FINITE DIFFERENCES

Assume that we have a table of values $x_1, y_1, i = 0, 1, 2, \ldots\ldots, n$ of any function $y = f(x)$, the values of x being equally spaced, i.e. $x_1 = x_0 + i\,h\,i = 0, 1, 2, \ldots, n$. Suppose that we are required to recover the values of $f(x)$ for some intermediate values of x, or to obtain the derivate of $f(x)$ for some x in the range $x_0 \le x \le x_n$. The methods for the solution to these problems are based on the concept of the 'differences' of a function which we now proceed to define.

Forward Differences

If $y_0, y_1, y_2, \ldots, y_n$ denote a set of values of y, then $y_1 - y_0, y_2 - y_1, \ldots, y_n - y_{n-1}$ are called the differences of y. Denoting these differences by $\Delta y_0, \Delta y_1, \ldots, \Delta y_{n-1}$ respectively, we have

$$\Delta y_0 = y_1 - y_0,\ \Delta y_1 = y_2 - y_1, \ldots, \Delta y_{n-1} = y_n - y_{n-1},$$

Where Δ is called the forward difference operator and $\Delta y_0, \Delta y_1, \ldots,$ are called first forward differences. The differences of the first forward differences are called second forward differences and are denoted by $\Delta^2 y_1, \Delta^2 y_1, \ldots$ Similarly, one can define third forward differences, forth forward differences, ect. Thus,

$$\Delta^2 y_0 = \Delta y_1 - \Delta y_0 = y_2 - y_1 - (y_1 - y_0)$$

$$= y_2 - 2y_1 + y_0,$$

$$\Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_0 = y_3 - 2y_2 + y_1 - (y_2 - 2y_1 + y_0)$$

$$= y_3 - 3y_2 + 3y_1 - y_0.$$

$$\Delta^4 y_0 = \Delta^3 y_1 - \Delta^3 y_0 = y_4 - 3y_3 + 3y_2 - y_1 - (y_3 - 3y_2 - 3y_1 - y_0)$$

$$= y_4 - 4y_3 + 6y_2 - 4y_1 + y_0.$$

It is therefore clear that any higher-order difference can easily be expressed in terms of the ordinates, since the coefficients occurring on the right side are the binomial coefficients.

Table shows how the forward differences of all orders can be formed.

Table: Forward Difference Table

x	y	Δ	Δ^2	Δ^3	Δ^4	Δ^5	Δ^6
x_0	y_0						
		Δy_0					
x_1	y_1		$\Delta^2 y_0$				
		Δy_1		$\Delta^3 y_0$			
x_2	y_2		$\Delta^2 y_1$		$\Delta^4 y_0$		
		Δy_2		$\Delta^3 y_1$		$\Delta^5 y_0$	
x_3	y_3		$\Delta^2 y_2$		$\Delta^4 y_1$		$\Delta^6 y_0$
		Δy_3		$\Delta^3 y_2$		$\Delta^5 y_1$	
x_4	y_4		$\Delta^2 y_3$		$\Delta^4 y_2$		
		Δy_4		$\Delta^3 y_3$			
x_5	y_5		$\Delta^2 y_4$				
		Δy_5					
x_6	y_6						

Backward Differences

The differences $y_1 - y_0, y_2 - y_1, \dots, y_n - y_{n-1}$ are called first backward differences if they are denoted by $\nabla y_1, \nabla y_2, \dots, \nabla y_n$ respectively, so that$\nabla y_1 = y_1 - y_0, \nabla y_2 = y_2 - y_1, \dots, \nabla y_n = y_n - y_{n-1}$, where ∇ is called the backward difference operator. In a similar way, one can define backward differences of higher orders. Thus we obtain

$$\nabla^2 y_2 = \nabla y_2 - \nabla y_1 = y_2 - y_1 - (y_1 - y_0) = y_2 - 2y_1 + y_0$$

$$\nabla^3 y_3 = \Delta^3 y_3 - \Delta^3 y_2 = y_3 - 3y_2 + 3y_1 - y_0, etc$$

With the same values of x and y as in table, a backward difference table can be formed:

Table: Backward Difference Table

x	y	∇	∇^2	∇^3	∇^4	∇^5	∇^6
x_0	y_0						
		∇y_1					
x_1	y_1		$\nabla^2 y_2$				
		∇y_2		$\nabla^3 y_3$			
x_2	y_2		$\nabla^2 y_3$		$\nabla^4 y_4$		
		∇y_3		$\nabla^3 y_4$		$\nabla^5 y_5$	
x_3	y_3		$\nabla^2 y_4$		$\nabla^4 y_5$		$\nabla^6 y_6$
		∇y_4		$\nabla^3 y_5$		$\nabla^5 y_6$	
x_4	y_4		$\nabla^2 y_5$		$\nabla^4 y_6$		
		∇y_5		$\nabla^3 y_6$			
x_5	y_5		$\nabla^2 y_6$				
		∇y_6					
x_6	y_6						

Central Differences

The central difference operator δ is defined by the relations

$$y_1 - y_0 = \delta y_{1/2}, \quad y_2 - y_1 = \delta y_{3/2}, \ldots, \quad y_n - y_{n-1} = \delta y_{n-1/2}.$$

Similarly, higher-order central differences can be defined. With the values of x and y as in the preceding two tables, a central difference table can be formed:

Table: Central Difference Table

x	y	δ	δ^2	δ^3	δ^4	δ^5	δ^6
x_0	y_0						
		$\delta y_{1/2}$					
x_1	y_1		$\delta^2 y_1$				
		$\delta y_{3/2}$		$\delta^3 y_{3/2}$			
x_2	y_2		$\delta^2 y_2$		$\delta^4 y_2$		
		$\delta y_{5/2}$		$\delta^3 y_{5/2}$		$\delta^5 y_{5/2}$	
x_3	y_3		$\delta^2 y_3$		$\delta^4 y_3$		$\delta^6 y_3$
		$\delta y_{7/2}$		$\delta^3 y_{7/2}$		$\delta^5 y_{7/2}$	
x_4	y_4		$\delta^2 y_4$		$\delta^4 y_4$		
		$\delta y_{9/2}$		$\delta^3 y_{9/2}$			
x_5	y_5		$\delta^2 y_5$				
		$\delta y_{11/2}$					
x_6	y_6						

It is clear from the three tables that in a definite numerical case, the same numbers occur in the same positions whether we use forward, backward or central differences. Thus we obtain

$$\Delta y_0 = \nabla y_1 = \delta y_{1/2}, \qquad \Delta^3 y_2 = \Delta^3 y_5 = \delta^3 y_{7/2}, \ldots.$$

Symbolic Relations and Separation of Symbols

Difference formula can easily be established by symbolic methods, using the shift operator E and the averaging or the mena operator μ, in addition to the operators, Δ, ∇ and δ already defined.

The averaging operator μ is defined by the equation:

$$\mu y_r = \frac{1}{2}\left(y_{r+1/2} + y_{r-1/2}\right).$$

The shift operator E is defined by the equation:

$$Ey_r = y_{r+1},$$

Which shows that the effect of E is to shift the functional value y_r to the next higher value y_{r+1}. A second equation with E gives

$$E^2 y_r = E(Ey_r) = Ey_{r+1} = y_{r+2},$$

and in general

$$E^n y_r = y_{r+n}.$$

It is now easy to derive a relationship between Δ and E, for we have

$$\Delta y_0 = y_1 - y_0 = Ey_0 - y_0 = (E-1)y_0$$

and hence

$$\Delta \equiv E - 1 \quad or\ E \equiv 1 + \Delta.$$

We can now express any higher-order forward difference in terms of the given function values. For example,

$$\Delta^3 y_0 = (E-1)^3 y_0 = (E^3 - 3E^2 + 3E - 1)y_0 = y_3 - 3y_2 + 3y_1 - y_0.$$

From the definitions, the following relations can easily be established:

$$\nabla = 1 - E^{-1}$$

$$\delta = E^{1/2} - E^{-1/2},$$

$$\mu = (1/2)\left(E^{1/2} + E^{-1/2}\right), \mu^2 = 1 + (1/4)\delta^2$$

$$\Delta = \nabla E = \delta E^{1/2}.$$

As an example, we prove the relation $\mu^2 \equiv 1 + (1/4)\delta^2$. We have, by definitions,

$$\begin{aligned}
\mu y_r &= \frac{1}{2}\left(y_{r+1/2} + y_{r-1/2}\right) \\
&= \frac{1}{2}\left(E^{1/2} y_r + E^{-1/2} y_r\right) \\
&= \frac{1}{2}\left(E^{1/2} + E^{-1/2}\right) y_r
\end{aligned}$$

