Unit-6

Special Stochastic Processes

6.1

Poisson Process

There are many practical situations where the random times of occurrences of some specific events are of primary interest. For example, we may want to study the times at which components fail in a large system or the times at which jobs enter the queue in a computer system or the times of arrival of phone calls at an exchange or the times of emission of electrons from the cathode of a vacuum tube. In these examples, our main interest may not be the event itself but the sequence of random time instants at which the events occur. An ensemble of discrete sets of points from the time domain called a **point process** is used to model and analyse phenomena such as the ones mentioned above. An independent increments point process, $i.e.$, a point process with the property that the number of occurrences in any finite collection of non overlapping time intervals are independent r.vs, leads to a Poisson process.

Definition: If $X(t)$ represents the number of occurrences of a certain event in $(0, t)$, then the discrete random process $\{X(t)\}$ is called the ***Poisson process***, provided the following postulates are satisfied:

(i) $P[1 \text{ occurrence in } (t, t+\Delta t)] = \lambda\Delta t + o(\Delta t)$

(ii) $P[0 \text{ occurrence in } (t, t+\Delta t)] = 1 - \lambda\Delta t + o(\Delta t)$

(iii) $P[2 \text{ or more occurrences in } (t, 1+\Delta t) = o(\Delta t)$

(iv) $X(t)$ is independent of the number of occurrences of the event in any interval prior and after the interval $(0, t)$.

(v) The probability that the event occurs a specified number of times in $(t_0, t_0 + t)$ depends only on t, but not on t_0.

Probability Law for the Poisson Process $\{X(t)\}$

Let λ be the number of occurrences of the event in unit time.

Let $P_n(t) = P\{X(t) = n\}$

$\therefore\ P_n(t + \Delta t) = P\{X(t + \Delta t) = n\}$

$= P\{(n-1)$ calls in $(0, t)$ and 1 call in $(t, t + \Delta t)\}$

$+P\ \{n$ calls in $(0, t)$ and no call in $(t, t + \Delta t)\}$

$= P_{n-1}(t)\lambda\Delta t + P_n(t)(1 - \Delta t)$ (by the postulates (i) and (ii))

$$\therefore \quad \frac{P_n(t+\Delta t) - P_n(t)}{\Delta t} = \lambda\{P_{n-1}(t) - P_n(t)\}$$

Taking the limits as $\Delta t \longrightarrow 0$

$$\frac{d}{dt}P_n(t) = \lambda\{P_{n-1}(t) - P_n(t)\} \qquad \text{... (1)}$$

Let the solution of the equation (1) be

$$P_n(t) = \frac{(\lambda t)^n}{n!} f(t) \qquad \text{... (2)}$$

Differentiating (2) with respect to t,

$$P_n'(t) = \frac{\lambda^n}{n!}\{nt^{n-1}f(t) + t^n f'(t)\} \qquad \text{... (3)}$$

Using (2) and (3) in (1),

$$\frac{\lambda^n}{n!}t^n f'(t) = -\lambda\frac{(\lambda t)^n}{n!}f(t)$$

$i.e.,$ $\qquad f'(t) = -\lambda f(t)$

Integrating, $\qquad f(t) = ke^{-\lambda t} \qquad \text{... (4)}$

Taking $n = 0$ in (2) , we get $P_0(t) = f(t)\ \ \forall\ t$

$\therefore\ f(0) = P_0(0)$

$= P\{X(0) = 0\}$

$= P$ {no event occurs in $(0,0)\} = 1$ ... (5)

Using (5) in (4), we get $k = 1$ and hence

$$f(t) = e^{-\lambda t} \qquad \dots (6)$$

Using (6) in (2),

$$P_n(t) = P\{X(t) = n\} = \frac{e^{-\lambda t}(\lambda t)^n}{n!}, \;\; n = 0,,1\,,2, \dots, \dots$$

Thus the probability distribution of $X(t)$ is the Poisson distribution with parameter λt.

