Basics of Inductor and capacitor

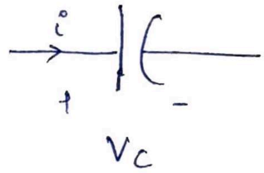
Q) A resistor is an energy storing storing element or energy dissipating element.

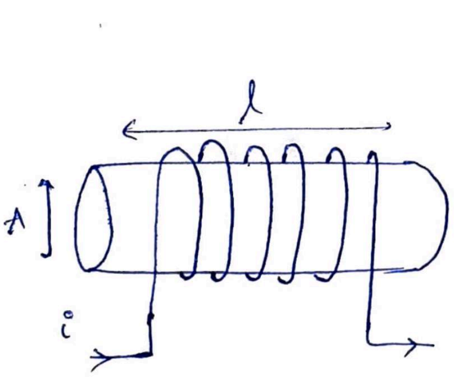
Ans. Energy dissipating element

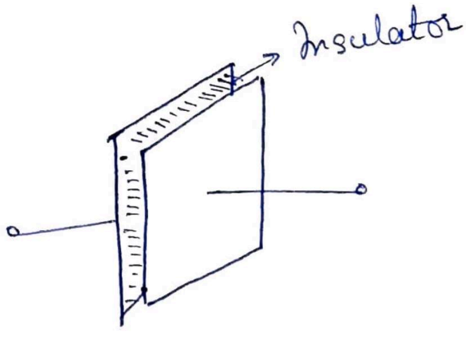
It dissipates energy in the form of heat.

- Inductor and capacitor are energy storing elements.

Inductor	Capacitor







A cylindrical core with a conducting wire wound on it.

Two conducting plates with an insulator (or) dielectric in between

$$L = \frac{N^2 \mu A}{l}$$

$N \rightarrow$ No. of turns of wire

$A \rightarrow$ Cross sectional area of core

$l \rightarrow$ length of core

$$C = \frac{\varepsilon A}{d}$$

ε - permittivity of dielectric medium

A - Cross sectional area of plates

d - Distance between the plates.

$$C = \frac{q}{v}$$

q - charge on plates

v - voltage across the plates

Units: Henry

Unit: farad

$$V_L = L \frac{di_L}{dt}$$

$$i_c = C \frac{dV_c}{dt}$$

for DC supply,

$V_L = L(0) = 0V$

* $\Rightarrow$ short circuit

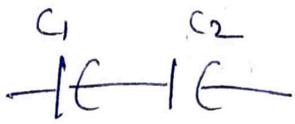
Inductor behaves as short circuit for DC supply

for DC supply,

$i_c = C(0) = 0A$

* $\Rightarrow$ Capacitor behaves as open circuit for DC supply

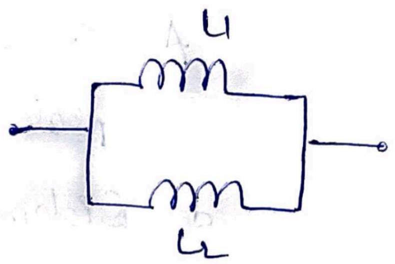
series :

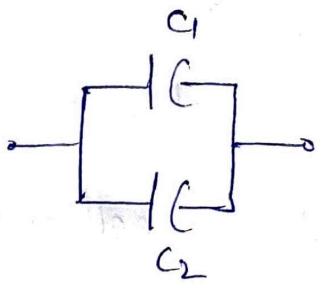


$L_{eq} = L_1 + L_2$

$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2}$

parallel:





$L_{eq} = \frac{L_1 L_2}{L_1 + L_2}$

(or) $\frac{1}{L_{eq}} = \frac{1}{L_1} + \frac{1}{L_2}$

$C_{eq} = C_1 + C_2$

Energy is stored by inductor in the form of magnetic field.

$W = \frac{1}{2} L i^2$.

Energy is stored in the form of electric field by capacitor.

$W = \frac{1}{2} C V^2$.

UNIT-2

TRANSCIENT ANALYSIS OF FIRST ORDER CIRCUITS

first order circuit:

- It is a circuit comprising of R and C or R and L.
- If we consider a purely resistive circuit and apply Kirchoff's laws, we will get only algebraic equations.
- If we apply KVL or KCL for RC or RL circuits, the differential equations are obtained. The differential equations resulting from analysis of RC and RL circuits are of first order. Hence, the circuits are collectively called as first order circuits.

"A first order circuit is characterized by first order differential equations."

Transcient analysis:

- The output of any system or network is consists of two parts. i, Transcient response
ii, Steady state response.

Let c(t) be the output of a network in time do-
- main.

$$\therefore \; c(t) = C_{tr}(t) + C_{ss}(t)$$

- The response of the circuit when $t=0^+$ i.e just after ~~at~~ the instant of switching is called transcient response and analyzing the circuit during this timeperiod is called as transcient analysis.

$$C_{tr}(t) = \lim_{t \to 0^+} c(t).$$

(Transcient response becomes zero as time becomes large).

- Steadystate response is that part of the response of ~~t~~ which remains after the transcients have died out.

$$C_{ss}(t) = \lim_{t \to \infty} c(t).$$

Initial conditions of capacitor :

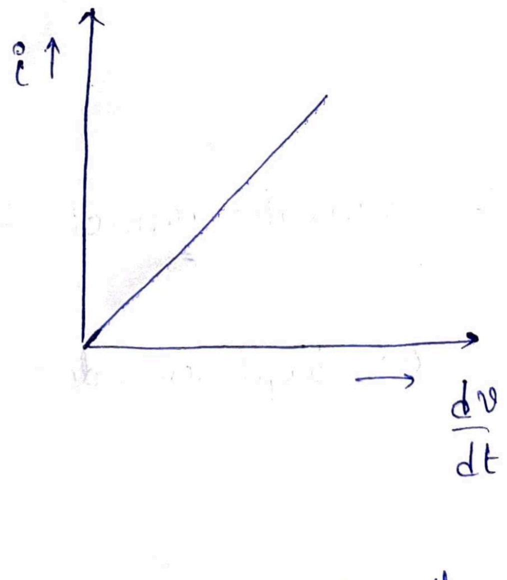
from $i = \frac{dq}{dt}$

$$i = \frac{d(Cv)}{dt} \quad (\because \text{ for capacitor } q = Cv)$$

$$\boxed{i = C\frac{dv}{dt}} \longrightarrow (1)$$

- This is the current voltage relationship of a capacitor.

- A capacitor which satisfies eq (1) is called as a linear capacitor.

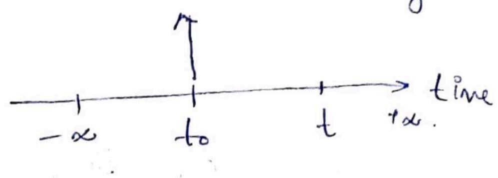


for a nonlinear capacitor, the plot of i,v relationship is not a straight line.

. Here, we consider linear capacitors only from eq ①,

$$v = \frac{1}{C}\int_{-\infty}^{t} i\,dt$$

t_0 is the instant of switching



$$v = \frac{1}{C}\int_{t_0}^{t} i\,dt + v(t_0) \longrightarrow ②$$

where, $v(t_0) = \frac{q(t_0)}{C}$ is the voltage across the capacitor at time t_0 or till t_0 or from $-\infty$ to t_0.

" This is called as initial condition of the capacitor "

{ Eq ② shows that the capacitor voltage depends on the past history of the capacitor current. Hence capacitor has memory. }

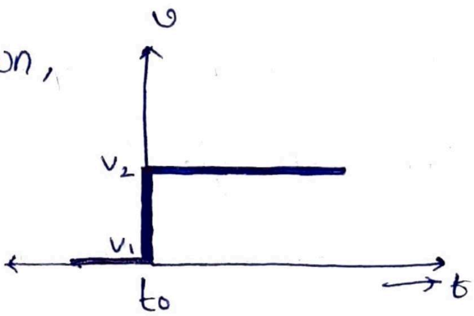
* " [The voltage across a capacitor cannot change abruptly] "

– A capacitor resists an abrupt or sudden change in voltage across it.

Consider a signal as shown,

from, $i = C\frac{dv}{dt}$

at $t=t_0$, voltage is raised from V_1 to V_2.



$\therefore dv = V_2 - V_1$ but $dt = 0$ ie $t_2 - t_1 =$

$$\therefore \quad i = C\frac{dv}{dt} = C\frac{(V_2 - V_1)}{0} = \infty.$$

$\therefore$ If the capacitor has to follow this sudden change in its voltage, it requires infinite amount of current which is not possible

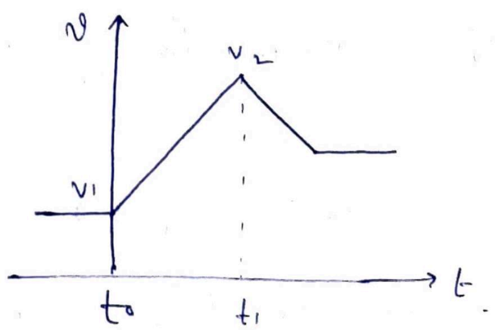
So, a capacitor cannot follow sudden or abrupt changes in voltage. $\Rightarrow V_c(0^-) = V_c(0^+)$

- In other words, capacitor is having inertia to voltage changes.

- Conversely, current through capacitor can change ~~its~~ instantaneously.

{- Suppose, if the voltage signal is as shown,

A capacitor can follow this kind of gradual changes in voltage.



}

Initial conditions of an inductor:

from the current voltage relationship of inductor,

$$v = L\frac{di}{dt}$$

$$\Rightarrow di = \frac{1}{L} v\, dt$$

$$\Rightarrow i = \int_{-\infty}^{t} \frac{1}{L} v\, dt$$

$$= \frac{1}{L}\int_{-\infty}^{t} v\, dt$$

$$i = \frac{1}{L}\int_{t_0}^{t} v(t)\, dt + i(t_0).$$

where i_0 or $i(t_0)$ is the total current for $-\infty < t < t_0$

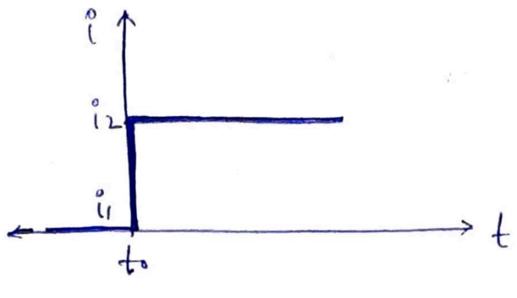
and also $i(-\infty) = 0\,A$.

"This is called as initial condition of inductor."

* "[The current through an inductor cannot change abruptly]"

- An inductor opposes an abrupt or sudden change in the current flowing through it.

Consider a signal as shown,



from $v = L\dfrac{di}{dt}$

At $t = t_0$, current is raised on changed abruptly from i_1 to i_2.

$\therefore\ di = i_2 - i_1$, but $dt = 0$

~~but dt = to - to = 0.~~

$$\therefore\ v = L\frac{di}{dt} = L\frac{(i_2 - i_1)}{0} = \infty$$

So, if the inductor has to follow this sudden change in current, it requires an infinite amount of voltage, which is impossible.

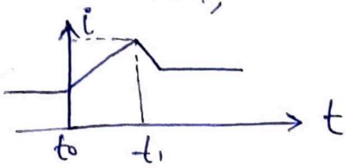
- So, an inductor cannot follow abrupt on sudden changes in current. $\Rightarrow i_L(0^-) = i_L(0^+)$

- In other words, it is having inertia to current changes.

- Conversely, it can follow sudden changes in voltage across it.

{ - Suppose if the current signal is as shown,

An Inductor can follow this kind of gradual changes in current.



}.

Example problems:

Q1) If a 10 μF capacitor is connected to a voltage source with $v(t) = 50\sin 2000t$ V, determine the current through the capacitor.

Sol) Given $C = 10\,\mu F$,

$$v(t) = 50\sin 2000t.$$

$$i = C\frac{dv(t)}{dt}$$

$$= 10\mu \times \frac{d}{dt}(50\sin 2000t)$$

$$= 10\mu \times 50\,(2000)\cos 2000t$$

$$= 10^{6} \times 10^{-6}\cos 2000t$$

$$= \cos 2000t.$$

Q2) Determine the voltage across a 2 μF capacitor if the current through it is $i(t) = 6e^{-3000t}$ mA. Assume that the initial capacitor voltage is i, zero ii, 5V.

Sol) Given, $C = 2\mu F$, $i(t) = 6e^{-3000t}$ mA

i, $v(0) = 0V$.

from $v = \frac{1}{C}\int_0^t i\,dt + v(0)$

$$v = \frac{1}{2\times10^{-6}}\int_0^t (6e^{-3000t}\times10^{-3})\cdot dt.$$

$$= \frac{\times10^{-3}\times 6}{2\times10^{-6}\times(-3000)}\, e^{-3000t}\Big|_0^t$$

$$= -(e^{-3000t})\Big|_0^t$$

$$= -\left(e^{-3000t} - e^{0}\right)$$

$$= (1-e^{-3000t})\ V.$$

ii, $v(0) = 5V$

from $v = \frac{1}{C}\int_0^t i\,dt + v(0)$

$$= \frac{1}{2\times10^{-6}}\int_0^t (6e^{-3000t}\times10^{-3})\,dt + 5$$

$$= \frac{6\times10^{-3}}{2\times10^{-6}\times(-3000)}\, e^{-3000t}\Big|_0^t + 5$$

$$= (1-e^{-3000t}) + 5$$

$$= (6-e^{-3000t})\ V.$$

Natural & Forced response of RC, RL circuits

- Natural response of a circuit refers to the behaviour (in terms of voltage and currents) of the circuit itself, with no external sources of excitation.

