03-03-2023

1. OSCILLATIONS

* Periodic Motion :-

→ A motion which repeats itself after equal intervals of time is called "Periodic motion" or "Harmonic motion."

Ex :- Spin of earth, Motion of a satellite around a planet, vibration of atoms in molecules etc.

* Oscillatory Motion :-

A Body or a particle is said to be possess 'oscillatory' or 'vibratory motion' if it moves 'back' and 'forth' repeatedly about the mean position.

Ex :- Pendulum of clock, prongs of a tuning fork, Motion of piston of an engine etc.

(i) Periodic time :- Time taken for one oscillation.

(ii) Frequency :- No. of oscillations in one second. '$n = \frac{1}{T}$'

(iii) Displacement :-

→ The Distance of the particle in any direction from the equilibrium position at any instant is called 'Displacement' of the particle at that instant.

(iv) Amplitude :-

→ The maximum displacement or distance b/w the equilibrium position & the extreme position is known as 'Amplitude' 'a' of the oscillation.

(v) Phase :-

→ The phase of an oscillatory particle at any instant defines "the state of the particle as regards its position and direction of motion at that instant."

(vi) Restoring force :-

→ In the equilibrium position of oscillating particle, no net force acts on it. When the particle is displaced from its equilibrium position, a periodic force acts on it such a direction as to bring the particle to its equilibrium position. This is called Restoring Force.

* Simple Harmonic Motion:-

→ It is defined as "the motion of an oscillatory particle which is acted upon by a restoring force which is directly proportional to displacement but opposite to it in direction."

→ The following are characteristics of S.H.M :-

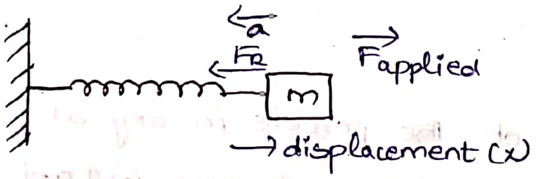
(a) "The motion is periodic."

(b) "The motion is along a straight line about the mean or equilibrium position."

(c) "The acceleration is proportional to displacement".

(d) "Acceleration is directed towards the mean or equilibrium position".

* Simple Harmonic oscillator :-



$a \propto -x$

$F \propto -x$

$F = -kx \longrightarrow (1)$

$F = ma \longrightarrow (2)$

From (1) & (2),

$\Rightarrow ma = -kx$

$\Rightarrow m\dfrac{d^2x}{dt^2} = -kx$

$\Rightarrow \dfrac{d^2x}{dt^2} + \left(\dfrac{k}{m}\right)x = 0$

Let $\dfrac{k}{m} = \omega^2$

$\Rightarrow \boxed{\dfrac{d^2x}{dt^2} + \omega^2 x = 0} \longrightarrow (3)$

General solution,

$x = c \cdot e^{\alpha t} \longrightarrow (4)$

$$\left.\begin{aligned} &\Rightarrow \frac{dx}{dt} = c\alpha \cdot e^{\alpha t} \\ &\Rightarrow \frac{d^2x}{dt^2} = c\alpha^2 e^{\alpha t} \end{aligned}\right\} \longrightarrow (5)$$

Substitute (4), (5) in eqn (3)

$\Rightarrow c\alpha^2 e^{\alpha t} + \omega^2(c \cdot e^{\alpha t}) = 0$

$\Rightarrow c \cdot e^{\alpha t}(\alpha^2 + \omega^2) = 0.$

Here, $ce^{\alpha t} \neq 0$, $\alpha^2 + \omega^2 = 0.$

constant

$\Rightarrow \alpha^2 = -\omega^2$

$\Rightarrow \alpha = \pm\sqrt{-\omega^2}$

$\Rightarrow \alpha = \pm i\omega$

$x = ce^{\alpha t}$

$\Rightarrow x = ce^{\pm i\omega t}$

$\Rightarrow x = c_1 e^{i\omega t} + c_2 e^{-i\omega t}$

$\Rightarrow x = c_1[\cos\omega t + i\sin\omega t] + c_2[\cos\omega t - i\sin\omega t]$.

$\Rightarrow x = \cos\omega t\,(c_1+c_2) + i\sin\omega t\,(c_1-c_2)$.

Complex analysis

$e^{i\theta} = \cos\theta + i\sin\theta$

$e^{-i\theta} = \cos\theta - i\sin\theta$.

Let $c_1+c_2 = A\sin\theta$, $\quad i(c_1-c_2) = A\cos\theta$.

$\Rightarrow x = A\cos\omega t\sin\theta + i\sin\omega t\cos\theta$.

$\therefore \boxed{x = a\sin(\omega t + \phi)}$.

This is the solution of equation of S.H.O.

