2.4

Length of Arc of a Curve (Rectification)

In this section, we consider the process of determining the length of arc of a curve between two specified points on it.

The only length we know is how to compute the length of line segment. If the endpoints of the line are (x_1, y_1) and (x_2, y_2) then the length of the line is given by the distance formula:

$$s = \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

We can break up a smooth curve over an interval $[a, b]$ into a number n of such segments, whose total length will be

$$s = \sum_{k=1}^{n} \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2} = \sum_{k=1}^{n} \sqrt{1 + \left(\frac{\Delta y_k}{\Delta x_k}\right)^2}\, \Delta x_k .$$

By increasing the number of segments, we obtain increasingly better approximations to the smooth curve, and hence our notion of arc length can be expressed by the limit:

$$s = \lim_{n \to \infty} \sum_{k=1}^{n} \sqrt{1 + \left(\frac{\Delta y_k}{\Delta x_k}\right)^2}\, \Delta x_k$$

as $n \to \infty$ and $\Delta x_k \to 0$, this limit becomes an integral expression giving us a definition for the arc length s of the curve over the integral $[a, b]$.

$$s = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

Alternatively, if our curve is expressed in the form $x = g(y)$ where g is a smooth function of y over the interval $[c, d]$ on the $Y-$axis, then the arc length of the curve over that interval becomes

$$s = \int_c^d \sqrt{1 + [g'(x)]^2}\, dy = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy.$$

So we have the following formulae.

Curves Given in Cartesian Form

Consider a curve $y = f(x)$. Let s denote the length of the arc of the curve between two points $x = a$ and $x = b$. Then, s is determined by using the following formula:

$$s = \int_a^b \left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{1}{2}} dx \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Remarks

1) If the equation of the curve is given in the form $x = f(y)$, then the formula (1) with x and y interchanged is employed to find the length of the arc of the curve between $y = a$ and $y = b$; that is

$$s = \int_a^b \left\{1 + \left(\frac{dx}{dy}\right)^2\right\}^{\frac{1}{2}} dy \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

2) If the equation of the curve is specified in the parametric form $x = x(t), y = y(t)$, then (1) yields the following formula for the length s of the arc of the curve between the points corresponding to $t = t_1$ and $t = t_2$, $t_1 < t_2$:

$$s = \int_{t_1}^{t_2} \left\{ \left(\frac{dx}{dt} \right)^2 + \left(\frac{dy}{dt} \right)^2 \right\}^{\frac{1}{2}} dt \ldots\ldots\ldots\ldots (3)$$

Curves Given in Polar Form

Next, consider a polar curve $r = f(\theta)$. The length s of the arc of the curve between two points corresponding to $\theta = \theta_1$ and $\theta = \theta_2$, $\theta_1 < \theta_2$, is determined by the formula:

$$s = \int_{\theta_1}^{\theta_2} \left\{ r^2 + \left(\frac{dr}{d\theta} \right)^2 \right\}^{\frac{1}{2}} d\theta \ldots\ldots\ldots\ldots (4)$$

Also, the length s of the arc of the curve between two points corresponding to $r = r_1$ and $r = r_2$, $r_1 < r_2$, is determined by the formula:

$$s = \int_{r_1}^{r_2} \left\{ 1 + r^2 \left(\frac{d\theta}{dr} \right)^2 \right\}^{\frac{1}{2}} dr \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (5)$$

Problem 1: Find the length of the arc of the curve $y^2 = x^3$ from the origin to the point (4,8).

Solution: From the given equation, we find that

$$2y\frac{dy}{dx} = 3x^2, \text{ which yields } \left(\frac{dy}{dx}\right)^2 = \frac{9}{4}\frac{x^4}{y^2} = \frac{9}{4}\frac{x^4}{x^3} = \frac{9}{4}x$$

For the arc of the given curve from the origin (0,0) to the point (4,8), x increases from 0 to 4. Therefore, by formula (1), this length of this arc is given by

$$s = \int_0^4 \left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{1}{2}} dx = \int_0^4 \left(1+\frac{9}{4}x\right)^{\frac{1}{2}} dx$$

Length of Arc of a curve

$$= \left[\frac{\left\{1+\left(\frac{9}{4}\right)x\right\}^{\frac{3}{2}}}{\left(\frac{9}{4}\right)\left(\frac{3}{2}\right)}\right]_0^4 = \frac{8}{27}\left[\left\{1+\left(\frac{9}{4}\right).4\right\}^{\frac{3}{2}} - 1\right]$$

$$= \frac{8}{27}\left(10^{\frac{3}{2}} - 1\right) = \frac{8}{27}\left(10\sqrt{10} - 1\right)$$

Problem 2: Find the length of the arc of the curve $y = \log\sec x$ between the points for which $x = 0$ and $x = \frac{\pi}{3}$.

Solution: From the given equation, we find that $\frac{dy}{dx} = \frac{1}{\sec x}(\sec x \tan x) = \tan x$.

Therefore, the required arc-length is

$$s = \int_0^{\frac{\pi}{3}} \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{1}{2}} dx = \int_0^{\frac{\pi}{3}} \left(1 + \tan^2 x\right)^{\frac{1}{2}} dx = \int_0^{\frac{\pi}{3}} \sec x dx$$

$$= \left[\log\left|\sec x + \tan x\right|\right]_0^{\frac{\pi}{3}} = \log\left(\sec\frac{\pi}{3} + \tan\frac{\pi}{3}\right) - \log\left(\sec 0 + \tan 0\right)$$

$$= \log\left(2 + \sqrt{3}\right) - \log\left(1 + 0\right) = \log\left(2 + \sqrt{3}\right)$$

Problem 3: Find the arc-length of the curve $y = \log\frac{e^x-1}{e^x+1}$ from the point corresponding to $x = 1$ to the point corresponding to $x = 2$.

Solution: From the given equation, we find that

$$\frac{dy}{dx} = \frac{1}{\left(e^x-1\right)/\left(e^x+1\right)}\cdot\frac{e^x(e^x+1)-(e^x-1)e^x}{(e^x+1)^2}$$

$$= \frac{e^x+1}{e^x-1}\cdot\frac{2e^x}{(e^x+1)^2} = \frac{2e^x}{\left(e^x-1\right)\left(e^x+1\right)} = \frac{2e^x}{e^{2x}-1}$$

That gives

$$1+\left(\frac{dy}{dx}\right)^2 = 1+\frac{4e^{2x}}{\left(e^{2x}-1\right)^2} = \frac{(e^{2x}-1)^2+4e^{2x}}{(e^{2x}-1)^2}$$

$$= \frac{e^{4x}-2e^{2x}+1+4e^{2x}}{(e^{2x}-1)^2} = \frac{e^{4x}+2e^{2x}+1}{(e^{2x}-1)^2} = \frac{(e^{2x}+1)^2}{(e^{2x}-1)^2}$$

$$= \frac{\left\{e^x\left(e^x+e^{-x}\right)\right\}^2}{\left\{e^x\left(e^x-e^{-x}\right)\right\}^2} = \frac{\left(e^x+e^{-x}\right)^2}{\left(e^x-e^{-x}\right)^2}$$

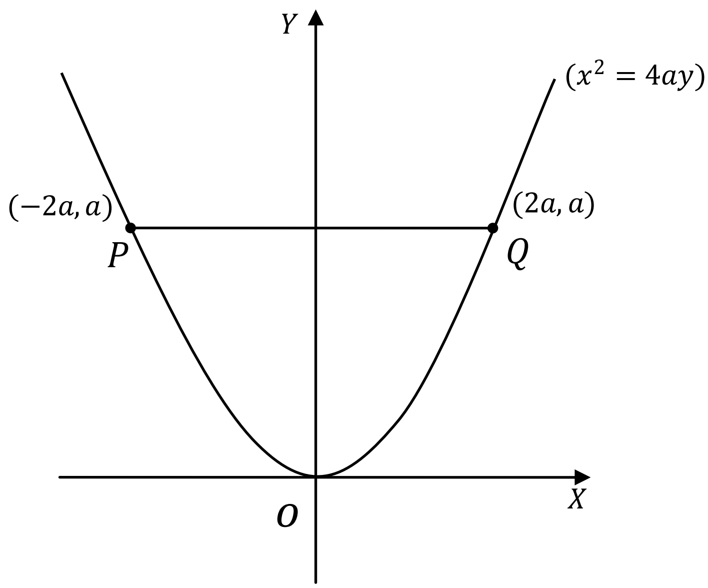
Now, we find that the required arc-length is given by

$$s = \int_1^2 \left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{1}{2}} dx = \int_1^2 \left(\frac{e^x+e^{-x}}{e^x-e^{-x}}\right) dx$$

$$= \left[\log\left(e^x-e^{-x}\right)\right]_1^2 = \log\left(e^2-e^{-2}\right)-\log(e-e^{-1})$$

$$= \log \frac{e^2 - e^{-2}}{e - e^{-1}} = \log \frac{\left(e + e^{-1}\right)\left(e - e^{-1}\right)}{\left(e - e^{-1}\right)} = \log\left(e + e^{-1}\right) = \log\left(e + \frac{1}{e}\right)$$

Problem 4: Find the arc-length of the parabola $x^2 = 4ay$ from its vertex to one extremity of the latus rectum.



The ends of the latus rectum of the given parabola are $(\pm 2a, a)$. See above figure. Therefore for the arc between the vertex (origin) $\boldsymbol{O}$ and one extremity Q of the latus rectum, x increases from 0 to $2a$. Because of symmetry of the curve about the y-axis, the arc-length between $\boldsymbol{O}$ and Q may be taken as the required arc-length.

From the given equation, we find that $4a\dfrac{dy}{dx} = 2x$, or $\dfrac{dy}{dx} = \dfrac{x}{2a}$.

Therefore, the required arc-length is

$$s = \int_0^{2a} \left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{1}{2}} dx = \int_0^{2a} \left\{1+\left(\frac{x}{2a}\right)^2\right\}^{\frac{1}{2}} dx$$

$$= \int_0^{\frac{\pi}{4}} \left(1+\tan^2\theta\right)^{\frac{1}{2}} \left(2a\sec^2\theta\right) d\theta$$, on setting $x = 2a tan\theta$

$$= 2a\int_0^{\frac{\pi}{4}} \sec^3\theta d\theta$$

..........................(1)

We note that

$$\int \sec^3\theta d\theta = \int \sec\theta\sec^2\theta d\theta = (\sec\theta)(\tan\theta) - \int (\sec\theta\tan\theta)\tan\theta d\theta$$

on integration by parts

$$= (\sec\theta)(\tan\theta) - \int \sec\theta\tan^2\theta d\theta$$

$$= (\sec\theta)(\tan\theta) - \int \sec\theta(\sec^2\theta - 1) d\theta$$

$$= (\sec\theta)(\tan\theta) - \int \sec^3\theta d\theta + \int \sec\theta d\theta$$

or

$$2\int \sec^3\theta d\theta = (\sec\theta)(\tan\theta) + \int \sec\theta d\theta = (\sec\theta)(\tan\theta) + \log(\sec\theta + \tan\theta)$$

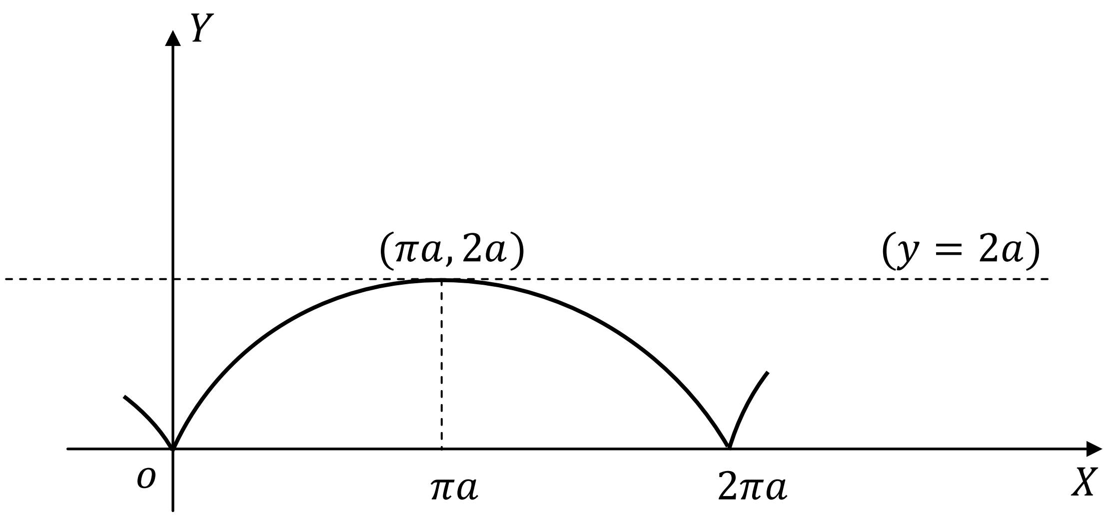
Therefore, expression (1) becomes

$$s = 2a\int_0^{\frac{\pi}{4}} \sec^3\theta = a\Big[(\sec\theta)(\tan\theta) + \log(\sec\theta + \tan\theta)\Big]_0^{\frac{\pi}{4}}$$

$$= a\left[\sqrt{2} + \log\left(\sqrt{2} + 1\right)\right]$$

Problem 5: Find the arc-length of an arc of the cycloid $x = a(t - \sin t), y = a(1 - \cos t)$.

Solution:



The cycloid $x = a(t - sint), y = a(1 - cost)$

The given curve is shown above figure. By using formula (3), we obtain the required arc-length as

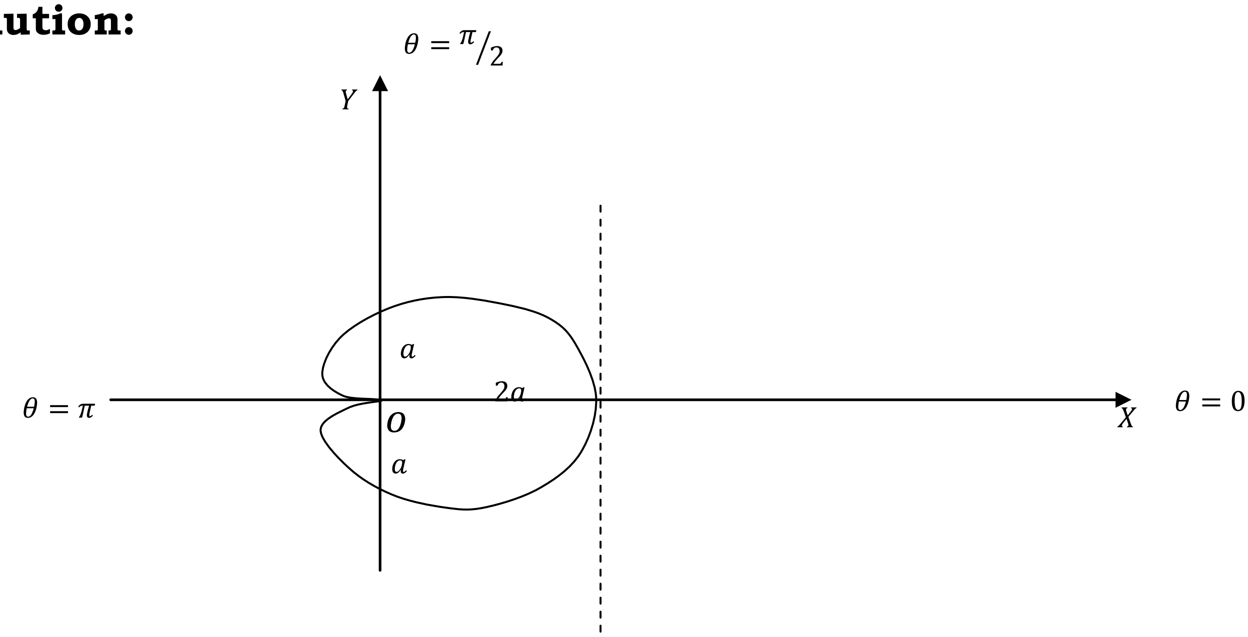
$$s = \int_0^{2\pi} \left\{ \left(\frac{dx}{dt} \right)^2 + \left(\frac{dy}{dt} \right)^2 \right\}^{\frac{1}{2}} dt = \int_0^{2\pi} a \left\{ (1 - \cos t)^2 + \sin^2 t \right\}^{\frac{1}{2}} dt$$

$$= a \int_0^{2\pi} \left(1 - 2\cos t + \cos^2 t + \sin^2 t \right)^{\frac{1}{2}} dt = a \int_0^{2\pi} \sqrt{2}\sqrt{1 - \cos t}\, dt$$

$$= 2a\int_0^{2\pi} \sin\left(\frac{t}{2}\right) dt = -4a\left[\cos\frac{t}{2}\right]_0^{2\pi} = -4a\left(\cos\pi - \cos 0\right) = 8a$$

Problem 6: Obtain the total length of the cardioid $r = a(1 + \cos\theta)$.

Solution:



The cardioid $r = a(1 + cos\theta)$

The given curve is shown in above figure. We note that the curve is symmetrical about the initial line $\theta = 0$, and for the upper half of the curve, θ varies from 0 to π; Therefore, by formula (4), the length of the upper half of the curve is

$$s = \int_0^{\pi} \left\{ r^2 + \left(\frac{dr}{d\theta} \right)^2 \right\}^{\frac{1}{2}} d\theta$$

$$= \int_0^{\pi} \left\{ a^2 (1+\cos\theta)^2 + a^2 \sin^2\theta \right\}^{\frac{1}{2}} d\theta ,$$

using the given equation

$$= \int_0^{\pi} a\left\{1+\cos^2\theta+2\cos\theta+\sin^2\theta\right\}^{\frac{1}{2}} d\theta$$

$$= \int_0^{\pi} a\sqrt{2}\left(1+\cos\theta\right)^{\frac{1}{2}} d\theta = a\sqrt{2}\int_0^{\pi}\left(2\cos^2\frac{\theta}{2}\right)^{\frac{1}{2}} d\theta$$

$$= 2a\int_0^{\pi}\cos\left(\frac{\theta}{2}\right)d\theta = 4a\int_0^{\frac{\pi}{2}}\cos\varphi d\varphi,\text{ where } \varphi = \frac{\theta}{2}$$

$$= 4a\sin\frac{\pi}{2} = 4a$$

Consequently, the total length of the given cardioid is $8a$.

EXERCISE

1. Find the length of the arc of the cantenary $y = \frac{a}{2}\left(e^{x/a} + e^{-x/a}\right)$ from $x = 0, y = a$ to the point (x, y).
2. Find the length of the quadrant of the curve $x = a\cos^3\theta$ and $y = a\sin^3\theta$. From $\theta = 0$ to $\theta = \frac{\pi}{2}$.
3. Find the arc length of the curve $y = x^{3/2}$ from $x = 0$ to $x = 4$.
4. Find the arc length of the curve parameterized by the equations $x = \cos 2t\,, y = \sin 2t\,, 0 \le t \le \frac{\pi}{2}$.
5. Find the length of the arc of $x = a(\cos\theta + \theta\sin\theta)$, $y = a(\sin\theta - \theta\cos\theta)$ from $\theta = 0$ to b.