- This response will die out with time.

- Forced response of a circuit refers to the behaviour of the circuit, when it is excited by some external means, either an independent voltage source or current source.

Natural response of first order circuits.

→ There are two types of first order circuits (ie RC, RL) and also there are two different ways to excite them.

i, By initial conditions of the storage elements in the circui

In these so called sourcefree circuits, we assume that energy is initially stored in the inductive or capacitive element.

The energy causes the current to flow in the circuit and is gradually dissipated in the resistors.

- By definition, these source free circuits are free from independent sources. But, they may have dependent sources.

ii, By independent sources.

Source free RC ckts.

source free RL ckt

forced response of RC ckt

forced response of RL ckt

Source free RC ckt:

{ A source free RC ckt occurs when its dc supply is suddenly disconnected. Now, the energy stored in the capacitor is released to the resistors }.

- Consider a series combination of R and initially charged capacitor, as shown.

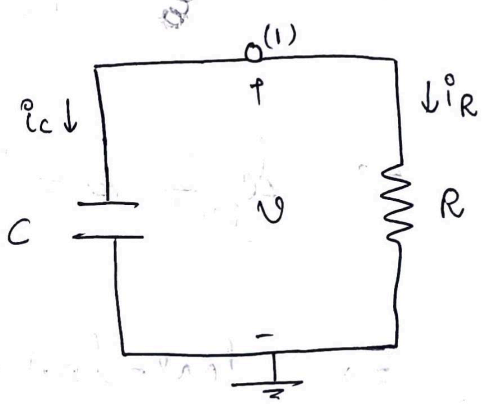


fig: A source free RC ckt.

- Since the capacitor is initially charged, we assume that at time ~~t=to~~ $t=0$, the initial voltage is,

$$v(0) = V_0. \longrightarrow (1)$$

with the corresponding value of energy stored as,

$$w(0) = \frac{1}{2} C V_0^2.$$

Apply KCL at node 1, from the above ckt

$$i_c + i_R = 0$$

$$\Rightarrow C \frac{dv}{dt} + \frac{v}{R} = 0$$

$$\Rightarrow C\frac{dv}{dt} = -\frac{v}{R}$$

$$\Rightarrow \frac{dv}{v} = -\frac{1}{RC}\,dt$$

Integrating on bothsides.

$$\Rightarrow \ln v = -\frac{t}{RC} + A'$$

(A' is a constant
Let $A' = \ln A$)

$$\Rightarrow \ln v = -\frac{t}{RC} + \ln A$$

$$\Rightarrow \cancel{\frac{\ln v}{\ln A}}$$

$$\Rightarrow \ln v - \ln A = -\frac{t}{RC}$$

$$\Rightarrow \ln\frac{v}{A} = -\frac{t}{RC}$$

$$\Rightarrow \frac{v(t)}{A} = e^{-t/RC}$$

$$\Rightarrow v(t) = Ae^{-t/RC} \longrightarrow (2)$$

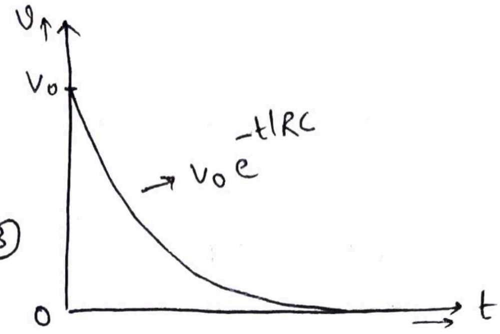
from initial condition, from eq (1),

$$V_0 = v(0) = Ae^{-0/RC} = A(1) = A$$

$$\Rightarrow \boxed{A = V_0}$$

from eq (2),

$$v(t) = V_o e^{-t/RC} \longrightarrow (3)$$



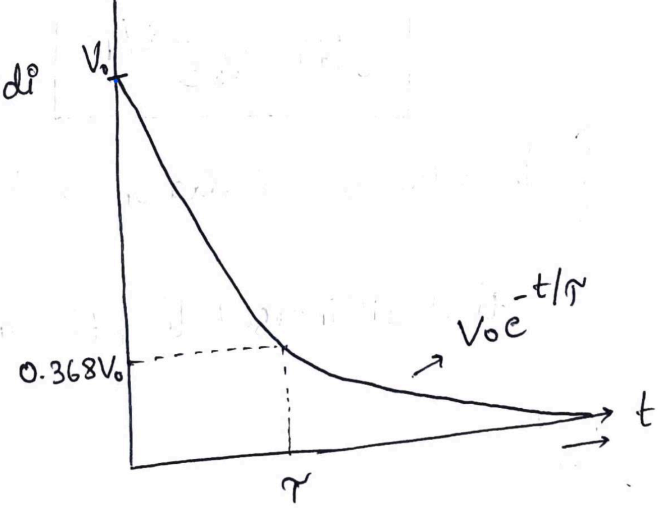
This shows that the voltage response of the RC circuit is an exponential decay of the initial voltage.

{ Therefore, since the response is due to the initial energy stored and the physical characteristics of the circuit and not due to some external voltage or current source, it is called as the natural response of the circuit. }

from the natural response curve of RC ckt,

At $t=0$

$\Rightarrow V = V_o \Rightarrow$ Initial condition



As t increases, voltage decreases towards zero.

- The speed with which the voltage decreases is expressed in terms of "time constant", denoted by 'τ'

" The time constant of a circuit is the time required for the response to decay by a factor 1/e or 36.8 percent of it's initial value. " $\Rightarrow$ $\boxed{\tau = RC}$

$\Rightarrow$ At $t = \tau$, eq ② becomes

$$v(t)\Big|_{t=\tau} = v(\tau) = V_0 e^{-\tau/RC}$$

$$= V_0 e^{-RC/RC}$$

$$= V_0 e^{-1}$$

$$= V_0 (0.368)$$

$$= 0.368\, V_0$$

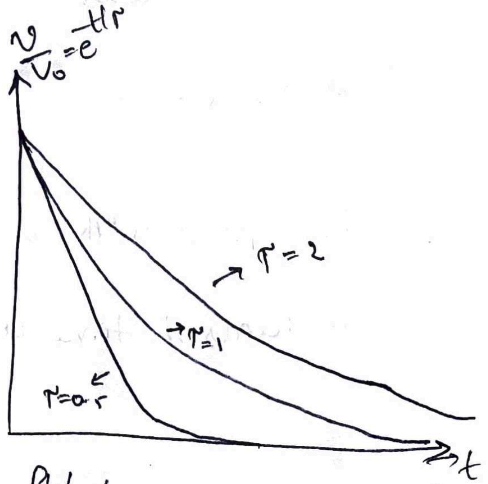
$\therefore$ Writing eq ② in terms of time constant,

$$\boxed{v(t) = V_0 e^{-t/\tau}} \longrightarrow (4)$$

{ If a tangent is drawn to the above curve (voltage curve), that will intersect the time axis at $t = \tau$ }

Note:

- Smaller the time constant, faster will be the response of the system.



Plot of $\frac{v}{V_0}$ vs t for various time constants.

- If the time constant is large, it will give slower response because it takes longer to reach steady state.

from,

$$v(t) = V_0 e^{-t/\tau}$$

$$\Rightarrow i_R(t) = \frac{v(t)}{R} = \frac{V_0}{R} e^{-t/\tau}$$

- Power dissipated in the resistor,

$$p(t) = v i_R = \left(V_0 e^{-t/\tau}\right)\left(\frac{V_0}{R} e^{-t/\tau}\right)$$

$$p(t) = \frac{V_0^2}{R} e^{-2t/\tau}$$

- Energy absorbed by the resistor upto time t is,

$$w_R(t) = \int_0^t p\,dt$$

$$= \int_0^t \frac{V_0^2}{R} e^{-2t/\tau}\,dt$$

$$= \frac{V_0^2}{R}\int_0^t e^{-2t/\tau}\,dt = \frac{V_0^2}{R}\left(\frac{e^{-2t/\tau}}{-2/\tau}\right)\Bigg|_0^t$$

$$= \frac{V_0^2\tau}{-2R}\left(e^{-2t/\tau}\Big|_0^t\right)$$

$$= \frac{-(RC)V_0^2}{2R}\left(e^{-2t/\tau}\Big|_0^t\right) \quad (\because \tau = RC)$$

$$\Rightarrow w_R(t) = \frac{1}{2} C V_0^2 \left(1 - e^{-2t/\tau}\right)$$

$$\therefore \boxed{w_R(t) = \frac{1}{2} C V_0^2 \left(1 - e^{-2t/\tau}\right)}$$

Note: As $t \to \infty \Rightarrow w_R(t) = \frac{1}{2} C V_0^2 (1 - 0)$

$$= \frac{1}{2} C V_0^2 = w_C(0)$$

$\Rightarrow$ As time becomes large, resistor dissipates the entire initial energy present in the capacitor.

Keypoints:

1. The initial voltage $v(0) = V_0$ across the capacitor

2. The time constant τ.

Source free RL circuit:

Consider a series combination of a resistor and an inductor as shown.

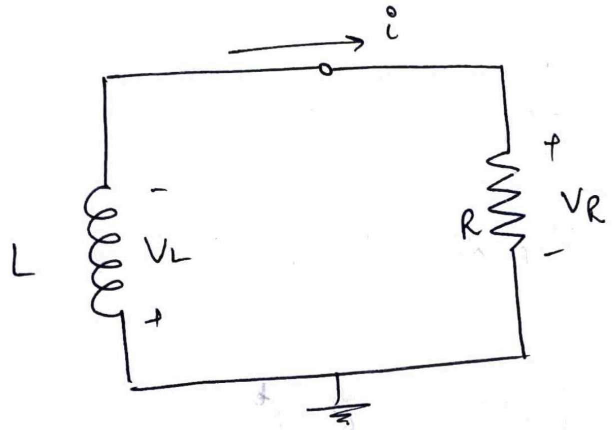


fig 1

Our goal is to determine the circuit response, here we consider the current through inductor as the required response.

- We assume that the inductor is having some initial current I_0.

$\therefore$ At $t=0 \Rightarrow i(0) = I_0$.

The corresponding energy stored in the inductor is,

$$w(0) = \frac{1}{2} L I_0^2$$

Apply KVL in circuit 1.

$$V_L + V_R = 0$$

$$\Rightarrow L\frac{di}{dt} + iR = 0$$

$$\Rightarrow \frac{di}{dt} + i\frac{R}{L} = 0$$

$$\Rightarrow \frac{di}{i} = -\frac{R}{L}\,dt$$

Integrating on bothsides.

$$\Rightarrow \int \frac{di}{i} = -\frac{R}{L}\int_0^t 1 \cdot dt$$

$$\Rightarrow \ln i = -\frac{R}{L}t + A' \quad \left(\text{where } A' \text{ is a constant. Let } A' = \ln A\right)$$

$$\Rightarrow \ln i = -\frac{R}{L}t + \ln A$$

$$\Rightarrow \ln i - \ln A = -\frac{R}{L}t$$

$$\Rightarrow \ln \frac{i}{A} = -\frac{R}{L}t \Rightarrow i = Ae^{-\frac{Rt}{L}} \longrightarrow (5)$$

We know that, from initial condition,

at $t=0 \Rightarrow i = I_0$

$$\Rightarrow i(0) = I_0 = Ae^{-\frac{R(0)}{L}}$$

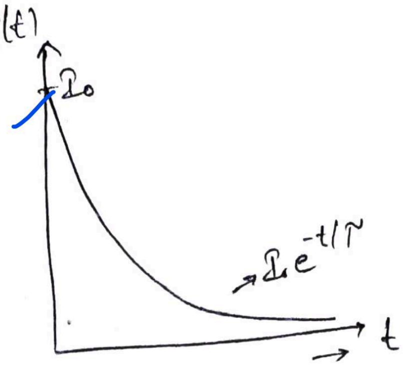
$$= Ae^{-0} = A(1)$$

$$\Rightarrow \boxed{A = I_0}$$

Now, from (5),

$$i(t) = A e^{-\frac{Rt}{L}}$$

$$\boxed{i(t) = I_0 e^{-\frac{R}{L}t}} \longrightarrow (6)$$



from eq (6),

we can write, time constant $\boxed{\tau = \frac{L}{R}.}$

∴ Eq (6) can be re-written as,

$$\boxed{i(t) = I_0 e^{-t/\tau}} \longrightarrow (7)$$

⇒ voltage across the resistor,

$$v_R(t) = iR$$
$$= I_0 R e^{-t/\tau}$$

& The power dissipated in the resistor,

$$P = v_R i = (I_0 R e^{-t/\tau})(I_0 e^{-t/\tau})$$

$$P = I_0^2 R e^{-2t/\tau}$$

& The energy absorbed by the resistor is,

$$w_R(t) = \int_0^t p\,dt = \int_0^t I_0^2 R e^{-2t/\tau}\,dt$$

$$= I_0^2 R \int_0^t e^{-2t/\tau} dt = I_0^2 R \left(\frac{e^{-2t/\tau}}{-2/\tau} \right) \Big|_0^t$$

$$= -\frac{1}{2} \tau I_0^2 R e^{-2t/\tau} \Big|_0^t$$

$$= -\frac{1}{2} \tau I_0^2 R (e^{-2t/\tau} - e^0)$$

$$= -\frac{1}{2} \tau I_0^2 R (e^{-2t/\tau} - 1)$$

$$= \frac{1}{2} \tau I_0^2 R (1 - e^{-2t/\tau})$$

$$= \frac{1}{2} \frac{L}{R} \times I_0^2 R (1 - e^{-2t/\tau}) \quad (\because \tau = \frac{L}{R})$$

$$w_R(t) = \frac{1}{2} L I_0^2 (1 - e^{-2t/\tau}).$$

Note:

As $t \to \infty$, $\Rightarrow w_R(t) = \frac{1}{2} L I_0^2 (1 - e^{-\infty})$

$$w_R(t) = \frac{1}{2} L I_0^2$$

$\Rightarrow$ As the time becomes large, resistor dissipates the entire initial energy present in the inductor

Key points:

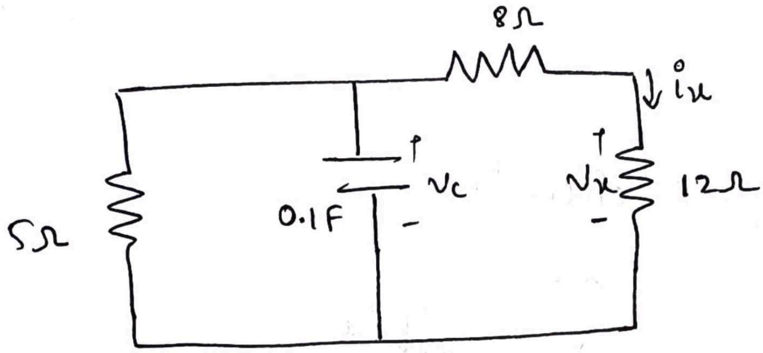
1) The initial current $i(0) = I_0$ through the inductor

2) The time constant τ of the circuit.

"Unit for time constant is sec."

Example problems :

Q1) find v_c, v_x and i_x, for $t>0$, if $v_c(0) = 15V$



Sol) firstly, find R_{eq}, C_{eq}.

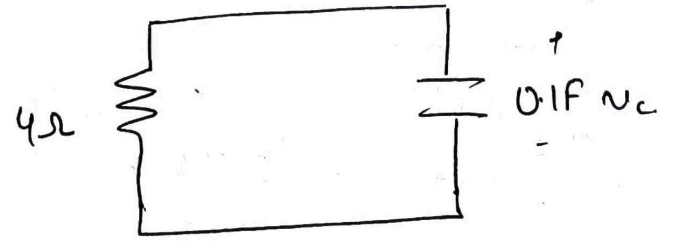
8Ω, 12Ω are in series, $\Rightarrow 8+12 = 20\,\Omega$

20Ω, 5Ω are in parallel $20 \parallel 5 \Rightarrow \frac{20\times 5}{20+5} = \frac{20\times 5}{25}$

$= 4\Omega$

$C_{eq} = C = 0.1F$

∴ Equivalent circuit becomes.



from this equivalent circuit, $\tau = R_{eq} C_{eq}$

$= 4 \times 0.1$

$= 0.4$ sec

Now,

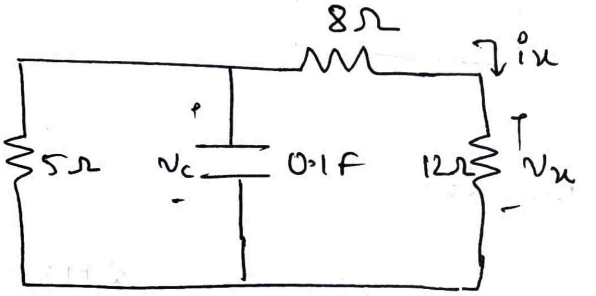
voltage across capacitor $v_c = V_0 e^{-t/RT}$

$= 15 e^{-t/0.4}$

$= 15 e^{-t \times \frac{10}{4}}$

$$\boxed{v = v_c = 15 e^{-2.5t}} \quad V$$

from the given circuit,



Apply voltage divider rule across the series combination of 8Ω, 12Ω to get the value of v_x.

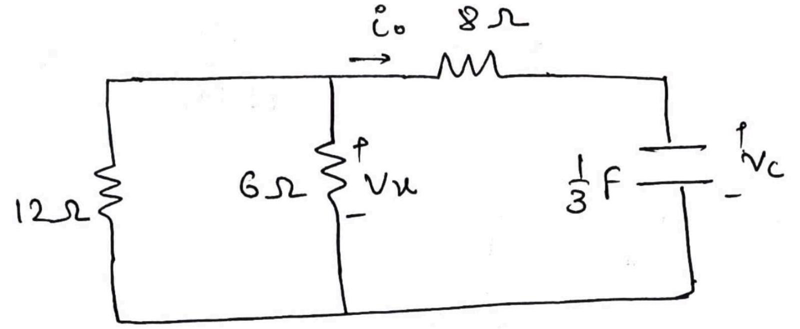
$$\therefore v_x = v_c \times \frac{12}{8+12}$$

$$= 15 e^{-2.5t} \times \frac{12}{20}$$

$$\boxed{v_x = 9 e^{-2.5t} \; V.}$$

$$i_x = \frac{v_x}{R} = \frac{9 e^{-2.5t}}{12} = \boxed{0.75 e^{-2.5t} \; A. = i_x}$$

Q2) Let $v_c(0) = 30V$, Determine v_c, v_x and i_o for $t \geq 0$



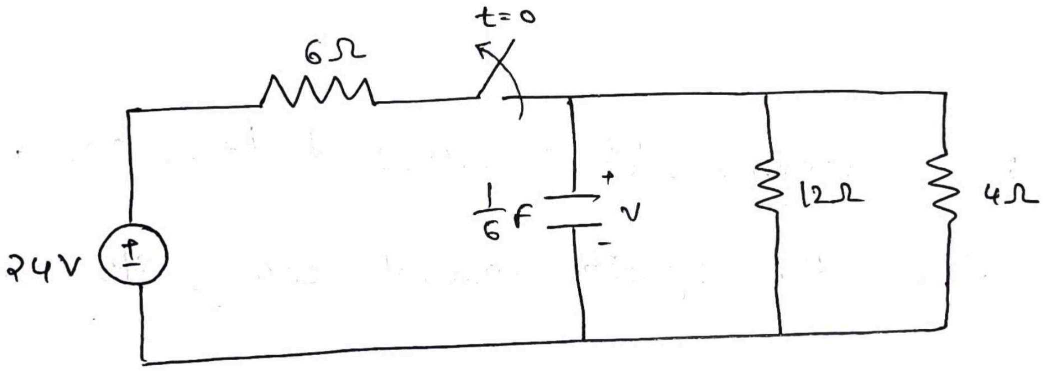
Ans:

$v_c = 30e^{-0.25t}$ V

$v_x = 10e^{-0.25t}$ V

$i_o = -2.5e^{-0.25t}$ A

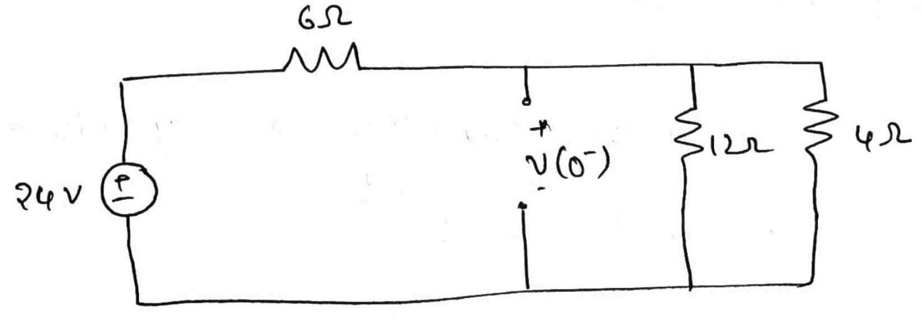
Q3) If the switch in the below circuit opens at $t=0$, find $v(t)$ for $t \geq 0$ and $w_c(0)$.



Sol) Initially switch is kept closed.

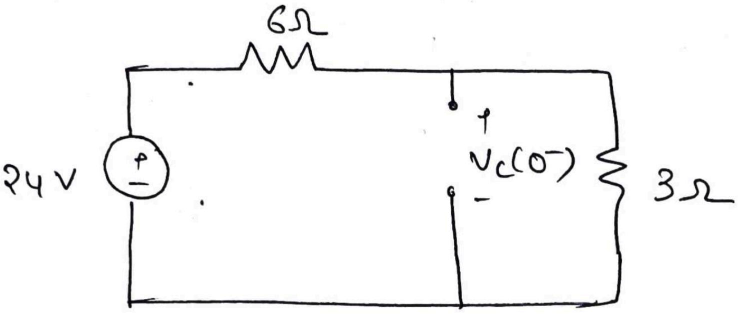
A capacitor behaves as an open circuit for dc

The circuit is like this for $t<0$.



$\therefore$ $V_c(0^-)$ is the voltage across parallel combination of $12, 4\,\Omega$

$$12 \,||\, 4 \Rightarrow \frac{12 \times 4}{12+4} = 3\,\Omega$$



$$V_c(0^-) = \frac{3}{6+3} \times 24 = \frac{3}{9} \times 24$$

$$= 8V$$

Now, when switch is closed at $t=0$, so, at $t=0^+$, since the capacitor cannot change its voltage instantaneously, $V_c(0^+) = V_c(0^-)$

$$= 8V.$$

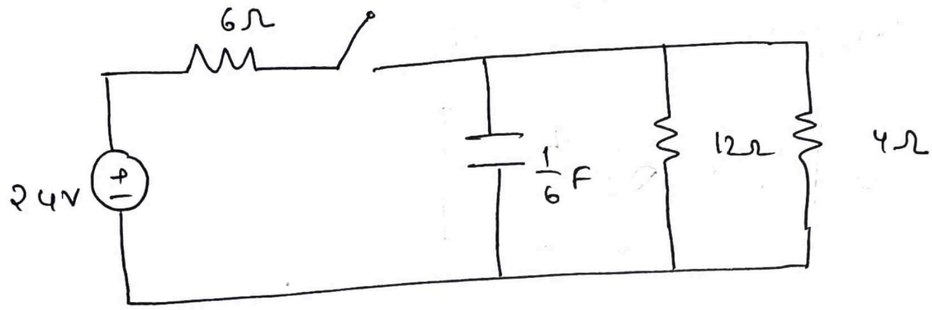
$\therefore$ Initial condition (or) initial voltage across capacitor $V(0) = 8V = V_0$.

$\therefore$ Voltage across capacitor at any instant $t \geq 0$ is $V_c(t) = V_0 e^{-\frac{t}{\tau}}$

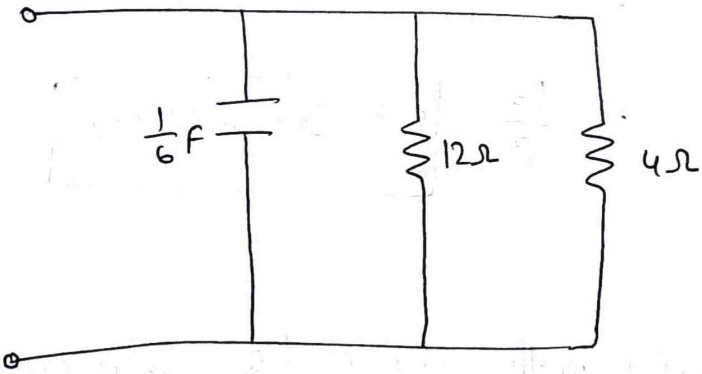
$$= V_0 e^{-t/RC}$$

$$V_c(t) = 8\, e^{-t/(RC)}. \rightarrow (1)$$

At $t=0$, switch is opened. So, for $t \geq 0$, the circuit will be as shown.



