UNIT - II

Random Variables

(a) Random Variables

(b) Types of R.V $\left[\begin{array}{l}\to D.R.V \\ \to C.R.V\end{array}\right.$

(c) Prob. Distributions $\left[\begin{array}{l}\to D.P.D \\ \to C.P.D\end{array}\right.$

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(d) Problems on D.P.D

(a) Mean (b) Variance (c) S.D (d) $F(x)$

(e) $P(a \le X)$ (f) $P(X \le b)$ (g) $P(a \le X \le b)$

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(e) Problems on C.P.D

(a) Mean (b) Variance (c) S.D (d) Median

(e) Mode (f) $F(x)$ (g) $P(a \le X)$ (h) $P(X \le b)$

(i) $P(a \le X \le b)$

(X, Y)

(f) Bivariate R.V $\left[\begin{array}{l}\to D.B.R.V \\ \to C.B.R.V\end{array}\right.$

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(g) Problems on D.B.R.V $\left[\begin{array}{l}\to J.P.M.F \\ \to M.P.M.F \\ \to C.P.M.F \\ \to X \,\&\, Y \text{ inde or not}\end{array}\right.$

(h)* problems on C.B.R.V
- J.p.d.f
- M.p.d.f
- C.p.d.f
- X & Y indep or not

(i)* covariance

(j)* correlation coefficient

Random variable

A real variable X whose values are determined by the outcome of a random experiment is called R.V

Ex: ① Tossing two coins $n(S) = 4$

$S = \{HH, HT, TH, TT\}$.
$\quad\; E1 \quad E2 \quad E3 \quad E4$

Define $X : S \to R$ → $X(S) =$ no. of heads

$X(E1) = 2$

$X(E2) = 1$

$X(E3) = 1$

$X(E4) = 0$

$\boxed{X = \{0, 1, 2\}}$

$[0, 2]$

② Tossing three coins $2^3 = n(S) = 8$

$S = \{HHH, HHT, HTH, HTT, TTT, TTH, THT, THH\}$
$\quad E1 \quad E2 \quad E3 \quad E4 \quad E5 \quad E6 \quad E7 \quad E8$

Define $X : S \to R$ of $X(S) =$ no of Tails

$X(E_1) = 0$

$X(E_2) = 1$

$X(E_3) = 1$

$X(E_4) = 2$

$X(E_5) = 3$

$X(E_6) = 2$

$X(E_7) = 2$

$X(E_8) = 1$

X Takes The values

$= \{0, 1, 2, 3\}$

X	0	1	2	3
$P(X)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

X (0, 3)

X

R.V

DRV

X Takes The integers b/w The given interval

✓ Two coins Toss

Dice Toss

CRV

X Takes The all possible values b/w The interval

Temp, R.F, Ht, Wt

5.6

X	0	1	2	3
$P(x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

$P(1 \le x \le 3)$

$P(x \ge 3)$

$P(X \le 3)$

$P(x)$ p.m.f $\left\{ \begin{array}{l} \to \sum P(x) = 1 \\ \to 0 \le p \le 1 \end{array} \right.$

$[0, 3]$

$\left\{ \begin{array}{l} X \quad \text{DRV} \\ \quad | \\ P(x) \quad \text{pmf} \end{array} \right. \left| \begin{array}{l} \sum p = 1 \\ 0 \le p \le 1 \end{array} \right.$

$(X, P(x))$ D. P. D

$$\boxed{(\text{D.R.V}, \ \text{P.M.F}) = \text{D.P.D}}$$

R.V

D.R.V $X = \{0, 1, 2, 3\}$

P.M.F $P(x)$

$\sum P = 1$

$0 \le P \le 1$

$(X, P(x))$
D.P.D

C.R.V X

P.d.f $f(x)$

$\int_{-\infty}^{\infty} f(x)\,dx = 1$

$0 \le f \le 1$

$(X, f(x))$
C.P.D

DPD

X	0	1	2	3
$P(x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

M
V
S
$F(x)$
P

$P(X \le 2)$ $P(X \le 3)$

$P(X \ge 2)$ $P(2 \le X \le 3)$

Expectation, mean, variance, SD of a DPD

$(X, P(x))$ D.P.D $\qquad X$ D.R.V $\qquad \Sigma P = 1$

$P(x)$ P.M.F $\qquad 0 \le P \le 1$

X	x_0	x_1	x_2	x_3
$P(x)$	p_0	p_1	p_2	p_3

① Expectation: $E(x)$

$$E(x) = \sum_{i=1}^{n} x_i p_i$$

$$= x_0 p_0 + x_1 p_1 + x_2 p_2 + \cdots + x_n p_n$$

② Mean : μ

$$\mu = \frac{\sum_{i=1}^{n} x_i p_i}{\sum_{i=1}^{n} p_i} = \sum_{i=1}^{n} x_i p_i = E(x)$$

$$\mu = E(x) = \sum_{i=1}^{n} x_i p_i$$

NOTE: $E(x^2) = \sum_{i=1}^{n} x_i^2 p_i$

③ Variance: σ^2

$$\sigma^2 = E(x^2) - [E(x)]^2$$

④ Standard Deviation : σ

$$\sigma = \sqrt{\text{variance}}$$

Note:

① C.D.F :- $F(x)$

X	x_0	x_1	x_2	--	x_n
$P(x)$	P_0	P_1	P_2	--	P_n

$\Sigma P = 1$

X	$F(x) = P(X \leq x)$
x_0	P_0
x_1	$P_0 + P_1$
x_2	$P_0 + P_1 + P_2$
x_3	$P_0 + P_1 + P_2 + P_3$
- - -	- - - ......
x_n	$P_0 + P_1 + P_2 + \cdots + P_n = 1$

① For The D.P.D Table

X	0	1	2	3	4	5	6
$P(x)$	0	$2K$	$2K$	$3K$	K^2	$2K^2$	$7K^2+K$

Then find

① K ② $E(X) = \mu$ ③ σ^2 ④ σ ⑤ $F(x)$

✓ ⑥ $P(X \le 3)$ ⑦ $P(X \ge 4)$ ⑧ $P(1 \le X \le 5)$

⑨ $P(0 \le X \le 6)$

Ans ① Since $P(x)$ is p.m.f $\begin{cases} \to ① \ \Sigma P = 1 \\ \to ② \ 0 \le P \le 1 \end{cases}$

$$0 + 2K + 2K + 3K + K^2 + 2K^2 + 7K^2 + K = 1$$

$$10K^2 + 8K - 1 = 0$$

$$K = 0.1099 \qquad K = -0.9099$$

Here $\boxed{K = 0.1099}$ $\because \ 0 \le P \le 1$

② Mean $\boxed{E(x) = \mu = \Sigma x P(x)}$

$$\mu = 0(0) + 1(2K) + 2(2K) + 3(3K)$$

$$+ 4(K^2) + 5(2K^2) + 6(7K^2 + K)$$

$$= 56K^2 + 21K$$

$$= 56(0.1099)^2 + 21(0.1099)$$

$$\boxed{\mu = 2.9842 = E(X)}$$

Now $E(X^2) = \Sigma x^2 p(x)$

$$= 0^2(0) + 1^2(2K) + 2^2(2K) + 3^2(3K)$$
$$+ 4^2(K^2) + 5^2(2K^2) + 6^2(7K^2 + K)$$

$$= 11.8635$$

$$\boxed{E(X^2) = 11.8635}$$

③ Variance: σ^2

$$\sigma^2 = E(X^2) - [E(X)]^2$$

$$\sigma^2 = 11.8635 - (2.9842)^2$$

$$\boxed{\sigma^2 = 2.9580}$$

④ Standard Deviation σ

$$\sigma = \sqrt{\text{Variance}}$$

$$\sigma = \sqrt{2.9580}$$

$$\boxed{\sigma = 1.7198}$$

⑤ C.D.F: $F(x) = P(X \le x)$

X	$F(x) = P(X \le x)$
0	0
1	$0 + 2K = 2(0.1099) =$
2	$0 + 2K + 2K = 4(0.1099) =$
3	$0 + 2K + 2K + 3K = 7(0.1099) =$
4	$0 + 2K + 2K + 3K + K^2 =$
5	$0 + 2K + 2K + 3K + K^2 + 2K^2 =$
6	$0 + 2K + 2K + 3K + K^2 + 2K^2 + 7K^2 + K = 10K^2 + 8K = \underline{\underline{1}}$

