1.7

Matrix Representations of Linear Transformation

In this module we introduce a way of representing a linear transformation between general vector spaces by a matrix. We lead up to this discussion by looking at the information below that is necessary to represent a linear transformation by a matrix.

Definition: Let $\boldsymbol{U}$ be a vector space with basis $\boldsymbol{B} = \{u_1, . . . , u_n\}$ and let $\boldsymbol{u}$ be a vector in U. We know that there exist unique scalars $a_1, ... , a_n$ such that

$$\boldsymbol{u} = a_1\boldsymbol{u}_1 + ... + a_n\boldsymbol{u}_n$$

The column vector $u_{\boldsymbol{B}} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$ is called the **coordinate vector of u** relative to this basis. The scalars $a_1, ... , a_n$ are called the **coordinates of** $\boldsymbol{u}$ relative to this basis.

Note: We will use a column vector from coordinate vectors rather than row vectors. The theory develops most smoothly with this convention.

Example: Find the coordinate vector of $\boldsymbol{u} = (4,5)$ relative to the following bases $\boldsymbol{B}$ and $\boldsymbol{B}'$ of R^2:

(a) The standard basis, $B = \{(1,0),(0,1)\}$ and

(b) $B' = \{(2,1),(-1,1)\}$.

Solution:

(a) By observation, we see that

$$(4,\ 5) = 4(1,\ 0) + 5(0,\ 1)$$

Thus $\boldsymbol{u}_B = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$. The given representation of **u** is, in fact, relative to the standard basis.

(b) Let us now find the coordinate vector of **u** relative to B', a basis that is not the standard basis. Let

$$(4,\ 5) = a_1(2,\ 1) + a_2\ (-1,\ 1)$$

Thus

$$(4,\ 5) = (2a_1, a_1) + (-a_2, a_2)$$
$$(4,\ 5) = (2a_1 - a_2, a_1 + a_2)$$

Comparing components leads to the following system of equations.

$$2a_1 - a_2 = 4$$

$$a_1 + a_2 = 5$$

This system has the unique solution

$$a_1 = 3\ ,\ \ a_2 = 2$$

$$\text{Thus } u_{B'} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

Definition: Let $\boldsymbol{B} = \{u_1, . \ . \ . \ , u_n\}$ and $\boldsymbol{B'} = \{u'_1, . \ . \ . \ , u'_n\}$ be bases for a vector space U. Let the coordinate vectors of $u_1, . \ . \ . \ , u_n$ relative to the basis $\boldsymbol{B'} = \{u'_1, . \ . \ . \ , u'_n\}$ be $(u_1)_{B'}, \ . \ . \ . \ ., (u_n)_{B'}$. The matrix P, having these vectors as columns , plays a central role in our discussion. It is called the **transition matrix from the basis B to the basis B'.**

Transition matrix $\boldsymbol{P}$ $= [(u_1)_{B'}, \ . \ . \ . \ ., (u_n)_{B'}]$.

Theorem: Let $\boldsymbol{B} = \{u_1, . \ . \ . \ , u_n\}$ and $\boldsymbol{B'} = \{u'_1, . \ . \ . \ , u'_n\}$ be bases for a vector space U. If **u** is a vector in U having coordinate vectors $\boldsymbol{u}_B$ and $\boldsymbol{u}_{B'}$ relative to these bases, then

$$\boldsymbol{u}_{B'} = P\boldsymbol{u}_B$$

where P is the transition matrix from B to B':

$$P = [(u_1)_{B'}, \ . \ . \ . \ ., (u_n)_{B'}].$$

Proof: Since $\{u'_1, . \ . \ . \ . \ , u'_n\}$ is a basis for U, each of the vectors $u_1, . \ . \ . \ , u_n$ can be expressed as a linear combination of these vectors.

