Unit-2 SETS AND RELATIONS

2.1

SETS

In this unit we study the fundamental discrete structure on which all other discrete structures are built, namely, *the set*. The following are some discrete structures built from sets: *combinations*, *unordered collection of objects used extensively in counting*; *relations*, *sets of ordered pairs that represent relationships between objects*; *graphs*, sets *of vertices and edges that connect vertices*; *finite state machines*, *used to model computing machines*.

The concept of a set is fundamental and it unifies mathematics. It has revolutionized mathematical thinking and it enables us to express ourselves in clear and concise terms.
The foundation of set theory was laid by the eminent German mathematician *Georg Cantor* (1845-1918).
An axiomatic approach of development of the set theory and questions of a philosophical nature are avoided. The presentation is informal, formal proofs are indicated which use the notation and the rules of inference of the predicate calculus.
Set: A ***set*** is a well-defined collection of objects. The objects of a set are called ***elements*** or ***members*** of a set. A collection (of objects) is well-defined if there is no ambiguity in determining whether or not a given object (what so ever) belongs to the collection.
Sets are denoted by capital letters and their elements by lowercase letters. If an object x is an element of a set A, then we write $x \in A$; otherwise $x \notin A$.

There are two methods of defining sets
Listing method: A set can sometimes be described by listing its elements within braces.
Example 1: The set V of all Vowels in the English alphabet is given by
$\boldsymbol{V = \{a, e, i, o, u\}}$

A set with a large number of elements that follow a definite pattern often described using ellipses (…) by listing a few elements at the beginning. For example, the set O of odd positive integers can be listed as $\{1,3,5, \dots, \dots\}$.

Set builder notation: Another way of describing a set by using the set builder notation. We characterize all those elements in the set by stating the property or properties they must have to be members.

Example 2: Let M be the set of all months of the year with exactly 30 days. Then $\boldsymbol{M}$ in set builder notation is

$$M = \{x | x \text{ is a month of the year with exactly 30 days}\}$$

Also, M in the listing method is

$$M = \{\text{April, June, September, November}\}$$

The following sets, each denoted by a bold faced letter, play an important role in discrete mathematics;

$\boldsymbol{N} = \{1,2,3, \dots, \dots\}$, the set of ***natural numbers***

$\boldsymbol{Z} = \{\dots, \dots, -2, -1, 0, 1, 2, \dots, \dots\}$, the set of ***integers***

$\boldsymbol{Q} = \left\{\frac{p}{q} | p, q \in \boldsymbol{Z}, q \neq 0\right\}$, the set of ***rational numbers***

$\boldsymbol{R}$: The set of ***real number***

In general a set can be defined or characterized by a predicate .If $P(x)$ is the predicate then $\{x|P(x)\}$ defines a set. An element a belongs to the set $\{x|P(x)\}$ if $P(a)$ is true; otherwise a does not belongs to the set. This statement is written symbolically as

$$\forall y(P(y) \Leftrightarrow y \in \{x|P(x)\})$$

If $A = \{x|P(x)\}$, then the set A is called an ***extension* of $(\boldsymbol{x})$** , (and A is said to be specified by the predicate $P(x)$) and $\forall y(y \in A \Leftrightarrow y \in \{x|x \in A\})$

Let $P(x)$ and $Q(x)$ be any two predicates defined over a common universe of discourse denoted by U. If for every assignment of values to x from U, the resulting propositions have the same truth values, then the predicates $P(x)$ and $Q(x)$ are said to be ***equivalent to each other over $\boldsymbol{E}$***. Then we write $P(x) \Leftrightarrow Q(x)$. The definition of the (tautological) implication can be extended in the same way.
All the implications and equivalences of the propositional calculus given in module 1.3 can be considered as implications and equivalences of the predicate calculus.
Throughout this module the proofs of set identities are established using the implications and equivalences of predicate calculus, assuming that sets are specified by predicates.

Equality of sets: Two sets A and B are ***equal*** (***extensionally equal***, written $A = B$) if they have the same elements. That is, $A = B$ if and only if $\forall x(x \in A \leftrightarrow x \in B)$.

Let the sets A and B be extensions of the predicates $P(x)$ and $Q(x)$ respectively. If $P(x) \Leftrightarrow Q(x)$, then $A = B$, that is if two predicates are equivalent then they have the same extension and the two sets specified by equivalent predicates are equal. *This is the analogy between the equality of sets and the equivalence of predicates*.

Note:

1. The sets $\{1,3,5\}, \{3,5,1\}$ and $\{1,3,3,5,1,5,5,5\}$ are all equal, because they have the same elements.
2. The order in which the elements of a set are listed does not matter and also it does not matter if an element of set is listed more than once.

There are two special sets, one of which includes every set *under discussion* while the other is included in every set *under discussion*.

Universal set: It is always possible to choose a special set U such that every set under discussion is a subset of U. Such a set is called a **Universal set**. Thus, $A \subseteq U$ for every set A.That is, every element $x \in U$, $i.e.$, $\forall\, x\ (x \in U)$ is identically true.

Further, U can be specified as $U = \{x|P(x) \vee \sim P(x)\}$, where $P(x)$ is any predicate.

Empty set: The set containing no elements is called an **empty set** or **null set** and is denoted by ϕ.
It follows from the definition of an empty set $\emptyset$, for any $x, x \in \emptyset$ is a contradiction, $i.e.$, $\forall x(x \in \emptyset)$ is a contradiction.
Further, $\emptyset$ can be specified as $\emptyset = \{x|P(x) \wedge \sim P(x)\}$ where $P(x)$ is any predicate. It is easy to see that, all such sets are equal to $\emptyset$. This shows that there is a unique empty set.

