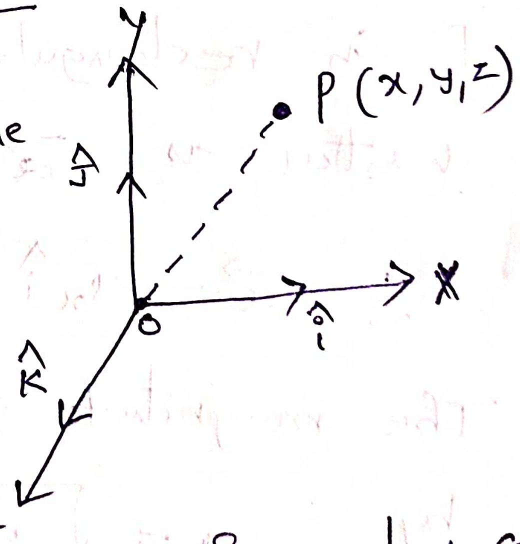
UNIT- I (Mathematical physics)

In electromagnetics, Some fundamental quantities are Scalars and while Some other are Vectors. These Scalars and Vectors are can be functions of position and time. They can be completely described using an appropriate co-ordinate system. They are (i) Cartesian co-ordinate System (ii) cylindrical co-ordinate system (iii) Spherical co-ordinate system

Cartesian Co-ordinate System



An orthogonal system is one in which the co-ordinates are mutually perpendicular. In Cartesian co-ordinate system, a point 'P' in the Space has co-ordinates x, y and z. OP is a line drawn from origin to the point 'P'. Here $\vec{OP}$ is a position Vector. $\therefore \vec{OP} = \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$$\therefore \boxed{r = |\vec{r}| = \sqrt{x^2+y^2+z^2}}$$

The position of the particle 'p' represented by the position Vector $\vec{op} = \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$.

The instantaneous Velocity of 'p' in Cartesian Coordinate system is given by $\vec{v} = \frac{d\vec{r}}{dt}$

$$\therefore \vec{v} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k} = v_x\hat{i} + v_y\hat{j} + v_z\hat{k}$$

The Magnitude of Velocity of particle 'v' can be written as. $V = \sqrt{V_x^2 + V_y^2 + V_z^2}$

Similarly, the instantaneous acceleration of 'p' in rectangular co-ordinate System can be written as $\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt} V_x\hat{i} + \frac{d}{dt} V_y\hat{j} + \frac{d}{dt} V_z\hat{k}$

$$\therefore \vec{a} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k}$$

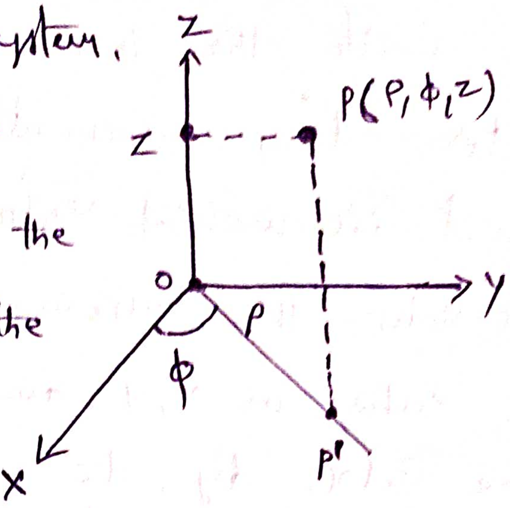
The magnitude of acceleration (a) of p is given by $a = \sqrt{a_x^2 + a_y^2 + a_z^2}$.

The ranges of co-ordinates Variables are. i.e for x, y, z are:

$$-\infty < x < \infty$$
$$-\infty < y < \infty$$
$$-\infty < z < \infty.$$

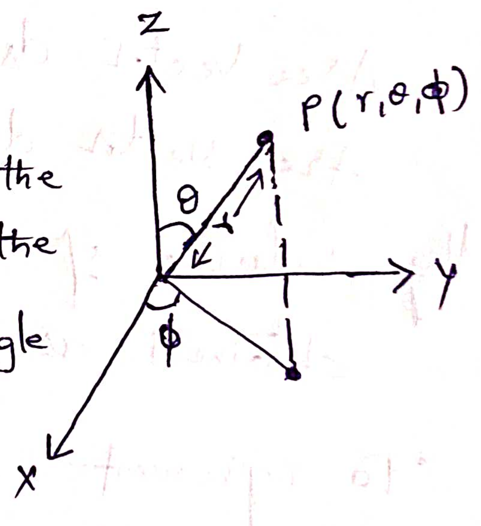
Ex:- A body moving (Car) in one direction is exampl

Cylindrical co-ordinate System:



In cylindrical co-ordinate system, the co-ordinates are ρ, ϕ, z. 'ρ' is the distance from the origin to the projection of the point 'p' on xy plane. 'ϕ' be the angle made by 'x-axis' with this projection vector. Here 'z' is same as that of Cartesian co-ordinate system. Here 'p' is a point having coordinates (ρ, ϕ, z), then its projection on XY plane may be denoted by pp'.

