Digital Signal Processing

UNIT-01

Signal:- Any physical quantity vary with time / space and have two or more independent variables and that carries some information.

Ex: 1) Speech signal : Used in telephone, radio

2) Biomedical signal : ECG, MRI scan, EEG, city scan...

3) Sound signal : Music, .mp3's

4) Video signal : 3D , Image signal : 2D

5) Radar signal : Used to determine the range ~~of~~ target.

Signal Processing :- To extract useful information carried out by a signal.

⇒ It is concerned with the mathematical representation ~~of~~ a signal and the algorithmic operations carried out on it to extract the information present.

Signal Processing →
- Analog signal Processing (ASP)
- Digital signal Processing (DSP)
- Mixed signal processing (ASP + DSP)

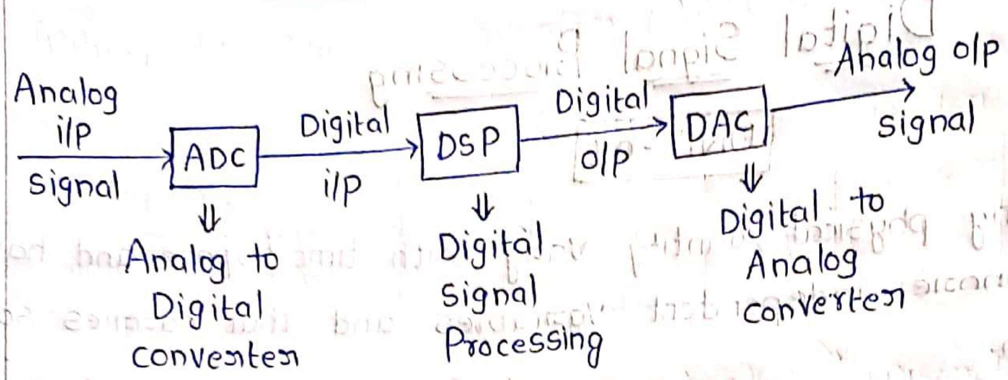
Analog Signal Processing :-

Processing of analog signals to extract / modify to get meaningful information is said to be ASP.

Analog i/p signal → [Analog s/g Processing] → Analog o/p signal

Digital Signal Processing :-

Processing of digital signals inorder to analyze / extract / modify to get meaningful info is said to be DSP.



Advantages of DSP:-

1. **Accuracy**: No. of bits used to represent the signal samples and the coefficients describing the processing operation.

   Accuracy ↑ as no. of bits to represent a given sample also ↑

2. **Flexibility**: DSP systems can be programmed/reprogrammed to perform variety of functions without modifying the hardware.

3. **Drift**:- DSP systems have no drift in performance with age/temperature.

4. **Adaptability**:- It can adopt to characteristics of the i/p or the processing task changes.

5. **Reproducability**: Identical performance from a DSP system is obtained since there are no variations due to component tolerance.

⇒ Copy/reproduce several times over without any degradation in the signal quality.

6. **Multiplexing**: Time sharing of digital signal processor.

⇒ Digital processing allows the sharing of a given processor, among a no. of signals by time sharing, so that we can reduce the cost of processing per signal.

7. <u>Data logging</u>: Signals are stored for indefinitely without any loss of information.

8. <u>Low frequency capability</u>: DSP can handle lower frequency signal very efficiently

⇒ ASP can also handles very low frequency signal but with very large capacitors & inductors.

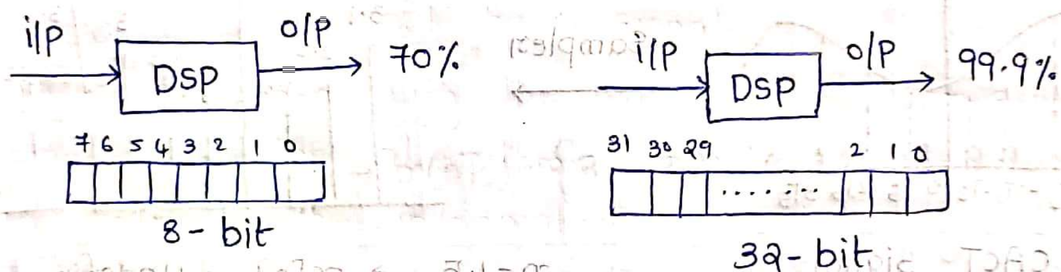
<u>Disadvantages of DSP</u>:-

1. <u>Increase system complexibty</u>: It requires preprocessing units like ADC, DAC

2. <u>Speed & cost</u>: Speed of ASP > Speed of DSP

cost of DSP ↑ as the no. ~~of~~ bits required to implemen the signal ↑

3. <u>Finite word length</u>:



<u>Applications of DSP</u>:-

Space , <u>Medical</u> : ECG signal analysis, CT scan, MRI scan , ultra sound , Diagnostic imaging.

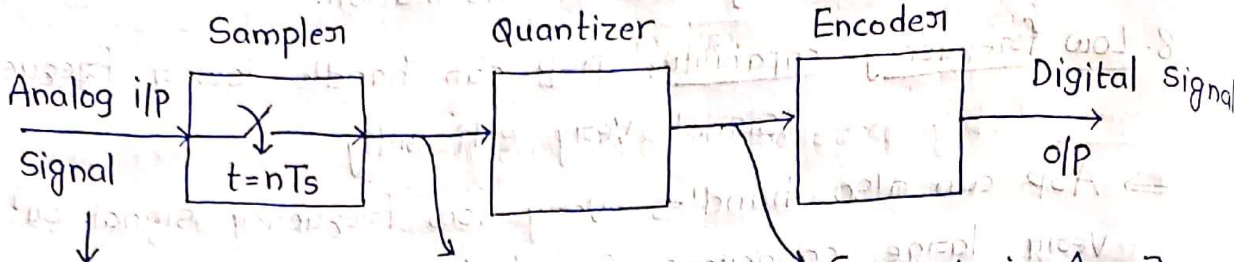
<u>Commercial</u>: Movie special effects, Video conferencing

<u>Telephone</u>: Voice , filtering

<u>Military</u>: Radar signaling , Sonar , missile's , communications.

<u>Scientific</u>: Earthquake recording , spectral analysis.

Analog to Digital Converter (ADC):-

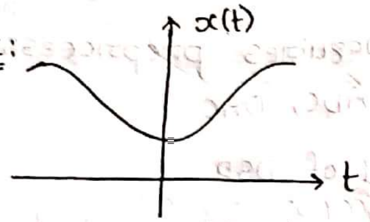


[Continuous in time and in Amplitude]

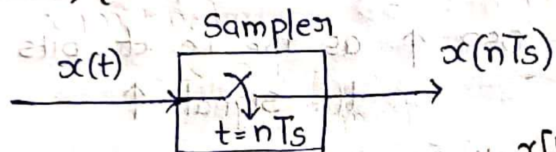
[Continuous in Amp Discrete in time]

[Discrete in Amp Discrete in time]

Ex:



① Sampler

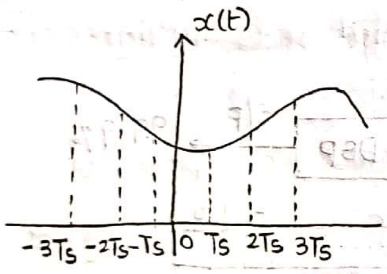


$n=0 \Rightarrow t=0$

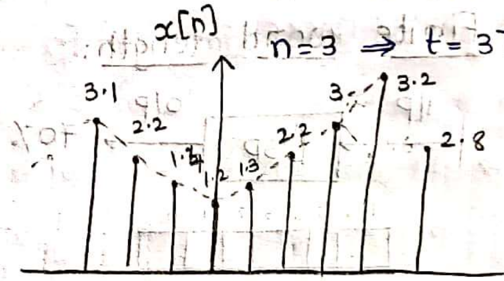
$n=1 \Rightarrow t=1T_s$

$n=2 \Rightarrow t=2T_s$

$n=3 \Rightarrow t=3T_s$



Sampler $\longrightarrow$



CADT Signal

CACT Signal

$n=1.5 \Rightarrow x[n]$ = undefined but not zero.

$$x[n] = \{3.1, 2.2, 1.4, \underset{\uparrow}{1.2}, 1.3, 2.2, 3, 3.2, 2.8\}$$

② Quantizer

Quantizer $\Longrightarrow$ Quantizer levels $\{1, 1.5, 2, 2.5, 3\}$

(5 levels)

$\downarrow$

It will assign the input sequence samples to its nearest quantization levels.

i/p of quantizer $x[n] = \{3.1, 2.2, 1.4, \underset{\uparrow}{1.2}, 1.3, 2.2, 3, 3.2, 2.8\}$

o/p of quantizer $x[n] = \{3, 2, 1.5, \underset{\uparrow}{1}, 1.5, 2, 3, 3, 3\}$

$\uparrow$

Both discrete in time & Amplitude.

③ **Encoder** : Discrete Sequence $\xrightarrow{i/P}$ [ ] $\xrightarrow{o/P}$ Digital Sequence [0/1]

i/p of encoder $x[n] = \{3, 2, 1.5, \underset{\uparrow}{1}, 1.5, 2, 3, 3, 3\}$

$(3)_{10} = (11)_2$

$(2)_{10} = (10)_2$

$(1.5)_{10} = (01.10)_2$

$(1)_{10} = (01)_2$

$(1.5)_{10} = (01.10)_2$

$$\begin{array}{r|l} 2 & 3 \\ 2 & 2 \quad 1 \\ & 1 \quad 0 \end{array} \qquad \begin{array}{r|l} 2 & 3 \\ & 1 \quad 1 \end{array}$$

$1_{10} = 01_2$

$(1.5)_{10} = 01.10$

	Integer	Fraction
$0.5 \times 2 = 1$	1	0

Output : {11 10 01.10 01 0110 10 11 11 11}

Overall Steps:-

Analog signal

CACT ↓

[Sampler]

CADT ↓ $x(nTs)$ $\longrightarrow$ $x(nTs) = \{3.1, 2.2, 1.4, \underset{\uparrow}{1.2}, 1.3, 2.2, 3, 3.2, 2.8\}$

[Quantizer]

DADT ↓ $x[n]$ $\longrightarrow$ $x[n] = \{3, 2, 1.5, \underset{\uparrow}{1}, 1.5, 2, 3, 3, 3\}$

[Encoder]

↓ Digital signal $\longrightarrow$ = {1110011001011010111111}

Basic operations on discrete time sequence / signals:-

1. Signal Addition $\quad z[n] = x[n] + y[n]$
2. Signal multiplication $\quad z[n] = x[n] \cdot y[n]$
3. Scalar Addition $\quad y[n] = k + x[n]$
4. Scalar multiplication $\quad y[n] = k \cdot x[n]$

Transformation of independent Variable (time):

1. Time shifting
2. Time Scaling $\begin{cases} \rightarrow \text{Decimation in time (Compression)} \\ \rightarrow \text{Interpolation in time (Expansion)} \end{cases}$
3. Time reversal

① **Time Shifting:-** $x[n] \longrightarrow x[n \pm n_0]$

if n_0 → +ve ⇒ right shift $= x[n+n_0]$ Delayed Version of x[n]

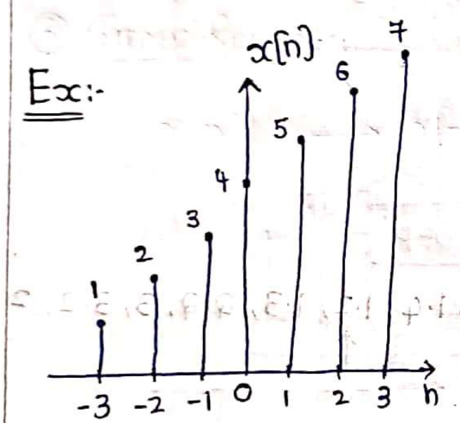
→ -ve ⇒ Left shift $= x[n+n_0]$ Advanced Version of $x[n]$

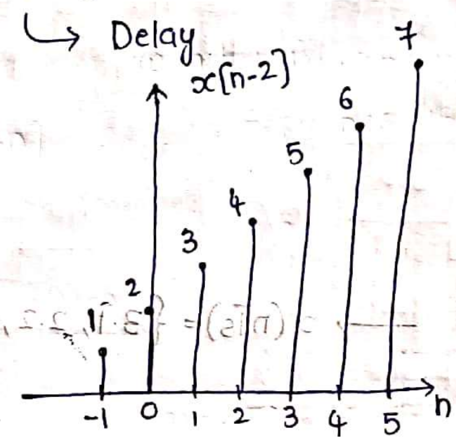
$$y[n] = x[n \pm n_0]$$

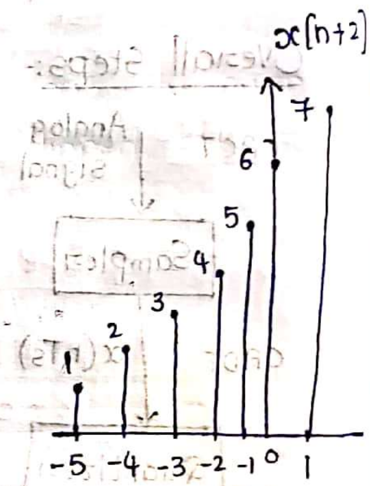
→ Advancement

↳ Delay

Ex:-







$x[n] = \{1, 2, 3, \underset{\uparrow}{4}, 5, 6, 7\}$

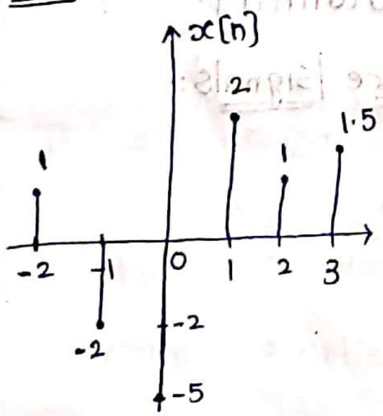
Delay (Right Shif)

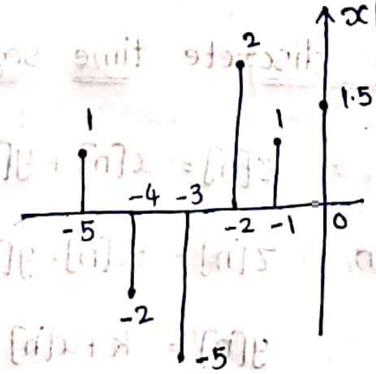
$n-2 \Rightarrow n = +2$

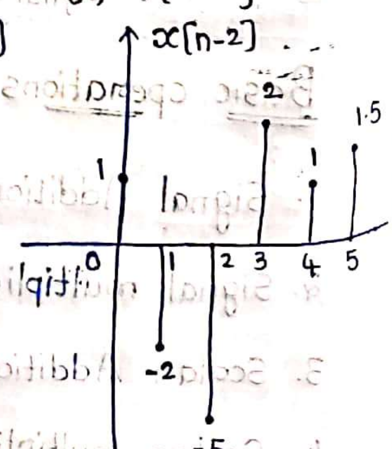
Advancement (Left Shift)

$n+2 = 0 \Rightarrow n = -2$

Ex:- $x[n] = \{1, -2, \underset{\uparrow}{-5}, 2, 1, 1.5\}$ draw for $x[n+3]$, $x[n-2]$







$n+3 = 0 \Rightarrow n = -3$

Advanced by 3 units

$n-2 = 0 \Rightarrow n = 2$

Delayed by 2 units

② **Time Scaling:-**

→ Decimation (Discarding s/g samples)

→ Interpolation (Inserting s/g samples)

- → 0 interpolation
- → 1 unity interpolation
- → Average interpolation

$\alpha > 1$ - Decimation

(α-1) Samples get discarded

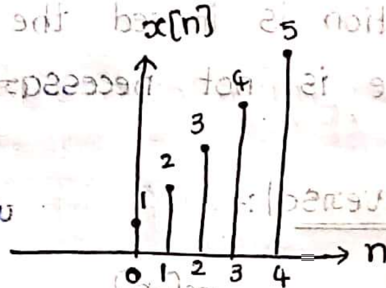
$\alpha < 1$ = Interpolation

(α-1) samples get added.

$\frac{n}{\alpha}$ ← = Area of $x[n]$ / samples in $x[n]$

* Decimation of discrete time sequence will leads to loss of signal information.

Ex:- $x[n] = \{\underset{\uparrow}{\overset{0}{1}}, \overset{1}{2}, \overset{2}{3}, \overset{3}{4}, \overset{4}{5}\}$



$x[2n] = \{\underset{\uparrow}{\overset{0}{1}}, \overset{1}{3}, \overset{2}{5}\}$

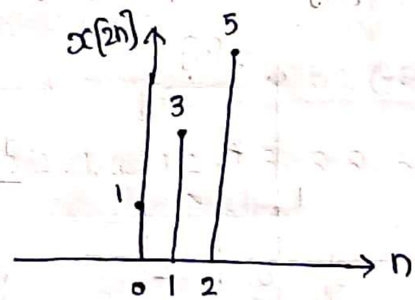
$x[n] = 1, \quad n=0 \qquad \frac{0}{2} = 0 \longrightarrow$ int

$= 2, \quad n=1 \qquad \frac{1}{2} = 0.5$

$= 3, \quad n=2 \qquad \frac{2}{2} = 1 \longrightarrow$ int

$= 4, \quad n=3 \qquad \frac{3}{2} = 1.5$

$= 5, \quad n=4 \qquad \frac{4}{2} = 2 \longrightarrow$ int



Decimation:

$x[n] = \{1,2,3,4,5\} \Rightarrow x[2n] = \{1,3,5\}$

$x[n] = \{1,-6,2,-3,7,9,12\} \Rightarrow x[3n] = \{1,-3,12\}$

$x[n] = \{1,2,3,4,5,6,7,8,9,6,7,5,4,3,2,1,1\} \Rightarrow x[4n] = \{1,5,9,4,1\}$

Interpolation:-

$x[n] = \{\underset{\uparrow}{1},2,3,4,5\} \Rightarrow x\left[\frac{n}{2}\right] = \{\underset{\uparrow}{1},0,2,0,3,0,4,0,5\}$

* By interpolation we don't lose any info, it is an expansion/ insertion of information.

Take $x[n] = \{1, 2, 8, 4, 6\}$

(i) $\{1, 2, 8, 4, 6\} \xrightarrow[\text{i.e } n \to 2n]{\text{Decimated by 2}} \{1, 8, 6\} \xrightarrow[n \to \frac{n}{2}]{\text{Interpolated by 2}} \{1, 0, 8, 0, 6\}$
$\uparrow$ (at 1 in each sequence)

(ii) $\{1, 2, 8, 4, 6\} \xrightarrow[n \to \frac{n}{2}]{\text{Interpolation}} \{1, 0, 2, 0, 8, 0, 4, 0, 6\}$

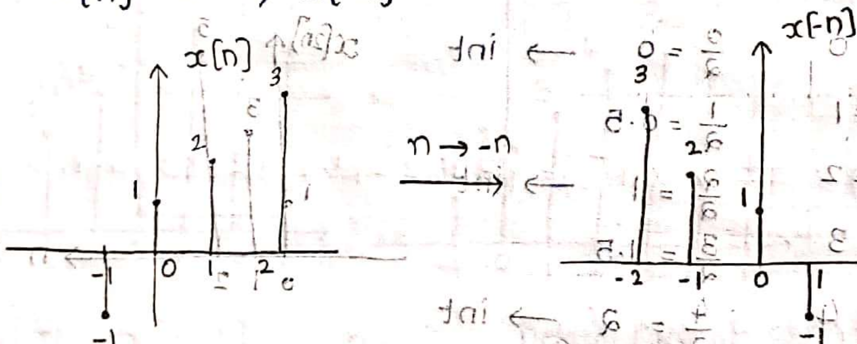
$\downarrow$ Decimated by 2

$\{1, 2, 8, 4, 6\}$

* Decimation is indeed the inverse of interpolation but the converse is not necessarily true.

