LAPLACE TRANSFORM

$$X(\omega) = \int_{-\infty}^{\infty} x(t) \cdot e^{-j\omega t} dt$$

Laplace transform is the generalization of FT.

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-\sigma t} e^{-j\omega t} dt$$

$$X(\sigma + j\omega) = \int_{-\infty}^{\infty} x(t) e^{-(\sigma + j\omega)t} dt$$

$$\sigma + j\omega = S$$

$$X(S) = \int_{-\infty}^{\infty} x(t) e^{-St} dt$$

$$|X(\omega)| < \infty$$

$$\left| \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \right| < \infty$$

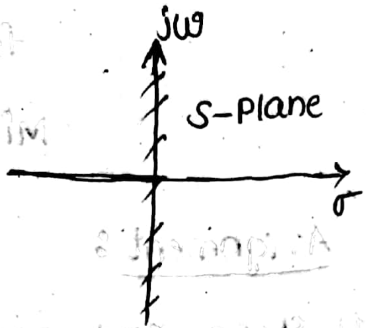
$$\left| \int_{-\infty}^{\infty} x(t) dt \right| < \infty$$ — Absolutely integrable

$\sigma - ? \rightarrow X(S) \rightarrow$ exist

$\sigma ? \quad |X(S)| < \infty$

— Region of convergence

$$X(S) = \int_{-\infty}^{\infty} x(t) e^{-St} dt$$



$$|X(S)| < \infty$$

$$\left| \int_{-\infty}^{\infty} x(t) e^{-(\sigma + j\omega)t} dt \right| < \infty$$

$$\underline{\int_{-\infty}^{\infty} |x(t) e^{-\sigma t}| dt < \infty}$$

Range of 'σ' / Re(S)

Region of Convergence: The range of values of s (Re(s)) (σ) for which the Laplace Transform X(s) should exist

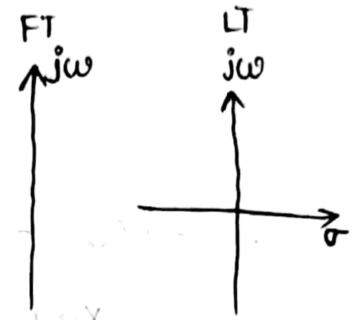
i.e., $|X(s)| < \infty \Rightarrow \int_{-\infty}^{\infty} |x(t)e^{-\sigma t}| dt < \infty$

Relation b/w FT and LT:

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

$$= \int_{-\infty}^{\infty} x(t) e^{-(\sigma + j\omega)t} dt$$

$$= \int_{-\infty}^{\infty} [x(t) \cdot e^{-\sigma t}] \cdot e^{-j\omega t} dt$$



$$X(s) = FT[x(t) \cdot e^{-\sigma t}]$$

* If $\sigma = 0$ then LT = FT $\Rightarrow X(s) = X(\omega)$

Laplace Transform of basic signals:

→ $x(t) = e^{-at} u(t)$

$$X(s) = \int_{-\infty}^{\infty} [e^{-at} u(t)] e^{-st} dt$$

$\sigma + a > 0$

$\Rightarrow \sigma > -a$

$Re\{s\} > Re\{-a\}$ if 'a' is complex

↓

Region of convergence (ROC)

$$= \int_{0}^{\infty} e^{-at} (1) e^{-st} dt$$

$$= \int_{0}^{\infty} e^{-(s+a)t} dt$$

$$= \left[\frac{e^{-(s+a)t}}{-(s+a)} \right]_0^{\infty} = \left[\frac{e^{-(\sigma + j\omega + a)t}}{-(s+a)} \right]_0^{\infty}$$

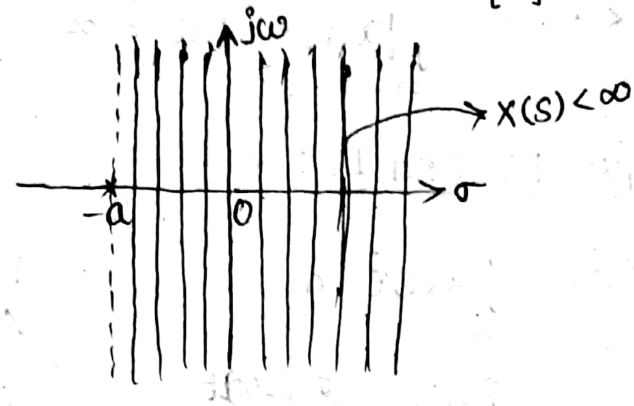
$$= \cancel{-1}\left[\frac{1}{\cancel{-}(s+a)}\right] = \left[\frac{e^{-(\sigma+a)t} \cdot e^{-j\omega t}}{-(s+a)} \right]_0^{\infty}$$

$$= \frac{1}{s+a}$$

$$= 0 - \frac{1}{-(s+a)}$$

$$X(s) = \frac{1}{s+a}$$

$$e^{-at}u(t) \xleftrightarrow{LT} \frac{1}{s+a}, \quad \text{ROC} \quad \sigma > \cancel{a}\, Re(-a), \quad Re\{s\} > Re\{-a\}$$



$\rightarrow x(t) = -e^{-at}u(-t)$

$$X(s) = \int_{-\infty}^{\infty} -e^{-at}u(-t).e^{-st}\,dt$$

$$= \int_{-\infty}^{0} -e^{-at}(1)e^{-st}\,dt$$

$$= -\int_{-\infty}^{0} e^{-(a+s)t}\,dt$$

$$= \left[\frac{e^{-(s+a)t}}{-(s+a)}\right]_0^{-\infty}$$

$$= \left.\frac{e^{-(\sigma+j\omega+a)t}}{-(s+a)}\right]_0^{-\infty}$$

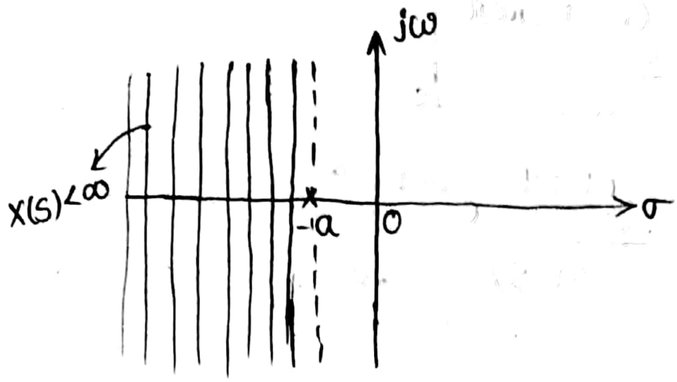
$$= \left.\frac{e^{-(\sigma+a)t}\cdot e^{-j\omega t}}{-(s+a)}\right]_0^{-\infty}$$

$$= \frac{1}{s+a}$$

$\sigma + a < 0$ $\left(e^{-(\sigma+a)t\cdot\infty} = 0 \text{ when } \sigma+a<0\right)$