* Characteristics of S.H.O :-

① Displacement :- (x)

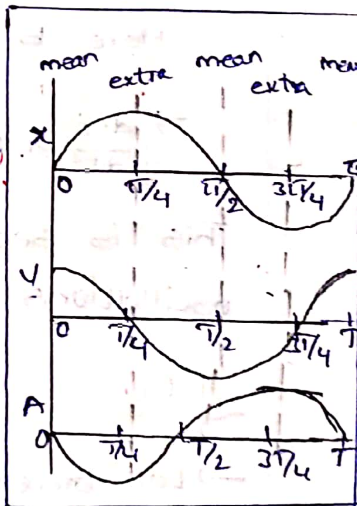
$x = A\sin(\omega t + \theta)$.

② Velocity :- (v)

$v = \frac{dx}{dt} = A\omega\cos(\omega t+\theta) = A\omega\sqrt{1-\sin^2(\omega t+\theta)} = \omega\sqrt{A^2-x^2}$

$v_{max} = \omega A$ [x is 0] at mean or equilibrium position.

$v_{min} = 0$ [x = A].



③ Acceleration :- (A)

$A = \frac{dv}{dt} = -A\omega^2\sin(\omega t+\theta) = -\omega^2 x$.

④ Periodicity :- (T)

$x = A\sin[\omega(t + \frac{2\pi}{\omega}) + \theta]$

$x = A\sin[\omega t + 2\pi + \theta]$

$x = A\sin[\omega t + \theta]$.

$\therefore T = \frac{2\pi}{\omega} \Rightarrow T = 2\pi\sqrt{\frac{x}{a}}$ (or) $T = 2\pi\sqrt{\frac{m}{k}}$.

⑤ Frequency :- $F = \frac{1}{T}$.

⑥ Epoch :- Putting $\omega = 2\pi/T$ in x, v, A, & $\phi = 0$ for simplicity.

T	0	$\frac{T}{4}$	$\frac{T}{2}$	$3T/4$	T
x	0	a	0	$-a$	0
dx/dt	ωa	0	$-\omega a$	0	ωa
d^2x/dt^2	0	$-\omega^2 a$	0	$\omega^2 a$	0

* Damped oscillations :-

$\rightarrow$ $F_R \propto -x$

$F_f \propto -v$

$\Rightarrow F_R = -\mu x$

$F_f = -rv = -r\frac{dx}{dt}$

Now, $F = ma$

Here, $F = -\mu x - r\frac{dx}{dt}$

$a = \frac{d^2x}{dt^2}$

$\Rightarrow m\left(\frac{d^2x}{dt^2}\right) = -\mu x - r\frac{dx}{dt}$

$\Rightarrow \frac{d^2x}{dt^2} = -\frac{\mu}{m}x - \frac{r}{m}\frac{dx}{dt}$

$\Rightarrow \boxed{\frac{d^2x}{dt^2} + \left(\frac{r}{m}\right)\frac{dx}{dt} + \left(\frac{\mu}{m}\right)x = 0} \longrightarrow (1)$

Let $\frac{r}{m} = 2b$, $\frac{\mu}{m} = \omega^2$.

Here 'b' is Damping coefficient.

$\Rightarrow \boxed{\frac{d^2x}{dt^2} + 2b\frac{dx}{dt} + \omega^2 x = 0} \longrightarrow (2)$

This is second order differential eq^n for Damped oscillations.

Solution :-

$\rightarrow$ Let General solution is $x = A \cdot e^{\alpha t}$. $\longrightarrow (3)$

$\Rightarrow \frac{dx}{dt} = A \cdot \alpha e^{\alpha t}$.

$\Rightarrow \frac{d^2x}{dt^2} = A\alpha^2 e^{\alpha t}$.

Substitute in eq^n (2)

$\Rightarrow A \cdot \alpha^2 e^{\alpha t} + 2b\, A\alpha e^{\alpha t} + \omega^2 A \cdot e^{\alpha t} = 0$.

$\Rightarrow A e^{\alpha t}[\alpha^2 + 2b\alpha + \omega^2] = 0$

Here, $A e^{\alpha t} \neq 0$ [$\because$ Arbitary constant :- A].

i.e, $\alpha^2 + 2b\alpha + \omega^2 = 0$.

$$\alpha = \frac{-2b \pm \sqrt{4b^2 - 4\omega^2}}{2}$$

$$\alpha = \frac{\cancel{2}\left[-b \pm \sqrt{b^2-\omega^2}\right]}{\cancel{2}}$$

$$\alpha = -b \pm \sqrt{b^2-\omega^2}$$

Substitute α value in eq^n 3.

$$\Rightarrow \boxed{x = A_1 \cdot e^{(-b+\sqrt{b^2-\omega^2})t} + A_2 \cdot e^{(-b-\sqrt{b^2-\omega^2})t}}$$

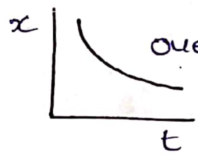
$\therefore$ This is the "solution for Damped oscillations."

