2.2

BETA AND GAMMA FUNCTIONS

Introduction

Beta and Gamma functions are improper integrals which are commonly encountered in many science and engineering applications. These functions are used in evaluating many definite integrals. In this module, we study some simple properties of the beta function and the gamma function. These functions are defined in the form of an integral and all the additional properties of these functions are derived from the integral representation.

Definition: The Beta function is defined by

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx \qquad m,n > 0.$$

The Gamma function is defined by

$$\Gamma(n) = \int_0^\infty x^{n-1}e^{-x}dx \qquad n > 0.$$

Note: The Beta function defined above converges if $m,n > 0$ and the Gamma function defined above converges if $n > 0$.

Properties and results of Beta and Gamma functions

1. $\beta(m,n) = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}dx$

Proof: $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$.

Put $x = \frac{y}{1+y}$. Hence $1 - x = 1 - \frac{y}{1+y} = \frac{1}{1+y}$.

When $x = 0, y = 0$. When $x \to 1, y \to \infty$. Also $dx = \frac{dy}{(1+y)^2}$.

Therefore, $\beta(m,n) = \int_0^\infty \frac{y^{m-1}}{(1+y)^{m-1}(1+y)^{n-1}(1+y)^2}dy$

$$= \int_0^\infty \frac{y^{m-1}}{(1+y)^{m+n}}dy$$

$$= \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}dx$$

2. $\beta(m,n) = 2\int_0^{\pi/2} (\sin x)^{2m-1}(\cos x)^{2n-1}dx.$

Proof: $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$.

Put $x = \sin^2 t$. Hence $dx = 2\sin t\cos t\, dt$.

When $x = 0, t = 0$ and when $x = 1, t = \frac{\pi}{2}$.

Therefore, $\beta(m,n) = \int_0^{\pi/2} (\sin^2 t)^{m-1}(\cos^2 t)^{n-1} 2\sin t\cos t dt$

$$= 2\int_0^{\pi/2} (\sin x)^{2m-1}(\cos x)^{2n-1}dx.$$

3. $\beta(m,n) = \beta(n,m)$.

Proof: $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx.$

Put $x = 1 - y$. Hence $1 - x = y$. Also $dx = -dy$.

So when $x = 0, y = 1$ and when $x = 1, y = 0$.

Therefore, $\beta(m,n) = \int_1^0 (1-y)^{m-1}y^{n-1}(-dy)$

$$= \int_0^1 y^{n-1}(1-y)^{m-1}dy$$

$$= \beta(n,m).$$

4. $\beta(m,n) = \beta(m+1,n) + \beta(m,n+1)$.

Proof: $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$

$$= \int_0^1 x^{m-1}(1-x)^{n-1}(x+1-x)dx$$

$$= \int_0^1 x^m(1-x)^{n-1}dx + \int_0^1 x^{m-1}(1-x)^n dx$$

$$= \beta(m+1,n) + \beta(m,n+1).$$

5. $\Gamma(n+1) = n\Gamma(n)$.

Proof: $\Gamma(\mathrm{n}) = \int_0^\infty x^{n-1}e^{-x}dx.$

So $\Gamma(\mathrm{n}+1)=\int_{0}^{\infty} x^{n} e^{-x} d x$

$$=\lim_{a \to \infty}\left[\int_{0}^{a} x^{n} e^{-x} d x\right]$$

$$=\lim_{a \to \infty}\left[\left[-x^{n} e^{-x}\right]_{0}^{a}+\int_{0}^{a} n e^{-x} x^{n-1} d x\right]$$

$$=\lim_{a \to \infty}\left[-x^{n} e^{-x}\right]_{0}^{a}+n \int_{0}^{\infty} x^{n-1} e^{-x} d x$$

$$=n \Gamma(n)\left(\text{since } \lim_{a \to \infty}\left(-a^{n} e^{-a}\right)=0\right).$$

6. $\Gamma(1)=1$.

Proof: $\Gamma(\mathrm{n})=\int_{0}^{\infty} x^{n-1} e^{-x} d x$.

So $\Gamma(1)=\int_{0}^{\infty} e^{-x} d x$

$$=\left[-e^{-x}\right]_{0}^{\infty}$$
$$=[0-(-1)]=1.$$

7. $\Gamma(\mathrm{n}+1)=\mathrm{n}!$, where n is a positive integer.

