3.7

STOKES'S THEOREM

The Green's theorem derived in last module can be written in a vector form different as $\oint_C (\boldsymbol{V}.\boldsymbol{n})ds = \iint_R (\nabla.\boldsymbol{V})dx\,dy$. Let C be a curve in two dimenstions which is written in the parametric form $\boldsymbol{r} = \boldsymbol{r}(s)$. Then, the unit tangent vector to C is given by

$$\boldsymbol{T} = \frac{dx}{ds}\boldsymbol{i} + \frac{dy}{ds}\boldsymbol{j}$$

Let $\boldsymbol{V}$ be written in the form $\boldsymbol{V} = g\,\boldsymbol{i} - f\,\boldsymbol{j}$.

Then,

$$\boldsymbol{V}.\boldsymbol{T} = (g\,\boldsymbol{i} - f\,\boldsymbol{j}).\left(\frac{dx}{ds}\boldsymbol{i} + \frac{dy}{ds}\boldsymbol{j}\right) = g\frac{dx}{ds} - f\frac{dy}{ds}.$$

By Green's theorem, we have

$$\oint_C \boldsymbol{V}.d\boldsymbol{r} = \oint_C \mathbf{V}.\mathbf{T}\,\mathrm{ds} = \oint_C g\,dx - f\,dy = \iint_R -\left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}\right)dx\,dy$$

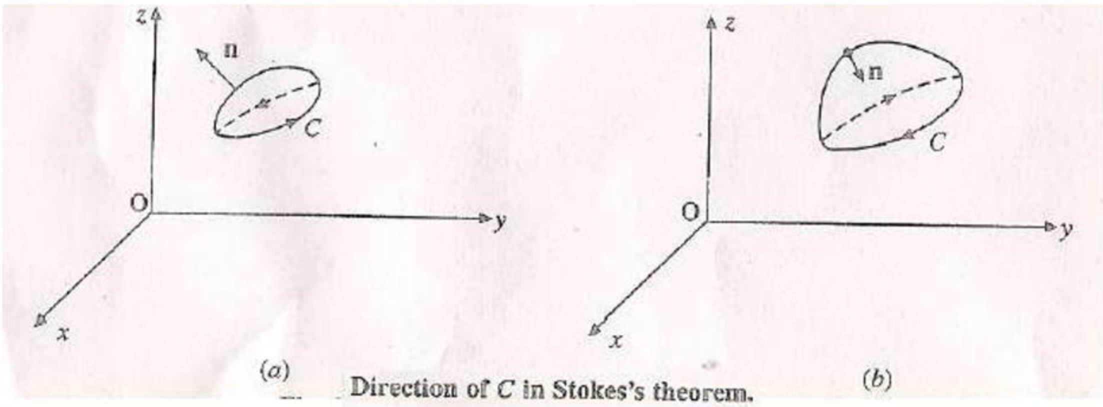
$$= \iint_R (\nabla \times \mathbf{V}).\mathbf{k}\,dx\,dy.$$

This result can be considered as a particular case of the Stokes's theorem. Extension of the Green's theorem to three dimensions can be done under the following generalizations.

i. The closed curve C enclosing R in the plane $\longrightarrow$ the closed curve C bounding an open smooth orientable surface S(open two sided surface).

ii. The unit normal $\boldsymbol{n}$ to $C \longrightarrow$ the outward or inward normal $\boldsymbol{n}$ to S.

iii. Counter clockwise direction of $C \longrightarrow$ the direction of C is governed by the direction of the normal $\boldsymbol{n}$ to S. If $\boldsymbol{n}$ is taken as outward normal, then C is oriented as right handed screw and if $\boldsymbol{n}$ is taken as inward normal, then C is oriented as left handed screw (figure (a),(b)).



Direction of C in Stokes's theorem.

Theorem (Stokes's theorem)

Let S be a piecewise smooth orientable surface bounded by a piecewise smooth simple closed curve C. Let $\boldsymbol{V}(x, y, z) = v_1(x, y, z)\boldsymbol{i} + v_2(x, y, z)\boldsymbol{j} + v_3(x, y, z)\boldsymbol{k}$ be a vector function which is continuous and has continuous first order partial derivatives in a domain which contains S. If C is traversed in the positive direction, then

$$\oint_C \boldsymbol{V}.d\boldsymbol{r} = \oint_C (\mathbf{V}.\mathbf{T})\mathrm{ds} = \iint_S (\nabla \times \mathbf{V}).\mathbf{n}\,\mathrm{dA}$$

Where $\boldsymbol{n}$ is the unit normal vector to S in the direction of orientation of C.

