4.5

Linear Recurrence relations

A wide variety of recurrence relations occur in models. Some of these recurrence relations can be solved using iteration or some other **adhoc** technique. However, one important class of recurrence relations can be explicitly solved in a systematic way. These are recurrence relations that express terms of a sequence as linear combination of previous terms.

Linear homogeneous recurrence relation of degree $\boldsymbol{k}$

A linear homogeneous recurrence relation of degree k is a recurrence relation of the form

$$\boldsymbol{a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k},}$$

where $c_1, c_2, \dots, c_k$ are real numbers and $c_k \neq 0$.

A recurrence relation is ***linear*** if a_n is the sum of previous terms of the sequence each multiplied by a function of n. The recurrence relation is ***homogeneous*** because no terms occur that are not multiples of the a_js. The coefficients of the terms of the sequence are all ***constants*** rather than functions n. The **degree** is k because a_n is expressed in terms of the previous k terms of the sequence.

Example 1: The recurrence relation $P_n = (1.11)P_{n-1}$ is a linear homogeneous recurrence relation of degree one. The recurrence relation $f_n = f_{n-1} + f_{n-2}$ is a linear homogeneous recurrence relation of degree two. The recurrence relation $a_n = a_{n-1} + a_{n-2}^2$ is not linear. The recurrence relation $H_n = 2H_{n-1} + 1$ is not homogeneous.

Note: A sequence $\{a_n\}$ satisfying the recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$$

where $c_1, c_2, \dots, c_k$ are real constants, is uniquely determined whenever k initial conditions $a_0 = c_0, a_1 = c_1, \dots, a_{k-1} = c_{k-1}$ are given.

Solving linear homogeneous recurrence relations with constant coefficients

Lemma 1: ***A sequence $\{a_n\}$ defined by $a_n = r^n$ is a solution of the linear homogeneous recurrence relation of degree k with constant coefficients***

$$\boldsymbol{a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k},} \qquad \ldots (1)$$

if and only if r is a solution of the equation

$$\boldsymbol{r^k - c_1 r^{k-1} - c_2 r^{k-2} - \cdots - c_{k-1} r - c_k = 0} \qquad \ldots (2)$$

Proof: A sequence $\{a_n\}$ defined by $a_n = r^n$ is a solution of equation (1) if and only if it satisfies equation (1), *i.e.,*

$$r^n - c_1 r^{n-1} - c_2 r^{n-2} - \cdots - c_{k-1} r^{n-k+1} - c_k r^{n-k} = 0$$

Because, we are looking for nonzero solutions, $r \neq 0$, cancelling r^{n-k} on both sides, we get

$$r^k - c_1 r^{k-1} - c_2 r^{k-2} - \cdots - c_{k-1} r - c_k = 0 \qquad \ldots (3)$$

Thus, $a_n = r^n$ is a solution of equation (1) if and only if r is a solution of equation (3).

Hence the result

The equation (3) is called the ***characteristic equation*** of the recurrence relation (1) and the roots of the equation (3) are called ***characteristic roots*** of the recurrence relation (1).

We now consider linear homogeneous recurrence relation of degree two. We consider the case when there are two distinct characteristic roots.

Theorem 1: ***Let c_1 and c_2 be real numbers. Suppose that $r^2 - c_1 r - c_2 = 0$ has two distinct roots r_1 and r_2. Then the sequence $\{a_n\}$ is a solution of the recurrence relation***

$$\boldsymbol{a_n = c_1 a_{n-1} + c_2 a_{n-2}} \qquad \ldots (1)$$

if and only if $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n,\ n = 0, 1, 2, \ldots$, where α_1 and α_2 are constants.

Proof: We have that r_1 and r_2 are the distinct roots of the equation $r^2 - c_1 r - c_2 = 0$, where c_1 and c_2 are real numbers. Therefore, $r_i^2 = c_1 r_i + c_2, \;\; i = 1, 2$.

Let $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$, where α_1 and α_2 are constants. Then

$$
\begin{aligned}
c_1 a_{n-1} + c_2 a_{n-2} &= c_1(\alpha_1 r_1^{n-1} + \alpha_2 r_2^{n-1}) + c_2(\alpha_1 r_1^{n-2} + \alpha_2 r_2^{n-2}) \\
&= \alpha_1 r_1^{n-2}(c_1 r_1 + c_2) + \alpha_2 r_2^{n-2}(c_1 r_2 + c_2) \\
&= \alpha_1 r_1^{n-2}.r_1^2 + \alpha_2 r_2^{n-2}.r_2^2 = \alpha_1 r_1^n + \alpha_2 r_2^n = a_n
\end{aligned}
$$

This shows that the sequence $\{a_n\}$, where $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ is a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$

Conversely, suppose that the sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1\, a_{n-1} + c_2\, a_{n-2}$.

Let $a_0 = k_0$ and $a_1 = k_1$ be the initial conditions of the recurrence relation.

We will now show that there are constants α_1 and α_2 such that the sequence $\{a_n\}$ with $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ satisfies these initial conditions.

