Vector Spaces

* Let $(u_1, u_2, u_3, \cdots u_n)$ be a sequence of n Real numbers. the set of all such sequences is called n-space and it is denoted by $\mathbb{R}^n$.

where u_1 is the first component of $(u_1, u_2, \cdots u_n)$
u_2 " Second " " $(u_1, u_2, \cdots u_n)$.
$\vdots$

Eg ①. $(1,2,3,4)$, $(-1, 3/4, 0, 5)$ are elements of $\mathbb{R}^4$. where $\mathbb{R}^4$ is the set of (or) collection of "all sets of four ordered real numbers.

2) $(-1, 2, 0, 7/8, 9)$ is the element of $\mathbb{R}^5$.

* If two elements of $\mathbb{R}^n$ are equal if their corresponding components are equal.

i.e Let $u = (u_1, u_2, u_3, \cdots u_n)$ and $v = (v_1, v_2, v_3, \cdots v_n)$ be two elements of $\mathbb{R}^n$. then we say that u and v are equal if $u_1 = v_1$, $u_2 = v_2$, $u_3 = v_3$, $u_4 = v_4$ $\cdots$ $u_n = v_n$.

* Let $u = (u_1, u_2, \cdots u_n)$ and $v = (v_1, v_2, v_3, \cdots v_n)$ be the elements of $\mathbb{R}^n$ and let 'c' be the Scalar. then the Addition and Scalar multiplication are performed as

$$u + v = (u_1 + v_1, u_2 + v_2, \cdots, u_n + v_n)$$
$$cu = (cu_1, cu_2, cu_3, \cdots cu_n).$$

i.e to add two elements of $\mathbb{R}^n$, we add corresponding components.
to multiply an element of $\mathbb{R}^n$, we multiply every component by that scalar.

Binary operation:- Let 's' be a non empty set then the mapping $f: S\times S \to S$ is called as binary operation on S.

ie $a, b \in S \Rightarrow a * b \in S$ then $*$ is called as binary operation on S.

Eg:- $(N,+)$, $(N,\cdot)$, $(Q,+)$, $(Q^*,\cdot)$, ...

Field:- Let 'F' be a set having at least two elements 0 & 1 $(0 \neq 1)$ together with two operations '+' (addition) and '$\cdot$' (multiplication) then $(F,+,\cdot)$ is said to be a field if it satisfied the following properties.

for any three element $a, b, c \in F$.

1) Addition and Multiplication are Closed
ie $a+b \in F$ and $ab \in F$.

2) Addition & Multiplication are Associative.
ie $(a+b)+c = a+(b+c)$ & $(ab)c = a(bc)$.

3). Addition & Multiplication are Commutative.
ie $a+b = b+a$ & $ab = ba$.

4) The Multiplicative operation distributive over addition.
ie $a(b+c) = ab+ac$.

5). Additive Identity:-
ie $0+a = a+0 = a$.
Where '0' is the Additive Identity

6) Multiplicative Identity:-
i.e $1.a = a.1 = a$.
Where '1' is the Multiplicative Identity

7) Every element has additive Inverse.
$\exists\ -a \in F \Rightarrow a+(-a) = (-a)+a = 0$.

8) Every non zero element has Multiplicative Inverse.
i.e If $a \neq 0 \in F$, then $\exists\ a^{-1} \in F \Rightarrow a.a^{-1} = a^{-1}.a = 1$.

Eg:- $(Q,+,\cdot)$, $(R,+,\cdot)$ & $(C,+,\cdot)$ are all fields.

and $(Z,+,\cdot)$ is not a field, because every non zero element except -1 & 1 has no multiplicative Inverse.

Vector Space :- Let 'V' be a non empty set whose elements are named as vectors and $(F,+,\cdot)$ be a field whose elements are named as scalars then the set 'V' is said to be a vector space over a field F, if it satisfies the following properties.

For a, b and $c \in V$ and $\alpha, \beta \in F$.

I 1) Vector addition is closed.
i.e $a, b \in V \Rightarrow a+b \in V$.

2) Vector Addition is Commutative
i.e $a, b, c \in V \Rightarrow (a+b) = (b+a)$.

3). Vector Addition is Associative
i.e $(a+b)+c = a+(b+c)$.

4) Existence of an additive Identity
$\exists\ 0 \in V \ni a+0 = 0+a = a$.

5) Existence of additive Inverse
$\exists\ -\bar{a} \in V \ni a+(-a) = (-a)+a = 0$.

II 1) Scalar multiplication is closed on V over field F.
i.e $\bar{a} \in V, \alpha \in F \Rightarrow \bar{a}\alpha \in V$.

2) $\alpha(\bar{a}+b) = \bar{a}\alpha + \bar{b}\alpha, \forall \alpha \in F, a, b \in V$.

3). $(\alpha+\beta)\bar{a} = \bar{a}\alpha + \bar{a}\beta, \forall \alpha, \beta \in F, a \in V$.

4). $(\alpha\beta)\bar{a} = \alpha(\beta\bar{a}), \forall \alpha, \beta \in F, a \in V$.

5). $1 \cdot \bar{a} = \bar{a} \cdot 1 = \bar{a} \quad \forall a \in V$.

* The Elements of the field F are Called Scalars.

* The Elements of the Vector Space V' are Called Vectors.

VECTOR SPACE – I

In this chapter we shall study an important algebraic system known as **vector space.** First we define two compositions internal composition and external composition.

INTERNAL COMPOSITION:- Let V be a set. If $a \circ b \in$ V and $a \circ b$ is unique for all $a, b \in$ V, then '$\circ$' is said to be an **internal composition** in the set V. *i.e* a binary operation on V is called an **internal composition** in V.

EXTERNAL COMPOSITION:- Let V and F be two sets. If $a \circ \alpha \in$ V and $a \circ \alpha$ is unique for all $a \in$ F and $\alpha \in$ V, then '$\circ$' is said to be an external composition in V over F.

VECTOR SPACE:- Let V be a non-empty set whose elements are called vectors. Let (F, +, ·) be a field whose elements are called scalars. Then V is said to be a vector space over the field F, if

(1) There is defined an internal composition in V, called addition of vectors denoted by '+' for which (V, +) is an abelian group.

i.e (*i*) '+' is closed:- $\forall \alpha, \beta \in V, \alpha + \beta \in V$.

(*ii*) '+' is associative:- $\forall \alpha, \beta, \gamma \in V, (\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$

(*iii*) Existence of additive identity:- For each $\alpha \in V$, $\exists\ \overline{0} \in V \ni \alpha + \overline{0} = \overline{0} + \alpha = \alpha$. Here $\overline{0}$ is called zero vector in V.

(*iv*) Existence of additive inverse:- For $\alpha \in V\ \exists -\alpha \in V \ni \alpha + (-\alpha) = (-\alpha) + \alpha = \overline{0}$. Here $-\alpha$ is the additive inverse of α. *i.e* every element in V has additive inverse in V.

(v) Commutative law:- $\forall \alpha, \beta \in V, \alpha + \beta = \beta + \alpha$

$$\boxed{\therefore (V, +) \text{ is an abelian group}}$$

(2) There is defined an external composition in V over the field F, called scalar multiplication denoted by '·' *i.e* $a\alpha \in V\ \forall a \in F, \forall \alpha \in V$.

(3) The above two compositions satisfies the following properties.

(*i*) $a(\alpha + \beta) = a\alpha + a\beta \quad \forall a \in F, \forall \alpha, \beta \in V$

(*ii*) $(a + b)\alpha = a\alpha + b\alpha \quad \forall a, b \in F, \forall \alpha \in V$

(iii) $(ab)\alpha = a(b\alpha)$ $\forall\ a, b \in F, \forall\ \alpha \in V$

(v) $1 \cdot \alpha = \alpha\ \forall\ \alpha \in V$ where 1 is the unity element in F.[ANU S96, S97, M99, M2000, A01, A02, S02, M06, M07,

Note:- 1.If V is a vector space over the field F, then we simply say that V(F) is a vector space. Sometimes if the field F is known, then we simply say that V is a vector space.

2. If F = R, then V(R) is called a ' Real vector space'.

3. If F = C, then V(C) is called a 'Complex vector space'.

4. If V=$\{\overline{0}\}$ where $\overline{0}$ is the zero vector in V,then V is a vector space over the field F.

5. In a vector space two types of zero elements come into operation. One is the zero vector ($\overline{0}$) in V and the other is the zero scalar (0) in the field F.

NULL SPACE OR ZERO VECTOR SPACE:- The vector space having only one vector $\overline{0}$ is called a **Null space** or **Zero vector space.**

THE n-DIMENSIONAL VECTORS OR ORDERED n-TUPLE:- Let F be a field. If $a_1, a_2, \ldots\ldots\ldots, a_n \in$ F, then the point $(a_1, a_2, \ldots\ldots\ldots, a_n)$ is called the n-dimensional vector or an ordered n - tuple. The set of n-tuples denoted by F^n or V_n.

$\therefore F^n = V_n = \{(a_1, a_2, \ldots\ldots\ldots, a_n) / a_1, a_2, \ldots\ldots\ldots, a_n \in F\}$

EQUALITY OF VECTORS OR EQUALITY OF n-TUPLES:- Let $\alpha = (a_1, a_2, \ldots\ldots, a_n)$, $\beta = (b_1, b_2, \ldots\ldots\ldots, b_n)$ be two vectors in F^n. Then α and β are said to be equal if $a_1 = b_1, a_2 = b_2, \ldots\ldots, a_n = b_n$. It is denoted by $\alpha = \beta$.

ADDITION OF VECTORS OR ADDITION OF n-TUPLES:- Let $\alpha = (a_1, a_2, \ldots\ldots, a_n)$, $\beta = (b_1, b_2, \ldots\ldots\ldots, b_n)$ be two vectors in F^n . Then the sum(addition) of α and β is defined as $\alpha + \beta = (a_1+b_1, a_2+b_2, \ldots\ldots, a_n+b_n)$.

SCALAR MULTIPLICATION OF A VECTOR (n-TUPLE) OR MULTIPLICATION OF A VECTOR(n-TUPLE) BY A SCALAR:- If $\alpha = (a_1, a_2, \ldots\ldots\ldots, a_n) \in F^n$ and $c \in$ F, then the multiplication of α by c is defined as $c\alpha = (ca_1, ca_2, \ldots\ldots\ldots, ca_n)$.

GENERAL PROPERTIES OF A VECTOR SPACE:

Theorem: Let V(F) be a vector space and 0, $\overline{0}$ be the zero scalar and zero vector respectively. Then

(i) $a\overline{0} = \overline{0}$ $\forall\ a \in F$ [ANU A91, S98

(ii) $0\alpha = \overline{0}$ $\forall\ \alpha \in V$ [ANU A91

(iii) $a(-\alpha) = -(a\alpha) \quad \forall\, a \in \mathbf{F}, \forall\, \alpha \in \mathbf{V}$

(iv) $(-a)\alpha = -(a\alpha) \quad \forall\, a \in \mathbf{F}, \forall\, \alpha \in \mathbf{V}$

(v) $(-a)(-\alpha) = a\alpha \quad \forall\, a \in \mathbf{F}, \forall\, \alpha \in \mathbf{V}$

(vi) $a(\alpha - \beta) = a\alpha - a\beta \quad \forall\, a \in \mathbf{F}, \forall\, \alpha, \beta \in \mathbf{V}$

(vii) $(a - b)\alpha = a\alpha - b\alpha \quad \forall\, a, b \in \mathbf{F}, \forall\, \alpha \in \mathbf{V}$

(viii) $a \in \mathbf{F}, \alpha \in \mathbf{V}$ **and** $a\alpha = \bar{0} \Rightarrow a = 0$ **or** $\alpha = \bar{0}$ **[ANU S96, M96**

Proof: Let V(F) be a vector space and $\bar{0}$ be the zero vector in V, 0 be the zero scalar in F.

(i) <u>To prove that $a\bar{0} = \bar{0} \ \forall\, a \in F$</u>:

Since $\bar{0} + \bar{0} = \bar{0} \Rightarrow a(\bar{0} + \bar{0}) = a\bar{0} \quad \forall\, a \in F$

$\Rightarrow a\bar{0} + a\bar{0} = a\bar{0} \quad \forall\, a \in F$ [$\because$ V is a vector space]

$\Rightarrow a\bar{0} + a\bar{0} = a\bar{0} + \bar{0} \quad \forall\, a \in F$ [$\because$ $\bar{0}$ is the zero vector in V]

$\Rightarrow a\bar{0} = \bar{0} \quad \forall\, a \in F$ [$\because$ (V, +) is an abelian group and in a group cancelation laws hold]

$$\boxed{\therefore\, a\bar{0} = \bar{0} \ \forall\, a \in F}$$

(ii) <u>To prove that $0\alpha = \bar{0} \ \forall\, \alpha \in V$</u>:

Since $0 + 0 = 0 \Rightarrow (0 + 0)\alpha = 0\alpha \quad \forall\, \alpha \in V$

$\Rightarrow 0\alpha + 0\alpha = 0\alpha \quad \forall\, \alpha \in V$ [$\because$ V is a vector space]

$\Rightarrow 0\alpha + 0\alpha = 0\alpha + \bar{0} \quad \forall\, \alpha \in V$ [$\because$ $\bar{0}$ is the zero vector in V]

$\Rightarrow 0\alpha = \bar{0} \quad \forall\, \alpha \in F$ [$\because$ (V, +) is an abelian group and in a group cancelation laws hold]

$$\boxed{\therefore\, 0\alpha = \bar{0} \ \forall\, \alpha \in V}$$

(iii) <u>To prove that $a(-\alpha) = -(a\alpha) \ \forall\, a \in \mathbf{F}, \forall\, \alpha \in \mathbf{V}$</u> :

Since $\alpha + (-\alpha) = \bar{0} \ \forall\, \alpha \in V \Rightarrow a[\alpha + (-\alpha)] = a\bar{0} \quad \forall\, a \in F, \forall\, \alpha \in V$

$\Rightarrow a\alpha + a(-\alpha) = \bar{0} \quad \forall\, a \in F, \forall\, \alpha \in V$ [$\because$ V is a vector space]

$\Rightarrow a\alpha + a(-\alpha) = a\alpha + [-(a\alpha)] \quad \forall\, a \in F, \forall\, \alpha \in V$

$\Rightarrow a(-\alpha) = -(a\alpha) \quad \forall\, a \in F, \forall\, \alpha \in V$ [by L.C.L]

$$\boxed{\therefore\, a(-\alpha) = -(a\alpha) \ \forall\, a \in F, \forall\, \alpha \in V}$$

(iv) To prove that $(-a)\alpha = -(a\alpha)\ \forall\ a \in F, \forall\ \alpha \in V$:

Since $a + (-a) = 0\ \forall\ a \in F \Rightarrow [a + (-a)]\alpha = 0\alpha\quad \forall\, a \in F, \forall\ \alpha \in V$

$$\Rightarrow a\alpha + (-a)\alpha = \bar{0}\ \forall\, a \in F, \forall\ \alpha \in V \text{ [$\because$ V is a vector space]}$$

$$\Rightarrow a\alpha + (-a)\alpha = a\alpha + [-(a\alpha)]\quad \forall\, a \in F, \forall\ \alpha \in V$$

$$\Rightarrow a(-\alpha) = -(a\alpha)\quad \forall\ a \in F, \forall\ \alpha \in V \qquad \text{[by L.C.L]}$$

$$\boxed{\therefore (-a)\alpha = -(a\alpha)\ \forall\ a \in F, \forall\ \alpha \in V}$$

(v) To prove that $(-a)(-\alpha) = a\alpha\ \forall\ a \in F, \forall\ \alpha \in V$: Let $a \in F$, $\alpha \in V$

Now $(-a)(-\alpha) = -[(-a)\alpha]$ [By (iii)]

$= -[-(a\alpha)]$ [By (iv)]

$= a\alpha$

$$\boxed{\therefore (-a)(-\alpha) = a\alpha\ \forall\ a \in F, \forall\ \alpha \in V}$$

(vi) To prove that $a(\alpha - \beta) = a\alpha - a\beta\ \forall\ a \in F, \forall\ \alpha, \beta \in V$: Let $a \in F$, $\alpha, \beta \in V$

Now $a(\alpha - \beta) = a[\alpha + (-\beta)] = a\alpha + a(-\beta)$ [$\because$ V is a vector space]

$= a\alpha + [-(a\beta)]$ [By (iii)]

$= a\alpha - a\beta$

$$\boxed{\therefore a(\alpha - \beta) = a\alpha - a\beta\ \forall\ a \in F, \forall\ \alpha, \beta \in V}$$

(vii) To prove that $(a - b)\alpha = a\alpha - b\alpha\ \forall\ a, b \in F, \forall\ \alpha \in V$: Let $a, b \in F$, $\alpha \in V$

Now $(a - b)\alpha = [a + (-b)]\alpha = a\alpha + [(-b)\alpha]$ [$\because$ V is a vector space]

$= a\alpha + [-(b\alpha)]$ [By (iv)]

$= a\alpha - b\alpha$

$$\boxed{\therefore (a - b)\alpha = a\alpha - b\alpha\ \forall\ a, b \in F, \forall\ \alpha \in V}$$

(viii) To prove that $a\alpha = \bar{0} \Rightarrow a = 0$ or $\alpha = \bar{0}\ \ \forall\ a \in F, \forall\ \alpha \in V$:

Let $a \in F$ and $\alpha \in V$ such that $a\alpha = \bar{0}$

If $a = 0$, then there is nothing to prove.

Suppose $a \neq 0$

$\because 0 \neq a \in$ F, F is field $\Longrightarrow \exists\, a^{-1} \in F \ni aa^{-1} = a^{-1}a = 1$ where 1 is the unity element in F.

$\because a\alpha = \bar{0} \Rightarrow a^{-1}(a\alpha) = a^{-1}\bar{0} \Rightarrow (a^{-1}a)\alpha = \bar{0}$ [$\because$ V is a vector space]

$\Rightarrow 1.\alpha = \bar{0} \Rightarrow \alpha = \bar{0}$ [$\because$ V is a vector space]

$$\boxed{\therefore a\alpha = \bar{0} \Rightarrow a = 0 \text{ or } \alpha = \bar{0}}$$

Theorem: Let V(F) be a vector space.

(*i*) If $a, b \in$ F and $\alpha \in$ V, $\alpha \neq \bar{0}$, then $a\alpha = b\alpha \Rightarrow a = b$

(*ii*) If $a \in$ F, $a \neq 0$ and $\alpha, \beta \in$ V, then $a\alpha = a\beta \Rightarrow \alpha = \beta$

Proof: Let V(F) be a vector space.

(*i*) Let $a, b \in$ F and $\bar{0} \neq \alpha \in$ V such that $a\alpha = b\alpha$

Claim: To prove that $a = b$

$\because a\alpha = b\alpha \Rightarrow a\alpha - b\alpha = \bar{0} \Rightarrow (a - b)\alpha = \bar{0}$

$\Rightarrow a - b = 0$ or $\alpha = \bar{0}$

$\Rightarrow a - b = 0$ [$\because \alpha \neq \bar{0}$]

$\Rightarrow a = b$

$$\boxed{\therefore a, b \in \text{F and } \bar{0} \neq \alpha \in \text{V and } a\alpha = b\alpha \Rightarrow a = b}$$

(*ii*) Let $0 \neq a \in$ F and $\alpha, \beta \in$ V such that $a\alpha = a\beta$

Claim: To prove that $\alpha = \beta$

$\because a\alpha = a\beta \Rightarrow a\alpha - a\beta = \bar{0} \Rightarrow a(\alpha - \beta) = \bar{0}$

$\Rightarrow a = 0$ or $\alpha - \beta = \bar{0}$

$\Rightarrow \alpha - \beta = \bar{0}$ [$\because a \neq 0$]

$\Rightarrow \alpha = \beta$

$$\boxed{\therefore 0 \neq a \in \text{F and } \alpha, \beta \in \text{V and } a\alpha = a\beta \Rightarrow \alpha = \beta}$$

<u>PROBLEMS</u>

1. Show that the set V_n of all ordered n - tuples over a field F is a vector space w.r.t addition of n – tuples as addition of vectors and multiplication of an n – tuple by a scalar as scalar multiplication. [ANU M99,

Solution: $V_n = \{(a_1, a_2, \ldots\ldots, a_n) \,/\, a_1, a_2, \ldots\ldots, a_n \in F\}$ where F is a field.

Now we define the addition of vectors (+) in V_n and scalar multiplication (·) in V_n over F as follows.

For $\alpha = (a_1, a_2, \ldots\ldots, a_n)$, $\beta = (b_1, b_2, \ldots\ldots, b_n)$ in V_n and $c \in F$

$\alpha + \beta = (a_1, a_2, \ldots\ldots, a_n) + (b_1, b_2, \ldots\ldots, b_n) = (a_1+b_1, a_2+b_2, \ldots\ldots, a_n+b_n) \in V_n$
[$\because$ each $a_i, b_i \in F$ and F is field $\Rightarrow a_i + b_i \in F$ for $1 \leq i \leq n$]

$c\alpha = c(a_1, a_2, \ldots\ldots, a_n) = (ca_1, ca_2, \ldots\ldots, ca_n) \in V_n$
[$\because$ each $a_i \in F$, $c \in F$ & F is field $\Rightarrow ca_i \in F$ for $1 \leq i \leq n$]

Claim: To show that $(V_n, +, \cdot)$ is a vector space over the field F.

(1) By the definition, clearly addition of vectors (+) is an internal composition in V_n

To prove that $(V_n, +)$ is an abelian group:-

(*i*) '+' is closed:- Since '+' is an internal composition in V_n, so for all $\alpha, \beta \in V_n$, $\alpha + \beta \in V_n$

(*ii*) '+' is associative:- Let $\alpha = (a_1, a_2, \ldots\ldots, a_n)$, $\beta = (b_1, b_2, \ldots\ldots, b_n)$, $\gamma = (c_1, c_2, \ldots, c_n)$ be three vectors in V_n.

$$\begin{aligned}
\text{Now } (\alpha + \beta) + \gamma &= [(a_1, a_2, \ldots\ldots, a_n) + (b_1, b_2, \ldots\ldots, b_n)] + (c_1, c_2, \ldots, c_n) \\
&= (a_1+b_1, a_2+b_2, \ldots\ldots, a_n+b_n) + (c_1, c_2, \ldots, c_n) \\
&= \big((a_1+b_1) + c_1, (a_2+b_2) + c_2, \ldots\ldots\ldots, (a_n+b_n) + c_n\big) \\
&= \big(a_1 + (b_1 + c_1), a_2 + (b_2 + c_2), \ldots\ldots\ldots, a_n + (b_n + c_n)\big) \\
&= (a_1, a_2, \ldots\ldots, a_n) + \big((b_1 + c_1), (b_2 + c_2), \ldots\ldots\ldots, (b_n + c_n)\big) \\
&= (a_1, a_2, \ldots\ldots, a_n) + [(b_1, b_2, \ldots\ldots, b_n) + (c_1, c_2, \ldots, c_n)] \\
&= \alpha + (\beta + \gamma)
\end{aligned}$$

$\therefore$ '+' is associative on V_n.

(*iii*) Existence of identity:- Let '0' be the additive identity in the field F.

Then $\bar{0} = (0, 0, \ldots\ldots, 0) \in V_n$. Let $\alpha = (a_1, a_2, \ldots\ldots, a_n) \in V_n$.

$$\begin{aligned}
\text{Now } \alpha + \bar{0} &= (a_1, a_2, \ldots\ldots, a_n) + (0, 0, \ldots\ldots, 0) = (a_1 + 0, a_2 + 0, \ldots\ldots, a_n + 0) \\
&= (a_1, a_2, \ldots\ldots, a_n) = \alpha
\end{aligned}$$

Similarly $\bar{0} + \alpha = \alpha$. $\quad \therefore \alpha + \bar{0} = \bar{0} + \alpha = \alpha \quad \forall \alpha \in V_n \qquad \therefore \bar{0}$ is the zero vector in V_n

(*iv*) Existence of inverse:- Let $\alpha = (a_1, a_2, \ldots, a_n) \in V_n$.

Then $a_1, a_2, \ldots, a_n \in F$ & F is field $\Rightarrow -a_1, -a_2, \ldots, -a_n \in F$

$$\Rightarrow -\alpha = (-a_1, -a_2, \ldots, -a_n) \in V_n$$

Now $\alpha + (-\alpha) = (a_1, a_2, \ldots, a_n) + (-a_1, -a_2, \ldots, -a_n)$

$$= \big(a_1 + (-a_1), a_2 + (-a_2), \ldots, a_n + (-a_n)\big) = (a_1 - a_1, a_2 - a_2, \ldots, a_n - a_n)$$

$$= (0,0,\ldots,0) = \bar{0}$$

Similarly $-\alpha + \alpha = \bar{0}$. $\qquad \therefore \alpha + (-\alpha) = -\alpha + \alpha = \bar{0} \quad \forall \alpha \in V$.

$\therefore$ Every element in V_n has additive inverse.

(*v*) '+' is Commutative:- Let $\alpha = (a_1, a_2, \ldots, a_n)$, $\beta = (b_1, b_2, \ldots, b_n) \in V_n$.

$\alpha + \beta = (a_1, a_2, \ldots, a_n) + (b_1, b_2, \ldots, b_n) = (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n)$

$= (b_1 + a_1, b_2 + a_2, \ldots, b_n + a_n) \qquad [\because$ each $a_i, b_j \in F$ & F is field$]$

$= (b_1, b_2, \ldots, b_n) + (a_1, a_2, \ldots, a_n) = \beta + \alpha$

$\therefore$ '+' is commutative in V_n

$$\boxed{\therefore (V_n, +) \text{ is an abelian group}}$$

(2)By the definition, clearly scalar multiplication $(\cdot)$ is an external composition in V_n over F. *i.e* $a\alpha \in V_n \quad \forall a \in F, \forall \alpha \in V_n$.

(3) To prove that remaining all properties of vector space:-

(*i*) Let $a \in F$ and $\alpha = (a_1, a_2, \ldots, a_n)$, $\beta = (b_1, b_2, \ldots, b_n) \in V_n$.

Now $a(\alpha + \beta) = a[(a_1, a_2, \ldots, a_n) + (b_1, b_2, \ldots, b_n)] = a(a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n)$

$$= \big(a(a_1 + b_1), a(a_2 + b_2), \ldots, a(a_n + b_n)\big)$$

$$= (aa_1 + ab_1, aa_2 + ab_2, \ldots, aa_n + ab_n)$$

$$= (aa_1, aa_2, \ldots, aa_n) + (ab_1, ab_2, \ldots, ab_n)$$

$$= a(a_1, a_2, \ldots, a_n) + a(b_1, b_2, \ldots, b_n) = a\alpha + b\beta$$

$$\boxed{\therefore a(\alpha + \beta) = a\alpha + b\beta \quad \forall a \in F, \forall \alpha, \beta \in V_n.}$$

(ii) Let $a, b \in$ F and $\alpha = (a_1, a_2, \ldots\ldots, a_n) \in V_n$.

Now $(a + b)\alpha = (a + b)\,(a_1, a_2, \ldots\ldots, a_n) = \big((a + b)a_1, (a + b)a_2, \ldots\ldots, (a + b)a_n\big)$

$$= (aa_1+ba_1, aa_2+ba_2, \ldots\ldots, aa_n+ba_n)$$

$$= (aa_1, aa_2, \ldots\ldots, aa_n) + (ba_1, ba_2, \ldots\ldots, ba_n)$$

$$= a(a_1, a_2, \ldots\ldots, a_n) + b(a_1, a_2, \ldots\ldots, a_n) = a\alpha + b\alpha$$

$$\boxed{\therefore (a + b)\alpha = a\alpha + b\alpha \quad \forall\, a, b \in F, \forall\, \alpha \in V_n\,.}$$

(iii) Let $a, b \in$ F and $\alpha = (a_1, a_2, \ldots\ldots, a_n) \in V_n$.

Now $(ab)\alpha = (ab)(a_1, a_2, \ldots\ldots, a_n) = \big((ab)a_1, (ab)a_2, \ldots\ldots, (ab)a_n\big)$

$$= \big(a(ba_1), a(ba_2), \ldots\ldots, a(ba_n)\big)$$

$$= a(ba_1, ba_2, ba_n)$$

$$= a\big(b(a_1, a_2, \ldots\ldots, a_n)\big) = a(b\alpha)$$

$$\boxed{\therefore (a + b)\alpha = a\alpha + b\alpha \quad \forall\, a, b \in F, \forall\, \alpha \in V_n\,.}$$

(iv) Let $1 \in$ F be the unity element in F and $\alpha = (a_1, a_2, \ldots\ldots, a_n) \in V_n$.

Now $1 \cdot \alpha = 1 \cdot (a_1, a_2, \ldots\ldots, a_n) = (1a_1, 1a_2, \ldots\ldots, 1a_n) = (a_1, a_2, \ldots\ldots, a_n) = \alpha$.

$$\boxed{\therefore 1\alpha = \alpha \quad \forall\, \alpha \in V_n}$$

$\therefore (V_n, +, \cdot)$ is a vector space over the field F.

Eg ①: Show that the triad (x_1, x_2, x_3) in $\{R\}$ i.e. where x_1, x_2, x_3 be the real numbers forms a vector space over the field of Real numbers. Define,

$$(x_1, x_2, x_3) + (y_1, y_2, y_3) = (x_1+y_1, x_2+y_2, x_3+y_3)$$

& $$c(x_1, x_2, x_3) = (cx_1, cx_2, cx_3) \quad \forall c \in R.$$

Sol:- Let us consider,

$$V = \{\alpha, \beta, \gamma, \ldots\}$$

where $\alpha = (x_1, x_2, x_3)$, $\beta = (y_1, y_2, y_3)$ & $\gamma = (z_1, z_2, z_3)$.
$\forall x_i, y_i, z_i \in R$.

Given that, $(x_1, x_2, x_3) + (y_1, y_2, y_3) = (x_1+y_1, x_2+y_2, x_3+y_3)$

$c(x_1, x_2, x_3) = (cx_1, cx_2, cx_3)$.

To prove $(V, +, \cdot)$ is a Vector Space:-

By the given data, clearly, Addition of vector (+) is an Internal Composition in V_3

i) closure property:- Let $\alpha, \beta \in V$.

Consider, $\alpha+\beta = (x_1, x_2, x_3) + (y_1, y_2, y_3)$

$= (x_1+y_1, x_2+y_2, x_3+y_3) \in V$.

$\therefore$ If $\alpha, \beta \in V$ then $\alpha+\beta \in V$.

$\therefore$ closure is exists.

ii) Commutative property:-

Consider, $\alpha+\beta = (x_1, x_2, x_3) + (y_1, y_2, y_3)$

$= (x_1+y_1, x_2+y_2, x_3+y_3)$

$= (y_1+x_1, y_2+x_2, y_3+x_3)$

$= (y_1, y_2, y_3) + (x_1, x_2, x_3)$

$= \beta+\alpha$.

$\therefore \alpha+\beta = \beta+\alpha$.

$\therefore$ Commute property is exists

iii) Associative property:- Let $\alpha, \beta, \gamma \in V$.

Consider,

$(\alpha+\beta)+\gamma = ((x_1, x_2, x_3) + (y_1, y_2, y_3)) + (z_1, z_2, z_3)$

$= (x_1+y_1, x_2+y_2, x_3+y_3) + (z_1, z_2, z_3)$

$= ((x_1+y_1)+z_1, (x_2+y_2)+z_2, (x_3+y_3)+z_3)$

$= (x_1+(y_1+z_1), x_2+(y_2+z_2), x_3+(y_3+z_3))$

$= (x_1, x_2, x_3) + (y_1+z_1, y_2+z_2, y_3+z_3)$.

$= \alpha + (\beta+\gamma)$.

$\therefore (\alpha+\beta)+\gamma = \alpha + (\beta+\gamma)$.

$\therefore$ Associative property is exists.

iv). Exestance of Identity :-

Let '0' be the an additive Identity in the field f.

Now, $\bar{0} = (0,0,0)$ & $\alpha = (x_1, x_2, x_3)$.

Consider, $\alpha + \bar{0} = (x_1, x_2, x_3) + (0,0,0)$

$= (x_1+0, x_2+0, x_3+0) = (x_1, x_2, x_3) = \alpha$.

$\therefore \alpha + \bar{0} = \alpha$.

Illy $\bar{0} + \alpha = \alpha$.

$\therefore \alpha + \bar{0} = \alpha = \bar{0} + \alpha$. $\forall \alpha \in V$.

$\therefore \bar{0}$ is the zero vector in V.

v). Exestance of Inverse :-

Let $\alpha = (x_1, x_2, x_3)$ where $x_1, x_2, x_3 \in f(R)$

$\Rightarrow -x_1, -x_2, -x_3 \in f(R)$

$\therefore -\alpha = (-x_1, -x_2, -x_3) \in V$.

Consider, $\alpha + (-\alpha) = (x_1, x_2, x_3) + (-x_1, -x_2, -x_3)$

$= (x_1 + (-x_1), x_2 + (-x_2), x_3 + (-x_3))$

$= (x_1 - x_1, x_2 - x_2, x_3 - x_3)$

$= (0,0,0) = \bar{0}$.

$\therefore \alpha + (-\alpha) = \bar{0}$.

Illy $(-\alpha) + \alpha = \bar{0}$.

$\therefore \alpha + (-\alpha) = (-\alpha) + \alpha = \bar{0}$.

'$-\alpha$' is the additive Inverse of. α.

$\therefore$ Every element in V has additive Inverse.

$(V_3, +)$ is an Abelian group.

2). By the defination, Clearly Scalar multiplication is an external Composition in V_3 over field F of Real numbers.

i.e $a\alpha \in V \quad \forall\ a \in F, \alpha \in V$.

i.e $a(x_1, x_2, x_3) = (ax_1, ax_2, ax_3) \in V$.