Note: *We have assumed that the rate of occurrence of the event λ is a constant, but it can be function of t also. When λ is a constant, the process is called a* ***homogeneous Poisson process****. Unless specified otherwise, the Poisson process will be assumed homogeneous.*

Second Order Probability Function of a Homogeneous Poisson Process:

$P[X(t_1) = n_1,\; X(t_2) = n_2]$

$= P[X(t_1) = n_1]P[X(t_2) = n_2 | X(t_1) = n_1], t_2 > t_1$

$= P[X(t_1) = n_1]P$ [the event occurs $(n_2 - n_1)$ times in the interval of $(t_2 - t_1)$]

$$= \frac{e^{-\lambda t_1}(\lambda t_1)^{n_1}}{n_1!} \frac{e^{-\lambda(t_2-t_1)}\{\lambda(t_2-t_1)\}^{n_2-n_1}}{(n_2-n_1)!}, \text{ if } n_2 \geq n_1$$

$$= \begin{cases} \dfrac{e^{-\lambda t_2}\, \lambda^{n_2}\;\, t_1^{n_1}\, (t_2-t_1)^{n_2-n_1}}{n_1!\, (n_2-n_1)!}, & n_2 \geq n_1 \\ 0, & otherwise \end{cases}$$

Proceeding similarly, we can get the third order probability function as

$P[X(t_1) = n_1, X(t_2) = n_2,\; X(t_3) = n_3]$

$$= \begin{cases} \frac{e^{-\lambda t_3}\lambda^{n_3}\, t_1^{n_1}\,(t_2-t_1)^{n_2-n_1}(t_3-t_2)^{n_3-n_2}}{n_1!\,(n_2-n_1)!\,(n_3-n_2)!}, & n_3 \geq\; n_2 \geq n_1 \\ 0, & otherwise \end{cases}$$

Mean and Autocorrelation of the Poisson Process

The probability law of the Poisson process $\{X(t)\}$ is the same as that of a Poisson distribution with parameter λt.

$$E\{X(t)\} = Var\{X(t)\} = \lambda t$$

$$\therefore \;\; \lambda t = Var\{X(t)\} = E(\{X^2(t)\}) - E(\{X(t)\})^2$$

$$\lambda t = E(\{X^2(t)\}) - \lambda^2 t^2$$

$$\therefore \; E\{X^2(t)\} = \lambda t + \lambda^2 t^2 \qquad \ldots (1)$$

$$R_{xx}(t_1, t_2) = E\{X(t_1)X(t_2)\}$$

$$= E[X(t_1)\,\{X(t_2) - X(t_1) + X(t_1)\}]$$

$$= E[X(t_1)\,\{X(t_2) - X(t_1)\}] + E\{X^2(t_1)\}$$

$$= E[X(t_1)]\;\; E[X(t_2) - X(t_1)] + E\{X^2(t_1)\}$$

since $\{X(t)\}$ is a process of independent increments.

$= \lambda t_1 \lambda(t_2 - t_1) + \lambda t_1 + \lambda^2 t_1^2$, if $t_2 \geq t_1$ [by (1)]

$= \lambda^2 t_1 t_2 + \lambda t_1$, if $t_2 \geq t_1$

or $R_{xx}(t_1, t_2) = \lambda^2 t_1 t_2 + \lambda\, min(t_1, t_2)$

$$C_{xx}(t_1, t_2) \; = R_{xx}(t_1, t_2) - E\{X(t_1\}E\{X(t_2)\}$$

$$= \lambda^2 t_1 t_2 + \lambda t_1 - \lambda^2 t_1 t_2$$

$= \lambda t_1$, if $t_2 \geq t_1$

or $\qquad = \lambda \min(t_1, t_2)$

$r_{xx}(t_1, t_2) = \frac{C_{xx}(t_1,t_2)}{\sqrt{Var\,\{X(t_1)\}}\sqrt{Var\{X(t_2)\}}} = \frac{\lambda t_1}{\sqrt{\lambda t_1\, \lambda t_2}} = \sqrt{\frac{t_1}{t_2}}$, if $t_2 \geq t_1$

Note: *Poisson process is not a stationary process.*

Properties of Poisson Process:

1. The Poisson process is a **Markov process**.

Proof: Consider $P[X(t_3) = n_3 | X(t_2) = n_2,\ X(t_1) = n_1]$

$$= \frac{P[X(t_1)=n_1, X(t_2)=n_2, X(t_3)=n_3]}{P[X(t_1)=n_1, X(t_2)=n_2]}$$

$$= \frac{e^{-\lambda(t_3-t_2)}\lambda^{n_3-n_2}(t_3-t_2)^{n_3-n_2}}{(n_3-n_2)!}$$

[Refer to the second and third order probability functions of the Poisson process]

$$= P[X(t_3) = n_3 | X(t_2) = n_2]$$

This means that the conditional probability distribution of $X(t_3)$ given all the past values $X(t_1) = n_1, X(t_2) = n_2$ depends only on the most recent value $X(t_2) = n_2$.

That is, the Poisson process possesses the Markov property. Hence the result.

2. **Additive Property :**
   Sum of two independent Poisson processes is a Poisson process.

   (**See P1** for proof)

3. Difference of two independent Poisson processes is not a Poisson process

   (**See P2** for proof)

4. The interarrival time of a Poisson process $i.e.$, interval between two successive occurrences of a Poisson process with parameter λ has an exponential distribution with mean $\frac{1}{\lambda}$.

   (**See P3** for proof)

5. If the number of occurrences of an event E in an interval of length t is a Poisson process $\{X(t)\}$ with parameter λt and if each occurrence of E has a constant probability p of being recorded and the recordings are independent of each other then the number $N(t)$ of the recorded occurrences in t is also a Poisson process with parameter $\lambda p t$.

   (**See P4** for proof)

Example 1: Suppose that customers arrive at a bank according to a Poisson process with a mean rate of 3 per minute; find the probability during a time interval of 2 min $(\boldsymbol{i})$ exactly 4 customers arrive and $(\boldsymbol{ii})$ more than 4 customers arrive.

Solution: Mean of the Poisson process $= \lambda t$

Mean arrival rate $=$ mean number of arrivals per minute (unit time) $= \lambda$

Given $\lambda = 3$. We have

$$P\{X(t) = k\} = \frac{e^{-\lambda t}(\lambda t)^k}{k!}$$

$$\therefore \qquad P\{X(2) = 4\} = \frac{e^{-6}6^4}{4!} = 0.133$$

$$P\{X(2) > 4\} = 1 - [P\{X(2) = 0\} + P\{X(2) = 1\} + P\{X(2) = 2\} + P\{X(2) = 3\} + P\{X(2) = 4\}]$$

$$= 1 - \sum_{k=0}^{4} \frac{e^{-6}6^k}{k!} = 0.715$$

Example 2: A machine goes out of order, whenever a component fails. The failure of this part follows a Poisson process with a mean rate of 1 per week. Find the probability that 2 weeks have elapsed since last failure. If there are 5 spare parts of this component in an inventory and that the next supply is not due in 10 weeks, find the probability that the machine will not be out of order in the next 10 weeks.

Solution:

(i) Here the unit times is 1 week
Mean failure rate $=$ mean number of failure in a week $= \lambda = 1$.
P {no failures in the 2 weeks since last failure} $= P\{X(2) = 0\}$

$$= \frac{e^{-2\lambda}(2\lambda)^0}{0!} = e^{-2} = 0.135$$

(ii) There are only 5 spare parts and the machine should not go out of order in the next 10 weeks.
P{for this event} $= P\{X(10) \leq 5\}$

$$= \sum_{k=0}^{5} \frac{e^{-10}10^k}{k!} = 0.068$$

Example 3: If $\{N_1(t)\}$ and $\{N_2(t)\}$ are 2 independent Poisson processes with parameters λ_1 and λ_2 respectively, show that

$$\boldsymbol{P[N_1(t) = k | \{N_1(t) + N_2(t) = n\}] = {}^nC_k\, p^k q^{n-k},}$$ **where**

$$\boldsymbol{p = \frac{\lambda_1}{\lambda_1+\lambda_2}} \text{ and } \boldsymbol{q = \frac{\lambda_2}{\lambda_1+\lambda_2}}$$

