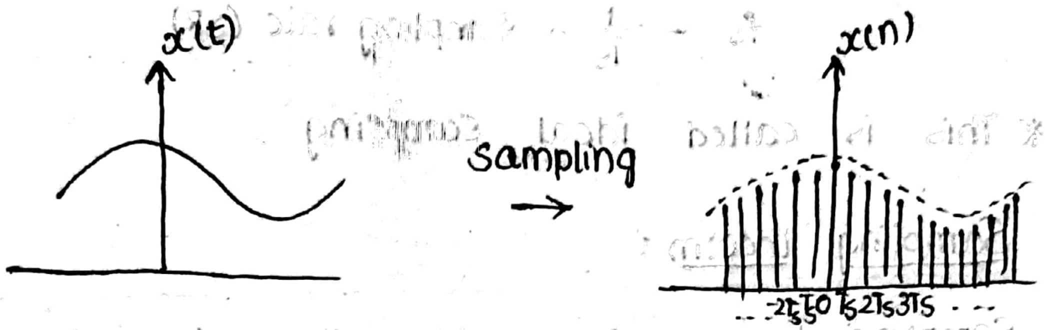
# Sampling theorm

**Sampling**: It is the process of converting a CTS to DTS




$$x(n) = x(t)\big|_{t=nT_s}$$

$T_s$ = sampling interval

$n$ = integer

**Sampling interval**: The time interval b/w two subsequent samples is known as sampling interval denoted with $T_s$.

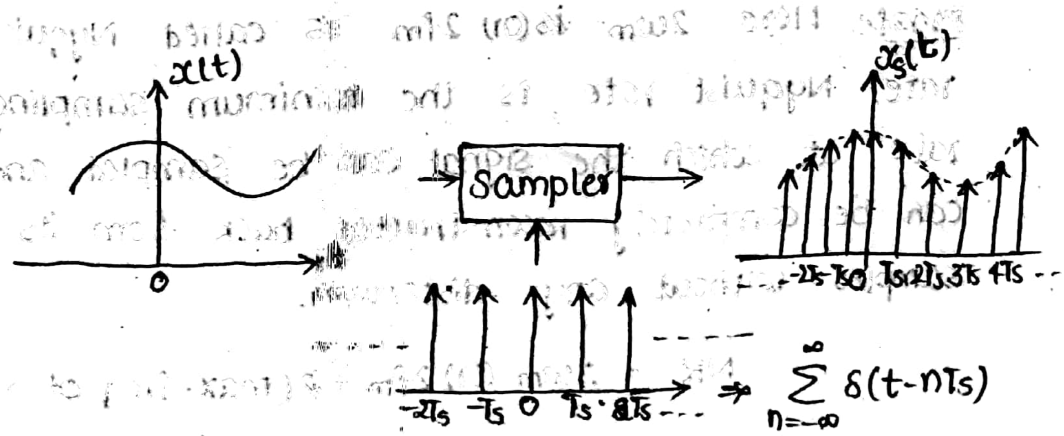
$$f_s = \frac{1}{T_s}$$

$f_s \rightarrow$ sampling rate

units – samples/sec

We need to consider that sampling rate ($f_s$) must be high inorder to reconstruct the original signal from its samples.

**Time domain representation of sample signal**:



$$x_s(t) = [x(t)]\left[\sum_{n=-\infty}^{\infty} \delta(t-nT_s)\right]$$

$T_s$ – sampling interval (SI)

$f_s$ – $\frac{1}{T_s}$ – sampling rate (SR)

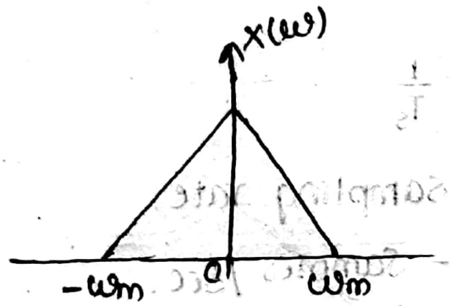
* This is called ideal sampling

## Sampling theorm :

### Sampling theorm for Band limited signals :

**Theorm:** Any signal $x(t)$ which is band limited to $\omega_m$ Hz (i.e., $X(\omega) = 0$ for $|\omega| > \omega_m$) can be completely reconstructed back from its sampled signal taken at a rate $\omega_s \geq 2\omega_m$ (or) $f_s \geq 2f_m$ (or) $\frac{1}{T_s} \geq 2f_m$ where $\omega_s = 2\pi f_s$

**Band limited signal :**




↳ signal Band width

$2\omega_m$

spectral width

$\omega_m$ – max frequency component

**Proof:** Here $2\omega_m$ (or) $2f_m$ is called Nyquist rate. Nyquist rate is the minimum sampling rate at which the signal can be sampled and can be completely reconstructed back from its samples without any distortion.

$$NR = 2\omega_m \text{ (or) } 2f_m = 2(\text{max. freq of signal})$$

**Nyquist interval** : The time interval b/w two adjacent samples when the sampling rate is Nyquist rate.

$$NI = \frac{1}{NR} = \frac{1}{2\omega_m} \text{ or } \frac{1}{2f_m}$$

Based on Nyquist rate sampling is divided into 3 cases:

**case(i)**: over sampling

$$\omega_s > NR \quad \text{i.e.,} \quad \omega_s > 2\omega_m \text{ (or) } f_s > 2f_m$$

**case(ii)**: Critical sampling

$$\omega_s = NR \; ; \quad \omega_s = 2\omega_m \text{ (or) } f_s = 2f_m$$

**case(iii)**: Under sampling

$$\omega_s < NR \; ; \quad \omega_s < 2\omega_m \text{ (or) } f_s < 2f_m$$

**Freq. Domain representation of sampled signal** :

$$x_s(t) = x(t) \sum_{n=-\infty}^{\infty} \delta(t-nT_s)$$

Apply FT

$$FT[x_s(t)] = FT\left[(x(t)) \cdot \left(\sum_{n=-\infty} \delta(t-nT_s)\right)\right]$$

$$X_s(\omega) = \cancel{\tfrac{1}{2\pi}}\left[FT[x(t)] * FT\left[\sum_{n=-\infty}^{\infty} \delta(t-nT_s)\right]\right]\frac{1}{2\pi}$$

$$FT\left[\sum_{n=-\infty}^{\infty} \delta(t-nT_s)\right] = \frac{2\pi}{T_s}\sum_{n=-\infty}^{\infty}\delta(\omega-n\omega_s)$$

$$X_s(\omega) = \left[X(\omega) * \frac{2\pi}{T_s}\sum_{n=-\infty}^{\infty}\delta(\omega-n\omega_s)\right]\frac{1}{2\pi}$$

