Mechanics of Materials-II

DEFLECTION OF BEAMS

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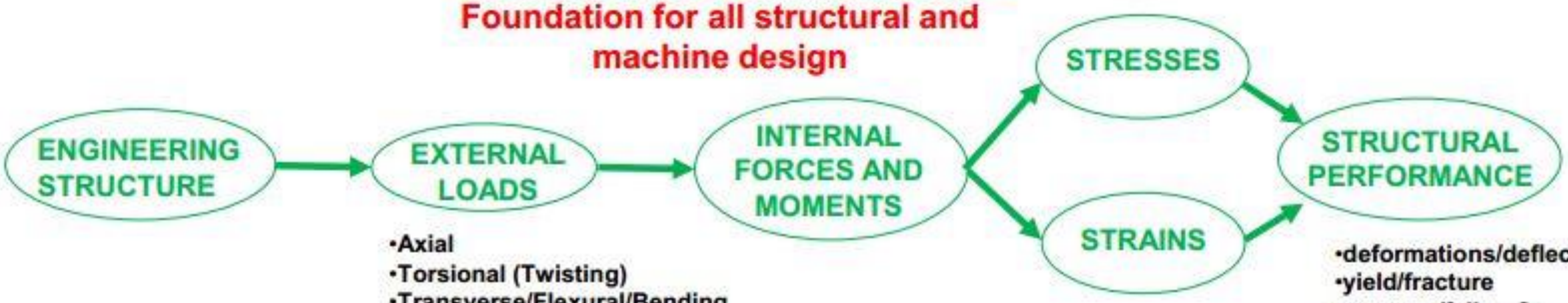
Introduction: MOM

Foundation for all structural and machine design



- Axial
- Torsional (Twisting)
- Transverse/Flexural/Bending
- Combined Loading

- deformations/deflections
- yield/fracture
- success/failure?

Structural Performance Considerations:

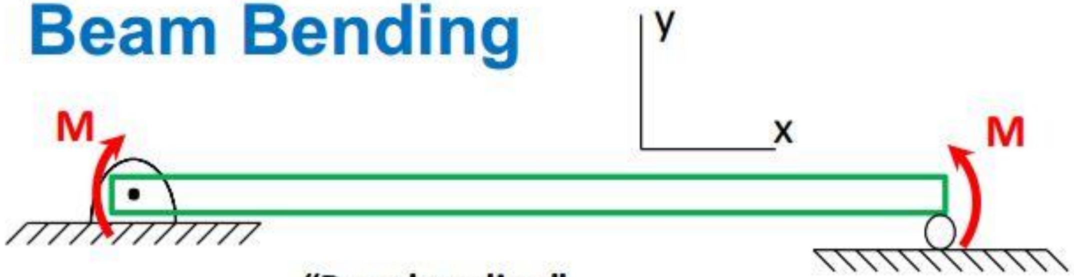
- no normal stress failure
- no shear stress failure
- no excessive deflections
- no buckling

There are numerous examples of requirements for an engineering structure to stay within a specified deflection under a given load.

Research some on your own

Pure Bending

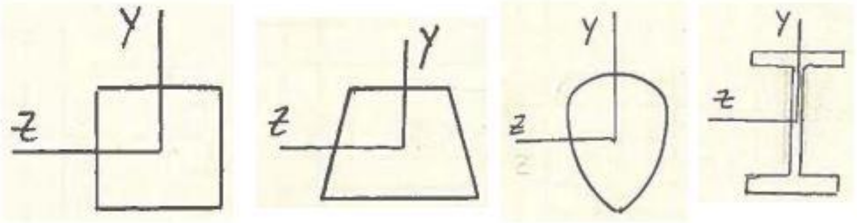
Beam Bending



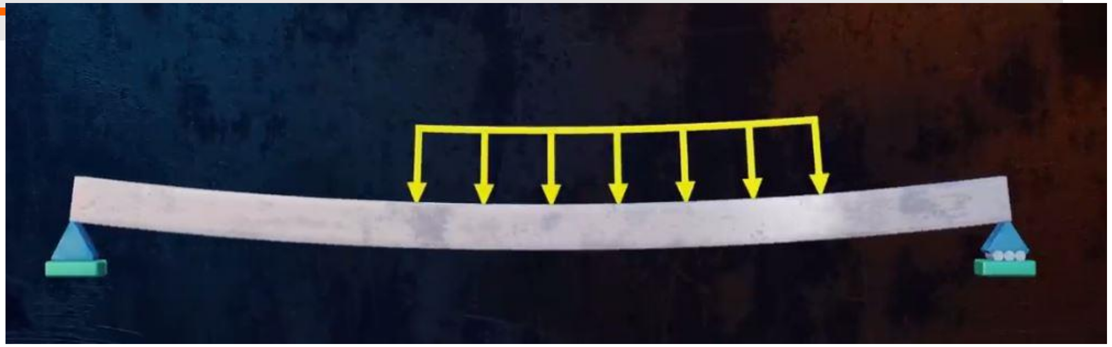
"Pure bending"
Flexure under constant bending moment
No shear force

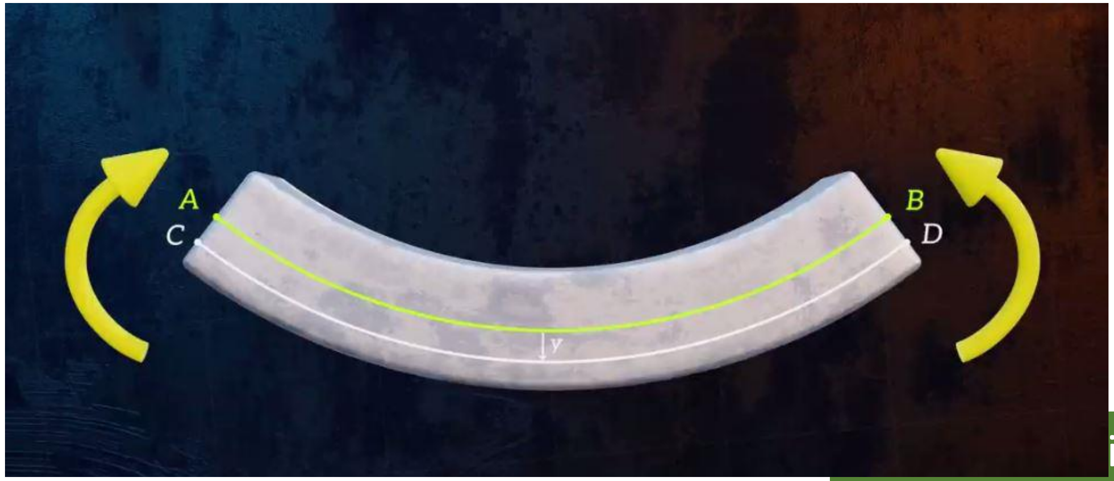
Assumptions:

- Symmetric about x-y plane (plane of bending)

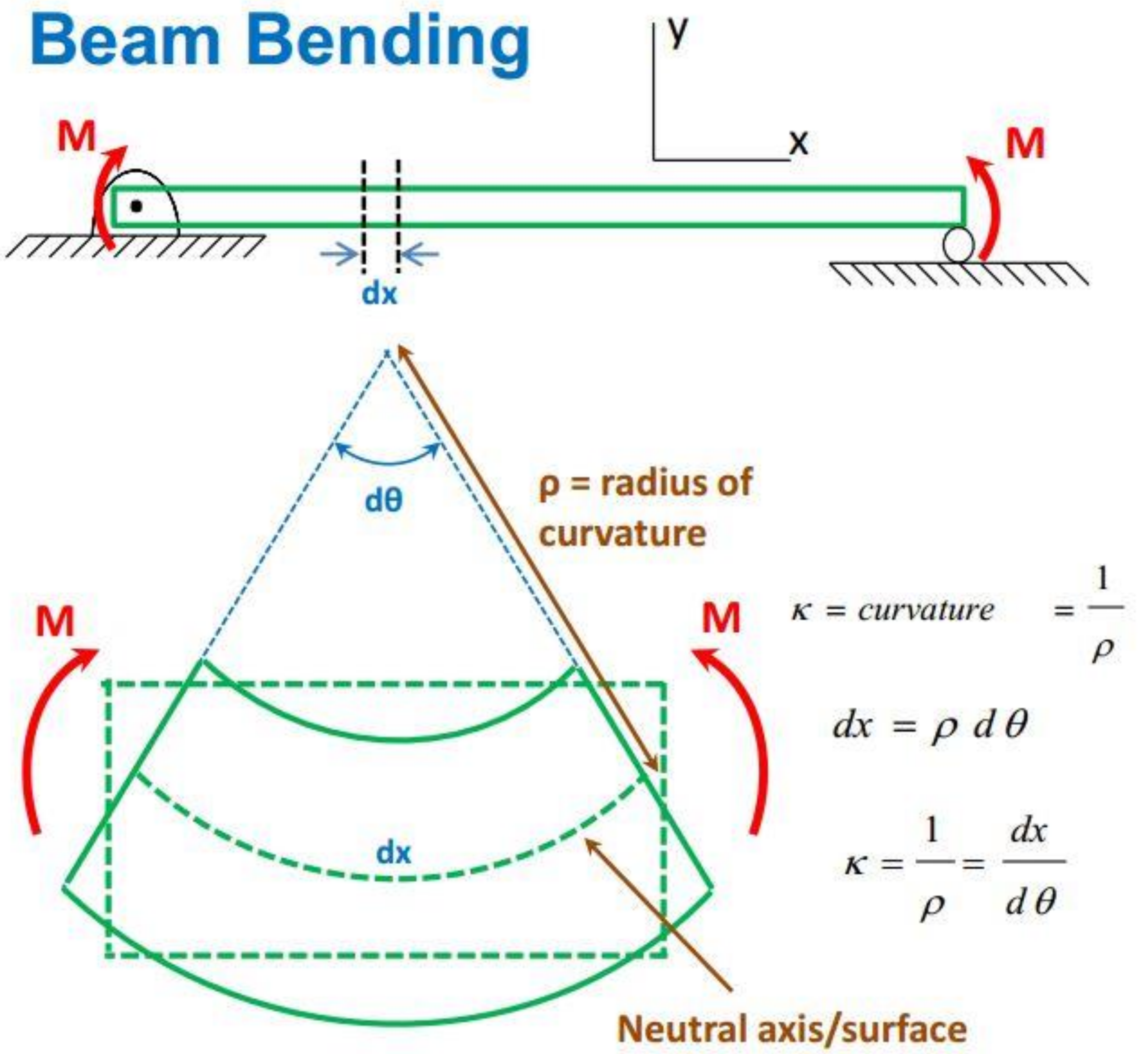


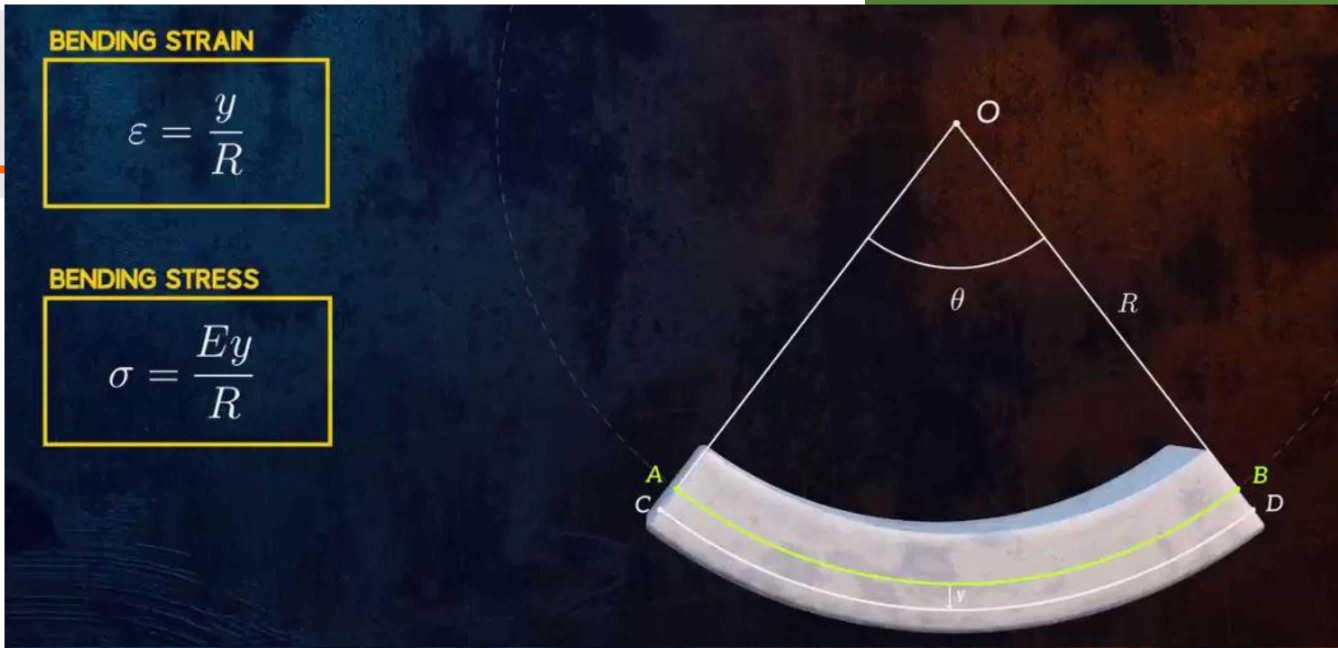
- Plane sections remain plane
- No twisting
- No buckling
- Small deflections

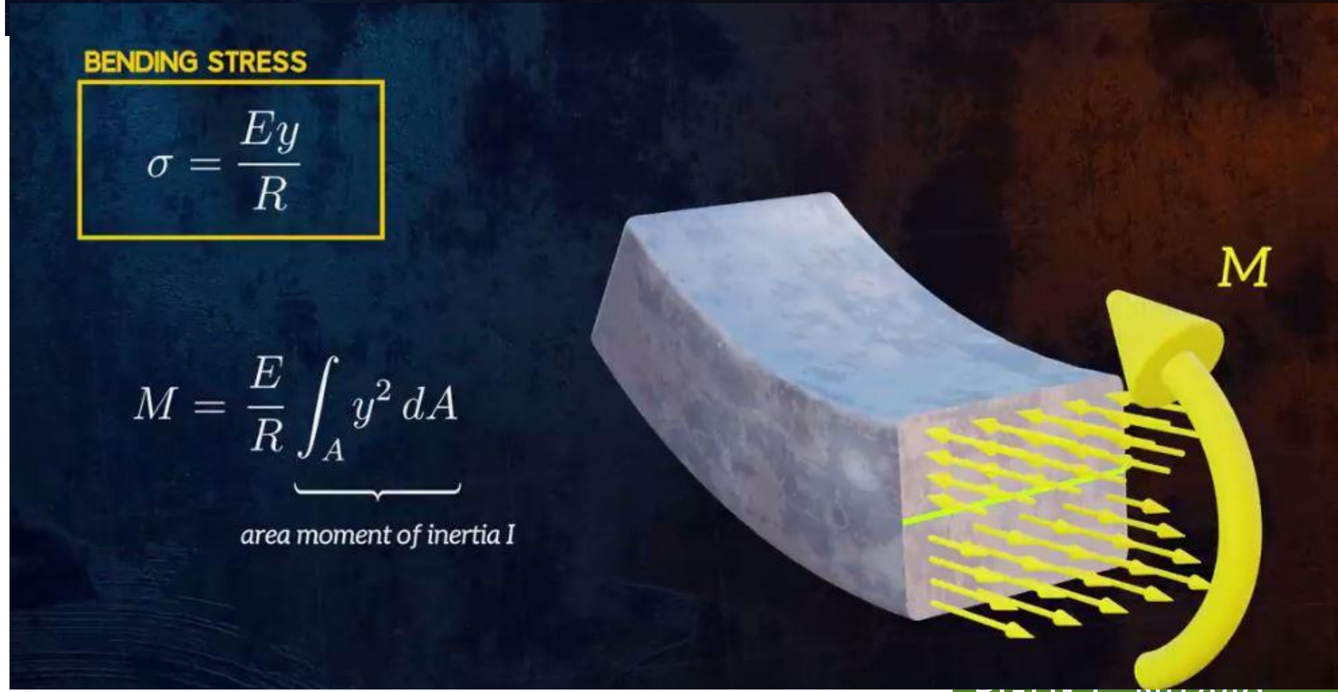




Pure Bending

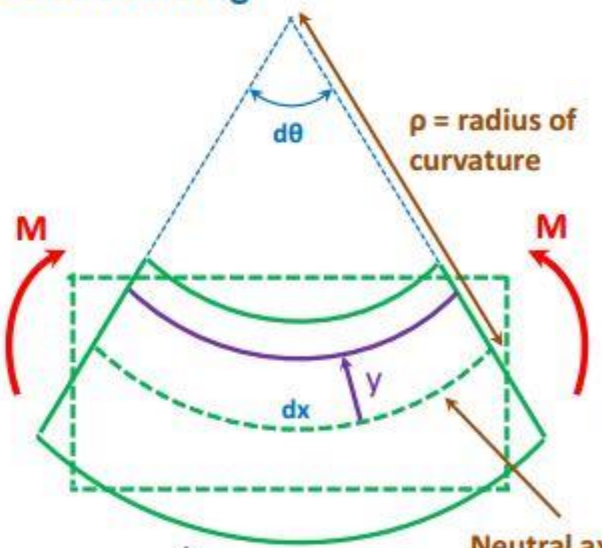






Pure Bending

Beam Bending



Recall: Normal Strain

Elongation per unit length

$$\varepsilon = \frac{\delta}{L}$$

$$\kappa = curvature = \frac{1}{\rho}$$

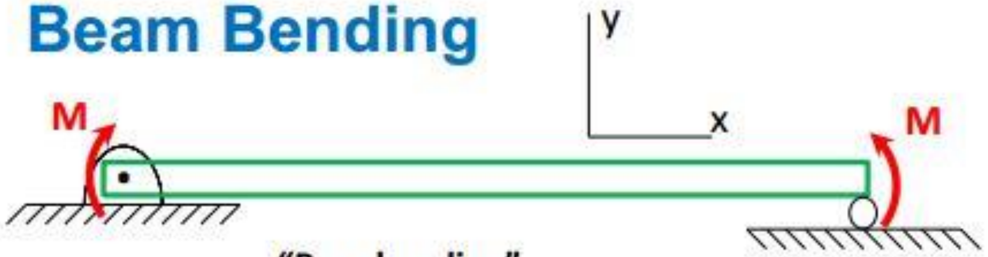
$$dx = \rho\, d\theta$$

$$(\rho - y)\,d\theta = dx - \frac{y}{\rho}dx \qquad d\theta = \frac{dx}{\rho}$$

Strain-Curvature Relationship

$$\varepsilon_x = \frac{-\frac{y}{\rho}\cancel{dx}}{\cancel{dx}} = -\frac{y}{\rho} = -\kappa\, y$$

Beam Bending



"Pure bending"

Flexure under constant bending moment

No shear force

Strain-Curvature Relationship

$$\varepsilon_x = -\frac{y}{\rho} = -\kappa\, y$$

Strain Sign Convention

(+) elongation

(-) shortening

Strain is proportional to curvature and varies linearly with distance, y, from the neutral axis.

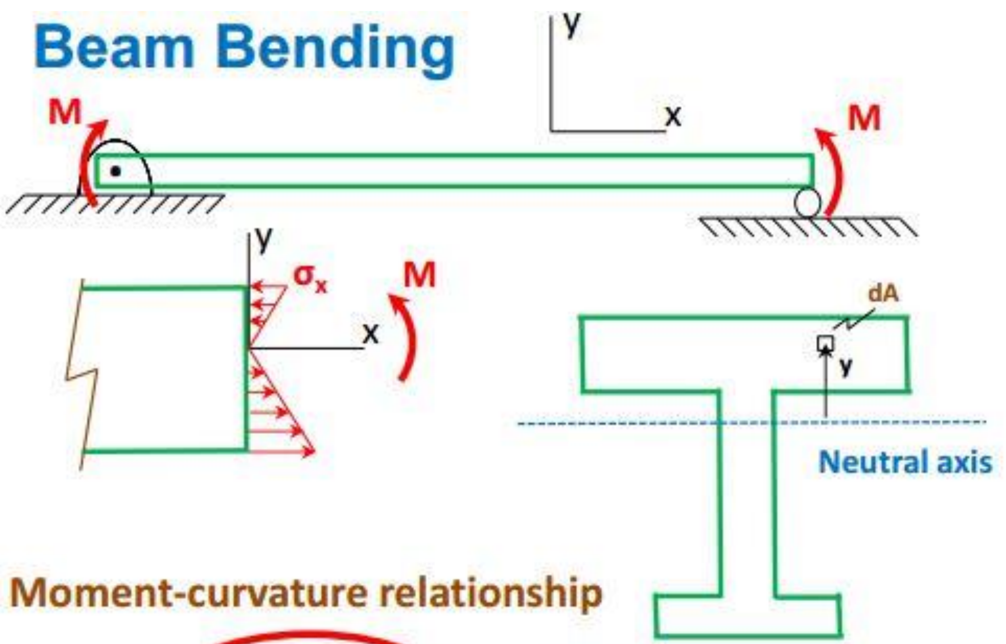
Independent of material

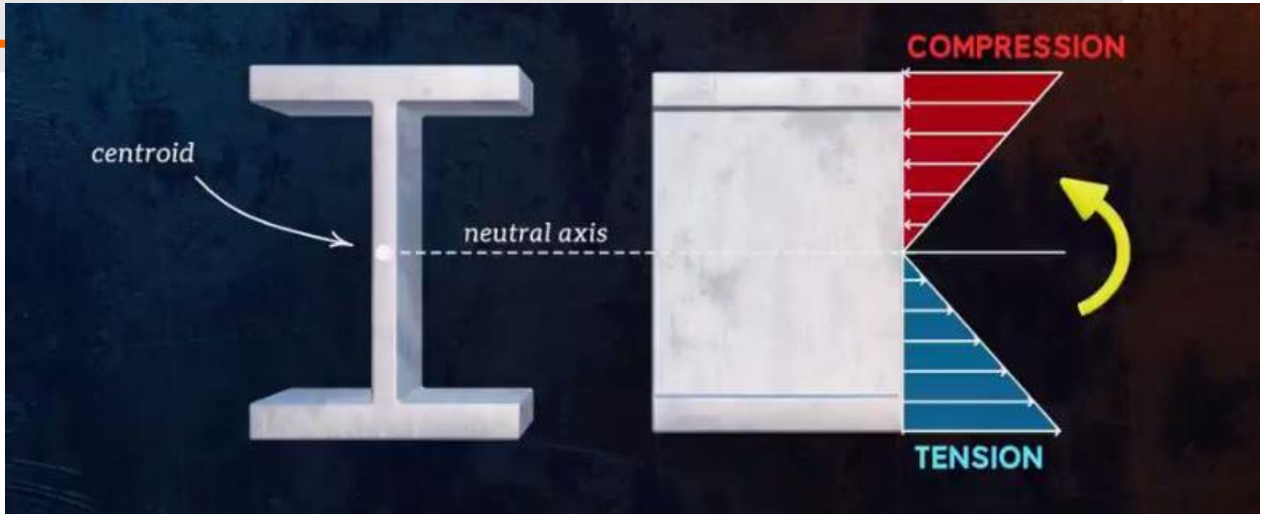
Note: There are strains in the y and z direction due to Poisson's effect, but no stresses because the beam is free to deform laterally.

Therefore pure bending in beams produces uniaxial stress.

We'll start looking at the stresses next time!

Pure Bending





Moment-curvature relationship

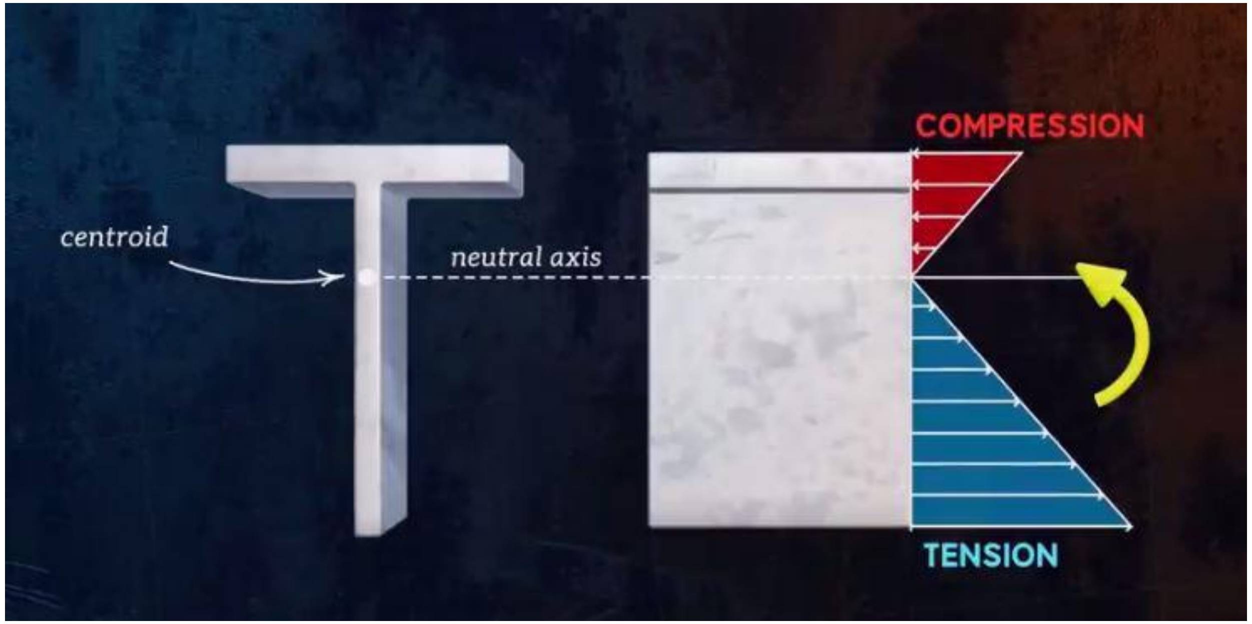
$$\kappa = \frac{1}{\rho} = \frac{M}{EI}$$

Curvature is proportional to moment

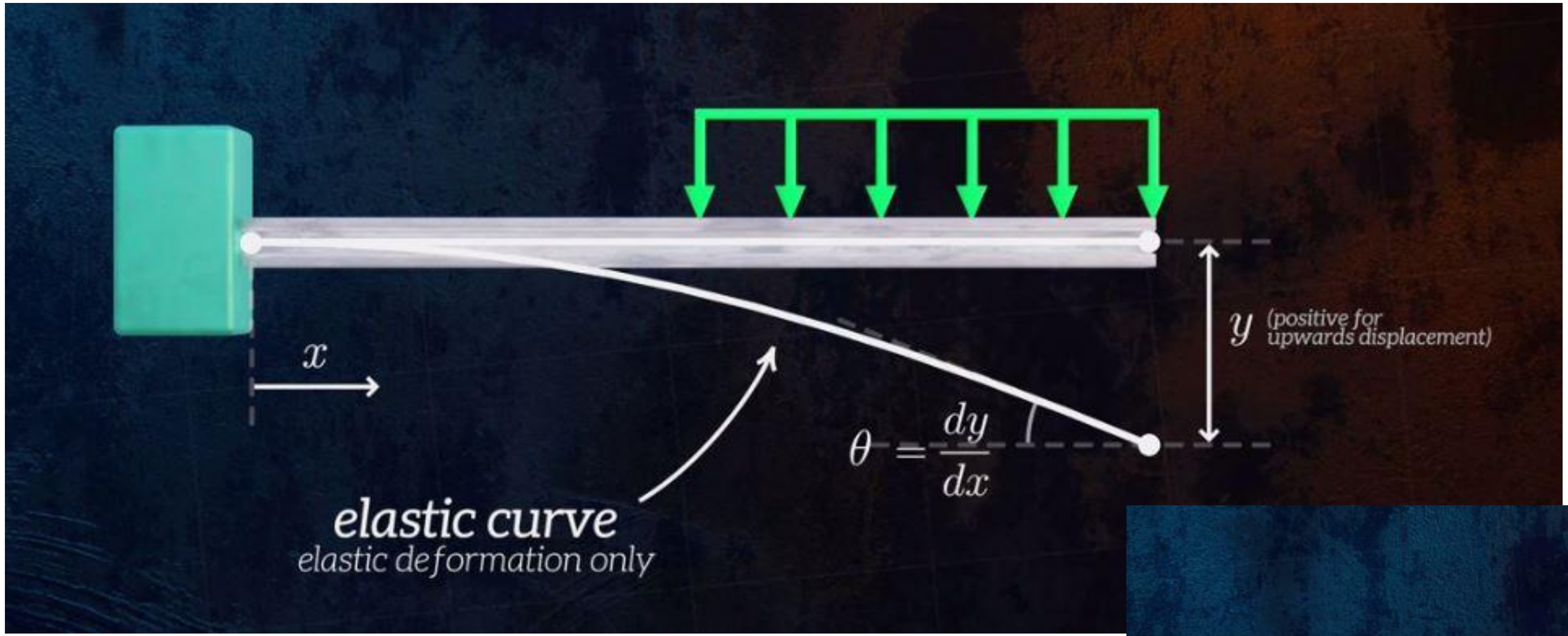
EI ≡ Flexural Rigidity

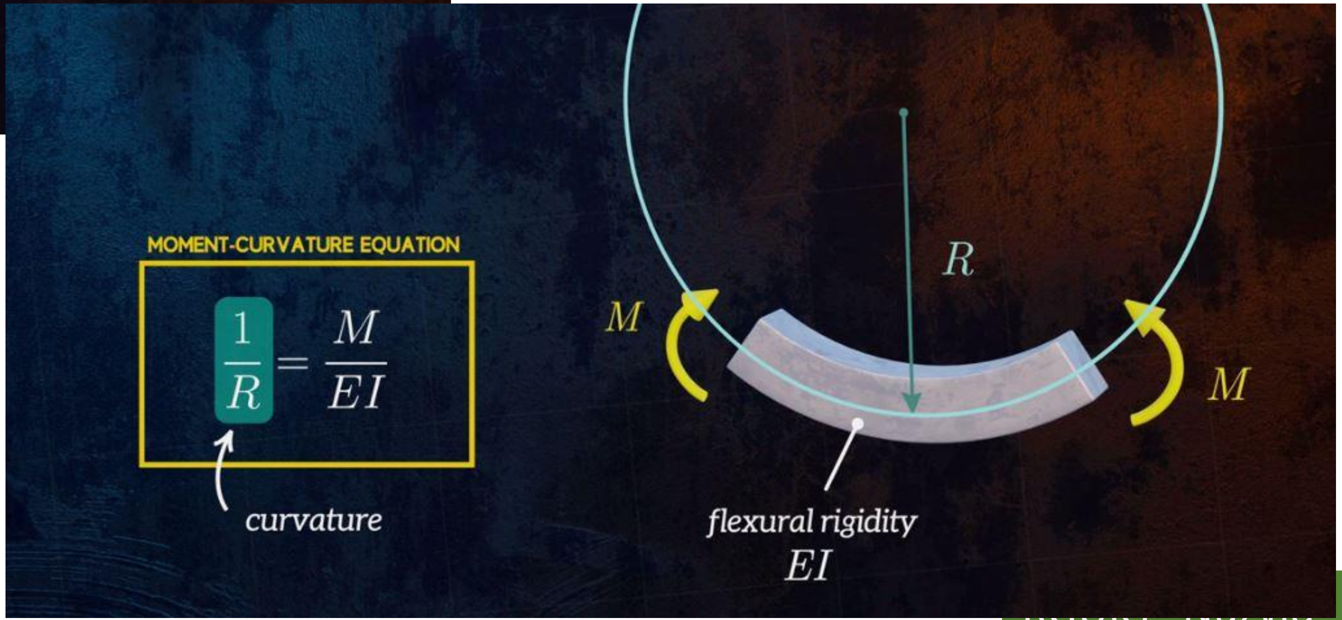
Resistance of the beam to bending for a given curvature

↑EI ↑M



Deflection of Beam Due to Bending





Deflection of Beam Due to Bending

Moment-curvature relationship

$$\frac{1}{\rho} = \frac{M}{EI}$$

Curvature equation
(see any standard calculus textbook)

$$\frac{1}{\rho} = \frac{\left(\frac{d^2 y}{dx^2}\right)}{\left[1 + \underbrace{\left(\frac{dy}{dx}\right)}_{0}^{2}\right]^{\frac{3}{2}}}$$

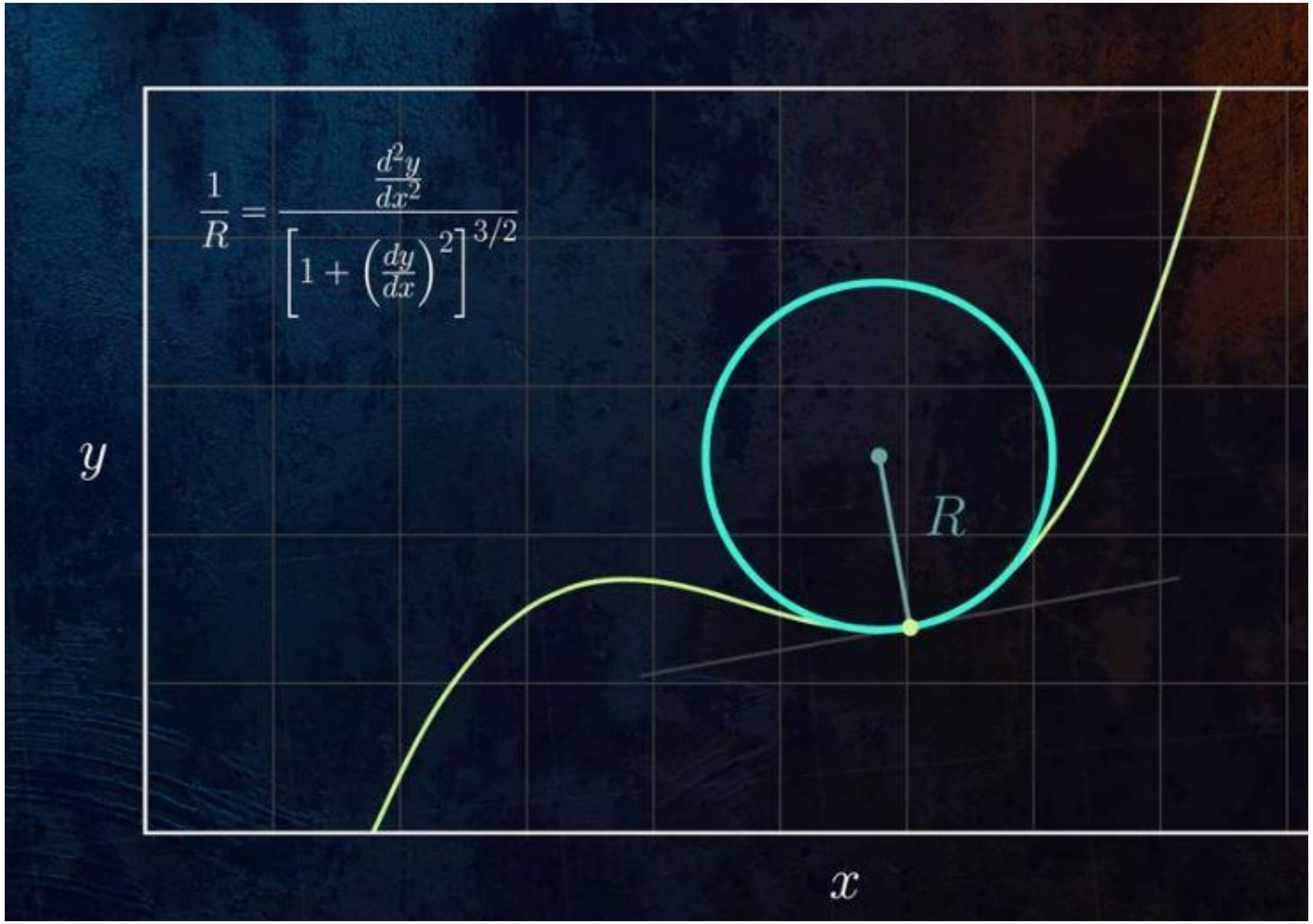
assume small deformations

$$square \quad of \quad \frac{dy}{dx} << 1$$

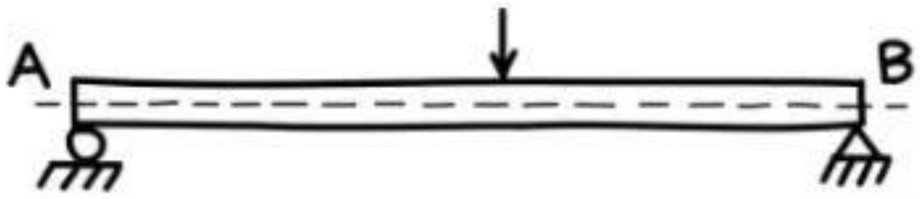
$$\frac{1}{\rho} \approx \frac{d^2 y}{dx^2}$$

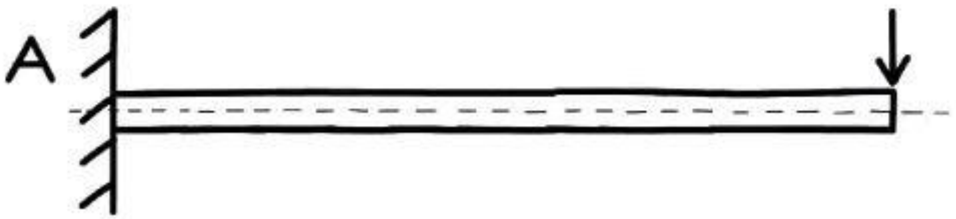
Differential equation for the elastic curve of a beam