In terms of components of $\boldsymbol{V}$ we have

$$\oint_C [v_1(x,y,z)dx + v_2(x,y,z)dy + v_3(x,y,z)dz] = \iint_S (\nabla \times \mathbf{V}).\mathbf{n}\, \mathrm{dA}$$

Proof: We shall prove the theorem for the special case when the equation of the surface can be written simultaneously in the forms

$$z = f(x,y),\ y = g(x,z),\ \text{and } x = h(y,z)$$

Where f , g , h are continuous functions and have continuous first order partial derivatives.

Let the surface S be oriented upward. Consider the case when the equation of the surface is written as $z = f(x,y)$. If we write $g(x,y,z) = z - f(x,y) = 0$ then the unit normal is given by

$$\boldsymbol{n} = \frac{-(\partial f/\partial x)\boldsymbol{i} - (\partial f/\partial y)\boldsymbol{j} + \boldsymbol{k}}{\sqrt{1 + (\partial f/\partial x)^2 + (\partial f/\partial y)^2}}$$

Also $\boldsymbol{n} = \cos\alpha\, \boldsymbol{i} + \cos\beta\, \boldsymbol{j} + \cos\gamma\, \boldsymbol{k}$, where α,β,γ are the angles which the unit normal makes with the positive directions of X, Y and Z axis respectively. Comparing, we have

$$\frac{\cos\alpha}{-(\partial f/\partial x)} = \frac{\cos\beta}{-(\partial f/\partial y)} = \frac{\cos\gamma}{1} \dots\dots\dots\dots\dots\dots\dots (A)$$

We need to show that

$$\oint_C [v_1(x,y,z)dx + v_2(x,y,z)dy + v_3(x,y,z)dz]$$

$$= \iint_S [\left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right)\cos\alpha + \left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}\right)\cos\beta + \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right)\cos\gamma]dA$$

Using the equation of the surface as $z = f(x, y)$, we shall prove that

$$\oint_C v_1(x, y, z)dx = \iint_S \left(\frac{\partial v_1}{\partial z} \cos\beta - \frac{\partial v_1}{\partial y} \cos\gamma \right) dA \text{.........(1)}$$

Let R be the projection of S and C^* be the projection of the bounding curve C on the XY – plane. Then,

$$\oint_C v_1(x, y, z)dx = \oint_{C^*} v_1[x, y, f(x, y)]dx$$

$$= \iint_R -\frac{\partial}{\partial y} v_1[x, y, f(x, y)]\, dxdy$$

(By Green's theorem)

$$= -\iint_R \left(\frac{\partial v_1}{\partial y} + \frac{\partial v_1}{\partial z} \frac{\partial f}{\partial y} \right) dxdy$$

$$= -\iint_R \left[\frac{\partial v_1}{\partial y} - \frac{\partial v_1}{\partial z} \frac{\cos\beta}{\cos\gamma} \right] dxdy \qquad \text{(By (A))}$$

$$= -\iint_S \left[\frac{\partial v_1}{\partial y} - \frac{\partial v_1}{\partial z} \frac{\cos\beta}{\cos\gamma} \right] \cos\gamma\, dA$$

since $dA = \frac{dx\ dy}{\boldsymbol{n}.\boldsymbol{k}} = \frac{dxdy}{\cos\gamma}$. Hence,

$$\oint_C v_1(x, y, z)dx = -\iint_S \left(\frac{\partial v_1}{\partial y} \cos\gamma - \frac{\partial v_1}{\partial z} \cos\beta \right) dA$$