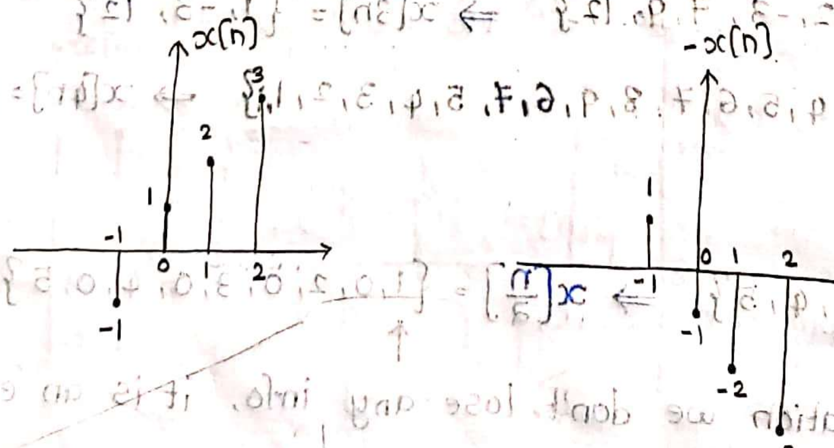
③ <u>Time</u> <u>Reversal</u>:- w.r.t imaginary axis, the given sequence is flipped off.

$x[n] \longrightarrow x[-n]$



④ Amplitude Reversal:-

$x[n] \longrightarrow -x[n]$ wrt real axis, sequence is flipped off.



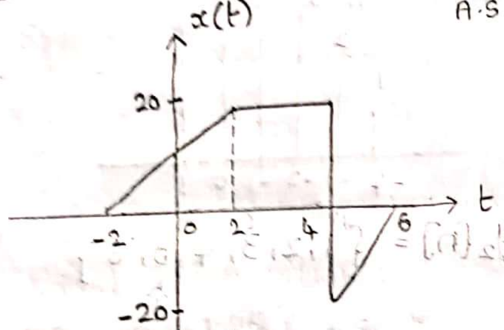
Transformation on independent variable (time 'n') :-

Method - 1 : Shifting, Scaling, Reversal

$\boxed{\text{Traverse from right to left}}$

Ex:- C.T signal : $y(t) = 5x(-2t+8)$

5 → A.S, $-$ → T.R, 2 → T.Scaling, $+8$ → T.shifting



$y(t) = 5x(-2t+8)$

$x(t)$

$\downarrow$ Shifting $\quad t \to t+8$

$x(t+8)$

$\downarrow$ Scaling $\quad t \to 2t$

$x(2t+8)$

$\downarrow$ Reversal $\quad t \to -t$

$x(-2t+8)$

$\downarrow$ Amplitude Scaling

$5x(-2t+8)$

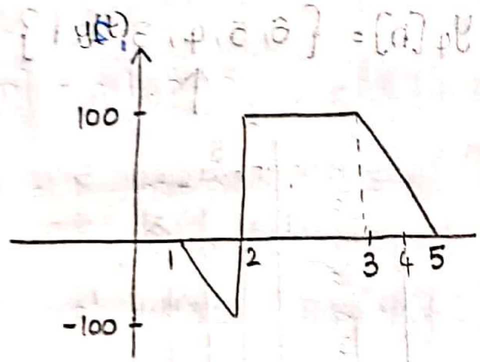
$\frac{-2-8}{-2} = 5$

$\frac{0-8}{-2} = 4$

$\frac{2-8}{-2} = 3$

$\frac{4-8}{-2} = 2$

$\frac{6-8}{-2} = 1$



Method - 2 : Reversal, scaling, shifting

$\boxed{\text{Traverse from left to right}}$

$y(t) = 5x(-2t+8)$
$= 5x(-2(t-4))$

$x(t)$

$\downarrow$ Reversal $\quad t \to -t$

$x(-t)$

$\downarrow$ Scaling $\quad t \to 2t$

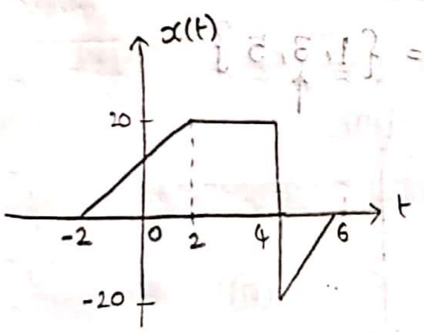
$x(-2t)$

$\downarrow$ shifting $\quad t \to t-4$

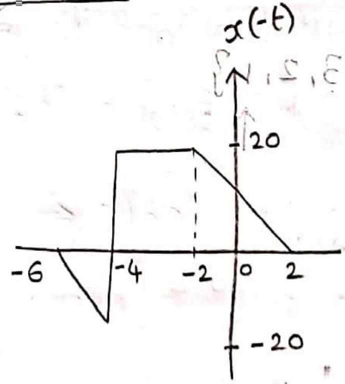
$x(-2(t-4))$

$\downarrow$ A.S

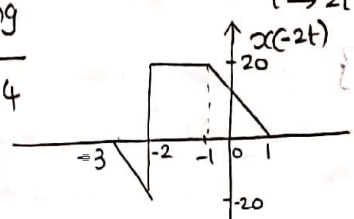
$5x(-2(t-4))$



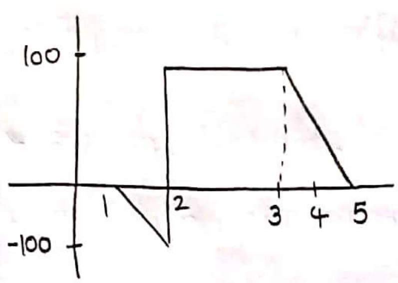
$\xrightarrow[\text{Reversal}]{t \to -t}$



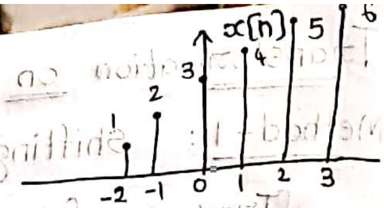
$\downarrow$ Scaling $\quad t \to 2t$



$\xleftarrow[t \to t-4]{\text{shifting}}$

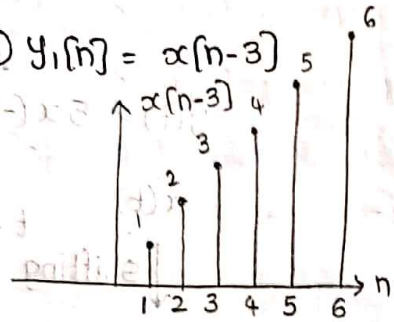


Pb Let $x[n] = \{1, 2, 3, 4, 5, 6\}$ (arrow under 3)



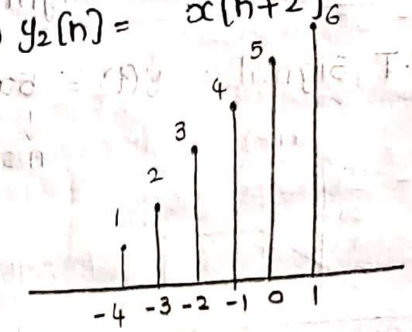
Find and sketch the following signals.

① $y_1[n] = x[n-3]$



$y_1[n] = \{0, 1, 2, 3, 4, 5, 6\}$
$\uparrow$ (under 0)

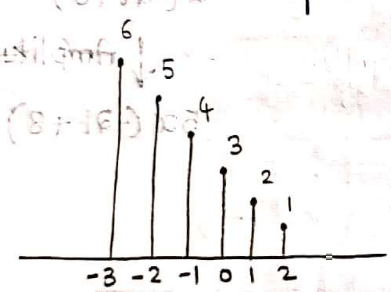
② $y_2[n] = x[n+2]$



$y_2[n] = \{1, 2, 3, 4, 5, 6\}$
$\uparrow$ (under 5)

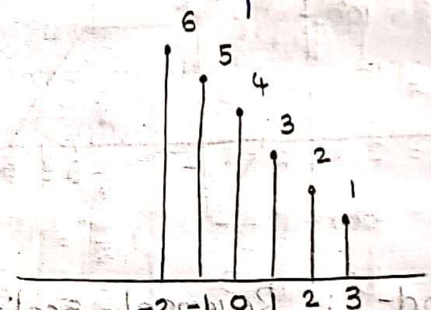
③ $y_3[n] = x[-n]$

$y_3[n] = \{6, 5, 4, 3, 2, 1\}$
$\uparrow$ (under 3)



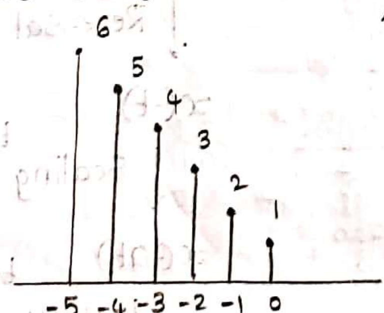
④ $y_4[n] = x[-n+1]$

$y_4[n] = \{6, 5, 4, 3, 2, 1\}$
$\uparrow$ (under 4)



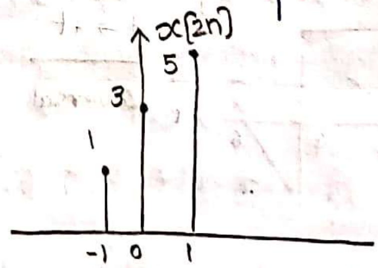
⑤ $y_5[n] = x[-n-2]$

$y_5[n] = \{6, 5, 4, 3, 2, 1\}$
$\uparrow$ (under 1)



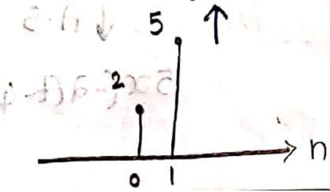
⑥ $y_6[n] = x[2n]$

$y_6[n] = \{1, 3, 5\}$
$\uparrow$ (under 3)



⑦ $y_7[n] = x[3n-1]$

$y_7[n] = \{2, 5\}$
$\uparrow$ (under 5)



$$x[n] \longrightarrow x\left[n-\frac{M}{N}\right]$$

what are the operations we have to perform on $x[n]$ to get $x\left[n-\frac{M}{N}\right]$

$$x[n] \longrightarrow \boxed{\text{Interpolation by 'N'}} \xrightarrow{x\left[\frac{n}{N}\right]} \boxed{\text{Delay by } M} \xrightarrow{x\left[\frac{n-M}{N}\right]} \boxed{\text{Decimate by } N} \xrightarrow{x\left[\frac{Nn-M}{N}\right]}$$

$$= x\left[n-\frac{M}{N}\right]$$

$*$ To retain information, first do interpolation then decimation.

Pb: $x[n] = e^{-n/2}$

(1) $W[n] = 2x\left[\frac{5n}{3}\right]$

$n=0 \Rightarrow W[0] = 2x[0] = 2e^{-0} = 2$

$n=1 \Rightarrow W[1] = 2x\left[\frac{5}{3}\right] = 0$

$\hookrightarrow$ Not an integer

$n=2 \Rightarrow W[2] = 2x\left[\frac{10}{3}\right] = 0$

$n=3 \Rightarrow W[3] = 2x\left[\frac{15}{3}\right] = e^{-5/2}$

$\vdots$

$x[n]=0$ if 'n' is not an integer

$$\therefore W[n] = 2x\left[\frac{5n}{3}\right] = \begin{cases} 2e^{-\frac{5n}{6}} & \text{'n' is multiple of '3'} \\ 0 & \text{else} \end{cases}$$

(2) $y[n] = x[2n]$

$n=0 \Rightarrow y[0] = x[0] = 1$

$n=1 \Rightarrow y[1] = x[2] = e^{-1}$

$n=2 \Rightarrow y[2] = x[4] = e^{-2}$

$n=3 \Rightarrow y[3] = x[6] = e^{-3}$

$\vdots$

$$y[n] = x[2n] = e^{-n}$$

(3) $y[n] = x[n^2]$

$n=0 \Rightarrow y[0] = x[0] = 1$

$n=1 \Rightarrow y[1] = x[1] = e^{-1/2}$

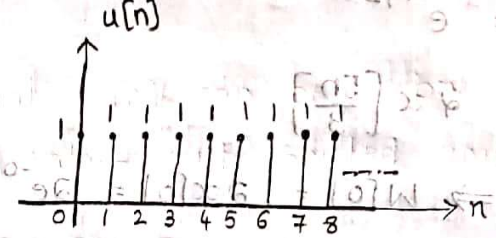
$n=2 \Rightarrow y[2] = x[4] = e^{-2}$

$\vdots$

$y[n] = x[n^2] = e^{-n^2/2}$

Basic discrete time signals / sequence:-

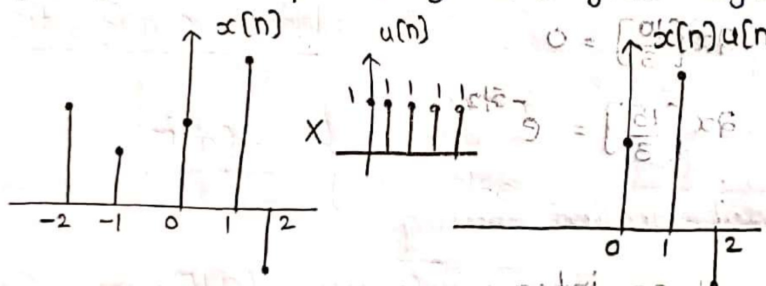
① **Unit step signal:**



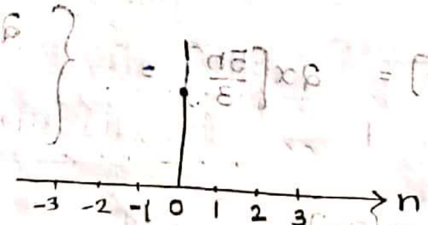
$$u[n] = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$$

$$u[n] = \{\underset{\uparrow}{1}, 1, 1, 1, \dots\}$$

$\Rightarrow$ Any signal multiplied by $u[n]$ gives right-sided signal.



② **Unit impulse:**



$$\delta[n] = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$$

$$\delta[n] = \{\dots, 0, 0, \underset{\uparrow}{1}, 0, 0, \dots\}$$

Main application: To find characterization of discrete system.

Properties of impulse function:

* i. **Time scaling:**

$$\boxed{\delta[kn] = \delta[n]}, \quad k \text{ is an integer}$$

Proof: $\delta[n] = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$

$$n \rightarrow kn \qquad \delta[kn] = \begin{cases} 1 & kn=0 \Rightarrow n=0 \\ 0 & kn \neq 0 \Rightarrow n \neq 0 \end{cases}$$

* ii) unit impulse function $\delta[n]$ & unit step function $u[n]$ are related as

$$\boxed{\delta[n] = u[n] - u[n-1]}$$

iii) Proof: $u[n] = \begin{cases} 1 & n \geq 0 \\ 0 & n<0 \end{cases}$

$$u[n] = \{\underset{\uparrow}{1},1,1,1,\dots\} \qquad u[n-1] = \{\underset{\uparrow}{0},1,1,1,1,\dots\}$$

$$u(n) = \delta[n] + \delta[n-1] + \delta[n-2] + \cdots$$

$$u[n] = \sum_{k=0}^{\infty} \delta[n-k] \quad , \quad u[n-1] = \sum_{k=1}^{\infty} \delta[n-k]$$

$$u[n] - u[n-1] = \sum_{k=0}^{\infty} \delta[n-k] - \sum_{k=1}^{\infty} \delta[n-k]$$

$$= \delta[n] + \cancel{\sum_{k=1}^{\infty} \delta[n-k]} - \cancel{\sum_{k=1}^{\infty} \delta[n-k]}$$

$$= \delta[n]$$

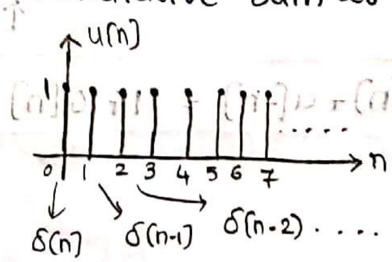
$$\boxed{u[n] - u[n-1] = \delta[n]}$$

* iii) Multiplication property:

$$\boxed{x[n]\,\delta[n-k] = x[k]\,\delta[n-k]}$$

* iv) Unit step signal is expressed as cummulative sum ~~of~~ $\delta[n]$

$$\boxed{u[n] = \sum_{k=0}^{\infty} \delta[n-k]}$$



* V) Sum property:

$$\boxed{\sum_{n=-\infty}^{\infty} \delta[n] = 1}$$

Proof: $\displaystyle\sum_{n=-\infty}^{\infty} \delta[n] = \sum_{n=-\infty}^{-1} \delta[n] + \delta[0] + \sum_{n=1}^{\infty} \delta[n]$

$= 0 + 1 + 0$

$= 1$

* vi) Sifting property :

$$\boxed{\sum_{n=-\infty}^{\infty} x[n]\,\delta[n-k] = x[k]}$$

<u>Proof</u>:

$$\sum_{n=-\infty}^{\infty} x[n]\,\delta[n-k] = \sum_{n=-\infty}^{\infty} x[k]\,\delta[n-k] \quad \text{multiplication property}$$

$$= x[k] \sum_{n=-\infty}^{\infty} \delta[n-k] \quad \text{Sum property } (\because =1)$$

$$= x[k].$$

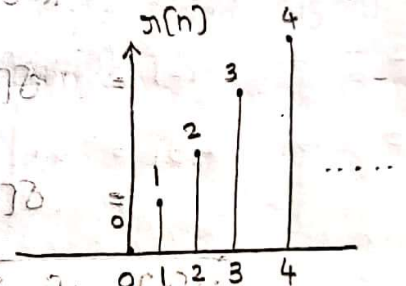
* vii) Signal decomposition:

Any arbitrary signal $x[n]$ can be represented as a summation of the signal values with shifted unit impulses.