This requires that $k_0 = a_0 = \alpha_1 + \alpha_2 \;\; ; \;\; k_1 = a_1 = \alpha_1 r_1 + \alpha_2 r_2$

Now, $\alpha_2 = k_0 - \alpha_1$ and so $k_1 = \alpha_1 r_1 + (k_0 - \alpha_1) r_2 \Longrightarrow k_1 = \alpha_1(r_1 - r_2) + k_0$

This shows that $\alpha_1 = \frac{k_1 - k_0 r_2}{r_1 - r_2}$, $(r_1 \neq r_2)$ and $\alpha_2 = k_0 - \alpha_1 = \frac{k_0 r_1 - k_1}{r_1 - r_2}$

Therefore, with these values of α_1 and α_2, the sequence $\{a_n\}$ with $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ satisfy the two initial conditions.

Now, $\{a_n\}$ and $\{\alpha_1 r_1^n + \alpha_2 r_2^n\}$ are both solutions of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$, when $n = 0$ and $n = 1$. It is known that a sequence satisfying the recurrence relation is uniquely determined by initial conditions.

Therefore, a_n must be equal to $\alpha_1 r_1^n + \alpha_2 r_2^n$, for all nonnegative integers n.

Thus, a solution of equation (1) must be of the form $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$, where α_1 and α_2 are constants. Hence the theorem

Example 2: Find the solution of the recurrence relation $\boldsymbol{a_n = a_{n-1} + 2a_{n-2}}$ with $\boldsymbol{a_0 = 2}$ and $\boldsymbol{a_1 = 7}$?

Solution: The given recurrence relation is $a_n = a_{n-1} + 2a_{n-2}$

This is a linear homogeneous recurrence relation with constant coefficients of degree two. The characteristic equation is

$$r^2 = r + 2, \textit{ i.e.}, r^2 - r - 2 = 0 \Longrightarrow (r+1)(r-2) = 0$$

The characteristic roots are $r = -1$ and $r = 2$.

Therefore, the sequence $\{a_n\}$ is a solution to the recurrence relation iff

$$a_n = \alpha_1(-1)^n + \alpha_2 2^n, \text{ where } \alpha_1 \text{ and } \alpha_2 \text{ are constants.}$$

The given initial conditions are $a_0 = 2$ and $a_1 = 7$

$$\textit{i.e.}, a_0 = 2 = \alpha_1 + \alpha_2\,,\ a_1 = 7 = -\alpha_1 + 2\alpha_2.$$

Solving for α_1, α_2 , we get $\alpha_1 = -1$ and $\alpha_2 = 3$.

The required solution is $a_n = -(-1)^n + 3.2^n$,*i.e.*, $a_n = (-1)^{n+1} + 3.2^n$, for all nonnegative integers n.

Fibonacci sequence: The sequence of Fibonacci numbers are:
$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, \dots$, …. and they satisfy the recurrence relation:
$f_n = f_{n-1} + f_{n-2},\ n \geq 2$, with initial conditions $f_0 = 0$ and $f_1 = 1$.

Example 3: Find an explicit formula for the Fibonacci numbers.

Solution: The sequence of Fibonacci numbers satisfies the recurrence relation

$$f_n = f_{n-1} + f_{n-2},\ \ n \geq 2,\ \ f_0 = 0 \text{ and } f_1 = 1.$$

This is a linear homogenous recurrence relation with constant coefficients of degree two. Its characteristic equation is $r^2 = r + 1$, *i.e.*, $r^2 - r - 1 = 0$. Solving for r we get, $= \frac{1 \pm \sqrt{5}}{2}$. The characteristic roots are $r_1 = \frac{1+\sqrt{5}}{2}$ and $r_2 = \frac{1-\sqrt{5}}{2}$.
Therefore, the sequence $\{f_n\}$ is a solution of the recurrence relation iff

$f_n = \alpha_1 \left(\frac{1+\sqrt{5}}{2}\right)^n + \alpha_2 \left(\frac{1-\sqrt{5}}{2}\right)^n$, where α_1 and α_2 are constants.

To find α_1 and α_2, we use the initial conditions $f_0 = 0$ and $f_1 = 1$.

$$f_0 = 0 = \alpha_1 + \alpha_2 \quad ; \quad f_1 = 1 = \alpha_1 \left(\frac{1+\sqrt{5}}{2}\right) + \alpha_2 \left(\frac{1-\sqrt{5}}{2}\right)$$

Solving we get, $\alpha_1 = \frac{1}{\sqrt{5}}, \ \ \alpha_2 = -\frac{1}{\sqrt{5}}$.

Thus, the solution for the given recurrence relation is

$$f_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

Therefore, the Fibonacci numbers are given by

$$\boldsymbol{f_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n}$$

for all nonnegative integers n

Note: Theorem 1 is not applicable when there is one characteristic root of multiplicity two. If r_0 is a root of multiplicity two of the characteristic equation then nr_0^n is another solution besides r_0^n. The following theorem shows this case.

Theorem 2: ***Let c_1 and c_2 be real numbers with $c_2 \neq 0$. Suppose that $r^2 - c_1 r - c_2 = 0$ has only one root r_0. A sequence $\{a_n\}$ is a solution of the recurrence relation***

$$\boldsymbol{a_n = c_1 a_{n-1} + c_2 a_{n-2}}$$

if and only if

$$\boldsymbol{a_n = \alpha_1 r_0^n + \alpha_2 n r_0^n,}$$

for all nonnegative integers n, where α_1 and α_2 are constants.

