3.6

Divergence Theorem of Gauss

The Green's theorem already derived and it can be written in a vector form $\boldsymbol{V} = g\boldsymbol{i} - f\boldsymbol{j}$. Let C be a curve in two dimensions which is written in the parametric form $\boldsymbol{r} = \boldsymbol{r}(s)$. Then, the unit tangent and unit normal vectors to C be given by

$$\boldsymbol{T} = \frac{dx}{ds}\,\boldsymbol{i} + \frac{dy}{ds}\,\boldsymbol{j},\ \ \boldsymbol{n} = \frac{dy}{ds}\,\boldsymbol{i} - \frac{dx}{ds}\,\boldsymbol{j}.$$

Then,

$$f\,dx + g\,dy = \left(f\,\frac{dx}{ds} + g\,\frac{dy}{ds}\right)ds = (g\,\boldsymbol{i} - f\,\boldsymbol{j}).\left(\frac{dy}{ds}\,\boldsymbol{i} - \frac{dx}{ds}\,\boldsymbol{j}\right)ds$$

$$= (\boldsymbol{V}.\boldsymbol{n})ds$$

Also

$$\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} = \left(i\,\frac{\partial}{\partial x} + j\,\frac{\partial}{\partial y}\right).(g\,\boldsymbol{i} - f\,\boldsymbol{j}) = \nabla.\boldsymbol{V}$$

Hence, Green's theorem can be written in a vector form as

$$\oint_C (\boldsymbol{V}.\boldsymbol{n})ds = \iint_R (\nabla.\boldsymbol{V})dx\,dy$$

This result is a particular case of the Gauss's divergence theorem. Extension of the Green's theorem to three dimensions can be done under the following generalizations.

i. A region R in the plane $\rightarrow$ a three dimensional solid D.

ii. The closed curve C enclosing R in the plane $\rightarrow$ the closed surface S enclosing the solid D.

iii. The unit outer normal $\boldsymbol{n}$ to C $\rightarrow$ the unit outer normal $\boldsymbol{n}$ to S.

iv. A vector field $\boldsymbol{V}$ in the plane $\rightarrow$ a vector field $\boldsymbol{V}$ in the three dimensional space.

v. The line integral $\oint_C (\boldsymbol{V}.\boldsymbol{n})ds$ $\rightarrow$ a surface integral $\iint_S (\boldsymbol{V}.\boldsymbol{n})dA$.

vi. The double integral $\iint_R \nabla.\boldsymbol{V}\, dx\, dy$ $\rightarrow$ a triple (volume) integral $\iiint_D \nabla.\boldsymbol{V}\, dV$.

The above generalizations give the following divergence theorem.

Theorem: (Divergence theorem of Gauss) Let D be a closed and bounded region in the three dimensional space whose boundary is a piecewise smooth surface S that is oriented outward. Let $\boldsymbol{V}(x,y,z) = v_1(x,y,z)\boldsymbol{i} + v_2(x,y,z)\boldsymbol{j} + v_3(x,y,z)\boldsymbol{k}$ be a vector field for which v_1, v_2 and v_3 are continuous and have continuous first order partial derivatives in some domain containing D.

Then, $\iint_S (\boldsymbol{V}.\boldsymbol{n})dA = \iiint_D \nabla.\boldsymbol{V}\, dV = \iiint_D div\,(\boldsymbol{V})dV$

Where $\boldsymbol{n}$ is the outer unit normal vector to S.

