3.3

Connectivity

Many problems can be modeled with paths formed by travelling along the edges of graphs. Problems of efficiently planning routes for mail delivery, garbage pickup, diagnostics in computer networks can be solved using models that involve paths in graphs.

Paths

A **path** is a sequence of edges that begins at a vertex of a graph and travels from vertex to vertex along edges of the graph.

The following is a formal definition of a path in an undirected graph:

A path in an undirected graph: Let G be an undirected graph and n be a nonnegative integer. A **path of length n** from a vertex u to a vertex v in G is a sequence of n edges $e_1, e_2, \dots, e_n$ of G such that e_1 is associated with $\{x_0, x_1\}$, e_2 is associated with $\{x_1, x_2\}$, and so on, finally e_n is associated with $\{x_{n-1}, x_n\}$, where $x_0 = u$ and $x_n = v$. Such a path is said to *pass through* the vertices $x_1, x_2, \dots, x_{n-1}, x_n$ or *traverse* the edges $e_1, e_2, \dots, e_n$. If the graph G is simple, then we denote this path by its sequence of vertices $x_0, x_1, x_2, \dots, x_n$.

Note that a path of length *zero* consists of a single vertex

Circuit: A path is a **circuit** if it begins and ends at the same vertex, (*i.e.*, $= v$) and the length $n > 0$.

Simple path: A path or a circuit is said to be **simple** if it does not contain the same edge more than once.

Example 1: In the following simple undirected graph (figure 1), a, d, c, f, e is a simple path from a to e of length 4, because the edges $\{a, d\}, \{d, c\}, \{c, f\}, \{f, e\}$ are all distinct.

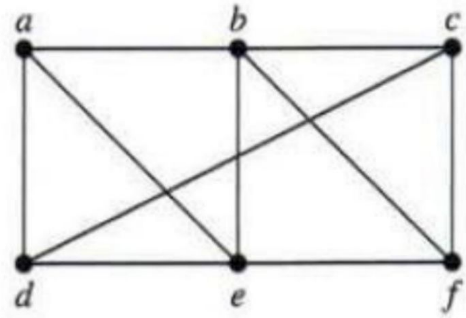


Figure 1

Notice that d, e, c, a is not a path, because $\{e, c\}$ is not an edge.

The path b, c, f, e, b is a circuit of length 4, because it is a path of length 4 that begins and ends at b.

Note that a, b, e, f, c, d, e, b is a path of length 7 and it is not simple because it has the edge $\{b, e\}$ occurs twice in the path.

The following is the definition of a path in a digraph.

A path in a digraph: Let G be a digraph and n be a nonnegative integer .A path of length n from u to v in G is a sequence of edges $e_1, e_2, \dots, e_n$ of G such that e_1 is associated with the directed edge (x_0, x_1), e_2 with (x_1, x_2), and so on, e_n is associated with (x_{n-1}, x_n), where $x_0 = u$ and $x_n = v$. When there are no multiple edges in the digraph G, this path is denoted by its vertex sequence $x_0, x_1, x_2, \dots, x_n$.

A path of length greater than zero that begins and ends at the same vertex is called a **circuit.**

A path or circuit is said to be **simple** if it does not contain the same edge more than once.

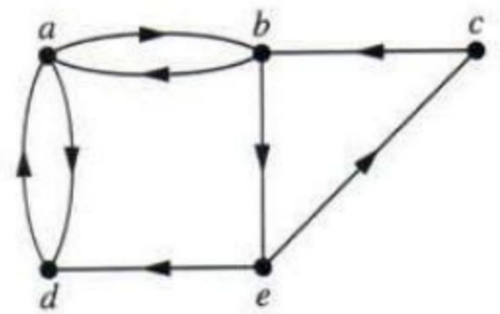
Note:

(1) The alternate terminology for path, circuit and simple path are *walk*, *closed walk* and *trail* respectively.

(2) There may be more than one path from the initial vertex u to the terminal vertex v.

(3) Paths represent useful information in many graph models.

Example 2: In the following digraph



(i) a, b, e, c, b is a path of length 4 and it is simple.

(ii) a, b, e, d, a is a circuit of length 4 and it is simple.

(iii) c, b, a, b, e, c, b is a circuit of length 6 but it is not simple, because it contains the edge (c, b) twice.

(iv) a, d, e, c is not a path, because (d, e) is not an edge in the digraph

Connectedness in undirected graphs

When does a computer network have the property that every pair of computers can share information, if messages can be sent through one or more intermediate computers? If computers are represented by vertices and communication links represent edges then the computer network represents an undirected graph and the question now becomes: Is there a path between every pair of vertices in the graph?

Undirected connected graph: An undirected graph G is said to be **connected** if there is a path between every pair of distinct vertices of G.