Example 4: Find the solution of the recurrence relation

$$\boldsymbol{a_n = 6a_{n-1} - 9a_{n-2}, \quad a_0 = 1, \ a_1 = 6.}$$

Solution: The given recurrence relation is a linear homogenous recurrence relation with constant coefficients of degree two. Its characteristic equation is

$r^2 = 6r - 9,\ i.e.,\ r^2 - 6r + 9 = 0 \Longrightarrow (r-3)^2 = 0.$ The characteristic root 3 and its multiplicity is 2. Therefore, the solution of the given recurrence relation is

$$a_n = \alpha_1 3^n + \alpha_2 n 3^n \text{ , where } \alpha_1 \text{ and } \alpha_2 \text{ are constants.}$$

To evaluate α_1 and α_2 we use the initial conditions.

Take $n = 0,\ a_0 = 1 = \alpha_1$ and take $n = 1, a_1 = 6 = 3\alpha_1 + 3\alpha_2$

Solving, we get $\alpha_1 = 1$ and $\alpha_2 = 1$. Thus, the solution of the given recurrence relation with the initial conditions is $a_n = 3^n + n3^n$, for all nonnegative integers n

The following are general result when the roots are distinct.

Theorem 3: ***Let $c_1, c_2, \dots, c_k$ be real numbers. Suppose that the characteristic equation $r^k - c_1 r^{k-1} - c_2 r^{k-2} - \cdots - c_k = 0$ has k distinct roots $r_1, r_2, \dots, r_k$. Then a sequence $\{a_n\}$ is a solution of the recurrence relation***

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} \quad , \ c_k \neq 0$$

if and only

$$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n + \cdots + \alpha_k r_k^n \,,$$

for $n = 0, 1, 2, \dots$, where $\alpha_1, \alpha_2, \dots, \alpha_k$ are constants.

Example 5: Find the solution of the recurrence relation

$$a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3}$$

with the initial conditions $a_0 = 2, a_1 = 5$ and $a_2 = 15$

Solution: The given recurrence relation is a linear homogenous recurrence relation of degree 3 with constant coefficients. The characteristic equation is

$$r^3 = 6r^2 - 11r + 6 \text{ , } i.e.,\ r^3 - 6r^2 + 11r - 6 = 0$$

Notice that $r = 1$ satisfies the characteristic equation and so $r - 1$ is a factor. Then$(r-1)(r^2 - 5r + 6) = 0$ and $(r-1)(r-2)(r-3) = 0$.

The characteristic roots are $r = 1, 2, 3$ and they are all distinct. Therefore, the solutions to this recurrence relation are of the form

$$a_n = \alpha_1 . 1^n + \alpha_2 . 2^n + \alpha_3 . 3^n$$

To find the constants α_1, α_2 and α_3 we use the initial conditions.

$$a_0 = 2 = \alpha_1 + \alpha_2 + \alpha_3$$

$$a_1 = 5 = \alpha_1 + 2\alpha_2 + 3\alpha_3$$

$$a_2 = 15 = \alpha_1 + 4\alpha_2 + 9\alpha_3$$

Solving these simultaneous equations we get, $\alpha_1 = 1, \alpha_2 = -1$ and $\alpha_3 = 2$. Therefore, the unique solution to the given recurrence relation and the given initial conditions is the sequence $\{a_n\}$ with

$$a_n = 1 - 2^n + 2.3^n$$

for all nonnegative integers n

The following is the most general result related to linear homogenous recurrence relation with constant coefficients, *allowing the characteristic equation to have multiple roots.*

For each root r of multiplicity m of the characteristic equation, the general solution has a summand of the form $P(n)r^n$, where $P(n)$ is a polynomial of degree $m - 1$.

Theorem 4: ***Let $c_1, c_2, \ldots, c_k$ be real numbers. Suppose that the characteristic equation***

$$\boldsymbol{r^k - c_1 r^{k-1} - c_2 r^{k-2} - \cdots - c_k r^k = 0}$$

has t disticnt roots $r_1, r_2, \ldots, r_t$ with multiplicities $m_1, m_2, \ldots, m_t$ respectively, so that $m_i \geq 1$ for $i = 1, 2, \ldots, t$ and $m_1 + m_2 + \cdots + m_t = k$.

Then a sequence $\{a_n\}$ is a solution of the recurrence relation

$$\boldsymbol{a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}}$$

if and only if

$$\boldsymbol{a_n = \left(\alpha_{1,0} + \alpha_{1,1}\, n + \alpha_{1,2}\, n^2 + \cdots + \alpha_{1,m_1-1}\, n^{m_1-1}\right) r_1^n}$$
$$\boldsymbol{+\left(\alpha_{2,0} + \alpha_{2,1}\, n + \alpha_{2,2}\, n^2 + \cdots + \alpha_{2,m_2-1}\, n^{m_2-1}\right) r_2^n}$$
$$\boldsymbol{+ \cdots + \left(\alpha_{t,0} + \alpha_{t,1} n + \alpha_{t,2} n^2 + \cdots + \alpha_{t,m_t-1} n^{m_t-1}\right) r_t^n}$$

for $\boldsymbol{n = 0, 1, 2, \ldots, \ldots}$***, where*** $\boldsymbol{\alpha_{i,0}}$ ***are constants for*** $\boldsymbol{1 \le i \le t}$ ***and*** $\boldsymbol{0 \le j \le m_i - 1.}$

Example 6: Suppose that the roots of the characteristic equation of linear homogenous recurrence relation of degree 6 with constant coefficients are $2, 2, 2, 5, 5$ and 9. What is the form of the general solution?