Hence

$$\mu = \frac{1}{2}\left(E^{1/2} + E^{-1/2}\right)$$

and

$$\begin{aligned}
\mu^2 &= \frac{1}{4}\left(E^{1/2} + E^{-1/2}\right)^2 \\
&= \frac{1}{4}\left(E + E^{-1} + 2\right) \\
&= \frac{1}{4}\left[\left(E^{1/2} - E^{-1/2}\right)^2 + 4\right] \\
&= \frac{1}{4}\left(\delta^2 + 4\right).
\end{aligned}$$

We therefore have

$$\mu = \sqrt{1 + \frac{1}{4}\delta^2}.$$

Finally, we define the operator D such that

$$Dy(x) = \frac{d}{dx} y(x).$$

To relate D to E, we start with the Taylor's series

$$y(x+h) = y(x) + hy'(x) + \frac{h^2}{2!} y''(x) + \frac{h^3}{3!} y'''(x) + \cdots$$

This can be written in the symbolic form

$$Ey(x) = \left(1 + hD + \frac{h^2D^2}{2!} + \frac{h^2D^2}{3!} + \cdots \right) y(x).$$

Since the series in the brackets is the expansion of e^{hD}, we obtain the interesting result

$$E \equiv e^{hD}.$$

Using the relation $\Delta \equiv E - 1$, a number of useful identities can be derived. This relation is used to separate the effect of E into that of the powers of Δ and this method of separation is called the method of separation of symbols.

DETECTION OF ERRORS BY USE OF DIFFERENCE TABLES

Difference tables can be used to check errors in tabular values. Suppose that there is an error of +1 in a certain tabular value. As higher differences are formed, the error spreads out fanwise, and is at the same time, considerably magnified, as shown in table.

Table: Detection of Errors using Difference Table

y	Δ	Δ^2	Δ^3	Δ^4	Δ^5
0					
	0				
0		0			
	0		0		
0		0		0	
	0		0		1

0		0		1	
	0		1		-5
0		1		-4	
	1		-3		10
1		-2		6	
	-1		3		-10
0		1		-4	
	0		-1		5
0		0		1	
	0		0		-1
0		0		0	
	0		0		
0		0			
	0				
0					

This table shows the following characteristics:

i. The effect of the error increases with the order of the differences.
ii. The errors in any one column are the binomial coefficients with alternating signs.
iii. The algebraic sum of the errors in any difference column is zero, and
iv. The maximum error occurs opposite the function value containing the error. These facts can be used to detect errors by difference tables. We illustrate this by means of an example.

Example: Consider the following difference table

x	y	Δ	Δ^2	Δ^3	Δ^4
1	3010				
		414			

2	3424		-36		
		378		-39	
3	3802		-75		178
		303		139	
4	4105		64		-271
		367		-132	
5	4472		-68		181
		299		49	
6	4771		-19		-46
		280		3	
7	5051		-16		
		264			
8	5315				

The term -271 in the fourth difference column has fluctuations of 449 and 452 on either side of it. Comparison with above Table suggests that there is an error of -45 in the entry for $x = 4$. The correct value of y is therefore $4105 + 45 = 4150$, which shows that the last-two digits have been transposed, a very common form of error. The reader is advised to form a new difference table with this correction, and to check that the third differences are now practically constant.

If an error is present is a given data, the differences of some order will become alternating in sign. Hence, higher-order differences should be formed till the error is revealed as in the above example. If there are errors in several tabular values, then it is not easy to detect the errors by differencing.

DIFFERENCES OF A POLYNOMIAL

Let $y(x)$ be a polynomial of the nth degree so that

$$y(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_n.$$

Then we obtain

$$y(x+h) - y(x) = a_0[(x+h)^n - x^n] + a_1[(x+h)^{n-1} - x^{n-1}] + ..$$

$$= a_0(nh)x^{n-1} + a'_1 x^{n-2} + \cdots + a'_n,$$

where $a'_1, a'_2, \ldots, a'_n$ are the new coefficients.

