Matrices.

Matrices are named by J.J. Solvester. and Devellopped by Arthur Caylay.

Matrix A rectangular array of 'm.n' numbers can be arranged in m rows (horizontal lines) and 'n' columns (vertical lines). is called a matrix. of order $m \times n$.

These numbers (m.n) are called the elements (or) entries of the matrix enclosed in brackets. [ ] (or) ( ) (or) || ||.

Generally matrices are denoted by capital letters A, B, C, ...

Generally an $m \times n$ matrix can be written as.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}_{m \times n}$$

The compact form of the above matrix can be written as.

$A = [a_{ij}]_{m \times n}$ where $1 \le i \le m$
$1 \le j \le n$.

* The element a_{ij} belongs to ith row and jth column.

Eg: Suppose a matrix having 3 rows and 5 columns. then the order of the matrix is 3×5.

Types of Matrices:

1) **Row matrix** A matrix having only one row is called a row matrix

Eg: 1) $(1 \quad 2 \quad 3)_{1\times3}$.

2). **Column matrix** A matrix having only one column is called a column matrix. Eg 1) $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}_{3\times1}$.

3). **Rectangular matrix** A matrix, ~~having~~ In which the no. of rows is not equal to the no. of columns is called a rectangular matrix.

(or)

$A = [a_{ij}]_{m \times n}$ $\forall$, $m \neq n$

Eg: 1) $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}_{2\times3}$.

4). **Square matrix** A matrix, In which the no. of rows is equal to the no. of columns is called a square matrix.

$A = (a_{ij})_{m \times n}$ $\forall$ $m = n$.

Eg: 1) $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}_{3\times3}$.

5). principal diagonal (or) diagonal :-

In a square matrix, the diagonal from first row of the first element to the last row of the last element is called the principal diagonal.

The elements lying along the principle diagonal are called diagonal elements.

eg: 1). $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ 2). $B = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}_{3\times3}$ diagonal elements are 1, 5, 9.

diagonal elements = 1, 4

6). Trace of a matrix :-

The sum of the diagonal elements of a square matrix A is called the trace of the matrix A. It is denoted by Tr(A).

eg: 1). $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ 2). $B = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$

$Tr(A) = 1+4 = 5$ $Tr(B) = 1+5+9 = 15$.

(*) property :-

If A and B are two matrices of the same order and λ be any scalar, then.

1). $Tr(\lambda A) = \lambda \cdot Tr(A)$ 2). $Tr(A+B) = Tr(A) + Tr(B)$.

3). $Tr(AB) = Tr(BA)$ *4). $Tr(I_n) = n$.

7) Null matrix (or) Zero matrix

In a matrix all the elements are zero is called a Null matrix (or) zero matrix. It is denoted by 'O'.

eg: 1) $O = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}_{2\times2}$ 2) $O = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{3\times3}$ 3). $O = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{2\times3}$.

8). Lower triangular matrix :-

In a square matrix all the elements above the principal diagonal are zero's is called lower triangular matrix.

(or).

A square matrix $A = [a_{ij}]_{n\times n}$ is said to be a lower triangular matrix if $a_{ij} = 0 \; \forall \; i<j$.

eg: $A = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 6 \end{pmatrix}$

9). Upper triangular matrix:-

In a Square matrix all the elements below the principal diagonal are zero's is called an upper triangular matrix.

(or).

A Square matrix $A = [a_{ij}]$ is said to be an upper triangular matrix if $a_{ij} = 0 \ \forall \ i > j$

eg:- $\begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{pmatrix}$

10). Triangular matrix:-

A Square matrix is said to be triangular matrix. if it is Either a Lower triangular matrix (or) an upper triangular matrix is Known as triangular matrix.

11). Diagonal matrix:-

If each non diagonal elements of a Square matrix. is equal to Zero. is called a diagonal matrix.

(or).

A Square matrix is Said to be a diagonal matrix If it is both a lower triangle matrix and an upper triangle matrix

(or).

A Square matrix $A = [a_{ij}]$ is said to be diagonal matrix if $a_{ij} = 0 \ \forall \ i \neq j$

Eg:- $A = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$,i.e, $dig(a\,b\,c) = A$

12). Scalar matrix:-

A Diagonal matrix is said to a Scalar matrix if all the diagonal elements are equal.

(or).

In a square matrix, Each non diagonal elements are equal ~~and~~ to ~~dia~~ Zero, and diagonal elements are equal.

Eg:- $A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}_{3\times 3}$.

13). Unit matrix (or). Identity matrix :-

A Diagonal matrix is said to be a unit matrix. if all the diagonal elements is equal to Unity (one).

(or).

In a Square matrix each non diagonal elements are equal to zero and the diagonal elements equal to One. is Called a Unit Matrix (or) Identity matrix.

It is denoted by. I_n of order 'n'.

Eg: $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $I_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

14). Real matrix :-

A matrix in which all the elements are Real. is Called a real matrix.

Eg: $A = \begin{pmatrix} 1 & 2 & -3 \\ \sqrt{2} & \frac{1}{2} & -7 \end{pmatrix}$ is a real matrix.

15). Complex matrix :-

A matrix in which atleast one element is Imaginary is called a Complex ~~Number~~ matrix.

Eg: $A = \begin{pmatrix} 5i & 2 & -3 \\ \sqrt{2} & \frac{1}{2} & -4 \end{pmatrix}$ is a Complex matrix.

Equality of matrices :-

Two matrices A & B are said to be equal if and only if ($A = [a_{ij}]$, $B = [b_{ij}]$)

(i) they have the same order (A & B).

(ii). Each element of A is equal to the corresponding elements of B. i.e $a_{ij} = b_{ij}$ for every i and j.

Eg: $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then find a, b, c, d values if $A = B$.

Sol: Given $A = B$ $\therefore$ $a = 1$, $b = 2$, $c = 3$, $d = 4$.

Addition of two matrices :-

If two matrices A & B have the same order. then the matrix 'A+B' is obtained by adding the corresponding elements of A and B.
. the order of A+B is also same of the two matrices A and B.

i.e If $A = [a_{ij}]_{m \times n}$, $B = [b_{ij}]_{m \times n}$

then $A+B = [a_{ij} + b_{ij}]_{m \times n}$

egr $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}_{2\times 2}$, $B = \begin{pmatrix} 4 & 2 \\ 5 & 0 \end{pmatrix}$, $A+B = \begin{pmatrix} 5 & 4 \\ 8 & 4 \end{pmatrix}_{2\times 2}$

Properties :-

1) $A+B = B+A$ (Commutative law)
2). $(A+B)+C = A+(B+C)$.(Associative law)
3) $A+O = O+A = A$ (Existance of Identity)
4) $A+(-A) = (-A)+A = O$.(Existance of Inverse)

Subtraction of Matrices (or) Difference of two matrices :-

If two matrices A & B have same order. then the matrix 'A-B' is obtained by subtracting the ~~corresponding elements of 'A' and 'B'.~~
each element of B from the corresponding element of 'A'.

ie. If $A = [a_{ij}]_{m \times n}$, $B = [b_{ij}]_{m \times n}$ then

$A-B = [a_{ij} - b_{ij}]_{m \times n}$.

egr $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}_{2\times 2}$, $B = \begin{pmatrix} 4 & 2 \\ 0 & 5 \end{pmatrix}$

$A-B = \begin{pmatrix} 1-4 & 2-2 \\ 3-0 & 4-5 \end{pmatrix} = \begin{pmatrix} -3 & 0 \\ 3 & -1 \end{pmatrix}_{2\times 2}$

Multiplication of matrix by a Scalar :- (Note points [illegible])

If $A = [a_{ij}]_{m\times n}$ be a matrix of order $m \times n$ and K is any Scalar. then the matrix 'KA' is obtained by multiplying the each element of A by 'K'.

Eg:- $A = \begin{pmatrix} 2 & 3 & 4 \\ 1 & 2 & 5 \end{pmatrix}_{2\times 3}$, $4A = \begin{pmatrix} 8 & 12 & 16 \\ 4 & 8 & 20 \end{pmatrix}_{2\times 3}$.

Multiplication of matrices :-

If two matrices A and B are said to be Conformable for the product AB.

If the no. of Columns in the first matrix A is equal to the no. of rows in the Second matrix 'B'.

i.e. $A = [a_{ij}]_{m\times n}$, $B = [b_{jk}]_{n\times p}$. then the product AB is defined as $AB = [C_{ik}]_{m\times p}$.

where $C_{ik} = \sum_{j=1}^{n} a_{ij} \cdot b_{jk}$.

Eg:- $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}_{2\times 2}$, $B = \begin{pmatrix} 4 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}_{2\times 3}$, $AB = \begin{pmatrix} 4 & 2 & 1 \\ 12 & 6 & 3 \end{pmatrix}_{2\times 3}$.

Note :-

1). If the product of AB is defined then the product 'BA' need not be defined.

2). If the product AB and BA are defined. then they need not be equal i.e $AB \neq BA$.

properties :-

1) Matrices multiplication is distributive under matrix Addition

i.e $A(B+C) = AB + AC$.

2). Matrices multiplication is Associative

i.e $(AB)C = A(BC)$.

Idempotent matrix:-

A Square matrix A is said to be Idempotent if $A^2 = A$.

Eg:- $A = \begin{pmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{pmatrix}$

$A^2 = \begin{pmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{pmatrix}\begin{pmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{pmatrix}$

$= \begin{pmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{pmatrix} = A$.

$\therefore A^2 = A$.

$\therefore$ A is an Idempotent matrix.

Involutory matrix:-

~~A~~ Square matrix A is said to be Involutary matrix If $A^2 = I$.

Eg:- let $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$A^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$.

$\therefore A^2 = I$. , A is an Involutary matrix.

Nillpotent matrix:-

A Square matrix A is said to be Nillpotent matrix. If $\exists$ a +ve Integer 'm' $\ni A^m = 0$. (i) If m is aleast +ve Integer $\ni A^m = 0$. then m is called Index of the Nillpotent matrix 'A'.

Eg:- 1) $A = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$ then $A^2 = 0$.

$\therefore$ A is a Nill potent matrix with Index '2'.

(2). $A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 6 \\ -1 & -1 & -3 \end{pmatrix}$ then $A^2 = O_{3\times3}$

$\therefore$ A is a Nill potent matrix with Index '2'.

Transpose of a matrix :-

If the rows and Columns of a matrix A are Inter-changed then the resultant matrix is Called transpose matrix of the original matrix 'A'.

It is denoted by A^T (or) A'.

Eg:- 1) $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}_{2\times 3}$, $A^T = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}_{3\times 2}$.

Note :- If the order of A is $m\times n$ then the order of A^T is $n\times m$.

i.e $A_{m\times n} \Rightarrow A^T_{n\times m}$.

properties :-

* If A^T and B^T are the transpose matrices of A and B resp. and K is a Scalar. then

(1) $(A^T)^T = A$ (2). $(A+B)^T = A^T + B^T$

(3) $(KA)^T = K.A^T$ (4) $(AB)^T = B^T . A^T$.

Symmetric matrix :-

A Square matrix A is said to be symmetric If $A^T = A$

(or).

A square matrix $A = [a_{ij}]$ is said to be Symmetric if $a_{ij} = a_{ji}$.

eg:- $A = \begin{pmatrix} 1 & -2 & 0 \\ -2 & 2 & 6 \\ 0 & 6 & 3 \end{pmatrix}$ ∴ A is a Symmetric matrix

$A^T = \begin{pmatrix} 1 & -2 & 0 \\ -2 & 2 & 6 \\ 0 & 6 & 3 \end{pmatrix} = A$.

Skew - Symmetric matrix :-

A Square matrix A is said to be Skew Symmetric if $A^T = -A$.

(or)

A Square matrix $A = [a_{ij}]$ is said to be Skew Symmetric if $a_{ij} = -a_{ji}$.

Eg:- $A = \begin{pmatrix} 0 & 1 & -2 \\ -1 & 0 & 3 \\ 2 & -3 & 0 \end{pmatrix}$, $A^T = \begin{pmatrix} 0 & -1 & 2 \\ 1 & 0 & -3 \\ -2 & 3 & 0 \end{pmatrix}$

$= -\begin{pmatrix} 0 & 1 & -2 \\ -1 & 0 & 3 \\ 2 & -3 & 0 \end{pmatrix} = -A$

$\therefore A^T = -A$

$\therefore$ A is a Skew Symmetric matrix.

Submatrix :-

Suppose A is any matrix of order $m \times n$. then a matrix is obtained by ~~deleting~~ deleting (or) ~~Some~~ elemenating Some rows ~~and~~ (&) Columns (or) both Rows and Columns is called a Submatrix of the given matrix 'A'.

Eg:- If $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}_{2\times3}$. then 1) $\begin{pmatrix} 2 & 3 \\ 5 & 6 \end{pmatrix}$ 2) $\begin{pmatrix} 1 & 2 \\ 4 & 5 \end{pmatrix}$

3) $[1\ 2\ 3]$ 4) $[4\ 5]$ ...... are Submatrices of A

Minor of an element :-

Suppose A is a Square matrix of order 'n'. then the determinent of a Square matrix of order $(n-1)$ is obtained by deleting (or) elemenating i^{th} row and j^{th} Column is Called minor of the element a_{ij}. It is dinoted by M_{ij}.

Eg:- $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$.

M_{11} = minor of a_{11} = $\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = a_{22}\cdot a_{33} - a_{23}\cdot a_{32}$.

M_{12} = Minor of a_{12} = $\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} = a_{21}\cdot a_{33} - a_{23}\cdot a_{31}$.

Co-factor of an element :-

Suppose $A = [a_{ij}]$ is a Square matrix of order 'n'. then the Co-factor of an element a_{ij} is $(-1)^{i+j}\cdot M_{ij}$. It is dinoted by A_{ij}.

Eg:- $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}_{3\times 3}$

A_{11} = Cofactor of $a_{11} = (-1)^{1+1} M_{11}$

$= (-1)^2 \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = a_{22} \cdot a_{33} - a_{23} \cdot a_{32}$.

A_{12} = Co-factor of $a_{12} = (-1)^{1+2} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$.

$= (-1)(a_{21} \cdot a_{33} - a_{23} \cdot a_{31})$.

Adjoint of a matrix :-

The transe-pose of the matrix formed by the cofactors of the corresponding elements in the given matrix 'A' is called Adjoint matrix of 'A'.

It is denoted by 'adj A'.

Eg: let $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

Let $A_{11}, A_{12}, A_{13}, A_{21}, A_{22}, A_{23}, A_{31}, A_{32}, A_{33}$ are the cofactors of the elements $a_{11}, a_{12}, a_{13}, a_{21}, a_{22}, a_{23}, a_{31}, a_{32}$ & a_{33}.

then the cofactor matrix of $A = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}$

$\therefore \text{Adj } A = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}^T$.

$= \begin{pmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{pmatrix}$

Note :- (1). $(\text{adj } A)^T = \text{adj } A^T$.

(2) For any scalar K, $(\text{adj } KA) = K^{n-1} \text{adj } A$. (n is order of A.

Determinent of a Square matrix:-

The Sum of the product of the elements of any row (or) Column with their Corresponding Cofactors is called the Determinent of the Square matrix 'A'.

It is denoted by $|A|$ (or) $\det A$.

Eg: let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

$$\therefore |A| \text{ (or) } \det A = a_{11}A_{11} + a_{12}A_{12} + a_{13}\cdot A_{13}.$$

$$= a_{11}A_{11} + a_{21}A_{21} + a_{31}A_{31} \ldots \text{etc.}$$

properties:-

1) $|A| = |A^T|$.

2). If every element of a row (or) Column of the matrix 'A' is Zero. then $|A| = 0$.

3). The Determinent of a Square matrix changes its Sign when two rows (or) Columns are Inter changed.

4). If two rows (or) Columns of a matrix A are identical then $|A| = 0$.

5). If all the elements of a row (or) Column of a Square Matrix A are multiplied a number 'K'. then the determinent of the resulting matrix is equal to 'K' times of the determinent of the original matrix. Eg: $\begin{vmatrix} Ka_1 & b_1 & c_1 \\ Ka_2 & b_2 & c_2 \\ Ka_3 & b_3 & c_3 \end{vmatrix} = K \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

6). If each element in a row (or) Column of a Square matrix A is the Sum of two terms.

then its determinent Can be expressed as the Sum of the Determinents of two Square matrices of the same order.

* If the corresponding elements of two rows (or) columns of a Square matrix are in the same ratio, then the determinent of that matrix is Zero.

eg:- $\begin{vmatrix} a_1 & b_1 & c_1 \\ ka_1 & kb_1 & kc_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0.$

7). The sum of the product of the elements of any row (or) Column of a square matrix with the cofactors of the corresponding elements of any other row (or) column is zero.

8). If A and B are two square matrices of the same order then $|AB| = |A| \cdot |B|$.

9). $\det(KA) = K^n \det A$. where n is the order of the matrix.

10). $\det I = 1$.

11). If A is triangular matrix. then

$\det A = a_{11}\, a_{22}\, a_{33} \cdots\cdots a_{nn}$.

= product of diagonal elements.

Note :- 1). The determinent of an Idempotent is always zero

2). the determinent of an Involutory matrix is always -1.

Singular matrix :-

A square matrix 'A' is said to be singular. If $|A| = 0$.

eg:- 1) If $A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$, then $|A| = 0$. $\therefore$ A is a singular matrix

Non Singular matrix :-

A square matrix 'A' is said to be Non singular. If $|A| \neq 0$.

eg:- 1). If $A = \begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix}$ then $|A| = 2 - 12 = -10 \neq 0$.

$\therefore$ A is a non singular matrix.

Inverse of a matrix :-

If A is any square matrix and if there exists another square matrix 'B'. such that $AB = BA = I$.

then B is Inverse of the matrix 'A'.

It is denoted by A^{-1}.

$\therefore A^{-1} = B$.

Note :- 1) Non Square matrix Can't posses inverse

2). Every Square matrix need not have an Inverse.

3). Only Non Singular matrix having the Inverse matrix.

(4). $(A^T)^{-1} = (A^{-1})^T$ (5). $(A^{-1})^{-1} = A$. (6) $(AB)^{-1} = B^{-1}.A^{-1}$.

(7). $(A^n)^{-1} = (A^{-1})^n$.

Results :- (1). Inverse of a Square matrix is Unique if it exist.

(2). If A & B are two Non Singular matrices then $(AB)^{-1} = B^{-1}.A^{-1}$.

(3). The Inverse of the transpose is equal to the transpose of the Inverse. i.e $(A^{-1})^T = (A^T)^{-1}$.

(4). If A is a Non Singular matrix, then $A^{-1} = \frac{\text{Adj } A}{\det A}$.

(5). If A & B are Non Singular Square matrices having the same order then $\text{Adj}(AB) = (\text{Adj } A).(\text{Adj } B)$.

(6). Every Square matrix Can be expressed as the Sum of a Symmetric and Skew Symmetric matrices uniquely

(7). Inverse of a Non Singular ~~matrix~~ Symmetric matrix A is Symmetric.

(8). If A is any Symmetric matrix then Adj A is also Symmetric

~~(9)~~.

Orthogonal Matrix :- A Square matrix A is said to be Orthogonal If $A^T.A = A.A^T = I$.

i.e $A^T = A^{-1}$.

Result (1). If A and B are two orthogonal matrices. then AB and BA are also orthogonal.

(2). An inverse of an orthogonal matrix is orthogonal and its transpose is also orthogonal.

i.e If A is orthogonal then A^{T} & A^{-1} are also orthogonal

(3). If $A = \text{diag}(a\ b\ c)$ then $A^{-1} = \text{diag}(a^{-1}\ b^{-1}\ c^{-1})$.

Minor of the matrix:- Let 'A' be an $m \times n$ matrix. The determinant of a square submatrix of A is called a minor of the matrix.

If the order of the square sub matrix is 't' then its determinant is called a minor of order 't'

Eg:- $$A = \begin{pmatrix} 2 & 1 & 1 \\ 3 & 1 & 2 \\ 1 & 2 & 3 \\ 5 & 6 & 7 \end{pmatrix}_{4\times 3}.$$

Sub matrix $B = \begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix}_{2\times 2}$ is a submatrix of order 2.

$$|B| = \begin{vmatrix} 2 & 1 \\ 3 & 1 \end{vmatrix} = 2 - 3 = -1$$. is a minor of order '2'.

Rank of a Matrix:-

If A is non-null matrix, and 'r' is the rank of matrix A. If

(i). Every $(r+1)^{th}$ Order minor of A is Zero.

(ii). $\exists$ at least one r^{th} order minor of A is non Zero.

It is denoted by $\rho(A)$.

Note:- (1). Rank of a matrix is Unique.

(2). Every matrix will have a rank

*(3). If A is a matrix of order $m \times n$. then rank of A $\leq$ min. of (m,n).

Eg: If $A = (1\ 2\ 3)_{1\times3}$
$\therefore \rho(A) \leq \min(1,3)$
$\Rightarrow \rho(A) = 1$.

(4). If $\rho(A) = r$, then Every minor of A of order $(r+1)^{th}, (r+2)^{th}, (r+3)^{th}$ .... is Zero, (if exist).

(5). If A is a matrix of order 'n' & A is a Non-singular $(|A| \neq 0)$. then $\rho(A) = n$.

(6). If A is a matrix of order $m \times n$,
If Every k^{th} order minor $(k<m, k<n)$ is 0.
then $\rho(A) < K$

(7). If A is a matrix of order $m \times n$.
If Every k^{th} order minor $(k<m, k<n)$ is Non zero. then $\rho(A) \geq K$.

(8). Rank of an Identity matrix I_n is 'n'

(9). If A is Non Null matrix then $\rho(A) \geq 1$.

(10). If A is Null matrix then $\rho(A) = 0$.

Equivalence of matrices :-

If B is obtained from A after a finite Chain of elementary transformations then B is said to be equivalent to A.

Symbolically It is dinoted by $B \sim A$.

Result :(1):- If A and B are two equovalent matrices, then $\rho(A) = \rho(B)$.

(2). If two matrices A and B have the same size and same rank, then the two matrices A and B are equivalent.

Zero row and Non Zero row :-

If all the elements in a row of a matrix are Zero. Then it is called a Zero row.

If there exist at least one Non Zero element in a row of a matrix, then it is called a Non Zero row.

Elementary matrix :-

A matrix obtained by elementary transformations is called elementary matrix.

Note :- Elementary transformations does not change the rank of the matrix.

Different methods to find the rank of the matrix :-

Method (1) :-

The rank can be determined by finding the heighest Order non-vanishing minor of the given matrix

disappear Suddenly

appear Suddenly

[i.e By using determinent findout the rank of the matrix of $|A| = 0$ then by taking It's Submatrix & find its 'det'.]

Method (2) :- Echelon form of the matrix :-

A matrix is said to be Echelon form If

(i). Zero rows, If any, are below any non zero row.

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{array}{l} \rightarrow \text{Non zero rows} \\ \\ \rightarrow \text{Zero row} \end{array}$$

(ii). The no. of Zero's before the first non Zero element in a row is less than the no. of such Zero's in the next row.

(iii). The first non-Zero entry in each non Zero is equal to 1. Eg:- $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 9 \\ 0 & 0 & 1 \end{bmatrix}$.

Note :- 1) In Echelon form. The no. of non Zero rows is equal to the rank of the given matrix.

2). In Echelon form to find the rank of the matrix, We can apply only elementary row-operations But we can't apply elementary column operations.

1.12 ZERO ROW AND NON - ZERO ROW

If all the elements in a row of a matrix are zeros, then it is called a zero row and if there is at least one non-zero element in a row, then it is called a non-zero row.

Methods to find the Rank of a Matrix

The following three methods are used for finding the rank of a matrix.

1. Echelon form or Triangular form
2. Normal form
3. Normal form of the type PAQ

1.13 ECHELON FORM OF A MATRIX

A matrix is said to be in Echelon form if it has the following properties.

(*i*) Zero rows, if any, are below any non-zero row.

(*ii*) The first non-zero element in each non-zero row is equal to 1.

(*iii*) The number of zero rows before the first non-zero element of a row is less than the number of such zeros in the next row.

Note: The condition (*ii*) is optional.

Important Result : The number of non-zero rows in the row echelon form of A is the rank of A.

e.g. 1. $\begin{bmatrix} 1 & 0 & 0 & 2 & 3 & 0 \\ 0 & 1 & 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ is a row echelon form.

2. $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ is in row echelon form.

3. $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ is in row echelon form.

SOLVED EXAMPLES

Example 1 : Reduce the matrix $A=\begin{pmatrix}1&2&3&0\\2&4&3&2\\3&2&1&3\\6&8&7&5\end{pmatrix}$ **into echelon form and hence find its rank.** **[JNTU (A) June 2016 (R13)]**

Solution : Given matrix is $A=\begin{bmatrix}1&2&3&0\\2&4&3&2\\3&2&1&3\\6&8&7&5\end{bmatrix}$

Applying $R_2 \to R_2-2R_1$, $R_3 \to R_3-3R_1$, and $R_4 \to R_4-6R_1$, we get

$$A\sim\begin{bmatrix}1&2&3&0\\0&0&-3&2\\0&-4&-8&3\\0&-4&-11&5\end{bmatrix}$$

$$\sim\begin{bmatrix}1&2&3&0\\0&-4&-8&3\\0&0&-3&2\\0&-4&-11&5\end{bmatrix}\text{(Applying } R_2 \leftrightarrow R_3\text{)}$$

$$\sim\begin{bmatrix}1&2&3&0\\0&-4&-8&3\\0&0&-3&2\\0&0&-3&2\end{bmatrix}\text{ (Applying } R_4 \to R_4-R_2\text{)}$$

$$\sim\begin{bmatrix}1&2&3&0\\0&-4&-8&3\\0&0&-3&2\\0&0&0&0\end{bmatrix}\text{(Applying } R_4 \to R_4-R_3\text{)}$$

This is in Echelon form and the number of non-zero rows is 3.

$\therefore$ Rank $(A)=\rho(A)=3$.

Example 2 : Reduce the matrix to Echelon form and find its rank.

$$\begin{bmatrix}-1&-3&3&-1\\1&1&-1&0\\2&-5&2&-3\\-1&1&0&1\end{bmatrix}$$

[JNTU (A) June 2011 (Set No. 4)]

Solution : Given matrix is $A = \begin{bmatrix} -1 & -3 & 3 & -1 \\ 1 & 1 & -1 & 0 \\ 2 & -5 & 2 & -3 \\ -1 & 1 & 0 & 1 \end{bmatrix}$

Applying $R_2 \to R_2 + R_1$, $R_3 \to R_3 + 2R_1$ and $R_4 \to R_4 - R_1$, we get

$$A \sim \begin{bmatrix} -1 & -3 & 3 & -1 \\ 0 & -2 & 2 & -1 \\ 0 & -11 & 8 & -5 \\ 0 & 4 & -3 & 2 \end{bmatrix}$$

Applying $R_3 \to 2R_3 - 11R_2$ and $R_4 \to R_4 + 2R_2$, we get

$$A \sim \begin{bmatrix} -1 & -3 & 3 & -1 \\ 0 & -2 & 2 & -1 \\ 0 & 0 & -6 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Applying $R_4 \to 6R_4 + R_3$, we get

$$A \sim \begin{bmatrix} -1 & -3 & 3 & -1 \\ 0 & -2 & 2 & -1 \\ 0 & 0 & -6 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

This is in Echelon form. Number of non-zero rows is 4.

$\therefore$ Rank of $A = \rho(A) = 4$.

Note : The rank of a product of two matrices can not exceed the rank of either matrix.

Example 3 : Define the rank of the matrix and find the rank of the following matrix

$$\begin{bmatrix} 2 & 1 & 3 & 5 \\ 4 & 2 & 1 & 3 \\ 8 & 4 & 7 & 13 \\ 8 & 4 & -3 & -1 \end{bmatrix}$$

[JNTU May 2005S, 2005, 2006S (Set No.2), Sep. 2008 (Set No.2), (A) June 2013 (Set No. 3)]

Solution : Rank of a Matrix : A number 'r' is said to be the rank of the matrix A if it possess the following two properties :

(*i*) There exist at least one minor of order 'r' which is non-zero.

(*ii*) All minors of order $(r+1)$ if they exists are zeros.

Let $A = \begin{bmatrix} 2 & 1 & 3 & 5 \\ 4 & 2 & 1 & 3 \\ 8 & 4 & 7 & 13 \\ 8 & 4 & -3 & -1 \end{bmatrix}$

$$\sim \begin{bmatrix} 2 & 1 & 3 & 5 \\ 0 & 0 & -5 & -7 \\ 0 & 0 & -5 & -7 \\ 0 & 0 & -15 & -21 \end{bmatrix} \text{(Applying } R_2 \to R_2 - 2R_1\text{; } R_3 \to R_3 - 4R_1\text{; } R_4 \to R_4 - 4R_1\text{)}$$

$$\sim \begin{bmatrix} 2 & 1 & 3 & 5 \\ 0 & 0 & -5 & -7 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{(Applying } R_3 \to R_3 - R_2\text{; } R_4 \to R_4 - 3R_2\text{)}$$

This matrix is in Echelon form.

Rank of A = number of non - zero rows = 2

Example 4 : Find the value of k such that the rank of $\begin{bmatrix} 1 & 2 & 3 \\ 2 & k & 7 \\ 3 & 6 & 10 \end{bmatrix}$ is 2.

[JNTU 2006 (Set No. 4)]

Solution : Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & k & 7 \\ 3 & 6 & 10 \end{bmatrix}$

Given rank of A = $\rho(A) = 2$.

$\therefore \ |A| = 0$

$\Rightarrow 1(10k - 42) - 2(20 - 21) + 3(12 - 3k) = 0$ or $k = 4$

Example 5 : Find the value of k if the Rank of Matrix A is 2 where $A = \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & k & 0 \end{bmatrix}$

[JNTU (H) June 2011 (Set No. 4)]

Solution : Given matrix is $A = \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & k & 0 \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & k & 0 \end{bmatrix} \text{(Applying } R_1 \leftrightarrow R_2\text{)}$$

$$\sim\begin{bmatrix}1 & 0 & 1 & 1\\0 & 1 & -3 & -1\\0 & 1 & -3 & -1\\0 & 1 & k-1 & -1\end{bmatrix}\text{(Applying } R_3-3R_1, R_4-R_1\text{)}$$

For the rank (A) to be equal to 2, we must have 3 rows identical.

$\therefore\ k-1=-3 \Rightarrow k=-2$

Example 6 : Find the value of k such that the rank of $A=\begin{bmatrix}1 & 1 & -1 & 1\\1 & -1 & k & -1\\3 & 1 & 0 & 1\end{bmatrix}$ is 2.

[JNTU (H) Jan. 2012 (Set No. 2)]

Solution : Given $A=\begin{bmatrix}1 & 1 & -1 & 1\\1 & -1 & k & -1\\3 & 1 & 0 & 1\end{bmatrix}$

$$\sim\begin{bmatrix}1 & 1 & -1 & 1\\0 & -2 & k+1 & -2\\0 & -2 & 3 & -2\end{bmatrix}\text{(Applying } R_2-R_1, R_3-3R_1\text{)}$$

$$\sim\begin{bmatrix}1 & 1 & -1 & 1\\0 & -2 & k+1 & 2\\0 & 0 & 2-k & 0\end{bmatrix}\text{(Applying } R_3-R_2\text{)}$$

Since the rank of A is 2, there will be only two non-zero rows.

$\therefore$ Third row must be a zero row $\Rightarrow 2-k=0 \Rightarrow k=2$

Example 7 : Find the rank of $\begin{bmatrix}2 & -4 & 3 & -1 & 0\\1 & -2 & -1 & -4 & 2\\0 & 1 & -1 & 3 & 1\\4 & -7 & 4 & -4 & 5\end{bmatrix}$.

[JNTU 2008, (K)Nov. 2009S (Set No.3)]

Solution : Given $A=\begin{bmatrix}2 & -4 & 3 & -1 & 0\\1 & -2 & -1 & -4 & 2\\0 & 1 & -1 & 3 & 1\\4 & -7 & 4 & -4 & 5\end{bmatrix}$

Interchanging R_1 and R_2, we get

$$A\sim\begin{bmatrix}1 & -2 & -1 & -4 & 2\\2 & -4 & 3 & -1 & 0\\0 & 1 & -1 & 3 & 1\\4 & -7 & 4 & -4 & 5\end{bmatrix}$$

Applying $R_2 \to R_2 - 2R_1$ and $R_4 \to R_4 - 4R_1$, we get

$$A \sim \begin{bmatrix} 1 & -2 & -1 & -4 & 2 \\ 0 & 0 & 5 & 7 & -4 \\ 0 & 1 & -1 & 3 & 1 \\ 0 & 1 & 8 & 12 & -3 \end{bmatrix}$$

Applying $R_2 \leftrightarrow R_4$, we get

$$A \sim \begin{bmatrix} 1 & -2 & -1 & -4 & 2 \\ 0 & 1 & 8 & 12 & -3 \\ 0 & 1 & -1 & 3 & 1 \\ 0 & 0 & 5 & 7 & -4 \end{bmatrix}$$

Applying $R_3 \to R_3 - R_2$, we get

$$A \sim \begin{bmatrix} 1 & -2 & -1 & -4 & 2 \\ 0 & 1 & 8 & 12 & -3 \\ 0 & 0 & -9 & -9 & 4 \\ 0 & 0 & 5 & 7 & -4 \end{bmatrix}$$

Applying $R_4 \to 9R_4 + 5R_3$, we get

$$A \sim \begin{bmatrix} 1 & -2 & -1 & -4 & 2 \\ 0 & 1 & 8 & 12 & -3 \\ 0 & 0 & -9 & -9 & 4 \\ 0 & 0 & 0 & 18 & -16 \end{bmatrix}$$

This is in Echelon form.

Number of non-zero rows is 4. Thus, rank of A = $\rho(A) = 4$.