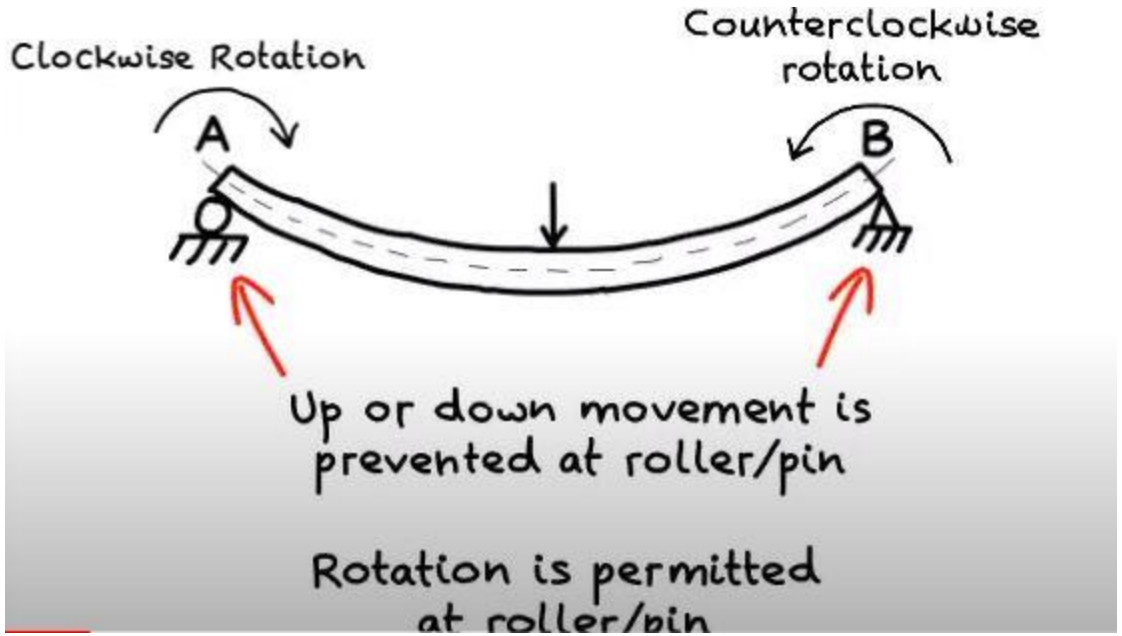
$$E\,I \frac{d^2 y}{dx^2} = M(x)$$

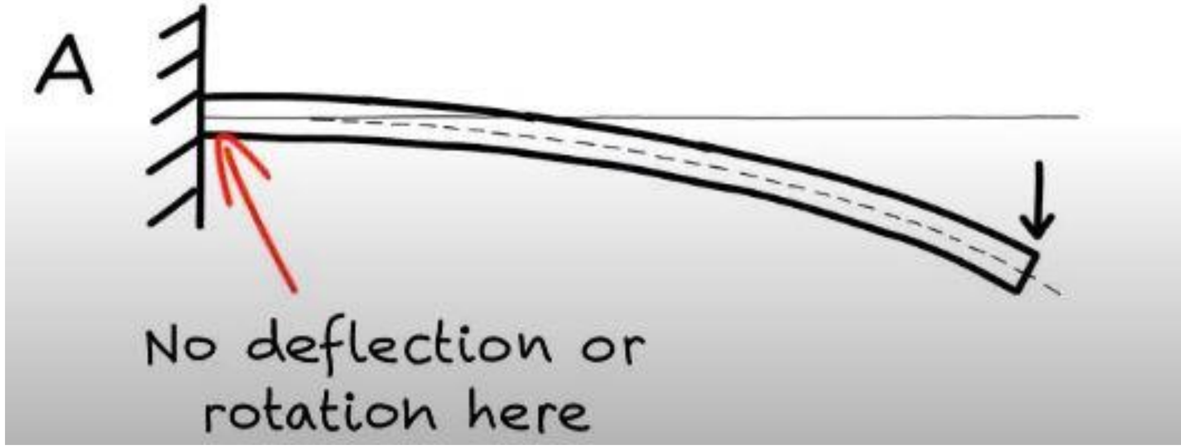


Deflection of Beam –Elastic Curve

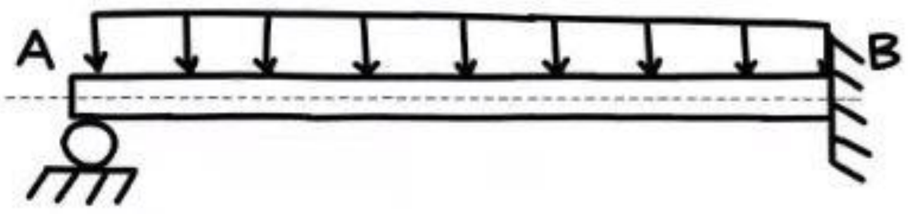


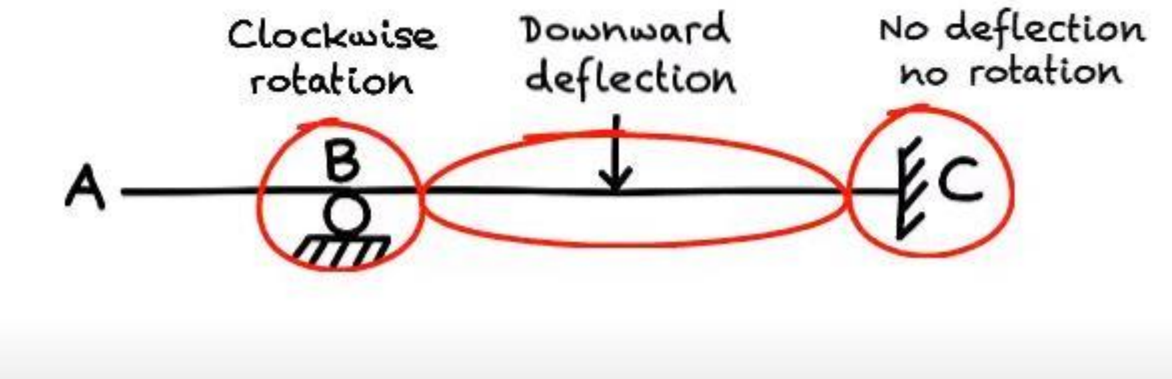


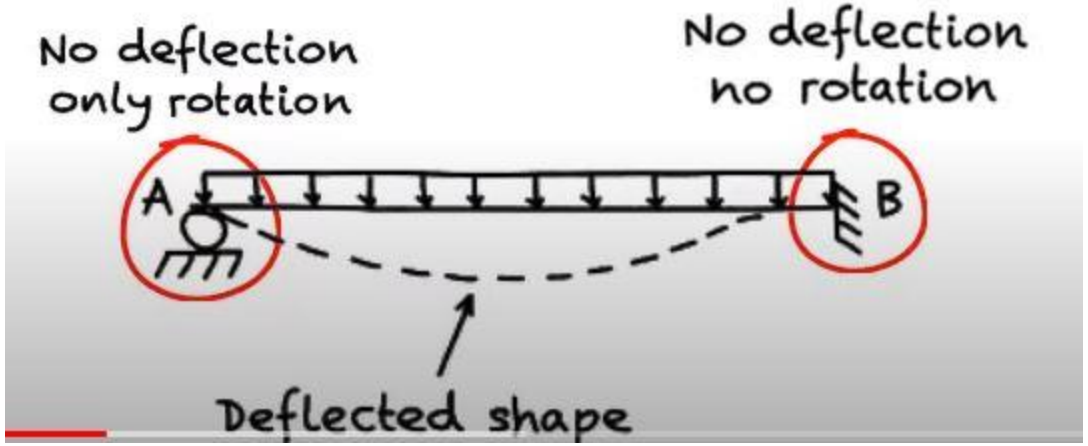


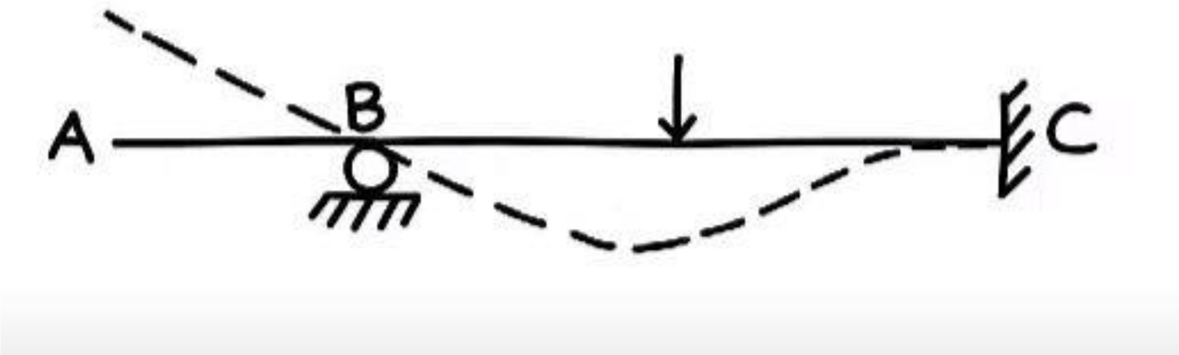


Deflection of Beam –Elastic Curve

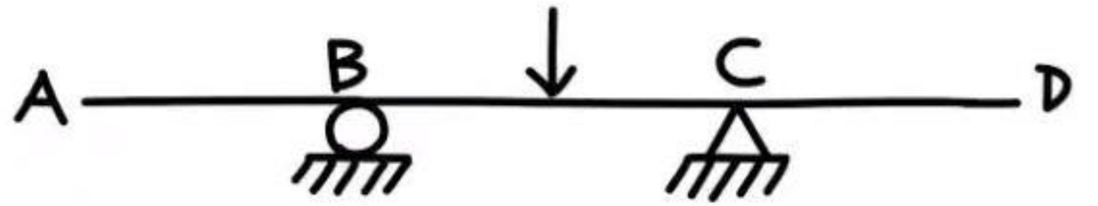


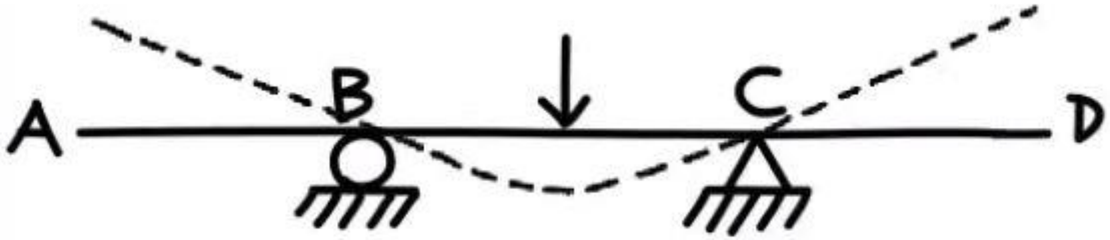


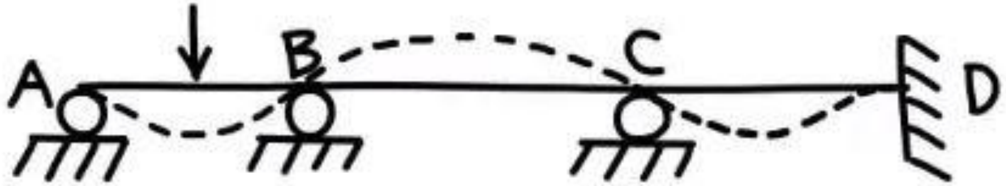




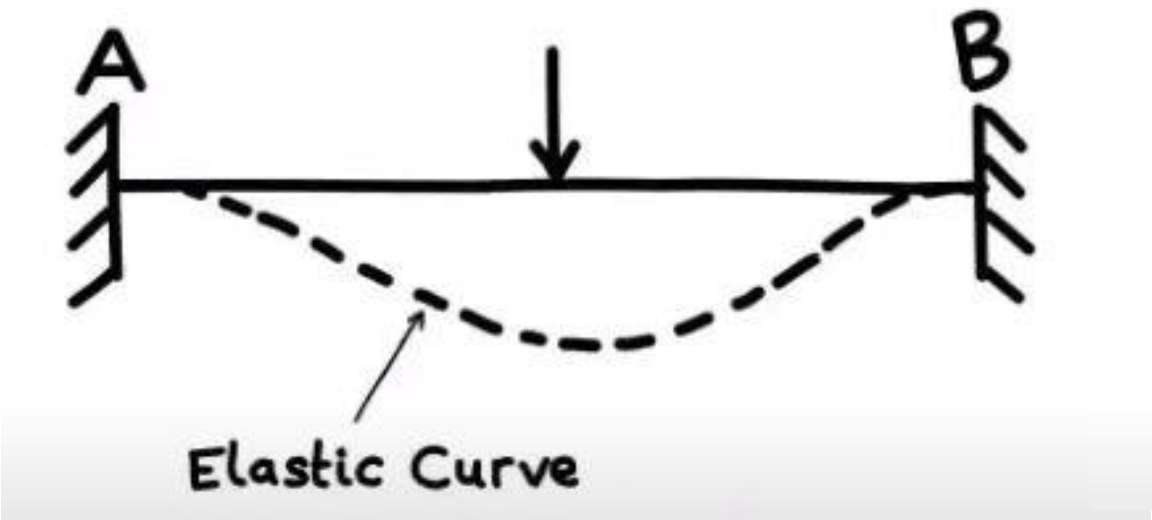
Deflection of Beam –Elastic Curve

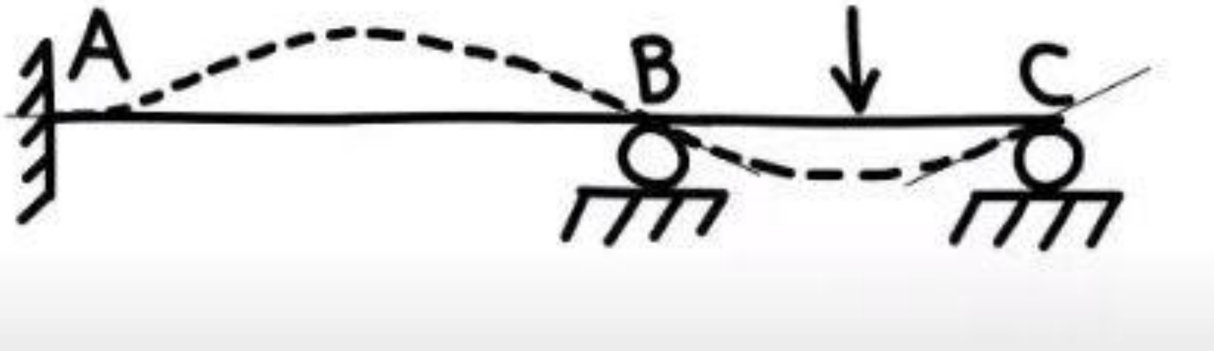


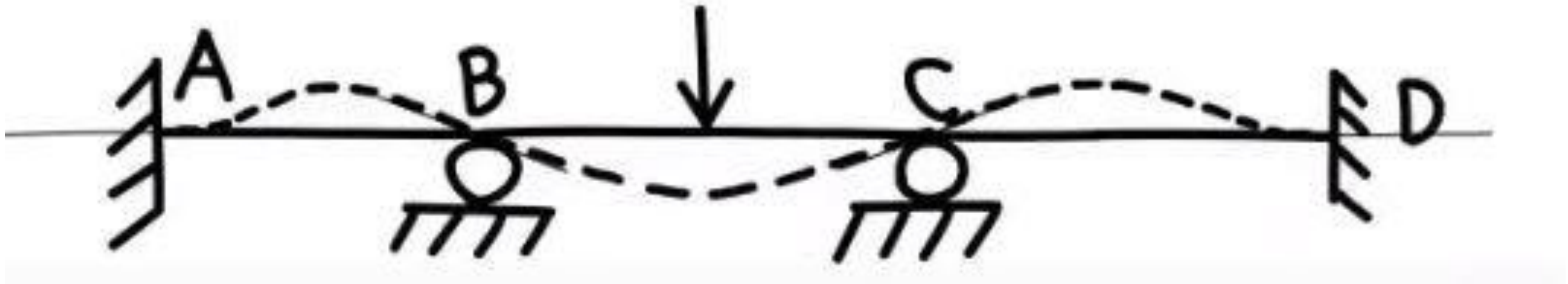




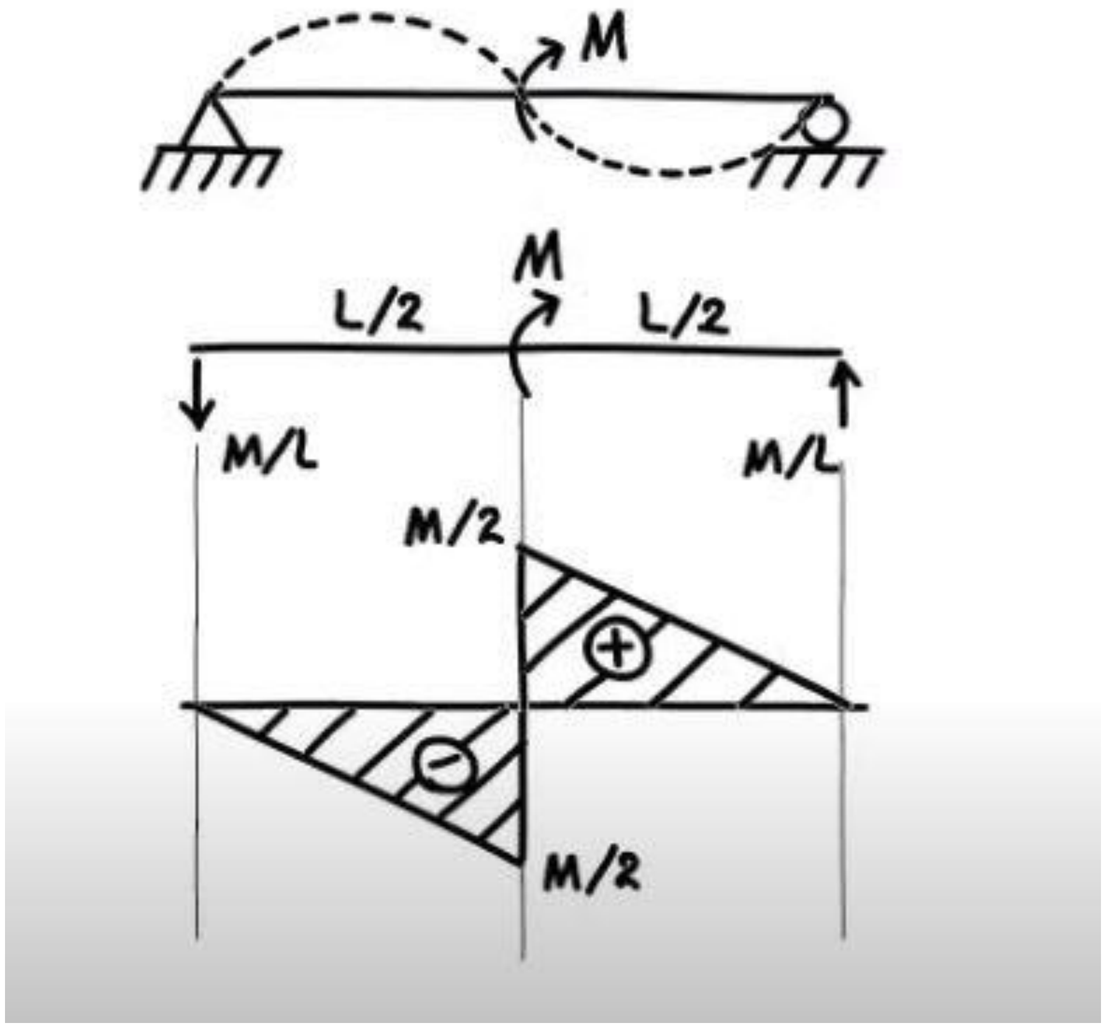
Deflection of Beam –Elastic Curve

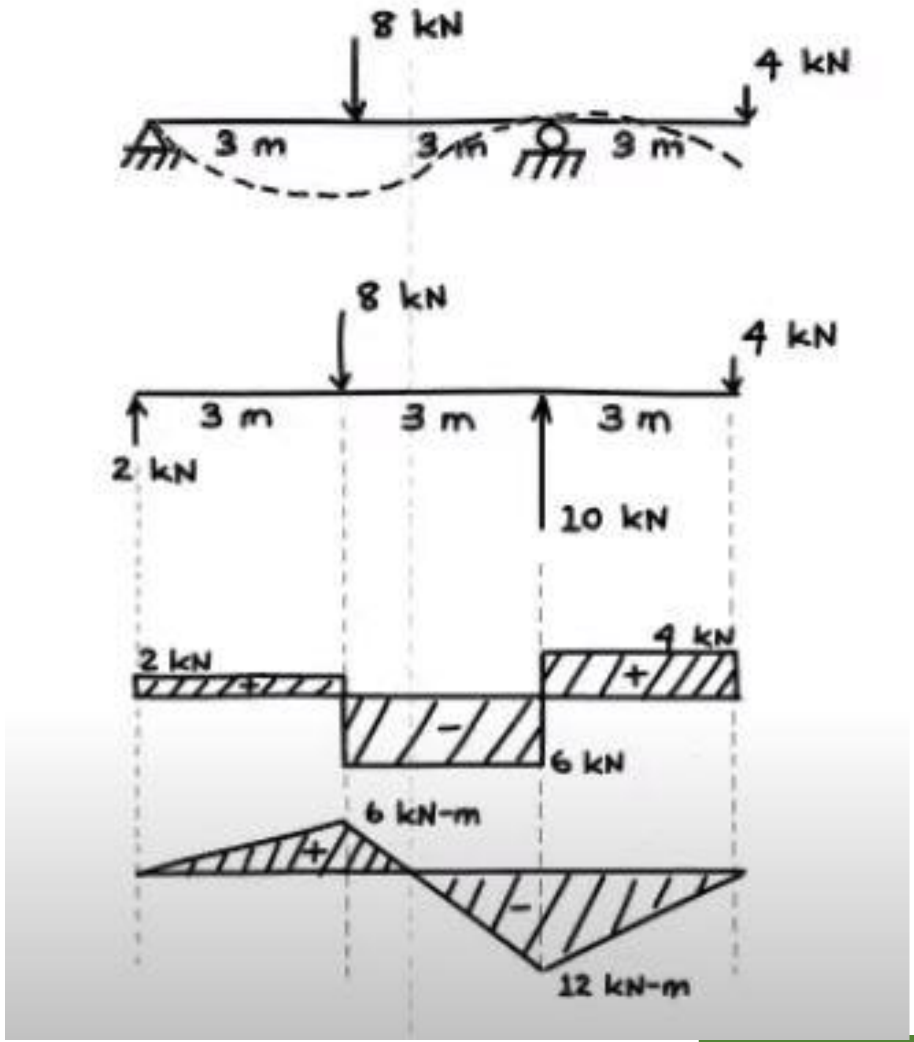




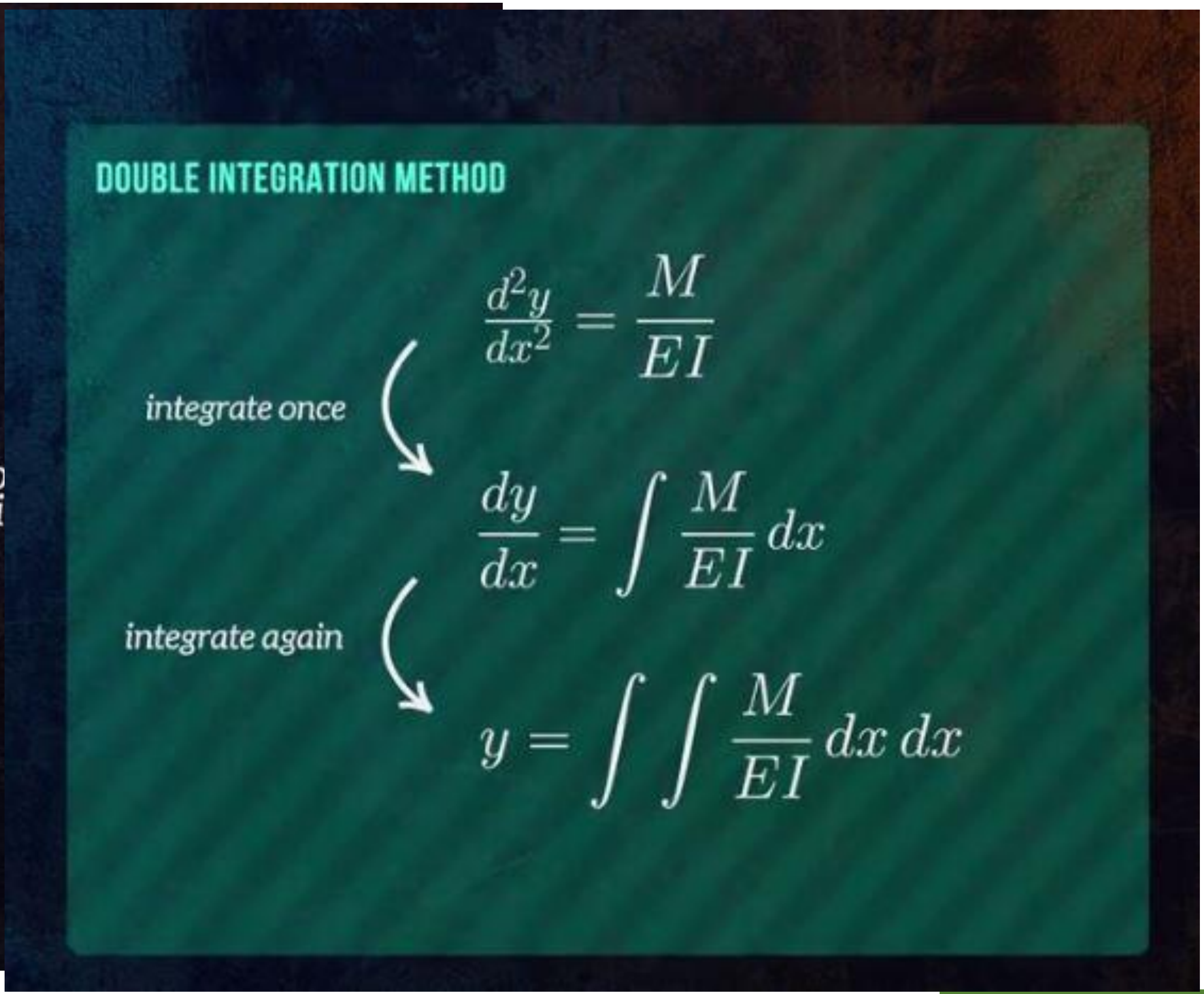


Deflection of Beam –Elastic Curve

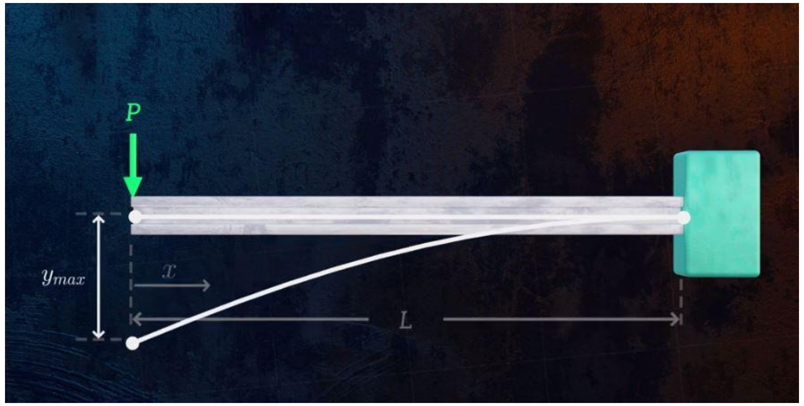


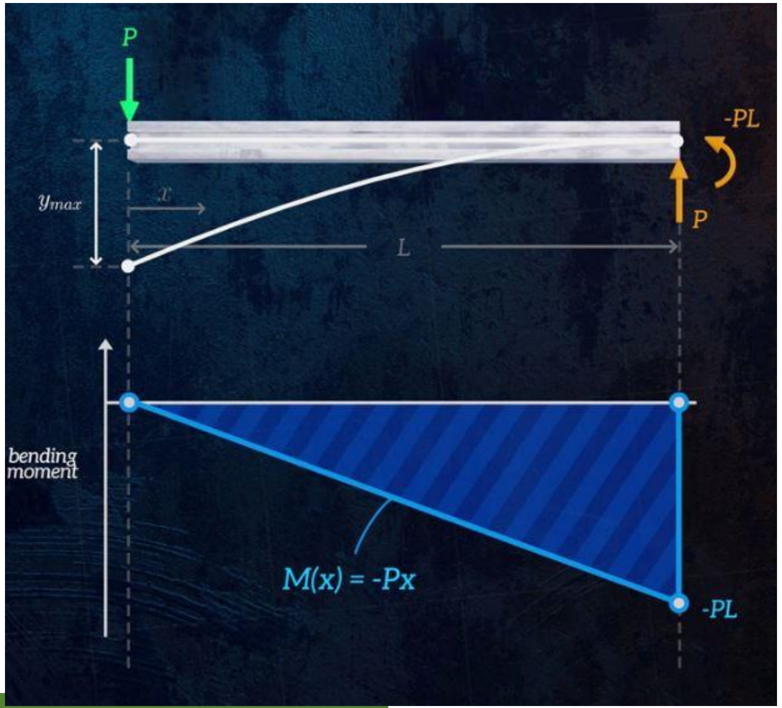


Double Integration Method



Double Integration Method





$$\frac{d^2y}{dx^2} = \frac{M(x)}{EI}$$

integrate once

$$\frac{d^2y}{dx^2} = \frac{M(x)}{EI}$$

$$\frac{dy}{dx} = \frac{1}{EI}\left(-\frac{P}{2}x^2 + C_1\right)$$

$$y = \frac{1}{EI}\left(-\frac{P}{6}x^3 + C_1x + C_2\right)$$

- At $x = L \rightarrow \frac{dy}{dx} = 0$

$$C_1 = \frac{P}{2}L^2$$

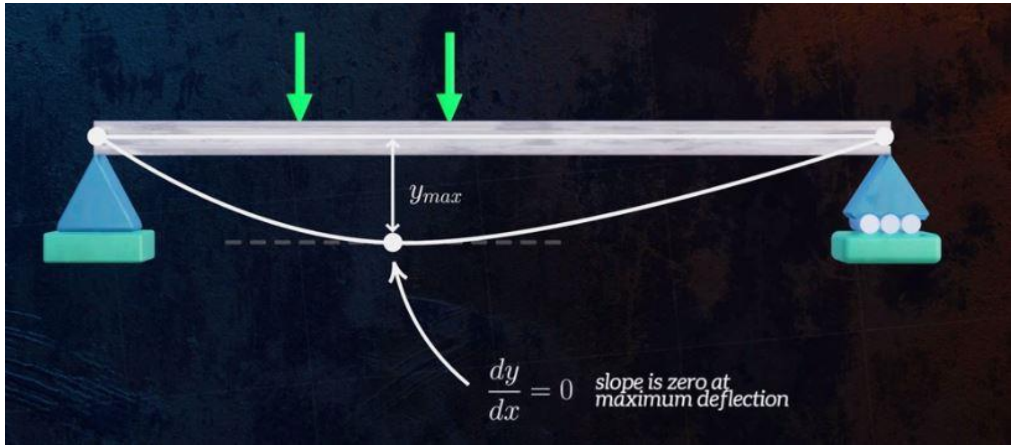
- At $x = L \rightarrow y = 0$

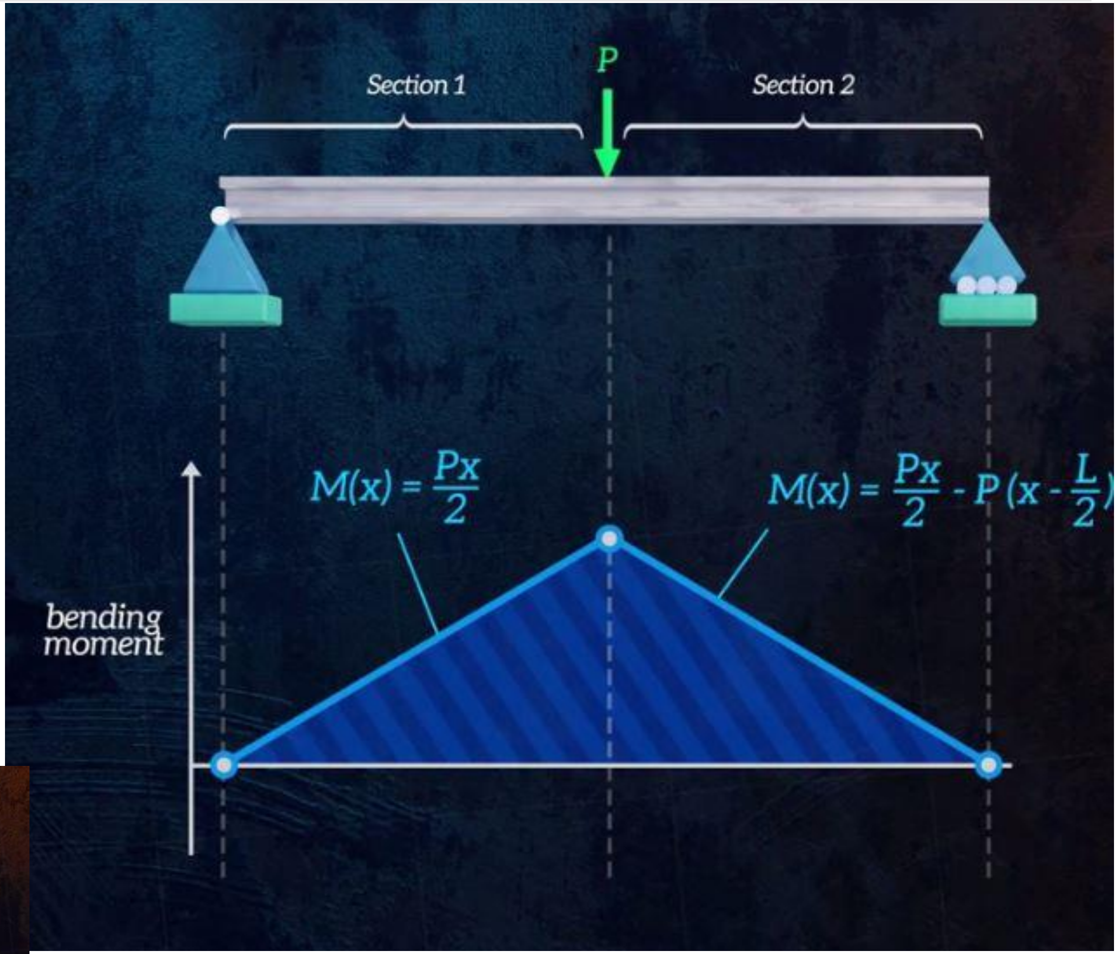
$$0 = \frac{1}{EI}\left(-\frac{P}{6}L^3 + C_1L + C_2\right)$$

$$y = \frac{P}{6EI}(-x^3 + 3L^2x - 2L^3)$$

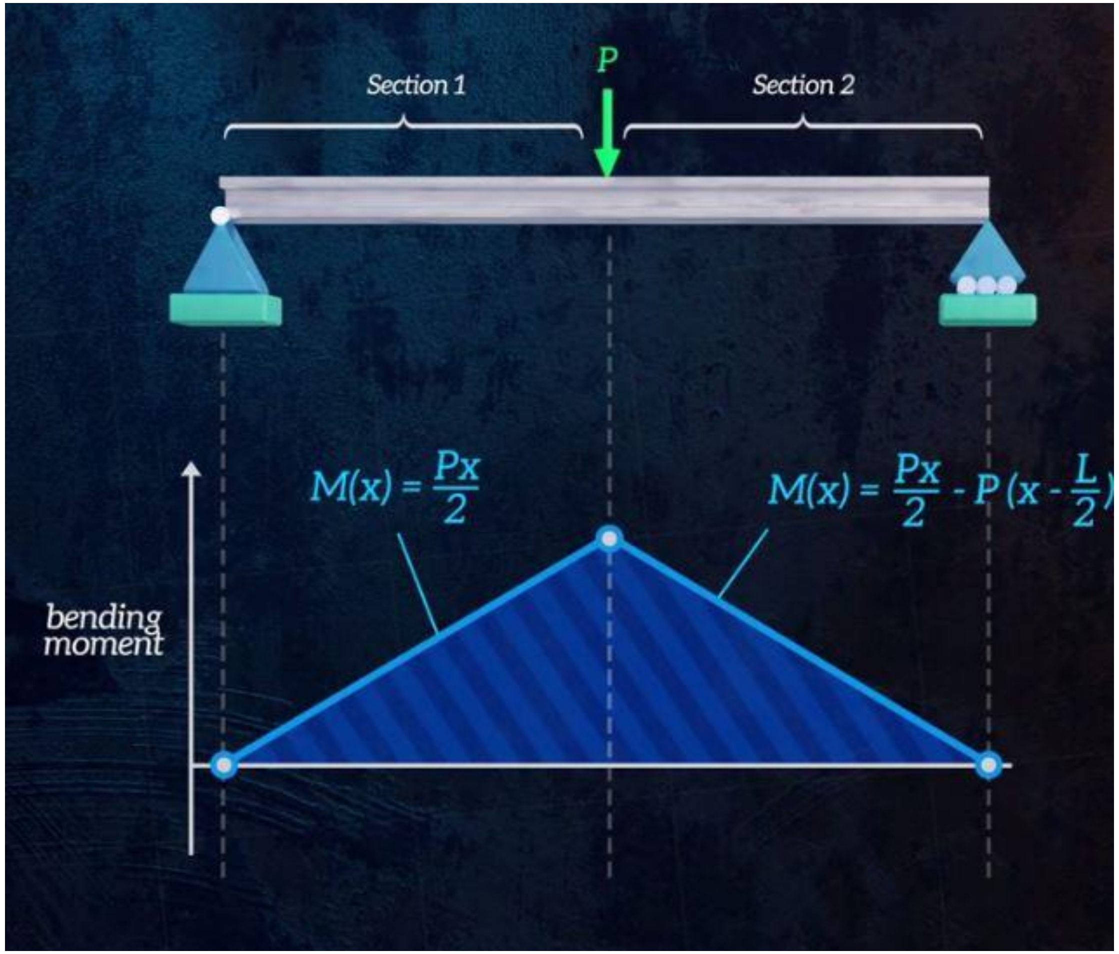
$$y_{max} = y(0) = \frac{-PL^3}{3EI}$$

Double Integration Method





Double Integration Method



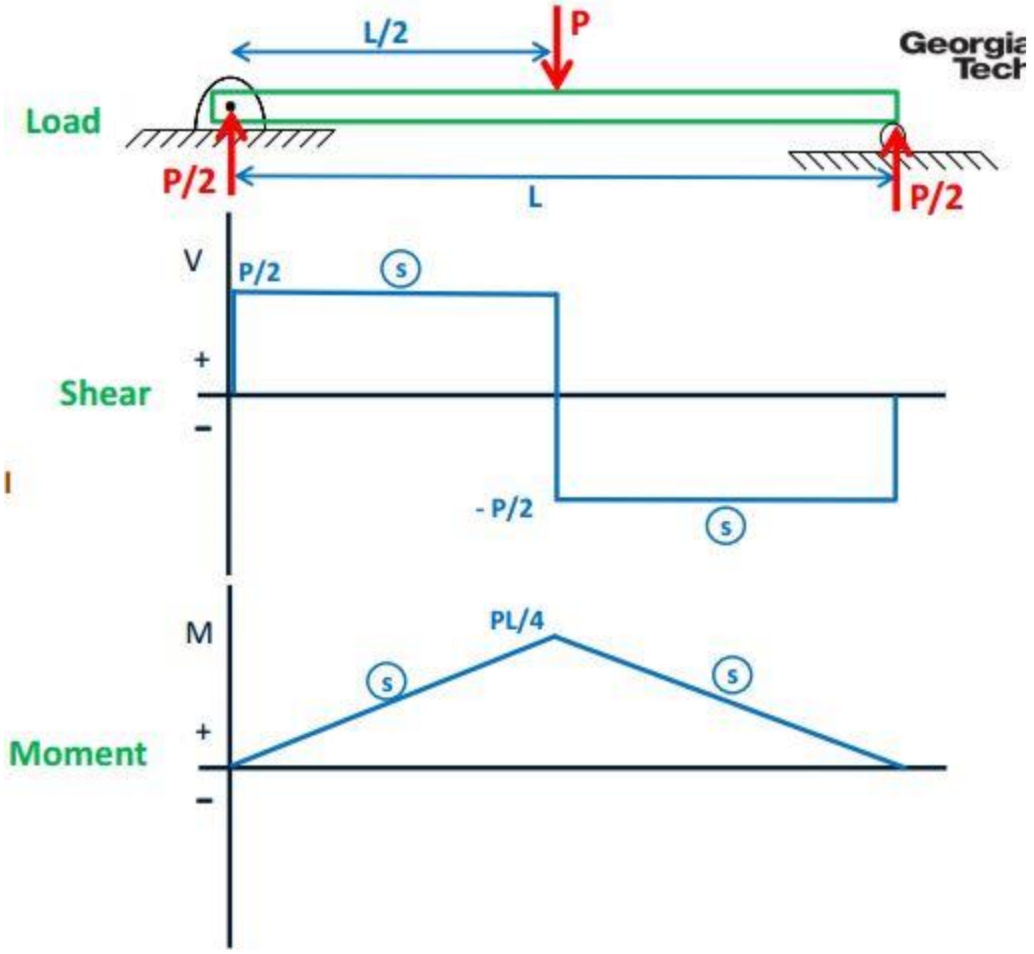
$$y_1 = \frac{1}{EI}\left(\frac{P}{12}x^3 + C_1x + C_2\right) \qquad (0 \le x \le \tfrac{L}{2})$$

$$y_2 = \frac{1}{EI}\left(\frac{-P}{12}x^3 + \frac{PL}{4}x^2 + C_3x + C_4\right) \qquad (\tfrac{L}{2} \le x \le L)$$

- $y_1(0) = 0 \rightarrow C_2 = 0$
- $\theta_1(L/2) = \theta_2(L/2) \rightarrow C_3 = C_1 - \dfrac{PL^2}{8}$
- $y_1(L/2) = y_2(L/2) \rightarrow C_4 = \dfrac{PL^3}{48}$
- $y_2(L) = 0 \rightarrow C_1 = \dfrac{-PL^2}{16}$

$$y_{max} = y(L/2) = -\frac{PL^3}{48EI}$$

Double Integration Method



$$-q = \frac{dV}{dx} = EI\frac{d^4y}{dx^4}$$

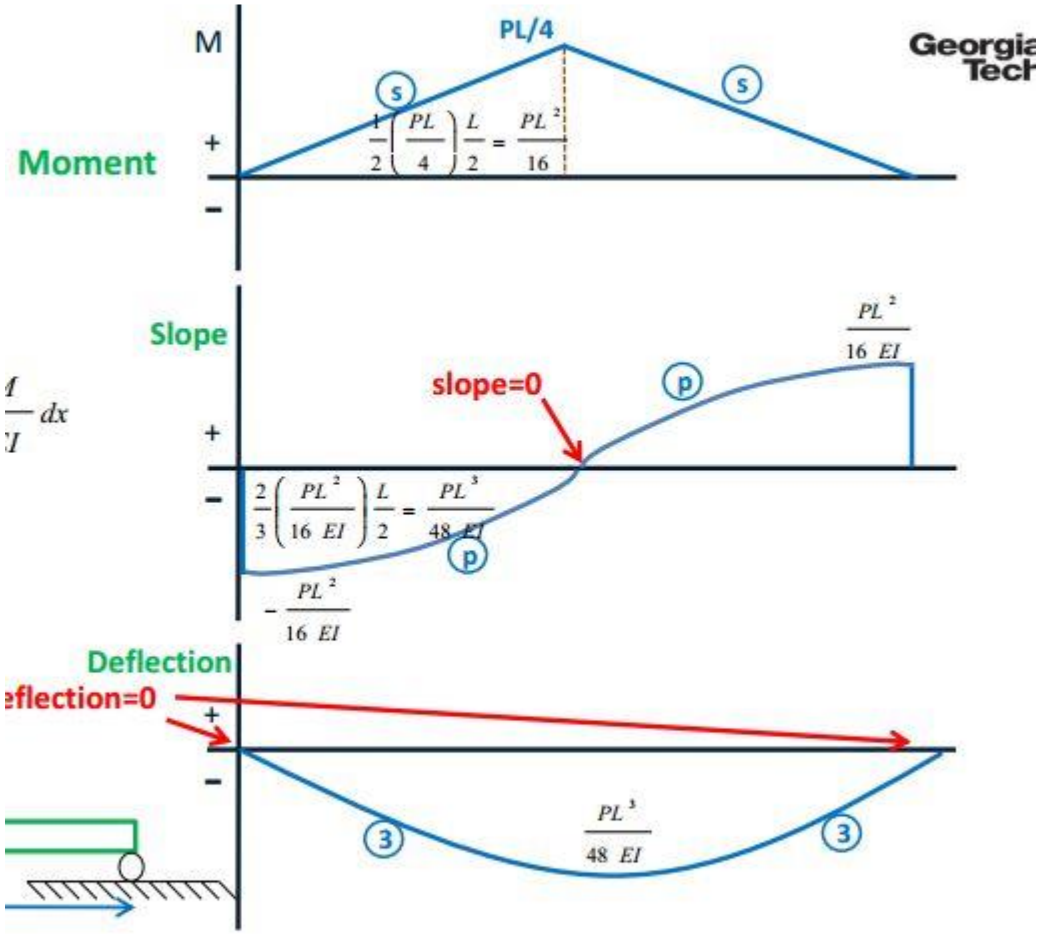
constant EI

$$V = \frac{dM}{dx} = EI\frac{d^3y}{dx^3}$$

constant EI

$$EI\frac{d^2y}{dx^2} = M(x)$$

Double Integration Method

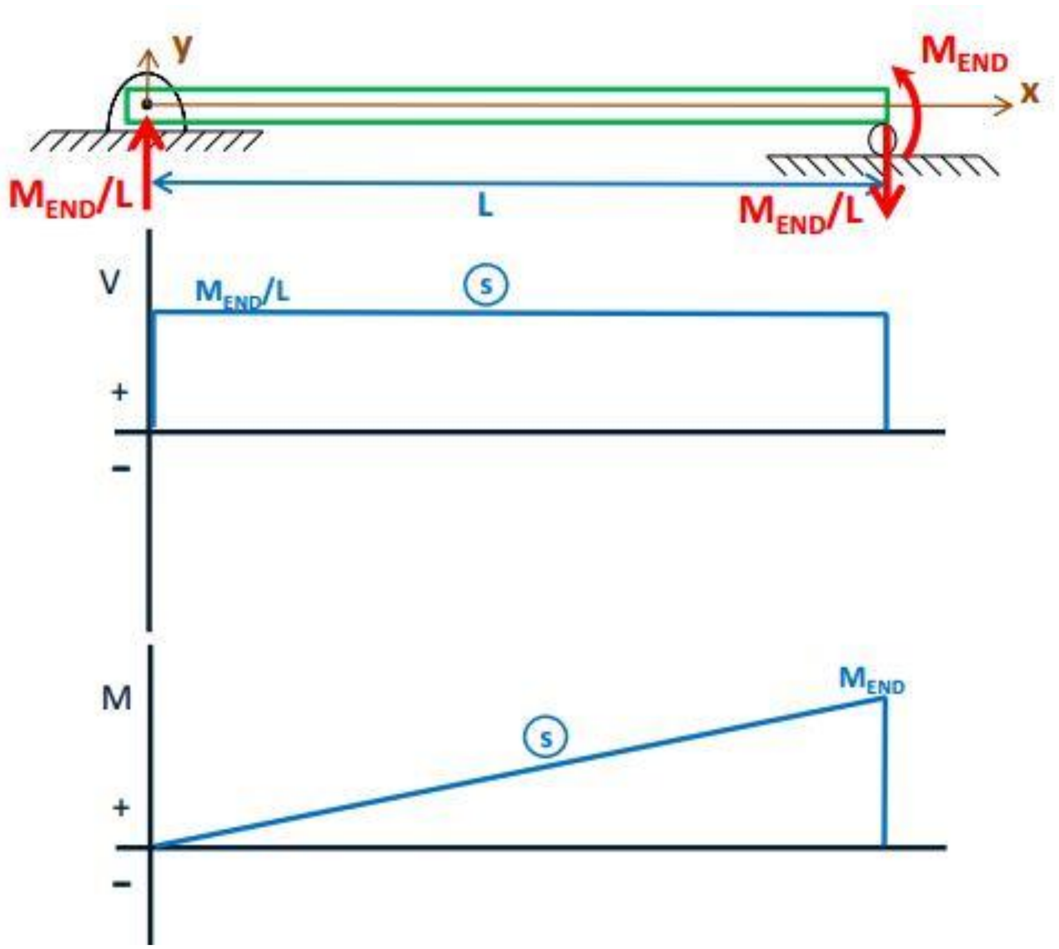


$$E\,I\,\frac{d^2y}{dx^2} = M(x)$$

$$slope = \frac{dy}{dx} = \int \frac{M}{EI}\,dx$$

$$deflection = y = \iint \frac{M}{EI}\,dx\ dx$$

Double Integration Method



$$EI\frac{d^2y}{dx^2} = M(x) = \frac{M_{END}}{L}x$$

$$EI\frac{dy}{dx} = \frac{M_{END}}{L}\frac{x^2}{2} + C_1$$

$$EI\ y = \frac{M_{END}}{2L}\frac{x^3}{3} + C_1x + C_2$$

What are the boundary conditions to find the constants of integration?

$$y(0) = 0$$

$$y(L) = 0$$

Double Integration Method

$$EI\frac{d^2y}{dx^2} = M(x) = \frac{M_{END}}{L}x$$

$$EI\frac{dy(x)}{dx} = \frac{M_{END}}{L}\frac{x^2}{2} - \frac{M_{END}L}{6}$$

$$EI\ y(x) = \frac{M_{END}}{2L}\frac{x^3}{3} - \frac{M_{END}L}{6}x$$

Worksheet: Assume E and I for the beam below are constant. Find the deflection of the beam as a function of x. Determine the max deflection and where it occurs.