Case-1 :- 'over damped' ($b^2 > \omega^2$)

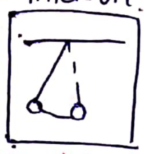
$b^2 - \omega^2 = +ve$

$b > \sqrt{b^2-\omega^2}$

$A_1 e^{-ve} + A_2 \cdot e^{-ve(more)}$



Thick oil



Dead beat

Case-2 :- 'critical damped' ($b^2 = \omega^2$).

$b^2 - \omega^2 = 0$, $b^2 - \omega^2 = h \to 0$.

Now, $x = A_1 e^{(-b+h)t} + A_2 e^{(-b-h)t}$

$\Rightarrow x = e^{-bt}[A_1 \cdot e^{ht} + A_2 \cdot e^{-ht}]$.

$\Rightarrow x = e^{-bt}[A_1(1+ht) + A_2(1-ht)]$.

$\Rightarrow x = e^{-bt}[(A_1+A_2) + ht(A_1-A_2)]$.

$\Rightarrow x = e^{-bt}[(A_1+A_2) + h(A_1-A_2)t]$.

Let $A_1 + A_2 = P$, $h(A_1-A_2) = q$.

$\Rightarrow \boxed{x = e^{-bt}(p+qt)}$ if $t\uparrow \Rightarrow e^{-bt}\downarrow \Rightarrow x\downarrow$

Ex:- Pointers [Analog voltmeter, ammeter, Galvanometer].

Case-3 : Under damped $(b^2 < \omega^2)$ .. $\beta = \sqrt{\omega^2 - b^2}$

$$b^2 - \omega^2 = -ve$$

Let $\sqrt{b^2 - \omega^2} = i\beta$, substitute in solution.

$$\Rightarrow x = A_1 \cdot e^{(-b+i\beta)t} + A_2 \cdot e^{(-b-i\beta)t}$$

$$\Rightarrow x = e^{-bt}[A_1 e^{i\beta t} + A_2 \cdot e^{-i\beta t}].$$

$$\Rightarrow x = e^{-bt}[A_1(\cos\beta t + i\sin\beta t) + A_2(\cos\beta t - i\sin\beta t)].$$

$$\Rightarrow x = e^{-bt}[\cos\beta t\,(A_1 + A_2) + i\sin\beta t\,(A_1 - A_2)].$$

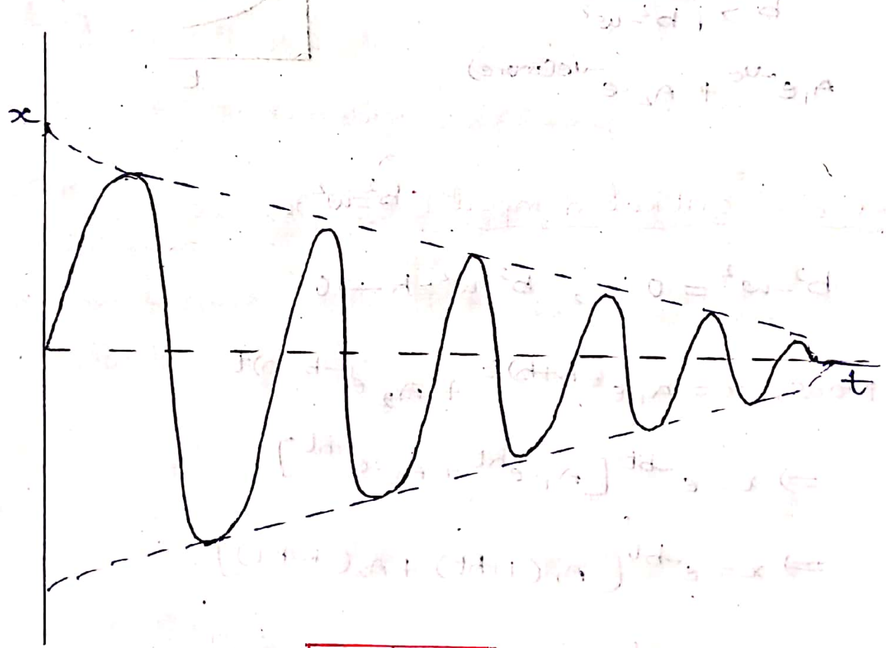
Let $A_1 + A_2 = a\sin\theta$ and

$i(A_1 - A_2) = a\cos\theta$

$$\Rightarrow x = e^{-bt}[\cos\beta t\,(a\sin\theta) + \sin\beta t\,(a\cos\theta)].$$

$$\Rightarrow x = ae^{-bt}[\sin(\beta t + \theta)].$$

$$\Rightarrow \boxed{x = ae^{-bt}\sin(\beta t + \theta)}.$$



Here, Amplitude $\boxed{A = ae^{-bt}}$.

Time period, $T = \frac{2\pi}{\beta}$

$$\therefore \boxed{T = \frac{2\pi}{\sqrt{\omega^2 - b^2}}}$$

* Forced oscillations :-

→ We know that

$$F_R = -\mu x$$

$$F_F = -r\frac{dx}{dt}$$

'p': Applied force frequency
(or)
Driving force Frequency.

