2.4

Transitive closure

In this module we discuss composition of binary relation, Boolean matrices and transitive closure of a relation on a set.

Composition of Binary Relation

Let A, B and C be sets. Let R be a relation from A to B and S be a relation from B to C. Then a relation, written as $\boldsymbol{R \circ S}$, is called a **composition** of R and S, and is defined as

$$\boldsymbol{R \circ S = \{(x, z) | x \in A \wedge z \in C \wedge (\exists y)(y \in B \wedge (x, y) \in R \wedge (y, z) \in S)\}}$$

The operation of obtaining $R \circ S$ from R and S is called **composition** of relations.

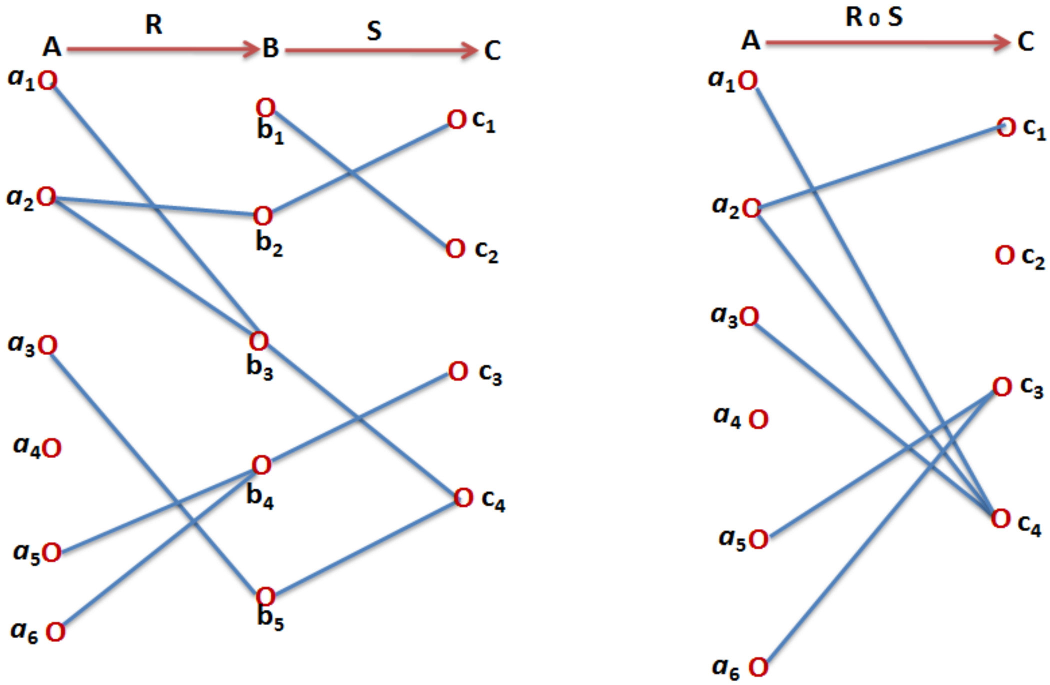
Note:

(i) $R \circ S = \phi$,if the intersection of range of R and the domain of S is empty.

(ii) $R \circ S \neq \phi$, if there exists at least one ordered pair $(x, y) \in R$ such that $y \in B$ is a first component in an ordered pair of S.

(iii) Domain of $R \circ S$ is a subset of A and its range is a subset of C. In fact, the $D(R \circ S) \subseteq D(R)$ and $\boldsymbol{R}(R \circ S) \subseteq \boldsymbol{R}(S)$.

From the graphs of R and S we can easily construct the graph of $R \circ S$.



Relations R, S and R∘S

The operation of composition of relations produces a relation from two relations. Therefore, the operation of composition is a binary operation. The same operation can be applied again to produce other relations. For example let R, S and P be relations from A to B, from B to C and from C to D respectively. Then we can also form $(R \circ S) \circ P$, which is a relation from A to D. Similarly, we can also form $R \circ (S \circ P)$, which is also a relation from A to D.

Lemma: The operation of composition on relations is associative.

