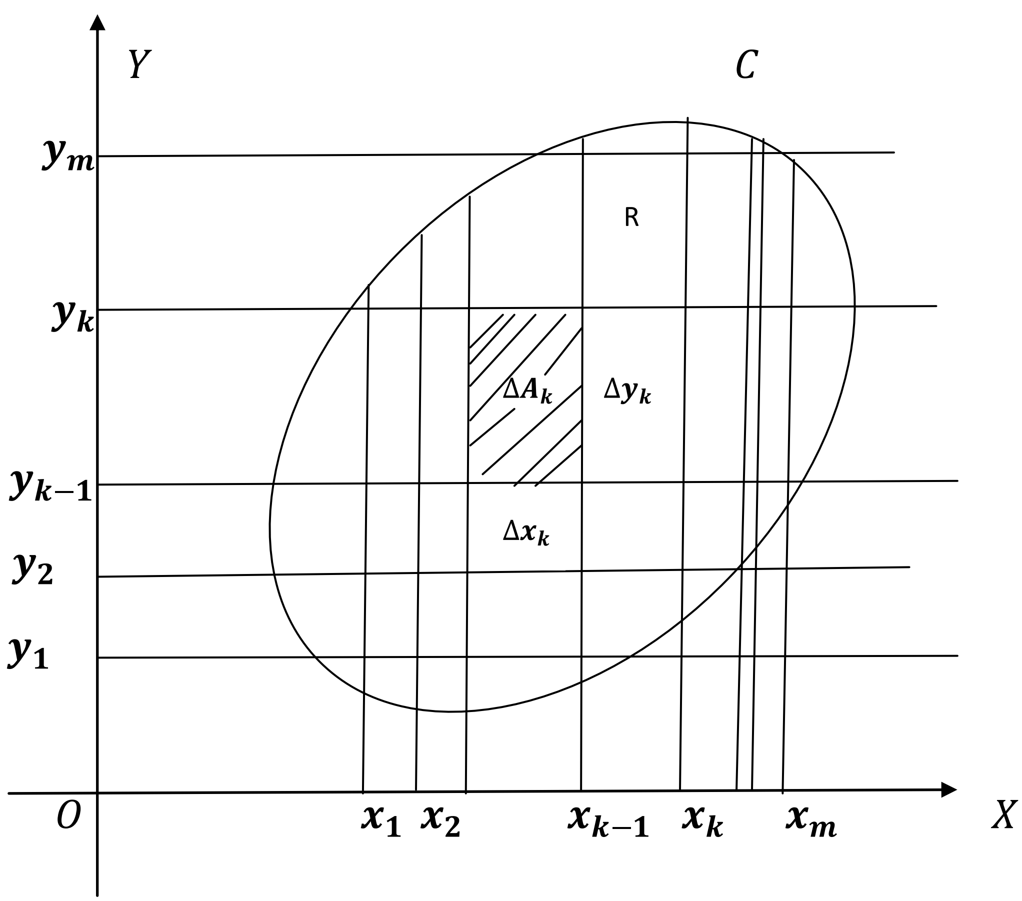
2.5

Double Integrals

In this module, we discuss methods of evaluating integrals of functions of two variables over a suitable region in $\mathbb{R}^2$. Integrals of a function of two variables over a region in $\mathbb{R}^2$ are called double integrals.

Let $f(x, y)$ be a continuous function in a simply connected, closed and bounded region R in a two dimensional space $\mathbb{R}^2$, bounded by a simple closed curve.



Region R for double integral

Subdivide the region R by drawing lines $x = x_k, y = y_k, k = 1, 2, \dots, m,$ parallel to the coordinate axes. Number the rectangles which are inside R from 1 to n. In each such rectangle, take an arbitrary point, say (ξ_k, η_k) in the kth rectangle and form the sum $J_k = \sum_{k=0}^{n} f(\xi_k, \eta_k)\Delta A_k$, where $\Delta A_k = \Delta x_k \Delta y_k$ is the area of the kth rectangle $d_k = \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}$ is the length of the diagonal of this rectangle. The maximum length of the diagonal, that is max d_k of the subdivisions is also called the norm of the subdivision. For different values of n, say $n_1, n_2, \dots, n_m, \dots$, we obtain a sequence of sums $J_{n_1}, J_{n_2}, \dots, J_{n_m}, \dots$. Let $n \to \infty$, such that the length of the largest diagonal $d_k \to 0$. If $\lim_{n\to\infty} J_n$ exists, independent of the choice of the subdivision and the point (ξ_k, η_k), then we say that $f(x, y)$ is integrable over R. This limit is called the double integral of $f(x, y)$ over R and is denoted by

$$J = \iint_R f(x, y)dxdy.$$

Let $f(x, y)$ be a continuous function in $\mathbb{R}^2$ defined on a closed rectangle $R = \{(x, y) | a \leq x \leq b \text{ and } c \leq y \leq d\}$. For any fixed $x \in [a, b]$ consider the integral $\int_c^d f(x, y)dy$.

