UNIT - 3

01-02-2023

* Taylor series :

→ suppose 'f(x)' be an 'infinitely many times' differentiable function in the neighbourhood of 'x_0' then

$$\boxed{f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + (x-x_0)^2 + \frac{f'''(x_0)}{3!}(x-x_0)^3}$$

Ex :

$f(x) = e^x$ $f(0) = 1$

$f'(x) = e^x$ $f'(0) = 1$

$f''(x) = e^x$ $f''(0) = 1$

$f'''(x) = e^x$ $f'''(0) = 1$

$\therefore e^x = 1 + 1(x) + \frac{1}{2!}x^2 + \frac{1}{3!}x^3$.

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \text{---} + \frac{x^n}{n!}$

Ex :

Sinx :

$P_1(x) :- x$

$P_2(x) :- x - \frac{x^3}{3!}$

$P_3(x) :- x - \frac{x^3}{3!} + \frac{x^5}{5!}$

$P_4(x) :- x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$

$P_5(x) :- x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!}$

$P_6(x) :- x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!}$

$P_7(x) :- x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \frac{x^{13}}{13!}$

$P_8(x) :- x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \frac{x^{13}}{13!} \text{---} + \frac{x^{202300}}{(202300)!}$

$P_9(x) :- x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \frac{x^{13}}{13!} \text{---} + \frac{x^{(\text{Your mobile})}}{(\text{Your mobile})!}$

* Ex:- $f(x)=x^2+5x+6$, about $x=2$.

$f(x)=f(2)+\frac{f'(2)}{1!}(x-2)+\frac{f''(2)}{2!}(x-2)^2+\frac{f'''(2)}{3!}(x-2)^3+$ ———

$\therefore f(x)=20+9(x-2)+\frac{2}{2!}(x-2)^2+0$ ———

* Taylor series for two variable function:-

→ Let $f(x,y)$ be a two variable function x and y. whose domain is $D\subseteq\mathbb{R}^2$ and which has continuous first order partial derivatives upto $(n+1)^{th}$ order in the neighbourhood of $(x_0,y_0)\in D$ then at any point (x_0+h, y_0+k) is belongs to neighbourhood of (x_0,y_0) then

$$\boxed{\begin{aligned}f(x_0+h, y_0+k)&=f(x_0,y_0)+\left(h\cdot\frac{\partial}{\partial x}+k\cdot\frac{\partial}{\partial y}\right)\cdot f(x_0,y_0)\\&+\frac{1}{2!}\left(h\cdot\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^2\cdot f(x_0,y_0)+\frac{1}{3!}\times\\&\left(h\cdot\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^3 f(x_0,y_0)+\text{———}\\&\text{———}+\frac{1}{n!}\left(h\cdot\frac{\partial}{\partial x}+k\cdot\frac{\partial}{\partial y}\right)^n f(x_0,y_0)+R_n.\end{aligned}}$$

where, $R_n=\frac{1}{(n+1)!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^{n+1}f(x_0,y_0)$

Suppose, $x_0+h=x$, $y_0+k=y$ $\Rightarrow h=x-x_0$, $\Rightarrow k=y-y_0$.

Linear approximation

$$\rightarrow f(x,y)=f(x_0,y_0)+\left((x-x_0)\cdot\frac{\partial f}{\partial x}+(y-y_0)\cdot\frac{\partial f}{\partial y}\right)+\left((x-x_0)^2\cdot\frac{\partial^2 f}{\partial x^2}+2(x-x_0)\cdot(y-y_0)\cdot\frac{\partial^2 f}{\partial y\,\partial x}+(y-y_0)^2\frac{\partial^2 f}{\partial y^2}\right)+\text{———}$$
$$\text{———}+\frac{1}{n!}\left((x-x_0)^n\frac{\partial^n f}{\partial x^n}+{}^nC_1(x-x_0)^{n-1}(y-y_0)\cdot\frac{\partial^n f}{\partial x\,\partial y}+\text{———}(y-y_0)^n\frac{\partial^n f}{\partial y^n}\right)+R_n.$$

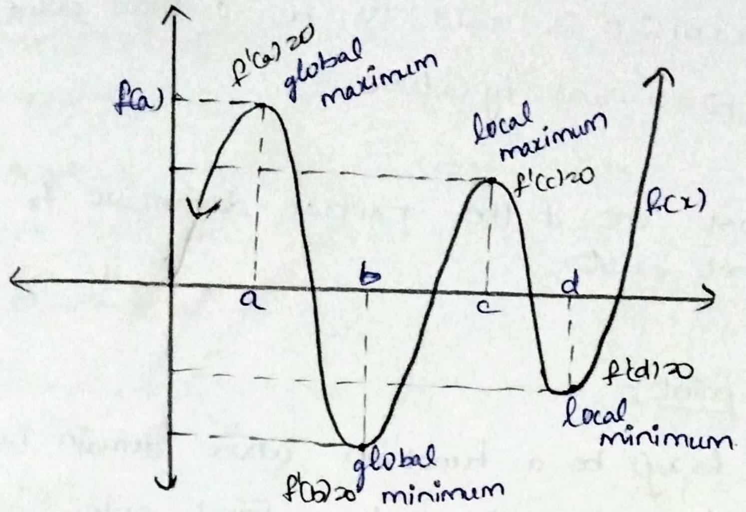
$\therefore f(x,y)=P_n(x,y)+R_n(x,y)$.

