## 1.3

## Linear Combination of Vectors

Observe that any vector $(a, b, c)$ in the vector space can be written as

$$(a, b, c) = a(1,0,0) + b(0,1,0) + c(0,0,1)$$

The vector $(1,0,0), (0,1,0)$ and $(0,1,0)$ in some sense characterize the vector space $\mathbb{R}^3$. We pursue this approach to understanding vector spaces in terms of certain vectors that represent the whole space.

**Definition:** Let $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m$ be vectors in a vector space $V$. We say that $v$, a vector in $V$, is a linear combination of $v_1, v_2, \dots, v_m$ if there exists scalars of $c_1, c_2, \dots, c_m$ such that '$v$' can be written as

$$v = c_1 v_1 + c_2 v_2 + \cdots + c_m v_m$$

**Example:** The vector (5,4,2) is a linear combination of the vectors (1,2,0) (3,1,4) and (1,0,3). Since it can be written as

$(5,4,2) = (1,2,0) + 2(3,1,4) - 2(1,0,3)$.

**DEFINITION:** The vectors $v_1, v_2, \dots \dots \dots, v_m$ are said to span a vector space if every vector in the space can be expressed as a linear combination of these vectors.

A spanning set of vectors in a sense defines the vector space, since every vector in the space can be obtained from this set.

We have developed the mathematics for looking at a vector space in terms of a set of vectors that spans the space. It is also useful to be able to do the converse, namely to use a set of vectors to generate a vector space.

**THEOREM:** Let $v_1, v_2, \dots, v_m$ be vectors in a vector space $V$. Let $U$ be the set consisting of all linear combinations of $v_1, v_2, \dots, v_m$. Then $U$ is a subspace of $V$ spanned by the vectors $v_1, v_2, \dots, v_m$. $U$ is said to be the vector space generated by $v_1, v_2, \dots, v_m$.

**Proof:** Let $u_1 = a_1 v_1 + \cdots + a_m v_m$ and $u_2 = b_1 v_1 + \cdots + b_m v_m$ be arbitrary elements of $U$. Then

$$u_1 + u_2 = (a_1 v_1 + \cdots + a_m v_m) + (b_1 v_1 + \cdots + b_m v_m)$$

$$= (a_1 + b_1) v_1 + (a_m + b_m) v_m .$$

$u_1 + u_2$ is a linear combination of $v_1, v_2, \dots, v_m$. Thus $u_1 + u_2$ is in $U$. vector addition.

Let '$c$' be an arbitrary scalar. Then

$$cu_1 = c(a_1 v_1 + \cdots + a_m v_m) = ca_1 v_1 + \cdots + ca_m v_m$$

$cu_1$ is a linear combination of $v_1, v_2, \dots, v_m$. Therefore $cu_1$ is in $U$. $U$ is closed under scalar multiplication. Thus $U$ is a subspace of $v$.

By the definition of $U$, every vector in $U$ can be written as a linear combination of $v_1, v_2, \dots, v_m$. Thus $v_1, v_2, \dots, v_m$ span $U$.

**Problem 1:** Determine whether or not the vector $(-1,1,5)$ is a linear combination of the vectors $(1,2,3)(0,1,4)$ and $(2,3,6)$

**Solution:** We examine the identity

$C_1\,(1,2,3) + C_2\,(0,1,4) + C_3\,(2,3,6) = (-1,1,5)$

Can we find scalars $C_1$ , $C_2$ and $C_3$ such that this identity holds?

Using the operations of addition and scalar multiplication we get

$(C_1 + 2C_3\,, 2C_1 + C_2 + 3C_3\,, 3C_1 + 4\,C_2 + 6C_3\,) = (-1,,5)$

Equating components leads to the following system of linear equations.

$C_1 + 2C_3 = -1$

$2C_1 + C_2 + 3C_3 = 1$

$3C_1 + 4\,C_2 + 6C_3 = 5$

It can be shown that this system of equations has the unique solution.

