3.3

Characteristic Function

In some cases m.g.f. does not exist. For example, consider the p.m.f. given by

$$p(x) = \begin{cases} \frac{6}{\pi^2 x^2}, & x = 1, 2, 3, \dots \\ 0, & otherwise \end{cases}$$

Its m.g.f. is given by

$$M_X(t) = \sum_{x=1}^{\infty} e^{tx} p(x) = \frac{6}{\pi^2} \sum_{x=1}^{\infty} \frac{e^{tx}}{X^2},$$

which is divergent. Thus, $M_X(t)$ does not exist. A more serviceable function that the m.g.f. is the **characteristic function.**

Characteristic function: The characteristic function of a.r.v. X is defined by

$$\phi_X(t) = E\left[e^{itX}\right] = \begin{cases} \int e^{itx} f(x) dx & if\ X\ is\ a\ c.r.v. with\ p.d.f\ f(x) \\ \sum_x e^{itx}\ p(x) & if\ X\ is\ a\ d.r.v.\ with\ p.m.f\ p(x) \end{cases}$$

where $i = \sqrt{-1}$, the imaginary number.

Note:

1. $|\emptyset_X(t)| = \left|E\left(e^{itX}\right)\right| \leq E\left(\left|e^{itX}\right|\right) = E\left(\sqrt{cos^2 tX + sin^2 tX}\right) = E(1) = 1$

   Since $|\emptyset_X(t)| \leq 1$, $\emptyset_X(t)$ always exists for any **probability distribution**.

2. $\emptyset_X(t) = E\left[e^{itX}\right] = E\left[1 + (it)X + \frac{(it)^2}{2!}X^2 + \frac{(it)^3}{3!}X^3 + \cdots\right]$

$$= 1 + (it)E(X) + \frac{(it)^2}{2!}E(X^2) + \frac{(it)^3}{3!}E(X^3) + \cdots$$

$$= 1 + (it)\mu_1' + \frac{(it)^2}{2!}\mu_2' + \frac{(it)^3}{3!}\mu_3' + \cdots$$

where $\mu_r' = E(X^r) = r^{th}$ moment abut origin for $r = 1, 2, \ldots$

3. If $\emptyset_X(t)$ is given, then the r^{th} moment about origin is given b

$$\boldsymbol{\mu_r' = coefficient\ of\ \frac{(it)^r}{r!}\ \text{in}\ \emptyset_X(t).}$$

Properties:

1. $\boldsymbol{\emptyset_X(0) = 1}$

Proof: $\emptyset_X(t) = E\left[e^{itX}\right] = E(1)$ when $t = 0$

$$= 1$$

Thus, $\emptyset_X(0) = 1$

2. $\boldsymbol{|\emptyset_X(t)| \leq 1}$ **for all real $\boldsymbol{t}$.**

Proof: $|\emptyset_X(t)| = \left|E\left(e^{itX}\right)\right| \leq E\left(\left|e^{itX}\right|\right) = E\left(\sqrt{cos^2tX + sin^2tX}\right) = E(1) = 1$

$\Longrightarrow |\emptyset_X(t)| \leq 1$ for all real t

3. $\boldsymbol{\emptyset_X(t)}$ **continuous function of $\boldsymbol{t}$ in** $(-\infty, \infty)$.

Proof: For $h \neq 0$,

$|\emptyset_X(t+h) - \emptyset_X(t)| = \left|E\left(e^{i(t+h)X}\right) - E\left(e^{itX}\right)\right| = \left|E\left(e^{i(t+h)X} - e^{itX}\right)\right|$

$= \left|E\left\{e^{itX}\left(e^{ihX} - 1\right)\right\}\right| \leq E\left(\left|e^{itX}\right|\left|e^{ihX} - 1\right|\right) = E\left(\left|e^{ihX} - 1\right|\right) \longrightarrow 0$ as $h \longrightarrow 0$

Thus $\lim\limits_{h \to 0} |\emptyset_X(t+h) - \emptyset_X(t)| = 0$

$\Longrightarrow \lim\limits_{h \to 0} \emptyset_X(t+h) = \emptyset_X(t)$

$\Longrightarrow \emptyset_X(t)$ is a continuous function of t in $(-\infty, \infty)$.