Proof: We have $\Gamma(\mathrm{n}+1)=\mathrm{n} \Gamma(\mathrm{n})$ (by Property 5)

$$=n(n-1) \Gamma(n-1)$$

$$=n(n-1) \ldots 2.1. \Gamma(1)$$

$= n!$ (using Property 6).

8. $\Gamma(\mathrm{n}) = 2\int_0^\infty e^{-y^2} y^{2n-1} dy.$

Proof: We have $\Gamma(\mathrm{n}) = \int_0^\infty e^{-x} x^{n-1} dx.$

Put $x = y^2$. Hence $dx = 2y\, dy$.

Therefore $\Gamma(\mathrm{n}) = \int_0^\infty e^{-y^2} (y^2)^{n-1} 2y\, dy = 2\int_0^\infty e^{-y^2} y^{2n-1} dy.$

9. $\beta(m,n) = \frac{\Gamma(\mathrm{m})\Gamma(\mathrm{n})}{\Gamma(\mathrm{m+n})}.$

Proof: We have (by Property 8)

$$\Gamma(\mathrm{m}) = 2\int_0^\infty e^{-y^2} y^{2m-1} dy \text{ and}$$

$$\Gamma(\mathrm{n}) = 2\int_0^\infty e^{-x^2} x^{2n-1} dx$$

$$\text{Now, } \Gamma(\mathrm{m})\Gamma(\mathrm{n}) = 4\int_0^\infty e^{-y^2} y^{2m-1} dy \int_0^\infty e^{-x^2} x^{2n-1} dx$$

$$= 4\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} x^{2n-1} y^{2m-1} dx dy.$$

Put $x = r\cos\theta$ and $y = r\sin\theta$. Hence $|J| = r$.

In the double integral the region of integration is the entire first quadrant. In this region r varies from 0 to ∞ and θ varies from 0 to $\frac{\pi}{2}$.

So $\Gamma(\mathrm{m})\Gamma(\mathrm{n}) = 4\int_0^\infty \int_0^{\pi/2} e^{-r^2} r^{2m+2n-2} (\cos\theta)^{2n-1} (\sin\theta)^{2m-1} |J| d\theta dr$

$$= 4\int_0^\infty \int_0^{\pi/2} e^{-r^2} r^{2m+2n-1} (\cos\theta)^{2n-1} (\sin\theta)^{2m-1} d\theta dr$$

$$= 4\int_0^\infty e^{-r^2} r^{2m+2n-1} dr \int_0^{\pi/2} (\cos\theta)^{2n-1} (\sin\theta)^{2m-1} d\theta$$

$$= 4\int_0^\infty e^{-r^2} (r^2)^{m+n-1} \frac{1}{2} d(r^2) \int_0^{\pi/2} (\cos\theta)^{2n-1} (\sin\theta)^{2m-1} d\theta$$

$= 4\left[\frac{1}{2}\Gamma(\mathrm{m}+\mathrm{n})\right]\left[\frac{1}{2}\beta(m,n)\right]$ [using definition of $\Gamma(\mathrm{n})$ and Property 2]

$= \Gamma(\mathrm{m}+\mathrm{n})\beta(\mathrm{m},\mathrm{n})$.

Therefore, $\beta(m,n) = \frac{\Gamma(\mathrm{m})\Gamma(\mathrm{n})}{\Gamma(\mathrm{m}+\mathrm{n})}$.

10. $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

Proof: We know that

$$\beta(m,n) = 2\int_0^{\pi/2} (\sin x)^{2m-1} (\cos x)^{2n-1} dx \text{ [refer Property 2].}$$

Now, $\beta\left(\frac{1}{2},\frac{1}{2}\right) = 2\int_0^{\pi/2} dx = 2[x]_0^{\pi/2} = \pi.$

$\Rightarrow \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(1)} = \pi$ (using Property 9).

$\Rightarrow \left[\Gamma\left(\frac{1}{2}\right)\right]^2 = \pi$ ($\because \Gamma(1) = 1$).

