Syllabus:-

Artificial variables: Big-M method, Sensitivity analysis, Duality problems, Economic interpretation of Simplex Tableau, Computer software for solving LPP.

Big-M method:-

If constraints are of "=" or "≥" type, a new variable called "Artificial variable" will be introduced in each of such constraint.

If Objective fn. is maximisation type, coefficient of artificial variable in objective function is "-M". Otherwise, it is "M", where M is a very large value.

For "≤" ⇒ Add Slack Variable (+S)

"≥" ⇒ Subtract Surplus variable (-S) and Add Artificial variable (+A).

Problems

① Solve: Minimize $Z = 7x_1 + 15x_2 + 20x_3$

Subject to $2x_1 + 4x_2 + 6x_3 \geq 24$

$3x_1 + 9x_2 + 6x_3 \geq 30$

$x_1, x_2, x_3 \geq 0$

Sol:- Step-1:- Express the given LP in Std. form

Minimize $Z = 7x_1 + 15x_2 + 20x_3 + (0 \times S_1) + (0 \times S_2) + MA_1 + MA_2$

Subject to $2x_1 + 4x_2 + 6x_3 - S_1 + A_1 = 24$

$3x_1 + 9x_2 + 6x_3 - S_2 + A_2 = 30$

$x_1, x_2, x_3, S_1, S_2, A_1, A_2 \geq 0$

Step-2:- To find IBFS

To find IBFS, let us substitute the values of decision variables and Surplus variables as zeroes.

This will result in the following: ②

$$Z_{min} = (7\times0)+(15\times0)+(20\times0)+(0\times S_1)+(0\times S_2) + MA_1 + MA_2$$

$$= MA_1 + MA_2$$

$$(2\times0)+(4\times0)+(6\times0)-0+A_1=24 \Rightarrow A_1=24$$

$$(3\times0)+(9\times0)+(6\times0)-0+A_2=30 \Rightarrow A_2=30$$

<u>Step-3:</u>- To perform optimality test by BigM method (or) Artificial variable technique.

C_{B_i}	c_j / Basis	7 x_1	15 x_2	20 x_3	0 S_1	0 S_2	M A_1	M A_2	b_j	$\theta = \frac{b_j}{\text{Key Column}}$
M	A_1	2	4	6	-1	0	1	0	24	$\frac{24}{4}=6$
M	A_2	3	(9)	6	0	-1	0	1	30	$\frac{30}{9}=3.33 \rightarrow$
	Z_j	5M	13M	12M	-M	-M	M	M	54M	
	$C_j - Z_j$	7-5M	15-13M	20-12M	M	M	0	0	-	

Key row

↑ Key Column

leaving variable - A_2

Entering variable - x_2

[∵ $C_j - Z_j$ has max. negative value for the variable "x_2", x_2 entering variable.]

<u>Second Simplex Table.</u>

C_{B_i}	c_j / Basis	7 x_1	15 x_2	20 x_3	20 S_1	0 S_2	M A_1	M A_2	Sol.	$\theta = \frac{\text{Sol}}{\text{Key Column}}$
M	A_1	2/3	0	(10/3)	-1	4/9	1	$\cancel{-4/9}$	32/3	$\frac{32/3}{10/3}=3.2 \rightarrow$
15	x_2	1/3	1	2/3	0	-1/9	0	$\cancel{1/9}$	10/3	$\frac{10/3}{2/3}=5$
	Z_j	$\frac{15+2M}{3}$	15	$\frac{30+10M}{3}$	-M	$\frac{4}{9}M-15$	M	$\cancel{\frac{15-4M}{9}}$	$\frac{150+32M}{3}$	
	$C_j - Z_j$	$-\frac{2M}{3}+2$	0	$-\frac{10M}{3}+10$	M	$\frac{5}{3}-\frac{4M}{9}$	0	$\cancel{\frac{5M-15}{9}}$	-	

↑

For x_2: New value $= \dfrac{\text{Old Value}}{\text{Key element}}$

$x_1 = \frac{3}{9} = \frac{1}{3}$; $x_2 = \frac{9}{9} = 1$; $x_3 = \frac{6}{9} = \frac{2}{3}$

$S_1 = \frac{0}{9} = 0$; $S_2 = \frac{-1}{9}$; $A_1 = \frac{0}{9} = 0$; $A_2 = \frac{1}{9}$

$b_j = \frac{30}{9} = \frac{10}{3}$

(3) For A_1: New Value = Old Value $-\left[\frac{\text{Key Row Ele.} \times \text{Key Column Ele.}}{\text{Key Ele.}}\right]$.

$$x_1 = 2-\left[\cancel{3}\times\left(\frac{4}{\cancel{9}_3}\right)\right] = \frac{2}{3}$$

$$x_2 = 4-\left[9\times\frac{4}{9}\right] = 0$$

$$x_3 = 6-\left[6\times\frac{4}{9}\right] = \frac{10}{3}$$

$$S_1 = -1-\left[0\times\frac{4}{9}\right] = -1$$

$$S_2 = 0-\left[(-1)\times\frac{4}{9}\right] = \frac{4}{9}$$

$$A_1 = 1-\left[0\times\frac{4}{9}\right] = 1$$

$$\cancel{A_2 = 0-\left[1\times\frac{4}{9}\right] = \frac{-4}{9}}$$

$$b_j = 24-\left[\cancel{30}^{10}\times\frac{4}{\cancel{9}_3}\right] = \frac{72-40}{3} = \frac{32}{3}$$

Third Simplex table

C_j 7 15 20 0

C_B Basis x_1 x_2 S_1 S_2 A_1 A_2 Sol.

