Multiple Integrals

Double Integrals:

1. Evaluate $\int_0^2\int_0^3 xy\,dx\,dy$.

Sol:

$$\frac{x^2}{2}\Big|_0^2 \;\frac{y^2}{2}\Big|_0^3$$

$$= \tfrac{1}{2}(4-0)\;\tfrac{1}{2}(9-0)$$

$$= 9$$

2. Evaluate $\int_0^3\int_1^2 xy(1+x+y)\,dy\,dx$.

$$= \int_0^3\int_1^2 xy\,dx\,dy + \int_0^3\int_1^2 x^2\,dx\,y\,dy + \int_0^3\int_1^2 x\,dx\,y^2\,dy$$

$$= \frac{x^2}{2}\Big|_1^2 \frac{y^2}{2}\Big|_0^3 + \frac{x^3}{3}\Big|_1^2 \frac{y^2}{2}\Big|_0^3 + \frac{x^2}{2}\Big|_1^2 \cdot \frac{y^3}{3}\Big|_0^3$$

$$= \tfrac{1}{2}[4-1]\cdot\tfrac{1}{2}[9-0] + \tfrac{1}{3}[8-1]\,\tfrac{1}{2}[9-0] + \tfrac{1}{2}(4-1)\tfrac{1}{3}(27)$$

$$= \frac{27}{4} + \frac{21}{2} + \frac{27}{2}$$

$$= \frac{27}{4} + \frac{48}{2}$$

$$= \cancel{\frac{54+48}{4}} \quad \frac{27+96}{4} = \frac{123}{4}.$$

96
27
123

3. Solve $\int_0^2\int_0^x y\,dy\,dx$.

clearly y depends on x and x is independent variable
$\therefore$ x varying from $0 \to 2$
y " " $0 \to x$

$$\int_0^2 \frac{y^2}{2}\Big|_0^x dx$$

$$\int_0^2 \tfrac{1}{2}(x^2-0)\,dx = \frac{x^3}{6}\Big|_0^2 = \tfrac{1}{6}[8-0]$$

$$= 4/3.$$

4. Solve. $\int_0^1 \int_x^{\sqrt{x}} (x^2+y^2)\, dx\, dy$.

$y \to$ dependent & varying from $x \to \sqrt{x}$

$x \to$ in " " " $0 \to 1$

$$= \int_0^1 \left[\int_x^{\sqrt{x}} (x^2+y^2)\, dy \right] dx$$

$$= \int_0^1 \left[x^2 y + \frac{y^3}{3} \right]_x^{\sqrt{x}} dx$$

$$= \int_0^1 \left[x^{5/2} + \frac{x^{3/2}}{3} - x^3 - \frac{x^3}{3} \right] dx$$

$$= \left[\frac{x^{7/2}}{7/2} + \frac{x^{5/2}}{3 \cdot \frac{5}{2}} - \frac{x^4}{4} - \frac{x^4}{12} \right]_0^1$$

$$= \left[\frac{2}{7}(1) + \frac{2}{15}(1) - \frac{1}{4}(1) - \frac{1}{12}(1) \right]$$

$$= \left[\frac{2}{7} + \frac{2}{15} - \frac{1}{4} - \frac{1}{12} \right]$$

$= 3/35$.

$= 0.0857$

5. $\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dy\, dx}{1+y^2+x^2}$.

$y \to$ dep $\to 0 \to \sqrt{1+x^2}$

$x \to$ indep $\to 0 \to 1$.

$$= \int_0^1 \left[\int_0^{\sqrt{1+x^2}} \frac{1}{1+y^2+x^2}\, dy \right] dx.$$

$$= \int_0^1 \left[\int_0^{\sqrt{1+x^2}} \frac{1}{\left(\sqrt{1+x^2}\right)^2 + y^2}\, dy \right] dx$$

$$= \int_0^1 \left[\frac{1}{\sqrt{1+x^2}} \tan^{-1} \frac{y}{\sqrt{1+x^2}} \Bigg|_0^{\sqrt{1+x^2}} \right] dx$$

$$\int_0^1 \frac{1}{\sqrt{1+x^2}} (\tan^{-1} 1 - \tan^{-1} 0)\, dx$$

$$= \int_0^1 \frac{1}{\sqrt{1+x^2}} \, \pi/4 \, dx.$$

$$= \pi/4 \ \sinh^{-1}\left(\frac{x}{1}\right)\Big|_0^1$$

$$= \pi/4 \left[(\sinh^{-1} 1)\right]$$

6. Solve $\int_0^a \int_0^{\sqrt{a^2-y^2}} \left(\sqrt{a^2-x^2-y^2}\right) dx\, dy.$ $x \to$ dep $\to 0 \to \sqrt{a^2-y^2}$
$y \to$ indep $\to 0 \to a$

$$= \int_0^a \left[\int_0^{\sqrt{a^2-y^2}} \left(\sqrt{a^2-x^2-y^2}\right) dx \right] dy. \qquad \sqrt{\left(\sqrt{a^2-y^2}\right)^2 - x^2}$$

$$= \int_0^a \left[\frac{x}{2}\sqrt{a^2-y^2-x^2} + \frac{\left(\sqrt{a^2-y^2}\right)^2}{2} \sin^{-1}\left(\frac{x}{\sqrt{a^2-y^2}}\right)\right]_0^{\sqrt{a^2-y^2}} dy.$$

$$= \int_0^a (0-0) + \frac{a^2-y^2}{2}(\sin^{-1} 1 - \sin^{-1} 0)\, dy$$

$$= \int_0^a \frac{a^2-y^2}{2} (\pi/2)\, dy$$

$$= \pi/4 \int_0^a a^2 - y^2 \, dy$$

$$= \pi/4 \left(a^2 y - \frac{y^3}{3}\right)\Big|_0^a$$

$$= \pi/4 \left[a^3 - \frac{a^3}{3}\right].$$

$$= \frac{2a^3\pi}{4\times 3} = a^3 \pi/6$$

7. Evaluate : $\int_0^4\int_0^{x^2} e^{y/x}\,dx\,dy$:

$x \to$ indep
$y \to$ indep

$$\int_0^4\left[\int_0^{x^2} e^{y/x}\,dy\right]dx$$

$$= \int_0^4 \left.\frac{e^{y/x}}{1/x}\right|_0^{x^2} dx$$

$$= \int_0^4 x\left(e^x - e^0\right)dx$$

$$= \int_0^4 xe^x\,dx - \int_0^4 x\,dx$$

$$= e^x(x-1)\Big|_0^4 - \frac{x^2}{2}\Big|_0^4$$

$$= 3e^4 - e^0(0-1) - \frac{1}{2}(16-0)$$

$$= 3e^4 + 1 - 8$$

$$= \underline{\underline{3e^4 - 7}}$$

All eqs except 11 upto eq 14.

eq 4: $\int_{x=0}^{a}\int_{y=0}^{b}(x^2+y^2)\,dy\,dx$

$$\int_{x=0}^{a}\left(x^2y + \frac{y^3}{3}\right)_0^b dx$$

$$\int_{x=0}^{a}\left(x^2b + \frac{b^3}{3}\right)dx = \left(\frac{x^3b}{3} + \frac{b^3x}{3}\right)_0^a = \frac{a^3b}{3} + \frac{ab^3}{3}.$$

eq 7: $\int_0^1\int_0^1 \frac{dx\,dy}{\sqrt{(1-x^2)(1-y^2)}}$

Sol: $\int_0^1 \frac{1}{\sqrt{1-y^2}} (\sin^{-1}x)_0^1 \, dy$

$\sin^{-1}(1) - \sin^{-1}(0)$
$= \pi/2$.

$\int_0^1 \frac{1}{\sqrt{1-y^2}} \frac{\pi}{2} \, dy$

$\frac{\pi}{2} (\sin^{-1}y)_0^1 = \pi^2/4$.

eg8: $\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dy\,dx}{1+x^2+y^2}$

$= \int_0^1 \left[\int_0^{\sqrt{1+x^2}} \frac{1}{(\sqrt{1+x^2})^2+y^2} dy \right] dx$.

$= \int_0^1 [$ see 5th sum before pg

eg9: Evaluate $\int_0^2 \int_0^x e^{x+y} \, dy \, dx$ $\qquad e^{x+y} = e^x . e^y$

$\int_0^2 \left[\int_0^x e^{x+y} dy \right] dx = \int_0^2 \left(e^{x+y} \right)_0^x dx$

$= \int_0^2 (e^{2x} - e^x) \, dx$

$= \frac{e^{2x}}{2} - e^x \Big|_0^2$

$= \frac{e^4}{2} - e^2 - \frac{1}{2} + 1$

$= \frac{e^4}{2} - e^2 + 1/2$

$= \frac{e^4 - 2e^2 + 1}{2}$.

eg10: $\int_0^{\pi/2} \int_{-1}^1 x^2 y^2 \, dx \, dy$

$\frac{x^3}{3} \Big|_0^{\pi/2} \frac{y^3}{3} \Big|_{-1}^1$

$\frac{\pi^3}{24} \cdot \left(\frac{1}{3} + \frac{1}{3} \right) . = \frac{\pi^3}{36}$

ii. $$\int_0^4 \int_{y^2/4}^{y} \frac{y}{x^2+y^2}\,dx\,dy$$

$$= \int_0^4 \left[\int_{y^2/4}^{y} \frac{y}{x^2+y^2}\,dx \right] dy$$

$$\int \frac{1}{x^2+a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + c$$

$$= \int_0^4 \left[y \cdot \frac{1}{y} \tan^{-1}\left(\frac{x}{y}\right) \right]_{y^2/4}^{y} dy$$

$$= \int_0^4 \left[\tan^{-1}(1) - \tan^{-1}\left(\frac{y}{4}\right) \right] dy$$

$$= \int_0^4 \left(\pi/4 - \tan^{-1}(y/4) \right) dy$$

$$= \pi/4 \, y \Big|_0^4 - \int_0^4 \tan^{-1}(y/4)\,dy$$

$$= \frac{\pi}{4}(4-0) - \int_0^4 1 \cdot \tan^{-1}(y/4)\,dy.$$

$$\int uv = uv_1 - \int u'v_1$$

$$= \tan^{-1}(y/4)(y) - \int_0^4 \frac{1}{1+(y/4)^2}(1/4) \cdot y\,dy$$

$$= y\tan^{-1}(y/4)\Big|_0^4 - \int_0^4 \frac{4y}{16+y^2}\,dy$$

$$= 4\tan^{-1}(1) - 2\int_0^4 \frac{2y}{y^2+16}\,dy$$

$$= 4 \cdot \frac{\pi}{4} - 2\log|y^2+16|\Big|_0^4$$

$$\int uv = \pi - 2(\log 32 - \log 16)$$

$$\therefore = \pi - (\pi - 2\log 32 + 2\log 16)$$

$$= +2\log 32 - 2\log 16.$$

$$= 2\log 32/16$$

$$= \underline{2\log 2}$$

13 (ii) $\int_0^1\int_0^{x^2} e^{y/x}\, dy\, dx$

$$\int_0^1\left[\int_0^{x^2} e^{y/x}\, dy\right] dx = \int_0^1\left[\frac{e^{y/x}}{1/x}\Big|_0^{x^2}\right] dx$$

$$= \int_0^1 x(e^x-1)\, dx$$

$$= e^x(x-1) - \frac{x^2}{2}\Big|_0^1$$

$$= -1/2 - (-1) + 0$$

$$= 1/2 \,//.$$

eg14:

$$\int_0^1\int_x^{\sqrt{x}} x^2y^2(x+y)\, dy\, dx$$

$$= \int_0^1\left[\int_x^{\sqrt{x}} (xy^2+y^3)\, dy\right] dx$$

$$= \int_0^1 x^2\left[\frac{xy^3}{3} + \frac{y^4}{4}\right]_x^{\sqrt{x}} dx.$$

$$= \int_0^1 x^2\left[\frac{x}{3}(x^{3/2}-x^3) + \frac{1}{4}(x^2-x^4)\right] dx$$

