4.6

Generating Functions:

Generating function for a sequence: The **generating function** (or **ordinary generating function**) for the sequence $\{a_n\}_{n=0}^{\infty}, i.e., a_0, a_1, a_2, \dots, a_n, \dots$ of real number is the infinite series

$$\boldsymbol{G(x) = a_0 + a_1 x + a_2 x + \cdots + a_n x^n \dots} \qquad \dots (1)$$

$$\boldsymbol{i.e., G(x) = \sum_{n=0}^{\infty} a_n x^n}$$

Example 1: The generating functions for the sequences $\{a_n\}_{n=0}^{\infty}$ with

i. $a_n = 3$, ii. $a_n = n + 1$ and iii. $a_n = 2^n$ are

Solution:

$$i. \sum_{n=0}^{\infty} 3x^n \quad , \quad ii. \sum_{n=0}^{\infty} (n+1) x^n \quad and \quad iii. \sum_{n=0}^{\infty} 2^n x^n$$

respectively.

Generating functions for a finite sequence:

Define the generating function of a finite sequence $a_0, a_1, a_2, \dots, a_n$ of real numbers by extending it by setting $a_k = 0$ for $k = n + 1, n + 2, \dots$.
The generating function of this infinite sequence $\{a_n\}_{n=0}^{\infty}$ is a polynomial of degree n, since no terms of the form $a_k x^k$ with $k > n$ ocuurs, $i.e.$,

$$G(x) = a_0 + a_1 x + a_2 x + \cdots + a_n x^n$$

Example 2: Write down the generating function for the finite sequence $\mathbf{1, 1, 1, 1, 1, 1}$.

Solution: The generating function for 1,1,1,1,1,1 is

$$G(x) = 1 + x + x^2 + x^3 + x^4 + x^5$$

Note that $\frac{x^6-1}{x-1} = 1 + x + x^2 + x^3 + x^4 + x^5$, when $x \neq 1$. Therefore, $G(x) = \frac{x^6-1}{x-1}$ is the generating function for the sequence 1,1,1,1,1,1.

Note: The RHS of the equation (1) is a formal power series in x. The letter x does not represent any thing. The various powers x^n of x are simply used to keep track of the corresponding terms a_n of the sequence. The convergence/divergence of the series is of no interest to us (at present).

Example 3: Let $\boldsymbol{m}$ be a positive integer and let $\boldsymbol{a_k} = {}^{m}C_k$, $\boldsymbol{k = 0, 1, 2, \ldots, m}$. What is the generating function for the sequence $\boldsymbol{a_0, a_1, \ldots a_m}$?

Solution: The generating function for the finite sequence $a_0, a_1, a_2, \ldots, a_n$ is

$$G(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_m x^m$$

$$= {}^{m}C_0 + {}^{m}C_1\, x + {}^{m}C_2\, x^2 +, .. + {}^{m}C_m\, x^m$$

$$= (1 + x)^m$$

Example 4:

i. $f(x) = 1 + 2x + 3x^2 + \cdots + (n+1)x^n + \cdots$ is the generating function for the sequence $\{n\}_{n=1}^{\infty}$ of positive integers.

ii. The function $g(x) = \frac{1}{1-x}$ is the generating function for the sequence 1,1,1, ... since $\frac{1}{1-x} = 1 + x + x^2 + \cdots$ for $|x| < 1$.

iii. The function $h(x) = \frac{1}{1-ax}$ is the generating function for the sequence $1, a, a^2, a^3, \ldots$,since $\frac{1}{1-ax} = 1 + ax + a^2x^2 + a^3x^3 + \cdots$ when $|ax| < 1$ or $|x| < \frac{1}{|a|}, a \neq 0$.

Equality of generating functions:

Two generating functions $f(x) = \sum_{n=0}^{\infty} a_n x^n$ and $g(x) = \sum_{n=0}^{\infty} b_n x^n$ are **equal** if $a_n = b_n \ \forall n = 0,1,2, \ldots$

Addition and Multiplication of generating functions:

Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ and $g(x) = \sum_{n=0}^{\infty} b_n x^n$ be two generating functions. Then

$$\boldsymbol{f(x) + g(x) = \sum_{n=0}^{\infty} (a_n + b_n) x^n}$$

$$\boldsymbol{f(x) g(x) = \sum_{n=0}^{\infty} \left(\sum_{j=0}^{n} a_j b_{n-j} \right) x^n}$$

Example 5: Let $f(x) = \frac{1}{(1-x)^2}$. Find the coefficients $a_0, a_1, a_2, \ldots$ in the expansion of $f(x) = \sum_{n=0}^{\infty} a_n x^n$.