Solution: Given that there are three roots. The root 2 with multiplicity three, the root 5 with multiplicity two and the root 9 with multiplicity one. Therefore, the general solution is of the form

$a_n = \left(\alpha_{1,0} + \alpha_{1,1}\, n + \alpha_{1,2}\, n^2\right)2^n + \left(\alpha_{2,0} + \alpha_{2,1}\, n\right)5^n + \alpha_{3,0}\, 9^n$, for $n = 0, 1, 2, \ldots$

Example 7: Find the solution to the recurrence relation

$$\boldsymbol{a_n = -3a_{n-1} - 3a_{n-2} - a_{n-3}}$$

with initial conditions $\boldsymbol{a_0 = 1, a_1 = -2,}$and $\boldsymbol{a_2 = -1.}$

Solution: The given recurrence relation is a linear homogenous recurrence relation of degree 3 with constant coefficients. The characteristic equation is

$$r^3 = -3r^2 - 3r - 1$$

i.e., $r^3 + 3r^2 + 3r + 1 = 0$, $i.e.,$ $(r+1)^3 = 0$ and $r = (-1)$ (repeated thrice). Thus, the characteristic equation has only one root $r = -1$ with multiplicity three. The solutions of the given recurrence relation is of the form

$$a_n = (\alpha + \beta n + rn^2)(-1)^n$$

where α, β and γ are constants. We evaluate α, β and γ using the initial conditions.

Taking $n = 0$; $a_0 = 1 = \alpha$

Taking $n = 1$; $a_1 = -2 = -\alpha - \beta - \gamma$

Taking $n = 2$; $a_2 = -1 = \alpha + 2\beta + 4\gamma$

Solving the above three simultaneous equations we get,

$$\alpha = 1, \beta = 3 \text{ and } \gamma = -2$$

Therefore, the unique solution to the given recurrence relation with the given initial conditions is the sequence $\{a_n\}$, where $a_n = (1 + 3n - 2n^2)(-1)^n$.

Linear Nonhomogeneous Recurrence Relations with Constants Coefficients

Consider a *linear nonhomogeneous recurrence relation with constant coefficients* of the form

$$\boldsymbol{a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + F(n),}$$

where $c_1, c_2, \dots, c_k$ are real numbers and $F(n)$ is a function, not identically zero, depending only on n.

The recurrence relation

$$\boldsymbol{a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}}$$

is called the ***associated homogenous recurrence relation***.

Example 8: Each of the recurrence relations

$$a_n = a_{n-1} + 2^n$$

$$a_n = a_{n-1} + a_{n-2} + n^2 + n + 1$$

$$a_n = 3a_{n-1} + n3^n$$

$$\text{and} \qquad a_n = a_{n-1} + a_{n-2} + a_{n-3} + n!$$

is a linear nonhomogeneous recurrence relation with constant coefficients. The associated homogeneous recurrence relations are

$a_n = a_{n-1},\ a_n = a_{n-1} + a_{n-2},\ a_n = 3a_{n-1}$ and $a_n = a_{n-1} + a_{n-2} + a_{n-3}$ respectively.

The key fact about linear nonhomogeneous recurrence relation with constant coefficients is that ***every solution is the sum a solution of the associated linear homogeneous recurrence relation and particular solution***.

Theorem 5: ***If*** $\left\{\boldsymbol{a_n^{(p)}}\right\}$ ***is a particular solution of the linear nonhomogeneous recurrence relation with constant coefficients***

$$\boldsymbol{a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + F(n)} \qquad \dots (1)$$

then every solution of (1) ***is of the form*** $\left\{\boldsymbol{a_n}^{(p)} + \boldsymbol{a_n}^{(h)}\right\}$***, where*** $\boldsymbol{a_n}^{(h)}$ ***is a solution of the associated homogeneous recurrence relation***

$$\boldsymbol{a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}} \dots \qquad \dots (2)$$

Proof: Because $a_n^{(p)}$ is a particular solution of (1),

$$a_n^{(p)} = c_1 a_{n-1}^{(p)} + c_2 a_{n-2}^{(p)} + \cdots + c_k a_{n-k}^{(p)} + F(n) \qquad \dots (3)$$

Suppose that $\{b_n\}$ be any solution of (1). Then

$$b_n = c_1 b_{n-1} + c_2 b_{n-2} + \cdots + c_k b_{n-k} + F(n) \qquad \dots (4)$$

Subtracting (3) from (4), we get

$$b_n - a_n^{(p)} = c_1\left(b_{n-1} - a_{n-1}^{(p)}\right) + c_2\left(b_{n-2} - a_{n-2}^{(p)}\right) + \cdots + c_k\left(b_{n-k} - a_{n-k}^{(p)}\right)$$

This shows that $\left\{b_n - a_n^{(p)}\right\}$ is a solution of (2), say $a_n^{(h)}$.