$3+\frac{3}{2}$

$$= \int_0^1\left[\frac{x^{9/2}}{3} - \frac{x^6}{3} + \frac{x^4}{4} - \frac{x^6}{4}\right] dx$$

$$= \frac{x^{11/2}}{\frac{11}{2}\cdot 3} - \frac{x^7}{21} + \frac{x^5}{20} - \frac{x^7}{28}\Big|_0^1$$

$$= \frac{2}{33}(1) - \frac{1}{21}(1) + \frac{1}{20}(1) - \frac{1}{28}(1) = 0.027$$

eg15:

$$\int_0^a\int_0^{\sqrt{a^2-x^2}} \sqrt{a^2-x^2-y^2}\, dy\, dx$$

$$\int_0^a\left[\int_0^{\sqrt{a^2-x^2}} \sqrt{}\ dy\right] dx$$

Same as -6 before 2 pgs.

1. Evaluate $\iint_R y\,dy\,dx$ where R is the region bounded by the parabolas $y^2 = 4x$ and $x^2 = 4y$

Sol: Given parabolas $y^2 = 4x$ and $x^2 = 4y$

$x^4 = 4(4x)$

$x^4 - 16x = 0$

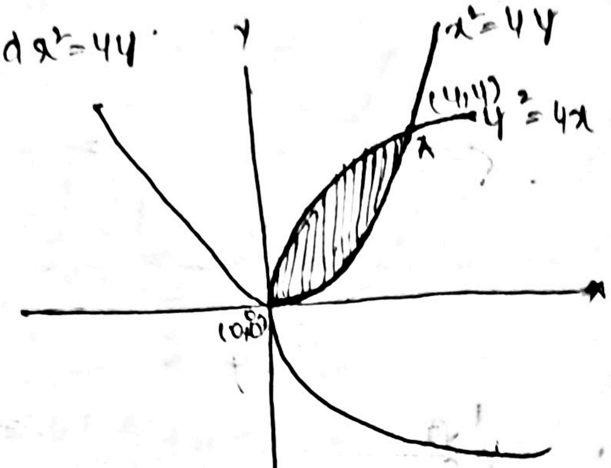
$x^3 = 64$

$x = 4$

If $x = 4 \Rightarrow y = \pm 4$

pt is in Q_1

$\therefore A(4,4)$



$\therefore$ Pt of intersection of axis and parabolas

$O(0,0)$ and $A(4,4)$

let us consider x is indep and x varying from 0 to 4. and y is dep and y " " from $\frac{x^2}{4}$ (OA) lower part to $2\sqrt{x}$ (AO) upper part

(If y indep and x dep then $2\sqrt{y} \to \frac{y^2}{4}$ ~~$\frac{x^2}{4} \to 2\sqrt{x}$~~)

Given $\iint_R y\,dy\,dx$

$$\int_0^4 \left[\int_{x^2/4}^{2\sqrt{x}} y\,dy \right] dx$$

$$= \int_0^4 \left[\frac{y^2}{2} \right]_{x^2/4}^{2\sqrt{x}} dx$$

$$= \frac{1}{2}\left[\int_0^4 \left(4x - x^4/16\right) dx\right]$$

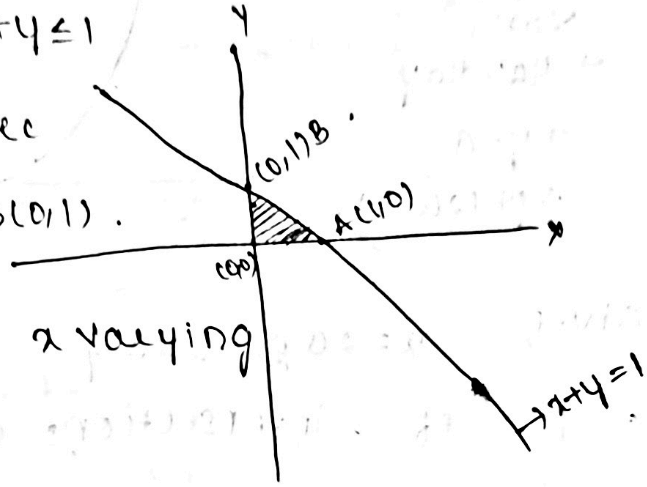
$= \frac{1}{2}\left[\frac{4x^2}{2} - \frac{x^5}{80}\right]_0^4$

$= 48/5.$

2. Evaluate $\iint (x^2+y^2)\,dx\,dy$ in the +ve co-ordinate for which $x+y \le 1$

Given curve is $x+y \le 1$

and pt of intersec are $O(0,0), A(1,0), B(0,1)$.



let x indep and x varying from $0 \to 1$

and y dep and y varying from 0 to $1-x$.

$x+y=1$

$y = 1-x$

$$\int_0^1\int_0^{1-x} (x^2+y^2)\,dx\,dy$$

$$\int_0^1\left[\int_0^{1-x} (x^2+y^2)\,dy\right]dx$$

$$\int_0^1\left[x^2y+\frac{y^3}{3}\right]_0^{1-x} dx$$

$$\int_0^1\left[x^2(1-x)+\frac{(1-x)^3}{3}\right]dx$$

$$\cancel{\int_0^1\left[x^2 - x^3\right.}$$

$$\frac{x^3}{3} - x^4/4 + 1/3\left[\frac{(1-x)^4}{-4}\right]\Bigg|_0^1$$

$$\left(\frac{1}{3}-\frac{1}{4}+0\right) - \left(0-0-\frac{1}{12}\right) = 1/6$$

3. Evaluate $\iint_R xy\,dx\,dy$ where R is the region bounded by x-axis of $x=2a$ and curve $x^2=4ay$.

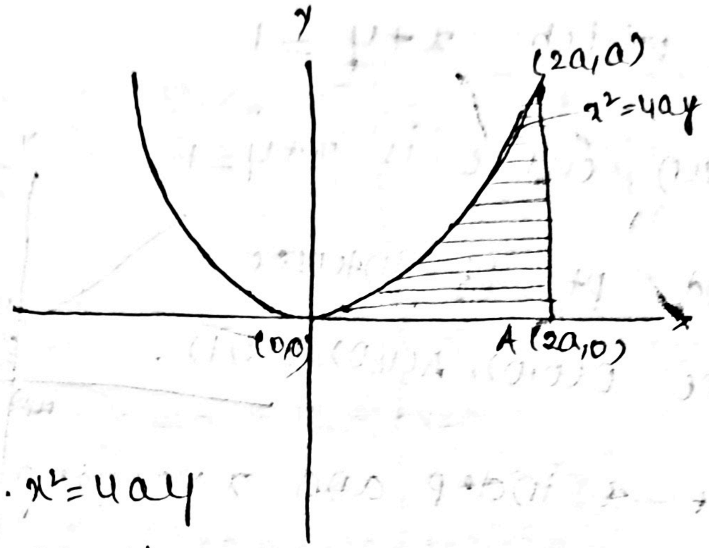
$x^2 = 4ay$

~~x=2a~~

$\Rightarrow 4a^2 = 4ay$

$\Rightarrow y = a$

$\therefore$ pt $(2a, a)$.



Given $x = 2a$ & $x^2 = 4ay$

$\therefore$ pt of intersections are $O(0,0)$, $A(2a,0)$, $B(2a,a)$.

If x is indep then x varying from $0 \to 2a$.
y is dep " y " " $0 \to \frac{x^2}{4a}$

Given $\int_0^{2a}\int_0^{x^2/4a} xy\,dx\,dy$

$$\int_0^{2a}\left[\int_0^{x^2/4a} xy\,dy\right]dx.$$

$$\int_0^{2a}\left[x\frac{y^2}{2}\right]_0^{x^2/4a} dx$$

$$\int_0^{2a}\left[\frac{x}{2}\left(\frac{x^4}{16a^2}\right)\right]dx.$$

$$\frac{1}{32a^2}\int_0^{2a} x^5\,dx$$

$$\frac{1}{32a^2}\left.\frac{x^6}{6}\right|_0^{2a} \Rightarrow \frac{1}{32a^2}\cdot\frac{1}{6}\cdot 64a^6 = \frac{a^4}{3}$$

4. Find $\iint (x+y)^2 \, dx\, dy$ over the area bounded by the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

circle $x^2+y^2=a^2$

$x^2+y^2=a^2$
$y^2=a^2-x^2$
$y=\pm\sqrt{a^2-x^2}$

If x indep it varies from $\to -a \to a$
y dep " " " $\to -\sqrt{a^2-x^2} \to \sqrt{a^2-x^2}$

If only Q_1 is considered.

x indep $0 \to a$
y dep $0 \to \sqrt{a^2-x^2}$

Sol:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

$\frac{x^2}{a^2}=1$ If $y=0$

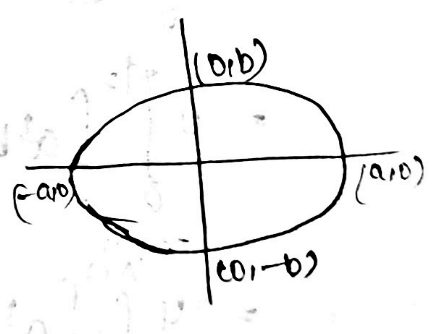
$\Rightarrow x=\pm a.$

$\frac{y^2}{b^2}=1-\frac{x^2}{a^2}$

$y^2=\frac{b^2}{a^2}(a^2-x^2)$

$\Rightarrow y=\pm\frac{b}{a}\sqrt{a^2-x^2}$

Given curve is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$



Pt of intersec are

$(-a,0), (a,0), (0,b), (0,-b).$

Let $x \to$ indep $\longrightarrow -a \to +a$
$y \to$ dep $\longrightarrow -\frac{b}{a}\sqrt{a^2-x^2} \longrightarrow \frac{b}{a}\sqrt{a^2-x^2}.$

$$= \int_{-a}^{a}\int_{-\frac{b}{a}\sqrt{a^2-x^2}}^{\frac{b}{a}\sqrt{a^2-x^2}} (x^2+y^2+2xy)\,dx\,dy.$$

$$= \int_{-a}^{a}\left[\int_{-b/a\sqrt{a^2-x^2}}^{b/a\sqrt{a^2-x^2}} (x^2+y^2+2xy)\,dy\right]dx.$$

$$2\int_{-a}^{a}\left[2\int_{0}^{b/a\sqrt{a^2-x^2}}(x^2+y^2)\,dy\right]dx \qquad \boxed{\because \int xy\,dx\,dy=0}$$

$$4\int_{0}^{a}\left[x^2y+y^3/3\right]_{0}^{b/a\sqrt{a^2-x^2}}dx$$

$$4\int_{0}^{a}\left[x^2\frac{b}{a}\sqrt{a^2-x^2}+\frac{1}{3}\left[\frac{b^3}{a^3}(a^2-x^2)^{3/2}\right]\right]dx.$$

let $x=a\sin\theta$.

