Unit 5: Laplace transform

* we know in forier transform,

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt. \quad \text{or} \quad X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t} dt.$$

Now substitute $(\sigma + j\omega)$ in $(j\omega)$

$$X(\sigma + j\omega) \Rightarrow \int_{-\infty}^{\infty} x(t)\, e^{-(\sigma + j\omega)t}\, dt$$

$$= \int_{-\infty}^{\infty} x(t)\, e^{-\sigma t}\, e^{-j\omega t}\, dt$$

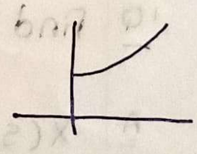
Now in laplace transform,

$$X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt \quad \{\sigma + j\omega = s\}$$

* we convert unstable signal into stable signal by adding σ to $j\omega$. This is Laplace.

Here σ = Real part and $j\omega$ = imaginary part.

Ex: $e^{2t} u(t)$, this is a unstable signal

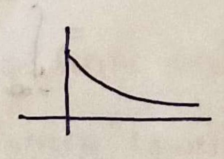


$\rightarrow e^{2t} u(t) \cdot e^{-\sigma t}$ or $e^{(2-\sigma)t} u(t)$

{adding σ to the power}

$= e^{2t} u(t)\, e^{-\sigma t}$ Let $\sigma = 4$

$= e^{2t} u(t)\, e^{-4t}$



$= e^{-2t} u(t)$ {converted into stable singl}

* Condition for Laplace transform:

$\boxed{|X(S)| < \infty}$ $\{ \text{i.e} \left| \int_{-\infty}^{\infty} x(t) e^{-St} dt \right| < \infty \}$

$$\left| \int_{-\infty}^{\infty} x(t) e^{-(\sigma + j\omega)t} dt \right| < \infty$$

* Region of convergence:

For what values of 'σ' the Laplace transform exist it is called Region of convergence.

* Relation b/w forier transform & laplace transform

$$LT\{x(t)\} = \int_{-\infty}^{\infty} x(t) e^{-St} dt$$

$$= \int_{-\infty}^{\infty} x(t) e^{-\sigma t} e^{-j\omega t} dt$$

$$= FT\{x(t) e^{-\sigma t}\}$$

* If $\sigma = 0$, laplace transform = forier transform

Q Find the laplace transform of $x(t) = e^{-at} u(t)$

A $X(S) = \int_{-\infty}^{\infty} x(t) e^{-St} dt$

$$= \int_{-\infty}^{\infty} e^{-at} u(t) e^{-St} dt$$

$$= \int_{-\infty}^{\infty} e^{-at} u(t) e^{-(\sigma + j\omega)t} dt \qquad t > 0$$

$$= \int_{0}^{\infty} e^{-(a+\sigma+j\omega)t} dt = \frac{e^{-(a+\sigma+j\omega)t}}{-(a+\sigma+j\omega)} = \frac{e^{-(a+\sigma)t} e^{-j\omega}}{-(S+a)}$$

$$= \int_0^{\infty} e^{-(a+\sigma)t} e^{-j\omega t}\, dt$$

{ to substitute upper limit an to have a finite value. we should satisfy the condition $\boxed{\sigma + a > 0}$

$$= \left. \frac{e^{-(a+\sigma)t} e^{-j\omega t}}{-(s+a)} \right|_0^{\infty}$$

$$= \left[\frac{e^{-\infty} e^{-j\omega(t)}}{-(s+a)} - \frac{e^{0} e^{-j\omega(0)}}{-(s+a)} \right]$$

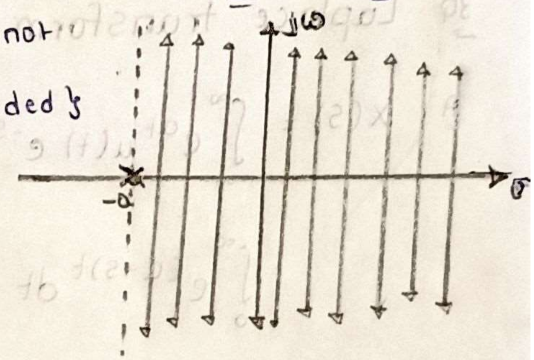
$e^{-j\omega} = \cos\omega + j\sin\omega$

$\boxed{\sigma > -a}$

ROC $\sigma > -a$

$$= 0 - \frac{1}{-(s+a)}$$

{ $-a$ not included }



$$= \boxed{\frac{1}{s+a}}, \quad \boxed{\sigma + a > 0}$$

$$\boxed{\sigma > -a}$$

2 Q $x(t) = -e^{-at} u(-t)$, find laplace transform.

A $X(s) = \int_{-\infty}^{\infty} x(t) e^{-st}\, dt$

$$= \int_{-\infty}^{\infty} -e^{-at} u(-t) e^{-st}\, dt$$

$$= -\int_{-\infty}^{\infty} e^{-at} u(-t) e^{-st}\, dt \qquad -t > 0 \qquad t < 0$$

$$= -\int_{-\infty}^{0} e^{-at} e^{-st}\, dt = -\int_{-\infty}^{0} e^{-(a+s)t}\, dt$$

$\boxed{a + \sigma < 0}$

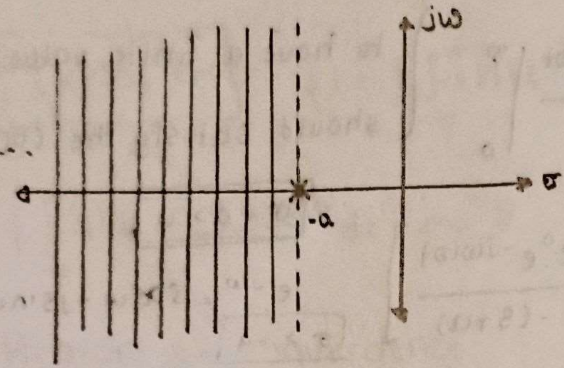
$$= -\left[\int_{-\infty}^{0} e^{-(a+\sigma)t} e^{-j\omega t} \right] = -\left[\frac{e^{-(a+\sigma)t} e^{-j\omega t}}{-(s+a)} \right]_{-\infty}^{0}$$

$$= -\left[\frac{1}{-(s+a)} - 0 \right]$$

for getting finite value, when we substitute lower limit $(-\infty)$, we should satisfy condition $\boxed{\sigma + a < 0}$ $\boxed{\sigma < -a}$

$$= \boxed{\frac{1}{s+a}, \quad \sigma < -a}$$

ROC , $\sigma < -a$



{-a is not included}

It is important to draw ROC.