Proof: In terms of the components of V, the left and right hand sides, cab be written as

$$\iint_S (\boldsymbol{V}.\boldsymbol{n})dA = \iint_S v_1(\boldsymbol{i}.\boldsymbol{n})dA + \iint_S v_2(\boldsymbol{j}.\boldsymbol{n})dA + \iint_S v_3(\boldsymbol{k}.\boldsymbol{n})dA$$

$$\iiint_D \nabla . \boldsymbol{V}\, dV = \iiint_D \frac{\partial v_1}{\partial x}\, \mathrm{dV} + \iiint_D \frac{\partial v_2}{\partial y}\, \mathrm{dV} + \iiint_D \frac{\partial v_3}{\partial z}\, \mathrm{dV}$$

Where $dV = dx\, dy\, dz$. To prove the divergence theorem it is sufficient to show that

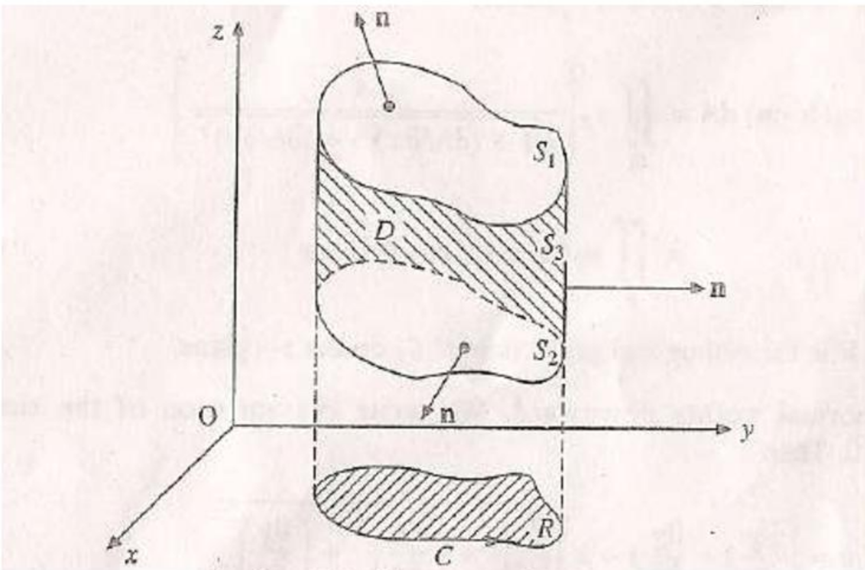
$$\iint_S v_1(\boldsymbol{i}.\boldsymbol{n})dA = \iiint_D \frac{\partial v_1}{\partial x}\mathrm{dV}, \ldots\ldots\ldots\ldots\ldots (1)$$

$$\iint_S v_2(\boldsymbol{j}.\boldsymbol{n})dA = \iiint_D \frac{\partial v_2}{\partial y}\mathrm{dV}, \ldots\ldots\ldots\ldots\ldots (2) \text{ and}$$

$$\iint_S v_3(\boldsymbol{k}.\boldsymbol{n})dA = \iiint_D \frac{\partial v_3}{\partial z}\mathrm{dV} \ldots\ldots\ldots\ldots\ldots (3)$$

We shall prove Eq. (3). The other results are proved in a similar manner.

We shall prove the theorem for the special case of the region D whose bounding surface can be written as follows



Surface S in divergence theorem.

Top surface $S_1 : z = h(x, y), (x, y)$ in R

Bottom surface $S_2 : z = g(x, y), (x, y)$ in R

Side (vertical) surface $S_3 : g(x, y) \leq z \leq h(x, y), (x, y)$ in R

Where R is the orthogonal projection of S in the xy plane.

Now, $\iiint\limits_D \frac{\partial v_3}{\partial x}\, dV = \iint\limits_R \left[\int_{g(x,y)}^{h(x,y)} \frac{\partial v_3}{\partial x}\, dz\right] dx\, dy$

$$= \iint\limits_R \left[v_3\big(x, y, h(x, y)\big) - v_3\big(x, y, g(x, y)\big)\right] dx\, dy$$

..............(4)

We write

$$\iint\limits_S v_3(\boldsymbol{k}.\boldsymbol{n})dA = \iint\limits_{S_1} v_3(\boldsymbol{k}.\boldsymbol{n})dA + \iint\limits_{S_2} v_3(\boldsymbol{k}.\boldsymbol{n})dA + \iint\limits_{S_3} v_3(\boldsymbol{k}.\boldsymbol{n})dA.$$

We evaluate the surface integrals on the right hand side separately.