$\sigma < Re\{-a\}$

ROC : $Re\{s\} < Re\{-a\}$



$\rightarrow x(t) = e^{at} u(+t)$

$\rightarrow x(t) = -e^{at} u(-t)$

1. $X(S) = \int_{-\infty}^{\infty} [e^{at} u(t)] e^{-st} dt$

$$= \int_{0}^{\infty} e^{at} e^{-st} dt$$

$$= \int_{0}^{\infty} e^{(a-s)t} dt$$

$$= \left. \frac{e^{(a-s)t}}{(a-s)} \right]_{0}^{\infty}$$

$$= \left. \frac{e^{(a-(\sigma+j\omega))t}}{(a-s)} \right]_{0}^{\infty}$$

$$= 0 - \frac{1}{a-s} = \frac{1}{s-a}$$

2. $X(S) = \int_{-\infty}^{\infty} -e^{+at} u(-t) e^{-st} dt$

$$= -\int_{-\infty}^{0} e^{at} \cdot e^{-st} dt$$

$$= -\int_{-\infty}^{0} e^{(a-s)t} dt$$

$$\left. \frac{e^{(a-s)t}}{(a-s)} \right]_{\infty}^{\infty}$$

$$= \left.\frac{e^{(a-(\sigma+j\omega))t}}{a-s}\right]_0^{-\infty}$$

$$= \left.\frac{e^{(a-\sigma)t} \cdot e^{-j\omega t}}{(a-s)}\right]_0^{-\infty}$$

$$= 0 - \frac{1}{a-s}$$

$$= \frac{1}{s-a}$$

$$\sigma < Re(a)$$

$$e^{-at}u(t) \xleftrightarrow{LT} \frac{1}{s+a} \; ; \; Re\{s\} > Re\{-a\}$$

$$-e^{-at}u(-t) \xleftrightarrow{LT} \frac{1}{s+a} \; ; \; Re\{s\} < Re\{-a\}$$

$$e^{at}u(t) \xleftrightarrow{LT} \frac{1}{s-a} \; ; \; Re\{s\} > Re\{a\}$$

$$-e^{at}u(-t) \xleftrightarrow{LT} \frac{1}{s-a} \; ; \; Re\{s\} < Re\{a\}$$

Ex:-

$$e^{3t}u(-t) \xleftrightarrow{LT} \frac{-1}{s-3} \; ; \; \sigma < 3$$

$$e^{-4t}u(t) \xleftrightarrow{LT} \frac{1}{s+4} \; ; \; Re(s) > -4$$

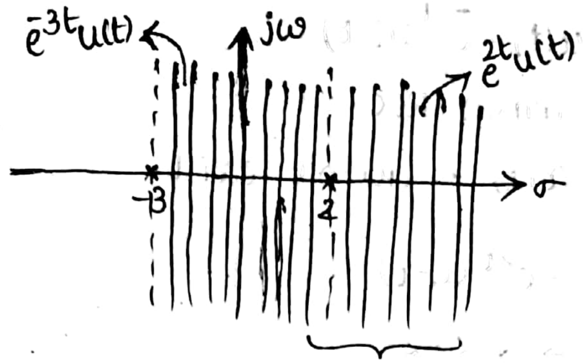
ex:-

$$x(t) = e^{2t}u(t) + e^{-3t}u(t)$$

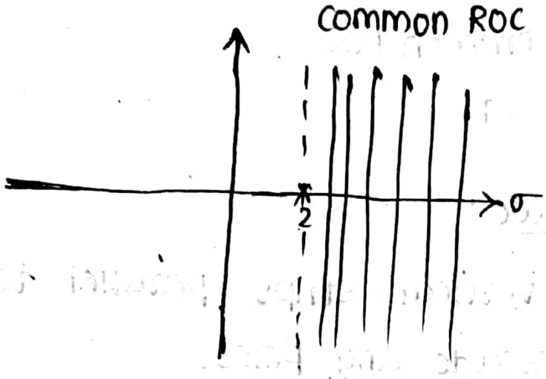
Here $s=2$ & $s=-3$ are poles

$$X(s) = LT\left[e^{2t}u(t)\right] + LT\left[e^{-3t}u(t)\right]$$

$$= \frac{1}{s-2} + \frac{1}{s+3} \; ; \; \sigma > 2 \;\&\; \sigma > -3$$



common ROC $\sigma > 2$



ex:- $x(t) = e^{2t}u(t) + e^{6t}u(t)$

$\sigma > 2 \qquad \sigma > 6$

ROC : $\sigma > 6$

$x(t) = e^{2t}u(t) + e^{6t}u(t) + e^{10t}u(t)$

$\sigma > 2 \qquad \sigma > 6 \qquad \sigma > 10$

ROC : $\sigma > 10$

ex:- $x(t) = e^{-2t}u(t) + e^{3t}u(-t)$

$X(S) = \frac{-1}{S+2} - \frac{1}{S-3}$

$\sigma < -2 \qquad \sigma < 3$

common ROC : $\sigma < -2$

$x(t) = e^{-2t}u(t) + e^{3t}u(-t) + e^{-6t}u(-t)$

$\sigma < -6 \rightarrow$ ROC

$x(t) = e^{-2t}u(-t) + e^{3t}u(-t) + e^{-6t}u(-t) + e^{2t}u(-t)$

$\sigma < -6 \rightarrow$ ROC

ex:- $x(t) = e^{-2t}u(t) - e^{2t}u(-t)$

$X(S) = \frac{1}{S+2} + \frac{1}{S-2}$

$\sigma > -2 \qquad \sigma < 2 \qquad$ ROC: $-2 < \sigma < 2$

$x(t) = e^{2t}u(t) - e^{-2t}u(-t)$

NO common ROC

LT of $x(t)$ — doesn't exist.

ex:- $e^{2t}u(t) - e^{-2t}u(-t)$

$\sigma > 2$ & $\sigma < -2$

NO common ROC

NO LT.

Properties of ROC :

1. ROC contains vertical strips parallel to $j\omega$ axis.
2. ROC doesnot include any poles.

For example $x(t) = e^{-2t}u(t)$

$$X(S) = \frac{1}{S+2}$$

ROC : $\sigma > -2$

$\sigma \neq -2$

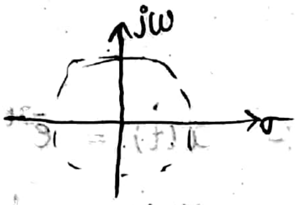
3. ROC of a finite duration absolutely integrable signal is entire s-plane.

ex:- $x(t) = \delta(t)$

4. ROC of a causal signal (right sided) is $Re(s) >$ the largest pole. (or) ROC is right of right most pole.
5. ROC of an anticausal signal (left sided) is $Re(s) <$ the smallest pole (or) ROC is left of leftmost pole.
6. ROC of a Non causal signal (both sided) is $\sigma_1 < Re(s) < \sigma_2$ (or) ROC exists inbetween the poles.
7. ROC of stable systems include $j\omega$ axis.