20 x_3 64+18 $\frac{40}{15}-\frac{15}{5} = \frac{40-45}{15} = \frac{-5}{15}$

2

x_2

	C_j	7	15	20	0	0	M	M	
C_B	Basis	x_1	x_2	x_3	S_1	S_2	A_1	A_2	Sol.
20	x_3	1/5	0	1	-3/10	2/15	-	-	16/5
15	x_2	1/5	1	0	1/5	-1/5	-	-	6/5
	Z_j	7	15	20	-3	-1/3	-	-	82
	C_j-Z_j	0	0	0	3	1/3			

For x_2 row: $x_1 = \frac{1}{3}-\left[\frac{2}{3}\times\frac{2/\cancel{3}}{10/\cancel{3}}\right] = \frac{3}{15} = \frac{1}{5}$

$$x_2 = 1-\left[0\times\frac{2/3}{10/3}\right] = 1$$

$$x_3 = \frac{2}{3}-\left[\frac{10}{3}\times\frac{2/3}{10/3}\right] = 0$$

$$S_1 = 0-\left[-1\times\frac{2/\cancel{3}}{10/\cancel{3}}\right] = \frac{1}{5}$$

$$S_2 = \frac{-1}{9}-\left[\frac{4}{9}\times\frac{2/\cancel{3}}{10/\cancel{3}}\right] = \frac{-1}{45}$$

$$Sol. = \frac{10}{3}-\left[\frac{32}{3}\times\frac{2/\cancel{3}}{10/\cancel{3}}\right] = \frac{10}{3}-\frac{32}{15} = \frac{50-32}{15} = \frac{6}{5}$$

Solution: $x_1 = 0$ $S_1 = 0$
$x_2 = 6/5$ $S_2 = 0$
$x_3 = 16/5$

$Z_{min} = 82$

Sensitivity Analysis – LPP:- [Pannerselvam]

In many situations, the parameters and characteristics of a LP model may change over a period of time. Also, the analyst may be interested to know the effect of changing the parameters and characteristics of the model on optimality. This kind of sensitivity analysis can be carried out in the following ways:

1. Making changes in RHS constants of constraints
2. Making changes in the objective function coefficients.
3. Adding a new constraint.
4. Adding a new variable.

1. Making Changes in RHS constants of constraints:-

The RHS constant of one or more constraints of a LP model may change over a period of time.
The changes bring in the following results:
(a) Same set of basic variables with modified RHS constants in the optimal table.
(b) Different set of basic variables in the optimal table.

Problem

① Maximize $Z = 6x_1 + 8x_2$

Subject to: $5x_1 + 10x_2 \leq 60$
$4x_1 + 4x_2 \leq 40$
$x_1, x_2 \geq 0$

Sol:- Step-1:- Express the given LP problem in std. form

Maximize $Z = 6x_1 + 8x_2 + (0 \times S_1) + (0 \times S_2)$

Subject to: $5x_1 + 10x_2 + S_1 = 60$
$4x_1 + 4x_2 + S_2 = 40$
$S_1, S_2, x_1 \;\&\; x_2 \geq 0$

Step-2:- To find IBFS.

To find IBFS, let us substitute the values of decision variables ~~and~~ as zeroes.

This will result in the following:

- $Z_{max} = (6\times0)+(8\times0)+(0\times S_1)+(0\times S_2)=0$

$(5\times0)+(10\times0)+S_1=60 \Rightarrow S_1=60$

$(4\times0)+(4\times0)+S_2=40 \Rightarrow S_2=40$

Step-3:- To perform optimality test

Initial Simplex table:-

Key Row - Leaving

Key Column - Entering

C_{B_i}	C_j / Basis	6 / x_1	8 / x_2	0 / S_1	0 / S_2	b_j	$\theta = \frac{b_j}{\text{Key Column}}$
0	S_1	5	10 (Key Element)	1	0	60	$\frac{60}{10}=6$ Key Row
0	S_2	4	4	0	1	40	$\frac{40}{4}=10$
	Z_j	0	0	0	0	0	
	$C_j - Z_j$	6	8	0	0	-	

↑ Key Column.

Leaving variable $= S_1$; Entering Variable $= x_2$.

C_{B_i}	C_j / Basis	6 / x_1	8 / x_2	0 / S_1	0 / S_2	b_j	$\theta = \frac{b_i}{\text{Key Column}}$
8	x_2	$\frac{1}{2}$	1	$\frac{1}{10}$	0	6	$\frac{6}{1/2}=12$
0	S_2	2 (Key Element)	0	$-\frac{2}{5}$	1	16	$\frac{16}{2}=8$ Key Row
	Z_j	4	8	$\frac{4}{5}$	0	48	
	$C_j - Z_j$	2	0	$-\frac{4}{5}$	0	~~48~~	

↑ Key Column

New Value = Old Value / Key Element

S_2: $NV = OV - \left[\frac{KRE \times KCE}{KE}\right]$

C_{B_i}	C_j / Basis	6 / x_1	8 / x_2	0 / S_1	0 / S_2	b_j	$\theta = \frac{b_j}{}$
8	x_2	0	1	$\frac{1}{5}$	$-\frac{1}{4}$	2	
6	x_1	1	0	$-\frac{1}{5}$	$\frac{1}{2}$	8	
	Z_j	6	8	$\frac{2}{5}$	1	64	
	$C_j - Z_j$	0	0	$-\frac{2}{5}$	-1	—	

