2.6

Functions of Random Variables

The previous modules discussed basic properties of events defined in a given sample space and the random variables used to represent those events. The fundamental assumption that was made in those modules is that events can always be defined by **random variables**. However, in many applications, the events are functions of other events. For example, the time until a complex system fails is a function of the time to failure of the individual components that make up the system. This means that the random variable used to represent the time to failure of the complex system is a function of the random variables used to represent the times to failure of the component parts of the system. This module deals with functions of random variables. Because of the complexity involved in computing the c.d.fs and p.d.fs of multiple random variables, the discussion is restricted to functions of at most two random variables.

Functions of One Random Variable: Let X be a r.v. with p.d.f. (or p.m.f.) $f_X(x)$ and c.d.f. $F_X(x)$. Let Y be the new random variable that is a function of X. That is,

$$Y = g(X)$$

Then we are interested in computing p.d.f (or p.m.f.) $f_Y(y)$ and c.d.f. $F_Y(y)$ of Y.

For example, let $Y = X + 5$. Then

$$F_Y(y) = P(Y \leq y) = P[X + 5 \leq y] = P[X \leq y - 5] = F_X(y - 5)$$

Linear Functions: Consider the function $g(X) = aX + b$, where a and b are constants. The c.d.f of Y is given by

$$F_Y(y) = P\{Y \leq y\} = P[aX + b \leq y]$$

$$= P\left[X \leq \frac{y-b}{a}\right] = F_X\left(\frac{y-b}{a}\right)$$

where a is positive .The p.d.f. of Y is given by

$$f_Y(y) = \frac{d}{dy}\left(F_Y(y)\right) = \frac{d}{dy}\left(F_X\left(\frac{y-b}{a}\right)\right) = \left(\frac{d}{du}\left(F_X(u)\right)\right)\left(\frac{du}{dy}\right)$$

where $u = \frac{y-b}{a}$ and $\frac{du}{dy} = \frac{1}{a}$. Thus,

$$f_Y(y) = \left(\frac{1}{a}\right) f_X(u) = \left(\frac{1}{a}\right) f_X\left(\frac{y-b}{a}\right)$$

If $a < 0$, we have,

$$F_Y(y) = P(Y \le y) = P(aX + b \le y) = P(aX \le y - b)$$

$$= P\left(X \ge \frac{y-b}{a}\right) = 1 - \left\{P\left[X \le \frac{y-b}{a}\right] - P\left[X = \frac{y-b}{a}\right]\right\} \quad (\because a < 0)$$

The change in sign on the second line arises from the fact that a is negative. If X is continuous, $P\left[X = \frac{(y-b)}{a}\right] = 0$. Thus, the c.d.f and p.d.f for the case of negative a are given by

$$F_Y(y) = 1 - P\left[X \le \frac{y-b}{a}\right]$$

$$= 1 - F_X\left(\frac{y-b}{a}\right)$$

Therefore, $f_Y(y) = \frac{d}{dy}\left(F_Y(y)\right) = -\left(\frac{1}{a}\right) f_X\left(\frac{y-b}{a}\right)$

Therefore, the general p.d.f. of Y is given by

$$f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)$$

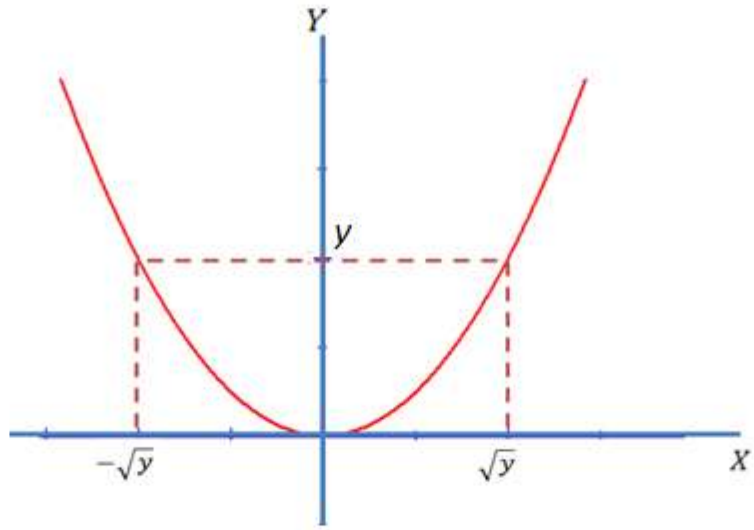
Example 1: Find the p.d.f of Y in terms of the p.d.f of X if $Y = 2X + 7$.

