26-01-2023

4. LINEAR ALGEBRA.

* concepts:-

vector space } Basis
subspaces

$V \to W \Rightarrow$ Linear transmissions $\longleftrightarrow$ Eigen values

$\updownarrow$ relation b/w

Matrix

* vector spaces:-

Ex:-

→ set of all solutions of 'x+y=0' is

$S = \{ (x,y) \in \mathbb{R}^2 / y = -x \}$

$S_1 = \{ (x,-x) / x \in \mathbb{R} \}$

$(1,-1) \in S_1 \qquad (2,-2) \in S_1$

$\Rightarrow (1,-1) + (2,-2) = (3,-3) \in S_1$

$\Rightarrow \alpha(1,-1) = (\alpha, -\alpha) \in S_1$

→ solutions of 'x+y=1' is

$S_2 = \{ (x, 1-x) / x \in \mathbb{R} \}$

$(1,0) \in S_2, \quad (-1,2) \in S_2$

$\Rightarrow (1,0) + (-1,2) = (0,2) \notin S_2$

$\Rightarrow \alpha(1,0) = (\alpha, 0) \notin S_2$.

→ Let V be a Non Empty set

$+ : V \times V \to V$ | + is defined, scalar multiplication is defined.

$* : V \times V \to V$ |

N → '0' is not an element

Z → A Id, A In, M Id, M In ✗
(0, a, 1, $\frac{1}{a}$)

Q → $0 \in Q$

$a \in Q_1, \ -a \in Q_1$

$1 \in Q_1$

$a \neq 0, \ a \in Q, \ \frac{1}{a} \in Q$

∴ scalar set is

$(Q, +, \cdot)$ Field

$(R, +, \cdot)$ Field

$(C, +, \cdot)$ Field

Scalars:-

Field

$(F, +, \cdot)$

(i) Additive Identity :- '0'

(ii) Additive Inverse :- '$-a$'

(iii) Multiplicative Identity :- '1'

(iv) Multiplicative Inverse :- '$\frac{1}{a}$'

(v) commutative propeity :- '$a+b=b+a$', '$ab=ba$'

(vi) Associative :- '$(a+b)+c = a+(b+c)$' [under Addition]

'$a(bc) = (ab)c$' [under multiplication]

(vii) Distributive :- '$a~~x~~(b+c) = ab+ac$'.

* → Let V be a non empty set + is defined on V

$V = \mathbb{R}^2$

$F = \mathbb{R}$.

scalar multiplication :- $F \times V \longrightarrow V$

$\alpha(a,b) = (\alpha a, \alpha b)$ ✓

$\alpha(a,b) = (\alpha a, b)$ → Fails at '0'.

$0(a,b) = (0 \times a, b) = (0, b)$

* Ex:-

→ V = set of all solutions of the O.D.E

$\frac{dy}{dx} + 2y = 0.$

+, •   Field → Real Number.

$\Rightarrow \frac{dy}{dx} + 2y = 0$

$\Rightarrow \frac{dy}{dx} = -2y$

$\Rightarrow \int \frac{dy}{-2y} = \int dx$

$\Rightarrow -\frac{1}{2} \ln y = x + c$

$\Rightarrow \ln y = -2x + c$

$\Rightarrow y = e^{-2x} * c = ce^{-2x}$

$\Rightarrow V = \{ ce^{-2x} / c \in \mathbb{R} \}$

$v_1 = c_1 e^{-2x}$, $v_2 = c_2 e^{-2x}$

Add: $v_1 + v_2 = (c_1 + c_2) e^{-2x} = ke^{-2x} \in V$

Mul: $\alpha v = \alpha (ce^{-2x}) = (\alpha c) e^{-2x} = ke^{-2x} \in V$ $[\alpha \in \mathbb{R}]$.

It satisfies 'Add' & 'Mul'.

Assos: $v_1 + (v_2 + v_3) = (v_1 + v_2) + v_3$

comm: $v_1 + v_2 = v_2 + v_1$

Ad. Id: $0 = 0e^{-2x} \in V$

Ad. Inv: $ce^{-2x} \in V \;\exists\; -ce^{-2x} \in V$.

Distr: $\alpha (v_1 + v_2) = \alpha v_1 + \alpha v_2$

$\alpha (\beta v) = (\alpha \beta) v$

$1 v = v$

* vector space over the field $V(\mathbb{R})$:

→ vector space over the field $V(\mathbb{R})$.

$V = \mathbb{R}^2 = \{ (x, y) / x, y \in \mathbb{R} \}$

$v_1 = (x_1, y_1)$ & $v_2 = (x_2, y_2)$

$\oplus : V \times V \longrightarrow V$

$(x_1, y_1) \oplus (x_2, y_2) = (x_1 + y_2, y_1 + x_2)$

$\Rightarrow v_1 \oplus v_2 = (1, 2) \oplus (3, 5) = (1 + 5, 2 + 3) = (6, 5)$

$\Rightarrow v_2 \oplus v_1 = (3, 5) \oplus (1, 2) = (3 + 2, 5 + 1) = (5, 6)$.

$\therefore v_1 \oplus v_2 \neq v_2 \oplus v_1$

* Ex:

$U = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| \ ad - bc = 0 \right\} \subset M_2(\mathbb{R})$.

$U = \{ A \in M_2(\mathbb{R}) \mid \det(A) = 0 \}$

Is $U(\mathbb{R})$ a vector space?

A) $U_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $U_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

$\det(U_1) = \det(U_2) = 0$

$U_1, U_2 \in U$.

$U_1 + U_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$, $\because \det(I) = 1 \neq 0$

$\therefore U_1 + U_2 \notin U$

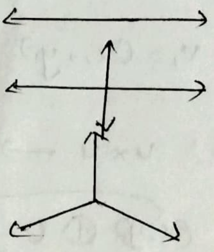
$\therefore U$ is not a vector space.

* Vector spaces & Examples

$\mathbb{R}(\mathbb{R})$ – vector space – line

$\mathbb{R}^2(\mathbb{R})$ – vector space. – plane

$\mathbb{R}^3(\mathbb{R})$ – vector space – space



$\vdots$

$\mathbb{R}^n(\mathbb{R})$ – vector space.

$\mathbb{R}^n = \{ (x_1, x_2, \ldots, x_n) / x_i \in \mathbb{R}, i = 1, 2, \ldots, n \}$

Ex:

① V = set of all polynomials.

$V = \{ a_0 + a_1 x + \ldots + a_n x^n / a_i \in \mathbb{R}, i \in \mathbb{N} \}$

$\therefore V(\mathbb{R})$ is a vector space

$V(\mathbb{C})$ not a vector space.

② V = set of all real valued functions, $V(\mathbb{R})$ is a vector space.

③ V = set of all $m \times n$ matrices of same size with real entries $= M_{m \times n}(\mathbb{R})$

$\therefore V(\mathbb{R})$ is a vector space.

④ V = set of all real valued sequences

$x\{n\}$, $y\{n\}$ $\{x(n)+y(n)\}$

$\therefore V(\mathbb{R})$ is a vector space

⑤ V = set of all complex Numbers.

$\mathbb{F}=\mathbb{R}$ (or) $\mathbb{F}=\mathbb{Q}$ (or) $\mathbb{F}=\mathbb{C}$.

$\therefore \mathbb{C}(\mathbb{R})$ is a vector space

$\mathbb{C}(\mathbb{Q})$ is a vector space

$\mathbb{C}(\mathbb{C})$ is a vector space.

⑥ $\mathbb{P}_2$ = set of all polynomials of degree less than or equal to 2.

$\mathbb{P}_2(\mathbb{R})$ is

$\mathbb{P}_2 \subset V$ = set of all polynomials.

$\therefore \mathbb{P}_2$ is a subspace of V.

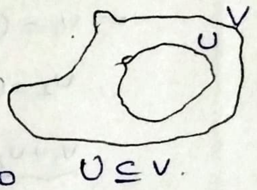
Exercise

① $\frac{dy}{dx} + ey = 0$

② $\frac{dy}{dx} + 2y = x$.

* Subspace of a vector space :-

→ Let $V(\mathbb{F})$ be a vector space & $U \subseteq V$. If $U(\mathbb{F})$ is a vector space then U is said to be subspace of V.



Ex :-

① $f(x)=0$, is a constant function & also $U \subseteq V$. a continous function.

$U = \{ f(x) \in C[a,b] \,/\, f(a) = 0 \}$
f continuous

$f \in U$ $\quad$ $g \in U$

$f(a)=0$ $\quad$ $g(a)=0$

① $(f+g)(a) = f(a)+g(a) = 0+0=0$.

② $(\alpha f)(a) = \alpha(f(a)) = \alpha(0) = 0$.

③ $0(a) = 0$.

$\therefore$ It is a subspace to set of continuous functions.

② $V=\mathbb{R}$, $F=\mathbb{R}$.

$\mathbb{R}(\mathbb{R})$

$U \subset \mathbb{R}$

$U = \{0\}$

$U = [-1, 1]$

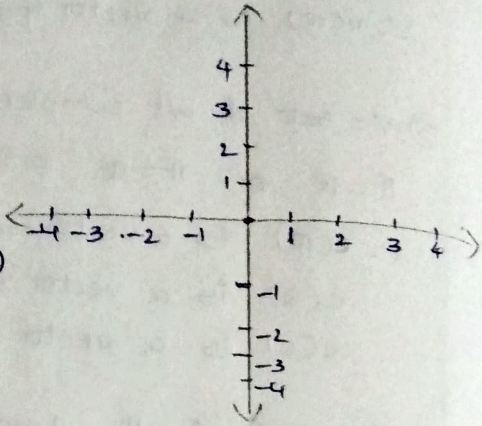
$0.9 + 0.8 = 1.7 \notin U$

$2 \times (1) \notin U$.

$\therefore$ Not a vector space.

