UNIT - I

Probability

1. Sets
2. Venn Diagrams
3. Algebra of Sets
4. Probability
5. Probability axioms
6. Theorems on probability
7. Problems on probability
8. Conditional probability
9. Conditional prob" properties
10. Problems on Conditional prob
11. Independent Events
12 Total probability
13 problems on Total probability
14 Baye's Theorem
15 Applications of Baye's Theorem

Unit-1

Probability

1.1

Algebra of Sets and Counting Methods

The algebra of sets and counting methods are useful in understanding the basic concepts of *probability*. These concepts are briefly reviewed from the point of view of probability.

Sets and Elements of sets: The fundamental concept in the study of the probability is the set.
A set is a well defined collection of objects and denoted by upper case English letters. The objects in a set are known as **elements** and denoted by lower case letters. A set can be written in two ways. Firstly, if the set has a finite number of elements, we may list the elements, separated by commas and enclosed in brackets. For example, a set A with elements $1, 2, 3, 4, 5$ and 6, it may be written as

$$A = \{1, 2, 3, 4, 5, 6\}$$

Secondly, the set may be described by a statement or a rule. Then A may be written as

$$A = \{x \mid x \text{ is a natural number less than or equal to } 6\}$$

If x is an element of the set A, we write $x \in A$. If x is not a element of the set A, then we write $x \notin A$.

Equal Sets: Two sets A and B are said to be ***equal*** or ***identical*** if they have exactly the same elements and we write as $A = B$

Subset: If every element of the set A belong to the set B, $i.e.$, if $x \in A \implies x \in B$, then we say that A is a ***subset*** of B and we write $A \subseteq B$ (A is contained in B) or $B \supseteq A$ (B contians A). If $A \subseteq B$ and $B \subseteq A$, then $A = B$.

Null set: A ***null*** or an ***empty set*** is one which does not contain any element at all and denoted by $\emptyset$.

Note:

1. Every set is a subset it self
2. An empty set is a subset of every set.
3. A set containing only one elements is conceptually different from the element itself .
4. In all applications of set theory, especially in probability theory, we shall have a fixed set S (say), given in advance and we shall be concerned only with subsets of S. This set is referred to universal set.

1) Union or sum:

$$A \cup B = \{x \mid x \in A \ \ or \ \ x \in B\}$$

$$\bigcup_{i=1}^{n} A_i = \{x \mid \ x \in A_i \quad for\ at\ least\ one\ i = 1, 2, \dots, n\}$$

2) Intersection or Product:

$$A \cap B = \{x \mid \ x \in A \ \ and \ \ x \in B\}$$

$$\bigcap_{i=1}^{n} A_i = \{x \mid x \in A_i \ \ for\ all\ i = 1, 2, \dots, n\}$$

If $A \cap B = \emptyset$, then we say that A and B are **disjoint sets.**

3) Relative Difference: $A - B = \{x \mid \ \ x \in A \ \ \ and \ \ x \notin B\}$

4) Complement: $\bar{A} = S - A$

Algebra of Sets:

If A, B and C are subsets of a universal set S, then the following laws hold:

Commutative laws: $A \cup B = B \cup A, \ \ A \cap B = B \cap A$

Associative laws: $(A \cup B) \cup C = A \cup (B \cup C), (A \cap B) \cap C = A \cap (B \cap C)$

Distributive laws:

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C), A \cup (B \cap C) = (A \cup B) \cap (A \cap C)$

Complementary laws: $A \cup \bar{A} = S, A \cap \bar{A} = \emptyset, A \cup S = S, A \cap S = A$

Difference laws: $A - B = A \cap \bar{B} = A - (A \cap B) = (A \cup B) - B,$

$A - (B - C) = (A - B) \cup (A - C), (A \cup B) - C = (A - C) \cup (B - C),$

$(A \cap B) \cup (A - B) = A, (A \cap B) \cap (A - B) = \emptyset$

De – Morgan's laws:

$\overline{A \cup B} = \bar{A} \cap \bar{B}, \overline{A \cap B} = \bar{A} \cup \bar{B}$

$\overline{\bigcup_{i=1}^{n} A_i} = \bigcap_{i=1}^{n} \bar{A}_i$ and $\overline{\bigcap_{i=1}^{n} A_i} = \bigcup_{i=1}^{n} \bar{A}_i$

Involution law: $\overline{(\bar{A})} = A$

Idempotent law: $A \cup A = A, A \cap A = A$

1.2

Basic Concepts in Probability

Introduction to uncertainty

Every day we have been coming across statements like the ones mentioned below:

1. Probably it will rain tonight.
2. It is quiet likely that there will be a good yield of paddy this year.
3. Probably I will get a first class in the examination.
4. India might win the cricket series against Australia
   and so on.

In all the above statements some element of uncertainty or chance is involved. A numerical measure of uncertainty is provided by a very important branch of statistics known as **Theory of Probability**. In the words of *Prof. Ya-Lin-Chou*: *Statistics is the science of decision making with calculated risks in the face of uncertainty.*

History of Probability

The history of probability suggests that its theory developed with the study of *games of chance*, such as *rolling of dice*, *drawing a card from a pack of cards*, etc. Two French gamblers had once decided that any one person who will first get a 'particular point' will win the game. If the game is stopped before reaching that point, the question is how to share the stake. This and similar other problems were then posed by the great French mathematician *Blaise Pascal*, who after consulting another great French mathematician *Pierre de Fermat*, gave the solution of the problems and then laid down a strong foundation of probability. Later on, another French mathematician, *Laplace*, improved the definition of probability.

Coins, Dice and Playing Cards: The basic concepts in probability are better explained using *coins*, *dice* and *playing cards*. The knowledge of these is very much useful in solving problems in probability.

Coin: A coin is round in shape and it has two sides. One side is known as ***head (H)*** and the other is known as ***tail (T).*** When a coin is tossed, the side on the top is known as the result of the toss.

Die: A die is cube in shape in which length, breadth and height are equal. It has six faces which have same area and numbered from 1 to 6. The plural of die is dice. When a die is thrown, the number on the top face is the result of the throw.

Pack of Cards: A pack of cards 52 cards. It is divided into four suits called *spades*, *clubs*, *hearts* and *diamonds*. Spades and clubs are black; hearts and diamonds are red in colour. Each suit consists of 13 cards, of which *nine* cards are numbered from 2 to 10, an ace, jack, queen and king. We shuffle the cards and then take a card from the top which is the result of selecting a card.

Basic Concepts in Probability

The following basic concepts are very important in understanding the definitions of the probability:

Experiment: The process of making an observation or measurement and observation about a phenomenon is known as an ***experiment***.

Example1: Sitting in the balcony of the house and watching the movement of clouds in the sky is an experiment.

Example2: For given values of pressure (P), measuring the corresponding values of volume (V) of a gas and observing that $P \cdot V = k$(constant) is an experiment. The experiments are of two types:

Deterministic experiment: If an experiment produces the same result when it is conducted several times under identical conditions, then the experiment is known as ***determinant experiment***.
All the experiments in physical and engineering sciences are deterministic.

Random Experiment: If an experiment produces different results even though it is conducted several times under identical conditions, then the experiment is known as ***random experiment***. All the experiments in social sciences are random.

Trial: Conducting a random experiment once is known as a ***trial***.

Outcome: A result of a random experiment in a trial is known as an ***outcome***.
Outcomes are denoted by lowercase letters $a, b, c, d, e, \dots$.

Equally Likely Outcomes: Outcomes of a random experiment are said to be ***equally likely*** if all have the same chance of occurrence. Getting a H and T in a balanced coin are equally likely. The outcomes $1,2,3,4,5$ and 6 are equally likely if the die is a cube.

Sample space: The set of all possible outcomes of a random experiment is known as a ***sample space*** and denoted by **S**.

Event: A subset of the sample space is known as an ***event***.
The events are denoted by uppercase letters $\mathrm{A}, \mathrm{B}, \mathrm{C}$ etc.

Happening of an event: We say that an event happens (or occurs) if any one outcome in it happens (or occurs).

Elementary Event: A singleton set consisting an outcome of a random experiment is known as an ***elementary event***.

Favorable outcomes: The outcomes in an event are known as ***favorable outcomes*** or ***cases*** of that event.

Impossible Event: An event with no outcome in it is known as ***impossible event*** and is denoted by $\boldsymbol{\phi}$.

Certain or Sure Event: An event consisting of all possible outcomes of a random experiment is known as ***certain*** or ***sure event*** and it is same as the sample space.

Exhaustive Events: The events in a sample space are said to be ***exhaustive*** if their union is equal to the sample space. The events $A_1, A_2, \dots, A_n$ in S are said to be exhaustive if

$$\bigcup_{i=1}^{n} A_i = S$$

Mutually Exclusive Events: Two or more events in the sample space are said to be ***mutually exclusive*** if the happening of one of them precludes the happening of the others. Mathematically two events A and B in S are said to be mutually exclusive if $A \cap B = \phi$.

Example3: Consider a random experiment of tossing a coin. The possible outcomes are H and T. Thus, the sample space is given by $S = \{H, T\}$ and $n(S) = 2$ where $n(S)$ is the total number of outcomes in S.