On S_1: The outward normal points upward. We write the equation of the surface as $f(x, y, z) = z - h(x, y) = 0$.

Then

$$\boldsymbol{n} = \frac{-\frac{\partial h}{\partial x}\boldsymbol{i} - \frac{\partial h}{\partial y}\boldsymbol{j} + \boldsymbol{k}}{\sqrt{1 + \left(\frac{\partial h}{\partial x}\right)^2 + \left(\frac{\partial h}{\partial y}\right)^2}}$$

So that $\boldsymbol{k}.\boldsymbol{n} = 1\Big/\sqrt{1 + \left(\frac{\partial h}{\partial x}\right)^2 + \left(\frac{\partial h}{\partial y}\right)^2}$. Hence,

$$\iint_{S_1} v_3(\boldsymbol{k}.\boldsymbol{n})dA = \iint_{S_1} v_3\left[\frac{dA}{\sqrt{1+\left(\frac{\partial h}{\partial x}\right)^2+\left(\frac{\partial h}{\partial y}\right)^2}}\right]$$

$$= \iint_{R} v_3\big(x, y, h(x, y)\big)dx\, dy \ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

where R is the orthogonal projection of S_1 on the $XY-$ plane.

On S_2: The outward normal points download. We write the equation of the surface as $f(x, y, z) = g(x, y) - z = 0$.

Then

$$\boldsymbol{n} = \left[\frac{\partial g}{\partial x}\boldsymbol{i} + \frac{\partial g}{\partial y}\boldsymbol{j} - \boldsymbol{k}\right] \Big/ \sqrt{1 + \left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2}$$

So that $\boldsymbol{k}.\boldsymbol{n} = -1 \Big/ \sqrt{1 + \left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2}$. Hence,

$$\iint_{S_2} v_3(\boldsymbol{k}.\boldsymbol{n})dA = \iint_{S_2} -v_3\left[\frac{dA}{1+\left(\frac{\partial g}{\partial x}\right)^2+\left(\frac{\partial g}{\partial y}\right)^2}\right]$$

$$= -\iint_{R} v_3\big(x, y, g(x, y)\big)dx\, dy \ldots\ldots\ldots\ldots\ldots(6)$$

where, again, R is the orthogonal projection of S_2 on the $XY-$ plane.

On S_3: Since the surface is vertical, the outward normal $\boldsymbol{n}$ is perpendicular to $\boldsymbol{k}$, that is, $\boldsymbol{k}.\boldsymbol{n} = 0$.

Therefore, $\iint_{S_3} v_3(\boldsymbol{k}.\boldsymbol{n})dA = 0$ $\qquad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7)$

Adding equations (5) to (7), we obtain

$$\iint_S v_3(\boldsymbol{k}.\boldsymbol{n})dA = \iint_R \left[v_3\big(x,y,h(x,y)\big) - v_3\big(x,y,g(x,y)\big)\right]\text{dx dy}. \qquad \text{.......................(8)}$$

From equations (4) and (8), we obtain

$$\iint_S v_3(\boldsymbol{k}.\boldsymbol{n})dA = \iiint_D \frac{\partial v_3}{\partial z} dV.$$

Expressing the bounding surface of the region D in a suitable manner, similar to the particular case given in figure, equations (1) and (2) can be proved.

Remarks:

1. The given domain D can be subdivided into finitely many special regions such that each such region can be described in the required manner. In the proof of the divergence theorem, the special region D has a vertical surface. This type of region is not required in the proof. The region may have a vertical surface on a part of the region, the other part may be simply a curve. Also, the region may not have any vertical surface. For example, the region bounded by a sphere or an ellipsoid has no vertical surface. The divergence theorem holds in all these cases. The divergence theorem also holds for the region D bounded by two closed surfaces.

