Operations Research

* operations research is "a scientific approach to decision making, which seeks to determine how best to design and operate a system, under conditions required the allocation of scarce resources.
* provides a set of algorithms that act as tools for effective problem solving and decision making.
* it is a scientific approach to decision making that seeks to determine how best to operate a system under conditions of allocating scarce resources

Topics covered:

1. Linear programming - Formulations.
2. Linear programming - Solution.
3. Duality and sensitivity analysis
4. Transportation problem.
5. Assignment problem.
6. Dynamic programming.
7. Deterministic inventory Models.

1. Linear Programming

Formulations.

example1 :-

- Consider a Small manufacturer making 2 products A and B.

- Two Resources R_1 and R_2 are required to make these products.
- Each unit of product A requires 1 unit of R_1 and 3 units of R_2.
- each unit of product B requires 1 unit of R_1 and 2 units of R_2.
- The manufacturer has 5 units of R_1 and 12 units of R_2 available.
- The manufacturer also makes a profit of Rs 6 per unit of product A sold and Rs 5 per unit of product B sold.

sol

	R_1	R_2	profit
A	1	3	Rs 6
B	1	2	Rs 5
	5	12	

let x – Number of units of A produced

y – Number of units of B produced.

This is called formulation

profit Maximize $\rightarrow Z = 6x + 5y$.

$x + y \leq 5$

$3x + 2y \leq 12$.

$x, y \geq 0$ (a person can't produce –ve quantity of both x & y)

Terminology :-

- The problem variables x_1 and x_2 (x, y) are called decision variables and they represent the solution or the output decision from the problem.

- The profit function that the manufacturer wishes to increase, represents the objective of making the decisions on the production quantities and is called the objective function.

$$Z_{max} = 6x + 5Y \rightarrow OF.$$

- The conditions matching the resource availability and resource requirement are called constraints.
  → These usually limit (or restrict) the values the decision variables can take.

$$\left.\begin{array}{l} x+Y \le 5 \\ 3x+2Y \le 12 \end{array}\right\} C.S.t.$$

- We have also explicitly stated that the decision variable should take non negative values.
  → This is true for all linear programming problems.
  → This is called non negativity restriction.

- The problem that we have written down in algebraic form represents the mathematical model of the given system and is called the problem Formulation.

* The problem Formulation has the following steps:

1. identifying the decision variables.
2. Writing the objective function.
3. Writing the constraints.
4. Writing the non-negative restrictions

* in the above formulation, the objective function & the constraints are linear.
  - therefore the model that we formulated is a linear programming problem.

* A linear programming has:-
  - a linear objective function.
  - linear constraints &.
  - the non negative constraints on all decision variables.

Example 2 :- production planning problem

- Let us consider a company making a single product.
- The estimated demand for the product for the next four months are 1000, 800, 1200, 900 respectively.
- The company has a regular time capacity of 800 per month and an over time capacity of 200 per month.
- The cost of regular time production is RS 20 per unit and the cost of over-time production is RS 25 per unit.
- The company can carry inventory to the next month and the holding cost is RS 3/unit/month.

. The demand has to be met every month. Formulate a linear programming problem for the above situation.

Sol 1000

800

1200

900

Decision Variables:

Let x_j be the quantity produced using regular time production in Month j.

Let y_j be the quantity produced using over time production in month j.

Let I_j be the quantity carried at the end of month j to the next month.

Objective Function:

Minimize: $20\sum_{j=1}^{4} x_j + 25\sum_{j=1}^{4} y_j + 3\sum_{j=1}^{3} I_j$.

