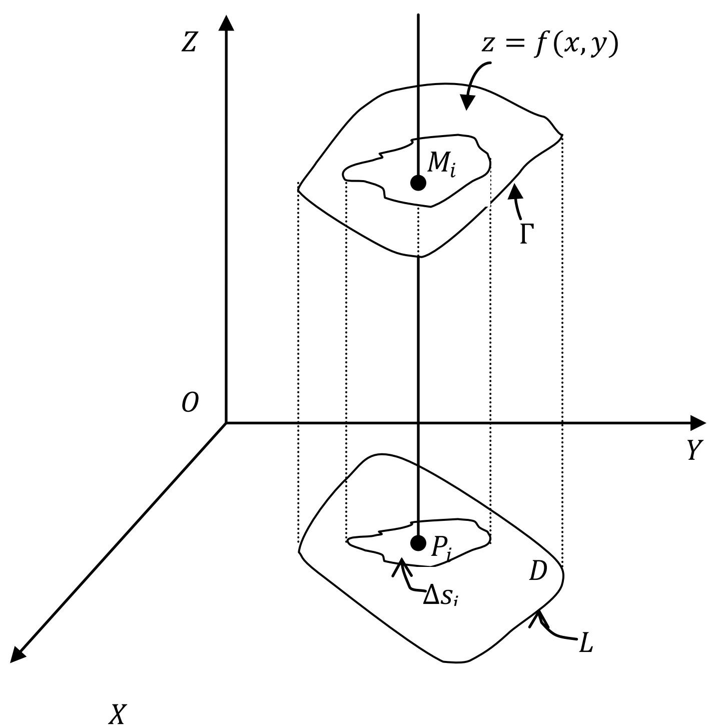
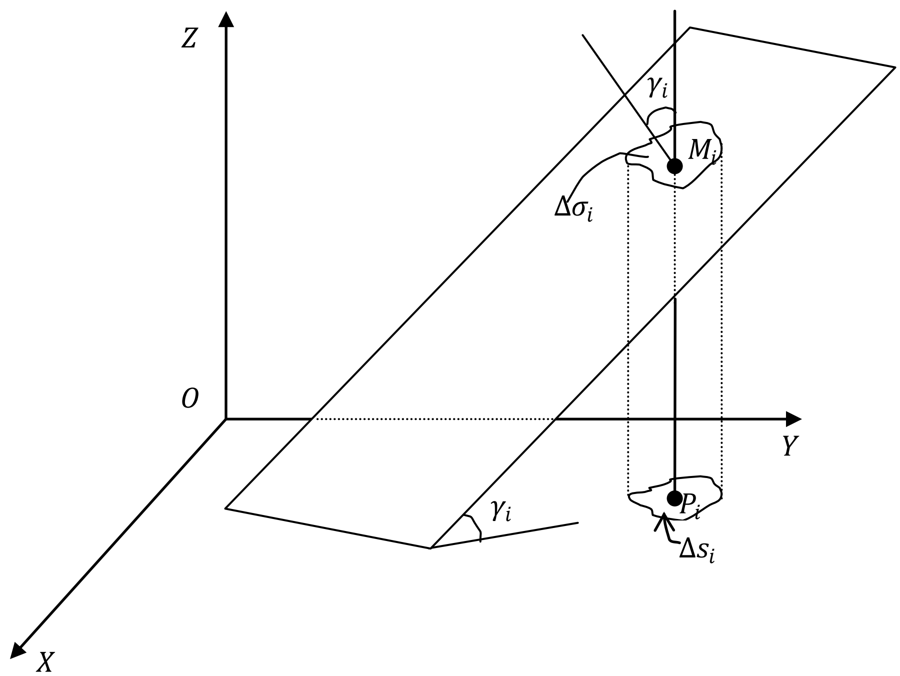
2.7

SURFACE AREA AND VOLUME BY USING DOUBLE INTEGRALS

COMPUTING THE AREA OF A SURFACE

Let it be required to compute the area of a surface bounded by a curve Γ (given in the figure below); the surface is defined by the equation $z = f(x, y)$, where the function $f(x, y)$ is continuous and has continuous partial derivatives. Denote the projection of Γ on the $XY-$plane by L. Denote by D the domain on the $XY-$plane bounded by the curve L.