How do we find max deflection and where it occurs?

$$EI\frac{d^2y}{dx^2} = M(x) = \frac{M_{END}}{L}x$$

$$EI\frac{dy(x)}{dx} = \frac{M_{END}}{L}\frac{x^2}{2} - \frac{M_{END}L}{6}$$

$$EI\ y(x) = \frac{M_{END}}{2L}\frac{x^3}{3} - \frac{M_{END}L}{6}x$$

Find the max deflection

The max deflection occurs at

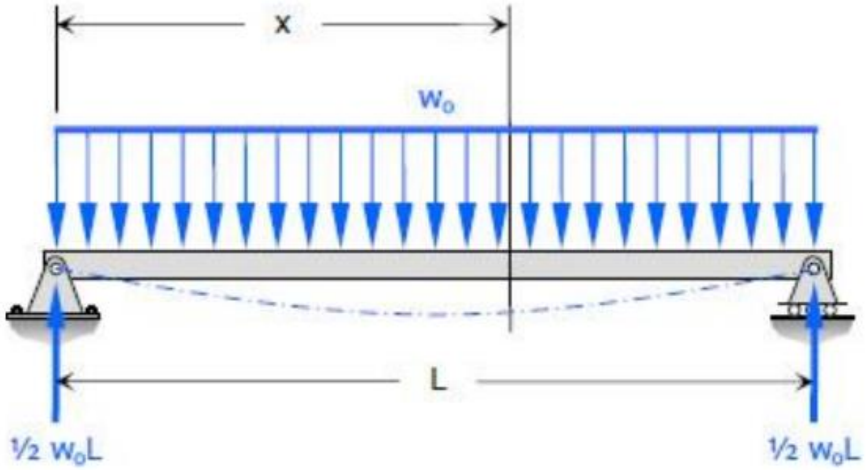
$$x = \frac{L}{\sqrt{3}} = 0.577\ L$$

ANS

The max deflection is $-\dfrac{M_{END}L^2}{9\sqrt{3}\ EI}$

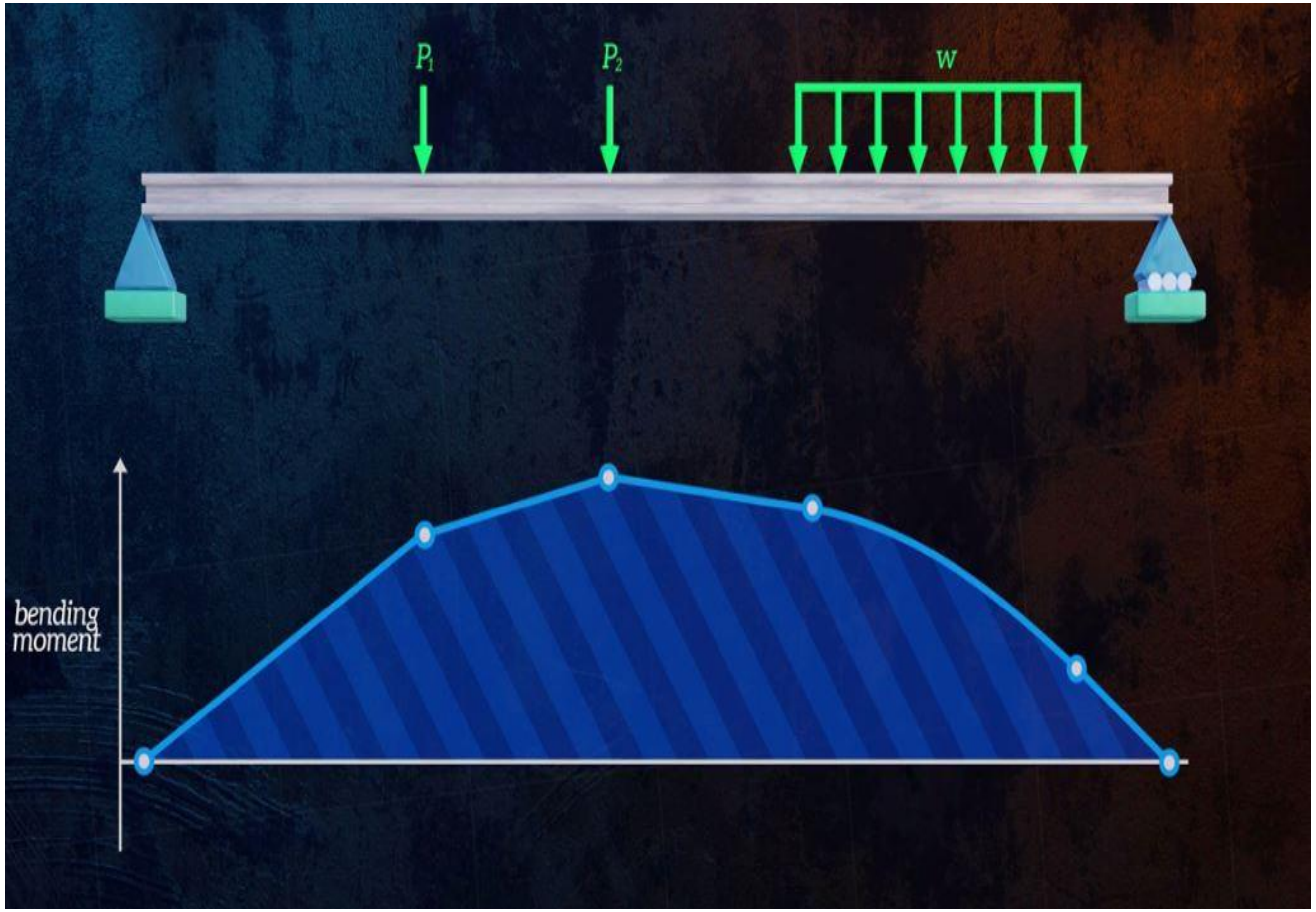
and occurs at $x = \dfrac{L}{\sqrt{3}} = 0.577\ L$

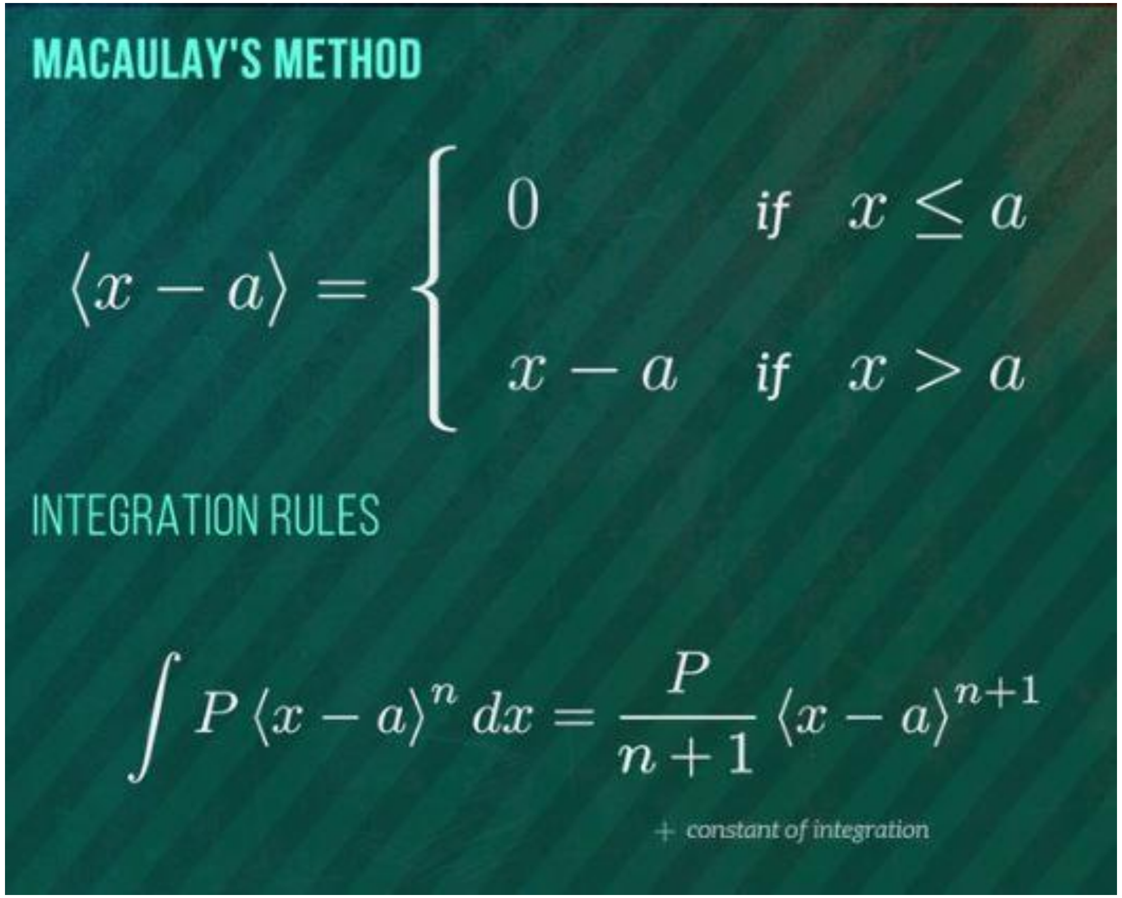
Double Integration Method



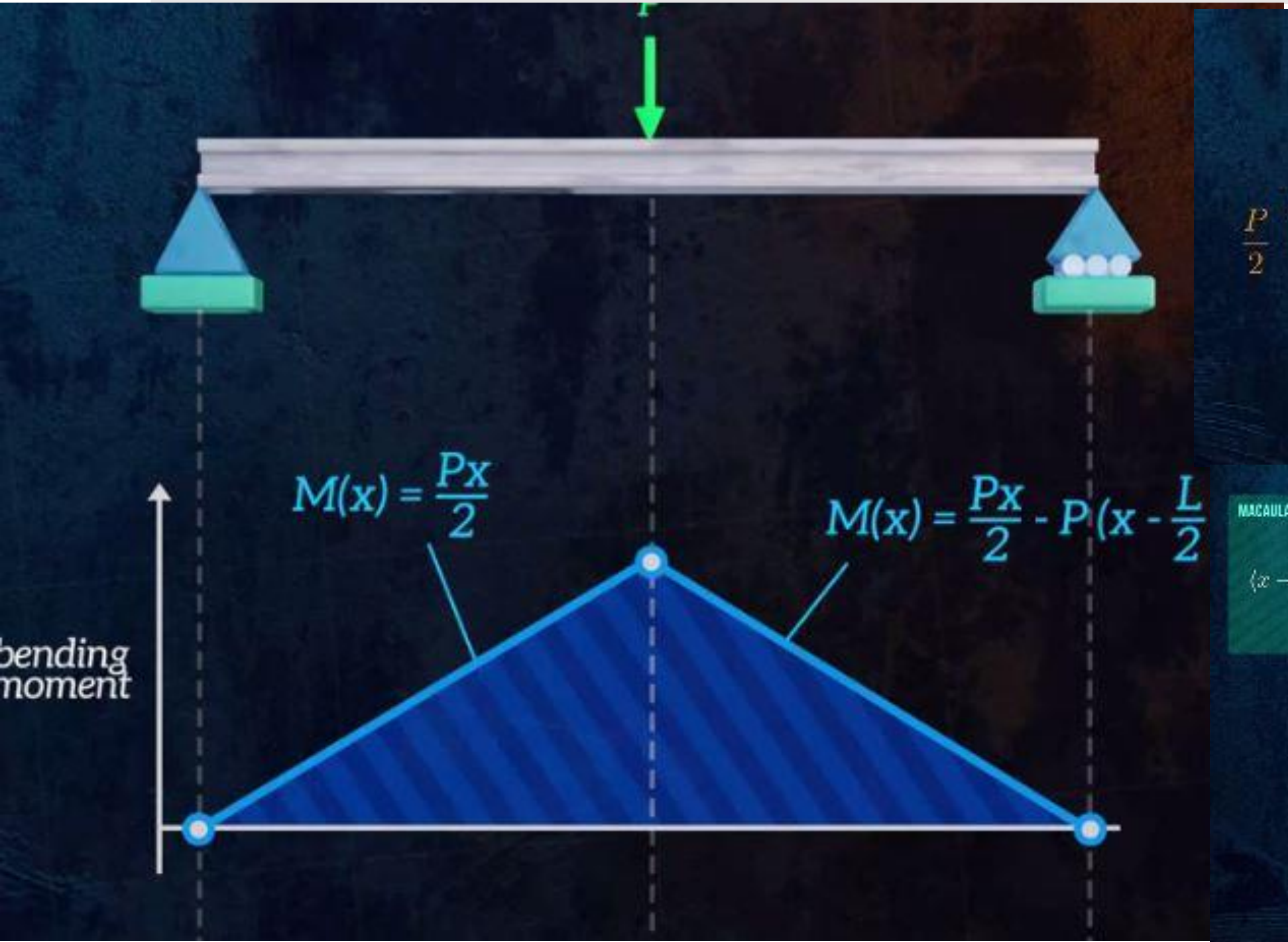
Double Integration Method

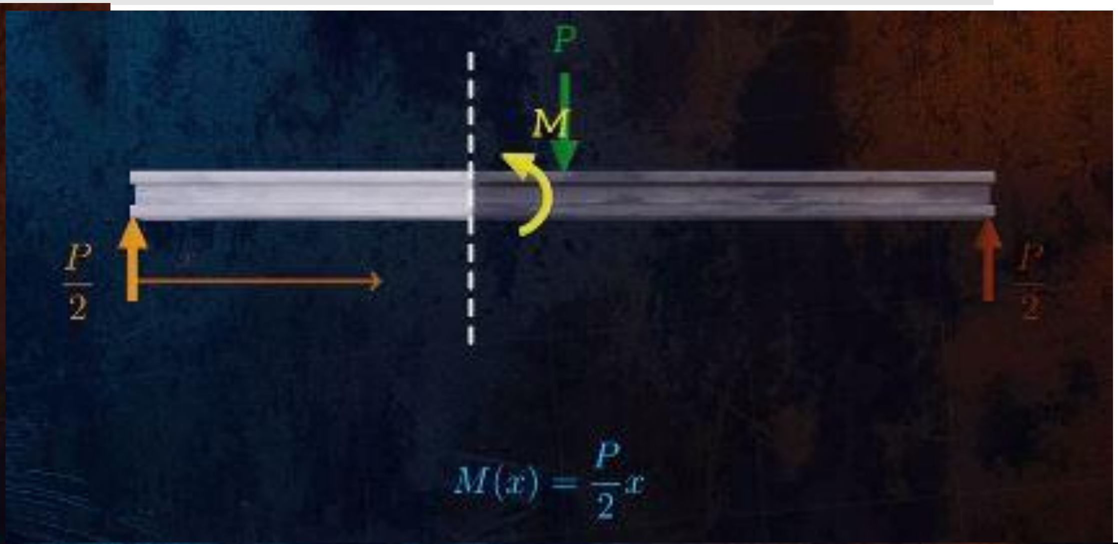
Macaulay's Method- Singularity Function

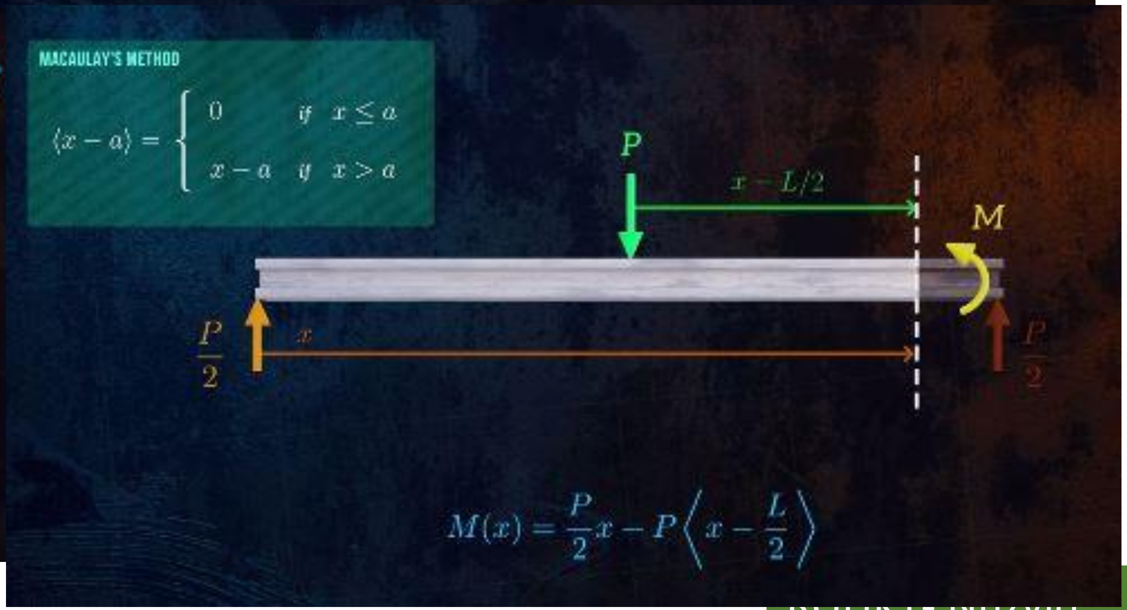




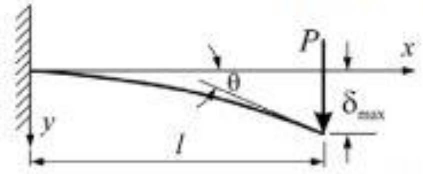
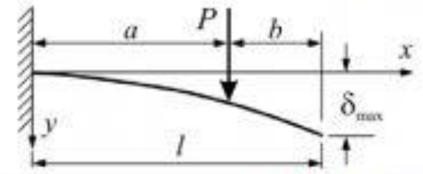
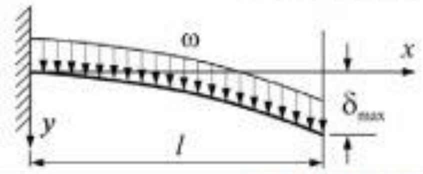
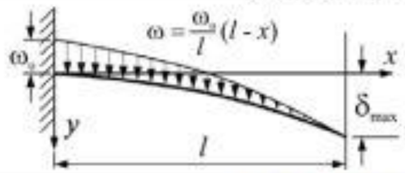
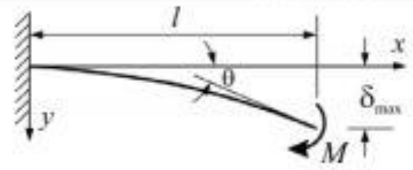
Macaulay’s Method- Singularity Function



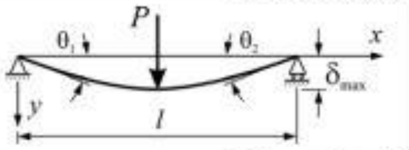
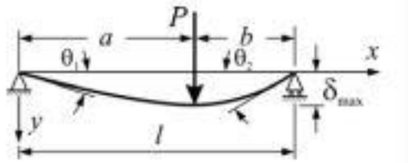
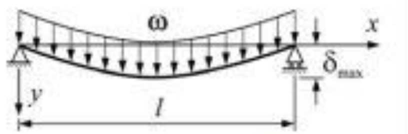
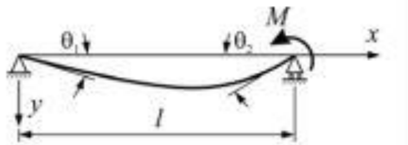
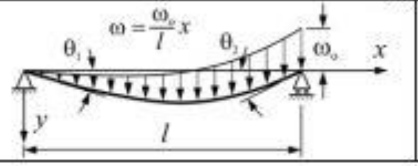




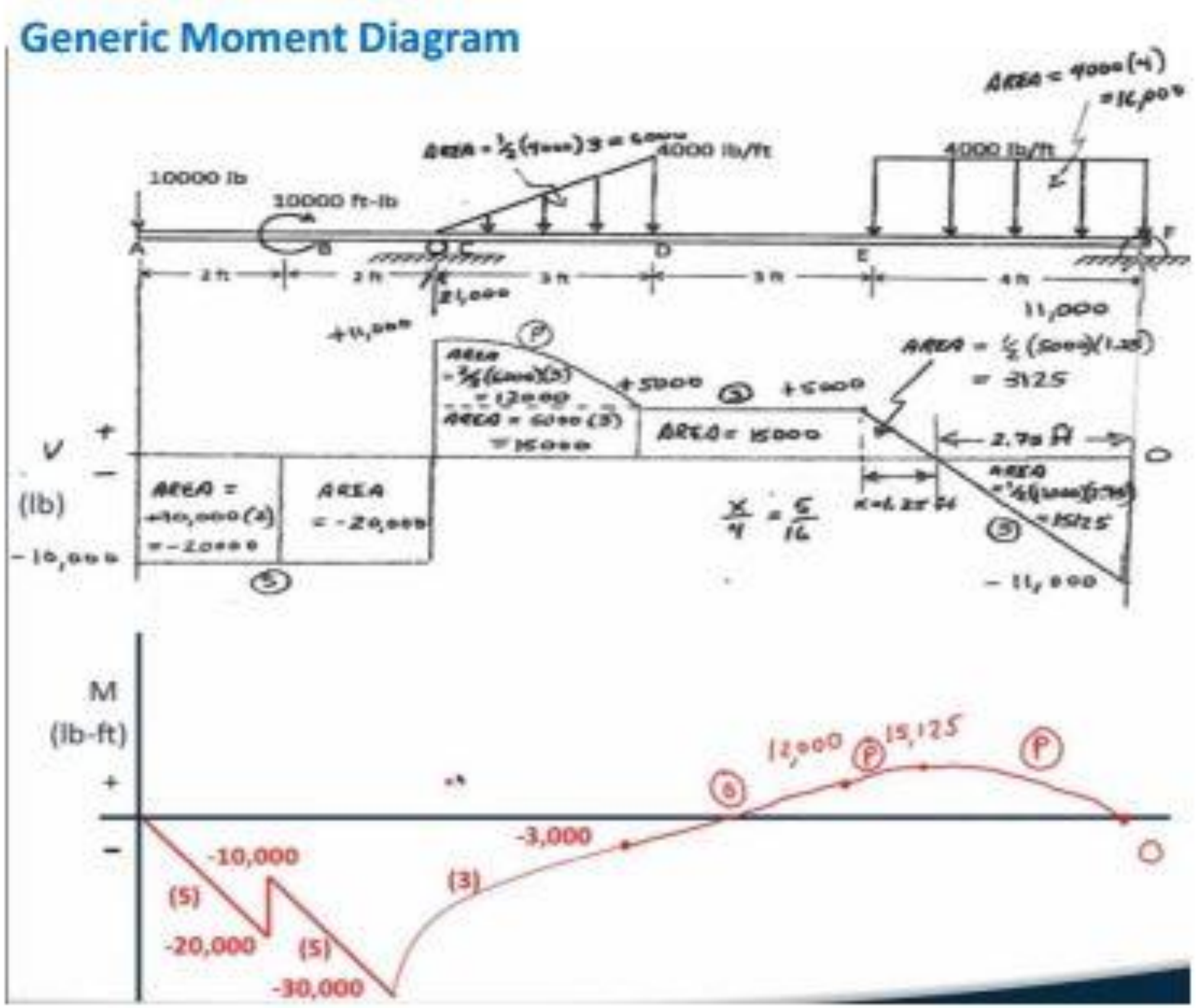
Double Integration Method- Assignment

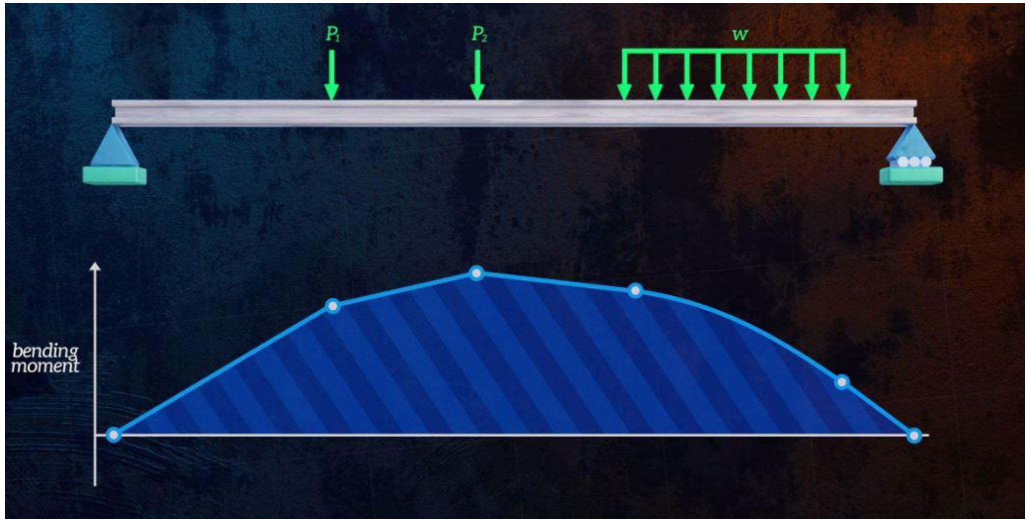
BEAM TYPE	SLOPE AT FREE END	DEFLECTION AT ANY SECTION IN TERMS OF x	MAXIMUM DEFLECTION
1. Cantilever Beam – Concentrated load P at the free end			
	$\theta = \frac{Pl^2}{2EI}$	$y = \frac{Px^2}{6EI}(3l - x)$	$\delta_{max} = \frac{Pl^3}{3EI}$
2. Cantilever Beam – Concentrated load P at any point			
	$\theta = \frac{Pa^2}{2EI}$	$y = \frac{Px^2}{6EI}(3a - x)$ for $0 < x < a$ $y = \frac{Pa^2}{6EI}(3x - a)$ for $a < x < l$	$\delta_{max} = \frac{Pa^2}{6EI}(3l - a)$
3. Cantilever Beam – Uniformly distributed load ω (N/m)			
	$\theta = \frac{\omega l^3}{6EI}$	$y = \frac{\omega x^2}{24EI}(x^2 + 6l^2 - 4lx)$	$\delta_{max} = \frac{\omega l^4}{8EI}$
4. Cantilever Beam – Uniformly varying load: Maximum intensity ω_o (N/m)			
	$\theta = \frac{\omega_o l^3}{24EI}$	$y = \frac{\omega_o x^2}{120lEI}(10l^3 - 10l^2x + 5lx^2 - x^3)$	$\delta_{max} = \frac{\omega_o l^4}{30EI}$
5. Cantilever Beam – Couple moment M at the free end			
	$\theta = \frac{Ml}{EI}$	$y = \frac{Mx^2}{2EI}$	$\delta_{max} = \frac{Ml^2}{2EI}$

Double Integration Method- Assignment

BEAM TYPE	SLOPE AT ENDS	DEFLECTION AT ANY SECTION IN TERMS OF x	MAXIMUM AND CENTER DEFLECTION
6. Beam Simply Supported at Ends – Concentrated load P at the center			
	$\theta_1 = \theta_2 = \dfrac{Pl^2}{16EI}$	$y = \dfrac{Px}{12EI}\left(\dfrac{3l^2}{4} - x^2\right)$ for $0 < x < \dfrac{l}{2}$	$\delta_{max} = \dfrac{Pl^3}{48EI}$
7. Beam Simply Supported at Ends – Concentrated load P at any point			
	$\theta_1 = \dfrac{Pb(l^2 - b^2)}{6lEI}$ $\theta_2 = \dfrac{Pab(2l - b)}{6lEI}$	$y = \dfrac{Pbx}{6lEI}\left(l^2 - x^2 - b^2\right)$ for $0 < x < a$ $y = \dfrac{Pb}{6lEI}\left[\dfrac{l}{b}(x-a)^3 + \left(l^2 - b^2\right)x - x^3\right]$ for $a < x < l$	$\delta_{max} = \dfrac{Pb\left(l^2 - b^2\right)^{3/2}}{9\sqrt{3}\,lEI}$ at $x = \sqrt{\left(l^2 - b^2\right)/3}$ $\delta = \dfrac{Pb}{48EI}\left(3l^2 - 4b^2\right)$ at the center, if $a > b$
8. Beam Simply Supported at Ends – Uniformly distributed load ω (N/m)			
	$\theta_1 = \theta_2 = \dfrac{\omega l^3}{24EI}$	$y = \dfrac{\omega x}{24EI}\left(l^3 - 2lx^2 + x^3\right)$	$\delta_{max} = \dfrac{5\omega l^4}{384EI}$
9. Beam Simply Supported at Ends – Couple moment M at the right end			
	$\theta_1 = \dfrac{Ml}{6EI}$ $\theta_2 = \dfrac{Ml}{3EI}$	$y = \dfrac{Mlx}{6EI}\left(1 - \dfrac{x^2}{l^2}\right)$	$\delta_{max} = \dfrac{Ml^2}{9\sqrt{3}\,EI}$ at $x = \dfrac{l}{\sqrt{3}}$ $\delta = \dfrac{Ml^2}{16EI}$ at the center
10. Beam Simply Supported at Ends – Uniformly varying load: Maximum intensity ω_o (N/m)			
	$\theta_1 = \dfrac{7\omega_o l^3}{360EI}$ $\theta_2 = \dfrac{\omega_o l^3}{45EI}$	$y = \dfrac{\omega_o x}{360lEI}\left(7l^4 - 10l^2x^2 + 3x^4\right)$	$\delta_{max} = 0.00652\dfrac{\omega_o l^4}{EI}$ at $x = 0.519l$ $\delta = 0.00651\dfrac{\omega_o l^4}{EI}$ at the center

Macaulay's Method





Macaulay's Method

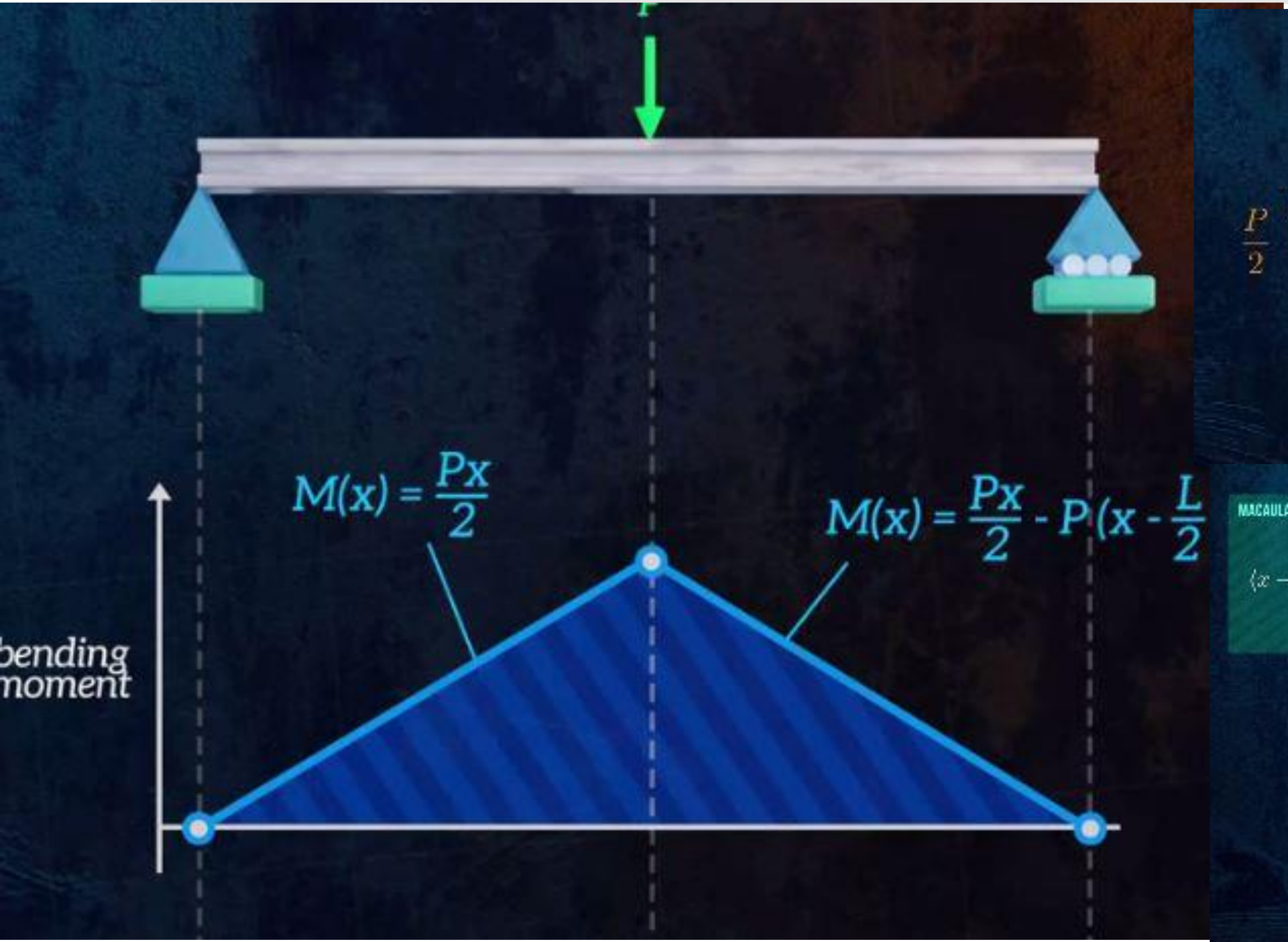
Used to write a single moment equation involving discontinuous functions without the need for matching conditions

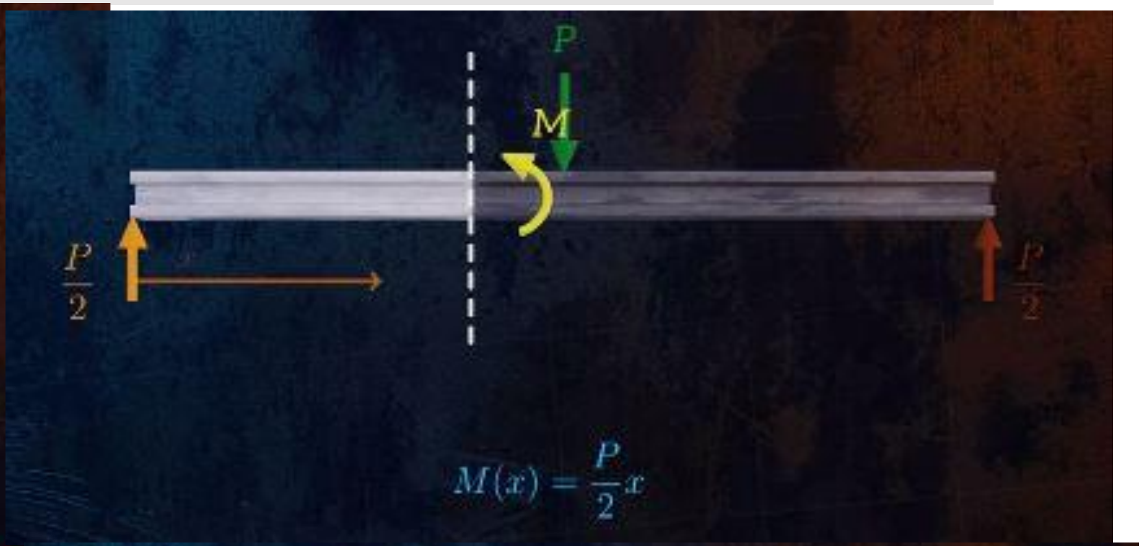
$$\langle x - x_0 \rangle^n = \begin{cases} 0 & for \quad x < x_0 \\ 1 & for \quad n = 0 \quad and \quad x \geq x_0 \\ (x - x_0)^n & for \quad n = 0 \quad and \quad x \geq x_0 \end{cases}$$

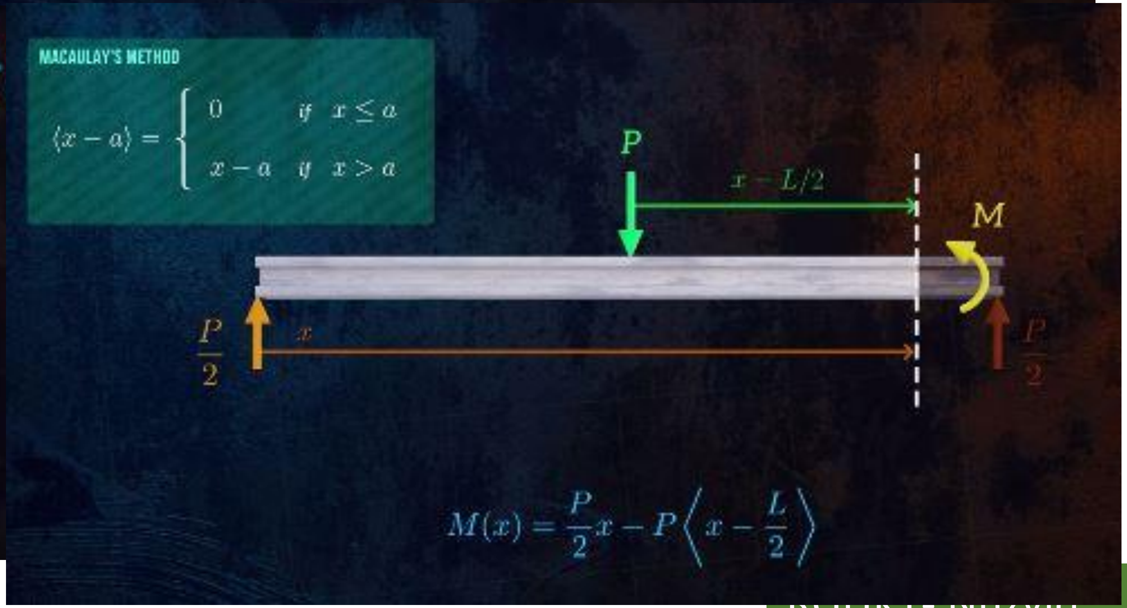
$$\frac{d}{dx}\langle x - x_0 \rangle^n = n \langle x - x_0 \rangle^{n-1} \qquad for \quad n \geq 1$$

$$\int \langle x - x_0 \rangle^n dx = \frac{1}{n+1} \langle x - x_0 \rangle^{n+1} + C \quad for \quad n \geq 0$$