3). To prove remaining all properties of Vector Space :-

(i) Let $a, b \in F$, $\alpha, \beta \in V$.

Now, $a(\alpha+\beta) = a(x_1+y_1, x_2+y_2, x_3+y_3)$

$= (a(x_1+y_1), a(x_2+y_2), a(x_3+y_3))$

$= (ax_1+ay_1, ax_2+ay_2, ax_3+ay_3)$

$= (ax_1, ax_2, ax_3) + (ay_1, ay_2, ay_3)$

$= a(x_1, x_2, x_3) + a(y_1, y_2, y_3)$

$= a\alpha + a\beta$.

$\therefore a(\alpha+\beta) = a\alpha + a\beta, \quad \forall\ a \in F, \alpha, \beta \in V$.

ii) Let $a, b \in F$ and $\alpha = (x_1, x_2, x_3) \in V$.

Now, $(a+b)\alpha = (a+b)(x_1, x_2, x_3)$

$= ((a+b)x_1, (a+b)x_2, (a+b)x_3)$

$= (ax_1+bx_1, ax_2+bx_2, ax_3+bx_3)$

$= (ax_1, ax_2, ax_3) + (bx_1, bx_2, bx_3)$

$= a(x_1, x_2, x_3) + b(x_1, x_2, x_3)$

$= a\alpha + b\alpha$.

$\therefore (a+b)\alpha = a\alpha + b\alpha. \quad \forall\ a, b \in F, \alpha \in V$.

(iii). Let $a, b \in F$ and $\alpha = (x_1, x_2, x_3) \in V$.

Now, $(ab)\alpha = (ab)(x_1, x_2, x_3)$

$= ((ab)x_1, (ab)x_2, (ab)x_3)$

$= (a(bx_1), a(bx_2), a(bx_3))$

$= a(bx_1, bx_2, bx_3)$

$= a(b(x_1, x_2, x_3)) = a(b\alpha)$.

$\therefore (ab)\alpha = a(b\alpha)$. $\forall\ a, b \in F,\ \alpha \in V$.

(iv). Let $1 \in F$ be the unit element in 'F'.

and $\alpha = (x_1, x_2, x_3) \in V$.

Now, $1 \cdot \alpha = 1(x_1, x_2, x_3)$

$= (1 \cdot x_1, 1 \cdot x_2, 1 \cdot x_3)$

$= (x_1, x_2, x_3)$

$= \alpha$.

$\therefore 1 \cdot \alpha = \alpha$. $\forall\ 1 \in F,\ \alpha \in V$.

$\therefore (V_3, +, \cdot)$ is a Vector Space over field of Real numbers.

Eg ②. Let 'S' be a non empty set and F be a field. Let 'V' be the set of all functions from S into F. If addition and Scalar multiplication are defined as

$(f+g)(x) = f(x) + g(x)\ \ \forall\ x \in S$.

& $(cf)(x) = c \cdot f(x)\ \ \forall\ x \in S$, where $f, g \in V$ & $c \in F$.

then S.T. V is a Vector Space.

Sol: Let 'S' be a non empty set & V be the set of all functions from S into F,

i.e $V = \{ f \mid f : S \to F$ is a function $\}$.

Given that, the addition of Vectors & Scalar multiplication on V. as follows.

$(f+g)(x) = f(x) + g(x)$,

& $(cf)(x) = c \cdot f(x)$ $\forall\ x \in S,\ f, g \in V$ & $c \in F$.

To prove that V is a Vector Space :-

1) By the definition, clearly Vector addition (+) is an Internal composition on V.

i.e $\forall\ f, g \in V \Rightarrow f+g \in V$.

i) Closure property:-

$f, g \in V \Rightarrow f+g \in V.$

$\therefore$ closure property is exists

ii) Commutative property:-

Let, $f, g \in V$ & $x \in S$.

Now, $(f+g)(x) = f(x) + g(x)$

$= g(x) + f(x)$

$= (g+f)(x)$

$\therefore (f+g)(x) = (g+f)(x). \ \forall f, g \in V$ & $x \in S$.

$\therefore$ Commute property exists.

iii) Associative property:-

Let $f, g, h \in V$ & $x \in S$.

Now, $((f+g)+h)(x) = (f+g)(x) + h(x)$

$= (f(x) + g(x)) + h(x)$

$= f(x) + (g(x) + h(x))$ | $\because f(x), g(x)$ & $h(x) \in F$ & F is field

$= (f+(g+h))(x)$

$\therefore ((f+g)+h)(x) = (f+(g+h))(x)$

i.e $(f+g)+h) = f+(g+h). \ \forall f, g, h \in V$ & $x \in S$.

$\therefore$ Associative property exists.

iv) Existance of Identity:-

Let '0' be the zero element in F, and defined as

$\bar{0} : S \to F \ \ni \bar{0}(x) = 0 \ \forall x \in S$., clearly $\bar{0} \in V$.

Let $f \in V$, and $x \in S$.

Now, $(f+\bar{0})(x) = f(x) + \bar{0}(x)$

$= f(x) + 0$

$= f(x)$

$\therefore f+\bar{0} = f$

$^{||}$ly $\bar{0}+f = f$ $\therefore f+\bar{0} = f = \bar{0}+f \ \forall f \in V.$

$\therefore \bar{0}$ is the additive Identity in V.

(v). Existance of Inverse :- Let $f \in V$.

Now, we define, $-f : S \to F$ as $(-f)(x) = -f(x)\ \forall\ x \in S$

Clearly, $-f \in V$. & $x \in S$.

Now, $(f + (-f))(x) = f(x) + (-f)(x)$

$= f(x) + [-f(x)]$

$= f(x) - f(x)$

$= 0$.

$= \bar{0}(x)$.

$\therefore f + (-f) = \bar{0}$.

Wly $(-f) + f = \bar{0}$.

$\therefore f + (-f) = (-f) + f = \bar{0}\ \forall\ f \in V$.

$\therefore -f$ is the additive Inverse of 'f'.

$\therefore$ Every element in V has additive Inverse of f.

$\therefore (V, +)$ is an Abelian Group.

2) By the defination, Clearly Scalar multiplication (.) is an external Composition.

I.e $c \in F$, $f \in V \Rightarrow cf \in V$.

3). To prove that the remaining all properties of Vector Spaces-

i) Let $a \in F$ and $f, g \in V$. & $x \in S$.

Now, $[a(f+g)](x) = a(f+g)(x)$

$= a(f(x) + g(x))$ | $\because$ by distributive property

$= af(x) + ag(x)$.

$= (af + ag)(x)$.

$\therefore a(f+g) = af + ag\ \forall\ a \in F, f, g \in V$ & $x \in S$.

ii). Let $a, b \in F$, and $f \in V$. and $x \in S$.

Now, $[(a+b)f](x) = (a+b)f(x)$

$= af(x) + bf(x)$ | $\because$ by distributive.

$= (af + b.f)(x)$

$\therefore (a+b)f = af + bf\ \forall\ a, b \in F$, & $f \in V$, $x \in S$.

(iii). Let $a, b \in F$ & $f \in V$, $x \in S$.

Now, $[(ab)f](x) = (ab) \cdot f(x)$

$= a \cdot (b \cdot f(x))$

$= a(bf)(x)$

$\therefore (ab)f = a(bf)$. $\forall a, b \in F, f \in V$.

(v). Let '1' be the unit element in F. & $f \in V$., $x \in S$.

Now, $(1 \cdot f)(x) = 1 \cdot f(x)$

$= f(x)$

$\therefore 1 \cdot f = f \quad \forall f \in S$.

$\therefore (V, +, \cdot)$ is a Vector Space over the field F.

Eg ③ :- V is the Set of all $m \times n$ matrices with real entries and R is the field of real numbers. "Addition of matrices" is the Internal Composition and "Multiplication of a matrix by a real number" an external Composition in V. Show that V(R) is a Vector Space.

Sol:- Let us Consider,

$V = \{ \alpha, \beta, \gamma, \ldots \}$, where $\alpha = [a_{ij}]_{m \times n}$, $\beta = [b_{ij}]_{m \times n}$, $\gamma = [c_{ij}]_{m \times n}$ $\forall$ a's, b's, c's $\in R$.

I) Addition of ~~two~~ matrices is an Internal composition.

i). Closure property :-

Addition of two matrices is a matrix.

i.e If $\alpha, \beta \in V \Rightarrow \alpha + \beta \in V$.

Consider, $\alpha + \beta = [a_{ij}] + [b_{ij}] = [d_{ij}] \in V$.

$\therefore$ closure property is exist.

ii). Commutative property :- If $\alpha, \beta \in V$ then $\alpha + \beta = \beta + \alpha$.

Consider, $\alpha + \beta = [a_{ij}] + [b_{ij}]$

$= [a_{ij} + b_{ij}]$

$= [b_{ij} + a_{ij}]$ $\quad [\because$ real number's commute.

$= [b_{ij}] + [a_{ij}]$

$= \beta + \alpha$.

$\therefore \alpha + \beta = \beta + \alpha$.

$\therefore$ commutative property is exist.

iii) Associative property :-

If $\alpha, \beta, \gamma \in V$ then $(\alpha+\beta)+\gamma = \alpha+(\beta+\gamma)$.

Consider, $(\alpha+\beta)+\gamma = ([a_{ij}]+[b_{ij}])+[c_{ij}]$

$= ([a_{ij}+b_{ij}]+c_{ij})$

$= ([a_{ij}]+[b_{ij}+c_{ij}])$ | $\because$ real numbers are Associative.

$= [a_{ij}]+[b_{ij}+c_{ij}]$

$= \alpha+(\beta+\gamma)$.

$\therefore (\alpha+\beta)+\gamma = \alpha+(\beta+\gamma)$.

Associative property is exists.

iv). Existence of Identity :-

If $O=[0_{ij}]_{m\times n}$ is the null matrix then

$\alpha+0 = [a_{ij}]+[0_{ij}] = [a_{ij}+0_{ij}] = [a_{ij}] = \alpha$.

|||ly $0+\alpha = \alpha$,

$\therefore \alpha+0 = \alpha = 0+\alpha$.

$\therefore$ the Null matrix is the additive Identity in V.

v). Existence of Inverse :- If $\alpha = [a_{ij}]$ then $a_{ij} \in R$

$\Rightarrow -a_{ij} \in R$

$\therefore -\alpha = [-a_{ij}]_{m\times n} \in V$.

Consider, $\alpha+(-\alpha) = [a_{ij}]+[-a_{ij}]$

$= [a_{ij}-a_{ij}]$

$= [0_{ij}] = 0$.

$\therefore \alpha+(-\alpha) = 0$.

|||ly $(-\alpha)+\alpha = 0$

$\therefore \alpha+(-\alpha) = (-\alpha)+\alpha = 0$.

$\therefore (-\alpha)$ is the additive Inverse of α, in V.

$\therefore (V,+)$ is an Abelian group.

II). By the defination, clearly multiplication of a matrix by a scalar is an external composition in V.

i.e $x \in R, \alpha \in V \Rightarrow x\alpha = x[a_{ij}] = (xa_{ij}) \in V$.

$\therefore x \in R, \alpha \in V \Rightarrow x\alpha \in V$.

III To prove that the Remaining all properties of a Vector Space:-

i) If $x, y \in \mathbb{R}$ & $\alpha, \beta \in V$ then

$x(\alpha+\beta) = x\alpha + x\beta$

Consider, $x(\alpha+\beta) = x([a_{ij}]+[b_{ij}])$

$= x[a_{ij}+b_{ij}]$

$= [x(a_{ij}+b_{ij})] = [xa_{ij}+xb_{ij}]$

$= [xa_{ij}] + [xb_{ij}]$

$= x[a_{ij}] + x[b_{ij}]$

$= x\alpha + x\cdot\beta$

$\therefore x(\alpha+\beta) = x\alpha + x\beta$.

ii) $(x+y)\alpha = x\alpha + y\alpha$.

Consider, $(x+y)\alpha = (x+y)[a_{ij}]$

$= [(x+y)a_{ij}]$

$= [xa_{ij} + ya_{ij}]$

$= [xa_{ij}] + [ya_{ij}]$

$= x[a_{ij}] + y[a_{ij}]$.

$= x\alpha + y\alpha$

$\therefore (x+y)\alpha = x\alpha + y\alpha$.

iii). $(xy)\alpha = x(y\alpha)$.

Consider, $(xy)\alpha = (xy)[a_{ij}]$

$= [(xy)a_{ij}]$

$= [x(ya_{ij})]$

$= x(y[a_{ij}])$

$= x\cdot(y\cdot\alpha)$

$\therefore (xy)\alpha = x(y\alpha)$.

iv). $1\cdot\alpha = \alpha$.

Consider, $1\cdot\alpha = 1\cdot[a_{ij}]$

$= [1\cdot a_{ij}] = [a_{ij}] = \alpha$.

$\therefore 1\cdot\alpha = \alpha$.

$\therefore V(\mathbb{R})$ is a Vector Space.

5. Show that the set C is the set of all complex number form say vector space over the field of real numbers.

Sol:

$C = \{\alpha, \beta, \gamma, \ldots\}$

Then $\alpha = a_1 + ib_1$, $\beta = a_2 + ib_2$, $\gamma = a_3 + ib_3$; where $a_i, b_i \in R$

i) i. $\alpha + \beta = (a_1 + ib_1)(a_2 + ib_2)$

$= (a_1 + a_2) + i(b_1 + b_2) \in C$

$\therefore$ C is closure w.r. to '+'.

ii. $(\alpha+\beta)+\gamma = \{(a_1+a_2) + i(b_1+b_2)\} + (a_3 + ib_3)$

$= ((a_1+a_2) + a_3) + i[(b_1+b_2) + b_3]$

$= [a_1 + (a_2 + a_3)] + i[b_1 + (b_2 + b_3)]$

$= (a_1 + ib_1)[(a_2 + a_3) + i(b_2 + b_3)]$

$= \alpha + (\beta + \gamma)$

$\therefore$ C is associative w.r. to '+'.

iii. $\bar{0} = 0 + i(0)$

$\alpha + \bar{0} = (a_1 + ib_1) + (0 + i(0))$

$= (a_1 + 0) + i(b_1 + 0)$

$= a_1 + ib_1 = \alpha$

$\therefore \bar{0}$ is the identity in C w.r. to '+'.

(ii). let $a \in F$ & $\alpha \in V$

$a\alpha = a(a_1 + ib_1) = (aa_1) + i(ab_1) \in C$

(iii). let $a, b \in F$, $\alpha, \beta \in C$

① $a.(\alpha+\beta) = a[(a_1 + a_2) + i(b + b_2)]$

$= [aa_1 + aa_2) + i(ab_1 + ab_2)]$

$= (aa_1 + iab_1) + (aa_2 + iab_2)$

$= a\alpha + a\beta$

② $(a+b)\alpha = (a+b)(a_1 + ib_1)$

$= (a+b)a_1 + i(a+b)b_1$

$= (aa_1 + ba_1) + i(ab_1 + bb_1)$

$= (aa_1 + iab_1) + (ba_1 + ibb_1)$

$= a\alpha + b\alpha$

③ $(ab)\alpha = (ab)(a_1 + ib_1) = (ab)a_1 + i(ab)b_1$

$= a(ba_1) + ia(bb_1)$

$= a[ba_1 + i(bb_1)]$

$= a(b\alpha)$

④ $1.\alpha = 1.(a_1 + ib_1) = a_1 + ib_1 = \alpha$

$\therefore$ $C(R)$ is a vector space.

5M

6. let 'v' be the set of all pairs (x, y) of real numbers and 'F' be the field of real numbers.

Define $(x, y) + (x_1, y_1) = (x + x_1, y + y_1)$

$c(x, y) = (cx, y)$

Show that $V(F)$ is not a vectorspace.

Sol:- $V = \{\alpha, \beta, \gamma, \ldots\}$

then $\alpha = (x_1, y_1)$ $\beta = (x_2, y_2)$ $\gamma = (x_3, y_3)$ where $x_i, y_i \in R$.

let $a, b \in F$ & $\alpha, \beta \in V$

①. $a(\alpha+\beta) = a[(x_1, y_1) + (x_2, y_2)]$

$= a[(x_1 + x_2), (y_1 + y_2)]$

$= (a(x_1 + x_2), y_1 + y_2)$

$= (ax_1 + ax_2, y_1 + y_2)$

$a\alpha + a\beta = a(x_1, y_1) + a(x_2 y_2)$

$= (ax_1, y_1) + (ax_2, y_2)$

$= (ax_1 + ax_2, y_1 + y_2)$

$a(\alpha+\beta) = a\alpha + a\beta$

(2) $a\alpha + b\alpha = a(x_1, y_1) + b(x_1, y_1)$

$= (ax_1, y_1) + (bx_1, y_1)$

$= [(ax_1 + bx_1), y_1 + y_1)]$

$= [ax_1 + bx_1, y_1 + y_1]$

$= (ax_1 + bx_1, 2y_1)$

$a\alpha + b\alpha = (ax_1 + bx_1, 2y_1)$

$(a+b)\alpha \neq a\alpha + b\alpha$

$\therefore$ v(f) is not a vector speare.

7. let 'v' be the set of all pairs (x, y) of real numbers & 'R' be the field of realnumbers. Define $(x_1, y_1) + (x_2 y_2) = (x_1 + x_2, 0)$ $c(x, y_1) = (cx, y_1)$ show that V(R) is not a vectorspace.

Sol:- $V = \{\alpha, \beta, \gamma, \ldots\ldots\}$

$\alpha = (x_1, y_1)$ $\beta = (x_2, y_2)$ $\gamma = (x_3, y_3)$; where $x_i, y_i \in R$

let $a, b \in F$

(1) $a(\alpha+\beta) = a[(x, y) + (x_2 y_2)]$

$= a(x_1 + x_2, 0)$

$= (a(x_1 + x_2), 0)$

$= (ax_1 + ax_2, 0)$

$a\alpha + a\beta = a(x_1, y_1) + a(x_2 y_2)$

$= (ax_1, y_1) + (ax_2, y_2)$

$= (ax_1 + ax_2, 0)$

$a(\alpha+\beta) = a\alpha + a\beta$

(2) $(a+b)\alpha = (a+b)(x_1, y_1) = ((a+b)x_1, y_1)$

$= (ax_1 + bx_1, y_1)$

$a\alpha + b\alpha = a(x_1, y_1) + b(x_1, y_1)$

$= (ax_1, y_1) + (bx_1, y_1)$

$= (ax_1 + bx_1, 0)$

$(a+b)\alpha \neq a\alpha + b\alpha$

$V(R)$ is not a vector space.

8\. let V be the set of all pairs of real numbers and the field of real numbers Define $(x_1, y_1) + (x_2, y_2) = (3y_1 + 3y_2, -x_1 - x_2)$

$c(x_1, y_1) = (3cy_1, -cx_1)$

show that $V(F)$ is not a vector space.

Sol: $V = \{\alpha, \beta, \gamma, \dots\}$ $\alpha = (x_1, y_1)$ $\beta = (x_2, y_2)$ $\gamma = (x_3, y_3)$ where $x_i, y_i \in R$

let $a, b \in F$

$$a(\alpha+\beta) = a[(x_1, y_1) + (x_2, y_2)]$$

$$= a(3y_1 + y_2, -x_1 - x_2)$$

$$= 3a(-x_1 - x_2), -a(3y_1 + 3y_2)$$

$$= (-3ax_1 - 3ax_2, -3ay_1 - 3ay_2)$$

$$a\alpha + a\beta = a(x_1, y_1) + a(x_2, y_2)$$

$$= (3ay_1, -3ax_1) + (3ay_2, -ax_2)$$

$$= (-3ax_1 - 3ax_2, -3ay_1 - 3ay_2)$$

$$a(\alpha+\beta) = a\alpha + a\beta$$

$$(a+b)\alpha = (a+b)(x_1, y_1)$$

$$= (3(a+b)y_1, -(a+b)x_1)$$

$$= (3ay_1 + 3by_1, -ax_1 - bx_1)$$

$$a\alpha + b\alpha = a(x_1, y_1) + b(x_1, y_1)$$

$$= (3ay_1, -ax_1) + (3aby_1, -bx_1)$$

$$= (-3ax_1 - 3bx_1, -3ay_1 - 3by_1)$$

$$(a+b)\alpha \neq a\alpha + b\alpha$$

$\therefore V(F)$ is not a vectorspace.

(or)

$V = \{\alpha, \beta, \gamma, \dots\}$

Then $\alpha = (x_1, y_1)$, $\beta = (x_2, y_2)$, $\gamma = (x_3, y_3)$ where $x_i, y_i \in R$

$$1 \cdot \alpha = 1(x_1, y_1)$$

$$= (3y_1, -x_1)$$

$$\neq \alpha$$

$$1 \cdot \alpha \neq \alpha$$

$\therefore V(R)$ is not a vector space.

9. Show that let 'v' be the set of all pairs of real numbers and 'f' be the field of real numbers Define $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$ $c(x, y) = (c^2 x, c^2 y)$ show that $V(F)$ is not a vector space.

Sol: $V = \{\alpha, \beta, \gamma, \ldots\}$

then $\alpha = (x_1, y_1)$, $\beta = (x_2, y_2)$, $\gamma = (x_3, y_3)$ where $x_i, y_i \in R$

let $a, b \in R \Rightarrow$ ① $a(\alpha+\beta) = a[(x_1, y_1) + (x_2, y_2)]$

$$= a[(x_1+x_2), (y_1+y_2)]$$

$$= a^2(x_1+x_2), a^2(y_1+y_2)$$

$$= (a^2x_1 + a^2x_2, a^2y_1 + a^2y_2)$$

$$a\alpha + a\beta = a(x_1, y_1) + a(x_2, y_2)$$

$$= (a^2x_1, a^2y_1) + (a^2x_2, a^2y_2)$$

$$= (a^2x_1 + a^2x_2, a^2y_1 + a^2y_2)$$

$$a(\alpha+\beta) = a\alpha + a\beta$$

② $(a+b)\alpha = (a+b)(x_1, y_1)$

$$= ((a+b)^2 x_1, (a+b)^2 y_1)$$

$$a\alpha + b\alpha = a(x_1, y_1) + b(x_1, y_1)$$

$$= (a^2x_1, a^2y_1) + (b^2x_1, b^2y_1)$$

$$= (a^2x_1 + b^2x_1, a^2y_1 + b^2y_1)$$

$(a+b)\alpha \neq a\alpha + b\alpha$

$\therefore$ $V(F)$ is not a vector space.

10. let 'v' be the set of all ~~phase~~ pairs of realnumbers and 'f' be the field of realnumbers. Define $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$

$$c(x, y) = (|c|x, |c|y)$$

Show that $V(F)$ is not a vector space.

Sol: $V = \{\alpha, \beta, \gamma, \ldots\}$

Then, $\alpha = (x_1, y_1)$, $\beta = (x_2, y_2)$, $\gamma = (x_3, y_3)$; where $x_i, y_i \in R$

let $a, b \in F$

$$a(\alpha+\beta) = a[(x_1, y_1) + (x_2, y_2)]$$

$$= a(x_1 + x_2, y_1 + y_2)$$

$$= (|a|(x_1 + x_2), |a|(y_1 + y_2))$$

$= (|a| x_1 + |a| x_2 \ , |a| y_1 + |a| y_2)$

$a\alpha + a\beta = a(x_1, y_1) + a(x_2, y_2)$

$= (|a| x_1 , |a| y_1) + (|a| x_2 , |a| y_2)$

$= (|a| x_1 + |a| x_2 \ , |a| y_1 + |a| y_2)$

$a(\alpha + \beta) = a\alpha + a\beta$

$(a+b)\alpha = (a+b)(x_1, y_1)$

$= (|a+b| x_1 , |a+b| y_1)$

$a\alpha + b\alpha = a(x_1, y_1) + b(x_2, y_2)$

$= (|a| x_1 , |a| y_1) + (|b| x_1 , |b| y_1)$

$= (|a| x_1 + |b| x_1 , |a| y_1 + |b| y_1)$

$(a+b)\alpha \neq a\alpha + b\alpha$

$\therefore$ V(R) is not vector space.

VECTOR SUBSPACE

Definition (Subspace): Let V(F) be a vector space and let $W \subseteq V$. Then W is said to be a subspace of V if W itself is a vector space over F with the same operations of vector addition and scalar multiplication in V.

Note:- 1. If W(F) is a subspace of V(F), then W is a subgroup of V w.r.t the addition of vectors (+) in V.

2. Let V(F) be a vector space. The zero vector space $\{\bar{0}\} \subseteq V$ and $V \subseteq V$. Therefore $\{\bar{0}\}$ and V are the trivial or improper subspaces of V.

3. Any subspace of V other than $\{\bar{0}\}$ and V are called non-trivial or proper subspaces of V.

Theorem:- The necessary-sufficient condition for a non-empty subset W of a vector space V(F) to be a subspace of V is that (i) $\alpha \in W, \beta \in W \Rightarrow \alpha - \beta \in W$ (ii) $a \in F, \alpha \in W \Rightarrow a\alpha \in W$ [ANU J2016,N2019

Proof: Let W be a non-empty subset of a vector space V(F).

Suppose W is a subspace of V.

Claim: To prove that (i) $\alpha \in W, \beta \in W \Rightarrow \alpha - \beta \in W$ (ii) $a \in F, \alpha \in W \Rightarrow a\alpha \in W$

(i) Since W is a subspace of V $\Rightarrow$ (w, +) is a subgroup of (V, +)

$\Rightarrow$ (W, +) is a group

$\Rightarrow$ if $\alpha \in W$, $\beta \in W$, then $\alpha - \beta \in W$

$$\boxed{\therefore \alpha \in W, \beta \in W \Rightarrow \alpha - \beta \in W}$$

(*ii*) Since W is a subspace of V $\Rightarrow$ W is closed under scalar multiplication.

$\Rightarrow$ If $a \in F$, $\alpha \in W$, then $a\alpha \in W$

$$\boxed{\therefore a \in F, \alpha \in W \Rightarrow a\alpha \in W}$$

Conversely suppose that W satisfies the conditions (*i*) & (*ii*).

Claim: To prove that W is a subspace of V. It is enough to prove that W itself a vector space over the field F with the same operations in V.

(1) By condition (*i*), $\alpha \in W, \alpha \in W \Rightarrow \alpha - \alpha \in W \Rightarrow \bar{0} \in W$
$[\because \alpha \in W \,\&\, W \subseteq V \Rightarrow \alpha \in V \,\&\, \alpha - \alpha = \bar{0} \in V]$

$\therefore$ The zero vector in V is also the zero vector in W.

(2) By condition (*i*), $\bar{0} \in W, \alpha \in W \Rightarrow \bar{0} - \alpha \in W \Rightarrow -\alpha \in W$
$[\because \bar{0}, \alpha \in W \,\&\, W \subseteq V \Rightarrow \bar{0}, \alpha \in V \,\&\, \bar{0} - \alpha = -\alpha \in V]$

$\therefore$ The inverse $-\alpha$ of α in V is also in W.

$\therefore$ Additive inverse of each element of W is also in W

(3) Let $\alpha, \beta \in W$. Since $\beta \in W \Rightarrow -\beta \in W$ [By (2)]

$\because \alpha \in W, -\beta \in W \Rightarrow \alpha - (-\beta) \in W$ [From condition (*i*)]

$\Rightarrow \alpha + \beta \in W$

$\therefore \alpha, \beta \in W \Rightarrow \alpha + \beta \in W$. Hence W is closed under vector addition (+). *i.e* '+' is an internal composition in W.

(4) Since $W \subseteq V$, then every element in W is also an element in V. So W satisfies associative and commutative laws w.r.t vector addition in V.

$$\boxed{\therefore (W, +) \text{ is an abelian group}}$$

(5) By the condition (*ii*), Clearly W is closed under scalar multiplication. *i.e* External composition $(\cdot)$ exist in W.

(6) The remaining all properties of vector space are hold in W as $W \subseteq V$.

i.e (*i*) $a(\alpha + \beta) = a\alpha + a\beta \;\; \forall\, a \in F, \forall\, \alpha, \beta \in W$

(*ii*) $(a + b)\alpha = a\alpha + b\alpha \; \forall\, a, b \in F, \forall\, \alpha \in W$

(iii) $(ab)\alpha = a(b\alpha)$ $\forall$ $a, b \in$ F, $\forall$ $\alpha \in$ W

(iv) $1\alpha = \alpha$ where 1 is the unity element in F.

$\therefore$ (W, +, $\cdot$) is a vector space over the field F under the composition in V.

$\boxed{\therefore \text{ W is a subspace of V}}$

Theorem:- The necessary-sufficient condition for a non-empty subset W of a vector space V(F) to be a subspace of V is that (i) $\alpha \in$ W, $\beta \in$ W $\Rightarrow \alpha + \beta \in$ W (ii) $a \in$ F, $\alpha \in$ W $\Rightarrow a\alpha \in$ W

Proof: Let W be a non-empty subset of a vector space V(F).

Suppose W is a subspace of V.

Claim: To prove that (i) $\alpha \in$ W, $\beta \in$ W $\Rightarrow \alpha + \beta \in$ W (ii) $a \in$ F, $\alpha \in$ W $\Rightarrow a\alpha \in$ W

(i) Since W is a subspace of V $\Rightarrow$ (w, +) is a subgroup of (V, +)

$\Rightarrow$ W is closed under vector addition (+) in V

$\Rightarrow$ if $\alpha \in$ W, $\beta \in$ W, then $\alpha + \beta \in$ W

$\boxed{\therefore \alpha \in \text{W}, \beta \in \text{W} \Rightarrow \alpha + \beta \in \text{W}}$

(ii) Since W is a subspace of V $\Rightarrow$ W is closed under scalar multiplication.

$\Rightarrow$ If $a \in$ F, $\alpha \in$ W, then $a\alpha \in$ W

$\boxed{\therefore a \in \text{F}, \alpha \in \text{W} \Rightarrow a\alpha \in \text{W}}$

Conversely suppose that W satisfies the conditions (i) & (ii).

Claim: To prove that W is a subspace of V. It is enough to prove that W itself a vector space over the field F with the same operations in V.

(1) By condition (i), Clearly W is closed under vector addition (+) in V. So vector addition (+) is an internal composition in W

(2) Let $-1 \in$ F and $\alpha \in$ W. Then by condition (ii), $(-1)\alpha \in$ W $\Rightarrow -\alpha \in$ W
[$\because$ $\alpha \in$ W & W $\subseteq$ V $\Rightarrow \alpha \in$ V & $(-1)\alpha = -\alpha$]

$\therefore$ The inverse $-\alpha$ of α in V is also in W.

$\therefore$ Additive inverse of each element of W is also in W

(3) Let $\alpha \in$ W $\Rightarrow -\alpha \in$ W [By (2)]

$\because \alpha \in W, -\alpha \in W \Rightarrow \alpha + (-\alpha) \in W$ [From condition (i)]

$\Rightarrow \bar{0} \in W$ [$\because \alpha \in W$ & $W \subseteq V \Rightarrow \alpha \in V$ & $\alpha + (-\alpha) = \bar{0} \in V$]

$\therefore$ The additive identity $\bar{0}$ in V is also in W. Hence $\bar{0}$ is the additive identity in W.

(4) Since $W \subseteq V$, then every element in W is also an element in V. So W satisfies associative and commutative laws w.r.t vector addition in V.

$\boxed{\therefore (W, +) \text{ is an abelian group}}$

(5) By the condition (ii), Clearly W is closed under scalar multiplication. *i.e* External composition $(\cdot)$ exist in W.

(6) The remaining all properties of vector space are hold in W as $W \subseteq V$.

i.e (i) $a(\alpha + \beta) = a\alpha + a\beta \;\; \forall a \in F, \forall \alpha, \beta \in W$

(ii) $(a + b)\alpha = a\alpha + b\alpha \; \forall a, b \in F, \forall \alpha \in W$

(iii) $(ab)\alpha = a(b\alpha) \; \forall a, b \in F, \forall \alpha \in W$

(iv) $1\alpha = \alpha$ where 1 is the unity element in F.

$\therefore (W, +, \cdot)$ is a vector space over the field F under the composition in V.

$\boxed{\therefore W \text{ is a subspace of } V}$

Theorem:- The necessary-sufficient condition for a non-empty subset W of a vector space V(F) to be a subspace of V is that $a, b \in F$ and $\alpha, \beta \in W \Rightarrow a\alpha + b\beta \in W$. [ANU M98,M2000,M06,M09,M11,M15,May17, N18

Solution: Let W be a non-empty subset of a vector space V(F).

Suppose W is a subspace of V.

$\Rightarrow$ W is closed under vector addition and scalar multiplication in V

Claim: To prove that $a, b \in F$ and $\alpha, \beta \in W \Rightarrow a\alpha + b\beta \in W$

Let $a, b \in F$ and $\alpha, \beta \in W$

$\because a \in F, \alpha \in W \Rightarrow a\alpha \in W$ [$\because$ W is closed under scalar multiplication]

$\because b \in F, \beta \in W \Rightarrow b\beta \in W$ [$\because$ W is closed under scalar multiplication]

$\because a\alpha \in W, b\beta \in W \Rightarrow a\alpha + b\beta \in W$ [$\because$ W is closed under vector addition]

$\boxed{\therefore a, b \in F \text{ and } \alpha, \beta \in W \Rightarrow a\alpha + b\beta \in W}$

Conversely suppose that $a, b \in$ F and $\alpha, \beta \in$ W $\Rightarrow a\alpha + b\beta \in$ W.

Claim: To prove that W is a subspace of V. It is enough to prove that W itself a vector space over the field F with the same operations in V.

(1) Taking $a = 1$, $b = 1$ in F and $\alpha, \beta \in$ W.

$\therefore$ By our converse assumption, we have $1\alpha + 1\beta \in$ W $\Rightarrow \alpha + \beta \in$ W
[$\because \alpha, \beta \in$ W & W $\subseteq$ V $\Rightarrow \alpha, \beta \in$ V & hence $1\alpha = \alpha$, $1\beta = \beta$]

$\therefore \alpha, \beta \in$ W $\Rightarrow \alpha + \beta \in$ W. Hence W is closed under vector addition (+) in V. $i.e$ '+' is an internal composition in W.

