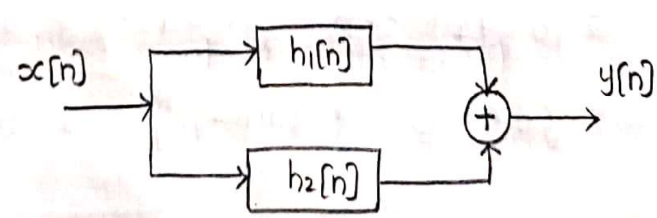
$$H[e^{j\omega}] = \left(1-\frac{1}{4}e^{-j\omega}\right)\left(g[0]+g[1]e^{-j\omega}\right)$$

$$H[e^{j\omega}] = g[0] - \frac{1}{4}e^{-j\omega}g[0] + g[1]e^{-j\omega} - \frac{1}{4}g[1]e^{-j2\omega}$$

(4) An LTI system with impulse response $h_1[n] = \left(\frac{1}{3}\right)^n u[n]$ is connected in parallel with another LTI system with impulse response $h_2[n]$. The resulting parallel interconnection has the frequency response

$$H[e^{j\omega}] = \frac{-12 + 5e^{-j\omega}}{12 - 7e^{-j\omega} + e^{-j2\omega}}$$; Determine $h_2[n]$

Ans:- Given, $h_1[n]$ and $h_2[n]$ are connected in parallel (addition)



$$h_1[n] \longleftrightarrow H_1[e^{j\omega}]$$

$$h_2[n] \longleftrightarrow H_2[e^{j\omega}]$$

$$H[e^{j\omega}] = H_1[e^{j\omega}] + H_2[e^{j\omega}]$$

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$$h_1[n] = \left(\frac{1}{3}\right)^n u[n]$$

$$H_1[e^{j\omega}] = \frac{1}{1-\left(\frac{1}{3}\right)e^{-j\omega}}$$

$$\frac{1}{1-\left(\frac{1}{3}\right)e^{-j\omega}} + H_2[e^{j\omega}] = \frac{-12+5e^{-j\omega}}{12-7e^{-j\omega}+e^{-j2\omega}}$$

$$H_2[e^{j\omega}] = \frac{-12+5e^{-j\omega}}{12-7e^{-j\omega}+e^{-j2\omega}} - \frac{1}{1-\left(\frac{1}{3}\right)e^{-j\omega}}$$

$$H_2[e^{j\omega}] = \frac{-12+5e^{-j\omega}}{12-7e^{-j\omega}+e^{-j2\omega}} - \frac{3}{3-e^{-j\omega}}$$

$$= \frac{(-12+5e^{-j\omega})(3-e^{-j\omega}) - 3(12-7e^{-j\omega}+e^{-j2\omega})}{(12-7e^{-j\omega}+e^{-j2\omega})(3-e^{-j\omega})}$$

$(12-7x+x^2)(3-x)$

$(x^2-3x-4x+12)(3-x)$

$(x-4)(x-3)(3-x)$

$-(x-4)(x-3)(x-3)$

$$= \frac{-36+15e^{-j\omega}+12e^{-j\omega}-5e^{-j\omega 2}-36+21e^{-j\omega}-3e^{-j2\omega}}{36-21e^{-j\omega}+3e^{-j2\omega}-12e^{-j\omega}+7e^{-j2\omega}-e^{-3j\omega}}$$

$$= \frac{-72+48e^{-j\omega}-8e^{-j2\omega}}{36-33e^{-j\omega}+10e^{-j2\omega}-e^{-j3\omega}}$$

$$= \frac{-72+48e^{-j\omega}-8e^{-j2\omega}}{-(e^{-j\omega}-4)(e^{-j\omega}-3)(e^{-j\omega}-3)}$$

Let $e^{-j\omega} = x$

$$= \frac{-72+48x-8x^2}{-(x-4)(x-3)(x-3)} = \frac{8x^2-48x+72}{(x-4)(x-3)(x-3)}$$

By using partial fractions,

$$\frac{8x^2-48x+72}{(x-4)(x-3)^2} = \frac{A}{x-4} + \frac{Bx+C}{(x-3)^2} \rightarrow 8x^2-48x+72 = A(x-3)^2 + (Bx+C)(x-4)$$

$$8x^2-48x+72 = Ax^2+9A-6Ax+Bx^2-4Bx+Cx-4C$$

Comparing coefficients, $A+B = 8$, $-6A-4B+C = -48$

$$9A-4C = 72$$

$$-6A-4B+\frac{9A-72}{4} = -48 \qquad C = \frac{9A-72}{4}$$

$$-24A-16B+9A-72 = -192$$

$$-15A-16B = -120$$

$$15A+15B = 120$$

$$-B = 0 \Rightarrow \boxed{B=0} \; ; \; \boxed{A=8} \; ; \; \boxed{C=0}$$

$$\therefore \frac{A}{x-4} = \frac{8}{e^{-j\omega}-4} = H_2[e^{j\omega}]$$

$$H_2[e^{j\omega}] = \frac{8}{-4+e^{-j\omega}} = \frac{\cancel{8}\,2}{\cancel{4}\left(\frac{1}{4}e^{-j\omega}-1\right)} = \frac{-2}{1-\frac{1}{4}e^{-j\omega}}$$