⑥ $P(X \leq 3)$

$P(X \leq 3) = P(X=0) + P(X=1) + P(X=2) + P(X=3)$

$= 0 + 2K + 2K + 3K$

$= 7K$

$= 7(0.1099)$

$\boxed{P(X \leq 3) = 0.7616}$

⑦ $P(X \geq 4)$

$P(X \geq 4) = P(X=4) + P(X=5) + P(X=6)$

$= K^2 + 2K^2 + 7K^2 + K$

$= 10K^2 + K$

$= 10(0.1099)^2 + (0.1099)$

$\boxed{P(X \geq 4) = 0.2306}$

⑧ $P(1 \le x \le 5)$

$$P(1 \le x \le 5) = P(x=1) + P(x=2) + P(x=3)$$
$$+ P(x=4) + P(x=5)$$

$$= 2k + 2k + 3k + k^2 + 2k^2$$

$$= 3k^2 + 7k$$

$$= 3(0.1099)^2 + 7(0.1099)$$

$$\boxed{P(1 \le x \le 5) = 0.8055}$$

⑨ $P(0 \le x \le 6)$

$$P(0 \le x \le 6) = P(x=0) + P(x=1) + P(x=2)$$
$$+ P(x=3) + P(x=4) + P(x=5) + P(x=6)$$

$$= 0 + 2k + 2k + 3k + k^2 + 2k^2 + 7k^2 + k$$

$$= 10k^2 + 8k = 10(0.1099)^2$$
$$+ 8(0.1099)$$

$$\boxed{P(0 \le x \le 6) = 1}$$

① A random variable X has the following

X	-2	-1	0	1	2	3
$P(x)$	0.1	K	0.2	$2K$	0.3	K

Then find

① K ② Mean ③ Variance ④ S.D

⑤ C.D.F ⑥ $P(-2 \leq X \leq 3)$ ⑦ $P(X \leq 3)$

⑧ $P(X > 1)$ ⑨ $P(X \leq 1)$ ⑩ $P(0 < X < 2)$

Continuous probability Distribution (C.P.D)

Mean, variance, S.D, Median, mode

Mean $E(x)$ (or) μ

$$\mu = E(x) = \int_{-\infty}^{\infty} x \, f(x) \, dx$$

here $E(x^2) = \int_{-\infty}^{\infty} x^2 f(x) \, dx$

variance σ^2

$$\sigma^2 = E(x^2) - [E(x)]^2$$

Standard Deviation σ

$$\sigma = \sqrt{\text{variance}}$$

Median M

$$\int_{a}^{M} f(x) \, dx = \int_{M}^{b} f(x) \, dx = \frac{1}{2}$$

solving for M

Mode

Mode is the value of x for which $f(x)$ is maximum at 'x'. Mode is then given by $f'(x)=0$ and $f''(x)<0$ for $a<x<b$

$$f(x)$$
$$\downarrow$$
$$f'(x) \qquad [2, 3]$$
$$\downarrow$$
$$f''(x)<0 \text{ for } a<x<b$$
$$\downarrow$$
$$f'(x)=0 \quad \text{solve for } x$$

we get mode

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* *
Cumulative Distributive function (C.D.F): $F(x)$

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(x)\,dx$$

properties of C.D.F (or) $F(x)$

① $0 \le F(x) \le 1$

② $F(-\infty)=0$; $F(\infty)=1$

* ③ $P(a \le X \le b) = \int_a^b f(x)\,dx = F(b)-F(a)$

④ $\frac{d}{dx}F(x) = F'(x) = f(x) \quad \forall x \in R$

① The p.d.f $f(x)$ of a c.r.v is given by

$$f(x) = \begin{cases} kx^3 & \text{for } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases} \qquad \boxed{[0,1]}$$

Then find the value of

① k ② μ ③ σ^2 ④ σ ⑤ median

⑥ mode ⑦ $F(x)$ ⑧ $P\left(\frac{1}{4} \le x \le \frac{3}{4}\right)$

⑨ $P\left(x \ge \frac{2}{3}\right)$ ⑩ $P\left(x \le \frac{1}{3}\right)$

so ① Since $f(x)$ is p.d.f $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\int_0^1 f(x)\,dx = 1$$

$$k\int_0^1 x^3\,dx = 1$$

$$k\left[\frac{x^4}{4}\right]_0^1 = 1$$

$$\boxed{k = 4}$$

② Mean $\mu = E(x) = \int_{-\infty}^{\infty} x\, f(x)\, dx$

$$\mu = E(x) = \int_0^1 x\, f(x)\, dx$$

$$\boxed{f(x) = 4x^3 \quad 0<x<1}$$

$$= 4\int_0^1 x \cdot x^3\, dx$$

$$= 4\left[\frac{x^5}{5}\right]_0^1$$

$$\boxed{E(x) = \frac{4}{5}}$$

Now $E(x^2) = \int_{-\infty}^{\infty} x^2\, f(x)\, dx$

$$= 4\int_0^1 x^2 \cdot x^3\, dx$$

$$= 4\left[\frac{x^6}{6}\right]_0^1 = \frac{2}{3}$$

$$\boxed{E(x^2) = \frac{2}{3}}$$

③ variance σ^2

$$\sigma^2 = E(x^2) - [E(x)]^2$$

$$= \frac{2}{3} - \frac{16}{25}$$

$$\boxed{\sigma^2 = \frac{2}{75}}$$

④ Standard Deviation σ

$$\sigma = \sqrt{\text{variance}}$$

$$\boxed{\sigma = \sqrt{\frac{2}{75}} =}$$

$a=0,\ b=1,\ f(x)=4x^3$

⑤ median:(M)

$$\int_a^M f(x)\,dx = \int_M^b f(x)\,dx = \frac{1}{2}$$

$$4\int_0^M x^3\,dx = 4\int_M^1 x^3\,dx = \frac{1}{2}$$

consider $4\int_0^M x^3\,dx = \frac{1}{2}$

$$4\left[\frac{x^4}{4}\right]_0^M = \frac{1}{2}$$

$$M^4 = \frac{1}{2}$$

$$M = \left(\frac{1}{2}\right)^{\frac{1}{4}}$$

$$\boxed{M =}$$

⑥ Mode

$f(x) = 4x^3 \quad \forall x \in [0,1]$

$f'(x) = 12x^2$

$\boxed{f''(x) = 24x > 0} \quad \forall x \in [0,1]$

Hence Mode does not exist

$f(x)$
$\downarrow$
$f'(x)$
$\downarrow$
$f''(x) < 0$
$\downarrow$
$f'(x) = 0$
solve for x

⑦ $F(x)$ (or) c.D.F $\quad \forall x \in [0,1]$

$$\boxed{F(x) = P(X \le x) = \int_{-\infty}^{x} f(x)\,dx}$$

$$F(x) = P(X \le x) = \int_{0}^{x} f(x)\,dx$$

$$F(x) = 4\int_{0}^{x} x^3\,dx$$

$$= 4\left[\frac{x^4}{4}\right]_0^x$$

$$\boxed{F(x) = x^4 \;;\; \forall x \in [0,1]}$$

⑧ $P\left(\frac{1}{4} \le X \le \frac{3}{4}\right)$ $\qquad f(x)\ x \in [0,1]$

$$P\left(\frac{1}{4} \le X \le \frac{3}{4}\right) = \int_{1/4}^{3/4} f(x)\,dx$$

$$= 4\int_{1/4}^{3/4} x^3\,dx = \left[x^4\right]_{1/4}^{3/4}$$

$$= \left(\frac{3}{4}\right)^4 - \left(\frac{1}{4}\right)^4$$

$$\boxed{P\left(\frac{1}{4} \le x \le \frac{3}{4}\right) = \frac{80}{256}}$$

$$\boxed{F(x) = x^4}$$

(or)

$$P\left(\frac{1}{4} \le x \le \frac{3}{4}\right) = \int_{\frac{1}{4}}^{3/4} f(x)\,dx = F\left(\frac{3}{4}\right) - F\left(\frac{1}{4}\right)$$

$$= \left(\frac{3}{4}\right)^4 - \left(\frac{1}{4}\right)^4$$

$$= \frac{80}{256}$$

⑨ $P\left(x \ge \frac{2}{3}\right)$: $f(x)$; $\forall\, x \in [0,1]$

$$P\left(\frac{2}{3} \le x\right) = \int_{\frac{2}{3}}^{1} 4x^3\,dx$$

$$= 4\left[\frac{x^4}{4}\right]_{\frac{2}{3}}^{1} = \left[x^4\right]_{\frac{2}{3}}^{1}$$

$$= 1 - \frac{2^4}{3^4} =$$

(10) $P(X \le \frac{1}{3})$

$$P(X \le \tfrac{1}{3}) = P(0 \le X \le \tfrac{1}{3})$$

$$= \int_0^{\frac{1}{3}} f(x)\,dx = 4\int_0^{\frac{1}{3}} x^3\,dx = \frac{1}{81}$$

(or)

$$= F(\tfrac{1}{3}) - F(0)$$

$$= \left(\tfrac{1}{3}\right)^4 - 0$$

$$\boxed{P(X \le \tfrac{1}{3}) = \frac{1}{81}}$$

(2) let $f(x) = K(1-x^2)$ for $0 < x < 1$ Then find

(1) K (2) μ (3) σ^2 (4) σ (5) M (6) mode

(7) $F(x)$ (8) $P(\frac{1}{3} \le x \le \frac{1}{4})$ (9) $P(X \le 1)$

(10) $P(X \ge 0)$ (11) $P(x < \frac{1}{2})$ (12) $x > \frac{1}{3})$

$$\sum_{i=1}^{10} x_i = 110 \; ; \qquad \sum_{i=1}^{10} x_i^2 = 1540$$

$$E(x) = \frac{\sum_{i=1}^{10} x_i}{10} = \frac{110}{10} = 11$$

$$E(x^2) = \frac{\sum_{i=1}^{10} x_i^2}{10} = \frac{1540}{10} = 154$$

Now $\quad \sigma^2 = E(x^2) - [E(x)]^2$

$$= 154 - (11)^2$$

$$= 154 - 121$$

$$\boxed{\sigma^2 = 33}$$

① Let x denote the minimum of two numbers that appear when a pair of fair dice thrown once. Determine the