ANSWERS

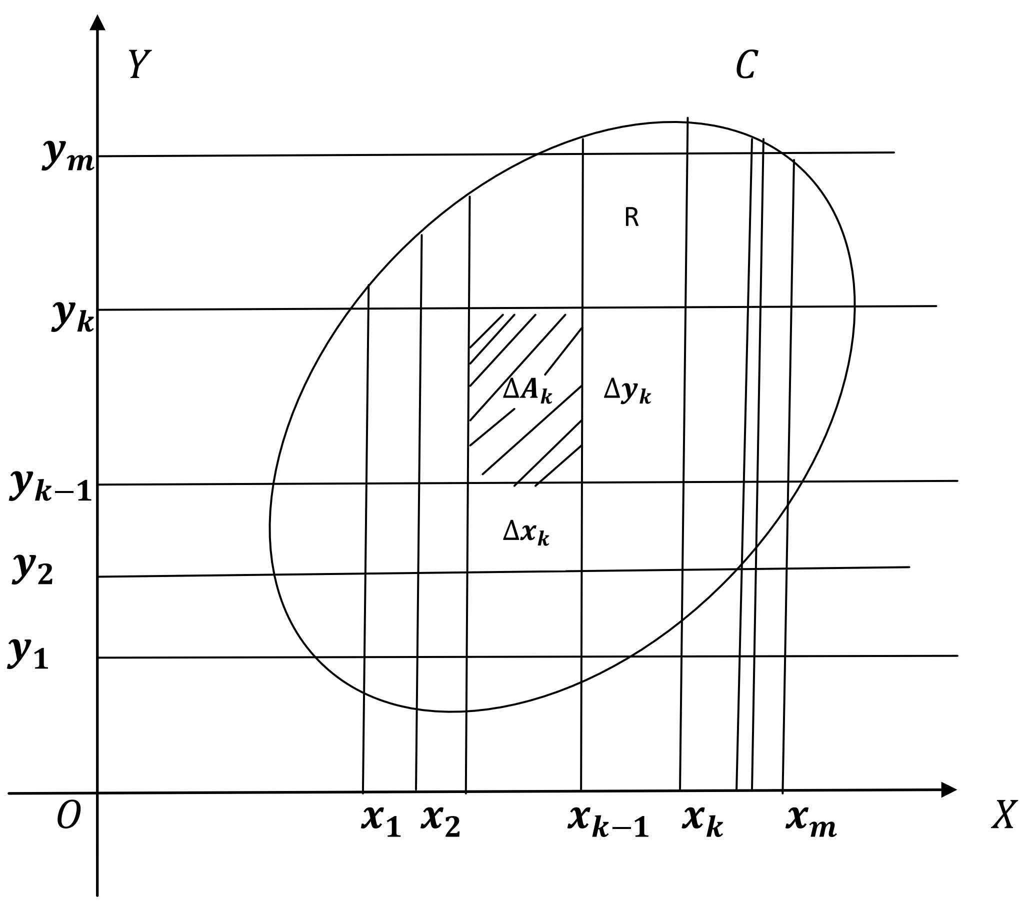
1. $a\sinh\left(\frac{x}{a}\right)$
2. $\frac{3a}{2}$
3. ≈ 9.07
4. π
5. $\frac{ab^2}{2}$

2.5

Double Integrals

In this module, we discuss methods of evaluating integrals of functions of two variables over a suitable region in $\mathbb{R}^2$. Integrals of a function of two variables over a region in $\mathbb{R}^2$ are called double integrals.

Let $f(x, y)$ be a continuous function in a simply connected, closed and bounded region R in a two dimensional space $\mathbb{R}^2$, bounded by a simple closed curve.



Region R for double integral

Subdivide the region R by drawing lines $x = x_k, y = y_k, k = 1, 2, \dots, m,$ parallel to the coordinate axes. Number the rectangles which are inside R from 1 to n. In each such rectangle, take an arbitrary point, say (ξ_k, η_k) in the kth rectangle and form the sum $J_k = \sum_{k=0}^{n} f(\xi_k, \eta_k)\Delta A_k$, where $\Delta A_k = \Delta x_k \Delta y_k$ is the area of the kth rectangle $d_k = \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}$ is the length of the diagonal of this rectangle. The maximum length of the diagonal, that is max d_k of the subdivisions is also called the norm of the subdivision. For different values of n, say $n_1, n_2, \dots, n_m, \dots$, we obtain a sequence of sums $J_{n_1}, J_{n_2}, \dots, J_{n_m}, \dots$. Let $n \to \infty$, such that the length of the largest diagonal $d_k \to 0$. If $\lim_{n\to\infty} J_n$ exists, independent of the choice of the subdivision and the point (ξ_k, η_k), then we say that $f(x, y)$ is integrable over R. This limit is called the double integral of $f(x, y)$ over R and is denoted by

$$J = \iint_R f(x, y)dxdy.$$

Let $f(x, y)$ be a continuous function in $\mathbb{R}^2$ defined on a closed rectangle $R = \{(x, y) | a \le x \le b \text{ and } c \le y \le d\}$. For any fixed $x \in [a, b]$ consider the integral $\int_c^d f(x, y)dy$.

The value of this integral depends on x and we get a new function of x. This can be integrated with respect to x and we get

$\int_a^b \left[\int_c^d f(x,y)dy \right] dx$. This is called an iterated integral.

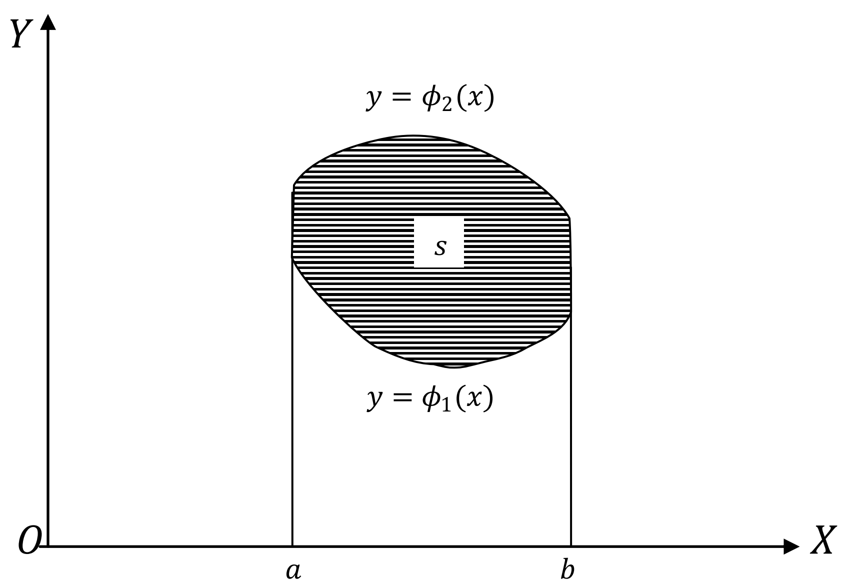
Similarly we can define another integral $\int_c^d \left[\int_a^b f(x,y)dx \right] dy$.

For continuous functions $f(x,y)$ we have

$$\iint_R f(x,y)dxdy = \int_a^b \left[\int_c^d f(x,y)dy \right] dx = \int_c^d \left[\int_a^b f(x,y)dx \right] dy$$

two continuous functions on $[a,b]$ then

$$\iint_s f(x,y)dxdy = \int_a^b \left[\int_{\phi_1(x)}^{\phi_2(x)} f(x,y)dy \right] dx.$$



The iterated integral in the right hand side is also written in the form

$$\int_a^b dx \int_{\phi_1(x)}^{\phi_2(x)} f(x,y)dy$$

Similarly if $S = \{(x,y)|c \le y \le d \ and \ \phi_1(y) \le x \le \phi_2(y)\}$ then

$$\iint_S f(x,y)dxdy = \int_c^d \left[\int_{\phi_1(x)}^{\phi_2(x)} f(x,y)dx \right] dy$$

If S cannot be written in neither of the above two forms we divide S into finite number of subregions such that each of the sub regions can be represented in one of the above forms and we get the double integral over S by adding the integrals over these sub regions. Hence to evaluate $\iint_S f(x,y)dxdy$ we first convert it to an iterated integral of the two forms given above.

Note 1: $\iint_S dxdy$ represents the area of the region S.

Note 2: In an iterated integral the limits in the first integral are constants and if the limits in the second integral are functions of x then we must first integrate with respect to y and the integrand will become a function of x alone. This is integrated with respect to x.

If the limits of the second integral are functions of y then we must first integrate with respect to x and the integrand will become a function of y alone. This is integrated with respect to y.

Properties of double integrals

1. If $f(x,y)$ and $g(x,y)$ integrable functions, then
$$\iint_R [f(x,y) \pm g(x,y)]dxdy = \iint_R f(x,y)dxdy \pm \iint_R g(x,y)dxdy.$$
2. $\iint_R kf(x,y)dxdy = k\iint_R f(x,y)dxdy$, where k is any real constant.
3. When $f(x,y)$ is integrable, then $|f(x,y)|$ is also integrable, and
$$\left| \iint_R f(x,y)dxdy \right| \leq \iint_R |f(x,y)|dxdy.$$
4. $\iint_R f(x,y)dxdy = f(\xi,\eta)A$, where A is the area of the region R and (ξ,η) is any arbitrary point in R. This result is called the mean value theorem of the double integrals.
If $m \leq f(x,y) \leq M$ for all (x,y) in R, then
$$mA \leq \iint_R f(x,y)dxdy \leq MA.$$
5. If $0 < f(x,y) \leq g(x,y)$ for all (x,y) in R, then
$$\iint_R f(x,y)dxdy \leq \iint_R g(x,y)dxdy.$$
6. If $f(x,y) \geq 0$ for all (x,y) in R, then
$$\iint_R f(x,y)dxdy \geq 0.$$

Application of double integrals

Double integrals have large number of applications. We state some of them.

1. If $f(x,y)=1$, then $\iint_R dxdy$ gives the area A of the region R.

   For example, if R is the rectangle bounded by the lines $x=a, x=b, y=c$ and $y=d$, then $A=\int_c^d\int_a^b dxdy = \int_c^d\left[\int_a^b dx\right]dy=(b-a)\int_c^d dy=(b-a)(d-c)$ gives the area of the rectangle.
2. If $z=f(x,y)$ is a surface, then

   $$\iint_R zdxdy \text{ or } \iint_R f(x,y)dxdy$$

   gives the volume of the region beneath the surface $z=f(x,y)$ and above the $x-y$ plane.

   For example:

   if $z=\sqrt{a^2-x^2-y^2}$ and $R: x^2+y^2\le a^2$, then

   $$V=\iint_R \sqrt{a^2-x^2-y^2}\,dxdy$$

   gives the volume of the hemisphere $x^2+y^2+z^2=a^2\ge 0$.

3. Let $f(x,y)=\rho(x,y)$ be a density function (mass per unit area) of a distribution of mass in the $x-y$ plane. Then

   $$M=\iint_R f(x,y)dxdy$$

   give the total mass of R.
4. Let $f(x,y)=\rho(x,y)$ be a density function. Then

$$\bar{x} = \frac{1}{M}\iint_R xf(x,y)dxdy, \bar{y} = \frac{1}{M}\iint_R yf(x,y)dxdy$$

give the coordinates of the centre of gravity $(\bar{x}, \bar{y})$ of the mass M in R.

5. Let $f(x,y) = \rho(x,y)$ be a density function. Then

$$I_x = \iint_R y^2 f(x,y)dxdy \text{ and } I_y = \iint_R x^2 f(x,y)dxdy$$

give the moments of inertia of the mass in R about the X-axis and the Y-axis respectively, whereas $I_0 = I_x + I_y$ is called the moment of inertia of the mass in R about the origin. Similarly,

$$I_y = \iint_R (x-a)^2 f(x,y)dxdy \text{ and } I_x = \iint_R (y-b)^2 f(x,y)dxdy$$

give the moment of inertia of the mass in R about the lines $x = a$ and $y = b$ respectively.

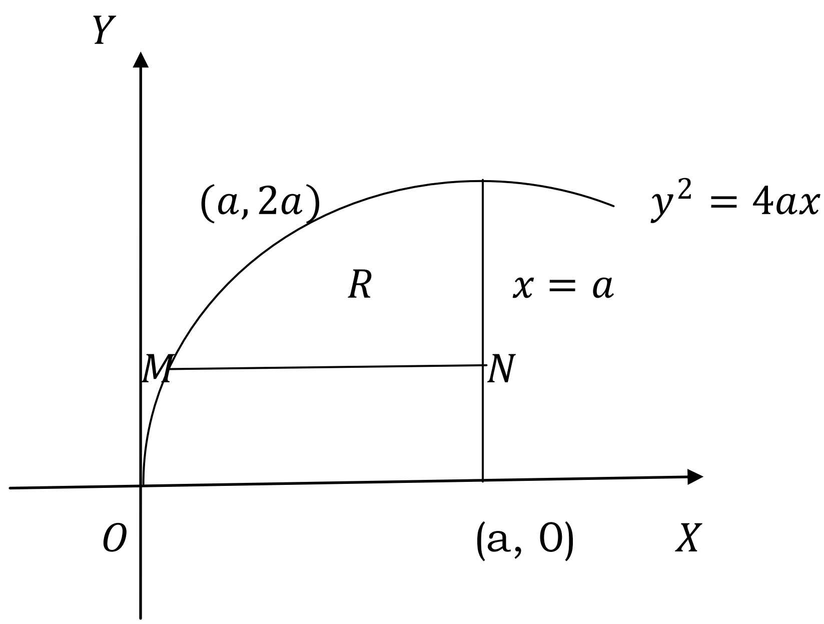
6. $\frac{1}{A}\iint_R f(x,y)dxdy$ gives the average value of $f(x,y)$ over R, where A is the area of the region R.

Change of order of Integration

In the evaluation of double integrals the computational work can often be reduced by interchanging the order of integration. While using this method, one has to bear in mind that the change of order of integration generally changes the limits of integration; the new limits are to be determined by examining the geometrical region on which the integration is being carried out.

Example: Evaluate $\int_0^a \int_0^{2\sqrt{xa}} x^2 dydx, a > 0,$ by changing the order of integration.

In the given integral, x increases from 0 to a, for each x, y increases from 0 to $2\sqrt{xa}$. Hence the lower value of y lies on the X-axis and the upper value on the upper part of the parabola $y^2 = 4ax$. Therefore the region R of integration is bounded by the X-axis, the line and the arc of the parabola $y^2 = 4ax$ in the first quadrant. This region is shown in figure.



We observe that, in R, y increases from 0 from $2a$, and, for each y, x varies from a point M on the parabola $y^2 = 4ax$ to a point N on the line $x = a$; that is x increases from $(y^2/4a)$ to a. Hence

$$\int_0^a \int_0^{2\sqrt{xa}} x^2 dy dx = \int_{y=0}^{2a} \left\{ \int_{x=y^2/4a}^{a} x^2 dx \right\} dy = \int_0^{2a} \frac{1}{3} \left\{ a^3 - \left(\frac{y^2}{4a} \right)^3 \right\} dy$$

$$= \frac{1}{3} [a^3 - \frac{1}{64a^3} \frac{y^7}{7}]_0^{2a} = \frac{1}{3} \{a^3(2a) - \frac{1}{64a^3} \frac{(2a)^7}{7}\}$$

$$= \frac{4}{7} a^4.$$

Problem 1: Evaluate the following repeated integrals:

i. $\int_1^4 \int_0^{\sqrt{4-x}} xydydx$

ii. $\int_0^5 \int_0^{y^2} x(x^2+y^2)dxxy$

Solution: By using the meaning of repeated integrals, we find:

i. $\int_1^4 \int_0^{\sqrt{4-x}} xydydx = \int_{x=1}^4 \left\{ \int_{y=0}^{\sqrt{4-x}} xydy \right\} dx$

$= \int_{x=1}^4 x \left\{ \left[\frac{y^2}{2} \right]_{y=0}^{\sqrt{4-x}} \right\} dx,$ on evaluating the inner integral with x held fixed.

$= \int_1^4 x \left\{ \frac{1}{2}(4-x) \right\} dx = \left[x^2 - \frac{x^3}{6} \right]_1^4$

$= \left(16 - \frac{64}{6}\right) - \left(1 - \frac{1}{6}\right) = \frac{32}{6} - \frac{5}{6} = \frac{27}{6} = \frac{9}{2}.$

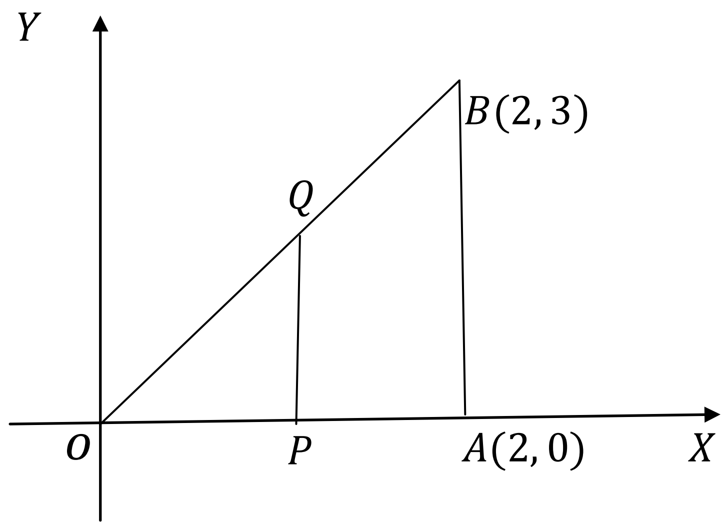
ii. $\int_0^5 \int_0^{y^2} x(x^2+y^2)dxxy = \int_{y=0}^5 \left\{ \int_{x=0}^{y^2} \left(x^3 + xy^2\right) dx \right\} dy$

$= \int_{y=0}^5 \left\{ \left[\frac{x^4}{4} + y^2\left(\frac{x^2}{2}\right) \right]_{x=0}^{y^2} \right\} dy,$ on evaluating the inner integral with y held fixed

$= \int_{y=0}^5 \left\{ \frac{(y^2)^4}{4} + y^2 \frac{(y^2)^2}{2} \right\} dy = \left[\frac{1}{4} . \frac{y^9}{9} + \frac{1}{2} . \frac{y^7}{7} \right]_0^5 = \frac{5^7}{4}\left(\frac{25}{9} + \frac{2}{7}\right).$

Problem 2: If $\Re$ is the triangular region with vertices $(0,0),(2,0),(2,3)$, evaluate $\iint_{\Re} x^2 y^2 dxdy$.

Solution:



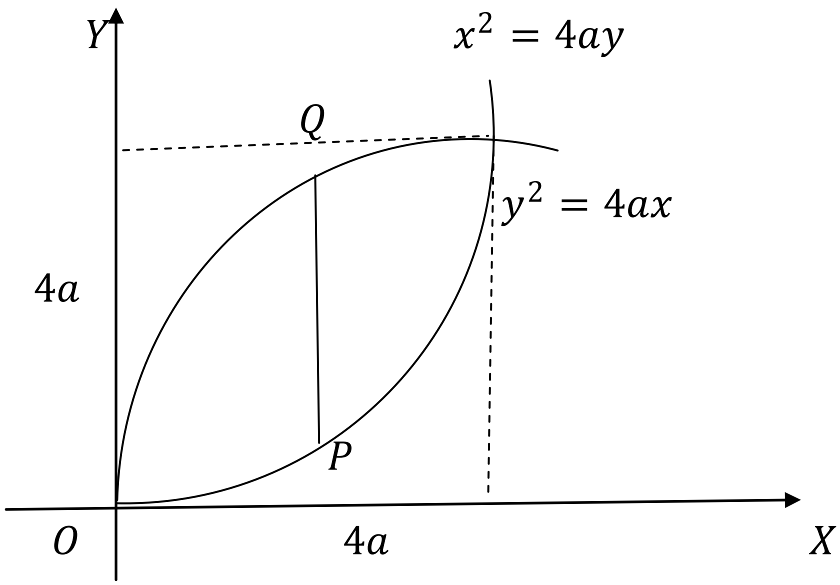
The region where in the integration is to be carried out is shown in figure. We note that, in this region, x increases from 0 to 2 and for each x, y increases from a point P on the X-axis to a point Q on the line OB. For the point P, We have $y=0$. The equation of OB is $y=(3/2)x$. Therefore, for the point Q, we have $y=(3/2)x$. Thus, for each x, y increases from 0 to $(3/2)x$. Hence

$$\iint_{\Re} x^2 y^2 dxdy = \int_{x=0}^{2}\left\{\int_{y=0}^{(3/2)x} x^2 y^2 dy\right\}dx = \int_0^2\left\{x^2\left[\frac{y^3}{3}\right]_0^{(3/2)x}\right\}dx$$

$$= \int_0^2 x^2\left\{\frac{1}{3}.\frac{27}{8}x^3\right\}dx = \frac{9}{8}\left[\frac{x^6}{6}\right]_0^2 = \frac{3}{16}.2^6 = 12.$$

Problem 3: Evaluate $\iint_{\Re} y\,dxdy,$ where $\Re$ is the region bounded by the parabolas $y^2 = 4ax$ and $x^2 = 4ay, a > 0$.