Solution: We have $g(x) = \frac{1}{1-x} = 1 + x + x^2 + \cdots = \sum_{n=0}^{\infty} a_n x^n$

Now:

$$f(x) = \frac{1}{(1-x)^2} = \frac{1}{(1-x)} . \frac{1}{(1-x)} = \left(\sum_{n=0}^{\infty} x^n \right) \left(\sum_{m=0}^{\infty} x^m \right)$$

$$= \sum_{n=0}^{\infty} \left(\sum_{j=0}^{n} a_j b_{n-j} \right) x^n = \sum_{n=0}^{\infty} \left(\sum_{j=0}^{n} 1.1 \right) x^n$$

$$= \sum_{n=0}^{\infty}\left(\sum_{j=0}^{n} 1\right)x^n = \sum_{n=0}^{\infty}(n+1)x^n \;\; (\because \sum_{j=0}^{n} 1 = n+1)$$

Therefore, the coefficients of $f(x)$ are $a_n = n+1, n = 0,1,2,\ldots$

Extended binomial coefficients:

Let u be a real number and k be a nonnegative integer. Then the **extended binomial coefficients** $\binom{\boldsymbol{u}}{\boldsymbol{k}}$ is defined by

$$\binom{\boldsymbol{u}}{\boldsymbol{k}} = \begin{cases} \dfrac{\boldsymbol{u(u-1)(u-2)\ldots(u-k+1)}}{\boldsymbol{k}!} & \boldsymbol{if\ k} > 0 \\ \boldsymbol{1} & \boldsymbol{if\ k \leq 0} \end{cases}$$

Example 6:

(i). $\binom{-2}{3} = \frac{(-2)(-2-1)(-2-2)}{3!} = \frac{(-2)(-3)(-4)}{3!} = -4$

(ii). $\binom{\frac{1}{2}}{4} = \frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\left(\frac{1}{2}-3\right)}{4!} = \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{4!} = -\frac{5}{128}$

The following is a useful formula for extended binomial coefficients when the top parameter is a negative integer. If the top parameter u is a negative integer then the extended binomial coefficient can be expressed in terms of an **ordinary binomial coefficient.**

Theorem 1: If $\boldsymbol{n}$ is a positive integer then

$$\binom{-\boldsymbol{n}}{\boldsymbol{r}} = (-\mathbf{1})^r \; {}^{n+r-1}C_r$$

Proof:

$$\binom{-n}{r} = \frac{(-n)(-n-1)(-n-2)\ldots(-n-r+1)}{r!}$$

$$= (-1)^r \frac{n(n+1)(n+2)\ldots(n+r-1)}{r!}$$

$$= (-1)^r \frac{(n+r-1)(n+r-2)\ldots(n+1)n}{r!}$$

$$= (-1)^r \binom{n+r-1}{r}$$

$$= (-1)^r \; {}^{n+r-1}C_r$$

The extended binomial Theorem

Let $\boldsymbol{x}$ be a real number with $|\boldsymbol{x}| < 1$ and let $\boldsymbol{u}$ be a real number. Then

$$(\mathbf{1} + \boldsymbol{x})^{\boldsymbol{u}} = \sum_{\boldsymbol{k}=\mathbf{0}}^{\infty} \binom{\boldsymbol{u}}{\boldsymbol{k}} \boldsymbol{x}^{\boldsymbol{k}}$$

Remark: If u is a positive integer, the extended Binomial Theorem reduces to Binomial Theorem, (since $\binom{u}{k} = 0$ if $k > u$).

Example 7: Find the generating functions for $(\mathbf{1} + \boldsymbol{x})^{-\boldsymbol{n}}$ and $(\mathbf{1} - \boldsymbol{x})^{-\boldsymbol{n}}$ where $\boldsymbol{n}$ is a positive integer, using the extended Binomial theorem.

Solution: By the extended Binomial Theorem, we have

$$(1 + x)^{-n} = \sum_{k=0}^{\infty} \binom{-n}{k} x^k$$

$$= \sum_{k=0}^{\infty} (-1)^k \, {}^{(n+k-1)}C_k \; x^k$$

Thus, the generating function for $(1+x)^{-n}$ is

$$\sum_{k=0}^{\infty} (-1)^k \, {}^{(n+k-1)}C_k \; x^k$$

Replacing x by $-x$, we get the generating function for $(1-x)^n$. It is given by

$$\sum_{k=0}^{\infty} (-1)^k \, {}^{(n+k-1)}C_k \; (-x)^k = \sum_{k=0}^{\infty} {}^{(n+k-1)}C_k \; x^k$$