That is $a_n^{(h)}$ is a solution of the associated homogeneous linear equation. Consequently, $b_n - a_n^{(p)} = a_n^{(h)}$ *i.e.*, $b_n = a_n^{(h)} + a_n^{(p)}$

Hence the theorem

Note: We see that the key to solving (1) is finding a particular solution. Then every solution is a sum of this particular solution and a solution of (2). Although there is no general method for finding such a particular solution that works for every function $F(n)$, there are techniques that work for certain types of functions of $F(n)$, such as polynomials and powers of constants. The following is a related theorem:

Theorem 6: ***Suppose that $\{a_n\}$ satisfies the linear nonhomogeneous recurrence relation***

$$\boldsymbol{a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + F(n),} \qquad \ldots (1)$$

where $c_1, c_2, \ldots, c_k$ are real numbers, and

$$\boldsymbol{F(n) = (b_0 + b_1 n + b_2 n^2 + \cdots + b_t n^t) s^n}$$

where $b_0, b_1, \ldots, b_t$ and s are real numbers.

(i) When s is not a root of the characterstic equation of the associated homogenous recurrence relation, there is a particular solution of (1) of the form

$$\boldsymbol{(p_0 + p_1 n + p_2 n^2 + \cdots + p_t n^t) s^n}$$

(ii) When s is a root of the characterstic equation and its multiplicity is m, there is a particular solution of (1) of the form

$$\boldsymbol{n^m (p_0 + p_1 n + p_2 n^2 + \cdots + p_t n^t) s^n}$$

Remark: Care must be taken when $s = 1$, in particular when

$$F(n) = b_0 + b_1 n + b_2 n^2 + \cdots + b_t n^t,$$

then the parameter s takes the value $s = 1$.

Example 9: a) Find all solutions of the recurrence relation $\boldsymbol{a_n = 2a_{n-1} + 3^n}$

b) Find the solution of the recurrence relation with initial condition $\boldsymbol{a_1 = 5}$

Solution: a) We have $a_n = 2a_{n-1} + 3^n$

It is a linear nonhomogeneous recurrence relation with constant coefficients. The associated homogeneous recurrence relation for a_n is

$$a_n = 2a_{n-1}$$

The characteristic equation is $r = 2$ and the characteristic root is 2.

Therefore, $a_n^{(h)} = \alpha.2^n$, where α is a constant.

We have $F(n) = 3^n$. Note that 3 is not a characteristic root. Therefore

$$a_n^{(p)} = p.3^n$$

Substituting in the given recurrence relation we get

$$p.3^n = 2.p.3^{n-1} + 3^n$$

$\Longrightarrow\ (p-1)3^n = 2p3^{n-1}\ \ \Longrightarrow 3(p-1) = 2p\ \Longrightarrow p = 3$

Thus, $a_n^{(p)} = 3.3^n = 3^{n+1}$

The general solution of the given recurrence relation is

$$a_n = a_n^{(h)} + a_n^{(p)} = \alpha.2^n + 3^{n+1}$$

b) To obtain α we use the initial condition

Taking $n = 1$; $a_1 = 5 = \alpha.2 + 3^2 \Longrightarrow 2\alpha = -4 \ \Longrightarrow\ \alpha = -2.$

∴ The solution of the given recurrence relation with the given initial condition is

$$a_n = (-2).2^n + 3^{n+1} \quad \textit{i.e.,} \quad a_n = -2^{n+1} + 3^{n+1}$$

Example 10: Find all solutions of the recurrence relation

$$\boldsymbol{a_n = 4a_{n-1} - 4a_{n-2} + (n+1)2^n.}$$

Solution: We have $a_n = 4a_{n-1} - 4a_{n-2} + (n+1)2^n$

It is a linear homogeneous recurrence relation with constant coefficients. The associated homogeneous recurrence relation is

$$a_n = 4a_{n-1} - 4a_{n-2}$$

The characteristic equation is $r^2 = 4a - 4$ *i.e.,* $(r-2)^2 = 0 \quad \Longrightarrow r = 2$ (twice)

Note that the characteristic root 2 has multiplicity 2.

Therefore, $a_n = (\alpha_1 + \alpha_2 n).2^n$, where α is a constant.

We have $F(n) = (n+1)2^n$.

Because 2 is a characterstic root with multiplicity $m = 2$ and $F(n) = (n+1)2^n$,

$$a_n^{(p)} = n^m(p_0 + p_1 n)2^n = n^2(p_0 + p_1 n)2^n = (p_0 n^2 + p_1 n^3)2^n$$

Substituting in the given recurrence relation we get

$$(p_0 n^2 + p_1 n^3)2^n = 4(p_0(n-1)^2 + p_1(n-1)^3)2^{n-1}$$
$$-4[p_0(n-2)^2 + p_1(n-2)^3]2^{n-2} + (n+1)2^n$$

$$\Rightarrow\ p_0 n^2 + p_1 n^3 = 2p_0(n-1)^2 + 2p_1(n-1)^3 - p_0(n-2)^2 - p_1(n-2)^3 + n + 1$$

$$\Rightarrow\ p_0 n^2 + p_1 n^3 = 2p_0(n^2 - 2n + 1) + 2p_1(n^3 - 3n - 1) - p_0(n^2 - 4n + 4)$$
$$-p_1(n^3 - 6n^2 + 12n - 8) + n + 1$$

Equating the coefficients of n^2 on both sides, we get

$$\Rightarrow\ 0 = (-6p_1 + 1)n + (6p_1 - 2p_0 + 1)$$

$$\Rightarrow\ -6p_1 + 1 = 0,\ \ 6p_1 - 2p_0 + 1 = 0$$

$$\Rightarrow\ p_1 = \tfrac{1}{6},\ \ p_0 = 1$$

Therefore, $a_n^{(p)} = (p_0 n^2 + p_1 n^3)2^n = \left(n^2 + \frac{n^3}{6}\right)2^n$

The general solution of the given recurrence relation is

$$a_n = a_n^{(h)} + a_n^{(p)} = (\alpha_1 + \alpha_2 n)2^n + \left(n^3 + \frac{n^3}{6}\right)2^n$$

i.e., $a_n = \left(\alpha_1 + \alpha_2 n + n^2 + \frac{n^3}{6}\right)2^n$, where α_1 and α_2 are constants.