Solution: From the results obtained above,

$$F_Y(y) = F_X\left(\frac{y-7}{2}\right)$$

and $$f_Y(y) = \left(\frac{1}{2}\right) f_X\left(\frac{y-7}{2}\right)$$

Power Functions: Consider the quadratic function $Y = X^2$. The plot of Y against X is shown in the following figure where we see that for one value of Y there are two values of X, namely $\sqrt{Y}$ and $-\sqrt{Y}$.



Thus, the c.d.f of Y is given by

$$F_Y(y) = P[Y \leq y] = P[X^2 \leq y]$$

$$= P\left[|X| \leq \sqrt{y}\right], \quad y > 0$$

$$= P\left[-\sqrt{y} \leq X \leq \sqrt{y}\right]$$

$$= F_X\left(\sqrt{y}\right) - F_X\left(-\sqrt{y}\right)$$

The p.d.f of Y is given by

$$f_Y(y) = \frac{dF_Y(y)}{dy} = \frac{d}{dy}\left[F_X\left(\sqrt{y}\right) - F_X\left(-\sqrt{y}\right)\right]$$

Let $u = \sqrt{y} = y^{\frac{1}{2}}$. Thus, $\frac{du}{dy} = \frac{1}{2} y^{-\frac{1}{2}}$ and

$$f_Y(y) = \frac{d}{dy}\left[F_X\left(\sqrt{y}\right) - F_X\left(-\sqrt{y}\right)\right]$$

$$= \frac{d}{du}\left(F_X(u)\right)\frac{du}{dy} + \frac{d}{du}\left(F_X(-u)\right)\frac{du}{dy}$$

$$=\frac{1}{2}y^{-\frac{1}{2}}\left[\frac{d}{du}\left(F_X(u)\right)+\frac{d}{du}\left(F_X(-u)\right)\right]$$

$$=\frac{1}{2}y^{-\frac{1}{2}}\left[f_X\left(\sqrt{y}\right)+f_X\left(-\sqrt{y}\right)\right]$$

$$=\frac{f_X(\sqrt{y})+f_X(-\sqrt{y})}{2\sqrt{y}} \quad ,y>0$$

If $f_X(x)$ is an even function, then $f_X(x)=f_X(-x)$ and $F_X(-x)=1-F_X(x)$. Thus, we have

$$f_Y(y)=\frac{f_X(\sqrt{y})+f_X(-\sqrt{y})}{2\sqrt{y}}=\frac{2f_X(\sqrt{y})}{2\sqrt{y}}=\frac{f_X(\sqrt{y})}{\sqrt{y}}$$

Example2: Find the p.d.f of the random variable $Y=X^2$, where X is the standard normal random variable.

Solution: Since the p.d.f. of X is given by $f_X(x)=\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$, which is an even function, we know that

$$F_Y(y)=F_X\left(\sqrt{y}\right)-F_X\left(-\sqrt{y}\right)$$

$$=2F_X\left(\sqrt{y}\right)-1$$

Therefore, if we let $u=\sqrt{y}$, then

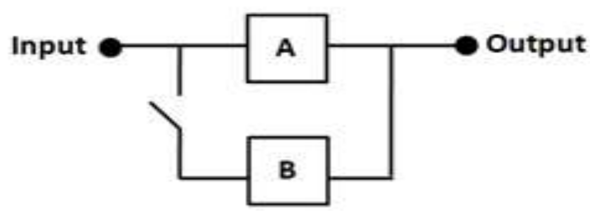
$$f_Y(y)=\frac{dF_Y(y)}{dy}=2\frac{dF_X(u)}{du}\frac{1}{2\sqrt{y}}$$

