4.3

Permutations and Combinations

Many counting problems can be solved by finding the number of ways to arrange a specified number of distinct elements of a set of a particular size, where the order of these elements matters. Many other counting problems can be solved by finding the number of ways to select a particular number of elements from a set of particular size where the order does not matter. In this module we will develop methods for such counting problems through permutations and combinations.

Permutations

A **permutation** of a set A of distinct objects is an ordered arrangement of the objects of A.

The following is the concept of an ordered arrangement of some elements of a set.

An ordered arrangement of r elements of a set A is called an $\boldsymbol{r}$ **− permutation** of A.

The number of $r-$ permutation of a set with n elements is denoted by ${}^{n}P_{r}$ or $P(n,r)$. We can find ${}^{n}P_{r}$ using the product rule.

Theorem1: If $\boldsymbol{n}$ is a positive integer and $\boldsymbol{r}$ is an integer with $\boldsymbol{1 \leq r \leq n}$, then there are

$${}^{n}P_{r} = n(n-1)(n-2)\ldots(n-r+1)$$

$\boldsymbol{r-}$ **Permutations of a set with $\boldsymbol{n}$ elements**.

Proof: The first element of the $r-$ permutation can be chosen in n ways, because there are n elements in the set. The second element can be chosen in $n-1$ ways, because there are $n-1$ elements left in the set after using the element in the first position. Similarly, there are $n-2$ ways to chose the third element, and so on, until there are exactly $n-(r-1)=n-r+1$ ways to choose the r^{th} element. By the product rule, there are

$$n(n-1)(n-2)\ldots(n-r+1)$$

$r-$ permutations of the set. Thus,

$${}^{n}P_{r}=n(n-1)(n-2)\ldots(n-r+1)$$

Note:

1. ${}^{n}P_{n}=n(n-1)(n-2)\ldots3.2.1=n!$, where n is a positive integer.
2. ${}^{n}P_{0}=1$, when n is a nonnegative integer, because there is exactly one way to order zero elements. That is, there is exactly one list with no elements in it, namely the empty set.

Corollary 1: If n and r are integers with $0\le r\le n$, then

$${}^{n}P_{r}=\frac{n!}{(n-r)!}$$

Proof: When n and r are integers with $1\le r\le n$, by Theorem 1, we have

$${}^{n}P_{r}=n(n-1)(n-2)\ldots(n-r+1)$$

$$=\frac{n(n-1)(n-2)\ldots(n-r+1).(n-r).(n-r-1).(n-r-2)\ldots.3.2.1}{(n-r).(n-r-1).(n-r-2)\ldots.3.2.1}$$

$$=\frac{n!}{(n-r)!}$$

We see that ${}^{n}P_{0}=\frac{n!}{(n-0)!}=\frac{n!}{n!}=1$, where n is a nonnegative integer. Thus, the formula for $P(n,r)$ also holds when $r=0$.

Example 1: How many permutations of the letters A , B, C , D, E, F, G and H contain the string . (See **P1**)

Example 2: How many ways are there to select a first – prize winner, a second – prize winner, and a third – prize winner from 100 different people who have entered a contest. (See **P1**)

Example 3: A group contains n men and n women. How many ways are there to arrange these people in a row if the men and women alternate? (See **P2**)

Combinations

We now discuss the counting of unordered selections of objects. Many counting problems can be solved by finding the number of subsets of a particular size of a set with n elemens, where n is a positive integer.

r − Combination: An **r − Combination** of elements of a set A is an unordered selection of r elements from the set A. That is, an r −combination of a set is simply a subset of the set with r elements.

The number of r −combinations of a set with n distinct elements is denoted by ${}^nC_r$ or $\binom{n}{r}$ or $C(n,r)$ and is called a **binomial coefficient**.

Theorem 2: The number of r −combinations of a set with n elements, where n is a nonnegative integer and r is an integer with $0 \le r \le n$ is given by

$${}^nC_r = \frac{n!}{r!(n-r)!}$$

Proof: The r −permutation of the set can be obtained by forming ${}^nC_r$ r −combinations of the set, and then ordering the elements of each r −combination, which can be done in ${}^rP_r$ ways. Therefore, ${}^nP_r = {}^nC_r . {}^rP_r$.