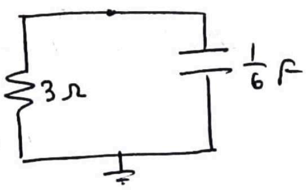
⇓



So, now, $R_{eq} = 12 \| 4 = \frac{12 \times 4}{12+4} = \frac{12 \times 4}{16} = 3\Omega$

$C_{eq} = C = \frac{1}{6} F$

∴ Equivalent ckt is



$\therefore \tau = RC = 3 \times \frac{1}{6} = \frac{1}{2}$ sec $= 0.5$ sec

Now, from ①,

$$V_c(t) = 8e^{-t/\tau}$$

$$= 8e^{-t/0.5}$$

$$= 8e^{-\frac{t\times 10}{5}}$$

$$V_c(t) = 8e^{-2t} \text{ V}$$

$$\Rightarrow \boxed{v(t) = 8e^{-2t} \text{ V}}$$

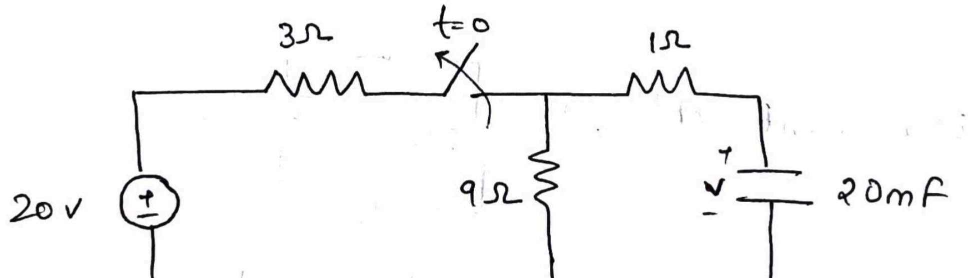
$$w_c(0) = \frac{1}{2} C V_0^2$$

$$= \frac{1}{2} \times \frac{1}{6} \times 8^2$$

$$= \frac{16}{3} = \boxed{5.33 \text{ J} = w_c(0)}$$

(Q4) The switch in the circuit has been closed for long time, it is opened at $t=0$. find $v(t)$ for $t \geq 0$.

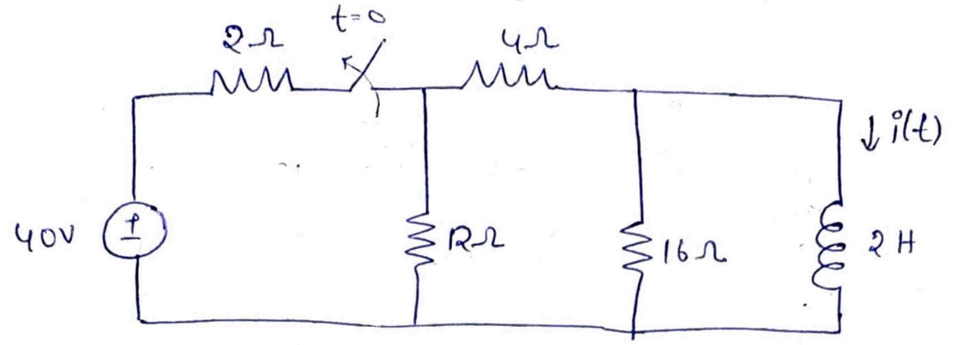
Calculate the initial energy stored in the capacitor



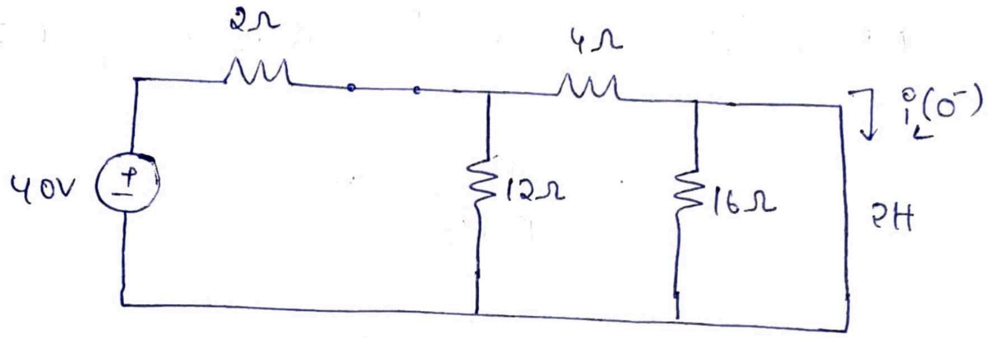
Ans: $v(t) = 15e^{-5t}$ V , $w_c(0) = 2.25$ J

Q5) The switch in the circuit has been closed for a long time.

At $t=0$, the switch is opened. Calculate $i(t)$ for $t>0$.

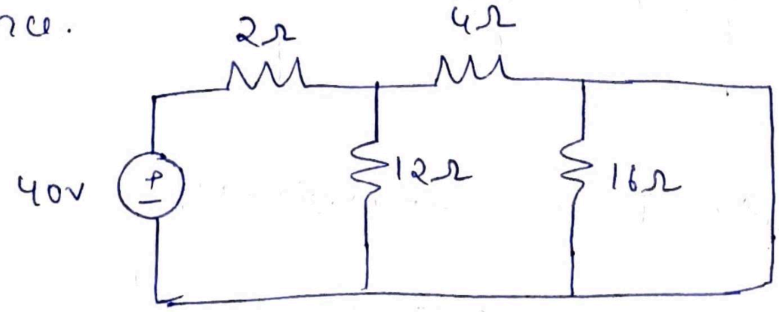


Sol) Before instant of switching, the ckt is as shown.



Inductor behaves as short circuit to dc. Hence, it is replaced by a short circuit in the above circuit

To find $i_L(0^-)$ ie current through inductor before switching, we have to find equivalent resistance.



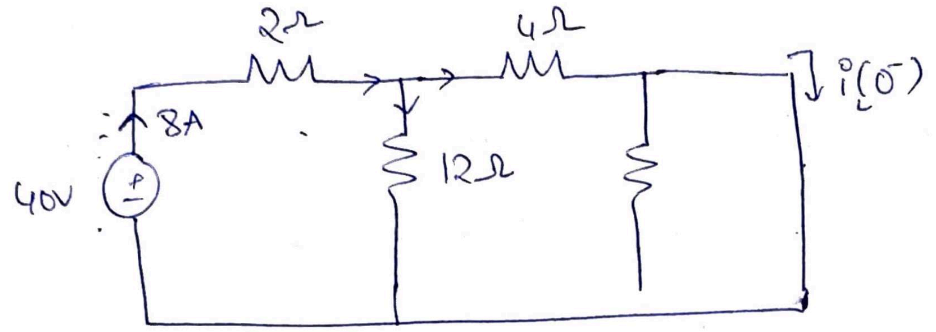
16Ω is neglected. , 4Ω, 12Ω are in parallel

$$= \frac{4\times 12}{4+12} = 3\Omega$$

3Ω, and 2Ω are in series.

$\therefore$ Total current supplied by the 40V source

is $\frac{40}{2+3} = \frac{40}{5} = 8A$.



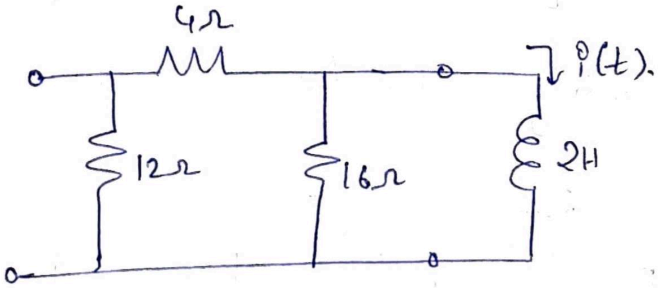
Now, $i_L(0^-)$ = Current flowing through 4Ω resistor

$$= \frac{12}{12+4} \times 8$$

$$= \frac{12}{16} \times 8$$

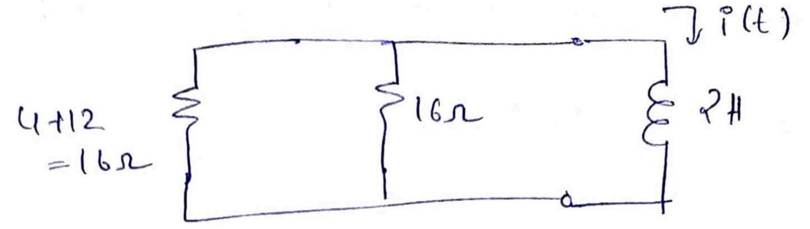
$$\boxed{i_L(0^-) = 6A.}$$

After instead of switching, the ckt is as shown.



for finding τ, we have to find R_{eq}

(R_{eq} across the inductor terminals)



$$R_{eq} = 16 \| 16 = 8\,\Omega$$

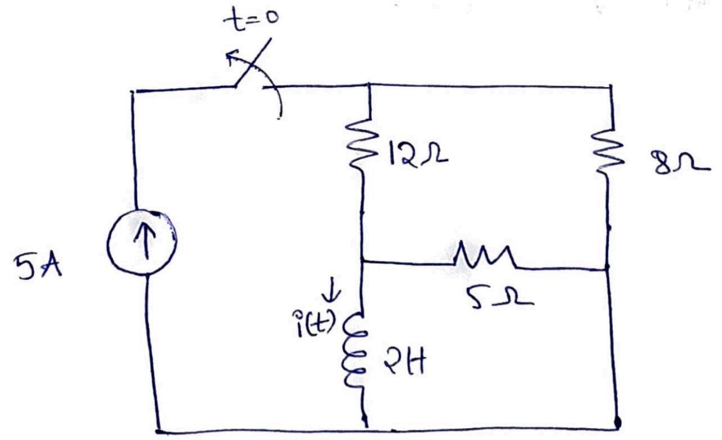
$$\therefore \tau = \frac{L}{R_{eq}} = \frac{2}{8} = \frac{1}{4} \text{ sec}$$

$$\therefore i(t) = i_L(t) = i(0)\, e^{-t/\tau}$$

$$= 6\, e^{-t/\frac{1}{4}}$$

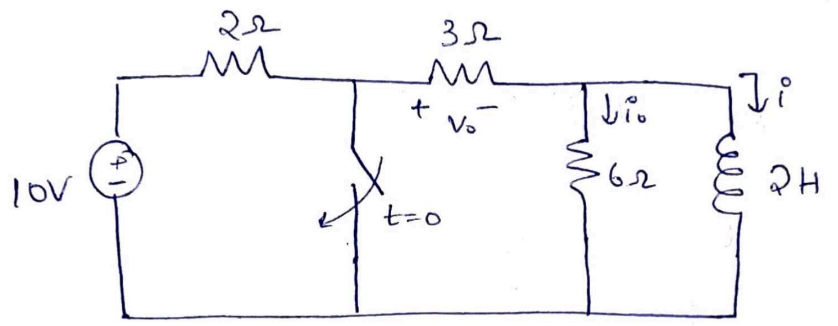
$$= 6 e^{-4t} \text{ A}$$

Q6) for the circuit shown, find $i(t)$, for $t > 0$.



Ans, $2e^{-2t}$ A , $t > 0$.

Q7) find i_o, V_o and i for all time, assuming that the switch was open for a long time.



Ans: $i_o(t) = \begin{cases} 0A, & t<0 \\ -\frac{2}{3}e^{-t}\,A, & t>0 \end{cases}$

$i(t) = \begin{cases} 2A, & t<0 \\ 2e^{-t}\,A, & t>0 \end{cases}$

$V_o(t) = \begin{cases} 6V, & t<0 \\ 4e^{-t}\,V, & t>0. \end{cases}$

Forced response of RC circuit:

i, Impulse function

ii, Step function

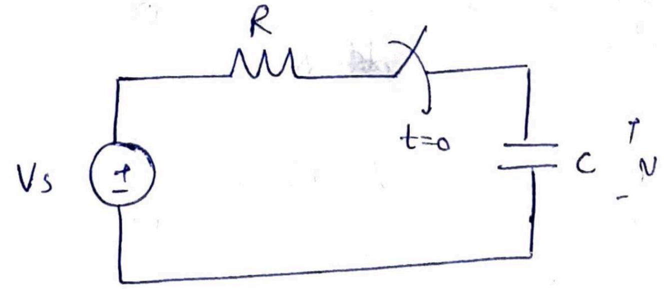
iii, Sinusoidal function.

Step response of an RC circuit:

When a dc source is suddenly applied to RC circuit, the voltage (or) current can be modelled as a step function & the response is called as step response.

"The step response of a circuit is its behaviour when the excitation is the step function, which may be a voltage (or) a current source."

Consider an RC ckt as shown,



V_s is a constant dc source

V_0 is the initial voltage across capacitor.

The capacitor voltage is taken as the circuit response to be determined.

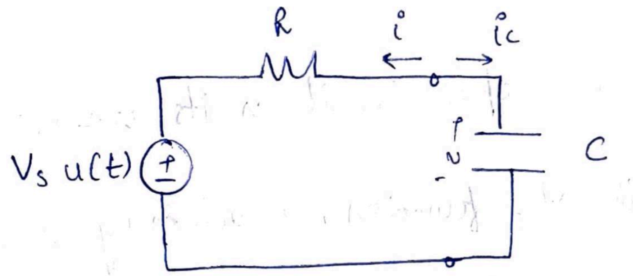
- Since the voltage across capacitor cannot be changed instantly for sudden changes in supply,

$$v_c(0^-) = v_c(0^+) = V_0.$$

$v_c(0^-)$ – voltage across capacitor just before switching

$v_c(0^+)$ – voltage " " " after ".

After closing of switch, the ckt will be as shown.