$dx=a\cos\theta\,d\theta$.

as $x\to 0 \to \theta\to 0$

as $x\to a \to \theta\to \pi/2$

$$=4\int_{0}^{\pi/2}\left[a^2\sin^2\theta\,\frac{b}{a}\sqrt{a^2-a^2\sin^2\theta}+\frac{b^3}{3a^3}(a^2-a^2\sin^2\theta)^{3/2}\right]a\cos\theta\,d\theta$$

$$=4\int_{0}^{\pi/2}\left(a^2b\sin^2\theta\cos^2\theta\,d\theta+\frac{b^3}{3a^2}\cos\theta(a^2\cos^2\theta)^{3/2}d\theta\right)$$

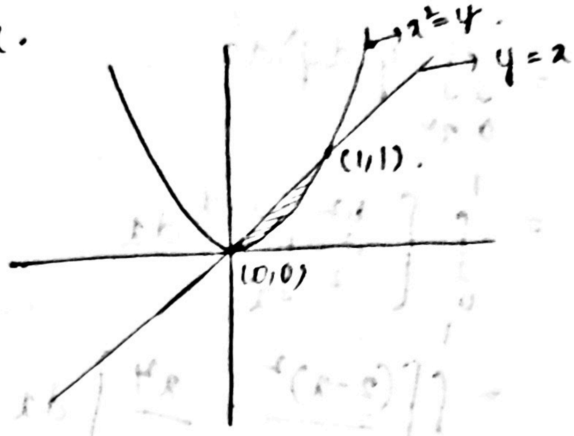
$$=4\int_{0}^{\pi/2}\left[a^3b\sin^2\theta\cos^2\theta\,d\theta+\frac{b^3}{3a^2}(a\cos\theta)^3\cos\theta\,d\theta\right]$$

$$=4\int_{0}^{\pi/2}\left[a^3b\sin^2\theta\cos^2\theta\,d\theta+\frac{ab^3}{3}\cos^4\theta\,d\theta\right].$$

$$=\theta\int_{0}^{\pi/2}a^3b\sin^2 2\theta+\int_{0}^{\pi/2}\frac{ab^3}{12}\cos^4\theta\,d\theta$$

$$=a^3b$$

eg18: Q. Evaluate $\iint xy(x+y)\,dx\,dy$ over the region R bounded by $y=x^2$ and $y=x$.



$y=x^2 \quad (\because y=x$

$x=x^2$

$x(x-1)=0$

$x=0;\ x=1$

If $x=1 \Rightarrow y=1$.

x indep $\to 0 \to 1$

y dep $\to x^2 \to x$

$$\int_0^1\int_{x^2}^{x} (x^2y+xy^2)\,dx\,dy.$$

$$\int_0^1 \left[x^2\frac{y^2}{2}+x\frac{y^3}{3}\right]_{x^2}^{x} dx$$

$$\int_0^1 \left[\frac{x^4}{2}+\frac{x^4}{3}-\frac{x^6}{2}-\frac{x^7}{3}\right] dx$$

$$\frac{x^5}{10}+\frac{x^5}{15}-\frac{x^7}{14}-\frac{x^8}{24}\Big|_0^1$$

$$\frac{1}{10}+\frac{1}{15}-\frac{1}{14}-\frac{1}{24}$$

$$\frac{15+10}{150}-\frac{1}{2}\left(\frac{1}{7}+\frac{1}{12}\right) = \frac{25}{150}-\frac{1}{2}\left(\frac{7+12}{84}\right)$$

$$= \frac{1}{6}-\frac{19}{168}$$

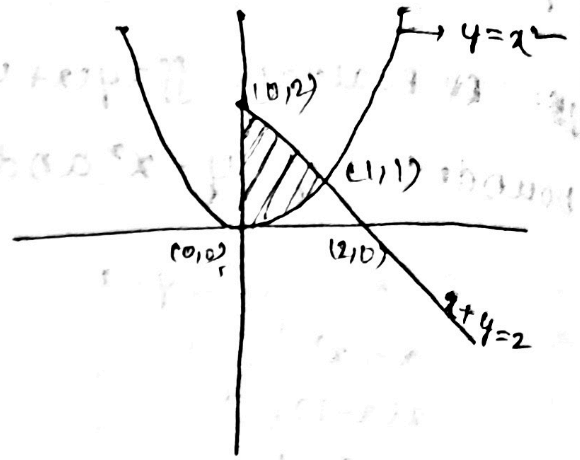
$$= \frac{1}{6}\left(1-\frac{19}{28}\right)$$

$$= \frac{28-19}{168} = \frac{9}{168} = \frac{3}{56}$$

(23) Evaluate $\iint_R y\,dx\,dy$, where R is the domain bounded by y-axis, the curve $y=x^2$ and $x+y=2$ in Q_1

$\rightarrow 0 \rightarrow 1$ — x indep

$x^2 \rightarrow 2-x$ — y dep



$$= \int_0^1 \left(\int_{x^2}^{2-x} y \, dy \right) dx$$

$$= \int_0^1 \left[\frac{y^2}{2} \right]_{x^2}^{2-x} dx$$

$$= \int_0^1 \left[\frac{(2-x)^2}{2} - \frac{x^4}{2} \right] dx$$

$$= \left. \frac{-(2-x)^3}{6} - \frac{x^5}{10} \right|_0^1$$

$$= -\frac{1}{6} - \frac{1}{10} + \frac{4}{3}$$

$$= \frac{-10-6}{60} + \frac{4}{3}$$

$$= \frac{-\cancel{16}^{\cancel{8}\,4}}{\cancel{60}_{\cancel{30}\,15}} + \frac{4}{3}$$

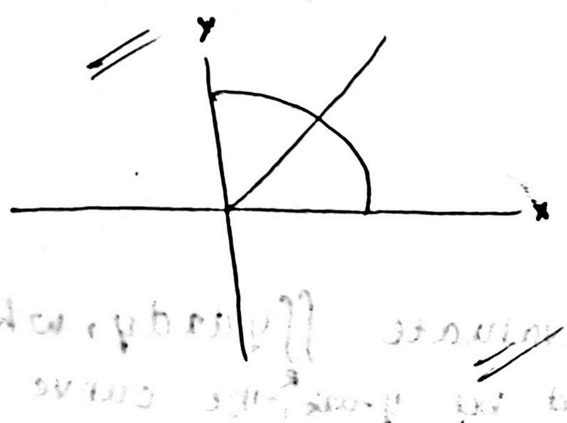
$$\Rightarrow \frac{-4}{3}\left(\frac{1}{5} - 1\right) = \frac{-4}{3}\left(\frac{-4}{5}\right) = \frac{16}{15}$$

Double integral in polar form:

1. Evaluate $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} \, dx \, dy$ (or)

" " $\int_0^\infty \int_0^{\pi/2} e^{-r^2} \, r \, dr \, d\theta$.

Sol: By changing into Polar form

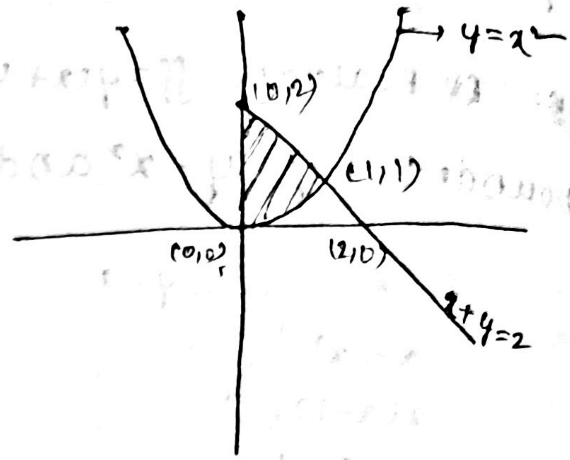


Put $x = r\cos\theta$

$y = r\sin\theta$

$\therefore x^2+y^2 = r^2$

$\to 0 \to 1$ — x indep
$x^2 \to 2-x$ — y dep



$$= \int_0^1 \left(\int_{x^2}^{2-x} y \, dy \right) dx$$

$$= \int_0^1 \left[\frac{y^2}{2} \right]_{x^2}^{2-x} dx$$

$$= \int_0^1 \left[\frac{(2-x)^2}{2} - \frac{x^4}{2} \right] dx$$

$$= \left. -\frac{(2-x)^3}{6} - \frac{x^5}{10} \right|_0^1$$

$$= -\frac{1}{6} - \frac{1}{10} + \frac{4}{3}$$

$$= \frac{-10-6}{60} + \frac{4}{3}$$

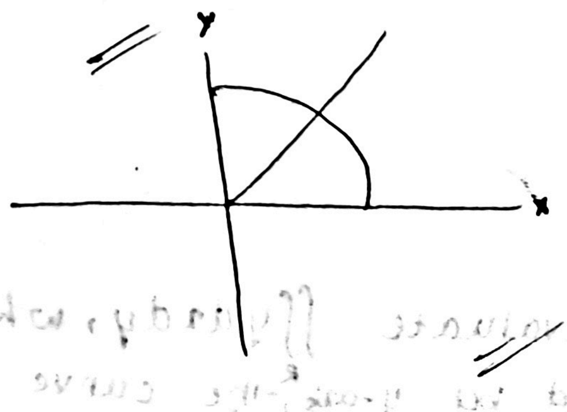
$$= \frac{-16}{60} + \frac{4}{3}$$

$$\Rightarrow \frac{-4}{3}\left(\frac{1}{5} - 1\right) = -\frac{4}{3}\left(\frac{-4}{5}\right) = \frac{16}{15}$$

Double integral in polar form:

1. Evaluate $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} \, dx \, dy$ (or)

" " $\int_0^\infty \int_0^{\pi/2} e^{-r^2} \, r \, dr \, d\theta$.



Sol: By changing into Polar form

Put $x = r\cos\theta$

$y = r\sin\theta$

$\therefore x^2+y^2 = r^2$

and $dxdy = r\,dr\,d\theta$

as $x \to 0 \to \infty$
$y \to 0 \to \infty$

After changing into polar from.