② ⓑ $\mathbb{R}^2(\mathbb{R})$:- [Subspaces]

$U \subset \mathbb{R}^2$



ⓘ $U = \{(0,0)\} = \{\bar{0}\}$

ⓘⓘ $U = \{(x,0) / x \in \mathbb{R}\}$

$\therefore$ It is subspace of $\mathbb{R}^2(\mathbb{R})$

ⓘⓘⓘ $U = \{(x, x^2) / x \in \mathbb{R}\}$

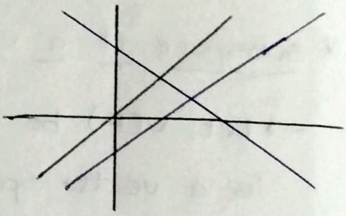
$(1,1) \in U$, $(2,4) \in 'U'$

$\Rightarrow (1,1) + (2,4) = (3,5) \notin U$

But $(3, 3^2) = (3,9)$

$\therefore$ It is a subset but not subspace.

"Sum of any two points on parabola may not lie on parabola."



ⓘⓥ $U = \{(x, mx+c) / x \in \mathbb{R}\}$.

For $x=0 \Rightarrow (0,c)$. $(0,0)$ Not present.

$U_m = \{(x, mx) / x \in \mathbb{R}\}$

$\Rightarrow u_1 = (x_1, mx_1)$

$u_2 = (x_2, mx_2)$

$\underline{v_1 + u_2} = ((x_1 + x_2), m(x_1 + x_2)) \in 'U_m'$

$\Rightarrow \underline{\alpha v} = \alpha(x, mx) = (\alpha x, m(\alpha x)) \in 'U_m'$

$\Rightarrow x=0$, $(x, mx) = (0, m(0)) = (0,0) \in 'U_m'$

ⓥ All lines passing through the origin is a subspace to $\mathbb{R}^2$. [ex-iii].

③ $\mathbb{R}^3(\mathbb{R})$ [subspaces].

Exercise:

① $W = \{(x, y, x+y) / x, y \in \mathbb{R}\} \subset \mathbb{R}^3$

② Is $W(\mathbb{R})$ a subspace of $\mathbb{R}^3(\mathbb{R})$?

② $W = \{(x, x, y) / a, b \in \mathbb{R}\} \subset \mathbb{R}^3$

(or)

(α, α, β)

(or)

(a, a, b)

Is it subspace to $\mathbb{R}^3$?

* **Ex:-**

(1) $W = W_1 \cup W_3 =$ First quadrant $\cup$ Third quadrant.

$W = \{ (x,y) \in \mathbb{R}^2 / xy \geq 0 \}$

Let $v_1 = 1, 2$ & $v_2 = (2, -1)$

$v_1 + v_2 = (1, 2) + (2, -1) = (3, -1) \notin W_1 \cup W_3$.

Now $W_1 \cup W_2 \cup W_3$

$\Rightarrow W_1 \cup W_2 \cup W_3 \cup W_4$.

(2) $W = \{ (x, |x|) ; \forall x \in \mathbb{R} \}$.

Let $v_1 = 1, 1$ $\quad v_2 = +2, +2 = +2, 2$.

$\alpha v_1 = 2(1, 1) = 2, 2$

fails

$\alpha v_2 = -2(2, 2) = -4, 4$.

(3) $W = \{ (x, x+1) ; \forall x \in \mathbb{R} \}$.

Let $x = 0$ $\qquad$ fails.

$\Rightarrow (0, 0+1) = (0, 1) \notin (0, 0)$.

* **Note :-**

$$\left.\begin{array}{l} v_1 + u_2 \in V \\ \alpha v \in V \end{array}\right\} \Leftrightarrow \underline{u_1 + \alpha v_2 \in V}, \quad \forall v_1, v_2 \in V \ \& \ \alpha \in F.$$

Ex:-

$\rightarrow U = \{ A \in M_2(\mathbb{R}) / A^T = A \}$.

$$A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}, \quad A^T = A$$

$A \in U$, $B \in U$ i.e, $A^T = A$, $B^T = B$.

$v_1 + \alpha v_2 = A + \alpha B$, $\alpha \in \mathbb{R}$.

$(A + \alpha B)^T = A^T + (\alpha B)^T$

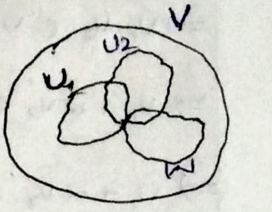
$= A^T + \alpha \cdot B^T$

$= A + \alpha B$

$\therefore A + \alpha B \in U$. $\qquad \therefore U(\mathbb{R})$ is a subspace of $M_2(\mathbb{R})$ with standard addition & scalar multiplication

Theorem :-

* → Let $V(\mathbb{F})$ be a vector space and $U \subseteq V$. U is a subspace of V if and only if $u_1 + \alpha u_2 \in U$ for every $u_1, u_2 \in U$ and $\alpha \in \mathbb{F}$.

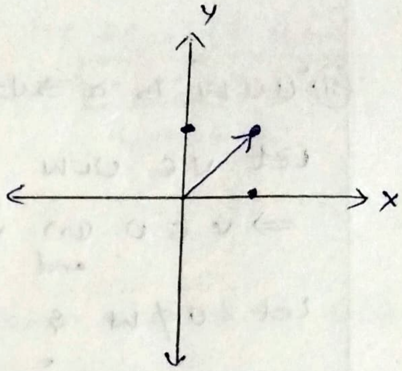


* Let $V(F)$ be a vectorspace & $U(F)$ & $W(F)$ are subspaces of V.

(i) Is $U \cup W$ a subspace of V?

(ii) $U \cap W$ a subspace of V?

A) Let $V = \mathbb{R}^2$.



Subspaces:

$U = \{ (x, 0) \in \mathbb{R}^2 / x \in \mathbb{R} \}$.

$W = \{ (0, y) \in \mathbb{R}^2 / y \in \mathbb{R} \}$

$(1, 0) \in U \cup W$,

$(0, 1) \in U \cup W$.

Now, $(1,0) + (0,1) = (1,1) \notin U \cup W$.

Let $V = \mathbb{R}^3$.

Subspaces :

$U = \{ (x, 0, 0) / x \in \mathbb{R} \} \subset \mathbb{R}^3$

$W = \{ (x, y, 0) / x, y \in \mathbb{R} \} \subset \mathbb{R}^3$

(i) $U \cup W = W$

$\therefore U \cup W$ is a subspace of $\mathbb{R}^3$

(ii) $U \cap W = U$

$\therefore U \cap W$ is a subspace of $\mathbb{R}^3$.

* Theorem :-

→ Let $V(F)$ be a vector space & $U(F)$, $W(F)$ are subspaces of V.

(1) $U \cap W$ is a subspace of V.

(2) $U \cup W$ is a subspace of V if & only if $U \subseteq W$ (or) $W \subseteq U$.

Proof :-

(i) $v_1, v_2 \in U \cap W$, $\alpha \in F$

$\Rightarrow v_1, v_2 \in U$ & $v_1, v_2 \in W$.

$\Rightarrow v_1 + \alpha v_2 \in U$ & $v_1 + \alpha v_2 \in W$ [$\because$ U is a subspace
W is a subspace]

$\Rightarrow v_1 + \alpha v_2 \in U \cap W$

$\therefore U \cap W$ is a subspace of V.

(ii) '$U \cup W$' is a subspace where 'U', 'W' are subspaces.

Let $v \in U \cup W$.

$\Rightarrow v \in U$ (or) $v \in W$.
and

Let $U \not\subseteq W$ & $W \not\subseteq U$.

Assume that '$U \not\subseteq W$', '$W \not\subseteq U$'.

Let $v_1 \in U \cup W$ such that '$v_1 \in U$' & '$v_1 \notin W$'. $\longrightarrow$ (1)

Let $v_2 \in U \cup W$ such that '$v_2 \notin U$' & '$v_2 \in W$'. $\longrightarrow$ (2)

$v_1 + \alpha v_2 \in U \cup W$ [$\because U \cup W$ is a subspace].

$v_1 + \alpha v_2 \in U$ (or) $v_1 + \alpha v_2 \in W$.

Case-i :

$v_1 + \alpha v_2 \in U$

$\Rightarrow v_1 \in U$ & $v_1 + \alpha v_2 \in U$

$\Rightarrow -v_1 + (v_1 + \alpha v_2) \in U$

$\Rightarrow \alpha v_2 \in U$

$\Rightarrow v_2 \in U$ [$\alpha = 1$]

But $v_2 \notin U$. [From (2)]

Case-ii :

$v_1 + \alpha v_2 \in W$. , $v_2 \in W$.

$\Rightarrow v_1 + v_2 \in W$ [$\alpha = 1$]

$\Rightarrow (-1) v_2 + (v_1 + v_2) \in W$

$\Rightarrow \cancel{-v_2} + v_1 + \cancel{v_2} \in W$

$\Rightarrow v_1 \in W$

But $v_1 \notin W$. [From (1)]

$\therefore$ This is due to our wrong assumption.

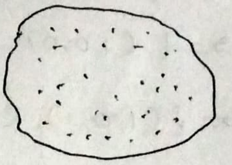
$\therefore$ Either $U \subseteq W$ (or) $W \subseteq U$.

$\therefore$ If $U \cup W$ is a subspace then either $U \subseteq W$ (or) $W \subseteq U$.

* Linear combinations :-

→ Let $V(\mathbb{F})$, then linear combination of

v_1, v_2, v_3 is given by.



$\Rightarrow \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 \quad , \quad \alpha_i \in \mathbb{F}$

→ Let $A = \{v_1, v_2, v_3\}$.

Linear span $(A) = LS(A)$

$= \{\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 /$
$\alpha_i \in \mathbb{F},\ i = 1, 2, 3\}$

$v_1 = (1, 2)$, $v_2 = (3, 4)$

$\alpha_1 = 3 \quad \alpha_2 = -1$

$\Rightarrow 3(1,2) + (-1)(3,4)$

$\Rightarrow \boxed{\alpha(1,2) + \alpha_2(3,4)}$

$\alpha_1, \alpha_2 \in \mathbb{R}$.

* Question :-

$S = \{(1,0), (0,1)\}$, what is the linear span of S, $LS(S) = ?$

A) For $S = \{(1,0)\}$ $\quad LS(S) = ?$

$\Rightarrow LS(S) = \{\alpha(1,0) / \alpha \in \mathbb{R}\}$

$= \{(\alpha, 0) / \alpha \in \mathbb{R}\}$

$=$ x-axis.

