NEWTON'S INTERPOLATION FORMULAE

Interpolation

The statement

$$y = f(x), \quad x_0 \leq x \leq x_n$$

means: Corresponding to every value of x in the range $x_0 \leq x \leq x_n$, there exists one or more values of y. Assuming that $f(x)$ is single-valued and continuous and that it is known explicitly, then the values of $f(x)$ correseponding to certain given values of x, say $x_0, x_1, \ldots, x_n$ can easily be computed and tabulated. The central problem of numerical analysis is the converse one: Given the set of tabular values $(x_0, y_0), (x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ satisfying the relation $y = f(x)$ where the explicit nature of $f(x)$ is not known, it is required to find a simpler function, say $\emptyset(x)$, such that $f(x)$ and $\emptyset(x)$ agree at the set of tabulated points. Such a process is called interpolation. If $\emptyset(x)$ is a polynomial, then the process is called polynomial interpolation and $\emptyset(x)$ is a called the interpolating polynomial. Similarly, different types of interpolation arise depending on whether $\emptyset(x)$ is a finite trigonometric series, series of Bessel functions, etc. In this chapter, we shall be concerned with polynomial interpolation only. As a justification for the approximation of an unknown function by means of a polynomial, we state here, without proof, a famous theorem due to Weierstrass (1885): if $f(x)$ is continuous in $x_0 \leq x \leq x_n$, then given any $\varepsilon > 0$, there exists a polynomial $P(x)$ such that

$|f(x) - P(x)| < \varepsilon$, for all x in (x_0, x_n).

This means that it is possible to find a polynomial $P(x)$ whose graph remains within the region bounded by $y = f(x) - \varepsilon$ and $y = f(x) + \varepsilon$ for all x between x_0 and x_n, however small ε may be.

When approximating a given function $f(x)$ by means of polynomial $\emptyset(x)$, one may be tempted to ask: (i) How should the closeness of the approximation be measured? and (ii) What is the criterion to decide the best polynomial approximation to the function? Answers to these question, important though they are for the practical problem of interpolation, are outside the scope of this course and will not be attempted here. We will, however, derive in the next section a formula for finding the error associated with the approximation of a tabulated function by means of a polynomial.

ERRORS IN POLYNOMIAL INTERPOLATION

Let the function $y(x)$, defined by the $(n+1)$ points $(x_i, y_i), i = 0,1,2, \ldots, n$, be continuous and differentiable $(n+1)$ times, and let $y(x)$ be approximated by a polynomial $\emptyset_n(x)$ of degree not exceeding n such that

$$\emptyset_n(x_i) = y_i, \qquad i = 0,1,2, \ldots, n \qquad (1)$$

If we now use $\emptyset_n(x)$ to obtain approximate values of $y(x)$ at some points other than those defined by (1), what would be the accuracy of this approximation? Since the expression $y(x) - \emptyset_n(x)$ vanishes for $x = x_0, x_1, x_2, \ldots, x_n$, we put

$$y(x) - \emptyset_n(x) = L\pi_{n+1}(x), \qquad (2)$$

where

$$\pi_{n+1}(x) = (x - x_0)(x - x_1), \dots, (x - x_n) \qquad (3)$$

And L is to be determined such that Eq.(2) holds for any intermediate values of x, say $x = x', x_0 < x' < x_n$. Clearly,

$$L = \frac{y(x') - \emptyset_n(x')}{\pi_{n+1}(x')} \ . \qquad (4)$$

We construct a function $F(x)$ such that

$$F(x) = y(x) - \emptyset_n(x) - L\pi_{n+1}(x), \qquad (5)$$

where L is given by Eq. (4) above,

It is clear that

$$F(x_0) = F(x_1) = \cdots = F(x_n) = F(x') = 0,$$

that is, $F(x)$ vanishes $(n+2)$ times in the interval $x_0 \le x \le x_n$; consequently, by the repeated application of Rolle's theorem(Let $f(x)$ be a function which is in n times differentiable on $[a, b]$. If $f(x)$ vanishes at the $(n+1)$ distinct points $x_0, x_1, \dots \dots x_n$ in (a, b), then there exists a number ξ in (a, b) such that $f^n(\xi) = 0$.). $F'(x)$ must vanish $(n+1)$ times, $F''(x)$ must vanish n times, etc., in the interval $x_0 \le x \le x_n$. In particular, $F^{(n+1)}(x)$ must vanish once in the interval.

