2.5.

Posets

In this module we introduce the concepts of Posets and totally ordered set. We illustrate the method of drawing a Hasse diagram of a finite poset and study Lexicographic ordering. At the end we discuss the technique of topological sorting.

We often use relations to order some or all of the elements of sets. For example (i) we order words using the relation containing pairs of words (x, y), where the word x comes before the word y in the dictionary, (ii) we schedule projects using the relation consisting of pairs (x, y), where x, y are tasks in a project such that the task x must be completed before the task y begins, (iii) we order the elements of a set of real numbers using the relation containing the pairs (x, y) where x is less than y. If we add all the pairs of the form (x, x) to these relations, we obtain the relation that is reflexive, antisymmetric and transitive. These are the properties that characterize relations used to order the elements.

Partial order and Poset

A relation R on a set A is called a ***partial ordering*** or a ***partial order*** if it is *reflexive*, *antisymmetric* and *transitive*. A set A together with a partial order R defined on it, denoted by $(\boldsymbol{A}, \boldsymbol{R})$ is called a ***Partially ordered set*** or ***Poset*** in short.

Examples

i) The relation "less than or equal to $(\leq)$ " and "greater than or equal to $(\geq)$" are partial orders on any nonempty subset A of $\boldsymbol{R}$. Therefore, $(A, \leq)$ and $(A, \geq)$ are posets, where A is a nonempty subset of $\boldsymbol{R}$.
ii) The divisibility relation (i.e., divides (|)) is a partial order on any nonempty subset A on $\boldsymbol{N}$. Therefore, $(A, |)$ is a poset, where A is a nonempty subset of $\boldsymbol{N}$.
iii) Let A be a set and $P(A)$ be its power set. The inclusion relation $(\subseteq)$ is a partial order on $P(A)$. Therefore, $(P(A), \subseteq)$ is poset, where A is a set.

Notation: Customarily a partial order is denoted by $\preccurlyeq$.

This notation is used because the relation $\leq$ is the most familiar example of a partial order on $\boldsymbol{R}$ and the symbol $\preccurlyeq$ is similar to $\leq$.

The notation $x \prec y$ denotes that $x \preccurlyeq y$ and $x \neq y$.

Comparability in a poset

Let $(A, \preccurlyeq)$ be a poset. Two elements $a, b \in A$ are said to be ***comparable*** if ***either*** $\boldsymbol{a \preccurlyeq b}$ ***or*** $\boldsymbol{b \preccurlyeq a}$. The elements $a, b \in A$ are ***incomparable*** if ***neither*** $\boldsymbol{a \preccurlyeq b}$ ***nor*** $\boldsymbol{b \preccurlyeq a}$.

Example 1: In the poset $(P(A), \subseteq)$, where $A = \{1, 2, 3\}$; $\{1\}$ and $\{1, 3\}$are comparable (because$\{1\} \subseteq \{1, 3\}$) and $\{1, 2\}, \{1, 3\}$ are incomparable because neither set is a subset of the other.

The adjective ***partial*** in the poset means that there may be pairs of elements which are incomparable. If every pair of elements in a poset are comparable then the ordering relation is called a ***total order***.

Totally ordered set: If $(A, \preccurlyeq)$ be a poset and every two elements of A are comparable then $(A, \preccurlyeq)$ is called a ***totally ordered set*** or ***linearly ordered set*** or ***chain*** and $\preccurlyeq$ is called a ***total order*** or a ***linear order***.

Example 2: Let A be any nonempty subset of $\boldsymbol{R}$. The poset $(A, \leq)$ is a totally ordered set because for any $a, b \in A$, we have either $a \leq b$ or $b \leq a$.

Example 3: Let $A = \{1, 2, 3, 4\}$. The poset $(A, |)$ is not a chain because A contains elements that are incomparable such as 2 and 3.

Consider the totally ordered set $(\boldsymbol{N}, \leq)$ and $\boldsymbol{N} \times \boldsymbol{N}$. Define a relation $\preccurlyeq$ on $\boldsymbol{N} \times \boldsymbol{N}$ as follows. For any $(x_1, y_1), (x_2, y_2) \in \boldsymbol{N} \times \boldsymbol{N}$, define

$$\boldsymbol{(x_1, y_1) \preccurlyeq (x_2, y_2)} \textbf{ either if } \boldsymbol{x_1 < x_2} \textbf{ or if both } \boldsymbol{x_1 = x_2} \textbf{ and } \boldsymbol{y_1 \leq y_2}$$

Example 4: $(\boldsymbol{N} \times \boldsymbol{N}, \preccurlyeq)$ **is a totally ordered set** **(See P1 for solution)**

Lexicographic order

The words in a dictionary are listed in alphabetic or lexicographic order which is based on the ordering of the letters in the alphabet. This is a special case of an ordering of strings on a set constructed from a partial ordering on the set. The following is such a construction in any poset.

Let $(A_1, \preccurlyeq_1)$ and $(A_2, \preccurlyeq_2)$ be two Posets. Consider the Cartesian product $A_1 \times A_2$. The ***lexicographic ordering*** $\preccurlyeq$ on $A_1 \times A_2$ is defined by

$$(\boldsymbol{x_1}, \boldsymbol{y_1}) \preccurlyeq (\boldsymbol{x_2}, \boldsymbol{y_2}) \Leftrightarrow (\boldsymbol{x_1} \prec_{\boldsymbol{1}} \boldsymbol{x_2}) \vee \big((\boldsymbol{x_1} = \boldsymbol{x_2}) \wedge (\boldsymbol{y_1} \preccurlyeq_{\boldsymbol{2}} \boldsymbol{y_2})\big)$$

It may be verified that $\preccurlyeq$ is a partial order. Therefore, $(A_1 \times A_2, \preccurlyeq)$ is a poset.

