5.4

Linear Systems with Random Inputs

Mathematically, a system is a functional relationship between the input $x(t)$ and the output $y(t)$. Usually this relationship is written as $y(t) = f[x(t)]$, $-\infty < t < \infty$.

If we assume that $x(t)$ represents a sample function of a random process $\{X(t)\}$, the system produces an output or response $y(t)$ and the ensemble of the output functions forms a random process $\{Y(t)\}$. The process $\{Y(t)\}$ can be considered as the output of the system or transformation f with $\{X(t)\}$ as the input. The system is completely specified by the operator f.

We recall that $X(t)$ actually means $X(s, t)$, where $s \in S$ (sample space). If the system operates only on the variable t treating s as a parameter, it is called a ***deterministic system***. If the system operates on both t and s, it is called ***stochastic***. We shall consider only deterministic systems in our study.

Definitions: If $f[a_1X_1(t) \pm a_2X_2(t)] = a_1f[X_1(t)] \pm a_2f[X_2(t)]$, then f is called a **linear system**.

If $Y(t+h) = f[X(t+h)]$, where $Y(t) = f[X(t)]$, f is called a **time-invariant system** or $\boldsymbol{X(t)}$ **and** $\boldsymbol{Y(t)}$ **are said to form a *time-invariant system*.**

If the output $Y(t_1)$ at a given time $t = t_1$ depends only on $X(t_1)$ and not on any other past or future values of $X(t)$, then the system f is called a **memoryless system.**

If the value of the output $Y(t)$ at $t = t_1$ depends only on the past values of the input $X(t)$, $t \le t_1$, i.e., $Y(t_1) = f[X(t); t \le t_1]$, then the system is called a **causal system.**

System in the Form of Convolution

Very often in electrical systems, the output $Y(t)$ is expressed as a convolution of the input $X(t)$ with a system weighting function $h(t)$, i.e., the input-output relationship will be of form

$$Y(t) = \int_{-\infty}^{\infty} h(u) X(t-u) du \qquad \text{... (1)}$$

Unit Impulse Response of the System

The unit impulse function $\delta(t-a)$ is defined as

$$\delta(t-a) = \begin{cases} \frac{1}{\epsilon} & if \;\; a - \frac{\epsilon}{2} \le t \le a + \frac{\epsilon}{2} \\ 0 & otherwise \end{cases}$$

where $\epsilon \longrightarrow 0$.

Let $\phi(t)$ be some bounded function of t such that it can be considered as a constant in a small interval of length ϵ.

Then $\int_{-\infty}^{\infty} \phi(t)\delta(t-a)dt = \int_{a-\frac{\varepsilon}{2}}^{a+\frac{\varepsilon}{2}} \phi(t)\frac{1}{\varepsilon}dt$

$$= \frac{\phi(a)}{\varepsilon}\int_{a-\frac{\varepsilon}{2}}^{a+\frac{\varepsilon}{2}} dt = \frac{\phi(a)}{\varepsilon}\varepsilon = \phi(a)$$

Thus, $\int_{-\infty}^{\infty} \phi(t)\delta(t-a)dt = \phi(a)$.

If we take $a = 0$, we get

$$\int_{-\infty}^{\infty} \phi(t)\delta(t)dt = \phi(0) \qquad \text{... (2)}$$

Put $X(t) = \delta(t)$ in (1), then

$$\begin{aligned} Y(t) &= \int_{-\infty}^{\infty} h(u)\delta(t-u)du \\ &= \int_{-\infty}^{\infty} h(t-u)\delta(u)du \;\; \text{(by the property of the convolution)} \\ &= h(t-0) \text{, by (2)} \\ &= h(t) \end{aligned}$$

Thus if the input of the system is the unit impulse function, then the output or response is the system weighting function. Hence the system weighting function $h(t)$ will be hereafter called ***unit impulse response function***.