Example 4: Consider a random experiment of tossing two coins (or two tosses of a coin). The sample space is given by $S = \{H, T\} \times \{H, T\} = \{HH, HT, TH, TT\}$ and

Mathematical (or classical or A priori) definition of probability

Let S be a sample space associated with a random experiment. Let A be an event in S. We make the following assumptions on S:

(i) It is discrete and finite
(ii) The outcomes in it are equally likely

Then the probability of happening (or occurrence) of the event A is defined by

$$P(A) = \frac{Number\ of\ outcomes\ in\ A}{Number\ of\ outcomes\ in\ S} = \frac{n(A)}{n(S)}$$

Probability axioms

① $0 \le P(E) \le 1$

② $P(S) = 1$

③ $P(E^c) = 1 - P(E)$

④ If $A \cap B = \phi$ then $P(A \cup B) = P(A) + P(B)$

1.4

Theorems in Probability

In this module, we shall prove some theorems which help us to evaluate the probabilities of some complicated events in a rather simple way. In proving these theorems, we shall follow the axiomatic approach based on the three axioms given in axiomatic definition of probability in module 1.3 on definitions of probability.

In a problem on probability, we are required to evaluate probability of certain statements. These statements can be expressed in terms of set notation and whose probabilities can be evaluated using theorems in probability. Let A and B be two events in S. Certain statements in set notation are given in the following table.

S. No.	Statement	Set notation
1.	At least one of the events A or B occurs	$A \cup B$
2.	Both the events A and B occur	$A \cap B$
3.	Neither A nor B occurs	$\bar{A} \cap \bar{B}$
4.	Event A occurs and B does not occur	$A \cap \bar{B}$
5.	Exactly one of the events A or B occurs	$(\bar{A} \cap B) \cup (A \cap \bar{B}) = A \Delta B$
6.	Not more than one of the events A or B occurs	$(A \cap \bar{B}) \cup (\bar{A} \cap B) \cup (\bar{A} \cap \bar{B})$
7.	If event A occurs, so does B	$A \subset B$
8.	Events A and B are mutually exclusive	$A \cap B = \phi$
9.	Complement of event A	$\bar{A}$
10.	Sample space	S

Theorems on Probability

Theorem 1: Probability of the impossible event is zero, i.e., $P(\phi) = 0$.

Proof: We know that $S \cup \phi = S \Rightarrow P\ (S) = P(S \cup \phi)$

$$\Rightarrow P(S) = P(S) + P\ (\phi) \text{ (Axiom 3)}$$

$$\Rightarrow P(\phi) = 0$$

Theorem 2: Probability of the complementary event $\bar{A}$ of A is given by $P(\bar{A}) = 1 - P(A)$.

Proof: Since A and $\bar{A}$ are mutually exclusive events in S,

$$A \cup \bar{A} = S \Rightarrow P(A \cup \bar{A}) = P(S) \Rightarrow P(A) + P(\bar{A}) = 1 \text{ (Axioms 2 and 3)}$$

$$\Rightarrow P(\bar{A}) = 1 - P(A)$$

Corollary 1: $\mathbf{0 \leq P(A) \leq 1}$

Proof: We have $P(A) = 1 - P(\bar{A}) \leq 1$ ($\because P(\bar{A}) \geq 0$, by Axiom 1)

Further, $P(A) \geq 0$ (by Axiom 1). Therefore, $0 \leq P(A) \leq 1$

Corollary 2: $\boldsymbol{P(\phi) = 0}$

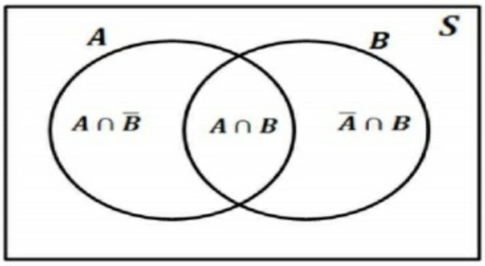
Proof: Since $\phi = \bar{S}$, $P(\phi) = P(\bar{S}) = 1 - P(S) = 1 - 1 = 0$ (by Axiom 2)

$$\Rightarrow P(\phi) = 0$$

Theorem 3: For any two events $\boldsymbol{A}$ and $\boldsymbol{B}$, we have

(i) $\boldsymbol{P(\bar{A} \cap B) = P(B) - P(A \cap B)}$ **(ii)** $\boldsymbol{P(A \cap \bar{B}) = P(A) - P(A \cap B)}$

Proof:



(i) From the Venn diagram, we have,

$$B = (A \cap B) \cup (\bar{A} \cap B),$$

where $(\bar{A} \cap B)$ and $(A \cap B)$ are mutually exclusive events. Hence by Axiom 3,

$$P(B) = P(A \cap B) + P(\bar{A} \cap B)$$

$$\boldsymbol{\Rightarrow P(\bar{A} \cap B) = P(B) - P(A \cap B)}$$

(ii) Similarly, we have,

$$A = (A \cap B) \cup (A \cap \bar{B}),$$

where $(A \cap B)$ and $(A \cap \bar{B})$ are mutually exclusive events. Hence by Axiom 3

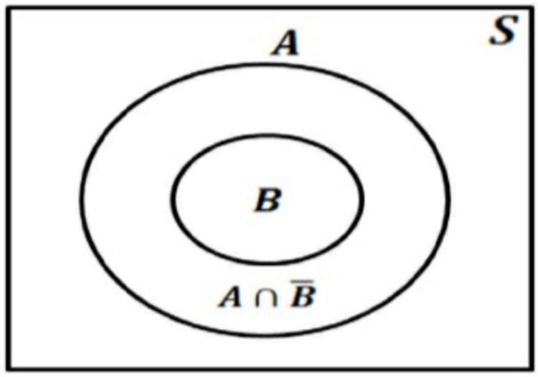
$$P(A) = P(A \cap B) + P(A \cap \bar{B})$$

$$\boldsymbol{\Rightarrow P(A \cap \bar{B}) = P(A) - P(A \cap B)}$$

Theorem 4: If $B \subset A$, then

(i) $\boldsymbol{P(A \cap \bar{B}) = P(A) - P(B)}$ **(ii)** $\boldsymbol{P(B) \leq P(A)}$

Proof:



(i) If $B \subset A$, then B and $A \cap \bar{B}$ are mutually exclusive events and

$$A = B \cup (A \cap \bar{B})$$

$$\Rightarrow P(A) = P(B) + P(A \cap \bar{B}) \text{ (Axiom 3)}$$

$$\Rightarrow P(A \cap \bar{B}) = P(A) - P(B)$$

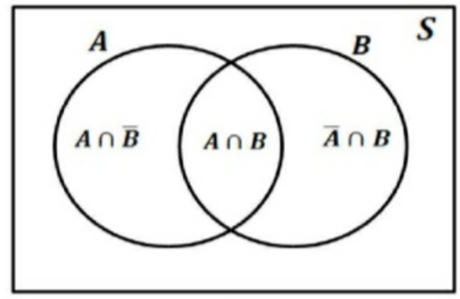
(ii) We have $P(A \cap \bar{B}) \geq 0$ (Axiom 1). Hence $P(A) - P(B) \geq 0 \Rightarrow P(B) \leq P(A)$.

Thus, $B \subset A \Rightarrow P(B) \leq P(A)$.

Theorem 5: Addition Theorem of Probability for Two Events:

Let A and B be any two events in S. Then

$$\boldsymbol{P(A \cup B) = P(A) + P(B) - P(A \cap B)}$$



Proof: From Venn diagram, we have

$$A \cup B = A \cup (\bar{A} \cap B)$$

where A and $\bar{A} \cap B$ are mutually exclusive events in S.

$\therefore P(A \cup B) = P(A) + P(\bar{A} \cap B)$ (Axiom 3)

$= P(A) + P(B) - P(A \cap B)$ (From Theorem 3)

Thus, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

Note:

1. If A and B are mutually exclusive events then $A \cap B = \phi$ and hence $P(A \cap B) = P(\phi) = 0$. Thus, if A and B are mutually exclusive events, then $P(A \cup B) = P(A) + P(B)$.
6. The addition theorem of probability for three events is given by

$$\boldsymbol{P(A \cup B \cup C) =}$$

$$\boldsymbol{P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)}$$

This can be proved first by taking $A \cup B$ as one event and C as second event and repeated application of Theorem 5

$$P(A \cup B \cup C) = P((A \cup B) \cup C) = P(A \cup B) + P(C) - P((A \cup B) \cap C)$$

$$= P(A \cup B) + P(C) - P((A \cap C) \cup (B \cap C))$$

$$= P(A) + P(B) - P(A \cap B) + P(C) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$$

$$= P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$$

① A bag contains 12 balls numbered from 1 to 12. If a ball is drawn randomly what is the prob. of having a ball with a number which is multiple of either 2 or 3?