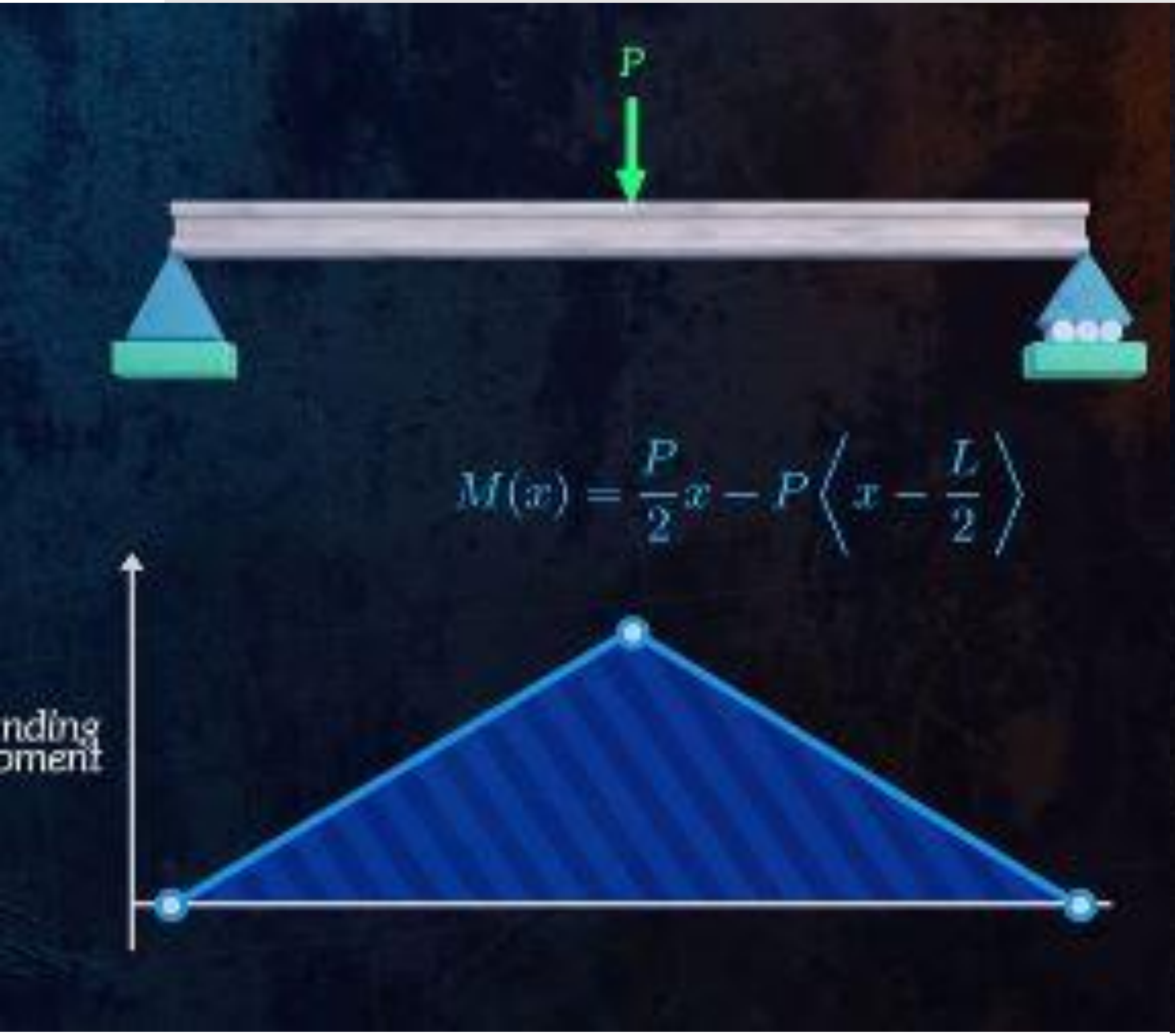
Macaulay's Method- Singularity Function







Macaulay's Method- Singularity Function



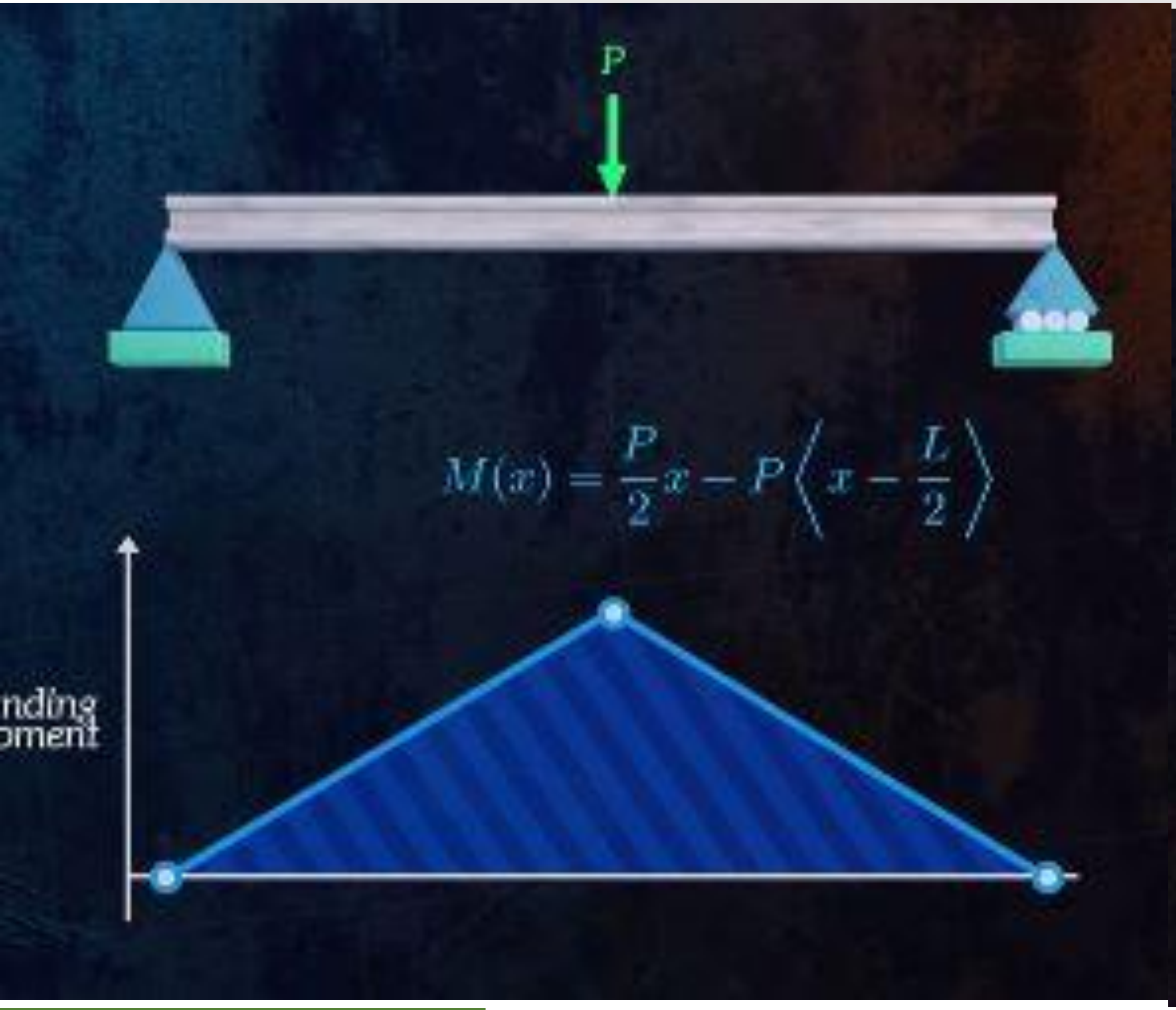
$$M(x) = \frac{P}{2}x - P\left\langle x - \frac{L}{2} \right\rangle$$

$$\frac{d^2y}{dx^2} = \frac{1}{EI}\left(\frac{P}{2}x - P\left\langle x - \frac{L}{2} \right\rangle\right)$$

$$\frac{dy}{dx} = \frac{1}{EI}\left(\frac{P}{4}x^2 - \frac{P}{2}\left\langle x - \frac{L}{2} \right\rangle^2 + C_1\right)$$

$$y = \frac{1}{EI}\left(\frac{P}{12}x^3 - \frac{P}{6}\left\langle x - \frac{L}{2} \right\rangle^3 + C_1x + C_2\right)$$

Macaulay’s Method- Singularity Function



$$y = \frac{1}{EI}\left(\frac{P}{12}x^3 - \frac{P}{6}\left\langle x - \frac{L}{2} \right\rangle^3 + C_1 x + C_2 \right)$$

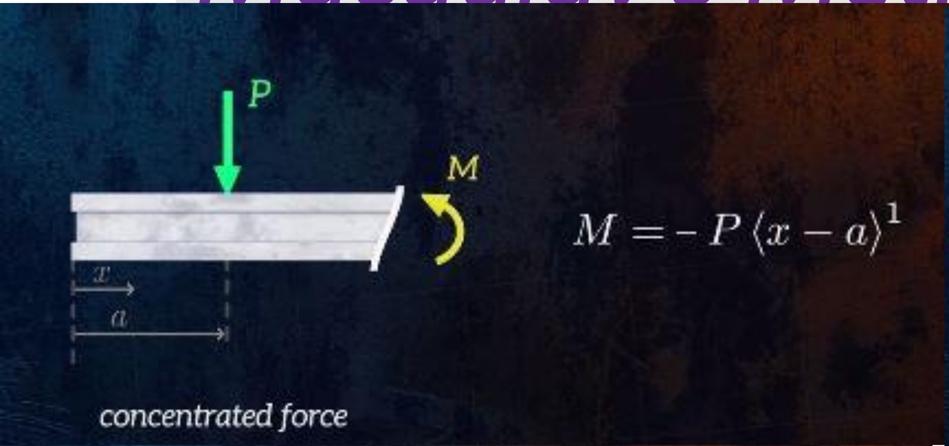
$$y(0) = 0 \rightarrow C_2 = 0$$

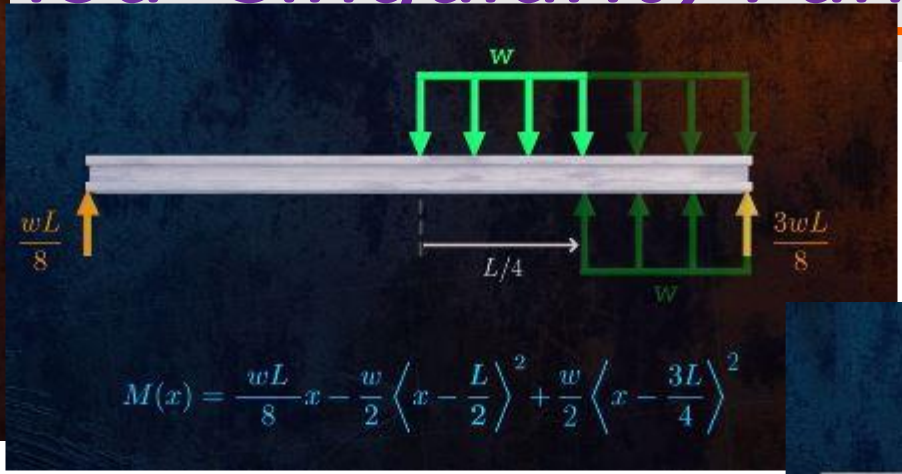
$$y(L) = 0 \rightarrow C_1 = -\frac{3PL^2}{48}$$

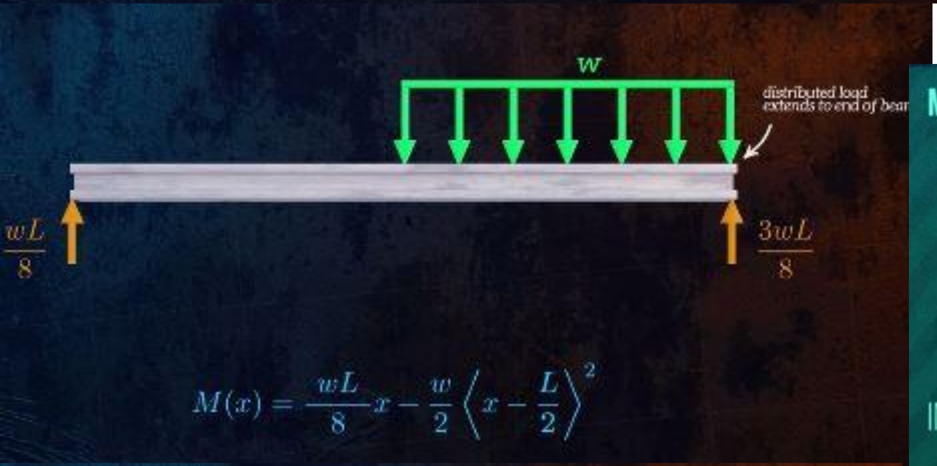
$$\rightarrow \theta_{max} = \theta(0) = -\frac{PL^2}{16EI}$$

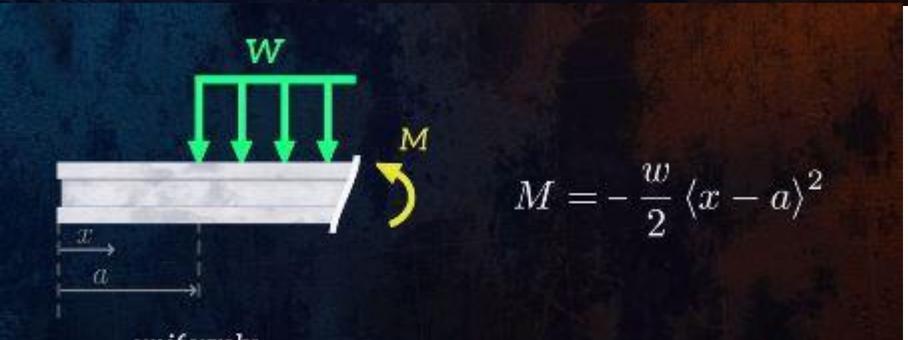
$$\rightarrow y_{max} = y(L/2) = -\frac{PL^3}{48EI}$$

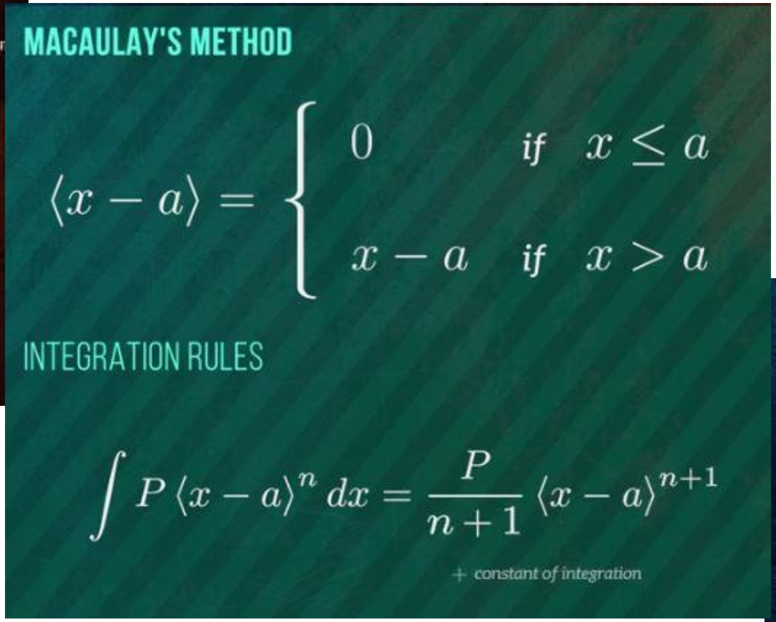
Macaulay's Method- Singularity Function

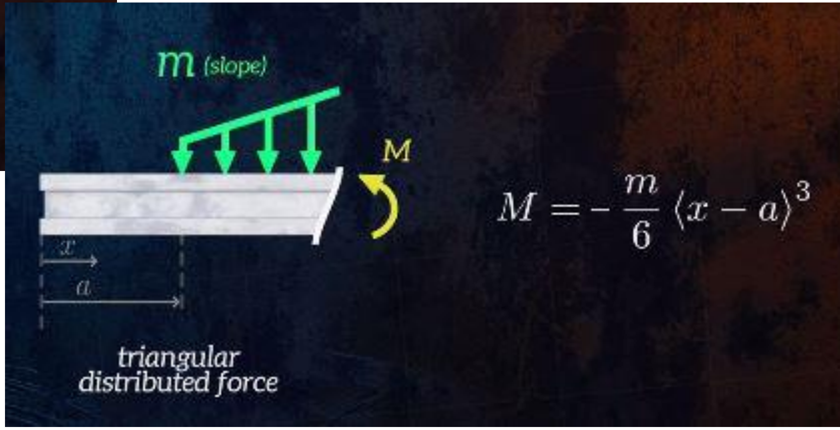


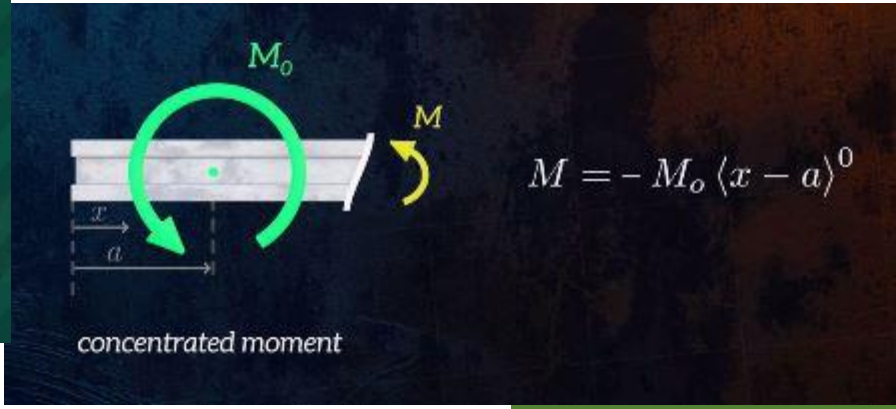






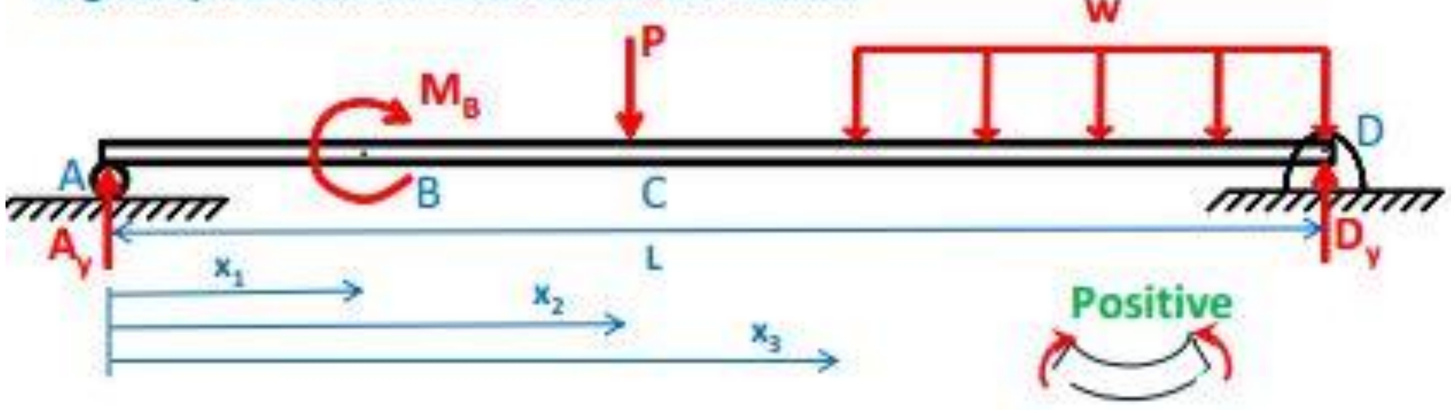






Macaulay’s Method- Singularity Function

Singularity Functions for Beam Deflection Problems



$$M_1 = A_y x \qquad 0 < x < x_1$$

$$M_2 = A_y x + M_B \qquad x_1 < x < x_2$$

$$M_3 = A_y x + M_B - P(x - x_2) \qquad x_2 < x < x_3$$

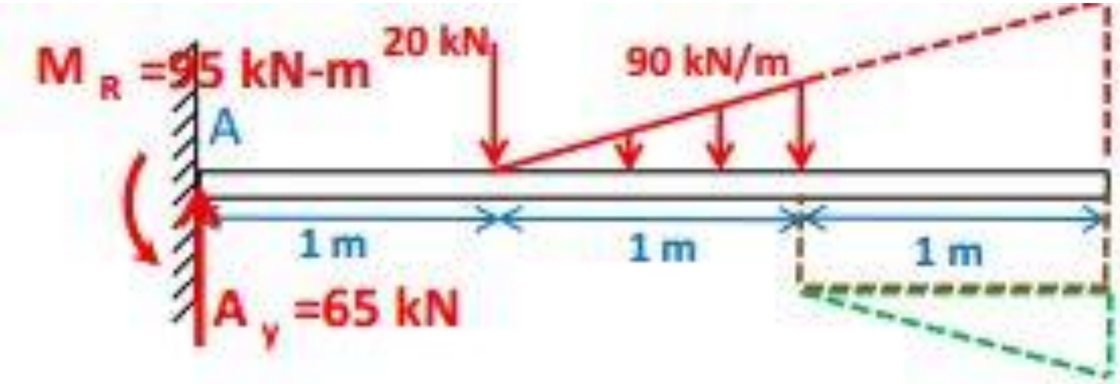
$$M_4 = A_y x + M_B - P(x - x_2) - w(x - x_3)\frac{(x - x_3)}{2} \qquad x_3 < x < L$$

$$M = A_y x + M_B\langle x - x_1\rangle^0 - P\langle x - x_2\rangle^1 - \frac{w}{2}\langle x - x_3\rangle^2 \qquad 0 \le x \le L$$

Macaulay's Method- Singularity Function

Macaulay's Method- Singularity Function

Macaulay’s Method- Singularity Function



$$\int \langle x - x_0 \rangle^n dx = \frac{1}{n+1} \langle x - x_0 \rangle^{n+1} + C \quad for \quad n \geq 0$$

$$E\,I \frac{d^2 y}{dx^2} = M(x) = -95 \frac{\langle x-0 \rangle^0}{0!} + 65 \frac{\langle x-0 \rangle^1}{1!} - 20 \frac{\langle x-1 \rangle^1}{1!} - \frac{90}{1} \frac{\langle x-1 \rangle^3}{3!} + \frac{90}{1} \frac{\langle x-2 \rangle^3}{3!} + 90 \frac{\langle x-2 \rangle^2}{2!}$$

$$EI \frac{dy}{dx} = -95 \frac{x^1}{1!} + 65 \frac{x^2}{2!} - 20 \frac{\langle x-1 \rangle^2}{2!} - 90 \frac{\langle x-1 \rangle^4}{4!} + 90 \frac{\langle x-2 \rangle^4}{4!} + 90 \frac{\langle x-2 \rangle^3}{3!}$$

Integrate again to find deflection equation

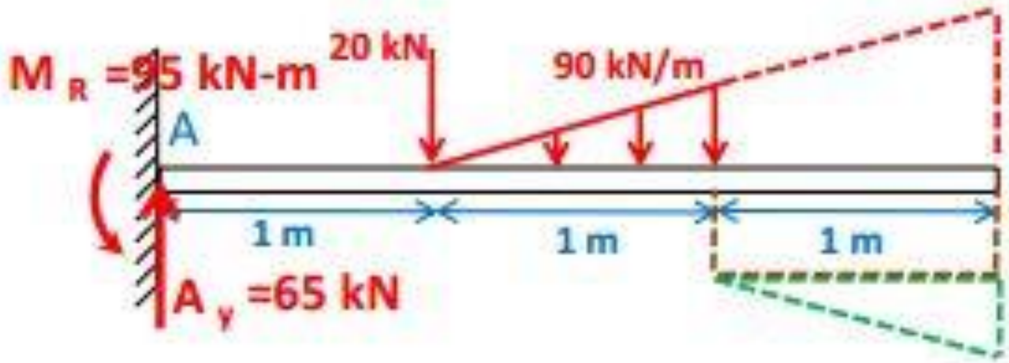
$$EI\ y = -95 \frac{x^2}{2!} + 65 \frac{x^3}{3!} - 20 \frac{\langle x-1 \rangle^3}{3!} - 90 \frac{\langle x-1 \rangle^5}{5!} + 90 \frac{\langle x-2 \rangle^5}{5!} + 90 \frac{\langle x-2 \rangle^4}{4!} + \cancel{C_2}$$

Boundary Condition: $y(0) = 0 = C_2$

Macaulay's Method- Singularity Function

Worksheet: Write the moment equation using singularity functions, integrate to find the deflection equation, and determine the deflection at the free end for the beam shown below made of steel with a modulus of elasticity, E = 200 GPa, and an area of moment of inertia for the cross section, I = 130 x 10⁶ mm⁴

Georg Te



applied moment: $M_o \dfrac{\langle x-a\rangle^0}{0!}$

point load: $P_b \dfrac{\langle x-b\rangle^1}{1!}$

distributed load: $w \dfrac{\langle x-c\rangle^2}{2!}$

ramp load: $\dfrac{\Delta w}{\Delta x}\dfrac{\langle x-d\rangle^3}{3!}$

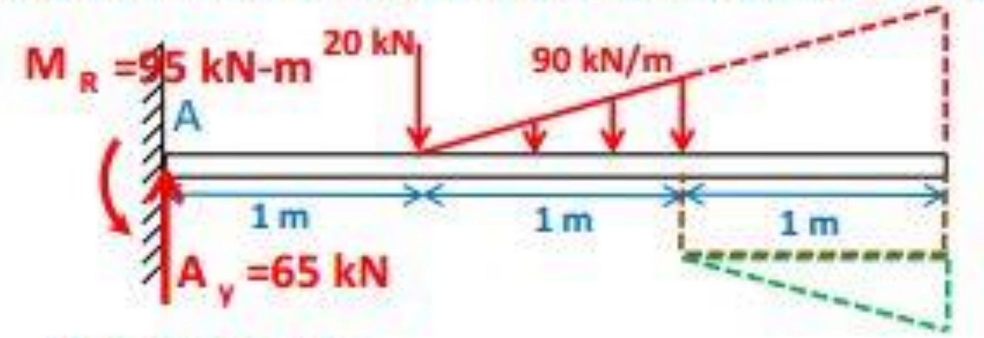
Write the moment equation using singularity functions

$$EI\frac{d^2y}{dx^2} = M(x) = -95\frac{\langle x-0\rangle^0}{0!} + 65\frac{\langle x-0\rangle^1}{1!} - 20\frac{\langle x-1\rangle^1}{1!} - \frac{90}{1}\frac{\langle x-1\rangle^3}{3!} + \frac{90}{1}\frac{\langle x-2\rangle^3}{3!} + 90\frac{\langle x-2\rangle^2}{2!}$$

ANS

Macaulay's Method- Singularity Function

Worksheet: Write the moment equation using singularity functions, integrate to find the deflection equation, and determine the deflection at the free end for the beam shown below made of steel with a modulus of elasticity, E = 200 GPa, and an area of moment of inertia for the cross section, I = 130 x 10^6 mm^4



$$y(3) = -6.97 \ mm$$

ANS

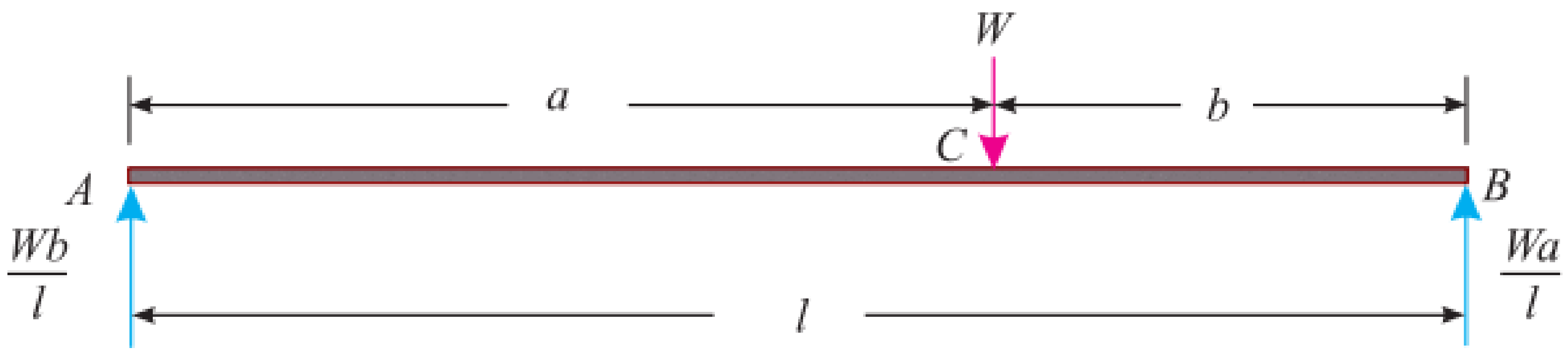
deflection equation

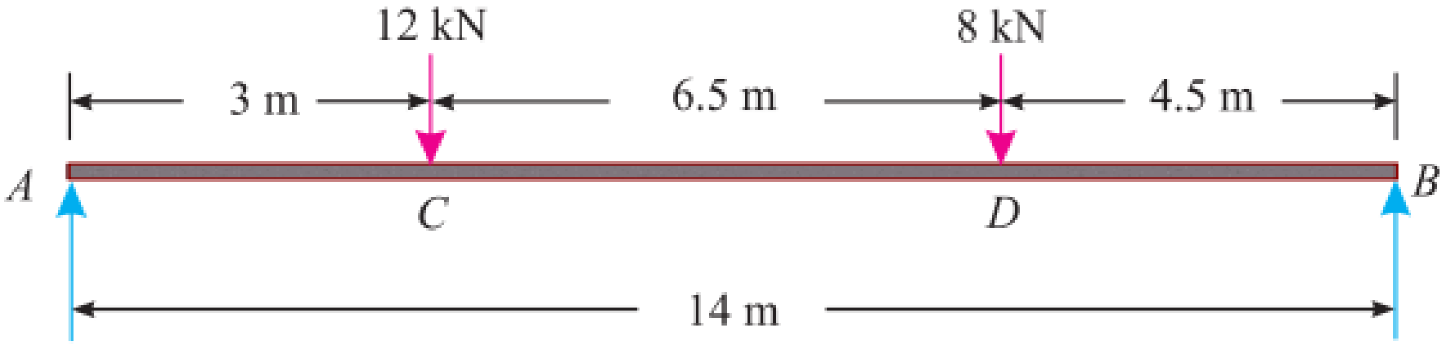
$$EI\ y = -95\frac{x^2}{2!} + 65\frac{x^3}{3!} - 20\frac{\langle x-1\rangle^3}{3!} - 90\frac{\langle x-1\rangle^5}{5!} + 90\frac{\langle x-2\rangle^5}{5!} + 90\frac{\langle x-2\rangle^4}{4!}$$

$$y(3) = \frac{1}{EI}\left\{-95\frac{3^2}{2!} + 65\frac{3^3}{3!} - 20\frac{\langle 3-1\rangle^3}{3!} - 90\frac{\langle 3-1\rangle^5}{5!} + 90\frac{\langle 3-2\rangle^5}{5!} + 90\frac{\langle 3-2\rangle^4}{4!}\right\}$$

$$y(3) = \frac{1}{200\ GPa\left(\frac{1\ kN/mm^2}{GPa}\right)(130\ x\ 10^6\ mm^4)}\left[-427.5 + 292.5 - 26.67 - 24 + 0.75 + 3.75\right] kN\cdot m^3\left(\frac{1\ x\ 10^9\ mm^3}{m^3}\right)$$

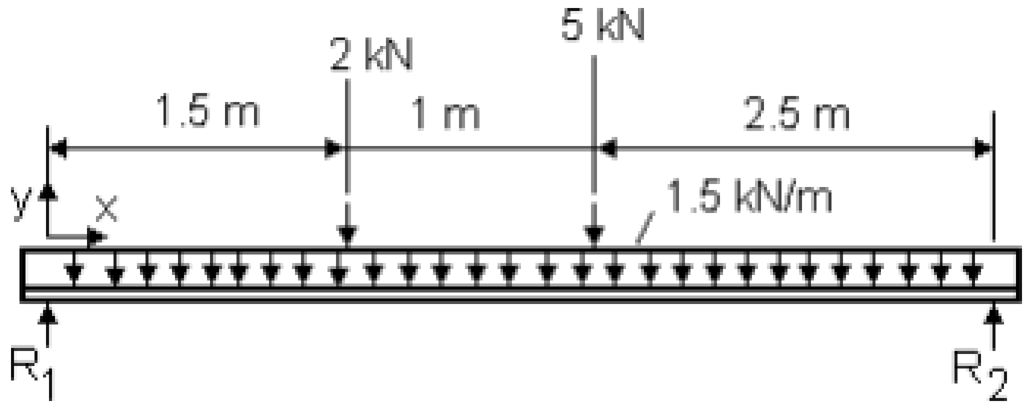
Macaulay's Method- Assignment

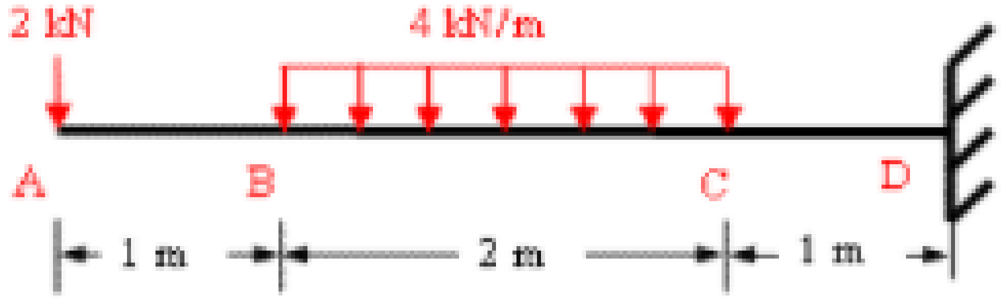




Macaulay's Method- Assignment

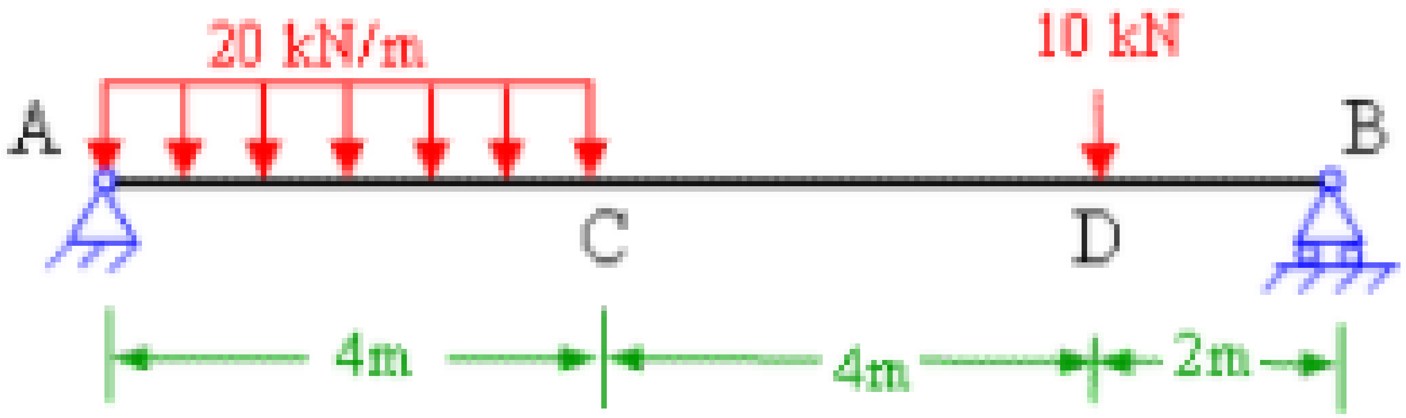
Find the deflection of the beam shown at the centre position. The flexural stiffness is 18 MNm^2. **(1.6 mm)**





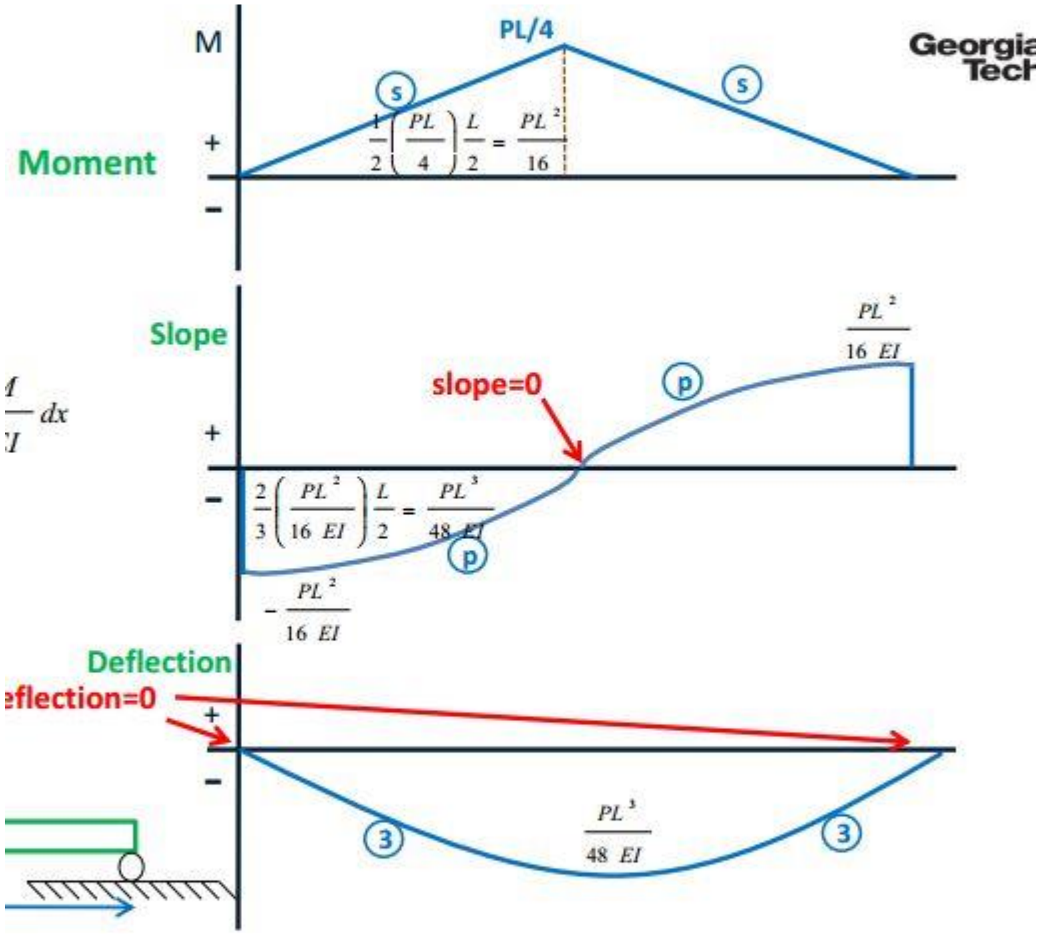
$$EI\ (d^2y/dx^2) = -2[x] - 4[x-1][x-1]/2 + 4[x-3][x-3]/2$$

Macaulay's Method- Assignment



$$EI\,(d^2y/dx^2) = 66[x] - 20[x]^2/2 + 20[x-4]^2/2 - 10[x-8]$$

Moment Area Method



$$E\,I\,\frac{d^2y}{dx^2} = M(x)$$

$$slope = \frac{dy}{dx} = \int \frac{M}{EI}\,dx$$

$$deflection = y = \iint \frac{M}{EI}\,dx\;dx$$

Methods

- **DOUBLE INTEGRATION** METHOD
- **MACAULAY'S** METHOD
- **MOMENT-AREA** METHOD
- **SUPERPOSITION** METHOD
- **CASTIGLIANO'S** THEOREM

$$\int_{x_1}^{x_2} \frac{d^2y}{dx^2}\,dx = \int_{x_1}^{x_2} \frac{M}{EI}\,dx$$

$$\left[\frac{dy}{dx}\right]_{x_1}^{x_2} = \int_{x_1}^{x_2} \frac{M}{EI}\,dx$$

$$\theta_2 - \theta_1 = \frac{1}{EI}\int_{x_1}^{x_2} M\,dx$$

$$\int_{x_1}^{x_2} \frac{d^2y}{dx^2}\,x\,dx = \int_{x_1}^{x_2} \frac{M}{EI}\,x\,dx$$

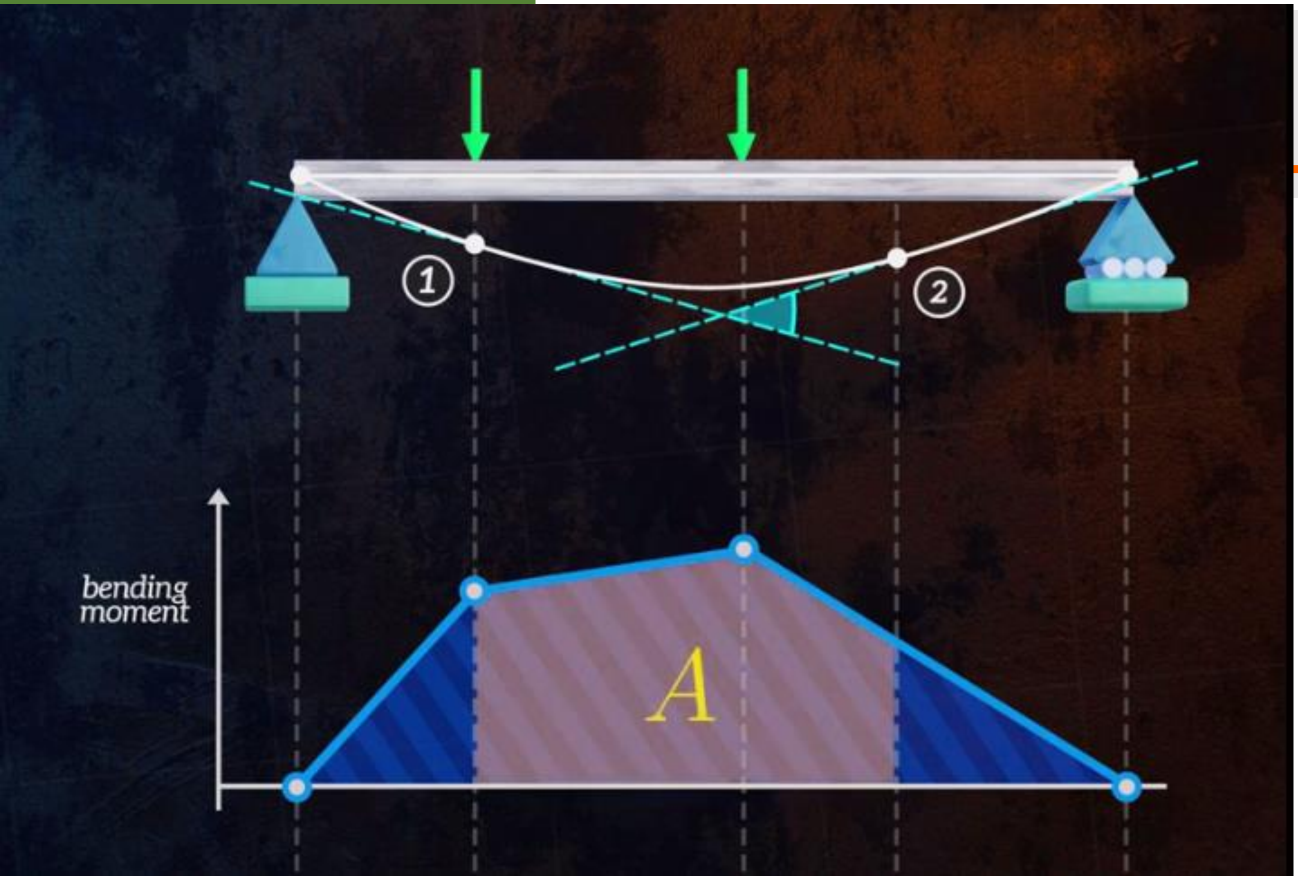
$$t_{2/1} = \frac{1}{EI}\int_{x_1}^{x_2} M\,x\,dx$$

