5.3

Power Spectral Density Function

So far we have been able to characterize a stochastic process by its mean, autocorrelation function, and covariance function. All these functions deal with time domain. We have not studied anything about the **spectral** (or **frequency domain**) properties of the process. For a deterministic signal $y(t)$, it is well known that its spectral properties are contained in its **Fourier transform** $Y(\omega)$, which is given by

$$\boldsymbol{Y}(\omega) = \int_{-\infty}^{\infty} \boldsymbol{y}(\boldsymbol{t}) \boldsymbol{e}^{-i\omega \boldsymbol{t}} \boldsymbol{dt}$$

Conversely, given $Y(\omega)$ we can recover $y(t)$ by means of the inverse Fourier transform

$$\boldsymbol{y}(\boldsymbol{t}) = \frac{\mathbf{1}}{\mathbf{2}\boldsymbol{\pi}} \int_{-\infty}^{\infty} \boldsymbol{Y}(\omega) \boldsymbol{e}^{i\omega \boldsymbol{t}} \boldsymbol{dt}$$

Thus, $Y(\omega)$ provides a complete description of $y(t)$ and vice -versa. Unfortunately, the same argument cannot be applied to a stochastic process $X(t)$ because the Fourier transform may not exist for most sample functions of the process. One of the conditions for the function $y(t)$ to be Fourier transformable is that is must be absolutely integrable,$i.e.$,

$$\int_{-\infty}^{\infty} |y(t)| dt < \infty$$

Recall that for stationary process the autocorrelation function $R(\tau)$ is bounded $i.e.$, $|R(\tau)| \le R(0) = E[X^2(t)]$ (see property 2 of R(τ) in module 5.2). Thus, instead of working directly with stochastic process $X(t)$, we work with its autocorrelation function which is bounded and hence absolutely integrable. We shall now give mathematical definition of power spectral density function of a stationary process.

Power spectral density function: If $\{X(t)\}$ is a stationary process (either in the strict sense or wide sense) with autocorrelation function $R(\tau)$, then the Fourier transform of $R(\tau)$ is called the **power spectral density function** of $\{X(t)\}$ and it is denoted by $\boldsymbol{S_{xx}(\omega)}$ or $\boldsymbol{S_x(\omega)}$ or $\boldsymbol{S(\omega)}$.

Thus, $$\boldsymbol{S(\omega)=\int_{-\infty}^{\infty} R(\tau)e^{-i\omega\tau}d\tau} \qquad \text{... (1)}$$

Sometimes ω is replaced by $2\pi f$, where f is the frequency variable, in which case the power spectral density function will be a function of f, denoted by $S(f)$.

Then $$\boldsymbol{S(f)=\int_{-\infty}^{\infty} R(\tau)e^{-i2\pi f\tau}d\tau} \qquad \text{... (2)}$$

Note: Equation (1) or (2) is sometimes called the **Wiener Khinchine relation**. We shall mostly follow the definition (1) and denote the power spectral density as a function of ω only.

Given the power spectral density function $S(\omega)$, the autocorrelation function $R(\tau)$ is given by the Fourier inverse transform of $S(\omega)$.

$i.e.$, $$R(\tau)=\frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)e^{i\tau\omega}d\omega \qquad \text{... (3)}$$

(or) $$R(\tau)=\int_{-\infty}^{\infty} S(f)e^{i2\pi\tau f}df \qquad \text{... (4)}$$

If $\{X(t)\}$ and $\{Y(t)\}$ are two jointly stationary random processes with cross-correlation function $R_{xy}(\tau)$, then the Fourier transform of $R_{xy}(\tau)$ is called the **crosspower spectral density** of $\{X(t)\}$ and $\{Y(t)\}$, denoted as $\boldsymbol{S_{xy}(\omega)}$.

$i.e.$, $$S_{xy}(\omega)=\int_{-\infty}^{\infty} R_{xy}(\tau)e^{-i\omega\tau}d\tau$$

Properties of Power Spectral Density Function:

1. The value of the spectral density function at zero frequency is equal to the total area under the graph of the autocorrelation function. By putting $\omega = 0$ in (1) or $f = 0$ in (2), we get

   $$S(0) = \int_{-\infty}^{\infty} R(\tau) d\tau$$, which is the given property.

