# 2.6

# DOUBLE INTEGRALS CONTINUED

## Change of variables

In the evaluation of repeated integrals, the computational work can often be reduced by changing the variables of integration to some other appropriate variables. The procedure followed in this regard in respect of double integrals is explained below.

Suppose $x$ and $y$ are related to two variables $u$ and $v$ through relations of the form $x = x(u, v), y = y(u, v)$, or $u = u(x, y), v = v(x, y)$. Suppose also that $x, y, u, v$ are such that the Jacobain

$$J = \frac{\partial(x,y)}{\partial(u,v)} \neq 0.$$

Then it can be proved that (-we omit the proof, see our video to get some idea of the following)

$$\iint_{\Re} f(x, y)dxdy = \iint_{\bar{\Re}} \varphi(u, v)Jdudv \qquad (1)$$

Hence $\Re$ is the region in which $(x, y)$ vary, $\overline{\Re}$ is the corresponding region in which $(u, v)$ vary, and $\varphi(u, v) = f\{x(u, v), y(u, v)\}$.

Once the double integral with respect to $x$ and $y$ is changed to a double integral with respect to $u$ and $v$ by using the formula (1), the later integral can be evaluated by expressing it in terms of repeated integrals with appropriate limits of integration.

**Double Integral in Polar form**

As a special case of formula (1), we can obtain the relation connecting a double integral in Cartesian form and the corresponding double integral in polar form.

Let $(r, \theta)$ be the polar coordinates of a point $(x, y)$. Then $x = r\cos\theta\,, y = r\sin\theta$, so that

$$J = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r.$$

Hence, for $(u, v) = (r, \theta)$, formula (1) becomes

$$\iint\limits_{\Re} f(x,y)dxdy = \iint\limits_{\overline{\Re}} \varphi(r,\theta) r dr d\theta \tag{2}$$

Here $\overline{\Re}$ is the region in which $(r, \theta)$ vary as $(x, y)$ vary in $\Re$, and $\varphi(r, \theta) = f(r\cos\theta\,, r\sin\theta)$.

The formula (2) is particularly useful when the region $\Re$ is bounded (in part or whole) by a circle centred at the origin. Observed that when $(x, y)$ are changed to $(r, \theta)$, $dxdy$ is changed to $rdrd\theta$.

## Computation of Area

Let us recall the double integral expression for $f(x,y) \equiv 1$, this expression reads

$$\int_A dA \equiv \iint_{\Re} dxdy = \int_a^b \int_{y_1(x)}^{y_2(x)} dxdy \tag{3}$$

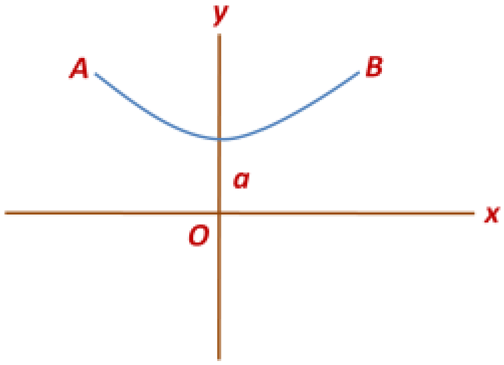
The integral $\int_A dA$ represents the total area $A$ of the plane region $\Re$ over which the repeated integrals are taken. Thus, expression (3) given above may be used to compute the area $A$. We note that $dxdy$ is the plane area element $dA$ in the Cartesian form. By taking $f(x,y) \equiv 1$. We obtain the following formula for area in polar coordinates:

$$\iint_{\Re} dxdy = \iint_{\bar{\Re}} rdrd\theta \tag{4}$$

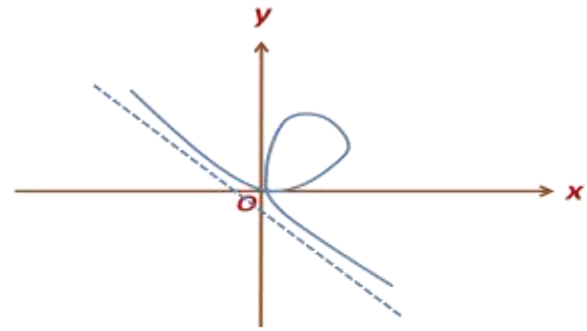
We observe that $rdrd\theta$ is the plane area element in polar form.