Solution:



Solving the given equations, we find that the two parabolas intersect at the points (0,0) and $(4a, 4a)$. Therefore, the region bounded by these parabolas is as shown in figure. In this region, x increases from 0 to $4a$, and, for each x, y increases from a point P on the parabola $x^2 = 4ay$ to a point Q on the parabola $y^2 = 4ax$. We find that, at $P, y = (x^2/4a)$ and, at $Q, y = \sqrt{4ax}$, Hence

$$\iint_{\Re} y\,dxdy = \int_{x=0}^{4a} \left\{ \int_{y=x^2/4a}^{\sqrt{4ax}} y\,dy \right\} = \int_0^{4a} \frac{1}{2} \left\{ (4ax) - \left(\frac{x^2}{4a} \right)^2 \right\} dx$$

$$= \frac{1}{2}\left[2ax^2 - \frac{1}{16a^2}\left(\frac{x^5}{5}\right)\right]_0^{4a} = \frac{1}{2}\left\{32a^3 - \frac{1}{16a^2}.\frac{(4a)^5}{5}\right\}$$

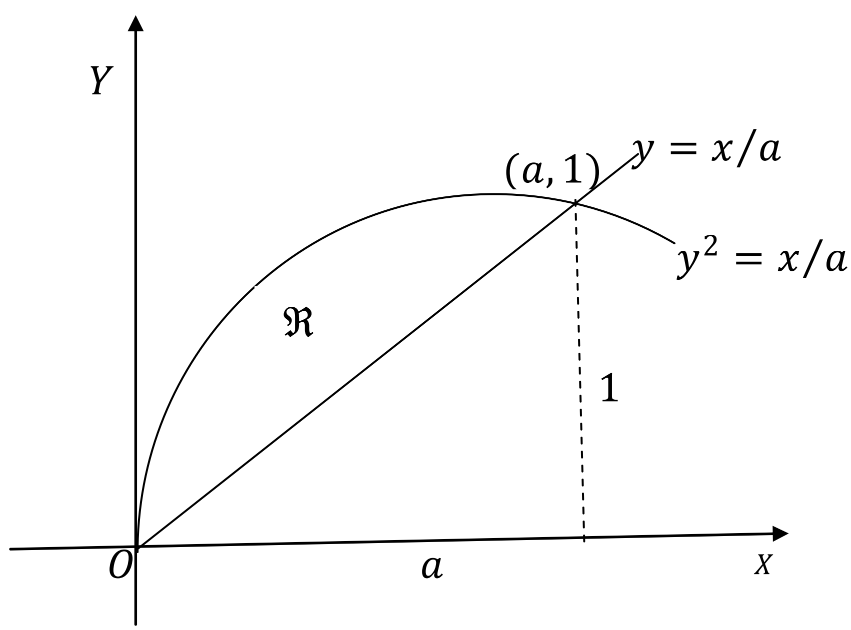
$$= \frac{1}{2}\left\{32a^3 - \frac{64}{5}a^3\right\} = \frac{48}{5}a^3.$$

Problem 4: Change the order of integration in the integral

$$\int_0^a \int_{x/a}^{\sqrt{x/a}} \left(x^2+y^2\right) dydx, a>0$$

and hence evaluate it.

Solution: In the given integral, x increases from 0 to a, and, for each x, y varies from $y = x/a$ to $y = \sqrt{x/a}$. Hence the lower value of y lies on he curve $y = x/a$ (which is a straight line) and the upper value of y lies on the curve $y^2 = x/a$(which is a parabola). We check that the line $y = x/a$ and the parabola $y^2 = x/a$ intersect at the point $(0,0)$ and $(a,1)$. The region $\Re$ of integration is therefore bounded by the line $y = x/a$ and the parabola $y^2 = x/a$ between the origin and the point $(a,1)$. This region is shown in figure.



From the above figure, we observe that in $\Re$, y increases from 0 to 1 and, for each y, x varies from a point on the parabola $y^2 = x/a$ to a point on the line $y = x/a$; that is, each y with $0 \le x \le 1, x$ varies from ay^2 to ay. Hence

$$\int_0^a \int_{x/a}^{\sqrt{x/a}} (x^2 + y^2) dy dx = \int_{y=0}^{1} \left\{ \int_{x=ay^2}^{ay} \left(x^2 + y^2 \right) dx dy \right\}$$

$$= \int_0^1 \left\{ \left[\frac{x^3}{3} + y^2 x \right]_{x=ay^2}^{ay} \right\} dy$$

$$= \int_0^1 \left[\frac{1}{3} \left(a^3 y^3 - a^3 y^6 \right) + y^2 \left(ay - ay^2 \right) \right] dy$$

$$= \frac{a^3}{3} \left(\frac{1}{4} - \frac{1}{7} \right) + a \left(\frac{1}{4} - \frac{1}{5} \right) = \frac{a^3}{28} + \frac{a}{20}.$$

Problem 5: Evaluate $I = \int_0^{\pi}\int_0^{a\cos\theta} r\sin\theta\, dr\, d\theta$

Solution:

$$I = \int_0^{\pi}\int_0^{a\cos\theta} r\sin\theta\, dr\, d\theta$$

$$I = \int_0^{\pi} \sin\theta \left[\frac{r^2}{2}\right]_0^{a\cos\theta} d\theta$$

$$= \frac{1}{2}\int_0^{\pi} a^2\cos^2\theta\sin\theta\, d\theta$$

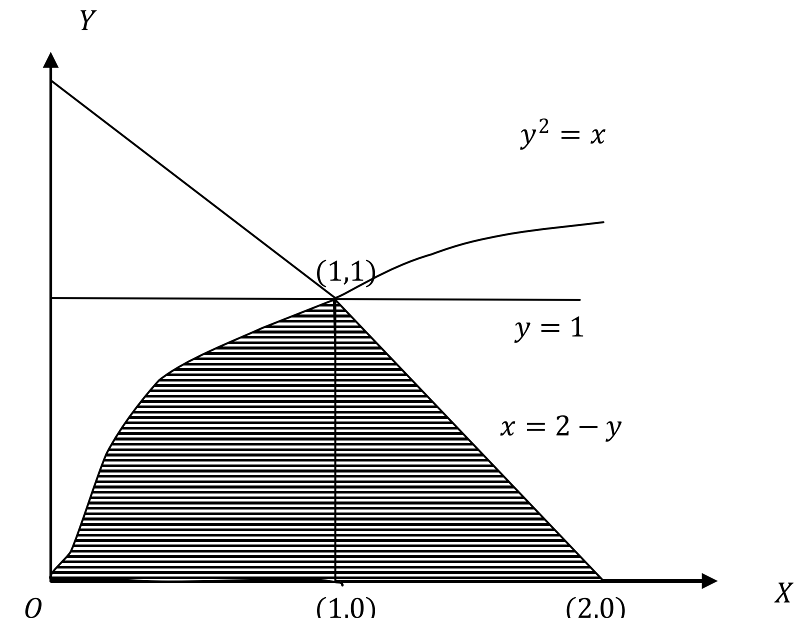
$$= -\frac{a^2}{2}\int_0^{\pi}\cos^2\theta\, d\left(\cos\theta\right)$$

$$= -\frac{a^2}{6}\left[\cos^3\theta\right]_0^{\pi} = \frac{a^2}{3}.$$

Problem 6: Evaluate

$I = \iint_D xydydx$ where D is the region bounded by the curve $x = y^2, x = 2 - y, y = 0$ and $y = 1$.

Solution: The given region bounded by the curves is shown in the figure.



In this region x varies from 0 to 2. When $0 \leq x \leq 1$ for fixed x, y varies from 0 to $\sqrt{x}$. When $1 \leq x \leq 2, y$ varies from 0 to $2 - x$.

Therefore the region D can be subdivided into two regions D_1 and D_2 as shown in the figure.

$$\therefore \iint_D xydydx = \iint_{D_1} xydydx + \iint_{D_2} xydydx$$

In this region D_1 for fixed x, y varies from $y = 0$ to $y = \sqrt{x}$ and for fixed y, x varies from $x = 0$ to $x = 1$. Similarly for the region D_2, the limit of integration for y is $y = 0$ to $y = 2 - x$.

$$\iint_D xydydx = \int_0^1 \int_0^{\sqrt{x}} xy\, dy\, dx + \int_1^2 \int_0^{2-x} xy\, dy\, dx$$

$$= \int_0^1 \left[\frac{xy^2}{2}\right]_0^{\sqrt{x}} dx + \int_1^2 \left[\frac{xy^2}{2}\right]_0^{2-x} dx$$

$$= \frac{1}{2}\int_0^1 x^2\, dx + \frac{1}{2}\int_1^2 x(2-x)^2\, dx$$

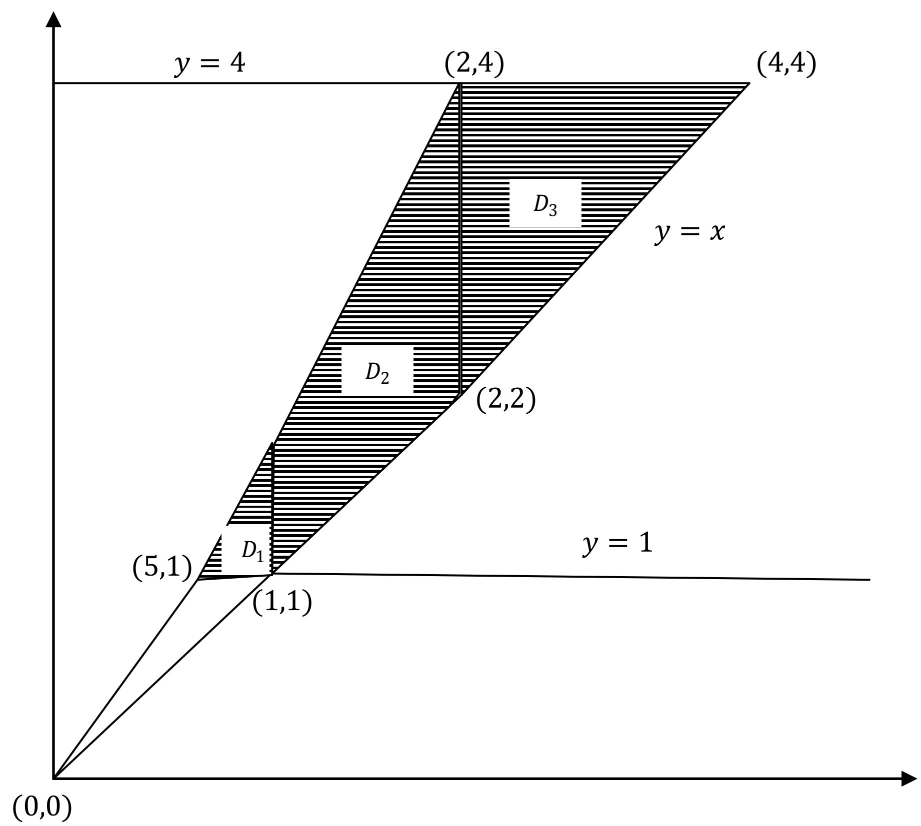
$$= \left[\frac{x^3}{6}\right]_0^1 + \frac{1}{2}\left[2x^2 + \frac{x^4}{4} - \frac{4x^3}{3}\right]_1^2$$

$$= \frac{3}{8}.$$

Problem 7: Change the order of integration in the integral

$I = \int_1^4 \int_{y/2}^{y} f(x, y)\, dx\, dy.$

Solution: The region of integration D is bounded by the lines $x = \frac{y}{2}; x = y; y = 1$ and $y = 4$. The region is a quadrilateral as shown in the figure



In this region x varies from $\frac{1}{2}$ to 4.

When $\frac{1}{2} \le x \le 1, y$ varies from 1 to $2x$.

When $1 \le x \le 2, y$ varies from x to $2x$.

When $2 \le x \le 4, y$ varies from x to 4.

Hence for changing the order of integration we must divide D into sub regions D_1, D_2, D_3 as shown in the figure.

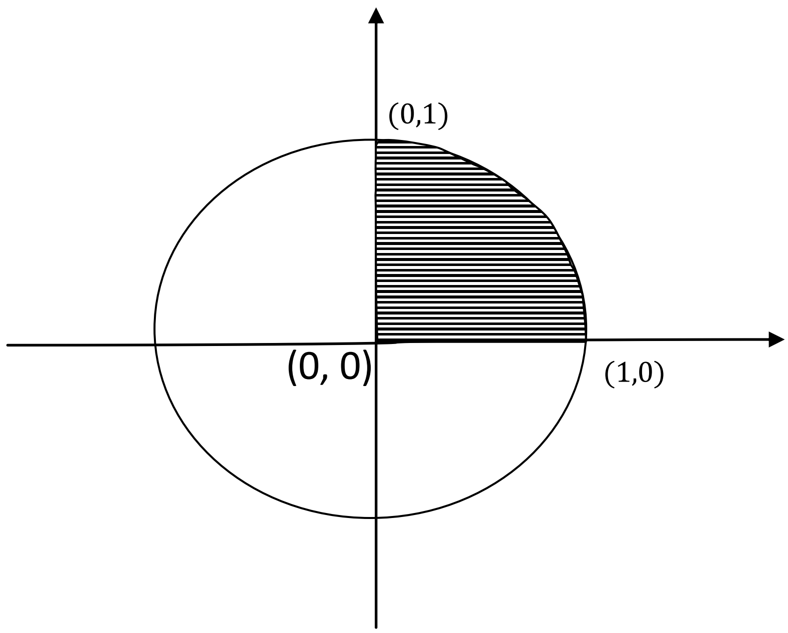
$$\therefore \quad I = \iint_D f(x,y)dxdy$$

$$= \iint_{D_1} f(x,y)dxdy + \iint_{D_2} f(x,y)dxdy + \iint_{D_3} f(x,y)dxdy$$

$$= \int_{1/2}^{1}\int_{1}^{2x} f(x,y)dy\,dx + \int_{1}^{2}\int_{x}^{2x} f(x,y)dy\,dx +$$
$$\int_{2}^{4}\int_{x}^{4} f(x,y)dy\,dx\ .$$

Problem 8: Evaluate $\int_0^1 \int_0^{\sqrt{1-x^2}} y^2 \; dy\, dx$ by interchanging the order of integration.

Solution: The region is bounded by the line$y = 0(X - \text{axis})$; the unit circle $x^2 + y^2 = 1$; the line $x = 0(Y - \text{axis})$ and the line $x = 1$. Hence the region of integration is the positive quadrant of the unit circle $x^2 + y^2 = 1$ and it is given in the figure.



In this region y varies from 0 to 1 and for a fixed y, x varies from 0 to $\sqrt{1-y^2}$.

$$\int_0^1 \int_0^{\sqrt{1-x^2}} y^2 \; dy\, dx = \int_0^1 \int_0^{\sqrt{1-y^2}} y^2 \; dy\, dx$$

$$= \int_0^1 y^2 [x]_0^{\sqrt{1-y^2}} \, dy$$

$$= \int_0^1 y^2 \sqrt{1-y^2} dy$$

$$= \int_0^{\pi/2} \sin^2\theta \cos^2\theta \, d\theta \quad (\text{putting } y = \sin\theta)$$

$$= \left(\frac{1.1}{4.2}\right)\frac{\pi}{2} = \frac{\pi}{16}.$$

Exercise

1) Evaluate $\int_1^4 \int_{\sqrt{y}}^2 (x^2 + y^2)dxdy$ by changing the order of integration.

2) By changing the order of integration evaluate $\int_0^3 \int_1^{\sqrt{4-y}} (x + y)dxdy$.

3) Change the order of integration in the integral $\int_0^a \int_{\frac{x^2}{a}}^{2a-x} xydydx$ and evaluate.

4) Evaluate $I = \iint_D e^{\frac{y}{x}} dxdy$ where D is the region bounded by the straight lines $y = x$; $y = 0$ and $x = 1$.

5) Evaluate $\iint_D (x^2 + y^2)dxdy$ where D is the region bounded by $y = x^2, x = 2$ and $x = 1$.

6) Evaluate $I = \int_0^\infty \int_x^\infty \frac{e^{-y}}{y} dydx$.

7) Find the area of the circle $x^2 + y^2 = r^2$ by using double integral.

8) Find the area of the region D bounded by the parabolas $y = x^2$ and $x = y^2$.

9) Evaluate $I = \int_0^{\frac{\pi}{2}} \int_0^{\infty} \frac{r}{(r^2+a^2)^2} dr d\theta.$

Answers

1) $\frac{1026}{105}$

2) $\frac{241}{60}$

3) $\frac{9a^4}{24}$

4) $\frac{1}{2}(e-1)$

5) $\frac{1286}{105}$

6) 1

7) πr^2

8) $\frac{1}{3}$

9) $\frac{\pi}{4a^2}$

2.6

DOUBLE INTEGRALS CONTINUED

Change of variables

In the evaluation of repeated integrals, the computational work can often be reduced by changing the variables of integration to some other appropriate variables. The procedure followed in this regard in respect of double integrals is explained below.

Suppose x and y are related to two variables u and v through relations of the form $x = x(u,v), y = y(u,v)$, or $u = u(x,y), v = v(x,y)$. Suppose also that x, y, u, v are such that the Jacobain

$$J = \frac{\partial(x,y)}{\partial(u,v)} \neq 0.$$

Then it can be proved that (-we omit the proof, see our video to get some idea of the following)

$$\iint_{\Re} f(x,y)dxdy = \iint_{\bar{\Re}} \varphi(u,v)Jdudv \qquad (1)$$

Hence $\Re$ is the region in which (x,y) vary, $\bar{\Re}$ is the corresponding region in which (u,v) vary, and $\varphi(u,v) = f\{x(u,v), y(u,v)\}$.

Once the double integral with respect to x and y is changed to a double integral with respect to u and v by using the formula (1), the later integral can be evaluated by expressing it in terms of repeated integrals with appropriate limits of integration.

Double Integral in Polar form

As a special case of formula (1), we can obtain the relation connecting a double integral in Cartesian form and the corresponding double integral in polar form.

Let (r,θ) be the polar coordinates of a point (x,y). Then $x = r\cos\theta\,, y = r\sin\theta$, so that

$$J = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r.$$

Hence, for $(u,v) = (r,\theta)$, formula (1) becomes

$$\iint_{\Re} f(x,y)dxdy = \iint_{\overline{\Re}} \varphi(r,\theta) r dr d\theta \qquad (2)$$

Here $\overline{\Re}$ is the region in which (r,θ) vary as (x,y) vary in $\Re$, and $\varphi(r,\theta) = f(r\cos\theta\,, r\sin\theta)$.

The formula (2) is particularly useful when the region $\Re$ is bounded (in part or whole) by a circle centred at the origin. Observed that when (x,y) are changed to (r,θ), $dxdy$ is changed to $rdrd\theta$.

Computation of Area

Let us recall the double integral expression for $f(x,y) \equiv 1$, this expression reads

$$\int_A dA \equiv \iint_{\Re} dxdy = \int_a^b \int_{y_1(x)}^{y_2(x)} dxdy \qquad (3)$$

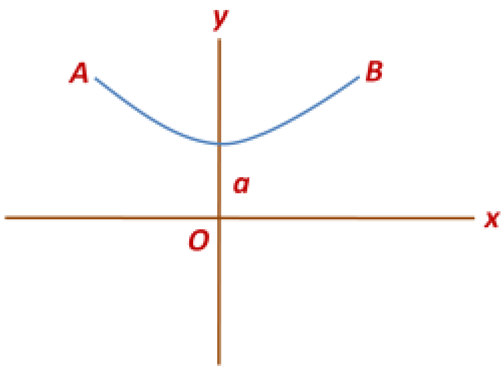
The integral $\int_A dA$ represents the total area A of the plane region $\Re$ over which the repeated integrals are taken. Thus, expression (3) given above may be used to compute the area A. We note that $dxdy$ is the plane area element dA in the Cartesian form. By taking $f(x,y) \equiv 1$. We obtain the following formula for area in polar coordinates:

$$\iint_{\Re} dxdy = \iint_{\bar{\Re}} rdrd\theta \qquad (4)$$

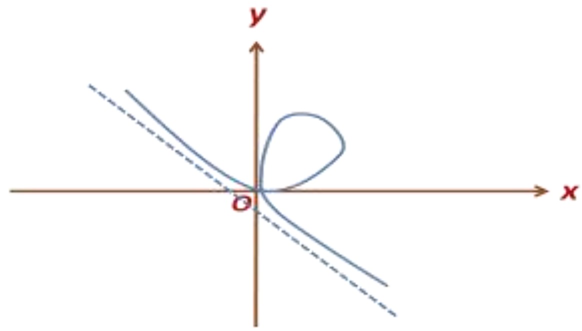
We observe that $rdrd\theta$ is the plane area element in polar form.

