VECTOR CALUCULUS

Vector differentiation :-

Vector :-

A vector is a quantity which has both magnitude and direction

Scalar :-

A Scalar is a quantity which has only magnitude but their is no direction

Unit Vector :-

A vector of magnitude one then it is called Unit vector.

if $\bar{a} = a_1\bar{i} + a_2\bar{j} + a_3\bar{k}$ is a vector then the unit vector is denoted by "$\hat{a}$" and it is defined

$$\hat{a} = \frac{\bar{a}}{|\bar{a}|} = \frac{a_1\bar{i} + a_2\bar{j} + a_3\bar{k}}{\sqrt{a_1^2 + a_2^2 + a_3^2}}$$

where, $|\bar{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$

→ If $\bar{a}$ & $\bar{b}$ are any two vector then $\bar{a}\cdot\bar{b} = |\bar{a}||\bar{b}|\cos\theta$

position Vector :-

Let 'O' be a fixed point in the space whose quadrants are (0,0,0) and 'P' be any point in the space whose quadrants are (x, y, z) then '$\bar{OP}$' is called the position vector of P with reference to the origin.

It is simply denoted by

$$\boxed{\bar{OP} = \bar{x}i + \bar{y}j + \bar{z}k}$$

→ If $\bar{a}$ and $\bar{b}$ are any two vector then

$$\bar{a}\cdot\bar{b} = \bar{b}\cdot\bar{a}$$

→ If $\bar{a} = a_1 i + a_2 j + a_3 k$

$\bar{b} = b_1 i + b_2 j + b_3 k$

$$\boxed{\bar{a}\cdot\bar{b} = a_1b_1 + a_2b_2 + a_3b_3}$$

→ If $\bar{a}$ and $\bar{b}$ are two vector then $\bar{a}\times\bar{b} = |\bar{a}||\bar{b}|\sin\theta\,\hat{n}$

where 'θ' is the angle b/w the two vectors and '$\hat{n}$' is the normal unit vector to the plane.

→ $\bar{a}$ and $\bar{b}$ are any two vector

$$\boxed{\bar{a}\times\bar{b} = -(\bar{b}\times\bar{a})}$$

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→ If $\bar{a} = a_1 i + a_2 j + a_3 k$

$\bar{b} = b_1 i + b_2 j + b_3 k$

Then $\bar{a} \times \bar{b} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

→ If the two vectors $\bar{a}$ and $\bar{b}$ are perpendicular then $\bar{a} \cdot \bar{b} = 0$

→ If $\bar{a} = a_1 i + a_2 j + a_3 k$

then, $\bar{a} \cdot \bar{a} = a_1^2 + a_2^2 + a_3^2 = |\bar{a}|^2$

→ If $\bar{a} = a_1 i + a_2 j + a_3 k$

$\bar{b} = b_1 i + b_2 j + b_3 k$

$\bar{c} = c_1 i + c_2 j + c_3 k$

→ Then scalar triple product

$$[\bar{a}\ \bar{b}\ \bar{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

→ The $[\bar{a}\ \bar{b}\ \bar{c}] = \bar{a} \cdot (\bar{b} \times \bar{c})$

$[\bar{a}\ \bar{b}\ \bar{c}] = [\bar{b}\ \bar{c}\ \bar{a}] = [\bar{c}\ \bar{a}\ \bar{b}]$

$[\bar{a}\ \bar{a}\ \bar{c}] = 0$

In a determinant two rows are identical it's value is '0'.

→ If $\bar{a}, \bar{b}, \bar{c}$ are any three vectors

$\bar{a} \times (\bar{b} \times \bar{c}) = (\bar{a} \cdot \bar{c})\bar{b} - (\bar{a} \cdot \bar{b})\bar{c}$

→ If the two vectors $\bar{a}$ and $\bar{b}$ are parallel than $\bar{a} \times \bar{b} = 0$

→ If i, j, k be mutually perpendicular unit along the co-ordinate axis

* $i \cdot i = j \cdot j = k \cdot k = 1$
* $i \cdot j = j \cdot k = k \cdot i = 0$
* $i \times i = j \times j = k \times k = 0$
* $i \times j = k$, $j \times k = i$, $k \times i = j$

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Point Function :-

A variable quantity whose values at any point in a space depends upon the position of the point is called the "Point Function".

There are two types of point function :-

* Scalar point function.
* Vector point function.

Scalar Point function :-

Let R be the region of the space at each point of which a scalar $\phi = \phi(x, y, z)$ is given then 'ϕ' is called a "Scalar Point function".

Vector point function :-

Let R be the region of the space at each point of which a vector $\bar{v} = v(x, y, z)$ is given then '$\bar{v}$' is called a "Vector Point function".

Gradient of a scalar function

(i) Vector differential operator :-

The vector differential operator is denoted by "∇". We read it as Del (or) Grad. And it is defined as

$$\boxed{\nabla = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}}$$

(ii) Gradient of a scalar function :-

Let $\phi = \phi(x, y, z)$ be a function in the scalar field. Then the gradient of scalar function is defined as

$$\boxed{\nabla\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\cdot\frac{\partial\phi}{\partial z}}$$

'ϕ' is a scalar point function then grad ϕ is a vector point function.

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Problems

1. Find the Gradient of the scalar point function where $\phi = x^3 + y^3 + z^3 + 3xyz$

Given,

the Scalar function is

$$\phi = x^3 + y^3 + z^3 + 3xyz$$

$$\nabla\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$$

$$\frac{\partial\phi}{\partial x} = 3x^2 + 3yz \quad ; \quad \frac{\partial\phi}{\partial y} = 3y^2 + 3xz \quad ; \quad \frac{\partial\phi}{\partial y} = 3z^2 + 3xy$$

$$\nabla\phi = (3x^2 + 3yz)\,i + (3y^2 + 3xz)\,j + (3z^2 + 3xy)\,k$$

2. Find gradϕ where $\phi = xyz^2 + xy^2z + x^2yz$ at $(1,1,1)$

Given, Scalar function is

$$\phi = xyz^2 + xy^2z + x^2yz$$

$$\nabla\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$$

$$\frac{\partial\phi}{\partial x} = yz^2 + y^2z + 2xyz$$

$$\frac{\partial\phi}{\partial y} = xz^2 + 2xyz + x^2z$$

$$\frac{\partial\phi}{\partial z} = 2xyz + xy^2 + x^2y$$

$$\nabla\phi = (yz^2 + y^2z + 2xyz)\,i + (xz^2 + 2xyz + x^2z)\,j + (2xyz + xy^2 + x^2y)\,k$$

$$(\nabla\phi)_{(1,1,1)} = 4i + 4j + 4k$$

$$\therefore (\nabla\phi)_{(1,1,1)} = 4i + 4j + 4k$$

3. Find '$\nabla\phi$' where $\phi = x^2y + xy^2 + z^2$ at the point $(1,1,-2)$

Given, scalar function is

$$\phi = x^2y + xy^2 + z^2$$

$$\frac{\partial\phi}{\partial x} = 2xy + y^2 \quad \frac{\partial\phi}{\partial y} = x^2 + 2xy \quad \frac{\partial\phi}{\partial z} = 2z$$

$$\nabla\phi = (2xy + y^2)\,i + (x^2 + 2xy)\,j + (2z)\,k$$

$$(\nabla\phi)_{(1,1,-2)} = 3i + 3j - 4k$$

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4. If $a = x+y+z$, $b = x^2+y^2+z^2$, $c = xy+yz+zx$ then S.T $[\nabla a \ \nabla b \ \nabla c] = 0$

$$a = x+y+z$$

$$\nabla a = i\cdot\frac{\partial a}{\partial x} + j\frac{\partial a}{\partial y} + k\frac{\partial a}{\partial z}$$

$$\frac{\partial a}{\partial x} = 1 \ ; \ \frac{\partial a}{\partial y} = 1 \ ; \ \frac{\partial a}{\partial z} = 1$$

$$\nabla a = i + j + k$$

$$b = x^2+y^2+z^2$$

$$\nabla b = i\cdot\frac{\partial b}{\partial x} + j\frac{\partial b}{\partial y} + k\frac{\partial b}{\partial z}$$

$$\frac{\partial b}{\partial x} = 2x \ ; \ \frac{\partial b}{\partial y} = 2y \ ; \ \frac{\partial b}{\partial z} = 2z$$

$$\nabla b = 2xi + 2yj + 2zk.$$

$$c = xy+yz+zx$$

$$\nabla c = i\cdot\frac{\partial c}{\partial x} + j\frac{\partial c}{\partial y} + k\frac{\partial c}{\partial z}$$

$$\frac{\partial c}{\partial x} = y+z \quad \frac{\partial c}{\partial y} = x+z \quad \frac{\partial c}{\partial z} = y+x$$

$$\nabla c = (y+z)i + (x+z)j + (x+y)k$$

$$[\nabla a \ \nabla b \ \nabla c] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ 2x & 2y & 2z \\ y+z & x+z & x+y \end{vmatrix}$$

$$= 1[(2xy+2y^2)-(2xz+2z^2)]$$

$$- 1[(2x^2+2xy)-[2yz+2z^2)] + 1[(2x^2+2xz)-[2y^2+2yz]]$$

$$= 2xy+2y^2-2xz-2z^2-[2x^2+2xy-2yz+2z^2]+2x^2+2xz-2y^2-2yz$$

$$= \cancel{2xy}+\cancel{2y^2}-\cancel{2xz}-\cancel{2z^2}-\cancel{2x^2}+\cancel{2xy}+\cancel{2yz}+\cancel{2z^2}+\cancel{2x^2}+\cancel{2xz}-\cancel{2y^2}-\cancel{2yz}$$

$$= 0$$

5. If $\bar{r} = xi + yj + zk$ and $r = |\bar{r}| = \sqrt{x^2+y^2+z^2}$ then find the values of ① ∇r ② $\nabla \log r$ ③ $\nabla\left(\frac{1}{r}\right)$ ④ $\nabla(r^n)$ ⑤ $\nabla[f(r)]$

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Given:

$$r = \sqrt{x^2+y^2+z^2} \quad (S.O.B.S)$$

$$r^2 = x^2+y^2+z^2 \rightarrow (1)$$

diff eq (1) w.r.t x

$$2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$$

Similarly $\frac{\partial r}{\partial y} = \frac{y}{r}$

$$\frac{\partial r}{\partial z} = \frac{z}{r}$$

(i) $\nabla r = \frac{\bar{r}}{r}$

w.k.T $\nabla r = i.\frac{\partial r}{\partial x} + j\frac{\partial r}{\partial y} + z\frac{\partial r}{\partial z}$

$$= i\left(\frac{x}{r}\right) + j\left(\frac{y}{r}\right) + k\left(\frac{z}{r}\right)$$

$$= \frac{1}{r}[ix + jy + kz]$$

$$\Rightarrow \frac{1}{r}(\bar{r}) = \frac{\bar{r}}{r}$$

(ii) $\nabla \log r = \frac{\bar{r}}{r^2} = i.\frac{\partial}{\partial x}\log r + j\frac{\partial}{\partial y}\log r + k.\frac{\partial}{\partial z}\log r$

$$= i\left[\frac{1}{r}.\frac{\partial r}{\partial x}\right] + j\left[\frac{1}{r}\frac{\partial r}{\partial y}\right] + k\left[\frac{1}{r}\frac{\partial r}{\partial z}\right]$$

$$= i\left[\frac{1}{r}.\frac{x}{r}\right] + j\left[\frac{1}{r}\frac{y}{r}\right] + k\left[\frac{1}{r}\frac{z}{r}\right]$$

$$= \frac{1}{r^2}[xi + yj + zk]$$

$$= \frac{1}{r^2}(\bar{r})$$

$$= \frac{\bar{r}}{r^2}$$

(iii) $\nabla\left(\frac{1}{r}\right) = i.\frac{\partial}{\partial x}\left(\frac{1}{r}\right) + j\frac{\partial}{\partial y}\left(\frac{1}{r}\right) + k\frac{\partial}{\partial y}\left(\frac{1}{r}\right)$

