6.3

Processes Depending on Stationary Gaussian Process

Square law detector process

If $\{X(t)\}$ is a zero mean stationary Gaussian process and if $Y(t) = X^2(t)$, then $\{Y(t)\}$ is called a **square law detector process**.

$$E\{Y(t)\} = E\{X^2(t)\} = Var\{X(t)\} = R_{xx}(0)$$

$$R_{yy}(t_1, t_2) = E\{Y(t_1)Y(t_2)\} = E\{X^2(t_1)X^2(t_2)\}$$

$$= E\{X^2(t_1)\}E\{X^2(t_2)\} + 2E^2\{X(t_1)X(t_2)\}$$

[Since X and Y are jointly normal, $E(X^2Y^2) = E(X^2)E(Y^2) + 2E^2(XY)$]

$$= R_{xx}^2(0) + 2R_{xx}^2(\tau) \text{ [since } X(t) \text{ is stationary]}$$

Since the RHS is a function of τ, LHS is also a function of $\tau = t_1 - t_2$.

$i.e.$,

$$R_{yy}(\tau) = R_{xx}^2(0) + 2R_{xx}^2(\tau)$$

Therefore, $\{Y(t)\}$ is also a stationary process (at least in the wide-sense).

We note that $E\{Y^2(t)\} = R_{yy}(0) = 3R_{xx}^2(0)$

$Var\{Y(t)\} = E\{Y^2(t)\} - (E\{Y(t)\})^2 = 3R_{xx}^2(0) - R_{xx}^2(0) = 2R_{xx}^2(0)$ and $C_{yy}(\tau) = 2R_{xx}^2(\tau)$

Power spectral density of $\{Y(t)\}$ is given by

$$S_{yy}(\omega) = \int_{-\infty}^{\infty} R_{yy}(\tau)\, e^{-i\omega\tau}\, d\tau = \int_{-\infty}^{\infty} \{R_{xx}^2(0) + 2R_{xx}^2(\tau)\} e^{-i\tau\omega} d\tau$$

$$= 2\pi R_{xx}^2(0)\delta(\omega) + 2F\{R_{xx}(\tau)R_{xx}(\tau)\} \qquad \dots(1)$$

[since $F^{-1}\{2\pi m^2\delta(\omega)\} = m^2$]

Consider $F^{-1}\{S_{xx}(\omega) * S_{xx}(\omega)\}$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} S_{xx}(\omega-\alpha)S_{xx}(\alpha)e^{i\tau\omega}d\alpha d\omega \qquad \ldots.(2)$$

Put $\omega - \alpha = \beta$ and $\alpha = \gamma$ $i.e.$, $\omega = \beta + \gamma$, $\alpha = \gamma$

Then, from calculus,

$$d\omega d\alpha = \begin{vmatrix} \frac{\partial\omega}{\partial\beta} & \frac{\partial\omega}{\partial\gamma} \\ \frac{\partial\alpha}{\partial\beta} & \frac{\partial\alpha}{\partial\gamma} \end{vmatrix} d\beta d\gamma = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} d\beta d\gamma = d\beta d\gamma \qquad \ldots(3)$$

Using (3) in (2), we get

$$F^{-1}\{S_{xx}(\omega) * S_{xx}(\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} S_{xx}(\beta)S_{xx}(\gamma)e^{i\tau(\beta+\gamma)}d\beta d\gamma$$

$$= 2\pi\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xx}(\beta)e^{i\tau\beta}d\beta\right)\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xx}(\gamma)e^{i\tau\gamma}d\gamma\right)$$

$$= 2\pi R_{xx}(\tau)R_{xx}(\tau) \qquad \ldots(4)$$

Using (4) in (1),

$$S_{yy}(\omega) = 2\pi R^2{}_{xx}(0)\delta(\omega) + \frac{1}{\pi}S_{xx}(\omega) * S_{xx}(\omega)$$

Example 1: If $\{Y(t)\}$ is the square law detector process and if $Z(t) = Y(t) - E\{Y(t)\}$, show that the spectral density of $\{Z(t)\}$ is given by $S_{zz}(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} S_{xx}(\alpha) S_{xx}(\omega - \alpha) d\alpha$, where $S_{xx}(\omega)$ is the input spectral density.