$C_1 = 1, C_2 = 2, C_3 = -1.$

Thus the vector $(-1,1,5)$ has the following linear combination of the vectors $(1,2,3)(0,1,4)$ and $(2,3,6)$

$(-1,1,5) = (1,2,3) + 2(0,1,4) - 1(2,3,6).$

**Problem 2:** Express the vector (4,5,5) as a linear combination of the vectors (1,2,3), (−1,1,4) and (3,3,2)

**Solution:** Examine the following indentify for values of $C_1$, $C_2$ and $C_3$.

$C_1\,(1,2,3) + C_2\,(-1,1,4) + C_3\,(3,3,2) = (4,5,5)$

We get $(C_1 - C_2 + 3C_3,\ 2C_1 + C_2 + 3C_3,\ 3C_1 + 4\,C_2 + 2C_3\ ) = (4,5,5)$

Equating components leads to the following system of linear equations.

$C_1 - C_2 + 3C_3 = 4$

$2C_1 + C_2 + 3C_3 = 5$

$3C_1 + 4\,C_2 + 2C_3 = 5$

This system of equations has many solutions,

$C_1 = -2r + 3, \qquad C_2 = r - 1, \qquad C_3 = r$

Thus the vector can be expressed in many ways as a linear combination of the vectors (1,2,3), (−1,1,4) and (3,3,2)

$(4,5,5) = (-2r + 3)\,(1,2,3) + (r - 1)\,(-1,1,4) + r(3,3,2)$

For example,

$r = 3$ gives $(4,5,5) = -3\,(1,2,3) + 2(-1,1,4) + 3(3,3,2)$

$r = -1$ gives $(4,5,5) = 5\,(1,2,3) - 2(-1,1,4) - (3,3,2)$.

**Problem 3:** Show that the vector $(3,-4,-6)$ cannot be expressed as a linear combination of the vectors $(1,2,3)$ $(-1,-1,-2)$ and $(1,4,5)$

**Solution:** Consider the identity

$$C_1\,(1,2,3) + C_2\,(-1,-1,-2) + C_3\,(1,4,5) = (3,-4,-6)$$

This identity leads to the following system of linear equations.

$$C_1 - C_2 + C_3 = 3$$

$$2C_1 - C_2 + 4C_3 = -4$$

$$3C_1 - 2\,C_2 + 5C_3 = 6$$

This system has no solution. Thus $(3,-4,-6)$ is not a linear combination of the vectors

$(1,2,3)$ $(-1,-1,-2)$ and $(1,4,5)$.

**Problem 4:** Show that the vectors $(1,2,0), (0,1,-1)$ and $(1,1,2)$ span $\mathbb{R}^3$.

**Solution:** Let $(x, y, z)$ be an arbitrary element of $\mathbb{R}^3$.

We have to determine whether we can write $(x, y, z) = C_1\,(1,2,0) + C_2\,(0,1,-1) + C_3\,(1,1,2)$.

Multiply and add the vectors to get

$(x, y, z) = (C_1 + C_3, 2C_1 + C_2 + C_3, -C_2 + 2C_3)$

Thus, $\quad C_1 + C_3 = x$

$2C_1 + C_2 + C_3 = y$

$-C_2 + 2C_3 = z$

This system of equations in the variables $C_1, C_2$ and $C_3$ is solved by the method of Gauss-Jordon elimination. It is found to have the solution

$C_1 = 3x - y - z,$

$C_2 = -4x + 2y + z,$

$C_3 = -2x + y + z.$

We can write an arbitrary vector of $\mathbb{R}^3$ as a linear combination of these vectors as follows.

$$\begin{aligned}(x, y, z) &= (3x - y - z)\,(1,2,0) + (-4x + 2y + z)\,(0,1,-1)\\ &\quad + (-2x + y + z)\,(1,1,2).\end{aligned}$$

The vectors $(1,2,0), (0,1,-1)$ and $(1,1,2)$ span $\mathbb{R}^3$.

**Problem 5:** Let $v_1$and $v_2$ span a subspace $U$ of a vector space $V$. Let $k_1$ and $k_2$ be non-zero scalars. Show that $k_1 v_1$ and $k_2 v_2$ also span $U$.

