3.5

Probability Generating Function

Let X be a non-negative integer valued random variable with p.m.f. $p(x) = P(X = x)$. Then the **probability generating function** (p.g.f.) of X is defined by

$$\boldsymbol{G_X(t) = E[t^X] = \sum_{x=0}^{\infty} t^x p(x)}$$

where $-1 \leq t \leq 1$ is a dummy variable.

Advantages:

1. It is easy to compute.
2. Moments and some probabilities can be obtained easily.
3. The p.m.f. can be obtained easily from p.g.f.
4. It is easy to handle with sum of independent r.vs.

Effect of linear transformation of p.g.f:

Theorem 1: Let $\boldsymbol{X}$ be a discrete random variable with p.g.f. $\boldsymbol{G_X(t)}$. Let $\boldsymbol{Y = a + bX}$ where $\boldsymbol{a}$ and $\boldsymbol{b}$ are real constants. Then $\boldsymbol{G_Y(t) = t^a G_X\left(t^b\right)}$

Proof: By the definition of probability generating function, we have, $G_X(t) = E[t^X]$. Then

$$G_Y(t) = E\left[t^{(a+bX)}\right] = E[t^a t^{bX}] = t^a E[(t^b)^X] = t^a G_X(t^b)$$

$$\Longrightarrow G_Y(t) = t^a G_X(t^b)$$

Theorem 2: Additive Property: If X and Y are independent random variables, then for constants a, b, we have

$$\boldsymbol{G_{(aX+bY)}(t) = G_X(t^a) + G_Y(t^b)}$$

Proof: $G_{aX+bY}(t) = E[t^{aX+bY}]$ (by P.g.f.)

$$= E[(t^a)^X (t^b)^Y]$$

$$= E[(t^a)^X] E[(t^b)^Y] \quad (\because X \& Y \text{ are indpendent.})$$

$$= G_X(t^a) G_Y(t^b)$$

Thus, $G_{aX+bY}(t) = G_X(t^a) G_Y(t^b)$

Note: In particular, if $a = b = 1$, then $\boldsymbol{G_{X+Y}(t) = G_X(t) G_Y(t)}$

Generalization: If $X_1, X_2, \dots, X_n$ are independent random variables, then

$$\boldsymbol{G_{(X_1+\dots+X_n)}(t) = G_{X_1}(t) G_{X_2}(t) \dots G_{X_n}(t)}$$

Relationship between p.g.f. and m.g.f :

The p.g.f. and m.g.f. of a random variable X are defined by $G_X(t) = E[t^X]$ and $M_X(t) = E[e^{tX}]$ respectively.

Now, $M_X(t) = E[e^{tX}] = E[(e^t)^X] = G_X(e^t)$

$$\Longrightarrow \boldsymbol{M_X(t) = G_X(e^t)}$$

Further, $G_X(t) = E[t^X] = E\left[e^{ln(t^X)}\right] = E[e^{X \ln t}] = M_X(ln\, t)$

$$\Longrightarrow \boldsymbol{G_X(t) = M_X(ln\, t)}$$

Theorem 3: p.m.f. from p.g.f : Let $G_X(t)$ be the p.g.f. of a discrete r.v. X that can take the values 0,1,2,…. Then the p.m.f. of X is given by

$$\boldsymbol{p(x) = P(X = x) = \frac{1}{x!} G_X^{(x)}(t)\Big|_{t=0}}$$

Proof: By definition, we have

$$G_X(t) = E(t^X) = \sum_{x=0}^{\infty} t^x p(x)$$

$$= P(X = 0)t^0 + P(X = 1)t^1 + P(X = 2)t^2 + \cdots + P(X = x)t^x + \cdots$$

It can be observed that the coefficient of t^x in $G_X(t)$ is $P(X = x)$. To obtain coefficient of t^x, differentiate $G_X(t)$, x times and substitute $t = 0$. Thus,

$$G_X^{(x)}(t) = x(x-1)(x-2)\dots 2 \cdot 1 \cdot P(X = x) + (x+1)(x) + \cdots 2 \cdot 1 \cdot t \cdot P(X = x+1) + \cdots$$

When $t = 0$, all terms after the first vanish. Thus,

$$\boldsymbol{P(X = x) = \frac{1}{x!} G_X^{(x)}(t)\Big|_{t=0} = \frac{1}{x!} G_X^{(x)}(0)}$$