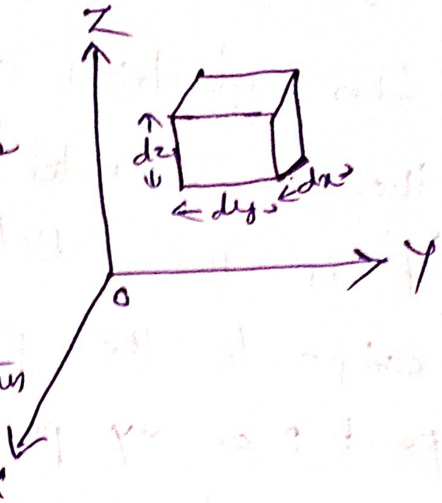
Spherical Co-ordinate System:



'r' is shortest distance between origin to the point 'p'. 'θ' is the angle made by z-axis with the position vector $\vec{r}$. 'ϕ' is the angle made by position vector with x-axis.

Cartesian Coordinate (or) Rectangular Co-ordinate System

To write the incremental length vector $\vec{dl}$, incremental area vector and incremental volume vector, consider the incremental length of a cube in X, Y and Z-direction be dx, dy, dz in each of the 3- coordinates.



$$\therefore \text{length } \vec{dl} = dx\,\hat{a}_x + dy\,\hat{a}_y + dz\,\hat{a}_z$$

$\hat{a}_x$, $\hat{a}_y$, $\hat{a}_z$ be the unit vectors along x, y and z-directions.

Area vector ds in x direction $= dy \cdot dz$

Area vector ds in y direction $= dx\,dz$

Area vector ds in z direction $= dx\,dy$.

Ily volume of Cartesian coordinate system can be obtained as Volume $= dx\,dy\,dz$.

To represent any vector in Cartesian coordinate system which is extending from (x_1, y_1, z_1) to $(x_2\ y_2\ z_2)$;

$$\vec{A} = (x_2 - x_1)\,\hat{a}_x + (y_2 - y_1)\,\hat{a}_y + (z_2 - z_1)\,\hat{a}_z$$

Unit vector in the direction of $\vec{A}$ can be obtained as $\hat{A} = \dfrac{\vec{A}}{|\vec{A}|}$

$$\therefore \hat{A} = \frac{(x_2-x_1)\hat{a}_x + (y_2-y_1)\hat{a}_y + (z_2-z_1)\hat{a}_z}{\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}}$$

Ex:-1 Specify the unit vector extending from the origin towards the point G (−2, −2, 1)

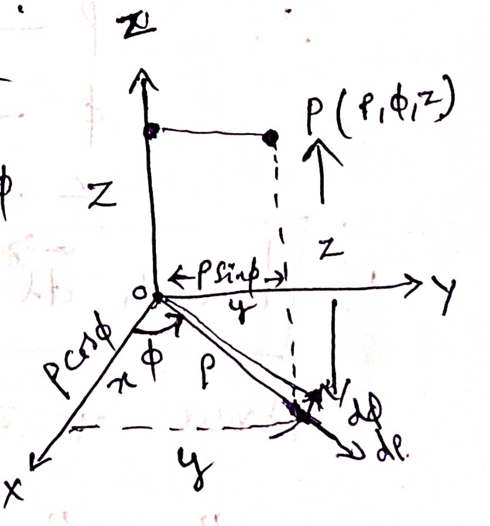
Sol:- Vector is from O (0,0,0) to G(−2, −2, 1)

$$\vec{G} = (-2-0)\hat{a}_x + (-2-0)\hat{a}_y + (1-0)\hat{a}_z = -2\hat{a}_x - 2\hat{a}_y + \hat{a}_z$$

$$\therefore G = |\vec{G}| = \sqrt{2^2+2^2+1^2} = \sqrt{9} = 3$$

unit vector $\hat{G}$ is given by, $\hat{G} = \frac{\vec{G}}{|\vec{G}|} = \frac{-2\hat{a}_x - 2\hat{a}_y + \hat{a}_z}{3}$

2. **Cylindrical Co-ordinate System:**



From the figure the component of ρ along x-direction is $x = \rho\cos\phi$

$y = \rho\sin\phi$, $z = z$

From these equations, we get

$x^2 = \rho^2\cos^2\phi$ and $y^2 = \rho^2\sin^2\phi$

$\therefore x^2+y^2 = \rho^2 \Rightarrow \boxed{\rho = \sqrt{x^2+y^2}}$

Similarly from $x = \rho\cos\phi$ and $y = \rho\sin\phi$, we have

$\tan\phi = \frac{y}{x} \Rightarrow \boxed{\phi = \tan^{-1}(y/x)}$ and

$\boxed{z = z}$

To find the lengths, area and volume in cylindrical Co-ordinate system, we have to understand how the incremental lengths varies.

If the incremental length in the direction of 'ρ' be '$d\rho$', if we increase the length in z-direction, it will be 'dz', if we increase the lengths in the direction ϕ, it will be calculated as using the formula,

$$\text{angle} = \frac{\text{arc}}{\text{radius}} \Rightarrow d\phi = \frac{l}{\rho} \Rightarrow \boxed{l = \rho\, d\phi}$$

$\therefore$ The incremental length in the direction of 'ϕ' will be $\rho\, d\phi$ ($\because$ arc lengths)

$\therefore$ $\rho \rightarrow d\rho$

$\phi \rightarrow \rho\, d\phi$

$z \rightarrow dz$.

$$\therefore \boxed{\text{lengths } \vec{dl} = d\rho\, \hat{a}_\rho + \rho\, d\phi\, \hat{a}_\phi + dz\, \hat{a}_z}$$