In arbitrary fashion, divide D into n elementary subdomains $\Delta s_1, \Delta s_2, \ldots \Delta s_n$. In each subdomain Δs_i take a point $P_i(\xi_i, \eta_i)$. To the point P_i there will correspond, on the surface, a point

$$M_i[\xi_i, \eta_i, f(\xi_i, \eta_i)]$$

Through M_i draw a tangent plane to the surface. Its equation is of the form

$$z - z_i = f'_x(\xi_i, \eta_i)(x - \xi_i) + f'_y(\xi_i, \eta_i)(y - \eta_i) \tag{1}$$

In this plane, pick out a subdomain $\Delta\sigma_i$ which is projected onto the $XY-$plane in the form of a subdomain Δs_i. Consider the sum of the sub domains $\Delta\sigma_i$:

$$\sum_{i=1}^{n} \Delta\sigma_i$$

We shall call the limit σ of this sum, when the greatest of the diameters of the subdomains $\Delta\sigma_i$ approaches zero, the area of the surface; that is, by definition we set

$$\sigma = \lim_{diam\ \Delta\sigma_i \to 0} \sum_{i=1}^{n} \Delta\sigma_i \qquad (2)$$

Now let us calculate the area of the surface. Denote by γ_i the angle between the tangent plane and the $XY-$plane. Using a familiar formula of analytic geometry we can write

$$\Delta s_i = \Delta\sigma_i cos\gamma_i$$

or

$$\Delta\sigma_i = \frac{\Delta s_i}{cos\,\gamma_i} \qquad (3)$$

The angle γ_i is at the same time the angle between the $Z-$axis and the perpendicular to the plane (1). Therefore, by equation (1) and the formula of analytic geometry we have

$$cos\gamma_i = \frac{1}{\sqrt{1 + f'^2_x(\xi_i, \eta_i) + f'^2_y(\xi_i, \eta_i)}}$$

Hence,

$$\Delta\sigma_i = \sqrt{1 + f'^2_x(\xi_i, \eta_i) + f'^2_y(\xi_i, \eta_i)}\, \Delta s_i$$

Putting this expression into formula (2), we get

$$\sigma = \lim_{diam\ \Delta s_i \to 0} \sum_{i=1}^{n} \sqrt{1 + f'^2_x(\xi_i, \eta_i) + f'^2_y(\xi_i, \eta_i)}\, \Delta s_i$$

Since the limit of the integral sum on the right side of the last equation is, by definition, the double integral $\iint_D \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dxdy$ we finally get

$$\sigma = \iint_D \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dxdy \tag{4}$$

This is the formula use to compute the area of the surface $z = f(x, y)$.