(a) If RHS constants of constraints 1 & 2 are changed from 60 & 40 to 40 & 20 respectively, then

$$\begin{bmatrix}\text{Basic Variables} \\ \text{in the optimal} \\ \text{table}\end{bmatrix} = \begin{bmatrix}\text{Technological coefficients} \\ \text{columns in the optimal} \\ \text{table w.r.t the basic} \\ \text{variables in initial table}\end{bmatrix} \begin{bmatrix}\text{New RHS} \\ \text{Constants}\end{bmatrix}$$

$$\begin{bmatrix} x_2 \\ x_1 \end{bmatrix} = \begin{bmatrix} 1/5 & -1/4 \\ -1/5 & 1/2 \end{bmatrix} \begin{bmatrix} 40 \\ 20 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \quad \begin{pmatrix} (\frac{1}{5} \times 40) + [(-\frac{1}{4}) \times 20] \\ (-\frac{1}{5} \times 40) + (\frac{1}{2} \times 20) \end{pmatrix}$$

$x_1 = 2$ & $x_2 = 3$

$\because$ These values are non-negative, revised solution is feasible and optimal.

Obj. fn. value is $Z_{max} = (6 \times 2) + (8 \times 3) = 36$.

2. Changes in the Objective function coefficients:-

In reality, the profit or cost coefficients of the objective function undergo changes over a period of time. Under such situation, revised optimum solution can be obtained from optimal table of original problem by following certain steps. Also, the range of coefficient of a variable in the objective function over which the optimality is unaffected, can be known.

(a) Determination of range of C_1 of basic variable x_1;
(b) Determination " " C_2 of non-basic variable x_2; and
(c) Check the optimality.

3. Adding a new constraint:-

Sometimes, a new constraint may be added to an existing LP model. Under such situation, each of the basic variables in the new constraint is substituted with the corresponding expression based on current optimal table. This will yield a modified version of the new constraint in terms of only the current non-basic variables.

(a) If new constraint is satisfied by values of current basic variables, constraint is said to be redundant one. So, optimality of original problem will not be affected even after including the new constraint into existing model.

If new constraint is not satisfied by values of current basic variables, the optimality of original problem will be affected. So, modified version of new constraint is to be augmented to optimal table of original problem and iterated till optimality is reached.

Ex:- Maximize $Z = 6x_1 + 8x_2$

Subject to $5x_1 + 10x_2 \le 60$

$4x_1 + 4x_2 \le 40$

x_1 & $x_2 \ge 0$

Sol:- $x_1 = 8$

$x_2 = 2$

(a) New constraint: $7x_1 + 2x_2 \le 65$.

If we sub $x_1 = 8$ & $x_2 = 2$ in new constraint, then the constraint is satisfied which shows that the new $((7\times 8)+(2\times 2) = 60 \le 65)$ constraint is redundant constraint & doesn't affect optimality of original problem.

*b) New constraint: $6x_1 + 3x_2 \le 48$ x

Sub $x_1 = 8$ & $x_2 = 2$ in new constraint.

$(6\times 8)+(3\times 2) = 48+6 = 54 > 48$

The new constraint is not satisfied. So, the modified form of the new constraint in terms of only non-basic variables is obtained.

Std. form of new constraint: $6x_1 + 3x_2 + S_3 = 48$

From optimal table of original problem, we have,

$x_2 + \frac{1}{5}S_1 - \frac{1}{4}S_2 = 2$

$x_1 - \frac{1}{5}S_1 + \frac{1}{2}S_2 = 8$

Sub these expressions in std. form to get eq. in S_1, S_2 & S_3

4. **Adding a new variable:-**

In a problem like product mix problem, over a period of time, a new product may be added to existing product mix.

The following items are to be determined after incorporating the data of the new variable (new product).

$C_j - Z_j$ Value:

$$C_j - Z_j = C_j - [CB]_{1\times m} \begin{bmatrix} \text{Technological coeff.'s} \\ \text{of optimal table} \\ \text{wrt basic variables} \\ \text{of initial table} \end{bmatrix}_{m\times m} \times \begin{bmatrix} \text{Constraint coeffs} \\ \text{of new variable} \end{bmatrix}_{m\times 1}$$

where, m - no. of constraints in problem.

If $C_j - Z_j$ value of new variable indicates optimality as per nature of optimization (max. or min.), optimality of problem after including the new variable is not affected. Otherwise, the constraint coefficients of the new variable are to be computed.

The Constraint coefficients (Technological coefficients) of the column Corresponding to the new variable:

$$\begin{bmatrix} \text{Revised constraint} \\ \text{coefficients of the} \\ \text{new variable} \end{bmatrix}_{m\times 1} = \begin{bmatrix} \text{Technological coeff.s of} \\ \text{optimal table wrt the} \\ \text{basic variables of the} \\ \text{initial table} \end{bmatrix}_{m\times m} \times \begin{bmatrix} \text{Constraints coeff.s} \\ \text{of new variable} \end{bmatrix}_{m\times 1}$$

These coefficients are incorporated in the current optimal table and the necessary number of iterations is to be carried out from the current till the optimality is reached.