3Q Laplace transform of $e^{at}u(t)$

A $X(s) = \int_{-\infty}^{\infty} e^{at}u(t)\, e^{-st}.dt$

$= \int_{0}^{\infty} e^{(a-s)t}\, dt$

$= \int_{0}^{\infty} e^{-(s-a)t}\, dt$

$= \left. \frac{e^{-(s-a)t}}{-(s-a)} \cancel{dt} \right|_{0}^{\infty}$

$= \left. \frac{e^{-(\sigma+j\omega-a)t}}{-(s-a)} \right|_{0}^{\infty} = \left. \frac{e^{-\underline{(\sigma-a)}t}\, e^{-j\omega t}}{-(s-a)} \right|_{0}^{\infty}$ $\underline{\sigma-a} > 0$

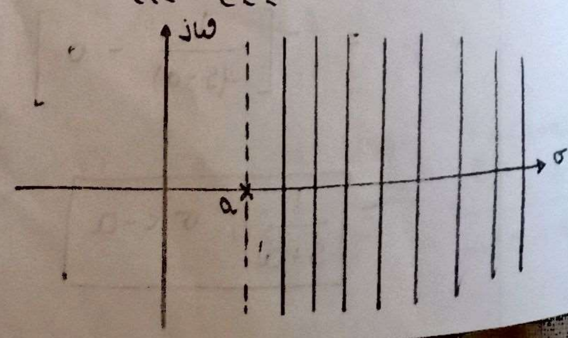
$= \frac{e^{-\infty} e^{-j\omega t}}{-(s-a)} - \frac{e^{0} e^{-j\omega(0)}}{-(s-a)}$ { condition for finite value $\sigma - a > 0$ }

$\therefore \boxed{\sigma > a}$ {a not included}

$= 0 - \frac{1}{-(s-a)}$

$= \boxed{\frac{1}{s-a};\ \sigma > a}$

ROC $\sigma > a$



4Q Laplace transform $x(t) = -e^{at}u(-t)$

A $X(S) = \int_{-\infty}^{\infty} -e^{at}\, u(-t)\, e^{-st} \cdot dt$

$= -\int_{-\infty}^{0} e^{at} e^{-st} \cdot dt$

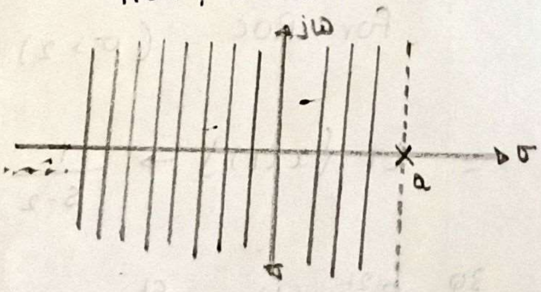
$= -\int_{-\infty}^{0} e^{-(s-a)t}\, dt$

$= -\left.\dfrac{e^{-(s-a)t}}{-(s-a)}\right|_{-\infty}^{0}$

$= -\left.\dfrac{e^{-(\sigma+j\omega-a)t}}{-(s-a)}\right|_{-\infty}^{0} = -\left.\dfrac{e^{-(\sigma-a)t}\, e^{-j\omega t}}{-(s-a)}\right|_{-\infty}^{0}$

condition for finite value is .. $\sigma - a < 0$ $\boxed{\sigma < a}$

ROC, $\sigma < a$



$= -\left[\dfrac{1}{-(s-a)} - 0\right]$

$= \boxed{\dfrac{1}{s-a};\ \sigma < a}$

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* Important formulas of Laplace Transform :

* $LT\{e^{-at}u(t)\} \longrightarrow \dfrac{1}{s+a},\ \sigma > -a$

* $LT\{-e^{-at}u(-t)\} \longrightarrow \dfrac{1}{s+a},\ \sigma < -a$

* $LT\{e^{at}u(t)\} \longrightarrow \dfrac{1}{s-a},\ \sigma > a$

* $LT\{-e^{at}u(-t)\} \longrightarrow \dfrac{1}{s-a},\ \sigma < a$

1Q Laplace transform of $e^{3t}u(-t)$

A From $LT\{-e^{at}u(-t)\} \to \frac{1}{s-a}$ $\sigma < a$

then $LT\{e^{at}u(-t)\} \to \frac{-1}{s-a}$, $\sigma < a$

so $LT\{e^{3t}u(-t)\} \to \frac{-1}{s-3}$, $\sigma < 3$

2Q Laplace transform of $x(t) = e^{2t}u(t) + e^{-3t}u(t)$

A From $LT\{e^{at}u(t)\} \to \frac{1}{s-a}$ $\sigma > a$ and

$LT\{e^{-at}u(t)\} \to \frac{1}{s+a}$ $\sigma > -a$ }

$LT\{x(t)\} \to \frac{1}{s-2} + \frac{1}{s+3}$ $\{\sigma > 2$ & $\sigma > -3\}$

for ROC $(\sigma > 2) \cap (\sigma > -3) = \sigma > 2$

$= LT\{x(t)\} \to \frac{1}{s-2} + \frac{1}{s+3}$ $(\sigma > 2)$

3Q $e^{2t}u(t) + e^{6t}u(t) + e^{10t}u(t) = x(t)$, LT ?