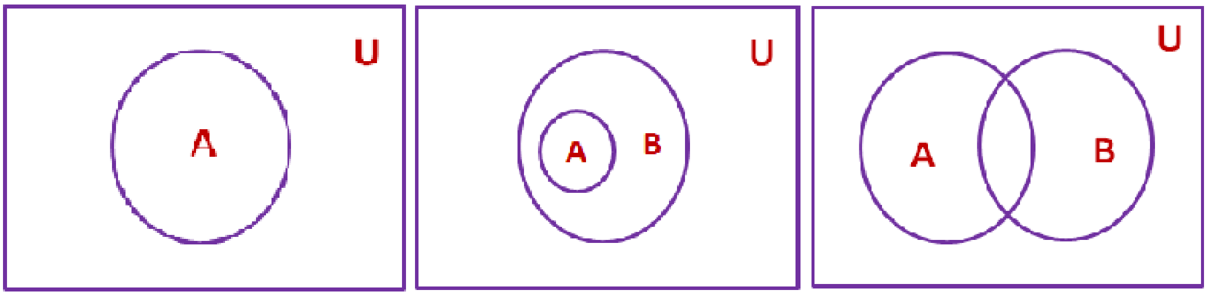
Example 3: The set of all positive integers that are greater than their squares is the empty set.

A set with one element is called a **singleton set.**

Subset: Let A and B be sets. The set A is said to be a **subset** of B
or A *is* **included** in B (written as $A \subseteq B$ and called as **inclusion**) if and only if every element of A is also an element of B.
We see that $A \subseteq B$ if and only if the quantification $\forall x(x \in A \longrightarrow x \in B)$ is true.

Venn diagrams: Sets and relationships between sets can be represented graphically using Venn diagrams, named after the English mathematician *John Venn* (1845 – 1918).

In Venn diagrams the universal set U which contains all the objects under consideration is represented by a rectangle. (Note that the universal set varies depending on the context).Inside this rectangle, circles (or other closed geometrical figures) are used to represent sets.



$A \subseteq B$

A and B may have common elements

Theorem 1: For every set S,

(i) $\boldsymbol{\phi \subseteq S}$ **and (ii) $\boldsymbol{S \subseteq S}$**

Proof: Let S be a set.

(i) To show that $\phi \subseteq S$, we must show that $\forall x(x \in \phi \longrightarrow x \in S)$ is true. Since ϕ has no elements, $x \in \phi$ is always false. Recall that a conditional with a false hypothesis is guaranteed to be true. Therefore, $\forall x(x \in \phi \longrightarrow x \in S)$ is true. Thus, $\phi \subseteq S$, i.e., ***empty set is a subset of every set.***
Note: This is an example of a vacuous proof.

(ii) Note that $\forall x(x \in S \longrightarrow x \in S)$ is true. Therefore, $S \subseteq S$, i.e., ***every set is a subset of itself.***

Theorem 2: For any sets $\boldsymbol{A, B}$ and $\boldsymbol{C}$, $(\boldsymbol{A \subseteq B}) \wedge (\boldsymbol{B \subseteq C}) \Longrightarrow (\boldsymbol{A \subseteq C})$

Proof: We have,
$(A \subseteq B) \wedge (B \subseteq C)$ if and only if $\forall x(x \in A \longrightarrow x \in B) \wedge \forall x(x \in B \longrightarrow x \in C)$
We now use the following (tautological) implication of the predicate calculus
$\forall x(x \in A \longrightarrow x \in B) \wedge \forall x(x \in B \longrightarrow x \in C) \Longrightarrow \forall x(x \in A \longrightarrow x \in C)$
(by hypothetical syllogism) ,Thus, $(A \subseteq B) \wedge (B \subseteq C) \Longrightarrow (A \subseteq C)$

Note: The set inclusion is reflexive and transitive

Theorem 3: For any sets $\boldsymbol{A}$ and $\boldsymbol{B}$, $\boldsymbol{A} = \boldsymbol{B} \Leftrightarrow \big((\boldsymbol{A} \subseteq \boldsymbol{B}\ ) \wedge (\boldsymbol{B} \subseteq \boldsymbol{A})\big)$

Proof: Let A and B be sets specified by the predicates $P(x)$ and $Q(x)$ respectively. We have the equivalence from the predicate calculus

$$\forall x\big((P(x) \longrightarrow Q(x)) \wedge (Q(x) \longrightarrow P(x))\big) \Leftrightarrow \forall x(P(x) \leftrightarrow Q(x))$$

Therefore

$$A = B \Leftrightarrow \forall x(x \in A \leftrightarrow x \in B) \Leftrightarrow \forall x\big((x \in A \longrightarrow x \in B) \wedge (x \in B \longrightarrow x \in A)\big)$$

$$\Leftrightarrow \big((A \subseteq B\ ) \wedge (B \subseteq A)\big)$$

Proper subset: Let A and B be sets. The set A is said to be a **proper subset** of B, written as $A \subset B$ if $A \subseteq B$ and there must exist an element x of B that is not an element of A. That is, $A \subset B$ iff $\forall x(x \in A \to x \in B) \wedge \exists x(x \in B \wedge\ x \notin A)$ is true. $A \subset B$ is also called a **proper inclusion**.

Note: A proper inclusion is not reflexive; however, it is transitive, $i.e.$,

$$(A \subset B) \wedge (B \subset C) \Longrightarrow (A \subset C)$$

Cardinality of a set: Let S be a set. If there are exactly n distinct elements in S, where n is a nonnegative integer, then we say that S is a ***finite* set** and that n is the *cardinality* of S. The cardinality is denoted by $|\boldsymbol{S}|$ or $\boldsymbol{n}(\boldsymbol{S})$ or $\boldsymbol{k}(\boldsymbol{S})$

A set is said to be ***infinite*** if it is not finite.

Example 4: If $\boldsymbol{S}$ is the set of letters in the English alphabet, then $\boldsymbol{S}$ in finite and $|\boldsymbol{S}| = \boldsymbol{26}$.

Note that $|\emptyset| = 0$.

The power set: Given a set S, the power set of S is the set of all subsets of the set S. The power set of S is denoted by $\boldsymbol{P}(\boldsymbol{S})$ or $\boldsymbol{2^s}$.

The following theorem is well known and its proof is already given under mathematical induction.

Theorem 4: If S is a finite set with n element, then its power set $P(S)$ has 2^n elements.