'ω': object frequency.
[Internal]

External force = 'F sinpt'

Now, 'F = ma'

$$\Rightarrow m\left(\frac{d^2x}{dt^2}\right) = -\mu x - r\frac{dx}{dt} + F\sin pt$$

$$\Rightarrow \frac{d^2x}{dt^2} + \left(\frac{r}{m}\right)\frac{dx}{dt} + \left(\frac{\mu}{m}\right)x = \left(\frac{F}{m}\right)\sin pt \longrightarrow (A)$$

Let '$\frac{r}{m} = 2b$', '$\frac{\mu}{m} = \omega^2$', '$\frac{F}{m} = f$'

$$\Rightarrow \boxed{\frac{d^2x}{dt^2} + 2b\frac{dx}{dt} + \omega^2 x = f\sin pt} \longrightarrow (1)$$

This is the second order diff eqⁿ for Forced oscillations.

General solution :- $x = A\sin(pt-\theta)$ ⟶ (2)

Here, A & θ are arbitrary constants.

Now,
$$\left.\begin{aligned}\frac{dx}{dt} &= Ap\cos(pt-\theta)\\ \frac{d^2x}{dt^2} &= -Ap^2\sin(pt-\theta)\end{aligned}\right\} \longrightarrow (3)$$

Substitute eqⁿs (2) & (3) in (1)

$$\Rightarrow -Ap^2\sin(pt-\theta) + 2b\,Ap\cos(pt-\theta) + \omega^2 A\sin(pt-\theta) = f\sin(pt-\theta+\theta)$$

$$\Rightarrow -Ap^2\sin(pt-\theta) + 2b\,Ap\cos(pt-\theta) + \omega^2 A\sin(pt-\theta) = f\sin(pt-\theta)\cos\theta + f\cos(pt-\theta)\sin\theta.$$

$$\Rightarrow A(\omega^2-p^2)\sin(pt-\theta) + 2bAp\cos(pt-\theta) = f\sin(pt-\theta)\cos\theta + f\cos(pt-\theta)\sin\theta$$

⟶

compare $\sin(pt-\theta)$ & $\cos(pt-\theta)$ terms on both sides.

$$A(\omega^2-p^2) = f\cos\theta \longrightarrow (4)$$

$$2bAp = f\sin\theta. \longrightarrow (5)$$

Now $(4)^2 + (5)^2$

$$\Rightarrow A^2(\omega^2-p^2)^2 + 4b^2A^2p^2 = f^2$$

$$\Rightarrow A^2 = \frac{f^2}{(\omega^2-p^2)^2+4b^2p^2}.$$

∴ Amplitude of the Forced oscillations is

$$\boxed{A = \frac{f}{\sqrt{(\omega^2-p^2)^2+4b^2p^2}}} \longrightarrow (6)$$

Now, eq^n $\frac{(5)}{(4)}$.

$$\Rightarrow \frac{f\sin\theta}{f\cos\theta} = \frac{2bAp}{A(\omega^2-p^2)}$$

$$\Rightarrow \tan\theta = \frac{2bp}{(\omega^2-p^2)} \quad \therefore \boxed{\theta = \tan^{-1}\frac{2bp}{(\omega^2-p^2)}} \longrightarrow (7)$$

Here, 'θ' is the 'Phase' of the forced oscillations.

Now, substitute eq^n (6) in eq^n (2)

$$\therefore \boxed{x = \frac{f}{\sqrt{(\omega^2-p^2)^2+4b^2p^2}}\cdot\sin(pt-\theta)}$$

This is the 'solution' for Forced oscillations.

case-(i) :-

If '$p \ll \omega$'

$\boxed{A \simeq \frac{f}{\omega^2}}$, $\theta \simeq \tan^{-1}(0) = 0°$ $\boxed{\theta = 0°}$

case-(ii) :-

If '$p = \omega$' (Ideal case).

$\boxed{A \simeq \frac{f}{2bp}}$ $\theta = \tan^{-1}(\infty) = \frac{\pi}{2}$.

case-(iii) :-

If '$p \gg \omega$'

$\boxed{A \simeq \frac{f}{p^2}}$

$\theta \simeq \tan^{-1}(-0) = \pi$.

* Resonance :-

Amplitude Resonance :-

$$\rightarrow \quad A = \frac{f}{\sqrt{(\omega^2-p^2)^2 + 4b^2p^2}} \longrightarrow (1)$$

$$\theta = \tan^{-1}\left[\frac{2bp}{(\omega^2-p^2)}\right].$$

'Amax', when '$(\omega^2-p^2)^2 + 4b^2p^2$' is minimum.

$$\Rightarrow \frac{d}{dp}\left[(\omega^2-p^2)^2 + 4b^2p^2\right] = 0,$$

$$\Rightarrow 2(\omega^2-p^2)(-2p) + 8b^2p = 0.$$

$$\Rightarrow 4\cancel{p}(\omega^2-p^2) = 8b^2\cancel{p}$$

$$\Rightarrow \omega^2-p^2 = 2b^2$$

$$\Rightarrow p^2 = \omega^2-2b^2$$

$$\therefore \boxed{p = \sqrt{\omega^2-2b^2}} \longrightarrow (2)$$

It is with 'Damping'.

