NEWTON-RAPHSON METHOD

This method is generally used to improve the result obtained by one of the previous methods. Let x_0 be an approximate root of $f(x)=0$ and let $x_1 = x_0 + h$ be the correct root so that $f(x_1) = 0$. Expanding $f(x_0 + h)$ by Taylor's series, we obtain

$$f(x_0) + hf'(x_0) + \frac{h^2}{2!}f''(x_0) + \cdots = 0.$$

Neglecting the second and higher order derivatives, we have

$$f(x_0) + hf'(x_0) = 0,$$

which gives

$$h = -\frac{f(x_0)}{f'(x_0)}.$$

A better approximation than x_0 is therefore given by x_1, where

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

Successive approximations are given by $x_2, x_3, \ldots, x_{n+1}$, where

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \tag{1}$$

which is the Newton-Raphson formula.

Convergence

If we compare equation (1) with the relation

$$x_{n+1} = \emptyset(x_n)$$

of the iterative method we obtain

$$\emptyset(x) = x - \frac{f(x)}{f'(x)},$$

which gives

$$\emptyset'(x) = \frac{f(x)f''(x)}{[f'(x)]^2}. \tag{2}$$

To examine the convergence we assume that $f(x), f'(x)$ and $f''(x)$ are continuous and bounded on any interval containing the root $x = \xi$ of the equation $f(x) = 0$. If ξ is a simple root, then $f'(x) \neq 0$. Further since $f'(x)$ is continuous, $|f'(x)| \geq \varepsilon$ for some $\varepsilon > 0$ in a suitable neighbourhood of ξ. Within this neighbourhood we can select an interval such that $|f(x)f''(x)| < \epsilon^2$ and this is possible since $f(\xi) = 0$ and since $f(x)$ is continuously twice differentiable. Hence, in this interval we have

$$|\emptyset'(x)| < 1. \tag{3}$$

Therefore by iteration theorem, the Newton-Raphson formula (1) converges, provided that the initial approximation x_0 is chosen sufficiently close to ξ. When ξ is a multiple root, the Newton-Raphson method still converges but slowly.

Rate of Convergence

To obtain the rate of convergence of the method, we note that $f(\xi) = 0$ so that Taylor's expansion gives

$$f(x_n) + (\xi - x_n)f'(x_n) + \frac{1}{2}(\xi - x_n)^2 f''(x_n) + \cdots = 0,$$

from which we obtain

$$-\frac{f(x_n)}{f'(x_n)} = (\xi - x_n) + \frac{1}{2}(\xi - x_n)^2 \frac{f''(x_n)}{f'(x_n)} \tag{4}$$

From (1) and (4), we have

$$x_{n+1} - \xi = \frac{1}{2}(x_n - \xi)^2 \frac{f''(x_n)}{f'(x_n)} \tag{5}$$

Setting $\xi_n = x_n - \xi,$

Equation (5) gives

$$\varepsilon_{n+1} \approx \frac{1}{2}\varepsilon_n^2 \frac{f''(\xi)}{f'(\xi)}, \tag{6}$$

so that the Newton-Raphson process has a second-order or quadratic convergence.

Geometric Representation

Geometrically, the method consists in replacing the part of the curve between the point $[x_0, f(x_0)]$ and the $x-$axis by means of the tangent to the curve at the point, and is described graphically in figure. It can be used for solving both algebraic and transcendental equations and it can also be used when the roots are complex.

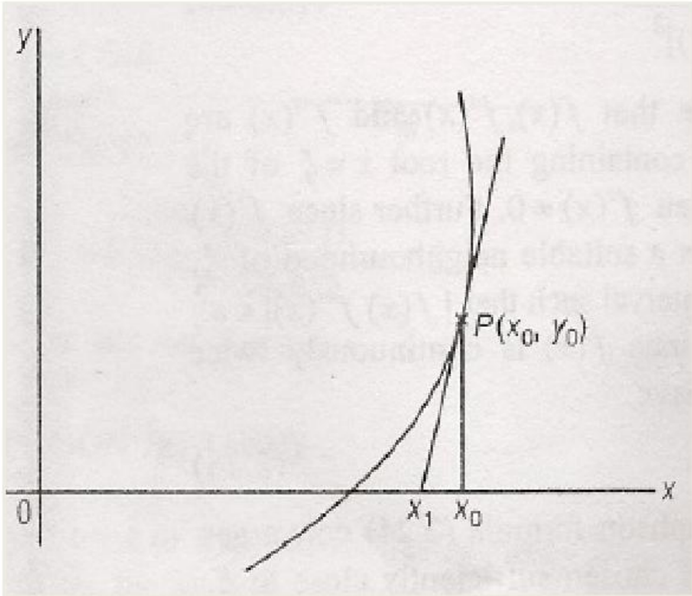

Figure: Newton-Raphson method

Example 1

Use the Newton-Raphson method to find a root of the equation $x^3 - 2x - 5 = 0$.