This implies that

$$^{n}C_{r}=\frac{^{n}P_{r}}{^{r}P_{r}}=\frac{\frac{n!}{(n-r)!}}{\frac{r!}{(r-r)!}}=\frac{n!}{r!(n-r)!}$$

$$i.e.,\ \boldsymbol{^{n}C_{r}=\frac{n!}{r!(n-r)!}}\ \text{or}\ \boldsymbol{^{n}C_{r}=\frac{n(n-1)(n-2)\ldots(n-r+1)}{r!}}$$

Corollary 2: Let n and r be nonnegative integers with $r \leq n$ then $^{n}C_{r}={}^{n}C_{n-r}$

Proof: From Theorem 2, we have

$$^{n}C_{r}=\frac{n!}{r!(n-r)!}\quad ;\quad ^{n}C_{n-r}=\frac{n!}{(n-r)!(n-(n-r))!}=\frac{n!}{r!(n-r)!}$$

Therefore, $^{n}C_{r}={}^{n}C_{n-r}$.

A ***combinatorial proof*** of an identity is a proof that uses counting arguments to prove that both sides of the identity count the same objects but in different ways.

Many identities involving binomial coefficients can be proved using combinatorial proofs. The following is a combinatorial proof of Corollary 2:

Proof: let S be a set with n elements. Every subset A of S with r elements corresponds to a subset of S with $n-r$ elements, namely $A'(=S-A)$ Consequently, the number of subsets of S with r elements $(0 \leq r \leq n)$ is equal to the number of subsets of with $n-r$ elements. Thus,

$$^{n}C_{r}={}^{n}C_{n-r}$$

Example 4: How many bit strings of length n contain exactly r 1s?

Solution: The position of r 1s in a bit string of length n form an r- combination of the set $\{1, 2, \ldots, n\}$. Hence there are ${}^nC_r$ bit strings of length n that contain exactly r 1s.

Example 5: How many bit strings of length 10 contain

(a) Exactly four 1s **(b) at most four 1s**

(c) at least four 1s **(d) an equal number of 0s and 1s**

(See **P3**)

Theorem 3: Binomial Theorem

Let x and y be variables and let n be a nonnegative integer. Then

$$(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^{n-r} y^r$$

$$= \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{r}x^{n-r}y^r + \cdots + \binom{n}{n-1}xy^{n-1} + \binom{n}{n}y^n$$

Proof: We give a combinatorial proof.

First we note that $(x+y)^n = \underbrace{(x+y)(x+y)\ldots(x+y)}_{n\ times}$

The terms in the product when it is expanded are of the form $x^{n-r}y^r$ for $r = 0, 1, 2, \ldots, n$. Note that to obtain such a term it is necessary to choose $n - r$ xs so that the other r terms in the product are ys. The number of such terms is $\binom{n}{n-r}$ which is same as $\binom{n}{r}$. Thus the coefficient of $x^{n-r}y^r$ is $\binom{n}{r}$.

This proves the theorem.

We can prove some useful identities using the binomial theorem.

Corollary 3: Let n be a nonnegative integer. Then

$$\sum_{r=0}^{n} \binom{n}{r} = 2^n$$

Proof: Using binomial theorem with $x = 1$ and $y = 1$, we see that

$$2^n = (1+1)^n = \sum_{r=0}^{n} \binom{n}{r} 1^{n-r} 1^r = \sum_{r=0}^{n} \binom{n}{r}$$

Hence the result

Corollary 4: Let n be a positive integer. Then

$$\sum_{r=0}^{n} (-1)^r \binom{n}{r} = 0$$

Proof: Let n be a positive integer. Using binomial theorem with $x = 1$ and $y = -1$, we see that

$$0 = 0^n = (1+(-1))^n = \sum_{r=0}^{n} \binom{n}{r} 1^{n-r} (-1)^r = \sum_{r=0}^{n} (-1)^r \binom{n}{r}$$