For $S = \{(1,0), (0,1)\}$

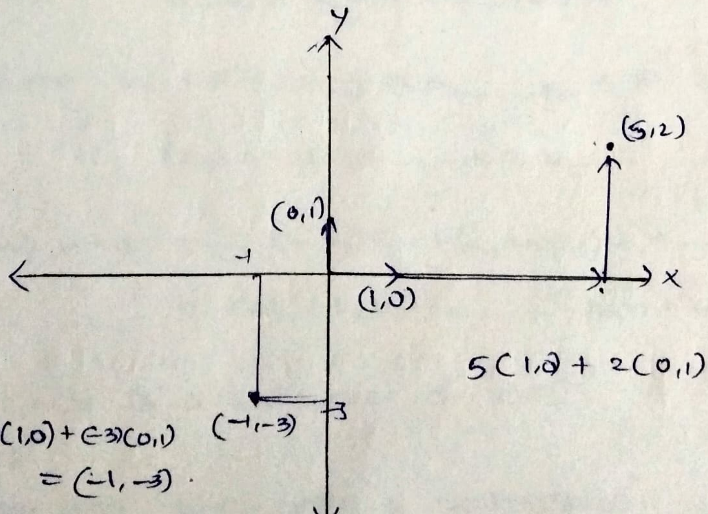
$\Rightarrow LS(S) = \{\alpha_1(1,0) + \alpha_2(0,1)\} \alpha_1, \alpha_2 \in \mathbb{R}\}$

$= \{(\alpha_1, 0) + (0, \alpha_2) / \alpha_1, \alpha_2 \in \mathbb{R}\}$.

$= \{(\underset{x}{\alpha_1}, \underset{y}{\alpha_2}) / \alpha_1, \alpha_2 \in \mathbb{R}\}$

$= \{(x, y) / x, y \in \mathbb{R}\}$

$\therefore LS(S) = \mathbb{R}^2$.



$5(1,0) + 2(0,1) = (5,2)$

$(-1)(1,0) + (-3)(0,1)$
$= (-1,-3)$.

* Linear combination of vectors:

→ Linear combination of $v_1, v_2, \ldots v_n$.

$$= \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n \ , \ \alpha_i \in \mathbb{F}$$

* Question:

$v = (1, 2, 5)$, $v_1 = (1, -1, 3)$, $v_2 = (5, 2, 1)$.

Is this possible to express v as a linear combination of v_1 & v_2?

A) If $\exists \alpha_1, \alpha_2$ such that.

$$v = \alpha_1 v_1 + \alpha_2 v_2$$

$$\Rightarrow (1,2,5) = \alpha_1 (1,-1,3) + \alpha_2 (5,2,1)$$

$$\Rightarrow \alpha_1 + 5\alpha_2 = 1$$
$$-\alpha_1 + 2\alpha_2 = 2$$
$$3\alpha_1 + \alpha_2 = 5$$

$$\begin{vmatrix} 1 & 2 & 5 \\ 1 & -1 & 3 \\ 5 & 2 & 1 \end{vmatrix} = (-1-6) - 2(1-15) + 5(2+7)$$
$$= -7 + 28 + 45 = \cancel{58}\ 66$$

* Linear span:

→ Let $S = \{u_1, u_2, \ldots, u_n\} \subset V$ and the Linear span(S),

$L(S) = \{\alpha_1 u_1 + \alpha_2 u_2 + \ldots + \alpha_n u_n \,/\, \alpha_i \in \mathbb{F},\ i = 1, 2, \ldots n\}$.

$$LS(S) \subseteq V$$

Is $LS(S)$ a subspace of V?

→ Assume $u \in LS(S) \Rightarrow \exists\ \alpha_1, \alpha_2 \ldots \alpha_n$ such that

$$u = \alpha_1 v_1 + \alpha_2 u_2 + \ldots + \alpha_n v_n.$$

Assume $v \in LS(S) \Rightarrow \exists\ \beta_1, \beta_2 \ldots \beta_n \in \mathbb{F}$ such that

$$v = \beta_1 u_1 + \beta_2 u_2 + \ldots + \beta_n u_n.$$

Now, $u + \alpha v = (\alpha_1 + \alpha\beta_1) u_1 + (\alpha_2 + \alpha\beta_2) u_2 + \ldots + (\alpha_n + \alpha\beta_n) u_n$

$$= k_1 u_1 + k_2 u_2 + \ldots + k_n u_n \in LS(S).$$

$\therefore LS(S)$ is a subspace of $V(\mathbb{F})$.

* Note:

'Linear span' gives you a 'vectorspace'.

* Ex:

Let $S=\{(1,2,1),(3,1,0)\}$, Find $LS(S)=?$

A) $LS(S)=\{\alpha.(1,2,1)+\beta(3,1,0) / \alpha,\beta \in \mathbb{R}\}$

$=\{(\alpha+3\beta)+(2\alpha+\beta)+(\alpha+(0)\beta) / \alpha,\beta \in \mathbb{R}\}$

$=\{(x,y,z)\in \mathbb{R}^3 / \text{condition } x-3y+5z=0\}$

$x=\alpha+3\beta \Rightarrow 3\beta=x-z \Rightarrow \beta=\frac{x-z}{3}$

$y=2\alpha+\beta \Rightarrow y=2z+\frac{x-z}{3} \Rightarrow 3y=6z+x-z \Rightarrow \underline{x+5z-3y=0}$

$z=\alpha \Rightarrow \boxed{\alpha=z}$

* Question:

$S=\{1-x, 1+x^2, 1\}\subseteq P_2$. what is $LS(S)$?

A) $LS(S)=\{\alpha(1-x)+\alpha_2(1+x^2)+\alpha_3(1) / \alpha_1,\alpha_2,\alpha_3 \in \mathbb{R}\}$

$=\{(\alpha_1+\alpha_2+\alpha_3)+(-\alpha_1+0)x+\alpha_2 x^2 / \alpha_1,\alpha_2,\alpha_3 \in \mathbb{R}\}$

compare

$=\{a+bx+cx^2 / a,b,c \text{ are } \in \mathbb{R}\}$

$a=\alpha_1+\alpha_2+\alpha_3 \Rightarrow a=-b+c+\alpha_3 \Rightarrow \alpha_3=a+b-c$

$b=-\alpha_1 \Rightarrow \alpha_1=-b$

$c=\alpha_2 \Rightarrow \alpha_2=c$

Here, a, b, c are arbitary constants

$\therefore LS(S)=$ Entire P_2.

* Linear span:

→ set of all linear combinations is called Linear span.

* Linear combinations:

→ Multiplying vectors with some constants are called 'Linear combinations'.

* Ex: $\overset{v_1}{(5,6)} \to \overset{v_2}{(1,2)}\ \overset{v_3}{(1,0)}$. can express v_1 a Linear combination of v_2 & v_3?

A) given, $v_1 = (5,6)$

$v_2 = (1,2)$

$v_3 = (1,0)$

$\Rightarrow (5,6) = \alpha(1,2) + \beta(1,0)$

$(5,6) = (\alpha, 2\alpha) + (\beta, 0)$

Now, $\alpha + \beta = 5 \quad \Rightarrow \beta = 5 - 3 = 2$

$2\alpha + 0\beta = 6 \quad \Rightarrow \alpha = \frac{6}{2} = 3$

$\therefore v_1$ can be expressed as a Linear combination of v_2 & v_3.

$\therefore v_1 = 3v_2 + 2v_3$.

$\therefore$ It is 'Linearly dependent'

* Linearly Dependent:

→ If a vector can be express as a Linear combination of other vectors then it is said to be 'Linearly Dependent'.

* Question:

Is $S = \{\overset{v_1}{(1,0)}, \overset{v_2}{(0,1)}\}$ linearly dependent?

A) $v_1 = (1,0)$ & $v_2 = (0,1)$

Now, $v_1 = \alpha v_2$

$\Rightarrow (1,0) = \alpha(0,1)$

$\Rightarrow (1,0) = (0,\alpha)$

Here, $1 = 0$

$0 = \alpha$.

$\therefore$ First element can't express as multiple of second element.

Now, $v_2 = \alpha v_1$

$\Rightarrow (0,1) = \alpha(1,0)$

$\Rightarrow (0,1) = (\alpha, 0)$

$0 = \alpha$

$1 = 0$

$\therefore$ Second element can't express as multiple of First element.

$\therefore$ It is 'linearly Independent'.

② Is $S=\{(1,0),(0,0)\}$ Linearly Dependent?
$\quad\quad\quad u_1 \quad\quad u_2$

A) $v_1 = \alpha u_2$ | $u_2 = \alpha v_1$

$\Rightarrow (1,0) = \alpha(0,0)$ | $\Rightarrow (0,0) = \alpha(1,0)$

$\Rightarrow (1,0) = (0,0)$ | $\Rightarrow (0,0) = (\alpha,0)$.

But $u_2 = \alpha u_1$ take $(\alpha=0)$.

$(0,0) = 0(1,0)$.

$\therefore S$ is Linearly dependent.

$\alpha u_2 + \beta u_1 = 0$

$1(0,0) + 0(1,0) = (0,0)$.

$\therefore$ It is possible only in Linearly dependent.

* Note :-

→ Any set which contains zero vector is L.D - [Linearly dependent].

③ Is $S=\{1-x, 1+x^2\} \subseteq P_2$ a linearly Dependent or Independent?

A) $v_1 = \alpha v_2$

$\Rightarrow (1-x) = \alpha(1+x^2)$

$\Rightarrow 1-x+0x^2 = \alpha + 0x + \alpha x^2$

Here $\alpha = 1$

$\left.\begin{array}{l}-1=0\\ \alpha=0\end{array}\right\}$ Not possible

$u_2 = \alpha v_1$

$\Rightarrow 1+x^2 = \alpha(1-x)$

$\Rightarrow 1+0x+x^2 = \alpha - \alpha x + 0x^2$

Here $\alpha = 1$

$\left.\begin{array}{l}\alpha=0\\ 0=1\end{array}\right\}$ Not possible.

$\therefore S$ is Linearly Independent.

④ Is $S=\{1-x, 1+x^2, 3+2x^2-x\}$ a linearly Dependent or Independent?
$\quad\quad\quad v_1 \quad\quad u_2 \quad\quad v_3$

A) $3+2x^2-x = 1(1-x) + 2(1+x^2)$.