$$=\frac{1}{\sqrt{y}}f_X\left(\sqrt{y}\right)$$

$$=\frac{1}{\sqrt{2\pi y}}e^{-\frac{y}{2}},\ y>0$$

Sum of Two Independent Random variables

Consider two independent continuous random variables X and Y. We are interested in computing the c.d.f and p.d.f of their sum $g(X,Y) = S = X + Y$. The random variable S can be used to model the reliability of systems with stand-by connections, as shown in *fig*.1 In such systems, the component A whose time-to-failure is represented by the random variable X is the primary component, and the component B whose time-to-failure is represented by the random variable Y is the backup component that is brought into operation when the primary component fails. Thus, S represents the time until the system fails, which is the sum of the lifetimes of both components.



$fig.1$

Their c.d.f. can be obtained as follows:

$$F_S(s) = P[S \le s] = P[X + Y \le s] = \iint_D f_{XY}(x,y)\,dxdy$$

where D is the set $D = \{(x,y) | x + y \le s\}$, which is the area to the left of the line $s = x + y$ as shown in *fig.* 2.

Thus,

$$F_S(s) = \int_{-\infty}^{\infty}\int_{-\infty}^{s-y} f_{XY}(x,y)\,dxdy = \int_{-\infty}^{\infty}\int_{-\infty}^{s-y} f_X(x)\,f_Y(y)\,dxdy$$

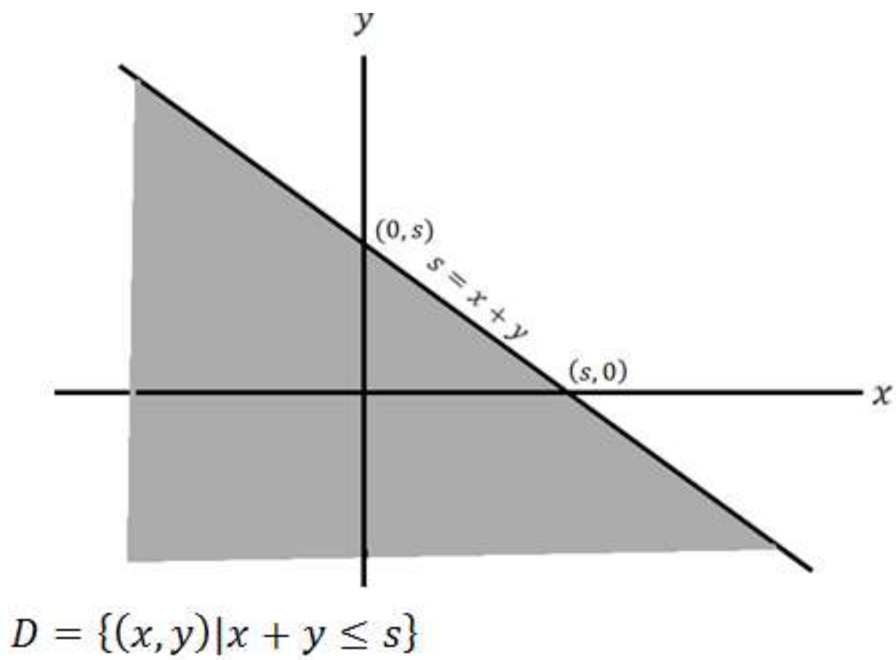
$$= \int_{-\infty}^{\infty}\left\{\int_{-\infty}^{s-y} f_X(x)\,dx\right\} f_Y(y)\,dy$$

$$= \int_{-\infty}^{\infty} F_X(s-y)\,f_Y(y)\,dy$$

The p.d.f. of S is obtained by differentiating the c.d.f. , as follows:

$$
\begin{aligned}
f_S(s) = \frac{d}{ds} F_S(s) &= \frac{d}{ds} \int_{-\infty}^{\infty} F_X(s-y)\, f_Y(y)\, dy \\
&= \int_{-\infty}^{\infty} \frac{d}{ds} F_X(s-y)\, f_Y(y)\, dy \\
&= \int_{-\infty}^{\infty} f_X(s-y)\, f_Y(y)\, dy
\end{aligned}
$$

where we have assumed that we can interchange differentiation and integration. The expression on the right-hand side is a well-known result in signal analysis