Computation of moments using p.g.f:

In the derivation of moments, we use *Taylor's expansion*:

Suppose $f(x)$ has derivatives of all orders at $x = a$. The Taylor's expansion of $f(x)$ at the point $x = a$ is givne by

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^n(a)}{n!}(x-a)^n + \cdots$$

$$\Rightarrow f(x) = \sum_{i=0}^{\infty} \frac{f^i(a)}{i!}(x-a)^i$$

The Taylor's expansion of $f(t) = t^X$ about $t = 1$ is given by

$$t^X = 1 + X(t-1) + X(X-1)\frac{(t-1)^2}{2!} + X(X-1)(X-2)\frac{(t-1)^3}{3!} + \cdots$$

$$\Rightarrow G_X(t) = E[t^X]$$

$$= 1 + (t-1)E(X) + \frac{(t-1)^2}{2!}E[X(X-1)] + \frac{(t-1)^3}{3!}E[X(X-1)(X-2)] + \cdots$$

Differentiating (1) w.r.t., t r times and setting $t = 1$, we get

$$G_X^{(r)}(t)\Big|_{t=1} = E[X(X-1)\dots(X-r+1)]$$

$$\Rightarrow \boldsymbol{E[X(X-1)\dots(X-r+1)] = G_X^{(r)}(1)} \qquad \dots(2)$$

which is known as $\boldsymbol{r}^{\text{th}}$ **factorial moment of $\boldsymbol{X}$**. Using these, we can find the moments about origin as follows:

If $r = 1$ in (2), we have

$$\boldsymbol{\mu_1' = E(X) = G_X^{(1)}(1)}$$

If $r = 2$ in (2), we have

$$E[X(X-1)] = E[X^2 - X] = E(X^2) - E(X) = G_X^{(2)}(1)$$

$$\Rightarrow E(X^2) = G_X^{(2)}(1) + E(X) = G_X^{(2)}(1) + G_X^{(1)}(1)$$

Thus, the second moment about origin is given by

$$\boldsymbol{\mu_2' = E(X^2) = G_X^{(2)}(1) + G_X^{(1)}(1)}$$

Similarly, we can find any moment about origin.

Computation of mean and variance using p.g.f:

Theorem 4: If the r.v. X has p.g.f. $\boldsymbol{G_X(t)}$, then the mean and variance of $\boldsymbol{X}$ are given by

$$\boldsymbol{\mu = E(X) = G_X^{(1)}(1)} \text{ and}$$

$$\boldsymbol{\sigma^2 = V(X) = G_X^{(2)}(1) + G_X^{(1)}(1) - \left[G_X^{(1)}(1)\right]^2}$$

respectively.

Proof: From the above, we have

$$\mu_1' = G_X^{(1)}(1),\ \mu_2' = G_X^{(2)}(1) + G_X^{(1)}(1)$$

Thus , the mean $\mu = \mu_1' = G_X^{(1)}(1)$ and variance $\sigma^2 = \mu_2' - (\mu_1')^2$

$$\Longrightarrow \sigma^2 = G_X^{(2)}(1) + G_X^{(1)}(1) - \left(G_X^{(1)}(1)\right)^2$$

Convolution formula:

Theorem 5: If X and Y are independent integer-valued random variables with $\boldsymbol{P(X=x)=p_1(x)}$ and $\boldsymbol{P(Y=y)=p_2(y)}, \boldsymbol{x=0,1,2,\ldots}$ and $\boldsymbol{y=0,1,2,\ldots}$, then

$$\boldsymbol{P(X+Y=z)=p(z)=\sum_{x=0}^{z} p_1(x)p_2(z-x)}$$

Proof: We have ,

$$G_X(t) = \sum_{x=0}^{\infty} t^x p_1(x) \quad and \quad G_Y(t) = \sum_{y=0}^{\infty} t^y p_2(y)$$

Now, $\quad G_{X+Y}(t) = G_X(t)G_Y(t) \quad$ (Since X and Y are independt)

$$= \left(\sum_{x=0}^{\infty} t^x p_1(x)\right)\left(\sum_{y=0}^{\infty} t^y p_2(y)\right)$$

$$\Longrightarrow G_{X+Y}(t) = \sum_{x=0}^{\infty}\sum_{y=0}^{\infty} p_1(x)p_2(y)t^{x+y} \qquad \ldots(1)$$