(2) Taking $a = 0$, $b = 0$ in F and $\alpha, \alpha \in$ W.

$\therefore$ By our converse assumption, we have $0\alpha + 0\alpha \in$ W $\Rightarrow \bar{0} + \bar{0} \in$ W $\Rightarrow \bar{0} \in$ W
[$\because \alpha \in$ W & W $\subseteq$ V $\Rightarrow \alpha \in$ V & hence $0\alpha = \bar{0}$]

$\therefore$ The zero vector $\bar{0}$ in V is also in W. So $\bar{0}$ is the zero vector in W.

(3) Taking $a = -1$, $b = 0$ in F and $\alpha, \alpha \in$ W.

$\therefore$ By our converse assumption, we have $(-1)\alpha + 0\alpha \in$ W $\Rightarrow -\alpha + \bar{0} \in$ W $\Rightarrow -\alpha \in$ W
[$\because \alpha \in$ W & W $\subseteq$ V $\Rightarrow \alpha \in$ V & hence $(-1)\alpha = -\alpha$, $0\alpha = \bar{0}$]

$\therefore$ The additive inverse $-\alpha$ of α in V is also in W.

$\therefore$ Every element in W has additive inverse in W.

(4) Since W $\subseteq$ V, then every element in W is also an element in V. So W satisfies associative and commutative laws w.r.t vector addition in V.

$\boxed{\therefore \text{(W, +) is an abelian group}}$

(5) Taking $a, b = 0$ in F and $\alpha, \alpha \in$ W.

$\therefore$ By our converse assumption, we have $a\alpha + 0\alpha \in$ W $\Rightarrow a\alpha + \bar{0} \in$ W $\Rightarrow a\alpha \in$ W
[$\because \alpha \in$ W & W $\subseteq$ V $\Rightarrow \alpha \in$ V & hence $0\alpha = \bar{0}$]

$\therefore a \in$ F, $\alpha \in$ W $\Rightarrow a\alpha \in$ W. Hence W is closed under scalar multiplication ($\cdot$) in V. $i.e$ '$\cdot$' is an external composition in W.

(6) The remaining all properties of vector space are hold in W as W $\subseteq$ V.

$i.e$ (i) $a(\alpha + \beta) = a\alpha + a\beta \quad \forall\, a \in$ F, $\forall\, \alpha, \beta \in$ W

(ii) $(a + b)\alpha = a\alpha + b\alpha\ \forall\, a, b \in$ F, $\forall\, \alpha \in$ W

(iii) $(ab)\alpha = a(b\alpha)\ \forall\, a, b \in$ F, $\forall\, \alpha \in$ W

(iv) $1\alpha = \alpha$ where 1 is the unity element in F.

$\therefore$ (W, +, $\cdot$) is a vector space over the field F under the composition in V.

$$\boxed{\therefore \text{W is a subspace of V}}$$

IInd proof: Let W be a non-empty subset of a vector space V(F).

Suppose W is a subspace of V.

$\Rightarrow$ W is closed under vector addition and scalar multiplication in V

Claim: To prove that $a, b \in$ F and $\alpha, \beta \in$ W $\Rightarrow a\alpha + b\beta \in$ W

Let $a, b \in$ F and $\alpha, \beta \in$ W

$\because a \in$ F, $\alpha \in$ W $\Rightarrow a\alpha \in$ W [$\because$ W is closed under scalar multiplication]

$\because b \in$ F, $\beta \in$ W $\Rightarrow b\beta \in$ W [$\because$ W is closed under scalar multiplication]

$\because a\alpha \in$ W, $b\beta \in$ W $\Rightarrow a\alpha + b\beta \in$ W [$\because$ W is closed under vector addition]

$$\boxed{\therefore a, b \in \text{F and } \alpha, \beta \in \text{W} \Rightarrow a\alpha + b\beta \in \text{W}}$$

Conversely suppose that $a, b \in$ F and $\alpha, \beta \in$ W $\Rightarrow a\alpha + b\beta \in$ W.

Claim: To prove that W is a subspace of V. It is enough to prove that (i) $\alpha \in$ W, $\beta \in$ W $\Rightarrow \alpha + \beta \in$ W (ii) $a \in$ F, $\alpha \in$ W $\Rightarrow a\alpha \in$ W

(i) Taking $a = 1$, $b = 1$ in F and $\alpha, \beta \in$ W.

$\therefore$ By our converse assumption, we have $1\alpha + 1\beta \in$ W $\Rightarrow \alpha + \beta \in$ W
[$\because \alpha, \beta \in$ W & W $\subseteq$ V $\Rightarrow \alpha, \beta \in$ V & hence $1\alpha = \alpha$, $1\beta = \beta$]

$$\boxed{\therefore \alpha \in \text{W}, \beta \in \text{W} \Rightarrow \alpha + \beta \in \text{W}}$$

(ii) First we taking $a = 0$, $b = 0$ in F and $\alpha, \alpha \in$ W.

$\therefore$ By our converse assumption, we have $0\alpha + 0\alpha \in$ W $\Rightarrow \bar{0} + \bar{0} \in$ W $\Rightarrow \bar{0} \in$ W
[$\because \alpha \in$ W & W $\subseteq$ V $\Rightarrow \alpha \in$ V & hence $0\alpha = \bar{0}$]

$\therefore$ The zero vector $\bar{0}$ in V is also in W. So $\bar{0}$ is the zero vector in W.

Now we taking $a, b = 0$ in F and $\alpha, \alpha \in$ W.

$\therefore$ By our converse assumption, we have $a\alpha + 0\alpha \in$ W $\Rightarrow a\alpha + \bar{0} \in$ W $\Rightarrow a\alpha \in$ W

$$\boxed{\therefore a \in \text{F}, \alpha \in \text{W} \Rightarrow a\alpha \in \text{W}}$$

$\therefore$ By the known theorem W is a subspace of V

Theorem:- A non-empty subset W of a vector space V(F) is a subspace of V iff $a \in F$ and $\alpha, \beta \in W \Rightarrow a\alpha + \beta \in W$. [ANU M2017,O2017

Solution: Let W be a non-empty subset of a vector space V(F).

Suppose W is a subspace of V.

$\Rightarrow$ W is closed under vector addition and scalar multiplication in V

Claim: To prove that $a \in F$ and $\alpha, \beta \in W \Rightarrow a\alpha + \beta \in W$

Let $a, b \in F$ and $\alpha, \beta \in W$

$\because a \in F, \alpha \in W \Rightarrow a\alpha \in W$ [$\because$ W is closed under scalar multiplication]

$\because a\alpha \in W, \beta \in W \Rightarrow a\alpha + \beta \in W$ [$\because$ W is closed under vector addition]

$$\boxed{\therefore a \in F \text{ and } \alpha, \beta \in W \Rightarrow a\alpha + \beta \in W}$$

Conversely suppose that $a \in F$ and $\alpha, \beta \in W \Rightarrow a\alpha + \beta \in W$.

Claim: To prove that W is a subspace of V. It is enough to prove that W itself a vector space over the field F with the same operations in V.

(1) Taking $a = 1$ in F and $\alpha, \beta \in W$.

$\therefore$ By our converse assumption, we have $1\alpha + \beta \in W \Rightarrow \alpha + \beta \in W$
[$\because \alpha, \beta \in W$ & $W \subseteq V \Rightarrow \alpha, \beta \in V$ & hence $1\alpha = \alpha$]

$\therefore \alpha, \beta \in W \Rightarrow \alpha + \beta \in W$. Hence W is closed under vector addition (+) in V. *i.e* '+' is an internal composition in W.

(2) Taking $a = -1$ in F and $\alpha, \alpha \in W$.

$\therefore$ By our converse assumption, we have $(-1)\alpha + \alpha \in W \Rightarrow -\alpha + \alpha \in W \Rightarrow \bar{0} \in W$
[$\because \alpha \in W$ & $W \subseteq V \Rightarrow \alpha \in V$ & hence $-\alpha + \alpha = \bar{0}$]

$\therefore$ The zero vector $\bar{0}$ in V is also in W. So $\bar{0}$ is the zero vector in W.

(3) Taking $a = -1$ in F and $\alpha, \bar{0} \in W$.

$\therefore$ By our converse assumption, we have $(-1)\alpha + \bar{0} \in W \Rightarrow -\alpha + \bar{0} \in W \Rightarrow -\alpha \in W$
[$\because \alpha \in W$ & $W \subseteq V \Rightarrow \alpha \in V$ & hence $(-1)\alpha = -\alpha$]

$\therefore$ The additive inverse $-\alpha$ of α in V is also in W.

$\therefore$ Every element in W has additive inverse in W.

(4) Since $W \subseteq V$, then every element in W is also an element in V. So W satisfies associative and commutative laws w.r.t vector addition in V.

$\boxed{\therefore (W, +) \text{ is an abelian group}}$

(5) Taking $a \in F$ and $\alpha, \bar{0} \in W$.

$\therefore$ By our converse assumption, we have $a\alpha + \bar{0} \in W \Rightarrow a\alpha \in W$

$\therefore a \in F, \alpha \in W \Rightarrow a\alpha \in W$. Hence W is closed under scalar multiplication $(\cdot)$ in V. i.e '·' is an external composition in W.

(6) The remaining all properties of vector space are hold in W as $W \subseteq V$.

i.e (i) $a(\alpha + \beta) = a\alpha + a\beta \;\; \forall\, a \in F, \forall\, \alpha, \beta \in W$

(ii) $(a + b)\alpha = a\alpha + b\alpha \; \forall\, a, b \in F, \forall\, \alpha \in W$

(iii) $(ab)\alpha = a(b\alpha) \; \forall\, a, b \in F, \forall\, \alpha \in W$

(iv) $1\alpha = \alpha$ where 1 is the unity element in F.

$\therefore (W, +, \cdot)$ is a vector space over the field F under the composition in V.

$\boxed{\therefore W \text{ is a subspace of } V}$

IInd proof: Let W be a non-empty subset of a vector space V(F).

Suppose W is a subspace of V.

$\Rightarrow$ W is closed under vector addition and scalar multiplication in V

Claim: To prove that $a, b \in F$ and $\alpha, \beta \in W \Rightarrow a\alpha + \beta \in W$

Let $a, b \in F$ and $\alpha, \beta \in W$

$\because a \in F, \alpha \in W \Rightarrow a\alpha \in W$ [$\because$ W is closed under scalar multiplication]

$\because a\alpha \in W, \beta \in W \Rightarrow a\alpha + \beta \in W$ [$\because$ W is closed under vector addition]

$\boxed{\therefore a \in F \text{ and } \alpha, \beta \in W \Rightarrow a\alpha + \beta \in W}$

Conversely suppose that $a \in F$ and $\alpha, \beta \in W \Rightarrow a\alpha + \beta \in W$.

Claim: To prove that W is a subspace of V. It is enough to prove that (i) $\alpha \in W, \beta \in W \Rightarrow \alpha + \beta \in W$ (ii) $a \in F, \alpha \in W \Rightarrow a\alpha \in W$

(i) Taking $a = 1$ in F and $\alpha, \beta \in W$.

$\therefore$ By our converse assumption, we have $1\alpha + \beta \in W \Rightarrow \alpha + \beta \in W$
[$\because \alpha, \beta \in W$ & $W \subseteq V \Rightarrow \alpha, \beta \in V$ & hence $1\alpha = \alpha$]

$$\boxed{\therefore \alpha \in W, \beta \in W \Rightarrow \alpha + \beta \in W}$$

(*ii*) First we taking $a = -1$ in F and $\alpha, \alpha \in W$.

$\therefore$ By our converse assumption, we have $(-1)\alpha + \alpha \in W \Rightarrow -\alpha + \alpha \in W \Rightarrow \bar{0} \in W$
[$\because \alpha \in W$ & $W \subseteq V \Rightarrow \alpha \in V$ & hence $-\alpha + \alpha = \bar{0}$]

$\therefore$ The zero vector $\bar{0}$ in V is also in W. So $\bar{0}$ is the zero vector in W.

Now we taking $a \in F$ and $\alpha, \bar{0} \in W$

$\therefore$ By our converse assumption, we have $a\alpha + \bar{0} \in W \Rightarrow a\alpha \in W$

$$\boxed{\therefore a \in F, \alpha \in W \Rightarrow a\alpha \in W}$$

$\therefore$ By the known theorem W is a subspace of V

PROBLEMS

(1) Show that $W = \{(x, y, z) / x, y, z \in R$ and $ax + by + cz = 0\}$ is a subspace of $V_3(R)$ where a, b, c are three fixed elements in R. [ANU M2014,O2017

Solution: Given $W = \{(x, y, z) / x, y, z \in R$ and $ax + by + cz = 0\}$ where a, b, c are three fixed elements in R.

Clearly $W \subseteq V_3(R)$ and $a.0 + b.0 + c.0 = 0 \Rightarrow (0, 0, 0) \in W$

$\therefore$ W is a non-empty subset of $V_3(R)$.

Claim: To show that W is a subspace of $V_3(R)$. It is enough to show that $p, q \in R$ and $\alpha, \beta \in W \Rightarrow p\alpha + q\beta \in W$

Let $p, q \in R$ and $\alpha = (x_1, y_1, z_1), \beta = (x_2, y_2, z_2)$ in W.

$\because \alpha, \beta \in W \Rightarrow ax_1+by_1+cz_1 = 0$ and $ax_2+by_2+cz_2 = 0$

Now $p\alpha + q\beta = p(x_1, y_1, z_1) + q(x_2, y_2, z_2) = (px_1, py_1, pz_1) + (qx_2, qy_2, qz_2)$

$= (px_1+qx_2, py_1+qy_2, pz_1+qz_2)$

Now $a(px_1+qx_2) + b(py_1+qy_2) + c(pz_1+qz_2) = (apx_1+aqx_2)+(bpy_1+bqy_2)+(cpz_1+cqz_2)$

$= (apx_1+bpy_1+cpz_1) + (aqx_2+bqy_2+cqz_2)$

$= p(ax_1+by_1+cz_1) + q(ax_2+by_2+cz_2)$

$= p.0 + q.0 = 0 + 0 = 0$

$\therefore p, q \in R$ and $\alpha, \beta \in W \Rightarrow p\alpha + q\beta \in W$

$\boxed{\therefore \text{W is a subspace of } V_3(R).}$

(2) If F is a field then show that $W = \{(x, y, 0) / x, y \in F\}$ is a subspace of $V_3(F)$. [ANU S99, N18,

Solution: Given $W = \{(x, y, 0) / x, y \in F\}$ where F is a field.

Clearly $W \subseteq V_3(F)$ and $(0, 0, 0) \in W$.

$\therefore$ W is a non-empty sub set of $V_3(F)$.

Claim: To show that W is a subspace of $V_3(F)$. It is enough to show that $a, b \in F$ and $\alpha, \beta \in W \Rightarrow a\alpha + b\beta \in W$.

Let $a, b \in F$ and $\alpha = (x_1, y_1, 0), \beta = (x_2, y_2, 0)$ in W

Now $a\alpha + b\beta = a(x_1, y_1, 0) + b(x_2, y_2, 0) = (ax_1, ay_1, 0) + (bx_2, by_2, 0)$

$= (ax_1+bx_2, ay_1+by_2, 0) \in W$

$[\because a, b \in F \ \& \ x_1, x_2, y_1, y_2 \in F \Rightarrow ax_1+bx_2 \in F \ \& \ ay_1+by_2 \in F]$

$\therefore a, b \in F$ and $\alpha, \beta \in W \Rightarrow a\alpha + b\beta \in W$

$\boxed{\therefore \text{W is a subspace of } V_3(R).}$

(3) Show that $W = \{(x, 2y, 3z) / x, y, z \in R\}$ is a subspace of $V_3(R)$.

Solution: Given $W = \{(x, 2y, 3z) / x, y, z \in R\}$

Clearly $W \subseteq V_3(R)$ and $(0, 0, 0) = (0, 2(0), 3(0)) \in W$

$\therefore$ W is a non-empty subset of $V_3(R)$.

Claim: To show that W is a subspace of $V_3(R)$. It is enough to show that $a, b \in R$ and $\alpha, \beta \in W \Rightarrow a\alpha + b\beta \in W$

Let $a, b \in R$ and $\alpha = (x_1, 2y_1, 3z_1), \beta = (x_2, 2y_2, 3z_2)$ in W

$\therefore a\alpha + b\beta = a(x_1, 2y_1, 3z_2) + b(x_2, 2y_2, 3z_2) = (ax_1, 2ay_1, 3az_1) + (bx_2, 2by_2, 3bz_2)$

$= (ax_1+bx_2, 2ay_1+2by_2, 3az_1+3bz_2) = (ax_1+bx_2, 2(ay_1+by_2), 3(az_1+bz_2)) \in W$

$[\because a, b \in R \ \& \ x_1, x_2, y_1, y_2, z_1, z_2 \in R \Rightarrow ax_1+bx_2, ay_1+by_2, az_1+bz_2 \in R]$

$\therefore a, b \in F$ and $\alpha, \beta \in W \Rightarrow a\alpha + b\beta \in W$

$\boxed{\therefore \text{W is a subspace of } V_3(R).}$

(4) Let R be the field of real numbers and W = {(x, y, z) / x, y, z are rational numbers}. Is W a subspace of $V_3(R)$.

Solution: Given W = {(x, y, z) / $x, y, z \in Q$} and R is the field of real numbers.

Now $a\alpha = \sqrt{7}(2, 3, 4) = (2\sqrt{7}, 3\sqrt{7}, 4\sqrt{7}) \notin$ W [$\because 2\sqrt{7}, 3\sqrt{7}, 4\sqrt{7}$ are not rational

$\therefore$ W is not closed under scalar multiplication in $V_3(R)$

$\boxed{\therefore \text{W is not a subspace of } V_3(R)}$

(5)Let V be a vector space of all polynomials in an indeterminate x over the field F. Let W be a subset of V consisting of all polynomials of degree $\leq n$. Then show that W is a subspace of V.

Solution: Given V = {$f(x) = \sum a_i x^i$ / each $a_i \in$ F,F is field} is a vector space over F.

Given W = {$f(x) = \sum_{i=0}^{n} a_i x^i$ / each $a_i \in$ F, deg$f(x) \leq n$}

Clearly W is a non-empty subset of V

Claim: To show that W is a subspace of V. It is enough to show that $a, b \in$ F and $f(x), g(x) \in$ W $\Rightarrow af(x) + bg(x) \in$ W

Let $a, b \in$ F and $f(x), g(x) \in$ W

$\because f(x), g(x) \in$ W $\Rightarrow f(x), g(x)$ are polynomials over F of degree $\leq n$

$\Rightarrow af(x) + bg(x)$ is a polynomial over F of degree $\leq n$

$\Rightarrow af(x) + bg(x) \in$ W

$\therefore a, b \in$ F and $f(x), g(x) \in$ W $\Rightarrow af(x) + bg(x) \in$ W

$\boxed{\therefore \text{W is a subspace of } V_3(R).}$

(6) Let V be the set of all nxn matrices and F be the field. If W is the subset of nxn symmetric matrices in V, then show that W is a subspace of V(F).

Solution: Given V is the set of all nxn matrices whose entries are elements in the field F.

Let W be the subset of nxn symmetric matrices in V.

We know that $O_{nxn} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}_{m\times n}$ is a symmetric matrix.

[$\because$ $(i-j)^{th}$ element of O = $(j-i)^{th}$ element of O]

$\therefore O_{n\times n} \in$ W and hence W $\neq \phi$

Claim: To show that W is a subspace of V. It is enough to show that $a, b \in$ R and A,B $\in$ W $\Rightarrow aA + bB \in$ W.

Let $a, b \in$ F and A, B $\in$ W where A = $[a_{ij}]_{n\times n}$, B = $[b_{ij}]_{n\times n}$

$\because$ A, B $\in$ W $\Rightarrow$ A, B are $n\times n$ symmetric matrices $\Rightarrow a_{ij} = a_{ji}$ and $b_{ij} = b_{ji}$ .... (1)

Now $aA + bB = a[a_{ij}]_{n\times n} + b[b_{ij}]_{n\times n} = [aa_{ij}]_{n\times n} + [bb_{ij}]_{n\times n} = [aa_{ij} + bb_{ij}]_{n\times n}$

$\therefore$ The $(i-j)^{th}$ element of $aA + bB = aa_{ij} + bb_{ij} = aa_{ji} + bb_{ji}$ [From (1)]

$= (j-i)^{th}$ element of $aA + bB$

$\therefore aA + bB$ is $n\times n$ symmetric matrix $\Rightarrow aA + bB \in$ W

$\therefore a, b \in$ F and A, B $\in$ W $\Rightarrow aA + bB \in$ W

$\therefore$ W is a subspace of V.

(7) Let F be a field and A be a $m\times n$ matrix over F. $F_{1\times m}$ is the set of all $1\times m$ matrices defined over F forming the vector space $F_{1\times m}$(F).
Define W = {X = $[x_1, x_2, \ldots\ldots, x_n] \in F_{1\times m}$ / XA = $O_{1\times n}$}. Prove that W is a subspace of $F_{1\times m}$(F).

Solution: Given A is a $m\times n$ matrix over the field F.

Given $F_{1\times m}$ is the set of all $1\times m$ matrices over D forming the vector space $F_{1\times m}$(F).

Define W = {X = $[x_1, x_2, \ldots\ldots, x_n] \in F_{1\times m}$ / XA = $O_{1\times n}$}

Clearly $O_{1\times m} \in$ W [$\because O_{1\times m}A = O_{1\times n}$]. Hence W $\neq \phi$

$\therefore$ W is a non-empty subset of $F_{1\times m}$

Claim: To show that W is a subspace of $F_{1\times m}$. It is enough to show that $a, b \in$ F and X, Y $\in$ W $\Rightarrow aX + bY \in$ W.

Let $a, b \in$ F and X, Y $\in$ W.

$\because$ X, Y $\in$ W $\Rightarrow$ XA = $O_{1\times n}$ and YA = $O_{1\times n}$

$\therefore (aX+bY)A = (aX)A+(bY)A = a(XA)+b(YA) = aO_{1\times n} + bO_{1\times n} = O_{1\times n} + O_{1\times n} = O_{1\times n}$

$\therefore aX+bY \in$ W $\forall\ a, b \in$ F and X, Y $\in$ W

$\therefore$ W is a subspace of $F_{1\times m}$

LINEAR COMBINATION OF VECTORS

Definition: Let V(F) be a vector space. If $\alpha_1, \alpha_2, \ldots, \alpha_n \in V$ then any vector $\gamma = a_1\alpha_1 + a_2\alpha_2 + \ldots + a_n\alpha_n$ where $a_1, a_2, \ldots, a_n \in F$ is called a linear combination of the vectors $\alpha_1, \alpha_2, \ldots, \alpha_n$.

Note: $\gamma = a_1\alpha_1 + a_2\alpha_2 + \ldots + a_n\alpha_n \in V(F)$

LINEAR SPAN OF A SET

Definition: Let S be a non-empty subset of a vector space V(F). The linear span of S is the set of all possible linear combination of all possible finite subsets of S. It is denoted by L(S). **[ANU S96, J08, J11,N19**

$\therefore L(S) = \{\gamma \in V \,/\, \gamma = a_1\alpha_1 + a_2\alpha_2 + \ldots + a_n\alpha_n \,,\, a_1, a_2, \ldots, a_n \in F, \alpha_1, \alpha_2, \ldots, \alpha_n \in S\}$

$= \{\gamma \in V \,/\, \gamma = \sum a_i\alpha_i \,,\, a_i \in F \;\&\; \alpha_i \in S\}$

Note: 1. If $x \in L(S)$, then x = linear combination of elements of S.

2. S may be a finite set but L(S) is infinite.

3. L(S) is said to be generated or spanned by S

4. If S is an empty subset of V, then $L(S) = \{\bar{0}\}$

5. $S \subseteq L(S)$

Ex.1. *Express the vector* $\alpha = (1, -2, 5)$ *as a linear combination of the vectors* $e_1 = (1,1,1), e_2 = (1,2,3)$ *and* $e_3 = (2, -1, 1)$

Sol: Let $\alpha = (1, -2, 5) = x\,(1,1,1) + y\,(1,2,3) + z\,(2, -1, 1)$

$$= (x + y + 2z,\ x + 2y - z,\ x + 3y + z)$$

Hence $x + y + 2z = 1,\ x + 2y - z = -2,\ x + 3y + z = 5$

Reducing to echelon form we get $\alpha + \beta + 2\gamma = 1,\ \beta - 3\gamma = -3,\ 5\gamma = 10$

These equations are consistant and have a solution given by $\alpha = -6, \beta = 3, \gamma = 2$

ALGEBRA OF SUBSPACES

Theorem: The intersection of any two subspaces W_1 and W_2 of a vector space V(F) is also a subspace (OR) If W_1 and W_2 are two subspaces of a vector space V(F), then $W_1 \cap W_2$ is also a subspace of V(F). [ANU S02, M06, M09, M10

Proof: Let W_1 and W_2 be two subspaces of a vector space V(F).

$\Rightarrow W_1 \subseteq V,\ W_2 \subseteq V$ and $\bar{0} \in W_1,\ \bar{0} \in W_2 \Rightarrow W_1 \cap W_2 \subseteq V$ and $\bar{0} \in W_1 \cap W_2$

$\therefore W_1 \cap W_2$ is a non-empty subset of V.

Claim: To prove that $W_1 \cap W_2$ is a subspace of V. It is enough to prove that $a, b \in F$ and $\alpha, \beta \in W_1 \cap W_2 \Rightarrow a\alpha + b\beta \in W_1 \cap W_2$.

Let $a, b \in F$ and $\alpha, \beta \in W_1 \cap W_2$

$\because \alpha, \beta \in W_1 \cap W_2 \Rightarrow \alpha, \beta \in W_1$ and $\alpha, \beta \in W_2$

$\because a, b \in F$ and $\alpha, \beta \in W_1 \Rightarrow a\alpha + b\beta \in W_1$ [$\because W_1$ is a subspace of V]

$\because a, b \in F$ and $\alpha, \beta \in W_2 \Rightarrow a\alpha + b\beta \in W_2$ [$\because W_2$ is a subspace of V]

$\because a\alpha + b\beta \in W_1$ and $a\alpha + b\beta \in W_2 \Rightarrow a\alpha + b\beta \in W_1 \cap W_2$

$$\boxed{\therefore a, b \in F \text{ and } \alpha, \beta \in W_1 \cap W_2 \Rightarrow a\alpha + b\beta \in W_1 \cap W_2}$$

$\therefore W_1 \cap W_2$ is a subspace of V.

Important note: The union of two subspaces of a vector space V(F) need not be a subspace of V(F).

For example $W_1 = \{(0, x, 0) / x \in R\}$ and $W_2 = \{(0, 0, y) / y \in R\}$ are two subspaces of $V_3(R)$.

$\therefore W_1 \cup W_2 = \{(0, x, 0), (0, 0, y) / x, y \in R\}$.

Let $\alpha = (0, 2, 0) \in W_1$, $\beta = (0, 0, 3) \in W_2$. Then $\alpha, \beta \in W_1 \cup W_2$

But $\alpha + \beta = (0, 2, 0) + (0, 0, 3) = (0, 2, 3) \notin W_1 \cup W_2$

$\therefore W_1 \cup W_2$ is not a subspace of $V_3(R)$.

Theorem: The union of two subspaces is a subspace iff one is contained in the other. (OR) If W_1 and W_2 are two subspaces of a vector space V(F), then $W_1 \cup W_2$ is also a subspace of V iff $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$. [ANU O92, S97, S01, J05, M06, J06, J07, M08, M10, J13,

Proof: Let W_1 and W_2 be two subspaces of a vector space V(F).

Suppose $W_1 \cup W_2$ is subspace of V(F).

Claim: To prove that $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$.

If possible let $W_1 \nsubseteq W_2$ and $W_2 \nsubseteq W_1$

$\because W_1 \nsubseteq W_2 \Rightarrow \exists\, x \in W_1$ and $x \notin W_2$

$\because W_2 \nsubseteq W_1 \Rightarrow \exists\, y \in W_2$ and $y \notin W_1$

$\because x \in W_1, y \in W_2 \Rightarrow x, y \in W_1 \cup W_2 \Rightarrow x + y \in W_1 \cup W_2$ [$\because W_1 \cup W_2$ is subspace of V]

$\Rightarrow x + y \in W_1$ or $x + y \in W_2$

If $x + y \in W_1$, then $x + y, x \in W_1 \Rightarrow (x + y) - x \in W_1$ [$\because W_1$ is a subspace of V]

$\Rightarrow y \in W_1$

It is contradiction to $y \notin W_1$.

If $x + y \in W_2$, then $x + y, y \in W_2 \Rightarrow (x + y) - y \in W_2$ [$\because W_1$ is a subspace of V]

$\Rightarrow x \in W_2$

It is contradiction to $x \notin W_2$.

$\therefore$ Our assumption is wrong. Hence either $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$

Conversely suppose that $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$.

Claim: To prove that $W_1 \cup W_2$ is subspace of V(F).

If $W_1 \subseteq W_2 \Rightarrow W_1 \cup W_2 = W_2$, is a subspace of V

If $W_2 \subseteq W_1 \Rightarrow W_1 \cup W_2 = W_1$, is a subspace of V

$\therefore W_1 \cup W_2$ is subspace of V(F).

Theorem: The intersection of any family of subspaces of a vector space is also a subspace.

Proof: Let $\{W_i\}_{i \in I}$ be a family of subspaces of a vector space V(F) and $W=\bigcap_{i\in I} w_i$.

Then each $W_i \subseteq V$ and $\bar{0} \in W_i\ \forall\ i \in I \Rightarrow \bigcap_{i\in I} w_i \subseteq V$ and $\bar{0} \in \bigcap_{i\in I} w_i \Rightarrow W \subseteq V$ and $\bar{0} \in W$

$\therefore$ W is a non-empty subset of V

Claim: To prove that W is a subspace of V. It is enough to prove that $a, b \in F$ and $\alpha, \beta \in W \Rightarrow a\alpha + b\beta \in W$

Let $a, b \in F$ and $\alpha, \beta \in W \Rightarrow a, b \in F$ and $\alpha, \beta \in \bigcap_{i\in I} w_i$

$\Rightarrow a, b \in F$ and $\alpha, \beta \in W_i\ \forall\ i \in I$

$\Rightarrow a\alpha + b\beta \in W_i\ \forall\ i \in I$ [$\because$ each W_i is a subspace of V]

$\Rightarrow a\alpha + b\beta \in \bigcap_{i\in I} w_i$

$\Rightarrow a\alpha + b\beta \in W$

$\therefore a, b \in F$ and $\alpha, \beta \in W \Rightarrow a\alpha + b\beta \in W$ $\therefore$ W is a subspace of V.

Hence the intersection of any family of subspaces of a vector space V is also a subspace of V.

Subspace generated by a subset:- Let V(F) be a vector space and $S \subseteq V$. A subspace W of V containing S is said to be smallest subspace of V containing S or the subspace of V generated by S or spanned by S if U is any subspace of V containing S then $W \subseteq U$. It is denoted by {S} or (S) or <S>.

Note: 1. Let V(F) be a vector space and $S \subseteq V$. It can be easily seen that the intersection of all subspaces of V(F) containing S is the subspaces of V generated by S.

2. Let V(F) be a vector spaces and $S \subseteq V$. If V = <S>, then we say that V is spanned by S.

LINEAR SUM OF TWO SUBSPACES

Definition: Let W_1 and W_2 be two subspaces of a vector space V(F). Then the set $\{\alpha_1 + \alpha_2 / \alpha_1 \in W_1, \alpha_2 \in W_2\}$ is called the linear sum of two subspaces W_1 and W_2. It is denoted by $W_1 + W_2$.

$$\therefore W_1 + W_2 = \{\alpha_1 + \alpha_2 / \alpha_1 \in W_1, \alpha_2 \in W_2\}$$

Theorem: If W_1 and W_2 are any two subspaces of a vector space V(F), then (i) $W_1 + W_2$ is a subspace of V (ii) $W_1 \subseteq W_1 + W_2$ and $W_2 \subseteq W_1 + W_2$. [M2013]

Proof: Let W_1 and W_2 be two subspaces of a vector space $V(F)$.

By the def $W_1 + W_2 = \{\alpha_1 + \alpha_2 / \alpha_1 \in W_1, \alpha_2 \in W_2\} \subseteq V$

$\because \bar{0} \in W_1$ and $\bar{0} \in W_2 \Rightarrow \bar{0} + \bar{0} = \bar{0} \in W_1 + W_2$

$\therefore W_1 + W_2$ is a non-empty subset of V.

Claim *i*: To prove that $W_1 + W_2$ is a subspace of V. It is enough to prove that $a, b \in F$ and $\alpha, \beta \in W_1 + W_2 \Rightarrow a\alpha + b\beta \in W_1 + W_2$.

Let $a, b \in F$ and $\alpha, \beta \in W_1 + W_2$

$\because \alpha \in W_1 + W_2 \Rightarrow \alpha = \alpha_1 + \alpha_2$ where $\alpha_1 \in W_1$ and $\alpha_2 \in W_2$

$\because \beta \in W_1 + W_2 \Rightarrow \beta = \beta_1 + \beta_2$ where $\beta_1 \in W_1$ and $\beta_2 \in W_2$

$\because a, b \in F$ and $\alpha_1, \beta_1 \in W_1 \Rightarrow a\alpha_1 + b\beta_1 \in W_1$ [$\because W_1$ is a subspace of V] .... (1)

$\because a, b \in F$ and $\alpha_2, \beta_2 \in W_2 \Rightarrow a\alpha_2 + b\beta_2 \in W_2$ [$\because W_2$ is a subspace of V] .... (2)

Now $a\alpha + b\beta = a(\alpha_1 + \alpha_2) + b(\beta_1 + \beta_2) = (a\alpha_1 + a\alpha_2) + (b\beta_1 + b\beta_2)$

$= (a\alpha_1 + b\beta_1) + (a\alpha_2 + b\beta_2)$

$\in W_1 + W_2$ [From (1) & (2)]

$\boxed{\therefore a, b \in F \text{ and } \alpha, \beta \in W_1 + W_2 \Rightarrow a\alpha + b\beta \in W_1 + W_2}$

$\therefore W_1 + W_2$ is a subspace of V.