IDTFT

$$\boxed{h_2[n] = -2\left(\frac{1}{4}\right)^n u[n]}$$

(5) Consider a discrete time LTI system with frequency response

$$H[e^{j\omega}] = \left[\frac{1+e^{-j2\omega}+4e^{-j4\omega}}{1+\frac{1}{2}e^{-j2\omega}}\right] \times e^{-j(\omega-\pi/4)}$$

for $-\pi < \omega \le \pi$

Determine the Fourier transform of the output if the input is $x[n] = \cos\left(\frac{\pi}{2}n\right)$.

Ans:- If the input is sinusoidal,

$$y[n] = A\,\left|H[e^{j\omega_0}]\right|\cos(\omega_0 n+\theta(\omega_0))$$

Here, $x(n) = \cos\left(\frac{\pi}{2}n\right) \Rightarrow \omega_0 = \frac{\pi}{2}$

$$H[e^{j\pi/2}] = \frac{1+e^{-j2\left(\frac{\pi}{2}\right)}+4e^{-j4\left(\frac{\pi}{2}\right)}}{1+\frac{1}{2}e^{-j2\left(\frac{\pi}{2}\right)}} \times e^{-j\left(\frac{\pi}{2}-\frac{\pi}{4}\right)}$$

$$H[e^{j\pi/2}] = \frac{1-1+4}{1-\frac{1}{2}} \times e^{-j\pi/4}$$

$$= \frac{4}{1/2} e^{-j\pi/4} = 8e^{-j\pi/4} = 8\left(\cos\frac{\pi}{4} - j\sin\frac{\pi}{4}\right)$$

$$|H[e^{j\pi/2}]| = 8$$

$$\angle H[e^{j\omega}] = \tan^{-1}\left(\frac{-8/\sqrt{2}}{8/\sqrt{2}}\right) = \frac{-\pi}{4}$$

$\cos\left(\frac{\pi}{2}n - \frac{\pi}{4}\right) \leftrightarrow \pi e^{-j\pi/4}\left[\delta(\omega-\frac{\pi}{2}) + \delta(\omega+\frac{\pi}{2})\right]$

$$\therefore \quad y[n] = 8\cos\left(\frac{\pi}{2}n - \frac{\pi}{4}\right)$$

$$\cos\omega_0 n \xleftrightarrow{DTFT} \pi\left[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)\right]$$

$$\cos\frac{\pi}{2}n \longleftrightarrow \pi\left[\delta(\omega-\frac{\pi}{2}) + \delta(\omega+\frac{\pi}{2})\right]$$

$$8\cos\left(\frac{\pi}{2}n - \frac{\pi}{4}\right) \longleftrightarrow 8\pi\left[\delta\left(\omega-\frac{\pi}{2}-\frac{\pi}{4}\right) + \delta\left(\omega+\frac{\pi}{2}-\frac{\pi}{4}\right)\right]$$

$$\therefore \quad \boxed{Y[e^{j\omega}] = 8\pi\left[\delta\left(\omega-\frac{3\pi}{4}\right) + \delta\left(\omega+\frac{\pi}{4}\right)\right]}$$

Convergence of DTFT:-

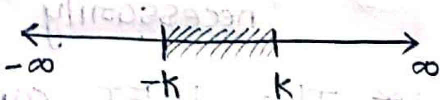
$$x[n] \xleftrightarrow{DTFT} X[e^{j\omega}]$$

$$X[e^{j\omega}] = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

infinite summation series
↳ may / may not converge.

→ $X[e^{j\omega}]$ exists if the above series converges

$$X_K[e^{j\omega}] = \sum_{n=-K}^{K} x[n]e^{-j\omega n}$$



$$\lim_{K\to\infty} X_K[e^{j\omega}] = \lim_{K\to\infty}\sum_{n=-K}^{K} x[n]e^{-j\omega n} = X[e^{j\omega}]$$

$$\lim_{K\to\infty}\left[X[e^{j\omega}] - X_K[e^{j\omega}]\right] = 0$$

$$\lim_{K\to\infty} X_K[e^{j\omega}] = X[e^{j\omega}]$$

⇒ Uniform convergence is guaranteed if $x[n]$ is absolutely summable

$$\underbrace{\sum_{n=-\infty}^{\infty}|x[n]|}_{\text{infinite series}} < \underset{\text{finite value}}{\infty}$$

infinite series
summation should lead to finite value.