If There is 'No Damping', $p = \sqrt{\omega^2}$ $\therefore$ '$p=\omega$'

substitute (2) in (1)

$$\Rightarrow A_{max} = \frac{f}{\sqrt{[\omega^2-(\omega^2-2b^2)]^2 + 4b^2(\omega^2-2b^2)}}$$

$$\Rightarrow A_{max} = \frac{f}{\sqrt{4b^4+4b^2\omega^2-8b^4}}$$

$$\Rightarrow A_{max} = \frac{f}{\sqrt{4b^2\omega^2-4b^4}}$$

$$\Rightarrow A_{max} = \frac{f}{2b\sqrt{\omega^2-b^2}}.$$

$$\therefore \boxed{A_{max} = \frac{f}{2b\sqrt{\omega^2-b^2}}} \quad \text{(or)} \quad \boxed{A_{max} = \frac{f}{2b\sqrt{p^2+b^2}}}$$

By adding $(+b^2, -b^2)$

$\Rightarrow$ If No damping, '$A_{max} = \infty$' $[\because b=0]$

* Graph :-

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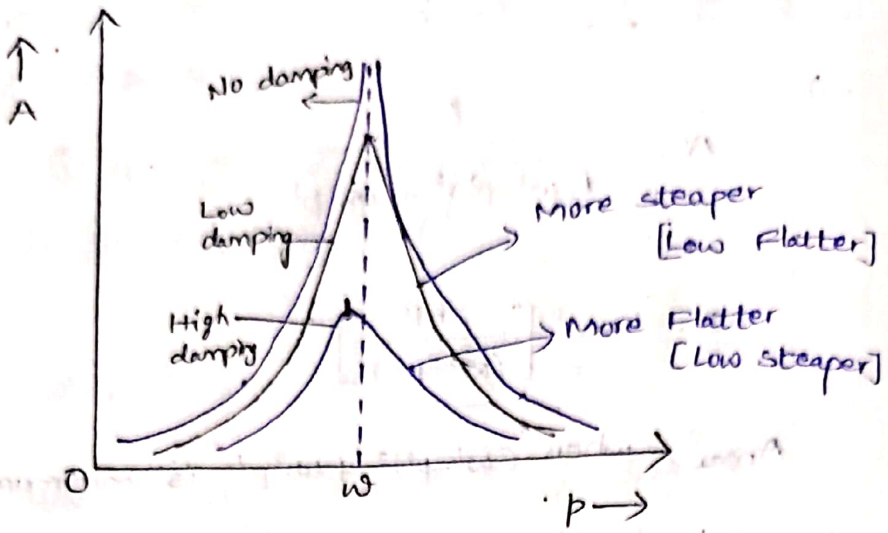

→ In High damping, peak is tries to shift origin. [Moves towards origin].

* Velocity Resonance :-

$$x = \frac{f}{\sqrt{(\omega^2-p^2)^2+4b^2p^2}} \sin(pt-\theta).$$

Velocity, $u = \frac{dx}{dt} = \underbrace{\frac{fp}{\sqrt{(\omega^2-p^2)^2+4b^2p^2}}}_{u_{max}} \cos(pt-\theta).$

$\Rightarrow u = u_{max} \sin(pt-\theta+\frac{\pi}{2}).$

$$\Rightarrow u_{max} = \frac{fp}{\sqrt{(\omega^2-p^2)^2+4b^2p^2}}$$

$(\omega^2-p^2)^2+4b^2p^2$ is minimum.

$\Rightarrow \boxed{p = \sqrt{\omega^2-2b^2}}$

u_{max} with damping.

→ If NO damping, $b=0$ $p=\omega$.

Case-i :- If $p \gg \omega$, $u_{max} \approx \frac{f}{p} \approx \frac{F}{mp}$.

Case-ii :- If $p \ll \omega$, $u_{max} \approx \frac{fp}{\omega^2} \approx \frac{Fpm}{m\mu} \approx \frac{Fp}{\mu}$

~~Case-iii~~

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* Quality Factor :- (Q)

$$\boxed{Q = 2\pi \times \frac{\text{Total Energy of the oscillation}}{\text{Energy lost per one cycle}}}$$

$$\Rightarrow Q = 2\pi \frac{E}{PT} \quad ; \quad P = \frac{E}{\tau}$$

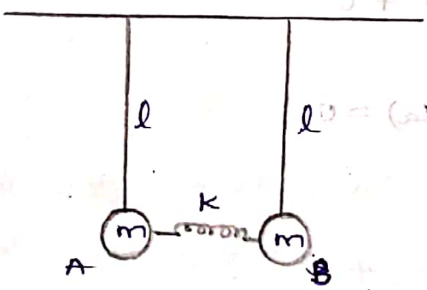
$$\Rightarrow Q = 2\pi \cdot \frac{\cancel{E}\tau}{\cancel{E}T}$$