$fig.2$

called the **convolution integral**. Thus, we find that the p.d.f of the sum S of two independent random variables X and Y is the convolution of the p.d.fs of the two random variables; that is,

$$f_S(s) = f_X(s) f_Y(s)$$

Example 3: Find the p.d.f. of the sum of X and Y if the two random variables are independent random variables with the common p.d.f.

$$\boldsymbol{f_X(u) = f_Y(u)} = \begin{cases} \boldsymbol{\frac{1}{4}} & \boldsymbol{0 < u < 4} \\ \boldsymbol{0} & \textbf{otherwise} \end{cases}$$

Solution: The limits of integration of the p.d.f of $S = X + Y$ can be computed with the aid of $fig.3$ When $0 \le s \le 4$ (see $fig.3\ (a)$ where $f_Y(s-x)$ is shown in dashed lines),

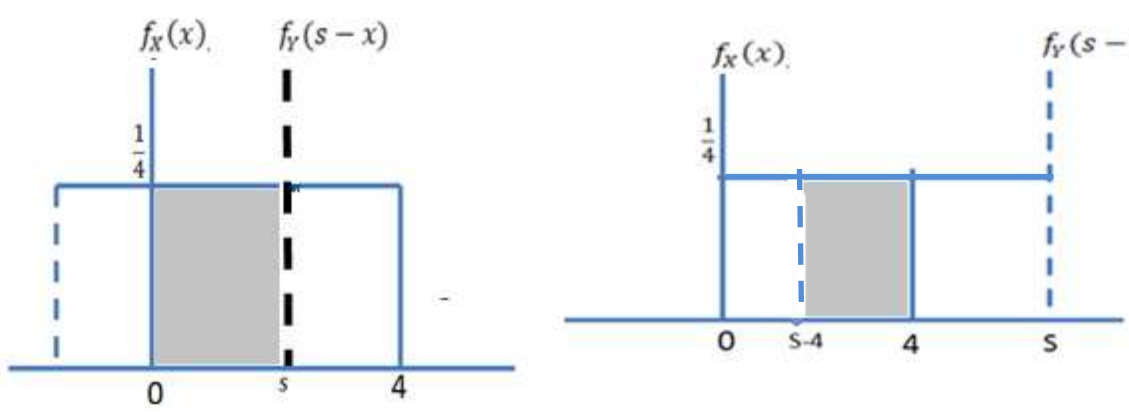
$$f_S(s) = \int_0^s \frac{1}{16} dy = \frac{s}{16}$$

For $4 < s < 8$ (see $fig.3\ (b)$), we obtain

$$f_S(s) = \int_{s-4}^{4} dy = \frac{8-s}{16}$$

Thus ,

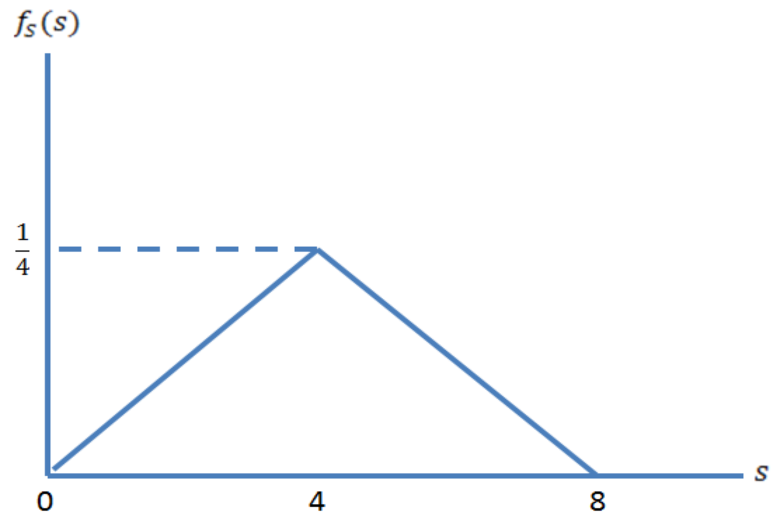
$$f_S(s) = \begin{cases} \frac{s}{16} & ,\ 0 \le s \le 4 \\ \frac{8-s}{16} & ,\ 4 < s < 8 \\ 0 & ,\ \text{otherwise} \end{cases}$$