2. In terms of the components of $\boldsymbol{V}$, divergence theorem can be written as

$$\iint_S v_1\, dy\, dz + v_2\, dz\, dx + v_3\, dx\, dy$$

$$= \iiint_D \left(\frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}\right) dx\, dy\, dz$$

or $\iint_S (v_1 \cos\alpha + v_2 \cos\beta + v_3 \cos\gamma) dA$

$$= \iiint_D \left(\frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}\right) dx\, dy\, dz.$$

Example: Use the divergence theorem to evaluate

$\iint_S (\boldsymbol{V}.\boldsymbol{n})dA$, where $\boldsymbol{V} = x^2 z\,\boldsymbol{i} + y\,\boldsymbol{j} - xz^2\boldsymbol{k}$ and S is the boundary of the region bounded by the parabolic $z = x^2 + y^2$ and the plane $z = 4y$.

Solution: We have

$$\iint_S (\boldsymbol{V}.\boldsymbol{n})dA = \iiint_D \nabla.\boldsymbol{V}\, dV = \iiint_D (2xz + 1 - 2xz)dV = \iiint_D dV$$

$$= \int_{y=0}^{4} \int_{x=-\sqrt{4y-y^2}}^{\sqrt{4y-y^2}} \int_{z=x^2+y^2}^{4y} dz\, dx\, dy$$

Since the projection of S on the xy plane is $x^2 + y^2 = 4y$. Therefore,

$$\iint_S (\boldsymbol{V}.\boldsymbol{n})dA = \int_{y=0}^{4} \int_{x=-\sqrt{4y-y^2}}^{\sqrt{4y-y^2}} (4y - x^2 - y^2)\, dx\, dy$$

$$= \int_{y=0}^{4} \left[2(4y - y^2)(4y - y^2)^{1/2} - \frac{2}{3}(4y - y^2)^{3/2}\right] dy$$

$$= \int_{y=0}^{4} \frac{4}{3}(4y - y^2)^{3/2}\, dy = \frac{4}{3}\int_{y=0}^{4} [4 - (y-2)^2]^{3/2}\, dy$$

Set $y - 2 = 2\sin t$. We obtain

$$\iint_S (\boldsymbol{V}.\boldsymbol{n})dA = \frac{4}{3}\int_{-\pi/2}^{\pi/2} 16\ \cos^4 t\ dt = \frac{4}{3}(32)\left(\frac{3}{4}.\frac{1}{2}.\frac{\pi}{2}\right) = 8\pi.$$

Green's Identities (formulas)

Divergence theorem can be used to prove some important identities, called Green's identities which are of use in solving partial differential equations. Let f and g be scalar function which are continuous and have continuous first and second order partial derivatives in some region of the three dimensional space. Let S be a piecewise smooth surface bounding a domain D in this region. Let the functions f and g be such that $\boldsymbol{V} = f\ grad\ g$. Then, we have

$$\nabla.(f\nabla g) = f\nabla^2 g + \nabla f.\nabla g$$

By divergence theorem, we obtain

$$\iint_S (\boldsymbol{V}.\boldsymbol{n})dA = \iint_S f(\nabla g.\boldsymbol{n})dA = \iiint_D \nabla.(f.\nabla g)dV$$

$$= \iiint_D (f\nabla^2 g + \nabla f.\nabla g)dV.$$

Now, $\nabla g.\boldsymbol{n}$ is the directional derivative of g in the direction of the unit normal vector $\boldsymbol{n}$. Therefore, it can be denoted by $\partial g/\partial n$. We have the Green's first identity as

$$\iint_S f(\nabla g.\boldsymbol{n})dA = \iint_S \mathrm{f}\frac{\partial g}{\partial n}\,\mathrm{dA} = \iiint_D (f\nabla^2 g + \nabla f.\nabla g)dV \ldots\ldots.(1)$$

Interchanging f and g, we obtain

$$\iint_S f(\nabla f.\boldsymbol{n})dA = \iint_S \mathrm{f}\frac{\partial f}{\partial n}\,\mathrm{dA} = \iiint_D (g\nabla^2 f + \nabla g.\nabla f)dV.$$