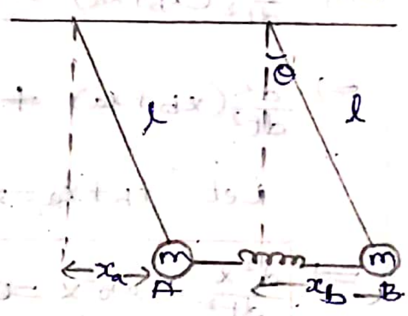
$$\Rightarrow Q = 2\pi \cdot \frac{\tau}{T} = \frac{2\pi}{T} \cdot \tau = \omega\tau .$$

$$\therefore \boxed{Q = \omega\tau}$$

* Coupled oscillations :-

$(x_b > x_a)$

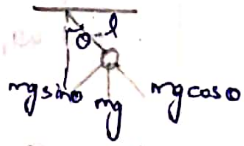




① Restoring / Returning force due to 'gravity' :-

$$-mg\sin\theta .$$

$$\Rightarrow -mg\left(\frac{x_b}{l}\right).$$



② Restoring force due to 'spring' :-

$$-K(x_b - x_a).$$

For Bob (B) :-

$$ma = F$$

$$\Rightarrow m\frac{d^2x_b}{dt^2} = -mg\left(\frac{x_b}{l}\right) - K(x_b - x_a).$$

For Bob (A) :-

$$\Rightarrow m\frac{d^2x_a}{dt^2} = -mg\left(\frac{x_a}{l}\right) + K(x_b - x_a).$$

For B :- $\frac{d^2x_b}{dt^2} = \left(\frac{-g}{l}\right)(x_b) - \left(\frac{k}{m}\right)(x_b - x_a)$.

For A :- $\frac{d^2x_a}{dt^2} = \left(\frac{-g}{l}\right) x_a + \left(\frac{k}{m}\right)(x_b - x_a)$.

We know $\omega_0 = \sqrt{\frac{g}{l}}$ $\Rightarrow \omega_0^2 = \frac{g}{l}$

substitute ω_0 in above equations.

$$\Rightarrow \frac{d^2x_b}{dt^2} = -\omega_0^2 x_b - \left(\frac{k}{m}\right)(x_b - x_a). \longrightarrow (1)$$

$$\Rightarrow \frac{d^2x_a}{dt^2} = -\omega_0^2 x_a + \left(\frac{k}{m}\right)(x_b - x_a). \longrightarrow (2)$$

Here eq^n (1) & (2) are not satisfying S.H.O
[$\because$ Two constants are in two terms].

Now, (1) + (2) :-

$$\Rightarrow \frac{d^2}{dt^2}(x_b + x_a) = -\omega_0^2 (x_b + x_a) + 0.$$

$$\Rightarrow \frac{d^2}{dt^2}(x_b + x_a) + \omega_0^2 (x_b + x_a) = 0.$$

Let $x_b + x_a = x$

$$\Rightarrow \boxed{\frac{d^2x}{dt^2} + \omega_0^2 x = 0.} \longrightarrow (3)$$

$\therefore$ eq^n (3) satisfying S.H.O.

$$\Rightarrow \omega_1 = \omega_0 = \sqrt{\frac{g}{l}}$$

Now, (1) − (2) :-

$$\Rightarrow \frac{d^2}{dt^2}(x_b - x_a) = -\omega_0^2 (x_b - x_a) - 2\left(\frac{k}{m}\right)(x_b - x_a).$$

$$\Rightarrow \frac{d^2}{dt^2}(x_b - x_a) + \left(\omega_0^2 + \frac{2k}{m}\right)(x_b - x_a) = 0.$$

Let $x_b - x_a = x'$

$$\Rightarrow \boxed{\frac{d^2x'}{dt^2} + \left(\omega_0^2 + \frac{2k}{m}\right) x' = 0.} \longrightarrow (4)$$

$\therefore$ This eq^n (4) satisfies S.H.O

$$\Rightarrow \omega_2 = \omega_0^2 + \frac{2k}{m} = \sqrt{\frac{g}{l} + \frac{2k}{m}}$$

case-(i) :-

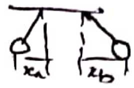
→ if $x_a = x_b$, eqⁿ ④ vanished [∵ $x_b - x_a = 0$]. i.e $x' = 0$.

only eqⁿ ③ present.

∴ $\boxed{\omega_1 = \omega_0 = \sqrt{\frac{g}{l}}}$, 'Normal mode' frequency.

→ It is also called 'Fundamental mode' (or) 'Inphase mode'. [No phase difference].

case-(ii) :- ($x_a = -x_b$).