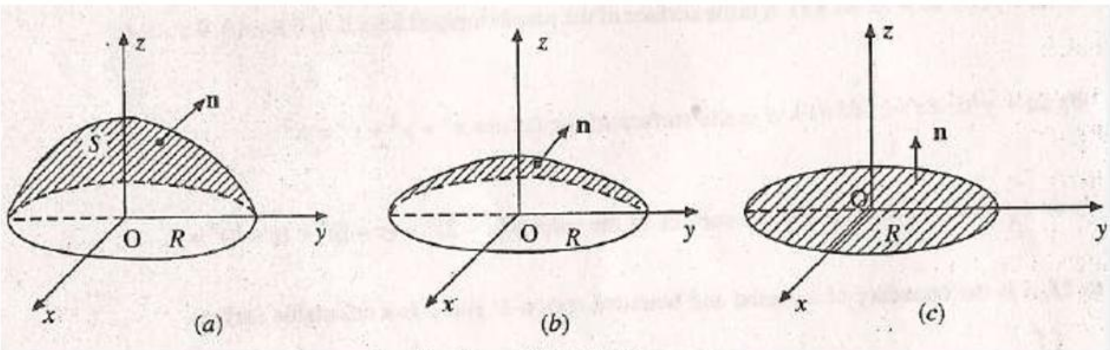
$$= \iint_S \left(\frac{\partial v_1}{\partial z} \cos\beta - \frac{\partial v_1}{\partial y} \cos\gamma \right) dA$$

and the result in equation (1) is proved.

Similarly, assuming the equation of the surface as $y = g(x, z)$ and $x = h(y, z)$ we can prove the equality of the terms corresponding to the components $v_2(x, y, z)$ and $v_3(x, y, , z)$.

Remarks:

1. As in divergence theorem, the theorem holds if the given surface S can subscribed into finitely many special surfaces such that each of these surfaces can be described in the required manner.
2. Stokes's theorem states that the value of the surface integral is same for any surface as long as the bounding curve, bounding the projection R on any coordinate plane, is the same curve C. Hence, in the degenerate case, when S coincides with R, we can take $\boldsymbol{n} = \boldsymbol{k}$ or $\boldsymbol{j}$ or $\boldsymbol{i}$ depending on whether the projection is taken on the $XY -$ plane or $XZ -$ plane or $YZ -$ plane (below figure).



Application of Stoke's theorem.

Example: Verify Stokes's theorem for the vector field $\boldsymbol{V} = (3x - y)\boldsymbol{i} - 2yz^2\boldsymbol{j} - 2y^2z\boldsymbol{k}$, where S is the surface of the sphere $x^2 + y^2 + z^2 = 16, z > 0$.

Solution:

Consider projection of S on the x-y plane. The projection is the circular region $x^2 + y^2 \leq 16$, $z = 0$ and the bounding curve C is the circle $z = 0, x^2 + y^2 = 16$. We have

$$\oint_C V.dr = \oint_C (3x - y)dx - 2yz^2dy - 2y^2zdz = \oint_C (3x - y)dx$$

Since $z = 0$. Setting $x = 4\cos\theta,\ y = 4\sin\theta$, we obtain

$$\oint_C (3x - y)dx = \int_0^{2\pi} 4(3\cos\theta - \sin\theta)(-4\sin\theta)d\theta$$

$$= -16\int_0^{2\pi}\left[\frac{3}{2}\sin 2\theta - \frac{1}{2}(1 - \cos 2\theta)\right]d\theta$$

$$= 16\left(\frac{1}{2}\right)2\pi = 16\pi.$$

$$\text{Now, } \nabla\times\boldsymbol{V} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x - y & -2yz^2 & -2y^2z \end{vmatrix}$$

$$= \boldsymbol{i}(-4yz + 4yz) - \boldsymbol{j}(0) + \boldsymbol{k}(1) = \boldsymbol{k}$$

$$\boldsymbol{n} = \frac{2(x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k})}{2\sqrt{x^2 + y^2 + z^2}} = \frac{1}{4}(x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}).\ (\nabla\times V).\boldsymbol{n} = \frac{z}{4}$$

$$\text{Therefore, } \iint_S (\nabla\times\boldsymbol{V}).\boldsymbol{n}dA = \iint_S \frac{z}{4}dA = \iint_R \frac{z}{4}\frac{dx\,dy}{\boldsymbol{n}.\boldsymbol{k}} = \iint_R \frac{z}{4}\frac{dx\,dy}{(z/4)}$$