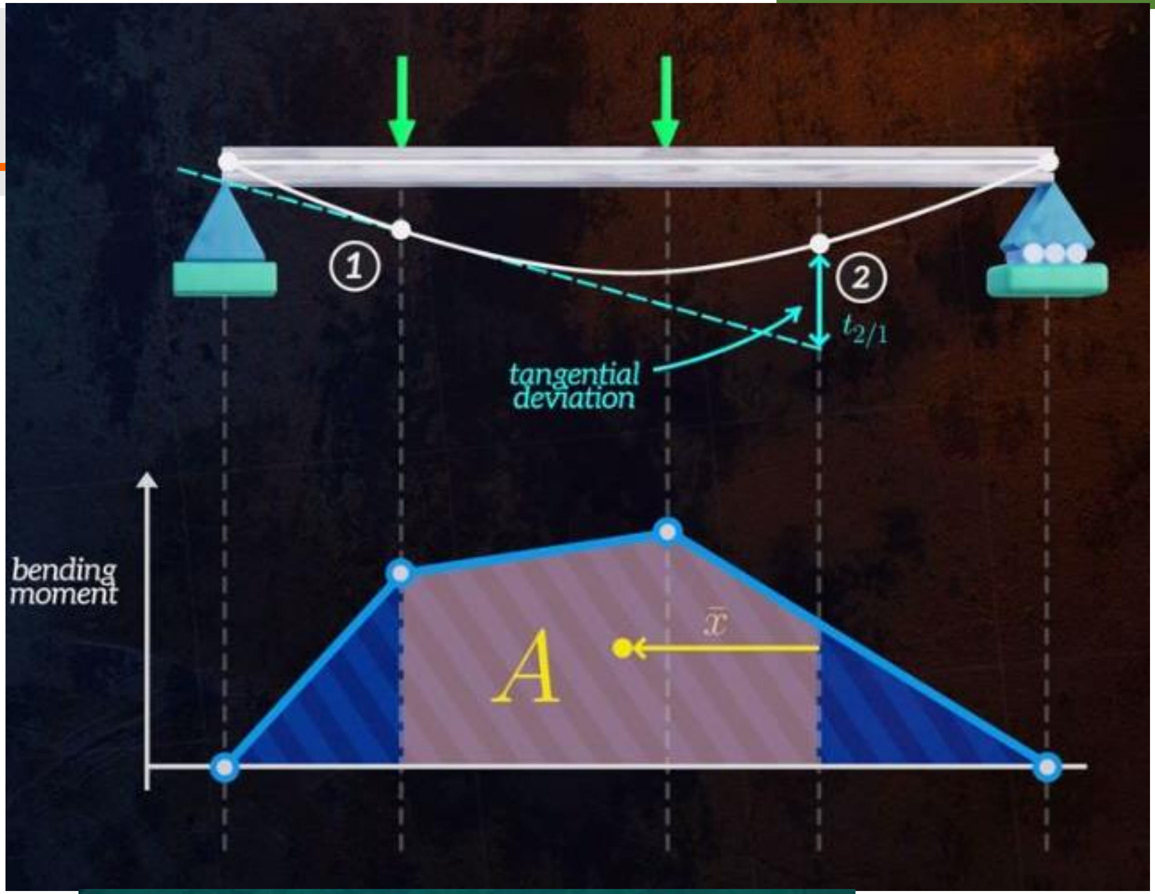
MOMENT-AREA METHOD
FIRST THEOREM

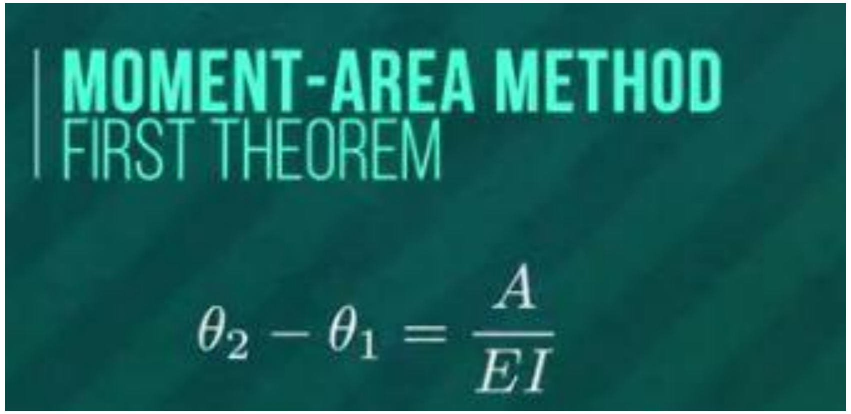
$$\theta_2 - \theta_1 = \frac{A}{EI}$$

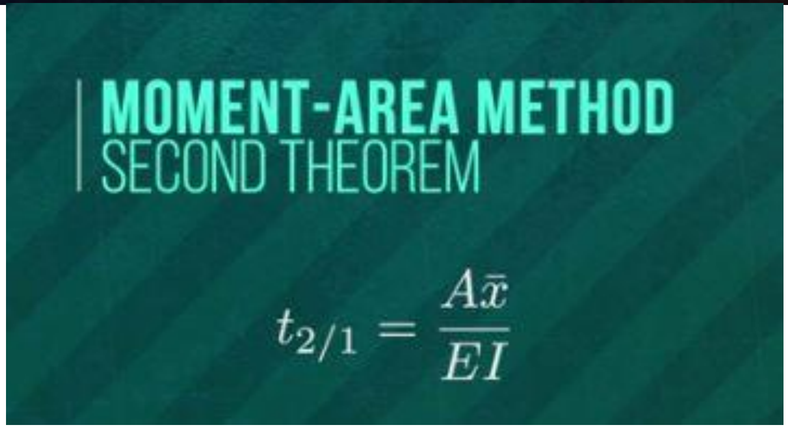
MOMENT-AREA METHOD
SECOND THEOREM

$$t_{2/1} = \frac{A\bar{x}}{EI}$$

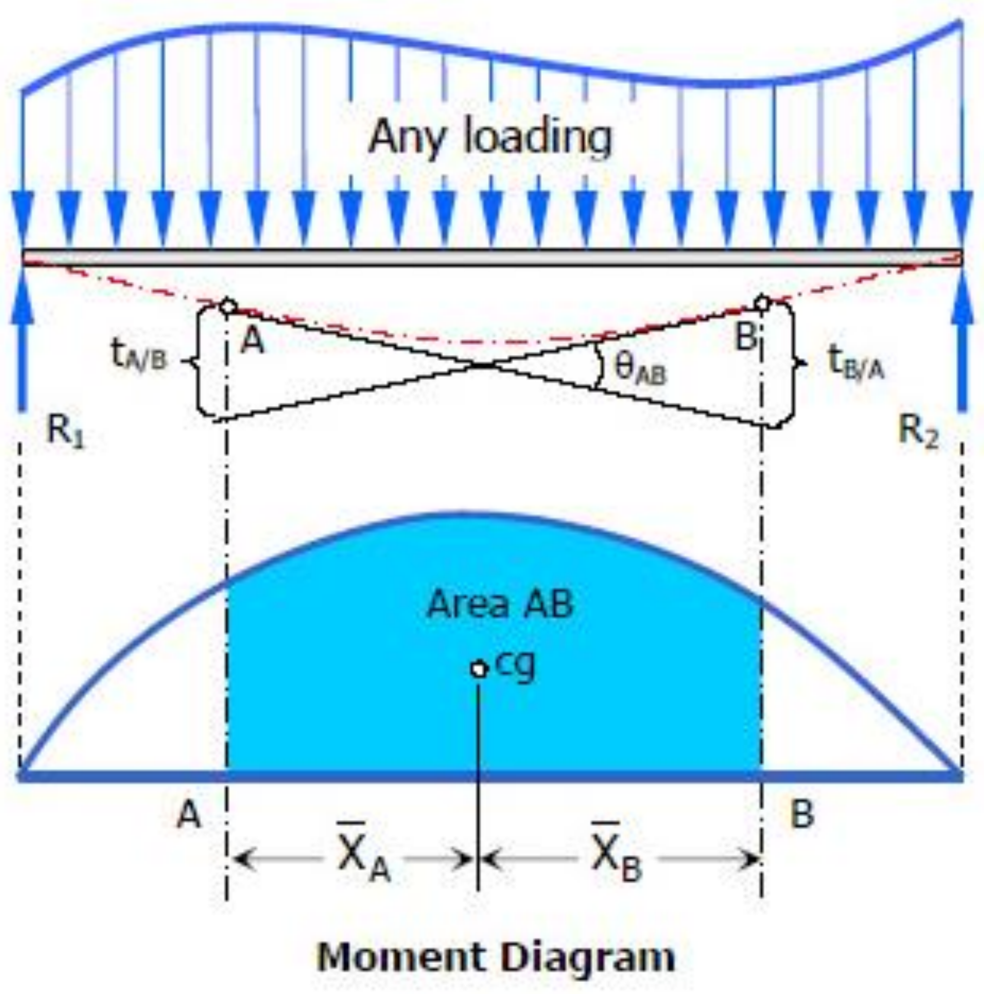

1
2
bending moment
A


1
2
$t_{2/1}$
tangential deviation
bending moment
A
$\bar{x}$


MOMENT-AREA METHOD
FIRST THEOREM
$\theta_2 - \theta_1 = \frac{A}{EI}$


MOMENT-AREA METHOD
SECOND THEOREM
$t_{2/1} = \frac{A\bar{x}}{EI}$

Moment Area Method



Theorem-I

The change in **slope between the tangents drawn to the elastic curve at any two points A and B** is equal to the product of 1/EI multiplied by the area of the moment diagram between these two points.

$$\theta_{AB} = \frac{1}{EI}(Area_{AB})$$

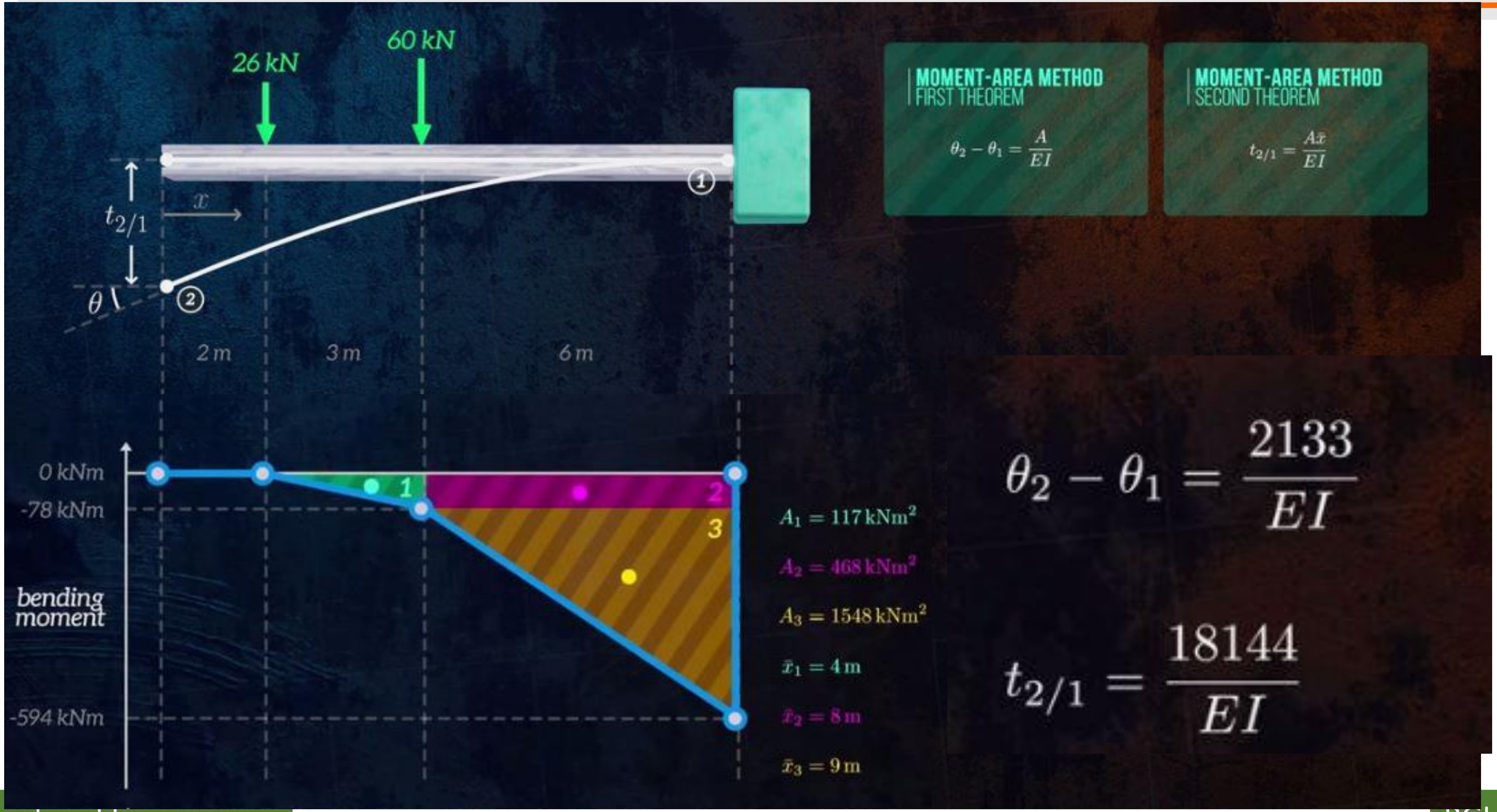
Theorem-II

The deviation **of any point B relative to the tangent drawn** to the elastic curve at any other point A, in a direction perpendicular to the original position of the beam, is equal to the product of 1/EI multiplied by the moment of an area about B of that part of the moment diagram between points A and B.

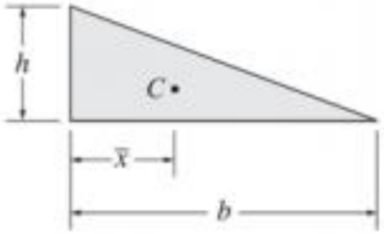
$$t_{B/A} = \frac{1}{EI}(Area_{AB}) \cdot \bar{X}_B$$

$$t_{A/B} = \frac{1}{EI}(Area_{AB}) \cdot \bar{X}_A$$

Moment Area Method- Cantilever Beam



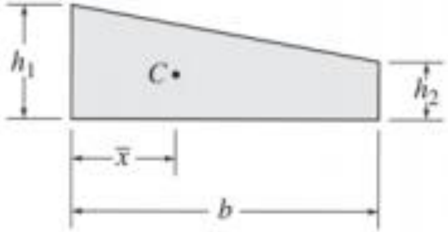
Moment Area Method- Area formula



$A = \frac{1}{2}bh$

$\bar{x} = \frac{1}{3}b$

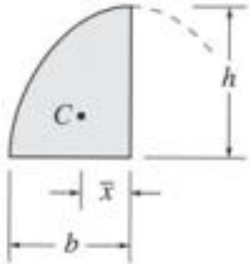
Triangle



$A = \frac{1}{2}b\,(h_1 + h_2)$

$\bar{x} = \frac{b\,(2h_2 + h_1)}{3\,(h_1 + h_2)}$

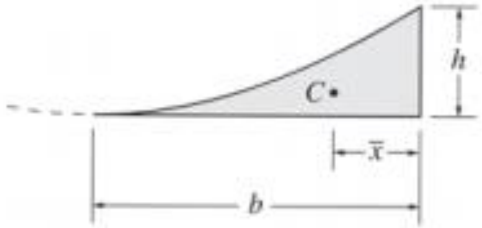
Trapezoid



$A = \frac{2}{3}bh$

$\bar{x} = \frac{3}{8}b$

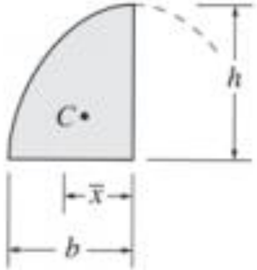
Semi Parabola



$A = \frac{1}{3}bh$

$\bar{x} = \frac{1}{4}b$

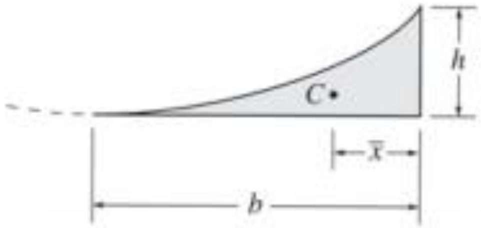
Parabolic spandrel



$A = bh\,(\frac{n}{n+1})$

$\bar{x} = \frac{b\,(n+1)}{2\,(n+2)}$

Semi-segment of *n*th degree curve



$A = bh\,(\frac{1}{n+1})$

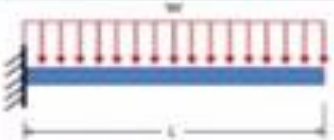
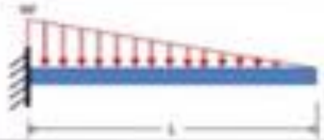
$\bar{x} = \frac{b}{(n+2)}$

Spandrel of *n*th degree curve

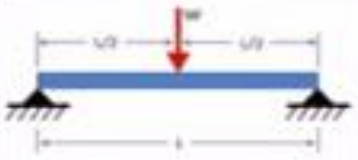
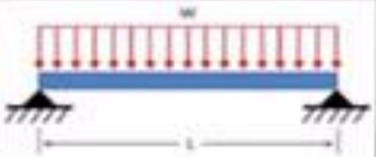
Moment Area Method- Area formula

SR. NO.	TYPE OF BEAM	MAX. BM	SLOPE	DEFLECTON
1		M	$\theta = \frac{ML}{EI} = \frac{ML}{EI}$	$\delta = \theta \times \frac{L}{2} = \frac{ML^2}{2EI}$
2		WL	$\theta = \frac{ML}{2EI} = \frac{WL^2}{2EI}$	$\delta = \theta \times \frac{2L}{3} = \frac{WL^3}{3EI}$

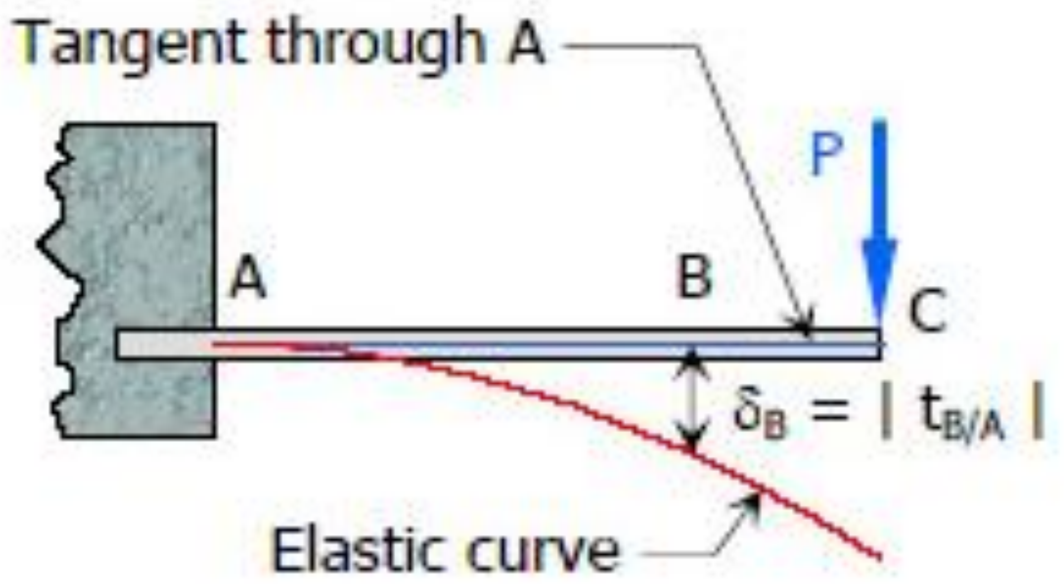
Moment Area Method- Area formula

SR. NO.	TYPE OF BEAM	MAX. BM	SLOPE	DEFLECTON
3		$\frac{WL^2}{2}$	$\theta = \frac{ML}{3EI} = \frac{WL^3}{6EI}$	$\delta = \theta \times \frac{3L}{4} = \frac{WL^4}{8EI}$
4		$\frac{WL^2}{6}$	$\theta = \frac{ML}{4EI} = \frac{WL^3}{24EI}$	$\delta = \theta \times \frac{4L}{5} = \frac{WL^4}{30EI}$

Moment Area Method- Area formula

5		$\frac{WL}{4}$	$\theta=\frac{ML}{4EI}=\frac{WL^2}{16EI}$	$\delta=\theta\times\frac{L}{3}=\frac{WL^3}{48EI}$
6		$\frac{WL^2}{8}$	$\theta=\frac{ML}{3EI}=\frac{WL^3}{24EI}$	$\delta=\theta\times\frac{5L}{16}=\frac{5WL^4}{384EI}$

Moment Area Method- SSB – Problem-2

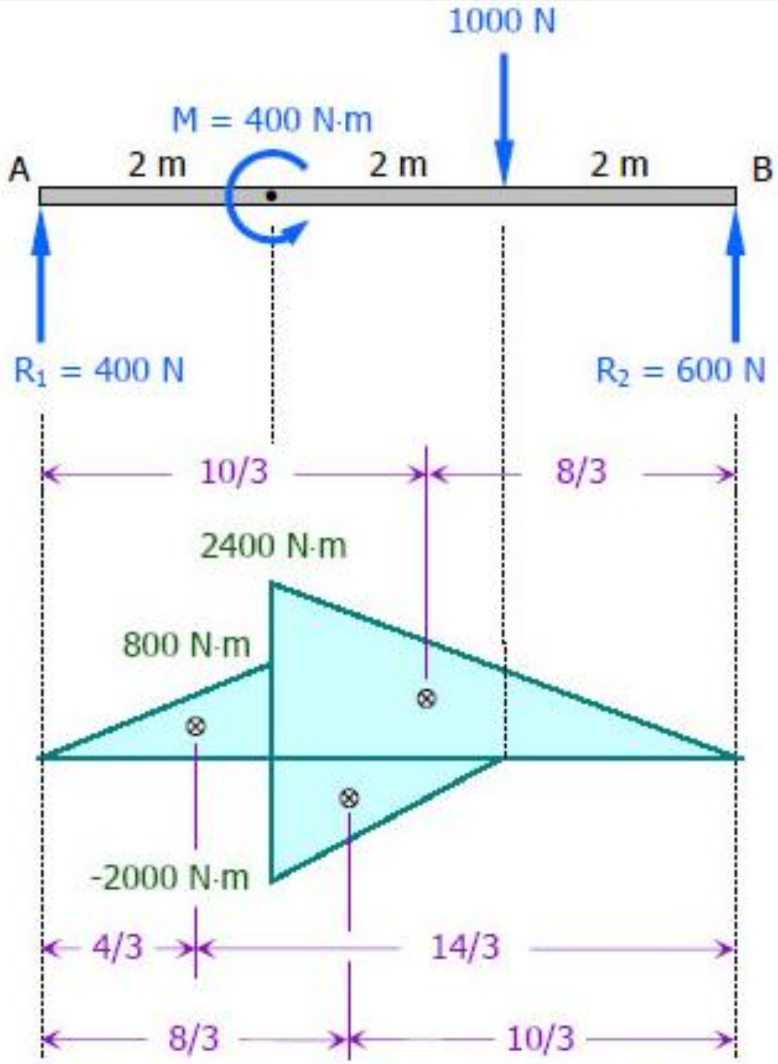


- Generally, the **tangential deviation t is not equal to the beam deflection.**
- In cantilever beams, however, the tangent drawn to the elastic curve at the wall is horizontal and **coincidence therefore with the neutral axis of the beam.**
- The **tangential deviation in this case is equal to the deflection of the beam** as shown below.

Moment Area Method- Moment By Parts

Moment Area Method- Moment By Parts

Moment Area Method- Moment By Parts



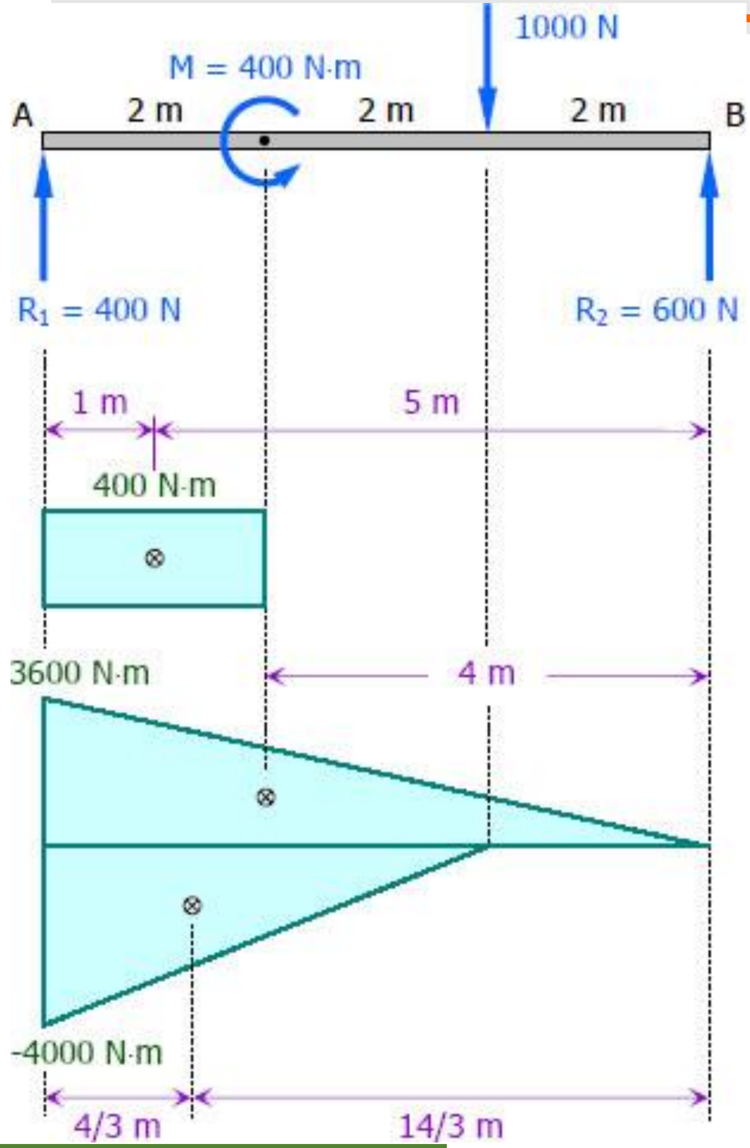
$$(Area_{AB})\bar{X}_A = \tfrac{1}{2}(2)(800)(\tfrac{4}{3}) + \tfrac{1}{2}(4)(2400)(\tfrac{10}{3}) - \tfrac{1}{2}(2)(2000)(\tfrac{8}{3})$$

$$(Area_{AB})\bar{X}_A = 11\,733.33\ \mathrm{N\cdot m^3}$$

$$(Area_{AB})\bar{X}_B = \tfrac{1}{2}(2)(800)(\tfrac{14}{3}) + \tfrac{1}{2}(4)(2400)(\tfrac{8}{3}) - \tfrac{1}{2}(2)(2000)(\tfrac{10}{3})$$

$$(Area_{AB})\bar{X}_B = 9\,866.67\ \mathrm{N\cdot m^3}$$

Moment Area Method- Moment By Parts



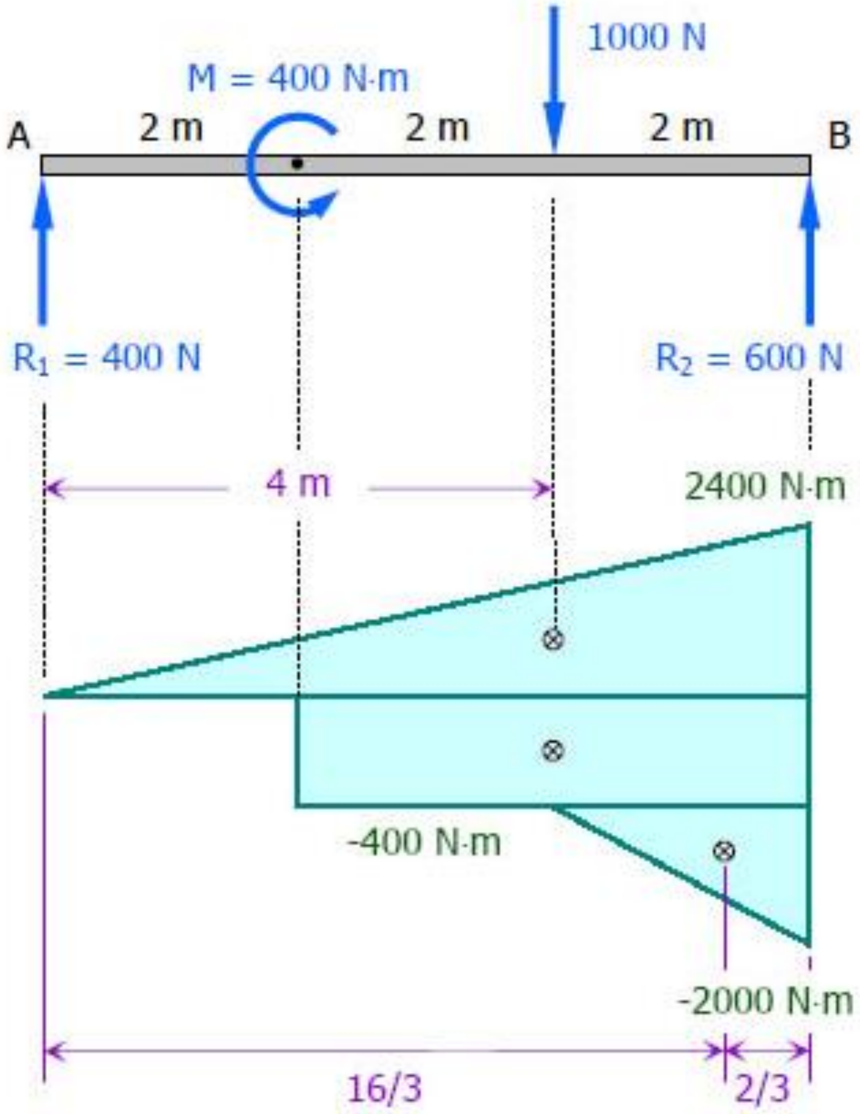
$$(Area_{AB})\bar{X}_A = 400(2)(1) + \tfrac{1}{2}(6)(3600)(2) - \tfrac{1}{2}(4)(4000)(\tfrac{4}{3})$$

$$(Area_{AB})\bar{X}_A = 11\,733.33\,\mathrm{N \cdot m^3}$$

$$(Area_{AB})\bar{X}_B = 400(2)(5) + \tfrac{1}{2}(6)(3600)(4) - \tfrac{1}{2}(4)(4000)(\tfrac{14}{3})$$

$$(Area_{AB})\bar{X}_B = 9\,866.67\,\mathrm{N \cdot m^3}$$

Moment Area Method- Moment By Parts



$$(Area_{AB})\,\bar{X}_A = \tfrac{1}{2}(6)(2400)(4) - 400(4)(4) - \tfrac{1}{2}(2)(2000)(\tfrac{16}{3})$$

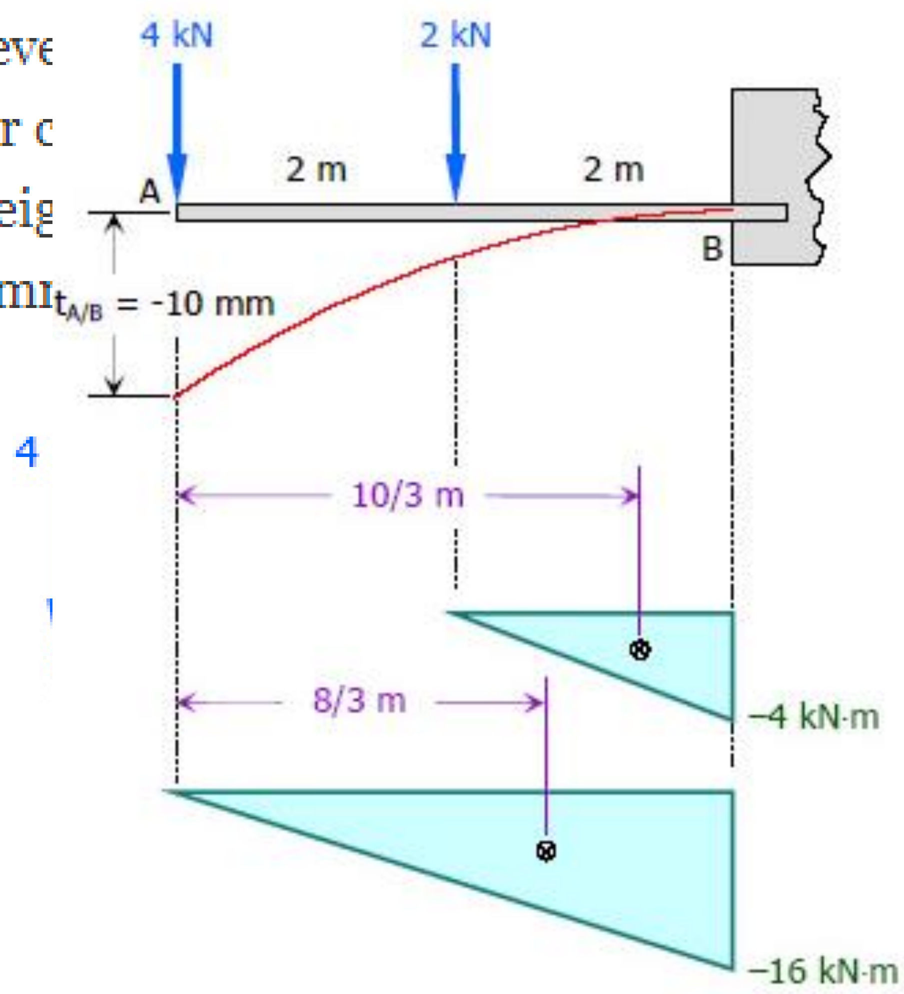
$$(Area_{AB})\bar{X}_A = 11\,733.33\,\mathrm{N\cdot m^3}$$

$$(Area_{AB})\bar{X}_B = \tfrac{1}{2}(6)(2400)(2) - 400(4)(2) - \tfrac{1}{2}(2)(2000)(\tfrac{2}{3})$$

$$(Area_{AB})\bar{X}_B = 9\,866.67\,\mathrm{N\cdot m^3}$$

Moment Area Method- SSB – Problem-2

The cantileve
rectangular c
Find the heig
exceed 10 mı



$$t_{A/B} = \frac{1}{EI}(Area_{AB})\,\bar{X}_A$$

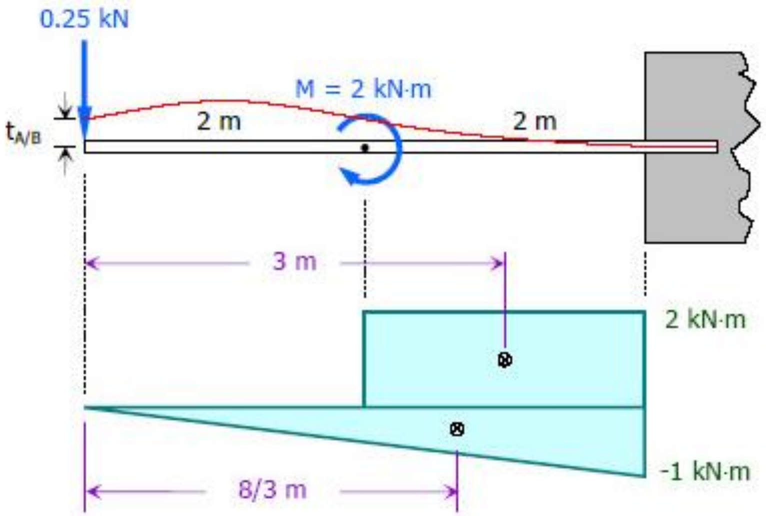
$$-10 = \frac{1}{10\,000\left(\frac{50h^3}{12}\right)}\left[-\tfrac{1}{2}(2)(4)(\tfrac{10}{3}) - \tfrac{1}{2}(4)(16)(\tfrac{8}{3})\right](1000$$

$$-10 = \frac{3}{125\,000h^3}\left[-\frac{296}{3}\right](1000^4)$$

$$h^3 = \frac{-296(1000^4)}{125\,000(-10)}$$

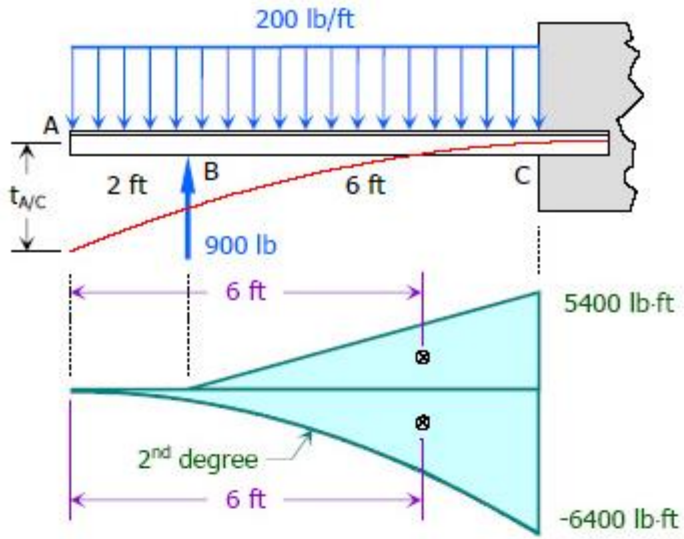
$$h = 618.67\text{ mm} \qquad \textit{answer}$$

Moment Area Method- SSB – Problem-2



$$EI\,t_{A/B} = (Area_{AB})\,\bar{X}_A$$
$$EI\,t_{A/B} = 2(2)(3) - \tfrac{1}{2}(4)(1)(\tfrac{8}{3})$$
$$EI\,t_{A/B} = \tfrac{20}{3} = 6.67\ \text{kN}\cdot\text{m}^3$$

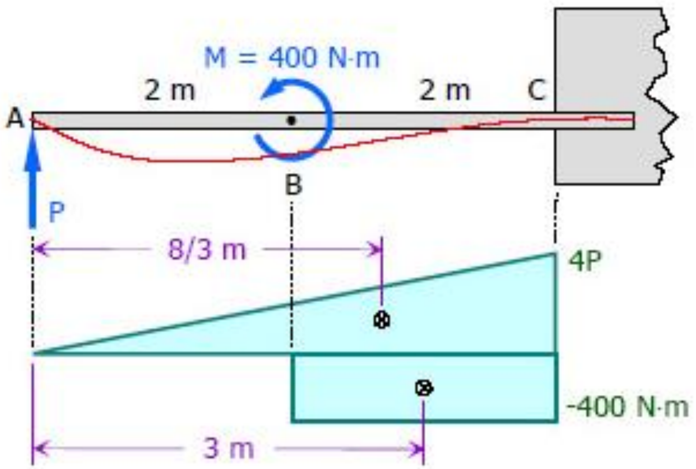
∴ EIδ = 6.67 kN·m³ upward *answer*



$$t_{A/C} = \frac{1}{EI}(Area_{AB})\,\bar{X}_A$$
$$t_{A/C} = \frac{1}{(1.5\times10^6)(60)}\left[\,\tfrac{1}{2}(6)(5400)(6) - \tfrac{1}{3}(8)(6400)(6)\,\right](12^3)$$
$$t_{A/C} = -0.09984\ \text{in}$$

∴ The free end will move by 0.09984 inch downward. *answer*

Moment Area Method- Cantilever



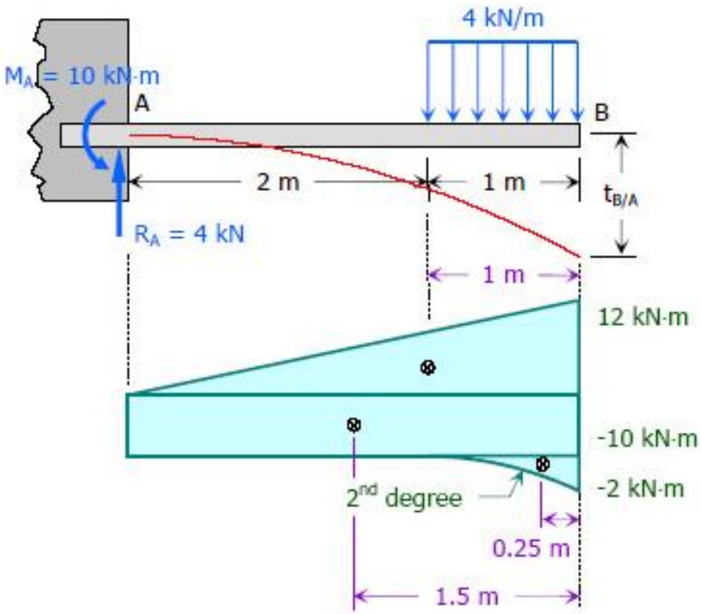
$$\frac{1}{EI}(Area_{AC})\,\bar{X}_A = 0$$

$$\frac{1}{EI}\left[\,\tfrac{1}{2}(4)(4P)(\tfrac{8}{3}) - 2(400)(3)\,\right] = 0$$

$$P = 112.5\ \text{N} \qquad answer$$

$$R_A = 4(1) = 4\,\text{kN}$$

$$M_A = 4(1)(2.5) = 10\,\text{kN}\cdot\text{m}$$

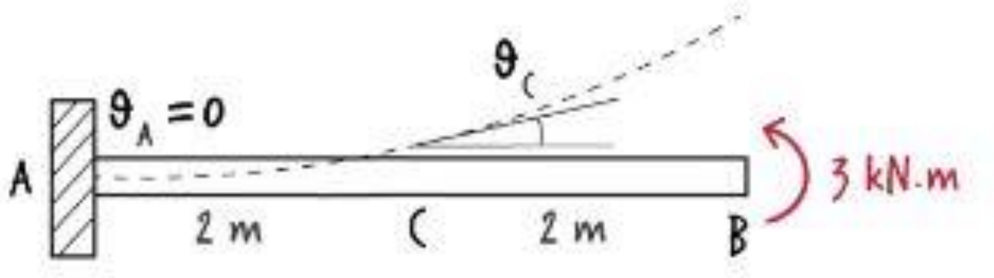


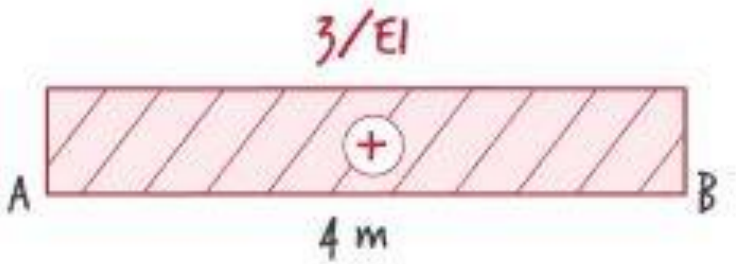
$$t_{B/A} = \frac{1}{EI}(Area_{AB})\,\bar{X}_B$$

$$t_{B/A} = \frac{1}{69\,000\left[\dfrac{50(150^3)}{12}\right]}\left[\,\tfrac{1}{2}(3)(12)(1) - 3(10)(1.5) - \tfrac{1}{3}(1)(2)(0.25)\,\right](1000^4)$$

$$t_{B/A} = -28\,\text{mm}$$

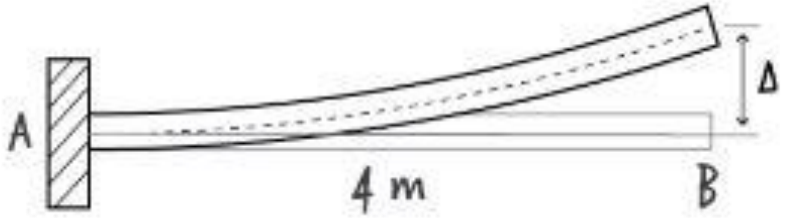
Moment Area Method- SSB – Problem-2

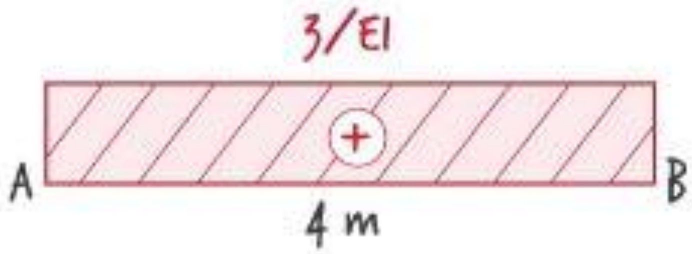


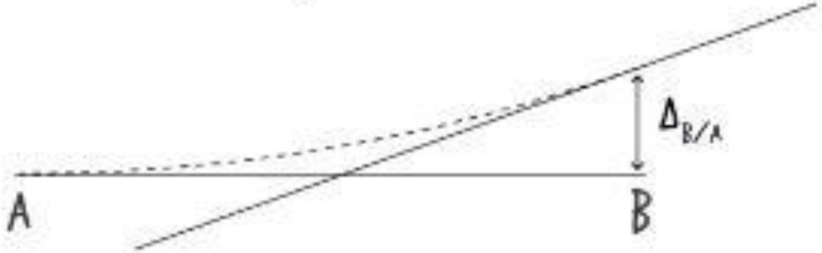


$$\vartheta_C - \vartheta_A = \int_0^2 \frac{M}{EI} dx$$

$$\vartheta_C - \vartheta_A = \frac{6}{EI} \Rightarrow \boxed{\vartheta_C = \frac{6}{EI}}$$