$\Rightarrow -\frac{\bar{r}}{r^3}$

$$= i\frac{\partial r}{\partial x}\frac{-1}{r^2} + j\frac{\partial r}{\partial y}\frac{-1}{r^2} + k\frac{\partial r}{\partial z}.\frac{-1}{r^2}$$

$$= \left[i\frac{x}{r}\frac{-1}{r^2} + j\frac{y}{r}\frac{-1}{r^2} + k\frac{z}{r}\frac{-1}{r^2}\right]$$

$$= \frac{-1}{r^3}[xi + yj + zk]$$

$$= \left[-\frac{\bar{r}}{r^3}\right] \quad \therefore \nabla\left(\frac{1}{r}\right) = -\frac{\bar{r}}{r^3}$$

(iv) $\nabla(r^n)$

$$= i\frac{\partial}{\partial x}(r^n) + j\frac{\partial}{\partial y}(r^n) + k\frac{\partial}{\partial z}(r^n)$$

$$= i(nr^{n-1})\frac{\partial r}{\partial x} + j(n.r^{n-1})\frac{\partial r}{\partial y} + k\left(n.r^{n-1}\frac{\partial r}{\partial z}\right)$$

$$= i\,n.\frac{r^n}{r}\left[i.\frac{x}{r} + y/r.j + k\frac{z}{r}\right]$$

$$= \frac{n.r^n}{r^2}[xi + yj + kz]$$

$$= \frac{n.r^n}{r^2}.\bar{r} = n.r^{n-2}\bar{r}$$

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(v) $\nabla[f(r)] = i\cdot\frac{\partial}{\partial x}f(r) + j\frac{\partial}{\partial y}f(r) + k\frac{\partial}{\partial z}f(r)$

$= i\, f'(r)\frac{\partial r}{\partial x} + j\, f'(r)\frac{\partial r}{\partial y} + k\frac{\partial r}{\partial z}f'(r)$

$= f'(r)\left[i\cdot\frac{x}{r} + j\frac{y}{r} + k\frac{z}{r}\right]$

$= \frac{f'(r)}{r}[\bar{r}]$

(6) Find the gradient 'f' where $f = x^2yz^3$ at the point $(1,1,-2)$ and also find (i) $|\nabla f|$

Given,

$f = x^2yz^3$

$\nabla f = i\cdot\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z}$

$\frac{\partial f}{\partial x} = 2xyz^3 \;;\; \frac{\partial f}{\partial y} = x^2z^3 \;;\; \frac{\partial f}{\partial z} = 3x^2yz^2$

$\nabla f = (2xyz^3)\,i + (x^2z^3)\,j + (3x^2yz^2)\,k$

$\nabla f = -16i + (-8j) + 12k$

$\nabla f = -16i - 8j + 12k$

$|\nabla f| = \sqrt{(16)^2 + (8)^2 + (12)^2}$

$= \sqrt{256 + 64 + 144}$

$= \sqrt{464}$

$|\nabla f| = \sqrt{464}$

Normal Unit Vector to the surface :

If 'ϕ' is a scalar point function then '$\nabla\phi$' is also called " Normal vector to the surface".

Then the normal unit vector to the surface is given by $\boxed{\frac{\nabla f}{|\nabla f|}}$

Directional Derivative :

The directional derivative of a scalar point function " ϕ" at the point " p" in the direction of the vector $\vec{a}$ is given by $\boxed{\nabla\phi\cdot\frac{\bar{a}}{|\bar{a}|} = \nabla\phi\cdot\hat{a}}$

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① Find the directional derivative of $f = xy + yz + zx$ in the direction of the vector $i + 2j + 2k$ at the point $(1,2,0)$

Given,

$$f = xy + yz + zx$$

$$\nabla f = i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} + k\frac{\partial f}{\partial z}$$

$$\frac{\partial f}{\partial x} = y + z \;;\; \frac{\partial f}{\partial y} = x + z \;;\; \frac{\partial f}{\partial z} = x + y$$

$$\nabla f = (y+z)i + (x+z)j + (x+y)k$$

$$(\nabla f)_{(1,2,0)} = 2i + j + 3k$$

The given vector $\bar{a} = i + 2j + 2k$

$$|\bar{a}| = \sqrt{1+4+4} = \sqrt{9} = 3$$

$$\hat{a} = \frac{i + 2j + 2k}{3}$$

$(\nabla \phi \cdot \hat{a})$ The directional derivative of f in the direction of $\bar{a}$

$$\nabla f \cdot \hat{a} = \nabla f \cdot \frac{\bar{a}}{|\bar{a}|}$$

$$= 2i + j + 3k \cdot \frac{i + 2j + 2k}{3}$$

$$= \frac{1}{3}[2 + 2 + 6]$$

$$\nabla f \cdot \hat{a} = \frac{10}{3}$$

$$\therefore \nabla f \cdot \hat{a} = \frac{10}{3}$$

② Find the directional derivative of $f = 2xy + z^2$ at the point $(1,-1,3)$ in the direction of $i + 2j + 3k$

Given,

$$f = 2xy + z^2$$

$$\nabla f = i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} + k\frac{\partial f}{\partial z}$$

$$\frac{\partial f}{\partial x} = 2y \quad \frac{\partial f}{\partial y} = 2x \quad \frac{\partial f}{\partial z} = 2z$$

$$\nabla f = 2yi + 2xj + 2zk$$

$$(\nabla f)_{(1,-1,3)} = -2i + 2j + 6k$$

The given vector is $= i + 2j + 3k$

$$\bar{a} = i + 2j + 3k$$

$$|\bar{a}| = \sqrt{1+4+9}$$

$$= \sqrt{14}$$

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The directional derivative of f in the direction of line $\bar{a}$ is

$$\nabla f \cdot \hat{a} = \nabla f \cdot \frac{\bar{a}}{|\bar{a}|}$$

$$= -2i + 2j + 6k \cdot \frac{i + 2j + 3k}{\sqrt{14}}$$

$$= \frac{1}{\sqrt{14}}(-2 + 4 + 18)$$

$$\nabla f \cdot \hat{a} = \frac{20}{\sqrt{14}}$$

(3) Find the directional derivative of $f = 4xy^2 + 2x^2yz$ at the point $P(1,2,3)$ in the direction of the line $\overline{PQ}$ where $Q = (5,0,4)$

Given, scalar function is

$$f = 4xy^2 + 2x^2yz$$

$$\nabla f = i \frac{\partial f}{\partial x} + j \frac{\partial f}{\partial y} + k \frac{\partial f}{\partial z}$$

$$\frac{\partial f}{\partial x} = 4y^2 + 4xyz ; \frac{\partial f}{\partial y} = 8xyz + 2x^2z ; \frac{\partial f}{\partial z} = 2x^2y$$

$$\nabla f = (4y^2 + 4xyz)i + (8xy + 2x^2z)j + (2x^2y)k$$

$$[\nabla f]_{(1,2,3)} = 40i + 22j + 4k$$

$$\overline{PQ} = \overline{OQ} - \overline{OP} \quad \text{(Position vector)}$$

$$\overline{PQ} = (5i + 0j + 4k) - (i + 2j + 3k)$$

$$\overline{PQ} = 4i - 2j + k$$

$$\overline{PQ} = a = 4i - 2j + k \Rightarrow |\bar{a}| = \sqrt{21}$$

$$\nabla f \cdot \hat{a} = \nabla f \cdot \frac{\bar{a}}{|\bar{a}|}$$

$$= (40i + 22j + 4k) \cdot \frac{(4i - 2j + k)}{\sqrt{21}}$$

$$= \frac{1}{\sqrt{21}}[160 - 44 + 4]$$

$$= \frac{1}{\sqrt{21}}[164 - 44]$$

$$\nabla f \cdot \hat{a} = \frac{120}{\sqrt{21}}$$

(4) $\phi = xy + yz + zx$ at the point $P(1,2,-1)$ in the direction of the line $\overline{PQ}$ where $Q = (1,2,3)$

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Given,

$$\phi = xy + yz + zx$$

$$\nabla\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\cdot\frac{\partial\phi}{\partial z}$$

$$\frac{\partial\phi}{\partial x} = y+z \;;\; \frac{\partial\phi}{\partial y} = x+z \;;\; \frac{\partial\phi}{\partial z} = x+y$$

$$\nabla\phi = (y+z)i + (x+z)j + (x+y)k$$

$$[\nabla\phi]_{(1,2,-1)} = i + 3k$$

position vector $(\overline{PQ}) = \overline{OQ} - \overline{OP}$

$$P = (1,2,-1)$$

$$Q = (1,2,3)$$

$$= (i + 2j + 3k) - (i + 2j - k)$$

$$\overline{PQ} = \bar{a} = 4k$$

$$|\bar{a}| = \sqrt{16} = 4$$

$$\therefore \nabla\phi\cdot\hat{a} = \nabla\phi\cdot\frac{\bar{a}}{|\bar{a}|}$$

$$= (i + 3k)\cdot\frac{4k}{4}$$

$$= \frac{12}{4} \; 3$$

$$\boxed{\nabla\phi\cdot\hat{a} = 3}$$

5. Find the directional derivative of the function $xyz^2 + xz$ at the point $(1,1,1)$ in the direction of the normal to the surface $3xy^2 + y - z$ at the point $(0,1,1)$

Let us take,

$$\phi = xyz^2 + xz$$

$$\nabla\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$$

$$\frac{\partial\phi}{\partial x} = yz^2 + z \;;\; \frac{\partial\phi}{\partial y} = xz^2 \;;\; \frac{\partial\phi}{\partial z} = 2xyz + x$$

$$\nabla\phi = (yz^2 + z)i + (xz^2)j + (2xyz + x)k$$

$$(\nabla\phi)_{(1,1,1)} = 2i + j + 3k$$

The given second surface is

$$f = 3xy^2 + y - z$$

The normal vector to the surface

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$$\nabla f = i\cdot\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} + k\frac{\partial f}{\partial z}$$

$$\frac{\partial f}{\partial x} = 3y^2 \quad ; \quad \frac{\partial f}{\partial y} = 6xy+1 \quad ; \quad \frac{\partial f}{\partial z} = -1$$

$$\nabla f = (3y^2)\,i + (6xy+1)\,j + (-1)\,k$$

$(\nabla f)_{(0,1,1)} = 3i + j - k$ Let it be take it is $\bar{a}$

$$\bar{a} = 3i + j - k$$

$$|\bar{a}| = \sqrt{9+1+1}$$
$$= \sqrt{11}$$

The directional derivative of the given function in the direction of normal vector to the surface is given by

$$\nabla\phi \cdot \frac{\nabla f}{|\nabla f|} = \nabla\phi \cdot \frac{\bar{a}}{|\bar{a}|}$$
$$= (2i + j + 3k)\,\frac{(3i + j - k)}{\sqrt{11}}$$
$$= \frac{6+1-3}{\sqrt{11}}$$
$$\nabla\phi \cdot \frac{\nabla f}{|\nabla f|} = \frac{4}{\sqrt{11}}$$

6. Find the directional derivative of the function $xy^2 + yz^3$ at the point $(2,-1,1)$ in the direction of normal vector to the surface $x\log z - y^2 + y$ at the point $(-1,2,1)$

Let us take,

$$\phi = xy^2 + yz^3$$

$$\nabla\phi = i\cdot\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$$

$$\frac{\partial\phi}{\partial x} = y^2 \quad ; \quad \frac{\partial\phi}{\partial y} = 2xy + z^3 \quad ; \quad \frac{\partial\phi}{\partial z} = 3yz^2$$

$$\nabla\phi = y^2 i + (2xy + z^3)\,j + (3yz^2)\,k$$

$(\nabla\phi)_{(-1,2,1)} = \cancel{4i - 3j + 6k} \rightarrow (\nabla\phi)_{(2,-1,1)} = i - 3j - 3k$