ex:- 1. $L[\delta(t)]$

$$X(S) = \int_{-\infty}^{\infty} \delta(t) \cdot e^{-st} dt$$



$$= \delta(0) e^{-s(0)}$$

$X(S) = 1$ ROC : entire s plane

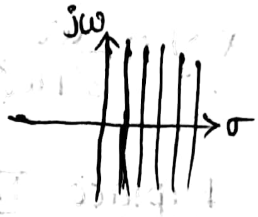
2. LT of $u(t)$

$$X(S) = \int_{-\infty}^{\infty} u(t)\cdot e^{-St}\, dt$$

$$= \int_{0}^{\infty} e^{-St} dt$$

$$LT[u(-t)] = \frac{-1}{S} \quad \boxed{\sigma < 0}$$

$$= \left. \frac{e^{-St}}{-S} \right]_0^{\infty}$$



$$X(S) = \frac{1}{S} \qquad \boxed{\sigma > 0}$$

3. LT of $e^{-a|t|}$

$$e^{-a|t|} = \begin{cases} e^{-at} u(t) & t \geq 0 \\ e^{at} u(-t) & t \leq 0 \end{cases}$$

$e^{-at}u(t)$ $e^{at}u(-t)$

$t > 0$ $t < 0$

$$e^{-a|t|} = e^{-at} u(t) + e^{at} u(-t)$$

$$= \frac{1}{S+a} - \frac{1}{S-a}$$

$$= \frac{S-a-S-a}{(S+a)(S-a)}$$

$$= \frac{-2a}{S^2 - a^2}$$

$$\sigma > -a \quad \sigma < a$$

$$\text{ROC}: -a < \sigma < a$$

4. LT of $[\cos\omega_0 t]\, u(t)$ (or) $x(t) = \cos\omega_0 t\ ;\ t \geq 0$

$$\cos\omega_0 t = \frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}$$

$$LT[\cos\omega_0 t \cdot u(t)] = LT\left[\frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}\right] u(t)$$

$$= \frac{1}{2}\left\{ L\left[e^{j\omega_0 t} u(t)\right] + L\left[e^{-j\omega_0 t} u(t)\right]\right\}$$

$$= \frac{1}{2}\left[\frac{1}{s-j\omega_0} + \frac{1}{s+j\omega_0}\right]$$

$$= \frac{s}{s^2+\omega_0^2} \; ; \text{ ROC } \sigma > 0$$

$\rightarrow x(t) = \sin\omega_0 t \, u(t)$

$$X(s) = \frac{\omega_0}{s^2+\omega_0^2} \; ; \sigma > 0$$

Inverse Laplace Transform :

$$LT\left[e^{-3t}u(t)\right] \xleftrightarrow{LT} \frac{1}{s+3} \; ; \sigma > -3$$

$$ILT\left[\frac{1}{s+3}\right] \xleftrightarrow{ILT} e^{-3t}u(t)$$

$$ILT\left[\frac{1}{s-3}\right] \xleftrightarrow{ILT} \begin{cases} \sigma > 3 \rightarrow e^{3t}u(t) \\ \sigma < 3 \rightarrow -e^{3t}u(-t) \end{cases}$$

ex:- ILT of $\left(\frac{1}{s+4}\right)$

case(i) : $\sigma > -4$

$$x(t) = e^{-4t}u(t)$$

case(ii): $\sigma < -4$

$$x(t) = -e^{-4t}u(-t)$$

ex:- ILT of $X(s) = \frac{1}{s-2} + \frac{1}{s+3}$

case(i) : $\sigma > 2$ $\quad x(t) = e^{2t}u(t) + e^{-3t}u(t)$

case(ii) : $\sigma < -3$ $\quad x(t) = e^{2t}[-u(-t)] + e^{-3t}[-u(-t)]$

case(iii): $\underbrace{-3 < \sigma < 2}$ $\quad x(t) = e^{2t}[-u(-t)] + e^{-3t}u(t)$

$\sigma > -3 \quad \sigma < 2$

ex:- $X(s) = \frac{1}{(s-2)(s+2)}$;

$$= \frac{A}{S-2} + \frac{B}{S+2}$$

$$\Rightarrow 1 = A(S+2) + B(S-2)$$

$$S = -2 \Rightarrow B = \frac{-1}{4}$$

$$S = 2 \Rightarrow A = \frac{1}{4}$$

$$= \frac{1}{4(S-2)} - \frac{1}{4(S+2)}$$

case(i) : $\sigma > 2$ $\quad x(t) = \frac{1}{4}\left[e^{2t}u(t) + e^{-2t}u(t)\right]$

case(ii) : $\sigma < -2$ $\quad x(t) = \frac{1}{4}\left[e^{2t}(-u(-t)) + e^{-2t}(-u(-t))\right]$

case(iii) : $-2 < \sigma < 2$ $\quad x(t) = \frac{1}{4}\left[e^{2t}(-u(-t)) + e^{-2t}u(t)\right]$

ex:- $X(S) = \frac{S}{(S+2)(S-3)}$; ROC : $-2 < \{Re(S)\} < 3$

$$X(S) = \frac{A}{S+2} + \frac{B}{S-3} = \frac{2/5}{S+2} + \frac{3/5}{S-3}$$

$$x(t) = \frac{2}{5}e^{-2t}u(t) + \frac{3}{5}e^{3t}(-u(-t))$$

$Re(S) > -2$ $\quad Re(S) < 3$

ex:- $X(S) = \frac{5(S+5)}{S(S+3)(S+7)}$; $\sigma > 0$

$X(S) = \frac{4S}{(S+3)(S+8)}$; $\sigma > -3$

Properties of Laplace Transform:

1. Linearity property:

If $x_1(t) \xleftrightarrow{LT} X_1(S)$; ROC : R_1

$x_2(t) \xleftrightarrow{LT} X_2(S)$; ROC : R_2

$ax_1(t) + bx_2(t) \xleftrightarrow{LT} aX_1(S) + bX_2(S)$; $R_1 \cap R_2$

2. Time Shifting:

$x(t) \xleftrightarrow{LT} X(S)$; $\overset{ROC}{R}$

$x(t-t_0) \xleftrightarrow{LT} e^{-st_0} X(S)$; R

$x(t+t_0) \xleftrightarrow{LT} e^{st_0} X(S)$; R

3. Time scaling:

$x(t) \xleftrightarrow{LT} X(S)$; R

$x(at) \xleftrightarrow{LT} \frac{1}{|a|} X\left(\frac{S}{a}\right)$; $|a|R$

4. ~~Frequency~~ S-domain Shifting:

$x(t) \xleftrightarrow{LT} X(S)$; ROC : R

$e^{at} x(t) \xleftrightarrow{LT} X(S-a)$; ROC : $R + Re\{a\}$

$e^{-at} x(t) \xleftrightarrow{LT} X(S+a)$; ROC : $R + Re\{-a\}$

Pb: LT of $e^{at} u(t)$ using F.S property

$u(t) \xleftrightarrow{LT} \frac{1}{S}$; $\sigma > 0$

$$X(s) = \frac{1}{s}$$

$$X(s-a) = \frac{1}{s-a} \quad ; \quad ROC: \sigma > 0 + Re(a)$$
$$\sigma > Re(a)$$