Let

$$u_1 = c_{11}u'_1 + \ . \ . \ . + c_{n1}u'_n$$
$$\vdots$$
$$u_n = c_{1n}u'_1 + \ . \ . \ . + c_{nn}u'_n$$

If $u = a_1u_1 + . \ . \ . \ . + a_nu_n$, we get

$$\boldsymbol{u} = a_1u_1 + . \ . \ . \ . + a_nu_n$$

$$= a_1(c_{11}u'_1 + \ . \ . \ . + c_{n1}u'_n) + . \ . \ . \ . + a_n(c_{1n}u'_1 + \ . \ . \ . + c_{nn}u'_n)$$

$= (a_1 c_{11} + \ . \ . \ . + a_n c_{1n}) u'_1 + . \ . \ . \ . + (a_1 c_{n1} + \ . \ . \ . + a_n c_{nn}) u'_n$

The coordinate vector of **u** relative to B' can therefore be written

$$\boldsymbol{u}_{B'} = \begin{bmatrix} (a_1 c_{11} + . . . + a_n c_{1n}) \\ \vdots \\ (a_1 c_{n1} + . . . + a_n c_{nn}) \end{bmatrix} = \begin{bmatrix} c_{11} . . . c_{1n} \\ \vdots \\ c_{n1} . . . c_{nn} \end{bmatrix} \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$$

$$= [(u_1)_{B'}, \ . \ . \ . \ . , (u_n)_{B'}] \boldsymbol{u}_B$$

proving the theorem.

Example: Consider the bases $B = \{(1,2), (3,-1)\}$ and $B' = \{(1,0), (0,1)\}$ of R^2. If **u** is a vector such that $\boldsymbol{u}_B = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$, find $\boldsymbol{u}_{B'}$.

Solution: We express the vectors of B in terms of the vectors of B' to get the transition matrix.

$$(1,2) = 1(1,0) + 2(0,1)$$

$$(3,-1) = 3(1,0) - 1(0,1)$$

The coordinate vectors of $(1,2)$ and $(3,-1)$ are $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ -1 \end{bmatrix}$

.

The transition matrix P is thus

$$P=\begin{bmatrix}1 & 3\\ 2 & -1\end{bmatrix}$$

(Observer that the columns of P are the vectors of the basis.) We get

$$\boldsymbol{u}_{B'}=\begin{bmatrix}1 & 3\\ 2 & -1\end{bmatrix}\begin{bmatrix}3\\ 4\end{bmatrix}=\begin{bmatrix}15\\ 2\end{bmatrix}$$

Let f be any function. We know that f is defined if its effect on *every* element of the domain is known. This is usually done by means of an equation that gives that effect of the function on an arbitrary element in the domain. For example, consider the function f defined by

$$f(x)=\sqrt{x-3}$$

The domain of f is $x\geq 3$. The above equation gives the effect of f on every element in this interval. For example, $f(7)=2$. Similarly, a linear transformation T is defined if its value at every vector in the domain is known. However, unlike a general function, we will see that if we know the effect of the linear transformation on a finite subset of the domain (a basis), it will be automatically defined on all elements of the domain.

Theorem: Let $T:U\to V$ be a linear transformation. Let $\{u_1,\ldots\ldots,u_n\}$ be a basis for U. T is defined by its effect on the base vectors, namely by $T(u_1),.....T(u_n)$. The range of T is spanned by $\{T(u_1),.....T(u_n)\}$.

Thus, defining a linear transformation on a basis defined it on the whole domain.

Proof: Let u be an element of U. Since $\{u_1,.....,u_n\}$ is a basis for U, there exist scalars $a_1,.....a_n$ such that

$$u = a_1u_1 + ... + a_nu_n$$

The linearity of T gives

$$T(u) = T(a_1u_1 + ... + a_nu_n)$$

$$= a_1T(u_1) + ... + a_nT(u_n)$$

Therefore $T(u)$ is known if $T(u_1),.....T(u_n)$ are known.