If $(\boldsymbol{A_1}, \preccurlyeq_{\boldsymbol{1}})$ **and** $(\boldsymbol{A_2}, \preccurlyeq_{\boldsymbol{2}})$ **be totally ordered sets then so is** $(\boldsymbol{A_1} \times \boldsymbol{A_2}, \preccurlyeq)$

Example 5: In the totally ordered set $(\boldsymbol{N} \times \boldsymbol{N}, \preccurlyeq)$, where $\preccurlyeq$ is the lexicographic ordering constructed from the usual $\leq$ relation on $\boldsymbol{N}$,

$$(2,2) \preccurlyeq (2,1), (3,1) \preccurlyeq (1,5), (2,2) \preccurlyeq (2,2), (3,2) \preccurlyeq (1,1), (4,9) \preccurlyeq (4,11), \ldots$$

Generalization of Lexicographic ordering

Let $(A, \preccurlyeq_1)$ be a totally ordered set. Let n be a given natural number and

$$P = A \cup A^2 \cup A^3 \cup \ldots . UA^n = \bigcup_{i=1}^{n} A^i$$

That is, P consists of strings of elements of A of length less than or equal to n.
A string of length p may be considered as a p-tuple .
Define a total ordering $\preccurlyeq$ on P, called **lexicographic ordering,** as follows:

Let $(a_1, a_2, \ldots, a_p)$ and $(b_1, b_2, \ldots, b_p, \ldots, b_q)$ with $p \leq q$ be any two elements of P. Now,

$$(a_1, a_2, \ldots, a_p) \preccurlyeq (b_1, b_2, \ldots, b_p, \ldots, b_q)$$

if any of the following hold:

i) $(a_1, a_2, \dots., a_p) = (b_1, b_2, \dots, b_p)$
ii) $a_1 \neq b_1$ and $a_1 \prec_1 b_1$ in $(A \preccurlyeq_1)$
iii) $a_i = b_i, i = 1,2, \dots, k \ (k < p)$ and $a_{i+1} \neq \ b_{i+1}$ and $a_{i+1} \prec_1 b_{i+1}$ in $(A, \preccurlyeq_1)$

If none of these conditions are satisfied, then

$$(b_1, b_2, \dots, b_q) \preccurlyeq (a_1, a_2, \dots, a_p)$$

Example 6: Let $A = \{a, b, c, \dots, z\}$ be the lower case English alphabet and there is a linear ordering on A denoted by $\leq$, where $a \leq b \leq c \leq \cdots \leq z$. Let

$$P = A \cup A^2 \cup A^3 \cup A^4 \cup A^5$$

That is ,P consists of all *words* or strings of 5 or fewer than 5 letters from A. The lexicographic ordering in P is same as that used in dictionaries

For example, ante ≼ axe (by ii), zebra ≼ zero (Here “zero” and “zebra” are compared and the conditions (i), (ii) and (iii) are not satisfied) and
zerm ≼ zero (by (iii))

Note: Instead of using ≼ to denote lexicographic ordering, it is customary to use the terminology such as *lexically less than or equal to* or *lexically greater than*

Application of lexicographic ordering

Lexicographic ordering is used in **sorting character data** on a computer

Hasse diagrams

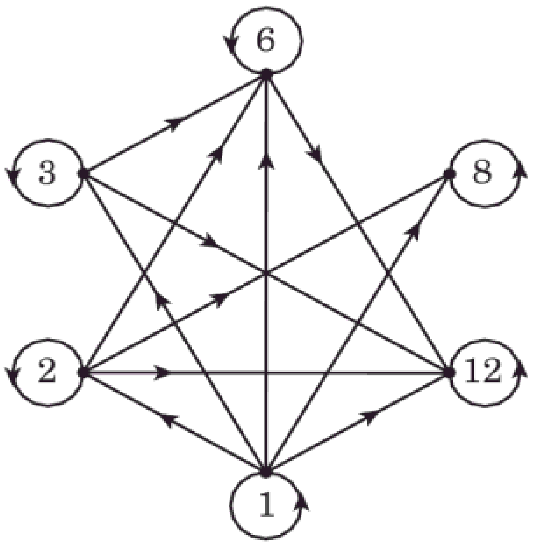
We can simplify the diagraphs of a finite poset by omitting many of its edges. For instance, since a partial order is reflexive, each vertex has a loop, which we can delete. In addition, drop all edges implied by transitivity (For example if the digraph of a poset contains edges (a, b) and (b, c) , then it has the edge (a, c) , which we can omit. Finally, draw the remaining edges ***upward*** and ***drop all arrows***. The resulting diagram is called the ***Hasse diagram,*** named after the twentieth – century German mathematician **Helmut Hasse** $(1898 - 1979)$.

Example 7: Draw the Hasse diagram for the poset $(A, |)$, where $A = \{1, 2, 3, 6, 8, 12\}$ and | denotes the divisibility relation.