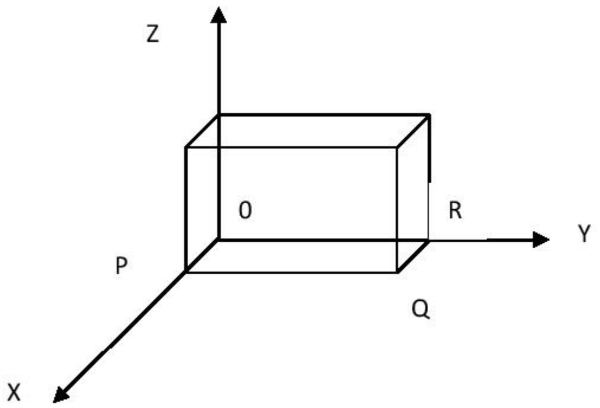
$$= \iint_R dx\, dy = 16\pi.$$

which is the area of the circular region in the XY – plane. Hence, Stokes's theorem is proves.

Problem 1: Using Stokes's Theorem, evaluate $\int_s (\text{curl}\boldsymbol{f}).\boldsymbol{n}\, ds$ for $\boldsymbol{f} = (y - z + 2)\boldsymbol{i} + (yz + 4)\boldsymbol{j} - xz\boldsymbol{k}$, where S is the surface of the cube formed by the planes $x = 0, y = 0, x = 2, y = 2$ and $z = 2$ with its bottom removed.

Solution:

The curve C of the given surface is the square $OPQR$ in the XY −plane, where $O = (0,0), P = (2,0), Q = (2,2), R = (0,2)$; see in figure. We note that C lies in the XY −plane, so that $z \equiv 0$ on the whole of C, x =constant on PQ and RO, and y =constant on OP and QR



Therefore, by using stokes's theorem, we get

$$\int_s (\text{curl}\boldsymbol{f}).\boldsymbol{n}\, ds = \int_C \boldsymbol{f}.\,d\boldsymbol{r}$$

$$= \int_{OP} f_1\, dx + \int_{PQ} f_2\, dy + \int_{QR} f_1\, dx + \int_{R0} f_2\, dy$$

$$= \int_{OP} (y - z + 2)\, dx + \int_{PQ} (yz + 4)\, dy + \int_{QR} (y - z + 2)\, dx + \int_{R0} (yz + 4)\, dy$$

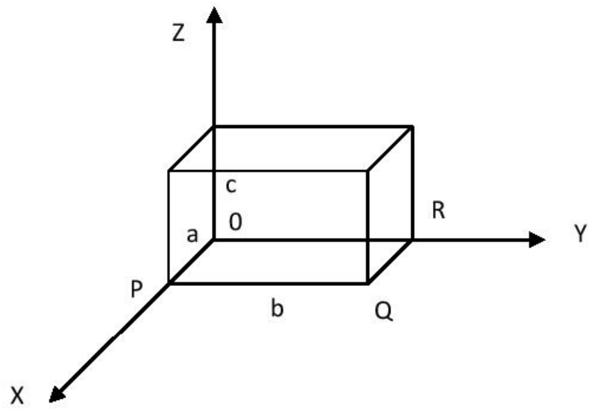
using the given **f**

$$= \int_0^2 2\, dx + \int_0^2 4\, dy + \int_2^0 4\, dx + \int_2^0 4\, dy$$

$$= -4.$$

Problem 2: Verify Stokes's Theorem for the vector field $\boldsymbol{f} = (x^2 - y^2)\boldsymbol{i} + 2xy\boldsymbol{j}$ over the rectangular box bounded by the planes $x = 0, x = a; y = 0, y = b; z = 0, z = c$ with the face $z = 0$ removed.