Constraints:

$$x_1 + y_1 \geq 1000$$

$$x_1 + y_1 \cancel{+ I_1} = 1000 + I_1$$

$$I_1 + x_2 + y_2 = 800 + I_2$$

$$I_2 + x_3 + y_3 = 1200 + I_3$$

$$I_3 + x_4 + y_4 = 900$$

$$x_j, y_j, I_j \geq 0$$

$$x_j \leq 800 \ \forall j$$

$$y_j \leq 200 \ \forall j$$

second way.

eliminating I_1, I_2, I_3.

$$x_1 + y_1 \geq 1000$$

$$x_1 + y_1 - 1000 + x_2 + y_2 \geq 800$$

$$x_1 + y_1 - 1000 + x_2 + y_2 - 800 \geq 1200$$
$$+ x_3 + y_3$$

$$x_1 + y_1 + x_2 + y_2 + x_3 + y_3 - 1000 - 800 - 1200 + x_4 + y_4 = 900$$

Re writing objective function:

$$\Rightarrow \quad 20 \sum_{j=1}^{4} x_j + 25 \sum_{j=1}^{4} y_j + 3(x_1 + y_1 - 800) +$$
$$3(x_1 + y_1 + x_2 + y_2 - 1800) + 3(x_1 + y_1 + x_2 + y_2 +$$
$$x_3 + y_3 - 3000) +$$
$$3(x_1 + y_1 + x_2 + y_2 + x_3 + y_3 + x_4 + y_4 - 3900)$$

$$x_j \leq 800$$
$$y_j \leq 200$$

* <u>objective</u> <u>Functions</u> are <u>2</u> <u>types</u> :-

1. Maximization.
2. Minimization.

Example 3 :- (cutting stock problem)

consider a big sheet roll from which steel sheets of the same lengths but different width have to be cut. Let us assume that the roll is 20 inch wide and the following sizes have to be cut.

1. 9 inch 511 numbers.
2. 8 inch 301 numbers.
3. 7 inch 263 numbers.
4. 6 inch 383 numbers

It is assumed that all the cut sheets have the same length (say 25 inches). Only one dimensional cutting is only allowed.

The problem is to cut the sheets in such a way as to minimize wastage of material.

20/3/23

$$\text{Max } z = 3x_1 + 9x_2 + x_3$$

s.t.c $\quad x_1 + 2x_2 = 0 \Rightarrow 1\cdot x_1 + 2x_2 + 0\cdot x_3 = 0$

$\quad 2x_2 + x_3 = 1 \rightarrow 0 + 2x_2 + 1\cdot x_3 = 1$

$x_1 = 0$
$x_3 = 1$
$$\begin{array}{c} \begin{array}{cc} x_1 & x_3 \end{array} \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{array} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

pivot element

c_{Bi}	basis	c_j / b	3 / x_1	9 / x_2	1 / x_3	$\theta = b/x_{rj}$
3	x_1	0	1	$\boxed{2}$	0	$0/2 = 0$
1	x_3	1	0	2	1	$1/2 = 0.5$
$z_j - c_j$			$3-3 = 0$	$8-9 = -1$	$1-1 = 0$	

outgoing vector $= x_1$ (we take this because min)

$\hookrightarrow$ entering vector

x_1 is replaced with ~~$x_1/2$~~ x_2.

~~$x_1 = x_1/2$~~

$x_1 = x_1/2$, $x_3 = x_3 - x_1 = 2 - 2 = 0$

c_{bi}	basis	c_j / b	3 / x_1	9 / x_2	1 / x_3	
9	x_2	0	$1/2$	1	0	
1	x_3	$1-0=1$	-1	0	1	
$z_j - c_j$			$\frac{9}{2} - 1 - 3 = 1/2$	$9-9 = 0$	$1-1 = 0$	≥ 0

$\boxed{x_1 = 0, x_2 = 0, x_3 = 1}$ then $z = 1$.

$\hookrightarrow$ optimal solution.