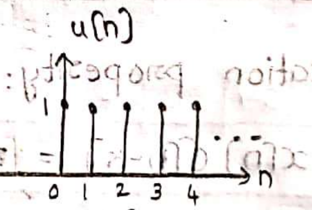
$$\boxed{x[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k]}$$

③ <u>Unit Ramp Signal</u> :

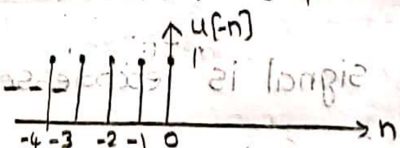
$$r[n] = \begin{cases} n & n \geq 0 \\ 0 & \text{else} \end{cases}$$



$$\boxed{r[n] = n\,u[n]}$$

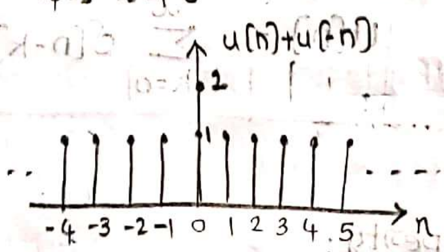


$u[n] = \{\underset{\uparrow}{1}, 1, 1, 1, \ldots\}$

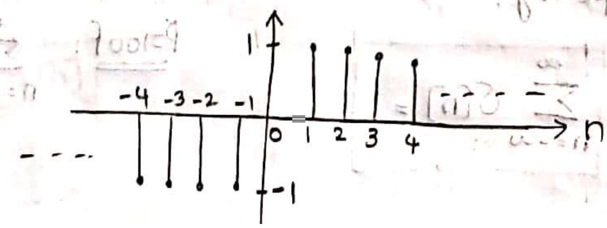


$u[-n] = \{\ldots, 1, 1, 1, 1, \underset{\uparrow}{1}\}$

$$\boxed{u[n] + u[-n] = 1 + \delta[n]}$$



$u[n] - u[-n] =$



$u[n] = \{\underset{\uparrow}{1}, 1, 1, \ldots\ldots\} \Rightarrow n \rightarrow 0$ to ∞

$u[-n] = \{\ldots\ldots, 1, 1, \underset{\uparrow}{1}\} \Rightarrow n \rightarrow -\infty$ to 0

$u[n-1] = \{\underset{\uparrow}{0}, 1, 1, \ldots\ldots\} \Rightarrow n \rightarrow 1$ to ∞

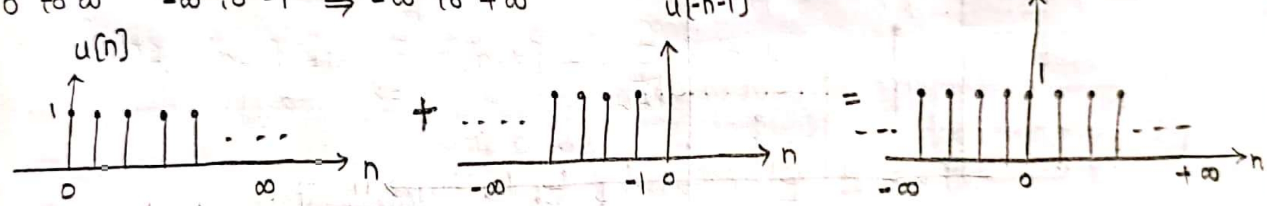
$u[-n-1] = \{\ldots\ldots, 1, 1, 1, \underset{\uparrow}{0}\} \Rightarrow n \rightarrow -\infty$ to -1

$u[-n+n_0] \Rightarrow n \rightarrow -\infty$ to n_0

$u[-n-n_0] \Rightarrow n \rightarrow -\infty$ to $-n_0$

$u[n] + u[-n-1] = 1$

0 to ∞ $-\infty$ to -1 $\Rightarrow -\infty$ to $+\infty$



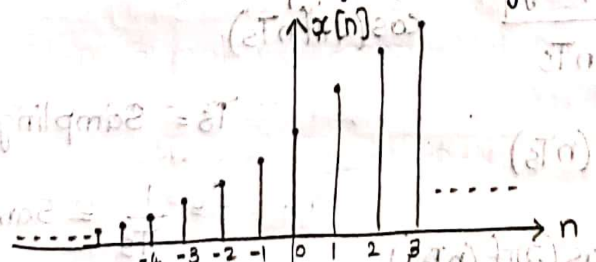
④ Discrete time real exponential signal:

$$x[n] = C \cdot r^n \quad ; \quad -\infty < n < \infty$$

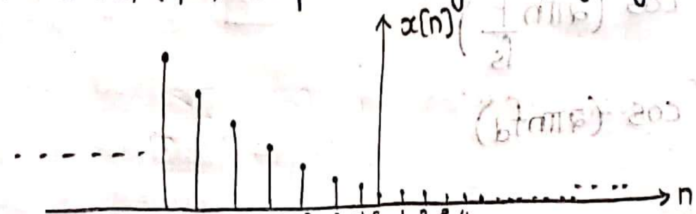
$$C \;\&\; r \in \mathbb{R}$$

Depending on 'r' values, signals are classified as

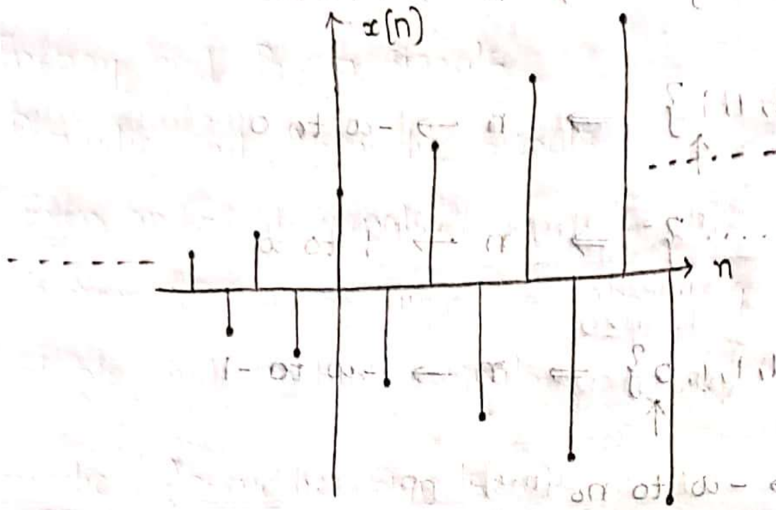
(i) if $r > 1$, exponentially growing signal



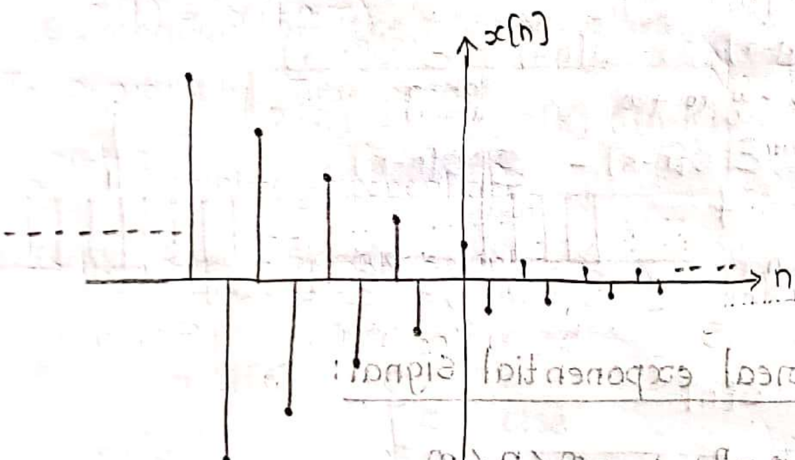
(ii) if $0 < r < 1$, exponentially decaying signal



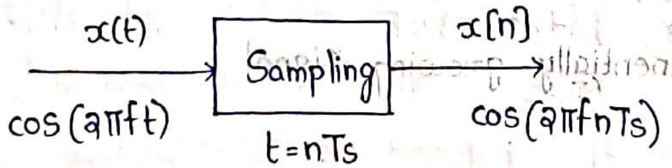
(iii) if $r < -1$, exponentially growing signal with alternate signs.



(iv) $-1 < r < 0 \Rightarrow$ exponentially decaying signal with alternate signs.



⑤ <u>Discrete time sinusoidal signals:</u>



Ts = Sampling interval

$f = \frac{1}{Ts}$ = Sampling frequency

$$x(t)\Big|_{t=nTs} = x(nTs)$$

$$= \cos(2\pi f(nTs))$$

$$= \cos\left(2\pi n \frac{f}{fs}\right)$$

$$= \cos(2\pi n f_d)$$

$\omega = 2\pi f_d$

$\Omega = 2\pi f$

$f_d = \dfrac{f}{f_s}$

<u>Analog</u>	<u>Digital</u>
f (frequency)	f_d (Digital frequency)
Ω (Angular frequency)	ω (Angular digital frequency)

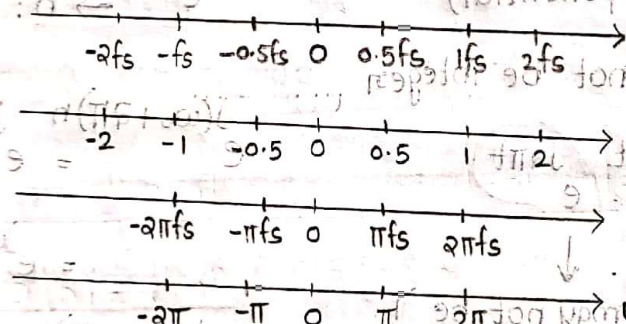
$$f_d \left(\frac{\text{cycles}}{\text{samples}}\right) = \frac{f}{f_s} \; \frac{(\text{cycles/sec})}{(\text{Samples/sec})}$$

$$\omega \left(\text{radians/sample}\right) = \frac{\Omega}{f_s} \; \frac{(\text{radians/sec})}{(\text{samples/sec})} = 2\pi f_d$$

$x(t) = A\cos(2\pi f t)$ → [Sampling] → $x[n] = A\cos(\omega_0 n)$

$= A\cos(\omega t)$ $\quad T_s, f_s$ $\quad$ Discrete

Continuous



Analog frequency (fs)

$f_d = \dfrac{f}{f_s}$ Digital frequency(f_d)

$\Omega = 2\pi f$ (Analog angular freq)

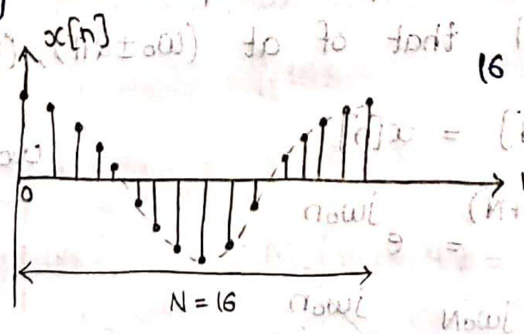
$\omega = 2\pi f_d$ (Digital angular freq)

(Pb) $x[n] = 2\cos(0.125\pi n)$

$$= 2\cos\left(\frac{2\pi n}{16}\right) = A\cos(2\pi n f_d) \quad \therefore f_d = \frac{1}{16} \frac{\text{cycles}}{\text{Samples}}$$

$\Rightarrow \boxed{N = 16}$

16 samples per one cycle.



⑥ <u>Complex exponential Signal:</u>

Classification of Discrete time Signals:-

1. Energy and Power signals
2. Periodic and Aperiodic signals
3. Even and Odd signals
4. Summable, Square summable, absolutely summable signals.
5. Stable and unstable signals.

1. Periodic / Aperiodic signals:-

CTS

$x(t) = x(t \pm T)$

T = periodicity / Time period

$e^{j\omega_0 t}$ (complex exponential)

↳ t: may / may not be integer

$e^{j(\omega_0+2\pi)t} = e^{j\omega_0 t} \cdot e^{j2\pi t}$

$e^{j2\pi t}$ ↓ may / may not be '1'.

DTS

$x[n] = x[n \pm N]$

N = Time period of D.T signal

$e^{j\omega_0 n}$ → n: Always an integer

$e^{j(\omega_0+2\pi)n} = e^{j\omega_0 n} \cdot e^{j2\pi n}$ ↳ Always = 1

$= e^{j\omega_0 n}$

$$\boxed{e^{j(\omega_0+2\pi)n} = e^{j\omega_0 n}}$$

* Discrete time complex exponential $(e^{j\omega_0 n})$ at frequency ω_0 are identical that of at $(\omega_0 \pm 2\pi), (\omega_0 \pm 4\pi), \ldots$

$x[n+N] = x[n]$

$e^{j\omega_0(n+N)} = e^{j\omega_0 n}$

$\cancel{e^{j\omega_0 n}} \cdot e^{j\omega_0 N} = \cancel{e^{j\omega_0 n}}$

$e^{j\omega_0 N} = 1$

$e^{j\omega_0 N} = e^{j2\pi m}$ → integer

$\omega_0 N = 2\pi m$

$$\boxed{N = \left(\frac{2\pi}{\omega_0}\right) m}$$

N → No. of samples / Time period of DTS

m → Smallest integer which makes 'N' as an integer

(1) $x[n] = 5\sin\left(\frac{3\pi}{5}\right)n$ $\quad A = 5,\ \omega_0 = \frac{3\pi}{5}$

$$N = \left(\frac{2\pi}{\omega_0}\right)m = \frac{2\pi}{3\pi/5}m = \frac{10}{3}m$$

$\boxed{N = 10}$ for $m = 3$

(2) $x[n] = e^{j5\pi n}$

$$N = \frac{2\cancel{\pi}}{5\cancel{\pi}}m = \frac{2}{5}m$$

$m = 5 \Rightarrow \boxed{N = 2}$

(3) $x[n] = 3\sin(5n)$

$\omega_0 = 5$

$$N = \frac{2\pi}{5}m$$

$m = 5 \Rightarrow N = 2\pi \rightarrow$ Not an integer

Aperiodic DTS.

(4)
$$x[n] = \overset{P}{x_1[n]} + \overset{P}{x_2[n]} - \overset{P}{x_3[n]} + \overset{P}{x_4[n]}$$

$x_1[n] \downarrow \omega_{01} \downarrow N_1$, $\quad x_2[n] \downarrow \omega_{02} \downarrow N_2$, $\quad x_3[n] \downarrow \omega_{03} \downarrow N_3$, $\quad x_4[n] \downarrow \omega_{04} \downarrow N_4$

If all signals are periodic, then $x[n]$ is periodic

$\boxed{\text{Time period } N = \text{LCM}(N_1, N_2, N_3, N_4)}$

(5) $x[n] = 4\sin\left(\frac{3\pi}{5}\right)n + 2\cos\left(\frac{5\pi}{4}\right)n - 3\sin\left(\frac{7\pi}{2}\right)n$

$\omega_{01} = \frac{3\pi}{5}$ $\quad \omega_{02} = \frac{5\pi}{4}$ $\quad \omega_{03} = \frac{7\pi}{2}$

$$N_1 = \frac{2\pi}{3\pi/5}m_1 = \frac{10}{3}m_1 \qquad N_2 = \frac{2\cancel{\pi}}{5\cancel{\pi}/4}m_2 = \frac{8}{5}m_2 \qquad N_3 = \frac{2\cancel{\pi}}{7\cancel{\pi}/2}m_3 = \frac{4}{7}m_3$$

$m_1 = 3 \Rightarrow N_1 = 10$

$m_2 = 5 \Rightarrow N_2 = 8$

$m_3 = 7 \Rightarrow N_3 = 4$

$N = \text{LCM}(N_1, N_2, N_3) = \text{LCM}(10, 8, 4)$

$$\begin{array}{r|l} 2 & 10, 8, 4 \\ \hline 2 & 5, 4, 2 \\ \hline & 5, 2, 1 \end{array}$$

$\boxed{N = 40}$

(or)

$$\omega_0 = \text{GCD}(\omega_{01}, \omega_{02}, \omega_{03}) = \text{GCD}\left(\frac{3\pi}{5}, \frac{5\pi}{4}, \frac{7\pi}{2}\right) = \frac{\text{GCD}(3\pi, 5\pi, 7\pi)}{\text{LCM}(5, 4, 2)}$$

$$= \frac{\pi}{20} \Rightarrow N = \frac{2\cancel{\pi}}{\cancel{\pi}/20}m = 40m$$

$m = 1 \Rightarrow \boxed{N = 40}$

(6) $x[n] = \cos^2 \frac{\pi}{4} n$

$x[n] = \dfrac{1 + \cos \frac{2\pi}{4}}{2} = \frac{1}{2} + \frac{1}{2} \cos \frac{\pi}{2} n$

$\omega_{01} = 2\pi, \quad \omega_{02} = \pi/2$

$N_1 = \frac{2\pi}{2\pi} m \quad N_2 = \frac{2\cancel{\pi}}{\cancel{\pi}/2} m$

$N_1 = 1 \quad N_2 = 4 \Rightarrow N = LCM(1,4)$

$\boxed{N = 4}$

Problems:-

(1) $x[n] = \cos(0.01\pi n)$

$N = \frac{2\cancel{\pi}}{0.01\cancel{\pi}} m = 200m$

$\boxed{N = 200}$

(2) $x[n] = \cos\left(\pi \frac{30}{105} n\right)$

$N = \frac{2\cancel{\pi}}{\frac{30}{105}\cancel{\pi}} m = \frac{\cancel{2} \times 105}{\cancel{30}\,15} m$ 7

$\boxed{N = 7}$

(3) $x(n) = \cos 3\pi n$

$N = \frac{2\cancel{\pi}}{3\cancel{\pi}} m = \frac{2}{3} m$

$\boxed{N = 2}$

(4) $x(n) = \sin 3n$

$N = \frac{2\pi}{3} \rightarrow$ Not an integer

Aperiodic DTS

(5) $x[n] = \sin\left(\frac{62\pi}{10} n\right)$

$N = \frac{2\cancel{\pi}}{62\cancel{\pi}} \times 10\, m = \frac{10}{31} m$

$\boxed{N = 10}$

(6) $x[n] = \cos(0.5n + 30^\circ)$

$N = \frac{2\pi}{0.5} \rightarrow$ Not an integer

Aperiodic DTS.

(7) $x[n] = \sin^2\left(\frac{\pi}{4} n\right)$

$x[n] = \dfrac{1 - \cos \frac{\pi}{2} n}{2}$

$= \frac{1}{2} - \frac{1}{2} \cos \frac{\pi}{2} n$

$N_1 = 1 \quad N_2 = \frac{2\cancel{\pi}}{\cancel{\pi}/2} m = 4$

$N = LCM(1,4) \Rightarrow \boxed{N = 4}$

(8) $x[n] = \cos \frac{\pi}{2} n + \sin \frac{\pi}{8} n$

$N_1 = \frac{2\cancel{\pi}}{\cancel{\pi}} (2) m = 4$

$N_2 = \frac{2\cancel{\pi}}{\cancel{\pi}} (8) m = 16$

$N = LCM(4,16) \Rightarrow \boxed{N = 16}$

(9) $x[n] = 2\cos\frac{\pi}{2}n \sin\frac{\pi}{8}n$

$= 2\cos A \sin B$

$= \sin\left(\frac{\pi}{2}+\frac{\pi}{8}\right)n - \sin\left(\frac{\pi}{2}-\frac{\pi}{8}\right)n$

$= \sin\left(\frac{5\pi}{8}\right)n - \sin\left(\frac{3\pi}{8}\right)n$

$\sin(A+B) - \sin(A-B) = 2\cos A \sin B$

$\sin(A+B) + \sin(A-B) = 2\sin A \cos B$

$\cos(A+B) + \cos(A-B) = 2\cos A \cos B$

$\cos(A-B) - \cos(A+B) = 2\sin A \sin B$

$\omega_{01} = \frac{5\pi}{8}$ $\quad \omega_{02} = \frac{3\pi}{8}$

$N_1 = \frac{2\cancel{\pi}}{5\cancel{\pi}}(8)m$ $\quad N_2 = \frac{2\cancel{\pi}}{3\cancel{\pi}}(8)m$

$N_1 = \frac{16}{5}m$ $\quad N_2 = \frac{16}{3}m \Rightarrow N = \text{LCM}\left(\frac{16}{5}, \frac{16}{3}\right)$

$N_1 = 16$ $\quad N_2 = 16$ $\quad \boxed{N = 16}$

2. Even and Odd Signals:-

CTS	DTS	
$x(t) = x(-t)$	$x[n] = x[-n]$	$\longrightarrow$ Even
$x(t) = -x(-t)$	$x[n] = -x[-n]$	$\longrightarrow$ Odd

Note:- An odd signal must be '0' at $n=0$.