Sol. $n(S) = 12$

let A be the Event whose are multiples of 2 $= \{2, 4, 6, 8, 10, 12\}$

let B be the Event whose are multiples of 3 $= \{3, 6, 9, 12\}$

$A \cap B = \{6, 12\}$

$P(A) = \frac{6}{12}$, $P(B) = \frac{4}{12}$, $P(A \cap B) = \frac{2}{12}$

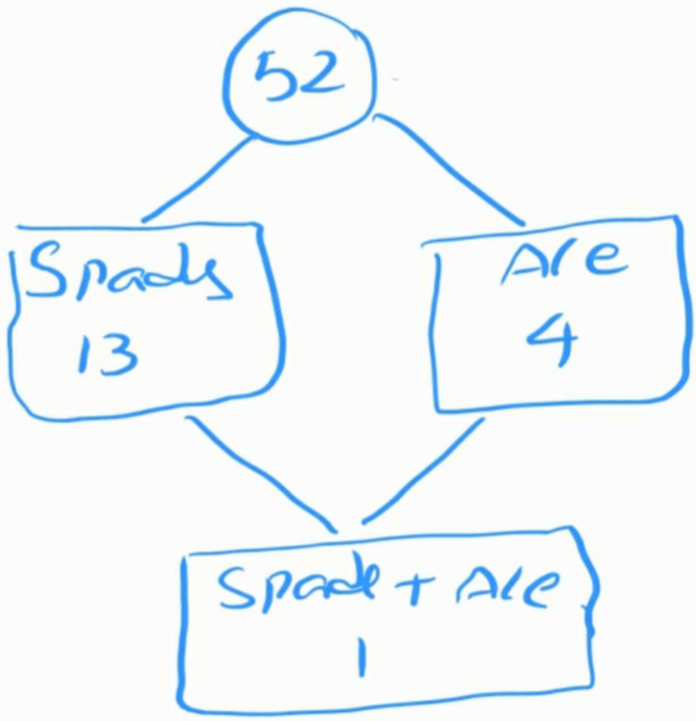
P(multiple of either 2 or 3) =

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= \frac{6}{12} + \frac{4}{12} - \frac{2}{12} = \frac{8}{12} = \frac{2}{3}$$

(2) A card is drawn from a well shuffled deck. what is the prob that it is either Spade or an ace

Sol



$1/52$

$13/52$

$4/52$

$1/52$

$n(S) = 52$

let A be the Event whose cards are Spades $n(A) = 13$

let B be the Event whose cards are Aces $n(B) = 4$

Here $n(A \cap B) = 1$

$P(A) = \frac{13}{52}$, $P(B) = \frac{4}{52}$,

$P(A \cap B) = \frac{1}{52}$

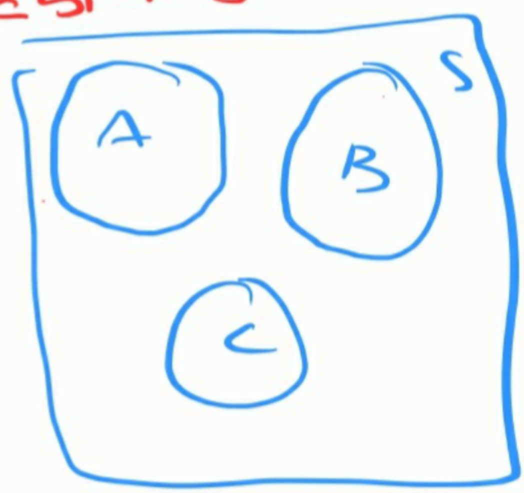
P(getting either spade or Ace)

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= \frac{13}{52} + \frac{4}{52} - \frac{1}{52} = \frac{16}{52} = \frac{4}{13}$

GATE-2010

③ 3 students A, B, C are in running race, A & B have the same probl of winning, and each is twice as likely to win as C. Then find the probability that B or C win



Sol. Here

$A \cup B \cup C = S$

$A \cap B \cap C = \phi$

$P(A \cup B \cup C) = P(S)$

$$\boxed{P(A) + P(B) + P(C) = 1}$$

Given
$$\left.\begin{aligned} P(A) &= P(B) \\ P(A) &= 2P(C) \\ P(B) &= 2P(C) \end{aligned}\right\}$$

$$2P(C) + 2P(C) + P(C) = 1$$

$$5P(C) = 1$$

$$P(C) = \frac{1}{5}$$

$$P(A) = \frac{2}{5}, \quad P(B) = \frac{2}{5}$$

$$\begin{aligned} P(B \text{ or } C) &= P(B \cup C) \\ &= P(B) + P(C) - P(B \cap C) \\ &= \frac{2}{5} + \frac{1}{5} - 0 \end{aligned}$$

$$\boxed{P(B \text{ or } C) = 3/5}$$

③ If $P(A) = \frac{1}{2}$, $P(B) = \frac{1}{3}$

$P(A \cap B) = \frac{1}{5}$ Then find

(1) $P(A \cup B)$ (2) $P(A^c \cap B)$ (3) $P(A \cap B^c)$

(4) $P(A^c \cap B^c)$ (5) $P(A^c \cup B^c)$ (6) $P(A^c)$

(7) $P(B^c)$

Sol: Given $P(A) = \frac{1}{2}$, $P(B) = \frac{1}{3}$

$P(A \cap B) = \frac{1}{5}$

① $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= \frac{1}{2} + \frac{1}{3} - \frac{1}{5} = \frac{19}{30}$

② $P(A^c \cap B) = P(B) - P(A \cap B) =$

③ $P(A \cap B^c) = P(A) - P(A \cap B) =$

④ $P(A^c \cap B^c) = P((A \cup B)^c)$

$= 1 - P(A \cup B)$

$=$

(5) $P(A^c \cup B^c) = 1 - P(A \cap B) =$

(6) $P(A^c) = 1 - P(A) =$

(7) $P(B^c) = 1 - P(B) =$

Solve the following problems

(1) From a city 3 news papers A, B, C are published. A is read by 20%, B is read by 16%, C is read by 14%, Both A and B read by 8%, both A and C read by 5%, both B and C read by 4%, and all three read by 2%, what is the percentage of the population that read at least one paper.

(2) A die is thrown randomly then wt is the prob. of getting a number which is multiple of either 2 or 3?

Example 7 : In a group there are 3 men and 2 women. Three persons are selected at random from this group. Find the probability that one man and two women or two men and one women are selected.

Solution : Let S be the sample space associated with the selection of 3 persons out of 5.

$\therefore \quad n(S) = {}^5C_3 = 10$

Let E_1 be the event of selecting 1 man and 2 women

$\therefore \quad n(E_1) = {}^3C_1 \times {}^2C_2$

Let E_2 be the event of selecting 2 men and a women

$\therefore \quad n(E_2) = {}^3C_2 \times {}^2C_1 = 6$

But $E_1 \cap E_2 = \phi$. i.e., E_1, E_2 are mutually exclusive.

Now $E_1 \cup E_2$ is the event of selecting 1 man and 2 women or 2 men or 1 woman.

$$\therefore \quad P(E_1 \cup E_2) = P(E_1) + P(E_2) = \frac{n(E_1)}{n(S)} + \frac{n(E_2)}{n(S)} = \frac{3}{10} + \frac{6}{10} = \frac{9}{10}$$

Example 8 : 60 boys and 20 girls are there in a class. Half of the boys and half of the girls of the class play cricket. Find the probability of the selected person to be a "boy" or "a girl who plays cricket".

Solution : Class (S) consists of 60 boys and 20 girls $\Rightarrow n(S) = 80$

Let E_1 be the event of selecting a boy and E_2 be the event of selecting a girl who play cricket. Half of the boys and half of the girls play cricket.

$\therefore \quad n(E_1) = 30 \quad$ and $\quad n(E_2) = 10$

$$\therefore \; P(E_1) = \frac{n(E_1)}{n(S)} = \frac{30}{80} \text{ and } P(E_2) = \frac{n(E_2)}{n(S)} = \frac{10}{80}$$

Also E_1 or $E_2 = E_1 \cup E_2$

is the event of selecting a 'boy' or a 'girl' who plays cricket and $E_1 \cap E_2 = \phi$

$$\therefore \quad P(E_1 \text{ or } E_2) = P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$$

$$= \frac{30}{80} + \frac{10}{80} - 0 = \frac{40}{80} = \frac{1}{2}$$

Conditional probability:

If E_1 and E_2 are any two event in a sample space 'S' and $P(E_1) \neq 0$ then the event E_2 occurs only after the event E_1 occurs, then the probability of E_2 given by E_1 is called conditional probability and it is denoted by $P\left(\frac{E_2}{E_1}\right)$ and is defined as

$$\boxed{P\left(\frac{E_2}{E_1}\right) = \frac{P(E_1 \cap E_2)}{P(E_1)}} \quad * (i)$$

Illy $P\left(\frac{E_1}{E_2}\right) = \frac{P(E_1 \cap E_2)}{P(E_2)}$ * (ii)

Note: ① $P(E_1 \cap E_2) = P(E_1)\, P\left(\frac{E_2}{E_1}\right)$

② $P(E_1 \cap E_2) = P(E_2)\, P\left(\frac{E_1}{E_2}\right)$

Properties: let A, B, C are any Events in S

$$P\left(\frac{A}{B}\right) = \frac{P(A \cap B)}{P(B)}$$

① If $A \subseteq S$ $\quad A \cap S = A$

$$P\left(\frac{A}{S}\right) = \frac{P(A \cap S)}{P(S)} = \frac{P(A)}{1} = P(A)$$

$$\boxed{P\left(\frac{A}{S}\right) = P(A)}$$

② $$P\left(\frac{A}{A}\right) = \frac{P(A \cap A)}{P(A)} = \frac{P(A)}{P(A)} = 1$$

$$\boxed{P\left(\frac{A}{A}\right) = 1}$$

③ $$P\left(\frac{\phi}{A}\right) = \frac{P(\phi \cap A)}{P(A)} = \frac{P(\phi)}{P(A)} = 0$$

$$\boxed{P\left(\frac{\phi}{A}\right) = 0}$$

④ $P\left(\frac{A}{B^c}\right) = \frac{P(A\cap B^c)}{P(B^c)} = \boxed{\frac{P(A) - P(A\cap B)}{1 - P(B)}}$

