Unit-4

Counting

4.1

Counting Principles

Combinatorics, the study of arrangements of objects, is an important part of discrete structures. Enumeration, the counting of objects with certain properties, is an important part of Combinatorics. Counting is used to determine the complexity of algorithms and it is also required whether there are enough internet protocol addresses to meet the demand. Further, counting techniques are extensively used when probabilities of events are computed.

In this module we introduce the basic methods of counting. These methods serve as the foundation for almost all counting techniques.

Basic counting principles: There are two basic counting principles: (i) the ***product rule*** and (ii) *the **sum rule**.*

The product rule applies when a procedure is made up of separate tasks.

The Product Rule: Suppose that a procedure can be broken down into a sequence of two tasks. If there are n_1 ways to do the first task and for each of these ways of doing the first task, there are n_2 ways to do the second task, then there are $n_1 n_2$ ways to do the procedure.

An extended version of the product rule is often useful.
Suppose that a procedure is carried out by performing the tasks $T_1, T_2, \dots, T_m$ in sequence. If each task T_i , $i = 1,2, \dots, n$, can be done is n_i ways, (regardless of how the previous tasks were done), then there are $n_1 n_2 \dots n_m$ ways to carry out the procedure.

This can be proved by mathematical induction from the product rule of two tasks.

Example 1: How many bit strings of length seven are there?

Solution: Each of the seven bits can be chosen in two ways, because each bit is either 0 or 1. Therefore, by the product rule there are $\underbrace{2 \times 2 \times \ldots \times 2}_{7\ times} = 2^7 = 128$ different bit strings of length seven.

Example: **Counting Functions**

How many functions are there from a set with m elements to another set with n elements.

Solution: A function corresponds to a choice of one of the n elements in the codomain for each of the m elements in the domain. By product rule there are $\underbrace{n \cdot n \cdot \ldots \cdot n}_{m\ times} = n^m$ functions from a set with m elements to a set with n elements.

Example 2: **Counting one-to-one Functions**

How many one-to-one functions are there from a set with m elements to another set with n elements?

Solution: First note that there are no one-to-one functions when $m > n$.
Let $m \leq n$. Let the elements of the domain be $a_1, a_2, \ldots, a_m$. There are n ways to choose the value of the function at a_1. Since the function is one-to-one, the value of the function at a_2 can be chosen in $n - 1$ ways. In general, the value of the function at a_k, having chosen the values of $a_1, a_2, \ldots, a_{k-1}$, can be chosen in $n - (k - 1) = n - k + 1$ ways. By the product rule, there are

$$n(n - 1)(n - 2) \ldots (n - m + 1)$$

one-to-one functions from a set with m elements to another set with n elements.

Note: The product rule is often phrased in terms of sets as given below:
If $A_1, A_2, \ldots, A_m$ are finite sets, then the number of elements in the Cartesian product of these sets is the product of the number of elements in each set. To relate this to the product rule, note that the task of choosing an element in the

Cartesian product $A_1 \times A_2 \times \ldots \times A_m$ is done by choosing an element in A_1, an element in A_2 , …, and an element in A_m. By the product rule,

$$|A_1 \times A_2 \times \ldots \times A_m| = |A_1| \cdot |A_2| \cdot \ldots \cdot |A_m|$$

We now introduce the sum rule.

Sum Rule: If a task can be done in either in one of n_1 ways or in one of n_2 ways, where none of the set of n_1 ways is the same as any of the set of n_2 ways, then there are $n_1 + n_2$ ways to do the task.

The following is the extended version of the sum rule:
Suppose that a task can be done in one of n_1 ways, in one of n_2 ways,…,or in one of n_m ways, where none of the set of n_i ways of doing the task is the same as any of the set of n_j ways, for all i and j with $1 \leq i < j \leq m$. Then the number of ways to do the task is

$$n_1 + n_2 + \cdots + n_m.$$

Example 3: A student can choose a computer project from one of three lists. The three lists contain 23,15 and 19 possible projects, respectively. No project is on more than one list. How many possible projects are there to choose from?

Solution: The student can choose a project by selecting from the first list, the second list, or the third list. Because no project is on more than one list, by sum rule there are 23+15+19=57 ways to choose a project.

Note: The sum rule is often phrased in terms of sets.
If $A_1, A_2, \ldots, A_m$ are pairwise disjoint finite sets, then the number of elements in the union of the sets is the sum of the number of elements in the sets. To relate this to the sum rule, note that there are $|A_i|$ ways to choose an element from A_i , $i = 1,2, \ldots, m$. Because the sets are disjoint, when we select an element from one of the sets A_i , we do not also select an element from a different set A_j. By sum rule (because we cannot select an element from two of these sets at the same time) the number of ways to choose an element from one of the sets, which is the number of elements in the union, is

$$|A_1 \cup A_2 \cup \ldots \cup A_m| = |A_1| + |A_2| + \cdots + |A_m|.$$

Many counting problems can be solved using both of the above rules in combination.

Example 4: In a version of the computer language *BASIC*, the name of a variable is a string of one or two alphanumeric characters, where uppercase and lowercase letters are not distinguished. Moreover, a variable name must begin with a letter and must be different from five strings of two characters that are reserves for programming use. How many different variable names are there in this version of BASIC?

(An alphanumeric character is either one of 26 English letters or one of 10 digits)

Solution: Let v be the number of variable names in this version of BASIC. Let v_1 and v_2 be number of variable names of one character long and two characters long respectively. By the sum rule, $v = v_1 + v_2$.

Note that $v_1 = 26$, because a one character variable name must be a letter.

Further, by the product rule there are $26 \cdot 36$ strings of length two that begin with a letter and end with an alphanumeric character. However, five of these are excluded, so $v_2 = 26 \cdot 36 - 5 = 931$.