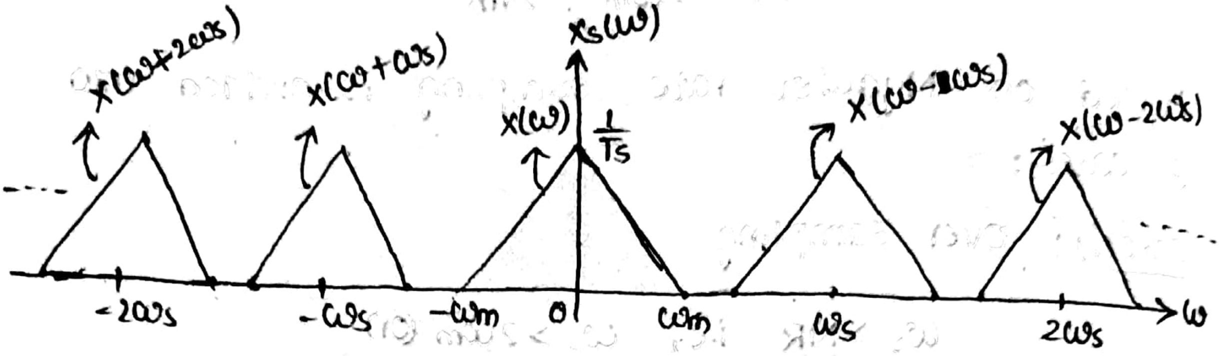
$$X_s(\omega) = \frac{1}{T_s}\left[X(\omega) * \sum_{n=-\infty}^{\infty}\delta(\omega-n\omega_s)\right]$$

$$= \frac{1}{T_s}\sum_{n=-\infty}^{\infty} X(\omega-n\omega_s)$$

*

$$\boxed{X_s(\omega) = \frac{1}{T_s}\sum_{n=-\infty}^{\infty} X(\omega-n\omega_s)}$$

$$X_s(\omega) = \frac{1}{T_s}\Big[\dots + X(\omega+2\omega_s) + X(\omega+\omega_s) + X(\omega) + X(\omega-\omega_s) + X(\omega-2\omega_s) + \dots\Big]$$

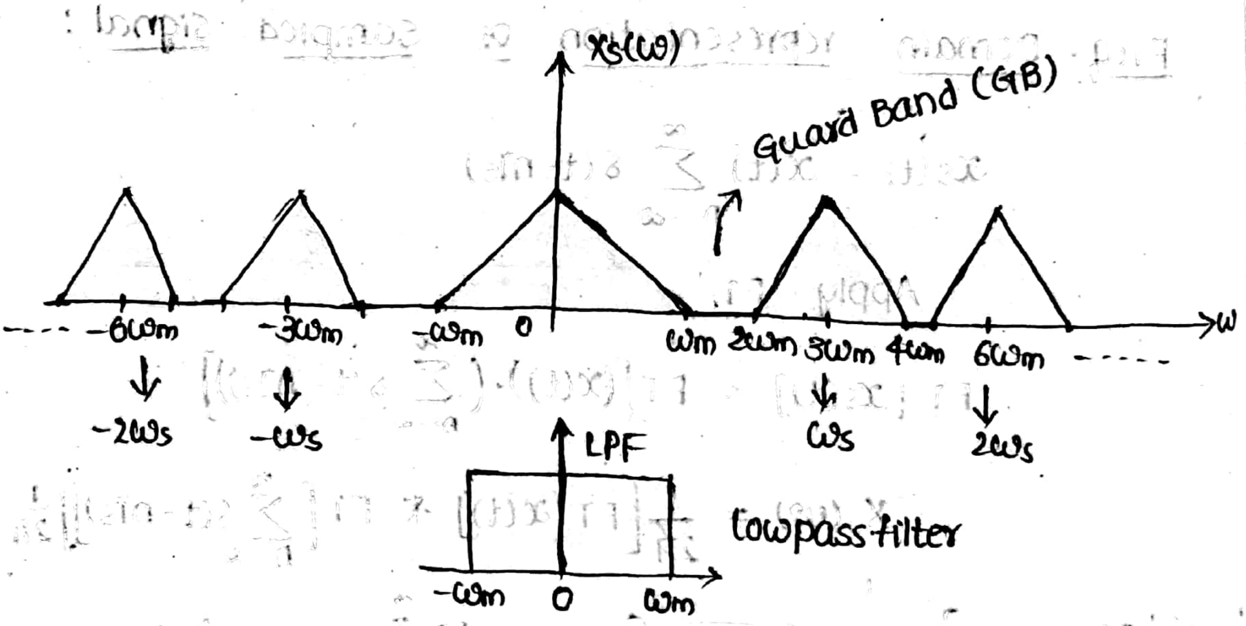



* FT of sampled signal is infinite sum of shifted replicas of spectrum of original signal

<u>Case(i)</u>: over sampling :

$\omega_s \geq 2\omega_m$ $\qquad X_s = \frac{1}{T_s}\sum_{n=-\infty}^{\infty} X(\omega - n(3\omega_m))$

let $\omega_s = 3\omega_m$




* In oversampling the seperation b/w signals is large so that the filtering is easier

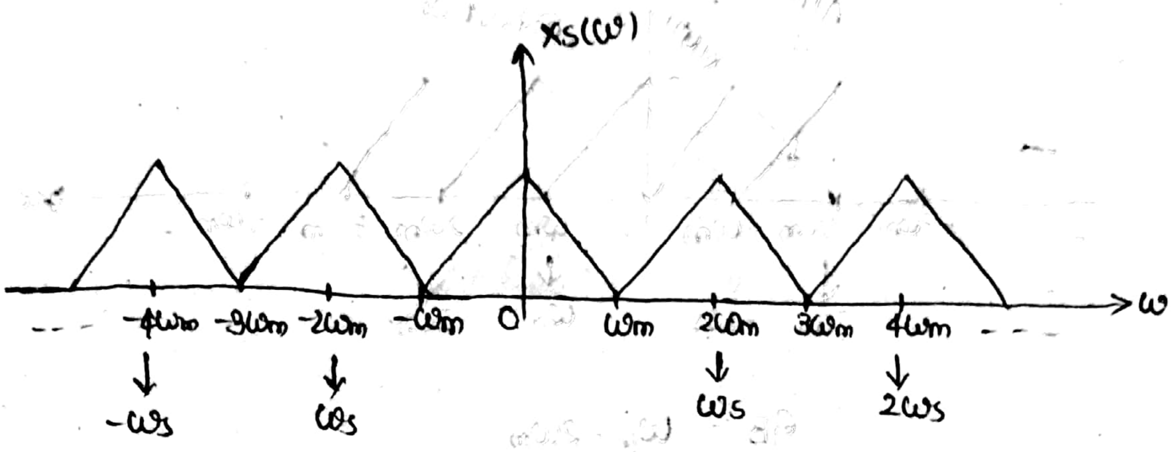
$$\boxed{GB - \omega_s - 2\omega_m}$$

$= 3\omega_m - 2\omega_m$

$= \omega_m$

case(ii): Critical sampling :