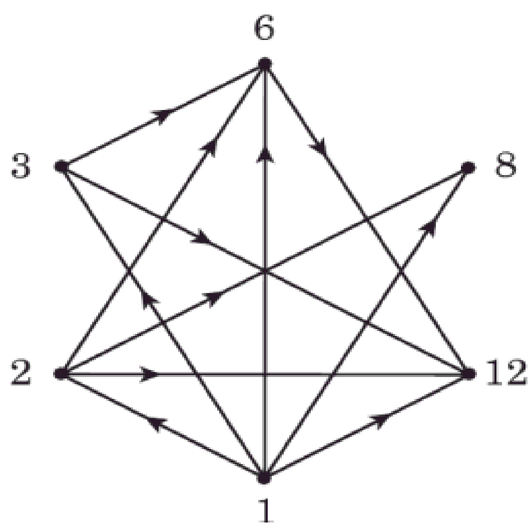
Solution: The ordered pairs in the partial order are
$(1,1), (1,2), (1,3), (1,6), (1,8), (1,12), (2,2), (2,6), (2,8), (2,12), (3,3), (3,6), (3,12)$

$(6,6), (6,12), (8,8), (12,12)$

The diagraph of the poset is

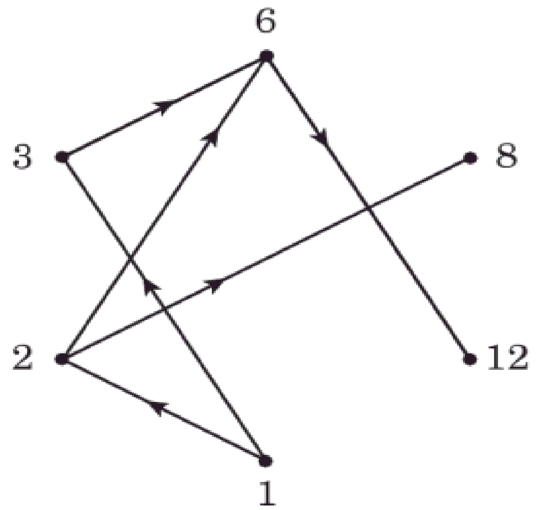


Step 1: Delete the loop at each vertex

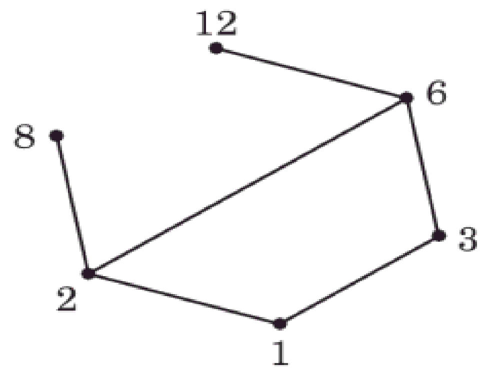


Step 2: Delete all the edges implied by transitivity. These are

$(1,6), (1,8), (1,12), (2,12), (3,12)$



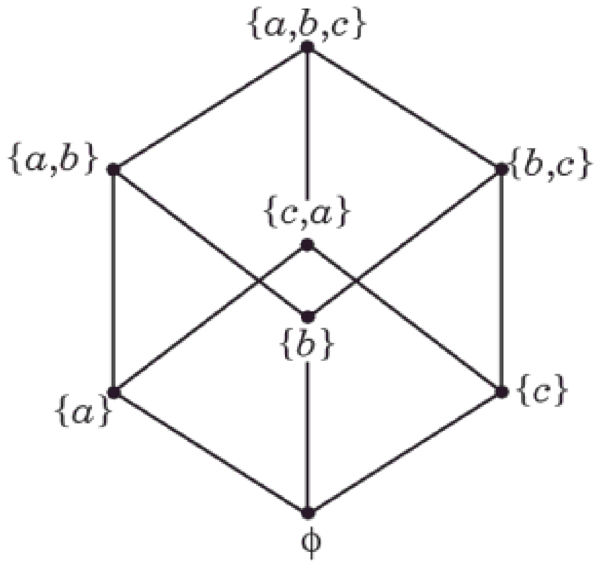
Step 3: Omit all arrows and arrange all edges pointing upward to obtain the Hasse diagram.



Example 8: **Draw Hasse diagram of the poset $(\boldsymbol{P}(\boldsymbol{A}), \subseteq)$, where $\boldsymbol{A} = \{\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}\}$**

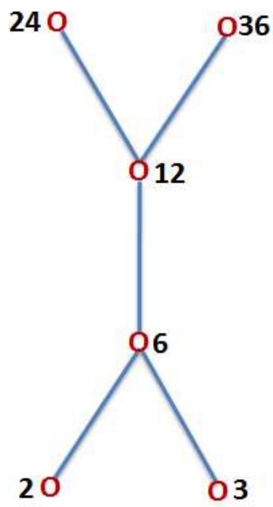
Solution: We have $A = \{a, b, c\}$ and
$P(A) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{c, a\}, \{a, b, c\}\}$

Following the steps 1-3, produces the Hasse diagram of the poset $(\boldsymbol{P}(\boldsymbol{A}), \subseteq)$



Example 9: Draw the Hasse diagram of the poset $(A, |)$, where $A = \{2, 3, 6, 12, 24, 36\}$

Solution: We have $A = \{2,3,6,12,24,36\}$. Following the steps 1-3, produces the Hasse diagram of $(A, |)$,



Hasse diagram of divides relation

Extremal elements: An element a in a poset $(A, \preccurlyeq)$ is a **maximal element**, if there is *no element* b in A such that $a \prec b$. An element a in a poset $(A, \preccurlyeq)$ is a **minimal element**, if there is no element b in A such that $b \prec a$.

The maximal and minimal elements in a finite poset are the *top* and *bottom* elements in its Hasse diagram.

The poset $(A, |)$, where $A = \{1, 2, 3, 6, 8, 12\}$ has two maximal elements 8 and 12 and one minimal element 1.
The poset $(A, |)$ where $A = \{2, 3, 6, 12, 24, 36\}$ has two maximal elements 24 and 36 and two minimal elements 2 and 3.