Claim *ii*: To prove that $W_1 \subseteq W_1 + W_2$ and $W_2 \subseteq W_1 + W_2$.

$\because \alpha_1 \in W_1$ and $\bar{0} \in W_2 \Rightarrow \alpha_1 + \bar{0} \in W_1 + W_2 \Rightarrow \alpha_1 \in W_1 + W_2$

$\therefore \alpha_1 \in W_1 \Rightarrow \alpha_1 \in W_1 + W_2$ $\therefore W_1 \subseteq W_1 + W_2$

Similarly $W_2 \subseteq W_1 + W_2$

Problem: Let W_1 and W_2 be subspaces of a vector space V such that $W_1 + W_2 = V$ and $W_1 \cap W_2 = \{\bar{0}\}$. Prove that for each vector α in V there are unique vectors $\alpha_1 \in W_1$, $\alpha_2 \in W_2$ such that $\alpha = \alpha_1 + \alpha_2$.

Solution: Given W_1, W_2 are subspaces of a vector space V such that $W_1 + W_2 = V$ and $W_1 \cap W_2 = \{\bar{0}\}$.

Claim: To prove that $\forall\, \alpha \in V\ \exists$ unique vectors $\alpha_1 \in W_1, \alpha_2 \in W_2 \ni \alpha = \alpha_1 + \alpha_2$

Let $\alpha \in V \Rightarrow \alpha \in W_1 + W_2$ [$\because W_1 + W_2 = V$]

$\Rightarrow \alpha = \alpha_1 + \alpha_2$ where $\alpha_1 \in W_1$ and $\alpha_2 \in W_2$

If possible let $\alpha = \beta_1 + \beta_2$ where $\beta_1 \in W_1$ and $\beta_2 \in W_2$

$\therefore \alpha_1 + \alpha_2 = \beta_1 + \beta_2 \Rightarrow \alpha_1 - \beta_1 = \beta_2 - \alpha_2$

$\because \alpha_1 \in W_1, \beta_1 \in W_1 \Rightarrow \alpha_1 - \beta_1 \in W_1$ [$\because$ W_1 is a subspace of V]

$\because \alpha_2 \in W_2, \beta_2 \in W_2 \Rightarrow \beta_2 - \alpha_2 \in W_2$ [$\because$ W_2 is a subspace of V]

$\Rightarrow \alpha_1 - \beta_1 \in W_2$ [$\because \alpha_1 - \beta_1 = \beta_2 - \alpha_2$]

$\therefore \alpha_1 - \beta_1 \in W_1$ and $\alpha_1 - \beta_1 \in W_2 \Rightarrow \alpha_1 - \beta_1 \in W_1 \cap W_2$ and hence $\beta_2 - \alpha_2 \in W_1 \cap W_2$

$\Rightarrow \alpha_1 - \beta_1 = \bar{0}$ and $\beta_2 - \alpha_2 = \bar{0}$ [$\because W_1 \cap W_2 = \{\bar{0}\}$]

$\Rightarrow \alpha_1 = \beta_1$ and $\beta_2 = \alpha_2$

Hence $\alpha = \alpha_1 + \alpha_2$ is unique

LINEAR COMBINATION OF VECTORS

Definition: Let V(F) be a vector space. If $\alpha_1, \alpha_2, \ldots, \alpha_n \in V$ then any vector $\gamma = a_1\alpha_1 + a_2\alpha_2 + \ldots + a_n\alpha_n$ where $a_1, a_2, \ldots, a_n \in F$ is called a linear combination of the vectors $\alpha_1, \alpha_2, \ldots, \alpha_n$.

Note: $\gamma = a_1\alpha_1 + a_2\alpha_2 + \ldots + a_n\alpha_n \in V(F)$

LINEAR SPAN OF A SET

Definition: Let S be a non-empty subset of a vector space V(F). The linear span of S is the set of all possible linear combination of all possible finite subsets of S. It is denoted by L(S). **[ANU S96, J08, J11,N19**

$\therefore L(S) = \{\gamma \in V / \gamma = a_1\alpha_1 + a_2\alpha_2 + \ldots + a_n\alpha_n, a_1, a_2, \ldots, a_n \in F, \alpha_1, \alpha_2, \ldots, \alpha_n \in S\}$

$= \{\gamma \in V / \gamma = \sum a_i\alpha_i , a_i \in F \,\&\, \alpha_i \in S\}$

Note: 1. If $x \in L(S)$, then x = linear combination of elements of S.

2. S may be a finite set but L(S) is infinite.

3. L(S) is said to be generated or spanned by S

4. If S is an empty subset of V, then $L(S) = \{\bar{0}\}$

5. $S \subseteq L(S)$

Theorem: Let S be a non-empty subset of a vector space V(F), then L(S) is a subspace of V. [ANU M2012, M2017, O18,N19

Proof: Let S be a non-empty subset of a vector space V(F).

Then by the def, $L(S) = \{\gamma \in V / \gamma = \sum a_i\alpha_i , a_i \in F \,\&\, \alpha_i \in S\}$

Clearly $L(S) \subseteq V$ and $L(S) \neq \phi$ [$\because$ S is non-empty]

Claim: To prove that L(S) is a subspace of V. It is enough to prove that $a, b \in F$ and $\alpha, \beta \in L(S) \Rightarrow a\alpha + b\beta \in L(S)$.

Let $a, b \in F$ and $\alpha, \beta \in L(S)$

$\because \alpha \in L(S) \Rightarrow \alpha = a_1\alpha_1 + a_2\alpha_2 + \ldots + a_m\alpha_m$ where $a_1, a_2, \ldots, a_m \in F, \alpha_1, \alpha_2, \ldots, \alpha_m \in S$

$\because \beta \in L(S) \Rightarrow \beta = b_1\beta_1 + b_2\beta_2 + \ldots + b_n\beta_n$ where $b_1, b_2, \ldots, b_n \in F, \beta_1, \beta_2, \ldots, \beta_n \in S$

Now $a\alpha + b\beta = a(a_1\alpha_1 + a_2\alpha_2 + \ldots + a_m\alpha_m) + b(b_1\beta_1 + b_2\beta_2 + \ldots + b_n\beta_n)$

$= (aa_1)\alpha_1 + (aa_2)\alpha_2 + \ldots + (aa_m)\alpha_m + (bb_1)\beta_1 + (bb_2)\beta_2 + \ldots + (bb_n)\beta_n$

= Linear combination of elements of $S \in L(S)$

$\boxed{\therefore a, b \in F \text{ and } \alpha, \beta \in L(S) \Rightarrow a\alpha + b\beta \in L(S)}$

$\therefore$ L(S) is a subspace of V

Theorem: If S is a non-empty subset of a vector space V(F), then L(S) = <S> *i.e* L(S) is equal to the subspace generated by S.

(OR)

If S is a non-empty subset of a vector space V(F), then the linear span L(S) is the intersection of all subspaces of V which contain S *i.e* L(S) is the smallest subspace of V containing S. [S96, J08, M02, J11, M12

Proof: Let S be a non-empty subset of a vector space V(F).

Then by the def, $L(S) = \{\gamma \in V / \gamma = \sum a_i\alpha_i, a_i \in F \;\&\; \alpha_i \in S\}$

Clearly $L(S) \subseteq V$ and $L(S) \neq \phi$ [$\because$ S is non-empty]

Claim: To prove that L(S) = <S>. It is enough to prove that L(S) is the smallest subspace of V containing S.

<u>To prove that L(S) is a subspace of V</u>: Let $a, b \in F$ and $\alpha, \beta \in L(S)$

$\because \alpha \in L(S) \Rightarrow \alpha = a_1\alpha_1 + a_2\alpha_2 + \ldots + a_m\alpha_m$ where $a_1, a_2, \ldots, a_m \in F, \alpha_1, \alpha_2, \ldots, \alpha_m \in S$

$\because \beta \in L(S) \Rightarrow \beta = b_1\beta_1 + b_2\beta_2 + \ldots + b_n\beta_n$ where $b_1, b_2, \ldots, b_n \in F, \beta_1, \beta_2, \ldots, \beta_n \in S$

Now $a\alpha + b\beta = a(a_1\alpha_1 + a_2\alpha_2 + \ldots + a_m\alpha_m) + b(b_1\beta_1 + b_2\beta_2 + \ldots + b_n\beta_n)$

$= (aa_1)\alpha_1 + (aa_2)\alpha_2 + \ldots + (aa_m)\alpha_m + (bb_1)\beta_1 + (bb_2)\beta_2 + \ldots + (bb_n)\beta_n$

= Linear combination of elements of $S \in L(S)$

$\boxed{\therefore a, b \in F \text{ and } \alpha, \beta \in L(S) \Rightarrow a\alpha + b\beta \in L(S)}$

$\therefore$ L(S) is a subspace of V

<u>TO prove that $S \subseteq L(S)$</u>: Let $\alpha \in S$

Now $\alpha = 1\cdot\alpha$ = Linear combination of elements of S $\in$ L(S)

$\therefore \alpha \in S \Rightarrow \alpha \in L(S)$ $\qquad \therefore S \subseteq L(S)$

To prove that L(S) is the smallest subspace of V containing S:

Let W be a subspace of V containing S.

Now we prove that $L(S) \subseteq W$.

Let $\alpha \in L(S) \Rightarrow \alpha$ = Linear combination of elements of S

$$\Rightarrow \alpha = a_1\alpha_1 + a_2\alpha_2 + \ldots + a_m\alpha_m \text{ where } a_1, a_2, \ldots, a_m \in F,\ \alpha_1, \alpha_2, \ldots, \alpha_m \in S$$

$\because \alpha_1, \alpha_2, \ldots, \alpha_m \in S$ and $S \subseteq W \Rightarrow \alpha_1, \alpha_2, \ldots, \alpha_m \in W$

$$\Rightarrow a_1\alpha_1 + a_2\alpha_2 + \ldots + a_m\alpha_m \in W \ [\because W \text{ is a subspace of } V]$$

$$\Rightarrow \alpha \in W$$

$\therefore \alpha \in L(S) \Rightarrow \alpha \in W$ $\qquad \therefore L(S) \subseteq W$

$\therefore$ L(S) is the smallest subspace of V containing S. Hence L(S) =<S>

Theorem: If S is a non-empty subset of a vector space V(F), then (*i*) S is a subspace of V $\Leftrightarrow$ L(S) = S [ANU M12, M15, J15,N19
(*ii*) L(L(S)) = L(S) [ANU S98, J11, J14, M15, J15

Proof: Let S be a non-empty subset of a vector space V(F).

(*i*) Suppose S is a subspace of V.

Claim: To prove that L(S) = S *i.e* To prove that L(S) $\subseteq$ S and S $\subseteq$ L(S)

Let $\alpha \in S$

Then $\alpha = 1\cdot\alpha$ = Linear combination of elements of S $\in$ L(S)

$\therefore \alpha \in S \Rightarrow \alpha \in L(S)$ $\qquad \therefore S \subseteq L(S)$ .... (1)

Let $\beta \in L(S) \Rightarrow \beta$ = Linear combination of elements of S

$$\Rightarrow \beta = b_1\beta_1 + b_2\beta_2 + \ldots + b_n\beta_n \text{ where } b_1, b_2, \ldots, b_n \in F,\ \beta_1, \beta_2, \ldots, \beta_n \in S$$

$\because \beta_1, \beta_2, \ldots, \beta_n \in S$ and S is a subspace of V $\Rightarrow b_1\beta_1 + b_2\beta_2 + \ldots + b_n\beta_n \in S$

$$\Rightarrow \beta \in S$$

$\therefore \beta \in L(S) \Rightarrow \beta \in S$ $\qquad \therefore L(S) \subseteq S$ .... (2)

$\therefore$ From (1) & (2), L(S) = S

Conversely suppose that L(S) = S

Claim: To prove that S is a subspace of V

We know that L(S) is a subspace of V $\Rightarrow$ S is a subspace of V [$\because$ L(S) = S]

(*ii*) We know that L(S) is a subspace of V.

Claim: To prove that L(L(S)) = L(S) *i.e* To P.T L(L(S)) $\subseteq$ L(S), L(S) $\subseteq$ L(L(S))

Let $\alpha \in$ L(S)

Then $\alpha = 1{\cdot}\alpha$ = Linear combination of elements of L(S) $\in$ L(L(S))

$\therefore \alpha \in$ L(S) $\Rightarrow \alpha \in$ L(L(S)) $\therefore$ L(S) $\subseteq$ L(L(S)) .... (1)

Let $\beta \in$ L(L(S)) $\Rightarrow \beta$ = Linear combination of elements of L(S)

$\Rightarrow \beta = b_1\beta_1 + b_2\beta_2 + \ldots\ldots + b_n\beta_n$ where $b_1, b_2, \ldots, b_n \in F$, $\beta_1, \beta_2, \ldots, \beta_n \in$ L(S)

$\because \beta_1, \beta_2, \ldots, \beta_n \in$ L(S) and L(S) is a subspace of V $\Rightarrow b_1\beta_1 + b_2\beta_2 + \ldots\ldots + b_n\beta_n \in$ L(S)

$\Rightarrow \beta \in$ L(S)

$\therefore \beta \in$ L(L(S)) $\Rightarrow \beta \in$ L(S) $\therefore$ L(L(S)) $\subseteq$ L(S) .... (2)

$\therefore$ From (1) & (2), L(L(S)) = L(S)

Theorem: If S and T are two subsets of a vector space V(F), then
(*i*) S $\subseteq$ T $\Rightarrow$ L(S) $\subseteq$ L(T) [ANU S01, J12, M16, O17
(*ii*) S $\subseteq$ L(T) $\Rightarrow$ L(S) $\subseteq$ L(T)

Proof: Let S and T be two subsets of a vector space V(F).

(*i*) Suppose S $\subseteq$ T

Claim: To prove that L(S) $\subseteq$ L(T)

Let $\alpha \in$ L(S) $\Rightarrow \alpha$ = Linear combination of elements of S

$\Rightarrow \alpha = a_1\alpha_1 + a_2\alpha_2 + \ldots\ldots + a_n\alpha_n$ where $a_1, a_2, \ldots, a_n \in F$, $\alpha_1, \alpha_2, \ldots, \alpha_n \in S$

$\because \alpha_1, \alpha_2, \ldots, \alpha_n \in$ S and S $\subseteq$ T $\Rightarrow \alpha_1, \alpha_2, \ldots, \alpha_n \in$ T

$\therefore \alpha = a_1\alpha_1 + a_2\alpha_2 + \ldots\ldots + a_n\alpha_n$ = Linear combination of elements of T $\in$ L(T)

$\therefore \alpha \in$ L(S) $\Rightarrow \alpha \in$ L(T) $\therefore$ L(S) $\subseteq$ L(T)

(*ii*) Suppose S $\subseteq$ L(T)

Claim: To prove that L(S) $\subseteq$ L(T)

$\because$ S $\subseteq$ L(T) $\Rightarrow$ L(S) $\subseteq$ L(L(T)) [From(*i*)]

$\Rightarrow$ L(S) $\subseteq$ L(T) [$\because$ L(L(T)) = L(T)]

Theorem: If W_1 and W_2 are any two subspaces of a vector space V(F), then $W_1 + W_2$ is a subspace of V and $W_1 + W_2 = \langle W_1 \cup W_2 \rangle = L(W_1 \cup W_2)$
[ANU M94, M97, J08, S09, J12, M13, O18

Proof: Let W_1 and W_2 be two subspaces of a vector space V(F).

By the def $W_1 + W_2 = \{\alpha_1 + \alpha_2 / \alpha_1 \in W_1, \alpha_2 \in W_2\} \subseteq V$

$\because \bar{0} \in W_1$ and $\bar{0} \in W_2 \Rightarrow \bar{0} + \bar{0} = \bar{0} \in W_1 + W_2$

$\therefore W_1 + W_2$ is a non-empty subset of V.

Claim: To prove that $W_1 + W_2$ is a subspace of V and $W_1 + W_2 = \langle W_1 \cup W_2 \rangle$. It is enough to prove that $W_1 + W_2$ is a smallest subspace of V containing $W_1 \cup W_2$.

<u>To prove that $W_1 + W_2$ is a subspace of V</u>: Let $a, b \in F$ and $\alpha, \beta \in W_1 + W_2$

$\because \alpha \in W_1 + W_2 \Rightarrow \alpha = \alpha_1 + \alpha_2$ where $\alpha_1 \in W_1$ and $\alpha_2 \in W_2$

$\because \beta \in W_1 + W_2 \Rightarrow \beta = \beta_1 + \beta_2$ where $\beta_1 \in W_1$ and $\beta_2 \in W_2$

$\because a, b \in F$ and $\alpha_1, \beta_1 \in W_1 \Rightarrow a\alpha_1 + b\beta_1 \in W_1$ [$\because W_1$ is a subspace of V] .... (1)

$\because a, b \in F$ and $\alpha_2, \beta_2 \in W_2 \Rightarrow a\alpha_2 + b\beta_2 \in W_2$ [$\because W_2$ is a subspace of V] .... (2)

Now $a\alpha + b\beta = a(\alpha_1 + \alpha_2) + b(\beta_1 + \beta_2) = (a\alpha_1 + a\alpha_2) + (b\beta_1 + b\beta_2)$

$= (a\alpha_1 + b\beta_1) + (a\alpha_2 + b\beta_2) \in W_1 + W_2$ [From (1) & (2)]

$\boxed{\therefore a, b \in F \text{ and } \alpha, \beta \in W_1 + W_2 \Rightarrow a\alpha + b\beta \in W_1 + W_2}$

$\therefore W_1 + W_2$ is a subspace of V.

<u>To prove that $W_1 \cup W_2 \subseteq W_1 + W_2$</u>:

$\because \alpha_1 \in W_1$ and $\bar{0} \in W_2 \Rightarrow \alpha_1 + \bar{0} \in W_1 + W_2 \Rightarrow \alpha_1 \in W_1 + W_2$

$\therefore \alpha_1 \in W_1 \Rightarrow \alpha_1 \in W_1 + W_2$ $\therefore W_1 \subseteq W_1 + W_2$

Similarly $W_2 \subseteq W_1 + W_2$ $\therefore W_1 \cup W_2 \subseteq W_1 + W_2$

<u>To prove that $W_1 + W_2$ smallest subspace of V containing $W_1 \cup W_2$</u>:

Let W be any subspace of V containing $W_1 \cup W_2$

$\because W_1 \cup W_2 \subseteq W \Rightarrow W_1 \subseteq W$ and $W_2 \subseteq W$

Let $\gamma \in W_1 + W_2 \Rightarrow \gamma = \alpha + \beta$ where $\alpha \in W_1, \beta \in W_2$

$\because \alpha \in W_1$ and $W_1 \subseteq W \Rightarrow \alpha \in W$

$\because \beta \in W_2$ and $W_2 \subseteq W \Rightarrow \beta \in W$

$\because \alpha \in W, \beta \in W \Rightarrow \alpha + \beta \in W$ [$\because$ W is a subspace]

$\Rightarrow \gamma \in W$

$\therefore \gamma \in W_1 + W_2 \Rightarrow \gamma \in W$ $\therefore W_1 + W_2 \subseteq W$

$\therefore W_1 + W_2$ is a smallest subspace of V containing $W_1 \cup W_2$.

$\therefore W_1 + W_2 = \langle W_1 \cup W_2 \rangle = L(W_1 \cup W_2)$

Theorem: If S and T are two subsets of a vector space V(F), then $L(S \cup T) = L(S) + L(T)$ [ANU S98, J11, J12, M14, M16, O17

Proof: Let S and T be two subsets of a vector space V(F).

Claim: To prove that $L(S \cup T) = L(S) + L(T)$

Let $\gamma \in L(S \cup T)$

$\Rightarrow \gamma =$ Linear combination of elements of $S \cup T$

$= a_1\alpha_1 + a_2\alpha_2 + \ldots.. + a_m\alpha_m + b_1\beta_1 + b_2\beta_2 + \ldots.. + b_n\beta_n$

Where $a_1, a_2, \ldots, a_m, b_1, b_2, \ldots, b_n \in F$ and $\alpha_1, \alpha_2, \ldots, \alpha_m \in S, \beta_1, \beta_2, \ldots, \beta_n \in T$

$\because a_1, a_2, \ldots, a_m \in F$ and $\alpha_1, \alpha_2, \ldots, \alpha_m \in S \Rightarrow a_1\alpha_1 + a_2\alpha_2 + \ldots.. + a_m\alpha_m \in L(S)$

$\because b_1, b_2, \ldots, b_n \in F$ and $\beta_1, \beta_2, \ldots, \beta_n \in T \Rightarrow a_1\beta_1 + a_2\beta_2 + \ldots.. + a_n\beta_n \in L(T)$

$\therefore \gamma = (a_1\alpha_1 + a_2\alpha_2 + \ldots.. + a_m\alpha_m) + (b_1\beta_1 + b_2\beta_2 + \ldots.. + b_n\beta_n) \in L(S) + L(T)$

$\therefore \gamma \in L(S \cup T) \Rightarrow \gamma \in L(S) + L(T)$ $\therefore L(S \cup T) \subseteq L(S) + L(T)$ .... (1)

Let $\gamma \in L(S) + L(T)$

$\Rightarrow \gamma = \alpha + \beta$ where $\alpha \in L(S), \beta \in L(T)$

$\because \alpha \in L(S) \Rightarrow \alpha =$ Linear combination of elements of S

$\because \beta \in L(T) \Rightarrow \beta =$ Linear combination of elements of T

$\therefore \gamma = \alpha + \beta$

$=$ Linear combination of elements of S + Linear combination of elements of T

$=$ Linear combination of elements of $(S \cup T) \in L(S \cup T)$

$\therefore \gamma \in L(S) + L(T) \Rightarrow \gamma \in L(S \cup T)$ $\therefore L(S) + L(T) \subseteq L(S \cup T)$ .... (2)

From (1) & (2), $L(S \cup T) = L(S) + L(T)$

PROBLEMS

(1) Express the vector $\alpha = (1, -2, 5)$ as a linear combination of the vectors $\alpha_1 = (1, 1, 1), \alpha_2 = (1, 2, 3), \alpha_3 = (2, -1, 1)$ [ANU M04,J06,M07,M14,J15,N19

1.28. LINEAR DEPENDENCE OF VECTORS

Definition. *Let V (F) be a vector space. A finite subset* $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$ *of vectors of V is said to be a linearly dependent (L.D.) set if there exist scalars* $\alpha_1, \alpha_2, \dots, \alpha_n \in F$, *not all zero, such that* $a_1\alpha_1 + a_2\alpha_2 + \dots + a_n\alpha_n = \bar{O}$.

1.29. LINEAR INDEPENDENCE OF VECTORS

Definition. Let V (F) be a vector space. A finite subset $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$ of vectors of V is said to be linearly independent (L.I,) if every relation of the form

$$a_1\alpha_1 + a_2\alpha_2 + \dots + a_n\alpha_n = \bar{O},\ a_i\text{'s} \in F$$

$$\Rightarrow a_1 = 0, a_2 = 0, \dots, a_n = 0.$$

PROBLEMS

(1) Show that (1, 2, 1), (3, 1, 5) in R^3 are L.I [ANU S96,

Solution: Let $\alpha = (1, 2, 1)$, $\beta = (3, 1, 5)$ in R^3.

Claim: To show that α, β are L.I. *i.e* To S.T $a\alpha + b\beta = \bar{0} \Rightarrow a = 0, b = 0$

Let $a, b \in R$ such that $a\alpha + b\beta = \bar{0}$

$\Rightarrow a(1, 2, 1) + b(3, 1, 5) = (0, 0, 0) \Rightarrow (a+3b, 2a+b, a+5b) = (0, 0, 0)$

$\Rightarrow a+3b = 0 \rightarrow (1)$, $2a+b = 0 \rightarrow (2)$, $a+5b = 0 \rightarrow (3)$

$(3) - (1) \Rightarrow 2b = 0 \Rightarrow b = 0$ From(1), $a = 0$

$\therefore a = 0, b = 0.$ $\therefore \alpha, \beta$ are L.I

(2) Show that the vectors (1, 3, 2), (1, − 7, − 8), (2, 1, − 1) of $V_3(R)$ are linearly dependent. [ANU S2000, M13, J14

Solution: Let $\alpha = (1, 3, 2)$, $\beta = (1, -7, -8)$, $\gamma = (2, 1, -1)$ in $V_3(R)$

Claim: To S.T α, β, γ are L.D.

Let $a, b, c \in R$ such that $a\alpha + b\beta + c\gamma = \bar{0}$

$\Rightarrow a(1, 3, 2) + b(1, -7, -8) + c(2, 1, -1) = (0, 0, 0)$

$\Rightarrow (a + b + 2c, 3a - 7b + c, 2a - 8b - c) = (0, 0, 0)$

$\Rightarrow a + b + 2c = 0 \rightarrow (1), 3a - 7b + c = 0 \rightarrow (2), 2a - 8b - c = 0 \rightarrow (3)$

Solving (1),(2) & (3)

$(3) - 2(1) \Rightarrow -10b - 5c = 0 \Rightarrow 2b + c = 0 \rightarrow (4)$

$(2) - 3(1) \Rightarrow -10b - 5c = 0 \Rightarrow 2b + c = 0 \rightarrow (5)$

Since (4) & (5) are same, so $c = -2b$ where b is any non-zero real number

Take $b = 1$, then $c = -2$ From (1), $a = -b - 2c = -1 + 4 = 3$

$\therefore a = 3, b = 1, c = -2$ $\therefore \alpha, \beta, \gamma$ are L.D

II[nd] method: Write the coefficient matrix of the equations and reducing to echelon form,

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 3 & -7 & 1 \\ 2 & -8 & -1 \end{bmatrix} \begin{matrix} R_2 \longrightarrow R_2 - 3R_1 \\ R_3 \longrightarrow R_3 - 2R_1 \end{matrix} \sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & -10 & -5 \\ 0 & -10 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & -10 & -5 \\ 0 & -1 & -5 \end{bmatrix} R_3 \longrightarrow R_3 - R_2$$

$$\sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & -10 & -5 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore \rho(A) = 2 \neq$ The no. of unknowns.

$\therefore$ The system of equations have a non-zero solution.

The system is $a + b + 2c = 0 \longrightarrow (4), -10b - 5c = 0 \Rightarrow 2b + c = 0 \longrightarrow (5)$

Take $b = 1$, then $c = -2$ From (1), $a = -b - 2c = -1 + 4 = 3$

$\therefore a = 3, b = 1, c = -2$ $\therefore \alpha, \beta, \gamma$ are L.D

(3) Test the linear independence of (1,1,2,4), (2,−1,−5,2), (1,−1,−4,0) and (2, 1, 1, 6) in $R^4(R)$. [ANU J11, J12, M17

Solution: Let $\alpha = (1, 1, 2, 4)$, $\beta = (2, -1, -5, 2)$, $\gamma = (1, -1, -4, 0)$ and $\delta = (2, 1, 1, 6)$ in $R^4(R)$.

Claim: Test the L.I of $\alpha, \beta, \gamma, \delta$ in $R^4(R)$

Let $a, b, c, d \in R$ such that $a\alpha + b\beta + c\gamma + d\delta = \bar{0}$

$\Rightarrow a(1, 1, 2, 4) + b(2, -1, -5, 2) + c(1, -1, -4, 0) + d(2, 1, 1, 6) = (0, 0, 0, 0)$

$\Rightarrow (a + 2b + c + 2d, a - b - c + 2d, 2a - 5b - 4c + d, 4a + 2b + 6d) = (0, 0, 0, 0)$

$\Rightarrow a + 2b + c + 2d = 0 \longrightarrow (1)$, $a - b - c + d = 0 \longrightarrow (2)$, $2a - 5b - 4c + d = 0 \longrightarrow (3)$,

$4a + 2b + 6d = 0 \longrightarrow (4)$

Write the coefficient matrix of the equations and reducing to echelon form,

$$A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 1 & -1 & -1 & 1 \\ 2 & -5 & -4 & 1 \\ 4 & 2 & 0 & 6 \end{bmatrix} \begin{matrix} R_2 \longrightarrow R_2 - R_1 \\ R_3 \longrightarrow R_3 - 2R_1 \\ R_4 \longrightarrow R_4 - 4R_1 \end{matrix} \sim \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & -3 & -2 & -1 \\ 0 & -9 & -6 & -3 \\ 0 & -6 & -4 & -2 \end{bmatrix} \begin{matrix} R_3 \longrightarrow R_3 - 3R_2 \\ R_4 \longrightarrow R_4 - 2R_2 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & -3 & -2 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore \rho(A) = 2 \neq$ The no. of unknowns.

$\therefore$ The system of equations have a non-zero solution.

The system is $a + 2b + c + 2d = 0 \longrightarrow (5)$, $-3b - 2c - d = 0 \longrightarrow (6)$

Take $b = 1$, $c = 1$, then $d = -5$ From (1), $a = -2b - c - 2d = -3 + 10 = 7$

$\therefore a = 7, b = 1, c = 1, d = -5$ $\therefore \alpha, \beta, \gamma, \delta$ are L.D

(4) Examine the following vectors are L.I or L.D (1, 2, 1, −1), (0, 1, −1, 2), (2, 1, 0, 3), (1, 1, 0, 0) [ANU J12

Solution: Let $\alpha = (1, 2, 1, -1)$, $\beta = (0, 1, -1, 2)$, $\gamma = (2, 1, 0, 3)$ and $\delta = (1, 1, 0, 0)$ in $R^4(R)$.

Claim: Test the L.I of $\alpha, \beta, \gamma, \delta$ in $R^4(R)$

Let $a, b, c, d \in R$ such that $a\alpha + b\beta + c\gamma + d\delta = \bar{0}$

$\Rightarrow a(1, 2, 1, -1) + b(0, 1, -1, 2) + c(2, 1, 0, 3) + d(1, 1, 0, 0) = (0, 0, 0, 0)$

$\Rightarrow (a + 2c + d, 2a + b + c + d, a - b, -a + 2b + 3c) = (0, 0, 0, 0)$

$\Rightarrow a + 2c + d = 0 \longrightarrow (1)$, $2a + b + c + d = 0 \longrightarrow (2)$, $a - b = 0 \longrightarrow (3)$,

$-a + 2b + 3c = 0 \longrightarrow (4)$

Write the coefficient matrix of the equations and reducing to echelon form,

$$A = \begin{bmatrix} 1 & 0 & 2 & 1 \\ 2 & 1 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ -1 & 2 & 3 & 0 \end{bmatrix} \begin{matrix} R_2 \longrightarrow R_2 - 2R_1 \\ R_3 \longrightarrow R_3 - R_1 \\ R_4 \longrightarrow R_4 + R_1 \end{matrix} \sim \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & -1 & -2 & -1 \\ 0 & 2 & 5 & 1 \end{bmatrix} \begin{matrix} R_3 \longrightarrow R_3 + R_2 \\ R_4 \longrightarrow R_4 - 2R_2 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & -5 & -2 \\ 0 & 0 & 11 & 3 \end{bmatrix} R_4 \longrightarrow 5R_4 + 11R_3 \sim \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & -5 & -2 \\ 0 & 0 & 0 & 7 \end{bmatrix}$$

$\therefore \rho(A) = 4 =$ The no. of unknowns

$\therefore$ The system of equations have zero solution only.

$\therefore a = 0, b = 0, c = 0, d = 0$ $\therefore \alpha, \beta, \gamma, \delta$ are L I

(5) Examine the following vectors are L.I or L.D (1, 2, −1, 1), (0, 1, −1, 2), (2, 1, 0, 3), (1, 1, 0, 0)

Solution: Let $\alpha = (1, 2, -1, -1)$, $\beta = (0, 1, -1, 2)$, $\gamma = (2, 1, 0, 3)$ and $\delta = (1, 1, 0, 0)$ in $R^4(R)$.

Claim: Test the L.I or L.D of $\alpha, \beta, \gamma, \delta$ in $R^4(R)$

Let $a, b, c, d \in R$ such that $a\alpha + b\beta + c\gamma + d\delta = \bar{0}$

$\Rightarrow a(1, 2, -1, 1) + b(0, 1, -1, 2) + c(2, 1, 0, 3) + d(1, 1, 0, 0) = (0, 0, 0, 0)$

$\Rightarrow (a + 2c + d, 2a + b + c + d, -a - b, a + 2b + 3c) = (0, 0, 0, 0)$

$\Rightarrow a + 2c + d = 0 \longrightarrow (1)$, $2a + b + c + d = 0 \longrightarrow (2)$, $-a - b = 0 \longrightarrow (3)$,

$a + 2b + 3c = 0 \longrightarrow (4)$

Write the coefficient matrix of the equations and reducing to echelon form,

$$A = \begin{bmatrix} 1 & 0 & 2 & 1 \\ 2 & 1 & 1 & 1 \\ -1 & -1 & 0 & 0 \\ 1 & 2 & 3 & 0 \end{bmatrix} \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 + R_1 \\ R_4 \to R_4 - R_1 \end{matrix} \sim \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & -1 & 2 & 1 \\ 0 & 2 & 1 & -1 \end{bmatrix} \begin{matrix} R_3 \to R_3 + R_2 \\ R_4 \to R_4 - 2R_2 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 7 & 1 \end{bmatrix} R_4 \to R_4 + 7R_3 \sim \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & -5 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\therefore \rho(A) = 4 =$ The no. of unknowns

$\therefore$ The system of equations have zero solution only.