$\omega_s = 2\omega_m$

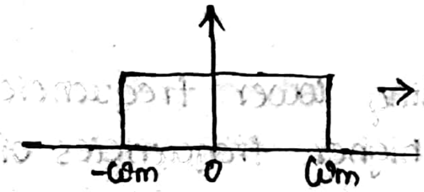



$GB = \omega_s - 2\omega_m$

$= 2\omega_m - 2\omega_m$

$= 0$

$X_s = \frac{1}{T_s} \sum_{n=-\infty}^{\infty} X(\omega - n(2\omega_m))$




→ Ideal LPF → Reconstruction of signal is easy

* In critical sampling the original signal can be reconstructed only when we use Ideal LPF (which are practically not realisable).

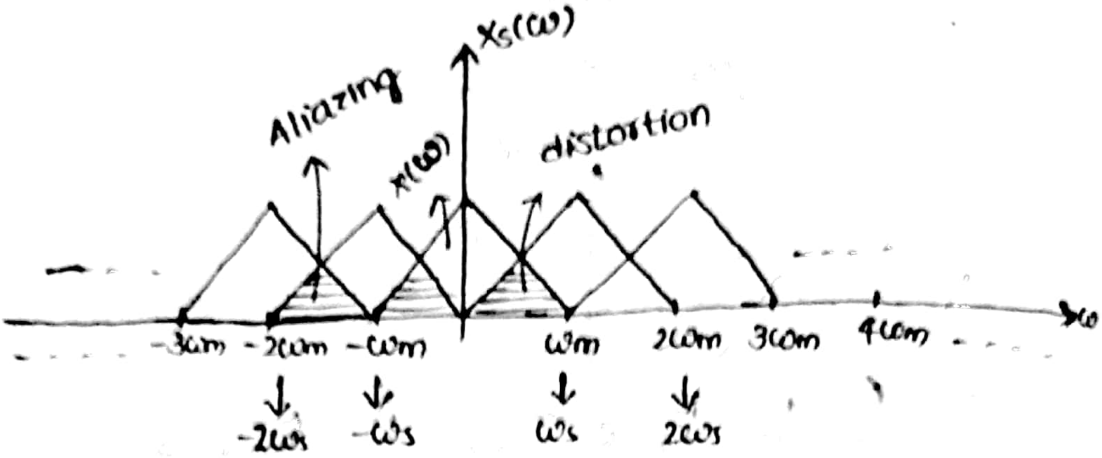
Practical LPF → Distortion occurs

case(iii): Under sampling

$\omega_s < 2\omega_m$

let $\omega_s = \omega_m$

$X_s(\omega) = \frac{1}{T_s}\left[\sum_{n=-\infty}^{\infty} X(\omega - n(\omega_m))\right]$




$$GB = \omega_s - 2\omega_m$$

$$= \omega_m - 2\omega_m$$

$$= -\omega_m$$

<u>Aliazing</u> : overlapping of replicas with original signal.

* In undersampling lower frequencies of $X_s(\omega)$ overlaps with higher frequencies of $X(\omega)$. This overlapping leads to distortion and it is known as aliazing.

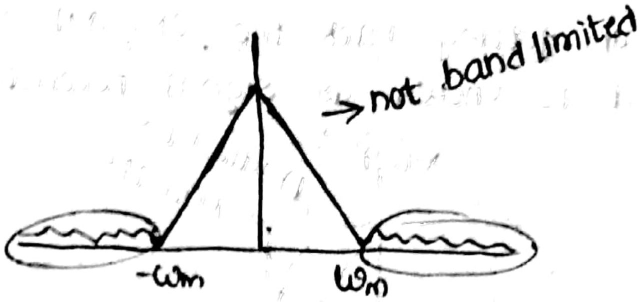
<u>Effects of Undersampling</u>:

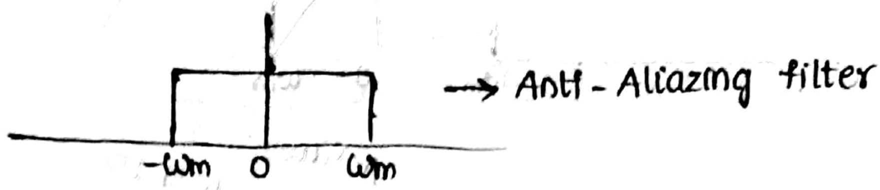
<u>Aliazing</u> :

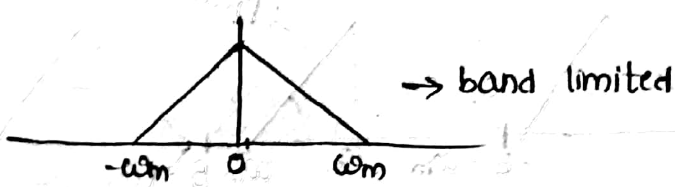
When $\omega_s < 2\omega_m$ lower frequencies of $X_s(\omega)$ overlap with higher frequencies of original signal. due to overlapping original signal will be lost and cannot be reconstructed back.

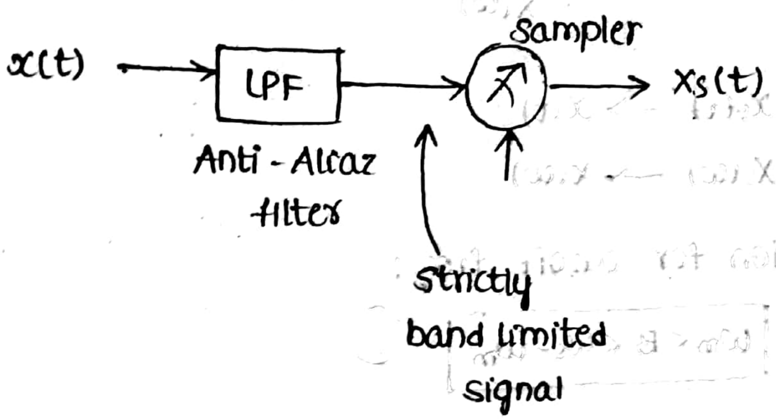
To avoid aliazing sampling rate must be greater than the Nyquist rate.