DUALITY:-

A generalized format of LPP:

Maximize or minimize $Z = C_1x_1 + C_2x_2 + \dots + C_nx_n$

Subject to
$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \le, = \text{or} \ge b_1$$
$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \le, = \text{or} \ge b_2$$
$$\vdots \qquad\qquad\qquad \vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n \le, = \text{or} \ge b_m$$

where $x_1, x_2, \ldots, x_n \geq 0$.

Let this problem be called as a primal LPP. If constraints in the primal problem are too many, then the time taken to solve the problem is expected to be higher. Under such situation, the primal LPP can be converted into its dual Linear Pro. Pro. which requires relatively lower time to solve.

Formulation of Dual Problem:

The primal problem can be written as:

Maximize or minimize $Z = c_1x_1 + c_2x_2 + \ldots + c_nx_n$

$$\text{Subject to } a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \gtreqless b_1 \longleftarrow y_1$$

$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \gtreqless b_2 \longleftarrow y_2$$

$$a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n \gtreqless b_i \longleftarrow y_i$$

$$a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \gtreqless b_m \longleftarrow y_m$$

where $x_1, x_2, \ldots, x_n \geq 0$.

The variable "y_i" is called as the dual variable associated with the constraint "i".

OBJECTIVE FUNCTION:

No. of variables in Dual problem = No. of constraints in primal problem

+ Objective function of dual problem is constructed by adding multiples of RHS constants of constraints of primal problem with respective dual variables.

CONSTRAINTS:

No. of constraints in dual problem = No. of variables in primal problem.

+ Each dual constraint corresponds to each primal variable.

LHS of dual constraint corresponding to jth primal variable is sum of multiples of LHS constraint coefficients of the jth primal variable with the corresponding dual variables.

The RHS constant of dual constraint corresponding to the jth primal variable is obj. fn. coeff. of jth primal variable.

Guidelines for dual formulation

Type of problem	Obj. Fn.	Constraint type	Nature of variab
Primal	Max.	$\leq$	Restricted in sig
Dual	Min.	$\geq$	"
Primal	Min.	$\geq$	"
Dual	Max.	$\leq$	"
Primal	Max.	$=$	"
Dual	Min.	$\geq$	UnRestricted in sig
Primal	Min.	$=$	Restricted in sig
Dual	Max.	$\leq$	Unrestricted in sig
Primal	Max.	$\leq$	Unrestricted in sig
Dual	Min.	$=$	Restricted in sign
Primal	Min.	$\geq$	Unrestricted in sig
Dual	Max.	$=$	Restricted in sign

Problem

① Form the dual of the following problem:

Maximize $Z = 4x_1 + 10x_2 + 25x_3$

Subject to
$$2x_1 + 4x_2 + 8x_3 \leq 25$$
$$4x_1 + 9x_2 + 8x_3 \leq 30$$
$$6x_1 + 8x_2 + 2x_3 \leq 40$$
$$x_1, x_2 \text{ \& } x_3 \geq 0$$

Sol:- Primal problem:

Maximize $Z = \boxed{4}\, x_1 + \boxed{10}\, x_2 + \boxed{25}\, x_3$

Subject to
$$\boxed{\begin{matrix}2\\4\\6\end{matrix}}\, \begin{matrix}x_1 +\\ x_1 +\\ x_1 +\end{matrix} \boxed{\begin{matrix}4\\9\\8\end{matrix}}\, \begin{matrix}x_2 +\\ x_2 +\\ x_2 +\end{matrix} \boxed{\begin{matrix}8\\8\\2\end{matrix}}\, \begin{matrix}x_3 \leq 25 \leftarrow y_1\\ x_3 \leq 30 \leftarrow y_2\\ x_3 \leq 40 \leftarrow y_3\end{matrix}$$
$$x_1, x_2, x_3 \geq 0$$

Corresponding DUAL problem:

Minimize $Y = 25y_1 + 30y_2 + 40y_3$

Subject to
$$2y_1 + 4y_2 + 6y_3 \geq 4$$
$$4y_1 + 9y_2 + 8y_3 \geq 10$$
$$8y_1 + 8y_2 + 2y_3 \geq 25$$

$y_1, y_2, y_3 \geq 0$

② Form the dual of the following primal problem.

$$\text{Minimize } Z = 20x_1 + 40x_2$$

Subject to

$$2x_1 + 20x_2 \geq 40$$
$$20x_1 + 3x_2 \geq 20$$
$$4x_1 + 15x_2 \geq 30$$
$$x_1 \;\&\; x_2 \geq 0$$

Sol:- DUAL problem:

$$\text{Maximize } Y = 40y_1 + 20y_2 + 30y_3$$

Subject to

$$2y_1 + 20y_2 + 4y_3 \leq 20$$
$$20y_1 + 3y_2 + 15y_3 \leq 40$$
$$y_1, y_2 \;\&\; y_3 \geq 0$$

③ Form the dual of the following primal problem.

$$\text{Maximize } Z = 4x_1 + 10x_2 + 25x_3$$

Subject to

$$2x_1 + 4x_2 + 8x_3 = 25$$
$$4x_1 + 9x_2 + 8x_3 = 30$$
$$6x_1 + 8x_2 + 2x_3 = 40$$
$$x_1, x_2 \;\&\; x_3 \geq 0$$

Sol:- DUAL problem:

$$\text{Minimize } Y = 25y_1 + 30y_2 + 40y_3.$$

Subject to

$$2y_1 + 4y_2 + 6y_3 \geq 4$$
$$4y_1 + 9y_2 + 8y_3 \geq 10$$
$$8y_1 + 8y_2 + 2y_3 \geq 25$$

$y_1, y_2 \;\&\; y_3$ $\cancel{\geq 0}$ - Unrestricted in sign

(4) Form the dual of the following primal problem.