Summary of some generating functions for certain sequences

$\boldsymbol{a_k}$	$\boldsymbol{G(x)}$ **:Generating for the sequence** $\{\boldsymbol{a_k}\}_{\boldsymbol{k=0}}^{\infty}$
$\frac{1}{k!}$	$\sum_{k=0}^{\infty} \frac{x^n}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = e^x$
$\frac{(-1)^{k+1}}{k}$	$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k!} x^k = x - \frac{x^2}{2!} + \frac{x^3}{3!} - \cdots = ln(1+x)$
1	$\sum_{k=0}^{\infty} x^k = 1 + x + x^2 + \cdots = \frac{1}{1-x}$
a^k	$\sum_{k=0}^{\infty} a^k x^k = 1 + ax + (ax)^2 + \cdots = \frac{1}{1-ax}$
1 if $r\|k$; 0 otherwise	$\sum_{k=0}^{\infty} x^{rk} = 1 + x^r + x^{2r} + \cdots = \frac{1}{1-x^r}$
$k+1$	$\sum_{k=0}^{\infty} (k+1)x^k = 1 + 2x + 3x^2 + \cdots = \frac{1}{(1-x)^2}$
${}^{(n+k-1)}C_k$	$\sum_{k=0}^{\infty} {}^{(n+k-1)}C_k \; x^k = 1 + {}^{n}C_1 \, x + {}^{(n+1)}C_2 \, x^2 + \cdots = \frac{1}{(1-x)^n}$

$(-1)^k \; {}^{(n+k-1)}C_k$	$\sum_{k=0}^{\infty}(-1)^n \; {}^{(n+k-1)}C_k \, x^k = 1 - {}^nC_1 \, x + {}^{(n+1)}C_2 \, x^2 - \cdots = \frac{1}{(1+x)^n}$
${}^{(n+k-1)}C_k \, x^k$	$\sum_{k=0}^{\infty} {}^{(n+k-1)}C_k \, a^k x^k = 1 + {}^nC_1 \, (ax) + {}^{(n+1)}C_2 \, (ax)^2 + \cdots = \frac{1}{(1-ax)^n}$
1 , if $k \le n$; 0 , otherwise	$\sum_{k=0}^{n} x^k = 1 + x + x^2 + \cdots + x^n = \frac{1-x^{n+1}}{1-x}$
${}^nC_k$	$\sum_{k=0}^{n} {}^nC_k \, x^k = 1 + {}^nC_1 \, x + {}^nC_2 \, x^2 + \cdots = (1+x)^n$
${}^nC_k \, a^k$	$\sum_{k=0}^{n} {}^nC_k \, a^k x^k = 1 + {}^nC_1 \, ax + {}^nC_2 \, (ax)^2 + \cdots = (1+ax)^n$

Counting problems and Generating Functions

Generating functions can be used to solve a wide variety of counting problems.

Example 8: Find the number of solutions of

$$\boldsymbol{e_1 + e_2 + e_3 = 17}$$

where $\boldsymbol{e_1, e_2}$ and $\boldsymbol{e_3}$ are nonnegative integers with $\boldsymbol{2 \le e_1 \le 5, 3 \le e_2 \le 6}$ and $\boldsymbol{4 \le e_3 \le 7}$.

Solution: The number of solutions with the given constraints is the coefficient of x^{17} in the expansion of

$$(x^2 + x^3 + x^4 + x^5)(x^3 + x^4 + x^5 + x^6)(x^4 + x^5 + x^6 + x^7)$$

This is so, since we obtain a term equal to x^{17} in the product by taking a term in the first sum x^{e_1}, a term in the second sum x^{e_2} and a term in the third sum x^{e_3}, where the exponents e_1, e_2 and e_3 satisfy the equation (1) and the given constraints.

The coefficient of x^{17} in this product is $1 + 1 + 1 = 3$
(The products $x^4x^6x^7$, $x^5x^5x^7$, $x^5x^6x^6$)

Proving Identities using Generating Functions

Example 9: Use generating function to show that

$$\sum_{k=0}^{n} \left({}^{n}C_{k}\right)^{2} = {}^{2n}C_{n}$$

where n is a positive integer?

Solution: Note that by the Binomial Theorem ${}^{2n}C_{n}$ is the coefficient of x^n in $(1+x)^{2n}$, now

$$(1+x)^{2n} = (1+x)^n(1+x)^n$$

$$= \left({}^{n}C_{0} + {}^{n}C_{1}x + {}^{n}C_{2}x^2 + \cdots {}^{n}C_{n}x^n\right)^2$$

Equating the coefficient x^n on both sides ,we get

$$ {}^{2n}C_{n} = {}^{n}C_{0}\cdot{}^{n}C_{n} + {}^{n}C_{1}\cdot{}^{n}C_{n-1} + {}^{n}C_{2}\cdot{}^{n}C_{n-2} + \cdots + {}^{n}C_{n}\cdot{}^{n}C_{0}$$

$$= {}^{n}C_{0}\cdot{}^{n}C_{0} + {}^{n}C_{1}\cdot{}^{n}C_{1} + {}^{n}C_{2}\cdot{}^{n}C_{2} + \cdots + {}^{n}C_{n}\cdot{}^{n}C_{n}$$

$$(\because {}^{n}C_{r} = {}^{n}C_{r-1})$$

$$= \sum_{k=0}^{n} \left({}^{n}C_{k}\right)^{2}$$

Hence the result.

