THE ITERATION METHOD

We have so far discussed root-finding methods for algebraic and transcendental equations, which require the interval in which the root lies. We now describe methods which require one or more starting values of x. These values need not necessarily bracket the root. The first is the iteration method, which requires one starting value of x.

To describe this method for finding the roots of the equation $f(x) = 0$, we rewrite this equation in the form

$$x = \emptyset(x). \tag{1}$$

There are many ways of doing this. For example, the equation $x^3 + x^2 - 1 = 0$ can be expressed as either of the forms:

$$x = (1+x)^{-\frac{1}{2}},\ x = (1-x^3)^{\frac{1}{2}}, x = (1-x^2)^{\frac{1}{3}}, \ldots$$

Let x_0 be an approximate value of the desired root ξ. Substituting it for x on the right side of (1), we obtain the first approximation

$$x_1 = \phi(x_0)$$

The successive approximations are then given by

$$x_2 = \phi(x_1), x_3 = \phi(x_2), \ldots,\ x_n = \phi(x_{n-1}).$$

A number of questions now arise:

(i) Does the sequence of approximations $x_0, x_1, \ldots, x_n$, always converge to some number ξ?

(ii) If it does, will ξ be a root of the equation $x = \phi(x)$?

(iii) How should we choose ϕ in order that the sequence $x_0, x_1, \ldots, x_n$ converges to the root?

The answer to the first question is negative. As an example, we consider the equation

$$x = 10^x + 1.$$

If we take $x_0 = 0, x_1 = 2, x_2 = 101, x_3 = 10^{101} + 1$, etc, and as n increases, x_n increases without limit. Hence, the sequence $x_0, x_1, x_2, \ldots, x_n$ does not always converge and, in Theorem below, we state the conditions which are sufficient for the convergence of the sequence.

The second question is easy to answer, for consider the equation

$$x_{n+1} = \phi(x_n), \tag{2}$$

which gives the relation between the approximations at the nth and $(n+1)$th stages. As n increases, the left side tends to the root ξ. Hence, in the limit, we have $\xi = \phi(\xi)$ which shows that ξ is a root of the equation $x = \phi(x)$.

The answer to the third question is contained in the following theorem:

Theorem: Let $x = \xi$ be a root of $f(x) = 0$ and let I be an interval containing the point $x = \xi$. Let $\phi(x)$ and $\phi'(x)$ be continuous in I, where $\phi(x)$ is defined by the equation $x = \phi(x)$ which is equivalent to $f(x) = 0$. Then if $|\phi'(x)| < 1$ for all x in I, the sequence of approximations $x_0, x_1, x_2, \dots, x_n$ defined by (2) converges to the root ξ, provided that the initial approximation x_0 is chosen in I.

Proof: Since ξ is a root of the equation $x = \phi(x)$, we have

$$\xi = \phi(\xi) \tag{3}$$

From (2)

$$x_1 = \phi(x_0) \tag{4}$$

Subtraction gives

$$\xi - x_1 = \phi(\xi) - \phi(x_0)$$

By using the Lagrange's mean value theorem (If $f(x)$ is continuous in $[a, b]$ and $f'(x)$ exists in (a, b), then there exists at least one value of x, say ξ, between a and b such that $f'(\xi) = \frac{f(b)-f(a)}{b-a}$, $a < \xi < b$), the right-hand side can be written as $(\xi - x_0)\phi'(\xi_0)$, $x_0 < \xi_0 < \xi$. Hence we obtain

$$\xi - x_1 = (\xi - x_0)\phi'(\xi_0), \quad x_0 < \xi_0 < \xi \tag{5}$$

Similarly we obtain

$$\xi - x_2 = (\xi - x_1)\phi'(\xi_1), \quad x_1 < \xi_1 < \xi \tag{6}$$

$$(\xi - x_3) = (\xi - x_2)\phi'(\xi_2), \quad x_2 < \xi_2 < \xi \tag{7}$$

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$$(\xi - x_{n+1}) = (\xi - x_n)\phi'(\xi_n), \quad x_n < \xi_n < \xi \tag{8}$$

If we let

$$|\phi'(\xi_i)| \le k < 1, \quad \text{for all } i \tag{9}$$

Then Equations (6)-(8) give

$$|\xi - x_1| \le |\xi - x_0|, \quad |\xi - x_2| \le |\xi - x_1|, \ldots,$$

which show that each successive approximation remains in I provided that the initial approximation is chosen in I. Now, multiplying Equations (6) to (8) and simplifying, we obtain

$$\xi - x_{n+1} = (\xi - x_0)\phi'(\xi_0)\phi'(\xi_1)\ldots\phi'(\xi_n), \tag{10}$$

