UNIT - IV

CIRCUIT ANALYSIS USING LAPLACE TRANSFORM

- The laplace transform is a very powerful tool for analyzing circuits with sinusoidal or non sinusoidal inputs.

- We use laplace transformation method to transform the circuit from time domain to s-domain (or frequency domain, obtain the solution, and apply the inverse laplace transform to the result to transform it back to time domain.

- Laplace transform is significant because,

  i) It can be applied to a wide variety of inputs

  ii) It provides an easy way to solve circuit problems involving initial conditions, because it allows us to work with algebraic equations instead of differential equations.

  iii) It is capable of providing us, the total response of the circuit comprising both the natural and forced responses, in one single operation.

Definition of laplace transform:

Given a function $f(t)$, its laplace transform, is denoted by $F(s)$ ~~and~~ as $L[f(t)]$, it is given by,

$$L[f(t)] = F(s) = \int_{0^-}^{\infty} f(t)\, e^{-st}\, dt$$

where s is a complex variable given by,

$$s = \sigma + j\omega.$$

{ ↳ Lower limit 0^-, indicates a time just before $t=0$ }

" The laplace transform is an integral transformation of a function $f(t)$ from the time domain into complex frequency domain, giving $F(s)$.

Laplace Transform pairs

$\delta(t)$	-	1
$u(t)$	-	$\frac{1}{s}$
e^{-at}	-	$\frac{1}{s+a}$
t	-	$\frac{1}{s^2}$
t^n	-	$\frac{n!}{s^{n+1}}$
$t \cdot e^{-at}$	-	$\frac{1}{(s+a)^2}$
$t^n \cdot e^{-at}$	-	$\frac{n!}{(s+a)^{n+1}}$
$\sin \omega t$	-	$\frac{\omega}{s^2+\omega^2}$
$\cos \omega t$	-	$\frac{s}{s^2+\omega^2}$
$e^{-at} \sin(\omega t)$	-	$\frac{\omega}{(s+a)^2+\omega^2}$
$e^{-at} \cos(\omega t$	-	$\frac{s+a}{(s+a)^2+\omega^2}$

$\sin(\omega t+\theta)$ - $\dfrac{s\sin\theta + \omega\cos\theta}{s^2+\omega^2}$

$\cos(\omega t+\theta)$ - $\dfrac{s\cos\theta - \omega\sin\theta}{s^2+\omega^2}$

Properties

Properties of laplace Transforms

	$f(t)$	$F(S)$	Property
$\rightarrow$	$a_1 f_1(t) + a_2 f_2(t)$	$a_1 F_1(s) + a_2 F_2(s)$	Linearity
$\rightarrow$	$f(at)$	$\frac{1}{a} F\left(\frac{s}{a}\right)$	Scaling
$\rightarrow$	$f(t-a)\,u(t-a)$	$e^{-as} F(s)$	Time shift
$\rightarrow$	$e^{-at} f(t)$	$F(s+a)$	Frequency shift
$\rightarrow$	$\frac{df(t)}{dt}$	$sF(s) - f(0^-)$	Time differentiation.
	$\frac{d^2 f(t)}{dt^2}$	$s^2 F(s) - sf(0^-) - f'(0^-)$	"
	$\frac{d^3 f(t)}{dt^3}$	$s^3 F(s) - s^2 f(0^-) - s f'(0^-) - f''(0)$	"
$\rightarrow$	$\int_0^t f(t)\, dt$	$\frac{1}{s} F(s)$	Time integration
$\rightarrow$	$t f(t)$	$-\frac{d}{ds} F(s)$	Frequency differentiation
$\rightarrow$	$f(0^+)$	$\lim_{s \to \infty} sF(s)$	Initial value
$\rightarrow$	$f(\infty)$	$\lim_{s \to 0} sF(s)$	Final value
$\rightarrow$	$f_1(t) * f_2(t)$	$F_1(s) * F_2(s)$	Convolution.

Circuit Element models:

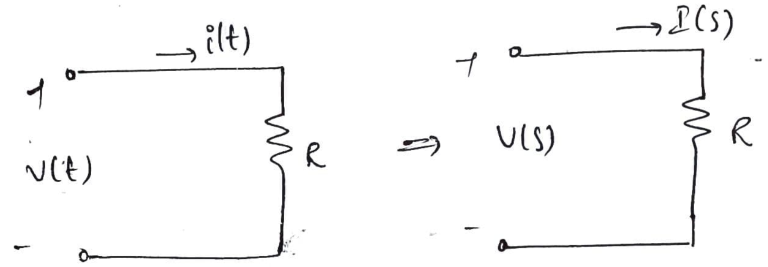
Resistor:

For a resistor, the voltage - current relationship in the time domain is,

$$v(t) = R\,i(t)$$

Taking laplace transforms, we get,

$$\boxed{V(s) = R\,I(s)}$$



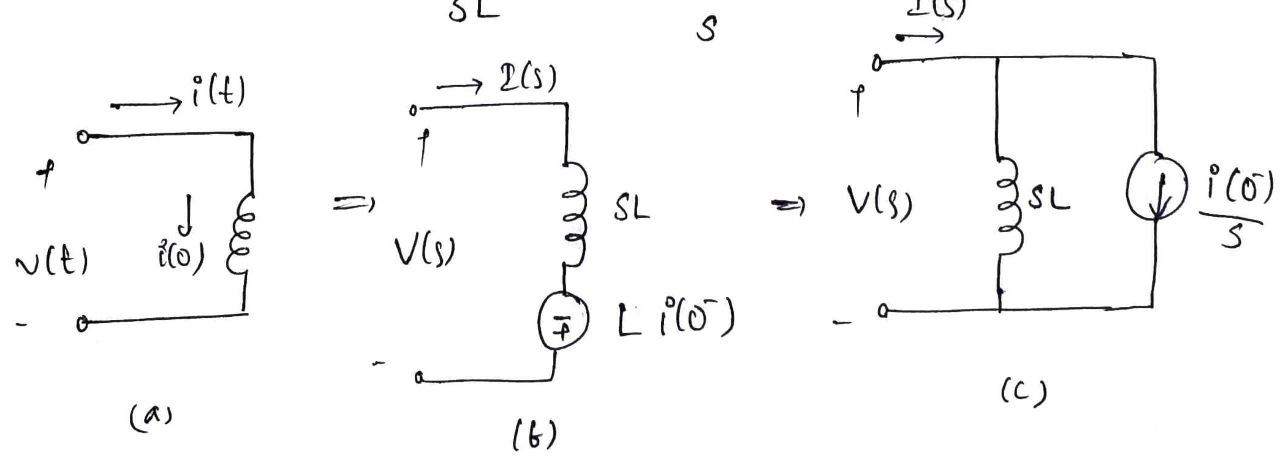
Inductor:

$$v(t) = L\,\frac{di(t)}{dt}$$

Taking laplace transform on both sides,

$$V(s) = L\left[s\,I(s) - i(0^-)\right] = sL\,I(s) - L\,i(0^-)$$

$$\Rightarrow I(s) = \frac{1}{sL}V(s) + \frac{i(0^-)}{s}$$



where fig (a) is time domain circuit, (b, c) are s-domain circuits. for inductor.

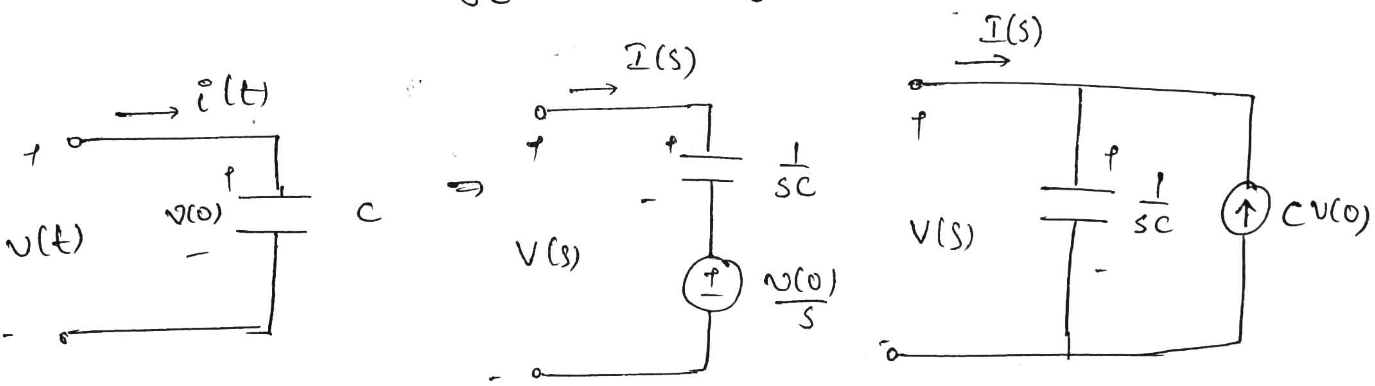
Capacitor:

$$i(t) = C\frac{dv(t)}{dt}$$

Take laplace transform,

$$I(s) = C\left[sV(s) - v(0^-)\right] = sCV(s) - Cv(0^-)$$

$$\Rightarrow V(s) = \frac{1}{sC}I(s) + \frac{v(0^-)}{s}$$



(a) (b) (c)

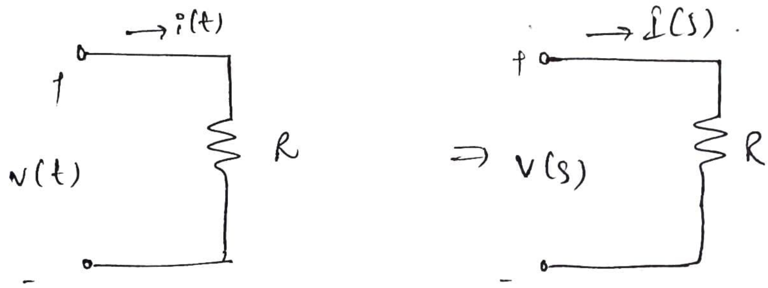
where (a) is time domain equivalent of capacitor and

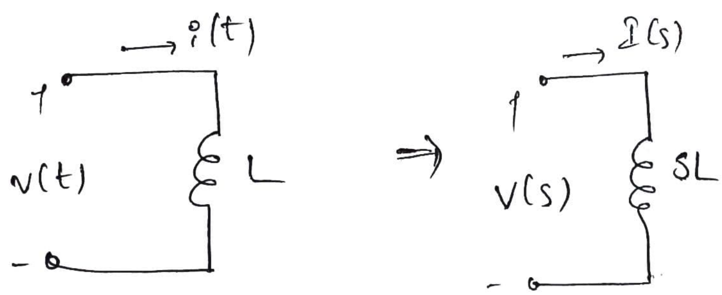
b, c are s domain " " ".

