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2. METHODS OF PROOF

* Module-2.1 :-

* Tautological Implication :-

→ If $a \to b$ is a Tautology then 'a tautological imply b' / 'a imply b'

→ If '$a \Rightarrow b$'

'a imply b' (or) 'b is implied by a'.

Note points :-

① '$\Rightarrow$' is Meta ~~logical~~ language symbol.

② '$a \Rightarrow b$' is not a formula [wff].

[∵ $\Rightarrow$ is not a Logical operator].

③ '$a \Rightarrow b$' is just a relation [Transitive relation].

If $a \Rightarrow b$, $b \Rightarrow c$ then $a \Rightarrow c$.

④ If '$a \Rightarrow b$' then 'a' is Hypothesis (or) anticident (or) premix and 'b' is consequence (or) conclusion.

⑤ '$a \equiv b$', if and only if '$a \Rightarrow b$' and '$b \Rightarrow a$':

$$\downarrow$$

Proof :- $p \leftrightarrow q$

$$T \quad T$$

'$a \equiv b$' when $a \leftrightarrow b$ is a Tautology.

$\Rightarrow (a \to b) \wedge (b \to a)$ is a Tautology.

$$\downarrow T \qquad \downarrow T$$

∴ $a \Rightarrow b$, $b \Rightarrow a$. then $a \equiv b$.

⑥ If '$T_0 \Rightarrow a$' then 'a' is a 'tautology.'

⑦ Let $P_1, P_2, \text{---} P_n$ are n propositions.

$P_1 \wedge P_2 \wedge \text{--} \wedge P_n \Rightarrow q$.

* **Methods :-**

I :- The truth value of antecedent 'T' leads the truth value of conclusion is also 'T'.

(or)

Conclusion is True whenever Hypothesis is True.

$$\underset{T}{a} \Rightarrow \underset{T}{b} \quad (or) \quad \underset{T}{a} \rightarrow \underset{T}{b} \text{ is a Tautology}$$

II :- The Assumption of the truth value of conclusion 'F' leads the truth value of hypothesis is also 'F'.

(or)

The Hypothesis is False whenever conclusion is 'F'.

$$a \Rightarrow b \quad (or) \quad \underset{F}{a} \rightarrow \underset{F}{b} \text{ is Tautology.}$$

[∵ T→F is F].

* **Deduction (or) Formal Proof :-**

→ The process of Derivation of conclusion [conclusion is also a proposition] from a given set of proportions by using acceptable rules is called **Deduction (or) Formal proof**.

→ Those rules are called '**Inference rules**'.

* **Argument :-**

→ An Argument is a sequence of propositions in which the last proposition is a conclusion & The preceding propositions are called premices (or) Hypothesis.

→ Let $P_1, P_2, ---, P_n$ be the given propositions and q be the conclusion.

→ Corresponding Argument is denoted by

$$\boxed{P_1, P_2 ---, P_n \vdash C}$$

* Representation of Argument :-

Generally

→ The Argument is represented as follows.

$$\left.\begin{array}{c} P_1 \\ P_2 \\ \vdots \\ P_n \end{array}\right\} \text{premises}$$

$$\therefore C \rightarrow \text{conclusion}$$

* Types of Arguments :-

① valid Argument

② Invalid / Falsi Argument

① valid Argument :-

→ The Argument is said to be a valid Argument if all premises $[h_1 \wedge h_2 \wedge \ldots \wedge h_n]$ jointly implied to conclusion.

$$\underset{T}{h_1 \wedge h_2 \wedge \ldots \wedge h_n} \vdash \underset{T}{C}.$$

(or)

conclusion is 'True' whenever All premises are 'True'.

(or)

$(h_1 \wedge h_2 \wedge \ldots \wedge h_n) \rightarrow c$ is a tautology.

Notation :-

→ Generally, a valid argument is represented as follows :-

$$h_1 \wedge h_2, \ldots, h_n \Rightarrow c.$$

$$\boxed{h_1, h_2, \ldots, h_n \Rightarrow c}$$

→ In this case, we say that "c follows Logically from a set of premises $\{h_1, h_2, \ldots, h_n\}$."

② Invalid / Falsi Argument :-

→ An Argument which is not a valid is called 'Invalid / Falsi Argument'.

Notation :-

$h_1, h_2, ---, h_n \not\Rightarrow c.$

* Inference Theory :-

→ The Analysis of an argument by using acceptable rules is called 'Inference Theory'.

→ Those rules are called 'Inference Rules.'

Methods to test an argument valid or not :-

→ There are 3 methods to test a given argument whether it is valid or not.

→ They are

① 'Truth Table Technic'

② 'Rules of Inference'

③ 'Indirect Method of proof.'

① Method - 1 :-

* Determine whether the conclusion 'c' is valid in the following when h_1, h_2, h_3 are premises.

ⓐ $h_1 : p \vee q$; $h_2 : p \rightarrow r$; $h_3 : q \rightarrow r$; $c : r$.

ⓑ $h_1 : \sim p$; $h_2 : p \vee q$; $c : p \wedge q$.

A) ⓐ

p	q	r	$h_1 : p \vee q$	$h_2 : p \rightarrow r$	$h_3 : q \rightarrow r$	$c : r$
T	T	T	T	T	T	T ✓
T	T	F	T	F	F	F
T	F	T	T	T	T	T ✓
T	F	F	T	F	T	F
F	T	T	T	T	T	T ✓
F	T	F	T	T	F	F
F	F	T	F	T	T	T
F	F	F	F	T	T	F

Here critical rows [Premises] are True and conclusion is also T.

∴ It is a valid Argument.

We notice that there are 3 critical rows in this Truth Table.

Also Notice that the conclusion 'c' has truth value 'T' for Each corresponding critical row.

so that the given arguement is valid.

(b)

p	q	$h_1: \sim p$	$h_2: p \vee q$	$c: p \wedge q$
T	T	F	T	T
T	F	F	T	F
F	T	$\boxed{T}$	$\boxed{T}$	F ×
F	F	T	F	F

Here, we identified one critical row and corresponding conclusion is False.

$\therefore$ It is 'Not valid Argument'.