$fig\ 3(a)$ $fig\ 3(b)$

$Fig.3$: Convolution of p.d.fs (a) $0 \le s \le 4$ and (b) $4 \le s \le 8$

The p.d.f of $S = X + Y$ is illustrated in the following figure.



Example 4: The time X between consecutive snowstorms in winter is a random variable with the p.d.f.

$$\boldsymbol{f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}}$$

Assume it has not snowed up until now. What is the p.d.f. of the time U until the second snowstorm?

Solution: Let X be the random variable that denotes the time until the first snowstorm from the reference time, and let Y be the random variable that denotes the time between the first snowstorm and the second snowstorm. If we assume that the times between snowstorms are independent, then X and Y are independent and identically distributed random variables. That is, the p.d.f of Y is given by

$$f_Y(y) = \begin{cases} \lambda e^{-\lambda y}, & y \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

Thus, $U = X + Y$, and the p.d.f. of U is given by

$$f_U(u) = \int_0^{\infty} f_X(x) f_Y(u - x) dx$$

Since $f_X(x) = 0$ when $x < 0, f_Y(u - x) = 0$ when $u - x < 0$ (or $x > u$). Thus, the range of interest in the integration is $0 \leq x \leq u$, and we obtain

$$\begin{aligned} f_U(u) &= \int_0^u f_X(x) f_Y(u-x)dx \\ &= \int_0^u \lambda e^{-\lambda x} \lambda e^{-\lambda(u-x)} dx = \lambda^2 e^{-\lambda u} \int_0^u dx \\ &= \lambda^2 u e^{-\lambda u} \quad u \geq 0 \end{aligned}$$

This is the Erlang – 2 distribution.

Note: A random variable X is said to follow Erlang-k distribution if its p.d.f. is given by

$$f(x) = \begin{cases} \frac{\lambda^k x^{k-1} e^{-\lambda x}}{(k-1)!} &, \quad k = 1,2,\ldots;\ \lambda > 0; x \geq 0 \\ 0 &, \quad x < 0 \end{cases}.$$

Sum of Two Discrete Random Variables

The examples above deal with continuous random variables.
Let $Z = X + Y$, where X and Y are discrete random variables. Then the p.m.f of Z is given by

$$\begin{aligned} p_Z(z) = P[Z = z] = P[X + Y = z] &= \sum_{k \leq z} P[X = k, Y = z - k] \\ &= \sum_{k \leq z} p_{XY}[k, z - k] \end{aligned}$$

If X and Y are independent random variables, then the p.m.f. of Z is the convolution of the p.m.f of X and the p.m.f of Y. That is,

$$p_Z(z) = \sum_{k \leq z} p_{XY}(k, z - k) = \sum_{k \leq z} p_X(k) p_Y(z - k)$$

Sum of Two Independent Binomial Random Variables

Let X and Y be two independent binomial random variables with parameters (n, p) and (m, p), respectively and their sum be Z; that is, $Z = X + Y$. Then the p.m.f of Z is given by

$$\begin{aligned} p_Z(z) &= P[X+Y=z] \\ &= \sum_{k=0}^{n} P[X=k, Y=z-k] = \sum_{k=0}^{n} P[X=k]P[Y=z-k] \\ &= \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} \binom{m}{z-k} p^{z-k} (1-p)^{m-z+k} \\ &= p^z (1-p)^{n+m-z} \sum_{k=0}^{n} \binom{n}{k} \binom{m}{z-k} \end{aligned}$$

Using the combinatorial identity $\binom{n+m}{z} = \sum_{k=0}^{n} \binom{n}{k}\binom{m}{z-k}$, we obtain

$$p_Z(z) = \binom{n+m}{z} p^z (1-p)^{n+m-z}$$

This result shows that the sum of two independent binomial random variables with parameters (n, p) and (m, p) is a binomial random variable with parameter $(n+m, p)$.