Subtracting the two results, we obtain the Green's second identity as

$$\iint_S (f\nabla g - g\nabla f).\boldsymbol{n}\, dA = \iint_S \left(\mathrm{f}\,\frac{\partial g}{\partial n} - \mathrm{g}\,\frac{\partial \mathrm{f}}{\partial \mathrm{n}}\right) \mathrm{dA}$$

$$= \iiint_D (f\nabla^2 g - g\nabla^2 f)dV.$$

Let $f = 1$ in equation (1) then, we obtain

$$\iint_S \nabla g.\boldsymbol{n}\, dA = \iint_S \frac{\partial g}{\partial n}\,\mathrm{dA} = \iiint_D \nabla^2 g\, dV.$$

If g is a harmonic function, then $\nabla^2 g = 0$

Therefore,

$$\iint_S \nabla g.\boldsymbol{n}\, dA = \iint_S \frac{\partial g}{\partial n}\,\mathrm{dA} = 0.$$

This equation gives a very important property of the solutions of Laplace equation, that is of harmonic functions. It states that if $g(x, y, z)$ is a harmonic function, that is, it is a solution of the equation

$$\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} + \frac{\partial^2 g}{\partial z^2} = 0$$

Then, the integral of the normal derivative of g over any piecewise smooth closed orientable surface is zero.

Problem 1: Let $\boldsymbol{F} = 2x\boldsymbol{i} + y^2\boldsymbol{j} + z^2\boldsymbol{k}$ and S be the unit sphere defined by $x^2 + y^2 + z^2 = 1$. Evaluate $\iint_s \boldsymbol{F}.\boldsymbol{n}\, dA$

Solution: By Gauss theorem,

$$\iiint_W (div\, \boldsymbol{F})dV = \iint_S \boldsymbol{F}.\boldsymbol{n}dA$$

Where W is the ball bounded by the sphere. The integral on the left is

$$\iint_S \boldsymbol{F}.\boldsymbol{n}dA = \iiint_W (div\, \boldsymbol{F})dV$$

$$= 2\iiint_W (1 + y + z)dV$$

$$= 2\iiint_W dV + 2\iiint_W y\, dV + 2\iiint_W z\, dV$$

By symmetry,

$$\iiint_W y\, dV = \iiint_W z\, dV = 0$$

Thus

$$\iint_S \boldsymbol{F}.\boldsymbol{n}dA = 2\iiint_W (1 + y + z)dV = 2\iiint_W dV = \frac{8\pi}{3}.$$

Problem 2: Calculate the flux of $\boldsymbol{V}(x,y,z) = x^3\boldsymbol{i} + y^3\boldsymbol{j} + k^3\boldsymbol{k}$ outward through the unit sphere $x^2 + y^2 + z^2 = 1$.

Solution: We know that if $\boldsymbol{V}$ is the velocity field of a fluid, then div($\boldsymbol{V}$) gives the flux per unit volume.

So by Gauss theorem, the flux is

$$\iiint_W (div\ \boldsymbol{F}) dxdydz = \iiint_W 3(x^2 + y^2 + z^2) dxdydz$$

Using spherical coordinates, the triple integral becomes

$$\iiint_W (div\ \boldsymbol{F}) dxdydz = \int_0^{2\pi} \int_0^{\pi} \int_0^1 3\rho^4 \sin\phi\ d\rho d\phi d\theta$$

$$= \int_0^{2\pi} \int_0^{\pi} \left[\frac{3\rho^5}{5}\right]_0^1 \sin\phi\ d\phi d\theta$$

$$= \frac{3}{5} \int_0^{2\pi} [-\cos\phi]_0^{\pi} d\theta$$

$$= \frac{3}{5} \int_0^{2\pi} 2 d\theta$$

$$= \frac{6}{5} [\theta]_0^{2\pi} = \frac{12\pi}{5}\ .$$

Problem 3: Using divergence theorem to evaluate $\iint_S (x+z)dydz + (y+z)dzdx + (x+y)dxdy$, where S is the surface of the sphere $x^2+y^2+z^2=4$