## A List of Curves

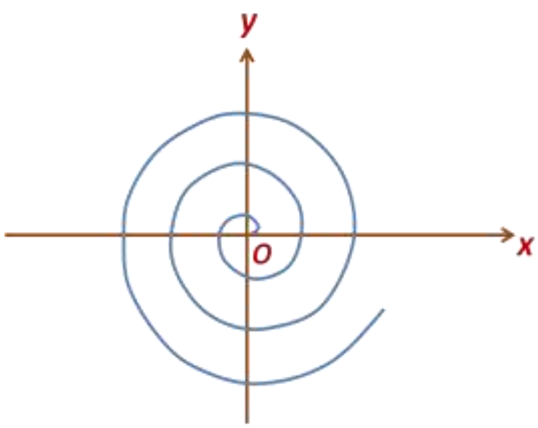
The following list of curves will be useful to find the limits of integration of some problems.



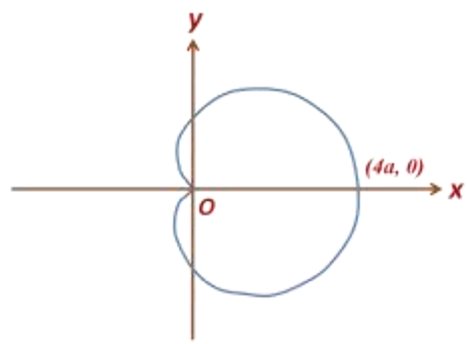

Centenary, $y = a \cosh (x/a)$



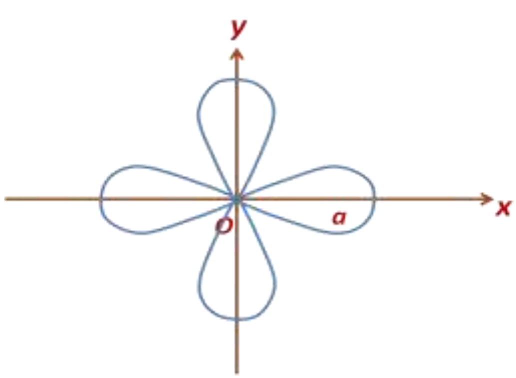

Folium of Descartes, $x^3 + y^3 = 3axy$.
Parametric equations: $x = 3at/(1+t^3)$,
$y = 3at^2/(1+t^3)$.



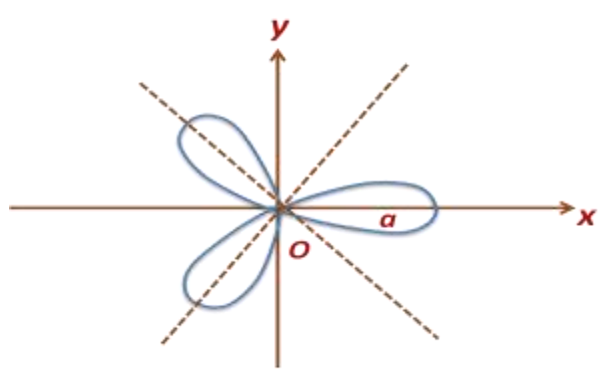

Spiral of Archimedes, $r = a\theta$



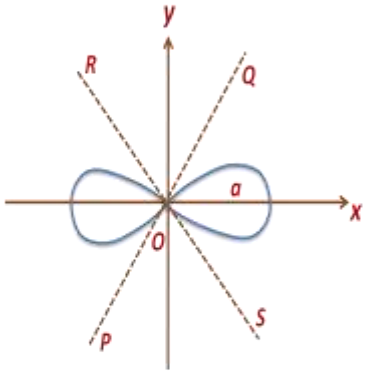

Cardioid, $r = 2a (1 + \cos \theta)$. (Locus of a point on a circle of radius a rolling on the outside of a fixed circle of radius a. The fixed circle has centre at (a, 0) and touches y – axis at the origin.



