1.9

The Rank of a Matrix

In this module the student is introduced to the concept of the rank of a matrix. Rank enables one to relate matrices to vectors, and vice versa. Rank is a unifying tool that enables us to bring together many of the concepts discussed in the course. Solutions to certain systems of linear equations and invertibility of a matrix all come together under the umbrella of rank.

Definition: Let A be an $m \times n$ matrix. The rows of A may be viewed as row vectors $\boldsymbol{r}_1, \dots, \boldsymbol{r}_m$, and the columns as column vectors $\boldsymbol{c}_1, \dots, \boldsymbol{c}_n$. Each row vector will have n components, and each column vector will have m components. The row vectors will span a subspace of $\boldsymbol{R}^n$ called the **row space** of A, and the column vectors will span a subspace of $\boldsymbol{R}^m$ called the **column space** of A.

Example: Consider the matrix

$$\begin{bmatrix} 1 & 2 & -1 & 2 \\ 3 & 4 & 1 & 6 \\ 5 & 4 & 1 & 0 \end{bmatrix}$$

The row vectors of A are

$$\boldsymbol{r}_1 = (1,2,-1,2) \quad \boldsymbol{r}_2 = (3,4,1,6) \quad \boldsymbol{r}_3 = (5,4,1,0)$$

These vectors span a subspace of $\boldsymbol{R}^4$ called the row space of A.

The column vectors of A are

$$\boldsymbol{c}_1 = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} \quad \boldsymbol{c}_2 = \begin{bmatrix} 2 \\ 4 \\ 4 \end{bmatrix} \quad \boldsymbol{c}_3 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \quad \boldsymbol{c}_4 = \begin{bmatrix} 2 \\ 6 \\ 0 \end{bmatrix}$$

These vectors span a subspace of $\boldsymbol{R}^3$ called the column space of A.

Theorem: The row space and the column space of a matrix A have the same dimension.

Proof: Let $\boldsymbol{u}_1, \dots, \boldsymbol{u}_m$ be the row vectors of A. The ith vector is

$$\boldsymbol{u}_i = (a_{i1}, a_{i2}, \dots, a_{in})$$

Let the dimension of the row space be s. Let the vectors $\boldsymbol{v}_1, \dots \boldsymbol{v}_s$ form a basis for the row space. Let the jth vector of this set be

$$\boldsymbol{v}_j = \left(b_{j1}, b_{j2}, \dots, b_{jn}\right)$$

Each of the row vectors of A is a linear combination of $\boldsymbol{v}_1, \dots \boldsymbol{v}_s$. Let

$$\boldsymbol{u}_1 = c_{11}\boldsymbol{v}_1 + c_{12}\boldsymbol{v}_2 + \cdots + c_{1s}\boldsymbol{v}_s$$

$$\vdots$$

$$\boldsymbol{u}_m = c_{m1}\boldsymbol{v}_1 + c_{m2}\boldsymbol{v}_2 + \cdots + c_{ms}\boldsymbol{v}_s$$

Equating the ith components of the vectors on the left and right, we get

$$a_{1i} = c_{11}b_{1i} + c_{12}b_{2i} + \cdots + c_{1s}b_{si}$$

$$\vdots$$

$$a_{mi} = c_{m1}b_{1i} + c_{m2}b_{2i} + \cdots + c_{ms}b_{si}$$

This may be written

$$\begin{bmatrix} a_{1i} \\ \vdots \\ a_{mi} \end{bmatrix} = b_{1i}\begin{bmatrix} c_{11} \\ \vdots \\ c_{m1} \end{bmatrix} + b_{2i}\begin{bmatrix} c_{12} \\ \vdots \\ c_{m2} \end{bmatrix} + \cdots + b_{si}\begin{bmatrix} c_{1s} \\ \vdots \\ c_{ms} \end{bmatrix}$$

This implies that each column vector of A lies in a space spanned by a single set of s vectors. Since s is the dimension of the row space of A, we get

$$\dim(\text{column space of } A) \leq \dim(\text{row space of } A)$$

By similar reasoning, we can show that

$$\dim(\text{row space of } A) \leq \dim(\text{column space of } A)$$

Combining these two results we see that

$$\dim(\text{row space of } A) = \dim(\text{column space of } A)$$

proving the theorem.

Definition: The dimension of the row space and the column space of a matrix A is called the **rank** of A. The rank of A is denoted $\operatorname{rank}(A)$.

EXAMPLE: Determine the rank of the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 2 & 5 & 8 \end{bmatrix}$$

Solution: We see by inspection that the third row of A is a linear combination of the first two rows:

$$(2,5,8) = 2(1,2,3) + (0,1,2)$$

Hence the three rows of A are linearly dependent. The rank of A must be less than 3. Since (1,2,3) is not a scalar multiple of (0,1,2), these two vectors are linearly independent. These vectors form a basis for the row space of A. Thus $\text{rank}(A) = 2$.