$\theta \to 0 \to \pi/2$
$r \to 0 \to \infty$

$$\therefore \int_0^{\infty}\int_0^{\pi/2} e^{-r^2}\, r\,dr\,d\theta$$

$$(-1/2)\int_0^{\infty} e^{-r^2}(-2r\,dr) \int_0^{\pi/2} d\theta$$

$$-1/2 \int_0^{\infty} d(e^{-r^2})\,dr.\ \theta\Big|_0^{\pi/2}$$

$$-1/2\left(e^{-r^2}\right)\Big|_0^{\infty}\left(\pi/2 - 0\right)$$

$$-\pi/4\left(e^{-\infty} - e^{-0}\right)$$

$$-\pi/4\,(0-1)$$

$$\pi/4$$

2. Evaluate $\displaystyle\int_0^{\pi}\int_0^{a(1+\cos\theta)} r^2\cos\theta\,dr\,d\theta.$

$\theta \to$ indep $\to 0 \to \pi$
$r \to$ dep $\to 0 \to a(1+\cos\theta)$

$$= \int_0^{\pi}\left[\int_0^{a(1+\cos\theta)} r^2\cos\theta\,dr\right] d\theta$$

$$= \int_0^{\pi}\left[\frac{r^3}{3}\cos\theta\right]_0^{a(1+\cos\theta)} d\theta$$

$$= \int_0^{\pi}\frac{\cos\theta}{3}\left(a(1+\cos\theta)\right)^3 d\theta.$$

$$= \frac{a^3}{3}\int_0^{\pi}\left(1+3\cos\theta+3\cos^2\theta+\cos^3\theta\right)\cos\theta\,d\theta.$$

$\because \cos\theta$ is even

$$= \frac{a^3}{3}\left[2\int_0^{\pi/2} \cos\theta + 3\cos^2\theta + 3\cos^3\theta + \cos^4\theta\right].$$

$$= \frac{2a^3}{3}\left[\sin\theta + 3\left(\tfrac{1}{2}\tfrac{\pi}{2}\right) + 3\left(\tfrac{2}{3}\right) + \tfrac{3}{4}\,\tfrac{1}{2}\tfrac{\pi}{2}\right]_0^{\pi/2}$$

$$= \frac{2a^3}{3}\left[(1-0) + \frac{3\pi}{4} + 2 + \frac{3\pi}{16}\right]$$

$$= \frac{2a^3}{3}\left[3 + \frac{3\pi + 12\pi}{16}\right]$$

$$= \frac{2a^3}{3}\left[3 + \frac{15\pi}{16}\right]$$

$$= 2a^3 + \frac{5\pi a^3}{8}$$

Evaluate. $\displaystyle\int_0^{\pi/4}\int_0^{a\sin\theta} \frac{r\,dr\,d\theta}{\sqrt{a^2-r^2}}$

$r \to$ dep

$\theta \to$ indep

$$= \left(-\tfrac{1}{2}\right)\int_0^{\pi/4}\left[\int_0^{a\sin\theta} \frac{-2r}{\sqrt{a^2-r^2}}\,dr\right]d\theta$$

$$= \left(-\tfrac{1}{2}\right)\int_0^{\pi/4}\left[\frac{(a^2-r^2)^{1/2}}{1/2}\right]_0^{a\sin\theta} d\theta$$

$$= -\int_0^{\pi/4}\left[(a^2 - a^2\sin^2\theta)^{1/2} - (a^2-0)^{1/2}\right]d\theta$$

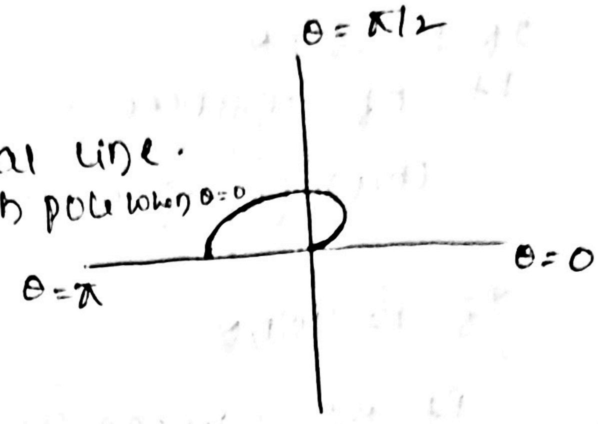
$$= -a\int_0^{\pi/4}\left[\cos\theta - 1\right]d\theta$$

$$= -a\left[\sin\theta - \theta\right]_0^{\pi/4}$$

$$= -a\left[\frac{1}{\sqrt{2}} - \pi/4\right] = a\left(\pi/4 - 1/\sqrt{2}\right)$$

4. Evaluate $\iint r\,dr\,d\theta$ over the cardioid $a(1-\cos\theta)$.

above the Initial line



curve is symm to initial line.

and curve passing through pole when $\theta = 0$

$\theta \to 0 \to \pi$

$r \to 0 \to a(1-\cos\theta)$.

let $\theta \to$ indep $\to 0 \to \pi$

$r \to$ dep $\to 0 \to a(1-\cos\theta)$.

Given $\displaystyle\int_0^{\pi}\int_0^{a(1-\cos\theta)} r\,dr\,d\theta$

$$\int_0^{\pi}\left[\int_0^{a(1-\cos\theta)} r\,dr\right]d\theta$$

$$\int_0^{\pi}\left[\frac{r^2}{2}\right]_0^{a(1-\cos\theta)} d\theta$$

$$\int_0^{\pi}\left[\frac{a^2(1-\cos\theta)^2}{2}\right]d\theta.$$

$$\frac{a^2}{2}\int_0^{\pi}\left[1+\cos^2\theta-2\cos\theta\right]_0^{\pi} d\theta$$

$$\frac{a^2}{2}\left[\theta-2\sin\theta\Big|_0^{\pi}\right]+\frac{a^2}{2}\left(2\int_0^{\pi/2}\cos^2\theta\,d\theta\right)$$

$$= \frac{a^2}{2}\left[\pi\right] + \cancel{2}\cdot\frac{a^2}{2}\left[1/\cancel{2}\cdot\pi/2\right]$$

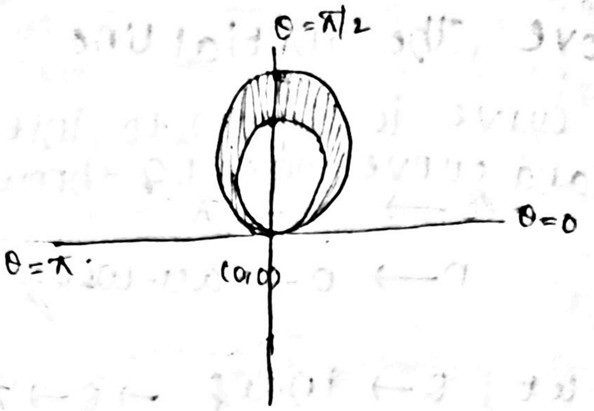
$$= \frac{a^2}{2}\left[\pi+\pi/2\right] = \frac{3\pi a^2}{4}.$$

5. Evaluate $\iint r^3 \, dr \, d\theta$ over the area included by the circles $r = 2\sin\theta$ and $r = 4\sin\theta$.

If $r = 2\sin\theta$

Pt of intersec

$(0,0), (\pi/2, 2), (\pi, 0)$



If $r = 4\sin\theta$

Pt of intersec

$(0,0); (\pi/2, 4), (\pi, 0)$.

$\theta \to 0 \to \pi$.

$r \to 2\sin\theta \to 4\sin\theta$

Given,

$$\int_0^{\pi}\left[\int_{2\sin\theta}^{4\sin\theta} r^3 \, dr\right] d\theta$$

$$\int_0^{\pi} \left.\frac{r^4}{4}\right|_{2\sin\theta}^{4\sin\theta} d\theta.$$

$$\int_0^{\pi} \frac{1}{4}(256\sin^4\theta - 16\sin^4\theta)\, d\theta.$$

$$\frac{1}{\cancel{4}}\int_0^{\pi} \cancel{240}^{\,60} \sin^4\theta \, d\theta$$

$$60\int_0^{\pi} \sin^4\theta \, d\theta.$$

$$60 \cdot 2\int_0^{\pi/2} \sin^4\theta \, d\theta.$$

$$\cancel{120}^{\,\cancel{30}\,15}\left[\frac{3}{\cancel{4}} \cdot \frac{1}{\cancel{2}} \cdot \frac{\pi}{2}\right] = \frac{45\pi}{2}.$$

6. Evaluate $\iint \frac{r\,dr\,d\theta}{\sqrt{a^2+r^2}}$ over 1 loop of the curve
$r^2 = a^2\cos 2\theta$.

Symm to pole and initial line

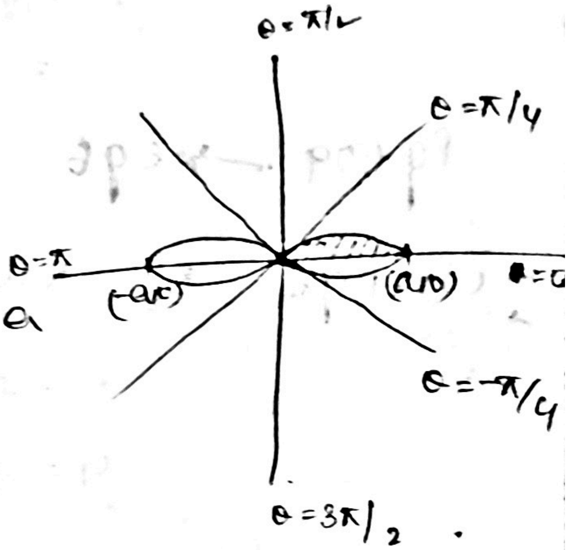
If $r=0$ $\cos 2\theta = 0 \Rightarrow 2\theta = \pm\pi/2$

$\theta = \pm\pi/4$.

Pt $(0,\pi/4),(0,-\pi/4)$

If $\theta=0 \Rightarrow r^2=a^2 \Rightarrow r=\pm a$

$(a,0),(-a,0)$.