If $A\cap B = \phi \Rightarrow P(A\cap B) = 0$

Then $\boxed{P\left(\frac{A}{B^c}\right) = \frac{P(A)}{1-P(B)}}$

⑤ $P\left(\frac{A^c}{B}\right) = \frac{P(A^c\cap B)}{P(B)} = \boxed{\frac{P(B) - P(A\cap B)}{P(B)}}$

If $A\cap B = \phi \Rightarrow P(A\cap B) = P(\phi) = 0$

Then $\boxed{P\left(\frac{A^c}{B}\right) = 1}$

⑥ $P\left(\frac{A^c}{B^c}\right) = \frac{P(A^c\cap B^c)}{P(B^c)} = \frac{P(A\cup B)^c}{P(B^c)}$

Demorgan laws

$(A\cup B)^c = A^c \cap B^c$

$P(A^c) = 1 - P(A)$

$$\boxed{P\left(\frac{A^c}{B^c}\right) = \frac{1 - P(A \cup B)}{1 - P(B)}}$$

||ly $$P\left(\frac{B^c}{A^c}\right) = \frac{1 - P(A \cup B)}{1 - P(A)}$$

(7) If $\underline{A \cap B = \Phi}$ Then $P\left(\frac{A}{A \cup B}\right)$

$$P\left(\frac{A}{A \cup B}\right) = \frac{P(A \cap (A \cup B))}{P(A \cup B)}$$

$$= \frac{P\{[(A \cap A) \cup (A \cap B)]}{P(A \cup B)}$$

$$= \frac{P(A \cup (A \cap B))}{P(A \cup B)}$$

$$= \frac{P(A) + P(A \cap B) - P(A \cap (A \cap B))}{P(A) + P(B) - P(A \cap B)}$$

$$= \frac{P(A) + 0 - 0}{P(A) + P(B) - 0}$$

$$\boxed{P\left(\frac{A}{A \cup B}\right) = \frac{P(A)}{P(A) + P(B)}}$$

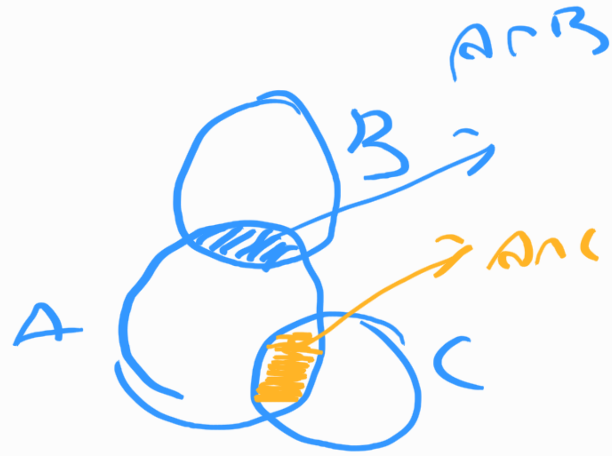
8 If $\underline{B \cap C = \Phi}$ Then $P\left(\frac{B \cup C}{A}\right)$

$$P\left(\frac{B \cup C}{A}\right) = \frac{P((B \cup C) \cap A)}{P(A)}$$

$$= \frac{P\left[(A \cap B) \cup (A \cap C)\right]}{P(A)}$$

$$= \frac{P(A \cap B) + P(A \cap C)}{P(A)}$$

$$= \frac{P(A \cap B)}{P(A)} + \frac{P(A \cap C)}{P(A)}$$



$B \cap C = \Phi$

$(A \cap B) \cap (A \cap C) = \Phi$

$= P(B/A) + P(C/A)$

$$\boxed{P\left(\frac{B\cup C}{A}\right) = P\left(\frac{B}{A}\right) + P\left(\frac{C}{A}\right)}$$

(9) If $A \subseteq B$ Then C is any event

$A \subseteq B \Rightarrow A\cap C \subseteq B\cap C$

$\Rightarrow P(A\cap C) \leq P(B\cap C)$

$$\Rightarrow \frac{P(A\cap C)}{P(C)} \leq \frac{P(B\cap C)}{P(C)}.$$

$$\Rightarrow P\left(\frac{A}{C}\right) \leq P\left(\frac{B}{C}\right)$$

$$\boxed{\begin{array}{l}\text{If } A \subseteq B \text{ Then } C \\ P\left(\frac{A}{C}\right) \leq P\left(\frac{B}{C}\right)\end{array}}$$

(10) If $A \cap B = \phi$ Then

$$P\left(\frac{A}{B}\right) = \frac{P(A \cap B)}{P(B)}$$

$$= \frac{0}{P(B)} = 0$$

$$\boxed{\text{If } A \cap B = \phi \quad P\left(\frac{A}{B}\right) = 0 \qquad \text{Illy } P\left(\frac{B}{A}\right) = 0}$$

Problems on Conditional probability

① If $P(A^c) = \frac{3}{8}$, $P(B^c) = \frac{1}{2}$ and

$P(A \cap B) = \frac{1}{4}$ Then find

① $P\left(\frac{A}{B}\right)$ ② $P\left(\frac{B}{A}\right)$ ③ $P\left(\frac{A^c}{B^c}\right)$

④ $P\left(\frac{B^c}{A^c}\right)$ ⑤ $P\left(\frac{A^c}{B}\right)$ ⑥ $P\left(\frac{A}{B^c}\right)$

Sol: Here $P(A^c) = \frac{3}{8}$, $P(B^c) = \frac{1}{2}$, $P(A \cap B) = \frac{1}{4}$

$$P(A^c) = 1 - P(A)$$

$$P(A) = 1 - P(A^c)$$

$$= 1 - 3/8 = \frac{5}{8}$$

$$P(B) = 1 - P(B^c) = 1 - \frac{1}{2} = \frac{1}{2}$$

Hence $P(A) = \frac{5}{8}$, $P(B) = \frac{1}{2}$, $P(A \cap B) = \frac{1}{4}$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= \frac{5}{8} + \frac{1}{2} - \frac{1}{4} = \frac{7}{8}$$

① $P\left(\frac{A}{B}\right) = \frac{P(A\cap B)}{P(B)} = \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2}$

② $P\left(\frac{B}{A}\right) = \frac{P(A\cap B)}{P(A)} = \frac{\frac{1}{4}}{5/8} = \frac{2}{5}$

③ $P\left(\frac{A^c}{B^c}\right) = \frac{P(A^c\cap B^c)}{P(B^c)} = \frac{P((A\cup B)^c)}{P(B^c)}$

$= \frac{1-P(A\cup B)}{1-P(B)} = \frac{1-\frac{7}{8}}{1-\frac{1}{2}} = \frac{\frac{1}{8}}{\frac{1}{2}} = \frac{1}{4}$

④ $P\left(\frac{B^c}{A^c}\right) = \frac{P(A^c\cap B^c)}{P(A^c)} = \frac{P((A\cup B)^c)}{P(A^c)}$

$= \frac{1-P(A\cup B)}{1-P(A)} = \frac{1-7/8}{1-\frac{5}{8}} = \frac{1/8}{3/8} = \frac{1}{3}$

⑤ $P\left(\frac{A^c}{B}\right) = \frac{P(A^c\cap B)}{P(B)} = \frac{P(B)-P(A\cap B)}{P(B)}$

$= 1 - \frac{P(A\cap B)}{P(B)} = 1 - P\left(\frac{A}{B}\right) = 1-\frac{1}{2} = \frac{1}{2}$

⑥ $P\left(\frac{A}{B^c}\right) = \frac{P(A \cap B^c)}{P(B^c)} = \frac{P(A) - P(A \cap B)}{1 - P(B)}$

$= \frac{\frac{5}{8} - \frac{1}{4}}{1 - \frac{1}{2}} = \frac{\frac{3}{8}}{\frac{1}{2}} = \frac{3}{4}$

Solve

② If $P(A) = \frac{1}{2}$, $P(B) = \frac{1}{3}$

$P(A \cap B) = \frac{1}{5}$ Then find

① $P\left(\frac{A}{B}\right)$ ② $P\left(\frac{B}{A}\right)$ ③ $P\left(\frac{A^c}{B^c}\right)$

④ $P\left(\frac{B^c}{A^c}\right)$ ⑤ $P\left(\frac{A^c}{B}\right)$ ⑥ $P\left(\frac{A}{B^c}\right)$

③ let a die be rolled and $A = \{1, 3, 5\}$, $B = \{2, 3\}$, $C = \{2, 3, 4, 5\}$ be the events associated with the random experiment. Then find $P\left(\frac{A}{B}\right)$, $P(B/A)$, $P(B/C)$, $P\left(\frac{C}{B}\right)$, $P\left(\frac{A}{C}\right)$, $P(C/A)$

Sol. $S = \{1, 2, 3, 4, 5, 6\}$ $n(S) = 6$

$A = \{1, 3, 5\}$, $B = \{2, 3\}$ $C = \{2, 3, 4, 5\}$

$A \cap B = \{3\}$, $B \cap C = \{2, 3\}$, $C \cap A = \{3, 5\}$

$P(A) = \frac{3}{6}$, $P(B) = \frac{2}{6}$, $P(C) = \frac{4}{6}$

$P(A \cap B) = \frac{1}{6}$ $P(B \cap C) = \frac{2}{6}$, $P(C \cap A) = \frac{2}{6}$