Aliazing occurs due to (i) If sampling rate is lower than NR (ii) If the signal is not band limited to finite range.




No spectrum in real time is band limited (infinite length spectrum). To make infinite length spectrum as band limited continous signal is passed through a LPF before sampling.

To avoid aliazing :

(i) sampling rate should be greater than NR.

(ii) signal should be passed through Anti-Aliazing filter.

Pb:- $x(t) = \cos(500\pi t) + \cos(350\pi t) + \sin(200\pi t)$

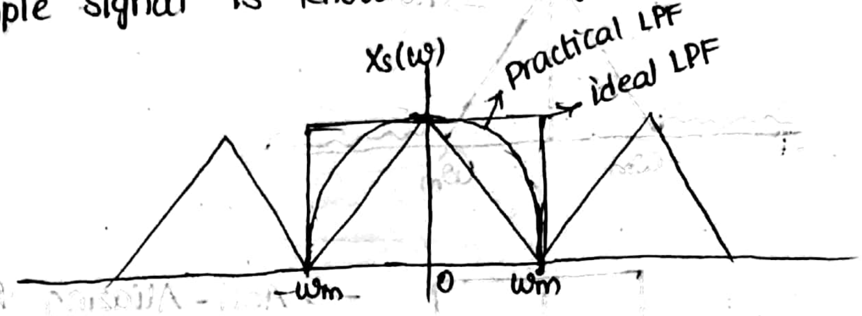
$\omega_1 = 500\pi$ $\quad \omega_2 = 350\pi$ $\quad \omega_3 = 200\pi$

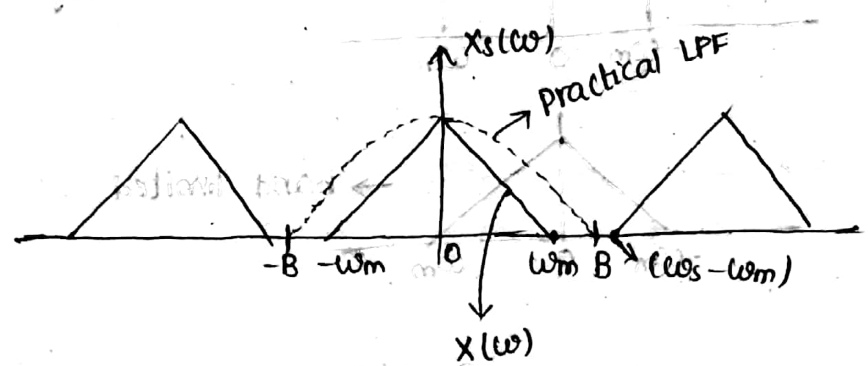
$\omega_m = 500\pi$

NR $= 2\omega_m = 2(500\pi) = 1000\pi$
$= 500\,Hz$

## Signal reconstruction / recovery :

The process of getting back the original signal from the sample signal is known as signal reconstruction.




$$x_s(t) \rightarrow x(t)$$

$$X_s(\omega) \rightarrow X(\omega)$$

condition for cutoff freq :

$$\boxed{\omega_m < B < \omega_s - \omega_m}$$

for over sampling, $\omega_s = 3\omega_m$

$$\omega_m < B < 2\omega_m$$


$$x_r(t) = x_s(t) * h(t)$$

$$= \left[x(t)\sum_{n=-\infty}^{\infty}\delta(t-nT_s)\right] * h(t)$$

$$= \left[\sum_{n=-\infty}^{\infty}x(nT_s)\,\delta(t-nT_s)\right] * h(t)$$

$$x(t)\delta(t-t_0) = x(t_0)\delta(t-t_0)$$

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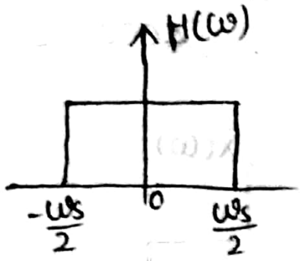
$$x_r(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\, h(t-nT_s)$$

↓

Time domain representation of Reconstructed signal.

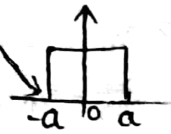



$\xrightarrow{IFT}$ $h(t) = ?$

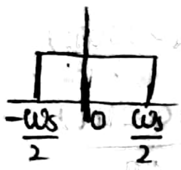
↳ sampling / sinc

$$h(t) = f_s \operatorname{sinc}(f_s t)$$

$$\frac{\sin at}{\pi t} \overset{FT}{\longleftrightarrow}$$




$$\frac{\sin \frac{\omega_s}{2} t}{\pi t} \overset{FT}{\longleftrightarrow}$$




$$\frac{\sin\left(\frac{\cancel{2}\pi f_s t}{\cancel{2}}\right)}{\pi t} = f_s \frac{\sin(\pi f_s t)}{\pi f_s t}$$

$$= f_s \operatorname{sinc}(f_s t)$$

$$x_r(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\, f_s \operatorname{sinc}(f_s(t-nT_s))$$

$$* \quad x_r(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\, f_s \operatorname{sinc}(f_s t - n)$$

This shows that original signal can be reconstructed by weighing each sample by sinc function centred at sampling time and their summing.

⇒

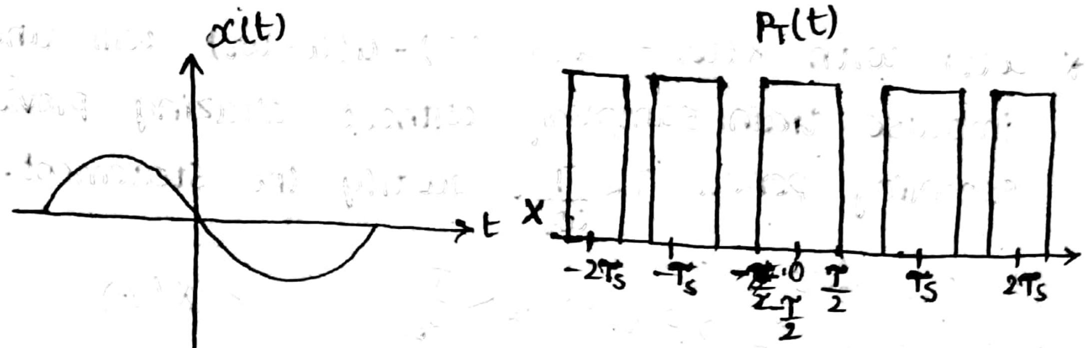
Sampling

- Ideal
- Practical
  - Natural sampling
  - Flat-top sampling

**Natural Sampling:**

* In Ideal sampling we use impulses which are not practically generated
* In Natural sampling generation of pulse train is possible.