Minimum of Two Independent Random Variables

Consider two independent continuous random variables X and Y. We are interested in a random variable U that is the minimum of X and Y; that is, $U = \min(X, Y)$. The random variable U can be used to represent the reliability of systems with series connections, as shown in $fig. 4$ .Such systems are operational as long as all components are operational. The first component to fail causes the system to fail. Thus, if in the example shown in $fig. 4$, the times-to-failure are represented by the random variables X and Y, then S represents the time until the system fails, which is the minimum of the lifetimes of the two components.

The c.d.f. of U can be obtained as follows:

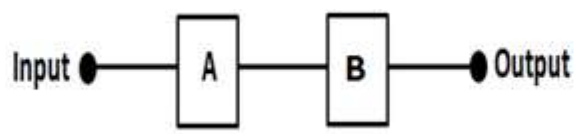
$$F_U(u) = P[U \leq u] = P[\min(X, Y) \leq u] = P[(X \leq u, X \leq Y) \cup (Y \leq u, X > Y)]$$

Since $P[A \cup B] = P[A] + P[B] - P[A \cap B]$, we have that $F_U(u) = F_X(u) + F_Y(u) - F_{XY}(u, u)$. Also, since X and Y are independent, we obtain the c.d.f. and p.d.f. of U as follows:

$$F_U(u) = F_X(u) + F_Y(u) - F_{XY}(u, u) = F_X(u) + F_Y(u) - F_X(u)F_Y(u)$$

$$f_U(u) = \frac{d}{du} F_U(u) = f_X(u) + f_Y(u) - f_X(u)F_Y(u) - F_X(u)f_Y(u)$$

$$= f_X(u)\{1 - F_Y(u)\} + f_Y(u)\{1 - F_X(u)\}$$



$fig.4$

$fig.4$: Series connection modeled by random variable U

Example 5: Assume that $U = \min(X, Y)$, where X and Y are independent random variables with the respective p.d.fs

$$\boldsymbol{f_X(x) = \lambda e^{-\lambda x} \quad x \geq 0}$$

$$\boldsymbol{f_Y(y) = \mu e^{-\mu y} \quad y \geq 0}$$

where $\lambda > 0$ and $\mu > 0$. What is the p.d.f. of U?

Solution: We first obtain the c.d.fs of X and Y, which are as follows:

$$F_X(x) = P[X \leq x] = \int_0^x \lambda e^{-\lambda w}\, dw = 1 - e^{-\lambda x}$$

$$F_Y(y) = P[Y \leq y] = \int_0^y \mu e^{-\mu w}\, dw = 1 - e^{-\mu y}$$

Thus, the p.d.f of U is given by

$$f_U(u) = f_X(u)\{1 - F_Y(u)\} + f_Y(u)\{1 - F_X(u)\}$$

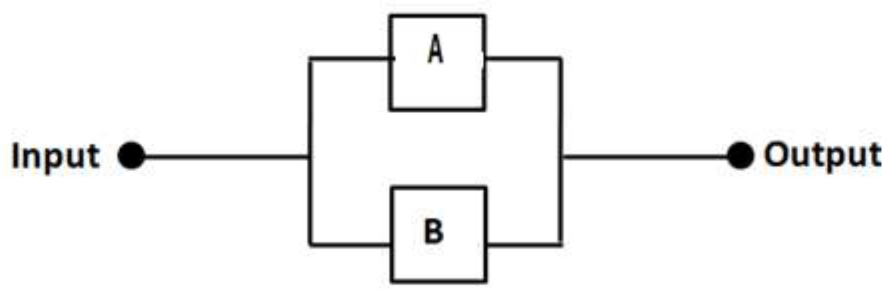
$$= \lambda e^{-\lambda u} e^{-\mu u} + \mu e^{-\mu u} e^{-\lambda u}$$

$$= (\lambda + \mu) e^{-(\lambda+\mu)u} \ , \ u \geq 0$$

This is exponential distribution with mean $\frac{1}{\lambda+\mu}$.