Minimize $Z = 20x_1 + 40x_2$

Subject to
$$2x_1 + 20x_2 = 40$$
$$20x_1 + 3x_2 = 20$$
$$4x_1 + 15x_2 = 30$$
$$x_1 \ \& \ x_2 \geq 0$$

Sol:- DUAL problem:

Maximize $Y = 40y_1 + 20y_2 + 30y_3$

Subject to
$$2y_1 + 20y_2 + 4y_3 \leq 20$$
$$20y_1 + 3y_2 + 15y_3 \leq 40$$

$y_1, y_2 \ \& \ y_3$ - Unrestricted in sign.

(5) Form the dual of the following primal problem.

Minimize $Z = 5x_1 + 8x_2$

Subject to
$$4x_1 + 9x_2 \geq 100$$
$$2x_1 + x_2 \leq 20 \Rightarrow -2x_1 - x_2 \geq -20$$
$$2x_1 + 5x_2 \geq 120$$
$$x_1 \ \& \ x_2 \geq 0$$

Sol:- DUAL problem:

Maximize $Y = 100y_1 - 20y_2 + 120y_3$

Subject to
$$4y_1 - 2y_2 + 2y_3 \leq 5$$
$$9y_1 - y_2 + 5y_3 \leq 8.$$
$$y_1, y_2 \ \& \ y_3 \geq 0$$

(5) Form the dual of the following primal problem.

Minimize $Z = 2x_1 + 6x_2$

Subject to
$$9x_1 + 3x_2 \geq 20$$
$$2x_1 + 7x_2 = 40$$
$$x_1 \ \& \ x_2 \geq 0$$

Sol:- Modified form of primal problem:

Minimize $Z = 2x_1 + 6x_2$

Subject to $9x_1' + 3x_2' \geq 20$

$2x_1 + 7x_2 \geq 40$

$2x_1 + 7x_2 \leq 40 \Rightarrow -2x_1 - 7x_2 \geq -40$

$x_1 \; \& \; x_2 \geq 0$

DUAL problem:

Maximize $Y = 20y_1 + 40y_2' - 40y_2''$

Subject to $9y_1 + 2y_2' - 2y_2'' \leq 2$

$3y_1 + 7y_2' - 7y_2'' \leq 6$

$y_1, y_2' \; \& \; y_2'' \geq 0$

(4) Consider the following LPP and solve it using its dual solution.

Minimize $Z = 40x_1 + 30x_2 + 25x_3$

Subject to $4x_1 + 2x_2 + 5x_3 \geq 30$

$3x_1 + 6x_2 + x_3 \geq 20$

$x_1 + 3x_2 + 6x_3 \geq 36$

$x_1, x_2 \; \& \; x_3 \geq 0$

Sol:- Let $y_1, y_2 \; \& \; y_3$ be the dual variables wrt constraints 1, 2 & 3 respectively of primal problem as shown.

Primal problem:

Minimize $Z = 40x_1 + 30x_2 + 25x_3$

Subject to $4x_1 + 2x_2 + 5x_3 \geq 30$

$3x_1 + 6x_2 + x_3 \geq 20$

$x_1 + 3x_2 + 6x_3 \geq 36$

$x_1, x_2 \; \& \; x_3 \geq 0$

DUAL problem:

Maximize $Y = 30y_1 + 20y_2 + 36y_3$

Subject to $4y_1 + 3y_2 + y_3 \leq 40$

$2y_1 + 6y_2 + 3y_3 \leq 30$

$5y_1 + y_2 + 6y_3 \leq 25$

$y_1, y_2 \; \& \; y_3 \geq 0$

Canonical form of DUAL problem:

Maximize $Z = 30y_1 + 20y_2 + 36y_3 + (0 \times S_1) + (0 \times S_2) + (0 \times S_3)$

Subject to $4y_1 + 3y_2 + y_3 + S_1 = 40$

$2y_1 + 6y_2 + 3y_3 + S_2 = 30$

$5y_1 + y_2 + 6y_3 + S_3 = 25$

$y_1, y_2, y_3, S_1, S_2 \& S_3 \geq 0$

Initial Simplex Table:-

C_{B_i}	Basis	C_j 30 y_1	20 y_2	36 y_3	0 S_1	0 S_2	0 S_3	Sol	$\theta = \frac{Sol}{KC}$
0	S_1	4	3	1	1	0	0	40	$\frac{40}{1} = 40$
0	S_2	2	6	3	0	1	0	30	$\frac{30}{3} = 10$
0	S_3	5	1	KE 6	0	0	1	25	$\frac{25}{6} = 4.1 \rightarrow$ KR
	Z_j	0	0	0	0	0	0		
	$C_j - Z_j$	30	20	36	0	0	0		
				↑ KC					

Leaving variable = S_3 ; Entering variable = y_3

Key element = 6

∵ All $(C_j - Z_j) \geq 0$, Optimality is not reached.