Thus, any two computers in the network can communicate if and only if the graph of the network is connected.

Example 3: The graph G given in figure 1 is connected, because there is a path between every pair of distinct vertices.

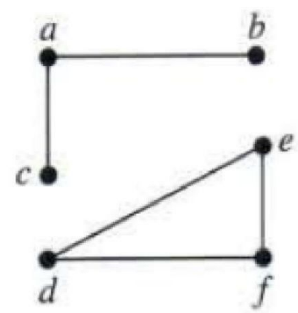
The following is a useful result:

Theorem 1: In a connected undirected graph, there is a simple path between every pair of vertices.

Connected component: A **connected component** of a graph G is a connected subgraph of G that is not a subgraph of another connected subgraph of G. That is, a connected component of a graph G is a maximal connected subgraph of G.

If a graph G is not connected then it has two or more connected components that are disjoint and have G as their union. That is, if G is not connected then G is partitioned into connected components.

Example 4: The following graph is not connected, because there is no path between the vertices a and d.



G

Its connected subgraphs are the subgraphs H_1 and H_2 , where

$$H_1 = (V_1, E_1), V_1 = \{a, b, c\}, E_1 = \{\{a, b\}, \{a, c\}\} \text{ and}$$
$$H_2 = (V_2, E_2), V_2 = \{d, e, f\},\ E_2 = \{\{d, e\}, \{d, f\}, \{e, f\}\}.$$

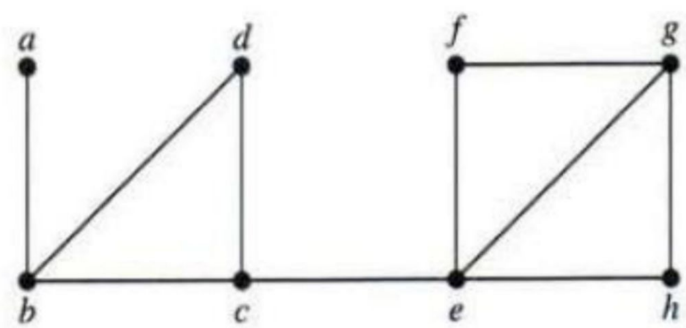
Notice that H_1 and H_2 are maximal connected sub graphs. Therefore, H_1, H_2 are components and G is partitioned into H_1 and H_2.

Cut vertex: The removal of a vertex a and all edges incident with it in a graph G produces a subgraph of G with more connected components than in G. Then such a vertex a is called a **cut vertex** or a **articulation point.**

Note that the removal of a cut vertex from a connected graph produces a subgraph that is not connected.

Cut edge: An edge whose removal produces a graph with more connected components than the original graph is called a **cut edge** or a **bridge**.

Example 5: Find all cut vertices and all cut edges in the following graph.



Solution: The removal of the vertex b and all edges that incident with b, i.e., $\{b, a\}$, $\{b, d\}$, $\{b, c\}$ disconnects the graph. (Note that remove the edges that incident with b only but not the other end vertices namely a, d and c). Therefore b is a cut vertex. Similarly c and e are also cut vertices.

Removal of the edge $\{a, b\}$ (but not the end points a, b) disconnects the graph. Therefore $\{a, b\}$ is a cut edge. Similarly $\{c, e\}$ is also a cut edge.

Connectedness in Digraphs

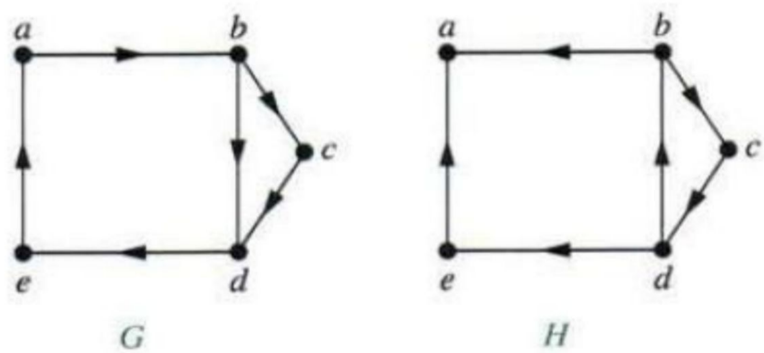
The following are two notions of connectedness in digraphs:

Weakly connectedness: A digraph G is **weakly connected** if there is a path between every pair of distinct vertices in the underlying undirected graph. That is, a digraph G is weakly connected if the undirected graph of G obtained by ignoring the directions of the edges is connected.

Strongly connectedness: A digraph G is **strongly connected** if for every two distinct vertices a and b in G, there is a path from a to b as well as a path from b to a.

Note: Every strongly connected digraph is also weakly connected, but not conversely.