If the equation of the surface is given in the form

$x = \mu(y, z)$ or in the form $y = \chi(x, z)$

then the corresponding formulas for calculating the surface area are of the form

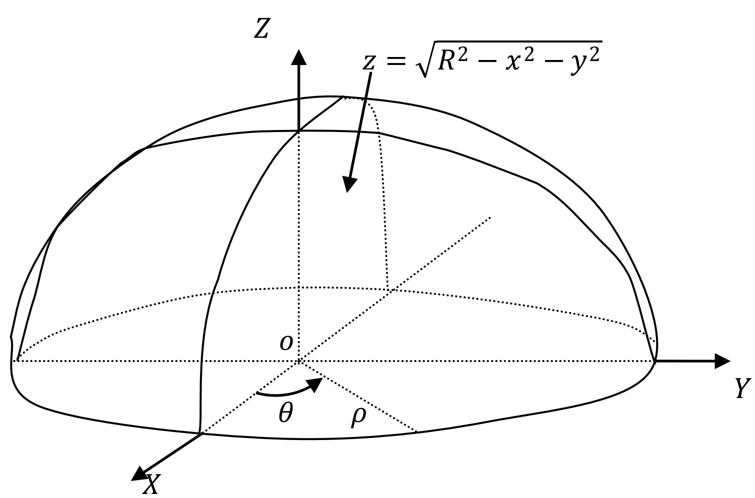
$$\sigma = \iint_{D'} \sqrt{1+\left(\frac{\partial x}{\partial y}\right)^2+\left(\frac{\partial x}{\partial z}\right)^2}\, dydz \qquad (4')$$

$$\sigma = \iint_{D''} \sqrt{1+\left(\frac{\partial y}{\partial x}\right)^2+\left(\frac{\partial y}{\partial z}\right)^2}\, dxdz \qquad (4'')$$

where D' and D'' are the domains in the YZ-plane and the XZ-plane in which the given surface is projected.

Example: Compute the surface area σ of the sphere $x^2+y^2+z^2=R^2$

Solution: Compute the surface are of the upper half of the sphere $z=\sqrt{R^2-x^2-y^2}$



In this case

$$\frac{\partial z}{\partial x} = -\frac{x}{\sqrt{R^2 - x^2 - y^2}}$$

$$\frac{\partial z}{\partial y} = -\frac{y}{\sqrt{R^2 - x^2 - y^2}}$$

Hence,

$$\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} = \sqrt{\frac{R^2}{R^2 - x^2 - y^2}} = \frac{R}{\sqrt{R^2 - x^2 - y^2}}$$

The domain of integration is defined by the condition

Thus, by formula (4) we will have

$$\frac{1}{2}\sigma = \int_{-R}^{R} \left(\int_{-\sqrt{R^2 - x^2}}^{\sqrt{R^2 - x^2}} \frac{R}{\sqrt{R^2 - x^2 - y^2}} dy \right) dx$$

To compute the double integral obtain let us make the transformation to polar coordinates. In polar coordinates the boundary of the domain of integration is determined by the equation $\rho = R$. Hence,

$$\sigma = 2\int_{0}^{2\pi} \left(\int_{0}^{R} \frac{R}{\sqrt{R^2 - \rho^2}} \rho d\rho \right) d\theta = 2R \int_{0}^{2\pi} \left[-\sqrt{R^2 - \rho^2} \right]_0^R d\theta$$

$$= 2R \int_{0}^{2\pi} R d\theta = 4\pi R^2 .$$

Computing the Volume of a Solid

Recall that

1. If $f(x, y) = 1$, then $\iint_R dxdy$ gives the area A of the region R.
2. If $z = f(x, y)$ is a surface, then
$$\iint_R zdxdy \text{ or } \iint_R f(x, y)dxdy$$
gives the volume of the region beneath the surface $z = f(x, y)$ and above the XY- plane.

Example: Evaluate the volume of the sphere

$x^2 + y^2 + z^2 = a^2$.

Solution: The given sphere is $z = \sqrt{a^2 - x^2 - y^2}$

The volume of the upper half of the sphere is

$$\iint_{x^2+y^2\le a^2} \sqrt{a^2 - x^2 - y^2}dx\,dy$$

By changing to polar coordinates.

i.e substitute $x = r\cos\theta\,, y = r\sin\theta\,, dx\,dy = r\,dr\,d\theta$

$$\iint_{x^2+y^2\le a^2} \sqrt{a^2 - x^2 - y^2}dx\,dy = \int_0^{2\pi}\int_0^a \sqrt{a^2 - r^2}\; r\,dr\,d\theta$$

$$= \int_0^{2\pi}\left[\left(-\frac{1}{2}\right)\int_0^a \sqrt{a^2 - r^2}.d(-r)^2\right]d\theta$$

$$= \int_0^{2\pi}\frac{1}{3}a^3\,d\theta = \frac{2}{3}\pi a^3.$$

Therefore the volume of the sphere is $2\left(\left(\frac{2}{3}\pi a^3\right)\right) = \frac{4}{3}\pi a^3$.