From $LT\{e^{at}u(t)\} \to \frac{1}{s-a}$ $\sigma > a$

and $LT\{e^{at}\}$ we have

$LT\{x(t)\} = \frac{1}{s-2} + \frac{1}{s-6} + \frac{1}{s-10}$ $\{(\sigma > 2) \cap (\sigma > 6) \cap (\sigma > 10)$ $= \boxed{\sigma > 10}$

4Q Laplace transform of $x(t) = e^{-2t}u(-t) + e^{3t}u(-t)$

A From $LT\{-e^{-at}u(-t)\} \to \frac{1}{s+a}$, $\sigma < -a$

and $LT\{-e^{at}u(-t)\} \to \frac{1}{s-a}$, $\sigma < a$

$$LT\{x(t)\} = \frac{-1}{s+2} - \frac{1}{s-3} \quad \{(\sigma < -2) \cap (\sigma < 3) = \boxed{\sigma < -}$$

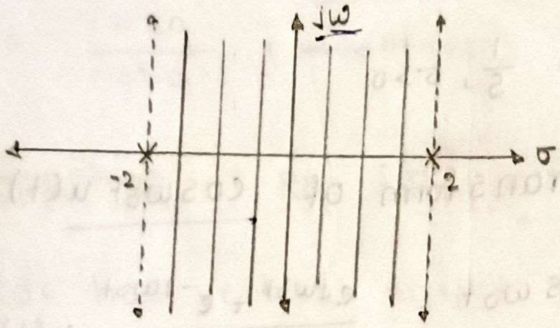
5Q Laplace transform of $x(t) = e^{-2t}u(t) - e^{2t}u(-t)$

A From $LT\{e^{-at}u(t)\} \rightarrow \frac{1}{s+a}, \ \sigma > -a$

and $LT\{-e^{at}u(-t)\} \rightarrow \frac{1}{s-a}, \ \sigma < a$

$$LT\{x(t)\} \rightarrow \frac{1}{s+2} + \frac{1}{s-2} \quad \{(\sigma > -2) \cap (\sigma < 2) = -2 \text{ to } 2\}$$

ROC $(-2 < \sigma < 2)$



6Q Find Laplace transform of $x(t) = e^{2t}u(t) - e^{-2t}u(-t)$

- from $LT\{e^{at}u(t)\} \rightarrow \frac{1}{s-a}, \ \sigma > a$ and

$LT\{-e^{-at}u(-t)\} \rightarrow \frac{1}{s+a}, \ \sigma < -a$

so, $LT\{x(t)\} \rightarrow \frac{1}{s-2} + \frac{1}{s+2} \quad \{(\sigma > 2) \cap (\sigma < -2) = \phi\}$

No common ROC, then no laplace transform.

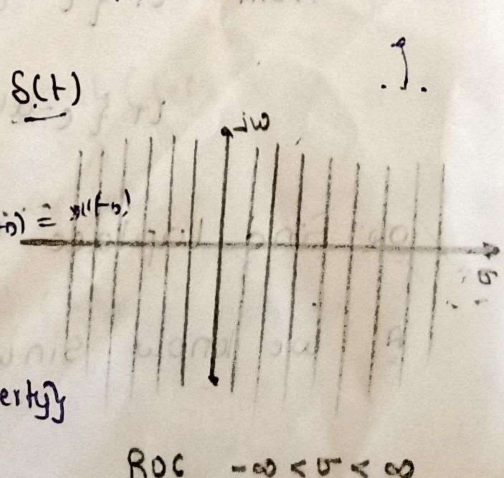
7Q Find Laplace transform of $\delta(t)$

$$X(s) = \int_{-\infty}^{\infty} \delta(t) e^{-st} \, dt$$

$e^{-st}\delta(t-0)$ $\quad e^{-s(0)}$ $\quad \int_{-\infty}^{\infty} x(t)\delta(t-t_0) = x(t_0)$

$$= e^{-s(0)}$$

$\boxed{=1}$ {from sampling property}



ROC $-\infty < \sigma < \infty$

ROC entire s-plane.

8Q Find laplace transform of $u(t)$.

A $X(S) = \int_{-\infty}^{\infty} u(t) e^{-st} dt$

$= \int_{0}^{\infty} e^{-st} dt = \frac{e^{-st}}{-s}\Big|_0^{\infty} = \boxed{\frac{1}{s}, \sigma > 0}$

$\{ e^{-\sigma t} e^{-j\omega t} / -s \,|_0^{\infty} \}$ $\quad \frac{e^{-s(\infty)}}{-s} - \frac{e^{-s(0)}}{-s} = \frac{-1}{-s} = \frac{1}{s}$

* $LT\{u(t)\} \longrightarrow \frac{1}{s}, \sigma > 0$

9Q Find laplace transform of $\cos\omega_0 t\, u(t) = x(t)$

A we know $\cos\omega_0 t = \frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2} \cdot u(t)$

$x(t) = \frac{1}{2}\left[e^{j\omega_0 t} u(t) + e^{-j\omega_0 t} u(t) \right]$ $\quad \sigma > 0 \quad \sigma < 0$

$\sigma > j\omega_0 \quad \sigma < j\omega_0$

$0 + j\omega_0 \quad 0 - j\omega_0$

$X(S) = \frac{1}{2}\left[\frac{1}{s - j\omega_0} + \frac{1}{s + j\omega_0} \right] = \boxed{\frac{s}{s^2 + \omega_0^2}}$ $\boxed{\sigma > 0}$