Example 6: Are the following digraphs strongly connected? Are they weakly connected?



Solution: Notice the following in G:

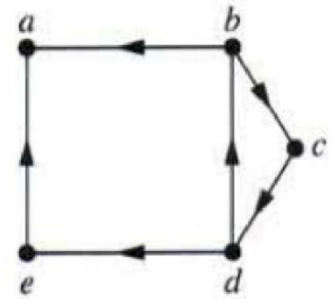
Vertex to vertex	Path
a to b	a, b
b to a	b, d, e, a
a to c	a, b, c
c to a	c, d, e, a
a to d	a, b, c, d
d to a	d, e, a
b to c	b, c
c to b	c, d, e, a, b
b to d	b, d
d to b	d, e, a, b
c to d	c, d
d to c	d, e, a, b, c

Thus for every two distinct vertices , y ;G has a path from x to y as well as a path from y to x. Thus G is strongly connected. Therefore, G is also weakly connected.

The digraph H is weakly connected, because the undirected graph derived from H by ignoring the directions of the edges of G is connected. Further, it is not strongly connected because there is no directed path from b to a

Strongly connected component: A subgraph of a digraph G that is strongly connected but not contained in a larger strongly connected subgraph ($i.e.$, maximal strongly connected subgraph) is called a **strongly connected component** or **strong component** of G

Example 7: The following graph H has three strongly connected components, consisting the vertex a; the vertex e ; and the subgraph consisting of the vertices b, c, d and edges (b, c),(c, d) and (d, b)

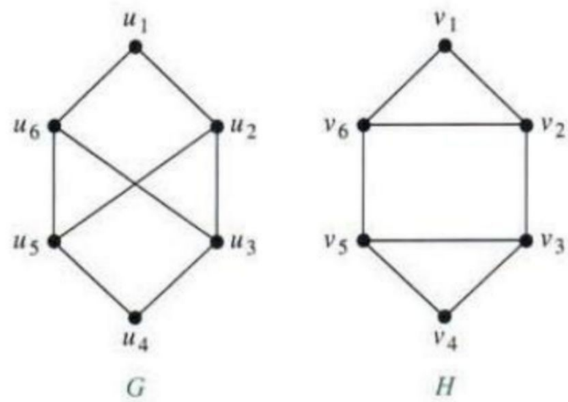


H

Paths and Isomorphism:

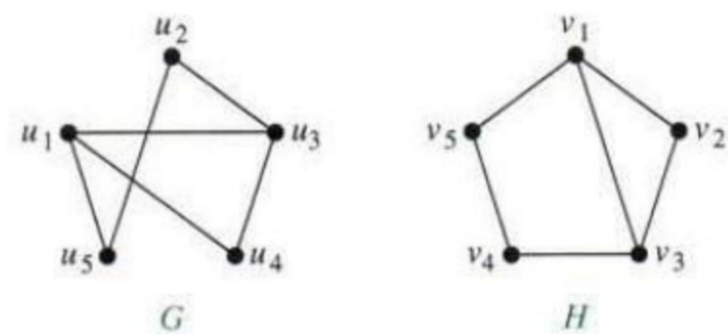
Paths and circuits can help to determine whether two graphs are isomorphic . Paths can be used to construct mapping that may be isomorphisms. A useful isomorphic invariant for simple graphs is the existence of a simple circuit of length k, where k is a natural number greater than 2.

Example 8: Determine whether the following graphs G and H are isomorphic.



Solution: Both the graphs G and H are undirected graphs with six vertices and eight edges. Both have the degree sequence 3,3,3,3,2,2. Notice that H has a simple circuit v_1, v_2, v_6, v_1 of length 3. Observe that G has no simple circuit of length 3. It may be noted that all simple circuits in G have length atleast 4. Therefore G is not isomorphic to H, because the existence of a simple circuit of length 3 is an isomorphic invariant.

Example 9: Determine whether the following graphs G and H are isomorphic.



Solution: Both the graphs G and H are undirected graphs with five vertices and six edges .Both have the degree sequence 3,3,2,2,2. Further ,both have a simple circuit of length 3, a simple circuit of length 4 and a simple circuit of length 5. Because all these isomorphic invariants agree, the graphs $G$and H may be isomorphic. Observe that the circuit of length 5 in H; $v_3, v_2, v_1, v_5, v_4, v_3$ in which the vertex v_2 of degree 2 is trapped between two vertices of degree 3, $i.e.$, $v_3$and v_1. Notice that the circuit of length 5 in G; $u_1, u_4, u_3, u_2, u_5, u_1$ in a similar circuit with the same characteristics .These paths guide us to set up the following bijection f from the vertex set of G to the vertex set of H.

$$f(u_1) = v_3, f(u_4) = v_2, f(u_3) = v_1, f(u_2) = v_5, f(u_5) = v_4$$

Let A_G and A_H be the adjacency matrices of G and H respectively, w.r.t. the ordering of the vertices.

$$u_1, u_4, u_3, u_2, u_5 \text{ in } G \text{ and } v_3, v_2, v_1, v_5, v_4 \text{ in } H$$

We notice that

$$A_G = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix} = A_H$$

Thus $G \cong H$

Counting Paths Between Vertices

The number of paths between two vertices in a graph can be determined using its adjacency matrix.