$\therefore a = 0, b = 0, c = 0, d = 0$ $\therefore \alpha, \beta, \gamma, \delta$ are L.I

(6) Determine whether the following sets of vectors in $R^3(R)$ are L.D or L.I (*i*){(1, −2, 1), (2, 1, −1), (7, −4, 1)} (*ii*) {(−1, 2, 1), (3, 0, −1), (−5, 4, 3)}; [ANU J13,

Solution: (*i*) Let $S = \{(1, -2, 1), (2, 1, -1), (7, -4, 1)\} \subseteq R^3(R)$

Claim: To show that S is L.I or L.D

Let $a, b, c \in R$ such that $a(1, -2, 1) + b(2, 1, -1) + c(7, -4, 1) = (0, 0, 0)$

$\Rightarrow (a + 2b + 7c, -2a + b - 4c, a - b + c) = (0, 0, 0)$

$\Rightarrow a + 2b + 7c = 0 \rightarrow (1),\ -2a + b - 4c = 0 \rightarrow (2),\ a - b + c = 0 \rightarrow (3)$

Write the coefficient matrix of the equations and reducing to echelon form,

$$A = \begin{bmatrix} 1 & 2 & 7 \\ -2 & 1 & -4 \\ 1 & -1 & 1 \end{bmatrix} \begin{matrix} R_2 \rightarrow R_2 + 2R_1 \\ R_3 \rightarrow R_3 - R_1 \end{matrix} \sim \begin{bmatrix} 1 & 2 & 7 \\ 0 & 5 & 10 \\ 0 & -3 & -6 \end{bmatrix} R_3 \rightarrow 5R_3 + 3R_2 \sim \begin{bmatrix} 1 & 2 & 7 \\ 0 & 5 & 10 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore \rho(A) = 2 \neq$ The no. of unknowns

$\therefore$ The system of equations have a non-zero solution.

$\therefore$ The equations are $a + 2b + 7c = 0 \rightarrow (4),\ 5b + 10c = 0 \Rightarrow b + 2c = 0 \rightarrow (5)$

Take $c = 1$, then from (5), $b = -2$ from (4), $a = -2b - 7c = 4 - 7 = -3$

$\therefore a = -3, b = -2, c = 1$ $\therefore$ S is L.D

(ii) Let $S = \{(-1, 2, 1), (3, 0, -1), (-5, 4, 3)\} \subseteq R^3(R)$

Claim: To show that S is L.I or L.D

Let $a, b, c \in R$ such that $a(-1, 2, 1) + b(3, 0, -1) + c(-5, 4, 3) = (0, 0, 0)$

$\Rightarrow (-a + 3b - 5c, 2a + 4c, a - b + 3c) = (0, 0, 0)$

$\Rightarrow -a + 3b - 5c = 0 \rightarrow (1),\ 2a + 4c = 0 \rightarrow (2),\ a - b + 3c = 0 \rightarrow (3)$

Write the coefficient matrix of the equations and reducing to echelon form,

$$A = \begin{bmatrix} -1 & 3 & -5 \\ 2 & 0 & 4 \\ 1 & -1 & 3 \end{bmatrix} \begin{matrix} R_2 \rightarrow R_2 + 2R_1 \\ R_3 \rightarrow R_3 + R_1 \end{matrix} \sim \begin{bmatrix} -1 & 3 & -5 \\ 0 & 6 & -6 \\ 0 & 2 & -2 \end{bmatrix} R_3 \rightarrow 3R_3 - R_2$$

$$\sim \begin{bmatrix} -1 & 3 & -5 \\ 0 & 6 & -6 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore \rho(A) = 2 \neq$ The no. of unknowns

$\therefore$ The system of equations have a non-zero solution.

$\therefore$ The equations are $-a + 3b - 5c = 0 \rightarrow (4),\ 6b - 6c = 0 \Rightarrow b - c = 0 \rightarrow (5)$

Take $c = 1$, then from (5), $b = 1$ from (4), $a = 3b - 5c = 3 - 5 = -2$

$\therefore a = -2, b = 1, c = 1$ $\therefore$ S is L.D

(7) Show that the vectors (1, 2, 0), (0, 3, 1), (−1, 0, 1) are L.I [ANU J14,

Solution: Let $\alpha = (1, 2, 0), \beta = (0, 3, 1), \gamma = (-1, 0, 1)$

Claim: To S.T α, β, γ are L.I.

Let $a, b, c \in R$ such that $a\alpha + b\beta + c\gamma = \bar{0}$

$\Rightarrow a(1, 2, 0) + b(0, 3, 1) + c(-1, 0, 1) = (0, 0, 0)$

$\Rightarrow (a - c, 2a + 3b, b + c) = (0, 0, 0)$

$\Rightarrow a - c = 0 \rightarrow (1), 2a + 3b = 0 \rightarrow (2), b + c = 0 \rightarrow (3)$

$(1) + (3) \Rightarrow a + b = 0 \rightarrow (4)$ $\quad (2) - 2(4) \Rightarrow b = 0$ $\quad$ from (4) & (3), $a = 0, c = 0$

$\therefore a = 0, b = 0, c = 0$ $\qquad \therefore \alpha, \beta, \gamma$ are L.I

(8) Show that the vectors (1, 0, −1), (2, 1, 3), (−1, 0, 0), (1, 0, 1) are L.D [ANU M96, J13, M14,

Solution: Solution: Let $\alpha = (1, 0, -1), \beta = (2, 1, 3), \gamma = (-1, 0, 0), \delta = (1, 0, 1)$

Claim: To S.T $\alpha, \beta, \gamma, \delta$ are L.D.

Let $a, b, c, d \in R$ such that $a\alpha + b\beta + c\gamma + d\delta = \bar{0}$

$\Rightarrow a(1, 0, -1) + b(2, 1, 3) + c(-1, 0, 0) + d(1, 0, 1) = (0, 0, 0)$

$\Rightarrow (a + 2b - c + d, b, -a + 3b + d) = (0, 0, 0)$

$\Rightarrow a + 2b - c + d = 0 \rightarrow (1), b = 0 \rightarrow (2), -a + 3b + d = 0 \rightarrow (3)$

$\Rightarrow a - c + d = 0, b = 0, -a + d = 0 \Rightarrow 2a - c = 0, b = 0, a = d$

Take $d = 1$, then $a = 1$ and $c = 2$

$\therefore a = 1, b = 0, c = 2, d = 1$ $\qquad \therefore \alpha, \beta, \gamma, \delta$ are L.D

(9) Prove that the four vectors $\alpha = (1, 0, 0), \beta = (0, 1, 0), \gamma = (0, 0, 1), \delta = (1, 1, 1)$ in $V_3(C)$ form L.D set but any three of them are L.I. [ANU M17

Solution: Given $\alpha = (1, 0, 0), \beta = (0, 1, 0), \gamma = (0, 0, 1), \delta = (1, 1, 1)$ in $V_3(C)$

Claim: (*i*) To show that $\alpha, \beta, \gamma, \delta$ are L.D. (*ii*) To show that α, β, γ are L.I

(*i*) Let $a, b, c, d \in C$ such that $a\alpha + b\beta + c\gamma + d\delta = \bar{0}$

$\Rightarrow a(1, 0, 0) + b(0, 1, 0) + c(0, 0, 1) + d(1, 1, 1) = (0, 0, 0)$

$\Rightarrow (a + d, b + d, c + d) = (0, 0, 0)$

$\Rightarrow a + d = 0, b + d = 0, c + d = 0 \Rightarrow a = -d, b = -d, c = -d$

Take $d = 1$, then $a = b = c = -1$ $\qquad \therefore \alpha, \beta, \gamma, \delta$ are L.D

(*ii*) Let $a, b, c \in C$ such that $a\alpha + b\beta + c\gamma = \bar{0}$

$\Rightarrow a(1, 0, 0) + b(0, 1, 0) + c(0, 0, 1) = (0, 0, 0)$

$\Rightarrow (a, b, c) = (0, 0, 0)$

$\Rightarrow a = 0, b = 0, c = 0$ $\therefore \alpha, \beta, \gamma$ are L.I

Similarly we can show that any there of $\alpha, \beta, \gamma, \delta$ are L.I

(10) If α, β, γ are L.I vectors of V(R), then show that $\alpha + \beta, \beta + \gamma, \gamma + \alpha$ are also L.I. [ANU J13, M15, M16

Solution: Let α, β, γ are L.I vectors in V(F).

Claim: To show that $\alpha + \beta, \beta + \gamma, \gamma + \alpha$ are L.I

Let $a, b, c \in$ R such that $a(\alpha + \beta) + b(\beta + \gamma) + c(\gamma + \alpha) = \bar{0}$

$\Rightarrow (a\alpha + a\beta) + (b\beta + b\gamma) + (c\gamma + c\alpha) = \bar{0}$

$\Rightarrow (a + c)\alpha + (a + b)\beta + (b + c)\gamma = \bar{0}$

$\Rightarrow a + c = 0, a + b = 0, b + c = 0$

$\Rightarrow a = 0, b = 0, c = 0$

$\therefore \alpha + \beta, \beta + \gamma, \gamma + \alpha$ are L.I

(11) Show that in the vector space $V_3(R)$, the vectors (1, 2, 3), (−2, 1, 4), $(-1, -\frac{1}{2}, 0)$ form linearly dependent set.[ANU A92

Solution: Let S = $\{(1, 2, 3), (-2, 1, 4), (-1, -\frac{1}{2}, 0)\} \subseteq V_3(R)$

Claim: To show that S is L.D

Let $a(1, 2, 3) + b(-2, 1, 4) + c(-1, -\frac{1}{2}, 0) = (0, 0, 0)$

$\Rightarrow (a - 2b - c, 2a + b - \frac{1}{2}c, 3a + 4b) = (0, 0, 0)$

$\Rightarrow a - 2b - c = 0 \longrightarrow (1), 2a + b - \frac{1}{2}c = 0 \longrightarrow (2), 3a + 4b = 0 \longrightarrow (3)$

$2(2) - (1) \Rightarrow 3a + 4b = 0 \longrightarrow (4)$

$\because$ (3) & (4) are same, So take $b = 3$, then $a = -4$ and from (1), $c = -4 - 6 = -10$

$\therefore a = -4, b = 3, c = 10$ $\therefore$ S is linearly dependent subset of $V_3(R)$.

(12) Show that the vectors (1, 0, −1), (1, 2, 1), (0, −3, 2) are L.I in $V_3(R)$ [ANU M03

Solution: Let $\alpha = (1, 0, -1), \beta = (1, 2, 1), \gamma = (0, -3, 2)$ in $V_3(R)$

Claim: To show that α, β, γ are L.I

Let $a, b, c \in$ R such that $a\alpha + b\beta + c\gamma = \bar{0}$

Basis of a Vector Space :-

A Subset S of a Vector Space $V(F)$ is said to be the basis of V, if

i) S is Linearly Independent

ii) The Linear Span of S is V. ie $L(S) = V$.

* Note :- A Vector Space has more than one basis.

Eg ①. Show that the set of n vectors $e_1 = (1,0,0,\cdots 0)$, $e_2 = (0,1,0,\cdots 0)$, $e_3 = (0,0,1,\cdots 0)$ $\cdots$ $e_n = (0,0,0,\cdots 1)$ is a basis of $V_n(F)$.

Sol:- Let us Consider, $S = \{e_1, e_2, e_3 \cdots e_n\}$

where, $e_1 = (1,0,0,\cdots 0)$, $e_2 = (0,1,0 \cdots 0)$ $\cdots$ $e_n = (0,0,0,\cdots 1)$

1) To verify S is Linearly Independent :-

Consider,

$$a_1e_1 + a_2e_2 + a_3e_3 + \cdots + a_ne_n = \bar{0} \quad \forall\, a_1, a_2, \cdots a_n \in F$$

$$\Rightarrow a_1(1,0,0,\cdots 0) + a_2(0,1,0,\cdots 0) + a_3(0,0,1,\cdots 0) + \cdots + a_n(0,0,0\cdots 1) = \bar{0}.$$

$$\Rightarrow (a_1, a_2, a_3, \cdots a_n) = (0,0,0,\cdots 0).$$

$\therefore a_1 = 0,\ a_2 = 0,\ a_3 = 0, \cdots\ a_n = 0.$

$\therefore S$ is a Linear Independent.

2). To Verify $L(S) = V_n(F)$:-

Let $\alpha = (a_1, a_2, a_3, \cdots a_n) \in V_n(F)$.

Now, $\alpha = (a_1, a_2, a_3, \cdots a_n)$

$= a_1(1,0,0,\cdots 0) + a_2(0,1,0\cdots 0) + a_3(0,0,1,\cdots 0) + \cdots + a_n(0,0,0,\cdots 1)$

$= a_1e_1 + a_2e_2 + a_3e_3 + \cdots + a_ne_n.$

$\therefore \alpha =$ Linear Combination of elements of S.

$\Rightarrow \alpha \in L(S)$.

$\therefore$ If $\alpha \in V_n(F) \Rightarrow \alpha \in L(S)$

$\therefore V_n(F) \subset L(S)$

But, $L(S) \subset V_n(F)$.

$\therefore L(S) = V_n(F)$.

$\therefore S$ is a basis of $V_n(F)$.

Note 1). The Set $S = \{e_1, e_2, e_3, \cdots e_n\}$ is called the Standard basis of $V_n(F)$ (or) F^n.

2). The Set $\{(1,0,0), (0,1,0), (0,0,1)\}$ is the Standard basis of $V_3(R)$ (or) R^3.

3). The Set $\{(1,0), (0,1)\}$ is the Standard basis of $V_2(R)$ (or) R^2.

* Finite Dimensional Vector Space:-

A Vector Space $V(F)$ is said to be finite Dimensional, if it has a finite basis.

(or)

A Vector Space $V(F)$ is said to be finite dimensional, if there is a finite Subset 'S' in V such that $L(S) = V$.

* A Vector Space is not finite dimensional is called an Infinite dimensional vector Space

Eg:- 1) The Vector Space $V_n(F)$ of n - tuples is a finite dimensional Vector Space.

2) The Vector Space $F(x)$ of all polynomials over the field F is not a finite dimensional Vector Space.

Note:- 1) Every finite dimensional Vector Space has a basis.

2) If $V(F)$ is a finite dimensional Vector Space, then any two bases of V have the Same number of elements.

i.e $S_1 = \{\alpha_1, \alpha_2, \cdots \alpha_m\}$, $S_2 = \{\beta_1, \beta_2, \cdots \beta_n\}$ then $m = n$

This is called as Invariance theorem.

* 3). Let $V(F)$ be a finite dimensional Vector Space & $S = \{\alpha_1, \alpha_2, \cdots \alpha_n\}$ a Linearly Independent Subset of V, Then Either 'S' itself a basis of V. (or) It Can be extended to form a basis of V

i.e Every linearly Independent subset of a finite dimensional Vector Space $V(F)$ is either a basis of V (or) It Can be extended to form a basis of V.

**4). Every basis is a Spanning set but Every Spanning set is not a basis.

Dimension of a Vector Space :-

Let $V(F)$ be a finite dimensional Vector space. The number of elements in any basis of V is called the Dimension of V and It is denoted by '$\dim V$'.

Note:- 1) The dimension of zero Vector $\{\bar{0}\}$ is zero.

2) The dimension of a Vector Space $V_n(F)$ is n.

Eg ①. If $S = \{(1,0,0),(0,1,0),(0,0,1)\}$ is a basis of $V_3(R)$.

$\therefore \dim V = 3 =$ no. of elements of S.

Note:- 1) Every set of $(n+1)$ (or) more vectors in an 'n'-dimensional Vector Space is Linearly dependent (L.D)

** 2) Let $V(F)$ be a finite dimensional Vector Space of dimension 'n'. Then any set of 'n' Linearly Independent Vectors in V forms a basis of V.

* 3). If a vector Spanned by a set of 'n' vectors and if $m > n$, then any set of m Vectors is Linearly dependent

4). If a Vector Space Spanned by 'K' Vectors then there are at most 'K' Independent vectors

5). Let $V(S)$ be a finite dimensional Vector Space of dimension 'n'. Let S be a set of n Vectors of V Such that $L(S)=V$ (generates V) then 'S' is a basis of $V(F)$.

problems.

(1). S.T $\alpha_1 = (1,0,0)$, $\alpha_2 = (0,1,0)$, $\alpha_3 = (1,1,1)$ form a basis of $C^3(C)$.

Sol G.T $\alpha_1 = (1,0,0)$, $\alpha_2 = (0,1,0)$, $\alpha_3 = (1,1,1)$.

Let $S = \{\alpha_1, \alpha_2, \alpha_3\} \subset C^3(C)$

We know that $\dim C^3(C) = 3$ i.e no. of elements in S.

Now, to prove that S is a basis of $C^3(C)$, It is enough to prove that S is linearly Independent.

Consider, $a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3 = \bar{0}$ $\forall a_1, a_2, a_3 \in C$.

$\Rightarrow a_1(1,0,0) + a_2(0,1,0) + a_3(1,1,1) = (0,0,0)$

$\Rightarrow (a_1 + a_3, a_2 + a_3, a_3) = (0,0,0)$

$\therefore a_1 + a_3 = 0$ —— (1)

$a_2 + a_3 = 0$ —— (2)

& $a_3 = 0$. —— (3).

On sub: eq (3) in eq (1) & (2), we get

$a_1 = 0$, $a_2 = 0$.

$\therefore a_1 = a_2 = a_3 = 0$.

$\therefore$ S is a L.I subset of $C^3(C)$.

(2). S.T the vectors $(1,2,1)$, $(2,1,0)$ $(1,1,2)$ form a basis for $R^3(R)$.

Sol Let $S = \{\alpha_1, \alpha_2, \alpha_3\} \subset R^3(R)$

where $\alpha_1 = (1,2,1)$, $\alpha_2 = (2,1,0)$, $\alpha_3 = (1,1,2)$.

We know that $\dim R^3(R) = 3$.

Now, to prove that 'S' is linearly Independent.

Consider, $a\alpha_1 + b\alpha_2 + c\alpha_3 = \bar{0}$ $\forall a, b, c \in R$.

$\Rightarrow a(1,2,1) + b(2,1,0) + c(1,1,2) = (0,0,0)$

$\Rightarrow (a+2b+c, 2a+b+c, a+2c) = (0,0,0)$

$\therefore a+2b+c = 0$.

$2a+b+c = 0$

$a+2c = 0$.

The above system of eq's can be written in matrix form as $AX = 0$.

i.e. $\begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & -1 \\ 1 & 0 & 2 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

Consider,

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & -1 \\ 1 & 0 & 2 \end{pmatrix}$$

$R_2 \to R_2 - 2R_1$

$R_3 \to R_3 - R_1$

$$\sim \begin{pmatrix} 1 & 2 & 1 \\ 0 & -3 & -3 \\ 0 & -2 & 1 \end{pmatrix}$$

$R_3 \to 3R_3 - 2R_2$

$$A \sim \begin{pmatrix} 1 & 2 & 1 \\ 0 & -3 & -3 \\ 0 & 0 & 9 \end{pmatrix}$$

$\therefore$ $\rho(A) = 3 =$ no. of variables.

$\therefore$ the given system is consistent so having a zero solution for $a = 0$, $b = 0$, $c = 0$.

$\therefore$ S is L.I subset of $R^3(R)$.

$\therefore$ $\dim R^3(R) = 3$ & S is L.I subset of $R^3(R)$.

$\therefore$ S is a basis of $R^3(R)$.

③. S.T the vectors $(1,1,2), (1,2,5), (5,3,4)$ of $R^3(R)$ do not form a basis of $R^3(R)$.

Soln: Let $S = \{\alpha_1, \alpha_2, \alpha_3\} \subset R^3(R)$

Where $\alpha_1 = (1,1,2)$, $\alpha_2 = (1,2,5)$, $\alpha_3 = (5,3,4)$.

We know that, $\dim R^3(R) = 3$.

To prove that S is not a basis of $R^3(R)$. It is enough to prove that S is linear dependent:

Consider, $a\alpha_1 + b\alpha_2 + c\alpha_3 = \bar{0}$ where $a, b, c \in R$.

$\Rightarrow a(1,1,2) + b(1,2,5) + c(5,3,4) = (0,0,0)$.

$\Rightarrow (a+b+5c,\ a+2b+3c,\ 2a+5b+4c) = (0,0,0)$

$\therefore$ $a+b+5c = 0$.

$a+2b+3c = 0$

$2a+5b+4c = 0$.

The above eq's can be written in matrix form as $AX = 0$.

Consider, $A = \begin{pmatrix} 1 & 1 & 5 \\ 1 & 2 & 3 \\ 2 & 5 & 4 \end{pmatrix}$

$R_2 \to R_2 - R_1$, $R_3 \to R_3 - 2R_1$

$$\sim \begin{pmatrix} 1 & 1 & 5 \\ 0 & 1 & -2 \\ 0 & 3 & -6 \end{pmatrix}$$

$R_3 \to R_3 - 3R_2$

$$A \sim \begin{pmatrix} 1 & 1 & 5 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix}$$

$\therefore$ Rank of $A = 2 \neq$ no. of variables.

$\therefore$ the given system has a non zero solution.

$\therefore$ the set S is linearly dependent.

$\therefore$ S is not a bases of $R^3(R)$.

7) S.T the vectors $\alpha_1 = (1,1,1)$, $\alpha_2 = (-1,1,1)$, $\alpha_3 = (1,0,-1)$ form a bases of $R^3(R)$ and express $(4,5,6)$ in terms of $\alpha_1, \alpha_2, \alpha_3$.

Sol: Let $S = \{\alpha_1, \alpha_2, \alpha_3\}$.

where $\alpha_1 = (1,1,1)$, $\alpha_2 = (-1,1,1)$, $\alpha_3 = (1,0,-1)$.

We know that $R^3(R) = 3$.

1) Now, to prove that S is a bases of $R^3(R)$:-

It is enough to prove that S is L.I.

Consider,

$$a\alpha_1 + b\alpha_2 + c\alpha_3 = \bar{0} \quad \forall\ a, b, c \in R.$$

$$\Rightarrow a(1,1,1) + b(-1,1,1) + c(1,0,-1) = (0,0,0)$$

$$\Rightarrow (a-b+c,\ a+b,\ a+b-c) = (0,0,0)$$

$\therefore$ $a - b + c = 0$ — (1)

$a + b = 0$ — (2)

$a + b - c = 0$ — (3).

on solving eq (1) & (3), we get.

$$\begin{array}{l} a - \cancel{b} + \cancel{c} = 0 \\ a + \cancel{b} - \cancel{c} = 0 \\ \hline a = 0. \end{array}$$

on sub: $a = 0$ in eq (2), we get $b = 0$.

on sub: eq (2) in eq (3), we get $c = 0$.

$\therefore a = b = c = 0$. $\therefore$ S is a L.I subset of $R^3(R)$

$\therefore$ S is a bases of $R^3(R)$.

2) To express that (4, 5, 6) in terms of $\alpha_1, \alpha_2, \alpha_3$:-

Let $\alpha = (4, 5, 6) \in R^3(R)$.

Consider, $\alpha = x\alpha_1 + y\alpha_2 + z\alpha_3$

$\therefore (4, 5, 6) = x(1,1,1) + y(-1,1,1) + z(1,0,-1)$ — (a)

$\Rightarrow (4, 5, 6) = (x-y+z,\ x+y,\ x+y-z)$

$\therefore x-y+z = 4$ — (4)

$x+y = 5$ — (5)

$x+y-z = 6$ — (6).

On solving eq (4) & (6), we get.

$$\begin{array}{r} x - \cancel{y} + \cancel{z} = 4 \\ x + \cancel{y} - \cancel{z} = 6 \\ \hline 2x = 10 \end{array}$$

$\therefore x = 5$.

on sub. $x = 5$ in eq (5), we get

$y = 0$.

on sub. ~~eq~~ (5) in eq (6), we get

$5 - z = 6$.

$z = -1$.

$\therefore$ the values of x, y & z

are $x = 5, y = 0, z = -1$.

$\therefore$ the values of x, y & z are sub. in eq (a), we get

$(4, 5, 6) = 5(1,1,1) + 0(-1,1,1) - 1(1,0,-1)$.

$\therefore \alpha = 5\alpha_1 + 0\alpha_2 - 1\alpha_3$.

This is the req. Linear combination.

H.W.

(6). S.T. the vectors $\alpha_1 = (1,1,1)$, $\alpha_2 = (-1,1,0)$, $\alpha_3 = (1,0,-1)$ form a basis of $R^3(R)$ and express (4, 5, 6) in terms of $\alpha_1, \alpha_2, \alpha_3$.

(7). S.T. {(1,1,1,1), (0,1,1,1), (0,0,1,1), (0,0,0,1)} is a basis of $V_4(R)$. Express $\alpha = (2,3,4,1)$ as a linear combination of these basis.

Soln

Let $S = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\} \in V_4(R)$.

where $\alpha_1 = (1,1,1,1)$, $\alpha_2 = (0,1,1,1)$, $\alpha_3 = (0,0,1,1)$, $\alpha_4 = (0,0,0,1)$

We know that, $\dim V_4(R) = 4$.

1) To prove that S is a basis of $V_4(R)$:-

It is enough to prove that S is linearly independent.

Now, Consider,

$a\alpha_1 + b\alpha_2 + c\alpha_3 + d\alpha_4 = \bar{0}$ $\forall a, b, c, d \in R$

$\Rightarrow a(1,1,1,1)+b(0,1,1,1)+c(0,0,1,1)+d(0,0,0,1)=(0,0,0,0)$.

$\Rightarrow (a, a+b, a+b+c, a+b+c+d) = (0,0,0,0)$

$\therefore a=0$ —— ①

$a+b=0$ —— ②

$a+b+c=0$ —— ③

$a+b+c+d=0$ —— ④.

On solving the above eqs, we get

$a=0, b=0, c=0, d=0$.

$\therefore S$ is L.I subset of $V_4(R)$.

$\therefore \dim V_4(R)=4$ & S is L.I subset of $V_4(R)$.

$\therefore S$ is a basis of $V_4(R)$.

2) <u>To express $\alpha=(2,3,4,1)$ as L.C of basis</u> :-

Consider, $\alpha = p\alpha_1 + q\alpha_2 + r\alpha_3 + s\alpha_4$, $\forall p, q, r, s \in R$.

$\Rightarrow (2,3,4,1) = p(1,1,1,1)+q(0,1,1,1)+r(0,0,1,1)+s(0,0,0,1)$.
—— ⑤.

$\Rightarrow (2,3,4,1) = (p, p+q, p+q+r, p+q+r+s)$.

$\therefore p=2$ —— ⑥

$p+q=3$ —— ⑦

$p+q+r=4$ —— ⑧

$p+q+r+s=1$ —— ⑨.

On solving eq ⑥, ⑦, ⑧ & ⑨, we get.

$p=2, q=1, r=1, s=-3$

On subs: the above values in eq ⑤, we get

$(2,3,4,1) = 2(1,1,1,1)+1(0,1,1,1)+1(0,0,1,1)+(-3)(0,0,0,1)$.

i.e $\alpha = 2\alpha_1 + 1\alpha_2 + 1\alpha_3 - 3\alpha_4$.

$\therefore (2,3,4,1)$ Can be expressed as a linear Combination of vectors of S.

Co-ordinates :-

Let $V(F)$ be a finite dimensional Vector space and $B = \{\alpha_1, \alpha_2, \alpha_3, \dots \alpha_n\}$ be an ordered basis for V. If $\alpha \in V$ then α can be uniquely expressed as $\alpha = a_1\alpha_1 + a_2\alpha_2 + \dots + a_n\alpha_n$ where $a_1, a_2, a_3, \dots a_n \in F$. then the scalars $a_1, a_2, \dots a_n$ are called co-ordinates of α relative to the ordered bases B.

The n-tupple $(a_1, a_2, a_3, \dots a_n)$ is called the n-tupple of coordinates of 'α' relative to the ordered 'basis B.

The matrix $X = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$ is called the co-ordinate matrix of 'α' relative to the ordered basis B. It is denoted by $[\alpha]_B$.

Eg ①. Find the co-ordinate matrix of the vector $(2,1)$ of $V_2(R)$ in the ordered basis $(1,0), (1,1)$.

Soln Let $S = \{(1,0) (1,1)\}$. is a basis of $V_2(R)$.

To find the coordinate matrix of $(2,1)$:-

Consider, $(2,1) = a(1,0) + b(1,1)$

$\Rightarrow (2,1) = (a+b, b)$.

$\therefore a+b = 2$ — ①

$b = 1$. — ②

on subs b in eq ①, we get

$a = 1$.

$\therefore a = 1, b = 1$.

$\therefore$ The co-ordinates matrix of $(2,1)$ is $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Eg ② S.T the set $\{(1,0,0), (1,1,0), (1,1,1)\}$ is a basis of $C^3(C)$. Hence find the co-ordinates of the vector $\{3+4i, 6i, 3+7i\}$ in $C^3(C)$.

Soln Let $S = \{\alpha_1, \alpha_2, \alpha_3\} \in C^3(C)$.

where $\alpha_1 = (1,0,0)$, $\alpha_2 = (1,1,0)$, $\alpha_3 = (1,1,1)$.

We know that $\dim C^3(C) = 3$.

1) to prove that S is a basis of $C^3(C)$:-

It is enough to prove that S is L.I.

Consider,

$a\alpha_1 + b\alpha_2 + c\alpha_3 = \bar{0}$. $\forall$ $a, b, c \in C$.

$\Rightarrow a(1,0,0) + b(1,1,0) + c(1,1,1) = (0,0,0)$

$\Rightarrow (a+b+c,\ b+c,\ c) = (0,0,0)$

$\therefore a+b+c = 0$

$b+c = 0$

$c = 0$.

On solving the above eqns, we get

$a=0,\ b=0,\ c=0$.

$\therefore S$ is L.I subset of $C^3(C)$.

$\therefore \dim C^3(C) = 3$ & S is L.I subset of $C^3(C)$.

$\therefore S$ is a basis of $C^3(C)$.

2) to find the co-ordinates of 'α' relative to the basis S:-

let $\alpha = (x, y, z) \in C^3(C)$

where $x = 3+4i$, $y = 6i$, $z = 3+7i$.

Consider, $\alpha = a\alpha_1 + b\alpha_2 + c\alpha_3$, $\forall$ $a, b, c \in C$.

$\Rightarrow (x, y, z) = a(1,0,0) + b(1,1,0) + c(1,1,1)$.

$\Rightarrow (x, y, z) = (a+b+c,\ b+c,\ a)$

$\therefore a+b+c = x$

$b+c = y$

$c = z$.

on solving above eqns, we get

$a = x-y$, $b = y-z$, $c = z$.

$\therefore$ The coordinates of $\alpha = (x, y, z)$ are $x-y,\ y-z,\ z$.

i.e the co-ordinates of $\alpha = (3+4i, 6i, 3+7i)$ are $3-2i, -3-i, 3+7i$

Eg ⑤ Find the Coordinates of $(2i, 3+4i, 5i)$ and $(2i, 3+4i, 5)$ w.r.t the basis $S = \{(1,0,0), (1,1,0), (1,1,1)\}$ of $C^3(C)$.

Soln

G.T $S = \{(1,0,0), (1,1,0), (1,1,1)\}$ is a basis of $C^3(C)$.

Let $\alpha = (x, y, z) \in C^3(C)$ & $\alpha_1 = (1,0,0)$, $\alpha_2 = (1,1,0)$, $\alpha_3 = (1,1,1)$

Consider,

$$\alpha = a\alpha_1 + b\alpha_2 + c\alpha_3 \quad \forall\, a, b, c \in C.$$

$$\Rightarrow (x, y, z) = a(1,0,0) + b(1,1,0) + c(1,1,1)$$

$$\Rightarrow (x, y, z) = (a+b+c,\ b+c,\ c)$$

$$\therefore\ a+b+c = x,$$
$$b+c = y$$
$$c = z.$$

On solving the above eqn, we get

$$a = x-y,\quad b = y-z,\quad c = z.$$

$\therefore$ the coordinates of $\alpha = (x, y, z)$ are $x-y,\ y-z,\ z$.

& the " " $\alpha = (2i, 3+4i, 5i)$ are $-3-2i,\ 3-i,\ 5i$.

$\therefore$ the coordinates of $\alpha = (2i, 3+4i, 5)$ are $-3-2i,\ -2+4i,\ 5$.

4. Find the coordinates of $(6i, 7, 8i)$ w.r.t to the basis $S = \{(1, 0, 0), (1, 1, 0), (1, 1, 1)\}$ of $C^3(C)$.

Solution: Given $S = \{(1, 0, 0), (1, 1, 0), (1, 1, 1)\}$ is a basis of $C^3(C)$.

Let $\alpha = (x, y, z) \in C^3(C)$

Claim: To find the coordinates of α relative to the basis S.

$\alpha = a\alpha_1 + b\alpha_2 + c\alpha_3$ where $a, b, c \in C$

$\Rightarrow (x, y, z) = a(1, 0, 0) + b(1, 1, 0) + c(1, 1, 1)$

$\Rightarrow (x, y, z) = (a + b + c, b + c, c)$

$\Rightarrow a + b + c = x, b + c = y, c = z \Rightarrow a = x - y, b = y - z, c = z$

$\therefore$ The coordinates of $\alpha = (x, y, z)$ are $x - y, y - z, z$

$\therefore$ The coordinates of $\alpha = (6i, 7, 8i)$ are $6i - 7, 7 - 8i, 8i$

5. Find the coordinates of (2, 3, 4, –1) w.r.t to the basis B = {(1, 1, 1, 2), (1, –1, 0, 0), (0, 0, 1, 1), (0, 1, 0, 0)} of $V_4(R)$. [ANU M11, M17

Solution: Given B = {(1, 1, 1, 2), (1, –1, 0, 0), (0, 0, 1, 1), (0, 1, 0, 0)} is a basis of $V_4(R)$.