Hence the result

Note: From this corollary it follows that

$$\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \cdots$$

Corollary 5: Let n be a nonnegative integer. Then

$$\sum_{r=0}^{n} 2^r \binom{n}{r} = 3^n$$

Proof: Using binomial theorem with $x = 1$ and $y = 2$, we see that

$$3^n = (1+2)^n = \sum_{r=0}^{n} \binom{n}{r} 1^{n-r} 2^r = \sum_{r=0}^{n} 2^r \binom{n}{r}$$

Hence the result

Pascal's identity

The binomial coefficients satisfy many different identities. The following is the one of the most important identity.

Theorem 4: Pascal's identity

Let n and r be positive integers with $r \leq n$. Then

$$\binom{n}{r} + \binom{n}{r-1} = \binom{n+1}{r}$$

Proof: let T be a set containing $n+1$ elements. Let a be an element of T and let $S = T - \{a\}$. Clearly, there are $\binom{n+1}{r}$ subsets of T with r elements. Note that a subset of T with r elements either contains a together with $r-1$ elements of S or contains r elements of S and does not contain a.

Because there are $\binom{n}{r-1}$ subsets each containing $r-1$ elements, there are $\binom{n}{r-1}$ subsets of T containing a and each containing r elements. Further, there are $\binom{n}{r}$ subsets of T not containing a and each containing r elements and these are the subsets of S each containing r elements. Consequently,

$$\binom{n+1}{r} = \binom{n}{r} + \binom{n}{r-1}$$

Note: Pascal's identity, together with the initial conditions $\binom{n}{0} = \binom{n}{n} = 1$ for all positive integers n, can be used to recursively define binomial coefficients. This recursive definition is useful in the computation of binomial coefficients because only addition, and not multiplication, of integers is needed to use this recursive definition.

Some other identities of the binomial coefficients

Theorem 5: Vandermonde's Identity

Let $\boldsymbol{m}, \boldsymbol{n}$ and $\boldsymbol{r}$ be nonnegative integers with $\boldsymbol{r}$ not exceeding either $\boldsymbol{m}$ or $\boldsymbol{n}$. Then

$$\binom{\boldsymbol{m}+\boldsymbol{n}}{\boldsymbol{r}} = \sum_{\boldsymbol{k}=0}^{\boldsymbol{r}} \binom{\boldsymbol{m}}{\boldsymbol{r}-\boldsymbol{k}}\binom{\boldsymbol{n}}{\boldsymbol{k}}$$

Proof: Suppose that there are m items in one set and n items in a second set. Then the total number of ways to pick r elements from the union of these sets is $\binom{m+n}{r}$.

Another way to pick r elements from the union is to pick k elements from the first set and then $r-k$ elements from the second set, where k is an integer with $0 \leq k \leq r$. This can be done in $\binom{m}{k}\binom{n}{r-k}$ ways, using the product rule. Therefore, the total number of ways to pick r elements from union is

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{k}\binom{n}{r-k}$$

Let $j = r-k$. Then $k = r-j$ and

$$\sum_{k=0}^{r} \binom{m}{k}\binom{n}{r-k} = \sum_{j=0}^{r} \binom{m}{r-j}\binom{n}{j}$$

Thus,

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{r-k}\binom{n}{k}$$

Hence the result

Corollary 6: Let n be a nonnegative integer. Then

$$\binom{2n}{n} = \sum_{r=0}^{n} \binom{n}{r}^2$$

Proof: We have Vandermonde's identity

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{r-k}\binom{n}{k}$$

Taking $m = n = r$, we get

$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{n-k}\binom{n}{k} = \sum_{k=0}^{n} \binom{n}{k}^2 \text{, } because \binom{n}{n-k} = \binom{n}{k}$$

$$= \sum_{r=0}^{n} \binom{n}{r}^2$$

Hence the result

Theorem 6: Let n and r be nonnegative integers with $r \leq n$ then

$$\binom{n+1}{r+1} = \sum_{j=r}^{n} \binom{j}{r}$$

Proof: We give the combinatorial proof. By Example 4, the left-hand side, $\binom{n+1}{r+1}$, counts the number of bit strings of length $(n+1)$ containing $(r+1)$ 1s.