Area at z plane $= \rho\, d\rho\, d\phi$

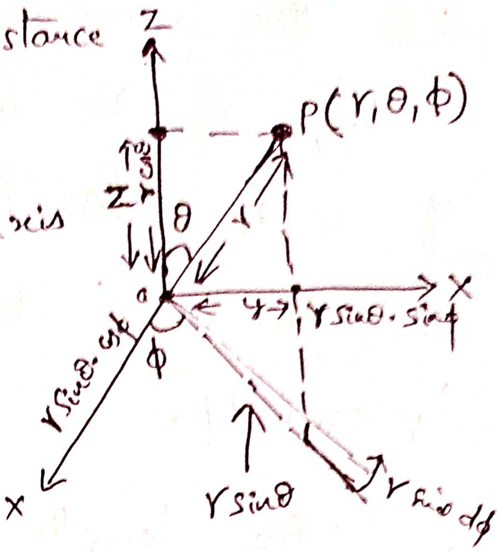
" " ρ-plane $= \rho\, d\phi\, dz$

" " ϕ-plane $= d\rho\, dz$

$$\therefore \boxed{\text{volume} = \rho\, d\rho\, d\phi\, dz}.$$

Spherical Co-ordinate System:

$r \rightarrow$ length
$\theta, \phi \rightarrow$ angles.



In this system, r is the distance of the point 'p' from origin.

'θ' is the angle made by z-axis with the position vector 'r'.

'ϕ' is the angle made by the line joining to the projection of the point and with the x-axis.

From the figure, it is clear that horizontal component of 'r' is '$r\cos\theta$' (i.e along z-axis) and the vertical component of 'r' is, $r\sin\theta$. (i.e in the xy plane, since xy plane is $\perp$er to z-axis).

Since $r\sin\theta$ makes an angle 'ϕ' with x-direction,

The x-component of '$r\sin\theta$' along x-axis $= r\sin\theta.\cos\phi$

lly The y-component of $r\sin\theta$ along y-axis $= r\sin\theta.\sin\phi$

$\therefore$ $x = r\sin\theta.\cos\phi$ — ①

$y = r\sin\theta.\sin\phi$ — ②

$z = r\cos\theta.$ — ③

$\therefore$ by taking ①^2+②^2+③^2 $\Rightarrow$ $r^2 = x^2+y^2+z^2$

$\Rightarrow \boxed{r = \sqrt{x^2+y^2+z^2}}$

From ③, we have $r\cos\theta = z$

$$\cos\theta = \frac{z}{r} \Rightarrow \theta = \cos^{-1}\left(\frac{z}{r}\right)$$

$$\therefore \boxed{\theta = \cos^{-1}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)}$$

From ① & ②, $\tan\phi = y/x \Rightarrow \boxed{\phi = \tan^{-1}(y/x)}$

If the incremental change in the direction of 'r' is 'dr'; if the incremental change in the direction of 'θ' is $r\,d\theta$. ($\because$ angle $(d\theta) = \frac{\text{arc length}}{\text{radius}}$) and if the incremental change in the direction 'φ' be '$r \sin\theta\, d\phi$'. $\therefore$

$$\boxed{\begin{matrix} r \rightarrow dr \\ \theta \rightarrow r\,d\theta \text{ and } \phi \rightarrow r\sin\theta\, d\phi \end{matrix}}$$

$\therefore$ The length $\vec{dl} = dr\,\hat{a}_r + r\,d\theta\,\hat{a}_\theta + r\sin\theta\, d\phi\,\hat{a}_\phi$.

Area :

r-plane	θ-plane	φ plane
$r^2 \sin\theta\, d\theta\, d\phi$	$r\,dr \sin\theta\, d\phi$	$r\,dr \cdot d\theta$.

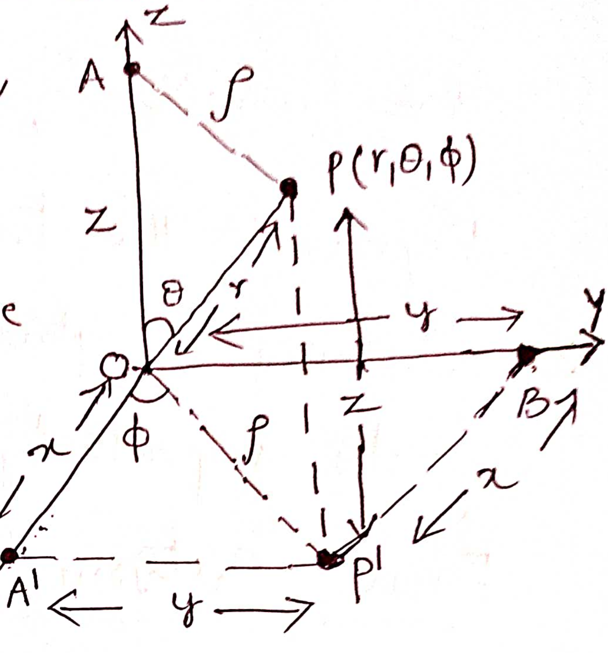
Volume : $r^2 dr \sin\theta\, d\theta\, d\phi$

$\therefore$ Length, Area and volume Calculated in Spheri-
-cal Co-ordinate system.

Transformations

From Cartesian to Spherical Co-ordinate System:

Consider a point $P(r, \theta, \phi)$ in Spherical Co-ordinate system. Draw the projection of the point 'P' on to the x-y-plane as shown.