→ Eqⁿ ③ vanished [∵ $x_b + x_a = x = 0$].

→ only Eqⁿ ④ present.

$$\boxed{\omega_2 = \sqrt{\omega_0^2 + \frac{2k}{m}} = \sqrt{\frac{g}{l} + \frac{2k}{m}}}$$

→ It is called as "second Normal mode frequency" (or) "outerphase mode".

General solution :-

$$X = x_b + x_a = C \cos\omega_1 t \longrightarrow ⑤$$

$$X' = x_b - x_a = D \cos\omega_2 t \longrightarrow ⑥$$

⑤ + ⑥

$\Rightarrow 2x_b = C\cos\omega_1 t + D\cos\omega_2 t$

$\Rightarrow x_b = \frac{C}{2}\cos\omega_1 t + \frac{D}{2}\cos\omega_2 t$

⑤ − ⑥

$\Rightarrow x_a = \frac{C}{2}\cos\omega_1 t - \frac{D}{2}\cos\omega_2 t$.

At Initial condition [t=0].

$\Rightarrow x_b = \frac{C}{2} + \frac{D}{2}$ and $x_a = \frac{C}{2} - \frac{D}{2}$.

$\Rightarrow x_b = \frac{C}{2} + \frac{D}{2} = A_0$. [say].

$\Rightarrow x_a = \frac{C}{2} - \frac{D}{2} = 0$. [∵ $C = D = A_0$].

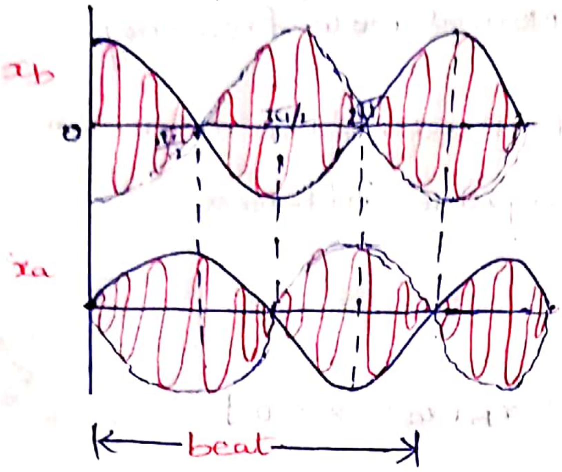
$\Rightarrow x_b = \frac{A_0}{2}[\cos\omega_1 t + \cos\omega_2 t]$

$\Rightarrow x_a = \frac{A_0}{2}[\cos\omega_1 t - \cos\omega_2 t]$.

$\Rightarrow x_b = A_0 \cos\left(\frac{\omega_2 - \omega_1}{2}\right)t \cdot \cos\left(\frac{\omega_2 + \omega_1}{2}\right)t \longrightarrow ⑦$

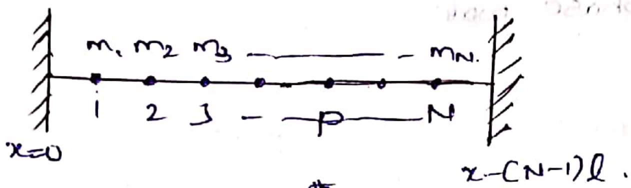
$$\Rightarrow x_a = A_0 \sin\left(\frac{\omega_2-\omega_1}{2}\right)t \cdot \sin\left(\frac{\omega_2+\omega_1}{2}\right)t \longrightarrow (8)$$

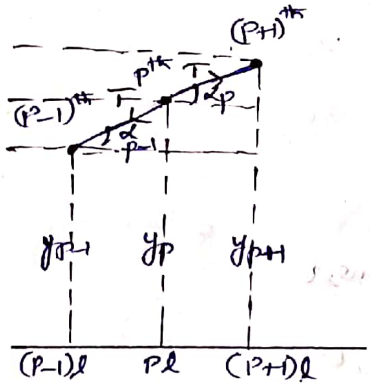
Here (7) eq^n & eq^n (8) are solutions.



* N- coupled oscillations :-

[small oscillations].





$$F_p = -T\sin\alpha_{p-1} + T\sin\alpha_p .$$

Here α is very small

$$\sin\alpha = \tan\alpha$$

$$\Rightarrow F_p = -T\tan\alpha_{p-1} + T\tan\alpha_p \text{ (keypoint)}$$

Here

$$\tan\alpha_{p-1} = \frac{y_p - y_{p-1}}{l} \quad \text{and} \quad \tan\alpha_p = \frac{y_{p+1} - y_p}{l}$$

$$\Rightarrow F_p = \frac{-T}{l}[y_p - y_{p-1}] + \frac{T}{l}[y_{p+1} - y_p]$$

$$\Rightarrow F_p = \frac{T}{l}[-y_p + y_{p-1} + y_{p+1} - y_p].$$

$\Rightarrow F_p - \frac{T}{l}[y_{p+1} + y_{p-1} \mp 2y_p]$

$\Rightarrow m\left[\frac{d^2y_p}{dt^2}\right] = \frac{T}{l}[y_{p+1} + y_{p-1} \mp 2y_p]$ $\left\{\because F = ma \Rightarrow F_p = m\left[\frac{d^2y_p}{dt^2}\right]\right\}$

$\Rightarrow \boxed{\frac{d^2y_p}{dt^2} = \frac{T}{ml}[y_{p+1} + y_{p-1} \mp 2y_p]} \longrightarrow (1)$

This is D.E for N.C.O.

