Transcient Analysis Of Second Order Circuits

- Second order circuits consist of two energy storage elements
- These are known as second-order circuits because their responses are described by differential equations that contain second order derivatives.
- Examples of second order circuits are RLC circuits, in which three kinds of passive elements are present.

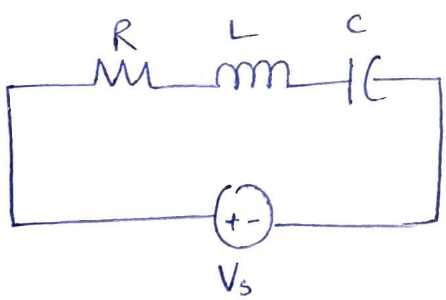


fig. series RLC circuit

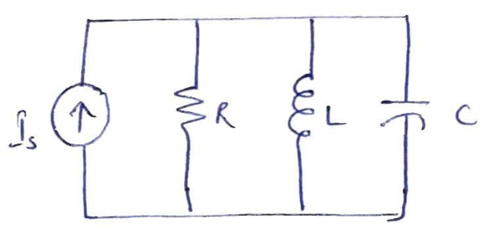
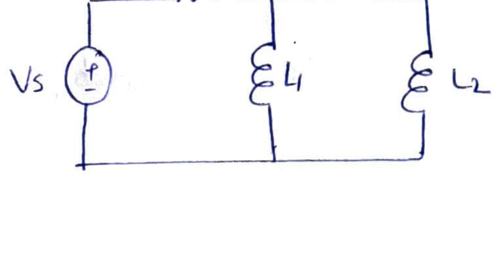
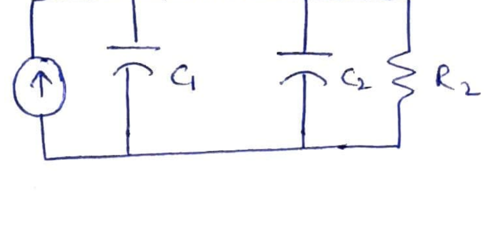


fig. Parallel RLC circuit

- Other examples are RC and RL circuits, ~~the~~ as shown.





*Note:

Second order circuits may have two storage elements of different type (or) the same type (ie RL (or) RC), provided elements of the same type cannot be represented by an equivalent single element. *

" A second-order circuit is characterized by a second order differential equation. It consists of resistors and the equivalent two energy storage elements. "

- Analysis of second order circuits is similar to that of first order circuits.

- In second order ~~order~~ circuits, also, firstly, the circuit analysis is done when it is excited by its initial conditions

- This is the natural response of RLC circuits (or) second order circuits.

- Later, forced response of second order circuits is done by ~~excited~~ exciting the circuits with independent sources.

This will be giving natural and forced response of second order circuits.

Initial & final values of second order circuits:

$$i_L(0^-) = i_L(0^+)$$

$$v_c(0^-) = v_c(0^+)$$

{ In finding initial conditions, we first focus on those variables that cannot change abruptly, i.e., capacitor voltage & inductor current, by applying the above two equations. }

- Polarity of voltage $v(t)$ across the capacitor & the direction of the current $i(t)$ through the inductor must be carefully handled

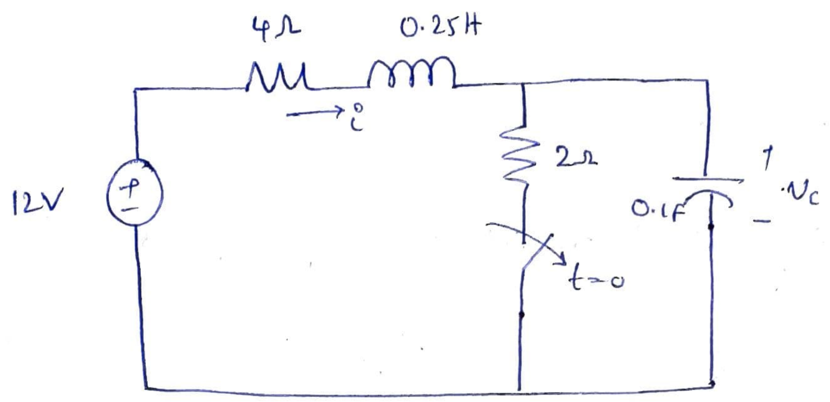
- Here 0^- ⇒ the instant just before switching

  0^+ ⇒ the instant just after switching

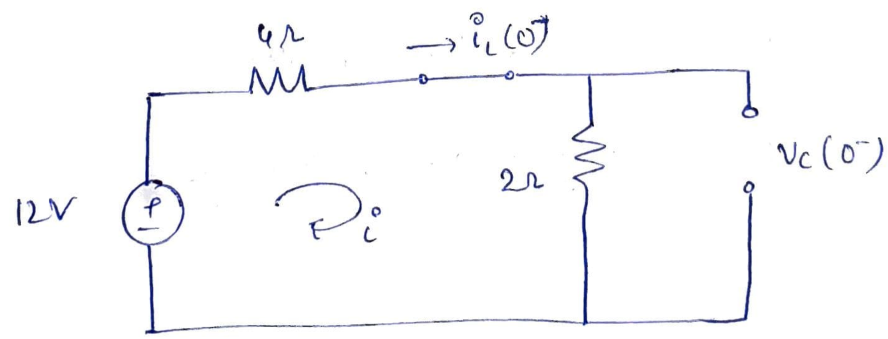
Q1) The switch in the below circuit has been closed for a long time. It is opened at $t=0$. find, (a) $i_L(0^+)$, $v_C(0^+)$

(b) $\frac{di_L(0^+)}{dt}$, $\frac{dv_C(0^+)}{dt}$

(c) $i_L(\infty)$, $v_C(\infty)$



Sol) (a) Since the switch has been closed for long time, the inductor and capacitor reaches their dc steady state i.e., inductor becomes shorted, & capacitor becomes open circuited. So, at $t=0^-$, the circuit will be as shown.



$i = i_L(0^-)$

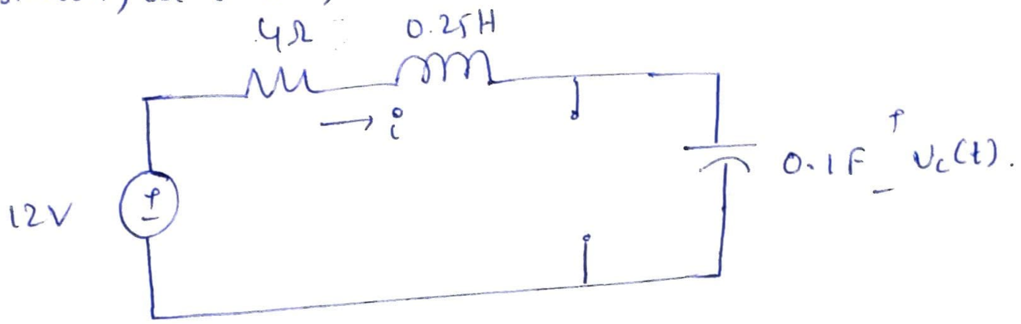
Apply KVL in the above circuit,

$12 = 4i + 2i \Rightarrow 6i = 12 \Rightarrow \boxed{i = i_L(0^-) = 2A}$

$$v_c(0^-) = 2(i) = 2(2) = 4\,V$$

$$\Rightarrow \boxed{V_c(0^-) = 4\,V}$$

b) At $t=0$, the switch is opened ~~closed~~, hence the circuit will be as shown, at $t=0^+$,



- The current flowing through inductor after the instant of switching, $i_L(0^+) = i_L(0^-) = 2A$

Here in this circuit, since inductor and capacitor both are in series, $i_L(0^+) = i_c(0^+) = 2A$.

- The voltage across capacitor just after instant of switching,

$$V_c(0^+) = v_c(0^-) = 4V.$$

Note: The current through inductor cannot be changed abruptly, $\Rightarrow i_L(0^-) = i_L(0^+)$

The voltage across capacitor cannot be changed abruptly, $\Rightarrow V_c(0^-) = V_c(0^+)$

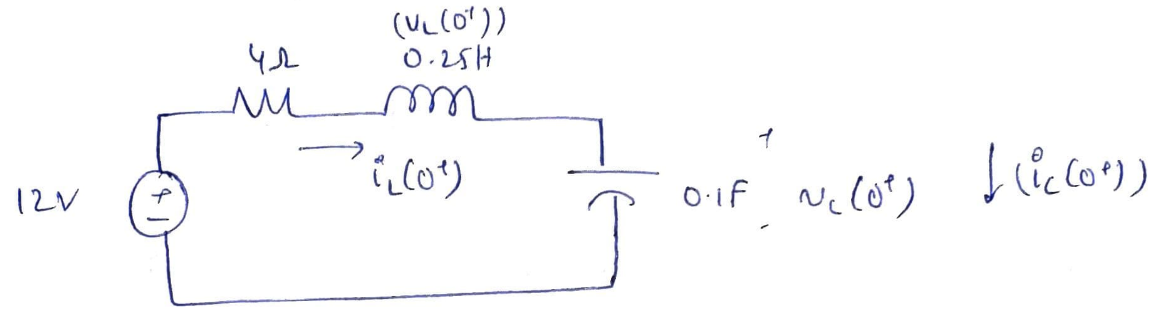
Now, we need to calculate $\frac{di_L(0^+)}{dt}$, $\frac{dv_C(0^+)}{dt}$.

from,

$$v_L = L\frac{di_L}{dt}$$

$$\Rightarrow v_L(0^+) = L\frac{di_L(0^+)}{dt}$$

from the circuit, at $t = 0^+$



$$v_L(0^+) = 12 - 4(i_L(0^+)) - v_c(0^+)$$

$$\Rightarrow v_L(0^+) = 12 - 4(2) - 4$$

$$= 0\ V$$

Now,

From, $v_L(0^+) = L\frac{di_L(0^+)}{dt}$

$$\Rightarrow \frac{di_L(0^+)}{dt} = \frac{v_L(0^+)}{L} = \frac{0}{0.25}$$

$$\Rightarrow \boxed{\frac{di_L(0^+)}{dt} = 0\ A/sec}$$

from,

$$i_c = C\frac{dv_c}{dt}$$

$$\Rightarrow i_c(0^+) = C\frac{dv_c(0^+)}{dt}$$

$$\Rightarrow \frac{dv_c(0^+)}{dt} = \frac{i_c(0^+)}{C}$$

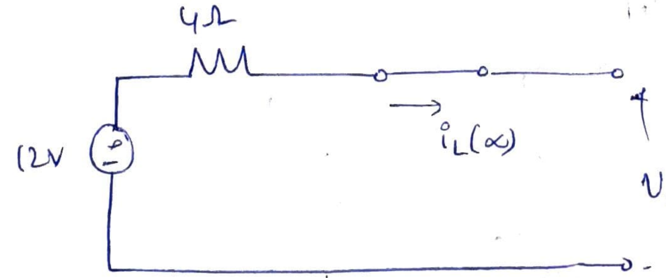
from the previous circuit, $i_c(0^+) = i_L(0^+) = 2A$

$$\Rightarrow \frac{dv_c(0^+)}{dt} = \frac{2}{0.1} = 20 \text{ V/s}$$

$$\Rightarrow \boxed{\frac{dv_c(0^+)}{dt} = 20 \text{ V/sec}}$$

c) As $t \to \infty$, the circuit reaches its steady state value

That means, inductor is replaced by short circuit & capacitor is replaced by open circuit. Hence, as $t \to \infty$, the circuit will be as shown,



Since the circuit is open, $\boxed{i_L(\infty) = 0A}$

& $\boxed{V_c(\infty) = 12V}$.

Simple process to solve numericals on initial conditions of II order circuits:

i, Draw the circuit for $t=0^-$

(Keep in mind the steady state of inductor and capacitor)

find $i_L(0^-)$ and $v_C(0^-)$

ii. At $t=0$, switching is done, draw the circuit.

iii, At $t=0^+$, draw the circuit.

To find $\frac{di_L(0^+)}{dt}$:

from $V_L = L\frac{di_L}{dt}$,

we have $V_L(0^+) = L\frac{di_L(0^+)}{dt}$

$$\Rightarrow \boxed{\frac{di_L(0^+)}{dt} = \frac{V_L(0^+)}{L}} \text{ A/sec}$$

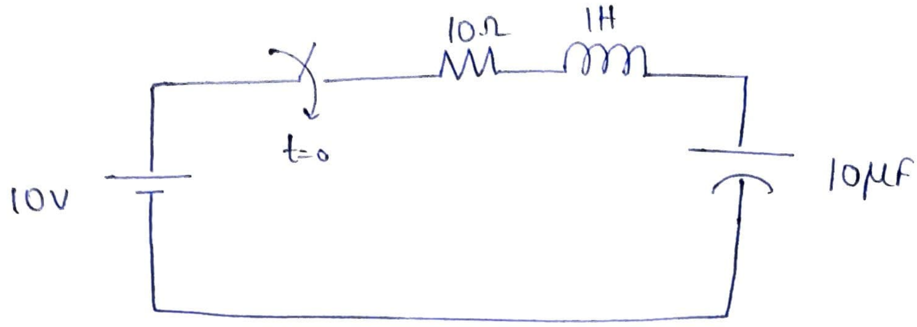
To find $\frac{dv_C(0^+)}{dt}$:

from $i_C(t) = C\frac{dV_C}{dt}$

we have $i_C(0^+) = C\frac{dv_C(0^+)}{dt}$

$$\Rightarrow \boxed{\frac{dv_C(0^+)}{dt} = \frac{i_C(0^+)}{C}} \text{ V/sec.}$$

Q1) Find $i(0^+)$, $\frac{di(0^+)}{dt}$, and $\frac{d^2i(0^+)}{dt^2}$ from the circuit, if the switch is closed at $t=0$.



Sol) - At $t=0^-$, the switch is kept opened, so, current flowing through the loop is 0A

∴ i_L - current flowing through inductor at $t=0^-$

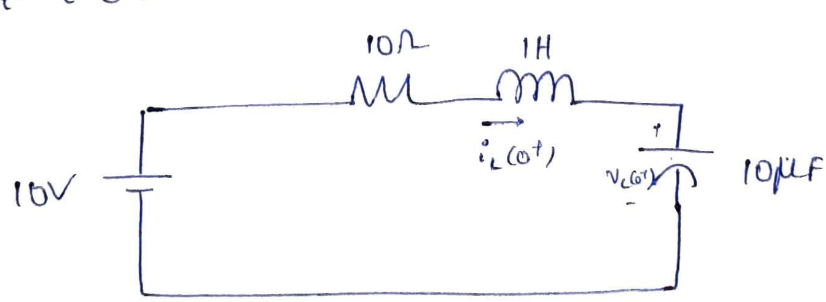
ie $i_L(0^-) = 0A$.

v_C - voltage across capacitor at $t=0^-$

ie $v_C(0^-) = 0V$, since there will not be any voltage drop in the capacitor.

- At $t=0$, switch is closed.

- At $t=0^+$, the circuit will be as shown.



We know that,

$$i_L(0^+) = i_L(0^-)$$

and $v_c(0^+) = v_c(0^-)$

$\therefore$ $i_L(0^+) = i_L(0^-) = 0\,A$.

Now, from $v_L(0^+) = L\dfrac{di_L(0^+)}{dt}$

$$\Rightarrow \frac{di_L(0^+)}{dt} = \frac{v_L(0^+)}{L}$$

Apply KVL in the circuit,

$$10 = v_R(0^+) + v_L(0^+) + v_C(0^+)$$

$$= i(0^+)R + v_L(0^+) + v_C(0^+)$$

$$= 0(10) + v_L(0^+) + 0$$

$$\Rightarrow v_L(0^+) = 10V$$

$$\therefore \frac{di_L(0^+)}{dt} = \frac{v_L(0^+)}{L} = \frac{10}{1} = 10\ A/sec$$

- Apply KVL in the circuit,

$$V = V_R + V_L + V_C$$

$$v = Ri + L\frac{di}{dt} + C\int i\,dt$$

Apply differentiation,

$$0 = R\frac{di}{dt} + L\frac{d^2 i}{dt^2} + Ci$$

$$0 = R\frac{di(0^+)}{dt} + L\frac{d^2 i(0^+)}{dt^2} + C\,i(0^+)$$

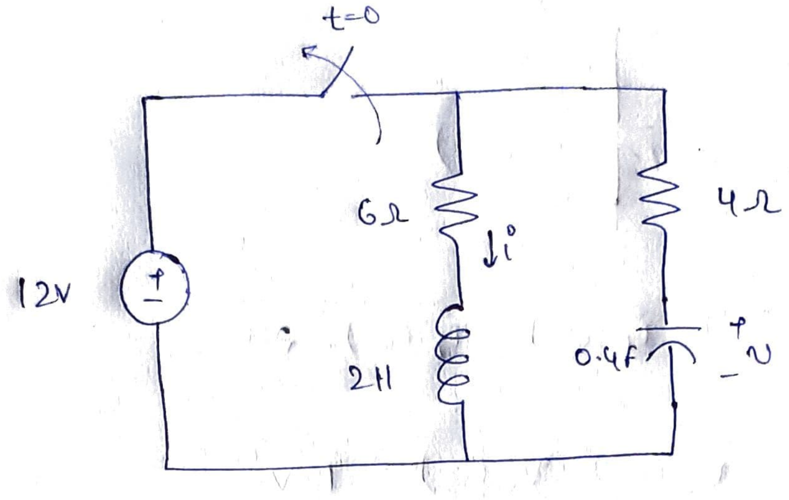
$$0 = 10R(10) + 1\left(\frac{d^2 i(0^+)}{dt}\right) + 10\mu(0)$$

$$\Rightarrow \quad \frac{d^2 i_L(0^+)}{dt^2} = -100$$

Q1) For the circuit, find, (a) $i(0^+)$, & $v(0^+)$

(b) $\frac{di(0^+)}{dt}$, $\frac{dv(0^+)}{dt}$

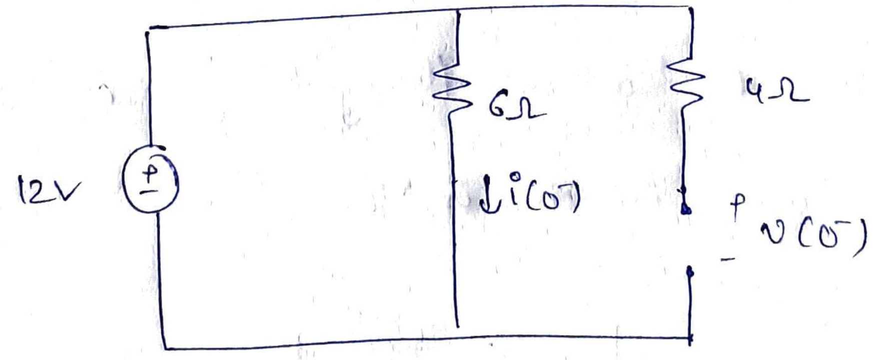
(c) $i(\infty)$, $v(\infty)$.



Sol) Initially switch is kept closed for a long time.

By the time $t=0^-$, the circuit has reached its steady state.

