UNIT-04 FAST FOURIER TRANSFORM (FFT)

→ FFT is an efficient algorithm to compute the DFT with reduced computation.

→ FFT ⇒ widely used applications
- Spectrum Analysis
- Convolution
- Correlation
- Linear filtering

→ FFT is a computational algorithm.

→ FFT is invented by Cooley and Tukey in 1965.

→ FFT algorithms → Decimation in time (DIT) / → Decimation in Frequency (DIF)

Computational Complexity of Direct computation of DFT:-

Consider an N-point DFT,

$$X[K] = \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}Kn} \qquad 0 \leq K \leq N-1$$

$$X[K] = \underbrace{\{X[0], X[1], X[2], \ldots\ldots, X[N-1]\}}_{\text{DFT coefficients}}$$

Let us consider $N=8$,

$$X[K] = \sum_{n=0}^{7} x(n)\, e^{-j\frac{2\pi}{8}Kn} \quad ; \quad 0 \leq K \leq 7$$

$$X[K] = \underline{x[0]}\, e^{\underline{\underline{-j\frac{2\pi}{8}\cdot K\cdot 0}}} + \underline{x[1]}\, e^{\underline{\underline{-j\frac{2\pi}{8}\cdot K\cdot 1}}} + \underline{x[2]}\, e^{\underline{\underline{-j\frac{2\pi}{8}\cdot K\cdot 2}}} +$$
$$\underline{x[3]}\, e^{\underline{\underline{-j\frac{2\pi}{8}\cdot K\cdot 3}}} + \underline{x[4]}\, e^{\underline{\underline{-j\frac{2\pi}{8}\cdot K\cdot 4}}} + \underline{x[5]}\, e^{\underline{\underline{-j\frac{2\pi}{8}\cdot K\cdot 5}}} +$$
$$\underline{x[6]}\, e^{\underline{\underline{-j\frac{2\pi}{8}\cdot K\cdot 6}}} + \underline{x[7]}\, e^{\underline{\underline{-j\frac{2\pi}{8}\cdot K\cdot 7}}}$$

Double line ═══ ⇒ Complex exponential term

Single line ─── ⇒ it may be real / complex term

$X[k]$ has 7 additions and 8 multiplications.

① For each value of 'k', $0 \le k \le 7$, the 8-point DFT requires eight complex multiplication and $8-1=7$ complex additions.

$\therefore$ For each value of k, $0 \le k \le N-1$, the N-point DFT requires 'N' $\Rightarrow$ Complex multiplications

'(N-1)' $\Rightarrow$ Complex additions

② For each complex multiplication requires

4 $\rightarrow$ real multiplications

and 2 $\rightarrow$ real additions

For each complex addition requires two real additions.

$\therefore$ For each value of 'k'

Number of real multiplications $= 4N$

Number of real additions $= 2N + 2(N-1)$

$= 4N - 2$

③ For all values of 'k', $0 \le k \le 7$

the 8-point DFT requires

$8 \times 8 = 64$ complex multiplications

$8 \times 7 = 56$ complex additions

$\therefore$ For all values of 'k', $0 \le k \le N-1$

the N-point DFT requires

$N \times N = N^2$ complex multiplications

$N \times (N-1) \Rightarrow$ Complex additions

④ Each complex multi requires $\rightarrow$ 4 real multi

$\rightarrow$ 2 real multi

Each complex add requires $\rightarrow$ 2 real add

$\therefore$ For N-point DFT $\rightarrow$ No. of real multi $= 4N^2$

$\rightarrow$ No. of real add $= 2N^2 + 2N(N-1)$

$$2N^2 + 2N(N-1) = 2N^2 + 2N^2 - 2N = 4N^2 - 2N$$

Ex:- 1024 - point DFT

Here, $N = 1024$

N^2 complex multiplications $\rightarrow 1024 \times 1024$

$= 1048576 \approx 1$ million

$N(N-1)$ complex additions $\rightarrow 1024 \times (1024-1)$

$= 1024 \times 1023 \approx 1$ million

$$X[K] = \{X[0], X[1], X[2], \ldots\ldots, X[1023]\}$$

Symmetry and periodicity properties of Twiddle factor:

$$W_N = e^{-j\frac{2\pi}{N}}$$

① Symmetry property : $W_N^{K+\frac{N}{2}} = e^{-j\frac{2\pi}{N}(K+\frac{N}{2})}$

$$= e^{-j\frac{2\pi}{N}K} \cdot \underbrace{e^{-j\frac{2\pi}{\cancel{N}}\cdot\frac{\cancel{N}}{2}}}_{=-1}$$

$$W_N^{K+\frac{N}{2}} = -e^{-j\frac{2\pi}{N}K}$$

$$\boxed{W_N^{K+\frac{N}{2}} = -W_N^K}$$

Symmetry about $\frac{N}{2}$ on -ve side

② Periodicity property : $W_N^{K+N} = e^{-j\frac{2\pi}{N}(K+N)}$

$$= e^{-j\frac{2\pi}{N}K} \cdot \underbrace{e^{-j\frac{2\pi}{\cancel{N}}\cdot\cancel{N}}}_{=+1}$$