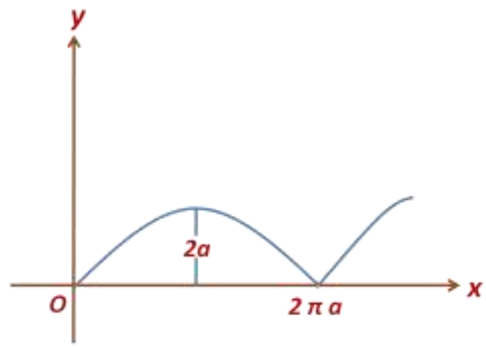

Four leaved rose, $r = a \cos (2\theta)$



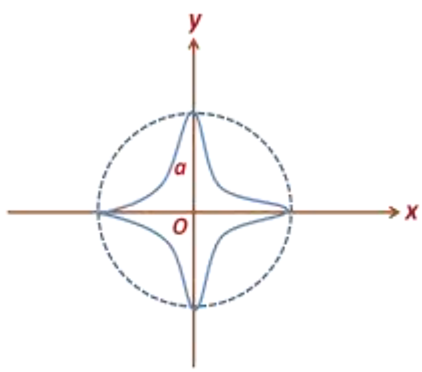

Three leaved rose, $r = a \cos (3\theta)$



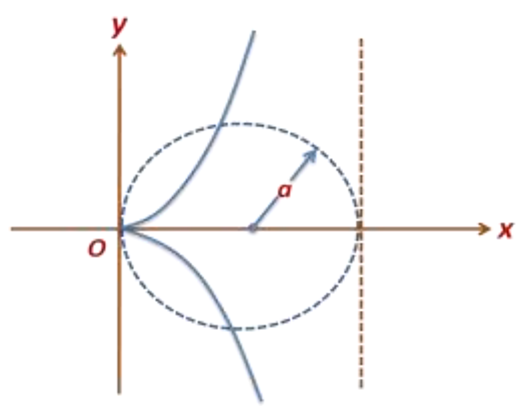

Lemniscate. Polar from: $r^2 = a^2 \cos(2\theta)$.
Rectangular coordinates form: $(x^2 + y^2)^2 = a^2 (x^2 - y^2)$



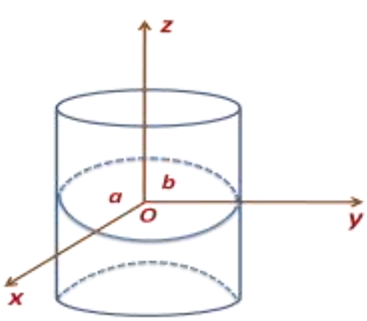

**Cycloid. Parametric from:** $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$. **(Locus of a point P on** *a* **circle of radius a rolling along** $x$ – *axis*. Initially, the circle has centre at $(0, a)$ and touches $x$ – *axis* at the origin).



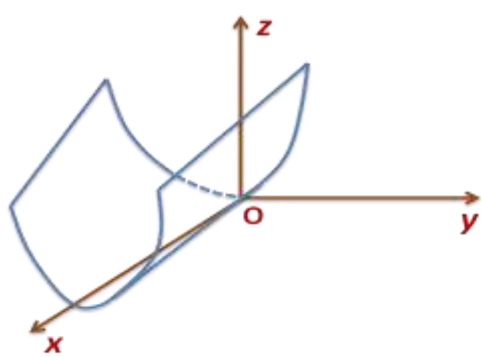

Four cusp hypocycloid. Parametric form:
$x = a\cos^3\theta$, $y = a\sin^3\theta$. Rectangular co – ordinates form: $x^{2/3} + y^{2/3} = a^{2/3}$.
(Locus of a point P on a circle of radius $a/4$, rolling on the inside of a circle of a circle of radius a).



