30-01-2023.

6. NUMERICAL ANALYSIS

* **syllabus:-**

① "Root finding methods for algebraic functions and transidental functions"

ⓐ 'Bisection method'

ⓑ 'Regular falisi method'

ⓒ 'Newton Raphson method.'

② "Interpolation"

ⓐ 'Newton's forward interpolation.'

ⓑ 'Newton's backward interpolation.'

ⓒ 'Gauss central difference Interpolation.'

* **Example :-**

$f(x) = x^2 + 5x + 6$

What are roots of $f(x)$?

A) 2, 3.

* **Ex:-**

$g(x) = x^4 - 5x^2 + 6$. $\quad (x^2)^2 - 5x^2 + 6 \quad t = x^2.$

what are the roots of $g(x)$? $\quad \Rightarrow t^2 - 5t + 6$

A) $\pm\sqrt{2}, \pm\sqrt{3}$.

* **Ex:-**

$\sin x = 2$, what are real roots?

A) No real roots.

* Ex: Find roots of $f(x) = x^3 - x^2 + x + 1$?

A)
$$\begin{array}{c|cccc} 1 & 1 & -1 & 1 & 1 \\ & 0 & 1 & 0 & -1 \\ \hline & 1 & 0 & 1 & 0 \end{array}$$

$f(x) = (x^2+1)(x-1)$

$= (x-1)(x+i)(x+i)$

$f(x)$ has only one real root i.e, $x = 1$.

$f(x)$ has two complex roots they are $i, -i$.

* **Analytical method:**

$x^2 - 5x + 6 = 0$

$\Rightarrow x^2 - 2x - 3x + 6 = 0$

$\Rightarrow (x-2)(x-3) = 0$

$\therefore x = 2, x = 3$

(or)

$ax^2 + bx + c = 0$

$\Rightarrow x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

* **Fundamental Theorem of Algebra:**

→ suppose / Let $p(x)$ be a polynomial of degree n.

i.e, $p(n): a_n x^n + a_{n-1} x^{n-1} + \_\_\_ + a_0$, a_i's are constants

$a_n \neq 0$ then $p(n)$ has atmost 'n' roots.

Ex:

→ For polynomial of degree - 100, there exists atmost 100 roots.

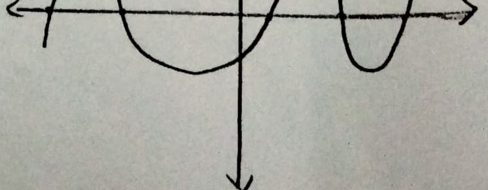
→ Can you give a function which does not have any real (or) complex roots?

* **Root (or) solution:**

$f: \mathbb{R} \to \mathbb{R}$

$f: D \subseteq \mathbb{R} \to \mathbb{R}$

5 roots.



* Roots:-

Roots

- complex
- Real roots
  - +ve
  - −ve

* Ex:-

$f(x) = x^3 + x + 1$ is a polynomial of degree '3'. then

(a) $f(x)$ has exactly 3 real roots.

(b) $f(x)$ has exactly 3 complex roots

(c) $f(x)$ has at least one real root.

(d) $f(x)$ has 2 real roots and 1 complex roots.

Which one is correct?

A) Note:-

→ $P(x) = a_n x^n + a_{n-1} x^{n-1} + \_\_ + a_0$ be a polynomial of degree 'n'. where $a_i \in \mathbb{R}$.

$p(n) \in \mathbb{R}[n]$

degree $(p(n)) = 3$.

By Fundamental Theorem of Algebra, $f(x)$ has atmost 3 roots

3 roots

complex	Real roots
2	1
0	3

deg $(P(n)) = 5$

Complex	Real roots
2	3
4	1
0	5

* Ex:- x^5+3x^2-x+1.

A) 5 roots

complex	real	
2	3	— case (i)
4	1	— case (ii)
0	5	— case (iii).

case-(i) :-

3 real roots

+ve	-ve
0	3

For x^5+3x^2-x+1

At $x=0$, $f(0)=1$

$x=1$, $f(1)=4$

$x=2$, $f(2)=47$

$\therefore$ It lies b/w 0 to $-\infty$.