Solution: Required conditional probability

$$= \frac{P[\{N_1(t)=k\} \cap \{N_1(t)+N_2(t)=n\}]}{P\{N_1(t)+N_2(t)=n\}}$$

$$= \frac{P[\{N_1(t)=k\} \cap \{N_2(t)=n-k\}]}{P\{N_1(t)+N_2(t)=n\}}$$

$$= \frac{\frac{e^{-\lambda_1 t}(\lambda_1 t)^k}{k!}\frac{e^{-\lambda_2 t}(\lambda_2 t)^{n-k}}{(n-k)!}}{\frac{e^{-(\lambda_1+\lambda_2)t}\{(\lambda_1+\lambda_2)t\}^n}{n!}}$$

(by independence and additive property)

$$= \frac{n!}{k!(n-k)!}\frac{(\lambda_1 t)^k(\lambda_2 t)^{n-k}}{\{(\lambda_1+\lambda_2)t\}^n}$$

$$= {}^nC_k\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^k\left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{n-k}$$

$$= {}^nC_k\, p^k q^{n-k}$$

Example 4: **If customers arrive at a counter in a accordance with a Poison process with a mean rate of 2 per minute, find the probability that the interval between 2 consecutive arrivals is (i) more than 1 min, (ii) between 1 min and 2 min and (iii) 4 min or less.**

Solution: Refer to property 4 of Poisson process.

The interval T between 2 consecutive arrivals follows an exponential distribution with parameter $\lambda = 2$.

(i) $P(T>1) = \int_1^\infty 2e^{-2t}dt = e^{-2} = 0.135$

(ii) $P(1<T<2) = \int_1^2 2e^{-2t}dt = e^{-1} - e^{-2} = 233$

(iii) $P(T\le 4) = \int_0^4 2e^{-2t}dt = 1 - e^{-8} = 0.999$

Example 5: A radioactive source emits particles at a rate of 5 per minute in accordance with Poisson process. Each particle emitted has a probability 0.6 of being recorded. Find the probability that 10 particles are recorded in $4-min$ period.

Solution: Refer to property 5 Poisson processes.
The number of recorded particles $N(t)$ follows a Poisson process with parameter λp. Here $\lambda = 5$ and $p = 0.6$

$$\therefore P\{N(t) = k\} = \frac{e^{-\lambda pt}(\lambda pt)^k}{k!} = \frac{e^{-3t}(3t)^k}{k!}$$

$$\therefore P\{N(4) = 10\} = \frac{e^{-12}(12)^{10}}{10!} = 0.104$$

Example 6: The number of accidents in a city follows a Poisson process with a mean of 2 per day and the number X_i of people involved in the i^{th} accident has the distribution (independent) $P\{X_i = k\} = \frac{1}{2^k}(k \geq 1)$. Find the mean and variance of the number of people involved in accidents per week.

Solution: The mean and variance of the distribution

$P\{X_i = k\} = \frac{1}{2^k}$, $k = 1, 2, 3, \ldots, \ldots$ can be obtained as 2 and 2.

Let the number of accidents on any day be assumed as n.

The number of people involved in these accidents be $X_1, X_2, \ldots X_n$.

$X_1, X_2, \ldots X_n$ are independent and identically distributed $r.vs$ with mean 2 and variance 2.

Therefore, by central limit theorem, $(X_1 + X_2 + \cdots + X_n)$ follows a normal distribution with mean $2n$ and variance $2n$, $i.e.$, the total number of people involved in all the accidents on a day with n accidents is $2n$.