Note: $P(\emptyset) = \{\emptyset\}$, $P\big(P(\emptyset)\big) = \{\emptyset, \{\emptyset\}\}$

$$P\Big(P\big(P(\emptyset)\big)\Big) = \Big\{\emptyset,\ \{\emptyset\}, \{\{\emptyset\}\},\ \{\emptyset, \{\emptyset\}\}\Big\}$$

Indexed set: Let $\boldsymbol{I} = \{i_1, i_2, i_3, \dots, \dots\}$. If A be a family of sets $A = \{A_{i_1}, A_{i_2}, A_{i_3}, \dots, \dots\}$ such that for any $i_j \in \boldsymbol{I}$ there exists a set $A_{i_j} \in A$ and $A_{i_j} = A_{i_k}$ iff $i_j = i_k$ then A is called ***indexed set,*** $\boldsymbol{I}$ is the ***index set*** and any subscript i_j in A_{i_j} is called an ***index***.

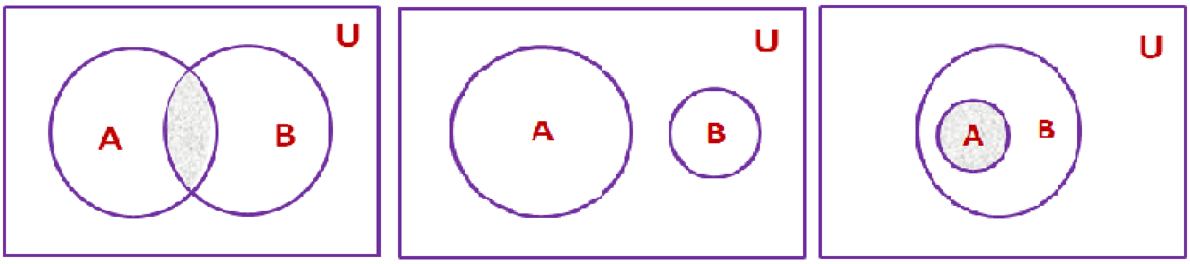
An indexed family of sets can also be written as $A = \{A_i\}_{i\in \boldsymbol{I}}$

Set Operations

Sets can be combined in several ways to get new sets. Note that we find a close relationship between logic operations and set operations.

Intersection: Let A and B be sets. The *intersection* of the sets A and B, denoted by $\boldsymbol{A} \cap \boldsymbol{B}$, is the set containing elements in both A and B. That is,

$$\boldsymbol{A} \cap \boldsymbol{B} = \{\boldsymbol{x} | \boldsymbol{x} \in \boldsymbol{A} \wedge \boldsymbol{x} \in \boldsymbol{B}\}$$



$A \cap B$ $\quad$ $A \cap B = \emptyset$ $\quad$ $A \cap B = A$

Note: It is easy to see that $A \cap B \subseteq A$ and $A \cap B \subseteq B$

Theorem 5: For any sets $\boldsymbol{A}$,$\boldsymbol{B}$ and $\boldsymbol{C}$

i) $\boldsymbol{A \cap B = B \cap A}$
ii) $\boldsymbol{A \cap A = A}$
iii) $\boldsymbol{A \cap \emptyset = \emptyset}$
iv) $\boldsymbol{A \cap (B \cap C) = (A \cap B) \cap C}$

Proof:

(i) $A \cap B = \{x|\ x \in A \land x \in B\}$

$= \{x|\ x \in B \land x \in A\}$ (by the commutativity of the predicate calculus)

$= B \cap A$

(ii) $A \cap A = \{x|\ x \in A \land x \in A\} = \{x|\ x \in A\} = A$
(iii) $A \cap \emptyset = \{x|\ x \in A \land x \in \emptyset\} = \{x|\ x \in \emptyset\} = \emptyset \qquad (\because\ \ p \land F_0 = F_0)$

(iv) $A \cap (B \cap C) = \{x|\ x \in A \land\ x \in (B \cap C)\}$

$= \{x|\ x \in A\ \land (x \in B\ \land x \in C)\}$

$= \{x|\ (x \in A\ \land x \in B) \land x \in C\}$

(by the associativity for the predicate calculus)

$= \{x|\ x \in (A \cap B) \land x \in C\}$

$= (A \cap B) \cap C$

this completes the proof of the theorem.

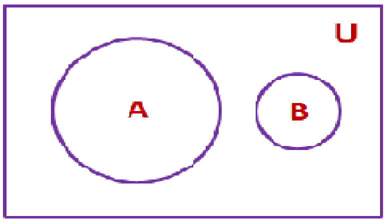
For any indexed set $A = \{A_i\}_{i \in \boldsymbol{I}}$

$$\bigcap_{i \in \boldsymbol{I}} A_i = \{x|\ x \in A_i\ for\ all\ i \in \boldsymbol{I}\ \}$$

For $\boldsymbol{I} = \boldsymbol{I_n} = \{1, 2, 3, \dots, n\}$, we write

$$A_1 \cap A_2 \cap \dots \cap A_n = \bigcap_{i=1}^{n} A_i = \bigcap_{i \in \boldsymbol{I_n}} A_i$$

Disjoint sets: Two sets are said to be *disjoint* if and only if their intersection is the empty set. That is, two sets A and B are disjoint iff $A \cap B = \emptyset$



Disjoint Sets

A collection of sets is called a ***disjoint collection*** if every pair of sets in the collection are disjoint .The sets in a disjoint collection are said to be ***mutually disjoint***.

Let $A = \{A_i\}_{i \in \boldsymbol{I}}$ be an indexed set. The set A is a disjoint collection iff $A_i \cap A_j = \emptyset$ for $i, j \in \boldsymbol{I}, i \neq j$.