Solution:

Note: $Z(t)$ is called the **fluctuation of the square law detector.**

$$E\{Z(t)Z(t-\tau)\} = E[\{Y(t) - E[Y(t)]\}\{Y(t-\tau) - E[Y(t-\tau)]\}]$$

$$= E\{Y(t)Y(t-\tau)\} - E\{Y(t)\}E\{Y(t-\tau)\} \text{ (simplification!)}$$

$$i.e., \; R_{zz}(\tau) = R_{yy}(\tau) - E\{Y(t)\}E\{Y(t-\tau)\}$$

$$= R_{xx}^2(0) + 2R_{xx}^2(\tau) - R_{xx}^2(0)$$

(By square law detector process)

$$= 2R_{xx}^2(\tau)$$

$$\Longrightarrow R_{zz}(\tau) = 2R_{xx}^2(\tau)$$

Taking Fourier transforms,

$$S_{zz}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R_{zz}(\tau)e^{-i\omega\tau}d\tau = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2R_{xx}^2(\tau)e^{-i\omega\tau}d\tau$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} S_{xx}(\alpha)S_{xx}(\omega-\alpha)d\alpha$$

Two Important Results

We now consider two important results which will be used in the discussion of other processes depending *on stationary Gaussian process*, that will follow.

Result 1: If X and Y are two normal r.vs with zero means, variances σ_1^2 and σ_2^2 and correlation coefficient r, then the probability that they are of the same sign $=\frac{1}{2}+\frac{1}{\pi}sin^{-1}(r)$ and the probability that they are of opposite sign $=\frac{1}{2}-\frac{1}{\pi}sin^{-1}(r)$.

Result 2: If X and Y are two normal r.vs with zero means, variances σ_1^2 and σ_2^2 and correlation coefficient r, then $E\{|XY|\}=\frac{2}{\pi}\sigma_1\sigma_2(\cos\alpha+\alpha\sin\alpha)$, where $\sin\alpha=r$.

Full wave linear detector process

If $\{X(t)\}$ is a zero mean stationary Gaussian process and if $Y(t)=|X(t)|$, then $\{Y(t)\}$ is called **a full wave linear detector process**.

$$E\{Y(t)\}=E\{|X(t)|\}=\int_{-\infty}^{\infty}|x|\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{x^2}{2\sigma^2}}dx$$

$$=\frac{1}{\sigma}\sqrt{\frac{2}{\pi}}\int_0^{\infty}xe^{-\frac{x^2}{2\sigma^2}}dx$$

$$=\frac{1}{\sigma}\sqrt{\frac{2}{\pi}}\sigma^2\int_0^{\infty}e^{-t}dt\ ,putting\ \frac{x^2}{2\sigma^2}=t$$

$$=\sigma\sqrt{\frac{2}{\pi}}=\sqrt{\frac{2R_{xx}(0)}{\pi}}$$

$$R_{yy}(t_1,t_2)=E\{Y(t_1)Y(t_2)\}=E\{|X(t_1)X(t_2)|\}$$

$$=\frac{2}{\pi}\sigma^2(cos\,\alpha+\alpha\,sin\,\alpha)\quad \text{(using Result 2)}$$

where $\sin\alpha=r=\frac{C\{X(t_1),X(t_2)\}}{\sigma^2}$

$$=\frac{E\{X(t_1)\ X(t_2)\}}{\sigma^2}$$

$$= \frac{R_{xx}(t_1 - t_2)}{\sigma^2} \qquad \text{[since } \{X(t)\} \text{ is stationary]}$$

Therefore, $\{Y(t)\}$ is wide-sense stationary, with

$R_{yy}(\tau) = \frac{2}{\pi} R_{xx}(0)(\cos\alpha + \alpha\sin\alpha)$, where $sin\,\alpha = \frac{R_{xx}(\tau)}{R_{xx}(0)}$