Therefore, there are $v = v_1 + v_2 = 26 + 931 = 957$ different names of variables in this version of BASIC.

The Inclusion-Exclusion Principle

Suppose that a task can be done in n_1 or n_2 ways, but that some of the set of n_1 ways to do the task are the same as some of n_2 ways to do the task. In this situation, we cannot use the sum rule to count the number of ways to do the task. Adding the number of ways to do the tasks in these two ways leads to an overcount, because the ways to do the task in the two ways that are common are counted twice. To correctly count the number of ways to do the task, we add the number of ways to do it in one way and the number of ways to do it in the other way, and then subtract the number ways to do the task in both among the set of n_1 ways and the set of n_2 ways. This technique is called the ***principle of inclusion -exclusion*** or ***subtraction principle for counting***.

We can phrase this principle in terms of sets.
Let A and B be finite sets. There are $|A|$ and $|B|$ ways to select an element from A and B respectively. The number of ways to select an element from $A \cup B$ is the sum of the number of ways to select an element from A and B, minus the number of ways to select an element from $A \cap B$. That is

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Example 5: How many bit strings of length eight either start with a 1 bit or end with the two bits 00?

Solution: Let A and B be the set of bit strings of length eight that start with 1 and end with 00 respectively. Then $A \cap B$ consists of all bit strings that start with 1 and end with 00. Required to find $|A \cup B|$.

Note that, we can construct a bit string of length eight that starts with 1 in $2^7 = 128$ ways. This follows by the product rule, because the first bit can be chosen in one way and each of the other seven bits can be choosen in two ways. Thus, $|A| = 128$.

We can construct a bit string of length eight that ends with 00 in $2^6 = 64$ ways. This follows by the product rule, because each of the first six bits can be chosen in two ways and the last two bits in only one way. Thus, $|B| = 64$.

We can construct a bit string of length eight that begins with 1 and ends with 00 in $2^5 = 32$ ways. This follows by the product rule, because the first bit, the last two bits can be chosen in only one way and each of the five bits in between first bit and last two bits can be chosen in two ways. Thus, $|A \cap B| = 32$.

By the principle of inclusion- exclusion

$$|A \cup B| = |A| + |B| - |A \cap B| = 128 + 64 - 32 = 160$$

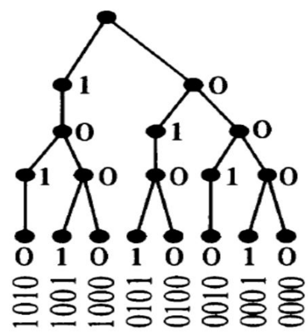
Thus, the number of bit strings of length eight that begin with 1 or that end with 00 is 160.

Tree Diagrams

Counting problems can be solved using ***tree diagrams***. A tree consists of a root, a number of branches leaving the root, and possible additional branches leaving the end points of other branches. To use trees in counting, we use a branch to represent each possible choice. We represent the possible outcomes by the leaves.

Example 6: How many bit strings of length four do not have two consecutive 1s?

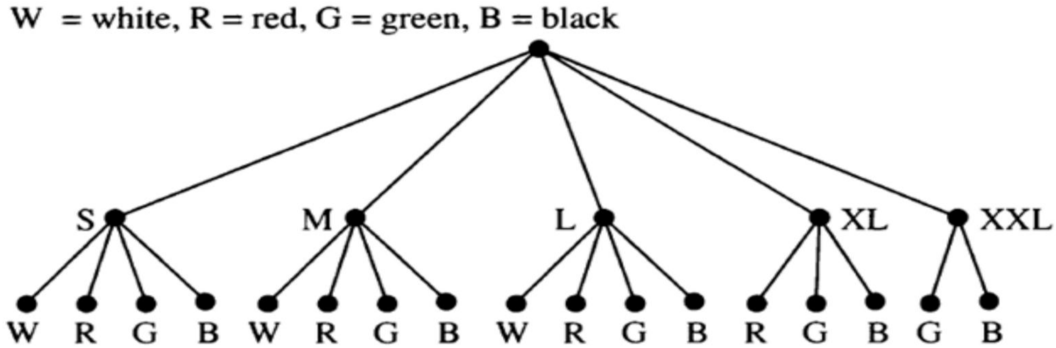
Solution:



The tree diagram displays all bit strings of length four without two consecutive 1s. There are eight bit strings of length four without two consecutive 1s.

Example 7: Suppose that **I love India** T-shirts come in five different sizes: S, M, L, XL and XXL. Suppose that each size comes in four colors, white, red, green and black, except for XL which comes only in red, green and black, and XXL which comes only in green and black. How many different T shirts does a souvenir shop have to stock to have at least one of each available size and color of the T-shirt?

Solution:



The tree diagram displays all possible size and color pairs. The shop owner need to stock 17 different T-shirts.

Example 8: How many positive integers not exceeding 1000 are divisible by 7 or 11

Solution: Let A be the set of positive integers not exceeding 1000 that are divisible by 7 and let B be the set of positive integers not exceeding 1000 that are divisible by 11. Then $A \cup B$ is the set of positive integers not exceeding 1000 that are divisible by 7 or 11, and $A \cap B$ is the set that are divisible by 7 and 11. Then

$$|A| = \left\lfloor \frac{1000}{7} \right\rfloor = 142 \qquad ; \qquad |B| = \left\lfloor \frac{1000}{7} \right\rfloor = 90$$

Note that the positive integers divisible by 7 and 11 are divisible by $7 \cdot 11$, because 7 and 11 are relatively prime. Therefore,

$$|A \cap B| = \left\lfloor \frac{1000}{7 \cdot 11} \right\rfloor = 12$$

By the principle of inclusion-exclusion

$$|A \cup B| = |A| + |B| - |A \cap B| = 142 + 90 - 12 = 220$$

There are 220 positive integers not exceeding 1000 that are divisible by 7 or 11.