Second Simplex Table:-

C_{B_i}	Basis	C_j 30 y_1	20 y_2	36 y_3	0 S_1	0 S_2	0 S_3	Sol	$\theta = \frac{Sol}{KC}$
0	S_1	19/6	17/6	0	1	0	$-1/6$	$\frac{215}{6}$	$\frac{215/6}{17/6} = \frac{215}{17} = 12.65$
0	S_2	$-1/2$	KE 11/2	0	0	1	$-1/2$	35/2	$\frac{35}{11} = 3.18$ KR
36	y_3	5/6	1/6	1	0	0	1/6	25/6	25
	Z_j	30	6	36	0	0	6	150	
	$C_j - Z_j$	0	14	0	0	0	-6		
			↑ KC						

$\frac{215}{6} - \frac{595}{66}$

~~590~~ 295

~~1770~~

y_3:- New values $= \dfrac{\text{Old Values}}{KE}$

$y_1 = \frac{5}{6}$; $y_2 = 1/6$; $y_3 = \frac{6}{6} = 1$; $S_1 = 0$; $S_2 = 0$; $S_3 = 1/6$

Sol $= \frac{25}{6}$

S_1:- $NV = OV - \left[KRE \times \dfrac{KCE}{KE}\right]$

$y_1 = 4 - \left[5 \times \frac{1}{6}\right] = \frac{19}{6}$; $y_2 = 3 - \left[1 \times \frac{1}{6}\right] = \frac{17}{6}$

$y_3 = 1 - \left[6 \times \frac{1}{6}\right] = 0$; $S_1 = 1 - \left[0 \times \frac{1}{6}\right] = 1$

$S_2 = 0 - \left[0 \times \frac{1}{6}\right] = 0$; $S_3 = 0 - \left[1 \times \frac{1}{6}\right] = -\frac{1}{6}$

Sol $= 40 - \left[25 \times \frac{1}{6}\right] = \frac{240 - 25}{6} = \frac{215}{6}$

S_2:- $y_1 = 2 - \left[5 \times \frac{1}{2}\right] = -\frac{1}{2}$; $y_2 = 6 - \left[1 \times \frac{1}{2}\right] = \frac{11}{2}$

$y_3 = 3 - \left[6 \times \frac{1}{2}\right] = 0$; $S_1 = 0 - \left[0 \times \frac{1}{2}\right] = 0$; $S_2 = 1 - \left[0 \times \frac{1}{2}\right] = 1$

$S_3 = 0 - \left[1 \times \frac{1}{2}\right] = -\frac{1}{2}$; Sol $= 30 - \left[25 \times \frac{1}{2}\right] = \frac{35}{2}$

Third Simplex Table:-

	c_j	30	20	36	0	0	0		
C_{B_i}	Basis	y_1	y_2	y_3	S_1	S_2	S_3	Sol	$\theta = \frac{Sol}{KC}$
0	S_1	113/33	0	0	1	−17/33	1/11	295/4	385/113 = 7.83
20	y_2	−1/11	1	0	0	2/11	−4/11	35/11	−35
36	y_3	KE 28/33	0	1	0	−1/33	2/11	40/11	$\frac{30}{7}$ = 4.28 KR
	Z_j	316/11	20	36	0	28/11	54/11	2140/11	
	$C_j - Z_j$	14/11	0	0	0	−28/11	−54/11		
		↑ KC							

y_2:- $NV = \dfrac{OV}{KE}$; $y_1 = \dfrac{-1/2}{11/2} = -1/11$; $y_2 = \dfrac{11/2}{11/2} = 1$; $y_3 = \dfrac{0}{11/2} = 0$;

$S_1 = \dfrac{0}{11/2} = 0$; $S_2 = \dfrac{1}{11/2} = 2/11$; $S_3 = \dfrac{-1/2}{11/2} = -\frac{1}{11}$;

Sol $= \dfrac{35/2}{11/2} = \dfrac{35}{11}$

S_1:- $NV = OV - \left[KRE \times \dfrac{KCE}{KE}\right]$ $y_1 = \frac{19}{6} - \left[-\frac{1}{2} \times \dfrac{17/33}{11/2}\right] = \frac{19}{6} + \frac{17}{66} = \frac{113}{33}$

$y_2 = 0$; $y_3 = 0 - \left[0 \times \frac{17}{33}\right] = 0$; $S_1 = 1 - \left[0 \times \frac{17}{33}\right] = 1$;

$S_2 = 0 - \left[1 \times \frac{17}{33}\right] = -\frac{17}{33}$; $S_3 = -\frac{1}{6} - \left[-\frac{1}{2} \times \frac{17}{33}\right] = \frac{1}{11}$; Sol $= \frac{215}{6} - \left[\frac{35}{2} \times \frac{17}{33}\right]$
$= 295/11$

$\underline{y_3:-}$ $y_1 = \frac{5}{6} - \left[-\frac{1}{\cancel{2}} \times \frac{1/\cancel{33}}{11/\cancel{2}}\right] = \frac{5}{6} - \left[-\frac{1}{2} \times \frac{1}{33}\right] = \frac{55+1}{66} = \frac{\cancel{56}^{28}}{\cancel{66}_{33}}$

$y_2 = \frac{1}{6} - \left[\frac{11}{2} \times \frac{1}{33}\right] = 0$; $y_3 = 1 - \left[0 \times \frac{1}{33}\right] = 1$;

$s_1 = 0 - \left[0 \times \frac{1}{33}\right] = 0$; $s_2 = 0 - \left[1 \times \frac{1}{33}\right] = -\frac{1}{33}$; $s_3 = \frac{1}{6} - \left[-\frac{1}{2} \times \frac{1}{33}\right]$