$\Longrightarrow \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$

11. $\Gamma(\mathrm{n}) = \int_0^1 \left[\log\left(\frac{1}{x}\right)\right]^{n-1} dx.$

Proof: We have $\Gamma(\mathrm{n}) = \int_0^{\infty} e^{-x}x^{n-1}dx.$

Put $x = \log\left(\frac{1}{y}\right)$. Hence $dx = -\left(\frac{1}{y}\right)dy.$

When $x = 0, y = 1$ and when $x = \infty, y = 0.$

Therefore, $\Gamma(\mathrm{n}) = \int_1^0 y\left[\log\left(\frac{1}{y}\right)\right]^{n-1}\left(\frac{-1}{y}\right)dy = \int_0^1 \left[\log\left(\frac{1}{x}\right)\right]^{n-1} dx.$

12. $2^{2n-1}\Gamma(\mathrm{n})\Gamma\left(\mathrm{n}+\frac{1}{2}\right) = \Gamma(2\mathrm{n})\sqrt{\pi}.$ (This is known as duplication formula).

Proof: Let $I = \int_0^{\pi/2} \sin^{2\mathrm{n}} x\, dx$. We claim that $\int_0^{\pi/2} \sin^{2\mathrm{n}} 2x\, dx = I.$

$$\int_0^{\pi/2} \sin^{2\mathrm{n}} 2x\, dx = \frac{1}{2}\int_0^{\pi} \sin^{2\mathrm{n}} y\, dy \quad \text{(putting } 2x = y\text{)}$$

$$= \int_0^{\pi/2} \sin^{2n} y\, dy \quad (\because \sin^{2n}(\pi - y) = \sin^{2n} y)$$

$$= I.$$

Taking $I = \int_0^{\pi/2} \sin^{2n} x\, dx$

$$= \int_0^{\pi/2} (\sin x)^{2(n+1/2)-1} (\cos x)^{2(1/2)-1} dx$$

$$= \frac{1}{2}\beta\left(n + \frac{1}{2}, \frac{1}{2}\right) \quad \text{(by Property 2)}$$

$$= \frac{\Gamma\left(n+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma(n+1)} \quad \text{(by Property 2)}$$

$$= \frac{\Gamma\left(n+\frac{1}{2}\right)\sqrt{\pi}}{2\Gamma(n+1)} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}$$

Now, taking $I = \int_0^{\pi/2} \sin^{2n} 2x\, dx$

$$= \int_0^{\pi/2} 2^{2n} (\sin x)^{2n} (\cos x)^{2n} dx$$

$$= 2^{2n-1} \frac{\Gamma\left(n+\frac{1}{2}\right)\Gamma\left(n+\frac{1}{2}\right)}{\Gamma(2n+1)} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(ii)}$$

From (i) and (ii) we get

$$\frac{\Gamma\left(n+\frac{1}{2}\right)\sqrt{\pi}}{2\Gamma(n+1)} = 2^{2n-1} \frac{\Gamma\left(n+\frac{1}{2}\right)\Gamma\left(n+\frac{1}{2}\right)}{\Gamma(2n+1)}.$$

$$\Rightarrow \Gamma(2n+1)\sqrt{\pi} = 2^{2n}\Gamma(n+1)\Gamma\left(n+\frac{1}{2}\right).$$

$$\Rightarrow 2n\Gamma(2n)\sqrt{\pi} = 2^{n} n\Gamma(n)\Gamma\left(n+\frac{1}{n}\right) \quad \text{(using 5).}$$

$$\Rightarrow 2^{2n-1}\Gamma(n)\Gamma\left(n+\frac{1}{2}\right) = \Gamma(2n)\sqrt{\pi}.$$

13. $\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) = \sqrt{2}\pi.$

Proof: Put $n = \frac{1}{4}$ in the duplication formula.

Then $\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) = \frac{\Gamma\left(\frac{1}{2}\right)\sqrt{\pi}}{2^{-1/2}}$

$$= \sqrt{2}\pi \quad \left(\because \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}\right)$$

Problem 1: Evaluate $\int\limits_0^\infty x^6 e^{-3x}\,dx$.

Solution: Put $y = 3x$. Hence $dy = 3dx$.

Now, $\int\limits_0^\infty x^6 e^{-3x}\,dx = \int\limits_0^\infty \left(\frac{\mathrm{y}}{3}\right)^6 \mathrm{e}^{-\mathrm{y}} \left(\frac{\mathrm{dy}}{3}\right)$

$$= \left(\frac{1}{3}\right)^7 \int\limits_0^\infty y^6 e^{-y}\,dy$$

$$= \left(\frac{1}{3}\right)^7 \Gamma(7) == \left(\frac{1}{3}\right)^7 6! = \frac{80}{243}.$$

Problem 2: Prove that $\int_0^\infty \frac{e^{-st}}{\sqrt{t}} dt = \sqrt{\frac{\pi}{s}}$, where $s > 0$.