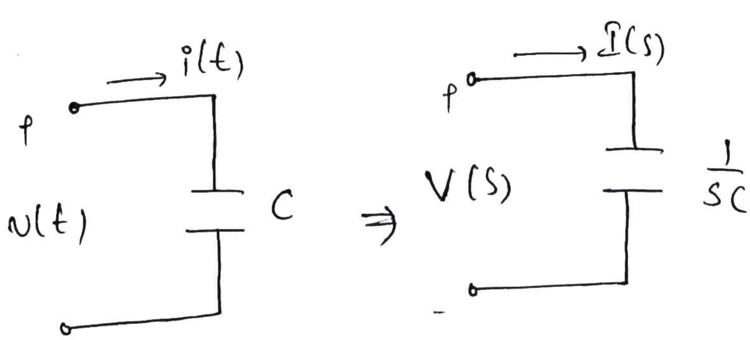
Without Initial conditions:

If we assume zero initial conditions for the inductor & capacitor, then the previous equations get reduced to,

		Impedance.
Resistor	: $V(S) = R\,I(S)$	$Z(S) = \frac{V(S)}{I(S)} = R$
Inductor	: $V(S) = SL\,I(S)$	$Z(S) = \frac{V(S)}{I(S)} = SL$
Capacitor	: $V(S) = \frac{1}{SC} I(S).$	$Z(S) = \frac{V(S)}{I(S)} = \frac{1}{SC}.$





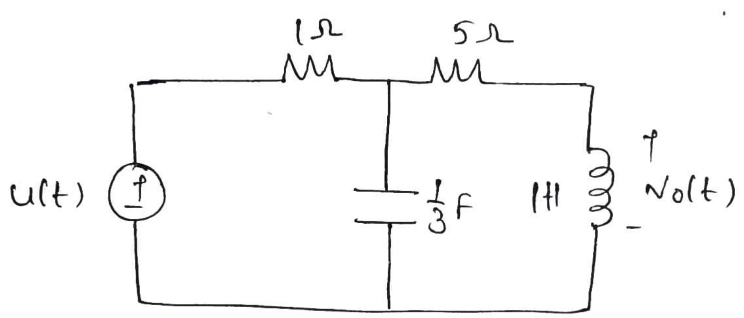


[Impedance is defined as the ratio of voltage transform to the current transform under zero initial conditions, i.e $Z(S) = \frac{V(S)}{I(S)}$]

Admittance in s-domain is the reciprocal of the impedance,

$$Y(s) = \frac{1}{Z(s)} = \frac{I(s)}{V(s)}$$

Q1) find $v_o(t)$ in the below circuit, assuming zero initial conditions.



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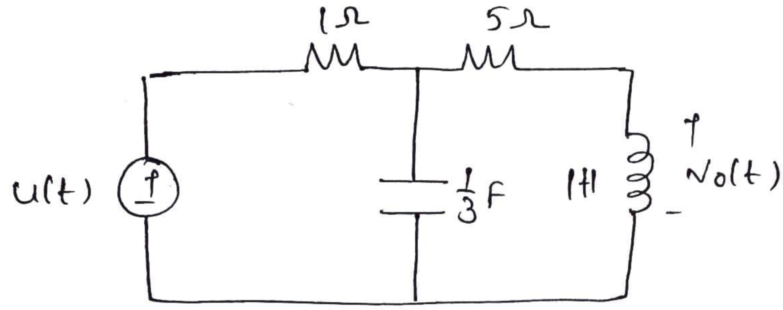


}

* Steps in applying the laplace-transform :

1) Transform the circuit from time domain to s-domain

2) Solve the circuit using nodal analysis, mesh analysis, source transformation, superposition or any circuit analysis technique with which we are familiar

3) Take the inverse laplace transform of the solution & obtain the solution in the time domain. *

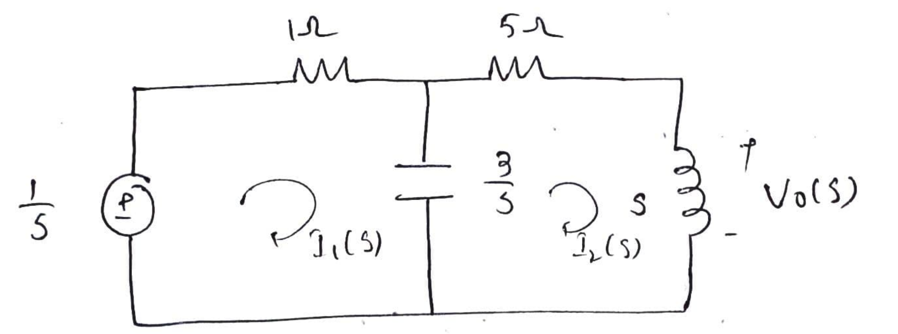
Sol) 1) Transform the circuit from time domain to s-domain

$$u(t) \xrightarrow{LT} \frac{1}{s}$$

Z_L ~~in KH~~ $\longrightarrow sL$ here $s(1) = s$

$Z_C \longrightarrow \frac{1}{sC}$ here $\frac{1}{s(\frac{1}{3})} = \frac{3}{s}$.

Now, the circuit is redrawn in s-domain, as below,



Let the loop currents be $I_1(s)$, $I_2(s)$.

Apply mesh analysis,

<u>loop1</u> :

$$\frac{1}{s} = 1(I_1(s)) + \frac{3}{s}(I_1(s) - I_2(s))$$

$$\frac{1}{s} = I_1(s)\left(1+\frac{3}{s}\right) - \frac{3}{s} I_2(s) \rightarrow (1)$$

<u>loop2</u> :

$$5 I_2(s) + s(I_2(s)) + \frac{3}{s}(I_2(s) - I_1(s)) = 0$$

$$\Rightarrow -\frac{3}{s} I_1(s) + \left(5+s+\frac{3}{s}\right) I_2(s) = 0$$

$$\Rightarrow \frac{3}{s} I_2(s) = \left(5+s+\frac{3}{s}\right) I_1(s)$$

$$\Rightarrow I_1(s) = \frac{s+3+\frac{3}{s}}{\frac{3}{s}} I_2(s)$$

$$I_1(s) = \frac{s^2+5s+3}{3} I_2(s). \quad \rightarrow (2)$$

Sub eq (2) in eq (1),

$$\Rightarrow \frac{1}{s} = \left(1+\frac{3}{s}\right)\left(\frac{s^2+5s+3}{3}\right) I_2(s) - \frac{3}{s} I_2(s)$$

$$\Rightarrow \frac{1}{s} = \left(\frac{s+3}{s}\right)\left(\frac{s^2+5s+3}{3}\right) I_2(s) - \frac{3}{s} I_2(s)$$

$$\Rightarrow 1 = \frac{(s+3)(s^2+5s+3)}{3} I_2(s) - \frac{3}{\cancel{s}} I_2(s).$$

$$\Rightarrow 3 = \left[(s+3)(s^2+5s+3) - 3\times 3\right] I_2(s)$$

$$\Rightarrow 3 = (s^3+8s^2+18s+9-9)\, I_2(s)$$

$$\Rightarrow 3 = (s^3+8s^2+18s)\, I_2(s). \Rightarrow I_2(s) = \frac{3}{s^3+8s^2+18s}$$

$$\Rightarrow \cancel{I_2(s)} = \cancel{\frac{s^3+8s^2+18s}{3}}$$

we need to find $v_o(t)$.

from the circuit in laplace domain,

$$V_o(s) = I_2(s) \times (sL)$$

$$\Rightarrow V_o(s) = I_2(s) \times (s \times 1)$$

$$= s\, I_2(s).$$

$$\Rightarrow V_o(S) = S \times \frac{3}{S^3+8S^2+18S}$$

$$= S \times \frac{3}{S(S^2+8S+18)}$$

$$= \frac{3}{S^2+8S+18}$$

$$= \frac{3}{S^2+2(S)(4)+4^2-4^2+18}$$

$$= \frac{3}{(S+4)^2+2}$$

$$\Rightarrow V_o(S) = \frac{3}{(S+4)^2+(\sqrt{2})^2} = \frac{3}{\sqrt{2}} \times \left\{\frac{\sqrt{2}}{(S+4)^2+(\sqrt{2})^2}\right\}$$

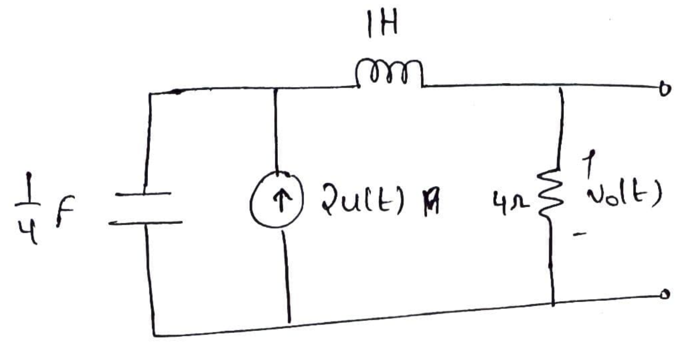
It is of the form $\frac{\omega}{(S+a)^2+\omega^2}$, where $\omega = \sqrt{2}$, $a=4$

It is the laplace transform of $e^{-at}\sin\omega t$

$$\therefore V_o(t) = L^{-1}[V_o(S)] = \frac{3}{\sqrt{2}} L^{-1}\left[\frac{\sqrt{2}}{(S+4)^2+(\sqrt{2})^2}\right]$$

$$\boxed{V_o(t) = \frac{3}{\sqrt{2}} \cdot e^{-4t} \sin\sqrt{2}t \; V.} \quad , t \geq 0.$$

Q2) Determine $v_o(t)$, assuming zero initial conditions.