The value of this integral depends on x and we get a new function of x. This can be integrated with respect to x and we get

$\int_a^b \left[\int_c^d f(x,y)dy \right] dx$. This is called an iterated integral.

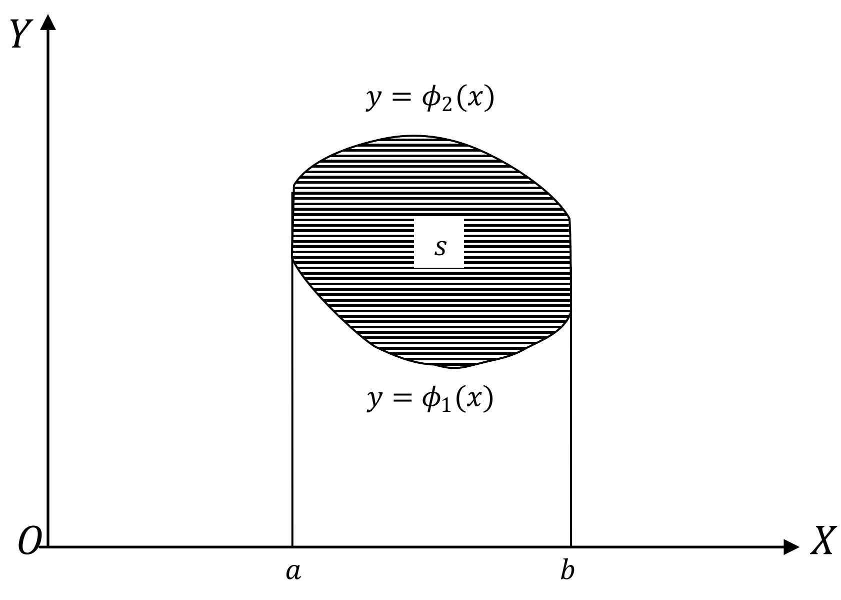
Similarly we can define another integral $\int_c^d \left[\int_a^b f(x,y)dx \right] dy$.

For continuous functions $f(x,y)$ we have

$$\iint_R f(x,y)dxdy = \int_a^b \left[\int_c^d f(x,y)dy \right] dx = \int_c^d \left[\int_a^b f(x,y)dx \right] dy$$

two continuous functions on $[a,b]$ then

$$\iint_s f(x,y)dxdy = \int_a^b \left[\int_{\phi_1(x)}^{\phi_2(x)} f(x,y)dy \right] dx.$$



The iterated integral in the right hand side is also written in the form

$$\int_a^b dx \int_{\phi_1(x)}^{\phi_2(x)} f(x,y)dy$$

Similarly if $S = \{(x,y)|c \le y \le d \ and \ \phi_1(y) \le x \le \phi_2(y)\}$ then

$$\iint_S f(x,y)dxdy = \int_c^d \left[\int_{\phi_1(x)}^{\phi_2(x)} f(x,y)dx \right] dy$$

If S cannot be written in neither of the above two forms we divide S into finite number of subregions such that each of the sub regions can be represented in one of the above forms and we get the double integral over S by adding the integrals over these sub regions. Hence to evaluate $\iint_s f(x,y)dxdy$ we first convert it to an iterated integral of the two forms given above.

Note 1: $\iint_s dxdy$ represents the area of the region S.

Note 2: In an iterated integral the limits in the first integral are constants and if the limits in the second integral are functions of x then we must first integrate with respect to y and the integrand will become a function of x alone. This is integrated with respect to x.

If the limits of the second integral are functions of y then we must first integrate with respect to x and the integrand will become a function of y alone. This is integrated with respect to y.

Properties of double integrals

1. If $f(x,y)$ and $g(x,y)$ integrable functions, then
$$\iint_R [f(x,y) \pm g(x,y)]dxdy = \iint_R f(x,y)dxdy \pm \iint_R g(x,y)dxdy.$$
2. $\iint_R kf(x,y)dxdy = k\iint_R f(x,y)dxdy$, where k is any real constant.
3. When $f(x,y)$ is integrable, then $|f(x,y)|$ is also integrable, and
$$\left|\iint_R f(x,y)dxdy\right| \leq \iint_R |f(x,y)|dxdy.$$
4. $\iint_R f(x,y)dxdy = f(\xi,\eta)A$, where A is the area of the region R and (ξ,η) is any arbitrary point in R. This result is called the mean value theorem of the double integrals.
If $m \leq f(x,y) \leq M$ for all (x,y) in R, then
$$mA \leq \iint_R f(x,y)dxdy \leq MA.$$
5. If $0 < f(x,y) \leq g(x,y)$ for all (x,y) in R, then
$$\iint_R f(x,y)dxdy \leq \iint_R g(x,y)dxdy.$$
6. If $f(x,y) \geq 0$ for all (x,y) in R, then
$$\iint_R f(x,y)dxdy \geq 0.$$

Application of double integrals

Double integrals have large number of applications. We state some of them.