Solution:



For the given box, the curve C is the rectangle formed by the lines $x = 0, x = a$ and $y = 0, y = b$. Let us denotes boundary lines by OR, PQ, OP and RQ respectively. See above figure. Then, since C is in the XY −plane.

$$\int_s \boldsymbol{f}.d\boldsymbol{r} = \int_C (x^2 - y^2)dx + 2xy\,dy$$

$$= \int_{OP}[(x^2 - y^2)dx + 2xy\,dy]$$

$$+ \int_{PQ}[(x^2 - y^2)dx + 2xy\,dy]$$

$$+ \int_{QR}[(x^2 - y^2)dx + 2xy\,dy]$$

$$+ \int_{RO}[(x^2 - y^2)dx + 2xy\,dy]$$

$$= \int_0^a x^2 dx + \int_0^b 2aydy + \int_a^0 (x^2 - b^2)dx + \int_b^0 0.d\,y$$

$$= \frac{1}{3}a^3 + ab^2 - \frac{1}{3}a^3 + ab^2 = 2ab^2 \qquad (1)$$

Now, we find that

$$\text{curl}\boldsymbol{f} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 - y^2 & 2xy & 0 \end{vmatrix} = 4y\boldsymbol{k}$$

Therefore, if S is the surface of the given box, we have

$$\int_S (\text{curl}\boldsymbol{f}).\boldsymbol{n}\ dS = 4\int_S y\boldsymbol{k}.\boldsymbol{n}\ dS$$

We note that S is made up of five faces of the box, and $\boldsymbol{k}.\boldsymbol{n} = 0$ on all faces except the upper face $z = c$. on the upper face x varies from 0 to a and y varies from 0 to b , and $\boldsymbol{n} = \boldsymbol{k}$.

$$\int_S (\text{curl}\boldsymbol{f}).\boldsymbol{n}\ dS = 4\int_{x=0}^{a}\int_{y=0}^{b} y\, dy\, dx = 2ab^2 \qquad (2)$$

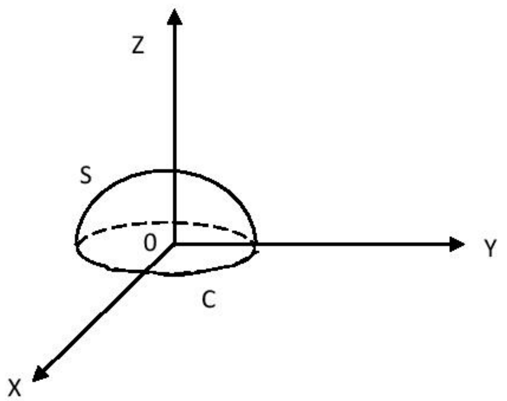
From (1) and (2)

$$\int_s \boldsymbol{f}.d\boldsymbol{r} = \int_S (\text{curl}\boldsymbol{f}).\boldsymbol{n}\ dS.$$

Thus, the Stokes's theorem is verified in the given case.

Problem 3: Verify Stokes's Theorem for $\boldsymbol{f} = y\boldsymbol{i} + z\boldsymbol{j} + x\boldsymbol{k}$ for the upper part of the sphere $x^2 + y^2 + z^2 = a^2$.

Solution:



The curve C of the given surface is the circle $x^2 + y^2 = a^2$ in the XY −plane. Therefore, the parametric equations of C are $x = a\cos t\,, y = a\sin t\,, z = 0; 0 \le t \le 2\pi$. Hence,

$$\int_C \boldsymbol{f}.\,d\boldsymbol{r} = \int_C (y\boldsymbol{i} + z\boldsymbol{j} + x\boldsymbol{k}).\,(dx\boldsymbol{i} + dy\boldsymbol{j} + dz\boldsymbol{k}) = \int_C y\,dx$$

because $z = 0$ on C

$$= \int_{\boldsymbol{0}}^{\boldsymbol{2\pi}} (a\ \sin t)(-a\sin t\, dt)$$

$$= -4a^2 \int_0^{\pi/2} \sin^2 t\, dt = -\pi a^2 \qquad (1)$$

The given surface S for which C is the upper part of the sphere $x^2 + y^2 + z^2 = a^2$

Therefore, on $z^2 = a^2 - x^2 - y^2,\ z > 0$

Now, $$\operatorname{curl}\boldsymbol{f} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{vmatrix} = -(\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k})$$

$$\boldsymbol{n} = \frac{2(x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k})}{2\sqrt{x^2+y^2+z^2}} = \frac{1}{a}(x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k})$$

$$\text{curl}\boldsymbol{f}.\boldsymbol{n} = -\frac{z}{a}.$$

Therefore, $\iint_S (\text{curl}\boldsymbol{f}).\boldsymbol{n}dA = \iint_S -\frac{z}{a}dA = \iint_R -\frac{z}{a}\frac{dx\ dy}{\boldsymbol{n}.\boldsymbol{k}} = \iint_R -\frac{z}{a}\frac{dx\ dy}{(z/a)}$

$$= -\iint_R dxdy$$

$$= -\int_0^{2\pi}\int_0^a r dr d\theta = -\pi a^2. \qquad (2)$$

By polar coordinates

From (1) and (2) , we note that

$$\int_C \boldsymbol{f}.d\boldsymbol{r} = \int_S (\text{curl}\boldsymbol{f}).\boldsymbol{n}dS$$

Thus, Stokes's theorem is verified.