Sol) 1) Convert the circuit from time domain to s-domain,

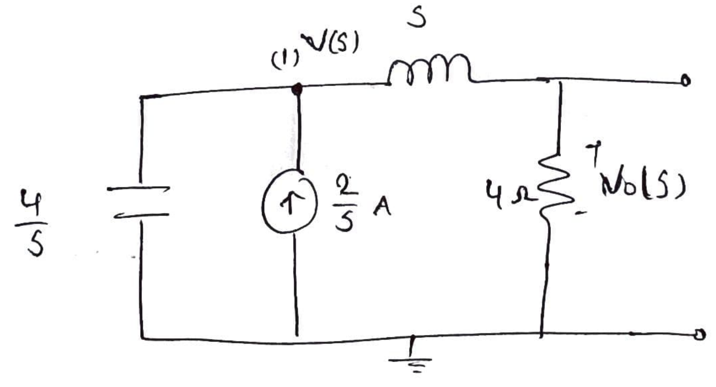
$2u(t) \xrightarrow{LT} \frac{2}{s}$

$Z_L \longrightarrow sL$ here $s(1) = s$

$Z_C \longrightarrow \frac{1}{sC}$ here $\frac{1}{s\left(\frac{1}{4}\right)} = \frac{4}{s}$

$Z_R \longrightarrow Z_R = R$ here 4.

Circuit in s-domain:



Apply nodal analysis at node 1. Let the unknown voltage at node 1 be V(S).

(Assume all currents are outgoing except the incoming current source).

$$\frac{2}{s} = \frac{V(s)}{\frac{4}{s}} + \frac{V(s)}{(s+4)}$$

$$\Rightarrow \frac{2}{s} = \left(\frac{s}{4} + \frac{1}{s+4}\right) V(s)$$

$$\Rightarrow \frac{2}{s} = \frac{s^2+4s+4}{(s+4)4} V(s)$$

$$\Rightarrow V(s) = \frac{8(s+4)}{s(s^2+4s+4)}$$

Current flowing through 4Ω is, (let it be $I(s)$)

$$I(s) = \frac{V(s)}{(s+4)} = \frac{\frac{8(s+4)}{s(s^2+4s+4)}}{(s+4)}$$

$$= \frac{8}{s(s^2+4s+4)}$$

$$\therefore V_o(s) = I(s) \times 4 = \frac{8}{s(s^2+4s+4)} \times 4$$

$$= \frac{32}{s(s^2+4s+4)}$$

$$= 32\left[\frac{A}{s} + \frac{Bs+C}{s^2+4s+4}\right]$$

$$\left\{ \frac{A}{s} + \frac{Bs+C}{s^2+4s+4} = \frac{1}{s(s^2+4s+4)} \right.$$

$$\Rightarrow A(s^2+4s+4) + (Bs+C)s = 1$$

$$\Rightarrow (A+B)s^2 + (4A+C)s + 4A = 1$$

Comparing s^2, s, constant co-efficients,

s^2: $A+B=0$ $\Rightarrow A=-B$

s: $4A+C=0$ $\Rightarrow 4A=-C$

constant: $4A=1$ $\Rightarrow \boxed{A=\frac{1}{4}}$

$\Rightarrow$ from,

$$C=-4A$$

$$= -4 \times \frac{1}{4} = -1$$

$$\Rightarrow \boxed{C=-1}$$

and from $B=-A$

$$\boxed{B=-\frac{1}{4}}$$

$\}$

$$\therefore \quad V_o(s) = 32\left[\frac{\frac{1}{4}}{s} + \frac{-\frac{1}{4}s-1}{s^2+4s+4}\right]$$

$$= \frac{8}{s} + \frac{-8s-32}{(s+2)^2}$$

$$= \frac{8}{s} + \frac{-8s}{(s+2)^2} - \frac{32}{(s+2)^2}$$

$$= \frac{8}{s} - 8\,\frac{s+2-2}{(s+2)^2+0^2} - \frac{32}{(s+2)^2}$$

$$= \frac{8}{s} - 8\,\frac{(s+2)}{(s+2)^2+0^2} + \underbrace{\frac{16}{(s+2)^2} - \frac{32}{(s+2)^2}}$$

$$= \frac{8}{s} - 8 \times \frac{s+2}{(s+2)^2+0^2} - \frac{16}{(s+2)^2}$$

$$v_o(t) = L^{-1}\left[V_o(s)\right]$$

$$= L^{-1}\left[\frac{8}{s} - 8 \times \frac{s+2}{(s+2)^2+0^2} - \frac{16}{(s+2)^2}\right]$$

$$= L^{-1}\left[\frac{8}{s}\right] - 8\,L^{-1}\left[\frac{s+2}{(s+2)^2+0^2}\right] - 16\,L^{-1}\left[\frac{1}{(s+2)^2}\right]$$

$$= 8\,u(t) - 8 \times e^{-2t}\cos 0t - 16t \cdot e^{-2t}$$

$$\left[\because\ L^{-1}\left[\frac{1}{s}\right] = u(t)\right.$$

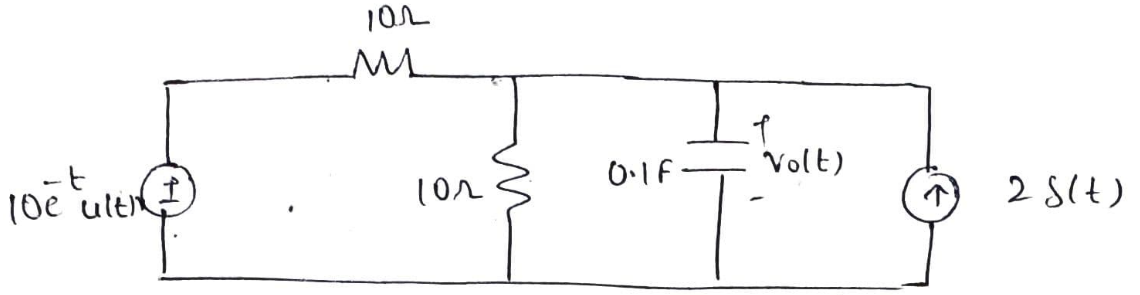
$$L^{-1}\left[\frac{s+a}{(s+a)^2+\omega^2}\right] = e^{-at}\cos\omega t$$

$$\left.L^{-1}\left[\frac{1}{(s+a)^2}\right] = e^{-at}\,t\right]$$

$$\Rightarrow \boxed{v_o(t) = 8\left[1 - e^{-2t} - 2te^{-2t}\right]u(t)\ V}$$

$$\text{(or) } v_o(t) = 8\left(1 - e^{-2t} - 2te^{-2t}\right) V,\ t \geq 0$$

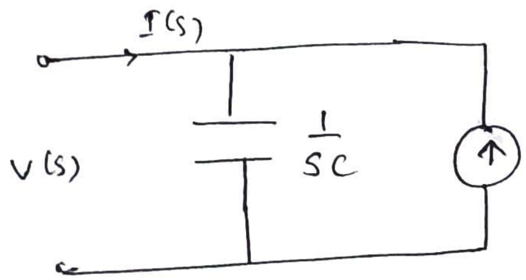
Q3) Find $v_o(t)$ in the circuit, assume $v_o(0) = 5V$



Sol) Convert all the elements into s-domain,

$$e^{-t}u(t) \xrightarrow{LT} \frac{1}{s+1}$$

capacitor is having some initial voltage, so it can be modelled as current source in parallel, ie the initial condition is included in the form of a current source.

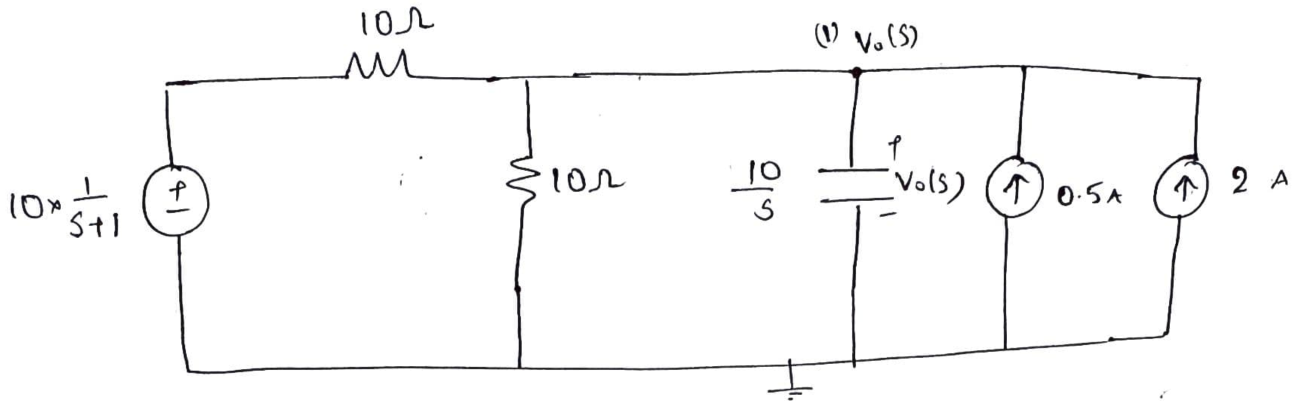


$CV_o = 0.1 \times 5 = 0.5A.$

$$\frac{1}{sC} \longrightarrow \frac{1}{s \times 0.1} = \frac{10}{s}.$$

$$\delta(t) \xrightarrow{LT} 1.$$

Now, circuit can be re-drawn in s-domain,



If we apply nodal analysis, we have to identify the no. of nodes.

Here there are 2 nodes, bottom node is taken as reference node, voltage is 0V.

Top node is having a voltage of Vo(S), from the circuit.

∴ Apply KCL at node 1.