$P_n(x,y)$ ↓ polynomial of x,y with degree 'n'

$R_n(x,y)$ ↓ Remainder

$\therefore R_n=\frac{1}{(n+1)!}\left[(x-x_0)\frac{\partial}{\partial x}+(y-y_0)\frac{\partial}{\partial y}\right]^{n+1}\cdot f(x,y).$

* Applications of Derivatives:-



g_{max} :- $f(x) \leq f(a)\ \forall\ x \in$ Domain.

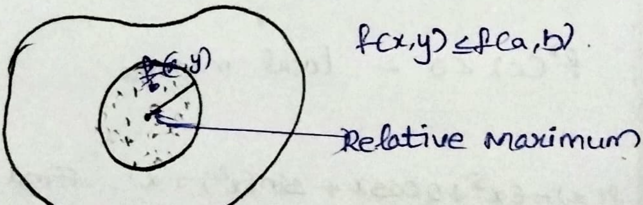
l_{max} :- $f(x) \leq f(c)\ \forall\ x \in N_\epsilon(c)$.

g_{min} :- $f(x) \geq f(b)\ \forall\ x \in$ Domain.

l_{min} :- $f(x) \geq f(d)\ \forall\ x \in N_\epsilon(d)$.

* Relative & Absolute Maximum of two variable function:-

→ Let '$f(x,y)$' be a (~~continuous~~) function in the domain D. suppose/ Let '$(a,b) \in D$'. suppose '$f(x,y) \leq f(a,b)$' $\forall$ $(x,y) \in$ some neighbourhood of '(a,b)'. then we say that $f(x,y)$ has 'Relative maximum' at (a,b). and $f(a,b)$ is called 'Relative Maximum value'



$f(x,y) \leq f(a,b)$.

→ if $f(x,y) \leq f(a,b)\ \forall\ (x,y) \in D$ then $f(x,y)$ has 'Absolute maximum' at (x,y).

* Relative minimum & Absolute minimum:-

→ Let $f(x,y)$ be a function where Domain $D \subseteq \mathbb{R}^2$ & $(a,b) \in D$, if $f(x,y) \geq f(a,b)\ \forall\ (x,y) \in$ some neighbourhood of (a,b) then $f(x,y)$ has relative minimum at (a,b).

Absolute minimum:-

→ If $f(x,y) \geq f(a,b)\ \forall\ (x,y) \in D$. then $f(x,y)$ has Absolute minimum at (a,b).

* critical point :-

→ Suppose $f(x,y)$ be a function where Domain is D, Let $(a,b) \in D$ is said to be 'critical point' if '$f_x(a,b)=0$' and '$f_y(a,b)=0$'

(or)

Atleast one of the partial derivative 'f_x' (or) 'f_y' doesnot exist.

* Saddle point :-

→ Let $f(x,y)$ be a function whose domain is D. & $(a,b) \in D$, $f(x,y)$ has First order partial derivatives (a,b) and $f_x(a,b)=0$ & $f_y(a,b)=1$ [i.e (a,b) is critical point] is said to be 'saddle point' if every neighbourhood of a,b contains atleast two points (x_1,y_1), (x_2,y_2) such that "$f(x_1,y_1) \geq f(a,b)$" & $f(x_2,y_2) \leq f(a,b)$."

- i.e, $f(x,y)$ has doesnot have 'relative maximum' & 'relative minimum' in every neighbourhood of (a,b).

Second Derivative Test :-

$f''(c) > 0$ - local maxima.

$f''(c) < 0$ - local minima.

* $f(x) = 6x^5 + 2\cos x + \sin(x^2) + x$. Find the critical points of $f(x)$?

* Algorithm to find Relative Extrema :-

→ Let $f(x,y)$ be a function which has partial derivatives upto 2nd order.

step-1 :- Find critical point of $f(x,y)$ ~~using~~ such that $f_x(a,b)=0$ & $f_y(a,b)=0$. ⟶ ①

step-2 :- Find $D(x,y) = f_{xx}(x,y) \cdot f_{yy}(x,y) - f_{xy}(x,y)$.

step-3 :- classification :-

step-3 : ~~calgulat~~ classification:

→ suppose (a,b) be critical point of $f(x,y)$. [∵ From (1)]

ⓐ '$D(a,b)>0$' & '$f_{xx}(a,b)<0$' then $f(x,y)$ has "relative maximum" at (a,b)

ⓑ '$D(a,b)>0$' and '$f_{xx}(a,b)>0$' then $f(x,y)$ has "relative minimum" at (a,b)

ⓒ '$D(a,b)<0$' ~~an~~, then $f(x,y)$ has 'saddle point'.

ⓓ '$D(a,b)=0$' then the test can't determine anything.

* Flowchart :