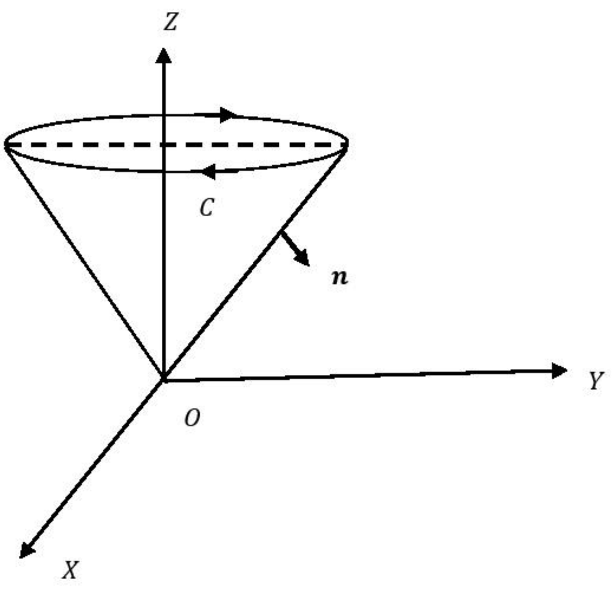
Problem 4: Evaluate $\oint_C 2y^3 dx + x^3 dy + z dz$ where C is the trace of the cone $z = \sqrt{x^2 + y^2}$ intersected by the plane $z = 4$ and S is the surface of the cone below $z = 4$.

Solution:

We have $\boldsymbol{V} = 2y^3\boldsymbol{i} + x^3\boldsymbol{j} + z\boldsymbol{k}$ and

$$\text{curl } \boldsymbol{V} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2y^3 & x^3 & z \end{vmatrix} = \boldsymbol{i}(0) - \boldsymbol{j}(0) + \boldsymbol{k}(3x^2 - 6y^2)$$

If the outward normal to S is taken, then it points downwards. Then, the orientation of C is taken as given in following figure. Alternately, if the inward normal to S is takes, then C is oriented in the counter clockwise direction.



Let $f(x, y, z) = \sqrt{x^2 + y^2} - z = 0$ be taken as the equation of the surface. Then, the normal and unit normal are given by

$$\nabla f = \frac{x\boldsymbol{i}+y\boldsymbol{j}}{\sqrt{x^2+y^2}} - k = \frac{x\boldsymbol{i}+y\boldsymbol{j}-z\boldsymbol{k}}{z} \text{ and}$$

$$\boldsymbol{n} = \frac{\nabla f}{|\nabla f|} = \frac{(x\boldsymbol{i}+y\boldsymbol{j}-z\boldsymbol{k})/z}{\sqrt{\frac{x^2+y^2+z^2}{z^2}}} = \frac{x\boldsymbol{i}+y\boldsymbol{j}-z\boldsymbol{k}}{\sqrt{2}z}$$

except at the origin.

We have

$$\iint_S (\nabla \times \boldsymbol{V}).\boldsymbol{n}\, dA = \iint_S -\frac{(3x^2-6y^2)}{\sqrt{2}} dA = -\iint_R \frac{(3x^2-6y^2)}{\sqrt{2}} \frac{dxdy}{(-1/\sqrt{2})}$$