Assume all currents are outgoing except 0.5 A, 2A which are incoming.

$$\frac{V_o(S) - \frac{10}{S+1}}{10} + \frac{V_o(S)}{10} + \frac{V_o(S)}{\frac{10}{S}} = 0.5 + 2$$

$$\Rightarrow V_o(S) - \frac{10}{S+1} + V_o(S) + V_o(S) \times S = 10(2.5)$$

$$\Rightarrow \quad V_o(S)\left(S+2\right) = 25 + \frac{10}{S+1}$$

$$\Rightarrow \quad V_o(S) = \frac{25 + \frac{10}{S+1}}{S+2}$$

$$= \frac{25(S+1) + 10}{(S+1)(S+2)}$$

$$= \frac{25S + 35}{(S+1)(S+2)}$$

$$\left\{ \frac{25s+35}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2} \right.$$

$$\Rightarrow 25s+35 = A(s+2) + B(s+1)$$

s term co-efficients:

$25 = A+B \rightarrow (1)$

s^0 coefficients

$35 = 2A+B \rightarrow (2)$

from (1), $A = -B+25$

Sub A value in eq (2)

$$35 = 2(25-B) + B$$

$$\Rightarrow 35 = 50 - B$$

$$\Rightarrow \boxed{B = +15}$$

$$\Rightarrow A = 25 - B = 25 - 15$$

$$\left. \Rightarrow \boxed{A = 10} \right\}$$

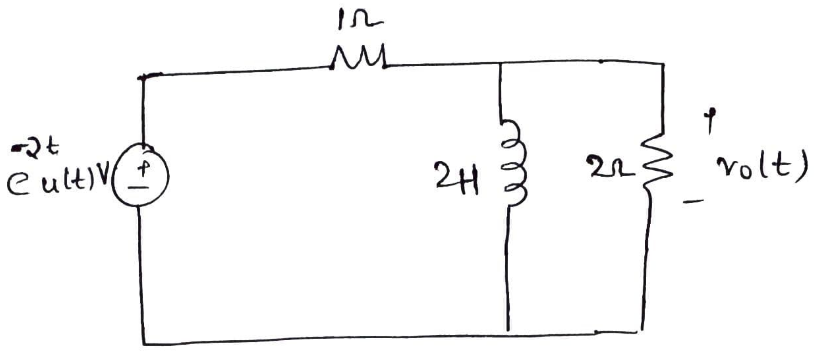
$$\therefore V_o(s) = \frac{10}{s+1} + \frac{15}{s+2}$$

$$v_o(t) = L^{-1}\left[\frac{10}{s+1} + \frac{15}{s+2}\right] = L^{-1}\left[\frac{10}{s+1}\right] + L^{-1}\left[\frac{15}{s+2}\right]$$

$$= 10\, L^{-1}\left[\frac{1}{s+1}\right] + 15\, L^{-1}\left[\frac{1}{s+2}\right]$$

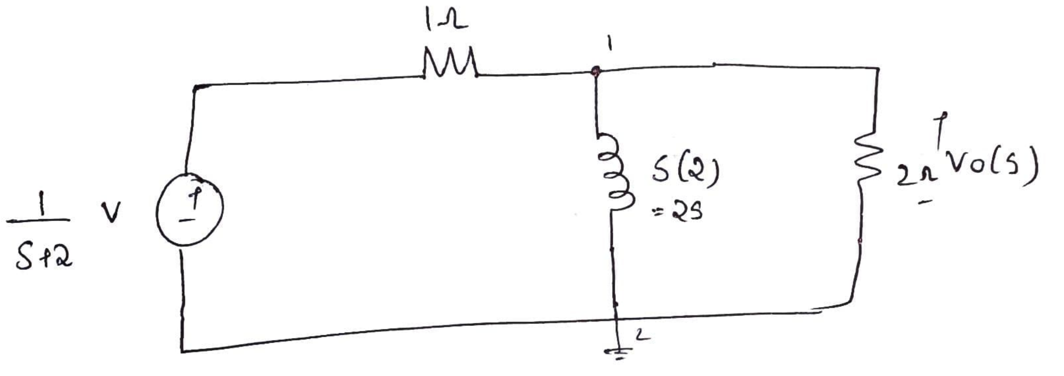
$$\Rightarrow \boxed{v_o(t) = (10e^{-t} + 15e^{-2t})\, u(t)\ .\ V}$$

Q4) Find $v_o(t)$ in the shown circuit.



Ans: $\left(\frac{4}{5}e^{-2t} + \frac{8}{15}e^{-t/3}\right)u(t)$.

Sol) Transform the circuit into s-domain



To find $V_o(s)$, we can used nodal or mesh analysis

If we use nodal analysis, we have two nodes in the circuit,

Top node is 1 & Bottom node is 2.

Top node voltage, let it be $V_1(s)$. From the circuit,

we can see that $V_1(s) = V_o(s)$

And bottom node is taken as reference node, so, its voltage is 0V.

Apply KCL at node 1, and let all the currents be outgoing (or) leaving the node.

$$\frac{V_o(s) - \frac{1}{s+2}}{1} + \frac{V_o(s)}{2s} + \frac{V_o(s)}{2} = 0$$

$$\Rightarrow V_o(s)\left[1 + \frac{1}{2s} + \frac{1}{2}\right] = \frac{1}{s+2}$$

$$\Rightarrow V_o(s)\left[\frac{3s+1}{2s}\right] = \frac{1}{s+2}$$

$$\Rightarrow V_o(s) = \frac{2s}{(s+2)(3s+1)}$$

$$\left\{ \frac{2s}{(s+2)(3s+1)} = \frac{A}{s+2} + \frac{B}{3s+1} \right.$$

$$\Rightarrow 2s = A(3s+1) + B(s+2)$$

$$\Rightarrow 2s = s(3A+B) + (A+2B)$$

Comparing s^0, s^1 term co-efficients

$\underline{s^0}$: $0 = A + 2B$ & $\underline{s^1}$: $2 = 3A + B$

$\Rightarrow A = -2B$

$\Rightarrow 2 = 3(-2B) + B$

$\Rightarrow 2 = -6B + B$

$$\Rightarrow \boxed{B = -\frac{2}{5}}$$

$$\therefore \boxed{A = \frac{4}{5}} \quad \}$$

$$\Rightarrow V_o(s) = \frac{4/5}{s+2} + \frac{-2/5}{3s+1}$$

$$= \frac{4}{5}\left(\frac{1}{s+2}\right) + \frac{-2}{5}\left(\frac{1}{3(s+1/3)}\right)$$

$$= \frac{4}{5}\left(\frac{1}{s+2}\right) + \frac{-2}{15}\left(\frac{1}{s+1/3}\right)$$

$$v_o(t) = L^{-1}\left(V_o(s)\right)$$

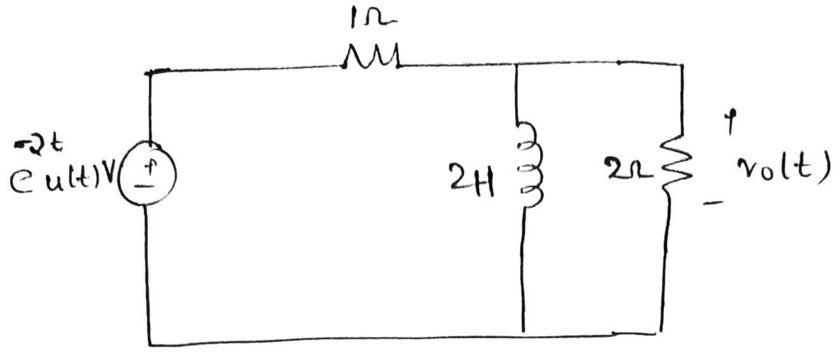
$$= L^{-1}\left[\frac{4}{5}\left(\frac{1}{s+2}\right) - \frac{2}{15}\left(\frac{1}{s+1/3}\right)\right]$$

$$= \frac{4}{5}L^{-1}\left(\frac{1}{s+2}\right) - \frac{2}{15}L^{-1}\left(\frac{1}{s+1/3}\right)$$

$$= \frac{4}{5} \times e^{-2t} - \frac{2}{15} e^{-t/3}$$

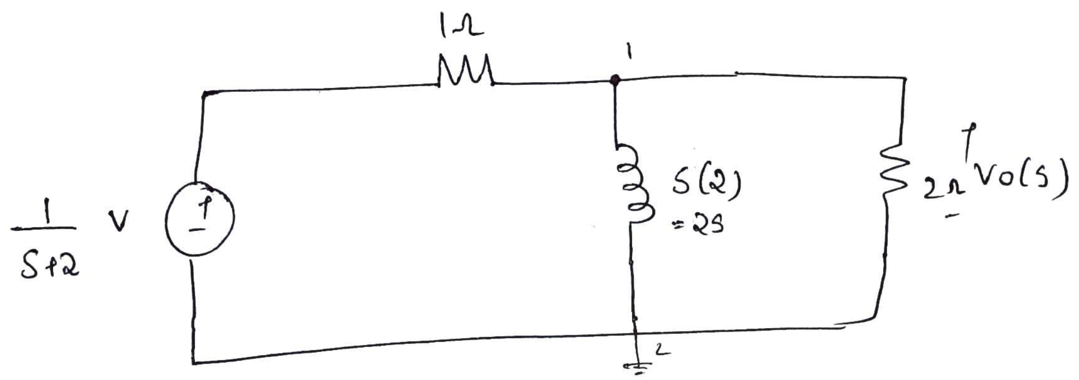
$$\therefore \boxed{v_o(t) = \left(\frac{4}{5}e^{-2t} - \frac{2}{15}e^{-t/3}\right)u(t)} \ V.$$

Q4) Find $v_o(t)$ in the shown circuit.



Ans: $\left(\frac{4}{5}e^{-2t} + \frac{8}{15}e^{-t/3}\right)u(t)$.

Sol) Transform the circuit into s-domain



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If we use nodal analysis, we have two nodes in the circuit,

Top node is 1 & Bottom node is 2.

Top node voltage, let it be $V_1(S)$. From the circuit,

we can see that $V_1(S) = V_o(S)$

And bottom node is taken as reference node, so, its

voltage is 0V.

Apply KCL at node 1, and let all the currents be outgoing (or) leaving the node.

$$\frac{V_o(s) - \frac{1}{s+2}}{1} + \frac{V_o(s)}{2s} + \frac{V_o(s)}{2} = 0$$

$$\Rightarrow V_o(s)\left[1 + \frac{1}{2s} + \frac{1}{2}\right] = \frac{1}{s+2}$$

$$\Rightarrow V_o(s)\left[\frac{3s+1}{2s}\right] = \frac{1}{s+2}$$

$$\Rightarrow V_o(s) = \frac{2s}{(s+2)(3s+1)}$$

$$\left\{ \frac{2s}{(s+2)(3s+1)} = \frac{A}{s+2} + \frac{B}{3s+1} \right.$$

$$\Rightarrow 2s = A(3s+1) + B(s+2)$$

$$\Rightarrow 2s = s(3A+B) + (A+2B)$$

Comparing s^0, s^1 term co-efficients

s^0: $0 = A + 2B$ & s^1: $2 = 3A + B$

$\Rightarrow A = -2B$

$\Rightarrow 2 = 3(-2B) + B$

$\Rightarrow 2 = -6B + B$

$\Rightarrow \boxed{B = -\frac{2}{5}}$

$\therefore \boxed{A = \frac{4}{5}}$ $\}$

$$\Rightarrow V_o(s) = \frac{4/5}{s+2} + \frac{-2/5}{3s+1}$$

$$= \frac{4}{5}\left(\frac{1}{s+2}\right) + \frac{-2}{5}\left(\frac{1}{3(s+1/3)}\right)$$

$$= \frac{4}{5}\left(\frac{1}{s+2}\right) + \frac{-2}{15}\left(\frac{1}{s+1/3}\right)$$