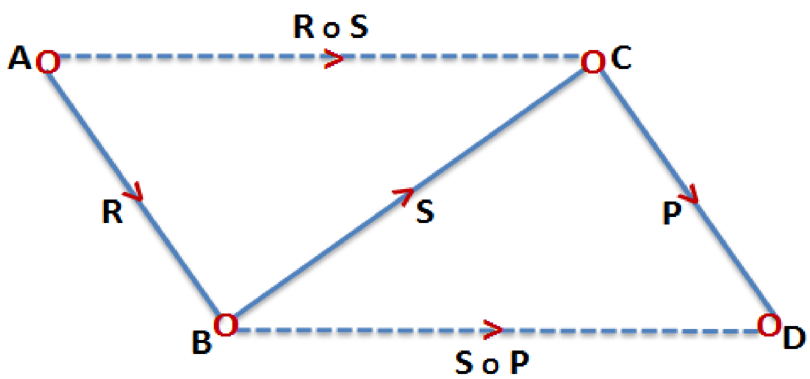
$i.e.$, if R, S and P be any relations defined above then

$$(\boldsymbol{R} \circ \boldsymbol{S}) \circ \boldsymbol{P} = \boldsymbol{R} \circ (\boldsymbol{S} \circ \boldsymbol{P})$$

Proof: Assume that $(R \circ S) \circ P$ is nonempty. Let $(x, y) \in R$, $(y, z) \in S$ and $(z, w) \in P$. This means that $(x, z) \in R \circ S$ and $(x, w) \in (R \circ S) \circ P$. Also note that $(y, w) \in S \circ P$ and $(x, w) \in R \circ (S \circ P)$.

This shows that $(R \circ S) \circ P \subseteq R \circ (S \circ P)$. In a similar manner we show that $R \circ (S \circ P) \subseteq (R \circ S) \circ P$. Thus, $(R \circ S) \circ P = R \circ (S \circ P)$. Therefore, the operation of composition on relations is associative.

The above result follows from the following partial graph



Associatively of composition of relations

Since the composition of relations is associative, we may delete parenthesis, so that $R \circ (S \circ P) = (R \circ S) \circ P = R \circ S \circ P$.

Example 1: Let $\boldsymbol{A = \{1, 2, 3, 4, 5\}}$.Let $\boldsymbol{R}$ and $\boldsymbol{S}$ be relations on $\boldsymbol{A}$ defined by

$$\boldsymbol{R = \{(1,2), (2,2), (3,4)\}}$$

$$\boldsymbol{S = \{(4,2), (2,5), (3,1), (1,3)\}}$$

Find $\boldsymbol{R \circ R}$, $\boldsymbol{S \circ S}$, $\boldsymbol{R \circ S}$, $\boldsymbol{S \circ R}$, $\boldsymbol{R \circ (S \circ R)}$, $\boldsymbol{(R \circ S) \circ R}$ and $\boldsymbol{R \circ R \circ R}$.

Solution:

(i) $R \circ R = \{(1,2), (2,2), (3,4)\} \circ \{(1,2), (2,2), (3,4)\} = \{(1,2), (2,2)\}$

(ii) $S \circ S = \{(1,3), (2,5), (3,1), (4,2)\} \circ \{(1,3), (2,5), (3,1), (4,2)\}$

$= \{(1,1), (3,3), (4,5)\}$

(iii) $R \circ S = \{(1,2), (2,2), (3,4)\} \circ \{(1,3), (2,5), (3,1), (4,2)\}$

$= \{(1,5), (2,5), (3,2)\}$

(iv) $S \circ R = \{(1,3),(2,5),(3,1),(4,2)\} \circ \{(1,2),(2,2),(3,4)\}$

$= \{(1,4),(3,2),(4,2)\}$

Note that $R \circ S \neq S \circ R$

(v) $(R \circ S) \circ R = \{(1,5),(2,5),(3,2)\} \circ \{(1,2),(2,2),(3,4)\} = \{(3,2)\}$

(vi) $R \circ (S \circ R) = \{(1,2),(2,2),(3,4)\} \circ \{(1,4),(3,2),(4,2)\}$

$= \{(3,2)\} = (R \circ S) \circ R$

(vii) $R \circ R \circ R$

$= \{(1,2),(2,2),(3,4)\} \circ \{(1,2),(2,2)\}$

$= \{(1,2),(2,2)\}$

Boolean Matrix

A matrix whose entries are bits (i.e., 0 and 1) is called a **Boolean Matrix**. That is $M = (m_{ij})_{m\times n}$ is a Boolean matrix if $m_{ij} = 0 \ or \ 1$ for all i and j.