Problem 1: Compute the area of that part of the surface of the cone $x^2+y^2=z^2$ which is cut out by the cylinder $x^2+y^2=2ax$.

Solution: The equation of the surface of the upper half of the cone is $z=\sqrt{x^2+y^2}$

$$\therefore \frac{\partial z}{\partial x}=\frac{x}{\sqrt{x^2+y^2}}, \qquad \therefore \frac{\partial z}{\partial y}=\frac{y}{\sqrt{x^2+y^2}}$$

$\therefore$ The domain of integration is defined by

$$x^2+y^2\le 2ax \Rightarrow (x-a)^2+y^2\le a^2$$

$\therefore$ Surface area of upper half cone

$$=\iint_D \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dydx$$

$$\text{Total surface area} = 2\int_0^{2a}\int_{-\sqrt{a^2-(x-a)^2}}^{\sqrt{a^2-(x-a)^2}} \sqrt{1+\frac{x^2}{x^2+y^2}+\frac{y^2}{x^2+y^2}}\,dydx$$

$$=2\int_0^{2a}\int_{-\sqrt{a^2-(x-a)^2}}^{\sqrt{a^2-(x-a)^2}} \sqrt{\frac{2(x^2+y^2)}{x^2+y^2}}\,dydx$$

$$=4\int_0^{2a}\int_0^{\sqrt{a^2-(x-a)^2}} \sqrt{2}\,dydx$$

$$= 4\sqrt{2}\int_0^{2a} [y]_0^{\sqrt{a^2-(x-a)^2}}\, dx$$

$$= 4\sqrt{2}\int_0^{2a} \sqrt{a^2-(x-a)^2}\,dx$$

$$= 4\sqrt{2}\left[\frac{x-a}{2}\sqrt{a^2-(x-a)^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{x-a}{a}\right)\right]_0^{2a}$$

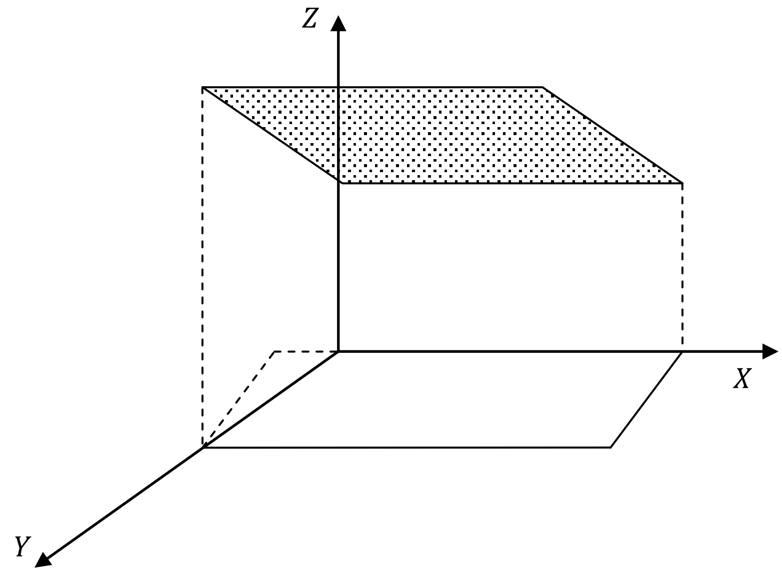
$$= 4\sqrt{2}\left[\frac{a^2}{2}\sin^{-1}1-\frac{a^2}{2}\sin^{-1}(-1)\right]$$

$$= 4\sqrt{2}\,\frac{a^2}{2}\left[\frac{\pi}{2}+\frac{\pi}{2}\right]$$

$$= 2\sqrt{2}\pi a^2 .$$

Problem 2: Find the surface area of $2x + 3y - z = 1$ in the region $[0,1] \times [0,1]$.