5. <u>Time</u> <u>Reversal</u> :

$$x(t) \xleftrightarrow{LT} X(s) \; ; \; ROC : R$$

$$x(-t) \xleftrightarrow{LT} X(-s) \; ; \; ROC : -R$$

6. <u>Conjugation</u> <u>property</u> :

$$x(t) \xleftrightarrow{LT} X(s) \quad ; \; ROC : R$$

$$x^*(t) \xleftrightarrow{LT} X^*(s^*) \; ; \; ROC : R$$

7. <u>convolution</u> <u>in</u> <u>time</u> :

$$x_1(t) \xleftrightarrow{LT} X_1(s) \quad ; \; ROC : R_1$$

$$x_2(t) \xleftrightarrow{LT} X_2(s) \quad ; \; ROC : R_2$$

$$x_1(t) * x_2(t) \xleftrightarrow{LT} X_1(s) \cdot X_2(s) \; ; \; ROC : R_1 \cap R_2$$

8. <u>Time</u> <u>Differentiation</u> :

$$x(t) \xleftrightarrow{LT} X(s) \; ; \; ROC : R$$

$$\frac{d}{dt}[x(t)] \xleftrightarrow{LT} s X(s) \; ; \; ROC : R$$

$$\frac{d^n}{dt^n}[x(t)] \xleftrightarrow{LT} s^n X(s)$$

9. <u>Integration</u> <u>in</u> <u>time</u> :

$$x(t) \xleftrightarrow{LT} X(s) \; ; \; ROC : R$$

$$\int_{-\infty}^{t} x(\tau)\,d\tau \xleftrightarrow{LT} \frac{X(s)}{s} \; ; \; ROC : R \cap \sigma > 0$$

10. <u>Differentiation</u> <u>in</u> <u>s-domain</u> :

$$x(t) \xleftrightarrow{LT} X(s) \; ; \; ROC : R$$

$$t \cdot x(t) \xleftrightarrow{LT} (-1)\frac{d}{dt}[X(s)] \; ; \; ROC : R$$

$$t^n \cdot x(t) \xleftrightarrow{LT} (-1)^n \frac{d^n}{ds^n}[X(s)] \; ; \text{ ROC : R}$$

11. Division with 't' / Integration with s-domain:

$$x(t) \xleftrightarrow{LT} X(s) \; ; \text{ ROC : R}$$

$$\frac{x(t)}{t} \xleftrightarrow{LT} \int_s^{\infty} X(s) \; ; \text{ ROC : R}$$

Initial value and Final value theorms:

Initial value theorm: (IVT)

If $x(t) \xleftrightarrow{LT} X(s)$

then $x(0) = \lim_{s\to\infty} sX(s)$

conditions:

* It is applicable only for causal signals i.e., $x(t) = 0 \; ; t<0$
* $x(t)$ must not contain any impulse terms or higher order singularities. (Discontinuities)

Final value theorm: (FVT)

If $x(t) \xleftrightarrow{LT} X(s)$

then $x(\infty) = \lim_{s\to 0} sX(s)$

conditions: * Poles must lie left half of the s-plane
* Applicable for causal signals.
* The signal must be absolutely integrable (or) FVT is applicable to stable systems.
* $\sin\omega t$, $\cos\omega t$ responses are considered as marginally stable systems.

LT < bi $= \int_{-\infty}^{\infty}$; uni $= \int_{0}^{\infty}$

Proof: (IVT)

For causal signals consider unilateral LT

$$X(s) = \int_0^{\infty} x(t)\, e^{-st}\, dt$$

$$L\left[\frac{d}{dt}[x(t)]\right] = SX(S) - x(0)$$

$$\int_0^\infty \frac{d}{dt}[x(t)]e^{-st}\,dt = SX(S) - x(0)$$

Apply $\lim_{S\to\infty}$ on both sides

$$\lim_{S\to\infty}\int_0^\infty \frac{d}{dt}[x(t)]e^{-st}\,dt = \lim_{S\to\infty}[SX(S) - x(0)]$$

$$\int_0^\infty \frac{d}{dt}[x(t)] \lim_{S\to\infty}(e^{-st})dt = \left[\lim_{S\to\infty} SX(S)\right] - x(0)$$

$$0 = \lim_{S\to\infty} SX(S) - x(0)$$

$$\Rightarrow x(0) = \lim_{S\to\infty} SX(S)$$

<u>Proof</u> : (FVT)

$$X(S) = \int_0^\infty x(t)e^{-st}\,dt$$

$$L\left[\frac{d}{dt}[x(t)]\right] = SX(S) - \cancel{x(\infty)}\ x(0)$$

$$\int_0^\infty \frac{d}{dt}[x(t)]e^{-st}\,dt = SX(S) - x(0)$$

Apply $\lim_{S\to 0}$ on both sides

$$\lim_{S\to 0}\int_0^\infty \frac{d}{dt}[x(t)]e^{-st}\,dt = \lim_{S\to 0}[SX(S) - x(0)]$$

$$\int_0^\infty \frac{d}{dt}[x(t)] \lim_{S\to 0}(e^{-st})dt = \left[\lim_{S\to 0} SX(S)\right] - x(0)$$

$$\int_0^\infty d[x(t)] = \lim_{S\to 0} SX(S) - x(0)$$

$$[x(t)]_0^\infty = \lim_{S\to 0} SX(S) - x(0)$$

$$x(\infty) - \cancel{x(0)} = \lim_{S\to 0} SX(S) - \cancel{x(0)}$$

$$x(\infty) = \lim_{S\to 0} SX(S)$$

Signals & Systems

Imp. Properties of Laplace Transforms

1. Time scaling

Property : $x(t) \longleftrightarrow X(s)$, ROC = R

$x(at) \longleftrightarrow \frac{1}{|a|} X\left(\frac{s}{a}\right)$, ROC = |a| R

Proof : When a = +ve

$$L[x(at)] = \int_{-\infty}^{\infty} x(at)\, e^{-st}\, dt$$

$\left(\text{Let } at = t' \Rightarrow t = \frac{t'}{a} \Rightarrow dt = \frac{dt'}{a} \right.$

$\left. \text{Limits } -\infty \text{ to } +\infty \right)$

$$= \int_{-\infty}^{\infty} x(t') \cdot e^{-s\left(\frac{t'}{a}\right)} \frac{dt'}{a}$$

$$= \frac{1}{a} \int_{-\infty}^{\infty} x(t') \cdot e^{-\left(\frac{s}{a}\right)t'} dt'$$

$$= \frac{1}{a} X\left(\frac{s}{a}\right)$$

When a = -Ve (say -a)

$$L[x(at)] = \int_{-\infty}^{\infty} x(at) \cdot e^{-st} dt$$

$\left(\text{Let, } -at = t' \Rightarrow t = \frac{-t'}{a} \right.$

$\Rightarrow dt = \frac{-dt'}{a}$

$\left. \text{limits } +\infty \text{ to } -\infty \right)$

$$= \int_{+\infty}^{-\infty} x(t') \cdot e^{-\left(\frac{s}{-a}\right)\cdot\left(\frac{-t'}{a}\right)} \cdot \left(\frac{-dt'}{a}\right)$$

$$= -\int_{+\infty}^{-\infty} x(t') \cdot e^{-\left(\frac{s}{a}\right)t'} \frac{dt'}{a}$$

$$= \frac{1}{a} \int_{-\infty}^{\infty} x(t') \cdot e^{-(s/a)t'} dt'$$

$$= \frac{1}{a} X\left(\frac{s}{a}\right)$$

$\therefore$ from both cases,

$$LT[x(at)] = \frac{1}{|a|} X\left(\frac{s}{a}\right)$$

Time Shifting

Property : $x(t) \longleftrightarrow X(s)$, ROC : R

$x(t-t_0) \longleftrightarrow X(s) \cdot e^{-st_0}$, ROC : R

$x(t+t_0) \longleftrightarrow X(s) \cdot e^{st_0}$

Proof: $$\mathcal{L}[x(t-t_0)] = \int_{-\infty}^{\infty} x(t-t_0) \cdot e^{-st} dt$$