* Valid Argument	Arg Repres.	Implication	Name
① $p \wedge q \Rightarrow p$	$\dfrac{p \wedge q}{\therefore p}$	$p \wedge q \Rightarrow p$	Simplification
② $p \wedge q \Rightarrow q$	$\dfrac{p \wedge q}{\therefore q}$	$p \wedge q \Rightarrow q$	
③ $p \Rightarrow p \vee q$	$\dfrac{p}{\therefore p \vee q}$	$p \Rightarrow p \vee q$	Disjunctive Addition
④ $q \Rightarrow p \vee q$	$\dfrac{q}{\therefore p \vee q}$	$q \Rightarrow p \vee q$	
⑤ $\sim p \Rightarrow p \to q$	$\dfrac{\sim p}{\therefore p \to q}$	$\sim p \Rightarrow p \to q$	
⑥ $q \Rightarrow p \to q$	$\dfrac{q}{\therefore p \to q}$	$q \Rightarrow p \to q$	
⑦ $\sim(p \to q) \Rightarrow p$	$\dfrac{\sim(p \to q)}{\therefore p}$	$\sim(p \to q) \Rightarrow p$	
⑧ $\sim(p \to q) \Rightarrow \sim q$	$\dfrac{\sim(p \to q)}{\therefore \sim q}$	$\sim(p \to q) \Rightarrow \sim q$	
⑨ $p, q \Rightarrow p \wedge q$	$\dfrac{p \quad q}{\therefore p \wedge q}$	$((p \wedge q)) \Rightarrow (p \wedge q)$	conjunction addition
~~⑩~~			
⑩ $\sim p, p \vee q \Rightarrow q$	$\dfrac{\sim p \quad p \vee q}{\therefore q}$	$\sim p \wedge (p \vee q) \Rightarrow q$	Disjunctive syllogism
⑪ $p, p \to q \Rightarrow q$	$\dfrac{p \quad p \to q}{\therefore q}$	$p \wedge (p \to q) \Rightarrow q$	Modus ponens (or) Law of detachment (or) Method of Affirming
~~⑫~~			
⑫ $\sim q, p \to q \Rightarrow \sim p$	$\dfrac{\sim q \quad p \to q}{\therefore \sim p}$	$\sim q \wedge (p \to q) \Rightarrow \sim p$	Modus Tollens (or) Method of denying
⑬			

(13) $p \to q,\ q \to r \Rightarrow p \to r$

$$\begin{array}{l} p \to q \\ q \to r \\ \hline \therefore p \to r \end{array}$$

$(p \to q) \wedge (q \to r) \Rightarrow (p \to r)$ Hypothetical syllogism.

(14) $p \to q,\ q \to s,$
$p \vee r \Rightarrow q \vee s$

$$\begin{array}{l} p \to q \\ r \to s \\ p \vee r \\ \hline \therefore q \vee s \end{array}$$

$(p \to q) \wedge (r \to s) \wedge (p \vee r) \Rightarrow (q \vee s)$ constructive ~~Dilemma~~ Dilemma

(15) $p \to q,\ r \to s,$
$\sim q \vee \sim s \Rightarrow \sim p \vee \sim p$

$$\begin{array}{l} p \to q \\ r \to s \\ \sim q \vee \sim s \\ \hline \therefore \sim p \vee \sim s \end{array}$$

$(p \to q) \wedge (r \to s) \wedge (\sim q \vee \sim s) \Rightarrow \sim p \vee \sim s.$ Destructive Dilemma

*Method-2 :- Rules of Inference.

→ There are three rules of inference in proposition logic. They are.

Rule P :- A premise that may be introduced at any stage or any point in the derivation.

Rule T :- A formula 's' may be introduced (~~at any~~ point) in a derivation, if 's' follows logically from any one or more of the preceding formulas in the derivation.

Rule CP :- (Rule of conditional proof) (or) Deduction Theorem :-

→ If we can derive 's' from 'r' and a set of premises, then we can derive $r \to s$ from the set of premises alone.

→ It is generally used if the conclusion is of the form. $r \to s$. In such cases, 'r' is taken as an additional premise and 's' is derived from the given premises and r.

* Exercise :-

7] Test the validity of following argument.

ⓒ If the advertisement is successful, then the sales of the product will go up. Either the advertisement is successful or the production of the product will be stopped. The sales of the product will not go up. Therefore the production of the product will be stopped?

A) First we symbolise the given meta language statements as follows:-

p: The advertisement is successful.

q: The sales of the product will ~~be stopped~~ go up

r: The production of the product will be stopped

$\sim q$: The sales of the product will not go up

The given premises are $p \to q$, $p \vee r$, $\sim q$.
and the conclusion is r.

∴ The given argument represented as follows

$$\begin{array}{c} p \to q \\ p \vee r \\ \underline{\sim q} \\ \therefore r \end{array}$$

The Derivation of above argument is as follows

① $\sim q$	rule P	given premise
② $p \to q$	rule P	given premise
③ $\sim p$	rule T	(①, ② modus tollens)
④ $p \vee r$	rule P	given premise
⑤ r	rule T	(③, ④ Disjunctive syllogism)

Hence given argument is valid.

* Exercise.

7) Test the validity of following argument.

(d) If Ram is clever, then prem is well-behaved. If Joe is good, then sam is bad and prem is not well-behaved. If Lal is educated, then Joe is good or Ram is clever. Hence, if Lal is educated and prem is not well-behaved, then sam is bad.

A) Let, p: Ram is clever

q: prem is well behaved.

r: Joe is good.

s: sam is bad

t: lal is educated.

$$\frac{p \rightarrow q, \quad r \rightarrow (s \wedge \sim q), \quad t \rightarrow (r \vee p)}{\therefore (t \wedge \sim q) \rightarrow s}$$

$(t \wedge \sim q) \rightarrow s$

The new set of premises are

$p \rightarrow q$

$r \rightarrow (s \wedge \sim q)$

$t \rightarrow (r \vee p)$

$t \wedge \sim q$

$\therefore s$

Derivation:-

①	$t \wedge \sim q$	P	(Additional premise)
②	t	rule T	(① simplification).
③	$t \rightarrow (r \vee p)$	rule P	(given premise)
④	$r \vee p$	rule T	(②, ③ modus ponens)
⑤	$r \rightarrow (s \wedge \sim q)$	rule P.	(given premise)
⑥	$p \rightarrow q$	rule p	(given premise)
⑦	$(s \wedge \sim q) \vee q$	rule T	(④, ⑤, ⑥ constructive dynama).

8 $(s \vee q) \wedge (\sim q \vee q)$ rule T (Disjunctive law)

9 $(s \vee q) \wedge T_0$ rule T (Inverse Law)

10 $s \vee q$ rule T (Identity Law)

11 $\sim q$ rule T (1 simplification)

12 s rule T (10, 11 Disjunctive syllogism).

13 $(t \wedge \sim q) \rightarrow s$ rule CP (Additional premise $\rightarrow s$).