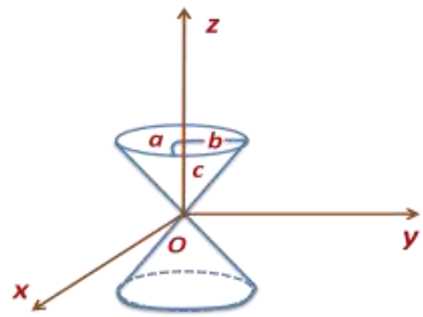

Cissoid of Diocles, $y^2 = x^3 / (2a - x)$



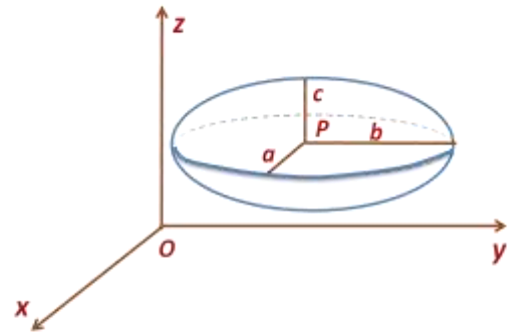

**Elliptic cylinder** $(x^2/a^2) + (y^2/b^2) = 1$, **where** $a$, $b$ **are the** *semi – axis* **of the elliptic cross section. When** $a = b$, **we get the circular cylinder** $x^2 + y^2 = a^2$.



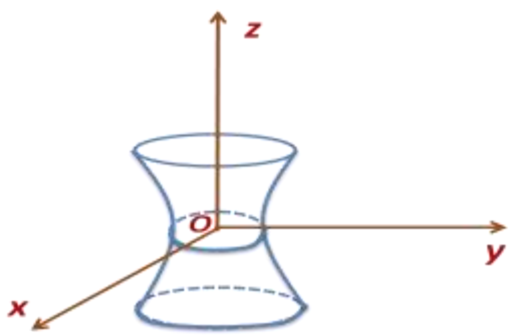

Parabolic cylinder $z = y^2$



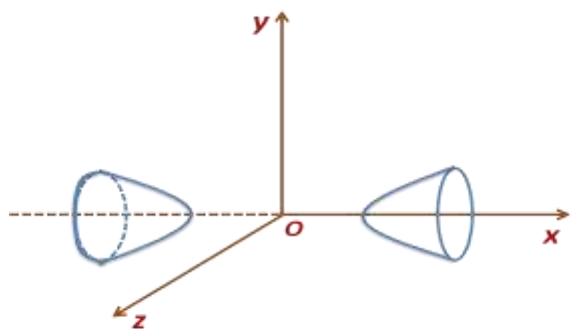

Ellipitic cone (with z – axis as axis $(x^2 / a^2) + (y^2/b^2) = (z^2/c^2)$.
When $a = b$, we get a right circular cone.



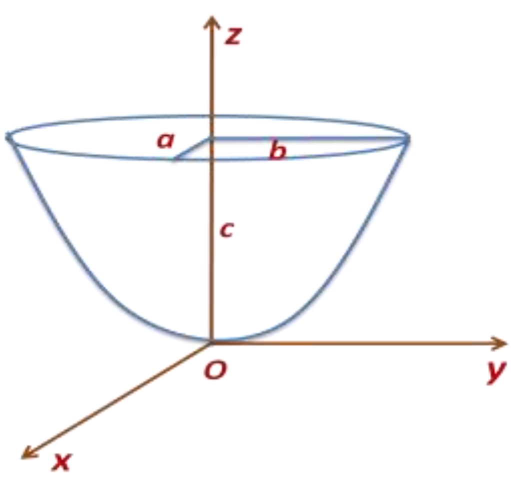

Ellipsoid with centre at $P(x_0, y_0, z_0)$ and semi – axis $a, b, c$.