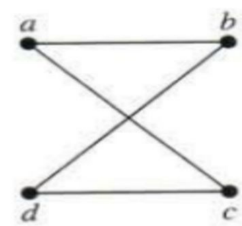
Theorem 2: Let $G = (V, E)$ be any graph with $|V| = n$. Let A be the adjacency matrix of G w.r.t. the ordering of vertices $v_1, v_2, v_3, \dots, v_n$. The number of different paths of length r from v_i to v_j equals the (i, j)th entry of A^r (where r is a natural number)

Proof: Follows by mathematical induction.
In the above theorem, directed or undirected edges, multiple edges and loops are allowed in G

Note:

1. In the above theorem A^r denotes the matrix multiplication $A.A.A \dots A(r\ times)$. It is not Boolean product.
2. The above theorem can be used to find the length of the shortest path between two vertices and it can be used to determine whether a graph is connected.

Example 10: How many paths of length *four* are there from a to d in the following simple graph G



Solution: The adjacency matrix of G, w.r.t. the ordering of vertices as a, b, c, d is

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}.$$

To determine the number of paths of length 4 between two vertices of G, we have to compute A^4 . Now

$$\mathbf{A}^2 = \begin{bmatrix} 2 & 0 & 0 & 2 \\ 0 & 2 & 2 & 0 \\ 0 & 2 & 2 & 0 \\ 2 & 0 & 0 & 2 \end{bmatrix}, \ \mathbf{A}^3 = \begin{bmatrix} 0 & 4 & 4 & 0 \\ 4 & 0 & 0 & 4 \\ 4 & 0 & 0 & 4 \\ 0 & 4 & 4 & 0 \end{bmatrix} \text{ and } \mathbf{A}^4 = \begin{bmatrix} 8 & 0 & 0 & 8 \\ 0 & 8 & 8 & 0 \\ 0 & 8 & 8 & 0 \\ 8 & 0 & 0 & 8 \end{bmatrix}$$

The number of paths of length 4 from a to d in $(1,4)^{th}$ entry in A^8, $i.e.$, 8

By inspection, we see the following 8 paths from a to d

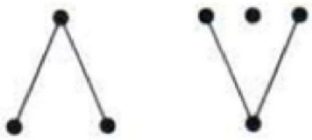
$$a, b, a, b, d \ ; \ a, b, a, c, d \ ; \ a, b, d, b, d \ ; \ a, b, d, c, d$$

$$a, c, a, b, d \ ; \ a, c, a, c, d \ ; a, c, d, b, d \ ; \ a, c, d, c, d$$

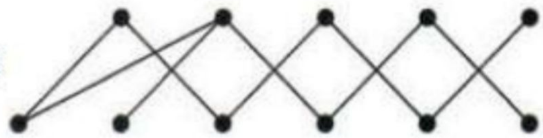
P1:

Determine whether the following graphs are connected.

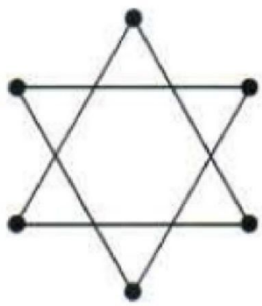
i



ii

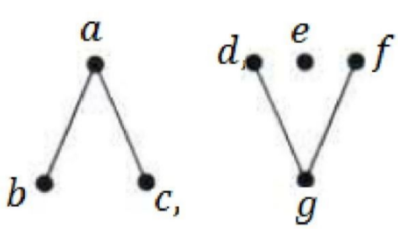


iii



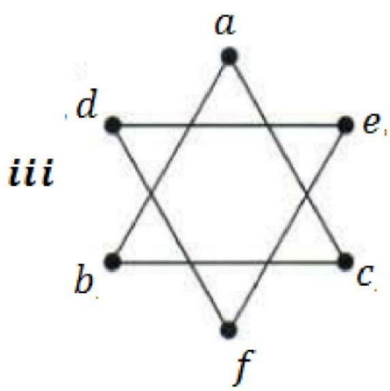
Solution:

i



It is not connected ,because there is no path between the vertices a and d.