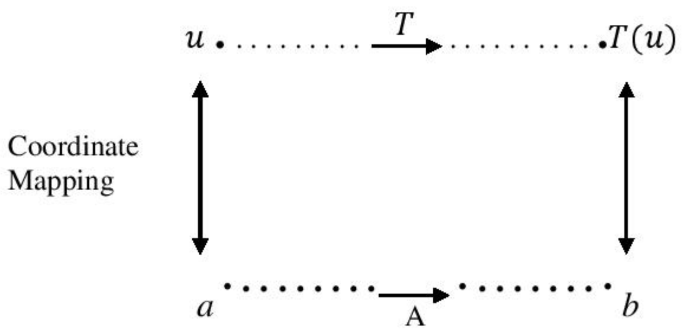
Further, $T(u)$ may be interpreted to be an arbitrary element in the range of T. It can be expressed as a linear combination of $T(u_1),.....T(u_n)$. Thus $\{T(u_1),.....T(u_n)\}$ spans the range of T.

From now onwards, we will represent the elements of U and V by coordinate vectors, and T by a matrix A that defines a transformation of coordinate vectors. The matrix A is constructed by finding the effect of T on basis vectors.

Theorem: Let U and V be vector spaces with bases $B = \{u_1,.....u_n\}$ and $B' = \{v_1,.....,v_m\}$. Let $T : U \rightarrow V$ be a linear transformation. If u is a vector in U with image $T(u)$ having coordinate vectors a and b relative to these bases, then

$b = Aa$ Where $A = \left[T(u_1)_{B'} \ldots . T(u_n)_{B'} \right]$

The Matrix A thus defines a transformation of coordinate vectors of U in the "same way" as T transforms the vectors of U . See the figure below. A is called the **matrix representation of *T*** (or **matrix of *T***) with respect the bases B and B'.



Figure

Proof: Let $u = a_1u_1 + \ . \ . \ . \ + a_nu_n$.
Using the linearity of T, we can write

$$\begin{aligned} T(u) &= T(a_1u_1 + \ . \ . \ . + a_nu_n) \\ &= a_1T(u_1) + \ . \ . \ . + a_nT(u_n) \end{aligned}$$

Let the effect of T on the basis vectors of U be

$$\begin{aligned} T(u_1) &= c_{11}v_1 + \ . \ . \ . + c_{1m}v_m \\ T(u_2) &= c_{21}v_1 + \ . \ . \ . + c_{2m}v_m \end{aligned}$$

$$\vdots$$

$$T(u_n) = c_{n1}v_1 + \ . \ . \ . + c_{nm}v_m$$

Thus

$$T(u) = a_1(c_{11}v_1 + \ . \ . \ . + c_{1m}v_m) + \ . \ . \ . + a_n(c_{n1}v_1 + \ . \ . \ . + c_{nm}v_m)$$

$$= (a_1c_{11} + \ . \ . \ . + a_nc_{n1})v_1 + \ . \ . \ . + (a_1c_{1m} + \ . \ . \ . + a_nc_{nm})v_m$$

The coordinate vector of $T(u)$ is therefore

$$b = \begin{bmatrix} (a_1c_{11} + \ . \ . \ . + a_nc_{n1}) \\ \vdots \\ (a_1c_{1m} + \ . \ . \ . + a_nc_{nm}) \end{bmatrix} = \begin{bmatrix} c_{11} \ \dots \ c_{n1} \\ \vdots \\ c_{1m} \ \dots \ c_{nm} \end{bmatrix} \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = Aa$$

proving the theorem.

Importance of Matrix Representation

The fact that every linear transformation can now be represented by a matrix means that all the theoretical mathematics of these vector spaces and their linear transformation can be undertaken in terms of the vector spaces R^n and matrices. A second reason is a computational one. The elements of R^n and matrices can be manipulated on computers. Thus general vector spaces and their linear transformations can be discussed on computers through these representations.

Relation between Matrix Representations

We have seen that the matrix representation of a linear transformation depends upon the bases selected. When linear transformations arise in applications, a goal is often to determine a simple matrix representation. At this time we discuss how matrix representations of linear operators relative to different bases related. We remind the reader that if A and B are square matrices of the same size, then B is said to be **similar** to A if there exists an invertible matrix P such that

$$B = P^{-1}AP$$

The transformation of the matrix A into the matrix B in this manner is called **similarity transformation.** We now find the matrix representations of a linear operator relative to two bases are similar matrices.