Note: The number of positive integers not exceeding 1000 that are not divisible 7 and not divisible 11 are $1000 - 220 = 780$.

The principle of inclusion-exclusion for three finite sets is given below:

Theorem 1: If A, B and C are three finite sets, then

$$|\boldsymbol{A} \cup \boldsymbol{B} \cup \boldsymbol{C}| = |\boldsymbol{A}| + |\boldsymbol{B}| + |\boldsymbol{C}| - |\boldsymbol{A} \cap \boldsymbol{B}| - |\boldsymbol{A} \cap \boldsymbol{C}| - |\boldsymbol{B} \cap \boldsymbol{C}| + |\boldsymbol{A} \cap \boldsymbol{B} \cap \boldsymbol{C}|$$

Proof: $|A \cup B \cup C| = |A \cup (B \cup C)| = |A| + |B \cup C| - |A \cap (B \cup C)|$

$$= |A| + |B| + |C| - |B \cap C| - |(A \cap B) \cup (A \cap C)|$$

(By the principle of inclusion- exclusion and distributive law)

Now, $|(A \cap B) \cup (A \cap C)| = |A \cap B| + |A \cap C| - |(A \cap B) \cap (A \cap C)|$

(By the principle of inclusion- exclusion)

$$= |A \cap B| + |A \cap C| - |A \cap B \cap C|$$

Thus,

$$|A \cup B \cup C| = |A| + |B| + |C| - |B \cap C| - \{|A \cap B| + |A \cap C| - |A \cap B \cap C|\}$$

$$= |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

Hence the result.

Example 9: A total of 1232 students who have taken a course in Spanish, 879 have taken a course in French, and 114 have taken a course in Russian. Further, 103 have taken courses in both Spanish and French, 23 have taken courses in both Spanish and Russian, and 14 have taken courses in both French and Russian. If 2092 students have taken at least one of Spanish, French and Russian, then how many students have taken a course in all three languages?

Solution: Let S, F and R be the sets of students who have taken Spanish, French and Russian. Then, $|S| = 1232$, $|F| = 879$, $|R| = 114$. Further,

$|S \cap F| = 103$, $|S \cap R| = 23$, $|F \cap R| = 14$ and $|S \cup F \cup R| = 2092$.

By the principle of inclusion- exclusion,

$$|S \cup F \cup R| = |S| + |F| + |R| - |S \cap F| - |S \cap R| - |F \cap R| + |S \cap F \cap R|$$

i.e., $2092 = 1232 + 879 + 114 - 103 - 23 - 14 + |S \cap F \cap R|$

and so $|S \cap F \cap R| = 2092 - 2085 = 7$.

Therefore, there are seven students who have taken courses in all the three languages.

Theorem 2: The principle of inclusion-exclusion for n finite sets

If $A_1, A_2, \dots, A_n$ be finite sets then

$$
\begin{aligned}
&|\boldsymbol{A_1} \cup \boldsymbol{A_2} \cup \dots \cup \boldsymbol{A_n}| \\
&\qquad = \sum_{1 \le i \le n} |\boldsymbol{A_i}| - \sum_{1 \le i < j \le n} |\boldsymbol{A_i} \cap \boldsymbol{A_j}| + \sum_{1 \le i < j < k \le n} |\boldsymbol{A_i} \cap \boldsymbol{A_j} \cap \boldsymbol{A_k}| - \cdots \\
&\qquad + (-\mathbf{1})^{\boldsymbol{n}+\mathbf{1}} |\boldsymbol{A_1} \cap \boldsymbol{A_2} \cap \dots \cap \boldsymbol{A_n}|
\end{aligned}
$$

(The result follows by mathematical induction)

An alternate form of the principle of inclusion - exclusion

This form can be used to solve problems that ask for the number of elements in a set A that have none of n properties $P_1, P_2, \dots, P_n$.

Let A_i be the subset of A containing the elements that have the property P_i, $1 \le i \le n$.

The number of elements of A with all the properties $P_{i_1}, P_{i_2}, \dots, P_{i_k}$ is denoted by $N(P_{i_1} P_{i_2} \dots P_{i_k})$. That is,

$$
|A_{i_1} \cap A_{i_2} \cap \dots \cap A_{i_k}| = N(P_{i_1} P_{i_2} \dots P_{i_k})
$$

If the number of elements with none of the properties $P_1, P_2, \dots, P_n$ is denoted by $N(P_1' P_2' \dots P_n')$ and the number of elements in A by N then

$$
N(P_1' P_2' \dots P_n') = |A| - |A_1 \cup A_2 \cup \dots \cup A_n|
$$

$$= N - \left[\sum_{1 \le i \le n} |A_i| - \sum_{1 \le i < j \le n} |A_i \cap A_j| + \sum_{1 \le i < j < k \le n} |A_i \cap A_j \cap A_k| - \cdots + (-1)^{n+1} |A_1 \cap A_2 \cap \ldots \cap A_n| \right]$$

Thus,

$$\boldsymbol{N(P'_1, P'_2, P_3, \ldots, P'_4) = N - \sum_{1 \le i \le n} N(P_i) + \sum_{1 \le i < j \le n} N(P_i P_j) - \sum_{1 \le i < j < k \le n} N(P_i P_j P_k) + \cdots + (-1)^n N(P_1 P_2 \ldots P_n)}$$

Application 1: The principle of inclusion – exclusion can be used to find the *number of primes not exceeding a specified positive integer.*

Example 10: Find the number primes not exceeding 100.
Recall that a composite number is divisible by a prime not exceeding its square root. Therefore, to find the number of primes not exceeding 100, first note the composite integers not exceeding 100 must have a prime factor not exceeding $\sqrt{100} = 10$.