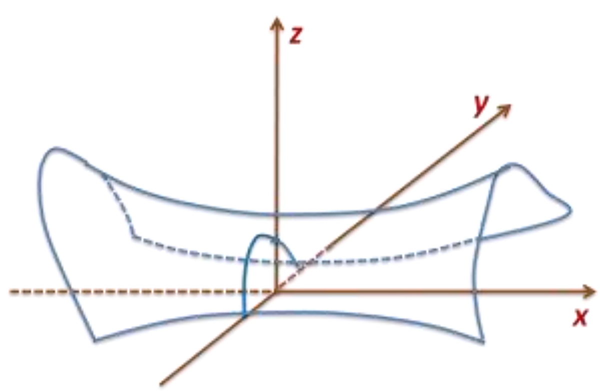

Hyperboloid of one sheet, $(x^2/a^2) + (y^2/b^2) - (z^2/c^2) = 1$.




Hyperboloid of two sheet, $(x^2/a^2) - (y^2/b^2) - (z^2/c^2) = 1$.



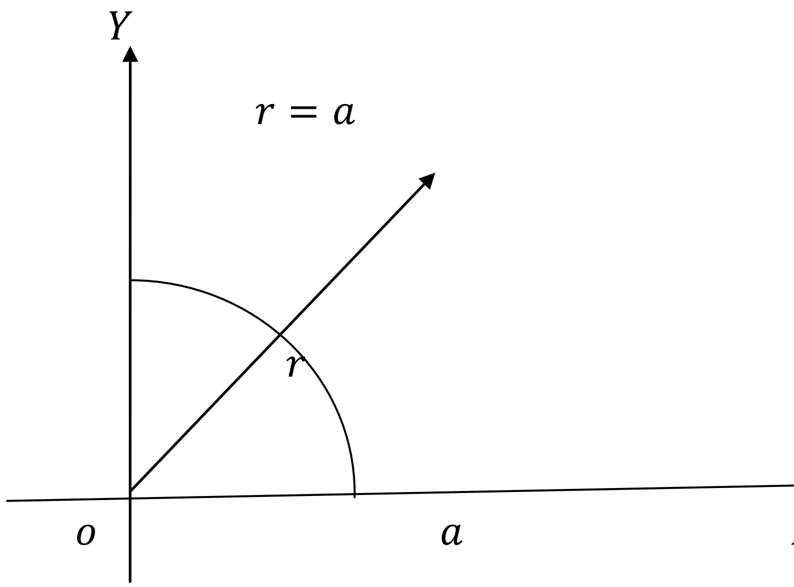

Hyperboloid of one sheet, $(x^2/a^2) + (y^2/b^2) = (z / c)$.




Hyperbolic paraboloid. $(x^2/a^2) - (y^2/b^2) = (z / c)$.

**Problem 1:** Evaluate the double integral $\iint xy\,dx\,dy$ over the positive quadrant bounded by the circle $x^2 + y^2 = a^2$.

**Solution:**




In the positive quadrant bounded by the circle $x^2 + y^2 = a^2$, the radial distance $r$ varies from 0 to $a$ and the polar angle $\theta$ varies from 0 to $\frac{\pi}{2}$. Therefore,

$$\iint xydxdy = \int_{r=0}^{a} \int_{\theta=0}^{\pi/2} (r\cos\theta)(r\sin\theta)(rdrd\theta)$$

$$= \int_0^a r^3 dr \times \int_0^{\pi/2} \sin\theta\cos\theta d\theta = \left[\frac{1}{4}r^4\right]_0^a \times \left[\frac{1}{2}\sin^2\theta\right]_0^{\pi/2}$$

$$= \frac{1}{8}a^4.$$

**Problem 2:** Evaluate the integral $I = \int_0^a \int_{y=0}^{\sqrt{a^2-x^2}} y^2\sqrt{x^2+y^2}\,dydx$ transforming to polar coordinates.