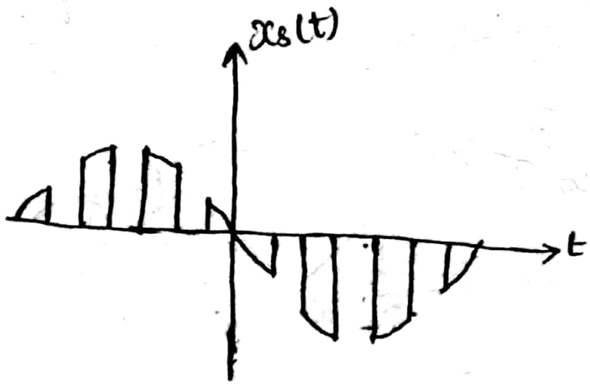
Natural sampling:




Sampling period $T_s$

Pulse width '$\tau$'

$\Downarrow$




$$x_s(t) = x(t) \cdot P_T(t)$$

$P_T(t)$ — Periodic signal with period $T_s$

$$T = T_s$$

$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_s t}$$

$$C_n = \frac{1}{T}\int_T x(t)\, e^{-jn\omega_s t}\, dt$$

$$= \frac{1}{T_s}\int_{-\tau/2}^{\tau/2} (A)\, e^{-jn\omega_s t}\, dt$$

$$= \frac{1}{T_s}\left(A\,\frac{e^{-jn\omega_s t}}{-jn\omega_s}\right)_{-\tau/2}^{\tau/2}$$

$$C_n = \frac{A\tau}{T_s}\,\mathrm{sinc}(nf_s\tau)$$

$$P_T(t) = \sum_{n=-\infty}^{\infty}\left(\frac{A\tau}{T_s}\,\mathrm{sinc}(nf_s\tau)\right)\cdot e^{jn\omega_s t}$$

$$x_s(t) = x(t)\cdot P_T(t)$$

$$* \quad x_s(t) = x(t)\left[\sum_{n=-\infty}^{\infty} \frac{A\tau}{T_s}\,\mathrm{sinc}(nf_s\tau)\, e^{jn\omega_s t}\right]$$

Time Domain of sampled signal

Apply F.T on both sides

$$X_s(\omega) = \sum_{n=-\infty}^{\infty} \frac{A\tau}{T_s}\,\mathrm{sinc}(nf_s\tau)\; F.T\left[x(t)\, e^{jn\omega_s t}\right]$$

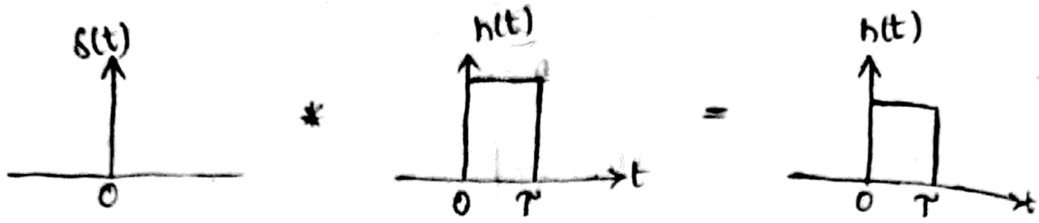
$$FT\left[x(t)\, e^{jn\omega_0 t}\right] = X(\omega-\omega_0)$$

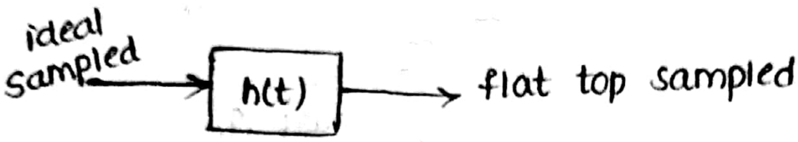
$$* \quad \Rightarrow X_s(\omega) = \sum_{n=-\infty}^{\infty} \frac{A\tau}{T_s}\,\mathrm{sinc}(nf_s\tau)\, X(\omega - n\omega_s)$$

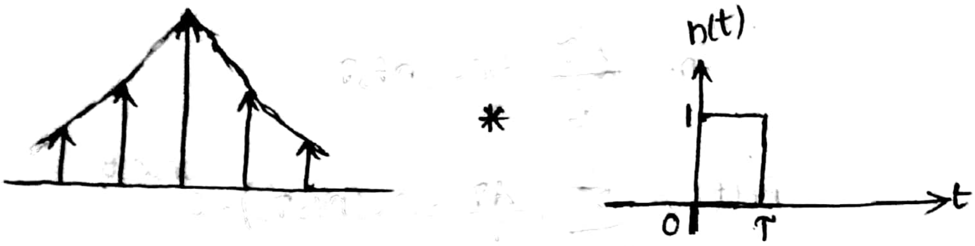
Frequency Domain of sampled signal

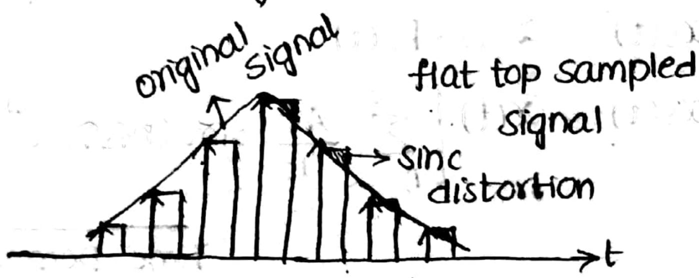
# Flat - top sampling :

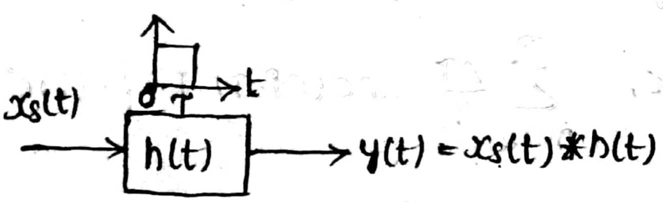
$$\delta(t) * h(t) = h(t)$$




$$y(t) = x_s(t) * h(t)$$

$$y(t) = \left[ x(t) \sum_{n=-\infty}^{\infty} \delta(t - nT_s) \right] * h(t)$$

convolution property

$$x_1(t) * x_2(t) = X_1(\omega) \cdot X_2(\omega)$$

$$Y(\omega) = FT\left[x(t)\sum_{n=-\infty}^{\infty}\delta(t-nT_s)\right] \cdot FT\left[h(t)\right]$$

$$FT\left[\sum_{n=-\infty}^{\infty}\delta(t-nT_s)\right]$$

$$= \frac{2\pi}{T_s}\sum_{n=-\infty}^{\infty}\delta(\omega-n\omega_s)$$

$$FT\left[x(t)\cdot\sum_{n=-\infty}^{\infty}\delta(t-nT_s)\right]$$

$$= \frac{1}{2\pi}\left[FT(x(t)) * FT\left[\sum_{n=-\infty}^{\infty}\delta(t-nT_s)\right]\right]$$