Since $dxdy = (\boldsymbol{n}.\boldsymbol{k})dA$. Therefore, substituting

$x = r\cos\theta, y = r\sin\theta$, we obtain

$$\iint_S (\nabla \times \boldsymbol{V}).\boldsymbol{n}\, dA = \iint_R (3x^2 - 6y^2)dxdy$$

$$= \int_{r=0}^{4} \int_{2\pi}^{0} (3\cos^3\theta - 6\sin^2\theta)\, r^3 dr\, d\theta$$

$$= \frac{3}{2}\int_{r=0}^{4} \int_{2\pi}^{0} [(1+\cos 2\theta) - 2(1-\cos 2\theta)] r^3 dr\, d\theta$$

$$= \frac{3}{2}\int_{0}^{4} \int_{2\pi}^{0} (3\cos 2\theta - 1) r^3\, drd\theta$$

$$= \frac{3}{2}\left[\frac{r^4}{4}\right]_0^4 \left[\frac{3\sin 2\theta}{2} - \theta\right]_{2\pi}^{0} = 192\pi.$$

The bounding curve C is given by $x^2 + y^2 = 16, z = 4$. Now, setting $x = 4\cos\theta, y = 4\sin\theta$, we obtain

$$\oint_C 2y^3 dx + x^3 dy + zdz = \oint_C 2y^3 dx + x^3 dy$$

$$= \int_{2\pi}^{0} 64[2\ \sin^3\theta\,(-4\sin\theta) + \cos^3\theta\,(4\cos\theta)]d\theta$$

$$= -256 \int_0^{2\pi} [\cos^4 \theta - 2\sin^4 \theta] d\theta$$

$$= -1024 \int_0^{2\pi} (\cos^4 \theta - 2\sin^4 \theta) d\theta$$

$$= -1024 \left[\frac{3}{2} \cdot \frac{1}{2} \cdot \frac{\pi}{2} - 2\left(\frac{3}{2} \cdot \frac{1}{2} \cdot \frac{\pi}{2}\right)\right] = 192\pi.$$

Hence, Stokes's theorem is verified.

Exercise

I. In the problems below, verify the Stokes's theorem. Assume that the surface S is oriented upward.

1. $\boldsymbol{V} = x^3\boldsymbol{i} + x^2y\boldsymbol{j}$. C is the boundary of the rectangle whose sides are $x = 0, x = 3, y = 0, y = 4$ in the plane $z = 0$.
2. $\boldsymbol{V} = z\boldsymbol{i} + x\boldsymbol{j} + z\boldsymbol{k}$. S is the portion of the sphere $x^2 + y^2 + z^2 = 9$ above the $XY -$ plane.
3. $\boldsymbol{V} = z\boldsymbol{i} + (2x + z)\boldsymbol{j} + x\boldsymbol{k}$. C is the boundary of the triangle with vertices at $(1, 0, 0), (0, 2, 0)$ and $(0, 0, 3)$.

II. In problems, evaluate the integral $\iint_S (\nabla \times \boldsymbol{V}).\boldsymbol{n}\, dA$ by Stokes's theorem.

4. $\boldsymbol{V} = (x^2 - y^2)\boldsymbol{i} + (y^2 - x^2)\boldsymbol{j} + z\boldsymbol{k}$. S is the portion of the surface $x^2 + y^2 - 2by + bz = 0$, b constant, whose boundary lies in the $XY -$ plane.
5. $\boldsymbol{V} = (x + y)\boldsymbol{i} + (y + z)\boldsymbol{j} + (z + x)\boldsymbol{k}$. S is the portion of the cone $z = \sqrt{x^2 + y^2}$ for $x^2 + y^2 \leq 4$.

III. In problems, evaluate $\oint_C \boldsymbol{V}.d\boldsymbol{r}$ using the Stokes's theorem. Assume C is oriented in the counter clockwise direction as viewed from above.

6. $\boldsymbol{V} = 3y\boldsymbol{i} + 4z\boldsymbol{j} + 2x\boldsymbol{k}$. C is the intersection of the surface of the sphere $x^2 + y^2 + z^2 = 16, x \geq 0$ and the cylinder $y^2 + z^2 = 4$.
7. $\boldsymbol{V} = (3x + 2z)\boldsymbol{i} + (x + 3y)\boldsymbol{j} + (2y - 3z)\boldsymbol{k}$. C is the curve of intersection of the plane $6x + 3y + 4z = 12$ with the coordinate planes.

Answers

I.

1. 72
2. 9π
3. -1

II.

4. $2\pi b^3$
5. -4π

III.

6. -16π
7. 22