***ii*.** It is connected ; because there is a path between every pair of distinct vertices



It is not connected ,because there is no path between the vertices a and d

2. **How many connected components does each of the above graphs have? For each graph ,find its connected components.**

Solution:

i. There are 3 components .They are

$H_1 = (V_1, E_1)$ where $V_1 = \{a, b, c\}$, $E_1 = \{\{a, b\}, \{a, c\}\}$

$H_2 = (V_2, E_2)$ where $V_2 = \{d, g, f\}$, $E_2 = \{\{d, g\}, \{d, f\}\}$

$H_3 = (V_3, E_3)$ where $V_3 = \{e\}$, $E_3 = \emptyset$

ii. There is one connected component $i.e$, the whole graph , because it is connected

iii. There are two connected components . They are

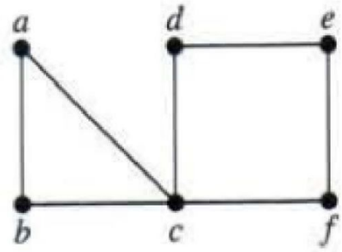
$H_1 = (V_1, E_1)$ where $V_1 = \{a, b, c\}$, $E_1 = \{\{a, b\}, \{a, c\}, \{b, c\}\}$

$H_2 = (V_2, E_2)$ where $V_2 = \{d, e, f\}$, $E_2 = \{\{d, e\}, \{d, f\}, \{e, f\}\}$

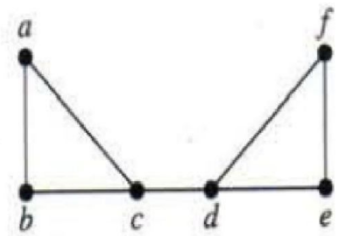
P2:

Find all cut vertices and cut edges of the following graphs

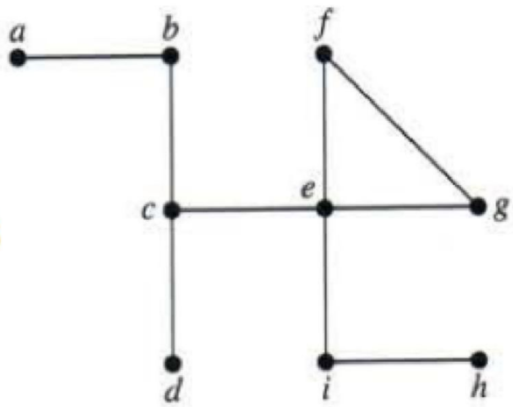
i



ii



$iii.$



Solution:

$\boldsymbol{i}$. The removal of the vertex c and all edges that incident with c disconnects the graph. Therefore, c is the cut vertex. Further , it is the only cut vertex. Note that the removal of any edge in this graph is not producing a graph with more connected components than the original graph. Therefore , there are no cut edges(bridges) in this graph.

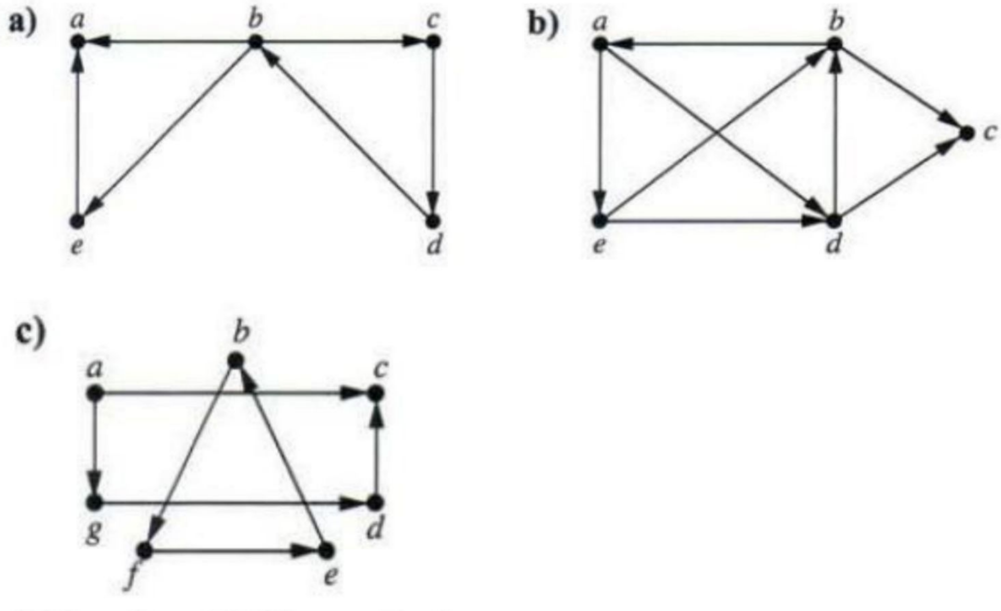
$\boldsymbol{ii}$. The cut vertices are c, d.The cut edge is $\{c, d\}$. The removal of the edge $\{c, d\}$ (not the vertices c and d) disconnects the graph into two connected components.