$\theta \to -\pi/4 \to \pi/4$ indep

$r \to 0 \to a\sqrt{\cos 2\theta}$. dep.

$$= \frac{1}{2}\int_{-\pi/4}^{\pi/4}\left(\int_0^{a\sqrt{\cos 2\theta}} \frac{2r}{\sqrt{a^2+r^2}}\,dr\right)d\theta$$

$$= 1/2\int_{-\pi/4}^{\pi/4}\left[\frac{(a^2+r^2)^{1/2}}{1/2}\right]\Bigg|_0^{a\sqrt{\cos 2\theta}} d\theta.$$

$$= \int_{-\pi/4}^{\pi/4} (a^2+a^2\cos 2\theta)^{1/2} - a\,d\theta.$$

$$= a\int_{-\pi/4}^{\pi/4} (1+\cos 2\theta)^{1/2} - a\,d\theta.$$

$$= a\int_{-\pi/4}^{\pi/4} (\sqrt{2}\cos\theta - 1)\, d\theta$$

$$= a\left(\sqrt{2}\sin\theta - \theta\right)\Big|_{-\pi/4}^{\pi/4}$$

$$= a\left(2 - \pi/2\right) \text{ cmleans}$$

Pg 509 → eg 6

Change of variables in double integrals:

see Pg 506

$$x = f(u,v)$$
$$y = g(u,v).$$

$$\iint_R f(x,y)\,dx\,dy = \iint_{R'} f\left[f(u,v), g(u,v)\right] |J|\, du\, dv.$$

$$|J| = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}.$$

$r\cos\theta = x$
$r\sin\theta = y$

$$\iint_R f(x,y)\,dx\,dy = \iint_{R'} f(r,\theta)|J|\,dr\,d\theta$$

$$|J| = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & -r\cos\theta \end{vmatrix}.$$

$$= r\cos^2\theta + r\sin^2\theta$$

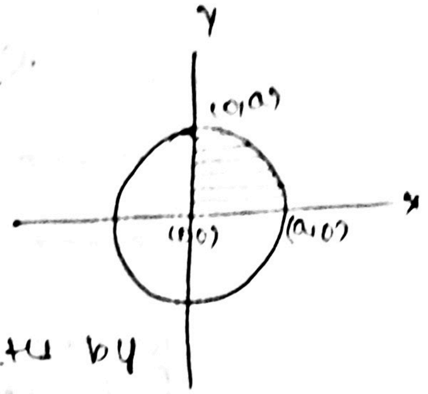
$$= r$$

1. Evaluate the following integral by transforming into polar co-ordinates

$$\int_0^a \int_0^{\sqrt{a^2-x^2}} y\sqrt{x^2+y^2}\,dx\,dy.$$

$x \to$ indep $\to 0 \to a$

$y \to$ dep $\to 0 \to \sqrt{a^2-x^2}$



$$y = \sqrt{a^2-x^2}$$

$$x^2+y^2 = a^2.$$

changing into polar co-ordinates by taking $x = r\cos\theta$; $y = r\sin\theta$

$$dx\,dy = r\,dr\,d\theta.$$

$$\therefore x^2+y^2 = r^2$$

$$r \to 0 \to a$$

$$\theta \to 0 \to \pi/2.$$

$$\int_0^{\pi/2}\int_0^a r\sin\theta\sqrt{r^2}\,r\,dr\,d\theta$$

$$\int_0^a r^3\,dr \int_0^{\pi/2}\sin\theta\,d\theta$$

$$\left.\frac{r^4}{4}\right|_0^a \left(-\cos\theta\right)\Big|_0^{\pi/2}$$

$$\frac{a^4}{4}\left(-\cos\pi/2+\cos 0\right)$$

$$= \frac{a^4}{4}$$

2. Evaluate $\displaystyle\int_0^2\int_0^{\sqrt{2x-x^2}} \frac{x\,dx\,dy}{\sqrt{x^2+y^2}}$ by changing to p.c.

$x \to$ indep. $\to 0 \to 2$

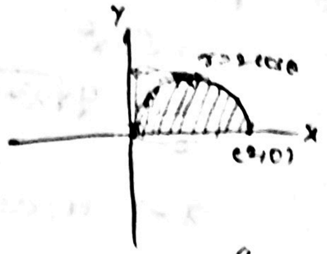
$y \to$ dep $\to 0 \to \sqrt{2x-x^2}.$

$$y = \sqrt{2x-x^2}$$

$$x^2+y^2 = 2x.$$

By changing into Polar co-ordinates

Put $x = r\cos\theta$; $y = r\sin\theta$.



$dx\,dy = r\,dr\,d\theta$.

$x^2+y^2 = r^2$.

$x^2+y^2 = 2x$
$r^2 = 2r\cos\theta$
$r = 2\cos\theta$

$r \to 0 \to 2\cos\theta$.

$\theta \to 0 \to \pi/2$.

$x^2+y^2 = 2x$
$r^{\cancel{2}} = 2\cancel{r}\cos\theta$
$r = 2\cos\theta$

$$\int_0^{\pi/2}\int_0^{2\cos\theta} \frac{r\cos\theta\, r\,dr\,d\theta}{r}$$

$$\int_0^{\pi/2}\left[\int_0^{2\cos\theta} r\cos\theta\,dr\right]d\theta$$

$$\int_0^{\pi/2}\left[\frac{r^2}{2}\cos\theta\right]_0^{2\cos\theta} d\theta$$

$$\int_0^{\pi/2}\left[2\cos^3\theta\right]d\theta.$$

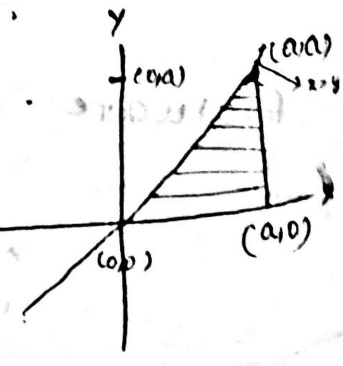
$= 2\cdot 2/3$

$= 4/3$.

3. Evaluate $\displaystyle\int_0^a\int_y^a \frac{x\,dx\,dy}{x^2+y^2}$

$x \to$ dep $\to y \to a$.

$y \to$ indep $\to 0 \to a$.



$\therefore \boxed{x = y}$

$\because r\cos\theta = r\sin\theta$

$\therefore \theta = \pi/4$.

// changing into polar co-ordinates
Put $x = r\cos\theta,\ y = r\sin\theta$
$dx\,dy = r\,dr\,d\theta$ //

// $x = a$
$r\cos\theta = a$
$r = a\sec\theta$ //

$\therefore\ \theta \to 0 \to \pi/4.$

$r \to 0 \to a\sec\theta.$

$$\int_0^{\pi/4}\int_0^{a\sec\theta} \frac{r\cos\theta\, r\,dr\,d\theta}{r^2}$$

$$= \int_0^{\pi/4}\left[\int_0^{a\sec\theta} \cos\theta\, dr\right] d\theta.$$

$$= \int_0^{\pi/4} \cos\theta\ r\Big|_0^{a\sec\theta} d\theta$$

$$= \int_0^{\pi/4} \cancel{\cos\theta}\,(a\cancel{\sec\theta})\, d\theta$$

$$= a\theta\Big|_0^{\pi/4}$$

$$= a\pi/4$$

4. By changing into polar form Evaluate

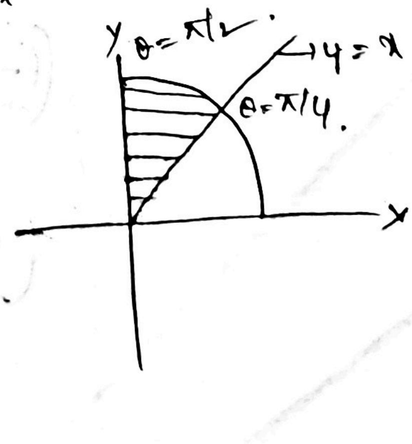
$$\int_0^1\int_x^{\sqrt{2-x^2}} \frac{x\,dx\,dy}{x^2+y^2}$$

Sol: $y \to \text{dep} \to x \to \sqrt{2-x^2}$

$x \to \text{indep} \to 0 \to 1$

$y = \sqrt{2-x^2}$ & $y = x.$

$x^2+y^2 = 2.$



By changing.
Put $x = r\cos\theta;\ y = r\sin\theta.$
$dx\,dy = r\,dr\,d\theta.$

$\theta \to \pi/4 \to \pi/2$

$r \to 0 \to \sqrt{2}\quad (\because r^2 = 2).$

$$\int_{\pi/4}^{\pi/2}\int_0^{\sqrt{2}} \frac{r\cos\theta}{r^2}\, r\,dr\,d\theta = \sin\theta\Big|_{\pi/4}^{\pi/2}\ r\Big|_0^{\sqrt{2}}.$$

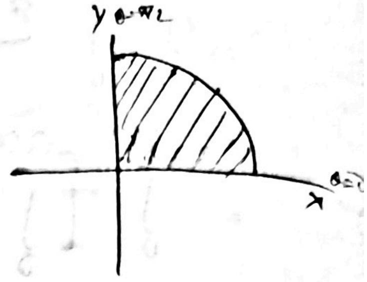
$$= \sqrt{2}-1$$

Evaluate.

5. $$\int_0^a \int_0^{\sqrt{a^2-x^2}} (x^2y+y^2)\,dx\,dy.$$

dep $\to y \to 0 \to \sqrt{a^2-x^2}$

indep $\to x \to 0 \to a$.



$y = \sqrt{a^2-x^2}$

$x^2+y^2=a^2$

By changing

Put $x = r\cos\theta;\ y = r\sin\theta$.

$dx\,dy = r\,dr\,d\theta$.

$\therefore r \to 0 \to a$ indep

$\theta \to 0 \to \pi/2$. indep

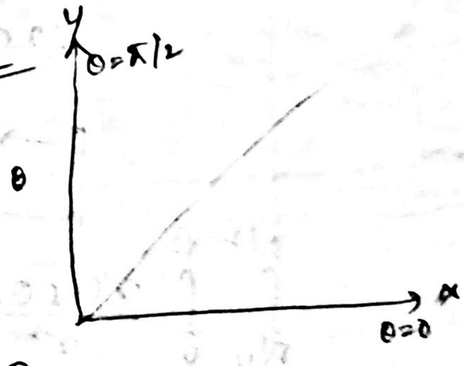
$$= \int_0^{\pi/2}\int_0^a (r^2\cos^2\theta\, r\sin\theta + r^2\sin^2\theta)\, r\,dr\,d\theta.$$

$$= \int_0^{\pi/2} (1-\sin^2\theta)\sin\theta + \int_0^a r^4\,dr + \int_0^a r^3\,dr \int_0^{\pi/2} \sin^2\theta\,d\theta.$$

$$= \left(\left(-\cos\theta\right)\Big|_0^{\pi/2} + \frac{2}{3}\right)\frac{r^5}{5}\Big|_0^a + \frac{r^4}{4}\Big|_0^a \frac{1}{2}\frac{\pi}{2}.$$

$$\left(1-\frac{2}{3}\right)\frac{a^5}{5} + \frac{a^4}{4}\frac{\pi}{4}$$

$$\frac{a^5}{15} + \frac{\pi a^4}{16}$$



6. $$\int_0^\infty\int_0^\infty \frac{dx\,dy}{(x^2+y^2+a^2)^2}$$

$x, y \to$ indep $\to 0 \to \infty$.

// $x = r\cos\theta$; $y = r\sin\theta$

$x^2+y^2 = r^2$ // $\quad r\,dr\,d\theta = dx\,dy$.

$\theta \to 0 \to \pi/2$.