Theorem: Let U be a vector space with bases B and B'. Let P be the transformation matrix from B' to B. If T is a linear operator on U, having matrix A with respect to the first basis and A' with respect to the second basis , then

$$A' = P^{-1}AP$$

Proof: Consider a vector u in U. Let its coordinate vector relative to B and $B^{'}$ be a and a'. The coordinate vectors of $T(u)$ are Aa and $A'a'$. Since P is the transition matrix from B' to B, we know that

$$a = Pa^{'} \text{ and } Aa = P(A^{'}a^{'})$$

This second equation may be rewritten

$$P^{-1}Aa = A'a'$$

Substituting $a = Pa'$ into this equation gives

$$P^{-1}APa^{'} = A'a'$$

This effect of the matrices $P^{-1}AP$ and A' as transformations on an arbitrary coordinate vector a' is the same. Thus these matrices are equal.

Applications of Linear Transformation

A specific application of linear maps is for geometric transformations, such as those performed in computer graphics, where the translation, rotation and scaling of 2D or 3D objects is performed by the use of a transformation matrix. For Example:

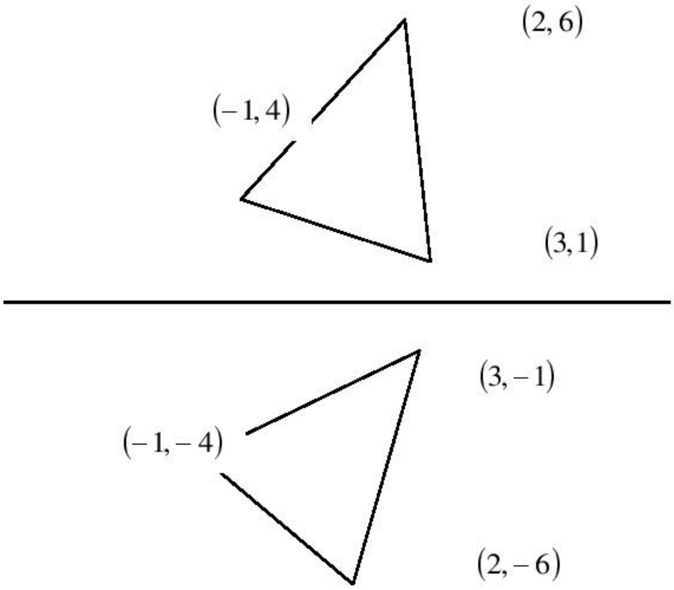
1. Reflection with respect to x-axis:

$$L: R^2 \to R^2, L\left(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\right) = A\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} u_1 \\ -u_2 \end{bmatrix}.$$

For example, the reflection for the triangle with vertices $(-1, 4), (3, 1), (2, 6)$ is

$$L\left(\begin{bmatrix} -1 \\ 4 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ -4 \end{bmatrix}, L\left(\begin{bmatrix} 3 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ -1 \end{bmatrix}, L\left(\begin{bmatrix} 2 \\ 6 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ -6 \end{bmatrix}.$$

The plot is given below.