If N denotes number of people involved in accidents on any day, then

$P\{N = 2n\} = P\{X(t) = n\}$ [where $X(t)$ is the number of accidents]

$$= \frac{e^{-2t}(2t)^n}{n!} \text{ (by data)}$$

$$\therefore \qquad E(N) = \sum_{n=0}^{\infty} \frac{2ne^{-2t}(2t)^n}{n!}$$

$$= 2E\{X(t)\} = 4t$$

$$Var\{N\} = E\{N^2\} - E^2(N)$$

$$= \sum_{n=0}^{\infty} \frac{4n^2e^{-2t}(2t)^n}{n!} - 16t^2$$

$$= 4E\{X^2(t)\} - 16t^2$$

$$= 4[Var\{X(t)\} + E^2\{X(t)\}] - 16t^2$$

$$= 4[2t + 4t^2] - 16t^2 = 8t$$

Therefore, mean and variance of the number of people involved in accidents per week are 28 and 56 respectively.

Example 7: If T_n is the r.v denoting the time of occurrence of the n^{th} event in a Poisson process with parameter λ, show that the distribution function $F_n(t)$ of T_n is given by

$$F_n(t) = \begin{cases} 1 - \sum_{k=0}^{n-1} \frac{(\lambda t)^k}{k!} e^{-\lambda t} & , \quad if \quad t \geq 0 \\ 0 & , \quad if \quad t < 0 \end{cases}$$

Deduce the density function $f_n(t)$ of T_n

Solution:

$$F_n(t) = P\{T_n \le t\} = 1 - P\{T_n > t\}$$

when $T_n > t$, $i.e.$, the time of occurrence of the n^{th} event $> t$, $(n-1)$ or less events must have occurred in $(0, t)$

$$\therefore \quad F_n(t) = 1 - P\{X(t) \le n-1\}$$

$$= 1 - \sum_{k=0}^{n-1} \frac{(\lambda t)^k}{k!} e^{-\lambda t}, \text{ when } t \ge 0$$

Differentiating both sides with respect to t and noting that $F_n'(t) = f_n(t)$

$$f_n(t) = -\sum_{k=0}^{n-1} \left\{ -\lambda \frac{(\lambda t)^k}{k!} e^{-\lambda t} + \lambda \frac{(\lambda t)^{k-1}}{(k-1)!} e^{-\lambda t} \right\}$$

$$= \lambda e^{-\lambda t} \sum_{k=0}^{n-1} \left\{ \frac{(\lambda t)^k}{k!} e^{-\lambda t} - \frac{(\lambda t)^{k-1}}{(k-1)!} \right\}$$

$$= \lambda e^{-\lambda t} \left[1 + \left\{ \frac{\lambda t}{1!} - 1 \right\} + \left\{ \frac{(\lambda t)^2}{2!} - \frac{\lambda t}{1!} \right\} + \ldots + \left\{ \frac{(\lambda t)^{n-1}}{(n-1)!} - \frac{(\lambda t)^{n-2}}{(n-2)!} \right\} \right]$$

$$= \frac{\lambda^n t^{n-1} e^{\lambda t}}{(n-1)!}, \quad t \ge 0$$

Example 8 : If $\{X(t)\}$ is a Poisson process, prove that

$$\boldsymbol{P\{X(s) = r | X(t) = n\} = \binom{n}{r} \left(\frac{s}{t}\right)^r \left(1 - \frac{s}{t}\right)^{n-1}} \textbf{ where } \boldsymbol{s} < t$$

Solution:

$$P\{X(s) = r | X(t) = n\} = \frac{P[\{X(s)=r\} \cap \{X(t)=n\}]}{P\{X(t)=n\}}$$

$$= \frac{P\{X(s)=r \cap X(t-s)=n-r\}}{P\{X(t)=n\}}$$

$$= \frac{P\{X(s)=r\}P\{X(t-s)=n-r\}}{P\{X(t)=n\}} \quad \text{(by independence)}$$

$$= \frac{\dfrac{e^{-\lambda s}(\lambda s)^r}{r!} \dfrac{e^{-\lambda(t-s)}\left[\lambda(t-s)^{n-r}\right]}{(n-r)!}}{\dfrac{e^{-\lambda t}(\lambda t)^n}{n!}}$$

$$= \frac{n!}{\boldsymbol{r!}\,(n-r)!} \frac{s^r (t-s)^{n-r}}{t^n} = {}^nC_r \left(\frac{s}{t}\right)^r \left(1-\frac{s}{t}\right)^{n-r}$$