Example 8 : Find the rank of matrix $\begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$ **[JNTU 2008 (Set No.4)]**

(or) Find the rank of the matrix $A = \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$ by reducing to echelon form

[JNTU (H) Dec. 2017]

Solution : Let $A = \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix} \text{(Applying } R_1 \leftrightarrow R_2\text{)}$$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \end{bmatrix} \text{(Applying } R_3 - 3R_1, R_4 - R_1\text{)}$$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{(Applying } R_4 - R_2, R_3 - R_2\text{)}$$

This is in Echelon form. The number of non-zero rows is 2.

$\therefore$ Rank of the matrix = 2.

Example 9 : Find the rank of $\begin{pmatrix} 1 & 4 & 3 & -2 & 1 \\ -2 & -3 & -1 & 4 & 3 \\ -1 & 6 & 7 & 2 & 9 \\ -3 & 3 & 6 & 6 & 12 \end{pmatrix}$

[JNTU (H) June 2009 (Set No.1), (A) Nov. 2010 (Set No. 4)]

System of Linear Equations:-

An equation of the form

$$a_1x_1 + a_2x_2 + a_3x_3 + \cdots + a_nx_n = b \quad \text{--- (1)}$$

is called a linear equation in n-unknowns.

Where $x_1, x_2, x_3, \ldots x_n$ are unknowns and $a_1, a_2, a_3, \ldots a_n$ & b are constants.

Consider a system of m-linear equations in n-unknowns $x_1, x_2, x_3, \ldots x_n$ as given below.

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n = b_3 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n = b_m \end{array}\right\} \text{--- (2)}$$

Where all a's & b's are constants.

The above system of equations can be written in matrix form as $AX = B$.

i.e
$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_m \end{bmatrix}$$

Where $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$ is called Co-efficient matrix

$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}$ is called Variable matrix.

$B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_m \end{bmatrix}$ is called Constant matrix.

* The matrix $[A \; B]$ is called an Augmented matrix.

$$\therefore (A \; B) = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} & b_m \end{pmatrix}.$$

Gauss Elimination Method:-

Consider the System of equations.

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3.$$

The above System of equations Can be written in matrix form as $AX = B$.

i.e $\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$

Consider the Augmented matrix as $[A\ B]$. i.e

$$(A\ B) = \begin{pmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{pmatrix}$$

The above matrix Can be reduce to either echelon form (or) triangular form, by using row operations only. then after the System Can be written as.

$$\begin{pmatrix} a_1 & b_1 & c_1 \\ 0 & \alpha_2 & \beta_2 \\ 0 & 0 & \beta_3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} d_1 \\ \gamma_2 \\ \gamma_3 \end{pmatrix}$$

The above System Can be written as.

$$a_1x + b_1y + c_1z = d_1 \quad \text{--- (1)}$$
$$\alpha_2 y + \beta_2 z = \gamma_2 \quad \text{--- (2)}$$
$$\beta_3 z = \gamma_3.$$
$$\boxed{z = \gamma_3/\beta_3}$$

On Subs. z Value in eq (2) we get the value of y

$|||^{ly}$ on Sub. y & z Values in eq (1), we get the Value of x.

$\therefore$ The req. Solution is $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

(1) Solve the system of equations by using the Gauss elimination method.

$2x_1 + x_2 + x_3 = 10$, $3x_1 + 2x_2 + 3x_3 = 18$, $x_1 + 4x_2 + 9x_3 = 16$.

Sol:- Given system equations are

$$2x_1 + x_2 + x_3 = 10$$
$$3x_1 + 2x_2 + 3x_3 = 18$$
$$x_1 + 4x_2 + 9x_3 = 16$$

The above system of eqn can be written in matrix form as $AX = B$.

$$\begin{pmatrix} 2 & 1 & 1 \\ 3 & 2 & 3 \\ 1 & 4 & 9 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 10 \\ 18 \\ 16 \end{pmatrix}$$

Consider, the Augmented matrix as

$$(A\ B) = \begin{pmatrix} 2 & 1 & 1 & 10 \\ 3 & 2 & 3 & 18 \\ 1 & 4 & 9 & 16 \end{pmatrix}$$

Apply, $R_2 \to 2R_2 - 3R_1$
$R_3 \to 2R_3 - R_1$.

$$\sim \begin{pmatrix} 2 & 1 & 1 & 10 \\ 0 & +1 & 3 & 6 \\ 0 & 7 & 17 & 22 \end{pmatrix}$$

Apply, $R_3 \to R_3 - 7R_2$

$$(A\ B) \sim \begin{pmatrix} 2 & 1 & 1 & 10 \\ 0 & 1 & 3 & 6 \\ 0 & 0 & -4 & -20 \end{pmatrix}$$

This is in Echelon form.

$\therefore$ the above matrix can be written in the form as

$$\begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & -4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 10 \\ 6 \\ -20 \end{pmatrix}$$

$$\therefore 2x_1 + x_2 + x_3 = 10 \quad \text{--- (1)}$$
$$x_2 + 3x_3 = 6 \quad \text{--- (2)}$$
$$-4x_3 = -20$$
$$\therefore \underline{x_3 = 5}$$

on Subt. x_3 in eq (2), we get

$x_2 + 3(5) = 6$.

$x_2 = 6 - 15 = -9$.

$\therefore x_2 = -9$.

on Sub. x_2 & x_3 in eq (1), we get

$2x_1 - 9 + 5 = 10$

$2x_1 - 4 = 10$

$2x_1 = 14$

$x_1 = 7$

$\therefore$ the req. Solution is $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 7 \\ -9 \\ 5 \end{pmatrix}$

1.27 SOLUTION OF LINEAR SYSTEMS - DIRECT METHODS

The solution of a linear system of equations can be found out by numerical methods known as **direct method** and **iterative methods.** We will discuss the Gauss elimination method and Gauss-Seidel iteration method.

1.27.1 Gaussian Elimination Method

This method of solving a system of n linear equations in n unknowns consists of eliminating the coefficients in such a way that the system reduces to upper triangular system which may be solved by backward substitution . We discuss the method here for $n = 3$. The method is analagous for $n > 3$.

Consider the system

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned}\right\} \quad \text{... (1)}$$

The augmented matrix of this system is

$$[A, B] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{bmatrix} \qquad .... (2)$$

Performing $R_2 \to R_2 - \frac{a_{21}}{a_{11}} R_1$ and $R_3 \to R_3 - \frac{a_{31}}{a_{11}} R_1$, we get

$$[A, B] \sim \begin{bmatrix} a_{11} & a_{12} & a_{13} & b_1 \\ 0 & \alpha_{22} & \alpha_{23} & \beta_2 \\ 0 & \alpha_{32} & \alpha_{33} & \beta_3 \end{bmatrix} \qquad .... (3)$$

where $\alpha_{22} = a_{22} - a_{12}\left(\frac{a_{21}}{a_{11}}\right)$; $\alpha_{23} = a_{23} - a_{13}\left(\frac{a_{21}}{a_{11}}\right)$;

$$\alpha_{32} = a_{32} - \left(\frac{a_{31}}{a_{11}}\right) a_{12}; \quad \alpha_{33} = a_{33} - \left(\frac{a_{31}}{a_{11}}\right) a_{13};$$

$$\beta_2 = b_2 - \left(\frac{a_{21}}{a_{11}}\right) b_1; \quad \beta_3 = b_3 - \left(\frac{a_{31}}{a_{11}}\right) b_1$$

Here we assume $a_{11} \neq 0$

We call $\frac{-a_{21}}{a_{11}}, \frac{-a_{31}}{a_{11}}$ as **multipliers for the first stage**. a_{11} **is called first pivot.**

Now applying $R_3 \to R_3 - \frac{\alpha_{32}}{\alpha_{22}}(R_2)$, we get

$$[A, B] \sim \begin{bmatrix} a_{11} & a_{12} & a_{13} & b_1 \\ 0 & \alpha_{22} & \alpha_{23} & \beta_2 \\ 0 & 0 & \gamma_{33} & \Delta_3 \end{bmatrix} \qquad .... (4)$$

where $\gamma_{33} = \alpha_{33} - \left(\frac{\alpha_{32}}{\alpha_{22}}\right) \alpha_{23}$; $\Delta_3 = \beta_3 - \left(\frac{\alpha_{32}}{\alpha_{22}}\right) \beta_2$

We have assumed $\alpha_{22} \neq 0$.

Here the **multiplier** is $-\frac{\alpha_{32}}{\alpha_{22}}$ and **new pivot** is α_{22}.

The augmented matrix (4) corresponds to an upper triangular system which can be solved by backward substitution. The solution obtained is exact.

Note : If one of the elements $a_{11}, \alpha_{22}, \alpha_{33}$ are zero the method is modified by rearranging the rows, so that the pivot is non- zero.

This procedure is called **partial pivoting**. If this is impossible then the matrix is singular and the system has no solution.

SOLVED EXAMPLES

Example 1 : Solve the equations

$2x_1 + x_2 + x_3 = 10;\ 3x_1 + 2x_2 + 3x_3 = 18;\ x_1 + 4x_2 + 9x_3 = 16$

using Gauss-Elimination method.

Solution : The Augmented matrix of the given system is $[A \mid B] = \begin{bmatrix} 2 & 1 & 1 & 10 \\ 3 & 2 & 3 & 18 \\ 1 & 4 & 9 & 16 \end{bmatrix}$

Performing $R_2 \to 2R_2 - 3R_1$ and $R_3 \to 2R_3 - R_1$, we get

$$[A \mid B] \sim \begin{bmatrix} 2 & 1 & 1 & 10 \\ 0 & 1 & 3 & 6 \\ 0 & 7 & 17 & 22 \end{bmatrix}$$

Applying $R_3 \to R_3 - 7R_2$, we get $[A \mid B] \sim \begin{bmatrix} 2 & 1 & 1 & 10 \\ 0 & 1 & 3 & 6 \\ 0 & 0 & -4 & -20 \end{bmatrix}$

This Augmented matrix corresponds to the following upper triangular system.

$2x_1 + x_2 + x_3 = 10;\ x_2 + 3x_3 = 6;\ -4x_3 = -20$

$\Rightarrow x_3 = 5, x_2 = -9, x_1 = 7$ by backward substitution.

$\therefore$ The solution is $x_1 = 7, x_2 = -9, x_3 = 5$

Example 2 : Solve the system of equations :

$3x + y - z = 3;\ 2x - 8y + z = -5;\ x - 2y + 9z = 8$ **using Gauss elimination method.**

Solution : The Augmented matrix is $[A \mid B] = \begin{bmatrix} 3 & 1 & -1 & 3 \\ 2 & -8 & 1 & -5 \\ 1 & -2 & 9 & 8 \end{bmatrix}$

Performing $R_2 \to R_2 - \frac{2}{3}R_1;\ R_3 \to R_3 - \frac{1}{3}R_1$, we get

$$[A, B] \sim \begin{bmatrix} 3 & 1 & -1 & 3 \\ 0 & -26/3 & 5/3 & -7 \\ 0 & -7/3 & 28/3 & 7 \end{bmatrix}$$

Performing $R_3 \to R_3 - \frac{7}{26} R_2, [A \mid B] \sim \begin{bmatrix} 3 & 1 & -1 & 3 \\ 0 & -26/3 & 5/3 & -7 \\ 0 & 0 & 693/78 & 231/26 \end{bmatrix}$

From this, we get

$3x + y - z = 3\,;\ \frac{-26}{3} y + \frac{5}{3} z = -7\,,\ \frac{693}{78} z = \frac{231}{26}$

$\Rightarrow z = 1, y = 1$ and $x = 1$.

$\therefore$ The solution is $x = 1, y = 1, z = 1$.

Example 3 : Solve the equations $x + y + z = 6;\ 3x + 3y + 4z = 20;\ 2x + y + 3z = 13$ using Gauss elimination method. **[JNTU (H) Dec. 2018]**

Solution : The augmented matrix of the system is $[A, B] = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 3 & 3 & 4 & 20 \\ 2 & 1 & 3 & 13 \end{bmatrix}$

Performing the operations $R_2 \to R_2 - 3R_1$ and $R_3 \to R_2 - 2R_1$, we get

$$[A, B] \sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 0 & 1 & 2 \\ 0 & -1 & 1 & 1 \end{bmatrix}$$

Using the operation $R_2 \leftrightarrow R_3$, we get $[A, B] \sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$

This corresponds to the upper triangular system $x + y + z = 6;\ -y + z = 1;\ z = 2$

$\therefore$ By backward substitution, we get $x = 3, y = 1, z = 2$.

Example 4 : Solve the system of equations $3x + y + 2z = 3,\quad 2x - 3y - z = -3,$ $x + 2y + z = 4$ using Gauss elimination method. **[JNTU 2008, (K) Nov.2009S (Set No.3)]**

Solution : The given system of equation can be written in the matrix form as AX = B.

where A $= \begin{bmatrix} 3 & 1 & 2 \\ 2 & -3 & -1 \\ 1 & 2 & 1 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $B = \begin{bmatrix} 3 \\ -3 \\ 4 \end{bmatrix}$

The Augumented matrix of the given system is

$$[A,B] = \begin{bmatrix} 3 & 1 & 2 & 3 \\ 2 & -3 & -1 & -3 \\ 1 & 2 & 1 & 4 \end{bmatrix}$$

$$[A,B] \sim \begin{bmatrix} 1 & 2 & 1 & 4 \\ 2 & -3 & -1 & -3 \\ 3 & 1 & 2 & 3 \end{bmatrix} \text{(Operating } R_1 \leftrightarrow R_3\text{)}$$

$$[A,B] \sim \begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & -7 & -3 & -11 \\ 0 & -5 & -1 & -9 \end{bmatrix} \text{(Operating } R_2 \to R_2 - 2R_1 \text{ and } R_3 \to R_3 - 3R_1)$$

$$[A,B] \sim \begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & -7 & -3 & -11 \\ 0 & 0 & 8 & -8 \end{bmatrix} \text{(Operating } R_3 \to 7R_3 - 5R_2)$$

This augmented matrix corresponds to the following upper triangular system.

$$x + 2y + z = 4 \quad \text{... (1)}$$

$$-7y - 3z = -11 \quad \text{... (2)}$$

$$8z = -8$$

By back substitution, we have

$$z = -1 \quad \text{... (3)}$$

Substituting equation (3) in equation (2), we get

$$-7y = -11 - 3 \Rightarrow y = 2$$

Now from (1), we have

$$x + 4 - 1 = 4 \Rightarrow x = 1.$$

$\therefore$ The solution is $x = 1, y = 2, z = -1$.

Example 5 : Solve the system of equations $x + 2y + 3z = 1, 2x + 3y + 8z = 2, x + y + z = 3$.

[JNTU 2008 (Set No.4)]

Solution : The given non-homogeneous linear system of equations are

$x + 2y + 3z = 1;\ 2x + 3y + 8z = 2;\ x + y + z = 3.$

These can be written in matrix form as $AX = B$

where $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 8 \\ 1 & 1 & 1 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

The Augmented matrix is

$$[A,B] = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 2 & 3 & 8 & 2 \\ 1 & 1 & 1 & 3 \end{bmatrix}$$

Applying $R_2 - 2R_1, R_3 - R_1$, we get

$$[A,B] \sim \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & -1 & 2 & 0 \\ 0 & -1 & -2 & 2 \end{bmatrix}$$

Applying $R_3 - R_2$, we get

$$[A, B] \sim \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & -4 & 2 \end{bmatrix}$$

Rank of [A, B] = 3. Using the same operations, rank (A) = 3.

$\therefore$ The system of equations is consistent.

From the above Augmented matrix, we have

$$-4z = 2 \Rightarrow z = \frac{-1}{2}$$

$$-y + 2z = 0 \Rightarrow -y - 1 = 0 \Rightarrow y = -1$$

and $x + 2y + 3z = 1 \Rightarrow x - 2 - \frac{3}{2} = 1 \Rightarrow x = \frac{9}{2}$

$\therefore$ The solution is $x = \frac{9}{2}$, $y = -1$, $z = -\frac{1}{2}$

Example 6 : Solve the equations

$$3x + y + 2z = 3; 2x - 3y - z = -3; x + 2y + z = 4$$

Using Gauss elimination method **[JNTU 2008S (Set No.4)]**

Solution : The given system can be written as AX = B

where $A = \begin{bmatrix} 3 & 1 & 2 \\ 2 & -3 & -1 \\ 1 & 2 & 1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 3 \\ -3 \\ 4 \end{bmatrix}$

The Augmented matrix is

$$[A,B] = \begin{bmatrix} 3 & 1 & 2 & 3 \\ 2 & -3 & -1 & -3 \\ 1 & 2 & 1 & 4 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 1 & 4 \\ 2 & -3 & -1 & -3 \\ 3 & 1 & 2 & 3 \end{bmatrix} (\text{Applying } R_1 \leftrightarrow R_3)$$

$$\sim \begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & -7 & -3 & -11 \\ 0 & -5 & -1 & -9 \end{bmatrix} (\text{Applying } R_2 - 2R_1, R_3 - 3R_1)$$

$$\sim \begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & -7 & -3 & -11 \\ 0 & 0 & 8 & -8 \end{bmatrix} (\text{Applying } 7R_3 - 5R_2)$$

This corresponds to upper triangular system.

This gives $8z = -8 \Rightarrow z = -1$

$7y + 3z = 11 \Rightarrow 7y = 11 - 3z = 14 \Rightarrow y = 2$

$x + 2y + z = 4 \Rightarrow x = 4 - 2y - z = 4 - 4 + 1 = 1$

$\therefore$ The solution is $x = 1, y = 2, z = -1$

Example 7: Express the following system in matrix form and solve by Gauss Elimination method.

$$2x_1 + x_2 + 2x_3 + x_4 = 6;\ 6x_1 - 6x_2 + 6x_3 + 12x_4 = 36$$
$$4x_1 + 3x_2 + 3x_3 - 3x_4 = -1;\ 2x_1 + 2x_2 - x_3 + x_4 = 10$$

[JNTU 2008, 2008S, (H), (A) 2009, (K) 2009S, (K)May 2010 (Set No.]

Solution : The Augmented matrix of the given equations is

$$[A,B] = \begin{bmatrix} 2 & 1 & 2 & 1 & 6 \\ 6 & -6 & 6 & 12 & 36 \\ 4 & 3 & 3 & -3 & -1 \\ 2 & 2 & -1 & 1 & 10 \end{bmatrix}$$

Performing $R_2 \to \frac{R_2}{6}$, we get

$$[A,B] \sim \begin{bmatrix} 2 & 1 & 2 & 1 & 6 \\ 1 & -1 & 1 & 2 & 6 \\ 4 & 3 & 3 & -3 & -1 \\ 2 & 2 & -1 & 1 & 10 \end{bmatrix}$$

Performing $R_1 \to R_2$ gives,

$$[A,B] \sim \begin{bmatrix} 1 & -1 & 1 & 2 & 6 \\ 2 & 1 & 2 & 1 & 6 \\ 4 & 3 & 3 & -3 & -1 \\ 2 & 2 & -1 & 1 & 10 \end{bmatrix}$$

Performing $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 4R_1$, and $R_4 \to R_4 - 2R_1$, we get

$$[A,B] \sim \begin{bmatrix} 1 & -1 & 1 & 2 & 6 \\ 0 & 3 & 0 & -3 & -6 \\ 0 & 7 & -1 & -11 & -25 \\ 0 & 4 & -3 & -3 & -2 \end{bmatrix}$$

Performing $R_3 \to 3R_3 - 7R_2$, $R_4 \to 3R_4 - 4R_2$, we obtain

$$[A,B] \sim \begin{bmatrix} 1 & -1 & 1 & 2 & 6 \\ 0 & 3 & 0 & -3 & -6 \\ 0 & 0 & -3 & -12 & -33 \\ 0 & 0 & 9 & 3 & 18 \end{bmatrix}$$

Performing $R_4 \to R_4 - 3R_3$, we obtain

$$[A,B] \sim \begin{bmatrix} 1 & -1 & 1 & 2 & 6 \\ 0 & 3 & 0 & -3 & -6 \\ 0 & 0 & -3 & -12 & -33 \\ 0 & 0 & 0 & 39 & 117 \end{bmatrix}$$

This corresponds to the upper triangular system.

$$x_1 - x_2 + x_3 + 2x_4 = 6 \quad \text{... (1)}$$

$$3x_2 - 3x_4 = -6 \quad \text{... (2)}$$

$$-3x_3 - 12x_4 = -33 \quad \text{... (3)}$$

$$39x_4 = 117 \quad \text{... (4)}$$

By back substitution,

(4) $\Rightarrow$ $x_4 = 3$

Substituting the value of x_4 in (3) and (2), we get

$x_3 = -1$ and $x_2 = 1$

Now substituting the values of x_2, x_3 and x_4 in (1), we get

$x_1 - 1 - 1 + 6 = 6 \Rightarrow x_1 = 2$

$\therefore$ The solution is $x_1 = 2,\ x_2 = 1,\ x_3 = -1,\ x_4 = 3$

EXERCISE 1.7

1. Use the Gauss Elimination method to solve
   $x+2y-3z=9, 2x-y+z=0, 4x-y+z=4$ **[JNTU (H) May 2018]**

Homogenious Linear Equations :-

An equation of the form

$a_1x_1 + a_2x_2 + a_3x_3 + \cdots + a_nx_n = 0$ is called a homogenious linear equations in n-unknowns.

where, $x_1, x_2, x_3, \cdots x_n$ are n-unknowns (Variables)

$a_1, a_2, a_3, \cdots a_n$ & b are Constants.

Test for Consistancy of System of Homogenious Linear equations :-

Consider, a system of m-homogenious equations in n-unknowns $x_1, x_2, x_3, \cdots x_n$ is given below.

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = 0.$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = 0.$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n = 0$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n = 0.$$

The above system of equations can be written in matrix form as $AX = 0$ where A is the Co-efficient matrix.

* $\rho(A)$ & $\rho(A\ B)$ is same $\therefore$ the given system is always consistant.

i.e $\rho(A) = \rho(A\ B)$.

procedure Step(1): Reduce the matrix 'A' to Echelon form.

$\rho(A) = r$ & no. of Variables $= n$.

Step(2) a) If $\rho(A) = n$ then the system has a trivial solution (or) zero solution.

b) If $\rho(A) < n$ then the given system having non trivial solution (or) infinitely many solution i.e $n-r$ linearly Independent solutions.

* The given system $AX = 0$ has a zero (or) trivial solution then A is a Non Singular matrix i.e $|A| \neq 0$. i.e Linearly dependant.

* The given system $AX = 0$ having non trivial solution then the matrix A is a Singular matrix i.e $|A| = 0$. i.e Linearly Independant

Eg ①. Solve the System of equations Completely

$x+2y+3z=0$; $3x+4y+4z=0$, $7x+10y+12z=0$.

Sol: Given System of eq's are

$x+2y+3z=0$

$3x+4y+4z=0$

$7x+10y+12z=0$.

the above System of eq's can be written as matrix form as

$AX=0$.

ie $\begin{pmatrix} 1 & 2 & 3 \\ 3 & 4 & 4 \\ 7 & 10 & 12 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

Consider,

$$A=\begin{pmatrix} 1 & 2 & 3 \\ 3 & 4 & 4 \\ 7 & 10 & 12 \end{pmatrix}$$

Apply, $R_2 \to R_2-3R_1$

$R_3 \to R_3-7R_1$

$$\sim\begin{pmatrix} 1 & 2 & 3 \\ 0 & -2 & -5 \\ 0 & -4 & -9 \end{pmatrix}$$

$R_3 \to R_3-2R_2$

$$A \sim \begin{pmatrix} 1 & 2 & 3 \\ 0 & -2 & -5 \\ 0 & 0 & 1 \end{pmatrix}$$

$\therefore \rho(A)=$ no. of non zero rows $=3$.

$n=$ no. of variables $=3$.

$\therefore \rho(A)=n$.

$\therefore$ the given System has a trivial (or) zero Solution.

Now, Solve $AX=0$.

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & -2 & -5 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\therefore x+2y+3z=0$ —①

$-2y-5z=0$ —②

$z=0$.

on Sub: z value in eq ②, we get

$y=0$

on Sub: y & z value in eq ①, we get

$x=0$.

$\therefore$ the Req. Solution is $X=\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.

Eg ②:- Find all the Solutions of the System of equations

$x+2y-z=0,\ 2x+y+z=0,\ x-4y+5z=0.$

Sol: Given System of eq's are

$x+2y-3z=0,\ 2x+y+z=0,\ x-4y+5z=0.$

The above System of eq's Can be written in matrix form as

$AX=0.$

i.e $\begin{pmatrix}1 & 2 & -3\\ 2 & 1 & 1\\ 1 & -4 & 5\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\begin{pmatrix}0\\ 0\\ 0\end{pmatrix}$

Consider,

$A=\begin{pmatrix}1 & 2 & -3\\ 2 & 1 & 1\\ 1 & -4 & 5\end{pmatrix}$

Apply, $R_2 \to R_2 - 2R_1$

$R_3 \to R_3 - R_1.$

$\sim \begin{pmatrix}1 & 2 & -3\\ 0 & -3 & 3\\ 0 & -6 & 6\end{pmatrix}$

$R_3 \to R_3 - 2R_2$

$A \sim \begin{pmatrix}1 & 2 & -3\\ 0 & -3 & 3\\ 0 & 0 & 0\end{pmatrix}$

$\therefore \rho(A)=2,\ n=3.$

$\therefore \rho(A)<n.$

$\therefore$ the given System has Non trivial (or) Infinitely many Solutions.. i.e $n-r=3-2=1$ linearly Independent Solutions.

Now, Solve $AX=0.$

i.e $\begin{pmatrix}1 & 2 & -3\\ 0 & -3 & 3\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\begin{pmatrix}0\\ 0\\ 0\end{pmatrix}$

$\therefore\ x+2y-3z=0.$ —①

$-3y+3z=0.$

$\Rightarrow \cancel{-3}y=\cancel{-3}z$

$\therefore y=z$ —②

Let $z=k$ $\therefore$ from eq ②, $y=k$

on Sub: y & z values in eq ①, we get.

$x+2k-3k=0$

$x-k=0 \Rightarrow x=k$

$\therefore$ The req. Solution is $X=\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\begin{pmatrix}k\\ k\\ k\end{pmatrix}=k\begin{pmatrix}1\\ 1\\ 1\end{pmatrix}$

1.25 CONSISTENCY OF SYSTEM OF HOMOGENEOUS LINEAR EQUATIONS

Consider a system of m homogeneous linear equations in n unknowns, namely

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \ldots\ldots + a_{1n}x_n = 0 \\ a_{21}x_1 + a_{22}x_2 + \ldots\ldots + a_{2n}x_n = 0 \\ a_{31}x_1 + a_{32}x_2 + \ldots\ldots + a_{3n}x_n = 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = 0 \end{array}\right\} \quad \ldots (1)$$

We write

$$A = \begin{bmatrix} a_{11} & a_{12} \ldots a_{1n} \\ a_{21} & a_{22} \ldots a_{2n} \\ \vdots & \\ a_{m1} & a_{m2} \ldots a_{mn} \end{bmatrix}_{m\times n}, \; X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1}, \; O = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{m\times 1} \quad \ldots (2)$$

Then (1) can be written as $AX = O$ which is the matrix equation.

Here A is called the *coefficient matrix*. It is clear that

$x_1 = 0 = x_2 = x_3 = \ldots . = x_n$ is a solution of (1) *i.e.*, $X = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ is a solution of (*2*).

This is called **trivial solution** of $AX = O$.
This $AX = O$ is always **consistent** (*i.e.*) it has a solution.
The ***trivial solution*** is called the **zero solution**.
A zero solution is not linearly independent (*i.e.*), it is linearly dependent.

Theorem : The number of linearly independent solutions of the linear system $AX = O$ is $n-r$, r being the rank of the matrix $A_{m\times n}$ and n being the number of variables.

Note :

1. If A is a non-singular matrix. (*i.e.*, det $A \neq 0$) then the linear system $AX = O$ has only the zero solution.
2. The system $AX = O$ possesses a non-zero solution if and only if A is a singular matrix.

Working Rule for Finding the Solutions of the Equation $AX = O$:

Let rank of $A = r$ and rank of $[A/B] = r_1$.

Since all b's are zero, $r = r_1$, then

I.(*i*) If $r = n$(number of variables) $\Rightarrow$ the system of equations have only trivial solution (*i.e.*, zero solution).

(*ii*) If $r < n \Rightarrow$ the system of equations have an infinite number of non-trivial solutions, we shall have $n - r$ linearly independent solutions.

To obtain infinite solutions, set $(n - r)$ variables any arbitrary value and solve for the remaining unknowns.

If the number of equations is less than the number of unknowns, it has a non-trivial solution. The number of solutions of the equation $AX = O$ will be infinite.

II. If the number of equations is less than number of variables, the solution is always other than a trivial solution.

III. If the number of equations = number of variables, the necessary and sufficient condition for solutions other than a trivial solution is that the determinant of the coefficient matrix is zero.

SOLVED EXAMPLES

Example 1 : Solve completely the system of equations

$$x + y - 2z + 3w = 0;\ x - 2y + z - w = 0$$
$$4x + y - 5z + 8w = 0;\ 5x - 7y + 2z - w = 0$$

Solution : The given system of equations in matrix form is

$$AX = \begin{bmatrix} 1 & 1 & -2 & 3 \\ 1 & -2 & 1 & -1 \\ 4 & 1 & -5 & 8 \\ 5 & -7 & 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = O$$

Applying $R_2 - R_1$, $R_3 - 4R_1$ and $R_4 - 5R_1$, we get $A \sim \begin{bmatrix} 1 & 1 & -2 & 3 \\ 0 & -3 & 3 & -4 \\ 0 & -3 & 3 & -4 \\ 0 & -12 & 12 & -16 \end{bmatrix}$

Applying $R_4 \to \dfrac{R_4}{4}$, we get $A \sim \begin{bmatrix} 1 & 1 & -2 & 3 \\ 0 & -3 & 3 & -4 \\ 0 & -3 & 3 & -4 \\ 0 & -3 & 3 & -4 \end{bmatrix}$

Applying $R_3 - R_2$ and $R_4 - R_2$, we get $A \sim \begin{bmatrix} 1 & 1 & -2 & 3 \\ 0 & -3 & 3 & -4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

This is in the Echelon form. We have

rank A = the number of non-zero rows in this Echelon form = 2

Since rank $A(= 2)$ is less than the number of unknowns (= 4), therefore, the given system has infinite number of non-trivial solutions.

$\therefore$ Number of independent solutions = 4 – 2 = 2.

Now, we shall assign arbitrary values to 2 variables and the remaining 2 variables shall be found in terms of these. The given system of equations is equivalent to

$$\begin{bmatrix} 1 & 1 & -2 & 3 \\ 0 & -3 & 3 & -4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

This gives the equations $x + y - 2z + 3w = 0, -3y + 3z - 4w = 0$

Taking $z = \lambda$ and $w = \mu$, we see that $x = \lambda - \frac{5}{3}\mu, y = \lambda - \frac{4}{3}\mu, z = \lambda, w = \mu$ constitutes the general solution of the given system.

Example 2 : Solve the system of equations
$x+y-3z+2w=0;\ 2x-y+2z-3w=0; 3x-2y+z-4w=0; -4x+y-3z+w=0.$

Solution : The given homogeneous system of linear equations can be expressed as

$$\begin{bmatrix} 1 & 1 & -3 & 2 \\ 2 & -1 & 2 & -3 \\ 3 & -2 & 1 & -4 \\ -4 & 1 & -3 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad ...(1)$$

i.e., $AX = O$

Consider $A = \begin{bmatrix} 1 & 1 & -3 & 2 \\ 2 & -1 & 2 & -3 \\ 3 & -2 & 1 & -4 \\ -4 & 1 & -3 & 1 \end{bmatrix}$.

We will reduce this matrix to Echelon form using elementary row operations and find its rank.

Applying $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 3R_1$ and $R_4 \to R_4 + 4R_1$, we get

$$A \sim \begin{bmatrix} 1 & 1 & -3 & 2 \\ 0 & -3 & 8 & -7 \\ 0 & -5 & 10 & -10 \\ 0 & 5 & -15 & 9 \end{bmatrix}$$

Performing $\frac{R_3}{-5}$, we get $A \sim \begin{bmatrix} 1 & 1 & -3 & 2 \\ 0 & -3 & 8 & -7 \\ 0 & 1 & -2 & 2 \\ 0 & 5 & -15 & 9 \end{bmatrix}$

Performing $R_4 \to R_4 - 5R_3$, we get $A \sim \begin{bmatrix} 1 & 1 & -3 & 2 \\ 0 & -3 & 9 & -7 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & -5 & -1 \end{bmatrix}$

Performing $R_3 \to 3R_3 + R_2$, we get $A \sim \begin{bmatrix} 1 & 1 & -3 & 2 \\ 0 & -3 & 8 & -7 \\ 0 & 0 & 3 & -1 \\ 0 & 0 & -5 & -1 \end{bmatrix}$

Performing $R_4 \to 3R_4 + 5R_3$, we get $A \sim \begin{bmatrix} 1 & 1 & -3 & 2 \\ 0 & -3 & 8 & -7 \\ 0 & 0 & 3 & -1 \\ 0 & 0 & 0 & -8 \end{bmatrix}$

This is in Echelon form. The number of non-zero rows = 4

$\therefore$ Rank $A = 4$. Here number of variables $= n = 4$

We have number of non-zero solutions is $n - r = 4 - 4 = 0$

$\therefore$ There are no non-zero solutions.

$\therefore x = y = z = w = 0$ is the only solution.

Example 3 : Show that the only real number λ for which the system $x + 2y + 3z = \lambda x;\ 3x + y + 2z = \lambda y\ ;\ 2x + 3y + z = \lambda z$ has non-zero solution is 6 and solve them, when $\lambda = 6$.