Note:

(i) A poset may have more than one maximal element and more than one minimal element.
(ii) A poset need not have any maximal or minimal elements. For example the poset $(\boldsymbol{Z}, \leq)$ has no maximal or minimal elements.
(iii) A poset may have a maximal element but no minimal elements or a minimal element but no maximal elements. For example $(\boldsymbol{Z}^{-}, \leq)$ has a maximal element but no minimal elements, where as the poset $(\boldsymbol{N}, \leq)$ has a minimal element but no maximal elements.

Although an arbitrary poset need not have a minimal element, *every non empty finite poset has at least one minimal element*. We state this result as a lemma.

Lemma 1: Every finite nonempty poset $(\boldsymbol{A}, \preccurlyeq)$ has at least one minimal element.

Two special extremal elements are the *greatest* and the *least elements*.

Greatest and least elements: Let $(A, \preccurlyeq)$ be a poset. If there exists an element $a \in A$ such that $a \preccurlyeq x$, for all $x \in A$, then a is called the **least element** of A. If these exists an element $b \in A$ such that $x \preccurlyeq b$ for all $x \in A$, then b is called the **greatest element** of A.

Note:

i. The least element (greatest element) if exists is unique and they are the *bottom most* (*topmost*) elements in the Hasse diagram of a finite poset.
ii. The least element and the greatest element of a poset are usually denoted by 0 and 1 respectively

The poset $(A, |)$ where $A = \{1, 2, 3, 6, 8,12\}$ has no greatest element, but has the least element 1.

The poset $(A, |)$ where $A = \{ 2,3, 6, 12, 24, 36\}$ has no least element and has no greatest element.

Topological Sorting

Suppose that a project is made up of n different tasks say $t_1, t_2, t_3, \dots, t_n$ .Some tasks can be completed only after others have been finished. To find an order R for these tasks, we set up a partial order on the set of tasks so that $t_i R\ t_j, i \neq j$, iff the task t_j cannot be started until the task t_i has been completed. To produce a schedule for the project, we need to produce an order for all n tasks that is *compatible* with this partial order.

A total ordering $\preccurlyeq$ is said to be *compatible* with the partial ordering R, if

$$\boldsymbol{a \preccurlyeq b} \textbf{ whenever } \boldsymbol{aRb}$$

Constructing a compatible total ordering from a given partial ordering is called **topological sorting.**

The following is the procedure of a topological sorting and it works for any finite nonempty poset.
To define a total ordering on the poset $(A,\ R)$ where A has n elements , first choose a minimal element, say a_1 (such an element exists by Lemma 1). Now note that $(A - \{a_1\}, R)$ is also a poset (where R is the restriction on $A - \{a_1\}$). If it is nonempty choose a minimal element, say a_2 ,of this poset and we have $a_1 \prec a_2$.Remove a_2 from $A - \{a_1\}$, if $A - \{a_1\ a_2\}$ is nonempty continue the procedure. Because A is finite this process must terminate.

The end product is a sequence of elements $a_1, a_2, \dots, a_n$ and the desired total ordering $\preccurlyeq$ is defined by

$$\boldsymbol{a_1 \prec a_2 \prec a_3 \prec a_4 \prec \cdots \prec a_n}$$

This total ordering $\preccurlyeq$ is compatible with the original partial ordering R.

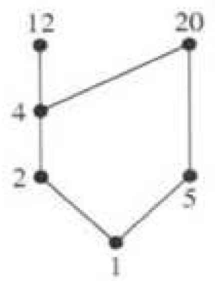
Application: Topological sorting has an application to the scheduling of projects.

Example 10: Topologically sort the elements of the poset $(A, |)$, where

$$\boldsymbol{A = \{1, 2, 4, 5, 12, 20\}}$$

Find a compatible total ordering for the poset $(A, |)$

Solution: The Hasse diagram of the poset $(\boldsymbol{A}, |)$ is



Step 1: Choose a minimal element. This must be 1 (because 1 is the only minimal element.

Step -2: Extract 1 and obtain $A - \{1\} = \{2, 4, 5, 12, 20\}$ **.** Now, There are two minimal elements, namely 2 and 5, select 5 .

Repeat step – 2: Extract 5 and obtain $A - \{1, 5\} = \{2,4, 12, 20\}$ .The minimal element at this stage is 2.

Repeat Step – 2: Extract 2 and obtain $A - \{1, 5, 2\} = \{4, 12, 20\}$. The minimal element at this stage is 4.

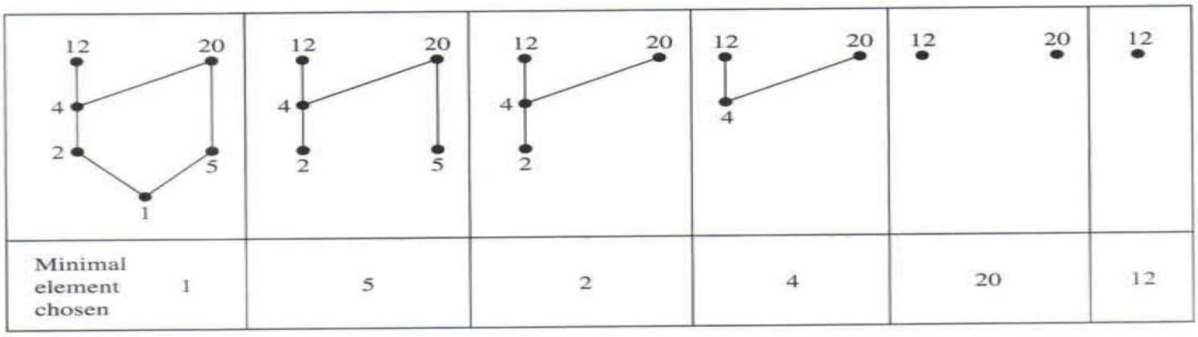
Repeat Step – 2: Extract 4 and obtain $A - \{1, 5, 2, 4\} = \{12, 20\}$**.** Either of the two can be a minimal element, select 20.