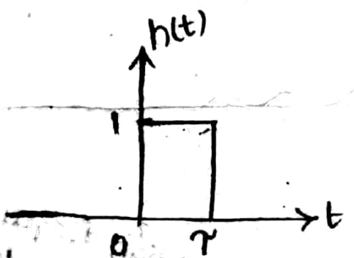
$$= \frac{1}{2\pi}\left[X(\omega) * \frac{2\pi}{T_s}\sum_{n=-\infty}^{\infty}\delta(\omega-n\omega_s)\right]$$

$$= \frac{1}{T_s}\sum_{n=-\infty}^{\infty}X(\omega-n\omega_s)$$

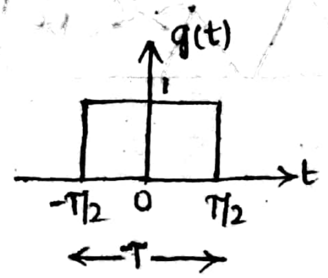
$$Y(\omega) = \left(\frac{1}{T_s}\sum_{n=-\infty}^{\infty}X(\omega-n\omega_s)\right)H(\omega)$$

$$Y(\omega) = H(\omega)\, f_s\sum_{n=-\infty}^{\infty}X(\omega-n\omega_s)$$

Frequency Domain



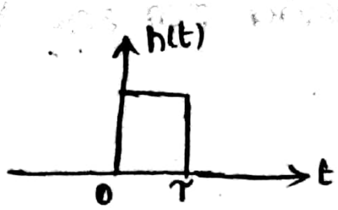

$\xrightarrow{FT}$ ? $H(\omega)$




$$\overset{FT}{\rightleftharpoons} \tau\, Sa\left(\frac{\omega T}{2}\right)$$

$$\tau\, \mathrm{sinc}\left(\frac{\omega T}{2\pi}\right) = G(\omega)$$

$$h(t) = g(t-\tau/2)$$

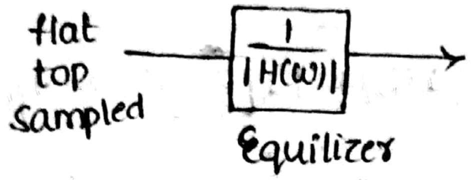



$$\overset{FT}{\rightleftharpoons} e^{-j\omega\tau/2}\cdot G(\omega)$$

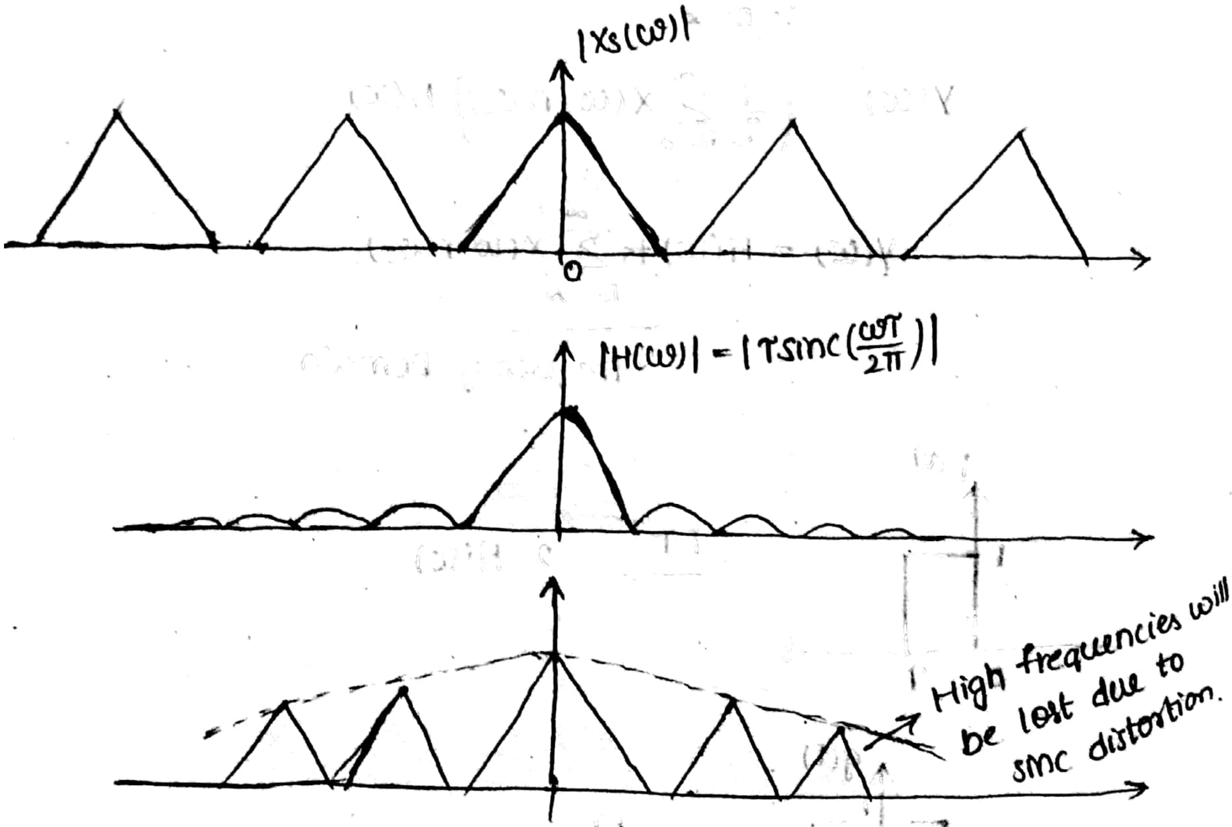
$$H(\omega) = e^{-j\omega\tau/2}\,\tau\,\mathrm{sinc}\left(\frac{\omega T}{2\pi}\right)$$

* Because of maintaining constant amplitude levels we introduce a magnitude distortion of $\tau \operatorname{sinc}\left(\frac{\omega\tau}{2\pi}\right)$ and a phase delay of $-\tau/2$ which is known as aperture effect.

* To cancel this distortion flat top sampled signal is passed through an equilizer.