Solution: Put $st = u$. Hence $s\,dt = du$.

$$\text{So } \int_0^\infty \frac{e^{-st}}{\sqrt{t}} dt = \frac{1}{\sqrt{s}} \int_0^\infty e^{-u} u^{-\frac{1}{2}} du$$

$$= \frac{1}{\sqrt{s}} \Gamma\left(\frac{1}{2}\right) = \sqrt{\frac{\pi}{s}}.$$

Problem 3: Evaluate $\int_0^1 (x \log x)^3 dx$.

Solution: Put $x = e^{-y}$. Hence $\log x = -y$.

When $x = 0, y = \infty$ and when $x = 1, y = 0$.

Further $dx = -e^{-y} dy$.

So $\int_0^1 (\mathrm{x} \log \mathrm{x})^3 dx = \int_\infty^0 \mathrm{e}^{-3\mathrm{y}} (-\mathrm{y})^3 (-\mathrm{e}^{-\mathrm{y}} \mathrm{dy})$

$$= \int_\infty^0 \mathrm{e}^{-4\mathrm{y}} \mathrm{y}^3 \mathrm{dy}.$$

Now, put $4y = t$. Hence $4dy = dt$.

Therefore, $\int_\infty^0 e^{-4y} y^3 dy = \int_\infty^0 \mathrm{e}^{-\mathrm{t}} \left(\frac{\mathrm{t}}{4}\right)^3 \left(\frac{1}{4}\right) \mathrm{dt}$

$$= \frac{1}{256} \int_\infty^0 \mathrm{e}^{-\mathrm{t}} \mathrm{t}^3 \mathrm{dt}$$

$$= -\frac{1}{256} \int_0^\infty \mathrm{e}^{-\mathrm{t}}\, \mathrm{t}^3 \mathrm{dt}$$

$$= -\frac{1}{256} \Gamma(4) = -\frac{1}{256} (3!) = -\frac{3}{128}.$$

Problem 4: Prove that $\int_0^\infty \frac{x^8(1-x^6)}{(1+x)^{24}}\,dx = 0.$

Solution: $I = \int_0^\infty \frac{x^8}{(1+x)^{24}}\,dx - \int_0^\infty \frac{x^{14}}{(1+x)^{24}}\,dx$

$$= \int_0^\infty \frac{x^{9-1}}{(1+x)^{9+15}}\,dx - \int_0^\infty \frac{x^{15-1}}{(1+x)^{15+9}}\,dx$$

$$= \beta(9,15) - \beta(15,9) \quad \text{(by Property 1)}$$

$$= 0 \quad [\text{since } \beta(m,n) = \beta(n,m)].$$

Problem 5: Prove that $\int_0^{\pi/2} \frac{d\theta}{\sqrt{\sin\,\theta}} \int_0^{\pi/2} \sqrt{\sin\theta}\,d\theta = \pi$

Solution: $\int_0^{\pi/2} \sqrt{\sin\theta}\,d\theta = \int_0^{\pi/2} (\sin\theta)^{1/2}(\cos\theta)^0 d\theta$

$$= \int_0^{\pi/2} (\sin\theta)^{2(3/4)-1}(\cos\theta)^{2(1/2)-1} d\theta$$

$$= \frac{1}{2}\beta\left(\frac{3}{4},\frac{1}{2}\right)$$

$$= \frac{1}{2}\left(\frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{5}{4}\right)}\right)$$

$$= \frac{1}{2}\left(\frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\frac{1}{4}\Gamma\left(\frac{1}{4}\right)}\right)$$

and $\int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{sin\theta}} = \int_0^{\frac{\pi}{2}} (\sin\theta)^{-\frac{1}{2}}(\cos\theta)^0\,d\theta$

$$= \int_0^{\frac{\pi}{2}} (\sin\theta)^{2\left(\frac{1}{4}\right)-1}\,(\cos\theta)^{2\left(\frac{1}{2}\right)-1} d\theta$$

$$= \frac{1}{2}\beta\left(\frac{1}{4},\frac{1}{2}\right)$$

$$= \frac{1}{2}\frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{3}{4}\right)}$$