Solution: Given $\iint_S (x+z)dydz + (y+z)dzdx + (x+y)dxdy$

Here $v_1 = x+z,\ v_2 = y+z,\ v_3 = x+y$

$$\therefore\ \frac{\partial v_1}{\partial x} = 1, \frac{\partial v_2}{\partial y} = 1, \frac{\partial v_3}{\partial z} = 0$$

By Gauss's divergence theorem,

$$\iint_S v_1 dydz + v_2 dzdx + v_3 dxdy = \iiint_V \left(\frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial x} + \frac{\partial v_3}{\partial x}\right) dxdydz$$

$$= \iiint_V 2dxdydz$$

$$= 2V$$

$$= 2\left[\frac{4}{3}\pi(2)^3\right] = \frac{64\pi}{3}.$$

($\because$ volume of sphere is $\frac{4}{3}\pi r^3$, here $r = 2$)

Problem 4: Evaluate $\iint_S \boldsymbol{V}.\boldsymbol{n}dA$, if $\boldsymbol{V} = xy\boldsymbol{i} + z^2\boldsymbol{j} + 2yz\boldsymbol{k}$ over the tetrahedron bounded by $x = 0, y = 0, z = 0$ and the plane $x + y + z = 1$.

Solution: Given $\boldsymbol{V} = xy\boldsymbol{i} + z^2\boldsymbol{j} + 2yz\boldsymbol{k}$

then div$\boldsymbol{V} = y + 2y = 3y$

By Gauss's divergence theorem

$$\iint_S \boldsymbol{V}.\boldsymbol{n}dA = \iiint_V (div\ \boldsymbol{V})dV$$

$$= \iiint_V 3ydxdydz$$

$$= \int_{x=0}^{1}\int_{y=0}^{1-x}\int_{z=0}^{1-x-y} 3y\ dxdydz$$

$$= 3\int_{x=0}^{1}\int_{y=0}^{1-x} y[z]_0^{1-x-y}dxdy$$

$$= 3\int_{x=0}^{1}\int_{y=0}^{1-x} y(1-x-y)dxdy$$

$$= 3\int_{x=0}^{1}\left[\frac{y^2}{2}-\frac{xy^2}{2}-\frac{y^3}{3}\right]_0^{1-x} dx$$

$$= 3\int_{x=0}^{1}\left[\frac{(1-x)^2}{2}-\frac{x(1-x)^2}{2}-\frac{(1-x)^3}{3}\right] dx$$

$$= \int_{x=0}^{1}\left[\frac{(1-x)^2}{2}-\frac{(1-x)^3}{3}-\frac{x(1+x^2-2x)}{2}\right] dx$$

$$= 3\left[\frac{(1-x)^3}{-6}+\frac{(1-x)^4}{12}-\frac{1}{2}\left(\frac{x^2}{2}+\frac{x^4}{4}-\frac{2x^3}{3}\right)\right]_0^1$$

$$= 3\left[-\frac{1}{2}\left(\frac{1}{2}+\frac{1}{4}-\frac{2}{3}\right)+\frac{1}{6}-\frac{1}{12}\right]$$

$$= 3\left[-\frac{1}{2}\frac{(6+3-8)}{12}+\frac{1}{12}\right]$$

$$= 3\left[-\frac{1}{24}+\frac{1}{12}\right] = \frac{1}{8}.$$

Problem 5: Verify Gauss's divergence theorem for $\boldsymbol{V} = (x^3 - yz)\boldsymbol{i} - 2x^2y\boldsymbol{j} + z\boldsymbol{k}$ taken over the surface of the cube bounded by the planes $x = y = z = 1$ and coordinate plane.

