Unit - IV

Linear Programming:

Linear Programming Problems (LPP): Introduction to Linear Programming (LP), Illustration of LP Problems, Formulation exercises on LP problem, Graphical method of solving LPP, Simplex method, Unboundedness, Multiple Optimum Solutions, Degeneracy and Cycling Problems.

Introduction:-

[Panneerselvam]

Linear Programming is a mathematical programming technique to optimize ~~param~~ performance under a set of resource constraints as specified by an organisation.

Concept of LP Model:

Model of any LPP will contain: Objective function, set of constraints and non-negativity restrictions. Each of these contain one or more of the following:

- Decision variables
- Objective function coefficients
- Technological coefficients
- Availability of resources

Assumptions in LP:-

1. Linearity:- Amount of resource required for a given activity level is directly proportional to the level of that activity.

Ex:- If no of hours req for a particular m/c = 5 hrs/unit, then total " " on that m/c to produce 10 units = 50 hours

2. Divisibility:- Fractional values of the decision variables are permitted

3. Non-negativity:- Decision variables are permitted to have only the values ~~of~~ which are greater than or equal to zero.

4. Additivity:- Total o/p for a given combination of activity levels is the algebraic sum of the o/p of each individual process

Properties of Linear Programming Solution:-

Feasible Solution:-

- If all the constraints of given LP model are satisfied by solution of the model, then that solution is known as feasible solution.
- Several such solutions are possible for a given LP model.

Optimal Solution:-

- If there is no other superior solution to the solution obtained for a given LP model, then the solution obtained is "Optimum Solution."
- It is the best feasible solution.
- It is one of the feasible solutions.

Unique Optimal Solution:-

- For a given problem, only one solution will give the best objective value

Multiple Optimal Solution:-

- The problem will have more than one solution with the best objective value

Unbounded Solution:-

- For some LP model, the objective function value can be increased/decreased infinitely without any limitation. Such solution is known as "Unbounded Solution."

Infeasible Solution:-

- If there is no combination of the values of the decision variable satisfying all the constraints of LP model, then that model is said to have infeasible solution

Corner point Optimal Solution:-

- In the feasible region, the optimal solution lies at any one of the corner points of the region.

Degenerate Solution:-

- In LP problems, intersection of two constraints will define a corner point of the feasible region. But, if more than two constraints pass through any one of corner points of feasible region, excess constraints will not serve any purpose, and therefore act as redundant constraints. Under such situation, degeneracy will occur.
- There will not be any improvement in objective function, even after some iterations are carried out in Simplex method

LP Structure:-

1. Objective Function
2. Decision Variables
3. Constraints

1. **Objective Function:-** It is the function of the decision variables that the decision maker wants to maximize or minimize.

2. **Decision Variables:-**

- Completely describe the decisions to be made.
- Key parameters which are to be calculated

Ex:- Let, x_1 = No. of Chairs, x_2 = No. of Tables } to be produced

$x_1, x_2 \Rightarrow$ decision variable

$x_1, x_2, \ldots, x_n \geq 0$

3. **Constraints:-**

- Constraints show the restrictions on the values of decision variables.
- For maximisation $\Rightarrow$ $\leq$ type constraint
- For minimisation $\Rightarrow$ $\geq$ type constraint
- $\leq$ (or) $\geq$ (or) $=$ (Mixed Constraints)

Formulation of LP:-

① A firm produces 3 products. These products are produced on 3 diff. m/c's. The time req. to manufacture 1 unit of each of the 3 products & the daily capacity of the 3 machines are given in the table below

Machine	Time per Unit (minutes) Product 1	Product 2	Product 3	Machine Capacity (min/day)
M_1	2	3	2	440
M_2	4	-	3	470
M_3	2	5	-	430

It is required to determine the daily no. of units to be manufactured for each product. The profit per unit for product 1, 2 & 3 is Rs. 4, 3 & 6 respectively. It is assumed that all the products produced are consumed in the market.

Sol:- Step-1:- Identification of decision variables

Here, the key decision is to identify the daily no. of units to be manufactured for each product.

Let, x_1, x_2 & x_3 be the daily no. of units to be produced of product 1, 2 & 3 respectively.

Step-2:- Writing the objective function

The objective is to maximise the profit earned by selling products 1, 2 & 3 in the market. So, the objective function can be written as:

$$\text{Maximize } Z = 4x_1 + 3x_2 + 6x_3$$

Step-3:- Writing the constraints

M_1: $2x_1 + 3x_2 + 2x_3 \leq 440$ — I

M_2: $4x_1 + 3x_3 \leq 470$ — II

M_3: $2x_1 + 5x_2 \leq 430$ — III

The constraints are due to the m/c capacity available per day. So, the constraints can be written as above.

Step-4:- Writing the non-negative constraints

All the decision variables are non-negative values i.e.,

$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$

Step-5:- Summarize the result

Maximize $Z = 4x_1 + 3x_2 + 6x_3$

Subject to: (i) $2x_1 + 3x_2 + 2x_3 \leq 440$ (M_1)

(ii) $4x_1 + 3x_3 \leq 470$ (M_2)

(iii) $2x_1 + 5x_2 \leq 430$ (M_3)

(iv) $x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$

② A person wants to decide the constituents of a diet which will fulfill his daily requirements of proteins, fats & carbohydrates at the min. cost. The choice is to be made from 4 different types of food. The yields per unit of these foods are given below.

Food type	Yield per Unit: Proteins	Fats	Carbohydrates	Cost per unit (Rs)
1	3	2	6	45
2	4	2	4	40
3	8	7	7	85
4	6	5	4	65
Min. requirement	800	200	700	

Formulate LP model for the problem.

Sol:- Step-1:- Identification of decision variables

The key decision is to find yield per unit of 4 diff types of food to fulfil the daily requirements.

Let x_1, x_2, x_3, x_4 are the yield per unit of food types 1,2,3 & 4 respectively.

Step-2:- Writing the objective function

The objective is to minimize the cost of to fulfil the daily requirements of proteins, fats & carbohydrates

The objective function here is minimisation & written

Minimise $Z = 45x_1 + 40x_2 + 85x_3 + 65x_4$

Step-3:- Writing the constraints.

The constraints are due to minimum requirement of food type.

(i) $3x_1+4x_2+8x_3+6x_4 \geq 800$ (Proteins)

(ii) $2x_1+2x_2+7x_3+5x_4 \geq 200$ (Fats)

(iii) $6x_1+4x_2+7x_3+4x_4 \geq 700$ (Carbohydrates)

Step-4:- Writing the non-negative constraints.

All the decision variables are non-negative constraints i.e.,

$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$ & $x_4 \geq 0$.

Step-5:- Writing the result.

Minimize $Z = 45x_1+40x_2+85x_3+65x_4$

Subject to: (i) $3x_1+4x_2+8x_3+6x_4 \geq 800$.