Properties

1. *If a system is such that its input $X(t)$ and its output $Y(t)$ are related by a convolution integral, i.e., if $Y(t)=\int_{-\infty}^{\infty} h(u)X(t-u)du$, then the system is a linear time-invariant system.*

Proof: Let $X(t) = a_1X_1(t) + a_2X_2(t)$. Then

$$\begin{aligned} Y(t) &= \int_{-\infty}^{\infty} h(u)\left[a_1X_1(t-u)+a_2X_2(t-u)\right]du \\ &= a_1Y_1(t)+a_2Y_2(t) \end{aligned}$$

Therefore, the system is linear. If $X(t)$ is replaced by $X(t+h)$, then

$$\int_{-\infty}^{\infty} h(u)X\left(\overline{t+h}-u\right)du = Y(t+h)$$

Therefore, the system is time-invariant.

Note: If $h(t)$ is absolutely integrable, viz., $\int_{-\infty}^{\infty}|h(t)|dt<\infty$, then the system is said to be *stable* in the sense that every bounded input gives a bounded output.

In addition, if $h(t) = 0$, when $t < 0$, the system is said to be **causal.**

2. *If the input to a time-invariant, stable linear system is a WSS process, then the output will also be a WSS process.* (For proof **see P1**)
3. *If $\{X(t)\}$ is a WSS process and if $Y(t)=\int_{-\infty}^{\infty} h(u)X(t-u)du$, then*
   (i) $R_{xy}(\tau) = R_{xx}(\tau) * h(-\tau)$ *and*
   (ii) $R_{yy}(\tau) = R_{xy}(\tau) * h(\tau)$, *where $*$ denotes convolution. Also*
   (iii) $S_{xy}(\omega) = S_{xx}(\omega)H^*(\omega)$ *and*
   (iv) $S_{yy}(\omega) = S_{xx}(\omega)|H(\omega)|^2$

(For proof **see P2**)

4. *If $\{X(t)\}$ is a WSS process and if $Y(t)=\int_{-\infty}^{\infty} h(u)X(t-u)du$, then*

$$R_{yy}(\tau) = R_{xx}(\tau) * K(\tau)$$

where $K(t) = h(t)h(-t) = \int_{-\infty}^{\infty} h(u)h(t+u)\,du$ (For proof **see P3**)

5. *The power spectral densities of the input and output processes in the system are connected by the relation*

$$S_{yy}(\omega) = |H(\omega)|^2 S_{xx}(\omega),$$

where $H(\omega)$ is the Fourier transform of unit impuse response function $h(t)$.

(For proof **see P4**)

Example 1: The short-time moving average of a process $\{X(t)\}$ is defined as $Y(t) = \frac{1}{T}\int_{t-T}^{t} X(s)\,ds$. Prove that $X(t)$ and $Y(t)$ are related by means of a convolution type integral. Find the unit impulse response of the system also.

Solution: We have $Y(t) = \frac{1}{T}\int_{t-T}^{t} X(s)\,ds$ ... (1)

Putting $s = t - u$ and treating t as a parameter, (1) becomes

$$Y(t) = \frac{1}{T}\int_{0}^{T} X(t-u)\,du \quad \text{... (2)}$$

Let us define the unit impulse response of the system as follows:

$$h(t) = \begin{cases} \frac{1}{T} & , \quad for \;\; 0 \le t \le T \\ 0 & , \quad otherwise \end{cases}$$

Then (2) can be expressed as

$$Y(t) = \int_{-\infty}^{\infty} h(u)X(t-u)\,du$$

which is a convolution type integral.

Example 2: If the input $x(t)$ and the output $y(t)$ are connected by the differential equation $T\frac{dy(t)}{dt} + y(t) = x(t)$, then prove that they can be related by means of a convolution type integral. Assume that $x(t)$ and $y(t)$ are zero for $t \le 0$.