$$v_o(t) = L^{-1}\left(V_o(s)\right)$$

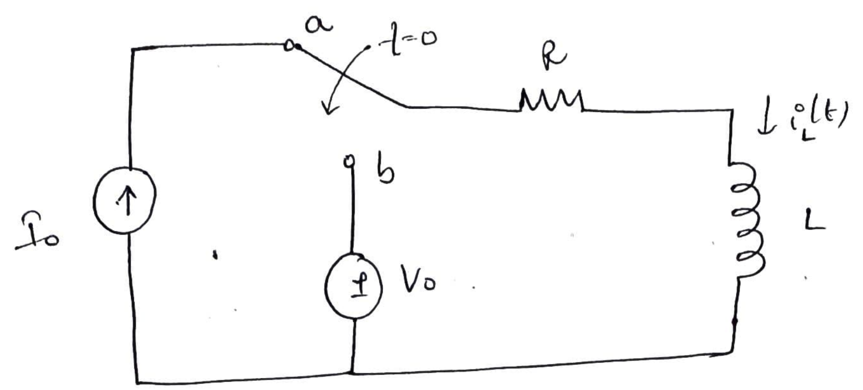
$$= L^{-1}\left[\frac{4}{5}\left(\frac{1}{s+2}\right) - \frac{2}{15}\left(\frac{1}{s+1/3}\right)\right]$$

$$= \frac{4}{5}L^{-1}\left(\frac{1}{s+2}\right) - \frac{2}{15}L^{-1}\left(\frac{1}{s+1/3}\right)$$

$$= \frac{4}{5} \times e^{-2t} - \frac{2}{15}e^{-t/3}$$

$$\therefore \boxed{v_o(t) = \left(\frac{4}{5}e^{-2t} - \frac{2}{15}e^{-t/3}\right)u(t)} \ V.$$

5Q) In the circuit shown, switch is moved from position a to position b at $t=0$. Find $i_L(t)$ for $t>0$. Initial value of current through inductor is I_0.

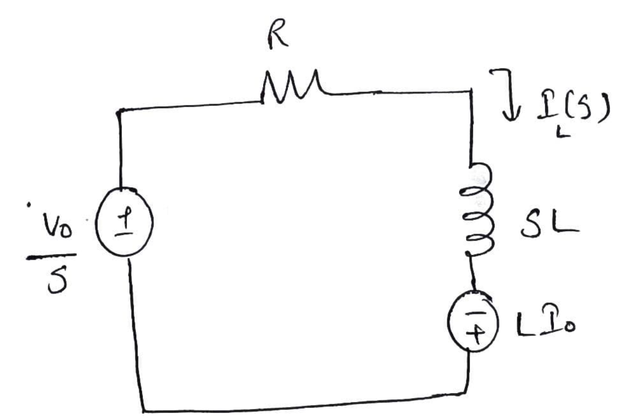


Sol) Given,

$i_L(0) = I_0$:

At $t=0$, ~~circuit~~ switch is connected to 'b' terminal,

for $t>0$, the given circuit in time domain can be converted into s-domain, as shown below.



Current circulating in the loop is same $I_L(s)$.

Apply KVL in the loop,

$$-\frac{V_0}{s} + I_L(s)R + SLI_L(S) - LI_0 = 0$$

$$\Rightarrow I_L(s)\left(R+sL\right) = \frac{V_0}{s} + LI_0$$

$$\Rightarrow I_L(s) = \frac{\frac{V_0}{s} + LI_0}{R+sL}$$

$$= \frac{LI_0 s + V_0}{s(R+sL)}$$

$$\left\{ \frac{LI_0 s + V_0}{s(R+sL)} = \frac{A}{s} + \frac{B}{sL+R} \right.$$

$$\Rightarrow LI_0 s + V_0 = A(sL+R) + Bs = s(AL+B) + AR$$

Compare s^1, s^0 ~~coefficients~~ co-efficients,

$\underline{s^0}$: $V_0 = AR$

$$\Rightarrow \boxed{A = \frac{V_0}{R}}$$

$\underline{s^1}$: $LI_0 = AL + B$

$$\Rightarrow LI_0 = \frac{V_0}{R}L + B$$

$$\Rightarrow \boxed{B = L\left(I_0 - \frac{V_0}{R}\right)} \Bigg\}$$

$$\Rightarrow I_L(s) = \frac{\frac{V_0}{R}}{s} + \frac{L\left(I_0 - \frac{V_0}{R}\right)}{sL+R}$$

$$= \frac{V_0}{R}\left(\frac{1}{s}\right) + \frac{L\left(I_0 - \frac{V_0}{R}\right)}{L\left(s + \frac{R}{L}\right)}$$

$$= \frac{V_0}{R}\left(\frac{1}{s}\right) + \left(I_0 - \frac{V_0}{R}\right)\frac{1}{\left(s + \frac{R}{L}\right)}$$

Now,

$$i_L(t) = L^{-1}\{I_L(s)\}$$

$$= L^{-1}\left\{\frac{V_0}{R}\left(\frac{1}{s}\right) + \left(I_0 - \frac{V_0}{R}\right)\frac{1}{s+\frac{R}{L}}\right\}$$

$$= \frac{V_0}{R}L^{-1}\left\{\frac{1}{s}\right\} + \left(I_0 - \frac{V_0}{R}\right)L^{-1}\left\{\frac{1}{s+\frac{R}{L}}\right\}$$

$$= \frac{V_0}{R}u(t) + \left(I_0 - \frac{V_0}{R}\right)\cdot e^{-R/L\, t}$$

$$\boxed{i_L(t) = \frac{V_0}{R} + \left(I_0 - \frac{V_0}{R}\right)e^{-\frac{Rt}{L}} \quad A} \quad ; t \geq 0.$$

final value of inductor current $i_L(\infty)$ can be found out by using final value theorem

$$i_L(\infty) = \lim_{s\to 0} s\, I_L(s) = \lim_{s\to 0}\left\{s\left(\frac{1}{s}\frac{V_0}{R} + \left(I_0 - \frac{V_0}{R}\right)\times\frac{1}{s+R/L}\right)\right\}$$

$$= \lim_{s\to 0}\left(\frac{V_0}{R} + \left(I_0 - \frac{V_0}{R}\right)\times\frac{s}{s+\frac{R}{L}}\right)$$

$$= \frac{V_0}{R} + \left(I_0 - \frac{V_0}{R}\right)\times\frac{0}{\frac{R}{L}}$$

$$= \frac{V_0}{R}$$

$\therefore \quad i_L(\infty) = \frac{V_0}{R}$ A.

from the solution, $i_L(t)$ is,

$$i_L(t) = \left(I_0 - \frac{V_0}{R}\right)e^{-t/\tau} + \frac{V_0}{R} \quad ; t \geq 0 \qquad \left(\text{where } \tau = \frac{R}{L}\right)$$

This equation can be re-written as,

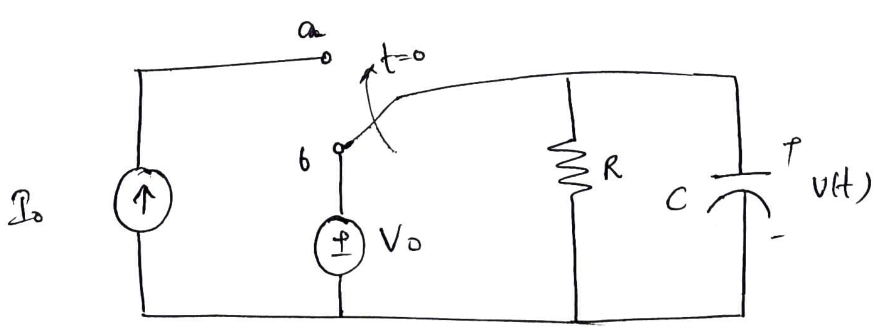
$$i_L(t) = I_0 e^{-t/\tau} + \frac{V_0}{R}\left(1 - e^{-t/\tau}\right) \quad , t \geq 0$$

In this equation, first term represents natural response & second term represents forced response.

If there is no initial energy, then $I_0 = 0$

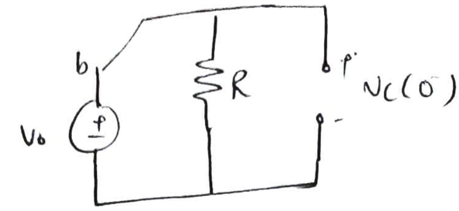
$\Rightarrow i_L(t) = \frac{V_0}{R}\left(1 - e^{-t/\tau}\right)$, which is due to the presence of step input alone.

6Q) The switch in the below circuit is moved from position b to a at $t=0$. Prior to that, it has been in position b for a long time. Determine $v(t)$ for $t>0$.

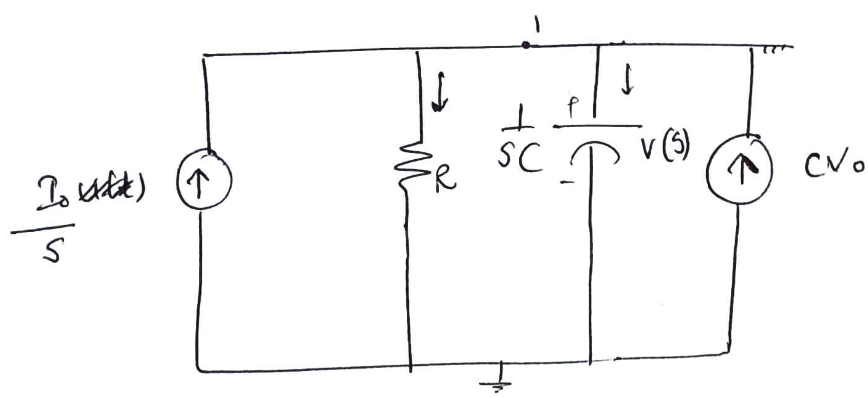


Sol) from the circuit, with switch in position (b), we can calculate

$v_c(0)$ ie $v_c(0) = V_o$.



After $t=0$, ie after switching, time domain circuit is transformed to s-domain circuit, is as shown. It includes the initial condition of capacitor too. { CV_o is the current source added due to initial condition of capacitor }



~~Now, transform~~ the

Apply nodal analysis, we have two nodes in the circuit. Top node voltage is V(S), from the circuit and bottom node is taken as reference node so, it is 0V

Assume the current through R and $\frac{1}{sC}$ as outgoing

Apply KCL at node 1,

$$\frac{I_o}{s} + CV_o = \frac{V(s)}{R} + \frac{V(s)}{1/sC}$$

$$\Rightarrow \left(\frac{I_0}{S} + CV_0\right) = V(S)\left(\frac{1}{R} + SC\right)$$

$$\Rightarrow V(S) = \frac{\frac{I_0}{S} + CV_0}{\frac{1}{R} + SC}$$

$$= \frac{R\,(I_0 + SCV_0)}{S(1 + SRC)}$$

$$\left\{ \frac{R(I_0 + SCV_0)}{1+SRC} = \frac{A}{S} + \frac{B}{1+SRC} \right.$$

$$\Rightarrow RI_0 + SRCV_0 = A(1+SRC) + BS$$

Comparing s^0, s^1 term co-efficients on bs.