This method, based on the definition, is not practical for determining the ranks of larger matrices. We shall give a more systematic method for finding the rank of a matrix. The following theorem, which paves the way for the method, tells us that the rank of a matrix that is in reduced echelon form is immediately known.

Theorem: The nonzero row vectors of a matrix A that is in reduced echelon form are a basis for the row space of A. The rank of A is the number of nonzero row vectors.

Proof: Let A be an $m \times n$ matrix with nonzero row vectors of $\boldsymbol{r}_1, \dots \boldsymbol{r}_t$. Consider the identity

$$k_1\boldsymbol{r}_1 + k_2\boldsymbol{r}_2 + \cdots + k_t\boldsymbol{r}_t = 0$$

where $k_1, \dots, k_t$ are scalars.

The first nonzero element of $\boldsymbol{r}_1$ is 1. $\boldsymbol{r}_1$ is the only one of the row vectors to have a nonzero number in this component.

Thus, on adding the vectors $k_1\boldsymbol{r}_1, k_2\boldsymbol{r}_2, \dots, k_t\boldsymbol{r}_t$, we get a vector whose first component is k_1. On equating this vector to zero, we get $k_1 = 0$. The identity then reduces to

$$k_2\boldsymbol{r}_2 + \cdots + k_t\boldsymbol{r}_t = 0$$

The first nonzero element of $\boldsymbol{r}_2$ is 1, and it is the only one of these remaining row vectors with a nonzero number in this component. Thus $k_2 = 0$. Similarly, $k_3, \dots, k_t$ are all zero. The vectors $\boldsymbol{r}_1, \dots, \boldsymbol{r}_t$ are therefore linearly independent. These vectors span the row space of A. They thus form a basis for the row space of A. The dimension of the row space is t. The rank of A is t, the number of nonzero row vectors in A.

Theorem: Let A and B be row equivalent matrices. Then A and B have the same row space. Rank(A) = rank(B).

Proof: Since A are B are row equivalent, the rows of B can be obtained from the rows of A through a sequence of elementary row operations. Therefore each row of B is a linear combination of the rows of A. Thus the row space of B is contained in the row space of A.

In the same way, the rows of A can be obtained from the rows of B through a sequence of elementary row operations, implying that the row space of A is contained in the row space of B.

It follows that the row spaces of A and B are equal. Since their row spaces are equal, their ranks must be equal.

The next result brings the last two results together to give a method for finding a basis for the row space of a matrix and the rank of the matrix.

Theorem: Let E be a reduced echelon form of a matrix A. The nonzero row vectors of E form a basis for the row space of A. The rank of A is the number of nonzero row vectors in E.

The concept of rank plays an important role in understanding the behavior of systems of linear equations. We have seen how systems of linear equations can have a unique solution, many solutions, or no solutions at all. These situations can be conveniently categorized in terms of the ranks of the augmented matrix and the matrix of coefficients.

THEOREM 5.17

Consider a system of m equations in n variables.

(a) If the augmented matrix and the matrix of coefficients have the same rank r and $r = n$, the solution is unique.
(b) If the augmented matrix and the matrix of coefficients have the same ran r and $r < n$, there are many solutions.
(c) If the augmented matrix and the matrix of coefficients do not have the same rank, a solution does not exist.

PROOF: Let the system of equations be

$$a_{11}x_1 + \cdots + a_{1n}x_n = b_1$$
$$\vdots$$
$$a_{m1}x_1 + \cdots + a_{mn}x_n = b_m$$

This system can be written

$$x_1 \begin{bmatrix} a_{11} \\ \vdots \\ a_{m1} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

That is

$$x_1\boldsymbol{a}_1 + \cdots + x_n\boldsymbol{a}_n = \boldsymbol{b} \qquad \text{-------} \qquad (1)$$

Let us now look at the three possibilities.

(a) Since the ranks of the matrix of coefficients and augmented matrix are the same, $\boldsymbol{b}$ must be linearly dependent on $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$. Furthermore, since the rank is $\boldsymbol{n}$, the vectors $\boldsymbol{a}_1, \ldots \boldsymbol{a}_n$ are linearly independent and thus form a basis for the column space of the augmented matrix. Therefore Equation (1) has a unique solution; the solution to the system is unique.

(b) Since the ranks of the matrix of coefficients and augmented matrix are the same, $\boldsymbol{b}$ must be linearly dependent on $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$. However since rank < $\boldsymbol{n}$, the vectors $\boldsymbol{a}_1, \ldots \boldsymbol{a}_n$ are linearly dependent. $\boldsymbol{b}$ can therefore be expressed in more than one way as a linear combination of $\boldsymbol{a}_1, \ldots \boldsymbol{a}_n$. Thus Equation (1) has many solutions; the solution to the system exists but is not unique.

(c) Since the rank of the augmented matrix is not equal to the rank of the matrix of coefficients, $\boldsymbol{b}$ is linearly independent of $\boldsymbol{a}_1, \ldots \boldsymbol{a}_n$. Thus Equation (1) has no solution; a solution to the system does not exist.