$r \to 0 \to \infty$.

$$\int_0^{\pi/2}\int_0^{\infty} \frac{r\,dr\,d\theta}{(r^2+a^2)^2}$$

$$= \frac{1}{2}\int_0^{\infty} \frac{2r\,dr}{(r^2+a^2)^2} \int_0^{\pi/2} d\theta.$$

$$= \frac{1}{2} \left.\frac{-1}{r^2+a^2}\right|_0^{\infty} \left.\theta\right|_0^{\pi/2}$$

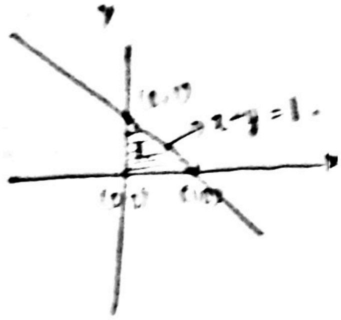
$$= -1/2\left[0 - 1/a^2\right) (\pi/2 - 0)$$

$$= \frac{1}{2a^2}\frac{\pi}{2}$$

$$= \pi/4a^2 //.$$

1. By using the transformations $x+y=u$, $y=uv$

S.T. $\int_0^1\int_0^{1-x} e^{\frac{y}{x+y}}\,dx\,dy = \frac{1}{2}(e-1)$.



Clearly x is indep $\to 0 \to 1$

y is dep $\to 0 \to 1-x$.

$\therefore y = 1-x$

$\Rightarrow x+y=1$ i.e., st. line.

By using Jacobian transforms,

$$\iint_R f(x,y)\,dx\,dy = \iint_{R'} f(u,v)\,|J|\,du\,dv.$$

$$|J| = \frac{\partial(x,y)}{\partial(u,v)}.$$

Given $x+y=u$; $y=uv$.

$\Rightarrow x = u(1-v)$.

$$|J| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$\begin{vmatrix} 1-v & -u \\ v & u \end{vmatrix}$$

$$= u-uv+uv$$

$$= \underline{\underline{u}}$$

Given, $f(x,y) = e^{\frac{y}{x+y}} = e^{\frac{uv}{u}} = e^{v} = f(u,v)$

$x = u(1-v)$; $y = uv$

when $x = 0 \Rightarrow v = 1$; $u = 0$

$y = 0 \Rightarrow v = 0$; $u = 0$

Since $x + y = 1$.

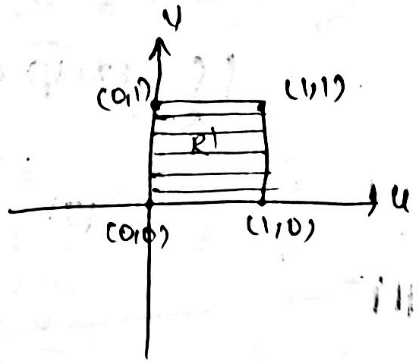
$\therefore \boxed{u = 1}$

$\therefore$ u varying from $0 \to 1$

v " " $0 \to 1$.

Pt of intersec $(0,0), (0,1), (1,0), (1,1)$.

$\therefore \cancel{\iint_R f(x,y)\,dx\,dy}$



$$\therefore \iint_R e^{\frac{y}{x+y}}\,dx\,dy$$

$$= \int_0^1\int_0^1 e^{v}\, u\,du\,dv$$

$$= e^{v}\Big|_0^1 \; \frac{u^2}{2}\Big|_0^1$$

$$= (e-1)\frac{1}{2}(1)$$

$$= \tfrac{1}{2}(e-1)$$

2. Evaluate $\iint_R \left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right) dx\,dy$. over the 1st co-ordinate of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ by using transformations $x = au$; $y = bv$.

Sol: $\iint_R f(x,y)\,dx\,dy = \iint_{R'} F(u,v)\,|J|\,du\,dv$.

Given $x = au$; $y = bv$.

$$|J| = \begin{vmatrix} a & 0 \\ 0 & b \end{vmatrix}$$

$$= ab.$$

$$f(x,y) = 1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}$$

$$= 1 - \frac{a^2u^2}{a^2} - \frac{b^2v^2}{b^2}$$

$$= 1 - u^2 - v^2 = F(u,v).$$

$$\iint_R \left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right) dx\,dy = \iint_{R'} (1 - u^2 - v^2)\, ab\, du\,dv$$

By changing into polar form:

$$u = r\cos\theta;\quad v = r\sin\theta$$

$$du\,dv = r\,dr\,d\theta.$$

$$\theta \to 0 \to \pi/2$$

$$r \to 0 \to 1 \quad (u^2 + v^2 = 1).$$

$$ab \int_0^{\pi/2} \int_0^1 (1 - r^2)\, r\, dr\, d\theta.$$

$$= ab \; \theta \Big|_0^{\pi/2} \left(\frac{r^2}{2} - \frac{r^4}{4}\right)\Big|_0^1$$

$$= ab \frac{\pi}{2} \left(\frac{1}{2} - \frac{1}{4}\right).$$

$$= \frac{\pi ab}{2} \left(\frac{1}{4}\right)$$

$$= \frac{\pi ab}{8}.$$

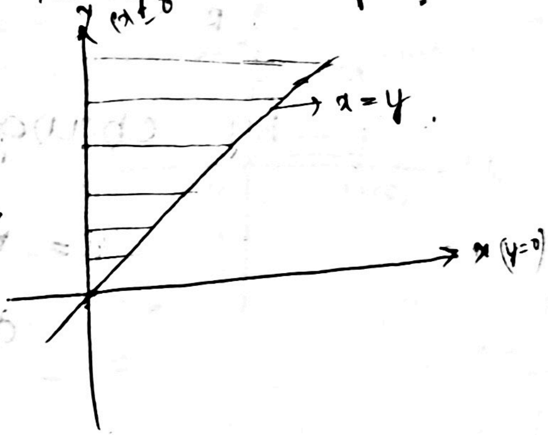
change of order of integration:

1. By change of order of integration,

Evaluate $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y} \, dx \, dy$.

$y \to$ dep $\to x \to \infty$.
$x \to$ indep $\to 0 \to \infty$

By change of order of integration, we have to change x to dep and y to indep.



i.e., $y \to$ indep $\to 0 \to \infty$
$x \to$ dep $\to 0 \to y$.

$$\int_0^\infty \int_0^y \frac{e^{-y}}{y} \, dx \, dy.$$

$$= \int_0^\infty \left[\int_0^y \frac{e^{-y}}{y} \, dx \right] dy$$

$$= \int_0^\infty \left[\int_0^y dx \right] \frac{e^{-y}}{y} \, dy$$

$$= \int_0^\infty x \Big|_0^y \frac{e^{-y}}{y} \, dy$$

$$= \int_0^\infty \cancel{y} \frac{e^{-y}}{\cancel{y}} \, dy$$

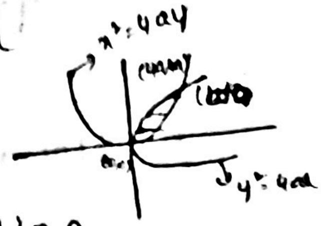
$$= \frac{e^{-y}}{-1} \Big|_0^\infty = -(0-1)$$

$$= 1.$$

2. By change of order of integration Evaluate $\int_0^{4a}\int_{x^2/4a}^{2\sqrt{ax}} dy\,dx.$

$y \to \text{dep} \to x^2/4a \to 2\sqrt{ax}$

$x \to \text{indep} \to 0 \to 4a.$



$y = \frac{x^2}{4a} \Rightarrow x^2 = 4ay \;|\; y = 2\sqrt{ax} \Rightarrow y^2 = 4ax$

By changing order of integration

i.e., $y \to \text{indep} \to 0 \to 4a.$

$x \to \text{dep} \to \frac{y^2}{4a} \to 2\sqrt{ay}.$

Given $\int_0^{4a}\int_{x^2/4a}^{2\sqrt{ax}} dx\,dy$

$$= \int_{y=0}^{4a}\left(\int_{x=y^2/4a}^{2\sqrt{ay}} dx\right) dy$$

$$= \int_0^{4a} (x)\Big|_{y^2/4a}^{2\sqrt{ay}} dy$$

$$= \int_0^{4a}\left(2\sqrt{ay} - \frac{y^2}{4a}\right) dy,$$

$$= 2\sqrt{a}\,\frac{y^{3/2}}{3/2} - \frac{1}{4a}\frac{y^3}{3}\Big|_0^{4a}$$

$$= \frac{4}{3}\sqrt{a}(4a)^{3/2} - \frac{1}{12a}\,64a^3.$$

$$= \frac{4}{3}\times 8a^2 - \frac{a^2 16}{3} = \frac{16a^2}{3}.$$

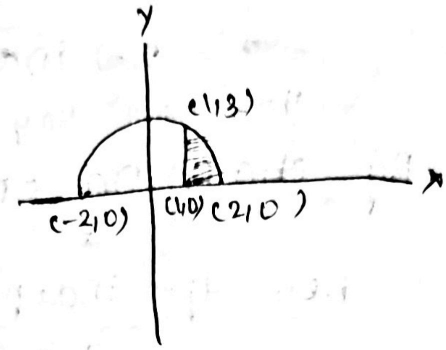
3. By C. of. O. of I. Evaluate $\int_0^3\int_1^{\sqrt{4-y}} (x+y)\,dx\,dy$.

$x \to$ dep $\to 1 \to \sqrt{4-y}$

$y \to$ indep $\to 0 \to 3$.

$x = 1$ and $x = \sqrt{4-y}$

$x^2 + y = 4$



After changing.

$x \to$ indep $\to 1 \to 2$.

$y \to$ dep $\to 0 \to 4-x^2$.

$$\int_1^2\int_0^{4-x^2} (x+y)\,dx\,dy$$

$$= \int_1^2 xy + \frac{y^2}{2}\Big|_0^{4-x^2} dx$$

$$= \int_1^2 x(4-x^2) + \frac{(4-x^2)^2}{2}\,dx$$

expand and do.

$$= \frac{4x^2}{2} - \frac{x^4}{4} + \frac{1}{2}\cdot\frac{(4-x^2)^3}{3(-2x)}\Big|_1^2.$$

$$= 2(3) - \frac{1}{4}(16-1) - \frac{1}{12}\left(-\frac{27}{1}\right).$$

wrong

$$= 6 - \frac{15}{4} + \frac{27}{12}$$

$$= 6 - \frac{15}{4} + \frac{9}{4}$$

$$= \frac{18}{4}$$

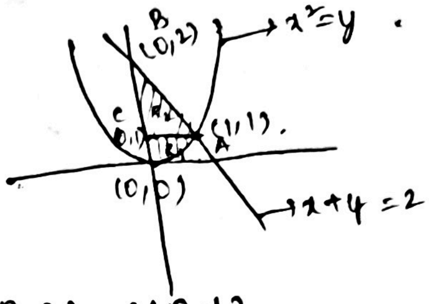
ANS: $\frac{241}{60}$.

*4. By change of o. of i. Evaluate

$$\int_0^1 \int_{x^2}^{2-x} xy\,dx\,dy.$$

Sol: $x \to$ indep $\to 0 \to 1$

$y \to$ dep $\to x^2 \to 2-x$.



from $x^2 = y$ & $x+y=2$.

$x^2+x-2=0$

$x = -2, 1$

Pt of intersec O(0,0) A(1,1), B(0,2), C(0,1).

By changing,

R

~~$y \to$ indep $\to 0 \to 1$.~~

~~$x \to$ dep $\to 0 \to \sqrt{y}$.~~

from the fig Area of OAB = (OACO + CABC) areas.

i) In the area of OACO,

By COOOI,

$y \to$ indep $\to 0 \to 1$.