4. $\emptyset_X(-t) = \overline{\emptyset_X(t)}$ **, $i.e$, $\emptyset_X(-t)$ is the complex conjugate of $\emptyset_X(t)$.**

Proof: Here $\overline{\emptyset_X(t)} = \overline{E[e^{itX}]} = E[\cos tX - i\sin tX]$

$\Longrightarrow$ $\emptyset_X(-t) = E[\cos(-tX) + i\sin(-tX)] = E[\cos tX - i\sin tX] = \overline{\emptyset_X(t)}$

Thus, $\emptyset_X(-t) = \overline{\emptyset_X(t)}$

5. If the p.d.f. is even $i.e.$, $f(-x) = f(x)$, then the characteristic function is real valued and even function of t.

Proof: We know that , $\phi_X(t) = E\left(e^{itX}\right) = \int_{-\infty}^{\infty} e^{itX} f(x)dx$

Let $x = -y \Longrightarrow dx = -dy$. Then

$$\phi_X(t) = \int_{\infty}^{-\infty} e^{-ity} f(-y)(-dy) = \int_{-\infty}^{\infty} e^{-ity} f(y)dy \qquad (\because\ f\ is\ an\ even\ function)$$

$$= E\left[e^{-itX}\right] = \emptyset_X(-t)$$

Thus, $\emptyset_X(t) = \emptyset_X(-t)$

$\Longrightarrow$ $\emptyset_X(t)$ is an even function of t.

Further, $\overline{\emptyset_X(t)} = \emptyset_X(-t)$ (by property 4)

$= \emptyset_X(t)$ (Since $\emptyset_X(t)$ is even function)

Thus, $\emptyset_X(t)$ is real.

6. If X is a r.v. with characteristic function $\emptyset_X(t)$ and $\mu'_r = E(X^r)$ exists, then

$$\boldsymbol{\mu'_r = (-i)^r \frac{d^r}{dt^r}\left(\emptyset_X(t)\right)\Big|_{t=0}}$$

Proof:

$$\frac{d^r}{dt^r}\left(\emptyset_X(t)\right) = \frac{d^r}{dt^r}\left(E\left(e^{itX}\right)\right) = i^r E\left[X^r e^{itX}\right] = i^r E(X^r)$$

Now, $\frac{d^r}{dt^r}\left(\emptyset_X(t)\right)\Big|_{t=0} = i^r E(X^r)$ and $\mu'_r = E(X^r) = \frac{1}{i^r}\frac{d^r}{dt^r}\left(\emptyset_X(t)\right)\Big|_{t=0}$.

Thus, $\mu'_r = (-i)^r \frac{d^r}{dt^r}\left(\emptyset_X(t)\right)\Big|_{t=0}$

7. Effect of change of origin and scale .

Let $U = \frac{X-a}{h}$ where a and h are constants.

Then $\emptyset_U(t) = E\left[e^{itU}\right] = E\left[e^{it\left(\frac{X-a}{h}\right)}\right] = e^{-\frac{ita}{h}} E\left[e^{i\left(\frac{t}{h}\right)X}\right]$

$$\Longrightarrow \emptyset_U(t) = e^{-\frac{ita}{h}} \emptyset_X\left(\frac{t}{h}\right)$$

8. If $X_1, X_2, \dots, X_n$ are independent, then

$$\boldsymbol{\emptyset_{X_1+\dots+X_n}(t) = \emptyset_{X_1}(t)\,\emptyset_{X_2}(t)\dots\emptyset_{X_n}(t)}$$

Proof:

$$\emptyset_{X_1+\dots+X_n}(t) = E\left[e^{it(X_1+X_2+,\dots+X_n)}\right] = E\left[e^{itX_1}.e^{itX_2}\dots e^{itX_n}\right]$$

$$(\because X_1, X_2, \dots, X_n \text{ are independent})$$

$$= E\left[e^{itX_1}\right].E\left[e^{itX_2}\right]\dots E\left[e^{itX_n}\right]$$

$$= \emptyset_{X_1}(t).\emptyset_{X_2}(t)\dots\emptyset_{X_n}(t)$$

$$\Longrightarrow \emptyset_{X_1+\dots+X_n}(t) = \emptyset_{X_1}(t).\emptyset_{X_2}(t)\dots\emptyset_{X_n}(t)$$

Note: Converse need not be true.

Uniqueness Theorem for Characteristic Functions:

The characteristic function uniquely determines the distribution. That is,
A necessary and sufficient condition for two distributions with p.d.fs $f_1(.)$ and $f_2(.)$ to be identical is that their characteristic function $\emptyset_1(t)$ and $\emptyset_2(t)$ are identical.

Example 1: If $X \sim B(n, p)$, find its characteristic function and hence obtain its mean and variance.