(ii) $2x_1+2x_2+7x_3+5x_4 \geq 200$.

(iii) $6x_1+4x_2+7x_3+4x_4 \geq 700$.

(iv) $x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0$.

Graphical Method:- [Prem Kumar]

This method provides a pictorial representation of solution process and a great deal of insight into the basic concepts used in solving large LP problems.

Steps: 1. Represent the given problem in mathematical form

2. Draw the x_1 and x_2 - axes.

3. Plot each of the constraint on the graph.

4. Identify the feasible region that satisfies all the constraints simultaneously.

5. Use iso-profit function line approach. For this, plot the objective function by assuming $Z=0$.

Problems

① Solve the following:

Maximize $Z = 3x + 2y$

Subject to: (i) $-2x + 3y \leq 9$

(ii) $3x - 2y \leq -20$

(iii) $x \geq 0, y \geq 0$

Sol:- O(0,0) is default solution for $\leq$ type constraints.

Constraint-1:- $-2x + 3y \leq 9$

Writing equation: $-2x + 3y = 9$

If $y = 0$, $-2x + (3 \times 0) = 9 \Rightarrow x = -9/2 = -4.5$

$(x, y) = (-4.5, 0)$

If $x = 0$, $-(2 \times 0) + 3y = 9 \Rightarrow y = 9/3 = 3$

$(x, y) = (0, 3)$

Constraint-2:- $3x - 2y \leq -20$

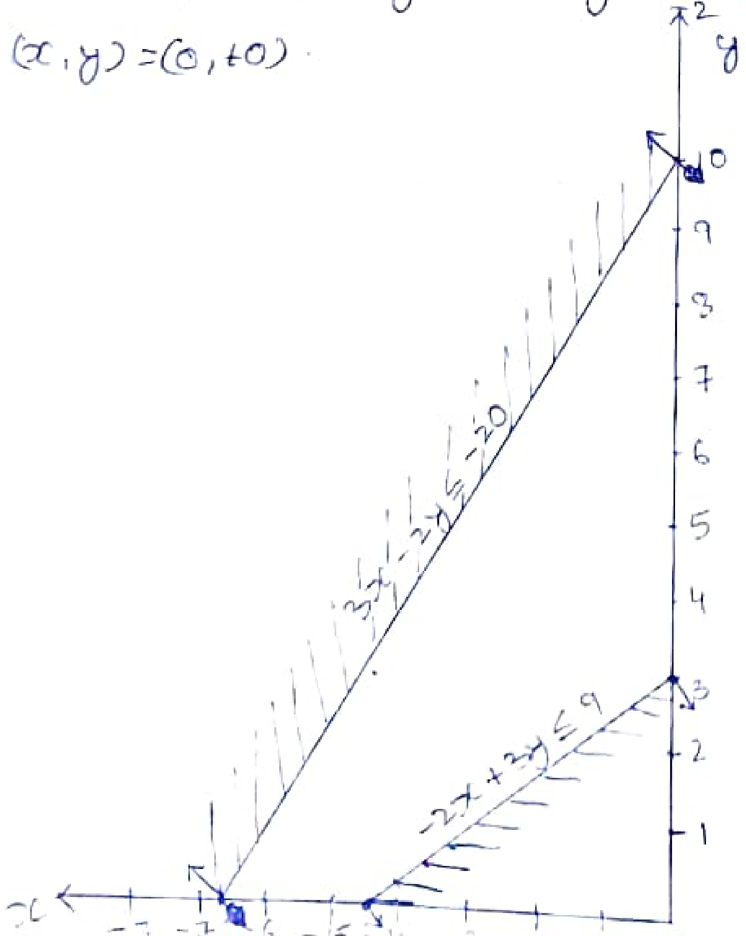
Writing equation: $3x - 2y = -20$

If $y = 0$, $3x - (2 \times 0) = -20 \Rightarrow x = \frac{-20}{3} = -6.667$

$(x, y) = (-6.67, 0)$

If $x = 0$, $(3 \times 0) - 2y = -20 \Rightarrow y = \frac{-20}{-2} = 10$

$(x, y) = (0, 10)$



From the above graph, we can conclude that there is no common region bounded by the given constraints.