$\underline{s^0}$: $RI_0 = A$

$$\Rightarrow \boxed{A = RI_0}$$

$\underline{s^1}$: $RCV_0 = ARC + B$

$$RCV_0 = (RI_0)RC + B$$

$$\left. \Rightarrow \boxed{B = RC\,(V_0 - RI_0)} \right\}$$

$$\Rightarrow V(S) = \frac{RI_0}{S} + \frac{RC(V_0 - RI_0)}{1+SRC}$$

Now, $v(t) = L^{-1}[V(S)] = L^{-1}\left[\frac{RI_0}{S} + \frac{RC(V_0 - RI_0)}{1+SRC}\right]$

$$\Rightarrow v(t) = RI_0\, L^{-1}\left[\frac{1}{S}\right] + RC(V_0 - RI_0)\, L^{-1}\left[\frac{1}{1+SRC}\right]$$

$$\Rightarrow v(t) = RI_0 \, \mathcal{L}^{-1}\left[\frac{1}{s}\right] + \frac{RC(V_0 - RI_0)}{RC} \, \mathcal{L}^{-1}\left[\frac{1}{s+\frac{1}{RC}}\right]$$

$$\Rightarrow v(t) = RI_0 \, u(t) + (V_0 - RI_0) \, e^{-\frac{1}{RC}t}$$

$$\Rightarrow \boxed{v(t) = RI_0 + (V_0 - RI_0) \, e^{-t/\tau}} \quad (\because \tau = RC) \;, t \geq 0$$

TRANSFER FUNCTION :

- The transfer function of a network describes how the output behaves in respect to the input. It specifies the transfer from the input to the output in s-domain, assuming zero initial conditions.

- Transfer function is helpful in finding the network response, determining network stability and network synthesis.

" Transfer function (say $H(s)$) is the ratio of output response (say $Y(s)$) to the input excitation (say $X(s)$), assuming all initial conditions are zero. ".

$$\therefore \quad H(s) = \frac{Y(s)}{X(s)}.$$

- The transfer function depends on what we define as input and output. Since the input and output can be either voltage (or) current at any element (or) point in the circuit, there are four possible transfer functions.

Units

$H(s)$ = Voltage gain $= \dfrac{V_o(s)}{V_i(s)}$ No units.

$H(s)$ = Current gain $= \dfrac{I_o(s)}{I_i(s)}$ No units

$H(s)$ = Impedance $= \dfrac{V(s)}{I(s)}$ Ohms.

$H(s)$ = Admittance $= \dfrac{I(s)}{V(s)}$. Mho.

→ from $H(s) = \dfrac{Y(s)}{X(s)}$, we generally assume that $X(s)$ and $Y(s)$ are known.

Suppose, if we know the input $X(s)$ and transfer function $H(s)$, then output can be calculated by,

$$Y(s) = H(s) \cdot X(s).$$

and then $y(t) = L^{-1}[Y(s)]$.

* A special case when the input is the unit impulse function, $x(t) = \delta(t) \Rightarrow X(s) = 1$,

Then $Y(s) = H(s) \cdot 1 = H(s)$.

$$\Rightarrow y(t) = h(t).$$

The term $h(t)$ represents the unit impulse response.

It is the time-domain response of the network to a unit impulse.

Therefore, transfer function can be ~~said~~ defined as the Laplace Transform of the unit impulse response of the network. This is another way of defining it

- from this we can say that, if we find out the impulse response of the network, ~~to~~ then we can obtain the response of the network to any input signal using, $Y(s) = H(s) \cdot X(s)$. equation, and inverse laplace transform. (or) by using convolution integral.

Q1) The output of a linear systems is $y(t) = 10e^{-t} \cos 4t\, u(t)$, when the input is $x(t) = e^{-t} u(t)$. Find the transfer function of the systems and its impulse response.

Sol) Given, $x(t) = e^{-t} u(t)$, $y(t) = 10e^{-t} \cos 4t\, u(t)$.

and we know, $H(s) = \dfrac{Y(s)}{X(s)}$

$\therefore X(s) = L[e^{-t} u(t)]$ and $Y(s) = L[y(t)] = L[10e^{-t} \cos 4t\, u(t)]$

$= \dfrac{1}{s+1}$ $= 10 \dfrac{(s+1)}{(s+1)^2+4^2}$

$$\therefore H(s) = \frac{Y(s)}{X(s)}$$

$$= \frac{10 \dfrac{s+1}{(s+1)^2+4^2}}{\dfrac{1}{s+1}}$$

$$= 10 \frac{(s+1)^2}{(s+1)^2+4^2}$$

Q2) The transfer function of a linear system is,

$H(s) = \dfrac{2s}{s+6}$. find the output $y(t)$ due to the input

$e^{-3t}u(t)$ and its impulse response.

Sol) Given, $H(s) = \dfrac{2s}{s+6}$

and $x(t) = e^{-3t}u(t)$.

$$X(s) = L[x(t)] = L[e^{-3t}u(t)]$$

$$= \frac{1}{s+3}$$

$$\therefore \quad H(s) = \frac{Y(s)}{X(s)}$$

$$\Rightarrow Y(s) = H(s) \cdot X(s)$$

$$= \frac{2s}{s+6} \times \frac{1}{s+3}$$

$$= \frac{2s}{(s+6)(s+3)}$$

$$\left\{ \frac{2s}{(s+6)(s+3)} = \frac{A}{s+6} + \frac{B}{s+3} \Rightarrow 2s = A(s+3) + B(s+6) \right.$$

s' terms $\Rightarrow \quad 2 = A+B$

constants $\Rightarrow \quad 0 = 3A + 6B \Rightarrow A = -2B$

$\therefore$ from $2 = A+B \Rightarrow 2 = -2B+B \Rightarrow \boxed{B=-2}$ & $\left.\boxed{A=4}\right\}$

$$= \frac{4}{S+6} + \frac{-2}{S+3}$$

$$\therefore y(t) = L^{-1}[Y(S)]$$

$$= 4 L^{-1}\left[\frac{1}{S+6}\right] - 2 L^{-1}\left[\frac{1}{S+3}\right]$$

$$= 4 e^{-6t} - 2 e^{-3t}. \quad , t \geq 0.$$

* Impulse response is the inverse laplace transform of $H(s)$.

$$h(t) = L^{-1}[H(s)]$$

$$= L^{-1}\left[\frac{2s}{S+6}\right]$$

$$= L^{-1}\left[\frac{2(S+6-6)}{S+6}\right]$$

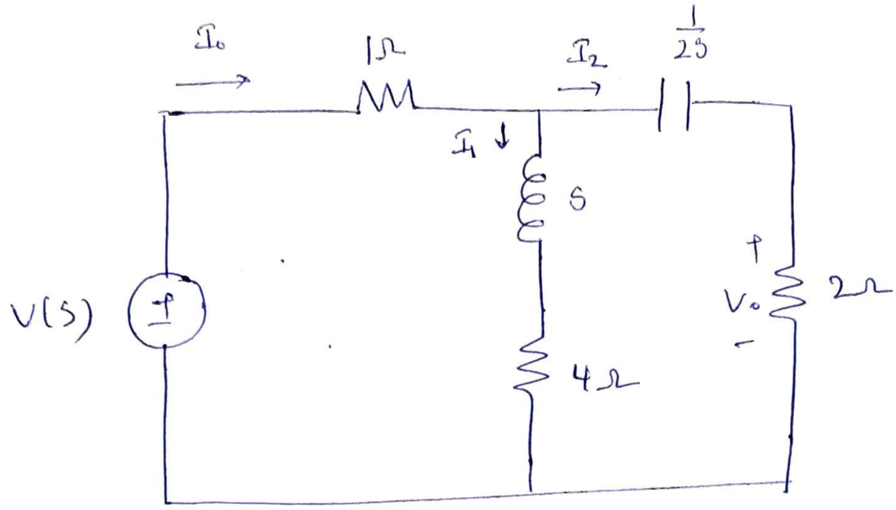
$$= 2 L^{-1}\left[\frac{(S+6)}{S+6} - \frac{6}{S+6}\right]$$

$$= 2 L^{-1}\left[1 - 6 \cdot \frac{1}{S+6}\right]$$

$$= 2\left\{L^{-1}[1] - 6 L^{-1}\left[\frac{1}{S+6}\right]\right\}$$

$$\boxed{\therefore h(t) = 2\delta(t) - 12 e^{-6t} u(t).}$$

Q3) Determine the transfer function $H(S) = \frac{V_o(S)}{I_o(S)}$ in the belowckt.



