Assignment Problem:-

It may be defined as "Given n facilities and n jobs and given the effectiveness of each facility for each job, the problem is to assign each facility to one and only one job so as to optimize the given measure of effectiveness."

Jobs

		1	2	......	n	a_i Supply
Facilities	1	C_{11}	C_{12}	......	C_{1n}	1
	2	C_{21}	C_{22}	......	C_{2n}	1
	⋮	⋮	⋮	⋮	⋮	⋮
	n	C_{n1}	C_{n2}	......	C_{nn}	1
Demand b_j		1	1	......	1	

Here, C_{ij} - Cost of assigning i^{th} facility to j^{th} job

x_{ij} - Assignment of i^{th} facility to j^{th} job

$x_{ij} = 1$, if i^{th} facility is assigned to j^{th} job

$= 0$, otherwise

Objective: To make assignments that minimize the total assignment cost or maximize the total associated gain.

<u>Hungarian method for solution of Assignment problems:-</u>

- Suggested by Hungary
- Also called Reduced Matrix Method (or) Flood's technique.

The method consists of the following steps:

<u>Step-1</u>:- To check whether the given problem is of <u>std. type</u> or not.

Minimisation type is considered to be the std. type of problem. Cost, time, idle time & wastage are to be minimised.

If the problem is of maximisation type, then convert it into equivalent minimisation matrix. This can be done by following any one of these methods:

(i) Multiply all the elements of matrix with '-'.

(ii) Identify highest element in matrix & subtract all the other elements from it.

Profit, Sales volume, Sales revenue, Productivity etc, are to be maximised.

Step-2:- To check whether the given matrix is a square matrix or not.

A matrix is said to be square matrix when no. of rows are equal to no. of columns.

If the given matrix is a non-square matrix i.e., no. of rows ≠ no. of columns, convert this non-square matrix into square matrix by adding a dummy row or a dummy column with zero elements.

Step-3:- Matrix reduction

The objective is to get atleast one zero in each row and in each column. This can be achieved by the following substeps.

(a) Row reduction:

The objective here is to get atleast one zero in each row. This can be achieved by identifying the smallest element in the row & subtract it from all the other elements.

(b) Column reduction:

The objective here is to get atleast one zero in each column. This can be achieved by identifying the smallest element in the column &

subtract it from all the other elements.

Step-4:- To check whether an optimal assignment can be ~~done~~ made in the reduced matrix or not.

For assignment, the following notations are used:

$\boxed{0}$ - Assigned zero; $\boxed{\otimes}$ - Unassigned zero

Procedure: 1. Identify the rows which has only one zero in it.

2. Assign that zero. It will lead to cancellation of zeroes in that row & in that column.

3. Continue the procedure until no further allocation is possible.

4. Identify the columns which has only one zero in it.

5. Assign that zero. It will lead to cancellation of zeroes in that row & in that column.

6. Continue the procedure until no further allocation is possible.

In case, there is no row or column containing single unmarked zero, assign any unmarked zero arbitrarily.

Repeat above steps 1 to 6 till one of the following two things occur:

(i) There is one assignment in each row & each column. Then, the current feasible solution is an optimal solution. ~~The min~~

(ii) There is some row/column without assignment. The optimal assignment cannot be made.

Step-5:- To find the min. no. of lines crossing all zeroes

This consists of the following substeps:

Sub step-1: Mark the row which is unassigned.

Sub step-2: Now, mark the column in which the unassigned zero exists.

Sub step-3: Now, mark the row of assigned zero in the above column.

Substep-4: Repeat the procedure until no further marking is possible.

Substep-5: Now, draw lines passing through unmarked rows and marked columns. This gives min. no. of lines crossing all zeroes. If this number is equal to order of matrix, then it is an optimal solution otherwise go to step 6.

Step-6:- Iterate towards the optimal solution. Examine the uncovered elements. Select the smallest element and subtract it from all the uncovered elements. Also, add this smallest element to every element that lies at intersection of two lines. Leave the remaining elements as such. This yields 2nd BFS.