$u_3 = \alpha u_1 + \beta u_2$

$\Rightarrow u_3 = 1(u_1) + 2(v_2)$ $\quad \alpha=1, \beta=2, \gamma=-1$

$\Rightarrow \alpha v_1 + \beta u_2 + (-1) u_3 = 0$.

Here there is a Non-zero solution set

$\therefore$ It is $\emptyset$ linearly dependent.

* **Linearly Dependent:**

→ A subset $S = \{v_1, v_2, ---, v_n\}$ of a v.s is said to be Linearly dependent if there exist scalars $\alpha_1, \alpha_2 --- \alpha_n$ such that $\alpha_1 v_1 + \alpha_2 v_2 + --- + \alpha_n v_n = 0$ but not all α_i are zero. [atleast one α is Non zero].

* **Linearly Independent:**

→ A subset $S = \{v_1, v_2, ---, v_n\}$ is said to be 'linearly independent' if $\alpha_1 v_1 + \alpha_2 v_2 + --- + \alpha_n v_n = 0$ then $\alpha_1 = \alpha_2 = --- = \alpha_n = 0$. [Must be zero].

Q) Prove that $S = \{1-x, 1+x^2, 3+2x^2-x\}$ is Linearly-dependent?

A) Let $\alpha, \beta, \gamma \in \mathbb{R}$.

$\Rightarrow \alpha u_1 + \beta u_2 + \gamma u_3 = \bar{0}$.

$\Rightarrow \alpha(1-x) + \beta(1+x^2) + \gamma(3+2x^2-x) = 0 + 0x + 0x^2$.

$\Rightarrow \alpha - \alpha x + \beta + \beta x^2 + 3\gamma + 2\gamma x^2 - \gamma x = 0 + 0x + 0x^2$

$\Rightarrow (\alpha + \beta + 3\gamma) + (-\alpha - \gamma)x + (\beta + 2\gamma)x^2 = 0 + 0x + 0x^2$

compare coefficients on both sides.

$\Rightarrow \alpha + \beta + 3\gamma = 0 \longrightarrow (1)$

$\Rightarrow -\alpha - \gamma = 0 \longrightarrow (2)$

$\Rightarrow \beta + 2\gamma = 0 \longrightarrow (3)$

(1) + (2) $\Rightarrow$
$$\begin{array}{r} \alpha + \beta + 3\gamma = 0 \\ -\alpha + 0 - \gamma = 0 \\ \hline \beta + 2\gamma = 0 \end{array} \longrightarrow (4)$$

From (3) & (4), $\boxed{\beta = -2\gamma}$

From (2) $\boxed{\alpha = -\gamma}$

If $\gamma = 1$, then $\alpha = -1$, $\beta = -2$.

$\Rightarrow \alpha u_1 + \beta u_2 + \gamma u_3 = 0$

$\Rightarrow (-1)u_1 + (-2)u_2 + 1(u_3) = 0$.

$\therefore$ Here 3 are Non-zeros, it is linearly dependent

⑤ Find $S = \left\{ \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix} \right\} \subset M_2(\mathbb{R})$ is Linearly
$\quad\quad\quad u_1 \quad\quad\quad u_2 \quad\quad\quad u_3$
dependent or independent?

A) $\alpha v_1 + \beta u_2 + \gamma u_3 = 0$

$$\Rightarrow \alpha \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} + \beta \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} + \gamma \begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} \alpha & 2\alpha \\ 0 & \alpha \end{bmatrix} + \begin{bmatrix} 3\beta & 0 \\ 0 & \beta \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \gamma & 2\gamma \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Equate component wise.

$\alpha + 3\beta + 0 = 0.$

$2\alpha + 0\beta + 0\gamma = 0 \Rightarrow \boxed{\alpha = 0}$

$0 + 0 + \gamma = 0 \Rightarrow \boxed{\gamma = 0}$

$\alpha + \beta + 2\gamma = 0. \Rightarrow 0 + \beta + 2(0) = 0 \Rightarrow \boxed{\beta = 0.}$

$\therefore$ Here there is only one solution $(0, 0, 0)$
$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \alpha \; \beta \; \gamma$

$\therefore$ S is 'Linearly independent'. – NO Non zero solutions.

* Linear span:-

→ Let $S = \{u_1, u_2, u_3\}$. is a subspace and

$LS(S) = \{ \alpha v_1 + \alpha u_2 + \alpha u_3 \,/\, \alpha_i \in \mathbb{R} \}$

$\therefore$ $LS(S)$ = vector space generated by u_1, v_2, u_3.

* Ex:- $S = \{ (1,0,0), (0,0,1) \subset \mathbb{R}^3$, $LS(S)$?

A) $LS(S) = \{ (\alpha)(1,0,0) + \beta(0,0,1) / \alpha, \beta \in \mathbb{R} \}$

$= \{ (\alpha, 0, 0) + (0, 0, \beta) / \alpha, \beta \in \mathbb{R} \}$

$= \{ (\alpha, 0, \beta) / \alpha, \beta \in \mathbb{R} \}$
$\quad\quad \downarrow \quad \downarrow \quad \downarrow$
$\quad\quad x \quad y \quad z$

$\therefore LS(S) = xz$ plane $[\because y = 0]$

$\therefore$ 'S' generates 'xz plane'.

* Question:

Let $B = \{ (1,0,0), (0,1,0), (0,0,1) \} \subset \mathbb{R}^3$. what is the Linear space generated by B?

A) $LS(B) = \{ \alpha(1,0,0) + \beta(0,1,0) + \gamma(0,0,1) / \alpha, \beta, \gamma \in \mathbb{R} \}$

$= \{ (\alpha,0,0) + (0,\beta,0) + (0,0,\gamma) / \alpha, \beta, \gamma \in \mathbb{R} \}$

$= \{ \underset{(x,y,z)}{(\alpha, \beta, \gamma)} \} = \mathbb{R}^3$

$\therefore$ 'B' generates Entire '$\mathbb{R}^3$ space'

* Basis set:

→ ~~A smallest~~ Let $B \subset V(\mathbb{F})$ be a vector space & $B \subset V$

B is said to 'Basis' of vector space $V(\mathbb{F})$ if

(i) '$LS(B) = V$' (ii) 'B is linearly independent'

Ex:

$S = \{ (1,0,0), (0,1,0), (1,1,0) \} \subseteq \mathbb{R}^3$, $LS(S) = ?$

→ $LS(S) = \{ (\alpha,0,0), (0,\beta,0), (\gamma,\gamma,0) / \alpha, \beta, \gamma \in \mathbb{R} \}$

$= \{ ((\alpha+\gamma) + (\beta+\gamma) + 0) / \alpha, \beta, \gamma \in \mathbb{R} \}$

$= \{ (\alpha+\gamma, \beta+\gamma, 0) / \alpha, \beta, \gamma \in \mathbb{R} \}$

$= \{ (x,y,0) / x, y \in \mathbb{R} \}$ Here $x = \alpha+\gamma \in \mathbb{R}$, $y = \beta+\gamma \in \mathbb{R}$

$\therefore LS(S) = xy$-plane

Let $S_2 = \{ (1,0,0), (0,1,0) \}$ $[v_1 \,\&\, v_2]$.

S_1 It is Linearly dependent $[\because u_1 + u_2 = u_3]$.

S_2 is Linearly Independent.

Here S_2 is also generates xy-plane.

$\therefore$ S is Basis set.

* Basis set of $\mathbb{R}^2$ - space :

$V = \mathbb{R}^2$, $\mathbb{F} = \mathbb{R}$.

Basis set of $\mathbb{R}^2$,

$\Rightarrow B = \{(1,0), (0,1)\}$ < Linearly Independent / It spans $\mathbb{R}^2$

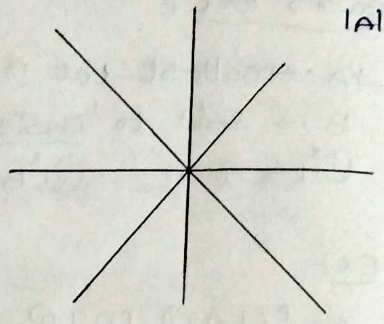
$\{(1,0),(0,1)\}$ is standard basis of $\mathbb{R}^2$.

* Question :

Prove that $B = \{(1,1), (1,-1)\}$ is a Basis of $\mathbb{R}^2$.

A) $\alpha(1,1) + \beta(1,-1) = (0,0)$.

$AX = 0$, if $|A| \neq 0$.



$\Rightarrow (\alpha+\beta, \alpha-\beta) = (0,0)$.

$\alpha + \beta = 0$

$\alpha - \beta = 0$.

$\Rightarrow \alpha = 0, \ \beta = 0$.

The linear system has unique solution, i.e, $(0,0)$

$\therefore$ It is Linearly independent.

Let $(x,y) \in \mathbb{R}^2$

$\Rightarrow (x,y) = \alpha(1,1) + \beta(1,-1)$

$\Rightarrow (x,y) = (\alpha+\beta, \alpha-\beta)$.

$\alpha + \beta = x$

$\alpha - \beta = y$

$2\alpha = x + y$ $\Rightarrow \boxed{\alpha = \frac{x+y}{2}}$, $\boxed{\beta = \frac{x-y}{2}}$

$\therefore (x,y) = \underbrace{\left(\frac{x+y}{2}\right)}_{\alpha}(1,1) + \underbrace{\left(\frac{x-y}{2}\right)}_{\beta}(1,-1)$ - spans entire $\mathbb{R}^2$.

Ex

① $(3,7) = 5(1,1) + (-2)(1,-1)$

$(3,7) = (5,5) + (-2,+2) = (3,7)$

② $(\sqrt{2}, 5) = \left(\frac{\sqrt{2}+5}{2}\right)(1,1) + \left(\frac{\sqrt{2}-5}{2}\right)(1,-1)$.

$= \left(\frac{\sqrt{2}+5}{2}, \frac{\sqrt{2}+5}{2}\right) + \left(\frac{\sqrt{2}-5}{2}, -\left(\frac{\sqrt{2}-5}{2}\right)\right)$

$= \frac{\cancel{2}(\sqrt{2}+5)}{\cancel{2}} + 0 = (\sqrt{2}, 5)$

$\therefore$ $\boxed{L(S) = \mathbb{R}^2 \quad B \text{ spans } \mathbb{R}^2.}$

$\therefore B = \{(1,1), (+1,-1)\}$ is a Basis set, Hence proved.