Now, $x=0 \Rightarrow y_0 = 0$

$x = (N+1) \Rightarrow y_{N+1} = 0$.

General sol^n:

$y_p = A_p \cos(\omega t + \phi)$.

$\Rightarrow y_{p+1} = A_{p+1} \cos(\omega t + \phi)$.

$\Rightarrow y_{p-1} = A_{p-1} \cos(\omega t + \phi)$

$\Rightarrow y_p = A_p \cos(\omega t + \phi)$.

$\Rightarrow \frac{d^2y_p}{dt^2} = \frac{d}{dt}\left[\frac{dy_p}{dt}\right]$.

$\Rightarrow \frac{d^2y_p}{dt^2} = -A_p\omega^2\cos(\omega t + \phi)$.

(2)

Substitute (2) in (1).

$\Rightarrow -A_p\omega^2\cos(\omega t+\phi) = \frac{T}{ml}[A_{p+1}\cos(\omega t+\phi) + A_{p-1}\cos(\omega t+\phi) - 2A_p\cos(\omega t+\phi)]$

$\Rightarrow -A_p\omega^2[\cancel{\cos(\omega t+\phi)}] = \left[\frac{T}{ml}(A_{p+1} + A_{p-1} - 2A_p)\right]\cancel{\cos(\omega t+\phi)}$

$\Rightarrow -A_p\omega^2 = \frac{T}{ml}[A_{p+1} + A_{p-1} - 2A_p]$.

$\Rightarrow -\frac{\omega^2 ml}{T} A_p = A_{p+1} + A_{p-1} - 2A_p$

$\Rightarrow \left[2 - \frac{\omega^2 ml}{T}\right]A_p = A_{p+1} + A_{p-1}$ $\left[\because \frac{T}{ml} = \omega_0^2\right]$.

$\Rightarrow 2 - \frac{\omega^2}{\omega_0^2} = \frac{A_{p+1} + A_{p-1}}{A_p}$.

$\therefore \boxed{\frac{A_{p+1} + A_{p-1}}{A_p} = 2 - \frac{\omega^2}{\omega_0^2}} \longrightarrow (3)$

Here Particle p is purely independent on Frequency

Let, $A_p = c \sin p\theta$ [Amp]

$A_{p+1} = c \sin(p+1)\theta$

$A_{p-1} = c \sin(p-1)\theta$

Substitute in eqⁿ (3), [Numerator part]

$\Rightarrow A_{p+1} + A_{p-1} = c \sin(p+1)\theta + c \sin(p-1)\theta$

$\Rightarrow A_{p+1} + A_{p-1} = c[\sin(p+1)\theta + \sin(p-1)\theta]$

$\Rightarrow A_{p+1} + A_{p-1} = 2c \sin(p)\theta \cos\theta$.

Now, Numerator & Denominator

$$\Rightarrow \frac{A_{p+1} + A_{p-1}}{A} = \frac{2c \cancel{\sin p\theta} \cos\theta}{c \cancel{\sin p\theta}} = 2\cos\theta \longrightarrow (4)$$

From (3) & (4)

$$\Rightarrow \frac{2 - \omega^2}{\omega_0^2} = 2\cos\theta.$$

Amplitude, $A_p = c \sin p\theta$.

$A_p = c \sin p\theta = 0$

$(N+1)\theta = n\pi$

$$\Rightarrow \theta = \frac{n\pi}{N+1}$$

$$\therefore \boxed{A_p = c \sin\left(\frac{np\pi}{N+1}\right)}$$

$\Rightarrow 2\omega_0^2 - \omega^2 = 2\omega_0^2 \cos\theta$.

$\Rightarrow 2\omega_0^2(1 - \cos\theta) = \omega^2$

$\Rightarrow 2\omega_0^2\left(2\sin^2\frac{\theta}{2}\right) = \omega^2$

$\Rightarrow \omega^2 = 4\omega_0^2 - \sin^2\frac{\theta}{2}$

$\Rightarrow \omega = 2\omega_0 \sin\left(\frac{\theta}{2}\right)$.

$$\Rightarrow \omega = 2\omega_0 \sin\left(\frac{n\pi}{2(N+1)}\right)$$

$$\therefore \boxed{\omega = 2\omega_0 \sin\left(\frac{n\pi}{2(N+1)}\right)} \rightarrow \text{Frequency.}$$