Even part of $x[n]$: $E\{x[n]\} = \frac{x[n] + x[-n]}{2}$

Odd part of $x[n]$: $O\{x[n]\} = \frac{x[n] - x[-n]}{2}$

Properties of discrete time even & Odd signals:-

1) If $x_1[n]$ is an odd signal $[x_1[n] = -x_1[-n]]$ and $x_2[n]$ is an even signal $[x_2[n] = x_2[-n]]$

ⓐ $x_3[n] = \overset{O}{x_1[n]}\,\overset{E}{x_2[n]} \Rightarrow$ Odd signal

ⓑ $x_4[n] = \overset{O}{x_1[n]} \cdot \overset{O}{x_2[n]} \Rightarrow$ Even signal

ⓒ $x_5[n] = \overset{E}{x_1[n]} \cdot \overset{E}{x_2[n]} \Rightarrow$ Even signal

2) If $x[n]$ is an even signal, then

$$\sum_{n=-\infty}^{\infty} x[n] = x[0] + 2\sum_{n=1}^{\infty} x[n]$$

3) If $x[n]$ is an odd signal, then

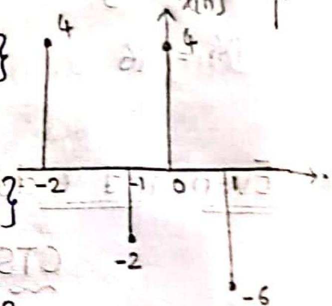
$$\sum_{n=-\infty}^{\infty} x[n] = 0$$

4) If $x[n]$ is an arbitrary signal with even & odd parts denoted by $x_e[n] = E\{x[n]\}$

$x_o[n] = O\{x[n]\}$, then

$$\sum_{n=-\infty}^{\infty} x^2[n] = \sum_{n=-\infty}^{\infty} x_e^2[n] + \sum_{n=-\infty}^{\infty} x_o^2[n]$$

Pb: Determine the even & odd parts of the $x[n] = \{4, -2, 4, -6\}$



$$x(n) = \{4, -2, \underset{\uparrow}{4}, -6\} = \{4, -2, \underset{\uparrow}{4}, -6, 0\}$$

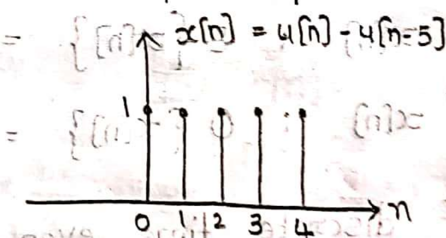
$$x[-n] = \{-6, \underset{\uparrow}{4}, -2, 4\} = \{0, -6, \underset{\uparrow}{4}, -2, 4\}$$

$$x_e(n) = \frac{x(n) + x[-n]}{2} = \{2, -4, 4, -4, 2\}$$

$$x_o(n) = \frac{x(n) - x(-n)}{2} = \{2, 2, 0, -2, -2\}$$

Pb: Find out and sketch the even & odd parts of $x[n] = u[n] - u[n-5]$

$$x[n] = u[n] - u[n-5]$$



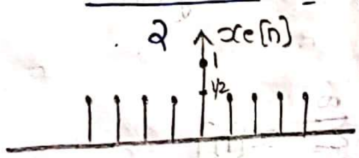
$$x[n] = \{\underset{\uparrow}{1}, 1, 1, 1, 1\}$$

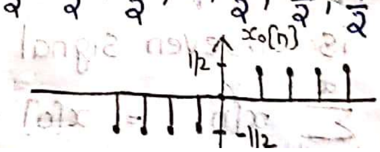
$$= \{0, 0, 0, 0, \underset{\uparrow}{1}, 1, 1, 1, 1\}$$

$$x[-n] = \{1, 1, 1, 1, \underset{\uparrow}{1}\} = \{1, 1, 1, 1, \underset{\uparrow}{1}, 0, 0, 0, 0\}$$

$$x_e[n] = \frac{x[n] + x[-n]}{2} = \{\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, 1, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}\}$$

$$x_o[n] = \frac{x[n] - x[-n]}{2} = \{\tfrac{-1}{2}, \tfrac{-1}{2}, \tfrac{-1}{2}, \tfrac{-1}{2}, 0, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}\}$$





3. Energy & Power Signals:-

Energy of a discrete time signal $\rightarrow$ $E_x = \sum_{n=-\infty}^{\infty} |x[n]|^2$

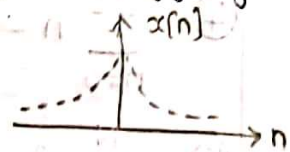
Power of a discrete time signal $\Rightarrow$ $P_x = \underset{N\to\infty}{Lt} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2$

$$\text{Power} = \frac{\text{Energy}}{\text{time}}$$

* Power is the time avg of energy.

Energy signal : if $0 < E_x < \infty$ (E_x is finite) and $P_x = 0$ $\Big\}$ $x[n]$ = Energy s/g

$\Rightarrow$ Signal amplitude tends to zero as $|n| \rightarrow \infty$



Power signal : if $E_x = \infty$ and $0 < P_x < \infty$ (P_x is finite) $\Big\}$ $x[n]$ = Power s/g.

→ There are signals that are neither energy nor power signals.
→ A signal can't be both energy & power signals.

Note :-

① Power is the time average of energy ; the averaging is over an infinitely large interval, a signal with finite energy has zero power and a signal with finite power has infinite energy.
∴ A signal can't be both energy & power s/g.

② All practical signals have finite energy $\Rightarrow$ Energy s/g

③ It is impossible to generate a true power signal in practice.
$P = \frac{E}{T}$, such a signal has infinite duration & infinite energy.

④ All finite periodic signals are power signals but not all power signals are periodic

Ex:- unit step signal → Power s/g
↳ but not periodic

Important results:-

Ex:- $1 + a + a^2 + a^3 + \cdots = \frac{1}{1-a}$

$a + ar + ar^2 + \cdots = \frac{a}{1-r}$

① $\sum_{n=0}^{\infty} a^n = \frac{1}{1-a}$; $|a| < 1$

② $\sum_{n=0}^{N} a^n = \frac{1 - a^{N+1}}{1-a}$

③ $\sum_{n=0}^{N} (1)^n = N+1$ Ex:- $\sum_{n=0}^{2} (1)^n = 1+1+1 = 3 \Rightarrow N+1$
$2+1=3$

④ $\sum_{n=0}^{N} n = \frac{N(N+1)}{2}$

⑤ $\sum_{n=0}^{N} n^2 = \frac{N(N+1)(2N+1)}{6}$

⑥ $\sum_{n=0}^{N} n^3 = \left(\frac{N(N+1)}{2}\right)^2$

⑦ $\sum_{n=a}^{b} 1 = (b-a)+1$ Ex:- $\sum_{n=-2}^{2} 1 = 1+1+1+1+1 = 5$
$= (2-(-2))+1 = 5$

Q) Determine the values of power (P_x) and Energy (E_x) for the following signals:

(1) $x_1[n] = \left(\frac{1}{2}\right)^n u[n]$

$|x_1(n)|^2 = \left(\frac{1}{2}\right)^{2n} u^2[n] = \left(\frac{1}{4}\right)^n u[n]$

Energy : $E_x = \sum_{n=-\infty}^{\infty} |x_1[n]|^2$

$= \sum_{n=-\infty}^{\infty} \left(\frac{1}{4}\right)^n u[n]$

$\left[\because u[n] = \begin{cases} 1, & n \ge 0 \\ 0, & n < 0 \end{cases} \quad \sum_{n=0}^{\infty} a^n = \frac{1}{1-a}\right]$

$= \sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^n$

$= \frac{1}{1-1/4} = \underline{4/3}$ (finite)

Power : $P_x = \underset{N\to\infty}{Lt} \frac{1}{2N+1} \sum_{n=-N}^{N} |x_1[n]|^2$

$= \underset{N\to\infty}{Lt} \frac{1}{2N+1} \sum_{n=-N}^{N} \left(\frac{1}{4}\right)^n u[n]$

$= \underset{N\to\infty}{Lt} \frac{1}{2N+1} \sum_{n=0}^{N} \left(\frac{1}{4}\right)^n$

$= \underset{N\to\infty}{Lt} \frac{1}{2N+1} \times \left[\frac{1-\left(\frac{1}{4}\right)^{N+1}}{1-\left(\frac{1}{4}\right)}\right]$

$P_x = 0\ W$

$E_x = \frac{4}{3} J$ (finite)
$P_x = 0\ W$
$\Bigg\}$ $\therefore x_1[n]$ is a energy signal.

(2) $x_2[n] = e^{j\left(\frac{\pi}{2}n + \frac{\pi}{8}\right)}$

$|x_2[n]|^2 = 1$

Energy : $E_{x_2} = \sum_{n=-\infty}^{\infty} |x_2[n]|^2$

$= \sum_{n=-\infty}^{\infty} (1)^2 = \infty$

Power : $P_{x2} = \underset{N\to\infty}{Lt} \frac{1}{2N+1} \sum_{n=-N}^{N}$

$= \underset{N\to\infty}{Lt} \frac{1}{2N+1} (N-(-N)+1)$

$= \underset{N\to\infty}{Lt} \frac{1}{2N+1} (2N+1) = 1$

$\therefore$ Power Signal with $\underline{P_x = 1 W}$

(3) $x_3[n] = \cos\left(\frac{\pi}{4}n\right)$

$|x_3[n]|^2 = \cos^2 \frac{\pi}{4}n = \frac{1+\cos 2\left(\frac{\pi}{4}n\right)}{2} = \frac{1}{2} + \frac{1}{2}\cos\left(\frac{\pi}{2}n\right)$

$$E_x = \sum_{n=-\infty}^{\infty} |x_3(n)|^2$$

$$= \sum_{n=-\infty}^{\infty} \left(\frac{1}{2} + \frac{1}{2}\cos\frac{\pi}{2}n\right)$$

$$= \sum_{n=-\infty}^{\infty} \frac{1}{2} + \sum_{n=-\infty}^{\infty} \frac{1}{2}\cos\frac{\pi}{2}n = \infty + \infty = \infty$$

$$P_x = \underset{N\to\infty}{Lt} \frac{1}{2N+1} \sum_{n=-N}^{N} |x_3(n)|^2$$

$$= \underset{N\to\infty}{Lt} \frac{1}{2N+1} \sum_{n=-N}^{N} \left(\frac{1}{2} + \frac{1}{2}\cos\frac{\pi}{2}n\right)$$

$$= \underset{N\to\infty}{Lt} \frac{1}{2N+1}\left[\frac{1}{2}(2N+1) + \frac{1}{2}(0)\right]$$

$$= \underset{N\to\infty}{Lt} \frac{1}{2}\left[\frac{2\cancel{N}+1}{2\cancel{N}+1}\right] = \frac{1}{2}$$

$\therefore$ $x_3(n)$ is a power signal with $\underline{\underline{P_x = \frac{1}{2} W}}$

(4) $x_4(n) = A$

$$|x_4(n)|^2 = A^2$$

$$E_x = \sum_{n=-\infty}^{\infty} |x_4(n)|^2 = \sum_{n=-\infty}^{\infty} A^2$$

$$= A^2(\infty) = \infty$$

$$P_x = \underset{N\to\infty}{Lt} \frac{1}{2N+1} \sum_{n=-N}^{N} |x_4(n)|^2$$

$$= \underset{N\to\infty}{Lt} \frac{1}{2N+1} \sum_{n=-N}^{N} A^2$$

$$= A^2 \underset{N\to\infty}{Lt} \frac{1}{2N+1} \sum_{n=-N}^{N} (1)$$

$$= A^2 \underset{N\to\infty}{Lt} \frac{1}{2\cancel{N}+1} (2\cancel{N}+1)$$

$$= A^2$$

$\therefore$ $x_4(n)$ is a power signal with power $\underline{A^2}$ W

(5) $x_5[n] = u[n]$

$$|x_5[n]|^2 = u^2[n] = u[n]$$

$$E_x = \sum_{n=-\infty}^{\infty} |x_5(n)|^2$$

$$= \sum_{n=-\infty}^{\infty} u[n] = \sum_{n=0}^{\infty} 1 = \infty$$

$$P_x = \underset{N\to\infty}{Lt} \frac{1}{2N+1} \sum_{n=-N}^{N} |x_5(n)|^2$$

$$= \underset{N\to\infty}{Lt} \frac{1}{2N+1} \sum_{n=-N}^{N} u[n]$$

$$= \underset{N\to\infty}{Lt} \frac{1}{2N+1} \sum_{n=0}^{N} 1$$

$$= \underset{N\to\infty}{Lt} \frac{1}{2N+1} (N+1) = \underset{N\to\infty}{Lt} \frac{N+1}{2N+1} \quad (\because \text{L-Hopital's rule})$$

$$= \underset{N\to\infty}{Lt} \frac{1}{2} = \frac{1}{2}$$

$\therefore$ Power signal with $P_x = \frac{1}{2} W$

(6) $x_6[n] = nu[n] = r[n]$

$$|x_6[n]|^2 = n^2u^2[n] = n^2u[n]$$

$$E_x = \sum_{n=-\infty}^{\infty} n^2u[n] = \sum_{n=0}^{\infty} n^2 = \infty$$

$$P_x = \underset{N\to\infty}{Lt} \frac{1}{2N+1} \sum_{n=-N}^{N} n^2u(n)$$

$$= \underset{N\to\infty}{Lt} \frac{1}{2N+1} \sum_{n=0}^{N} n^2$$

$$= \underset{N\to\infty}{Lt} \frac{1}{\cancel{2N+1}} \cdot \frac{N(N+1)(\cancel{2N+1})}{6} = \infty$$

$\therefore$ Neither Energy nor Power Signal. $\left.\begin{array}{l} E_x = \infty \\ P_x = \infty \end{array}\right\}$ NENP

$$\boxed{x[n] = nu[n] = r[n] \rightarrow \text{NENP}}$$

(7) $x_7[n] = \delta[n]$

$$|x_7[n]|^2 = \delta^2[n] = \delta[n]$$

$$E_x = \sum_{n=-\infty}^{\infty} \delta[n] = 1$$

$$P_x = \underset{N\to\infty}{Lt} \frac{1}{2N+1} \sum_{n=-N}^{N} \delta[n]$$

$$= \underset{N\to\infty}{Lt} \frac{1}{2N+1} (1) = \frac{1}{\infty} = 0$$

$\therefore$ Energy signal with $E_x = 1\,J$

(8) $x[n] = \begin{cases} n, & 0 \le n \le 5 \\ 0 & \text{else} \end{cases}$

$$E_x = \sum_{n=-\infty}^{\infty} n^2 = \sum_{n=0}^{5} n^2 = \frac{5(5+1)(2(5)+1)}{6}$$

$$= \frac{5 \times \cancel{6} \times 11}{\cancel{6}} = 55\,J$$

$$P_x = \underset{N\to\infty}{Lt} \frac{1}{2N+1} \sum_{n=-N}^{N} n^2$$

$$= \underset{N\to\infty}{Lt} \frac{1}{2N+1} \sum_{n=0}^{5} n^2 = \frac{1}{\infty} \times 55 = 0$$

$\therefore$ Energy signal with $E_x = 55\,J$

Note:- All finite length signals / Sequences are Energy Signals.

(9) Consider the discrete time signal $x[n] = \delta[n+2] - \delta[n-2]$
Calculate E_y for $y[n] = \sum_{k=-\infty}^{\infty} x[k]$

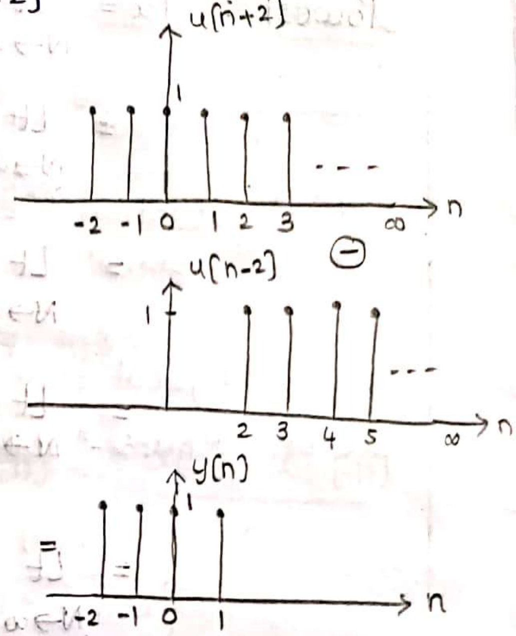
Given, $x[n] = \delta[n+2] - \delta[n-2]$

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]$$

$$x[k] = \delta[k+2] - \delta[k-2]$$

$$y[n] = \sum_{k=-\infty}^{\infty} \delta[k+2] - \delta[k-2]$$

$$y[n] = \sum_{k=-\infty}^{\infty} \delta[k+2] - \sum_{k=-\infty}^{\infty} \delta[k-2]$$



$$y[n] = u[n+2] - u[n-2]$$

-2 to $+2$

$$= \begin{cases} 1 & -2 \le n \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Energy : $E_y = \sum_{n=-\infty}^{\infty} |y[n]|^2$

$$= \sum_{n=-2}^{1} 1 = (1+2)+1$$

$$= \underline{4\,J} \Rightarrow \boxed{\text{Energy} : E_y = 4\,J}$$

Power : $P_y = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} (1)$

$$= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-2}^{1} (1) = \frac{1}{\infty} = \underline{\underline{0}}$$

(10) $x[n] = j^{n/4}$

$$x[n] = e^{\frac{jn\pi}{2}\cdot\frac{1}{4}} = e^{j\frac{\pi}{8}n} \qquad \left[j = e^{\frac{j\pi}{2}} \Rightarrow j^{n/4} = e^{\frac{j\pi}{2}\left(\frac{n}{4}\right)} = e^{j\frac{\pi}{8}n} \right]$$

$$|x[n]|^2 = 1$$

Energy : $E_x = \sum_{n=-\infty}^{\infty} 1 = \infty$

Power : $\lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} (1) = \lim_{N\to\infty} \frac{1}{\cancel{2N+1}} \cancel{(2N+1)} = \underline{\underline{1}}$

$\therefore$ Power signal with $\underline{P_x = 1\,W}$

(11) $x[n] = (\sqrt{j})^n + (\sqrt{j})^{-n}$ $\qquad \left[j = e^{\frac{j\pi}{2}} \right]$

$$x[n] = j^{n/2} + j^{-n/2}$$

$$= e^{jn\pi/4} + e^{-jn\pi/4} = 2\cos\left(\frac{\pi}{4}n\right)$$

Power : $P_x = \underset{N\to\infty}{Lt} \frac{1}{2N+1} \sum_{n=-N}^{N} \left(2\cos\frac{\pi}{4}n\right)^2$

$= \underset{N\to\infty}{Lt} \frac{1}{2N+1} \sum_{n=-N}^{N} \left[4\cos^2\frac{\pi}{4}n\right]$

$= \underset{N\to\infty}{Lt} \frac{1}{2N+1} \sum_{n=-N}^{N} \left[\cancel{4}^{2}\left(\frac{1+\cos 2\left(\frac{\pi}{4}n\right)}{\cancel{2}}\right)\right]$

$= \underset{N\to\infty}{Lt} \frac{1}{2N+1} \sum_{n=-N}^{N} \left(2+2\cos\frac{\pi}{2}n\right)$

$= \underset{N\to\infty}{Lt} \frac{1}{2N+1} \sum_{n=-N}^{N} 2 + \underset{N\to\infty}{Lt} \frac{1}{2N+1} \sum_{n=-N}^{N} 2\cos\frac{\pi}{2}n$

$= \underset{N\to\infty}{Lt} \frac{1}{\cancel{2N+1}} 2(\cancel{2N+1}) + \underset{N\to\infty}{Lt} \frac{1}{2N+1} 2(0)$

$P_x = \underline{\underline{2\ W}}$

Energy : $E_x = \sum_{n=-\infty}^{\infty} \left(2+2\cos\frac{\pi}{2}n\right)$

$= \infty$

$\therefore$ Power signal with $\underline{\underline{P_x = 2\ W}}$

4. **Conjugate Symmetric / Anti-Symmetric Signals :-**

Conjugate Symmetric Signals

$\boxed{x[n] = x^*[-n]}\ \forall n$

⇒ A real conjugate symmetric signal is said to be even s/g.

$x[n] = x[-n]$

Conjugate Anti-Symmetric Signals

$\boxed{x[n] = -x^*[-n]}\ \forall n$

⇒ A real conjugate Anti-Symmetric s/g is said to be odd s/g. $x[n] = -x[-n]$

Complex discrete time signal : $x[n] = x_{cs}[n] + x_{cA}[n]$

$$x_{cs}(n) = \frac{x[n] + x^*[-n]}{2} \qquad x_{cAS}(n) = \frac{x(n) - x^*[-n]}{2}$$

$$x_e[n] = \frac{x[n] + x[-n]}{2} \qquad x_o[n] = \frac{x[n] - x[-n]}{2}$$

Q) $x[n] = \{0, 1+j4, -2+j3, \underset{\uparrow}{2-j4}, -6-j5, 7, -j3\}$

$$x_{cs}[n] = \frac{x(n) + x^*[-n]}{2} \qquad x_{cAS}[n] = \frac{x[n] - x^*[-n]}{2}$$

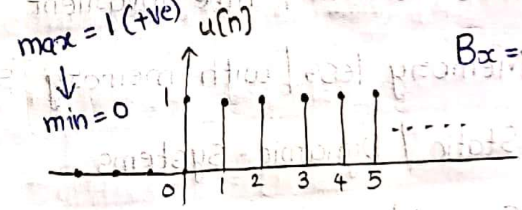
$x[-n] = \{-j3, 7, -6-j5, \underset{\uparrow}{2-j4}, -2+j3, 1+j4, 0\}$

$x^*[-n] = \{j3, 7, -6+j5, \underset{\uparrow}{2+j4}, -2-j3, 1-j4, 0\}$

$x_{cs}[n] = \{j1.5, 4+j2, -4+j4, \underset{\uparrow}{2}, -4-j4, 4-j2, -j1.5\}$

$x_{cAS}[n] = \{-j1.5, -3+j2, 2-j, \underset{\uparrow}{-j4}, -2-j, 3+j2, -j1.5\}$

5. **Bounded Signals:-**



A discrete time signal $x[n]$ is said to be bounded if each of its samples is of magnitude less than or equal to a finite +ve number B_x i.e, $\boxed{|x[n]| < B_x < \infty}$

Ex:- $u[n]$ is bounded slg with $B_x = 1$.