* Basis set of $\mathbb{P}^2$:-

→ For $\mathbb{P}_2$, $a+bx+cx^2$

$B = \{1, x, x^2\}$.

$LS(B) = \mathbb{P}_2$, B is Linearly Independent.

$\text{Dim}(\mathbb{P}_2) = 3$ [∵ 3 elements].

$\text{Dim}(\mathbb{P}_2(\mathbb{R})) = 3$

$\text{Dim}(\mathbb{P}_n(\mathbb{R})) = n+1$.

* Question :-

$M_2(\mathbb{R}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} / a, b, c, d \in \mathbb{R} \right\}$, what is the $\dim(M_2(\mathbb{R}))$?

A) For x - 1 dim

(x, y) - 2 dim

(x, y, z) - 3 dim

(x, x, y) - 2 dim.

For given $M_2(\mathbb{R})$, $\dim(M_2(\mathbb{R})) = 4$.

$B = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

We need to know all four elements in the Matrix.

∴ $\dim(M_2(\mathbb{R})) = 4$.

• $S = \{ A \in M_2(\mathbb{R}) / A^T = A \}$

→ To construct a symmetrix matrix, we need to know 3 elements only.

$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $B = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

* **Subset of a linearly Dependent/Independent Sets:**

Ex: $S = \{(1,1),(2,2)\}$ is Linearly Dependent.

$S_1 = \{(1,1)\} \subset S$

Here S_1 is Linearly Independent.

Ex: $S_2 = \{(1,0),(2,0),(3,0)\}$ is Linearly Dependent.

$\{(1,0),(3,0)\} \subset S_2$

Note:

① Every superset of 'linearly Dependent' set is linearly Dependent.

'S' is 'L.D' & $S \subset S_1$ then 'S_1' is 'L.D'

② Every subset of 'linearly independent set' is linearly independent.

'W' is 'L.I' & $W_1 \subset W$ then 'W_1' is 'L.I'.

③ Let $S = \{v_1, v_2, \ldots, v_n\}$ is 'L.I' & '$x \in LS(S)$'

x can be 'uniquely expressed' as a 'linear combination' of v_i's.

④ Dimension: No. of elements in Basis set.

$\dim(\mathbb{R}^2(\mathbb{R})) = 2 \longrightarrow$ Basis $= \{(1,0),(0,1)\}$, $F = \mathbb{R}$

$\dim(\mathbb{R}^2(\mathbb{Q})) = \infty \longrightarrow$ Basis $= \{(1,0),(0,1),(\sqrt{2},0)$, $F = \mathbb{Q}$

* **Questions:**

① $W = \{(x,y,z) \in \mathbb{R}^3 \mid x+y-z=0,\ x-y+2z=0\}$? Find Basis set of W?

A) W is a subspace of $\mathbb{R}^3$.

$x+y-z=0$

$x-y+2z=0$

$2x+z=0$

$\Rightarrow \boxed{z=-2x}$

$x-y+2z=0$

$x-y-4x=0$

$\boxed{y=-3x}$

$\dim(W) = 1$ [for line]

$\therefore W = \{(x,-3x,-2x) / x \in \mathbb{R}\}$

Basis set $(B) = \{(1,-3,-2)\}$

$\therefore$ '$\dim(W) = 1$'

(2) $U=\{(x,y,z)\in\mathbb{R}^3 / 2x-y+z=0\}$. What is the Basis set of U? and $\dim(U)=?$

A) $2x-y+z=0 \Rightarrow z=y-2x$

$U=\{(x,y,z)\in\mathbb{R}^3 / z=y-2x\}$

$U=\{(x,y,y-2x) / x,y\in\mathbb{R}\} = \{(x,0,-2x)+(0,y,y) / x,y\in\mathbb{R}\}$

$=\{x(1,0,-2)+y(0,1,1) / x,y\in\mathbb{R}\}$

$\therefore$ Basis of $U = B = \{(1,0,-2),(0,1,1)\}$

$\therefore \dim(U)=2$.

(3) $U=\left\{(x,y,z) \,/\, \begin{array}{l} x+y+z=0 \\ 2x-y+z=0 \\ 3x+3z=0 \end{array}\right\}$

A) $$\begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 1 \\ 3 & 0 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$Ax=0$. has unique solution if $|A|\neq 0$.

$Ax=0$ has infinetly many solutions if $|A|=0$.

Now $|A| = 1(-3-0)-1(6-3)+1(0+3) = -3-3+3=-3 \neq 0$.

$\therefore Ax=0$ has zero solution only. i.e, $x=0$ is the only solution of $Ax=0$.

$\therefore U=\{(0,0,0)\}$.

Basis set of $U=\{\ \}=\phi$.

$\therefore \dim(U)=0$.

(4) $W=\{A\in M_n(\mathbb{R}) / Tr(A)=0$, 1st row sum of A is zero$\}$. Find $\dim(W)$.

A) $\underline{2\times 2}$: $b=a \Rightarrow a+b=0$

$d=a \Rightarrow a+d=0$.

$a\begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix}+c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}=\begin{bmatrix} a & -a \\ c & -a \end{bmatrix} \rightarrow 2$

$\underline{2\times 2}$

$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\begin{bmatrix} 3 & -3 \\ 2 & -3 \end{bmatrix}$

$\begin{bmatrix} a & -a \\ c & -a \end{bmatrix}$

$\underline{3\times 3}$

$\underline{3\times 3}$: $\begin{bmatrix} a & b & -a-b \\ c & d & e \\ f & g & -a-d \end{bmatrix} \rightarrow 7$

$a+b+c=0$

$c=-b-a$

$\therefore W\subset M_3(\mathbb{R})$. $\therefore \dim(W)=n^2-2$.

* Note:

① $C(\mathbb{R})$

$a+ib,\ a,b \in \mathbb{R}$

$a(1) + b(i)$

Basis of $C(\mathbb{R}) = \{(1, i)\}$

Dim of $C(\mathbb{R}) = 2$.

② $C(C)$

Basis of $C(C) = \{1\}$

Dim of $C(C) = 1$.

③ $C^2(\mathbb{R})$:

$C^2(\mathbb{R})$ is a vector space

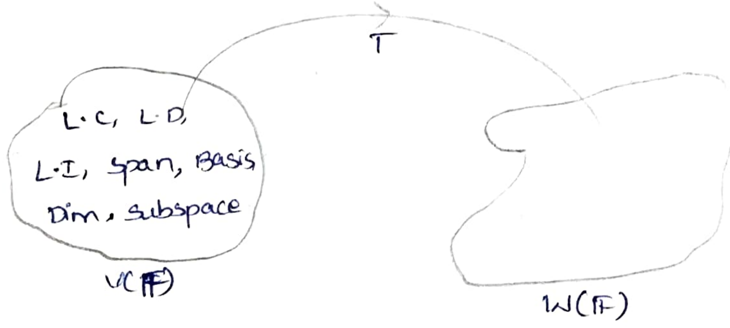
$(a+ib,\ c+id)$

$= (a,0) + (ib,0) + (0,c) + (0,id)$

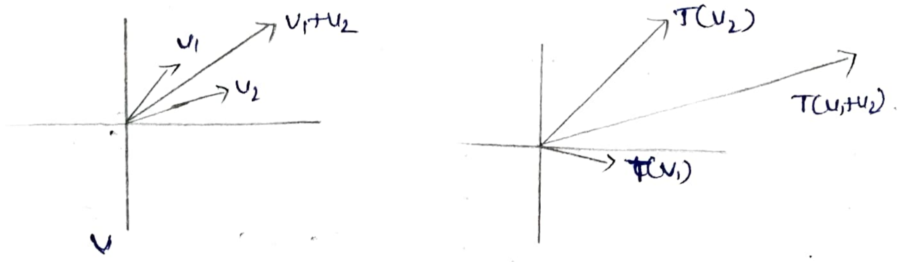
$= a(1,0) + ib(i,0) + c(0,1) + d(0,i)$

$\therefore$ Basis of $C^2(\mathbb{R}) = \{(1,0), (i,0), (0,1), (0,i)\}$

* Linear Transformation:



$T: V(F) \longrightarrow W(F)$



$T: V(F) \longrightarrow W(F)$ such that $\forall\ v_1, v_2 \in V$.

① $\boxed{T(v_1+v_2) = T(v_1) + T(v_2)}$

② $\boxed{T(\alpha v) = \alpha(T v)}$, $\alpha \in F$.

A function which satisfies these two conditions is called Linear transformation.

* Definition :-

→ A mapping $T: V(\mathbb{F}) \longrightarrow W(\mathbb{F})$ is said to be a linear transformation, if it satisfies the below conditions

(i) $T(u_1+u_2) = T(u_1) + T(u_2)$

(2) $T(\alpha u) = \alpha(Tu)$

* Questions :-

(1) $T: \mathbb{R}^2(\mathbb{R}) \longrightarrow \mathbb{R}(\mathbb{R})$ defined by $T(x,y) = x^2+y$. Is T a linear transformation?

A) Let $T(1,1) = 1^2+1 = 2 = T(u_1)$

$T(1,2) = 1^2+2 = 3 = T(u_2)$

$\therefore T(u_1+u_2) = T[(1,1)+(1,2)]$

$= T[(2,3)]$

$= 2^2+3 = 7$

$\therefore T(u_1+u_2) \neq T(u_1)+T(u_2)$.

$\Rightarrow 7 \neq 2+3$.

∴ No, it is not a linear transformation.

Note :-

→ Linear transformation is possible for linear eq^{ns} but not for squares, cubes etc.

→ In case of $T: \mathbb{R}^2(\mathbb{R}) \longrightarrow \mathbb{R}(\mathbb{R})$, if $T(x,y) = ax+by$ then only it is a Linear transformation.

→ Image of zero vector must be zero.

(2) $T: \mathbb{R}^2 \to \mathbb{R}$ defined by $T(x,y) = 2x+3y+5$. Is T a L.T?

Let

A) $T(u_1) = T(1,1) = 2(1)+3(1)+5 = 10$

$T(u_2) = T(1,2) = 2(1)+3(2)+5 = 13$.