* **Entailment & Provability:-**

$h_1, h_2, h_3, \ldots h_n.$

$h_1, h_2, h_3, \ldots h_n \vDash C.$

$h_1, h_2, \ldots, h_n$ entail C.

$h_1, h_2, \ldots h_n$ prove C

$h_1, h_2 \ldots, h_n \vdash C.$

→ showing validity using Truthtable (Method-1) is called 'Entailment'.

→ showing validity using Method-2 is 'Provability'.

* **soundness and completeness:-**

$h_1, h_2, \ldots, h_n \Rightarrow C.$

→ If $h_1, h_2, \ldots, h_n \vdash C$ then $h_1, h_2 \ldots, h_n \vDash C.$

→ showing validity using Logical Laws is called 'Soundness'.

→ If $h_1, h_2, \ldots h_n \vDash C$ then $h_1, h_2, \ldots h_n \vdash C.$

→ 'completeness' is track from 'Truthtable' to 'Laws of Logic'.

* **Consistent and Inconsistent:-**

→ consider premises $h_1, h_2, \ldots, h_n$ said to be consistent if. $h_1 \wedge h_2 \wedge h_3 \ldots \wedge h_n \rightarrow T.$

→ when their conjunction is 'satisfiable' then it is 'consistent'.

→ when the conjunction is 'contradiction' then it is 'Inconsistent'.

* Exercise :-

5] show that the following sets of premises are inconsistent.

(a) $a \rightarrow (b \rightarrow c)$, $d \rightarrow (b \wedge \sim c)$, and $a \wedge d$.

A) claim :- To prove that given set of premises are inconsistent

→ It is enough to prove that their conjunction

$$(a \rightarrow (b \rightarrow c)) \wedge (d \rightarrow (b \wedge \sim c)) \wedge (a \wedge d) \Rightarrow F_0.$$

(or)

$$(a \rightarrow (b \rightarrow c)), (d \rightarrow (b \wedge \sim c)), (a \wedge d) \Rightarrow F_0.$$

step-1 :-

→ The derivation is as follows:-

(1) $a \wedge d$ P [given premise].

(2) a T [(1), simplification].

(3) $a \rightarrow (b \rightarrow c)$ P [given premise]

(4) $b \rightarrow c$ T [(2), (3) modus ponens].

(5) $\sim b \vee c$ T [$b \rightarrow c \equiv \sim b \vee c$].

(6) $\sim (b \wedge \sim c)$ T [Demorgan].

(7) $d \rightarrow (b \wedge \sim c)$ P [given premise]

(8) $\sim d$ T [(6), (7) Modus ponens]

(9) d T [(1) simplification]

(10) $d \wedge \sim d$ T [(8), (9) conjunction addition]

(11) F_0. T [Inverse Law].

so that conjunction of all given premises is a contradiction. Hence given premises are inconsistent.

(b) $p \to q,\ q \vee r \to s,\ s \to \sim p,\ p \wedge \sim r.$

A)

①	$p \wedge \sim r$	P	[given premise]
②	p	T	[①, simplification].
③	$p \to q$	P	[given premise].
④	q	T	[②, ③ Modus ponens].
⑤	$q \vee r$	T	[④, Disjunctive addition].
⑥	$q \vee r \to s$	P	[given premise].
⑦	s	T	[⑤, ⑥ modus ponens].
⑧	$s \to \sim p$	P	[given premise].
⑨	$\sim p$	T	[⑦, ⑧ Modus ponens].
⑩	$p \wedge \sim p$	T	[②, ⑨ conjunctive addition].
⑪	F_0	T	[Inverse Law].

so that given premises are Inconsistent.

* **Method-3 : Indirect method of proof:-**

→ The notion of Inconsistency is used in the derivation of an argument is called Indirect method of proof.

→ $h_1, h_2, \ldots, h_n \vdash c$.

$$\begin{array}{l} h_1 \\ h_2 \\ \vdots \\ h_n \\ \hline \therefore c \end{array}$$

In the Indirect method of proof we will add $\sim c$.

$$\begin{array}{l} h_1 \\ h_2 \\ \vdots \\ h_n \\ \sim c \\ \hline \therefore F_0 \end{array}$$

$h_1 \wedge h_2 \wedge \ldots \wedge h_n \wedge \sim c \equiv F_0$.

h_1	h_2	...	h_n	$\sim c$	c
					F
					F
T	T	T	T	F	F

* Exercise :-

g] show the following using Indirect method.

(a) $p \to (q \to \sim r), \sim s \to q, \sim t \wedge (p \vee t) \Rightarrow (r \to s)$.

A) Given premises are $p \to (q \to \sim r), \sim s \to q, \sim t \wedge (p \vee t) \Rightarrow (r \to s)$.

→ For applying Indirect method we will use $\sim C$ as an additional premise. and then show that the new set of premises are Inconsistent.

Here The new set of premises are.

$p \to (q \to \sim r), (\sim s \to q), (\sim t \wedge (p \vee t)) \Rightarrow (r \to s), \sim C$.

$\sim C : \sim (r \to s)$.

~~Steps~~ :-

(1)	$\sim t \wedge (p \vee t)$	P	[given premise].
(2)	p	T	[(1), Disjunctive syllogism].
(3)	$p \to (q \to \sim r)$	P	[given premise].
(4)	$q \to \sim r$	T	[(2), (3) Modus ponens].
(5)	$\sim s \to q$	P	[given premise].
(6)	$\sim s \to \sim r$	T	[Hypothatical syllogism].
(7)	$r \to s$	T	[$p \to q \equiv \sim q \to \sim p$].
(8)	$\sim (r \to s)$	T	[Additional premise]
(9)	$(r \to s) \wedge \sim (r \to s)$	T	[(7). (8) conjunctive Add^n].
(10)	F_0	T	[Inverse law].

∴ The new set of premices are Inconsistent.

∴ Given Argument is a valid Argument. By Indirect method.

2.2 :- METHODS OF PROOF

* **Quantifier** :-
  → There are two types of quantifiers:
  ① Universal Quantifier [$\forall$ for all]
  ② Existential Quantifier [$\exists$ there exists].

* **Proofs** :- [Terminology].

① **Informal Proof (or) Proof (or) Mathematical proof** :-

→ 'Proof' is an argument of 'mathematical statement'.

→ All statements are 'mathematical related'.

→ We use only 'mathematical words', 'mathematical symbols' (or) 'mathematical expressions'.

→ If the truthness is 'False' then it is called **Disproof**.