The above one is any example for type I degeneracy.

assignment

Max $z = 6x_1 + 8x_2$

s.t. c

$5x_1 + 10x_2 \leq 60.$

$4x_1 + 4x_2 \leq 40.$

$x_1, x_2 \geq 0.$

Ans:- $x_1 = 2$

$x_2 = 8$

$z_{max} = 36.$

$12 + 24$

36

* charne's perturbation method : Type II degeneracy :-

1. Rename the column vector of A in such a way that initial basis B consists of first m column vectors a_j of A.
2. Let a_k be the entering vector compute min $\left\{ \frac{x_{i1}}{x_{ik}}, x_{ik} > 0 \right\}$ for all i, which tie of min $\left\{ \frac{b}{x_{ik}}, k > 0 \right\}$ occurs.

If min $\left\{ \frac{x_{i1}}{x_{ik}}, x_{ik} > 0 \right\} \forall$ i occurs for one & only value of i say i = s, then as if the departing vector. If this minimum occurs for more than one value of i, then go to next step.

3. compute min $\left\{ \frac{x_{i2}}{x_{ik}}, x_{ik} > 0 \right\} \forall$ i for which tie occurs in proceed until we get one unique non negative for outgoing vector.

eg:-

	x_1	x_2	x_3	s_1	s_2	a_1	θ
s_1	1	2	3	1	0	0	1.5
s_2	2	1	0	0	0	1	1.5
a_1	5 x_{13}	4 x_{23}	1	0	1	0	2

$z_j - c_j$

let say this is entiring vector.

entiring vector $k=3$.

Here $i = 1, 2$. (because both are same) and these are outgoing vector.

<u>New table</u> :-

s_1	s_2	a_1	x_1	x_2	x_3
$1^{x_{11}}$	$0^{x_{12}}$	$0^{x_{13}}$	$1^{x_{14}}$	$2^{x_{15}}$	$3^{x_{16}}$
$0^{x_{21}}$	$0^{x_{22}}$	1	2	1	0
0	1	0	5	4	1

$\Rightarrow \min \left\{ \frac{x_{i1}}{x_{ik}} \right\}$ → belogs to new table / → belongto old table.

$\Rightarrow \min \left\{ \frac{x_{11}}{x_{13}}, \frac{x_{21}}{x_{23}} \right\}$

$\Rightarrow \min \left\{ \frac{1}{5}, \frac{0}{4} \right\}$

$\Rightarrow \min \{ 1/5, 0 \}$

s_1 s_2 → outgoing vector.

$\therefore$ s_2 will be outgoing vector.

Example

max $z = 5x_1 + 3x_2$

s.t.c $x_1 + x_2 \le 2 \Rightarrow x_1 + x_2 + s_1 = 2$

$5x_1 + 2x_2 \le 10. \Rightarrow 5x_1 + 2x_2 + s_2 = 10$

$3x_1 + 8x_2 \le 12 \Rightarrow 3x_1 + 8x_2 + s_3 = 12$

$x_1, x_2 \ge 0.$

$\left.\begin{array}{l} s_1 = 2 \\ s_2 = 10 \\ s_3 = 12 \end{array}\right\}$ basic variables

	c_j		5	3	0	0	0	
c_{Bi}	Basis	b.	x_1	x_2	$\cancel{x_3}$ s_1	s_2	s_3	$\theta = b/x_{rj}$
0	s_1	2	1	1	1	0	0	$2/1 = 2$
10	s_2	10	5	2	0	1	0	$10/5 = 2$ } tie
0	s_3.	12	3	8	0	0	1	$12/3 = 4$
	$z_j - c_j$.		-5	-3	0	0	0.	

↑ entering vector.

→ pivot element.

{ * pivot element is 1 and other elements such as 1(s_1) & 3 (s_3) becomes zero. }

Here $i = 1, 2$

$k = 1$.

new table

	s_1	s_2	s_3	x_1	x_2.
			0	1	1
s_1	1	0	0	5	2
s_2	0	1	0	3	8
	0	0	1		

Now we have to find

$$\min \left\{ \frac{x_{i1}}{x_{ik}} \right\}$$

$$\min \left\{ \frac{x_{11}}{x_{11}} , \frac{x_{21}}{x_{21}} \right\}$$

$$\min \left\{ \frac{1}{1} , \frac{0}{5} \right\}$$

$$\min \{ 1 , \boxed{0} \}$$

s_1 $\boxed{s_2}$ → outgoing vector.