Solution:



The equation of the surface has the form

$z = 1 + 2x + 3y$

$$\therefore \frac{\partial z}{\partial x} = 2, \qquad \therefore \frac{\partial z}{\partial y} = 3$$

The region $D = [0,1] \times [0,1]$

$$\therefore \text{Surface area} = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dydx$$

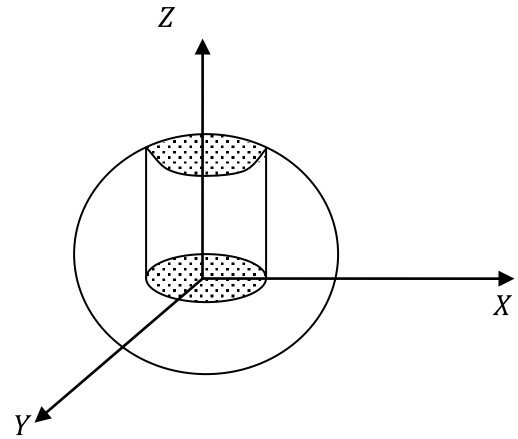
$$= \int_0^1 \int_0^1 \sqrt{1+4+9}\, dydx$$

$$= \int_0^1 \int_0^1 \sqrt{14}\, dydx$$

$$= \sqrt{14}.$$

Problem 3: Find the surface area of the portion of the unit sphere above $z = \frac{4}{5}$

Solution:



Unit sphere is $x^2 + y^2 + z^2 = 1$

$$\Rightarrow z = \sqrt{1 - x^2 - y^2}$$

$$\frac{\partial z}{\partial x} = -\frac{x}{\sqrt{1-x^2-y^2}}, \quad \frac{\partial z}{\partial y} = -\frac{y}{\sqrt{1-x^2-y^2}}$$

$$\frac{4}{5} = \sqrt{1 - x^2 - y^2} \Rightarrow \frac{16}{25} = 1 - x^2 - y^2 \Rightarrow x^2 + y^2 = \frac{9}{25}$$

Circle of radius is $\frac{3}{5}$

We have to find surface area of $z = \sqrt{1 - x^2 - y^2}$ over $x^2 + y^2 = \frac{9}{25}$

Domain of the radius is $\frac{3}{5}$

$$\therefore \text{ Surface area } = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dydx$$

$$= \iint_D \sqrt{\frac{1}{1 - x^2 - y^2}}\, dydx$$

Transformation to polar co coordinates. In polar coordinates the boundary of the domain of integration is determined by the equation

$$r = \frac{3}{5}$$

Let $x = r cos\theta, \quad y = r sin\theta$

$\therefore$ Surface area $= \int_0^{2\pi} \int_0^{\frac{3}{5}} \frac{1}{\sqrt{1-r^2}} r dr d\theta$