Let this point be given $x = \xi$, $x_0 \le \xi \le x_n$. On differentiating equation (5) $(n+1)$ times with respect to x and putting $x = \xi$, we obtain

$$0 = y^{(n+1)}(\xi) - L(n+1)!$$

so that

$$L= \frac{y^{(n+1)}(\xi)}{(n+1)!} \qquad (6)$$

Comparison of (4) and (6) yields the results

$$y(x') - \phi_n(x') = \frac{y^{(n+1)}(\xi)}{(n+1)!}\pi_{n+1}(x').$$

Dropping the prime on x', we obtain

$$y(x) - \phi_n(x) = \frac{\pi_{n+1}(x)}{(n+1)!}y^{(n+1)}(\xi), \quad x_0 \le \xi \le x_n, \qquad (7)$$

which is the required expression for the error . Since $y(x)$ is, generally, unknown and hence we do not have any information concerning $y^{(n+1)}(x)$, formula (7) is almost useless in practical computations. On the other hand, it is extremely useful in theoretical work in different branches of numerical analysis. In particular, we will use it to determine errors in Newton's interpolating formulae.

Newton's formulae for interpolation

Given the set of $(n+1)$ values, viz., $(x_0, y_0), (x_1, y_1), (x_2, y_2) \ldots\ldots\ldots\ldots (x_n, y_n)$, of x and y, it is required to find $y_n(x)$, a polynomial of the nth degree such that y and $y_n(x)$ agree at the tabulated points. Let the values of x be equidistant, i.e. let

$$x_i = x_0 + ih, \ i = 0,1,2, \ldots\ldots\ldots, n$$

Since $y_n(x)$ is a polynomial of the nth degree, it may be written as

$$y_n(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + a_3(x - x_0)(x - x_1)(x - x_2) + \cdots + a_3(x - x_0)(x - x_1)(x - x_2) \ldots .(x - x_{n-1}).$$

Imposing now the condition that y and $y_n(x)$ should agree at the set of tabulated points, we obtain

$$a_0 = y_0;\; a_1 = \frac{y_1 - y_0}{x_1 - x_0} = \frac{\Delta y_0}{h}; a_2 = \frac{\Delta^2 y_0}{h^2 2!}; a_3 = \frac{\Delta^3 y_0}{h^3 3!}; \ldots ; a_n = \frac{\Delta^n y_0}{h^n n!};$$

Setting $x = x_0 + ph$ and substituting for $a_0, a_1, \ldots .., a_n,$ then

$$y_n(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \cdots + \frac{p(p-1)(p-2)\ldots .(p-n+1)}{n!}\Delta^n y_0,$$

which is Newton's forward difference interpolation formula and is useful for interpolation near the beginning of a set of tabular values.

To find the error committed in replacing the function $y(x)$ by means of the polynomial $y_n(x)$, we use formula (7) to obtain

$$y(x) - y_n(x) = \frac{(x-x_0)(x-x_1)\ldots\ldots(x-x_n)}{(n+1)!} y^{(n+1)}(\xi), x_0 < \xi < x_n \ldots\ldots\ldots (8)$$

As remarked earlier we do not have any information concerning $y^{(n+1)}(x)$, and therefore formula (8) is useless in practice. Nevertheless, if $y^{(n+1)}(x)$ does not vary too rapidly in the interval, a useful estimate of the derivative can be obtained in the following way, expanding $y(x+h)$ by Taylor's series we obtain

$$y(x+h) = y(x) + hy'(x) + \frac{h^2}{2!}y''(x) + \cdots$$

Neglecting the terms containing h^2 and higher powers of h, this gives

$$y'(x) \approx \frac{1}{h}[y(x+h) - y(x)] = \frac{1}{h}\Delta y(x).$$

Writing $y'(x)$ as $Dy(x)$ where $D \equiv d/dx$, the differentiation operator, the above equation gives the operator relation

$D \equiv \frac{1}{h}\Delta$ and so $D^{n+1} \equiv \frac{1}{h^{n+1}}\Delta^{n+1}$.