Now $E\{Y^2(t)\} = R_{yy}(0) = \frac{2}{\pi} R_{xx}(0)\left\{0 + \frac{\pi}{2} 1\right\}$,

Since $\sin\alpha = \frac{R_{xx}(0)}{R_{xx}(0)} = 1$ and $\alpha = \frac{\pi}{2}$

$\therefore E\{Y^2(t)\} = R_{xx}(0)$ and $Var\{Y(t)\} = \left(1 - \frac{2}{\pi}\right) R_{xx}(0)$

Half-wave linear detector process

If $\{X(t)\}$ is a zero mean stationary Gaussian process and if

$$Z(t) = \begin{cases} X(t) & , \quad for\ X(t) \geq 0 \\ 0 & , \quad for\ X(t) < 0 \end{cases}$$

then $\{Z(t)\}$ is called a **half-wave linear detector process**.

$Z(t)$ can be rewritten as $Z(t) = \frac{1}{2}\{X(t) + |X(t)|\}$

$\therefore E\{Z(t)\} = \frac{1}{2}[E\{X(t)\} + E\{|X(t)|\}]$

$$= \frac{1}{2}\left[0 + \sqrt{\frac{2}{\pi}} R_{xx}(0)\right] \qquad \text{(refer to the full wave linear detector process)}$$

$$= \sqrt{\frac{R_{xx}(0)}{2\pi}}$$

$$E\{Z(t)Z(t-\tau)\} = E[E\{Z(t)Z(t-\tau)|X(t)X(t-\tau)\}] \qquad \dots(1)$$

Now $Z(t)Z(t-\tau)|X(t)X(t-\tau) = \frac{1}{2}\{X(t)X(t-\tau) + |X(t)X(t-\tau)|\}$ (or) $= 0$

The first value is assumed, when $X(t)X(t-\tau) > 0$, $i.e.$, when $X(t)$ and $X(t-\tau)$ are both positive or both negative.

$\therefore$ P\{The first value of assumed\} $= \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$

Similarly, P\{the second value is assumed\} $= \frac{1}{2}$

$$\therefore E\{Z(t)Z(t-\tau)|X(t)X(t-\tau)\} = \frac{1}{4}\{X(t)X(t-\tau) + |X(t)X(t-\tau)|\} \qquad \dots (2)$$

Using (2) in (1), we get,

$$E\{Z(t)Z(t-\tau)\} = \frac{1}{4}\left\{E\{X(t)X(t-\tau)\} + E\{|X(t)X(t-\tau)|\}\right\}$$

$$= \frac{1}{4}\left\{R_{xx}(\tau) + R_{yy}(\tau)\right\}$$

[where $\{Y(t)\}$ is the full-wave linear detector process]

i.e., $R_{zz}(\tau) = \frac{1}{4}\left[R_{xx}(\tau) + \frac{2}{\pi}R_{xx}(0)(\cos\alpha + \alpha\sin\alpha)\right]$, where $\sin\alpha = \frac{R_{xx}(\tau)}{R_{xx}(0)}$

Therefore, the process $\{Z(t)\}$ is wide-sense stationary.

Now $E\{Z^2(t)\} = R_{zz}(0) = \frac{1}{2}R_{xx}(0)$

$$\therefore Var\{Z(t)\} = E\{Z^2(t)\} - (E\{Z(t)\})^2$$

$$= \frac{1}{2}R_{xx}(0) - \frac{1}{2\pi}R_{xx}(0) = \frac{1}{2}\left(1 - \frac{1}{\pi}\right)R_{xx}(0)$$

Hard limiter process

If $\{X(t)\}$ is a zero mean stationary Gaussian process and if

$$Y(t) = \begin{cases} 1 & for\ X(t) \geq 0 \\ -1 & for\ X(t) < 0 \end{cases}$$

then $\{Y(t)\}$ is called a **hard limiter process** or **ideal limiter process**.