Sol) from the circuit, $V_o(S) = 2\, I_2(S)$. $\rightarrow$ ①

$I_2(S)$ can be calculated by current division rule,

$$I_2(S) = I_o(S) \times \frac{S+4}{(S+4) + \left(\frac{1}{2S} + 2\right)} \rightarrow ②$$

Sub ② in ①,

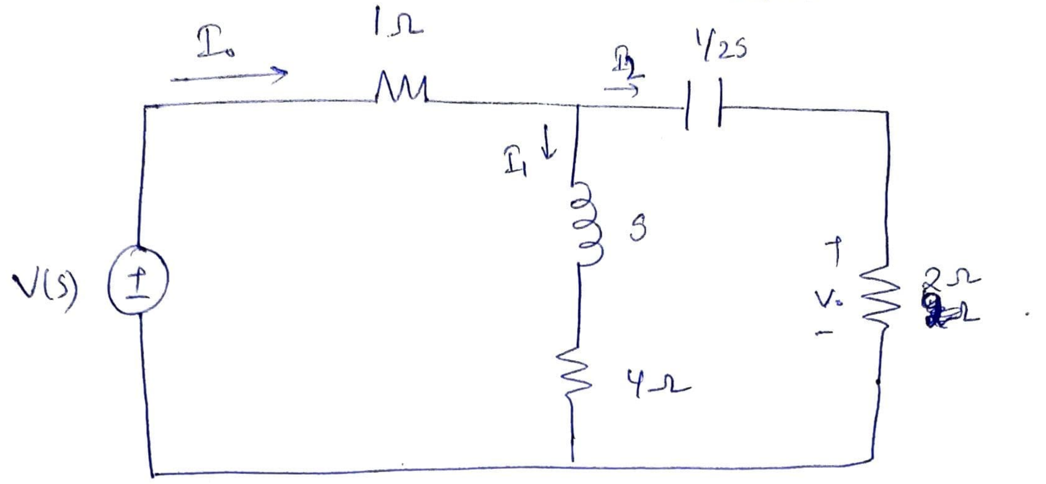
$$V_o(S) = 2 \times I_o(S) \times \frac{S+4}{(S+4) + \left(\frac{1}{2S} + 2\right)}$$

$$\Rightarrow \frac{V_o(S)}{I_o(S)} = \frac{2(S+4)}{(S+4) + \left(\frac{1}{2S} + 2\right)}$$

$$= \frac{4S(S+4)}{2S^2 + 8S + 1 + 4S}$$

$$\boxed{\frac{V_o(S)}{I_o(S)} = \frac{4S(S+4)}{2S^2 + 12S + 1}}$$

Q4) find the transfer function $H(s) = \frac{I_1(s)}{I_o(s)}$ in the below ckt

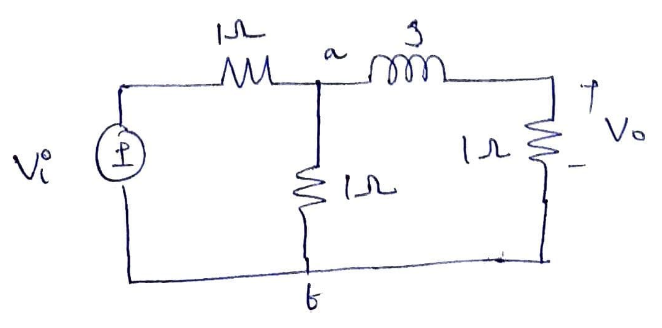


Sol) from the circuit, $I_1 = I_o \times \dfrac{\frac{1}{2s} + 2}{(s+4) + \left(\frac{1}{2s} + 2\right)}$

$$\Rightarrow \frac{I_1(s)}{I_o(s)} = \frac{\frac{1}{2s} + 2}{(s+4) + \left(\frac{1}{2s} + 2\right)}$$

$$\boxed{\frac{I_1(s)}{I_o(s)} = \frac{1+4s}{2s^2 + 12s + 1}}$$

Q5) for the s-domain circuit shown, find (a): the transfer function $H(s) = \frac{V_o}{V_i}$, (b) the impulse response, (c) the response when $v_i(t) = u(t)$ V, d) the response when $v_i(t) = 8\cos 2t$ V.



Sol) a) TF Transfer function, $\frac{V_o}{V_i}$:

Current supplied by V_i voltage source is, let it be I,

$$I = \frac{V_i}{Z_{eq}}$$

$$= \frac{V_i}{1+(1 \,//\, s+1)}$$

$$= \frac{V_i}{1+\left(\frac{1\times(s+1)}{1+s+1}\right)}$$

$$= \frac{V_i}{1+\frac{s+1}{s+2}}$$

$$I = \frac{s+2}{2s+3} V_i \quad \longrightarrow (1)$$

$V_o = I_o(1)$ (let the current flowing 1Ω, be I_o, whose voltage drop is V_o)

$$= (I)\times \frac{(1)}{1+(s+1)} \quad (1)$$

$$= \left(\frac{s+2}{2s+3} V_i\right) \times \frac{1}{s+2} \quad (\because \text{from } (1))$$

$$\Rightarrow \frac{V_o}{V_i} = \frac{s+2}{2s+3} \times \frac{1}{s+2}$$

$$\Rightarrow \boxed{\frac{V_o(S)}{V_i(S)} = \frac{1}{2S+3}}$$

(b) Impulse response :

Impulse response , $h(t) = L^{-1}[H(S)]$

$$= L^{-1}\left[\frac{V_o(S)}{V_i(S)}\right]$$

$$= L^{-1}\left[\frac{1}{2S+3}\right]$$

$$= \frac{1}{2} L^{-1}\left[\frac{1}{S+3/2}\right]$$

$$\boxed{h(t) = \frac{1}{2} e^{-3/2\, t} . u(t).}$$

(c) response when $v_i(t) = u(t).V$.

from $H(S) = \frac{V_o(S)}{V_i(S)} = \frac{1}{2S+3}$

Given, $v_i(t) = u(t) .V \Rightarrow V_i(S) = \frac{1}{S}. \quad V.$

$$\Rightarrow H(S) = \frac{1}{2S+3} \Rightarrow \frac{V_o(S)}{V_i(S)} = \frac{1}{2S+3}.$$

$$\Rightarrow V_o(s) = \frac{1}{2s+3} \cdot V_i(s)$$

$$= \frac{1}{2s+3} \times \frac{1}{s}$$

$$= \frac{1}{s(2s+3)}$$

$$\left\{ \frac{1}{s(2s+3)} = \frac{A}{s} + \frac{B}{2s+3} \right.$$

$$\Rightarrow 1 = A(2s+3) + Bs \Rightarrow 1 = s(2A+B) + 3A$$

Comparing s^0 terms, $1 = 3A \Rightarrow \boxed{A = \frac{1}{3}}$

Comparing s terms, $0 = 2A + B$

$$\Rightarrow 0 = 2 \times \frac{1}{3} + B$$

$$\left. \Rightarrow \boxed{B = -\frac{2}{3}} \right\}$$

$$\Rightarrow V_o(s) = \frac{\frac{1}{3}}{s} + \frac{-2/3}{2s+3}$$

$$= \frac{1}{3} \times \frac{1}{s} - \frac{2}{3} \times \frac{1}{2} \times \frac{1}{s + 3/2}$$

$$\therefore v_o(t) = L^{-1}[V_o(s)] = \frac{1}{3} L^{-1}\left[\frac{1}{s}\right] - \frac{1}{3} L^{-1}\left[\frac{1}{s+3/2}\right]$$

$$\therefore v_o(t) = \frac{1}{3} u(t) - \frac{1}{3} e^{-3/2\, t}$$

$$\boxed{v_o(t) = \frac{1}{3}\left(1 - e^{-3/2\, t}\right) u(t) \text{ . } V}$$

(d) when $v_i(t) = 8\cos 2t$,

$$\Rightarrow V_i(s) = 8 \times \frac{s}{s^2+2^2}$$

$$= 8 \times \frac{s}{s^2+4}$$

$$V_i(s) = \frac{8s}{s^2+4}$$

from $H(s) = \frac{V_o(s)}{V_i(s)} = \frac{1}{2s+3}$

$$\Rightarrow V_o(s) = \frac{1}{2s+3} \times V_i(s)$$

$$= \frac{1}{2s+3} \times \frac{8s}{s^2+4}$$

$$= \frac{8s}{(2s+3)(s^2+4)}$$

$$\left\{ \frac{8s}{(2s+3)(s^2+4)} = \frac{A}{2s+3} + \frac{Bs+C}{s^2+4} \right.$$

$$\Rightarrow 8s = A(s^2+4) + (Bs+C)(2s+3)$$

$$\Rightarrow 8s = s^2(A+2B) + s(3B+2C) + (4A+3C)$$

Comparing s^2, s^1, s^0 co-efficients on bothsides,

$s^2 \Rightarrow 0 = A+2B$, $s^1 \Rightarrow 8 = 3B+2C$, $s^0 \Rightarrow 0 = 4A+3C$.

$\Rightarrow A = -2B$

$\Rightarrow C = \frac{8-3B}{2}$

$\Rightarrow 0 = 4(-2B) + 3\left(\frac{8-3B}{2}\right)$

$$\Rightarrow 0 = -8B + \frac{24 - 9B}{2}$$

$$\Rightarrow 0 = -16B + 24 - 9B$$

$$0 = -25B + 24$$

$$\Rightarrow \boxed{B = \frac{24}{25}}$$

$$\Rightarrow A = -2B = -2 \times \frac{24}{25}$$

$$\boxed{A = -\frac{48}{25}}$$

and $C = \dfrac{8 - 3B}{2}$

$$= \frac{8 - 3\left(\frac{24}{25}\right)}{2}$$

$$= \frac{25 \times 8 - 3 \times 24}{2 \times 25}$$

$$= \frac{8(25 - 3 \times 3)}{2 \times 25}$$

$$= \frac{\cancel{8}^{4} \times 16}{\cancel{2} \times 25} = \frac{\cancel{48}\ 64}{25}$$

$$\therefore \boxed{C = \frac{64}{25}}$$

$$\Rightarrow V_o(s) = \frac{-48/25}{2s+3} + \frac{\frac{24}{25}s + \frac{64}{25}}{s^2+4}.$$

$$= \frac{-48}{25} \times \frac{1}{2} \cdot \frac{1}{s+3/2} + \frac{24}{25} \cdot \frac{s}{s^2+4} + \frac{64}{25} \cdot \frac{1}{s^2+4}$$

$$V_o(t) = \mathcal{L}^{-1}\left[V_o(s)\right]$$

$$= -\frac{24}{25} \mathcal{L}^{-1}\left[\frac{1}{s+3/2}\right] + \frac{24}{25} \mathcal{L}^{-1}\left[\frac{s}{s^2+4}\right]$$

$$+ \frac{\cancel{16}\,32}{25} \mathcal{L}^{-1}\left[\frac{\cancel{1}\,2}{s^2+4}\right]$$

$$= \frac{-24}{25} e^{-\frac{3}{2}t} + \frac{24}{25}\cos 2t + \frac{\cancel{16}\,32}{25}\sin 2t.$$

$$\boxed{V_o(t) = \frac{24}{25}\left(-e^{-3/2\,t} + \cos 2t + \frac{4}{3}\sin 2t\right) u(t).\ V}$$

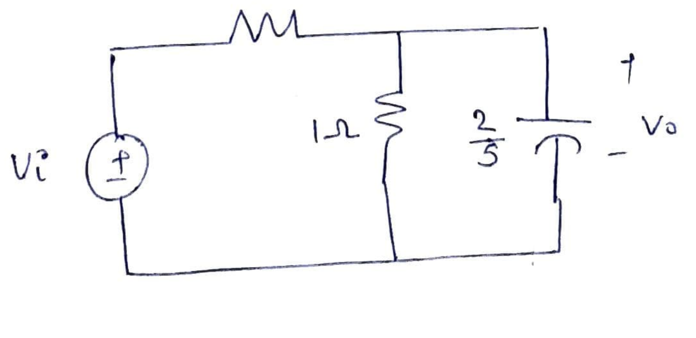
Q6) for the s-domain circuit shown, find,

(a) transfer function, $H(s) = \frac{V_o}{V_i}$

(b) impulse response

(c) the response when $v_i(t) = u(t)$ V

d) the response when $v_i(t) = 8\cos 2t$ V.