Let $t-t_0 = t' \Rightarrow dt = dt'$ Limits : $-\infty$ to $+\infty$

$t = t'+t_0$

$$= \int_{-\infty}^{\infty} x(t') \cdot e^{-s(t'+t_0)} \cdot dt'$$

$$= \int_{-\infty}^{\infty} x(t') \cdot e^{-st'} \cdot e^{-st_0} dt'$$

$$= e^{-st_0} \left[\int_{-\infty}^{\infty} x(t') \cdot e^{-st'} dt' \right]$$

$$= e^{-st_0} \cdot X(s)$$

$$\therefore \; x(t-t_0) \longleftrightarrow e^{-st_0} \cdot X(s)$$

3. Frequency shifting: Property:

$x(t) \longleftrightarrow X(s)$, ROC: R

$e^{at} x(t) \longleftrightarrow X(s-a)$, ROC: R+I

$e^{-at} x(t) \longleftrightarrow X(s+a)$, ROC: R+I

Proof:

$$\mathcal{L}[e^{at} x(t)] = \int_{-\infty}^{\infty} (e^{at} \cdot x(t)) e^{-st} dt$$

$$= \int_{-\infty}^{\infty} x(t) \cdot e^{-(s-a)t} dt$$

$$= X(s-a)$$

Ily, $\mathcal{L}[e^{-at} x(t)] = X(s+a)$

$$\therefore \; e^{\pm at} x(t) \longleftrightarrow X(s \mp a)$$

$t.$ Convolution in time:

property: $x_1(t) \longleftrightarrow X_1(s)$ $\quad ROC = R_1$

$x_2(t) \longleftrightarrow X_2(s)$ $\quad ROC = R_2$

$x_1(t) * x_2(t) \longleftrightarrow X_1(s) \cdot X_2(s)$ $\quad ROC = R_1 \cap R_2$

Proof: $$L[x_1(t) * x_2(t)] = \int_{-\infty}^{\infty} [x_1(t) * x_2(t)]\, e^{-st}\, dt$$

$$= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x_1(\tau)\, x_2(t-\tau)\, d\tau\right] e^{-st}\, dt$$

$\left[\text{let } t-\tau = t' \Rightarrow t = t'+\tau \right.$
$\left. \qquad \text{limits: } t' : -\infty \text{ to } \infty \right]$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_1(\tau)\, x_2(t')\, d\tau \cdot e^{-s(t'+\tau)}\, dt'$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_1(\tau) \cdot x_2(t') \cdot d\tau \cdot e^{-st'} \cdot e^{-s\tau}\, dt'$$

$$= \left(\int_{-\infty}^{\infty} x_1(\tau) \cdot e^{-s\tau}\, d\tau\right)\left(\int_{-\infty}^{\infty} x_2(t') \cdot e^{-st'}\, dt'\right)$$

$$= X_1(s) \cdot X_2(s)$$

$$\Rightarrow L[x_1(t) * x_2(t)] = X_1(s) \cdot X_2(s)$$

5. Time Differentiation

Property:
$$x(t) \longleftrightarrow X(s) \quad , \text{ROC} = R$$
$$\frac{d}{dt}(x(t)) \longleftrightarrow s \cdot X(s) \quad , \text{ROC} = R$$

Proof: $L\left[\frac{d}{dt} x(t)\right] = \int_{-\infty}^{\infty} \frac{d}{dt}(x(t)) \cdot e^{-st}\, dt$

$$= \int_{-\infty}^{\infty} e^{-st} \cdot d(x(t))$$

$$= e^{-st} \int_{-\infty}^{\infty} d\, x(t) - \int_{-\infty}^{\infty} \int d(x(t)) \frac{d}{dt}(e^{-st})\, dt$$

$$= e^{-st} \cdot x(t) \Big]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} x(t) \cdot (-s) \cdot e^{-st}\, dt$$

$$= (0 - 0) + s \underbrace{\int_{-\infty}^{\infty} x(t) \cdot e^{-st}\, dt}_{X(s)}$$

$$= s \cdot X(s)$$

$$\therefore L\left[\frac{d}{dt}(x(t))\right] = s \cdot X(s).$$

Ily, $L\left[\frac{d^n}{dt^n}(x(t))\right] = s^n \cdot X(s)$

6. Integration in time

property: $x(t) \longleftrightarrow X(s)$, ROC = R

$$\int_{-\infty}^{t} x(\tau) d\tau \longleftrightarrow \frac{X(s)}{s}, \quad ROC = R \cap Re(s) > 0$$

Proof: Consider,

$$x(t) * u(t) = \int_{-\infty}^{\infty} x(\tau)\, u(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{t} x(\tau).(1)\, d\tau + \int_{t}^{+\infty} 0\, d\tau$$

$$\Rightarrow x(t) * u(t) = \int_{-\infty}^{t} x(\tau)\, d\tau$$

$$\Rightarrow L[x(t) * u(t)]$$

$$= L\left[\int_{-\infty}^{t} x(\tau)\, d\tau\right]$$

$$= L[x(t)] \cdot L[u(t)]$$

$$= X(s) \cdot \frac{1}{s}$$

$$\Rightarrow L\left[\int_{-\infty}^{t} x(\tau)\, d\tau\right] = \frac{X(s)}{s}$$

Differentiation in frequency (Multiplication with 't')

Property: $x(t) \longleftrightarrow X(s)$, ROC = R

$$t \cdot x(t) \longleftrightarrow -\frac{d}{ds} X(s), \quad ROC = R$$

$$t^n\, x(t) \longleftrightarrow (-1)^n \frac{d^n}{ds^n} X(s), \quad ROC = R$$

Proof:

$$X(s) = \int_{-\infty}^{\infty} x(t). e^{-st}\, dt$$

$$\Rightarrow \frac{d}{ds}[X(s)] = \int_{-\infty}^{\infty} x(t). \frac{d}{ds}(e^{-st})\, dt$$

$$= \int_{-\infty}^{\infty} x(t). e^{-st}(-t)\, dt$$

$$-\frac{d}{ds}[X(s)] = \int_{-\infty}^{\infty} [t \cdot x(t)]. e^{-st}\, dt$$

$$= L[x(t).t]$$

$$\Rightarrow L[t \cdot x(t)] = (-1) \frac{d}{ds}(X(s))$$

$$\text{lly} \quad L[t^n x(t)] = (-1)^n \frac{d^n}{ds^n}[X(s)]$$

8. Integration in freq / Division with 't'

property:

$$x(t) \longleftrightarrow X(s) \quad , ROC = R$$

$$\frac{x(t)}{t} \longleftrightarrow \int_s^{\infty} X(s)\, ds \quad , ROC = R$$

Proof:

$$X(s) = \int_{-\infty}^{\infty} x(t) \cdot e^{-st}\, dt$$

$$\int_s^{\infty} X(s)\, ds = \int_s^{\infty} \int_{-\infty}^{\infty} x(t) \cdot e^{-st}\, dt$$

$$= \int_{-\infty}^{\infty} x(t) \int_s^{\infty} e^{-st}\, dt\, ds$$

$$= \int_{-\infty}^{\infty} x(t) \cdot \left(\frac{e^{-st}}{-t}\right)_s^{\infty} ds$$