[JNTU 2005-May, 2006, 2006S (Set No. 1), Sep. 2008 (Set No.1)]

Solution : Given system can be expressed as $AX = O$ where

$$A = \begin{bmatrix} 1-\lambda & 2 & 3 \\ 3 & 1-\lambda & 2 \\ 2 & 3 & 1-\lambda \end{bmatrix},\quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } O = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Here number of variables $= n = 3$.

The given system of equations possess a non-zero (non-trivial) solution, if

Rank of A < number of unknowns *i.e.*, Rank of $A < 3$

For this we must have det. $A = 0$

$$\therefore \begin{vmatrix} 1-\lambda & 2 & 3 \\ 3 & 1-\lambda & 2 \\ 2 & 3 & 1-\lambda \end{vmatrix} = 0$$

Applying $R_1 \to R_1 + R_2 + R_3$, we have $\begin{vmatrix} 6-\lambda & 6-\lambda & 6-\lambda \\ 3 & 1-\lambda & 2 \\ 2 & 3 & 1-\lambda \end{vmatrix} = 0$

$$\Rightarrow (6-\lambda) \begin{vmatrix} 1 & 1 & 1 \\ 3 & 1-\lambda & 2 \\ 2 & 3 & 1-\lambda \end{vmatrix} = 0$$

Applying $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$, we get $(6-\lambda)\begin{vmatrix} 1 & 0 & 0 \\ 3 & -2-\lambda & -1 \\ 2 & 1 & -1-\lambda \end{vmatrix} = 0$

$\Rightarrow (6-\lambda)\left[(-2-\lambda)(-1-\lambda)+1\right] = 0$

$\Rightarrow (6-\lambda)(\lambda^2 + 3\lambda + 3) = 0 \Rightarrow \lambda = 6$ is the only real value and other values are complex.

When $\lambda = 6$, the given system becomes

$$\begin{bmatrix} -5 & 2 & 3 \\ 3 & -5 & 2 \\ 2 & 3 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R_2 \to 5R_2 + 3R_1, R_3 \to 5R_3 + 2R_1 \Rightarrow \begin{bmatrix} -5 & 2 & 3 \\ 0 & -19 & 19 \\ 0 & 19 & -19 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R_3 \to R_3 + R_2 \Rightarrow \begin{bmatrix} -5 & 2 & 3 \\ 0 & -19 & 19 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow -5x + 2y + 3z = 0$ and $-19y + 19z = 0 \Rightarrow y = z$

Since Rank of A < number of unknowns (Rank of $A = 2$, number of unknowns = 3), therefore, the given system has infinite number of non-trivial solutions.

Let $z = k \Rightarrow y = k$ and $-5x + 2k + 3k = 0 \Rightarrow x = k$

$\therefore$ $x = k, y = k, z = k$ is the solution.

Example 4 : Solve the system of equations $x + 3y - 2z = 0$; $2x - y + 4z = 0$; $x - 11y + 14z = 0$. **[JNTU 2002, (K) May 2010 (Set No. 2), (A) Nov. 2010, Dec 2013 (Set No. 2)]**

Solution : Let $A = \begin{bmatrix} 1 & 3 & -2 \\ 2 & -1 & 4 \\ 1 & -11 & 14 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, O = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Then the given system can be written as $AX = O$

Now $A = \begin{bmatrix} 1 & 3 & -2 \\ 2 & -1 & 4 \\ 1 & -11 & 14 \end{bmatrix}$

Applying $R_3 - R_1$ and $R_2 - 2R_1$, we get $A \sim \begin{bmatrix} 1 & 3 & -2 \\ 0 & -7 & 8 \\ 0 & -14 & 16 \end{bmatrix}$

Applying $R_3 - 2R_2$, we get $A \sim \begin{bmatrix} 1 & 3 & -2 \\ 0 & -7 & 8 \\ 0 & 0 & 0 \end{bmatrix}$

Thus the matrix is in Echelon form. Number of non-zero rows is 2.

$\therefore$ The rank of matrix is 2.

Since number of variables is 3, this will have 3–2 = 1 non-zero solution.

The corresponding equations are, $x + 3y - 2z = 0$ and $-7y + 8z = 0$

Let $z = k$. Then $y = \frac{8}{7}k$ and $x = -3y + 2z = \frac{-24}{7}k + 2k = \frac{-10k}{7}$

$$\therefore \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{-10}{7}k \\ \frac{8}{7}k \\ k \end{bmatrix} = \frac{k}{7}\begin{bmatrix} -10 \\ 8 \\ 7 \end{bmatrix} \text{ i.e. } X = \frac{k}{7}\begin{bmatrix} -10 \\ 8 \\ 7 \end{bmatrix}.$$

which is the general solution of the given system.

Example 5 : Determine whether the following equations will have a non-trivial solution if so solve them. $4x+2y+z+3w=0$, $6x+3y+4z+7w=0$, $2x+y+w=0$.

[JNTU May 2006, 2006S (Set No.2)]

Solution : Given equations will have a non-trivial solution since the number of equations is less than the number of unknowns.

In other words the system will have a non-trivial solution if and only if the rank r of the coefficient matrix is less than n the number of unknowns.

Given equations can be written in the matrix form as follows :

$$AX = \begin{bmatrix} 4 & 2 & 1 & 3 \\ 6 & 3 & 4 & 7 \\ 2 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = O$$

$$\text{or} \quad AX = \begin{bmatrix} 1 & 2 & 4 & 3 \\ 4 & 3 & 6 & 7 \\ 0 & 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} z \\ y \\ x \\ w \end{bmatrix} = O \text{ (Interchanging the variables } x \text{ and } z)$$

We shall now reduce the coefficient matrix A to the Echelon form by applying only elementary row operations.

$$A \sim \begin{bmatrix} 1 & 2 & 4 & 3 \\ 0 & -5 & -10 & -5 \\ 0 & 1 & 2 & 1 \end{bmatrix} \text{ (Applying } R_2 - 4R_1)$$

$$\sim \begin{bmatrix} 1 & 2 & 4 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 1 & 2 & 1 \end{bmatrix} \left(\text{Applying } \frac{R_2}{-5}\right)$$

$$\sim \begin{bmatrix} 1 & 2 & 4 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ (Applying } R_3 - R_2)$$

Thus the matrix A has been reduced to Echelon form.

$\therefore$ Rank (A) = Number of non-zero rows = 2 < 4 (unknowns)

Hence the given system will have 4 – 2 i.e., 2 linearly independent solutions.

Now the given system of equations is equivalent to

$$AX = \begin{bmatrix} 1 & 2 & 4 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} z \\ y \\ x \\ w \end{bmatrix} = O$$

i.e., $z + 2y + 4x + 3w = 0$

and $y + 2x + w = 0$

Choose $x = k_1$ and $w = k_2$. Then solving these two equations, we get

$y = -2x - w = -2k_1 - k_2$ and $z = -4x - 2y - 3w = -4k_1 - 2(-2k_1 - k_2) - 3k_2 = -k_2$.

$\therefore$ The solution is

$x = k_1, y = -2k_1 - k_2, z = -k_2$ and $w = k_2$, where k_1 and k_2 are arbitrary constants.

Example 6 : Solve the system of equations
$x + y + w = 0, y + z = 0, x + y + z + w = 0, x + y + 2z = 0.$

[JNTU 2008, (H) June 2009 (Set No.1)]

Solution : The equations can be written in matrix form as $Ax = O$,

$$\text{where } A = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 0 \end{bmatrix}, \; X = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}, \; O = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{Consider } A = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & -1 \end{bmatrix} \text{(Applying } R_3 \to R_3 - R_1, R_4 \to R_4 - R_1)$$

$$\sim \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \text{(Applying } R_4 - 2R_3)$$

$$\sim \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \text{(Applying } R_1 + R_4)$$

$\therefore$ Rank (A) = 4 and Number of variables = 4

Therefore, there is no non-zero solution.

Hence $x = y = z = w = 0$ is the only solution.

Example 7 : Solve the system $2x - y + 3z = 0, 3x + 2y + z = 0$ **and** $x - 4y + 5z = 0$.

[JNTU 2008S (Set No.1)]

Solution : The Given system can be written as $AX = O$

$$\text{where } A = \begin{bmatrix} 2 & -1 & 3 \\ 3 & 2 & 1 \\ 1 & -4 & 5 \end{bmatrix}, \; X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } O = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Consider $A \sim \begin{bmatrix} 1 & -4 & 5 \\ 3 & 2 & 1 \\ 2 & -1 & 3 \end{bmatrix}$ (Applying $R_3 \leftrightarrow R_1$)

$\sim \begin{bmatrix} 1 & -4 & 5 \\ 0 & 14 & -14 \\ 0 & 7 & -7 \end{bmatrix}$ (Applying $R_2 - 3R_1$ and $R_3 - 2R_1$)

$\sim \begin{bmatrix} 1 & -4 & 5 \\ 0 & 14 & -14 \\ 0 & 0 & 0 \end{bmatrix}$ (Applying $2R_3 - R_2$)

$\sim \begin{bmatrix} 1 & -4 & 5 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$ (Applying $\frac{R_2}{14}$)

Since the number of non-zero rows is 2, we have rank (A) = 2

Here number of variables = 3

The system will have $n - r = 3 - 2 = 1$ non-zero solution.

From the matrix, we have $y - z = 0 \Rightarrow y = z$

Let $y = z = k$. Then

$x - 4y + 5z = 0 \Rightarrow x = 4y - 5z = 4k - 5k = -k$

$\therefore$ The solution is given by $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -k \\ k \\ k \end{bmatrix} = k \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$

Example 10 : Determine the values of λ for which the following set of equations may possess non-trivial solution:

$3x_1 + x_2 - \lambda x_3 = 0,\ 4x_1 - 2x_2 - 3x_3 = 0,\ 2\lambda x_1 + 4x_2 + \lambda x_3 = 0$

For each permissible value of λ, determine the general solution.

Solution : The given system of equations is equivalent to the matrix equation,

$$AX = \begin{bmatrix} 3 & 1 & -\lambda \\ 4 & -2 & -3 \\ 2\lambda & 4 & \lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = O$$

The given system possess non-trivial solution, if rank of $A <$ number of unknowns.

i.e., Rank of $A < 3$

$$i.e., \quad \begin{vmatrix} 3 & 1 & -\lambda \\ 4 & -2 & -3 \\ 2\lambda & 4 & \lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad 3(-2\lambda + 12) - 1(4\lambda + 6\lambda) - \lambda(16 + 4\lambda) = 0$$

$$\Rightarrow \quad -4\lambda^2 - 32\lambda + 36 = 0 \Rightarrow \lambda^2 + 8\lambda - 9 = 0$$

$$\Rightarrow \quad (\lambda + 9)(\lambda - 1) = 0 \quad \therefore \lambda = -9 \quad \text{or} \quad \lambda = 1$$

Case I. For $\lambda = -9$, the given system reduces to

$$3x_1 + x_2 + 9x_3 = 0$$
$$4x_1 - 2x_2 - 3x_3 = 0$$
$$-18x_1 + 4x_2 - 9x_3 = 0$$

Now rank of $A = 2 < 3$ (number of variables) $\left(\because \begin{vmatrix} 3 & 1 \\ 4 & -2 \end{vmatrix} = -10 \neq 0\right)$

$\therefore$ System has infinite number of solutions.

$\therefore$ Number of independent solutions $= 3 - 2 = 1$.

Let $x_1 = 2K$ and from the first two equations, we get

$$x_2 + 9x_3 = -6K \text{ and } -2x_2 - 3x_3 = -8K$$

On solving $x_2 = 6K$ and $x_3 = \dfrac{-4}{3}K$, we get

$x_1 = 2K$, $x_2 = 6K$ and $x_3 = \frac{-4}{3}K$ as the general solution of the given system.

Case II. For $\lambda = 1$, the given system reduces to

$$3x_1 + x_2 - x_3 = 0$$
$$4x_1 - 2x_2 - 3x_3 = 0$$
$$2x_1 + 4x_2 + x_3 = 0$$

Now rank of $A = 2 < 3$ (number of variables)

Hence the system has infinite number of solutions.

$\therefore$ Number of independent solutions $= 3 - 2 = 1$.

Let $x_1 = K$ and from the first two equations, we get

$$x_2 - x_3 = 3K \text{ and } -2x_2 - 3x_3 = -4K$$

On solving, $x_2 = -K$ and $x_3 = 2K$, where K is a constant

$\therefore$ $x_1 = K$, $x_2 = -K$ and $x_3 = 2K$ is the general solution of the given system.

Example 11 : Solve $x_1 + 2x_3 - 2x_4 = 0;\ 2x_1 - x_2 - x_4 = 0;\ x_1 + 2x_3 - x_4 = 0;$ $4x_1 - x_2 + 3x_3 - x_4 = 0.$

Solution : Writing the given system of equations in matrix form $AX = O$ ...(1), we have

$$A = \begin{bmatrix} 1 & 0 & 2 & -2 \\ 2 & -1 & 0 & -1 \\ 1 & 0 & 2 & -1 \\ 4 & -1 & 3 & -1 \end{bmatrix},\quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix},\quad O = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

We will perform elementary row operations on A on L.H.S. We will perform same row operations on O in right side But the matrix O will remain unaltered because of these row operations.

Applying $R_2 \to R_2 \to 2R_1$, $R_3 \to R_3 - R_1$, $R_4 \to R_4 - 4R_1$, we get

$$\begin{bmatrix} 1 & 0 & 2 & -2 \\ 0 & -1 & -4 & 3 \\ 0 & 0 & 0 & 1 \\ 0 & -1 & -5 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Applying $R_4 \to R_4 - R_2$, we get

$$\begin{bmatrix} 1 & 0 & 2 & -2 \\ 0 & -1 & -4 & 3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Applying $R_3 \leftrightarrow R_4$, we get

$$\begin{bmatrix} 1 & 0 & 2 & -2 \\ 0 & -1 & -4 & 3 \\ 0 & 0 & -1 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Here we can write system of equations as

$$\left.\begin{aligned} x_1 + 2x_3 - 2x_4 &= 0 \\ -x_2 - 4x_3 + 3x_4 &= 0 \\ -x_3 + 4x_4 &= 0 \\ x_4 &= 0 \end{aligned}\right\} \quad \ldots(2)$$

Solving (2), we get $x_4 = 0,\ x_3 = 0,\ x_2 = 0$ and $x_1 = 0$.

Note : In the above form of the matrix A, number of non-zero rows is equal to 4. I rank is 4. Number of variables is equal to 4. The number of non-zero solutions $= n - r = 0$

Hence $x_1 = 0 = x_2 = x_3 = x_4$ is the only solution.

Example 12 : Show that the only real number λ for which the system $x + 2y + 3z = \lambda x;\ 3x + y + 2z = \lambda y\ ;\ 2x + 3y + z = \lambda z$ has non-zero solution is 6 and solve them, when $\lambda = 6$.

[JNTU 2005-May, 2006, 2006S (Set No. 1), Sep. 2008 (Set No.1)]

Solution : Given system can be expressed as $AX = O$ where

$$A = \begin{bmatrix} 1-\lambda & 2 & 3 \\ 3 & 1-\lambda & 2 \\ 2 & 3 & 1-\lambda \end{bmatrix},\quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } O = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Here number of variables $= n = 3$.

The given system of equations possess a non-zero (non-trivial) solution, if

Rank of $A <$ number of unknowns *i.e.*, Rank of $A < 3$

For this we must have det. $A = 0$

$$\therefore \begin{vmatrix} 1-\lambda & 2 & 3 \\ 3 & 1-\lambda & 2 \\ 2 & 3 & 1-\lambda \end{vmatrix} = 0$$

Applying $R_1 \to R_1 + R_2 + R_3$, we get $\begin{vmatrix} 6-\lambda & 6-\lambda & 6-\lambda \\ 3 & 1-\lambda & 2 \\ 2 & 3 & 1-\lambda \end{vmatrix} = 0$

$$\Rightarrow (6-\lambda) \begin{vmatrix} 1 & 1 & 1 \\ 3 & 1-\lambda & 2 \\ 2 & 3 & 1-\lambda \end{vmatrix} = 0$$

Applying $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$, we get $(6-\lambda)\begin{vmatrix} 1 & 0 & 0 \\ 3 & -2-\lambda & -1 \\ 2 & 1 & -1-\lambda \end{vmatrix} = 0$

$$\Rightarrow (6-\lambda)\,[(-2-\lambda)\,(-1-\lambda)+1] = 0$$

$\Rightarrow\ (6-\lambda)\,(\lambda^2 + 3\lambda + 3) = 0 \Rightarrow \lambda = 6$ is the only real value and other values are complex.

When $\lambda = 6$, the given system becomes

$$\begin{bmatrix} -5 & 2 & 3 \\ 3 & -5 & 2 \\ 2 & 3 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\sim\begin{bmatrix}-5 & 2 & 3\\ 0 & -19 & 19\\ 0 & 19 & -19\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}\text{(Applying } R_2\to 5R_2+3R_1, R_3\to 5R_3+2R_1\text{)}$$

$$\sim\begin{bmatrix}-5 & 2 & 3\\ 0 & -19 & 19\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}\text{(Applying } R_3\to R_3+R_2\text{)}$$

$\Rightarrow -5x+2y+3z=0$ and $-19y+19z=0 \Rightarrow y=z$

Since Rank of A < number of unknowns (Rank of $A=2$, number of unknowns = 3), therefore, the given system has infinite number of non-trivial solutions.

Let $z=k\Rightarrow y=k$ and $-5x+2k+3k=0\Rightarrow x=k$

$\therefore\ x=k, y=k, z=k$ is the solution.

Example 13 : Solve completely the system of equations :

$x+3y-2z=0, 2x-y+4z=0, x-11y+14z=0.$

[JNTU (A) June 2009, (K) May 2010 (Set No. 2)]

Solution : Taking $A=\begin{pmatrix}1 & 3 & -2\\ 2 & -1 & 4\\ 1 & -11 & 14\end{pmatrix}; X=\begin{pmatrix}x\\ y\\ z\end{pmatrix}; O=\begin{pmatrix}0\\ 0\\ 0\end{pmatrix}$, we get

$AX=O$ is the matrix form of equations.

Consider $A=\begin{pmatrix}1 & 3 & -2\\ 2 & -1 & 4\\ 1 & -11 & 14\end{pmatrix}$

$$\sim\begin{pmatrix}1 & 3 & -2\\ 0 & -7 & 8\\ 0 & -14 & 16\end{pmatrix}\text{ (Applying } R_2-2R_1, R_3-R_1\text{)}$$

$$\sim\begin{pmatrix}1 & 3 & -2\\ 0 & -7 & 8\\ 0 & -7 & 8\end{pmatrix}\text{ (Applying } \frac{R_3}{2}\text{)}$$

$$\sim\begin{pmatrix}1 & 3 & -2\\ 0 & -7 & 8\\ 0 & 0 & 0\end{pmatrix}.\text{ (Applying } R_3-R_2\text{)}$$

This is in Echelon form.

$\therefore\ \rho(A)=2=r.$

n = number of unknowns = 3.

$\therefore\ r<n$

$\therefore$ Number of linearly independent solutions $= n - r = 3 - 2 = 1$

$\therefore$ The above system has infinite solutions.

Now solve $AX = O$.

$$i.e., \begin{pmatrix} 1 & 3 & -2 \\ 0 & -7 & 8 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \quad x + 3y - 2z = 0 \qquad \text{... (1)}$$

$$-7y + 8z = 0 \Rightarrow y = \frac{8}{7}z. \qquad \text{... (2)}$$

From (1) and (2), we have

$$x + \frac{24}{7}z - 2z = 0 \Rightarrow x + \frac{10z}{7} = 0$$

$$\Rightarrow x = -\frac{10}{7}z$$

Taking $z = c$, we get $x = -\frac{10}{7}c$; $y = \frac{8}{7}c$

$$\therefore \; X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -\frac{10}{7}c \\ \frac{8}{7}c \\ c \end{pmatrix} = \frac{c}{7}\begin{pmatrix} -10 \\ 8 \\ 7 \end{pmatrix}$$

This gives the solution of the system of equations.

Example 14 : Determine b such that the system of homogeneous equations $2x + y + 2z = 0$, $x + y + 3z = 0$, $4x + 3y + bz = 0$ has trivial and non trivial solutions. Find the Non trivial solution. **[JNTU (K) July 2011 (Set No. 1)]**

Solution : Given homogeneous system of equations are

$$2x + y + 2z = 0, \; x + y + 3z = 0, \; 4x + 3y + bz = 0$$

Taking $A = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 1 & 3 \\ 4 & 3 & b \end{bmatrix}$ $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $O = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

We get $AX = 0$

$$\det A = 2(b-9) - 1(b-12) + 2(3-4)$$

$$= 2b - 18 - b + 12 - 2$$

$$= b - 8$$

$\det A = 0 \Rightarrow b = 8$.

Then the system of equations will have non-trivial solution.

When $b = 8$, $\begin{bmatrix} 2 & 1 & 2 \\ 1 & 1 & 3 \\ 4 & 3 & 8 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 2 & 1 & 2 \\ 0 & 1 & 4 \\ 0 & 1 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ (Applying $2R_2 - R_1; R_3 - 2R_1$)

$\Rightarrow \begin{bmatrix} 2 & 1 & 2 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ (Applying $R_3 - R_2$)

This gives the equations

$y + 4z = 0 \Rightarrow y = -4z$

and $2x + y + 2z = 0$

$\Rightarrow \quad 2x - 4z + 2z = 0$

$\Rightarrow \quad 2x = 2z$

$\Rightarrow \quad x = z$

Let $z = k$. Then $y = -4k$ and $x = k$

For different values of k, we get infinite solutions, when $b = 8$

$x = 0 = y = z$ is the only solution (trivial solution)

Example 15 : Solve completely the system of equations:

$x + 2y + 3z = 0$; $3x + 4y + 4z = 0; 7x + 10y + 12z = 0$ **[JNTU (A) Jan 2014 (Set No. 2)]**

Solution : Taking $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 4 \\ 7 & 10 & 12 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $O = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

We get the system of equations as $AX = O$

Consider $\quad A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 4 \\ 7 & 10 & 12 \end{bmatrix}$

$R_2 - 3R_1; R_3 - 7R_1 \quad \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -5 \\ 0 & -4 & -9 \end{bmatrix}$

$R_3 - 2R_2 \quad \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -5 \\ 0 & 0 & 1 \end{bmatrix}$

This is in Echelon form. No. of non-zero rows is 3.

$\therefore$ The rank of $A = 3$

$\therefore$ No. of variables $= 3$

$\therefore$ No. of non-zero solutions $= 3 - 3 = 0$

$\therefore$ $x = 0, y = 0, z = 0$ is the only solution.

EXERCISE 1.6

1. Solve completely the system of equations:

   (i) $x + 2y + 3z = 0;\ 3x + 4y + 4z = 0;\ 7x + 10y + 12z = 0.$

   (ii) $4x + 2y + z + 3w = 0;\ 6x + 3y + 4z + 7w = 0;\ 2x + y + w = 0.$

   [JNTU 2004S, Sep. 2008 (Set No.2)]

   (iii) $3x + 4y - z - 6w = 0;\ 2x + 3y + 2z - 3w = 0;\ 2x + y - 14z - 9w = 0;$
   $x + 3y + 13z + 3w = 0.$ **[JNTU 2002 (Set No. 3)]**

   (iv) $2x - 2y + 5z + 3w = 0;\ 4x - y + z + w = 0;\ 3x - 2y + 3z + 4w = 0;$
   $x - 3y + 7z + 6w = 0.$

   (v) $x + y = 0; y + z = 0; z + x = 0$ **[JNTU (K) Feb. 2015 (Set No. 1)]**

2. Show that the system of equations

   $2x_1 - 2x_2 + x_3 = \lambda x_1,\ 2x_1 - 3x_2 + 2x_3 = \lambda x_2,\ -x_1 + 2x_2 = \lambda x_3$

   can possess a non-trivial solution only if $\lambda = 1, \lambda = -3$.

   Obtain the general solution in each case. **[JNTU 2003S (Set No. 1)]**

3. Determine whether the vectors $(1,2,3),(2,3,4),(3,4,5)$ are linearly dependent or not.

   [JNTU (H) Aug. 2017 (R15)]

4. Find the value of α such that the vectors $(1,1,0),(1,\alpha,0)$ and $(1,1,1)$ are linearly dependent

   [JNTU (H) May 2018]

ANSWERS

1. (i) $x = 0, y = 0, z = 0$ (ii) $x = 1, y = -21 - m, z = -m, w = m$

   (iii) $x = 11k_1 + 6k_2, y = -8k_1 - 3k_2, z = k_1, w = k_2$

   (iv) $x = \frac{5}{9}k,\ y = 4k,\ z = \frac{7}{9}k,\ w = k$

2. $\lambda = 1, x_1 = 2t - s, x_2 = t, x_3 = s;\ \lambda = -3, x_1 = t, x_2 = -2t, x_3 = t$

Non Homogenious linear equations:-

An equation of the form

$a_1x_1 + a_2x_2 + a_3x_3 + \cdots + a_nx_n = b$ is called a Non Homogenious linear equations in n- unknowns. where $x_1, x_2 \ldots x_n$ are variables $a_1, a_2, \ldots a_n$ & b are constants.

Test for Consistancy of Non Homogenious equations:-

(by Rank method – Gauss Elimination Method)

Consider, m- non homogenious linear equations in n- unknowns as

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n = b_3$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n = b_m$$

the above system of eq's can be written in matrix form as

$$AX = B.$$

Step(1): Reduce the Augmented matrix $(A\ B)$ in to the Echolon form.

then, ~~let~~ the Rank of $(A\ B) = R$.

Rank of $A = r$.

& no. of unknowns (variables) $= n$.

Step(2):- 1) If $\rho(A) \neq \rho(A\ B)$ $(R \neq r)$ then the given system has no solution i.e the system is Inconsistant.

2) If $\rho(A) = \rho(A\ B)$ $(R = r)$ then the given system has consistant & has a solution.

a) If $\rho(A) = \rho(A\ B) = n$ then the given system is consistant and has a unique solution.

b) If $\rho(A) = \rho(A\ B) < n$ then the given system is consistant. and having an Infinite no. of solutions. i.e $n-R$ linearly Independent solutions.

c) If the given system has linearly Independent solution then $|A| = 0$. i.e the matrix A is a singular matrix.

eg ① Solve the System of equations:

$x+2y+3z=1$, $2x+3y+8z=2$, $x+y+z=3$.

Soln Given System of equations are:

$x+2y+3z=1$

$2x+3y+8z=2$

$x+y+z=3$.

the above System of equations can be written in matrix form as $AX=B$.

i.e $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 8 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

Consider, the Augmented matrix of

$(A\ B) = \begin{pmatrix} 1 & 2 & 3 & 1 \\ 2 & 3 & 8 & 2 \\ 1 & 1 & 1 & 3 \end{pmatrix}$

Apply, $R_2 \to R_2 - 2R_1$.

$R_3 \to R_3 - R_1$.

$\sim \begin{pmatrix} 1 & 2 & 3 & 1 \\ 0 & -1 & 2 & 0 \\ 0 & -1 & -2 & 2 \end{pmatrix}$

$R_3 \to R_3 - R_2$

$(A\ B) \sim \begin{pmatrix} 1 & 2 & 3 & 1 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & -4 & 2 \end{pmatrix}$

This is in Echelon form. $\therefore \rho(A\ B) = 3 =$ no. of non zero rows.

Now, $A=\begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & 2 \\ 0 & 0 & -4 \end{pmatrix}$, $B=\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$

$\therefore \rho(A) = 3 =$ no. of non zero rows.

$\therefore \rho(A) = \rho(A\ B) = n = 3$.

$\therefore$ The given System is Consistant & has a Unique Solution.

Now, Solve $AX=B$.

i.e $\begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & 2 \\ 0 & 0 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$

$\therefore x+2y+3z=1$. —①

$-y+2z=0$ —②

$-4z=2$

$\therefore z=-\frac{1}{2}$

on Sub. z in eq ②, we get

$-y+2(-\frac{1}{2})=0$

$\Rightarrow -y-1=0$.

$\therefore y=-1$.

ON Sub. the Values of y & z in eq ①, we get

$x+2(-1)+3(-\frac{1}{2})=1$.

$x-2-\frac{3}{2}=1$

$x-\frac{7}{2}=1$

$x=1+\frac{7}{2}=\frac{9}{2}$.

$\therefore$ the req. Solution is

$X=\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 9/2 \\ -1 \\ -1/2 \end{pmatrix}$

Q(2):- Solve the System of equations Completely,

$x+y+z=3, \quad 3x-5y+2z=8. \quad 5x-3y+4z=14.$

Sol: Given System of eq's.

$x+y+z=3$

$3x-5y+2z=8$

$5x-3y+4z=14.$

the above System of eq's can be written in matrix form as

$AX=B.$

i.e $\begin{pmatrix} 1 & 1 & 1 \\ 3 & -5 & 2 \\ 5 & -3 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 3 \\ 8 \\ 14 \end{pmatrix}$

Consider, the Augmented matrix

$(AB)=\begin{pmatrix} 1 & 1 & 1 & 3 \\ 3 & -5 & 2 & 8 \\ 5 & -3 & 4 & 14 \end{pmatrix}$

$R_2 \to R_2-3R_1$

$R_3 \to R_3-5R_1$

$\sim \begin{pmatrix} 1 & 1 & 1 & 3 \\ 0 & -8 & -1 & -1 \\ 0 & -8 & -1 & -1 \end{pmatrix}$

$R_3 \to R_3-R_2$

$\sim \begin{pmatrix} 1 & 1 & 1 & 3 \\ 0 & -8 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$\therefore$ this is an Echelon form.

$\therefore \rho(AB) =$ no. of non zero rows $= 2.$

Now, $A=\begin{pmatrix} 1 & 1 & 1 \\ 0 & -8 & -1 \\ 0 & 0 & 0 \end{pmatrix}$ & $B=\begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix}$

$\rho(A)=2.$ & $n=3.$

$\therefore \rho(A)=\rho(AB)<n.$

$\therefore$ the given System is Consistant & having an infinite no. of Solutions. Now, Solve $AX=B.$

i.e $\begin{pmatrix} 1 & 1 & 1 \\ 0 & -8 & -1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix}$

$x+y+z=3 \quad -(1)$

$-8y-z=-1 \quad -(2)$

$\therefore 8y+z=1$

let $z=k.$

from eq(2), $8y+k=1$

$8y=1-k$

$\therefore y=\frac{1}{8}(1-k)=\frac{1}{8}-\frac{k}{8}.$

On Subs. y & z Values in eq (1), we get.

$x+\frac{1}{8}-\frac{k}{8}+k=3.$

$\Rightarrow x+\frac{7k}{8}=3-\frac{1}{8}$

$\Rightarrow x=\frac{7k}{8}+\frac{23}{8}.$

$\therefore x=\frac{1}{8}(7k+23)=\frac{7}{8}k+\frac{23}{8}.$

$\therefore$ the req. solution is $X=\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} \frac{7}{8}k+\frac{23}{8} \\ -\frac{k}{8}+\frac{1}{8} \\ k \end{pmatrix}=\begin{pmatrix} \frac{7}{8}k \\ -\frac{k}{8} \\ k \end{pmatrix}+\begin{pmatrix} \frac{23}{8} \\ \frac{1}{8} \\ 0 \end{pmatrix}=k\begin{pmatrix} \frac{7}{8} \\ -\frac{1}{8} \\ 1 \end{pmatrix}+\begin{pmatrix} \frac{23}{8} \\ \frac{1}{8} \\ 0 \end{pmatrix}$

Example 3 : Discuss for what values of λ, μ the simultaneous equations $x+y+z=6$, $x+2y+3z=10$, $x+2y+\lambda z=\mu$ have (*i*) no solution (*ii*) a unique solution (*iii*) an infinite number of solutions. **[JNTU 2001, 2002S, 2004S (Set No. 1), 2005 (Set No.3)]**

Solution : The matrix form of given system of equations is

$$AX=\begin{bmatrix}1&1&1\\1&2&3\\1&2&\lambda\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}6\\10\\\mu\end{bmatrix}=B$$

We have the augmented matrix is $[A/B]=\begin{bmatrix}1&1&1&6\\1&2&3&10\\1&2&\lambda&\mu\end{bmatrix}$

Applying $R_2\to R_2-R_1$ and $R_3\to R_3-R_1$, we get $[A/B]\sim\begin{bmatrix}1&1&1&6\\0&1&2&4\\0&1&\lambda-1&\mu-6\end{bmatrix}$

Applying $R_3\to R_3-R_2$, we get $[A/B]\sim\begin{bmatrix}1&1&1&6\\0&1&2&4\\0&0&\lambda-3&\mu-10\end{bmatrix}$

Case I. Let $\lambda\neq 3$ then rank of $A=3$ and rank of $[A/B]=3$, so that they have same rank. Then the system of equations is consistent. Here the number of unknowns is 3 which is same as the rank of A. The system of equations will have a unique solution. This is true for any value of μ.

Thus if $\lambda\neq 3$ and μ has any value, the given system of equations will have a unique solution.

Case II. Suppose $\lambda=3$ and $\mu\neq 10$, then we can see that rank of $A=2$ and rank of $[A/B]=3$. Since the ranks of A and $[A/B]$ are not equal, we say that the system of equations has no solution (inconsistent).

Case III. Let $\lambda=3$ and $\mu=10$. Then we have rank of A = rank of $[A/B]=2$.

$\therefore$ The given system of equations will be consistent.

But here the number of unknowns $=3>$ rank of A.

Hence the system has infinitely many solutions.