$$W_N^{K+N} = e^{-j\frac{2\pi}{N}K}$$

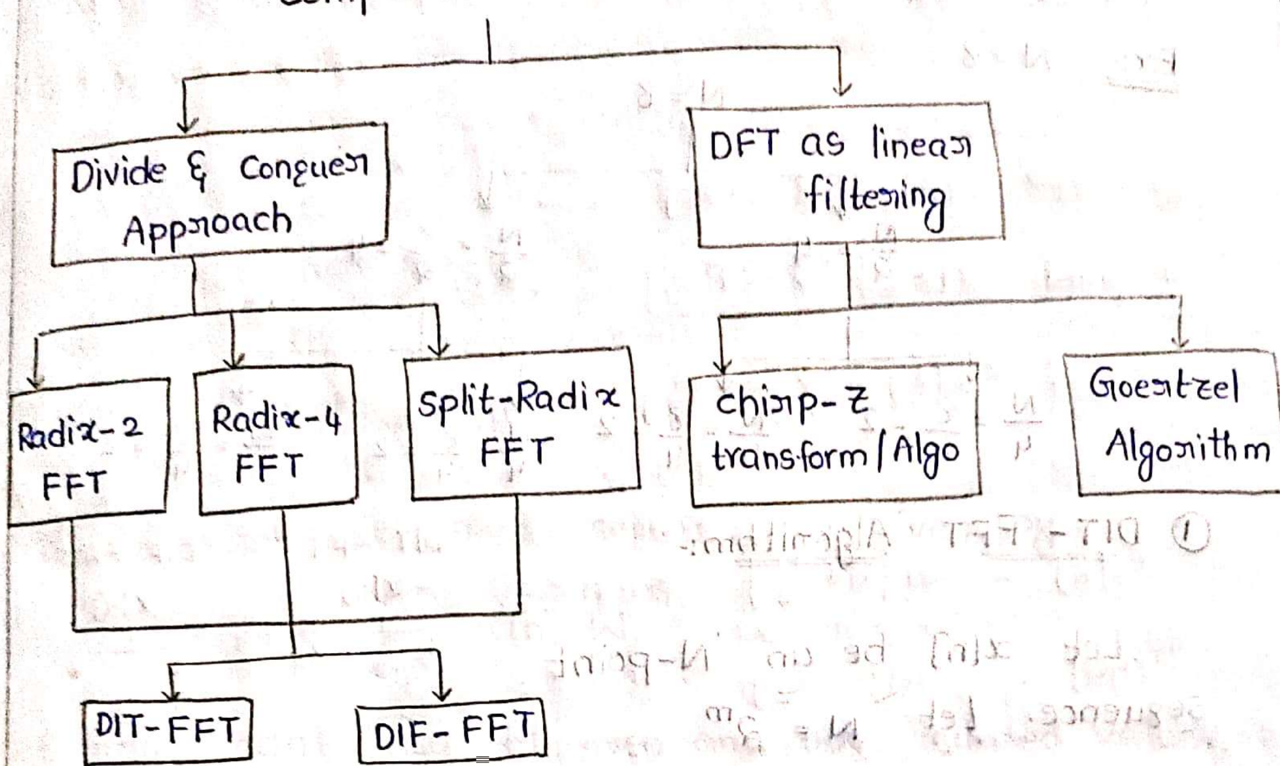
$$\boxed{W_N^{K+N} = W_N^K}$$

Symmetry about N on +ve side

③ $W_N^2 = e^{-j\frac{2\pi}{N}\cdot 2} = e^{-j\frac{2\pi}{(N/2)}} = W_{N/2}$

$$\boxed{W_N^2 = W_{N/2}}$$

Computation of DFT



RADIX-2 FFT Algorithm:-

DIT FFT
DIT FFT } Divide & Conquer Approach

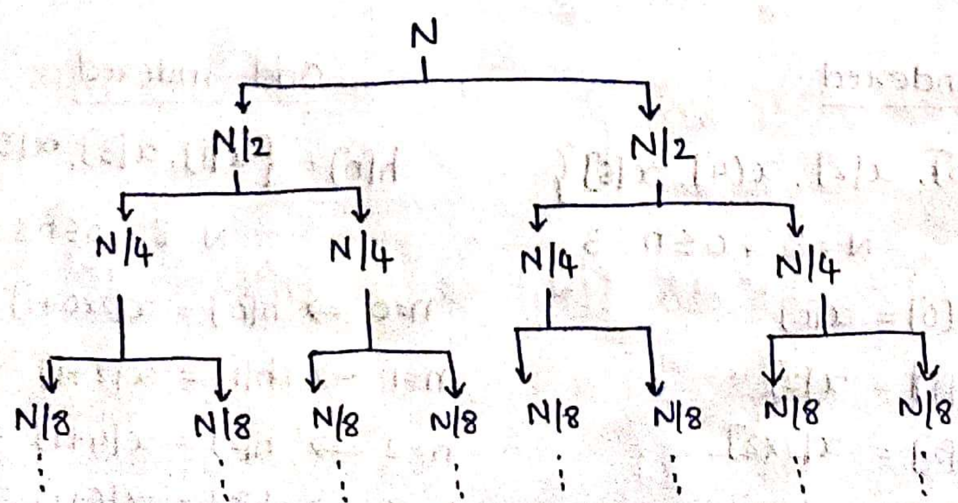
Signal length 'N' $\quad N = \underbrace{r_1 \; r_2 \; r_3 \cdots\cdots r_m}_{m\text{-factor}}$

If these factors are equal,

i.e $\quad r_1 = r_2 = r_3 = \cdots\cdots = r_m = r$

then $\quad N = r^m$

$\quad \hookrightarrow r =$ Radix of FFT Algorithm



until 2-point DFT's are obtained.

Ex:- $N = 8$

$N = 8$

$\frac{N}{2} = 4$ $\quad$ $\frac{N}{2} = \frac{8}{2} = 4$

$\frac{N}{4} = \frac{8}{4} = 2$ $\quad$ $\frac{N}{4} = \frac{8}{4} = 2$ $\quad$ $\frac{N}{4} = \frac{8}{4} = 2$ $\quad$ $\frac{N}{4} = \frac{8}{4} = 2$

① DIT- FFT Algorithm:-

DIT FFT: $x[n]$ $\quad$ DIF FFT: $X(k)$

Let $x[n]$ be an N-point sequence. Let $N = 2^m$

$\Rightarrow$ In DIT- FFT algorithm, the sequence $x[n]$ is decimated [broken] into two $\frac{N}{2}$ point sequence.

N-point

$\frac{N}{2}$ point Even indexed values of $x[n]$ $\quad$ $\frac{N}{2}$ point Odd indexed values of $x[n]$.

$$x[n] = \{x[0], x[1], x[2], x[3], x[4], x[5], x[6], x[7]\}$$

$\hookrightarrow$ 8-point sequence.

Even indexed

$g[n] = \{x[0], x[2], x[4], x[6]\}$

$N=8$, $0 \le n \le 3$

$n=0 \Rightarrow g[0] = x[0]$

$n=1 \Rightarrow g[1] = x[2]$

$n=2 \Rightarrow g[2] = x[2\times2]$

$n=3 \Rightarrow g[3] = x[2\times3]$

Odd indexed

$h[n] = \{x[1], x[3], x[5], x[7]$

$N=8$, $0 \le n \le 3$

$n=0 \Rightarrow h[0] = x[2\times0+1] = x[1]$

$n=1 \Rightarrow h[1] = x[2+1] = x[3]$

$n=2 \Rightarrow h[2] = x[4+1] = x[5]$

$n=3 \Rightarrow h[3] = x[6+1] = x[7]$

For $N=8$, $g(n) = \{x[0], x[2], x[4], x[6]\}$

$h(n) = \{x[1], x[3], x[5], x[7]\}$

$$x(n) \xleftrightarrow[N]{DFT} X[K] \qquad 0 \le k \le N-1$$

$$g(n) \xleftrightarrow[N/2]{DFT} G[K] \qquad 0 \le k \le \frac{N}{2}-1$$

$$h(n) \xleftrightarrow[N/2]{DFT} H[K] \qquad 0 \le k \le \frac{N}{2}-1$$

The N-point DFT ~~of~~ a sequence is given by

$$X[K] = \sum_{n=0}^{N-1} x(n)\, W_N^{nK} \qquad 0 \le k \le N-1$$

Splitting $x[n]$ into its even and odd indexed values,

$$X[K] = \sum_{n=0}^{\frac{N}{2}-1} x[2n]\, W_N^{2nk} + \sum_{n=0}^{\frac{N}{2}-1} x[2n+1]\, W_N^{(2n+1)\cdot K}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} x[2n]\, W_N^{2nk} + \sum_{n=0}^{\frac{N}{2}-1} x[2n+1]\, W_N^{2nk} \cdot W_N^{k}$$

$$X[K] = \sum_{n=0}^{\frac{N}{2}-1} x[2n]\, W_N^{2nk} + W_N^{k} \sum_{n=0}^{\frac{N}{2}-1} x[2n+1]\, W_N^{2nk}$$

We know that, $W_N^{2nk} = W_{N/2}^{nk}$

$$X[K] = \sum_{n=0}^{\frac{N}{2}-1} x[2n]\, W_{N/2}^{nk} + W_N^{k} \sum_{n=0}^{\frac{N}{2}-1} x[2n+1]\, W_{N/2}^{nk}$$

$$X[K] = \sum_{n=0}^{\frac{N}{2}-1} \underbrace{g(n)}_{N/2}\, W_{N/2}^{nk} + W_N^{k} \sum_{n=0}^{\frac{N}{2}-1} \underbrace{h(n)}_{N/2}\, W_{N/2}^{nk}$$

$$\boxed{\underbrace{X[K]}_{\text{N-point}} = \underbrace{G[K]}_{\frac{N}{2}\text{ point}} + W_N^{k}\, \underbrace{H[K]}_{\frac{N}{2}\text{-point}} \qquad 0 \le k \le N-1}$$

Where $G[k]$ & $H[k]$ are $\frac{N}{2}$ point DFTs of $g(n)$ & $h(n)$ respectively.

$$\boxed{\begin{array}{ll} N\text{-point} & X[k] = X[k+N] \\ \frac{N}{2}\text{-point} & X[k] = X[k+\frac{N}{2}] \\ \frac{N}{4}\text{-point} & X[k] = X[k+\frac{N}{4}] \end{array}}$$