(i) D.P.D (ii) Expectation (iii) Variance

Sol Two Dice are thrown once

Total no. of out comes $= n(S) = 36$

Sample Space $S = \begin{Bmatrix} (1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) \\ (2,1) & (2,2) & (2,3) & (2,4) & (2,5) & (2,6) \\ (3,1) & (3,2) & (3,3) & (3,4) & (3,5) & (3,6) \\ (4,1) & (4,2) & (4,3) & (4,4) & (4,5) & (4,6) \\ (5,1) & (5,2) & (5,3) & (5,4) & (5,5) & (5,6) \\ (6,1) & (6,2) & (6,3) & (6,4) & (6,5) & (6,6) \end{Bmatrix}$

x Takes The minimum number $= \{1, 2, 3, 4, 5, 6\}$

For 1 minimum favourable cases

$= (1,1) (1,2) (1,3) (1,4) (1,5) (1,6) (2,1), (3,1)$

$(4,1) (5,1) (6,1)$

$P(x=1) = \frac{11}{36}$

For 2 minimum favourable cases

$\{(2,2) (2,3) (2,4) (2,5) (2,6) (6,2) (5,2)$

$(4,2) (3,2)$

$$P(X=2) = \frac{9}{36}$$

IIly $$P(X=3) = \frac{7}{36}$$

$$P(X=4) = \frac{5}{36}$$

$$P(X=5) = \frac{3}{36}$$

$$P(X=6) = \frac{1}{36}$$

① D.P.D

X	1	2	3	4	5	6
$P(X=x)$	$\frac{11}{36}$	$\frac{9}{36}$	$\frac{7}{36}$	$\frac{5}{36}$	$\frac{3}{36}$	$\frac{1}{36}$

② Expected Value

$$E(X) = \Sigma x \, P(x)$$

$$= 1\left(\frac{11}{36}\right) + 2\left(\frac{9}{36}\right) + 3\left(\frac{7}{36}\right) + 4\left(\frac{5}{36}\right)$$

$$+ 5\left(\frac{3}{36}\right) + 6\left(\frac{1}{36}\right) = \frac{91}{36} = 2{,}5278$$

$$E(x^2) = \sum x^2 P(x)$$

$$= 1^2\left(\frac{11}{36}\right) + 2^2\left(\frac{9}{36}\right) + 3^2\left(\frac{7}{36}\right) + 4^2\left(\frac{5}{36}\right)$$

$$+ 5^2\left(\frac{3}{36}\right) + 6^2\left(\frac{1}{36}\right) = 8.3611$$

③ Variance

$$\sigma^2 = E(x^2) - [E(x)]^2$$

$$= 8.3611 - (2.5278)^2$$

$$\boxed{\sigma^2 = 1.9713}$$

$$\sigma = \sqrt{1.9713}$$

$$\boxed{\sigma = 1.404}$$

① For the continuous probability function

$f(x) = K x^2 e^{-x}$ when $x \geq 0$

find (1) K (2) Mean (3) Variance

✓ (4) $P(X \leq 1)$ (5) $P(0 \leq x \leq 2)$

(6) $P(X \geq 2)$ (7) $F(x)$ C.D.F

So

$f(x) = K x^2 e^{-x^2}$ $\quad x \geq 0$

$0 \leq x < \infty$

$x \in [0, \infty)$

Here $f(x)$ is p.d.f

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\int_{0}^{\infty} f(x)\,dx = 1$$

$$K \int_{0}^{\infty} x^2 e^{-x}\,dx = 1$$

$$K \left[x^2 \frac{e^{-x}}{(-1)} - 2x \frac{e^{-x}}{(-1)^2} + 2 \frac{e^{-x}}{(-1)^3} \right]_0^{\infty} = 1$$

$$K[0 + 2] = 1 \qquad \boxed{K = \frac{1}{2}}$$

Hence $\boxed{f(x) = \frac{1}{2} x^2 e^{-x}}$ $x \in [0, \infty)$

② Mean $E(x) = \mu = \int_{-\infty}^{\infty} x f(x) dx$

$$= \int_0^{\infty} x f(x) dx$$

$$= \frac{1}{2} \int_0^{\infty} x \cdot x^2 e^{-x} dx$$

$$= \frac{1}{2} \int_0^{\infty} x^3 e^{-x} dx$$

$$= \frac{1}{2} \left[x^3 \frac{e^{-x}}{(-1)} - 3x^2 \frac{e^{-x}}{(-1)^2} + 6x \frac{e^{-x}}{(-1)^3} - 6 \frac{e^{-x}}{(-1)^4} \right]_0^{\infty}$$

$$= \frac{1}{2} [0 + 6]$$

$$\boxed{\mu = E(x) = 3}$$

Now $E(x^2) = \int_{-\infty}^{\infty} x^2 f(x)\, dx$

$$= \int_{0}^{\infty} x^2 f(x)\, dx$$

$$= \int_{0}^{\infty} x^2 \, \frac{1}{2} x^2 e^{-x}\, dx$$

$$= \frac{1}{2} \int_{0}^{\infty} x^4 e^{-x}\, dx$$

$$= \frac{1}{2} \left[x^4 \frac{e^{-x}}{(-1)} - 4x^3 \frac{e^{-x}}{(-1)^2} + 12x^2 \frac{e^{-x}}{(-1)^3} - 24x \frac{e^{-x}}{(-1)^4} + 24 \frac{e^{-x}}{(-1)^5} \right]_0^{\infty}$$

$$= \frac{1}{2} \{0 + 24\} = 12$$

$$\boxed{E(x^2) = 12}$$

③ variance σ^2

$$\sigma^2 = E(X^2) - \{E(X)\}^2$$

$$= 12 - 3^2$$

$$\boxed{\sigma^2 = 3}$$

④ S.D $\sigma = \sqrt{3} = 1.732$

⑤ $F(x)$ $\boxed{x \in [0, \infty)}$

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(x)\,dx$$

$$F(x) = P(X \le x) = \int_{0}^{x} f(x)\,dx$$

$$= \frac{1}{2}\int_0^x x^2 e^{-x}\,dx$$

$$= \frac{1}{2}\left[x^2 \frac{e^{-x}}{(-1)} - 2x\frac{e^{-x}}{(-1)^2} + 2\frac{e^{-x}}{(-1)^3} \right]_0^x$$