A List of Curves

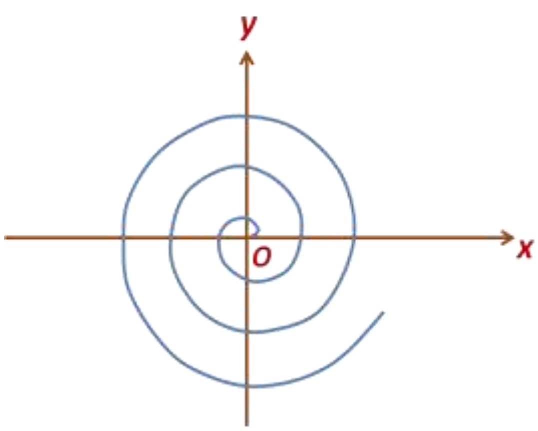
The following list of curves will be useful to find the limits of integration of some problems.



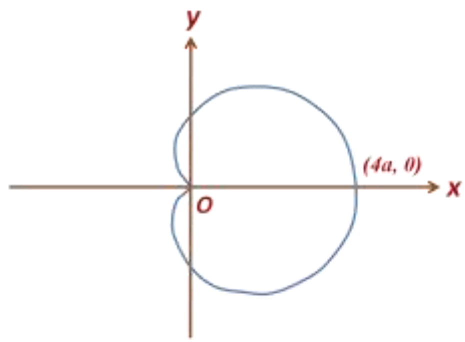
Centenary, $y = a \cos h\ (x/a)$



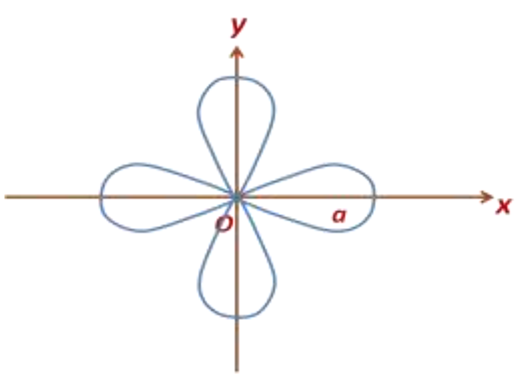
Folium of Descartes, $x^3 + y^3 = 3axy$.
Parametric equations: $x = 3at/(1+t^3)$,
$y = 3at^2/(1+t^3)$.



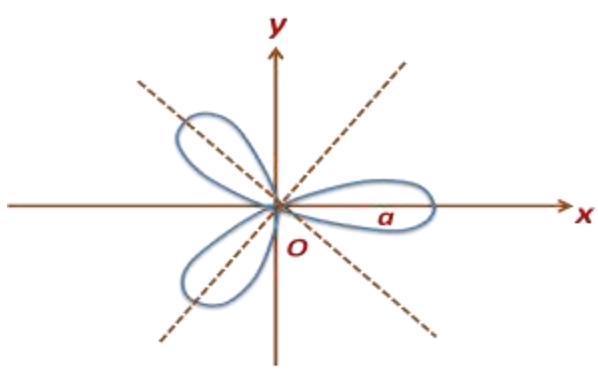
Spiral of Archimedes, $r = a\theta$



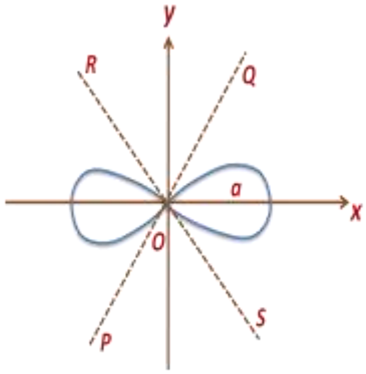
Cardioid, $r = 2a\ (1 + \cos \theta)$. (Locus of a point on a circle of radius a rolling on the outside of a fixed circle of radius a. The fixed circle has centre at (a, 0) and touches y – axis at the origin.



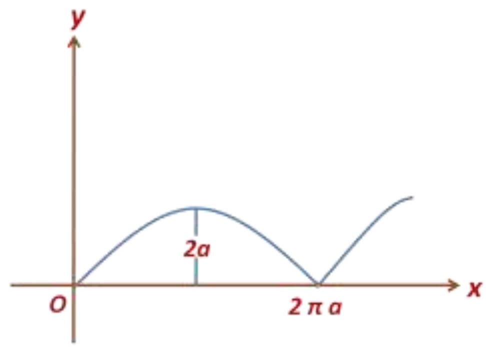
Four leaved rose, $r = a \cos (2\theta)$



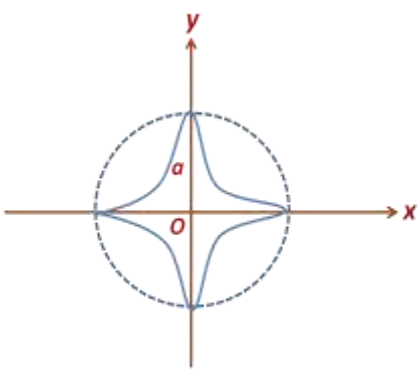
Three leaved rose, $r = a \cos (3\theta)$



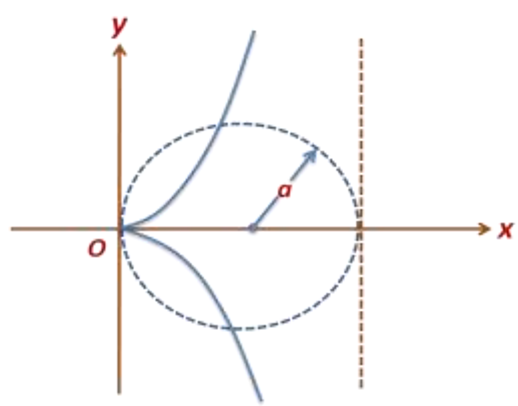
Lemniscate. Polar from: $r^2 = a^2 \cos(2\theta)$.
Rectangular coordinates form: $(x^2 + y^2)^2 = a^2 (x^2 - y^2)$



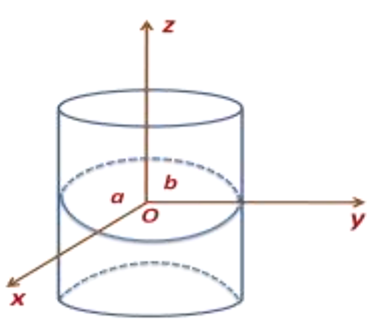
Cycloid. Parametric from: $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$. **(Locus of a point P on** a **circle of radius a rolling along** $x - axis$. Initially, the circle has centre at $(0, a)$ and touches $x - axis$ at the origin).



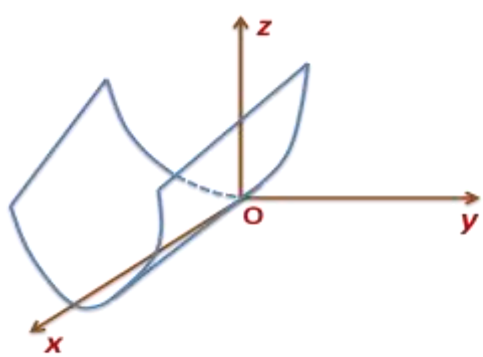
Four cusp hypocycloid. Parametric form:
$x = a \cos^3 \theta$, $y = a \sin^3 \theta$. Rectangular co – ordinates form: $x^{2/3} + y^{2/3} = a^{2/3}$.
(Locus of a point P on a circle of radius $a/4$, rolling on the inside of a circle of a circle of radius a).



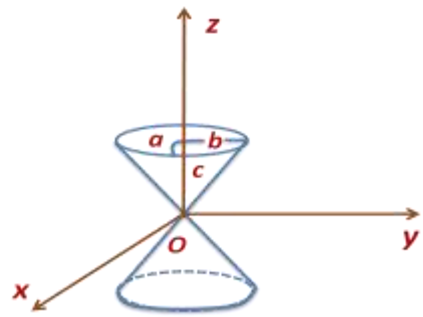
Cissoid of Diocles, $y^2 = x^3 / (2a - x)$



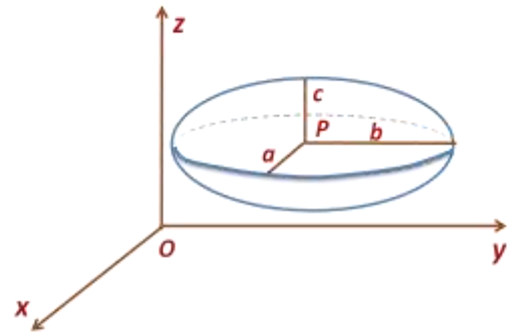
Elliptic cylinder $(x^2/a^2) + (y^2/b^2) = 1$, **where** a, b **are the** $semi - axis$ **of the elliptic cross section. When** $a = b$, **we get the circular cylinder** $x^2 + y^2 = a^2$.



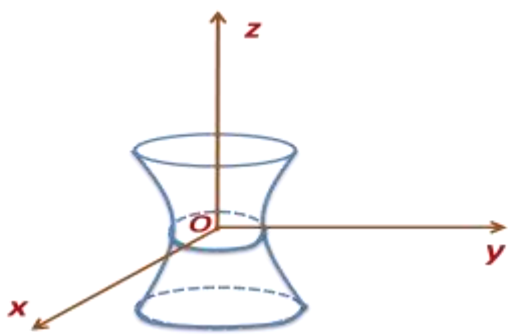
Parabolic cylinder $z = y^2$



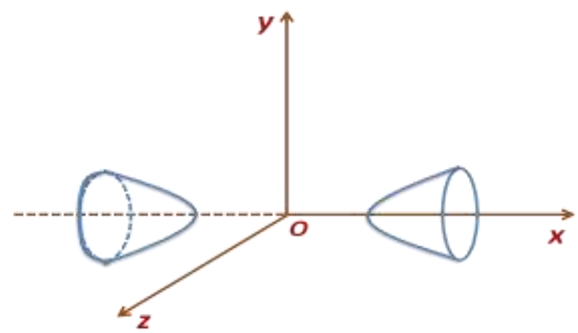
Ellipitic cone (with z – axis as axis
$(x^2 / a^2) + (y^2/b^2) = (z^2/c^2)$.
When $a = b$, we get a right circular cone.



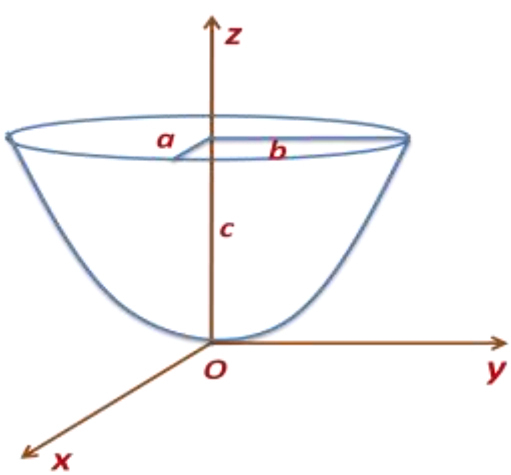
Ellipsoid with centre at $P\,(x_0, y_0, z_0)$ and semi – axis a, b, c.



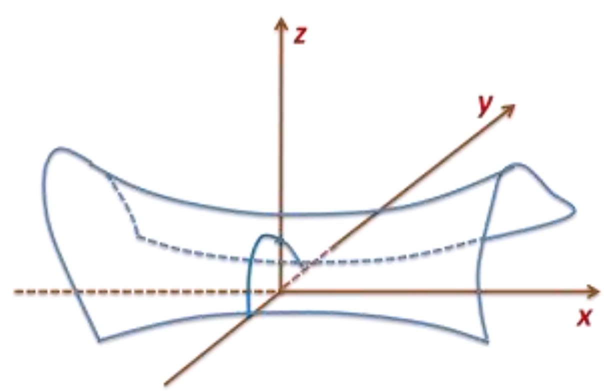
Hyperboloid of one sheet,
$(x^2/a^2) + (y^2/b^2) - (z^2/c^2) = 1$.



Hyperboloid of two sheet,
$(x^2/a^2) - (y^2/b^2) - (z^2/c^2) = 1$.



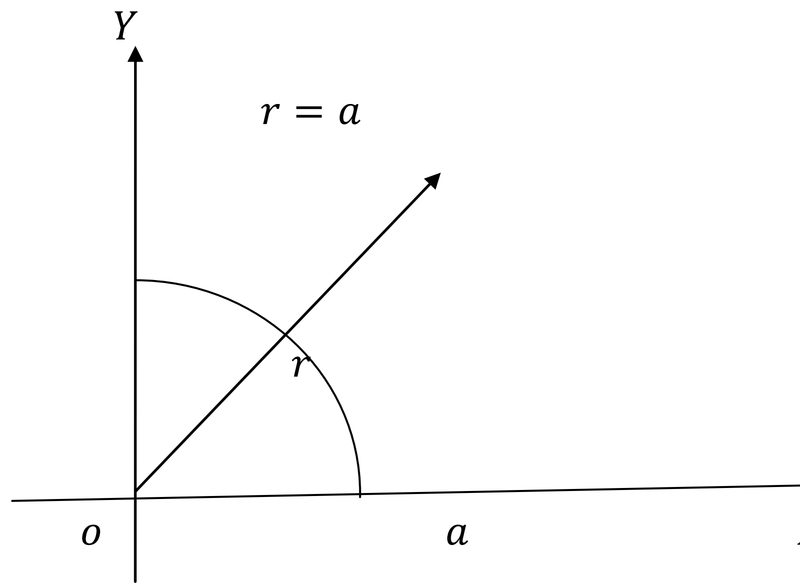
Hyperboloid of one sheet,
$(x^2/a^2) + (y^2/b^2)=(z / c)$.



Hyperbolic paraboloid.
$(x^2/a^2) - (y^2/b^2) = (z / c)$.

Problem 1: Evaluate the double integral $\iint xy\,dx\,dy$ over the positive quadrant bounded by the circle $x^2+y^2=a^2$.

Solution:



In the positive quadrant bounded by the circle $x^2+y^2=a^2$, the radial distance r varies from 0 to a and the polar angle θ varies from 0 to $\frac{\pi}{2}$. Therefore,

$$\iint xydxdy=\int_{r=0}^{a}\int_{\theta=0}^{\pi/2}(r\cos\theta)(r\sin\theta)(rdrd\theta)$$

$$=\int_0^a r^3dr\times\int_0^{\pi/2}\sin\theta\cos\theta d\theta=\left[\frac{1}{4}r^4\right]_0^a\times\left[\frac{1}{2}\sin^2\theta\right]_0^{\pi/2}$$

$$=\frac{1}{8}a^4.$$

Problem 2: Evaluate the integral $I = \int\limits_{0}^{a} \int\limits_{y=0}^{\sqrt{a^2-x^2}} y^2\sqrt{x^2+y^2}\,dydx$ transforming to polar coordinates.

Solution: In the given integral, x increases from 0 to a and, for each x, y varies from 0 to $\sqrt{(a^2 - x^2)}$. Thus, the lower value of y lies on the $X-$ axis and the upper value of y lies on the curve $y = \sqrt{(a^2 - x^2)}$, or $y^2 = a^2 - x^2$, or $x^2 + y^2 = a^2$, which is the circle of radius a centred at the origin. Therefore the region $\Re$ of integration is the region in the first quadrant bounded by the circle $x^2 + y^2 = a^2$,

We note that in $\Re$, θ varies from 0 from $\frac{\pi}{2}$ and, for each θ, r varies from 0 to a. Hence,

$$I = \int_0^a \int_0^{\sqrt{a^2-x^2}} y^2\sqrt{x^2+y^2}$$

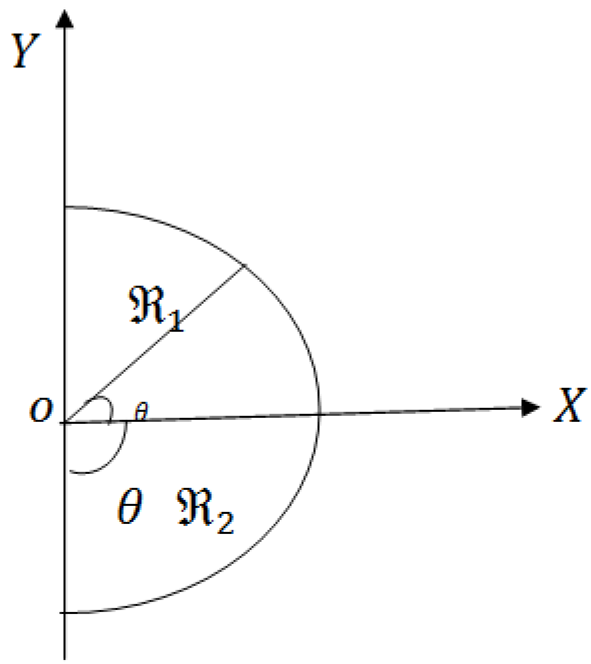
$$= \int_{\theta=0}^{\frac{\pi}{2}} \int_{r=0}^{a} (r^2 \sin^2\theta)\, r(rdrd\theta)$$

$$= \left\{\int_0^a r^4 dr\right\} \times \left\{\int_0^{\frac{\pi}{2}} \sin^2\theta d\theta\right\}$$

$$= \frac{a^5}{5} . \frac{1}{2} . \frac{\pi}{2} = \frac{\pi}{20} a^5.$$

Problem 3: If $\Re$ is the region $x^2+y^2 \le a^2, x \ge 0$, evaluate $\iint\limits_{\Re}(x+y)dxdy.$

Solution:



The given region $\Re$ is bounded by the $Y-$ axis and the right side of the circle $x^2+y^2=a^2$ as shown in figure. We observe that $\Re$ consists of two parts $\Re_1$ and$\Re_2$. In $\Re_1, \theta$

increases from 0 to $\frac{\pi}{2}$, and in $\Re_2, \theta$ increases from $\frac{3\pi}{2}$ to 2π. In both parts, r varies from 0 to a. Therefore,

$$\iint\limits_{\Re}(x+y)dxdy$$

$$=\iint_{\Re_1}(x+y)dxdy+\iint\limits_{\Re_2}(x+y)dxdy$$

$$=\int_{\theta=0}^{\pi/2}\int_{\theta=0}^{a}\left(r\cos\theta+r\sin\theta\right)rdrd\theta+\int_{\theta=3\pi/2}^{2\pi}\int_{r=0}^{a}\left(r\cos\theta+r\sin\theta\right)rdrd\theta$$

$$=\int_{0}^{a}r^{2}dr\times\left\{\int_{0}^{\pi/2}\left(\cos\theta+\sin\theta\right)d\theta+\int_{3\pi/2}^{2\pi}\left(\cos\theta+\sin\theta\right)\right\}$$

$$=\frac{a^{3}}{3}\left\{\left(1+1\right)+\left(1-1\right)\right\}=\frac{2}{3}a^{3}.$$

Problem 4: Using repeated integrals, find the area bounded by the arc of the ellipse $x^2/a^2 + y^2/b^2 = 1$ in the first quadrant.