: we should consider $\{\sigma > 0\}$ real part only

{not imaginary part such as $\sigma > j\omega_0$, $\sigma > -j\omega_0$ }

from $LT\{e^{at} u(t)\} \longrightarrow \frac{1}{s-a}, \sigma > a$

$LT\{e^{-at} u(t)\} \longrightarrow \frac{1}{s+a}, \sigma > -a$

10Q Find laplace transform of $\sin\omega_0 t\, u(t) = x(t)$

A we know $\sin\omega_0 t = \frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j} \cdot u(t)$

$$x(t) = \frac{1}{2j}\left[e^{j\omega_0 t}u(t) - e^{-j\omega_0 t}u(t)\right]$$

$$X(S) = \frac{1}{2j}\left[\frac{1}{S-j\omega_0} - \frac{1}{S+j\omega_0}\right] = \frac{1}{2j}\frac{2j\omega_0}{S^2+\omega_0^2} = \boxed{\frac{\omega_0}{S^2+\omega_0^2}}$$

$\boxed{\sigma > 0}$

Q Find Laplace transform of $e^{-a|t|}$

A
$$e^{-a|t|} = \begin{cases} e^{at}u(-t) \rightarrow \frac{-1}{S-a}, & \sigma > a \\ e^{-at}u(t) \rightarrow \frac{1}{S+a}, & \sigma > -a \end{cases}$$

$$= \frac{-2a}{S^2-a^2} \left\{ \cancel{-a < \sigma < a} \right\} \quad \sigma > a$$

* Properties of ROC {Region of ~~corres~~ convergence}

(i) ROC has vertical stripes parallel to $j\omega$ axis.

(ii) ROC doesnot include poles

(iii) ROC of an absolutely integrable with finite duration signal is entire S-plane

(iv) ROC of causal signal (Right sided), is 'σ' greater than the largest pole or right of right most pole.

v) ROC of an anti-causal signal (left sided), is 'σ' less than the smallest pole or left of leftmost pole

vi) ROC of non-causal signal (both sided) exist between the poles.

** (vii) ROC of stable system includes $j\omega$ axis.

* Inverse Laplace transform:

$e^{at}u(t) \rightarrow \frac{1}{s-a}$

$e^{-at}u(-t) \rightarrow \frac{-1}{s-a}$

Q. Find the inverse L transform of $1/s+4$.

A. $ILT\{1/s+4\}$

$\sigma > -4$

$\sigma < -4$

for $\sigma > -4$ and $\sigma < -4$

$\downarrow$ $\qquad\qquad$ $\downarrow$

$e^{-4t}u(t)$ $\qquad\qquad$ $-e^{-4t}u(-t)$

Q. Find the inverse L transform of $x(s) = \frac{1}{s-2} + \frac{1}{s+3}$

A. case-i : $\sigma > 2$ $\quad \{\sigma > 2 \;\&\; \sigma > -3\}$

$\sigma > 2 \quad \sigma > -3$

$\sigma < 2 \quad \sigma < -3$

$-3, 2$

$x(t) = e^{2t}u(t) + e^{-3t}u(t)$

case-ii : $\sigma < -3$ $\quad \{\sigma < 2 \;\&\; \sigma < -3\}$

$x(t) = -e^{2t}u(-t) - e^{-3t}u(-t)$

case-iii : $-3 < \sigma < 2$

$x(t) = -e^{2t}u(-t) + e^{-3t}u(t)$.

Q. Find the inverse laplace transform of $x(s) = \frac{1}{(s-2)(s+2)}$

A. $\frac{1}{(s-2)(s+2)} = \frac{A}{(s-2)} + \frac{B}{s+2}$

$= \frac{1}{4}\left[\frac{1}{s-2} - \frac{1}{s+2}\right]$

(i) $\sigma > 2$

$\frac{1}{4}\left[e^{2t}u(t) + e^{-2t}u(t)\right] \rightarrow$ causal

(ii) $\sigma < -2$, anti-causal

$\frac{1}{4}\left[-e^{2t}u(-t) + e^{-2t}u(-t)\right]$

(iii) $-2 < \sigma < 2$, non-causal

$\frac{1}{4}\left[-e^{2t}u(-t) + e^{-2t}u(t)\right]$

Q. Find inverse laplace transform of the following

(i) $X(S) = \dfrac{S}{(S+2)(S-3)}$, $\underline{-2 < \sigma < 3}$

(ii) $X(S) = \dfrac{5(S+5)}{S(S+3)(S+7)}$; $\underline{\sigma > 0}$

(iii) $X(S) = \dfrac{4S}{(S+3)(S+8)}$, $\underline{\sigma > -3}$.

* Imp formula : If eq is in the form $\dfrac{Px+q}{(ax+b)(cx+d)}$

then $p \quad a \quad c$ (x - coefficients)

$\quad\;\; q \quad b \quad d$ (constants)

then $\dfrac{-\left[\dfrac{Pb-qa}{ad-bc}\right]}{ax+b} + \dfrac{\left[\dfrac{Pd-cq}{ad-bc}\right]}{cx+d}$

(i) we have $X(S) = \dfrac{S}{(S+2)(S-3)}$ $\begin{cases} P=1,\ q=0, \\ a=1,\ b=2,\ c=1,\ d=-3 \end{cases}$

$= \dfrac{-\left[\dfrac{1(2)-0(1)}{-3-2}\right]}{S+2} + \dfrac{\dfrac{1(-3)-1(0)}{-3-2}}{S-3} = \dfrac{2}{5(S+2)} + \dfrac{3}{5(S-3)}$

we have $-2 < \sigma < 3$

$$= \frac{2}{5}\left[\frac{1}{S+2} \quad \sigma > -2\right] + \frac{3}{5}\left[\frac{1}{S-3} \quad \sigma < 3\right]$$