$$= \int_0^{2\pi} \left[-\sqrt{1 - r^2}\right]_0^{\frac{3}{5}} d\theta$$

$$= \int_0^{2\pi} \left(-\sqrt{1 - \frac{9}{25}} + 1\right) d\theta$$

$$= \int_0^{2\pi} \left(-\frac{4}{5} + 1\right) d\theta$$

$$= \int_0^{2\pi} \frac{1}{5} d\theta$$

$$= \frac{1}{5} [\theta]_0^{2\pi}$$

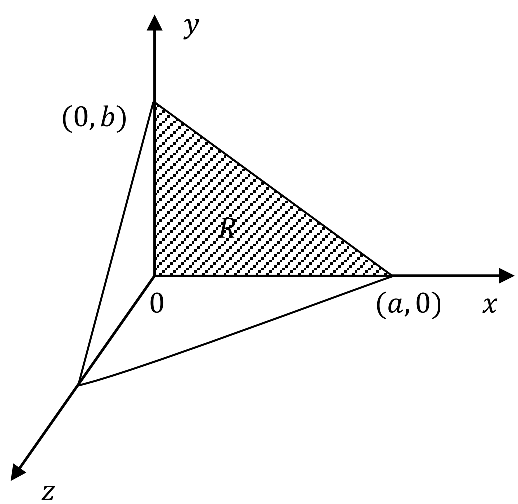
$$= \frac{2\pi}{5}.$$

Problem 4: Find the volume of the tetrahedron bounded by the coordinate surfaces $x = 0, y = 0$ and $z = 0$ and the plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

Solution:

The volume of the tetrahedron $(V) = \iint_R z\,dx\,dy$

$$V = \int_0^a \int_0^{b-\frac{bx}{a}} c\left(1 - \frac{y}{b} - \frac{x}{a}\right) dy\,dx$$



$$= c\int_0^a \left[y - \frac{y^2}{2b} - \frac{xy}{a} \right]_0^{b-\frac{bx}{a}} dx$$

$$= c\int_0^a \left(\frac{bx^2}{2a^2} - \frac{bx}{a} + \frac{b}{2} \right) dx$$

$$= c\left[\frac{bx^3}{6a^2} - \frac{bx^2}{2a} + \frac{b}{2}x \right]_0^a = \frac{abc}{6}.$$

Problem 5: A circular hole of a radius b is made centrally through a sphere of radius a. Find the volume of the remaining of the sphere.

Solution:

Let the centre of the sphere be at the origin and let the axis of the hole be along the z-axis. The volume V of the sphere is $\frac{4}{3}\pi a^3$ and that of the circular hole is obtained as follows.

Volume of the upper-half of the hole $= \iint_R f(x,y)dxdy$

$$= \iint_R zdxdy$$

where z is obtained from the equation $x^2+y^2+z^2=a^2$ and R is the circle in the $XY-$plane.

i.e $\quad x^2+y^2=b^2$

$\therefore$The volume V_1 of the circular hole is

$$V_1 = 2\iint_R \sqrt{a^2-x^2-y^2}\,dxdy$$

where R is $x^2+y^2=b^2$ changing into polar coordinates

$$\therefore V_1 = 2\int_0^{2\pi}\int_0^b \sqrt{a^2-r^2}\,rdrd\theta = \int_0^{2\pi}\left[\frac{(a^2-r^2)^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^b d\theta$$

$$= \frac{-2}{3} \int_0^{2\pi} \left[(a^2 - b^2)^{\frac{3}{2}} - a^3 \right] d\theta$$

$$= \frac{2}{3} \int_0^{2\pi} \left[a^3 - (a^2 - b^2)^{\frac{3}{2}} \right] d\theta$$

$$= \frac{2}{3} \left[a^3 - (a^2 - b^2)^{\frac{3}{2}} \right] [\theta]_0^{2\pi}$$