If $\sum_{n=-\infty}^{\infty}|x[n]| < \infty$ then $|X[e^{j\omega}]| = \left|\sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}\right|$

$$\leq \sum_{n=-\infty}^{\infty}|x[n]|\cdot\underbrace{|e^{-j\omega n}|}_{=1}$$

$$|X[e^{j\omega}]| \leq \sum_{n=-\infty}^{\infty}|x[n]| < \infty \longrightarrow (*)$$

Equ (*) is the sufficient condition for the existence of the Fourier transform $X[e^{j\omega}]$

Absolute summable	Square summable
$\sum_{n=-\infty}^{\infty}\lvert x[n]\rvert < \infty$	$\sum_{n=-\infty}^{\infty}\lvert x[n]\rvert^2$

Note:- * An absolute summable sequence always has a finite Energy but a finite energy sequence is not necessarily absolute summable.

* The DTFT can also be defined for a certain class of sequences that are neither absolutely summable nor square summable. Ex:- Unit step $u[n]$, Sinusoidal sequence $\sin\omega n, \cos\omega n$ ...

complex exponential sequence ($e^{j\omega n}$)...

* If $x[n]$ is absolutely summable then its DTFT exist

↳ This condition is only a sufficient condition.

$$|X[e^{j\omega}]| \leq \sum_{n=-\infty}^{\infty} |x[n]| < \infty$$

⇒ Sequences: not absolutely summable but its DTFT exists.

Ex:- $x[n] = \frac{1}{n} u[n-1]$

$$\sum_{n=-\infty}^{\infty} |x[n]| = \text{does not converge}$$

$$\sum_{n=-\infty}^{\infty} \left(\frac{1}{n}\right) u[n-1] = \sum_{n=1}^{\infty} \frac{1}{n} = \underbrace{\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \infty}_{\Rightarrow \text{ does not converge}}$$

Square Summable $\sum_{n=-\infty}^{\infty} |x[n]|^2 = \sum_{n=-\infty}^{\infty} \left|\frac{1}{n} u[n-1]\right|^2$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \underbrace{\frac{1}{1} + \frac{1}{4} + \frac{1}{8} + \cdots}_{\Rightarrow \text{ Converges.}}$$

$\frac{1}{n} u[n-1] \Rightarrow$ not absolutely summable

$\frac{1}{n^2} \Rightarrow$ Converges

∴ Absolute summable is not a necessary condition but it is a sufficient condition.

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 \Rightarrow \text{Energy} \longrightarrow \left(\frac{1}{n^2}\right) \Rightarrow \text{Energy}$$

* An absolutely summable sequence is also square summable

Energy → finite

Convergence | Existence of DTFT :-

$$x[n] \xleftrightarrow{DTFT} X[e^{j\omega}]$$

$$X[e^{j\omega}] = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$ infinite summation series (may/may not converge)

$x[n]$ DT signal $\longrightarrow$ Absolutely summable

$$\underbrace{\sum_{n=-\infty}^{\infty} |x[n]|}_{\text{finite}} < \infty$$

(sufficient condition for existence of DTFT)

⇒ If $x[n]$ is not absolutely summable then we have to check for square summable

$x[n] \longrightarrow$ Square summabe

$$\underbrace{\sum_{n=-\infty}^{\infty} |x[n]|^2}_{\text{finite}} < \infty$$

(sufficient condition for existence of DTFT).

⇒ If $x[n]$ is not absolutely summable and not square summable DTFT may exists.

Ex:- $u[n]$, $e^{j\omega_0 n}$, α^n (where α is any Value), $\cos \omega_0 n$, $\sin \omega_0 n$.

but DTFT exists for above sequences by incorporation $\delta(\omega)$ ~~along~~ analog freq function.

⇒ An absolute summable sequence $\sum_{n=-\infty}^{\infty} |x[n]| < \infty$

⇒ A square square summable sequence $\sum_{n=-\infty}^{\infty} |x[n]|^2 < \infty$

⇒ But converse is not true

All abs summable are square summable.

But all square summable are not abs summable abs Summable $\longrightarrow$ Square Summable

Square summable $\not\longrightarrow$ Absolute summable

Ex :- $x[n] = \frac{1}{n} u[n-1]$

Absolute summable $\Rightarrow \sum_{n=-\infty}^{\infty} |x[n]| = \sum_{n=-\infty}^{\infty} \frac{1}{n} u[n-1]$

$= \sum_{n=1}^{\infty} \frac{1}{n}$

Not a finite value, so not abs summable as it won't converges.