$\downarrow$ ILT

$$= \frac{2}{5}\left[e^{-2t}u(t)\right] + \frac{3}{5}\left[-e^{3t}u(-t)\right]$$

we have the inverse Laplace transform of $\frac{S}{(S+2)(S-3)}$

$$= \frac{2}{5}\left[e^{-2t}u(t)\right] - \frac{3}{5}\left[e^{3t}u(-t)\right]$$

2) $X(S) = \dfrac{5(S+5)}{S(S+3)(S+7)}$, $\sigma > 0$

A $X(S) = \dfrac{5S+25}{S(S+3)(S+7)} = \dfrac{A}{S} + \dfrac{B}{S+3} + \dfrac{C}{S+7}$

$$= \frac{A(S+3)(S+7) + B(S)(S+7) + C(S)(S+3)}{S(S+3)(S+7)} = \frac{5S+25}{S(S+3)S}$$

$=$ If $S = -3$, $\cancel{B}(\cancel{20}) = 5(-3)+25$

$B(-12)$

$-12B = 10 \rightarrow B = \frac{-10}{12} = \frac{-5}{6}$

$=$ If S $= -7$, $C(-7)(-7+3) = 5(-7)+25$

$= C(-7)(-4) = \cancel{20} - 10$

$= C = \frac{\cancel{20}}{\cancel{28}} = \frac{\cancel{8}}{\cancel{7}} \frac{-10}{28} = \frac{-5}{14}$

If $S = 0$, $A(\cancel{25}) = 25.$

$A = \frac{25}{21}$

$$= \frac{5}{7}\left(\frac{1}{s}\right) + \frac{-5}{6}\left(\frac{1}{s+3}\right) + \frac{25}{21}\left(\frac{1}{s+7}\right).$$

$$= \frac{25}{21}\left(\frac{1}{s}\right) - \frac{5}{14}\left(\frac{1}{s+7}\right) - \frac{5}{6}\left(\frac{1}{s+3}\right)$$

$$X(s) = \frac{25}{21}\left(\frac{1}{s}\right) - \frac{5}{6}\left(\frac{1}{s+3}\right) - \frac{5}{14}\left(\frac{1}{s+7}\right) \quad \sigma > 0$$

By inverse Laplace transform,

$$x(t) = \frac{25}{21}\left[e^{0t}u(t)\right] - \frac{5}{6}\left[e^{-3t}u(t)\right] - \frac{5}{14}\left[e^{-7t}u(t)\right]$$

$$x(t) = 5\left[\frac{5}{21}u(t) - \frac{1}{6}e^{-3t}u(t) - \frac{1}{14}e^{-7t}u(t)\right]$$

3) $X(s) = \dfrac{45}{(s+3)(s+8)}$ $\quad \sigma > -3$

A: $p = 4,\ q = 0$

$a = 1,\ b = 3,\ c = 1,\ d = 8$

partial fractions are $\dfrac{-\left[\dfrac{4(3) - (0)(1)}{8-3}\right]}{s+3} + \dfrac{\left[\dfrac{4(8) - 1(0)}{8-3}\right]}{s+8}$

$$= \frac{-\left[\frac{12}{5}\right]}{s+3} + \frac{\left[\frac{32}{5}\right]}{s+8} = \frac{32}{5}\left[\frac{1}{s+8}\right] - \frac{12}{5}\left[\frac{1}{s+3}\right]$$

$\sigma > -3$

By inverse forier transform

$$x(t) = \frac{32}{5}\left[e^{-8t}u(t)\right] - \frac{12}{5}\left[e^{-3t}u(t)\right]$$

$$x(t) = \frac{4}{5}\left[8e^{-8t}u(t) - 3e^{-3t}u(t)\right]$$

Properties of Laplace transform.

* **Linearity**: $x(t) \longleftrightarrow X(s)$, R_1 ROC

$y(t) \longrightarrow Y(s)$, R_2 ROC

$a\,x(t) + b\,y(t) \longrightarrow a X(s) + b Y(s)$; $R_1 \cap R_2$ ROC

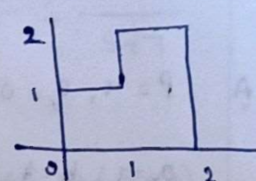
* **Time shifting**: $x(t) \longrightarrow X(s)$, R ROC

$x(t-t_0) \longrightarrow e^{-st_0} X(s)$, R ROC

* **Shifting s-domain**: $x(t) \longrightarrow X(s)$, R ROC

$e^{at} x(t) \longrightarrow X(s-a)$, $R + \text{Realpart}\{a\}$

Q Find the laplace transform of the following signal



A → $u(t) + u(t-1) - 2u(t-2)$

→ $\frac{1}{s} + e^{-s}\frac{1}{s} - 2e^{-2s}\frac{1}{s}$ {Time shifting}

$= \frac{1}{s}\left[1 + e^{-s} - 2e^{-2s}\right]$ $\boxed{\sigma > 0}$

Q Find the laplace transform of $e^{-3(t-2)}u(t-2)$

A $LT\{e^{-at}u(t)\} \longrightarrow \frac{1}{s+a}$ $\sigma > -a$

$LT\{e^{-3t}u(t)\} \rightarrow \frac{1}{s+3}$ $\sigma > -3$

$LT\{e^{-3(t-2)}u(t-2)\} = e^{-2s}\frac{1}{s+3}$, $\sigma > -3$

* Q **Time scaling**: $x(t) \longrightarrow X(s)$, R , ROC

$x(at) \rightarrow \frac{1}{|a|} X\left(\frac{s}{a}\right)$, $\{aR,$ ROC$\}$

* Time reversal: $x(t) \longrightarrow X(S)$, R ROC

$x(-t) \longrightarrow X(-S)$, $-R$ ROC

* Convolution in time :