Example 5: Show that $\boldsymbol{A \subseteq B}$ iff $\boldsymbol{A \cap B = A}$

Solution: We have $A \subseteq B$ iff $\forall x(x \in A \longrightarrow x \in B)$

Note that for any x $(x \in A \longrightarrow x \in B) \equiv (x \in A \wedge x \in B) \leftrightarrow x \in A$

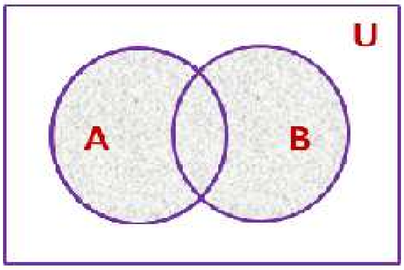
This follows from the equivalence $p \longrightarrow q \equiv \big((p \wedge q) \leftrightarrow p\big)$ (verify!)

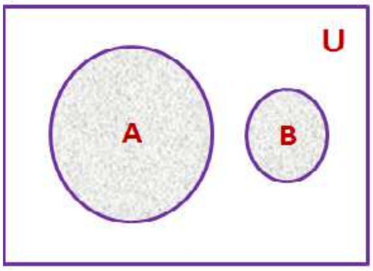
Thus, $A \subseteq B$ iff $\forall x \, \big((x \in A \wedge x \in B) \leftrightarrow x \in A\big)$

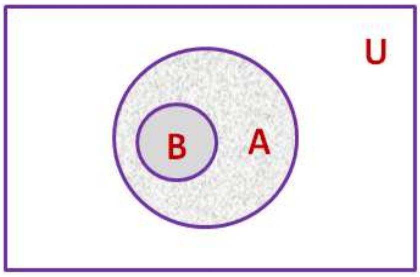
i.e., $A \subseteq B$ iff $A \cap B = A$

Union: Let A and B be sets. The *union* of the sets A and B is denoted by $\boldsymbol{A \cup B}$, is the set that contains those elements that are either in$A$ or in B or in both. That is,

$$\boldsymbol{A \cup B = \{x \mid x \in A \vee x \in B\}}$$







$\boldsymbol{A} \cup \boldsymbol{B}$

$\boldsymbol{A} \cup \boldsymbol{B}$**,** where A and B are disjoint

$A \cup B = B$

Theorem 6: For any sets $\boldsymbol{A}, \boldsymbol{B}$ and $\boldsymbol{C}$

i) $\boldsymbol{A} \cup \boldsymbol{B} = \boldsymbol{B} \cup \boldsymbol{A}$

ii) $\boldsymbol{A} \cup \emptyset = \boldsymbol{A}$

iii) $\boldsymbol{A} \cup \boldsymbol{A} = \boldsymbol{A}$

iv) $\boldsymbol{A} \cup (\boldsymbol{B} \cup \boldsymbol{C}) = (\boldsymbol{A} \cup \boldsymbol{B}) \cup \boldsymbol{C}$

Proof: Let A, B and C be any sets

i) $A \cup B = \{x|\ x \in A \vee x \in B\}$

$= \{x|\ x \in B \vee x \in A\}$ (by the commutativity of the predicate calculus)

$= B \cup A$

ii) $A \cup \emptyset = \{x|\ x \in A \vee x \in \emptyset\} = \{x|\ x \in A \vee F_0\} = \{x|\ x \in A\} = A$

iii) $A \cup A = \{x|\ x \in A \vee x \in A\}$

$= \{x|\ x \in A\}$ (by the idempotent law of the predicate calculus)

$= A$

iv) $A \cup (B \cup C) = \{x|\ x \in A \vee x \in (B \cup C)\} = \{x|\ x \in A \vee (x \in B \vee x \in C)\}$

$= \{x|\ (x \in A \vee x \in B) \vee x \in C\}$

(by the associativity of the predicate calculus)

$= (A \cup B) \cup C$

Thus the theorem is proved

For any indexed set $A = \{A_i\}_{i \in \boldsymbol{I}}$

$$\bigcup_{i \in \boldsymbol{I}} A_i = \{x |\ x \in A_i\ for\ at\ least\ one\ i \in \boldsymbol{I}\ \}$$

For $\boldsymbol{I} = \boldsymbol{I_n} = \{1, 2, 3, \dots, n\}$, we write

$$A_1 \cup A_2 \cup \dots \cup A_n = \bigcup_{i=1}^{n} A_i = \bigcup_{i \in \boldsymbol{I_n}} A_i$$

Theorem 7: Distributive laws:

For any sets $\boldsymbol{A}, \boldsymbol{B}$ and $\boldsymbol{C}$

i) $\boldsymbol{A} \cap (\boldsymbol{B} \cup \boldsymbol{C}) = (\boldsymbol{A} \cap \boldsymbol{B}) \cup (\boldsymbol{A} \cap \boldsymbol{C})$

ii) $\boldsymbol{A} \cup (\boldsymbol{B} \cap \boldsymbol{C}) = (\boldsymbol{A} \cup \boldsymbol{B}) \cap (\boldsymbol{A} \cup \boldsymbol{C})$

Proof:

$A \cap (B \cup C) = \{x |\ x \in A \wedge x \in (B \cup C)\}$

$= x | x \in A \wedge (x \in B\ \vee x \in C)$

$= \{x | (x \in A\ \wedge x \in B) \vee (x\ \in A\ \wedge x \in C)\}$

(By distributivity of the predicate calculus)

$= \{x | x \in (A \cap B) \vee x \in (A \cap C)\}$

$= (A \cap B) \cup (A \cap C)$

The other distributive law can be proved on similar lines.

Let A be a family of indexed set over an index set I such that

$$A = \{A_1, A_2, A_3, \dots, \dots\} = \{A_i\}_{i \in I}$$

The associative laws and distributive laws can be generalized in the following way

$$B \cup \left(\bigcap_{i \in I} A_i \right) = \bigcap_{i \in I} (B \cup A_i)$$

$$B \cap \left(\bigcup_{i\in I} A_i\right) = \bigcup_{i\in I}(B \cap A_i)$$

The above identities can be proved by the mathematical induction.