$s_3 = \frac{1}{6} + \frac{1}{66} = \frac{12}{66} = \frac{2}{11}$; $Sol = \frac{25}{6} - \left[\frac{35}{2} \times \frac{1}{33}\right]$

$= \frac{25}{6} - \frac{35}{66} = \frac{275-35}{66} = \frac{\cancel{240}^{40}}{\cancel{66}_{11}} = \frac{40}{11}$

Fourth Simplex Table:-

C_B:	C_j / Basis	30 y_1	20 y_2	36 y_3	0 S_1	0 S_2	0 S_3	Sol
0	S_1	0	0	$-113/28$	1	$-11/28$	$-9/14$	$85/7$
20	y_2	0	1	$3/28$	0	$55/308$	$-1/14$	$275/77$
30	y_1	1	0	$33/28$	0	$-1/28$	$3/14$	$30/7$
	Z_j	30	20	$525/14$	0	$5/2$	5	200
	$C_j - Z_j$	0	0	$-3/2$	0	$-5/2$	-5	

$\because$ All $(C_j - Z_j) \leq 0$, Optimality is reached.

Determination of solution of Primal:

Basic variable in initial table of dual pro.	S_1	S_2	S_3
$-(C_j - Z_j)$ of final table of dual pro.	0	$5/2$	5
Corresponding primal variable	x_1	x_2	x_3

Optimal solution: $x_1 = 0$

$x_2 = 5/2$

$x_3 = 5$

$Z_{min} = 200.$

(2) Use duality to solve the following problem:

Maximize $Z = 2x_1 + x_2$

Subject to $x_1 + 2x_2 \le 10$

$x_1 + x_2 \le 6$

$x_1 - x_2 \le 2$

$x_1 - 2x_2 \le 1$

$x_1 \,\&\, x_2 \ge 0.$

Sol:- Step-1:- Let $y_1, y_2, y_3 \,\&\, y_4$ be the dual variables w.r.t constraint 1, 2, 3 & 4 respectively of primal problem as shown.

Primal problem:

Maximize $Z = 2x_1 + x_2$

Subject to $x_1 + 2x_2 \le 10$

$x_1 + x_2 \le 6$

$x_1 - x_2 \le 2$

$x_1 - 2x_2 \le 1$

$x_1 \,\&\, x_2 \ge 0$

DUAL problem:

Minimize $Z = 10y_1 + 6y_2 + 2y_3 + y_4$

Subject to $y_1 + y_2 + y_3 + y_4 \ge 2$

$2y_1 + y_2 - y_3 - 2y_4 \ge 1$

$y_1, y_2, y_3 \,\&\, y_4 \ge 0.$

Canonical form of DUAL problem:

Minimize $Z = 10y_1 + 6y_2 + 2y_3 + y_4 + (0 \times S_1) + (0 \times S_2) + MA_1 + MA_2$

Subject to $y_1 + y_2 + y_3 + y_4 - S_1 + A_1 = 2$

$2y_1 + y_2 - y_3 - 2y_4 - S_2 + A_2 = 1$

$y_1, y_2, y_3, y_4, S_1, S_2, A_1 \,\&\, A_2 \ge 0$

Step-2:- To find IBFS

Let us substitute $y_1, y_2, y_3, y_4, S_1 \,\&\, S_2$ as zeroes.

$Z_{min} = MA_1 + MA_2 = M(A_1 + A_2)$

$0 + 0 + 0 + 0 - 0 + A_1 = 2 \Rightarrow A_1 = 2$

$(2 \times 0) + 0 - 0 - (2 \times 0) - 0 + A_2 = 1 \Rightarrow A_2 = 1.$

Step-3:- To perform optimality test.

Initial Simplex Table:-

	C_j	10	6	2	1	0	0	M	M		
C_{B_i}	Basis	y_1	y_2	y_3	y_4	S_1	S_2	A_1	A_2	Sol	$\theta = \frac{Sol}{KC}$
M	A_1	1	1	1	1	-1	0	1	0	2	$\frac{2}{1}=2$
M	A_2	KE 2	1	-1	-2	0	-1	0	1	1	$\frac{1}{2}$ → KR
	Z_j	3M	2M	0	-M	-M	-M	M	M	3M	
	$C_j - Z_j$	10-3M	6-2M	2	1+M	M	M	0	0	—	
		↑ KC									

$\because C_j - Z_j \ngtr 0$, the optimality is not reached.

Entering variable $= y_1$

Leaving variable $= A_2$.

Second simplex Table:-

	C_j	10	6	2	1	0	0	M	M		
C_{B_i}	Basis	y_1	y_2	y_3	y_4	S_1	S_2	A_1	A_2	Sol	$\theta = \frac{Sol}{KC}$
M	A_1	0	$\frac{1}{2}$	KE $\frac{3}{2}$	2	-1	$\frac{1}{2}$	1	—	$\frac{3}{2}$	$\frac{3}{2} \div \frac{3}{2} = 1$ → KR
10	y_1	1	$\frac{1}{2}$	$-\frac{1}{2}$	-1	0	$-\frac{1}{2}$	0	—	$\frac{1}{2}$	$\frac{\frac{1}{2}}{-\frac{1}{2}} = -1$
	Z_j	10	$\frac{10+M}{2}$	$\frac{3M-10}{2}$	2M-10	-M	$\frac{M-10}{2}$	M	—	$\frac{2M+10}{2}$	
	$C_j - Z_j$	0	$\frac{2-M}{2}$	$\frac{14-3M}{2}$	11-2M	M	$\frac{10-M}{2}$	0			
				↑ KC							

For y_1:- $NV = \frac{OV}{KE}$

$y_1 = \frac{2}{2} = 1$; $y_2 = \frac{1}{2}$; $y_3 = -\frac{1}{2}$; $y_4 = -\frac{2}{2} = -1$;

$S_1 = \frac{0}{2} = 0$; $S_2 = -\frac{1}{2}$; $A_1 = \frac{0}{2} = 0$; $A_2 = —$; Sol $= \frac{1}{2}$

For A_1:- $NV = OV - \left[KRE \times \frac{KCE}{KE}\right]$.