Let $Z = X + Y$. Then

$$G_Z(t) = E[t^Z] = \sum_{z=0}^{\infty} t^z p(z) \qquad \ldots(2)$$

From (1) and(2), we have

$$\sum_{z=0}^{\infty} t^z p(z) = \sum_{x=0}^{\infty}\sum_{y=0}^{\infty} p_1(x)p_2(y)t^{x+y}$$

$$\Rightarrow \sum_{z=0}^{\infty} t^z p(z) = \sum_{z=0}^{\infty}\left(\sum_{x=0}^{z} p_1(x)p_2(z-x)\right)t^z$$

$$\Rightarrow p(z) = \sum_{x=0}^{z} p_1(x)p_2(z-x) \text{ ,for } z = 0,1,2,\ldots$$

Example 1: If $\boldsymbol{X \sim B(n,p)}$, then find the p.g.f. of $\boldsymbol{X}$ and hence obtain its mean and variance.

Solution: Since $X \sim B(n,p)$, its p.m.f. is given by

$$p(x) = \binom{n}{x} p^x q^{n-x}, x = 0,1,\ldots,n, \qquad 0 < p < 1, q = 1-p$$

The p.g.f. of X is given by

$$G_X(t) = E[t^X] = \sum_{x=0}^{n} t^x p(x) = \sum_{x=0}^{n} t^x \binom{n}{x} p^x q^{n-x}$$

$$= \sum_{x=0}^{n} \binom{n}{x} (tp)^x q^{n-x} = (q+tp)^n$$

$$\Rightarrow G_X(t) = (q+tp)^n$$

Differentiating both sides w.r.t., t we get

$$G_X^{(1)}(t) = n(q+tp)^{n-1}p$$

$\Rightarrow \mu = \text{mean} = {\mu_1}' = G_X^{(1)}(1) = np$ and variance is given by

$$\sigma^2 = G_X^{(2)}(1) + G_X^{(1)}(1) - \left[G_X^{(1)}(1)\right]^2$$

But $G_X^{(2)}(t) = np(n-1)(q+tp)^{n-1}p$

$$\Rightarrow G_X^{(2)}(t) = n(n-1)p^2 = n^2p^2 - np^2$$

Therefore, $\sigma^2 = n^2p^2 - np^2 + np - n^2p^2 = np(1-p) \Rightarrow \sigma^2 = npq$

Example 2: If $X\sim P(\lambda)$, then find the p.g.f. of X and hence obtain its mean and variance.

Solution: Since $X\sim P(\lambda)$, its p.m.f. is given by

$$p(x) = \frac{e^{-\lambda}\lambda^x}{x!},\ x = 0,1,\dots \text{ and } \lambda > 0$$

The p.g.f. of X is given by

$$G_X(t) = E[t^X] = \sum_{x=0}^{\infty} t^x p(x) = \sum_{x=0}^{\infty} t^x \frac{e^{-\lambda}\lambda^x}{x!} = e^{-\lambda}\sum_{x=0}^{\infty}\frac{(t\lambda)^x}{x!} = e^{-\lambda}e^{t\lambda} = e^{\lambda(t-1)}$$

$\Longrightarrow G_X(t) = e^{\lambda(t-1)}$

Differentiating both sides w.r.t. t, we get

$G_X^{(1)}(t) = e^{\lambda(t-1)}\lambda$ and $G_X^{(2)}(t) = e^{\lambda(t-1)}\lambda^2$

Thus, $G_X^{(1)}(1) = \lambda$ and $G_X^{(2)}(1) = \lambda^2$

Hence, the mean and variance are given by $\mu = G_X^{(1)}(1) = \lambda$

and $\sigma^2 = G_X^{(2)}(1) + G_X^{(1)}(1) - \left[G_X^{(1)}(1)\right]^2 = \lambda^2 + \lambda - \lambda^2 = \lambda$ respectively.

Example 3: If $X\sim NB(r,p)$, then find the p.g.f. of X and hence obtain its mean and variance.