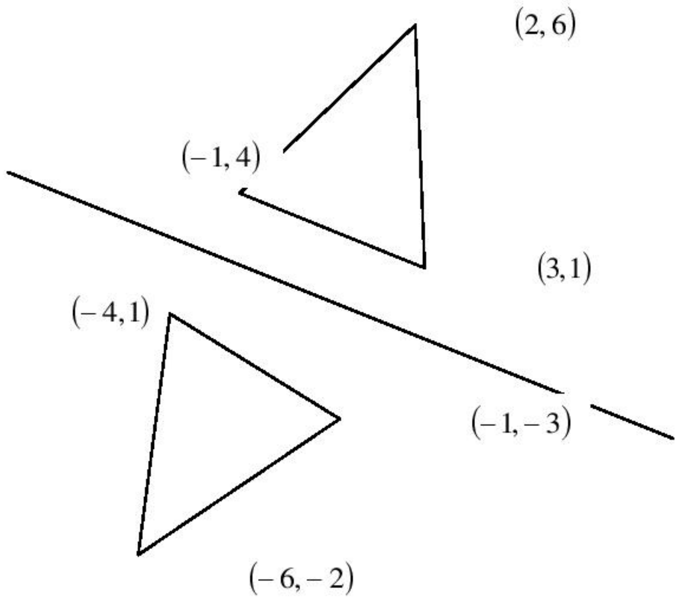
2. Reflection with respect to $y = -x$**:**

$$L: R^2 \to R^2, L\left(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\right) = A\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} -u_2 \\ -u_1 \end{bmatrix}.$$

Thus, the reflection for the triangle with vertices $(-1,4),(3,1),(2,6)$ is

$$L\left(\begin{bmatrix} -1 \\ 4 \end{bmatrix}\right) = \begin{bmatrix} -4 \\ 1 \end{bmatrix}, L\left(\begin{bmatrix} 3 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ -3 \end{bmatrix}, L\left(\begin{bmatrix} 2 \\ 6 \end{bmatrix}\right) = \begin{bmatrix} -6 \\ -2 \end{bmatrix}.$$

The plot is given below



3. Rotation:

$$L: R^2 \to R^2, L\left(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\right) = A\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

For example, as $\theta = \pi/2$,

$$A = \begin{bmatrix} \cos(\pi/2) & -\sin(\pi/2) \\ \sin(\pi/2) & \cos(\pi/2) \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Thus, the rotation for the triangle with vertices $(0,0), (1,0), (1,1)$ is

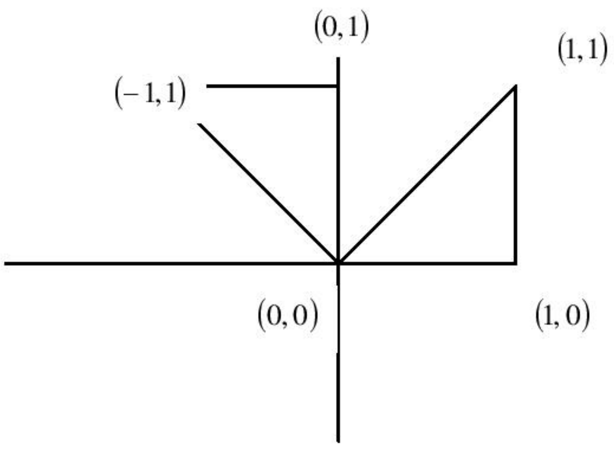
$$L\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},.$$

$$L\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

and

$$L\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

The plot is given below.



Problem 1: Consider the linear transformation $T:R^3 \to R^2$ defined as follows on basis vectors of R^3. Find $T(1,-2,3)$.

$$T(1,0,0)=(3,-1)\ ,\ T(0,1,0)=(2,1),\ T(0,0,1)=(3,0)$$

Solution: Since T is defined on basis vectors of R^3, it is defined on the whole space. To find, $T(1,-2,3)$, express the vector$(1,-2,3)$ as a linear combination of the basis vectors and use the linearity of T.

$$T(1,-2,3)=T\big(1(1,0,0)-2(0,1,0)+3(0,0,1)\big)$$

$$=1T(1,0,0),-2T(0,1,0)+3T(0,0,1)$$

$$=1(3,-1)-2(2,1)+3(3,0)$$

$$=(8,-3)$$

Problem 2: Let $T: U \to V$ be a linear transformation. T is defined relative to bases $B = \{u_1, u_2, u_3\}$ and $B^{'} = \{v_1, v_2\}$ of U and V as follows

$$T(u_1) = 2v_1 - v_2$$
$$T(u_2) = 3v_1 + 2v_2$$
$$T(u_3) = v_1 - 4v_2$$

Find the matrix representation of T with respect to these bases and use this matrix to determine the image of the vector $u = 3u_1 + 2u_2 - u_3$.