So all are 3 real -ve roots

Sign Rule :-

$p(x)$:- x^5+3x^2-x+1

$$\underset{+0}{+\;+}\;\;\underset{+1}{+\;-}\;\;\underset{+1}{-\;+}$$

$\therefore$ $p(x)$ has atmost '2' +ve real roots.

$$P(-x):-\;\;\underset{+1}{-\;+}\;\;\underset{+0}{+\;+}\;\;\underset{+0}{+\;+}$$

$\therefore$ $p(x)$ has atmost '1' -ve real roots.

$\therefore$ $p(x)$ has atmost '3' real roots i.e, $p(x)$ has atleast '2' complex roots.

$\therefore$ $p(x)$ has atleast two complex roots and atmost 4 complex roots. But not no complex root. [$\because$ By sine rule] ..

$\therefore$ By sine rule, $p(x)$ has 1 Negative real root (or) two positive real roots.

* Ex:- $p(x) = x^6 + x^5 + 3x^2 - x + 1$

A) sign Rule:-

$p(x)$:- $+ \quad + \quad + \quad - \quad +$

$+0 \quad +0 \quad +1 \quad +1$

$\therefore$ $p(x)$ has atmost '2' +ve roots.

$p(-x)$:- $+ \quad - \quad + \quad + \quad +$

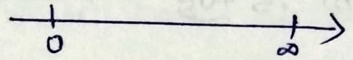
$+1 \quad +1 \quad 0 \quad 0$

$\therefore$ $p(x)$ has atmost '2' −ve roots.

$\therefore$ $p(x)$ has at most '4' real roots.

i.e, $p(x)$ has at least '2' complex roots [By F.T.A].

$(x^6 + x^5 + 3x^2 - x + 1) > x \quad \forall x \in [0, \infty)$.



$p(0) = 1 \neq 0$

$p(1) = +ve$

$p(2) = +ve$

$\}$ No real '+ve roots'

Here $x^6 + 3x^2 - x + 1 > x^5$.

* Ex:- $f(x) = x^4 + x^2 + 1$

A) By sign rule:-

$f(x)$:- $+ \quad + \quad +$ ($+0 \quad +0$) → No +ve roots

$f(-x)$:- $+ \quad + \quad +$ ($+0 \quad +0$) → No −ve roots

$\}$ No real roots.

* Ex:- $P(x) = x^{20} + x^{10} - 1$.

A) By sign rule:-

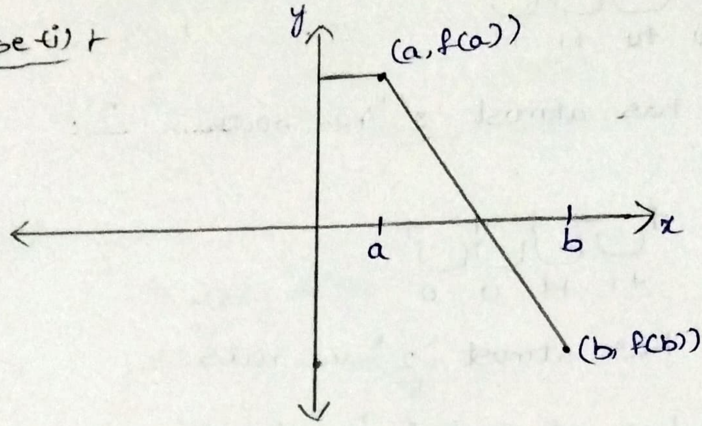
$f(x)$:- $+ \quad + \quad -$ ($+0 \quad +1$) 1 +ve real root

$f(-x)$:- $+ \quad + \quad -$ ($+0 \quad +1$) 1 −ve real root.

* Mean value Theorem :-

→ suppose 'f(x)' be a continuous function on [a,b], if 'f(a). f(b)' < 0 then f(x) has a root in the [a,b].

case (i) :-



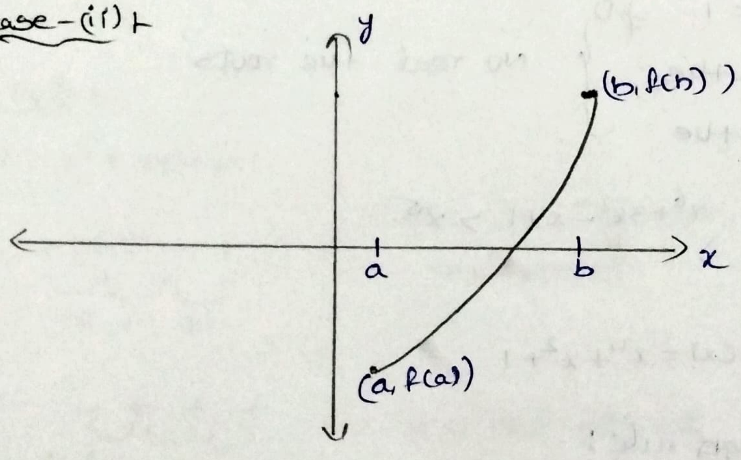
⇒ 'f(a)· f(b) < 0'