Example 4 : Find whether the following equations are consistent, if so solve them.
$x+y+2z=4$; $2x-y+3z=9$; $3x-y-z=2$

[JNTU May 2005S, 2005 (Set No.1), (A) Dec. 2013 (Set No. 4)]

Solution : The given equations can be written in the matrix form as $\begin{bmatrix} 1 & 1 & 2 \\ 2 & -1 & 3 \\ 3 & -1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 4 \\ 9 \\ 2 \end{bmatrix}$

i.e. AX = B

The augmented matrix $[A|B]=\begin{bmatrix} 1 & 1 & 2 & 4 \\ 2 & -1 & 3 & 9 \\ 3 & -1 & -1 & 2 \end{bmatrix}$

$$[A \mid B] \sim \begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & -3 & -1 & 1 \\ 0 & -4 & -7 & -10 \end{bmatrix} \text{(Applying } R_2 \to R_2 - 2R_1 \text{ and } R_3 \to R_3 - 3R_1)$$

$$[A \mid B] \sim \begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & -3 & -1 & 1 \\ 0 & 0 & -17 & -34 \end{bmatrix} \text{(Applying } R_3 \to 3R_3 - 4R_2)$$

Since Rank of A = 3 and Rank of $[A \mid B] = 3$ $\therefore$ Rank of A = Rank of $[A \mid B]$

The given system is consistent. So it has a solution.

Since Rank of A = Rank of $[A/B]$ = number of unknowns,

$\therefore$ The given system has a unique solution.

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & -3 & -1 \\ 0 & 0 & -17 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 4 \\ 1 \\ -34 \end{bmatrix}$$

$\Rightarrow$ $x+y+2z=4$ .... (1)

$-3y-z=1$ .... (2)

$-17z=-34$ or $z=2$ .... (3)

Substituting $z=2$ in (2) $\Rightarrow -3y-2=1 \Rightarrow -3y=3 \Rightarrow y=-1$

Substituting $y=-1, z=2$ in (1), we get

$x-1+4=4 \Rightarrow x=1$

$\therefore$ $x=1, y=-1, z=2$ is the solution.

Example 5 : Find whether the following system of equations are consistent. If so solve them.

$x + 2y + 2z = 2;$ $3x - 2y - z = 5;$ $2x - 5y + 3z = -4;$ $x + 4y + 6z = 0$

[JNTU May 2005S, Sep. 2008, (H) June 2010 (Set No. 4), (A) Dec. 2013 (Set No. 4)]

Solution : The given equations can be written in the matrix form as AX = B

$$(i.e.) \begin{bmatrix} 1 & 2 & 2 \\ 3 & -2 & -1 \\ 2 & -5 & 3 \\ 1 & 4 & 6 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 2 \\ 5 \\ -4 \\ 0 \end{bmatrix}$$

The Augmented matrix $[A,B]=\begin{bmatrix}1 & 2 & 2 & 2\\ 3 & -2 & -1 & 5\\ 2 & -5 & 3 & -4\\ 1 & 4 & 6 & 0\end{bmatrix}$

Applying $R_2 \to R_2-3R_1,\ R_3 \to R_3-2R_1;\ R_4 \to R_4-R_1$, we get $[A,B]\sim\begin{bmatrix}1 & 2 & 2 & 2\\ 0 & -8 & -7 & -1\\ 0 & -9 & -1 & -8\\ 0 & 2 & 4 & -2\end{bmatrix}$

Applying $R_3 \to 8R_3-9R_2$ and $R_4 \to 4R_4+R_2$, we get

$$[A,B]\sim\begin{bmatrix}1 & 2 & 2 & 2\\ 0 & -8 & -7 & -1\\ 0 & 0 & 55 & -55\\ 0 & 0 & 9 & -9\end{bmatrix}$$

$$\sim\begin{bmatrix}1 & 2 & 2 & 2\\ 0 & -8 & -7 & -1\\ 0 & 0 & 1 & -1\\ 0 & 0 & 1 & -1\end{bmatrix}\left(\text{Applying } \frac{R_3}{55}, \frac{R_4}{9}\right)$$

$$\sim\begin{bmatrix}1 & 2 & 2 & 2\\ 0 & -8 & -7 & -1\\ 0 & 0 & 1 & -1\\ 0 & 0 & 0 & 0\end{bmatrix}(\text{Applying } R_4 \to R_4-R_3)$$

Since Rank of A = 3 and Rank of $[A,B]=3$, we have Rank of A = Rank of $[A,B]$.

$\therefore$ The given system is consistent and it has solution.

Since Rank of A = Rank of $[A,B]$ = number of unknowns

$\therefore$ The given system has a unique solution.

We have $\begin{bmatrix}1 & 2 & 2\\ 0 & -8 & -7\\ 0 & 0 & 1\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}2\\ -1\\ -1\\ 0\end{bmatrix}$

$\Rightarrow\ x+2y+2z=2$ .... (1), $-8y-7z=-1$ .... (2) and $z=-1$

Put $z=-1$ in (2) $\Rightarrow\ -8y+7=-1\ \Rightarrow\ -8y=-8 \Rightarrow y=1$

Put $y=1, z=-1$ in (1) $\Rightarrow\ x+2-2=2\ \Rightarrow\ x=2$

$\therefore\ x=2, y=1, z=-1$ is the solution.

Example 6 : Find the value of λ for which the system of equations $3x - y + 4z = 3$, $x + 2y - 3z = -2$, $6x + 5y + \lambda z = -3$ will have infinite number of solutions and solve them with that λ value. [JNTU May 2005S]

Solution : The given system of equations can be written in the matrix form as $AX = B$

i.e. $$\begin{bmatrix} 3 & -1 & 4 \\ 1 & 2 & -3 \\ 6 & 5 & \lambda \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ -3 \end{bmatrix}$$

The Augmented matrix is $[A,B] = \begin{bmatrix} 3 & -1 & 4 & 3 \\ 1 & 2 & -3 & -2 \\ 6 & 5 & \lambda & -3 \end{bmatrix}$

Applying $R_2 \to 3R_2 - R_1, R_3 \to R_3 - 2R_1$, $[A,B] \sim \begin{bmatrix} 3 & -1 & 4 & 3 \\ 0 & 7 & -13 & -9 \\ 0 & 7 & \lambda - 8 & -9 \end{bmatrix}$

Applying $R_3 \to R_3 - R_2$, $[A,B] \sim \begin{bmatrix} 3 & -1 & 4 & 3 \\ 0 & 7 & -13 & -9 \\ 0 & 0 & \lambda + 5 & 0 \end{bmatrix}$

If $\lambda = -5$, Rank of A = 2 and Rank of $[A,B] = 2$

Number of unknowns = 3

$\therefore$ Rank of A = Rank of $[A,B]$ $\neq$ number of unknowns

Hence when $\lambda = -5$, the given system is consistent and it has an infinite number of solutions.

If $\lambda = -5$ the given system becomes $\begin{bmatrix} 3 & -1 & 4 \\ 0 & 7 & -13 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -9 \\ 0 \end{bmatrix}$

$3x - y + 4z = 3$ ..... (1) and $7y - 13z = -9$ ..... (2)

Let $z = k \cdot$ Then from (2), we get

$7y - 13k = -9 \Rightarrow 7y = 13k - 9 \Rightarrow y = (13k - 9)/7$

Substituting the value of y in (1), we get

$$3x - \frac{1}{7}(13k - 9) + 4k = 3 \Rightarrow 3x = \frac{13}{7}k - 4k + 3 - \frac{9}{7}$$

$$\Rightarrow 3x = -\frac{15}{7}k + \frac{12}{7} \Rightarrow x = \frac{1}{7}(-5k + 4)$$

$\therefore$ The solution is $x = \frac{1}{7}(-5k + 4), y = \frac{1}{7}(13k - 9), z = k$.

Example 7 : Prove that the following set of equations are consistent and solve them.

$$3x + 3y + 2z = 1; \; x + 2y = 4; \; 10y + 3z = -2; \; 2x - 3y - z = 5$$

[JNTU April 2007, Aug. 2007, 2008, (H) June 2009, (K) 2009S, (K) May 2010 (Set No. 1)]

Solution : The given system of equations can be written in the matrix form as follows:

$$AX = \begin{bmatrix} 3 & 3 & 2 \\ 1 & 2 & 0 \\ 0 & 10 & 3 \\ 2 & -3 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ -2 \\ 5 \end{bmatrix} = B$$

The Augmented matrix of the given equations is

$$[A|B] = \begin{bmatrix} 3 & 3 & 2 & 1 \\ 1 & 2 & 0 & 4 \\ 0 & 10 & 3 & -2 \\ 2 & -3 & -1 & 5 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 0 & 4 \\ 3 & 3 & 2 & 1 \\ 0 & 10 & 3 & -2 \\ 2 & -3 & -1 & 5 \end{bmatrix} \text{[Applying } R_1 \leftrightarrow R_2\text{]}$$

$$\sim \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & -3 & 2 & -11 \\ 0 & 10 & 3 & -2 \\ 0 & -7 & -1 & -3 \end{bmatrix} \text{[Applying } R_2 - 3R_1 \text{ and } R_4 - 2R_1\text{]}$$

$$\sim \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 1 & -2/3 & 11/3 \\ 0 & 10 & 3 & -2 \\ 0 & -7 & -1 & -3 \end{bmatrix} \left[\text{Applying } \frac{R_2}{-3}\right]$$

$$\sim \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 1 & -2/3 & 11/3 \\ 0 & 0 & 29/3 & -116/3 \\ 0 & 0 & -17/3 & 68/3 \end{bmatrix} \text{[Applying } R_3 - 10R_2 \text{ and } R_4 + 7R_2\text{]}$$

$$\sim \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 1 & -2/3 & 11/3 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & -17/3 & 68/3 \end{bmatrix} \left[\text{Applying } \frac{3}{29} R_2\right]$$

$$\sim\begin{bmatrix}1 & 2 & 0 & 4\\ 0 & 1 & -2/3 & 11/3\\ 0 & 0 & 1 & -4\\ 0 & 0 & 0 & 0\end{bmatrix} \left[\text{Applying } R_4 + \frac{17}{3}R_3\right]$$

Thus the matrix [A|B] has been reduced to Echelon form.

$\therefore$ Rank [A|B] = no. of non-zero rows = 3

By the same row operations, we have

$$A \sim \begin{bmatrix}1 & 2 & 0\\ 0 & 1 & -2/3\\ 0 & 0 & 1\\ 0 & 0 & 0\end{bmatrix}$$

$\therefore$ Rank (A) = 3

Since Rank (A) = Rank [A|B] = 3, therefore the given equations are consistent.

Also rank (A) = 3 = no. of unknowns.

Hence the given equations have unique solution.

The given equations are equivalent to the equations

$$x + 2y = 4;\ y - \frac{2}{3}z = \frac{11}{3};\ z = -4$$

On solving these equations, we get

$$x = 2, y = 1, z = -4.$$

Example 8 : Test for consistency and hence solve the system :

$x + y + z = 6, x - y + 2z = 5, 3x + y + z = 8, 2x - 2y + 3z = 7$ **[JNTU 2008S (Set No.3)]**

Solution : Consider $A = \begin{bmatrix}1 & 1 & 1\\ 1 & -1 & 2\\ 3 & 1 & 1\\ 2 & -2 & 3\end{bmatrix}, B = \begin{bmatrix}6\\5\\8\\7\end{bmatrix}, X = \begin{bmatrix}x\\y\\z\end{bmatrix}$

Then the system of equations can be written as $AX = B$

The Augmented matrix of the given equations is

$$[A/B] = \begin{bmatrix}1 & 1 & 1 & 6\\ 1 & -1 & 2 & 5\\ 3 & 1 & 1 & 8\\ 2 & -2 & 3 & 7\end{bmatrix}$$

$$\sim\begin{bmatrix}1 & 1 & 1 & 6\\ 0 & -2 & 1 & -1\\ 0 & -2 & -2 & -10\\ 0 & -4 & 1 & -5\end{bmatrix} (\text{Applying } R_2 - R_1, R_3 - 3R_1, R_4 - 2R_1)$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & -2 & 1 & -1 \\ 0 & -2 & -2 & -10 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{(Applying } R_4 - 2R_3\text{)}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & -2 & 1 & -1 \\ 0 & 0 & -3 & -9 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{(Applying } R_3 - R_2\text{)}$$

Thus the matrix [A/B] has been reduced to Echelon form.

$\therefore$ Rank [A/B] = no. of non-zero rows = 3

By applying same operations, we have Rank (A) = 3.

Since Rank (A) = Rank [A/B], therefore the given equations are consistent.

Hence the given equations have a unique solution.

From above Echelon form, we get

$-3z = -9 \Rightarrow z = 3$

$-2y + z = 1 \Rightarrow -2y = -y \Rightarrow y = 2$

$x + y + z = 6 \Rightarrow x = 6y - 3 = 6 - 2 - 3 = 1$

$\therefore$ The solution is $x = 1, y = 2, z = 3$.

Example 9 : Show that the equations $x - 4y + 7z = 14$, $3x + 8y - 2z = 13$, $7x - 8y + 26z = 5$ are not consistent. **[JNTU 2008S (Set No.3)]**

Solution : Consider $A = \begin{bmatrix} 1 & -4 & 7 \\ 3 & 8 & -2 \\ 7 & -8 & 26 \end{bmatrix}, B = \begin{bmatrix} 14 \\ 13 \\ 5 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$.

Then the given system of equations can be written as AX = B.

The Augmented matrix of the given equations is

$$[A/B] = \begin{bmatrix} 1 & -4 & 7 & 14 \\ 3 & 8 & -2 & 13 \\ 7 & -8 & 26 & 5 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -4 & 7 & 14 \\ 0 & 20 & -23 & -29 \\ 0 & 20 & -23 & -93 \end{bmatrix} \text{(Operating } R_2 - 3R_1 \text{ and } R_3 - 7R_1\text{)}$$

$$\sim \begin{bmatrix} 1 & -4 & 7 & 14 \\ 0 & 20 & -23 & -29 \\ 0 & 0 & 0 & -64 \end{bmatrix} \text{(Operating } R_3 - R_2\text{)}$$

Number of non-zero rows is 3. $\therefore$ Rank $(A, B) = 3$.

We can observe that number of non-zero rows is 2. $\therefore$ Rank (A) = 2.

Since Rank (A) $\neq$ Rank [A/B], the system is inconsistent.

Example 10 : Test for the consistency of $x+y+z=1, x-y+2z=1, x-y+2z=5, 2x-2y+3z=1$.
[JNTU 2008S (Set No.4), (A) Nov. 2010 (Set No. 3)]

Solution : Given system can be written as AX = B

where $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ 1 & -1 & 2 \\ 2 & -2 & 3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 1 \\ 1 \\ 5 \\ 1 \end{bmatrix}$

The Augmented matrix is

$$[A,B] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 2 & 1 \\ 1 & -1 & 2 & 5 \\ 2 & -2 & 3 & 1 \end{bmatrix}$$

Applying $R_2 - R_1, R_3 - R_1, R_4 - 2R_1$, we get

$$[A,B] \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & 1 & 0 \\ 0 & -2 & 1 & 4 \\ 0 & -4 & 1 & -1 \end{bmatrix}$$

Applying $R_3 - R_2, R_4 - R_2$, we get

$$[A,B] \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & 4 \\ 0 & -2 & 0 & -1 \end{bmatrix}$$

Applying $R_3 \leftrightarrow R_4$, we get

$$[A,B] \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & 1 & 0 \\ 0 & -2 & 0 & -1 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

We can see that number of non-zero rows = 4.

$\therefore$ Rank [A,B] = 4.

But $A \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & 1 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ (Applying Same operations)

$\therefore$ Rank (A) = 3.

Since Rank (A) $\neq$ Rank [A,B], therefore the system is not consistent.

Example 19 : Find the values of 'a' and 'b' for which the equations, $x+y+z=3, x+2y+2z=6,\ \ x+9y+az=b$ have

(*i*) No solution (*ii*) A unique solution (*iii*) Infinite number of solutions.

[JNTU (H) June 2012]

Solution : Given equations are $x+y+z=3, x+2y+3z=6, x+9y+az=b$

Taking $A=\begin{bmatrix}1 & 1 & 1\\ 1 & 2 & 3\\ 1 & 9 & a\end{bmatrix}, X=\begin{bmatrix}x\\ y\\ z\end{bmatrix}$ and $B=\begin{bmatrix}3\\ 6\\ b\end{bmatrix}$

The given equations can be written as $AX=B$

Augmented matrix is $[A, B]=\begin{bmatrix}1 & 1 & 1 & 3\\ 1 & 2 & 3 & 6\\ 1 & 9 & a & b\end{bmatrix}$

$$\sim\begin{bmatrix}1 & 1 & 1 & 3\\ 0 & 1 & 2 & 3\\ 0 & 8 & a-1 & b-3\end{bmatrix} \text{(Applying } R_2-R_1; R_3-R_1\text{)}$$

$$\sim\begin{bmatrix}1 & 0 & -1 & 0\\ 0 & 1 & 2 & 3\\ 0 & 0 & a-17 & b-27\end{bmatrix} \text{(Applying } R_1-R_2; R_3-8R_2\text{)}$$

Case I : Let $a=17, b=27$.

Then rank of A $=$ rank of $[A, B]=2$.

No. of variables $=3$

Since rank (A) $=$ rank $[A \mid B]=2<3$

$\therefore$ The system is consistent and will have infinite number of solutions.

Case II : Let $a=17, b \neq 27$.

Rank of A $=2$ and Rank of $[A, B]=3$

$\therefore$ The system is inconsistant. (no solution)

Case III : $a \neq 17, b \neq 27$

Rank A $=$ Rank $[A, B]=3=$ no. of variables

$\therefore$ The system will be consistent and there will be unique solution.

Example 21 : Test for consistency and if consistent solve the system,

$5x+3y+7t=4;\ 3x+26y+2t=9;\ 7x+2y+10t=5$ [JNTU (A) June. 2011 (Set No. 4)]

(or) Show that the system of equations

$5x+3y+7z=4, 3x+26y+2z=9, 7x+2y+10z=5$ is consistent and hence solve it.

[JNTU (H) Dec. 2017]

Solution : Given system of equations is

$5x+3y+7t=4; 3x+26y+2t=9; 7x+2y+10t=5$

Let $A=\begin{bmatrix}5 & 3 & 7\\ 3 & 26 & 2\\ 7 & 2 & 10\end{bmatrix} X=\begin{bmatrix}x\\ y\\ z\end{bmatrix}$ and $B=\begin{bmatrix}4\\ 9\\ 5\end{bmatrix}$

Then given system is of the form $AX=B$

The Augmented matrix is

$$[A\,|\,B]=\begin{bmatrix}5 & 3 & 7 & 4\\ 3 & 26 & 2 & 9\\ 7 & 2 & 10 & 5\end{bmatrix}$$

$$\sim\begin{bmatrix}5 & 3 & 7 & 4\\ 0 & 121 & -11 & 33\\ 0 & -11 & 1 & -3\end{bmatrix} \text{(Applying } 5R_2-3R_1, 5R_3-7R_1\text{)}$$

$$\sim\begin{bmatrix}5 & 3 & 7 & 4\\ 0 & 11 & -1 & 3\\ 0 & -11 & 1 & -3\end{bmatrix} \text{(Applying } \frac{R_2}{11}\text{)}$$

$$\sim\begin{bmatrix}5 & 3 & 7 & 4\\ 0 & 11 & -1 & 3\\ 0 & 0 & 0 & 0\end{bmatrix} \text{(Applying } R_3+R_2\text{)}$$

Number of non-zero rows = 2

Rank A = 2 = Rank (A | B)

$\therefore$ The given system is consistent.

Number of variables = 3. So number of solutions is infinite.

From the matrix

$5x+3y+7z=4$

$11y-z=3$

Let $z = k$. Then $11y = 3 + k \Rightarrow y = \dfrac{3+k}{11}$

$$5x = 4 - 3y - 7z = 4 - 3\left(\frac{3+k}{11}\right) - 7k$$

$$= \frac{44 - 9 - 3k - 77k}{11} = \frac{53 - 80k}{11} \Rightarrow x = \frac{53 - 80k}{55}$$

This will give infinite number of solutions.

Example 22 : Show that the equation $3x + 4y + 5z = a, 4x + 5y + 6z = b$ and $5x + 6y + 7z = 0$ do not have a solution unless $a + c = 2b$ **[JNTU (H) June 2015]**

Given system of equations are $3x + 4y + 5z = a, 4x + 5y + 6z = b$; $5x + 6y + 7z = 0$

Solution : Let $A = \begin{bmatrix} 3 & 4 & 5 \\ 4 & 5 & 6 \\ 5 & 6 & 7 \end{bmatrix}; X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}; B = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

Then the given system is of the form $AX = B$

$$\Rightarrow \begin{bmatrix} 3 & 4 & 5 \\ 4 & 5 & 6 \\ 5 & 6 & 7 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 3 & 4 & 5 \\ 4 & 5 & 6 \\ 0 & -2 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ b \\ 3c - 5a \end{bmatrix} \quad (\text{Applying } 3R_3 - 5R_1)$$

$$\Rightarrow \begin{bmatrix} 3 & 4 & 5 \\ 0 & -1 & -2 \\ 0 & -2 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ 3b - 4a \\ 3c - 5a \end{bmatrix} \quad (\text{Applying } 3R_2 - 4R_1)$$

$$\Rightarrow \begin{bmatrix} 3 & 4 & 5 \\ 0 & -1 & -2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ 3b - 4a \\ 3a - 6b + 3c \end{bmatrix} \quad (\text{Applying } R_3 - 2R_2)$$

From the matrix we can have $3a + 3c = 6b \Rightarrow a + c = 2b$

Hence the result.

Example 23 : Discuss the consistency of the system of equations

$2x + 3y + 4z = 11; x + 5y + 7z = 15; 3x + 11y + 13z = 25$ **[JNTU (H) Dec. 2016]**

Solution : Writing the above equations in matrix form $AX = B$

$$\text{where } A = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 5 & 7 \\ 3 & 11 & 13 \end{bmatrix}; X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}; B = \begin{bmatrix} 11 \\ 15 \\ 25 \end{bmatrix}$$

$$\text{Augmented matrix is } [A|B] = \begin{bmatrix} 2 & 3 & 4 & 11 \\ 1 & 5 & 7 & 15 \\ 3 & 11 & 13 & 25 \end{bmatrix} \quad \text{... (1)}$$

$$\sim \begin{bmatrix} 1 & 5 & 7 & 15 \\ 2 & 3 & 4 & 11 \\ 3 & 11 & 13 & 25 \end{bmatrix} \text{(Applying } R_1 \leftrightarrow R_2\text{)}$$

$$\sim \begin{bmatrix} 1 & 5 & 7 & 15 \\ 0 & -7 & -10 & -19 \\ 0 & -4 & -8 & -20 \end{bmatrix} \text{(Applying } R_2 - 2R_1; R_3 - 3R_1\text{)}$$

$$\sim \begin{bmatrix} 1 & 5 & 7 & 15 \\ 0 & -7 & -10 & -19 \\ 0 & 1 & 2 & 5 \end{bmatrix} \text{(Applying } \frac{R_3}{-4}\text{)}$$

$$\sim \begin{bmatrix} 1 & 5 & 7 & 15 \\ 0 & -7 & -10 & -19 \\ 0 & 0 & 4 & 16 \end{bmatrix} \text{(Applying } 7R_3 + R_2\text{)}$$

$$\sim \begin{bmatrix} 1 & 5 & 7 & 15 \\ 0 & -7 & -10 & -19 \\ 0 & 0 & 1 & 4 \end{bmatrix} \text{(Applying } \frac{R_3}{4}\text{)} \quad \text{... (2)}$$

This is in echelon form. Number of non-zero rows is 3.

$\therefore$ Rank $[A|B] = 3$.

Using the same operations, rank of $A = 3$.

Since Rank of $[A|B] = \text{rank}[A]$, therefore, the system of equations is consistent and solution exists.

Since the rank of $A =$ rank of $[A|B] = 3 =$ number of unknowns, the solution will be unique.

From the matrix, we have

$z = 4$

$-7y - 10z = -19 \Rightarrow -7y - 40 = -19 \Rightarrow -7y = 21 \Rightarrow y = -3$

and $x + 5y + 7z = 15$

$\Rightarrow x - 15 + 28 = 15 \Rightarrow x = 2$

$\therefore\ x = 2, y = -3, z = 4$ is the unique solution.

Example 24 : Find for what values of λ the equations $x + y + z = 1$, $x + 2y + 4z = \lambda$, $x + 4y + 10z = \lambda^2$ have a solution and solve them completely in each case.

Solution : The given system can be expressed as $AX = B$

$$i.e.\ \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 4 & 10 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ \lambda \\ \lambda^2 \end{bmatrix} \qquad ...(1)$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, we get

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 3 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ \lambda - 1 \\ \lambda^2 - 1 \end{bmatrix}$$

Applying $R_3 \to R_3 - 3R_2$, we get

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ \lambda - 1 \\ \lambda^2 - 3\lambda + 2 \end{bmatrix}$$

Now the given equations will be consistent if and only if, $\lambda^2 - 3\lambda + 2 = 0$, *i.e.*, iff $\lambda = 1$ or $= 2$.

Case I. If $\lambda = 1$ then we have $A \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$ so that rank of $A = 2$.

$[A \mid B] \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ so that rank of $[A|B] = 2$ and the two ranks are equal.

Then we have the system of equations is consistent.

We can write $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

Writing as linear equations, we get $x + y + z = 1, y + 3z = 0$

Let $z = k$. Then $y = -3z = -3k$ and $x = 1 - y - z \Rightarrow x = 1 + 3k - k = 2k + 1$

Thus $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2k+1 \\ -3k \\ k \end{bmatrix} = k\begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} + 1\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ where k is a parameter.

In this case the system will have infinite number of solutions.

Case II. $\lambda = 2$. Here $A \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$ so that the rank of A is 2

and $[A|B] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ and rank $[A/B] = 2$.

Here we have rank A = rank $[A/B]$ and the system will be consistent. Here no. of unknowns $= 3 >$ rank of A. Hence the number of solutions is infinite.

The system can be written as $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \Rightarrow x+y+z=1$ and $y+3z=1$

Take $z = c \Rightarrow y = 1-3c$ and $x = 1-y-z = 1-1+3c-c = 2c$

$\therefore$ The general solution is $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2c \\ 1-3c \\ c \end{bmatrix} = c\begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

Where c is a parameter. The system has infinitely many solutions.

Example 25 : Solve the following system of equations $2p+q+2r+s=6$; $6p-6q+6r+12s=36$; $4p+3q+3r-3s=1$; $2p+2q-r+s=10$?

[JNTU (K) Feb. 2011 (Set No. 1)]

Solution : Given systemof equations is $2p+q+2r+s=6; 6p-6q+6r+12s=36;$ $4p+3q+3r-3s=1;$ $2p+2q-r+s=10.$

Taking $A = \begin{bmatrix} 2 & 1 & 2 & 1 \\ 6 & -6 & 6 & 12 \\ 4 & 3 & 3 & -3 \\ 2 & 2 & -1 & 1 \end{bmatrix}$, $X = \begin{bmatrix} p \\ q \\ r \\ s \end{bmatrix}$, $B = \begin{bmatrix} 6 \\ 36 \\ 1 \\ 10 \end{bmatrix}$

the equations will be of the form AX = B.

Consider $[A,B] = \begin{bmatrix} 2 & 1 & 2 & 1 & 6 \\ 6 & -6 & 6 & 12 & 36 \\ 4 & 3 & 3 & -3 & 1 \\ 2 & 2 & -1 & 1 & 10 \end{bmatrix}$

$\sim \begin{bmatrix} 2 & 1 & 2 & 1 & 6 \\ 0 & -9 & 0 & 9 & 18 \\ 0 & 1 & -1 & -5 & -11 \\ 0 & 1 & -3 & 0 & 4 \end{bmatrix}$ (Applying $R_2 - 3R_1; R_3 - 2R_1; R_4 - R_1$)

$$\sim \begin{bmatrix} 2 & 1 & 2 & 1 & 6 \\ 0 & 1 & 0 & -1 & -2 \\ 0 & 1 & -1 & -5 & -11 \\ 0 & 1 & -3 & 0 & 4 \end{bmatrix} \text{(Applying } \frac{R_2}{-9}\text{)}$$

$$\sim \begin{bmatrix} 2 & 0 & 2 & 2 & 8 \\ 0 & 1 & 0 & -1 & -2 \\ 0 & 0 & -1 & -4 & -9 \\ 0 & 0 & -3 & 1 & 6 \end{bmatrix} \text{(Applying } R_1 - R_2; R_3 - R_2; R_4 - R_2\text{)}$$

$$\sim \begin{bmatrix} 2 & 0 & 0 & -6 & -10 \\ 0 & 1 & 0 & -1 & -2 \\ 0 & 0 & -1 & -4 & -9 \\ 0 & 0 & 0 & 11 & 33 \end{bmatrix} \text{(Applying } R_1 + 2R_3; R_4 - 3R_3\text{)}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & -3 & -5 \\ 0 & 1 & 0 & -1 & -2 \\ 0 & 0 & -1 & -4 & -9 \\ 0 & 0 & 0 & 1 & 3 \end{bmatrix} \text{(Applying } \frac{R_1}{2}, \frac{R_4}{11}\text{)}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 3 \end{bmatrix} \text{(Applying } R_1 + 3R_4; R_2 + R_4; R_3 + 4R_4\text{)}$$

From the matrix, $p = 4$; $q = 1$; $r = -3$; $s = 3$ is the solution

Example 26 : Solve the system of equations
$x + y + 2z = 4;\ 3x + y - 3z = -4;\ 2x - 3y - 5z = -5$ **[JNTU (K) Feb. 2011 (Set No. 4)]**

Solution : Take $A = \begin{bmatrix} 1 & 1 & 2 \\ 3 & 1 & -3 \\ 2 & -3 & -5 \end{bmatrix} X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} B = \begin{bmatrix} 4 \\ -4 \\ -5 \end{bmatrix}$.

Then the given system can be written as AX = B.

Consider, $[A, B] = \begin{bmatrix} 1 & 1 & 2 & 4 \\ 3 & 1 & -3 & -4 \\ 2 & -3 & -5 & -5 \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & -2 & -9 & -16 \\ 0 & -5 & -9 & -13 \end{bmatrix} \text{(Applying } R_2 - 3R_1; R_3 - 2R_1 \text{)}$$

$$\sim \begin{bmatrix} 2 & 0 & -5 & -8 \\ 0 & -2 & -9 & -16 \\ 0 & 0 & 27 & 54 \end{bmatrix} \text{(Applying } 2R_1 + R_2; 2R_3 - 5R_2 \text{)}$$

$$\sim \begin{bmatrix} 2 & 0 & -5 & -8 \\ 0 & -2 & -9 & -16 \\ 0 & 0 & 1 & 2 \end{bmatrix} \text{(Applying } \frac{R_3}{27} \text{)}$$

$$\sim \begin{bmatrix} 2 & 0 & 0 & 2 \\ 0 & -2 & 0 & 2 \\ 0 & 0 & 1 & 2 \end{bmatrix} \text{(Applying } R_1 + 5R_3; R_2 + 9R_3 \text{)}$$

From the above matrix, we have

$2x = 2 \Rightarrow x = 1$

$-2y = 2 \Rightarrow y = -1$

$z = 2$

$\therefore$ The solution is $x = 1, y = -1, z = 2$.

Example 27 : Solve the following system of equations using Gauss Jordan method $x + y + z = 3, 2x - y + 3z = 16, 3x + y - z = -3$ **[JNTU (K) June 2011 (Set No. 1)]**

Solution : Given system of equations is $x + y + z = 3;\ 2x - y + 3z = 16,\ 3x + y - z = -3$

Given system can be written in the form of the matrices.

Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 3 \\ 3 & 1 & -1 \end{bmatrix}$ $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $B = \begin{bmatrix} 3 \\ 16 \\ -3 \end{bmatrix}$

Then matrix form of the equations is AX=B

Augumented matrix is $[A, B] = \begin{bmatrix} 1 & 1 & 1 & 3 \\ 2 & -1 & 3 & 16 \\ 3 & 1 & -1 & -3 \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & -3 & 1 & 10 \\ 0 & -2 & -4 & -12 \end{bmatrix} \text{(Applying } R_2 - 2R_1; R_3 - 3R_1 \text{)}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & -3 & 1 & 10 \\ 0 & 1 & 2 & 6 \end{bmatrix} \text{(Applying } \frac{R_3}{-2}\text{)}$$

$$\sim \begin{bmatrix} 3 & 0 & 4 & 19 \\ 0 & -3 & 1 & 10 \\ 0 & 0 & 7 & 28 \end{bmatrix} \text{(Applying } 3R_1 + R_2; 3R_3 + R_2\text{)}$$

From the matrix, we have

$7z = 28 \Rightarrow z = 4$

$-3y + z = 10 \Rightarrow -3y = 6 \Rightarrow y = -2$

$3x + 4z = 19 \Rightarrow 3x = 3 \Rightarrow x = 1$

$\therefore$ The solution is $x = 1, y = -2, z = 4$.

Example 28 : Investigate for what values of a,b the equations $x + 2y + 3z = 4, x + 3y + 4z = 5, x + 3y + az = b$ have

(*i*) no solution (*ii*) a unique solution

(*iii*) an infinite number of solutions? **[JNTU (K) June 2011 (Set No. 3)]**

Solution : Given systemof equations are $x + 2y + 3z = 4; x + 3y + 4z = 5,$ $x + 3y + az = b$.

Given systemof equations can be written in the matrix from AX = B

$$\text{with } A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 4 \\ 1 & 3 & a \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 4 \\ 5 \\ b \end{bmatrix}$$

Consider the augumented matrix

$$[A, B] = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 5 \\ 1 & 3 & a & b \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & a-3 & b-4 \end{bmatrix} \text{(Applying } R_3 - R_1; R_2 - R_1\text{)}$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & a-4 & b-5 \end{bmatrix} \text{(Applying } R_3 - R_2\text{)}$$

Case I : Suppose $a = 4$ and $b = 5$. Then rank (A) = rank [A, B] = 2, no of variables = 3. The systemwill have infinite number of solutions.

Case II : $a = 4, b \neq 5$ then rank (A) = 2 and rank [A, B] = 3.
The system of equations will be inconsistant.