Repeat Step – 2: Extract 20 and obtain $A - \{1, 5, 2, 4, 20\} = \{12\}$ Now, it is the last element left.

This produces a total ordering $1 \prec 5 \prec 2 \prec 4 \prec 20 \prec 12$.

Note: It is one possible order for the tasks.

The steps used by this sorting are illustrated in the figure.



Note:

Computer scientists use the terminology *Topological sorting* and mathematician use the terminology *Linearization of partial ordering* for the same thing. In mathematics topology deals with the geometry. In computer science, a topology is any arrangement of objects that can be connected with edges.

P1:

Consider the totally ordered set $(N, \leq)$ and $N \times N$. Define a relation $\preccurlyeq$ on $N \times N$ as follows. For any $(x_1, y_1), (x_2, y_2) \in N \times N$, define

$$(x_1, y_1) \preccurlyeq (x_2, y_2) \text{ either if } x_1 < x_2 \text{ or if both } x_1 = x_2 \text{ and } y_1 \leq y_2$$

Show that $(N \times N, \preccurlyeq)$ is a totally ordered set

Solution:

For any $(x, y) \in \boldsymbol{N} \times \boldsymbol{N}$, we have $(x, y) \preccurlyeq (x, y)$ because $x = x$ and $y \leq y$. Therefore, $\preccurlyeq$ is reflexive.

If $(x_1, y_1) \preccurlyeq (x_2, y_2)$ and $(x_2, y_2) \preccurlyeq (x_1, y_1)$ then the possibility $x_1 < x_2$ and $x_2 < x_1$ is not possible. Therefore, $(x_1 = x_2 \ and \ y_1 \leq y_2)$ and $(x_2 = x_1 \ and \ y_2 \leq y_1)$ imply $x_1 = x_2$ and $y_1 = y_2$. Thus, $(x_1, y_1) = (x_2, y_2)$. This proves that $\preccurlyeq$ is antisymmetric.

$(x_1, y_1) \preccurlyeq (x_2, y_2)$ and $(x_2, y_2) \preccurlyeq (x_3, y_3)$

$\Longrightarrow (x_1 < x_2$ or $(x_1 = x_2 \ and \ y_1 \leq y_2))$ and $(x_2 < x_3$ or $(x_2 = x_3 \ and \ y_2 \leq y_3))$

$\Longrightarrow x_1 < x_3$ or $(x_1 = x_3 \ and \ y_1 \leq y_3)$ (How ?)

$\Longrightarrow (x_1, y_1) \preccurlyeq (x_3, y_3)$

This proves that $\preccurlyeq$ is Transitive. Thus, $(\boldsymbol{N} \times \boldsymbol{N}, \preccurlyeq)$ is a poset.

Let (x_1, y_1) and (x_2, y_2) be any two elements of $\boldsymbol{N} \times \boldsymbol{N}$. Since $x_1, x_2 \in \boldsymbol{N}$, by law of trichotomy $x_1 < x_2$ or $x_1 = x_2$ or $x_2 < x_1$.

If $x_1 < x_2$ then $(x_1, y_1) \preccurlyeq (x_2, y_2)$

If $x_2 < x_1$ then $(x_2, y_2) \preccurlyeq (x_1, y_1)$

Let $x_1 = x_2$. For any $y_1, y_2 \in \boldsymbol{N}$, we have either $y_1 \leq y_2$ or $y_2 \leq y_1$, because $(\boldsymbol{N}, \leq)$ is a chain. If $y_1 \leq y_2$ then $(x_1, y_1) \preccurlyeq (x_2, y_2)$. If $y_2 \leq y_1$ then $(x_2, y_2) \preccurlyeq (x_1, y_1)$. This shows that any two elements of $\boldsymbol{N} \times \boldsymbol{N}$ are comparable. Thus, $\preccurlyeq$ is a total order. Therefore, $(\boldsymbol{N} \times \boldsymbol{N}, \preccurlyeq)$ is a totally ordered set.

P2:

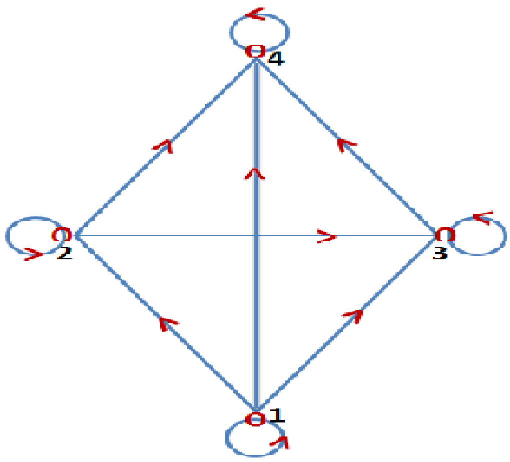
Draw the Hasse diagram for the *less than or equal* to relation defined on the set $A = \{1, 2, 3, 4\}$.