From the triangle 'OPA'

$$\sin\theta = \frac{\rho}{r} \Rightarrow \rho = r\sin\theta \quad -(1)$$

$$\cos\theta = \frac{z}{r} \Rightarrow z = r\cos\theta \quad -(2)$$

$$\tan\theta = \frac{\rho}{z} \Rightarrow \theta = \tan^{-1}\left(\frac{\rho}{z}\right) = \tan^{-1}\left[\frac{\sqrt{x^2+y^2}}{z}\right]; \text{ since}$$

From the figures, we have $\rho^2 = x^2+y^2 \Rightarrow \boxed{\rho = \sqrt{x^2+y^2}}$ $-(3)$

and $r^2 = z^2+\rho^2 \Rightarrow \boxed{r = \sqrt{z^2+\rho^2}}$ $-(4)$

put (3) in (4), we get $r = \sqrt{z^2+x^2+y^2} = \sqrt{x^2+y^2+z^2}$

$$\therefore \boxed{r = \sqrt{x^2+y^2+z^2}} \; -(5); \quad \boxed{\theta = \tan^{-1}\left[\frac{\sqrt{x^2+y^2}}{z}\right]} \; -(6)$$

Similarly from the Δ^{le} $OP'A'$

$$\cos\phi = \frac{x}{\rho} \Rightarrow \boxed{x = \rho\cos\phi}; \quad \sin\phi = \frac{y}{\rho} \Rightarrow \boxed{y = \rho\sin\phi}$$

$$\tan\phi = \frac{y}{x} \Rightarrow \boxed{\phi = \tan^{-1}(y/x)} \quad -⑦$$

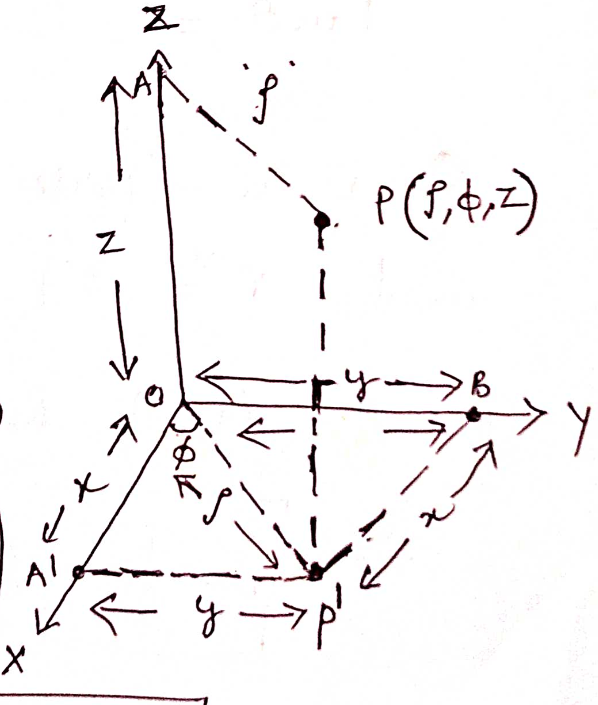
$\therefore$ Equations ⑤, ⑥ and ⑦, i.e,

$$r = \sqrt{x^2+y^2+z^2}, \quad \theta = \tan^{-1}\frac{\sqrt{x^2+y^2}}{z}, \quad \phi = \tan^{-1}(y/x)$$

represents the transformations of co-ordinates from Cartesian to Spherical and Spherical to Cartesian co-ordinates respectively.

From Cartesian to cylindrical co-ordinate System:

Consider a point $P(\rho, \phi, z)$ in a cylindrical co-ordinate system. The relationship between the values of Cartesian co-ordinates to cylindrical co-ordinates are obtained from the figure.



From $\triangle^{le}$ $OP'P$, $\sin\phi = \frac{y}{\rho}$

$$\Rightarrow y = \rho\sin\phi \quad -①$$

$$\cos\phi = \frac{x}{\rho} \Rightarrow x = \rho\cos\phi \quad -②$$

$$\tan\phi = \frac{y}{x} \Rightarrow \boxed{\phi = \tan^{-1}(y/x)} -③$$

$$\boxed{z = z} - ④$$

$$①^2 + ②^2 \Rightarrow x^2+y^2 = \rho^2 \Rightarrow \boxed{\rho = \sqrt{x^2+y^2}}$$

$\therefore$ $\rho = \sqrt{x^2+y^2}$, $\phi = \tan^{-1}(y/x)$, $z = z$ $\;-⑤$

$x = \rho\cos\phi$, $y = \rho\sin\phi$, $z = z$ $\;-⑥$

are transformations of co-ordinates from cartesian to cylinder and vice-versa.

problems ① If $\vec{A} = 2\hat{i}+3\hat{j}+\hat{k}$, then find the unit vector of $\vec{A}$?

Sol:- Given $\vec{A} = 2\hat{i}+3\hat{j}+\hat{k}$

WKT $\hat{a} = \frac{\vec{A}}{|\vec{A}|} = \frac{2\hat{i}+3\hat{j}+\hat{k}}{\sqrt{4+9+1}} = \frac{1}{\sqrt{14}}(2\hat{i}+3\hat{j}+\hat{k})$

② Convert $(2,3,5)$ into Spherical co-ordinates.