$= \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots$

$= \infty$

Square summable $\Rightarrow \boxed{\sum_{n=-\infty}^{\infty} |x(n)|^2 = \sum_{n=1}^{\infty} \left(\frac{1}{n^2}\right)}$

$= \frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \cdots$ fastly converges

$= \frac{\pi^2}{6}$ (finite value)

$\therefore$ It is a square summable sequence.

Ex :- $u[n]$ $\rightarrow$ Not absolutely summable
$\rightarrow$ Not Square Summable
} But DTFT exists for $u[n]$.

Abs summable $\Rightarrow \sum_{n=-\infty}^{\infty} |x[n]| = \sum_{n=-\infty}^{\infty} u[n]$

$= \sum_{n=0}^{\infty} u[n] = 1 + 1 + 1 + \cdots$

$= \infty$ (not abs sum)

Square Summable $\Rightarrow \sum_{n=-\infty}^{\infty} |x[n]|^2 = \sum_{n=-\infty}^{\infty} (u[n])^2$

$= \sum_{n=0}^{\infty} u[n]^2 = 1^2 + 1^2 + 1^2 + \cdots$

$= \infty$ (not square sum)

$u[n] \rightarrow$ Neither absolutely summable nor square summable.

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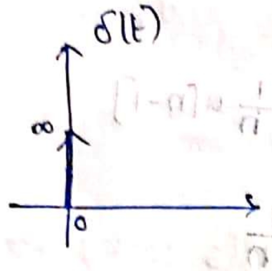
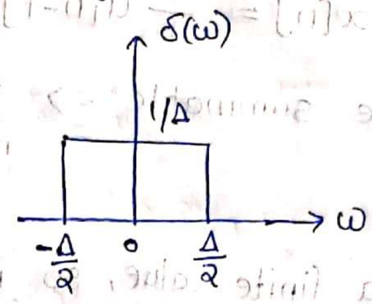
$x[n] = u[n]$

↳ appear to be analog delta function in frequency domain $\delta(\omega)$.



$$\int \delta(t)\, dt = 1$$

$\Longleftrightarrow$



$$\frac{1}{\Delta}\left[\frac{\Delta}{2} - \left(-\frac{\Delta}{2}\right)\right] = \frac{1}{\cancel{\Delta}}(\cancel{\Delta})$$

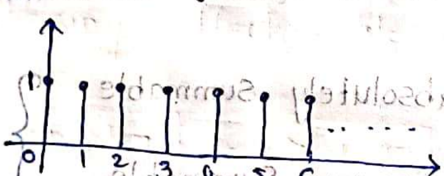
$$= 1 \text{ (Area)}$$

$$\boxed{\frac{1}{\Delta}\cdot\Delta = 1 \text{ (area)} \;;\; \int_{0^-}^{0^+} \delta(\omega)\, d\omega = 1}$$

↓ freq is continuous

Defining an Analog Delta function.

$x[n] = u[n]$



$x[0] = 1$

$x[1] = 1$

$$X[e^{j\omega}] = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} x[n]\, e^{-j\omega n}$$

$$= x(0)\, e^{j\omega 0} + x[1]\, e^{-j\omega} + x[2]\, e^{-j2\omega} + \cdots$$

$$X[e^{j\omega}] = 1 + e^{-j\omega} + e^{-j2\omega} + e^{-j3\omega} + \cdots$$

$$X[e^{j\omega}] = \frac{1}{1 - e^{-j\omega}} + \boxed{\text{analog delta function}}$$

↓ average value of $u[n]$

Avg $u[n] = 1/2$

$$\frac{1}{2} \xleftrightarrow{DTFT} \frac{1}{\cancel{2}} \sum \cancel{2}\pi \, \delta[\omega+2\pi k]$$

$$\frac{1}{2} \xleftrightarrow{DTFT} \sum_{k=-\infty}^{\infty} \pi \, \delta(\omega+2\pi k)$$

$$\boxed{\therefore \; u[n] \xleftrightarrow{DTFT} \frac{1}{1-e^{-j\omega}} + \underbrace{\sum_{k=-\infty}^{\infty} \pi \, \delta(\omega+2\pi k)}_{\text{avg value = Analog delta func}}}$$

Up Sampler :-

$$x[n] \longrightarrow x\left[\frac{n}{L}\right] \qquad n=0, \pm 1, \pm 2, \pm 3 \ldots$$

$$y[n] = \begin{cases} x\left[\frac{n}{L}\right] &, \; n=0, \pm 1, \pm 2, \pm 3 \ldots \\ 0 &, \; \text{otherwise} \end{cases}$$