1. If $f(x,y) = 1$, then $\iint_R dxdy$ gives the area A of the region R.

   For example, if R is the rectangle bounded by the lines $x = a, x = b, y = c$ and $y = d$, then $A = \int_c^d \int_a^b dxdy = \int_c^d \left[\int_a^b dx\right] dy = (b-a)\int_c^d dy = (b-a)(d-c)$ gives the area of the rectangle.
2. If $z = f(x,y)$ is a surface, then

   $$\iint_R zdxdy \text{ or } \iint_R f(x,y)dxdy$$

   gives the volume of the region beneath the surface $z = f(x,y)$ and above the $x - y$ plane.

   For example:

   if $z = \sqrt{a^2 - x^2 - y^2}$ and $R: x^2 + y^2 \leq a^2$, then

   $$V = \iint_R \sqrt{a^2 - x^2 - y^2}\, dxdy$$

   gives the volume of the hemisphere $x^2 + y^2 + z^2 = a^2 \geq 0$.

3. Let $f(x,y) = \rho(x,y)$ be a density function (mass per unit area) of a distribution of mass in the $x - y$ plane. Then

   $$M = \iint_R f(x,y)dxdy$$

   give the total mass of R.
4. Let $f(x,y) = \rho(x,y)$ be a density function. Then

$$\bar{x} = \frac{1}{M}\iint_R xf(x,y)dxdy, \bar{y} = \frac{1}{M}\iint_R yf(x,y)dxdy$$

give the coordinates of the centre of gravity $(\bar{x}, \bar{y})$ of the mass M in R.

5. Let $f(x,y) = \rho(x,y)$ be a density function. Then

$$I_x = \iint_R y^2 f(x,y)dxdy \text{ and } I_y = \iint_R x^2 f(x,y)dxdy$$

give the moments of inertia of the mass in R about the X-axis and the Y-axis respectively, whereas $I_0 = I_x + I_y$ is called the moment of inertia of the mass in R about the origin. Similarly,

$$I_y = \iint_R (x-a)^2 f(x,y)dxdy \text{ and } I_x = \iint_R (y-b)^2 f(x,y)dxdy$$

give the moment of inertia of the mass in R about the lines $x = a$ and $y = b$ respectively.

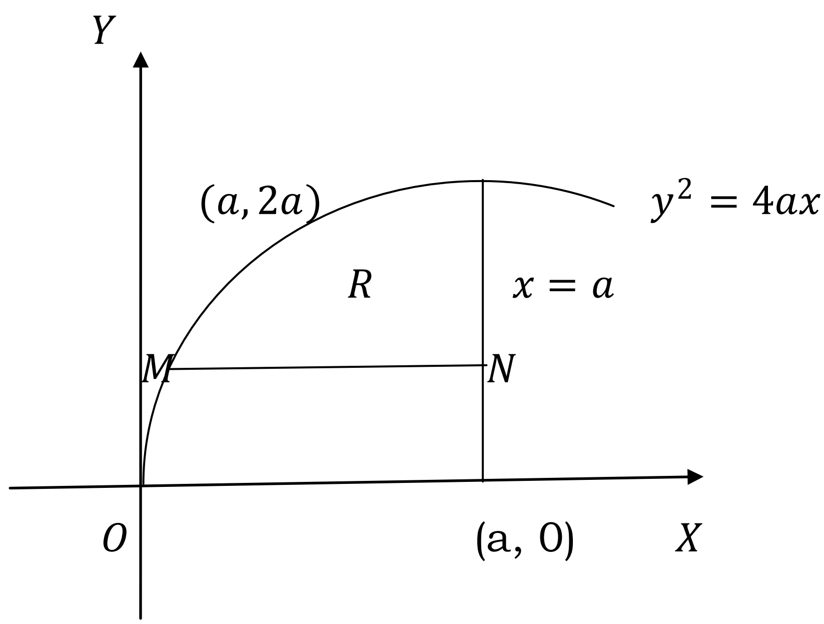
6. $\frac{1}{A}\iint_R f(x,y)dxdy$ gives the average value of $f(x,y)$ over R, where A is the area of the region R.

Change of order of Integration

In the evaluation of double integrals the computational work can often be reduced by interchanging the order of integration. While using this method, one has to bear in mind that the change of order of integration generally changes the limits of integration; the new limits are to be determined by examining the geometrical region on which the integration is being carried out.

Example: Evaluate $\int_0^a \int_0^{2\sqrt{xa}} x^2 dydx, a > 0,$ by changing the order of integration.

In the given integral, x increases from 0 to a, for each x, y increases from 0 to $2\sqrt{xa}$. Hence the lower value of y lies on the X-axis and the upper value on the upper part of the parabola $y^2 = 4ax$. Therefore the region R of integration is bounded by the X-axis, the line and the arc of the parabola $y^2 = 4ax$ in the first quadrant. This region is shown in figure.