Let $\alpha = (2, 3, 4, -1) \in V_4(R)$

Claim: To find the coordinates of α relative to the basis B.

Suppose $\alpha = a(1, 1, 1, 2) + b(1, -1, 0, 0) + c(0, 0, 1, 1) + d(0, 1, 0, 0)$

$\Rightarrow (2, 3, 4, -1) = (a + b, a - b + d, a + c, 2a + c)$

$\Rightarrow a + b = 2$ ...... (1), $a - b + d = 3$ .... (2), $a + c = 4$ .... (3), $2a + c = -1$ ..... (4)

(4) – (1) $\Rightarrow a = -5$, From (3), $c = 4 - a = 4 + 5 = 9$,

From (1), $b = 2 - a = 2 + 5 = 7$ From (2), $d = 3 - a + b = 3 + 5 + 7 = 15$

$\therefore$ The coordinates of $\alpha = (2, 3, 4, -1)$ are –5, 7, 9, 15i

6. Find the coordinates of the vector (2, 1, – 6) of $R^3(R)$ relative the ordered basis {(1, 1, 2), (3, –1, 0), (2, 0, –1)}

Solution: Given B = {(1, 1, 2), (3, –1, 0), (2, 0, –1)} is a basis of $R^3(R)$.

Let $\alpha = (2, 1, -6) \in R^3(R)$.

Claim: To find the coordinates of α relative to the basis B.

Suppose $\alpha = a(1, 1, 2) + b(3, -1, 0) + c(2, 0, -1)$

$\Rightarrow (2, 1, -6) = (a + 3b + 2c, a - b, 2a - c)$

$\Rightarrow a + 3b + 2c = 2$ ..... (1), $a - b = 1$ ...... (2), $2a - c = -6$ ...... (3)

(1) + 2(3) = $5a + 3b = -10$ .... (4) (4) + 3(2) $\Rightarrow 8a = -7 \Rightarrow a = -\frac{7}{8}$

From (2), $b = a - 1 = -\frac{7}{8} - 1 = -\frac{15}{8}$ From (3), $c = 2a + 6 = -\frac{7}{4} + 6 = \frac{17}{4}$

$\therefore$ The coordinates of $\alpha = (2, 1, -6)$ are $-\frac{7}{8}, -\frac{15}{8}, \frac{17}{4}$

(OR)

Write the augmented matrix [A|B] and reducing to echelon form,

$$[A|B] = \left[\begin{array}{ccc|c} 1 & 3 & 2 & 2 \\ 1 & -1 & 0 & 1 \\ 2 & 0 & -1 & -6 \end{array}\right] \begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 2R_1 \end{array} \sim \left[\begin{array}{ccc|c} 1 & 3 & 2 & 2 \\ 0 & -4 & -2 & -1 \\ 0 & -6 & -5 & -10 \end{array}\right] R_3 \to R_3 - \frac{6}{4}R_1$$

$$\sim \left[\begin{array}{ccc|c} 1 & 3 & 2 & 2 \\ 0 & -4 & -2 & -1 \\ 0 & 0 & -2 & -\frac{17}{2} \end{array}\right]$$

$\therefore a + 3b + 2c = 2 \Rightarrow a = 2 - 3b - 2c = 2 + \frac{45}{8} - \frac{17}{2} = -\frac{7}{8}$

$4b + 2c = 1 \Rightarrow 4b = 1 - 2c = 1 - \frac{17}{2} = -\frac{15}{2} \Rightarrow b = -\frac{15}{8}$

$2c = \frac{17}{2} \Rightarrow c = \frac{17}{4}$

$\therefore$ The coordinates of $\alpha = (2, 1, -6)$ are $-\frac{7}{8}, -\frac{15}{8}, \frac{17}{4}$

Dimension of a Subspace:- (12).

* Let V(F) be a finite dimensional Vector space of dimension 'n'. and W be the subspace of V. then W is a finite dimensional Vector space with $\dim W \le n$.

i.e $\dim W \le \dim V$.

and also $V = W$ iff $\dim V = \dim W$.

** Let W_1 and W_2 be two subspaces of a finite dimensional Vector space V(F). then $\dim(W_1+W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2)$

problems:-

(1). find a basis for the subspace spanned by the Vectors $(1,2,0)$, $(-1,0,1)$ and $(0,2,1)$ in $V_3(R)$.

Sol:- Let W be the subspace of $V_3(R)$ spanned by the vectors $\alpha = (1,2,0)$, $\beta = (-1,0,1)$ and $\gamma = (0,2,1)$.

To find a basis of W:-

Since, $\alpha + \beta = (1,2,0) + (-1,0,1) = (0,2,1) = \gamma$.

$\therefore \{\alpha, \beta, \gamma\}$ is L.D.

$\therefore$ the subspace spanned by $\{\alpha,\beta,\gamma\}$ = the subspace spanned by $\{\alpha,\beta\}$

Consider, $a\alpha + b\beta = \delta$

$\Rightarrow a(1,2,0) + b(-1,0,1) = (0,0,0)$

$\Rightarrow (a-b, 2a, b) = (0,0,0)$

$\therefore a - b = 0$

$2a = 0$ | $b = 0$.

$\therefore a = 0$

$\therefore a = 0$ & $b = 0$.

$\therefore \{\alpha, \beta\}$ is L.I.

Hence It is a basis of W spanned by $\{\alpha,\beta,\gamma\}$

(or)

2^{nd} Method

Given Vectors are $(1,2,0)$, $(-1,0,1)$, $(0,2,1)$.

Arrange the given vectors as rows of a matrix and reducing it to the Echelon form.

Consider, $A = \begin{pmatrix} 1 & 2 & 0 \\ -1 & 0 & 1 \\ 0 & 2 & 1 \end{pmatrix}$

$R_2 \to R_2 + R_1$,

$\therefore A \sim \begin{pmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \\ 0 & 2 & 1 \end{pmatrix}$

$R_3 \to R_3 - R_2$

$\sim \begin{pmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

$\therefore A \sim \begin{pmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

Hence, the two non zero rows $(1,2,0), (0,2,1)$ form the L.I set and it forms a basis of W.

$\therefore$ Basis of $W = \{(1,2,0), (0,2,1)\}$ and $\dim W = 2$.

2. If W is a subspace of $V_4(R)$ generated by the vectors $(1,-2,5,-3)$, $(2,3,1,-4)$ and $(3,8,-3,-5)$. Then find the basis of W and its dimension.

Soln Let W be the subspace of $V_4(R)$ generated by the vectors $(1,-2,5,-3)$, $(2,3,1,-4)$ and $(3,8,-3,-5)$.

To find a basis of W & its dimension :-

Arrange the given vectors as rows of a matrix and reducing it to Echelon form.

Consider, $A = \begin{pmatrix} 1 & -2 & 5 & 3 \\ 2 & 3 & 1 & -4 \\ 3 & 8 & -3 & -5 \end{pmatrix}$

$R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 3R_1$

$\sim \begin{pmatrix} 1 & -2 & 5 & 3 \\ 0 & 7 & -9 & 2 \\ 0 & 14 & -18 & 4 \end{pmatrix}$

$R_3 \to R_3 - 2R_2$

$\sim \begin{pmatrix} 1 & -2 & 5 & 3 \\ 0 & 7 & -9 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

Hence the two non zero rows $(1,-2,5,-3)$ & $(0,7,-9,2)$ form the least L.I set and it forms a basis of W.

$\therefore$ Basis of $W = \{(1,-2,5,-3),(0,7,-9,2)\}$ and

$\dim W = 2.$

(3). If W_1 and W_2 are two subspaces of $V_4(R)$ generated by the sets $\{(1,1,-1,2),(2,1,3,0),(3,2,2,2)\}$ and $\{(1,-1,0,1),(-1,1,0,-1)\}$. Find $\dim W_1$, $\dim W_2$, and $\dim(W_1+W_2)$. Hence find $\dim(W_1\cap W_2)$.

Soln Let W_1 and W_2 be the two ~~sets of~~ subspaces of $V_4(R)$ generated by

$S_1 = \{(1,1,-1,2),(2,1,3,0),(3,2,2,2)\}$ and

$S_2 = \{(1,-1,0,1),(-1,1,0,-1)\}$. resp.

To find ~~dim~~ W_1

Arrang the vectors of S_1, as rows of a matrix and reduced it to the Echelon form.

$$A = \begin{pmatrix} 1 & 1 & -1 & 2 \\ 2 & 1 & 3 & 0 \\ 3 & 2 & 2 & 2 \end{pmatrix}$$

$R_2 \to R_2 - 2R_1,\ R_3 \to R_3 - 3R_1$

$$\sim \begin{pmatrix} 1 & 1 & -1 & 2 \\ 0 & -1 & 5 & -4 \\ 0 & -1 & 5 & -4 \end{pmatrix}$$

$R_3 \to R_3 - R_2$

$$\sim \begin{pmatrix} 1 & 1 & -1 & 2 \\ 0 & -1 & 5 & -4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Hence, the two non zero rows $(1,1,-1,2)$ & $(0,-1,5,4)$ form the least L.I set and it forms a basis of W_1.

$\therefore$ Basis of $W_1 = \{(1,1,-1,2),(0,-1,5,-4)\}$ and

$\dim W_1 = 2.$

To find dim W_2 :-

Arranging the vectors of S_2 as rows of a matrix and and reducing to Echelon form.

Consider, $B = \begin{pmatrix} 1 & -1 & 0 & 1 \\ -1 & 1 & 0 & -1 \end{pmatrix}$

$R_2 \to R_2 + R_1$

$\sim \begin{pmatrix} 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

Hence the non zero row $(1\ -1\ 0\ 1)$ form the least L.I set and it forms a basis of W_2.

$\therefore$ Basis of $W_2 = \{(1,-1,0,1)\}$ and $\dim W_2 = 1$.

To find $\dim(W_1 + W_2)$:-

We know that $W_1 + W_2$ is spanned by $W_1 \cup W_2$

Arranging the vectors of S_1 and S_2 as rows of a matrix and reducing to echelon form.

Consider,

$$C = \begin{pmatrix} 1 & 1 & -1 & 2 \\ 2 & 1 & 3 & 0 \\ 3 & 2 & 2 & 2 \\ 1 & -1 & 0 & 1 \\ -1 & 1 & 0 & -1 \end{pmatrix}$$

$R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 3R_1$

$R_4 \to R_4 - R_1$, $R_5 \to R_5 + R_1$

$$\sim \begin{pmatrix} 1 & 1 & -1 & 2 \\ 0 & -1 & 5 & -4 \\ 0 & -1 & 5 & -4 \\ 0 & -2 & 1 & -1 \\ 0 & 2 & -1 & 1 \end{pmatrix}$$

$R_3 \to R_3 - R_2$, $R_5 \to R_5 + R_4$

$$\sim \begin{pmatrix} 1 & 1 & -1 & 2 \\ 0 & -1 & 5 & -4 \\ 0 & 0 & 0 & 0 \\ 0 & -2 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$R_4 \to R_4 - 2R_2$

$$\sim \begin{pmatrix} 1 & 1 & -1 & 2 \\ 0 & -1 & 5 & -4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -9 & 7 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$R_3 \leftrightarrow R_4$

$$C \sim \begin{pmatrix} 1 & 1 & -1 & 2 \\ 0 & -1 & 5 & -4 \\ 0 & 0 & -9 & 7 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Hence the three non zero rows $(1,1,-1,2)$, $(0,-1,5,-4)$ and $(0,0,-9,7)$ form the least L.I set and it forms a basis of $W_1 + W_2$.

$\therefore$ Basis of $W_1+W_2 = \{(1,1,-1,2),(0,-1,5,-4),(0,0,-9,7)\}$ and

$\dim(W_1+W_2) = 3$.

Now,

By the dimension theorem

$\dim(W_1+W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2)$.

$\therefore \dim(W_1 \cap W_2) = \dim W_1 + \dim W_2 - \dim(W_1+W_2)$

$= 2+1-3 = 0$.

$\therefore \dim(W_1 \cap W_2) = 0$.

5. If W_1 and W_2 are two subspaces of $V_4(R)$ generated by the sets $\{(1, 1, -1, 2), (2, 1, 3, 0), (3, 2, 2, 2)\}$ and $\{(1, -1, 0, 1), (-1, 1, 0, -1)\}$. Find dim $(W_1 + W_2)$. [ANU S97, M11, J14,

Solution: Let W_1 and W_2 be two subspaces of $V_4(R)$ generated by $S_1 = \{(1, 1, -1, 2), (2, 1, 3, 0), (3, 2, 2, 2)\}$ & $S_2 = \{(1, -1, 0, 1), (-1, 1, 0, -1)\}$ respectively.

Claim: To find dim $(W_1 + W_2)$.

We know that $W_1 + W_2$ is spanned by $W_1 \cup W_2$

Arranging the vectors of S_1 and S_2 as rows of a matrix and reducing to echelon form

$$C = \begin{bmatrix} 1 & 1 & -1 & 2 \\ 2 & 1 & 3 & 0 \\ 3 & 2 & 2 & 2 \\ 1 & -1 & 0 & 1 \\ -1 & 1 & 0 & -1 \end{bmatrix} \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \\ R_4 \to R_4 - R_1 \\ R_5 \to R_5 + R_1 \end{matrix} \sim \begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & -1 & 5 & -4 \\ 0 & -1 & 5 & -4 \\ 0 & -2 & 1 & -1 \\ 0 & 2 & -1 & 1 \end{bmatrix} \begin{matrix} R_3 \to R_3 - R_2 \\ R_5 \to R_5 - R_4 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & -1 & 5 & -4 \\ 0 & 0 & 0 & 0 \\ 0 & -2 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & -1 & 5 & -4 \\ 0 & -2 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} R_3 \to R_3 - 2R_1 \sim \begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & -1 & 5 & -4 \\ 0 & 0 & -9 & 7 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Hence the three non-zero rows $(1, 1, -1, 2)$, $(0, -1, 5, -4)$, $(0, 0, -9, 7)$ form the least L.I set and it forms a basis of $W_1 + W_2$.

$\therefore$ Basis of $W_1 + W_2 = \{(1, 1, -1, 2), (0, -1, 5, -4), (0, 0, -9, 7)\}$

$\therefore$ dim $(W_1 + W_2) = 3$

6. If W_1 and W_2 are two subspaces of R^4 generated by the sets $\{(1, 1, 0, -1), (1, 2, 3, 0), (2, 3, 3, -1)\}$ and $\{(1, 2, 2, -2), (2, 3, 2, -3), (1, 3, 4, -3\}$ respectively. Find the dimensions of (*i*) W_1 (*ii*) W_2 (*iii*) $(W_1 + W_2)$ (*iv*) $W_1 \cap W_2$ [ANU J15,

Solution: Let W_1 and W_2 be two subspaces of R^4 generated by $S_1 = \{(1, 1, 0, -1), (1, 2, 3, 0), (2, 3, 3, -1)\}$ & $S_2 = \{(1, 2, 2, -2), (2, 3, 2, -3), (1, 3, 4, -3)\}$ respectively.

Claim: To find dim W_1, dim W_2, dim $(W_1 + W_2)$ and dim $(W_1 \cap W_2)$

To find dim W_1:

Arranging the vectors of S_1 as rows of a matrix and reducing to echelon form,

$$A = \begin{bmatrix} 1 & 1 & 0 & -1 \\ 1 & 2 & 3 & 0 \\ 2 & 3 & 3 & -1 \end{bmatrix} \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 2R_1 \end{matrix} \sim \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 1 & 3 & 1 \end{bmatrix} R_3 \to R_3 - R_2 \sim \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Hence the two non-zero rows $(1, 1, 0, -1)$, $(0, 1, 3, 1)$ form the least L.I set and it forms a basis of W_1.

$\therefore$ Basis of $W = \{(1, 1, 0, -1), (0, 1, 3, 1)\}$ and dim $W = 2$

To find dim W_2:

Arranging the vectors of S_1 as rows of a matrix and reducing to echelon form,

$$B = \begin{bmatrix} 1 & 2 & 2 & -2 \\ 2 & 3 & 2 & -3 \\ 1 & 3 & 4 & -3 \end{bmatrix} \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - R_1 \end{matrix} \sim \begin{bmatrix} 1 & 2 & 2 & -2 \\ 0 & -1 & -2 & 1 \\ 0 & 1 & 2 & -1 \end{bmatrix} R_3 \to R_3 + R_2 \sim \begin{bmatrix} 1 & 2 & 2 & -2 \\ 0 & -1 & -2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Hence the two non-zero rows (1, 2, 2, − 2), (0, − 1, − 2, 1) form the least L.I set and it forms a basis of W_2.

$\therefore$ Basis of W = {(1, 2, 2, − 2), (0, − 1, − 2, 1)} and dim W = 2

To find dim ($W_1 + W_2$): We know that $W_1 + W_2$ is spanned by $W_1 \cup W_2$

Arranging the vectors of S_1 and S_2 as rows of a matrix and reducing to echelon form

$$C = \begin{bmatrix} 1 & 1 & 0 & -1 \\ 1 & 2 & 3 & 0 \\ 2 & 3 & 3 & -1 \\ 1 & 2 & 2 & -2 \\ 2 & 3 & 2 & -3 \\ 1 & 3 & 4 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 1 & 2 & 2 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} R_3 \longrightarrow R_3 - R_1 \sim \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & -1 & -2 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} R_3 \longrightarrow R_3 - R_2 \\ R_4 \longrightarrow R_4 + R_2 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} R_4 \longrightarrow R_4 + R_1 \sim \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Hence the three non-zero rows (1, 1, 0, − 1), (0, 1, 3, 1), (0, 0, − 1, − 2) form the least L.I set and it forms a basis of $W_1 + W_2$.

$\therefore$ Basis of $W_1 + W_2$ = {(1, 1, 0, − 1), (0, 1, 3, 1), (0, 0, − 1, − 2)} and

dim $(W_1 + W_2) = 3$

To find dim ($W_1 \cap W_2$):

$\therefore$ By the dimension theorem dim $(W_1 + W_2)$ = dim W_1 + dim W_2 − dim $(W_1 \cap W_2)$

$\Rightarrow$ dim $(W_1 \cap W_2)$ = dim W_1 + dim W_2 − dim $(W_1 + W_2) = 2 + 2 - 3 = 1$

$\therefore$ dim $(W_1 \cap W_2) = 1$

7. V is the vector space generated by the polynomials $\alpha = x^3 + 2x^2 - 2x + 1$, $\beta = x^3 + 3x^2 - x + 4$, $\gamma = 2x^3 + x^2 - 7x - 7$ over R. Find a basis of V and its dimension.[ANU J11,

Solution: Given V is the polynomial space generated by $\{\alpha, \beta, \gamma\}$ where $\alpha = x^3 + 2x^2 - 2x + 1$, $\beta = x^3 + 3x^2 - x + 4$, $\gamma = 2x^3 + x^2 - 7x - 7$

Claim: To find a basis of V and its dimension.

The coordinates of α w.r.t to the base $\{x^3, x^2, x, 1\}$ is (1, 2, − 2, 1)

Similarly, the coordinates of β, γ w.r.t to the base $\{x^3, x^2, x, 1\}$ are (1, 3, − 1, 4), (2, 1, − 7, − 7) respectively.

Arranging the coordinates of α, β, γ as rows in a matrix and reducing to echelon form,

$$A = \begin{bmatrix} 1 & 2 & -2 & 1 \\ 1 & 3 & -1 & 4 \\ 2 & 1 & -7 & -7 \end{bmatrix} \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 2R_1 \end{matrix} \sim \begin{bmatrix} 1 & 2 & -2 & 1 \\ 0 & 1 & 1 & 3 \\ 0 & -3 & -3 & -9 \end{bmatrix} R_3 \to R_3 + 3R_2 \sim \begin{bmatrix} 1 & 2 & -2 & 1 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Hence the two non-zero rows (1, 2, − 2, 1), (0, 1, 1, 3) form the least L.I set and the basis of V is formed by these coordinate vectors.

∴ Basis of V = $\{\alpha, \delta\}$ where $\alpha = x^3 + 2x^2 - 2x + 1$, $\delta = x^2 + x + 3$ and dim V = 2

8. V is the vector space of polynomials over R. W_1 and W_2 are the subspaces generated by $\{x^3 + x^2 - 1, x^3 + 2x^2 + 3x, 2x^3 + 3x^2 + 3x - 1\}$ and $\{x^3 + 2x^2 + 2x - 2, 2x^3 + 3x^2 + 2x - 3, x^3 + 3x^2 + 4x - 3\}$ respectively. Find (i) dim ($W_1 + W_2$) (ii) dim ($W_1 \cap W_2$).

Solution: Let $S_1 = \{x^3 + x^2 - 1, x^3 + 2x^2 + 3x, 2x^3 + 3x^2 + 3x - 1\}$ and $S_2 = \{x^3 + 2x^2 + 2x - 2, 2x^3 + 3x^2 + 2x - 3, x^3 + 3x^2 + 4x - 3\}$ generates W_1, W_2 respectively.

Claim: To find dim ($W_1 + W_2$) and dim ($W_1 \cap W_2$)

To find dim W_1:

The coordinates of vectors of S_1 are {(1, 1, 0, − 1), (1, 2, 3, 0), (2, 3, 3, − 1)} w.r.t to the basis $\{x^3, x^2, x, 1\}$.

Arranging the coordinates of vectors of S_1 as rows of a matrix and reducing to echelon form,

$$A = \begin{bmatrix} 1 & 1 & 0 & -1 \\ 1 & 2 & 3 & 0 \\ 2 & 3 & 3 & -1 \end{bmatrix} \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 2R_1 \end{matrix} \sim \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 1 & 3 & 1 \end{bmatrix} R_3 \to R_3 - R_2 \sim \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Hence the two non-zero rows of (1, 1, 0, − 1), (0, 1, 3, 1) form the least L.I set and the basis of W_1 is formed by these coordinate vectors.

∴ Basis of W = $\{\alpha, \beta\}$ where $\alpha = x^3 + x^2 - 1$, $\beta = x^2 + 3x + 1$ and dim W = 2

To find dim W_2:

The coordinates of vectors of S_2 are {(1, 2, 2, − 2), (2, 3, 2, − 3), (1, 3, 4, − 3)} w.r.t to the basis $\{x^3, x^2, x, 1\}$.

Arranging the coordinates of vectors of S_2 as rows of a matrix and reducing to echelon form,

$$B = \begin{bmatrix} 1 & 2 & 2 & -2 \\ 2 & 3 & 2 & -3 \\ 1 & 3 & 4 & -3 \end{bmatrix} \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - R_1 \end{matrix} \sim \begin{bmatrix} 1 & 2 & 2 & -2 \\ 0 & -1 & -2 & 1 \\ 0 & 1 & 2 & -1 \end{bmatrix} R_3 \to R_3 + R_2 \sim \begin{bmatrix} 1 & 2 & 2 & -2 \\ 0 & -1 & -2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Hence the two non-zero rows (1, 2, 2, − 2), (0, − 1, − 2, 1) form the least L.I set and the basis of W_2 is formed by these coordinate vectors.

$\therefore$ Basis of W = $\{\gamma, \delta\}$ where $\gamma = x^3 + 2x^2 + 2x - 2$, $\delta = -x^2 - 2x + 1$ and dim W = 2

To find dim ($W_1 + W_2$): We know that $W_1 + W_2$ is spanned by $W_1 \cup W_2$

Arranging the vectors of S_1 and S_2 as rows of a matrix and reducing to echelon form

$$C = \begin{bmatrix} 1 & 1 & 0 & -1 \\ 1 & 2 & 3 & 0 \\ 2 & 3 & 3 & -1 \\ 1 & 2 & 2 & -2 \\ 2 & 3 & 2 & -3 \\ 1 & 3 & 4 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 1 & 2 & 2 & -2 \\ 0 & -1 & -2 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} R_3 \longrightarrow R_3 - R_1 \sim \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 1 & 1 & 2 & -1 \\ 0 & -1 & -2 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} R_3 \longrightarrow R_3 - R_2 \\ R_4 \longrightarrow R_4 + R_2 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} R_4 \longrightarrow R_4 + R_1 \sim \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Hence the three non-zero rows (1, 1, 0, − 1), (0, 1, 3, 1), (0, 0, − 1, − 2) form the least L.I set and the basis of $W_1 + W_2$ is formed by these coordinate vectors.

$\therefore$ Basis of $W_1 + W_2 = \{\alpha, \beta, \theta\}$ where $\alpha = x^3 + x^2 - 1$, $\beta = x^2 + 3x + 1$, $\theta = -x - 2$

$\therefore$ dim ($W_1 + W_2$) = 3

To find dim ($W_1 \cap W_2$):

$\therefore$ By the dimension theorem dim ($W_1 + W_2$) = dim W_1 + dim W_2 − dim ($W_1 \cap W_2$)

$\Rightarrow$ dim ($W_1 \cap W_2$) = dim W_1 + dim W_2 − dim ($W_1 + W_2$) = 2 + 2 − 3 = 1

$\therefore$ dim ($W_1 \cap W_2$) = 1

9. Let W_1 and W_2 be two subspaces of R^4 defined by $W_1 = \{(a, b, c, d) / b - 2c + d = 0\}$, $W_2 = \{(a, b, c, d) / a = d, b = 2c\}$. Find the basis and dimension of (*i*) W_1 (*ii*) W_2 (*iii*) $W_1 \cap W_2$ and hence find dim ($W_1 + W_2$). [ANU M11, M12, M16, O18

Solution: Given $W_1 = \{(a, b, c, d) / b - 2c + d = 0\}$ and $W_2 = \{(a, b, c, d) / a = d, b = 2c\}$ are two subspaces of R^4

To find dim W_1:

Let $(a, b, c, d) \in W_1$ Then $b - 2c + d = 0 \Rightarrow b = 2c - d$

Now $(a, b, c, d) = (a, 2c - d, c, d) = a(1, 0, 0, 0) + c(0, 2, 1, 0) + d(0, -1, 0, 1)$

$\therefore$ The set $S_1 = \{(1, 0, 0, 0), (0, 2, 1, 0), (0, -1, 0, 1)\}$ spans W_1.

Clearly S_1 is L.I subset of W_1. Hence S_1 is a basis of W_1 and dim $W_1 = 3$

To find dim W_2:

Let $(a, b, c, d) \in W_2$ Then $a = d,\ b = 2c$

Now $(a, b, c, d) = (d, 2c, c, d) = c(0, 2, 1, 0) + d(1, 0, 0, 1)$

$\therefore$ The set $S_2 = \{(0, 2, 1, 0), (1, 0, 0, 1)\}$ spans W_2.

Clearly S_2 is L.I subset of W_2. Hence S_2 is a basis of W_2 and dim $W_2 = 2$.

To find dim $(W_1 \cap W_2)$:

$W_1 \cap W_2 = \{(a, b, c, d) \,/\, b - 2c + d = 0,\ a = d,\ b = 2c\}$

Let $(a, b, c, d) \in W_1 \cap W_2$

Then $b - 2c + d = 0,\ a = d,\ b = 2c \Rightarrow 2c - 2c + d = 0,\ a = d,\ b = 2c \Rightarrow a = d = 0,\ b = 2c$

Now $(a, b, c, d) = (0, 2c, c, 0) = c(0, 2, 1, 0)$

$\therefore$ The set $S_3 = \{(0, 2, 1, 0)\}$ spans $W_1 \cap W_2$

Clearly S_3 is L.T subset of $W_1 \cap W_2$. Hence S_3 is a basis of W_2 and $\dim(W_1 \cap W_2) = 1$.

10. If W_1 and W_2 be two subspaces of $V_4(R)$ defined by $W_1 = \{(a, b, c, d) \,/\, b + c + d = 0\}$, $W_2 = \{(a, b, c, d) \,/\, a + b = 0,\ c = 2d\}$. Find the basis and dimension of (*i*) W_1 (*ii*) W_2 (*iii*) $W_1 \cap W_2$ and hence find dim $(W_1 + W_2)$. [O17,

Solution: Given $W_1 = \{(a, b, c, d) \,/\, b + c + d = 0\}$ and
$W_2 = \{(a, b, c, d) \,/\, a + b = 0,\ c = 2d\}$ are two subspaces of R^4

To find dim W_1:

Let $(a, b, c, d) \in W_1$ Then $b + c + d = 0 \Rightarrow b = -c - d$

Now $(a, b, c, d) = (a, -c - d, c, d) = a(1, 0, 0, 0) + c(0, -1, 1, 0) + d(0, -1, 0, 1)$

$\therefore$ The set $S_1 = \{(1, 0, 0, 0), (0, -1, 1, 0), (0, -1, 0, 1)\}$ spans W_1.

Clearly S_1 is L.I subset of W_1. Hence S_1 is a basis of W_1 and dim $W_1 = 3$

To find dim W_2:

Let $(a, b, c, d) \in W_2$ Then $a + b = 0,\ c = 2d \Rightarrow b = -a,\ c = 2d$

Now $(a, b, c, d) = (a, -a, 2d, d) = a(1, -1, 0, 0) + d(0, 0, 2, 1)$

$\therefore$ The set $S_2 = \{(1, -1, 0, 0), (0, 0, 2, 1)\}$ spans W_2.

Clearly S_2 is L.I subset of W_2. Hence S_2 is a basis of W_2 and dim $W_2 = 2$.

To find dim $(W_1 \cap W_2)$:

$W_1 \cap W_2 = \{(a, b, c, d) \,/\, b + c + d = 0,\ a + b = 0,\ c = 2d\}$

Let $(a, b, c, d) \in W_1 \cap W_2$

Then $b + c + d = 0,\ a + b = 0,\ c = 2d \Rightarrow -a + 2d + d = 0,\ b = -a, c = 2d$

$$\Rightarrow a = 3d, b = -3d, c = 2d$$

Now $(a, b, c, d) = (3d, -3d, 2d, d) = d(3, -3, 2, 1)$

$\therefore$ The set $S_3 = \{(3, -3, 2, 1)\}$ spans $W_1 \cap W_2$

Clearly S_3 is L.T subset of $W_1 \cap W_2$. Hence S_3 is a basis of W_2 and $\dim(W_1 \cap W_2) = 1$.

Coset :- Let W is a Subspace of a Vector space $V(F)$ and $\alpha \in V$.

Then, the set $W+\alpha = \{x+\alpha \,/\, x \in W\}$ is Called the right coset of W in V generated by α.

The set $\alpha + W = \{\alpha + x \,/\, x \in W\}$ is Called the left coset of W in V. generated by α.

Note :- 1) Since $(W,+)$ is a Subgroup of the Abelian group $(V,+)$

$\therefore$ By Commutative property, $x+\alpha = \alpha + x$, $\forall\, \alpha \in V$, $x \in W$.

Hence $W+\alpha$ is Called Simply a coset of W in V, generated by α.

2) $\forall\, x \in W$, $x+W = W$

3) $W+\alpha = W+\beta \Leftrightarrow \alpha - \beta \in W$.

Quotient Set :- Let 'W' be a Subspace of a Vector Space $V(F)$

Then the set of all Cosets of W in V is Called the Quotient set.

It is denoted by $\frac{V}{W}$.

$$\therefore \frac{V}{W} = \{ W+\alpha \,/\, \alpha \in W \}.$$

Linear Transformations

3.1. VECTOR SPACE HOMOMORPHISM

Definition. *Let U and V be two vector spaces over the same field F. Thus the mapping $f: U \to V$ is called a homomorphism from U into V if*

(*i*) $f(\alpha+\beta) = f(\alpha)+f(\beta)\ \forall\ \alpha,\beta \in U$ (*ii*) $f(a\alpha) = af(\alpha)\ \forall a \in F; \forall\ \alpha \in U$

Note. 1. If *f* is onto function then V is called the homomorphic image of *f*.

2. If f is one - one onto function then f is called an isomorphism. Thus it is said that U is isomorphic to V denoted by $U \cong V$.

3. The two conditions of homomorphism are combined into a single condition, called the linear property to define the linear transformation as below.

3.2. LINEAR TRANSFORMATION

Definition. *Let U(F) and V(F) be two vector spaces. Then the function. $T: U \to V$ is called a linear transformation of U into V if* $T(a\alpha+b\beta) = aT(\alpha)+bT(\beta)\ \forall a,b \in F; \alpha,\beta \in U$.

Clearly the vector space homomorphism is equivalent to linear transformation.

Linear Operator : Definition. *If* $T: U \to U$ *(i.e T transforms U into itself) then T is called a linear operator on U.*

Linear Functional : Definition. *If* $T: U \to F$ *(i.e. T transforms U into the field F) then T is called a linear functional on U.*

3.3. ZERO TRANSFORMATION

Theorem. ***Let U(F) and V(F) be two vector spaces. Let the mapping*** $T: U \to V$ ***be defined by*** $T(\alpha) = \hat{O}\ \forall\ \alpha \in U$ ***where*** $\hat{O}$ ***(zero crown) is the zero vector of V. Then T is a linear transformation.***

Proof. For $a,b \in F$ and $\alpha,\beta \in U \Rightarrow a\alpha+b\beta \in U$ ($\because$ U is V.S.)

By definition we have $T(a\alpha+b\beta) = \hat{O} = a\hat{O}+b\hat{O} = aT(\alpha)+bT(\beta)$

$\therefore$ By the definition of linearity T is a linear transformation.

Such a L.T. is called the zero transformation and is denoted by O.

3.4. IDENTITY OPERATOR

Theorem. ***Let V(F) be a vector spaee and the mapping*** $I: V \to V$ ***be defined by*** $I(\alpha) = \alpha\ \ \forall\ \alpha \in V$. ***Then, I is a linear operator from V into itself.***

Proof. $a, b \in F$ and $\alpha, \beta \in V \Rightarrow a\alpha + b\beta \in V$ $\quad (\because$ V is L.S.)