**Solution:** Let $v$ be a vector in

Since $v_1$ and $v_2$ span $U$. There exists scalars $a$ and $b$ such that

$$v = a\, v_1 + b\, v_2$$

we can write

$$v = \frac{a}{k_1}(k_1 v_1) + \frac{b}{k_2}(k_2 v_2)$$

Thus the vectors $k_1 v_1$ and $k_2 v_2$ span $U$.

**Problem 6:** Let '$U$' be the subspace generated by the vectors $(1, 2, 0)$ and $(-3,1,2)$. Let $V$ be the subspace of $\mathbb{R}^3$ generated by the vectors $(-1,5,2)$ and $(4,1,-2)$. Show that $U = V$.

**Solution**: Let '$u$' be a vector in $U$. Let us show that $u$ is in $V$.

Since $u$ is in $U$, there exists scalars $a$ and $b$ such that

$$u = a\,(1, 2, 0) + b\,(-3,1,2)$$

$$= (a - 3b, 2a + b, 2b)$$

Let us see if we can write u as a linear combination of $(-1,5,2)$ and $(4,1,-2)$

$$u = p(-1,5,2) + q\,(4,1,-2)$$

$$= (-p + 4q, 5p + q, 2p - 2q)$$

Such $p$ and $q$ would have to satisfy

$$-p + 4q = a - 3b$$

$$5p + q = 2a + b$$

$$2p - 2q = 2b.$$

This system of eqs has unique solution $p = \frac{a+b}{3}, q = \frac{a-2b}{3}$.

Thus $u$ can be written as

$$p = \frac{a+b}{3}(-1,5,2) + \frac{a-2b}{3}(4,1,-2).$$

Therefore $u$ is a vector in $V$. Conversely, let $v$ be a vector in $V$. Similar to the above we can show that $v$ is in $U$. Therefore $U = V$.

**Exercise**

1. Let $U$ be the vector space generated by the functions $f(x) = x + 1$ and $g(x) = 2x^2 - 2x + 3$. Show that the function $h(x) = 6x^2 - 10x + 5$ lies in $U$.

2. In the following sets of vectors, determine whether the first vector is a linear combination of the other vectors.

(a)$(-3,3,7)$; $(1,-1,2), (2,1,0), (-1,2,1)$

(b)$(0,10,8)$; $(-1,2,3), (1,3,1), (1,8,5)$

3. Determine whether the following vectors span $\mathbb{R}^3$.

(a) $(2,1,0), (-1,3,1), (4,5,0)$

(b)$(1,2,1), (-1,3,0), (0,5,1)$

4. Give three other vectors in the subspace of $\mathbb{R}^3$generated by the vectors $(1,2,3), (1,2,0)$.

5. Let $U$ be the subspace of $\mathbb{R}^3$generated by the vectors $(3,-1,2)$ and $(1,0,4)$. Let $V$ be the subspace of $\mathbb{R}^3$generated by the vectors $(4,-1,6)$ and $(1,-1,-6)$. Show that $U = V$.

6. In each of the following, determine whether the first function is a linear combination of the functions that follow:

(a)$f(x) = 3x^2 + 2x + 9$; $g(x) = x^2 + 1, h(x) = x + 3$

(b) $f(x) = x^2 + 4x + 5$; $g(x) = x^2 + x - 1, h(x) = x^2 + 2x + 1$

7. Let $v$, $v_1$ and $v_2$. be vectors in a vector space $V$. Let v be a linear combination of $v_1$ and $v_2$. If $c_1$ and $c_2$ are nonzero

scalars, show that $v$ is also a linear combination of $c_1v_1$ and $c_2v_2$.

**Answers**

2. (a) $(-3,3,7) = 2(1,-1,2) - (2,1,0) + 3(-1,2,1)$

(b) $(0,10,8) = (2-c)(-1,2,3) + (2-2c)(1,3,1) + c(1,8,5)$, whether c is any real number

3. (a) Span (b) Do not span

4. e.g., $(1,2,3) + (1,2,0) = (2,4,3), (1,2,3) - (1,2,0)$

$= (0,0,3), 2(1,2,3) = (2,4,6)$.