Solution: Here $f(x) = x^3 - 2x - 5$ and $f'(x) = 3x^2 - 2$. Hence equation (1) gives:

$$x_{n+1} = x_n - \frac{x_n^3 - 2x_n - 5}{3x_n^2 - 2} \qquad \text{(i)}$$

Choosing $x_0 = 2$, we obtain $f(x_0) = -1$ and $f'(x_0) = 10$. Putting $n = 0$ in (i), we obtain

$$x_1 = 2 - \left(-\frac{1}{10}\right) = 2.1$$

Now,

$$f(x_1) = (2.1)^3 - 2(2.1) - 5 = 0.061,$$

and

$$f'(x_1) = 3(2.1)^2 - 2 = 11.23$$

Hence

$$x_2 = 2.1 - \frac{0.061}{11.23} = 2.094568.$$

This example demonstrates that Newton-Raphson method converges more rapidly than the methods described in the previous sections, since this requires fewer iterations to obtain a specified accuracy. But since two function evaluations are required for each iteration, Newton-Raphson method requires more computing time.

Example 2

Find a root of the equation $x \sin x + \cos x = 0$.

Solution: We have

$$f(x) = x \sin x + \cos x \text{ and } f'(x) = x \cos x.$$

The iteration formula is therefore

$$x_{n+1} = x_n - \frac{x_n \sin x_n + \cos x_n}{x_n \cos x_n}.$$

With $x_0 = \pi$, the successive iterates are given below

n	x_n	$f(x_n)$	x_{n+1}
0	3.1416	-1.0	2.8233
1	2.8233	-0.0662	2.7986
2	2.7986	-0.0006	2.7984
3	2.7984	0.0	2.7984

Example 3

Find a real root of the equation $x = e^{-x}$, using the Newton-Raphson method.

Solution: We write the equation in the form

$$f(x) = xe^x - 1 = 0$$

Let $x_0 = 1$. Then

$$x_1 = 1 - \frac{e-1}{2e} = \frac{1}{2}\left(1 + \frac{1}{e}\right) = 0.6839397$$

Now

$$f(x_1) = 0.3553424, \text{and} \quad f'(x_1) = 3.337012,$$

so that

$$x_2 = 0.6839397 - \frac{0.3553424}{3.337012} = 0.5774545.$$

Proceeding in this way, we obtain

$$x_3 = 0.5672297 \quad \text{and} \quad x_4 = 0.5671433.$$

Example 3

Using Newton-Raphson method, derive formulas to find

(i). $\frac{1}{N}$, (ii). $N^{\frac{1}{q}}, N > 0,\ q$ integer hence find $1/18, (18)^{\frac{1}{3}}$ to four decimals. Use suitable initial approximation.

Solution:

(i). Let $x = \frac{1}{N}, f(x) = \frac{1}{x} - N = 0$. We have $f'(x) = -\frac{1}{x^2}$.

Newton-Raphson method gives

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$
$$= x_k - \frac{\left(\frac{1}{x_k} - N\right)}{\left(-\frac{1}{x_k^2}\right)} = 2x_k - Nx_k^2.$$

We have $N = 18, x = \frac{1}{18}$. Let $x_0 = 0.05$.

The method gives

$$x_{k+1} = 2x_k - 18x_k^2, k = 0, 1, 2, \ldots$$

We obtain $x_1 = 2x_0 - 18x_0^2 = 0.55$

$$x_2 = 2x_1 - 18x_1^2 = 0.05555,$$

$$x_3 = 2x_2 - 18x_2^2 = 0.055555.$$
$$\therefore \frac{1}{18} \approx 0.0556 .$$

(ii). Let $x = N^{\frac{1}{q}}, f(x) = x^q - N = 0$. We have $f'(x) = q\, x^{q-1}$

Newton-Raphson method gives

$$\begin{aligned} x_{k+1} &= x_k - \frac{f(x_k)}{f'(x_k)} \\ &= x_k - \frac{x_k^q - N}{q x_k^{q-1}} = \frac{(q-1)x_k^q + N}{q x_k^{q-1}}. \end{aligned}$$

We have $N = 18, q = 3$. Let $x_0 = 2.5$

The method gives

$$x_{k+1} = \frac{2x_k^3 + 18}{3x_k^2}, k = 0, 1, 2, \ldots$$

We obtain, $x_1 = \frac{2x_0^3 + 18}{3x_0^2} = 2.62667,$

$$x_2 = \frac{2x_1^3 + 18}{3x_1^2} = 2.62075,$$

$$x_3 = \frac{2x_2^2 + 18}{3x_2^2} = 2.62074.$$

$$\therefore (18)^{\frac{1}{3}} \approx 2.6207.$$

Exercise

1. Use Newton-Raphson method to obtain a root, correct to three decimal places, of the following equations.
   a. $\sin x = 1 - x$
   b. $x^3 - 5x + 3 = 0$

c. $x^4 + x^2 - 80 = 0$

d. $x + \mathrm{l\ og}x = 2$

2. Using the Newton-Raphson method, find the root of the equation $f(x) = e^x - 3x$ that lies between 0 and 1.

3. Perform four iterations of the Newton-Raphson to obtain the approximate value of $(17)^{1/3}$ starting with the initial approximation 2.

4. Using Newton-Raphson method to find the smallest positive root of the equation $f(x) = x^3 - 5x + 1 = 0$.

Answers

1.
   a. 0.511
   b. 0.657
   c. 2.908
   d. 1.756
2. 0.619061
3. 2.571282
4. 0.201640