By definition we have $I(a\alpha + b\beta) = a\alpha + b\beta = a\,I(\alpha) + b\,I(\beta)$ (by def.)

$\therefore$ I is a L. T from V into itself and I is called the **Identity Operator.**

3.5. NEGATIVE OF TRANSFORMATION

Theorem. *Let U (F) and V (F) be two vector spaces and* $\mathbf{T : U \rightarrow V}$ *be a linear transformation. Then the mapping* $(-T)$ *defined by* $(-T)(\alpha) = -T(\alpha)\ \forall\ \alpha \in U$ *is a linear transformation.*

Proof. $a, b \in F$ and $\alpha, \beta \in U \Rightarrow a\alpha + b\beta \in U$ $\quad (\because$ U is V.S.)

Now by definition $(-T)(a\alpha + b\beta) = -[T(a\alpha + b\beta)]$

$= -[a\,T(\alpha) + b\,T(\beta)] = -a\,T(\alpha) - b\,T(\beta)$

$= a[-T(\alpha)] + b[-T(\beta)] = a[(-T)(\alpha)] + b[(-T)(\beta)]$

$\Rightarrow -T$ is a linear transformation.

PROPERTIES OF LINEAR TRANSFORMATIONS

3.6. Theorem. *Let* $\mathbf{T : U \rightarrow V}$ *is a linear transformation from the vector space U(F) to the vector space V(F). Then (i)* $T(\bar{O}) = \hat{\hat{O}}$*, where* $\bar{O} \in U$ *and* $\hat{\hat{O}} \in V$

(ii) $T(-\alpha) = -T(\alpha)\ \forall\ \alpha \in U$ *(iii)* $T(\alpha - \beta) = T(\alpha) - T(\beta)\ \forall\ \alpha, \beta \in U$

(iv) $T(a_1\alpha_1 + a_2\alpha_2 \ldots + a_n\alpha_n) = a_1 T(\alpha_1) + a_2 T(\alpha_2) \ldots + a_n T(\alpha_n)\ \forall\ a_i \in F$ *and* $\alpha's \in U$.

Proof. (*i*) $\alpha, \bar{O} \in U \Rightarrow T(\alpha), T(\bar{O}) \in V$

Now $T(\alpha) + T(\bar{O}) = T(\alpha + \bar{O})$ (T is L.T.) $= T(\alpha) = T(\alpha) + \hat{\hat{O}}$ $\quad (\bar{O} \in V)$

By cancellation law $T(\bar{O}) = \hat{\hat{O}}$

(*ii*) $T(-\alpha) = T(-1 . \alpha) = (-1)\,T(\alpha) = -T(\alpha)$

(*iii*) $T(\alpha - \beta) = T[\alpha + (-1)\beta] = T(\alpha) + (-1)T(\beta) = T(\alpha) - T(\beta)$ $\quad (\because$ T is L.T.)

(*iv*) For $n = 1$, $T(a_1\alpha_1) = a_1\,T(\alpha_1)$ $\quad (\because$ T is L.T.)

$n = 2$, $T(a_1\alpha_1 + a_2\alpha_2) = a_1 T(\alpha_1) + a_2 T(\alpha_2)$

Let this be true for $n = m$

$\therefore T(a_1\alpha_1 + a_2\alpha_2 + a_m\alpha_m) = a_1 T(\alpha_1) + a_2 T(\alpha_2) \ldots + a_m T(\alpha_m)$ ... (1)

Now $T[a_1\alpha_1 + \ldots + a_m\alpha_m) + a_{m+1}\alpha_{m+1}] = T(a_1\alpha_1 + \ldots + a_m\alpha_m) + T(a_{m+1}\alpha_{m+1})$

$= a\,T(\alpha_1) + \ldots + a_m T(\alpha_m) + a_{m+1} T(\alpha_{m+1})$ $\quad \therefore$ The relation is true for $n = m+1$

Hence it is true for all integral values of n.

3.8. Theorem. *Let* $S = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ *and* $S' = \{\beta_1, \beta_2, \ldots, \beta_n\}$ *be two ordered bases of n, dimensional vector space V(F). Let* $\{a_1, a_2, \ldots, a_n\}$ *be an ordered set of n scalars such that* $\alpha = a_1\alpha_1 + a_2\alpha_2, \ldots, a_n\alpha_n$ *and* $\beta = a_1\beta_1 + a_2\beta_2, \ldots, a_n\beta_n$. *Show that* $T(\alpha) = \beta$ *where T is the linear operator on V defined by* $T(\alpha_i) = \beta_i$, $i = 1, 2, \ldots, n$.

PROBLEMS

1. If the mapping T: $V_3(R) \longrightarrow V_2(R)$ is defined by $T(x, y, z) = (x - y, x - z)$, then show that T is a linear transformation.[ANU M13, M16, O18,

Solution: Given T: $V_3(R) \longrightarrow V_2(R)$ is a mapping defined by $T(x, y, z) = (x - y, x - z)$

Claim: To show that T is a L.T. *i.e* To S.T $T(a\alpha + b\beta) = aT(\alpha) + bT(\beta)\ \forall\ a, b \in R$ & $\alpha, \beta \in V_3(R)$

Let $a, b \in R$ and $\alpha = (x_1, y_1, z_1), \beta = (x_2, y_2, z_2) \in V_3(R)$

Then $T(\alpha) = (x_1 - y_1, x_1 - z_1)$ and $T(\beta) = (x_2 - y_2, x_2 - z_2)$

$a\alpha + b\beta = a(x_1, y_1, z_1) + b(x_2, y_2, z_2) = (ax_1 + bx_2, ay_1 + by_2, az_1 + bz_2)$

Now $T(a\alpha + b\beta) = T(ax_1 + bx_2, ay_1 + by_2, az_1 + bz_2)$

$$= \big((ax_1 + bx_2) - (ay_1 + by_2), (ax_1 + bx_2) - (az_1 + bz_2)\big)$$

$= \left(a(x_1 - y_1) + b(x_2 - y_2),\ a(x_1 - z_1) + b(x_2 \quad z_2)\right)$

$= a(x_1 - y_1,\ x_1 - z_1) + b(x_2 - y_2,\ x_2 - z_2) = a\mathrm{T}(\alpha) + b\mathrm{T}(\beta)$

$\therefore$ $\mathrm{T}(a\alpha + b\beta) = a\mathrm{T}(\alpha) + b\mathrm{T}(\beta)\ \forall\ a, b \in \mathrm{R}\ \&\ \alpha, \beta \in V_3(\mathrm{R})$

$\therefore$ T is a linear transformation from $V_3(\mathrm{R})$ to $V_2(\mathrm{R})$.

2. If the mapping T: $V_3(\mathrm{R}) \longrightarrow V_3(\mathrm{R})$ is defined by $\mathrm{T}(x, y, z) = (x + 2y - z, y + z, x + y - 2z)$, then show that T is a linear transformation. [ANU J14,

Solution: Given T: $V_3(\mathrm{R}) \longrightarrow V_2(\mathrm{R})$ is a mapping defined by

$\mathrm{T}(x, y, z) = (x + 2y - z,\ y + z,\ x + y - 2z)$

Claim: To show that T is a L.T. *i.e* To S.T $\mathrm{T}(a\alpha + b\beta) = a\mathrm{T}(\alpha) + b\mathrm{T}(\beta)\ \forall\ a, b \in \mathrm{R}$ $\&\ \alpha, \beta \in V_3(\mathrm{R})$

Let $a, b \in \mathrm{R}$ and $\alpha = (x_1, y_1, z_1)$, $\beta = (x_2, y_2, z_2) \in V_3(\mathrm{R})$

Then $\mathrm{T}(\alpha) = (x_1 + 2y_1 - z_1,\ y_1 + z_1,\ x_1 + y_1 - 2z_1)$.

$\mathrm{T}(\beta) = (x_2 + 2y_2 - z_2,\ y_2 + z_2,\ x_2 + y_2 - 2z_2)$

$a\alpha + b\beta = a(x_1, y_1, z_1) + b(x_2, y_2, z_2) = (ax_1 + bx_2,\ ay_1 + by_2,\ az_1 + bz_2)$

Now $\mathrm{T}(a\alpha + b\beta) = \mathrm{T}(ax_1 + bx_2,\ ay_1 + by_2,\ az_1 + bz_2)$

$= \big((ax_1 + bx_2) + 2(ay_1 + by_2) - (az_1 + bz_2),\ (ay_1 + by_2) + (az_1 + bz_2),$
$(ax_1 + bx_2) + (ay_1 + by_2) - 2(az_1 + bz_2)\big)$

$= \big(a(x_1 + 2y_1 - z_1) + b(x_2 + 2y_2 - z_2),\ a(y_1 + z_1) + b(y_2 + z_2),\ a(x_1 + y_1 - 2z_1) +$
$b(x_2 + y_2 - 2z_2)\big)$

$= a(x_1 + 2y_1 - z_1,\ y_1 + z_1,\ x_1 + y_1 - 2z_1) + b(x_2 + 2y_2 - z_2,\ y_2 + z_2,\ x_2 + y_2 - 2z_2)$

$= a\mathrm{T}(\alpha) + b\mathrm{T}(\beta)$

$\therefore$ $\mathrm{T}(a\alpha + b\beta) = a\mathrm{T}(\alpha) + b\mathrm{T}(\beta)\ \forall\ a, b \in \mathrm{R}\ \&\ \alpha, \beta \in V_3(\mathrm{R})$

$\therefore$ T is a linear transformation from $V_3(\mathrm{R})$ to $V_3(\mathrm{R})$.

3. If the mapping T: $V_3(\mathrm{R}) \longrightarrow V_2(\mathrm{R})$ is defined by $\mathrm{T}(x_1, x_2, x_3) = (x_1 - x_2, x_1 + x_3)$, then show that T is a linear transformation.[ANU M96, S02,

Sol: Given T: $V_3(\mathrm{R}) \longrightarrow V_2(\mathrm{R})$ is a mapping defined by $\mathrm{T}(x_1, x_2, x_3) = (x_1 - x_2,\ x_1 + x_3)$

Claim: To show that T is a L.T. *i.e* To S.T $\mathrm{T}(a\alpha + b\beta) = a\mathrm{T}(\alpha) + b\mathrm{T}(\beta)\ \forall\ a, b \in \mathrm{R}$ $\&\ \alpha, \beta \in V_3(\mathrm{R})$

Let $a, b \in \mathrm{R}$ and $\alpha = (x_1, x_2, x_3)$, $\beta = (y_1, y_2, y_3) \in V_3(\mathrm{R})$

Then $\mathrm{T}(\alpha) = (x_1 - x_2, x_1 + x_3)$ and $\mathrm{T}(\beta) = (y_1 - y_2, y_1 + y_3)$

$a\alpha + b\beta = a(x_1, x_2, x_3) + b(y_1, y_2, y_3) = (ax_1 + by_1, ax_2 + by_2, ax_3 + by_3)$

Now $\mathrm{T}(a\alpha + b\beta) = \mathrm{T}(ax_1 + by_1, ax_2 + by_2, ax_3 + by_3)$

$$= \big((ax_1 + by_1) - (ax_2 + by_2), (ax_1 + by_1) + (ax_3 + by_3)\big)$$

$$= \big(a(x_1 - x_2) + b(y_1 - y_2), a(x_1 + x_3) + b(y_1 + y_3)\big)$$

$$= a(x_1 - x_2, x_1 + x_3) + b(y_1 - y_2, y_1 + y_3) = a\mathrm{T}(\alpha) + b\mathrm{T}(\beta)$$

$\therefore \mathrm{T}(a\alpha + b\beta) = a\mathrm{T}(\alpha) + b\mathrm{T}(\beta)\ \forall\ a, b \in \mathrm{R}\ \&\ \alpha, \beta \in V_3(\mathrm{R})$

$\therefore$ T is a linear transformation from $V_3(\mathrm{R})$ to $V_2(\mathrm{R})$.

4. If the mapping T: $V_3(\mathrm{R}) \longrightarrow V_3(\mathrm{R})$ is defined by $\mathrm{T}(a, b, c) = (2a+b+2c, 2a - b, -a - 2b + 2c)$, then show that T is a linear transformation.[ANU M03,

Solution: Given T: $V_3(\mathrm{R}) \longrightarrow V_2(\mathrm{R})$ is a mapping defined by

$\mathrm{T}(a, b, c) = (2a + b + 2c, 2a - b, -a - 2b + 2c)$

Claim: To show that T is a L.T. *i.e* To S.T $\mathrm{T}(a\alpha + b\beta) = a\mathrm{T}(\alpha) + b\mathrm{T}(\beta)\ \forall\ a, b \in \mathrm{R}$ $\&\ \alpha, \beta \in V_3(\mathrm{R})$

Let $a, b \in \mathrm{R}$ and $\alpha = (a_1, b_1, c_1)$, $\beta = (a_2, b_2, c_2) \in V_3(\mathrm{R})$

Then $\mathrm{T}(\alpha) = (2a_1 + b_1 + 2c_1, 2a_1 - b_1, -a_1 - 2b_1 + 2c_1)$

$\mathrm{T}(\beta) = (2a_2 + b_2 + 2c_2, 2a_2 - b_2, -a_2 - 2b_2 + 2c_2)$

$a\alpha + b\beta = a(a_1, b_1, c_1) + b(a_2, b_2, c_2) = (aa_1 + ba_2, ab_1 + bb_2, ac_1 + bc_2)$

Now $\mathrm{T}(a\alpha + b\beta) = \mathrm{T}(aa_1 + ba_2, ab_1 + bb_2, ac_1 + bc_2)$

$= \big(2(aa_1 + ba_2) + (ab_1 + bb_2) + 2(ac_1 + bc_2), 2(aa_1 + ba_2) - (ab_1 + bb_2), -(aa_1 + ba_2) - 2(ab_1 + bb_2) + 2(ac_1 + bc_2)\big)$

$= \big(a(2a_1 + b_1 + 2c_1) + b(2a_2 + b_2 + 2c_2), a(2a_1 - b_1) + b(2a_2 - b_2), a(-a_1 - 2b_1 + 2c_1) + b(-a_2 - 2b_2 + 2c_2)\big)$

$= a(2a_1 + b_1 + 2c_1, 2a_1 - b_1, -a_1 - 2b_1 + 2c_1) + b(2a_2 + b_2 + 2c_2, 2a_2 - b_2, -a_2 - 2b_2 + 2c_2)$

$= a\mathrm{T}(\alpha) + b\mathrm{T}(\beta)$

$\therefore$ $T(a\alpha + b\beta) = aT(\alpha) + bT(\beta)$ $\forall$ $a, b \in R$ & $\alpha, \beta \in V_3(R)$

$\therefore$ T is a linear transformation from $V_3(R)$ to $V_3(R)$.

5. If the mapping T: $V_3(R) \longrightarrow V_2(R)$ is defined by $T(a_1,a_2,a_3) = (a_1 + a_2, a_1 - a_3)$, then show that T is a linear transformation.[ANU J03,

Sol: Given T: $V_3(R) \longrightarrow V_2(R)$ is a mapping defined by $T(a_1,a_2,a_3) = (a_1 + a_2, a_1 - a_3)$

Claim: To show that T is a L.T. *i.e* To S.T $T(a\alpha + b\beta) = aT(\alpha) + bT(\beta)$ $\forall$ $a, b \in R$ & $\alpha, \beta \in V_3(R)$

Let $a, b \in R$ and $\alpha = (a_1, a_2, a_3)$, $\beta = (b_1, b_2, b_3) \in V_3(R)$

Then $T(\alpha) = (a_1 + a_2, a_1 - a_3)$ and $T(\beta) = (b_1 + b_2, b_1 - b_3)$

$a\alpha + b\beta = a(a_1, a_2, a_3) + b(b_1, b_2, b_3) = (aa_1 + bb_1, aa_2 + bb_2, aa_3 + bb_3)$

Now $T(a\alpha + b\beta) = T(aa_1 + bb_1, aa_2 + bb_2, aa_3 + bb_3)$

$$= \left((aa_1 + bb_1) + (aa_2 + bb_2), (aa_1 + bb_1) - (aa_3 + bb_3)\right)$$

$$= \left(a(a_1 + a_2) + b(b_1 + b_2), a(a_1 - a_3) + b(b_1 - b_3)\right)$$

$$= a(a_1 + a_2, a_1 - a_3) + b(b_1 + b_2, b_1 - b_3) = aT(\alpha) + bT(\beta)$$

$\therefore$ $T(a\alpha + b\beta) = aT(\alpha) + bT(\beta)$ $\forall$ $a, b \in R$ & $\alpha, \beta \in V_3(R)$

$\therefore$ T is a linear transformation from $V_3(R)$ to $V_2(R)$.

6. If the mapping T: $V_3(R) \longrightarrow V_2(R)$ is defined by $T(a_1,a_2,a_3) = (3a_1 - 2a_2 + a_3, a_1 - 3a_2 - 2a_3)$, then show that T is a linear transformation.[ANUM2000,

Sol: Given T: $V_3(R) \longrightarrow V_2(R)$ is a mapping defined by

$T(a_1,a_2,a_3) = (3a_1 - 2a_2 + a_3, a_1 - 3a_2 - 2a_3)$

Claim: To show that T is a L.T. *i.e* To S.T $T(a\alpha + b\beta) = aT(\alpha) + bT(\beta)$ $\forall$ $a, b \in R$ & $\alpha, \beta \in V_3(R)$

Let $a, b \in R$ and $\alpha = (a_1, a_2, a_3)$, $\beta = (b_1, b_2, b_3) \in V_3(R)$

Then $T(\alpha) = (3a_1 - 2a_2 + a_3, a_1 - 3a_2 - 2a_3)$ and $T(\beta) = (3b_1 - 2b_2 + b_3, b_1 - 3b_2 - 2b_3)$

$a\alpha + b\beta = a(a_1, a_2, a_3) + b(b_1, b_2, b_3) = (aa_1 + bb_1, aa_2 + bb_2, aa_3 + bb_3)$

Now $T(a\alpha + b\beta) = T(aa_1 + bb_1, aa_2 + bb_2, aa_3 + bb_3)$

$=\left(3(aa_1 + bb_1) - 2(aa_2 + bb_2) + (aa_3 + bb_3), (aa_1 + bb_1) - 3(aa_2 + bb_2) - 2(aa_3 + bb_3)\right)$

$= \left(a(3a_1 - 2a_2 + a_3) + b(3b_1 - 2b_2 + b_3),\ a(a_1 - 3a_2 - 2a_3) + b(b_1 - 3b_2 - 2b_3)\right)$

$= a(3a_1 - 2a_2 + a_3,\ a_1 - 3a_2 - 2a_3) + b(3b_1 - 2b_2 + b_3,\ b_1 - 3b_2 - 2b_3) = a\mathrm{T}(\alpha) + b\mathrm{T}(\beta)$

$\therefore$ $\mathrm{T}(a\alpha + b\beta) = a\mathrm{T}(\alpha) + b\mathrm{T}(\beta)\ \forall\ a, b \in \mathrm{R}\ \&\ \alpha, \beta \in V_3(\mathrm{R})$

$\therefore$ T is a linear transformation from $V_3(\mathrm{R})$ to $V_2(\mathrm{R})$.

7. Show that the function T: $V_2(\mathrm{R}) \longrightarrow V_2(\mathrm{R})$ defined by $\mathrm{T}(x, y) = (0, y)$ is a linear transformation.[ANU S98, M06,

Solution: Given a mapping T: $V_2(\mathrm{R}) \longrightarrow V_2(\mathrm{R})$ defined by $\mathrm{T}(x, y) = (0, y)$

Claim: To show that T is a L.T. *i.e* **To S.T $\mathrm{T}(a\alpha + b\beta) = a\mathrm{T}(\alpha) + b\mathrm{T}(\beta)\ \forall\ a, b \in \mathrm{R}$ & $\alpha, \beta \in V_2(\mathrm{R})$**

Let $a, b \in \mathrm{R}$ and $\alpha = (x_1, y_1)$, $\beta = (x_2, y_2) \in V_2(\mathrm{R})$

Then $\mathrm{T}(\alpha) = (0, y_1)$ and $\mathrm{T}(\beta) = (0, y_2)$

$a\alpha + b\beta = a(x_1, y_1) + b(x_2, y_2) = (ax_1 + bx_2, ay_1 + by_2)$

Now $\mathrm{T}(a\alpha + b\beta) = \mathrm{T}(ax_1 + bx_2, ay_1 + by_2)$

$= \left(0, ay_1 + by_2\right)$

$= (0, ay_1) + (0, by_2)$

$= a(0, y_1) + b(0, y_2) = a\mathrm{T}(\alpha) + b\mathrm{T}(\beta)$

$\therefore$ $\mathrm{T}(a\alpha + b\beta) = a\mathrm{T}(\alpha) + b\mathrm{T}(\beta)\ \forall\ a, b \in \mathrm{R}\ \&\ \alpha, \beta \in V_2(\mathrm{R})$

$\therefore$ T is a linear transformation from $V_2(\mathrm{R})$ to $V_2(\mathrm{R})$.

8. Test whether the function T: $\mathrm{R}^2 \longrightarrow \mathrm{R}^2$ defined by $\mathrm{T}(x_1, x_2) = (1 + x_1, x_2)$ is a linear transformation.[ANU M02,

Solution: Given a mapping T: $\mathrm{R}^2 \longrightarrow \mathrm{R}^2$ defined by $\mathrm{T}(x_1, x_2) = (1 + x_1, x_2)$

Claim: To show that T is L.T or not

Let $a, b \in \mathrm{R}$ and $\alpha = (x_1, x_2)$, $\beta = (y_1, y_2) \in \mathrm{R}^2$

Then $\mathrm{T}(\alpha) = (1 + x_1, x_2)$ and $\mathrm{T}(\beta) = (1 + y_1, y_2)$

$a\alpha + b\beta = a(x_1, x_2) + b(y_1, y_2) = (ax_1 + by_1, ax_2 + by_2)$

Now $\mathrm{T}(a\alpha + b\beta) = \mathrm{T}(ax_1 + by_1, ax_2 + by_2) = \left(1 + ax_1 + by_1\ , ax_2 + by_2\right)$

$a\mathrm{T}(\alpha) + b\mathrm{T}(\beta) = a(1 + x_1, x_2) + b(1 + y_1, y_2) = (a + b + ax_1 + by_1, ax_2 + by_2)$

$\therefore T(a\alpha + b\beta) \neq aT(\alpha) + bT(\beta)\ \forall\ a, b \in R\ \&\ \alpha, \beta \in R^2$

$\therefore$ T is not a linear transformation from R^2 to R^2.

9. Show that the function T: $R^3 \to R^2$ defined by $T(x, y, z) = (|x|, 0)$ is not a linear transformation.[ANU M04, O17

Solution: Given a mapping T: $R^3 \to R^2$ defined by $T(x, y, z) = (|x|, 0)$

Claim: To show that T is not a L.T. *i.e* To S.T $T(a\alpha + b\beta) \neq aT(\alpha) + bT(\beta)$ $\forall\ a, b \in R\ \&\ \alpha, \beta \in R^3$

Let $a, b \in R$ and $\alpha = (x_1, y_1, z_1), \beta = (x_2, y_2, z_2) \in R^3$

Then $T(\alpha) = (|x_1|, 0)$ and $T(\beta) = (|x_2|, 0)$

$a\alpha + b\beta = a(x_1, y_1, z_1) + b(x_2, y_2, z_2) = (ax_1 + bx_2, ay_1 + by_2, az_1 + bz_2)$

Now $T(a\alpha + b\beta) = T(ax_1 + bx_2, ay_1 + by_2, az_1 + bz_2) = (|ax_1 + bx_2|, 0)$

$aT(\alpha) + bT(\beta) = a(|x_1|, 0) + b(|x_2|, 0) = (a|x_1| + b|x_2|, 0)$

$\therefore T(a\alpha + b\beta) \neq aT(\alpha) + bT(\beta)\ \forall\ a, b \in R\ \&\ \alpha, \beta \in R^2$ $[\because |ax_1 + bx_2| \neq a|x_1| + b|x_2|]$

$\therefore$ T is not a linear transformation from $R^3$to R^2.

10. The mapping T: $V_3(R) \to V_1(R)$ is defined by $T(a, b, c) = a^2 + b^2 + c^2$. Can T is a linear transformation? [ANU M17, N19

Solution: Given T: $V_3(R) \to V_1(R)$ is defined by $T(a, b, c) = a^2 + b^2 + c^2$.

Claim: Test the linearity of T

Let $p, q \in R$ and $\alpha = (a, b, c), \beta = (x, y, z) \in V_3(R)$

$T(\alpha) = a^2 + b^2 + c^2$ and $T(\beta) = x^2 + y^2 + z^2$.

$p\alpha + q\beta = p(a, b, c) + q(x, y, z) = (pa + qx, pb + qy, pc + qz)$

$T(p\alpha + q\beta) = T(pa + qx, pb + qy, pc + qz) = (pa + qx)^2 + (pb + qy)^2 + (pc + qz)^2$

$pT(\alpha) + qT(\beta) = p(a^2 + b^2 + c^2) + q(x^2 + y^2 + z^2)$

$\therefore T(p\alpha + q\beta) \neq pT(\alpha) + qT(\beta)$

$\therefore$ T is not a linear transformation from $V_3(R)$ to $V_1(R)$.

SOME PARTICULAR TRANSFORMATIONS:

Theorem: Let U(F) and V(F) be two vector spaces over the same field F. Then the mapping T: U $\to$ V defined by $T(\alpha) = \hat{0}\ \forall\ \alpha \in U$ where $\hat{0}$ (zero crown) is the zero vector of V is a linear transformation.[ANU S97,

DETERMINATION OF LINEAR TRANSFORMATION

3.7. Theorem. *Let U(F) and V(F) be two vector spaces and* $S = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ *be a basis of U. Let* $\{\delta_1, \delta_2, \ldots, \delta_n\}$ *be a set of n vectors in V. Then there exists a unique linear transformation* $T : U \to V$ *such that* $T(\alpha_i) = \delta_i$ *for i = 1, 2, ... n.*

PROBLEMS

1. Describe explicitly the linear transformation T: $R^2 \to R^2$ such that T(2, 3) = (4, 5) and T(1, 0) = (0, 0). [ANU J15, J17,

Solution: Given a L.T T: $R^2 \to R^2$ such that T(2, 3) = (4, 5) and T(1, 0) = (0, 0)

Consider S = $\{(2, 3), (1, 0)\} \subseteq R^2$

Claim: To find T. For this first we prove that S is a basis of R^2

To S.T S is L.I: Suppose $a(2, 3) + b(1, 0) = (0, 0)$ where $a, b \in R$

$\Rightarrow (2a + b, 3a) = (0, 0) \Rightarrow 2a + b = 0, 3a = 0 \Rightarrow a = 0, b = 0$

$\therefore$ S is L.I subset of R^2.

To S.T L(S) = R^2: Let $(x, y) \in R^2$ and $(x, y) = a(2, 3) + b(1, 0) = (2a + b, 3a)$

$\Rightarrow 2a + b = x,\ 3a = y \ \Rightarrow a = \frac{y}{3},\ b = x - \frac{2y}{3} = \frac{3x-2y}{3}$

$\therefore (x, y) = \frac{y}{3}(2, 3) + \frac{3x-2y}{3}(1, 0)$. *i.e* L(S) = R^2 $\therefore$ S is a basis of R^2.

To find T: Now $T(x, y) = T[\frac{y}{3}(2, 3) + \frac{3x-2y}{3}(1, 0)] = \frac{y}{3}T(2, 3) + \frac{3x-2y}{3}T(1, 0)$

$$= \frac{y}{3}(4, 5) + \frac{3x-2y}{3}(0, 0) \quad = \left(\frac{4y}{3}, \frac{5y}{3}\right).$$

$\therefore$ This is the required transformation.

2. Find a linear transformation $T: R^2 \to R^2$ such that $T(1, 0) = (1, 1)$ and $T(0, 1) = (-1, 2)$. Prove that T maps the square with vertices (0, 0), (1, 0), (1, 1), (0, 1) into a parallelogram. [ANU M09, M11

Solution: Given a L.T $T: R^2 \to R^2$ such that $T(1, 0) = (1, 1)$ and $T(0, 1) = (-1, 2)$

Consider $S = \{(1, 0), (0, 1)\} \subseteq R^2$

Claim: To find T. For this first we prove that S is a basis of R^2

Clearly S is a basis of R^2. Which is called standard basis of R^2.

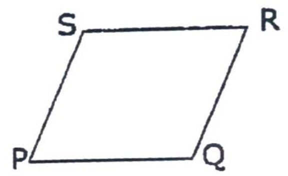
<u>**TO FIND T**</u>**:** Now $T(x, y) = T[x(1, 0) + y(0, 1)] = x\,T(1, 0) + y\,T(0, 1)$ [$\because$ T is a L.T]

$$= x(1, 1) + y(-1, 2) = (x - y, x + 2y)$$

$\therefore$ This is the required transformation.

(*ii*) Let A(0, 0), B(1, 0), C(1, 1), D(0, 1) be the vertices of a square.

Let P, Q, R, S be the T-images of A, B, C, D respectively.



Then $P = T(0, 0) = (0, 0)$

$Q = T(1, 0) = (1 - 0, 1 + 2(0)) = (1, 1)$

$R = T(1, 1) = (1 - 1, 1 + 2(1)) = (0, 3)$

$S = T90, 1) = (0 - 1, 0 + 2(1)) = (-1, 2)$

Claim: To S.T PQRS is parallelogram.

$PQ = \sqrt{(1-0)^2 + (1-0)^2} = \sqrt{1+1} = \sqrt{2}$

$RS = \sqrt{(-1-0)^2 + (2-3)^2} = \sqrt{1+1} = \sqrt{2}$

$QR = \sqrt{(0-1)^2 + (3-1)^2} = \sqrt{1+4} = \sqrt{5}$

$PS = \sqrt{(-1-0)^2 + (2-0)^2} = \sqrt{1+4} = \sqrt{5}$

Midpoint of $PR = (\frac{0+0}{2}, \frac{0+3}{2}) = (0, \frac{3}{2})$ and Midpoint of $QS = (\frac{-1+1}{2}, \frac{2+1}{2}) = (0, \frac{3}{2})$

$\therefore$ PQ = RS, QR = PS and Midpoint of PR = Midpoint of QS.

$\therefore$ PQRS is a parallelogram.

3. Find $T(x, y, z)$ where $T: R^3 \to R$ is defined by $T(1,1,1) = 3$, $T(0,1,-2) = 1$, $T(0, 0, 1) = -2$. [ANU M17, N18, N19

Solution: Given $T: R^3 \longrightarrow R$ defined by $T(1, 1, 1) = 3$, $T(0, 1, -2) = 1$, $T(0, 0, 1) = -2$

Consider $S = \{(1, 1, 1), (0, 1, -2), (0, 0, 1)\} \subseteq R^3$

Claim: To find T. For this first we prove that S is a basis of R^3

<u>To S.T S is L.I</u>: Let $a(1, 1, 1) + b(0, 1, -2) + c(0, 0, 1) = (0, 0, 0)$

$\Rightarrow (a, a + b, a - 2b + c) = (0, 0, 0)$

$\Rightarrow a = 0, a + b = 0, a - 2b + c = 0 \Rightarrow a = 0, b = 0, c = 0$

$\therefore$ S is L.I subset of R^3.

<u>To S.T L(S) = R³</u>: Let $(x, y, z) \in R^3$

Let $(x, y, z) = a(1, 1, 1) + b(0, 1, -2) + c(0, 0, 1)$

$\Rightarrow (x, y, z) = (a, a + b, a - 2b + c)$

$\Rightarrow a = x, a + b = y, a - 2b + c = z \Rightarrow a = x, b = y - x, c = z - a + 2b = -3x + 2y + z$

$\therefore (x, y, z) = x(1, 1, 1) + (y - x)(0, 1, -2) + (-3x + 2y + z)(0, 0, 1)$

$\therefore$ S spans R^3. *i.e* $L(S) = R^3$

<u>To find T</u>: $T(x, y, z) = T[x(1, 1, 1) + (y - x)(0, 1, -2) + (-3x + 2y + z)(0, 0, 1)]$

$$= xT(1, 1, 1) + (y - x)T(0, 1, -2) + (-3x + 2y + z)T(0, 0, 1)$$

$$= x(3) + (y - x)(1) + (-3x + 2y + z)(-2)$$

$$= 8x - 3y - 2z$$

SUM AND PRODCT OF LINEAR TRANSFORMATIONS:

Theorem: If T_1 and T_2 are two linear transformations from a vector space U(F) into a Vector space V(F), then the mapping $T_1 + T_2$ defined by $(T_1 + T_2)(\alpha) = T_1(\alpha) + T_2(\alpha)\ \forall\, \alpha \in U$ is a linear transformation.

Proof: Let $T_1: U \longrightarrow V$ and $T_2: U \longrightarrow V$ be two L.T's.

Given a mapping $T_1 + T_2: U \longrightarrow V$ defined by $(T_1 + T_2)(\alpha) = T_1(\alpha) + T_2(\alpha)\ \forall\, \alpha \in U$

Claim: To P.T $T_1 + T_2$ is a L.T. *i.e* To P.T $(T_1 + T_2)(a\alpha + b\beta)\ \forall\, a, b \in F\ \&\ \alpha, \beta \in U$

$\forall\, a, b \in F\ \&\ \alpha, \beta \in U$,

$(T_1 + T_2)(a\alpha + b\beta) = T_1(a\alpha + b\beta) + T_2(a\alpha + b\beta)$ [By the def $T_1 + T_2$]

$= [aT_1(\alpha) + bT_1(\beta)] + [aT_2(\alpha) + bT_2(\beta)]$ [$\because T_1, T_2$ are L.T S]

$= a[T_1(\alpha) + T_2(\alpha)] + b[T_1(\beta) + T_2(\beta)]$

$= a(T_1 + T_2)(\alpha) + b(T_1 + T_2)(\beta)$

$\therefore T_1 + T_2$ is a L.T from U into V.