Therefore,

$$\int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{\sin\theta}} \int_0^{\frac{\pi}{2}} \sqrt{\sin\theta} = \frac{1}{2}\frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{3}{4}\right)} \cdot \frac{1}{2}\frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\frac{1}{4}\Gamma\left(\frac{1}{4}\right)}$$

$$= \pi.$$

Exercise

1. Given that $\int_0^\infty \frac{x^{\rho-1}}{1+x} = \frac{\pi}{\sin \rho\pi}$. Show that $\Gamma(\rho).\Gamma(1-\rho) = \frac{\pi}{\sin \rho\pi}$.
2. Evaluate the following improper integrals
   (i) $\int_0^\infty \sqrt{x}\, e^{-x^2} dx$
   (ii) $\int_0^\infty e^{-x^3} dx$ in terms of Gamma functions.
3. Use Beta and Gamma functions, to evaluate the integral

   $I = \int_{-1}^{1} (1-x^2)^n\, dx$, where '$n$' is a positive integer.

4. Express $\int_0^1 x^m (1-x^\rho)^n\, dx$ in terms of Beta function and hence evaluate the integral $\int_0^1 x^{\frac{3}{2}}(1-\sqrt{x})^{\frac{1}{2}} dx$.
5. Evaluate $\int_0^\infty 2^{-9x^2} dx$ using the Gamma function.
6. Show that $\int_0^{\frac{\pi}{2}} \sqrt{\tan x} = \frac{\pi}{\sqrt{2}}$.
7. Evaluate $\int_0^{\frac{\pi}{2}} \frac{dx}{\sqrt{\sin x}}$.
8. Evaluate $\int_0^1 x^n (\ln x)^m\, dx$.
9. Evaluate $\int_0^a \frac{x^{\frac{3}{2}}}{\sqrt{a^2-x^2}}\, dx$.

Answers

2. (i). $\frac{1}{2}\Gamma\left(\frac{3}{4}\right)$ Hint: put $x = \sqrt{t}$.

   (ii). $\frac{1}{3}\Gamma\left(\frac{1}{3}\right)$ Hint: put $x = t^{\frac{1}{3}}$.

3. $\frac{2^{2n+1}(n!)^2}{(2n+1)!}$ (Put $1 + x = 2t$).

4. $\frac{512}{3465}$ (Put $x^p = y$).

5. $\frac{1}{6}\sqrt{\frac{\pi}{\ln 2}}$ (Put $9x^2 \ln 2 = y$).

7. $\frac{1}{2}\beta\left(\frac{1}{4}, \frac{1}{2}\right)$.

8. By substituting $x = e^{-t}$: $\frac{(-1)^m}{(n+1)^{m+1}}\Gamma(m+1)$.

9. By substituting $x = asin\theta$: $\frac{1}{2}a^{\frac{3}{2}}\,\beta\left(\frac{5}{4}, \frac{1}{2}\right)$.

2.3

List of Standard Integrals

1) $\int x^n dx = \frac{x^{n+1}}{n+1}$ when $n \neq -1$

2) $\int \frac{dx}{x} = \log x$

3) $\int e^x dx = e^x$

4) $\int \sin x dx = -\cos x$

5) $\int \cos x dx = \sin x$

6) $\int \sec^2 x dx = \tan x$

7) $\int \operatorname{cosec}^2 x dx = -\cot x$

8) $\int \sec x \tan x dx = \sec x$

9) $\int \operatorname{cosec} x \cot x dx = -\operatorname{cosec} x$

10) $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x$

11) $\int \frac{dx}{1+x^2} = \tan^{-1} x$

12) $\int \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1} x$

13) $\int \sinh x dx = \cosh x$

14) $\int \cosh x dx = \sinh x$

15) $\int \frac{dx}{\sqrt{1+x^2}} = \sinh^{-1} x = \log\left(x + \sqrt{x^2+1}\right)$

16) $\int \frac{dx}{\sqrt{x^2-1}} = \cosh^{-1} x = \log\left(x + \sqrt{x^2-1}\right)$

17) $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right)$

18) $\int \frac{dx}{\sqrt{x^2-a^2}} = \cosh^{-1}\left(\frac{x}{a}\right)$

19) $\int \frac{dx}{\sqrt{a^2+x^2}} = \sinh^{-1}\left(\frac{x}{a}\right)$

20) $\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$

21) $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\log\left(\frac{x-a}{x+a}\right)$

22) $\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\log\left(\frac{a+x}{a-x}\right)$

