4.2

Pigeon hole Principle

Suppose that a flock of 20 pigeons flies into a set of 19 pigeonholes to roost. Because there are 20 pigeons but only 19 pigeonholes, at least one of these 19 pigeonholes must have at least two pigeons in it. If each pigeonhole had at most one pigeon in it, at most 19 pigeons, one per hole, could be accommodated. This illustrates a general principle called the **pigeonhole principle**, which states that if there are more pigeons than pigeonholes, then there must be at least one pigeonhole with at least two pigeons in it.

Theorem 1: The Pigeonhole Principle

If $\boldsymbol{k}$ is a positive integer and $\boldsymbol{k+1}$ or more objects are placed in $\boldsymbol{k}$ boxes, then there is at least one box containing two or more objects.

Proof: We give the proof by contraposition. Suppose that none of the k boxes contains more than one object. Then the total number of objects would be at most k. This is a contradiction, because there are at least $k + 1$ objects. Hence the theorem.

The pigeonhole principle is also called the **Dirichlet drawer principle**. It is named after the nineteenth century German mathematician Dirichlet (1805-1859), who often used this principle in his work.

The pigeonhole principle can be used to prove a useful result about functions.

Theorem 2: A function $\boldsymbol{f}$ from a set with $\boldsymbol{k+1}$ or more elements to a set with $\boldsymbol{k}$ elements is not one-to-one.

Proof: Suppose that for each element b in the codomain of f we have a box that contains all elements x of the domain of f such that $f(x) = b$. Thus, we have k boxes and $k + 1$ or more elements of the domain are to be placed in these k boxes. By the pigeonhole principle at least one box receives two or more elements of the domain. That is f cannot be one – to – one.

Example 1: Show that for every positive integer n there is a multiple of n that has only $0s$ and $1s$ in its decimal expansion.

Solution: Let n be a positive integer. Consider that $n + 1$ integers

$$1, \ 11, \ \ 111, \dots, 111 \dots 1$$

where the last integer in this list is the integer with $n + 1, 1s$ in its decimal expansion. First note that the difference of any two in this list has a decimal expansion consisting entirely of $0s$ and $1s$. It is known that there are n possible remainders, $0, 1, 2, \dots, n - 1$, when an integer is divided by n. Because there are $n + 1$ integers in the list, by the pigeonhole principle there must be two integers say p, q in the list with the same remainder say $r \in \{0, 1, \dots, n - 1\}$ when divided by n. Let $p < q$. Then

$$q = tn + r, \ \ p = sn + r$$

for some positive integer $t, \ s$ and $q - p = (t - s)n$

Thus, a multiple of n has only $0s$ and $1s$ in its expansion. Hence the result.

The following example shows how the pigeonhole principle is used:

Example 2: In any group of 27 English words there must be at least two that begin with the same letter, because there are 26 letters in the English alphabet.

The generalized Pigeonhole Principle:

The pigeonhole principle states that there must be at least two objects in the same box when there are more objects than boxes. However, even more can be said when the number of objects exceed a multiple of the number of boxes.

The Generalized Pigeonhole Principle:

If N objects are palced into k boxes, then there is at least one box receiving at least $\lceil N/k \rceil$ objects.

Proof: The proof is given by the method of contradiction. Suppose that none of the boxes receives more than $\lceil N/k \rceil - 1$ objects. Then, the total number of objects is at most

$$k\left(\left\lceil\frac{N}{k}\right\rceil - 1\right) < k\left(\left(\frac{N}{k} + 1\right) - 1\right) = N\,,$$

where the inequality $\left\lceil\frac{N}{k}\right\rceil < \frac{N}{k} + 1$ is used in the above. This is a contradiction, because there are a total of N objects. Hence the theorem.

Example 3: What is the minimum number of students required in a discrete structures class to be sure that at least six will receive the same grade, if there are five possible grades $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D}$ and $\boldsymbol{E}$?

Solution: The minimum number of students required to ensure that at least six students receive the same grade is the smallest positive integer N such that $\left\lceil\frac{N}{5}\right\rceil = 6$, The smallest such integer is $N = 5.5 + 1 = 26$. Thus, 26 is the minimum number of students required to ensure that at least six students will receive the same grade.