(i) $P\left(\frac{A}{B}\right) = \frac{P(A \cap B)}{P(B)} = \frac{1/6}{2/6} = \frac{1}{2}$

(ii) $P\left(\frac{B}{A}\right) = \frac{P(A \cap B)}{P(A)} = \frac{1/6}{3/6} = \frac{1}{3}$

(iii) $P\left(\frac{B}{C}\right) = \frac{P(B \cap C)}{P(C)} = \frac{2/6}{4/6} = \frac{1}{2}$

(iv) $P(C/B) = \dfrac{P(B \cap C)}{P(B)} = \dfrac{2/6}{2/6} = 1$

(v) $P(A/C) = \dfrac{P(A \cap C)}{P(C)} = \dfrac{2/6}{4/6} = \dfrac{1}{2}$

(vi) $P(C/A) = \dfrac{P(A \cap C)}{P(A)} = \dfrac{2/6}{3/6} = \dfrac{2}{3}$

Independent Events

If the occurence of the Event E_2 is not effected by the occurence (or) non occurence of the event E_1 then the Event E_2 is said to be independent of E_1 and is defined by

$$* \boxed{P\left(\frac{E_2}{E_1}\right) = P(E_2)} *$$

lly $P\left(\frac{E_1}{E_2}\right) = P(E_1)$

Note: ① Suppose E_1 and E_2 are two independent events in 'S'

inde def
(i) $P\left(\frac{E_2}{E_1}\right) = P(E_2)$

cond. p def
(ii) $P\left(\frac{E_2}{E_1}\right) = \frac{P(E_1 \cap E_2)}{P(E_1)}$

LHS Equal

Hence RHS Equal

$$P(E_2) = \frac{P(E_1 \cap E_2)}{P(E_1)}$$

$$\boxed{P(E_1 \cap E_2) = P(E_1)\, P(E_2)}$$

let A and B are two independent Events

(i) $P\left(\frac{A}{B}\right) = P(A)$

(ii) $P\left(\frac{B}{A}\right) = P(B)$

(iii) $P(A \cap B) = P(A)\, P(B)$

A, B independent Events	A, B Disjoint Exclusive Even
$P(A\cap B) = P(A)\,P(B)$	$P(A\cap B) = P(\phi) = 0$
$P(A\cup B) = P(A) + P(B) - P(A)\,P(B)$	$P(A\cup B) = P(A) + P(B)$

② If $E_1\ E_2\ E_3 \ldots E_n$ are n mutually independent Events

$$\left\{\begin{array}{l} P(E_1\cap E_2) = P(E_1)\,P(E_2) \\ P(E_2\cap E_3) = P(E_2)\,P(E_3) \\ P(E_i\cap E_j) = P(E_i)\,P(E_j)\ \forall i,j \end{array}\right.$$

$$P(E_1\cap E_2\cap E_3) = P(E_1)\,P(E_2)\,P(E_3)$$

$$P(E_1\cap E_2\cap \ldots E_n) = P(E_1)\,P(E_2) \ldots P(E_n)$$

(3) If E_1 E_2 E_3 are 3 mutually independent events

$$\boxed{\begin{aligned} P(E_1 \cup E_2 \cup E_3) &= P(E_1) + P(E_2) + P(E_3) - P(E_1)P(E_2) - P(E_2)P(E_3) \\ &\quad - P(E_3)P(E_1) + P(E_1)P(E_2)P(E_3) \end{aligned}}$$

we know that

$$P(A) = 1 - P(A^c)$$

$$P(E_1 \cup E_2 \cup E_3) = 1 - P((E_1 \cup E_2 \cup E_3)^c)$$

$$= 1 - P(E_1^c \cap E_2^c \cap E_3^c)$$

$$= 1 - P(E_1^c)\, P(E_2^c)\, P(E_3^c)$$

$$\boxed{P(E_1 \cup E_2 \cup E_3) = 1 - P(E_1^c)\, P(E_2^c)\, P(E_3^c)}$$

only when E_1, E_2, E_3 are independent

① A can hit a target 3 times in 5 shots, B hits the target 2 times in 5 shots, C hits the target 3 times in 4 shots. Find the prob" of target being hit when all of them try:

Sol Here A, B, C are mutually independent events

A can hit a target 3 times in 5 shots

$$P(A) = \frac{3}{5} ; P(A^c) = 1 - P(A) = 1 - \frac{3}{5} = \frac{2}{5}$$

B can hit a target 2 times in 5 shots

$$P(B) = \frac{2}{5}, \quad P(B^c) = \frac{3}{5}$$

C can hit a target 3 times in 4 shots

$$P(C) = \frac{3}{4}, \quad P(C^c) = \frac{1}{4}$$

When all of them try

$$P(A \cup B \cup C) = 1 - P(A^c)\,P(B^c)\,P(C^c)$$

$$= 1 - \frac{2}{5} \times \frac{3}{5} \times \frac{1}{4} = \frac{47}{50}$$

② The probabilities that students A, B, C, D solve a problem are $\frac{1}{3}$, $\frac{2}{5}$, $\frac{1}{5}$, $\frac{1}{4}$ respectively. If all of them to solve the problem. What is the prob. that the problem is solved.

③ In a certain town 40% have brown hair, and 25% have brown eyes and 15% have both brown hair and brown eyes. A person is selected at randomly from the town

(i) If he has brown hair (A), what is the prob. that he has brown eyes (B) also

$$P\left(\frac{B}{A}\right)$$

(ii) If he has brown eyes (B), determine the prob. that he does not have brown hair (A).

$$P\left(\frac{A^c}{B}\right)$$

Sy. let A be the Event whose are Brown hair $P(A) = \frac{40}{100}$

let B be the Event whose are Brown eyes $P(B) = \frac{25}{100}$

Now $P(A \cap B) = \frac{15}{100}$

Now $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= \frac{40}{100} + \frac{25}{100} - \frac{15}{100} = \frac{50}{100} = \frac{1}{2}$

i $P(B/A)$:

$P(B/A) = \frac{P(A \cap B)}{P(A)} = \frac{15/100}{40/100} = \frac{15}{40} = 3/8$

(ii) $P\left(\frac{A^c}{B}\right)$

$P\left(\frac{A^c}{B}\right) = \frac{P(A^c \cap B)}{P(B)} = \frac{P(B) - P(A \cap B)}{P(B)}$

$= \frac{\frac{25}{100} - \frac{15}{100}}{\frac{25}{100}} = \frac{10}{25} = \frac{2}{5}$

Total probability

$S = \{E_1\ E_2\ E_3\ E_4\}$

mutually exclusive events

exhaustive events

$E_i \cap E_j = \phi \quad \forall i, j$

$E_1 \cap E_2 \cap E_3 \cap E_4 = \phi$

$E_1 \cup E_2 \cup E_3 \cup E_4 = S$

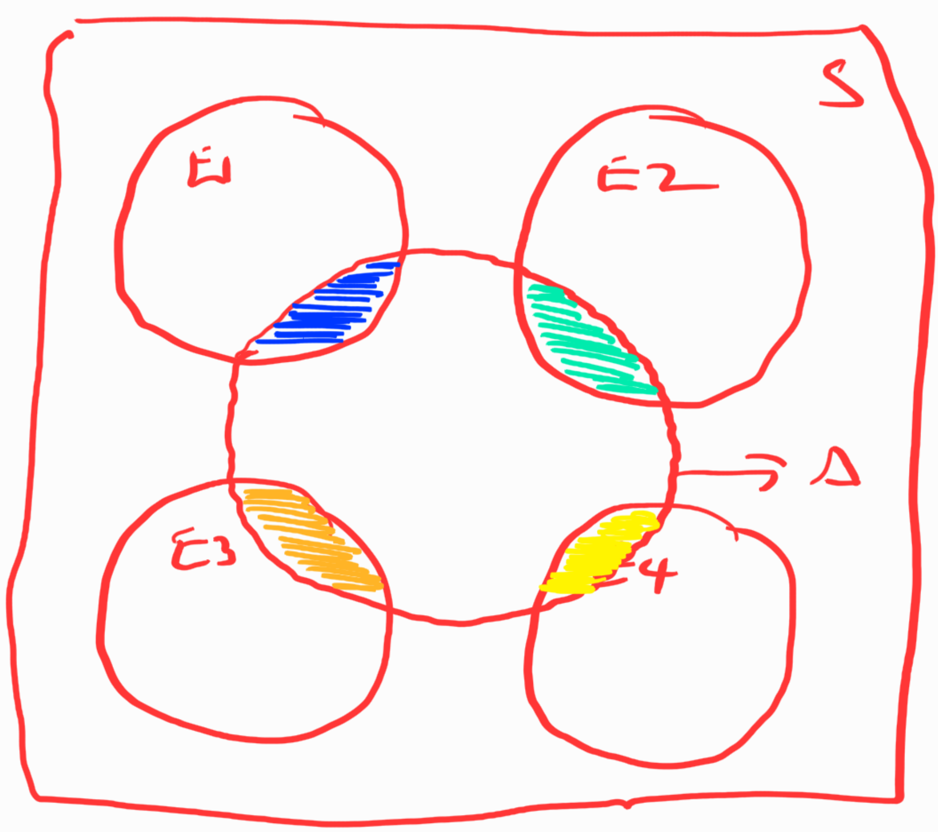
Let A be any other event in S which is intersect with E_i

$$\left.\begin{array}{l} A \cap E_1 \\ A \cap E_2 \\ A \cap E_3 \\ A \cap E_4 \end{array}\right\}$$

key

$(A \cap E_i) \cap (A \cap E_j) = \phi$

$\forall i, j$



NOW

$$\boxed{\begin{aligned}P(A) = {} & P(E_1)\,P(A/E_1) + P(E_2)\,P(A/E_2) \\ & + P(E_3)\,P(A/E_3) + P(E_4)\,P(A/E_4)\end{aligned}}$$

$$\boxed{P(A) = \sum_{i=1}^{n} P(E_i)\,P(A/E_i)}$$

Total probability

Total probability : If $E_1, E_2, \ldots, E_n$ are n mutually exclusive and exhaustive events in a sample space S and A be any other event which is intersect with E_i where $i = 1, 2, \ldots, n$