(v – voltage across capacitor)

Apply KCL at the node,

$$C\frac{dv}{dt} + \frac{v - V_s u(t)}{\cancel{dt}\, R} = 0$$

$$\Rightarrow \frac{dv}{dt} + \frac{v - V_s u(t)}{RC} = 0$$

$$\Rightarrow \frac{dv}{dt} + \frac{v}{RC} - \frac{V_s}{RC} u(t) = 0$$

$$\Rightarrow \frac{dv}{dt} + \frac{v}{RC} = \frac{V_s}{RC} u(t)$$

$$\left\{ \begin{array}{ll} u(t) = 1 & \text{for } t > 0 \\ \quad = 0 & \text{for } t < 0 \end{array} \right\}$$

So, for $t > 0$, the above eq becomes.

$$\frac{dv}{dt} + \frac{v}{RC} = \frac{V_s}{RC}.$$

$$\Rightarrow \frac{dv}{dt} = \frac{V_s - v}{RC}$$

$$\Rightarrow \frac{dv}{V_s - v} = \frac{dt}{RC}$$

Integrating on bothsides

$$\Rightarrow \int_{V_0}^{v(t)} \frac{dv}{V_s - v} = \int_0^t \frac{dt}{RC}$$

$$\Rightarrow -1 \times \left(\ln (V_s - v) \right) \Big|_{V_0}^{v(t)} = + \frac{t}{RC} \Big|_0^t$$

$$\Rightarrow \ln (V_s - v) \Big|_{V_0}^{v(t)} = -\left(\frac{t}{RC}\right) \Big|_0^t$$

$$\Rightarrow \ln (V_s - v(t)) - \ln (V_s - V_0) = -\frac{t}{RC}$$

$$\Rightarrow \ln\left(\frac{V_s - v(t)}{V_s - V_o}\right) = -\frac{t}{RC}$$

$$\Rightarrow \frac{V_s - v(t)}{V_s - V_o} = e^{-t/RC}$$

$$\Rightarrow \frac{v(t) - V_s}{V_o - V_s} = e^{-t/RC}$$

$$\Rightarrow v(t) - V_s = (V_o - V_s)\, e^{-t/RC}$$

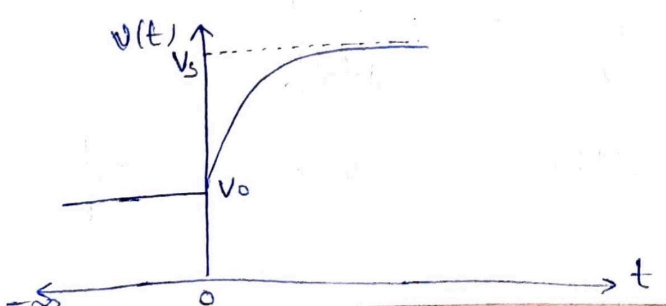
$$\Rightarrow v(t) = V_s + (V_o - V_s)\, e^{-t/RC}$$

$$\Rightarrow v(t) = V_s + (V_o - V_s)\, e^{-t/\tau}, \quad (\because \tau = RC) \rightarrow (1)$$

$$t > 0.$$

$$\therefore v(t) = \begin{cases} V_o & , t < 0 \\ V_s + (V_o - V_s)\, e^{-t/\tau} & , t > 0. \end{cases}$$

This is the complete response of RC circuit to a sudden application of a dc voltage source, when capacitor is assumed to be charged initially



If the capacitor is uncharged initially, then $V_o = 0V$

$$\therefore \quad v(t) = \begin{cases} 0 \,, & t<0 \\ V_s(1-e^{-t/\tau}) \,, & t>0 . \end{cases}$$

from this, $i(t)$, current through capacitor can be calculated, $i(t) = C\dfrac{dv(t)}{dt}$

$$= C \frac{d}{dt}\left(V_s(1-e^{-t/\tau})\right)$$

$$= C\left(0 - (V_s e^{-t/\tau}) \times \left(\frac{-1}{\tau}\right)\right)$$

$$= \frac{C \cdot V_s e^{-t/\tau}}{\tau}$$

$$= \frac{C V_s}{RC} \cdot e^{-t/\tau}$$

$$= \frac{V_s}{R} e^{-t/\tau}$$

$$\Rightarrow i(t) = \frac{V_s}{R} e^{-t/\tau} \cdot u(t)$$

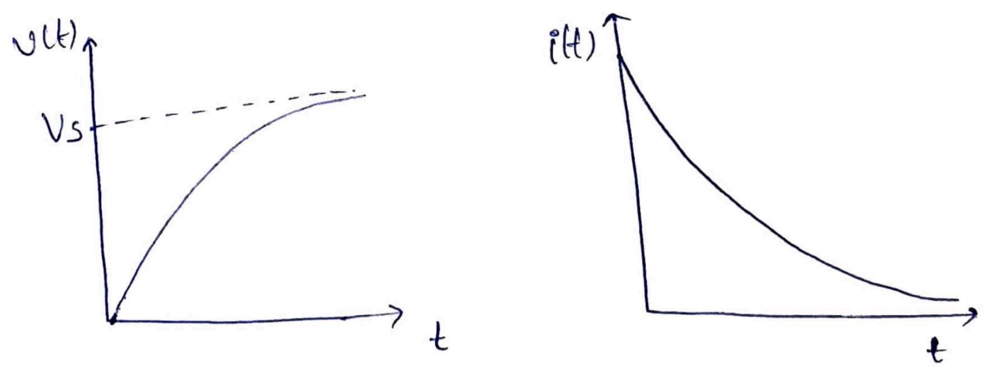


fig: voltage & current waveforms through uncharged capacitor.

from ①,

The complete response of the circuit, is the sum of natural response and forced response,

So, eq ① can also be written as,

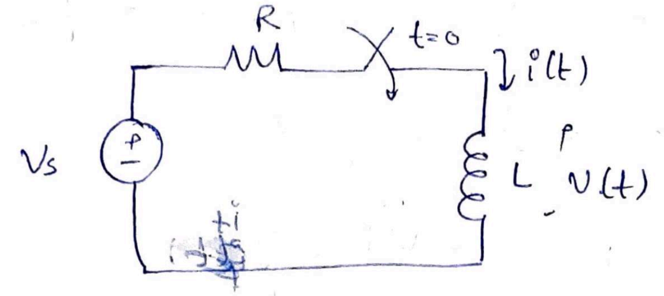
$$v(t) = v(\infty) + [v(0) - v(\infty)]\, e^{-t/\tau}$$

where $v(0)$ is the initial voltage at $t=0^+$ and $v(\infty)$ is the final or steady state value //

Forced response of RL circuit :–

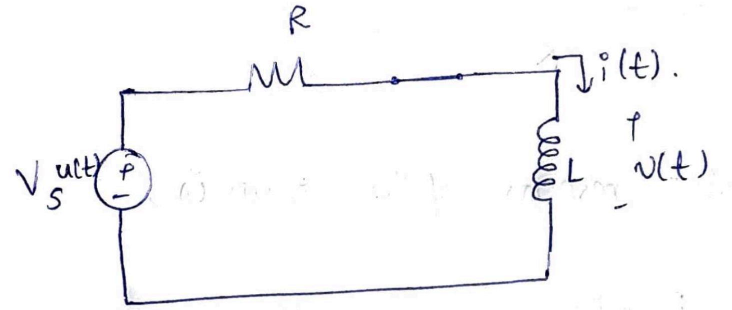
Step response of RL circuit:

Consider an RL circuit as shown,



Switch is closed at $t=0$,

then circuit is as shown.



Here, the circuit response is studied in terms of inductor current.

As it is already known, that circuit response is the sum of natural and forced responses.

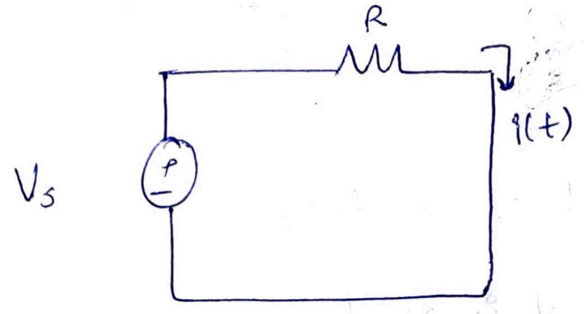
$\therefore \; i(t) = i_n + i_f$ (i_n - natural response current

i_f - forced response current)

- As, natural response is always a decaying exponential, $i_n = A e^{-t/\tau}$, here $\tau = \frac{L}{R}$

- forced response is the circuit's response after long time after switching. As the inductor behaves as short circuit

$\therefore$ ~~So~~, long time after the application of a dc source, the circuit is as shown, for $t \to \infty$.



$$\therefore i_f = \frac{V_s}{R}$$

$\therefore$ Complete response of the circuit is,

$$i(t) = i_n + i_f$$

$$= A e^{-t/\tau} + \frac{V_s}{R} \longrightarrow (1)$$

To find out the value of constant A, we require initial condition i.e., $i(0)$ of inductor.

Let $i(0) = I_0$, the initial current through inductor

At $t=0$, $i(0) = I_0$

$$\Rightarrow A e^{-0/\tau} + \frac{V_s}{R} = I_0$$

$$\Rightarrow A(1) = I_0 - \frac{V_s}{R}$$

$\Rightarrow A = I_0 - \frac{V_s}{R}$

Substitute it in eq ①,

$$i(t) = \left(I_0 - \frac{V_s}{R}\right) e^{-t/\tau} + \frac{V_s}{R}$$

$$\boxed{i(t) = \frac{V_s}{R} + \left(I_0 - \frac{V_s}{R}\right) e^{-t/\tau}}$$

This is the complete response of RL circuit.

It can be re-written as,

$$\boxed{i(t) = i(\infty) + \left[i(0) - i(\infty)\right] e^{-t/\tau}}$$

where $i(0)$ - Initial values of i i.e., current through inductor.

$i(\infty)$ - final value of i i.e, current through inductor.

If $I_0 = 0$ i.e., there is no initial energy stored in inductor,

then $i(t) = \begin{cases} 0 \,, & t < 0 \\ \frac{V_s}{R}\left(1 - e^{-t/\tau}\right) \,, & t > 0. \end{cases}$

or simply,

$$i(t) = \frac{V_s}{R}\left(1 - e^{-t/\tau}\right) \cdot u(t).$$

Now, voltage across inductor,

$$V(t) = L\frac{di}{dt}$$

$$= L\frac{d}{dt}\left(\frac{V_s}{R}(1-e^{-t/\tau})\right)$$

$$= L\left(0 - \frac{V_s}{R}e^{-t/\tau} \times -\frac{1}{\tau}\right)$$

$$= \frac{V_s L}{R\tau}e^{-t/\tau}$$

$$= \frac{V_s L}{R \times \frac{L}{R}}e^{-t/\tau}$$

$$= V_s e^{-t/\tau} \cdot u(t).$$

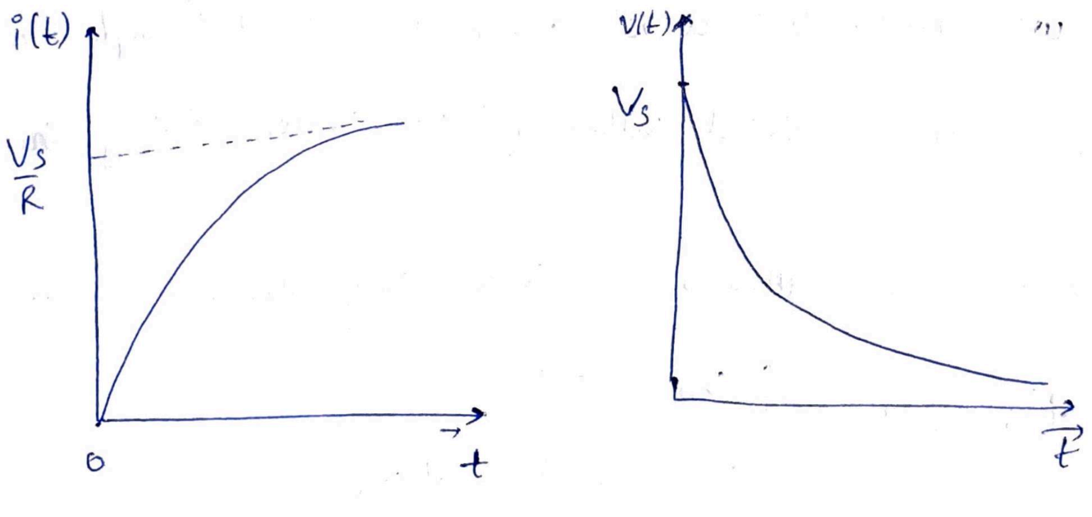
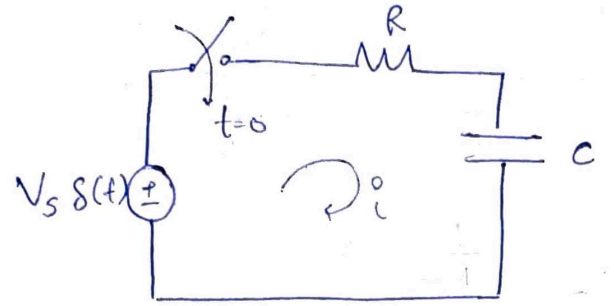


fig: Current & voltage across inductor with no initial energy stored.