Since $|\phi'(\xi_i)| < k$, the above equation becomes

$$|\xi - x_{n+1}| \le k^{n+1}|\xi - x_0| \tag{11}$$

As $n \to \infty$ the right-hand side of (11) tends to zero, and it follows that the sequence of approximation $x_0, x_1, \ldots,$ converges to the root ξ if $k < 1$. The method can be represented graphically as follows. By sketching the line $y = x$ and the curve $y = \phi(x)$ and considering the way in which the approximations x_i are obtained, a geometrical significance of the method is obtained and this is shown in figures 1 to 4.

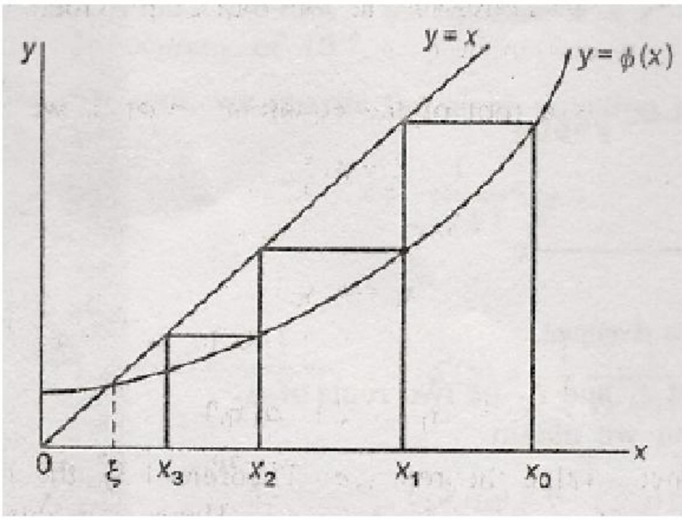

Figure 1 Convergence of $x_{n+1} = \phi(x_n)$, when $|\phi'(x)| < 1$.

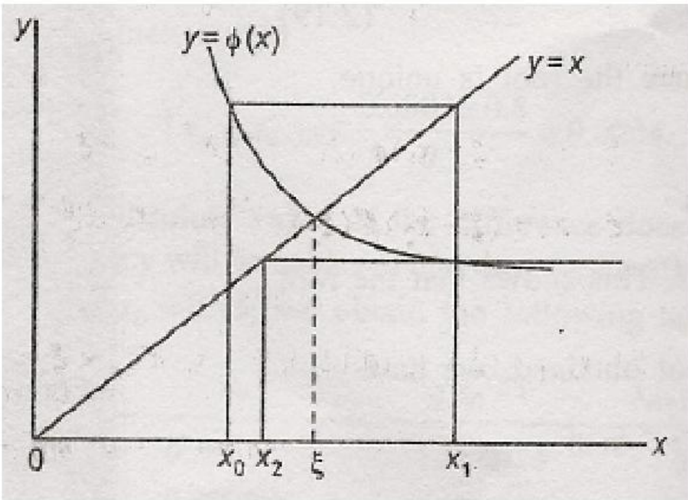

Figure 2 $|\phi'(x)| < 1$ but $\phi'(x) < 1$. The process is convergent but the approximations oscillate about the exact value.