Solution: By Gauss's divergence theorem, we have

$$\iint_S \boldsymbol{V}.\boldsymbol{n}dA = \iiint_V (div\ \boldsymbol{V})dV$$

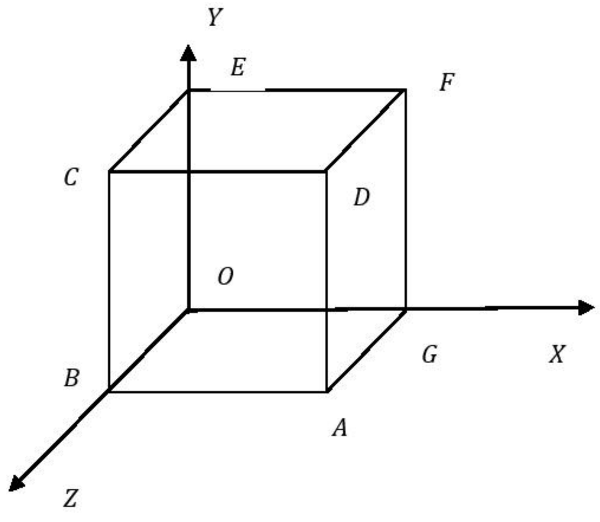
$$\text{RHS} = \int_0^1 \int_0^1 \int_0^1 (3x^2 - 2x^2 + 1)\, dxdydz$$

$$= \int_0^1 \int_0^1 \int_0^1 (x^2 + 1)\, dxdydz$$

$$= \int_0^1 \int_0^1 \left[\frac{x^3}{3} + x\right]_0^1 dydz$$

$$= \int_0^1 \int_0^1 \frac{4}{3} dydz = \frac{4}{3}\int_0^1 [y]_0^1\, dz = \frac{4}{3}\int_0^1 dz = \frac{4}{3}.$$

Verification: we will calculate the value of $\iint_S \boldsymbol{V}.\boldsymbol{n}dA$ over the six faces of the cube.



(i) For $S_1 = AGFD$; unit outward drawn normal $\boldsymbol{n} = \boldsymbol{i}$
$x = 1,\ dA = dydz,\ 0 \le y \le 1, 0 \le z \le 1$

$$\therefore \quad \iint_{S_1} \boldsymbol{V}.\boldsymbol{n}dA = \int_{z=0}^{1}\int_{y=0}^{1}(1-yz)dydz$$
$$= \int_{z=0}^{1}\left[y-\frac{y^2}{2}z\right]_0^1 dz$$
$$= \int_{z=0}^{1}\left(1-\frac{z}{2}\right)dz$$
$$= \left[z-\frac{z^2}{4}\right]_0^1 = 1-\frac{1}{4}=\frac{3}{4}.$$

(ii) For $S_2 = OECB$; unit outward drawn normal $\boldsymbol{n} = -\boldsymbol{i}$
$x = 0, dA = dydz, 0 \le y \le 1, 0 \le z \le 1$

$$\therefore \quad \iint_{S_2} \boldsymbol{V}.\boldsymbol{n}dA = \int_{z=0}^{1}\int_{y=0}^{1} yzdydz = \int_{z=0}^{1}\left[\frac{y^2z}{2}\right]_0^1 dz$$

$$= \int_{z=0}^{1}\frac{z}{2}dz = \frac{1}{2}\left[\frac{z^2}{2}\right]_0^1 = \frac{1}{4}.$$

(iii) For $S_3 = ADCB$; unit outward drawn normal $\boldsymbol{n} = \boldsymbol{k}$
$z = 1, dA = dxdy,\ 0 \le x \le 1, 0 \le y \le 1$

$$\therefore \quad \iint_{S_3} \boldsymbol{V}.\boldsymbol{n}dA = \int_{y=0}^{1}\int_{x=0}^{1} dxdy = 1.$$