Case III : $a \neq 4, b \neq 5$ then rank (A) = rank [A, B] = 3.
The system will have a unique solution.

EXERCISE 1.5

1. Solve the system of equations
   (*i*) $x - 4y + 7z = 8;\ 3x + 8y - 2z = 6;\ 7x - 8y + 26z = 31$.
   (*ii*) $10x - y - z = 13, x + 10y + z = 36, -x - y + 10z = 35$ **[JNTU (K) Feb. 2015 (Set No. 3)]**
2. Show that the equations $x + 2y - z = 3$, $3x - y + 2z = 1$, $2x - 2y + 3z = 2$, $x - y + z = -1$ are consistent and solve them. **[JNTU 2003S (Set No. 3)]**
3. Solve completely the equations $x + y + z = 3$, $3x - 5y + 2z = 8$, $5x - 3y + 4z = 14$.
4. Solve the equations $\lambda x + 2y - 2z = 1;\ 4x + 2\lambda y - z = 2;\ 6x + 6y + \lambda z = 3$ for all values of λ.
5. Solve $x - y + 2z + t - 2 = 0;\ 3x + 2y + t - 1 = 0;\ 4x + y + 2z + 2t - 3 = 0$
6. Test for consistency and solve
   $5x + 3y + 2z = 4;\ 3x + 5y + 2z = 9;\ 7x + 2y + 2z = 4$. **[JNTU (A) Dec. 2018]**
7. Solve $x - 2y - 5z = -9; 3x - y + 2z = 5;\ 2x + 3y - z = 3; 4x - 5y + z = -3$.
8. Solve $x + 2y + z = 14;\ 3x + 4y + z = 11;\ 2x + 3y + z = 11$.
9. Solve $x - y + 2z = 4;\ 3x + y + 4z = 6;\ x + y + z = 1$.
10. Solve $x + y + z = 6;\ x - y + 2z = 5;\ 2x - 2y + 3z = 7$. **[JNTU2000, Sup. 2008 (Set No. 3)]**
11. Test for consistency and solve
    (*i*) $2x + 3y + 7z = 5;\ 3x + y - 3z = 12;\ 2x + 19y - 47z = 32$ **[JNTU 2003S (Set No. 4)]**
    (*ii*) $2x + y + 5z = 4, 3x - 2y + 2z = 2, 5x - 8y - 4z = 1$ **[JNTU (A) June 2013]**
12. Investigate for what values of λ and μ the simultaneous equations $2x + 3y + 5z = 9$, $7x + 3y - 2z = 8$, $2x + 3y + \lambda z = \mu$ have
    (*i*) no solution (*ii*) a unique solution (*iii*) an infinite number of solutions.

Eigen values & Eigen vectors :-

Consider the following n - homogenious equations in n unknowns

$$(a_{11}-\lambda)x_1 + a_{12}x_2 + \dots + a_{1n}x_n = 0.$$
$$a_{21}x_1 + (a_{22}-\lambda)x_2 + \dots + a_{2n}x_n = 0.$$
$$\dots\dots\dots$$
$$a_{n1}x_1 + a_{n2}x_2 + \dots + (a_{nn}-\lambda)x_n = 0.$$

the above system of equations can be written in matrix form as.

$(A-\lambda I)X = 0$ where λ is parameter.

* these equations will have a Non-trivial Solutions if and only if $(A-\lambda I)$ is Singular.

i.e $|A-\lambda I| = 0.$ $\Rightarrow$ $\begin{vmatrix} a_{11}-\lambda & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & a_{23} & \cdots & a_{2n} \\ \vdots & & & & \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{vmatrix} = 0.$

$\Rightarrow (-1)^n\lambda^n + a_{n-1}\lambda^{n-1} + a_{n-2}\lambda^{n-2} + \dots + a_2\lambda^2 + a_1\lambda + \dots = 0.$

Note: In this chapter we are consider only Square matrices.

Definations :-

1) **Characteristic matrix** the matrix $(A-\lambda I)$ is called the characteristic matrix.

2). **Characteristic polynomial :-**

the polynomial $|A-\lambda I|$ ~~([illegible])~~ is called characteristic polynomial. in λ of degree n.

3). **Characteristic equation :-**

the equation $|A-\lambda I| = 0$ is called the characteristic equation.

** → Eigenvalue (or) characteristic roots (or) Latent roots (or) proper values :-

the roots of the characteristic equation $|A-\lambda I| = 0$ are called the Eigen values (or) characteristic roots (or) Latent roots (or) proper values.

→ Eigen Vector :-

If λ is a characteristic root of a matrix A, then ∃ a non zero vector X such that $AX = \lambda X$ is called a characteristic vector (or) Eigen vector of A corresponding to the characteristic root λ.

Note: Eigen vector must be a Non Zero vector.

eg: 1) Let $A = \begin{pmatrix} 5 & 4 \\ 1 & 2 \end{pmatrix}$, $X = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

then $AX = \begin{pmatrix} 5 & 4 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$= \begin{pmatrix} 5-4 \\ 1-2 \end{pmatrix}$

$= \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$= 1 \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 1 \cdot X$.

$\therefore AX = 1X$ i.e $AX = \lambda X$ where $\lambda = 1$.

2) Let $A = \begin{pmatrix} 5 & 4 \\ 1 & 2 \end{pmatrix}$, $X = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$

$AX = \begin{pmatrix} 5 & 4 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 20+4 \\ 4+2 \end{pmatrix} = \begin{pmatrix} 24 \\ 6 \end{pmatrix} = 6\begin{pmatrix} 4 \\ 1 \end{pmatrix}$

$= 6(X)$

i.e $AX = 6 \cdot X$. $= 6 \cdot X$.

In above two examples.

$\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ & $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$ are eigen vectors of A

and 1, 6 are the eigen values of A.

and 1, 6 are the eigen values of A.

procedure to find Eigen Values and Eigen Vectors:-

Step 1) Write down the characteristic matrix of A i.e $A - \lambda I$..

2). Write down the characteristic equation of A i.e $|A - \lambda I| = 0$.

3). Find the roots of the characteristic eq. $|A - \lambda I| = 0$. these roots are known as Characteristic roots (or) Eigen Values.

4) Corresponding each eigen Value, we Consider the homogenious System. $(A - \lambda I) X = 0$.

5). Solve the above Systems for X - which is the required eigen Vector of A. Corresponding to Eigen Value 'λ'.

Note:- 1). Sum of the Eigen Values = Sum of the elements in principle diagonal.

2). product of Eigen Values = Determinent of the matrix.

3) In Upper triangular matrix Eigen Values is equal to principle diagonal elements.

* The no. of linearly Independent solutions of m-homogenious linear equations in n-Variables of $AX = 0$ is '$n - r$'.

where 'r' is the rank of the matrise.

SOLVED EXAMPLES

Example 1 : Find the characteristic roots of the matrix $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$

[JNTU 2008, (H) June 2009 (Set No.4)]

Solution : Given matrix is $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$

The characteristic equation of A is $|A - \lambda I| = 0$

i.e., $\begin{vmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{vmatrix} = 0$

$\Rightarrow$ $(2-\lambda)[(3-\lambda)(2-\lambda)-2] - 2[2-\lambda-1] + 1[2-3+\lambda] = 0$

$\Rightarrow$ $\lambda^3 - 7\lambda^2 + 11\lambda - 5 = 0$, on simplification

$\Rightarrow$ $(\lambda-1)(\lambda^2 - 6\lambda + 5) = 0$

$\Rightarrow$ $(\lambda-1)(\lambda-1)(\lambda-5) = 0 \quad \therefore \lambda = 1, 1, 5$

Hence the characteristic roots of A are 1, 1, 5.

Example 2 : Find the eigen values and the corresponding eigen vectors of $\begin{pmatrix} 5 & 4 \\ 1 & 2 \end{pmatrix}$.

[JNTU (K) Nov. 2009S (Set No.3)]

Solution : Let $A = \begin{pmatrix} 5 & 4 \\ 1 & 2 \end{pmatrix}$

Its characteristic matrix = $A - \lambda I = \begin{pmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{pmatrix}$

Characteristic equation of A is $|A - \lambda I| = 0$

i.e., $\begin{vmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{vmatrix} = 0$ ... (1)

i.e., $(5-\lambda)(2-\lambda) - 4 = 0$

On simplification, we get $\lambda^2 - 7\lambda + 6 = 0$

$\Rightarrow (\lambda-1)(\lambda-6) = 0 \quad \therefore \lambda = 1, 6$ ... (2)

The roots of the equation are $\lambda = 1, 6$. Hence the eigen values of the matrix A are 1,6.

Consider the system $\begin{pmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ ... (3)

To get the eigen vector X corresponding to each eigen value λ, we have to solve the above system.

Eigen vector corresponding to $\lambda = 1$.

Put $\lambda = 1$ in the system (3), we get

$$\begin{pmatrix} 4 & 4 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow 4x_1 + 4x_2 = 0$$
$$x_1 + x_2 = 0$$

This implies that $x_2 = -x_1$.

Taking $x_1 = \alpha$, we get $x_2 = -\alpha$.

$\therefore \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ where $\alpha \neq 0$ is a scalar.

Hence $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is an eigen vector of A corresponding to the eigen value $\lambda = 1$.

Eigen vector corresponding to the eigen value $\lambda = 6$.

Put $\lambda = 6$ in (3). We get $\begin{pmatrix} -1 & 4 \\ 1 & -4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\therefore -x_1 + 4x_2 = 0$ and $x_1 - 4x_2 = 0$

This implies that $x_1 = 4x_2$.

Taking $x_2 = \alpha$, we get $x_1 = 4\alpha$

$$\therefore \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 4\alpha \\ \alpha \end{pmatrix} = \alpha\begin{pmatrix} 4 \\ 1 \end{pmatrix}$$

Hence $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$ is an eigen vector of A corresponding to the eigen value $\lambda = 6$.

Example 3 : **Find the eigen values and the corresponding eigen vectors of**

$$A = \begin{pmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{pmatrix}$$

[JNTU 1995, 2005S, 2006, 2008S (Set No.1)]

Solution : If X is an eigen vector of A corresponding to the eigen value λ of A, we have $(A - \lambda I)X = O$

$$i.e., \begin{pmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & 0-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad ...(1)$$

The characteristic equation of A is

$$|A - \lambda I| = 0 \ (i.e.) \begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{vmatrix} = 0$$

Expanding by R_1, we get

$$(-2-\lambda)[(1-\lambda)(-\lambda) - 12] - 2[-2\lambda - 6] - 3[-4 + 1 - \lambda] = 0.$$

$$\Rightarrow -(2+\lambda)\left[\lambda^2-\lambda-12\right]+4(\lambda+3)+3(\lambda+3)=0$$
$$\Rightarrow -(\lambda+2)(\lambda-4)(\lambda+3)+7(\lambda+3)=0$$
$$\Rightarrow (\lambda+3)\left[-(\lambda+2)(\lambda-4)+7\right]=0$$
$$\Rightarrow -(\lambda+3)(\lambda^2-2\lambda-15)=0$$
$$\Rightarrow (\lambda+3)(\lambda+3)(\lambda-5)=0$$

$\therefore$ The eigen values of A are $-3, -3, 5$.

Eigen vector of A corresponding to $\lambda = -3$

Put $\lambda = -3$ in (1). We get $\begin{pmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

The Augmented matrix of the system is $\left(\begin{array}{ccc|c} 1 & 2 & -3 & 0 \\ 2 & 4 & -6 & 0 \\ -1 & -2 & 3 & 0 \end{array}\right)$

Performing $R_2 - 2R_1, R_3 + R_1$, we get $\left(\begin{array}{ccc|c} 1 & 2 & -3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$

Hence we have

$x_1 + 2x_2 - 3x_3 = 0 \Rightarrow x_1 = -2x_2 + 3x_3$

$0 = 0$

$0 = 0$

Thus taking $x_2 = \alpha$ and $x_3 = \beta$, we get $x_1 = -2\alpha + 3\beta;\ x_2 = \alpha;\ x_3 = \beta$

Hence $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \alpha\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + \beta\begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$ is an eigen vector corresponding to $\lambda = -3$.

(Here we are getting $\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$ as eigen vectors of A corresponding to $\lambda = -3$. A linear combination of these two vectors is also an eigen vector of A corresponding to $\lambda = -3$).

Eigen vector corresponding to $\lambda = 5$

Putting $\lambda = 5$ in (1), we get $\begin{pmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

Consider the Augmented matrix of the system

$$\left(\begin{array}{ccc|c} -7 & 2 & -3 & 0 \\ 2 & -4 & -6 & 0 \\ -1 & -2 & -5 & 0 \end{array}\right)$$

Performing $R_1 \leftrightarrow R_3$ and $R_1 \to (-)R_1$, we get

$$\left(\begin{array}{ccc|c} 1 & 2 & 5 & 0 \\ 2 & -4 & -6 & 0 \\ -7 & 2 & -3 & 0 \end{array}\right)$$

Performing $R_2 - 2R_1$, $R_3 + 7R_1$, we get $\left(\begin{array}{ccc|c} 1 & 2 & 5 & 0 \\ 0 & -8 & -16 & 0 \\ 0 & 16 & 32 & 0 \end{array}\right)$

Performing $\dfrac{R_2}{-8}$, we get $\left(\begin{array}{ccc|c} 1 & 2 & 5 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 16 & 32 & 0 \end{array}\right)$

Performing $R_1 - 2R_2$, $R_3 - 16R_2$, we get $\left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$

This implies that

$$x_1 + x_3 = 0$$
$$x_2 + 2x_3 = 0$$
$$0 = 0$$

Taking $x_3 = \alpha_1$, we get $x_1 = -\alpha_1$; $x_2 = -2\alpha_1$.

Thus $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -\alpha_1 \\ -2\alpha_1 \\ \alpha_1 \end{pmatrix} = \alpha_1 \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}$

$\therefore$ The eigen vector of A corresponding to $\lambda = 5$ is $(-1, -2, 1)^T$.

Thus the eigen values of A are -3, -3 and 5.

Hence the eigen vector corresponding to $\lambda = -3$ is $\alpha(-2, 1, 0)^T + \beta(3, 0, 1)^T$ and the eigen vector corresponding to $\lambda = 5$ is $\alpha_1(-1, -2, 1)^T$.

An observation :

Here again sum of the eigen values of A is $-3 -3 + 5 = -1$ and this is same as trace of A.

The product of the eigen values is $(-3)(-3)5 = 45$ and this is same as the determinant of A.

Example 4 : Find the characteristic roots of the matrix $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ and the corresponding eigen vectors . [JNTU May 2005S, (A) Nov. 2010, (H) May 2012]

Solution : The characteristic equation of A is $|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{vmatrix} = 0$

$$\Rightarrow (6-\lambda)[(3-\lambda)^2 - 1] + 2[-2(3-\lambda) + 2] + 2[2 - 2(3-\lambda)] = 0$$

$$\Rightarrow (6-\lambda)[9 + \lambda^2 - 6\lambda - 1] + 2[-6 + 2\lambda + 2] + 2[2 - 6 + 2\lambda] = 0$$

$$\Rightarrow (6-\lambda)[\lambda^2 - 6\lambda + 8] + 2[2\lambda - 4] + 2[2\lambda - 4] = 0$$

$$\Rightarrow \lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0 \Rightarrow (\lambda - 2)(\lambda^2 - 10\lambda + 16) = 0$$

$$\Rightarrow (\lambda - 2)(\lambda - 2)(\lambda - 8) = 0 \Rightarrow \lambda = 2, 2, 8$$

$\therefore$ The eigen values of A are 2, 2, 8.

The eigen vector of A corresponding to $\lambda = 2$.

The eigen vector corresponding to $\lambda = 2$ is given by $(A - 2I)X = O$

$$\Rightarrow \begin{bmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Applying $R_2 \to 2R_2 + R_1, R_3 \to 2R_3 - R_1$, we get

$$\begin{bmatrix} 4 & -2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow 4x_1 - 2x_2 + 2x_3 = 0 \Rightarrow 2x_1 - x_2 + x_3 = 0$

Let $x_2 = k_1, x_3 = k_2$. Then

$2x_1 - k_1 + k_2 = 0 \Rightarrow 2x_1 = k_1 - k_2 \Rightarrow x_1 = \frac{k_1}{2} - \frac{k_2}{2}$

$\therefore X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{k_1}{2} - \frac{k_2}{2} \\ k_1 \\ k_2 \end{bmatrix} = k_1 \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix} + k_2 \begin{bmatrix} -1/2 \\ 0 \\ 1 \end{bmatrix}$ is the eigen vector of A corresponding to $\lambda = 2$ where k_1 and k_2 are arbitrary constants (both are not equal to zero simultaneously).

The eigen vector of A corresponding to $\lambda = 8$

The eigen vector corresponding to eigen value $\lambda = 8$ is given by $(A - 8I)X = O$

$$\Rightarrow \begin{bmatrix} -2 & -2 & 2 \\ -2 & -5 & -1 \\ 2 & -1 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Applying $R_2 \to R_2 - R_1, R_3 \to R_3 + R_1$, we get

$$\begin{bmatrix} -2 & -2 & 2 \\ 0 & -3 & -3 \\ 0 & -3 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Applying $R_3 \to R_3 - R_2$, we get

$$\begin{bmatrix} -2 & -2 & 2 \\ 0 & -3 & -3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow -2x_1 - 2x_2 + 2x_3 = 0 \Rightarrow x_1 + x_2 - x_3 = 0$ and $-3x_2 - 3x_3 = 0 \Rightarrow x_2 + x_3 = 0$

Put $x_3 = k$. Then $x_2 + k = 0 \Rightarrow x_2 = -k$ and $x_1 - k - k = 0 \Rightarrow x_1 - 2k = 0 \Rightarrow x_1 = 2k$

$\therefore X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2k \\ -k \\ k \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} k$ is the eigen of vector of A corresponding to $\lambda = 8$ where k is any non - zero arbitrary constant.

Note 1 : One eigen value $\Rightarrow$ number of eigen vectors.
Note 2 : One eigen vector $\Rightarrow$ one eigen value.
i.e., for one eigen vector there corresponds one eigen value only.

Example 5 : Find the eigen values and the corresponding eigen vectors of $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ **[JNTU May 2006, (H) June 2011 (Set No. 1)]**

Solution : Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

The characteristic equation of A is $|A - \lambda I| = 0$

i.e., $\begin{vmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{vmatrix} = 0$

i.e., $\begin{vmatrix} -\lambda & 0 & 1 \\ \lambda & -\lambda & 1 \\ 0 & \lambda & 1-\lambda \end{vmatrix} = 0$ (Applying $C_1 - C_2$ and $C_2 - C_3$)

i.e., $\begin{vmatrix} -\lambda & 0 & 1 \\ 0 & -\lambda & 2 \\ 0 & \lambda & 1-\lambda \end{vmatrix} = 0$ (Applying $R_2 \to R_2 + R_1$)

i.e., $-\lambda\,[-\lambda\,(1-\lambda) - 2\lambda] = 0$ [Expanding by C_1]

i.e., $\lambda^2\,(1-\lambda+2) = 0$

or $\lambda^2\,(3-\lambda) = 0$

$\therefore \lambda = 0, 0, 3$

To find the eigen vectors for the corresponding eigen values *i.e.*, 0, 0, 3.
We will consider the matrix equation $(A - \lambda I)\,X = O$

i.e., $\begin{bmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ .... (1)

<u>**Eigen vector of A corresponding to $\lambda = 0$**</u>

By putting $\lambda = 0$, the matrix equation (1) will become

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

i.e., $x_1 + x_2 + x_3 = 0$

Choose $x_2 = \alpha$ and $x_3 = \beta$. Then $x_1 = -(\alpha + \beta)$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \alpha \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Hence the eigen vectors of A corresponding to eigen value $\lambda = 0$ are $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

A linear combination of these two vectors is also an eigen vector of A corresponding to $\lambda = 0$.

Eigen vector of A corresponding to $\lambda = 3$

Putting $\lambda = 3$ in (1), we get

$$\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad -2x_1 + x_2 + x_3 = 0$$
$$x_1 - 2x_2 + x_3 = 0$$
$$x_1 + x_2 - 2x_3 = 0$$

On solving the first two equations by cross-multiplication, we get

$$x_1 = k,\ x_2 = k,\ x_3 = k.$$

Hence the eigen vector of A corresponding to eigen value $\lambda = 1$ is $\begin{bmatrix} k \\ k \\ k \end{bmatrix}$

By putting $k = 1$, we get the simplest eigen vector as $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

Thus the eigen values and eigen vectors of A are 0,0,3 and $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

Example 6 : Find the eigen values and the corresponding eigen vectors of the matrix $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$.

[JNTU May 2006 (Set No. 3), 2008, (H) 2009 (Set No. 2), (A) May 2012 (Set No.3)]

Solution : Let $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$

The characteristic equation of A is $|A-\lambda I|=0$

$$i.e.,\quad \begin{vmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{vmatrix} = 0$$

On expanding, we get

$\lambda^3 - 18\lambda^2 + 45\lambda = 0$ *i.e.* $\lambda(\lambda^2 - 18\lambda + 45) = 0$

or $\lambda(\lambda-3)(\lambda-15)=0$

$\therefore\ \lambda = 0, 3, 15$

To find eigen vectors for the corresponding eigen values 0, 3 and 15, we will consider the matrix equation $(A-\lambda I)X=0$

$$i.e.,\quad \begin{bmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \text{.... (1)}$$

Eigen vector corresponding to eigen value $\lambda = 0$

Putting $\lambda = 0$ in (1), we obtain

$$\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow 8x_1 - 6x_2 + 2x_3 = 0$

$-6x_1 + 7x_2 - 4x_3 = 0$

$2x_1 - 4x_2 + 3x_3 = 0$

Solving the first two equations by cross-multiplication, we get

$$\frac{x_1}{24-14} = \frac{-x_2}{-32+12} = \frac{x_3}{56-36}$$

$$i.e.,\quad \frac{x_1}{10} = \frac{x_2}{20} = \frac{x_3}{20} \quad \text{or} \quad \frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{2} = k$$

$\therefore\ x_1 = k,\ x_2 = 2k,\ x_3 = 2k$

Choosing $k = 1$, we get

$x_1 = 1,\ x_2 = x_3 = 2$

$\therefore$ The eigen vector corresponding to eigen value $\lambda = 0$ is $\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$.

Eigen vector corresponding to $\lambda = 3$

Putting $\lambda = 3$ in (1), we get

$$\begin{bmatrix} 5 & -6 & 2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow 5x_1 - 6x_2 + 2x_3 = 0$$
$$-6x_1 + 4x_2 - 4x_3 = 0$$
$$2x_1 - 4x_2 + 0.x_3 = 0$$

Solving any of the above two equations, we get

$x_1 = -2, x_2 = -1, x_3 = 2$ (taking $k = 1$)

$\therefore$ The eigen vector corresponding to eigen value $\lambda = 3$ is $\begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix}$

Eigen vector corresponding to $\lambda = 15$

Putting $\lambda = 15$ in (1), we get

$$\begin{bmatrix} -7 & -6 & 2 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow -7x_1 - 6x_2 + 2x_3 = 0,\ -6x_1 - 8x_2 - 4x_3 = 0,\ 2x_1 - 4x_2 - 12x_3 = 0.$

Solving any two of the above equations by cross multiplication, we get

$x_1 = 2, x_2 = -2, x_3 = 1.$

$\therefore$ The eigen vector corresponding to eigen value $\lambda = 15$ is $\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$.

Hence the eigen values of A are 0, 3, 15 and the corresponding eigen vectors of A are

$$\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}.$$

Example 7 : Find the Eigen values and the corresponding Eigen vectors of the matrix $\begin{bmatrix} 2 & 2 & 0 \\ 2 & 5 & 0 \\ 0 & 0 & 3 \end{bmatrix}$. **[JNTU Sep. 2008, (H) Dec. 2011 (Set No. 2)]**

Solution : Let $A = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 5 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

The characteristic equation of 'A' is $|A - \lambda I| = 0$.

i.e., $$\begin{vmatrix} 2-\lambda & 2 & 0 \\ 2 & 5-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (2-\lambda)\,[(5-\lambda)\,(3-\lambda)] - 2\,(2(3-\lambda)) = 0$$

$$\Rightarrow (2-\lambda)\,(15 - 5\lambda - 3\lambda + \lambda^2) - 4\,(3-\lambda) = 0$$

$$\Rightarrow 30 - 10\lambda - 6\lambda + 2\lambda^2 - 15\lambda + 5\lambda^2 + 3\lambda^2 - \lambda^3 - 12 + 4\lambda = 0$$

$$\Rightarrow -\lambda^3 + 10\lambda^2 - 27\lambda + 18 = 0$$

$$\Rightarrow \lambda^3 - 10\lambda^2 + 27\lambda - 18 = 0$$

We observe that $\lambda = 1$ is a root.

Dividing with $(\lambda - 1)$, we get

$$(\lambda - 1)\;(\lambda^2 - 9\lambda + 18) = 0$$

i.e., $(\lambda - 1)\;(\lambda^2 - 6\lambda - 3\lambda + 18) = 0$

i.e., $(\lambda - 1)\,[\lambda\,(\lambda - 6) - 3\,(\lambda - 6)] = 0$

or $(\lambda - 1)\,(\lambda - 3)\,(\lambda - 6) = 0$

$\therefore\ \lambda = 1, 3, 6$

Synthetic Division

1	1	–10	27	–18
	0	1	–9	18
	1	–9	18	0

Thus the Eigen values of 'A' are 1, 3, 6.

Case 1: The Eigen vector corresponding to $\lambda = 1$ is given by

$$\begin{bmatrix} 1 & 2 & 0 \\ 2 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow x_1 + 2x_2 = 0$

$2x_1 + 4x_2 = 0$ ... (1)

and $2x_3 = 0 \Rightarrow x_3 = 0$

Let $x_2 = k$. Then

(1) $\Rightarrow x_1 + 2k = 0 \Rightarrow x_1 = -2k$

$\therefore$ The Eigen vector corresponding to $\lambda = 1$ is $X_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$.

Case 2 : The Eigen vector corresponding to $\lambda = 3$ is given by

$$\begin{bmatrix} -1 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow -x_1 + 2x_2 = 0 \Rightarrow 2x_2 = x_1$ ... (1)

and $2x_1 + 2x_2 = 0 \Rightarrow x_1 + x_2 = 0$ ... (2)

From (1) and (2), we get $3x_2 = 0 \Rightarrow x_2 = 0$

From (1), we get $x_1 = 0$

Let $x_3 = k$.

The Eigen vector corresponding to $\lambda = 3$ is $X_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Case 3 : The Eigen vector corresponding to $\lambda = 6$ is given by

$$\begin{bmatrix} -4 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow -4x_1 + 2x_2 = 0$

$2x_1 - x_2 = 0$. $\therefore 2x_1 = x_2$

and $-3x_3 = 0$

Let $x_1 = k$. Then $x_2 = 2x_1 = 2k$

$\therefore$ The eigen vector corresponding to $\lambda = 6$ is $X_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$, and the Eigen vectors corresponding to $\lambda = 1, 3, 6$ are $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$.

Example 8 : Find the eigen values and the corresponding eigen vectors of $\begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$.

[JNTU (H) June 2010 (Set No.4)]

Solution : Given matrix is $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$

The characteristic equation of A is $|A - \lambda I| = 0$

i.e., $\begin{vmatrix} 1-\lambda & 0 & -1 \\ 1 & 2-\lambda & 1 \\ 2 & 2 & 3-\lambda \end{vmatrix} = 0$ [Expand by R_1]

$\Rightarrow (1-\lambda)[(2-\lambda)(3-\lambda) - 2] - 1[2 - 2(2-\lambda)] = 0$

$$\Rightarrow (1-\lambda)(\lambda^2-5\lambda+4)-1[2\lambda-2]=0$$

$$\Rightarrow (1-\lambda)(\lambda^2-5\lambda+4)+2(\lambda-1)=0$$

$$\Rightarrow (1-\lambda)[\lambda^2-5\lambda+4-2]=0$$

$$\Rightarrow (1-\lambda)(\lambda^2-5\lambda+2)=0$$

$$\Rightarrow \lambda=1,\ \lambda=\frac{5\pm\sqrt{25-8}}{2}=\frac{5\pm\sqrt{17}}{2}$$

$\therefore \quad \lambda=1, \dfrac{5+\sqrt{17}}{2}, \dfrac{5-\sqrt{17}}{2}$ are the characteristic (eigen) values.

Eigen Vector Corresponding to eigen value $\lambda = 1$

Let X_1 be the corresponding eigen vector. It is given by

$(A-I)X_1 = O$

$$\Rightarrow \begin{bmatrix} 0 & 0 & -1 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 & 0 & -1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ (Applying } R_3 \to R_3-2R_2)$$

$\Rightarrow \ x_3=0, x_1+x_2+x_3=0.$

Let $x_2=k_1$. Then $x_3=-x_2=-k_1$.

$$\therefore \ X_1=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ k_1 \\ -k_1 \end{bmatrix}=k_1\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

Hence the eigen vector corresponding to $\lambda = 1$ is $\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$

Example 9 : Find the eigen values and the corresponding eigen vectors of $\begin{bmatrix} 3 & 2 & 2 \\ 1 & 2 & 2 \\ -1 & -1 & 0 \end{bmatrix}$

[JNTU (H) Jan. 2012 (Set No. 1)]

Solution : Given matrix is $A=\begin{bmatrix} 3 & 2 & 2 \\ 1 & 2 & 2 \\ -1 & -1 & 0 \end{bmatrix}$

Characteristic equation of A is $|A-\lambda I|=0$

i.e., $\begin{vmatrix} 3-\lambda & 2 & 2 \\ 1 & 2-\lambda & 2 \\ -1 & -1 & -\lambda \end{vmatrix}=0$

$\Rightarrow (3-\lambda)[(2-\lambda)(-\lambda)+2]-2[-\lambda+2]+2[-1+(2-\lambda)]=0$

$\Rightarrow (3-\lambda)(\lambda^2-2\lambda+2)+2\lambda-4+2-2\lambda=0$

$\Rightarrow (3-\lambda)(\lambda^2-2\lambda+2)-2=0$

$\Rightarrow 3\lambda^2-6\lambda+6-\lambda^3+2\lambda^2-2\lambda-2=0$

$\Rightarrow -\lambda^3+5\lambda^2-8\lambda+4=0$ or $\lambda^3-5\lambda^2+8\lambda-4=0$

$\Rightarrow (\lambda-1)(\lambda^2-4\lambda+4)=0 \Rightarrow (\lambda-1)(\lambda-2)^2=0$

Synthetic Division

$$\begin{array}{c|cccc} 1 & 1 & -5 & 8 & -4 \\ & 0 & 1 & -4 & 4 \\ \hline & 1 & -4 & 4 & \boxed{0} \end{array}$$

$\therefore\ \lambda=2,2,1$ are the roots.

Thus the eigen values of A are $2,2,1$.

Eigen vector corresponding to $\lambda=2$

The eigen vectors of A corresponding to $\lambda=2$ are given by

$(A-\lambda I)X=O$ *i.e.,* $(A-2I)X=O$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 2 \\ 1 & 0 & 2 \\ -1 & -1 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 2 \\ 1 & 0 & 2 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ (Applying } R_3+R_1)$$

$\Rightarrow x_1+2x_2+2x_3=0$ ... (1) $x_1+2x_3=0$ ... (2) and $x_2=0$ ... (3)

Now (1) $\Rightarrow x_1=-2x_3\ [\because\ x_2=0]$

Taking $x_3=k$, we have $x_1=-2k$

The eigen vector corresponding to $\lambda=2$ is $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} -2k \\ 0 \\ k \end{bmatrix}=k\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$;

$X_1=\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$ is the eigen vector corresponding to $\lambda=2$

Eigen vector corresponding to $\lambda = 1$

$$\begin{bmatrix} 2 & 2 & 2 \\ 1 & 1 & 2 \\ -1 & -1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$\Rightarrow 2x_1 + 2x_2 + 2x_3 = 0$

i.e., $x_1 + x_2 + x_3 = 0$ ... (1)

$x_1 + x_2 + 2x_3 = 0$ ... (2)

and $-x_1 - x_2 - x_3 = 0 \Rightarrow x_1 + x_2 + x_3 = 0$ ... (3)

(1) and (3) are same. $(2)-(1)$ gives $x_3 = 0$

Take $x_2 = k \Rightarrow x_1 = -k$

$\therefore X_2 = \begin{bmatrix} -k \\ k \\ 0 \end{bmatrix} = -k\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$; $X_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ is eigen vector corresponding to $\lambda = 1$.

Example 10 : Find the eigen values and Eigen vectors of $A = \begin{bmatrix} 2 & 1 \\ 4 & 5 \end{bmatrix}$

[JNTU (H) May 2012, (A) Nov. 2012 (Set No. 2)]

Solution : Characteristic equation is $(A - \lambda I) = 0 \Rightarrow \begin{bmatrix} 2-\lambda & 1 \\ 4 & 5-\lambda \end{bmatrix} = 0$

$\Rightarrow (2-\lambda)(5-\lambda) - 4 = 0$

$\Rightarrow \lambda^2 - 7\lambda + 10 - 4 = 0$

$\Rightarrow \lambda^2 - 7\lambda + 6 = 0$

$\Rightarrow (\lambda - 6)(\lambda - 1) = 0$

$\Rightarrow \lambda = 6,1$ are the eigen values.