Solution:

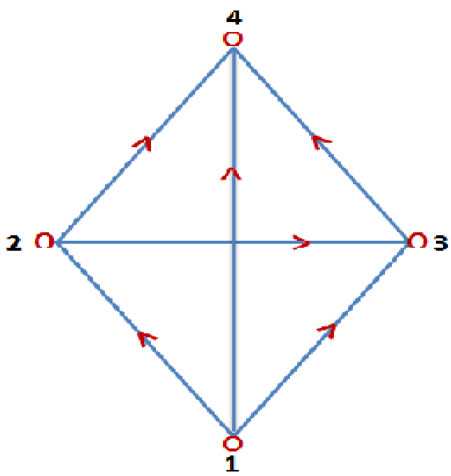
Let R be a relation *less than or equal to* on $A = \{1,2,3,4\}$. Therefore

$R = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,3), (3,4), (4,4)\}$

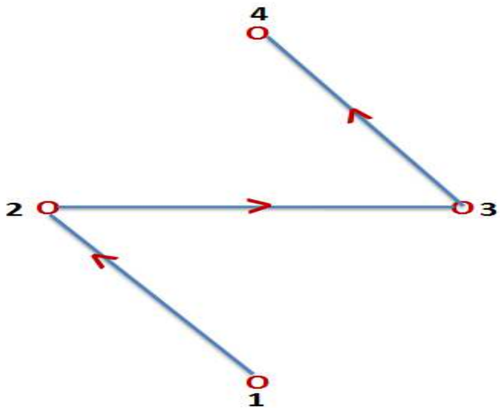
The diagraph of the poset (A, R) is



Step 1: Delete the loop at each vertex



Step 2 : Delete all the edges implied by transitivity . These are $(1,4), (1,3), (2,4)$



Step 3 : Omit all arrows and arrange all edges point upward to obtain the Hasse diagram.

It is a chain

P3:

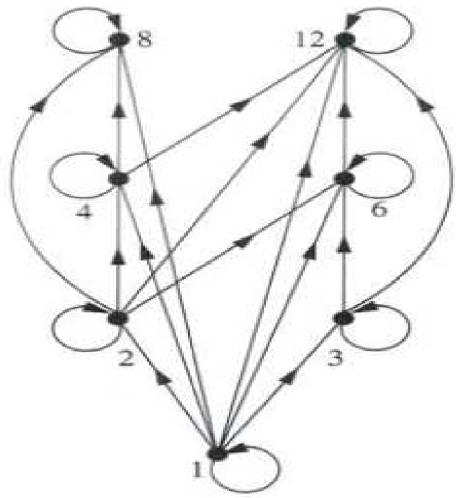
Draw the Hasse diagram representing the partial ordering $\{(\boldsymbol{a},\boldsymbol{b})|\boldsymbol{a}\ \boldsymbol{divides}\ \boldsymbol{b}\}$ on $\boldsymbol{A}=\{\mathbf{1},\mathbf{2},\mathbf{3},\mathbf{4},\mathbf{6},\mathbf{8},\mathbf{12}\}$.

Solution:

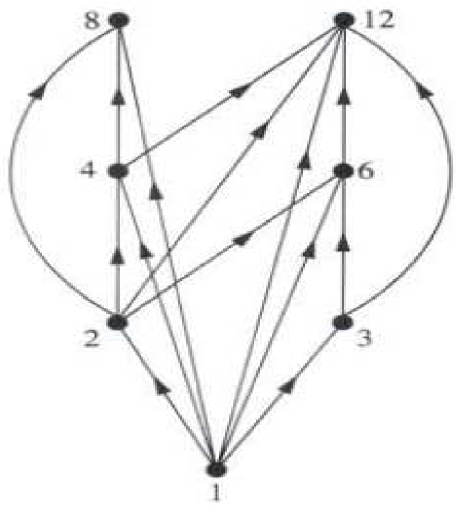
Let R be the partial ordering on A $i.e.$, $R=\{(a,b)|a,b\in A, a|b\}$

$$R=\left\{\begin{matrix}(1,1),(1,2),(1,3),(1,4),(1,6),(1,8),(1,12),(2,2),(2,4),(2,6),(2,8),\\(2,12),(3,3),(3,6),(3,12),(4,4),(4,8),(4,12),(6,6),(6,12),(8,8),(12,12)\end{matrix}\right\}$$

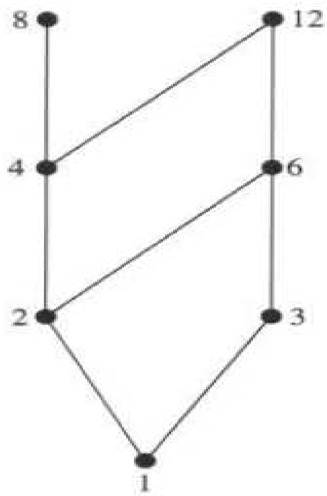
The digraph for this poset (A,R) is



Remove all loops then the diagram is



Delete all the edges implied by the transitivity. These are $(1,4), (1,6), (1,8), (1,12), (2,8), (2,12)$ and $(3,12)$. Arrange all edges pointing upward and delete all arrows to obtain the following Hasse diagram of (A, R).

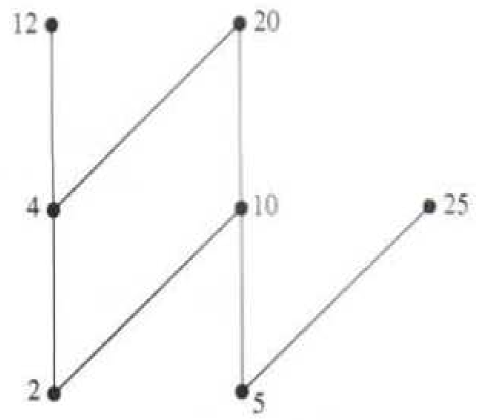


P4:

Which elements of the poset $(A, |)$ are maximal and which are minimal, defined on the set $A = \{2, 4, 5, 10, 12, 20, 25\}$.