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$$= \int_{-\infty}^{\infty} \left(\frac{x(t)}{t}\right) \cdot e^{-st}\, dt$$

$$= L\left[\frac{x(t)}{t}\right]$$

$$\Rightarrow L\left[\frac{x(t)}{t}\right] = \int_s^{\infty} X(s)\, ds.$$

9. Initial value theorem:

St: $x(t) \longleftrightarrow X(s)$

$$x(0) = \lim_{s \to \infty} s.X(s)$$

Conditions: → Applicable only when $x(t) = 0$, $t < 0$ (Ca

→ $x(t)$ must not contain any impulse or higher order singularities at $\underline{t=0}$ (discontinuities)

Proof: We know that,

$$\frac{d\,x(t)}{dt} \longleftrightarrow s.X(s) - x(0^-) \quad \text{(for unilateral Laplace transform)}$$

and, $$L\left[\frac{d}{dt}x(t)\right] = \int_{0^-}^{\infty} \frac{d}{dt}x(t) \cdot e^{-st}\,dt$$

$0^-\ 0\ 0^+ \rightarrow t$

$$= \int_{0^-}^{0^+} \frac{d}{dt}x(t) \cdot e^{-st}\,dt + \int_{0^+}^{\infty} \frac{d}{dt}x(t) \cdot e^{-st}\,dt$$

$$= \int_{0^-}^{0^+} \frac{d}{dt}x(t) + \int_{0^+}^{\infty} \frac{d}{dt}x(t) \cdot e^{-st}\,dt$$

(from 0^- to 0^+ $e^{-st} = 1$)

$$L\left[\frac{d}{dt}(x(t))\right] = [x(t)]_{0^-}^{0^+} + \int_{0^+}^{\infty} \frac{d}{dt}x(t) \cdot e^{-st}\,dt$$

$$\Rightarrow s.X(s) - \cancel{x(0^-)} = x(0^+) - \cancel{x(0^-)} + \int_{0^+}^{\infty} \frac{d}{dt}x(t) \cdot e^{-st}\,dt$$

Apply $\lim_{s \to \infty}$ on both sides

$$\Rightarrow \lim_{s \to \infty} s.X(s) = x(0^+) + \int_{0^+}^{\infty} \frac{d}{dt}x(t) \cdot (0)\,dt$$

$$\Rightarrow x(0) = \lim_{s \to \infty} s.X(s)$$

1. Final value Theorem:

$$x(t) \longleftrightarrow X(s)$$

St:

$$x(\infty) = \lim_{t \to \infty} x(t) = \lim_{s \to 0} s.X(s)$$

Conditions: → Applicable only when $x(t)=0, t<0$

→ $s.X(s)$ must have poles in left half of s-plane

Proof: We know that,

$$\frac{dx(t)}{dt} \longleftrightarrow s\,X(s) - x(0^-)$$

$$\therefore \quad s.X(s) - x(0^-) = \int_{0^-}^{\infty} \frac{dx(t)}{dt} \cdot e^{-st}\, dt$$

$$\lim_{s \to 0} s\,X(s) - \lim_{s \to 0} x(0^-) = \int_{0^-}^{\infty} \frac{d}{dt} x(t) \cdot \lim_{s \to 0} e^{-st}\, dt$$

$$\Rightarrow \lim_{s \to 0} s\,X(s) - x(0^-) = \int_{0^-}^{\infty} \frac{d\,x(t)}{dt} (1)\, dt$$

$$\Rightarrow \lim_{s \to 0} s.X(s) - x(0^-) = \left[x(t)\right]_{0^-}^{\infty}$$

$$\Rightarrow \lim_{s \to 0} s.X(s) - \cancel{x(0^-)} = x(\infty) - \cancel{x(0^-)}$$

$$\therefore \quad x(\infty) = \lim_{s \to 0} s\,X(s)$$

Other properties

11. Linearity:

$$x_1(t) \longleftrightarrow X_1(s) \quad , ROC: R_1$$

$$x_2(t) \longleftrightarrow X_2(s) \quad , ROC: R_2$$

$$a\,x_1(t) + b\,x_2(t) \longleftrightarrow a.X_1(s) + b\,X_2(s) \quad , ROC = R_1 \cap R_2$$

*12. Time Reversal:

$$x(t) \longleftrightarrow X(s) \quad , ROC: R$$

$$x(-t) \longleftrightarrow X(-s) \quad , ROC: -R$$

13 Conjugation

$$x(t) \longleftrightarrow X(s) \quad , ROC: R$$

$$x^*(t) \longleftrightarrow X^*(s^*) \quad , ROC = R$$

Q) LT of $e^{-6t}\sin\omega_0 t\, u(t)$

$$x(t) \xleftrightarrow{LT} X(s)$$

$$e^{-at}x(t) \xleftrightarrow{LT} X(s+a)$$

$$x(t) = \sin\omega_0 t\, u(t) \xleftrightarrow{LT} \frac{\omega_0}{s^2+\omega_0^2} = X(s); \quad ROC: \sigma>0$$

$$e^{-at}\sin\omega_0 t\, u(t) \xleftrightarrow{LT} \frac{\omega_0}{(s+a)^2+\omega_0^2} \quad ROC: \sigma>-a$$

$$e^{-6t}\sin\omega_0 t\, u(t) \xleftrightarrow{LT} \frac{\omega_0}{(s+6)^2+\omega_0^2}; \quad ROC: \sigma>-6$$

Q) LT of $-e^{-\alpha t}\cos\omega_0 t\, u(-t)$

$$x(t) = \cos\omega_0 t\, u(-t)$$

$$X(s) = \frac{-s}{s^2+\omega_0^2}$$

$$X(s+\alpha) = \frac{(s+\alpha)}{(s+\alpha)^2+\omega_0^2}; \quad \sigma<-\alpha$$