Join and Meet

The ***Join*** of the Boolean matrices $M = (m_{ij})_{m\times n}$ and $N = (n_{ij})_{m\times n}$, denoted by $\boldsymbol{M} \vee \boldsymbol{N}$, is defined by $\boldsymbol{M} \vee \boldsymbol{N} = (\boldsymbol{m_{ij}} \vee \boldsymbol{n_{ij}})_{\boldsymbol{m\times n}}$

The ***Meet*** of the Boolean matrices $M = (m_{ij})_{m\times n}$ and $N = (n_{ij})_{m\times n}$, denoted by $\boldsymbol{M} \wedge \boldsymbol{N}$, is defined by $\boldsymbol{M} \wedge \boldsymbol{N} = (\boldsymbol{m_{ij}} \wedge \boldsymbol{n_{ij}})_{\boldsymbol{m\times n}}$

Example 2: Let $\boldsymbol{M} = \begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} \end{bmatrix}$ and $\boldsymbol{N} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} & \mathbf{1} \end{bmatrix}$ then find $\boldsymbol{M} \vee \boldsymbol{N}$ and $\boldsymbol{M} \wedge \boldsymbol{N}$.

Solution: $M \vee N = \begin{bmatrix} 1 \vee 0 & 0 \vee 0 & 1 \vee 1 \\ 0 \vee 1 & 1 \vee 0 & 0 \vee 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

$$M \wedge N = \begin{bmatrix} 1 \wedge 0 & 0 \wedge 0 & 1 \wedge 1 \\ 0 \wedge 1 & 1 \wedge 0 & 0 \wedge 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Note: If R is a relation from a finite set A to a finite set B then its relation matrix M_R is a Boolean matrix

If R and S are relations from a finite set A to a finite set B with respective relation matrices M_R and M_S then

$$\boldsymbol{M_{R\cup S} = M_R \vee M_S}$$

$$\boldsymbol{M_{R\cap S} = M_R \wedge M_S}$$

Boolean Product

The ***Boolean product*** of the Boolean matrices $P = (p_{ij})_{m\times n}$ and $Q = (q_{ij})_{n\times p}$, denoted by $\boldsymbol{P \circ Q}$ (or $\boldsymbol{P \odot Q}$) is the matrix $R = (r_{ij})_{m\times p}$, where

$$\boldsymbol{r_{ij} = (p_{i1} \wedge q_{1j}) \vee (p_{i2} \wedge q_{2j}) \vee \dots \vee (p_{in} \wedge q_{nj}) = \bigvee_{k=1}^{n} (p_{ik} \wedge q_{kj})},$$

$$i = 1,2,3,\dots,m \;;\; j = 1,2,3,\dots,p$$

where $p_{ik} \wedge q_{kj}$ and $\bigvee_{k=1}^{n}$ indicate **bit-ANDing** and **bit- ORing** respectively.

Example 3: let $\boldsymbol{P = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}}$ and $\boldsymbol{= \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 0 \end{bmatrix}}$, then find $\boldsymbol{P \circ Q}$ and $\boldsymbol{Q \circ P}$, if it is defined.