$x \to$ dep $\to 0 \to \sqrt{y}$-

ii) In the area CABC,

By COOOI, $y \to$ indep $\to 1 \to 2$.

$x \to$ dep $\to 0 \to (2-y)$.

$$\int_0^1 \int_{x^2}^{2-x} xy\,dx\,dy = \int_0^1 \int_0^{\sqrt{y}} xy\,dx\,dy + \int_1^2 \int_0^{2-y} xy\,dx\,dy.$$

$$\rightarrow \int_0^1 \frac{x^2}{2}\Big|_0^{\sqrt{y}} y\,dy + \int_1^2 \frac{x^2}{2}\Big|_0^{2-y} y\,dy.$$

$$= \frac{1}{2}\left[\int_0^1 y^2 dy + \int_1^2 (2-y)^2 y\,dy\right]$$

$$= \frac{1}{2}\frac{y^3}{3}\Big|_0^1 + \left(\frac{4y^2}{2} + \frac{y^4}{4} - 4\frac{y^3}{3}\right)\Big|_1^2$$

$$= \frac{1}{6} + \frac{2(3)}{2} + \frac{1}{4}\frac{(15)}{2} - \frac{4}{3}\frac{(7)}{2}.$$

$$= 36 + \left(\frac{1}{6} + \frac{15}{8} - \frac{28}{6}\right)$$

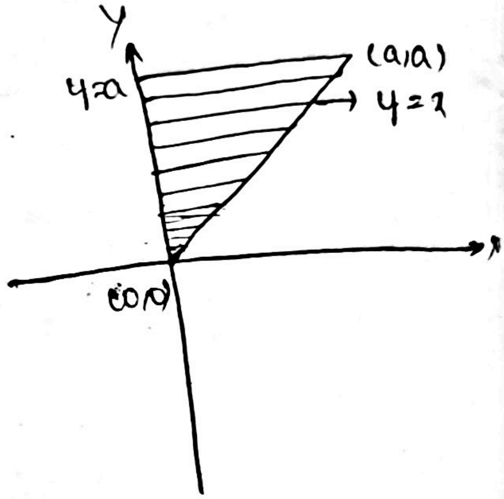
$$= \cancel{4/13}\ 3/8 \text{ (check ans)}.$$

1. By changing into order of integration.

$$\int_0^a \int_x^a (x^2+y^2)\,dx\,dy$$

$x \to$ indep $\to 0 \to a$.

$y \to$ dep $\to x \to a$



By cooo I,

let $y \to$ indep $\to 0 \to a$.

$x \to$ dep $\to 0 \to y$.

Given $\int_0^a \int_x^a (x^2+y^2)\,dx\,dy$

$$= \int_0^a \left(\int_0^y (x^2+y^2)\,dx\right)dy$$

$$= \int_0^a \left.\frac{x^3}{3} + xy^2\right|_0^y dy$$

$$= \int_0^a \frac{y^3}{3} + y^3\,dy$$

$$= \left.\frac{y^4}{3\times 4} + \frac{y^4}{4}\right|_0^a$$

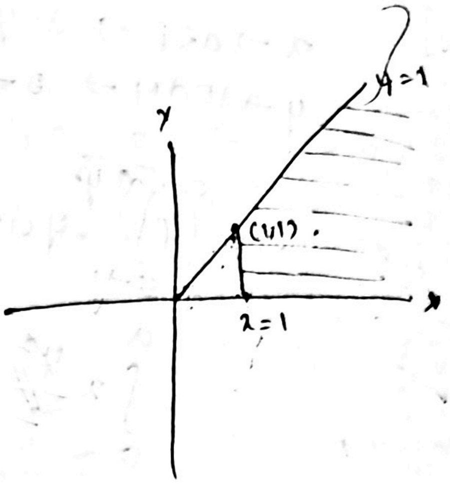
$$= \frac{a^4}{12} + \frac{a^4}{4}$$

$$= \frac{a^4}{3}$$

3. By C.O.O.O.I $\int_0^\infty \int_1^y y e^{-y^2/x} \, dx\, dy.$

$x \to$ dep $\to 1 \to y$

$y \to$ indep $\to 0 \to \infty.$



By C.O.O.O.I

$x \to$ indep $\to 1 \to \infty$

$y \to$ dep $\to 0 \to x.$

$$\int_1^\infty \left(\frac{-x}{2}\right) \left(\int_0^x \left(\frac{-2}{x}\right) y \, e^{-y^2/x} dy \right) dx$$

$$\int_1^\infty \left(\frac{-x}{2}\right) \int_0^x d\left(e^{-y^2/x}\right) dy \, dx$$

$$\int_1^\infty \left(\frac{-x}{2}\right) e^{-y^2/x} \Big|_0^x \, dx$$

$$\int_1^\infty \frac{-x}{2} \left(e^{-x} - 1\right) dx$$

$$= \frac{+1}{2} \int_0^\infty \left(x - x e^{-x}\right) dx = \frac{1}{2}\left[\frac{x^2}{2} - \left(e^{-x}(x+1)\right)\right]\Big|_1^\infty$$

$$= 1/2\,(0 - e^{-1}(2)).$$

$$= 1/e$$

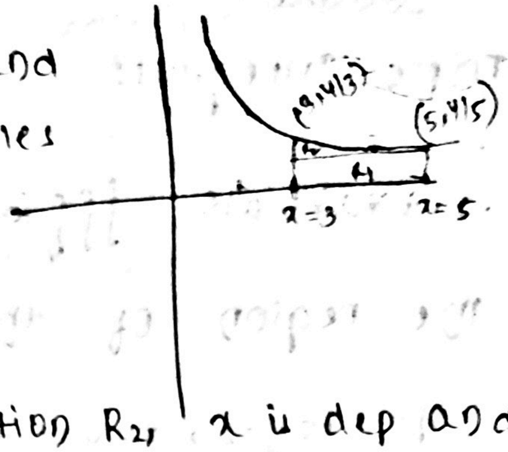
4. Evaluate $\int_3^5 \int_0^{4/x} xy \, dx\, dy$ by C.O.O.O.I.

$x \to$ indep $\to 3 \to 5$

$y \to$ dep $\to 0 \to 4/x.$

By C.O.O.O.I.

In the region R_1, x and y both are indep variables



and $y \to 0 \to 4/5$.

$x \to 3 \to 5$

In the portion R_2, x is dep and y is indep.

$$x \to 3 \to 4/y$$

$$y \to 4/5 \to 4/3.$$

$$R = R_1 + R_2$$

$$\therefore \int_3^5 \int_0^{4/x} xy\,dx\,dy = \int_3^5 \int_0^{4/5} xy\,dy\,dx + \int_{4/5}^{4/3} \int_3^{4/y} xy\,dy\,dx.$$

$$= \frac{x^2}{2}\Big|_3^5 \; \frac{y^2}{2}\Big|_0^{4/5} + \int_{4/5}^{4/3} \frac{x^2}{2}\Big|_3^{4/y} . y\,dy$$

$$= \left(\frac{25}{2} - \frac{9}{2}\right)\left(\frac{16}{50}\right) + \frac{1}{2}\int_{4/5}^{4/3} y\left(\frac{16}{y^2} - 9\right) dy$$

$$= 8\left(\frac{8}{25}\right) + \frac{1}{2}\left(16\log y - \frac{9y^2}{2}\right)\Big|_{4/5}^{4/3}.$$

$$= \frac{64}{25} + \frac{1}{2}\left(16\left(\log\left(\frac{4/3}{4/5}\right)\right) - \frac{9}{2}\left(\frac{16}{9} - \frac{16}{25}\right)\right)$$

$$= \frac{64}{25} + 8\log\frac{5}{3} - \frac{9 \times 16}{4}\left(\frac{16}{9 \times 25}\right)$$

$$= \frac{64}{25} + 8\log\frac{5}{3} - \cancel{36}^{\,4}\left(\frac{16}{\cancel{225}_{\,25}}\right).$$

$$= \frac{64}{25} + 8\log\frac{5}{3} - \frac{64}{25}.$$

$$= 8\log 5/3.$$

Triple Integrals:

1. Evaluate $\iiint_V (xy+yz+zx)\,dx\,dy\,dz$. where V is the region of the space bounded by $x=0$ to $x=1$, $y=0$ to $y=2$, $z=0$ to $z=3$.

Sol:

$$\int_0^1\int_0^2\int_0^3 (xy\,dx\,dy\,dz + yz\,dx\,dy\,dz + zx\,dx\,dy\,dz)$$

$$= \frac{x^2}{2}\Big|_0^1 \frac{y^2}{2}\Big|_0^2 z\Big|_0^3 + x\Big|_0^1 \frac{y^2}{2}\Big|_0^2 \frac{z^2}{2}\Big|_0^3 + \frac{x^2}{2}\Big|_0^1 y\Big|_0^2 \frac{z^2}{2}\Big|_0^3$$

$$= \frac{1}{2}\frac{4}{2}3 + 1\frac{4}{2}\frac{9}{2} + \frac{1}{2}2\frac{9}{2}$$

$$= 3 + 9 + \frac{9}{2}$$

$$= 12+\frac{9}{2} = \frac{24+9}{2} = \frac{33}{2}.$$

2. Evaluate. $\int_0^{\log 2}\int_0^{x}\int_0^{x+\log y} e^{x+y+z}\,dx\,dy\,dz.$

$$= \int_0^{\log 2}\int_0^{x}\left[\int_0^{x+\log y} e^{z}\,dz\right] e^{x+y}\,dx\,dy$$

$$= \int_0^{\log 2}\int_0^{x} e^{z}\Big|_0^{x+\log y} e^{x+y}\,dx\,dy.$$

$$= \int_0^{\log 2}\int_0^{x} \left(e^{x+\log y}-1\right)e^{x+y}\,dx\,dy$$

$$= \int_0^{\log 2}\left[\int_0^{x} e^{2x}\cdot e^{y}\,y - e^{x}e^{y}\,dy\right]dx.$$

$$= \int_0^{\log 2} \left[e^{2x} (ye^y - e^y)\Big|_0^x - e^x e^y \Big|_0^x \right] dx$$

$$= \int_0^{\log 2} \left[e^{2x}(xe^x - e^x + 1) - e^x(e^x - 1) \right] dx .$$

// $\int e^y y \to$ $\cancel{u = e^y ; \; v = y}$ $\qquad uv_1 - \int u' v_1$

$\cancel{u = e^y ; \; v_1 = \frac{y^2}{2}}$

$u = y ; \; v = e^y$

$u' = 1 ; \; v_1 = e^y$

$$\int uv = ye^y - \int e^y$$

$$= ye^y - e^y //$$

$$= \int_0^{\log 2} \left[xe^{3x} - e^{3x} + \cancel{e^{2x}} - \cancel{e^{2x}} + e^x \right] dx -$$

$$= \int_0^{\log 2} xe^{3x} - \frac{e^{3x}}{3}\Big|_0^{\log 2} + e^x \Big|_0^{\log 2}$$

$$= \frac{xe^{3x} - e^{3x}}{3}\Bigg|_0^{\log 2} - \frac{1}{3}\left[e^{3\log 2} - 1 \right]$$

$$+ (e^{\log 2} - 1)$$

// $\int x e^{3x}$

$u = x ; \; v = e^{3x}$

$u' = 1 ; \; v_1 = \frac{e^{3x}}{3}$

$\frac{xe^{3x}}{3} - \frac{e^{3x}}{3} .$

$$= \frac{1}{3}\left[\log 2 \, e^{3\log 2} - e^{3\log 2} + 1 \right] - \frac{1}{3}(7) + (1) .$$

$$= \frac{1}{3}\left[8\log 2 - 8 + 1 \right] - 7/3 + 1 .$$

$\frac{-14}{3} + 1$

$\frac{-14 + 3}{3}$

$$= 8/3 \log 2 - 7/3 - 7/3 + 1 .$$

$$= 8/3 \log 2 - 11/3 . \; \text{(check)}$$

3. $\int_0^1\int_0^{1-x}\int_0^{1-x-y} e^x \, dz\,dy\,dx$.