The given second surface is

$$f = x\log z - y^2 + y$$

The normal vector to the surface

$$f = x\log z - y^2 + y$$

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$$\nabla f = i\cdot\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} + k\frac{\partial f}{\partial z}$$

$$\frac{\partial f}{\partial x} = \log z \quad \Big| \quad \frac{\partial f}{\partial y} = -2y \quad \Big| \quad \frac{\partial f}{\partial z} = \frac{x}{z}$$

$$\nabla f = (\log z)i + (-2y)j + \left(\frac{x}{z}\right)k$$

$$(\nabla f)_{(-1,2,1)} = \log 1\, i + (-2(2))j + \left(\frac{-1}{1}\right)k$$

$$= -4j - k$$

Let it be take as $\bar{a}$

$$\bar{a} = -4j - k$$

$$|\bar{a}| = \sqrt{16+1}$$

$$= \sqrt{17}$$

The directional derivative of the given function in the direction of normal vector to the surface

$$\nabla\phi\cdot\frac{\nabla f}{|\nabla f|} = \nabla\phi\cdot\frac{\bar{a}}{|\bar{a}|}$$

$$= (4i - 3j + 6k)\,\frac{(-4j-k)}{\sqrt{17}} \rightarrow (i-3j-3k)\frac{(-4j+}{\sqrt{17}}$$

$$= \frac{-16-6}{\sqrt{17}} \qquad \nabla\phi\cdot\frac{\nabla f}{|\nabla f|} \rightarrow \frac{15}{\sqrt{17}}$$

$$\nabla\phi\cdot\frac{\nabla f}{|\nabla f|} = \frac{-22}{\sqrt{17}}$$

(7) Find the directional derivative of the function $\phi = xyz$ along the direction normal to the surface $x^2z + xy^2 + yz^3 = 3$ at the point $(1,1,1)$

Given,

$$\phi = xyz$$

$$\nabla\phi = i\cdot\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$$

$$\frac{\partial\phi}{\partial x} = yz \quad \frac{\partial\phi}{\partial y} = xz \quad \frac{\partial\phi}{\partial z} = xy$$

$$\nabla\phi = (yz)i + (xz)j + (xy)k$$

Let $f = x^2z + xy^2 + yz^3 - 3$

$$\nabla f = i\cdot\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} + k\frac{\partial f}{\partial z}$$

$$\frac{\partial f}{\partial x} = 2xz + y^2 \quad \Big| \quad \frac{\partial f}{\partial y} = 2xy + z^3 \quad \Big| \quad \frac{\partial f}{\partial z} = x^2 + 3yz^2$$

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$$\nabla f = (2xz+y^2)\,i + (2xy+z^3)\,j + (x^2+3yz^2)\,k$$

$$(\nabla f)_{(1,1,1)} = 3i + 3j + 4k = \bar{a}$$

$$|\bar{a}| = \sqrt{9+9+16} = \sqrt{34}$$

$$\therefore \nabla\phi \cdot \frac{\bar{a}}{|\bar{a}|} = (yz)i + (xz)j + (xy)k \cdot \frac{3i+3j+4k}{\sqrt{34}}$$

(8) Find the directional derivative of $\frac{1}{r}$ in the direction of $\bar{r} = xi + yj + zk$ at the point $(1,1,2)$

Given, $\phi = \frac{1}{r}$

$$\nabla\phi = i\cdot\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$$

$$= i\frac{\partial}{\partial x}\left(\frac{1}{r}\right) + j\frac{\partial}{\partial y}\left(\frac{1}{r}\right) + k\frac{\partial}{\partial z}\left(\frac{1}{r}\right)$$

$$= i\left(-\frac{1}{r^2}\right)\frac{\partial r}{\partial x} + j\left(-\frac{1}{r^2}\right)\frac{\partial r}{\partial y} + k\left(\frac{-1}{r^2}\right)\frac{\partial r}{\partial z}$$

$$= i\left(-\frac{1}{r^2}\right)\left[\frac{x}{r}\right] + j\left(-\frac{1}{r^2}\right)\left(\frac{y}{r}\right) + k\left(-\frac{1}{r^2}\right)\left(\frac{z}{r}\right)$$

We know that

$$\frac{\partial r}{\partial x} = \frac{x}{r},\ \frac{\partial r}{\partial y} = \frac{y}{r},\ \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$= -\frac{1}{r^3}\left[xi + yj + zk\right]$$

$$= -\frac{1}{r^3}\left[\bar{r}\right)$$

$$= \frac{-\bar{r}}{r^3}$$

Given,

$$\bar{a} = \bar{r} = xi + yj + zk$$

$$|\bar{r}| = \sqrt{x^2+y^2+z^2} = r$$

$$\hat{a} = \frac{\bar{r}}{|\bar{r}|} = \frac{xi+yj+zk}{r} = \frac{\bar{r}}{r}$$

The directional derivatives in the direction of $\bar{r}$ is given by

$$\nabla\phi \cdot \frac{\bar{r}}{|\bar{r}|} = \frac{-\bar{r}}{r^3}\cdot\frac{\bar{r}}{r} \qquad r^2$$

$$= \frac{-(x^2+y^2+z^2)}{r^4}$$

$$= \frac{-r^2}{r^4} = -\frac{1}{r^2} = \frac{-1}{x^2+y^2+z^2}$$

$$\nabla\phi \cdot \frac{\bar{r}}{|\bar{r}|} = -\frac{1}{6}$$

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Tangent Vector to the surface :-

Let $\bar{r} = xi + yj + zk$ where xyz are the functions of t then the tangent vector to the surface is given by $\left(\frac{d\bar{r}}{dt}\right)$

Unit tangent vector to the surface is given by $\dfrac{\left(\frac{d\bar{r}}{dt}\right)}{\left|\frac{d\bar{r}}{dt}\right|}$

(1) Find the directional derivative of the function $xy^2 + yz^2 + zx^2$ along the tangent to the curve $x = t$, $y = t^2$, $z = t^3$ at the point $(1,1,1)$

Given,

$$\phi = xy^2 + yz^2 + zx^2$$

$$\nabla\phi = i\cdot\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$$

$$\frac{\partial\phi}{\partial x} = y^2 + 2xz$$

$$\frac{\partial\phi}{\partial y} = 2xy + z^2$$

$$\frac{\partial\phi}{\partial z} = 2zy + x^2$$

$$\nabla\phi = (y^2 + 2xz)i + (2xy + z^2)j + (2zy + x^2)k$$

$$(\nabla\phi)_{(1,1,1)} = 3i + 3j + 3k$$

$$\bar{r} = xi + yj + zk$$

$$\bar{r} = ti + t^2j + t^3k$$

$$\frac{d\bar{r}}{dt} = i + 2tj + 3t^2k$$

$x = 1 \Rightarrow t = 1$

$y = t^2 \Rightarrow t^2 = 1 \Rightarrow t = \pm 1$

$z = t^3 \Rightarrow t^3 - 1 = 0 \Rightarrow t = 1$

roots are comp

$t = 1$

$$\frac{d\bar{r}}{dt} = i + 2j + 3k$$

$$\left|\frac{d\bar{r}}{dt}\right| = \sqrt{9+4+1} = \sqrt{14}$$

$$\therefore \nabla\phi\cdot\frac{\frac{d\bar{r}}{dt}}{\left|\frac{d\bar{r}}{dt}\right|} = (y^2 + 2xz)i + (2xy + z^2)j + (2zy + x^2)k \cdot \frac{i + 2j + 3k}{\sqrt{14}}$$

$$= (3i + 3j + 3k)\cdot\frac{i + 2j + 3k}{\sqrt{14}}$$

$$= \frac{3 + 6 + 9}{\sqrt{14}} = \frac{18}{\sqrt{14}}$$

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The maximum (Greatest) value of the directional derivative of the given function is given by "∇f" and its magnetude is given by $|\nabla f|$

(2) In what direction from the point $(1,3,2)$ in the directional derivative of $f = 2xz - y^2$ is maximum and what is its magnitude.

Given, $f = 2x3 - y^2$

$$\nabla f = i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z}$$

$$\frac{\partial f}{\partial x} = 2z \quad \frac{\partial f}{\partial y} = -2y \quad \frac{\partial f}{\partial z} = 2x$$

$$\nabla f = 2z\,i - 2y\,j + 2x\,k$$

$$(\nabla f)_{(1,3,2)} = 4i - 6j + 2k$$

$$|\nabla f| = \sqrt{16+36+4}$$

$$|\nabla f| = \sqrt{56}$$

$\therefore$ The directional derivative is maximum in the direction of the vector $4i - 6j + 2k$ and its magnitude is $\sqrt{56}$

(3) Find the greatest value of the directional derivative of the function $f = x^2yz^3$ at the point $(2,1,-1)$

Given, $f = x^2yz3$

$$\nabla f = i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} + k\frac{\partial f}{\partial z}$$

$$\frac{\partial f}{\partial x} = 2xyz^3 \quad \frac{\partial f}{\partial y} = x^2z^3 \quad \frac{\partial f}{\partial z} = 3x^2yz^2$$

$$\nabla f = (2xyz3)\,i + (x^2z3)\,j + (3x^2yz2)\,k$$

$$(\nabla f)_{(2,1,-1)} = -4i - 4j + 12k$$

$$|\nabla f| = \sqrt{16+16+144}$$

$$|\nabla f| = \sqrt{176}$$

$\therefore$ The directional derivative is maximum in the direction of the vector at the point $(2,1,-1)$ $-4i - 4j + 12k$ and magnitude $\sqrt{176}$

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(4) The temperature at any point in space is given by $f = xy + yz + zx$. Find the direction in which the temperature changes most rapidly with distance from the point $(1,1,1)$ and determine the maximum rate of change.

Given,

$$f = xy + yz + zx$$

$$\nabla f = i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} + k\frac{\partial f}{\partial z}$$

$$\frac{\partial f}{\partial x} = y+z \quad \frac{\partial f}{\partial y} = x+z \quad \frac{\partial f}{\partial z} = x+y$$

$$(\nabla f)_{(1,1,1)} = 2i + 2j + 2k$$

$$|\nabla f| = \sqrt{4+4+4}$$

$$|\nabla f| = \sqrt{12}$$

(5) The temperature at any point (x,y,z) in space is given by $f = x^2+y^2-z$. A mosquito located at $(1,1,2)$ desires to fly in such a direction that it will get warm as soon possible in what direction should it fly.

Given,

$$f = x^2 + y^2 - z$$

$$\nabla f = i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} + k\frac{\partial f}{\partial z}$$

$$\frac{\partial f}{\partial x} = 2x \quad \frac{\partial f}{\partial y} = 2y \quad \frac{\partial f}{\partial z} = -1$$

$$\nabla f = 2xi + 2yj - k$$

$$(\nabla f)_{(1,1,2)} = 2i + 2j - k$$

$$|\nabla f| = \sqrt{9} = 3$$

(6) Find the values of a, b, c so that the directional derivative of $f = axy^2 + byz + czx^3$ at $(1,2,-1)$ has maximum magnitude "64" in the direction parallel to z-axis.

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Given,

$$f = axy^2 + byz + cz^2x^3$$

$$\nabla f = i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} + k\frac{\partial f}{\partial z}$$

$$\nabla f = i\frac{\partial f}{\partial x} = ay^2 + 3cx^2z^2$$

$$\frac{\partial f}{\partial y} = 2axy + bz$$

$$\frac{\partial f}{\partial z} = by + 2cx^3z$$

$$\nabla f = (ay^2 + 3cx^2z^2)i + (2axy + bz)j + (by + 2cx^3z)k$$

$$(\nabla f)_{(1,2,-1)} = (4a + 3c)i + (4a - b)j + (2b - 2c)k$$

It is given, the directional derivative maximum in the direction parallel to z-axis [xy=0]

And its maximum value is given by

$$4a + 3c = 0 \qquad 4a - b = 0$$

$$4a = b$$

$$4\left(\frac{b}{4}\right) + 3c = 0 \qquad a = b/4$$

$$\boxed{c = -\frac{4a}{3}} \qquad \boxed{b = 4a}$$

$$|\nabla f| = \sqrt{(4a+3c)^2 + (4a-b)^2 + (2b-2c)^2} = 64$$

$$= \sqrt{\left(4a + 3\left(-\frac{4a}{3}\right)\right)^2 + (4a - 4a)^2 + \left(2(4a) - 2\left(-\frac{4a}{3}\right)\right)^2} = 64$$

$$\Rightarrow \sqrt{\left(8a + \frac{8a}{3}\right)^2} = 64$$

$$\Rightarrow 8a + \frac{8a}{3} = 64$$

$$\Rightarrow \frac{32a}{3} = 64 \times 2$$

$$\boxed{a = 6}$$

$$b = 4(6)$$

$$\boxed{b = 24}$$

$$c = \frac{-4(8)}{3}$$

$$\boxed{c = -8}$$

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Angle between two surfaces :-

Let $f(x,y,z)$ and $g(x,y,z)$ be two scalar point functions.