Solution: Since $X\sim P(\lambda)$, its p.m.f. is given by

$$p(x) = \binom{-r}{x} p^x(-q)^x, x = 0,1,2,\dots$$

The p.g.f. of X is given by

$$G_X(t) = E[t^X] = \sum_{x=0}^{\infty} t^x p(x) = \sum_{x=0}^{\infty} t^x \binom{-r}{x} p^r(-q)^x$$

$$= p^r \sum_{x=0}^{\infty}\binom{-r}{x}(-tq)^x = p^r(1-tq)^{-r}$$

$\Longrightarrow G_X(t) = p^r(1-tq)^{-r}$

$\Longrightarrow G_X^{(1)}(t) = p^r(-r)(1-tq)^{-(r+1)}(-q) = rqp^r(1-tq)^{-(r+1)}$

$\Longrightarrow G_X^{(2)}(t) = rqp^r\big(-(r+1)\big)(1-tq)^{-(r+2)}(-q) = r(r+1)q^2p^r(1-tq)^{-(r+2)}$

Thus, $G_X^{(1)}(t) = rqp^rp^{-(r+1)} = \frac{rq}{p}$ and

$G_X^{(2)}(t) = r(r+1)q^2p^rp^{-(r+2)} = (r^2+r)\frac{q^2}{p^2}$

$$\Longrightarrow G_X^{(2)}(t) = \frac{r^2q^2}{p^2} + \frac{rq^2}{p^2}$$

Thus, $\mu =$ mean $= G_X^{(1)}(1) = \frac{rq}{p}$ and

$\sigma^2 =$ variance $= G_X^{(2)}(1) + G_X^{(1)}(1) - \left[G_X^{(1)}(1)\right]^2$

$$= \frac{r^2q^2}{p^2} + \frac{rq^2}{p^2} + \frac{rq}{p} - \frac{r^2q^2}{p^2} = \frac{rq}{p^2}(q+p) \Longrightarrow \sigma^2 = \frac{rq}{p^2}$$

Example4: If $X \sim G(p)$, then find the p.g.f. of X and hence obtain its mean and variance.

Solution: Since $X \sim G(p)$, its p.m.f. is given by

$$p(x) = q^xp, \qquad x = 0,1,2,\ldots$$

The p.g.f. of X is given by

$$G_X(t) = E[t^X] = \sum_{x=0}^{\infty} t^xp(x) = \sum_{x=0}^{\infty} t^xq^xp = p\sum_{x=0}^{\infty}(tq)^x = \frac{p}{1-tq}$$

$\Longrightarrow G_X(t) = p(1-tq)^{-1}$

$G_X^{(1)}(t) = p(-1)(1-tq)^{-2}(-q) = pq(1-tq)^{-2}$

$$\Longrightarrow G_X^{(1)}(1) = \frac{pq}{p^2} = \frac{q}{p}$$

Now, $G_X^{(2)}(t) = pq(-2)(1-tq)^{-3}(-q) = 2pq^2(1-tq)^{-3}$

$$\Longrightarrow G_X^{(2)}(1) = \frac{2pq^2}{p^3} = \frac{2q^2}{p^2}$$

Hence, the mean μ and variance σ^2 of X are given by:

$\mu = G_X^{(1)}(1) = \frac{q}{p}$ and

$$\sigma^2 = G_X^{(2)}(1) + G_X^{(1)}(1) - \left[G_X^{(1)}(1)\right]^2$$

$$= \frac{2q^2}{p^2} + \frac{q}{p} - \frac{q^2}{p^2} = \frac{q^2}{p^2} + \frac{q}{p} = \frac{q}{p^2}(q+p) = \frac{q}{p^2}.$$

Example 5:The j.p.m.f. of (X, Y) is given in the following table. Prove or diprove

$G_{x+y}(t) = G_X(t)G_y(t)$ iff X and Y are independent.

X \ Y	0	1	2	Total
0	$\frac{1}{9}$	$\frac{2}{9}$	0	$\frac{1}{3}$
1	0	$\frac{1}{9}$	$\frac{2}{9}$	$\frac{1}{3}$
2	$\frac{2}{9}$	0	$\frac{1}{9}$	$\frac{1}{3}$
Total	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	1

Solution: Since $P(X = 1, Y = 2) = \frac{2}{9} \neq P(X = 1)P(Y = 2) = \frac{1}{3}\frac{1}{3} = \frac{1}{9}$, it follows that X and Y are not independent.