Example 11: Find the solution of the recurrence relation

$$\boldsymbol{a_n = 4a_{n-1} - 3a_{n-2} + 2^n + n + 3, \text{ with } a_0 = 1 \text{ and } a_1 = 4}$$

Solution: The given recurrence relation

$$a_n = 4a_{n-1} - 3a_{n-2} + 2^n + n + 3$$

is a linear nonhomogeneous recurrence relation with constant coefficients. The associated homogeneous recurrence relation is

$$a_n = 4a_{n-1} - 3a_{n-2}$$

The characteristic equation is $r^2 = 4r - 3$,

i.e., $(r-1)(r-3) = 0$, *i.e.,* the characteristic roots are $r = 1$ and 3.

Therefore, $a_n^{(h)} = \alpha.1^n + \beta.3^n = \alpha + \beta.3^n$, where α and β are constants.

We have $F(n) = 2^n + n + 3$. Notice that 2 is not a characteristic root and 1 is a characteristic root of multiplicity $m = 1$. Therefore,

$$a_n^{(p)} = p.2^n + n^m\,(q + rn)1^n = p.2^n + n(q + rn)$$

$$= p.2^n + qn + rn^2$$

Substituting in the given recurrence relation we get,

$$p.2^n + qn + rn^2$$

$$= 4(p.2^{n-1} + q(n-1) + r(n-1)^2) - 3(p.2^{n-2} + q(n-2) + r(n-2)^2) + 2^n + n + 3$$

Equating the coefficient of 2^n on both sides, we get

$$p = p - \frac{3}{4}p + 1 \Rightarrow p = -4$$

Equating the coefficient of n on both sides, we get

$$q = 4q - 8r - 3q + 12r + 1 \;\; \Rightarrow 4r + 1 = 0 \;\; \Rightarrow r = -\frac{1}{4}$$

Equating the constant terms on both sides we get

$$0 = -4q + 4r + 6q - 12r + 3 \;\; \Rightarrow 2q - 8r = -3 \Rightarrow q = -\frac{5}{2}$$

Thus, $a_n^{(p)} = -4.2^n + \frac{5}{2}n - \frac{n^2}{4}$

The general solution of the given recurrence equation is

$$a_n = a_n^{(h)} + a_n^{(p)} = \alpha + \beta.3^n - 4.2^n - \frac{5}{2}n - \frac{n^2}{4},$$

where α and β are constants. To evaluate α and β we use the initial conditions.

Taking $n = 0,\ a_0 = 1 = \alpha + \beta - 4$

Taking $n = 1,\ a_1 = 4\ = \alpha + 3\beta - 8 - \frac{5}{2} - \frac{1}{4}$

$\Longrightarrow\ \ \alpha + \beta = 5$ and $\alpha + 3\beta = \frac{59}{4}$

Solving the equations, we get $\alpha = \frac{1}{8}$ and $\beta = \frac{39}{8}$

The unique solution of the given recurrence relation is

$$a_n = \frac{1}{8} + \frac{39}{8}.3^n - 4.2^n - \frac{5}{2}n - \frac{n^2}{4}, \text{ for all nonnegative integers } n$$

P1:

Find the solution of the recurrence relation $\boldsymbol{a_n = 2a_{n-1} + 3a_{n-2}}$ with $\boldsymbol{a_0 = 0}$ and $\boldsymbol{a_1 = 1}$?

Solution: The given recurrence relation is $a_n = 2a_{n-1} + 3a_{n-2}$

This is a linear homogeneous recurrence relation with constant coefficients of degree two. The characteristic equation is

$$r^2 = 2r + 3, \textit{i.e.,}\ r^2 - 2r - 3 = 0 \Longrightarrow (r+1)(r-3) = 0$$

The characteristic roots are $r = -1$ and $r = 3$.

Therefore, the sequence $\{a_n\}$ is a solution to the recurrence relation iff

$a_n = \alpha_1(-1)^n + \alpha_2 3^n$, where α_1 and α_2 are constants.

The given initial conditions are $a_0 = 0$ and $a_1 = 1$

i.e., $a_0 = 0 = \alpha_1 + \alpha_2$, $a_1 = 1 = -\alpha_1 + 3\alpha_2$.

Solving for α_1, α_2 , we get $\alpha_1 = -\frac{1}{4}$ and $\alpha_2 = \frac{1}{4}$.

The required solution is $a_n = -\frac{1}{4}(-1)^n + \frac{1}{4}.3^n$

i.e., $a_n = (-1)^{n+1}\frac{1}{4} + \frac{1}{4}3^n$, for all nonnegative integers n.

