1.6

LINEAR TRANSFORMATIONS

A vector space has two operations defined on it, namely, addition and scalar multiplication. Linear transformations between vectors spaces are those functions that preserve these linear structures in the following sense.

DEFINITION: Let U and V be vector spaces. Let $\vec{u}$ and $\vec{v}$ be vectors in U and let c be a scalar. A function $T: U \rightarrow V$ is said to be linear transformation if

$$T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$$

$$T(c\vec{u}) = cT(\vec{u})$$

The first condition implies that T' maps the sum of two vectors into the sum of images of those vectors. The second condition implies that T' maps the scalar multiple of a vector into the same scalar multiple of the image. Thus the operations of addition and scalar multiplication are preserved under linear transformation.

THEOREM: Let V be a finite-dimensional vector space over the field F and let $\{\alpha_1, \dots, \alpha_n\}$ be ordered basis for V. Let W be a vector space over the same field F and let $\beta_1, \dots, \beta_n$ be any vectors in W. Then there is precisely one linear transformation T from V into W such that

$$\mathrm{T}(\alpha_j) = \beta_j \qquad \text{j= 1,…,n.}$$

Proof: To prove there is some linear transformation T with $T(\alpha_j) = \beta_j$ we proceed as follows. Given α in V, there is a unique n-tuple $(x_1, \ldots, x_n)$ such that

$$\alpha = x_1\alpha_1 + \cdots + x_n\alpha_n.$$

For this vector α we define

$$T(\alpha) = x_1\beta_1 + \cdots + x_n\beta_n.$$

Then T is a well-defined rule for associating with each vector α in V a vector $T(\alpha)$ in W. From the definition it is clear that $T(\alpha_j) = \beta_j$ for each j. To see that T is linear, let

$$\beta = y_1\alpha_1 + \cdots + y_n\alpha_n$$

be in V and let c be any scalar. Now

$$c\alpha + \beta = (cx_1 + y_1)\alpha_1 + \cdots + (cx_n + y_n)\alpha_n$$

and so by definition

$$T(c\alpha + \beta) = (cx_1 + y_1)\beta_1 + \cdots + (cx_n + y_n)\beta_n$$

on the other hand,

$$c(T(\alpha)) + T(\beta) = c\sum_{i=1}^{n} x_i\,\beta_i + \sum_{i=1}^{n} y_i\,\beta_i$$

$$= \sum_{i=1}^{n} (cx_i + y_i)\,\beta_i$$

and thus $$T(c\alpha + \beta) = c(T(\alpha)) + T(\beta).$$

If U is a linear transformation from V into W with $U(\alpha_j) = \beta_j$, $j = 1, \dots, n$, then for the vector $\alpha = \sum_{i=1}^{n} x_i \alpha_i$ we have

$$U(\alpha) = U\left(\sum_{i=1}^{n} x_i \alpha_i\right)$$

$$= \sum_{i=1}^{n} x_i (U(\alpha_i))$$

$$= \sum_{i=1}^{n} x_i \beta_i$$

so that U is exactly the rule T which we defined above. This shows that the linear transformation T with $T(\alpha_i) = \beta_i$ is unique.

The following theorem shows that any linear transformation maps the zero vector of the domain vector space to the zero vector of the co-domain vector space.

THEOREM: Let $T: U \to V$ be a linear transformation. Let 0_U and 0_V be the zero vectors of U and V. Then $T(\vec{0}_U) = \vec{0}_V$.

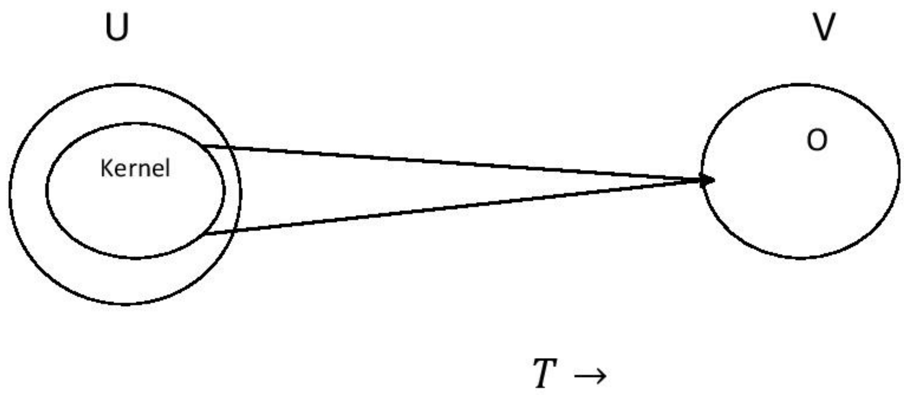
That is, a linear transformation maps a zero vector into a zero vector.