Impulse response of RC and RL circuit:

RC circuit:

Consider a series RC circuit as shown.



This series RC circuit is excited by an impulse function at $t=0$.

Apply KVL in the circuit,

$$V_s\,\delta(t) = i(t)\,R + \frac{1}{C}\int i(t)\,dt$$

To solve this equation, take laplace transform on bothsides,

$$V_s\,(1) = I(S)\cdot R + \frac{1}{C}\,\frac{I(S)}{S}$$

$$\Rightarrow V_s = \left(R + \frac{1}{CS}\right) I(S)$$

$$= \frac{R}{S}\left(S + \frac{1}{RC}\right) I(S)$$

$$\Rightarrow I(S) = \frac{V_s}{R} S \times \frac{1}{S + \frac{1}{RC}}$$

$$\Rightarrow I(s) = \frac{V}{R} \cdot \frac{s}{s+\frac{1}{RC}}$$

$$= \frac{V}{R} \cdot \frac{s+\frac{1}{RC}-\frac{1}{RC}}{s+\frac{1}{RC}}$$

$$= \frac{V}{R} \cdot \left(\frac{s+\frac{1}{RC}}{s+\frac{1}{RC}} - \frac{\frac{1}{RC}}{s+\frac{1}{RC}} \right)$$

$$\Rightarrow I(s) = \frac{V}{R}\left(1 - \frac{\frac{1}{RC}}{s+\frac{1}{RC}}\right)$$

$$\Rightarrow I(s) = \frac{V}{R} - \frac{V}{R^2C} \times \left(\frac{1}{s+\frac{1}{RC}}\right)$$

Take inverse laplace transform $\Rightarrow$ $i(t) = \frac{V}{R}\delta(t) - \frac{V}{R^2C} e^{-\frac{1}{RC}\cdot t}$

$$\left[\because L^{-1}\left\{\frac{1}{s+a}\right\} = e^{-at} \qquad L^{-1}\{1\} = \delta(t) \right]$$

$$\Rightarrow \boxed{i(t) = \frac{V}{R}\delta(t) - \frac{V}{R\tau} e^{-t/\tau}} \text{ A}$$

$(\because \tau = RC)$

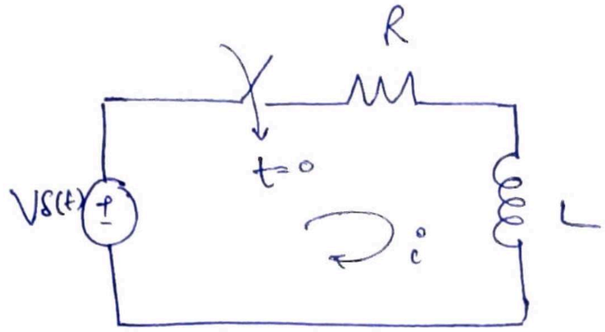
Now,

voltage across capacitor, $v(t) = \frac{1}{C}\int i(t)\,dt$

$$\Rightarrow v(t) = \frac{1}{C}\int\left(\frac{V}{R}\delta(t) - \frac{V}{R\tau}e^{-t/\tau}\right)dt$$

RL circuit:

Consider a series RL circuit as shown.



This series RL circuit is excited by an impulse function at $t=0$.

Apply KVL in the above circuit.

$$V\,\delta(t) = i(t)\cdot R + L\,\frac{di(t)}{dt}$$

Taking laplace transform on bothsides.

$$V(1) = I(S)\cdot R + L\,S\,I(S)$$

$$V = (R+SL)\,I(S)$$

$$\Rightarrow I(S) = \frac{V}{R+SL}$$

$$= \frac{1}{L}\,\frac{V}{S+\frac{R}{L}}$$

Taking inverse laplace transform,

$$i(t) = \frac{V}{L}\,e^{-\frac{R}{L}t} = \frac{V}{L}\,e^{-t/\tau} \quad (\because \tau = \tfrac{L}{R})$$

$$\therefore \boxed{i(t) = \frac{\varepsilon}{L} e^{-t/\tau}} \quad A$$

Voltage across inductor,

$$v(t) = L \frac{di(t)}{dt}$$

$$= L \frac{d}{dt}\left(\frac{\cancel{\varepsilon} V}{L} e^{-t/\tau}\right)$$

$$= L \times \frac{V}{L} \frac{d}{dt} e^{-t/\tau}$$

$$= V e^{-t/\tau} \times \left(-\frac{1}{\tau}\right)$$

$$= -\frac{V}{\tau} e^{-t/\tau}$$

$$= -\frac{V}{\frac{L}{R}} e^{-t/\tau}$$

$$\boxed{v(t) = \frac{-VR}{L} e^{-t/\tau}}$$

These are the equations for voltage across & current through inductor

Summary :

Source free response -

RC ckt - $v(t) = V_0 e^{-t/\tau}$; $\tau = RC$

RL ckt - $i(t) = I_0 e^{-t/\tau}$; $\tau = \frac{L}{R}$

forced response.

Step function as input:

RC:

$$v(t) = V_s + (V_0 - V_s) e^{-t/\tau} \quad ; t > 0$$

$$= V_0 \quad ; t < 0.$$

If $V_0 = 0$, then,

$$v(t) = V_s \left(1 - e^{-t/\tau}\right) \quad ; t > 0$$

$$= 0 \quad ; t < 0$$

RL:

$$i(t) = \frac{V_s}{R} + \left(I_0 - \frac{V_s}{R}\right) e^{-t/\tau} \quad ; t > 0$$

$$= I_0 \quad ; t < 0$$

If $I_0 = 0$,

$$i(t) = \frac{V_s}{R}\left(1 - e^{-t/\tau}\right) \quad ; t > 0$$

$$= 0 \quad ; t < 0$$

Problems

Q1) A resistance R and $5\mu F$ capacitor are connected in series across a 100V dc supply. Calculate the value of R such that the voltage across the capacitor becomes 50V in 5 sec after the circuit is switched on.

Sol) Given,

$C = 5\mu F$

$V_s = 100\,V$

$v(t) = 50V$

$\tau = RC$

$= R \times 5\mu F$

$t = 5\,sec$

from,

$$v(t) = V_s(1 - e^{-t/\tau})$$

$$\Rightarrow 50 = 100\left(1 - e^{-t/5\mu \times R}\right)$$

$$\Rightarrow \frac{50}{100} = 1 - e^{\frac{-t}{5\mu \times R}}$$

$$\Rightarrow 0.5 = 1 - e^{-t/5\mu \times R}$$

$$\Rightarrow e^{\frac{-t}{5\mu \times R}} = 1-0.5 = 0.5$$

$$\Rightarrow -\frac{t}{5\mu \times R} = \ln 0.5$$

$$\Rightarrow \frac{-5}{5\mu \times R} = \ln 0.5$$

$$\Rightarrow R = \frac{-1}{10^{-6} \times \cancel{R} \times \ln 0.5}$$

$$\boxed{R = 1.44\ M\Omega}$$

Q2) Calculate the time taken by a capacitor of $1\mu F$ and in series with a $1M\Omega$ resistance to be charged upto 80% of the final value.

Sol) Given, $C = 1\mu F$

$R = 1\ M\Omega$

$\tau = RC = 1\mu \times 1M$

$= 1$ sec

$v(t) = (80\%)\ V_S$

$\therefore v(t) = V_s(1-e^{-t/RC})$

$0.8\ V_s = V_s(1-e^{-t/1})$

$0.8 = 1-e^{-t} \Rightarrow e^{-t} = 1-0.8 = 0.2$

$\Rightarrow -t = \ln 0.2$

$\Rightarrow \boxed{t = 1.61\ \text{sec}}$

Q3) A $10\mu F$ capacitor is initially charged to 100 V d.c. It is then discharged through a resistance R ohms for 20 seconds when the potential difference across the capacitor is 50 V. Calculate the value of R.

Sol) Given, $C = 10\mu F$

$V_0 = 100V$

$t = 20 sec.$

$v(t) = 50V$

from, $v(t) = V_0 (e^{-t/\tau})$

$$\Rightarrow 50 = 100 (e^{-t/\tau})$$

$$\Rightarrow \frac{50}{100} = e^{-20/R \times 10\mu}$$

$$\Rightarrow \frac{1}{2} = e^{-20/(R \times 10\mu)}$$

$$\Rightarrow \frac{-20}{R \times 10\mu} = \ln 0.5$$

$$\Rightarrow R = \frac{-20}{10\mu \times \ln 0.5}$$

$$\boxed{R = 2.89\ M\Omega}$$

Q4) A dc constant voltage source feeds a resistance of 2000kΩ in series with a 5μF capacitor. find the time taken for the capacitor when the charge retained will be decayed to 50% of the initial value, the voltage source being short circuited. (Ans: $t = 6.94$ sec)

Sol)

$$v(t) = V_0 e^{-t/\tau}$$

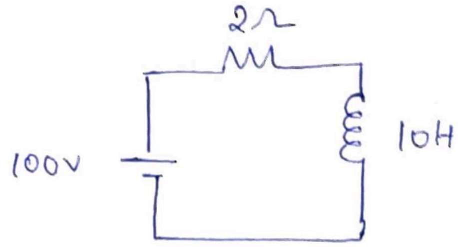
$$\Rightarrow C\,q(t) = C Q_0 e^{-t/\tau}$$

$$\Rightarrow \boxed{q(t) = Q_0 e^{-t/\tau}} \quad c$$

Q5) find the current in a series RL circuit having $R = 2\Omega$ and $L = 10H$ while a dc voltage of 100V is applied. what is the value of this current after 5 sec of switching on?

Sol)

$$\boxed{i(t) = I(\infty) + (I(0) - I(\infty))\, e^{-t/\tau}}$$



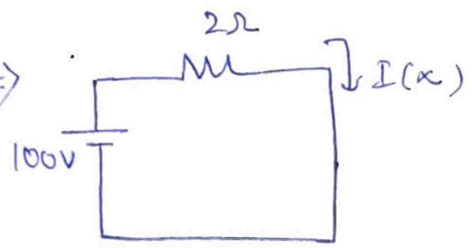
Given, $R = 2\Omega$

$L = 10H$

$V_s = 100V$

$(t = 5\ sec)$

$I(\infty) \Rightarrow$



$$I(\infty) = \frac{100}{2} = 50A$$

$I(0) = 0$ → since nothing is specified about initial condition of inductor

$$\tau = \frac{L}{R}$$

$$= \frac{10}{2}$$

$$= 5 \text{ sec}$$

$$i(t) = 50 + (0 - 50) e^{-t/5}$$

$$\boxed{i(t) = 50\left(1 - e^{-t/5}\right) A}$$

Now,

Value of current after 5 sec of switching,

$$i(5) = 50\left(1 - e^{-5/5}\right)$$

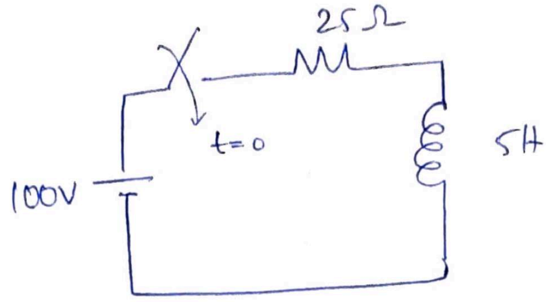
$$\boxed{i(5) = 31.6 A}$$

Q6) A series RL circuit has $R = 25\Omega$ and $L = 5H$. A dc voltage of 100V is applied at t=0. find (a) the equations for charging current, voltage across R and L, and (b) the current in the circuit 0.5 seconds later, (c) the time at which the drop across R and L are same.