∴ At $t=0^-$, the circuit is as shown,



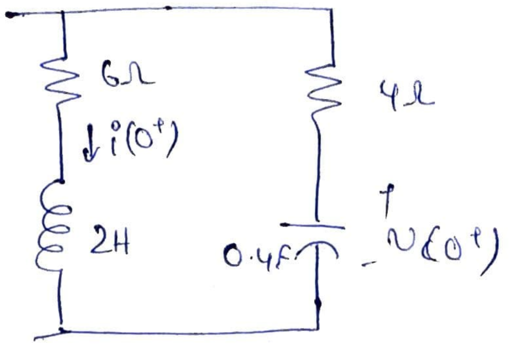
(Here $i = i_L$ & $v = v_C$)

Here, $i(0^-) = i_L(0^-) = \frac{12}{6} = 2A$

and $v_C(0^-) = 12V$. (because no current is flowing through 4Ω)

- At $t=0$, switch is operated.

- At $t=0^+$, the circuit will be as shown.



we know that, $i(0^+) = i_L(0^+) = i_L(0^-) = 2A$

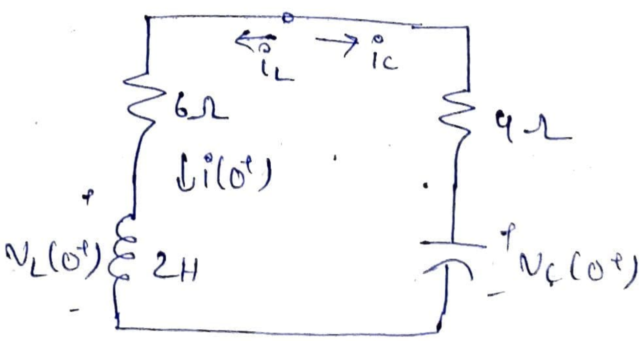
& $v(0^+) = v_C(0^+) = v_C(0^-) = 12V$

$\therefore$ $\boxed{i(0^+) = 2A, \quad v(0^+) = 12V}$

Now, $\dfrac{di(0^+)}{dt} = \dfrac{v_L(0^+)}{L}$ and $\dfrac{dv(0^+)}{dt} = \dfrac{i_C(0^+)}{C}$.

Let us consider the current flowing through capacitor is i_C & voltage across inductor is v_L

Now,



Apply KCL at node,

$\Rightarrow i_L + i_C = 0$

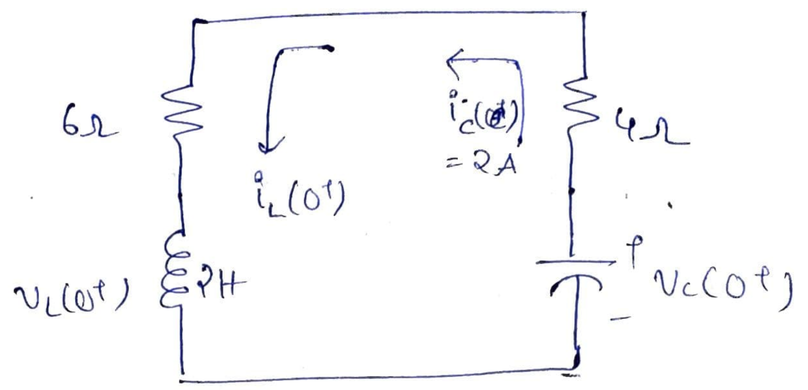
$\Rightarrow i_L(0^+) + i_C(0^+) = 0$

$\Rightarrow 2 + i_C(0^+) = 0$

$\Rightarrow i_C(0^+) = -2$

$\Rightarrow$ The direction of i_C is opposite to that, that is considered.

$\therefore$ The circuit with modified i_C direction,



Apply KVL in the loop,

$$6(i(0^+)) + 4(i(0^+)) + v_L(0^+) = v_C(0^+)$$

$$\Rightarrow 6(2) + 4(2) + v_L(0^+) = 12V$$

$$\Rightarrow 12 + 8 + v_L(0^+) = 12V$$

$$\Rightarrow v_L(0^+) = -8V$$

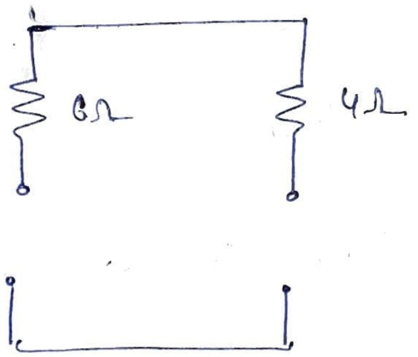
$$\therefore \frac{di_L(0^+)}{dt} = \frac{v_L(0^+)}{L} = \frac{-8}{2} = -4\ A/s.$$

and $\frac{dv_c(0^+)}{dt} = \frac{i_c(0^+)}{C} = \frac{-2}{0.4}$

$= -5 \text{ V/sec}$

$\therefore$ $\boxed{\begin{aligned} \frac{di(0^+)}{dt} &= -4 \text{ A/sec} \\ \frac{dv(0^+)}{dt} &= -5 \text{ V/sec} \end{aligned}}$

At $t \to \infty$, all the energy stored in the inductor and capacitor will become zero.

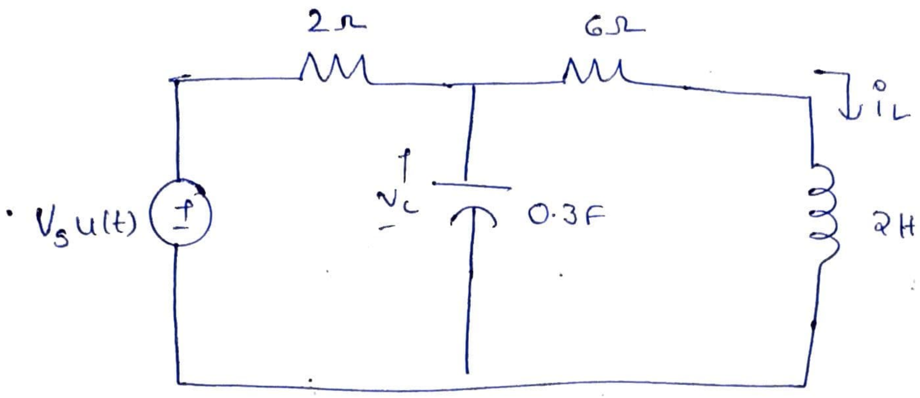


$\therefore$ $\boxed{i(\infty) = 0A \;\&\; V(\infty) = 0V}$

Q2) find; a) $i_L(0^+)$, $V_C(0^+)$

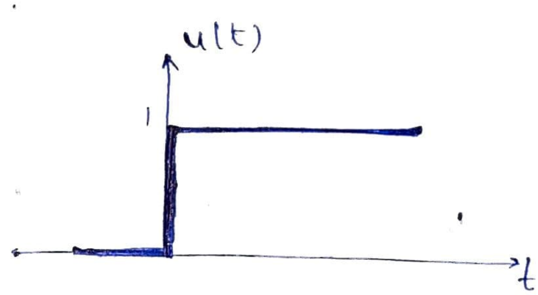
b) $\frac{di_L(0^+)}{dt}$, $\frac{dV_C(0^+)}{dt}$

c) $i_L(\infty)$, $V_C(\infty)$

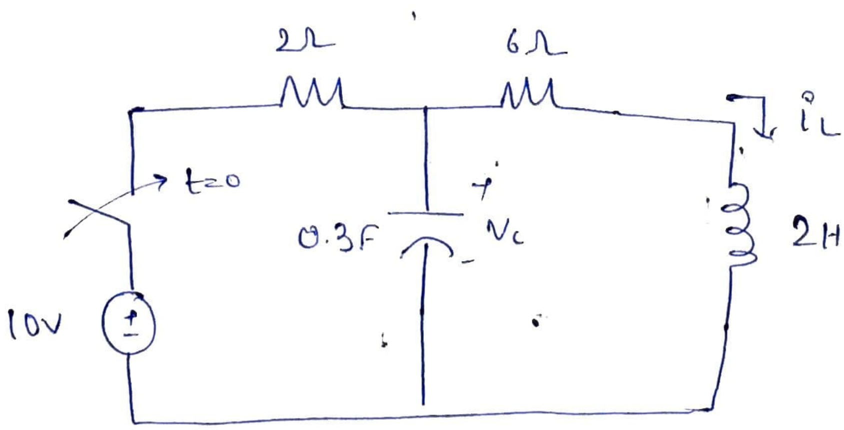


$V_S = 10V$.

Sol) $u(t) = \begin{cases} 1 & , t \geq 0 \\ 0 & , t < 0 \end{cases}$



∴ Circuit can be redrawn as shown below,

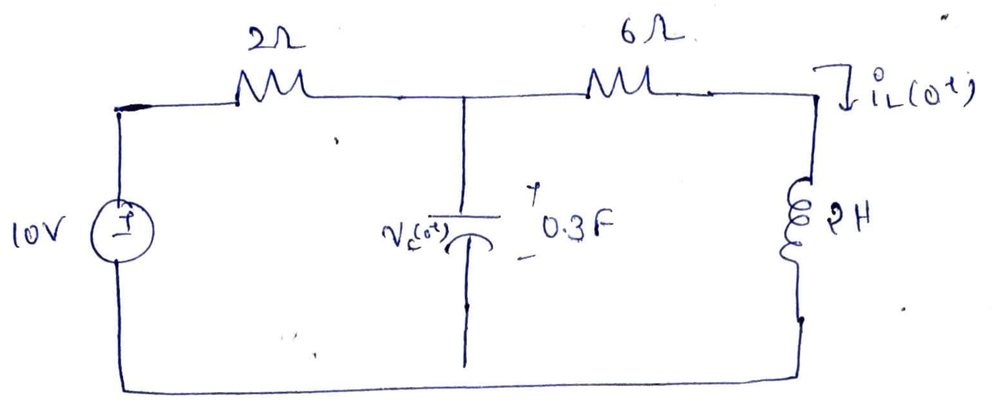


At $t=0^-$, since the switch has been opened & as there is no source included in the network,

$i_L(0^-) = 0A$ & $v_C(0^-) = 0V$

At $t=0^+$, switch is closed.

At $t=0^+$, the circuit is as shown,



we have, $i_L(0^+) = i_L(0^-) = 0A$

and $v_C(0^+) = v_C(0^-) = 0V$

$\therefore \boxed{i_L(0^+) = 0A, \quad v_C(0^+) = 0V}$

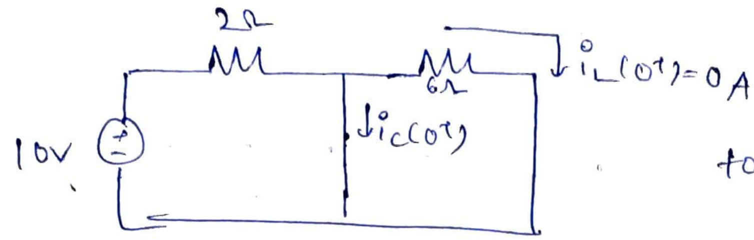
Now, $\frac{di_L(0^+)}{dt} = \frac{v_L(0^+)}{L}$ $\because$ there is no current flowing through inductor & also because it is in parallel with capacitor,

$V_L(0^+) = v_C(0^+) = 0V$

$\therefore \frac{di_L(0^+)}{dt} = \frac{0}{2} = \boxed{0 \text{ A/sec}}$

And $\frac{dv_c(0^+)}{dt} = \frac{i_c(0^+)}{C}$

{ the circuit can be understood as shown, at $t=0^+$,



total current $i = i_c(0^+) + i_L(0^+)$

$i = i_c(0^+) + 0$

$= i_c(0^+)$

$\therefore \quad i_c(0^+) = \frac{10}{2} = 5A$ }

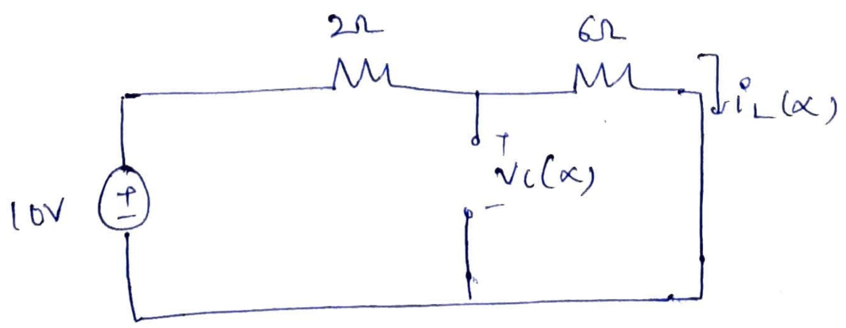
$$\therefore \quad \frac{dv_c(0^+)}{dt} = \frac{i_c(0^+)}{C} = \frac{5}{0.3}$$

$$= \frac{50}{3}$$

$$= \boxed{16.66 \text{ V/sec.}}$$

$\therefore \quad \frac{di_L(0^+)}{dt} = 0 \text{ A/sec}$ and $\frac{dv_c(0^+)}{dt} = 16.66 \text{ V/sec}$

At $t \to \infty$



Inductor is replaced by short circuit & capacitor by open circuit

$$i_L(\infty) = \frac{10}{2+6} = \frac{10}{8} = 1.25A$$

$V_c(\infty)$ is the voltage across 6Ω

$$V_c(\infty) = i \times R$$

$$= 1.25 \times 6$$

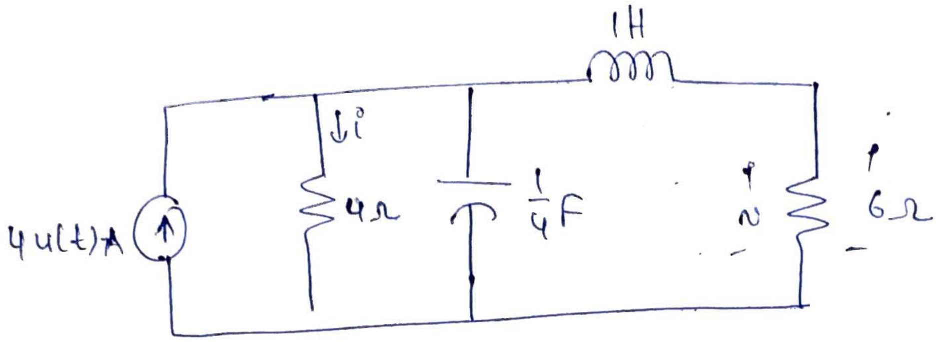
$$V_c(\infty) = \boxed{7.5V}$$

$$\therefore \boxed{i_L(\infty) = 1.25A \quad \text{and} \quad V_c(\infty) = 7.5V}$$

Q3) Determine : (a) $i(0^+)$, $v(0^+)$

(b) $\frac{di(0^+)}{dt}$, $\frac{dv(0^+)}{dt}$

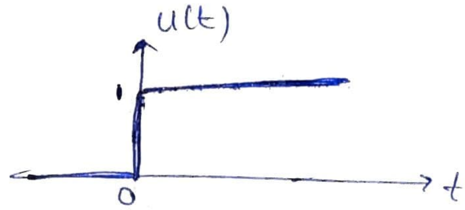
(c) $i(\infty)$, $v(\infty)$



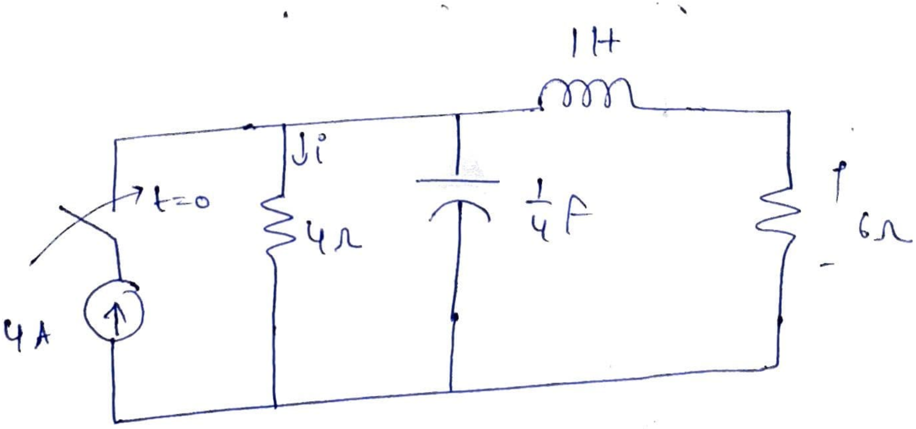
Sol) The current is given as, $4u(t)$ A

$u(t) = 1 \quad , t \geq 0$

$\quad\;\; = 0 \quad , t < 0$



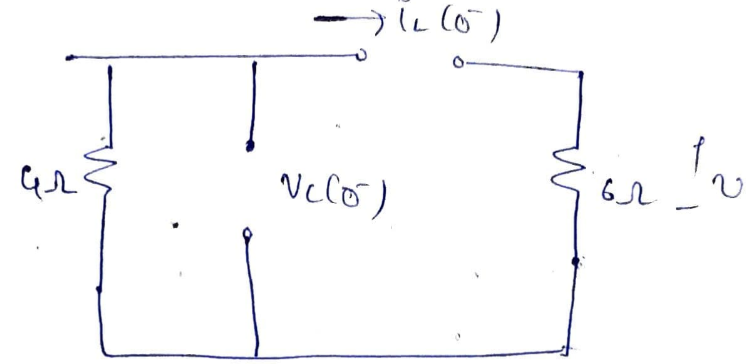
So, it can be modelled as a current source of 4A in series with a switch at , which is closed at $t=0$.



At $t=0^-$, there was no energy source in the network,

So, inductor & capacitor remained opened.

At $t=0^-$,

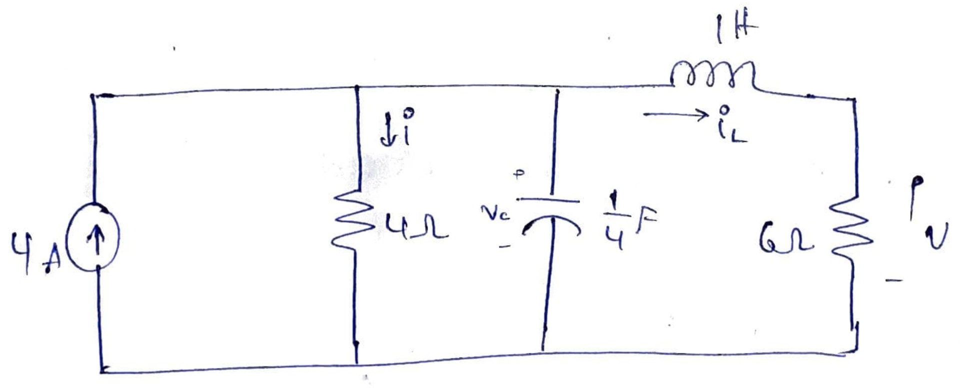


$i_L(0^-) = 0A$

$V_C(0^-) = 0V$.