We will show that the right-side counts the same objects by considering the cases corresponding to the possible locations of the final 1 (or the last 1) in a string with $(r+1)$ 1s. This last 1 must occur at positions $r+1, r+2, \ldots,$ or $n+1$.

Further, if the last 1 is the k^{th} bit in the string then there must be r 1s among the first $k-1$ positions. By Example4, there are $\binom{k-1}{r}$ such bit strings.

Now summing over k from $r+1$ to $n+1$, we get

$$\sum_{k=r+1}^{n+1} \binom{k-1}{r}$$

bit strings of length $n+1$ containing exactly $+1$ $1s$. Therefore

$$\binom{n+1}{r+1} = \sum_{k=r+1}^{n+1} \binom{k-1}{r}$$

Let $= k - 1$. Then

$$\sum_{k=r+1}^{n+1} \binom{k-1}{r} = \sum_{j=r}^{n} \binom{j}{r}$$

and

$$\binom{n+1}{r+1} = \sum_{j=r}^{n} \binom{j}{r}$$

Hence the result.

Permutations and combinations with repetitions

Counting permutations when repetitions of elements are allowed can be easily be done using the product rule.

Theorem 7: The number of r −permutations of a set of n objects with repetition allowed is n^r .

The following is for $r-$ combinations:

Theorem 8: There are ${}^{n+r-1}C_r = {}^{n+r-1}C_{n-1}$, r −combinations from a set with n elements when repetition of elements is allowed.

Proof: Each r- combination of a set with n elements when repetition is allowed can be represented by a list with $n-1$ bars and r stars. The $n-1$ bars are used to mark off n different cells, with the i^{th} cell containing a star for each time the i^{th} element of the set occurs in the combination.

For instance, a 6- combination of a set with four elements is represented with $4 - 1 = 3$ bars and 6- stars. Now,

★★|★| |★★★

represents the combination containing exactly two of the first element, one of the second element, none of the third element and three of the fourth element of the set.

Thus, each different list containing $n - 1$ bars and r- stars corresponding to an r- combination of the set with n- elements, when repetition is allowed. The number of such lists is ${}^{(n-1+r)}C_r$, because each list corresponding to a choice of r- positions to place the r- stars from the $n - 1 + r$ positions that contain r- stars and $n - 1$ bars. Note that

$$ {}^{(n-1+r)}C_r = {}^{(n-1+r)}C_{(n-1+r)-r} = {}^{(n-1+r)}C_{n-1} $$

Example 6: Suppose that a cookie shop has four different kinds of cookies. How many different ways can six cookies be chosen? Assume that only the type of cookies and not the individual cookies or the order in which they are chosen, matters.

Solution: The number of ways to choose six cookies is the number of 6-combinations of a set with 4 elements. By Theorem 8, we have

$$ {}^{(4+6-1)}C_6 = {}^{9}C_6 = {}^{9}C_3 = \frac{9.8.7}{1.2.3} = 84 $$

Thus, there are 84 different ways to choose the six cookies from four different kinds of cookies.

Theorem 6 can also be used to find the number of solutions of certain linear equations where the variables are integers subject to constraints.

Example 7: How many solutions does the equation $x_1 + x_2 + x_3 = 11$ have, where x_1, x_2 and x_3 are nonnegative integers?

Solution: Note that a solution corresponds to a way of selecting 11 items from a set with three elements so that x_1 items of type1, x_2 items of type2 and x_3 items of type3. Hence the number of solutions is the number of 11 combinations with repetition allowed from a set with three elements. By Theorem 8, there are

$$^{(3+11-1)}C_{11} = {}^{13}C_{11} = {}^{13}C_2 = \frac{13.12}{1.2} = 78 \text{ solutions.}$$

Note: The number of solutions of this equation can also be found when the variables are subject to constraints.

