## 5.5

## Random walk and Telegraph signal processes

**Random walk:** A random walk is derived from a sequence of Bernoulli trials as follows:

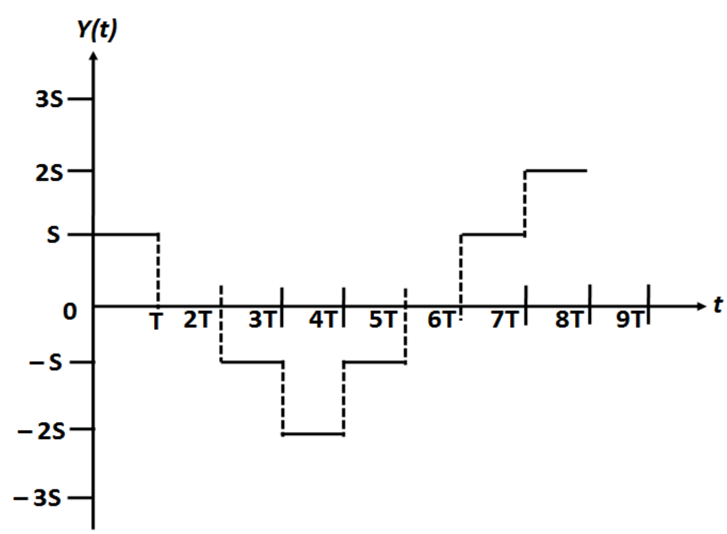
Consider a Bernoulli trial in which the probability of success is $p$ and the probability of failure is $1-p=q$. Assume that the experiment is performed every $T$ time units, and let the random variable $X_k$ denote the outcome of the $k^{th}$ trial. Furthermore, assume that the p.m.f of $X_k$ is as follows:

$$P_{X_k}(x)=\begin{cases} p & , \; x=1 \\ 1-p & , \; x=-1 \end{cases}$$

Finally, let the random variable $Y_n$ be defined as follows:

$$Y_n=\sum_{k=1}^{n} X_k \qquad n=1,\ 2,\ \ldots$$

where $Y_0=0$. If we use $X_k$ to model a process where we take a step to the right if the outcome of the $k^{th}$ trial is a success and a step to the left if the outcome is a failure, then the random variable $Y_n$ represents the location of the process relative to the starting point (or origin) at the end of the $n^{th}$ trial.




**A sample Path of the Random Walk**

The resulting trajectory of the process as it moves through the $xy$ plane, where the $x$ coordinate represents the time and the $y$ coordinate represents the location at a given time, is called a *one – dimensional random walk*. If we define the random process $Y(t) = Y_n,\ n \leq t < n+1$, then the above figure shows an example of the sample path of $Y(t)$, where the length of each step is $s$. It is a staircase with discontinuities at $t = kT, k = 1, 2, \ldots$

Suppose that at the end of the $n^{th}$ trial there are exactly $k$ success. Then there are $k$ steps to the right and $n-k$ steps to the left. Thus,

$$Y(nT) = ks - (n-k)s = (2k-n)s = rs$$

where $r = 2k - n$. This implies that $Y(nT)$ is a random variable that assumes values $rs$, where $r = n, n-2, n-4, \ldots, -n$. Since the event $\{Y(nT) = rs\}$ is the event $\{k\ successes\ in\ n\ trials\}$, where $k = \frac{(n+r)}{2}$, we have that

$$P[Y(nT) = rs] = P\left[\frac{n+r}{2} successes\ in\ n\ trials\right] = \binom{n}{\frac{n+r}{2}} p^{\frac{n+r}{2}} (1-p)^{\frac{n-r}{2}}$$

Note that $(n+r)$ must be an even number. Also, since $Y(nT)$ is the sum of $n$ independent Bernoulli random variables, its mean and variance are given as follows:

$$E[Y(nT)] = nE[X_k] = n[ps - (1-p)s] = (2p-1)ns$$

$$E[X_k^2] = ps^2 + (1-p)s^2 = s^2$$

$$Var[Y(nT)] = nVar[X_k] = n[s^2 - s^2(2p-1)^2] = 4p(1-p)ns^2$$

In the special case where $p = \frac{1}{2},\ E[Y(nT)] = 0$, and $Var[Y(nT)] = ns^2$.