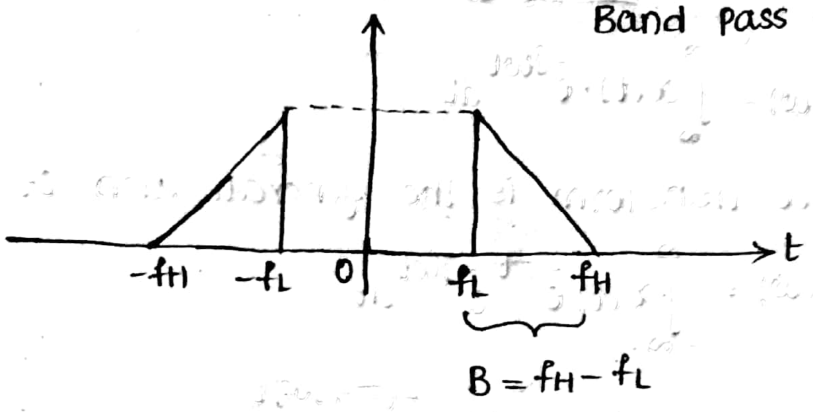

Equilization is a process of correcting channel distortion i.e., sinc distortion.




## Band Pass Signal ~~Sampling~~:

* A signal which has a band of frequencies from one non zero value to another non zero value is Band Pass signal.

Band pass Signal




<u>Band Pass sampling theorm</u> :

A band pass signal $x(t)$ can be recovered back from its sampled signal if the sampling frequency $f_s$

i.e., $\boxed{f_s \geq \frac{2f_H}{K}}$ where K is $K = \frac{f_H}{f_H - f_L} = \frac{f_H}{B}$

$\Rightarrow \boxed{f_s \geq 2B}$

* A Band Pass signal from 4KHz to 6KHz is sampled what is the smallest frequency to retain the original signal.

$$B = f_H - f_L$$
$$= 2KHZ$$

$$f_s \geq 2B$$

$$f_s \geq 4\,KHZ$$

$$Min\ f_s = 4\,KHZ$$

<u>Assignment</u> :

1) State and prove sampling theorm for Band limited signals.
2) Discuss the effects of Under Sampling.
3) Explain reconstruction of signals from samples using relevant expressions.

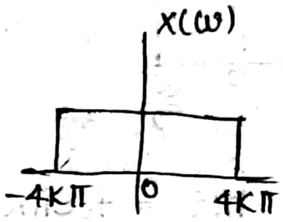
* Find the NR of

1. $x(t) = 1 + \cos(2000\pi t) + \sin(4000\pi t)$

$$\omega_m = 4000\pi$$

$$NR = 2\omega_m = 2(4000\pi) = 8000\pi \text{ rad/sec}$$
$$= 4000 \text{ Hz}$$

2. $x(t) = \dfrac{\sin(4000\pi t)}{\pi t}$




$$a = 4000\pi$$

$$\omega_m = 4000\pi$$

$$NR = 2\omega_m = 8000\pi \text{ rad/sec} = 2\pi f_m$$
$$f_m = 4000 \text{Hz or } 4\text{KHz}$$

3. $x(t) = \text{sinc}(2000\pi)$

$$\frac{\sin(2000\pi t)}{2000\pi}$$

$$\omega_m = 2000\pi$$

$$NR = 4000\pi \text{ rad/sec}$$
$$= 2\text{KHZ}$$

4. $x(t) = \text{sinc}^2(4000\pi t)$

$$\Rightarrow \frac{1}{2}(1 - \cos 2(4000\pi t))$$

$$\omega_m = 8000\pi$$

$$NR = 16000\pi \text{ rad/sec}$$
$$= 8000 \text{ HZ} = 8 \text{ KHZ}$$

* If NR of $x(t)$ is $\omega_0$, what is the NR of the following signals :

a) $x(t) + x(t-1)$

No change in frequency

$$x(t) + x(t-1) \xrightarrow{FT} X(\omega) + e^{-j\omega} X(\omega)$$

$NR = \omega_0$ no modification in $\omega$

b) $\frac{d}{dt}[x(t)]$

$\downarrow FT$

$j\omega X(\omega)$

$NR = \omega_0$

c) $x^2(t) = x(t)x(t)$

$\downarrow FT$

$X(\omega) * X(\omega)$

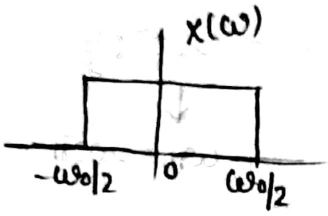
change in frequency due to convolution frequency will be doubled