**Solution:** In the given integral, $x$ increases from 0 to $a$ and, for each $x, y$ varies from 0 to $\sqrt{(a^2-x^2)}$. Thus, the lower value of $y$ lies on the $X-$ axis and the upper value of $y$ lies on the curve $y=\sqrt{(a^2-x^2)}$, or $y^2=a^2-x^2$, or $x^2+y^2=a^2$, which is the circle of radius $a$ centred at the origin. Therefore the region $\Re$ of integration is the region in the first quadrant bounded by the circle $x^2+y^2=a^2$,

We note that in $\Re$, $\theta$ varies from 0 from $\frac{\pi}{2}$ and, for each $\theta$, $r$ varies from 0 to $a$. Hence,

$$I = \int_0^a \int_0^{\sqrt{a^2-x^2}} y^2\sqrt{x^2+y^2}$$

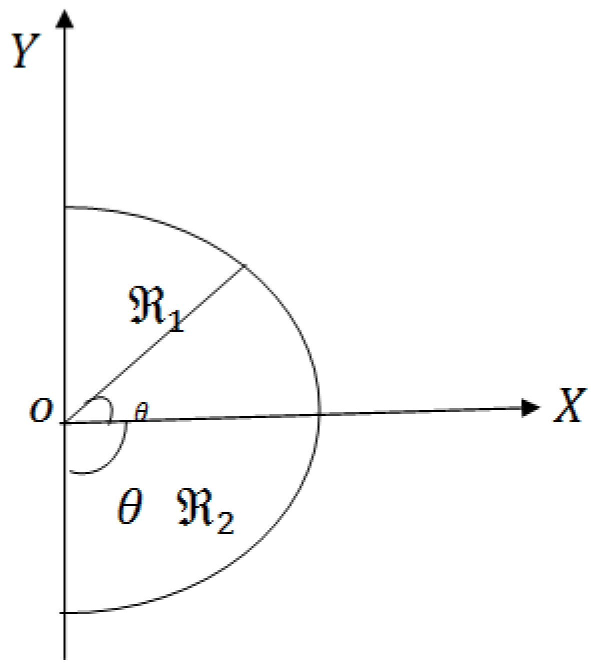
$$= \int_{\theta=0}^{\frac{\pi}{2}} \int_{r=0}^{a} (r^2\sin^2\theta)\, r(rdrd\theta)$$

$$= \left\{\int_0^a r^4 dr\right\} \times \left\{\int_0^{\frac{\pi}{2}} \sin^2\theta d\theta\right\}$$

$$= \frac{a^5}{5}.\frac{1}{2}.\frac{\pi}{2} = \frac{\pi}{20}a^5.$$

**Problem 3:** If $\Re$ is the region $x^2 + y^2 \leq a^2, x \geq 0$, evaluate $\iint_{\Re} (x+y)dxdy.$

**Solution:**




The given region $\Re$ is bounded by the $Y -$ axis and the right side of the circle $x^2 + y^2 = a^2$ as shown in figure. We observe that $\Re$ consists of two parts $\Re_1$ and$\Re_2$. In $\Re_1, \theta$

increases from 0 to $\frac{\pi}{2}$, and in $\Re_2, \theta$ increases from $\frac{3\pi}{2}$ to $2\pi$. In both parts, $r$ varies from 0 to $a$. Therefore,

$$\iint_{\Re} (x+y)dxdy$$

$$= \iint_{\Re_1} (x+y)dxdy + \iint_{\Re_2} (x+y)dxdy$$

$$= \int_{\theta=0}^{\pi/2} \int_{\theta=0}^{a} \left(r\cos\theta + r\sin\theta\right) r dr d\theta + \int_{\theta=3\pi/2}^{2\pi} \int_{r=0}^{a} \left(r\cos\theta + r\sin\theta\right) r dr d\theta$$

$$= \int_{0}^{a} r^{2} dr \times \left\{ \int_{0}^{\pi/2} \left(\cos\theta + \sin\theta\right) d\theta + \int_{3\pi/2}^{2\pi} \left(\cos\theta + \sin\theta\right) \right\}$$

$$= \frac{a^{3}}{3} \left\{ \left(1+1\right) + \left(1-1\right) \right\} = \frac{2}{3} a^{3}.$$

**Problem 4:** Using repeated integrals, find the area bounded by the arc of the ellipse $x^2/a^2 + y^2/b^2 = 1$ in the first quadrant.