$$= \frac{1}{2}\left(-x^2 e^{-x} - 2xe^{-x} - 2e^{-x} + 2 \right)$$

$$\boxed{F(x) = \frac{1}{2}\left[-x^2 e^{-x} - 2x e^{-x} - 2e^{-x} + 2\right]}$$

$x \in [0, \infty)$

CDF

⑥ $P(X \leq 1) = P(0 \leq X \leq 1) = F(1) - F(0)$

$$= \int_0^1 f(x)\,dx$$

$$= \frac{1}{2}\left[\left(-e^{-1} - 2e^{-1} - 2e^{-1} + 2\right)\right]$$

$$= \frac{1}{2}\left[-5e^{-1} + 2\right]$$

⑦ $P(0 \leq X \leq 2) = F(2) - F(0) = \int_0^2 f(x)\,dx$

$$= \frac{1}{2}\left[-4e^{-2} - 4e^{-2} - 2e^{-2} + 2\right]$$

$$= \frac{1}{2}\left[-10e^{-2} + 2\right]$$

$$= -5e^{-2} + 1$$

(c) $P(X \geq 2) = P(2 \leq X < \infty)$ $\quad x \in [0, \infty)$

$= F(\infty) - F(2)$ $\quad F(\infty) = 1$

$= 1 - \frac{1}{2}(-4e^{-2} - 4e^{-2} - 2e^{-2} + 2)$

$= 1 - \frac{1}{2}[-10e^{-2} + 2]$

$= 1 + 5e^{-2} - 1]$

$$\boxed{P(X \geq 2) = 5e^{-2}}$$

Mean properties

a, b, k constant
x variable

① $E(k) = k$ $\qquad E(6) = 6$

② $E(kx) = k\,E(x)$ $\qquad E(6x) = 6\,E(x)$

③ $E(x+k) = E(x) + k$ $\qquad E(x+6) = E(x) + 6$

④ $E(ax+b) = a\,E(x) + b$ $\qquad E(2x+3) = 2E(x) + 3$

Variance properties

① $V(k) = 0$

② $V(kx) = k^2\,V(x)$

③ $V(x+k) = V(x)$

④ $V(ax+b) = a^2\,V(x) + 0$

Note: $E(x) = \sum x\,P(x)$; $E(x) = \int_{-\infty}^{\infty} x\,f(x)\,dx$

Mean properties

① $E(k) = k$

$\boxed{\sum P = 1}$

we know that $E(x) = \sum x\,P(x)$

$E(k) = \sum k\,P(x$

$= k \sum P(x)$

$$\boxed{E(k) = k}$$

② $E(KX) = K\,E(X)$

$$E(X) = \sum x\, p(x)$$

$$E(KX) = \sum Kx\, p(x)$$

$$= K \sum x\, p(x)$$

$$\boxed{E(KX) = K\,E(X)}$$

③ $E(X+K) = E(X) + K$

$$E(X) = \sum x\, p(x)$$

$$E(X+K) = \sum (x+K)\, p(x)$$

$$= \sum x\, p(x) + \sum K\, p(x)$$

$$= \sum x\, p(x) + K \sum p(x)$$

$$\boxed{E(X+K) = E(X) + K}$$

④ $E(aX+b) = aE(X)+b$

$$E(X) = \sum x\, p(x)$$

$$E(aX+b) = \sum (ax+b)\, p(x)$$

$$= \sum ax\, p(x) + \sum b\, p(x)$$

$$= a\sum x\, p(x) + b\sum p(x)$$

$$\boxed{E(aX+b) = aE(X)+b}$$

Variance properties

$$V(X) = E(X^2) - [E(X)]^2$$

① $V(k) = 0$

$$\boxed{E(k) = k}$$

$$\boxed{E(k^2) = k^2}$$

$$V(X) = E(X^2) - [E(X)]^2$$

$$V(k) = E(k^2) - [E(k)]^2$$

$$= k^2 - k^2$$

$$\boxed{V(k) = 0}$$

② $\underline{V(KX) = K^2 V(X)}$

$$V(X) = E(X^2) - [E(X)]^2$$

$$V(KX) = E(K^2X^2) - [E(KX)]^2$$

$$= K^2 E(X^2) - [K E(X)]^2$$

$$= K^2 E(X^2) - K^2 [E(X)]^2$$

$$= K^2 \left[E(X^2) - [E(X)]^2\right]$$

$$\boxed{V(KX) = K^2 V(X)}$$

③ $\underline{V(X+K) = V(X)}$

$$V(X) = E(X^2) - [E(X)]^2$$

$$V(X+K) = E(X+K)^2 - [E(X+K)]^2$$

$$= E[X^2 + 2KX + K^2] - [E(X) + K]^2$$

$$= E(X^2) + 2K E(X) + K^2$$

$$- \left[(E(X))^2 + 2K E(X) + K^2\right]$$

$$= E(x^2) + 2k\cancel{E(x)} + \cancel{k^2} - [E(x)]^2 - 2k\cancel{E(x)} - \cancel{k^2}$$

$$V(x+k) = E(x^2) - [E(x)]^2$$

$$\boxed{V(x+k) = V(x)}$$

(4) $V(ax+b) = a^2 V(x)$

$$V(x) = E(x^2) - [E(x)]^2$$

$$V(ax+b) = E(ax+b)^2 - [E(ax+b)]^2$$

$$= E[a^2x^2 + 2abx + b^2] - [aE(x)+b]^2$$

$$= a^2 E(x^2) + 2ab E(x) + b^2 - [a^2(E(x))^2 + b^2 + 2ab E(x)]$$

$$= a^2 E(x^2) + \cancel{2ab E(x)} + \cancel{b^2} - a^2(E(x))^2 - \cancel{b^2} - \cancel{2ab E(x)}$$

$$V(aX+b) = a^2 E(X^2) - a^2 [E(X)]^2$$

$$= a^2 \left[E(X^2) - [E(X)]^2 \right]$$

$$\boxed{V(aX+b) = a^2 V(X)}$$

<u>Note</u>: $E(aX+b) = a E(X) + b$

$$E(X) = \int_{-\infty}^{\infty} x f(x)\, dx$$

$$E(aX+b) = \int_{-\infty}^{\infty} (ax+b) f(x)\, dx$$

$$= \int_{-\infty}^{\infty} ax f(x)\, dx + \int_{-\infty}^{\infty} b f(x)\, dx$$

$$= a \int_{-\infty}^{\infty} x f(x)\, dx + b \int_{-\infty}^{\infty} f(x)\, dx$$

$$= a E(X) + b$$

$$\boxed{E(aX+b) = a E(X) + b}$$

Note (2)

$\boxed{V(X) = E(X-\mu)^2}$ $\boxed{\mu = E(X)}$

$= E[X^2 + \mu^2 - 2X\mu]$

$= E(X^2) + \mu^2 - 2\mu E(X)$

$= E(X^2) + [E(X)]^2 - 2E(X)E(X)$

$= E(X^2) + [E(X)]^2 - 2[E(X)]^2$

$\boxed{V(X) = E(X^2) - [E(X)]^2}$

$E(X) = 2$, $V(X) = 3$ $E(X^2) = 4$

$E(2X+3) = 2E(X) + 3$

$= 2(2) + 3$

$= 7$

$V(3X) = 3^2 V(X)$

$= 9V(X) = 27$

$V(X+3) = V(X) + 3$ 3

3 6 9

① For The probl Distribution

x	-3	6	9
$P(x)$	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{1}{3}$

① $E(x)$ ② $E(x^2)$ ③ $E(2x+1)^2$

④ $V(5x)$

So ① $E(x) = \sum x\,P(x)$

$$= (-3)\left(\frac{1}{6}\right) + 6\left(\frac{1}{2}\right) + 9\left(\frac{1}{3}\right)$$

$$\boxed{E(x) = \frac{11}{2}}$$

② $E(x^2) = \sum x^2\,P(x)$

$$= (-3)^2\left(\frac{1}{6}\right) + (6)^2\left(\frac{1}{2}\right) + (9)^2\left(\frac{1}{3}\right)$$

$$\boxed{E(x^2) = \frac{93}{2}}$$

③ $E(2x+1)^2 = E\left[4x^2 + 4x + 1\right]$

$$= 4\,E(x^2) + 4\,E(x) + 1$$

$$= 4 \times \frac{93}{2} + 4 \times \frac{11}{2} + 1$$

$$\boxed{E(2X+1)^2 = 209}$$

(4) $V(5X)$

$$V(aX) = a^2\, V(X)$$

$$V(5X) = 5^2\, V(X)$$

$$= 25\, V(X)$$

$$= 25\, E(X^2) - [E(X)]^2$$

$$= 25\left[\frac{93}{2} - \left(\frac{11}{2}\right)^2\right]$$

$$\boxed{V(5X) = 406.25}$$

GATE

A player wins if he gets 5 on a single throw of a die, he loses if he gets 2 or 4; If he wins he gets Rs 50; If he loses he gets Rs 10; otherwise he has to pay Rs 15. Find the value of the game to the player; Is it favourable?

① A continuous R.V X has the Distribution function

CDF $\quad F(x) = \begin{cases} 0 & \text{if } x \le 1 \\ K(x-1)^4 & \text{if } 1 \le x \le 3 \\ 1 & \text{if } x > 3 \end{cases}$

Determine (i) $f(x)$ pdf (ii) K

Sol

we know that

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(x)\,dx$$

$$\boxed{\frac{d}{dx} F(x) = f(x)}$$

$$\therefore f(x) = \begin{cases} 0 & \text{if } x \le 1 \\ K\,4(x-1)^3 & \text{if } 1 \le x \le 3 \\ 0 & \text{if } x > 3 \end{cases}$$

$$\boxed{f(x) = 4K(x-1)^3 \quad \text{for } x \in [1,3]}$$

(ii) $f(x)$ is p.d.f

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\int_{1}^{3} f(x)\,dx = 1$$

$$4k \int_{1}^{3} (x-1)^3\,dx = 1$$

$$4k \left[\frac{(x-1)^4}{4}\right]_1^3 = 1$$

$$4k \left[\frac{2^4}{4}\right] = 1$$

$$\boxed{k = \frac{1}{16}}$$

$$\boxed{\text{Bivariate Random Variables}}$$

X

no of heads

Y

$\{HHH, HHT, HTH, HTT, TTT, THT, TTH, THH\}$

0

=

Experiment

✓ X ✓ Y

D C

✓ (X, Y) B·R·V 1B R V

Two coins toss $n(S) = 4$ out-comes

$S = \{ HH, HT, TH, TT \}$

$X = \{0, 1\}$ ✓ $Y = \{0, 1, 2\}$

{Takes 1st first coin head 1, " Tail 0}

{Takes the no of heads}

X	Y	(X, Y)	
1	2	(1, 2)	HH
1	1	(1, 1)	HT
0	1	(0, 1)	TH
0	0	(0, 0)	TT

$(X, Y) = \{ (0,0)\ (0,1), (0,2), (1,0)\ (1,1)\ (1,2) \}$

✓

✓ (X,Y) DBRV ✓ $X=\{0,1\}$ $Y=\{0,1,2\}$

$(X,Y)=\{(0,0)\ (0,1)\ (0,2)\ (1,0)\ (1,1)\ (1,2)\}$

JPMF Joint PMF

$P(x,y)$ JPMF ?