Solution: The given differential equation $y'(t) + \frac{1}{T}y(t) = \frac{1}{T}x(t)$ is a linear equation. Its solution is

$$y(t)e^{\frac{t}{T}} = \int \frac{1}{T}x(u)e^{\frac{u}{T}}du + c$$

i.e.,
$$y(t)e^{\frac{t}{T}} = \frac{1}{T}\int x(u)e^{-\frac{t-u}{T}}du + c$$

Since $y(0) = 0$,

$$y(t) = \frac{1}{T}\int_0^t x(u)e^{-\frac{t-u}{T}}du$$

(or)
$$y(t) = \frac{1}{T}\int_0^t x(t-u)e^{-\frac{u}{T}}du \qquad \dots (1)$$

Given:

$$x(t) = 0, \text{ for } t < 0$$

$$\therefore x(t-u) = 0, \text{ for } t < u$$

$\therefore$ (1) can be written as

$$y(t) = \frac{1}{T}\int_0^\infty x(t-u)e^{-\frac{u}{T}}du \qquad \dots (2)$$

Now if we define

$$h(t) = \begin{cases} \frac{1}{T}e^{-\frac{t}{T}} & , \quad for \;\; t \geq 0 \\ 0 & , \quad otherwise \end{cases}$$

(2) can be rewritten as

$$y(t) = \int_{-\infty}^{\infty} h(u)x(t-u)du$$

Hence the result.

Example 3: $\boldsymbol{X(t)}$ **is the input voltage to a circuit (system) and** $\boldsymbol{Y(t)}$ **is the output voltage.** $\boldsymbol{\{X(t)\}}$ **is a stationary random process with** $\boldsymbol{\mu_x = 0}$ **and** $\boldsymbol{R_{xx}(\tau) = e^{-\alpha|\tau|}}$**. Find** $\boldsymbol{\mu_y}$ **,** $\boldsymbol{S_{yy}(\omega)}$ **and** $\boldsymbol{R_{yy}(\tau)}$**, if the power transfer function is**

$$\boldsymbol{H(\omega) = \frac{R}{R + iL\omega}}$$

Solution: $Y(t) = \int_{-\infty}^{\infty} h(\alpha) X(t-\alpha) d\alpha$

$$\therefore E\{Y(t)\} = \int_{-\infty}^{\infty} h(\alpha) E\{X(t-\alpha)\} d\alpha = 0$$

Since $[E\{X(t-\alpha) = \mu_x = 0\}]$

$$S_{xx}(\omega) = \int_{-\infty}^{\infty} R_{xx}(\tau) e^{-i\omega\tau} d\tau$$

$$= \int_{-\infty}^{0} e^{\alpha\tau} e^{-i\omega\tau} d\tau + \int_{0}^{\infty} e^{-\alpha\tau} e^{-i\omega\tau} d\tau$$

$$= \left\{ \frac{e^{(\alpha - i\omega)\tau}}{\alpha - i\omega} \right\}_{-\infty}^{0} + \left\{ \frac{e^{-(\alpha + i\omega)\tau}}{-(\alpha + i\omega)} \right\}_{0}^{\infty}$$

$$= \frac{1}{\alpha - i\omega} + \frac{1}{\alpha + i\omega} = \frac{2\alpha}{\alpha^2 + \omega^2}$$

Now, $S_{yy}(\omega) = S_{xx}(\omega) |H(\omega)|^2$

$$= \frac{2\alpha}{\alpha^2 + \omega^2} \frac{R^2}{R^2 + L^2\omega^2}$$

$$= \frac{\{(2\alpha R^2 / (R^2 - L^2\alpha^2))\}}{\alpha^2 + \omega^2} + \frac{\{(2\alpha R^2 / (\alpha^2 - R^2/L^2))\}}{R^2 + L^2\omega^2} \quad \text{(by partial fractions)}$$

$$= \frac{2\alpha \left(\frac{R}{L}\right)^2}{\left(\frac{R}{L}\right)^2 - \alpha^2} \times \frac{1}{\alpha^2 + \omega^2} + \frac{2\alpha R^2 / L^2}{\alpha^2 - \left(\frac{R}{L}\right)^2} \times \frac{1}{\left(\frac{R}{L}\right)^2 + \omega^2}$$

$$= \lambda \frac{1}{\alpha^2 + \omega^2} + \mu \frac{1}{\left(\frac{R}{L}\right)^2 + \omega^2}, \text{ say}$$

$$\therefore R_{yy}(\tau)=\frac{\lambda}{2\pi}\int_{-\infty}^{\infty}\frac{e^{i\omega\tau}}{\alpha^2+\omega^2}d\omega+\frac{\mu}{2\pi}\int_{-\infty}^{\infty}\frac{e^{i\omega\tau}}{\left(\frac{R}{L}\right)^2+\omega^2}d\omega \quad \text{... (1)}$$