Solving recurrence relations using generating functions

Example 10: Solve the Fibonacci recurrence relation

$$\boldsymbol{F_n = F_{n-1} + F_{n-2}, \qquad F_1 = F_2 = 1}$$

by using generating function.

Solution: We have the recurrence relation

$$F_n = F_{n-1} + F_{n-2} \quad , F_1 = F_2 = 1$$

Put $n = 2$ then $F_2 = F_1 + F_0 \Longrightarrow F_0 = 0$

Let $G(x)$ be the generating function for the sequence $\{F_n\}_{n=0}^{\infty}$, i.e,

$$G(x) = \sum_{n=0}^{\infty} F_n x^n$$

$F_n = F_{n-1} + F_{n-2} \Longrightarrow F_n x^n = F_{n-1}x^n + F_{n-2}x^n$

$$\sum_{n=2}^{\infty} F_n x^n = x\sum_{n=2}^{\infty} F_{n-1}x^{n-1} + x^2\sum_{n=2}^{\infty} F_{n-2}x^{n-2}$$

$\Longrightarrow G_n(x) - F_1 x - F_0 = x(G_n(x) - F_0) + x^2 G_n(x)$

$\Longrightarrow G_n(x) - x = xG_n(x) + x^2 G_n(x)$

$\Longrightarrow G_n(x)(1 - x - x^2) = x$

$\Longrightarrow G_n(x) = \dfrac{x}{1 - x - x^2}$

Now, $1 - x - x^2 = (1 - \alpha\, x)(1 - \beta x)$, where $\alpha + \beta = 1, \alpha\beta = -1$.

$i.e., \alpha = \frac{1+\sqrt{5}}{2}, \beta = \frac{1-\sqrt{5}}{2}$

$$\frac{x}{1 - x - x^2} = \frac{x}{(1 - \alpha\, x)(1 - \beta x)} = \frac{A}{1 - \alpha\, x} + \frac{B}{1 - \beta x}$$

$\Longrightarrow x = A(1-\beta x) + B(1-\alpha x) = (A+B) - (\beta A + \alpha B)x$

$\Longrightarrow A + B = 0, \qquad \beta A + \alpha B = -1$

Solving we get, $A = \frac{1}{\sqrt{5}} = -B$(do it!)

Thus,

$$G(x) = \frac{1}{\sqrt{5}}\left[\frac{1}{1-\alpha x} - \frac{1}{1-\beta x}\right]$$

$i.e.,$

$$\sum_{n=0}^{\infty} F_n x^n = \frac{1}{\sqrt{5}}\left[\sum_{n=0}^{\infty} \alpha^n x^n - \sum_{n=0}^{\infty} \beta^n x^n\right] = \frac{1}{\sqrt{5}}\left(\sum_{n=0}^{\infty} (\alpha^n - \beta^n) x^n\right)$$

Equating the coefficients of x^n on both sides we get

$$F_n = \frac{1}{\sqrt{5}}(\alpha^n - \beta^n)$$

$$i.e., F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$

This is called **Binnet Formula for $\boldsymbol{F_n}$**.

Example 11: Solve the recurrence relation

$$\boldsymbol{a_n = 8a_{n-1} + 10^{n-1}, a_1 = 9}$$

by using generating function.

Solution*:* Let $G(x) = \sum_{n=0}^{\infty} a_n x^n$ be the generating function for the sequence $\{a_n\}_{n=0}^{\infty}$. Putting $n = 1$, in the given recurrence relation we get

$$a_1 = 8a_0 + 1 \Longrightarrow 9 = 8a_0 + 1 \Longrightarrow a_0 = 1$$

Multiply the given recurrence relation by x^n, we get

$$a_n x^n = 8a_{n-1}x^n + 10^{n-1}x^n$$

Sum both sides starting from $n = 1$

$$\sum_{n=1}^{\infty} a_n x^n = 8\sum_{n=1}^{\infty} a_{n-1}x^n + \sum_{n=1}^{\infty} 10^{n-1}x^n$$

$$\Longrightarrow G(x) - a_0 = 8x\sum_{n=1}^{\infty} a_{n-1}x^{n-1} + x\sum_{n=1}^{\infty} 10^{n-1}x^{n-1}$$

$$\Longrightarrow G(x) - 1 = 8x\sum_{n=0}^{\infty} a_n x^n + x\sum_{n=0}^{\infty} 10^n x^n$$