$T(0,0) = 5$

Not L.T.

$\therefore T(u_1+u_2) = T(2,3) = 2(2)+3(3)+5 = 18$.

$\therefore T(u_1+u_2) \neq T(u_1)+T(u_2)$. $[\because 18 \neq 23]$.

$\therefore T(\alpha u) = \alpha(Tu)$

Let $u(1,2)$ & $\alpha = 3$

$\Rightarrow T(3(1,2)) = T(3,6) = 29$

$T(u) = T(1,2) = 13$ & $\alpha(Tu) = 3(13)$

$= 39$.

$\therefore 39 \neq 29$.

∴ No it is not a L.T.

* Linear Transformation:-

$T: V(\mathbb{F}) \longrightarrow W(\mathbb{F})$

$v \in V, \quad T(v) \in W.$

$\Rightarrow T(v_1 + v_2) = T(v_1) + T(v_2)$

$\Rightarrow T(\alpha v) = \alpha T(v).$

Ex:-

$T: \mathbb{R}^2 \longrightarrow \mathbb{R}$

$T(x,y) = ax + by.$ ✓ . Linear Transmission

$T(x,y) = ax^2 + bx^2$ × Not "

$T(x,y) = ax + by + c.$ × Not "

Now, consider $T(\alpha v) = \alpha T(v)$

Let $\alpha = 0$, $\vec{v} \in V$, $\alpha v = 0v = \vec{0}$

$\Rightarrow T(\alpha v) = T(0v) = T(0) := \alpha(Tv) = 0\,T(\vec{v}) = \vec{0}_w.$

$\therefore \boxed{T(\vec{0}_v) = \vec{0}_w}$

Ex:-

① $T: \mathbb{R}^2 \longrightarrow \mathbb{R}.$

$T(x,y) = 2x + 3y - 5$

Let $(x,y) = (0,0)$

$\Rightarrow T(0,0) = 2(0) + 3(0) - 5 = -5 \neq 0.$

$\therefore$ T is not a Linear transformation. $[\because T(\vec{0}_v) \neq \vec{0}_w]$

② $T: \mathbb{R}^3 \rightarrow \mathbb{R}^2.$

Defined by $T(x,y,z) = [x+y, 2x+3z+1].$

Is T a linear transmission?

A) It's Not L.T.

$\Rightarrow T(0,0,0) = (0+0, 0+0+1) = (0,1) \neq (0,0).$

③ $T: \mathbb{R}^2 \to \mathbb{R}^2$"

Defined by $T(x,y) = (|x|, |x+y|)$. Is T a L.T?

A) $T(0,0) = |0|, |0+0| = (0,0)$.

$T[(x_1,y_1) + (x_2,y_2)] = T(x_1+x_2, y_1+y_2)$

$= (|x_1+x_2|, |x_1+x_2+y_1+y_2|)$

Now, $T(v_1) = (|x_1|, |x_1+y_1|)$

$T(v_2) = (|x_2|, |x_2+y_2|)$

$T(v_1) + T(v_2) = (|x_1|+|x_2|, |x_1+y_1|+|x_2+y_2|)$

$|x_1+x_2| = |x_1|+|x_2|$ is not true always.

$\therefore$ It is not a Linear transformation.

* ④ $T: \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x,y) = (x+y, y)$. Is it a L.T?

A) ① $T(v_1+v_2) = T(v_1) + T(v_2)$

② $T(\alpha v) = \alpha T(v)$.

$\Rightarrow T(v_1+v_2) = T((1,1)+(2,3)) = T(3,4) = (7,4)$

$T(v_1) = T(1,1) = (2,1)$ $\therefore T(v_1)+T(v_2) = (7,4)$.

$T(v_2) = T(2,3) = (5,3)$

$\Rightarrow T(\alpha v) = T(1(1,1)) = T(1,1) = (2,1)$ $\therefore$ It is a L.T

$\Rightarrow \alpha T(v) = (1) T(1,1) = T(1,1) = (2,1)$

* ⑤ $D: P_3(\mathbb{R}) \to P_2(\mathbb{R})$ defined by $D[P(x)] = \frac{d}{dx}[P(x)]$. Is it L.T?

A) $P(x), q(x) \in P_3$, $\alpha \in \mathbb{R}$

$\Rightarrow D(p+\alpha q) = \frac{d}{dx}[P(x) + \alpha q(x)]$.

$= \frac{d}{dx} P(x) + \frac{d}{dx}[\alpha q(x)]$

$= Dp + \alpha Dq$.

$\Rightarrow D(p) + \alpha D(q) = D(p) + \alpha D(q)$.

* Question :-

$T : \mathbb{R}^2 \to \mathbb{R}^2$.

$T(1,2) = (5,6)$ & $T(3,-1) = (0,1)$.

Is it possible to find such Linear transformation?

A) Step-1 :-

Does $\{(1,2), (3,-1)\}$ spans $\mathbb{R}^2$ & Is it L. Independent?

$\Rightarrow \alpha(1,2) + \beta(3,-1) = 0$

$\Rightarrow (\alpha + 3\beta) = 0 \quad \Rightarrow \alpha = -3\beta \quad \Rightarrow \alpha = -3(0) = 0.$

$\Rightarrow (2\alpha + (-1\beta)) = 0 \quad \Rightarrow 2(-3\beta) - \beta = 0 \quad \Rightarrow \beta = 0.$

$\therefore$ It is L. Independent

Let $(x,y) \in \mathbb{R}^2$

$\Rightarrow (x,y) = \alpha(1,2) + \beta(3,-1)$

$\Rightarrow T(x,y) = \alpha T(1,2) + \beta T(3,-1).$

Now. $x = \alpha + 3\beta$, $y = 2\alpha - \beta$.

$\Rightarrow 2x - y = 6\beta + \beta = 7\beta. \Rightarrow \boxed{\beta = \frac{2x-y}{7}}$

$\Rightarrow \alpha = x - 3\beta = x - 3\left(\frac{2x-y}{7}\right) = \boxed{\frac{x+3y}{7}}.$

$\therefore (x,y) = \left(\frac{x+3y}{7}\right)(1,2) + \left(\frac{2x-y}{7}\right)(3,-1)$

$\therefore$ It spans $\mathbb{R}^2$.

$T(x,y) = T(\alpha(1,2) + \beta(3,-1))$

$= T(\alpha(1,2)) + T(\beta(3,-1))$

$= \alpha T(1,2) + \beta T(3,-1)$

$= \alpha(5,6) + \beta(0,1)$. [$\because$ given]

$= \left(\frac{x+3y}{7}\right)(5,6) + \left(\frac{2x-y}{7}\right)(0,1)$.

$= \left(\frac{5x+15y}{7}, \frac{(6x+2x)+(18y-y)}{7}\right)$

$\because \boxed{T(x,y) = \left(\frac{5x+15y}{7}, \frac{8x+17y}{7}\right)}$

* Question :

$T(1,1,0) = (1,2,1)$ $T: \mathbb{R}^3 \to \mathbb{R}^3$.

$T(0,1,0) = (5,2,0)$.

What is the formula for $T(x,y,z)$?

A) $(x,y,z) = \alpha(1,2,1) + \beta(5,2,0)$

$= (\alpha, 2\alpha, \alpha) + (5\beta, 2\beta, 0)$

Now, $\Rightarrow \alpha + 5\beta = x \longrightarrow ①$

$2\alpha + 2\beta = y \longrightarrow ②$

$\alpha = z. \longrightarrow ③$

① $\Rightarrow z + 5\beta = x$ ② $\Rightarrow 2z + 2\left(\frac{x-z}{5}\right) = y$.

$\Rightarrow \beta = \frac{x-z}{5}$

$\therefore \alpha = z$ & $\beta = \frac{x-z}{5}$.

Here, we need three elements.

Theorem :-

→ Every linearly Independent set can be extended to its basis set.

→ $\{(1,1,0), (0,1,0), (0,0,1)\}$ is

$$\begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1(1) - 1(0) + 0(0)$$

$= 1 \neq 0$ $\therefore$ It is L.I. →

$\Rightarrow (x,y,z) = \alpha(1,1,0) + \beta(0,1,0) + \gamma(0,0,1).$

$= (\alpha, \alpha+\beta, \gamma)$

$\alpha = x, \quad \gamma = z, \quad \alpha+\beta = y$

$\Rightarrow \beta = y - x.$

$\therefore (x,y,z) = x(1,1,0) + \beta(y-x)(0,1,0) + z(0,0,1).$

$\therefore T(x,y,z) = xT(1,1,0) + (y-x)T(0,1,0) + zT(0,0,1).$

$= x(1,2,1) + (y-x)(5,2,0) + z(2,5,3).$

$= (x+5y-5x+2z, \; 2x+2y-2x+5z, \; x+0+3z).$

$\boxed{T(x,y,z) = (-4x+5y+2z, \; 2y+5z, \; x+3z)}$

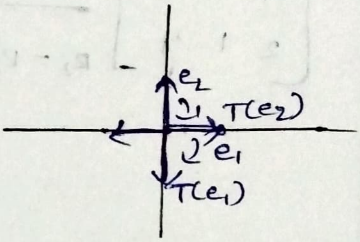
* Ex:

$T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$

$T(x,y) = (y, -x).$

standard Basis, $B = \{(1,0),(0,1)\}$

$\therefore B = \{e_1, e_2\}$



$T(e_1) = T(1,0) = (0,-1)$

$T(e_2) = T(0,1) = (1,0)$

* co-ordinate vectors:

$\rightarrow \mathbb{R}^2, \; B = \{(1,0),(0,1)\}$ $\quad v_B = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$

$(5,6) = 5(1,0) + 6(0,1)$

Ex: $B_1 = \{(1,1),(1,-1)\}$ $\quad v_{B_1} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \frac{11}{2} \\ \frac{-1}{2} \end{bmatrix}.$

$(5,6) = \alpha(1,1) + \beta(1,-1).$

$\Rightarrow \alpha+\beta = 5 \quad \Rightarrow 6+\beta+\beta = 5 \Rightarrow \beta = \frac{-1}{2}$

$\Rightarrow \alpha-\beta = 6 \quad \Rightarrow \alpha + \frac{1}{2} = 6 \Rightarrow \alpha = \frac{11}{2}$

Ex: $T: \mathbb{R}^2 \rightarrow \mathbb{R}^3$

$T(1,2) = (1,1,1)$

$T(1,1) = (0,1,1)$

$(x,y) = \alpha(1,2) + \beta(1,1)$:

$T(x,y) = T[\alpha(1,2) + \beta(1,1)]$.