Example 4: During a month of $\mathbf{30}$ days, a baseball team plays at least one game a day, but no more than $\mathbf{45}$ games. Show that there must be a period of some number of consecutive days during which the team must play exactly $\mathbf{14}$ days.

Solution: Let a_j be the number of games played on or before the j^{th} day of the month. Then $a_1, a_2, \dots, a_{30}$ is an increasing sequence of distinct positive integers, with $1 \le a_j \le 45$. Notice that $a_1 + 14,\; a_2 + 14, \dots, a_{30} + 14$ is an increasing sequence of distinct positive integers with $15 \le a_j + 14 \le 59$. Now, we have 60 positive integers

$$a_1, a_2, \dots, a_{30}, a_1 + 14, a_2 + 14, \dots, a_{30} + 14$$

which are less than or equal to 59. By pigeonhole principle , two of these integers are equal. Because $a_j,\; j = 1, 2, \dots, 30$ are all distinct and $a_j + 14,\; j = 1, 2, \dots, 30$ are all distinct, there must be indices i and j with

$$a_i = a_j + 14,\ \ i.e.,\ \ a_i - a_j = 14$$

This means that exactly 14 games were played from $(j+1)^{th}$ day to i^{th} day. Hence the result.

Example 5: Show that among $n+1$ positive integers not exceeding $2n$ there must be an integer that divides one of the other integers.

Solution: Let the $n+1$ positive integers, not exceeding $2n$, be $a_1, a_2, \ldots, a_{n+1}$. Let $a_j = 2^{k_j} q_j,\ \ j = 1, 2, \ldots, n+1$, where k_j is a nonnegative integer and q_j is an odd positive integer. Thus, we have $n+1$ odd positive integers $q_1, q_2, \ldots, q_{n+1}$ less than $2n$. Because there are only n odd positive integers less than $2n$, by pigeonhole principle, two of these $q_1, q_2, \ldots, q_{n+1}$ must be equal. Therefore, there are integers i and j such that $q_i = q_j$. Then $a_i = 2^{k_i} q_i$ and $a_j = 2^{k_j} q_i$. If $k_i < k_j$, then $a_i | a_j$; while if $k_i > k_j$, then $a_j | a_i$

Before going to the next theorem, we review some definitions.

Suppose that $a_1, a_2, \ldots, a_N$ is a sequence of real numbers. A **subsequence** of this sequence is a sequence of the form $a_{i_1}, a_{i_2}, \ldots, a_{i_m}$, where

$$1 \le i_1 < i_2 < \cdots < i_m \le N.$$

A sequence is called **strictly increasing** if each term is larger than the one that precedes it and is called **strictly decreasing** if each term is smaller than the one that precedes it.

Theorem 3: Every sequence of n^2+1 distinct real numbers contains a subsequence of length $n+1$ that is either strictly increasing or strictly decreasing.

Proof: Let $a_1, a_2, \ldots, a_{n^2+1}$, be a sequence of n^2+1 distinct real numbers. We associate with each term a_k an ordered pair (i_k, d_k), where i_k is the length of the longest increasing subsequence starting at a_k and d_k is the length of the longest decreasing subsequence starting at a_k. Notice that there are n^2+1 ordered pairs (i_k, d_k).

Assume the contrary. That is, assume that there are no increasing or decreasing subsequence of length $n+1$. Then i_k and d_k are both positive integers less than or equal to n, for $k = 1, 2, \dots, n^2+1$. By product rule there are n^2 possible ordered pairs (i_k, d_k). By the pigeonhole principle, two of these n^2+1 ordered pairs are equal. That is, there exist terms a_s and a_t, with $s < t$ such that $i_s = i_t$ and $d_s = d_t$. We will now show that this leads to contradictions. Because the terms of the sequence are distinct, we have either

$$a_s < a_t \text{ or } a_s > a_t$$

Let $a_s < a_t$. There is an increasing subsequence of length i_t beginning at a_t. Because, $a_s < a_t$, we have an increasing subsequence of length $i_t + 1 > i_s$ starting at a_s -a contradiction.
Let $a_s > a_t$. There is a decreasing subsequence of length i_s beginning at a_s. Because, $a_s > a_t$, we have a decreasing subsequence of length $i_s + 1 > i_t$ starting at a_t - a contradiction. Hence the theorem.

