4.1

Solution of Algebraic and Transcendental Equations: Bisection Method

Introduction

In scientific and engineering studies, a frequently occurring problem is to find the roots of equations of the form.

$$f(x) = 0 \tag{1}$$

If $f(x)$ is a quadratic, cubic or a biquadratic expression then algebraic formulae are available for expressing the roots in terms of the coefficients. For example, the roots of the quadratic equations $ax^2 + bx + c = 0$ are $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

On the other hand, when $f(x)$ is a polynomial of higher degree or an expression involving transcendental functions, algebraic methods are not available, and recourse must be taken to find the roots by approximate methods.

In the first four modules, we concerned with the description of several numerical methods for the solution of equations of the form (1), where $f(x)$ is algebraic or transcendental or a combination of both. Now, algebraic functions of the form

$$f_n(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n, \tag{2}$$

are called *polynomials* and we discuss some special methods for determining their roots. A non-algebraic function is called a *transcendental* function, for example,

$f(x) = \ln x^3 - 0.7, \phi(x) = e^{-0.5x} - 5x, \Psi(x) = \sin^2 x - x^2 - 2$, etc... The roots of (1) may be either real or complex. We discuss methods of finding a real root of algebraic or transcendental equations and also methods of determining all real and complex roots of polynomials.

THE BISECTION METHOD

This method is based on mean value theorem which states that if a function $f(x)$ is continuous between a and b, and $f(a)$ and $f(b)$ are of opposite signs, then there exists at least one root between a and b. For definiteness, let $f(a)$ be negative and $f(b)$ be positive. Then the root lies between a and b and let its approximate value be given by $x_0 = \frac{(a+b)}{2}$. If $f(x_0) = 0$, we conclude that x_0 is a root of the equation $f(x) = 0$. Otherwise, the root lies either between x_0 and b, or between x_0 and a depending on whether $f(x_0)$ is negative or positive. We designate this new interval as $[a_1, b_1]$ whose length is $|b-a|/2$. As before, this is bisected at x_1 and the new interval will be exactly half the length of the previous one. We repeat this process until the latest interval (which contains the root) is as small as desired, say ε. It is clear that the interval width is reduced by a factor of one-half at each step and at the end of the n^{th} step, the new interval will be $[a_n, b_n]$ of length $|b-a|/2^n$. We then have

$$\frac{|b-a|}{2^n} \leq \varepsilon,$$

which give on simplification

$$n \geq \frac{\log_e(|b-a|/\varepsilon)}{\log_e 2} \quad (3)$$

Inequality (3) gives the number of iterations required to achieve an accuracy ε. For example, if $|b - a| = 1$ and $\varepsilon = 0.001$, then it can be seen that

$$n \geq 10 \quad (4)$$

The method is shown graphically in the figure below.

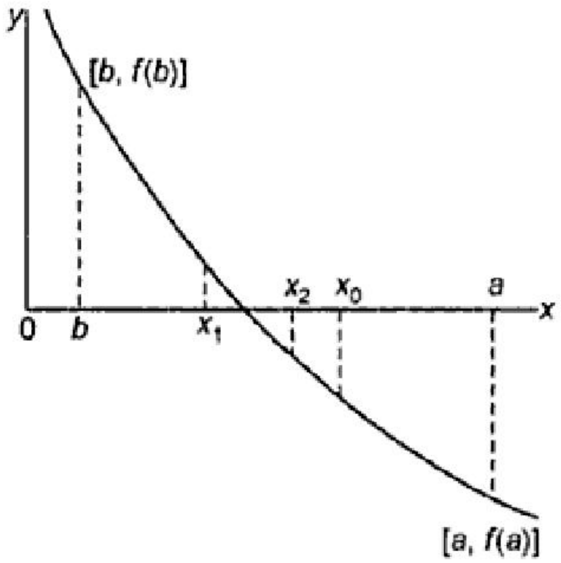

Figure: Graphical representation of the bisection method

It should be noted that this method always succeeds. If there are more roots than one in the interval, bisection method finds one of the roots. It can be easily programmed using the following computational steps:

1. Choose two real number a and b such that $f(a)f(b) < 0$.
2. Set $x_r = (a + b)/2$.
3.

a. If $f(a)f(x_r) < 0$, the root lies in the interval (a, x_r). Then, set $b = x_r$ and go to step 2 above.
b. If $f(a)f(x_r) > 0$, the root lies in the interval (x_r, b). Then, set $a = x_r$ and got to step 2.
c. If $f(a)f(x_r) = 0$, it means that x_r is a root of the equation $f(x) = 0$ and the computation may be terminated.

In practical problems, the roots may not be exact so that condition (c) above is never satisfied. In such a case, we need to adopt a criterion for deciding when to terminate the computations.

A convenient criterion is to compute the percentage error ε_r defined by

$$\varepsilon_r = \left|\frac{x_r' - x_r}{x_r'}\right| \times 100\% \tag{5}$$

where $x_r{}'$ is the new value of x_r. The computations can be terminated when ε_r becomes less than a prescribed tolerance, say ε_p. In addition, the maximum number of iterations may also be specified in advance.

Definitions

Given some value v and its approximation v_{approx}, the **absolute error** is

$$\epsilon = |v - v_{approx}|,$$

Where the vertical bars denote the absolute value. If: $v \neq 0$. the **relative error** is

$\eta = \frac{|v - v_{approx}|}{|v|} = \left|\frac{v - v_{approx}}{v}\right|$, and the **percent error** is

$$\delta = \frac{|v - v_{approx}|}{|v|} \times 100\% = \left|\frac{v - v_{approx}}{v}\right| \times 100\%$$

One commonly distinguishes between the relative error and the absolute error. The absolute error is the magnitude of the difference between the exact value and the approximation. The relative error is the absolute error divided by the magnitude of the exact value. The percent error is the relative error expressed in terms of per 100.