Proof: let $\vec{u}$ be a vector in U and let $T(\vec{u}) = \vec{v}$

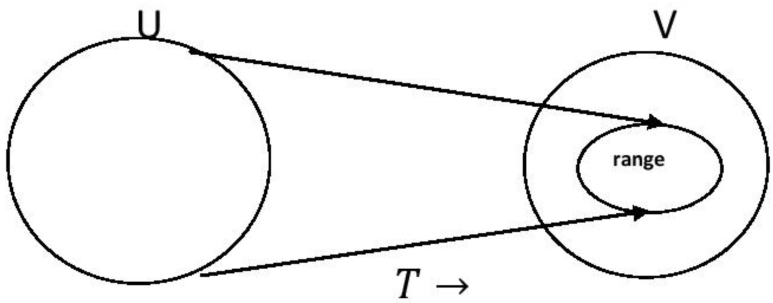
$T(\vec{0}_U) = T(0\,\vec{u}) = 0\,T(\vec{u}) = 0\vec{v} = \vec{0}_V$.

DEFINITION: Let $T: U \to V$ be a linear transformation. The set of vectors in U that are mapped into the zero vector of V is called the kernel of T. The kernel is denoted by $ker(T)$.

The set of vectors in V that are images of vectors in U is called the range of T. The range is denoted by $range(T)$. We illustrate these sets in the following figure.



All vectors in U that are mapped into 0.



All vectors in V that are images of vectors in U.

Whenever we introduce sets in linear algebra, we are interested in knowing whether they are vector spaces or not. We now find that the kernel and range are indeed vector spaces.

THEREM: Let $T: U \rightarrow V$ be a linear transformation.

a) The kernel of T' is a subspace of U
b) The range of T' is a subspace of V.

Proof: From the previous theorem, we know that the kernel is non empty since it contains the zero vector of U.

To prove that the kernel is a subspace of U, we show that it is closed under addition and scalar multiplication. First we prove closure under addition, Let $\overrightarrow{u_1}, \overrightarrow{u_2} \in \ker(T)$.

Then $T(\overrightarrow{u_1}) = T(\overrightarrow{u_2}) = 0$.

Now $T(\overrightarrow{u_1} + \overrightarrow{u_2}) = T(\overrightarrow{u_1}) + T(\overrightarrow{u_2}) = 0 + 0 = 0$.

Then vector $\overrightarrow{u_1} + \overrightarrow{u_2}$ is mapped into 0. Thus $\overrightarrow{u_1} + \overrightarrow{u_2}$ is in $\ker(T)$

Let us now show that $\ker(T)$ is closed under scalar multiplication. Let 'c' be a scalar.

$T(c\overrightarrow{u_1}) = cT(\overrightarrow{u_1}) = c0 = 0$.

Thus cu_1 is in $\ker(T)$.

The kernel is closed under addition and under scalar multiplication. It is a subspace of U.

(b) The previous theorem tells us that the range is non empty since it contains the zero vector of V.

To prove that the range is a subspace of V, we show that it is closed under addition and scalar multiplication. Let v_1 and v_2 be elements of $range(T)$. Thus $\exists$ vectors $\overrightarrow{u_1}$ and $\overrightarrow{u_2}$ in the domain U such that

$$T(\overrightarrow{u_1}) = \overrightarrow{v_1} \text{ and } T(\overrightarrow{u_2}) = \overrightarrow{v_2}$$

Now $T(\overrightarrow{u_1} + \overrightarrow{u_2}) = T(\overrightarrow{u_1}) + T(\overrightarrow{u_2}) = \overrightarrow{v_1} + \overrightarrow{v_2}$. The vector $\overrightarrow{v_1} + \overrightarrow{v_2}$ is the image of $\overrightarrow{u_1} + \overrightarrow{u_2}$. Thus $\overrightarrow{v_1} + \overrightarrow{v_2}$ is in the range. Let 'c' be a scalar. Then $T(c\overrightarrow{u_1}) = cT(\overrightarrow{u_1}) = c\overrightarrow{v_1}$

The vector $c\overrightarrow{v_1}$ is the image of $c\overrightarrow{u_1}$. Thus $c\overrightarrow{v_1}$ is in the range. The range is closed under addition and under scalar multiplication. It is a subspace of V.

The following theorem gives an important relationship between the "sizes" of the kernel and the range of a linear transformation.