$x(t) \longrightarrow X(S)$, R_1 ROC

$y(t) \longrightarrow Y(S)$, R_2 ROC

$x(t) * y(t) \longrightarrow X(S) \cdot Y(S)$, $\{R_1 \cap R_2$ ROC $\}$

** Differentiation in time:

$x(t) \longrightarrow X(S)$, R ROC

$\frac{d}{dt} x(t) \longrightarrow S \cdot X(S)$, R ROC

Proof :. $x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$ (F.T)

Inverse LT $\{\frac{d}{dt} x(t)\} = ILT\left\{\frac{1}{2\pi j} \int_{-\infty}^{\infty} X(S) e^{St} dS\right\}$ {I.T}

$\frac{d}{dt} \cdot x(t) = ILT\left\{\frac{1}{2\pi j} \int_{-\infty}^{\infty} X(S) \cdot S e^{St} dS\right\}$

$LT\left\{\frac{d}{dt} x(t)\right\} = \frac{1}{2\pi j} \int_{-\infty}^{\infty} X(S)\, S\, e^{St} \cdot dS$

$= S\, X(S).$

* Differentiation in S-domain :

$x(t) \longrightarrow X(S)$, R ROC

$t \cdot x(t) \longrightarrow \frac{-d}{dS} X(S)$, R ROC

Proof : $X(S) = \int_{-\infty}^{\infty} x(t) e^{-St} dt$

$$\frac{d}{ds}X(S) = \int_{-\infty}^{\infty} x(t) \; -t \; e^{-St} \, dt$$

$$\frac{d}{ds}X(S) = -t \int_{-\infty}^{\infty} x(t)\, e^{-St} \, dt$$

$$\frac{-d}{ds}X(S) \longrightarrow LT\{t x(t)\}$$

$$\longrightarrow LT\{t \, x(t)\}$$

* <u>Integration property</u>:

$$x(t) \longrightarrow X(S) \quad , R \text{ ROC}$$

$$\int_{-\infty}^{t} x(T)\, dT \longrightarrow \frac{X(S)}{S} \quad ; R \text{ ROC}$$

<u>Proof</u> : $x(t) * u(t) = \int_{-\infty}^{t} x(T)\, dT \longrightarrow \int_{-\infty} x(T)\, u(t-T)\, dT$

So $LT\left\{\int_{-\infty}^{t} x(T)\, dT\right\} = LT\{x(t) * u(t)\}$

$$= X(S) \cdot \frac{1}{S} = \frac{X(S)}{S}$$

* <u>Integration in frequency</u>:

$$x(t) \longrightarrow X(S)$$

$$\frac{x(t)}{t} \longrightarrow \int_{S}^{\infty} X(S)\, ds$$

<u>Proof</u>: $X(S) = \int_{-\infty}^{\infty} x(t)\, e^{-St}\, dt$

$$\int_{S}^{\infty} X(S) = \int_{-\infty}^{\infty} x(t) \left. \frac{e^{-St}}{-t} \right|_{S}^{\infty} dt.$$

$$\int_s^{\infty} X(s) = \int_{-\infty}^{\infty} x(t) \left(\frac{-e^{-st}}{-t} \right) dt$$

$$\int_s^{\infty} X(s) \rightarrow LT\{ x(t)/t \}$$

* Conjugate.

$x(t) \longrightarrow X(s)$

$x^*(t) \longrightarrow X^*(s^*)$

Proof : $X(s) = \int_{-\infty}^{\infty} x(t) e^{-(\sigma + j\omega)t} dt$

$$X(s^*) = \int_{-\infty}^{\infty} x(t)\, e^{-(\sigma - j\omega)t} dt$$

$$[X(s^*)]^* = \left[\int_{-\infty}^{\infty} x(t)\, e^{-(\sigma - j\omega)t} dt \right]^*$$

$$= \int_{-\infty}^{\infty} x^*(t)\, e^{-st} dt$$

$$= LT\{ x^*(t) \}$$

Q Find the laplace transform of unit ramp signal starting at $t = a$ i.e $\boxed{r(t-a)}$.

A we know, $X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$

$X(s) = \int_{-\infty}^{\infty} r(t)\, e^{-st} dt \quad \{ r(t) = t\, u(t) \}$

$= \int_{0}^{\infty} t\, e^{-st} dt$

we know, $\{ r(t) = t \cdot u(t) \}$,

$LT\{ u(t) \} \longrightarrow 1/s$

$t \cdot u(t) \rightarrow -\frac{d}{ds}\left(\frac{1}{s}\right)$

$r(t) \rightarrow \frac{1}{s^2}$

$r(t-a) \rightarrow e^{-as}/s^2$

Q $x(t) = e^{-3t} \frac{d}{dt}\left[e^{-(t+1)} u(t+1)\right]$

A $e^{-at} u(t) \rightarrow \frac{1}{s+a}, \sigma > -a$

$e^{-t} u(t) \rightarrow \frac{1}{s+1} \quad \sigma > -1$

$e^{-(t+1)} u(t+1) = e^{-s(-1)} \frac{1}{s+1} \quad \sigma > -1$

$\frac{d}{dt}\left[e^{-(t+1)} u(t+1)\right] = s \cdot e^{s} \frac{1}{s+1}, \quad \sigma > -1]$

$e^{-3t} \frac{d}{dt}\left[e^{-(t+1)} u(t+1)\right] = s e^{s} \frac{1}{s+1} \quad (s = s+3)$

$\rightarrow (s+3) e^{(s+3)} \frac{1}{s+4}$

LT $\{x(t)\}$ is $\boxed{(s+3) e^{(s+3)} \frac{1}{s+4}}$ $\{\sigma > -4\}$

Q Find LT of t sinat $u(t)$

A LT $\{\sin \omega_0 t \, u(t)\} \rightarrow \frac{\omega_0}{s^2 + \omega_0^2} \quad \sigma > 0$