$$= 8xG(x) + x\sum_{n=0}^{\infty} 10^n x^n$$

$$= 8xG(x) + \frac{x}{1-10x}$$

$$\Longrightarrow (1-8x)G(x) = 1 + \frac{x}{1-10x} = \frac{1-9x}{1-10x}$$

$$\Longrightarrow G(x) = \frac{1-9x}{(1-8x)(1-10x)} = \frac{1}{2}\left(\frac{1}{1-8x} + \frac{1}{1-10x}\right)$$

$$= \frac{1}{2}\left(\sum_{n=0}^{\infty} 8^n x^n + \sum_{n=0}^{\infty} 10^n x^n\right)$$

$$= \sum_{n=0}^{\infty} \frac{1}{2}(8^n + 10^n)\, x^n$$

Therefore, $a_n = \frac{1}{2}(8^n + 10^n)$

Example 12: Solve the recurrence relation

$$\boldsymbol{a_n - 3a_{n-1} = n, n \in N, a_0 = 1}$$

by using generating function.

Solution: We have the generating function

$$a_n - 3a_{n-1} = n, n \in \boldsymbol{N}, a_0 = 1$$

Let $G(x)$ be the generating function for the sequence $\{a_n\}_{n=0}^{\infty}$ $i.e.$,

$$G(x) = \sum_{n=0}^{\infty} a_n x^n$$

Now,

$$a_n - 3a_{n-1} = n \Rightarrow a_n x^n - 3a_{n-1}x^n = nx^n$$

$$\Rightarrow \sum_{n=1}^{\infty} a_n x^n - 3\sum_{n=1}^{\infty} a_{n-1}x^n = \sum_{n=1}^{\infty} nx^n$$

$$\Rightarrow G(x) - a_0 - 3x\sum_{n=1}^{\infty} a_{n-1}x^{n-1} = \sum_{n=0}^{\infty} nx^n$$

$$\Rightarrow G(x) - 1 - 3xG(x) = \sum_{n=0}^{\infty} nx^n$$

$$\Rightarrow G(x)(1-3x) - 1 = x + 2x^2 + 3x^3 + \cdots = x(1 + 2x + 3x^2 + \cdots)$$

$$= x.\frac{1}{(1-x)^2}$$

$$\Rightarrow G(x) = \frac{x}{(1-3x)(1-x)^2} + \frac{1}{1-3x} \qquad \ldots (1)$$

$$\frac{x}{(1-x)^2(1-3x)} = \frac{A}{1-x} + \frac{B}{(1-x)^2} + \frac{C}{1-3x}$$

$\Longrightarrow x = A(1-x)(1-3x) + B(1-3x) + C(1-x)^2$

Put $x = 1,\ \ 1 = B(-2) \Longrightarrow B = -\frac{1}{2}$

Put $x = \frac{1}{3},\ \frac{1}{3} = C.\frac{4}{9} \Longrightarrow C = \frac{3}{4}$

Put $x = 0, 0 = A + B + C \Longrightarrow A = -(B + C) = -\frac{1}{4}$

$$\therefore \ \frac{x}{(1-x)^2(1-3x)} = -\frac{1}{4}\frac{1}{1-x} - \frac{1}{2}\frac{1}{(1-x)^2} + \frac{3}{4}\frac{1}{1-3x}$$

From (1)

$$G(x) = \frac{1}{4}\frac{1}{1-x} - \frac{1}{2}\frac{1}{(1-x)^2} + \frac{7}{4}\frac{1}{1-3x}$$

$$\sum_{n=1}^{\infty} a_n x^n = -\frac{1}{4}\sum_{n=1}^{\infty} x^n - \frac{1}{2}\sum_{n=1}^{\infty} \binom{2+n-1}{n} x^n + \frac{7}{4}\sum_{n=1}^{\infty} 3^n x^n$$

Equating the coefficients of x^n on both sides we get

$$a_n = -\frac{1}{4} - \frac{1}{2}\,{}^{(n+1)}C_n + \frac{7}{4}3^n = -\frac{1}{4} - \frac{(n+1)}{2} + \frac{7}{4}3^n$$

$$= -\frac{3}{4} - \frac{n}{2} + \frac{7}{4}3^n$$

Generating functions can be used to solve a system of recurrence relation.