Sol:- Given Cartesian co-ordinates are $(2,3,5)$

WKT $r = \sqrt{x^2+y^2+z}$, $\theta = \tan^{-1}\frac{\sqrt{x^2+y^2}}{z}$, $\phi = \tan^{-1}\frac{y}{x}$

$\therefore r = \sqrt{4+9+25} = \sqrt{38}$

$\theta = \tan^{-1}\frac{\sqrt{4+9}}{5} \Rightarrow \theta = \tan^{-1}\left(\frac{\sqrt{13}}{5}\right) = \tan^{-1}(0.72) = 35.79$

$\phi = \tan^{-1}(y/x) = \tan^{-1}(3/2) = 56.30$

③ Find the cylindrical co-ordinates of the point $(3\hat{i}+4\hat{j}+\hat{k})$

Given $x=3, y=4, z=1$

WKT, $\rho = \sqrt{x^2+y^2}$, $\phi = \tan^{-1}(y/x)$, $z=z$

$\rho = \sqrt{3^2+4^2} \Rightarrow \rho = \sqrt{16+9} = \sqrt{25} = 5 \Rightarrow \boxed{\rho=5}$

$\phi = \tan^{-1}(4/3) = 53.13 \Rightarrow \boxed{\phi = 53.13}$

$\boxed{z=1}$

Table

From / To	Cartesian	Spherical	Cylindrical
Cartesian	$x = x$ $y = y$ $z = z$	$x = r\sin\theta \cdot \cos\phi$ $y = r\sin\theta\sin\phi$ $z = r\cos\theta$	$x = \rho\cos\phi$ $y = \rho\sin\phi$ $z = z$
Spherical	$r = \sqrt{x^2+y^2+z^2}$ $\theta = \tan^{-1}\frac{\sqrt{x^2+y^2}}{z}$ $\phi = \tan^{-1}(y/x)$	$r = r$ $\theta = \theta$ $\phi = \phi$	$r = \sqrt{\rho^2+z^2}$ $\theta = \tan^{-1}\left(\frac{\rho}{z}\right)$ $\phi = \phi$
Cylindrical	$\rho = \sqrt{x^2+y^2}$ $\phi = \tan^{-1}(y/x)$ $z = z$	$\rho = r\sin\theta$ $\phi = \phi$ $z = r\cos\theta$	$\rho = \rho$ $\phi = \phi$ $z = z$.

problem: Convert a point $(5, 60^\circ, 7)$ to Cartesian Co-ordinate

Sol:- Given $(5, 60^\circ, 7)$, it is a cylindrical system.

$\therefore \rho = 5, \phi = 60^\circ, z = 7$

Conversion of cylindrical to Cartesian is given by

$x = \rho\cos\phi, \; y = \rho\sin\phi, \; z = z$

$\therefore x = 5 \times \cos 60^\circ = 5 \times 1/2 = 2.5$

$y = 5 \times \sin 60^\circ = 5 \times \sqrt{3}/2 = 4.33$

$z = 7$

$\therefore (x, y, z) = (2.5, 4.33, 7)$

(2) Convert a point $(6, 45°, 60°)$ to Cartesian and then Cartesian to cylindrical?

Sol:- (a) The given system is in spherical

$\therefore (r, \theta, \phi) = (6, 45°, 60°)$

$x = r\sin\theta . \cos\phi, \quad y = r\sin\theta . \sin\phi, \quad z = r\cos\theta.$

$\therefore x = 6 \times \sin 45° \times \cos 60° = 6 \times \frac{1}{\sqrt{2}} \times \frac{1}{2} = \frac{3}{\sqrt{2}} = 2.12$

$y = 6 \times \sin 45° \times \sin 60° = 6 \times \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{2} = 3.67$

$z = 6 \times \cos 45° = 6 \times \frac{1}{\sqrt{2}} = 3\sqrt{2} = 4.24$

$\therefore (x, y, z) = (2.12, 3.67, 4.24)$

$\therefore$ Cartesian co-ordinates are $(x, y, z) = (2.12, 3.67, 4.24)$

(b) Cartesian to cylindrical

Given $(x, y, z) = (2.12, 3.67, 4.24)$

ie $x = 2.12, \; y = 3.67, \; z = 4.24$

$\rho = \sqrt{x^2+y^2}, \quad \phi = \tan^{-1}(y/x), \quad z = z$

$\therefore \rho = \sqrt{(2.12)^2 + (3.67)^2} = 4.25$

$\phi = \tan^{-1}\left(\frac{3.67}{2.12}\right) = 59.97$

$z = 4.24$

$\therefore$ cylindrical co-ordinates are $(\rho, \phi, z) = (4.25, 59.97, 4.24)$

Scalar field: A scalar quantity which can be specified at every point in a region of space is a scalar field.

Ex:- Electric potential is the example of scalar field.

Vector field: A vector quantity which can be specified at every point in a region (or) space is called vector field.

Ex:- Gravitational force field, Electric field due to a point charge

Gradient: The 'del' operator (∇) plays a very important role in vector calculus. It is not a vector.

$$\nabla = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}$$

Del (∇) is used to define gradient of a scalar field, divergence and curl of a vector field.

Let $\phi(x,y,z)$ be a scalar function of position of a scalar point of co-ordinates (x,y,z)

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Then Gradient of a Scalar function ϕ is defined as $grad(\phi) = \nabla\phi$

ie $\nabla\phi = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\phi$

$\therefore$ $Grad\,\phi = \nabla\phi = \left(\vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}\right)$ — ①

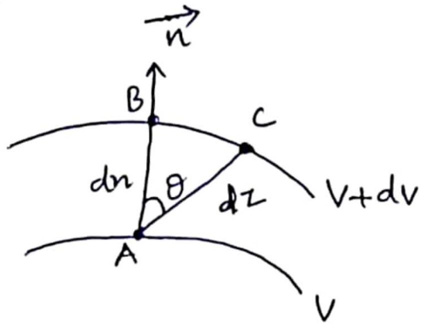
$\therefore$ ∇ is a Vector operator. When it is operated with a Scalar (ϕ), it converts the Scalar into a Vector.