$$E\{Y(t)\} = P\{X(t) \geq 0\} - P\{X(t) < 0\}$$

$$= 0$$

Now

$$Y(t)Y(t-\tau)=\begin{cases} \ \ 1\ , \text{ if } X(t)X(t-\tau)\geq 0 \\ -1\ \ , \text{ if } X(t)X(t-\tau)<0 \end{cases}$$

$i.e.,\ P\{Y(t)Y(t-\tau)=1\}=P\{X(t)X(t-\tau)\geq 0\}$

$$=\frac{1}{2}+\frac{1}{\pi}sin^{-1}(r_{xx}) \qquad \text{(by Result 1)}$$

$$=\frac{1}{2}+\frac{1}{\pi}sin^{-1}\left\{\frac{R_{xx}(\tau)}{R_{xx}(0)}\right\}$$

and $P\{Y(t)Y(t-\tau)=-1\}=P\{X(t)X(t-\tau)<0\}$

$$=\frac{1}{2}-\frac{1}{\pi}\sin^{-1}\left\{\frac{R_{xx}(\tau)}{R_{xx}(0)}\right\} \quad \text{(by Result 1)}$$

$$\therefore\ E\{Y(t)Y(t-\tau)\}=\frac{2}{\pi}\sin^{-1}\left\{\frac{R_{xx}(\tau)}{R_{xx}(0)}\right\}$$

$$i.e.,\ R_{yy}(\tau)=\frac{2}{\pi}\sin^{-1}\left\{\frac{R_{xx}(\tau)}{R_{xx}(0)}\right\} \qquad \text{....}(1)$$

Thus (1) is called the **arcsine law**

Therefore $\{Y(t)\}$ is wide-sense stationary.

Also $E\{Y^2(t)\}=1$ and $Var\{Y(t)\}=1$

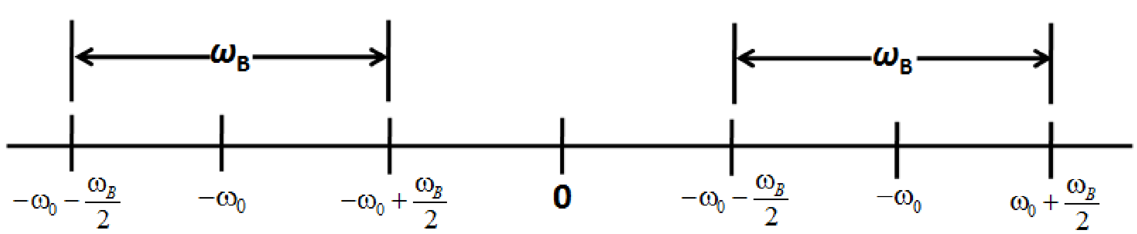
Band Pass Process (Signal)

If the power spectrum of a random process $\{X(t)\}$ is zero outside a certain band (an interval in the ω-axis),

i.e., $S_{xx}(\omega)\neq 0$, in $|\omega-\omega_0|\leq\frac{\omega_B}{2}$ and in $|\omega+\omega_0|\leq\frac{\omega_B}{2}$

and $S_{xx}(\omega)\ =0$, in $|\omega-\omega_0|>\frac{\omega_B}{2}$ and in $|\omega+\omega_0|>\frac{\omega_B}{2}$

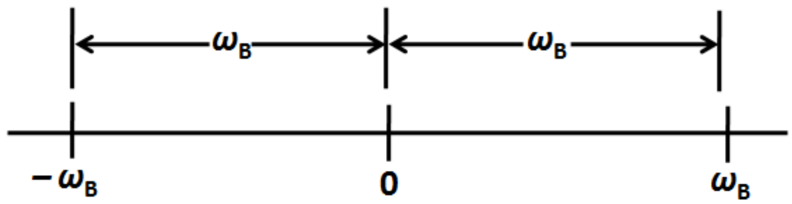
then $\{X(t)\}$ is called a **band pass process**.



If $S_{xx}(\omega) \neq 0$, in $|\omega| \leq \omega_B$ and

$S_{xx}(\omega) \;= 0$, in $|\omega| > \omega_B$

then $\{X(t)\}$ is called a low pass process or ideal low pass process.