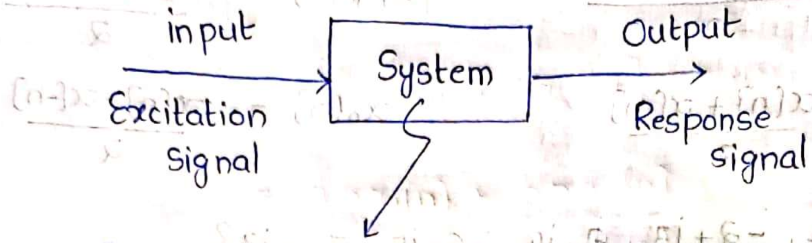
6. **Summable / Square Summable Signals:-**

Absolutely Summable slg's: A DTS is said to be absolutely summable if and only if $\boxed{\sum_{n=-\infty}^{\infty} |x(n)| < \infty}$ → finite value

Square Summable slg:- A DTS is said to be SSS if & only if $\boxed{\sum_{n=-\infty}^{\infty} |x(n)|^2 < \infty}$

* SSS has finite energy, $P=0 \Rightarrow$ Energy slg.

Discrete time systems :-



Physical entity which is characterized by its ability to accept inputs - Voltage, current, pressure, force, displacement.

$$x[n] \longrightarrow \boxed{T\{\cdot\}} \longrightarrow y[n]$$

$T\{\cdot\}$ ⇒ Transformation operation / System operation.

$$y[n] = T\{x[n]\}$$

Classification of systems :- → Based on how the system interacts with the input signals "MODELS"

1. Linear / Non-linear Systems
2. Time varient / Time invarient systems.
3. Memory less / with memory systems.
4. Static / Dynamic systems
5. Causal / Non-causal systems
6. Stable / Unstable systems.

1. Linear / Non-linear Systems :-

Linear System : A system which possesses the important property of superposition.

Superposition principle : The response resulting from several input signals can be computed as the sum of the responses resulting from each input signal acting alone.

$x_1[n]$, $x_2[n]$ $\longrightarrow$ $\boxed{\text{System}}$ $\longrightarrow$ $y_1[n]$, $y_2[n]$

$x_1[n] \longrightarrow y_1[n]$

$x_2[n] \longrightarrow y_2[n]$

$$x_1[n] + x_2[n] \longrightarrow y_1[n] + y_2[n]$$

① The response to $\{x_1[n] + x_2[n]\}$ is $\{y_1[n] + y_2[n]\}$ $\longrightarrow$ Additivity property

② The response to $a \cdot x[n]$ is $a \cdot y[n]$ $\longrightarrow$ Scaling / Homogenity property

a : arbitrary constant

① + ② $\Rightarrow$ Superposition principle

$$\boxed{\{a \cdot x_1[n] + b \cdot x_2[n]\} \longrightarrow \{a \cdot y_1[n] + b\, y_2[n]\}}$$

Superposition $\Rightarrow$ [Additivity + Homogenity] $\Rightarrow$ Linear System

<u>Note:-</u> * Zero input $\longrightarrow$ zero output ✓

Q) For each of the following i/p-o/p relationship, determine whether the following systems are linear / Non-linear.

(a) $y(n) = n\,x[n]$

$x(n) \longrightarrow \boxed{S} \longrightarrow y[n] = n\,x[n]$

$x[n] = \{\underset{\uparrow}{2}, 1, -3, 1, 4\}$

$y(n) = \{\underset{\uparrow}{0}, 1, -6, 3, 16\}$

n	$x(n)$	$y[n] = n\,x[n]$
0	2	$0\,x[0] = 0 \times 2 = 0$
1	1	$1\,x[1] = 1 \times 1 = 1$
2	-3	$2\,x[2] = 2 \times -3 = 6$
3	1	$3\,x[3] = 3 \times 1 = 3$
4	4	$4\,x[4] = 4 \times 4 = 16$

Consider two arbitrary signals,

$x_1[n] \longrightarrow y_1[n] = n\,x_1[n]$

$x_2[n] \longrightarrow y_2[n] = n\,x_2[n]$

$x_3(n) \longrightarrow y_3[n] = n\,x_3[n]$

Let x_3 be linear combination of $x_1[n]$ & $x_2[n]$

$$x_3[n] = a\,x_1[n] + b\,x_2[n]$$

a & b are arbitrary constants

LHS : $y_3[n] = x_3[n] \cdot x_3[n-1]$

$= (a x_1[n] + b x_2[n]) \cdot (a x_1[n-1] + b x_2[n-1])$

$= a^2 x_1[n] x_1[n-1] + ab\, x_1[n] x_2[n-1] + ab\, x_2[n] x_1[n-1]$

$+ b^2 x_2[n] x_2[n-1]$

$\neq$ RHS

$\therefore$ LHS $\neq$ RHS

So, the given system is non-linear system.

Q) For the system described by the following difference equation with the input $x[n]$ and output $y[n]$, determine which of the systems are linear / Non-linear.

Note :- A difference equation is non-linear, if a constant term is present or a term contains the product of the input and/or output.

(a) $y[n] - 2y[n-1] = 4x[n]$

$x_1[n] \longrightarrow y_1[n]$	$y_1[n] - 2y_1[n-1] = 4x_1[n] \rightarrow ①$
$x_2[n] \longrightarrow y_2[n]$	$y_2[n] - 2y_2[n-1] = 4x_2[n] \rightarrow ②$
$x_3[n] \longrightarrow y_3[n]$	$y_3[n] - 2y_3[n-1] = 4x_3[n] \rightarrow ③$

Multiplying ① x a & ② x b and then add ①a + ②b

$a y_1[n] - 2a y_1[n-1] = 4a x_1[n]$

$b y_2[n] - 2b y_2[n-1] = 4b x_2[n]$

$a y_1[n] + b y_2[n] - 2a y_1[n-1] - 2b y_2[n-1] = 4a x_1[n] + 4b x_2[n]$

$= 4[a x_1[n] + b x_2[n]]$

$= 4(x_3[n])$

$\therefore$ Linear System.

$y[n] - 2y[n-1] = 4x[n]$

$x_3[n] = a x_1[n] + b x_2[n]$, $y_3[n] = a y_1[n] + b y_2[n]$

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(b) $y[n] - 3n\,y[n-1] = x[n]$

$x_1[n] \rightarrow y_1[n]$ | $y_1[n] - 3n y_1[n-1] = x_1[n] \rightarrow ①$

$x_2[n] \rightarrow y_2[n]$ | $y_2[n] - 3n y_2[n-1] = x_2[n] \rightarrow ②$

$x_3[n] \rightarrow y_3[n]$ | $y_3[n] - 3n y_3[n-1] = x_3[n] \rightarrow ③$

Let x_3 be linear combination of x_1 & x_2,

$x_3[n] = a x_1[n] + b x_2[n]$, $y_3[n] = a y_1[n] + b y_2[n]$

$① \times a + ② \times b \Rightarrow a y_1[n] - 3an y_1[n-1] + b y_2[n] - 3bn y_2[n-1] = a x_1[n] + b x_2[n]$

$a y_1[n] + b y_2[n] - 3n[a y_1[n-1] + b y_2[n-1]] = x_3[n]$

$y_3[n] - 3n y_3[n-1] = x_3[n]$

Hence proved → ③

∴ Given system is Linear.

(c) $y[n] + 2y^2[n] = x[n]$

$x_1[n] \rightarrow y_1[n]$ | $y_1[n] + 2y_1^2[n] = x_1[n] \rightarrow ①$

$x_2[n] \rightarrow y_2[n]$ | $y_2[n] + 2y_2^2[n] = x_2[n] \rightarrow ②$

$x_3[n] \rightarrow y_3[n]$ | $y_3[n] + 2y_3^2[n] = x_3[n] \rightarrow ③$

Let x_3 be linear combination of x_1 & x_2,

$x_3[n] = a x_1[n] + b x_2[n]$, $y_3[n] = a y_1[n] + b y_2[n]$

$① \times a + ② \times b \rightarrow a y_1[n] + 2a y_1^2[n] + b y_2[n] + 2b y_2^2[n] = a x_1[n] + b x_2[n]$

(d) $y[n] - 2y[n-1] = 2^{x[n]} \cdot x[n]$

$x_1[n] \rightarrow y_1[n]$ | $y_1[n] - 2y_1[n-1] = 2^{x_1[n]} \cdot x_1[n] \rightarrow ①$

$x_2[n] \rightarrow y_2[n]$ | $y_2[n] - 2y_2[n-1] = 2^{x_2[n]} x_2[n] \rightarrow ②$

$x_3[n] \rightarrow y_3[n]$ | $y_3[n] - 2y_3[n-1] = 2^{x_3[n]} x_3[n] \rightarrow ③$

Let $x_3[n] = ax_1[n] + bx_2[n]$, $y_3[n] = ay_1[n] + by_2[n]$

$① \times a + ② \times b \Rightarrow ay_1[n] - 2ay_1[n-1] + by_2[n] - 2by_2[n-1] = a \cdot 2^{x_1[n]} \cdot x_1[n]$
$+ b2^{x_2[n]} \cdot x_2[n]$

$$\underbrace{ay_1[n] + by_2[n]}_{y_3[n]} - 2\big(\underbrace{ay_1[n-1] + by_2[n-1]}_{y_3[n-1]}\big) = a2^{x_1[n]} \cdot x_1[n] + b2^{x_2[n]} \cdot x_2[n]$$

$$y_3[n] - 2y_3[n-1] = a2^{x_1[n]} x_1[n] + bx_2[n]2^{x_2[n]}$$

LHS $\neq$ RHS ($\because$ From -③)

$\therefore$ Given system is non-linear.

(e) $y[n] + 2^n y[n-1] = x[n]$

$x_1[n] \rightarrow y_1[n]$ | $y_1[n] + 2^n y_1[n-1] = x_1[n] \rightarrow ①$

$x_2[n] \rightarrow y_2[n]$ | $y_2[n] + 2^n y_2[n-1] = x_2[n] \rightarrow ②$

$x_3[n] \rightarrow y_3[n]$ | $y_3[n] + 2^n y_3[n-1] = x_3[n] \rightarrow ③$

Let $x_3[n] = ax_1[n] + bx_2[n]$, $y_3[n] = ay_1[n] + by_2[n]$

$① \times a + ② \times b \rightarrow ay_1[n] + a2^n y_1[n-1] + by_2[n] + b2^n y_2[n-1] = ax_1[n]$
$+ bx_2[n]$

$$\underbrace{ay_1[n] + by_2[n]}_{y_3[n]} + \big(\underbrace{ay_1[n-1] + by_2[n-1]}_{y_3[n-1]}\big)2^n = x_3[n]$$

$$y_3[n] + 2^n y_3[n-1] = x_3[n]$$

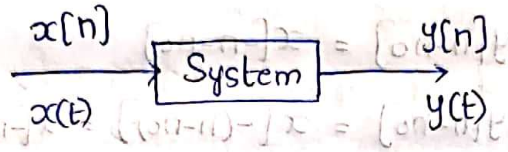
LHS = RHS (Hence proved - ③)

$\therefore$ Given system is Linear.

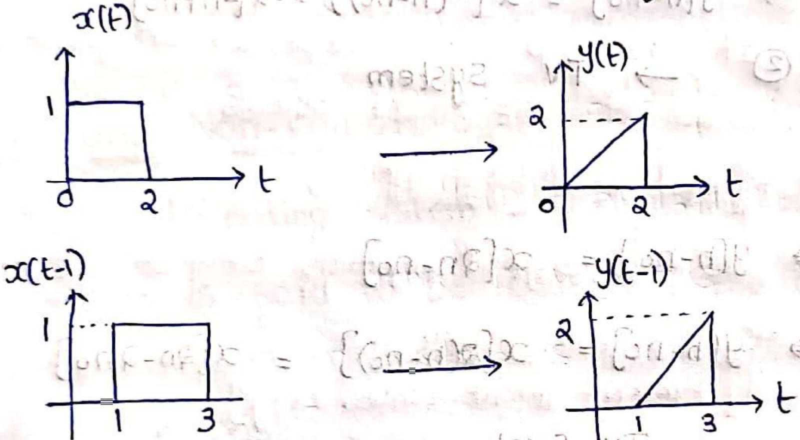
2. Time variant / Time invariant Systems:-
(shift varient) (shift invarient)

⇒ A system is time invariant, if the behaviour and characteristics of the system is fixed over time.

⇒ A system is said to be time invariant, if a time shift in the input signal causes an identical time shift in the output signal.

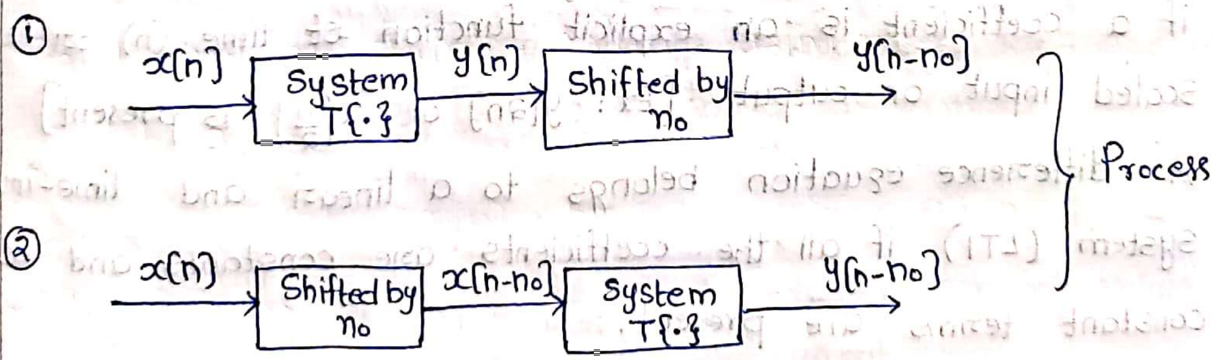


⇒ Time invarient System



If $x[n] \longrightarrow y[n]$

$x[n-n_0] \longrightarrow y[n-n_0]$



⇒ If the output is same in both processes, then the system is said to be time-invariant.

Q) For each of the following i/p-o/p relationships, determine whether the corresponding systems is time variant / invariant [TI].

(a) $y[n] = n x[n]$

① Shift in i/p by no $\Rightarrow y[n-n_0] = nx[n-n_0]$

$x[n] \rightarrow y[n]$

$x[n-n_0] \rightarrow y[n-n_0]$

② Shift in o/p by no $\Rightarrow y[n-n_0] = (n-n_0)x[n-n_0]$

① ≠ ② → Time variant System.

(b) $y[n] = x[-n]$

① shift in i/p $\Rightarrow y[n-n_0] = x[-n-n_0]$

② shift in o/p $\Rightarrow y[n-n_0] = x[-(n-n_0)] = x[-n+n_0]$

① ≠ ② → TV system.

(c) $y[n] = x[2n]$

① Shift in i/p $\Rightarrow y[n-n_0] = x[2n-n_0]$

② Shift in o/p $\Rightarrow y[n-n_0] = x[2(n-n_0)] = x[2n-2n_0]$

① ≠ ② → TV system.

Note:-

① A difference equation belongs to a time varying system if a coefficient is an explicit function of time (n) or a Scaled input or output [Ex: $y[2n]$ or $x[\frac{n}{2}]$ is present]

② A difference equation belongs to a linear and time-invariant system (LTI) if all the coefficients are constants and no constant terms are present.

Q) For the system described by the difference equation TV (or) TI

(a) $y[n] - 2y[n-1] = 4x[n]$

Since all the coefficients are constant, this difference equation belongs to Time invariant system.

(b) $y[n] - \underbrace{3n}_{} \, y[n-1] = x[n]$

$\downarrow$ explicit function of n $\Rightarrow$ So, time variant system.

(c) $y[n] + 2y^2[n] = 2x[n]$

$\downarrow 1 \quad \downarrow 2 \quad \downarrow 2$

All coefficients are constants $\Rightarrow$ So, TI system.

(d) $y[n] - 2y[n-1] = 2^{x[n]} \cdot x[n]$ $\Rightarrow$ Time invariant system.

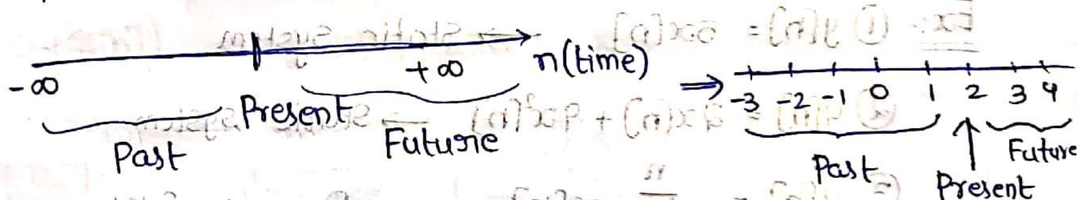
(e) $y[n] + 2^n y[n-1] = x[n]$

↳ Explicit function of n $\Rightarrow$ So, time variant system.

3. Causal and Non-causal Systems :-

↳ Non-anticipating system $\rightarrow$ Physically realizable systems.

$\Rightarrow$ A system is said to be causal if the output at any time, (to) 'n_0' depends only on the values of the input at the present and in the past.



Q) For each of the following i/p-o/p relationship, determine whether the corresponding system is causal or non-causal.

(a) $y[n] = nx[n]$

n	y[n]
-2	$-2x[-2]$
-1	$-x[-1]$
0	$0x[0]$
1	$1x[1]$
2	$2x[2]$

} Causal System

Output depends on present values of input.

(b) $y[n] = x[-n]$

$y[0] = x[0] \rightarrow$ Present $\qquad y[-1] = x[1] \rightarrow$ Future

$y[1] = x[-1] \rightarrow$ Past $\qquad \therefore$ Non-causal system.

(c) $y[n] = x[2n]$

$y[0] = x[0] \rightarrow$ Present So, non-Causal system.

$y[1] = x[2] \rightarrow$ Future

(d) $y[n] = x[n]\{\cos(n+1)\} \rightarrow$ Causal system

4. <u>Systems with memory and without memory</u> :-

with memory $\rightarrow$ dynamic system

without memory $\rightarrow$ Static system.

<u>Dynamic system</u>: If its o/p signal depends on past or future values of the input signals

o/p = f[past or future inputs]

<u>Static system</u>: If its o/p signal depends only on the present value of input signal.

o/p = f[Present input values only]

<u>Ex</u>:- ① $y[n] = 5x[n] \longrightarrow$ Static system

② $y[n] = 2x(n) + 9x^2[n] \longrightarrow$ Static system

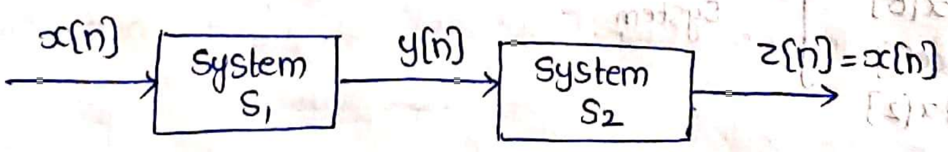
③ $y[n] = \sum_{k=-\infty}^{n} x[k] \longrightarrow$ Dynamic System

$\hookrightarrow$ Accumulator System (or) Summar system

④ $y[n] = x[n-1] \longrightarrow$ Dynamic System, depends on past values

$\hookrightarrow$ Delay system

5. <u>Invertible / Non-invertible Systems</u> :-



S_1 : Invertible system S_2 : Inverse system.