Because the only primes less than 10 are 2,3,5, and 7; the primes not exceeding 100 are: these four primes and the number of those primes greater 1 and not exceeding 100 that are divisible by none of 2,3,5,7. Now, we have to consider $A = \{2,3,5, \ldots, 100\}$, since 1 is neither prime nor composite and $N = |A| = 99$.

Let P_1 be the property that an integer is divisible by 2 let P_2 be the property that an integer is divisible by 3 ,let P_3 be the property that an integer is divisible by 5 , and let P_4 be the property that an integer is divisible by 7.

Thus, the number of primes not exceeding 100 is given by $4 + \ N(P_1{'}P_2{'}P_3{'}P_4{'})$.

By the principle of inclusion – exclusion

$$N(P_1{}'P_2{}'P_3{}'P_4{}') = N - [N(P_1) + N(P_2) + N(P_3) + N(P_4)] +$$
$$[N(P_1P_2) + N(P_1P_3) + N(P_1P_4) + N(P_2P_3) + N(P_2P_4) + N(P_1P_4)] +$$
$$[N(P_1P_2P_3) + N(P_1P_3P_4) + N(P_2P_3P_4)] + N(P_1P_2P_3P_4)$$

$$= 99 - \left[\left\lfloor\frac{100}{2}\right\rfloor + \left\lfloor\frac{100}{3}\right\rfloor + \left\lfloor\frac{100}{5}\right\rfloor + \left\lfloor\frac{100}{7}\right\rfloor\right] + \left\lfloor\frac{100}{2\cdot3}\right\rfloor + \left\lfloor\frac{100}{2\cdot5}\right\rfloor + \left\lfloor\frac{100}{2\cdot7}\right\rfloor + \left\lfloor\frac{100}{3\cdot5}\right\rfloor + \left\lfloor\frac{100}{3\cdot7}\right\rfloor +$$
$$\left\lfloor\frac{100}{5\cdot7}\right\rfloor - \left[\left\lfloor\frac{100}{2\cdot3\cdot5}\right\rfloor + \left\lfloor\frac{100}{2\cdot3\cdot7}\right\rfloor + \left\lfloor\frac{100}{2\cdot5\cdot7}\right\rfloor + \left\lfloor\frac{100}{3\cdot5\cdot7}\right\rfloor\right] + \left\lfloor\frac{100}{2\cdot3\cdot5\cdot7}\right\rfloor$$

$$= 99 - 50 - 33 - 20 - 14 + 16 + 10 + 7 + 6 + 4 + 2 - 3 - 2 - 1 - 0 + 0$$

$$= 21$$

Thus, there are $4 + 21 = 25$ primes not exceeding 100.

Note: The ***Sieve of Eratosthenes*** is used to find all primes not exceeding a specified positive integer.

Application 2: The principle of inclusion-exclusion can also be used to determine the *number of onto functions from a set with m elements to a set with n elements*.

We first consider the following example:

Example 11: How many onto functions are there from a set A with six elements to a set B with four elements.

Solution: Let $B = \{b_1, b_2, b_3, b_4\}$. Let F be the set of all function from A to B. Clearly, $|F| = |B|^{|A|} = 4^6$.

Let P_1, P_2, P_3 and P_4 be the properties that b_1, b_2, b_3 and b_4 are not in the range of the function, respectively. Now, a function is onto iff it has none of the properties P_1, P_2, P_3 or P_4. The number of onto functions is given by $N(P_1{}'P_2{}'P_3{}'P_4{}')$ and by the principle of inclusion-exclusion.

$$
\begin{aligned}
&N(P_1{}'P_2{}'P_3{}'P_4{}') \\
&\qquad = |F| - [N(P_1) + N(P_2) + N(P_3) + N(P_4)] \\
&\qquad + [N(P_1P_2) + N(P_1P_3) + N(P_1P_4) + N(P_2P_3) + N(P_2P_4) \\
&\qquad + N(P_3P_4)] - [N(P_1P_2P_3) + N(P_1P_2P_4) + N(P_1P_3P_4) + N(P_2P_3P_4)] \\
&\qquad - N(P_1P_2P_3P_4)
\end{aligned}
$$

Note that $N(P_i)$ is the number of functions that do not have b_i in their range, for $i = 1,2,3,4$.

Therefore, $N(P_i) = 3^6$ for all $i = 1,2,3,4$ and there are ${}^4C_1$ terms of this kind.

Further, $N(P_iP_j) = 2^6$ for $1 \leq i < j \leq 4$ and there are ${}^4C_2$ terms of this kind.

Similarly, $N(P_iP_jP_k) = 1^6$ for $1 \leq i < j < k \leq 4$ and there are ${}^4C_3$ terms of this kind and $N(P_1P_2P_3P_4) = 0$ because this term is the number of functions that have none of b_1, b_2, b_3, b_4 in their range. Clearly there are no such functions. Therefore, the number of onto functions from a set A with six elements to a set B with four elements is given by

$$
\begin{aligned}
N(P_1{}'P_2{}'P_3{}'P_4{}') &= 4^6 - {}^4C_1 \cdot 3^6 + {}^4C_2 \cdot 2^6 - {}^4C_3 . 1^6 + {}^4C_4 \cdot 0 \\
&= 4096 - 4 \cdot 729 + 6 \cdot 64 - 4 \\
&= 4096 - 2916 + 384 - 4 = 1560
\end{aligned}
$$

The following is the general result related to the number of onto functions from a set with m elements to the set with n elements, where $m \geq n$.

Theorem 3: Let m and n be positive integers with $m \geq n$.The number of onto functions from a set with m elements to a set with n elements is

$$\boldsymbol{n^m - {}^nC_1 (n-1)^m + {}^nC_2 (n-2)^m - \cdots + (-1)^{n-1}\, {}^nC_{n-1} 1^m}$$

Example 12: How many ways are there to assign five different jobs to four different employees if every employee is assigned at least one job?