2. The mean square value of a wide-sense stationary process is equal to the total area under the graph of the spectral density. By putting $\tau = 0$ in (4), we get

   $$E\left[X^2(t)\right] = R(0) = \int_{-\infty}^{\infty} S(f) df$$, which is the given property

3. The spectral density function of a real stochastic process is an even function **(For proof see P1).**
4. The spectral density of a process $\{X(t)\}$ is a real function of ω and non-negative **(For proof see P2).**
5. Spectral density of any WSS is non-negative $i.e.$, $S(\omega) \geq 0$ **(see example 9).**
6. The spectral density and autocorrelation function of a real WSS process form a Fourier cosine transform pair**(For proof see P3)**

Wiener-Khinchine Theorem

If $X_T(\omega)$ is the Fourier transform of the truncated stochastic process defined as

$$X_T(t) = \begin{cases} X(t) & for \quad |t| \leq T \\ 0 & for \quad |t| > T \end{cases}$$

Where $\{X(t)\}$ is a real WSS process with power spectral density function $S(\omega)$, then

$$S(\omega) = lim_{T\to\infty} \left[\frac{1}{2T} E\{|X_T(\omega)|^2\}\right]$$

Proof: **See P4**

Example 1: The autocorrelation function of the random telegraph signal process is given by $R(\tau) = a^2 e^{-2\gamma|\tau|}$. Determine the power density spectrum of the random telegraph signal.

Solution: $S(\omega) = \int_{-\infty}^{\infty} R(\tau) e^{-i\omega\tau} d\tau$

$$= a^2 \int_{-\infty}^{\infty} e^{-2\gamma|\tau|} (\cos\omega\tau - i\sin\omega\tau) d\tau$$

$$= 2a^2 \int_0^{\infty} e^{-2\gamma\tau} \cos\omega\tau d\tau$$

$$= \left[\frac{2a^2 e^{-2\gamma\tau}}{4\gamma^2 + \omega^2} (-2\gamma\cos\omega\tau + \omega\sin\omega\tau) \right]_0^{\infty}$$

$$= \frac{4a^2\gamma}{4\gamma^2 + \omega^2}$$

Example 2: The autocorrelation function of the Poisson increment process is given by

$$R(\tau) = \begin{cases} \lambda^2 & for \quad |\tau| > \epsilon \\ \lambda^2 + \frac{\lambda}{\epsilon}\left(1 - \frac{|\tau|}{\epsilon}\right) & for \quad |\tau| \le \epsilon \end{cases}$$

Prove that its spectral density function is given by

$$S(\omega) = 2\pi\lambda^2\delta(\omega) + \frac{4\lambda \sin^2\left(\frac{\omega\epsilon}{2}\right)}{\epsilon^2\omega^2}$$

Solution:

$$S(\omega)=\int_{-\varepsilon}^{\varepsilon}\left\{\lambda^2+\frac{\lambda}{\varepsilon}\left(1-\frac{|\tau|}{\varepsilon}\right)\right\}e^{-i\omega\tau}d\tau+\int_{-\infty}^{-\varepsilon}\lambda^2 e^{-i\omega\tau}d\tau+\int_{\varepsilon}^{\infty}\lambda^2 e^{-i\omega\tau}d\tau$$

$$=\frac{\lambda}{\varepsilon}\int_{-\varepsilon}^{\varepsilon}\left(1-\frac{|\tau|}{\varepsilon}\right)e^{-i\omega\tau}d\tau+\int_{-\infty}^{-\varepsilon}\lambda^2 e^{-i\omega\tau}d\tau+\int_{-\varepsilon}^{\varepsilon}\lambda^2 e^{-i\omega\tau}d\tau+\int_{\varepsilon}^{\infty}\lambda^2 e^{-i\omega\tau}d\tau$$

$$=\frac{\lambda}{\varepsilon}\int_{-\varepsilon}^{\varepsilon}\left(1-\frac{|\tau|}{\varepsilon}\right)e^{-i\omega\tau}d\tau+\int_{-\infty}^{\infty}\lambda^2 e^{-i\omega\tau}d\tau$$

$$=\frac{2\lambda}{\varepsilon}\int_{0}^{\varepsilon}\left(1-\frac{\tau}{\varepsilon}\right)\cos\omega\tau\, d\tau+F\left\{\lambda^2\right\}$$

where $F(\lambda^2)$ is the Fourier transform of λ^2.