At $t=0^+$, switching is done

At $t=0^+$, the circuit is as shown,

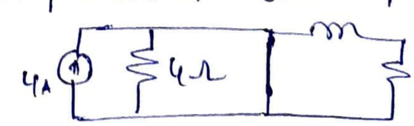


Now, $i_L(0^+) = i_L(0^-) = 0A$

$\Rightarrow v(0^+) = i_L(0^+) \times 6 = 0 \times 6 = \boxed{0V.}$

and, $V_c(0^+) = V_c(0^-) = 0V$

$\Rightarrow$ 4Ω resistance is in parallel with capacitor which is a short circuit ie

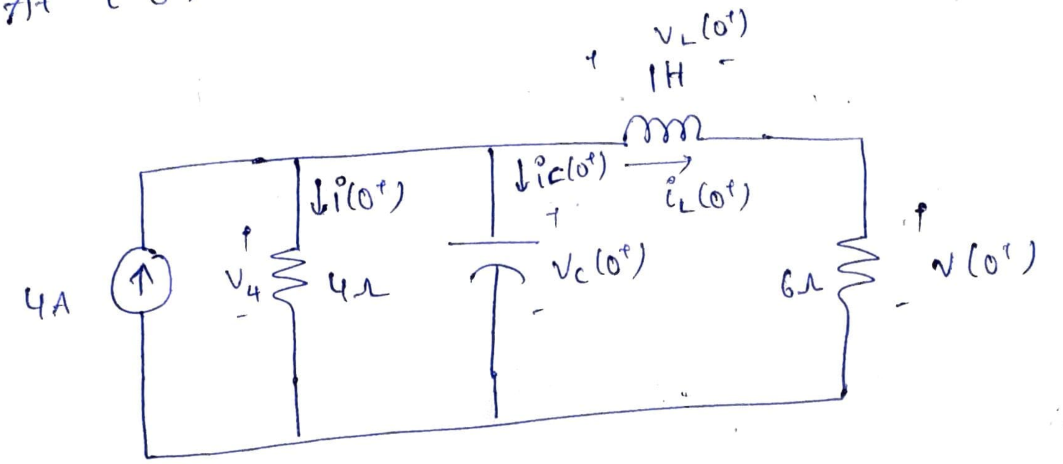


So, 4Ω can be neglected. i.e., current through 4Ω i.e $i(0^+) = \boxed{0A}$

$\therefore \boxed{i(0^+) = 0A \quad \& \quad v(0^+) = 0V}$

Now, at t=0 switch is closed as here, the signal (4A) is applied.

At $t=0^+$, the circuit will be as shown.



Now, $i(0^+) = \frac{1}{4} \times V_4(0^+)$

$$\frac{di(0^+)}{dt} = \frac{1}{4}\frac{dV_4(0^+)}{dt}$$

$$= \frac{1}{4}\frac{dV_c(0^+)}{dt} \quad (\because V_4 \text{ and } V_c \text{ are in parallel}) \rightarrow (1)$$

Now, $$\frac{dV_c(0^+)}{dt} = \frac{i_c(0^+)}{C} = \frac{4-(i(0^+)+i_L(0^+))}{C}$$

$$= \frac{4-(0+0)}{1/4}$$

$$= 4 \times 4 = 16 \text{ V/sec}$$

from (1), $\frac{di(0^+)}{dt} = \frac{1}{4}(16) = \boxed{4 \text{ A/sec}}$

And, $v(0^+) = i_L(0^+) \times 6$

$$\Rightarrow \frac{dv(0^+)}{dt} = 6\,\frac{di_L(0^+)}{dt} \longrightarrow (2)$$

from $\frac{di_L(0^+)}{dt} = \frac{V_L(0^+)}{L}$

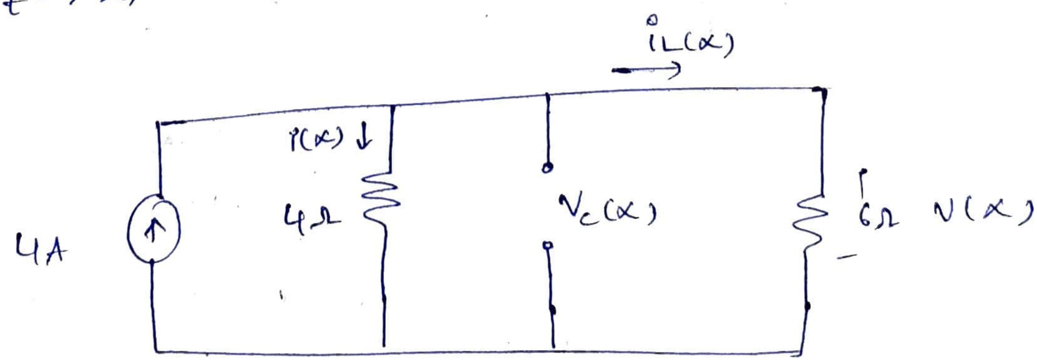
from the circuit, at $t=0^+$, inductor and resistor are in parallel, $V(0^+) = 0V \Rightarrow$ inductor is in parallel with capacitor & since $V_C(0^+) = 0V \Rightarrow V_L(0^+) = 0V$

$$\therefore \frac{di_L(0^+)}{dt} = \frac{0}{1} = 0 \text{ A/sec}$$

from (2) $\frac{dv(0^+)}{dt} = 6 \times 0 = \boxed{0 \text{ V/sec}}$

$$\therefore \boxed{\frac{di(0^+)}{dt} = 4 \text{ A/sec} \quad \text{and} \quad \frac{dv(0^+)}{dt} = 0 \text{ V/sec}}$$

As $t \to \infty$, the circuit will be as shown,



Now, $i(\infty)$ = current through 4Ω

($\because 4\Omega, 6\Omega$ are in parallel and the total current being supplied is $4A$)

$$= 4 \times \frac{6}{6+4}$$

$$= 4 \times \frac{6}{10}$$

$$\boxed{i(\infty) = 2.4\,A}$$

and $V(\infty) = i_L(\infty) \times 6$

$$= \left(4 \times \frac{4}{4+6}\right) \times 6$$

$$= \left(4 \times \frac{4}{10}\right) \times 6$$

$$= 1.6 \times 6$$

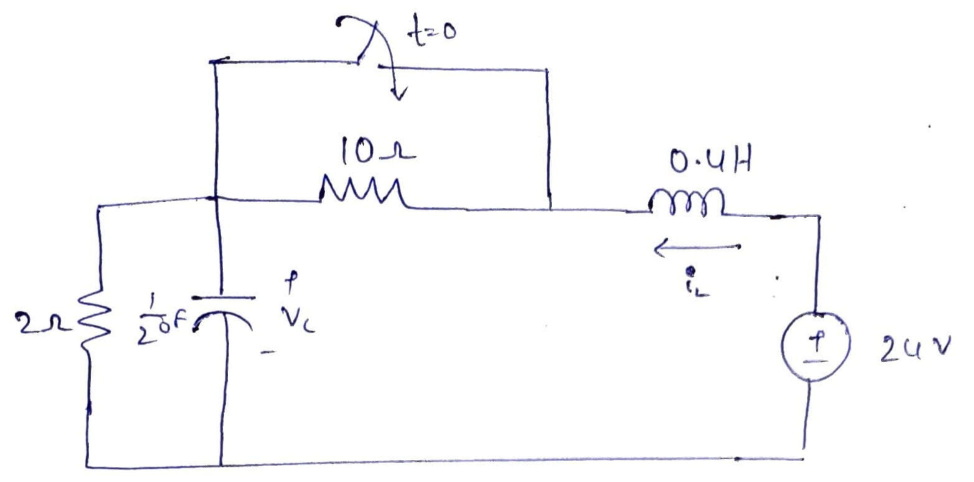
$$\boxed{V(\infty) = 9.6\,V}$$

$$\therefore \boxed{i(\infty) = 2.4A \quad , \quad V(\infty) = 9.6V}$$

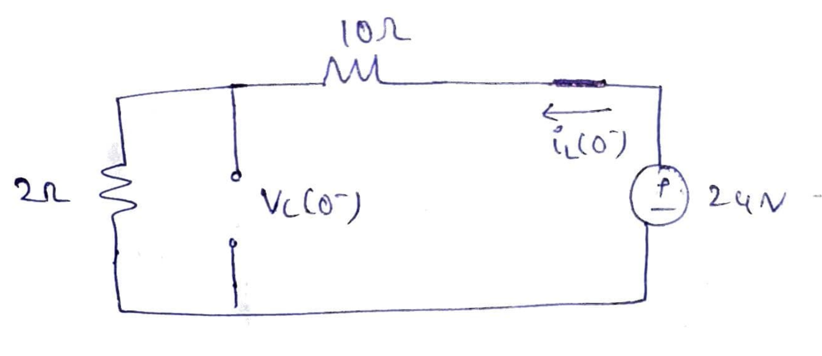
Q2 Sol) Calculate: (a) $i_L(0^+)$, $V_C(0^+)$

(b) $\frac{di_L(0^+)}{dt}$, $\frac{dV_C(0^+)}{dt}$

(c) $i_L(\infty)$, $v_C(\infty)$



Sol) At $t=0^-$,

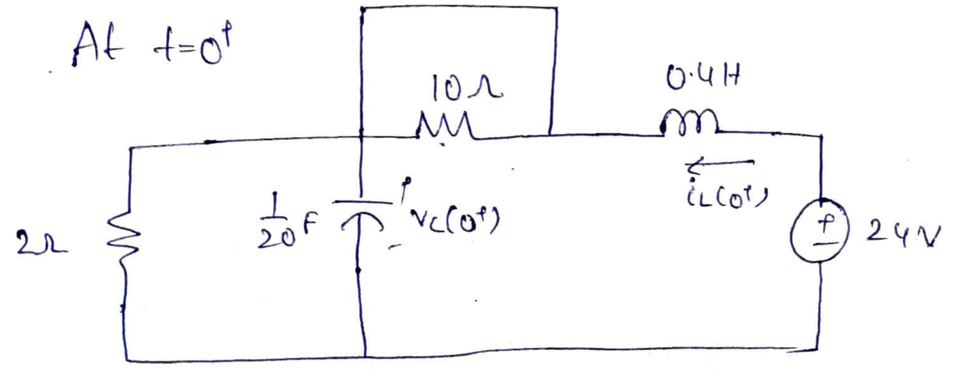


$$i_L(0^-) = \frac{24}{10+2} = \frac{24}{12} = \boxed{2A}$$

$V_C(0^-)$. = drop across 2Ω resistance $= i_L(0^-) \times 2$

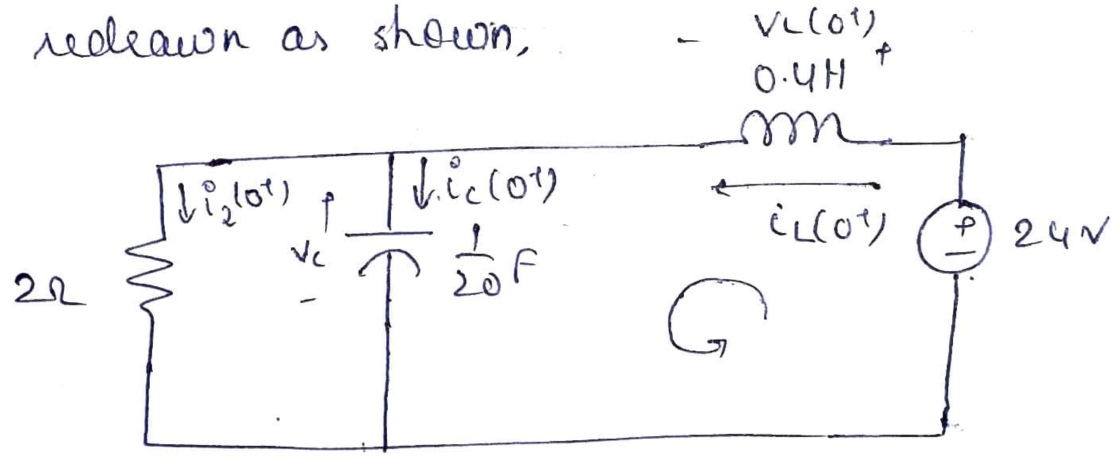
$= 2 \times 2 = \boxed{4V}$

At $t=0^+$



from theory, $i_L(0^+) = i_L(0^-) = \boxed{2A}$ & $V_C(0^-) = V_C(0^+) = \boxed{4V}$

Since 10Ω is shorted, it can be neglected. So, the circuit can be redrawn as shown,



we know, $V_C(0^+) = 4V$ $\Rightarrow$ voltage drop across 2Ω is also 4V

$\because$ Both are in parallel $\Rightarrow$ $2 \times i_2(0^+) = 4$

$\Rightarrow i_2(0^+) = 4$

Now $\Rightarrow$ $i_L(0^+) = i_C(0^+) + i_2(0^+)$

$\Rightarrow 2 = i_C(0^+) + 2 \Rightarrow i_C(0^+) = 0\ A$

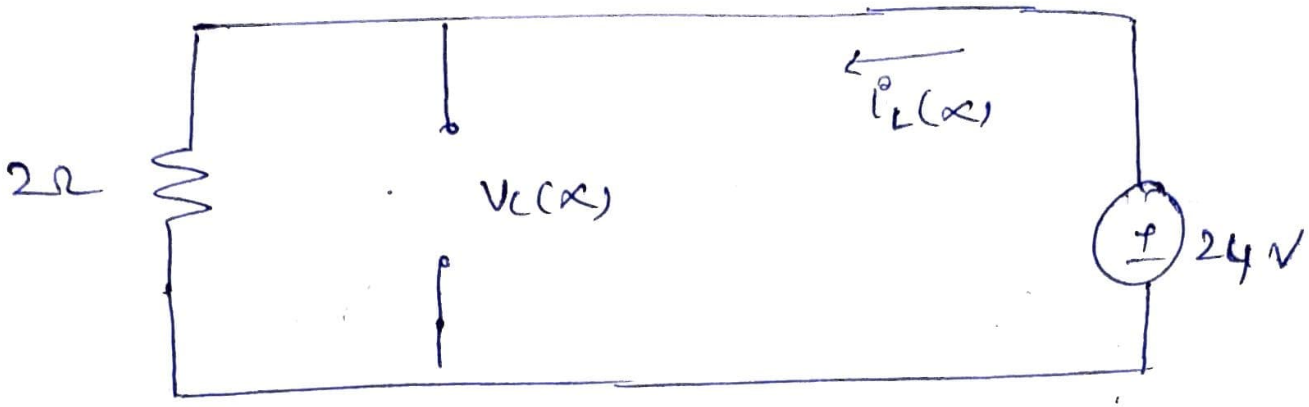
$$\therefore \frac{dV_C}{dt} = \frac{i_C(0^+)}{C} = \frac{0}{1/20} = \boxed{0\ V/s}$$

Apply KVL, in the loop, $24 = V_L(0^+) + V_C(0^+)$

$\Rightarrow V_L(0^+) = 24 - 4 = 20V$

$$\therefore \frac{di_L(0^+)}{dt} = \frac{V_L(0^+)}{L} = \frac{20}{0.4} = \boxed{50\ A/s.}$$

At $t \to \infty$



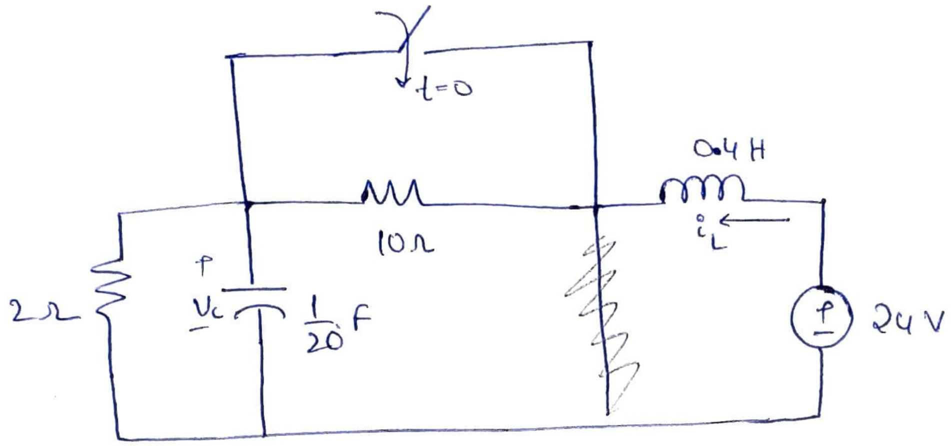
$$\therefore i_L(\infty) = \frac{24}{2} = \boxed{12A}$$

$$V_C(\infty) = \boxed{24V}$$

Q2) The switch in below circuit was open for a long time, but closed at $t=0$. Determine (a) $i_L(0^+)$, $v_C(0^+)$

(b) $\frac{di_L(0^+)}{dt}$, $\frac{dv_C(0^+)}{dt}$

(c) $i_L(\infty)$, $v_C(\infty)$

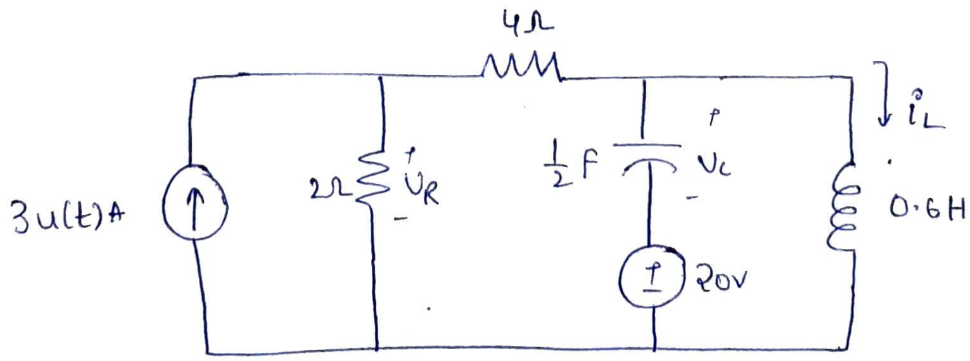


Ans: (a) 2A, 4V, (b) 50 A/s, 0 V/s (c) 12A, 24V

Q3) In the below circuit, calculate: (a) $i_L(0^+)$, $v_C(0^+)$, $v_R(0^+)$

(b) $\frac{di_L(0^+)}{dt}$, $\frac{dv_C(0^+)}{dt}$, $\frac{dv_R(0^+)}{dt}$

(c) $i_L(\infty)$, $v_C(\infty)$, $V_R(\infty)$

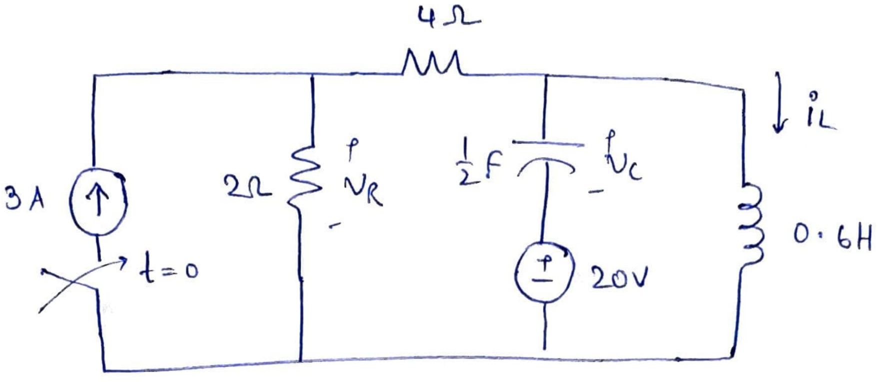


Ans: (a) 0A, −20V, 4V (b) $\frac{di_L(0^+)}{dt} = 0$ A/sec, $\frac{dv_C(0^+)}{dt} = 2$ V/sec, $\frac{dv_R(0^+)}{dt} = \frac{2}{3}$ V/s

(c) $i_L(\infty) = 1$A, $v_C(\infty) = -20$V, $v_R(\infty) = 4$ V

Sol> { Hint :

The circuit can be understood as shown below,

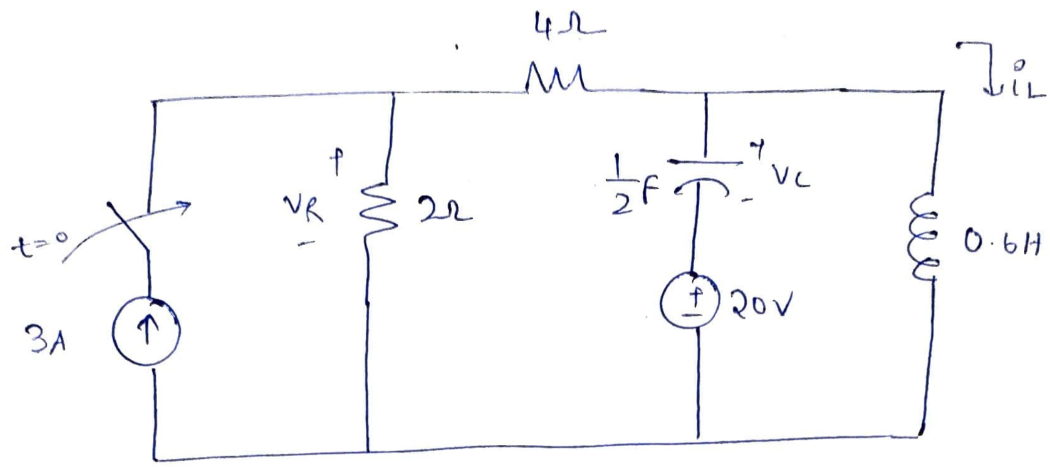


At t=0, 3A current source is introduced into the network.