$y_1 = 1 - \left[2 \times \frac{1}{2}\right] = 0$; $y_2 = 1 - \left[1 \times \frac{1}{2}\right] = \frac{1}{2}$; $y_3 = 1 - \left[-1 \times \frac{1}{2}\right] = \frac{3}{2}$

$y_4 = 1 - \left[-2 \times \frac{1}{2}\right] = 2$; $S_1 = -1 - \left[0 \times \frac{1}{2}\right] = -1$; $S_2 = 0 - \left[-1 \times \frac{1}{2}\right]$

$A_1 = 1 - \left[0 \times \frac{1}{2}\right] = 1$; Sol $= 2 - \left[1 \times \frac{1}{2}\right] = \frac{3}{2}$.

$\because c_j - z_j \neq 0$, the optimality is not reached.

Entering variable $= y_3$

Leaving variable $= A_1$

Third Simplex Table:-

C_{B_i}	C_j / Basis	10 y_1	6 y_2	2 y_3	1 y_4	0 S_1	0 S_2	M A_1	M A_2	Sol	$\theta = \frac{Sol.}{KC}$
2	y_3	0	$1/3$	1	$4/3$	$-2/3$	$1/3$	-	-	1	$\frac{1}{(4/3)} = \frac{3}{4}$
10	y_1	1	$2/3$	0	$5/3$	$-1/3$	$2/3$	-	-	1	$\frac{1}{(5/3)} = \frac{3}{5} \rightarrow$
	z_j	10	$\frac{22}{3}$	2	$\frac{58}{3}$	$\frac{-14}{3}$	$\frac{22}{3}$	-	-	12	
	$c_j - z_j$	0	$\frac{-4}{3}$	0	$\frac{-55}{3}$	$\frac{14}{3}$	$\frac{-22}{3}$	-	-	-	
					$\uparrow$ KC						

LV $= y_1$

EV $= y_4$

For y_3:- $NV = \frac{OV}{KE}$.

$y_1 = \frac{0}{3/2} = 0$; $y_2 = \frac{1/2}{3/2} = \frac{1}{3}$; $y_3 = \frac{3/2}{3/2} = 1$; $y_4 = \frac{2}{3/2} = \frac{4}{3}$;

$S_1 = \frac{-1}{3/2} = -\frac{2}{3}$; $S_2 = \frac{1/2}{3/2} = \frac{1}{3}$; $Sol = \frac{3/2}{3/2} = 1$

For y_1:- $NV = OV - \left[KRE \times \frac{KCE}{KE}\right]$

$y_1 = 1 - \left[0 \times \frac{-1/2}{3/2}\right] = 1$; $y_2 = \frac{1}{2} - \left[\frac{1}{2} \times \left(-\frac{1}{3}\right)\right] = \frac{4}{6} = \frac{2}{3}$;

$y_3 = -\frac{1}{2} - \left[\frac{\cancel{3}}{2} \times \left(-\frac{1}{\cancel{3}}\right)\right] = 0$; $y_4 = -1 - \left[2 \times \left(-\frac{1}{3}\right)\right] = \frac{5}{3}$;

$S_1 = 0 - \left[-1 \times \left(-\frac{1}{3}\right)\right] = -\frac{1}{3}$; $S_2 = -\frac{1}{2} - \left[\frac{1}{2} \times \left(-\frac{1}{3}\right)\right] = \frac{2}{3}$

$Sol = \frac{1}{2} - \left[\frac{\cancel{3}}{2} \times \left(-\frac{1}{\cancel{3}}\right)\right] = 1$

$\frac{1}{3} - \left[\frac{2}{3} \times \frac{4}{5}\right]$ $\quad -\frac{2}{3} - \left[-\frac{1}{3} \times \frac{4}{5}\right]$ $\quad \frac{-42}{15} \; 5$

$\frac{1}{3} - \left[\frac{2}{3} \times \frac{4}{5}\right]$

Fourth Simplex Table:-

C_{B_i}	C_j / Basis	10 y_1	6 y_2	2 y_3	1 y_4	0 S_1	0 S_2	M A_1	M A_2	Sol	$\theta = \frac{Sol.}{KC}$
2	y_3	$\frac{-1}{5}$	$\frac{-3}{5}$	1	0	$-2/5$	$-1/5$	-	-	$1/5$	
1	y_4	$3/5$	$2/5$	0	1	$-1/5$	$2/5$	-	-	$3/5$	
	z_j	-1	0	2	1	-1	0	-	-	1	
	$c_j - z_j$	11	6	0	0	1	0	-	-		

~~$y_3 = 2$; $y_4 = 1$; $z = 1$~~

$\because$ All $(c_j - z_j) \geq 0$, optimality is reached.