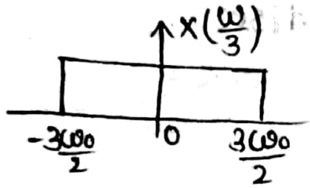
$NR = 2\omega_0$

d) $x(3t)$

$\downarrow FT$

$\frac{1}{3}X\left(\frac{\omega}{3}\right)$



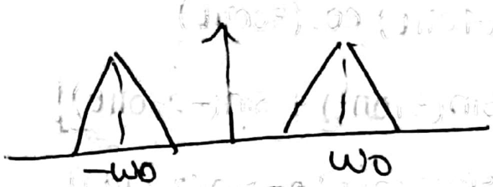

$\omega_m = \frac{3\omega_0}{2}$

$NR = 2\left(\frac{3\omega_0}{2}\right) = 3\omega_0$

e) $x(t)\cos\omega_0 t$

$\downarrow FT$

$\frac{1}{2}\left[X(\omega-\omega_0) + X(\omega+\omega_0)\right]$




$\downarrow$

$\omega + \omega_0$ ~~$= \omega_m$~~

$\frac{\omega_0}{2}$ max possible frequency

$= \frac{\omega_0}{2} + \omega_0$

$= \frac{3\omega_0}{2}$

$NR = 2\left(\frac{3\omega_0}{2}\right) = 3\omega_0$

* Find NR & NI of

1) $x(t) = 2\operatorname{sinc}(100\pi t)$

$\downarrow$

$\omega_m = 100\pi$ $\quad NR = 2\omega_m = 200\pi$ rad/sec

$NR = 200\pi \text{ rad/sec}$

$= 100 \text{ HZ}$

$NI = \frac{1}{NR}$

2) $x(t) = \text{sinc}(80\pi t)\,\text{sinc}(120\pi t)$

$= \frac{\sin(80\pi t)}{80\pi t} \cdot \frac{\sin(120\pi t)}{120\pi t}$

$= \frac{1}{(80\pi)(120\pi)t^2}\left[\sin(80\pi t)\,\sin(120\pi t)\right]$

$= \frac{1}{2t^2(80\pi)(120\pi)}\left[\cos(40\pi t) - \cos(200\pi t)\right]$

$\downarrow$

$\omega_m = 200\pi$

$NR = 2\omega_m = 400\pi \text{ rad/sec}$

$= 200 \text{ HZ}$

* a) $\text{rect}(300t)$

b) $-10\sin(40\pi t)\cos(300\pi t)$

$= -5\left[\sin(340\pi t) + \sin(-260\pi t)\right]$

$= -5\left[\sin(340\pi t) - \sin(260\pi t)\right]$

$\downarrow$

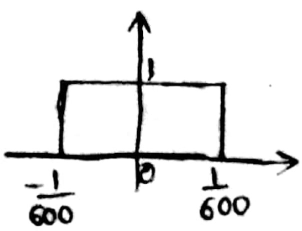
$\omega_m = 340\pi$

$NR = 2\omega_m = 680\pi \text{ rad/sec}$

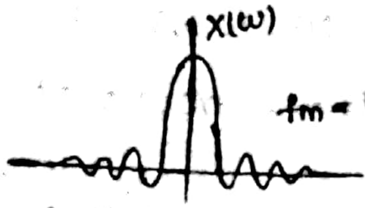
$= 340 \text{ HZ}$

a) $\text{rect}(300t)$

$= \text{rect}\left(\frac{t}{\frac{1}{300}}\right)$




$\xleftrightarrow{FT} \quad \frac{1}{300}\,\text{sinc}\left(\frac{\omega}{600}\right)$




$f_m = \text{infinite}$

$NR = 2\omega_m = \text{infinite}$

* a) $\text{sinc}^2(100\pi t)$

$$\Rightarrow \frac{\sin 100\pi t}{100\pi t} \cdot \frac{\sin(100\pi t)}{100\pi t}$$

frequency doubles

$\omega_m = 200\pi$

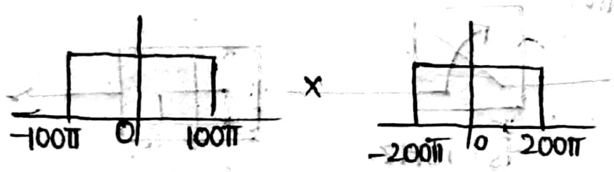
$NR = 2\omega_m = 400\pi$ rad/sec

b) $\text{sinc}(100\pi t) + \text{sinc}^2(50\pi t)$

$NR = 200\pi$ rad/sec (or) 100HZ

c) $\text{sinc}(100\pi t) * \text{sinc}(200\pi t)$

$\downarrow$ FT $\qquad\qquad$ $\downarrow$ FT




$\rightarrow \omega_m = 100\pi$

$NR = 200\pi$ rad/sec

d) $\text{sinc}(100\pi t) \cdot \text{sinc}(200\pi t)$

$$= \frac{1}{2}\left[\cos(100\pi t) - \cos(300\pi t)\right]$$

$\Rightarrow \omega_m = 300\pi \quad NR = 2\omega_m = 600\pi$ rad/sec

* A signal $x(t) = \sin(150\pi t)$ is idealy sampled at a) 100HZ b) ~~200~~ 150 HZ c) 300HZ for each case explain if you can receive the original signal.

Sol:-

$$\cancel{f_m = \frac{2\pi}{\omega_m}} \quad f_m = \frac{\omega_m}{2\pi} = 75\,HZ$$

In order to get original signal, fs must be greater than 2fm.

$NR = 150\,HZ$

$$\boxed{f_s > 2f_m}$$

$f_s > 150\,HZ$

a) $f_s = 100\,HZ \quad f_s < 150\,HZ$

signal reconstruction is possible

b) $f_s = 150\,Hz$

$f_s = 2f_m$

signal reconstruction is possible with ideal LPF

c) $f_s = 300\,Hz$

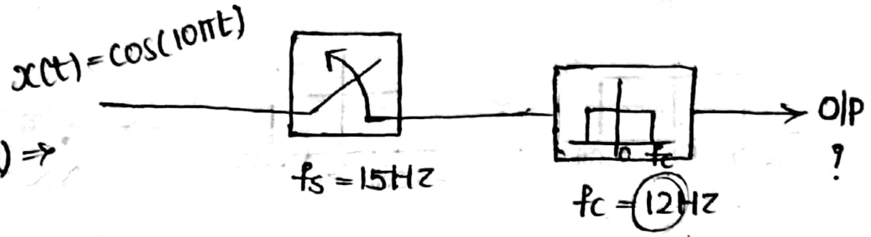
$f_s > 2f_m$

signal reconstruction is possible

(*) $x(t) = \cos(10\pi t)$ is ideally sampled at 15 Hz and it is passed through a LPF with cut off frequency 12 Hz. what frequencies will appear at the output?

Sol:-

Q) ⇒




** For any signal the sampled spectrum contains the frequencies $\pm(\omega_m \pm n\omega_s)$ (or) $\pm(f_m \pm nf_s)$

$f_m = 5\,Hz$, $f_s = 15\,Hz$

o/p : $f_m \pm nf_s$

$n=0 \Rightarrow f_m$ — $+f_m = 5\,Hz$ ✓ ; $-f_m = -5\,Hz$ ✓

$n=1 \Rightarrow f_m + f_s = 20\,Hz$

$f_m - f_s = -10\,Hz$ ✓

$n=2 \Rightarrow f_m + 2f_s = 5 + 2(15) = 35\,Hz$

$f_m - 2f_s = 5 - 2(15) = -25\,Hz$

o/p : 5 Hz & 10 Hz

since cut off freq = 12 Hz

The freq will be cut off after 12 Hz

so we take freq less than 12 Hz

* $x(t) = 2\cos(400\pi t) + 6\cos(640\pi t)$

$f_s = 500\,HZ$

$f_c = 400\,HZ$

$\omega_m = 640\pi$

$f_m = 320\,HZ$

O/P : $f_m \pm n f_s$

if $n = 0 \Rightarrow f_m$ $\begin{cases} +320 \checkmark \\ -320 \end{cases}$

$n = 1 \Rightarrow f_m + f_s = 820\,HZ$

$f_m - f_s = -180\,HZ \checkmark$

$n = 2 \Rightarrow f_m + 2f_s = 1320\,HZ$

$f_m - 2f_s = -680\,HZ$

O/P : 180 HZ & 320 HZ

* $x(t)$ with $X(\omega) = u(\omega + \omega_0) - u(\omega - \omega_0)$ will undergo impulse train sampling without aliazing provided that sampling period $T < \frac{\pi}{\omega_0}$. justify the statement.

without aliazing, $\omega_s > 2\omega_0 \Rightarrow \frac{2\pi}{T_s} > 2\omega_0 \Rightarrow T_s < \frac{\pi}{\omega_0}$

~~without aliazing~~ >

$> 2\omega_0$