$$= \frac{4\pi}{3} \left[a^3 - (a^2 - b^2)^{\frac{3}{2}} \right]$$

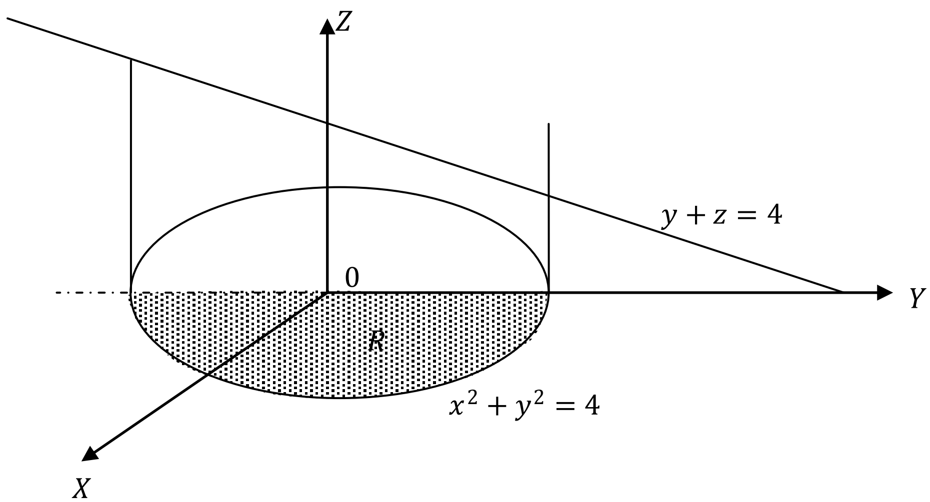
Volume of the remaining portion $= V - V_1$

$$= \frac{4}{3}\pi a^3 - \frac{4\pi}{3} \left[a^3 - (a^2 - b^2)^{\frac{3}{2}} \right]$$

$$= \frac{4\pi}{3} (a^2 - b^2)^{\frac{3}{2}}.$$

Problem 6: Find the volume bounded by the cylinder $x^2+y^2=4$, $y+z=4$ and $z=0$.

Solution:



The volume V of the plane $y+z=4$ and $z=0$ is

$$V=\iint_R z\,dx\,dy$$

$$=\iint_R (4-y)\,dx\,dy$$

where R is bounded by the $x^2+y^2=4$

$$\therefore V=\int_{-2}^{2}\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}}(4-y)\,dx\,dy$$

$$=\int_{-2}^{2}(4-y)\left[x\right]_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}}dy$$

$$= \int_{-2}^{2} (4-y).2\sqrt{4-y^2}\,dy$$

$$= 2\int_{-2}^{2} 4\sqrt{4-y^2}\,dy - 2\int_{-2}^{2} y\sqrt{4-y^2}\,dy$$

$$= 16\int_{0}^{2} \sqrt{4-y^2}\,dy - 0$$ ($\because y\sqrt{4-y^2}$ is odd function)

$$= 16\left[\frac{y}{2}\sqrt{4-y^2} + 2\sin^{-1}\frac{y}{2}\right]_0^2$$

$$= 16\left[2\sin^{-1}1\right] = 32\cdot\frac{\pi}{2} = 16\pi\,.$$

Exercise

1. Compute the area of that part of the plane $x + y + z = 2a$. Which lies in the first octant and is bounded by the cylinder $x^2 + y^2 = a^2$.
2. Compute the area of that part of the square of the cone $x^2 + y^2 = z^2$ which is cut by the cylinder $x^2 + y^2 = 2ax$.
3. Find the surface area of a solid that is the common part of two cylinders $x^2 + y^2 = a^2,\ y^2 + z^2 = a^2$.
4. Compute the volumes of solids bounded by the coordinate planes, the plane $2x + 3y - 12 = 0$ and the cylinder $z = \frac{1}{2}y^2$.
5. Compute the volumes of solids bounded by the following surfaces:
   a) $z = 0, x^2 + y^2 = 1, x + y + z = 3$.
   b) $x^2 + y^2 - 2ax = 0, z = 0, x^2 + y^2 = z^2$.
6. The base of a solid is the region in $XY -$ plane. That is bounded by the circle $x^2 + y^2 = a^2$. While the top of the solid is bounded by the paraboloid $az = x^2 + y^2$. Find the volume.
7. Find the volume common to the cylinders $x^2 + y^2 = a^2$ and $x^2 + z^2 = a^2$.

Answers

1. $\frac{\sqrt{3}}{4}\pi a^2$
2. $2\sqrt{2}\pi a^2$

3. $8a^2$

4. 16

5.

   a) 3π

   b) $\frac{32}{9}a^3$

6. $\frac{1}{2}\pi a^3$

7. $\frac{16a^3}{3}$