② Maximize $Z = 100x_1 + 60x_2$

Subject to: $5x_1 + 10x_2 \leq 50$

$8x_1 + 2x_2 \geq 16$

$3x_1 - 2x_2 \geq 6$

$x_1, x_2 \geq 0$

Sol:- Constraint-1:-

$5x_1 + 10x_2 \leq 50$

Writing eq.: $5x_1 + 10x_2 = 50$

If $x_2 = 0$, $5x_1 + (10 \times 0) = 50 \Rightarrow x_1 = 10$

$(x_1, x_2) = (10, 0)$

If $x_1 = 0$, $(5 \times 0) + 10x_2 = 50 \Rightarrow x_2 = 5$

$(x_1, x_2) = (0, 5)$

Constraint-2:-

$8x_1 + 2x_2 \geq 16$

Writing eq.: $8x_1 + 2x_2 = 16$

If $x_2 = 0$, $8x_1 + (2 \times 0) = 16 \Rightarrow x_1 = 2$

$(x_1, x_2) = (2, 0)$

If $x_1 = 0$, $(8 \times 0) + 2x_2 = 16 \Rightarrow x_2 = 8$

$(x_1, x_2) = (0, 8)$

Constraint-3:-

$3x_1 - 2x_2 \geq 6$

Writing eq: $3x_1 - 2x_2 = 6$

If $x_2 = 0$, $(3x_1) - (2 \times 0) = 6 \Rightarrow x_1 = 2$

$(x_1, x_2) = (2, 0)$

If $x_1 = 0$, $(3 \times 0) - 2x_2 = 6 \Rightarrow x_2 = -3$

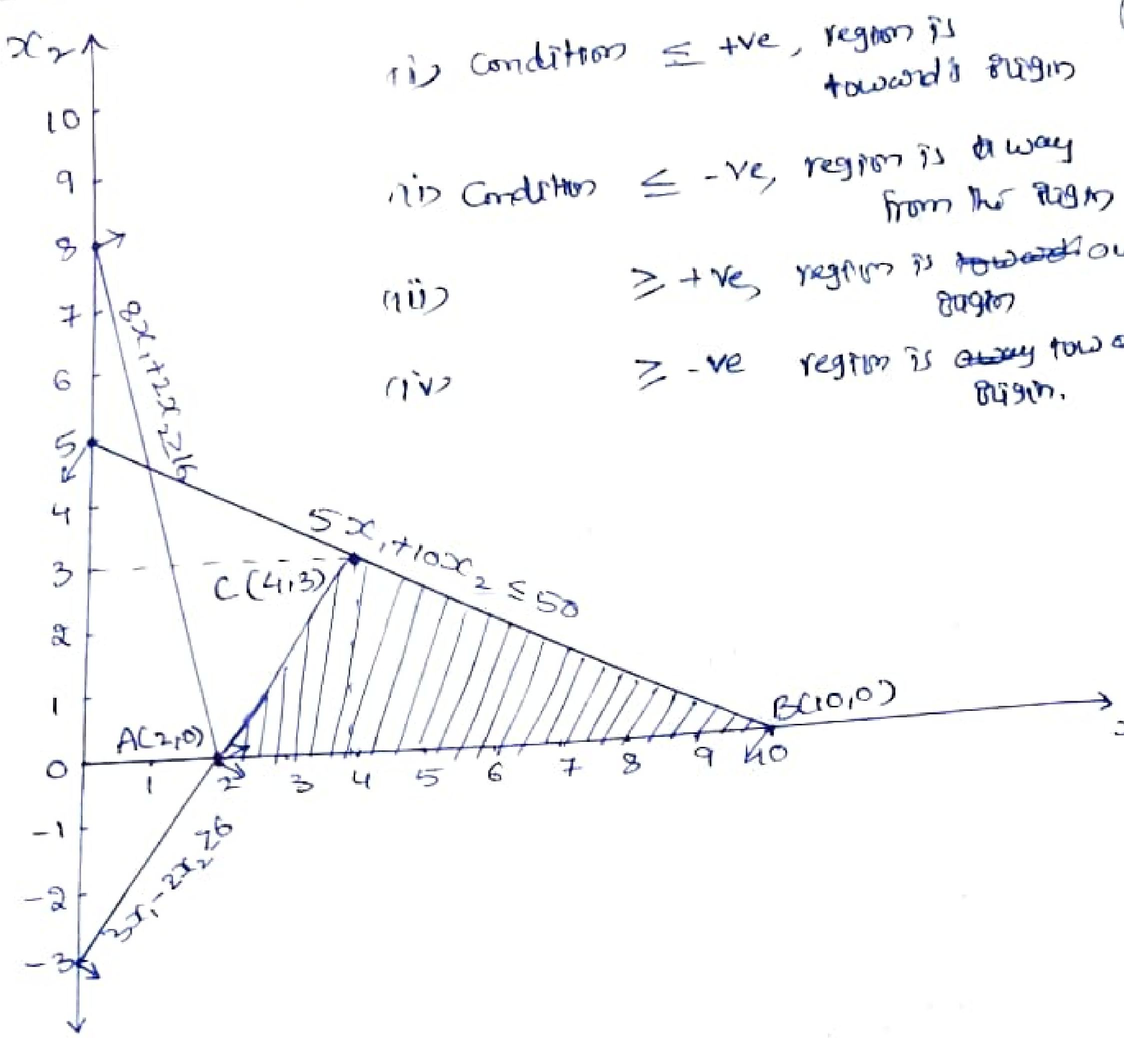
$(x_1, x_2) = (0, -3)$

(i) Condition $\leq +ve$, region is towards origin

(ii) Condition $\leq -ve$, region is away from the origin

(iii) $\geq +ve$, region is ~~toward~~ ow origin

(iv) $\geq -ve$ region is ~~away~~ towar origin.



$A(2,0) \Rightarrow Z_A = (100\times2)+(60\times0) = 200$

$B(10,0) \Rightarrow Z_B = (100\times10)+(60\times0) = 1000$

$C(4,3) \Rightarrow Z_C = (100\times4)+(60\times3) = 580$

The problem has multiple solutions.

Since, the maximum value of Z is 1000, which occurs at $B(10,0)$, the solution to the given problem is

$$x_1 = 10,\ x_2 = 0 \ \& \ Z_{max} = Z_B = 1000.$$

Simplex Method:-

Some definitions.

1. Basic Solution:- A solution obtained by setting any "n" variables (among m+n variables) equal to zero and solving for remaining "m" variables is called a Basic Solution.

"m" variables → Basic variables.

"n" variables → Non-basic variables.

2. Basic Feasible Solution:- It is a basic solution that also satisfies the non-negativity restrictions.

3. Non-degenerate Basic Feasible Solution:- It is a basic feasible solution in which all the "m" variables are positive (>0) and the remaining "n" variables are zero each.

4. Degenerate Basic Feasible Solution:- It is a basic feasible solution in which one or more of the "m" basic variables are equal to zero

5. Optimal Basic Feasible Solution:- It is the BFS that also optimizes the objective function.

Problems.

① Solve the following LPP:

Maximize $Z = 3x_1 + 4x_2$

Subject to: (i) $x_1 + x_2 \leq 450$.

(ii) $2x_1 + x_2 \leq 600$.

(iii) $x_1 \geq 0, x_2 \geq 0$.

Sol:- Step-1:- Express the given LP problem in standard form

To do this, the following conditions must be satisfied.

(i) All the decision variables must be +ve values

(ii) RHS values of constraints must be +ve values. If not convert them into +ve values by multiplying the entire constraint with "−".

(iii) Convert the constraints into equations. This can be done by adding slack variables when the constraint is of $\leq$ type & by subtracting surplus variable when the constraint is of $\geq$ type. Also when the constraint is of = type, add artificial variable.

Now, let us convert the given constraints into eq's.

(i) $x_1 + x_2 + S_1 = 450$

(ii) $2x_1 + x_2 + S_2 = 600$

(iii) $x_1 \geq 0, x_2 \geq 0, S_1 \geq 0, S_2 \geq 0$

∴ The standard form of the objective function can be written as:

Maximize $Z = 3x_1 + 4x_2 + (0 \times S_1) + (0 \times S_2)$

Step-2:- To find IBFS (Initial Basic feasible Solution)

To find IBFS, let us substitute the values of decision variables as zeroes.

This will result in the following:

$Z_{max} = (3\times0) + (4\times0) + (0\times S_1) + (0\times S_2) = 0$

$0 + 0 + S_1 = 450 \Rightarrow S_1 = 450$

$(2\times0) + 0 + S_2 = 600 \Rightarrow S_2 = 600$

Step-3:- To perform optimality test.

Initial Simplex Table

		C_j	3	4	0	0	
C_{B_i}	Basis	b_j	x_1	x_2	S_1	S_2	$\theta = \frac{b_j}{\text{Key Column}}$
0	S_1	450	1	1	1	0 →	$\frac{450}{1} = 450$
0	S_2	600	2	1	0	1	$\frac{600}{1} = 600$
	Z_j	0	0	0	0	0	
	$C_j - Z_j$	–	3	4	0	0	

↑ Key Column

$Z_i = \sum_{i=1}^{2} (C_{B_i})(a_{ij}) = (0\times\cancel{450})(0\times1) + (0\times2) = 0$

Optimality Condition: For max, $C_j - Z_j \leq 0$

For min, $C_j - Z_j \geq 0$

$$\text{New value} = \text{Old Value} - \frac{\text{Corr. Key Column} \times \text{Corr. Key row}}{\text{Key element}}$$

[C_{B_i} - Coefficients of Basic variables

C_j - " "

$C_j - Z_j$ - Index row or Net Evaluation Row]

[When no more +ve values remain in $C_j - Z_j$ row, the profit attained is max. & optimal solution is achieved.]