If the bandwith ω_B of a band pass process is small compared with the centre frequaency ω_0, the process is called **narrow band process** or **quasimonochromatic.**

If the power spectrum $S_{xx}(\omega)$ of a bandpass process $\{X(t)\}$ is an impulse function, then the process is called **monochromatic**.

Narrow-Band Gaussian Process

In communication system, information bearing signals are often narrow-band Gaussian processes. When such signals are viewed on an oscilloscope, they appear like a sine wave with slowly varying amplitude and phase. Hence a narrow-band Gaussian process $\{X(t)\}$ is often represented as

$$X(t) = R_X(t)\cos[\omega_0 \pm \theta_X(t)] \qquad \ldots(1)$$

$R_X(t)$ and $\theta_X(t)$, which are low pass processes, are called the **envelope** and phase of the process $\{X(t)\}$ respectively. (1) can be rewritten as

$$X(t) = [R_X(t)\cos\theta_X(t)]\cos\omega_0 t \mp [R_X(t)\cos\theta_X(t)]\sin\omega_0 t \qquad \ldots(2)$$

$R_X(t)\cos\theta_Y(t)$ is called the **inphase component** of the process $\{X(t)\}$ and denoted as $X_c(t)$ or $I(t)$. $R_X(t)\sin\theta_X(t)$ is called the **quadrature component** of $\{X(t)\}$ and denoted as $X_s(t)$ or $Q(t)$.

Both $X_c(t)$ and $X_s(t)$ are low pass processes.

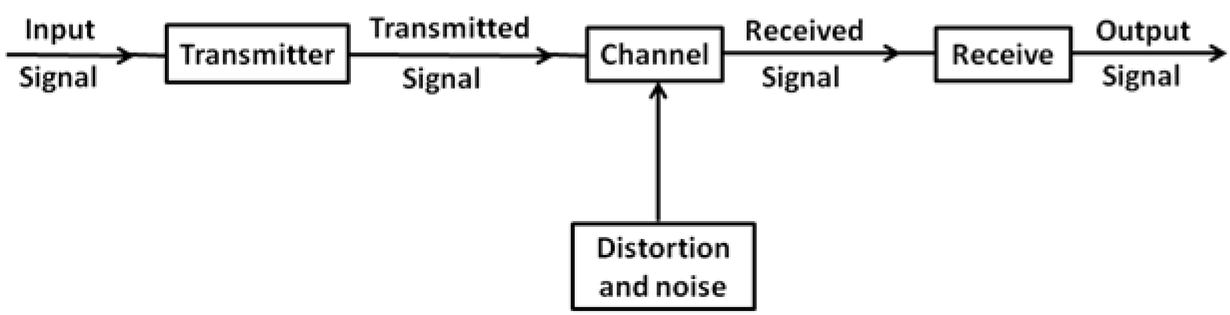
Property 1: The envelope of a narrow-band Gaussian process follows a Rayleigh distribution and the phase follows a uniform distribution in $(\mathbf{0}, \mathbf{2\pi})$.

We note that

$$\sqrt{{X_c}^2(t) + {X_s}^2(t)} = R_X(t) \text{ and } \tan^{-1}\left\{\frac{X_s(t)}{X_c(t)}\right\} = \theta_X(t)$$

If X and Y are two independent $N(0, \sigma)$ then $R = \sqrt{X^2 + Y^2}$ follows a Rayleigh distribution and $\phi = tan^{-1}\frac{Y}{X}$ follows a uniform distribution in $(0,2\pi)$.

Noise in Communication Systems



In communication systems, the message to be transmitted to a far-off location is first converted into an electrical waveform called input signal, before being sent into the transmitter. The transmitter processes and modifies the input signal for efficient transmission. The transmitter output is then sent through the channel which is just a medium such as *wire*, *coaxial cable* or *optical fibre*. The channel output or the received signal is then reprocessed by the receiver which sends out the output signal. The output signal is converted to its original form, namely the message.

When the message is communicated in this manner, the signal is not only distorted by the channel but also contaminated along the path by undesirable signals that are generally referred to by the term **noise**. The noise can come from many external and internal sources and take many forms.