Solution: (i) Since the number of columns in P is equal to the number of rows of Q, $P \circ Q$ is defined.

$$P \circ Q = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \circ \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} (1 \wedge 1) \vee (0 \wedge 1) \vee (1 \wedge 0) & (1 \wedge 0) \vee (0 \wedge 1) \vee (1 \wedge 0) \\ (0 \wedge 1) \vee (1 \wedge 1) \vee (0 \wedge 0) & (0 \wedge 0) \vee (1 \wedge 1) \vee (0 \wedge 0) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

(i) Since the number of columns in Q is equal to the number of rows of P, $Q \circ P$ is defined.

$$Q \circ P = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 0 \end{bmatrix} \circ \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} (1 \wedge 1) \vee (0 \wedge 0) & (1 \wedge 0) \vee (0 \wedge 1) & (1 \wedge 1) \vee (0 \wedge 0) \\ (1 \wedge 1) \vee (1 \wedge 0) & (1 \wedge 0) \vee (1 \wedge 1) & (1 \wedge 1) \vee (1 \wedge 0) \\ (0 \wedge 1) \vee (0 \wedge 0) & (0 \wedge 0) \vee (0 \wedge 1) & (0 \wedge 1) \vee (0 \wedge 0) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Boolean Power of a Boolean Matrix

Let M be an $m \times m$ boolean matrix and n is any nonnegative integer. The $\boldsymbol{n^{th}}$ **Boolean power of $\boldsymbol{M}$**, denoted by M^n or (sometimes $M^{[n]}$) defined recursively as follows:

$\boldsymbol{M^0 = I_m}$ (where I_m is the $m \times m$ identity matrix) $\boldsymbol{M^n = M^{n-1} \circ M,\ \ if\ n \geq 1}$

Example 4: if $\boldsymbol{M = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}}$ then compute $\boldsymbol{M^2}$ and $\boldsymbol{M^3}$

Solution:

$$M^2 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \qquad \text{(do it!)}$$

$$M^3 = M^2 \circ M = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \circ \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \text{ (do it!)}$$

Note: If A, B and C are Boolean matrices then

i) $A \vee A = A$, $A \wedge A = A$
ii) $A \vee B = B \vee A$, $A \wedge B = B \wedge A$
iii) $A \vee (B \vee C) = (A \vee B) \vee C$, $A \wedge (B \wedge C) = (A \wedge B) \wedge C$
iv) $A \wedge (B \vee C) = (A \wedge B) \vee (A \wedge C)$, $A \vee (B \wedge C) = (A \vee B) \vee (A \vee C)$
v) $A \circ (B \circ C) = (A \circ B) \circ C$

Matrix of the composite relation

Let $A = \{a_1, a_2, \dots, a_m\}, B = \{b_1, b_2, \dots, b_n\}$ and $C = \{c_1, c_2, \dots, c_p\}$

Let R be a relation from A to B and S be a relation from B to C then the relation matrices M_R and M_S are $m \times n$ and $n \times p$ (zero- one) matrices. The relation matrix of the relation $R \circ S$ can be obtained from the matrices M_R and M_S in the following manner

$$\boldsymbol{M_{R \circ S} = M_R \circ M_S}$$

Example 5: Let $\boldsymbol{A = \{a_1, a_2, a_3, a_4, a_5\}}$ and $\boldsymbol{R, S}$ be relations on $\boldsymbol{A}$ given by

$$\boldsymbol{R = \{(a_1, a_2), (a_2, a_2), (a_3, a_4)\}}$$

$$\boldsymbol{S = \{(a_1, a_3), (a_2, a_5), (a_3, a_1), (a_4, a_2)\}}$$

Compute $\boldsymbol{M_{R \circ S}}$ and $\boldsymbol{M_{S \circ R}}$

Solution:

To compute $M_{\boldsymbol{R \circ S}}$ and $M_{\boldsymbol{S \circ R}}$, we first write the relation matrices of R and S.

$$M_R = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad M_S = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Now, $M_{R \circ S} = M_R \circ M_S$

$$= \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \circ \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$M_{S \circ R} = M_S \circ M_R$

$$= \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \circ \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Note: Note that $R \circ S = \{(a_1, a_5), (a_2, a_5), (a_3, a_2)\}$

$$S \circ R = \{(a_1, a_4), (a_3, a_2), (a_4, a_2)\}$$

Converse of a relation

Let R be a relation from a set A to a set B . A relation $\overline{\boldsymbol{R}}$ from B to A is called the **converse of** . The ordered pairs of $\overline{R}$ are obtained by interchanging the elements of the ordered pairs of R. That is for $x, y \in R,\ x\, R\, y \iff y\, \overline{R}\, x$ $i.e.$, $(x, y) \in R$ iff $(y, x) \in \overline{R}$. From the definition it follows that $\overline{\overline{R}} = R$. The relation matrix $M_{\overline{R}}$ of $\overline{R}$ is obtained by interchanging the rows and columns of M_R. That is $M_{\overline{R}}$ is the transpose of M_R. That is, $\boldsymbol{M_{\overline{R}}} = (\boldsymbol{M_R})'$.Further, the graph of $\overline{R}$ is obtained from the graph of R by reversing the arrow on each arc.