② **Hypothesis (or) Premises** :-

→ The statements that are primarily use in the proof are called '**Hypothesis**' (or) '**Premises**.'

③ **Axioms (or) Postulates** :-

→ The statements that are assumed to be true. and that are used in proof are called '**Axioms**'

④ **Theorems (or) Facts (or) Results** :-

→ A mathematical statements which can be shown as True are called '**Theorems/Facts/Results**'.

Ex :- $p \to q$, $p \leftrightarrow q$.

→ A '**Theorem**' is a conditional (or) Biconditional statement which contains one or more premises/ Hypothesis.

5) * Lemmas:-

→ 'Lemma' is a minor theorem which is a stepping stone to develop a major theorem.

6] Corollary:-

→ 'Corollary' is a minor theorem proved as a consequence of major theorem.

7) Conjecture:-

→ 'Conjecture' is a mathematical statement that can't be proved.

→ Generally all conjectures are not Theorems.

* Note:-

→ A valid Mathematical argument is called a 'Theorem'.

* Methods of proof:-

1) vacuous proof:-

→ Major methods of proof are defined based on '$p \to q$'

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

→ A conditional statement whose 'Hypothesis is False' is gaurranted to be 'True'. This is called as "vacuous proof."

* Example :-

Let $p(n)$ be proposition "If n is an integer and $n>1$, then $n^2>n$." show that the proposition $p(0)$ is true?

A) For $P(0)$,

If 0 is an integer and $0>1$, then $0^2>0$.

Let $p: 0>1$

$q: 0^2>0$

Here the truth value of p is False.

$\therefore$ By vacuous proof, $P(0)$ is True.

② Trivial Proof :-

→ It is based on 'First' and 'Third' combinations' of Truth values of $P\rightarrow q$.

→ If 'q is true', then $p\rightarrow q$ is 'True' irrespective of Truthvalue of p.

P	q	$P\rightarrow q$
T	T	T ✓
T	F	T
F	T	T ✓
F	F	T

* Example-1 :-

Let $p(n)$ be the proposition "if a and b are positive integers with $a\geq b$ and n is an integer, then $a^n\geq b^n$. show that proposition $P(0)$ is true.

A) Given, if $a\geq b$ then $a^n\geq b^n$.

For $p(0)$, if $a\geq b$ then $a^0\geq b^0$. [a, b are +ve]

Here $p: a\geq b$

$q: a^0\geq b^0$.

Here $q: a^0\geq b^0$

$\Rightarrow q: 1\geq 1$ [True].

By Trivial proof, $P(0)$ is true. irrespective of Truthvalue of p.

③ Direct Proof :-

→ It is based on 'first combination' of $p \to q$.

→ If we are proving '$p \to q$' is 'True', Assume 'p' is 'True' and try to prove 'q' is 'True' then $p \to q$ is 'True'.

→ If p is True then q must be True for $p \to q$ is True, this is called 'Direct proof'.

* Algorithm for Direct proof :-

step-1 :- Assume proposition 'p' is True.

step-2 :- 'p' is tautologically implies to 'c_1' [$p \Rightarrow c_1$]

step-3 :- '$c_1 \Rightarrow c_2$'

step-4 :- '$c_2 \Rightarrow c_3$'

step-5 :- so on, '$c_n \Rightarrow q$'.

step-6 :- By Hypothetical syllogism, '$p \Rightarrow q$'.

* Exercise :-

1] prove by direct proof that, if an integer 'a' is such that '$a-2$' is divisible by '3', then 'a^2-1' is divisible by '3'?

A) Divides :- Divides is a Number Theory. [$a|b$].
→ if '$b = ak$', for some $k \in \mathbb{Z}$.
Then 'b' divides 'a' (or) 'a' is divided by 'b'

Given, p : $a-2$ is divisible by 3.
q : a^2-1 is divisible by 3.

Given Hypothesis is, a is an integer.

Here $a-2$ is divisible by 3 is an axiom. [∵ We assumed as True].

Then By Direct proof, We assume p is true

Thus $a-2$ is divisible by 3.

Now By Divides rule,

$a-2 = 3k$, for k is an integer.

Now consider a^2-1

$\Rightarrow a^2-1 = (a-1)(a+1)$

$= (a-1)(a-2+3)$

$= (a-1)(a-2) + 3(a-1)$

$= (a-1)3k + 3(a-1)$ $\quad [\because (a-2) = 3k]$.

$= 3\underbrace{[(a-1)k + (a-1)]}_{m}$

$= 3m$ $\quad [\because$ assume as $m]$.

$\Rightarrow a^2-1 = 3m$.

$\therefore a^2-1$ is divisible by 3.

That means q is also True.

$\therefore$ By Direct proof, If an integer a is such that $a-2$ is divisible by 3 then a^2-1 is divisible by 3.

* P-1:-

Give a direct proof of the theorem;

If m and n are both perfect squares, then mn is also a perfect square.

A) Given, If m and n are both perfect squares then mn is also a perfect square.

Assume p is true.

$\Rightarrow m = a^2$

$n = b^2$.

$\Rightarrow m \cdot n = a^2 \cdot b^2$

$= (a \cdot b)^2$ $\quad [\because a^m \times b^m = (ab)^m]$.

Thus q is true.

$\therefore$ By Direct proof, If m and n are both perfect squares then mn is also a perfect square.

* Indirect proof :-

① Proof by contrapositive.

② Proof by contradiction.

① Proof by contrapositive :-

→ We have, $\underset{T}{p \rightarrow q} \equiv \underset{T}{\sim q \rightarrow \sim p}$

Algorithm :-

step-1 :- Assume $\sim q$ is true.

step-2 :- ~~we~~ Follow the method of direct proof to show that $\sim p$ is true.

step-3 :- Then conclude that $\sim q \rightarrow \sim p$ is also true.

step-4 :- By proof of contrapositive, we conclude that $p \rightarrow q$ is true.

* Exercise:-

2) Prove by proof by contraposition that for any non-negative integers x, y, if $\sqrt{xy} \neq \frac{x+y}{2}$, then $x \neq y$.

A) Given Hypothesis/ data,

Let x, y are Non-negative integers.

$p: \sqrt{xy} \neq \frac{x+y}{2}$

$q: x \neq y$

Now, $\sim p: \sqrt{xy} = \frac{x+y}{2}$

$\sim q: x = y$.