Then

$$P(A) = P(E_1)\,P(A/E_1) + P(E_2)\,P(A/E_2) + \cdots + P(E_n)\,P(A/E_n)$$

$$\boxed{P(A) = \sum_{i=1}^{n} P(E_i)\,P(A/E_i)}$$

① 3 Machines A, B, C produce 40%, 30% and 30% Total no. of items of the factory. The percentage of defective items of those 3 machines 4%, 2%, 3%. If an item is selected at randomly find the prob. that the item is defective.

Sol- given A, B, C three machines

w.D be the defective item is each Event

A	B	C
40%	30%	30%
4%	2%	3%

$P(A) = \frac{40}{100}$,

$P(B) = \frac{30}{100}$

$P(C) = \frac{30}{100}$

Now $P(D/A) = \frac{4}{100}$, $P(D/B) = \frac{2}{100}$, $P\left(\frac{D}{C}\right) = \frac{3}{100}$

By Total probability theorem

$$P(D) = P(A)\,P(D/A) + P(B)\,P(D/B) + P(C)\,P(D/C)$$

$$= \frac{40}{100} \times \frac{4}{100} + \frac{30}{100} \times \frac{2}{100} + \frac{30}{100} \times \frac{3}{100}$$

$$= \frac{16}{1000} + \frac{6}{1000} + \frac{9}{1000}$$

$$= \frac{31}{1000} = 0.031$$

$$\boxed{P(D) = 0.031}$$

(2) 3 Companies A, B, C produce 20%, 25%, 55% cars respectively. It is known that 2%, 3%, 4% of the cars produced by A, B, C are defective. If a car is selected at randomly what is prob that it is defective:

Sol: $P(D) = \frac{67}{2000}$

$$P\left(\frac{B}{D}\right) =$$

$$= \frac{\frac{25}{100} \times \frac{3}{100}}{\frac{67}{2000}}$$

$$P\left(\frac{B}{D}\right) = \frac{15}{67}$$

lly $P\left(\frac{A}{D}\right)$ $\qquad$ $P\left(\frac{C}{D}\right)$

Bayes Theorem :

If $E_1 E_2 E_3$ —— E_n are n mutually exclusive and exhaustive events in a sample space S where $P(E_i)>0$ $i=1,2,\dots n$, and A is any other event in S which is intersecting with E_i $i=1,2,\dots n$ and $P(A)>0$ If E_i is any of the events of $E_1 E_2$ E_3 -- E_n where $P(E_1), P(E_2)$, ... $P(E_n)$ and $P(A/E_1)$, $P(A/E_2)$ ... $P(A/E_n)$ are known then

$$\boxed{P\left(\frac{E_i}{A}\right) = \frac{P(E_i)\,P(A/E_i)}{\sum_{i=1}^{n} P(E_i)\,P(A/E_i)}}$$

Proof. given $E_1 E_2 \cdots E_n$ are n

mutually exclusive and exhaustive

Events in S

(i) $E_i \cap E_j = \phi \quad \forall i, j$

(ii) $E_1 \cup E_2 \cup E_3 \cup \cdots \cup E_n = S$

$$\boxed{\bigcup_{i=1}^{n} E_i = S}$$

let A is another event which is

intersect $E_i \ \forall i$ in S

Here $A \subseteq S$

$$A \cap S = A$$

$$A = A \cap \bigcup_{i=1}^{n} E_i$$

$$= A \cap (E_1 \cup E_2 \cup \cdots \cup E_n)$$

$$= (A \cap E_1) \cup (A \cap E_2) \cup \cdots \cup (A \cap E_n)$$

$$A = \bigcup_{i=1}^{n} A \cap E_i$$

$$P(A) = \sum_{i=1}^{n} P(A \cap E_i) \longrightarrow (1)$$

Consider $P\left(\frac{A}{E_i}\right) = \frac{P(A \cap E_i)}{P(E_i)}$

$$P(A \cap E_i) = P(E_i)\, P\left(\frac{A}{E_i}\right) \rightarrow ②$$

Sub ② in ①

Hence $P(A) = \sum_{i=1}^{n} P(E_i)\, P\left(\frac{A}{E_i}\right) \rightarrow ③$

Consider $P\left(\frac{E_i}{A}\right) = \frac{P(A \cap E_i)}{P(A)}$

Sub ② & ③

$$\boxed{P\left(\frac{E_i}{A}\right) = \frac{P(E_i)\, P\left(\frac{A}{E_i}\right)}{\sum_{i=1}^{n} P(E_i)\, P\left(\frac{A}{E_i}\right)}}$$

Hence Theorem

Applications of Baye's Theorem

① In a factory Machine A produce 40% of out put and Machine B produce 60% of out put. on The avrge 9 items in 1000 produced by A is defective. and 1 item in 250 produced by B is defective. An item is drawn at randomly from a Day's out put & is defective. what is The probability That defective item is produced by B.

Sol. let A, B The Machines & D is The Defective

A	B
40%	60%
$\frac{9}{1000}$	$\frac{1}{250}$

$P(A) = \frac{40}{100}$, $P(B) = \frac{60}{100}$

$P(D/A) = \frac{9}{1000}$, $P(D/B) = \frac{1}{250}$

By Total probability Theorem

$$P(D) = P(A)\,P(D/A) + P(B)\,P(D/B)$$

$$= \frac{40}{100} \times \frac{9}{1000} + \frac{60}{100} \times \frac{1}{250} = 0.006$$

$$\boxed{P(D) = \frac{6}{1000}}$$

By Baye's Theorem

$$P\left(\frac{B}{D}\right) = \frac{P(B)\,P(D/B)}{P(D)}$$

$$= \frac{\cancel{60}}{\cancel{100}} \times \frac{1}{25\cancel{0}} \times \frac{100\cancel{0}}{6}$$

$$\boxed{P\left(\frac{B}{D}\right) = \frac{10}{25}} = \frac{2}{5}$$

$$\boxed{P\left(\frac{B}{D}\right) = \frac{2}{5}}$$

$$\text{lly} \quad P\left(\frac{A}{D}\right) = \frac{P(A)\,P(D/A)}{P(D)}$$

$$= \frac{\overset{2}{\cancel{40}}}{\cancel{100}} \times \frac{\overset{3}{\cancel{9}}}{\cancel{1000}} \times \frac{\cancel{1000}}{\underset{3}{\cancel{6}}} = \frac{3}{5}$$

$$\boxed{P\left(\frac{A}{D}\right) = \frac{3}{5}}$$

No

$$\boxed{P\left(\frac{A}{D}\right) + P\left(\frac{B}{D}\right) = 1}$$

Example 1: In a bolt factory machines A, B, C manufacutre respectively 25%, 35% and 40% of the total. Of their output $5, 4, 2$ percent are known to be defective bolts. A bolt is drawn at random from the product and is found to be defective. What are the probabilities that it was manufactured by

(i) Machine A.

(ii) Machine B or C

A	B	C
25%	35%	40%
5%	4%	2%

A, B, C are three machines

let D be the Defective

$P(A) = \frac{25}{100}$, $P(B) = \frac{35}{100}$,

$P(C) = \frac{40}{100}$

$P\left(\frac{D}{A}\right) = \frac{5}{100}$, $P\left(\frac{D}{B}\right) = \frac{4}{100}$, $P\left(\frac{D}{C}\right) = \frac{2}{100}$

Now By Total prob. the

$$P(D) = P(A)\,P(D/A) + P(B)\,P(D/B) + P(C)\,P(D/C)$$

$$= \frac{25}{100} \times \frac{5}{100} + \frac{35}{100} \times \frac{4}{100} + \frac{40}{100} \times \frac{2}{100}$$

$$= \frac{125}{10{,}000} + \frac{140}{10000} + \frac{80}{10{,}000}$$

$$\boxed{P(D) = \frac{345}{10{,}000}}$$

(i) By Bayes Theorem

$$P\left(\frac{A}{D}\right) = \frac{P(A)\ P(D|A)}{P(D)}$$

$$= \frac{25}{1\cancel{00}} \times \frac{5}{1\cancel{00}} \times \frac{1\cancel{0000}}{345}$$

$$\boxed{P\left(\frac{A}{D}\right) = \frac{125}{345}}$$

(ii)

Now $P\left(\frac{B \text{ or } C}{D}\right) = P\left(\frac{B \cup C}{D}\right)$

$$= P\left(\frac{B}{D}\right) + P\left(\frac{C}{D}\right)$$

Now $P\left(\frac{B}{D}\right) = \frac{P(B)\ P(D/B)}{P(D)}$

$$= \frac{35}{\cancel{100}} \times \frac{4}{\cancel{100}} \times \frac{\cancel{10000}}{345} = \frac{140}{345}$$

$\boxed{P(B/D) = \frac{140}{345}}$. |||ly $\boxed{P\left(\frac{C}{D}\right) = \frac{80}{345}}$

$$P\left(\frac{B \text{ or } C}{D}\right) = \frac{140}{345} + \frac{80}{345} = \frac{220}{345}$$

Note:

$$\boxed{P\left(\frac{A}{D}\right) + P\left(\frac{B}{D}\right) + P\left(\frac{C}{D}\right) = 1}$$

Example 4: The contents of urns I, II and III are respectively as follows:

1 white, 2 black and 3 red balls,

2 white, 1 black and 1 red balls, and

4 white, 5 black and 3 red balls.