Note: Some books define converse of $\boldsymbol{R}$ as the inverse of $\boldsymbol{R}$ and denote it as $\boldsymbol{R^{-1}}$. That is $\boldsymbol{R^{-1} = \{(y, x) | (x, y) \in R\}}$

Converse of a composite relation

Theorem: If $\boldsymbol{R}$ is a relation from $\boldsymbol{A}$ to $\boldsymbol{B}$ and $\boldsymbol{S}$ is a relation from $\boldsymbol{B}$ to $\boldsymbol{C}$ then $\boldsymbol{\overline{R \circ S} = \overline{S} \circ \overline{R}}$.

Proof: Now, $R \circ S$ is a relation from A to C and its converse $\overline{R \circ S}$ is a relation from C to A. Since $\overline{S}$ is a relation form C to A, $\overline{R}$ from B to A ; $\overline{S} \circ \overline{R}$ is a relation from C to A.

Let $x \in A$ and $z \in C$ be arbitrary elements. Suppose that there is an element $y \in B$ such that xRy and ySz. Then $x(R \circ S)z$ and so $z\left(\overline{R \circ S}\right)x$.

Further, $x\ R\ y$ and $y\ S\ z \implies z\ \overline{S}\ y$ and $y\ \overline{R}\ x \implies z\left(\overline{S} \circ \overline{R}\right)x$.

For any $x \in A$ and $z \in C$, $z\left(\overline{R \circ S}\right)x \implies x(R \circ S)z \implies \exists\ y \in B$ such that $x\ R\ y$ and $y\ S\ z \implies z\left(\overline{S} \circ \overline{R}\right)x$ (shown as above) .Therefore, $\overline{R \circ S} \subseteq \overline{S} \circ \overline{R}$

Conversely for any $x \in A$ and $z \in C$, $z\left(\overline{S} \circ \overline{R}\right)x \implies \exists\ y \in B$ such that $z\ \overline{S}y$ and $y\ \overline{R}\ x \implies x\ R\ y$ and $y\ Sz \implies x(R \circ S)\ z \implies z\ \left(\overline{R \circ S}\right)\ x$.Therefore, $\overline{S} \circ \overline{R} \subseteq \overline{R \circ S}$ .Thus, $\overline{R \circ S} = \overline{S} \circ \overline{R}$. Hence the result

The same rule can be expressed in terms of the relation matrices by saying that the **transpose of $M_{R \circ S}$ is the same as the matrix $M_{\overline{S} \circ \overline{R}}$** .The matrix $M_{\overline{S} \circ \overline{R}}$ can be obtained from the matrices $M_{\overline{S}}$ and $M_{\overline{R}}$, which in turn can be obtained from M_S and M_R.

Note: $\boldsymbol{M_{\overline{R \circ S}} = (M_{R \circ S})' = (M_R \circ M_S)' = M_S' \circ M_R' = M_{\overline{S}} \circ M_{\overline{R}} = M_{\overline{S} \circ \overline{R}}}$.

Example 6: If $\boldsymbol{M_R = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}}$ **and** $\boldsymbol{M_S = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}}$

Then show that $\boldsymbol{M_{\overline{R \circ S}} = M_{\overline{S}} \circ M_{\overline{R}}}$

Solution:

$$M_{R \circ S} = M_R \circ M_S = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

Now, $M_{\overline{R \circ S}} = (M_{R \circ S})' = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

We have, $M_{\overline{S}} = (M_S)' = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

$M_{\overline{R}} = (M_R)' = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$

Now, $M_{\overline{S}} \circ M_{\overline{R}} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \circ \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

Thus, $M_{\overline{R \circ S}} = M_{\overline{S} \circ \overline{R}}$.