Solution: Consider the assignment of jobs as a function from the set of five jobs to the set of four employees. An assignment where every employee gets at least one job is the same as an onto function from the set of jobs to the set of employees. *By Theorem* 3 there are

$$4^5 - {}^4C_1 . 3^5 + {}^4C_2 . 2^5 - {}^4C_3 . 1^5 = 1024 - 972 + 192 - 4 = 240$$

ways to assign the jobs so that each employee is assigned at least one job.

Application 3: The principle of inclusion-exclusion can be used to *count the permutations of* n *objects that leave no objects in their original positions*.

Derangements

A ***derangement*** is a permutation of objects that leaves no object in its original position.

The permutation $2\ 1\ 4\ 5\ 3$ is a derangement of $1\ 2\ 3\ 4\ 5$, because no number is left in its original position. The permutation $2\ 1\ 5\ 4\ 3$ is not a derangement of $1\ 2\ 3\ 4\ 5$, because this permutation leaves 4 fixed.
Let $\boldsymbol{D_n}$ be the number of derangement of n objects.

For example, $D_3 = 2$, because the derangements of $1\ 2\ 3$ are $2\ 3\ 1$ and $3\ 1\ 2$.

We will now derive a formula for D_n using the principle of inclusion-exclusion.

Theorem 4: The number of derangements of a set with $\boldsymbol{n}$ elements is

$$\boldsymbol{D_n = n!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!}\right]}$$

Proof: Let P be the set of all permutations of the set with n elements.Clearly $|P| = n!$. Let P_i be the property of fixing an element i in a permutation of P, $i = 1,2, \dots, n$. The number of derangements is the number of permutations of P having none of the properties P_i , for $i = 1,2, \dots, n$. That is

$$D_n = N(P_1{}'P_2{}' \dots P_n{}')$$

By the principle of inclusion-exclusion

$$
\begin{aligned}
D_n &= N(P_1{}'P_2{}' \dots P_n{}') \\
&= |P| - \sum_{1\le i\le n} N(P_i) + \sum_{1\le i<j\le n} N(P_iP_j) - \sum_{1\le i<j<k\le n} N(P_iP_jP_k) + \cdots \\
&+ (-1)^n N(P_1P_2 \dots P_n)
\end{aligned}
$$

Note that $N(P_i)$ is the number of permutations that fix the element i. If the element i is left in its original position, the remaining $n-1$ positions can be filled is $(n-1)!$ ways. Therefore, $N(P_i) = (n-1)!$. Similarly $N(P_iP_j) = (n-2)!$. In general $N(P_{i_1}P_{i_2} \dots P_{i_m}) = (n-m)!$.

Because there are ${}^nC_m$ ways to choose m elements from n elements,

$$\sum_{1\le i\le n} N(P_i) = {}^nC_1(n-1)!$$

$$\sum_{1\le i<j\le n} N(P_iP_j) = {}^nC_2(n-2)!$$

In general $(P_{i_1}P_{i_2} \dots P_{i_m}) = {}^nC_m(n-m)!$. Thus,

$$
\begin{aligned}
D_n &= n! - {}^nC_1.(n-1)! + {}^nC_2.(n-2)! - \cdots + (-1)^n\,{}^nC_n.0! \\
&= n! - \frac{n!}{1!(n-1)!}(n-1)! + \frac{n!}{2!(n-2)!}(n-2)! - \cdots + (-1)^n \frac{n!}{n!0!}0! \\
&= n!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \cdots + (-1)^n \frac{1}{n!}\right]
\end{aligned}
$$

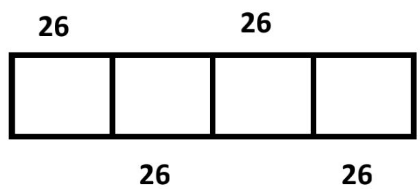
P1:

How many strings are there of lowercase English letters of length four or less?

Solution:

Number of strings of length 4

There are 26 choices for each of four places



By product rule there are $26 \times 26 \times 26 \times 26 = 4{,}56{,}976$ strings of length 4.

Similarly, there are

$$26 \times 26 \times 26 = 17{,}576 \text{ strings of length three}$$

$$26 \times 26 = 676 \text{ strings of length two}$$

$$26 \text{ strings of length one}$$

Note that there is one string of length zero called empty string:

It is customary to take this empty string in the counting when we want to find the number of strings of length n or less.

The number of strings of length four or less of lower case English letters
$= 4{,}56{,}976 + 17{,}576 + 6{,}756 + 26 + 1 = 4{,}75{,}255.$

P2:

Each user on a computer system has a password, which is six to eight character long, where each character is an uppercase English letter (A – Z) or a digit (0 – 9). Each password must contain at least one digit. How many possible passwords are there?

Solution:

Let P be the total number of possible passwords, and let P_6, P_7 and P_8 denote the number of passwords of length 6, 7, and 8 respectively. By the sum rule

$$P = P_6 + P_7 + P_8$$

To find P_6 , find the number of strings x of uppercase letters and digits that are six characters long and subtract from this, the number of strings y with no digits. By the product rule we see $x = 36^6$ and $y = 26^6$. Thus, $P_6 = 36^6 - 26^6$.

Similarly, $P_7 = 36^7 - 26^7$ and $P_8 = 36^8 - 26^8$.

Therefore $P = P_6 + P_7 + P_8 = 36^6 + 36^7 + 36^8 - (26^6 + 26^7 + 26^8)$

$$= 26{,}84{,}48{,}30{,}63{,}360$$

P3

How many positive integers between $\mathbf{100}$ and $\mathbf{999}$ inclusive

a. **are divisible by $\mathbf{3}$ or $\mathbf{4}$**
b. **are not divisible by either $\mathbf{3}$ or $\mathbf{4}$**
c. **are divisible by exactly one of $\mathbf{3}$ and $\mathbf{4}$**

Solution:

Let $U = \{100,101, \dots ,999\}$. Then $|U| = 900$.