P2:

Find the solution of the recurrence relation $\boldsymbol{a_n = -7a_{n-1} - 10a_{n-2}}$ with $\boldsymbol{a_0 = 3}$ and $\boldsymbol{a_1 = 3}$?

Solution: The given recurrence relation is $a_n = -7a_{n-1} - 10a_{n-2}$

This is a linear homogeneous recurrence relation with constant coefficients of degree two. The characteristic equation is

$$r^2 = -7r - 10, \textit{ i.e., } r^2 + 7r + 10 = 0 \Longrightarrow (r+5)(r+2) = 0$$

The characteristic roots are $r = -5$ and $r = -2$.

Therefore, the sequence $\{a_n\}$ is a solution to the recurrence relation iff

$$a_n = \alpha_1(-5)^n + \alpha_2(-2)^n, \text{ where } \alpha_1 \text{ and } \alpha_2 \text{ are constants.}$$

The given initial conditions are $a_0 = 3$ and $a_1 = 3$

$$\textit{i.e., } a_0 = 3 = \alpha_1 + \alpha_2\,,\ \ a_1 = 3 = -5\alpha_1 - 2\alpha_2.$$

Solving for α_1, α_2 , we get $\alpha_1 = -3$ and $\alpha_2 = 6$.

The required solution is

$$a_n = -3(-5)^n + 6(-2)^n, \text{ for all nonnegative integers } n.$$

P3:

Find the solution of the recurrence relation $\boldsymbol{a_n = 10a_{n-1} - 25a_{n-2}}$ **with** $\boldsymbol{a_0 = 3,\ a_1 = 4.}$

Solution: The given recurrence relation is a linear homogenous recurrence relation is with constant coefficients of degree two. Its characteristic equation is

$$r^2 = 10r - 25,\ i.e.,\ r^2 - 10r + 25 = 0 \Longrightarrow (r-5)^2 = 0.$$

The characteristic root 5 and its multiplicity is 2. Therefore, the solution of the given recurrence relation is

$$a_n = \alpha_1 5^n + \alpha_2 n\, 5^n \text{ , where } \alpha_1 \text{ and } \alpha_2 \text{ are constants.}$$

To evaluate α_1 and α_2 we use the initial conditions.

Take $n = 0,\ a_0 = 3 = \alpha_1$ and take $n = 1, a_1 = 4 = 5\alpha_1 + 5\alpha_2$

Solving, we get $\alpha_1 = 3$ and $\alpha_2 = -\frac{11}{5}$.

∴ The solution of the given recurrence relation with the given initial conditions is

$$a_n = 3.5^n - \frac{11}{5}.n\,5^n = \left(3 - \frac{11}{5}n\right)5^n \text{ , for all nonnegative integers } n$$

P4:

Find the solution of the recurrence relation

$$\boldsymbol{a_n = 7a_{n-1} - 13a_{n-2} - 3a_{n-3} + 18a_{n-4}},$$

with $\boldsymbol{a_0 = 5}$, $\boldsymbol{a_1 = 3, a_2 = 6}$ **and** $\boldsymbol{a_3 = -21}$

Solution: The given recurrence relation is a linear homogenous recurrence relation is with constant coefficients of degree 4. Its characteristic equation is

$$r^4 = 7r^3 - 13r^2 - 3r + 18$$

$$i.e., \quad r^4 - 7r^3 + 13r^2 + 3r - 18 = 0 \Longrightarrow (r+1)(r-2)(r-3)^2 = 0.$$

The characteristic roots are: $-1, 2$ with multiplicity one, and 3 with multiplicity 2.

Therefore, the solution of the given recurrence relation is

$$a_n = \alpha_1(-1)^n + \alpha_2 . 2^n + \alpha_3 . 3^n + \alpha_4 . n3^n ,$$

where $\alpha_1, \alpha_2, \alpha_3$ and α_4 are constants.

To evaluate $\alpha_1, \alpha_2, \alpha_3$ and α_4 we use the initial conditions.

Take $n = 0$, $a_0 = 5 = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$

Take $n = 1, a_1 = 3 = -\alpha_1 + 2\alpha_2 + 3\alpha_3 + 3\alpha_4$

Take $n = 2, a_2 = 6 = \alpha_1 + 4\alpha_2 + 9\alpha_3 + 18\alpha_4$

Take $n = 3, a_3 = -21 = -\alpha_1 + 8\alpha_2 + 27\alpha_3 + 81\alpha_4$

Solving, we get $\alpha_1 = 2 = \alpha_3$, $\alpha_2 = 1$ and $\alpha_4 = -1$

∴ The solution of the given recurrence relation with the given initial conditions is

$$a_n = 2(-1)^n + 2^n + 2.3^n - n3^n$$, for all nonnegative integers n

P6:

Find all solutions of the recurrence relation

$$\boldsymbol{a_n = 6a_{n-1} - 9a_{n-2} + 4(n+1)3^n.}$$

Solution: We have $a_n = 6a_{n-1} - 9a_{n-2} + 4(n+1)3^n$

It is a linear nonhomogeneous recurrence relation with constant coefficients. The associated homogeneous recurrence relation is $a_n = 6a_{n-1} - 9a_{n-2}$

The characteristic equation is $r^2 = 6r - 9$ *i.e.,* $(r-3)^2 = 0 \quad \Longrightarrow r = 3$ (twice)

Note that the characteristic root 3 has multiplicity 2.