We observe that, in R, y increases from 0 from $2a$, and, for each y, x varies from a point M on the parabola $y^2 = 4ax$ to a point N on the line $x = a$; that is x increases from $(y^2/4a)$ to a. Hence

$$\int_0^a \int_0^{2\sqrt{xa}} x^2 dy dx = \int_{y=0}^{2a} \left\{ \int_{x=y^2/4a}^{a} x^2 dx \right\} dy = \int_0^{2a} \frac{1}{3} \left\{ a^3 - \left(\frac{y^2}{4a} \right)^3 \right\} dy$$

$$= \frac{1}{3} [a^3 - \frac{1}{64a^3} \frac{y^7}{7}]_0^{2a} = \frac{1}{3} \{a^3 (2a) - \frac{1}{64a^3} \frac{(2a)^7}{7}\}$$

$$= \frac{4}{7} a^4.$$

Problem 1: Evaluate the following repeated integrals:

i. $\int_1^4 \int_0^{\sqrt{4-x}} xydydx$

ii. $\int_0^5 \int_0^{y^2} x(x^2+y^2)dxxy$

Solution: By using the meaning of repeated integrals, we find:

i. $$\int_1^4 \int_0^{\sqrt{4-x}} xydydx = \int_{x=1}^4 \left\{ \int_{y=0}^{\sqrt{4-x}} xydy \right\} dx$$

$$= \int_{x=1}^4 x \left\{ \left[\frac{y^2}{2} \right]_{y=0}^{\sqrt{4-x}} \right\} dx,$$ on evaluating the inner integral with x held fixed.

$$= \int_1^4 x \left\{ \frac{1}{2}(4-\mathrm{x}) \right\} \mathrm{dx} = \left[\mathrm{x}^2 - \frac{\mathrm{x}^3}{6} \right]_1^4$$

$$= \left(16 - \frac{64}{6}\right) - \left(1 - \frac{1}{6}\right) = \frac{32}{6} - \frac{5}{6} = \frac{27}{6} = \frac{9}{2}.$$

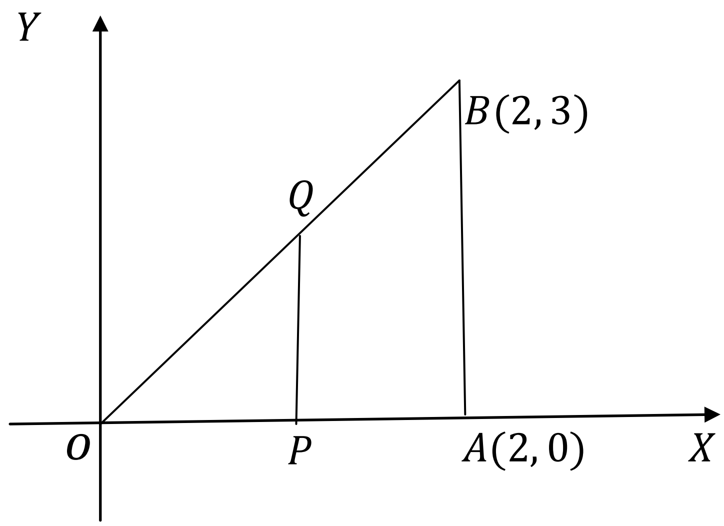
ii. $$\int_0^5 \int_0^{y^2} x(x^2+y^2)dxxy = \int_{y=0}^5 \left\{ \int_{x=0}^{y^2} \left(x^3 + xy^2 \right) dx \right\} dy$$

$$= \int_{y=0}^5 \left\{ \left[\frac{x^4}{4} + y^2 \left(\frac{x^2}{2} \right) \right]_{x=0}^{y^2} \right\} dy,$$ on evaluating the inner integral with y held fixed

$$= \int_{y=0}^5 \left\{ \frac{(y^2)^4}{4} + y^2 \frac{(y^2)^2}{2} \right\} dy = \left[\frac{1}{4} . \frac{y^9}{9} + \frac{1}{2} . \frac{y^7}{7} \right]_0^5 = \frac{5^7}{4} \left(\frac{25}{9} + \frac{2}{7} \right).$$

Problem 2: If $\Re$ is the triangular region with vertices $(0,0),(2,0),(2,3)$, evaluate $\iint_{\Re} x^2 y^2 dxdy$.