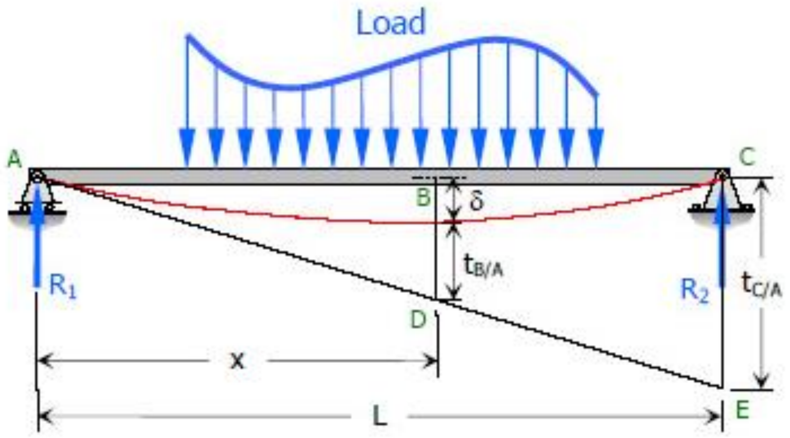






$$\Delta = \Delta_{B/A} = (2)\left(\frac{3}{EI}\right)(4) \Rightarrow \boxed{\Delta = \frac{24}{EI}}$$

Moment Area Method- SSB – Problem-2



Geometry of area-moment method for finding deformation δ in simply supported beam

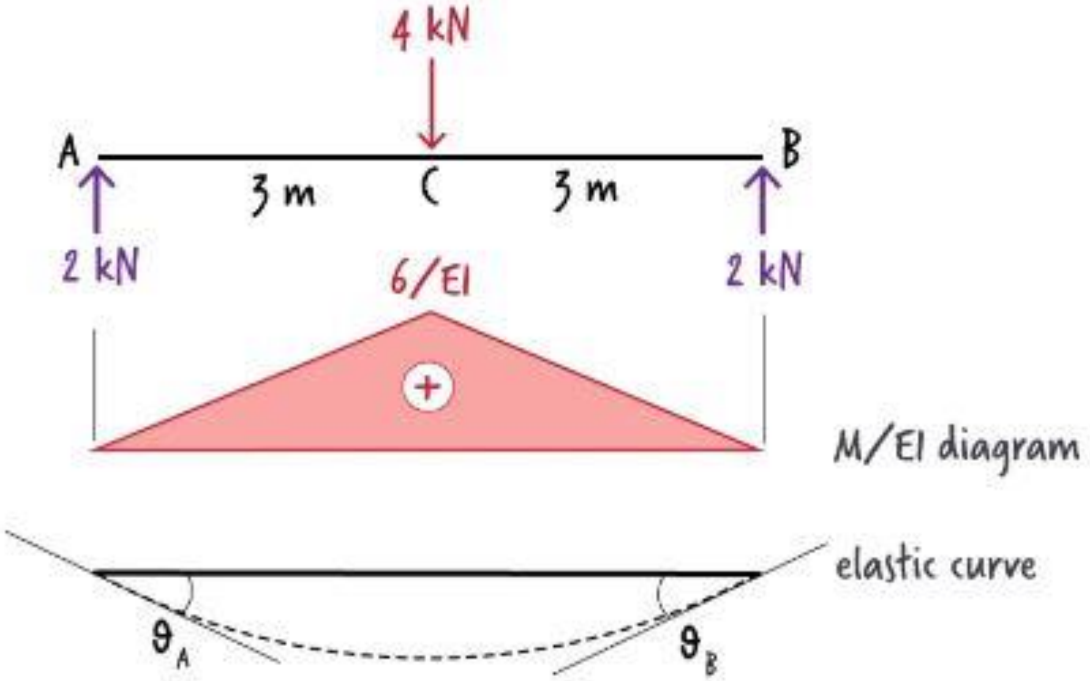
1. Compute $t_{C/A} = \dfrac{1}{EI}(Area_{AC})\,\bar{X}_C$

2. Compute $t_{B/A} = \dfrac{1}{EI}(Area_{AB})\,\bar{X}_B$

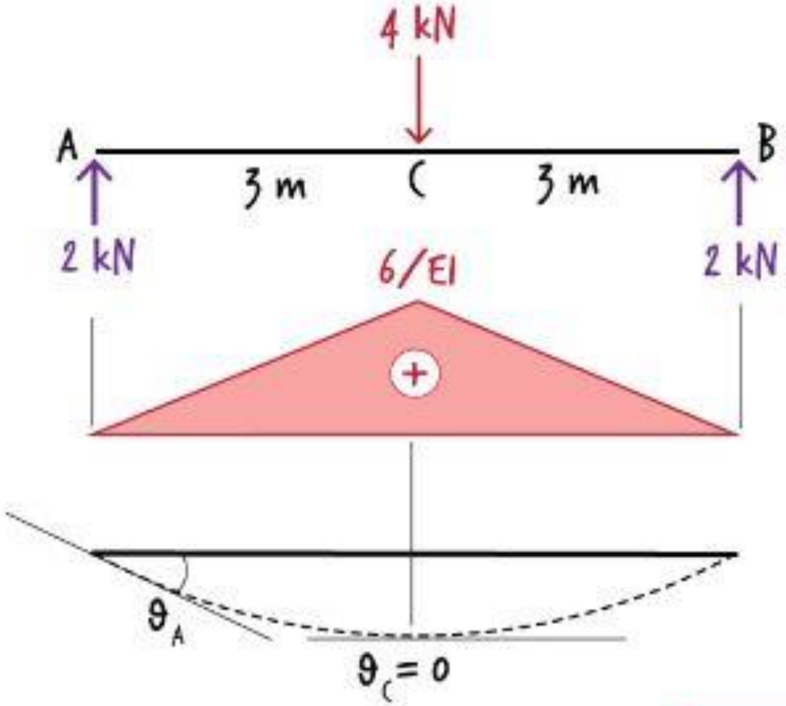
3. Solve δ by ratio and proportion (see figure above).

$$\frac{\delta + t_{B/A}}{x} = \frac{t_{C/A}}{L}$$

Moment Area Method- SSB – Problem-3

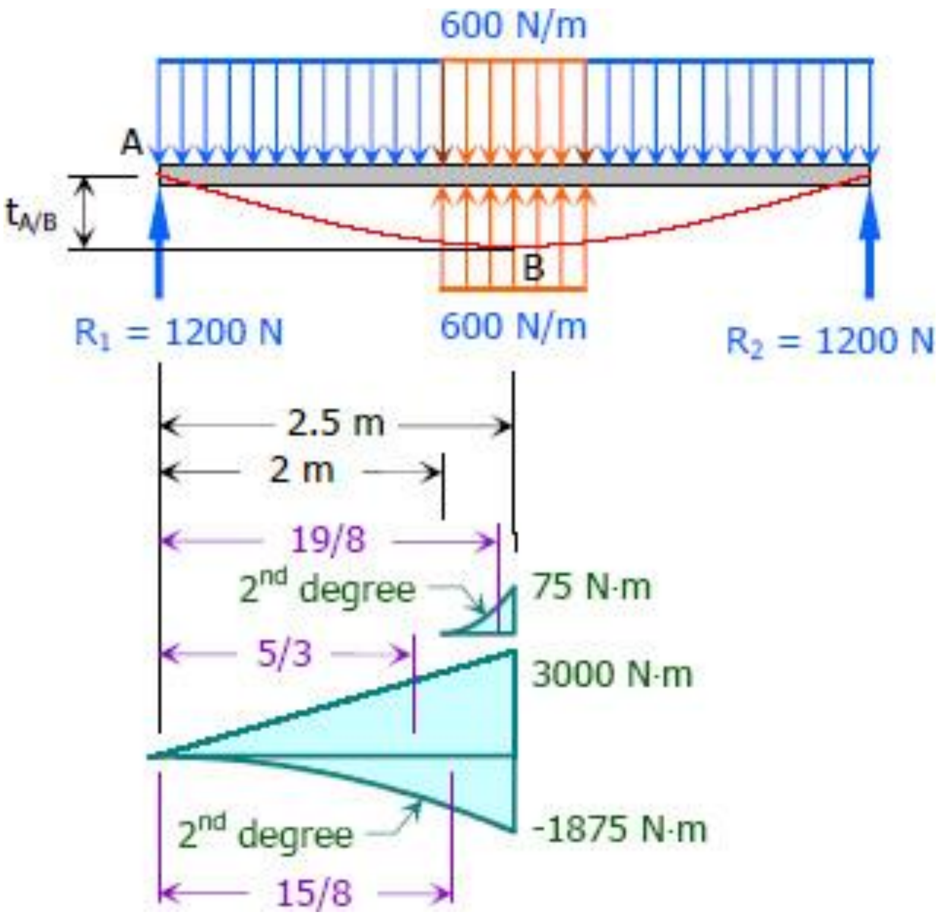


$$\theta_B - \theta_A = 6\left(\frac{6}{EI}\right)\left(\frac{1}{2}\right) = \frac{18}{EI}$$



$$\theta_C - \theta_A = 3\left(\frac{6}{EI}\right)\left(\frac{1}{2}\right) = \frac{9}{EI} \Rightarrow \boxed{\theta_A = -\frac{9}{EI}}$$

Moment Area Method- SSB – Problem-4



$$t_{A/B} = \frac{1}{EI}(Area_{AB})\,\bar{X}_A$$

$$t_{A/B} = \frac{1}{EI}\left[\,\tfrac{1}{2}(2.5)(3000)(\tfrac{5}{3}) + \tfrac{1}{3}(0.5)(75)(\tfrac{19}{8}) - \tfrac{1}{3}(2.5)(1875)(\tfrac{15}{8})\right]$$

$$t_{A/B} = \frac{3350}{EI}$$

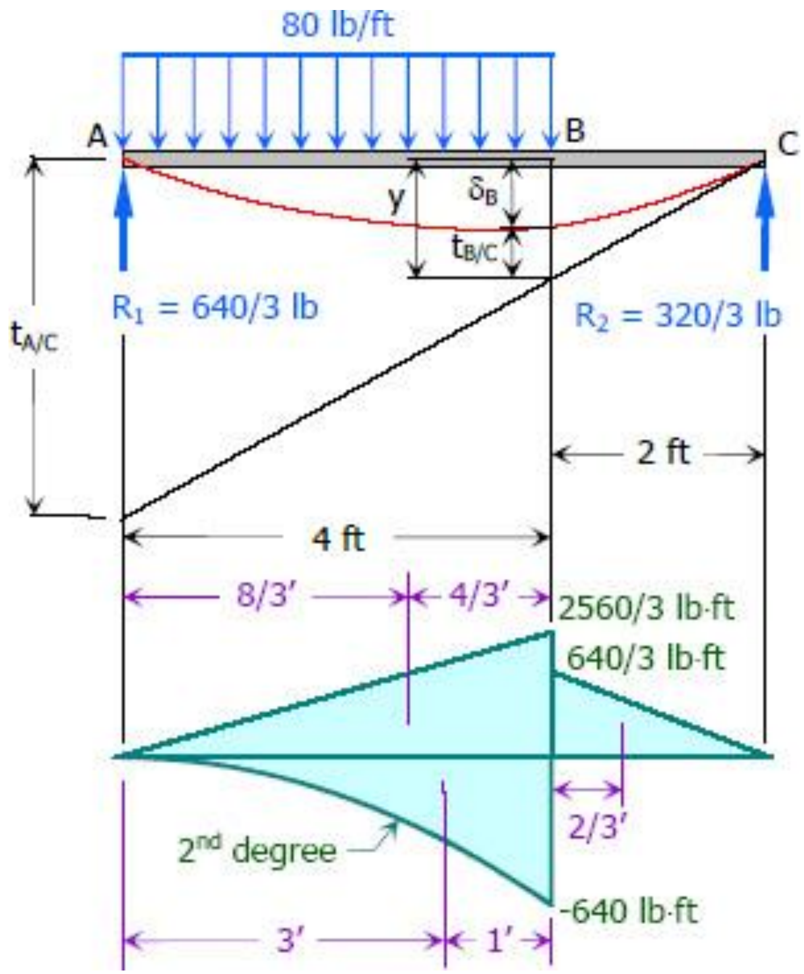
From the figure

$$\delta_{midspan} = t_{A/B}$$

Thus

$$EI\,\delta_{midspan} = 3350\ \mathrm{N}\cdot\mathrm{m}^3 \qquad answer$$

Moment Area Method- SSB – Problem-4



$$t_{A/C} = \frac{1}{EI}(Area_{AC})\,\bar{X}_A$$

$$t_{A/C} = \frac{1}{EI}\left[\tfrac{1}{2}(4)(2560/3)(\tfrac{8}{3}) + \tfrac{1}{2}(2)(\tfrac{640}{3})(4+\tfrac{2}{3}) - \tfrac{1}{3}(4)(640)(3)\right]$$

$$t_{A/C} = \frac{8960}{3EI}$$

$$t_{B/C} = \frac{1}{EI}(Area_{BC})\,\bar{X}_B$$

$$t_{B/C} = \frac{1}{EI}\left[\tfrac{1}{2}(2)(\tfrac{640}{3})(\tfrac{2}{3})\right]$$

$$t_{B/C} = \frac{1280}{9EI}$$

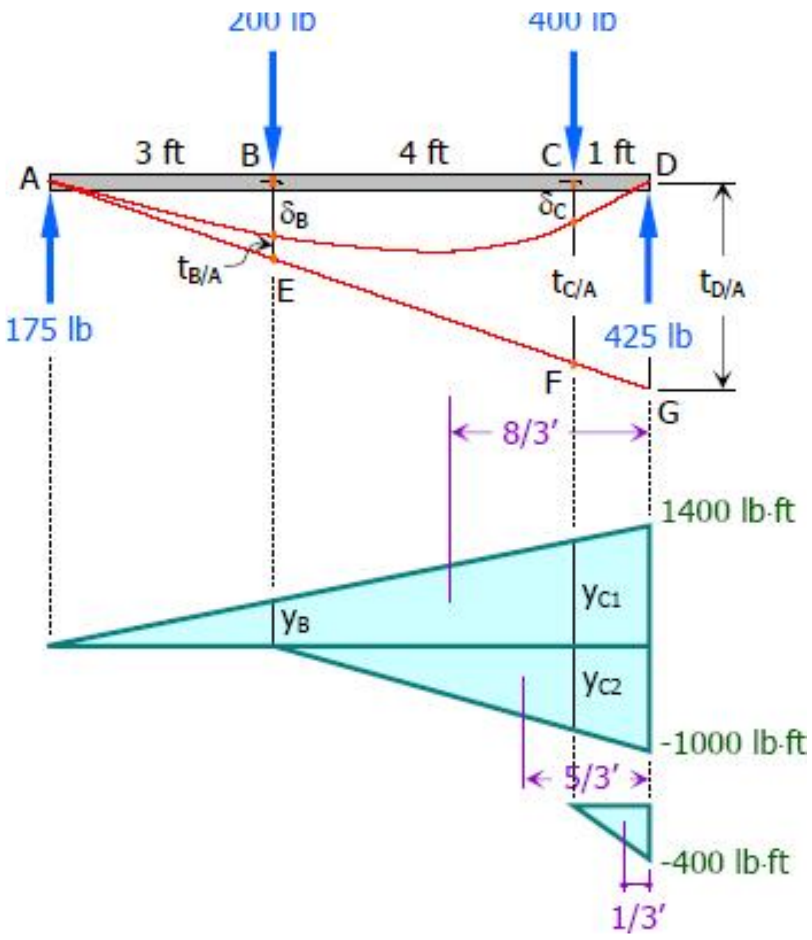
By ratio and proportion:

$$\frac{y}{2} = \frac{t_{A/C}}{6}$$

$$y = \frac{2}{6}\left(\frac{8960}{3EI}\right)$$

$$y = \frac{8960}{9EI}$$

Moment Area Method- SSB – Problem-4



$$\frac{y_{C1}}{7} = \frac{1400}{8}$$

$$y_{C1} = 1225 \text{ lb}$$

$$\frac{y_{C2}}{4} = \frac{-1000}{5}$$

$$y_{C2} = -800 \text{ lb}$$

$$\frac{y_B}{3} = \frac{1400}{8}$$

$$y_B = 525 \text{ lb}$$

$$EI\,t_{D/A} = (Area_{AD})\,\bar{X}_D$$

$$EI\,t_{D/A} = \tfrac{1}{2}(8)(1400)(\tfrac{8}{3}) - \tfrac{1}{2}(5)(1000)(\tfrac{5}{3}) - \tfrac{1}{2}(1)(400)(\tfrac{1}{3})$$

$$EI\,t_{D/A} = 10\,700 \text{ lb}\cdot\text{ft}^3$$

$$EI\,t_{C/A} = (Area_{AC})\,\bar{X}_C$$

$$EI\,t_{C/A} = \tfrac{1}{2}(7)(y_{C1})(\tfrac{7}{3}) - \tfrac{1}{2}(4)(y_{C2})(\tfrac{4}{3})$$

$$EI\,t_{C/A} = \tfrac{1}{2}(7)(1225)(\tfrac{7}{3}) - \tfrac{1}{2}(4)(800)(\tfrac{4}{3})$$

$$EI\,t_{C/A} = \frac{47\,225}{6} \text{ lb}\cdot\text{ft}^3$$

$$EI\,t_{B/A} = (Area_{AB})\,\bar{X}_B$$

$$EI\,t_{C/A} = \tfrac{1}{2}(3)(y_B)(1)$$

$$EI\,t_{C/A} = \tfrac{1}{2}(3)(525)(1)$$

$$EI\,t_{C/A} = \frac{1575}{2} \text{ lb}\cdot\text{ft}^3$$

By ratio and proportion:

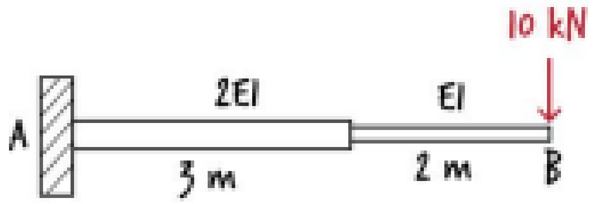
$$\frac{\bar{BE}}{3} = \frac{\bar{CF}}{7} = \frac{t_{D/A}}{8}$$

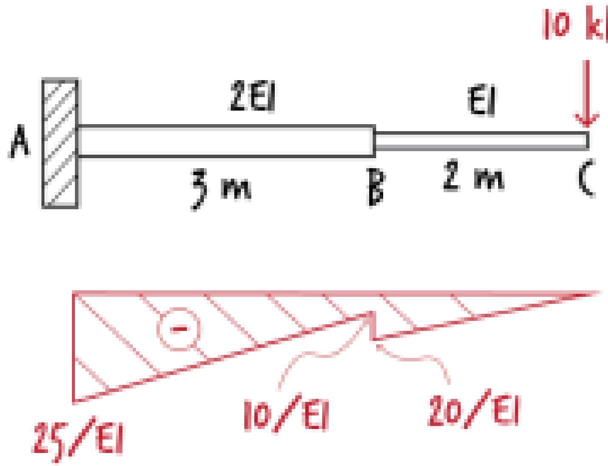
$$\bar{BE} = \tfrac{3}{8}t_{D/A} = \tfrac{3}{8}(10\,700) = \frac{8025}{2}$$

$$\bar{CF} = \tfrac{7}{8}t_{D/A} = \tfrac{7}{8}(10\,700) = \frac{18\,725}{2}$$

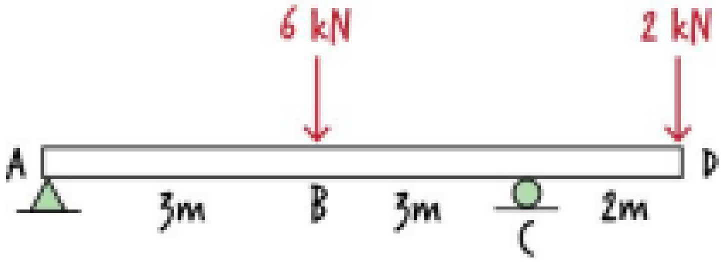
Moment Area Method- Assignment -3

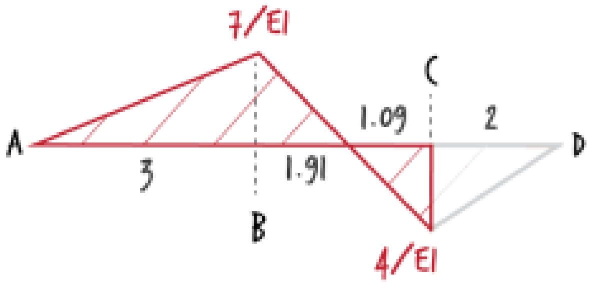
A) Determine the slope and deflection at the free end of the cantilever beam.

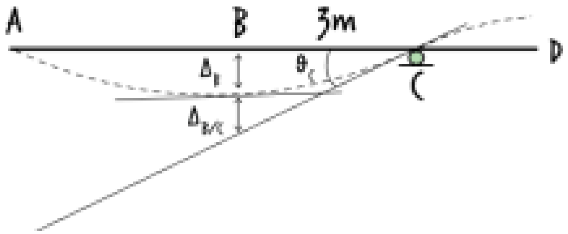


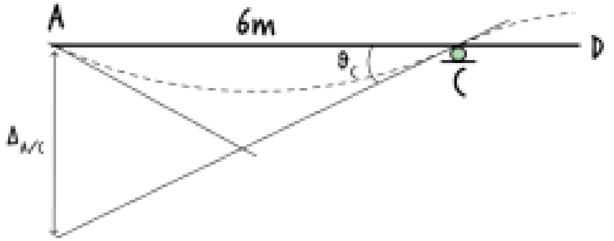


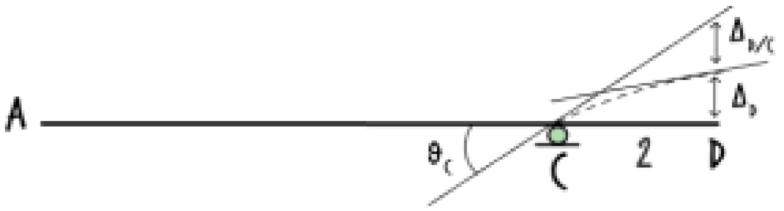
B) Determine the beam's deflection at B and D, and the slope of the elastic curve at C and D. The beam has a constant EI.



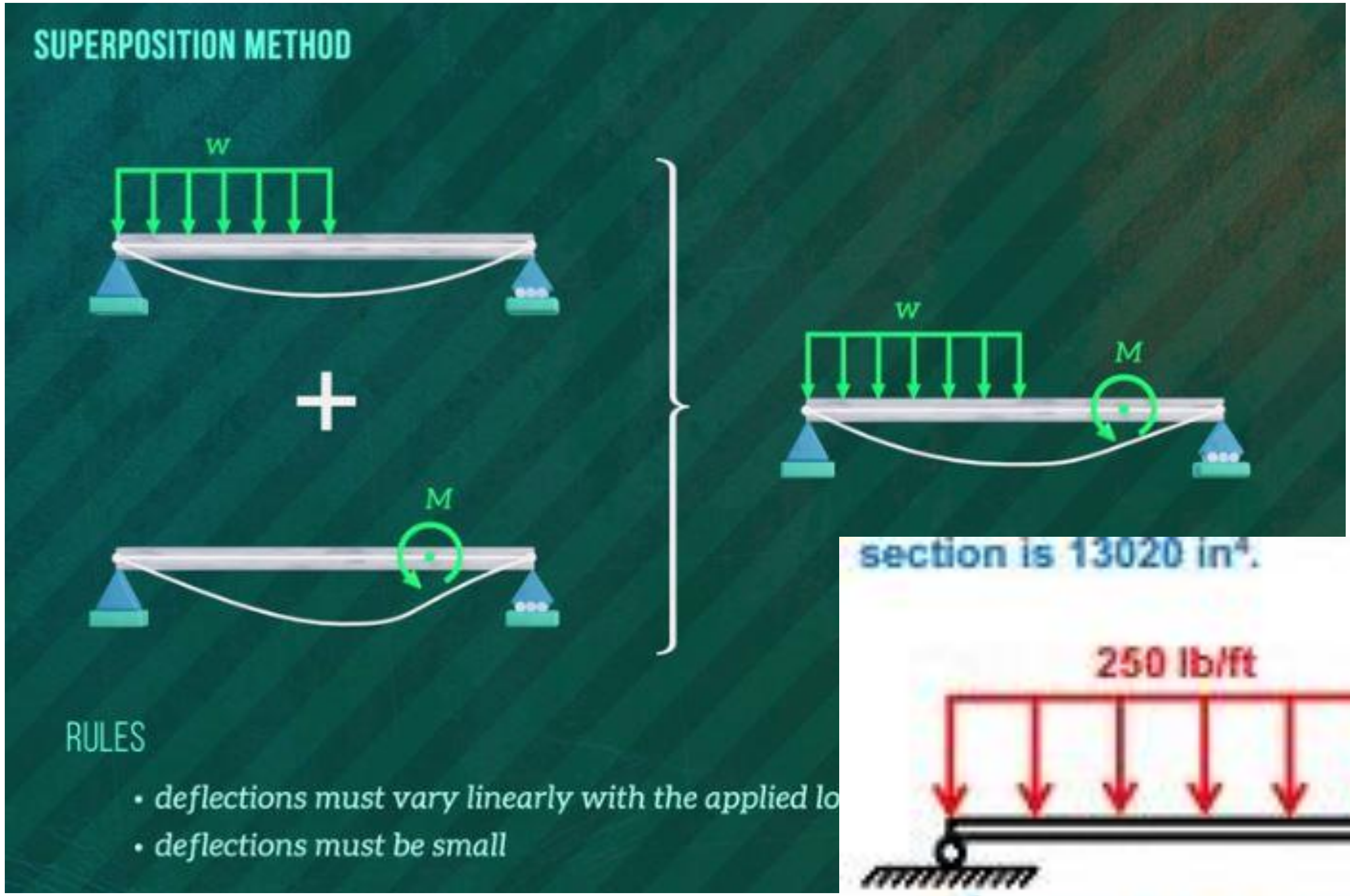


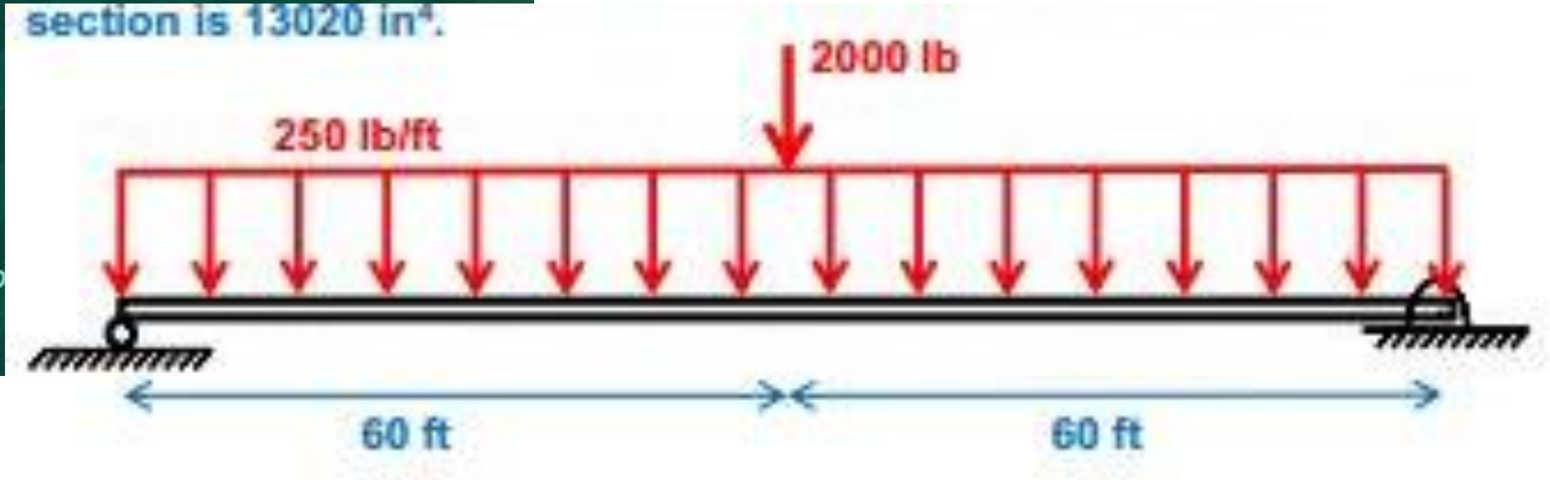






Method of Super Position

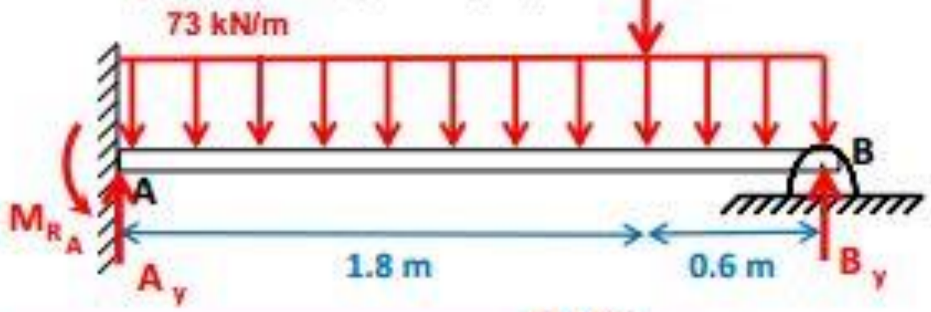




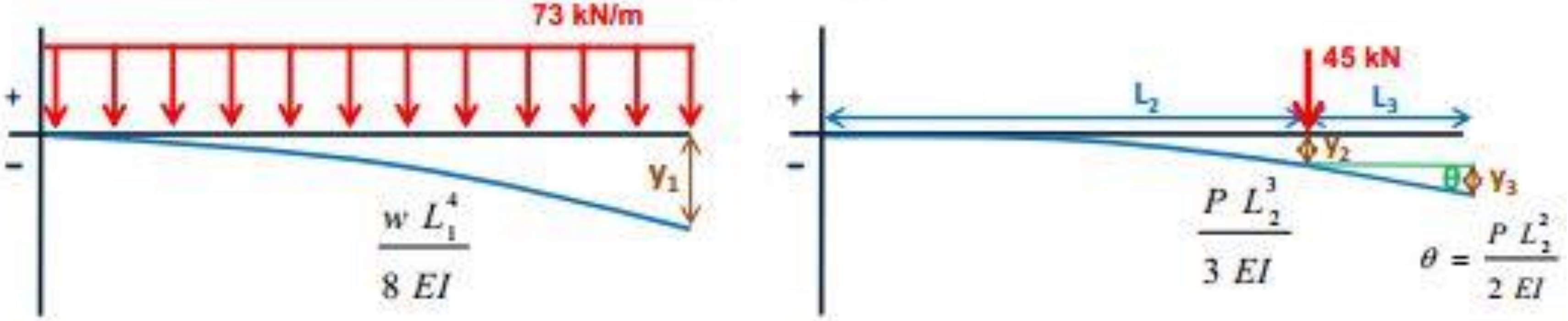
Method of Super Position

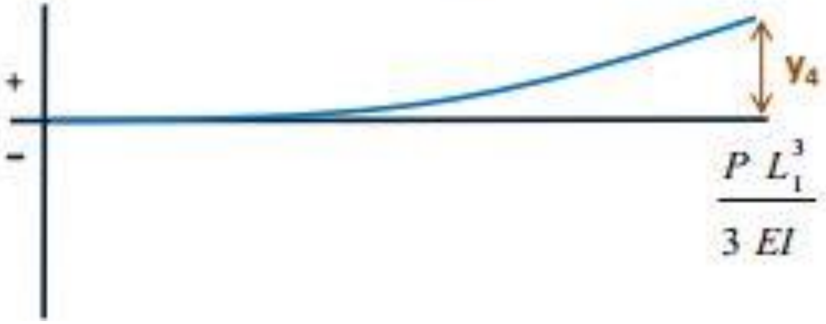
Method of Super Position

Deformation Equation or Compatibility Equation



Assumption:
Linear elastic



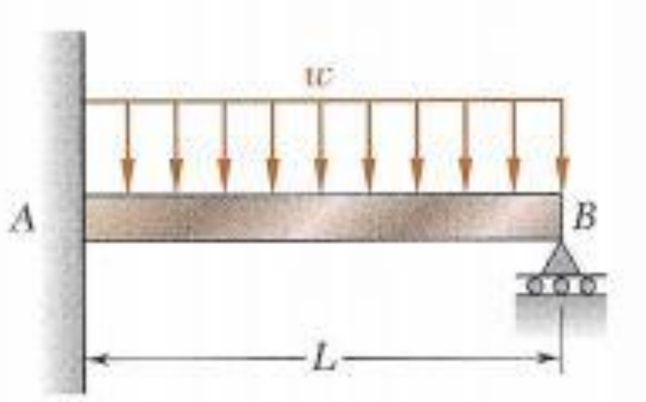


$$y_B = 0 = -y_1 - y_2 - y_3 + y_4$$

$$\frac{45 L_2^2}{2 EI}$$

$$0 = -\frac{73 L_1^4}{8 EI} - \frac{45 L_2^3}{3 EI} - L_3 \theta + \frac{B_y L_1^3}{3 EI}$$

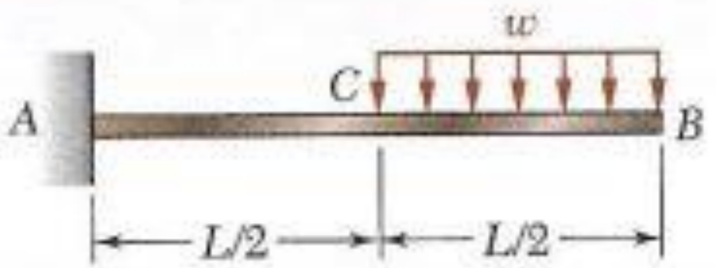
Method of Super Position

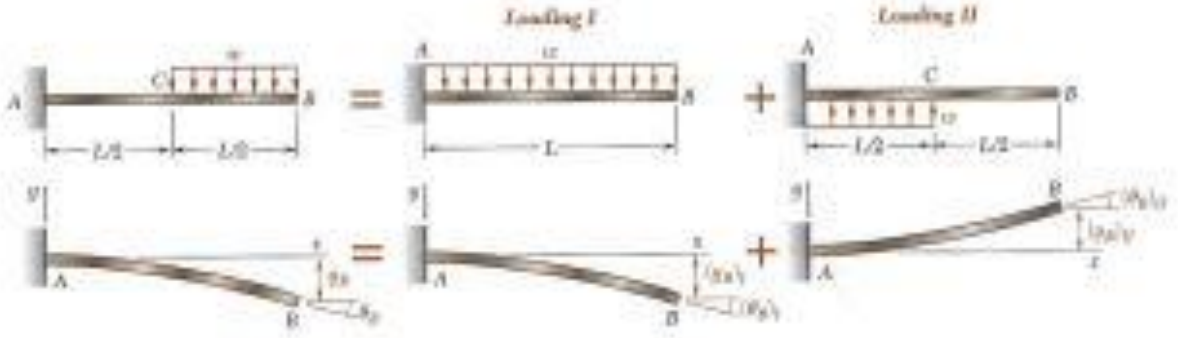


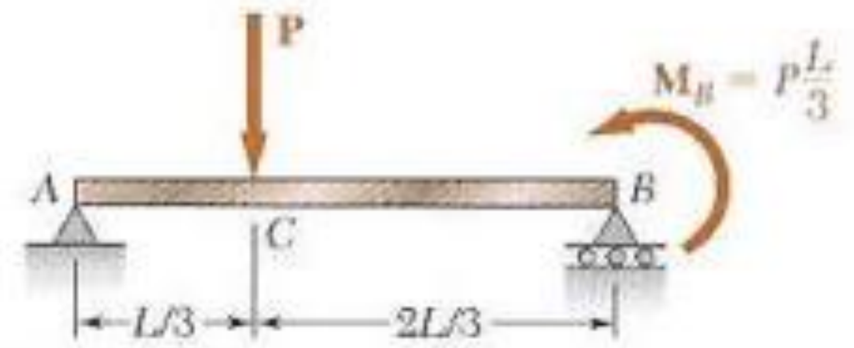
Treating the reaction at B as the redundant support, we have:

(a) (b) (c)

Method of Super Position







t C.

Method of Super Position

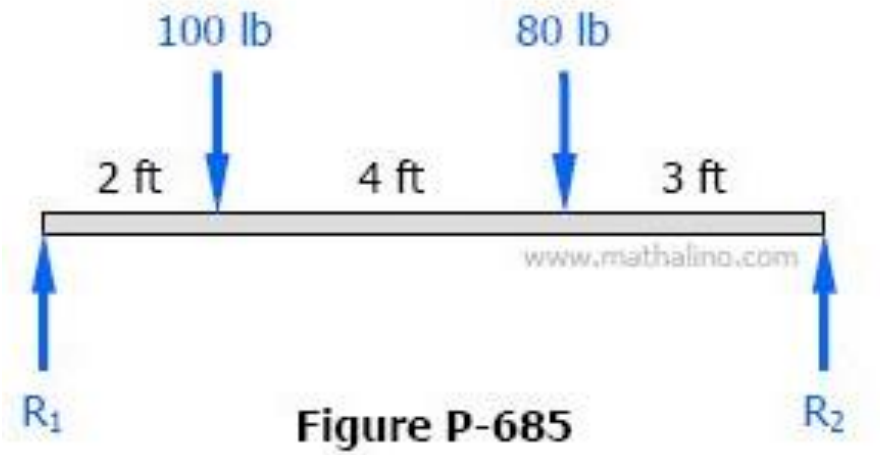


Figure P-685

Determine the midspan deflection of the beam shown in Fig. P-687 if E = 10 GPa and I = 20×10^6 mm^4.

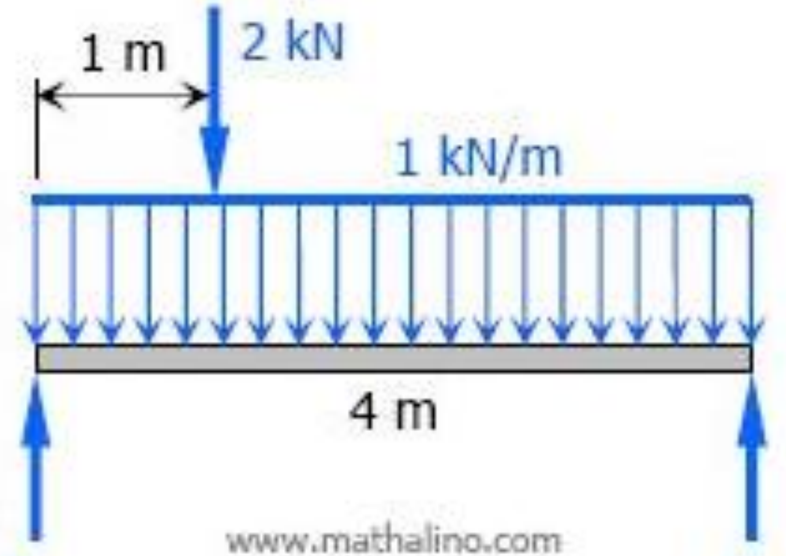
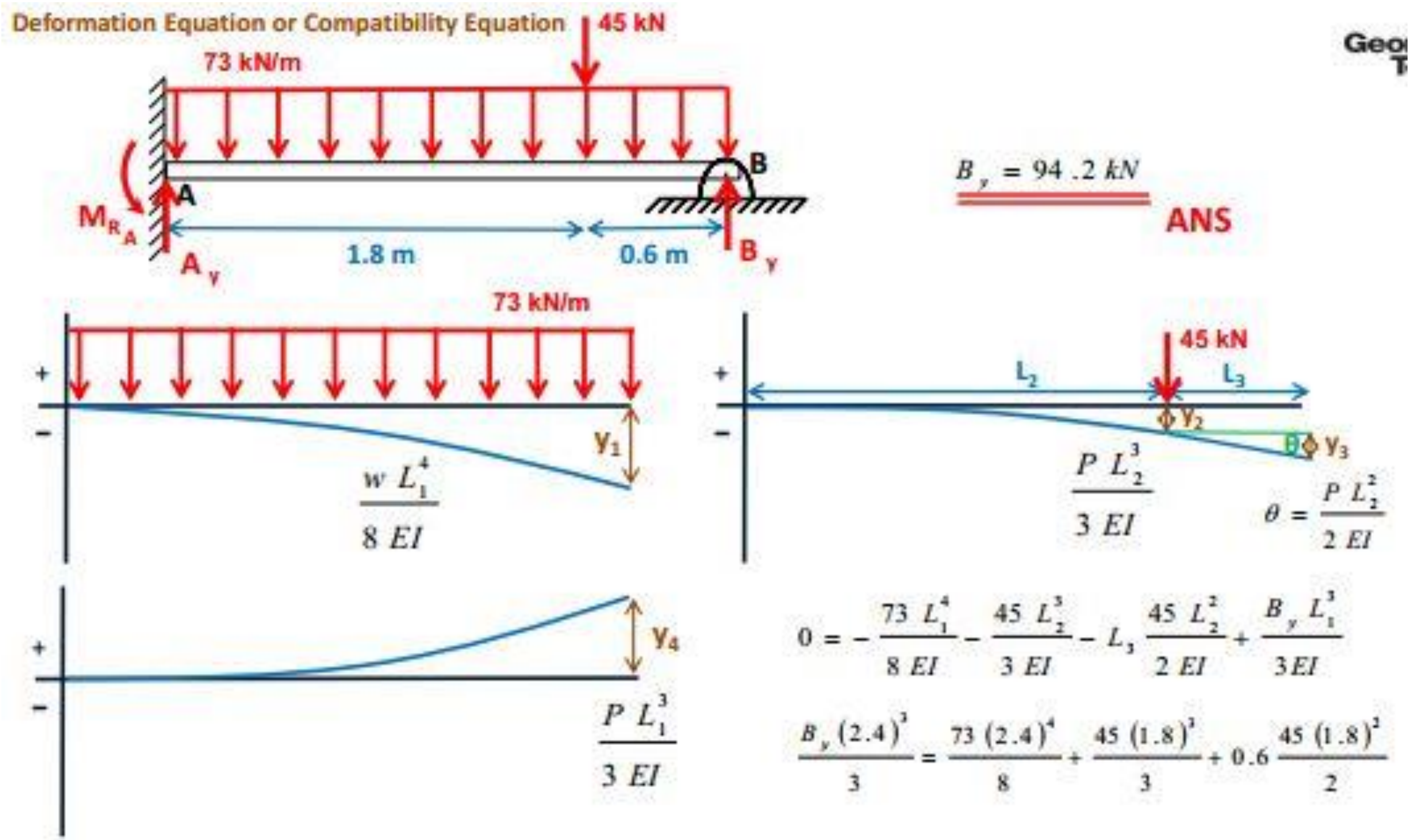
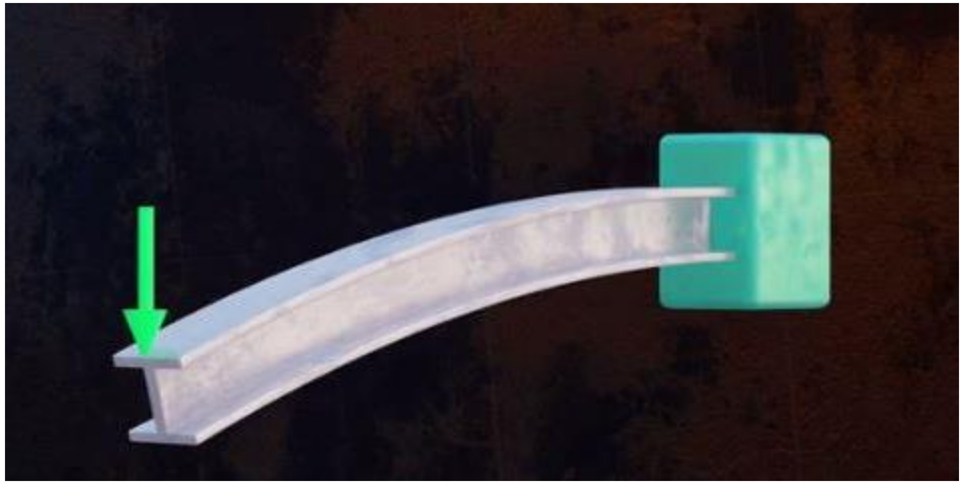


Figure P-687

Method of Super Position



Castigliano's Theorem /Strain Energy Method



$$\delta_i = \int \frac{\partial}{\partial P_i}\left(\frac{M^2}{2EI}\right) dx$$

$$U = \int_0^L \frac{M^2}{2EI} dx$$

$$\delta_i = \int \frac{2M \frac{\partial M}{\partial P_i}}{2EI} dx$$

$$\delta = \frac{\partial U}{\partial P} \quad \text{or} \quad \theta = \frac{\partial U}{\partial \bar{M}}$$

$$\delta = \int_0^L \left(\frac{\partial M}{\partial P}\right) \frac{M}{EI} dx \quad \text{and} \quad \theta = \int_0^L \left(\frac{\partial M}{\partial \bar{M}}\right) \frac{M}{EI} dx$$

Methods – Castigliano's Theorem (Strain Energy)

- For linearly elastic structures, the partial derivative of the strain energy with respect to an applied force (or couple) is equal to the displacement (or rotation) of the force (or couple) along its line of action.