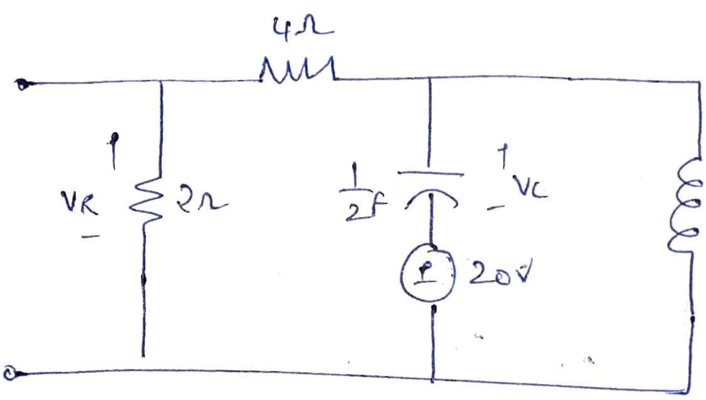
For t<0, the circuit has reached its steady state with switch open. At t=0, switch is closed }

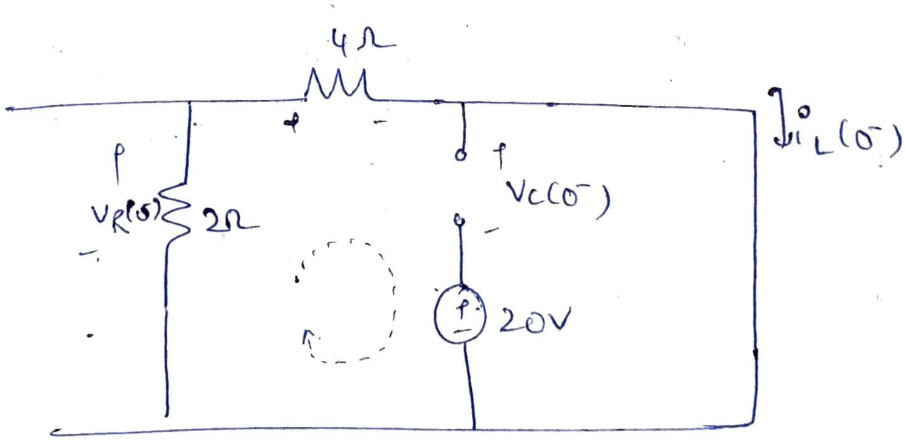
(a) for t<0, 3u(t) = 0 . So, the circuit will reach its steady state accordingly

Sol) Circuit can be re-drawn as shown.



At $t=0^-$, circuit will as shown.





Since source is not properly connected in the network, there will not be any current flowing through it.

$\therefore\ i_L(0^-) = 0A$

Apply KVL along the dotted line to find $V_C(0^-)$

$$i_L(0^-) \times 4 + V_C(0^-) + 20 = i_L(0^-) \times 2$$

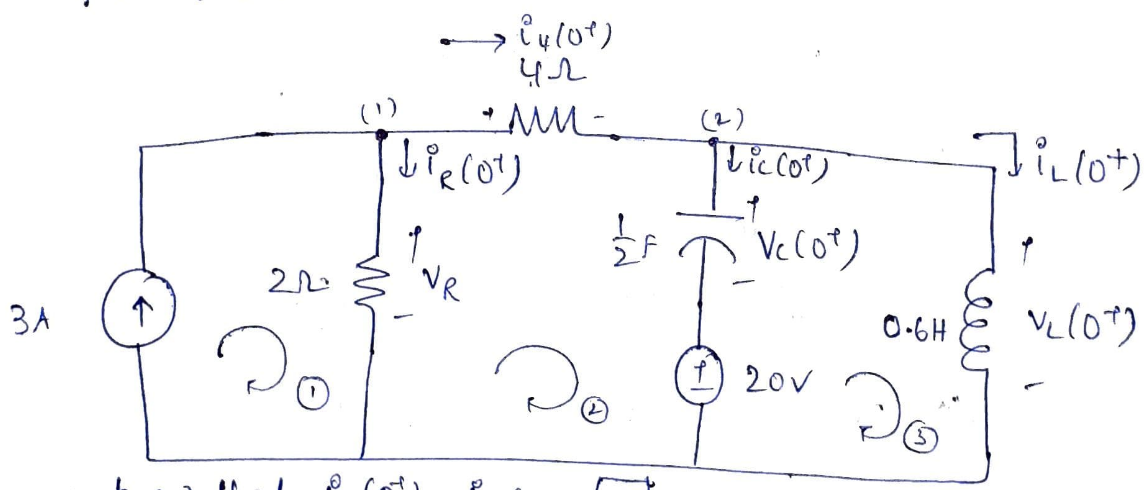
$$\Rightarrow 0 \times 4 + V_C(0^-) + 20 = 0 \times 2$$

$$\Rightarrow V_C(0^-) = -20V$$

$$V_R(0^-) = i_L(0^-) \times 2 = 0 \times 2 = 0V.$$

At $t=0$, $u(t)$ signal will exist $\Rightarrow$ 3A voltage source is included in the network.

At $t=0^+$, the circuit will be as shown.



we know that, $i_L(0^+) = i_L(0^-) = \boxed{0A}$, $V_C(0^+) = V_C(0^-) = \boxed{-20V}$

$\underline{\frac{di_L(0^+)}{dt}}$: Apply KVL in loop 3,

$$V_L(0^+) - 20 - V_C(0^+) = 0$$

$$V_L(0^+) = 20 + V_C(0^+)$$

$$= 20 + (-20) = 0V$$

$$\therefore V_L(0^+) = 0V$$

$$\therefore \frac{di_L(0^+)}{dt} = \frac{V_L(0^+)}{L} = \frac{0}{0.6} = \boxed{0\ A/sec.}$$

Let the voltage across 4Ω be V_4.

Apply KVL in second loop.

$$V_4(0^+) + V_C(0^+) + 20 - V_R(0^+) = 0$$

$$V_4(0^+) - 20 + 20 - V_R(0^+) = 0$$

$$\Rightarrow V_R(0^+) = V_4(0^+) \longrightarrow (1)$$

$V_R(0^+)$: Apply ~~KV~~ KCL at node 1

$$3 = i_R(0^+) + i_4(0^+)$$

$$\Rightarrow 3 = \frac{V_R(0^+)}{2} + \frac{V_4(0^+)}{4} \longrightarrow (2)$$

Sub (1) in (2)

$$\Rightarrow 3 = \frac{V_R(0^+)}{2} + \frac{V_R(0^+)}{4}$$

$$\Rightarrow 3 = \frac{3}{4} V_R(0^+) \Rightarrow \boxed{V_R(0^+) = 4V}$$

$\frac{dV_C(0^+)}{dt}$: Apply ~~KV~~ KCL at node 2,

$$i_4(0^+) = i_C(0^+) + i_L(0^+)$$

$$\Rightarrow \frac{V_4(0^+)}{4} = i_C(0^+) + i_L(0^+)$$

$$\Rightarrow \frac{4}{4} = i_C(0^+) + 0 \Rightarrow i_C(0^+) = 1\ A$$

$$\therefore \frac{dv_c(0^+)}{dt} = \frac{i_c(0^+)}{C}$$

$$= \frac{1}{1/2}$$

$$\boxed{\frac{dv_c(0^+)}{dt} = 2\ v/sec}$$

$\frac{dv_R(0^+)}{dt}$: Apply KVL in second loop.

$$v_4(0^+) + v_c(0^+) + 20 - v_R(0^+) = 0$$

Take differentiation,

$$\frac{dv_4(0^+)}{dt} + \frac{dv_c(0^+)}{dt} + 0 - \frac{dv_R(0^+)}{dt} = 0$$

$$\Rightarrow \frac{dv_4(0^+)}{dt} - \frac{dv_R(0^+)}{dt} = -\frac{dv_c(0^+)}{dt}$$

$$\Rightarrow \frac{dv_4(0^+)}{dt} - \frac{dv_R(0^+)}{dt} = -2 \longrightarrow (3)$$

from eq (2),

$$3 = \frac{v_R(0^+)}{2} + \frac{v_4(0^+)}{4}$$

Apply differentiation,

$$0 = \frac{1}{2} \times \frac{dv_R(0^+)}{dt} + \frac{1}{4}\frac{dv_4(0^+)}{dt} \longrightarrow (4)$$

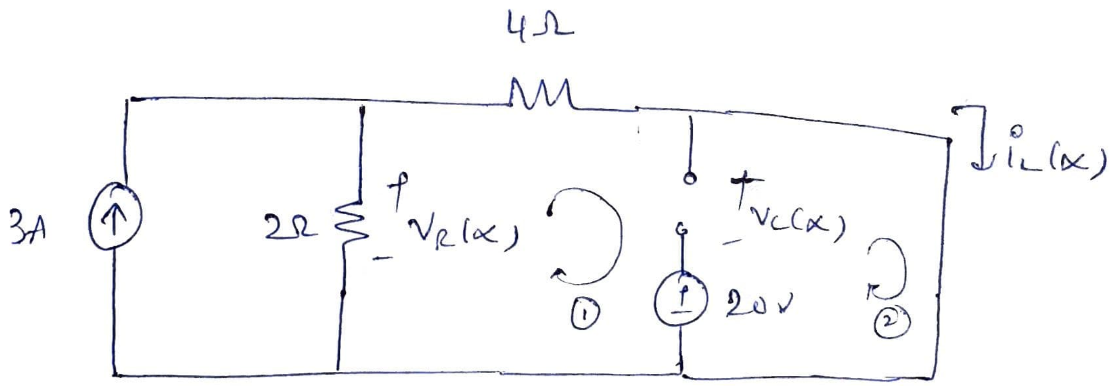
Substitute the value of $\frac{dv_4(0^+)}{dt}$ from eq (4) in eq (3)

$$\left(-2\frac{dv_R(0^+)}{dt}\right) - \frac{dv_R(0^+)}{dt} = -2$$

$$-(2+1)\frac{dv_R(0^+)}{dt} = -2$$

$$\boxed{\frac{dv_R(0^+)}{dt} = \frac{2}{3}\ \text{V/sec}}$$

As $t \to \infty$



Current flowing through 4Ω is $i_L(\infty) = 3 \times \frac{2}{2+4}$

$$= 3 \times \frac{2}{6}$$

$$= 1A$$

$$\therefore \boxed{i_L(\infty) = 1A}$$

$V_R(\infty) = i_R(\infty) \times 2$

$= \left(3 \times \frac{4}{2+4}\right) \times 2$

$= \left(3 \times \frac{4}{6}\right) \times 2$

$= 2 \times 2$

$\boxed{V_R(\infty) = 4V}$

$V_C(\infty)$:

Apply KVL in loop 1

$\cancel{V_R(\infty)}$.

$i_L(\infty) \times 4 + V_C(\infty) + 20 - i_R(\infty) \times 2 = 0$

$\Rightarrow 1 \times 4 + V_C(\infty) + 20 - 2 \times 2 = 0$

$\Rightarrow \boxed{V_C(\infty) = -20V}$

(or)

Apply KVL in second loop,

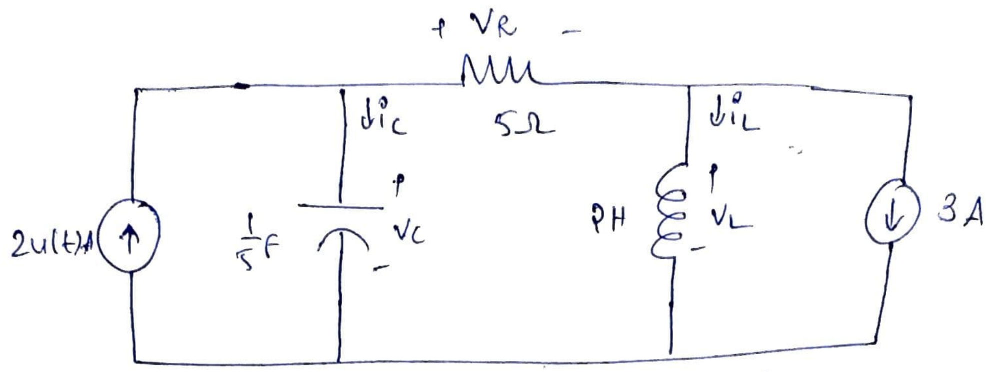
$-20 - V_C(\infty) = 0$

$\Rightarrow \boxed{V_C(\infty) = -20V}$

Q4) find (a) $i_L(0^+)$, $v_C(0^+)$, $v_R(0^+)$

(b) $\frac{di_L(0^+)}{dt}$, $\frac{dv_C(0^+)}{dt}$, $\frac{dv_R(0^+)}{dt}$

(c) $i_L(\infty)$, $v_C(\infty)$, $v_R(\infty)$

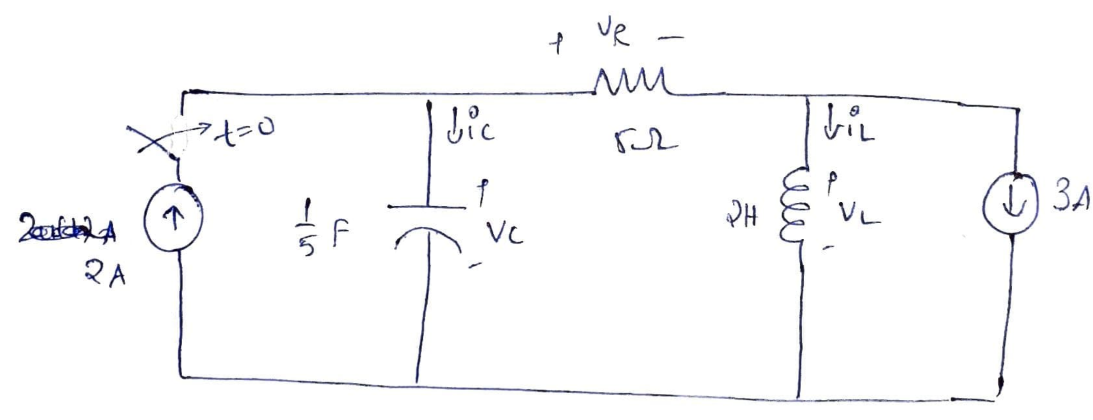


Ans: (a) −3A, 0V, 0V

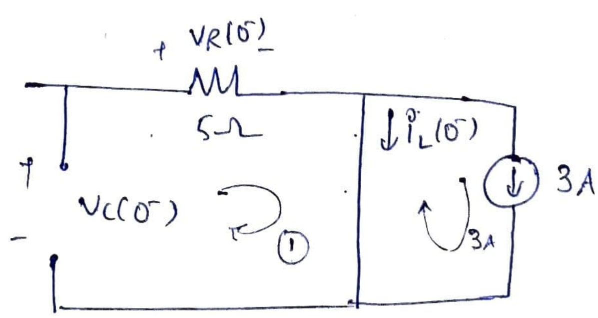
(b) 0A/s, 10V/s, 0V/s

(c) −1A, 10V, 10V.

Sol) Circuit can be redrawn as shown.



At $t=0^-$, the circuit is as shown.



$\therefore i_L(0^-) = -3A$ ($\because$ both directions i.e loop direction and $i_L(0^-)$ are opposite to each other)

~~and $V_C(0^-) = 0V$ $\because$ no current~~

Apply KVL in loop 1

$V_R(0^-) + 0 - V_C(0^-) = 0$

$\Rightarrow V_C(0^-) = V_R(0^-)$

$= i_R(0^-) \times 5\Omega$

($i_R(0^-)$ is 0A because the entire current will flow through short circuited path and no current will flow through 5Ω)

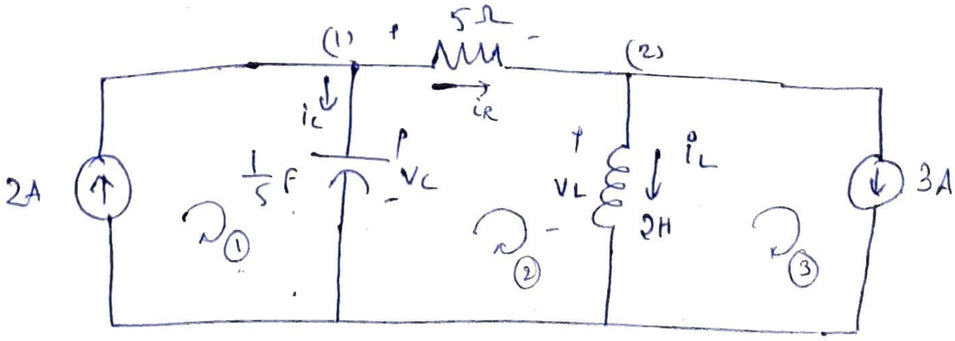
$= 0 \times 5$

$= 0V$

$\therefore V_C(0^-) = 0V$

At $t=0$; 2A source will be included in the network.

At $t=0^+$, the circuit will be as shown.