Solution:

The Hasse diagram of the poset is



Notice that there is no $b \in A$ such that

$12 \prec b$ $i.e.$, $12|b$, $20 \prec b$ $i.e.$, $20|b$ and $25 \prec b$ $i.e.$, $25|b$

Therefore, 12, 20 and 25 are maximal elements of the poset.

Further, notice that there is no $b \in A$ such that

$b \prec 2$, $i.e.$, $b|2$ and$b \prec 5$, $i.e.$, $b|5$

Therefore, 2 and 5 are minimal elements of the poset.

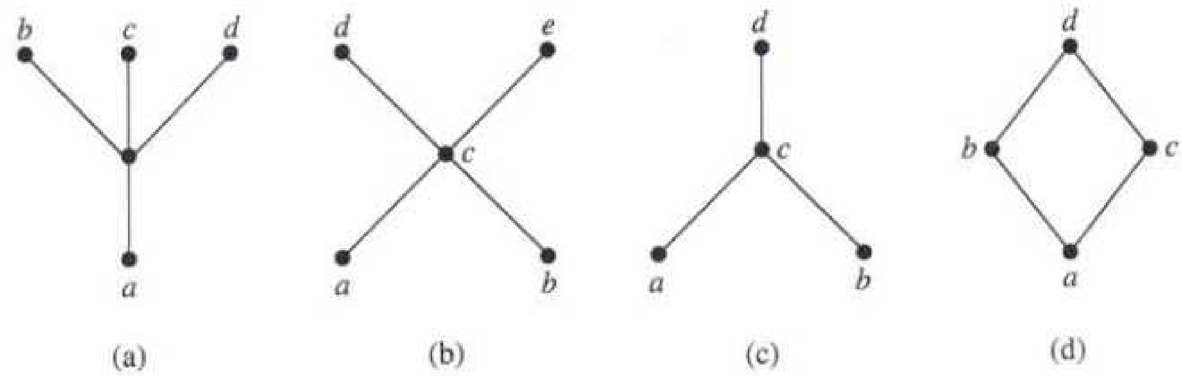
Note:

(i) There is no $a \in A$ such that $a \preccurlyeq x, \forall x \in A$, $i.e.$, $a|x$, $\forall x \in A$. Therefore, the poset has no least element.

(ii) There is no $a \in A$ such that , $x \preccurlyeq a, \forall x \in A$, $i.e.$, $x|a$, $\forall x \in A$. Therefore, the poset has no greatest element.

P5:

Determine whether the Posets represented by each of the following Hasse diagrams have a greatest element and least element:



Solution:

Recall the definition of least and greatest elements.

Poset	Least element	Greatest element
(a)	a	No
(b)	No	No
(c)	No	d
(d)	a	d

P6:

Let A be a nonempty set. Determine whether there is a greatest element and a least element in $(\boldsymbol{P}(\boldsymbol{A}), \subseteq)$.

Solution:

Notice that $\phi \subseteq X$, for every subset X of A. Therefore, ϕ is the least element of the poset. Further, $X \subseteq A$, for every subset X of A. Therefore, A is the greatest element of the poset.

P7:

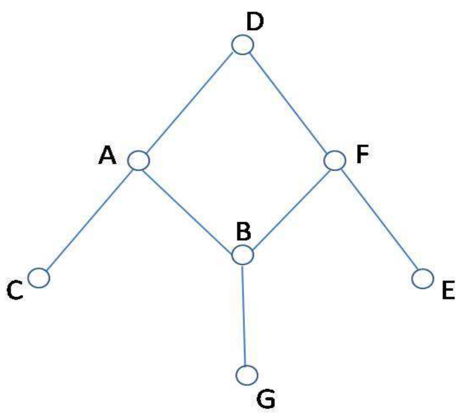
Seven tasks, A through G, comprise a project. Some of them can be started only after others are completed as indicated in the following table:

Task	Requires the completion of
A	B, C
B	G
C	None
D	A, F
E	None
F	B, E
G	None

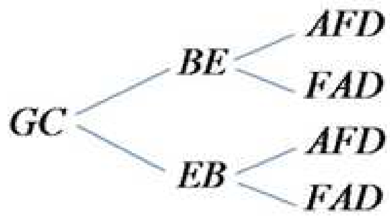
How many ways can the tasks be arranged sequentially, so that the prerequisites of each task will be completed before it started? List all of them.

Solution:

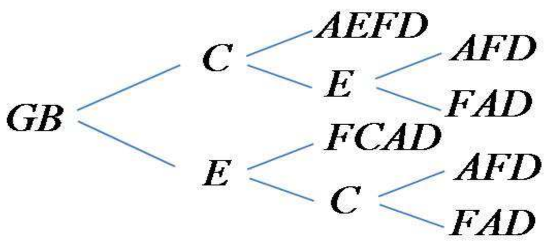
From the table, we draw the following Hasse diagram:



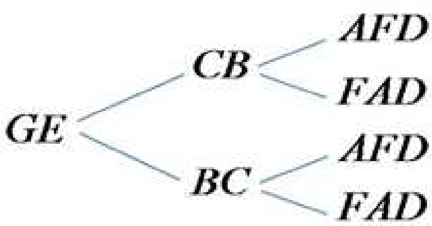
We topologically sort the elements of the poset and obtain 26 possible orders for the tasks. The following are the possible orders.