Therefore, $a_n = (\alpha_1 + \alpha_2 n)3^n$, where α is a constant.

We have $F(n) = 4(n+1)3^n$.

Because 3 is a characterstic root with multiplicity $m = 2$ and $F(n) = 4(n+1)3^n$,

$$a_n^{(p)} = n^m(p_0 + p_1 n)3^n = n^2(p_0 + p_1 n)3^n = (p_0 n^2 + p_1 n^3)3^n$$

Substitution in the given recurrence relation, we get

$$(p_0 n^2 + p_1 n^3)3^n = 6[p_0(n-1)^2 + p_1(n-1)^3]3^{n-1}$$
$$-9[p_0(n-2)^2 + p_1(n-2)^3]3^{n-2} + 4(n+1)3^n$$

$$\Longrightarrow \; p_0 n^2 + p_1 n^3 = 2p_0(n-1)^2 + 2p_1(n-1)^3 - p_0(n-2)^2 - p_1(n-2)^3 + 4n + 4$$

Equating the like terms on both sides and solving, we get

$$\Longrightarrow \; p_0 = \frac{2}{3}, p_1 = 4 \quad \text{(verify!)}$$

Therefore, $a_n^{(p)} = (p_0 n^2 + p_1 n^3)3^n = \left(4n^2 + \frac{2n^3}{3}\right)3^n$

The general solution of the given recurrence relation is

$a_n = a_n^{(h)} + a_n^{(p)} = (\alpha_1 + \alpha_2 n)3^n + \left(4n^2 + \frac{2n^3}{3}\right)3^n$, where α_1 and α_2 are constants.

4.5. Recurrence relations

Exercises:

1. Solve the recurrence relation together with the initial conditions given.
   a. $a_n = 2a_{n-1}$ for $n \geq 1, , a_0 = 3$
   b. $a_n = a_{n-1}$for $n \geq 1, , a_0 = 2$
   c. $a_n = 5a_{n-1} - 6a_{n-2}$ for $n \geq 2,\ a_0 = 1, a_1 = 0$
   d. $a_n = 4a_{n-1} - 4a_{n-2}$ for $n \geq 2,\ a_0 = 6, a_1 = 8$
   e. $a_n = -4a_{n-1} - 4a_{n-2}$ for $n \geq 2,\ a_0 = 0, a_1 = 1$
   f. $a_n = 4a_{n-2}$ for $n \geq 2$, $a_0 = 0$, $a_1 = 4$
   g. $a_n = \frac{a_{n-2}}{4}$ for $n \geq 2$, $a_0 = 1, a_1 = 0$
2. Find the solution to $a_n = 2a_{n-1} + a_{n-2} - 2a_{n-3}$ for $n = 3,4,5, \ldots$, with $a_0 = 3$, $a_1 = 6$ and $a_2 = 0$.
3. Find the solution to $a_n = 7a_{n-2} + 6a_{n-3}$ with $a_0 = 9$, $a_1 = 10$ and $a_2 = 32$.
4. Find the solution to $a_n = 5a_{n-2} - 4a_{n-4}$ with $a_0 = 3$, $a_1 = 2$, $a_2 = 6$ and $a_3 = 8$.
5. Find the solution to $a_n = 2a_{n-1} + 5a_{n-2} - 6a_{n-3}$ with $a_0 = 7$, $a_1 = -4$ and $a_2 = 8$.
6. Solve the recurrence relation $a_n = -3a_{n-1} - 3a_{n-2} - a_{n-3}$ with $a_0 = 5$, $a_1 = -9$ and $a_2 = 15$.
7. What is the general form of the solutions of a linear homogenous recurrence relation if its characteristic equation has roots $1,1,1,1,-2,-2,-2,3,3,-4$?

8. What is the general form of the particular solution guarantee to exist by Theorem 6 of the linear non homogeneous recurrence relation $a_n = 8a_{n-2} - 16a_{n-4} + F(n)$ if
   a. $F(n) = n^3$
   b. $F(n) = (-2)^n$
   c. $F(n) = n.2^n$
   d. $F(n) = n^2 4^n$
   e. $F(n) = (n^2 - 2)(-2)^n$
   f. $F(n) = n^4 2^n$
   g. $F(n) = 2$
9.
   a. Find all the solutions of the recurrence relation $a_n = 2a_{n-1} + 2n^2$
   b. Find the solutions of the recurrence relation in part(a) with the initial condition $a_1 = 4$.

10.
   a. Find all the solutions of the recurrence relation $a_n = -5a_{n-1} - 6a_{n-2} + 42.4^n$
   b. Find the solutions of this recurrence relation in with $a_1 = 56$ and $a_2 = 278$.

11. Find all solutions of the recurrence relation $a_n = 5a_{n-1} - 6a_{n-2} + 2^n + 3n$. (Hint: Look for a particular solution of the form $qn2^n + p_1 n + p_2$, where q, p_1, p_2 are constants.
12. Find the solution of recurrence relation $a_n = 2a_{n-1} + 3.2^n$.
13. Find all solutions of the recurrence relation $a_n = 7a_{n-1} - 16a_{n-2} + 12a_{n-3} + n.4^n$.with $a_0 = -2, a_1 = 0$ and $a_2 = 5$.