Second Simplex Table

		C_j	3	4	0	0
C_{B_i}	Basis	b_j	x_1	x_2	S_1	S_2
4	x_2	450	1	1	1	0
0 $(S_2 - x_2)$	S_2	150	1	0	-1	1
	Z_j	1800	4	4	4	0
	$C_j - Z_j$	-	-1	0	-4	0

Since, all elements $(C_j - Z_j)$ are either zero or negative, the second feasible solution is optimal.

Optimal Solution: $x_1 = 0$ $S_1 = 0$

$x_2 = 450$ $S_2 = 150$

$Z_{max} = 1800$

② Solve the following LPP using the Simplex method.

Maximize $Z = 12x_1 + 16x_2$

Subject to $10x_1 + 20x_2 \leq 120$

$8x_1 + 8x_2 \leq 80$

$x_1, x_2 \geq 0$

Sol:- Step-1:- Express the given LP problem in std. form. To do this, the following conditions must be satisfied.

(i) All the decision variables must be +ve values.

(ii) RHS values of constraints must be +ve values. If not convert them into +ve values by multiplying the entire constraint with "−".

(iii) Convert the constraints into equations.

(a) $10x_1 + 20x_2 + S_1 = 120$

(b) $8x_1 + 8x_2 + S_2 = 80$

(c) $x_1, x_2 \geq 0, S_1, S_2 \geq 0$

∴ The Std. form of the objective function can be written as:

Maximize $Z = 12x_1 + 16x_2 + (0 \times S_1) + (0 \times S_2)$

Step-2:- To find IBFS.

To find IBFS, let us substitute the values of decision variables as zeroes.

This will result in the following:

$$Z_{max} = (12\times0)+(16\times0)+(0\times S_1)+(0\times S_2)=0$$

$$0+0+S_1=120 \Rightarrow S_1=120$$

$$0+0+S_2=80 \Rightarrow S_2=80$$

Step-3:- To perform optimality test.

Initial Simplex Table

C_{B_i}	Basis	C_j / b_j	12 / x_1	16 / x_2	0 / S_1	0 / S_2	$\theta = \frac{b_j}{\text{Key Column}}$
0	S_1	120	10	20	1	0	$\frac{120}{20} = 6 \rightarrow$
0	S_2	80	8	8	0	1	$\frac{80}{8} = 10$
	Z_j	0	0	0	0	0	
	$C_j - Z_j$	–	12	16	0	0	
			↑	↑			

Key Column

Second Simplex Table

	C_{B_i}	Basis	C_j / b_j	12 / x_1	16 / x_2	0 / S_1	0 / S_2	$\theta = \frac{b_j}{\text{Key Column}}$
	16	x_2	6	1/2	1	1/20	0	$\frac{6}{(1/2)} = 12$
$(S_2 - 8x_2)$	0	S_2	32	4	0	-2/5	1	$\frac{32}{4} = 8 \rightarrow$
		Z_j	96	8	16	4/5	0	
		$C_j - Z_j$	–	4	0	-4/5	0	
				↑				

Third Simplex Table

	C_{B_i}	Basis	C_j / b_j	12 / x_1	16 / x_2	0 / S_1	0 / S_2
$(x_2 - \frac{x_1}{2})$	16	x_2	2	0	1	1/10	-1/8
	12	x_1	8	1	0	-1/10	1/4
		Z_j	128	12	16	2/5	1
		$C_j - Z_j$	–	0	0	-2/5	-1

$\because C_j - Z_j \le 0$ – Optimal.

Since, all $(C_j - Z_j) \le 0$, the Third feasible solution is optimal.

Optimal Solution; $x_1 = 8$ $S_1 = 0$

$x_2 = 2$ $S_2 = 0$

$Z_{max} = 128$.

(3) Solve the following LPP by Simplex method:

Maximize $Z = 2x_1 + x_2 - 3x_3 + 5x_4$

Subject to: (i) $x_1 + 7x_2 + 3x_3 + 7x_4 \le 46$.

(ii) $3x_1 - x_2 + x_3 + 2x_4 \le 8$.

(iii) $2x_1 + 3x_2 - x_3 + x_4 \le 10$.

(iv) $x_1, x_2, x_3, x_4 \ge 0$.

Sol:- Step-1:- Express the given LPP in std. form. To do this, the following conditions must be satisfied.

(i) All the decision variables must be +ve values.

(ii) RHS side values must be +ve values. If not, convert them into +ve values by multiplying the entire constraint with "–".

(iii) Convert the constraints into eq's.

(i) $x_1 + 7x_2 + 3x_3 + 7x_4 + S_1 = 46$

(ii) $3x_1 - x_2 + x_3 + 2x_4 + S_2 = 8$

(iii) $2x_1 + 3x_2 - x_3 + x_4 + S_3 = 10$

(iv) $x_1, x_2, x_3, x_4, S_1, S_2, S_3 \ge 0$.

$\therefore$ The std. form of the objective function can be written as:

Maximize $Z = 2x_1 + x_2 - 3x_3 + 5x_4 + (0 \times S_1) + (0 \times S_2) + (0 \times S_3)$

Step-2:- To find IBFS

To find IBFS, let us substitute the values of decision variables as zeroes.

This will result in the following:

$Z_{max} = (2 \times 0) + 0 - (3 \times 0) + (5 \times 0) + (0 \times S_1) + (0 \times S_2) + (0 \times S_3)$

$= 0$

$S_1 = 46$

$S_2 = 8$

$S_3 = 10$

Step-3:- To perform optimality test.

Initial Simplex Table

C_{B_i}	C_j → Basis	2 x_1	1 x_2	-3 x_3	5 x_4	0 S_1	0 S_2	0 S_3	b_j	$\theta = \frac{b_j}{\text{Key Column}}$
0	S_1	1	7	3	7	1	0	0	46	$\frac{46}{7} = 6.57$
0	S_2	3	-1	1	2	0	1	0	8	$\frac{8}{2} = 4$ →
0	S_3	2	3	-1	1	0	0	1	10	$\frac{10}{1} = 10$
	Z_j	0	0	0	0	0	0	0	0	
	$C_j - Z_j$	2	1	-3	5	0	0	0	–	
					↑ Key Column					

∵ All $(C_j - Z_j) \nleq 0$, IBFS is not optimal.