$\therefore$ s_2 → outgoing vector.

$s_2 = s_2/5 = 5/5 = 1$.

$s_1 = s_1 - \frac{1}{5} s_2$

$= 1 - \frac{1}{5}(5) = 0$.

$s_3 = s_3 - 3/5 \, s_2$

$= 3 - 3/5 (5) = 0$.

		c_j	5	3	0	0	0	
c_{Bi}	basis	b	x_1	x_2	s_1	s_2	s_3	θ
0	s_1	$2 - \frac{1}{5}(10) = 0$	0	$1 - \frac{2}{5} = 3/5$	1	$-1/5$	0	0 → outgoing vector
5	x_1	2	1	$2/5$	0	$1/5$	0	~~$6/34/5 = \frac{30}{34}$~~ 5
0	s_3	$12 - \frac{3}{5}(10) = 6$	0	$8 - \frac{3}{5}(2) = 34/5$	$-3/5$	0	1	$6/34/5 = 30/34$
	$z_j - c_j$		$5 - 5 = 0$	$2 - 3 = -1$	0	1	0	

↑ entiring vector

$s_1 = \frac{5}{3} s_1$

$x_1 = x_1 - \frac{2}{5}(s_1)$

$s_3 = s_3 - \frac{34}{3}(s_1)$

c_{Bi}	basis	c_j / b	5 / x_1	3 / x_2	0 / s_1	0 / s_2	0 / s_3	θ
3	x_2	0	0	1	5/3	-1/3	0	
5	x_1	$2-\frac{2}{3}(0)$	1	0	-2/3	+1/3	0	
0	s_3	6-0=6	0	0	$\frac{-179}{15}$	$\frac{34}{15}$	1	
	$z_j - c_j$		5-5 0	3-3 0	$5-\frac{10}{3}$ 5/3	-1+5/3 2/3	0 0	≥ 0

$\therefore x_1 = 2, x_2 = 0$

$z = 5x_1 + 3x_2$

$= 5(2) + 3(0)$

$\boxed{z = 10}$

assignment solution :-

Max $z = 6x_1 + 8x_2$

s.t.c

$5x_1 + 10x_2 \leq 60 \Rightarrow 5x_1 + 10x_2 + s_1 = 60$

$4x_1 + 4x_2 \leq 40 \Rightarrow 4x_1 + 4x_2 + s_2 = 40$

$x_1, x_2 \geq 0$

pivot element

c_{Bi}	basis	c_j / b	5 / x_1	3 / x_2	0 / s_1	0 / s_2	$\theta = b/x_{ij}$
0	s_1	60	5	10	1	0	60/5 = 12
0	s_2	40	4	4	0	1	40/4 = 10 → outgoing vector
	$z_j - c_j$		-5	-3	0	0	

$s_1 = \cancel{4}\, s_1 - \frac{5}{4}(s_2) = 5 - \frac{5}{4}(4) = 0.$

$\cancel{5 - 20 \cdot \frac{1}{4}\, s_1 - \frac{5}{4} s_2}.$

$s_2 = s_2/4.$

		c_j	5	3	0	0	
C_{Bi}	basis	b.	x_1	x_2	s_1	s_2	$\theta = b/x_{ij}$
0	s_1	10.	0	5	1	$-5/4$	
5	x_1	10.	1	1.	0	$1/4$	
	$z_j - c_j$		0.	2.	0.	$5/4$	

assignment :-

$$\max\ z = 6x_1 + 8x_2$$

s.t.c

$$5x_1 + 10x_2 \le 60 \Rightarrow 5x_1 + 10x_2 + s_1 = 60.$$
$$4x_1 + 4x_2 \le 40 \Rightarrow 4x_1 + 4x_2 + s_2 = 40.$$
$$x_1, x_2 \ge 0;$$

→ pivot element.

		c_j	6	8	0	0	
C_{Bi}	Basic	b	x_1	x_2	s_1	s_2	$\theta = b/x_{ij}$
0	s_1	60	5	10	1	0	$\frac{60}{10} = 6$ → outgoing vector.
0	s_2.	40	4	4	0	1	$\frac{40}{4} = 10.$
	$z_j - c_j$		-6	-8	0	0	

↳ entering vector.