We can prove that, by contour integration technique,

$$\int_{-\infty}^{\infty}\frac{e^{iaz}}{z^2+b^2}dz=\frac{\pi}{b}e^{-ab};\ a>0 \quad \text{... (2)}$$

Using (2) in (1)

$$R_{yy}(\tau)=\frac{\left(\frac{R}{L}\right)^2}{\left(\frac{R}{L}\right)^2-\alpha^2}e^{-\alpha|\tau|}+\frac{\left(\frac{R}{L}\right)^2\alpha}{\alpha^2-\left(\frac{R}{L}\right)^2}e^{-\left(\frac{R}{L}\right)|\tau|}$$

Example 4: Given that $Y(t)=\frac{1}{2\varepsilon}\int_{t-\varepsilon}^{t+\varepsilon}X(\alpha)d\alpha$, where $\{X(t)\}$ is a WSS process, prove that $S_{yy}(\omega)=\frac{\sin^2\epsilon\omega}{\epsilon^2\omega^2}S_{xx}(\omega)$ and hence find the relation between $R_{xx}(\tau)$ and $R_{yy}(\tau)$.

Solution: Putting $\alpha=t-u$, we get $Y(t)=\frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon}X(t-u)du$

If we define $h(t)$ as follows

$$h(t)=\begin{cases}\frac{1}{2\epsilon} & , \quad |t|\le\epsilon\\ 0 & , \quad |t|>\epsilon\end{cases}$$

then $Y(t)=\int_{-\infty}^{\infty}h(u)X(t-u)du$

$\therefore S_{yy}(\omega)=|H(\omega)|^2S_{xx}(\omega)$, where $H(\omega)=F\{h(t)\}$

$$=\int_{-\varepsilon}^{\varepsilon}\frac{1}{2\varepsilon}e^{-i\omega t}dt=\frac{\sin\varepsilon\omega}{\varepsilon\omega}$$

i.e., $S_{yy}(\omega)=\frac{\sin^2\epsilon\omega}{(\epsilon\omega)^2}S_{xx}(\omega)$

$$\therefore R_{yy}(\tau) = F^{-1}\left\{\frac{\sin^2 \epsilon\omega}{\epsilon^2\omega^2} S_{xx}(\omega)\right\} \qquad \text{(inverse Fourier transformation)}$$

$$= F^{-1}\left\{\frac{\sin^2 \epsilon\omega}{\epsilon^2\omega^2}\right\} * R_{xx}(\tau) \qquad \dots (1)$$

We can prove that

if $$R(\tau) = \begin{cases} 1 - \frac{|\tau|}{2\epsilon} & , \quad if \ \ |\tau| \le 2\epsilon \\ 0 & , \quad if \ |\tau| > 2\epsilon \end{cases}$$

then $S(\omega) = 2\epsilon \ \frac{\sin^2 \epsilon\omega}{\epsilon^2\omega^2}$

$$\therefore F^{-1}\left(\frac{\sin^2 \epsilon\omega}{\epsilon^2\omega^2}\right) = \begin{cases} \frac{1}{2\epsilon}\left(1 - \frac{|\tau|}{2\epsilon}\right) & , \quad if \ \ |\tau| \le 2\epsilon \\ 0 & , \quad if \ |\tau| > 2\epsilon \end{cases} \qquad \dots (2)$$

Using (2) in (1)

$$R_{yy}(\tau) = \frac{1}{2\varepsilon}\int_{-2\varepsilon}^{2\varepsilon}\left(1 - \frac{|u|}{2\varepsilon}\right)R_{xx}(\tau - u)\,du$$