Leaving variable = S_2

Entering variable = x_4

$$\text{New Value} = \text{Old Value} - \frac{\text{Key Row Value} \times \text{Key Column Value}}{\text{Key element}}$$

Second Simplex Table:-

C_{B_i}	C_j → Basis	2 x_1	1 x_2	-3 x_3	5 x_4	0 S_1	0 S_2	0 S_3	b_j	$\theta = \frac{b_j}{\text{Key Column}}$
($S_1 - 7x_4$) 0	S_1	$-19/2$	$21/2$	$-1/2$	0	1	$-7/2$	0	18	$18/(21/2) = 1.714$ →
5	x_4	$3/2$	$-1/2$	$1/2$	1	0	$1/2$	0	4	$4/(-1/2) = -8$
($S_3 - x_4$) 0	S_3	$1/2$	$7/2$	$-3/2$	0	0	$-1/2$	1	6	$6/(7/2) = 1.714$
	Z_j	$15/2$	$-5/2$	$5/2$	5	0	$5/2$	0	20	
	$C_j - Z_j$	$-11/2$	$7/2$	$-11/2$	0	0	$-5/2$	0	–	
			↑ Key Column							

∵ All $(C_j - Z_j) \nleq 0$, this solution is not optimal

Leaving variable = S_1

Entering variable = x_2

Third Simplex table

C_{B_i}	c_j / Basis	2 / x_1	1 / x_2	-3 / x_3	5 / x_4	0 / S_1	0 / S_2	0 / S_3	b_i
1	x_2	$-19/21$	1	$-1/21$	0	$2/21$	$-1/3$	0	$12/7$
5 $(x_4+\frac{21}{2}x_2)$	x_4	$\frac{22}{21}$	0	$\frac{10}{21}$	1	$+1/21$	$4/3$	0	$34/7$
0 $(S_3-\frac{7}{2}x_2)$	S_3	$11/3$	0	$-4/3$	0	$-1/3$	$2/3$	1	0
Rough:	z_j	$\frac{91}{21}$	1	$\frac{49}{21}$	5	$1/3$	$4/3$	0	
	c_j-z_j	$\frac{-49}{21}$	0	$\frac{-16}{3}$	0	$-1/3$	$-4/3$	0	

[Rough: $\frac{3}{2}+\frac{21}{2}\left(\frac{-19}{21}\right)=\frac{22}{2}=11$ $\quad \frac{1}{2}-\frac{21}{2}\left(\frac{-1}{21}\right)=1$

$\frac{-19}{21}+5\left(\frac{22}{21}\right)$ $\quad 1-\frac{21}{2}(0)=1$ $\quad 0-\frac{21}{2}\left(\frac{2}{21}\right)=-1$ $\quad \frac{1}{2}-\frac{21}{2}\left(\frac{-1}{3}\right)$

$0-\frac{21}{2}(0)$ $\quad \frac{3}{2}+\frac{1}{2}\left(\frac{-19}{21}\right)=\frac{63-19}{42}=\frac{44}{42}=\frac{22}{21}$

$\frac{-19+110}{21}$ $\quad \frac{1}{2}+\frac{1}{2}\left(\frac{-1}{21}\right)=\frac{21-1}{2\times 21}=\frac{20}{2\times 21}=\frac{10}{21}$ $\quad 1+\frac{1}{2}(0)=1$

$+\frac{91}{21}$ $\quad 0+\frac{1}{2}\left(\frac{2}{21}\right)$ $\quad 0+\frac{1}{2}\left(-\frac{1}{3}\right)$ $\quad \frac{1}{2}+\frac{1}{2}\left(-\frac{1}{3}\right)=\frac{3-1}{6}=\frac{2}{6}=\frac{1}{3}$

$\frac{-1}{21}+\frac{50}{21}$ $\quad 0+\frac{1}{2}(0)$

$\frac{1}{2}-\frac{7}{2}\left(\frac{-19}{21}\right)=\frac{3+19}{6}=\frac{22}{6}=\frac{11}{3}$ $\quad \frac{-3}{2}-\frac{7}{2}\left(\frac{-1}{21}\right)=\frac{-3}{2}+\frac{1}{6}=\frac{-9+1}{6}=\frac{-8}{6}$

$\frac{91}{42}$ $\quad 0-\frac{7}{2}(0)$ $\quad 0-\frac{7}{2}\left(\frac{2}{21}\right)$ $\quad -\frac{1}{2}-\frac{7}{2}\left(-\frac{1}{3}\right)=\frac{-3+7}{6}=\frac{4}{6}$

$\frac{49}{}$

$-3-\frac{49}{21}$ $\quad -8+\frac{1}{2}\left(\frac{12}{7}\right)=\frac{-56+6}{7}=\frac{50}{7}$ $\quad 4+\frac{1}{2}\left(\frac{12}{7}\right)$

$\frac{-63-49}{21}$ $\quad 6-\left(\frac{7}{2}\times\frac{12}{7}\right)$ $\quad \frac{12}{7}+5\left(\frac{34}{7}\right)$ $\quad 12+170$ $\quad \frac{182}{7}=26$

$\frac{-112}{21}=\frac{16}{3}$]

$\because (c_j-z_j)\le 0$ - Optimal - Third feasible solution

Optimal solution: $x_1=0,\ x_2=\frac{12}{7},\ x_3=0,\ x_4=\frac{34}{7}$

$S_1=0,\ S_2=0,\ S_3=0$

Max. $Z=26$

Graphical Method:-

③ Solve the following LP problem graphically:

$$\text{Minimize } Z = 20x_1 + 40x_2$$

$$\text{Subject to } x_1 + 2x_2 \le 40$$

$$3x_1 + x_2 \ge 30$$

$$4x_1 + 3x_2 \ge 60$$

$$x_1 \text{ \& } x_2 \ge 0$$

Sol:- Constraint-1:- $x_1 + 2x_2 \le 40$

Writing eq.: $x_1 + 2x_2 = 40$

If $x_2 = 0$, $x_1 + (2\times 0) = 40 \Rightarrow x_1 = 40$

$(x_1, x_2) = (40, 0)$

If $x_1 = 0$, $0 + 2x_2 = 40 \Rightarrow x_2 = 20$

$(x_1, x_2) = (0, 20)$

Constraint-2:- $3x_1 + x_2 \ge 30$

Writing eq.: $3x_1 + x_2 = 30$

If $x_2 = 0$, $3x_1 + 0 = 30 \Rightarrow x_1 = 10$

$(x_1, x_2) = (10, 0)$

If $x_1 = 0$, $(3\times 0) + x_2 = 30 \Rightarrow x_2 = 30$

$(x_1, x_2) = (0, 30)$

Constraint-3:- $4x_1 + 3x_2 \ge 60$

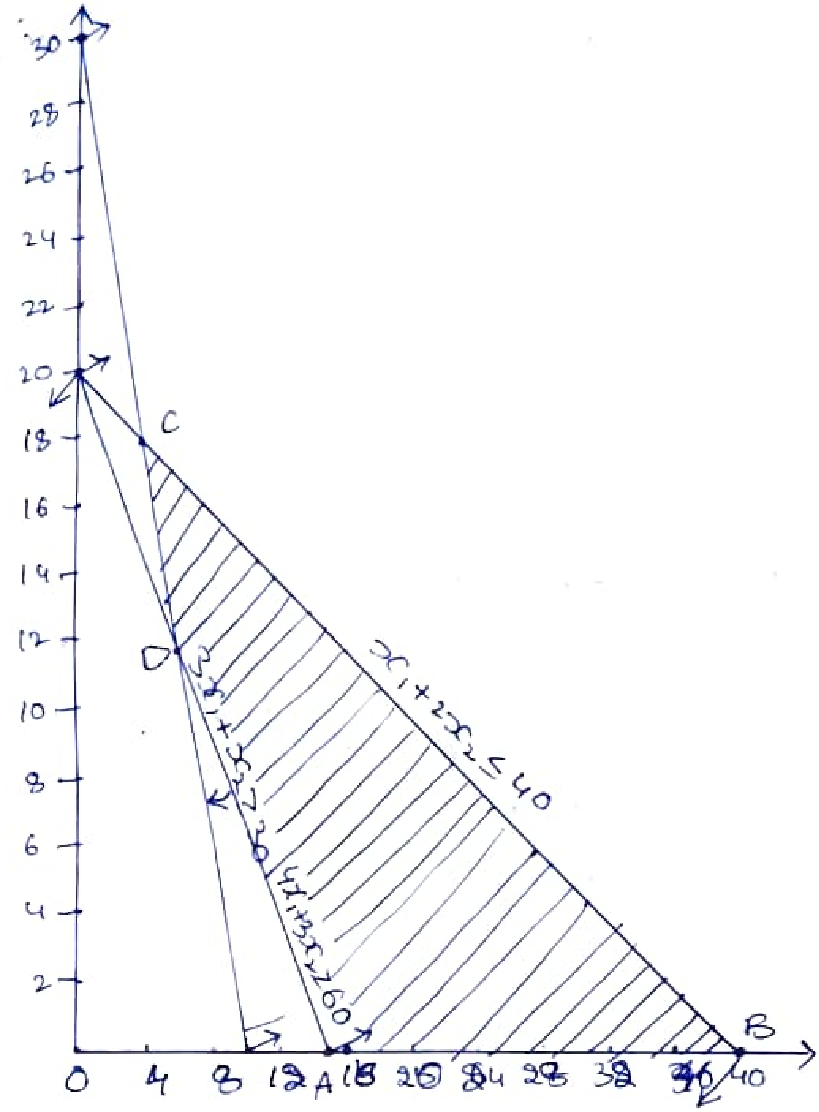
Writing eq.: $4x_1 + 3x_2 = 60$

If $x_2 = 0$, $4x_1 + (3\times 0) = 60 \Rightarrow x_1 = 15$

$(x_1, x_2) = (15, 0)$

If $x_1 = 0$, $(4\times 0) + 3x_2 = 60 \Rightarrow x_2 = 20$

$(x_1, x_2) = (0, 20)$



[To find C, int. of ① & ② constraints,

$$3\times(x_1+2x_2)=40\times 3 \qquad \cancel{18}+2x_2=40$$

$$3x_1+x_2=30 \qquad 2x_2=22$$

$$(-)\quad(-)\quad(-) \qquad x_2=11$$

$$5x_2=90$$

$$x_2=18$$

$$x_1+(2\times 18)=40 \Rightarrow x_1=40-36=4$$

At C, $(x_1,x_2)=(4,18)$

To find D, int. of ② & ③ constraints,

$$4\times(3x_1+x_2)=30\times 4 \qquad 3x_1+12=30$$

$$3\times(4x_1+3x_2)=60\times 3 \qquad 3x_1=18$$

$$(-)\quad(-)\quad - \qquad x_1=6$$

$$-5x_2=-60$$

$$x_2=12$$

At D, $(x_1,x_2)=(6,12)$]

$A(15,0) \Rightarrow Z_A = (20 \times 15) + (10 \times 0) = 300$

$B(40,0) \Rightarrow Z_B = (20 \times 40) + (10 \times 0) = 800$

$C(4,18) \Rightarrow Z_C = (20 \times 4) + (10 \times 18) = 80 + 180 = 260$

$D(6,12) \Rightarrow Z_D = (20 \times 6) + (10 \times 12) = 120 + 120 = 240$

The problem has multiple solutions.

Since, the min. value of Z is 240, which occurs at $D(6,12)$, the solution to the given problem is

$x_1 = 6, \ x_2 = 12, \ Z_{min} = 240. = Z_D$

Unbounded Solution:-

(4) Solve the following LP problem using graphical method.

Maximize $Z = 12x_1 + 25x_2$

Subject to $12x_1 + 3x_2 \geq 36$

$15x_1 - 5x_2 \leq 30$

$x_1 \ \& \ x_2 \geq 0$

Sol:- Constraint-1:- $12x_1 + 3x_2 \geq 36$

Writing eq.: $12x_1 + 3x_2 = 36$

If $x_2 = 0$, $12x_1 + (3 \times 0) = 36 \Rightarrow x_1 = 3$

$(x_1, x_2) = (3,0)$

If $x_1 = 0$, $(12 \times 0) + 3x_2 = 36 \Rightarrow x_2 = 12$

$(x_1, x_2) = (0,12)$

Constraint-2:- $15x_1 - 5x_2 \leq 30$

Writing eq.: $15x_1 - 5x_2 = 30$

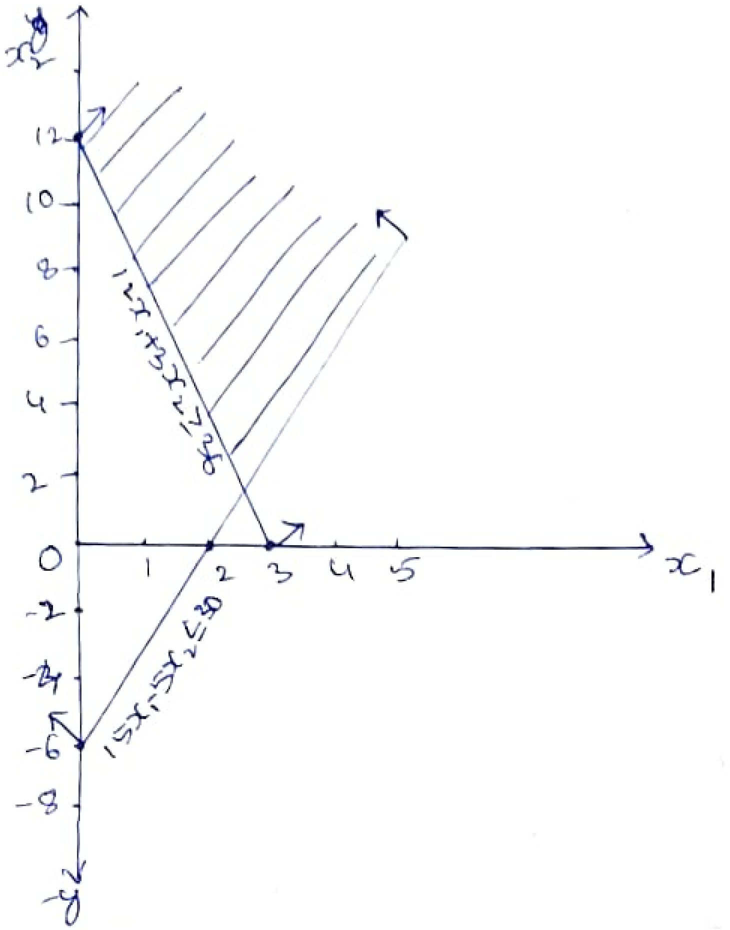
If $x_2 = 0$, $15x_1 - (5 \times 0) = 30 \Rightarrow x_1 = 2$