THEOREM: Let $T: U \rightarrow V$ be a linear transformation. Then $Dim\ker(T) + \dim range(T) = \dim domain(T)$

Proof: If $\ker(T) = U$, then $range(T) = \{\vec{0}\}$. Since the only vector space with dimension 0 is $\overrightarrow{\{0\}}$, we are done in this case.

Suppose that $\ker(T) \neq U$.

Let $\overrightarrow{u_1}, \overrightarrow{u_2}, \dots, \overrightarrow{u_m}$ be a basis for $\ker(T)$. Add vectors $\overrightarrow{u_{m+1}}, \dots, u_n$ to this set to get a basis $\overrightarrow{u_1}, \overrightarrow{u_2}, \dots, \overrightarrow{u_n}$ for U.

We shall show that $T(\overrightarrow{u_{m+1}}), \dots, T(\overrightarrow{u_n})$ form a basis for the range, thus proving the theorem.

Let $u \in U$.

Then we get scalars $a_1, a_2, \dots, a_n$ such that $\vec{u} = a_1\overrightarrow{u_1} + a_2\overrightarrow{u_2} + \dots + a_m\overrightarrow{u_m} + a_{m+1}\overrightarrow{u_{m+1}} + \dots + a_n\overrightarrow{u_n}$.

Thus

$$T(\vec{u}) = T(a_1\overrightarrow{u_1} + a_2\overrightarrow{u_2} + \cdots + a_m\overrightarrow{u_m} + a_{m+1}\overrightarrow{u_{m+1}} + \cdots + a_n\overrightarrow{u_n})$$

$$= a_1T(\overrightarrow{u_1}) + \cdots + a_mT(\overrightarrow{u_m}) + a_{m+1}T(\overrightarrow{u_{m+1}}) + \cdots + a_nT\overrightarrow{u_n}$$

$$= a_{m+1}T(\overrightarrow{u_{m+1}}) + \cdots + a_nT\overrightarrow{u_n}.$$

Since $T(\vec{u})$ represents an arbitrary vector in the range of T, the vectors $T(\overrightarrow{u_{m+1}}), \dots . T(\overrightarrow{u_n})$ span the range.

It remains to prove that these vectors are linearly independent. Consider the identity

$$b_{m+1}\, T(\overrightarrow{u_{m+1}}) + \cdots + b_n\, T(\overrightarrow{u_n}) = 0$$

$$\Longrightarrow T(b_{m+1}\overrightarrow{u_{m+1}} + \cdots + b_n\overrightarrow{u_n}) = 0$$

$$\Longrightarrow b_{m+1}\overrightarrow{u_{m+1}} + \cdots + b_n\overrightarrow{u_n} \in \ker(T).$$

$$\Longrightarrow b_{m+1}\overrightarrow{u_{m+1}} + \cdots + b_n\overrightarrow{u_n} = c_1\overrightarrow{u_1} + \cdots + c_m\overrightarrow{u_m}$$

$$\Longrightarrow c_1\overrightarrow{u_1} + \cdots + c_m\overrightarrow{u_m} - b_{m+1}\overrightarrow{u_{m+1}} - \cdots - b_n\overrightarrow{u_n} = c$$

Since the vectors $\overrightarrow{u_1}, \dots, \overrightarrow{u_m}, , \overrightarrow{u_{m+1}}, \dots, \overrightarrow{u_n}$ are a basis, they are linearly independent. Therefore, the coefficients are all zero.

$c_1 = 0, c_m = 0, b_{m+1} = 0, \dots, b_n = 0.$ So $T(\overrightarrow{u_{m+1}}), \dots, T(\overrightarrow{u_n})$ are linearly independent.

Therefore the set of vectors $T(\overrightarrow{u_{m+1}}), \dots, T(\overrightarrow{u_n})$ is a basis for the range.

TERMINOLOGY:

The kernel of a linear mapping T' is often called the null space. $Dim \ker(T)$ is called the nullity, and $\dim range(T)$ is called the rank of the transformation. The previous theorem is often referred to as the rank/nullity theorem and written in the following form. $Rank(T) + nullity(T) = \dim domain(T)$.

Problem 1: Prove that the following transformation $T: R^2 \to R^2$ is linear. T(x,y) = (2x, x+y)

Solution: We first show that T preserves addition. Let (x_1,y_1) and (x_2,y_2) be elements of R^2. Then

$T((x_1,y_1)+ (x_2,y_2)) = T(x_1+x_2,y_1+y_2)$ by vector addition

$= (2x_1+2x_2, x_1+x_2+y_1+y_2)$ by definition of T

$= (2x_1, x_1+ y_1)+(2x_2, x_2+ y_2)$ by vector addition

$= T(x_1,y_1)+T(x_2,y_2)$ by definition of T

Thus T preserves vector addition.