$\boldsymbol{iii}$. The cut vertices are :b, c, e, i and the cut edges are:

$$\{a, b\}, \{b, c\}, \{c, d\}, \{c, e\}, \{e, i\}, \{i, h\}$$

P3:

Determine whether each of these graphs is strongly connected and if not , whether it is weakly connected.



Solution:

a) The undirected graph derived from this graph by ignoring the directions of the edges is connected. Therefore it is a weakly connected graph. Notice that there is a path from a to b but there is no path from b to a. Therefore it is not strongly connected. Thus, it is a weakly connected graph, but not strongly connected.

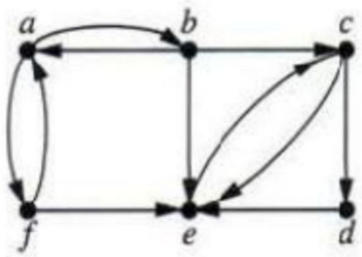
b) It is weakly connected, because the undirected graph derived from it by ignoring the directions of edges is connected. Notice that there is path from a to ; $a\,, d, c,$ but there is no path from c to a. Therefore, this graph is not strongly connected. Thus, it is a weakly connected graph, but not strongly connected.

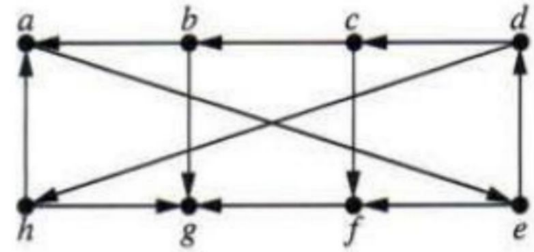
c) Notice that the undirected graph derived from this graph by ignoring the directions of the edges is not connected (because it is portioned into two connected components, one is a rectangle a, c, d, g and the other is a triangle b, e, f. Therefore, it is not weakly connected. Consequently it is not strongly

connected (If it is strongly connected, then it is weakly connected
a contradiction(because every strongly connected graph is a weakly connected).).It is neither weakly connected nor strongly connected.

P4:

Find the strongly connected components of each of these graphs.

a) 

b) 

Solution:

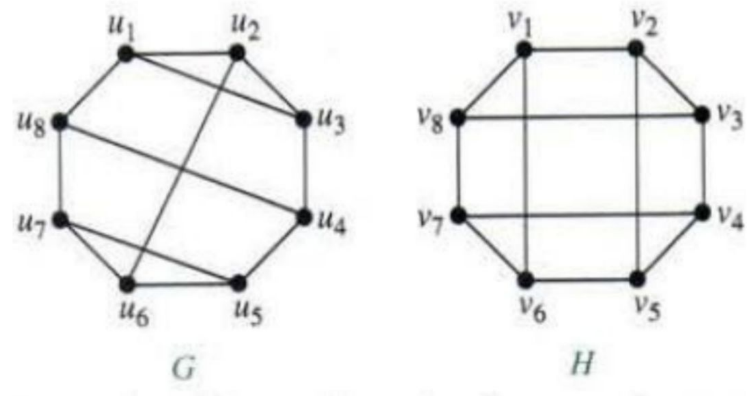
a) First note that there is a path from a to c ,but there is no path from c to a. Therefore it is not strongly connected.
Recall that a strongly connected component of a graph G is a maximal strongly connected subgraph of G
Notice that there are two strongly connected components H_1, H_2 in this graph, where
$H_1 = (V_1, E_1), V_1 = \{a, b, f\}$ and $E_1 = \{(a, b), (b, a), (a, f), (f, a)\}$
$H_2 = (V_2, E_2), V_2 = \{c, d, e\}$ and $E_2 = \{(c, e), (e, c), (c, d), (d, e)\}$

b) Notice that there is no path from g to any vertex. Therefore, $H_1 = (V_1, E_1)$, where $V_1 = \{g\}$ and $E_1 = \emptyset$ is a strongly connected component. Similarly $H_2 = (V_2, E_2)$,where $V_2 = \{f\}$ and $E_2 = \emptyset$ is a strongly connected component.
Notice that a, e, d, c, b, a is a simple circuit and a, e, d, h, a is another simple circuit; there is a interconnecting path a, e, d between these circuits. Thus every two vertices in these circuits are connected. This shows that the subgraph $H_3 = (V_3, E_3)$ is a strongly connected component where

$$V_3 = \{a, b, c, d, e, h\} \text{ and}$$
$$E_3 = \{(a, e), (e, d), (d, c), (c, b), (b, a), (d, h), (h, a)\}$$

P5:

Use paths either to show that these graphs are not isomorphic or to find an isomorphism between them.