$\updownarrow$ DTFT

$$Y[e^{j\omega}] = \sum_{n=-\infty}^{\infty} x\left[\frac{n}{L}\right] e^{-j\omega n} \quad , \; n=0, \pm 1, \pm 2, \ldots$$

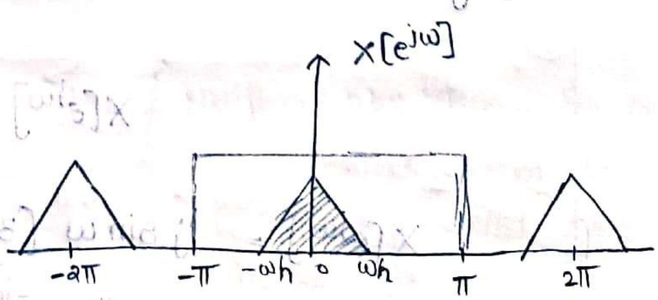
$$x\left[\frac{n}{L}\right] \neq 0 \; , \text{ only when } n=0, \pm 1, \pm 2, \ldots$$

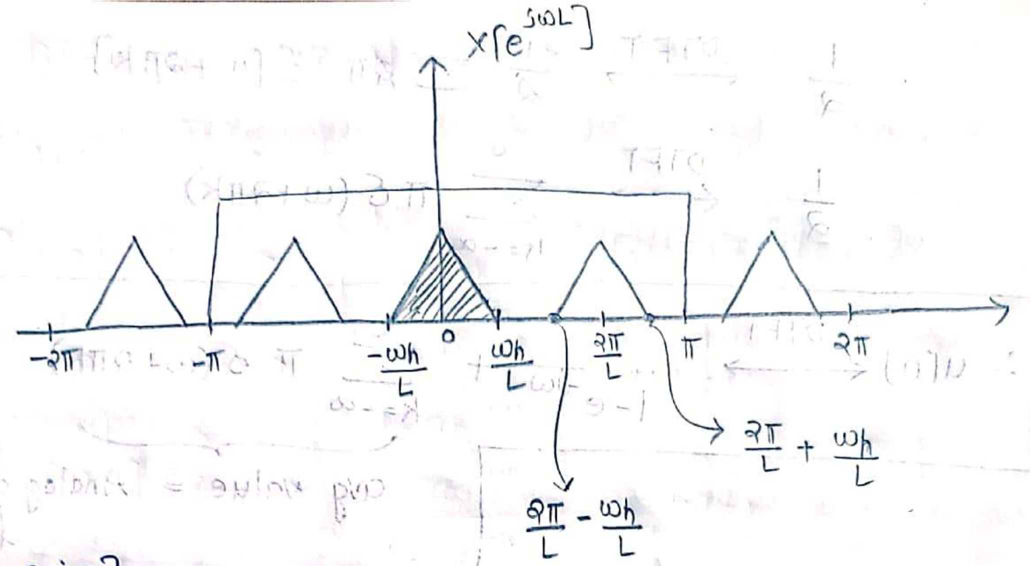
Let $\frac{n}{L} = m \implies n = mL$

$$\sum_{m=-\infty}^{\infty} x\left[\frac{n}{L}\right] e^{-j\omega n} \implies \sum_{m=-\infty}^{\infty} x[m] \, e^{-j\omega mL}$$

$$\left[\begin{array}{l} X[e^{j\omega}] = \sum x[n] e^{-j\omega n} \\ X[e^{j\omega L}] = \sum x[m] e^{-j\omega m} \\ X[e^{j\omega}] \underset{\text{due to up sampling}}{\implies} X[e^{j\omega L}] \end{array}\right]$$

$$Y[e^{j\omega}] = X[e^{jLm}]$$





For $X[e^{j\omega}]$, periodicity $\Rightarrow$ 0 to 2π

$-\pi$ to π only one spectrum is present.

Due to up sampling,

in the periodicity of $0 \to 2\pi$ (or) $-\pi \to \pi$ the spectrum gets 'compresses' and also other period spectrum also enters into the period of $-\pi$ to π.

$\Rightarrow$ This is aliasing effect. We use LPF to overcome this.

$$\longrightarrow \boxed{\text{upSampler}} \longrightarrow \boxed{\text{LPF}} \longrightarrow$$

cutoff freq

$\omega_L \geq \frac{\omega_h}{L}$

Q) Find IDTFT of $x[n] = ?$

$$X[e^{j\omega}] = j\sin\omega\,[3 + 4\cos\omega + 2\cos^2\omega]$$

Ans:- By definition, $x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X[e^{j\omega}]\,e^{j\omega n}\,d\omega$