External noise includes interfering signals from nearby sources, man-made noise generated by faulty contact switches for the electrical equipment, by ignition radiation, fluorescent lights, natural noise from lighting and extraterrestrial radiation etc. Internal noise results from thermal motion of electrons in conductors, random emission and diffusion or recombination of charged carriers in electronic devices. By careful engineering techniques, the effects of many unwanted signals can be eliminated or minimized. But there always remain certain inescapable random signals that set a limit to system performance, $i.e.$, on the efficiency of communication.

One of the main reasons for introducing probability theory in the study of *Signal Analysis* is the random nature of noise. Because of this randomness, it is usual to describe noise as a random process and hence in terms of a probabilistic model. Such a model describes the noise amplitude or any other parameter by means of a probability density function $f(x)$ [x represents voltage]. For many important types of noise, the density function can be determined theoretically and for others it has been estimated empirically.

Certain properties of noise, such as mean value, mean square value and the root-mean square value can be found by using the probability density function.

However the probability density function does not describe a noise waveform sufficiently so as to determine its effect on the performance of a communication system. To achieve this, it is necessary to know how the noise changes with time. This information is provided by a mean-square voltage spectrum, called the **power spectrum** or **spectral density**,that represents the distribution of signal power as a function of frequency.

Thermal Noise

Thermal noise is the noise because of the random motion of free electrons in conducting media such as a resistor. Thermal noise generated in resistors and semiconductors is assumed to be a zero mean, stationary Gaussian random process $\{N(t)\}$ with a power spectral density that is flat over a very wide range of frequencies, $i.e.$, the graph of $S_{NN}(\omega)$ is a straight line parallel to the ω-axis. Since $S_{NN}(\omega)$ contains all frequencies in equal amount, the noise is also called **white**

Gaussian noise or **simply white noise** in analogy to white light which consists of all colours.

It is customary to denote the constant spectral density of white noise by $\frac{N_0}{2}$ or $\frac{\eta}{2}$.

$i.e., S_{NN}(\omega) = \frac{N_0}{2}$

The autocorrelation function of the white noise is given by

$R_{NN}(\tau) = \frac{N_0}{2}\delta(\tau)$, since $\int_{-\infty}^{\infty} \frac{N_0}{2}\delta(\tau)e^{-i\omega\tau}d\tau = \frac{N_0}{2}$

The average power of the white noise $\{N(t)\}$ is given by

$$R_{NN}(0) = \int_{-\infty}^{\infty} S_{NN}(\omega)d\omega = \int_{-\infty}^{\infty} \frac{N_0}{2}d\omega \longrightarrow \infty$$

Therefore, the spectral density of $\{N(t)\}$ is not physically realisable. However, since the bandwidths of real processes are always finite and since

$$\int_{-\omega_B}^{\omega_B} S_{NN}(\omega)d\omega = N_0\omega_B < \infty$$

for any finite bandwidth, the spectral density $S_{NN}(\omega)$ can be used over finite bandwidths.

Band-limited white noise: Noise having a nonzero and constant spectral density over a finite frequency band and zero elsewhere is called **band-limited white noise.** $i.e.$, if $\{N(t)\}$ is a band-limited white noise then

$$S_{NN}(\omega) = \begin{cases} \frac{N_0}{2} & , \quad |\omega| \leq \omega_B \\ 0 & , \quad elsewhere \end{cases}$$

We give below a few properties of the band-limited white noise which can be easily verified by the reader.

1. $E\{N^2(t)\} = \frac{N_0\omega_B}{2\pi}$
2. $R_{NN}(\tau) = \frac{N_0\omega_B}{2\pi}\left(\frac{\sin\omega_B\tau}{\omega_B\tau}\right)$

3. $N(t)$ and $N\left(t+\frac{k\pi}{\omega_B}\right)$ are independent, where k is a nonzero integer.

Filters

Filtering is commonly used in electrical systems to reject undesirable signals and noise and to select the desired signal. A simple example of filtering occurs when we *tune* in a particular radio to *select* one of many signals.