Solution: The coordinate vectors of $T(u_1), T(u_2)$ and $T(u_3)$ are

$$\begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \end{bmatrix} \text{ and } \begin{bmatrix} 1 \\ -4 \end{bmatrix}$$

These vectors make up the columns of the matrix of T

$$A = \begin{bmatrix} 2 & 3 & 1 \\ -1 & 2 & -4 \end{bmatrix}$$

Let us now find the image of the vector $u = 3u_1 + 2u_2 - u_3$ using this matrix.

The coordinate vector of u is $a = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$.

We get

$$Aa = \begin{bmatrix} 2 & 3 & 1 \\ -1 & 2 & -4 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 11 \\ 5 \end{bmatrix}$$

$T(u)$ has coordinate vector $\begin{bmatrix} 11 \\ 5 \end{bmatrix}$. Thus $T(u) = 11v_1 + 5v_2$.

Problem 3: Consider the linear transformation $T: R^3 \rightarrow R^2$, defined by $T(x, y, z) = (x + y, 2z)$. Find the matrix of T with respect to the bases $\{u_1, u_2, u_3\}$ and $\{u'_1, u'_2\}$ of R^3 and R^2, where
$u_1 = (1,1,0), u_2 = (0,1,4), u_3 = (1,2,3) and\, u'_1 = (1,0), u'_2 = (0,2)$.

Use this matrix to find the image of the vector $u = (2,3,5)$

Solution: We find the effect of T on the basis vectors of R^3.

$$T(u_1) = T(1,1,0) = (2,0) = 2(1,0) + 0(0,2) = 2u'_1 + 0u'_2$$
$$T(u_2) = T(0,1,4) = (1,8) = 1(1,0) + 4(0,2) = 1u'_1 + 4u'_2$$
$$T(u_3) = T(1,2,3) = (3,6) = 3(1,0) + 3(0,2) = 3u'_1 + 3u'_2$$

The coordinate vector of $T(u_1), T(u_2)$ and $T(u_3)$ are thus $\begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \end{bmatrix}$, and $\begin{bmatrix} 3 \\ 3 \end{bmatrix}$. These vectors from the columns of the matrix of T.

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 0 & 4 & 3 \end{bmatrix}$$

Let us now use A to find the image of the vector $u = (2,3,5)$. We determine the coordinate vector of u. It can be shown that

$$\begin{aligned} u = (2,3,5) &= 3(1,1,0) + 2(0,1,4) - (1,2,3) \\ &= 3u_1 + 2u_2 + (-1)u_3 \end{aligned}$$

The coordinate vector of u is thus $a = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$. The coordinate vector of $T(u)$ is

$$b = Aa = \begin{bmatrix} 2 & 1 & 3 \\ 0 & 4 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$$

Therefore, $T(u) = 5u'_1 + 5u'_2 = 5(1,0) + 5(0,2) = (5,10)$.
We can check this result directly using the definition $T(x, y, z) = (x + 2y, 2z)$.
For $u = (2,3,5)$, this gives

$$T(u) = T(2,3,5) = (2 + 3, 2 \times 5) = (5,10).$$

Problem 4: Consider the linear operator $T(x, y) = (2x, x + y)$ on R^2. Find the matrix of T with respect to the standard basis $B = \{(1,0), (0,1)\}$ of R^2. Use the transformation $A^{'} = P^{-1}AP$ to determine the matrix $A^{'}$ with respect to the basis $B^{'} = \{(2,3), (1, -1)\}$.