The above equation can be written as

$$\Delta y(x) = a_0(nh)x^{n-1} + a'_1 x^{n-2} + \cdots + a'_n,$$

Which shows that the first difference of a polynomial of the nth degree is a polynomial of degree $(n-1)$. Similarly, the second difference will be a polynomial of degree $(n-2)$, and the coefficient of x^{n-2} will be $a_0 n(n-1)h^2$. Thus the nth difference is $a_0 n! h^n$, which is a constant. Hence the $(n+1)th$, and higher differences of a polynomial of nth degree will be zero. Conversely, if the nth difference of a tabulated function are constant and the $(n+1)$ th, $(n+2)$ th,..., differences all vanish, then the tabulated function represents a polynomial of degree n, It should be noted that these results hold good only if the values of x are equally spaced. The converse is important in numerical analysis since it enables us to approximate a function by a polynomial if its differences of some order become nearly constant.

Problem 1: Using the method of separation of symbols, show that $\Delta^n u_{x-n} = u_x - nu_{x-1} + \frac{n(n+1)}{2} u_{x-2} + \cdots + (-1)^n u_{x-n}$.

Solution: To prove this result, we start with the right-hand side. Thus,

$$u_x - n u_{x-1} + \frac{n(n-1)}{2} u_{x-2} + \cdots + (-1)^n u_{x-n}$$

$$= u_x - nE^{-1} u_x + \frac{n(n-1)}{2} E^{-2} u_x + \cdots + (-1)^n E^{-n} u_x$$

$$= \left[1 - nE^{-1} + \frac{n(n-1)}{2} E^{-2} + (-1)^n E^{-n} \right] u_x$$

$$= (1 - E^{-1})^n u_x$$

$$= \left(1 - \frac{1}{E} \right)^n u_x$$

$$= \left(\frac{E-1}{E} \right)^n u_x$$

$$= \frac{\Delta^n}{E^n} u_x$$

$$= \Delta^n E^{-n} u_x$$

$$= \Delta^n u_{x-n},$$

which is the left-hand side.

Problem 2: Show that

$$e^x \left(u_0 + x\Delta u_0 + \frac{x^2}{2!} \Delta^2 u_0 + \cdots \right) = u_0 + u_1 x + u_2 \frac{x^2}{2!} + \cdots$$

Solution: Now,

$$e^x \left(u_0 + x\Delta u_0 + \frac{x^2}{2!} \Delta^2 u_0 + \cdots \right) = e^x \left(1 + x\Delta + \frac{x^2 \Delta^2}{2!} + \cdots \right) u_0$$

$$= e^x e^{x\Delta} u_0 = e^{x(1+\Delta)} u_0$$

$$= e^{xE} u_0$$

$$= \left(1 + xE + \frac{x^2E^2}{2!} + \cdots\right) u_0$$

$$= u_0 + xu_1 + \frac{x^2}{2!} u_2 + \cdots,$$

which is the required result.

Problem 3: Evaluate $\Delta^2(\cos x)$

Solution: We know $\Delta f(x) = f(x+h) - f(x),\ h > 0$

$$\Delta^2(\cos x) = \Delta(\Delta \cos x)$$

$$= \Delta[\cos(x+h) - \cos x]$$

$$= \Delta\left[-2\sin\left(x + \frac{h}{2}\right)\sin\frac{h}{2}\right]$$

$$\left[\because \cos C - \cos D = -2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)\right]$$

$$= -2\sin\frac{h}{2}\ \Delta\left[\sin\left(x + \frac{h}{2}\right)\right]$$

$$= -2\sin\frac{h}{2}\ \Delta\left[-\cos\left(\frac{\pi}{2} + x + \frac{h}{2}\right)\right]$$

$$= 2\sin\frac{h}{2}\ \Delta\left[\cos\left(x + \frac{h+\pi}{2}\right)\right]$$

$$= 2\sin\frac{h}{2}\left[-2\sin\left\{(x+h) + \frac{\pi}{2}\right\}\sin\frac{h}{2}\right]$$

$$= -4\ \sin^2\frac{h}{2}\cos(x+h).$$

Problem 4: Evaluate $\Delta\left[\frac{f(x)}{g(x)}\right]$.