The angle between the two surfaces is equal to the angle between the normal vectors to the two surfaces. If 'θ' is the angle b/w the two scalar point functions then

$$\boxed{\cos\theta = \frac{\nabla f \cdot \nabla g}{|\nabla f||\nabla g|}}$$

① Find the angle between the two surfaces $xy^2z = 2$ and $x^2+y^2+z^2 = 16$ at the point $(2,1,1)$

Let,

$$f = xy^2z = 2$$

$$xy^2z - 2 = 0$$

$$\nabla f = y^2z\,i + 2xyz\,j + xy^2\,k$$

$$\bar{a} = (\nabla f)_{(2,1,1)} = i + 4j + 2k$$

let

$$g = x^2+y^2+z^2-16 = 0$$

$$\nabla g = 2x\,i + 2y\,j + 2z\,k$$

$$\bar{b} = (\nabla g)_{(2,1,1)} = 4i + 2j + 2k$$

$$|\bar{a}| = \sqrt{1+16+4} = \sqrt{21}$$

$$|\bar{b}| = \sqrt{16+4+4} = \sqrt{24}$$

$$\cos\theta = \frac{\bar{a}\cdot\bar{b}}{|\bar{a}|\cdot|\bar{b}|} = \frac{\nabla f \cdot \nabla g}{|\nabla f||\nabla g|}$$

$$\cos\theta = \frac{(i+4j+2k)(4i+2j+2k)}{\sqrt{21}\sqrt{24}}$$

$$= \frac{4+8+4}{\sqrt{21}\sqrt{24}}$$

$$\cos\theta = \frac{16}{\sqrt{21}\sqrt{24}}$$

$$\therefore \boxed{\theta = \cos^{-1}\left(\frac{16}{\sqrt{21}\sqrt{24}}\right)}$$

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(2) Find the angle b/w the two spheres $x^2+y^2+z^2=29$ and $x^2+y^2+z^2+4x-6y-8z-47=0$ at the point $(4,-3,2)$

(2) Let,

$$\nabla f = x^2+y^2+z^2-29=0$$

$$\nabla f = 2x\hat{i}+2y\hat{j}+2z\hat{k}$$

$$(\nabla f)_{(4,-3,2)} = 8\hat{i}-6\hat{j}+4\hat{k}$$

$$\bar{a} = \nabla f = 8\hat{i}-6\hat{j}+4\hat{k}$$

$$|\bar{a}| = \sqrt{64+36+16}$$

$$= \sqrt{116}$$

Let $g = x^2+y^2+z^2+4x-6y-8z-47$

$$\nabla g = (2x+4)\hat{i}+(2y-6)\hat{j}+(2z-8)\hat{k}$$

$$(\nabla g)_{(4,-3,2)} = 12\hat{i}+(-12\hat{j})+(-4)\hat{k}$$

$$= 12\hat{i}-12\hat{j}-4\hat{k}$$

$$\bar{b} = \nabla g = 12\hat{i}-12\hat{j}-4\hat{k}$$

$$|\bar{b}| = \sqrt{144+144-16}$$

$$= \sqrt{304}$$

$$\cos\theta = \frac{\bar{a}\cdot\bar{b}}{|\bar{a}|\cdot|\bar{b}|}$$

$$= \frac{(8\hat{i}-6\hat{j}+4\hat{k})(12\hat{i}-12\hat{j}-4\hat{k})}{\sqrt{116}\sqrt{304}}$$

$$= \frac{96+72-16}{\sqrt{116}\sqrt{304}}$$

$$\cos\theta = \frac{152}{\sqrt{116}\sqrt{304}}$$

$$\therefore \boxed{\theta = \cos^{-1}\frac{152}{\sqrt{116}\sqrt{304}}}$$

3 Find the values of a and b so that the surfaces (vectors) (m) $ax^2-byz=(a+2)x$ and $4x^2y+z^3=4$ may intersect orthogonally (perpendicularly) at the point $(1,-1,2)$.

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The given two surfaces are

$$f = ax^2 - by^2 - (a+2)x \rightarrow (1)$$

$$g = 4x^2y + z^3 - 4 \rightarrow (2)$$

These two curves intersect at the point $(1,-1,2)$

$\therefore$ The point satisfy the above equations

Substitute the point in eq (1)

$$a(1)^2 - b(-1)(2) - (a+2)1 = 0$$

$$a + 2b - a - 2 = 0$$

$$2b = 2$$

$$\boxed{b = 1}$$

It is given that two curves intersect orthogonally $(\theta = 90^\circ)$

$\therefore \quad \bar{n}_1 \cdot \bar{n}_2 = 0$

$$\bar{n}_1 = \nabla f = (2ax - a - 2)\,i + (-bz)\,j + (-by)\,k$$

$$[\nabla f]_{(1,-1,2)} = (2a - a - 2)\,i + (-b(2))\,j + ((-b)(-1))\,k$$

$$\bar{n}_1 = (a-2)\,i - 2b\,j + b\,k$$

$$\bar{n}_2 = \nabla g = (8xy)\,i + (4x^2)\,i + (3z^2)\,k$$

$$[\nabla g]_{(1,-1,2)} = -8i + 4j + 12k$$

$$\bar{n}_2 = -8i + 4j + 12k$$

Since $\bar{n}_1 \cdot \bar{n}_2 = 0$

$$\Rightarrow ((a-2)i - 2bj + bk) \cdot (-8i + 4j + 12k) = 0$$

$$\Rightarrow (a-2)(-8) - 8 + 12 = 0$$

$$\Rightarrow -8a + 16 - 8 + 12 = 0$$

$$\Rightarrow -8a + 20 = 0$$

$$8a = 20$$

$$\boxed{a = \frac{20}{8}}$$

$$\boxed{a = 5/2}$$

$$\therefore \quad a = \frac{5}{2} \; ; \; b = 1$$

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Find the angle between the normals to the surface $x^2 = yz$ at the point $(1,1,1)$ and $(2,4,1)$

Given,

$$f = x^2 - yz$$

$$\nabla f = (2x)i - zj - yk$$

$$(\nabla f)_{(1,1,1)} = 2i - j - k = a$$

$$(\nabla f)_{(2,4,1)} = 4i - j - 4k = b$$

$$|a| = \sqrt{4+1+1} = \sqrt{6}$$

$$|b| = \sqrt{16+16+1} = \sqrt{33}$$

$$\cos\theta = \frac{(2i-j-k)(4i-j-4k)}{\sqrt{6}\sqrt{33}}$$

$$= \frac{8+1+4}{\sqrt{6}\sqrt{33}}$$

$$\cos\theta = \frac{13}{\sqrt{6}\sqrt{33}}$$

$$\therefore \boxed{\theta = \cos^{-1}\left(\frac{13}{\sqrt{6}\sqrt{33}}\right)}$$

Find the angle b/w the normals to the surface $xy = z^2$ at the point $(4,1,2)$ and $(3,3,-3)$

The given surface is

$$f = xy - z^2$$

$$\nabla f = i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} + k\frac{\partial f}{\partial z}$$

$$\frac{\partial f}{\partial x} = y \quad \frac{\partial f}{\partial y} = x \quad \frac{\partial f}{\partial z} = -2z$$

$$\nabla f = yi + xj - 2zk$$

The normal vector to the surface at the point $(4,1,2)$ is

$$\bar{n}_1 = (\nabla f)_{(4,1,2)} = i + 4j - 4k$$

The normal unit vector to the surface at the point $(3,3,-3)$ is

$$n_2 = (\nabla f)_{(3,3,-3)} = 3i + 3j + 6k$$

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If 'θ' is the angle b/w the normals

then $\cos\theta = \dfrac{\bar{n}_1 \cdot \bar{n}_2}{|\bar{n}_1||\bar{n}_2|}$

$$= \frac{(i+4j-4k)\cdot(3i+3j+6k)}{\sqrt{1+16+16}\ \sqrt{9+9+36}}$$

$$= \frac{(1)(3)+4(3)-4(6)}{\sqrt{33}\ \sqrt{52}}$$

$$= \frac{3+12-24}{\sqrt{33}\ \sqrt{54}}$$

$$\cos\theta = \frac{-9}{\sqrt{33}\sqrt{54}}$$

$$\therefore\ \theta = \cos^{-1}\left(\frac{9}{\sqrt{33}\ \sqrt{54}}\right)$$

Divergence and curl of a vector point function

If '$\bar{F}$' is a vector point function then divergence of $\bar{F}$ is denoted by "$\nabla\cdot\bar{F}$" (or) div $\bar{F}$ and it is defined as $\text{div}\,\bar{F} = i\cdot\frac{\partial\bar{F}}{\partial x} + j\cdot\frac{\partial\bar{F}}{\partial y} + k\cdot\frac{\partial\bar{F}}{\partial z}$ which can be written as $\nabla\cdot\bar{F} = \sum i\cdot\frac{\partial\bar{F}}{\partial x}$

If $\bar{F} = F_1 i + F_2 j + F_3 k$ then divergent

$$\text{div}\cdot\bar{F} = \sum i\cdot\frac{\partial\bar{F}}{\partial x}$$

$$= \sum i\cdot\frac{\partial(F_1 i + F_2 j + F_3 k)}{\partial x}$$

$$= \sum i\left(\frac{\partial F_1}{\partial x} i + \frac{\partial F_2}{\partial x} j + \frac{\partial F_3}{\partial x} k\right)$$

$$\boxed{\text{div}\cdot\bar{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}}$$

If $\text{div}\cdot\bar{F} = 0$ then '$\bar{F}$' is called "solenoidal" vector.

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Curl :-

If $\bar{F}$ is a vector point function then curl $\bar{F}$ is denoted by "$\nabla \times \bar{F}$" (or) curl $\bar{F}$

then, $\nabla \times \bar{F} = i \times \frac{\partial \bar{F}}{\partial x} + j \times \frac{\partial \bar{F}}{\partial y} + k \times \frac{\partial \bar{F}}{\partial z}$ i.e

$$\nabla \times \bar{F} = \sum i \times \frac{\partial \bar{F}}{\partial x}$$

If $\bar{F} = F_1 i + F_2 j + F_3 k$ then curl $\bar{F}$

$$\text{curl}\cdot\bar{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

If curl $\bar{F} = \bar{0}$ then the vector '$\bar{F}$' is called "Irrotational vector"

If $\bar{F}$ is a vector point function then divergent $\bar{F}$ is a scalar point function and curl $\bar{F}$ is a vector point function.