Solution: In the region of integration x increases from 0 to a and, for each x, y increase from 0 to a point on the ellipse; i.e the point for which $y = b(1 - x^2/a^2)^{1/2}$. Hence, the required are is

$$A = \int_{x=0}^{a} \int_{y=0}^{b\left(1-x^2/a^2\right)^{1/2}} dydyx = \int_{x=0}^{a} \left\{ \int_{y=0}^{b\left(1-x^2/a^2\right)^{1/2}} dy \right\} dx$$

$$= \int_0^a b\left(1-\frac{x^2}{a^2}\right)^{1/2} dx = \frac{b}{a}\int_0^a \left(a^2-x^2\right)^{1/2} dx = \frac{b}{a}\int_0^{\pi/2} (a\cos\theta)(a\cos\theta d\theta),$$

on setting $x = a\sin\theta$

$$= ab \int_0^{\frac{\pi}{2}} \cos^2\theta d\theta = ab.\frac{1}{2}.\frac{\pi}{2} = \frac{\pi}{4} ab.$$

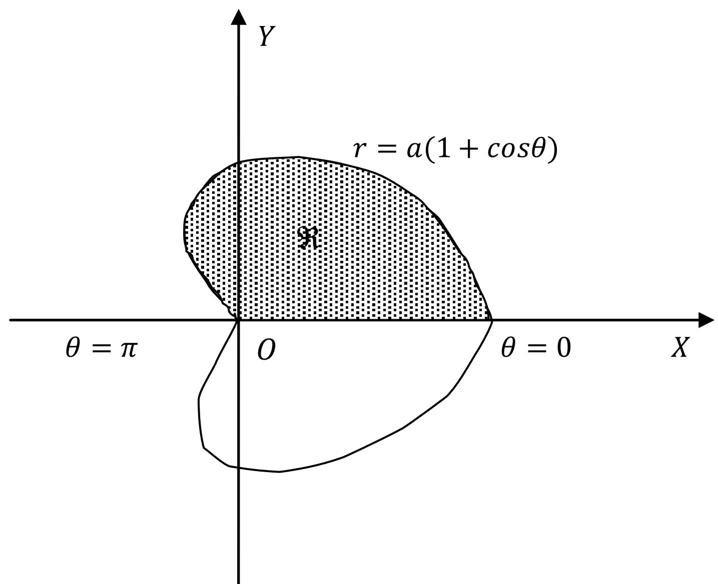
Problem 5: Find the areas enclosed by the following curves:

(i) Cardioids: $r = a(1 + cos\theta)$ between $\theta = 0$ and $\theta = \pi$.

(ii) One loop of the Lemniscate $r^2 = a^2\cos2\theta$.

Solution:

(i)



The given region is depicted in figure (shaded portion).In this region, θ varies from O to π and, for each θ, r varies from O to $a(1 + cos\theta)$.

Therefore, the required area is

$$A = \int_{\theta=0}^{\pi} \int_{r=0}^{a(1+cos\theta)} rdr\, d\theta$$

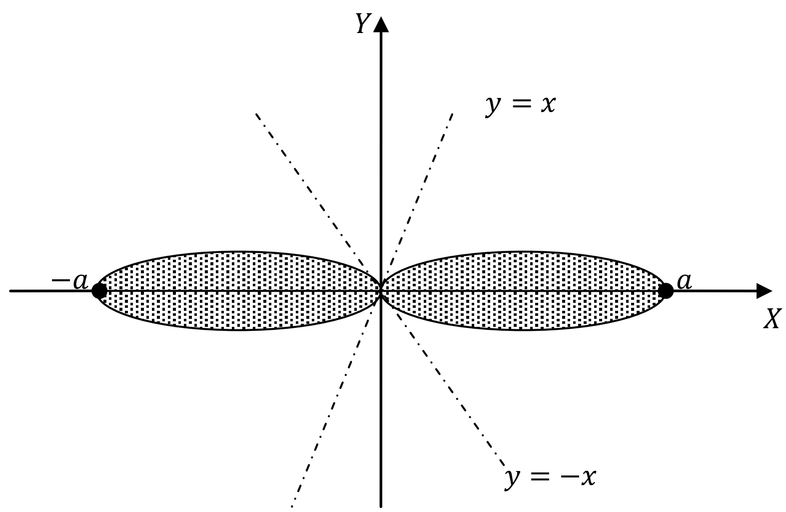
$$= \int_0^{\pi} [\frac{r^2}{2}]_0^{a(1+cos\theta)} d\theta$$

$$= \frac{a^2}{2}\int_0^{\pi}(1+cos\theta)^2 d\theta$$

$$= \frac{a^2}{2}\int_0^{\pi}\{1+2cos\theta+\frac{1}{2}(1+cos2\theta)\}\,d\theta$$

$$= \frac{a^2}{2}\left\{\pi+0+\frac{1}{2}(\pi+0)\right\} = \frac{3a^2}{4}\pi.$$

(ii)



The given Lemniscate is shown in figure. We note that the area enclosed by one loop of this curve is twice the area bounded by the first quadrant, for which θ increases O to $\frac{\pi}{4}$ and, r varies from O to $a\sqrt{cos2\theta}$. Hence the required area is

$$A = 2\int_{\theta=0}^{\pi/4}\int_{r=0}^{a\sqrt{cos2\theta}} r\ dr\, d\theta$$

$$= 2\int_0^{\pi/4}\left\{\left[\frac{r^2}{2}\right]_0^{a\sqrt{cos2\theta}}\right\}d\theta$$

$$= \int_{0}^{\pi/4} a^2 cos2\theta d\theta$$

$$= a^2 \left[\frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{4}} = \frac{a^2}{2}.$$

Exercise

1. Evaluate the following integrals by changing the Cartesian coordinates to polar coordinates:

   a. $\int_{-a}^{a}\int_{0}^{\sqrt{a^2-x^2}} \sqrt{x^2+y^2}\ dydx$

   b. $\int_{0}^{4a}\int_{y^2/4a}^{y} \frac{x^2-y^2}{x^2+y^2} dxdy$

2. Evaluate the integral $\int_{o}^{\pi}\int_{0}^{a} r^3 sin\theta cos\theta\ dr\ d\theta$ by changing the polar coordinates to Cartesian coordinates.
3. Find the area lying inside the circle $r = a\sin\theta$ and outside the cardioids $r = a(1 - cos\theta)$.
4. Find the area included between the curve $r = a(sec\theta + cos\theta)$ and its asymptote $r = asec\theta$.
5. Find the area bounded by the positive X-axis, the arc of the circle $x^2 + y^2 = a^2$ and the upper part of the line $y = x$.
6. Find the area enclosed by the parabola $y^2 = 4ax$ and the line $x + y = 3a$.
7. Find the area bounded by the parabola $y = 4x - x^2$ and the line $y = x$.
8. Find the area bounded between the circles $r = a$ and $r = 2a\ cos\theta$.
9. Find the area bounded by the circles $r = 2a\ sin\theta$ and $r = 2b\ sin\theta$, $b > a > 0$.

Answers

1.

a. $\left(\frac{1}{3}\right)\pi a^3$

b. $8(\frac{\pi}{2} - \frac{5}{3})a^2$

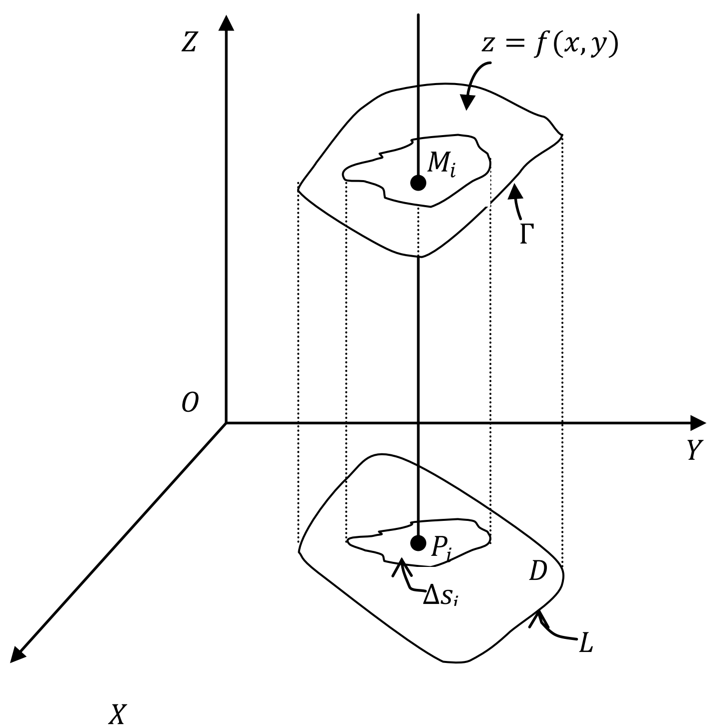
2. 0
3. $a^2(1 - \frac{\pi}{4})$
4. $\frac{5\pi a^2}{4}$
5. $\frac{\pi a^2}{8}$
6. $\frac{10}{3}a^2$
7. $\frac{9}{2}$
8. $a^2(\frac{\pi}{3} + \frac{\sqrt{3}}{2})$
9. $(b^2 - a^2)\pi$

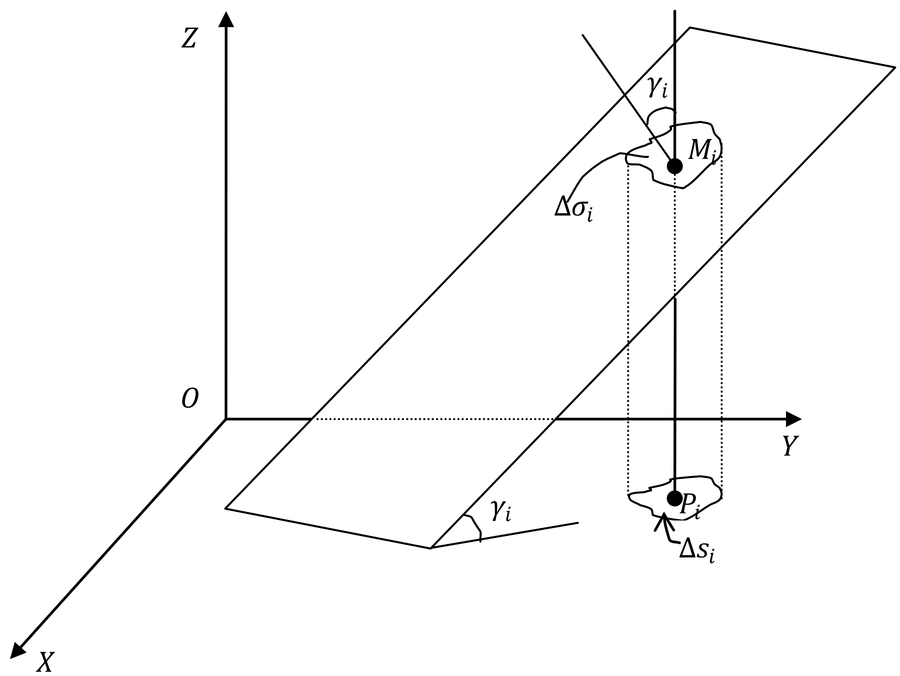
2.7

SURFACE AREA AND VOLUME BY USING DOUBLE INTEGRALS

COMPUTING THE AREA OF A SURFACE

Let it be required to compute the area of a surface bounded by a curve Γ (given in the figure below); the surface is defined by the equation $z = f(x, y)$, where the function $f(x, y)$ is continuous and has continuous partial derivatives. Denote the projection of Γ on the $XY-$plane by L. Denote by D the domain on the $XY-$plane bounded by the curve L.





In arbitrary fashion, divide D into n elementary subdomains $\Delta s_1, \Delta s_2, \ldots \Delta s_n$. In each subdomain Δs_i take a point $P_i(\xi_i, \eta_i)$. To the point P_i there will correspond, on the surface, a point

$$M_i[\xi_i, \eta_i, f(\xi_i, \eta_i)]$$

Through M_i draw a tangent plane to the surface. Its equation is of the form

$$z - z_i = f'_x(\xi_i, \eta_i)(x - \xi_i) + f'_y(\xi_i, \eta_i)(y - \eta_i) \tag{1}$$

In this plane, pick out a subdomain $\Delta\sigma_i$ which is projected onto the $XY-$plane in the form of a subdomain Δs_i . Consider the sum of the sub domains $\Delta\sigma_i$:

$$\sum_{i=1}^{n} \Delta\sigma_i$$

We shall call the limit σ of this sum, when the greatest of the diameters of the subdomains $\Delta\sigma_i$ approaches zero, the area of the surface; that is, by definition we set

$$\sigma = \lim_{diam\ \Delta\sigma_i \to 0} \sum_{i=1}^{n} \Delta\sigma_i \qquad (2)$$

Now let us calculate the area of the surface. Denote by γ_i the angle between the tangent plane and the $XY-$plane. Using a familiar formula of analytic geometry we can write

$$\Delta s_i = \Delta\sigma_i cos\gamma_i$$

or

$$\Delta\sigma_i = \frac{\Delta s_i}{cos\,\gamma_i} \qquad (3)$$

The angle γ_i is at the same time the angle between the $Z-$axis and the perpendicular to the plane (1). Therefore, by equation (1) and the formula of analytic geometry we have

$$cos\gamma_i = \frac{1}{\sqrt{1 + f'^2_x(\xi_i, \eta_i) + f'^2_y(\xi_i, \eta_i)}}$$

Hence,

$$\Delta\sigma_i = \sqrt{1 + f'^2_x(\xi_i, \eta_i) + f'^2_y(\xi_i, \eta_i)}\, \Delta s_i$$

Putting this expression into formula (2), we get

$$\sigma = \lim_{diam\ \Delta s_i \to 0} \sum_{i=1}^{n} \sqrt{1 + f'^2_x(\xi_i, \eta_i) + f'^2_y(\xi_i, \eta_i)}\, \Delta s_i$$

Since the limit of the integral sum on the right side of the last equation is, by definition, the double integral $\iint_D \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\, dxdy$ we finally get

$$\sigma = \iint_D \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\, dxdy \qquad (4)$$

This is the formula use to compute the area of the surface $z = f(x, y)$.

If the equation of the surface is given in the form

$$x = \mu(y, z) \text{ or in the form } y = \chi(x, z)$$

then the corresponding formulas for calculating the surface area are of the form

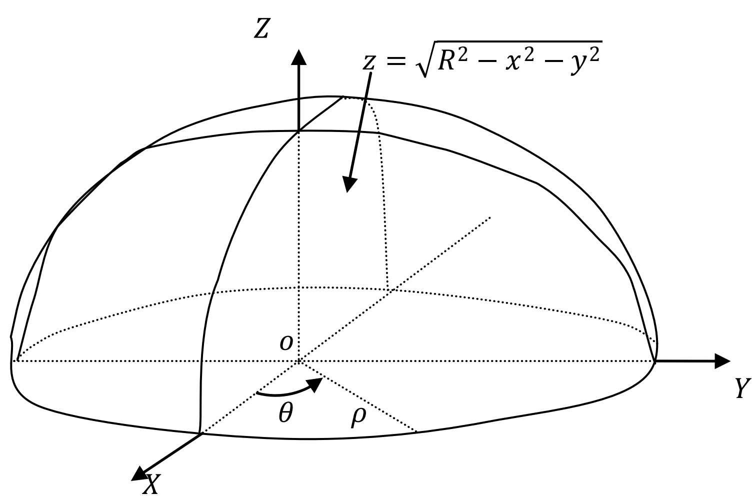
$$\sigma = \iint_{D'} \sqrt{1+\left(\frac{\partial x}{\partial y}\right)^2+\left(\frac{\partial x}{\partial z}\right)^2}\,dydz \tag{4'}$$

$$\sigma = \iint_{D''} \sqrt{1+\left(\frac{\partial y}{\partial x}\right)^2+\left(\frac{\partial y}{\partial z}\right)^2}\,dxdz \tag{4''}$$

where D' and D'' are the domains in the YZ-plane and the XZ-plane in which the given surface is projected.

Example: Compute the surface area σ of the sphere $x^2+y^2+z^2=R^2$

Solution: Compute the surface are of the upper half of the sphere $z=\sqrt{R^2-x^2-y^2}$



In this case

$$\frac{\partial z}{\partial x} = -\frac{x}{\sqrt{R^2 - x^2 - y^2}}$$

$$\frac{\partial z}{\partial y} = -\frac{y}{\sqrt{R^2 - x^2 - y^2}}$$

Hence,

$$\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2} = \sqrt{\frac{R^2}{R^2 - x^2 - y^2}} = \frac{R}{\sqrt{R^2 - x^2 - y^2}}$$

The domain of integration is defined by the condition

Thus, by formula (4) we will have

$$\frac{1}{2}\sigma = \int_{-R}^{R}\left(\int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} \frac{R}{\sqrt{R^2 - x^2 - y^2}}\,dy\right)dx$$

To compute the double integral obtain let us make the transformation to polar coordinates. In polar coordinates the boundary of the domain of integration is determined by the equation $\rho = R$. Hence,

$$\sigma = 2\int_0^{2\pi}\left(\int_0^R \frac{R}{\sqrt{R^2 - \rho^2}}\rho d\rho\right)d\theta = 2R\int_0^{2\pi}\left[-\sqrt{R^2-\rho^2}\right]_0^R d\theta$$

$$= 2R\int_0^{2\pi} R d\theta = 4\pi R^2 .$$

Computing the Volume of a Solid

Recall that

1. If $f(x, y) = 1$, then $\iint_R dxdy$ gives the area A of the region R.
2. If $z = f(x, y)$ is a surface, then
$$\iint_R zdxdy \text{ or } \iint_R f(x,y)dxdy$$
gives the volume of the region beneath the surface $z = f(x, y)$ and above the XY- plane.

Example: Evaluate the volume of the sphere

$x^2 + y^2 + z^2 = a^2$.

Solution: The given sphere is $z = \sqrt{a^2 - x^2 - y^2}$

The volume of the upper half of the sphere is

$$\iint_{x^2+y^2 \le a^2} \sqrt{a^2 - x^2 - y^2}dx\, dy$$

By changing to polar coordinates.

i.e substitute $x = r\cos\theta\, , y = r\sin\theta\, , dx\, dy = r\, dr\, d\theta$

$$\iint_{x^2+y^2 \le a^2} \sqrt{a^2 - x^2 - y^2}dx\, dy = \int_0^{2\pi} \int_0^a \sqrt{a^2 - r^2}\ r\, dr\, d\theta$$

$$= \int_0^{2\pi} \left[\left(-\frac{1}{2}\right) \int_0^a \sqrt{a^2 - r^2}. d(-r)^2\right] d\theta$$

$$= \int_0^{2\pi} \frac{1}{3} a^3\, d\theta = \frac{2}{3}\pi a^3.$$

Therefore the volume of the sphere is $2\left(\left(\frac{2}{3}\pi a^3\right)\right) = \frac{4}{3}\pi a^3$.

Problem 1: Compute the area of that part of the surface of the cone $x^2+y^2=z^2$ which is cut out by the cylinder $x^2+y^2=2ax$.