$x = \alpha + \beta$

$y = 2\alpha + \beta$:

$= T(\alpha(1,2)) + T(\beta(1,1))$.

$= \alpha(T(1,2)) + \beta(T(1,1))$.

$= \alpha(1,1,1) + \beta(0,1,1)$.

Now $= \alpha + 0\beta$

$\alpha + \beta$

$\alpha + \beta$

$= (\alpha, \alpha+\beta, \alpha+\beta)$.

consider $x = \alpha + \beta$

$y = 2\alpha + \beta$

$$\begin{bmatrix} 1 & 1 & x \\ 2 & 1 & y \end{bmatrix} \xrightarrow[R_2 = R_2 - 2R_1]{\cdot R_2(-1)} \simeq \begin{bmatrix} 1 & 1 & x \\ 0 & +1 & y+2x \end{bmatrix} \xrightarrow{R_1 = R_1 - R_2}$$

$$\simeq \begin{bmatrix} 1 & 0 & y-x \\ 0 & 1 & 2x-y \end{bmatrix}$$

$\therefore \alpha = y - x$

$\beta = 2x - y$

Now, $T(x,y) = (\alpha, \alpha+\beta, \alpha+\beta)$

$\Rightarrow T(x,y) = ((y-x), (y-x+2x-y), y-x+2x-y)$.

$\therefore T(x,y) = (y-x, x, x)$

verifying:-

$T(1,2) = (y-x, x, x)$

$= (2-1, 1, 1)$

$= (1,1,1)$

$T(1,1) = (1-1, 1, 1)$

$= (0,1,1)$.

* Note :

$\rightarrow T(2,2) = (1,1,1)$

$T(1,1) = (0,1,1)$

$\Rightarrow (2,2) = 2(1,1)$

$\Rightarrow T(2,2) \neq 2T(1,1)$

$\Rightarrow (1,1,1) \neq 2(0,1,1)$

$\therefore (1,1,1) \neq (0,2,2)$

→ We can take elements which are Linearly Independent only.

* Ex :

$\rightarrow T(9,2) = (2,0,2,3)$ & $T(0,1) = (0,0,0,0)$

$B = \{(9,2), (0,1)\}$

$\Rightarrow (x,y) = \alpha(9,2) + \beta(0,1)$

$9\alpha + 0\beta = x \quad \Rightarrow \alpha = \frac{x}{9}$

$2\alpha + \beta = y \quad \Rightarrow y = 2\left(\frac{x}{9}\right) + \beta$

$\Rightarrow \beta = y - \left(\frac{2x}{9}\right)$

Now, $T(x,y) = \alpha T(9,2) + \beta T(0,1)$

$= \frac{x}{9}(2,0,2,3) + \left(y - \frac{2x}{9}\right)(0,0,0,0)$

$= \left(\frac{2x}{9}, 0, \frac{2x}{9}, \frac{x}{3}\right) + (0,0,0,0)$

$\therefore T(x,y) = \left(\frac{2x}{9}, 0, \frac{2x}{9}, \frac{x}{3}\right)$

Verifying :

$\Rightarrow T(9,2) = \left(\frac{2(9)}{9}, 0, \frac{2(9)}{9}, \frac{9}{3}\right)$

$= (2,0,2,3)$

$\therefore$ Infinite solutions.

* Matrix Representation of vector coordinates:-

Ex:- $T: \mathbb{R}^2 \to \mathbb{R}^2$

$T(x,y) = (5x+3y,\ 2x-y)$

$$\begin{bmatrix} 5 & 3 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5x+3y \\ 2x-y \end{bmatrix}.$$

$$\therefore [T]_B = \begin{bmatrix} 5 & 3 \\ 2 & -1 \end{bmatrix}$$

$T(1,0) = (5(1)+3(0),\ 2(1)-0) = (5,2)$. [$1^{st}$ column]

$T(0,1) = (5(0)+3(1),\ 2(0)-1) = (3,-1)$ [2^{nd} column].

① Domain & codomain,

$B_1 = \{(1,1), (1,-1)\}$.

A) $[T]_{B_1}^{B_1 \to \text{codomain basis}}$ (or) $[T]_{B_1 \to B_1}$

$B_1 \to$ domain basis

$T(1,1) = (5x+3y,\ 2x-y)$

$\Rightarrow T(1,1) = (5(1)+3(1),\ 2(1)-1) = (5+3,\ 1) = (8,1)$.

$\Rightarrow (8,1) = \alpha(1,1) + \beta(1,-1)$.

$\alpha + \beta = 8 \quad \Rightarrow 1+\beta+\beta = 8 \quad \Rightarrow \beta = \frac{7}{2}$

$\alpha - \beta = 1 \quad \Rightarrow \alpha - \frac{7}{2} = 1 \frac{7}{2} \quad \Rightarrow \alpha = \frac{9}{2}$.

Now, $T(1,-1) = (5(1)+3(-1),\ 2(1)-(-1)) = (2,3)$.

$\Rightarrow (2,3) = a(1,1) + b(1,-1)$.

$a+b = 2 \quad \Rightarrow a = \frac{5}{2}$.

$a-b = 3 \quad \Rightarrow b = \frac{-1}{2}$

$$\therefore [T]_{B_1}^{B_1} = \begin{bmatrix} \alpha & a \\ \beta & b \end{bmatrix} = \begin{bmatrix} \frac{9}{2} & \frac{5}{2} \\ \frac{7}{2} & -\frac{1}{2} \end{bmatrix}.$$

* Ex:

$T(0,1) = ?$

$v = (0,1) = \alpha_1 (1,1) + \alpha_2 (1,-1)$

$\alpha_1 + \alpha_2 = 0 \quad \Rightarrow \alpha_1 = \frac{1}{2}$

$\alpha_1 - \alpha_2 = 1 . \quad \Rightarrow \alpha_2 = \frac{-1}{2}.$

$[v]_{B_1} = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}.$

$\therefore [T]_{B_2}^{B_1} \cdot [v]_{B_1}' = \begin{bmatrix} \frac{9}{2} & \frac{5}{2} \\ \frac{7}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}$

$= \left(\left(\frac{9}{2}\right)\left(\frac{1}{2}\right) + \left(\frac{5}{2}\right)\left(-\frac{1}{2}\right) \right), \left(\left(\frac{7}{2}\right)\left(\frac{1}{2}\right) + \left(-\frac{1}{2}\right)\left(-\frac{1}{2}\right) \right)$

$= \begin{bmatrix} 1 \\ 2 \end{bmatrix} = T[v]_{B_1}$

$T(v) = T(0,1) = 1(1,1) + 2(1,-1) = (3,-1).$

* Question:

$T: \mathbb{R}^3 \to \mathbb{R}^2$. defined by the following :-

$T(x,y,z) = (x-y+z, 2x-3y)$., $B_1 = \{(1,1,0), (1,0,1), (0,1,1)\}$.

and for $\mathbb{R}^2$, $B_2 = \{(1,1), (1,0)\}$. Write the matrix

Representation for it?

A) $T(x,y,z) = (x-y+z, 2x-3y)$.

$$\begin{bmatrix} 1 & -1 & 1 \\ 2 & -3 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x-y+z \\ 2x-3y+0z \end{bmatrix}.$$

$\therefore [T]_{B_1}^{B_2} = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -3 & 0 \end{bmatrix}.$

$T(1,1,0) = (1-1+0, 2(1)-3(1)) = (0,-1)$

$\cancel{T(1,1,0)} = (0,-1)$

$(0,-1) = \alpha(1,1) + \beta(1,0).$

$(0,-1) = (-1)(1,1) + (1)(1,0)$

$\therefore [T(1,1,0)]_{B_2} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$

Image of 2nd element

$T(1,0,1) = (x-y+z,\ 2x-3y)$

$= (1-0+1,\ 2-3(0)) = (2,2).$

Now, $(2,2) = a(1,1) + b(1,0).$

$a = 2$

$b = 0$

$(2,2) = 2(1,1) + 0(1,0).$

$\therefore [T(1,0,1)]_{B_2} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}.$

Image of 3rd Basis element

$T(0,1,1) = (0-1+1,\ 0-3) = (0,-3).$

Now, $(0,-3) = c(1,1) + d(1,0)$

$c+d = 0$

$c = -3$

$d = 3$

$\Rightarrow (0,-3) = -3(1,1) + 3(1,0).$

$\therefore [T(0,1,1)]_{B_2} = \begin{bmatrix} -3 \\ 3 \end{bmatrix}.$

$\therefore (T)_{B_1}^{B_2} = \begin{bmatrix} \alpha & a & c \\ \beta & b & d \end{bmatrix} = \begin{bmatrix} -1 & 2 & -3 \\ 1 & 0 & 3 \end{bmatrix}_{2\times 3}$

* Note :-

→ No. of collumns = Dimension of Domain

No. of rows = Dimension of co-domain.

Ex :- (1) $A_{m\times n}$

$\Rightarrow T: \mathbb{R}^n \to \mathbb{R}^m$

(2) $A_{2\times 2}$

$\Rightarrow T: \mathbb{R}^2 \to \mathbb{R}^2$

(3) $A_{4\times 4}$

$\Rightarrow T: \mathbb{R}^4 \to \mathbb{R}^4.$

* Ex:

$B = \{v_1, v_2, \ldots, v_n\}$. $\quad T: V \to W$, $\dim(V) = n$.

$B' = \{u_1, u_2, \ldots, u_n\}$. $\quad \dim(W) = m$.

A) $T(v_1) = \alpha_{11}(u_1) + \alpha_{21}(u_2) + \ldots \alpha_{m1}(u_n)$.

$T(v_2) = \alpha_{12}(u_1) + \alpha_{22}(u_2) + \ldots \alpha_{m2}(u_n)$.