* If the system has inverse system, then it is invertible system otherwise non-invertible.

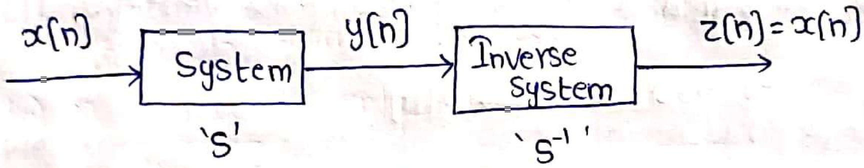
* Inverse system "undoes" what the given system $[S_i]$ does to input.

⇒ For a system to have an inverse [or to be invertible], distinct input must lead to distinct outputs.

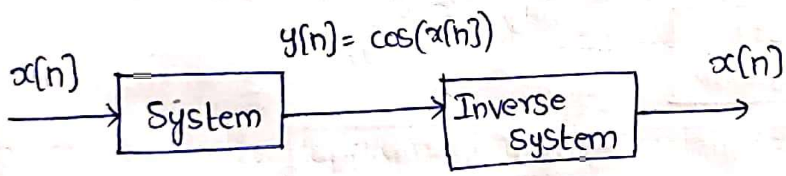
⇒ If a system produces identical output for two different inputs, it does not have an inverse.

→ For an invertible system, it is essential that every input have a unique output.

⇒ One to one mapping b/w input & output.



Ex:- $y[n] = \cos(x[n])$ $x[n] + 2\pi k$, $k = 0, \pm 1, \pm 2, \pm 3 \cdots$



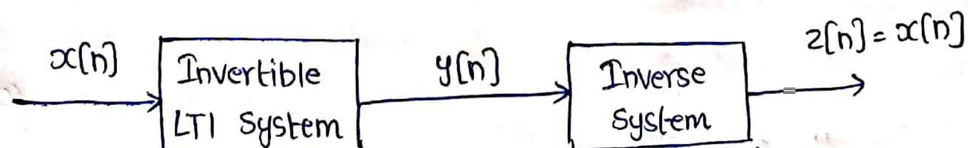
$$\cos(x[n]) = \cos(x[n] + 2\pi k)$$

$y[n] = \cos[x(n)]$ ⇒ Non-invertible System

$\because \cos(x[n]) = \cos(x(n) + 2\pi)$
$= \cos(x(n) - 2\pi)$

Bcoz different i/p's yield an identical output

Invertible LTI systems → difference equation



Difference equ of LTI I.S ⇒ $y[n] = x[n] - 0.5x[n-1]$
Difference equ of Inverse ⇒ $x[n] = y[n] - 0.5y[n-1]$
switching input with output

6. Stable / Unstable Systems:-

A stable system is the one in which small inputs leads to response that do not diverge.

Bounded input $\longrightarrow$ Bounded output

(BIBO stability)

$\Rightarrow$ A system is said to be BIBO stable if and only if, for every bounded input results in a bounded o/p.

$$|x[n]| < B_x < \infty \quad \forall n$$

$$|y[n]| < B_y < \infty$$

$$|x[n]| < B_x < \infty \implies |y[n]| < B_y < \infty$$

Analysis on Discrete time Systems :-

→ Physical phenomenon
→ Mathematical analysis (easy)

Solving difference equ by applying some specific inputs & with initial conditions.

- Impulse response
- represent any arbitrary signal in terms of time shifted impulses.
- Based on this compute the output of the given system.

* Any physical phenomenon in terms of LTI systems.
* Mathematical analysis is easy on LTI systems.

So, that's why we use LTI systems for discrete time system analysis.

Primary reasons:

⇒ LTI systems are amenable to analysis is that any such system possesses the superposition principle.

⇒ We can represent the input to an LTI system in terms of a linear combination of a set of basic signals.

∴ Output of the system could be computed as sum of individual responses.

$$x[n] \longrightarrow \boxed{\text{LTI System}} \longrightarrow y[n]$$

Analysis ⇒ Determining the response to some specified input

Solve the difference equ describing the system subjected to the specified i/p's & with initial conditions.

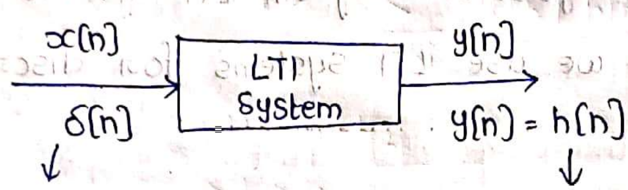
By characterizing an LTI system in terms of impulse response, defined as the o/p of an LTI system due to unit impulse slg applied at $n=0$.

Now, given the impulse response, we determine the response of an LTI system due to arbitrary signal by expressing the i/p as a weighted superposition of time shifted impulses.

⇒ By linearity and time invariance, the o/p signal must be a <u>weighted superposition</u> of time shifted impulse responses.

↳ CONVOLUTION sum of discrete time LTI system

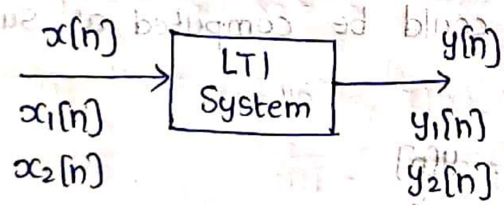
→ Impulse response



impulse i/p signal   Impulse response

$$x[n] \longrightarrow y[n]$$
$$\delta[n] \longrightarrow h[n]$$

Ex:-



$$x[n] = ax_1[n] + bx_2[n] \longrightarrow y[n] = ay_1[n] + by_2[n]$$

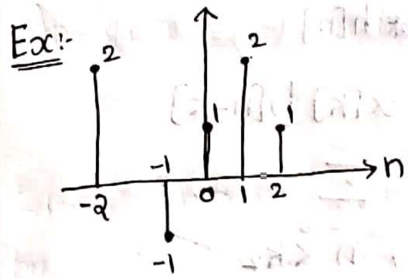
$$x[n] = a_1x_1[n] + a_2x_2[n] + a_3x_3[n] + \dots + a_N x_N[n]$$

$$y[n] = a_1y_1[n] + a_2y_2[n] + a_3y_3[n] + \dots + a_N y_N[n]$$

Input $\boxed{x[n] = \sum_{i=1}^{N} a_i x_i[n]}$

Output $\boxed{y[n] = \sum_{i=1}^{N} a_i y_i[n]}$

Ex:-

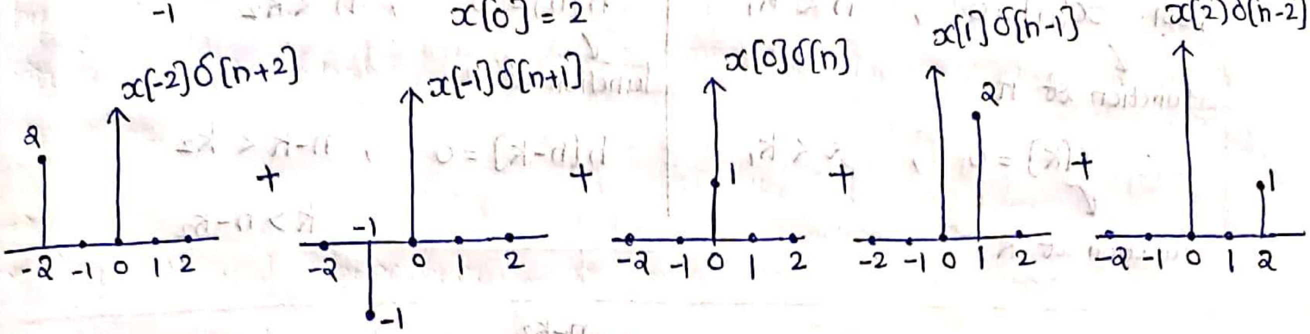


$$x[n] = \{2, -1, \underset{\uparrow}{1}, 2, 1\}$$

$x[-2] = 2$ $\quad x[1] = 2$

$x[-1] = -1$ $\quad x[2] = 1$

$x[0] = 2$



$$x[n] = x[-2]\delta[n+2] + x[-1]\delta[n+1] + x[0]\delta[n] + x[1]\delta[n-1] + x[2]\delta[n-2]$$

$$\boxed{\therefore x[n] = \sum_{k=-2}^{+2} x[k]\,\delta[n-k]}$$

If $x[n] \longrightarrow y[n]$

$\delta[n] \longrightarrow h[n]$

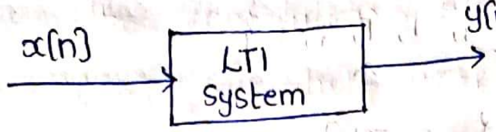
$\delta[n-k] \longrightarrow h[n-k]$ $\Rightarrow$ This is because of time invariance

$x[k]\delta(n-k] \longrightarrow x[k]h[n-k)$ $\Rightarrow$ Bcoz of Scaling

$$\underbrace{\sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k]}_{= x[n] \ (i/p)} \longrightarrow \underbrace{\sum_{k=-\infty}^{\infty} x[k]\,h[n-k]}_{= y[n] \ (o/p)} \Rightarrow \text{Bcoz of Linearity}$$

So, the response or output of an LTI system $\boxed{y[n] = \sum_{k=-\infty}^{\infty} x[k]\,h[n-k]}$ $\Rightarrow$ Convolution Sum of D.T LTI Systems.

$$\boxed{y[n] = x[n] * h[n]}$$



$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

For $x[n] = 0, \quad n < k_1$ (function of n) $\quad\big|\quad h[n] = 0, \quad n < k_2$ (function of n)

$x[k] = 0, \quad k < k_1$ (function of k) $\quad\big|\quad h[n-k] = 0, \quad n-k < k_2$

$k > n - k_2$

$$\boxed{\begin{aligned} \text{So, } y[n] &= x[n] * h[n] = \sum_{k=k_1}^{n-k_2} x[k]\, h[n-k] \;; \quad n \geq k_1 + k_2 \\ y[n] &= 0 \;; \quad n < k_1 + k_2 \end{aligned}}$$

$n - k_2 - k_1 \geq 0$

$n \geq k_1 + k_2$

Q) Find the convolution of two sequences, $x[n] = \begin{cases} 0 & n < -5 \\ \left(\frac{1}{2}\right)^n & n \geq -5 \end{cases}$

$$h[n] = \begin{cases} 0 & n < 3 \\ \left(\frac{1}{3}\right)^n & n \geq 3 \end{cases}$$

Here, $k_1 = -5$, $k_2 = 3$

we know that, $y[n] = x[n] * h[n] \;; \; n \geq k_1 + k_2 = -5 + 3 = -2$

$y[n] = 0 \;; \; n < k_1 + k_2$

$n < -5 + 3 = -2 \Rightarrow n < -2$

$$y[n] = \sum_{k=k_1}^{n-k_2} x[n]\, h[n-k] = \sum_{k=-5}^{n-3} \left(\frac{1}{2}\right)^k \left(\frac{1}{3}\right)^{n-k} \;; \; n \geq -2$$

$$y[n] = \sum_{k=-5}^{n-3} \left(\frac{1}{2}\right)^k \left(\frac{1}{3}\right)^{n-k} \;; \; n \geq -2$$

A change of variables is performed by letting $m = k + 5$

$k = m - 5$

As $k = -5 \Rightarrow m = -5 + 5 = 0$

As $k = n - 3 \Rightarrow m = n - 3 + 5 = n + 2$

$$\therefore \; y[n] = \sum_{m=0}^{n+2} \left(\frac{1}{2}\right)^{m-5} \left(\frac{1}{3}\right)^{n-m+5}$$

$$y[n] = \left(\frac{2}{3}\right)^5 \left(\frac{1}{3}\right)^n \sum_{m=0}^{n+2} \left(\frac{3}{2}\right)^m$$

$$= \left(\frac{2}{3}\right)^5 \left(\frac{1}{3}\right)^n \sum_{m=0}^{n+2} \left(\frac{3}{2}\right)^m$$

$$= \left(\frac{2}{3}\right)^5 \left(\frac{1}{3}\right)^n \left[\frac{\left(\frac{3}{2}\right)^0 - \left(\frac{3}{2}\right)^{n+2+1}}{1-\frac{3}{2}}\right] \qquad \left(\because \sum_{n=0}^{N} a^n = \frac{1-a^{N+1}}{1-a}\right)$$

$$= -2\left(\frac{2}{3}\right)^5 \left(\frac{1}{3}\right)^n \left[1 - \left(\frac{3}{2}\right)^{n+3}\right]$$

$$= -2\left(\frac{2}{3}\right)^5 \left[\left(\frac{1}{3}\right)^n - \left(\frac{1}{2}\right)^n \left(\frac{3}{2}\right)^3\right]$$

$$y[n] = -64\left(\frac{1}{3}\right)^{n+5} + 2\left(\frac{2}{3}\right)^2 \left(\frac{1}{2}\right)^n \; ; \; n \geq -2$$

Computation of convolution sum

- Analytical method
- Graphical method

Graphical method:-

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

1. Plotting : First plot $x[k]$ & $h[k]$ as a function of 'k' [not 'n'] bcoz summation is wrt 'k'.
2. Shifting : Shift $h[k]$ by 'n' units to obtain $h[k+n]$
3. Folding : Invert $h[k+n]$ about vertical axis [k=0 axis] to obtain $h[-k+n] = h[n-k]$
4. Multiplication & Addition : Next multiply $x[k]$ & $h[n-k]$ and add all the products to obtain $y[n]$.

$x[n] \rightarrow x[k]$

$h[n] \rightarrow h[k] \rightarrow h[k+n] \rightarrow h[-k+n]$

This procedure is repeated for each value of $n \Rightarrow -\infty$ to $+\infty$

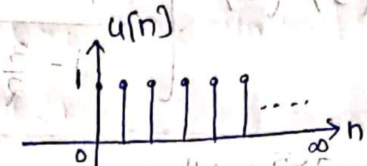
Q) Compute the convolution of sequences $y[n] = x[n] * h[n]$

(a) $x[n] = u[n]$; $h[n] = u[n]$

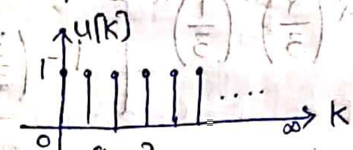
We know that, $y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$

① $y[n] = \sum_{k=-\infty}^{\infty} u[k]\, u[n-k]$ → function of k.

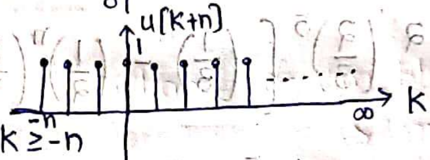
$$u[n] = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$$



$$u[k] = \begin{cases} 1 & k \geq 0 \\ 0 & k < 0 \end{cases}$$

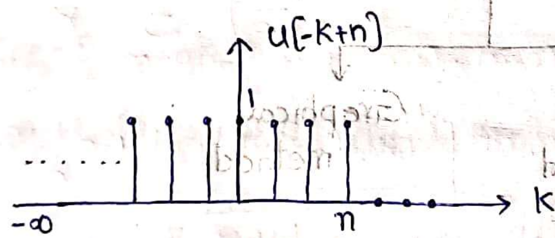


② Shifting: $u[k] \rightarrow u[k+n]$

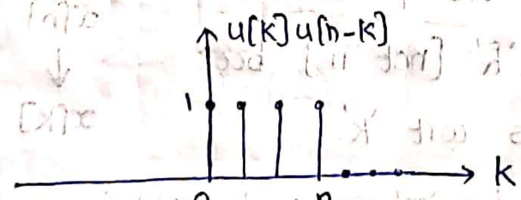


$$u[k+n] = \begin{cases} 1 & k+n \geq 0 \Rightarrow k \geq -n \\ 0 & k+n < 0 \Rightarrow k < -n \end{cases}$$

③ Folding, $u[k+n] \longrightarrow u[-k+n] = u[n-k] = \begin{cases} 1 & -\infty < k < n \\ 0 & k < n \end{cases}$



④ Multiply & add : $u[k]\, u[n-k] = \begin{cases} 1 & 0 \leq k \leq n \\ 0, & k < 0 \\ & k > n \end{cases}$



Summation

$$y[n] = \sum_{k=-\infty}^{\infty} u[k] . u[n-k] = \sum_{k=0}^{n} 1$$

$$= (n+1) \quad n \geq 0$$

$$y[n] = \begin{cases} n+1 & ; n \geq 0 \\ 0 & n < 0 \end{cases} = (n+1)\, u[n] \Rightarrow y[n] = (n+1)u[n]$$

$$\boxed{y[n] = r[n+1]}$$

So, $u[n] * u[n] = [n+1]\,u[n]$

$$\boxed{u[n] * u[n] = r[n+1]}$$

$n \cdot u[n] = r[n]$

$(n+1)\,u[n+1] = (n+1)\,u[n]$
$= r[n+1]$

(b) $x[n] = (0.8)^n u[n]$, $h[n] = (0.4)^n u[n]$

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

$$= \sum_{k=-\infty}^{\infty} (0.8)^k u[k] \cdot (0.4)^{n-k} u[n-k]$$

$$= (0.4)^n \sum_{k=0}^{n} (0.8)^k (0.4)^{-k}$$

$$= (0.4)^n \sum_{k=0}^{n} \left(\frac{0.8}{0.4}\right)^k = (0.4)^n \sum_{k=0}^{n} 2^k$$

$$= (0.4)^n \left[\frac{1-2^{n+1}}{1-2}\right]$$

$\therefore y[n] = \left[2(0.8)^n - (0.4)^n\right] u[n]$

$$\therefore y[n] = (0.4)^n \left[2^{n+1} - 1\right] = \left[2(0.8)^n - (0.4)^n\right] \quad ; n \geq 0$$

(c) $x[n] = h[n] = a^n u[n]$

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

$$= \sum_{k=-\infty}^{\infty} a^k u[k] \cdot a^{n-k} u[n-k]$$

$$= \sum_{k=0}^{n} a^k \cdot a^{n-k}$$

$$= \sum_{k=0}^{n} a^n = a^n \sum_{k=0}^{n} 1 = a^n (n+1) \quad ; n \geq 0$$

$$y[n] = \begin{cases} a^n (n+1) & ; n \geq 0 \\ 0 & ; n < 0 \end{cases}$$

$$y[n] = a^n (n+1) \cdot u[n]$$

(d) $x[n] = u[n-1] \quad h[n] = \alpha^n u[n-1]$

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

$$= \sum_{k=-\infty}^{\infty} u[k-1] \cdot \alpha^{n-k} u[n-k-1]$$

$$u[k-1] = \begin{cases} 1 & k-1 \geq 0 \Rightarrow k \geq 1 \\ 0 & k-1 < 0 \Rightarrow k < 1 \end{cases} \qquad u[n-k-1] = \begin{cases} 1 & n-k-1 \geq 0 \\ & k \leq n-1 \\ 0 & k > n-1 \end{cases}$$

$$\therefore y[n] = \sum_{k=1}^{n-1} \alpha^{n-k}$$

$n-k-1 \geq 0$ | $n-k-1 < 0$
$n-1 \geq k$ | $n-1 < k$
$k \leq n-1$ | $k > n-1$

$$= \alpha^n \sum_{k=1}^{n-1} \left(\frac{1}{\alpha}\right)^k$$

Change of Variables
$\Rightarrow m = k-1$

$$= \alpha^n \sum_{m=0}^{n-2} \left(\frac{1}{\alpha}\right)^{m+1} \qquad \left[\begin{array}{l} k=1 \Rightarrow m=0 \\ k=n-1 \Rightarrow m=n-2 \end{array} \right]$$

$$= \alpha^{n-1} \sum_{m=0}^{n-2} \left(\frac{1}{\alpha}\right)^m$$

$$= \alpha^{n-1} \left[\frac{1-\left(\frac{1}{\alpha}\right)^{n-2+1}}{1-\left(\frac{1}{\alpha}\right)} \right]$$

$$= \alpha^{n-1} \left[\frac{1-\left(\frac{1}{\alpha}\right)^{n-1}}{1-\left(\frac{1}{\alpha}\right)} \right] = \frac{\alpha^{n-1} - \cancel{\alpha^{n-1}} \frac{1}{\cancel{\alpha^{n-1}}}}{1-\frac{1}{\alpha}} = \frac{\alpha^n - \alpha}{\alpha - 1}$$