Solution: The equation of the surface of the upper half of the cone is $z=\sqrt{x^2+y^2}$

$$\therefore \frac{\partial z}{\partial x}=\frac{x}{\sqrt{x^2+y^2}}, \qquad \therefore \frac{\partial z}{\partial y}=\frac{y}{\sqrt{x^2+y^2}}$$

∴ The domain of integration is defined by

$$x^2+y^2\le 2ax \Rightarrow (x-a)^2+y^2\le a^2$$

∴ Surface area of upper half cone

$$=\iint_D \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dydx$$

Total surface area $=2\int_0^{2a}\int_{-\sqrt{a^2-(x-a)^2}}^{\sqrt{a^2-(x-a)^2}} \sqrt{1+\frac{x^2}{x^2+y^2}+\frac{y^2}{x^2+y^2}}\,dydx$

$$=2\int_0^{2a}\int_{-\sqrt{a^2-(x-a)^2}}^{\sqrt{a^2-(x-a)^2}} \sqrt{\frac{2(x^2+y^2)}{x^2+y^2}}\,dydx$$

$$=4\int_0^{2a}\int_0^{\sqrt{a^2-(x-a)^2}} \sqrt{2}\,dydx$$

$$= 4\sqrt{2}\int_0^{2a}\left[y\right]_0^{\sqrt{a^2-(x-a)^2}}\,dx$$

$$= 4\sqrt{2}\int_0^{2a}\sqrt{a^2-(x-a)^2}\,dx$$

$$= 4\sqrt{2}\left[\frac{x-a}{2}\sqrt{a^2-(x-a)^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{x-a}{a}\right)\right]_0^{2a}$$

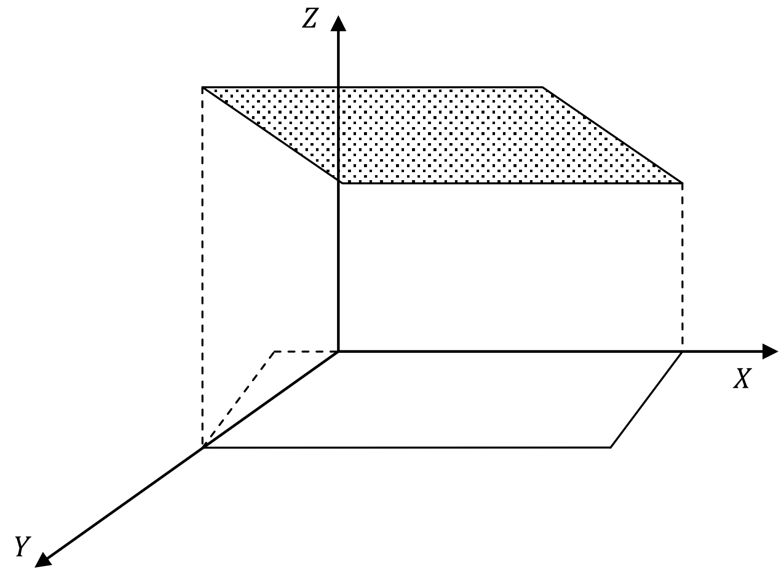
$$= 4\sqrt{2}\left[\frac{a^2}{2}\sin^{-1}1-\frac{a^2}{2}\sin^{-1}(-1)\right]$$

$$= 4\sqrt{2}\,\frac{a^2}{2}\left[\frac{\pi}{2}+\frac{\pi}{2}\right]$$

$$= 2\sqrt{2}\pi a^2.$$

Problem 2: Find the surface area of $2x + 3y - z = 1$ in the region $[0,1] \times [0,1]$.

Solution:



The equation of the surface has the form

$z = 1 + 2x + 3y$

$\therefore \dfrac{\partial z}{\partial x} = 2, \qquad \therefore \dfrac{\partial z}{\partial y} = 3$

The region $D = [0,1] \times [0,1]$

$$\therefore \text{Surface area } = \iint_D \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dydx$$

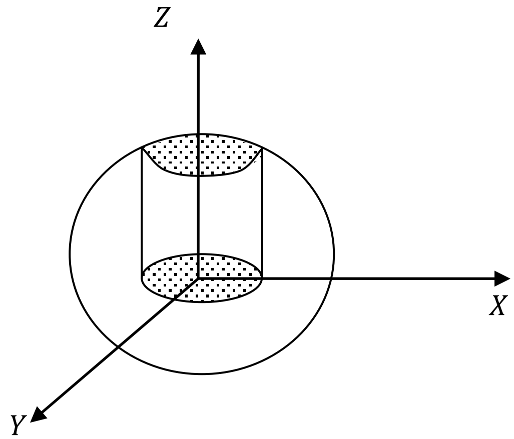
$$= \int_0^1\int_0^1 \sqrt{1+4+9}\,dydx$$

$$= \int_0^1\int_0^1 \sqrt{14}\,dydx$$

$$= \sqrt{14}.$$

Problem 3: Find the surface area of the portion of the unit sphere above $z = \frac{4}{5}$

Solution:



Unit sphere is $x^2 + y^2 + z^2 = 1$

$$\Rightarrow z = \sqrt{1 - x^2 - y^2}$$

$$\frac{\partial z}{\partial x} = -\frac{x}{\sqrt{1-x^2-y^2}}, \quad \frac{\partial z}{\partial y} = -\frac{y}{\sqrt{1-x^2-y^2}}$$

$$\frac{4}{5} = \sqrt{1 - x^2 - y^2} \Rightarrow \frac{16}{25} = 1 - x^2 - y^2 \Rightarrow x^2 + y^2 = \frac{9}{25}$$

Circle of radius is $\frac{3}{5}$

We have to find surface area of $z = \sqrt{1 - x^2 - y^2}$ over $x^2 + y^2 = \frac{9}{25}$

Domain of the radius is $\frac{3}{5}$

$$\therefore \text{Surface area} = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dydx$$

$$= \iint_D \sqrt{\frac{1}{1-x^2-y^2}}\,dydx$$

Transformation to polar co coordinates. In polar coordinates the boundary of the domain of integration is determined by the equation

$$r = \frac{3}{5}$$

Let $x = r\cos\theta, \quad y = r\sin\theta$

$\therefore$ Surface area $= \int_0^{2\pi} \int_0^{\frac{3}{5}} \frac{1}{\sqrt{1-r^2}}\, r dr d\theta$

$$= \int_0^{2\pi} \left[-\sqrt{1-r^2}\right]_0^{\frac{3}{5}} d\theta$$

$$= \int_0^{2\pi} \left(-\sqrt{1-\frac{9}{25}}+1\right) d\theta$$

$$= \int_0^{2\pi} \left(-\frac{4}{5}+1\right) d\theta$$

$$= \int_0^{2\pi} \frac{1}{5}\, d\theta$$

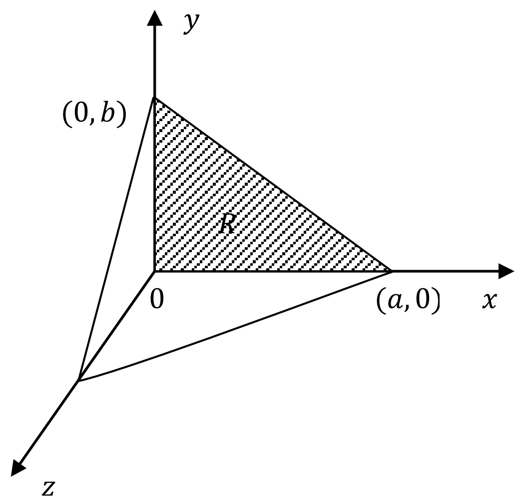
$$= \frac{1}{5}[\theta]_0^{2\pi}$$

$$= \frac{2\pi}{5}.$$

Problem 4: Find the volume of the tetrahedron bounded by the coordinate surfaces $x = 0, y = 0$ and $z = 0$ and the plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

Solution:

The volume of the tetrahedron $(V) = \iint_R z\,dx\,dy$



$$V = \int_0^a \int_0^{b-\frac{bx}{a}} c\left(1 - \frac{y}{b} - \frac{x}{a}\right) dy\,dx$$

$$= c\int_0^a \left[y - \frac{y^2}{2b} - \frac{xy}{a} \right]_0^{b-\frac{bx}{a}} dx$$

$$= c\int_0^a \left(\frac{bx^2}{2a^2} - \frac{bx}{a} + \frac{b}{2} \right) dx$$

$$= c\left[\frac{bx^3}{6a^2} - \frac{bx^2}{2a} + \frac{b}{2}x \right]_0^a = \frac{abc}{6}.$$

Problem 5: A circular hole of a radius b is made centrally through a sphere of radius a. Find the volume of the remaining of the sphere.

Solution:

Let the centre of the sphere be at the origin and let the axis of the hole be along the z-axis. The volume V of the sphere is $\frac{4}{3}\pi a^3$ and that of the circular hole is obtained as follows.

$$\text{Volume of the upper-half of the hole} = \iint_R f(x,y)\,dxdy$$

$$= \iint_R z\,dxdy$$

where z is obtained from the equation $x^2+y^2+z^2=a^2$ and R is the circle in the $XY-$plane.

i.e $\quad x^2+y^2=b^2$

$\therefore$The volume V_1 of the circular hole is

$$V_1 = 2\iint_R \sqrt{a^2-x^2-y^2}\,dxdy$$

where R is $x^2+y^2=b^2$ changing into polar coordinates

$$\therefore V_1 = 2\int_0^{2\pi}\int_0^{b} \sqrt{a^2-r^2}\,r\,dr\,d\theta = \int_0^{2\pi}\left[\frac{(a^2-r^2)^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^b d\theta$$

$$=\frac{-2}{3}\int_0^{2\pi}\left[(a^2-b^2)^{\frac{3}{2}}-a^3\right]d\theta$$

$$=\frac{2}{3}\int_0^{2\pi}\left[a^3-(a^2-b^2)^{\frac{3}{2}}\right]d\theta$$

$$=\frac{2}{3}\left[a^3-(a^2-b^2)^{\frac{3}{2}}\right][\theta]_0^{2\pi}$$

$$=\frac{4\pi}{3}\left[a^3-(a^2-b^2)^{\frac{3}{2}}\right]$$

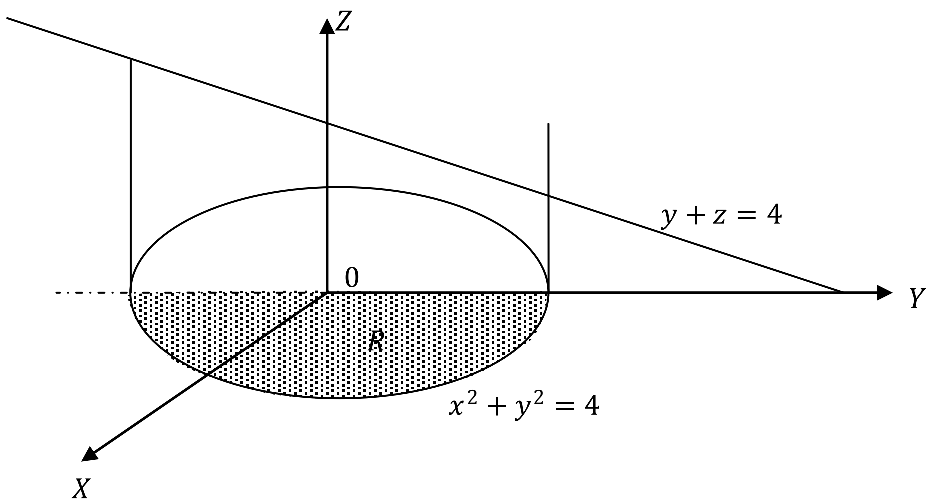
Volume of the remaining portion $= V - V_1$

$$=\frac{4}{3}\pi a^3-\frac{4\pi}{3}\left[a^3-(a^2-b^2)^{\frac{3}{2}}\right]$$

$$=\frac{4\pi}{3}(a^2-b^2)^{\frac{3}{2}}.$$

Problem 6: Find the volume bounded by the cylinder $x^2 + y^2 = 4$, $y + z = 4$ and $z = 0$.

Solution:



The volume V of the plane $y + z = 4$ and $z = 0$ is

$$V = \iint_R z\,dx\,dy$$

$$= \iint_R (4 - y)\,dx\,dy$$

where R is bounded by the $x^2 + y^2 = 4$

$$\therefore V = \int_{-2}^{2} \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} (4 - y)\,dx\,dy$$

$$= \int_{-2}^{2} (4 - y)\left[x\right]_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} dy$$

$$= \int_{-2}^{2} (4-y).2\sqrt{4-y^2}\,dy$$

$$= 2\int_{-2}^{2} 4\sqrt{4-y^2}\,dy - 2\int_{-2}^{2} y\sqrt{4-y^2}\,dy$$

$$=16\int_{0}^{2} \sqrt{4-y^2}\,dy - 0$$ ($\because y\sqrt{4-y^2}$ is odd function)

$$=16\left[\frac{y}{2}\sqrt{4-y^2} + 2\sin^{-1}\frac{y}{2}\right]_0^2$$

$$=16\left[2\sin^{-1}1\right] = 32\cdot\frac{\pi}{2} = 16\pi\,.$$

Exercise

1. Compute the area of that part of the plane $x + y + z = 2a$. Which lies in the first octant and is bounded by the cylinder $x^2 + y^2 = a^2$.
2. Compute the area of that part of the square of the cone $x^2 + y^2 = z^2$ which is cut by the cylinder $x^2 + y^2 = 2ax$.
3. Find the surface area of a solid that is the common part of two cylinders $x^2 + y^2 = a^2,\ y^2 + z^2 = a^2$.
4. Compute the volumes of solids bounded by the coordinate planes, the plane $2x + 3y - 12 = 0$ and the cylinder $z = \frac{1}{2}y^2$.
5. Compute the volumes of solids bounded by the following surfaces:
   a) $z = 0, x^2 + y^2 = 1, x + y + z = 3$.
   b) $x^2 + y^2 - 2ax = 0, z = 0, x^2 + y^2 = z^2$.
6. The base of a solid is the region in $XY -$ plane. That is bounded by the circle $x^2 + y^2 = a^2$. While the top of the solid is bounded by the paraboloid $az = x^2 + y^2$. Find the volume.
7. Find the volume common to the cylinders $x^2 + y^2 = a^2$ and $x^2 + z^2 = a^2$.

Answers

1. $\dfrac{\sqrt{3}}{4}\pi a^2$
2. $2\sqrt{2}\pi a^2$

3. $8a^2$

4. 16

5.

 a) 3π

 b) $\frac{32}{9}a^3$

6. $\frac{1}{2}\pi a^3$

7. $\frac{16a^3}{3}$

2.8

Computing Integrals Dependent on a Parameter

Consider an integral dependent on the parameter α:

$$I(\alpha) = \int_a^b f(x, \alpha) dx$$

We state without proof that if a function $f(x, \alpha)$ is continuous with respect to x over an interval $[a, b]$ and with respect to α over an interval $[\alpha_1, \alpha_2]$, then the function

$$I(\alpha) = \int_a^b f(x, \alpha) dx$$

is a continuous function on $[\alpha_1, \alpha_2]$. Consequently, the function $I(\alpha)$ may be integrated with respect to α over the interval $[\alpha_1, \alpha_2]$

$$\int_{\alpha_1}^{\alpha_2} I(\alpha)\, d\alpha = \int_{\alpha_1}^{\alpha_2} \left[\int_a^b f(x, \alpha) dx \right] d\alpha$$

The expression on the right is an iterated integral of the function $f(x, \alpha)$ over a rectangle situated in the plane $X\alpha$. We can change the order of integration in this integral:

$$\int_{\alpha_1}^{\alpha_2} \left[\int_a^b f(x, \alpha) dx \right] d\alpha = \int_a^b \left[\int_{\alpha_1}^{\alpha_2} f(x, \alpha) d\alpha \right] dx$$

This formula shows that for integration of an integral dependent on a parameter, it is sufficient to integrate the element of integration with respect to the parameter . This formula is also useful when computing definite integral

Example: Compute the integral

$$\int_0^\infty \frac{e^{-ax} - e^{-bx}}{x}\, dx \; (a > 0, b > 0)$$

This integral is not expressible in terms of the elementary function .To evaluate it , we consider another integral that may be readily computed:

$$\int_0^\infty e^{-\alpha x}\, dx = \frac{1}{\alpha} (\alpha > 0)$$

Integrating this equation between the limits $\alpha_1 = a$ and $\alpha_2 = b$, we get

$$\int_a^b \left[\int_0^\infty e^{-\alpha x}\, dx \right] d\alpha = \int_a^b \frac{d\alpha}{\alpha} = \ln\frac{b}{a}$$

changing the order of integration in the first integral, we rewrite this equation in the following form:

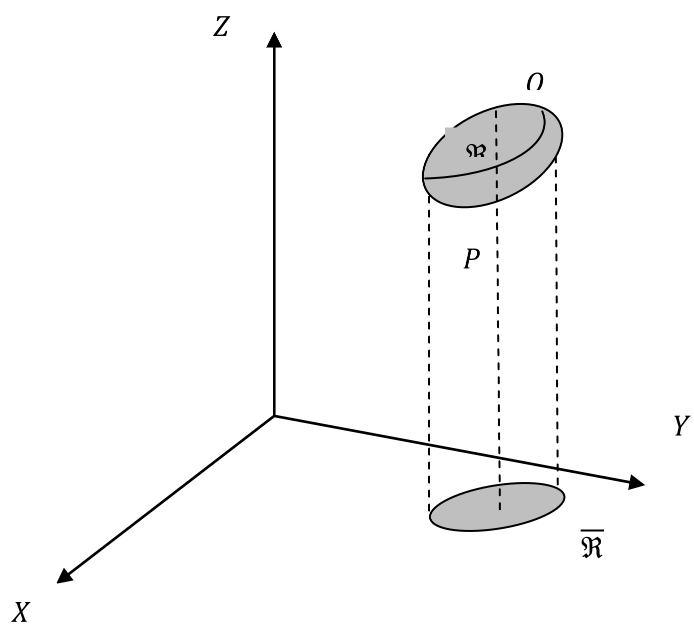
$$\int_0^\infty \left[\int_a^b e^{-\alpha x}\, d\alpha \right] dx = \ln\frac{b}{a}$$

whence, computing the inner integral, we get

$$\int_0^\infty \frac{e^{-ax} - e^{-bx}}{x}\, dx = \ln\frac{b}{a}.$$

Triple Integrals

The definition of a double integral can be extended to three-dimensions. For this purpose, let us consider a region $\Re$, in three dimensional spaces, bounded by a closed surface S whose Cartesian equation is of the form $\varphi(x, y, z) = 0$. Let $\overline{\Re}$ be the projection of $\Re$ on the $XY-$ plane. Then, as a point (x, y, z) varies over $\Re$, the corresponding point (x, y) varies over $\overline{\Re}$. Consider a line parallel to the $Z-$ axis and suppose this line cuts S at two points P and Q, P being below Q. Let z_1 and z_2 be the $Z-$ coordinates of P and Q respectively. Since P and Q lie on S, whose Cartesian equation is of the form $\varphi(x, y, z) = 0, z_1$ and z_2 are appropriate functions of $x, y; i.e. z_1 = z_1(x, y)$ and $z_2 = z_2(x, y)$.



Now, consider a function $f(x, y, z)$ defined over $\Re$ and S. On the line PQ, the coordinates (x, y) are held fixed so that $f(x, y, z)$ is a function of z only. Suppose we integrate this function with respect to z from $z_1(x, y)$ to $z_2(x, y)$. The resulting integral is a function of (x, y); let us denote it by $\psi(x, y)$. Thus,

$$\psi(x, y) = \int_{z_1(x,y)}^{z_2(x,y)} f(x, y, z)\, dz \tag{1}$$

Now, suppose we take the double integral of $\psi(x, y)$ over the plane region $\overline{\Re}$, and denote it by I. The integral I has following representation:

$$I = \int_a^b \left\{ \int_{y_1(x)}^{y_2(x)} \psi(x, y)\, dy \right\} dx \tag{2}$$

Substituting for $\psi(x, y)$ from (1) in (2), we get

$$I = \int_a^b \left\{ \int_{y_1(x)}^{y_2(x)} \left[\int_{z_1(x,y)}^{z_2(x,y)} f(x, y, z)\, dz \right] dy \right\} dx \tag{3}$$

Evidently, the integration covers the whole of the region $\Re$, and the integral is called the volume integral of $f(x, y, z)$ over the region $\Re$. It is denoted by $\int_V f(x, y, z) dV$; here, V stands for the volume of $\Re$. Expression (3) shown that this volume integral is represented in terms of three repeated integrals. For this reason a volume integral is also called a Triple integral and is denoted by

$$\iiint_{\Re} f(x, y, z) dxdydz \quad \text{..............................} \ (*)$$

Thus,

$$\int_V f(x,y,z) = \iiint_{\Re} f(x,y,z)dxdydz$$

$$= \int_a^b \int_{y_1(x)}^{y_2(x)} \int_{z_1(x,y)}^{z_2(x,y)} f(x,y,z)\, dzdydx \qquad (4)$$

Here, it is understood that, in the right hand side, the first integral sign is with respect to x, the second integral sign is with respect to y and the third integral sign is with respect to z. Thus, while evaluating the repeated integrals in (4), we have to first integrate $f(x,y,z)$ with respect to z from z_1 to z_2, keeping (x,y) fixed. Then we have to integrate the resulting function with respect to y from y_1 to y_2, keeping x fixed. Finally, we have to integrate the resulting function with respect to x from a to b. A similar procedure is adopted when the integrals with respect to x, y, z appear in other orders.