$i_L(0^+) = i_L(0^-) = \boxed{-3A}$

$V_C(0^+) = V_C(0^-) = \boxed{0V}$

$V_R(0^+) = i_R(0^+) \times 5$

Apply KVL in loop 2,

$V_R(0^+) + V_L(0^+) = V_C(0^+)$

$\Rightarrow V_R(0^+) + 0 = \cancel{V_C(0)}\ 0$ $\quad (\because V_L(0^+) = 0$, short circuit$)$ $\rightarrow (1)$

$\Rightarrow \boxed{V_R(0^+) = 0V}$

Apply ~~KV~~ KCL at node 1

$2 = i_C(0^+) + i_R(0^+)$

$2 = i_C(0^+) + \dfrac{V_R(0^+)}{5}$

$2 = i_C(0^+) + \dfrac{0}{5} \Rightarrow i_C(0^+) = 2A$

$\therefore \dfrac{dV_C(0^+)}{dt} = \dfrac{i_C(0^+)}{C} = \dfrac{2}{1/5} = \boxed{10\ V/sec}$

$\dfrac{di_L(0^+)}{dt} = \dfrac{V_L(0^+)}{L} = \dfrac{0}{2H} = \boxed{0\ A/sec}$

$(\because$ from $(1))$

Apply KCL at node 2,

$i_R = i_L + 3$ ie $i_R(0^+) = i_L(0^+) + 3 \Rightarrow \frac{V_R(0^+)}{5} = i_L(0^+) + 3$

Take differentiation,

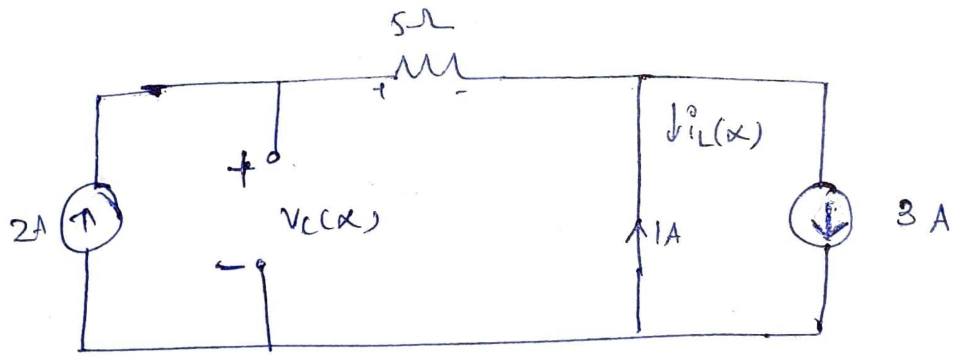
$$\frac{1}{5} \cdot \frac{dV_R(0^+)}{dt} = \frac{di_L(0^+)}{dt} + 0$$

$$\Rightarrow \frac{dV_R(0^+)}{dt} = 5\,\frac{di_L(0^+)}{dt}$$

$$= 5(0)$$

$$\boxed{\frac{dV_R(0^+)}{dt} = 0\,V/sec}$$

As $t \to \infty$, the circuit will be as shown.



$\therefore \boxed{i_L(\infty) = -1\,A}$

$V_C(\infty)$ = drop across 5Ω

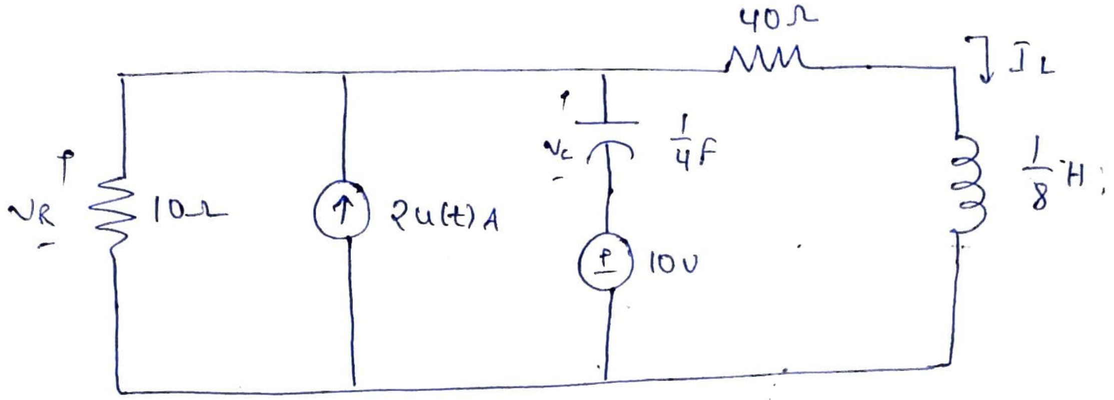
$= (2\,A) \times 5\Omega = \boxed{10V}$

$V_R(\infty)$ = drop across 5Ω $= \boxed{10V}$

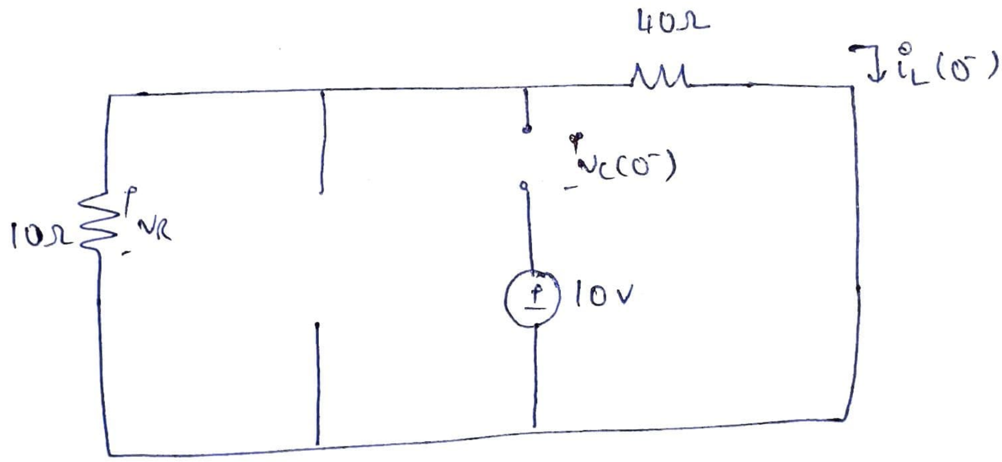
Q5) Calculate: (a) $i_L(0^+), v_C(0^+), v_R(0^+)$

(b) $di_L(0^+)/dt$, $dv_C(0^+)/dt$, $\frac{dv_R(0^+)}{dt}$

(c) $i_L(\infty), v_C(\infty), v_R(\infty)$



Sol) At $t=0^-$,



Since no source is properly connected in the n/w,

$\boxed{i_L(0^-) = 0\,A} \Rightarrow v_R(0^-) = i_R(0^-) \times 10$

$= i_L(0^-) \times 10$

$= 0 \times 10$

$v_R(0^-) = 0V$

Apply KVL,

$v_C(0^-) + 10 - v_R(0^-) = 0$

$\Rightarrow v_C(0^-) + 10 - i_L(0^-) \times 10 = 0$

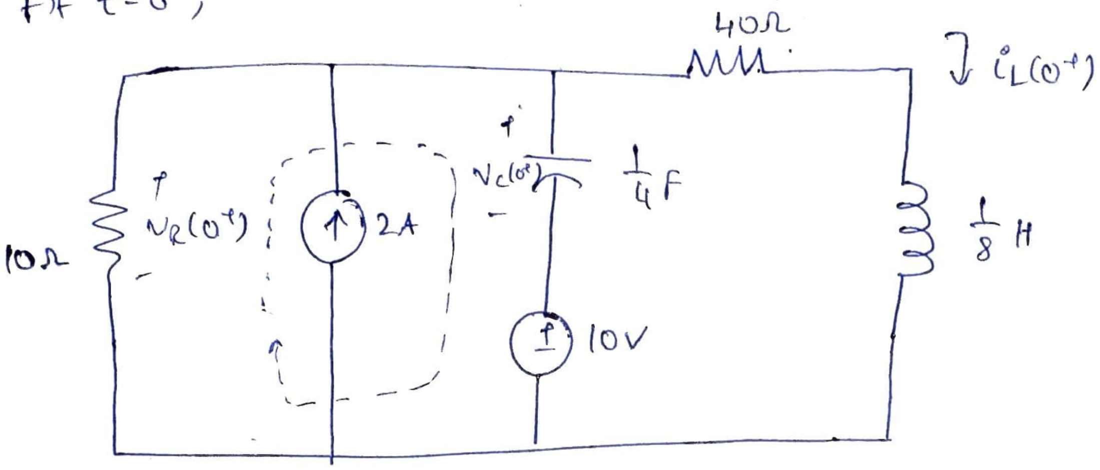
$\Rightarrow V_C(0^-) + 10 - (0 \times 10) = 0$

$\Rightarrow V_C(0^-) + 10 - 0 = 0$

$\Rightarrow \boxed{V_C(0^-) = -10\ V}$

Now, at $t=0$, 2A source will be included in the network.

At $t=0^+$,



from theory, $i_L(0^+) = i_L(0^-) = \boxed{0A}$

$V_C(0^+) = V_C(0^-) = \boxed{-10V}$

Apply KVL along the dotted line,

$V_C(0^+) + 10 - V_R(0^+) = 0$

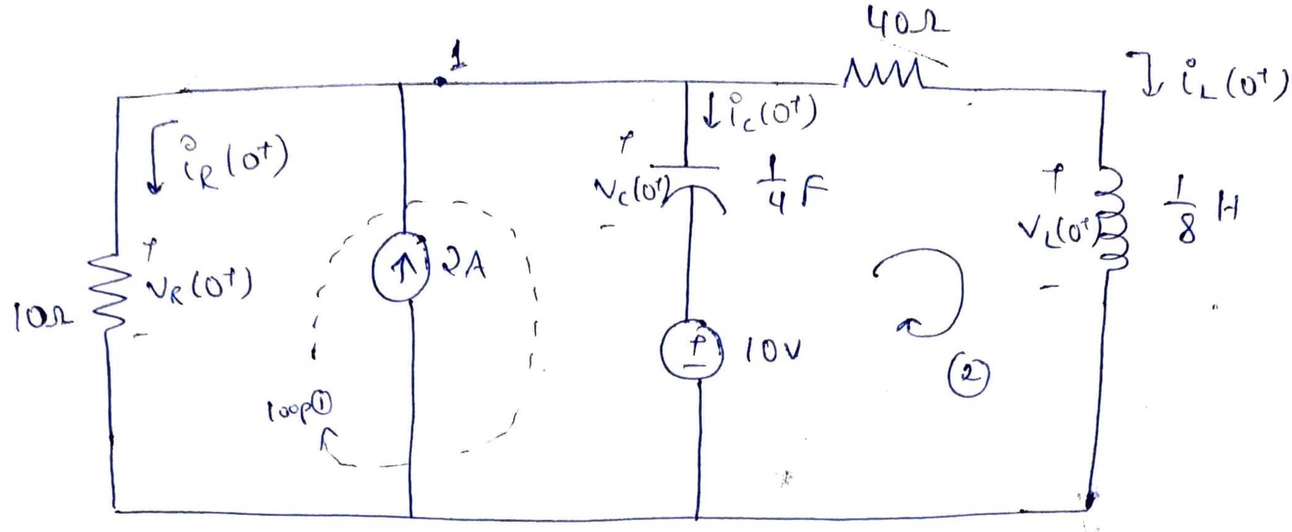
$\Rightarrow V_R(0^+) = V_C(0^+) + 10$

$= -10 + 10$

$= \boxed{0V}$

$\therefore \boxed{i_L(0^+) = 0A, \quad V_C(0^+) = -10V, \quad V_R(0^+) = 0V}$

$$\frac{di_L(0^+)}{dt}, \quad \underline{\frac{dv_C(0^+)}{dt}}, \quad \frac{dv_R(0^+)}{dt}:$$



Apply KVL in the last loop, ie loop 2

$$40(i_L(0^+)) + v_L(0^+) - 10 - v_C(0^+) = 0$$

$$\Rightarrow \quad 40(0) + v_L(0^+) - 10 - (-10) = 0$$

$$\Rightarrow \quad v_L(0^+) = 0\,V$$

$$\therefore \frac{di_L(0^+)}{dt} = \frac{v_L(0^+)}{L} = \frac{0}{\frac{1}{8}} = \boxed{0\,A/s = \frac{di_L(0^+)}{dt}}$$

Now, Apply KCL at node 1,

$$2 = i_R(0^+) + i_C(0^+) + i_L(0^+)$$

$$\left\{ \because v_R(0^+) = 0 \Rightarrow i_R(0^+) \times (10) = 0 \Rightarrow i_R(0^+) = 0\,A \right\}$$

$$\Rightarrow 2 = 0 + i_C(0^+) + 0$$

$\Rightarrow i_c(0^+) = 2\ A$

$\therefore \dfrac{dv_c(0^+)}{dt} = \dfrac{i_c(0^+)}{C}$

$= \dfrac{2}{1/4} = \boxed{8\ V/s. = \dfrac{dv_c(0^+)}{dt}}$

Now,

Apply KVL along the dotted lines, i.e loop1,

$v_R(0^+) = v_C(0^+) + 10$

$10 \times i_R(0^+) = \frac{1}{C}\int i_c(0^+)dt + 10$

Differentiate,

$\Rightarrow 10\ \dfrac{di_R(0^+)}{dt} = \frac{1}{C}\, i_c(0^+) + 0$

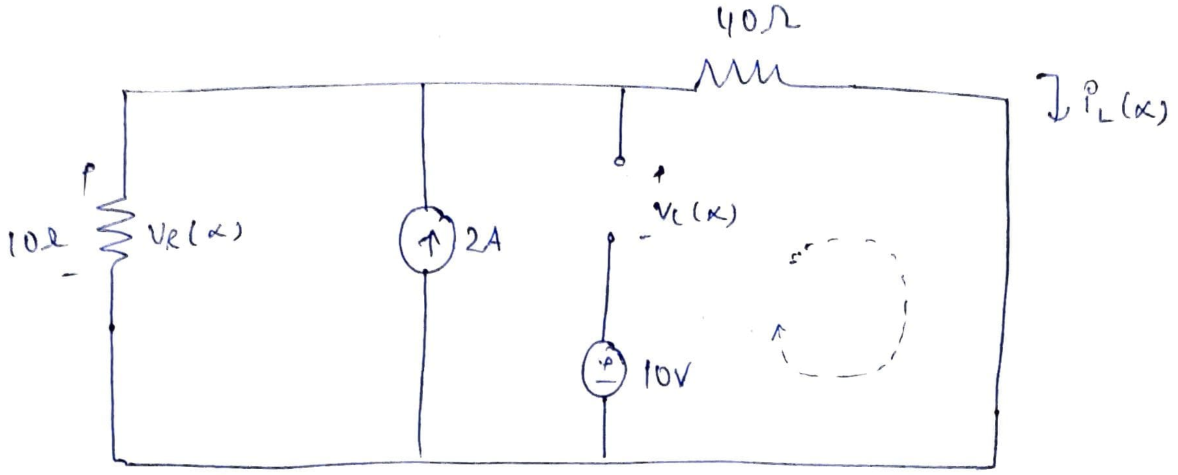
$\Rightarrow 10 \times \dfrac{di_R(0^+)}{dt} = \dfrac{1}{\frac{1}{4}}(2) + 0$

$\Rightarrow \dfrac{di_R(0^+)}{dt} = \dfrac{8}{10} = 0.8\ A/sec = \dfrac{di_R(0^+)}{dt}$

$v_R = i_R \times 10 \Rightarrow \dfrac{dv_R}{dt} = 10 \times \dfrac{di_R}{dt} = 10 \times \dfrac{8}{10}$

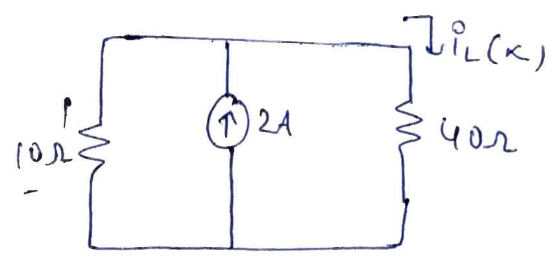
$\Rightarrow \boxed{\dfrac{dv_R}{dt} = 8\ V/sec}$

At $t \to \infty$,



ckt1

This circuit can be re-drawn as below to find $i_L(\infty)$



ckt 2

So, $i_L(\infty)$ is calculated from current division rule,

$$i_L(\infty) = 2 \times \frac{10}{10+40}$$

$$= 2 \times \frac{1}{5} = \boxed{0.4 \text{ A} = i_L(\infty)}$$

from circuit 1,

Apply KVL along the dotted line

$$40(i_L(\infty)) - 10 - V_C(\infty) = 0$$

$$\Rightarrow 40(0.40) - 10 - V_C(\infty) = 0$$

$$\Rightarrow \cancel{4-10} \Rightarrow 16 - 10 - V_C(\infty) = 0$$

$$\Rightarrow \boxed{V_C(\infty) = 6 \text{ V}}$$

$$V_R(\infty) = i_R(\infty) \times 10$$

$$= \left(2 \times \frac{40}{10+40}\right) \times 10$$

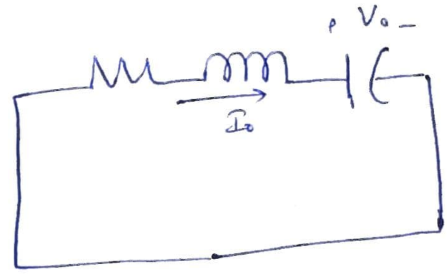
$$= \left(2 \times \frac{4}{5}\right) \times 10$$

$$\boxed{V_R(\infty) = 16V}$$

Source free series RLC circuit:

Consider a series RLC circuit as shown. The circuit is excited by the energy initially stored in the capacitor and inductor. The energy is represented by the initial capacitor voltage V_0 and initial inductor current I_0. Thus, at $t=0$,

$$v_c(0) = \frac{1}{C}\int_{-\infty}^{0} i\,dt = V_0 \rightarrow (1)$$



$$i_L(0) = I_0 \longrightarrow (2)$$

Applying KVL in the loop,

$$Ri + L\frac{di}{dt} + \frac{1}{C}\int_{-\infty}^{t} i\,dt = 0 \longrightarrow (3)$$

To eliminate the integral, we take differentiation with respect to t, and rearrange the terms.

$$L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \frac{1}{C}i = 0$$

$$\Rightarrow \frac{d^2i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{1}{LC}i = 0 \longrightarrow (4)$$

This is a second order differential equation & that's why the above shown RLC circuit is also called as second order circuit.

In order to solve this differential equation, we need two initial conditions. For example, initial value of i and its first derivative (or) initial values of some i and v.

from eq ②, we have initial value of current,

ie $i(0) = I_0$

from eq ③, $Ri + L\frac{di}{dt} + \frac{1}{C}\int_{-\infty}^{t} i\,dt = 0$

At $t=0 \Rightarrow Ri(0) + L\frac{di(0)}{dt} + v(0) = 0$

$$\Rightarrow Ri(0) + L\frac{di(0)}{dt} + V_0 = 0 \rightarrow ⑤$$

∴ from eq ⑤, initial value of derivative of 'i' can be calculated.

$$\Rightarrow L\frac{di(0)}{dt} = -(V_0 + Ri(0))$$

$$\frac{di(0)}{dt} = -\frac{1}{L}(V_0 + Ri(0))$$

$$\Rightarrow \frac{di(0)}{dt} = -\frac{1}{L}(V_0 + RI_0) \rightarrow ⑥$$

With the help of eq ② and eq ⑥, we can solve eq ④

we know that, from source free first order circuits the current or voltage, is either exponentially decaying or increasing function.