$= \int_0^1\int_0^{1-x} e^x \left[z \right]_0^{1-x-y} dx\,dy$

$= \int_0^1\int_0^{1-x} e^x (1-x-y)\, dy\, dx$

$= \int_0^1 \left[e^x \left[\int_0^{1-x} ((1-x)-y)\, dy \right] dx \right.$

$= \int_0^1 \left[e^x \left[(1-x)y - \frac{y^2}{2} \right] \Big|_0^{1-x} \right] dx$

$= \left. \int_0^1 e^x \left((1-x)^2 - \frac{(1-x)^2}{2} \right) dx \right]$.

$= \int_0^1 \frac{(1-x)^2 e^x}{2} dx$

$u = (1-x^2)$; $u' = -2(1-x)$; $u'' = 2$

$v = e^x$; $v_1 = \frac{e^x}{1}$; $v_2 = \frac{e^x}{1}$

$\frac{1}{2} \; (1-x^2)e^x + 2(1-x)e^x + 2e^x \Big|_0^1$

$= \frac{1}{2}\left[2e - 1 - 2 - 2) \right]$

$= e - 5/2$

(4) Evaluate $\int_0^{\pi/2}\int_0^{a\sin\theta}\int_0^{\frac{a^2-r^2}{2}} r \; dz\,dr\,d\theta$

$$= \int_0^{\pi/2}\int_0^{a\sin\theta} (z)\Big|_0^{\frac{a^2-r^2}{2}} r\,dr\,d\theta.$$

$$= \int_0^{\pi/2}\int_0^{a\sin\theta} \left(\frac{a^2-r^2}{2}\right) r\,dr\,d\theta.$$

$$= \frac{-1}{4}\int_0^{\pi/2}\left(\int_0^{a\sin\theta} (a^2-r^2)(-2r)\,dr\right)d\theta.$$

$$= \frac{-1}{4}\int_0^{\pi/2}\left[\frac{(a^2-r^2)^2}{2}\Big|_0^{a\sin\theta}\right]d\theta$$

$$= \frac{-1}{8}\int_0^{\pi/2} (a^2-a^2\sin^2\theta)^2 - a^4\,d\theta$$

$$= \frac{-a^4}{8}\int_0^{\pi/2} (\cos^4\theta - 1)\,d\theta.$$

$$= \frac{-a^4}{8}\left(\frac{3}{4}\cdot\frac{1}{2}\cdot\pi/2 - \theta\Big|_0^{\pi/2}\right)$$

$$= \frac{-a^4}{8}\left[\frac{3\pi}{16} - \pi/2\right].$$

$$= \frac{-a^4}{16}\left[\frac{3\pi}{8} - \pi\right]$$

$$= \frac{-a^4}{16}\left[\frac{-5\pi}{8}\right].$$

$$= \frac{5a^4\pi}{128}$$

Note: change of variables from Cartesian to Sperical Coordinates.

$$\int f(x,y,z)\,dx\,dy\,dz = \iiint_{R'} f(r,\theta,\phi)\,|J|\,dr\,d\theta\,d\phi.$$

$$|J| = \frac{\partial(x,y,z)}{\partial(r,\theta,\phi)}.$$

where $x = r\sin\theta\cos\phi$

$y = r\sin\theta\sin\phi$

$z = r\cos\theta$ then $dx\,dy\,dz = |J|\,dr\,d\theta\,d\phi$

$= r^2\sin\theta\,dr\,d\theta\,d\phi.$

If 4 co-ordinates of sphere are considered then

$r \to 0$ to a.

$\theta \to 0$ to π

$\phi \to 0$ to 2π

If 1st co-ordinate is considered.

$r \to 0 \to a$

$\theta \to 0 \to \pi/2$

$\phi \to 0 \to \pi/2$

5. Evaluate $\displaystyle\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}} \frac{1}{\sqrt{1-x^2-y^2-z^2}}\,dx\,dy\,dz.$

$z \to$ dep on x & y : $0 \to \sqrt{1-x^2-y^2}$

$y \to$ " " x : $0 \to \sqrt{1-x^2}$

$x \to$ indep : $0 \to 1$

$\therefore z = \sqrt{1-x^2-y^2}$

$x^2+y^2+z^2 = 1.$

By changing into Spherical Co-ordinates,

$x = r\sin\theta\cos\phi$

$y = r\sin\theta\sin\phi$

$z = r\cos\theta.$

and $dx\,dy\,dz = r^2\sin\theta\,dr\,d\theta\,d\phi.$

$r \to 0 \to a;\ \theta \to 0 \to \pi/2;\ \phi \to 0 \to \pi/2.$

$$\int_0^1\int_0^{\pi/2}\int_0^{\pi/2} \frac{r^2\sin\theta\, dr\, d\theta\, d\phi}{\sqrt{1-r^2}}$$

$$= \int_0^1 \frac{r^2}{\sqrt{1-r^2}}\,dr \int_0^{\pi/2} \sin\theta\, d\theta \int_0^{\pi/2} d\phi$$

Let $r = \sin t$ $\quad r \to 0 \to t \to 0$

$dr = \cos t\, dt$ $\quad r \to 1 \to t \to \pi/2.$

$$= \int_0^{\pi/2} \frac{\sin^2 t}{\cancel{\sqrt{1-\sin^2 t}}}\,\cancel{\cos t}\,dt \quad (-\cos\theta)\Big|_0^{\pi/2} (\phi)\Big|_0^{\pi/2}$$

$$= \tfrac{1}{2}\tfrac{\pi}{2}\ (1)\ (\tfrac{\pi}{2})$$

$$= \frac{\pi^2}{8}$$

6. Evaluate $\iiint xy^2z\, dx\,dy\,dz$. taken through the +ve octant of the sphere $x^2+y^2+z^2=a^2$

Sol:
$$\int_0^a\int_0^{\pi/2}\int_0^{\pi/2} r\sin\theta\cos\phi \cdot r^2\sin^2\theta\sin^2\phi\, r\cos\theta\ r^2\sin\theta\, dr\,d\theta\,d\phi$$

$$= \int_0^a r^6\,dr \int_0^{\pi/2} \sin^4\theta\cos\theta\, d\theta \int_0^{\pi/2} \sin^2\phi\cos\phi\, d\phi.$$

$$= \frac{r^7}{7}\Big|_0^a \quad \frac{\sin^5\theta}{5}\Big|_0^{\pi/2} \quad \frac{\sin^3\phi}{3}\Big|_0^{\pi/2}$$

$$= \frac{a^7}{7}\ \frac{1}{5}(1)\ \frac{1}{3}(1)$$

35
35

$$= \frac{a^7}{105}$$

7. Evaluate $\iiint \frac{xyz\, dx\,dy\,dz}{\sqrt{x^2+y^2+z^2}}$ taken throughout the volume of the sphere $x^2+y^2+z^2=a^2$.

$$= \int_0^a \int_0^\pi \int_0^{2\pi} \frac{r \sin\theta \cos\phi \; r \sin\theta \sin\phi \; r\cos\theta \; r^2 \sin\theta \, dr\, d\theta\, d\phi}{r}$$

$$= \int_0^a \int_0^\pi \int_0^{2\pi} r^4 \sin^3\theta \cos\theta \; \cos\phi \sin\phi \; dr\, d\theta\, d\phi$$

$$= \frac{r^5}{5}\Big|_0^a \; \frac{\sin^4\theta}{4}\Big|_0^\pi \cdot \frac{\sin^2\phi}{2}\Big|_0^{2\pi}$$

$$= \frac{a^5}{5} \; \frac{1}{4}(0) \; \frac{1}{2}(0)$$

$$= 0$$

8. Evaluate $\iiint_V \frac{dx\,dy\,dz}{(x+y+z+1)^3}$ taken over the Volume bounded by the plane $x=0, y=0, z=0$ & $x+y+z=1$

$x+y+z=1$

$z = 1-x-y$.

$z: \; 0 \longrightarrow 1-x-y$.

$y: \; 0 \longrightarrow 1-x \quad (\because x+y=1 \; z=0)$

$x: \; 0 \longrightarrow 1 \quad x=1 \; (\because y=0, z=0)$.

$$= \int_0^1 \int_0^{1-x} \left[\int_0^{1-x-y} \frac{dz}{(1+x+y+z)^3} \right] dx\, dy.$$

$$= \int_0^1 \int_0^{1-x} \frac{(1+x+y+z)^{-2}}{-2} \Bigg|_0^{1-x-y} dx\, dy$$

$$= -\frac{1}{2} \int_0^1 \int_0^{1-x} \left[(1+x+y+1-x-y)^{-2} - (1+x+y)^{-2} \right] dx\,dy$$

$$= -1/2 \int_0^1 \int_0^{1-x} \left[\frac{1}{4} - \frac{1}{(1+x+y)^2} \right] dy\, dx$$

$$= -1/2 \left[\int_0^1 \cancel{\int_0^{1-x}} \frac{1}{4} y - \frac{(1+x+y)^{-1}}{-1} \right]_0^{1-x} dx$$

$$= \frac{-1}{2} \left[\int_0^1 \left[\frac{1}{4}(1-x) + (1+\cancel{x}+1-\cancel{x})^{-1} - (1+x)^{-1} \right] dx \right.$$

$$= -1/2 \int_0^1 \left[\frac{1-x}{4} + \frac{1}{2} - \frac{1}{1+x} \right] dx$$

$$= -\frac{1}{2} \left[\frac{1}{4}\left(x - \frac{x^2}{2}\right) + \frac{1}{2} x \right] \Big|_0^1 + \frac{1}{2} \int_0^1 \frac{1}{1+x}$$

$$= \frac{-1}{8}\left(1 - \frac{1}{2}\right) - \frac{1}{4}(1) + \frac{1}{2} \log(1+x) \Big|_0^1$$

$$= \frac{-1}{16} - \frac{1}{4} + \frac{1}{2} \log 2$$

$$= 1/2 \log 2 - 5/16.$$