$$\boxed{\sum_x \sum_y P(x,y)=1}$$

MPMF Marginal PMF

$$P_1(x)=\sum_y P(x,y) \quad \text{MPMF of } X$$

$$P_2(y)=\sum_x P(x,y) \quad \text{MPMF of } Y$$

CPMF Conditional PMF

✓ $$P_{1|2}(x/y)=\frac{P(x,y)}{P_2(y)}=\frac{\text{JPMF}}{\text{MPMF of } Y}$$

$$P_{2|1}(y|x)=\frac{P(x,y)}{P_1(x)}=\frac{\text{JPMF}}{\text{MPMF of } X}$$

X and Y are independent or not

$$P(x, y) = P_1(x)\, P_2(y)$$

$$\text{JPMF} = (\text{MPMF of } X) \cdot (\text{MPMF of } Y)$$

① The JPMF of (X, Y) is given by

$$P(x,y) = \begin{cases} k(2x+y) & \text{for } x = 1, 2 \\ & \quad\;\; y = 1, 2 \\ 0 & \text{otherwise} \end{cases}$$

where k is constant

ⓐ Find K

ⓑ Find MPMF

ⓒ Find CPMF

ⓓ X & Y independent or not

So

$$\boxed{\sum_{x} \sum_{y} P(x,y) = 1}$$

$$k \sum_{x=1,2} \sum_{y=1,2} (2x+y) = 1$$

$(1,1) + (1,2) + (2,1) + (2,2) \rightarrow$ Rough

$$k[3+4+5+6] = 1$$

$$18k = 1 \Rightarrow \boxed{k = \frac{1}{18}}$$

MPMF

$$P_1(x) = \sum_{y} P(x,y)$$

$$P_1(x) = \sum_{y=1,2} \frac{1}{18}(2x+y)$$

$$= \frac{1}{18}\left[(2x+1) + (2x+2)\right]$$

$$\boxed{P_1(x) = \frac{1}{18}\{4x+3\}}$$

X	1	2
$P_1(x)$	$\frac{7}{18}$	$\frac{11}{18}$

$$P_2(y) = \sum_{x} P(x,y)$$

$$P_2(y) = \sum_{x=1,2} \frac{1}{18}(2x+y)$$

$$= \frac{1}{18}\left[(2+y) + (4+y)\right]$$

$$\boxed{P_2(y) = \frac{1}{18}[2y+6]}$$

Y	1	2
$P_2(y)$	$\frac{8}{18}$	$\frac{10}{18}$

CPMF

$$P_{1|2}(x|y) = \frac{P(x,y)}{P_2(y)} = \frac{\text{JPMF}}{\text{MPMF of } y}$$

$$= \frac{\frac{1}{18}(2x+y)}{\frac{1}{18}(2y+6)}$$

$$\boxed{P_{1|2}(x|y) = \frac{2x+y}{2y+6}}$$

$$P_{2|1}(y|x) = \frac{P(x,y)}{P_1(x)} = \frac{\text{JPMF}}{\text{MPMF of } x}$$

$$= \frac{\frac{1}{18}(2x+y)}{\frac{1}{18}(4x+3)}$$

$$\boxed{P_{2|1}(y|x) = \frac{2x+y}{4x+3}}$$

X & Y independent or not

$$P(x,y) = P_1(x)\, P_2(y)$$

$$\frac{1}{18}(2x+y) \neq \frac{1}{18}(4x+3)\,\frac{1}{18}(2y+6)$$

$\therefore$ X & Y are dependent

Example 1: A fair coin is tossed three times. Let X be a random variable that takes the value 0 if the first toss is a tail and the value 1 if the first toss is a head and Y be a random variable that defines the total number of heads in the three tosses. Then

i. Determine the joint, marginal and conditional mass functions of X and Y.

ii. Are X and Y independent?

sol: A fair coin is tossed three times $n(S) = 8$

$S = \{HHH, HTH, HHT, HTT, TTT, THT, TTH, THH\}$

outcomes	X	Y	(X, Y)
HHH	1	3	(1,3)
HTH	1	2	(1,2)
HHT	1	2	(1,2)
HTT	1	1	(1,1)
TTT	0	0	(0,0)
THT	0	1	(0,1)
TTH	0	1	(0,1)
THH	0	2	(0,2)

X Takes the values $\{0, 1\}$

Y Takes the values $\{0, 1, 2, 3\}$

$$(X, Y) = \{ (0,0)\ (0,1)\ (0,2)\ (0,3),\ (1,0)\ (1,1),\ (1,2)\ (1,3) \}$$

$P(0,0) = \frac{1}{8}$

$P(0,1) = \frac{2}{8}$

$P(0,2) = \frac{1}{8}$

$P(0,3) = 0$

$P(1,0) = 0$

$P(1,1) = \frac{1}{8}$

$P(1,2) = \frac{2}{8}$

$P(1,3) = \frac{1}{8}$

$$\boxed{\sum_x \sum_y P(x,y) = 1}$$

$$P(0,0) + P(0,1) + P(0,2) + P(0,3) + P(1,0) + P(1,1) + P(1,2) + P(1,3)$$

$$= \frac{1}{8} + \frac{2}{8} + \frac{1}{8} + 0 + 0 + \frac{1}{8} + \frac{2}{8} + \frac{1}{8}$$

$$= \frac{4}{8} + \frac{4}{8} = 1$$

$$\boxed{\sum_x \sum_y P(x,y) = 1}$$

$\Rightarrow P(x,y)$ is JPMF

MPMF

$$P_1(x) = \sum_y P(x,y)$$ ✓

$$P_2(y) = \sum_x P(x,y)$$

$X = \{0,1\}$ $\quad Y = \{0,1,2,3\}$

X	0	1
$P_1(x)$	$\frac{1}{2}$	$\frac{1}{2}$

✓

Y	0	1	2	3
$P_2(y)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

For $X = \{0,1\}$

$$P_1(0) = P(0,0) + P(0,1) + P(0,2) + P(0,3)$$

$$= \frac{1}{8} + \frac{2}{8} + \frac{1}{8} + 0 = \frac{4}{8} = \frac{1}{2}$$

$$P_1(1) = P(1,0) + P(1,1) + P(1,2) + P(1,3)$$

$$= 0 + \frac{1}{8} + \frac{2}{8} + \frac{1}{8} = \frac{4}{8} = \frac{1}{2}$$

For $Y = \{0,1,2,3\}$

✓ $P_2(0) = P(0,0) + P(1,0) = \frac{1}{8} + 0 = \frac{1}{8}$

✓ $P_2(1) = P(0,1) + P(1,1) = \frac{2}{8} + \frac{1}{8} = \frac{3}{8}$

✓ $P_2(2) = P(0,2) + P(1,2) = \frac{1}{8} + \frac{2}{8} = \frac{3}{8}$

$P_2(3) = P(0,3) + P(1,3) = 0 + \frac{1}{8} = \frac{1}{8}$

CPMF

$$P_{1|2}(x|y) = \frac{P(x,y)}{P_2(y)}$$

$$P_{2|1}(y|x) = \frac{P(x,y)}{P_1(x)}$$

$x = \{0, 1\}$

$Y = \{0, 1, 2, 3\}$

For $X : \{0, 1\}$

For 0

$$P_{1|2}(0|0) = \frac{P(0,0)}{P_2(0)} = \frac{\frac{1}{8}}{\frac{1}{8}} = 1$$

$$P_{1|2}(0|1) = \frac{P(0,1)}{P_2(1)} = \frac{\frac{2}{8}}{3/8} = 2/3$$

$$P_{1|2}(0|2) = \frac{P(0,2)}{P_2(2)} = \frac{1/8}{3/8} = \frac{1}{3}$$

$$P_{1|2}(0|3) = \frac{P(0,3)}{P_2(3)} = \frac{0}{1/8} = 0$$

For 1

$$P_{1|2}(1|0) = \frac{P(1,0)}{P_2(0)} = \frac{0}{\frac{1}{8}} = 0$$

$$P_{1|2}(1|1) = \frac{P(1,1)}{P_2(1)} = \frac{\frac{1}{8}}{3/8} = \frac{1}{3}$$

$$P_{1|2}(1|2) = \frac{P(1,2)}{P_2(2)} = \frac{\frac{2}{8}}{3/8} = \frac{2}{3}$$

$$P_{1|2}(1|3) = \frac{P(1,3)}{P_2(3)} = \frac{\frac{1}{8}}{\frac{1}{8}} = 1$$

<u>For $Y = \{0,1,2,3\}$</u>

$$\boxed{P_{2|1}(y|x) = \frac{P(x,y)}{P_1(x)}}$$

$Y = \{0,1,2,3\}$, $X = \{0,1\}$

<u>for 0</u>

$$P_{2|1}(0|0) = \frac{P(0,0)}{P_1(0)} = \frac{\frac{1}{8}}{\frac{1}{2}} = \frac{1}{4}$$

$$P_{2|1}(0|1) = \frac{P(1,0)}{P_1(1)} = 0$$

<u>for 1</u>

$$P_{2|1}(1|0) = \frac{P(0,1)}{P_1(0)} = \frac{\frac{2}{8}}{\frac{1}{2}} = \frac{1}{2}$$

$$P_{2|1}(1|1) = \frac{P(1,1)}{P_1(1)} = \frac{\frac{1}{8}}{\frac{1}{2}} = \frac{1}{4}$$

<u>for 2</u>

$$P_{2|1}(2|0) = \frac{P(0,2)}{P_1(0)} = \frac{\frac{1}{8}}{\frac{1}{2}} = \frac{1}{4}$$

$$P_{2|1}(2|1) = \frac{P(1,2)}{P_1(1)} = \frac{2/8}{1/2} = \frac{1}{2}$$

for 3

$$P_{2|1}(3|0) = \frac{P(0,3)}{P_1(0)} = 0$$

$$P_{2|1}(3|1) = \frac{P(1,3)}{P_1(1)} = \frac{\frac{1}{8}}{\frac{1}{2}} = \frac{1}{4}$$