LT $\{\sin at \, u(t)\} \rightarrow \frac{a}{s^2 + a^2} \quad \sigma > 0$

LT $\{t \cdot \sin at \, u(t)\} \rightarrow \frac{-d}{ds}\left[\frac{a}{s^2+a^2}\right]$

$$LT\{t\cdot \sin at\, u(t)\} \rightarrow -a\left[\frac{-1}{(s^2+a^2)^2}(2s)\right]$$

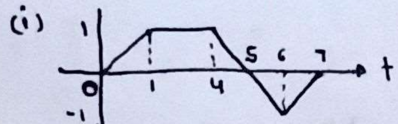
$$LT\{t\cdot \sin at\, u(t)\} \rightarrow \frac{+2as}{(s^2+a^2)^2} \quad \sigma > 0$$

Q Find LT of $e^{at}\cos\omega_0 t\, u(t)$

A $$LT\{\cos\omega_0 t\, u(t)\} \rightarrow \frac{s}{s^2+\omega_0^2} \quad \sigma > 0$$

$$LT\{e^{at}\cos\omega_0 t\, u(t)\} \rightarrow \frac{s-a}{(s-a)^2+\omega_0^2} \quad \sigma > a$$

Q Find LT of following signal.

(i) 

Q LT of $t^2 u(t-1)$

A $$LT\{u(t)\} \rightarrow \frac{1}{s}$$

$$LT\{u(t-1)\} \rightarrow e^{-s}\frac{1}{s}$$

$$LT\{t\cdot u(t-1)\} \rightarrow \frac{-d}{ds}\left[\frac{e^{-s}}{s}\right]$$

$$\rightarrow -\left(\frac{-se^{-s}-e^{-s}}{s^2}\right) = \frac{+e^{-s}(1+s)}{s^2}$$

$$LT\{t^2\cdot u(t-1)\} \rightarrow \frac{-d}{ds}\left[\frac{+e^{-s}(s+1)}{s^2}\right]$$

$$= \frac{+d}{ds}\left[\frac{e^{-s}(1+s)}{s^2}\right) \cdot \frac{d}{ds}\left[\frac{e^{-s}}{s^2}+\frac{e^{-s}}{s}\right]$$

$$= \frac{-s^2e^{-s}-e^{-s}2s}{s^4} + \frac{-se^{-s}-e^{-s}}{s^2}$$

$$= -\left[\frac{+e^{-s}(2s+s^2)}{s^4} + \frac{e^{-s}(1+s)}{s^2}\right]$$

Q $y(t) = x_1(t-2) * x_2(-t+3)$

$x_1(t) = e^{-2t}u(t)$

$x_2(t) = e^{-3t}u(t)$ Find laplace Transform.

A $LT\{x(t) * y(t)\} = X(s)Y(s)$

$x_1(t) = e^{-2t}u(t) \longrightarrow \frac{1}{s+2}, \quad \sigma > -2$

$x_2(t) = e^{-3t}u(t) \longrightarrow \frac{1}{s+2}, \quad \sigma > -3$

$x_1(t-2) \longrightarrow e^{-2s}\frac{1}{s+2}$

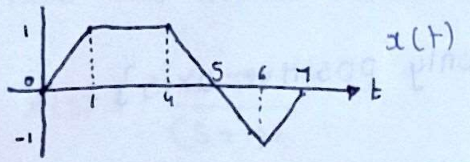
$x_2(-t+3) = x(-(t-3)) \longrightarrow \frac{e^{-3s}}{-s+3}$

$LT\{x_1(t-2)\} \cdot LT\{x_2(-t+3)\}$

$= \frac{e^{-2s}}{s+2} \cdot \frac{e^{-3s}}{-s+3} = \frac{e^{-5s}}{(s+2)(3-s)} = \frac{e^{-5s}}{-s^2-2s+3s+6}$

$= \frac{e^{-5s}}{-s^2+s+6}$

Q ~~$x(t) = (e^{-2t}u(t) +$~~ An input $x(t) = e^{-2t}u(t) + \delta(t-6)$ is applied to a L.T.I system with impulse response $h(t) = u(t)$. Find the output interms of

Q 

A The following signal can be written as

$$r(t) - r(t-1) - r(t-4) + 2r(t-6) - r(t-7) = x(t)$$

$$LT\{r(t)\} \longrightarrow \frac{1}{s^2}$$

$$LT\{r(t-1)\} \longrightarrow e^{-s} \cdot \frac{1}{s^2}$$

$$LT\{r(t-4)\} \longrightarrow e^{-4s} \frac{1}{s^2}$$

$$LT\{r(t-6)\} \longrightarrow e^{-6s} \frac{1}{s^2}$$

$$LT\{r(t-7)\} \longrightarrow e^{-7s} \frac{1}{s^2}$$

$$LT\{x(t)\} = \frac{1}{s^2} - e^{-s}/s^2 - \frac{e^{-4s}}{s^2} + 2e^{-6s}\frac{1}{s^2} - \frac{e^{-7s}}{s^2}$$

$$\rightarrow \frac{1}{s^2}\left[1 - e^{-s} - e^{-4s} + 2e^{-6s} - e^{-7s}\right]$$

$$LT\{x(t)\} \longrightarrow \frac{1}{s^2}\left[1 - e^{-s} - e^{-4s} + 2e^{-6s} - e^{-7s}\right]$$

* Bilateral laplace transform :-

$$X(S) = \int_{-\infty}^{\infty} x(t) e^{-St} dt \quad \{-\infty \text{ to } 0 \text{ to } \infty\}$$

* Unilateral laplace transform

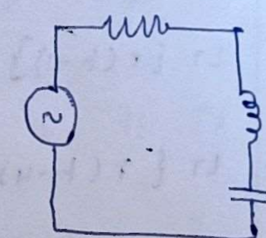
$$X(S) = \int_{0}^{\infty} x(t) e^{-St} dt \quad \{\text{only positive axis}\}.$$

→ Property of Bilateral laplace transform

$$LT\left\{\frac{d}{dt} x(t)\right\} = S X(S)$$

* Property of Unilateral laplace transform

$$LT\left\{\frac{d}{dt} x(t)\right\} = S X(S) - x(0)$$



$SX(S) - x(0)$

$x(0)$ is energy stored in inductor & capacitor.