One urn is chosen at random and two balls drawn. They happen to be white and red. What is the probability that they came from urns I, II, III?

$P(A) = \frac{1}{3} \quad P(B) = \frac{1}{3} \quad P(C) = \frac{1}{3}$ ✓

let D be the drawn two balls : one is white, one is red.

$$P(D/A) = \frac{{}^1C_1 \times {}^3C_1}{{}^6C_2} = \frac{1}{5}$$

$$P(D/B) = \frac{{}^2C_1 \times {}^1C_1}{{}^4C_2} = \frac{1}{3}$$

$$P(D|C) = \frac{{}^4C_1 \times {}^3C_1}{{}^{12}C_2} = \frac{12}{66}$$

Now

$$P(D) = P(A)\,P(D/A) + P(B)\,P(D/B) + P(C)\,P(D|C)$$

$$= \frac{1}{3}\left(\frac{1}{5}\right) + \frac{1}{3}\left(\frac{1}{3}\right) + \frac{1}{3}\left(\frac{12}{66}\right)$$

$$\boxed{P(D) = \frac{118}{495}}$$

Now $P\left(\frac{A}{D}\right) = \dfrac{P(A)\,P(D|A)}{P(D)}$

$$= \frac{1}{3} \times \frac{1}{5} \times \frac{495}{118} = \frac{33}{118}$$

$$P\left(\frac{B}{D}\right) = \frac{P(B)\,P(D|B)}{P(D)}$$

$$= \frac{1}{3} \times \frac{1}{3} \times \frac{495}{118} = \frac{55}{118}$$

$$P\left(\frac{C}{D}\right) = \frac{P(C)\,P(D|C)}{P(D)}$$

$$= \frac{1}{3} \times \frac{12}{66} \times \frac{495}{118} = \frac{30}{118}$$

$$\boxed{P\left(\frac{A}{D}\right) + P\left(\frac{B}{D}\right) + P\left(\frac{C}{D}\right) = \frac{33}{118} + \frac{55}{118} + \frac{30}{118} = \frac{118}{118} = 1}$$

$$\underset{\text{Total}}{495} \longrightarrow \underset{D}{118} \begin{cases} 33 \to \text{I} \\ 55 \to \text{II} \\ 30 \to \text{III} \end{cases}$$

③ Box 1 contains 1 white, 2 Red, 3 Green balls Box 2 contains 2 white, 3 red, 1 Green. Box 3 contains 3 white, 1 red, 2 Green.

one ball is is drawn from a box choosen at randomly. it is found to be red. Determine the probability that the ball is drawn from Box 2 and which is red!

→ let A, B, C are three boxes

Box I – A – 1W, 2R, 3G

Box II – B – 2W, 3R, 1G

Box III – C – 3W, 1R, 2G

let R be the red ball drawn

$P(A) = \frac{1}{3}, \quad P(B) = \frac{1}{3}, \quad P(C) = \frac{1}{3}$

$$P\left(\frac{R}{A}\right) = \frac{2C1}{6C1} = \frac{1}{3}$$

$$P\left(\frac{R}{B}\right) = \frac{3C1}{6C1} = \frac{1}{2}$$

$$P\left(\frac{R}{C}\right) = \frac{1C1}{6C1} = \frac{1}{6}$$

Hence By Total prob Th

$$P(R) = P(A)\,P(R/A) + P(B)\,P(R/B) + P(C)\,P(R/C)$$

$$= \frac{1}{3}\left(\frac{1}{3}\right) + \frac{1}{3}\left(\frac{1}{2}\right) + \frac{1}{3}\left(\frac{1}{6}\right)$$

$$\boxed{P(R) = \frac{1}{3}}$$

Now By Baye's Theorem

$$P\left(\frac{B}{R}\right) = \frac{P(B)\,P(R/B)}{P(R)}$$

$$= \frac{\cancel{\frac{1}{3}} \times \frac{1}{2}}{\cancel{\frac{1}{3}}} = \frac{1}{2}$$

$$\boxed{P(B/R) = \frac{1}{2}}$$

$||^{ly}$ $P\left(\frac{A}{R}\right) =$ $\qquad$ $P(C/R) =$

Solve The following problems

(d) $P(A \cup B) = 0.70$, ...

(e) $P(A \cup B) = 0.90$, $P(A \mid B) = 0.8$, $P(B) = 0.5$.

5. Give the correct label as answer like *a* or *b* etc., for the following questions :

(i) The probability of drawing any one spade card from a pack of cards is

(a) $\frac{1}{52}$ (b) $\frac{1}{13}$ (c) $\frac{4}{13}$ (d) $\frac{1}{4}$

(ii) The probability of drawing one white ball from a bag containing 6 red, 8 black, 10 yellow and 1 green balls is

(a) $\frac{1}{25}$ (b) 0 (c) 1 (d) $\frac{24}{25}$ (e) $\frac{15}{20}$

(iii) A coin is tossed three times in succession, the number of sample points in sample space is

(a) 6 (b) 8 (c) 3 (d) 9

(iv) In the simultaneous tossing of two perfect coins, the probability of having at least one head is

(a) $\frac{1}{2}$ (b) $\frac{1}{4}$ (c) $\frac{3}{4}$ (d) 1

(v) In the simultaneous tossing of two perfect dice, the probability of obtaining 4 as the sum of the resultant faces is

(a) $\frac{4}{12}$ (b) $\frac{1}{12}$ (c) $\frac{3}{12}$ (d) $\frac{2}{12}$

(vi) A single letter is selected at random from the word 'probability'. The probability that it is a vowel is

(a) $\frac{3}{11}$ (b) $\frac{2}{11}$ (c) $\frac{4}{11}$ (d) 0

(vii) An urn contains 9 balls, two of which are red, three blue and four black. Three balls are drawn at random. The chance that they are of the same colour is

(a) $\frac{5}{84}$ (b) $\frac{3}{9}$ (c) $\frac{3}{7}$ (d) $\frac{7}{17}$

(viii) A number is chosen at random among the first 120 natural numbers. The probability of the number chosen being a multiple of 5 or 15 is

(a) $\frac{1}{5}$ (b) $\frac{1}{8}$ (c) $\frac{1}{16}$ (d) $\frac{1}{9}$

(ix) If *A* and *B* are mutually exclusive events, then

(a) $P(A \cup B) = P(A) \cdot P(B)$, (b) $P(A \cup B) = P(A) + P(B)$, (c) $P(A \cup B) = 0$.

(x) If *A* and *B* are two independent events, the probability that both *A* and *B* occrs is $\frac{1}{8}$ and the probability that neither of them occurs is $\frac{3}{8}$. If $P(A) < P(B)$, then the probability of the occurrence of *A* is :

(a) $\frac{1}{2}$, (b) $\frac{1}{3}$, (c) $\frac{1}{4}$, (d) $\frac{1}{5}$.

(xi) If A and B are two events, the probability that exactly one of them occurs is given by :

(a) $P(A) + P(B) - 2P(A \cap B)$, (b) $P(A) + P(B) - P(A \cap B)$

(c) $P(\bar{A}) + P(\bar{B}) - 2P(\bar{A} \cap \bar{B})$, (d) $P(A \cap \bar{B}) + P(\bar{A} \cap B)$

(xii) If $P(A \cap B) = \frac{1}{2}$, $P(\bar{A} \cap \bar{B}) = \frac{1}{3}$, and $P(A) = P(B) = p$, then the value of p is given by :

(a) $\frac{1}{2}$, (b) $\frac{7}{8}$, (c) $\frac{1}{3}$, (d) $\frac{7}{12}$.

(xiii) If $P(A \cap B) = \frac{1}{2}$, $P(\bar{A} \cap \bar{B}) = \frac{1}{2}$ and $2P(A) = P(B) = p$, then the value of p is given by :

(a) $\frac{1}{4}$, (b) $\frac{1}{2}$, (c) $\frac{1}{3}$, (d) $\frac{2}{3}$.

(xiv) A and B are two independent events such that $P(A \cap \bar{B}) = \frac{3}{25}$ and $P(\bar{A} \cap B) = \frac{8}{25}$. If $P(A) < P(B)$, then $P(A)$ is :

(a) $\frac{1}{5}$, (b) $\frac{2}{5}$, (c) $\frac{3}{5}$, (d) $\frac{4}{5}$.