So, let us suppose, $i = Ae^{kt}$ → (7)

where A, K are constants to be determined eq (7) in eq (4).

$$\frac{d^2}{dt^2}(Ae^{kt}) + \frac{R}{L}\frac{d}{dt}(Ae^{kt}) + \frac{1}{LC}(Ae^{kt}) = 0$$

$$\Rightarrow Ak^2e^{kt} + \frac{AR}{L}ke^{kt} + \frac{Ae^{kt}}{LC} = 0$$

$$\Rightarrow Ae^{kt}\left(k^2 + \frac{R}{L}k + \frac{1}{LC}\right) = 0 \quad \rightarrow (8)$$

Either of the terms must be zero,

Ae^{kt} is the assumed solution. So, this term cannot be zero. Hence, second term can be made equal to zero.

$$k^2 + \frac{R}{L}k + \frac{1}{LC} = 0 \quad \rightarrow (9)$$

this is of the form, $ax^2 + bx + c = 0$, quadratic equation. → (10)

where, $x_1, x_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Comparing eq (9) and eq (10),

$$a = 1, \; b = \frac{R}{L}, \; c = \frac{1}{LC}$$

∴ Roots of eq (9), which is the required characteristic equation, are,

$$K_1, K_2 = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

Let $\alpha = \frac{R}{2L}$ and $\omega_0 = \frac{1}{LC}$

$$\boxed{\begin{aligned} \therefore\ K_1 &= -\alpha + \sqrt{\alpha^2 - \omega_0^2} \\ \text{and}\ K_2 &= -\alpha - \sqrt{\alpha^2 - \omega_0^2} \end{aligned}} \longrightarrow (11)$$

where ω_0 is known as resonant frequency or undamped natural frequency, in rad/sec.

α is known as damping factor

Now, eq (9) can be written in terms of α, ω_0 as,

$$k^2 + 2\alpha k + \omega_0^2 = 0$$

from eq (11), and eq (7),

eq (7) is $i = Ae^{kt}$.

eq (11) shows the values of K_1, K_2.

from eq (11) and eq (7), we can say that, eq (7) has two solutions, each with one value of K, i.e.,

$$i_1 = A_1 e^{k_1 t} \quad \text{and} \quad i_2 = A_2 e^{k_2 t} \longrightarrow (12)$$

Since eq (4) is a linear equation, any linear combination of the two distinct solutions i_1 and i_2 is also a solution of eq (4)

Thus, the natural response of series RLC circuit, is,

$$i(t) = A_1 e^{k_1 t} + A_2 e^{k_2 t} \longrightarrow (13)$$

A_1, A_2 constants can be determined from the initial conditions, ie $i(0)$ and initial value $\frac{di(0)}{dt}$.

- Now, from eq (11) i.e, roots of eq (9),

we can infer that there are three types of solutions.

i, If $\alpha > \omega_0$, we have overdamped system

$$\Rightarrow \left(\frac{R}{2L}\right)^2 > \left(\frac{1}{\sqrt{LC}}\right)^2$$

$\Rightarrow$ Roots k_1, k_2 are negative, real and distinct.

$$\therefore i(t) = A_1 e^{k_1 t} + A_2 e^{k_2 t}.$$

ii, If $\alpha = \omega_0$, we have ~~an~~ a critically damped system,

$$\Rightarrow \left(\frac{R}{2L}\right)^2 = \left(\frac{1}{\sqrt{LC}}\right)^2$$

$\Rightarrow$ Roots k_1, k_2 are real, negative and equal.

$$K_1 = K_2 = -\alpha = -\frac{R}{2L}$$

$$\therefore \cancel{i(t) = A_1 e^{-\alpha t} + A_2 e^{-\alpha t}}$$

$$\cancel{= (A_1 + A_2) e^{-\alpha t}}$$

$$\therefore i(t) = (A_2 + A_1 t) e^{-\alpha t}.$$

iii, If $\alpha < \omega_0$, we have an underdamped system.

$$\Rightarrow \left(\frac{R}{2L}\right)^2 < \left(\frac{1}{\sqrt{LC}}\right)^2$$

$\Rightarrow$ Roots K_1, K_2 are negative, complex and distinct.

$$\therefore \cancel{i(t) = e^{K_1 t}\left[C_1 \cos K_2 t + C_2 \sin K_2 t\right]}$$

$$i(t) = e^{-\alpha t}\left(B_1 \cos \omega_d t + B_2 \sin \omega_d t\right)$$

where ω_d - damping frequency,

$$\omega_d = \sqrt{\omega_0^2 - \alpha^2}$$

and $B_1 = A_1 + A_2$, $B_2 = j(A_1 - A_2)$.

Some important points:

1) - Damping, gives an idea about the behaviour of the network, which is the gradual loss of the initial stored energy.

- The damping effect is due to the presence of R, resistance.

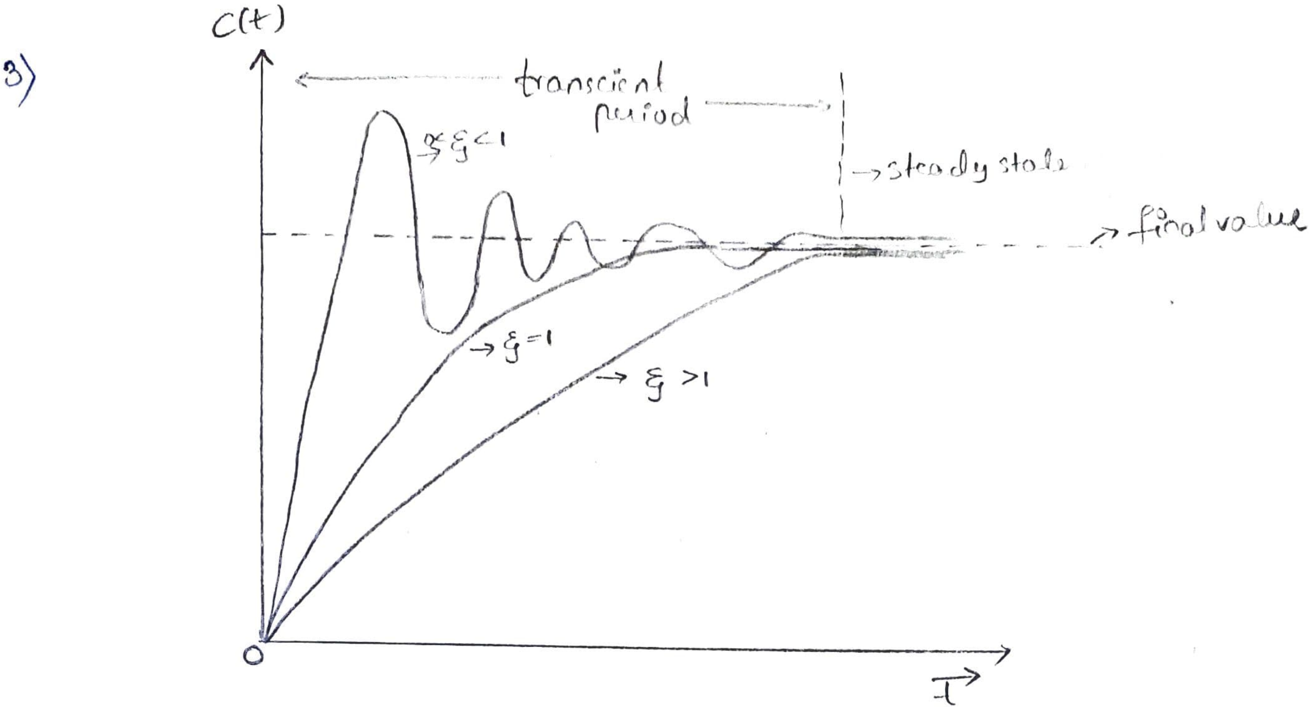
- The damping factor α determines the rate at which the response is damped.

- If $R = 0$ then $\alpha = \frac{R}{2L} = 0$, and we have an LC circuit with $\frac{1}{\sqrt{LC}}$ as the undamped natural frequency. This circuit is said to be lossless, because the dissipating (or damping element (R) is absent.

- Thus, by adjusting the value of R, the response may be made undamped, overdamped, critically damped (or underdamped.

2) Oscillatory response is possible due to the presence of the two types of storage elements. Having both L and C allows the flow of energy back and forth between the two.

3)



With same initial conditions, the overdamped has the longest settling time, because it takes longest time to dissipate the initial storedenergy.

If we desire fastest response without oscillation or ringing, the critically damped circuit is the right choice.

• Conclusion:

- If roots $K_1 K_2$ are

i, real, distinct and negative - overdamped system

ii, real, equal and negative - critically damped "

iii, complex, distinct, -ve realpart - underdamped "

iv) Imaginary roots. ~~complex~~ (~~purely~~), ie (jx) form - undamped "

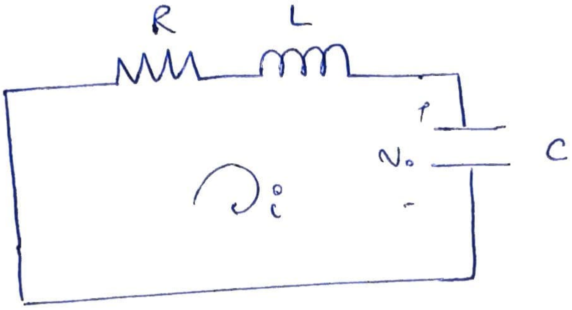
- $\alpha = \dfrac{R}{2L}$ Np/sec. (Nepers/sec).

- $\omega_0 = \dfrac{1}{\sqrt{LC}}$ rad/sec

- Roots of the characteristic equation,

$$K_1, K_2 = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$

Q1) If $R = 10\Omega$, $L = 5H$ and $C = 2mF$ in the below figure, find α, w_0, K_1 and K_2. What type of natural response will the circuit have?



Sol) Given, $R = 10\Omega$

$L = 5H$

$C = 2mF$

$\rightarrow \alpha = \dfrac{R}{2L} = \dfrac{10}{2(5)} = 1$

$\rightarrow w_0 = \dfrac{1}{\sqrt{LC}} = \dfrac{1}{\sqrt{5 \times 2m}} = \dfrac{1}{\sqrt{10 \times 10^{-3}}} = \dfrac{1}{10^{-2}} = 10^{2}$

$= 100$

$\rightarrow$ Roots are,

$K_1, K_2 = -\alpha \pm \sqrt{\alpha^2 - w_0^2} = -1 \pm \sqrt{1^2 - (100)^2}$

$= -1 \pm j9.95$

$\therefore K_1 = -1 + j9.95$, $K_2 = -1 - j9.95$

Since the roots are complex roots, the response of the system is underdamped.

Q2) Consider a series connection of R, L and C, with $R = 40\Omega$, $L = 4H$, $C = 1/4\ F$. Calculate the characteristic roots of the circuit. Is the natural response overdamped, underdamped (or) critically damped?

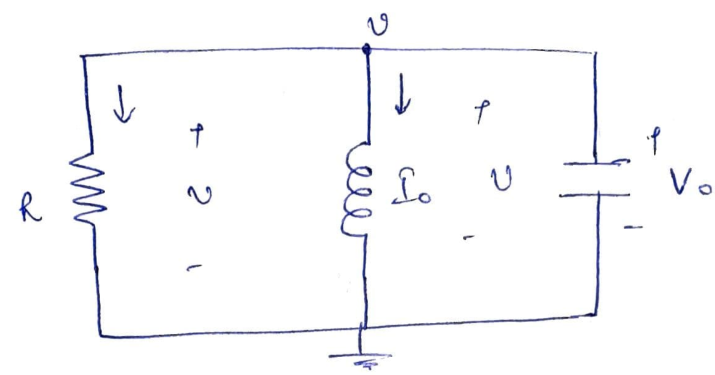
Ans: $S_1 = -0.101$, $S_2 = -9.899$

Hence the system is overdamped.

Source free parallel RLC circuit:

There are many practical applications of a parallel RLC circuit in communication networks and filter designs.

Consider a parallel RLC circuit as shown.



Assume, initial inductor current is I_o and initial capacitor voltage is V_o.

$$\therefore \quad i(0) = I_o = \frac{1}{L}\int_{\infty}^{0} v(t)\,dt \longrightarrow (1)$$

$$v(0) = V_o \longrightarrow (2)$$

- Since three elements are connected in parallel, they are having same voltage across them.

Assume that current through each element is leaving the top node.

$$\Rightarrow \frac{v}{R} + \frac{1}{L}\int_{-\infty}^{t} v\,dt + C\frac{dv}{dt} = 0 \longrightarrow (3)$$

Take differentiation on bothsides,

$$\Rightarrow \frac{1}{R}\frac{dv}{dt} + \frac{1}{L}v + C\frac{d^2v}{dt^2} = 0$$

$$\Rightarrow \frac{d^2v}{dt^2} + \frac{1}{RC}\frac{dv}{dt} + \frac{1}{LC}v = 0$$

$$\Rightarrow D^2v + \frac{1}{RC}Dv + \frac{1}{LC}v = 0$$

$$\Rightarrow \left(D^2 + \frac{1}{RC}D + \frac{1}{LC}\right)v = 0 \longrightarrow (4)$$

Now, the auxiliary equation is, or characteristic equation,

$$D^2 + \frac{1}{RC}D + \frac{1}{LC} = 0 \Rightarrow K^2 + \frac{1}{RC}K + \frac{1}{LC} = 0 \rightarrow (5)$$

Let the Roots be K_1, K_2.

$$\text{Roots of eq (5)} = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$= -\frac{\frac{1}{RC} \pm \sqrt{\left(\frac{1}{RC}\right)^2 - 4(1)\left(\frac{1}{LC}\right)}}{2(1)}$$

$$= -\frac{1}{2RC} \pm \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{4}{4(LC)}}$$

$$K_1, K_2 = -\frac{1}{2RC} \pm \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}$$

Let $\alpha = \frac{1}{2RC}$, $w_0 = \frac{1}{LC}$

$$\boxed{K_1, K_2 = -\alpha \pm \sqrt{\alpha^2 - w_0^2}}$$

where α is the damping factor (Np/sec)

w_0 is the resonant frequency (rad/sec)

Again, we get three possible solutions, depending upon on whether $\alpha > w_0$, $\alpha = w_0$ or $\alpha < w_0$.

i, If $\alpha > w_0$,

- Roots of the auxilary equation (or characteristic equation are real, distinct and negative
- The system is overdamped.

$$\therefore v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}.$$

$$\boxed{v(t) = A_1 e^{K_1 t} + A_2 e^{K_2 t}}$$

ii, If $\alpha = w_0$,

- Roots are real, equal and negative.
- the system is critically damped.

$$\therefore \boxed{v(t) = (A_1 + A_2 t) e^{-\alpha t}}$$

iii If $\alpha < \omega_0$,

- Roots are complex conjugates and have -ve real parts

- The system is underdamped.

$$\boxed{v(t) = e^{-\alpha t}(A_1 \cos\omega_d t + A_2 \sin\omega_d t)}$$

where $K_1, K_2 = -\alpha \pm j\omega_d$, $\omega_d = \sqrt{\omega_0^2 - \alpha^2}$.

→ The constants in each case, A_1 and A_2 must be determined using initial conditions.

For this, we need, $v_c(0)$ and $\dfrac{dv_c(0)}{dt}$.

$v_c(0)$ can be obtained from eq ②,

To determine the term $\dfrac{dv_c(0)}{dt}$, substitute eq ① and ② in eq ③, we get,

from eq ③, $\dfrac{v(t)}{R} + \dfrac{1}{L}\int v(t)\,dt + C\dfrac{dv(t)}{dt} = 0$

$$\Rightarrow \frac{V_0}{R} + I_0 + C\frac{dv(0)}{dt} = 0$$

$$\Rightarrow \boxed{\frac{dv(0)}{dt} = -\frac{(V_0 + RI_0)}{RC}}$$

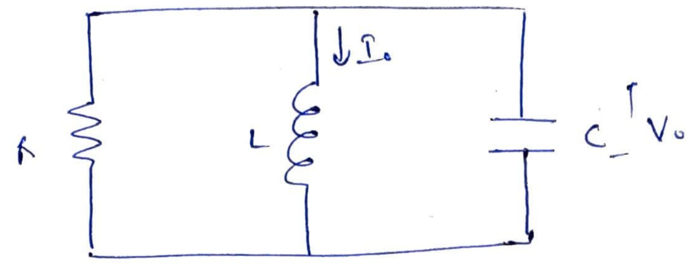
Note:

In RLC series circuit, we first find or calculate inductor current $i_L(t)$, whereas for parallel RLC circuit, we calculate capacitor voltage $v_C(t)$.

Q1) In the below parallel RLC circuit, find v(t) for $t>0$, assuming

$v_c(0) = 5V$, $i_L(0) = 0A$, $L = 1H$, $C = 10mF$.

Consider three cases : $R = 1.923\,\Omega$, $R = 5\,\Omega$, $R = 6.25\,\Omega$



Sol) Given, $v_c(0) = 5V$

$i_L(0) = 0A$

$L = 1H$

$C = 10mF$

{ formulae :

$\alpha = \frac{1}{2RC}$

$\omega_0 = \frac{1}{\sqrt{LC}}$

$\alpha > \omega_0 \rightarrow v(t) = A_1 e$

Case (i): If $R = 1.923\,\Omega$

$$\alpha = \frac{1}{2RC}$$

$$= \frac{1}{2 \times 1.923 \times 10 \times 10^{-3}}$$

$$= 26$$

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{1 \times 10 \times 10^{-3}}} = 10$$

Since $\alpha > \omega_0 \Rightarrow$ overdamped system.