Theorem 8: Absorption laws: For any sets A and B

i) $\boldsymbol{A \cup (A \cap B) = A}$
ii) $\boldsymbol{A \cap (A \cup B) = A}$

Proof:

$A \cup (A \cap B) = (A \cup A) \cap (A \cup B)$ (by distributive law)

$= A \cap (A \cup B)$ (by idempotent law)

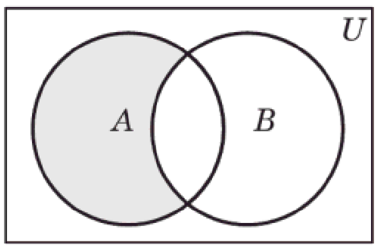
Now, $A \cup (A \cap B) = \{x|\ x \in A \vee\ (x \in A\ \wedge x \in B)\}$

$= \{x|\ x \in A\}$ (by absorption law of the predicate calculus)

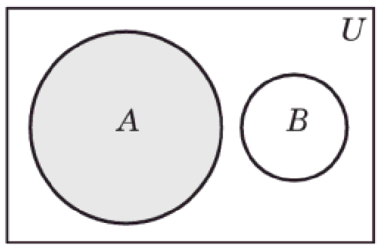
Thus, $A \cap (A \cup B) = A \cup (A \cap B) = A$

Difference of two sets: Let A and B sets. The ***difference*** of A and B is denoted by $\boldsymbol{A - B}$, is the set containing those elements that are in A but not in B. The difference of $A$and B is also called the ***complement of B with respect to A***.That is,

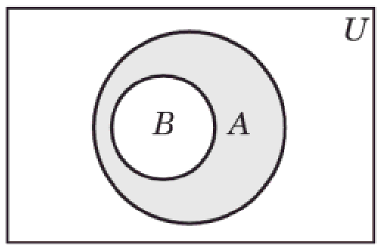
$$\boldsymbol{A - B = \{x|x \in A \wedge x \notin B\} = \{x|x \in A \wedge \sim(x \in B)\}}$$



$A - B$



$A - B = A$

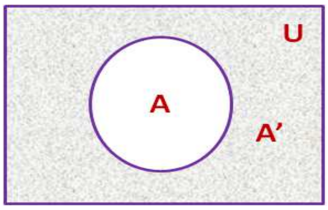


$A - B$

Note: It is easy to see that $A - B \subseteq A$

Complement of a set: Let U be the universal set. The ***complement*** of the set A is denoted by $'$, is the complement of A with respect to U, $i.e.$,

$$\boldsymbol{A' = U - A = \{x|x \in U \wedge x \notin A\} = \{x|x \notin A\} = \{x|\sim(x \in A)\}}.$$



Theorem 9: For any set A

i) $(\boldsymbol{A}')' = \boldsymbol{A}$
ii) $\boldsymbol{A} \cup \boldsymbol{A}' = \boldsymbol{U}$
iii) $\boldsymbol{A} \cap \boldsymbol{A}' = \emptyset$
iv) $\emptyset' = \boldsymbol{U}$ and
v) $\boldsymbol{U}' = \emptyset$

Proof:

i) $(A')' = \{x|\sim(x \in A')\} = \{x|\sim\big(\sim(x \in A)\big)\}$

$= \{x|\ x \in A\}$ (by double negation law of the predicate calculus)

$= A$

ii) $A \cup A' = \{x|\ x \in A \vee x \in A'\} = \{x|\ x \in A \vee \sim(x \in A)\} = U$
iii) $A \cap A' = \{x|\ x \in A \wedge x \in A'\} = \{x|\ x \in A \wedge \sim(x \in A)\} = \emptyset$
iv) $\emptyset' = \{x|\sim(x \in \emptyset)\} = \{x|\ x \in U\} = U$
v) $U' = (\emptyset')' = \emptyset$ (by (i))

Example 6: Show that (i) $\boldsymbol{A} - \boldsymbol{B} = \boldsymbol{A} \cap \boldsymbol{B}'$ (ii) $\boldsymbol{A} \subseteq \boldsymbol{B}$ iff $\sim\boldsymbol{B} \subseteq \sim\boldsymbol{A}$

Solution:

i) $A - B = \{x|\ x \in A \wedge x \notin B\} = \{x|\ x \in A \wedge x \in B'\} = A \cap B'$
ii) $A \subseteq B$ iff $\forall x(x \in A \longrightarrow x \in B) \equiv \forall x\big(\sim(x \in B) \longrightarrow \sim(x \in A)\big)$
(the conditional and its contrapositive are equivalent)

$\equiv \forall x(x \in B' \longrightarrow x \in A')$ iff $B' \subseteq A'$

Thus $A \subseteq B$ iff $B' \subseteq A'$

Theorem 10: De Morgan's laws: **For any sets A and B**

(i) $(\boldsymbol{A} \cap \boldsymbol{B})' = \boldsymbol{A}' \cup \boldsymbol{B}'$ **(ii)** $(\boldsymbol{A} \cup \boldsymbol{B})' = \boldsymbol{A}' \cap \boldsymbol{B}'$

Proof:

i) $(A \cap B)' = \{x |\ x \notin (A \cap B)\}$
$= \{x |\ \sim(x \in (A \cap B))\}$
$= \{x |\ \sim(x \in A \wedge x \in B)\}$
$= \{x |\ \sim(x \in A) \vee \sim(x \in B)\}$
(by De Morgan's law of the predicate calculus)
$= \{x |\ x \in A' \vee x \in B'\}$
$= A' \cup B'$

ii) $(A \cup B)' = \{x |\ x \notin (A \cup B)\}$
$= \{x |\ \sim(x \in (A \cup B))\}$
$= \{x |\ \sim(x \in A \vee x \in B)\}$
$= \{x |\ \sim(x \in A) \wedge \sim(x \in B)\}$
(by De Morgan's law for of the predicate calculus)
$= \{x |\ x \in A' \wedge x \in B'\}$
$= A' \cap B'$