We now show that T preserves scalar multiplication. Let c be a scalar.

$T(c(x_1,y_1))=T(cx_1,cy_1)$ by scalar multiplication of a vector

$=(2c\,x_1, cx_1+c\,y_1)$ by definition of T

$=c(2x_1, x_1+ y_1)$ by scalar multiplication of a vector

$=cT(x_1, y_1)$ by definition of T

Thus T preserves scalar multiplication. T is linear.

Problem 2: Let P_n be the vector space of real polynomial functions of degree $\leq$n. Show that the following transformation $T: P_2 \rightarrow P_1$is linear.

$$T\ (ax^2+bx+c)= (a+b)x+c$$

Solution: Let ax^2+bx+c and px^2+qx+r be arbitrary elements of P_2. Then

$T\ ((ax^2+bx+c)+(px^2+qx+r)) = T\ ((a+p)x^2+(b+q)x+(c+r)$ by vector addition

$= (a+p+b+q)x+(c+r)$ by definition of T

$=(a+b)x+c+(p+q)x+r$

$= T\ (ax^2+bx+c)+ T(px^2+qx+c\ )$ by definition of T

Thus T preserves addition

We now show that T preserves scalar multiplication. Let k be a scalar.

$T(k(ax^2+bx+c))=T(kax^2+kbx+kc)$ by scalar multiplication

$= (ka+kb)x+kc$ by definition of T

$=k((a+b)x+c)$

$=kT(ax^2+bx+c)$ by definition of T

T preserves scalar multiplication. Therefore, T is a linear transformation.

Problem 3: Find the kernel and range of the linear operator T(x,y,z)=(x,y,0)

Solution: Since the linear operator T maps $\mathbf{R}^3$ into $\mathbf{R}^3$, the kernel and range will both be subspaces of $\mathbf{R}^3$.

Kernel: ker(T) is the subset that is mapped into (0,0,0). We see that T(x,y,z) = (x,y,0)

= (0,0,0), if x=0,y=0

Thus ker(T) is the set of all vectors of the form (0,0,z). We express this as ker(T) ={(0,0,z)}

Geometrically, ker(T) is the set of all vectors that lie on the z-axis.

Range: The range of T is the set of all vectors of form (x,y,0). Thus range (T) = {(x,y,0)}

Range (T) is the set of all vectors that lie in the x-y plane.

Exercise

1. Prove that the following transformations $T: R^2 \rightarrow R$ are not linear.

(a) T(x, y) = y^2

(b) T(x, y) = x-3

2. Determine the kernel and range of each of the following transformations. Show that dim ker(T)+dim range (T)=dim domain (T) for each transformation.

(a) T(x, y, z) = (x, 0, 0) of $\boldsymbol{R}^3 \rightarrow \boldsymbol{R}^3$

(b) T(x, y, z) = (x + y, z) of $\boldsymbol{R}^3 \rightarrow \boldsymbol{R}^2$

(c) T(x, y) = (3x, x-y, y) of $\boldsymbol{R}^2 \rightarrow \boldsymbol{R}^3$

3. Let $T: U \rightarrow V$ be a linear mapping. Let v be a nonzero vector in V. Let W be the set of vectors in U such tat T(w)=v. Is W a subspace of U?

4. Let $T: U \rightarrow V$ be a linear transformation. Prove that

dim range (T) = dim domain (T)

if and only if T is one-to-one.

5 Let $T: U \rightarrow V$ be a linear transformation. Prove that T is one –to-one if and only if it preserves linear independence.

Answers

2. (a) The kernel is the set {(0,r,s)} and the range is the set {(a,0,0)}. Dim ker(T) =2, dim range(T) =1, and dim domain(T)= 3, so dim ker(T)+ dim range(T) =dim domain(T).

(b) The kernel is the set {(r,-r,0)} and the range is $\mathbf{R}^2$. Dim ker(T) =1, dim range(T)=2, and dim domain(T)= 3, so dim ker(T)+ dim range(T) = dim domain(T).

(c) The kernel is the zero vector and the range is the set {(3a,a-b,b)}. Dim ker(T) =0, dim range(T)= 2, so dim ker(T)+ dim range(T) = dim domain(T).

3. This set is not a subspace because it does not contain the zero vector.

4. T is one-to-one if and only if ker(T) is the zero vector if and only if dim ker(T) = 0 if and only if dim range (T)=dim domain(T).