$\therefore$ Roots of chr eq are $K_1, K_2 = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$

$$= -26 \pm \sqrt{26^2 - 10^2}$$

$$= -2, -50.$$

Response of an overdamped system is,

$$v_c(t) = A_1 e^{K_1 t} + A_2 e^{K_2 t}$$

$$= A_1 e^{-2t} + A_2 e^{-50t}. \longrightarrow (1)$$

Inorder to find A_1, A_2, we use initial conditions.

$$v_c(0) = A_1 e^0 + A_2 e^0 = 5$$

$$\Rightarrow A_1 + A_2 = 5 \longrightarrow (2)$$

and $$\frac{dv_c(0)}{dt} = -\frac{V_C(0) + R i_L(0)}{RC}$$

$$= -\frac{5 + R\ 1.923(0)}{1.923 \times 10m}$$

$$= 260$$

Differentiating eq (1),

$$\frac{dv_c(t)}{dt} = -2A_1 e^{-2t} - 50 A_2 e^{-50t} = 260$$

Now,

$$\frac{dV_C(0)}{dt} = -2A_1 e^0 - 50 A_2 e^0 = 260$$

$$\Rightarrow -2A_1 - 50 A_2 = 260 \longrightarrow (3)$$

Solving (2), (3) $\Rightarrow A_1 = 10.625$, $B = -5.625$

$\therefore$ from eq ①, the response of the given circuit is,

$$v_c(t) = 10.625e^{-2t} - 5.625e^{-50t}$$

<u>Condition</u>: $R = 5\Omega$

$$\alpha = \frac{1}{2RC} = \frac{1}{2\times5\times10\times10^{-3}} = 10$$

$$\omega_0 = \frac{1}{\sqrt{LC}} = 10$$

here, $\alpha = \omega_0 \Rightarrow$ system is critically damped ~~cos~~ we get a critically damped response.

$$k_1, k_2 = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$

$$= -\alpha \pm 0 \qquad (\because \alpha = \omega_0)$$

$$= -\alpha$$

$$= -10$$

$$\therefore v_c(t) = (A_1 + A_2t)\,\cancel{e^{-\alpha t}}\, e^{k_1t}$$

$$= (A_1 + A_2t)e^{-\alpha t}$$

$$= (A_1 + A_2t)e^{-10t} \longrightarrow ④$$

To find out A_1, A_2, we go for initial condition.

$$v_c(0) = (A_1 + A_2(0))\,e^{-10(0)} = 5$$

$$\Rightarrow A_1 = 5$$

and $$\frac{dv_c(0)}{dt} = -\frac{(v_c(0) + Ri_L(0))}{RC}$$

$$= -\frac{5+0}{5 \times 10 \times 10^{-3}}$$

$$= 100$$

Differentiating eq ④,

$$\cancel{\frac{dv_c(t)}{dt} = -10A_1e^{-10t}}.$$

$$\frac{dv_c(t)}{dt} = (A_1 + A_2t)(-10e^{-10t}) + e^{-10t}(A_2)$$

Now, $$\frac{dv_c(0)}{dt} = (A_1 + A_2(0))(-10e^{0}) + e^{-10(0)}(A_2)$$

$$\Rightarrow 100 = -10A_1 + A_2$$

$$\Rightarrow 100 = -10(5) + A_2$$

$$\Rightarrow 100 + 50 = A_2$$

$$\Rightarrow A_2 = 150$$

Sub A_1, A_2 in eq ④

$$v_c(t) = (5 + 150t)e^{-10t} \text{ V}.$$

Case (iii) : $R = 6.25\,\Omega$

$$\alpha = \frac{1}{2RC} = \frac{1}{2 \times 6.25 \times 10m}$$

$$= 8$$

$$\omega_0 = \frac{1}{\sqrt{LC}} = 10$$

here, $\alpha < \omega_0 \Rightarrow$ response is underdamped.

$\therefore$ Roots of the characteristic equation are,

$$K_1, K_2 = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$

$$= -8 \pm \sqrt{8^2 - 10^2}$$

$$= -8 \pm \sqrt{64 - 100}$$

$$= -8 \pm j6$$

$$\therefore v_c(t) = (A_1 \cos \omega_d t + A_2 \sin \omega_d t) e^{-\alpha t}$$

$$\omega_d = \sqrt{\omega_0^2 - \alpha^2} = \sqrt{10^2 - 8^2} = \sqrt{100 - 64} = \sqrt{36} = 6$$

$$\therefore v_c(t) = (A_1 \cos 6t + A_2 \sin 6t) e^{-8t} \rightarrow (5)$$

for A_1, A_2:

$$v_c(0) = 5 = (A_1 \cos 0 + A_2 \sin 0) e^0$$

$$= (A_1 + 0) 1$$

$$\Rightarrow A_1 = 5$$

$$\frac{dv_c(0)}{dt} = \frac{v_c(0) + R\, i_L(0)}{RC}$$

$$= \frac{5+0}{6.25 \times 10m}$$

$$= 80 \text{ V/sec}$$

Differentiate eq (5),

$$\frac{dv_c(t)}{dt} = e^{-8t}(-6A_1 \sin 6t + 6A_2 \cos 6t)$$

$$+ (A_1 \cos 6t + A_2 \sin 6t)e^{-8t} \times -8$$

$$= (-6A_1 \sin 6t + 6A_2 \cos 6t)e^{-8t}$$

$$- 8(A_1 \cos 6t + A_2 \sin 6t)e^{-8t}$$

Now,

$$\frac{dv_c(0)}{dt} = (-6A_1(0) + 6A_2(1))e^0$$

$$- 8(A_1(1) + A_2 (0))e^0$$

$$\Rightarrow 6A_2 - 8A_1 = 80$$

$$\Rightarrow 6A_2 - 8(5) = 80$$

$$\Rightarrow 6A_2 = 80 + 40 = 120$$

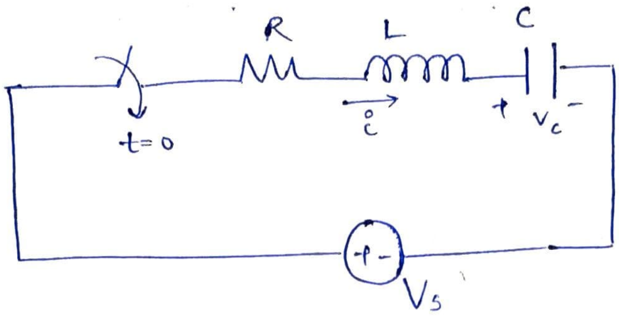
$$\Rightarrow A_2 = 20$$

$\therefore$ from (5), $v_c(t) = (5\cos 6t + 20 \sin 6t)e^{-8t}$.

* [By increasing R, degree of damping decreases & thereby response differs.]

Step response of series RLC circuit:

- Step response is obtained by sudden application of a dc source
- Consider a series RLC circuit as shown.



Apply KVL around the loop at $t>0$,

$$V_s = iR + L\frac{di}{dt} + \frac{1}{C}\int i\,dt \longrightarrow (1)$$

$$V_s = iR + L\frac{di}{dt} + v_c \longrightarrow (2)$$

Taking differentiation ~~of~~ for the ~~above~~ equation

$$\Rightarrow \cancel{0 = R\frac{di}{}}$$

Put, $i = C\frac{dv_c}{dt} \rightarrow (3)$

$$\Rightarrow V_s = RC\frac{dv_c}{dt} + LC\frac{d^2v_c}{dt} + v_c$$

$$\Rightarrow LC\frac{d^2v_c}{dt} + RC\frac{dv_c}{dt} + v_c = V_s$$

$$\Rightarrow \frac{d^2v_c}{dt} + \frac{R}{L}\frac{dv_c}{dt} + \frac{v_c}{LC} = \frac{V_s}{LC} \longrightarrow (4)$$

This equation (4) is of the same as form as in the case of source free RLC circuit. The auxiliary equation is same for both cases. Only the variable is different.

Hence the characteristic equation is not effected due to the presence of the dc source.

- The solution for eq. (4) has two components:

i, Natural response, $v_n(t)$

ii, forced response. , $v_f(t)$

$$\therefore v_c(t) = v_n(t) + v_f(t)$$

- The natural response is the solution obtained when $V_s = 0$.

So, It is same as that obtained in the case of series RLC source free network.

The natural response for different damping conditions is,

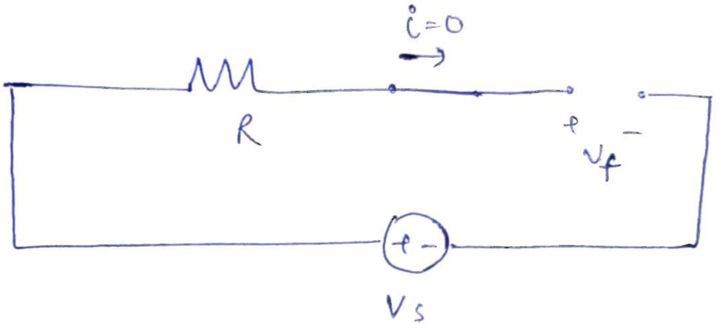
$$v_n(t) = A_1 e^{k_1 t} + A_2 e^{k_2 t} \quad \text{(Overdamped)}$$

$$v_n(t) = (A_1 + A_2 t) e^{-\alpha t} \quad \text{(Critically damped)}$$

$$v_n(t) = (A_1 \cos \omega_d t + A_2 \sin \omega_d t) e^{-\alpha t} \quad \text{(Underdamped)}$$

- The forced response is the steady state (or) final value of $v(t)$.

from the given circuit, as $t \to \infty$, capacitor and inductor reach their final steady states, i.e., capacitor becomes open circuited & inductor becomes short circuited. Hence the circuit will be as shown.



$$\therefore \; v_f(t) = v_c(\infty) = V_s.$$

Thus, the complete solution for different damping systems, is, systems with

$$v_c(t) = V_s + A_1 e^{s_1 t} + A_2 e^{s_2 t} \quad \text{(Overdamped)}$$

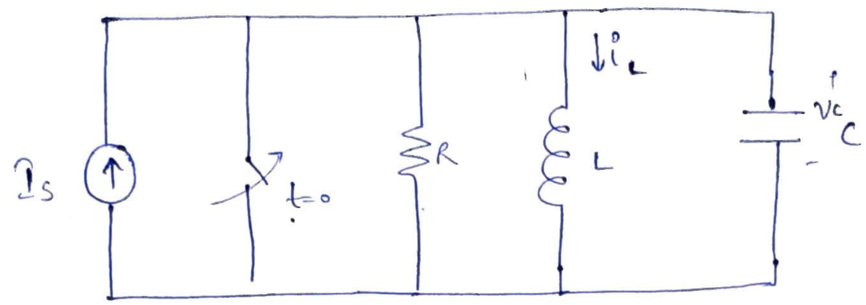
$$v_c(t) = V_s + (A_1 + A_2 t) e^{-\alpha t} \quad \text{(Critically damped)}$$

$$v_c(t) = V_s + (A_1 \cos \omega_d t + A_2 \sin \omega_d t) e^{-\alpha t} \quad \text{(Underdamped)}$$

The value of constants A_1, A_2 are obtained from initial conditions, $v_c(0)$ and $\frac{dv_c(0)}{dt}$.

Step response of a parallel RLC circuit:

Consider a parallel RLC circuit as shown.



The current through inductor (i_L) due to sudden application of a dc source is to be found out. Apply ~~KVL~~ KCL, for $t>0$, at the top node.

$$\frac{v_c}{R} + i_L + C\frac{dv_c}{dt} = I_s \longrightarrow (1)$$

Put $$v_c = L\frac{di_L}{dt} \longrightarrow (2)$$

$$\Rightarrow \frac{L}{R}\frac{di_L}{dt} + i_L + LC\frac{d^2 i_L}{dt} = I_s$$

$$\Rightarrow LC\frac{d^2 i_L}{dt} + \frac{L}{R}\frac{di_L}{dt} + i_L = I_s$$

$$\Rightarrow \frac{d^2 i_L}{dt} + \frac{1}{RC}\frac{di_L}{dt} + \frac{1}{LC}i_L = \frac{I_s}{LC} \longrightarrow (3)$$

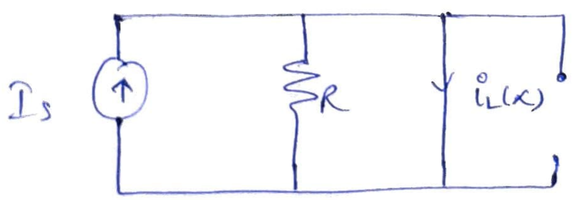
The characteristic equation of eq ③ is same as that in the case of the source free parallel RLC circuit.

Hence, the solution for this auxiliary equation will give the natural response of the given network & that response is same as that obtained in the source free RLC parallel ckt.

- the complete solution of eq③ consists of natural response $i_n(t)$ and forced response $i_f(t)$.

$$\therefore \; i_L(t) = i_n(t) + i_f(t)$$

- To find, $i_f(t)$, we have to find the circuit behaviour as $t \to \infty$, The circuit, at $t \to \infty$, will be as shown.



Inductor is replaced by short circuit and capacitor is replaced by open circuit.

From the above circuit, it is evident that,

$$i_L(\infty) = i_f(t) = I_s. \quad \text{ie source current}$$

$\therefore$ Complete solution of eq ③ is,

$$i_L(t) = I_s + A_1 e^{k_1 t} + A_2 e^{k_2 t} \quad (\text{Overdamped})$$

$$i_L(t) = I_s + (A_1 + A_2 t) e^{-\alpha t} \quad (\text{Critically damped})$$

$$i_L(t) = I_s + (A_1 \cos \omega_d t + A_2 \sin \omega_d t) e^{-\alpha t} \quad (\text{Underdamped}).$$

- The constants A_1, A_2 can be determined from the initial conditions for i_L and $\frac{di_L(0)}{dt}$.

General second order circuits & its solution:

Given a second order circuit, we determine its step response $x(t)$ (which may be voltage (or) current) by taking the following four steps:

(1) We first determine the initial conditions $x(0)$ and $\frac{dx(0)}{dt}$ and the final value $x(\infty)$.

(2) We find the natural response $x_n(t)$ by turning off independent sources and applying KVL and KCL.

Once a second order differential equation is obtained, we determine its characteristic roots. Depending on whether the response is overdamped, or critically damped (or) underdamped, we obtain the solution $x_n(t)$.

(3) We obtain the forced response as,

$$x_f(t) = x(\infty)$$

$x(\infty)$ is the final value, obtained in step 1.

(4) The total response is the sum of natural and forced response

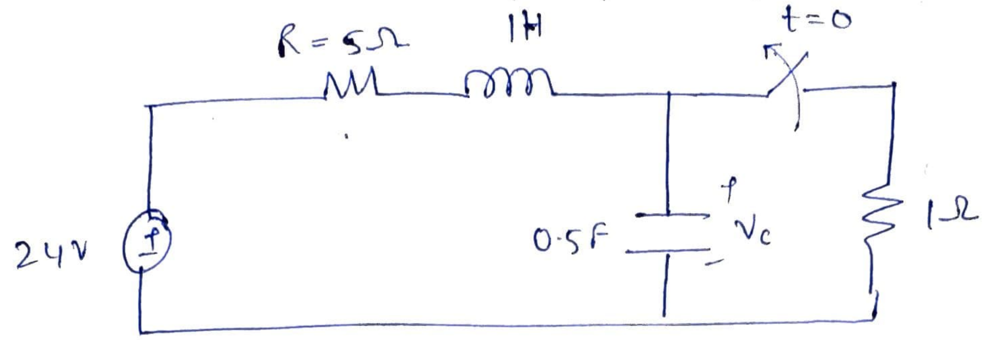
$$x(t) = x_n(t) + x_f(t)$$

Finally, inorder to determine the constants associated with natural response, we make use of the initial conditions

$x(0)$ and $\frac{dx(0)}{dt}$, determined in step 1.

- this general procedure can be used to find out the step response of any second order circuit.

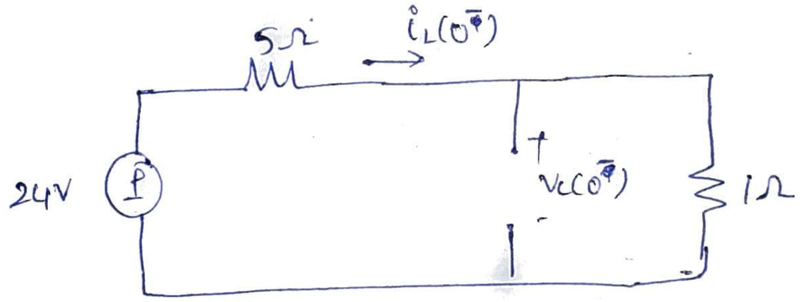
Q1) Find $v_c(t)$ and $i_L(t)$ for $t>0$, in the below circuit:



Sol) for $t=0^-$, switch has been closed from $-\infty$

So, the capacitor and inductor will reach their steady states

At $t=0^-$, circuit will be as shown.

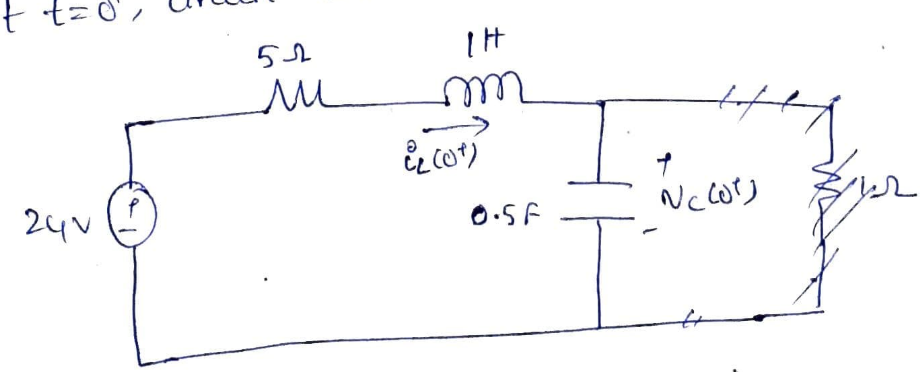


$$i_L(0^-) = \frac{24}{5+1} = \frac{24}{6} = 4A$$

$v_c(0^-)$ = drop across 1Ω = $i \times R = 4 \times 1 = 4V$

At $t=0$, switch is opened.

At $t=0^+$, circuit will be as shown.