Solution:



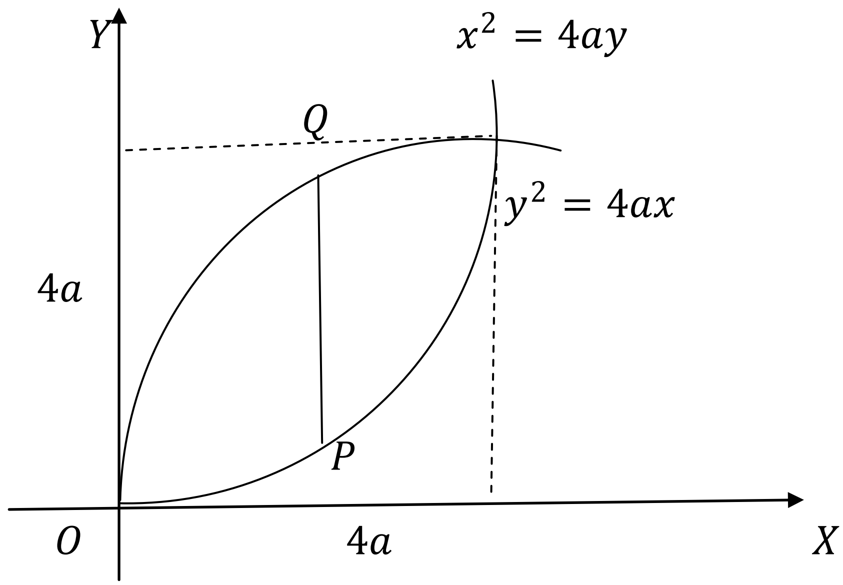
The region where in the integration is to be carried out is shown in figure. We note that, in this region, x increases from 0 to 2 and for each x, y increases from a point P on the X-axis to a point Q on the line OB. For the point P, We have $y = 0$. The equation of OB is $y = (3/2)x$. Therefore, for the point Q, we have $y = (3/2)x$. Thus, for each x, y increases from 0 to $(3/2)x$. Hence

$$\iint_{\Re} x^2 y^2 dxdy = \int_{x=0}^{2}\left\{\int_{y=0}^{(3/2)x} x^2 y^2 dy\right\}dx = \int_{0}^{2}\left\{x^2\left[\frac{y^3}{3}\right]_0^{(3/2)x}\right\}dx$$

$$= \int_0^2 x^2\left\{\frac{1}{3}.\frac{27}{8}x^3\right\}dx = \frac{9}{8}\left[\frac{x^6}{6}\right]_0^2 = \frac{3}{16}.2^6 = 12.$$

Problem 3: Evaluate $\iint_{\Re} y dx dy,$ where $\Re$ is the region bounded by the parabolas $y^2 = 4ax$ and $x^2 = 4ay, a > 0$.

Solution:



Solving the given equations, we find that the two parabolas intersect at the points (0,0) and $(4a, 4a)$. Therefore, the region bounded by these parabolas is as shown in figure. In this region, x increases from 0 to $4a$, and, for each x, y increases from a point P on the parabola $x^2 = 4ay$ to a point Q on the parabola $y^2 = 4ax$. We find that, at $P, y = (x^2/4a)$ and, at $Q, y = \sqrt{4ax}$, Hence

$$\iint_{\Re} y dx dy = \int_{x=0}^{4a} \left\{ \int_{y=x^2/4a}^{\sqrt{4ax}} y dy \right\} = \int_0^{4a} \frac{1}{2} \left\{ (4ax) - \left(\frac{x^2}{4a} \right)^2 \right\} dx$$

$$= \frac{1}{2}\left[2ax^2 - \frac{1}{16a^2}\left(\frac{x^5}{5} \right) \right]_0^{4a} = \frac{1}{2}\left\{ 32a^3 - \frac{1}{16a^2} \cdot \frac{(4a)^5}{5} \right\}$$

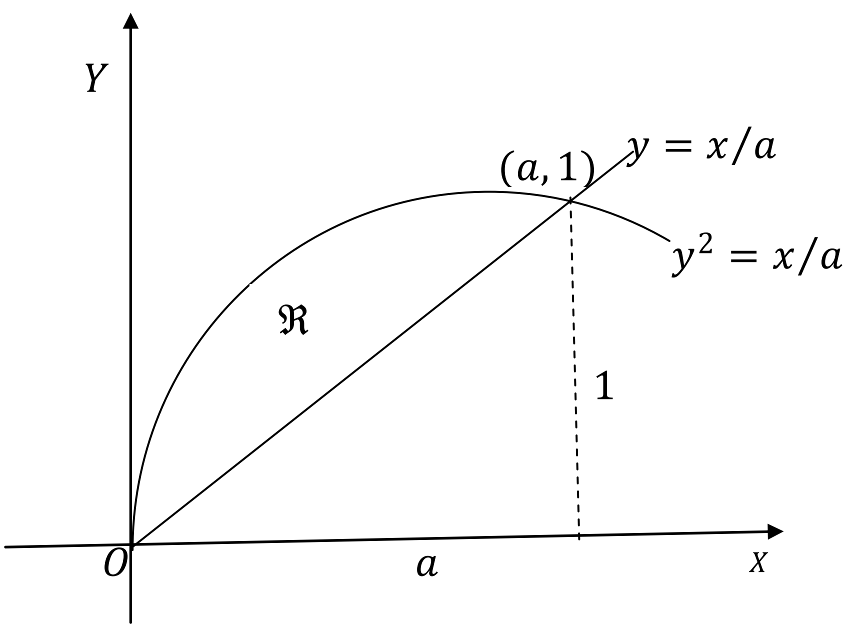
$$= \frac{1}{2}\left\{ 32a^3 - \frac{64}{5}a^3 \right\} = \frac{48}{5}a^3.$$

Problem 4: Change the order of integration in the integral

$$\int_0^a \int_{x/a}^{\sqrt{x/a}} \left(x^2+y^2\right) dydx, a>0$$

and hence evaluate it.