GC
BE
EB
AFD
FAD
AFD
FAD

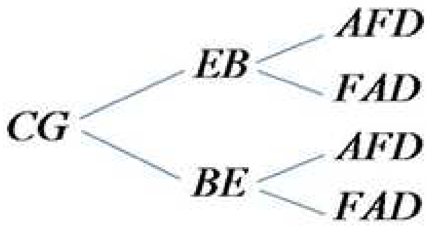
(4)


GB
C
E
AEFD
E
AFD
FAD
FCAD
C
AFD
FAD

(6)

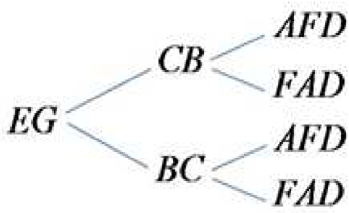

GE
CB
BC
AFD
FAD
AFD
FAD

(4)

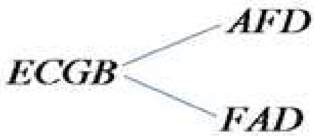

CG
EB
BE
AFD
FAD
AFD
FAD

(4)

(2)



(4)

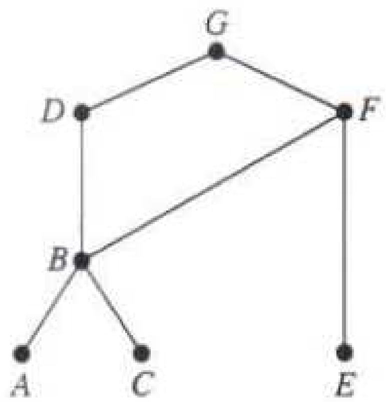


(2)

Therefore there are 26 possible orders.

P8:

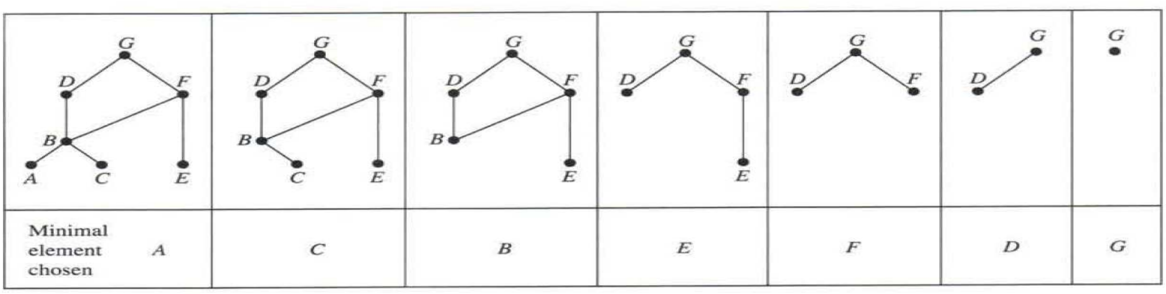
A development project at a computer company requires the completion of seven tasks. Some of these tasks can be started only after other tasks are finished. A partial ordering on tasks is set up by considering task $x \prec$ task y if task y cannot be started until task x has been completed. The Hasse diagram for the seven tasks with respect to this partial ordering is shown below.



Find an order in which these tasks can be carried out to complete the project.

Solution:

An ordering of the seven tasks can be obtained by topological sort. The steps are shown in the following figure.



The result of this sort is $A \prec C \prec B \prec E \prec F \prec D \prec G$

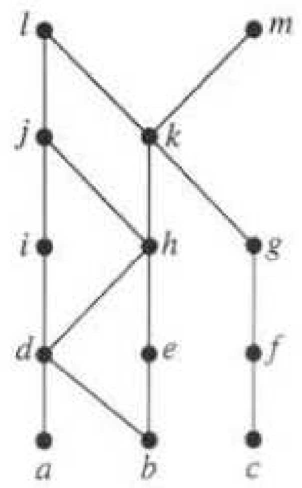
Note: It is one possible order for the tasks.

2.5. Posets

Exercise:

1. Let (S, R) be a poset. Show that $(S, \overline{R})$ is also a poset, where $\overline{R}$ is the inverse of R. The poset $(S, \overline{R})$ is called the ***dual*** of (S, R).
2. Find the duals of the following posets.
   a) $(\{0, 1, 2\}, \leq)$
   b) $(\boldsymbol{Z}, \geq)$
   c) $(P(\boldsymbol{Z}), \supseteq)$
   d) $(\boldsymbol{Z}^{+}, |)$
3. Which of the following pairs of elements are comparable in the poset $(\boldsymbol{Z}^{+}, |)$?
   a) $5, 15$
   b) $6, 9$
   c) $8, 16$
   d) $7, 7$
4. Find two incomparable elements in the following posets ?
   a) $(P(\{0, 1, 2\}), \subseteq)$
   b) $(\{1, 2, 4, 6, 8\}, |)$
5. Find the lexicographic ordering of the following n-tuples
   a) $(1, 1, 2), (1, 2, 1)$
   b) $(0, 1, 2, 3), (0, 1, 3, 2)$
   c) $(1, 0, 1, 0, 1), (0, 1, 1, 1, 0)$
6. Draw the Hasse diagram for divisibility on the set
   a) $\{1, 2, 3, 4, 5, 6, 7, 8\}$
   b) $\{1, 2, 3, 5, 7, 11, 13\}$
   c) $\{1, 2, 3, 6, 12, 24, 36, 48\}$
   d) $\{1, 2, 4, 8, 16, 32, 64\}$

7. Show that lexicographic order is a partial ordering on the Cartesian product of two sets.
8. Find a compatible total order for the poset with the Hasse diagram shown in the following figure.



9. Find a compatible total order for the divisibility relation on the set $\{1,2,3,6,8,12,24,36\}$
10. Find an ordering of the tasks of a software project if the Hasse diagram for the tasks of the project is as shown.