1. Find $\nabla \cdot \bar{F}$ and $\nabla \times \bar{F}$ where $\bar{F} = \nabla(x^3 + y^3 + z^3 - 3xyz)$

Given,

$$\bar{F} = \nabla(x^3 + y^3 + z^3 - 3xyz)$$

let $\bar{F} = \nabla \phi$

Here, $\phi = x^3 + y^3 + z^3 - 3xyz$

$$\nabla \phi = (3x^2 - 3yz) i + (3y^2 - 3xz) j + (3z^2 - 3xy) k$$

$F_1 = 3x^2 - 3yz$

$F_2 = 3y^2 - 3xz$

$F_3 = 3z^2 - 3xy$

$$\text{div}\cdot\bar{F} = \nabla \cdot \bar{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$\therefore \nabla \cdot \bar{F} = 6x + 6y + 6z$$

$$\text{Curl}\cdot\bar{F} = \nabla \times \bar{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x^2 - 3yz & 3y^2 - 3xz & 3z^2 - 3xy \end{vmatrix}$$

$$= i(-3x + 3x) + j(-3y + 3y) + k(-3z + 3z) = 0i + 0j + 0k$$

$$= \bar{0} \text{ (Irrotational vector)}$$

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(2) If $\bar{F} = \nabla\phi$ where $\nabla\phi = xy^2 i + 2x^2yz\,j - 3yz^2 k$ then find $\nabla\cdot\bar{F}$ and $\nabla\times\bar{F}$ at the point $(1,-1,1)$

Given, $\bar{F} = \nabla\phi$

$$\nabla\phi = xy^2 i + 2x^2yz\,j - 3yz^2 k$$

Here, $F_1 = xy^2$ $F_2 = 2x^2yz$ $F_3 = -3yz^2$

$$\nabla\cdot\bar{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$= y^2 + 2x^2z - 6yz$$

Given divergent at point $(1,-1,1)$

$$(\nabla\cdot\bar{F})_{(1,-1,1)} = 1 + 2 + 6$$

$$(\nabla\cdot\bar{F})_{(1,-1,1)} = +9$$

$$\text{curl}\cdot\bar{F} = \nabla\times\bar{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy^2 & 2x^2yz & -3yz^2 \end{vmatrix}$$

$$= i(-3z^2 - 2x^2y) - j(0-0) + k(4xyz - 2xy)$$

curl$\cdot\bar{F}$ at the point $(1,-1,1)$

$$= i(-3+2) + k(-4+2)$$

$$= i(-1) + k(-2)$$

$$= -i - 2k$$

$$(\nabla\times\bar{F})_{(1,-1,1)} = -(i + 2k)$$

(3) S.T. $3y^4z^2 i + z^3x^2 j - 3x^2y^2 k$ is it solenoidal

Let,

$$F = \nabla\phi = 3y^4z^2 i + z^3x^2 j - 3x^2y^2 k$$

Here, $F_1 = 3y^4z^2$ $F_2 = z^3x^2$ $F_3 = -3x^2y^2$

$$\nabla\cdot\bar{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$= 0$$

$$\therefore \boxed{\nabla\cdot\bar{F} = 0}$$

$\therefore$ the given vector is solenoidal.

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4. If $\bar{F} = (x+3y)i + (y-2z)j + (x-2z)k$ then find div. $\bar{F}$ and curl $\bar{F}$

Given,

$\bar{F} = \nabla\phi$

Here,

$\nabla\phi = (x+3y)i + (y-2z)j + (x-2z)k$

$F_1 = x+3y \quad F_2 = y-2z \quad F_3 = x-2z$

$\nabla\cdot\bar{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$

$\nabla\bar{F} = 1+1-2$

$\therefore \boxed{\nabla\bar{F} = 0}$ Solenoidal

$$\text{curl}\cdot\bar{F} = \nabla\times\bar{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+3y & y-2z & x-2z \end{vmatrix}$$

$\therefore i(0-2) - j(1-0) + k(0-3) \quad \cancel{i(0-2)}$

$= +2i - j - 3k$

$\nabla\times\bar{F} = (2i - j + k)$

$\therefore \nabla\times\bar{F} = 2i - j - 3k$

5. If $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ then find div $\bar{r}$ and curl $\bar{r}$

$\bar{r} = \nabla\phi$

Here, $F_1 = x \quad F_2 = y \quad F_3 = z$

$\text{div}\,\bar{r} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$

$= 1+1+1$

$\text{div}\,\bar{r} = 3$

$$\text{curl}\cdot\bar{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix}$$

$= i(0-0) - j(0-0) + k(0-0)$

$= \bar{0}$

The given vector is Irrotational.

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**6. If $\bar{F} = r^n \bar{r}$ then find div F and curl F

$\bar{r} = xi + yj + zk$, $r = |\bar{r}| = \sqrt{x^2+y^2+z^2}$

Given, $\bar{r} = xi + yj + zk$

and $r = |\bar{r}| = \sqrt{x^2+y^2+z^2}$

$$\frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$\bar{F} = r^n \cdot \bar{r} = r^n(xi + yj + zk)$$

$$\bar{F} = r^n x i + r^n y j + r^n z k$$

$$F_1 = r^n x \mid F_2 = r^n y \mid F_3 = r^n z$$

$$\text{div}\cdot\bar{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$= \frac{\partial}{\partial x}(r^n x) + \frac{\partial}{\partial y}(r^n y) + \frac{\partial}{\partial z}(r^n z)$$

$$\text{div}\bar{F} = 3r^n$$

$$\Rightarrow r^n + x\, n\, r^{n-1}\frac{\partial r}{\partial x} = \frac{\partial F_1}{\partial x}$$

$$\Rightarrow \frac{\partial F_1}{\partial x} = r^n + x \cdot n r^{n-1}\left(\frac{x}{r}\right)$$

$$\frac{\partial F_1}{\partial x} = r^n + n x^2 r^{n-2}$$

Similarly

$$\frac{\partial F_2}{\partial y} = r^n + n y^2 r^{n-2}$$

$$\frac{\partial F_3}{\partial z} = r^n + n z^2 r^{n-2}$$

$$\text{div}\cdot\bar{F} = 3r^n + n r^{n-2}(x^2 + y^2 + z^2)$$

$$= 3r^n + n r^{n-2} r^2$$

$$= 3r^n + n r^n$$

$$= r^n(3+n)$$

$$\therefore \boxed{\text{div}\bar{F} = (n+3) r^n}$$

If $n = -3$ then $\text{div}\bar{F} = 0$

Then $\bar{F}$ is called solenoidal vector.

$$\text{curl}\bar{F} = \nabla \times \bar{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ r^n x & r^n y & r^n z \end{vmatrix}$$

$$= i\left[\frac{\partial}{\partial y} r^n z - \frac{\partial}{\partial z} r^n y\right] - j\left(\frac{\partial}{\partial x} r^n z - \frac{\partial}{\partial z} r^n x\right) + k\left(\frac{\partial}{\partial x} r^n y - \frac{\partial}{\partial y} r^n x\right)$$

$$= \sum i\left[z\, n r^{n-1}\frac{\partial r}{\partial y} - y\, n r^{n-1}\frac{\partial r}{\partial z}\right]$$

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$= \sum i \left(nz r^{n-1} \frac{y}{r} - ny r^{n-1} \frac{z}{r} \right]$

$= \sum i \left(\frac{nyz r^{n-1}}{r} - \frac{nyz r^{n-1}}{r} \right)$

$\text{curl} \bar{F} = \bar{0}$

The given vector is Irrotational

$\therefore r^n \bar{r}$ is an Irrotational vector.

7. Find divergent of (i) $\text{div}\left(\frac{\bar{r}}{r^3}\right)$
(ii) $\text{div}\left(\frac{\bar{r}}{r}\right)$

we know that

$\text{div}\, \bar{F} = (n+3) r^n$

(i) $n = -3$

$\text{div}\, F = (-3+3) r^n$

$\text{div}\left(\frac{\bar{r}}{r^3}\right) = 0$

The given vector is solenoidal

(ii) $n = -1$

$\text{div}\, \bar{F} = (-1+3) r^n$

$\text{div}\, \bar{F} = 2r^n$

$\text{div}\left(\frac{\bar{r}}{r}\right) = 2r^{-1}$

$= \frac{2}{r}$

8. Find the values of a, b, c if the vector $\bar{F} = (2x+3y+az)i + (bx+2y+3z)j + (2x+cy+3z)k$ is an irrotational vector.

Given, $\bar{F}$ is an irrotational vector

$\therefore \boxed{\nabla \times \bar{F} = 0}$

Given, $\bar{F} = (2x+3y+az)i + (bx+2y+3z)j + (2x+cy+3z)k$

$F_1 = 2x+3y+az$

$F_2 = bx+2y+3z$

$F_3 = 2x+cy+3z$

$$\nabla \times \bar{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x+3y+az & bx+2y+3z & 2x+cy+3z \end{vmatrix}$$

$= i[c-3] - j[2-a] + k[b-3] = 0i + 0j + 0k$

$\boxed{a=2}$ $\boxed{b=3}$ $\boxed{c=3}$

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Scalar potential function :-

If $\bar{F}$ is irrotational then there exists a scalar function 'ϕ' such that $\bar{F} = \nabla\phi$ then the scalar point function ϕ is called a "scalar (Point) potential function".

① If $\bar{F} = (x^2 - yz)i + (y^2 - zx)j + (z^2 - xy)k$ is an irrotational vector and then find the scalar potential Function.

(B)

Given, $\bar{F} = \nabla\phi$

$F_1 = x^2 - yz \quad F_2 = y^2 - zx \quad F_3 = z^2 - xy$

$$\text{curl } \bar{F} = \nabla \times \bar{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 - yz & y^2 - zx & z^2 - xy \end{vmatrix}$$

$$= i(-x + x) - j(-y + y) + k(-z + z)$$

$\text{curl } \bar{F} = \bar{0}$

$\therefore$ Given vector is Irrotational vector.

$\bar{F}$ is an Irrotational vector

$$\bar{F} = \nabla\phi.$$

$$(x^2 - yz)i + (y^2 - zx)j + (z^2 - xy)k = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$$

$$\frac{d\phi}{dx} = x^2 - yz \rightarrow ① \quad \frac{d\phi}{\partial y} = y^2 - zx \rightarrow ② \quad \frac{\partial\phi}{\partial z} = z^2 - xy \rightarrow ③$$

eq ① Integrate w.r.t 'x'

$$\phi = \int (x^2 - yz)\,dx$$

$$\phi = \frac{x^3}{3} - xyz + c_1(y, z) \rightarrow ④$$

eq ④ Partial derivative w.r.t 'y' and eq eqn ④

$$\frac{d\phi}{dy} = -xz + \frac{\partial\phi_1}{\partial y} = y^2 - zx$$

$$\frac{\partial c_1}{\partial y} = y^2 \rightarrow ⑤$$

Now Integrate eqn ⑤ w.r.t 'y'

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$c_1 = \int y^2\, dy$

$c_1 = \frac{y^3}{3} + c_2(z) \rightarrow (6)$

sub (6) in (4)

$\phi = \frac{x^3}{3} - xyz + \frac{y^3}{3} + c_2(z) \rightarrow (7)$

Pa diff w.r.t 'z' and equate (3)

$\frac{d\phi}{dz} = -xy + \frac{dc_2}{dz} = z^2 - xy$

$\frac{dc_2}{dz} = z^2 \rightarrow (8)$

Integrate eq (8) w.r.t z

$c_2 = \frac{z^3}{3} + c \rightarrow (9)$

sub (9) in (7)

$$\boxed{\phi = \frac{x^3}{3} + \frac{y^3}{3} + \frac{z^3}{3} - xyz + c}$$

(or)

$\phi = \frac{x^3}{3} - xyz$

$\phi = \frac{y^3}{3} - xyz$

$\phi = \frac{z^3}{3} - xyz$

$\therefore \phi = \frac{x^3}{3} + \frac{y^3}{3} + \frac{z^3}{3} - xyz + c$

(2) Determine the constants so that $\bar{a} = (x+2y+az)i + (bx-3y-z)j + (4x+cy+2z)k$ is irrotational and find S.P.F