As an example, if the exact value is 50 and the approximation is 49.9, then the absolute error is 0.1 and the relative error is 0.1/50 = 0.002. The relative error is often used to compare approximations of numbers of widely differing size; for example, approximating the number 1,000 with an absolute error of 3 is, in most applications, much worse than approximating the number 1,000,000 with an absolute error of 3; in the first case the relative error is 0.003 and in the second it is only 0.000003.

Example 1

Find a real root of the equation $f(x) = x^3 - x - 1 = 0$.

Solution:

Since $f(1)$ is negative and $f(2)$ positive, a root lies between 1 and 2 and therefore we take $x_0 = \frac{3}{2}$. Then

$$f(x_0) = \frac{27}{8} - \frac{3}{2} = \frac{15}{8}, \text{ which is positive.}$$

Hence the root lies between 1 and 1.5 and we obtain

$$x_1 = \frac{1 + 1.5}{2} = 1.25$$

We find $f(x) = -19/64$, which is negative. We therefore conclude that the root lies between 1.25 and 1.5. If follows that

$$x_2 = \frac{(1.25 + 1.5)}{2} = 1.375$$

The procedure is repeated and the successive approximations are

$x_3 = 1.3125,\ x_4 = 1.34375,\ x_5 = 1.328125$, etc.

Example 2

Find the number of iterations required to find a root of $f(x) = x^3 - 5x + 1$ which lies in between 0 and 1 with 0.004 accuracy by using bisection method.

Solution: Given $f(x) = x^3 - 5x + 1$

Formula for finding number of iterations is $n \geq \frac{\log_e(|b-a|/\varepsilon)}{\log_e 2}$

Here $a = 0, b = 1$ and $\varepsilon = 0.004$.

$$\therefore \quad n \geq \frac{\log_e(|1-0|/0.004)}{\log_e 2}$$

$$\Rightarrow n \geq 7.96578$$

$\therefore$ number of iterations required to achieve the desired accuracy is 8.

Example 3

Find a real root of the equation $x^3 - 2x - 5 = 0$.

Solution:

Let $f(x) = x^3 - 2x - 5$.

Then $\quad f(2) = -1$ and $f(3) = 16$.

Hence a root lies between 2 and 3 we take

$$x_0 = \frac{(2+3)}{2} = 2.5$$

Since $f(x_0) = 5.6250$, we choose $[2, 2.5]$ as the new interval. Then

$$x_1 = \frac{(2+2.5)}{2} = 2.25 \text{ and } f(x_1) = 1.890625$$

Proceeding in this way, the following table is obtained.

n	a	b	x	$f(x)$
1	2	3	2.5	5.6250
2	2	2.5	2.25	1.8906
3	2	2.25	2.125	0.3457
4	2	2.125	2.0625	−0.3513
5	2.0625	2.125	2.09375	−0.0089
6	2.09375	2.125	2.10938	0.1668
7	2.09375	2.10938	2.10156	0.07856
8	2.09375	2.10156	2.09766	0.03471
9	2.09375	2.09766	2.09570	0.01286
10	2.09375	2.09570	2.09473	0.00195
11	2.09375	2.09473	2.09424	−0.0035
12	2.09424	2.09473		

At $n = 12$, it is seen that the difference between two successive iterates is 0.0005, which is less than 0.001. Thus this result agrees with condition given in (4).

Example 4

Find a positive root of the equation $xe^x = 1$, which lies between 0 and 1 with desired percentage error 0.05%.

Solution:

Let $f(x) = xe^x - 1$. Since $f(0) = -1$ and $f(1) = 1.718$, it follows that a root lies between 0 and 1. Thus $x_0 = 0.5$. Since $f(0.5)$ is negative, it follows that the root lies between 0.5 and 1. Hence the new root is 0.75 i.e., $x_1 = 0.75$. Using the values x_0 and x_1, we calculate ε_1:

$$\varepsilon_1 = \left|\frac{x_1 - x_0}{x_1}\right| \times 100 = 33.33\%$$

Again, we find that $f(0.75)$ is positive and hence the root lies between 0.5 and 0.75, $i.e.$, $x_2 = 0.625$. Now, the error is

$$\varepsilon_2 = \left|\frac{0.625 - 0.75}{0.625}\right| \times 100 = 20\%$$

Proceeding in this way, the following table is constructed where only the sign of the function value is indicated. The prescribed tolerance is 0.05%.

Iteration	a	b	x_r	Sign of $f(x_r)$	$E_r(\%)$
1	0	1	0.5	negative	–
2	0.5	1	0.75	positive	33.33
3	0.5	0.75	0.625	positive	20.00
4	0.5	0.625	0.5625	negative	11.11
5	0.5625	0.625	0.59375	positive	5.263
6	0.5625	0.593375	0.5781	positive	2.707
7	0.5625	0.5781	0.5703	positive	1.368
8	0.5625	0.5703	0.5664	negative	0.688
9	0.5664	0.5703	0.5684	positive	0.352
10	0.5664	0.5684	0.5674	positive	0.176
11	0.5664	0.5674	0.5669	negative	0.088
12	0.5669	0.5674	0.5671	negative	0.035

Thus after 12 iteration, the error, ε_r, finally satisfies the prescribed tolerance, viz, 0.05%. Hence the required root is 0.567 and it is easily seen that this value is correct to three decimal places.