Let A, B be the set of positive integers between 100 and 999 (both inclusive), which are divisible by 3, 4 respectively.

$$|A| = 300\,, |B| = 225$$

The set $A \cap B$ is the set of positive integers divisible by 3 and 4. Since 3 and 4 are relatively prime.

$$|A \cap B| = 75$$

a. The set $A \cup B$ is the set of positive integers divisible by 3 or 4.
   By the principle of inclusion- exclusion
   $$|A \cup B| = |A| + |B| - |A \cap B| = 300 + 225 - 75 = 450$$

b. The set of positive integers which are not divisible by either 3 or 4 is $(A \cup B)'$.
   We have
   $$(A \cup B)' = |U| - |A \cup B| = 900 - 450 = 450$$

c. The set of positive integers which are divisible by exactly one of 3 and 4 is given by
   $$(A - B) \cup (B - A)\ i.e., A \oplus B$$

   Now $|A \oplus B| = |A| + |B| - 2|A \cap B|$

   $$= 300 + 225 - 2 \cdot 75 = 375$$

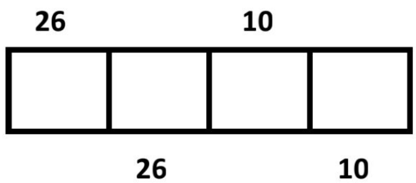
P4:

How many license plates can be made using either two or three letters followed by either two or three digits?

Solution:

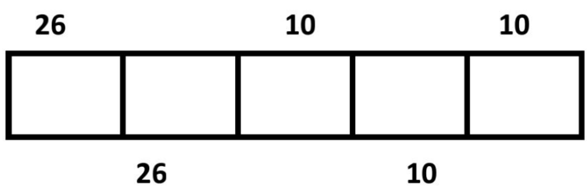
The following are four types of license plates:

(i) T_1: Two letters followed by two digits



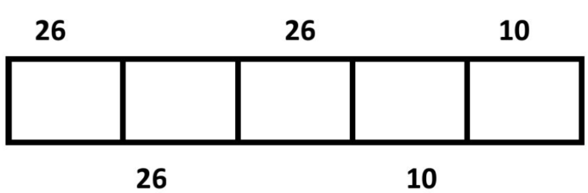
By product rule, $|T_1| = 26 \times 26 \times 10 \times 10 = 67600$

(ii) T_2: Two letters followed by three digits



By product rule, $|T_2| = 26 \times 26 \times 10 \times 10 \times 10 = 676000$

(iii) T_3: Three letters followed by two digits



By product rule, $|T_3| = 26 \times 26 \times 26 \times 10 \times 10 = 1757600$

(iv) T_4: Three letters followed by three digits

26	26	26	10	10	10

By product rule, $|T_4| = 26 \times 26 \times 26 \times 10 \times 10 \times 10 = 17576000$

The required number of license plates is

$|T_1 \cup T_2 \cup T_3 \cup T_4| = |T_1| + |T_2| + |T_3| + |T_4|$ (By sum rule)

Note that $T_i \cap T_j = \phi\ , i \neq j$

$= 57600 + 576000 + 1757600 + 17576000$

$= 2{,}00{,}77{,}200$

P5:

How many functions are there from the set $\{1, 2, 3, \ldots, n\}$, where n is a positive integer, to the set $\{0, 1\}$

a. **that are one-to-one**
b. **that assign 0 to both 1 and n**
c. **that assign 1 to exactly one of the positive integers less than n**

Solution:

The number of one-to-one function from a set with m elements to a set with n-elements is

$$n(n-1)(n-2)\ldots(n-m+1) \text{ if } m \leq n$$

$$0 \qquad\qquad \text{if } m > n$$

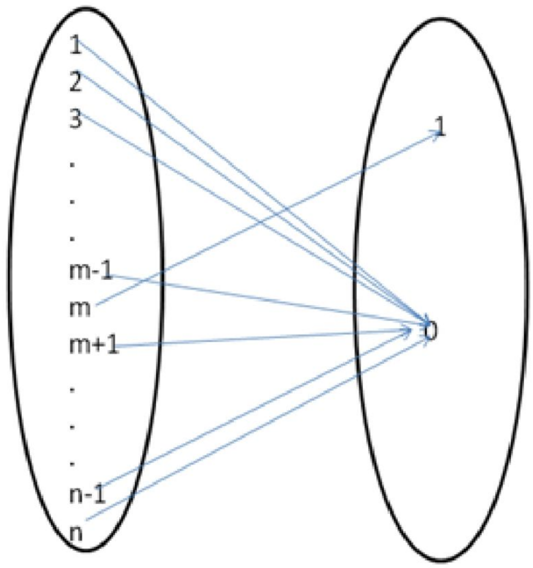
(a) The number of one-to-one function from the set $\{1,2,\ldots,n\}$ to a set $\{0,1\}$ is