The following hold for any relations R and S.

Theorem 1: For any relations R and S, the following hold:

(i) $\overline{\overline{R}} = R$

(ii) $R = S \Leftrightarrow \overline{R} = \overline{S}$

(iii) $R \subseteq S \Leftrightarrow \overline{R} \subseteq \overline{S}$

(iv) $\overline{R \cup S} = \overline{R} \cup \overline{S}$

(v) $\overline{R \cap S} = \overline{R} \cap \overline{S}$

Transitive closure: Consider the following distinct relations R_1, R_2, R_3 and R_4 on a set $A = \{a, b, c\}$

$$R_1 = \{(a,b), (a,c), (c,b)\}$$

$$R_2 = \{(a,b), (b,c), (c,a)\}$$

$$R_3 = \{(a,b),(b,c),(c,c)\}$$

$$R_4 = \{(a,b),(b,a),(c,c)\}$$

Denoting the composition of a relation R by itself as

$R \circ R = R^2$, $R \circ R \circ R = R^2 \circ R = R^3, \ldots,$ $R^{m-1} \circ R = R^m$

Now, $R_1^2 = \{(a,b)\}, R_1^3 = R_1^2 \circ R_1 = \emptyset$, $R_1^4 = R_1^3 \circ R_1 = \emptyset$, …

$R_2^2 = \{(a,c),(b,a),(c,b)\}$, $R_2^3 = R_2^2 \circ R_2 = \{(a,a),(b,b),(c,c)\}$
$R_2^4 = R_2^3 \circ R_2 = R_2$, $R_2^5 = R_2^4 \circ R_2 = R_2 \circ R_2 = R_2^2 = \{(a,c),(b,a),(c,b)\}$

$R_3^2 = \{(a,c),(b,c),(c,c)\}$, $R_3^3 = R_3^2 \circ R_3 = R_3^2$,
$R_3^4 = R_3^3 \circ R_3 = R_2^3 \circ R_3 = R_3^3 = R_3^2$
$R_4^2 = \{(a,a),(b,b),(c,c)\}, R_4^3 = R_4^2 \circ R_4 = R_4$, $R_4^4 = R_4^3 \circ R_4 = R_4 \circ R_4 = R_4^2$

Given a finite set A, containing n elements and a relation R on A, we can interpret $R^m, (m = 1, 2, 3, .., \ldots)$ in terms of its graph. From the examples given above, it is possible to say that ***there are at most n distinct powers of R, and R^m , $m > n$ can be expressed in terms of $R, R^2, \ldots, R^n$***. We now construct a relation on A given by

$$\boldsymbol{R^+ = R \cup R^2 \cup R^3 \cup \ldots \cup R^n}$$

This construction requires only a finite number of powers of R and these computations can easily be performed by using M_R, (the relation matrix of R) and Boolean multiplication of these matrices. Now,

$R_1^+ = R_1 \cup R_1^2 \cup R_1^3 = R_1$

$R_2^+ = R_2 \cup R_2^2 \cup R_2^3$
$= \{(a,b),(b,c),(c,a),(a,c),(b,a),(c,b),(a,a),(b,b),(c,c)\}$

$R_3^+ = R_3 \cup R_3^2 \cup R_3^3 = R_3 \cup R_3^2$

$= \{(a,b),(b,c),(c,c),(a,c)\}$

$R_4^+ = R_4 \cup R_4^2 \cup R_4^3 = R_4 \cup R_4^2$

$= \{(a,b),(b,a),(c,c),(a\ ,a),(b,b),(c,c)\}$

Notice that the relations R_1^+, R_3^+ and R_4^+ are all transitive and that

$R_i \subseteq R_i^+,\ i = 1,2,3,4$

Further, we see that R_i^+ is obtained from R_i $,(i = 1,2,3,4)$ by adding only those ordered pairs to R_i such that R_i^+ is transitive.

The following is the general definition of R^+.

Transitive Closure: Let A be a finite set with n elements and R be a relation on A. The relation

$$\boldsymbol{R^+ = R \cup R^2 \cup R^3 \cup \dots \cup R^n}$$

on A is called the **transitive closure** of R on A.