$$=\frac{2\lambda}{\varepsilon}\left[\left(1-\frac{\tau}{\varepsilon}\right)\frac{\sin\omega\tau}{\omega}+\frac{1}{\varepsilon}\left(\frac{-\cos\omega\tau}{\omega^2}\right)\right]_0^{\varepsilon}+F\left\{\lambda^2\right\} \qquad (\textit{Integeration by parts})$$

$$=\frac{2\lambda}{\varepsilon^2\omega^2}(1-\cos\omega\varepsilon)+F\left\{\lambda^2\right\}$$

$$=\frac{4\lambda\sin^2\left(\dfrac{\omega\varepsilon}{2}\right)}{\varepsilon^2\omega^2}+F\left\{\lambda^2\right\} \qquad .....(1)$$

The Fourier inverse transform of $S(\omega)$ is given by

$$R(\tau)=F^{-1}\left\{S(\omega)\right\}$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}S(\omega)e^{i\tau\omega}d\omega$$

Let us now find $R(\tau)$ corresponding to $S(\omega)=2\pi\lambda^2\delta(\omega)$, where $\delta(\omega)$ is the **unit impulse function**.

$i.e.,$ $\qquad R(\tau)=F^{-1}\left\{2\pi\lambda^2\delta(\omega)\right\}$

$$= \frac{2\pi\lambda^2}{2\pi}\int_{-\infty}^{\infty}\delta(\omega)e^{i\tau\omega}d\omega$$

$$= \lambda^2 \left[\text{since } \int_{-\infty}^{\infty}\phi(t)\delta(t)dt = \phi(0)\right]$$

Therefore, $F(\lambda^2) = 2\pi\lambda^2\delta(\omega)$ ...(2)

Inserting (2) in (1) the required result is obtained.

Example 3: Find the power spectral density function of a WSS process with autocorrelation function

$$\boldsymbol{R(\tau) = e^{-\alpha\tau^2}}$$

Solution:

$$S(\omega) = \int_{-\infty}^{\infty} e^{-\alpha\tau^2} e^{-i\omega\tau} d\tau$$

$$= e^{-\frac{\omega^2}{4\alpha}} \int_{-\infty}^{\infty} e^{-\alpha\left(\tau + \frac{i\omega}{2\alpha}\right)^2} d\tau$$

$$= \frac{1}{\sqrt{\alpha}} e^{-\frac{\omega^2}{4\alpha}} \int_{-\infty}^{\infty} e^{-x^2} dx \text{ , } putting\ \sqrt{\alpha}\left(\tau + \frac{i\omega}{2\alpha}\right) = x$$

$$= \sqrt{\frac{\pi}{\alpha}} e^{-\frac{\omega^2}{4\alpha}} \left[\text{since } \int_{-\infty}^{\infty} e^{-x^2} dx = \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}\right]$$

Example 4: A stochastic process $\{X(t)\}$ is given by $X(t) = A\cos pt + B\sin pt$, where A and B are independent r.vs such that $E(A) = E(B) = 0$ and $E(A^2) = E(B^2) = \sigma^2$. Find the power spectral density of the process.

Solution: The autocorrelation function of the given process can be found as

$$R(\tau) = \sigma^2 \cos p\tau$$

$$S(\omega) = \int_{-\infty}^{\infty} \sigma^2 \cos p\tau e^{-i\omega\tau} d\tau \quad \text{...(1)}$$

Consider $F^{-1}\{\pi\sigma^2[\delta(\omega + p) + \delta(\omega - p)]\}$

$$=\frac{1}{2\pi}\pi\sigma^2\int_{-\infty}^{\infty}\left[\delta(\omega+p)+\delta(\omega-p)\right]e^{i\tau\omega}d\omega$$
$$=\frac{\sigma^2}{2}\left[e^{-i\tau p}+e^{i\tau p}\right] \quad \left\{\text{since }\int_{-\infty}^{\infty}\phi(t)\delta(t-a)dt=\phi(a)\right\}$$
$$=\sigma^2\cos p\tau$$

$$\therefore F(\sigma^2\cos p\tau)=\pi\sigma^2[\delta(\omega+p)+\delta(\omega-p)] \qquad \ldots(2)$$

Using (2) in (1), we get,

$$S(\omega)=\pi\sigma^2\left[\delta(\omega+p)+\delta(\omega-p)\right]$$

Example 5: If $\boldsymbol{Y(t)=X(t+a)-X(t-a)}$, prove that $\boldsymbol{R_{yy}(\tau)=2R_{xx}(\tau)-R_{xx}(\tau+2a)-R_{xx}(\tau-2a)}$. Hence prove that $\boldsymbol{S_{yy}(\omega)=4\sin^2 a\omega\, S_{xx}(\omega)}$.