It is enough to prove $\sim q \rightarrow \sim p$

Assume $\sim q$ i.e $x = y$ is true.

consider L.H.S of P

$\Rightarrow \sqrt{xy} = \sqrt{x(x)} = \sqrt{x^2} = x$

consider R.H.S of P

$\Rightarrow \frac{x+y}{2} = \frac{x+x}{2} = \frac{2x}{2} = 2$

$\therefore$ L.H.S = R.H.S

$\therefore \sim p$ is True.

By Direct proof, $\sim q \rightarrow \sim p$.

By Proof by contraposition, $p \rightarrow q$ is true.

$\therefore$ If $\sqrt{xy} \neq \frac{x+y}{2}$ then $x \neq y$.

* P-31

Prove by a proof by contraposition that if $n = ab$, where a and b are positive integers, then $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$.

A) $p: n = ab$

$q: a \leq \sqrt{n}$ or $b \leq \sqrt{n}$

$\sim p : n \neq ab$

$\sim q : a \nleq \sqrt{n}$ and $b \nleq \sqrt{n}$

$\sim q : a > \sqrt{n}$ and $b > \sqrt{n}$.

Now Assume $a > \sqrt{n}$ and $b > \sqrt{n}$

$\Rightarrow ab > \sqrt{n} \cdot \sqrt{n}$ [Multiply both]

$\Rightarrow ab > \sqrt{n}^2$

$\Rightarrow ab > n$

$\Rightarrow ab \neq n$.

$\therefore \sim p$ is true.

$\therefore$ By Direct proof, $\sim q \rightarrow \sim p$ is true.

$\therefore$ By proof by contraposition $p \rightarrow q$ is true.

⑤ Proof by contradiction:-

→ Let $p \rightarrow q$ is given theorem, we need to prove it is true.

Algorithm for case-1 :- [Given theorem is in form of conditional statement]

step-1 :- Assume p is true and $\sim q$ is true.

i.e $p \wedge \sim q$ is true.

step-2 :- Follow the method of Direct proof, to show that $p \wedge \sim q \Rightarrow F_0$.

$P_1 \wedge P_2 \wedge \_\_\_ \wedge \sim C \Rightarrow F_0$

P_1
P_2
,
,
P_n
$\sim C$
$\therefore F_0$

step-3 :- Then conclude that $p \wedge \sim q$ is a contradiction.

step-4 :- Then conclude that, if p is true then q is true.

step-5 :- Therefore $p \rightarrow q$ is true.

Case-2 :- Given theorem is Non-conditional statement.

Algorithm :-

→ Let P be a non conditional statement.

step-1 :- Assume $\sim p$ is true.

step-2 :- Follow the method of direct proof, to show that $\sim p \Rightarrow F_0$. [$\sim p$ is contradiction]

step-3 :- Then conclude $\sim p$ is False.

step-4 :- Therefore P is true.

* P-3 :-

Prove by proof by contradiction that $\sqrt{2}$ is an irrational number?

A) Let P: $\sqrt{2}$ is an irrational number.

Case-2 :- Non conditional statement.

Assume $\sqrt{2}$ is a rational number.

Then $\sqrt{2} = \frac{a}{b}$ G.C.D. of a,b is (a,b).

Here $(a,b) = 1$.

$\Rightarrow \sqrt{2} = \frac{a}{b}$

$\Rightarrow \sqrt{2}\, b = a$

$\Rightarrow 2b^2 = a^2$

$\Rightarrow a^2$ is an even number.

$\Rightarrow a$ is also an even number.

Then $\exists\, c \in Z \Rightarrow a = 2c$.

$\Rightarrow a^2 = 4c^2$

Substitute $a^2 = 4c^2$

$\Rightarrow a^2 = 2b^2$

$\Rightarrow \cancel{4}^{2} c^2 = \cancel{2} b^2$

$\Rightarrow b^2 = 2c^2$

Thus b^2 is an even number, hence b is an even no.

Then $(a,b) \geq 2$ $\Rightarrow (a,b) \neq 1$ which is a contradiction

So, we conclude that $\sim p$ is False.

Hence p is True.

$\therefore \sqrt{2}$ is an irrational number.

* Exercise :-

4] Show that if n is an integer and n^3+5 is odd, then n is even, using.

ⓐ A proof by contraposition.

ⓑ A proof by contradiction.

A) Given data, n is an integer and

p : n^3+5 is odd.

q : n is even.

So, Given $p \to q$ is conditional statement.

ⓐ Proof by contradposition :-

Assume $\sim q$ is true i.e n is odd.

$\exists\ k \in \mathbb{Z} \Rightarrow n = 2k+1$

Consider n^3+5, & substitute $n=2k+1$

$\Rightarrow n^3+5 = ((2k)+1)^3+5$

$= 8k^3 + 1 + 12k^2 + 6k + 5$

$= 8k^3 + 12k^2 + 6k + 6$

$= 2(4k^3 + 6k^2 + 3k + 3)$

$= 2m$. [where $m = 4k^3+6k^2+3k+3$]

$=$ even.

Thus, n^3+5 is an even number.

By proof by contraposition, n^3+5 is odd.

$p \to q$ is true

⑤ Proof by contradiction :-

Case-1 :-

Assume n^3+5 is odd and n is odd.

since n is odd.

$\exists\, k \in \mathbb{Z} \Rightarrow n = 2k+1$

Substitute n in n^3+5 :

$\Rightarrow n^3+5 = (2k+1)^3 + 5$

$= 8k^3 + 12k^2 + 6k + 1 + 5$

$= 8k^3 + 12k^2 + 6k + 6$

$= 2(4k^3 + 6k^2 + 3k + 3)$

$= 2m.$

Thus n^3+5 is an even.

It is contradiction due to wrong Assumption.

By proof by contradiction, If p is True then $\sim q$ is False.

Thus if p is true and then q is also True.

$\therefore p \to q$ is True

⑥ Proof of equivalence :-

Case-1 :-

→ When the given theorem is in the form of Biconditional.

→ Let $p \leftrightarrow q$ (or) p iff q (or) p if and only if q be the given statement.

→ By equivalence formula,

$$\underset{T}{p \leftrightarrow q} \equiv \underset{T}{(p \to q)} \wedge \underset{T}{(q \to p)}$$

~~step-1~~ :- In order to prove $p \leftrightarrow q$ is true, follow the below steps.

Step-1 :- Prove $p \to q$ is True.

Step-2 :- Prove $q \to p$ is True.

Step-3:- Then conclude $(p \to q) \wedge (q \to p)$ is also true.

Step-4:- By equivalence, we can conclude that $p \leftrightarrow q$ is also true.