$\longleftrightarrow W_N^{k+N} = +W_N^k$

$\longleftrightarrow W_N^{k+\frac{N}{2}} = -W_N^k$

$$e^{-j\frac{2\pi}{N}(k+\frac{N}{2})} = e^{-j\frac{2\pi}{N}k} \cdot e^{-j\frac{2\pi}{N}\frac{N}{2}}$$

$$= e^{-j\frac{2\pi}{N}k} \cdot e^{-j\pi}$$

$$= -W_N^k$$

$$G[k] = G[k+\frac{N}{2}] \qquad 0 \le k \le \frac{N}{2}-1$$

$$H[k] = H[k+\frac{N}{2}], \qquad 0 \le k \le \frac{N}{2}-1$$

$$W_N^{k+N/2} = -W_N^k$$

* $$W_N^{k+\frac{N}{2}} = e^{-j\frac{2\pi}{N}(k+\frac{N}{2})}$$

$$= e^{-j\frac{2\pi}{N}k} \cdot e^{-j\frac{2\pi}{\cancel{N}}\cdot\frac{\cancel{N}}{2}}$$

$$= -e^{-j\frac{2\pi}{N}k} = -W_N^k$$

$$\boxed{W_N^{k+N/2} = -W_N^k}$$

* $$W_{N/2}^{k+\frac{N}{2}} = e^{-j\frac{2\pi}{N/2}(k+\frac{N}{2})}$$

$$= e^{-j\frac{2\pi}{N/2}k} \cdot e^{-j\frac{2\pi}{\cancel{N/2}}\cdot\cancel{N/2}}$$

$$\boxed{W_{N/2}^{k+\frac{N}{2}} = +W_{N/2}^k}$$

* $$X[k] = G[k] + W_N^k H[k] \qquad 0 \le k \le N-1 \longrightarrow ①$$

$$G[k] = G[k+\frac{N}{2}] \qquad 0 \le k \le \frac{N}{2}-1 \longrightarrow ②$$

$$H[k] = H[k+\frac{N}{2}] \qquad 0 \le k \le \frac{N}{2}-1 \longrightarrow ③$$

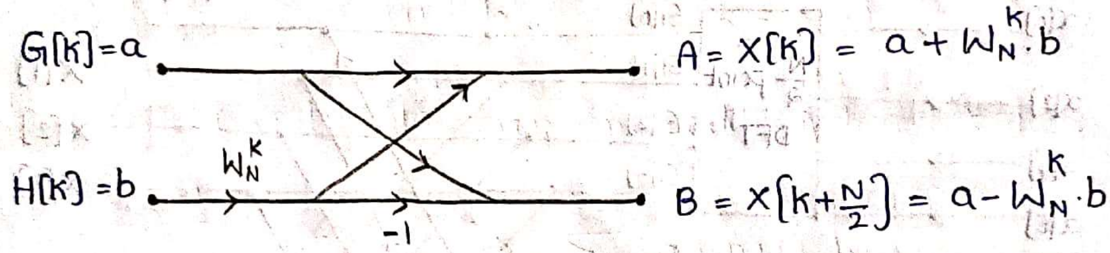
Substituting $k = k+\frac{N}{2}$ in eqn ①,

$$X\left[k+\tfrac{N}{2}\right] = \underbrace{G\left[k+\tfrac{N}{2}\right]}_{=G[k]} + \underbrace{W_N^{k+\frac{N}{2}}}_{=-W_N^k} \cdot \underbrace{H\left[k+\tfrac{N}{2}\right]}_{=H[k]} \qquad 0 \le k \le \tfrac{N}{2}-1$$

$$X\left[k+\tfrac{N}{2}\right] = G[k] - W_N^k H[k]$$

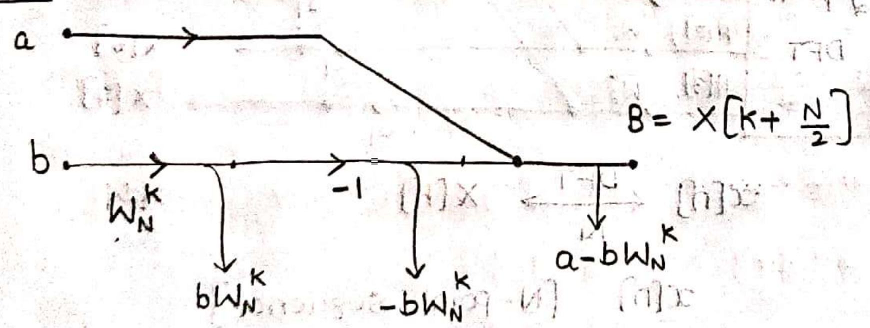
$$\boxed{\begin{aligned} X[k] &= G[k] + W_N^k H[k] \rightarrow \tfrac{N}{2}\text{ point} \quad 0 \le k \le \tfrac{N}{2}-1 \\ X\left[k+\tfrac{N}{2}\right] &= G[k] - W_N^k H[k] \rightarrow \tfrac{N}{2}\text{ point} \end{aligned}}$$

$$x(n) \xleftrightarrow{N} X[k]$$



Butterfly diagram

Ex:-



Let $N = 8$

$$X[k] = G[k] + W_N^k \cdot H[k] \qquad 0 \le k \le \tfrac{N}{2}-1$$

$$X[k] = G[k] + W_8^k \cdot H[k] \qquad 0 \le k \le 3$$

$$\left.\begin{aligned} k=0 &\Rightarrow X[0] = G[0] + W_8^0 H[0] \\ k=1 &\Rightarrow X[1] = G[1] + W_8^1 H[1] \\ k=2 &\Rightarrow X[2] = G[2] + W_8^2 H[2] \\ k=3 &\Rightarrow X[3] = G[3] + W_8^3 H[3] \end{aligned}\right\}$$

$$\longrightarrow X\left[k+\frac{N}{2}\right] = G[k] - W_N^k H[k] \qquad 0 \le k \le \frac{N}{2}-1$$

$N = 8$ $\qquad 0 \le k \le 3$

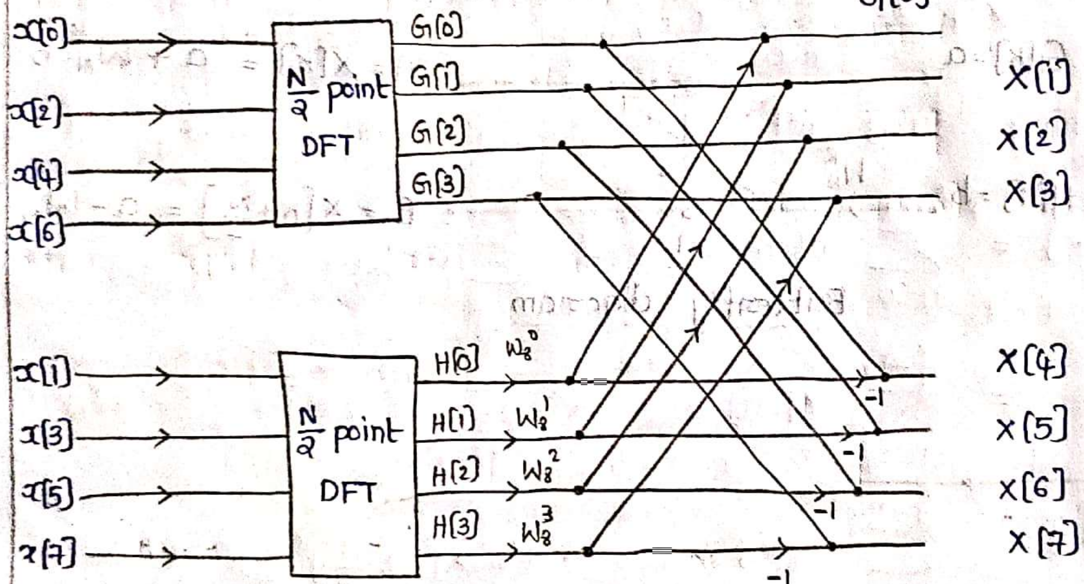
$$k=0 \Rightarrow X\left[0+\frac{8}{2}\right] = X[4] = G[0] - W_8^0 H[0]$$

$$k=1 \Rightarrow X\left[1+\frac{8}{2}\right] = X[5] = G[1] - W_8^1 H[1]$$

$$k=2 \Rightarrow X\left[2+\frac{8}{2}\right] = X[6] = G[2] - W_8^2 H[2]$$

$$k=3 \Rightarrow X\left[3+\frac{8}{2}\right] = X[7] = G[3] - W_8^3 H[3]$$

$G[0] = W_8^0 H[0] = X[0]$



DIT-FFT:

$$x[n] \xleftrightarrow[N]{DFT} X[k]$$

$x[n]$ [N-point sequence]

$g[n] = x[2n]$ ($\frac{N}{2}$ point sequence)

$h[n] = x[2n+1]$ ($\frac{N}{2}$ point sequence)

$p[n] = g[2n]$ ($\frac{N}{4}$-point)

$q[n] = g[2n+1]$ ($\frac{N}{4}$-point)

$r[n] = h[2n]$ ($\frac{N}{4}$-point)

$s[n] = h[2n+1]$ ($\frac{N}{4}$-point)

Decimation process can be repeated again and again until we are left with only 1-point sequence.