Symmetric Difference: Let A and B be two sets. The ***symmetric difference*** (or) ***Boolean sum*** of A and B is the set, denoted by $\boldsymbol{A} \oplus \boldsymbol{B}$, and it is defined by

$$\boldsymbol{A} \oplus \boldsymbol{B} = (\boldsymbol{A} - \boldsymbol{B}) \cup (\boldsymbol{B} - \boldsymbol{A})$$

Note: $\boldsymbol{A} \oplus \boldsymbol{B} = (\boldsymbol{A} - \boldsymbol{B}) \cup (\boldsymbol{B} - \boldsymbol{A}) = (\boldsymbol{A} \cap \boldsymbol{B}') \cup (\boldsymbol{A}' \cap \boldsymbol{B})$

Theorem 11: For any sets A, B and C

i) $A \oplus B = B \oplus A$ ($\oplus$ is commutative)
ii) $A \oplus \emptyset = A$
iii) $A \oplus A = \emptyset$
iv) $A \oplus (B \oplus C) = (A \oplus B) \oplus C$ ($\oplus$ is associative)

Example 7: Show that $(\boldsymbol{A} \oplus \boldsymbol{B}) = (\boldsymbol{A} \cup \boldsymbol{B}) - (\boldsymbol{A} \cap \boldsymbol{B})$

Solution: We have $(A \oplus B) = (A' \cap B) \cup (A \cap B')$

Now $(A \cup B) - (A \cap B) = (A \cup B) \cap (A \cap B)'$

$= (A \cup B) \cap (A' \cup B')$ (Demorgan's law)

$= (A \cap A') \cup (A \cap B') \cup (B \cap A') \cup (B \cap B')$(Distributive law)

$= (A \cap B') \cup (B \cap A') = (A - B) \cup (B - A) = (A \oplus B)$

Thus, $A \oplus B = (A \cup B) - (A \cap B)$

Therefore, $\boldsymbol{A} \oplus \boldsymbol{B} = (\boldsymbol{A} - \boldsymbol{B}) \cup (\boldsymbol{B} - \boldsymbol{A}) = (\boldsymbol{A} \cup \boldsymbol{B}) - (\boldsymbol{A} \cap \boldsymbol{B})$

Set Identities

If A and B are extensions of the predicates $P(x)$ and $Q(x)$ respectively in a universal set U, then $A \cup B$ and $A \cap B$ are the extensions of $P(x) \vee Q(x)$ and $P(x) \wedge Q(x)$ respectively. Similarly A' is the extension of $\sim P(x)$. The extension of $P(x) \longrightarrow Q(x)$ and $P(x) \longleftrightarrow Q(x)$ are respectively $A \cup B'$and $(A' \cup B) \cap (A \cup B')$. Thus, the new sets formed from the sets A and B can be interpreted in terms of extensions of formulas containing $P(x)$ and $Q(x)$.

The identities of set theory follow from the corresponding equivalences of predicate formulas. Also the inclusion of sets follows from the corresponding implications of predicates.
If we replace predicates by their extensions, $\wedge$ by $\cap$, $\vee$ by $\cup$ and $\sim$ by $'$ in any predicate formula then we obtain corresponding formulas of set theory. Also the equivalences and implications are replaced by equality and inclusion of sets. This technique has often been used to prove the identities other relations of set theory so far.
Some of the basic identities describe certain properties of the operations involved and are given special names. These properties describe an algebra called **set algebra** (or **algebra of sets**).We note that the propositional algebra and set algebra are particular cases of an abstract algebra called a **Boolean Algebra.**

This fact also explains the similarities between the operators in the propositional calculus and the operations of set theory.

For all identities listed in this module, the corresponding equivalences from the propositional calculus are also listed. Similar equivalences hold for the predicate calculus

Set algebra	Propositional algebra	Laws
$A \cup A = A; A \cap A = A$	$p \vee p \equiv p \quad ; p \wedge p \equiv p$	Idempotent laws
$A \cup B = B \cup A; A \cap B = B \cap A$	$p \vee q \equiv q \vee p \;\; ; p \wedge q \equiv q \wedge p$	Commutative laws
$A \cap (A \cup B) = A$ $A \cup (A \cap B) = A$	$p \wedge (p \vee q) \equiv p$; $p \vee (p \wedge q) \equiv p$	Absorption laws
$(A')' = A$	$\sim(\sim p) \equiv p$	Double negation law
$(A \cup B)' = A' \cap B'$ $(A \cap B)' = A' \cup B'$	$\sim(p \vee q) \equiv \sim p \wedge \sim q$; $\sim(p \wedge q) \equiv \sim p \vee \sim q$	De Morgan's laws
$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	$p \vee (q \vee r) \equiv (p \vee q) \vee r$ $p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r$	Associative laws
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$ $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$	Distributive laws
$A \cap U = A; A \cup \emptyset = A$	$p \wedge T_0 \equiv p \quad ; \quad p \vee F_0 \equiv p$	Identity laws
$A \cup U = U; A \cap \emptyset = \emptyset$	$p \vee T_0 \equiv T_0 \quad ; \quad p \wedge F_0 \equiv F_0$	Domination laws
$A \cup A' = U; A \cap A' = \emptyset$	$p \vee \sim p \equiv T_0 \quad ; \;\; p \wedge \sim p \equiv F_0$	Inverse laws

All the identities given above are presented in pairs except the double negation law. This pairing is done because a **principle of duality**, similar to the one given for propositional calculus also holds in the case of set algebra.(In fact the principle of duality holds in Boolean Algebra).Given any identity in the set algebra, one can obtain its *dual* $i.e.$, another identity by interchanging $\cup$ with $\cap$ and U with $\emptyset$.