Answers.

(a) $\frac{2}{s+4}$

(b) $2e^{-4t}u(t)$

(c) $\frac{1}{2}(1-e^{-4t})u(t)$ V

(d) $\frac{3}{2}\left(e^{-4t}+\cos 2t+\frac{1}{2}\sin 2t\right)u(t)$. V

For $t > 2$, the two functions overlap between $(t-2)$ and t, as in Fig. 15.37(b). Hence

$$i_o(t) = \int_{t-2}^{t} (1)e^{-\lambda}\,d\lambda = -e^{-\lambda}\Big|_{t-2}^{t} = -e^{-t} + e^{-(t-2)}$$
$$= (e^2 - 1)e^{-t}, \qquad t \ge 0 \tag{15.20.8}$$

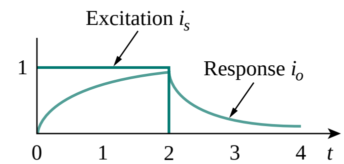


Figure 15.38 For Example 15.20; excitation and response.

From Eqs. (15.20.7) and (15.20.8), the response is

$$i_o(t) = \begin{cases} 1 - e^{-t}, & 0 \le t \le 2 \\ (e^2 - 1)e^{-t}, & t \ge 2 \end{cases} \tag{15.20.9}$$

which is the same as in Eq. (15.20.6). Thus, the response $i_o(t)$ along the excitation $i_s(t)$ is as shown in Fig. 15.38.

PRACTICE PROBLEM 15.20

Use convolution to find $v_o(t)$ in the circuit of Fig. 15.39(a) when the excitation is the signal shown in Fig. 15.39(b).

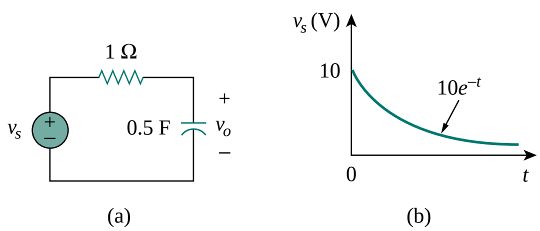


Figure 15.39 For Practice Prob. 15.20.

Answer: $20(e^{-t} - e^{-2t})$ V.

†15.8 APPLICATION TO INTEGRODIFFERENTIAL EQUATIONS

The Laplace transform is useful in solving linear integrodifferential equations. Using the differentiation and integration properties of Laplace transforms, each term in the integrodifferential equation is transformed. Initial conditions are automatically taken into account. We solve the resulting algebraic equation in the s domain. We then convert the solution back to the time domain by using the inverse transform. The following examples illustrate the process.

EXAMPLE 15.21

Use the Laplace transform to solve the differential equation

$$\frac{d^2v(t)}{dt^2} + 6\frac{dv(t)}{dt} + 8v(t) = 2u(t)$$

subject to $v(0) = 1$, $v'(0) = -2$.

Solution:

We take the Laplace transform of each term in the given differential equation and obtain

$$[s^2V(s) - sv(0) - v'(0)] + 6[sV(s) - v(0)] + 8V(s) = \frac{2}{s}$$

Substituting $v(0) = 1$, $v'(0) = -2$,

$$s^2V(s) - s + 2 + 6sV(s) - 6 + 8V(s) = \frac{2}{s}$$

or

$$(s^2 + 6s + 8)V(s) = s + 4 + \frac{2}{s} = \frac{s^2 + 4s + 2}{s}$$

Hence,

$$V(s) = \frac{s^2 + 4s + 2}{s(s+2)(s+4)} = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s+4}$$

where

$$A = sV(s)\Big|_{s=0} = \frac{s^2 + 4s + 2}{(s+2)(s+4)}\bigg|_{s=0} = \frac{2}{(2)(4)} = \frac{1}{4}$$

$$B = (s+2)V(s)\Big|_{s=-2} = \frac{s^2 + 4s + 2}{s(s+4)}\bigg|_{s=-2} = \frac{-2}{(-2)(2)} = \frac{1}{2}$$

$$C = (s+4)V(s)\Big|_{s=-4} = \frac{s^2 + 4s + 2}{s(s+2)}\bigg|_{s=-4} = \frac{2}{(-4)(-2)} = \frac{1}{4}$$

Hence,

$$V(s) = \frac{\frac{1}{4}}{s} + \frac{\frac{1}{2}}{s+2} + \frac{\frac{1}{4}}{s+4}$$

By the inverse Laplace transform,

$$v(t) = \frac{1}{4}(1 + 2e^{-2t} + e^{-4t})u(t)$$

PRACTICE PROBLEM 15.21

Solve the following differential equation using the Laplace transform method.

$$\frac{d^2v(t)}{dt^2} + 4\frac{dv(t)}{dt} + 4v(t) = e^{-t}$$

if $v(0) = v'(0) = 1$.

Answer: $(e^{-t} + 2te^{-2t})u(t)$.

EXAMPLE 15.22

Solve for the response $y(t)$ in the following integrodifferential equation.

$$\frac{dy}{dt} + 5y(t) + 6\int_0^t y(\tau)\,d\tau = u(t), \qquad y(0) = 2$$

Solution:

Taking the Laplace transform of each term, we get

$$[sY(s) - y(0)] + 5Y(s) + \frac{6}{s}Y(s) = \frac{1}{s}$$

Substituting $y(0) = 2$ and multiplying through by s,

$$Y(s)(s^2 + 5s + 6) = 1 + 2s$$

or

$$Y(s) = \frac{2s+1}{(s+2)(s+3)} = \frac{A}{s+2} + \frac{B}{s+3}$$

where

$$A = (s+2)Y(s)\Big|_{s=-2} = \frac{2s+1}{s+3}\bigg|_{s=-2} = \frac{-3}{1} = -3$$

$$B = (s+3)Y(s)\Big|_{s=-3} = \frac{2s+1}{s+2}\bigg|_{s=-3} = \frac{-5}{-1} = 5$$

Thus,

$$Y(s) = \frac{-3}{s+2} + \frac{5}{s+3}$$

Its inverse transform is

$$y(t) = (-3e^{-2t} + 5e^{-3t})$$

PRACTICE PROBLEM 15.22

Use the Laplace transform to solve the integrodifferential equation

$$\frac{dy}{dt} + 3y(t) + 2\int_0^t y(\tau)\,d\tau = 2e^{-3t}, \qquad y(0) = 0$$

Answer: $(-e^{-t} + 4e^{-2t} - 3e^{-3t})u(t)$.

†15.9 APPLICATIONS

So far we have considered three applications of Laplace's transform: circuit analysis in general, obtaining transfer functions, and solving linear integrodifferential equations. The Laplace transform also finds application in other areas in circuit analysis, signal processing, and control systems. Here we will consider two more important applications: network stability and network synthesis.

15.9.1 Network Stability

A circuit is *stable* if its impulse response $h(t)$ is bounded (i.e., $h(t)$ converges to a finite value) as $t \to \infty$; it is *unstable* if $h(t)$ grows without bound as $t \to \infty$. In mathematical terms, a circuit is stable when

$$\lim_{t\to\infty} |h(t)| = \text{finite} \tag{15.97}$$

1Q) Solve the differential equation using laplace transform method.

$$\frac{d^2 v(t)}{dt} + 4\frac{dv(t)}{dt} + 4v(t) = e^{t}, \quad \text{if } v(0) = v'(0) = 1$$

Sol) $\ast \left[\frac{df(t)}{dt} = sF(s) - f(0) \right.$

$$\left. \frac{d^2 f(t)}{dt^2} = s^2 F(s) - sf(0) - f'(0) \right] \ast$$

Given differential equation,

$$\frac{d^2 v(t)}{dt^2} + 4\frac{dv(t)}{dt} + 4v(t) = e^{-t}$$

$$\Rightarrow \left[s^2 V(s) - sv(0) \cancel{st} - v'(0) \right] + 4\left[sV(s) - v(0) \right] + 4V(s) = \frac{1}{s+1}$$

$$\Rightarrow \left[s^2 V(s) - s - 1 \right] + 4\left[sV(s) - 1 \right] + 4V(s) = \frac{1}{s+1}$$

$$\Rightarrow V(s)\left[s^2 + 4s + 4 \right] \cancel{+6} - s - 1 - 4 = \frac{1}{s+1}$$

$$\Rightarrow V(s)\left[s^2 + 4s + 4 \right] = \frac{1}{s+1} + s + 5$$

$$= \frac{1 + s^2 + s + 5s}{s+1}$$

$$= \frac{s^2 + 6s + 1}{s+1}$$

$$\Rightarrow V(s) = \frac{s^2 + 6s + 1}{(s+1)(s^2 + 4s + 4)}$$

$$V(s) = \frac{s^2+6s+6}{(s+1)(s+2)^2}$$

$$\left\{ \frac{s^2+6s+6}{(s+1)(s+2)^2} = \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{(s+2)^2}\right.$$

$$\Rightarrow s^2+6s+6 = A(s+2)^2 + B(s+1)(s+2) + C(s+1)$$

$$= A(s^2+4s+4) + B(s^2+3s+2) + C(s+1)$$

$$= s^2(A+B) + s(4A+3B+C)$$

$$+(4A+2B+C)$$

$s^2 \Rightarrow 1 = A+B \longrightarrow (1)$

$s \Rightarrow 6 = 4A+3B+C$ $\cancel{\rightarrow 4A = 3B+C-6}$ $\longrightarrow (2)$

$s^0 \Rightarrow 6 = 4A+2B+C \longrightarrow (3)$

from (2), (3) (2) − (3) $\Rightarrow 6-6 = (4A+3B+C) - (4A+2B+C)$

$$\Rightarrow \boxed{0 = B}$$

from ①, $A+B=1 \Rightarrow A+0=1$

$\Rightarrow \boxed{A=1}$

from ③ $4A+2B+C=6$

$\Rightarrow 4(1)+2(0)+C=6$

$\Rightarrow C=6-4$

$\boxed{C=2}$

$\therefore \quad V(s) = \dfrac{1}{s+1} + \dfrac{2}{(s+2)^2}$

$$\boxed{v(t) = \left(e^{-t} + 2te^{-2t}\right) u(t).\ V}$$