For example, $N=8$

$$\frac{N}{2} = \frac{8}{2} = 4 \qquad \frac{N}{2} = \frac{8}{2} = 4$$ stage ①

$$\frac{N}{4} = \frac{8}{4} = 2 \qquad \frac{N}{4} = \frac{8}{4} = 2 \qquad \frac{N}{4} = \frac{8}{4} = 2 \qquad \frac{N}{4} = \frac{8}{4} = 2$$ stage ②

$$\frac{N}{8} = 1 \qquad \frac{N}{8} = 1 \qquad \frac{N}{8} \qquad \frac{N}{8} \qquad \frac{N}{8} \qquad \frac{N}{8} \qquad \frac{N}{8} \qquad \frac{N}{8}$$ stage ③

$N = 2^m$ $\longrightarrow r_1\, r_2\, r_3 \cdots r_m$

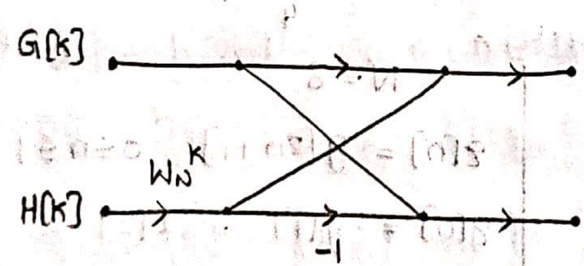
$r_1 = r_2 = \cdots = r_m$

$N = 8 = 2 \times 2 \times 2 = 2^3$

2^3: $3 \longrightarrow$ No. of stages; $2 \longrightarrow$ Radix

$$X[K] = G[K] + W_N^K H[K] \quad ; \quad 0 \le K \le \frac{N}{2} - 1$$

$$X\left[K + \frac{N}{2}\right] = G[K] - W_N^K H[K] \quad ; \quad 0 \le K \le \frac{N}{2} - 1$$



$X[K] = G[K] + W_N^K H[K]$

$X\left[K + \frac{N}{2}\right] = G[K] - W_N^K H[K]$

Butterfly diagram

Complex multiplication $= 1$ } For one butterfly
Complex Addition $= 2$ }

$x[n] \to g[n] = x[2n]$, $g[n] \to P[n] = g[2n]$, $Q[n] = g[2n+1]$

$x[n] \to h[n] = x[2n+1]$

$p[n] = g[2n] \longrightarrow$ even indexed value

$x[n] = \{x[0], x[1], x[2], x[3], x[4], x[5], x[6], x[7]\}$

- $g[n] = x[2n] = \{x[0], x[2], x[4], x[6]\}$
  - $p[n] = g[2n] = \{x[0], x[4]\}$
    - $x[0]$
    - $x[4]$
  - $q[n] = g[2n+1] = \{x[2], x[6]\}$
    - $x[2]$
    - $x[6]$
- $h[n] = x[2n+1] = \{x[1], x[3], x[5], x[7]\}$
  - $r[n] = h[2n] = \{x[1], x[5]\}$
    - $x[1]$
    - $x[5]$
  - $s[n] = h[2n+1] = \{x[3], x[7]\}$
    - $x[3]$
    - $x[7]$

$$p[n] = g[2n] \qquad 0 \le n \le \frac{N}{4} - 1$$

$$q[n] = g[2n+1] \qquad 0 \le n \le \frac{N}{4} - 1$$

Let $N = 8$

$p[n] = g[2n] \qquad 0 \le n \le 1$

$p[0] = g[0] = x[0]$

$p[1] = g[2] = x[4]$

$N = 8$

$q[n] = g[2n+1] \qquad 0 \le n \le 1$

$q[0] = g[1] = x[2]$

$q[1] = g[3] = x[6]$

$$G[k] = \sum_{n=0}^{\frac{N}{2}-1} g[n]\, W_{N/2}^{kn}$$

$$= \sum_{n=0}^{\frac{N}{4}-1} g[2n]\, W_{N/2}^{2nk} + \sum_{n=0}^{\frac{N}{4}-1} g[2n+1]\, W_{N/2}^{(2n+1)k}$$

$$G[k] = \sum_{n=0}^{\frac{N}{4}-1} \underbrace{g[2n]}_{=p[n]}\, W_{N/2}^{2nk} + W_{N/2}^{k} \sum_{n=0}^{\frac{N}{4}-1} \underbrace{g[2n+1]}_{=q[n]}\, W_{N/2}^{2nk}$$

$$G[K] = \underbrace{\sum_{n=0}^{\frac{N}{4}-1} p[n]\, W_{N/2}^{2nK}}_{P[K]} + W_{N/2}^{K} \underbrace{\sum_{n=0}^{\frac{N}{4}-1} q[n]\, W_{N/2}^{2nK}}_{= Q[K]}$$

$$* \quad G[K] = P[K] + W_{N/2}^{K} Q[K] \qquad 0 \le K \le \frac{N}{4}-1$$

$$\boxed{\begin{array}{l} g[n] \xleftrightarrow{N/2} G[K] \\ p[n] \xleftrightarrow{N/4} P[K] \\ q[n] \xleftrightarrow{N/4} Q[K] \end{array}}$$

$\because$ $P[K]$ and $Q[K]$ are periodic with period $N/4$

$$P[K] = P\left[K+\frac{N}{4}\right]$$

$$Q[K] = Q\left[K+\frac{N}{4}\right]$$

$$W_{N/2}^{K+\frac{N}{4}} = -W_{N/2}^{K}$$

$$G[K] = P[K] + W_{N/2}^{K} Q[K] \quad ; \; 0 \le K \le \frac{N}{4}-1$$

$$G\left[K+\frac{N}{4}\right] = P[K] - W_{N/2}^{K} Q[K] \quad ; \; 0 \le K \le \frac{N}{4}-1$$

Similarly,

$$r[n] = h[2n] \quad ; \; 0 \le n \le \frac{N}{4}-1$$

$$s[n] = h[2n+1] \quad ; \; 0 \le n \le \frac{N}{4}-1$$

$N=8 \Rightarrow r[n] = \{h[0], h[2]\} = \{x[1], x[5]\}$

$s[n] = \{h[1], h[3]\} = \{x[3], x[7]\}$

$$H[K] = R[K] + W_{N/2}^{K} S[K] \quad ; \; 0 \le K \le \frac{N}{4}-1$$

$$H\left[K+\frac{N}{4}\right] = R[K] - W_{N/2}^{K} S[K] \quad ; \; 0 \le K \le \frac{N}{4}-1$$

$N=8 \quad W_{N/2} = W_N^2 = W_8^2 \qquad W_N = e^{-j\frac{2\pi}{N}}$

$$W_8 = e^{-j\frac{2\pi}{8}} = e^{-j\frac{\pi}{4}}$$

$$W_8^2 = e^{-j\frac{\pi}{4}\cdot 2}$$

① $G[K] = P[K] + W_{N/2}^{K}\, Q[K] \qquad 0 \le K \le \frac{N}{4}-1$

$N=8, \quad K=0 \Rightarrow G[0] = P[0] + W_8^{0}\, Q[0]$

$K=1 \Rightarrow G[1] = P[1] + W_8^{2}\, Q[1]$

② $G\left[K+\frac{N}{4}\right] = P[K] - W_{N/2}^{K}\, Q[K] \qquad 0 \le K \le \frac{N}{4}-1$