* Case-2:-

→ Some of the mathematical theorems state several propositional ~~theo~~ statements are equivalent in the following form:-

$$P_1, P_2, \text{———}, P_n$$

$$P_1 \Leftrightarrow P_2 \Leftrightarrow P_3 \Leftrightarrow \text{———} \Leftrightarrow P_n.$$

(or)

$$P_1 \equiv P_2 \equiv P_3 \equiv \text{———} \equiv P_n.$$

→ $$\boxed{(P_1 \Leftrightarrow P_2 \Leftrightarrow P_3 \Leftrightarrow \text{——} \Leftrightarrow P_n) \equiv (P_1 \to P_2) \wedge (P_2 \to P_3) \wedge \text{———} \wedge (P_n \to P_1).}$$

Algorithm:-

Step-1:- Prove that $P_1 \to P_2$, $P_2 \to P_3$, ———, $P_n \to P_1$ are true.

Step-2:- Then $(P_1 \to P_2) \wedge (P_2 \to P_3) \wedge \text{———} \wedge (P_n \to P_1)$ are true.

Step-3:- Therefore, conclude that $P_1 \Leftrightarrow P_2 \Leftrightarrow P_3 \Leftrightarrow \text{———} \Leftrightarrow P_n$ i.e, $P_1, P_2, \text{———}, P_n$ all are equivalent.

* Exercise ;

6) Show that these statements about the integer x are equivalent:-

(a) $3x+2$ is even.

(b) $x+5$ is odd.

(c) x^2 is even.

Hypothesis ;

A) Let x is an integer and

P_1 : $3x+2$ is even.

P_2 : $x+5$ is odd.

P_3 : x^2 is even.

claim :- To prove that $P_1 \Leftrightarrow P_2 \Leftrightarrow P_3$.

(i) To prove that $P_1 \Leftrightarrow P_2$ is true :-

Now we can use proof by contraposition.

Assume ~~$\neg q$~~ / $\neg P_2$ i.e $x+5$ is even.

$\exists\ k \in Z \Rightarrow x+5 = 2k$

$\Rightarrow x = 2k-5$.

Now substitute x in $3x+2$.

$\Rightarrow 3x+2 = 3(2k-5)+2$

$= 6k-15+2$

$= 6k-13$.

Thus, clearly $3x+2$ is not even.

~~So~~ Thus $\neg P_2 \rightarrow \neg P_1$ is true.

Then By proof by contraposition, $P_1 \rightarrow P_2$ is true. $\longleftrightarrow$ (i)

(ii) To prove that $P_2 \Leftrightarrow P_3$ is true;-

Assume P_2 i.e ($x+5$ is odd) is true.

$\exists\ m \in Z \Rightarrow x+5 = 2m+1$

$\Rightarrow x = 2m+1-5$

$\Rightarrow x = 2m-4$

$\Rightarrow x = 2(m-2)$

$\Rightarrow x =$ even.

Thus, x is even. Then x^2 is also even.

By Direct proof, $P_2 \rightarrow P_3$ is True.

(iii) To prove that $P_3 \rightarrow P_1$ is true :-

Assume x^2 is an even.

Then x is also even.

$\exists\ \lambda \in \mathbb{Z} \ni x = 2\lambda$.

Consider, $3x+2 = 3(2\lambda)+2$

$= 6\lambda+2$

$= 2(3\lambda+1)$

$= 2\mu, \quad (\mu = 3\lambda+1)$

$\therefore$ $3x+2$ is an even.

By Direct proof, $P_3 \rightarrow P_1$ is True.

From (i), (ii), (iii),

we conclude that $P_1 \Leftrightarrow P_2 \Leftrightarrow P_3$.

$\therefore$ Hence proved.

(7) Method of Counter example (or) Method of disproof :-

→ Some times, mathematical statements are like following:

$\forall x \cdot (P(x)), \quad x \in D.$

$x \in 10.$ [Let].

$x_1, \text{———}, x_{10}.$

if (x_7) is False then it is counter example.

* P-6 :-

Show that the statement "Every positive integer is the sum of the squares of two integers" is False?

A) We cannot write 3 as sum of two squares.

$0^2=0,\ 1^2=1 \Rightarrow 0+1=1.$

$0^2=0,\ 2^2=4 \Rightarrow 0+4=4.$

$\therefore$ It is False statement.

$\therefore$ 3 is counter example here.

⑧ Proof by cases :-

→ sometimes, In mathematics the theorems are in conditional statement format but Hypothesis is Disjunction of propositions then we can use this method.

$$\overbrace{P_1 \vee P_2 \vee \cdots \vee P_n}^{T} \rightarrow q.$$

case-1 :- $\overset{T}{P_1} \rightarrow q$

case-2 :- $\overset{T}{P_2} \rightarrow q$

|

|

case-n :- $\underset{T}{P_n} \rightarrow q.$

*P-4 :-

Using a proof by cases, show that $|xy|=|x||y|$ where x and y are real numbers?

A) Given,

If $x, y \in \mathbb{R}$ then $|xy|=|x||y|$.

Here Hypothesis contains four cases.

x	y
+ve	-ve
-ve	+ve
+ve	+ve
-ve	-ve

$$|a| = \begin{cases} a, & a \geq 0 \text{ (a is non-negative)} \\ -a, & a < 0 \text{ (a is negative)}. \end{cases}$$

Case-1 :- Both are Non-Negative.

x, y are Non-negative

Then xy is also Non-negative.

Therefore, $|x| = x$; $|y| = y$ and $|xy| = xy$.

We have $|xy| = xy = |x|\,|y|$.

Case-2:- Both x, y are Negative:-

Then xy is positive.

Therefore $|xy| = xy = (-x)(-y) = |x|\,|y|$.

Case-3:- x is Non-negative & y is Negative.

x is Non-Negative then $|x| = x$.

y is Negative then $|y| = -y$.

$\therefore xy \le 0$.

Then $|xy| = -(xy) = (x)(-y) = |x|\,|y|$.

Case-4:- x is Negative & y is Non-negative.

x is Negative then $|x| = -x$.

y is Non-negative then $|y| = y$.

$\therefore xy \le 0$

Then $|xy| = -(xy) = (-x)(y) = |x|\,|y|$.

⑨ Exhaustive proof:-

→ It restricts the cases.

(i) No. of cases is small number.

(ii) All cases are particular / Individual.

* Example-10:-

Prove that $(n+1)^3 \ge 3^n$ if n is a positive integer with $n \le 4$?

A) Here, $n = 1, 2, 3, 4$.

$(n+1)^3 \geq 3^n$.