We thus obtain

$$y^{(n+1)}(x) \approx \frac{1}{h^{n+1}}\Delta^{n+1}y(x).$$

Equation (8) can therefore be written as

$$y(x) - y_n(x) = \frac{p(p-1)(P-2)\ldots.(p-n)}{(n+1)!}\Delta^{n+1}y(\xi)$$

in which form it is suitable for computation.

Instead of assuming $y_n(x)$ as in previous case, if we choose it in the form

$$\begin{aligned} y_n(x) = a_0 + a_1(x-x_n) &+ a_2(x-x_n)(x-x_{n-1}) \\ &+ a_3(x-x_n)(x-x_{n-1})(x-x_{n-2}) + \cdots \\ &+ a_3(x-x_n)(x-x_{n-1})(x-x_{n-2})\ldots.(x-x_1) \end{aligned}$$

and then impose the condition that y and $y_n(x)$ should agree at the tabulated points $x_n, x_{n-1}, \ldots., x_2, x_1, x_0$, we obtain (after some simplification)

$y_n(x) = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \cdots + \frac{p(p+1)\ldots(p+n-1)}{n!}\nabla^{\mathrm{n}} y_n$

where $p = (x - x_n)/h$

This is Newton's backward difference interpolation formula and it uses tabular values to the left of y_n . This formula is therefore useful for interpolation near the end of the tabular values.

It can be shown that the error in this formula may be written as

$$y(x) - y_n(x) = \frac{p(p+1)(P+2)\ldots.(p+n)}{(n+1)!}h^{n+1}y^{n+1}(\xi)$$

where $x_0 < \xi < x_n$ and $x = x_n + ph$.

The following examples illustrate the use of these formulae.

Example: Find the cubic polynomial which takes the following values; $y(1) = 24, y(3) = 120, y(5) = 336$, and $y(7) = 720$. Hence, or otherwise , obtain the value of $y(8)$.

We form the difference table:

x	y	Δ	Δ^2	Δ^3
1	24			
		96		

3	120		120	
		216		48
5	336		168	
		384		
7	720			

Here $h = 2$. With $x_0 = 1$, we have $x = 1 + 2p$ or $p = (x - 1)/2$. Substituting this value of p in Newton's forward difference formula, we obtain

$$\begin{aligned} y(x) &= 24 + \frac{x-1}{2}(96) + \frac{\left(\frac{x-1}{2}\right)\left(\frac{x-1}{2} - 1\right)}{2}(120) \\ &\quad + \frac{\left(\frac{x-1}{2}\right)\left(\frac{x-1}{2} - 1\right)\left(\frac{x-1}{2} - 2\right)}{6}(48) \\ &= x^3 + 6x^2 + 11x + 6. \end{aligned}$$

To determine $y(8)$, we observe that $p = 7/2$. Hence, the same formula gives:

$$y(x) = 24 + \frac{7}{2}(96) + \frac{\left(\frac{7}{2}\right)\left(\frac{7}{2} - 1\right)}{2}(120) + \frac{\left(\frac{7}{2}\right)\left(\frac{x-1}{2} - 1\right)\left(\frac{7}{2} - 2\right)}{6}(48) = 990.$$

Direct substitution in $y(x)$ yields the same value.

Note: This process of finding the value of y for some value of x outside the given range is called extrapolation and this example demonstrates the face that if a tabulated function is a polynomial, then both interpolation and extrapolation would give exact values.