$$X[e^{j\omega}] = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

$$X[e^{j\omega}] = j\sin\omega\,[3 + 4\cos\omega + 2\cos^2\omega]$$

$$= j\left[\frac{e^{j\omega}-e^{-j\omega}}{2j}\right]\left[3 + 4\left(\frac{e^{j\omega}+e^{-j\omega}}{2}\right) + 2\left(\frac{e^{j\omega}+e^{-j\omega}}{2}\right)^2\right.$$

$$X[e^{j\omega}] = \left[\frac{e^{j\omega} - e^{-j\omega}}{2}\right]\left[3 + 2e^{j\omega} + 2e^{-j\omega} + \frac{e^{j2\omega}}{2} + \frac{e^{-2j\omega}}{2} + 1\right]$$

$$X[e^{j\omega}] = \frac{1}{4}e^{j3\omega} + e^{j2\omega} + \frac{4}{7}e^{j\omega} + 0\cdot e^{j\omega 0} + \frac{9}{2}e^{-j\omega} - e^{-j2\omega} - \frac{1}{4}e^{-j2\omega}$$

$$X[e^{j\omega}] = x[-3]e^{j3\omega} + x[-2]e^{j2\omega} + x[-1]e^{j\omega} + x[0]e^{j\omega 0} + x[1]e^{-j\omega} + x[2]e^{-j2\omega} + x[3]e^{-j3\omega}$$

$$\therefore x[n] = \left\{\frac{1}{4}, 1, \frac{4}{7}, \underset{\uparrow}{0}, \frac{9}{2}, -1, -\frac{1}{4}\right\}$$

$\boxed{\text{Summary}}$

DTFT

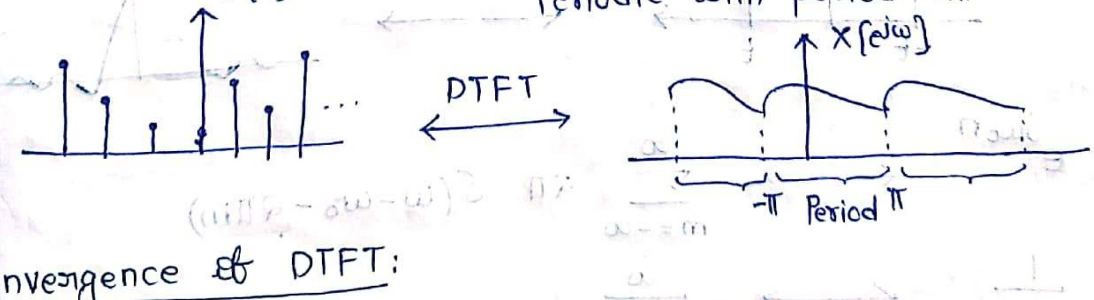
$$X[e^{j\omega}] = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \quad \text{(Analysis Equation)}$$

IDTFT

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X[e^{j\omega}]e^{j\omega n}\,d\omega \quad \text{(Synthesis Equation)}$$

<u>Periodicity of DTFT:</u>

$$X[e^{j(\omega+2\pi)}] = \underbrace{X[e^{j\omega}]}_{\text{Periodic with period } 2\pi}$$



<u>Convergence of DTFT:</u>

Abs Summable / Square Summable / Neither abs summable nor square summable

→ DTFT may exist.

→ Analog Delta func

Ex:- $u[n]$, $e^{j\omega_0 n}$, α^n, $\cos\omega_0 n$, $\sin\omega_0 n$

Basic DTFT's: $x[n] \longleftrightarrow X[e^{j\omega}]$

① $\delta[n] \xleftrightarrow{DTFT} 1$

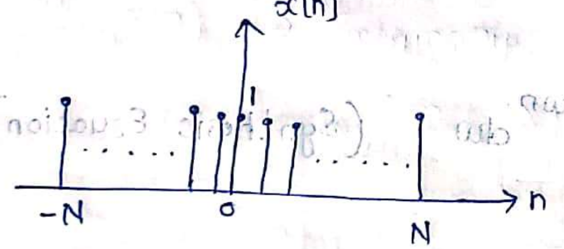
② $a^n u[n] \longleftrightarrow \dfrac{1}{1-ae^{-j\omega}}$; $|a|<1$

③ $a^n u[-n-1] \longleftrightarrow \dfrac{-1}{1-ae^{-j\omega}}$; $|a|<1$

④ $a^{-n} u[-n-1] \longleftrightarrow \dfrac{ae^{j\omega}}{1-ae^{j\omega}}$; $|a|<1$

⑤ $a^{|n|} \longleftrightarrow \dfrac{1-a^2}{1-2a\cos\omega+a^2}$; $|a|<1$

⑥ $u[n+N]-u[n-N-1] = \begin{cases} 1, & |n|\le N \quad (-N\le n\le N) \\ 0, & |n|\ge N \quad (\text{otherwise}) \end{cases}$