(e) $x[n] = r[n] = n u[n] \; ; \; h[n] = a^{-n} u[n-1] \; ; \; a < 1$

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

$$= \sum_{k=-\infty}^{\infty} k u[k] \cdot a^{-(n-k)} u[n-k-1]$$

$$u[k] = \begin{cases} 1 & k \geq 0 \\ 0 & k < 0 \end{cases} \qquad u[n-k-1] = \begin{cases} 1 & n-k-1 \geq 0 \\ & k \leq n-1 \\ 0 & k > n-1 \end{cases}$$

$$y(n) = \sum_{k=0}^{n-1} k a^{-n+k}$$

$$= a^{-n} \sum_{k=0}^{n-1} k a^{k}$$

$$= \frac{a}{(1-a)^2}\left[1 - na^{n-1} + (n-1)a^{n}\right] \cdot a^{-n}$$

Proof:-

$$\sum_{k=0}^{n-1} k a^{k} = S = 0 + a + 2a^2 + 3a^3 + \dots + (n-2)a^{n-2} + (n-1)a^{n-1} + 0$$

$$aS = 0 + a^2 + 2a^3 + 3a^4 + \dots + (n-2)a^{n-1} + (n-1)a^{n}$$

$$S - aS = \underbrace{a + a^2 + a^3 + a^4 + \dots + a^{n-2} + a^{n-1}}_{\hookrightarrow n-1 \text{ terms}} - (n-1)a^{n}$$

$$S(1-a) = \frac{a(1-a^{n+1})}{1-a} - (n-1)a^{n}$$

$$S = \frac{a(1-a^{n+1})}{(1-a)^2} - \frac{(n-1)a^{n}}{(1-a)}$$

$$= \frac{a(1-a^{n+1}) - (n-1)(1-a)a^{n}}{(1-a)^2}$$

$$= \frac{a - a^{n} - \left[(n-1)(a^{n} - a^{n+1})\right]}{(1-a)^2}$$

$$= \frac{a - a^{n} - \left[na^{n} - a^{n} - na^{n+1} + a^{n+1}\right]}{(1-a)^2}$$

$$= \frac{a - \cancel{a^{n}} - na^{n} + \cancel{a^{n}} + na^{n+1} - a^{n+1}}{(1-a)^2} = \frac{a - na^{n} + na^{n+1} - a^{n+1}}{(1-a)^2}$$

$$\sum_{k=0}^{n-1} k a^{k} = S = \frac{a(1 - na^{n-1} + na^{n} - a^{n})}{(1-a)^2} = \frac{a}{(1-a)^2}\left[1 - na^{n-1} + (n-1)a^{n}\right]$$

Q) Suppose the unit impulse response of an LTI system is a unit ramp, $h[n] = r[n] = nu[n]$. Compute the response of the system of the unit step input, $x[n] = u[n]$

Given, $x[n] = u[n] = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$

$h[n] = nu[n] = \begin{cases} n & n \geq 0 \\ 0 & n < 0 \end{cases}$

We know that, $y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$

$= \sum_{k=-\infty}^{\infty} u[k] \cdot (n-k)\, u[n-k]$

$u[k] = \begin{cases} 1 & k \geq 0 \\ 0 & k < 0 \end{cases}$ $\quad u[n-k] = \begin{cases} 1 & n-k \geq 0 \Rightarrow k \leq n \\ 0 & k > n \end{cases}$

Then, $y[n] = \sum_{k=0}^{n} 1\,(n-k) \;;\; n \geq 0$ $\quad \left(\because u[k]\,u[n-k] = 1 \text{ } 0 \text{ to } n\right)$

$= n + (n-1) + (n-2) + \cdots + 3 + 2 + 1 + 0 = \frac{n(n+1)}{2}$

$\therefore y[n] = \begin{cases} \frac{n(n+1)}{2} & n \geq 0 \\ 0 & n < 0 \end{cases}$

<u>Note:-</u>

① If $x[n] = \delta[n]$
$y[n] = h[n] \rightarrow$ Impulse response

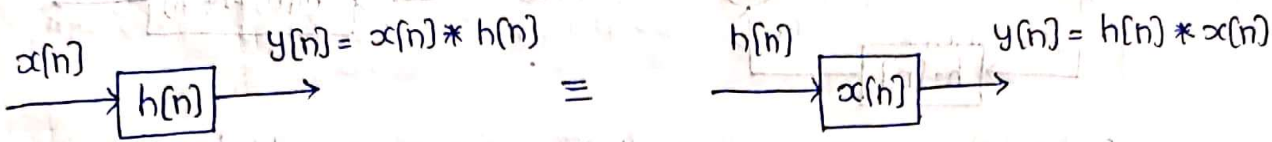
② If $x[n] = u[n]$
$y[n] = s[n] \rightarrow$ Unit step response

Properties of convolution sum:-

1. Commutative property:

$$x(n) * h[n] = h[n] * x(n]$$

$$\sum_{K=-\infty}^{\infty} x[K]\, h[n-K] = \sum_{K=-\infty}^{\infty} h[K]\, x[n-K]$$



Proof:- By definition, we know that

$$x[n] * h[n] = \sum_{K=-\infty}^{\infty} x[K]\, h(n-K]$$

By change of variables, Let $n-K = m$

$K = n-m$

As $K = -\infty \Rightarrow m = \infty$

$K = +\infty \Rightarrow m = -\infty$

$$x(n) * h[n] = \sum_{m=\infty}^{-\infty} x[n-m]\, h[m]$$

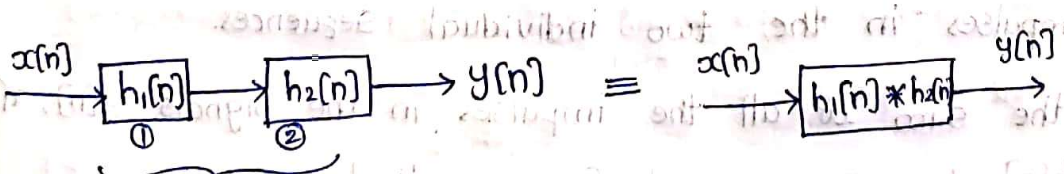
$$= \sum_{m=-\infty}^{\infty} h[m] \cdot x[n-m]$$

$$y[n] = h[n] * x[n]$$

$\therefore$ Convolution sum obeys commutative law.

2. Associative property:

$$x[n] * [h_1[n] * h_2[n]] = [x[n] * h_1[n]] * h_2[n]$$



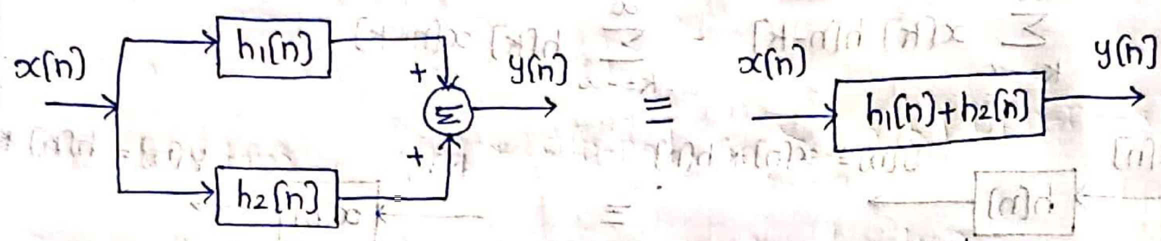
Cascaded

(Series connection)

↳ They get convolved

3. Distributive property:

$$x(n) * [h_1(n) + h_2(n)] = x(n) * h_1(n) + x(n) * h_2(n)$$



· If two systems are parallel → they get added.

4. Shift property:

If $x(n) * h(n) = y(n)$

then $x(n) * h(n-n_0) = y(n-n_0)$

(or)

then $x(n-n_0) * h(n) = y(n-n_0)$

(or)

then $x(n-n_1) * h(n-n_2) = y(n-n_1-n_2)$

5. Impulse convolution property:

Convolution of a signal $x(n)$ with a unit impulse signal results in the signal $x(n)$ itself.

$$x(n) * \delta(n) = x(n)$$

6. Sum property:

The sum of the impulses in a convolution, sum of two discrete time sequences is the product of the sum of the impulses in the two individual sequences.

Let the sum of all the impulses in the signals $y(n)$, $x(n)$ and $h(n)$ be S_y, S_x and S_h respectively.

$$\therefore S_y = \sum_{n=-\infty}^{\infty} y(n), \quad S_x = \sum_{n=-\infty}^{\infty} x(n), \quad S_h = \sum_{n=-\infty}^{\infty} h(n)$$

then, $\sum_{n=-\infty}^{\infty} y[n] = \sum_{n=-\infty}^{\infty} x[n] \cdot \sum_{n=-\infty}^{\infty} h[n]$

$$\boxed{S_y = S_x \cdot S_h}$$

Q) Show that

(a) The convolution of an odd signal and an even signal is an odd signal.

Let $x[n]$ be an odd signal,

$$x[n] = -x[-n]$$

Let $h[n]$ be an even signal,

$$h[n] = h[-n]$$

Then, $y[n] = \sum_{k=-\infty}^{\infty} -x[-n] \cdot h[-n-k]$

$$y[n] = x[n] * h[n]$$

$$= -x[-n] * h[-n]$$

$$= -\{x[-n] * h[-n]\} = -\{y[-n]\}$$

$y[n] = -y[-n] \Rightarrow$ Odd signal.

(b) The convolution of two odd signals is an even signal.

Let $x[n]$ and $h[n]$ be odd signals,

$$x[n] = -x[-n] \qquad h[n] = -h[-n]$$

$$y[n] = x[n] * h[n]$$

$$= -x[-n] * -h[-n]$$

$= x[-n] * h[-n] = y[-n] \longrightarrow$ Even signal

$$y[n] = y[-n]$$

(c) The convolution of two even signals is even signal.

Let $x[n]$ & $h[n]$ are two even signals

$$x[n] = x[-n] \quad , \quad h[n] = h[-n]$$

$x[n] * h[n] = x[-n] * h[-n]$

$y[n] = y[-n] \longrightarrow$ Even signal.

Q) Prove the following

(1) $x[n] * \delta[n-n_0] = x[n-n_0]$

By definition, $x[n] * \delta[n-n_0] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-n_0-k]$

By multiplication property of impulse function,

$$x[n]\,\delta[n-k] = x[k]\,\delta[n-k]$$

then $x[k]\,\delta[n-n_0-k] = x[n-n_0]\,\delta[n-n_0-k]$

By this property,

$$x[n] * \delta[n-n_0] = \sum_{k=-\infty}^{\infty} x[n-n_0]\,\delta[n-n_0-k]$$

$$= x[n-n_0] \underbrace{\sum_{k=-\infty}^{\infty} \delta[n-n_0-k]}_{=1}$$

$$\boxed{x[n] * \delta[n-n_0] = x[n-n_0]}$$

Hence proved.

(2) $x[n] * \delta[n+n_0] = x[n+n_0]$

By definition, $x[n] * \delta[n+n_0] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n+n_0-k]$

By multiplication property,

$$x[n]\,\delta[n-k] = x[k]\,\delta[n-k]$$

then $x[k]\,\delta[n+n_0-k] = x[n+n_0]\,\delta[n+n_0-k]$

By this property,

$$x[n] * \delta[n+n_0] = \sum_{k=-\infty}^{\infty} x[n+n_0]\,\delta[n+n_0-k]$$

$$= x[n+n_0] \underbrace{\sum_{k=-\infty}^{\infty} \delta[n+n_0-k]}_{=1}$$

$$= x[n+n_0]$$ // Hence proved.

(3) $x[n] * u[n] = \sum_{k=-\infty}^{n} x[k]$

By definition $\Rightarrow$ $x[n] * u[n] = \sum_{k=-\infty}^{\infty} x[k]u[n-k]$

$$u[n-k] = \begin{cases} 1 & ; \; n-k \geq 0 \Rightarrow k \leq n \\ 0 & \quad n-k < 0 \Rightarrow k > n \end{cases}$$

$$x[n] * u[n] = \sum_{k=-\infty}^{n} x[k] \underbrace{u[n-k]}_{=1}$$

$$= \sum_{k=-\infty}^{n} x[k] \;//$$ Hence proved

(4) $u[n] * u[n] = (n+1)u[n]$

By definition $\Rightarrow$ $u[n] * u[n] = \sum_{k=-\infty}^{\infty} u[k]\, u[n-k]$

$$u[k] = \begin{cases} 1 & k \geq 0 \\ 0 & k < 0 \end{cases} \qquad u[n-k] = \begin{cases} 1 & n-k \geq 0 \Rightarrow k \leq n \\ 0 & n-k < 0 \Rightarrow k > n \end{cases}$$

So, 1 for limits : 0 to n

$$u[n] * u[n] = \sum_{k=0}^{n} u(k)\, u(n-k)$$

$$= \sum_{k=0}^{n} (1) = \begin{cases} n+1 \; ; & n \geq 0 \\ 0 & \; n < 0 \end{cases}$$

$$u[n] * u[n] = \begin{cases} n+1 & ; \; n \geq 0 \\ 0 & \; n < 0 \end{cases} = (n+1)\, u(n) \;//$$ Hence proved.

CONVOLUTION SUM OF FINITE LENGTH SIGNALS:

$x[n]$, $h[n]$ } Finite in length then $y[n]$ = finite

$x[n] = \{\underset{\uparrow}{\overset{0}{1}}, \overset{1}{2}, \overset{2}{3}, \overset{3}{4}, \overset{4}{5}\}$ Length of $x[n]$ = 5 samples

$h[n] = \{\underset{\uparrow}{\overset{0}{-1}}, \overset{1}{3}, \overset{2}{2}, \overset{3}{-2}, \overset{4}{7}, \overset{5}{6}, \overset{6}{2}\}$ Length of $h[n]$ = 7 samples

$LE_x \rightarrow$ left end of x $\quad$ $LE_h \Rightarrow$ left end of h

$RE_x \rightarrow$ Right end of x $\quad$ $RE_h \rightarrow$ Right end of h

$LE_x = 0 \qquad LE_h = 0$

$RE_x = 4 \qquad RE_h = 6$

Length of $y(n)$ = No. of samples in $x(n)$ + No. of samples in $h(n)$ − 1

$= 5 + 7 - 1 = 11$

$LE_y = LE_x + LE_h = 0 + 0 = 0$

$RE_y = RE_x + RE_h = 4 + 6 = 10$

Methods:-

① Tabulation method to compute convolution sum of finite length sequence

② Multiplication method " " " "

1. Tabulation method:- $x(n) = \{1, 2, 3, 4, 5\}$

$h(n) = \{1, 1, 1, 1\}$

$h(n)$ \ $x(n)$	1	2	3	4	5
1	1×1 =1	1×2 =2	1×3 =3	1×4 =4	1×5 5
1	1	2	3	4	5
1	1	2	3	4	5
1	1	2	3	4	5

$y(n) = \{1, 1+2, 1+2+3, 1+2+3+4, 2+3+4+5, 3+4+5, 4+5, 5\}$

$y(n) = \{1, 3, 6, 10, 14, 12, 9, 5\}$

Length of $y[n]$ = Samples of $x[n]$ + Samples of $h[n]$ -1

$= 5+4-1 = 8$

	LE	RE
$x[n]$	0	4
$h[n]$	0	3
$y[n]$	0	7

$x[n] = LE_x, RE_x$, $h[n] = LE_h, RE_h$

$y[n] \Rightarrow LE_y = LE_x + LE_h$

$RE_y = RE_x + RE_h$

Q) $x[n] = \{\overset{0}{\underset{\uparrow}{2}}, \overset{1}{-1}, \overset{2}{3}\}$ $h[n] = \{\overset{0}{\underset{\uparrow}{1}}, \overset{1}{2}, \overset{2}{2}, \overset{3}{3}\}$

$LE_x = 0$ $LE_h = 0$ then $LE_y = 0+0=0$

$RE_x = 2$ $RE_h = 3$ $RE_y = 2+3 = 5$

Length of $y[n] = 3+4-1$

$= 7-1 = 6$ samples

$h[n]$ \ $x[n]$	2	-1	+3
1	2	-1	+3
2	4	-2	+6
2	4	-2	+6
3	6	-3	+9

$y[n] = \{\underset{\uparrow}{2}, 4-1, 4-2+3, 6-2+6, -3+6, 9\}$

$y[n] = \{\overset{0}{\underset{\uparrow}{2}}, \overset{1}{3}, \overset{2}{5}, \overset{3}{10}, \overset{4}{3}, \overset{5}{9}\}$

Sum property:- $S_y = S_x \cdot S_h$

$S_y = \sum_{n=-\infty}^{\infty} y[n]$, $S_h = \sum_{n=-\infty}^{\infty} h[n]$, $S_x = \sum_{n=-\infty}^{\infty} x[n]$

$x[n] = \{2, -1, 3\} \rightarrow S_x = 2-1+3 = 4$

$h[n] = \{1, 2, 2, 3\} \Rightarrow S_h = 1+2+2+3 = 8$

$y[n] = \{2, 3, 5, 10, 3, 9\} \Rightarrow S_y = 2+3+5+10+3+9 = 32$

$S_y = S_x \cdot S_h$

$S_y = 4 \cdot 8 = \underline{\underline{32}}$

2. Multiplication method:

$x[n] = \{4, 5, 6\}$ (arrow under 4) $h[n] = \{4, 4, 3, 2\}$ (arrow under the first 4)

$LE_x = 0$ $LE_h = 0$ then $LE_y = 0 + 0 = 0$

$RE_x = 2$ $RE_h = 3$ $RE_y = 2 + 3 = 5$

Length of $y[n]$ = Samples in $x[n]$ + Samples in $h[n]$ -1

$= 3 + 4 - 1 = \underline{6}$ samples

$$
\begin{array}{lcccccc}
x[n] & & & 4 & 4 & 3 & 2 \\
h[n] & & & \times & 4 & 5 & 6 \\
\hline
 & & & 24 & 24 & 18 & 12 \\
 & & 16 & 16 & 15 & 10 & \times \\
 & 16 & 16 & 12 & 8 & \times & \\
\hline
 & 16 & 36 & 56 & 47 & 28 & 12 \\
\hline
\end{array}
$$

$\therefore\ y[n] = \{16, 36, 56, 47, 28, 12\}$ (arrow under 16)

System response to periodic inputs:-

The response of a discrete time system to a periodic input with period N is also periodic with the same period N. $x[n] = x[n+N]$ → periodicity

Proof:-

$x[n] \rightarrow \boxed{h[n]} \rightarrow y[n] = x[n] * h[n]$

Using time shifting property of the convolution,

$y[n+N] = \underbrace{x[n+N]}$ $* \ h[n]$ → $x[n]$ be periodic with period N.

$x[n] = x[n+N]$

$y[n+N] = \underbrace{x[n] * h[n]}_{= y[n]}$

$\boxed{\therefore\ y[n+N] = y[n]}$ $\therefore$ The response is also periodic with period 'N'.

Ex:- $x[n] = \{\underset{\uparrow}{2}, 1, -3, 2, 1, -3, 2, 1, -3, \ldots\ldots_{\infty}\}$

$h[n] = \{\underset{\uparrow}{1}, 1\}$ — Find the convolution sum of $x[n]$ and $h[n]$ using multiplication method.

$x[n] \rightarrow LE_x = 0$ $\quad h[n] \rightarrow LE_h = 0$

$RE_x = \infty$ $\quad RE_h = 1$

Output : $y[n] \longrightarrow LE_y = 0 + 0 = 0$

$RE_y = \infty + 1 = \infty$

So, $y[n]$ starts at origin and ends at ∞.