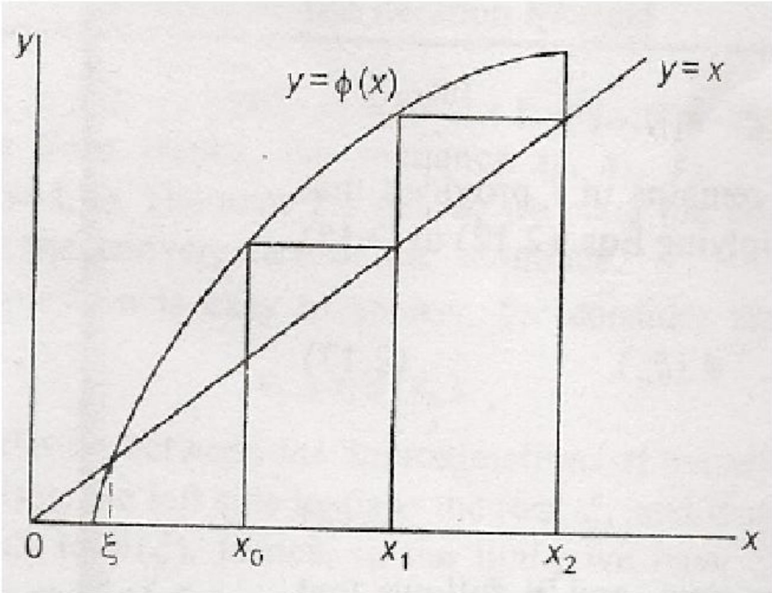

Figure 3 $\phi'(x) > 1$; the process is divergent.

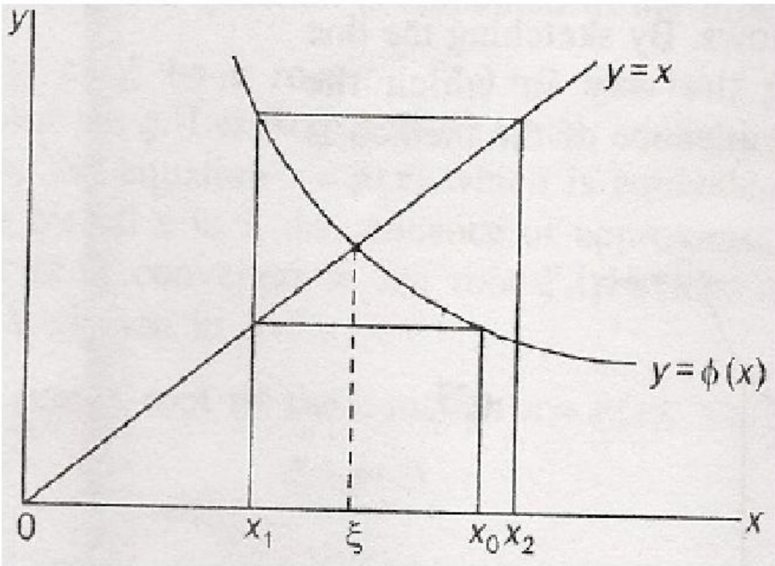

Figure 4 $|\phi'(x)| > 1$; the process is divergent.

Uniqueness

The root so obtained is unique. To prove this, let ξ_1 and ξ_2 be two roots of equation (1), i.e., let $\xi_1 = \phi(\xi_1)$ and $\xi_2 = \phi(\xi_2)$. Then we obtain

$$|\xi_1 - \xi_2| = |\emptyset(\xi_1) - \emptyset(\xi_2)| = |\phi'(\eta)||\xi_1 - \xi_2|, \qquad \eta \epsilon (\xi_1, \xi_2)$$

which further simplifies to

$$|\xi_1 - \xi_2|[1 - |\phi'(\eta)|] = 0. \tag{12}$$

Since $|\phi'(\eta)| < 1$, it follows that $\xi_1 = \xi_2$, and hence the root is unique.

Note: Again

$$\frac{d}{dx}[x - \phi(x)] = 1 - \phi'(x)$$

which is positive, since $\phi'(x) < 1$ in the interval I. This shows that the root obtained by this method is a simple root.

Error

To estimate the error or the approximate root obtained, we have

$$|\xi - x_n| = |\phi(\xi) - \phi(x_{n-1})| \le k|\xi - x_{n-1}|$$

$$= k|\xi - x_n + x_n - x_{n-1}|$$

$$\le k|\xi - x_n| + k|x_n - x_{n-1}|,$$

which gives

$$|\xi - x_n| \leq \frac{k}{1-k} |x_n - x_{n-1}| \leq \frac{k^n}{1-k} |x_1 - x_0|. \qquad (13)$$

In general, the speed of the iteration depends on the value of k; the smaller the value of k, the faster would be the convergence. If ϵ is the specified accuracy, i.e., if

$$|\xi - x_n| \leq \epsilon,$$

Then formula (13) gives

$$|x_n - x_{n-1}| \leq \frac{1-k}{k} \epsilon, \qquad (14)$$

which can be used to find the difference between two successive iterates necessary to achieve a specified accuracy. The following examples illustrate the application of this method.

Example 1

Find a real root of the equation $x^3 + x^2 - 1 = 0$ on the interval $[0, 1]$ with an accuracy of 10^{-4}.