Solution: Since $X \sim B(n, p)$, its p.m.f. is given by

$$p(x) = \binom{n}{x} p^x q^{n-x} \text{ for } x = 0, 1, 2, \dots, n$$

The characteristic function of X is given by

$$\phi_X(t) = E\left[e^{itX}\right] = \sum_{x=0}^{n} e^{itx} p(x) = \sum_{x=0}^{n} e^{itx} \binom{n}{x} p^x q^{n-x} = \sum_{x=0}^{n} \binom{n}{x} \left(pe^{it}\right)^x q^{n-x}$$

$\Rightarrow \emptyset_X(t) = \left(q + pe^{it}\right)^n$ and $\frac{d}{dt}\left(\emptyset_X(t)\right) = npi\left(q + pe^{it}\right)^{n-1} e^{it}$

The mean of X is given by

$$\mu = E(X) = \mu' = (-i)\frac{d}{dt}\left(\emptyset_X(t)\right)\Big|_{t=0} = (-i)\left[npi\left(q + pe^{it}\right)^{n-1} e^{it}\right]_{t=0}$$

$$= (-i)\, npi = np$$

Now, $\frac{d^2}{dt^2}\left(\emptyset_X(t)\right) = (npi)\frac{d}{dt}\left[\left(q + pe^{it}\right)^{n-1} e^{it}\right]$

$$= (npi)\left[(n-1)\left(q + pe^{it}\right)^{n-2} pie^{2it} + \left(q + pe^{it}\right)^{n-1} ie^{it}\right]$$

Thus, $\mu_2' = (-i)^2 \frac{d^2}{dt^2}\left(\emptyset_X(t)\right)\Big|_{t=0} = (-i)^2 (npi)[(n-1)pi + i]$

$$= (np)[np - p + 1] = np(np + q) = n^2p^2 + npq$$

$\Rightarrow \mu_2' = n^2p^2 + npq$

Therefore, the variance of X is given by

$$\sigma^2 = V(X) = \mu_2' - (\mu_1')^2 = n^2p^2 + npq - n^2p^2$$
$$\Longrightarrow \ \sigma^2 = npq.$$

Example 2: If $X \sim P(\lambda)$, find the characteristic function X and hence obtain its mean and variance.

Solution: Since $X \sim P(\lambda)$, its p.m.f. is given by

$$p(x) = \frac{e^{-\lambda}\lambda^x}{x!}\ , x = 0, 1, ,2 \ldots$$

The characteristic function of X is given by

$$\phi_X(t) = E\left[e^{itX}\right] = \sum_{x=0}^{\infty} e^{itx} p(x) = \sum_{x=0}^{\infty} e^{itx} \frac{e^{-\lambda}\lambda^x}{x!}$$

$$= e^{-\lambda} \sum_{x=0}^{\infty} \frac{\left(\lambda e^{it}\right)^x}{x!} \ = e^{-\lambda} e^{\lambda e^{it}} \ = e^{\lambda\left(e^{it}-1\right)}$$

$$\boldsymbol{\Rightarrow j_X(t) = e^{\lambda\left(e^{it}-1\right)}}$$

Now, $\frac{d}{dt}\left(\emptyset_X(t)\right) = e^{\lambda(e^{it}-1)}\lambda i e^{it}$

Thus, the mean is given by

$$\mu = \mu_1' = E(X) = (-i)\frac{d}{dt}\left(\phi_X(t)\right)\Big|_{t=0} = (-i)(\lambda i) = \lambda$$

$$\Longrightarrow \ \mu = \lambda$$

Now, $\frac{d^2}{dt^2}\left(\emptyset_X(t)\right) = (\lambda i)\frac{d}{dt}\left[e^{\lambda(e^{it}-1)}e^{it}\right]$

$$= (\lambda i)\left[e^{\lambda(e^{it}-1)}\lambda i e^{2it} + e^{\lambda(e^{it}-1)} i e^{it}\right]$$

Thus, μ_2' is given by

$$\mu_2' = (-i)^2 \frac{d^2}{dt^2}\left(\emptyset_X(t)\right)\Big|_{t=0} = (-i)^2(\lambda i)(\lambda i + i) = (-i)^2(i^2)\lambda(\lambda+1)$$

$$= \lambda(\lambda+1) = \lambda^2 + \lambda$$

Hence, the variance is given by $\sigma^2 = \mu_2' - (\mu_1')^2 = \lambda^2 + \lambda - \lambda^2 = \lambda \Longrightarrow \ \sigma^2 = \lambda$

Example 3: If $X \sim N(\mu, \sigma^2)$, find the characteristic function of X and hence obtain its mean and variance.