P1:

Prove or disprove: For any sets $\boldsymbol{A}, \boldsymbol{B} \subseteq \boldsymbol{U}$

(i) $\boldsymbol{P}(\boldsymbol{A} \cup \boldsymbol{B}) = \boldsymbol{P}(\boldsymbol{A}) \cup \boldsymbol{P}(\boldsymbol{B})$

(ii) $\boldsymbol{P}(\boldsymbol{A} \cap \boldsymbol{B}) = \boldsymbol{P}(\boldsymbol{A}) \cap \boldsymbol{P}(\boldsymbol{B})$

Solution:

(i) Let $A = \{1\}$, $B = \{2\}$

Therefore $A \cup B = \{1,2\}$, now $\{1,2\} \in P(A \cup B)$

Note that $P(A) \cup P(B) = \{\emptyset, \{1\}\} \cup \{\emptyset, \{2\}\} = \{\emptyset, \{1\}, \{2\}\}$

and $\{1,2\} \notin P(A) \cup P(B)$

hence $P(A \cup B) \neq P(A) \cup P(B)$

(ii) $X \in P(A \cap B) \Leftrightarrow X \subseteq A \cap B$

$\Leftrightarrow X \subseteq A$ and $X \subseteq B$

$\Leftrightarrow X \in P(A)$ and $X \in P(B)$

$\Leftrightarrow X \in P(A) \cap P(B)$

Therefore $P(A \cap B) = P(A) \cap P(B)$

P2:

Show that for any sets $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}$ and $\boldsymbol{D}$

$$(\boldsymbol{A} \cap \boldsymbol{B}) \cup \Big(\boldsymbol{B} \cap \big((\boldsymbol{C} \cap \boldsymbol{D}) \cup (\boldsymbol{C} \cap \boldsymbol{D}')\big)\Big) = \boldsymbol{B} \cap (\boldsymbol{A} \cup \boldsymbol{C})$$

Solution: $(A \cap B) \cup \Big(B \cap \big((C \cap D) \cup (C \cap D')\big)\Big)$

$= (A \cap B) \cup \Big(B \cap \big(C \cap (D \cup D')\big)\Big)$ (Distributive law)

$= (A \cap B) \cup \big(B \cap (C \cap U)\big)$

$= (A \cap B) \cup (B \cap C)$

$= (B \cap A) \cup (B \cap C)$ (Commutative law)

$= B \cap (A \cup C)$ (Distributive law)

P3:

Show that for any sets $\boldsymbol{A}$ and $\boldsymbol{B}$, $\boldsymbol{A} - (\boldsymbol{A} \cap \boldsymbol{B}) = \boldsymbol{A} - \boldsymbol{B}$

Solution: $A - (A \cap B) = A \cap (A \cap B)'$

$= A \cap (A' \cup B')$ (De Morgan's law)

$= (A \cap A') \cup (A \cap B')$

$= \emptyset \cup (A \cap B')$

$= A \cap B' = A - B$

P4:

Show that for any sets $\boldsymbol{A}, \boldsymbol{B}$ and $\boldsymbol{C}$

$$\left(\left((\boldsymbol{A} \cup \boldsymbol{B}) \cap \boldsymbol{C}\right)' \cup \boldsymbol{B}'\right)' = \boldsymbol{B} \cap \boldsymbol{C}$$

Solution:

$$\left(\left((A \cup B) \cap C\right)' \cup B'\right)' = \left(\left((A \cup B) \cap C\right)'\right)' \cap (B')' \quad \text{(De Morgan's law)}$$

$$= \left((A \cup B) \cap C\right) \cap B \quad \text{(Double negation law)}$$

$$= (A \cup B) \cap (C \cap B) \quad \text{(Associative law of intersection)}$$

$$= (A \cup B) \cap (B \cap C) \quad \text{(Commutative law of intersection)}$$

$$= \left((A \cup B) \cap B\right) \cap C \quad \text{(Associative law of intersection)}$$

$$= B \cap C \quad \text{(Absorption law)}$$

P5:

Simplify, using laws of set theory.

(i) $\boldsymbol{A \cap (B - A)}$
(ii) $\boldsymbol{(A \cap B) \cup (A \cap B \cap C' \cap D) \cup (A' \cap B)}$
(iii) $\boldsymbol{(A - B) \cup (A \cap B)}$
(iv) $\boldsymbol{A' \cup B' \cup (A \cap B \cap C')}$
(v) $\boldsymbol{A' \cup (A \cap B') \cup (A \cap B \cap C') \cup (A \cap B \cap C \cap D') \cup \ldots}$

Solution:

(i) $A \cap (B - A) = A \cap (B \cap A')$
$= A \cap (A' \cap B)$ (Commutative law)
$= (A \cap A') \cap B$ (Associative law)
$= \emptyset \cap B$
$= \emptyset$

(ii) $(A \cap B) \cup (A \cap B \cap C' \cap D) \cup (A' \cap B)$

$= (A \cap B) \cup (A' \cap B)$ (Absorption law)
$= (A \cup A') \cap B$ (Distributive law)
$= U \cap B$

$= B$

(iii) $(A - B) \cup (A \cap B) = (A \cap B') \cup (A \cap B)$

$= A \cap (B' \cup B)$ (Distributive law)
$= A \cap U$
$= A$

(vi) $A' \cup B' \cup (A \cap B \cap C')$

$= (A \cap B)' \cup (A \cap B \cap C')$ (De Morgan's law)

$= \big((A \cap B)' \cup (A \cap B)\big) \cap \big((A \cap B)' \cup C'\big)$ (Distributive law)

$= U \cap (A' \cup B' \cup C')$ (De Morgan's law)

$= (A' \cup B' \cup C')$

(V) First note that

$A' \cup (A \cap B') = (A' \cup A) \cap (A' \cup B')$ (Distributive law)

$= U \cap (A' \cup B')$

$= U \cap (A \cap B)'$ (De Morgan's law)

$= (A \cap B)' - - - - - - -(1)$

Now, $A' \cup (A \cap B') \cup (A \cap B \cap C')$

$= (A \cap B)' \cup (A \cap B \cap C')$

$= (A \cap B \cap C)'$ (using (1))

and $A' \cup (A \cap B') \cup (A \cap B \cap C') \cup (A \cap B \cap C \cap D')$

$= (A \cap B \cap C)' \cup (A \cap B \cap C \cap D')$

$= (A \cap B \cap C \cap D)'$ (using (1))

Continuing this process we get ,

$\boldsymbol{A' \cup (A \cap B') \cup (A \cap B \cap C') \cup (A \cap B \cap C \cap D') \cup \ldots = (A \cap B \cap C \cap D \cap \ldots)'}$.