The properties of transitive closure of a relation on a finite set are highlighted in the following theorem

Theorem 2: Let $\boldsymbol{R}$ be a relation on a finite set $\boldsymbol{A}$

(i) The *transitive closure $\boldsymbol{R^+}$ of $\boldsymbol{R}$ is transitive*
(ii) If $\boldsymbol{P}$ is any other transitive relation on $\boldsymbol{A}$ such that $\boldsymbol{R \subseteq P}$ then $\boldsymbol{R^+ \subseteq P}$. That is *$\boldsymbol{R^+}$ is the smallest transitive relation containing $\boldsymbol{R}$.*

Theorem 3: Let $\boldsymbol{R}$ be a relation on a set with $\boldsymbol{n}$ elements with its relation matrix $\boldsymbol{M_R}$. Let $\boldsymbol{R^+}$ be the transitive closure of $\boldsymbol{R}$ .Then
$\boldsymbol{M_{R^+} = M_R \vee M_{R^2} \vee M_{R^3} \vee \dots \vee M_{R^n} = M_R \vee (M_R)^2 \vee (M_R)^3 \vee \dots \vee (M_R)^n}$

For an illustration see P5

Applications of transitive closure of a relation

Transitive closures of relations have important applications in the following areas: Net works, synthetic analysis, fault detection and diagnosis in switching circuits.

P1:

Let $\boldsymbol{R}$ and $\boldsymbol{S}$ be two relations on N ,the set of Natural numbers

$$\boldsymbol{R} = \{(\boldsymbol{x}, \boldsymbol{2x}) |\ \boldsymbol{x} \in \boldsymbol{N}\} \quad , \quad \boldsymbol{S} = \{(\boldsymbol{x}, \boldsymbol{7x}) |\ \boldsymbol{x} \in \boldsymbol{N}\}$$

Find $\boldsymbol{R} \circ \boldsymbol{S}, \boldsymbol{R} \circ \boldsymbol{R}, \boldsymbol{R} \circ \boldsymbol{R} \circ \boldsymbol{R}$ and $\boldsymbol{R} \circ \boldsymbol{S} \circ \boldsymbol{R}$

Solution: We have $x \xrightarrow{\text{R}} 2x$ and $x \xrightarrow{\text{S}} 7x$

(i) $x \xrightarrow{\text{R}} 2x \xrightarrow{\text{S}} 14x$ and $R \circ S = \{(x, 14x) |\ x \in \boldsymbol{N}\}$

Further, $x \xrightarrow{\text{S}} 7x \xrightarrow{\text{R}} 14x$. Therefore, $S \circ R = \{(x, 14x) |\ x \in \boldsymbol{N}\} = R \circ S$

(ii) $x \xrightarrow{\text{R}} 2x \xrightarrow{\text{R}} 4x$. Therefore, $R \circ R = \{(x, 4x) |\ x \in \boldsymbol{N}\}$

(iii) $R \circ R \circ R = \{(x, 8x) |\ x \in \boldsymbol{N}\}$

(iv) $x \xrightarrow{R \circ S} 14x \xrightarrow{\text{R}} 28x$. Therefore, $R \circ S \circ R = \{(x, 28x) |\ x \in \boldsymbol{N}\}$

P2:

Let $\boldsymbol{E}$ be the identity relation on a set $\boldsymbol{A}$ and $\boldsymbol{R}$ be any relation on $\boldsymbol{A}$. Show that $\boldsymbol{S} = \boldsymbol{E} \cup \boldsymbol{R} \cup \overline{\boldsymbol{R}}$ is a compatibility relation.

Solution: Notice that S is a relation on A. For any $x \in A$, we have $(x, x) \in E$ and so $(x, x) \in S$. Thus S is reflexive.

If for $x, y \in A, x \neq y,\ (x, y) \in S$ then (x, y) belongs to either R or $\overline{R}$. If $(x, y) \in R$ then $(y, x) \in \overline{R}$ and so $(y, x) \in S$. A similar argument holds when $(x, y) \in \overline{R}$. This proves that $(y, x) \in S$, whenever $(x, y) \in S$. Thus S is symmetric.