We know that, $i_L(0^+) = i_L(0^-) = 4A$

and $v_C(0^+) = v_C(0^-) = 4V$

Now, the circuit is a series RLC circuit with a voltage source.

$$\alpha = \frac{R}{2L} = \frac{5}{2\times 1} = 2.5$$

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{1\times 0.25}} = 2$$

$$K_1, K_2 = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$

$$= -2.5 \pm \sqrt{2.5^2 - 2^2}$$

$$= -2.5 \pm 1.5$$

$$= -2.5+1.5, \quad -2.5-1.5$$

$$= -1, \quad -4.$$

$\therefore K_1, K_2 = -1, -4.$

$\therefore$ We have an overdamped response, because the roots are real and distinct

$$\therefore v(t) = V_f + (A_1 e^{-t} + A_2 e^{-4t})$$

where v_f = final voltage across a capacitor ie $v_C(\infty)$

$= 24V.$

$\therefore v_c(t) = 24 + A_1 e^{-t} + A_2 e^{-4t} \longrightarrow (1)$

To find A_1, A_2, we need initial conditions,

$v_c(0) = v_c(0^+) = 4 \text{ V}$

$\left\{ \begin{array}{l} \text{Put } t=0 \\ \text{in eq } (1) \end{array} \right\} \Rightarrow 24 + A_1 e^{-0} + A_2 e^{-0} = 4$

$\Rightarrow 24 + A_1 + A_2 = 4$

$\Rightarrow A_1 + A_2 = -20 \longrightarrow (2)$

Now, $\dfrac{dv_c(0)}{dt} = \dfrac{i_c(0)}{C} = \dfrac{i_L(0)}{C}$ ($\because$ R, L, C are in series)

$= \dfrac{i_L(0^+)}{C}$

$= \dfrac{4}{0.25}$

$= 16 \text{ v/sec}$

From eq (1), $\dfrac{dv_c(t)}{dt} = -A_1 e^{-t} - 4A_2 e^{-4t}$

Now, $\dfrac{dv_c(0)}{dt} = -A_1 e^{-0} - 4A_2 e^{-0}$

$\Rightarrow 16 = -A_1 - 4A_2 \longrightarrow (3)$

Solving eq (2), (3) for A_1, A_2

$\{$ add (2) and (3) $\Rightarrow A_1 + A_2 - A_1 - 4A_2 = -20 + 16$

$$-3A_2 = -4 \Rightarrow \boxed{A_2 = \frac{4}{3}}$$

Now, from eq (2), $A_1 + A_2 = -20$

$$\Rightarrow A_1 = -20 - A_2$$

$$= -20 - \frac{4}{3}$$

$$\boxed{A_1 = -\frac{64}{3}}$$

Sub A_1, A_2 in eq (1)

$$\therefore \boxed{v_c(t) = 24 + \frac{4}{3}\left(-16e^{-t} + e^{-4t}\right) V}$$

- Now, $i_L(t)$ is the current flowing through L, C as they are in parallel. Since, we know the voltage across capacitor & $i_L(t)$ is same as $i_C(t)$ as they are in parallel,

$$i_L(t) = i_C(t) = C\frac{dv_C(t)}{dt}$$

$$= C\frac{d}{dt}\left[24 + \frac{4}{3}\left(-16e^{-t} + e^{-4t}\right)\right]$$

$$= 0.25\left(-\frac{4}{3}\times 16\left(-e^{-t}\right) + \frac{4}{3}\left(-4e^{-4t}\right)\right)$$

$$= +\frac{16}{3}e^{-t} - \frac{4}{3}e^{-4t}$$

$$\therefore \boxed{i_L(t) = \frac{4}{3}\left(4e^{-t} - e^{-4t}\right)} \text{ A}$$

- Solve the same problem, for $R = 4\,\Omega$ & $R = 1\,\Omega$

answers:

$R = 4\,\Omega \Rightarrow$ $v_c(t) = 24 + (-19.5 + 57t)e^{-2t}$ V

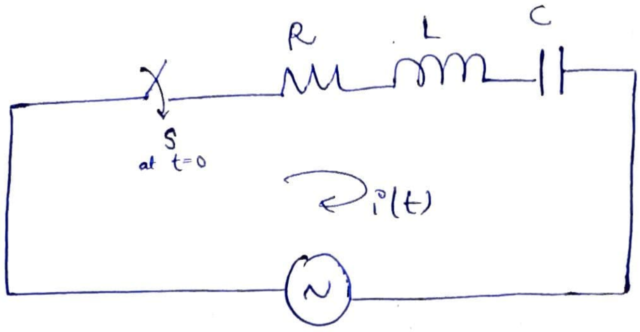
$i_L(t) = (4.5 - 28.5t)e^{-2t}$ A

$R = 1\,\Omega \Rightarrow$ $v_c(t) = 24 + (21.694 \sin 1.936t - 12 \cos 1.936t)e^{-0.5t}$ V

$i_L(t) = (3.1 \sin 1.936t + 12 \cos 1.936t)e^{-0.5t}$ A

Sinusoidal response of R-L-C circuit:

Consider a circuit consisting of resistance, inductance and capacitance in series as shown.



$V\cos(\omega t+\theta)$

- Switch S is closed at $t=0$
- At $t=0$, a sinusoidal voltage $V\cos(\omega t+\theta)$ is applied to the series RLC circuit, where,

V is the amplitude of the wave
θ is the phase angle of the wave.

Now, apply KVL in the loop, after switch S is closed.

Let $i(t)$ be the current flowing through the loop, then,

$$V\cos(\omega t+\theta) = Ri + L\frac{di}{dt} + \frac{1}{C}\int i\,dt \quad \longrightarrow (1)$$

Differentiating the above equation,

$$-\omega V\sin(\omega t+\theta) = R\frac{di}{dt} + L\frac{d^2 i}{dt^2} + \frac{1}{C}i.$$

$$\Rightarrow -V\omega\sin(\omega t+\theta) = L\frac{d^2 i}{dt^2} + R\frac{di}{dt} + \frac{i}{C}$$

$$\Rightarrow \frac{d^2 i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{1}{LC} i = -\frac{V\omega\sin(\omega t+\theta)}{L}$$

$$\Rightarrow \left(D^2 + \frac{R}{L}D + \frac{1}{LC}\right) i = -\frac{V\omega\sin(\omega t+\theta)}{L} \longrightarrow (2)$$

The solution for above differential equation consists of two parts, i) complementary function & ii) particular integral.

- The characteristic equation of eq (2) is same as that obtained in step response or DC response of series RLC circuit. So, complementary function also will be the same.

- The particular solution can be obtained by using undetermined co-efficients.

By assuming,

$$i_p = A\cos(\omega t+\theta) + B\sin(\omega t+\theta) \longrightarrow (3)$$

$$i_p' = -A\omega\sin(\omega t+\theta) + B\omega\cos(\omega t+\theta) \quad \left\{\text{ie } \frac{di_p}{dt}\right\} \rightarrow (4)$$

$$i_p'' = -A\omega^2\cos(\omega t+\theta) - B\omega^2\sin(\omega t+\theta) \quad \left\{\text{ie } \frac{d^2 i_p}{dt^2}\right\} \rightarrow (5)$$

Substituting eq ③, ④, ⑤ in eq ②, we have,

$$\left\{-A\omega^2\cos(\omega t+\theta)-B\omega^2\sin(\omega t+\theta)\right\}+\frac{R}{L}\left\{-A\omega\sin(\omega t+\theta)+B\omega\cos(\omega t+\theta)\right\}$$

$$+\frac{1}{LC}\left\{A\cos(\omega t+\theta)+B\sin(\omega t+\theta)\right\}=-\frac{V\omega}{L}\sin(\omega t+\theta) \rightarrow ⑥$$

Comparing $\sin(\omega t+\theta)$ terms and $\cos(\omega t+\theta)$ terms,

on bothsides,

sine:

$$-B\omega^2-A\frac{\omega R}{L}+\frac{B}{LC}=-\frac{V\omega}{L} \Rightarrow A\left(\frac{\omega R}{L}\right)+B\left(\omega^2-\frac{1}{LC}\right)=\frac{V\omega}{L}$$

~~cosine~~:

$$\Rightarrow A\left(\frac{\omega R}{L}\right)+B\left(\omega^2-\frac{1}{LC}\right)=\frac{V\omega}{L} \rightarrow ⑦$$

cosine:

$$-A\omega^2+B\frac{\omega R}{L}+\frac{A}{LC}=0$$

$$\Rightarrow A\left(\frac{1}{LC}-\omega^2\right)+B\left(\frac{\omega R}{L}\right)=0 \rightarrow ⑧$$

Solving eq ⑦, ⑧, we get the values of A and B,

$$A=\frac{V\times\frac{\omega^2 R}{L^2}}{\left[\left(\frac{\omega R}{L}\right)^2+\left(\omega^2-\frac{1}{LC}\right)^2\right]} \rightarrow ⑨$$

$$B = \frac{\left(\omega^2 - \frac{1}{LC}\right) V\omega}{L\left[\left(\frac{\omega R}{L}\right)^2 + \left(\omega^2 - \frac{1}{LC}\right)^2\right]} \rightarrow (10)$$

Substitute the values of A and B in eq (3) ie i_p,

$$i_p = \frac{V\frac{\omega^2 R}{L^2}}{\left[\left(\frac{\omega R}{L}\right)^2 + \left(\omega^2 - \frac{1}{LC}\right)^2\right]} \cos(\omega t + \theta)$$

$$+ \frac{\left(\omega^2 - \frac{1}{LC}\right) V\omega}{L\left[\left(\frac{\omega R}{L}\right)^2 + \left(\omega^2 - \frac{1}{LC}\right)^2\right]} \sin(\omega t + \theta) \rightarrow (11)$$

Putting $M\cos\phi = \dfrac{V\frac{\omega^2 R}{L^2}}{\left(\frac{\omega R}{L}\right)^2 + \left(\omega^2 - \frac{1}{LC}\right)^2} \rightarrow (12)$

and $M\sin\phi = \dfrac{V\left(\omega^2 - \frac{1}{LC}\right)\omega}{L\left[\left(\frac{\omega R}{L}\right)^2 + \left(\omega^2 - \frac{1}{LC}\right)^2\right]} \rightarrow (13)$

Now, from (12), (13), solve for M, ϕ.

$\phi \rightarrow$ divide eq (13) by eq (12)

$$\frac{M\sin\phi}{M\cos\phi} = \tan\phi = \frac{\left(\omega L - \frac{1}{\omega C}\right)}{R}$$

$$\phi = \tan^{-1}\left(\frac{\omega L - \frac{1}{\omega C}}{R}\right) \longrightarrow (14)$$

Squaring eq (12), eq (13) individually and adding, we get,

$M^2\cos^2\phi + M^2\sin^2\phi$

$$= \left\{\frac{V\,\frac{\omega^2 R}{L^2}}{\left(\frac{\omega R}{L}\right)^2 + \left(\omega^2 - \frac{1}{LC}\right)^2}\right\}^2 + \left\{\frac{V\left(\omega^2 - \frac{1}{LC}\right)\omega}{L\left[\left(\frac{\omega R}{L}\right)^2 + \left(\omega^2 - \frac{1}{LC}\right)^2\right]}\right\}^2$$

$$= \left\{\frac{V\,\frac{\omega^2 R}{L^2}}{\left(\frac{\omega}{L}\right)^2 R^2 + \left(\frac{1}{LC} - \omega^2\right)^2}\right\}^2 + \left\{\frac{V\left(\frac{\omega}{L}\right)\cdot\left(\frac{\omega}{L}\left(\omega L - \frac{1}{\omega C}\right)\right)}{\left[\left(\frac{\omega}{L}\right)^2 R^2 + \left(\frac{1}{LC} - \omega^2\right)^2\right]}\right\}^2$$

$$= \frac{\left\{V\,\frac{\omega^2 R}{L^2}\right\}^2}{\left\{\left(\frac{\omega}{L}\right)^2\left[R^2 + \left(\frac{1}{\omega C} - \omega L\right)^2\right]\right\}^2} + \frac{\left\{V\left(\frac{\omega}{L}\right)^2\left(\omega L - \frac{1}{\omega C}\right)\right\}^2}{\left\{\left(\frac{\omega}{L}\right)^2\left[R^2 + \left(\frac{1}{\omega C} - \omega L\right)^2\right]\right\}^2}$$

$$\Rightarrow \frac{V^2 R^2}{\left\{R^2 + \left(\frac{1}{\omega C} - \omega L\right)^2\right\}^2} + \frac{V^2\left(\omega L - \frac{1}{\omega C}\right)^2}{\left\{R^2 + \left(\frac{1}{\omega C} - \omega L\right)^2\right\}^2} = M^2 (1)$$

$$\Rightarrow \frac{V^2\left[R^2 + \left(\frac{1}{\omega C} - \omega L\right)^2\right]}{\left\{R^2 + \left(\frac{1}{\omega C} - \omega L\right)^2\right\}^2} = M^2$$

$$\Rightarrow \frac{V^2}{R^2 + \left(\frac{1}{\omega C} - \omega L\right)^2} = M^2$$

$$\Rightarrow M = \frac{V}{\sqrt{R^2 + \left(\frac{1}{\omega C} - \omega L\right)^2}} \longrightarrow \cancel{(18)}\ (15)$$

∴ The particular current becomes,

$$i_p = \frac{V}{\sqrt{R^2 + \left(\frac{1}{\omega C} - \omega L\right)^2}} \cos\left(\omega t + \theta + \tan^{-1}\frac{\left(\frac{1}{\omega C} - \omega L\right)}{R}\right).$$

Since the complementary function is similar to that of DC series RLC circuit, from the characteristic equation we have the solution as follows.

Characteristic equation, $D^2 + \frac{R}{L} D + \frac{1}{LC} = 0$

Roots of the characteristic equation are,

$$K_1, K_2 = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

$$\cancel{= -\alpha \pm}$$

$$= -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$

∴ i, For overdamped systems (ie $\alpha > \omega_0$), the complete response is,

$$i(t) = i_c + i_p$$

$$= A_1 e^{K_1 t} + A_2 e^{K_2 t} + \frac{V}{\sqrt{R^2 + \left(\frac{1}{\omega C} - \omega L\right)^2}} \cos\left(\omega t + \theta + \tan^{-1}\left(\frac{1}{\omega CR} - \frac{\omega L}{R}\right)\right)$$

ii, for critically damped systems (ie $\alpha = \omega_0$), the complete response is,

$$i(t) = i_c + i_p$$

$$= (A_1 + A_2 t)\, e^{-\alpha t} + \frac{V}{\sqrt{R^2 + \left(\frac{1}{\omega C} - \omega L\right)^2}} \cos\left(\omega t + \theta + \tan^{-1}\left(\frac{1}{\omega CR} - \frac{\omega L}{R}\right)\right)$$

iii, For underdamped systems, i.e ($\alpha < \omega_0$), we have the complete solution as,

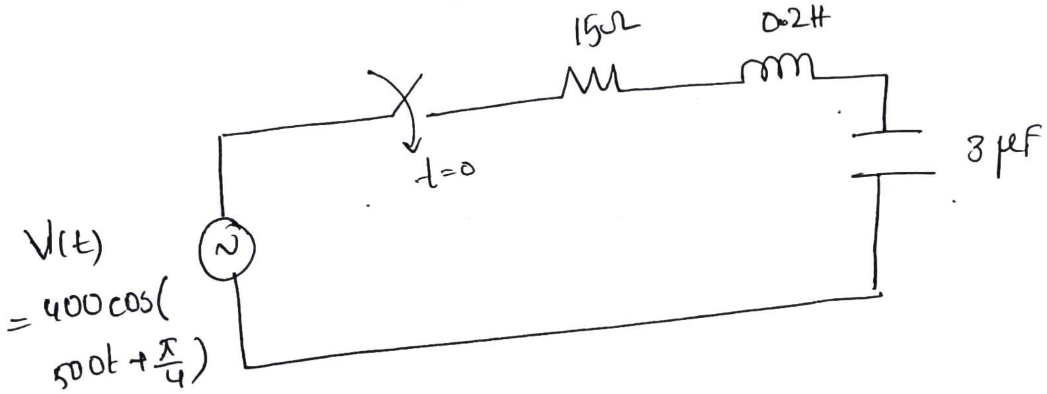
$$i(t) = i_c + i_p$$

$$= \left((A_1 \cos\omega_d t) + A_2 \sin\omega_d t \right) e^{-\alpha t}$$

$$+ \frac{V}{\sqrt{R^2 + \left(\frac{1}{\omega C} - \omega L\right)^2}} \cos\left(\omega t + \theta + \tan^{-1}\left(\frac{1}{\omega C R} - \frac{\omega L}{R}\right)\right).$$

where $\omega_d = \sqrt{\omega_0^2 - \alpha^2}$

Q1) In the circuit shown, determine the complete solution for the current, when the switch is closed at $t=0$. Applied voltage is $v(t) = 400\cos(500t + \frac{\pi}{4})$. Resistance $R = 15\,\Omega$, Inductance $L = 0.2H$ and capacitance $C = 3\,\mu F$.



Sol) At $t>0$, apply KVL in the loop, $R\,i(t) + L\frac{di(t)}{dt} + \frac{1}{C}\int i(t)\,dt = v(t)$

$$15\,i(t) + 0.2\frac{di(t)}{dt} + \frac{1}{3\times10^{-6}}\int i(t)\,dt = 400\cos(500t + \frac{\pi}{4})$$

$\Rightarrow$ Differentiate the above equation,

$$15\frac{di(t)}{dt} + 0.2\frac{d^2 i(t)}{dt^2} + \frac{i(t)}{3\times10^{-6}} = \frac{d}{dt}\left(400\cos(500t + \frac{\pi}{4})\right)$$

$$\Rightarrow \frac{15}{0.2}\frac{di(t)}{dt} + \frac{d^2 i(t)}{dt^2} + \frac{1}{0.2\times3\times10^{-6}}\,i(t) = \frac{-400\times500\sin(500t+\frac{\pi}{4})}{0.2}$$

$$\Rightarrow 75\frac{di(t)}{dt} + \frac{d^2 i(t)}{dt^2} + 16.7\times10^5\,i(t) = -10^6\sin(500t + \frac{\pi}{4})$$

$$\Rightarrow (D^2 + 75D + 16.7\times10^5)\,i(t) = -10^6\sin(500t + \frac{\pi}{4}).$$

Roots of the characteristic equation are,

$$D_1, D_2 = \frac{-75 \pm \sqrt{75^2 - 4(1)(16.5\times10^5)}}{2(1)}$$

$$\Rightarrow K_1, K_2 = -37.5 \pm j\,1290 \qquad = -\alpha \pm j\omega_d.$$

- The complementary solution is,

$$i_c = e^{-\alpha t}(A_1 \cos\omega_d t + A_2 \sin\omega_d t)$$

$$= e^{-37.5}(A_1 \cos 1290t + A_2 \sin 1290t)$$

- Particular solution is,

$$i_p = \frac{V}{\sqrt{R^2 + \left(\frac{1}{\omega C} - \omega L\right)^2}} \cos\left[\omega t + \theta + \tan^{-1}\left(\frac{1}{\omega C R} - \frac{\omega L}{R}\right)\right]$$

$$= \frac{400}{\sqrt{15^2 + \left(\frac{1}{500\times3\times10^{-6}} - 500\times0.2\right)^2}} \times$$

$$\cos\left[500t + \frac{\pi}{4} + \tan^{-1}\left(\frac{1}{500\times3\times10^{-6}\times15} - \frac{500\times0.2}{15}\right)\right]$$

$$i_p = 0.71 \cos\left(500t + \frac{\pi}{4} + 88.5^\circ\right)$$

The complete solution is,

$$i(t) = e^{-37.5t}(A_1 \cos 1290t + A_2 \sin 1290t) + 0.71 \cos(500t + 45° + 88.5°) \rightarrow (1)$$

for finding the values of A_1, A_2, we make use of

$i(0)$ and $\frac{di(0)}{dt}$.

At $t=0$, $i_L(0) = 0$ ($\because$ circuit is kept open initially)

Sub $t=0$ in eq (1)

$$i_L(0) = e^{-37.5 \times 0}(A_1 \cos 0 + A_2 \sin 0) + 0.71 \cos(0 + 133.5°) \text{ A}$$

$$0 = A_1 + 0.71 \cos(133.5°)$$

$$\Rightarrow A_1 = -0.71 \cos 133.5°$$

$$\boxed{A_1 = 0.49}$$

Now, differentiating eq (1),

$$\frac{di(t)}{dt} = \left[e^{-37.5t}(-A_1(1290) \sin 1290t + A_2(1290) \cos 1290t) + (-37.5)e^{-37.5t}(A_1 \cos 1290t + A_2 \sin 1290t)\right]$$

$$+ \left[0.71(-500) \sin(500t + 133.5°)\right]$$

Now, $$\frac{di(0)}{dt} = \left[e^0(0 + 1290A_2) - 37.5e^0 A_1\right] + \left[-0.71 \times 500 \sin 133.5°\right]$$

But, $$\frac{di(0)}{dt} = 1414$$

$$\Rightarrow \cancel{1414} \quad 1290\,A_2 - 37.5 \times 0.49 - 0.71\,500 \sin(133.5^\circ)$$

$$= 1414$$

$$\Rightarrow \boxed{A_2 = 1.31}$$

$\therefore$ The complete solution is,

$$i(t) = e^{-37.5t}(0.49\cos 1290t + 1.31 \sin 1290t)$$

$$+ 0.71\cos(500t + 133.5^\circ).$$