The Vector ($\nabla\phi$) is called Gradient of a Scalar.

physical Significance:

Consider two surfaces V and V+dv. Let the magnitude of 'V' be constant on these surfaces. Let 'dv' be the small change in V.

Let A be point on surface 'V' and B, C be the points on the surface V+dv.



The rate of change of function on the point 'A' along 'AB' is $\frac{dV}{dn}$

lly along 'AC', the rate of change of function $= \frac{dV}{dz}$.

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From figure, $\cos\theta = \frac{dn}{dz} \Rightarrow dn = dz\cos\theta.$

$$\therefore \frac{dv}{dn} = \frac{dv}{dz\cos\theta} \quad (\because dn = dz\cos\theta)$$

$$\Rightarrow \frac{dv}{dz} = \frac{dv}{dn}\cos\theta \quad \text{--- (2)}$$

It is obvious from the equation, that when $\theta = 0$, $\frac{dv}{dz}$ is maximum.

$$\therefore \text{ when } \theta = 0, \quad \frac{dv}{dz} = \frac{dv}{dn}$$

$\therefore$ The maximum rate of increase of a scalar function 'V' at any point is given in magnitude & direction by Vector $\left(\frac{dv}{dn}\right)\hat{n}$, where $\vec{n} \rightarrow$ unit norm Vector at that point

The Vector is defined as gradient of scalar field at that point and can be written as

$$\text{grad } V = \frac{dv}{dn}\vec{n} \quad \text{--- (3)}$$

Divergence: The dot product of an operator del (∇) with a Vector $\vec{A}$ is called divergence The divergence is a scalar.

Let $\vec{A}$ be a Vector function with co-ordinates (x, y, z) in a region of space.

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The divergence of $\vec{A}$ is given by

$$\nabla \cdot \vec{A} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\left(A_x\vec{i} + A_y\vec{j} + A_z\vec{k}\right)$$

$$\therefore \nabla \cdot \vec{A} = \left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right)$$

$$\therefore \text{div}\,\vec{A} = \nabla \cdot \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \quad — (1)$$

physical Significance:

The divergence of a Vector $\vec{V}$ is a measure of how much a Vector $\vec{V}$ spreads out from the point.

The Vector may have Positive divergence (arrows pointed out) and negative divergence (arrows pointed in) as shown.

(a) Positive divergence

(b) Divergence is negative.

A point of positive divergence is a Source.

A " "ve divergence " sink (drain)

If $\text{div}\,\vec{A} = 0$, then $\vec{A}$ is called Solenoidal Vector

i.e if $\nabla \cdot \vec{A} = 0$, then $\vec{A}$ = Solenoid (or) in Compressible

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Curl:- The cross product of an operator del (∇) with a Vector $\vec{A}$ is called as Curl.

'Curl' is a Vector.

Let $\vec{A}$ be a Vector with co-ordinates (x, y, z) in a region of space.

The curl of $\vec{A}$ is given by $(\nabla \times \vec{A})$.

$$\nabla \times \vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) - \hat{j}\left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\right) + \hat{k}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) \quad \text{— ①}$$

Physical Significance: The curl of a Vector function $\vec{V}$ $(\nabla \times \vec{V})$ is a measure of how much the Vector $\vec{V}$ rotates around that point.

When curl $V = 0$, it means that no rotation is attached with Vector $\vec{V}$, where as curl $V \neq 0$, it means that rotation is attached with Vector $\vec{V}$ as shown in figure.

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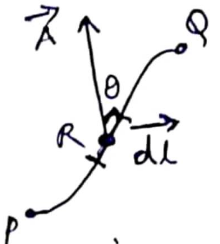
Integral Calculus

Line Integral :

The integration of a Vector along a curve is called its line integral. Consider a curve PQ, drawn between two points P and Q in Vector field as shown. Let 'dl' be the small element of the curve PQ at a point 'R'. Let 'A' be a Vector drawn from 'R' making an angle 'θ' with the direction of dl.



Then $\quad A \cdot dl = A\,dl \cos\theta = dl(A\cos\theta)$ — ①

This means that the value of $A \cdot dl$ at any point of curve is equal to product of small element 'dl' and the component of $A\cos\theta$ of 'A' along the direction 'dl'.

The Value of $A \cdot dl$ for complete curve 'PQ' can be obtained by integrating the equation ①,

Hence $\quad \int_P^Q A \cdot dl = \int_P^Q dl\,(A\cos\theta)$ — ②

Integral $\int_P^Q A \cdot dl$ is defined as the line integral of 'A' along the curve 'PQ'.

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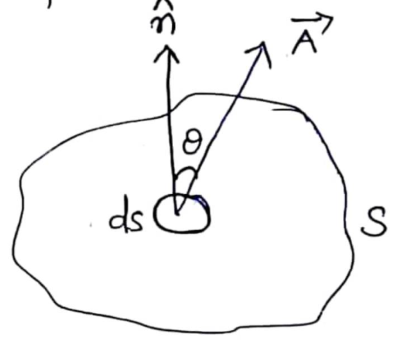
Example: If 'A' represents the force acting on a particle moving along a curve from 'P' to 'Q', then the line integral $\int_P^Q A \cdot dl$ represents the total work done by a force during the motion of a particle over its entire path from 'P' to 'Q'.

Surface integral:

Consider a simple surface 's' in a Vector field bounded by a curve as shown. Let ds be an infinite small element of the surface.