Solution:

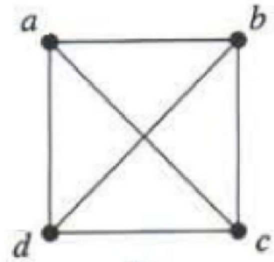
Both the graphs G and H are undirected graphs with 8 vertices and 12 edges. Both have the degree sequence 3,3,3,3,3,3,3,3. Notice that G has two simple circuits of length 3 (u_1, u_2, u_3, u_1 and u_5, u_6, u_7, u_5). Observe that H has no simple circuit of length 3. Therefore, G is not isomorphic to H,because the existence of a simple circuit of length 3 is an isomorphic invariant.

P7:

Find the number of paths of length 3 between two different vertices in K_4

Solution:

We have K_4



K_4

The adjacency matrix of K_4 w.r.t. the ordering of vertices a, b, c, d is

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

To determine the number of paths of length 3 between two vertices of K_4, we have to compute A^3. Now

$$\mathbf{A}^2 = \begin{bmatrix} 3 & 2 & 2 & 2 \\ 2 & 3 & 2 & 2 \\ 2 & 2 & 3 & 2 \\ 2 & 2 & 2 & 3 \end{bmatrix}, \qquad \mathbf{A}^3 = \begin{bmatrix} 6 & 7 & 7 & 7 \\ 7 & 6 & 7 & 7 \\ 7 & 7 & 6 & 7 \\ 7 & 7 & 7 & 6 \end{bmatrix}$$

The number of paths of length 3 between two different vertices in K_4 is 7

The number of circuits of length 3, from a to a is 6.They are

$$a, b, c, a \;;\quad a, c, b, a \,;\; a, d, c, a$$

$$a, c, d, a \,;\; a, d, b, a \;;\quad a, b, d, a$$

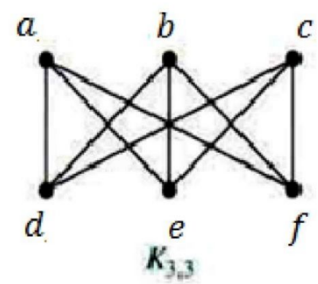
P8:

Find the number of paths of length 3 in $K_{3,3}$

i. **between any two adjacent vertices**
ii. **between any two nonadjacent vertices**

Solution:

We have $k_{3,3}$



The adjacency matrix of $K_{3,3}$, w.r.t. the ordering of vertices a, b, c, d, e, f.

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}$$

To determine the number of paths of length 3 between two vertices of $K_{3,3}$, we have to compute A^3.Now

$$A^2 = \begin{bmatrix} 3 & 3 & 3 & 0 & 0 & 0 \\ 3 & 3 & 3 & 0 & 0 & 0 \\ 3 & 3 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 3 & 3 \\ 0 & 0 & 0 & 3 & 3 & 3 \\ 0 & 0 & 0 & 3 & 3 & 3 \end{bmatrix}, \quad A^3 = \begin{bmatrix} 0 & 0 & 0 & 9 & 9 & 9 \\ 0 & 0 & 0 & 9 & 9 & 9 \\ 0 & 0 & 0 & 9 & 9 & 9 \\ 9 & 9 & 9 & 0 & 0 & 0 \\ 9 & 9 & 9 & 0 & 0 & 0 \\ 9 & 9 & 9 & 0 & 0 & 0 \end{bmatrix}$$

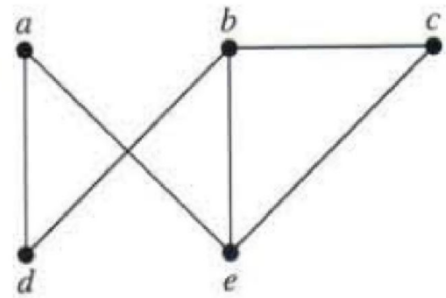
Each vertex of $\{a, b, c\}$ is adjacent to each vertex of $\{d, e, f\}$ and no two vertices of $\{a, b, c\}$ are adjacent and no two vertices of $\{d, e, f\}$ are adjacent.