Q) ILT of $X(s) = \frac{1}{s^2+2s+2}$

$$= \frac{1}{s^2+2s+1+1}$$

$$= \frac{1}{(s+1)^2+1}$$

$$ILT\left[\frac{1}{s^2+1}\right] = \sin t\, u(t)$$

$$s \to s+1 \quad ILT\left[\frac{1}{(s+1)^2+1}\right] = e^{-t}\sin t\, u(t)$$

Q) ILT of $X(s) = \frac{s}{s^2+2s+2}; \quad \sigma>-1$

$$= \frac{s}{s^2+2s+1+1}$$

$$= \frac{s}{(s+1)^2+1} = \frac{(s+1)-1}{(s+1)^2+1}$$

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$$= \frac{S+1}{(S+1)^2+1} - \frac{1}{(S+1)^2+1}$$

$$\downarrow \qquad \downarrow$$

$$= e^{-t}\cos t\, u(t) \quad e^{-t}\sin t\, u(t)$$

$$x(t) = e^{-t}\cos t\, u(t) - e^{-t}\sin t\, u(t)$$

$$x(t) = e^{-t}(\cos t - \sin t)\, u(t)$$

Q) ILT of $X(S) = \frac{e^{-2S}}{S^2+2S+2}$; $\sigma > -1$

$$= \frac{e^{-2S}}{(S+1)^2+1}$$

$$= e^{-2S}\left[\frac{1}{(S+1)^2+1}\right]$$

$\cancel{x(t) = \sin(t-2)\, u(t)}$

$$x(t-t_0) \xleftrightarrow{LT} e^{-st_0} X(S)$$

$$x(t-2) \xleftrightarrow{LT} e^{-2S} X(S)$$

$$X(S) = \frac{1}{(S+1)^2+1}$$

$$x(t) = e^{-t}\sin t\, u(t)$$

$$x(t-2) = e^{-(t-2)}\sin(t-2)\, u(t-2)$$

Q) LT of $-te^{-\alpha t}u(-t)$

$$t\,x(t) \xleftrightarrow{LT} (-1)\frac{d}{dS}[X(S)]$$

here, $x(t) = -e^{-\alpha t}u(t)$

$$X(S) = \frac{1}{S+\alpha} \; ; \; ROC : \sigma < -\alpha$$

$$LT\left[-te^{-\alpha t}u(-t)\right] = -\frac{d}{dS}[X(S)]$$

$$= -\frac{d}{dS}\left[\frac{1}{S+\alpha}\right] = -\left(\frac{-1}{(S+\alpha)^2}\right)$$

$$= \frac{1}{(S+\alpha)^2} \quad ROC : \sigma > -2$$

Q) LT of $e^{-5t} u(t-2)$

$$\downarrow$$

$$e^{-5(t-2+2)} u(t-2)$$

$$e^{-5(t-2)} u(t-2) \cdot e^{-5(2)}$$

$$= e^{-10}\left[e^{-5(t-2)} u(t-2)\right]$$

$$x(t) = e^{-5t} u(t)$$

$$X(S) = \frac{1}{S+5} \; ; \; \sigma > -5$$

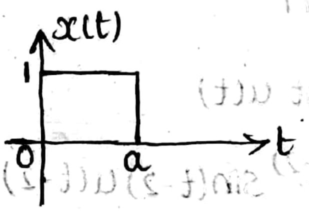
Property: $x(t-2) \xleftrightarrow{LT} e^{-2S} X(S)$

$$LT\left[e^{-5t} u(t-2)\right] = LT\left[e^{-10} x(t-2)\right]$$

$$= e^{-10} \cdot e^{-2S} X(S) = \frac{e^{-(10+2S)}}{S+5}$$

Laplace Transform based on wave form synthesis:

Find LT of wave form:



$$x(t) = \begin{cases} 1 \; ; & 0 < t < a \\ 0 \; ; & \text{otherwise} \end{cases}$$

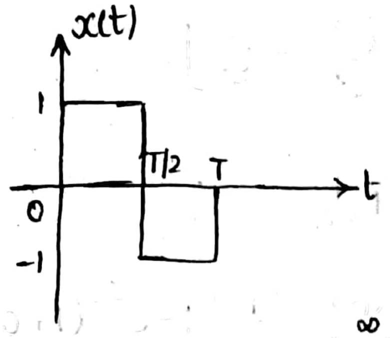
$$X(S) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

$$= \int_{0}^{a} 1 \cdot e^{-st} dt$$

$$= \left. \frac{e^{-st}}{-s} \right]_0^a$$

$$= \frac{e^{-as}}{-s} - \frac{1}{-s} = \frac{1 - e^{-as}}{s}$$

$$X(S) = \frac{1 - e^{-as}}{S} \; ; \text{ ROC : entire s-plane}$$



$$x(t) = \begin{cases} 1, & 0<t<T/2 \\ -1, & T/2<t<T \end{cases}$$

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}\,dt$$

$$= \int_{0}^{T/2} 1\cdot e^{-st}\,dt + \int_{T/2}^{T} -1\cdot e^{-st}\,dt$$

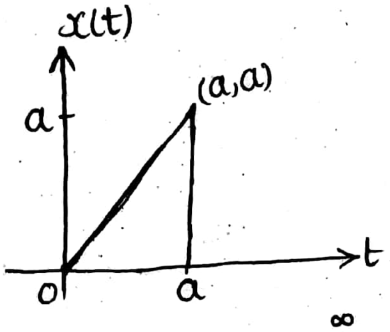
$$= \left.\frac{e^{-st}}{-s}\right]_0^{T/2} + (-)\left[\frac{e^{-st}}{-s}\right]_{T/2}^{T}$$

$$= \frac{e^{-sT/2}}{-s} + \frac{1}{s} + \frac{e^{-sT}}{s} - \frac{e^{-sT/2}}{s}$$

$$= -\frac{2e^{-sT/2}}{s} + \frac{1+e^{-sT}}{s}$$

$$= \frac{1}{s}\left(1+e^{-sT} - 2e^{-sT/2}\right)$$

$$= \frac{1}{s}\left(1-e^{-sT/2}\right)^2$$



$$x(t) = \begin{cases} t, & 0<t<a \\ 0, & \text{otherwise} \end{cases}$$

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}\,dt$$

$$= \int_{0}^{a} \underset{u}{t}\,\underset{v}{e^{-st}}\,dt$$

$$= t\left[\frac{e^{-st}}{-s}\right]_0^a - \int_0^a 1\cdot\left[\frac{e^{-st}}{-s}\right]dt$$

$$= a\left[\frac{e^{-as}}{-s} + \frac{a}{s}\right] - \left[\frac{e^{-ast}}{s^2}\right]_0^a$$