Example 13: Solve the following system of recurrence relations using the method of generating functions

$$a_{n+1} = -2a_n - 4b_n, \qquad \dots(1)$$

$$b_{n+1} = 4a_n + 6b_n, \qquad \dots(2)$$

$n = 0,1,2, \dots; a_0 = 1, b_0 = 0.$

Solution: Let $F(x)$ and $G(x)$ be the generating functions for the sequence $\{a_n\}_{n=0}^{\infty}$and $\{b_n\}_{n=0}^{\infty}$ respectively. Form the equations (1) and (2), when $n = 0,1,2,\dots$

$$a_{n+1}x^{n+1} = -2a_n x^{n+1} - 4b_n x^{n+1}$$

$$b_{n+1}x^{n+1} = 4a_n x^{n+1} + 6b_n x^{n+1}$$

$$\Longrightarrow \sum_{n=0}^{\infty} a_{n+1}x^{n+1} = -2x\sum_{n=0}^{\infty} a_n x^n - 4x\sum_{n=0}^{\infty} b_n x^n$$

$$\sum_{n=0}^{\infty} b_{n+1}x^{n+1} = 4x\sum_{n=0}^{\infty} a_n x^n + 6x\sum_{n=0}^{\infty} b_n x^n$$

$$\Longrightarrow F(x) - a_0 = -2xF(x) - 4xG(x)$$

$$G(x) - b_0 = 4xF(x) + 6xG(x)$$

$$\Longrightarrow (1+2x)F(x) + 4xG(x) = 1 \qquad \dots(3)$$

$$4x\,F(x) - (1-6x)G(x) = 0 \qquad \dots(4)$$

Solving for $F(x)$, we get

$F(x) = \frac{1-6x}{(1-2x)^2}$ (do it!)

Now, $\frac{1-6x}{(1-2x)^2} = \frac{A}{1-2x} + \frac{B}{(1-2x)^2}$

$i.e.,$ $\quad 1 - 6x = A(1-2x) + B \Longrightarrow A + B = 1, B = -2 \Longrightarrow A = 3, B = -2$

Hence $F(x) = \frac{3}{1-2x} - \frac{2}{(1-2x)^2}$

$$= 3\sum_{n=0}^{\infty} 2^n x^n - 2\sum_{n=0}^{\infty}(n+1)2^n x^n$$

$$= \sum_{n=0}^{\infty}(3.2^n - 2(n+1)2^n)x^n$$

$$= \sum_{n=0}^{\infty} 2^n(1-2n)x^n = \sum_{n=0}^{\infty} a_n x^n$$

From (4), we have $G(x) = \frac{4x}{1-6x}$ $F(x) = \frac{4x}{(1-2x)^2}$

Now, $\frac{4x}{(1-2x)^2} = -\frac{2}{1-2x} + \frac{2}{(1-2x)^2}$ (do it!)

$$= -2\sum_{n=0}^{\infty} 2^n x^n + 2\sum_{n=0}^{\infty}(n+1)2^n x^n$$

$$= \sum_{n=0}^{\infty} 2^n(-2+2n+2)x^n$$

$$= \sum_{n=0}^{\infty} 2^n(2n)x^n = \sum_{n=0}^{\infty} n.\, 2^{n+1}x^n = \sum_{n=0}^{\infty} b_n x^n$$

Thus , $a_n = 2^n(1-2n), b_n = n\, 2^{n+1}$.

P1:

Solve the recurrence relation $a_k = 3a_{k-1}$ for $k = 1, 2, 3, \ldots$ and with initial condition $a_0 = 2$ by using generating function.

Solution:

Let $G(x)$ be the generating function for the sequence $\{a_k\}_{k=0}^{\infty}$,

$$i.e., G(x) = \sum_{k=0}^{\infty} a_k x^k .$$

Multiplying the recurrence relation by x^k

$$a_k x^k - 3a_{k-1}x^k = 0$$

$$\Rightarrow \sum_{k=1}^{\infty} a_k x^{k+1} - 3x \sum_{k=1}^{\infty} a_{k-1} x^k = 0.$$

$$\Rightarrow G(x) - a_0 - 3xG(x) = 0 \Rightarrow (1 - 3x)G(x) = a_0$$

$$i.e., G(x)(1 - 3x) = 2 \Rightarrow G(x) = \frac{2}{1-3x} = 2\sum_{k=0}^{\infty} 3^k x^k \quad (\because \frac{1}{1-ax} = \sum_{k=0}^{\infty} a^k x^k)$$

Thus, $a_k = 2.3^k$

P2:

Using generating functions, solve the recurrence relation

$$\boldsymbol{a_n = 6a_{n-1} - 9a_{n-2},\ a_0 = 2 \text{ and } a_1 = 3.}$$

Solution: The given recurrence relation is

$$a_n = 6a_{n-1} - 9a_{n-2},\ a_0 = 2,\ a_1 = 3$$

Let $G(x)$ be the generating function the sequence $\{a_n\}_{n=0}^{\infty}$

$$i.e., \qquad G(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$\Rightarrow a_n x^n = 6a_{n-1}x^{n-1} - 9a_{n-2}x^n$$

$$\Rightarrow \sum_{n=2}^{\infty} a_n x^n = 6x \sum_{n=2}^{\infty} a_{n-1}x^{n-1} - 9x^2 \sum_{n=2}^{\infty} a_{n-2}x^{n-2}$$

$$\Rightarrow G(x) - a_1 x + a_0 = 6x(G(x) - a_0) - 9x^2 G(x)$$

$$\Rightarrow G(x) - 3x - 2 = 6x(G(x) - 2) - 9x^2 G(x)$$

$$\Rightarrow G(x)(1 - 6x - 9x^2) = 3x + 2 - 12x = 2 - 9x$$

$$\therefore G(x) = \frac{2 - 9x}{1 - 6x - 9x^2}$$

Now,

$$\frac{2 - 9x}{1 - 6x - 9x^2} = \frac{2 - 9x}{(1 - 3x)^2} = \frac{A}{1 - 3x} + \frac{B}{(1 - 3x)^2}$$

Therefore, $2 - 9x = A(1 - 3x) + B$

$$\Rightarrow A + B = 2, -3A = -9 \Rightarrow A = 3, B = -1$$

Thus,

$$G(x) = \frac{3}{1-3x} - \frac{1}{(1-3x)^2}$$

$i.e.,$

$$\sum_{n=0}^{\infty} a_n x^n = 3\sum_{n=0}^{\infty} 3^n x^n - \sum_{n=0}^{\infty} (n+1)3^n x^n$$

Equating the coefficients of x^n on both sides

$$a_n = 3^{n+1} - (n+1)3^n = 3^n(2-n)$$

$$\therefore\ a_n = 3^n(2-n), \qquad n = 0,1,2,\ldots$$

P3:

Solve the recurrence relation $\boldsymbol{a_n = 2a_{n-1} + 1, a_1 = 1}$ using generating function.

Solution: We have the recurrence relation

$$a_n = 2a_{n-1} + 1, a_1 = 1$$

The initial condition$a_1 = 1$, yields $a_0 = 0$. $(n = 1 \Longrightarrow a_1 = 2a_0 + 1 \Longrightarrow a_0 = 0)$

Let $G(x)$ be the generating function for the sequence $\{a_n\}_{n=0}^{\infty}$ $i.e.$,

$$G(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$a_n = 2a_{n-1} + 1 \Longrightarrow a_n x^n = 2a_{n-1}x^n + x^n$$

$$\Longrightarrow \sum_{n=1}^{\infty} a_n x^n = 2\sum_{n=1}^{\infty} a_{n-1}x^n + \sum_{n=1}^{\infty} x^n$$

$$\Longrightarrow G(x) - a_0 = 2x\sum_{n=1}^{\infty} a_{n-1}x^{n-1} + \sum_{n=1}^{\infty} x^n - 1$$

$$\Longrightarrow G(x) = 2x\sum_{n=0}^{\infty} a_n x^n + \sum_{n=0}^{\infty} x^n - 1 = 2xG(x) + \sum_{n=1}^{\infty} x^n - 1$$

$$\Longrightarrow (1 - 2x)G(x) = \frac{1}{1-x} - 1 = \frac{x}{1-x}$$

$\Longrightarrow G(x) = \frac{x}{(1-2x)(1-x)} \Longrightarrow G(x) = \frac{1}{1-2x} - \frac{1}{1-x}$ (by the method of partial fractions)

$$= \sum_{n=0}^{\infty} 2^n x^n - \sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} (2^n - 1)x^n$$

Thus,

$$a_n = 2^n - 1, \qquad n = 0,1,2, \dots$$

P4:

Solve the recurrence relation

$$\boldsymbol{a_{n+2} - 5a_{n+1} + 6a_n = 2, n = 0, 1, 2, \ldots, a_0 = 3, a_1 = 7}$$

using generating function.

Solution: We have the recurrence relation

$$a_{n+2} - 5a_{n+1} + 6a_n = 2, n = 0,1,2, \ldots, a_0 = 3, a_1 = 7$$

Let $G(x)$ be the generating function for the sequence $\{a_n\}_{n=0}^{\infty}$, $i.e.$,

$$G(x) = \sum_{n=0}^{\infty} a_n x^n$$

Multiply the recurrence relation by x^{n+2} we get

$$a_{n+2}x^{n+2} - 5a_{n+1}x^{n+2} + 6a_n x^{n+2} = 2x^{n+2}$$

$$\Longrightarrow \sum_{n=0}^{\infty} a_{n+2}x^{n+2} - 5\sum_{n=0}^{\infty} a_{n+1}x^{n+1} + 6\sum_{n=0}^{\infty} a_n x^{n+2} = 2\sum_{n=0}^{\infty} x^{n+2}$$