Solution: In the given integral, x increases from 0 to a, and, for each x, y varies from $y = x/a$ to $y = \sqrt{x/a}$. Hence the lower value of y lies on he curve $y = x/a$ (which is a straight line) and the upper value of y lies on the curve $y^2 = x/a$(which is a parabola). We check that the line $y = x/a$ and the parabola $y^2 = x/a$ intersect at the point $(0,0)$ and $(a,1)$. The region $\Re$ of integration is therefore bounded by the line $y = x/a$ and the parabola $y^2 = x/a$ between the origin and the point $(a,1)$. This region is shown in figure.



From the above figure, we observe that in $\Re$, y increases from 0 to 1 and, for each y, x varies from a point on the parabola $y^2 = x/a$ to a point on the line $y = x/a$; that is, each y with $0 \leq x \leq 1, x$ varies from ay^2 to ay. Hence

$$\int_0^a \int_{x/a}^{\sqrt{x/a}} (x^2 + y^2) dydx = \int_{y=0}^{1} \left\{ \int_{x=ay^2}^{ay} \left(x^2 + y^2 \right) dxdy \right\}$$

$$= \int_0^1 \left\{ \left[\frac{x^3}{3} + y^2 x \right]_{x=ay^2}^{ay} \right\} dy$$

$$= \int_0^1 \left[\frac{1}{3} \left(a^3 y^3 - a^3 y^6 \right) + y^2 \left(ay - ay^2 \right) \right] dy$$

$$= \frac{a^3}{3} \left(\frac{1}{4} - \frac{1}{7} \right) + a \left(\frac{1}{4} - \frac{1}{5} \right) = \frac{a^3}{28} + \frac{a}{20}.$$

Problem 5: Evaluate $I = \int_0^{\pi} \int_0^{a\cos\theta} r\sin\theta dr d\theta$

Solution:

$$I = \int_0^{\pi} \int_0^{a\cos\theta} r\sin\theta dr d\theta$$

$$I = \int_0^{\pi} \sin\theta \left[\frac{r^2}{2}\right]_0^{a\cos\theta} d\theta$$

$$= \frac{1}{2}\int_0^{\pi} a^2\cos^2\theta \sin\theta d\theta$$

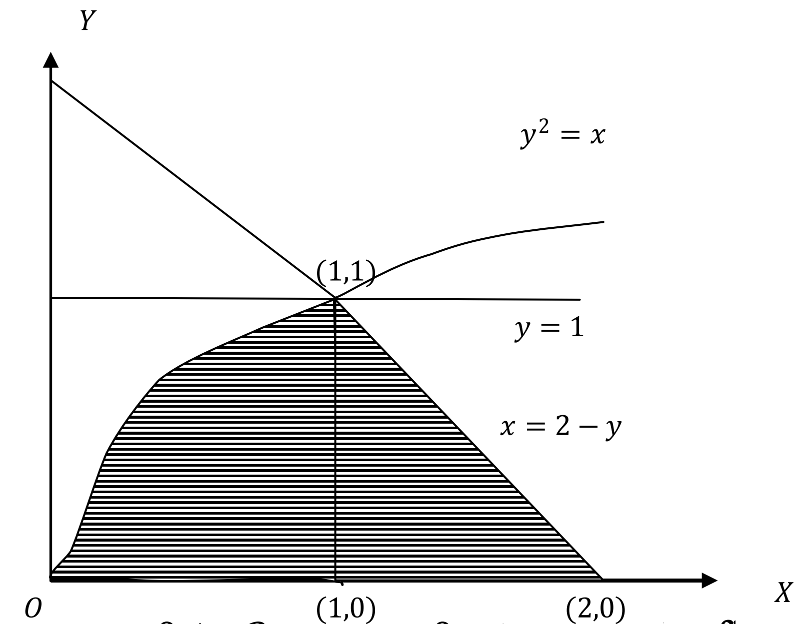
$$= -\frac{a^2}{2}\int_0^{\pi} \cos^2\theta d\left(\cos\theta\right)$$

$$= -\frac{a^2}{6}\left[\cos^3\theta\right]_0^{\pi} = \frac{a^2}{3}.$$

Problem 6: Evaluate

$I = \iint\limits_D xydydx$ where D is the region bounded by the curve $x = y^2, x = 2 - y, y = 0$ and $y = 1$.