Maximum of Two Independent Random Variables

Consider two independent continuous random variables X and Y. We are interested in the c.d.f. and p.d.f. of the random variable W that is the maximum of the two random variables; that is, $W = \max(X, Y)$. The random variable W can be used to represent the reliability of systems with parallel connections, as shown in



$fig. 5$

$fig. 5$: Parallel connection modeled by the random variable W

In such systems, we are interested in passing a signal between the two endpoints through either the component labeled A or the component labeled B. Thus, as long as one or both components are operational, the system is operational. This implies that the system is declared to have failed when both paths become unavailable. That is, the reliability of the system depends on the reliability of the last component to fail.

The c.d.f of W can be obtained by noting that if the greater of the two random variables is less than or equal to w, then the smaller random variable must also be less than or equal to w. Thus,

$$F_W(w) = P[W \leq w] = P[\max(X,Y) \leq w] = P[(X \leq w) \cap (Y \leq w)]$$

$$= F_{XY}(w,w)$$

Since X and Y are independent, we obtain the c.d.f and p.d.f of W as follows:

$$F_W(w) = F_{XY}(w,w) = F_X(w)F_Y(w)$$

$$f_W(w) = \frac{d}{dw}F_W(w) = f_X(w)F_Y(w) + F_X(w)f_Y(w)$$

Example 6: Assume that $W = \max(X,Y)$, where X and Y are independent random variables with the respective p.d.fs:

$$\boldsymbol{f_X(x) = \lambda e^{-\lambda x} \quad x \geq 0}$$

$$\boldsymbol{f_Y(y) = \mu e^{-\mu y} \quad y \geq 0}$$

where $\lambda > 0$ and $\mu > 0$. What is the pdf of W.

Solution: We first obtain the c.d.fs of X and Y, which are as follows:

$$F_X(x) = P[X \leq x] = \int_0^x \lambda e^{-\lambda z}\, dz = 1 - e^{-\lambda x}$$

$$F_Y(y) = P[Y \leq y] = \int_0^y \mu e^{-\mu z}\, dz = 1 - e^{-\mu y}$$

Thus, the p.d.f of W is given by

$$f_W(w) = f_X(w)F_Y(w) + F_X(w)f_Y(w)$$

$$= \lambda e^{-\lambda w}(1 - e^{-\mu w}) + \mu e^{-\mu w}\left(1 - e^{-\lambda w}\right)$$

$$= \lambda e^{-\lambda w} + \mu e^{-\mu w} - (\lambda + \mu)e^{-(\lambda+\mu)w} \quad w \geq 0$$

Note that the mean of W is $\frac{1}{\lambda} + \frac{1}{\mu} - \frac{1}{\lambda+\mu}$.

Two Functions of Two Random Variables

Let X and Y be two random variables with a given joint p.d.f $f_{XY}(x,y)$. Assume that U and W are two functions of X and Y; that is, $U = g(X,Y)$ and $W = h(X,Y)$. Sometimes it is necessary to obtain the joint p.d.f of U and W, $f_{UW}(u,w)$, in terms of the p.d.fs of X and Y.

It can be shown that $f_{UW}(u,w)$ is given by

$$f_{UW}(u,w) = \frac{f_{XY}(x_1,y_1)}{|J(x_1,y_1)|} + \frac{f_{XY}(x_2,y_2)}{|J(x_2,y_2)|} + \cdots + \frac{f_{XY}(x_n,y_n)}{|J(x_n,y_n)|}$$

where $(x_1,y_1),(x_2,y_2),\dots,(x_n,y_n)$ are real solutions of the equations $u = g(x,y)$ and $w = h(x,y)$; and $J(x,y)$ is called the **Jacobian** of the transformation $\{u = g(x,y), w = h(x,y)\}$ and defined by

$$J(x,y) = \begin{vmatrix} \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \\ \frac{\partial h}{\partial x} & \frac{\partial h}{\partial y} \end{vmatrix} = \frac{\partial g}{\partial x} \cdot \frac{\partial h}{\partial y} - \frac{\partial g}{\partial y} \cdot \frac{\partial h}{\partial x}$$

Example 7: Let $U = g(X,Y) = X + Y$ and $W = h(X,Y) = X - Y$. Find $f_{UW}(u,w)$.