$$\Longrightarrow \sum_{n=0}^{\infty} a_{n+2}x^{n+2} - 5x\sum_{n=0}^{\infty} a_{n+1}x^{n+1} + 6x^2\sum_{n=0}^{\infty} a_n x^{n} = 2x^2\sum_{n=0}^{\infty} x^{n}$$

$$\Longrightarrow G(x) - a_1 x - a_0 - 5x(G(x) - a_0) + 6x^2 G(x) = 2x^2 \frac{1}{1-x}$$

$$\Longrightarrow G(x)(1 - 5x + 6x^2) - 7x - 3 + 15x = \frac{2x^2}{1-x}$$

$$\Longrightarrow G(x)(1 - 5x + 6x^2) = \frac{2x^2}{1-x} + 3 - 8x$$

$$\Longrightarrow G(x) = \frac{2x^2}{(1-x)(1-5x+6x^2)} + \frac{3-8x}{1-5x+6x^2}$$

$$= \frac{2x^2 + (3 - 8x)(1 - x)}{(1 - x)(1 - 5x + 6x^2)}$$

$$\Rightarrow \sum_{n=0}^{\infty} a_n x^n = \frac{10x^2 - 11x + 3}{(1 - x)(1 - 2x)(1 - 3x)} = \frac{(5x - 3)(2x - 1)}{(1 - x)(1 - 2x)(1 - 3x)}$$

$$= \frac{3 - 5x}{(1 - x)(1 - 3x)} = \frac{2}{1 - 3x} + \frac{1}{1 - x} \quad (\,do\ it!)$$

$$= 2\sum_{n=0}^{\infty} 3^n x^n + \sum_{n=1}^{\infty} x^n$$

Equating the coefficients of x^n on both sides we get

$$a_n = 2.3^n + 1, n = 0,1,2, \ldots$$

4.6. Generating functions:

Exercise:

1. In how many different ways can eight identical cookies be distributed among three distinct children if each child receives atleast two cookies and no more than four cookies?

2. Find the coefficient of x^{10} in the power series of the following functions:
   a. $(1 + x^5 + x^{10} + x^{15} + \cdots)^3$
   b. $(x^4 + x^5 + x^6)(x^3 + x^4 + x^5 + x^6 + x^7)(1 + x + x^2 + x^3 + x^4 + \cdots)$

3. Use generating functions to solve the recurrence relation $a_k = 3a_{k-1} + 2$, with initial condition $a_0 = 1$.

4. Use generating functions to solve the recurrence relation

   $a_k = 5a_{k-1} - 6a_{k-2}$, with initial conditions $a_0 = 6$ and $a_1 = 30$.

5. Use generating functions to solve the recurrence relation $a_k = 4a_{k-1} - 4a_{k-2} + k^2$, with initial conditions $a_0 = 2$ and $a_1 = 5$.

6. How many integer solutions are there for the equation $c_1 + c_2 + c_3 + c_4 = 25$, if $0 \le c_i$, for all $1 \le i \le 4$.

7. Solve the following recurrence relations by the method of generating functions.

a. $a_{n+1} - a_n = 3^n \ , n \geq 0, a_0 = 1$

b. $a_{n+1} - a_n = n^2 \ , n \geq 0, a_0 = 1$

c. $a_n - 3a_{n-1} = 5^{n-1} \ , n \geq 1, a_0 = 1$

d. $a_{n+2} - 3a_{n+1} + 2a_n = 0 \ , n \geq 0, a_0 = 1, a_1 = 6.$

e. $a_{n+2} - 2a_{n+1} + a_n = 2^n \ , n \geq 0, a_0 = 1 \, , a_1 = 2$

8. Solve the following systems of recurrence relations.

a. $a_{n+1} = -2a_n - 4b_n \, , \ b_{n+1} = 4a_n + 6b_n \, , n \geq 0 \, , a_0 = 1 \, , b_0 = 0.$

b. $a_{n+1} = 2a_n - b_n + 2 \ , b_{n+1} = -a_n + 2b_n - 1 \, , n \geq 0 \, , a_0 = 0 \, , b_0 = 1$

9. Use generating functions to solve the following recurrence relations:

a. $a_n = 2a_{n-1} \, , a_0 = 1$

b. $a_n = a_{n-1} + 2 \, , a_1 = 1$

c. $a_n = 4a_{n-2} \, , a_0 = 2, a_1 = -8$

d. $a_n = 5a_{n-1} - 6a_{n-2} \, , a_0 = 4 \, , a_1 = 7$

e. $a_n = 3a_{n-1} + 4a_{n-2} - 12a_{n-3} \, , a_0 = 3, a_1 = -7, a_2 = 7$

f. $a_n = 6a_{n-1} - 12a_{n-2} + 8a_{n-3} \, , a_0 = 0, a_1 = 2, a_2 = -2$