$\vdots$

$T(v_n) = \alpha_{1n}(u_1) + \alpha_{2n}(u_2) + \ldots \alpha_{mn}(u_n)$.

$$\begin{bmatrix} \alpha_{11} & \alpha_{12} & \ldots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \ldots & \alpha_{2n} \\ \vdots & \vdots & & \vdots \\ \alpha_{m1} & \alpha_{m2} & \ldots & \alpha_{mn} \end{bmatrix}_{m \times n}$$

* Kernel of T:

→ $\{v \in V \mid T(v) = \vec{0}_W\} \subseteq V$.

$T(v_1) = \vec{0}_w$, $\quad T(v_2) = \vec{0}_w$.

$T(v_1 + \alpha v_2) = T(v_1) + \alpha T(v_2)$
$= \vec{0}_w + \alpha(\vec{0}_w)$
$= \vec{0}$

Note:

→ $\ker(T) \subseteq V$.

$\ker(T)$ is subspace of V.

$\dim[\ker(T)]$ = Nullity of T.

Ex:

$T: \mathbb{R}^3 \to \mathbb{R}^2$

$T(x,y,z) = (x+y, 2x-3y+z)$.

$\ker(T) = \{(x,y,z) \in \mathbb{R}^3 \mid T(x,y,z) = 0,0\}$.

① $T(x,y,z) = (x+y+z, 0)$
↓
we lost 2 dims.

② $T(x,y,z) = (0,0)$.
↓
We lost all dimns.
image is 0.

A) $x+y=0$ (or) $\{(x,y,z)\in\mathbb{R}^3 / x+y=0,\ 2x-3y+z=0\}$.

$2x-3y+z=0$.

From ① $y=-x$

$\Rightarrow 2x+3x+z=0$

$\Rightarrow 5x+z=0$

$\Rightarrow 5x=-z \Rightarrow z=-5x$

$\therefore \ker(T)=\{(x,y,z)\in\mathbb{R}^3 / y=-x \ \&\ z=-5x\}$.

$=\{(x,-x,-5x)/x\in\mathbb{R}\}$.

$\therefore \operatorname{Dim}(\ker T)=1$.

Now, $\ker(T)=\{(x,-x,-5x)/x\in\mathbb{R}\}$

$=\{x(1,-1,-5)/x\in\mathbb{R}\}$

$\therefore$ Matrix Represebtation:

$$A=\begin{bmatrix}1 & 1 & 0\\ 2 & -3 & 1\end{bmatrix}.$$

$T(x,y,z)=(x+y,\ 2x-3y+z)$.

$(1,0,0)=(1,2)$

$(0,1,0)=(1,-3)$

$(0,0,1)=(0,1)$

The solution space of $AX=0$ is Null space.

$$\underbrace{\begin{bmatrix}1 & 1 & 0\\ 2 & -3 & 1\end{bmatrix}}_{A}\underbrace{\begin{bmatrix}x\\ y\\ z\end{bmatrix}}_{X}=0.$$

* Image (or) Range of T:

$\text{Range}(T)=\{w\in W / T(v)=w.$ for some $v\in V\}$.

$\dim(\text{Range}(T))=$ Rank.

$\longrightarrow$

* 다
$T: \mathbb{R}^3 \to \mathbb{R}^2$.

$T(x,y,z) = (x+y,\ 2x-3y+z)$.

A) $\ker(T) = \{x(1,-1,-5) \mid x \in \mathbb{R}\}$. [we lost 1-dim].

$\text{Range}(T) = \mathbb{R}^2$

Ex
Preimage of $(5,3)$.

$\Rightarrow x+y=5$

$\Rightarrow 2x-3y+z=3$.

Let $z=0$

$\Rightarrow 2x-3y=3$

$\Rightarrow \cancel{2x}+2y=10 \longrightarrow ① \times ②$

$\cancel{2x}-3y=3$

$5y=7$

$y=\frac{7}{5}$.

$\Rightarrow x+\frac{7}{5}=5 \Rightarrow x=5-\frac{7}{5}=\frac{18}{5}$.

$\therefore T(x,y,z) = (x+y,\ 2x-3y+z)$.

$(\frac{18}{5}, \frac{7}{5}, 0) = (5,3)$.

$\Rightarrow (\frac{18}{5}+\frac{7}{5},\ 2(\frac{18}{5})-3(\frac{7}{5})+0)$

$\Rightarrow (\frac{25}{5},\ \frac{36}{5}-\frac{21}{5})$

$\Rightarrow (5,\ \frac{15}{5})$

$\Rightarrow (5,3)$.

$\therefore \text{Range}(T) = \mathbb{R}^2$.

* Rank-Nullity Theorem:

→ Let $T: U \to W$ is a linear transformation and $\dim(U) = n$.

then Rank-Nullity Theorem says that

$$\boxed{\text{Rank}(T) + \text{Nullity}(T) = \text{Dim}(U)}$$

→ In terms of Matrix,

$$\boxed{\text{Rank}(A) + \text{Nullity}(A) = \dim(\text{Domain}) \text{ (or) No. of columns of } A.}$$

* Problem:

If $\text{Rank}(A) = 3$, where A is a 5×4 Matrix. [Asked]

(a) what is the Nullity of A, $N(A)$?

(b) what is the Nullity of A^T, $N(A^T)$?

A) $\text{Rank}(A) + \text{Nullity}(A) = \text{No. of columns of } A$.

(a) → $3 + \text{Nullity}(A) = 4$

$\Rightarrow \text{Nullity}(A) = 4 - 3$

$\therefore N(A) = 1$.

(b) $\text{Rank}(A^T) + \text{Nullity}(A^T) = \text{No. of columns of } A^T$

$A^T_{4 \times 5}$

$\Rightarrow 3 + N(A^T) = 5$

$\Rightarrow N(A^T) = 5 - 3$

$\therefore N(A^T) = 2$.

* Problem :

$T(x,y) = (x+y,\ 2x+2y,\ 3x+3y)$; $T: \mathbb{R}^2 \to \mathbb{R}^3$.

Find the Rank & Nullity ?

A) $x+y=0 \Rightarrow x=-y \Rightarrow y=-x$.

$2x+2y=0 \Rightarrow x+y=0 \Rightarrow x=-y \Rightarrow y=-x$

$3x+3y=0 \Rightarrow x+y=0 \Rightarrow x=-y \Rightarrow y=-x$

$\therefore \ker(T) = \left\{ (x,y) \,\middle|\, \begin{array}{l} x+y=0 \\ 2x+2y=0 \\ 3x+3y=0 \end{array} \right\}$

$= \{(x,y) \in \mathbb{R}^2 \,/\, y=-x\}$.

$= \{(x,-x) / \cancel{x} \in \mathbb{R}^*\} = \{(x,-x) \,/\, x \in \mathbb{R}\}$

$\therefore \dim(\ker(T)) = 1$

Now,

$T(x,y) = (x+y,\ 2(x+y),\ 3(x+y))$

$= (x+y\ (1,2,3))$

$= (x+y)\ (1,2,3)$.

Range $(T) = \{(t, 2t, 3t) \,/\, t \in \mathbb{R}\}$.

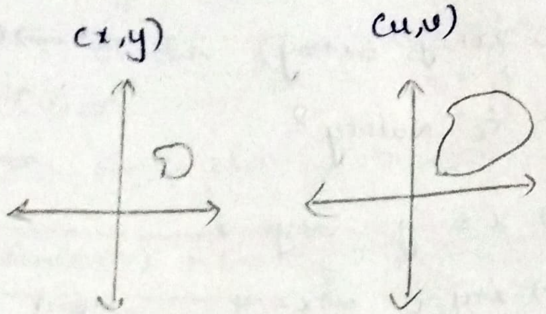
$\therefore \dim(\text{Range}(T)) = 1$.

$\therefore \dim(\text{Range}(T)) + \dim(\ker(T)) = \dim(V)$

$\Rightarrow$ Rank(T) + Nullity(T) = dim(V).

$\Rightarrow 1+1=2$.

* Jacobians :-



$u = f(x,y)$

$v = g(x,y)$

ex :-

$u = 2x - 3y$

$v = x + 5y$

→ Jacobians tells change when one coordinate plane Jumps to another.

→ (x, y) (r, θ)

$x = r\cos\theta$, $y = r\sin\theta$.

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}.$$

$$= r\cos^2\theta + r\sin^2\theta$$

$$\therefore \frac{\partial(x,y)}{\partial(r,\theta)} = r(1) = r.$$

Now, $$\frac{\partial(r,\theta)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \end{vmatrix} = \frac{1}{r} = \frac{1}{\sqrt{x^2+y^2}}$$

Note :-

→ $$\frac{\partial(x,y)}{\partial(u,v)} \cdot \frac{\partial(u,v)}{\partial(x,y)} = 1.$$

multiplication

Inverse to each other.

Problem:

$u = x^2+y^2$, $\quad v = 2xy$, $\quad \frac{\partial(x,y)}{\partial(u,v)} = ?$

A) $\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$ (or) $\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$

$$= \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} 2x & 2y \\ 2y & 2x \end{vmatrix} = 4x^2-4y^2.$$

$$\therefore \frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{4(x^2-y^2)} = \frac{1}{4\sqrt{u^2-v^2}}$$

$$\Rightarrow (x^2-y^2)^2 = (x^2+y^2)^2 - 4x^2y^2$$
$$= u^2 - v^2.$$

* Problem:-

$u = x^2+y^2$ $\qquad v = 2xy$

$x(s,t) = 2s+4t$. $\qquad y(s,t) = 3s^2+2t^2$.

Find $\frac{\partial(u,v)}{\partial(s,t)}$.

A) $\frac{\partial(u,v)}{\partial(s,t)} = \frac{\partial(u,v)}{\partial(x,y)} \times \frac{\partial(x,y)}{\partial(s,t)}$

$(u,v) \downarrow (x,y) \downarrow (s,t)$.

Now, $\frac{\partial(x,y)}{\partial(s,t)} = \begin{vmatrix} x_s & x_t \\ y_s & y_t \end{vmatrix}$

$$= \begin{vmatrix} 2 & 4 \\ 6s & 4t \end{vmatrix} = 8t - 24s.$$

$$\therefore \frac{\partial(u,v)}{\partial(s,t)} = 4(x^2-y^2)(8t-24s).$$