Problem 1: Find a basis for the row space of the following matrix A, and determine its rank.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 4 \\ 1 & 1 & 5 \end{bmatrix}$$

Solution: Use elementary row operations to find a reduced echelon form of the matrix A. We get

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 4 \\ 1 & 1 & 5 \end{bmatrix} \approx \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -2 \\ 0 & -1 & 2 \end{bmatrix} \approx \begin{bmatrix} 1 & 0 & 7 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

The two vectors $(1,0,7)$, $(0,1,-2)$ form a basis for the row space of A. $\text{Rank}(A) = 2$.

Problem 2: Find a basis for the column space of the following matrix A.

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 3 & -2 \\ -1 & -4 & 6 \end{bmatrix}$$

Solution: The transpose of A is

$$A^t = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 3 & -4 \\ -1 & -2 & 6 \end{bmatrix}$$

The column space of A becomes the row space of A^t. Let us find a basis for the row space of A^t. Compute a reduced echelon form of A^t.

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & -4 \\ 0 & -2 & 6 \end{bmatrix} \approx \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -3 \\ 0 & -2 & 6 \end{bmatrix} \approx \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

The nonzero row vectors of this echelon form, namely $(1,0,5)$, $(0,1,-3)$, are a basis for the row space of A^t. Write these vectors in column form to get a basis for the column space of A. The following vectors are a basis for the column space of A.

$$\begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}, \quad \begin{bmatrix} 2 \\ 4 \\ 4 \end{bmatrix}$$

Problem 3: Find a basis for the subspace V of $\boldsymbol{R}^4$ spanned by the vectors

$$(1,2,3,4), (-1,-1,-4,-2), (3,4,11,8)$$

Solution: We construct a matrix A having these vectors as row vectors.

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ -1 & -1 & -4 & -2 \\ 3 & 4 & 11 & 8 \end{bmatrix}$$

Determine a reduced echelon form of A. We get

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ -1 & -1 & -4 & -2 \\ 3 & 4 & 11 & 8 \end{bmatrix} \approx \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & -1 & 2 \\ 0 & -2 & 2 & -4 \end{bmatrix} \approx \begin{bmatrix} 1 & 0 & 5 & 0 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The nonzero vectors of this reduced echelon form, namely $(1,0,5,0)$ and $(0,1,-1,2)$, are a basis for the subspace V.

Exercise

1. Determine the ranks of the following matrices using the definition of rank.

   a. $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}$

   b. $\begin{bmatrix} 2 & 1 & 3 \\ 4 & 2 & 6 \\ 2 & 1 & 3 \end{bmatrix}$

   c. $\begin{bmatrix} 1 & 3 & 4 \\ -1 & 3 & 1 \\ 0 & 6 & 5 \end{bmatrix}$

2. Find the reduced echelon form for each of the following matrices. Use the echelon form to determine a basis for the row space and the rank of each matrix.

   a. $\begin{bmatrix} 1 & 2 & -1 \\ 2 & 5 & 2 \\ 0 & 2 & 9 \end{bmatrix}$

   b. $\begin{bmatrix} 1 & -3 & 2 \\ -2 & 6 & -4 \\ -1 & 3 & -2 \end{bmatrix}$

3. Find bases for the subspaces of $\boldsymbol{R}^3$ spanned by the following vectors.

   a. $(1,3,2), (0,1,4), (1,4,9)$

   b. $(1,-1,3), (1,0,1), (-2,1,-4)$

4. Find bases for both the row and column spaces of the following matrix A. Show that the dimensions of both row space and column space are the same.

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 3 \\ 1 & 4 & 6 \end{bmatrix}$$

5. Let A be a 3 X 4 matrix. Prove that the column vectors of A are linearly dependent.
6. Let A be a $n \times n$ invertible matrix. Prove that the row vectors of A form a basis for $\boldsymbol{R}^n$.
7. Let A be a $n \times n$ invertible matrix. Prove that the columns of A are linearly independent if and only if $\text{rank}(A) = n$.
8. If A and B are matrices of the same size, prove that $\text{rank}(A+B) \leq \text{rank}(A) + \text{rank}(B)$.

Answers

1.
   a. 2
   b. 1
   c. 2

2.
   a. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Basis $(1,0,0), (0,1,0)$ and $(0,0,1)$. Rank 3.
   b. $\begin{bmatrix} 1 & -3 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Basis $(1,-3,2)$. Rank 1.

3.

a. The given vectors are linearly independent so they are a basis for the space they span, which is $\boldsymbol{R}^3$.

b. $\begin{bmatrix} 1 & -1 & 3 \\ 1 & 0 & 1 \\ -2 & 1 & -4 \end{bmatrix} \approx \cdots \approx \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$. Vectors $(1,0,1)$ and $(0,1,-2)$ are a basis for the space spanned by the given vectors.

4. $A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 3 \\ 1 & 4 & 6 \end{bmatrix} \approx \cdots \approx \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Vectors $(1,0,0), (0,1,0)$ and $(0,0,1)$ are a basis for the row space of A.