$N=8, \quad G\left[K+\frac{N}{4}\right] = P[K] - W_N^{2K}\, Q[K]$

$K=0 \Rightarrow G\left[0+\frac{8}{4}\right] = G[2] = P[0] - W_8^{0}\, Q[0]$

$K=1 \Rightarrow G\left[1+\frac{8}{4}\right] = G[3] = P[1] - W_8^{2}\, Q[1]$

Similarly, $H[K] = R[K] + W_{N/2}^{K}\, S[K] \qquad 0 \le K \le \frac{N}{4}-1$

$N=8, \quad K=0 \Rightarrow H[0] = R[0] + W_N^{2\times 0}\, S[0] \qquad 0 \le K \le 1$

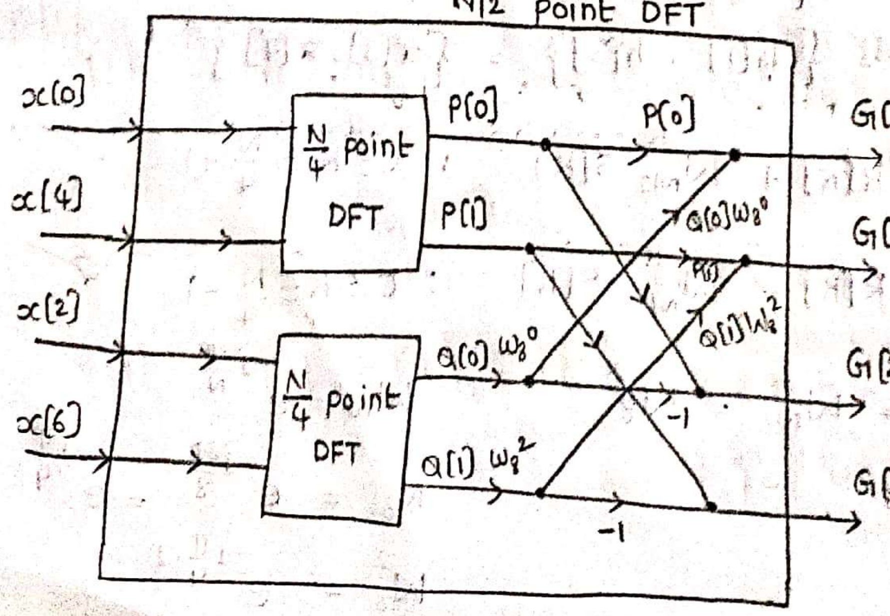
$K=1 \Rightarrow H[1] = R[1] + W_N^{2}\, S[1]$

Similarly, $H\left[K+\frac{N}{4}\right] = R[K] - W_{N/2}^{K}\, S[K] \qquad 0 \le K \le \frac{N}{4}-1$

$N=8, \quad H\left[K+\frac{N}{4}\right] = R[K] - W_N^{2K} \cdot S[K] \qquad 0 \le K \le 1$