SUM OF LINEAR TRANSFOR MATIONS: If T_1 and T_2 are two linear transformations from a vector space U(F) into a Vector space V(F), then the linear transformation $T_1 + T_2$ defined by $(T_1 + T_2)(\alpha) = T_1(\alpha) + T_2(\alpha)$ $\forall\ \alpha \in U$ is called sum of T_1 and T_2.

Theorem: If T: U(F) ⟶ V(F) is a linear transformation and $a \in F$. Then the mapping aT: U ⟶ V defined by $(aT)(\alpha) = a\,T(\alpha)$ $\forall\ \alpha \in U$ is a L.T.

Proof: Let T: U ⟶ V be a L.T and $a \in F$.

Given a mapping aT: U ⟶ V defined by $(aT)(\alpha) = aT(\alpha)$ $\forall\ a \in F$ & $\alpha \in U$.

Claim: To P.T aT is a L.T. *i.e* To P.T $(aT)(x\alpha + y\beta) = x\,(aT)(\alpha) + y\,(aT)(\beta)$ $\forall\, x, y \in F$ & $\alpha \in U$

$\forall\ x, y \in F$ & $\alpha \in U$, $(aT)(x\alpha + y\beta) = a\,T(x\alpha + y\beta)$ [By the def of aT]

$= a[xT(\alpha) + yT(\beta)]$ [$\because$ T is a L.T]

$= ax\,T(\alpha) + ay\,T(\beta)$

$= x(aT)(\alpha) + y(aT)(\beta)$

$\therefore aT$ is a L.T from U into V.

SCALAR MULTIPLE OF A LINEAR TRANSFORMATION: If T is a linear transformation from a vector space U(F) into a vector space V(F) and $a \in F$, then the linear transformation aT: U ⟶ V defined by $(aT)(\alpha) = a\,T(\alpha)$ $\forall\ \alpha \in U$ is called a scalar multiple of T or product of a and T.

Notation: The set of all linear transformations from a vector space U(F) into a vector space V(F) is denoted by L(U,V). Sometimes we denote this set by*Hom*(U,V).

TRANSFORMATIONS AS VECTORS:

Theorem: Let L(U, V) be the set of all linear transformations from a vector space U(F) into a vector space V(F). Then L(U, V) be a vector space relative to the operations of vector addition and scalar multiplication defined as

Definition (product of L.T'S): Let U(F), V(F) and W(F) be three vector spaces and $T: V \longrightarrow W$ and $H: U \longrightarrow V$ be two linear transformations. Then the composite function $T \circ H: U \longrightarrow W$ defined by $(T \circ H)(\alpha) = T[H(\alpha)] \ \forall \ \alpha \in U$ is a linear transformation. It is called product of linear transformations T and H. It is denoted by TH.

Important note: The product of T and H (*i.e* TH) is defined when the range of H is the domain of T.

Note: 1. Let U(F), V(F) and W(F) be three vector spaces and T_1, T_2 be two L.T'S from V to W and H_1, H_2 be two L.T's from U to V. Then (*i*) $T_1(H_1 + H_2) = T_1H_1 + T_2H_2$ (*ii*) $(T_1 + T_2)H_1 = T_1H_1 + T_2H_1$

(*iii*) $a(T_1H_1) = (aT_1)H_1 = T_1(aH_1)$.

2. If T is a linear operator on V_n, then $T^n(x_1, x_2, ..., x_n) = T^{n-1}[T(x_1, x_2, .., x_n)]$

ALGEBRA OF LINEAR OPERATORS:

Note: Let A, B, C be three linear operators on a vector space V(F). Also let $\hat{0}$ be the zero operator and I be the identity operator on V. Then

(*i*) $A\hat{0} = \hat{0}A = \hat{0}$ (*ii*) $AI = IA = A$ (*iii*) $A(B + C) = AB + AC$ (*iv*) $(A + B)C = AC + BC$
(*v*) $A(BC) = (AB)C$

PROBLEMS

1. Let T: $V_3(R) \to V_2(R)$ and H: $V_3(R) \to V_2(R)$ be the two linear transformations defined by $T(x, y, z) = (x - y, y + z)$ and $H(x,y,z) = (2x, y - z)$. Find (*i*) H + T (*ii*) *a*H

Solution: Given T: $V_3(R) \to V_2(R)$ and H: $V_3(R) \to V_2(R)$ are the two linear transformations defined by $T(x, y, z) = (x - y, y + z)$ and $H(x, y, z) = (2x, y - z)$

(*i*) $(H + T)(x, y, z) = H(x, y, z) + T(x, y, z) = (2x, y - z) + (x - y, y + z)$

$$= (3x - y, 2y)$$

(*ii*) $(aH)(x, y, z) = a(H)(x, y, z) = a(2x, y - z) = (2ax, ay - az)$

2. Let T_1: $R^3 \to R^2$ and T_2: $R^2 \to R^2$ be two linear transformations defined by $T_1(x, y, z) = (3x, 4y - z)$, $T_2(x, y) = (-x, y)$. Compute T_1T_2 and T_2T_1

Solution: Given T_1: $R^3 \to R^2$ and T_2: $R^2 \to R^2$ are two linear transformations defined by $T_1(x, y, z) = (3x, 4y - z)$, $T_2(x, y) = (-x, y)$.

(*i*) Since the range of T_2 *i.e* R^2 is not the domain of T_1 *i.e* R^3, So T_1T_2 is not defined.

(*ii*) Since the range of T_1 *i.e* R^2 is the domain of T_2 *i.e* R^2, So T_2T_1 is defined.

$\therefore (T_2T_1)(x, y, z) = T_2[T_1(x, y, z)] = T_2(3x, 4y - z) = (-3x, 4y - z)$

3. Define on R^2 linear operators H and T as follows $H(x, y) = (o, x)$ and $T(x,y) = (x, 0)$, then show that $TH = \hat{0}$, $HT \neq TH$ and $T^2 = T$ [ANU J04, J12

Solution: Given linear operators H: $R^2 \to R^2$ and T: $R^2 \to R^2$ as follows
$H(x, y) = (o, x)$ and $T(x,y) = (x, 0)$

(*i*) Since the range of T *i.e* R^2 is the domain of H *i.e* R^2, so HT is defined.

$\therefore (HT)(x, y) = H[T(x, y)] = H(x, 0) = (0, x)$

(*ii*) Since the range of H *i.e* R^2 is the domain of T *i.e* R^2, so TH is defined.

$\therefore$ (TH)(x, y) = T[H(x, y)] = T(o, x) = $(0, 0)$ = $\hat{O}(x, y)$

$\therefore$ TH = $\hat{O}$ and HT $\neq$ TH

(iii) $T^2(x, y) = T[T(x, y)] = T(x, 0) = (x, 0) = T(x, y)$ $\therefore T^2 = T$.

4. If T is a linear operator on R^2 defined by $T(x, y) = (x - y, y)$, then find the value of $T^2(x, y)$. [ANU S99,

Solution: Given a linear operator T: $R^2 \longrightarrow R^2$ defined by $T(x, y) = (x - y, y)$.

Now $T^2(x, y) = T[T(x, y)] = T(x - y, y) = (x - y - y, y) = (x - 2y, y)$.

5. Let T: $R^3 \longrightarrow R^2$ and H: $R^3 \longrightarrow R^2$ be defined by $T(x, y, z) = (3x, y + z)$ and $H(x, y, z) = (2x - z, y)$. Compute (i) T + H (ii) 4T – 5H (iii) TH (iv) HT [ANU J15,

Solution: Given T: $R^3 \longrightarrow R^2$ and H: $R^3 \longrightarrow R^2$ be defined by $T(x, y, z) = (3x, y + z)$ and $H(x, y, z) = (2x - z, y)$.

(i) $(T + H)(x, y, z) = T(x,y,z) + H(x,y,z) = (3x, y+z) + (2x - z, y) = (5x - z, 2y + z)$

(ii) $(4T - 5H)(x, y, z) = 4T(x, y, z) - 5H(x, y, z) = 4(3x, y + z) - 5(2x - z, y)$

$$= (12x, 4y + 4z) - (10x - 5z, 5y) = (2x + 5z, -y + 4z)$$

(iii) and (iv) both, TH and HT are not defined because the range of T is not equal to the domain of H and vice versa.

6. If T: $V_3 \longrightarrow V_3$ and S: $V_3 \longrightarrow V_3$ are two L.T's defined by $T(e_1) = e_1 + e_2$, $T(e_2) = e_3$, $T(e_3) = e_2 - e_3$ and $S(e_1) = e_3$, $S(e_2) = 2e_2 - e_3$, $S(e_3) = \bar{0}$ where $\{e_1, e_2, e_3\}$ is the standard basis of $V_3(R)$. Find (i) T + S (ii) 2T (iii) 3T – 2S (iv) ST (v) TS [ANU M14

Solution: Given T: $V_3 \longrightarrow V_3$ and S: $V_3 \longrightarrow V_3$ are two L.T's defined by $T(e_1) = e_1 + e_2$, $T(e_2) = e_3$, $T(e_3) = e_2 - e_3$ and $S(e_1) = e_3$, $S(e_2) = 2e_2 - e_3$, $S(e_3) = \bar{0}$ where $\{e_1,e_2,e_3\}$ is the standard basis of $V_3(R)$.

(i) $(T + S)(e_1) = T(e_1) + S(e_1) = (e_1 + e_2) + e_3 = e_1 + e_2 + e_3$

$(T + S)(e_2) = T(e_2) + S(e_2) = e_3 + (2e_2 - e_3) = 2e_2$

$(T + S)(e_3) = T(e_3) + S(e_3) = e_2 - e_3 + \bar{0} = e_2 - e_3$

(ii) $(2T)(e_1) = 2\,T(e_1) = 2(e_1 + e_2) = 2e_1 + 2e_2$

$(2T)(e_2) = 2\,T(e_2) = 2e_3$

$(2T)(e_3) = 2\,T(e_3) = 2(e_2 - e_3) = 2e_2 - 2e_3$

(iii) $(3T - 2S)(e_1) = 3T(e_1) - 2S(e_1) = 3\,(e_1 + e_2) - 2(e_3) = 3e_1 + 3e_2 - 2e_3$

$(3T - 2S)(e_2) = 3T(e_2) - 2S(e_2) = 3\,(e_3) - 2(2e_2 - e_3) = 5e_3 - 4e_2$

$(3T - 2S)(e_3) = 3T(e_3) - 2S(e_3) = 3\,(e_2 - e_3) - 2(\bar{0}) = 3e_2 - 3e_3$

(iv) $(ST)(e_1) = S[T(e_1)] = S(e_1 + e_2) = S(e_1) + S(e_2) = e_3 + (2e_2 - e_3) = 2e_2$

$(ST)(e_2) = S[T(e_2)] = S(e_3) = \bar{0}$

$(ST)(e_3) = S[T(e_3)] = S(e_2 - e_3) = S(e_2) - S(e_3) = (2e_2 - e_3) - \bar{0} = 2e_2 - e_3$

(v) $(TS)(e_1) = T[S(e_1)] = T(e_3) = e_2 - e_3$

$(TS)(e_2) = T[S(e_2)] = T(2e_2 - e_3) = 2T(e_2) - T(e_3) = 2(e_3) - (e_2 - e_3) = 3e_3 - e_2$

$(TS)(e_3) = T[S(e_3)] = T(\bar{0}) = \bar{0}$

7. Suppose S and T are L.T's on R^3 given by $S(x_1, x_2, x_3) = (x_2 + x_3,\ x_3 + x_1,\ x_1 + x_2)$ and $T(x_1, x_2, x_3) = (x_1 - x_2,\ x_2 - x_3,\ x_3 - x_1)$. Is ST = TS? [ANU O91,

Solution: Given L.T's T: $R^3 \longrightarrow R^3$ and S: $R^3 \longrightarrow R^3$ are defined by $S(x_1, x_2, x_3) = (x_2 + x_3,\ x_3 + x_1,\ x_1 + x_2)$ and $T(x_1, x_2, x_3) = (x_1 - x_2,\ x_2 - x_3,\ x_3 - x_1)$.

Since S and T are linear operators on R^3, So ST and TS are defined.

Now $(ST)(x_1, x_2, x_3) = S[T(x_1, x_2, x_3)] = S(x_1 - x_2,\ x_2 - x_3,\ x_3 - x_1)$

$= (x_2 - x_3 + x_3 - x_1,\ x_3 - x_1 + x_1 - x_2,\ x_1 - x_2 + x_2 - x_3)$

$= (x_2 - x_1,\ x_3 - x_2,\ x_1 - x_3)$

$(TS)(x_1, x_2, x_3) = T[S(x_1, x_2, x_3)] = T(x_2 + x_3,\ x_3 + x_1,\ x_1 + x_2)$

$= [(x_2 + x_3) - (x_3 + x_1), (x_3 + x_1) - (x_1 + x_2), (x_1 + x_2) - (x_2 + x_3)]$

$= (x_2 - x_1,\ x_3 - x_2,\ x_1 - x_3)$

$\therefore$ ST = TS.

8. If T: $V_3(R) \longrightarrow V_3(R)$ is a L.T's defined by $T(a, b, c) = (3a,\ a - b,\ 2a + b + c)$, then find $(T^2 - I)(T - 3I) = \hat{0}$.[ANU J04, J12

Solution: Given T: $V_3(R) \longrightarrow V_3(R)$ is a linear operator defined by

$T(a, b, c) = (3a,\ a - b,\ 2a + b + c)$,

We know that the identity operator on $V_3(R)$ is defined as $I(a, b, c) = (a, b, c)$.

$T^2(a, b, c) = T[T(a, b, c)] = T(3a,\ a - b,\ 2a + b + c)$

$= (3(3a),\ 3a - (a - b),\ 2(3a) + (a - b) + (2a + b + c))$

$= (9a,\ 2a + b,\ 9a + c)$

$(T^2 - I)(a, b, c) = T^2(a, b, c) - I(a, b, c) = (9a,\ 2a+b,\ 9a+c) - (a,b,c) = (8a,\ 2a,\ 9a)$

$(T - 3I)(a, b, c) = T(a, b, c) - 3I(a, b, c) = (3a,\ a - b,\ 2a + b + c) - 3(a, b, c)$

$$= (0,\ a - 4b,\ 2a + b - 2c)$$

$$[(T^2 - I)\ (T - 3I)](a, b, c) = (T^2 - I)[(T - 3I)(\ a, b, c)]$$

$$= (T^2 - I)(\ 0,\ a - 4b,\ 2a + b - 2c)$$

$$= \big(8(0),\ 2(0),\ 9(0)\big) = (0, 0, 0) = \hat{0}(a, b, c)$$

$\therefore (T^2 - I)(T - 3I) = \hat{0}$

9. Let $T: R^3 \longrightarrow R^2$ and $H: R^2 \longrightarrow R^3$ be two L.T's defined by $T(x, y, z) = (x - 3y - 2z, y - 4z)$ and $H(x, y) = (2x, 4x - y, 2x + 3y)$. Find HT and TH. Is product commutative?

Solution: Given $T: R^3 \longrightarrow R^2$ and $H: R^2 \longrightarrow R^3$ be two L.T's defined by $T(x, y, z) = (x - 3y - 2z,\ y - 4z)$ and $H(x, y) = (2x,\ 4x - y,\ 2x + 3y)$.

(i) Since the range of T $i.e$ R^2 is the domain of H $i.e$ R^2, so HT is defined.

$$(HT)(x, y, z) = H[T(x, y, z)] = H(x - 3y - 2z,\ y - 4z)$$

$$= \big(2(x - 3y - 2z),\ 4(x - 3y - 2z) - (y - 4z),\ 2(x - 3y - 2z) + 3(y - 4z)\big)$$

$$= (2x - 6y - 4z,\ 4x - 13y - 4z,\ 2x - 3y - 16z)$$

(ii) Since the range of H $i.e$ R^3 is the domain of T $i.e$ R^3, so TH is defined.

$$\therefore (TH)(x, y) = T[H(x, y)] = T(2x,\ 4x - y,\ 2x + 3y)$$

$$= \big(2x - 3(4x - y) - 2(2x + 3y),\ (4x - y) - 4(2x + 3y)\big)$$

$$= (-14x - 3y,\ -4x - 13y)$$

$\therefore$ HT $\neq$ TH. Hence the product is not commutative.

10. Give an example of a linear operator T on R^3 such that $T \neq \hat{0}$, $T^2 \neq \hat{0}$ but $T^3 = \hat{0}$.

Solution: Define a mapping $T: R^3 \longrightarrow R^3$ by $T(x, y, z) = (0, x, y)$ such that $T \neq \hat{0}$

First we show that T is a L.T

Let $a, b \in R$ and $\alpha = (x_1, y_1, z_1)$, $\beta = (x_2, y_2, z_2) \in R^3$

$$T(a\alpha + b\beta) = T(ax_1 + bx_2,\ ay_1 + by_2,\ ax_3 + bx_3)$$

$$= (0,\ ax_1 + bx_2,\ ay_1 + by_2) = a(0, x_1, y_1) + b(0, x_2, y_2) = aT(\alpha) + bT(\beta)$$

$\therefore$ T is a linear operator on R^3.

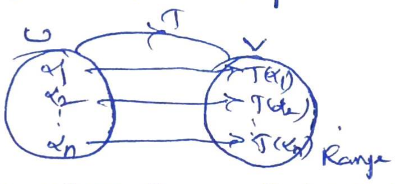
$T^2(x, y, z) = T[T(x, y, z)] = T(0, x, y) = (0, 0, x)$ $\qquad \therefore T^2 \neq \hat{0}$

$T^3(x, y, z) = T^2[T(x, y, z)] = T^2(0, x, y) = (0, 0, 0) = \hat{0}(x, y, z).$ $\qquad \therefore T^3 = \hat{0}$

Range and Null Space of a Linear Transformation:

Range:- Let $U(F)$ and $V(F)$ be two Vector Spaces. If $T: U \to V$ is a Linear transformation, then the set $\{T(\alpha) \mid \alpha \in U\}$ is Called the Range of T (or) Image of T. It is denoted by $R(T)$.

$$\therefore \quad R(T) = \{T(\alpha) \mid \alpha \in U\}.$$



Note:- 1) Range of T is a Subset of V. i.e $R(T) \subseteq V$.

2) Let $U(F)$ & $V(F)$ be two Vector Spaces. If $T: U \to V$ is a Linear transformation, then the range set $R(T)$ is a Subspace of $V(F)$.

Eg ①. Describe a Linearly transformation from $V_3(R)$ into $V_3(R)$ which has its range the subspace spanned by $(1,0,-1)$ and $(1,2,2)$.

Sol:- Let $S = \{(1,0,-1), (1,2,2)\} \subseteq V_3(R)$ which Spans $R(T)$

Let include a vector $(0,0,0) \in V_3(R)$ in this spanning set which will not effect the spanning property.

$$\therefore \quad S = \{(1,0,-1), (1,2,2), (0,0,0)\}.$$

Let $B = \{e_1, e_2, e_3\}$ be the Standard basis of $V_3(R)$ (or) R^3.

We know that, $\exists$ a L.T 'T' $\ni$ $T(e_i) = \delta_i$.

i.e $T(e_1) = (1,0,-1)$, $T(e_2) = (1,2,2)$, $T(e_3) = (0,0,0)$.

Now the vectors $T(e_1), T(e_2), T(e_3)$ Spans the Range of T.

i.e the Vectors $(1,0,-1), (1,2,2), (0,0,0)$ Spans the $R(T)$.

thus the Range of T is the Subspace of $V_3(R)$ Spanned by $\{(1,0,-1), (1,2,2)\}$.

$\therefore$ T is the required Linear transformation.

Now, If $\alpha \in V_3(R) \Rightarrow \alpha = (a,b,c)$ and $\alpha = ae_1 + be_2 + ce_3$.

$\therefore T(a,b,c) = T(ae_1+be_2+ce_3)$

$= a\,T(e_1) + bT(e_2) + c\cdot T(e_3)$

$= a\cdot(1,0,-1) + b(1,2,2) + c(0,0,0)$

$= (a+b,\ 2b,\ -a+2b)$

$\therefore T(a,b,c) = (a+b,\ 2b,\ -a+2b)$.

this is the required Linear Transformation.

Eg ②:- find a Linear transformation $T: R^3 \to R^4$ whose range is spanned by $\{(1,-1,2,3),(2,3,-1,0)\}$.

Soln Let $S = \{(1,-1,2,3),(2,3,-1,0)\} \subseteq R^4(R)$ which spans $R(T)$

Let us include a vector $(0,0,0,0) \in V_4(R)$ in this spanning set which will not effect the spanning property.

$\therefore S = \{(1,-1,2,3),(2,3,-1,0),(0,0,0,0)\}$.

Let $B = \{e_1, e_2, e_3\}$ be the standard basis of $V_3(R)$ (or) R^3.

By the know theorem $\exists$ a unique L.T $T: V_3(R) \to V_4(R)$ (or) $T: R^3 \to R^4 \ni T(e_1) = (1,-1,2,3)$, $T(e_2) = (2,3,-1,0)$ &

$T(e_3) = (0,0,0,0)$.

Now the vectors $T(e_1), T(e_2), T(e_3)$ spans the Range of T.

i.e the vectors $(1,-1,2,3),(2,3,-1,0)$ & $(0,0,0,0)$ spans the $R(T)$.

$\therefore T$ is the req. Linear Transformation.

If $\alpha \in R^3 \Rightarrow \alpha = (a,b,c)$ and $\alpha = ae_1+be_2+ce_3$.

$\therefore T(a,b,c) = T(ae_1+be_2+ce_3) = aT(e_1)+bT(e_2)+cT(e_3)$

$= a(1,-1,2,3)+b(2,3,-1,0)+c(0,0,0,0)$

$= (a+2b,\ -a+3b,\ 2a-b,\ 3a)$

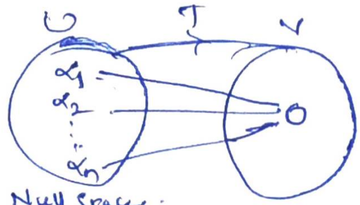
$\therefore T(a,b,c) = (a+2b,\ -a+3b,\ 2a-b,\ 3a)$

this is the req. Linear Transformation.

Null Space (or) Kernel :-

Let $U(F)$ and $V(F)$ be two Vector Spaces. If $T: U \to V$ is a linear transformation, then the set $\{\alpha \in U \mid T(\alpha) = \bar{0} \in V\}$ is called Null Space (or) Kernel of T. It is denoted by $N(T)$ (or) Ker T.

$\therefore N(T) = \text{Ker}\,T = \{\alpha \in U \mid T(\alpha) = \bar{0}\}$. where $\bar{0}$ is a zero vector in V.



Null Space.

Note:- 1) Null Space is a Subspace (Subset) of U i.e $N(T) \subseteq U$

2) Let $T: U(F) \to V(F)$ be a linear transformation. If U is finite Dimensional Vector Space then the range space R(T) is a finite Dimensional Vector Space of $V(F)$.

Dimension of Range and Kernel :-

1) Rank of a L.T (or) Dimension of Range :-

If $T: U(F) \to V(F)$ is a linear transformation, where U is finite Dimensional Vector Space, then the Dimension of Range Space $R(T)$ (no. of elements in $R(T)$) is called Rank of T. It is denoted by $\rho(T)$.

$\therefore$ Rank of $T = \rho(T) = \dim R(T)$.

Nullity of a L.T (or) Dimension of a Null Space :-

If $T: U(F) \to V(F)$ is a linear transformation, where U is a finite Dimensional Vector Space, then the Dimension of Null Space $N(T)$ is called Nullity of T. It is denoted by $\nu(T)$.

$\therefore$ Nullity of $T = \nu(T) = \dim N(T)$.

Rank - Nullity theorem (or) Dimension theorem :-

Statement :- If $U(F)$ & $V(F)$ be two Vector Spaces and $T: U \to V$ be a linear transformation. Let U be a finite Dimensional Vector Space, then

$$\rho(T) + \nu(T) = \dim U.$$

i.e $\quad \text{rank}(T) + \text{Nullity}(T) = \dim U.$

Eg ①:- Find the Kernel of the linear transformation $T: R^2 \to R^2$ defined by $T(1,0) = (1,1)$ & $T(0,1) = (-1,2)$.

Soln

G.T $T: R^2 \to R^2$ is a L.T defined by

$T(1,0) = (1,1)$ & $T(0,1) = (-1,2)$

$\therefore$ If $(x,y) \in R^2$ then

$$
\begin{aligned}
T(x,y) &= T[x(1,0) + y(0,1)] \\
&= xT(1,0) + yT(0,1) \\
&= x(1,1) + y(-1,2) \\
&= (x-y, x+2y)
\end{aligned}
$$

$\therefore T(x,y) = (x-y, x+2y)$

Now, by defination of Kernel of T.

$T(x,y) = \bar{0}$

i.e $(x-y, x+2y) = (0,0)$

On equating Corresponding elements, both sides, We get

$x - y = 0.$

$x + 2y = 0.$

on solving above two equations, We get

$x = 0, y = 0.$

$\therefore \text{Ker } T = \{(x,y) \in R^2 / T(x,y) = (0,0)\} = \{(0,0)\}.$

Eg ②. Find the Null space, range, Rank and Nullity of the linear transformation $T: R^2 \to R^3$ defined by
$T(x,y) = (x+y, x-y, y)$.

Soln

G.T $T: R^2 \to R^3$ is a L.T defined by

$T(x,y) = (x+y, x-y, y)$. Clearly $\dim R^2 = 2$

To find $R(T)$ & $\dim R(T)$:-

We know that,
$S = \{(1,0), (0,1)\}$ is the standard basis of R^2.

$\therefore T(1,0) = (1,1,0)$
$T(0,1) = (1,-1,1)$.

Let $S_1 = \{(1,1,0), (1,-1,1)\}$.

$\therefore$ Range of $T = R(T) =$ the subspace spanned by S_1.

$\therefore$ Arrange the vectors of S_1 as rows of a matrix & reducing it to Echelon form, we get

$$S_1 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix}$$

$R_2 \to R_2 - R_1$.

$$\sim \begin{pmatrix} 1 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix}$$

$\therefore$ this is in Echelon form.

Hence the two non-zero rows $(1,1,0), (0,-2,1)$ form the least L.I set and it forms a basis of $R(T)$

$\therefore$ Basis of $R(T) = \{(1,1,0), (0,-2,1)\}$.

& $\rho(T) = \dim R(T) = 2$.

To find $N(T)$ & $\dim N(T)$:-

Let $\alpha \in R^2$ & $\alpha = (x,y) \in N(T)$.

then $T(x,y) = \bar{0}$ ($\because$ by def.)

$\Rightarrow (x+y, x-y, y) = (0,0,0)$.

$\therefore x+y=0.$

$x-y=0$ & $y=0.$

On Sub. $y=0$ in above, we get $x=0.$

$\therefore x=y=0.$

$\therefore (x,y)=(0,0) = \bar{0}$

$\therefore N(T) = \{(x,y)\} = \{(0,0)\}$

$\therefore \nu(T) = \dim N(T) =$ Nullity of $T = 0.$

Eg ③: Verify the Rank-Nullity Theorem, for the linear transformation $T: R^3 \to R^3$ defined by

$T(x,y,z) = (x-y,\ 2y+z,\ x+y+z).$

Soln

Given $T: R^3 \to R^3$ defined by

$T(x,y,z) = (x-y,\ 2y+z,\ x+y+z).$

$\therefore$ clearly, $\dim R^3 = 3.$

To find dim $R(T)$:-

W.K.T $S = \{(1,0,0),(0,1,0),(0,0,1)\}$ is the standard basis of R^3.

Now, $T(1,0,0) = (1,0,1)$

$T(0,1,0) = (-1,2,1)$

$T(0,0,1) = (0,1,1).$

$\therefore R(T) =$ the subspace spanned by $S_1 = \{(1,0,1),(-1,2,1),(0,1,1)\}.$

Arranging the vectors of S_1, as rows of a matrix & reducing it to Echelon form, we get

$$S = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

$R_2 \to R_2 + R_1$

$$\sim \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 2 \\ 0 & 1 & 1 \end{pmatrix}$$

$R_3 \to 2R_3 - R_2$

$$\sim \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

$R_2 \to R_2/2,$

$$\therefore S \sim \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Hence the two non zero rows $(1,0,1), (0,1,1)$ form the least L.I set and it forms a bases of $R(T)$.

$\therefore$ Bases of $R(T) = \{(1,0,1), (0,1,1)\}$

Dim $R(T) = \rho(T) = 2$.

To find dim $N(T)$:-

Let $\alpha = (x,y,z) \ni T(\alpha) = \bar{0}$.

i.e $T(x,y,z) = (0,0,0)$

$\Rightarrow (x-y, 2y+z, x+y+z) = (0,0,0)$.

$\therefore x-y=0,$

$2y+z=0$

$x+y+z=0$.

On solving the above eq's, we get

$x=y, \quad 2y+z=0, \quad x+y+z=0$.

i.e $y=x$ & $2y=-z$.

$\therefore z=-2y=-2x$.

$\therefore (x,y,z) = (x,x,-2x) = x(1,1,-2)$.

this is a L.C. of vectors.

$\therefore$ Bases of $N(T) = \{(1,1,-2)\}$.

$\therefore$ dim $N(T) = 1 = \nu(T)$

Now, $\rho(T)+\nu(T) = 2+1 = 3 = \dim R^3$.

$\therefore$ Rank - Nullity theorem is Verified

Eg (4):- Verify the Rank-Nullity Theorem for the Linear map $T: V_4 \to V_3$ defined by $T(1,0,0,0) = (1,1,1)$, $T(0,1,0,0) = (1,-1,1)$, $T(0,0,1,0) = (1,0,0)$ & $T(0,0,0,1) = (1,0,1)$.

(or).

$T(e_1) = (1,1,1)$, $T(e_2) = (1,-1,1)$, $T(e_3) = (1,0,0)$ & $T(e_4) = (1,0,1)$.

(or)

Verify the Rank-Nullity Theorem for the Linear map $T: V_4 \to V_3$ defined by $T(e_1) = f_1 + f_2 + f_3$, $T(e_2) = f_1 - f_2 + f_3$, $T(e_3) = f_1$ and $T(e_4) = f_1 + f_3$ where $\{e_1, e_2, e_3, e_4\}$ & $\{f_1, f_2, f_3\}$ are Standard bases of V_4 & V_3 resp.

Sol:- G.T $T: V_4 \to V_3$ is a L.T and defined by

$T(1,0,0,0) = (1,1,1)$, $T(0,1,0,0) = (1,-1,1)$

$T(0,0,1,0) = (1,0,0)$, $T(0,0,0,1) = (1,0,1)$.

Clearly, $\dim V_4 = 4$.

To find $\dim R(T)$:-

$R(T)$ = the Subspace Spanned by

$S_1 = \{(1,1,1), (1,-1,1), (1,0,0), (1,0,1)\}$.

Arranging the Vectors of S_1 as rows of a matrix and reducing it to Echelon form, we get,

$$S_1 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$, $R_4 \to R_4 - R_1$.

$$\sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & -2 & 0 \\ 0 & -1 & -1 \\ 0 & -1 & 0 \end{pmatrix}$$

$R_3 \to 2R_3 - R_2$, $R_4 \to 2R_4 - R_2$.

$$\sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{pmatrix}$$

Hence the three non zero ~~vectors~~ $(1,1,1)$, $(0,-2,0)$, $(0,0,-2)$ form the Least L.I set and it forms a basis of $R(T)$.

$\therefore$ Basis of $R(T) = \{(1,1,1), (0,-2,0), (0,0,-2)\}$.

$\therefore$ dim $R(T) = \rho(T) = 3$.

To find dim $N(T)$:-

Let $\alpha = (x, y, z, t) \in R^4$ such that $T(\alpha) = \bar{0}$.

$\Rightarrow T(x,y,z,t) = (0,0,0,0)$

$\Rightarrow T[x(1,0,0,0) + y(0,1,0,0) + z(0,0,1,0) + t(0,0,0,1)] = (0,0,0,0)$

$\Rightarrow xT(1,0,0,0) + yT(0,1,0,0) + zT(0,0,1,0) + tT(0,0,0,1) = (0,0,0,0)$

$\Rightarrow x(1,1,1) + y(1,-1,1) + z(1,0,0) + t(1,0,1) = (0,0,0,0)$

$\Rightarrow (x+y+z+t,\ x-y,\ x+y+t) = (0,0,0)$

on equating corresponding elements, we get

$x+y+z+t = 0$.

$x-y = 0$

$x+y+t = 0$.

Write the co-efficient matrix of the above system (Homogeneous) and reducing it to Echelon form.

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}$$

$R_2 \to R_2 - R_1,\ R_3 \to R_3 - R_1$.

$$\sim \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & -1 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

This is on Echelon form. and

$\rho(A) = 3 \neq$ no. of unknowns.

$\therefore$ the given system have non zero solution.

the above system can be written as

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & -1 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\therefore$ $x+y+z+t = 0$. —— (1)

$2y+z+t = 0$ —— (2)

$z = 0$. —— (3).

Subs eq (3) in eq (2), we get

$t = -2y$. —(4).

Subs (3) & (4) in eq (1), we get.

$x + y + 0 - 2y = 0$.

$x - y = 0$.

$\therefore x = y$.

$\therefore x = y, \; t = -2y, \; w = 0$.

$\therefore (x, y, z, t) = (y, y, 0, -2y) = y(1, 1, 0, -2)$.

$\therefore$ The non zero vector forms a basis of $N(T)$

i.e $(1, 1, 0, -2)$ " " " .

$\therefore N(T) = \{(1, 1, 0, -2)\}$.

$\dim N(T) = 1 = \nu(T)$

Now,

$\rho(T) + \nu(T) = 3 + 1 = 4 = \dim V_4$.

$\therefore$ Rank - Nullity theorem verified