23) $\int \tan x dx = \log(\sec x)$

24) $\int \cot x dx = \log(\sin x)$

25) $\int \sec x dx = \log(\sec x + \cot x)$

26) $\int \operatorname{cosec} x dx = -\log(\operatorname{cosec} x + \cot x)$

27) $\int \sqrt{a^2 - x^2} dx = \frac{x\sqrt{a^2 - x^2}}{2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)$

28) $\int \sqrt{x^2 - a^2} dx = \frac{x\sqrt{x^2 - a^2}}{2} - \frac{a^2}{2}\cosh^{-1}\left(\frac{x}{a}\right)$

29) $\int \sqrt{a^2 + x^2} dx = \frac{x\sqrt{a^2 + x^2}}{2} + \frac{a^2}{2}\sinh^{-1}\left(\frac{x}{a}\right)$

Improper Integrals Involving a Parameter

Often, we come across integrals of the form

$$\emptyset(\alpha) = \int_{a(\alpha)}^{b(\alpha)} f(x, \alpha) dx$$

where α is a parameter and the integrand f is such that the integral cannot be evaluated by standard methods. We can evaluate some of these integrals by differentiating the integral with respect to the parameter, that is first obtain $\emptyset'(\alpha)$, evaluate the integral (that is integrate with respect to x) and then integrate $\emptyset'(\alpha)$, with respect to α. Note that f is a function of two variables x and α. When we differentiate f with respect to α, we treat x as a constant and denote the derivatives as $\partial f/\partial \alpha$ (partial derivate of f with respect to α). We assume that f, $\partial f/\partial \alpha$, $a(\alpha)$ and $b(\alpha)$ are continuous functions of α. We now present the formula which gives the derivative of $\emptyset(\alpha)$.

Theorem (Leibniz formula): If $a(\alpha), b(\alpha), f(x,\alpha)$ and $\partial f/\partial\alpha$ are continuous functions of α, then

$$\frac{d\emptyset}{d\alpha} = \int_{a(\alpha)}^{b(\alpha)} \frac{\partial f}{\partial\alpha}(x,\alpha)dx + f(b,\alpha)\frac{db}{d\alpha} - f(a,\alpha)\frac{da}{d\alpha}$$

Proof: Let $\Delta\alpha$ be an increment in α and Δa, Δb be the corresponding increments in a and b. We have

$$\Delta\emptyset = \emptyset(\alpha+\Delta\alpha) - \emptyset(\alpha) = \int_{a+\Delta a}^{b+\Delta b} f(x,\alpha+\Delta\alpha)dx - \int_a^b f(x,\alpha)dx$$

$$= \int_{a+\Delta a}^{a} f(x,\alpha+\Delta\alpha)dx + \int_a^b f(x,\alpha+\Delta\alpha)dx + \int_b^{b+\Delta b} f(x,\alpha+\Delta\alpha)dx - \int_a^b f(x,\alpha)dx$$

Or

$$\frac{\Delta\emptyset}{\Delta\alpha} = \int_{a+\Delta a}^{a} \frac{1}{\Delta\alpha} f(x,\alpha+\Delta\alpha)dx + \int_a^b \frac{1}{\Delta\alpha}[f(x,\alpha+\Delta\alpha) - f(x,\alpha)]dx + \int_b^{b+\Delta b} \frac{1}{\Delta\alpha} f(x,\alpha+\Delta\alpha)dx. \quad \text{------------------ (1)}$$

Using the mean value theorem of integrals

$$\int_{x_0}^{x_1} f(x)dx = (x_1 - x_0)f(\xi),\ x_0 < \xi < x_1 \quad \text{------- (2)}$$

we get

$$\int_{a+\Delta a}^{a} f(x,\alpha+\Delta\alpha)dx = -\Delta a f(\xi_1,\alpha+\Delta\alpha), a < \xi_1 < a + \Delta a \text{ --- (3)}$$

and

$$\int_b^{b+\Delta b} f(x,\alpha+\Delta\alpha)dx = \Delta b f(\xi_2,\alpha+\Delta\alpha), b < \xi_2 < b + \Delta b \quad \text{--- (4)}$$

Using the Lagrange mean value theorem, we get

$$f(x,\alpha+\Delta\alpha) - f(x,\alpha) = \Delta\alpha\frac{\partial f}{\partial\alpha}(x,\xi_3),\ \alpha < \xi_3 < \alpha + \Delta\alpha. \text{ --(5)}$$