Solution: $\Delta\left[\frac{f(x)}{g(x)}\right] = \frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}$

$$= \frac{f(x+h)g(x) - f(x)g(x+h)}{g(x)g(x+h)}$$

$$= \frac{g(x)f(x+h) - f(x)g(x) + f(x)g(x) - f(x)g(x+h)}{g(x)g(x+h)}$$

$$= \frac{g(x)[f(x+h) - f(x)] - f(x)[g(x+h) - g(x)]}{g(x)g(x+h)}$$

$$= \frac{g(x)\Delta f(x) - f(x)\Delta g(x)}{g(x)g(x+h)}.$$

Problem 5: Show that $\nabla \equiv E^{-1}\Delta$.

Solution:

We know $\Delta f(x) = f(x+h) - f(x)$ or $\Delta y_n = y_{n+1} - y_n$

$\nabla f(x) = f(x-h) - f(x)$ or $\nabla y_n = y_n - y_{n-1}$

$E^n y_r = y_{r+n}$.

Take $E^{-1}\Delta y_n = E^{-1}(y_{n+1} - y_n)$

$= E^{-1} y_{n+1} - E^{-1} y_n$

$= y_n - y_{n-1}$

$= \nabla y_n$

$\therefore E^{-1}\Delta \equiv \nabla$.

Problem 6: For the following data,
calculate $\Delta f(0.3), \Delta^2 f(0.2), \Delta^2 f(0.3)$ and $\Delta^3 f(0.1)$.

x	0.1	0.2	0.3	0.4	0.5
$f(x)$	1.40	1.56	1.76	2.00	2.28

Solution:

The difference table is obtained as

x	$f(x)$	Δf	$\Delta^2 f$	$\Delta^3 f$	$\Delta^4 f$
0.1	1.40				
		0.16			
0.2	1.56		0.04		
		0.20		0.0	
0.3	1.76		0.04		0.0
		0.24		0.0	
0.4	2.00		0.04		
		0.28			
0.5	2.28				

By table $\Delta f(0.3) = 0.24$

$$\Delta^2 f(0.2) = 0.04$$

$$\Delta^2 f(0.3) = 0.04$$

$$\Delta^3 f(0.1) = 0.$$

EXERCISE

1. Form a table of difference for the function $f(x) = x^3 + 5x - 7$ for$x = -1,0,1,2,3,4,5$. Continue the table to obtain $f(6)$ and $f(7)$.
2. Evaluate
   A. $\Delta^2 x^3$
   B. $\Delta[(x+1)(x+2)]$
   C. $\Delta(tan^{-1} x)$
3. Prove the following:
   a. $\Delta^n y_x = y_{x+n} - n_{C_1} y_{x+n-1} + n_{C_2} y_{x+n-2} - \cdots + (-1)^n y_x$
   b. $u_1 + u_2 + \cdots + u_n = n_{C_1} u_1 + n_{C_2} \Delta u_1 + \cdots + \Delta^{n-1} u_1$
4. Define the operators, Δ, ∇, δ and E,E^{-1} and show that
   i. $\Delta \equiv E\nabla$
   ii. $\nabla = E^{-1} \Delta$
   iii. $E \equiv 1 + \Delta$
   iv. $E^{-1} \equiv 1 - \nabla$
   v. $\Delta\nabla y_k = \nabla\Delta y_k = \delta^2 y_k$
   vi. $\Delta(y^2{}_k) = (y_k + \Delta y_{k+1})\Delta y_k$
   vii. $\Delta\left(\frac{1}{y_k}\right) = -\frac{\Delta y_k}{y_k y_{k+1}}$
5. Show that $E \equiv 1 + \Delta$ and $\Delta \equiv \nabla(1-\nabla)^{-1}$. Also, deduce that $1 + \Delta \equiv (E-1)\nabla^{-1}$.
6. Prove the following relations:
   i. $\nabla - \Delta = -\Delta\nabla$.
   ii. $\Delta + \nabla = \Delta/\nabla - \nabla/\Delta$
   iii. $\Delta(f_i\, g_i) = f_i \Delta g_i + g_{i+1} \Delta f_i$
   iv. $\Delta\left(\frac{1}{f_i}\right) = -\frac{\Delta f_i}{(f_i f_{i+1})}$.
7. If $f(x) = e^{ax}$, show that $\Delta^n f(x) = (e^{ah} - 1)^n e^{ax}$

ANSWERS

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2. (C) $\tan^{-1}\left[\frac{h}{1+x(x+h)}\right]$