Solution: The given region bounded by the curves is shown in the figure.



In this region x va n 0 to 2. When $0 \le x \le 1$ for fi x, y varies from 0 to $\sqrt{x}$. When $1 \le x \le 2, y$ varies from 0 to $2 - x$.

Therefore the region D can be subdivided into two regions D_1 and D_2 as shown in the figure.

$$\therefore \iint\limits_D xydydx = \iint\limits_{D_1} xydydx + \iint\limits_{D_2} xydydx$$

In this region D_1 for fixed x, y varies from $y = 0$ to $y = \sqrt{x}$ and for fixed y, x varies from $x = 0$ to $x = 1$. Similarly for the region D_2, the limit of integration for y is $y = 0$ to $y = 2 - x$.

$$\iint_D xydydx = \int_0^1 \int_0^{\sqrt{x}} xy\ dy\ dx + \int_1^2 \int_0^{2-x} xy\ dy\ dx$$

$$= \int_0^1 \left[\frac{xy^2}{2}\right]_0^{\sqrt{x}} dx + \int_1^2 \left[\frac{xy^2}{2}\right]_0^{2-x} dx$$

$$= \frac{1}{2}\int_0^1 x^2\ dx + \frac{1}{2}\int_1^2 x(2-x)^2\ dx$$

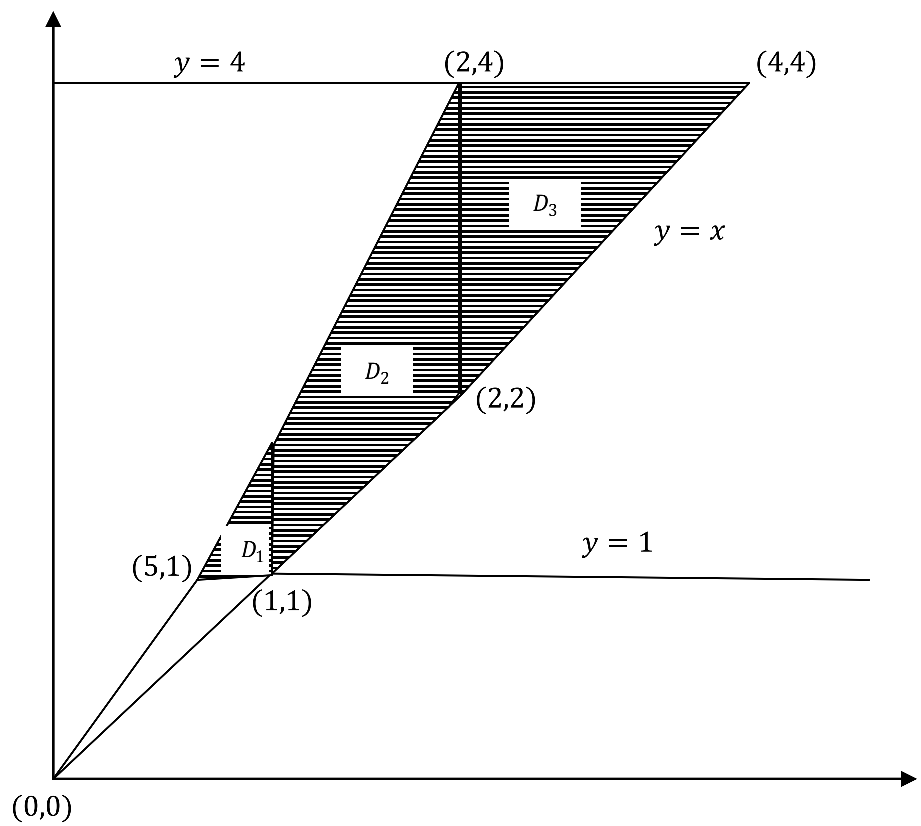
$$= \left[\frac{x^3}{6}\right]_0^1 + \frac{1}{2}\left[2x^2 + \frac{x^4}{4} - \frac{4x^3}{3}\right]_1^2$$

$$= \frac{3}{8}.$$

Problem 7: Change the order of integration in the integral

$I = \int_1^4 \int_{y/2}^{y} f(x, y)\, dx\, dy.$

Solution: The region of integration D is bounded by the lines $x = \frac{y}{2}; x = y; y = 1$ and $y = 4$. The region is a quadrilateral as shown in the figure



In this region x varies from $\frac{1}{2}$ to 4.

When $\frac{1}{2} \leq x \leq 1, y$ varies from 1 to $2x$.

When $1 \leq x \leq 2, y$ varies from x to $2x$.

When $2 \leq x \leq 4, y$ varies from x to 4.

Hence for changing the order of integration we must divide D into sub regions D_1, D_2, D_3 as shown in the figure.