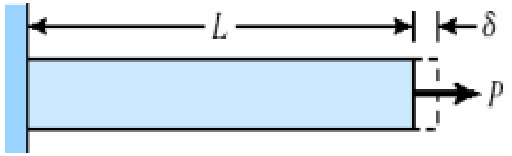
$$\delta = \frac{\partial U}{\partial P} \quad \text{or} \quad \theta = \frac{\partial U}{\partial \bar{M}}$$

Methods – Castigliano's Theorem (Strain Energy)

Table 5.3 (p193): **Energy and Deflection Equations**

Load Type (1)	Factors Involved (2)	Energy Equation Constant Factors (3)	General Energy Equation (4)	General Deflection Equation (5)
Axial	P, E, A	$U = \frac{P^2L}{2EA}$	$U = \int_0^L \frac{P^2}{2EA}\,dx$	$\Delta = \int_0^L \frac{P(\partial P/\partial Q)}{EA}\,dx$
Bending	M, E, I	$U = \frac{M^2L}{2EI}$	$U = \int_0^L \frac{M^2}{2EI}\,dx$	$\Delta = \int_0^L \frac{M(\partial M/\partial Q)}{EI}\,dx$
Torsion	T, G, K'	$U = \frac{T^2L}{2GK'}$	$U = \int_0^L \frac{T^2}{2GK'}\,dx$	$\Delta = \int_0^L \frac{T(\partial T/\partial Q)}{GK'}\,dx$
Transverse shear (rectangular section)	V, G, A	$U = \frac{3V^2L}{5GA}$	$U = \int_0^L \frac{3V^2}{5GA}\,dx$	$\Delta^a = \int_0^L \frac{6V(\partial V/\partial Q)}{5GA}\,dx$

Example: Axial Tension



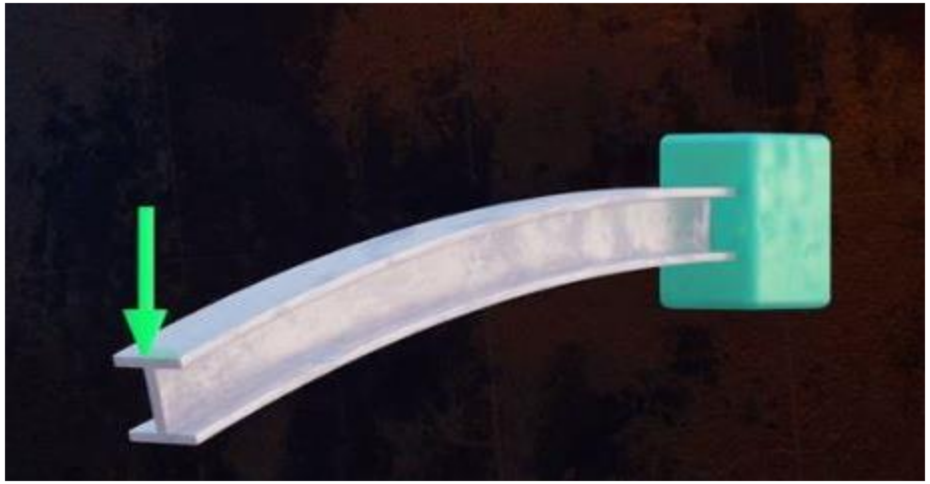
Stored Elastic Energy: $U = P\delta/2$

Case 1 from Table 5.1: $\delta = PL/AE$

gives: $U = P^2L/2EA$

For varying E and A: $U = \int_0^L \frac{P^2}{2EA}\,dx$

Castigliano's Theorem /Strain Energy Method



$$\delta = \frac{\partial U}{\partial P} \quad \text{or} \quad \theta = \frac{\partial U}{\partial \bar{M}}$$

Strain energy stored in the elementary area is

$$= \frac{\sigma^2}{2E} \times (volume)_{Elementary\ Area}$$

$$= \frac{\sigma^2}{2E} \times dx.dA$$

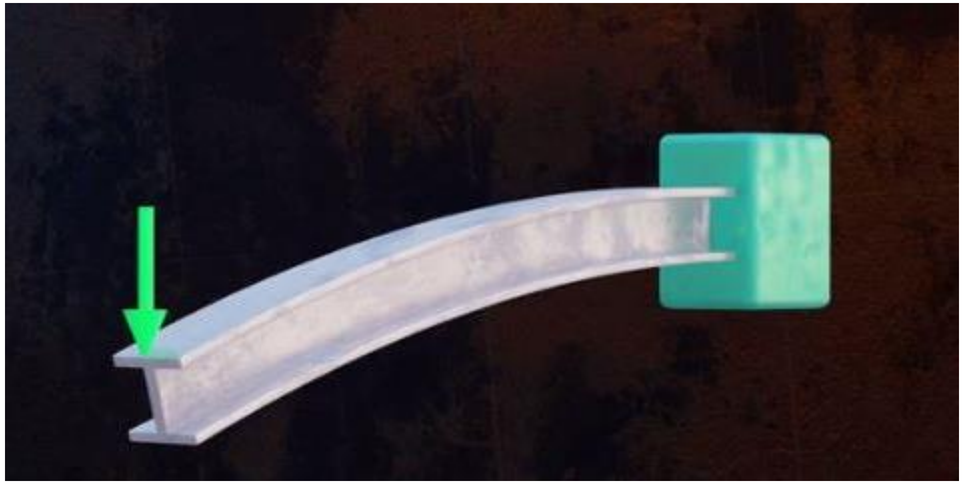
$$= \frac{M^2}{I^2} \cdot \frac{y^2}{2E} dx.dA$$

Strain energy stored in the beam between 1 and 2 is given as

$$dU = \frac{M^2}{2EI^2} dx \int y^2 dA$$

$$dU = \frac{M^2}{2EI^2} dx.I = \frac{M^2}{2EI} dx \qquad \left(I = \int y^2 dA\right)$$

Castigliano's Theorem /Strain Energy Method



$$U = \int_0^L \frac{M^2}{2EI} dx$$

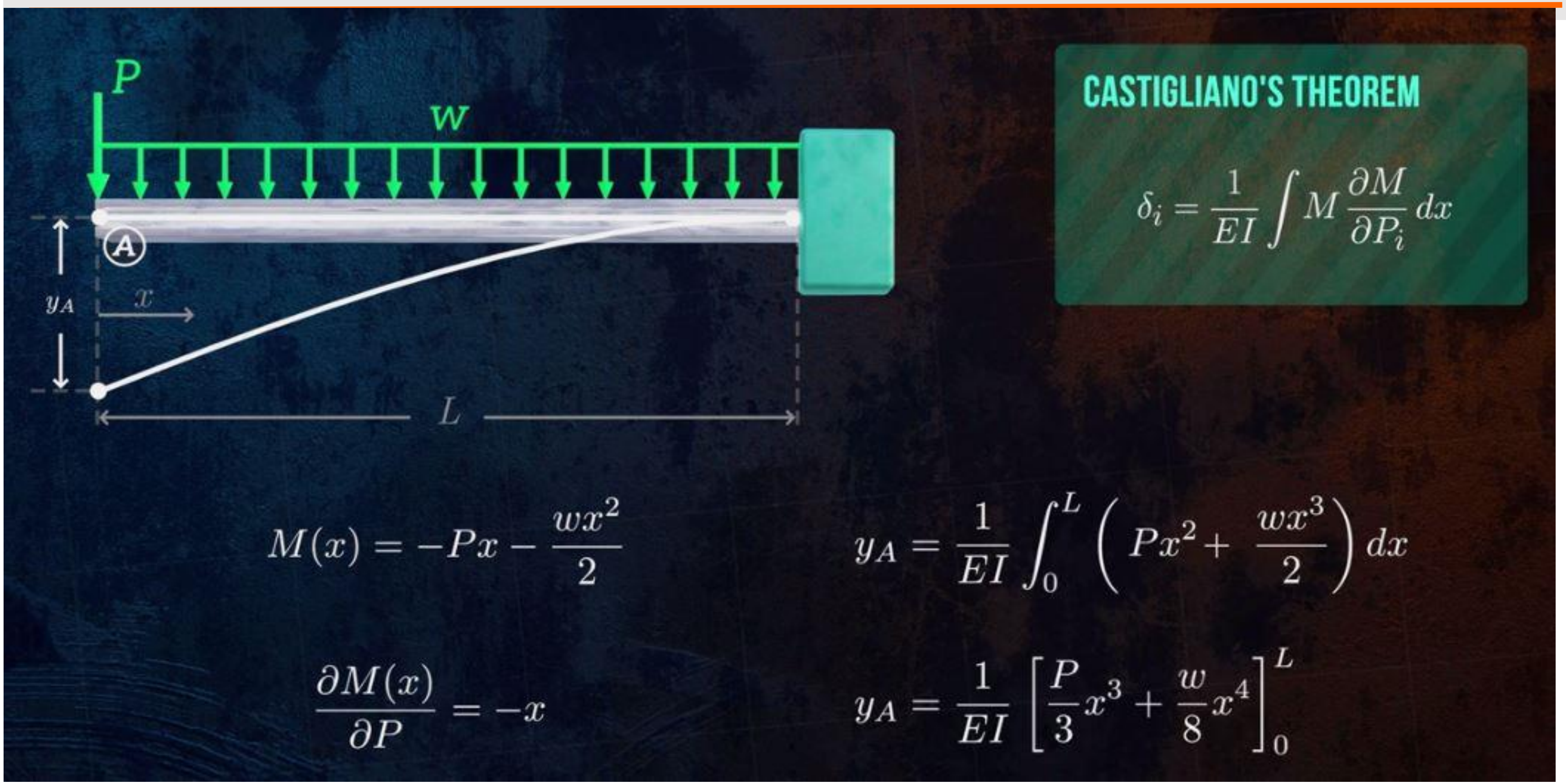
$$\delta = \frac{\partial U}{\partial P} \quad \text{or} \quad \theta = \frac{\partial U}{\partial \bar{M}}$$

$$\delta_i = \int \frac{\partial}{\partial P_i} \left(\frac{M^2}{2EI} \right) dx$$

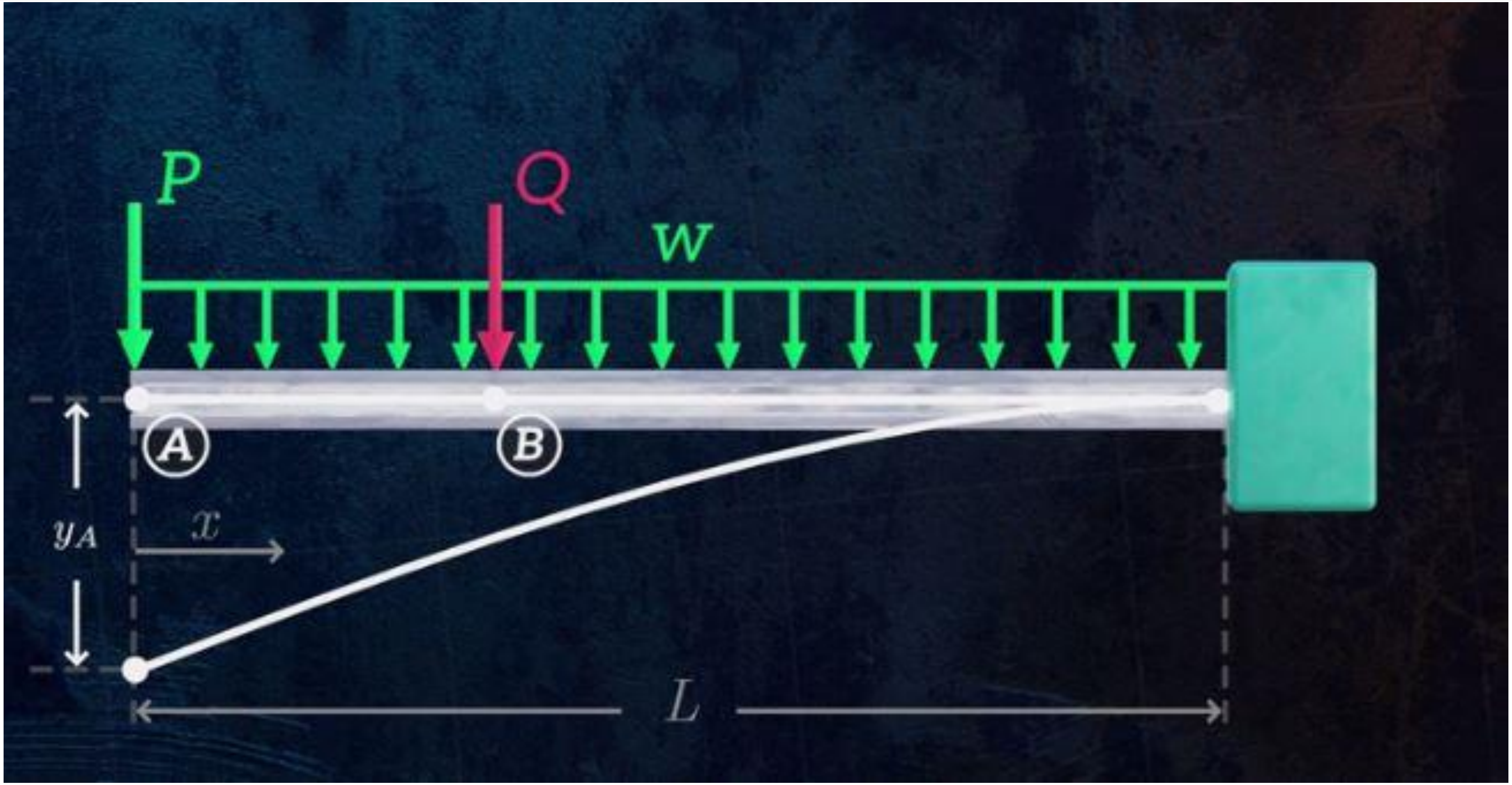
$$\delta_i = \int \frac{2M \frac{\partial M}{\partial P_i}}{2EI} dx$$

$$\delta = \int_0^L \left(\frac{\partial M}{\partial P} \right) \frac{M}{EI} dx \quad \text{and} \quad \theta = \int_0^L \left(\frac{\partial M}{\partial \bar{M}} \right) \frac{M}{EI} dx$$

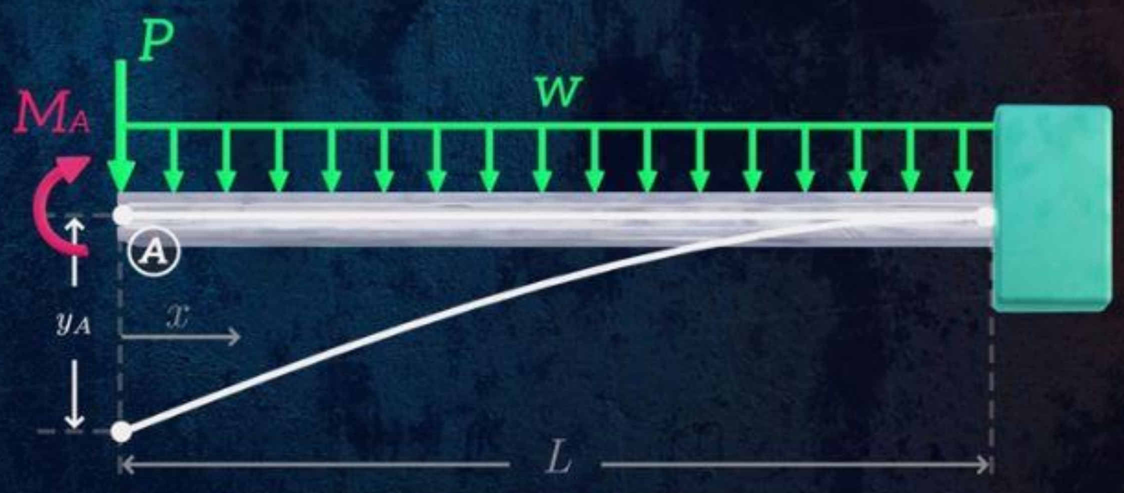
Castigliano's Theorem /Strain Energy Method



Castigliano's Theorem /Strain Energy Method



Castigliano's Theorem /Strain Energy Method



CASTIGLIANO'S THEOREM

$$\delta_i = \frac{1}{EI}\int M\frac{\partial M}{\partial P_i}\,dx$$

$$M(x) = M_A - Px - \frac{wx^2}{2}$$

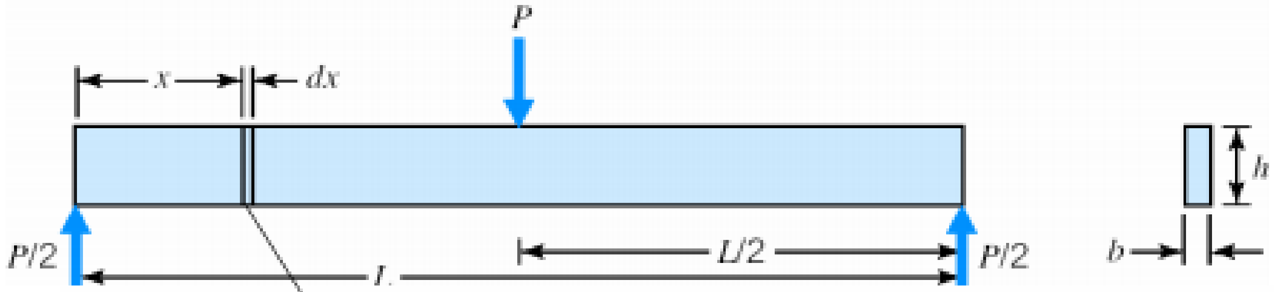
$$\frac{\partial M(x)}{\partial M_A} = 1$$

$$\theta_A = \frac{1}{EI}\left[M_A x - \frac{P}{2}x^2 - \frac{w}{6}x^3\right]_0^L$$

$$\theta_A = \frac{M_A L}{EI} - \frac{PL^2}{2EI} - \frac{wL^3}{6EI}$$

Set $M_A = 0$

Castigliano's Theorem (Strain Energy)



first compute **Energy**, **then Partial Derivative** to get deflection

Here 2 types of loading: Bending **and** Shear

magnitude @ x: $M = \frac{P}{2}x$ and $V = \frac{P}{2}$

Table 5.3

Load Type (1)	Factors Involved (2)	Energy Equation Constant Factors (3)	General Energy Equation (4)	General Deflection Equation (5)
Bending	M, E, I	$U = \frac{M^2L}{2EI}$	$U = \int_0^L \frac{M^2}{2EI}dx$	$\Delta = \int_0^L \frac{M(\partial M/\partial Q)}{EI}dx$
Transverse shear (rectangular	V, G, A	$U = \frac{3V^2L}{5GA}$	$U = \int_0^L \frac{3V^2}{5GA}dx$	$\Delta^* = \int_0^L \frac{6V(\partial V/\partial Q)}{5GA}dx$

1. Energy: here it has two components:

$$U = 2\int_0^{L/2} \frac{M^2}{2EI}dx + \int_0^L \frac{3V^2}{5GA}dx$$

$$= 2\int_0^{L/2} \frac{P^2x^2}{8EI}dx + \int_0^L \frac{3(P/2)^2}{5GA}dx$$

$$= \frac{P^2}{4EI}\int_0^{L/2} x^2\,dx + \frac{3P^2}{20GA}\int_0^L dx$$

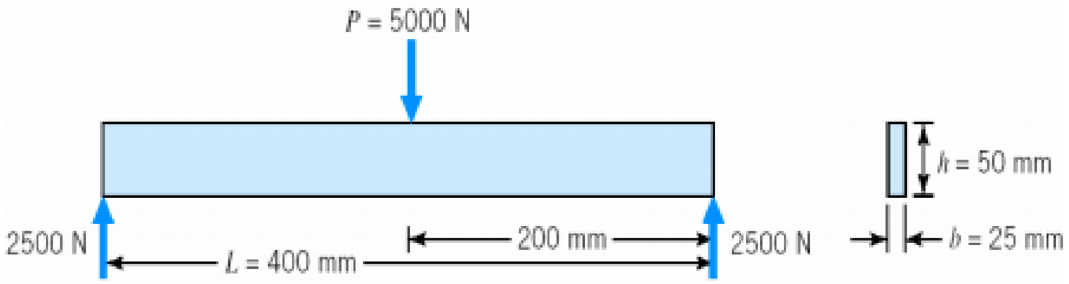
$$= \frac{P^2L^3}{96EI} + \frac{3P^2L}{20GA}$$

$(2^3=8)*3*4 = 96$

2. Partial Derivatives for deflection:

$$\delta = \frac{\partial U}{\partial P} = \frac{PL^3}{48EI} + \frac{3PL}{10GA}$$

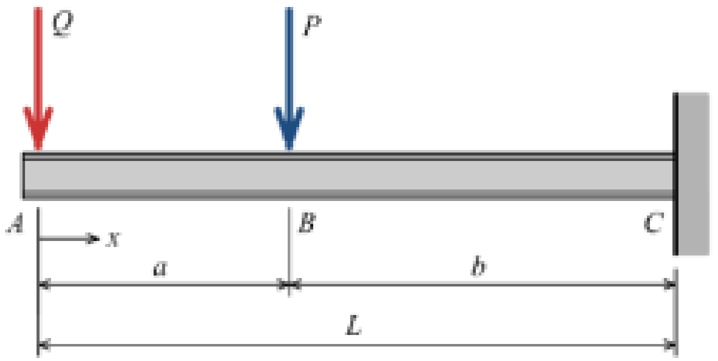
Castigliano's Theorem (Strain Energy)-Assignment

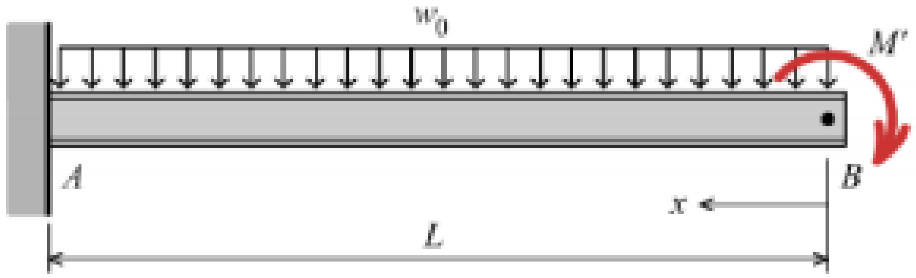


$$\delta = \frac{PL^3}{48EI} + \frac{3PL}{10GA}$$

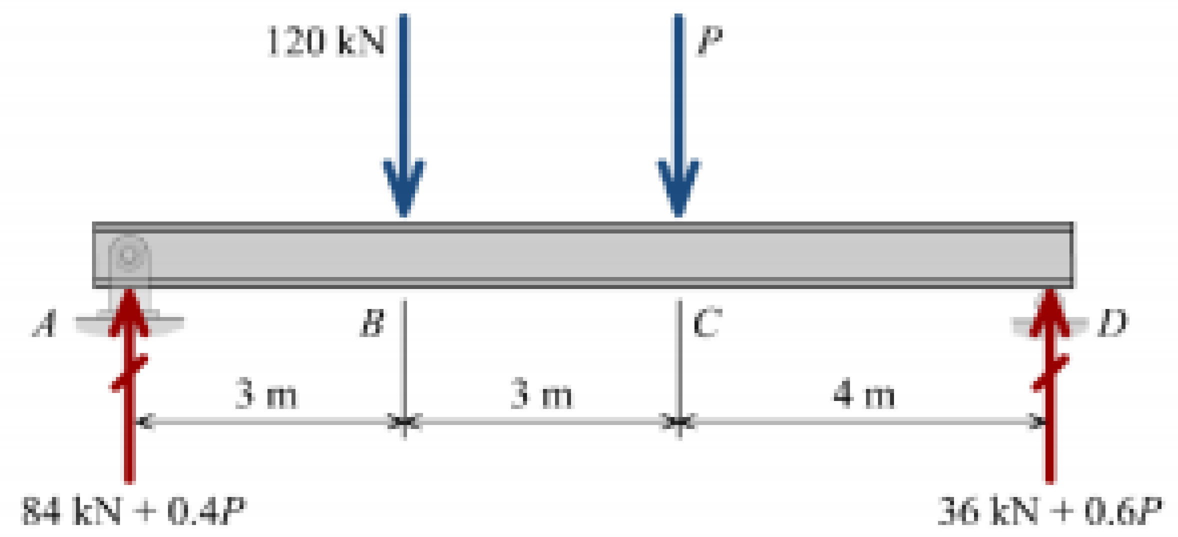
$$= \left[\frac{5000(0.400)^3}{48(207\times10^9)\left[\frac{25(50)^3}{12}\times10^{-12}\right]} + \frac{3(5000)(0.400)}{10(80\times10^9)(0.025)(0.050)}\right]\text{m}$$

$$= \left[(1.237\times10^{-4}) + \underbrace{(6.000\times10^{-6})}_{\text{Transverse shear contributes only <5\% to deflection}}\right]\text{m} = 1.297\times10^{-4}\ \text{m}$$

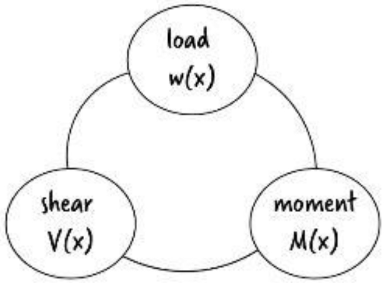




Castigliano's Theorem (Strain Energy)-Assignment

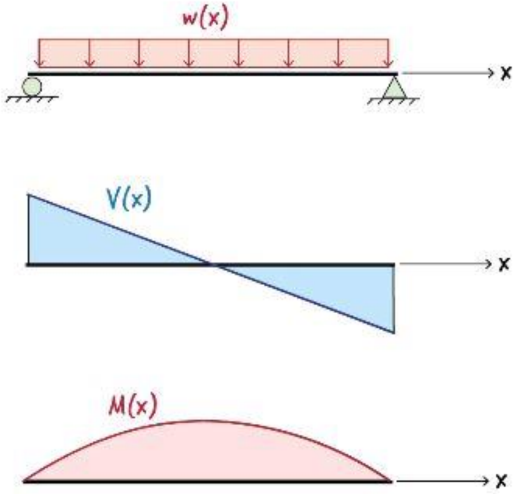


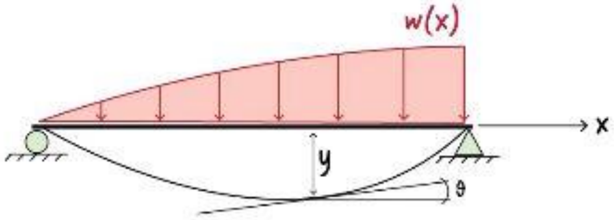
Conjugate Beam Method



$V(x)=\int w(x)\,dx$

$M(x)=\int V(x)\,dx=\iint w(x)\,d^2x$





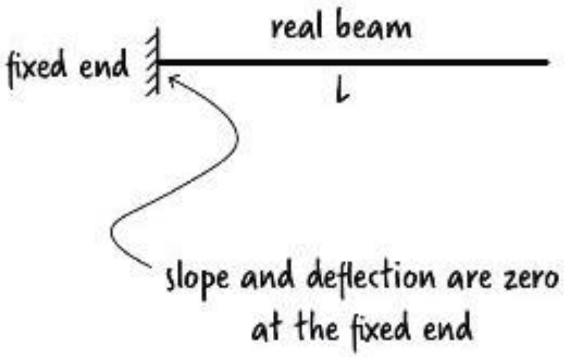
$V(x)=\int w(x)\,dx$

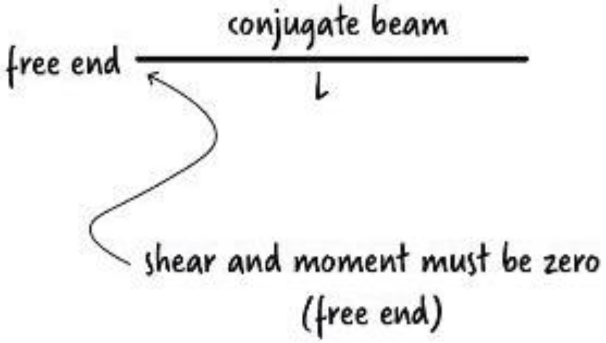
$\theta(x)=\int \frac{M}{EI}\,dx$

$M(x)=\iint w(x)\,d^2x$

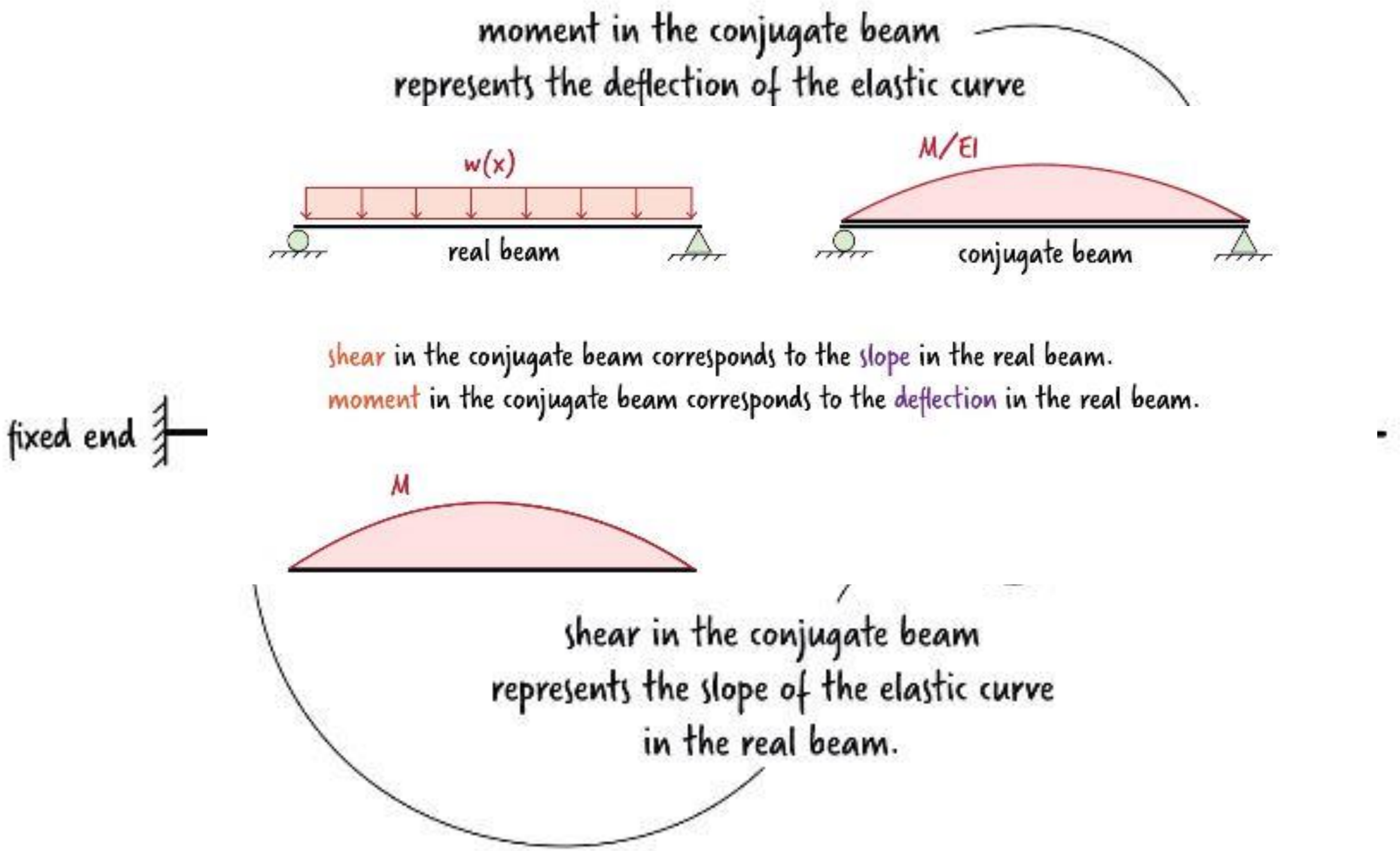
$y(x)=\iint \frac{M}{EI}\,d^2x$

M: bending moment
E: modulus of elasticity
I: moment of inertia

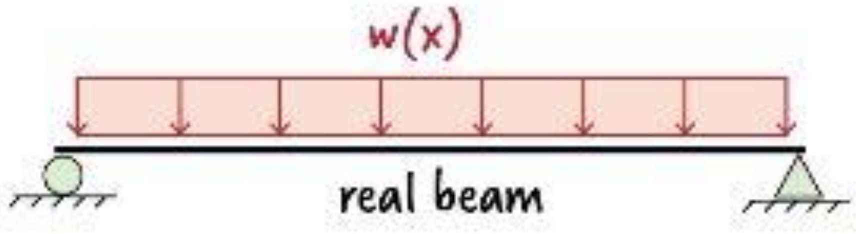


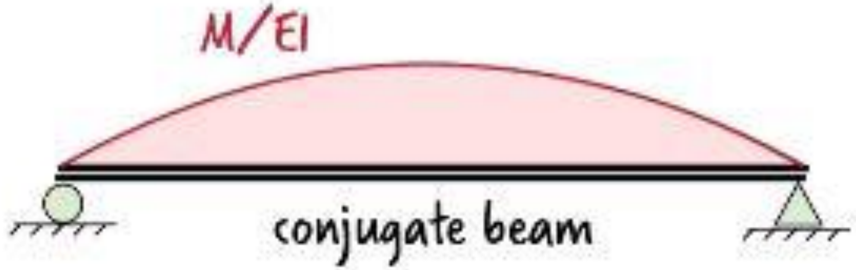


Methods – Conjugate Beam Method

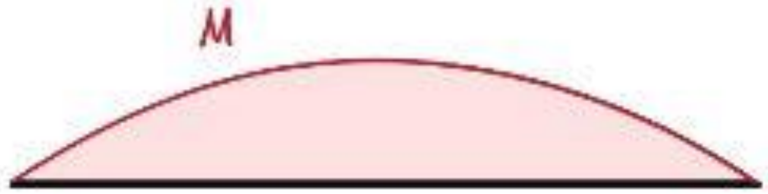


Methods – Conjugate Beam Method

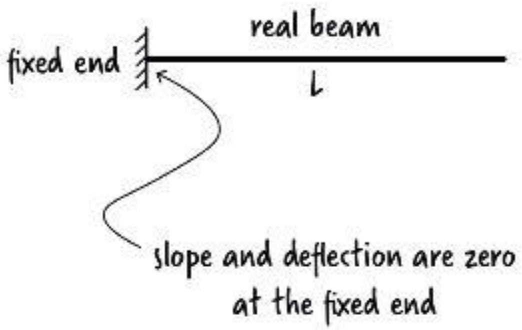


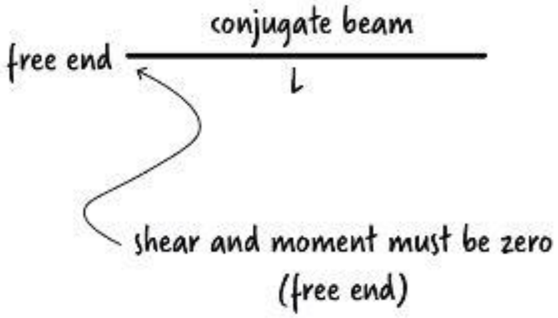


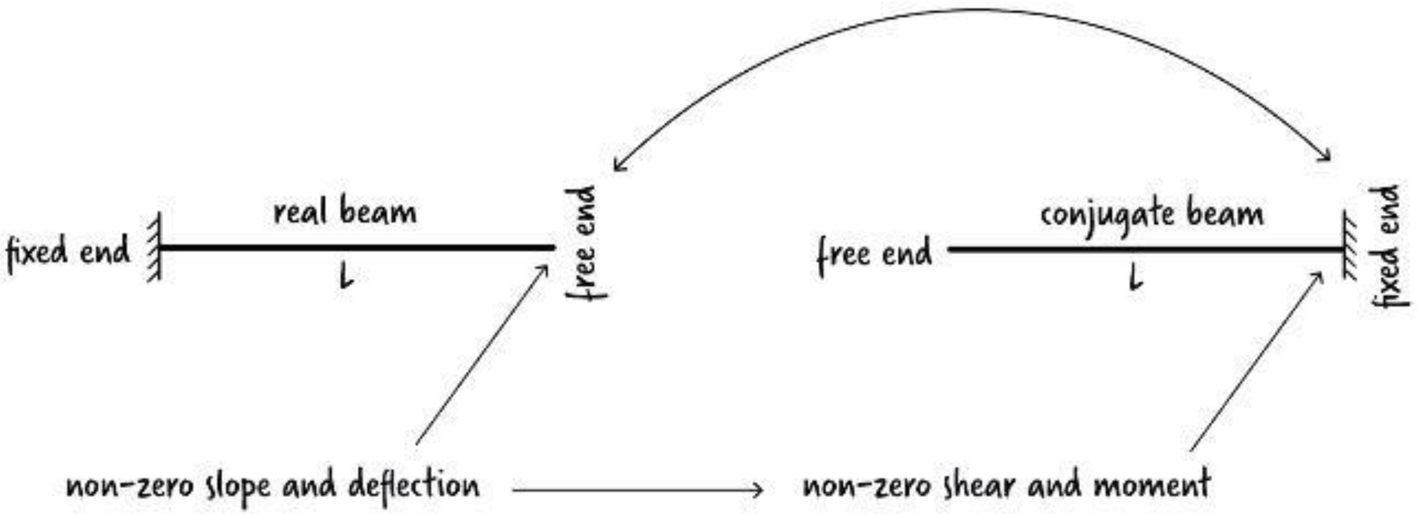
shear in the conjugate beam corresponds to the slope in the real beam.
moment in the conjugate beam corresponds to the deflection in the real beam.