P6:

For any sets $\boldsymbol{A}$ and $\boldsymbol{B}$

i) $\boldsymbol{A \oplus B = B \oplus A}$ (⊕ is commutative)
ii) $\boldsymbol{A \oplus \emptyset = A}$
iii) $\boldsymbol{A \oplus A = \emptyset}$
iv) $\boldsymbol{A \oplus A' = U}$
v) $\boldsymbol{A \oplus U = A'}$

Solution:

We have $A \oplus B = (A' \cap B) \cup (A \cap B')$

i) $A \oplus B = (A - B) \cup (B - A)$
$= (B - A) \cup (A - B)$ (Commutativity of ∪)
$= B \oplus A$

ii) $A \oplus \emptyset = (A' \cap \emptyset) \cup (A \cap \emptyset')$

$= \emptyset \cup (A \cap U) = \emptyset \cup A = A$

iii) $A \oplus A = (A' \cap A) \cup (A \cap A') = \emptyset \cup \emptyset = \emptyset$

iv) $A \oplus A' = (A \cap (A')') \cup (A' \cap A')$

$= (A \cap A) \cup A'$

$= A \cup A'$

$= U$

v) $A \oplus U = (A \cap U') \cup (A' \cap U)$

$= (A \cap \emptyset) \cup A'$

$= \emptyset \cup A'$

$= A'$

P7:

For any sets $\boldsymbol{A}, \boldsymbol{B}$ and $\boldsymbol{C}$

$$\boldsymbol{A} \oplus (\boldsymbol{B} \oplus \boldsymbol{C}) = (\boldsymbol{A} \oplus \boldsymbol{B}) \oplus \boldsymbol{C} \quad \textbf{(}\oplus\textbf{ is associative)}$$

Solution: $A \oplus (B \oplus C) = \big(A' \cap (B \oplus C)\big) \cup (A \cap (B \oplus C)')$

Now $A' \cap (B \oplus C) = A' \cap \big((B' \cap C) \cup (B \cap C')\big)$

$$= (A' \cap B' \cap C) \cup (A' \cap B \cap C')$$

and $(A \cap (B \oplus C)') = A \cap \big((B' \cap C) \cup (B \cap C')\big)'$

$$= A \cap \big((B' \cap C)' \cap (B \cap C')'\big) \qquad \text{(De Morgan's law)}$$

$$= A \cap \big((B \cup C') \cap (B' \cup C)\big)$$

(De Morgan's law and double negation law)

$$= A \cap \big((B \cap B') \cup (B \cap C) \cup (C' \cap B') \cup (C' \cap C)\big)$$

(Distributive law)

$$= A \cap (\emptyset \cup (B \cap C) \cup (C' \cap B') \cup \emptyset)$$

$$= A \cap \big((B \cap C) \cup (C' \cap B')\big)$$

$$= (A \cap B \cap C) \cup (A \cap B' \cap C')$$

Therefore,

$$A \oplus (B \oplus C) = (A' \cap B' \cap C) \cup (A' \cap B \cap C') \cup (A \cap B' \cap C') \cup (A \cap B \cap C)$$

Further, $(A \oplus B) \oplus C = C \oplus (A \oplus B)$ ($\because \oplus$ is commutative)

$= (C' \cap A' \cap B) \cup (C' \cap A \cap B') \cup (C \cap A' \cap B') \cup (C \cap A \cap B)$ (How ?)

$= (A' \cap B' \cap C) \cup (A' \cap B \cap C') \cup (A \cap B' \cap C') \cup (A \cap B \cap C)$

Thus, $A \oplus (B \oplus C) = (A \oplus B) \oplus C$

P8:

Show that the sets $\boldsymbol{A}$ and $\boldsymbol{B}$.

$$(\boldsymbol{A} \oplus \boldsymbol{B})' = \boldsymbol{A} \oplus \boldsymbol{B}'$$

Solution:

$(A \oplus B)' = \big((A - B) \cup (B - A)\big)'$ (Definition of symmetric difference)

$= \big((A \cap B') \cup (B \cap A')\big)'$

$= (A \cap B')' \cap (B \cap A')'$ (De Morgan's law)

$= (A' \cup (B')') \cap (B' \cup (A')')$

$= (A' \cup B) \cap (B' \cup A)$ (Double negation law)

$= (A' \cap B') \cup (B \cap B') \cup (A' \cap A) \cup (B \cap A)$ (Distributive law)

$= (A \cap B) \cup (A' \cap B')$

$= (A \cap (B')') \cup (A' \cap B') = A \oplus B'$

2.1 .SETS

Exercise

1. Prove that $\left[\left(A \cup (B \cap C)\right) \cap \left(A' \cup (B \cap C)\right)\right] \cap (B' \cup C') = \emptyset$

2. Show that $(A \cap B) \cup \left[B \cap \left((C \cap D) \cup (C \cap D')\right)\right] = B \cap (A \cup C)$

3. If $A \cup B = A \cup C$ and $A \cap B = A \cap C$, then prove that $B = C$.

4. Prove the following
   (a) $A \cup (B - A) = A \cup B$
   (b) $(A \cup B) \cap (A \cup C) = A \cup (B' \cup C')'$
   (c) $[C \cap (A \cup B)] \cup [(A \cup B) \cap C'] = A \cup B$
   (d) $(A \cup C) \cap [(A \cap B) \cup (C' \cap B)] = A \cap B$

5. Prove the following identities:
   (a) $A - (B - C) = (A - B) \cup (A \cap C)$
   (b) $A \cup (B - C) = (A \cup B) - (C - A)$
   (c) $A \cap (B - C) = (A \cap B) - (A \cap C)$
   (d) $(A - B) - C = A - (B \cup C) = (A - C) - B$

6. Write the following sets as a disjoint union.
   (a) $A \cup B$
   (b) $A \oplus B$
   (c) $A \cup B \cup C$.