$\Rightarrow (1+1)^3 \geq 3^1 \Rightarrow 2^3 \geq 3 \Rightarrow 8 \geq 3$.

$\Rightarrow (2+1)^3 \geq 3^2 \Rightarrow 3^3 \geq 9 \Rightarrow 9 \geq 9$

$\Rightarrow (3+1)^3 \geq 3^3 \Rightarrow 4^3 \geq 27 \Rightarrow 64 \geq 27$

$\Rightarrow (4+1)^3 \geq 3^4 \Rightarrow 5^3 \geq 81 \Rightarrow 125 \geq 81$

$\therefore$ In each case, Inequality is True.

$\therefore$ The result is True.

*P-5:-

Using a proof by exhaustion, show that there are no solutions in integers x and y of $x^2+3y^2=8$.

A) We notice that,

If $x^2 > 8$ (or) $3y^2 > 8$ then there are no solutions.

$x^2 \leq 8$ (or) $3y^2 \leq 8$

$\Rightarrow |x^2| \leq 8 < 9$ $\qquad \Rightarrow 3y^2 \leq 8 < 9$

$\Rightarrow |x^2| < 9$ $\qquad \Rightarrow 3y^2 < 9$

$\Rightarrow -3 < x < 3$ $\qquad \Rightarrow y^2 < 3$

$\Rightarrow x = -2, -1, 0, 1, 2$ $\qquad \Rightarrow |y|^2 \leq 3 \Rightarrow |y| < \sqrt{3}$.

$x^2 = 4, 1, 0$. $\qquad \Rightarrow |y| < 1.732$

Max, $x^2 = 4$ $\qquad \Rightarrow -1.732 < y < 1.732$.

Max $y^2 = 1$

Max $3y^2 = 3(1) = 3$.

*Exercise:-

5) Use a proof by contradiction to show that there is no rational number r for which $r^3+r+1=0$.

*** [Hint:- Assume $r = \frac{a}{b}$ is a root, where a & b are integers & a/b is in lowest terms. Obtain an eq^n involving integers by multiplying by b^3. Then look at whether a & b are each odd or even.]

A) Let $P(r)$: There is no rational number 'r' for which $r^3+r+1=0$.

Case-2:-

Assume there is a rational number 'r' for which $r^3+r+1=0$.

Then $r=\frac{a}{b}$, here $(a,b)=1$.

Then $r^3+r+1=0$ becomes $\frac{a^3}{b^3}+\frac{a}{b}+1=0$

$\therefore a^3+ab^2+b^3=0$

Thus sum of a^3, ab^2, b^3 is an even number.

Since $(a,b)=1$, then we have three possible cases.

Case-i:- Both a and b are odd

Case-ii:- a is even and b is odd.

Case-iii:- a is odd and b is even.

$E+E=E$

Case-i:-

$E+O=O$

Both a and b are odd.

$O+O=E$

$\Rightarrow a^3=a\cdot a\cdot a=\text{odd}$

$E\cdot E=E$

$\Rightarrow ab^2=a\cdot b\cdot b=\text{odd}$

$E\cdot O=E$

$\Rightarrow b^3=b\cdot b\cdot b=\text{odd}$.

$O\cdot O=O$.

$\therefore \underbrace{a^3+ab^2}_{E}+b^3=\text{odd}$

$+\ O\ =\text{odd}$

Case-ii:-

a is even and b is odd.

$\Rightarrow a^3=a\cdot a\cdot a=\text{even}$

$\Rightarrow ab^2=a\cdot \underbrace{b\cdot b}_{E\cdot O}=\text{even}$

$\Rightarrow b^3=b\cdot b\cdot b=\text{odd}$

$\therefore \underbrace{a^3+ab^2}_{E}+\underbrace{b^3}_{O}=\text{odd}$

$E+O$

Case - iii :-

a is odd and b is even.

$\Rightarrow a^3 = a \cdot a \cdot a = \text{odd}$

$\Rightarrow ab^2 = a \cdot b \cdot b = \text{even}$.

$\Rightarrow b^3 = b \cdot b \cdot b = \text{even}$

$\therefore \underbrace{a^3}_{\text{odd}} + \underbrace{ab^2 + b^3}_{\text{even}} = \text{odd}$.

odd + even = odd.

(10) Mathematical Induction :-

→ Sometimes, Theorems in Mathematics are as follows:-

$\forall x \, (P(x))$, $x \in N$ [Let].

Well ordering principle :-

→ A Natural number set (or) any subset of Natural number have least number.

Algorithm :-

Basis Step-1:- Prove $P(1)$ is true. [Basis set].

Here 1 means least value in corresponding Domain [ex:- 1 in $x \in N$].

Induction step:- Prove that, $P(k) \rightarrow P(k+1)$ is true.

$k \in N$.

That is, Assume $P(k)$ is true. & prove that $P(k+1)$ is true.

* Exercise :-

Q) Prove that 5 divides $n^5 - n$ whenever n is a nonnegative integer.

A) Here, Domain is $n \in \{0, 1, 2, \ldots\}$ whole Numbers.

Let $P(n) = 5$ divides $n^5 - n$.

→

~~Basis set~~ step : $P(0)$: 5 divides 0.

clearly $P(0)$ is true.

Inductive step :

Let k be the Nonnegative arbitary integer

Assume $P(k)$: $5 / k^5 - k$.

$\Rightarrow \exists\, m \in \mathbb{Z} \ni k^5 - k = 5m$.

claim :- To prove that, $P(k+1)$: $5 / (k+1)^5 - (k+1)$.

Now consider, $(k+1)^5 - (k+1)$

$\Rightarrow k^5 + 5k^4 + 10k^3 + 10k^2 + 5k + \cancel{1} - k - \cancel{1}$

$\Rightarrow (k^5 - k) + 5(k^4 + 2k^3 + 2k^2 + k)$

$\Rightarrow 5m + 5(k^4 + 2k^3 + 2k^2 + k)$.

$\Rightarrow 5(m + k^4 + 2k^3 + 2k^2 + k)$.

$\Rightarrow 5n$ $[\text{where, } n = m + k^4 + 2k^3 + 2k^2 + k] \in \mathbb{Z}$.

$\therefore (k+1)^5 - (k+1) = 5n$

Thus, $5 / (k+1)^5 - (k+1)$ is True.

Exercise I

3] Use a proof by contradiction to prove that the sum of an irrational number and a rational is irrational?

A) Let P be a statement that the sum of an irrational number and a rational number is irrational.

To prove this by proof by contradiction, we assume that P is false.

Then $\sim p$ is true. i.e, sum of an irrational number and a rational number is rational.