Example 8: How many solutions does the equation $x_1 + x_2 + x_3 = 11$ have, where x_1, x_2 and x_3 are nonnegative integers with $x_1 \geq 1, x_2 \geq 2$ and $x_3 \geq 3$?

Solution: A solution to the equation subject to these constraints corresponds to a selection of 11 items with x_1 items of type1, x_2 items of type2 and x_3 items of type3, where in addition, there is at least one item of type1, two items of type2, and three items of type3. Therefore, there are one item of type1, two items of type2, and three items of type3. Thus, 6 items are already chosen. Then select 5 additional items. By Theorem 8, this can be done in

$$^{(3+5-1)}C_5 = {}^7C_5 = {}^7C_2 = \frac{7.6}{1.2} = 21 \text{ ways.}$$

Thus, there are 21 solutions of the equation subject to the given constraints.

Permutations with indistinguishable objects

Some elements may be indistinguishable in counting problems. When this is the case, care must be taken to avoid counting things more than once.

Theorem 9: The number of different permutations of n objects, where there are n_1 distinguishable objects of type1, n_2 distinguishable objects of type2, … , and n_k distinguishable objects of type k, is

$$\frac{n!}{n_1!\,n_2!\dots n_k!}$$

Proof: To determine the number of permutations, first note that n_1 objects of type 1 can be placed among n positions ${}^{n}C_{n_1}$ in ways, leaving $n - n_1$ positions free. Then the objects of type 2 can be placed in ${}^{(n-n_1)}C_{n_2}$ ways, leaving $n - n_1 - n_2$ positions free. Continue placing objects of type 3, … , type $k - 1$, until at the last stage, n_k objects of type k can be placed in ${}^{(n-n_1-n_2-\cdots-n_k)}C_{n_k}$ ways. Hence, by the product rule, the total number of different permutations is

$$\begin{aligned}
&{}^{n}C_{n_1} \cdot {}^{(n-n_1)}C_{n_2} \cdot \ \cdots \ \cdot {}^{(n-n_1-n_2-\cdots-n_k)}C_{n_k} \\
&\quad = \frac{n!}{n_1!(n-n_1)!} \cdot \frac{(n-n_1)!}{n_2!(n-n_1-n_2)!} \cdot \ \cdots \ \cdot \frac{(n-n_1-n_2-\cdots-n_{k-1})!}{n_k!(n-n_1-n_2-\cdots-n_k)!} \\
&\quad = \frac{n!}{n_1!\,n_2!\ \cdots\ n_k!}
\end{aligned}$$

Example 9: How many different strings can be made from the letters in ABRACADABRA, using all the letters?

Solution: Note that the given word has 11 letters and it contains 5 A's, 2 B's, 2 R's, one C and one D.

The number of different strings can be made from the letters of the given word using all the letter is

$$\frac{11!}{5!\,2!\,2!\,1!\,1!} = 83{,}160$$

Distributing objects into boxes

Many counting problems can be solved by enumerating the ways. The objects can be placed into boxes. The objects can be either distinguishable (*i.e.,* different from each other) or indistinguishable (*i.e.,* considered identical).

Theorem 10: The number of ways to distribute $\boldsymbol{n}$ distinguishable objects into $\boldsymbol{k}$ distinguishable boxes so that $\boldsymbol{n_i}$ objects are placed into box $\boldsymbol{i}$, $\boldsymbol{i = 1, 2, \dots, k}$, equals

$$\frac{\boldsymbol{n!}}{\boldsymbol{n_1!\, n_2! \dots n_k!}}$$

Note: Counting the number of ways of placing n distinguishable objects into k distinguishable boxes is same as counting the number of n- combinations for a set of k elements when repetitions are allowed.