Solution: The effect of T on the vectors of the standard basis is

$$T(1,0) = (2,1) = 2(1,0) + 1(0,1)$$
$$T(0,1) = (0,1) = 0(1,0) + 1(0,1)$$

The matrix of T relative to the standard basis is

$$A = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$$

We now find P, the transition matrix from B' to B. Write the vectors of B' in terms of those of B.

$$(-2,3) = -2(1,0) + 3(0,1)$$
$$(1, -1) = 1(1,0) - (1(0,1)$$

The transition matrix is

$$P = \begin{bmatrix} -2 & 1 \\ 3 & -1 \end{bmatrix}$$

Therefore

$$A' = P^{-1}AP = \begin{bmatrix} -2 & 1 \\ 3 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 3 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ -10 & 6 \end{bmatrix}$$

Exercise

1) Let $T: U \to V$ be a linear transformation. Let T be defined relative to bases $\{u_1, u_2, u_3\}$ and $\{v_1, v_2, v_3\}$ of U and V as follows:

$$T(u_1) = v_1 + v_2 + v_3$$
$$T(u_2) = 3v_1 - 2v_2$$
$$T(u_3) = v_1 + 2v_2 - v_3.$$

Find the matrix of T with respect to these bases. Use this matrix to find the image of the vector $u = 3u_1 + 2u_2 - 5u_3$.

2) Find the matrices of the following linear operators on R^3 with respect to the standard basis of R^3. Use these matrices to find the images of the vector $(-1,5,2)$
   (a) $T(x, y, z) = (x, 2y, 3z)$
   (b) $T(x, y, z) = (x, 0,0)$

3) Consider the linear transformation $T: R^3 \to R^2$ defined by $T(x, y, z) = (x - y, x + z)$. Find the matrix of T with respect to the bases $\{u_1, u_2, u_3\}$ and $\{u'_1, u'_2\}$ of R^3 and R^2, where

$$u_1 = (1, -1,0), u_2 = (2,0,1)$$
$$u_3 = (1,2,1) \text{ and } u'_1 = (-1,0), u'_2 = (0,1).$$

Use this matrix to find the image of the vector $u = (3, -4,0)$.

4) Find the matrix of the differential operator D with respect to the basis $\{2x^2, x, -1\}$ of P_2. Use this matrix to find the image of $3x^2 - 2x + 4$.

5) Find the matrix of the following linear transformations with respect to the basis $\{x, 1\}$ of P_1 and $\{x^2, x, 1\}$of P_2.
   (a) $T(ax^2 + bx + c) = (b + c)x^2 + (b - c)x$ of P_2 into itself
   (b) $T(ax + b) = bx^2 + ax + b$ of P_1 into P_2.

6) Consider the linear operator $T(x, y) = (2x, x + y)$ on R^2. Find the matrix of T with respect to the standard basis of R^2. Use a similarity transformation to then find the matrix with respect to the basis $\{(1,1), (2,1)\}$ of R^2.

7) Let U, V *and* W be vector spaces with bases $B = \{u_1, . . . , u_n\}$, $B^{'} = \{v_1, ... , v_m\}$, and $B^{''} = \{w_1, ... , w_m\}$, be linear transformations. Let P be the matrix of T with respect to B *and* B', and Q be the matrix representation of L with respect to $B^{'}$*and* B''. Prove that the matrix of LoT with respect to B *and* B'' is QP.

8) Is it possible for two distinct linear transformations $T: U \rightarrow V$ and $L: U \rightarrow V$ to have the same matrix with respect to bases B *and* B' of U *and* V?

Answers

1) $\begin{bmatrix} 1 & 3 & 1 \\ 1 & -2 & 2 \\ 1 & 0 & -1 \end{bmatrix}, 4v_1 - 11v_2 + 8v_3$

2) (a) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}, (-1, 10, 6)$

(b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, (-1, 0, 0)$

3) $\begin{bmatrix} -2 & -2 & 1 \\ 1 & 3 & 2 \end{bmatrix}, (7, 3)$

4) $\begin{bmatrix} 0 & 0 & 0 \\ 4 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}, 6x - 2$

5) (a) $\begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$

(b) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$

6) $\begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 2 \\ 0 & 1 \end{bmatrix}$

8) No