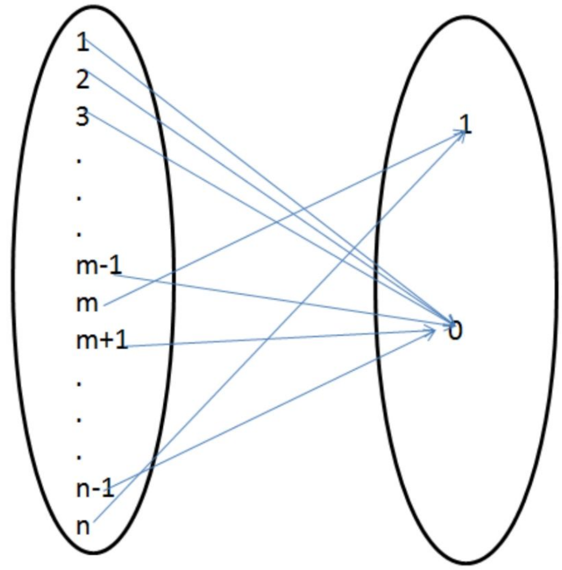
$$0 \qquad\qquad \text{if } n > 2, i.e., n \geq 3$$

$$2 \qquad\qquad \text{if } n = 1,2$$

(b) The number of functions that map 1 and n to 0 is equal to the number functions from the set $\{2,3,4,\ldots,n-1\}$ to the set $\{0,1\}$ is 2^{n-2} if $n > 1$. If $n = 1$ then the number functions that map 1 to 0 is 1.

(c) Notice that there are $n-1$ positive integers less n in $\{1,2,3,4,\ldots,n\}$. Suppose that we select one of positive integer less than n, say m. If only m is mapped to 1 and no others less than n then $1,2,\ldots,m-1,m+1,\ldots n-1$ must be mapped to 0 and n can be mapped either to 0 or 1.





Thus, we have two such functions. Since m can chosen in $n-1$ ways, the number of such functions is $2(n-1)$, (by product rule).

P6:

Palindrome: A **palindrome** is a string whose reversal is identical to the string**.**

Example: How many bit strings of length n are palindromes.

Solution:

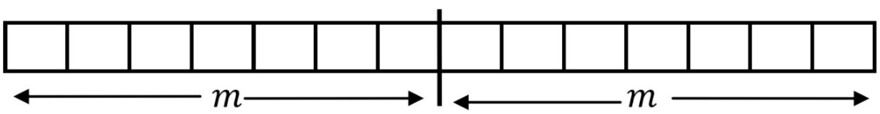
A palindrome is a string whose reversal is identical to the string.

For example, the bit string 0110110 of length 7 is a palindrome. The bit string 101101 of length 6 is a palindrome

Consider bit strings of length n.

We have two cases (i) n is even (ii) n is odd.

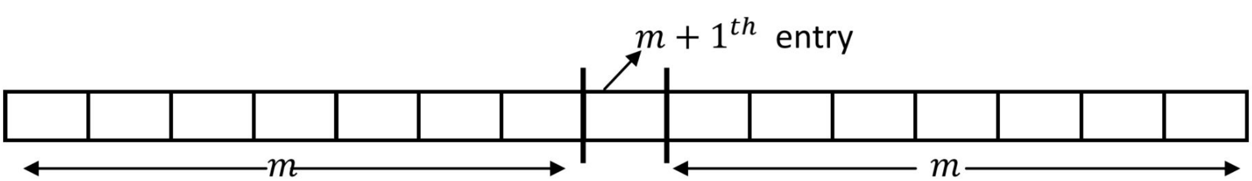
Case (i) Suppose that n is even say $n = 2m$, for some positive integer m.



$$n = 2m$$

A string of length $n = 2m$ can be seen as two strings of length m arranged side by side. If a (bit) string of length m in the left part is written in reverse order in the right part then we get a palindrome of length $n = 2m$. Thus, the number of palindromes of length $n = 2m$ is equal to the number of bit strings of length m, $i.e.$, 2^m that $2^{\frac{n}{2}}$,where n is even.

Case (ii) Suppose that n is odd, say $n = 2m + 1$, for some positive integer m.



A string of length $n = 2m + 1$ can be seen as two strings of length m arranged on either side of $(m + 1)^{\text{th}}$ entry, if the string of length m on the left side of

$(m+1)^{\text{th}}$ entry is written in the reverse order on the right side of the $(m+1)^{\text{th}}$ entry then we get a palindrome of length $n = 2m+1$. Thus,

The palindromes of length $n = 2m+1$

$$= 2 \times (\text{the number of bit strings of length } m)$$

$$= 2 \times 2^m = 2^{m+1} = 2^{\frac{n-1}{2}+1} = 2^{\frac{n+1}{2}}$$

Note: In the case of bit strings, the alphabet is $\{0,1\}$. If the alphabet is of order k then the number of palindromes of length n is

$k^{\frac{n}{2}}$ if n is even

$k^{\frac{n+1}{2}}$ if n is odd

P7:

How many positive integers not exceeding 100 are divisible either by 4 or by 6?

Solution:

Let A be the set of positive integers not exceeding 100 which are divisible by 4.

Let B be the set of positive integers not exceeding 100 which are divisible by 6.

Then

$$|A| = \left\lfloor \frac{100}{4} \right\rfloor = 25 \quad ; \qquad |B| = \left\lfloor \frac{100}{6} \right\rfloor = 16$$

The set of positive integers not exceeding 100 divisible by 4 and 6 is $A \cap B$. Then

$$|A \cap B| = \left\lfloor \frac{100}{lcm\{4,6\}} \right\rfloor = \left\lfloor \frac{100}{12} \right\rfloor = 8$$