Filtering actually means selecting carefully the transfer function $H(\omega)$ in a stable, linear, time-invariant system, so as to modify the spectral components of the input signal. The system function $H(\omega)$ or the linear system itself is referred to as filter, when it does the filtering.

The commonly used filters are narrow-band filters, $i.e.$, band pass and low pass filters.

If the system function $H(\omega)$ is defined as

$H(\omega) \neq 0$, for $\omega_0 - \frac{\varepsilon}{2} < \omega < \omega_0 + \frac{\varepsilon}{2}$ and $H(\omega) = 0$, otherwise

then the filter is called **a band pass filter**.

If $H(\omega) \neq 0$, for $-\frac{\varepsilon}{2} < \omega < \frac{\varepsilon}{2}$ and $H(\omega) = 0$, otherwise

then the filter is called **a low pass filter**.

The equation $S_{yy}(\omega) = S_{xx}(\omega)|H(\omega)|^2$ shows that the spectral properties of a signal can be modified by passing it through a linear time-invariant system with the appropriate transfer function. By carefully choosing $H(\omega)$, we can remove or filter out certain spectral components in the input. For example, let the input $X(t) = S(t) + N(t)$, where $S(t)$ is the signal of interest and $N(t)$ is an unwanted noise process. If the spectral densities of $\{S(t)\}$ and $\{N(t)\}$ are non-overlapping in the frequency domain, the noise $N(t)$ can be removed by passing $X(t)$ through a filter $H(\omega)$ that has a response of 1 for the range of frequencies occupied by the noise. But in most practical situations there is spectral overlap and the design of optimum filters to separate signal and noise is somewhat difficult. The discussion of this problem and the various optimum filters in common use such as matched filter and Wiener filter may be found in textbooks on Random Signal Analysis. It is beyond the scope of this syllabus.

Example 2: If $\{X(t)\}$ is a band limited process such that $S_{xx}(\omega) = 0$, when $|\omega| > \sigma$, prove that $2[R_{xx}(0) - R_{xx}(\tau)] \leq \sigma^2\tau^2 R_{xx}(0)$.

Solution: $R_{xx}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xx}(\omega)e^{i\tau\omega}d\omega$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xx}(\omega)\cos\tau\omega\, d\omega \qquad \text{(Since } s_{xx} \text{ is even)}$$

$$R_{xx}(0) - R_{xx}(\tau) = \frac{1}{2\pi}\int_{-\sigma}^{\sigma} S_{xx}(1-\cos\tau\omega)d\omega \qquad \text{(since}\{X(t)\}\text{ is band limited)}$$

$$= \frac{1}{2\pi}\int_{-\sigma}^{\sigma} S_{xx}(\omega)2\sin^2\left(\frac{\tau\omega}{2}\right)d\omega \qquad \dots (1)$$

From trigonometry, $|\sin\theta| \leq \theta$

$\therefore\ sin^2\theta \leq \theta^2$

$\therefore\ 2sin^2\left(\frac{\tau\omega}{2}\right) \leq \frac{\tau^2\omega^2}{2}$ ... (2)

Inserting (2) in (1)

$$R_{xx}(0) - R_{xx}(\tau) \leq \frac{1}{2\pi}\int_{-\sigma}^{\sigma} S_{xx}(\omega)\frac{\tau^2\omega^2}{2}d\omega$$

$$\leq \frac{\sigma^2\tau^2}{4\pi}\int_{-\sigma}^{\sigma} S_{xx}(\omega)d\omega$$

$$\leq \frac{\sigma^2\tau^2}{4\pi}\int_{-\infty}^{\infty} S_{xx}(\omega)d\omega$$

$i.e.,\quad R_{xx}(0) - R_{xx}(\tau) \leq \frac{\sigma^2\tau^2}{2}R_{xx}(0)$

Example 3: Consider a white Gaussian noise of zero mean and power spectral density $\frac{N_0}{2}$ applied to a law pass RC filter whose transfer function is $H(f) = \frac{1}{1+i2\pi fRC}$. Find the autocorrelation function of the output random process.