Let a is an irrational.

b is rational.

Then their sum a+b is rational.

Then we can write b & a+b as fractions,

Let $b=\frac{c}{d}$ and $a+b=\frac{m}{n}$ $(d, n \neq 0)$.

where c, d, m, n are integers.

since $a+b=\frac{m}{n}$

$\Rightarrow a+\frac{c}{d}=\frac{m}{n}$ $[\because b=\frac{c}{d}]$.

$\Rightarrow a=\frac{m}{n}+(\frac{-c}{d})$.

since the rational numbers are closed under addition,

so, $a=\frac{m}{n}+(\frac{-c}{d})$ is a rational number.

However, an assumptions said that a is an irrational.

so 'a' cannot be both rational and irrational.

This leads to contradiction.

$\therefore$ $\sim P$ is false & hence p is true.

$\therefore$ sum of irrational & rational is always irrational.

7) Use the mathematical induction to prove that

$$1^2+3^2+5^2+ \_\_\_\_ +(2n+1)^2 = \frac{(n+1)(2n+1)(2n+3)}{3}$$

whenever n is a non-negative integer.

A) Let $p(n)$ be proposition.

$$p(n): 1^2+3^2+5^2+ \_\_\_\_ + (2n+1)^2 = \frac{(n+1)(2n+1)(2n+3)}{3}$$

The domain of given propositional function is the set of nonnegative i.e, $n = \{0, 1, 2, \_\_\_\}$ integers.

Basis step:-

put $n=0$ in (1)

Then $1^2 = \frac{1(1)(3)}{3} = 1 \Rightarrow 1=1$.

Thus $p(0)$ is true.

Inductive step:-

Assume that $P(K)$ is true for an arbitrary non-negative integer K.

Then $1^2+3^2+5^2+ \_\_\_\_ +(2K+1)^2 = \frac{(K+1)(2K+1)(2K+3)}{3}$.

claim:-

To prove that $P(K+1)$ is true.

Then consider L.H.S of $P(K+1)$

$1^2+3^2+5^2+ \_\_\_\_ + (2K+1)^2+(2(K+1)+1)^2$

$= \frac{(K+1)(2K+1)(2K+3)}{3} + (2K+3)^2$ [by (2)]

$= \frac{(K+1)(2K+1)(2K+3) + 3(2K+3)^2}{3}$

$= \frac{(2K+3)[(K+1)(2K+1)+3(2K+3)]}{3}$

$= \frac{(2K+3)}{3}[2K^2+3K+1+6K+9]$

$= \frac{(2K+3)}{3}[2K^2+9K+10]$

$= \frac{(2K+3)}{3}[2K^2+4K+5K+10]$

$= \frac{(2K+3)}{3}[(K+2)(2K+5)]$.

$$= \frac{(k+2)(2k+3)(2k+5)}{3}$$

$$= \frac{[(k+1)+1][2(k+1)+1][2(k+1)+3]}{3}$$

This shows that $P(k+1)$ is true when $P(k)$ is true.

By P.M.I, $P(n)$ is true for all Nonnegative integers n.

8] Use the mathematical Induction to prove that for every positive integer n.

$$1\cdot2 + 2\cdot3 + \_\_ + n(n+1) = \frac{n(n+1)(n+2)}{3}$$

A) Let $P(n)$ be the proposition that

$$1\cdot2 + 2\cdot3 + \_\_ + n(n+1) = \frac{n(n+1)(n+2)}{3} \longrightarrow (1)$$

Also given that Domain of given propositional function is set of +ve integers.

<u>Basis step</u>:- Put $n=1$ in (1)

Then $1\cdot2 = \frac{1(1+1)(1+2)}{3} = \frac{2(\cancel{3})}{\cancel{3}}$

$\Rightarrow 2 = 2$

Thus $P(1)$ is true.

<u>Inductive step</u>:-

We assume that $P(k)$ is true for an arbitrary positive integer 'k'.

i.e, $1\cdot2 + 2\cdot3 + \_\_ + k(k+1) = \frac{k(k+1)(k+2)}{3} \longrightarrow (2)$

Then consider

$$\underbrace{1\cdot2 + 2\cdot3 + \_\_ + k(k+1)} + (k+1)(k+2)$$

$$= \frac{k(k+1)(k+2)}{3} + (k+1)(k+2) \quad [\text{by } (2)]$$

$$= \frac{k(k+1)(k+2) + 3(k+1)(k+2)}{3}$$

$$= \frac{(k+1)[(k+1)+1]\,[(k+1)+2]}{3}$$

This shows that $p(k+1)$ is true when $p(k)$ is true.

$\therefore$ By P.M.I, $p(n)$ is true for all +ve integers n.

10] Prove that if $A_1, A_2, \text{---}, A_n$ are subsets of a universal set U, then $\left(\bigcup_{k=1}^{n} A_k\right)' = \bigcap_{k=1}^{n} (A_k)'$

where A' denotes the complement of set A.

[This is an extended version of Demorgan's Laws].

A) Let $p(n) : \left(\bigcup_{k=1}^{n} A_k\right)' = \bigcap_{k=1}^{n} (A_k)'$

We need to prove that $p(n)$ by mathematical Induction

Basis step:-

$P(1)$ is True, $\Rightarrow A_1' = A_1'$.

Inductive step:-

Assume $p(k)$ is true.

$$\Rightarrow \left(\bigcup_{i=1}^{k} A_i\right)' = \bigcap_{i=1}^{k} (A_i)' \longrightarrow (1)$$

consider, $\left(\bigcup_{i=1}^{k+1} A_i\right)' = (A_1 \cup A_2 \cup A_3 \cup \text{---} \cup A_k \cup A_{k+1})'$

$$= [(A_1 \cup A_2 \cup A_3 \cup \text{---} \cup A_k) \cup A_{k+1}]'$$

$$= (A_1 \cup A_2 \cup A_3 \cup \text{---} \cup A_k)' \cap A_{k+1}'$$

$$= \left(\bigcup_{i=1}^{k} A_i\right)' \cap (A_{k+1})'$$

$$= \bigcap_{i=1}^{k} (A_i)' \cap (A_{k+1})' \quad [\text{by } (1)]$$

$$= \bigcap_{i=1}^{k+1} (A_i)'$$

This shows that $P(k+1)$ is true.

$\therefore$ By P.M.I, $P(n)$ is true for all +ve integer n.

$$\therefore \left(\bigcup_{k=1}^{n} A_k\right)' = \bigcap_{k=1}^{n} (A_k)'$$