Solution: $R_{yy}(\tau)=2R_{xx}(\tau)-R_{xx}(\tau+2a)-R_{xx}(\tau-2a)$

Taking Fourier transforms on both sides.

$$S_{yy}(\omega)=2S_{xx}(\omega)-\int_{-\infty}^{\infty}R_{xx}(\tau+2a)e^{-i\omega\tau}d\tau-\int_{-\infty}^{\infty}R_{xx}(\tau-2a)e^{-i\omega\tau}d\tau$$
$$=2S_{xx}(\omega)-e^{i2a\omega}\int_{-\infty}^{\infty}R_{xx}(u)e^{-i\omega u}du-e^{-i2a\omega}\int_{-\infty}^{\infty}R_{xx}(v)e^{-i\omega v}dv$$

(putting $\tau+2a=u$ in the first integral and $\tau-2a=v$ in the second integral)

$i.e.,\ S_{yy}(\omega)=2S_{xx}(\omega)-\{e^{i2a\omega}+e^{-i2a\omega}\}S_{xx}(\omega)$

$$=2(1-\cos 2a\omega)S_{xx}(\omega)$$

$$=4\sin^2 a\omega\, S_{xx}(\omega)$$

Example 6: If the process $\{\boldsymbol{X(t)}\}$ is defined as $\boldsymbol{X(t)=Y(t)Z(t)}$, where $\{\boldsymbol{Y(t)}\}$ and $\{\boldsymbol{Z(t)}\}$ are independent WSS processes, prove that

(i) $\boldsymbol{R_{xx}(\tau)=R_{yy}(\tau)R_{zz}(\tau)}$ **and**

(ii) $\boldsymbol{S_{xx}(\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty}S_{yy}(\alpha)S_{zz}(\omega-\alpha)d\alpha}$

Solution: $S_{xx}(\omega) = F\{R_{xx}(\tau)\} = F\{R_{yy}(\tau)R_{zz}(\tau)\}$ ....(1)

Consider $F^{-1}\left[\int_{-\infty}^{\infty} S_{yy}(\alpha)S_{zz}(w-\alpha)\,d\alpha\right]$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} S_{yy}(\alpha)S_{zz}(\omega-\alpha)e^{i\omega\tau}\,d\alpha\,d\omega$$

Putting $\alpha = y$ and $\omega - \alpha = z$, we get (from calculus)

$$d\alpha d\omega = \begin{vmatrix}\alpha_y & \alpha_z\\ \omega_y & \omega_z\end{vmatrix} dy\,dz = \begin{vmatrix}1 & 0\\ 1 & 1\end{vmatrix} dy\,dz$$

$$\begin{aligned}
\therefore F^{-1}&\left[\int_{-\infty}^{\infty} S_{yy}(\alpha)S_{zz}(\omega-\alpha)\,d\alpha\right]\\
&=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} S_{yy}(y)S_{zz}(z)e^{i(y+z)\tau}dydz\\
&=\frac{1}{2\pi}\int_{-\infty}^{\infty} S_{yy}(y)e^{iy\tau}dy\int_{-\infty}^{\infty} S_{zz}(z)e^{iz\tau}dz\\
&=F^{-1}\left\{S_{yy}(\omega)\right\}2\pi F^{-1}\left\{S_{zz}(\omega)\right\}\\
&=2\pi R_{yy}(\tau)R_{zz}(\tau)
\end{aligned}$$

$$\therefore F\left\{R_{yy}(\tau)R_{zz}(\tau)\right\}=\frac{1}{2\pi}\int_{-\infty}^{\infty} S_{yy}(\alpha)S_{zz}(\omega-\alpha)\,d\alpha \quad \text{.... (2)}$$

Using (2) in (1), we get $S_{xx}(\omega)$ in the required form.

Example 7: If the power spectral density of a WSS process is given by

$$\boldsymbol{S(\omega)} = \begin{cases}\dfrac{\boldsymbol{b}}{\boldsymbol{a}}(\boldsymbol{a}-|\boldsymbol{\omega}|) & , \quad |\boldsymbol{\omega}| \le \boldsymbol{a}\\ \boldsymbol{0} & , \quad |\boldsymbol{\omega}| > a\end{cases}$$

Find the autocorrelation function of the process.