Given,

$\bar{A} = (x+2y+az)i + (bx-3y-z)j + (4x+cy+2z)k$

Given $\bar{A}$ is irrotational vector

$$\boxed{\nabla \times \bar{A} = 0}$$

Here, $A_1 = x+2y+az$

$A_2 = bx-3y-z$

$A_3 = 4x+cy+2z$

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$$\nabla \times \bar{A} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+2y+az & bx-3y-z & 4x+cy+2z \end{vmatrix}$$

$$= i[c+1] + j[-4+a] + k[b-2]$$

$c+1=0$ $\quad a-4=0$ $\quad b-2=0$

$\boxed{c=-1}$ $\quad \boxed{a=4}$ $\quad \boxed{b=2}$

$$\bar{A} = (x+2y+4z)\,i + (2x-3y-z)\,j + (4x-y+2z)\,k$$

Now, we will find the scalar potential function

$$\frac{d\phi}{dx} = x+2y+4z \longrightarrow (1)$$

$$\frac{d\phi}{\partial y} = 2x-3y-z \longrightarrow (2)$$

$$\frac{\partial\phi}{\partial z} = 4x-y+2z \longrightarrow (3)$$

from eq (1)

Integrate eq (1) w.r.t 'x'

$$\phi = \int x+2y+4z\,dx$$

$$\phi = \frac{x^2}{2} + 2xy + 4xz + c_1(y,z) \rightarrow (4)$$

Diff partially eq (4) w.r.t y and equate eq (2)

$$\frac{d\phi}{\partial y} = \cancel{2x} + \frac{\partial c_1}{\partial y} = \cancel{2x} - 3y - z$$

$$\frac{\partial c_1}{\partial y} = -3y - z \longrightarrow (5)$$

Integrate w.r.t 'y' eq (5)

$$c_1 = -3\frac{y^2}{2} - zy + c_2(z)$$

$$c_1 = -\frac{3}{2}y^2 - zy + c_2(z) \rightarrow (6)$$

Now substitute eq (6) in (4)

$$\phi = \frac{x^2}{2} + 2xy + 4xz - \frac{3}{2}y^2 - zy + c_2(z) \longrightarrow (7)$$

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Now diff partially w.r.t 'z' and equate eq ③

$$\frac{\partial \phi}{\partial z} = 4x - y + \frac{\partial C_2}{\partial z} = 4x - y + 2z$$

$$\frac{\partial C_2}{\partial z} = 2z$$

Now Integrate w.r.t 'z'

$$C_2 = z^2 + c \rightarrow (8)$$

Substitute eq ⑧ in eq ④

$$\boxed{\phi = \frac{x^2}{2} + 2xy + 4xz - \frac{3}{2}y^2 - zy + z^2 + c}$$

(or)

Integrate eq ①, ② and ③ w.r.t x, y and z

$$\phi = \frac{x^2}{2} + 2xy + 4xz + c_1$$

$$\phi = 2xy - \frac{3y^2}{2} + yz + c_2$$

$$\phi = 4xz - yz + 2\cdot\frac{z^2}{2} + c_3$$

$$\phi = \frac{x^2}{2} - \frac{3y^2}{2} + z^2 + 2xy + 4xz - yz + c$$

Determine the constants A, B, C if the vector $\bar{F} = (2x+3y+az)i + (bx+2y+3z)j + (2x+cy+3z)k$ is irrotational and find the scalar potential function.

Given,

$$\bar{F} = (2x+3y+az)i + (bx+2y+3z)j + (2x+cy+3z)k$$

$$\nabla \times \bar{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x+3y+az & bx+2y+3z & 2x+cy+3z \end{vmatrix}$$

$$= i[c-3] - j[2-a] + k[b-3]$$

$a = 2$

$b = 3$

$c = 3$

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Here,

$$\frac{\partial \phi}{\partial x} = 2x+3y+2z \quad \frac{\partial \phi}{\partial y} = 3x+2y+3z \quad \frac{\partial \phi}{\partial z} = 2x+3y+3z$$

$\downarrow$ (1) $\quad$ $\downarrow$ (2) $\quad$ $\downarrow$ (3)

Integrate eq (1) w.r.t 'x'

$$\phi = 2\cdot\frac{x^2}{2} + 3xy + 2xz + c_1(y,z)$$

$$\phi = x^2 + 3xy + 2xz + c_1(y,z) \rightarrow (4)$$

Diff partially w.r.t y and equate eq (2)

$$\frac{d\phi}{dy} = 3x + \frac{\partial c_1}{\partial y} = 3x+2y+3z$$

$$\frac{\partial c_1}{\partial y} = 2y+3z \rightarrow (5)$$

Integrate eq (5) w.r.t. 'y'

$$c_1 = 2\cdot\frac{y^2}{2} + 3yz + c_2(z)$$

$$c_1 = y^2 + 3yz + c_2(z) \rightarrow (6)$$

Now substitute eq (6) in (4)

$$\phi = x^2 + 3xy + 2xz + y^2 + 3yz + c_2(z) \rightarrow (7)$$

Now diff partially w.r.t 'z' and eq (3)

$$\frac{\partial \phi}{\partial z} = 2x + 3y + \frac{\partial c_2}{\partial z} = 2x+3y+3z$$

$$\frac{\partial c_2}{\partial z} = 3z$$

Now Integrate w.r.t 'z'

$$c_2 = 3\frac{z^2}{2} \rightarrow (8)$$

Substitute eq (8) in eq (7)

$$\boxed{\phi = x^2 + 3xy + 2xz + y^2 + 3yz + \frac{3}{2}z^2 + c}$$

(or)

$$\phi = 2\cdot\frac{x^2}{2} + 3yx + 2xz \rightarrow x^2 + 3yx + 2xz + c_1$$

$$\phi = 3\cdot x\ 3yx + 2\cdot\frac{y^2}{2} + 3yz \rightarrow 3yx + y^2 + 3yz + c_2$$

$$\phi = 2xz + 3yz + 3\cdot\frac{z^2}{2} \rightarrow 2xz + 3yz + \frac{3}{2}z^2 + c_3$$

$$\phi = x^2 + 3xy + 2xz + y^2 + 3yz + \frac{3}{2}z^2 + c$$

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Show that the vector field
$\bar{F} = (2xyz^2)\hat{i} + (x^2z^2 + z\cos yz)\hat{j} + (2x^2yz + y\cos yz)$
is irrotational and find the scalar potential functional.

Given,

$$\bar{F} = (2xyz^2)\hat{i} + (x^2z^2 + z\cos yz)\hat{j} + (2x^2yz + y\cos yz)$$

Here,

$$F_1 = 2xyz^2$$
$$F_2 = x^2z^2 + z\cos yz$$
$$F_3 = 2x^2yz + y\cos yz$$

$$\nabla \times \bar{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xyz^2 & x^2z^2 + z\cos yz & 2x^2yz + y\cos yz \end{vmatrix}$$

$$= i\left[2x^2z + \cos yz + y(-\sin yz)z - [2x^2z + \cos yz + z(-\sin yz)y]\right]$$
$$- j\left[4xyz - 4xyz\right] + k\left[2xz^2 - 2xz^2\right]$$

$$= i\left[2x^2z + \cos yz - yz\sin yz - 2x^2z - \cos yz + yz\sin yz\right]$$

$$= \bar{0}$$

It is an irrotational vector.

Here,

$$\bar{F} = (2xyz^2)\hat{i} + (x^2z^2 + z\cos yz)\hat{j} + (2x^2yz + y\cos yz)k$$

$$\frac{d\phi}{dx} = 2xyz^2 \rightarrow (1) \qquad \frac{d\phi}{dy} = x^2z^2 + z\cos yz \rightarrow (2) \qquad \frac{\partial\phi}{\partial z} = 2x^2yz + y\cos yz \rightarrow (3)$$

Integrate eq (1) w.r.t 'x'

$$\phi = \int 2xyz^2\,dx$$

$$\phi = \cancel{2}\cdot\frac{x^2}{\cancel{2}}yz^2$$

$$\phi = x^2yz^2 + c_1(y,z) \rightarrow (4)$$

diff partially eq (4) w.r.t y and equate eq (2)

$$\frac{d\phi}{dy} = x^2z^2 + \frac{\partial c_1}{\partial y} = x^2z^2 + z\cos yz$$

$$\frac{\partial c_1}{\partial y} = z\cos yz \rightarrow (5)$$

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Integrate eq (5) w.r.t 'y'

$$c_1 = \int z\cos yz \, dy$$

$$c_1 = \sin yz + C_2(z) \rightarrow (6)$$

Now substitute eq (6) in eq (4)

$$\phi = x^2yz^2 + \sin yz + C_2(z) \rightarrow (7)$$

Now diff. partially w.r.t 'z' and equate eq (3)

$$\frac{\partial \phi}{\partial z} = 2x^2yz + y\cos yz + \frac{\partial C_2}{\partial z} = 2x^2yz + y\cos yz$$

$$\frac{\partial C_2}{\partial z} = 0$$

Integrate w.r.t 'z'

$$C_2 = 0 \rightarrow (8)$$

Substitute eq (8) in eq (7)

$$\phi = x^2yz^2 + \sin yz + C = 0$$

(or)

$$\frac{\partial \phi}{\partial x} = 2xyz^2$$

$$\frac{\partial \phi}{\partial y} = x^2yz^2 + z\cos yz$$

$$\frac{\partial \phi}{\partial z} = 2x^2yz + y\cos yz$$

$\phi = x^2yz^2$ $\quad$ $\phi = x^2yz^2 + \sin yz$ $\quad$ $\phi = x^2yz^2 + \sin yz$

$$\phi = x^2yz^2 + \sin yz + C$$

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Laplacian operator :

The laplacian operator is denoted by is "∇^2" and it is defined as $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$

If 'ϕ' is a scalar point function then $\nabla^2 \phi$ is defined as

$$\boxed{\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = \text{div}(\text{grad}\,\phi)}$$

Q Find the value of $\nabla^2 (r^n)$ where $r = \sqrt{x^2+y^2+z^2}$

$\bar{r} = xi + yj + zk$

W.K.T

$$\nabla^2 \phi = \text{div}(\text{grad}\,\phi)$$

$$\nabla^2 r^n = \text{div}(\text{grad}\, r^n)$$

$$= \nabla \cdot (\nabla r^n)$$

$$\nabla r^n = \sum i \frac{\partial}{\partial x} r^n$$

$$= \sum i \cdot n r^{n-1} \frac{\partial r}{\partial x}$$

$$= \sum i\, n r^{n-1} \frac{x}{r}$$

$$= \sum x i \cdot n r^{n-2}$$

$$= n r^{n-2} (xi + yj + zk)$$

$$\nabla r^n = n r^{n-2} (\bar{r})$$

$$\nabla (r^n) = n r^{n-2} x i + n r^{n-2} y j + n r^{n-2} z k$$

Here, $F_1 = n r^{n-2} x$

$F_2 = n r^{n-2} y$

$F_3 = n r^{n-2} z$

$$\text{div}\, \bar{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$\nabla \cdot (\nabla r^n) = \nabla^2 r^n = \frac{\partial}{\partial x}(n r^{n-2} x) + \frac{\partial}{\partial y}(n r^{n-2} y) + \frac{\partial}{\partial z}(n r^{n-2} z)$$

$$\frac{\partial}{\partial x}(n r^{n-2} x) = n r^{n-2} + x\, n(n-2) r^{n-3} \frac{\partial r}{\partial x}$$

$$= n r^{n-2} + x\, n(n-2) r^{n-3} \left(\frac{x}{r}\right)$$

$$\frac{\partial}{\partial x}(n r^{n-2} x) = n r^{n-2} + x^2 n(n-2) r^{n-4}$$

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Similarly,

$$\frac{\partial}{\partial y}(n r^{n-2} y) = n r^{n-2} + y^2 n(n-2) r^{n-4}$$

$$\frac{\partial}{\partial z}(n r^{n-2} z) = n r^{n-2} + z^2 n(n-2) r^{n-4}$$

$$\nabla^2 r^n = \nabla\cdot(\nabla r^n) = 3n r^{n-2} + n(n-2) r^{n-4}\left[x^2 + y^2 + z^2\right]$$

$$= 3n r^{n-2} + n(n-2) r^{n-4}(r^2)$$

$\therefore$

$$\nabla^2 r^n = 3n r^{n-2} + n(n-2) r^{n-2}$$

$$= n r^{n-2}\left[3 + n - 2\right]$$

$$\boxed{\nabla^2(r^n) = n(n+1) r^{n-2}}$$

(2) $\nabla^2 \log r$, $\nabla^2\left(\frac{1}{r}\right)$

W.K.T

$$\nabla^2 \phi = \nabla^2 \cdot \text{div}(\text{grad}\,\phi)$$

$$\nabla^2 \log r = \nabla(\nabla \log r)$$

$$\nabla \log r = \sum i \frac{\partial}{\partial x} \log r$$

$$= \sum i \cdot \frac{1}{r} \cdot \frac{\partial r}{\partial x}$$

$$= \sum i \cdot \frac{1}{r} \cdot \frac{x}{r}$$

$$= \sum i \cdot \frac{x}{r^2}$$

$$= \frac{1}{r^2} x i + \frac{1}{r^2} y j + \frac{1}{r^2} z k$$

Here, $F_1 = \frac{1}{r^2} x$; $F_2 = \frac{1}{r^2} y$; & $F_3 = \frac{1}{r^2} z$

$$\text{div}\,\bar{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$\nabla(\nabla \log r) = \nabla^2 \log r$$

$$= \frac{\partial}{\partial x}\left(\frac{1}{r^2} x\right) + \frac{\partial}{\partial x}\left(\frac{1}{r^2} y\right) + \frac{\partial}{\partial x}\left(\frac{1}{r^2} z\right)$$

$$\frac{\partial}{\partial x}\left(\frac{x}{r^2}\right) = \frac{r^2 - x \cdot 2r \cdot \frac{\partial r}{\partial x}}{r^4}$$

$$= \frac{r^2 - 2xr \cdot \frac{x}{r}}{r^4}$$

$$\frac{\partial}{\partial x}\left(\frac{x}{r^2}\right) = \frac{1}{r^2} - \frac{2x^2}{r^4}$$