Sol) Given, $R = 25\Omega$

$L = 5H$

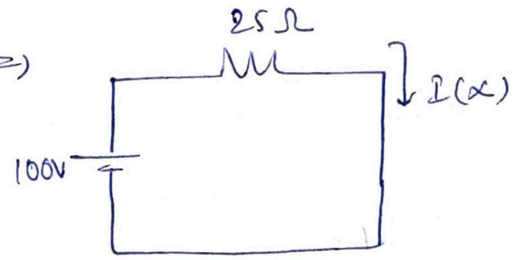
$V_s = 100V$



$i(t) = I(\infty) + (I(0) - I(\infty))\, e^{-t/\tau}$

$I(0) = 0$ (as nothing is specified about initial condition of inductor)

$I(\infty) \Rightarrow$



$I(\infty) = \dfrac{100}{25} = 4\ A$

$\tau = \dfrac{L}{R} = \dfrac{5}{25} = \dfrac{1}{5}$

$\therefore \rightarrow i_L(t) = 4 + (0 - 4)\, e^{-t/\frac{1}{5}}$

$\boxed{i_L(t) = 4(1 - e^{-5t})}$ A

$\rightarrow$ $\boxed{i_R(t) = i_L(t) = 4(1 - e^{-5t})}$ A

$\rightarrow V_L(t) = L \dfrac{di_L(t)}{dt}$

$= 5 \dfrac{d}{dt}\left(4(1 - e^{-5t})\right)$

$$= 5\left(-4 \times -5 \times e^{-5t}\right)$$

$$\boxed{V_L(t) = 100\, e^{-5t}} \ \text{V}$$

$$V_R(t) = i_R(t) \times R$$

$$= 25\left(4\left(1-e^{-5t}\right)\right)$$

$$\boxed{V_R(t) = 100\left(1-e^{-5t}\right)} \ \text{V}$$

ii, $i(t) = i_L(t) = i_R(t) = 4\left(1-e^{-5t}\right)$

at $t = 0.5 \Rightarrow i(0.5) = 4\left(1-e^{-5\times0.5}\right)$

$$= 4\left(1-e^{-2.5}\right)$$

$$\boxed{i(0.5) = 3.67\ A}$$

iii, $V_L = V_R$

$$\Rightarrow 100\, e^{-5t} = 100\left(1-e^{-5t}\right)$$

$$\Rightarrow e^{-5t} = 1 - e^{-5t}$$

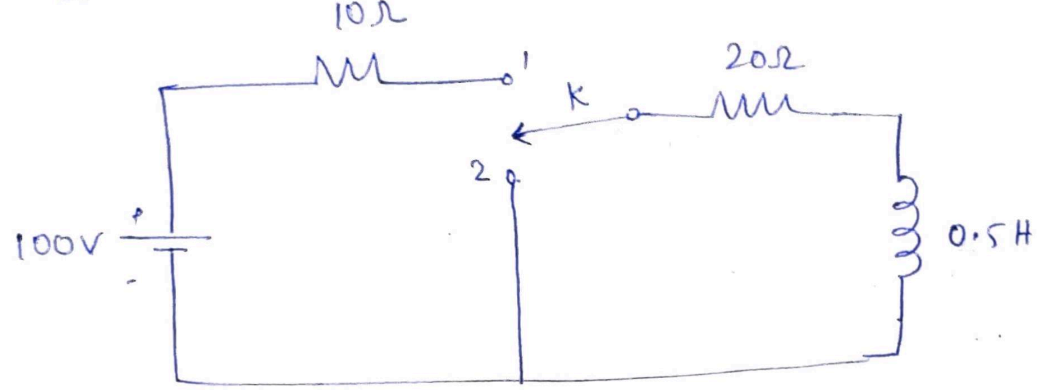
$$\Rightarrow 2e^{-5t} = 1$$

$$\Rightarrow e^{-5t} = \frac{1}{2} = 0.5$$

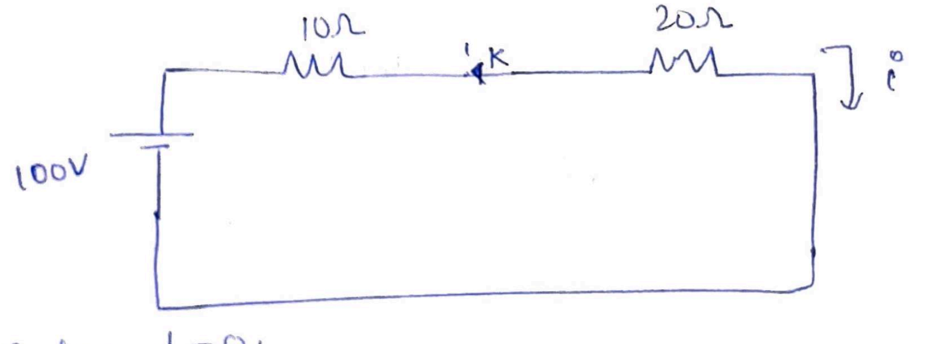
$$\Rightarrow -5t = \ln 0.5 = -0.693$$

$$\Rightarrow \boxed{t = 0.139\ \text{sec}}$$

Q7) In the circuit, switch K is kept first at position 1 and steady state condition is reached. At $t=0$, the switch is moved to position 2. Find the current in both the cases.



Sol) Initially, switch is at position 1, and steady state is reached. So, circuit will be as shown.

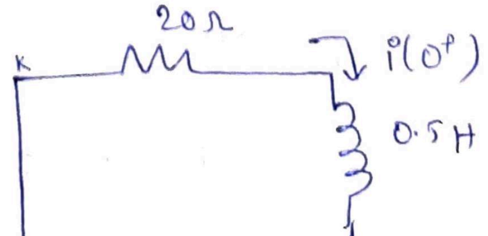


Before $t=0$,

$$i(0^-) = \frac{100}{20+10} = \frac{10}{3} A$$

At $t=0$, switch K is moved to position 2

So, the circuit will be as shown,



After switching, because, the inductor is having inertia to current changes,

$$i(0^+) = i(0^-) = \frac{10}{3} A$$

~~As $t \to \infty$, the circuit (when switch is at position 2) ~~will be~~ ~~becomes~~ as shown,~~

As $t \to \infty$, all the energy in the inductor will be dissipated through the resistor and finally it reaches to 0 J $\Rightarrow$ current flowing in the circuit = 0 A

$$\Rightarrow I(\infty) = 0 A$$

$$\therefore i(t) = I(\infty) + (I(0) - I(\infty)) e^{-t/\tau}$$

$$I(\infty) = 0 A$$

$$I(0) = I(0^+) = \frac{10}{3} A$$

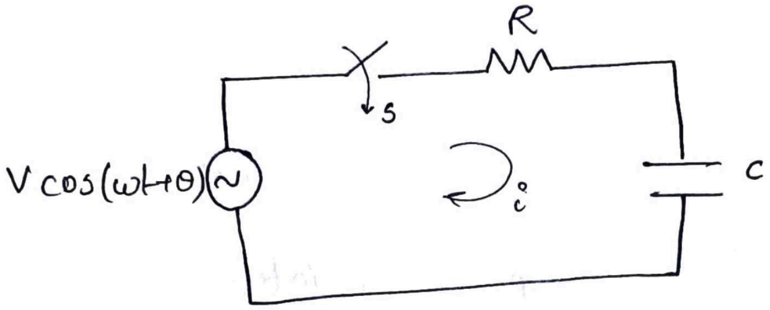
$$\tau = \frac{L}{R} = \frac{0.5}{20} = \frac{5}{200} = \frac{1}{40} \text{ sec}$$

$$\therefore i(t) = 0 + \left(\frac{10}{3} - 0\right) e^{-t/\frac{1}{40}}$$

$$\boxed{i(t) = \frac{10}{3} e^{-40t}} A$$

Sinusoidal response of RC circuit:

Consider a circuit consisting of resistance and capacitance in series as shown.



The switch S is closed at $t=0$.

· At $t=0$, a sinusoidal voltage $V\cos(\omega t+\theta)$ is applied to the RC circuit, where V is the amplitude of the wave

θ is the phase angle.

Apply KVL in the above circuit,

$$V\cos(\omega t+\theta) = Ri + \frac{1}{C}\int i\,dt$$

Apply differentiation on bothsides,

$$\Rightarrow -\omega V\sin(\omega t+\theta) = R\frac{di}{dt} + \frac{1}{C}i.$$

$$\Rightarrow -\omega V\sin(\omega t+\theta) = RDi + \frac{i}{C}$$

Dividing by R on bothsides

$$\Rightarrow -\frac{\omega V \sin(\omega t+\theta)}{R} = Di + \frac{1}{RC} i$$

$$\Rightarrow \left(D + \frac{1}{RC}\right) i = -\frac{V\omega}{R} \sin(\omega t+\theta) \longrightarrow (1)$$

The solution for above differential equation, consists of complementary function (i_c) and particular integral (i_p).

→ The complementary function, $i_c = ce^{-t/RC}$

→ The particular solution can be obtained by using undetermined co-efficients.

$$i_p = A\cos(\omega t+\theta) + B\sin(\omega t+\theta) \longrightarrow (2)$$

$$i_p' = -\omega A \sin(\omega t+\theta) + B\omega \cos(\omega t+\theta) \longrightarrow (3)$$

$$\left\{ \left(\text{ie } \frac{di_p}{dt} \right) \right\}$$

from (1),

$$\Rightarrow \left(D + \frac{1}{RC}\right) i = -\frac{V\omega}{R} \sin(\omega t+\theta)$$

$$\Rightarrow i' + \frac{i}{RC} = -\frac{V\omega}{R} \sin(\omega t+\theta)$$

$$\Rightarrow \left\{ -\omega A \sin(\omega t+\theta) + B\omega \cos(\omega t+\theta) \right\} + \frac{1}{RC} \left\{ A\cos(\omega t+\theta) + B\sin(\omega t+\theta) \right\} = -\frac{V\omega \sin(\omega t+\theta)}{R}$$

$(\because$ from (2), (3)$)$

from bothsides, comparing $\sin(\omega t+\theta)$ and $\cos(\omega t+\theta)$ terms,

we get,

$$-A\omega + \frac{B}{RC} = -\frac{V\omega}{R} \longrightarrow (4)$$

and $$B\omega + \frac{A}{RC} = 0 \longrightarrow (5)$$

Solving eq (4) and (5), and so writing the values of A, B,

$$A = \frac{VR}{R^2 + \left(\frac{1}{\omega C}\right)^2}$$

$$B = \frac{-V}{\omega C\left[R^2 + \left(\frac{1}{\omega C}\right)^2\right]}$$

Substitute the values of A, B in eq (2), (3),

from eq (2),

$$i_p = \frac{VR}{R^2+\left(\frac{1}{\omega C}\right)^2}\cos(\omega t+\theta) + \frac{(-V)}{\omega C\left[R^2+\left(\frac{1}{\omega C}\right)^2\right]}\sin(\omega t+\theta)$$

Let $$M\cos\phi = \frac{VR}{R^2+\left(\frac{1}{\omega C}\right)^2} \longrightarrow (6)$$

and $$M\sin\phi = \frac{V}{\omega C\left[R^2+\left(\frac{1}{\omega C}\right)^2\right]} \longrightarrow (7)$$

From ⑥, ⑦ , solving for M, ϕ.

To find M and ϕ, divide eq ⑥ ~~by~~ and eq ⑦.

$$\cancel{\frac{M\sin\phi}{M\cos\phi} = \tan\phi = \frac{VR}{\frac{V}{\omega C}}}$$

$$=$$

$$\Rightarrow \quad \frac{M\sin\phi}{M\cos\phi} = \frac{\frac{V}{\omega C}}{VR}$$

$$\Rightarrow \quad \tan\phi = \frac{1}{\omega CR}$$

and squaring eq ⑥ and eq ⑦ and adding them.

$$M^2\cos^2\phi + M^2\sin^2\phi$$

$$= \frac{V^2R^2}{\left[R^2 + \left(\frac{1}{\omega C}\right)^2\right]^2} + \frac{V^2}{(\omega C)^2\left[R^2 + \left(\frac{1}{\omega C}\right)^2\right]^2}$$

$$= \frac{V^2}{\left[R^2 + \left(\frac{1}{\omega C}\right)^2\right]^2}\left[R^2 + \frac{1}{(\omega C)^2}\right]$$

$$= \frac{V^2}{R^2 + \left(\frac{1}{\omega C}\right)^2}$$

$$\Rightarrow M^2(1) = \frac{V^2}{\left(R^2 + \left(\frac{1}{\omega C}\right)^2\right)}$$

$$\Rightarrow M = \frac{V}{\sqrt{R^2 + \frac{1}{(\omega C)^2}}}$$

∴ The particular current becomes,

$$i_p = M\cos\phi \cos(\omega t + \theta) \mp M\sin\phi \sin(\omega t + \theta)$$

$$= M\left[\cos\phi \cdot \cos(\omega t + \theta) \mp \sin\phi \cdot \sin(\omega t + \theta)\right]$$

$$= M \cos(\phi + \omega t + \theta)$$

$$= M \cos(\omega t + \theta + \phi)$$

$$i_p = \frac{V}{\sqrt{R^2 + \frac{1}{(\omega C)^2}}} \cos\left(\omega t + \theta + \tan^{-1}\frac{1}{\omega CR}\right)$$

∴ The complete solution for the current $i = i_c + i_p$

$$i = Ce^{-t/RC} + \frac{V}{\sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2}} \cos\left(\omega t + \theta + \cancel{\cos}\tan^{-1}\frac{1}{\omega CR}\right)$$

Since the capacitor does not allow sudden changes in voltages,

at $t=0$, $i = \frac{V}{R}\cos\theta$.

$$\therefore \frac{V}{R}\cos\theta = C + \frac{V}{\sqrt{R^2+\left(\frac{1}{wC}\right)^2}} \cos\left(\theta + \tan^{-1}\frac{1}{wCR}\right)$$

$$C = \frac{V}{R}\cos\theta - \frac{V}{\sqrt{R^2+\left(\frac{1}{wC}\right)^2}} \cos\left(\theta + \tan^{-1}\frac{1}{wCR}\right)$$

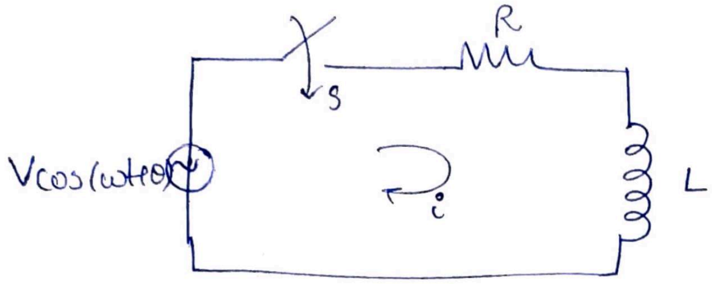
$\therefore$ The complete solution for the current is,

$$i = e^{-t/RC}\left[\frac{V}{R}\cos\theta - \frac{V}{\sqrt{R^2+\left(\frac{1}{wC}\right)^2}} \cos\left(\theta + \tan^{-1}\frac{1}{wCR}\right)\right]$$

$$+ \frac{V}{\sqrt{R^2+\left(\frac{1}{wC}\right)^2}} \cos\left(wt + \theta + \tan^{-1}\frac{1}{wCR}\right)$$

Sinusoidal response of RL circuit:

Consider a circuit consisting of resistance and inductance in series as shown.