(xv) A and B are two independent events such that $P(\bar{A}) = 0{\cdot}7$, $P(\bar{B}) = k$ and $P(A \cup B) = 0{\cdot}8$, then k is

(a) $\frac{5}{7}$, (b) 1, (c) $\frac{2}{7}$, (d) none of these

(xvi) The probability that a 3-card hand drawn at random and without replacement from an ordinary deck consists entirely of black cards is :

(a) $\frac{1}{17}$, (b) $\frac{2}{17}$, (c) $\frac{1}{8}$, (d) $\frac{3}{17}$, (e) $\frac{4}{17}$.

(xvii) What is the probability that a bridge hand contains one card of each denomination (*i.e.*, one ace, one king, one queen, ..., one three, one two) ?

(a) $\frac{13!}{13^{13}}$, (b) $\frac{4^{13}}{^{52}C_{13}}$, (c) $\frac{^{52}C_4}{^{52}C_{13}}$ (d) $\left(\frac{1}{13}\right)^{13}$ (e) $\frac{13^4}{^{52}C_{13}}$

(xviii) If the events S and T have equal probability and are independent with $P(S \cap T) = p > 0$, then $P(S)$

(a) $\sqrt{p}$, (b) p^2, (c) $\frac{p}{2}$, (d) p, (e) $2p$

(xix) The probability that both S and T occur, the probability that S occurs and T does not, and the probability that T occurs and S does not are all equal to p. The probability that either S or T occurs is :

(a) p, (b) $2p$, (c) $3p$, (d) $3p^2$, (e) p^3

(xx) Events S and T are independent with $P(S) < P(T)$, $P(S \cap T) = 6/25$, and $P(S \mid T) + P(T \mid S) = 1$. Then $P(S)$ is

(a) $\frac{1}{25}$, (b) $\frac{1}{5}$, (c) $\frac{6}{25}$, (d) $\frac{2}{5}$, (e) $\frac{3}{5}$

(xxi) An unbiased die is thrown two independent times. Given that the first throw resulted in an even number, the probability that the sum obtained is 8 is :

(a) $\frac{5}{36}$, (b) $\frac{1}{6}$, (c) $\frac{4}{21}$, (d) $\frac{7}{36}$, (e) $\frac{1}{3}$

6. Fill in the blanks :

   (i) Two events are said to be equally likely if ......

   (ii) A set of events is said to be independent if .....

   (iii) If $P(A) . P(B) . P(C) = P(A \cap B \cap C)$, then the events A, B, C are .....

   (iv) Two events A and B are mutually exclusive if $P(A \cap B) =$ ... and are independent if $P(A \cap B) =$ ...

   (v) The probability of getting a multiple of 2 in a throw of a dice is $1/2$ and of getting a multiple of 3 is $1/3$. Hence probability of getting a multiple of 2 or 3 is ......

   (vi) Let A and B be independent events and suppose the event C has probability 0 or 1. Then A, B and C are ...... events.

   (vii) If A, B, C are pairwise independent and A is independent of $B \cup C$, then, A, B, C are ...... independent.

   (viii) A man has tossed 2 fair dice. The conditional probability that he has tossed two sixes, given that he has tossed at least one six is ......

   (ix) Let A and B be two events such that $P(A) = 0{\cdot}3$ and $P(A \cup B) = 0{\cdot}8$. If A and B are independent events than $P(B) =$ ...

   (x) If $(1 + 3p)/3$, $(1 - p)/4$ and $(1 - 2p)/2$ are probabilities of three mutually exclusive events, then the set of all values of p is ...

   (xi) If A and B are two events, then $P(A \cup B) = P(A \cap B)$ if and only if relation between $P(A)$ and $P(B)$ is ...

   (xii) A bag contains tickets numbered 1 to 100. Ten tickets are drawn at random and arranged in ascending order. The probability that fourth and sixth ticket bear numbers 50 and 60 respectively is ......

...it otherwise.

QUIZ

1. If a coin is tossed twice, the probability of getting at least one head is

   (a) $\frac{1}{2}$ (b) $\frac{1}{4}$ (c) $\frac{3}{4}$ (d) none

2. The probability of getting a number greater than 2 or an even number in a single throw of a fair die is

   (a) $\frac{1}{3}$ (b) $\frac{2}{3}$ (c) $\frac{5}{6}$ (d) none

3. A bag contains 3 red balls, 4 white balls and 7 black balls. The probability of drawing a red or a black ball is

   (a) $\frac{2}{7}$ (b) $\frac{5}{7}$ (c) $\frac{3}{7}$ (d) $\frac{4}{7}$

4. The odds in favour of drawing a king or a diamond from a well shuffled pack are

   (a) 9 : 4 (b) 4 : 9 (c) 5 : 9 (d) 9 : 5

5. The probability that a leap year will have 53 Tuesdays is

   (a) $\frac{1}{7}$ (b) $\frac{3}{7}$ (c) $\frac{2}{7}$ (d) $\frac{5}{7}$

6. The probability of at least one of the events A and B occurs is 0.6. If A and B occurs simultaneously with probability .2 then $P(A') + P(B') =$

   (a) 0.4 (b) 1.2 (c) 0.8 (d) 1.4

7. A and B are two independent events such that $P(A' \cap B) = \frac{8}{25}$ and $P(A \cap B') = \frac{3}{25}$, then $P(A)$ is

   (a) $\frac{2}{5}$ (b) $\frac{4}{5}$ (c) $\frac{1}{5}$ (d) $\frac{3}{5}$

8. For two events A and B if $P(A) = P\left(\frac{A}{B}\right) = \frac{1}{4}$ and $P\left(\frac{B}{A}\right) = \frac{1}{2}$, then

   (a) A and B are mutually exclusive (b) A and B are independent

   (c) $P\left(\frac{A'}{B}\right) = \frac{3}{4}$ (d) none

9. If a coin is tossed 6 times in succession, the probability of getting at least one head is

(a) $\frac{1}{64}$ (b) $\frac{3}{32}$ (c) $\frac{63}{64}$ (d) none

10. If p_1 and p_2 are the probabilities of passing an examination respectively, then the probability of only one failing in the examination is

(a) $p_1 + p_2 - 2p_1 p_2$ (b) $p_1 + p_2$

(c) $p_1 + p_2 - p_1 p_2$ (d) none

11. A coin is tossed n times. The probability that the head will present itself an odd number of times is

(a) $\frac{1}{2^n}$ (b) $\frac{1}{2n}$ (c) $\frac{1}{2}$ (d) $\frac{4}{13}$

12. Two cards are drawn at random from a pack of 52 cards. The probability of these two being aces is

(a) $\frac{2}{{}^{52}C_2}$ (b) $\frac{1}{221}$ (c) $\frac{1}{663}$ (d) none

13. If a card is drawn from a well shuffled pack of 52 cards, then the probability that it is a spade or a queen is

(a) $\frac{17}{52}$ (b) $\frac{4}{13}$ (c) $\frac{13}{52}+\frac{4}{51}$ (d) none

14. If two balls are drawn from a bag containing 3 white 4 black and 5 red balls, then the probability that the balls drawn are of different colours is

(a) $\frac{47}{66}$ (b) $\frac{10}{33}$ (c) $\frac{5}{22}$ (d) $\frac{2}{11}$

15. Six boys and six girls sit round a table randomly. The probability that all the six girls sit together is

(a) $\frac{2}{77}$ (b) $\frac{1}{77}$ (c) $\frac{2! \times 6! \times 6!}{!1!}$ (d) $\frac{2! \times 6! \times 6!}{!2!}$

16. From 20 tickets marked 1, 2, ... 20 one is drawn at random. The probability that the number on it is a multiple of 2 or 3 is

(a) $\frac{16}{20}$ (b) $\frac{15}{20}$ (c) $\frac{14}{20}$ (d) $\frac{13}{20}$

17. 3 letters are written to different persons and addresses on the 3 envelopes are also written. Without looking at the addresses, the probability that the letters go into the right envelopes is

(a) 1 (b) $\frac{1}{6}$ (c) $\frac{1}{3}$ (d) none

18. The probability of solving a problem by three students A, B, C respectively are $\frac{1}{3}, \frac{1}{4}, \frac{1}{5}$. Then the probability that the problem will be solved is

(a) $\frac{59}{60}$ (b) $\frac{3}{5}$ (c) $\frac{2}{5}$ (d) none

19. If $P(A \cup B) = 0.75$ and $P(A \cap B) = 0.15$ then $P(\overline{A}) + P(\overline{B}) =$

(a) 0.9 (b) 0.6 (c) 1.25 (d) 1.1

20. If $P(A) = 0.4$, $P(A \cup B) = 0.7$, and A, B are independent events, then $P(B) =$

(a) $\frac{1}{3}$ (b) $\frac{1}{2}$ (c) $\frac{1}{4}$ (d) $\frac{1}{5}$

21. An experiment yields three mutually exclusive events A, B, C if $P(A) = 2\,P(B) = 3\,P(C)$, then $P(A)$

(a) $\frac{2}{11}$ (b) $\frac{3}{11}$ (c) $\frac{6}{11}$ (d) $\frac{5}{11}$

22. The chance that a non-leap year contains 53 Mondays is

(a) $\frac{1}{7}$ (b) $\frac{2}{7}$ (c) $\frac{1}{365}$ (d) $\frac{2}{365}$

ANSWERS

1. c	2. c	3. b	4. b
5. c	6. c & d	7. b & c	8. b & c
9. c	10. a	11. c	12. b
13. d	14. a	15. b	16. d
17. b	18. b	19. d	20. b
21. c	22. a		