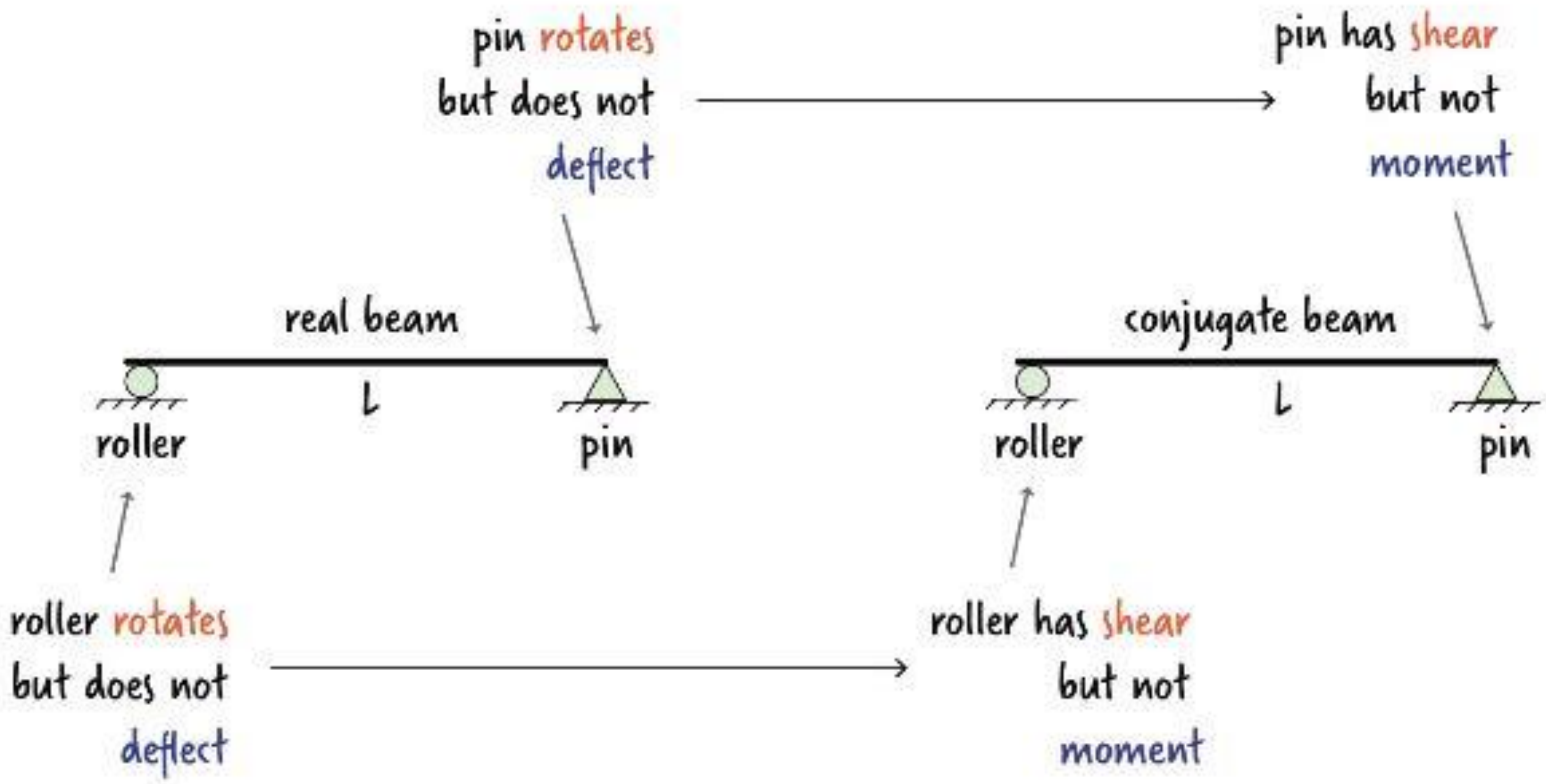
Methods – Conjugate Beam Method



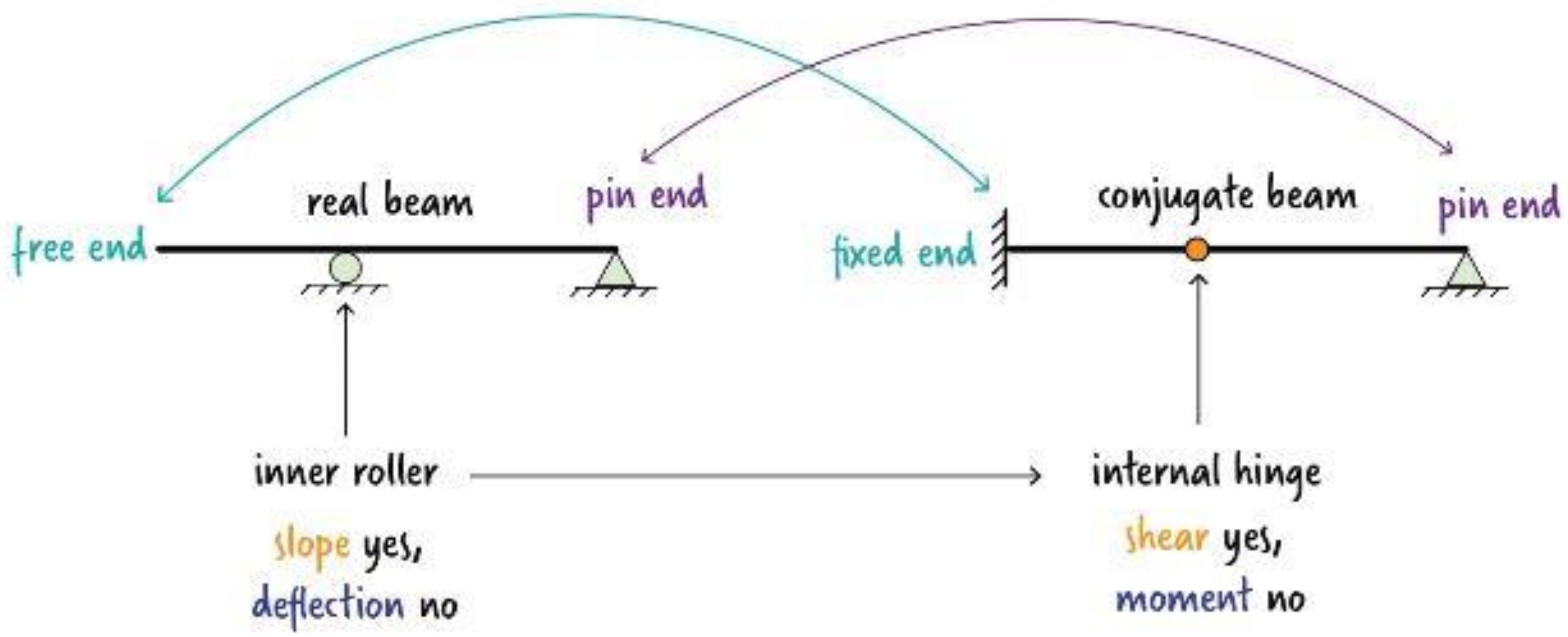




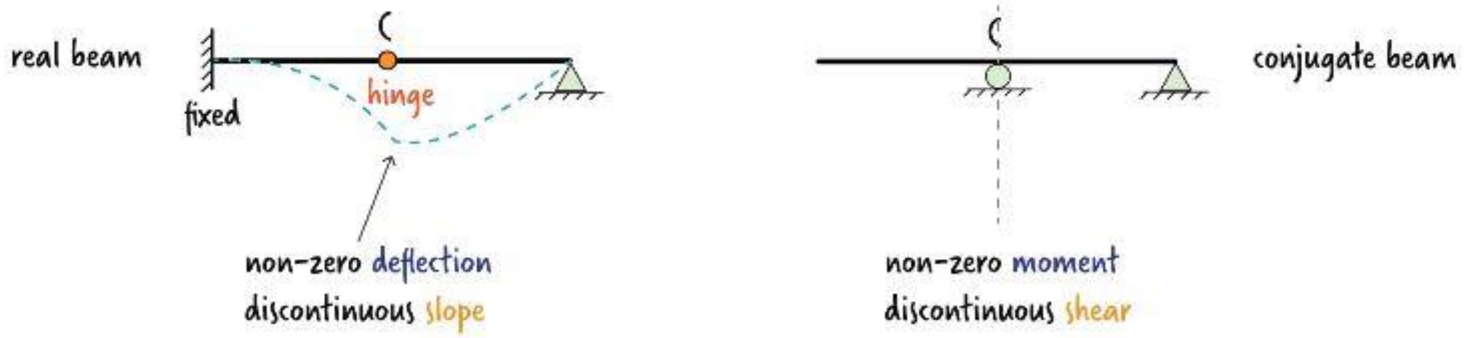
Methods – Conjugate Beam Method

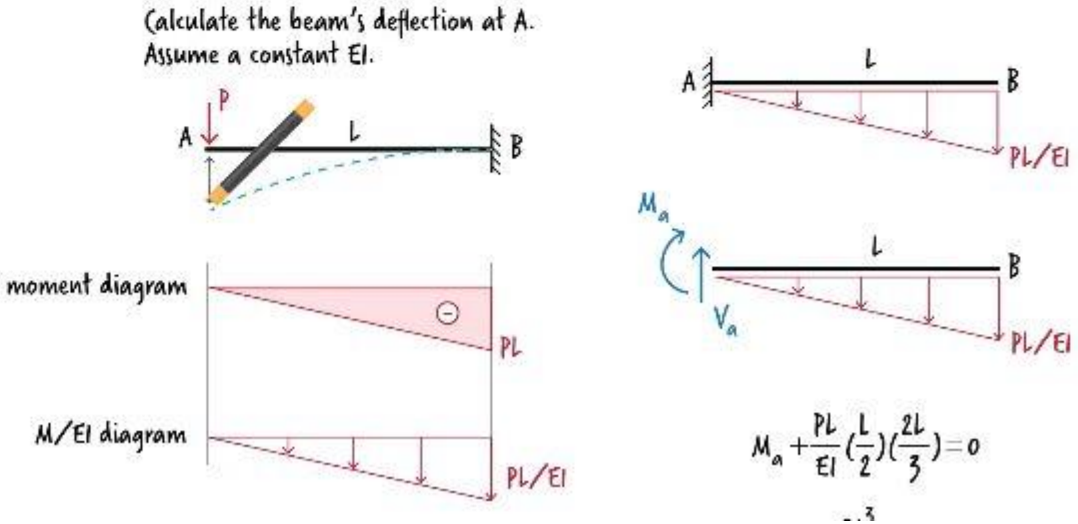


Methods – Conjugate Beam Method

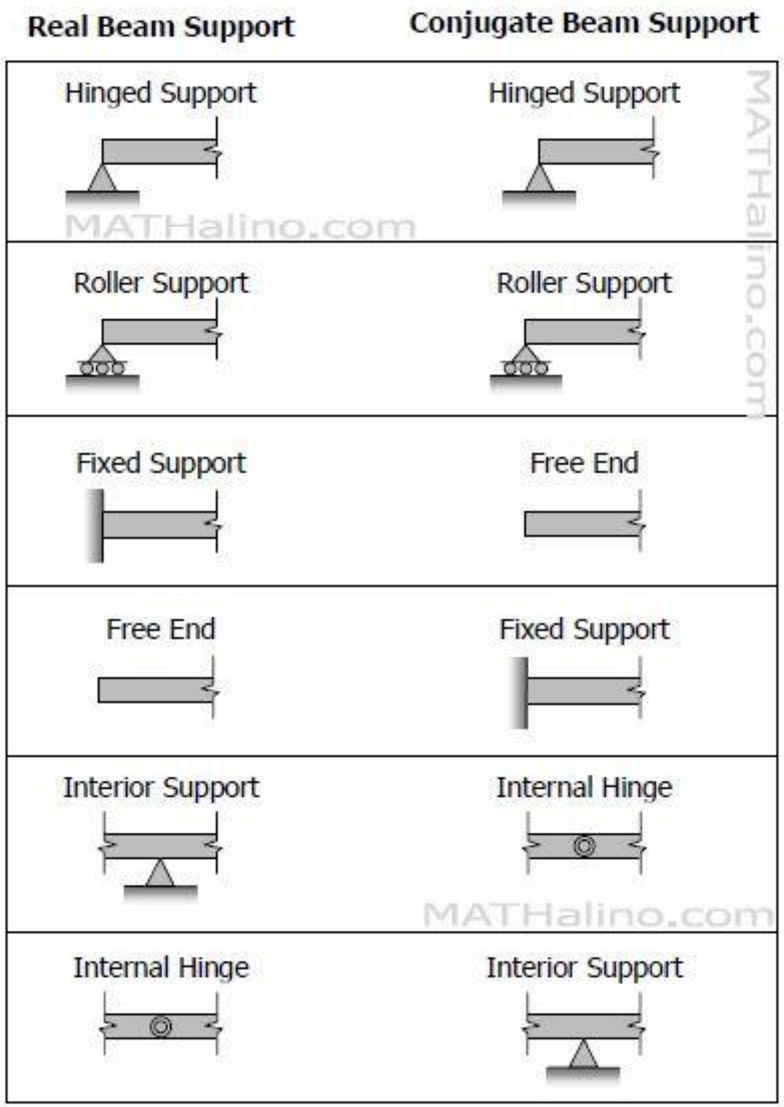


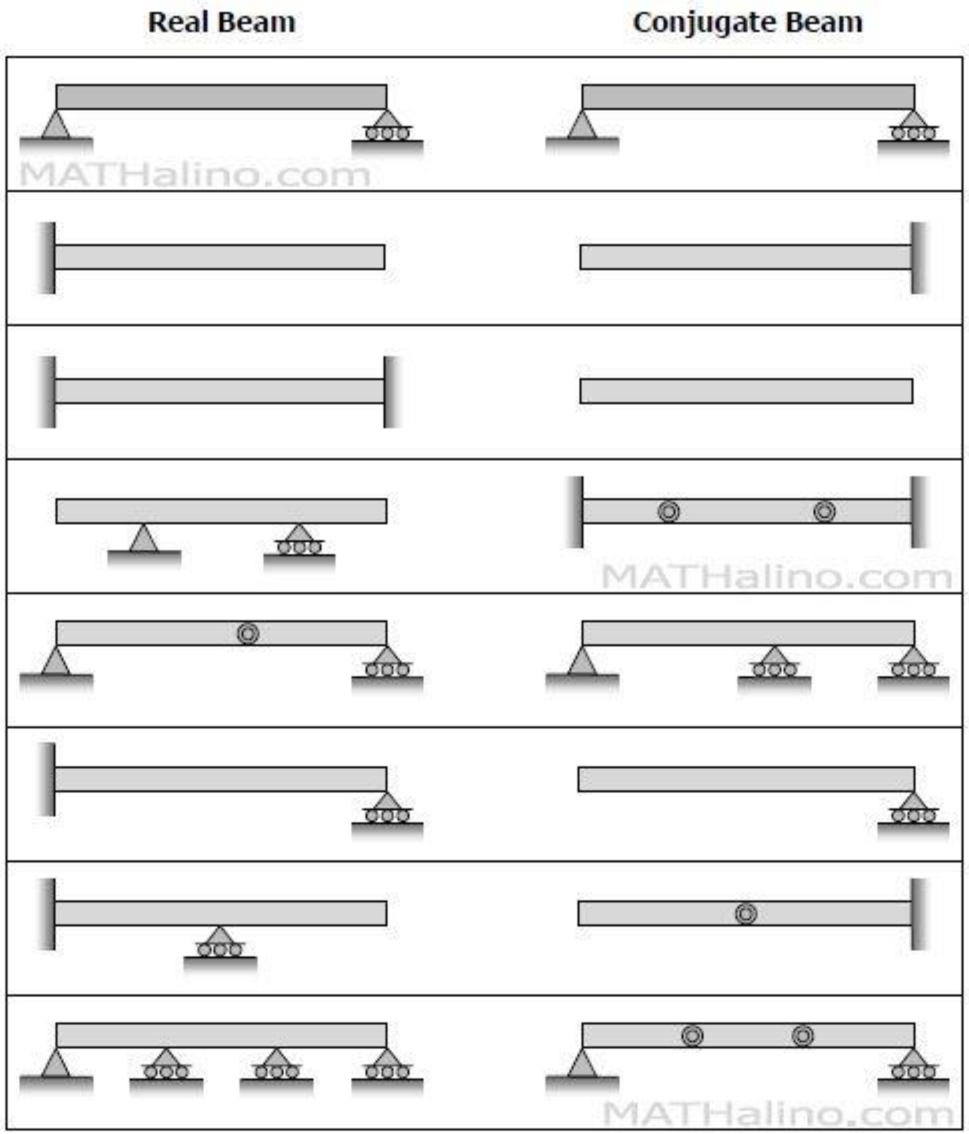
Methods – Conjugate Beam Method





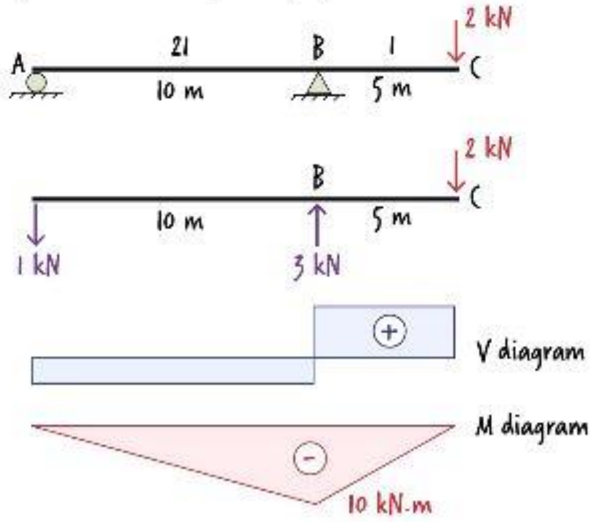
Methods – Conjugate Beam Method



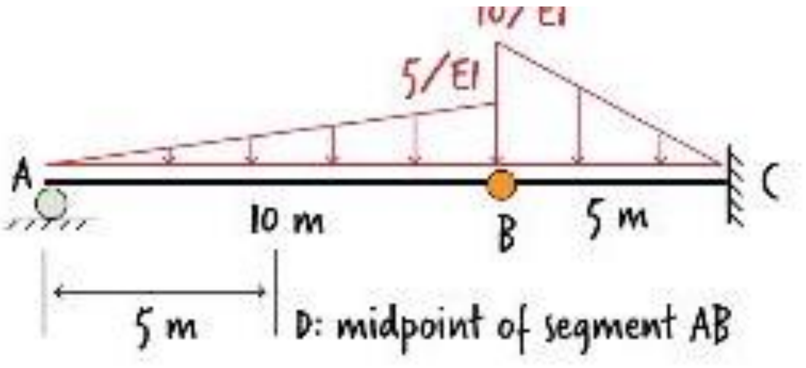


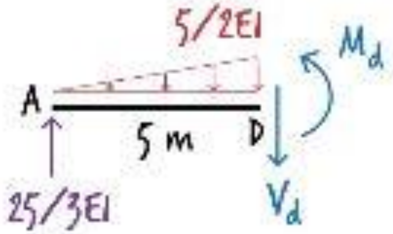
Methods – Conjugate Beam Method

Calculate the deflection at the midpoint of segment AB. Assume a constant E.



Conjugate Beam

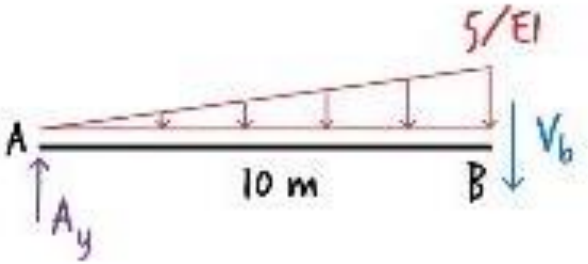




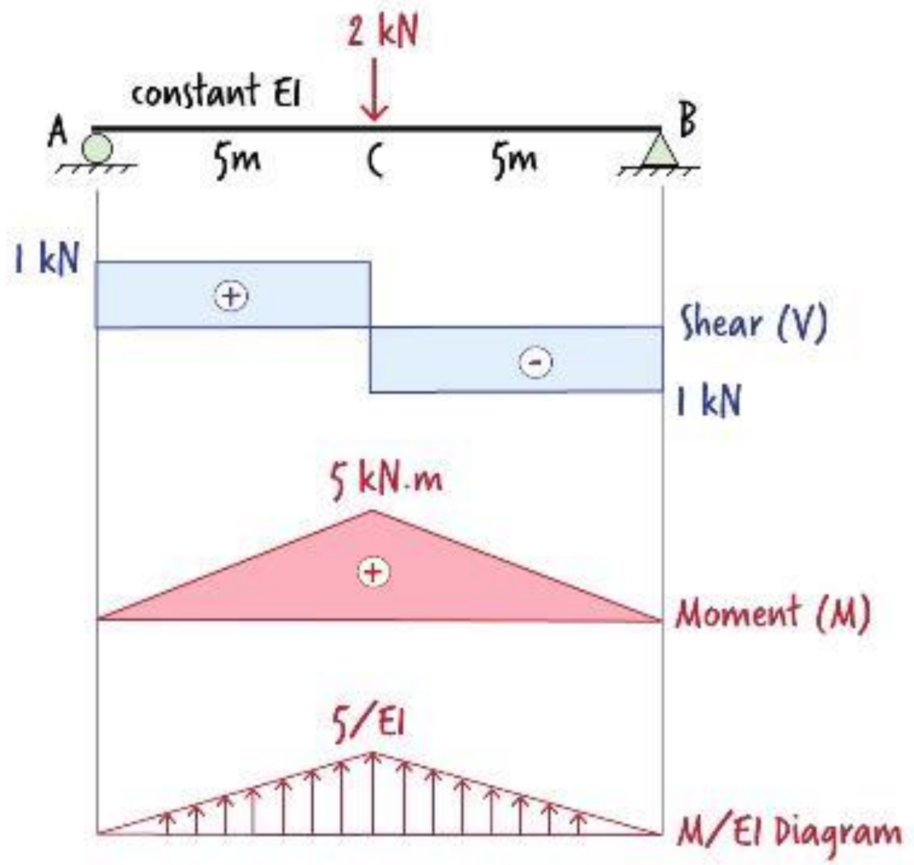
$$\frac{25}{3EI}-\frac{5}{2EI}\left(\frac{5}{2}\right)-V_d=0$$

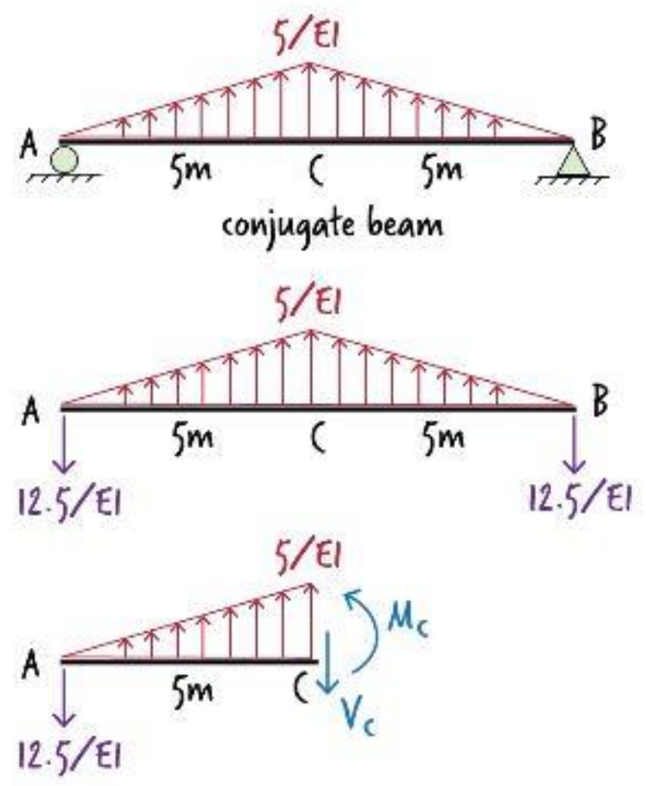
$V_d = 25/12EI$ slope at D

$$\frac{25}{3EI}(5)-\frac{5}{2EI}\left(\frac{5}{2}\right)\left(\frac{5}{3}\right)-M_d=0$$



Methods – Conjugate Beam Method

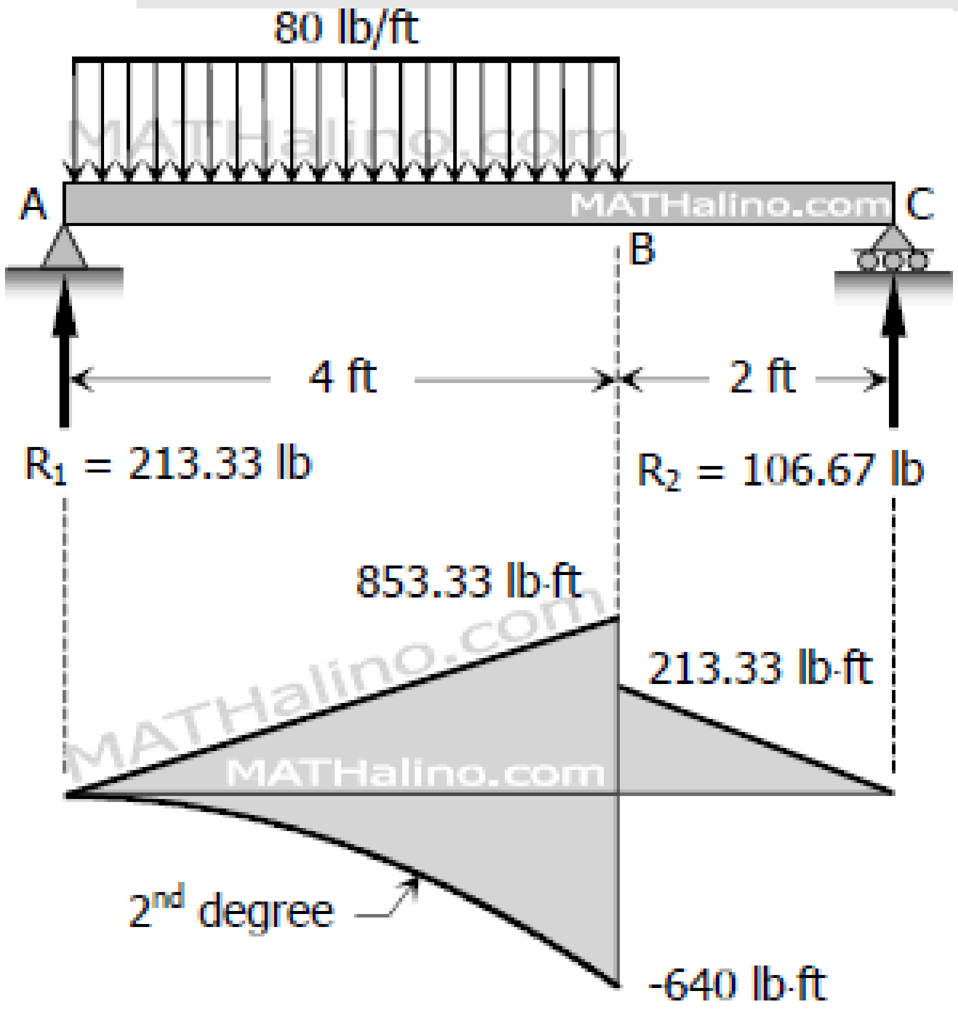




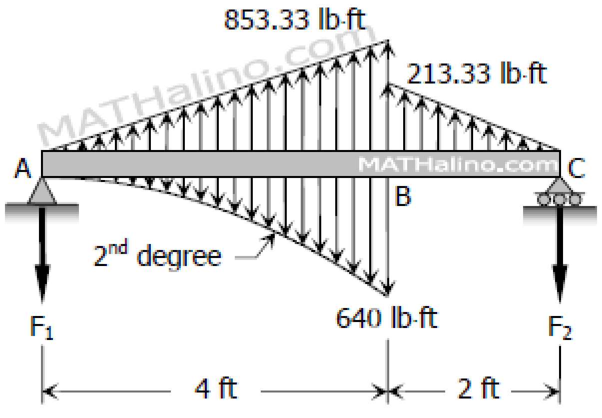
$$M_c = \frac{5}{EI}\left(\frac{5}{2}\right)\left(\frac{5}{3}\right) - \frac{12.5}{EI}(5) = \frac{-125}{3EI}$$

Methods – Conjugate Beam Method

Methods – Conjugate Beam Method



Moment Diagram by Parts



Conjugate Beam Loaded with M/EI Diagram

From the conjugate beam

$\Sigma M_A = 0$

$6F_2 + \frac{1}{3}(4)(640)[\frac{3}{4}(4)] = \frac{1}{2}(4)(853.33)[\frac{2}{3}(4)] + \frac{1}{2}(2)(213.33)[4 + \frac{1}{3}(2)]$

$F_2 = 497.77 \text{ lb} \cdot \text{ft}^2$

$M_B = \frac{1}{2}(2)(213.33[\frac{1}{3}(2)] - 2F_2$

$M_B = \frac{1}{2}(2)(213.33[\frac{1}{3}(2)] - 2(497.77)$

$M_B = -853.32 \text{ lb} \cdot \text{ft}^3$

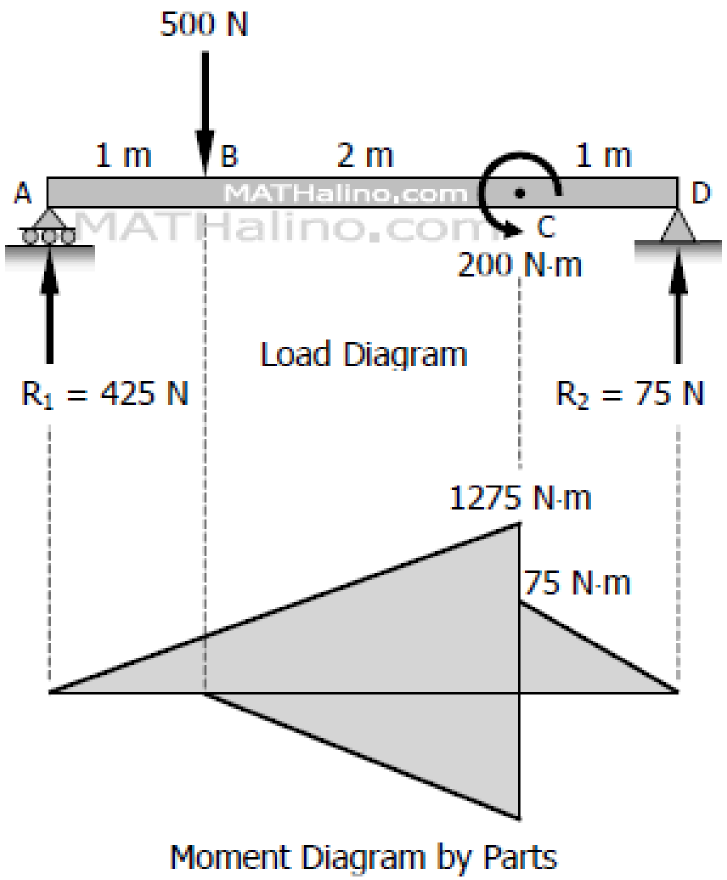
Thus, the deflection at B is

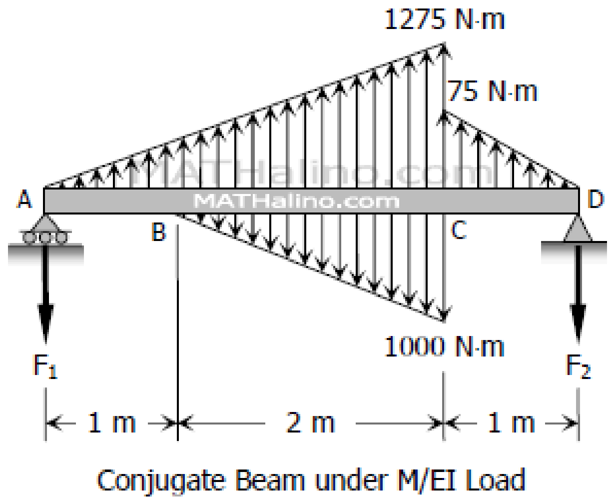
$EI\,\delta_B = M_B$

$EI\,\delta_B = -853.32 \text{ lb} \cdot \text{ft}^3$

$EI\,\delta_B = 853.32 \text{ lb} \cdot \text{ft}^3$ downward

Methods – Conjugate Beam Method





$$M_C = \frac{1}{2}(1)(75)[\tfrac{1}{3}(1)] - 1(F_2)$$
$$M_C = 12.5 - 1(404.17)$$
$$M_C = -391.67 \text{ N} \cdot \text{m}^3$$

Therefore, the deflection at C is

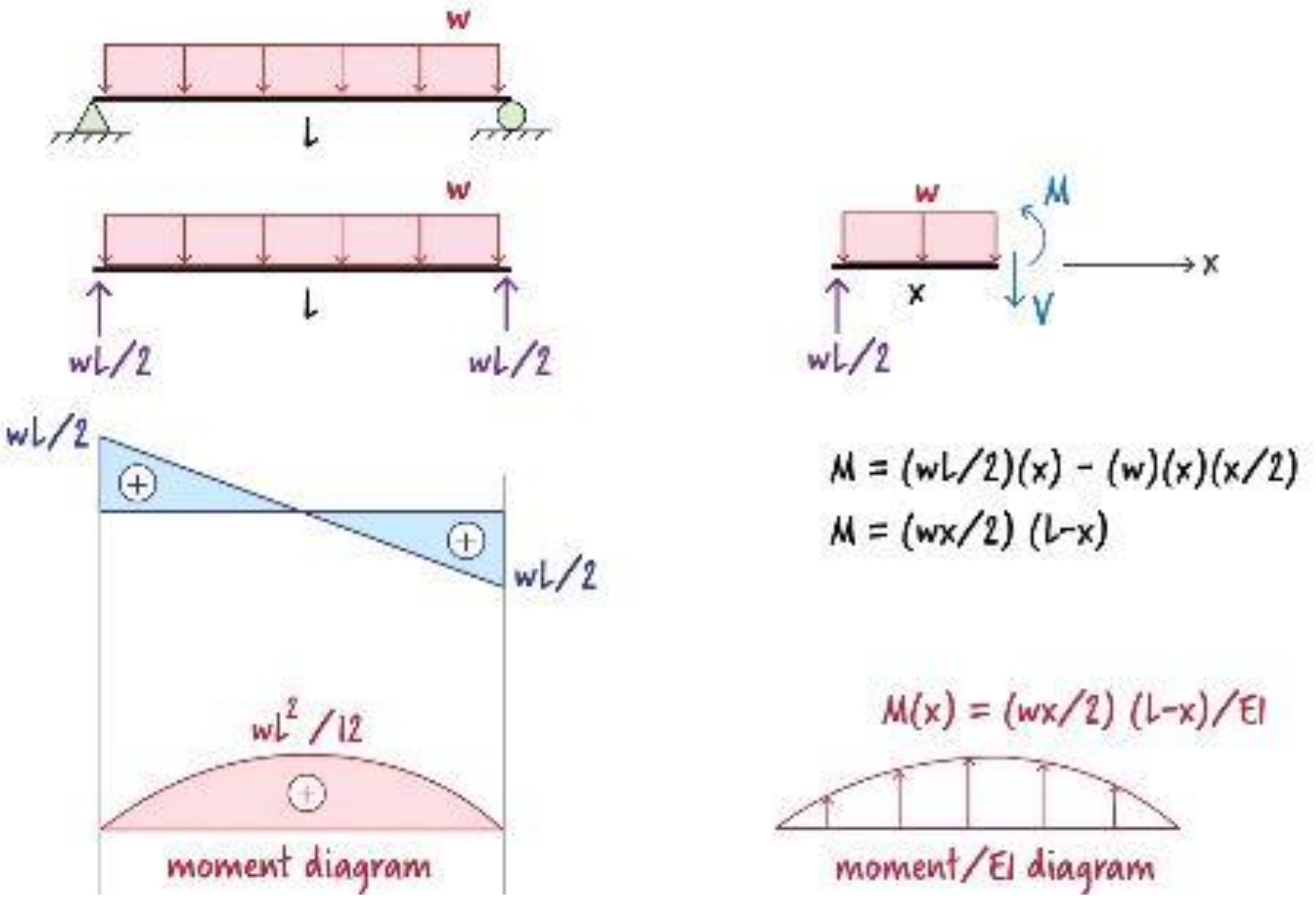
$$EI\,\delta_C = M_C$$
$$EI\,\delta_C = -391.67 \text{ N} \cdot \text{m}^3$$
$$EI\,\delta_C = 391.67 \text{ N} \cdot \text{m}^3 \quad \text{downward}$$

From the conjugate beam

$$\Sigma M_A = 0$$
$$4F_2 + \frac{1}{2}(2)(1000)[1 + \tfrac{2}{3}(2)] = \frac{1}{2}(3)(1275)[\tfrac{2}{3}(3)] + \frac{1}{2}(1)(75)[3 + \tfrac{1}{3}(1)]$$
$$F_2 = 404.17 \text{ N} \cdot \text{m}^3$$

Methods – Conjugate Beam Method

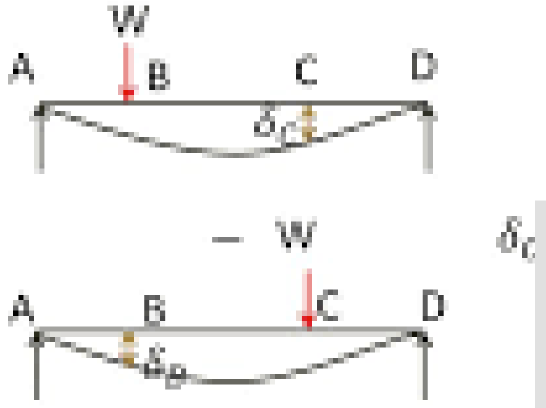
Determine the beam's deflection at the midpoint.



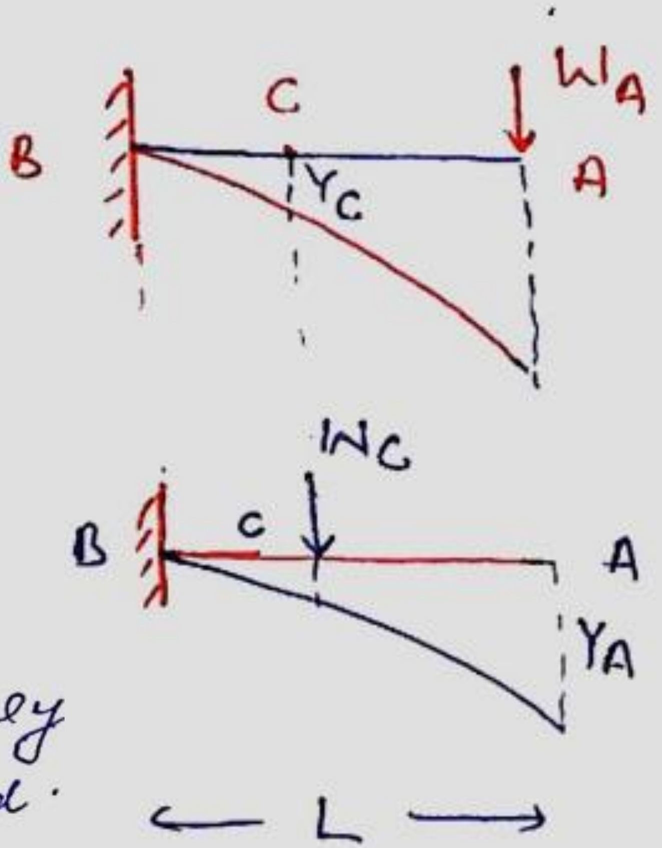
Application-Maxwell's Reciprocal Theorem

Maxwell's reciprocal theorem

In any beam (or) truss, the deflection at any point C due to load W at any point B is the same as the deflection at B due to the same load W applied at C.



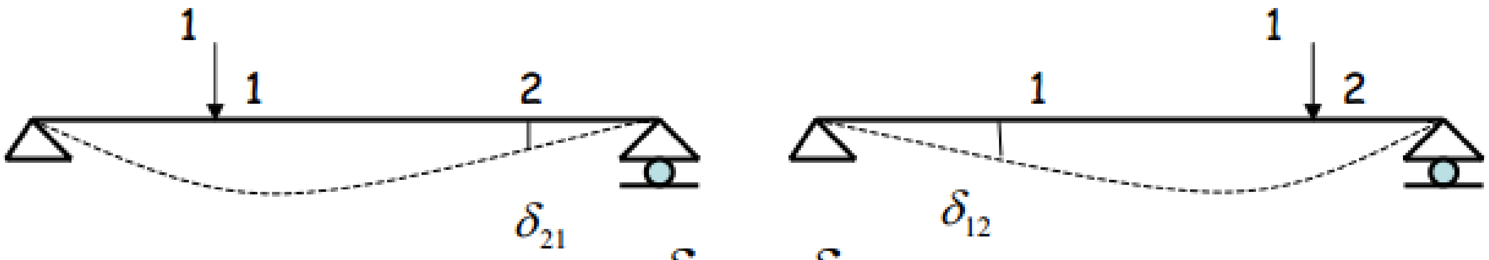
Maxwell's Reciprocal



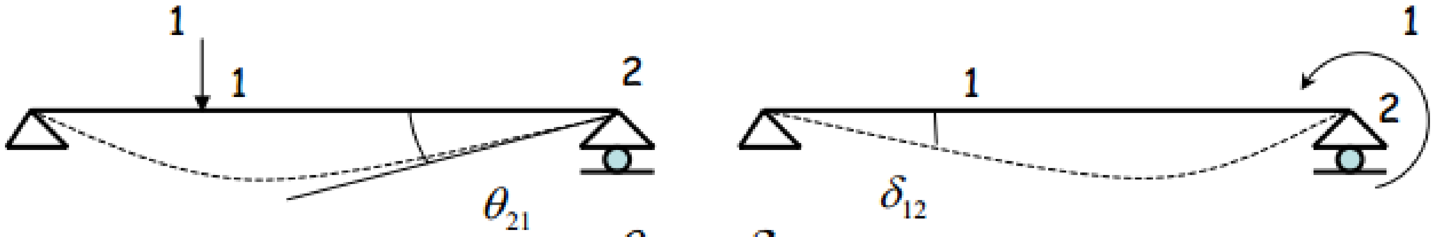
$$\boxed{W_A \times Y_A = W_C \times Y_C}$$

This theorem is a valid only for concentrated point load.

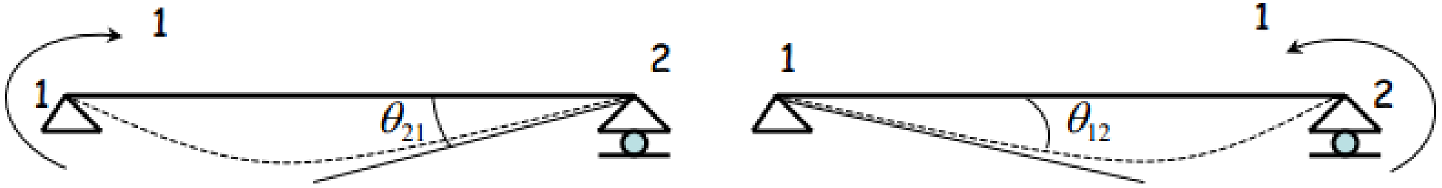
Application-Maxwell's Reciprocal Theorem



$$\delta_{21} = \delta_{12}$$



$$\theta_{21} = \delta_{12}$$



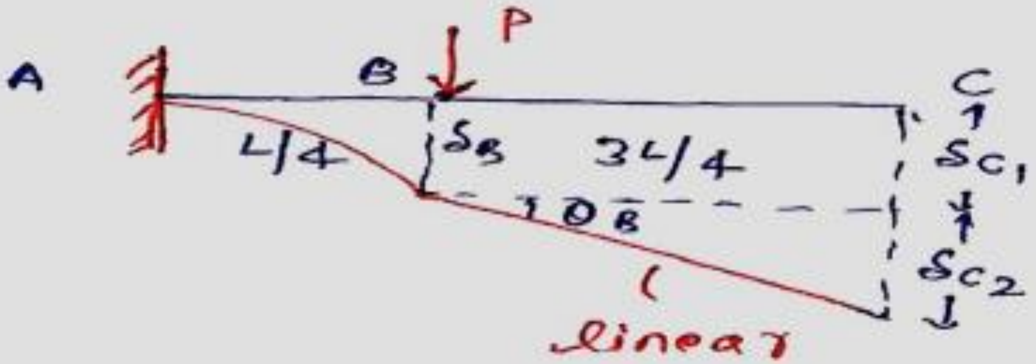
$$\theta_{21} = \theta_{12}$$

Consider two different loading on a linear elastic structure. The virtual work done by the forces of the first system acting through the displacements of the second system is equal to the virtual work done by the forces of the second system acting through the corresponding displacements of the first system. .

Application-Maxwell's Reciprocal Theorem

According to Maxwell's reciprocal theorem.

δ_B = deflection at C when same load is placed at B.



$\delta_B = (\delta_C)$ when load at B

$$\delta_C = \delta_{C_1} + \delta_{C_2} = \delta_B + \theta_B \times L_{BC}$$

$$\delta_B = \delta_C = \frac{P(L/4)^3}{3EI} + \frac{P(L/4)^2}{2EI} \times \frac{3L}{4} = \frac{11}{384}\frac{PL^3}{EI}$$

Deflection of Beams

Queries?