P1:

a. **How many permutations of the letters A , B, C , D, E, F, G and H contain the string ABC.**

b. **How many ways are there to select a first – prize winner, a second – prize winner, and a third – prize winner from 100 different people who have entered a contest.**

Solution:

a. Because ABC must occur as a block, we can find the number of permutations of six objects namely, the block ABC and the individual letter D, E, F, G and H. Because these six objects can occur in any order, there are $6! = 720$ permutations of letters $A\ B\ C\ D\ E\ F\ G\ H$ in which $A\ B\ C$ occurs as a block.

b. The number of ways to pick the three prize winners is the number of $3-$ permutations of a set with 100 elements. Therefore,

$$^{100}P_3 = 100.(100-1).(100-2).(100-3) = 100.99.98 = 9,70,200$$

P2:

A group contains n men and n women. How many ways are there to arrange these people in a row if the men and women alternate?

Solution:

It amounts to arrange them in a row of $2n$ places in two ways.

(i) Men occupying odd positions and Women occupying even positions. This can be done in $n!\,.\,n! = (n!)^2$ (by the product rule).

(ii) Women occupying odd positions and Men occupying even positions. This can be done in $n!\,.\,n! = (n!)^2$ (by the product rule).

The number of ways to arrange these $2n$ people in a row if the men and women sit alternately is $(n!)^2 + (n!)^2 = 2(n!)^2$.

P3:

How many bit strings of length 10 contain

(a) Exactly four 1s **(b) at most four 1s**

(c) at least four 1s **(d) an equal number of 0s and 1s**

Solution: It is known that there are ${}^{n}C_{r}$ bit strings of length n that contain exactly r 1s

(a) The number of bit strings of length 10 containing exactly four 1s is

$${}^{10}C_4 = \frac{10.9.8.7}{1.2.3.4} = 210$$

(b) The number of bit strings of length of length 10 containing at most four 1s is

$$\begin{aligned}{}^{10}C_0 + {}^{10}C_1 + {}^{10}C_2 + {}^{10}C_3 + {}^{10}C_4 &= 1 + 10 + \frac{10.9}{1.2} + \frac{10.9.8}{1.2.3} + \frac{10.9.8.7}{1.2.3.4} \\ &= 1 + 10 + 45 + 120 + 210 = 386\end{aligned}$$

(c) The number of bit strings of length 10 containing at least four 1s is

$$\begin{aligned}&{}^{10}C_4 + {}^{10}C_5 + {}^{10}C_6 + {}^{10}C_7 + {}^{10}C_8 + {}^{10}C_9 + {}^{10}C_{10} \\ &= {}^{10}C_4 + {}^{10}C_5 + {}^{10}C_4 + {}^{10}C_3 + {}^{10}C_2 + {}^{10}C_1 + {}^{10}C_0 \\ &= {}^{10}C_0 + {}^{10}C_1 + {}^{10}C_2 + {}^{10}C_3 + {}^{10}C_4 + {}^{10}C_4 + {}^{10}C_5 \\ &= 386 + 210 + \frac{10.9.8.7.6}{1.2.3.4.5} = 386 + 210 + 252 = 848\end{aligned}$$

(d) The number of bit strings of length 10 containing equal number of 0s and 1s is

${}^{10}C_5 = 252$

P4.

A coin is flipped 10 times where each flip comes up either heads or tails. How many possible outcomes

a) Are there in total?
b) Contain exactly two heads?
c) Contain at most three heads?
d) Contain the same number of heads and tails?

Solution:

a) The number of possible outcomes $= \underbrace{2 \times 2 \times \ldots \times 2}_{10\ times}$ (by product rule)

$$= 2^{10} = 1024$$

b) The number of possible outcomes containing exactly two heads $=$ the number of ways of selecting 2 places out of $10 = {}^{10}C_2 = \frac{10.9}{1.2} = 45$

c) The number of possible outcomes containing at most three heads is

$${}^{10}C_0 + {}^{10}C_1 + {}^{10}C_2 + {}^{10}C_3 + {}^{10}C_4 = 176$$

d) The number of possible outcomes containing the same number of heads and tails is ${}^{10}C_5 = 252$

P5:

Show that a nonempty set has the same number of subset with an odd number of elements as it does subsets with an even number of elements?