$\Big\updownarrow$ DTFT

$$\frac{\sin\left[\omega\left(N+\frac{1}{2}\right)\right]}{\sin(\omega/2)}$$

⑦ $e^{j\omega_0 n} \longleftrightarrow \sum_{m=-\infty}^{\infty} 2\pi\, \delta(\omega-\omega_0-2\pi m)$

⑧ $\dfrac{1}{2\pi} \longleftrightarrow \sum_{m=-\infty}^{\infty} \delta(\omega-2\pi m)$

⑨ $1 \longleftrightarrow 2\pi \sum_{m=-\infty}^{\infty} \delta(\omega-2\pi m)$

⑩ $u[n] \longleftrightarrow \dfrac{1}{1-e^{-j\omega}} + \pi \sum_{m=-\infty}^{\infty} \delta(\omega-2\pi m)$

⑪ $\frac{\omega_c}{\pi} \operatorname{sinc}\left[\frac{\omega_c n}{\pi}\right] \longleftrightarrow \sum_{m=-\infty}^{\infty} \operatorname{rect}\left[\frac{\omega - 2\pi m}{2\omega_c}\right]$

(or)

$\frac{\omega_c}{\pi} \frac{\sin(\omega_c n)}{\omega_c n} \longleftrightarrow \sum_{m=-\infty}^{\infty} \operatorname{rect}\left[\frac{\omega - 2\pi m}{2\omega_c}\right]$

Properties of DTFT:-

① $x_1[n] \longleftrightarrow X_1[e^{j\omega}]$, $x_2[n] \longleftrightarrow X_2[e^{j\omega}]$

$a x_1[n] + b x_2[n] \xleftrightarrow{DTFT} a X_1[e^{j\omega}] + b X_2[e^{j\omega}]$

② $x[n - n_0] \longleftrightarrow X[e^{j\omega}] e^{-j\omega n_0}$

③ $x[n] e^{j\omega_0 n} \longleftrightarrow X[e^{j(\omega-\omega_0)}]$

④ $x[-n] \longleftrightarrow X[e^{-j\omega}]$

⑤ $x\left[\frac{n}{m}\right] \longleftrightarrow X[e^{jm\omega}]$

⑥ $x[n] - x[n-1] \longleftrightarrow (1 - e^{-j\omega}) X[e^{j\omega}]$

⑦ $n \cdot x[n] \longleftrightarrow j \frac{dX[e^{j\omega}]}{d\omega}$

⑧ $x_1[n] * x_2[n] \longleftrightarrow X_1[e^{j\omega}] \cdot X_2[e^{j\omega}]$

⑨ $\sum_{K=-\infty}^{n} x[K] \longleftrightarrow \frac{1}{1-e^{-j\omega}} X[e^{j\omega}] + \pi X[e^{j0}] \sum_{m=-\infty}^{\infty} \delta(\omega - 2\pi m)$

⑩ $x_1[n] \cdot x_2[n] \longleftrightarrow \frac{1}{2\pi}\left[X_1[e^{j\omega}] * X_2[e^{j\omega}]\right]$

$\longleftrightarrow \frac{1}{2\pi} \int_{2\pi} X_1[e^{j\theta}] X_2[e^{j(\omega-\theta)}] d\theta$

(11) $x^*[n] \longleftrightarrow X^*[e^{-j\omega}]$

(i) $x[n] \rightarrow$ Real i.e., $x^*[n] = x[n]$

then $\underbrace{X[e^{-j\omega}] = X^*[e^{j\omega}]}_{\text{conjugate Symmetry}}$

(ii) $x[n] \Rightarrow$ Real & even i.e, $x[n] = x^*[n] = x[-n]$

then $X[e^{j\omega}] = X^*[e^{-j\omega}] = X[e^{-j\omega}]$

(real & even)

(iii) $x[n] \Rightarrow$ Real and Odd i.e, $x[n] = x^*[n] = -x[-n]$

then $X[e^{j\omega}] = \underbrace{X^*[e^{-j\omega}] = -X[e^{-j\omega}]}_{\text{Imaginary \& odd}}$

(iv) $x[n] \Rightarrow$ Imaginary i.e, $x[n] = -x^*[n]$

then $\underbrace{X[e^{j\omega}] = -X^*[e^{-j\omega}]}_{\text{conjugate Antisymmetry}}$

(v) $x[n]$ is $\Rightarrow$ Imaginary & Odd i.e, $x[n] = -x^*[n] = -x[-n]$

then $X[e^{j\omega}] = -X^*[e^{-j\omega}] = \underbrace{-X[e^{-j\omega}]}_{\text{Imaginary \& Odd}}$