$z_j - c_j = C_{Bi} \times x_i + C_{Bi} \times x_i$

In the entering vector, we have to make the pivot element as "1" and the remaining as "0"

For that

$s_1 = \frac{s_1}{10} \left[= \frac{10}{10} = 1\right]$

$s_2 = s_2 - \frac{4}{10} s_1 \left[= 4 - \frac{4}{10}(10) = 0\right]$

		c_j	6	8	0	0	
c_{Bi}	Basic	b	x_1	x_2	s_1	s_2	$\theta = b/x_{ij}$
8	x_2	6	1/2	1	1/10	0	$\frac{6}{1/2} = 12$
0	s_2	~~200~~ 16	2	0	-2/5	1	~~$\frac{200}{8}$ = 100~~ $\frac{16}{2} = 8$
$z_j - c_j$			-2	0	4/5	0	

$$s_2 = \frac{s_2}{2} = \frac{2}{2} = 1$$

$$x_2 = x_2 - \frac{1}{4}(s_2) \qquad \frac{1}{2} - \frac{1}{4}(2)$$

$$= \frac{1}{2} - \frac{1}{4}\left(\frac{2}{2}\right)$$

$$= \frac{1}{2} - \frac{1}{2} = 0$$

		c_j	6	8	0	0	
c_{Bi}	Basic	b	x_1	x_2	s_1	s_2	$\theta = b/x_{ij}$
8	x_2	2	0	1	0	-1/4	$\frac{2}{0}$
6	x_1	8	1	0	-1/5	1/2	$\frac{8}{-1/5} = -40$
$z_j - c_j$			0	0	-8/5	1	

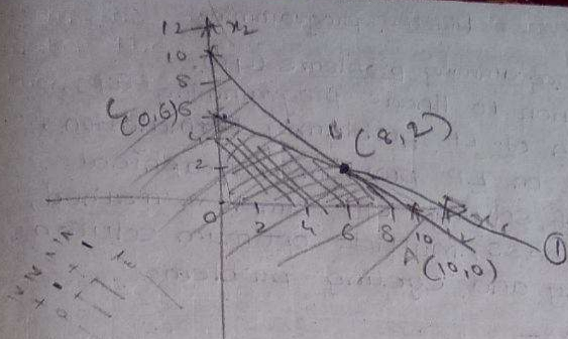
-8/5 $\quad \frac{8}{-1/5}$ -40

$$x_1 = \frac{x_1}{-5} = \frac{-5}{-5} = 1$$

$$x_2 = x_2 - 0(x_1) = 0 - 0(-5) = 0$$

$$x_2 = x_2$$

		c_j	6	8	0	0	
C_{Bi}	Basic	b	x_1	x_2	s_1	s_2	
8	x_2	2	0	1	0	$-1/4$	
0	s_1	$\frac{-8}{5}$					
$z_j - c_j$							



$5x_1 = 60 - 10x_2$

$x_1 = 12 - 2x_2 = 12 - 4 = 8$

$4(12 - 2x_2) + 4x_2 = 40$

$48 - 8x_2 + 4x_2 = 40$

$8 = 4x_2$

$\boxed{x_2 = 2}$

$Z_A = 60$

O.S

$\boxed{Z_B = 6(8) + 8(2) \overset{(8,2)}{=} 48 + 16 = 64}$

$Z_C = 8(6) = 48$