## Gambler's Ruin

The random walk described above assumes that the process can continue forever; in other words, it is unbounded. If the walk is bounded, then the ends of the walk are called **barriers**. These barriers can impose different characteristics on the process. For example, they can be ***reflecting barriers***, which means that on hitting them the walk turns around and continuous. They can also be ***absorbing barriers***, which means that the walk ends.

Consider the following random walk with absorbing barriers, which is generally referred to as the **gambler's ruin**. Suppose a gambler plays a sequence of independent games against an opponent. He starts out with $Rs\ k$, and in each game he wins $Rs\ 1$ with probability $p$ and loses $Rs\ 1$ with probability $q = 1 - p$. When $p > q$, the game is advantageous to the gambler either because he is more skilled than his opponent or the rules of the game favor him. If $p = q$, the game is fair; and if $p < q$, the game is disadvantageous to the gambler.

Assume that he gambler stops when he has a total of $Rs\ N$, which means he has additional $Rs\ (N - k)$ over his initial $Rs\ k$. (Another way to express this is that he plays against an opponent who starts out with $Rs\ (N - k)$ and the game stops when either player has lost all of his or her money.) We are interested in computing the probability $r_k$ that the player will be ruined (or he has lost all of his or her money) after starting with $Rs\ k$.

To solve the problem, we note that at the end of the first game, the player will have the sum of $Rs\ (k + 1)$ if he wins the game (with probability $p$) and the sum of $Rs\ (k - 1)$ if he loses the game (with probability $q$). Thus, if he wins the first game, the probability that he will eventually be ruined is $r_{k+1}$; and if he loses his first game, the probability that he will be ruined is $r_{k-1}$. There are two boundary conditions in this problem. First $r_0 = 1$, since he cannot gamble when he has no money. Second $r_N = 0$, since he cannot be ruined. Thus, we obtain the following:

$$r_k = qr_{k-1} + pr_{k+1} \quad 0 < k < N$$

Since $p + q = 1$, we obtain

$$(p + q)r_k = qr_{k-1} + pr_{k+1} \quad 0 < k < N$$

and we can write it as

$$p(r_{k+1} - r_k) = q(r_k - r_{k-1})$$

From this we obtain the following:

$$r_{k+1} - r_k = \frac{q}{p}(r_k - r_{k-1}) \quad 0 < k < N$$

Notice that $r_2 - r_1 = \frac{q}{p}(r_1 - r_0) = \frac{q}{p}(r_1 - 1)$ ,

$r_3 - r_2 = \frac{q}{p}(r_2 - r_1) = \left(\frac{q}{p}\right)^2 (r_1 - 1)$, and so on, we obtain the following:

$$r_{k+1} - r_k = \left(\frac{q}{p}\right)^k (r_1 - 1) \quad 0 < k < N$$

Now,

$$r_k - 1 = r_k - r_0 = (r_k - r_{k-1}) + (r_{k-1} - r_{k-2}) + \cdots + (r_1 - 1)$$

$$= \left[\left(\frac{q}{p}\right)^{k-1} + \left(\frac{q}{p}\right)^{k-2} + \cdots + 1\right](r_1 - 1)$$

Thus, $r_k - 1 = \begin{cases} \dfrac{1-\left(\frac{q}{p}\right)^k}{1-\left(\frac{q}{p}\right)} (r_1 - 1), & p \neq q \\ k(r_1 - 1) & , \quad p = q \end{cases}$

Recalling the boundary condition that $r_N = 0$ implies that

$$r_1 = \begin{cases} 1 - \dfrac{1-\left(\frac{q}{p}\right)}{1-\left(\frac{q}{p}\right)^N} & , \quad p \neq q \\ 1 - \dfrac{1}{N} & , \quad p = q \end{cases}$$