$(x_1, x_2) = (2,0)$

If $x_1 = 0$, $(15 \times 0) - 5x_2 = 30 \Rightarrow x_2 = -6$

$(x_1, x_2) = (0,-6)$

Constraints:-



In this figure, the solution space is not closed. This indicates unbounded nature. The obj. fn. can be increased to infinity.

Multiple optimum solution:-

(5) Solve the following LP problem using graphical method:

Maximize $Z = 20x_1 + 10x_2$

Subject to $10x_1 + 5x_2 \leq 50$

$6x_1 + 10x_2 \leq 60$

$4x_1 + 12x_2 \leq 48$

$x_1 \ \& \ x_2 \geq 0$

Sol:- Constrain-1:- $10x_1 + 5x_2 \leq 50$

Writing eq.: $10x_1 + 5x_2 = 50$

If $x_2 = 0$, $10x_1 + (5 \times 0) = 50 \Rightarrow x_1 = 5$

$(x_1, x_2) = (5, 0)$.

If $x_1=0$, $(10\times0)+5x_2=50 \Rightarrow x_2=10$

$(x_1,x_2)=(0,10)$

Constraint-2:- $6x_1+10x_2 \le 60$

Writing eq.: $6x_1+10x_2=60$

If $x_2=0$, $6x_1+0=60 \Rightarrow x_1=10$

$(x_1,x_2)=(10,0)$

If $x_1=0$, $0+10x_2=60 \Rightarrow x_2=6$

$(x_1,x_2)=(0,6)$

Constraint-3:- $4x_1+12x_2 \le 48$

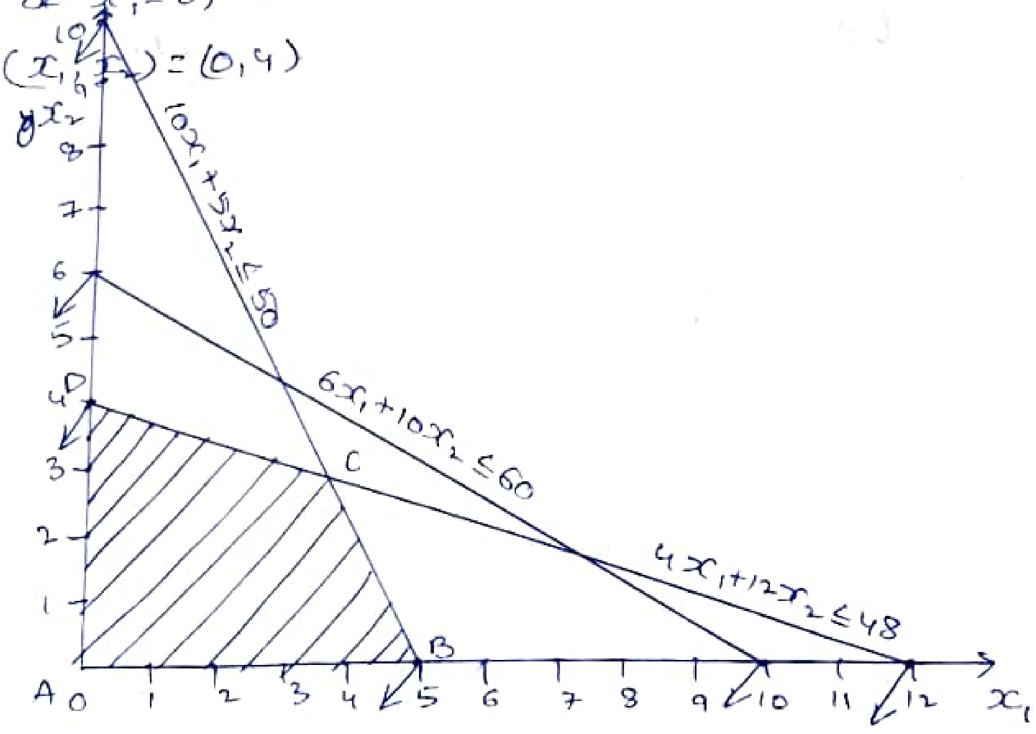
Writing eq.: $4x_1+12x_2=48$

If $x_2=0$, $4x_1+0=48 \Rightarrow x_1=12$

$(x_1,x_2)=(12,0)$

If $x_1=0$, $0+12x_2=48 \Rightarrow x_2=4$ $(x_1,x_2)=(0,4)$

$(x_1,x_2)=(0,4)$



[Point "C", Intersection of ① & ③ Constr.,

$4\times(10x_1+5x_2)=50\times4$

$10\times(4x_1+12x_2)=48\times10$

$-100x_2=-280$

$x_2=2.8$

$10x_1+(5\times2.8)=50$

$x_1=3.6$

$C=(3.6,2.8)$]

200
-480

50
14
36

$z_A = (20\times 0)+($

$z_B = (20\times 5)+(10\times 0) = 100$

$z_C = (20\times 3.6)+(10\times 2.8) = 72+28 = 100$

$z_D = (20\times 0)+(10\times 4) = 40$

The problem has multiple solutions.

Since, the max. value of Z is 100, which occurs at $B(5,0)$ & $C(3.6, 2.8)$, we have multiple optimum solutions for given obj. fn.

Degeneracy:-

(6) Solve the following LP problem graphically.