Solution: Let A be a set with n elements.

The number of subsets of A with k elements is ${}^nC_k$.

We have

$$(x+y)^n = {}^nC_0\,x^n + {}^nC_1\,x^{n-1}y + {}^nC_2\,x^{n-2}y^2 + \cdots + {}^nC_k\,x^{n-k}y^k + \cdots + {}^nC_n\,y^n$$

Put $x = 1, y = -1$. Then

$$0 = {}^nC_0 - {}^nC_1 + {}^nC_2 - {}^nC_3 + {}^nC_4 - {}^nC_5 + \cdots + (-1)^n\,{}^nC_n$$

From this we get

$${}^nC_0 + {}^nC_2 + {}^nC_4 + \cdots = {}^nC_1 + {}^nC_3 + {}^nC_5 + \cdots$$

The LHS gives the number of subsets of A with an even number of elements and the RHS gives the number of subsets of A with an odd number of elements.

4.3. Permutations and Combinations.

Exercise

1. Suppose that there are eight runners in a race. The winner receives a gold medal, the second – place finisher receivers a silver medal, and the third – place finisher receives a bronze medal. How many different ways are there to award these medals, if all possible outcomes of race can occur and there are no ties?
2. Suppose that a saleswoman has to visit eight different cities. She must begin her trip in a specified city, but she can visit the other seven cities in any order she wishes. How many possible orders can the saleswoman use when visiting these cities?
3. How many poker hands of five cards can be dealt from a standard deck of 52 cards? Also, how many are there to select 47 cards from a standard deck of 52 cards?
4. How many ways are there to select five players from a 10 member tennis team to make a trip to a match at another school?
5. Suppose that there are 9 faculty members in the mathematics department and 11 in the computer science department. How many ways are there to select a committee to develop a discrete mathematics course at a school if the committee is to consist of three faculty members from the mathematics department and four from the computer science department?
6. Find the value of each of these quantities.

   a. $P(6,3)$ b.$(6,5)$ c.$P(10,9)$ d. $P(8,5)$

7. Find the number of $5-$ permutations of a set with nine elements.
8. In how many different orders can five runners finish a race if no ties are allowed?
9. How many possibilities are there for the win, place, and show (first, second, and third) positions in a horse with 12 horses if all orders of finish are possible?

10. There are six different candidates for governor of a state. In how many different orders can the names of the candidates be printed on a ballot?
11. How many bit strings of length 12 contain
    a) Exactly three 1s ?
    b) At most three 1s?
    c) At least three 1s?
    d) An equal number of 0s and 1s?
12. A coin is flipped eight times where each flip comes up either heads or tails. How many possible outcomes
    a) Are there in total?
    b) Contain exactly three heads?
    c) Contain at least three heads?
    d) Contain the same number of head and tails?
13. How many bit strings of length 10 have
    a) Exactly three 0s?
    b) More 0s than 1s?
    c) At least seven 1s?
    d) At least three 1s?
14. How many permutations of the letters $ABCDEFG$ contain
    a) The string BCD?
    b) The string $CFGA$?
    c) The strings BA and GF?
    d) The string ABC and DE?
    e) The string ABC and CDE?
    f) The strings CBA and BED?
15. How many different strings can be made by reordering the letters of the word SUCCESS?
16. In how many different ways can five elements be selected in order from a set with three elements when repetition is allowed?
17. How many strings of six letters are there?
18. How many ways are there to assign three jobs to five employees if each employee can be given more than one job?

19. How many ways are there to select three unordered elements from a set with five elements when repetition is allowed?
20. How many solutions are there to the equation $x_1 + x_2 + x_3 + x_4 = 7$ where x_1 , x_2 , x_3 and x_4 are nonnegative integers?
21. How many different strings can be made from the letters in MISSISSIPPI, using all the letters?
22. How many different strings can be made from the letters in ABRACADABRA, using all the letters?
23. How many different strings can be made from the letters in AARDVARK, using all the letters, if all three $A\ s$ must be consective?