Solution: To find this root, we rewrite the given equation in the form

$$x = \frac{1}{\sqrt{x+1}}$$

Thus

$$\emptyset(x) = \frac{1}{\sqrt{x+1}}, \emptyset'(x) = -\frac{1}{2} \frac{1}{(x+1)^{\frac{3}{2}}}$$

And

$$\min_{[0,1]}|\emptyset'(x)| = \frac{1}{2\sqrt{8}} = k = 0.17\,678 < 0.2$$

Using (14) we then obtain

$$|x_n - x_{n-1}| < \frac{0.0001 \times 0.8}{0.2} = 0.0004.$$

Hence when the absolute value of the difference does not exceed 0.0004, the required accuracy will be achieved and then the iteration can be terminated.

Starting with $x_0 = 0.75,$ we obtain the following table:

n	x_n	$\sqrt{x_n + 1}$	$x_{n+1} = 1/\sqrt{x_n + 1}$
0	0.75	1.3228756	0.7559289
1	0.7559289	1.3251146	0.7546517
2	0.7546517	1.3246326	0.7549263

At this stage, we find that

$$|x_{n+1} - x_n| = 0.7549263 - 0.7546517 = 0.0002746,$$

which is less than 0.0004 . The iteration is therefore terminated and the root to the required accuracy is 0.7549.

Example 2

Find the root of the equation $2x = \cos\, x + 3$ correct to three decimal places.

Solution: We rewrite the equation in the form

$$x = \frac{1}{2}(\cos x + 3) \qquad (1)$$

So that

$$\phi(x) = \frac{1}{2}(\cos x + 3),$$

and

$$|\phi'(x)| = \left|\frac{\sin x}{2}\right| < 1.$$

Hence the iteration method can be applied to the equation (1) and we start with $x_0 = \frac{\pi}{2}$. The successive iterates are

$$x_1 = 1.5, \quad x_2 = 1.535, \quad x_3 = 1.518,$$

$$x_4 = 1.526, \quad x_5 = 1.522, \quad x_6 = 1.524,$$

$$x_7 = 1.523, \quad x_8 = 1.523889\ldots$$

Hence we take the solution as 1.524 correct to three decimal places.

Example 3

Use the method of iteration to find a positive root, between 0 and 1, of the equation $xe^x = 1$.

Solution: Writing the equation in the form

$$x = e^{-x} \qquad (1)$$

We find that $\phi(x) = e^{-x}$ and so $\phi'(x) = -e^{-x}$

Hence $|\phi'(x)| < 1$ for $x < 1$, which assures that the iterative process defined by the equation $x_{n+1} = \phi(x_n)$ will be convergent.

Starting with $x_0 = 1$, we find that the successive iterates are given by

$$x_1 = {}^{1}/_{e} = 0.3678794, \qquad x_2 = 0.6922006,$$

$$x_3 = 0.5004735, \qquad x_4 = 0.6062435,$$

$$x_5 = 0.5453957, \qquad x_6 = 0.5796123,$$

$$x_7 = 0.5601154, \qquad x_8 = 0.5711431,$$

$$x_9 = 0.5648793, \qquad x_{10,} = 0.5684284,$$

$$x_{11} = 0.5664147, \qquad x_{12} = 0.5675566,$$

$$x_{13} = 0.5669089, \qquad x_{14} = 0.5672762,$$

$$x_{15} = 0.5670679, \qquad x_{16} = 0.567186,$$

$$x_{17} = 0.567119, \qquad x_{18} = 0.567157,$$

$$x_{19} = 0.5671354, \qquad x_{20} = 0.5671477.$$

Exercise

1. Use the iteration method to find, correct to four significant figures, a real root of each of the following equations.
   a. $\cos x = 3x - 1$
   b. $x = (5 - x)^{1/3}$

c. $1 + x^2 = x^3$

d. $5x^3 - 20x + 3 = 0$

e. $x = 1 + \tan^{-1} x$.

2. Compute a root of the equation $e^x = x^2$ to an accuracy of 10^{-5}, using the iterative method.
3. By the iteration method, find the root of the equation $x^3 - 2x - 5 = 0$ which lies near to $x = 2$.
4. Evaluate $\sqrt{12}$ and $\frac{1}{\sqrt{12}}$ by the iteration method.

Answers

1.
   a. 0.6071
   b. 1.516
   c. 1.466
   d. 0.1514
   e. 2.1323
2. -0.70346
3. 2.09455
4. 3.46425 , 0.2887