Now the number of paths of length 3 in $K_{3,3}$, between any two adjacent vertices is 0 and the number of paths of length 3 in $K_{3,3}$ between any two non adjacent vertices is 9

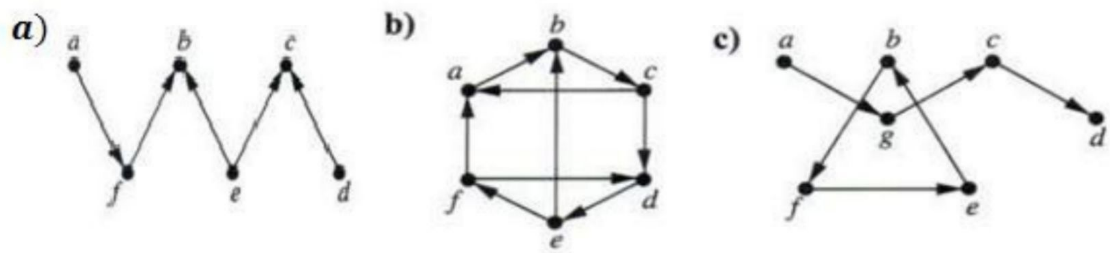
3.3

Exercise:

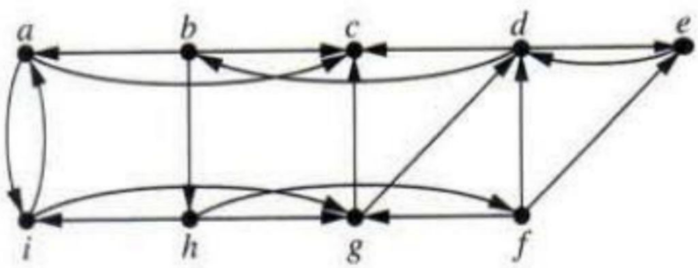
1. Does each of these lists of vertices form a path in the following following graph? Which paths are simple? Which are circuits? What are the lengths of those that are paths?
   a) a, e, b, c, b b)a, e, a, d, b, c, a c)e, b, a, d, b, e d)c, b, d, a, e, c



2. Determine whether each of these graphs is strongly connected and if not, whether it is weakly connected.

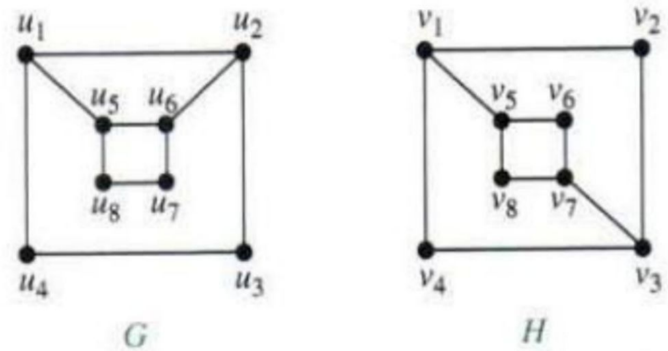


3. Find the strongly connected components of the following graph.

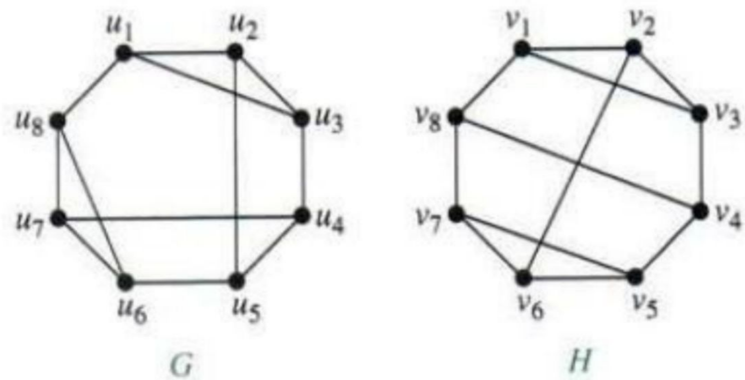


4. Use paths either to show that these graphs are not isomorphic or to find an isomorphism between these graphs

$a)$



b)

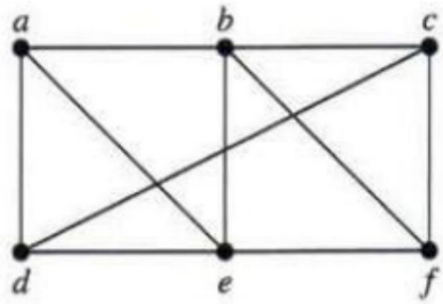


5. Find the number of paths of length n between two different vertices in K_4 if n is
   a) 2 b) 4 c)5

6. Find the number of paths of length n between any two adjacent vertices in $K_{3,3}$ if n is
   a) 2 b) 4 c)5

7. Find the number of paths of length n between any two non adjacent vertices in $K_{3,3}$ if n is
   b) 2 b) 4 c)5

8. Find the number of paths between c to d in the following graph of length
   a) 2 b) 3 c)4 d)5 e)6 f)7



9. Find the number of paths from a to e in the following directed graph of length
   b) 2 b) 3 c)4 d)5 e)6 f)7