By the principle of inclusion-exclusion, we have

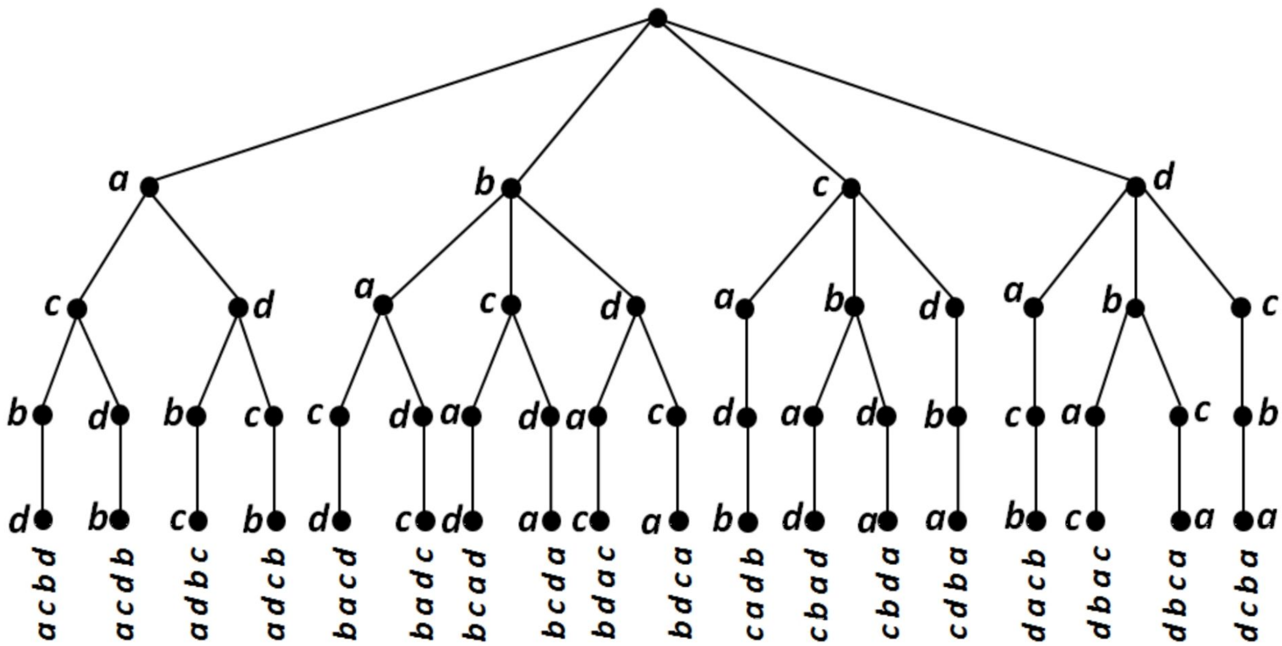
$$|A \cup B| = |A| + |B| - |A \cap B| = 25 + 16 - 8 = 33$$

Thus, the number of positive integers not exceeding 100 are divisible either by 4 or by 6 is 33.

P8:

Use a tree diagram to fine the number of ways to arrange the letters $\boldsymbol{a, b, c}$ and $\boldsymbol{d}$ such that $\boldsymbol{a}$ is not followed immediately by $\boldsymbol{b}$.

Solution:



The tree diagram displays arrangements of the letters a, b, c and d such that a is not followed immediately by b. There are 18 such arrangements.

4.1. Counting Principles.

Exercise:

1. How many different three-letters initials can people have?
2. How many different three-letters initials with none of the letters repeated can people have?
3. How many different three-letter initials are there that begin with an A?
4. How many bit stings are there of length eight?
5. How many bit strings of length ten both begin and end with a 1?
6. How many bit strings are there of length six or less?
7. How many bit strings with length not exceeding$n$, where n is a positive integer, consist entirely of $1's$?
8. How many strings are there of lowercase letters of length four or less?
9. How many strings are there of four lowercase letters that have the letter x in them?
10. How many positive integers less than 1000
    a. are divisible by 7?
    b. are divisible by 7 but not by 11?
    c. are divisible by both 7 and 11?
    d. are divisible by either 7 or 11?
    e. are divisible by exactly one of 7 and 11?
    f. are divisible by neither 7 nor 11?
    g. have distinct digits?
    h. have distinct digits and are even?

11. How many positive integers between 1000 and 9999 inclusive
    a. are divisible by 9?
    b. are even?
    c. have distinct digits?
    d. are not divisible by 3?
    e. are divisible by 5 or 7?
    f. are not divisible by either 5 or 7?

g. are divisible by 5 but not by 7?
h. are divisible by 5 and 7?

12. How many license plates can be made using either three digits followed by three letters or three letters followed by three digits?
13. How many license plates can be made using either two letters followed by four digits or two digits followed by four letters?
14. How many license plates can be made using either three letters followed by three digits or four letters followed by two digits?
15. How many strings of eight English letters are there
    a. that contain no vowels, if letters can be repeated?
    b. that contain no vowels, if letters cannot be repeated?
    c. that start with a vowel, if letters can be repeated?
    d. that start with a vowel, if letters cannot be repeated?
    e. that contain at least one vowel, if letters can be repeated?
    f. that contain exactly one vowel, if letters can be repeated?
    g. that start with X and contain at least one vowel, if letters can be repeated?
    h. that start and end with X and contain at least one vowel, if letters can be repeated?
16. How many different functions are there from a set with 10 elements to sets with the following number of elements?
    a. 2 $b.3$ $c.4$ $d.5$
17. How many one-to-one functions are there from a set with five elements to sets with the following number of elements?
    $a.4$ $b.5$ $c.6$ $d.7$
18. How many functions are there from the set $\{1,2,3,\dots,n\}$, where n is a positive integer, to the set $\{0,1\}$?
19. How many bit strings of length seven either begin with two $0's$ or end with three$1's$?
20. How many bit strings of length 10 either begin three $0's$ or end with two $0's$?
21. Use a tree diagram to find the number of bit strings of length four with no three consecutive $0's$.

22. How many onto functions are there from a set with seven elements to one with five elements?
23. How many ways are there to distribute six different toys to three different children such that each child gets at least one toy?
24. In how many ways can eight distinct balls be distributed into three distinct urns if each urn must contain at least one ball?
25. List all the derangements of $\{1,2,3,4\}$.
26. How many derangements are there of a set with seven elements?
27. How many derangements of $\{1,2,3,4,5,6\}$ end with the integers 1,2 and 3 in some order?