This surface element of area 'ds' can be repr-esented by area Vector 'ds'. Let $\hat{n}$ be a unit Vector drawn outward the surface in the direct-ion of 'ds', then $ds = \hat{n}\, ds$



Let 'A' be a Vector at the middle of 'ds' element making an angle 'θ' with $\hat{n}$.

Now, the scalar product $A \cdot ds = A \cdot \hat{n}\, ds$

$$= A\, ds \cos\theta .$$

This is called flux of Vector field 'A' across the elemental ~~(diarea)~~ area Vector 'ds'.

The total flux of the Vector field across the entire surface area 's' is given by

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$$\iint_S A \cdot ds = \iint_S A \cdot \hat{n}\, ds = \iint_S A \cos\theta\, ds .$$

This is defined as Surface integral.

<u>Ex:-</u> Let 'V' denotes Velocity Vector of a moving fluid. Consider a surface 's' in the fluid.

Now, the Surface integral, i.e $\iint_S V \cdot ds$

It represents the amount of fluid flowing per unit time normally to the surface.

The Surface integral is taken as '+ve' when the fluid flows out side the closed surface and "-ve" when the fluid flows into the closed surface.

<u>Volume integral</u>:

Consider a closed surface in space enclosing a Volume 'V'. Let 'A' be a Vector at a point in a small element of Volume 'dv', then the integral $\iiint_V A\, dv$ is called Volume integral of Vector 'A' over the surface.

<u>Stoke's theorem</u>

The Line integral of a Vector field 'A' along a closed curve is equal to the surface integral of the curl of a Vector 'A' taken over the surface 'S' surrounded by closed curve.

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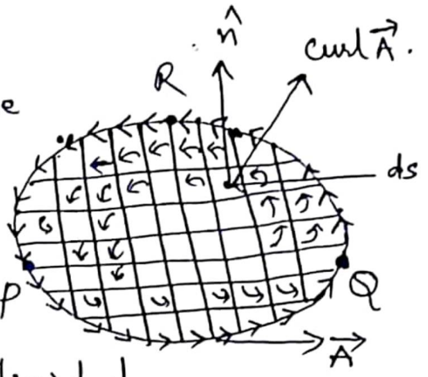
i.e $\oint A \cdot dr = \iint_S \text{curl} A \cdot ds = \iint_S (\nabla \times A) \cdot ds$

<u>proof</u>:- Consider a surface 'S' enclosed in a Vector field 'A'. The boundary of the surface is closed curve PQR.

The line integral 'A' around the curve PQR traced counter clockwise is $\oint A \cdot dr$ — ①



Let the entire surface be divided into no. of small loops of each area 'ds'. Let $\hat{n}$ be a unit Vector drawn out ward normal upon 'ds'.

The Vector area of the element is $\hat{n}\,ds = ds$.

Since the curl of a Vector at any point is the maximum line integral of the Vector per unit area along the boundary of infinitesimal area at that point, so the line integral of 'A' around the boundary of area 'ds' is (becomes) curl $A \cdot ds$

It applies to all surface elements.

Hence, the sum of the line integrals of 'A' around the boundaries of all area elements is given by

$$\iint_S \text{curl} A \cdot ds \quad — ②$$

From the figure, it is clear that the line integrals along the common sides of continuous elements

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mutually cancel because they are traversed in opposite directions. only the sides of element which lie in the periphery of the surface (inclosed curve) contribute to the line integral.

The sum of the line integrals on the boundary line of the curve is given by equation ②. It is also given by equation ①.

Hence $\oint A \cdot dr = \iint_S \text{curl } A \cdot ds = \iint (\nabla \times A)\, ds$.

Significance: stoke's theorem gives a method to convert a surface integral into a line integral and vice-versa. When a curl A = 0, the line integral of 'A' over the closed path is zero.

* Gauss - divergence theorem:

Statement: The surface integral of a vector field 'A' over a closed surface is equal to the volume integral of the divergence of a vector field 'A' over the volume 'V' enclosed by the closed surface.

i.e $\iint_S A \cdot ds = \iiint_V \text{div} A \cdot dv = \iiint_V (\nabla \cdot A)\, dv$

This theorem provides a method for connecting volume integrals to surface integrals.

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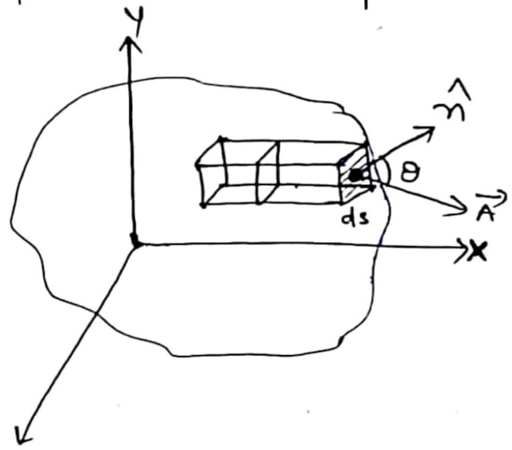
Proof:- Consider a closed surface 's' of any arbitrary shape in a vector field 'A'. Let 'v' be the volume of the surface. Let the whole volume may be assumed to be divided into a large no. of cubical volume elements.

We know that div A represents the amount of flux diverging per unit volume.