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Similarly,

$$\frac{\partial}{\partial y}\left(\frac{y}{r^2}\right) = \frac{1}{r^2} - \frac{2y^2}{r^4}$$

$$\frac{\partial}{\partial z}\left(\frac{z}{r^2}\right) = \frac{1}{r^2} - \frac{2z^2}{r^4}$$

$$\nabla \cdot \nabla(\log r) = \frac{3}{r^2} - \frac{2}{r^4}(x^2+y^2+z^2)$$

$$= \frac{3}{r^2} - \frac{2r^2}{r^4} \Rightarrow \frac{3}{r^2} - \frac{2}{r^2}$$

$$\boxed{\nabla^2 \log r = \frac{1}{r^2}}$$

(ii) $\nabla^2\left(\frac{1}{r}\right)$

$$\nabla^2 \frac{1}{\log r} = \nabla\left(\nabla \frac{1}{\log r}\right)$$

$$\nabla \frac{1}{\log r} = \sum i \frac{\partial}{\partial x} \frac{1}{r}$$

$$= \sum i \left(-\frac{1}{r^2}\right) \frac{\partial r}{\partial x}$$

$$\nabla \frac{1}{r} = -\frac{1}{r^3} x i - \frac{1}{r^3} y j - \frac{1}{r^3} z k$$

Here, $F_1 = -\frac{1}{r^3} x \quad F_2 = -\frac{1}{r^3} y \quad F_3 = -\frac{1}{r^3} z$

$$\text{div } \bar{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$\nabla\left(\nabla \frac{1}{r}\right) = \nabla^2 \frac{1}{r}$$

$$= \frac{\partial}{\partial x}\left(-\frac{x}{r^3}\right) + \frac{\partial}{\partial y}\left(-\frac{y}{r^3}\right) + \frac{\partial}{\partial z}\left(-\frac{z}{r^3}\right)$$

$$\frac{\partial}{\partial x}\left(-\frac{x}{r^3}\right) = \frac{-\left[r^3 - x\,3r^2 \frac{\partial r}{\partial x}\right]}{r^6} \Rightarrow \frac{-\left[r^3 - 3xr^{\cancel{2}}\frac{x}{\cancel{r}}\right]}{r^6}$$

$$\Rightarrow \frac{-\left[r^3 - 3x^2 r\right]}{r^6}$$

$$\Rightarrow -\frac{1}{r^3} + \frac{3x^2}{r^5}$$

Similarly,

$$\frac{\partial}{\partial x}\left(-\frac{y}{r^3}\right) = -\frac{1}{r^3} + \frac{3y^2}{r^5}$$

$$\frac{\partial}{\partial x}\left(-\frac{z}{r^3}\right) = -\frac{1}{r^3} + \frac{3z^2}{r^5}$$

$$\nabla^2 \frac{1}{r} = -\frac{3}{r^3} + \frac{3}{r^5}(x^2+y^2+z^2)$$

$$= -\frac{3}{r^3} + \frac{3}{r^5}(r^2)$$

$$= -\frac{3}{r^3} + \frac{3}{r^3}$$

$$\boxed{\nabla^2 \frac{1}{r} = 0}$$

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** Prove that $\nabla^2 f(r) = f''(r) + \frac{2}{r} f'(r)$

(4)

$$\nabla^2 \phi = \text{div}(\text{grad}\,\phi)$$

$$\therefore \nabla^2 f(r) = \text{div}(\text{grad}\, f(r))$$

$$= \nabla \cdot (\nabla f(r))$$

$$\nabla f(r) = \sum i \cdot \frac{\partial}{\partial x} f(r)$$

$$= \sum i \cdot f'(r) \cdot \frac{\partial r}{\partial x}$$

$$= \sum i \cdot f'(r) \cdot \frac{x}{r}$$

$$= \sum i \times \frac{f'(r)}{r}$$

$$\nabla f(r) = \frac{f'(r)}{r}[x\hat{i}] + \frac{f'(r)}{r}[y\hat{j}] + \frac{f'(r)}{r}[z\hat{k}]$$

Here,

$$F_1 = \frac{f'(r)}{r} x \quad | \quad F_2 = \frac{f'(r)}{r} y \quad | \quad F_3 = \frac{f'(r)}{r} z$$

$$\text{div}\, \bar{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$\nabla \cdot (\nabla f(r)) = \nabla^2 f(r)$$

$$= \frac{\partial}{\partial x}\left(\frac{f'(r)}{r} x\right) + \frac{\partial}{\partial y}\left(\frac{f'(r)}{r} y\right) + \frac{\partial}{\partial z}\left(\frac{f'(r)}{r} z\right)$$

$$\frac{\partial}{\partial x}\left(\frac{f'(r)}{r} x\right) = f'(r) \frac{\partial}{\partial x}\left(\frac{x}{r}\right) + \frac{x}{r} \frac{\partial}{\partial x}(f'(r))$$

$$= f'(r) \cdot \left[\frac{r - x\frac{\partial r}{\partial x}}{r^2}\right] + \frac{x}{r} f''(r) \frac{\partial r}{\partial x}$$

$$= f'(r) \frac{r}{r^2} - f'(r) \cdot \frac{x^2}{r^3} + \frac{x^2}{r^2} f''(r)$$

$$\frac{\partial}{\partial x}\left(\frac{f'(r)}{r} x\right) = \frac{f'(r)}{r} - f'(r) \cdot \frac{x^2}{r^3} + \frac{x^2}{r^2} f''(r)$$

Similarly

$$\frac{\partial}{\partial y}\left(\frac{f'(r)}{r} y\right) = \frac{f'(r)}{r} - f'(r) \cdot \frac{y^2}{r^3} + \frac{y^2}{r^2} f''(r)$$

$$\frac{\partial}{\partial z}\left(\frac{f'(r)}{r} z\right) = \frac{f'(r)}{r} - f'(r) \frac{z^2}{r^3} + \frac{z^2}{r^2} f''(r)$$

$$\nabla^2 f(r) = 3\frac{f'(r)}{r} - \frac{f'(r)}{r^3}[x^2 + y^2 + z^2] + \frac{f''(r)}{r^2}[x^2 + y^2 + z^2]$$

$$= 3\frac{f'(r)}{r} - \frac{f'(r)}{r^3}(r^2) + \frac{f''(r)}{r^2}(r^2)$$

$$= 3\frac{f'(r)}{r} - \frac{f'(r)}{r} + f''(r)$$

$$\therefore \boxed{\nabla^2 f(r) = \frac{2}{r} f'(r) + f''(r)}$$

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div $\nabla\left[r\nabla\left(\frac{1}{r^3}\right)\right]$

$$\nabla\left(\frac{1}{r^3}\right) = \sum i \frac{\partial}{\partial x}\left(\frac{1}{r^3}\right)$$

$$= \sum i\left(-\frac{3}{r^4}\right)\frac{\partial r}{\partial x}$$

$$= \sum i\left(-\frac{3}{r^4}\right)\left(\frac{x}{r}\right)$$

$$= \sum i\left(\frac{-3}{r^5}\right)x$$

$$r\nabla\left(\frac{1}{r^3}\right) = -\frac{3}{r^4}xi - \frac{3}{r^4}yj - \frac{3}{r^4}zk$$

Here,

$$F_1 = -\frac{3}{r^4}x \quad F_2 = -\frac{3}{r^4}y \quad F_3 = -\frac{3}{r^4}z$$

$$\text{div } \bar{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$\nabla\cdot\nabla\left(\frac{1}{r^3}\right) = \frac{\partial}{\partial x}\left(-\frac{3}{r^4}x\right) + \frac{\partial}{\partial y}\left(-\frac{3}{r^4}y\right) + \frac{\partial}{\partial z}\left(-\frac{3}{r^4}z\right)$$

$$\frac{\partial}{\partial x}\left(-3\cdot\frac{x}{r^4}\right) = \frac{-3\left(r^4 - x4r^{\cancel{3}}\frac{x}{\cancel{r}}\right)}{r^8}$$

$$= \frac{-3\left(r^4 - 4r^2x^2\right)}{r^8}$$

$$= \frac{-3r^4}{r^8} + \frac{12r^2x^2}{r^8}$$

$$\frac{\partial}{\partial x}\left(-3\cdot\frac{x}{r^4}\right) = -\frac{3}{r^4} + \frac{12x^2}{r^6}$$

$$\frac{\partial}{\partial x}\left(-3\frac{y}{r^4}\right) = -\frac{3}{r^4} + \frac{12y^2}{r^6}$$

$$\frac{\partial}{\partial x}\left(-3\frac{z}{r^4}\right) = -\frac{3}{r^4} + \frac{12z^2}{r^6}$$

$$\nabla\left(r\nabla\left(\frac{1}{r^3}\right)\right) = -\frac{9}{r^4} + \frac{12}{r^6}\left(x^2+y^2+z^2\right)$$

$$= -\frac{9}{r^4} + \frac{12}{r^4}$$

$$\boxed{\nabla\left(r\nabla\left(\frac{1}{r^3}\right)\right) = \frac{3}{r^4}}$$

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Scalar differential operator :

The scalar differential operator is denoted by "$(\bar{a}\cdot\nabla)$" and it is defined as

$$(\bar{a}\cdot\nabla) = (\bar{a}\cdot i)\frac{\partial}{\partial x} + (\bar{a}\cdot j)\frac{\partial}{\partial y} + (\bar{a}\cdot k)\frac{\partial}{\partial z}$$

If 'ϕ' is a scalar point function

① $(\bar{a}\cdot\nabla)\phi = (\bar{a}\cdot i)\frac{\partial\phi}{\partial x} + (\bar{a}\cdot j)\frac{\partial\phi}{\partial y} + (\bar{a}\cdot k)\frac{\partial\phi}{\partial z} = \sum(\bar{a}\cdot i)\frac{\partial\phi}{\partial x}$

② If $\bar{f}$ is a vector point function then

$$(\bar{a}\cdot\nabla)\bar{f} = (\bar{a}\cdot i)\frac{\partial\bar{f}}{\partial x} + (\bar{a}\cdot j)\frac{\partial\bar{f}}{\partial y} + (\bar{a}\cdot k)\frac{\partial\bar{f}}{\partial z} = \sum\bar{a}\cdot i\frac{\partial\bar{f}}{\partial x}$$