**Solution:** In the region of integration $x$ increases from 0 to $a$ and, for each $x$, $y$ increase from 0 to a point on the ellipse; i.e the point for which $y = b(1 - x^2/a^2)^{1/2}$. Hence, the required are is

$$A = \int_{x=0}^{a} \int_{y=0}^{b\left(1-x^2/a^2\right)^{1/2}} dydyx = \int_{x=0}^{a} \left\{ \int_{y=0}^{b\left(1-x^2/a^2\right)^{1/2}} dy \right\} dx$$

$$= \int_0^a b\left(1-\frac{x^2}{a^2}\right)^{1/2} dx = \frac{b}{a}\int_0^a \left(a^2-x^2\right)^{1/2} dx = \frac{b}{a}\int_0^{\pi/2} (a\cos\theta)(a\cos\theta d\theta),$$

on setting $x = a\sin\theta$

$$= ab\int_0^{\frac{\pi}{2}} \cos^2\theta d\theta = ab.\frac{1}{2}.\frac{\pi}{2} = \frac{\pi}{4}ab.$$

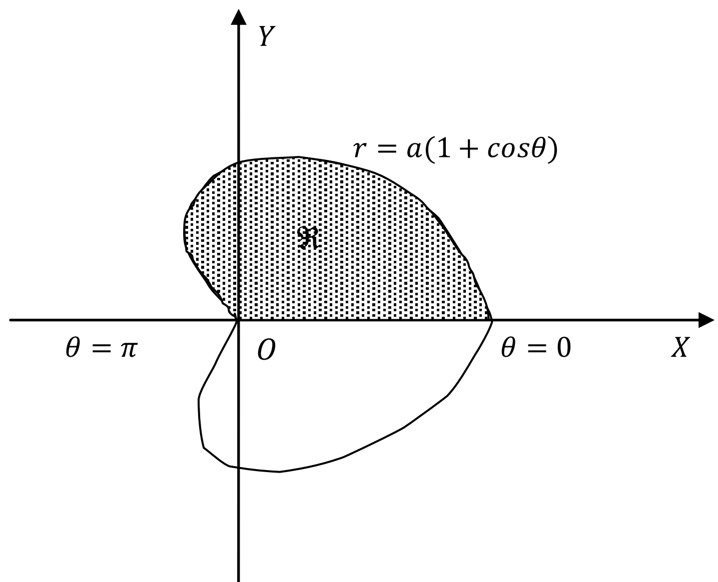
**Problem 5:** Find the areas enclosed by the following curves:

(i) Cardioids: $r = a(1 + cos\theta)$ between $\theta = 0$ and $\theta = \pi$ .

(ii) One loop of the Lemniscate $r^2 = a^2 \cos 2\theta$ .

**Solution:**

(i)




The given region is depicted in figure (shaded portion).In this region, $\theta$ varies from $O$ to $\pi$ and, for each $\theta$, $r$ varies from $O$ to $a(1 + cos\theta)$.