Eigen vector corresponding to $\lambda = 6$

It is given by $(A - \lambda I)X = O$

$$\Rightarrow \begin{bmatrix} 2-6 & 1 \\ 4 & 5-6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -4 & 1 \\ 4 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -4 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{(Applying } R_2 + R_1\text{)}$$

$\Rightarrow -4x_1 + x_2 = 0$

Let $x_1 = k \Rightarrow x_2 = 4k$

$$\therefore\ X = \begin{bmatrix} k \\ 4k \end{bmatrix} = k\begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

$\therefore \begin{bmatrix} 1 \\ 4 \end{bmatrix}$ is the eigen vector corresponding to $\lambda = 6$

Eigen value corresponding to $\lambda = 1$

It is given by $\begin{bmatrix} 2-1 & 1 \\ 4 & 5-1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} 1 & 1 \\ 4 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow x_1 + x_2 = 0$

Let $x_1 = k$. Then $x_2 = -k$

Thus $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} k \\ -k \end{bmatrix} = k\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\therefore\ X = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is the eigen vector corresponding to $\lambda = 1$

Example 11 : Compute the Eigen values and Eigen vectors of $\begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$

[JNTU (H) Dec. 2016]

Solution : The Characteristic equation of $A = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$ is $|A - \lambda I| = 0$

i.e., $\begin{vmatrix} 1-\lambda & 4 \\ 3 & 2-\lambda \end{vmatrix} = 0$

$\Rightarrow (1-\lambda)(2-\lambda) - 12 = 0$

$\Rightarrow \lambda^2 - 3\lambda - 10 = 0 \Rightarrow (\lambda - 5)(\lambda + 2) = 0$

$\therefore\ \lambda = 5$ and $\lambda = -2$ are the eigen values.

Eigen vector corresponding to $\lambda = 5$

Let X be the eigen vector corresponding to the eigen value λ of A.

Then we have $(A - \lambda I)X = O$.

So the eigen vector corresponding to $\lambda = 5$ is given by $(A - 5I)X = O$

$$\Rightarrow \begin{bmatrix} 1-5 & 4 \\ 3 & 2-5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -4 & 4 \\ 3 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow -x_1+x_2=0 \Rightarrow x_1=x_2$$

$\therefore \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is the eigen vector coorresponding to $\lambda=5$.

Eigen vector corresponding to $\lambda=-2$

It is given by $(A+2I)X=O$

$$\Rightarrow \begin{bmatrix} 1+2 & 4 \\ 3 & 2+2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 3 & 4 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow 3x_1+4x_2=0 \Rightarrow 3x_1=-4x_2, x_1=\frac{-4}{3}x_2$$

Take $x_2=k \Rightarrow x_1=-\frac{4}{3}k$

Eigen vector is $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\begin{bmatrix} \frac{-4}{3}k \\ k \end{bmatrix}=\begin{bmatrix} -4k \\ 3k \end{bmatrix}=\begin{bmatrix} -4 \\ 3 \end{bmatrix}$

$\therefore \begin{bmatrix} -4 \\ 3 \end{bmatrix}$ is the eigen vector corresponding to $\lambda=-2$

Example 12 : Find the Eigen values of the following system :

$8x-4y=\lambda x, 2x+2y=\lambda y$ **[JNTU (H) Dec. 2016]**

Solution : Given equations can be written as

$(8-\lambda)x-4y=0$

$2x+(2-\lambda)y=0$

Take $A=\begin{bmatrix} 8-\lambda & -4 \\ 2 & 2-\lambda \end{bmatrix}$

The eigen values of A are given by $|A-kI|=0$

$$\Rightarrow \begin{vmatrix} 8-\lambda-k & -4 \\ 2 & 2-\lambda-k \end{vmatrix} = 0$$

$$\Rightarrow (8-\lambda-k)(2-\lambda-k)+8=0$$

$$\Rightarrow 16-8\lambda-8k-2\lambda+\lambda^2+\lambda k-2k+\lambda k+k^2+8=0$$

$$\Rightarrow k^2-10k+2\lambda k+(24-10\lambda)=0$$

$$\Rightarrow k^2+k(2\lambda-10)+(24-10\lambda)=0.$$

This is a quadratic in k.

$$\therefore \quad k=\frac{(10-2\lambda)\pm\sqrt{(2\lambda-10)^2-4(1)(24-10\lambda)}}{2}$$

$$=\frac{2(5-\lambda)\pm\sqrt{4\lambda^2+100-40\lambda-96+40\lambda}}{2}$$

$$=\frac{2(5-\lambda)\pm\sqrt{4\lambda^2+4}}{2}=(5-\lambda)\pm\sqrt{\lambda^2+1}$$

Hence $(5-\lambda)+\sqrt{\lambda^2+1}$ and $(5-\lambda)-\sqrt{\lambda^2+1}$ are eigen values.

Example 13 : Find the characteristic roots and characteristic vectors of the matrix

$$A=\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$ **[JNTU 2003S, 2005S, (A) June 2009 (Set No. 2), (K) June 2011 (Set No. 4)]**

Solution : We are given $A=\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$

Characteristic equation of A is $|A-\lambda I|=0$

$$\Rightarrow \begin{vmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{vmatrix}=0$$

$$\Rightarrow \begin{vmatrix} 6-\lambda & 0 & 2 \\ -2 & 2-\lambda & -1 \\ 2 & 2-\lambda & 3-\lambda \end{vmatrix}=0 \quad \text{(Applying } C_2+C_3\text{)}$$

$$\Rightarrow \begin{vmatrix} 6-\lambda & 0 & 2 \\ -2 & 1 & -1 \\ 2 & 1 & 3-\lambda \end{vmatrix}=0 \quad \text{(Taking } (2-\lambda) \text{ common from } C_2\text{)}$$

$$\Rightarrow (2-\lambda)\left[(6-\lambda)(3-\lambda+1)+2(-2-2)\right]=0 \text{ [Expanding by } R_1 \text{]}$$

$$\Rightarrow (2-\lambda)\left[(6-\lambda)(4-\lambda)-8\right]=0$$

$$\Rightarrow (2-\lambda)(\lambda^2-10\lambda+16)=0 \Rightarrow (2-\lambda)(\lambda-2)(\lambda-8)=0 \quad \therefore \lambda=2,2,8$$

$\therefore$ The charectristic roots are $\lambda = 2, 2, 8$

To find the characteristic roots corresponding to $\lambda = 2$

The eigen vector for $\lambda=2$ is given by $(A-2I)X=0$

$$\Rightarrow \begin{bmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ (Applying } R_3+R_2\text{)}$$

$$\Rightarrow \begin{bmatrix} 4 & -2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ (Applying } 2R_2+R_1\text{)}$$

$$\Rightarrow 4x_1-2x_2+2x_3=0 \text{ or } 2x_1-x_2+x_3=0 \quad \dots (1)$$

Let $x_3=k_1$ and $x_2=k_2$

$$(1) \Rightarrow 2x_1 = x_2 - x_3 = k_2 - k_1$$

$$\Rightarrow x_1 = \frac{k_2-k_1}{2}$$

$$\text{Thus } X=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} \frac{k_2-k_1}{2} \\ k_2 \\ k_1 \end{bmatrix}=\frac{k_2}{2}\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}-\frac{k_1}{2}\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} \text{ (or) } \frac{k_2}{2}\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}+\frac{k_1}{2}\begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$$

$\therefore$ The charecteristic vectors relating to $\lambda = 2$ are given by $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$ or $\begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$

To Find Charecteristic Vector corresponding to $\lambda = 8$

The eigen vector for $\lambda=8$ is given by $(A-8I)X=0$

$$\Rightarrow \begin{bmatrix} -2 & -2 & 2 \\ -2 & -5 & -1 \\ 2 & -1 & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -2 & -2 & 2 \\ 0 & -3 & -3 \\ 0 & -3 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{(Applying } R_2 - R_1; R_3 + R_1\text{)}$$

$$\Rightarrow \begin{bmatrix} -1 & -1 & 1 \\ 0 & -3 & -3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{(Applying } \frac{R_1}{2}; R_3 - R_2\text{)}$$

$$\Rightarrow \begin{bmatrix} -1 & -1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{(Applying } \frac{R_2}{-3}\text{)}$$

$\Rightarrow -x_1 - x_2 + x_3 = 0$ ... (1) and $x_2 + x_3 = 0 \Rightarrow x_2 = -x_3$

From (1), we have $x_1 = -x_2 + x_3 = x_3 + x_3 = 2x_3$

Taking $x_3 = k$, we get

$$\text{Thus } X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2k \\ -k \\ k \end{bmatrix} = k \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

$\therefore$ The characteristic vector corresponding to $\lambda = 8$ is $\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$

Example 14 : Find the eigen values and eigen vectors of the matrix $A = \begin{bmatrix} 3 & 0 & 0 \\ 5 & 7 & 0 \\ 2 & 6 & 1 \end{bmatrix}$

[JNTU (K) Feb. 2011 (Set No. 2)]

Solution : We are given $A = \begin{bmatrix} 3 & 0 & 0 \\ 5 & 7 & 0 \\ 2 & 6 & 1 \end{bmatrix}$

Characteristic equation of A is $\begin{vmatrix} 3-\lambda & 0 & 0 \\ 5 & 7-\lambda & 0 \\ 2 & 6 & 1-\lambda \end{vmatrix} = 0$

$\Rightarrow (3-\lambda)[(7-\lambda)(1-\lambda)] = 0$ [Expanding by R_1]

$\Rightarrow \lambda = 3, 7, 1$ are characteristic (eigen values) values.

Important observation : Here the given matrix is a lower triangular matrix. For upper triangular, lower triangular and diagonal matrices, the eigen values are given by the diagonal elements. So the eigen values of A are 3, 7, 1 which are the diagonal elements.

To find the eigen vector corresponding to $\lambda = 1$

Its eigen vector is given by $(A-\lambda I)X = 0 \Rightarrow (A-I)X = 0$

$$\Rightarrow \begin{bmatrix} 2 & 0 & 0 \\ 5 & 6 & 0 \\ 2 & 6 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad 2x_1 = 0 \Rightarrow x_1 = 0$$

$$5x_1 + 6x_2 = 0 \Rightarrow x_2 = 0 \ (\because x_1 = 0)$$

and $2x_1 + 6x_2 = 0$

Let $x_3 = k$. Then

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ k \end{bmatrix} = k\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$\therefore \ X_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is the characteristic vector corresponding to $\lambda = 1$.

To Find the eigen vector corresponding to $\lambda = 3$

The eigen vector for $\lambda = 3$ is given by $(A-3I)X = 0$

$$i.e., \begin{bmatrix} 0 & 0 & 0 \\ 5 & 4 & 0 \\ 2 & 6 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow 5x_1 + 4x_2 = 0 \Rightarrow x_1 = \frac{-4}{5}x_2$$

and $2x_1 + 6x_2 - 2x_3 = 0 \Rightarrow \dfrac{-8}{5}x_2 + 6x_2 - 2x_3 = 0$

$$\Rightarrow \frac{22x_2}{5} - 2x_3 = 0 \Rightarrow 22x_2 = 10x_3$$

$$\Rightarrow x_3 = \frac{22}{10}, x_2 = \frac{11}{5}x_2$$

$$\therefore X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -\frac{4}{5}x_2 \\ x_2 \\ \frac{11}{5}x_2 \end{bmatrix} = \frac{-x_2}{5}\begin{bmatrix} 4 \\ -5 \\ -11 \end{bmatrix}$$

Thus $X_2 = \begin{bmatrix} 4 \\ -5 \\ -11 \end{bmatrix}$ is the eigen vector corresponding to $\lambda = 3$.

To Find eigen vector corresponding to $\lambda = 7$

The eigen vector for $\lambda = 7$ is given by $(A - 7I)X = 0$

i.e., $\begin{bmatrix} -4 & 0 & 0 \\ 5 & 0 & 0 \\ 2 & 6 & -6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow -4x_1 = 0, 5x_1 = 0$. So $x_1 = 0$

and $2x_1 + 6x_2 - 6x_3 = 0 \Rightarrow 6x_2 - 6x_3 = 0 \Rightarrow x_2 = x_3$ $(\because x_1 = 0)$

$\therefore\ X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

Thus $X_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ is the eigenvector corresponding $\lambda = 7$

Example 15 : Find the eigen values are Eigen vectors of $A = \begin{bmatrix} 2 & 1 \\ 4 & 5 \end{bmatrix}$

[JNTU (H) May 2012, (A) Nov. 2012 (Set No.2)]

Solution : Characteristic equation is $(A - \lambda I) = 0 \Rightarrow \begin{bmatrix} 2-\lambda & 1 \\ 4 & 5-\lambda \end{bmatrix} = 0$

$\Rightarrow (2-\lambda)(5-\lambda) - 4 = 0$

$\Rightarrow \lambda^2 - 7\lambda + 10 - 4 = 0$

$\Rightarrow \lambda^2 - 7\lambda + 6 = 0$

$\Rightarrow (\lambda - 6)(\lambda - 1) = 0$

$\Rightarrow \lambda = 6, 1$ are the eigen values.

Eigen vector corresponding to $\lambda = 6$

Let $(A - \lambda I)X = 0$

$\begin{bmatrix} 2-6 & 1 \\ 4 & 5-6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} -4 & 1 \\ 4 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$R_2 + R_1$ gives $\begin{bmatrix} -4 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow -4x_1 + x_2 = 0$

Let $x_1 = k \Rightarrow x_2 = 4k$

$\therefore X = \begin{bmatrix} k \\ 4k \end{bmatrix} = k\begin{bmatrix} 1 \\ 4 \end{bmatrix}$

$\therefore \begin{bmatrix} 1 \\ 4 \end{bmatrix}$ is the eigen vector corresponding to $\lambda = 6$

Eigen value corresponding to $\lambda = 1$

$$\begin{bmatrix} 2-1 & 1 \\ 4 & 5-1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 4 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$x_1 + x_2 = 0$

$x_1 = k \Rightarrow x_2 = -k$

$$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} k \\ -k \end{bmatrix} = k\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$\therefore X = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is the eigen vector corresponding to $\lambda = 1$

REVIEW QUESTIONS

1. Define eigen values and eigen vectors **(or)** Define latent roots and vectors.
2. Write the characteristic matrix and characteristic polynomial of the square matrix A.

EXERCISE 2.1

Determine the eigen values and the corresponding eigen vectors of the following matrices:

(1) $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ (2) (*a*) $\begin{bmatrix} -2 & 5 \\ -1 & 4 \end{bmatrix}$ (*b*) $\begin{bmatrix} 8 & -4 \\ 2 & 2 \end{bmatrix}$ **[JNTU 2004S (Set No. 2)]**

(3) $\begin{bmatrix} 2 & 0 & 0 \\ 3 & 1 & 0 \\ -1 & 2 & 3 \end{bmatrix}$ **[JNTU (H) June 2015**

(4) $\begin{bmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{bmatrix}$ **[JNTU (K) May 2016 (Set No. 3)]**

properties :-

1) The Sum of the Eigen values of a matrix is equal to its Trace (or) Sum of the diagonal elements.

proof: Consider $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$.

$\therefore$ the characteristic equation of A is $|A - \lambda I| = 0$.

i.e $\begin{vmatrix} a_{11}-\lambda & a_{12} & a_{13} \\ a_{21} & a_{22}-\lambda & a_{23} \\ a_{31} & a_{32} & a_{33}-\lambda \end{vmatrix} = 0.$

$\Rightarrow (a_{11}-\lambda)[(a_{22}-\lambda)(a_{33}-\lambda) - a_{23}\cdot a_{32}] - a_{12}[a_{21}(a_{33}-\lambda) - a_{23}\cdot a_{31}]$
$+ a_{13}[a_{21}\cdot a_{32} - a_{31}(a_{22}-\lambda)] = 0.$

$\Rightarrow -\lambda^3 + (a_{11} + a_{22} + a_{33})\lambda^2 + \dots = 0.$ —— ①.

Also If $\lambda_1, \lambda_2, \lambda_3$ be the Eigen values of A then

$|A - \lambda I| = (-1)^3 (\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3)$.

$\Rightarrow |A - \lambda I| = -\lambda^3 + \lambda^2(\lambda_1 + \lambda_2 + \lambda_3) + \dots$

$\therefore -\lambda^3 + \lambda^2(\lambda_1 + \lambda_2 + \lambda_3) + \dots = 0$ —— ②.

form ① & ② we get

$\lambda_1 + \lambda_2 + \lambda_3 = a_{11} + a_{22} + a_{33}$,

i.e. the Sum of the Eigen Values of a matrix is equal to its trace (or) Sum of the diagonal elements.

Hence proved

2). P.T. the product of the Eigen Values is equal to the determinant of the matrix.

proof. To prove that the product of the Eigen values is equal to the determinant of the matrix.

Let $A_{n\times n}$ be any square matrix such that $\lambda_1, \lambda_2, \lambda_3 - - - \lambda_n$ are the Eigen values.

$$\therefore |A-\lambda I| = (-1)^n (\lambda-\lambda_1)(\lambda-\lambda_2)(\lambda-\lambda_3) - - - (\lambda-\lambda_n).$$

put $\lambda = 0$.

$$\therefore |A| = (-1)^n (-\lambda_1)(-\lambda_2)(-\lambda_3) - - (-\lambda_n)$$

$$= (-1)^n . (-1)^n . (\lambda_1 . \lambda_2 . \lambda_3 - - \lambda_n)$$

$$= (-1)^{2n} (\lambda_1 . \lambda_2 . \lambda_3 . - - . \lambda_n)$$

$$|A| = \lambda_1 . \lambda_2 . \lambda_3 - - - \lambda_n.$$

$\therefore$ the product of the Eigen values is equal to the determinant of the matrix.

Hence proved.

3). The matrix A & A' (or) A^T have the same Eigen values.

proof. Let λ be the Eigen value of A.

We have $(A-\lambda I)^T = A^T - \lambda I^T = A^T - \lambda I.$

$$\therefore |(A-\lambda I)^T| = |A^T - \lambda I|$$

$$\Rightarrow |A-\lambda I| = |A^T - \lambda I| \qquad | \because |A^T| = |A|.$$

$$\therefore |A-\lambda I| = 0 \text{ iff. } |A' - \lambda I| = 0.$$

The det value remains same when rows & columns are interchange

i.e. λ is Eigen value of A iff. λ is the Eigen value of A^T.

Note: A & A^T have same Eigen values. But different Eigen vectors.

4). If $\lambda_1, \lambda_2, \lambda_3 \dots \lambda_n$ be the Eigen ~~vectors~~ Values of A then $\lambda_1^K, \lambda_2^K, \lambda_3^K, \dots \lambda_n^K$ are the eigen Values of A^K.

proof:- Let $\lambda_1, \lambda_2, \lambda_3, \dots \lambda_n$ be the Eigen Values of A. then ∃ and $x_1, x_2, x_3, \dots x_n$ be the Corresponding eigen Vectors of A

$\Rightarrow$ $$AX_1 = \lambda_1 x_1$$
$$AX_2 = \lambda_2 x_2$$
$$AX_3 = \lambda_3 x_3$$
$$\vdots$$
$$AX_n = \lambda_n x_n$$

i.e $AX_i = \lambda_i x_i \quad \forall\ i = 1, 2, 3, \dots$ ①

multiplying with 'A' on both sides

$$A(AX_i) = A(\lambda_i X_i).$$
$$\Rightarrow A^2 X_i = \lambda_i (AX_i)$$
$$= \lambda_i (\lambda_i X_i) \quad | \because \text{by ①}.$$
$$= \lambda_i^2 X_i$$
$$\therefore A^2 X_i = \lambda_i^2 X_i.$$

Again multiply with A on both sides.

$$A^3 X_i = \lambda_i^3 X_i \quad \forall\ i = 1, 2, 3, \dots$$

$\therefore$ Continuing this process upto K terms, we get

$$A^K X_i = \lambda_i^K X_i.$$

$\therefore \lambda_i^K$ is the eigen value of A^K

i.e $\lambda_1^K, \lambda_2^K, \lambda_3^K, \dots$ " , A^K.

Note:- The Eigen values of an Idempotent matrix are either 0 or 1.

Sol. Idempotent matrix $A^2 = A$.

where $AX = \lambda X$, & $A^2 X = \lambda^2 X$.

$\therefore \lambda^2 X = \lambda X$

$\Rightarrow (\lambda^2 - \lambda) X = 0$

$\Rightarrow \lambda(\lambda - 1) X = 0$

$\Rightarrow \lambda = 0, 1.$

Note:- S.T. If $\lambda_1, \lambda_2, \lambda_3, \dots \lambda_n$ are the latent roots of A then A^3 has the latent roots $\lambda_1^3, \lambda_2^3, \lambda_3^3, \dots \lambda_n^3$.

5). If $\lambda_1, \lambda_2, \lambda_3, \ldots \lambda_n$ are the Eigen values of A then $K\lambda_1, K\lambda_2, K\lambda_3, \ldots K\lambda_n$ " " " KA where K is a Non Zero Scalar.

proof:- Let $\lambda_1, \lambda_2, \lambda_3, \ldots \lambda_n$ are the Eigen values of A and then $\exists$ $x_1, x_2, x_3, \ldots x_n$ " " vectors of A $\ni$.

$$A x_1 = \lambda_1 x_1$$
$$A x_2 = \lambda_2 x_2$$
$$\vdots$$
$$A x_n = \lambda_n x_n$$

(or) $A x_i = \lambda_i x_i \; \forall \, i = 1, 2, \ldots n$.
multiply with K on both sides
$\therefore (KA) x_i = (K\lambda_i) x_i$

i.e $A x_i = \lambda_i x_i \quad \forall \; i = 1, 2, 3, \ldots n$.

multiply with 'K' on both sides

$$\therefore \; K(A x_i) = K(\lambda_i x_i)$$
$$\Rightarrow (KA). x_i = (K\lambda_i) x_i$$

$\Rightarrow$ $K\lambda_i$ is the Eigen value of KA

i.e $K\lambda_1, K\lambda_2, K\lambda_3, \ldots K\lambda_n$ be the Eigen Values of KA.

6). If λ is the Eigen value of a matrix A then $\lambda + K$ is the Eigen value of the matrix $A + KI$.

proof. Let λ be the Eigen value of A and x be the Corresponding Eigen vectors of A such that

$$AX = \lambda X. \quad —①$$

Now, $(A + KI) x = Ax + KIx$

$$= \lambda x + KIx$$
$$= (\lambda + K) x.$$

$$\therefore (A + KI) x = (\lambda + K) x.$$

$\therefore$ $\lambda + K$ is the Eigen value of the matrix $(A + KI)$.

7). If $\lambda_1, \lambda_2, \lambda_3, \ldots \lambda_n$ are the Eigen values of A then $(\lambda_1 - k), (\lambda_2 - k), (\lambda_3 - k), \ldots (\lambda_n - k)$ are the Eigen values of $(A - kI)$

~~proof~~: ~~[illegible]~~

Let $\lambda_1, \lambda_2, \lambda_3, \ldots \lambda_n$ are the Eigen values of A & $x_1, x_2, x_3, \ldots x_n$ be the corresponding Eigen vectors of A such that

$$A x_i = \lambda_i x_i \quad \forall\ i = 1, 2, 3, \ldots n.$$

adding $-k x_i$ on both sides:

$$\therefore A x_i - k x_i = \lambda_i x_i - k x_i$$

$$\Rightarrow (A - kI) x_i = (\lambda_i - k) x_i.$$

(or) Consider $(A - kI) x_i = A x_i - k x_i$
$= \lambda_i x_i - k x_i$
$= (\lambda_i - k) x_i$
$\therefore (A - kI) x_i = (\lambda_i - k) x_i$

$\therefore (\lambda_i - k)$ is the Eigen value of $A - kI$ $\forall\ i = 1, 2, 3, \ldots n$

i.e $(\lambda_1 - k), (\lambda_2 - k), (\lambda_3 - k), \ldots (\lambda_n - k)$ are the Eigen values of $A - kI$.

* 8) The characteristic roots of a triangular matrix are just the diagonal elements of the matrix.

(or).

the Eigen values of " " " " " " "

proof. Let us consider the triangular matrix $A = \begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$.

$\therefore$ the characteristic equation of A is $|A - \lambda I| = 0$.

i.e $\begin{vmatrix} a_{11} - \lambda & 0 & 0 \\ a_{21} & a_{22} - \lambda & 0 \\ a_{31} & a_{32} & a_{33} - \lambda \end{vmatrix} = 0.$

$\Rightarrow (a_{11} - \lambda)(a_{22} - \lambda)(a_{33} - \lambda) = 0.$

$\therefore \lambda = a_{11}, a_{22}, a_{33}$

$\therefore$ i.e a_{11}, a_{22}, a_{33} be the diagonal elements & Eigen values of A.

* llly we can prove the Eigen values of Diagonal matrix are the elements in the diagonal.

* 9) P.T. If λ is the Eigen value of a matrix A then $1/\lambda$ is the Eigen value of matrix A^{-1}. (or) P.T the Eigen value of A^{-1} are the reciprocals of the Eigen values of A.

proof. If X be the Eigen vector corresponding to λ.

then $AX = \lambda X$

pre multiplying A^{-1} on both sides we get

$$A^{-1}(AX) = A^{-1}(\lambda X).$$

$\Rightarrow (A^{-1}A)X = \lambda(A^{-1}X).$

$\Rightarrow IX = \lambda A^{-1}\cdot X. \Rightarrow X = \lambda\cdot A^{-1}X \Rightarrow \frac{1}{\lambda}\cdot X = A^{-1}X.$

$\therefore A^{-1}X = \frac{1}{\lambda}X.$

i.e. $\frac{1}{\lambda}$ is the Eigen Value of the matrix A^{-1}.

* **Note:** If λ is an Eigen Value of orthogonal matrix A, then $\frac{1}{\lambda}$ is also its Eigen Value.

proof:- If A is an orthogonal matrix then $AA^T = A^TA = I$

i.e $A^T = A^{-1}$.

$\boxed{\text{remaining proof same as above}}$

$\frac{1}{\lambda}$ is the E.V. of the matrix A^T. $[\because A^{-1} = A^T]$

But the Eigen values of A & A^T are same

$\therefore \frac{1}{\lambda}$ is the Eigen Value of A.

(10). If $\lambda_1, \lambda_2, \lambda_3, \ldots \lambda_n$ are the Eigen Values of A. find the Eigen values of the matrix $(A-\lambda I)^2$.

proof

proof of 7.

$\because (\lambda_1 - k), (\lambda_2 - k), (\lambda_3 - k), \ldots (\lambda_n - k)$ are Eigen values of $(A - kI)$.

$\Rightarrow (\lambda_1 - \lambda)(\lambda_2 - \lambda), (\lambda_3 - \lambda) \ldots (\lambda_n - \lambda)$ be the E.v. of $(A - \lambda I)$.

We know that If $\lambda_1, \lambda_2, \lambda_3, \ldots \lambda_n$ be the Eigen values of A then

$\lambda_1^2, \lambda_2^2, \lambda_3^2, \ldots \lambda_n^2$ are Eigen values of A^2.

$\therefore$ ~~Hence~~ the Eigen values of $(A-\lambda I)^2$ are $(\lambda_1-\lambda)^2 (\lambda_2-\lambda)^2, \ldots (\lambda_n-\lambda)^2$.

*** (11). If λ is the Eigen Value of a Non Singular matrix A, then $\frac{|A|}{\lambda}$ is an Eigen Value of the matrix adj A.

proof:- Let X be the Eigen Vector Corresponding to Eigen Value λ.

then $AX = \lambda X$.

premultiply ~~with~~ adj A on both sides.

$\therefore (\text{adj } A)(AX) = (\text{adj } A)(\lambda X) \cdot \Rightarrow (\text{adj} A) A \cdot X = \lambda\cdot(\text{adj} A)X.$

$\Rightarrow |A|\cdot X = \lambda\cdot \text{adj} A \cdot X \Rightarrow \frac{|A|}{\lambda}\cdot X = (\text{adj } A)X.$

$\therefore (\text{adj} A) X = \frac{|A|}{\lambda} . X.$

i.e $\frac{|A|}{\lambda}$ is an Eigen Value of adj A.

(12). If λ is Eigen Value of A then prove that the Eigen Value of $a_0 A^2 + a_1 A + a_2 I$ is $a_0 \lambda^2 + a_1 \lambda + a_2$.

proof: If X be the Eigen vector Corresponding to the Eigen Value λ, then $AX = \lambda X$. — ①.

premultiply eq① with 'A' on both sides.

$\therefore A(AX) = A(\lambda X)$

$\Rightarrow A^2 X = \lambda (AX)$

$= \lambda (\lambda X)$ | $\because$ by ①

$= \lambda^2 X$.

$\therefore A^2 X = \lambda^2 X$ — ②

$\therefore$ Consider, $(a_0 A^2 + a_1 A + a_2 I) X = a_0 A^2 X + a_1 AX + a_2 IX$

$= a_0 \lambda^2 X + a_1 \lambda X + a_2 X$ [$\because$ by ①&②

$= (a_0 \lambda^2 + a_1 \lambda + a_2) X$.

$\therefore (a_0 A^2 + a_1 A + a_2 I) X = (a_0 \lambda^2 + a_1 \lambda + a_2) X$.

$\therefore a_0 \lambda^2 + a_1 \lambda + a_2$ is the Eigen Value of $a_0 A^2 + a_1 A + a_2 I$.

(15) If A and B are n-rowed square matrices and if A is invertible show that $A^{-1}B$ and BA^{-1} have the same Eigen values.

proof:- Given that A is invertible.

$\therefore A^{-1}$ is exists.

Now $A^{-1}B = A^{-1} \cdot B I$.

$= A^{-1} \cdot B(A^{-1}A)$ | $\because A^{-1}A = I$

$= A^{-1}(BA^{-1})A$

$\therefore A^{-1}B = A^{-1}(BA^{-1})A$

this is in the form of $B = P^{-1}AP$.

We know that B & $P^{-1}AP$ ~~have~~ the same Eigen values.

$\therefore A^{-1}B$ & BA^{-1} have the same Eigen values.

problems based on properties

I. (1). Sum of Eigen values = Sum of the main diagonal elements

(2). product of the Eigen values = $|A|$.

Eg: 1) $A = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$ find Sum & product of the Eigen value of the matrix A where

2) $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$. 3). $A = \begin{bmatrix} 2 & -3 \\ 4 & -2 \end{bmatrix}$.

4). If the product of two Eigen values of the matrix $A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ is 6. find third Eigen value.

5). Two of the Eigen values of $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ are 2&8. find third E-value

II. property (2). the Eigen value of A = the Eigen value of A^T.

Eg: 1) If 2, 2, 3 are the Eigen value of $A = \begin{bmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{bmatrix}$. find the Eigen values of A^T.

2). If the Eigen values of the matrix $A = \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix}$ are 2, -2 then find the Eigen values of A^T.

III 3). the Eigen values of a Triangular matrix = Diagonal elements
(or)
" " Diagonal matrix = Diagonal elements

Eg: 1). find the Eigen values of $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$ without using the characteristic eq. Idea

2). find the Eigen values of $A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 3 & 0 \\ 0 & 4 & 4 \end{bmatrix}$.

IV. the Eige of A are $\lambda_1, \lambda_2, \lambda_3$ then Eigen values of A^{-1} are $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \frac{1}{\lambda_3}$
(or)
If λ is the Eigen value of $A \Rightarrow \frac{1}{\lambda}$ is the Eigen value of A^{-1}.

Eg: 1) Two of the Eigen values of $A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$ are 3 & 6. find the Eigen values of A^{-1}.

Soln: Sum of the ~~Eigen~~ diagonal Values = 3+5+3 = 11.
Let k be the third Eigen value $\therefore 3+6+k = 11 \Rightarrow k = 2$.

$\therefore$ the Eigen values of A^{-1} are $\frac{1}{2}, \frac{1}{3}, \frac{1}{6}$.

2). Find the Eigen values of the matrix $\begin{pmatrix} 1 & -2 \\ -5 & 4 \end{pmatrix}$. Hence form the matrix whose Eigen values are $\frac{1}{6}$ & -1.

Sol:- The Given matrix $A = \begin{bmatrix} 1 & -2 \\ -5 & 4 \end{bmatrix}$.

The Characteristic eq. of A is $|A-\lambda I| = 0 \Rightarrow \begin{vmatrix} 1-\lambda & -2 \\ -5 & 4-\lambda \end{vmatrix} = 0$

$\Rightarrow \lambda^2 - 5\lambda - 6 = 0 \Rightarrow (\lambda+1)(\lambda-6) = 0 \Rightarrow \lambda = 6$ (or) -1.

$\boxed{\because \text{ If } \lambda \text{ is E.Value of } A \text{ then } \lambda^{-1} \text{ is the Eigen Value } A^{-1}}$

$\therefore$ the matrix whose Eigen values are $\frac{1}{6}$ & -1 is A^{-1}.

$\therefore A^{-1} = \frac{1}{|A|} \operatorname{adj} A = \frac{1}{|A|} \begin{bmatrix} 4 & 2 \\ 5 & 1 \end{bmatrix} = \frac{1}{-6} \begin{bmatrix} 4 & 2 \\ 5 & 1 \end{bmatrix}$.

$\therefore A^{-1} = \frac{-1}{6} \begin{bmatrix} 4 & 2 \\ 5 & 1 \end{bmatrix}$.

3). Two Eigen values of the matrix $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$ are equal to 1 each. Find the Eigen values of A^{-1}. $\boxed{\lambda_1 = \lambda_2 = 1}$

4). Find the Eigen values of A^{-1} for $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 3 & 4 \\ 0 & 0 & 4 \end{bmatrix}$.

(v). If λ is an Eigen value of ~~.......~~ an Orthogonal matrix then $\frac{1}{\lambda}$ is ~~the~~ also an Eigen values of A.

Eg:- 1). the Eigen values of the given orthogonal matrix $A = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$ are $\frac{1+i}{\sqrt{2}}, \frac{1-i}{\sqrt{2}}$. S.T. $\frac{\sqrt{2}}{1+i}, \frac{\sqrt{2}}{1-i}$ is also Eigen Values of A.

(vi). If $\lambda_1, \lambda_2, \lambda_3, \ldots \lambda_n$ are the Eigen Values of A then $\lambda_1^k, \lambda_2^k, \lambda_3^k, \ldots \lambda_n^k$ are the Eigen values of A^k; $\forall k \in \mathbb{Z}^+$

Eg:- 1). If 1 & 2 are the Eigen values of a 2×2 matrix A what are the Eigen values of A^2 & A^{-1}.