Solution: The autocorrelation function

$$R(\tau) = F^{-1}\{S(\omega)\}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)e^{i\tau\omega}d\omega$$

$$= \frac{1}{2\pi}\int_{-a}^{a} \frac{b}{a}(a-|\omega|)e^{i\tau\omega}d\omega = \frac{1}{2\pi}\int_{-a}^{a} \frac{b}{a}(a-|\omega|)\cos\tau\omega\, d\omega$$

$$= \frac{1}{\pi}\int_{0}^{a} \frac{b}{a}(a-\omega)\cos\tau\omega\, d\omega$$

$$= \frac{b}{\pi a}\left\{(a-\omega)\frac{\sin\tau\omega}{\tau} - \frac{\cos\tau\omega}{\tau^2}\right\}_0^a \qquad \text{(integration by parts)}$$

$$= \frac{b}{\pi a\tau^2}(1-\cos a\tau)$$

$$= \frac{ab}{2\pi}\left(\frac{\sin a\frac{\tau}{2}}{a\frac{\tau}{2}}\right)^2$$

Example 8: The power spectral density function of a zero mean WSS process $\{\boldsymbol{X(t)}\}$ is given by

$$\boldsymbol{S(\omega) = \begin{cases} 1 & , \quad |\omega| < \omega_0 \\ 0 & , \quad elsewhere \end{cases}}$$

Find $\boldsymbol{R(\tau)}$ and show also that $\boldsymbol{X(t)}$ and $\boldsymbol{X\left(t+\frac{\tau}{\omega_0}\right)}$ are uncorrelated.

Solution:

We have $R(\tau) = F^{-1}\{S(\omega)\}$

$$i.e.,\ R(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)e^{i\tau\omega}d\omega = \frac{1}{2\pi}\int_{-\omega_0}^{\omega_0} e^{i\tau\omega}d\omega$$

$$= \frac{1}{2\pi}\left\{\frac{e^{i\tau\omega}}{i\tau}\right\}_{-\omega_0}^{\omega_0} = \frac{1}{2\pi i\tau}\left(e^{i\tau\omega_0} - e^{-i\tau\omega_0}\right)$$

$$= \frac{1}{\pi\tau}\sin\omega_0\tau$$

Now, $E\left\{X\left(t+\frac{\pi}{\omega_0}\right)X(t)\right\} = R\left(\frac{\pi}{\omega_0}\right) = \frac{\omega_0}{\pi^2} sin\left(\omega_0 \frac{\pi}{\omega_0}\right) = \frac{\omega_0}{\pi^2} sin\pi = 0$

Since the mean of the process is zero,

$$C\left\{X\left(t+\frac{\pi}{\omega_0}\right)X(t)\right\} = E\left\{X\left(t+\frac{\pi}{\omega_0}\right)X(t)\right\} = 0$$

Therefore, $X(t)$ and $X\left(t+\frac{\pi}{\omega_0}\right)$ are uncorrelated.

Example 9: Property (5) of power spectral density. Prove that the spectral density of any $\boldsymbol{WSS}$ process is non – negative. $\boldsymbol{i.e.}$, $\boldsymbol{S(w) \geq 0}$.

Solution: If possible, let $S(\omega) < 0$ at $\omega = \omega_0$. That is, let $S(\omega) < 0$ in $\omega_0 - \frac{\epsilon}{2} < \omega < \omega_0 + \frac{\epsilon}{2}$, where ϵ is very small. Let us assume that the system function of the convolution type linear system is

$$H(\omega) = \begin{cases} 1\,, & \omega_0 < \frac{\epsilon}{2} < \omega < \omega_0 + \frac{\epsilon}{2} \\ 0\,, & otherwise \end{cases}$$

Note: In this case, system is called a **narrow band filter**

Now $\quad S_{yy}(\omega) = |H(\omega)|^2 S_{xx}(\omega)$

$$= \begin{cases} S_{xx}(\omega)\,, & \omega_0 - \frac{\epsilon}{2} < \omega < \omega_0 + \frac{\epsilon}{2} \\ 0\,, & elsewhere \end{cases}$$

$E\{Y^2(t)\} = R_{yy}(0)$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{yy}(\omega)\, d\omega$$

$$= \frac{1}{2\pi}\int_{\omega_0-\frac{\varepsilon}{2}}^{\omega+\frac{\varepsilon}{2}} S_{xx}(\omega)\, d\omega$$

$$= \frac{\epsilon}{2\pi} S_{xx}(\omega_0)$$

[Since $S_{xx}(\omega_0)$ can be considered a contant $S_{xx}(\omega_0)$, as the band is narrow]

Since $E\{Y^2(t)\} \geq 0$, $S_{xx}(\omega_0) \geq 0$, which is contrary to our initial assumption. Therefore $S_{xx}(\omega) \geq 0$, since $\omega = \omega_0$ is arbitrary.