Thus, $r_k = \begin{cases} 1 - \dfrac{1-\left(\frac{q}{p}\right)^k}{1-\left(\frac{q}{p}\right)^N} = \dfrac{\left(\frac{q}{p}\right)^k - \left(\frac{q}{p}\right)^N}{1-\left(\frac{q}{p}\right)^N} & , \quad p \neq q \\ 1 - \dfrac{k}{N} & , \quad p = q \end{cases}$

**Example 1: A certain student wanted to travel during a break to visit his parents. The bus fare was $Rs\ 20$, but the student had only $Rs\ 10$. He figured out that there was a bar near by where people play card games for money. The student signed up for one where he could bet $Rs\ 1$ per game. If he won the game, he would gain $Rs\ 1$; but if he lost the game, he would lose his $Rs\ 1$ bet. If the probability that he won a game is $0.6$ independent of other games, what is the probability that he was not able to make the trip?**

**Solution:** We have, $k = 10$ and $N = 20$. Define $a = \frac{q}{p}$ ,where $p = 0.6$ and $q = 1 - p = 0.4$. Thus, $a = \frac{2}{3}$ and the probability that he was not able to make the trip is the probability that he was ruined given that he started with $k = 10$.

This is $r_{10}$, which is given by

$$r_{10} = \frac{\left(\frac{q}{p}\right)^{10} - \left(\frac{q}{p}\right)^{20}}{1 - \left(\frac{q}{p}\right)^{20}} = \frac{\left(\frac{2}{3}\right)^{10} - \left(\frac{2}{3}\right)^{20}}{1 - \left(\frac{2}{3}\right)^{20}} = 0.0170$$

Thus, there is only a very small probability that he will not make the trip.

## Semi Random and Random Telegraph signal process

If $N(t)$ represents the number of occurrences of a specific event in $(0, t)$ and $X(t) = (-1)^{N(t)}$, then $\{X(t)\}$ is called a **semi – random telegraph signal process**.

If $\{X(t)\}$ is a semi – random telegraph signal process, $\alpha$ is a r.v which is independent of $X(t)$ and which assumes the values $+1$ and $-1$ with equal probability and $Y(t) = \alpha X(t)$, then $\{Y(t)\}$ is called a **random telegraph signal process.**

**A semi-random telegraph signal process is evolutionary.**

It will be proved Module 6.1 that the distribution of $N(t)$ is Poison with mean $\lambda t$, where the probability of exactly one occurrence in a small interval of length $h$ is $\lambda h$.

In other words, the process $\{N(t)\}$ is a **Poisson process** with the probability law.

$$P\{N(t)=r\}=\frac{e^{-\lambda t}(\lambda t)^r}{r!}; \quad r=0,1,2,\dots$$

If $\{X(t)\}$ is the semi – random telegraph signal process, then as per the definition given above, $X(t)$ can take the values $+1$ and $-1$ only.

$$P\{X(t)=1\}=P\{N(t)\ is\ even\}$$

$$=P\{N(t)=0\}+P\{N(t)=2\}+P\{N(t)=4\}+\cdots+\cdots$$

(since the events are mutually exclusive)

$$=e^{-\lambda t}\left\{1+\frac{(\lambda t)^2}{2!}+\frac{(\lambda t)^4}{4!}+\cdots+\cdots\right\}$$

$$=e^{-\lambda t}\cos h\,\lambda t$$

$$P\{X(t)=-1\}=P\{N(t)\ is\ odd\}$$

$$=P\{N(t)=1\}+P\{N(t)=3\}+\cdots+\infty$$

(since the events are mutually exclusive)

$$=e^{-\lambda t}\left\{\frac{\lambda t}{1!}+\frac{(\lambda t)^3}{3!}+\cdots+\cdots\right\}$$

$$=e^{-\lambda t}\sin h\,\lambda t$$

$$\therefore \qquad E\{X(t)\}=e^{-\lambda t}(\cos h\,\lambda t-\sin h\lambda\,t)$$

$$=e^{-\lambda t}e^{-\lambda t}=e^{-2\lambda t}$$

Note that $E\{X(t)\}$ is not constant and it is a function of $t$.