We note that

$$\lim_{\Delta\alpha \to 0} \xi_1 = a, \lim_{\Delta\alpha \to 0} \xi_2 = b \text{ and} \lim_{\Delta\alpha \to 0} \xi_3 = \alpha.$$

Taking limits as $\Delta\alpha \to 0$ on both sides of equation (1) and using the results in equations (2) to (5), we obtain

$$\frac{d\emptyset}{d\alpha} = \int_a^b \frac{\partial f}{\partial \alpha}(x, \alpha)dx + f(b, \alpha)\frac{db}{d\alpha} - f(a, \alpha)\frac{da}{d\alpha}.$$

Remark

a) If the limits $a(\alpha)$ and $b(\alpha)$ are constants, then Liebniz formula becomes

$$\frac{d\emptyset}{d\alpha} = \int_a^b \frac{\partial f}{\partial \alpha}(x, \alpha)dx$$

b) If the integrand f is independent of α, then Liebniz formula becomes

$$\frac{d\emptyset}{d\alpha} = f(b)\frac{db}{d\alpha} - f(a)\frac{da}{d\alpha}$$

c) Leibniz formula is often used to evaluate certain types of improper integrals.

Problem 1: Evaluate the integral $\int_0^\infty \frac{e^{-\alpha x} \sin x}{x} dx$, $\alpha > 0$ and deduce that

(i) $\int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}$, (ii) $\int_0^\infty \frac{\sin ax}{x} dx = \frac{\pi}{2}, a > 0$.

Solution: Let $\emptyset(\alpha) = \int_a^\infty \frac{e^{-\alpha x} \sin x}{x} dx$. ----------- (1)

The limits of integration are independent of the parameter α. We obtain

$$\frac{d\emptyset}{d\alpha} = \int_0^\infty \frac{\partial}{\partial\alpha}\left[\frac{e^{-\alpha x} \sin x}{x}\right] dx = -\int_0^\infty \frac{x e^{-\alpha x} \sin x}{x} dx = -\int_0^\infty e^{-\alpha x} \sin x \, dx.$$

Using the result $\int e^{-\alpha x} \sin x \, dx = -\frac{e^{-\alpha x}}{1+\alpha^2}(\alpha \sin x + \cos x)$, we obtain

$$\frac{d\emptyset}{d\alpha} = \left[\frac{e^{-\alpha x}}{1+\alpha^2}(\alpha \sin x + \cos x)\right]_0^\infty = -\frac{1}{1+\alpha^2}.$$

Integrating with respect to α, we get

$\emptyset(\alpha) = -\tan^{-1}\alpha + c$, where c is the constant of integration.

From (1), we get the condition $\emptyset(\infty) = 0$. Hence,

$$\emptyset(\infty) = 0 = -tan^{-1}\infty + c \Rightarrow c = \frac{\pi}{2}.$$

Therefore, $\emptyset(\alpha) = \int_0^\infty \frac{e^{-\alpha x} sinx}{x} dx = \frac{\pi}{2} - \tan^{-1}\alpha$.

(i) Setting $\alpha = 0$, we obtain $\int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}$.

(ii) Substituting $x = ay$ on the left hand side of (i) we obtain $\int_0^\infty \frac{\sin x}{x} dx = \int_0^\infty \frac{\sin ay}{y} dy = \frac{\pi}{2}$.

Problem 2:

Using the result $\int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$, evaluate the integral $\int_0^\infty e^{-x^2}\cos(2\alpha x)dx$

Solution: Let $\phi(\alpha) = \int_0^\infty e^{-x^2}\cos(2\alpha x)dx$

The limits of integration are independent of the parameter α. Hence,

$$\frac{d\phi}{d\alpha} = \int_0^\infty \frac{\delta}{\delta\alpha}\left[e^{-x^2}\cos(2\alpha x)\right]dx = \int_0^\infty (-2x)e^{-x^2}\sin(2\alpha x)dx$$

$$= \left[e^{-x^2}\sin(2\alpha x)\right]_0^\infty - 2\alpha\int_0^\infty e^{-x^2}\cos(2\alpha x)dx = -2\alpha\phi$$

Integrating the differential equation $\frac{d\phi}{d\alpha} + 2\alpha\phi = 0$, we obtain $\phi(\alpha) = ce^{-\alpha^2}$

We get the condition $\phi(0) = \int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$