Now, $G_X(t) = G_Y(t) = \frac{1}{3}(1 + t + t^2)$

Let $Z = X + Y$. Then $Z = 0, 1, 2, 3, 4$. Let $p_i = P(Z = i), i = 0,1,2,3,4$.

$$p_0 = P(Z = 0) = P(X + Y = 0) = P(X = 0, Y = 0) = \frac{1}{9}$$

$$p_1 = P(Z = 1) = P(X + Y = 1) = P(X = 0, Y = 1) + P(X = 1, Y = 0) = \frac{2}{9} + 0 = \frac{2}{9}$$

$$p_2 = P(Z = 2) = P(X + Y = 2) = P(X = 0, Y = 2) + P(X = 1, Y = 1) + P(X = 2, Y = 0)$$

$$= 0 + \frac{1}{9} + \frac{2}{9} = \frac{3}{9}$$

$$p_3 = P(Z = 3) = P(X + Y = 3) = P(X = 1, Y = 2) + P(X = 2, Y = 1) = \frac{2}{9} + 0 = \frac{2}{9}$$

$$p_4 = P(Z = 4) = P(X + Y = 4) = P(X = 2, Y = 2) = \frac{1}{9}$$

The p.d.f. of $Z = X + Y$ is given by

$$G_{X+Y}(t) = \frac{1}{9} + \frac{2}{9}t + \frac{3}{9}t^2 + \frac{2}{9}t^3 + \frac{1}{9}t^4$$

$$\implies G_{X+Y}(t) = \frac{1}{9}(1 + 2t + 3t^2 + 2t^3 + t^4) = \left[\frac{1}{3}(1 + t + t^2)\right]^2$$

$\implies G_{X+Y}(t) = G_X(t)G_Y(t)$ but X and Y are not independent. Thus the statement is disproved.

Example 6: Can $G_X(t) = \frac{2}{1+t}$ be the p.d.f. of .r.v. X? Give reasons.

Solution: We have $G_X(1) = \frac{2}{1+1} = \frac{2}{2} = 1$

Further, $G_X(t) = \frac{2}{1+t} = 2(1 + t)^{-1} = 2(1 - t + t^2 - t^3 + \cdots)$

$$\implies G_X(t) = 2\sum_{x=0}^{\infty}(-1)^x t^x$$

Thus, $p(x) = P(X = x) = coef.of\ t^x$ in $G_X(t) = 2(-1)^x$

$$\implies p(x) = 2(-1)^x, x = 0, 1, 2, \ldots$$

Note that it takes negative values also. Hence, $G_X(t)$ is not a p.g.f.

Example 7 : A fair die is thrown $\boldsymbol{n}$ times. Let $\boldsymbol{S}$ be the total number of points. Show that $\boldsymbol{P(S = n + 5) = \binom{n+4}{5}\left(\frac{1}{6}\right)^{n}}$.

Solution: The p. g. f. of a single throw is given by:

$$G_X(t) = \sum_{x=1}^{6} t^x p(x) = \sum_{x=1}^{6} \frac{t^x}{6}$$

$$= \frac{1}{6}(t + t^2 + \cdots + t^6) = \frac{t}{6}(1 + t + \cdots + t^5) = \frac{t}{6}\frac{(1-t^6)}{1-t}$$

$$\Rightarrow\ G_X(t) = \frac{t}{6}(1 - t^6)\,(1 - t)^{-1}$$

Since the n throws are identical and independent,

$$G_S(t) = [G_S(t)]^n = \frac{t^n\left(1-t^6\right)^n(1-t)^{-n}}{6^n}$$

$$= \frac{t^n}{6^n}\sum_{j=0}^{n}\binom{n}{j}\left(-t^6\right)^j\sum_{k=0}^{\infty}\binom{n+k-1}{k}t^k$$

$$\Rightarrow G_S(t) = \frac{1}{6^n}\sum_{j=0}^{n}\sum_{k=0}^{\infty}(-1)^j\binom{n}{j}\binom{n+k-1}{k}\,t^{k+6j+n}$$

$$= \sum_{k=0}^{\infty} P\left(S = k + 6\,j + n\right)\,t^{k+6\,j+n}$$

where,

$$P(S = k + 6j + n) = \frac{1}{6^n} \sum_{j=0}^{n} (-1)^j \binom{n}{j} \binom{n+k-1}{k}$$

Now , $P(S = n + 5) = P(S = k + 6j + n)$ with $j = 0$ and $k = 5$

$$= \frac{1}{6^n} (-1)^0 \binom{n}{0} \binom{n+5-1}{5} = \frac{1}{6^n} \binom{n+4}{5}$$

$$\Longrightarrow P(S = n + 5) = \frac{1}{6^n} \binom{n+4}{5}$$