Therefore, S is reflexive and symmetric. This proves S is a compatibility relation on A.

P3:

Given the relation matrix $\boldsymbol{M_R}$ of a relation $\boldsymbol{R}$ on the set $\boldsymbol{A = \{a, b, c\}}$. Find the relation matrices of $\boldsymbol{\bar{R}, R \circ R \circ R}$ and $\boldsymbol{R \circ \bar{R}}$ on the set $\boldsymbol{A}$, where

$$\boldsymbol{M_R = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}}$$

Find the matrix of the transitive closure of $\boldsymbol{R}$

Solution: We have that R is a relation on the set $A = \{a, b, c\}$ and

$$M_R = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

Now, $M_{\bar{R}} = (M_R)' = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$

$$M_{R^2} = M_{R \circ R} = M_R \circ M_R = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \circ \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$M_{R^3} = M_{R \circ R \circ R} = M_{R^2 \circ R} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \circ \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = M_{R^2}$$

$$M_{R \circ \bar{R}} = M_{R \circ \bar{R}} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \circ \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

The matrix of the transitive closure R^+ of R is given by

$$M_{R^+} = M_R \vee M_{R^2} \vee M_{R^3} = M_R \vee M_{R^2}$$

That is $M_{R^+} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \vee \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

P4:

Two equivalence relations R and S are given by the relation matrices M_R and M_S, where $M_R = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $M_S = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

a) **Show that $R \circ S$ is not an equivalence relation**

b) **Obtain equivalence relations R_1 and R_2 on the set $A = \{1, 2, 3\}$ such that $R_1 \circ R_2$ is an equivalence relation.**

Solution:

a) We have $M_{R \circ S} = M_R \circ M_S = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \circ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

$R \circ S$ is not an equivalence relation, because $M_{R \circ S}$ is not a symmetric matrix.

b) Let R_1 be the identity relation on $A = \{1, 2, 3\}$ then $M_{R_1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Let R_2 be the universal relation on $A = \{1, 2, 3\}$ then $M_{R_2} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

It is known that the equality relation on a finite set A is the smallest equivalence relation on A and the universal relation on A is the largest equivalence relation on A. We see that

$$M_{R_1 \circ R_2} = M_{R_1} \circ M_{R_2} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \circ \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = M_{R_2}$$

This shows that $R_1 \circ R_2$ is an equivalence relation.

Note: There are five equivalence relations on the set $A = \{a, b, c\}$. They are corresponding to the following partitions

$P_1 = \{\{a\},\{b\},\{c\}\}$, $P_2 = \{\{a,b,c\}\}$, $P_3 = \{\{a\},\{b,c\}\}$, $P_4 = \{\{b\},\{a,c\}\}$

$P_5 = \{\{c\},\{a,b\}\}$.

Let R_1, R_2, R_3, R_4 and R_5 be the equivalence relations corresponding to the partitions P_1, P_2, P_3, P_4 and P_5 respectively.

$R_1 \circ R$ is an equivalence relation where $R \in \{R_1, R_2, R_3, R_4, R_5\}$

$R_2 \circ R$ is an equivalence relation where $R \in \{R_1, R_2, R_3, R_4, R_5\}$

Further, notice that $R_i \circ R_j$ is not an equivalence relation where $i, j \in \{3, 4, 5\}, i \neq j$

P5:

Find the matrix of the transitive closure of the relation $\boldsymbol{R}$ with

$$\boldsymbol{M_R} = \begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{1} & \mathbf{1} & \mathbf{0} \end{bmatrix}$$

Solution:

$M_{R^2} = M_{R \circ R} = M_R \circ M_R = (M_R)^2$ and $M_{R^3} = (M_R)^3$

Now, $(M_R)^2 = M_R \circ M_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \circ \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$

$$(M_R)^3 = (M_R)^2 \circ M_R = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \circ \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} = {M_R}^2$$

Now, we have

$$M_{R^+} = M_R \vee (M_R)^2 \vee (M_R)^3$$

$$= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \vee \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$