(iv) For $S_4 = OGFE$; unit outward drawn normal $\boldsymbol{n} = -\boldsymbol{k}$
$z = 0, dA = dxdy,\ 0 \le x \le 1,\ 0 \le y \le 1$

$$\therefore \quad \iint_{S_4} \boldsymbol{V}.\boldsymbol{n}dA = 0.$$

(v) For $S_5 = CDFE$; unit outward drawn normal $\boldsymbol{n} = \boldsymbol{j}$
$y = 1, dA = dxdz, 0 \le x \le 1, 0 \le z \le 1$

$$\therefore \quad \iint_{S_5} \boldsymbol{V}.\boldsymbol{n}dA = \int_{x=0}^{1}\int_{z=0}^{1} -2x^2\, dzdx$$
$$= \int_{x=0}^{1} -2x^2 [z]_0^1 dx$$
$$= \left[-\frac{2x^3}{3}\right]_0^1 = -\frac{2}{3}.$$

(vi) For $S_6 = OBAG$; unit outward drawn normal $\boldsymbol{n} = -\boldsymbol{j}$

$$y = 0, dA = dxdz, 0 \le x \le 1, 0 \le z \le 1$$

$$\therefore \quad \iint_{S_6} \boldsymbol{V}.\boldsymbol{n}dA = 0$$

$$\therefore \quad \iint_S \boldsymbol{V}.\boldsymbol{n}dA = \iint_{S_1} + \iint_{S_2} + \iint_{S_3} + \iint_{S_4} + \iint_{S_5} + \iint_{S_6}$$

$$= \frac{3}{4} + \frac{1}{4} + 1 + 0 - \frac{2}{3} + 0 = \frac{4}{3} = \text{RHS}$$

Hence Gauss's divergence theorem verified.

Exercise

1. Evaluate $\iint_s \boldsymbol{F}.\boldsymbol{n}dA$, where $\boldsymbol{F}(x,y,z) = xy^2\boldsymbol{i} + x^2y\boldsymbol{j} + y\boldsymbol{k}$ and S is the surface of the "can" W defined by $x^2 + y^2 \leq 1, -1 \leq z \leq 1$.
2. Use divergence theorem to evaluate the surface integrals $\iint_s \boldsymbol{V}.\boldsymbol{n}dA$ where $\boldsymbol{V} = 2x^3\boldsymbol{i} + 3y^3\boldsymbol{j} + z^3\boldsymbol{k}$ and D is the region bounded by $x^2 + y^2 + z^2 = 9$.
3. Evaluate the surface integral $\iint_s (yzdydz + zxdzdx + xydxdy)$, S is surface of the cube $0 \leq x \leq 1, 0 \leq y \leq 1, 0 \leq z \leq 1$.
4. Evaluate the surface integral $\iint_s (xdydz + ydzdx + zdxdy)$, S is surface of the sphere $(x-2)^2 + (y-2)^2 + (z-2)^2 = 4$.
5. Compute $\iint_s (ax^2 + by^2 + cz^2)\, dA$ over the surface of the sphere $x^2 + y^2 + z^2 = 1$.
6. Let D be the region bounded by the closed cylinder $x^2 + y^2 = 16, z = 0$ and $z = 4$. Verify the divergence theorem if $\boldsymbol{V} = 3x^2\boldsymbol{i} + 6y^2\boldsymbol{j} + z\boldsymbol{k}$.
7. Verify the divergence theorem for $\boldsymbol{V} = x^2\boldsymbol{i} + y^2\boldsymbol{j} + z^2\boldsymbol{k}$ over the surface S of the solid cut off by the plane $x + y + z = a$ in the first octant.
8. Verify the divergence theorem for $\boldsymbol{V} = (x^2 - 2yz)\boldsymbol{i} + (y^2 - 3zx)\boldsymbol{j} + (z^2 - xy)\boldsymbol{k}$ taken over the surface of the cube bounded by $0 \leq x \leq a,\ 0 \leq y \leq a,\ 0 \leq z \leq a$.

Answers

1. π
2. $\frac{5832}{5}\pi$
3. 0
4. 32π
5. $\frac{4\pi}{3}(a + b + c)$