Change of Variables

While evaluating triple integrals computational work can often be reduced by changing the variables x, y, z to some other appropriate variables u, v, w which are related to x, y, z and which are such that the Jacobian

$$J \equiv \frac{\partial(x,y,z)}{\partial(u,v,w)} \neq 0.$$

It can be proved that (we omit the proof)

$$\iiint_{\Re} f(x,y,z) = \iiint_{\bar{\Re}} \phi(u,v,w) J dudvdw \qquad (1)$$

Here, $\Re$ is the region in which (x, y, z) vary and $\bar{\Re}$ is the corresponding region in which (u, v, w) vary, and

$$\psi(u, v, w) = f\{x\,(u, v, w), y(u, v, w), z(u, v, w)\}$$

Once the triple integral with respect to (x, y, z) is changed to a triple integral with respect to (u, v, w) by using formula (1), the latter integral may be evaluated by expressing it in terms of repeated integrals with appropriate limits of integration.

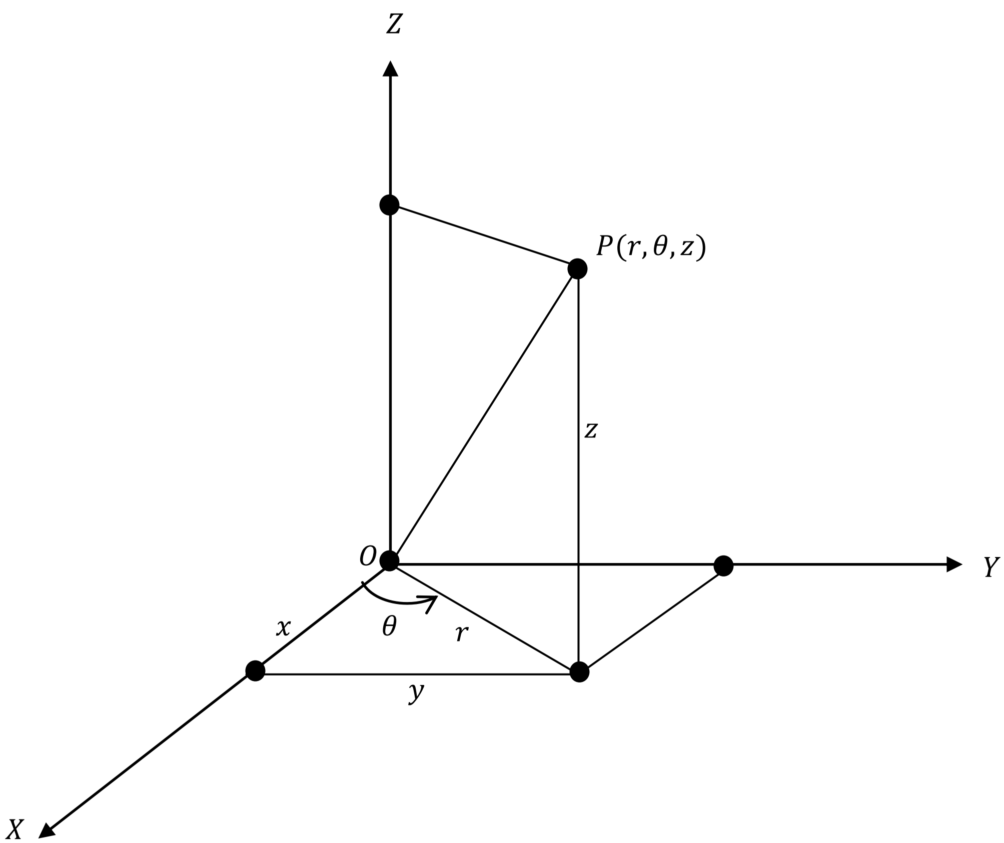
Two standard examples of (u, v, w) which are widely used in the evaluation of triple integrals are the cylindrical polar coordinates (r, θ, z) and the spherical polar coordinates (r, θ, ϕ). Let us specialize the expression (1) for these two cases.

Triple Integral in Cylindrical Polar Coordinates

Suppose (x, y, z) are related to three variables (r, θ, z) through the relations

$$x = r\cos\theta, \quad y = r\ \sin\theta\,, \quad z = z \tag{2}$$

Then (r, θ, z) are called cylindrical polar coordinates. These coordinates are depicted in following figure.



From expressions (2), we find that

$$J = \frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{vmatrix} = \begin{vmatrix} \cos\theta & -R\ \sin\theta & 0 \\ \sin\theta & R\cos\theta & 0 \\ 1 & 0 & 0 \end{vmatrix} = r \qquad (3)$$

Hence, for $(u, v, w) = (r, \theta, z)$, expression (1) becomes

$$\iiint_{\Re} f(x, y, z) = \iiint_{\Re} \varphi(r, \theta, z) r dr d\theta dz \qquad (4)$$

Here $\bar{\Re}$ is the region in which (r, θ, z) vary as (x, y, z) vary in $\Re$, and $\varphi(r, \theta, z) = f(x, y, z)$ with x, y, z given by (2). Observe that when (x, y, z) are changed to (r, θ, z) and $dxdydz$ changes to $rdrd\theta dz$.

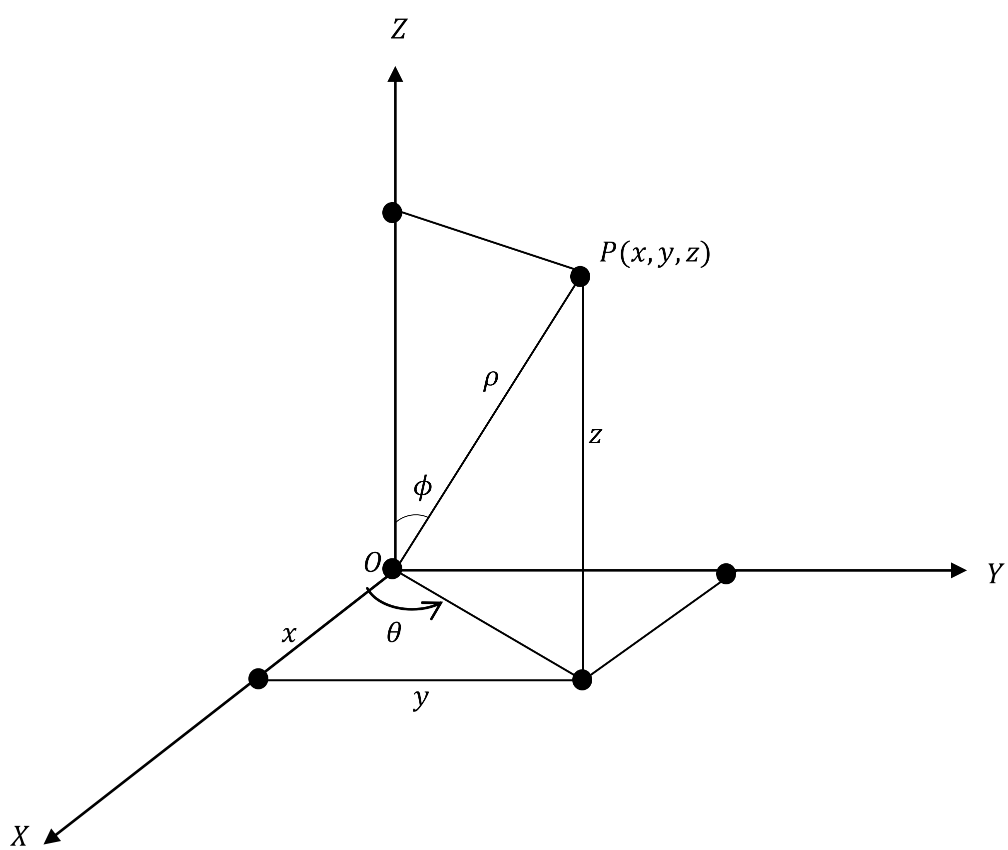
The formula (4) is particularly useful when the region $\Re$ is bounded by a cylindrical surface.

Triple Integral in Spherical Polar Coordinates

Suppose (x, y, z) are related to three variables $(\rho, \theta, \emptyset)$ through the relations

$$x = \rho \ \sin \emptyset \cos \theta, \quad y = \rho \ \sin \emptyset \sin \theta, \quad z = \rho \ \cos \emptyset \tag{5}$$

Then $(\rho, \theta, \emptyset)$ are called spherical polar coordinates. These coordinates are depicted in following figure.



From expressions (5), we find that

$$J = \frac{\partial(x,y,z)}{\partial(\rho,\theta,\emptyset)} = \begin{vmatrix} \frac{\partial x}{\partial\rho} & \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\emptyset} \\ \frac{\partial y}{\partial\rho} & \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\emptyset} \\ \frac{\partial z}{\partial\rho} & \frac{\partial z}{\partial\theta} & \frac{\partial z}{\partial\emptyset} \end{vmatrix}$$

$$= \begin{vmatrix} \sin\emptyset\cos\theta & -\rho\ \sin\phi\sin\theta & \rho\ \cos\ \theta\cos\ \emptyset \\ \sin\theta\sin\emptyset & \rho\ \cos\theta\sin\emptyset & \rho\ \sin\theta\cos\emptyset \\ \cos\phi & 0 & -\rho\ \sin\phi \end{vmatrix}$$

$$= \rho^2 \sin\phi. \tag{6}$$

Hence, for $(u, v, w) = (\rho, \theta, \emptyset)$, expression (1) becomes

$$\iiint_{\Re} f(x,y,z)dxdydz = \iiint_{\bar{\Re}} \phi(r,\theta,\phi)\rho^2 \sin\phi d\rho d\theta dz \tag{7}$$

Here $\bar{\Re}$ is the region in which $(\rho, \theta, \emptyset)$ vary as (x, y, z) vary in $\Re$, and $\varphi(\rho, \theta, \emptyset) = f(x, y, z)$ wirh x, y, z given by (5). Observe that when (x, y, z) are changed to $(\rho, \theta, \emptyset)$ and $dx\, dy\, dz$ changes to $\rho^2 \sin\phi\, d\rho d\theta d\emptyset$.

The formula (7) is particularly useful when the region $\Re$ is bounded by a spherical surface.

Computation of Volume:

Let us recall expression $(*)$. In the particular case where $f(x, y, z) \equiv 1$, this expression becomes

$$\int_V dV \equiv \iiint_{\Re} dx\, dy\, dz = \int_a^b \int_{y_1(x)}^{y_2(x)} \int_{Z_1(x,y)}^{Z_2(x,y)} dz\, dy\, dx \qquad (1)$$

The integral $\int_V dV$ represents the volume V of the region $\Re$. Thus, expression (1) may be used to compute V.

If (x, y, z) are changed to (u, v, w), we obtain the following expression for the volume.

$$\int_V dV \equiv \iiint_{\Re} dx\, dy\, dz = \int_{\overline{\Re}} J\, dudvdw \qquad (2)$$

Taking $(u, v, w) = (r, \theta, z)$ in the above expression, we obtain the following expression for volume in terms of cylindrical polar coordinates

$$\int_V dV \equiv \iiint_{\Re} r\, dr\, d\theta dz \qquad (3)$$

Similarly, we obtain the following expression for volume in terms of spherical polar coordinates $(\rho, \theta, \emptyset)$;

$$\int_V dV \equiv \iiint_{\Re} \rho^2 \sin\phi\, d\rho\, d\theta d\emptyset \qquad (4)$$

Observe that $rdrd\theta dz$ is the volume element in cylindrical polar coordinates (r, θ, z) and $\rho^2 \sin\phi\, d\rho\, d\theta\, d\emptyset$ is the volume element in spherical polar coordinates $(\rho, \theta, \emptyset)$.

Problem 1: Evaluate the following integrals:

i. $\int_0^1 \int_{y^2}^1 \int_0^{1-x} x \, dzdxdy$

ii. $\int_0^4 \int_0^{2\sqrt{z}} \int_0^{\sqrt{4z-x^2}} dydxdz$

Solution: we have

i.
$$\begin{aligned}
\int_0^1 \int_{y^2}^1 \int_0^{1-x} x \, dzdxdy &= \int_{y=0}^1 \left\{ \int_{x=y^2}^1 \left(\int_{z=0}^{1-x} x \, dz \right) dx \right\} dy \\
&= \int_{y=0}^1 \left\{ \int_{x=y^2}^1 x[z]_0^{1-x} dx \right\} dy \\
&= \int_{y=0}^1 \left\{ \int_{x=y^2}^1 x(1-x) \, dx \right\} dy \\
&= \int_0^1 \left\{ \left[\frac{x^2}{2} - \frac{x^3}{3} \right]_{y^2}^1 \right\} dy \\
&= \int_0^1 \left\{ \frac{1}{2}(1-y^4) - \frac{1}{3}(1-y^6) \right\} dy \\
&= \int_0^1 \left(\frac{1}{6} - \frac{1}{2} y^4 + \frac{1}{3} y^6 \right) dy \\
&= \frac{1}{6} - \frac{1}{2} \cdot \frac{1}{5} + \frac{1}{3} \cdot \frac{1}{7} = \frac{1}{35}.
\end{aligned}$$

ii.
$$\begin{aligned}
\int_0^4 \int_0^{2\sqrt{z}} \int_0^{\sqrt{4z-x^2}} xdydxdz &= \int_{z=0}^4 \left\{ \int_{x=0}^{2\sqrt{z}} \left(\int_{y=0}^{\sqrt{4z-x^2}} dy \right) dx \right\} dz \\
&= \int_{z=0}^4 \left\{ \int_{x=0}^{2\sqrt{z}} \left([y]_0^{\sqrt{4z-x^2}} \right) dx \right\} dz \\
&= \int_{z=0}^4 \left\{ \int_{x=0}^{2\sqrt{z}} \sqrt{4z - x^2} \, dx \right\} dz \\
&= \int_0^4 \left\{ \int_0^t \left(\sqrt{t^2 - x^2} \right) dx \right\} dz,
\end{aligned}$$

where $t = 2\sqrt{z}$

$$= \int_0^4 \left\{ \int_0^{\frac{\pi}{2}} (t \cos\theta)(t \cos\theta) \, d\theta \right\} dz,$$

where $x = t \sin\theta$

$$= \int_0^4 \left\{ t^2 \int_0^{\frac{\pi}{2}} \cos^2\theta \, d\theta \right\} dz$$

$$= \int_0^4 t^2 \left(\frac{1}{2}.\frac{\pi}{2}\right) dz = \frac{\pi}{4} \int_0^4 4z \, dz,$$

Using $t = 2\sqrt{z}$.

$$= \pi.\frac{4^2}{2} = 8\pi.$$

Problem 2: Evaluate the following integrals

i. $\int_{-1}^{1}\int_{0}^{z}\int_{x-z}^{x+z}(x+y+z)\,dy\,dx\,dz$

ii. $\int_{-c}^{c}\int_{-b}^{b}\int_{-a}^{a}(x^2+y^2+z^2)\,dx\,dy\,dz$

Solution: we have

$\int_{-1}^{1}\int_{0}^{z}\int_{x-z}^{x+z}(x+y+z)\,dy\,dx\,dz$

$$= \int_{-1}^{1}\left\{\int_{0}^{z}\left[\int_{x-z}^{x+z}(x+y+z)dy\right]dx\right\}dz$$

$$= \int_{-1}^{1}\left\{\int_{0}^{z}\left[(x+z)\int_{x-z}^{x+z}dy + \int_{x-z}^{x+z}ydy\right]dx\right\}dz$$

$$= \int_{-1}^{1}\left\{\int_{0}^{z}\left[(x+z)\{(x+z)-(x-z)\} + \frac{1}{2}\{(x+z)^2 - (x-z)^2\}\right]dx\right\}dz$$

$$= \int_{-1}^{1}\left\{\int_{0}^{z}[(x+z)(2z)+2zx]dx\right\}dz = \int_{-1}^{1}\left\{\int_{0}^{z}(4zx+2z^2)dx\right\}dz$$

$$= \int_{-1}^{1}\left\{4z\int_{0}^{z}x\,dx + 2z^2\int_{0}^{z}dx\right\}dz$$

$$= \int_{-1}^{1}\{4z(z^2/2)+2z^2(z)\}\,dz = \int_{-1}^{1}4z^3\,dz = 0.$$

ii. $\int_{-c}^{c}\int_{-b}^{b}\int_{-a}^{a}(x^2+y^2+z^2)\,dxdydz$

$$= \int_{-c}^{c}\left\{\int_{-b}^{b}\left[\int_{-a}^{a}(x^2+y^2+z^2)dx\right]dy\right\}dz$$

$$= \int_{-c}^{c}\left\{\int_{-b}^{b}\left[\frac{1}{3}(2a^3)+y^2(2a)+z^2(2a)\right]dy\right\}dz$$

$$= \int_{-c}^{c} \left\{ \frac{2}{3} a^3 (2b) + 2a \left(\frac{2b^3}{3} \right) + 2az^2 (2b) \right\} dz$$

$$= \frac{4a^3 b}{3} (2c) + \frac{4ab^3}{3} (2c) + 4ab \left(\frac{2c^3}{3} \right) = \frac{8}{3} abc (a^2 + b^2 + c^2).$$

Problem 3: Evaluate the following triple integrals:

i. $\iiint\limits_{\Re}(x+y+z)dxdydz$

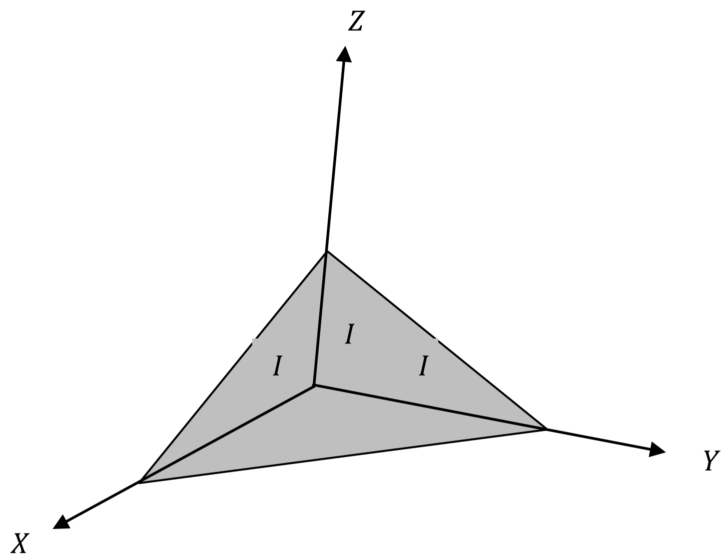
ii. $\iiint\limits_{\Re} zdxdydz$

iii. $\iiint\limits_{\Re} xyzdxdydz$

iv. $\iiint\limits_{\Re} \dfrac{dxdydz}{(1+x+y+z)^3}$

Here, $\Re$ is the region bounded by the planes $x = 0, y = 0, z = 0$ and $x + y + z = 1$.