(vi) $x[n] \Rightarrow$ Imaginary & even i.e, $x[n] = -x^*[n] = x[-n]$

then $X[e^{j\omega}] = -X^*[e^{-j\omega}] = \underbrace{X[e^{-j\omega}]}_{\text{Imaginary \& even}}$

(12) $x_e[n] \longleftrightarrow X_R[e^{j\omega}]$

$$\frac{x[n] + x[-n]}{2} \longleftrightarrow \frac{1}{2}\left[X[e^{j\omega}] + X[e^{-j\omega}]\right]$$

$$\longleftrightarrow \frac{1}{2}\left[X[e^{j\omega}] + X^*[e^{j\omega}]\right] = \frac{1}{\cancel{2}}\left[\cancel{2}X_R[e^{j\omega}]\right]$$

$\therefore x_e[n] \longleftrightarrow X_R[e^{j\omega}]$

$$\left[\begin{array}{l} \because x_e[n] \text{ is a even signal} \Rightarrow x_e[n] = x_e[-n] \\ \mathcal{F}\{x_e[n]\} = \mathcal{F}\{x_e[-n]\} \\ X_R[e^{j\omega}] = X_R[e^{-j\omega}] \end{array}\right]$$

(13) $x_o[n] \longleftrightarrow j X_I[e^{j\omega}]$

$$\frac{x[n]-x[-n]}{2} \longleftrightarrow \frac{1}{2}\left[X[e^{j\omega}] - X^*[e^{j\omega}]\right]$$

$$\longleftrightarrow \frac{1}{2}\left[2j\, X_I[e^{j\omega}]\right] = j X_I[e^{j\omega}]$$

$\therefore x_o[n] \longleftrightarrow j X_I[e^{j\omega}]$

$$\left[\begin{array}{l} \because x_o[n] \text{ is an odd signal} \\ x_o[n] = -x_o[-n] \\ \xrightarrow{\text{DTFT}} \\ j X_I[e^{j\omega}] = -j X_I[e^{-j\omega}] \\ X_I[e^{j\omega}] = -X_I[e^{-j\omega}] \end{array}\right]$$

(14) $$E_x = \sum_{n=-\infty}^{\infty} |x(n)|^2 = \boxed{\frac{1}{2\pi}\int_{2\pi} |X[e^{j\omega}]|^2 d\omega}$$

Important Results:

(1) $$X[e^{j0}] = \sum_{n=-\infty}^{\infty} x[n]$$

$\Rightarrow$ RHS: Sum of Samples of $x(n)$

LHS : $X[e^{j0}]$ $(\omega = 0)$ eigen value.

(2) $$X[e^{j\pi}] = \sum_{m=-\infty}^{\infty} (-1)^n x[n]$$

(3) $$\int_{-\pi}^{\pi} X[e^{j\omega}]\, d\omega = 2\pi \cdot x[0]$$

(4) $$\int_{-\pi}^{\pi} |X[e^{j\omega}]|^2 d\omega = 2\pi \cdot \sum_{n=-\infty}^{\infty} |x(n)|^2$$

(5) $$\int_{-\pi}^{\pi} \left|\frac{dX[e^{j\omega}]}{d\omega}\right|^2 d\omega = 2\pi \cdot \sum_{n=-\infty}^{\infty} |n \cdot x(n)|^2$$

Discrete time Fourier transform of periodic signals :-

DTFS representation of any periodic signal,

$$x[n] = \sum_{K=\langle N\rangle} X_K e^{jK\omega_0 n}$$

ω_0 : fundamental period

$\omega_0 = \frac{2\pi}{N}$

periodic signal with period N.

N : Period.

$$\mathcal{F}\{x[n]\} = \mathcal{F}\Big\{\sum_{K=\langle N\rangle} X_K e^{jK\omega_0 n}\Big\}$$

$$X[e^{j\omega}] = \sum_{K=\langle N\rangle} X_K \cdot \mathcal{F}\{e^{jK\omega_0 n}\}$$

$$X[e^{j\omega}] = \sum_{K=\langle N\rangle} X_K \, 2\pi \, \delta(\omega - K\omega_0) \quad ; 0 \le \omega \le 2\pi$$

Fourier transform of a periodic signal is an impulse train with impulses located at $\omega = K\omega_0$ each of strength $2\pi X_K$ and all impulses are seperated by 'ω_0'.

Signal Transmission through LTI system,

$$Y[e^{j\omega}] = H[e^{j\omega}] \cdot X[e^{j\omega}]$$

$|H[e^{j\omega}]|$ = magnitude response

$\angle H[e^{j\omega}]$ = phase response.

→ Linear & Non-linear phase

→ Phase & group delay

→ Ideal filter → $|H[e^{j\omega}]|$ → LPF, HPF, BPF, BSF,

→ ESD, PSD, $H[e^{j\omega}]$ All pass filters.

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