By multiplication method,

$x[n]$	2	1	-3	2	1	-3	2	1	-3	......	$n=\infty$
$h[n]$	1	1									
	2	1	-3	2	1	-3	2	1	-3	......	$n=\infty$
		2	1	-3	+2	2	-3	+2	2	-3 X	$n=\infty$
	2	3	-2	-1	3	-2	-1	3	-2	....	$n=\infty$

$\therefore\ y[n] = \{\underset{\uparrow}{2}, \underbrace{3, -2, -1}, \underbrace{3, -2, -1}, \underbrace{3, -2,} \ldots\ldots_{n=\infty}\}$

Startup effect

→ Except the value at origin, the response $y[n]$ is periodic with period $N=3$

Q) Let $x(n) = \{\underset{\uparrow}{2}, 1, -3, 2, 1, -3, 2, 1, -3, \ldots\}$

$h(n) = \{\underset{\uparrow}{1}, 1, 1\}$ Find the convolution sum by multiplication method?

$x[n] \longrightarrow LE_x = 0$ $\quad h(n) \rightarrow LE_h = 0$ $\quad$ then $y(n) \rightarrow LE_y = 0+0 = 0$

$RE_x = \infty$ $\quad RE_h = 2$ $\quad RE_y = \infty + 2 = \infty$

$x[n]$			2	1	-3	2	1	-3	2	1	-3	...	
$h(n)$							×	1	1	1			
			2	1	-3	2	1	-3	2	1	-3	...	
		2	1	-3	2	1	-3	2	1	-3	...		
	2	1	-3	2	1	-3	2	1	-3	...			
	2	3	0	0	0	0	0	0	0	-2	-3	...	

~~1 -3 2 1 -3~~ ×

Moving average system

Q) Let $x_p[n] = \{2, 1, -3\}$ described by one period with input $x[n]$ with period $N=3$ to the system whose impulse response is given as $h[n] = \{1, \underset{\uparrow}{1}\}$. Determine $y(n)$ for one period.

$y_1[n] = x_p(n) * h[n]$

$= \{2, 1, -3\} * \{1, 1\}$

$= \{\underset{\uparrow}{2}, 3, -2, | -3\} \rightarrow$ Length = 4

		2	1	-3
			1	1
		2	1	-3
	2	1	-3	×
	2	3	-2	-3

$x_p \Rightarrow N = 3$

	2	3	-2
	-3		
$y_p[n] =$	-1	3	-2

So, $y_p[n] = \{-1, 3, 2\} \Longrightarrow$ period $N=3$

$$x_p[n] = \{2,1,-3\} \longrightarrow \boxed{h[n]=\{1,1\}} \longrightarrow y_p[n] = \{-1,3,-2\}$$

$x[n] = \{2,1,-3,2,1,-3,\dots\}$ $\quad$ $y[n] = \{-1,3,-2,-1,3,-2\dots\}$

Note:- If i/p is periodic, o/p is also periodic

If i/p having a period of N then output also have period N = 3.

Q) $\underbrace{x_p[n] = \{1, \underset{\uparrow}{2}, 3\}}_{\text{one period of i/p}}$, $\underbrace{x[n] = \{1,2,3,1,2,3,\dots\}}_{\text{i/p to the system}}$, $\underbrace{h[n] = \{1,1,2,-1,2\}}_{\text{Impulse response of system.}}$

Periodicity of N = 3

$$y_1[n] = x_p(n) * h[n]$$

$$= \{\underset{\uparrow}{1},2,3\} * \{\underset{\uparrow}{1},1,2,-1,2\}$$

$$\begin{array}{ccccccc}
 & & 1 & 1 & 2 & -1 & 2 \\
 & & & & 1 & 2 & 3 \\
\hline
 & & 3 & 3 & 6 & -3 & 6 \\
 & 2 & 2 & 4 & -2 & 4 & \times \\
1 & 1 & 2 & -1 & 2 & \times & \\
\hline
1 & 3 & 7 & 6 & 6 & 1 & 6
\end{array}$$

$\Rightarrow y_1[n] = \{1, \underset{\uparrow}{3}, 7, 6, 6, 1, 6\}$

$\hookrightarrow$ Length = 7

$x_p \Rightarrow N=3 \qquad 1 \quad 3 \quad 7 \;|\; 6 \quad 6 \quad 1 \;|\; 6$

$$\begin{array}{ccc}
1 & 3 & 7 \\
6 & 6 & 1 \\
6 & & \\
\hline
13 & 9 & 8
\end{array} \rightarrow y_p[n] = \{\underset{\uparrow}{13}, 9, 8\}$$

$$x_p[n] = \{\underset{\uparrow}{1},2,3\} \longrightarrow \boxed{h[n]=\{1,1,2,-1,2\}} \longrightarrow y_p[n] = \{\underset{\uparrow}{13},9,8\}$$

with $\underline{\underline{N=3}}$

$x[n] = \{1,2,3,1,2,3,\dots\}$ $\quad$ $y[n] = \{\underset{\uparrow}{13},9,8,13,9,8,\dots\}$

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Relationship b/w LTI systems properties and impulse response

① LTI systems with memory and without memory

$\underbrace{\text{with memory}}_{\text{Dynamic}}$ $\quad$ $\underbrace{\text{without memory}}_{\text{static}}$

$h[n] \rightarrow$ impulse response

⇒ A DTS is said to be memoryless if and only if the impulse response $\boxed{h[n] = k\delta[n]}$ $\quad k = h[0]$ ↳ constant

⇒ If a DTS has an impulse response $h[n]$ that is not identically zero for $n \neq 0$ then that system has memory.

$$h[n] \neq 0 \quad \text{for } n \neq 0$$

② Causality for LTI system

⇒ A DTS is said to be causal if and only if $h[n] = 0$ for $n < 0$

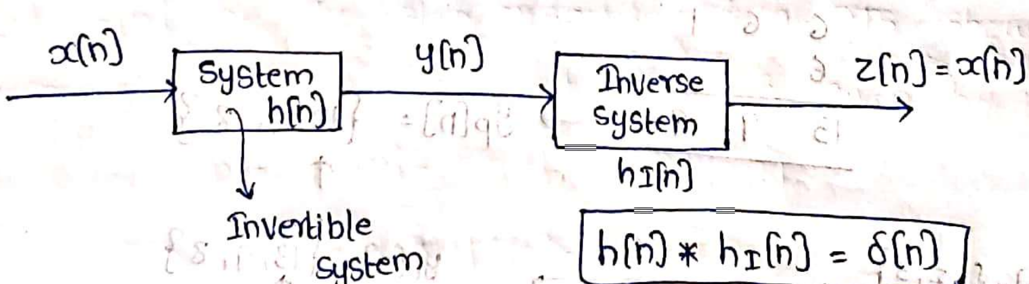
$$\boxed{h[n] = 0 \; ; \; n < 0} \quad \text{(past)}$$

③ Stability ~~of~~ an LTI system

⇒ A DTS is said to be stable if and only if its impulse response is absolutely summable.

i.e, $$\boxed{\sum_{n=-\infty}^{\infty} |h[n]| < \infty} \quad \text{(finite)}$$

④ Invertability for LTI system



$$\boxed{h[n] * h_I[n] = \delta[n]}$$

Property	conditions for impulse response
1. Memoryless	$h[n] = k\delta[n]$
2. Causality	$h[n] = 0$ for $n<0$
3. Stability	$\sum_{n=-\infty}^{\infty} \lvert h[n]\rvert < \infty$ (finite)
4. Invertibility	$h[n] * h_I[n] = \delta[n]$

Q) Consider a DTS with impulse response $h[n] = \alpha^n u[n]$. Determine whether the system is causal and/or stable.

Given, $h[n] = \alpha^n u[n] = \begin{cases} \alpha^n ; & n \geq 0 \\ 0 & n<0 \end{cases}$

Causal : $h[n] = 0$ for $n<0$ $\Rightarrow$ System is causal.

Stable : $\sum_{n=-\infty}^{\infty} |h[n]| \leq \infty \Rightarrow \sum_{n=-\infty}^{\infty} |\alpha^n u[n]|$

$$= \sum_{n=-\infty}^{\infty} |\alpha^n| \; |u[n]|$$

$$= \sum_{n=0}^{\infty} |\alpha^n| = \frac{1}{1-|\alpha|} , |\alpha|<1$$

If $|\alpha|<1$, Let $\alpha = 0.5$ so, $\frac{1}{1-|\alpha|} = \frac{1}{1-|0.5|} = 2$ (finite) (bounded)

when $|\alpha|<1$, the system will be stable.

If $|\alpha| = 1 \Rightarrow \frac{1}{1-|\alpha|} = \frac{1}{1-1} = \frac{1}{0} = \infty$ (unbounded)

when $|\alpha| = 1$, the system is unstable.

$|\alpha| < 1 \rightarrow$ Stable

$|\alpha| = 1 \rightarrow$ unstable

UNIT STEP RESPONSE OF LTI SYSTEM

$x[n] = \delta[n]$ $\quad\longrightarrow\quad$ **LTI System** $\quad\longrightarrow\quad$ $y[n] = h[n] \rightarrow$ unit impulse response

$x[n] = u[n]$ $\quad\longrightarrow\quad$ **LTI System** $\quad\longrightarrow\quad$ $y[n] = s[n] \rightarrow$ unit step response

$$y[n] = x[n] * h[n] \Rightarrow y[n] = u[n] * h[n]$$

$$= \sum_{k=-\infty}^{\infty} h[k]\, u[n-k]$$

$$u[n-k] = \begin{cases} 1 & ; k \le n \\ 0 & k > n \end{cases} \qquad = \sum_{k=-\infty}^{n} h[k]$$

$$* \quad \boxed{\therefore\ y[n] = s[n] = \sum_{k=-\infty}^{n} h[k]} \quad *$$

$$s[n] = h[n] + \underbrace{\sum_{k=-\infty}^{n-1} h[k]}_{s[n-1]}$$

$$\boxed{\begin{aligned} s[n] &= h[n] + s[n-1] \\ h[n] &= s[n] - s[n-1] \end{aligned}}$$

→ The step response of a DTS is the running sum of its impulses.

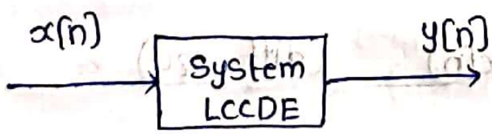
→ Conversely, the impulse response of a DTS is the first difference of its step response.

Q) Let $s[n]$ be the step response of a DTS. Show that the response $y[n]$ to the input $x[n]$ is given by

$$y[n] = [x[n] - x[n-1]] * s[n]$$

$$= x[n] * [s[n] - s[n-1]]$$

Difference Equations:- → Linear constant coefficient difference equation (LCCDE).



$$y(n) = f\{x(n], x(n-1], x(n-2], \ldots x(n-M], y(n-1], y(n-2], \ldots y(n-N]\}$$

$$y(n] = -\sum_{K=1}^{N} a_K \, y[n-K] + \sum_{l=0}^{M} b_l \, x[n-l]$$

$$\sum_{K=0}^{N} a_K \, y[n-K] = \sum_{l=0}^{M} b_l \, x[n-l]$$

⇒ Order of diff. equation : ORDER = Max [N, M]

⇒ Difference equation is said to be "NON - RECURSIVE"

if $a_k = 0$ for $k \neq 0$

Output y(n) is only defined in terms of the input signals.

⇒ NON- RECURSIVE systems are said to be FINITE LENGTH IMPULSE RESPONSE (FIR) system.

⇒ A difference equation is said to be "RECURSIVE" when y(n) includes weighted versions of the earlier outputs i.e., y(n) may be defined in terms of y(n-1), y(n-2), y(n-3), ...

⇒ RECURSIVE systems are said to be INFINITE LENGTH IMPULSE RESPONSE (IIR) system.

Causal LTI systems:-

→ Use difference equation as algorithm to find y(n) from x(n)

→ Initial conditions must be initial rest.

If $x(n) = 0$ for $n < n_0$ then $y(n) = 0$ for $n < n_0$

Then we call it as causal LTI system.

Ex:- $y(n) - a\,y(n-1) = b x(n)$ (diff equ)

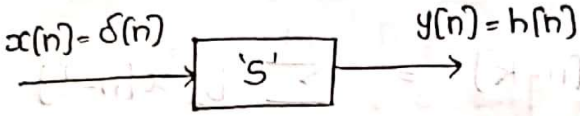
$y(n) - a y(n-1) = b x(n)$

This is REURSIVE diff equ ∵ Bcoz y(n) is defined in terms of y(n-1)

ORDER: Max [N, M] ⇒ $\left.\begin{matrix} M = 0 \\ N = 1 \end{matrix}\right\}$ Order = 1

Note:- Max delay which exists in a given diff equation = order.

Q) Find $h[n]$ for causal LTI system from diff equation:



$$y(n) = a y(n-1) + b x(n)$$

$$h(n) = a h(n-1) + b\delta(n)$$

($h(n) = 0$, $n<0$ for causal)

Initial rest $\Rightarrow$ $h[n]=0$ for $n<0$ bcoz $x[n]=0$ for $n<0$.

$n=0 \Rightarrow h[0] = a\underbrace{h[0-1]}_{=0} + b\underbrace{\delta[0]}_{=1}$

$h[0] = 0 + b = b$

$n=1 \Rightarrow h[1] = a\underbrace{h[1-1]}_{=b} + b\underbrace{\delta[1]}_{=0}$

$h[1] = ab$

$n=2 \Rightarrow h[2] = a\underbrace{h[2-1]}_{=ab} + b\underbrace{\delta[2]}_{=0}$

$h[2] = a^2 b$

$$\therefore h[n] = \begin{cases} a^n b & n \geq 0 \\ 0 & n<0 \end{cases}$$

Q) $y(n) = \frac{1}{2}x[n-1] + \frac{1}{2}x(n-2)$

This is NON-RECURSIVE diff. equation : Bcoz $y(n)$ depends only on values of input

ORDER: Max $(1,2) = 2$ (max delay)

Use diff equ to find $h(n)$ for causal LTI system;

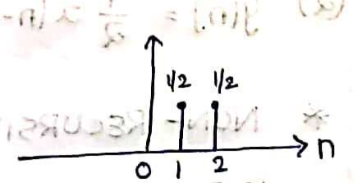
$$y[n] = \frac{1}{2}x[n-1] + \frac{1}{2}x[n-2]$$

$x(n) \rightarrow y(n)$

$\delta(n) \rightarrow h(n)$

$$h[n] = \frac{1}{2}\delta(n-1) + \frac{1}{2}\delta[n-2]$$

$$h[n] = \{\underset{\uparrow}{0}, \tfrac{1}{2}, \tfrac{1}{2}\}$$

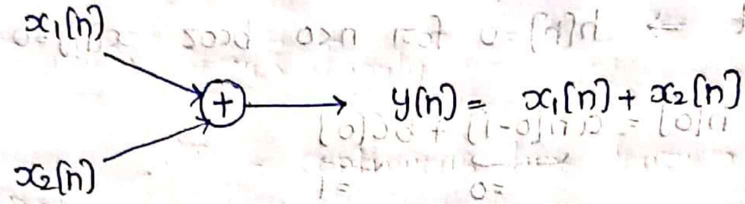


BLOCK DIAGRAM:-

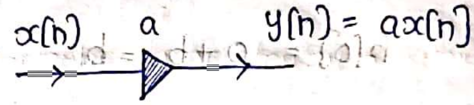
LCCDE : $\sum_{k=0}^{N} a_k\, y[n-k] = \sum_{l=0}^{M} b_l\, x[n-l]$

$\sum$ → sum, a_k → Scaling element, $y[n-k]$ → delay; $\sum$ → Sum, b_l → Scaling element, $x[n-l]$ → delay

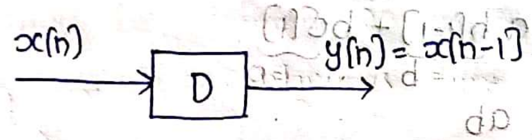
① **ADDER**:



② **Multiplier / scalar / Gain** :



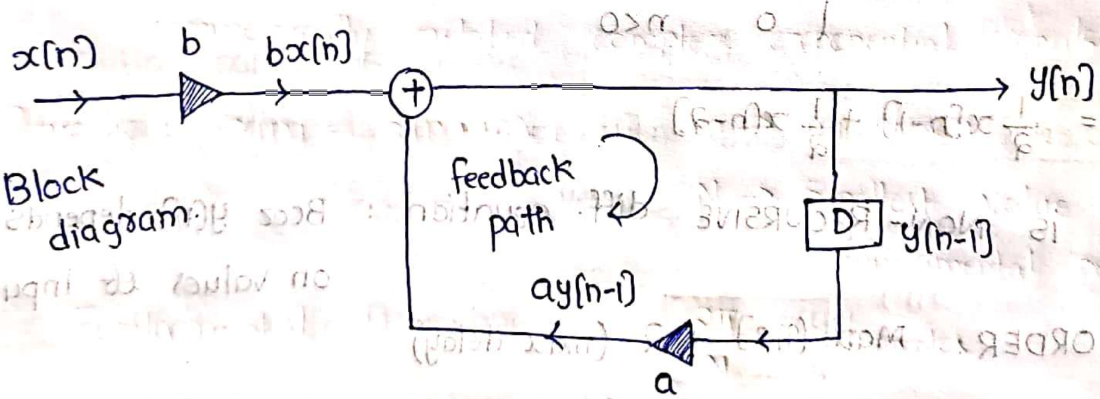
③ **Delay** :



Ex:- (1) $y(n) - ay(n-1) = bx[n]$

$y(n) = ay(n-1) + bx(n)$ $\qquad x(n) \longrightarrow y(n)$

$h(n) = ah(n-1) + b\delta(n)$ $\qquad \delta(n) \longrightarrow h(n)$



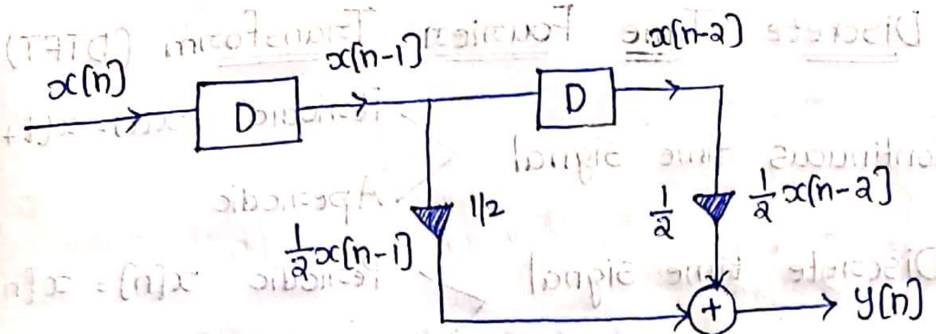
Block diagram

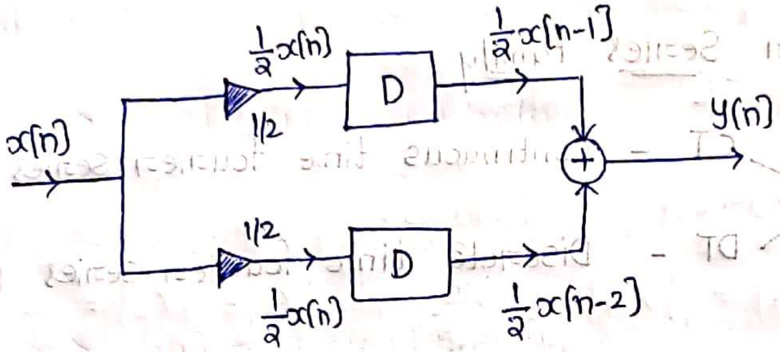
* When feedback path occurs, it is a RECURSIVE system & IIR system.

(2) $y(n) = \frac{1}{2} x(n-1) + \frac{1}{2} x(n-2)$

* NON-RECURSIVE system as it has no feedback path.

FIR system.


x(n)
D
x(n-1)
D
x(n-2)
1/2
1/2 x(n-1)
1/2
1/2 x(n-2)
+
y(n)


x(n)
1/2 x(n)
1/2
D
1/2 x(n-1)
+
y(n)
1/2
1/2 x(n)
D
1/2 x(n-2)