Solution:



The given region $\Re$ is shown in figure. In this region, z varies from a point in the $XY-$ plane to a point on the plane $x + y + z = 1$; that is z increases from 0 to $z = (1 - x - y)$. In the $xy-$ plane, y varies from a point on the $x-$ axis to a point on the line $x + y = 1$. That is, for $z = 0, y$ increases from 0 to $y = 1 - x$. For $y = 0, z = 0$ (that is, on the $X-$ axis), x varies from 0 to 1. Therefore:

i. $\iiint_{\Re}(x+y+z)dxdydz = \int_{x=0}^{1}\int_{y=0}^{1-x}\int_{z=0}^{1-x-y}(x+y+z)\,dz\,dy\,dx$

$$= \int_{x=0}^{1}\left\{\int_{y=0}^{1-x}\left[\int_{z=0}^{1-x-y}(x+y+z)\,dz\right]\,dy\right\}dx$$

$$= \int_{x=0}^{1}\left\{\int_{y=0}^{1-x}\left(\left[(x+y)z+\frac{z^2}{2}\right]_{z=0}^{1-x-y}\right)dy\right\}dx$$

$$= \int_{x=0}^{1}\left\{\int_{y=0}^{1-x}\left[(x+y)(1-x-y)+\frac{1}{2}(1-x-y)^2\right]dy\right\}dx$$

$$= \int_{x=0}^{1}\left\{\int_{y=0}^{1-x}\frac{1}{2}(1-x-y)(2x+2y+1-x-y)dy\right\}dx$$

$$= \int_{x=0}^{1}\left\{\int_{y=0}^{1-x}\frac{1}{2}[1-(x+y)^2]dy\right\}dx$$

$$= \frac{1}{2}\int_{0}^{1}\left\{\left[y-\frac{1}{3}(x+y)^3\right]_{y=0}^{1-x}\right\}dx = \frac{1}{2}\int_{0}^{1}\left\{(1-x)-\frac{1}{3}(1-\right.$$

$x3)dx$

$$= \frac{1}{6}\int_{0}^{1}(2-3x+x^3)dx = \frac{1}{6}\left(2-\frac{3}{2}+\frac{1}{4}\right) = \frac{1}{8}.$$

ii. $\iiint_{\Re} zdxdydz = \int_{x=0}^{1}\int_{y=0}^{1-x}\int_{z=0}^{1-x-y} z\,dz\,dy\,dx$

$$= \int_{x=0}^{1}\left\{\int_{y=0}^{1-x}\left[\int_{z=0}^{1-x-y} z\,dz\right]\,dy\right\}dx$$

$$= \int_{x=0}^{1}\left\{\int_{y=0}^{1-x}\frac{(1-x-y)^2}{2}dy\right\}dx$$

$$= \int_{x=0}^{1}\left\{\frac{1}{2}\left[-\frac{(1-x-y)^3}{3}\right]_{y=0}^{1-x}\right\}$$

$$= \frac{1}{6}\int_{0}^{1}(1-x)^3\,dx = -\frac{1}{6}.\left.\frac{(1-x)^4}{4}\right|_{0}^{1} = \frac{1}{24}.$$

iii. $\iiint_{\Re} xyzdxdydz = \int_{x=0}^{1}\int_{y=0}^{1-x}\int_{z=0}^{1-x-y} xyz\ \,dz\,dy\,dx$

$$= \int_{0}^{1} x\left\{\int_{0}^{1-x}\left[\int_{0}^{1-x-y} z\,dz\right]dy\right\}dx$$

$$= \int_{0}^{1} x\left\{\int_{0}^{1-x} y\left[\frac{1}{2}(1-x-y)^2\right]dy\right\}dx$$

$$= \frac{1}{2}\int_0^1 x\left\{\int_0^{1-x}[(1-x)^2 y - 2(1-x)y^2 + y^3]dy\right\}dx$$

$$= \frac{1}{2}\int_0^1 x\left\{\frac{1}{2}(1-x)^2(1-x)^2 - \frac{2}{3}(1-x)(1-x)^3 + \frac{1}{4}(1-x)^4\right\}dx$$

$$= \frac{1}{24}\int_0^1 x\,(1-x)^4 dx$$

$$= \frac{1}{24}\left\{-x\,\frac{(1-x)^5}{5}\bigg|_0^1 + \int_0^1 \frac{(1-x)^5}{5}dx\right\}$$

$$= \frac{1}{24}\left\{-\frac{(1-x)^6}{30}\right\}_0^1 = \frac{1}{24}\left(\frac{1}{30}\right) = \frac{1}{720}.$$

iv. $\iiint\limits_{\Re} \frac{dxdydz}{(1+x+y+z)^3}$

$$= \int_{x=0}^1\left\{\int_{y=0}^{1-x}\left[\int_{z=0}^{1-x-y}(1+x+y+z)^{-3}dz\right]dy\right\}dx$$

$$= \int_{x=0}^1\left\{\int_{y=0}^{1-x}\left[\frac{(1+x+y+z)^{-2}}{-2}\right]_{z=0}^{1-x-y}dy\right\}dx$$

$$= -\frac{1}{2}\int_{x=0}^1\left\{\int_{y=0}^{1-x}[\{(1+x+y+1-x-y)^{-2}\} -$$

1+x+y−2dydx

$$= -\frac{1}{2}\int_{x=0}^1\left\{\int_{y=0}^{1-x}\left[\frac{1}{2^2} - (1+x+y)^{-2}\right]dy\right\}dx$$

$$= -\frac{1}{2}\int_{x=0}^1\left\{\left[\frac{1}{4}y + \frac{1}{(1+x+y)}\right]_{y=0}^{1-x}\right\}dx$$

$$= -\frac{1}{2}\int_{x=0}^1\left\{\frac{1}{4}(1-x) + \left(\frac{1}{2} - \frac{1}{1+x}\right)\right\}dx$$

$$= -\frac{1}{2}\left[-\frac{1}{4}.\frac{(1-x)^2}{2} + \frac{1}{2}x - \log(1+x)\right]_0^1$$

$$= -\frac{1}{2}\left\{\frac{1}{8} + \frac{1}{2} - \log 2 + \log 1\right\} = \frac{1}{2}\left(\log 2 - \frac{5}{8}\right).$$

Problem 4: Evaluate $\mathrm{I} = \int_0^{\pi/2} \int_0^{a \sin\theta} \int_0^{(a^2-r^2)/a} r\, dz\, dr\, d\theta$, given in cylindrical polar coordinates (r, θ, z).

Solution: By using the meaning of repeated integrals, we find that

$$I = \int_{\theta=0}^{\frac{\pi}{2}} \left[\int_{r=0}^{a \sin\theta} \left\{ \int_{z=0}^{\frac{a^2-r^2}{a}} r\, dz \right\} dr \right] d\theta$$

$$= \int_{\theta=0}^{\pi/2} \left[\int_{r=0}^{a \sin\theta} r \left\{ [z]_0^{(a^2-r^2)/a} \right\} dr \right] d\theta$$

$$= \int_{\theta=0}^{\frac{\pi}{2}} \left[\int_{r=0}^{a \sin\theta} \left(\frac{a^2-r^2}{a} \right) r\, dr \right] d\theta$$

$$= \int_0^{\pi/2} \left\{ \left[a\frac{r^2}{2} - \frac{1}{a}\frac{r^4}{4} \right]_0^{a \sin\theta} \right\} d\theta$$

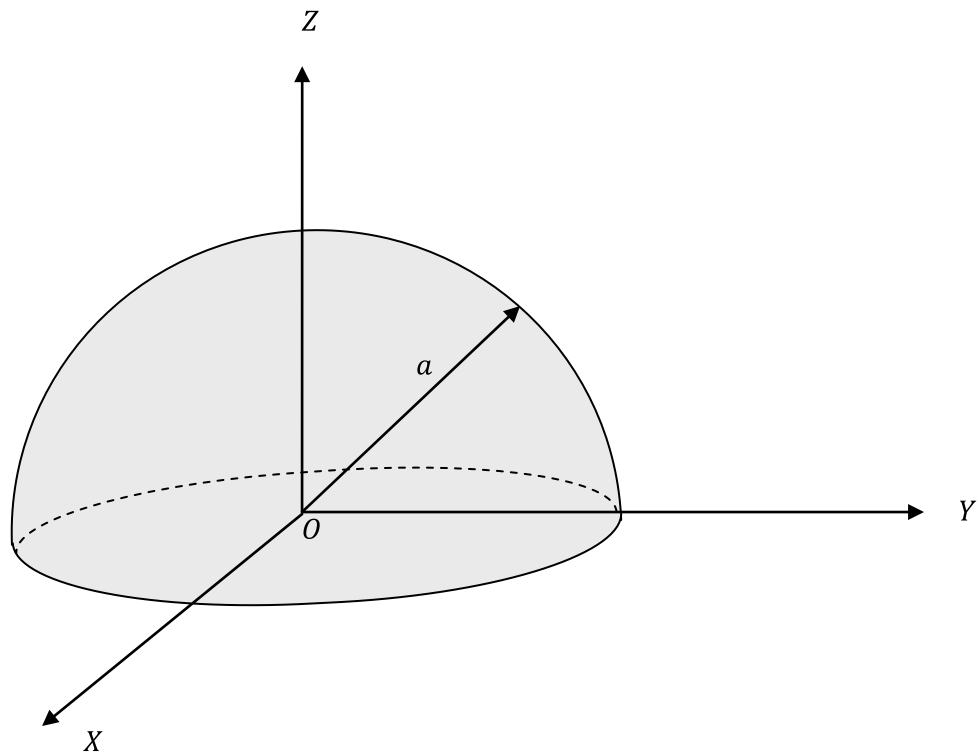
$$= \int_0^{\pi/2} \left\{ \frac{a^3}{2} \sin^2\theta - \frac{a^3}{4} \sin^4\theta \right\} d\theta$$

$$= \frac{a^3}{2} \int_0^{\pi/2} \sin^2\theta\, d\theta - \frac{a^3}{4} \int_0^{\pi/2} \sin^4\theta\, d\theta$$

$$= \frac{a^3}{2} . \frac{1}{2} . \frac{\pi}{2} - \frac{a^3}{4} . \frac{3}{4} . \frac{1}{2} . \frac{\pi}{2} = \frac{5\pi}{64} a^3.$$

Problem 5: Find the value of $\iiint z\,dx\,dy\,dz$ over the hemisphere $x^2+y^2+z^2 \le a^2, z \ge 0$.

Solution:



Let $(\rho, \theta, \emptyset)$ be the spherical polar coordinates. In the given hemisphere (shown in above figure), ρ increases from 0 to a, ϕ increases from 0 to $\pi/2$, and θ increases form 0 to 2π, Therefore,

$$\iiint z\,dx\,dy\,dz = \int_{\rho=0}^{a}\int_{\phi=0}^{\pi/2}\int_{\theta=0}^{2\pi}(\rho\cos\phi)\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

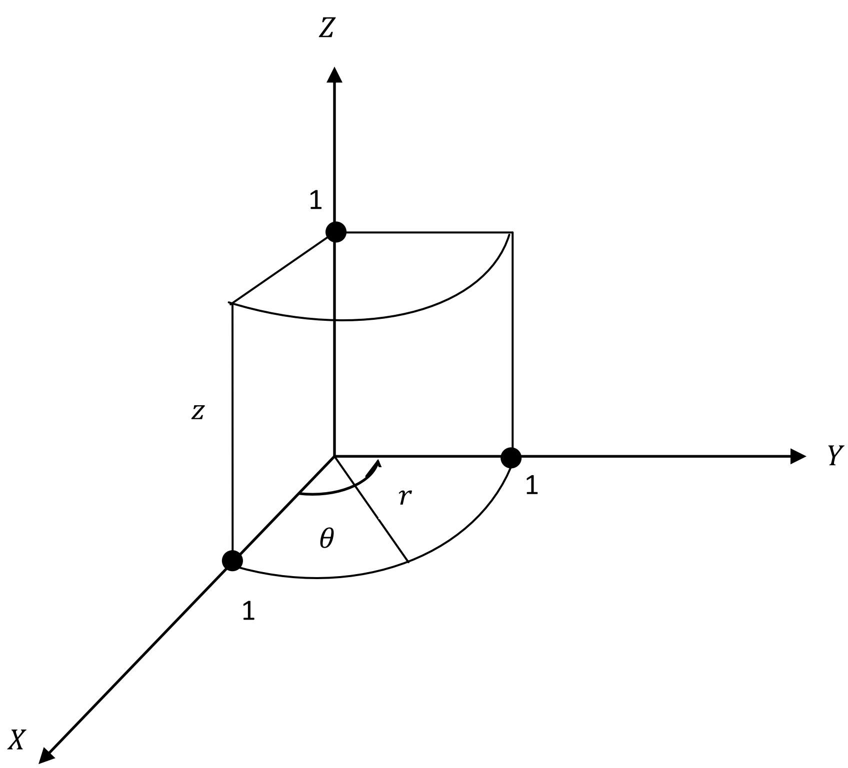
$$= \int_0^a \rho^3 d\rho \times \int_0^{\pi/2}\sin\phi\cos\phi\,d\phi \times \int_0^{2\pi} d\theta$$

$$= \frac{\pi a^4}{4}.$$

Problem 6: *If $\mathfrak{R}$ is the region bounded by the planes* $x = 0, y = 0, z = 1$ and the cylinder $x^2 + y^2 = 1$, evaluate the integral $\iiint_{\mathfrak{R}} xyzdxdydz$, by changing it to cylindrical polar coordinates.

Solution:

Let (r, θ, z) be cylindrical polar coordinates. In the given region (shown in following figure), r increases form 0 to $1, \theta$ increases form 0 to $\pi/2$ and z increases form 0 to 1. Therefore,



$$\iiint_{\mathfrak{R}} xyzdxdydz = \int_{r=0}^{1} \int_{\theta=0}^{\pi/2} \int_{z=0}^{1} (r\cos\theta)\,(r\sin\theta)\, z\,(r\, dr d\theta dz)$$

$$= \int_0^1 r^3 dr \times \int_0^{\frac{\pi}{2}} \sin\theta\cos\theta\, d\theta \times \int_0^1 z\, dz$$

$$= \frac{1}{4} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{16}.$$

Problem 7: Using triple integrals find the volume bounded by the surface $z^2 = a^2 - x^2$ and the planes $x = 0, y = 0$ and $y = b$.

Solution: Here, z varies from $z = 0$ to $z = a^2 - x^2$. For $z = 0$, x varies form 0 to a, and y varies from 0 to b.Therefore, the required volume is

$$V = \int_{x=0}^{a} \int_{y=0}^{b} \int_{z=0}^{a^2-x^2} dz\, dy\, dx$$

$$= \int_{x=0}^{x} \left[\int_{y=0}^{b} \left\{ \int_{z=0}^{a^2-x^2} dz \right\} dy \right] dx$$

$$= \int_{0}^{a} \left[\int_{0}^{b} (a^2 - x^2) dy \right] dx$$

$$= \int_{0}^{a} (a^2 - x^2) \{ [y]_0^b \} dx$$

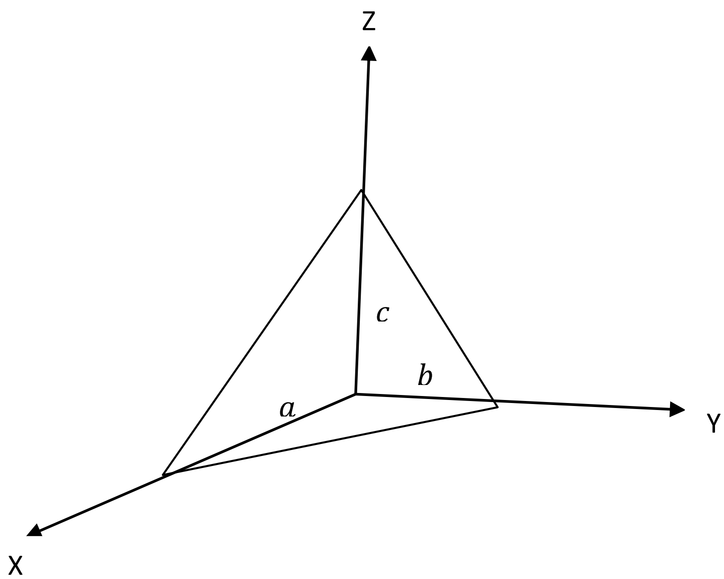
$$= \int_{0}^{a} b(a^2 - x^2) dx$$

$$= b \left[a^2 x - \frac{x^3}{3} \right]_0^a$$

$$= b \left(a^3 - \frac{a^3}{3} \right) = \frac{2}{3} b a^3.$$

Problem 8: Using triple integrals, find the volume of the tetrahedron bounded by the planes $x = 0, y = 0, z = 0$ and $(x/a) + (y/b) + (z/c) = 1$.

Solution:



The given tetrahedron is shown in above figure. In this tetrahedron, z varies from 0 to $c(1 - x/a - y/b)$. For $z = 0$, y varies from 0 to $b(1 - x/a)$. for $y = 0, z = 0$, x varies from 0 to a. Therefore, the required volume is

$$V = \int_{x=0}^{a} \int_{y=0}^{b\left(1-\frac{x}{a}\right)} \int_{z=0}^{c\left(1-\frac{x}{a}-\frac{y}{b}\right)} dz\, dy\, dx$$

$$= \int_{x=0}^{a} \left[\int_{y=0}^{b\left(1-\frac{x}{a}\right)} \left\{c\left(1-\frac{x}{a}-\frac{y}{b}\right)\right\} dy\right] dx$$

$$= c\int_{x=0}^{a} \left\{\left[\left(1-\frac{x}{a}\right)y - \frac{y^2}{2b}\right]_0^{b(1-x/a)}\right\} dx$$

$$= c\int_{x=0}^{a} \left\{b\left(1-\frac{x}{a}\right)^2 - \frac{1}{2b}.b^2\left(1-\frac{x}{a}\right)^2\right\} dx$$

$$= \frac{cb}{2} \int_{x=0}^{a} \left(1 - \frac{x}{a}\right)^2 dx$$

$$= \frac{cb}{2} \left\{ \left[\frac{\left(1 - \frac{x}{a}\right)^3}{3\left(-\frac{1}{a}\right)} \right]_0^a \right\} = \frac{1}{6} abc.$$

Exercise

1. Compute the integral $\int_0^1 \frac{x^a - x^b}{\log x} dx \quad a > b > -1.$
2. Compute the integral $\int_0^\infty (\frac{e^{-ax} - e^{-bx}}{x}) \sin x \, dx$.
3. Evaluate the integral $I = \iiint_{\Re} xydxdydz$ where $\Re$ is the positive octant of the sphere $x^2 + y^2 + z^2 = a^2$.
4. If $\Re$ is the region in the first octant bounded by the sphere $x^2 + y^2 + z^2 = a^2$, evaluate the integral $\iiint_{\Re} (x + y + z)dxdydz$ by changing it to spherical polar coordinates.
5. Using triple integrals, find the volume of the ellipsoid
$$\left(\frac{x^2}{a^2}\right) + \left(\frac{y^2}{b^2}\right) + \left(\frac{z^2}{c^2}\right) = 1.$$
6. Using triple integrals, find the volume of the sphere $x^2 + y^2 + z^2 = a^2$.
7. Compute $\iiint \frac{dx\ dy\ dz}{(x+y+z+1)^3}$ if the domain of integration is bounded by the coordinate planes and the plane $x + y + z = 1$.
8. Evaluate $\int_0^a \left[\int_0^x \left(\int_0^y xyz\ dz\right) dy\right] dx$.
9. Evaluate $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \frac{dx\ dy\ dz}{\sqrt{1-x^2-y^2-z^2}}$ by changing to spherical polar co-ordinates.
10. Evaluate $\iiint (x^2 + y^2)\, dx\, dy\, dz$ taken over the volume bounded by the XY-plane and the parabolaid $z = 9 - x^2 - y^2$ by using cylindrical polar coordinates.

11. Find the volume common to the cylinders $x^2 + y^2 = a^2$ and $x^2 + z^2 = a^2$ by triple integral.

12. Find the volume bounded by the XY-plane, the cylinder $x^2 + y^2 = 1$ and the plane $x + y + z = 3$ by triple integral.

Answers

1. $\log(\frac{b+1}{a+1})$
2. $\tan^{-1} b - \tan^{-1} a$
3. $\frac{1}{15} a^5$
4. $\frac{3\pi a^4}{16}$
5. $\frac{4\pi}{3} abc$
6. $\frac{4\pi}{3} a^3$
7. $\frac{\ln 2}{2} - \frac{5}{16}$
8. $\frac{a^6}{48}$.
9. $\frac{\pi^2}{8}$.
10. $\frac{243\pi}{2}$.
11. $\frac{16a^3}{3}$ Cubic units.
12. 3π Cubic units.