The switch S is closed at $t=0$.

At $t=0$, a sinusoidal voltage $V\cos(\omega t+\theta)$ is applied to the RL circuit, where V is the amplitude of the wave

θ is the phase angle

Apply KVL in the above circuit,

$$V\cos(\omega t+\theta) = Ri + L\frac{di}{dt}$$

$$\Rightarrow \frac{di}{dt} + \frac{R}{L}i = \frac{V}{L}\cos(\omega t+\theta)$$

$$\Rightarrow Di + \frac{R}{L}i = \frac{V}{L}\cos(\omega t+\theta)$$

$$\Rightarrow \left(D+\frac{R}{L}\right)i = \frac{V}{L}\cos(\omega t+\theta) \longrightarrow (1)$$

Equation ① consists of two parts in its solution, i.e., complementary function and particular integral.

The complementary function of the solution is,

$$i_c = Ce^{-t(R/L)}$$

The particular solution can be obtained by using undetermined co-efficients.

By assuming, $i_p = A\cos(\omega t+\theta) + B\sin(\omega t+\theta) \rightarrow ②$

$$i_p' = -A\omega\sin(\omega t+\theta) + B\omega\cos(\omega t+\theta) \rightarrow ③$$

Substituting ②, ③ in eq ①,

$$\{-A\omega\sin(\omega t+\theta) + B\omega\cos(\omega t+\theta)\} + \frac{R}{L}\{A\cos(\omega t+\theta)$$

$$+ B\sin(\omega t+\theta) = \frac{V}{L}\cos(\omega t+\theta)$$

Comparing $\cos(\omega t+\theta)$ terms and $\sin(\omega t+\theta)$ terms on bothsides

$$\Rightarrow \quad -A\omega + \frac{BR}{L} = 0 \rightarrow ④ \qquad \& \qquad +B\omega + \frac{AR}{L} = \frac{V}{L}. \rightarrow ⑤$$

Solving eq ④ and eq ⑤

$$A = V \cdot \frac{R}{R^2+(\omega L)^2} \qquad \& \qquad B = V \cdot \frac{\omega L}{R^2+(\omega L)^2}$$

Substituting values of A and B in eq (2),

$$i_p = V \cdot \frac{R}{R^2+(\omega L)^2} \cos(\omega t+\theta) + V \cdot \frac{\omega L}{R^2+(\omega L)^2} \sin(\omega t+\theta)$$

Put $$M\cos\phi = \frac{VR}{R^2+(\omega L)^2} \longrightarrow (6)$$

$$M\sin\phi = \frac{V \cdot \omega L}{R^2+(\omega L)^2} \longrightarrow (7)$$

Now, from eq (6), eq (7), solve for M, ϕ.

$$\frac{(7)}{(6)} \Rightarrow \frac{M\sin\phi}{M\cos\phi} = \frac{\frac{V\omega L}{R^2+(\omega L)^2}}{\frac{VR}{R^2+(\omega L)^2}}$$

$$\Rightarrow \tan\phi = \frac{\omega L}{R}$$

$$\Rightarrow \phi = \tan^{-1}\left(\frac{\omega L}{R}\right)$$

$$(7)^2 + (6)^2 \Rightarrow (M\sin\phi)^2 + (M\cos\phi)^2 =$$

$$\left(\frac{V\,\omega L}{R^2+(\omega L)^2}\right)^2 + \left(\frac{VR}{R^2+(\omega L)^2}\right)^2$$

$$\Rightarrow M^2(1) = \frac{V^2\left(R^2+(\omega L)^2\right)}{\left(R^2+(\omega L)^2\right)^2}$$

$$M = \frac{V}{\sqrt{R^2 + (\omega L)^2}}$$

$\therefore$ The particular current becomes,

$$i_p = \frac{V}{\sqrt{R^2 + (\omega L)^2}} \cos\left(\omega t + \theta - \tan^{-1}\frac{\omega L}{R}\right)$$

The complete solution for the current $i = i_c + i_p$

$$i = c e^{-t(R/L)} + \frac{V}{\sqrt{R^2 + (\omega L)^2}} \cos\left(\omega t + \theta - \tan^{-1}\frac{\omega L}{R}\right)$$

Since the inductor does not allow sudden changes in currents,

at $t = 0$, $i = 0$

$$\Rightarrow \quad i(0) = 0 = c e^{-0} + \frac{V}{\sqrt{R^2 + (\omega L)^2}} \cos\left(0 + \theta - \tan^{-1}\frac{\omega L}{R}\right)$$

$$\Rightarrow \quad C = -\frac{V}{\sqrt{R^2 + (\omega L)^2}} \cos\left(\theta - \tan^{-1}\frac{\omega L}{R}\right)$$

$\therefore$ The complete solution for the current is,

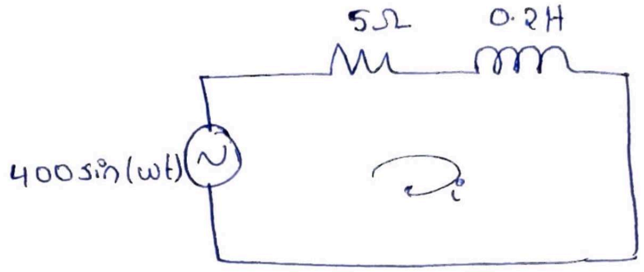
$$i = e^{-(R/L)t}\left[\frac{-V}{\sqrt{R^2 + (\omega L)^2}} \cos\left(\theta - \tan^{-1}\frac{\omega L}{R}\right)\right]$$

$$+ \frac{V}{\sqrt{R^2 + (\omega L)^2}} \cos\left(\omega t + \theta - \tan^{-1}\frac{\omega L}{R}\right)$$

Numericals on sinusoidal input to RL & RC circuits:

Q1) A 50 Hz 400V (peak value) sinusoidal voltage is applied at $t=0$ to a series RL circuit having resistance 5Ω & inductance $0.2H$. Obtain an expression of current at any instant 't'. Calculate the value of the transcient current 0.01 sec after switching on.

Sol)



Apply KVL,

$$Ri + L\frac{di}{dt} = 400\sin(2\pi \times 50t)$$

$$\Rightarrow Ri + L\frac{di}{dt} = 400\sin 314t \Rightarrow \frac{R}{L}i + \frac{di}{dt} = \frac{400}{L}\sin 314t$$

$$\Rightarrow \left(D+\frac{R}{L}\right)i = \frac{400}{0.2}\sin 314t$$

$$\Rightarrow \left(D+\frac{R}{L}\right)i = 2000\sin 314t$$

Complementary function for the above characteristic equation is, $i_c = Ce^{-(R/L)t} = Ce^{-(5/0.2)t} = Ce^{-25t}$

The particular solution is,

$$i_p = \frac{400}{\sqrt{5^2 + (2\pi \times 50 \times 0.2)^2}} \sin\left(314t + 0 - \tan^{-1} \frac{2 \times \pi \times 50 \times 0.2}{5}\right)$$

$$\left(\because i_p = \frac{V}{\sqrt{R^2 + (\omega L)^2}} \sin\left(\omega t + \theta - \tan^{-1} \frac{\omega L}{R}\right)\right)$$

$$i_p = 6.35 \sin(314t - 1.49)$$

$\therefore$ The complete solution is, $i = i_c + i_p$

$$i = Ce^{-25t} + 6.35 \sin(314t - 1.49)$$

From initial condition, $i(0) = 0$,

$\therefore$ At $t = 0$, $i(0) = 0$

$$\Rightarrow Ce^{-25(0)} + 6.35 \sin(314(0) - 1.49) = 0$$

$$\Rightarrow C(1) + 6.35(\sin(-1.49)) = 0$$

$$\Rightarrow \boxed{C = 6.33}$$

$$\therefore i(t) = 6.33\, e^{-25t} + 6.35 \sin(314t - 1.49)$$

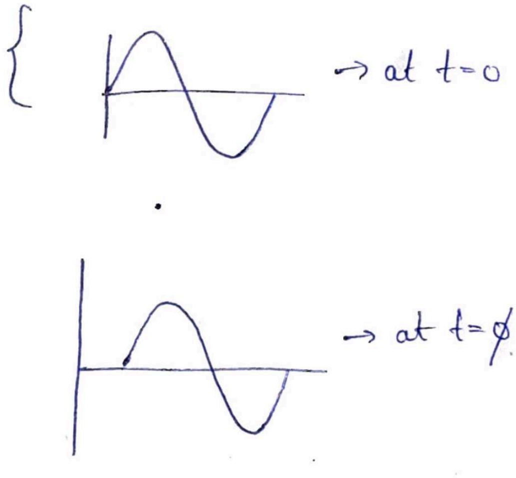
Now, the value of transcient current after 0.01 sec of switching, is given by complimentary function, whereas i_p refers to steady state part.

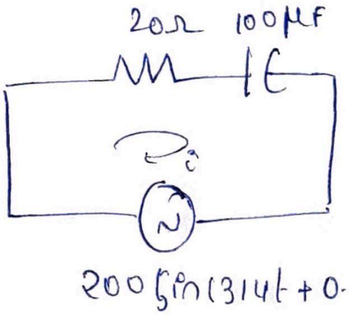
$$\therefore \quad i_c^{\circ}(0.01) = 6.33e^{-25(0.01)} = 4.93A$$

Q2) A series RC circuit has $R = 20\Omega$ and $C = 100\mu F$. A voltage $v = 200\sin 314t$ is applied at $t = 2.14$ msec. Obtain an expression for i. Also, find the value of current after time $t = 1$ msec from the switching instant.

Sol) * Switching is not done at $t=0$, but at $\phi = 2.14$ msec

$\therefore \phi = 0.673^c$





$f = 50\,Hz$

$T = \frac{1}{50} = 0.02$ sec $= 20$ msec.

$360^\circ = 2\pi^c$

$2\pi^c \rightarrow 20$ msec

$? \leftarrow 2.14$ m

$\frac{2\pi \times 2.14\,m}{20\,m} = 0.673$ rad

Apply KVL after switch is closed.

$$iR + \frac{1}{C}\int i\,dt = 200\sin(314t + 0.673)$$

$\Rightarrow$ Solution of this differential equation is,

$$i = i_c + i_p$$

i_c - complementary function

$$i_c = C_1 e^{-t/RC} = C_1 e^{-t/20\times 100\mu}$$

$$i_c = C_1 e^{-500t}$$

and i_p (particular function)

$$= \frac{V_m}{\sqrt{R^2+\left(\frac{1}{\omega C}\right)^2}} \sin\left(\omega t + \theta + \tan^{-1}\frac{1}{\omega CR}\right)$$

$$= \frac{200}{\sqrt{20^2+\frac{1}{(2\pi\times 50\times 100\mu)^2}}} \sin(\omega t + 0.672 + 1.59)$$

$$= \frac{200}{37.6} \sin(314t + 0.672 + 1.59)$$

∴ complete solution,

$$i = i_c + i_p$$

$$i = C_1 e^{-500t} + 5.32 \sin(314t + 0.672 + 1.01) \text{ A}$$

Now, to calculate the current through inductor just after switching $t = (2.14\text{ msec})^+$

$$i_c(0^+) \text{ ie } i_c(2.14\text{ msec})^+ = \frac{V \sin(\omega t)}{R}$$

$$= \frac{V \sin(\omega(2.14m))}{R}$$

$$= \frac{200 \sin(314 \times 0.00214)}{20}$$

$$= \frac{200 \sin(0.67196^c)}{20}$$

$2\pi^c - 360^\circ$

$0.67196^c - ?$

$$= \frac{200 \sin(38.49^\circ)}{20}$$

$$= 6.22 \text{ A}.$$

$\therefore i(0^+)$ ie $i(2.14 \text{ msec})^+ = 6.22 \text{A}$

Hence

$$6.22 = k(e^{-0}) + 5.32 \sin(0 + 0.672 + 1.01)$$

$$\Rightarrow 6.22 = k + (\sin(1.682))\, 5.32$$

$$\Rightarrow k = 0.93$$

$$\therefore i = 0.93 e^{-500t} + 5.32 \sin(314t + 1.682) \text{ A}$$

After 1 msec,

$$i_c = 0.93 e^{-500t} + 5.32 \sin(314 \times 1 \times 10^{-3} + 1.682)$$

$$\Rightarrow i_c = 5.414 \text{A}.$$