Solved Problems

Problem 1: Using Newton's forward difference formula, find the sum

$$S_n = 1^3 + 2^3 + 3^3 + \cdots + n^3$$

Solution: We have

$$S_{n+1} = 1^3 + 2^3 + 3^3 + \cdots + n^3 + (n+1)^3$$

Hence

$$S_{n+1} - S_n = (n+1)^3,$$

or

$$\Delta S_n = (n+1)^3.$$

It follows that

$$\Delta^2 S_n = \Delta S_{n+1} - \Delta S_n = (n+2)^3 - (n+1)^3 = 3n^2 + 9n + 7,$$

$$\Delta^3 S_n = 3(n+1)^2 + 9n + 7 - (3n^2 + 9n + 7) = 6n + 12$$

$$\Delta^4 S_n = 6(n+1) + 12 - (6n + 12) = 6.$$

Since $\Delta^5 S_n = \Delta^5 S_n = \cdots = 0, S_n$ is a fourth-degree polynomial in n. Further,

$$S_1 = 1,\ \Delta S_1 = 8,\ \Delta^2 S_1 = 19,\ \Delta^3 S_1 = 18,\ \Delta^4 S_1 = 6.$$

Newton's forward difference formula gives

$$\begin{aligned} S_n = 1 + (n-1)8 + \frac{(n-1)(n-2)}{2}(19) \\ + \frac{(n-1)(n-2)(n-3)}{6}(18) \\ + \frac{(n-1)(n-2)(n-3)(n-4)}{24}(6) \end{aligned}$$

$$= \frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{1}{4}n^2$$

$$= \left[\frac{n(n+1)}{2}\right]^2.$$

Problem 2: Values of x (in degrees) and $\sin x$ are given in the following table:

x(in degrees)	$\sin x$
15	0.2588190
20	0.3420201
25	0.4226183
30	0.5
35	0.5735764
40	0.6427876

Determine the value of $\sin 38^0$.

Solution: The backward difference tables is

x	$\sin x$	∇	∇^2	∇^3	∇^4	∇^5
15	0.2588190					
		0.0832011				
20	0.3420201		-0.0026029			
		0.0805982		-0.0006136		
25	0.4226183		-0.0032165		0.0000248	
		0.0773817		-0.0005838		0.0000041
30	0.5		-0.0038053		0.0000289	
		0.0735764		-0.0005599		
35	0.5735764		-0.0043652			
		0.0692112				
40	0.6427876					

To find $\sin 38^0$, we use Newton's backward difference formula with $x_n = 40$ and $x = 38$ This gives

$$p = \frac{x - x_n}{h} = \frac{38 - 40}{5} = -\frac{2}{5} = -0.4$$

Hence , using formula , we obtain

$$\begin{aligned}
&y(38) \\
&= 0.6427876 - 0.4(0.0692112) + \frac{-0.4(-0.4-1)}{2}(-0.0043652) \\
&+ \frac{(-0.4)(-0.4+1)(-0.4+2)}{6}(-0.0005599) \\
&+ \frac{(-0.4)(-0.4+1)(-0.4+2)(-0.4+3)}{24}(0.0000289) \\
&+ \frac{(-0.4)(-0.4+1)(-0.4+2)(-0.4+3)(-0.4+4)}{120}(0.0000041) \\
&= 0.6427876 - 0.02768448 + 0.00052382 + 0.00003583 \\
&- 0.0000120
\end{aligned}$$

$$= 0.6156614.$$

Problem 3: Find the missing term in the following table:

x	y
0	1
1	3
2	9
3	$-$
4	81

Explain why the result differs form $3^3 = 27$?

Solution: Since four points are given, the given data can be approximated by a third degree polynomial in x. Hence $\Delta^4 y_0 = 0$. Substituting $\Delta = E - 1$ and simplifying we get

$$E^4 y_0 - 4E^3 y_0 + 6E^2 y_0 - 4E y_0 + y_0 = 0.$$

Since $E^r y_0 = y_r$, the above equation becomes

$$y_4 - 4y_3 + 6y_2 - 4y_1 + y_0 = 0$$

Substituting for y_0, y_1, y_2 and y_4 in the above, we obtain

$$y_3 = 31.$$

The Tabulated function is 3^x and the exact value of $y(3)$ is 27. The error is due to the fact that the exponential function 3^x is approximated by means of a polynomial in x of degree 3.

Problem 4: The table below gives the values of $\tan x$ for $0.10 \leq x \leq 0.30$:

x	$y = \tan x$
0.10	0.1003
0.15	0.1511
0.20	0.2027
0.25	0.2553
0.30	0.3093

Find: (a) $\tan 0.12$, (b) $\tan 0.26$, (c) $\tan 0.40$ and (d) $\tan 0.50$.