X & Y are independent or not

$$P(x,y) \neq P_1(x)\ P_2(y)$$

$$P(1,2) \neq P_1(1)\ P_2(2)$$

$$\frac{2}{8} \neq \frac{1}{2} \times \frac{3}{8}$$

$\therefore$ X & Y are dependent

Continuous Bivariate Random Variable

(X, Y) is continuous B.R.V If X and Y both are continuous R.V

JPdf $f(x,y)$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$$

MPdf

$$f_1(x) = \int_{y=-\infty}^{\infty} f(x,y)\,dy$$

$$f_2(y) = \int_{x=-\infty}^{\infty} f(x,y)\,dx$$

CPdf

$$f_{1|2}(x|y) = \frac{f(x,y)}{f_2(y)} = \frac{\text{JPdf}}{\text{MPdf of } y}$$

$$f_{2|1}(y|x) = \frac{f(x,y)}{f_1(x)} = \frac{\text{JPdf}}{\text{MPdf of } x}$$

X and Y independent or not

$$f(x,y) = f_1(x)\cdot f_2(y)$$

$$\text{JPdf} = \text{MPdf of } x \cdot \text{MPdf of } y$$

① The j.p.d.f of (X, Y) is given by

$$f(x,y) = \begin{cases} k x^3 y^2 & \text{for } 0 < x < 2 \\ & \phantom{\text{for }} 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

Then find

① k ② m.p.d.f of X and Y

③ c.p.d.f of X and Y

④ X & Y are independent or not

Sol ① Given $f(x,y)$ is j.p.d.f

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y)\, dx\, dy = 1$$

$$k \int_{x=0}^{2} \int_{y=0}^{1} x^3 y^2\, dx\, dy = 1$$

$$k \int_{x=0}^{2} x^3 \left[\frac{y^3}{3}\right]_0^1 = 1$$

$$\frac{k}{3} \left[\frac{x^4}{4}\right]_0^2 = 1 \Rightarrow \boxed{k = \frac{3}{4}}$$

Hence $f(x) = \frac{3}{4} x^3 y^2$ for $0 < x < 2$
$0 < y < 1$

② mpdf

mpdf of x

$$f_1(x) = \int_{y=-\infty}^{\infty} f(x,y)\, dy$$

$$= \int_{y=0}^{1} \frac{3}{4} x^3 y^2\, dy$$

$$= \frac{3x^3}{4} \int_{y=0}^{1} y^2\, dy = \frac{3x^3}{4} \left[\frac{y^3}{3}\right]_0^1$$

$$\boxed{f_1(x) = \frac{x^3}{4}}$$

mpdf of y

$$f_2(y) = \int_{x=-\infty}^{\infty} f(x,y)\, dx$$

$$= \int_{x=0}^{2} \frac{3}{4} x^3 y^2\, dx$$

$$= \frac{3y^2}{4} \int_{x=0}^{2} x^3 \, dx$$

$$= \frac{3y^2}{4} \left[\frac{x^4}{4} \right]_0^2$$

$$\boxed{f_2(y) = 3y^2}$$

③ <u>cpdf</u>

cpdf of X

$$f_{1|2}(x|y) = \frac{f(x,y)}{f_2(y)} = \frac{\text{Jpdf}}{\text{mpdf of } y}$$

$$= \frac{\frac{3}{4} x^3 y^2}{3y^2}$$

$$\boxed{f_{1|2}(x|y) = \frac{x^3}{4}}$$

cpdf of y

$$f_{2|1}(y|x) = \frac{f(x,y)}{f_1(x)} = \frac{\text{Jpdf}}{\text{mpdf of } x}$$

$$= \frac{\frac{3}{4}x^3y^2}{\frac{x^3}{4}}$$

$$\boxed{f_{2|1}(y|x) = 3y^2}$$

④ X & Y are independent or not

$$f(x,y) = f_1(x)\, f_2(y)$$

$$\frac{3}{4}x^3y^2 = \frac{x^3}{4} \cdot 3y^2$$

$\therefore$ X & Y are independent

② The Jpdf of (X, Y) is given by

$$f(x,y) = \begin{cases} x e^{-x(y+1)} & 0 < x < \infty \\ & 0 < y < \infty \\ 0 & \text{otherwise} \end{cases}$$

Determine

(i) Marginal, conditional prob. Density function

(ii) X & Y are independent or not

Sol ① mpdf

mpdf of x

$$f_1(x) = \int_{y=-\infty}^{\infty} f(x,y)\, dy$$

$$= \int_{y=0}^{\infty} x e^{-x(y+1)}\, dy$$

$$= \int_{y=0}^{\infty} x\, e^{-xy} \cdot e^{-x}\, dy$$

$$= x e^{-x} \left[\frac{e^{-xy}}{-x} \right]_0^{\infty}$$

$$= -e^{-x}\left[e^{-\infty} - e^{0}\right]$$

$$= -e^{-x}\left[0-1\right]$$

$$\boxed{f_1(x) = e^{-x}}$$

mpdf of y

$$f_2(y) = \int_{x=-\infty}^{\infty} f(x,y)\, dx$$

$$= \int_{x=0}^{\infty} x e^{-x(y+1)}\, dx$$

$$= \left[x\,\frac{e^{-x(y+1)}}{-(y+1)} - 1\,\frac{e^{-x(y+1)}}{(y+1)^2} \right]_0^{\infty}$$

$$\boxed{e^{-\infty} = 0}$$

$$\boxed{e^{0} = 1}$$

$$= 0 - \left(0 - \frac{1}{(y+1)^2}\right)$$

$$\boxed{f_2(y) = \frac{1}{(y+1)^2}}$$

CPdf

CPdf of x

$$f_{1|2}(x|y) = \frac{f(x,y)}{f_2(y)} = \frac{\text{JPdf}}{\text{MPdf of } y}$$

$$= \frac{x e^{-x(y+1)}}{\frac{1}{(y+1)^2}}$$

$$\boxed{f_{1|2}(x|y) = x(y+1)^2 e^{-x(y+1)}}$$

CPdf of y

$$f_{2|1}(y|x) = \frac{f(x,y)}{f_1(x)} = \frac{\text{JPdf}}{\text{mpdf of } x}$$

$$= \frac{x e^{-x(y+1)}}{e^{-x}}$$

$$= x e^{-xy-\cancel{x}+\cancel{x}}$$

$$\boxed{f_{2|1}(y|x) = x e^{-xy}}$$

(2) X & Y are independent or not

$$f(x,y) \neq f_1(x)\, f_2(y)$$

$$x e^{-x(y+1)} \neq e^{-x} \cdot \frac{1}{(y+1)^2}$$

$\therefore$ X & Y are dependent

Mean and Variance of Bivariate R.V

(X, Y) be a CBRV

f(x, y) JPdf

$$\left.\begin{aligned} f_1(x) &= \int_{y=-\infty}^{\infty} f(x,y)\, dy \\ f_2(y) &= \int_{x=-\infty}^{\infty} f(x,y)\, dx \end{aligned}\right\} \text{mPdf}$$

For X

$$E(X) = \int_{x=-\infty}^{\infty} x\, f_1(x)\, dx$$

$$E(X^2) = \int_{x=-\infty}^{\infty} x^2\, f_1(x)\, dx$$

$$v(X) = E(X^2) - [E(X)]^2$$

For Y

$$E(Y) = \int_{y=-\infty}^{\infty} y \, f_2(y) \, dy$$

$$E(Y^2) = \int_{y=-\infty}^{\infty} y^2 \, f_2(y) \, dy$$

$$V(Y) = E(Y^2) - [E(Y)]^2$$

$$E(XY) = \int_{x=-\infty}^{\infty} \int_{y=-\infty}^{\infty} xy \, f(x,y) \, dx \, dy$$

Covariance of (X, Y) : $\mathrm{cov}(X,Y)$

$$\mathrm{cov}(X,Y) = E(XY) - E(X)\,E(Y)$$

Correlation coefficient of (X, Y): $\rho(X,Y)$

$$\rho(X,Y) = \frac{\mathrm{cov}(X,Y)}{\sqrt{V(X)}\,\sqrt{V(Y)}}$$

$$\boxed{\rho(X,Y) = \frac{cov(X,Y)}{\sqrt{V(X)}\sqrt{V(Y)}}}$$

$$-1 \le \rho(X,Y) \le 1$$

$\boxed{X, Y \text{ independent}}$

Sumnd

$$\left\{\begin{array}{ll} \rho(X,Y) = 0 \implies & \checkmark \\ cov(X,Y) = 0 & \checkmark \\ E(XY) = E(X)E(Y) & \checkmark \\ f(x,y) = f_1(x)\,f_2(y) & \checkmark \\ P(x,y) = P_1(x)\,P_2(y) & \end{array}\right.$$

1. If JPdf of (X, Y) is given by

$$f(x,y) = \begin{cases} 2-x-y & \text{for } 0<x<1 \\ & 0<y<1 \\ 0 & \text{otherwise} \end{cases}$$ Then

find

1. JPdf
2. MPdf
3. CPdf (X)
4. X & Y indep or not
5. $E(X)$ $E(X^2)$ $V(X)$
6. $E(Y)$ $E(Y^2)$ $V(Y)$
7. $E(XY)$
8. $cov(X, Y)$
9. $\rho(X, Y)$