Solution: Since $X \sim N(\mu, \sigma^2)$, its p.d.f. is given by

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} exp\left[-\frac{1}{2\sigma^2}(x-\mu)^2\right], -\infty < x, \mu < \infty, \sigma > 0$$

The characteristic function of X is given by

$$\phi_X(t) = E\left[e^{itX}\right] = \int_{-\infty}^{\infty} e^{itx} f(x)dx = \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{itx} exp\left[-\frac{1}{2\sigma^2}(x-\mu)^2\right]dx$$

Let $\frac{x-\mu}{\sigma} = z \Longrightarrow x = \mu + \sigma z \Longrightarrow dx = \sigma\, dz$

$$= \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{itx} exp\left[-\frac{1}{2\sigma^2}(x-\mu)^2\right]dx = \int_{-\infty}^{\infty} e^{it(\mu+\sigma z)} e^{-\frac{1}{2}z^2} dz$$

$$= \frac{e^{it\mu}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} exp\left[-\frac{1}{2}\left(z^2 - 2i\sigma zt\right)\right]dz$$

$$= \frac{e^{it\mu}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} exp\left[-\frac{1}{2}\left(z^2 - 2i\sigma zt + i^2\sigma^2t^2 - i^2\sigma^2t^2\right)\right]dz$$

$$= \frac{e^{it\mu - \frac{\sigma^2 t^2}{2}}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} exp\left[-\frac{1}{2}(z - i\sigma t)^2\right]dz$$

Let $z - i\sigma t = u \Longrightarrow dz = du$

$$= e^{it\mu - \frac{\sigma^2 t^2}{2}} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{u^2}{2}} du$$

$$= e^{it\mu - \frac{\sigma^2 t^2}{2}} \qquad \left(\because \quad \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{u^2}{2}} du = 1 \right)$$

$\Longrightarrow \; \emptyset_X(t) = e^{itu - \frac{\sigma^2 t^2}{2}}$

Now, $\frac{d}{dt}\left(\emptyset_X(t)\right) = e^{it\mu - \frac{\sigma^2 t^2}{2}}(i\mu - \sigma^2 t)$

Then $\mu_1' = (-i)\frac{d}{dt}\left(\emptyset_X(t)\right)\Big|_{t=0} = (-i)\,(i\mu) = \mu$

Thus, Mean $= E(X) = \mu$.

Now, $\frac{d^2}{dt^2}\left(\emptyset_X(t)\right) = e^{it\mu - \frac{1}{2}\sigma^2 t^2}(i\mu - \sigma^2 t)^2 + e^{it\mu - \frac{1}{2}t^2\sigma^2}(-\sigma^2)$

Thus, $\mu_2' = (-i)^2 \; \frac{d^2}{dt^2}\left(\emptyset_X(t)\right)\Big|_{t=0} = (-i)^2\,[i^2\mu^2 - \sigma^2] = (-1)(-\mu^2 - \sigma^2)$

$\Longrightarrow \; \mu_2' = \mu^2 + \sigma^2$

Hence, the variance is given by

$V(X) = \mu_2' - (\mu_1')^2 = \mu^2 + \sigma^2 - \mu^2 = \sigma^2$

$\Longrightarrow$ Variance $= V(X) = \sigma^2$

Finding p.m.f. (or p.d.f.) when characteristic function is known.

If X is a d.r.v. with characteristic function $\emptyset_X(t)$, then $\emptyset_X(t) = \sum P(X = j)e^{itj}$.

First write the characteristic function in this form and then identify the $P(X = j)$ which is the p.m.f. of the d.r.v. X.

Example 4: Find the p.m.f. of the d.r.v. X whose characteristic function is given by $\boldsymbol{\emptyset_X(t) = \left(q + pe^{it}\right)^n}$.

Solution: We have, $\emptyset_X(t) = \left(q + pe^{it}\right)^n$ and

$$\emptyset_X(t) = \left(q + pe^{it}\right)^n = \sum_{j=0}^{n} \binom{n}{j} \left(pe^{it}\right)^j q^{n-j}$$

$$= \sum_{j=0}^{n} \binom{n}{j} p^j q^{n-j} e^{itj} = \sum_{j=0}^{n} P(X=j) e^{itj}$$

$$= E\left[e^{itX}\right] \text{ where } P(X=j) = \binom{n}{j} p^j q^{n-j}$$

Thus p.m.f. is $p(j) = \binom{n}{j} p^j q^{n-j}$ for $j = 0,1,2, \dots, n$.