Step-7:- Repeat steps 4 to 6 until no. of lines crossing all zeroes become equal to order of matrix. This indicates that an optimum solution has been obtained. Total cost associated with this solution is obtained by adding original costs in the assigned cells.

Problems

(1) Four different jobs can be done on four different machines. The setup & take down time costs are assumed to be very high for changeovers. The matrix below gives the cost in rupees of producing job "i" on machine "j".

Machines

Jobs	M_1	M_2	M_3	M_4
J_1	5	7	11	6
J_2	8	5	9	6
J_3	4	7	10	7
J_4	10	4	8	3

How should the jobs be assigned to various machines so that total cost is minimized?

Sol:- Step-1:- To check whether the given problem is of std. type or not.

The cost is to be minimised.

∵ Minimisation type is std. type, the given problem is std. type.

Step-2:- To check whether the given problem is a square matrix or not.

No. of rows = 4

No. of columns = 4

∵ No. of rows = No. of columns, the given problem is a square matrix.

Step-3:- Matrix Reduction.

(a) Row reduction.

Objective is to get atleast one zero in each row.

Identify smallest element in row & subtract it from all the other elements.

	M_1	M_2	M_3	M_4
J_1	0	2	6	1
J_2	3	0	4	1
J_3	0	3	6	3
J_4	7	1	5	0

(b) Column reduction:

Objective is to get one zero in each column.
Identify smallest element in each column & subtract it from all other elements in that column.

	M_1	M_2	M_3	M_4
J_1	0	2	2	1
J_2	3	0	0	1
J_3	0	3	2	3
J_4	7	1	1	0

<u>Step-4</u>:- Assignment or allocation.

	M_1	M_2	M_3	M_4
J_1	$\boxed{0}$	2	2	1
J_2	3	$\boxed{0}$	$\cancel{0}$	1
J_3	$\cancel{0}$	3	2	3
J_4	7	1	1	$\boxed{0}$

First feasible solution

No of Rows = No of Allocations

<u>Step-5</u>:- To find min. no. of lines passing through all zeros. = No of Rows

	M_1	M_2	M_3	M_4
J_1	$\boxed{0}$	2	2	1
J_2	3	$\boxed{0}$	$\cancel{0}$	1
J_3	$\cancel{0}$	3	2	3
J_4	7	1	1	$\boxed{0}$

	M_1	M_2	M_3	M_4	
J_1	$\boxed{0}$	2	2	1	✓
J_2	3	$\boxed{0}$	$\cancel{0}$	1	
J_3	$\cancel{0}$	3	2	3	✓
J_4	7	1	1	$\boxed{0}$	

~~(?)~~ Crossing lines
~~?~~ passes 'Un- ~~?~~ Marked rows, Marked columns

No. of min. lines passing through all zero = 3
$\neq$
No. of rows = 4.

∴ The 1^{st} feasible solution is not optimal.

Step-6 :- Iterate towards optimality

	M_1	M_2	M_3	M_4
J_1	$\cancel{0}$	1	1	$\boxed{0}$
J_2	4	$\boxed{0}$	$\cancel{0}$	1
J_3	$\boxed{0}$	2	1	2
J_4	8	1	1	$\cancel{0}$

(URMC)

Second feasible solution

No. of min. lines passing through all zeroes = 3
No. of rows = 4

∴ The 2^{nd} feasible solution is not optimal.

	M_1	M_2	M_3	M_4
J_1	$\boxed{0}$	$\cancel{0}$	$\cancel{0}$	$\cancel{0}$
J_2	5	$\boxed{0}$	$\cancel{0}$	2
J_3	$\cancel{0}$	1	$\boxed{0}$	2
J_4	8	$\cancel{0}$	$\cancel{0}$	$\boxed{0}$

Third feasible solution.