To find $E\{X(t_1)X(t_2)\}$, we required the joint probability distribution of $\{X(t_1),X(t_2)\}$.

Now $P\{X(t_1)=1,X(t_2)=1\}=P\{X(t_1)=1|X(t_2)=1\}P\{X(t_2)=1\}$

$=P\{$even number of occurrences of the event in $(t_1-t_2)\}P\{X(t_2)=1\}$

$=e^{-\lambda\tau}\cos h\,\lambda\tau\times e^{-\lambda t_2}\cos h\,\lambda t_2$; where $\tau=t_1-t_2$

Similarly, $P\{X(t_1) = -1,\ X(t_2) = -1\}$

$$= e^{-\lambda\tau} \cos h\, \lambda\tau e^{-\lambda t_2} \sin h\, \lambda t_2$$

$P\{X(t_1) = 1,\ X(t_2) = -1\} = e^{-\lambda\tau} \sin h\lambda\tau\, e^{-\lambda t_2} \sin h\, \lambda t_2$

and $P\{X(t_1) = -1,\ X(t_2) = 1\} = e^{-\lambda\tau} \sin h\lambda\tau\, e^{-\lambda t_2} \sin h\, \lambda t_2$

Now $X(t_1)X(t_2) = 1$, if $\{X(t_1) = 1\ and\ X(t_2) = 1\}$ or

$$\{X(t_1) = -1\ \ and\ \ X(t_2) = -1\}$$

$\therefore\quad P\{X(t_1)X(t_2) = 1\} = e^{-\lambda(\tau+t_2)} \cos h\lambda\tau\, (\cos h\, \lambda t_2 + \sin h\, \lambda t_2)$

$$= e^{-\lambda\tau} \cos h\, \lambda\tau$$

and $P\{X(t_1)X(t_2) = -1\} = e^{-\lambda(\tau+t_2)} \sin h\, \lambda\tau(\cos h\, t_2 + \sin h\, \lambda t_2)$

$$= e^{-\lambda\tau} \sin h\, \lambda\tau$$

$\therefore R(t_1, t_2) = E\{X(t_1)X(t_2)\} = 1 \times e^{-\lambda\tau} \cos h\, \lambda\tau - 1 \times e^{-\lambda\tau} \sin h\, \lambda\tau = e^{-2\lambda\tau}$

$$= e^{-2\lambda(t_1-t_2)}$$

Although $R(t_1, t_2)$ is a function of $(t_1 - t_2)$, $E\{X(t)\}$ is not a constant.

Therefore, $\{\boldsymbol{X(t)}\}$ **is evolutionary**.

**A random telegraph signal processes is a WSS process.**

Let us now consider the random telegraph signal process $\{Y(t)\}$, where
$Y(t) = \alpha X(t)$.
By definition, $P(\alpha = 1) = \frac{1}{2}$ and $P(\alpha = -1) = \frac{1}{2}$

$\therefore\qquad E(\alpha) = 0$ and $E(\alpha^2) = 1$

Now $E\{Y(t)\} = E(\alpha) \times E\{X(t)\} = 0$ [since $\alpha$ and $X(t)$ are independent]

$E\{Y(t_1) \times Y(t_2)\} = E\{\alpha^2 X(t_1) \times X(t_2)\}$

$$= E(\alpha^2)E\{X(t_1)X(t_2)\} \qquad \text{(by independence)}$$

$$= e^{-2\lambda(t_1-t_2)}$$

$i.e.$, $R_{yy}(t_1, t_2) =$ a function of $(t_1 - t_2)$. Therefore, $\{Y(t)\}$ is a wide – sense stationary process.