Solution: The simple $RC -$ circuit for which the transfer function is given is a linear time – invariant system. The power spectral densities of the input $\{X(t)\}$ and the output $\{Y(t)\}$ of a linear system are connected by

$$S_{yy}(\omega) = S_{xx}(\omega)|H(\omega)|^2$$

In this problem the transfer function is expressed in terms of the frequency f.

Therefore, the above relation is

$$S_{yy}(f) = S_{xx}(f)|H(f)|^2$$

$$= \frac{1}{1+4\pi^2 f^2 R^2 C^2}\frac{N_0}{2} \qquad \text{(since the input is a white noise)}$$

$$\therefore \quad R_{yy}(\tau) = \frac{N_0}{2}\int_{-\infty}^{\infty}\frac{e^{i2\pi\tau f}}{1+4\pi^2 f^2 R^2 C^2}df$$

$$= \frac{N_0}{8\pi^2 R^2 C^2}\int_{-\infty}^{\infty}\frac{e^{i(2\pi\tau)f}df}{\left(\frac{1}{2\pi RC}\right)^2 + f^2} \qquad \dots (1)$$

Compare the integral in (1) with $\int_{-\infty}^{\infty}\frac{e^{imx}dx}{a^2+x^2} = \frac{\pi}{a}e^{-|m|\,a}$ ... (2)

Using (2) in (1)

$$R_{yy}(\tau) = \frac{N_0}{8\pi^2 R^2 C^2}\pi 2\pi RC\; e^{-|2\pi\tau| 2\pi RC}$$

$$= \frac{N_0}{4RC}e^{-\frac{|\tau|}{RC}}$$

The mean square value of $\{Y(t)\}$ is given by $E\{Y^2(t)\} = R_{yy}(0) = \frac{N_0}{4RC}$

Example 4: If $Y(t) = A\cos(\omega_0 t + \theta) + N(t)$, where A is a constant, θ is a random variable with a uniform distribution in $(-\pi, \pi)$ and $\{N(t)\}$ is a band limited Gaussian white noise with a power spectral density.

$$S_{NN}(\omega) = \begin{cases} \frac{N_0}{2} \;, & for \;\; |\omega - \omega_0| < \omega_B \\ 0 \;, & elsewhere \end{cases}$$

find the power spectral density of $\{Y(t)\}$. Assume that $N(t)$ and θ are independent.

Solution:

$$Y(t_1)Y(t_2) = \{A\cos(\omega_0 t_1 + \theta) + N(t_1)\}\{A\cos(\omega_0 t_2 + \theta) + N(t_2)\}$$

$$= A^2\cos(\omega_0 t_1 + \theta)\cos(\omega_0 t_2 + \theta) + N(t_1)N(t_2)$$

$$+A\cos(\omega_0 t_1 + \theta)\,N(t_2) + A\cos(\omega_0 t_2 + \theta)\,N(t_1)$$

$$\therefore \;\; R_{YY}(t_1, t_2) = E\{Y(t_1)Y(t_2)\}$$

$$= A^2 E\{\cos(\omega_0 t_1 + \theta)\cos(\omega_0 t_2 + \theta) + R_{NN}(t_1, t_2)\} +$$

$$AE\{\cos(\omega_0 t_1 + \theta)\}E\{N(t_2)\} + AE\{\cos(\omega_0 t_2 + \theta)\}\,E\{N(t_1)\}$$

(by independent)

$i.e., R_{YY}(\tau) = \frac{A^2}{2}\cos\omega_0\,\tau + R_{NN}(\tau)$ [since $\{N(t)\}$ is stationary]

$$\therefore \quad S_{YY}(\omega) = \frac{A^2}{2}\int_{-\infty}^{\infty}\cos\omega_0\tau e^{-i\omega\tau}d\tau + S_{NN}(\omega)$$

$$= \frac{\pi A^2}{2}\{\delta(\omega - \omega_0) + \delta(\omega + \omega_0)\} + S_{NN}(\omega)$$

Where $S_{NN}(\omega)$ is given.