(vii). The Eigen values of real Symmetric matrix A is real.

Eg:- S.T. the " " " $A = \begin{bmatrix} -2 & 2 \\ 2 & 1 \end{bmatrix}$ is real.

Sol:- Given A is a real Symmetric matrix

the Char. eq. is $|A-\lambda I| = 0$.

$\Rightarrow \lambda^2 + \lambda - 6 = 0 \therefore \lambda = 2, -3$. $\therefore$ the Eigen Values are real.

VIII). If d_1, d_2, d_3 be the Eigen values of A then
Kd_1, Kd_2, Kd_3 " " KA

Eg 1). If 1, 1, 5 are Eigen values of $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$ find the Eigen values of 5A.

2). Find the Eigen values of $A, A^4, 3A, A^{-1}, A-I$. if $A = \begin{bmatrix} 2 & 3 \\ 0 & 5 \end{bmatrix}$.

Sol the Eigen values of $A - I = \begin{bmatrix} 2 & 3 \\ 0 & 5 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 0 & 4 \end{bmatrix}$.

$\therefore$ the Eigen values are $\underline{1, 4}$.

Definition :- Conjugate matrix — I unit definitions.

20 theorem (1). The Eigen values of a real symmetric matrix are always real.

proof: Let λ be the Eigen value of a Real symmetric matrix A and X be the corresponding Eigen vector. $\ni$. $AX = \lambda X$. —— ①.

Take conjugate on both sides.

$\overline{AX} = \overline{\lambda X}$

$\Rightarrow \bar{A}\bar{X} = \bar{\lambda}\bar{X}$

Now, Take Transpose on both sides.

$(\bar{A}\bar{X})^T = (\bar{\lambda}\bar{X})^T$

$\Rightarrow \bar{X}^T (\bar{A})^T = \bar{\lambda}\bar{X}^T$.

$\Rightarrow \bar{X}^T A^T = \bar{\lambda}\bar{X}^T \quad | \because \bar{A} = A$ since A is a real symmetric matrix.

$\Rightarrow (\bar{X})^T \cdot A = \bar{\lambda}\bar{X}^T \quad | A^T = A$

post multiply by X, we get.

$(\bar{X})^T AX = \bar{\lambda}(\bar{X})^T X$. —— ②.

Premultiply $(\bar{X})^T$ to eq ①, we get

$(\bar{X})^T AX = (\bar{X})^T \lambda X$

$\therefore \bar{X}^T AX = \lambda \bar{X}^T X$ —— ③.

Apply ② - ③, we get $(\lambda - \bar{\lambda})(\bar{X}^T X) = 0$.

But $\bar{X}^T X \neq 0$. $\therefore \lambda - \bar{\lambda} = 0 \therefore \lambda = \bar{\lambda}$

$\therefore \lambda$ is a real Number Hence proved.

Marked $\lambda = \bar{\lambda}$

21 Theorem(2):- S.T. for a real symmetric matrix, the eigen vectors corresponding to two distinct eigen values are orthogonal.

proof:- Let λ_1, λ_2 be the two distinct eigen values of A and X_1, X_2 be the corresponding eigen vectors of A

$$\Rightarrow \quad AX_1 = \lambda_1 X_1 \quad \text{——— (1)}$$

$$AX_2 = \lambda_2 X_2 \quad \text{——— (2)}$$

To show that X_1 is orthogonal to X_2 ($X_1^T X_2 = 0$):-

premultiply eq (1) by X_2^T

$$X_2^T A X_1 = \lambda_1 X_2^T X_1$$

Taking Transpose on both sides, we get

$$(X_2^T A X_1)^T = (\lambda_1 X_2^T X_1)^T$$

$$\Rightarrow X_1^T A^T (X_2^T)^T = \lambda_1 \cdot X_1^T (X_2^T)^T$$

$$\Rightarrow X_1^T A X_2 = \lambda_1 \cdot X_1^T X_2 \quad \Big| \because A^T = A$$

——— (3)

premultiplying eq (2) by X_1^T we get

$$X_1^T A X_2 = \lambda_2 X_1^T X_2 \quad \text{——— (4)}$$

$\therefore$ Apply (3) − (4)

$$(\lambda_1 - \lambda_2) X_1^T X_2 = 0$$

Since $\lambda_1 - \lambda_2 \neq 0$

$$\therefore X_1^T X_2 = 0$$

$\therefore X_1$ is orthogonal to X_2.

Hence proved

Note :- Characteristic equation of A :- For 3×3 matrix

Model(1). the char. eq. of A is $|A - \lambda I| = 0$.

Model (2).
$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}.$$

∴ the characteristic eq. is

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0.$$

where,

S_1 = Sum of main diagonal elements $= a_{11} + a_{22} + a_{33}$.

S_2 = Sum of the minors of main diagonal elements

i.e
$$\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

S_3 = Determinent Value of A $= |A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

For 2×2 matrix
$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

∴ the characteristic eq. of A is
$$\lambda^2 - S_1\lambda + S_2 = 0.$$

where

S_1 = Sum of the main diagonal elements $= a_{11} + a_{22}$.

S_2 = Determinent Value of A $= |A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$.

IX problems:- $\lambda_1, \lambda_2 \dots \lambda_n \longrightarrow$ e.v. of A

$\lambda_1^k\ \lambda_2^k \dots \lambda_n^k \longrightarrow$ " A^k.

problem (1). If α & β are the eigen values of $\begin{bmatrix} 3 & -1 \\ -1 & 5 \end{bmatrix}$.

form the matrix whose eigen values are α^3 & β^3.

Ans $\underline{A^3}$ matrix.

problem on:- $\lambda_1 - k, \lambda_2 - k, \dots \lambda_n - k \xrightarrow{e.v.\ of} (A - KI)$

Eg:- form the matrix whose eigen values are $\alpha-5, \beta-5, \gamma-5$ where α, β, γ are the eigen values of

$$A = \begin{bmatrix} -1 & -2 & -3 \\ 4 & 5 & -6 \\ 7 & -8 & 9 \end{bmatrix}.$$

Ans. $A - kI = A - 5I = [\ \ ] - 5[\ \ ]$.

CAYLEY- HAMILTON THEOREM

We now give some difinitions and proceed to prove Cayley - Hamilton Theorem.

2.11 DEFINITIONS

1. Matrix Polynomial :

An expression of the form $F(x) = A_0 + A_1x + A_2x^2 + \cdots + A_mx^m, A_m \neq 0,$ where $A_0, A_1, A_2, \cdots, A_m$ are matrices each of order $n \times n$ over a field F, is called a matrix polynomial of degree m.

The symbol x is called indeterminate and will be assumed that it is commutative with every matrix coefficient.

The matrices themselves are matrix polynomials of zero degree.

2. Equality of Matrix Polynomials :

Two matrix polynomials are equal if and only if the coefficients of like powers of x are the same.

3. Addition and Multiplication of Polynomials :

Let $G(x) = A_0 + A_1x + A_2x^2 + \cdots + A_mx^m$ and
$H(x) = B_0 + B_1x + B_2x^2 + \cdots + B_kx^k$.

We define : If $m > k$, then

$G(x) + H(x) = (A_0 + B_0) + (A_1 + B_1)x + \ldots + (A_k + B_k)x^k + A_{k+1}x^{k+1} + \ldots + A_mx^m.$

Similarly, we have $G(x) + H(x)$ when $m = k$ and $m < k$.

Also $G(x) \cdot H(x) = A_0B_0 + (A_0B_1 + A_1B_0)x + (A_0B_2 + A_1B_1 + A_2B_0)x^2 + \ldots + A_mB_kx^{k+m}.$

Note that the degree of the product of two matric polynomials is less than or equal to the sum of their degree.

Theorem : Every square matrix, whose elements are polynomials in x, can be expressed as a matric polynomials in x of degree m, where m is the highest power of x having by any element of the matrix.

We illustrate the theorem by an example.

Consider the matrix $A = \begin{bmatrix} 2x & 3x^2+4 & x-x^2+x^3 \\ x^3-5 & 0 & 4+7x^2 \\ 6 & 8x^2 & -3x^2+2 \end{bmatrix}$

We write

$$A = \begin{bmatrix} 0+2x+0x^2+0x^3 & 4+0x+3x^2+0x^3 & 0+1x-1x^2+1x^3 \\ -5+0x+0x^2+x^3 & 0+0x+0x^2+0x^3 & 4+0x+7x^2+0x^3 \\ 6+0x+0x^2+0x^3 & 0+0x+8x^2+0x^3 & 2+0x-3x^2+0x^3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 4 & 0 \\ -5 & 0 & 4 \\ 6 & 0 & 2 \end{bmatrix} + \begin{bmatrix} 2 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 & 3 & -1 \\ 0 & 0 & 7 \\ 0 & 8 & -3 \end{bmatrix} x^2 + \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= A_0 + A_1x + A_2x^2 + A_3x^3$$

2.12 THE CAYLEY- HAMILTON THEOREM

Theorem: Every square matrix satisfies its own characteristic equation.

[JNTU 2002S]

Proof: Let A be an n-rowed square matrix. Then

$|A-\lambda I| = 0$ is the characteristic equation of A.

Let $|A-\lambda I| = (-1)^n[\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-1} + a_3\lambda^{n-2} + \ldots + a_n]$

Since all the elements of $A - \lambda I$ are at most of first degree in λ, all the elements of adj $(A - \lambda I)$ are polynomials in λ of degree $(n - 1)$ or less and hence adj.$(A - \lambda I)$ can be written as a matrix polynomials in λ.

Let adj. $(A-\lambda I) = B_0\lambda^{n-1} + B_1\lambda^{n-2} + \cdots + B_{n-2}\lambda^1 + B_{n-1}$, where $B_0, B_1, \ldots, B_{n-1}$ are n-rowed matrices.

Now $(A - \lambda I)$ adj$(A - \lambda I)$

$$= |A-\lambda I| I_n (A-\lambda I)(B_0\lambda^{n-1} + B_1\lambda^{n-2} + \ldots + B_{n-2}\lambda + B_{n-1})$$
$$= (-1)^n[\lambda^n + a_1\lambda^{n-1} + \ldots + a_n]I.$$

Comparing coefficients of like powers of λ, we obtain

$$-B_0 = (-1)^n I,$$
$$AB_0 - B_1 = (-1)^n a_1 I,$$
$$AB_1 - B_2 = (-1)^n a_2 I$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$AB_{n-1} = (-1)^n a_n I.$$

Premultiplying the above equations successively by $A^n, A^{n-1}, \ldots, I$ and adding, we obtain

$$0 = (-1)^n A^n + (-1)^n a_1 A^{n-1} + (-1)^n a_2 A^{n-2} + \cdots + (-1)^n a_n I$$

$$\Rightarrow (-1)^n [A^n + a_1 A^{n-1} + a_2 A^{n-2} + \ldots + a_n I] = 0$$

which implies that A satisfies its characteristic equation.

Applications of Cayley - Hamilton Theorem :

The important applications of Cayley - Hamilton theorem are

1. To find the inverse of a matrix.
2. To find higher powers of the matrix.

SOLVED EXAMPLES

Example 1 : If $A=\begin{bmatrix} 2 & 1 & 2 \\ 5 & 3 & 3 \\ -1 & 0 & -2 \end{bmatrix}$ verify Cayley- Hamilton theorem. Hence find A^{-1}.

Solution : The characteristic equation of A is $|A-\lambda I|=0$

$$i.e., \begin{vmatrix} 2-\lambda & 1 & 2 \\ 5 & 3-\lambda & 3 \\ -1 & 0 & -2-\lambda \end{vmatrix}=0$$

$i.e.$ $(2-\lambda)[-6-3\lambda+2\lambda+\lambda^2]-5[-2-\lambda]-1(3-6+2\lambda)=0$

$i.e.$ $(2-\lambda)(-6-\lambda+\lambda^2)+10+5\lambda+3-2\lambda=0$

$i.e.$ $-12-2\lambda+2\lambda^2+6\lambda+\lambda^2-\lambda^3+13+3\lambda=0$

$i.e.$ $-\lambda^3+3\lambda^2+7\lambda+1=0$

$i.e.$ $\lambda^3-3\lambda^2-7\lambda-1=0$

To Verify Cayley-Hamilton theorem, we have to show that

$$A^3-3A^2-7A-I=O.$$

That is, to prove that A satisfies $\lambda^3-3\lambda^2-7\lambda-1=0$, the characteristic equation of A.

$$\text{Now} \quad A^2=\begin{bmatrix} 2 & 1 & 2 \\ 5 & 3 & 3 \\ -1 & 0 & -2 \end{bmatrix}\begin{bmatrix} 2 & 1 & 2 \\ 5 & 3 & 3 \\ -1 & 0 & -2 \end{bmatrix}=\begin{bmatrix} 7 & 5 & 3 \\ 22 & 14 & 13 \\ 0 & -1 & 2 \end{bmatrix}$$

$$A^3=A^2\cdot A=\begin{bmatrix} 7 & 5 & 3 \\ 22 & 14 & 13 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 & 2 \\ 5 & 3 & 3 \\ -1 & 0 & -2 \end{bmatrix}=\begin{bmatrix} 36 & 22 & 23 \\ 101 & 64 & 60 \\ -7 & -3 & -7 \end{bmatrix}$$

$$\text{Now} \quad A^3-3A^2-7A-I=\begin{bmatrix} 36 & 22 & 23 \\ 101 & 64 & 60 \\ -7 & -3 & -7 \end{bmatrix}+\begin{bmatrix} -21 & -15 & -9 \\ -66 & -42 & -39 \\ 0 & 3 & -6 \end{bmatrix}$$

$$+\begin{bmatrix} -14 & -7 & -14 \\ -35 & -21 & -21 \\ 7 & 0 & 14 \end{bmatrix}+\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O$$

$\therefore$ This verifies Cayley – Hamilton theorem.

To find A^{-1}

$$A^3 - 3A^2 - 7A - I = O \Rightarrow A^{-1}(A^3 - 3A^2 - 7A - I) = O$$

$$\Rightarrow A^2 - 3A - 7I - A^{-1} = O \Rightarrow A^{-1} = A^2 - 3A - 7I$$

$$\therefore\ A^{-1} = \begin{bmatrix} 7 & 5 & 3 \\ 22 & 14 & 13 \\ 0 & -1 & 2 \end{bmatrix} + \begin{bmatrix} -6 & -3 & -6 \\ 15 & -9 & -9 \\ 3 & 0 & 6 \end{bmatrix} + \begin{bmatrix} -7 & 0 & 0 \\ 0 & -7 & 0 \\ 0 & 0 & -7 \end{bmatrix} = \begin{bmatrix} -6 & 2 & -3 \\ 7 & -2 & 4 \\ 3 & -1 & 1 \end{bmatrix}$$

Check : $AA^{-1} = I$

Example 2 : Find the inverse of the matrix $\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 2 & 1 & 2 \end{bmatrix}$ by using Cayley-Hamilton theorem.

Solution : Let $A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 2 & 1 & 2 \end{bmatrix}$

The characteristic equation of A is $|A - \lambda I| = 0$

$$i.e. \begin{vmatrix} 1-\lambda & -1 & 0 \\ 0 & 1-\lambda & 1 \\ 2 & 1 & 2-\lambda \end{vmatrix} = 0$$

$$i.e., (1-\lambda)[2-\lambda-2\lambda+\lambda^2-1]+[0-2]=0$$

$$i.e. (1-\lambda))(1-3\lambda+\lambda^2)-2=0 \text{ or } \lambda^3-4\lambda^2+4\lambda+1=0$$

By Cayley-Hamilton theorem, A satisfies its characteristic equation.

$$\therefore A^3 - 4A^2 + 4A + I = O$$

$$\Rightarrow A^{-1}(A^3 - 4A^2 + 4A + I) = O \quad (\because |A| = -1 \neq 0)$$

$$\Rightarrow A^2 - 4A + 4I + A^{-1} = O$$

$$\Rightarrow A^{-1} = -A^2 + 4A - 4I \qquad \text{... (1)}$$

Now $A^2 = A.A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 2 & 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 2 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -2 & -1 \\ 2 & 2 & 3 \\ 6 & 1 & 5 \end{bmatrix}$

$$\therefore\ A^{-1} = \begin{bmatrix} -1 & 2 & 1 \\ -2 & -2 & -3 \\ -6 & -1 & -5 \end{bmatrix} + \begin{bmatrix} 4 & -4 & 0 \\ 0 & 4 & 4 \\ 8 & 4 & 8 \end{bmatrix} + \begin{bmatrix} -4 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -4 \end{bmatrix} \text{ [by (1)]}$$

$$= \begin{bmatrix} -1 & -2 & 1 \\ -2 & -2 & 1 \\ 2 & 3 & -1 \end{bmatrix}$$

Check : $AA^{-1} = I$

Example 3: State Cayley-Hamilton theorem and use it to find the inverse of the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix}.$$ **[JNTU 2001]**

Solution : Cayley-Hamilton Theorem : Every square matrix satisfies its own characteristic equation.

Given $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix}$

The characteristic equation of A is $|A - \lambda I| = 0$ *i.e.* $\begin{vmatrix} 1-\lambda & 2 & 3 \\ 2 & -1-\lambda & 4 \\ 3 & 1 & -1-\lambda \end{vmatrix} = 0$

$\Rightarrow (1 - \lambda)\,[(1 + \lambda)^2 - 4] - 2[-2(1 + \lambda) - 12] + 3[2 + 3(1 + \lambda)] = 0$

$\Rightarrow \lambda^3 + \lambda^2 - 18\lambda - 40 = 0$

By Cayley-Hamilton theorem, we have

$A^3 + A^2 - 18A - 40I = O$

Multiplying with A^{-1} on both sides, we get

$A^2 + A - 18I = 40A^{-1} \quad \Rightarrow \quad A^{-1} = \frac{1}{40}\left[A^2 + A - 18I\right]$

We have $A^2 = \begin{bmatrix} 14 & 3 & 8 \\ 12 & 9 & -2 \\ 2 & 4 & 14 \end{bmatrix}$

$$\therefore \quad A^{-1} = \frac{1}{40}\left\{\begin{bmatrix} 14 & 3 & 8 \\ 12 & 9 & -2 \\ 2 & 4 & 14 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix} - \begin{bmatrix} 18 & 0 & 0 \\ 0 & 18 & 0 \\ 0 & 0 & 18 \end{bmatrix}\right\}$$

$$\Rightarrow \quad A^{-1} = \frac{1}{40}\begin{bmatrix} -3 & 5 & 11 \\ 14 & -10 & 2 \\ 5 & 5 & -5 \end{bmatrix}$$

Example 4 : Using Cayley-Hamilton theorem find the inverse and A^4 of the matrix

$$A = \begin{bmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{bmatrix}.$$ **[JNTU 2002, (H) May 2016]**

Solution : Let $A = \begin{bmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{bmatrix}$

The characteristic equation of A is given by $|A - \lambda I| = 0$

i.e. $\begin{vmatrix} 7-\lambda & 2 & -2 \\ -6 & -1-\lambda & 2 \\ 6 & 2 & -1-\lambda \end{vmatrix} = 0$

Performing $R_1 - R_3$ and $R_2 + R_3$, we get $\begin{vmatrix} 1-\lambda & 0 & \lambda-1 \\ 0 & 1-\lambda & 1-\lambda \\ 6 & 2 & -(1+\lambda) \end{vmatrix} = 0$

$$\Rightarrow (1-\lambda)^2 \begin{vmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 6 & 2 & -(1+\lambda) \end{vmatrix} = 0$$

$\Rightarrow (1-\lambda)^2[(-1-\lambda-2) - 1(-6)] = 0$

$\Rightarrow (1-\lambda)^2[3-\lambda] = 0 \Rightarrow \lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0$

By Cayley-Hamilton theorem, we must have

$A^3 - 5A^2 + 7A - 3I = O$ ...(1)

We have $A^2 = \begin{bmatrix} 25 & 8 & -8 \\ -24 & -7 & 8 \\ 24 & 8 & -7 \end{bmatrix}$ and $A^3 = \begin{bmatrix} 79 & 26 & -26 \\ -78 & -25 & 26 \\ 78 & 26 & -25 \end{bmatrix}$

To find A^{-1}, multiply with A^{-1} on both sides of (1). Then we get

$$A^{-1}[A^3 - 5A^2 + 7A - 3I] = O$$

$\Rightarrow A^2 - 5A + 7I - 3A^{-1} = O$

$\Rightarrow 3A^{-1} = A^2 - 5A + 7I$

$\Rightarrow A^{-1} = \frac{1}{3}[A^2 - 5A + 7I]$ ...(2)

Now $A^2 - 5A + 7I = \begin{bmatrix} 25 & 8 & -8 \\ -24 & -7 & 8 \\ 24 & 8 & -7 \end{bmatrix} - \begin{bmatrix} 35 & 10 & -10 \\ -30 & -5 & 10 \\ 30 & 10 & -5 \end{bmatrix} + \begin{bmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{bmatrix} = \begin{bmatrix} -3 & -2 & 2 \\ 6 & 5 & -2 \\ -6 & -2 & 5 \end{bmatrix}$

$\therefore A^{-1} = \frac{1}{3}\begin{bmatrix} -3 & -2 & 2 \\ 6 & 5 & -2 \\ -6 & -2 & 5 \end{bmatrix}$ [by (2)]

Multiplying (1) with A, we have

$$A^4 - 5A^3 + 7A^2 - 3A = O$$

$\Rightarrow A^4 = 5A^3 - 7A^2 + 3A$

$$= \begin{bmatrix} 395 & 130 & -130 \\ -390 & -125 & 130 \\ 390 & 130 & -125 \end{bmatrix} - \begin{bmatrix} 175 & 56 & -56 \\ -168 & -49 & 56 \\ 168 & 56 & -69 \end{bmatrix} + \begin{bmatrix} 21 & 6 & -6 \\ -18 & -3 & 6 \\ 18 & 6 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 241 & 80 & -80 \\ -240 & -79 & 80 \\ 240 & 80 & -79 \end{bmatrix}$$

Example 5 : Show that the matrix $A = \begin{bmatrix} 1 & -2 & 2 \\ 1 & -2 & 3 \\ 0 & -1 & 2 \end{bmatrix}$ satisfies its characteristic equation. Hence find A^{-1}.

[JNTU 2002, (A) May 2011, June 2016 (R13), Dec. 2017 Supl]

Solution : Characteristic equation of A is det $(A - \lambda I) = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & -2 & 2 \\ 1 & -2-\lambda & 3 \\ 0 & -1 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 0 & 2 \\ 1 & 1-\lambda & 3 \\ 0 & 1-\lambda & 2-\lambda \end{vmatrix} = 0 \text{ (Applying } C_2 \to C_2 + C_3\text{)}$$

$$\Rightarrow (1-\lambda) \begin{vmatrix} 1-\lambda & 0 & 2 \\ 1 & 1 & 3 \\ 0 & 1 & 2-\lambda \end{vmatrix} = 0 \text{ [Taking } (1-\lambda) \text{ common from } C_2\text{]}$$

$$\Rightarrow (1-\lambda)\left[(1-\lambda)(2-\lambda-3)+2\right] = 0 \qquad \text{[Expanding by } C_1\text{]}$$

$$\Rightarrow (1-\lambda)[\lambda^2 - 1 + 2] = 0 \Rightarrow (1-\lambda)(\lambda^2+1) = 0 \Rightarrow \lambda^3 - \lambda^2 + \lambda - 1 = 0$$

By Cayley - Hamilton theorem, we have $A^3 - A^2 + A - I = O$

Now $A = \begin{bmatrix} 1 & -2 & 2 \\ 1 & -2 & 3 \\ 0 & -1 & 2 \end{bmatrix}$ $\therefore A^2 = \begin{bmatrix} -1 & 0 & 0 \\ -1 & -1 & 2 \\ -1 & 0 & 1 \end{bmatrix}$, $A^3 = \begin{bmatrix} -1 & 2 & -2 \\ -2 & 2 & -1 \\ -1 & 1 & 0 \end{bmatrix}$

Substituting the values of A, A^2, A^3, we have

$$A^3 - A^2 + A - I = \begin{bmatrix} -1 & 2 & -2 \\ -2 & 2 & -1 \\ -1 & 1 & 0 \end{bmatrix} - \begin{bmatrix} -1 & 0 & 0 \\ -1 & -1 & 2 \\ -1 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & -2 & 2 \\ 1 & -2 & 3 \\ 0 & -1 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O$$

Hence A satisfies Cayley-Hamilton theorem.

Multiplying $A^3 - A^2 + A - I = O$ with A^{-1}, we get $A^2 - A + I = A^{-1}$

$$\therefore A^{-1} = \begin{bmatrix} -1 & 0 & 0 \\ -1 & -1 & 2 \\ -1 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & -2 & 2 \\ 1 & -2 & 3 \\ 0 & -1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 2 & -2 \\ -2 & 2 & -1 \\ -1 & 1 & 0 \end{bmatrix}$$

Example 6 : Show that the matrix $A = \begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix}$ **satisfies Cayley-Hamilton theorem.**

[JNTU (A) May 2013]

Solution : We have $|A - \lambda| = \begin{vmatrix} 0-\lambda & c & -b \\ -c & 0-\lambda & a \\ b & -a & 0-\lambda \end{vmatrix}$

$$= -\lambda(\lambda^2 + a^2) - c(c\lambda - ab) - b(ac + b\lambda)$$

$$= -\lambda^3 - \lambda(a^2 + b^2 + c^2)$$

$\therefore$ The characteristic equation of matrix A is given as

$$\lambda^3 + \lambda\ (a^2 + b^2 + c^2) = 0$$

To verify Cayley-Hamilton theorem, we have to prove that $A^3 + (a^2 + b^2 + c^2)A = O$

We have $A^2 = \begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix}\begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix} = \begin{bmatrix} -c^2-b^2 & ab & ac \\ ab & -c^2-a^2 & bc \\ ac & bc & -b^2-a^2 \end{bmatrix}$

$$\therefore A^3 = A^2 \cdot A = \begin{bmatrix} -c^2-b^2 & ab & ac \\ ab & -c^2-a^2 & bc \\ ac & bc & -b^2-a^2 \end{bmatrix}\begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -c^3-b^2c-a^2c & bc^2+b^3+a^2b \\ c^3+a^2c+b^2c & 0 & -ab^2-ac^2-a^3 \\ -bc^2-b^3-a^2b & ac^2+ab^2+c^3 & 0 \end{bmatrix}$$

$$= -(a^2+b^2+c^2)\begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix}$$

$$= -(a^2+b^2+c^2)\ A$$

$$\therefore A^3 + (a^2+b^2+c^2)A = O$$

Hence A satisfies Cayley-Hamilton theorem.

Example 7 : If $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$**, find the value of the matrix**

$$A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I.$$

Solution : The characteristic equation of A is $|A - \lambda I| = 0$

i.e. $\begin{vmatrix} 2-\lambda & 1 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 1 & 2-\lambda \end{vmatrix} = 0$ *i.e.* $\lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0$

$\therefore$ By Cayley–Hamilton theorem, $A^3 - 5A^2 + 7A - 3I = O$ ...(1)

We can rewrite the given expression as

$$A^5(A^3 - 5A^2 + 7A - 3I) + A(A^3 - 5A^2 + 8A - 2I) + I$$

$$= A[(A^3 - 5A^2 + 7A - 3I) + (A + I)] + I$$

$$= O + A(A^3 - 5A^2 + 7A - 3I) + A^2 + A + I, \text{ using (1)}$$

$$= A^2 + A + I$$

But $A^2 = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix}$

$\therefore$ Value of the given matrix $= A^2 + A + I$

$$= \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} + \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 8 & 5 & 5 \\ 0 & 3 & 0 \\ 5 & 5 & 8 \end{bmatrix}$$

Example 8 : Verify Cayley - Hamilton theorem for the matrix $A = \begin{bmatrix} 8 & -8 & 2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix}$

[JNTU 2005S, 2006S, (H) June 2011 (Set No. 3)]

Solution : The characteristic equation of A is $|A - \lambda I| = 0$

i.e. $\begin{vmatrix} 8-\lambda & -8 & 2 \\ 4 & -3-\lambda & -2 \\ 3 & -4 & 1-\lambda \end{vmatrix} = 0$

$\Rightarrow (8-\lambda)[(-3-\lambda)(1-\lambda) - 8] - 4[-8(1-\lambda) + 8] + 3[16 - 2(-3-\lambda)] = 0$

$\Rightarrow (8-\lambda)[\lambda^2 + 2\lambda - 11] - 4[8\lambda] + 3[2\lambda + 22] = 0 \Rightarrow \lambda^3 - 6\lambda^2 - \lambda + 22 = 0$

Cayley - Hamilton theorem states that every square matrix satisfies its characteristic equation.

To verify Cayley - Hamilton theorem, we have to prove that $A^3 - 6A^2 - A + 22I = O$

Now $A^2 = A \cdot A = \begin{bmatrix} 8 & -8 & 2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix}\begin{bmatrix} 8 & -8 & 2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix} = \begin{bmatrix} 38 & -48 & 34 \\ 14 & -15 & 12 \\ 11 & -16 & 15 \end{bmatrix}$

and $A^3 = A \cdot A^2 = \begin{bmatrix} 8 & -8 & 2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix}\begin{bmatrix} 38 & -48 & 34 \\ 14 & -15 & 12 \\ 11 & -16 & 15 \end{bmatrix} = \begin{bmatrix} 214 & -296 & 206 \\ 88 & -115 & 70 \\ 69 & -100 & 69 \end{bmatrix}$

Now $A^3 - 6A^2 - A + 22I$

$$=\begin{bmatrix}214 & -296 & 206\\ 88 & -115 & 70\\ 69 & -100 & 69\end{bmatrix}-6\begin{bmatrix}38 & -48 & 34\\ 14 & -15 & 12\\ 11 & -16 & 15\end{bmatrix}-\begin{bmatrix}8 & -8 & 2\\ 4 & -3 & -2\\ 3 & -4 & 1\end{bmatrix}+22\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}=O$$

Hence Cayley - Hamilton theorem is verified.

Example 9 : Verify Cayley -Hamilton theorem for the matrix $A=\begin{bmatrix}1 & 2 & 3\\ 2 & 4 & 5\\ 3 & 5 & 6\end{bmatrix}$. Hence find A^{-1}. **[JNTU 2005S, (A) May 2012 (Set No. 2)]**

Solution : The characteristic equation of A is $|A-\lambda I|=0$

i.e. $\begin{vmatrix}1-\lambda & 2 & 3\\ 2 & 4-\lambda & 5\\ 3 & 5 & 6-\lambda\end{vmatrix}=0$

$$\Rightarrow (1-\lambda)[(4-\lambda)(6-\lambda)-25]-2[2(6-\lambda)-15]+3[10-3(4-\lambda)]=0$$

$$\Rightarrow (1-\lambda)[\lambda^2-10\lambda-1]-2[-2\lambda-3]+3[3\lambda-2]=0$$

$$\Rightarrow -\lambda^3+10\lambda^2+\lambda+\lambda^2-10\lambda-1+4\lambda+6+9\lambda-6=0$$

$$\Rightarrow \lambda^3-11\lambda^2-4\lambda+1=0$$

Cayley - Hamilton theorem states that every square matrix satisfies its characteristic equation.

To verify Cayley - Hamilton theorem, we have to prove that $A^3-11A^2-4A+I=O$

$$A^2=A\cdot A=\begin{bmatrix}1 & 2 & 3\\ 2 & 4 & 5\\ 3 & 5 & 6\end{bmatrix}\begin{bmatrix}1 & 2 & 3\\ 2 & 4 & 5\\ 3 & 5 & 6\end{bmatrix}=\begin{bmatrix}14 & 25 & 31\\ 25 & 45 & 56\\ 31 & 56 & 70\end{bmatrix}$$

$$A^3=A\cdot A^2=\begin{bmatrix}14 & 25 & 31\\ 25 & 45 & 56\\ 31 & 56 & 70\end{bmatrix}\begin{bmatrix}1 & 2 & 3\\ 2 & 4 & 5\\ 3 & 5 & 6\end{bmatrix}=\begin{bmatrix}157 & 283 & 353\\ 283 & 510 & 636\\ 353 & 636 & 793\end{bmatrix}$$

A^3-11A^2-4A+I

$$=\begin{bmatrix}157 & 283 & 353\\ 283 & 510 & 636\\ 353 & 636 & 793\end{bmatrix}-11\begin{bmatrix}14 & 25 & 31\\ 25 & 45 & 56\\ 31 & 56 & 70\end{bmatrix}-4\begin{bmatrix}1 & 2 & 3\\ 2 & 4 & 5\\ 3 & 5 & 6\end{bmatrix}+\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}=O$$

To find A^{-1} :

$A^3-11A^2-4A+I=O \Rightarrow I=-A^3+11A^2+4A$

Multiplying by A^{-1}, we get

$A^{-1}=-A^2+11A+4I$

$$= \begin{bmatrix} -14 & -25 & -31 \\ -25 & -45 & -56 \\ -31 & -56 & -70 \end{bmatrix} + \begin{bmatrix} 11 & 22 & 33 \\ 22 & 44 & 55 \\ 33 & 55 & 66 \end{bmatrix} + \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 2 \\ -3 & 3 & -1 \\ 2 & -1 & 0 \end{bmatrix}$$

Example 10 : Using Cayley - Hamilton theorem, find

(*i*) A^8, if $A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$ **[JNTU July-2003]**

(*ii*) A^3, where $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ **[JNTU (A) Dec. 2017 Sup]**

Solution : (*i*) Given $A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$

Characteristic equation of A is $|A - \lambda I| = 0$

i.e. $\begin{vmatrix} 1-\lambda & 2 \\ 2 & -1-\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 - 5 = 0$

By Cayley - Hamilton theorem, A satisfies its characteristic equation. So we must have $A^2 = 5I$.

$$\therefore \quad A^8 = 5A^6 = 5(A^2)(A^2)(A^2)$$
$$= 5(5I)(5I)(5I) = 625I.$$

(*ii*) This is left as an exercise to the reader.