$K=0 \Rightarrow H\left[0+\frac{8}{4}\right] = H[2] = R[0] - W_8^{0}\, S[0]$

$K=1 \Rightarrow H\left[1+\frac{8}{4}\right] = H[3] = R[1] - W_8^{2}\, S[1]$

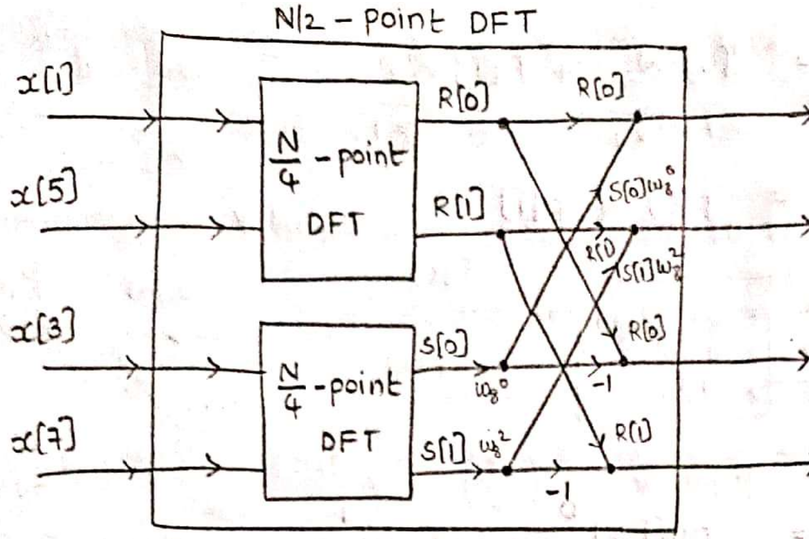


$G[0] = P[0] + W_8^{0}\, Q[0]$

$G[1] = P[1] + W_8^{2}\, Q[1]$

$G[2] = P[0] - W_8^{0}\, Q[0]$

$G[3] = P[1] - W_8^{2}\, Q[1]$



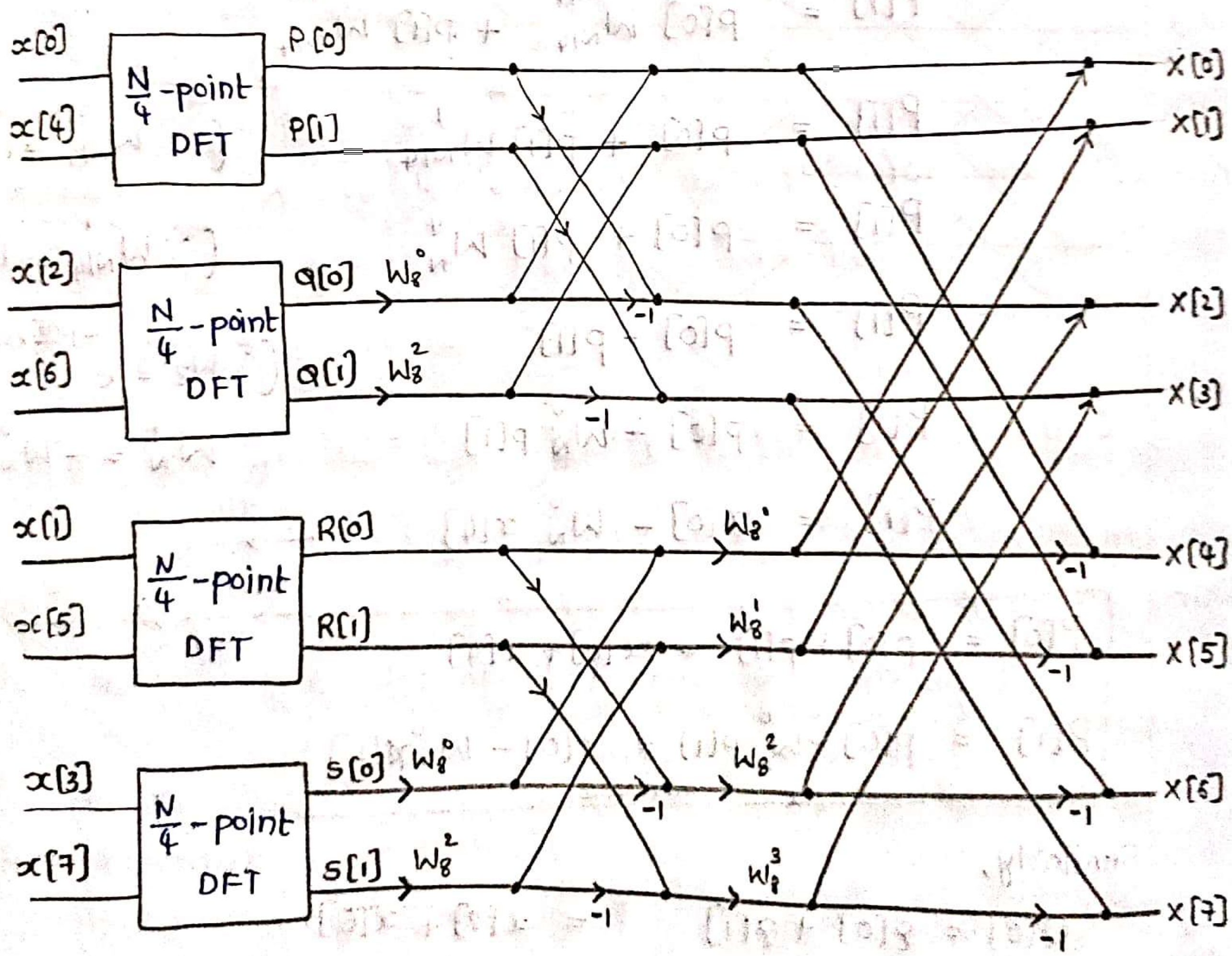
$H[0] = R[0] + S[0]\,W_8^0$

$H[1] = R[1] + S[1]\,W_8^2$

$H[2] = R[0] - S[0]\,W_8^0$

$H[3] = R[1] - S[1]\,W_8^2$

Signal flow graph of second stage DIT-FFT Algorithm for N = 8 :



For $N=8$, $\frac{N}{4} = 2$ point DFT

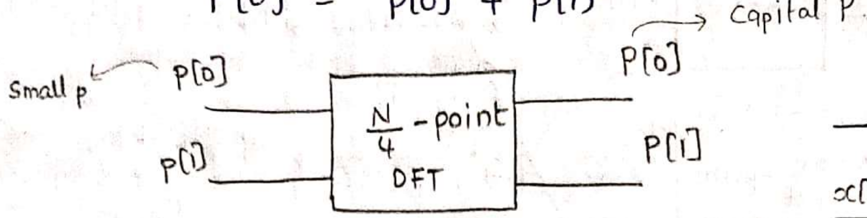
$$P[k] = \sum_{n=0}^{\frac{N}{4}-1} P[n]\, W_{N/4}^{nk} \qquad 0 \le k \le \frac{N}{4} - 1$$

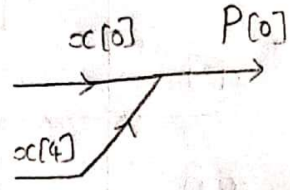
$$P[K] = \sum_{n=0}^{1} P[n]\, W_{N/4}^{nk} \qquad 0 \le k \le 2-1 \Rightarrow 0 \le K \le 1$$

$K=0 \Rightarrow P[0] = P[0]\underbrace{W_2^0}_{=1} + P[1]\underbrace{W_2^0}_{=1}$ $\qquad W_2 = e^{-j\frac{2\pi}{2}}$

$W_2^0 = e^{-j\frac{2\pi}{2}\times 0}$

$P[0] = p[0] + p[1]$





$K=1 \Rightarrow P[1] = p[0]W_2^0 + p[1]W_2^1$ $\qquad W_2^1 = e^{-j\frac{2\pi}{2}\times 1}$

$= e^{-j\pi} = -1$

$P[K] = \sum_{n=0}^{1} p[n]\, W_{N/4}^{nk}$

$P[1] = p[0]\,W_{N/4}^{0\times 1} + p[1]\,W_{N/4}^{1\times 1}$

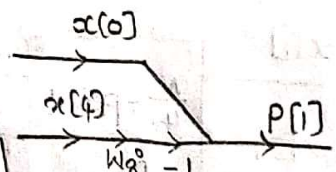
$P[1] = p[0] + p[1]\,W_{N/4}^{1}$ $\qquad (\because W_{N/4}^{0} = 1)$

$P[1] = p[0] + p[1]\,W_N^4$ $\qquad (\because W_{N/4}^{1} = W_N^4)$

$P[1] = p[0] - p[1]$ $\qquad (\because W_8^4 = e^{-j\frac{2\pi}{8}\times 4} = -1)$

$P[1] = p[0] - W_N^0\,p[1]$ $\qquad W_N^4 = -W_N^0$

$P[1] = x[0] - W_N^0\,x[4]$



$$\boxed{\begin{aligned} P[0] &= p[0] + p[1] = x[0] + x[4] \\ P[1] &= p[0] - W_N^0\,p[1] = x[0] - W_N^0 x[4] \end{aligned}}$$

Similarly,

$Q[0] = q[0] + q[1] \qquad = x[2] + x[6]$

$Q[1] = q[0] + W_{N/4}^{1}\,q[1] = q[0] + W_N^4\,q[1]$

$= x[2] - W_N^0\,x[6]$

Similarly, $R[0] = r[0] + r[1] = x[1] + x[5]$

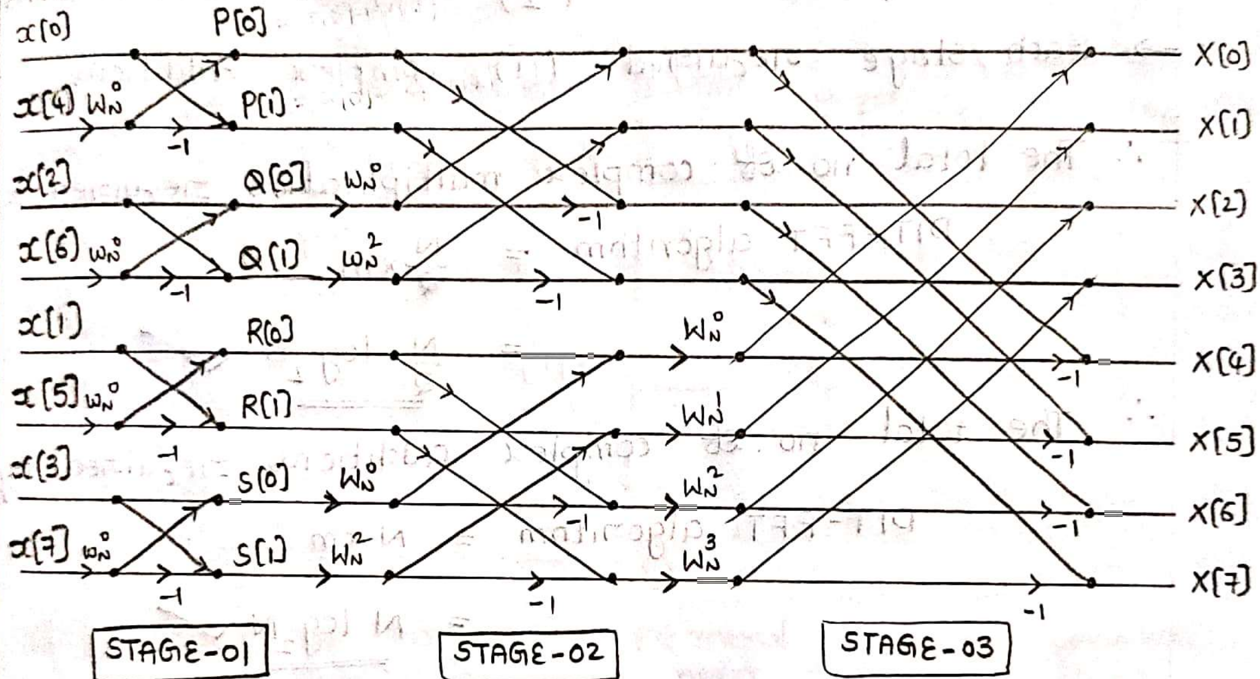
$R[1] = r[0] + W_{N/4}^{1} q[1] = x[2] - W_N^0 x[5]$

Similarly, $S[0] = s[0] + s[1] = x[3] + x[7]$

$S[1] = s[0] + W_{N/4}^{1} s[1] = s[0] + W_N^4 s[1]$

$= x[3] - W_N^0 x[7]$

* The overall signal flow graph of DIT-FFT for $N=8$,



$N=8$, $N = 2^m$ $8 = 2^m$

$2^3 = 2^m$ $\therefore m = 3 \longrightarrow$ No. of stages in DIT-FFT algorithm

* For $N = 2^m$, the decomposition can be performed $\boxed{m = \log_2 N \text{ times.}}$

For example, For $N=8$

$m = \log_2 8 = \log_2 2^3 = 3 \log_2 2 = \underline{\underline{3}}$

* Each decomposition is called a stage.