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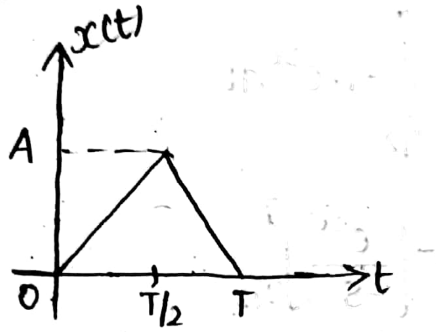
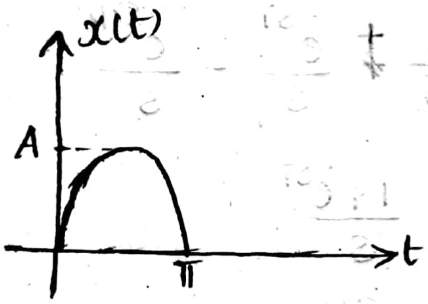
$$= \left[\frac{a e^{-as}}{-s} + \frac{a}{s}\right] - \left(\frac{e^{-as}}{s^2} - \frac{1}{s^2}\right)$$

$$= \frac{-a e^{-as}}{s} - \frac{e^{-as}}{s^2} + \frac{1}{s^2}$$

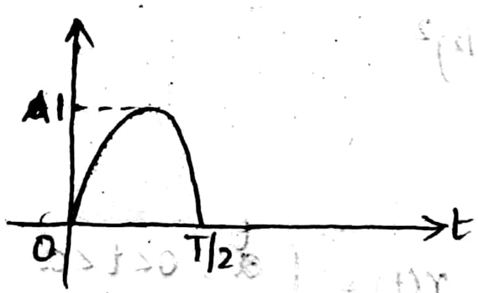
$$= \frac{1}{s^2}\left(1 - e^{-as} - as e^{-as}\right) = \frac{1}{s^2}\left[1 - e^{-as}(1+as)\right]$$



Ans: $X(s) = \frac{2A}{Ts^2}\left(1 - e^{-sT/2}\right)^2$



Ans: $X(s) = \frac{A}{s^2+1}\left(1 + e^{-\pi s}\right)$



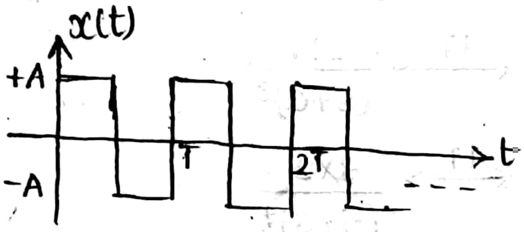
Laplace Transform of Periodic Signals:

If $X(S)$ is LT of $x(t)$ and $x(t)$ is a periodic signal with period T then LT of periodic signal $x(t)$ is given by

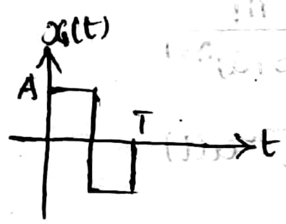
$$X(S) = \frac{1}{1-e^{-ST}} \underbrace{\int_0^T x_1(t)\, e^{-St}\, dt}$$

one fundamental Time period.

Q) LT of periodic square wave:



Fundamental Time period : T



$$\Rightarrow \frac{A}{S}\left[1-e^{-ST/2}\right]^2$$

$$X(S) = \frac{1}{1-e^{-ST}}\left(\frac{A}{S}(1-e^{-ST/2})^2\right)$$

$$= \frac{1}{\cancel{(1-e^{-ST/2})}(1+e^{-ST/2})}\ \frac{A}{S}(1-e^{-ST/2})^{\cancel{2}}$$

$$= \frac{A}{S}\left[\frac{1-e^{-ST/2}}{1+e^{-ST/2}}\right]$$

Q) LT of $t\cdot u(t)$

LT of $u(t)$ — $\frac{1}{S}$

$$x(t) \xrightarrow{LT} \frac{1}{S} = X(S)$$

$$t\cdot x(t) \longrightarrow -\frac{d}{dS}[X(S)]$$

$$= -\frac{d}{dS}\left[\frac{1}{S}\right] = \frac{-1}{S^2}$$

$$X(S) = \int_{-\infty}^{\infty} x(t) e^{-St}\, dt$$

$$= \int_{0}^{\infty} t e^{-St}\, dt$$

$$= \frac{1}{S^2}$$

$$t e^{-at} u(t) \xleftrightarrow{LT} \frac{1}{(S+a)^2}$$

$$t^2 e^{-at} u(t) \xleftrightarrow{LT} \frac{2 \times 1}{(S+a)^3}$$

$$t^3 e^{-at} u(t) \xleftrightarrow{LT} \frac{3 \times 2}{(S+a)^4}$$

$$\vdots$$

$$t^n e^{-at} u(t) \xleftrightarrow{LT} \frac{n!}{(S+a)^{n+1}}$$

Q) $ILT\left[\frac{1}{(S+5)^{10}}\right] \Rightarrow \frac{1}{9!} t^9 e^{-5t} u(t)$

$$\frac{t^n}{n!} e^{-at} u(t) \xleftrightarrow{LT} \frac{1}{(S+a)^{n+1}}$$

Q) $ILT\left[\frac{e^{-5S}}{(S+2)^{95}}\right] \Rightarrow x(t) = \frac{t^{94}}{94!} e^{-2t} u(t)$

$$x(t-5) = \frac{(t-5)^{94}}{94!} e^{-2(t-5)} u(t-5)$$

* <u>Problems on LT</u> :

1) Find the initial value of

a) $X(S) = \dfrac{3}{S^2+5S-1}$

b) $X(S) = \dfrac{2S+3}{S(S^2+5S+6)}$

a) $x(0) = \lim\limits_{S\to\infty} SX(S)$

$= \lim\limits_{S\to\infty} S \cdot \dfrac{3}{S^2+5S-1}$

$= \lim\limits_{S\to\infty} \cancel{S}\left(\dfrac{3}{S^{\cancel{2}}(1+5/S-1/S^2)}\right)$

$= \lim\limits_{S\to\infty} \dfrac{3/S}{1+\frac{5}{S}-\frac{1}{S^2}} = 0$

b) $x(0) = \lim\limits_{S\to\infty} \cancel{S}\left(\dfrac{2S+3}{\cancel{S}(S^2+5S+6)}\right)$

$$= \frac{\cancel{s}(2+3/s)}{s\cancel{s^2}(1+5/s+6/s^2)} = 0$$

* Find the final value of $X(s)$ =

1) $\frac{1}{s-2} \Rightarrow x(t) = e^{2t}u(t)$ unstable

Left Right

$\downarrow$ $s=2$ pole on right of s-plane – unstable system

$\frac{1}{s+2} \Rightarrow x(t) = e^{-2t}u(t)$ BS

Hence final value $= \infty$

2) $\frac{s-1}{s(s+1)} \Rightarrow x(\infty) = \lim_{s\to 0} sX(s)$

$$= \lim_{s\to 0} \cancel{s}\left(\frac{s-1}{\cancel{s}(s+1)}\right)$$

$$= \frac{-1}{+1} = -1$$

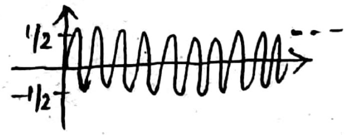
final value is -1

3) $\frac{1}{s^2+4}$

Poles: $s^2+4=0 \Rightarrow s = \pm 2j$

The poles lies on $j\omega$ axis, (imaginary axis) Hence final value is undefined. These systems are called marginally stable systems.

$x(t) = \frac{1}{2}\sin 2t$



* Find the initial and final values of $\frac{s+5}{s^2+3s+2}$

$$x(0) = \lim_{s\to\infty} sX(s)$$

$$= \lim_{s\to\infty} \cancel{s}\left(\frac{\cancel{s}(1+5/s)}{\cancel{s^2}(1+3/s+2/s^2)}\right) = 1$$