Sol

given $f(x,y) = 2-x-y \quad \begin{matrix} 0<x<1 \\ 0<y<1 \end{matrix}$

(2) mpdf

mpdf of x

$$f_1(x) = \int_{y=-\infty}^{\infty} f(x,y)\,dy$$

$$= \int_{y=0}^{1} [2-x-y]\,dy$$

$$= \left[2y - xy - \frac{y^2}{2}\right]_0^1$$

$$= 2-x-\frac{1}{2}$$

$$\boxed{f_1(x) = \frac{3}{2}-x}$$

Mpdf of y

$$f_2(y) = \int_{x=-\infty}^{\infty} f(x,y)\,dx$$

$$= \int_{x=0}^{1} [2-x-y]\,dx$$

$$= \left[2x - \frac{x^2}{2} - xy\right]_0^1$$

$$= 2 - \frac{1}{2} - y$$

$$\boxed{f_2(y) = \frac{3}{2} - y}$$

③ <u>cpdf</u>

cpdf of x

$$f_{1|2}(x|y) = \frac{f(x,y)}{f_2(y)}$$

$$= \frac{2-x-y}{\frac{3}{2}-y} = \frac{4-2x-2y}{3-2y}$$

cpdf of y

$$f_{2|1}(y|x) = \frac{f(x,y)}{f_1(x)}$$

$$= \frac{2-x-y}{\frac{3}{2}-x} = \frac{4-2x-2y}{3-2x}$$

(4) X & Y are independent (or) not

$$f(x,y) \neq f_1(x)\, f_2(y)$$

$$2-x-y \neq \left(\frac{3}{2}-x\right)\left(\frac{3}{2}-y\right)$$

$\therefore$ X & Y are dependent

(5) $E(X)$, $E(X^2)$, $V(X)$

$$E(X) = \int_{x=-\infty}^{\infty} x\, f_1(x)\, dx$$

$$= \int_{x=0}^{1} x\left(\frac{3}{2}-x\right) dx$$

$$= \int_{x=0}^{1} \left[\frac{3x}{2} - x^2\right] dx$$

$$= \left[\frac{3x^2}{4} - \frac{x^3}{3}\right]_0^1$$

$$= \frac{3}{4} - \frac{1}{3} = \frac{5}{12}$$

$$\boxed{E(X) = \frac{5}{12}}$$

$$E(x^2) = \int_{x=-\infty}^{\infty} x^2 \, f_1(x) \, dx$$

$$= \int_{x=0}^{1} x^2 \left[\frac{3}{2} - x\right] dx$$

$$= \left[\frac{3}{2}\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1$$

$$= \frac{1}{2} - \frac{1}{4}$$

$$\boxed{E(x^2) = \frac{1}{4}}$$

$$V(x) = E(x^2) - [E(x)]^2$$

$$= \frac{1}{4} - \frac{25}{144}$$

$$\boxed{V(x) = \frac{44}{576}}$$

⑥ $E(Y)$, $E(Y^2)$, $V(Y)$

$$E(Y) = \int_{y=-\infty}^{\infty} y \, f_2(y) \, dy$$

$$= \int_{y=0}^{1} y \left[\frac{3}{2} - y\right] dy$$

$$= \left[\frac{3}{2}\frac{y^2}{2} - \frac{y^3}{3}\right]_0^1$$

$$= \frac{3}{4} - \frac{1}{3} = \frac{5}{12}$$

$$\boxed{E(Y) = \frac{5}{12}}$$

lly $\boxed{E(Y^2) = \frac{1}{4}}$

$$\boxed{V(Y) = \frac{44}{576}}$$

⑦ $E(XY)$

$$E(XY) = \int_{x=-\infty}^{\infty} \int_{y=-\infty}^{\infty} xy \, f(x,y) \, dx \, dy$$

$$= \int_{x=0}^{1} \int_{y=0}^{1} xy \left[2 - x - y\right] dx \, dy$$

$$= \int_{x=0}^{1} \int_{y=0}^{1} \left[2xy - x^2 y - x y^2\right] dy \, dx$$

$$= \int_{x=0}^{1} \left[2x \frac{y^2}{2} - x^2 \frac{y^2}{2} - x \frac{y^3}{3}\right]_0^1 dx$$

$$= \int_{x=0}^{1} \left[x - \frac{x^2}{2} - \frac{x}{3}\right] dx$$

$$= \left[\frac{x^2}{2} - \frac{x^3}{6} - \frac{x^2}{6}\right]_0^1$$

$$= \frac{1}{2} - \frac{1}{6} - \frac{1}{6} = \frac{1}{6}$$

$$\boxed{E(XY) = \frac{1}{6}}$$

⑧ $\underline{\operatorname{cov}(X, Y)}$

$$\operatorname{cov}(X, Y) = E(XY) - E(X)\,E(Y)$$

$$= \frac{1}{6} - \frac{5}{12} \cdot \frac{5}{12}$$

$$\boxed{\operatorname{cov}(X, Y) = \frac{-1}{144}}$$

⑨ $\underline{\rho(X, Y)}$

$$\rho(X, Y) = \frac{\operatorname{cov}(X, Y)}{\sqrt{V(X)}\,\sqrt{V(Y)}}$$

$$= \frac{\frac{-1}{144}}{\sqrt{\frac{44}{576}}\,\sqrt{\frac{44}{576}}}$$

$$= \frac{-1}{144} \times \frac{576}{44}$$

$$\boxed{\rho(x,y) = \frac{-1}{11}} = \underline{-0.09}$$

$$\boxed{-1 \le \rho(x,y) \le 1}$$

$$\boxed{\rho(x,y) < 0}$$

x & y are opposite Direction

Discrete Bivariate Random Variable

① JPMF : $p(x,y)$ $\qquad \sum_x \sum_y p(x,y) = 1$

② MPMF $\qquad p_1(x) = \sum_y p(x,y)$

$$p_2(y) = \sum_x p(x,y)$$

③ CPMF $\qquad p_{1|2}(x|y) = \frac{p(x,y)}{p_2(y)}$

$$p_{2|1}(y|x) = \frac{p(x,y)}{p_1(x)}$$

④ X & Y independent

$$p(x,y) = p_1(x)\, p_2(y)$$

⑤ $E(X) = \sum_x x\, p_1(x)$

$$E(X^2) = \sum_x x^2\, p_1(x)$$

$$V(X) = E(X^2) - [E(X)]^2$$

⑥ $E(Y) = \sum_{y} y\, p_2(y)$

$E(Y^2) = \sum_{y} y^2\, p_2(y)$

$V(Y) = E(Y^2) - [E(Y)]^2$

⑦ $E(XY) = \sum_{x} \sum_{y} xy\, p(x,y)$

$E(X+Y) = \sum_{x} \sum_{y} (x+y)\, p(x,y)$

⑧ $\operatorname{cov}(X,Y) = E(XY) - E(X)\,E(Y)$

⑨ $\rho(X,Y) = \dfrac{\operatorname{cov}(X,Y)}{\sqrt{V(X)}\,\sqrt{V(Y)}}$

Example 2: The j.p.m.f of (X, Y) is given below:

$Y \backslash X$	-1	1
0	$\frac{1}{8}$	$\frac{3}{8}$
1	$\frac{2}{8}$	$\frac{2}{8}$

Find the correlation coefficient between X and Y

Sol: Given JPMF of (X, Y) is

$Y \backslash X$	-1	1	$P_2(y)$
0	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{4}{8}$
1	$\frac{2}{8}$	$\frac{2}{8}$	$\frac{4}{8}$
$P_1(x)$	$\frac{3}{8}$	$\frac{5}{8}$	1

MPMF of X

X	-1	1
$P_1(x)$	$\frac{3}{8}$	$\frac{5}{8}$

MPMF of Y

Y	0	1
$P_2(y)$	$\frac{4}{8}$	$\frac{4}{8}$

$X = \{-1, 1\}, \quad Y = \{0, 1\}$

$(X, Y) = \{(-1, 0)\ (-1, 1), (1, 0), (1, 1)\}$

(X, Y)	$(-1, 0)$	$(-1, 1)$	$(1, 0)$	$(1, 1)$
$P(x, y)$	$\frac{1}{8}$	$\frac{2}{8}$	$\frac{3}{8}$	$\frac{2}{8}$