Solution: The unique solution to the equations $u = x + y$ and $w = x - y$ is $x = \frac{u+w}{2}$ and $y = \frac{u-w}{2}$. Thus, there is only one set of solutions. Since

$$J(x,y) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2$$

we obtain

$$f_{UW}(u,w) = \frac{f_{XY}(x,y)}{|J(x,y)|} = \frac{1}{|-2|} f_{XY}\left(\frac{u+w}{2}, \frac{u-w}{2}\right) = \frac{1}{2} f_{XY}\left(\frac{u+w}{2}, \frac{u-w}{2}\right)$$

Application of the Transformation Method

Assume that $U = g(X, Y)$, and we are required to find the p.d.f. of U. We can use the above transformation method by defining an auxiliary function $W = X$ or $W = Y$ so we can obtain the joint p.d.f. $f_{UW}(u, w)$ of U and W. Then we obtain the required marginal p.d.f. $f_U(u)$ as follows:

$$f_U(u) = \int_{-\infty}^{\infty} f_{UW}(u, w)dw$$

Example 8: Find the p.d.f. of the random variable $\boldsymbol{U = X + Y}$, where the joint p.d.f. of $\boldsymbol{X}$ and $\boldsymbol{Y}$, $\boldsymbol{f_{XY}(x, y)}$, is given.

Solution: We define the auxiliary random variable $W = X$. Then the unique solution to the two equations is $x = w$ and $y = u - w$, and the Jacobian of the transformation is

$$J(x, y) = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial w}{\partial x} & \dfrac{\partial w}{\partial y} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = -1$$

Since there is only one solution to the equations, we have that

$$f_{UW}(u, w) = \frac{f_{XY}(w, u - w)}{|-1|} = f_{XY}(w, u - w)$$

$$f_U(u) = \int_{-\infty}^{\infty} f_{UW}(u, w)dw = \int_{-\infty}^{\infty} f_{XY}(w, u - w)dw$$

This reduces to the convolution integral. We obtained earlier when X and Y are independent.

Example 9: Find the p.d.f. of the random variable $U = X - Y$, where the joint p.d.f. of X and Y is given.

Solution: We define the auxiliary random variable $W = X$. Then the unique solution to the two equations is $x = w$ and $y = w - u$, and the Jacobian of the transformation is

$$J(x,y) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ 1 & 0 \end{vmatrix} = 1$$

Since there is only one solution to the equations, we have that

$$f_{UW}(u,w) = f_{XY}(w, w-u)$$

$$f_U(u) = \int_{-\infty}^{\infty} f_{UW}(u,w)dw = \int_{-\infty}^{\infty} f_{XY}(w, w-u)dw$$

Example 10: The joint p.d.f of two random variables X and Y is given by $f_{XY}(x,y)$. If we define the random variable $U = XY$, determine the p.d.f of U.

Solution: We define the auxiliary random variable $W = X$. Then the unique solution to the two equations is $x = w$ and $y = \frac{u}{x} = \frac{u}{w}$, and the Jacobian of the transformation is

$$J(x,y) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} \end{vmatrix} = \begin{vmatrix} y & x \\ 1 & 0 \end{vmatrix} = -x = -w$$

Since there is only one solution to the equations, we have that

$$f_{UW}(u,w) = \frac{f_{XY}(x,y)}{|J(x,y)|} = \frac{1}{|w|} f_{XY}\left(w, \frac{u}{w}\right)$$

$$f_U(u) = \int_{-\infty}^{\infty} f_{UW}(u,w)dw = \int_{-\infty}^{\infty} \frac{1}{|w|} f_{XY}\left(w, \frac{u}{w}\right) dw$$