$\therefore m = \log_2 N$ stages

Note:- Complex multiplication = 1 } For one butterfly.
Complex Additions = 2 }

Stage - 1 → 4 butterflies
Stage - 2 → 4 butterflies
Stage - 3 → 4 butterflies

⇒ Each stage required $\left(\frac{N}{2}\right)$ complex multiplications
→ Each stage required (N) complex Additions

∴ The total no. of complex multiplication required by DIT-FFT algorithm $= \frac{N}{2} \times m$

$$= \frac{N}{2} \log_2 N$$

∴ The total no. of complex additions required by DIT-FFT algorithm $= N \times m$

$$= N \log_2 N$$

Computation Advantages of DIT-FFT [N=8]

① No. of stages [m] $= \log_2 N = \log_2 8 = 3$

② No. of butterflies in each stage $= \frac{N}{2} = \frac{8}{2} = 4$

③ No. of complex multiplication in each stage $= \frac{N}{2} = \frac{8}{2} = 4$

④ No. of complex Additions " " " $= N = 8$

⑤ Total no. of complex multi $= \frac{N}{2} \log_2 N = 4 \times 3 = 12$

⑥ Total no. of complex Additions $= N \log_2 N = 8 \times 3 = 24$

⑦ Total no. of Real multi $= 4\left[\frac{N}{2} \log_2 N\right] = 4 \times 12 = 48$

4 (Total no. of complex multi)

⑧ Total no. of real Additions $= 2\left[\frac{N}{2}\log_2 N\right] + 2\left[N\log_2 N\right]$

$= 2\,[\text{complex multi}] + 2\,[\text{complex Addition}]$

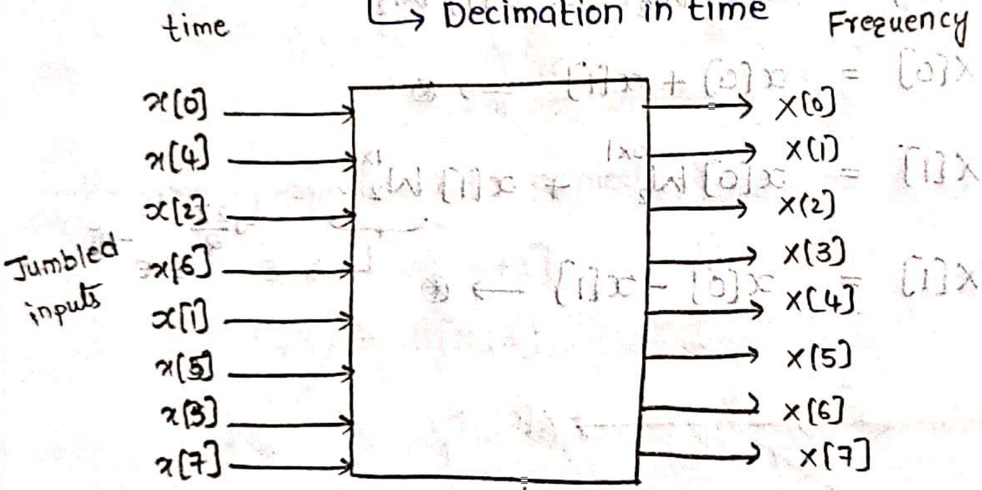
$= 2\left[4\log_2 8\right] + 2\left[8\log_2 8\right]$

$= 2\times 12 + 2\times 24 = 72$

Bit - reversal:-

DIT - FFT

↳ Decimation in time



x[n]	Index 'n'	Binary repre of 'n'	Bit reversal order	Decimal value 'n'	Sequence butterfly computation x(n)
x[0]	0	000	000	0	x[0]
x[1]	1	001	100	4	x[4]
x[2]	2	010	010	2	x[2]
x[3]	3	011	110	6	x[6]
x[4]	4	100	001	1	x[1]
x[5]	5	101	101	5	x[5]
x[6]	6	110	011	3	x[3]
x[7]	7	111	111	7	x[7]

①

(Pb) Compute the 2-point DFT $X[K]$ using DIT-FFT algorithm of a length-2 sequence. $x[n]$ $0 \le n \le 1$. Draw the butterfly diagram

Ans:- $x[n] = \{x[0], x[1]\}$

$N = 2$-point DFT

$$X[K] = \sum_{n=0}^{1} x[n] W_N^{nk} \qquad 0 \le K \le 1$$

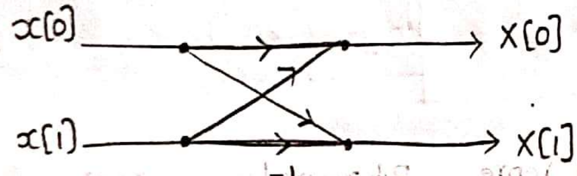
$K=0 \Rightarrow X[0] = x[0] W_2^{0\times0} + x[1] W_2^{0\times1}$

$X[0] = x[0] + x[1] \longrightarrow ⊛$

$K=1 \Rightarrow X[1] = x[0] W_2^{0\times1} + \underbrace{x[1] W_2^{1\times1}}$

$\hookrightarrow e^{-j\frac{2\pi}{2}} = e^{-j\pi} = -1$

$X[1] = x[0] - x[1] \longrightarrow ⊛$



② Compute the 4-point DFT $X(K)$ using DIT-FFT algorithm of length-4 sequence $x[n]$; $0 \le n \le 3$. Draw the butterfly diagram.

Ans:- $x[n] = \{x[0], x[1], x[2], x[3]\}$

$N = 4$-point DFT

Decimate the sequence $x[n]$ → even indexed values
→ Odd indexed values

$g[n] = x[2n] = \{x[0], x[2]\}$; $0 \le n \le 1$

$h[n] = x[2n+1] = \{x[1], x[3]\}$; $0 \le n \le 1$

By definition, 4-point DFT is given by

$$X[K] = \sum_{n=0}^{3} x[n] W_4^{nk}$$

$$X[K] = \sum_{n=0}^{1} x[2n]\, W_N^{2nK} + \sum_{n=0}^{1} x[2n+1]\, W_N^{(2n+1)k}$$

$$= \sum_{n=0}^{1} g(n)\, W_4^{2nK} + \sum_{n=0}^{1} h(n)\, W_4^{2nk} \cdot W_4^{k}$$

By using twiddle factor property, $W_4^{2nk} = W_{4/2}^{nk} = W_2^{nk}$

$$X[K] = \sum_{n=0}^{1} g(n)\, W_2^{nk} + W_4^{K} \sum_{n=0}^{1} h(n)\, W_2^{nk}$$

$$X[K] = G[K] + W_4^{K} H(K) \quad ; \; 0 \leq K \leq 3$$

2-point DFT'S

Both are periodic with period ~~of~~ 2

$$G[K] = G[K+2]$$

$$H(K) = H[K+2]$$

$$W_4^{K+2} = W_4^{K} \cdot W_4^{2} = W_4^{K} \cdot W_4^{2} = W_4^{K} \cdot e^{-j\frac{2\pi}{4} \cdot 2} = -W_4^{k}$$