X & Y independent (or) not

$$P(x, y) = P_1(x) \; P_2(y)$$

$$(x, y) = (-1, 0)$$

$$P(-1, 0) \neq P_1(-1) \; P_2(0)$$

$$\frac{1}{8} \neq \frac{3}{8} \times \frac{4}{8}$$

$\therefore$ X & Y are dependent

③ $E(X)$, $E(X^2)$, $V(X)$

X	-1	1
$P_1(x)$	$\frac{3}{8}$	$\frac{5}{8}$

$$E(X) = \Sigma x \, P_1(x)$$

$$= (-1)\left(\frac{3}{8}\right) + (1)\left(\frac{5}{8}\right)$$

$$= \frac{2}{8} = \boxed{\frac{1}{4} = E(X)}$$

$$E(X^2) = \sum x^2 \, p_1(x)$$

$$= (-1)^2\left(\frac{3}{8}\right) + (1)^2\left(\frac{5}{8}\right)$$

$$\boxed{E(X^2) = 1}$$

$$V(X) = E(X^2) - [E(X)]^2$$

$$= 1 - \frac{1}{16}$$

$$\boxed{V(X) = \frac{15}{16}}$$

(4) $E(Y)$, $E(Y^2)$, $V(Y)$

Y	0	1
$p_2(y)$	$\frac{4}{8}$	$\frac{4}{8}$

$$E(Y) = \sum y \, p_2(y)$$

$$= 0\left(\frac{4}{8}\right) + 1\left(\frac{4}{8}\right)$$

$$\boxed{E(Y) = \frac{1}{2}}$$

$$E(Y^2) = \sum y^2 \, p_2(y)$$

$$= 0^2\left(\frac{4}{8}\right) + 1^2\left(\frac{4}{8}\right)$$

$$\boxed{E(Y^2) = \frac{1}{2}}$$

$$V(Y) = E(Y^2) - [E(Y)]^2$$

$$= \frac{1}{2} - \frac{1}{4}$$

$$\boxed{V(Y) = \frac{1}{4}}$$

⑤ <u>$E(XY)$</u>

$$E(XY) = \sum_x \sum_y xy \; p(x,y)$$

$$= (-1)(0)\left(\frac{1}{8}\right) + (-1)(1)\left(\frac{2}{8}\right) + (1)(0)\left(\frac{3}{8}\right)$$

$$+ (1)(1)\frac{2}{8}$$

$$\boxed{E(XY) = 0}$$

⑥ <u>$\operatorname{cov}(X,Y)$</u>

$$\operatorname{cov}(X,Y) = E(XY) - E(X)E(Y)$$

$$= 0 - \frac{1}{4} \cdot \frac{1}{2}$$

$$\boxed{\operatorname{cov}(X,Y) = -\frac{1}{8}}$$

⑦ $\underline{\rho(X,Y)}$

$$\rho(X,Y) = \frac{\operatorname{cov}(X,Y)}{\sqrt{V(X)}\,\sqrt{V(Y)}}$$

$$= \frac{-\frac{1}{8}}{\sqrt{\frac{15}{16}}\,\sqrt{\frac{1}{4}}}$$

$$\boxed{\rho(X,Y) = \frac{-1}{\sqrt{15}}} < 0$$

$$\rho(X,Y) < 0$$

$\therefore$ X & Y are opposite Direction

Theorems

① $\text{cov}(X,Y) = E(XY) - E(X)E(Y)$

Proof By the def of covariance

$$\text{cov}(X,Y) = E\{(X-E(X))(Y-E(Y))\}$$

$$= E[XY - XE(Y) - YE(X) + E(X)E(Y)]$$

$$= E(XY) - \cancel{E(X)E(Y)} - E(Y)E(X) + \cancel{E(X)E(Y)}$$

$$\boxed{\text{cov}(X,Y) = E(XY) - E(X)E(Y)}$$

Note:

$$\text{cov}(aX,bY) = E(aX\,bY) - E(aX)E(bY)$$

$$= ab\,E(XY) - aE(X)\,bE(Y)$$

$$\text{cov}(aX,bY) = ab\,[E(XY) - E(X)E(Y)]$$

$$\boxed{\text{cov}(aX,bY) = ab\,\text{cov}(X,Y)}$$

$$\text{cov}(aX,aY) = a^2\,\text{cov}(X,Y)$$

② $E(X+Y) = E(X) + E(Y)$

Proof. let (X, Y) be a CBRV

$$E(X) = \int_{x=-\infty}^{\infty} x \, f_1(x) \, dx$$

where $f_1(x) = \int_{y=-\infty}^{\infty} f(x, y) \, dy$

$$E(Y) = \int_{y=-\infty}^{\infty} y \, f_2(y) \, dy$$

where $f_2(y) = \int_{x=-\infty}^{\infty} f(x, y) \, dx$

Now

$$E(X+Y) = \int_{x=-\infty}^{\infty} \int_{y=-\infty}^{\infty} (x+y) \, f(x, y) \, dx \, dy$$

$$= \int_{x=-\infty}^{\infty} \int_{y=-\infty}^{\infty} x \, f(x, y) \, dx \, dy + \int_{x=-\infty}^{\infty} \int_{y=-\infty}^{\infty} y \, f(x, y) \, dx \, dy$$

$$= \int_{x=-\infty}^{\infty} x \underbrace{\int_{y=-\infty}^{\infty} f(x, y) \, dy}_{f_1(x)} \, dx + \int_{y=-\infty}^{\infty} y \underbrace{\int_{x=-\infty}^{\infty} f(x, y) \, dx}_{f_2(y)} \, dy$$

$$= \int_{x=-\infty}^{\infty} x\, f_1(x)\,dx + \int_{y=-\infty}^{\infty} y\, f_2(y)\,dy$$

$$= E(X) + E(Y)$$

$$\boxed{E(X+Y) = E(X) + E(Y)}$$

③ If X and Y are independent then

prove that $E(XY) = E(X)\,E(Y)$

proof (X, Y) be a CBRV

$$E(XY) = \int_{x=-\infty}^{\infty} \int_{y=-\infty}^{\infty} xy\, f(x,y)\,dx\,dy$$

X & Y are independent

$$f(x,y) = f_1(x)\, f_2(y)$$

$$= \int_{x=-\infty}^{\infty} \int_{y=-\infty}^{\infty} xy\, f_1(x)\, f_2(y)\,dx\,dy$$

$$= \int_{x=-\infty}^{\infty} x\, f_1(x)\,dx \int_{y=-\infty}^{\infty} y\, f_2(y)\,dy$$

$$\boxed{E(XY) = E(X)\,E(Y)}$$

Problems model wise ① Discrete - 2 P

② Continuous - 1 P

③ Discrete Bivariate - 2 P

④ Continuous Bivariate - 1P

⑤ Correlation coeff - 2 P

— 8 Problems

Theorems - Mean properties - 4

Variance properties - 4

Covariance - 1

$E(X+Y)$, $E(XY)$ - 2

— 11 Theorems

Definitions - R.V

D.R.V - P.M.F $P(x)$

C.R.V - p.d.f $f(x)$

C.D.F $F(x)$

Bivariate R.V → D.B.R.V, → C.B.R.V

D.B.R.V	C.B.R.V
JPMF	JPdf
MPMF	MPdf
CPMF	CPdf
X & Y indep	X & Y indepen

Covariance, Correlation coeff

Problems on Discrete P.D

Continuous P.D

Discrete Bivariate R.V

Continuous Bivariate R.V

Discrete P.D

① For the Discrete P.D

X	1	2	3	4	5	6	7	8
$P(x)$	K	$2K$	$3K$	$4K$	$5K$	$6K$	$7K$	$8K$

Then Find ① K ② μ ③ σ^2 ④ σ ⑤ $F(x)$

⑥ $P(X \le 2)$ ⑦ $P(X \ge 7)$ ⑧ $P(1 \le X \le 7)$

⑨ $P(2 \le X \le 5)$

Continuous P.D

② The probability density function of a Random variable is given by

$$f(x) = \begin{cases} \frac{1}{2} \sin x & ; 0 \le x \le \pi \\ 0 & : \text{otherwise} \end{cases}$$

Then find

② μ ③ σ^2 ④ σ ⑤ Median ⑥ Mode

⑦ $F(x)$ ⑧ $P(0 \le X \le \frac{\pi}{2})$ ⑨ $P(X \ge \frac{\pi}{3})$

Discrete Bivariate R.V

③ A fair coin tossed 3 times, let X be denote a R.V that takes a value '0' if the first toss is Head and value '1' if the first toss is tail. and Y be a R.V that takes the no. of Tails Then find ① JPMF ② MPMF ③ CPMF

④ X & Y independent or not

④ The JPMF of (X, Y) is given by

$$P(x,y) = \begin{cases} K(x+3y) & \text{for } x = 1,2 \\ & \quad\;\; y = 1,2 \\ 0 & \text{otherwise} \end{cases}$$

Then find

① K ② MPMF ③ CPMF

④ X & Y are independent or not

Continuous Bivariate R.V

* * * (5) The JPdf of (X, Y) is given by

$$f(x,y) = \begin{cases} 2-x-y & \text{for } 0<x<2 \\ & \quad\;\; 0<y<1 \\ 0 & \text{otherwise} \end{cases}$$

Determine

(1) MPMF

(2) CPMF

(3) X & Y independent (or) not

(6) correlation coefficient (CBRV)

Example. Let (x, y) be a random vector with a p.d.f of

$$f(x,y) = \frac{1}{8}(6 - x - y); \quad 0 \le x \le 2; \quad 2 \le y \le 4.$$

find correlation coefficient of (X, Y)

(or) $\rho(X, Y)$

7 correlation coefficient (DBRV)

Suppose that X and Y have the following joint probability mass function:

	$f(x,y)$	1	2	3	$f_X(x)$
X	1	0.25	0.25	0	0.5
	2	0	0.25	0.25	0.5
	$f_Y(y)$	0.25	0.5	0.25	1

sothat $\mu_X = 3/2$, $\mu_Y = 2$, $\sigma_X = 1/2$, and $\sigma_Y = \sqrt{1/2}$

What is the correlation coefficient of X and Y?