Using this condition, we obtain $\phi(0) = \frac{\sqrt{\pi}}{2} = c$

Therefore, $\phi(\alpha) = \int_0^\infty e^{-x^2}\cos(2\alpha x)dx = \frac{\sqrt{\pi}}{2}e^{-\alpha^2}$

Problem 3: Evaluate the integral $\int_0^\infty \frac{\tan^{-1}(ax)}{x(1+x^2)}dx, a>0$ and $a \neq 1$

Solution:

Let $\emptyset(a) = \int_0^\infty \frac{\tan^{-1}(ax)}{x(1+x^2)} dx.$

We have

$$\frac{d\theta}{da} = \int_0^\infty \frac{\partial}{\partial a}\left[\frac{\tan^{-1}(ax)}{x(1+x^2)}\right]dx = \int_0^\infty \frac{dx}{(1+x^2)(1+a^2x^2)}$$

$$= \frac{1}{a^2-1}\int_0^\infty \left[\frac{a^2}{a^2x^2+1} - \frac{1}{1+x^2}\right]dx$$

$$= \frac{1}{a^2-1}[\{a\tan^{-1}(ax)\}_0^\infty - \{\tan^{-1}(x)\}_0^\infty] = \frac{\pi}{2}\left[\frac{a-1}{a^2-1}\right] = \frac{\pi}{2(a+1)}$$

Integrating with respect to a , we obtain

$$\emptyset(a) = \frac{\pi}{2}\ln(a+1) + c$$

We get the condition$\emptyset(0) = 0$. Using this condition, we obtain $\emptyset(0) = 0 = c$.

Therefore, $\emptyset(a) = \int_0^\infty \frac{\tan^{-1}(ax)}{x(1+x^2)} dx. = \frac{\pi}{2}\ln(a+1).$

EXERCISE

Evaluate the following integrals.

1. $\int_0^{\pi/2\alpha} \alpha \sin \alpha x dx$
2. $\int_{-\infty}^{\infty} x^2 e^{-x^2} dx, where \int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$
3. $\int_0^{\alpha^3} \cot^{-1}\left(x/\alpha^3 \right) dx$
4. $\int_0^{\pi} \frac{\cos x}{(a+b\cos x)^3} dx,$ given that $\int_0^{\pi} \frac{dx}{a+b\cos x} = \frac{\pi}{\sqrt{a^2-b^2}}, a>b>0.$
5. $\int_0^{\infty} \frac{dx}{(x^2+1)^{n+1}},$ n any positive integer.
6. $\int_0^{\infty} \frac{e^{-ax}-e^{-bx}}{x} dx, a>0, b>0.$
7. $\int_0^{1} \frac{x^a-x^b}{\log x} dx, a>b>-1.$

ANSWERS

1. 0
2. In $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$, let $x = \alpha y$ and define

   $\emptyset(\alpha) = \int_{-\infty}^{\infty} e^{-\alpha^2 y^2} dy = \frac{\sqrt{\pi}}{\alpha}$. Differentiate with respect to α and set $\alpha = 1$. We get $\emptyset = \frac{\sqrt{\pi}}{2}$.
3. $\frac{(\pi+2\ln 2)\alpha^3}{4}$

4. Differentiating $\int_0^{\pi} \frac{dx}{a+b\cos x} = \frac{\pi}{\sqrt{a^2-b^2}}$ with respect to b and readjusting the terms, we get $\int_0^{\pi} \frac{dx}{(a+b\cos x)^2} = \frac{\pi a}{(a^2-b^2)^{3/2}}$

Differentiate again with respect to b. We get

$\emptyset = \frac{-3\pi ab}{2(a^2-b^2)^{5/2}}$

5. Consider

$\phi(\alpha) = \int_0^{\infty} \frac{dx}{x^2+\alpha^2} = \frac{\pi}{2\sqrt{\alpha}}$. Differentiate n times with respect to α and $\alpha = 1$. We get $\emptyset(\alpha) = \frac{\pi}{2}\left[\frac{(2n)!}{2^{2n}(n!)^2}\right]$.

6. Find $\frac{d\emptyset}{da}$, integrate with respect to a, use $\emptyset(b) = 0$. We get $\emptyset(a) = \ln\left({}^{b}/_{a}\right)$.

7. Find $\frac{d\emptyset}{da}$, integrate with respect to a, use $\emptyset(b) = 0$. We get $\emptyset(a) = \ln\left[\frac{a+1}{b+1}\right]$.