Example 5: Find the p.m.f. of a d.r.v. X whose characteristic function is given by

$$\boldsymbol{\emptyset_X(t) = e^{\lambda(e^{it}-1)}}.$$

Solution: We have, $\emptyset_X(t) = e^{\lambda(e^{it}-1)}$

$$\emptyset_X(t) = e^{\lambda(e^{it}-1)} = e^{-\lambda} e^{\lambda e^{it}} = e^{-\lambda} \sum_{x=0}^{\infty} \frac{\left(\lambda e^{it}\right)^x}{x!} = \sum_{x=0}^{\infty} \frac{e^{-\lambda}\lambda^x}{x!} e^{itx}$$

$$= \sum_{x=0}^{\infty} P(X=x) e^{itx} = E\left[e^{itX}\right]$$

where $P(X=x) = \frac{e^{-\lambda}\lambda^x}{x!}$ for $x = 0,1,2, \dots$, which is Poisson distribution with parameter λ.

Theorem 1: If X is a continuous random variable with characteristic function $\emptyset_X(t)$, then its p.d.f. is given by

$$\boldsymbol{f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} \emptyset_X(t) dt}$$

Example 6: Find the p.d.f corresponding to the characteristic function

$$\boldsymbol{\emptyset_X(t) = e^{it\mu - \frac{1}{2}t^2\sigma^2}.}$$

Solution:

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\,itx} e^{it\mu - \frac{1}{2}t^2\sigma^2} dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\frac{1}{2}[t^2\sigma^2 - 2it(x-\mu)]} dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} exp\left[-\frac{1}{2}\left\{t\sigma - i\left(\frac{x-\mu}{\sigma}\right)\right\}^2 + \left(\frac{x-\mu}{\sigma}\right)^2\right] dt$$

$$= \frac{1}{2\pi} exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right]\int_{-\infty}^{\infty} exp\left[-\frac{1}{2}\left\{t\sigma - i\left(\frac{x-\mu}{\sigma}\right)\right\}^2\right] dt$$

Let $t\sigma - i\left(\frac{x-\mu}{\sigma}\right) = u \Rightarrow dt = \frac{du}{\sigma}$.

$$= \frac{1}{\sigma\sqrt{2\pi}} exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right]\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} exp\left(-\frac{u^2}{2}\right) dt$$

$$= \frac{1}{\sigma\sqrt{2\pi}} exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right]$$

Therefore, $f(x) = \frac{1}{\sigma\sqrt{2\pi}} exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right], -\infty < x < \infty$

$\Longrightarrow X \sim N(\mu, \sigma^2)$

Example 7: Find the p.d.f. corresponding to the characteristic function defined by

$$\emptyset(\boldsymbol{t}) = \begin{cases} \boldsymbol{1 - |t|} & , \quad \boldsymbol{|t| \le 1} \\ \boldsymbol{0} & , \quad |\boldsymbol{t}| > 1 \end{cases}$$

Solution: The p.d.f. of $f(x)$ is given by

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx}\emptyset(t)dt = \frac{1}{2\pi}\int_{-1}^{1} e^{-itx}\emptyset(t)dt$$

$$= \frac{1}{2\pi}\int_{-1}^{0} e^{-itx}(1+t)dt + \frac{1}{2\pi}\int_{0}^{1} e^{-itx}(1-t)dt$$

($\because$ for $-1 < t < 0, |t| = -t$ and for $0 < t < 1, |t| = t$)

Now,

$$\int_{-1}^{0} e^{-itx}(1+t)dt = \left[\frac{e^{-itx}}{-ix}(1+t)\right]_{-1}^{0} + \frac{1}{ix}\int_{-1}^{0} e^{-itx}dt$$

$$= -\frac{1}{ix} + \frac{1}{ix}\left[\frac{e^{-itx}}{-ix}\right]_{-1}^{0}$$

$$= -\frac{1}{ix} + \frac{1}{(ix)^2}\left(e^{ix} - 1\right)$$

Similarly,

$$\int_{0}^{1} e^{-itx}(1-t)dt = \frac{1}{ix} + \frac{1}{(ix)^2}\left(e^{-ix} - 1\right)$$

$$\therefore f(x) = \frac{1}{2\pi}\left[\frac{1}{(ix)^2}\left\{e^{ix} - 1 + e^{-ix} - 1\right\}\right] = \frac{1}{\pi x^2}\left(1 - \frac{e^{ix} + e^{-ix}}{2}\right)$$

$$\Longrightarrow f(x) = \frac{1}{\pi x^2}(1 - \cos x), -\infty < x < \infty$$