Hence, the flux diverging from element of volume

$dv = \text{div } A \cdot dv$



∴ The total flux coming out of the entire volume 'v' is given by $\iiint_V \text{div } A \cdot dv$ —①

Consider a small element of area 'ds' on the surface 's' as shown. Let $\hat{n}$ be the unit vector drawn normal to area 'ds'. If the vector field (or) vector A and outward normal $\hat{n}$ are at an angle 'θ', then the component of A along $\hat{n}$ is

$$A \cos\theta = A \cdot \hat{n}$$

The flux of 'A' through the surface element 'ds' is given by $(A \cdot \hat{n})\, ds = A \cdot ds$

Since, the flux is defined as the product of

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normal component of vector and surface area $\hat{}$,

The total flux through entire surface (S) is given by

$$\iint_S A \cdot ds \quad \text{--- (2)}$$

This must be equal to total flux diverging from the whole volume 'V' enclosed by the surface 'S'.

Hence, from (1) & (2), we get

$$\iint_S A \cdot ds = \iiint_V \text{div } A \cdot dv \quad \text{--- (3)}$$

This is Gauss theorem of divergence.

It may also be written as

$$\iint_S (A \cdot \hat{n})\, ds = \iiint_V (\nabla \cdot A)\, dv$$

<u>problems</u>: (1) find the curl of grad ϕ, where

$\phi = x\hat{i} + y\hat{j} + z\hat{k}$.

sol:- $\text{grad } \phi = \nabla \cdot \phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(\phi)$

$\therefore \text{grad } \phi = \left(\hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z}\right)$

$\text{Curl}(\text{grad } \phi) = \nabla \times (\nabla \phi) \Rightarrow$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial z} \end{vmatrix}$$

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$$= \hat{i}\left(\frac{\partial^2\phi}{\partial y\partial z} - \frac{\partial^2\phi}{\partial y\partial z}\right) - \hat{j}\left(\frac{\partial^2\phi}{\partial x\partial z} - \frac{\partial^2\phi}{\partial x\partial z}\right)$$

$$+ \hat{k}\left(\frac{\partial^2\phi}{\partial x\partial y} - \frac{\partial^2\phi}{\partial x\partial y}\right)$$

$$= 0$$

$\therefore$ curl grad $\phi = 0$.

② find the divergence of curl of a Vector $\vec{A}$,

where $\vec{A} = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$

Sol:- curl $A = \nabla \times A = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix}$

$$\nabla \times A = \hat{i}\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right) - \hat{j}\left(\frac{\partial A_3}{\partial x} - \frac{\partial A_1}{\partial z}\right) + \hat{k}\left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right)$$

div (curl A) $= \nabla \cdot (\nabla \times A)$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot \left[\hat{i}\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right) - \hat{j}\left(\frac{\partial A_3}{\partial x} - \frac{\partial A_1}{\partial z}\right) + \hat{k}\left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right)\right]$$

$$= \left[\cancel{\frac{\partial^2 A_3}{\partial x\partial y}} - \cancel{\frac{\partial^2 A_2}{\partial x\partial z}} - \cancel{\frac{\partial^2 A_3}{\partial x\partial y}} + \cancel{\frac{\partial^2 A_1}{\partial y\partial z}} + \cancel{\frac{\partial^2 A_2}{\partial x\partial z}} - \cancel{\frac{\partial^2 A_1}{\partial y\partial z}}\right]$$

$$= 0.$$

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Laplacian Operator:

Let 'ϕ' be a scalar potential function. if 'E' be the electric field intensity, then $E = -\nabla\phi$

$$\therefore \quad E = -\nabla\phi = -\text{grad}\,\phi \quad — ①$$

From Gauss law in Electrostatics,
The divergence of an electric field emerging from a closed surface is equal to $\frac{1}{\epsilon_0}$ times the charge density (ρ)

$$\left\{ \begin{array}{l} \int E \cdot ds = \frac{q}{\epsilon_0} \\ \therefore \oint E \cdot ds = \frac{\rho}{\epsilon_0}\int dv \\ \int(\nabla \cdot E)\, dv = \frac{\rho}{\epsilon_0}\int dv \\ \therefore \nabla \cdot E = \frac{\rho}{\epsilon_0}. \end{array} \right\}$$

i.e $\quad \nabla \cdot E = \frac{\rho}{\epsilon_0} \quad — ②$

put ① in ②, we get

$$\nabla \cdot (-\nabla\phi) = \frac{\rho}{\epsilon_0} \quad — ③$$

But $\nabla \cdot \nabla\phi = \text{div}(\text{grad}\,\phi)$

$$= \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\left(\hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}\right)$$

$$= \left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}\right)$$

$$\nabla \cdot (\nabla\phi) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\phi$$

$$\Rightarrow \quad \nabla^2\phi = \phi\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)$$

$$\therefore \quad \boxed{\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}}$$

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This is called as Laplacian operator (or) del-squared.

$$\therefore \nabla \cdot (\nabla \phi) = \nabla^2 \phi, \text{ where } \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \quad - (4)$$

From (3) & (4) $-\nabla^2 \phi = \rho/\epsilon_0$

$$\therefore \boxed{\nabla^2 \phi = \frac{-\rho}{\epsilon_0}}$$

This equation is known as Poisson's equation.

For a dielectric medium, the above equation becomes $\nabla^2 \phi = -\frac{\rho}{K\epsilon_0}$ ($\because K \rightarrow$ dielectric const.)

In a charge free region, volume charge density $\rho = 0$

$$\therefore \nabla^2 \phi = 0 \Rightarrow \boxed{\nabla^2 = 0}$$

This equation is called Laplace's equation.

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