Vector differential operator :-

The vector differential operator is denoted by "$(\bar{a}\times\nabla)$" and it is defined as

$$(\bar{a}\times\nabla) = (\bar{a}\times i)\frac{\partial}{\partial x} + (\bar{a}\times j)\frac{\partial}{\partial y} + (\bar{a}\times k)\frac{\partial}{\partial z}$$

If 'ϕ' is a scalar point function

① $(\bar{a}\times\nabla)\phi = (\bar{a}\times i)\frac{\partial\phi}{\partial x} + (\bar{a}\times j)\frac{\partial\phi}{\partial y} + (\bar{a}\times k)\frac{\partial\phi}{\partial z} = \sum(\bar{a}\times i)\frac{\partial\phi}{\partial x}$

② If $\bar{f}$ is vector point function then

$$(\bar{a}\times\nabla)\bar{f} = (\bar{a}\times i)\frac{\partial\bar{f}}{\partial x} + (\bar{a}\times j)\frac{\partial\bar{f}}{\partial y} + (\bar{a}\times k)\frac{\partial\bar{f}}{\partial z} = \sum\bar{a}\times i\frac{\partial\bar{f}}{\partial x}$$

(1) Find $(\bar{A}\cdot\nabla)\phi$ at the point $(1,-1,1)$ where $\bar{A} = 3xyz^2 i + 2xy^3 j - x^2yz k$, $\phi = 3x^2 - yz$

$$(\bar{A}\cdot\nabla)\phi = (\bar{A}\cdot i)\frac{\partial\phi}{\partial x} + (\bar{A}\cdot j)\frac{\partial\phi}{\partial y} + (\bar{A}\cdot k)\frac{\partial\phi}{\partial z}$$

$$= 3xyz^2\frac{\partial}{\partial x}(3x^2 - yz) + 2xy^3\frac{\partial}{\partial y}(3x^2 - yz) - x^2yz\frac{\partial}{\partial z}(3x^2 - yz)$$

$$= 3xyz^2(6x) + 2xy^3(-z) - x^2yz(-y)$$

$$(\bar{A}\cdot\nabla)\phi = 18x^2yz^2 - 2xy^3z + x^2y^2z$$

$$(\bar{A}\cdot\nabla)\phi_{(1,-1,1)} = -18 + 2 + 1$$

$$= -15$$

$$(\bar{A}\cdot\nabla)\phi_{(1,-1,1)} = -15$$

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(2) Find $(\bar{A}\times\nabla)\phi$ at the point $(1,-1,1)$ where $\bar{A} = yz^2 i - 3xz^2 j + 2xyz k$ and $\phi = xyz$

$$(\bar{A}\times\nabla)\phi = (\bar{A}\times i)\frac{\partial\phi}{\partial x} + (\bar{A}\times j)\frac{\partial\phi}{\partial y} + (\bar{A}\times k)\frac{\partial\phi}{\partial z}$$

$$(\bar{A}\times i) = yz^2(i\times i) - 3xz^2(j\times i) + 2xyz(k\times i)$$

$$= 3xz^2 k + 2xyz j$$

$$(\bar{A}\times j) = yz^2(i\times j) - 3xz^2(j\times j) + 2xyz(k\times j)$$

$$\Rightarrow yz^2(k) + 2xyz(-i) \Rightarrow yz^2 k - 2xyz i$$

$$(\bar{A}\times k) = yz^2(i\times k) - 3xz^2(j\times k) + 2xyz(k\times k)$$

$$= -yz^2 j - 3xz^2 i$$

$$(\bar{A}\times\nabla)\phi = 3xz^2 k + 2xyz j \frac{\partial}{\partial x}(xyz) + yz^2 k - 2xyz i \frac{\partial}{\partial y}(xyz)$$

$$+ (-yz^2 j - 3xz^2 i)\frac{\partial}{\partial z}(xyz)$$

$$= (3xz^2 k + 2xyz)(yz) + yz^2 k - 2xyz(xz) + (-yz^2 j - 3xz^2 i)(xy)$$

$$(\bar{A}\times\nabla)\phi_{(1,-1,1)} = (3k - 2j)(-1) + (-k + 2i)(1) + (j - 3i)(-1)$$

$$= -3k + 2j - k + 2i - j + 3i$$

$$(\bar{A}\times\nabla)\phi_{(1,-1,1)} = 5i + j - 4k$$

** (3) Find the directional derivative of $\nabla\cdot(\nabla\phi)$ at the point $(1,-2,1)$ in the direction of normal to the surface $xy^2 z = 3x + z^2$ where $\phi = 2x^3y^2z^4$

$\nabla\phi\cdot\frac{\bar{a}}{|\bar{a}|}$ $\nabla\phi\cdot\frac{\nabla f}{|\nabla f|}$

Given,

$$\phi = 2x^3y^2z^4$$

$$\nabla\phi\cdot\hat{a} \quad \frac{\partial\phi}{\partial x} = 6x^2y^2z^4 \quad \frac{\partial\phi}{\partial y} = 4x^3yz^4 \quad \frac{\partial\phi}{\partial z} = 8x^3y^2z^3$$

$$\nabla\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$$

$$= i(6x^2y^2z^4) + j(4x^3yz^4) + k(8x^3y^2z^3)$$

$$(\nabla\phi)_{(1,-2,1)} = 24i - 8j + 32k$$

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The direction of normal vector

$$\bar{F} = xy^2z - 3xz^2$$

$$\nabla F = \frac{dF_1}{dx} = y^2z - 3 \quad \frac{dF_2}{dy} = 2xyz \quad \frac{\partial F_3}{\partial z} = xy^2 - 6z$$

$$\nabla F = (y^2z - 3)i + (2xyz)j + (xy^2 - 6z)k$$

$$(\nabla f)_{(1,-2,1)} = i - 4j + 2k$$

$$|\nabla f| = \sqrt{1+16+4} = \sqrt{21}$$

$$= \nabla\phi \cdot \frac{\nabla f}{|\nabla f|}$$

$$= 24i - 8j + 32k \cdot \frac{i - 4j + 2k}{\sqrt{21}}$$

$$= \frac{24 + 32 + 64}{\sqrt{21}}$$

$$= \frac{120}{\sqrt{21}}$$

(4) Let $f = (x^2+y^2+z^2)^{-n}$ then find div grad f and determine the value of n if $\operatorname{div}(\nabla f) = 0$

Given,

$$f = (x^2+y^2+z^2)^{-n}$$

$$f = (r^2)^{-n}$$

$$f = r^{-2n}$$

$$\operatorname{div}(\operatorname{grad} f) = \nabla \cdot (\nabla f)$$

$$= \nabla^2 f(r)$$

W.K.T $\nabla^2 f(r) = f''(r) + \frac{2}{r} f'(r)$

$$f'(r) = (-2n) r^{-2n-1}$$

$$f''(r) = (-2n)(-2n-1) r^{-2n-2}$$

$$\nabla^2 f(r) = (-2n)(-2n-1) r^{-2n-2} + \frac{2}{r}(-2n) r^{-2n-1}$$

$$= \frac{2n(2n+1)}{r^{2n+2}} - \frac{4n}{r^{2n+2}}$$

$$= \frac{1}{r^{2n+2}}\left[2n(2n+1) - 4n\right]$$

$$= \frac{2n}{r^{2n+2}}\left[2n+1-2\right]$$

$$= \frac{2n(2n-1)}{r^{2n+2}}$$

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Given that $\operatorname{div}(\operatorname{grad} f) = 0$

$$\Rightarrow \frac{2n(2n-1)}{r^{2n+2}} = 0$$

$$(2n)(2n-1) = 0$$

$$n = 0 \quad n = \frac{1}{2}$$

(5) $$\operatorname{curl}\left(\bar{A} \times \frac{\bar{r}}{r^n}\right) = \frac{(2-n)\bar{A}}{r^n} + \frac{n(\bar{A}\cdot\bar{r})\bar{r}}{r^{n+2}}$$

$$\nabla \times \left(\bar{A} \times \frac{\bar{r}}{r^n}\right) = \nabla \times \left(\underbrace{r^{-n}}_{\phi} \underbrace{(\bar{A} \times \bar{r})}_{A}\right)$$

Formula:-

$$\operatorname{curl}(\phi \bar{A}) = \nabla\phi \times \bar{A} + \phi(\nabla \times \bar{A})$$

$$\nabla \times (r^{-n}(\bar{A} \times \bar{r})) = \nabla r^{-n} \times [\bar{A} \times \bar{r}] + r^{-n}[\nabla \times (\bar{A} \times \bar{r})]$$

$$= r^{-n}[\nabla \times (\bar{A} \times \bar{r})] + \nabla r^{-n} \times (\bar{A} \times \bar{r})$$

$$a \times (\bar{b} \times \bar{c}) = (\bar{a}\cdot\bar{c})\bar{b} - (\bar{a}\cdot\bar{b})\bar{c}$$

$$= r^{-n}[\nabla \times (\bar{A} \times \bar{r})] + \nabla r^{-n} \times (\bar{A} \times \bar{r})$$

$$= r^{-n}[(\nabla\cdot\bar{r})\bar{A} - (\nabla\cdot\bar{A})\bar{r}] + \nabla r^{-n} \times (\bar{A} \times \bar{r})$$

$$\bar{r} = xi + yj + zk$$

$$\bar{A} = A_1 i + A_2 j + A_3 k$$

(i) $\nabla\cdot\bar{r}$

$$\bar{r} = xi + yj + zk$$

$$F_1 = x \quad F_2 = y \quad F_3 = z$$

$$\nabla\cdot\bar{r} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z}$$

$$\nabla\cdot\bar{r} = 3$$

(ii) $(\nabla\cdot\bar{A})\bar{r} = (\bar{A}\cdot\nabla)\bar{r}$

$$= (\bar{A}\cdot i)\frac{\partial\bar{r}}{\partial x} + (\bar{A}\cdot j)\frac{\partial\bar{r}}{\partial y} + (\bar{A}\cdot k)\frac{\partial\bar{r}}{\partial z}$$

$$= (A\cdot i)i + (\bar{A}\cdot j)j + (\bar{A}\cdot k)k$$

$$= A_1 i + A_2 j + A_3 k$$

$$(\nabla\cdot\bar{A})\bar{r} = \bar{A}$$

(3) $$\nabla r^{-n} = \sum i \frac{\partial}{\partial x}(r^{-n})$$

$$= \sum i\left(-n r^{-n-1}\frac{\partial r}{\partial x}\right)$$

$$= \sum i\left(-n r^{-n-1}\frac{x}{r}\right)$$

$$\nabla r^{-n} = -n \cdot \frac{\sum xi}{r^{n+2}} = \frac{-n\bar{r}}{r^{n+2}}$$

$$\nabla \times \left(\frac{\bar{a} \times \bar{r}}{r^n}\right) = r^{-n}[(\nabla\cdot\bar{r})\bar{A} - (\nabla\cdot\bar{A})\bar{r}] + \nabla r^{-n} \times (\bar{A} \times \bar{r})$$

$$= r^{-n}[3\bar{A} - \bar{A}] - \frac{n\bar{r}}{r^{n+2}} \times (\bar{A} \times \bar{r})$$

$$= r^{-n} 2\bar{A} - \frac{n}{r^{n+2}}[(\bar{r}\cdot\bar{r})\bar{A} - (\bar{r}\cdot\bar{A})\bar{r}]$$

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$$= \frac{2\bar{A}}{r^n} - \frac{n\bar{A}}{r^n} + \frac{n(\bar{r}\cdot\bar{A})\bar{r}}{r^{n+2}}$$

$$= \frac{\bar{A}(2-n)}{r^n} + \frac{n(\bar{r}\cdot\bar{A})\bar{r}}{r^{n+2}}$$

$$\therefore \quad \nabla \times \left(\frac{\bar{a}\times\bar{r}}{r^n}\right) = \frac{(2-n)\bar{A}}{r^n} + \frac{n(\bar{r}\cdot\bar{A})\bar{r}}{r^{n+2}}$$

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