Therefore, the required area is

$$A = \int_{\theta=0}^{\pi} \int_{r=0}^{a(1+cos\theta)} rdr\, d\theta$$

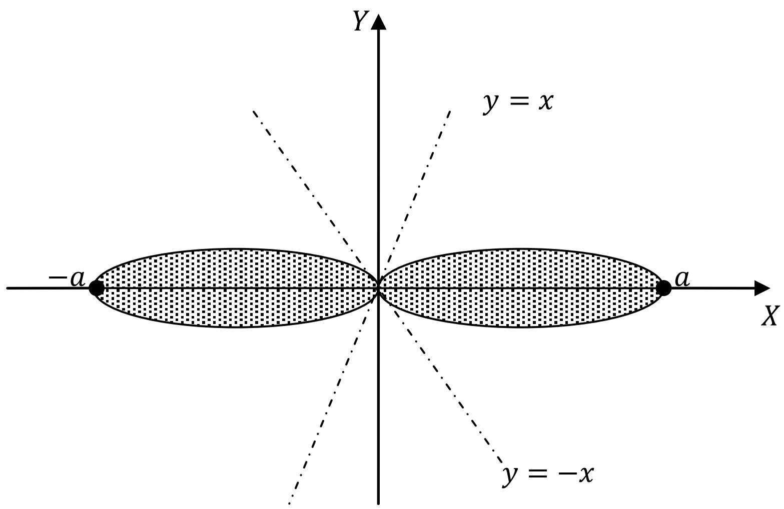
$$= \int_0^{\pi} [\frac{r^2}{2}]_0^{a(1+cos\theta)} d\theta$$

$$= \frac{a^2}{2} \int_0^{\pi} (1 + cos\theta)^2 d\theta$$

$$= \frac{a^2}{2} \int_0^{\pi} \{1 + 2cos\theta + \frac{1}{2}(1 + cos2\theta)\} d\theta$$

$$= \frac{a^2}{2} \left\{\pi + 0 + \frac{1}{2}(\pi + 0)\right\} = \frac{3a^2}{4}\pi.$$

(ii)




The given Lemniscate is shown in figure. We note that the area enclosed by one loop of this curve is twice the area bounded by the first quadrant, for which $\theta$ increases $O$ to $\frac{\pi}{4}$ and, $r$ varies from $O$ to $a\sqrt{cos2\theta}$. Hence the required area is

$$A = 2 \int_{\theta=0}^{\pi/4} \int_{r=0}^{a\sqrt{cos 2\theta}} r \ dr \, d\theta$$

$$= 2 \int_0^{\pi/4} \left\{ \left[\frac{r^2}{2}\right]_0^{a\sqrt{cos 2\theta}} \right\} d\theta$$

$$= \int_0^{\pi/4} a^2 cos2\theta d\theta$$

$$= a^2 \left[\frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{4}} = \frac{a^2}{2}.$$

## Exercise

1. Evaluate the following integrals by changing the Cartesian coordinates to polar coordinates:

   a. $\int_{-a}^{a}\int_{0}^{\sqrt{a^2-x^2}}\sqrt{x^2+y^2}\ dydx$

   b. $\int_{0}^{4a}\int_{y^2/4a}^{y}\frac{x^2-y^2}{x^2+y^2}dxdy$

2. Evaluate the integral $\int_{o}^{\pi}\int_{0}^{a} r^3 sin\theta cos\theta\ dr\ d\theta$ by changing the polar coordinates to Cartesian coordinates.
3. Find the area lying inside the circle $r = a\sin\theta$ and outside the cardioids $r = a(1 - cos\theta)$.
4. Find the area included between the curve $r = a(sec\theta + cos\theta)$ and its asymptote $r = asec\theta$.
5. Find the area bounded by the positive $X$-axis, the arc of the circle $x^2 + y^2 = a^2$ and the upper part of the line $y = x$ .
6. Find the area enclosed by the parabola $y^2 = 4ax$ and the line $x + y = 3a$.
7. Find the area bounded by the parabola $y = 4x - x^2$ and the line $y = x$.
8. Find the area bounded between the circles $r = a$ and $r = 2a\ cos\theta$ .
9. Find the area bounded by the circles $r = 2a\ sin\theta$ and $r = 2b\ sin\theta$ , $b > a > 0$.

## Answers

1.

a. $\left(\frac{1}{3}\right)\pi a^3$

b. $8(\frac{\pi}{2}-\frac{5}{3})a^2$

2. 0
3. $a^2(1-\frac{\pi}{4})$
4. $\frac{5\pi a^2}{4}$
5. $\frac{\pi a^2}{8}$
6. $\frac{10}{3}a^2$
7. $\frac{9}{2}$
8. $a^2(\frac{\pi}{3}+\frac{\sqrt{3}}{2})$
9. $(\mathrm{b}^2-\mathrm{a}^2)\pi$