Solution: The table of difference is

x	y	Δ	Δ^2	Δ^3	Δ^4
0.10	0.1003				
		0.0508			
0.15	0.1511		0.0008		
		0.0516		0.0002	
0.20	0.2027		0.0010		0.0002
		0.0526		0.0004	
0.25	0.2553		0.0014		
		0.0540			
0.30	0.3093				

a) To find $\tan(0.12)$, we have $0.12 = 0.10 + p(0.05)$, which gives $p = 0.4$. Hence Newton's forward difference interpolation formula gives,

$$\begin{aligned}\tan(0.12) &= 0.1003 + 0.4(0.0508) + \frac{0.4(0.4-1)}{2}(0.0008) \\ &+ \frac{0.4(0.4-1)(0.4-2)}{6}(0.0002) \\ &+ \frac{0.4(0.4-1)(0.4-2)(0.4-3)}{24}(0.0002) = 0.1205.\end{aligned}$$

b) To find $\tan(0.26)$, we have $0.26 = 0.30 + p(0.05)$, which gives $p = -0.8$. Hence Newton's backward difference interpolation formula gives

$\tan(0.26)$

$$= 0.3093 - 0.8(0.0540) + \frac{-0.8(-0.8+1)}{2}(0.0014)$$
$$+ \frac{-0.8(-0.8+1)(-0.8+2)}{6}(0.0004)$$
$$+ \frac{-0.8(-0.8+1)(-0.8+2)(-0.8+3)}{24}(0.0002)$$
$$= 0.2662.$$

Proceeding as in the above, we obtain

c) $\tan(0.40) = 0.4241$, and

d) $\tan(0.50) = 0.5543$.

The actual values, correct to four decimal places, of $\tan(0.12)$, $\tan(0.26)$, $\tan(0.40)$ and $\tan(0.50)$ are respectively 0.1206, $0.2660, 0.4228$ and 0.5463. Comparisons of the computed and actual values are fairly accurate whereas in the last-two cases (i.e. of extrapolation) the errors are quite considerable. The example therefore demonstrates the important result that if a tabulated function is other than a polynomial, then extrapolation very far from the table limits would be dangerous – although interpolation can be carried out very accurately.

EXERCISE

1. Construct the interpolating polynomial that fits the data

x	**0**	**0.1**	**0.2**	**0.3**	**0.4**	**0.5**
$f(x)$	-1.5	-1.27	-0.98	-0.63	-0.22	0.25

using Newton's forward or backward difference interpolation. Hence or otherwise estimate the values of $f(x)$ at $x = 0.15, 0.25$ and 0.45.

2. Using the Newton's backward difference interpolation, construct the interpolating polynomial that fits the data

x	0.1	0.3	0.5	0.7	0.9	1.1
$f(x)$	-1.699	-1.073	-0.375	0.443	1.429	2.631

Estimate the value of $f(x)$ at $x = 0.6$ and $x = 1.0$

3. The following data represents the function $f(x) = e^x$.

x	1	1.5	2.0	2.5
$f(x)$	2.7183	4.4817	7.3891	12.1825

Estimate the value of $f(2.25)$ using the

i. Newton's forward difference interpolation and
ii. Newton's backward difference interpolation.

Compare with the exact value.

4. The following data represents the function

$$f(x) = \cos(x + 1).$$

x	0.0	0.2	0.4	0.6
$f(x)$	0.5403	0.3624	0.1700	-0.0292

Estimate $f(0.5)$ using the Newton's backward difference interpolation. Compare with the exact value.

5. The following data represents the function $f(x) = (\cos x)/x$.

x (in radians)	0.1	0.2	0.3	0.4
$f(x)$	9.9500	4.9003	3.1845	2.3027

Calculate $f(0.12)$.

ANSWERS

1. $3x^2 + 2x - 1.5$; -1.1325; -0.8125; 0.0075
2. $x^3 + 3x - 2$; 0.016; 2
3. 9.5037; Exact value= 9.4877
4. 0.0708;Exact value= 0.0707
5. 8.5534,