* Initial Value Theorem :-

In a unilateral system laplace transform,

$$x(t) \xrightarrow{LT} X(S)$$

$$x(0) \longrightarrow \lim_{S\to\infty} S X(S)$$

Proof

we know $LT\left\{\frac{d}{dt} x(t)\right\} = SX(S) - x(0)$

$$\int_0^{\infty} \frac{d}{dt} x(t) e^{-St} dt = SX(S) - x(0)$$

Apply limit on both sides

$$\int_0^{\infty} \frac{d}{dt} x(t) \lim_{S\to\infty} e^{-St} dt = \lim_{S\to\infty} SX(S) - x(0)$$

$$\underset{s \to \infty}{Lt}\ s\cancel{g}\, X(s) - x(0) = 0 \qquad \{e^{-\infty} = 0\}$$

$$\Rightarrow x(0) = \underset{s \to \infty}{Lt}\ s\,[X(s)]$$

Q Find the initial value of $X(s) = \dfrac{5}{s^2 + 2s + 1}$

A $X(s) = \dfrac{5}{(s+1)^2}$

$$\underset{s \to \infty}{Lt}\ s\left[\frac{5}{(s+1)^2}\right] = \underset{s \to \infty}{Lt}\ \cancel{s}\left[\frac{5}{s^{\cancel{2}}(1+\frac{1}{s})^2}\right]$$

$$= \underset{s \to \infty}{Lt}\ \frac{5}{s(1+\frac{1}{s})^2} = \frac{5}{\infty} = \boxed{0}$$

$x(0) = 0$

Q Find the initial value of $X(s) = \dfrac{3s + 1}{s^2 + 5s + 6}$

A $X(s) = \dfrac{s(3 + \frac{1}{s})}{(s-1)(s+6)}$

$$X(s) = \frac{s(3 + 1/s)}{s^2(1 - 1/s)(1 + 6/s)}$$

$$\underset{s \to \infty}{Lt}\ \cancel{s}\left[\frac{\cancel{s}(3 + 1/s)}{\cancel{s^2}(1 - 1/s)(1 + 6/s)}\right]$$

$$\underset{s \to \infty}{Lt}\ \frac{3 + \frac{1}{s}}{(1 - 1/s)(1 + 6/s)} = \frac{3 + 1/\infty}{(1 - 1/\infty)(1 + 6/\infty)} = \boxed{3}$$

* Final value theorem:-

For unilateral laplace transform

$$x(t) \xrightarrow{L.T} X(s)$$

$$x(\infty) \longrightarrow \underset{s \to 0}{Lt}\ s\,X(s)$$

Proof

$$LT\left\{\frac{d}{dt}x(t)\right\} = sX(s) - x(0)$$

$$\int_0^{\infty}\frac{d}{dt}x(t)e^{-st}dt = sX(s) - x(0)$$

$\underset{s\to 0}{Lt}$ on both sides

$$\int_0^{\infty}\frac{d}{dt}x(t)\underset{s\to 0}{Lt}\, e^{-st}dt = \underset{s\to 0}{Lt}\, sX(s) - x(0)$$

$$\int_0^{\infty}\frac{d}{dt}x(t)\,dt = \underset{s\to 0}{Lt}\, sX(s) - x(0)$$

$$x(t)\Big|_0^{\infty} = \underset{s\to 0}{Lt}\, sX(s) - x(0)$$

$$x(\infty) - \cancel{x(0)} = \underset{s\to 0}{Lt}\, sX(s) - \cancel{x(0)}$$

$$\boxed{x(\infty) = \underset{s\to 0}{Lt}\, sX(s)}$$

* Conditions or cases for Final value Theorem:-

1) It is applicable only for causal signals.

*** 2) The poles of $X(s)$ must be on left side of s-plane (i.e the signal $x(t)$ is stable).

Q Find the final value of $X(s) = \frac{s-1}{s(s+1)}$

$$\rightarrow \underset{s\to 0}{Lt}\, \cancel{s}\left[\frac{s-1}{\cancel{s}(s+1)}\right) = \frac{0-1}{0+1} = -1$$

$x(\infty) = -1$

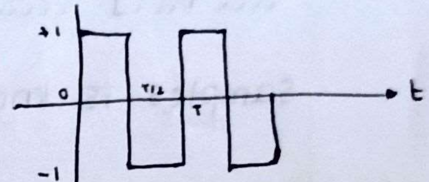
Q Find the final value of $X(s) = \dfrac{2}{(s-1)(s+2)}$

A Here poles are -2 & 1, as there is a pole in right side, so there is no final value.

* **Laplace Transform of periodic signal:** (not in exam)

A For a periodic signal, $X(s) = \dfrac{1}{1-e^{-sT}} \displaystyle\int_0^T x(t)\, e^{-st}\, dt$

Q Find the laplace transform of the graph



A $X(s) = \dfrac{1}{1-e^{-sT}} \displaystyle\int_0^T x(t)\, e^{-st}\, dt$

For time period 0 to T,

graph can be represented as

$x(t) = u(t) - 2u(t - T/2) + u(t-T)$ $\{LT(u(t)) = \frac{1}{s}\}$

$$X(s) = \frac{A}{1-e^{-sT}}\left[\frac{A}{s} - 2\left[e^{-s(\frac{T}{2})}\frac{A}{s}\right] + e^{-s(T)}\frac{A}{s}\right]$$

$$X(s) = \frac{A}{s(1-e^{-sT})}\left[A1 - 2e^{-s(\frac{T}{2})} + e^{-sT}\right]$$ //