Example 11 : If $A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$ verify Cayley-Hamilton theorem. Find A^4 and A^{-1} using Cayley-Hamilton theorem. **[JNTU Sep. 2006 (Set No. 4)]**

(or) Verify Cayley-Hamilton theorem for the matrix $A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$ and hence find A^4

Solution : Given $A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$

Characteristic equation of A is given by $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 2 & -1 \\ 2 & 1-\lambda & -2 \\ 2 & -2 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 3-\lambda & 0 & -\lambda \\ 2 & 1-\lambda & -2 \\ 2 & -2 & 1-\lambda \end{vmatrix} = 0 \quad (\text{Applying } R_1 \to R_1 + R_3)$$

$$\Rightarrow (3-\lambda)\left[(1-\lambda)^2-4\right]-\lambda[-4-2(1-\lambda)]=0$$

$$\Rightarrow -\lambda^3+3\lambda^2+3\lambda-9=0 \Rightarrow \lambda^3-3\lambda^2-3\lambda+9=0 \qquad \dots (1)$$

By Cayley-Hamilton theorem, matrix A should satisfy its characteristic equation.

i.e, $A^3-3A^2-3A+9I=O \qquad \dots (2)$

Now $A^2 = \begin{bmatrix}1 & 2 & -1\\ 2 & 1 & -2\\ 2 & -2 & 1\end{bmatrix}\begin{bmatrix}1 & 2 & -1\\ 2 & 1 & -2\\ 2 & -2 & 1\end{bmatrix} = \begin{bmatrix}3 & 6 & -6\\ 0 & 9 & -6\\ 0 & 0 & 3\end{bmatrix}$

and $A^3 = \begin{bmatrix}1 & 2 & -1\\ 2 & 1 & -2\\ 2 & -2 & 1\end{bmatrix}\begin{bmatrix}3 & 6 & -6\\ 0 & 9 & -6\\ 0 & 0 & 3\end{bmatrix} = \begin{bmatrix}3 & 24 & -21\\ 6 & 21 & -24\\ 6 & -6 & 3\end{bmatrix}$

$\therefore \quad A^3-3A^2-3A+9I$

$$= \begin{bmatrix}3 & 24 & -21\\ 6 & 21 & -24\\ 6 & -6 & 3\end{bmatrix} - 3\begin{bmatrix}3 & 6 & -6\\ 0 & 9 & -6\\ 0 & 0 & 3\end{bmatrix} - 3\begin{bmatrix}1 & 2 & -1\\ 2 & 1 & -2\\ 2 & -2 & 1\end{bmatrix} + 9\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$= \begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix} = O$$

Hence Cayley - Hamilton Theorem is verified.

To find A^{-1}.

Multiplying equation. (2) with A^{-1} on both sides, we get

$A^{-1}\,[A^3–3A^2\ 3A+9I] = A^{-1}(O)$

$\Rightarrow A^2-3A-3I+9A^{-1}=0$

$\Rightarrow 9A^{-1} = 3A+3I-A^2$

$\therefore\ A^{-1} = \frac{1}{9}(3A+3I-A^2)$

$$= \frac{1}{9}\left\{\begin{bmatrix}3 & 6 & -3\\ 6 & 3 & -6\\ 6 & -6 & 3\end{bmatrix} + \begin{bmatrix}3 & 0 & 0\\ 0 & 3 & 0\\ 0 & 0 & 3\end{bmatrix} - \begin{bmatrix}3 & 6 & -6\\ 0 & 9 & -6\\ 0 & 0 & 3\end{bmatrix}\right\}$$

$$= \frac{1}{9}\begin{bmatrix}3 & 0 & 3\\ 6 & -3 & 0\\ 6 & -6 & 3\end{bmatrix} = \begin{bmatrix}\frac{1}{3} & 0 & \frac{1}{3}\\ \frac{2}{3} & -\frac{1}{3} & 0\\ \frac{2}{3} & \frac{-2}{3} & \frac{1}{3}\end{bmatrix}$$

Now $A[A^3 - 3A^2 - 3A + 9I] = 0$

$\Rightarrow A^4 - 3A^3 - 3A^2 + 9A = 0$

$\Rightarrow A^4 = 3A^3 + 3A^2 - 9A$

$$= \begin{bmatrix} 9 & 72 & -63 \\ 18 & 63 & -72 \\ 18 & -18 & 9 \end{bmatrix} + \begin{bmatrix} 9 & 18 & -18 \\ 0 & 27 & -18 \\ 0 & 0 & 9 \end{bmatrix} - \begin{bmatrix} 9 & 18 & -9 \\ 18 & 9 & -18 \\ 18 & -18 & 9 \end{bmatrix}$$

$$\therefore \quad A^4 = \begin{bmatrix} 9 & 72 & -72 \\ 0 & 81 & -72 \\ 0 & 0 & 9 \end{bmatrix}.$$

Example 12 : Show that the Matrix $A = \begin{bmatrix} 1 & -2 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & 2 \end{bmatrix}$ satisfies its characteristic equation. Hence find A^{-1}. **[JNTU May 2007 (Set No. 3)]**

Solution : Given

$$A = \begin{bmatrix} 1 & -2 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & 2 \end{bmatrix}$$

The characteristic equation of A is $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & -2 & 2 \\ 1 & 2-\lambda & 3 \\ 0 & -1 & 2-\lambda \end{vmatrix} = 0$$

$\Rightarrow \quad (1 - \lambda)[(2 - \lambda)^2 + 3] + 2[(2 - \lambda) - 0] + 2(-1) = 0$

$\Rightarrow \quad \lambda^3 - 5\lambda^2 + 13\lambda - 9 = 0.$

By Cayley - Hamilton theorem, we must have

$A^3 - 5A^2 + 13A - 9I = O.$

Now $A^2 = \begin{pmatrix} 1 & -2 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} 1 & -2 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & 2 \end{pmatrix} = \begin{pmatrix} -1 & -8 & 0 \\ 3 & -1 & 14 \\ -1 & -4 & 1 \end{pmatrix}$; $A^3 = \begin{pmatrix} -9 & -14 & -26 \\ 2 & -22 & 31 \\ -5 & -7 & -12 \end{pmatrix}$

$\therefore \quad A^3 - 5A^2 + 13A - 9I$

$$= \begin{pmatrix} -9 & -14 & -26 \\ 2 & -22 & 31 \\ -5 & -7 & -12 \end{pmatrix} - \begin{pmatrix} -5 & -40 & 0 \\ 15 & -5 & 70 \\ -5 & -20 & 5 \end{pmatrix} + \begin{pmatrix} 13 & -26 & 26 \\ 13 & 26 & 39 \\ -0 & -13 & 26 \end{pmatrix} - \begin{pmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = O$$

$\therefore \quad A^3 - 5A^2 + 13A - 9I = O. \qquad ... (1)$

This verifies Cayley - Hamilton Theorem.

To find A^{-1}:

Multiplying (1) with A^{-1}, we get

$$A^2 - 5A + 13I - 9A^{-1} = 0$$

$$\Rightarrow A^{-1} = \frac{1}{9}(A^2 - 5A + 13I)$$

$$\therefore A^{-1} = \frac{1}{9}\begin{pmatrix} 7 & 2 & -10 \\ -2 & 2 & -1 \\ -1 & 1 & 4 \end{pmatrix}$$

Example 13 : Verify Cayley - Hamilton theorem and find the inverse of $\begin{bmatrix} 1 & 0 & 3 \\ 2 & -1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$.

[JNTU (H) June 2010 (Set No. 1)]

Solution : Let $A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & -1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$

The characteristic equation of A is $|A - \lambda I| = 0$

$$i.e., \begin{vmatrix} 1-\lambda & 0 & 3 \\ 2 & -1-\lambda & -1 \\ 1 & -1 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)(\lambda^2 - 1 - 1) + 3(-2 + 1 + \lambda) = 0$$

$$\Rightarrow (1-\lambda)(\lambda^2 - 2) + 3(\lambda - 1) = 0$$

$$\Rightarrow (1-\lambda)(\lambda^2 - 5) = 0$$

$$\Rightarrow \lambda^3 - \lambda^2 - 5\lambda + 5 = 0$$

By Cayley-Hamilton theorem, we must have $A^3 - A^2 - 5A + 5I = O$

Verification: $A^2 = \begin{bmatrix} 1 & 0 & 3 \\ 2 & -1 & -1 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 3 \\ 2 & -1 & -1 \\ 1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -3 & 6 \\ -1 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}$

$$A^3 = A^2.A = \begin{bmatrix} 4 & -3 & 6 \\ -1 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}\begin{bmatrix} 1 & 0 & 3 \\ 2 & -1 & -1 \\ 1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -3 & 21 \\ 9 & -8 & 1 \\ 5 & -5 & 5 \end{bmatrix}$$

Now $A^3 - A^2 - 5A + 5I$

$$= \begin{bmatrix} 4 & -3 & 21 \\ 9 & -8 & 1 \\ 5 & -5 & 5 \end{bmatrix} - \begin{bmatrix} 4 & -3 & 6 \\ -1 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix} - \begin{bmatrix} 5 & 0 & 15 \\ 10 & -5 & -5 \\ 5 & -5 & 5 \end{bmatrix} + \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

$\therefore$ Cayley-Hamilton theorem is verifed.

To find A^{-1}: We have $A^3 - A^2 - 5A + 5I = O$

Multiplying with A^{-1}, we get $A^2 - A - 5 + 5A^{-1} = O$

$\Rightarrow \quad 5A^{-1} = -A^2 + A + 5I$

$\Rightarrow \quad A^{-1} = \dfrac{1}{5}(-A^2 + A + 5I)$

$$\therefore \quad A^{-1} = \frac{1}{5}\left\{\begin{bmatrix} -4 & 3 & -6 \\ 1 & -2 & -6 \\ 0 & 0 & -5 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 3 \\ 2 & -1 & -1 \\ 1 & -1 & 1 \end{bmatrix} + \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}\right\}$$

$$= \frac{1}{5}\begin{bmatrix} 2 & 3 & -3 \\ 3 & 2 & -7 \\ 1 & -1 & 1 \end{bmatrix}$$

Example 14 : Verify Cayley Hamilton theorem and find the inverse of $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$.

[JNTU (H) Jan. 2012 (Set No. 3)]

(or) Find the characteristic equation of the matrix $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$ and hence find its inverse. Use Cayley-Hamilton theorem. **[JNTU (A) Dec. 2016]**

Solution : Given matrix is $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$

Characteristic equation of A is $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 1 & 3 \\ 1 & 3-\lambda & -3 \\ -2 & -4 & -4-\lambda \end{vmatrix} = 0$$

$\Rightarrow (1-\lambda)[(3-\lambda)(-4-\lambda) - 12] - 1[(-4-\lambda) - 6] + 3[-4 + 6 - 2\lambda] = 0$

$\Rightarrow (1-\lambda)(\lambda^2 + \lambda - 24) - 1(-\lambda - 10) + 3(-2\lambda + 2) = 0$

$\Rightarrow \lambda^2 + \lambda - 24 - \lambda^3 - \lambda^2 + 24\lambda + \lambda + 10 - 6\lambda + 6 = 0 \Rightarrow -\lambda^3 + 20\lambda - 8 = 0$.

By Cayley - Hamilton, we must have $-A^3 + 20A + 8I = O$.

Verification :

$$\text{Now } A^2 = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}\begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix} = \begin{bmatrix} -4 & -8 & -12 \\ 10 & 22 & 6 \\ 2 & 2 & 22 \end{bmatrix}$$

$$\text{and } A^3 = \begin{bmatrix} -4 & -8 & -12 \\ 10 & 22 & 6 \\ 2 & 2 & 22 \end{bmatrix}\begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix} = \begin{bmatrix} 12 & 20 & 60 \\ 20 & 52 & -60 \\ -40 & -80 & -88 \end{bmatrix}$$

$$\therefore\ -A^3 + 20A - 16I = \begin{bmatrix} -12 & -20 & -60 \\ -20 & -52 & 60 \\ 40 & 80 & 88 \end{bmatrix} + \begin{bmatrix} 20 & 20 & 60 \\ 20 & 60 & -60 \\ -40 & -80 & -80 \end{bmatrix} - \begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O$$

Hence Cayley - Hamilton theorem is verified.

Example 15 : Verify Cayley-Hamilton Theorem and find A^{-1} for $A = \begin{bmatrix} 3 & 0 & 0 \\ 5 & 4 & 0 \\ 3 & 6 & 1 \end{bmatrix}$

[JNTU (H) Dec. 2012]

Solution : We have $A = \begin{bmatrix} 3 & 0 & 0 \\ 5 & 4 & 0 \\ 3 & 6 & 1 \end{bmatrix}$

$\therefore$ The characteristic equation of A is $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 3-\lambda & 0 & 0 \\ 5 & 4-\lambda & 0 \\ 3 & 6 & 1-\lambda \end{vmatrix} = 0$$

$\Rightarrow (3-\lambda)\,[(4-\lambda)(1-\lambda)] = 0$

$\therefore\ \lambda = 3, 4, 1$ are the roots.

$\Rightarrow (3-\lambda)(\lambda^2 - 5\lambda + 4) = 0$

$\Rightarrow -\lambda^3 + 5\lambda^2 - 4\lambda + 3\lambda^2 - 15\lambda + 12 = 0$

$\Rightarrow -\lambda^3 + 8\lambda^2 - 19\lambda + 12 = 0$

By Cayley- Hamilton theorem, $-A^3 + 8A^2 - 19A + 12\,I = O$

Verification :

$$A^2 = \begin{bmatrix} 3 & 0 & 0 \\ 5 & 4 & 0 \\ 3 & 6 & 1 \end{bmatrix}\begin{bmatrix} 3 & 0 & 0 \\ 5 & 4 & 0 \\ 3 & 6 & 1 \end{bmatrix} = \begin{bmatrix} 9 & 0 & 0 \\ 35 & 16 & 0 \\ 42 & 30 & 1 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 9 & 0 & 0 \\ 35 & 16 & 0 \\ 42 & 30 & 1 \end{bmatrix}\begin{bmatrix} 3 & 0 & 0 \\ 5 & 4 & 0 \\ 3 & 6 & 1 \end{bmatrix} = \begin{bmatrix} 27 & 0 & 0 \\ 185 & 64 & 0 \\ 279 & 126 & 1 \end{bmatrix}$$

$$-A^3 + 8A^2 - 19A + 12\,I = \begin{bmatrix} -27 & 0 & 0 \\ -185 & -64 & 0 \\ -279 & -116 & -1 \end{bmatrix} + \begin{bmatrix} 72 & 0 & 0 \\ 280 & 128 & 0 \\ 336 & 240 & 8 \end{bmatrix} - \begin{bmatrix} 57 & 0 & 0 \\ 95 & 76 & 0 \\ 57 & -114 & 19 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O$$

Thus Cayley - Hamilton theorem is verified.

To find A^{-1}

We have $-A^3 + 8A^2 - 19A + 12\,I = O$

Multiplying with A^{-1}, we have $-A^2 + 8A - 19I + 12A^{-1} = 0$

$$\Rightarrow A^{-1} = \frac{1}{12}(A^2 - 8A + 19\,I)$$

$$= \frac{1}{12}\left[\begin{bmatrix} 9 & 0 & 0 \\ 35 & 16 & 0 \\ 42 & 30 & 1 \end{bmatrix} - \begin{bmatrix} 24 & 0 & 0 \\ 40 & 32 & 0 \\ 24 & 48 & 8 \end{bmatrix} + \begin{bmatrix} 19 & 0 & 0 \\ 0 & 19 & 0 \\ 0 & 0 & 19 \end{bmatrix}\right]$$

$$= \frac{1}{12}\begin{bmatrix} 4 & 0 & 0 \\ -5 & 3 & 0 \\ 18 & -18 & 12 \end{bmatrix}$$

Example 16 : Verify the Cayley-Hamilton theorem and find the characteristic roots, where $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$ **[JNTU (A)June 2009 (Set No.4)]**

Solution : Characteristic equation of A is $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 2 & 2 \\ 2 & 1-\lambda & 2 \\ 2 & 2 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} -1-\lambda & 0 & 1+\lambda \\ 2 & 1-\lambda & 2 \\ 2 & 2 & 1-\lambda \end{vmatrix} = 0 \text{ (Applying } R_1 - R_3)$$

$$\Rightarrow (\lambda+1)\begin{vmatrix} -1 & 0 & 1 \\ 2 & 1-\lambda & 2 \\ 2 & 2 & 1-\lambda \end{vmatrix} = 0 \text{ [Taking } (1+\lambda) \text{ common from } R_1]$$

$$\Rightarrow (1+\lambda)\begin{vmatrix} -1 & 0 & 0 \\ 2 & 1-\lambda & 4 \\ 2 & 2 & 3-\lambda \end{vmatrix} = 0 \text{ (Applying } C_3 + C_1)$$

$$\Rightarrow (1+\lambda)(-1)[(1-\lambda)(3-\lambda)-8] = 0$$

$$\Rightarrow (1+\lambda)(-\lambda^2+4\lambda+5) = 0 \Rightarrow -\lambda^3+3\lambda^2+9\lambda+5 = 0$$

$$\Rightarrow 1+\lambda = 0; -\lambda^2+4\lambda+5 = 0$$

$\Rightarrow \lambda = -1; 5; -1$ are charecteristic roots.

$\therefore$ Characteristic equation of A is $\lambda^3 - 3\lambda^2 - 9\lambda - 5 = 0$

Verification : By Cayley-Hamilton theorem, we must have $A^3 - 3A^2 - 9A - 5I = O$

L. H. S. $= A^3 - 3A^2 - 9A - 5I$

$$= \begin{bmatrix} 41 & 42 & 42 \\ 42 & 41 & 42 \\ 42 & 42 & 41 \end{bmatrix} - \begin{bmatrix} 27 & 24 & 24 \\ 24 & 27 & 24 \\ 24 & 24 & 27 \end{bmatrix} - \begin{bmatrix} 9 & 18 & 18 \\ 18 & 9 & 18 \\ 18 & 18 & 9 \end{bmatrix} - \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 41-27-9-5 & 42-24-18+0 & 42-24-18-0 \\ 42-24-18-0 & 41-27-9-5 & 42-24-18-0 \\ 42-24-18-0 & 42-24-18-0 & 41-27-9-5 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O = \text{R. H. S.}$$

Hence the Cayley - Hamilton theorem is satisfied.

Example 17 : Verify Cayley-Hamilton Theorem for the matrix $A = \begin{bmatrix} 3 & 4 & 1 \\ 2 & 1 & 6 \\ -1 & 4 & 7 \end{bmatrix}$. Hence find A^{-1} **[JNTU (K) June 2009 (Set No.1)]**

Solution : Let $A = \begin{pmatrix} 3 & 4 & 1 \\ 2 & 1 & 6 \\ -1 & 4 & 7 \end{pmatrix}$

Characteristic equation of A is $|A - \lambda I| = 0$.

$$\Rightarrow \begin{vmatrix} 3-\lambda & 4 & 1 \\ 2 & 1-\lambda & 6 \\ -1 & 4 & 7-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 4-\lambda & 0 & -6+\lambda \\ 2 & 1-\lambda & 6 \\ -1 & 4 & 7-\lambda \end{vmatrix} = 0 \text{ (Applying } R_1 - R_3)$$

$$\Rightarrow \lambda^3 - 11\lambda^2 + 122 = 0$$

Cayley - Hamilton Theorem says "every square matrix satisfies its characteristic equation".

Now we have to verify that $A^3 - 11A^2 + 122I = O$.

$$A^2 = \begin{pmatrix} 3 & 4 & 1 \\ 2 & 1 & 6 \\ -1 & 4 & 7 \end{pmatrix}\begin{pmatrix} 3 & 4 & 1 \\ 2 & 1 & 6 \\ -1 & 4 & 7 \end{pmatrix}$$

$$= \begin{pmatrix} 9+8-1 & 12+4+4 & 3+24+7 \\ 6+2-6 & 8+1+24 & 2+6+42 \\ -3+8-7 & -4+4+28 & -1+24+49 \end{pmatrix} = \begin{pmatrix} 16 & 20 & 34 \\ 2 & 33 & 50 \\ -2 & 28 & 72 \end{pmatrix}$$

$$A^3 = A^2 \cdot A = \begin{pmatrix} 16 & 20 & 34 \\ 2 & 33 & 50 \\ -2 & 28 & 72 \end{pmatrix}\begin{pmatrix} 3 & 4 & 1 \\ 2 & 1 & 6 \\ -1 & 4 & 7 \end{pmatrix}$$

$$= \begin{pmatrix} 48+40-34 & 64+20+136 & 16+120+238 \\ 6+66-50 & 8+33+200 & 2+198+350 \\ -6+56-72 & -8+28+288 & -2+168+504 \end{pmatrix} = \begin{pmatrix} 54 & 220 & 374 \\ 22 & 241 & 550 \\ -22 & 308 & 670 \end{pmatrix}$$

Now $A^3 - 11A^2 + 122I$

$$= \begin{pmatrix} 54 & 220 & 374 \\ 22 & 241 & 550 \\ -22 & 308 & 670 \end{pmatrix} - \begin{pmatrix} 176 & 220 & 374 \\ 22 & 363 & 550 \\ -22 & 308 & 792 \end{pmatrix} + \begin{pmatrix} 122 & 0 & 0 \\ 0 & 122 & 0 \\ 0 & 0 & 122 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = O$$

$\therefore$ Cayley-Hamilton theorem is verified.

Take $A^3 - 11A^2 + 122I = O$

Multiplying with A^{-1}, we get

$$\Rightarrow A^{-1} = \frac{-A^2 + 11A}{122} = \frac{1}{122}\begin{pmatrix} -17 & -24 & 23 \\ -20 & 22 & -16 \\ 9 & -16 & -5 \end{pmatrix}$$

Example 18 : For the matrix $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 3 & -1 & 1 \end{bmatrix}$ find A^{-1} by using Cayley-Hamilton theorem? **[JNTU (K) Feb. 2011 (Set No. 2)]**

Solution : $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 3 & -1 & 1 \end{bmatrix}$

The characteristic equation of A is

$$\begin{vmatrix} 1-\lambda & 2 & 1 \\ 0 & 1-\lambda & -1 \\ 3 & -1 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)[(1-\lambda)^2 - 1] - 2[0+3] + 1[0-3(1-\lambda)] = 0$$

$$\Rightarrow (1-\lambda)^3 - (1-\lambda) - 6 - 3(1-\lambda) = 0$$

$$\Rightarrow (1-\lambda)^3 - 4(1-\lambda) - 6 = 0$$

$$\Rightarrow 1 - 3\lambda + 3\lambda^2 - \lambda^3 - 4 + 4\lambda - 6 = 0$$

$$\Rightarrow -\lambda^3 + 3\lambda^2 + \lambda - 9 = 0$$

$$\Rightarrow \quad \lambda^3 - 3\lambda^2 - \lambda + 9 = 0$$

By Cayley-Hamilton theorem, A satisfies the characteristic equation

$$\therefore \quad A^3 - 3A^2 - A + 9I = O \qquad \text{... (1)}$$

Multiplying with A^{-1}, we get

$$A^2 - 3A - I + 9A^{-1} = O$$

Now $A^2 = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 3 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 3 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 3 & 0 \\ -3 & 2 & -2 \\ 6 & 4 & 5 \end{bmatrix}$

From (1), $A^{-1} = \dfrac{-A^2 + 3A + I}{9}$

$$= \frac{1}{9}\left\{ -\begin{bmatrix} 4 & 3 & 0 \\ -3 & 2 & -2 \\ 6 & 4 & 5 \end{bmatrix} + \begin{bmatrix} 3 & 6 & 3 \\ 0 & 3 & -3 \\ 9 & -3 & 3 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\} = \frac{1}{9}\begin{bmatrix} 0 & 3 & 3 \\ 3 & 2 & -1 \\ 3 & -7 & -1 \end{bmatrix}$$

Example 19 : State and prove Cayley-Hamilton theorem and Verify Cayley-Hamilton theorem for

$A = \begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$ and hence find A^{-1}? **[JNTU (K) Feb. 2011 (Set No. 3)]**

Solution : For theorem refer Theorem 2.12

Given matrix is $A = \begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

Characteristic equation of A is $\begin{vmatrix} 2-\lambda & -1 & 2 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{vmatrix} = 0$

$\Rightarrow \quad (2-\lambda)\left[(2-\lambda)^2-1\right]+1[(\lambda-2)+1]+2[1-2+\lambda]=0$

$\Rightarrow \quad (2-\lambda)(\lambda^2-4\lambda+3)+(\lambda-1)+2[1-2+\lambda]=0$

$\Rightarrow \quad 2\lambda^2-8\lambda+6-\lambda^3+4\lambda-3\lambda+3\lambda-3=0$

$\Rightarrow \quad -\lambda^3+6\lambda^2-8\lambda+3=0$

By Cayley-Hamilton theorem, we must have,

$-A^3+6A^2-8A+3I=O \qquad \dots (1)$

Verification :

$$A^2 = \begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{bmatrix}\begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 29 & -28 & 38 \\ -22 & 23 & -28 \\ 22 & -22 & 29 \end{bmatrix}$$

Substituting in LHS of (1), $-A^3+6A^2-8A+3I$

$$= -\begin{bmatrix} 29 & -28 & 38 \\ -22 & 23 & -28 \\ 22 & -22 & 29 \end{bmatrix} + 6\begin{bmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{bmatrix} - 8\begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} + 3\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O$$

$\therefore$ Cayley-Hamilton Theorem is verified.

Multiplying with A^{-1}, we get $-A^2+6A-8I+3A^{-1}=O$

$\Rightarrow A^{-1} = A^2-6A+8I$

$$= \begin{bmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{bmatrix} - \begin{bmatrix} 7 & -6 & 12 \\ -6 & 12 & -6 \\ 6 & -6 & 12 \end{bmatrix} + \begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix} = \begin{bmatrix} 8 & 0 & -3 \\ 1 & 2 & 0 \\ -1 & 1 & -3 \end{bmatrix}$$

Example 20 : Verify Cayley-Hamilton theorem and find A^{-1} for $A=\begin{bmatrix}-1 & -2 & 0\\ 1 & 0 & 2\\ 2 & 3 & 4\end{bmatrix}$

[JNTU (K) June 2011 (Set No. 1)]

Solution : Given matrix is $A=\begin{bmatrix}-1 & -2 & 0\\ 1 & 0 & 2\\ 2 & 3 & 4\end{bmatrix}$

Characteristic equation of A is $|A-\lambda I|=0$

i.e., $\begin{vmatrix}-1-\lambda & -2 & 0\\ 1 & -\lambda & 2\\ 2 & 3 & 4-\lambda\end{vmatrix}=0$

$\Rightarrow\ (-1-\lambda)[(-\lambda)(4-\lambda)-6]+2[(4-\lambda)-4]=0$

$\Rightarrow\ (-1-\lambda)(-4\lambda+\lambda^2-6)-2\lambda=0$

$\Rightarrow\ 4\lambda-\lambda^2+6+4\lambda^2-\lambda^3+6\lambda-2\lambda=0$

$\Rightarrow\ -\lambda^3+3\lambda^2+8\lambda+6=0$ ... (1)

By Cayley-Hamilton theorem, the characteristic equation is satisfied by A

$\therefore\ -A^3+3A^2+8A+6I=O$ ... (2)

Now $A^2=\begin{bmatrix}-1 & -2 & 0\\ 1 & 0 & 2\\ 2 & 3 & 4\end{bmatrix}\begin{bmatrix}-1 & -2 & 0\\ 1 & 0 & 2\\ 2 & 3 & 4\end{bmatrix}=\begin{bmatrix}-1 & 2 & -4\\ 3 & 4 & 8\\ 9 & 8 & 22\end{bmatrix}$

$A^3=\begin{bmatrix}-1 & 2 & -4\\ 3 & 4 & 8\\ 9 & 8 & 22\end{bmatrix}\begin{bmatrix}-1 & -2 & 0\\ 1 & 0 & 2\\ 2 & 3 & 4\end{bmatrix}=\begin{bmatrix}-5 & -10 & -12\\ 17 & 18 & 40\\ 43 & 48 & 104\end{bmatrix}$

Verification of Cayley-Hamilton Theorem :

$-A^3+3A^2+8A+6I=\begin{bmatrix}5 & 10 & 12\\ -17 & -18 & -40\\ -43 & -48 & -104\end{bmatrix}+\begin{bmatrix}-3 & 6 & -12\\ 9 & 12 & 24\\ 27 & 24 & 66\end{bmatrix}$

$+\begin{bmatrix}-8 & -16 & 0\\ 8 & 0 & 16\\ 16 & 24 & 32\end{bmatrix}+\begin{bmatrix}6 & 0 & 0\\ 0 & 6 & 0\\ 0 & 0 & 6\end{bmatrix}=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}=O$

$\therefore$ Cayley-Hamilton theorem is verified.

Multiplying (2) with A^{-1}, we get

$$-A^2+3A+8I+6A^{-1}=O \Rightarrow A^{-1}=\frac{1}{6}(A^2-3A-8I)$$

$$\therefore \; A^{-1}=\frac{1}{6}\left[\begin{bmatrix}-1&2&-4\\3&4&8\\9&8&22\end{bmatrix}+\begin{bmatrix}3&6&0\\-3&0&-6\\-6&9&-12\end{bmatrix}+\begin{bmatrix}-8&0&0\\0&-8&0\\0&0&-8\end{bmatrix}\right]=\frac{1}{6}\begin{bmatrix}-6&8&-4\\0&-4&2\\3&-1&2\end{bmatrix}$$

Verification :

$$AA^{-1}=\frac{1}{6}\begin{bmatrix}-1&-2&0\\1&0&2\\2&3&4\end{bmatrix}\begin{bmatrix}-6&8&-4\\0&-4&2\\3&-1&2\end{bmatrix}=\frac{1}{6}\begin{bmatrix}6&0&0\\0&6&0\\0&0&6\end{bmatrix}=I_3$$

Example 21: Verify Cayley-Hamilton theorem for $A=\begin{bmatrix}1&4\\2&3\end{bmatrix}$ and hence find A^{-1} and find $B=A^5-4A^4-7A^3+11A^2-A-10I$? **[JNTU (H) Aug. 2017 (R15)]**

Solution : Given $A=\begin{bmatrix}1&4\\2&3\end{bmatrix}$.

Characteristic equation of A is $\begin{vmatrix}1-\lambda&4\\2&3-\lambda\end{vmatrix}=0 \Rightarrow (1-\lambda)(3-\lambda)-8=0$

$\Rightarrow \lambda^2-4\lambda-5=O$

By Cayley-Hamilton theorem, A will satisfy its characteristic equation.

$\therefore A^2-4A-5I=0$ ... (1)

Now $A^2=\begin{bmatrix}1&4\\2&3\end{bmatrix}\begin{bmatrix}1&4\\2&3\end{bmatrix}=\begin{bmatrix}9&16\\8&17\end{bmatrix}$

$$A^2-4A-5I=\begin{bmatrix}9&16\\8&17\end{bmatrix}-4\begin{bmatrix}1&4\\2&3\end{bmatrix}-5\begin{bmatrix}1&0\\0&1\end{bmatrix}=\begin{bmatrix}0&0\\0&0\end{bmatrix}$$

$\therefore$ Cayley-Hamilton theorem is verified.

Hence multiplying (1) with A^{-1}, we get

$$A-4I-5A^{-1}=O$$

$$\Rightarrow \quad A^{-1}=\frac{A-4I}{5}=\frac{1}{5}\left\{\begin{bmatrix}1&4\\2&3\end{bmatrix}-\begin{bmatrix}4&0\\0&4\end{bmatrix}\right\}=\frac{1}{5}\begin{bmatrix}-3&4\\2&-1\end{bmatrix}$$

Now B $= A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I$

$= A^3(A^2 - 4A - 5I) - 2A^3 + 11A^2 - A - 10I$

$= -2A^3 + 11A^2 - A - 10I$, using (1)

$= -2A(A^2 - 4A - 5I) + 3A^2 - 11A - 10I$

$= 3(A^2 - 4A - 5I) + A - 5I$, using (1)

$=$ A $- 5I$, using (1)

$$= \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -4 & 4 \\ 2 & -2 \end{bmatrix}$$

REVIEW QUESTIONS

1. State Cayley-Hamilton Theorem. **[JNTU (A) Dec. 2016 (R13)]**
   (or) Write the statement of Cayley-Hamilton theorem.
2. State and prove Cayley - Hamilton Theorem.

EXERCISE 2.2

1. Verify Cayley-Hamilton theorem for A $= \begin{bmatrix} 3 & 2 \\ 1 & 5 \end{bmatrix}$
2. Show that the matrix $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$ satisfies Cayley-Hamilton theorem.
3. Show that $A = \begin{bmatrix} 1 & 1 & 2 \\ 3 & 1 & 1 \\ 2 & 3 & 1 \end{bmatrix}$ satisfies the characteristic equation. Hence find A^{-1}.
4. Show that $\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$ satisfies its characteristic equation. Hence find A^{-1}.
   [JNTU 2000S, 2002]
   (or) Using Cayley-Hamilton theorem find the inverse of $\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$
   [Hint : $-A^3 + 6A^2 - 9A + 4I = O$] **[JNTU (H) June 2014]**
5. Find the inverse of the matrix $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix}$ using Cayley-Hamilton theorem.
6. Using Cayley-Hamilton theorem find the inverse of A $= \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$ and also find A^{-3}.
   [JNTU (A) June 2014]
7. Using Cayley-Hamilton theorem, find the inverse of $A = \begin{bmatrix} 7 & -1 & 3 \\ 6 & 1 & 4 \\ 2 & 4 & 8 \end{bmatrix}$
   [JNTU (H) June 2015]