Example 6: Let (x_i, y_i, z_i), $i = 1, 2, \dots, 9$ be a set of nine distinct points with integer coordinates in xyz-space. Show that the midpoint of at least one pair of these points has integer coordinates.

Solution: Two integers x and y are said to be of the **same parity** if either both x and y are odd or both x and y are even.

The midpoint of the line segment joining the points (a, b, c) and (p, q, r) is $\left(\frac{a+p}{2}, \frac{b+q}{2}, \frac{c+r}{2}\right)$. It has integer coordinates if and only a and p have same parity, b and q have same parity, and c and r have same parity.

Note that there are eight possible triples of parity,

$$(e,e,e), (e,e,o), (e,o,e), (e,o,o), (o,e,e), (o,e,o), (o,o,e), (o,o,o),$$

where e and o respectively denote even and odd and each point (x_i, y_i, z_i), $i = 1, 2, \dots, 9$ has a parity triple. By the pigeonhole principle, at least two of the nine points have the same triple of partities. Thus, the midpoint of the line segment joining such points has integer coordinates.

Example 7:

a) **Show that if five integers are selected from the first eight positive integers, then there must be a pair of these integers with a sum equal to 9.**
b) **Is the conclusion in part (a) true if four integers are selected rather than five?**

Solution: Let $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$

a) Partition the set A into 4 subsets $\{1, 8\}, \{2, 7\}, \{3, 6\}$ and $\{4, 5\}$. Each consisting of two integers whose sum is 9. If five integers are selected from A then by pigeonhole principle, at least two must be from the same subset and the sum of these two integers is 9.
b) The conclusion in part (a) is false if four integers are selected.
   Take $A = \{1, 2, 3, 4\}$. Then the conclusion is false.

Example 8: In any group of six people each pair of individuals consists of two friends or two enemies. Prove that there are either 3 mutual friends or 3 mutual enemies.

Solution: Let A, B, C, D, E and F be six people. Consider A. Of the remaining five other people B, C, D, E and F, there are either three or more who are friends of A or three or more who are enemies of A. This follows from the generalized pigeon hole principle, because when five objects are divided into two sets, one of these sets has at least $\left\lceil \frac{5}{2} \right\rceil = 3$ objects.

In the first case, suppose that $B, C$and D are friends of A. If any two of these three are friends say B, C then A, B and C are mutual friends. Otherwise (*i.e.,* no two of B, C and D are friends) B, C and D are mutual enemies.

In the second case, suppose that that $B, C$and D are enemies of A. if any two of these three are enemies say B, C then A, B and C are mutual enemies. Otherwise (*i.e.,* no two of B, C and D are enemies) B, C and D are mutual friends.

Hence the result

Example 9: Show that in a group of 10 people (where any two people are either friends or enemies) there are either three mutual friends or four mutual enemies and there are either three mutual enemies or four mutual friends.

Solution: By symmetry we need to prove only the first statement,$i.e$, there are either 3 mutual friends or 4 mutual enemies.

Let A be one of the ten people. Of the remaining 9 other people, there are $\left\lceil\frac{9}{2}\right\rceil = 5$ or more who are enemies of A. Suppose that A has at most 5 enemies and at most 3 friends. Then the total people are $5 + 3 = 8$ people, which is a contradiction, because we have 9 people.Therefore A has at least 4 friends or at least 6 enemies in the remaining 9 people.

Case (i): Suppose that A has 4 friends, say B, C, D and E. If any two of these say, B, C ,are friends then we have found three (A,B and C) mutual friends. Otherwise $\{\mathrm{B}, \mathrm{C}, \mathrm{D}, \mathrm{E}\}$ is a set of four mutual enemies.

Case(ii): Let $\{\mathrm{B}, \mathrm{C}, \mathrm{D}, \mathrm{E}, \mathrm{F}, \mathrm{G}\}$ be the set of 6 enemies of A. It is known that among any six people there are either three mutual friends or three mutual enemies. These three mutual enemies form with A, a set of four mutual enemies. Hence the result.