Maximize $Z = 100x_1 + 50x_2$

Subject to $4x_1 + 6x_2 \le 24$

$x_1 \le 4$

$x_2 \le \frac{4}{3}$

$x_1, x_2 \ge 0$

Sol:- Constraint-1:- $4x_1 + 6x_2 \le 24$

Writing eq.: $4x_1 + 6x_2 = 24$

If $x_2 = 0$, $4x_1 + 0 = 24 \Rightarrow x_1 = 6$ $(x_1, x_2) = (6, 0)$

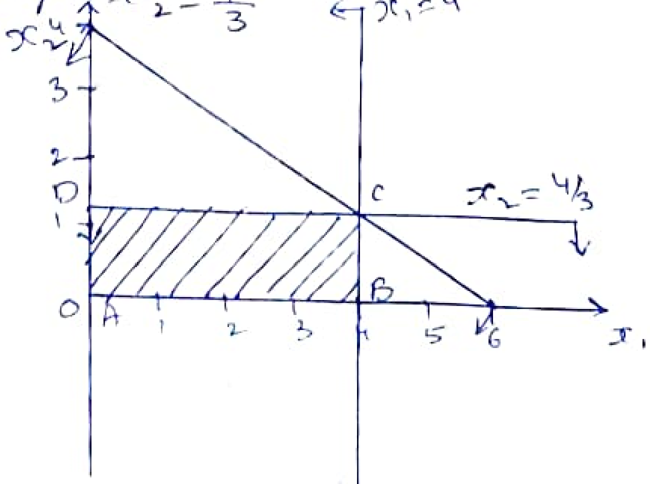
If $x_1 = 0$, $0 + 6x_2 = 24 \Rightarrow x_2 = 4$ $(x_1, x_2) = (0, 4)$

Constraint 2:- $x_1 \le 4$

Writing eq.: $x_1 = 4$

Constraint-3:- $x_2 \le \frac{4}{3}$

Writing eq.: $x_2 = \frac{4}{3}$



ABCD - feasible region.

In graph, at corner point "C", 3 lines intersect. This shows the presence of degeneracy in problem

At A, $Z_A = 0$

$B(4,0)$, $Z_B = (100 \times 4) + (50 \times 0) = 400$

$C(4, 1.33)$, $Z_C = (100 \times 4) + (50 \times 1.33) = 466.67$

$D(0, 1.33)$, $Z_D = (100 \times 0) + (50 \times 1.33) = 66.67$

The problem has multiple solutions.

Since, the max. value of Z is 466.67 which occurs at $C(4, 1.33)$, the solution to the given problem is,

$x_1 = 4$

$x_2 = 1.33$

$Z_{max} = Z_C = 466.67$

Simplex method:-

(4) Solve the following LPP using Simplex method:

Minimize $Z = 2x_1 - 3x_2 + 6x_3$

Subject to: $3x_1 - x_2 + 2x_3 \leq 7$

$2x_1 + 4x_2 \geq -12$

$-4x_1 + 3x_2 + 8x_3 \leq 10$

$x_1, x_2 \& x_3 \geq 0$

Sol:- Step 1:- Express the given LPP in std. form.

To do this, the following conditions must be satisfied.

(i) All the decision variables must be +ve values.

(ii) RHS values of constraints must be +ve values. If not convert them into +ve values by multiplying the entire eq. with "-".

Constrain-2: $2x_1 + 4x_2 \geq -12$

$-2x_1 - 4x_2 \leq 12$

(iii) Convert the constraints into eq's.

(i) $3x_1 - x_2 + 2x_3 + S_1 = 7$

(ii) $-2x_1 \mp 4x_2 \qquad + S_2 = 12$

(iii) $-4x_1 + 3x_2 + 8x_3 + S_3 = 10$

(iv) $x_1, x_2, x_3, S_1, S_2, S_3 \geq 0$.

∴ The std. form of obj. function can be written as:

Minimize $Z = 2x_1 - 3x_2 + 6x_3 + (0 \times S_1) + (0 \times S_2) + (0 \times S_3)$

Step-2:- To find IBFS.

To find IBFS, let us substitute the values of decision

variables as zeroes.

This will result in the following:

$Z\ min = (2\times0) - (3\times0) + (6\times0) + (0\times S_1) + (0\times S_2) + (0\times S_3) = 0$

$(3\times0) - 0 + (2\times0) + S_1 = 7 \Rightarrow S_1 = 7$

$-(2\times0) \mp (4\times0) + S_2 = 12 \Rightarrow S_2 = 12$

$-(4\times0) + (3\times0) + (8\times0) + S_3 = 10 \Rightarrow S_3 = 10$

Step-3:- To perform optimality test.

Initial Simplex Table

	C_j	2	-3	6	0	0	0		
C_{B_i}	Basis	x_1	x_2	x_3	S_1	S_2	S_3	b_j	$\theta = \frac{b_j}{\text{Key Column}}$
0	S_1	3	-1	2	1	0	0	7	$\frac{7}{-1} = -7$
0	S_2	-2	-4	0	0	1	0	12	$\frac{12}{-4} = -3$
0	S_3	-4	3	8	0	0	1	10	$\frac{10}{3} = 3.33$
	Z_j	0	0	0	0	0	0	0	
	$C_j - Z_j$	2	-3	6	0	0	0	-	

Key Column

Leaving Variable = S_3

Entering Variable = x_2

Key element = 3

∵ All $C_j - Z_j \ngeq 0$, optimality is not reached.

Second Simplex Table

C_{B_i}	C_j / Basis	2 x_1	−3 x_2	6 x_3	0 S_1	0 S_2	0 S_3	b_j	$\theta = \frac{b_j}{KC}$
0	S_1	$5/3$	0	$14/3$	1	0	$1/3$	$31/3$	$\frac{(31/3)}{(5/3)} = 6.2$
0	S_2	$-22/3$	0	$32/3$	0	1	$4/3$	$76/3$	$\frac{(76/3)}{(-22/3)} = -3$
−3	x_2	$-4/3$	1	$8/3$	0	0	$1/3$	$10/3$	$\frac{(10/3)}{(-4/3)} = +2$
	Z_j	4	−3	−8	0	0	−1	−10	
	$C_j - Z_j$	−2	0	14	0	0	1	–	

For x_2 : New Value = $\frac{\text{Old Value}}{\text{key element}}$

$x_1 = \frac{-4}{3}$ $\qquad S_1 = 0/3 = 0$

$x_2 = \frac{3}{3} = 1$ $\qquad S_2 = 0/3 = 0$

$x_3 = 8/3$ $\qquad S_3 = 1/3$

$b_j = 10/3$

For S_1 : New Value = Old value $- \left[\frac{\text{Key row element}}{\text{Key element}} \times \text{Key Column element}\right]$

$x_1 = 3 - \left(\frac{-4}{3} \times -1\right) = 3 - \frac{4}{3} = 5/3$

$x_2 = -1 - \left[\frac{-3}{3} \times -1\right] = -1 + 1 = 0$

$x_3 = 2 \cdot \left[\frac{8}{3} \times (-1)\right] = \frac{14}{3}$

$S_1 = 1 - \left[\frac{0}{3} \times (-1)\right] = 1$

$S_2 = 0 - \left[\frac{0}{3} \times (-1)\right] = 0$

$S_3 = 0 - \left[\frac{1}{3} \times (-1)\right] = \frac{1}{3}$

$b_j = 7 - \left[\frac{10}{3} \times (-1)\right] = \frac{31}{3}$