The optimum assignment is:

Job	M/C.	Cost.(Rs).
J_1	M_1	5
J_2	M_2	5
J_3	M_3	10
J_4	M_4	3

Total cost = ₹.23

② Find the optimum assignment schedule so as to minimise the man hours.

HW

Subordinates

Tasks	I	II	III	IV
A	8	26	17	11
B	13	28	4	26
C	38	19	18	15
D	19	26	24	10

Sol:- Step-1:- To check whether the given problem is std. type or not.

The man hours are to be minimised.

So, the given problem is std. type.

Step-2:- To check whether the given problem is square matrix or not.

No. of rows = No. of columns = 4. - Square Matrix.

Step-3:- Matrix reduction:

(a) Row reduction:

	I	II	III	IV
A	0	18	9	3
B	9	24	0	22
C	23	4	3	0
D	9	16	14	0

(b) Column reduction

	I	II	III	IV
A	0	14	9	3
B	9	20	0	22
C	23	0	3	0
D	9	12	14	0

Step-4:- Assignment

	I	II	III	IV
A	$\boxed{0}$	14	9	3
B	9	20	$\boxed{0}$	22
C	23	$\boxed{0}$	3	$\cancel{0}$
D	9	12	14	$\boxed{0}$

Man hour

The assignment:

Task	Subordinate	Manhours
A	I	8
B	III	4
C	II	19
D	IV	10

Total manhours = 41

H.W

③ A company has 1 surplus truck in each of the cities A, B, C, D & E and one deficit truck in each of the cities 1, 2, 3, 4, 5 & 6. The distance b/n cities in kms is shown in the matrix below. Find the assignment of trucks from cities in surplus to cities in deficit so that total distance covered by the vehicles is minimum.

Soln

	1	2	3	4	5	6
A	12	10	15	22	18	8
B	10	18	25	15	16	12
C	11	10	3	8	5	9
D	6	14	10	13	13	12
E	8	12	11	7	13	10

Sol:- Step-1:- To check whether the given problem is std. type or not.

Total distance covered by vehicles is to be minimised.

So, the given problem is std. type.

Step-2:- To check whether the given problem is square matrix or not. –

No. of rows = 5

No. of columns = 6. – Non-square matrix.

Convert this into a square matrix by adding a dummy row with zero elements.

	1	2	3	4	5	6
A	12	10	15	22	18	8
B	10	18	25	15	16	12
C	11	10	3	8	5	9
D	6	14	10	13	13	12
E	8	12	11	7	13	10
F	0	0	0	0	0	0

Step-3:- Matrix reduction

(a) Row reduction:

	I	II	III	IV	V	VI
A	4	2	7	14	10	0
B	0	8	15	5	6	2
C	8	7	0	5	2	6
D	0	8	4	7	7	6
E	1	5	4	0	6	3
F	0	0	0	0	0	0

(b) Column reduction not required.

Step-4:- Assignment

	1	2	3	4	5	6
A	4	2	7	14	10	$\boxed{0}$
B	$\boxed{0}$	8	15	5	6	2
C	8	7	$\boxed{0}$	5	2	6
D	$\cancel{0}$	8	4	7	7	6
E	1	5	4	$\boxed{0}$	6	3
F	$\cancel{0}$	$\boxed{0}$	$\cancel{0}$	$\cancel{0}$	$\cancel{0}$	$\cancel{0}$

1st feasible solution.

Step-5:- Assignment

	1	2	3	4	5	6
A	6	2	7	14	10	$\boxed{0}$
B	$\boxed{0}$	6	13	3	4	$\cancel{0}$
C	10	7	$\boxed{0}$	5	2	6
D	$\cancel{0}$	6	2	5	5	4
E	3	5	4	$\boxed{0}$	6	3
F	2	$\boxed{0}$	$\cancel{0}$	$\cancel{0}$	$\cancel{0}$	$\cancel{0}$

2nd FS

	1	2	3	4	5	6
A	6	$\boxed{0}$	5	12	8	$\cancel{0}$
B	$\cancel{0}$	4	11	1	2	$\boxed{0}$
C	12	7	$\boxed{0}$	5	2	8
D	$\boxed{0}$	4	$\cancel{0}$	3	3	4
E	5	5	4	$\boxed{0}$	6	5
F	4	$\cancel{0}$	$\cancel{0}$	$\cancel{0}$	$\boxed{0}$	2

3rd FS

The Assignment.