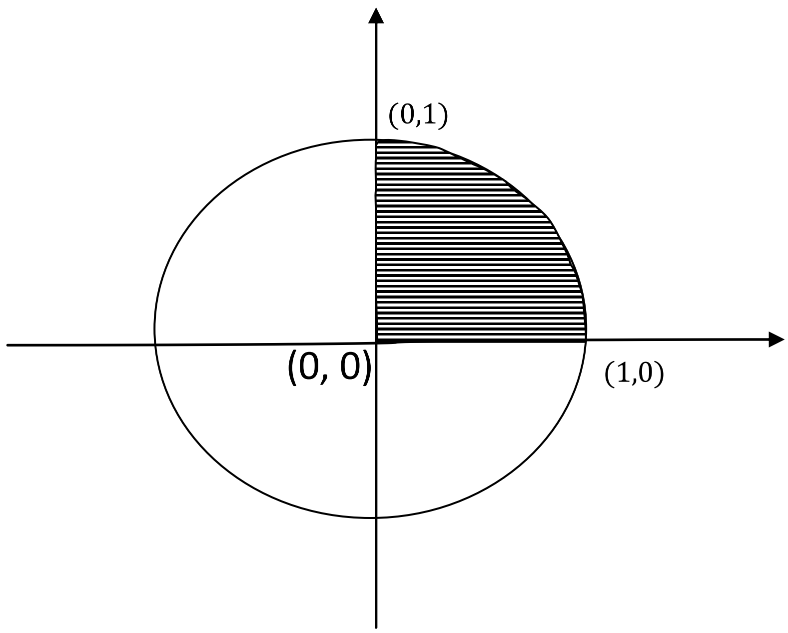
$$\therefore \quad I = \iint_D f(x,y)dxdy$$

$$= \iint_{D_1} f(x,y)dxdy + \iint_{D_2} f(x,y)dxdy + \iint_{D_3} f(x,y)dxdy$$

$$= \int_{1/2}^{1}\int_{1}^{2x} f(x,y)dy\,dx + \int_{1}^{2}\int_{x}^{2x} f(x,y)dy\,dx +$$
$$\int_{2}^{4}\int_{x}^{4} f(x,y)dy\,dx \,.$$

Problem 8: Evaluate $\int_0^1 \int_0^{\sqrt{1-x^2}} y^2 \; dy\, dx$ by interchanging the order of integration.

Solution: The region is bounded by the line$y = 0(X - \text{axis})$; the unit circle $x^2 + y^2 = 1$; the line $x = 0(Y - \text{axis})$ and the line $x = 1$. Hence the region of integration is the positive quadrant of the unit circle $x^2 + y^2 = 1$ and it is given in the figure.



In this region y varies from 0 to 1 and for a fixed y, x varies from 0 to $\sqrt{1-y^2}$.

$$\int_0^1 \int_0^{\sqrt{1-x^2}} y^2 \; dy\, dx = \int_0^1 \int_0^{\sqrt{1-y^2}} y^2 \; dy\, dx$$

$$= \int_0^1 y^2 [x]_0^{\sqrt{1-y^2}} \, dy$$

$$= \int_0^1 y^2 \sqrt{1-y^2} dy$$

$$= \int_0^{\pi/2} \sin^2\theta \cos^2\theta \, d\theta \quad (\text{putting } y = \sin\theta)$$

$$= \left(\frac{1.1}{4.2}\right)\frac{\pi}{2} = \frac{\pi}{16}.$$

Exercise

1) Evaluate $\int_1^4 \int_{\sqrt{y}}^2 (x^2 + y^2)dxdy$ by changing the order of integration.
2) By changing the order of integration evaluate $\int_0^3 \int_1^{\sqrt{4-y}} (x+y)dxdy.$
3) Change the order of integration in the integral $\int_0^a \int_{\frac{x^2}{a}}^{2a-x} xydydx$ and evaluate.
4) Evaluate $I = \iint_D e^{\frac{y}{x}} dxdy$ where D is the region bounded by the straight lines $y = x$; $y = 0$ and $x = 1$.
5) Evaluate $\iint_D (x^2 + y^2)dxdy$ where D is the region bounded by $y = x^2, x = 2$ and $x = 1$.
6) Evaluate $I = \int_0^\infty \int_x^\infty \frac{e^{-y}}{y} dydx.$
7) Find the area of the circle $x^2 + y^2 = r^2$ by using double integral.
8) Find the area of the region D bounded by the parabolas $y = x^2$ and $x = y^2$.

9) Evaluate $I = \int_0^{\frac{\pi}{2}} \int_0^{\infty} \frac{r}{(r^2+a^2)^2} dr d\theta$.

Answers

1) $\frac{1026}{105}$

2) $\frac{241}{60}$

3) $\frac{9a^4}{24}$

4) $\frac{1}{2}(e-1)$

5) $\frac{1286}{105}$

6) 1

7) πr^2

8) $\frac{1}{3}$

9) $\frac{\pi}{4a^2}$