$$X(K] = G[K] + W_4^{K} H[K] \qquad 0 \leq K \leq 1 \longrightarrow ①$$

$$X[K+2] = G[K] - W_4^{K} H[K] \qquad 0 \leq K \leq 1 \longrightarrow ②$$

Substitute $k=0,1$ in equ①,

$k=0 \Rightarrow \; X[0] = G[0] + W_4^{0} H[0]$

$K=1 \Rightarrow \; X(1) = G(1) + W_4^{1} H(1)$

Substitute $k=0,1$ in equ②,

$K=0 \Rightarrow \; X[2] = G[0] + W_4^{0} H[0]$

$K=1 \Rightarrow \; X[3] = G(1) - W_4^{1} H(1)$

Consider the 2-point DFT $G[K]$ & $H[K]$,

$$G[K] = \sum_{n=0}^{1} g(n)\, W_2^{nk} \qquad 0 \leq K \leq 1$$

$$G[0] = g[0] + g[1] = x[0] + x[2]$$

$$G[1] = g[0] + W_2^1 g[1]$$

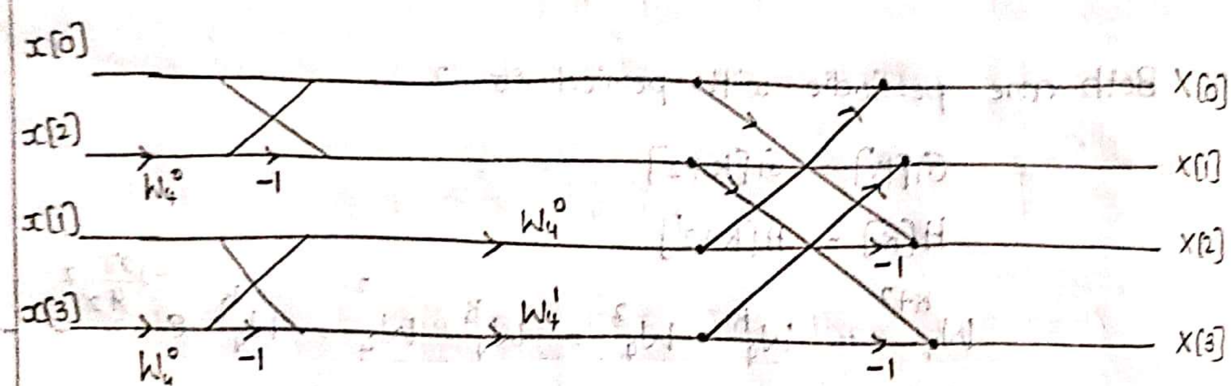
$$= g[0] - W_4^0 g[1] = x[0] - W_4^0 x[2]$$

Similarly, $H[K] = \sum_{n=0}^{1} h[n] W_2^{nk} \qquad 0 \le k \le 1$

$$H[0] = h[0] + h[1] = x[1] + x[3]$$

$$H[1] = h[0] + W_2^1 h[1]$$

$$H[1] = h[0] - W_4^0 h[1] = x[1] - W_4^0 x[3]$$



(Pb) Given $x[n] = \{0, 1, 2, 3, 4, 5, 6, 7\}$. Find the 8-point DFT $X[K]$ using DIT-FFT algorithm. Show all the intermediate results.

Ans:- Given, $N = 8$

$$W_N^K = e^{-j\frac{2\pi}{N}k}$$

$$W_8^0 = e^{-j\frac{2\pi}{8}\times 0} = 1$$

$$W_8^1 = e^{-j\frac{2\pi}{8}\times 1} = e^{-j\frac{\pi}{4}} = \cos\frac{\pi}{4} - j\sin\frac{\pi}{4} = 0.707 - j0.707$$

$$W_8^2 = e^{-j\frac{2\pi}{8}\times 2} = e^{-j\frac{\pi}{2}} = \cos\frac{\pi}{2} - j\sin\frac{\pi}{2} = -j$$

$$W_8^3 = e^{-j\frac{2\pi}{8}\times 3} = e^{-j\frac{3\pi}{4}} = \cos\frac{3\pi}{4} - j\sin\frac{3\pi}{4} = -0.707 - j0.707$$

(P6) Consider the 8-point DFT of the sequence $x[n] = \cos\left[\frac{\pi n}{2}\right]$ using the DIT-FFT algorithm. Show all the intermediate results. Given $x[n] = \{1, 0, -1, 0, 1, 0, -1, 0\}$ using DIT-FFT algorithm

Ans: Given, $N = 8$ $\quad x[n] = \cos\left(\frac{\pi n}{2}\right), \quad 0 \le n \le 7$

$$x[n] = \{1, 0, -1, 0, 1, 0, -1, 0\}$$

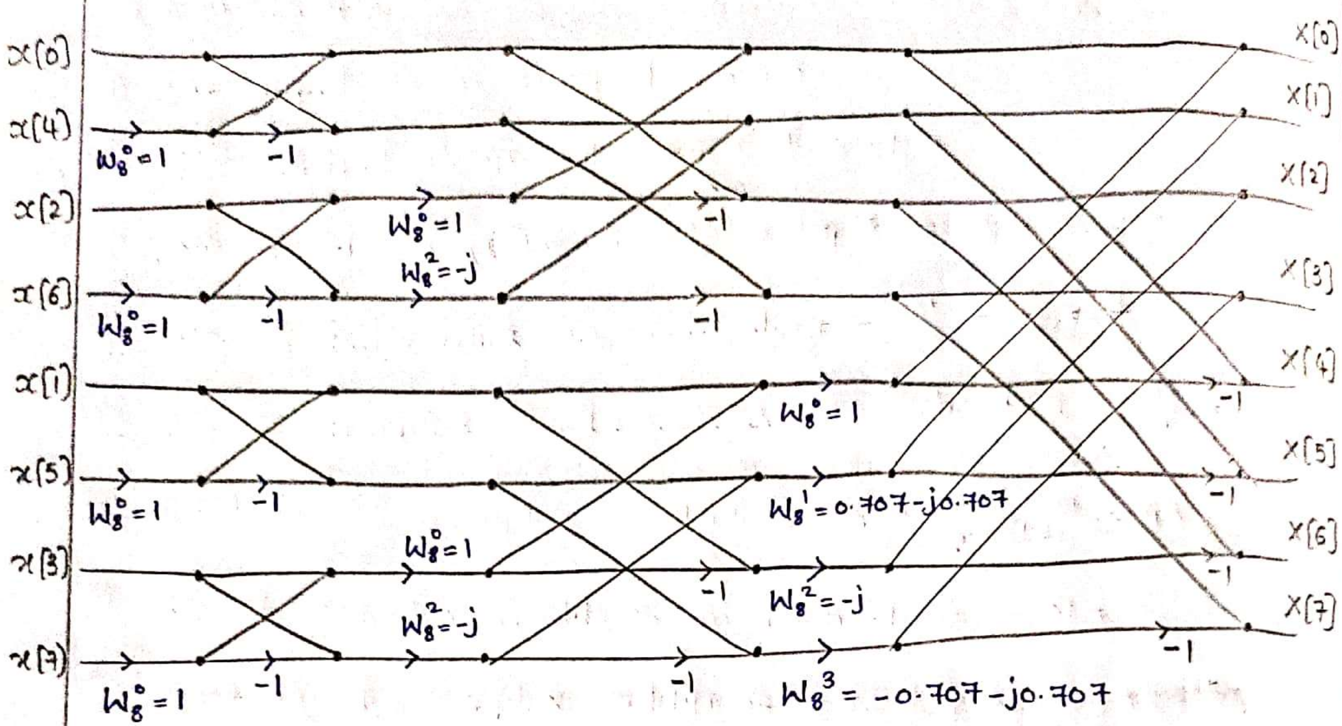
$$W_N^K = e^{-j\frac{2\pi}{N}K}$$

$W_8^0 = 1$

$W_8^1 = e^{-j\frac{\cancel{2}\pi}{\cancel{8}_4}\cdot 1} = e^{-j\frac{\pi}{4}} = 0.707 - j0.707$

$W_8^2 = e^{-j\frac{2\pi}{8}\cdot 2} = e^{-j\frac{\pi}{2}} = -j$

$W_8^3 = -0.707 - j0.707$



$$X[K] = \{0, 0, 4, 0, 0, 0, 4, 0\}$$

Decimation in Frequency - FFT Algorithm (DIF-FFT)