City with surplus	City with deficit	Distance covered.
A	2	10
B	6	12
C	3	3
D	1	6
E	4	7

Total distance covered = 38 Kms

(4) A company has a team of 4 salesmen & there are 4 districts where the company wants to start it's business. After taking into account the capabilities of salesmen & the nature of districts, the company estimates that the profit per day in ₹. for each salesman in each district is as given below:

District

Salesman	1	2	3	4
A	16	10	14	11
B	14	11	15	15
C	15	15	13	12
D	13	12	14	15

Find the assignment of salesman to various districts which will yield max. profit.

Sol:- Step-1:- To check whether the given problem is std. type or not.

Here, profit is to be maximised. So, it is not std. type

Convert it into std. type by subtracting ~~smallest~~ all elements in entire matrix from the highest element.

District

Salesman	1	2	3	4
A	0	6	2	5
B	2	5	1	1
C	1	1	3	4
D	3	4	2	1

Step-2:- To check whether the given matrix is a square matrix or not.

No. of rows = 4 = No. of columns - Square matrix.

Step-3:- Matrix reduction.

(a) Row reduction:

0	6	2	5
1	4	0	0
0	0	2	3
2	3	1	0

(b) Column reduction not required:

Step-4:- Assignment

$\boxed{0}$	6	2	5
1	4	$\boxed{0}$	~~0~~
~~0~~	$\boxed{0}$	2	3
2	3	1	$\boxed{0}$

The optimum assignment schedule:

Salesman	District	Profit
A	1	16
B	3	15
C	2	15
D	4	15

Total profit = ₹. 61

(5) 4 new machines M_1, M_2, M_3, M_4 are to be installed in a m/c shop. There are 5 vacant places available A, B, C, D & E. Because of limited space, machine M_2 can't be placed at C & M_3 can't be placed at A. C_{ij}, the assignment cost of machine "i" to place "j" in ₹ is shown below. Find the optimum assignment schedule.

Vacant place

	A	B	C	D	E
M_1	4	6	10	5	6
M_2	7	4	-	5	4
Machine M_3	-	6	9	6	2
M_4	9	3	7	2	3

Sol:- Step-1:- To check whether the given problem is std. type or not.

Cost is to be minimised.

So, the given problem is std. type.

Step-2:- To check whether the given matrix is square matrix or not.

No. of rows = 4

No. of columns = 5 - Non-square matrix.

Convert it into square matrix by adding a dummy row with zero elements.

	A	B	C	D	E
M_1	4	6	10	5	6
M_2	7	4	∞	5	4
M_3	∞	6	9	6	2
M_4	9	3	7	2	3
M_5	0	0	0	0	0

Step-3:- Matrix reduction.

(a) Row reduction

	A	B	C	D	E
M_1	0	2	6	1	2
M_2	3	0	∞	1	0
M_3	∞	4	7	4	0
M_4	7	1	5	0	1
M_5	0	0	0	0	0

(b) Column reduction not required.

Step 4:- Assignment.

	A	B	C	D	E
M_1	$\boxed{0}$	2	6	1	2
M_2	3	$\boxed{0}$	∞	1	$\cancel{0}$
M_3	∞	4	7	4	$\boxed{0}$
M_4	7	1	5	$\boxed{0}$	1
M_5	$\cancel{0}$	$\cancel{0}$	$\boxed{0}$	$\cancel{0}$	$\cancel{0}$

The optimum allocation:

Machine	Vacant Place	Cost
M_1	A	4
M_2	B	4
M_3	E	2
M_4	D	2

Total cost = ₹. 12