⇒ f(a) is '+ve' and f(b) is '-ve' → case(i)

(or)

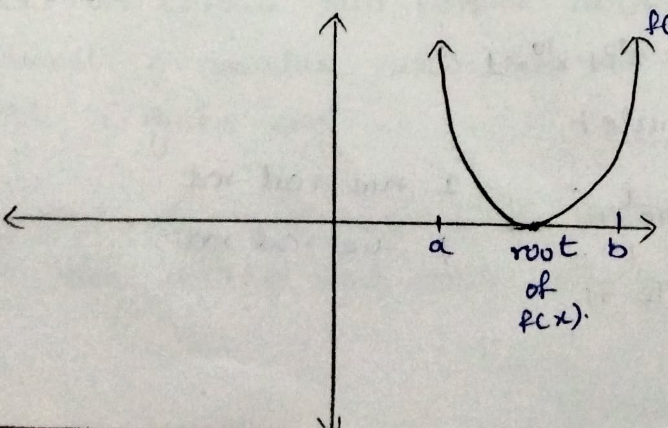
f(a) is '-ve' and f(b) is '+ve' → case(ii)

case-(ii) :-



converse :-

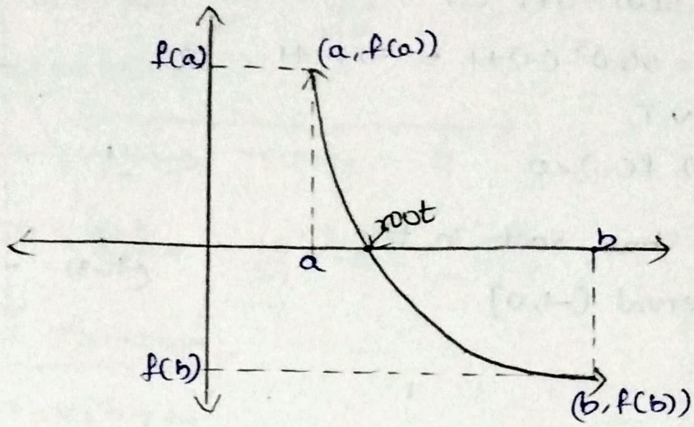
→ suppose 'f(x)' be a continuous function [a,b] if f(x) has root in [a,b], then 'f(a)· f(b)' is need not '-ve.'



f(a) is +ve

f(b) is +ve

* Bisection Method :-



* Ex :- $5x^3 - x + 1$.

A) ① $f(x)$ is the polynomial of degree '3'.

By 'F.T.A', $f(x)$ has atmost '3' roots.

② $f(x)$ be a real polynomial of degree odd then $f(x)$ must have at least one real root.

Possibilities

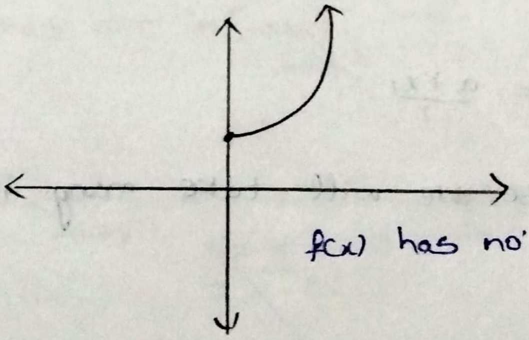
complex	real
2	1
0	3

③ By sign rule,

$f(x)$:- $\underbrace{+ \quad - \quad +}_{+1 \quad +1}$ $f(x)$ has atmost '2' +ve real roots

$\Rightarrow f(x) = 5x^3 - x + 1$

$(5x^3+1) - x > 0 \quad \forall \; x \in (0, \infty)$.



$f(x)$ has no +ve roots.

$f(-x)$:- $-5x^3 + x + 1$

$\underbrace{- \quad + \quad +}_{+1 \quad 0}$

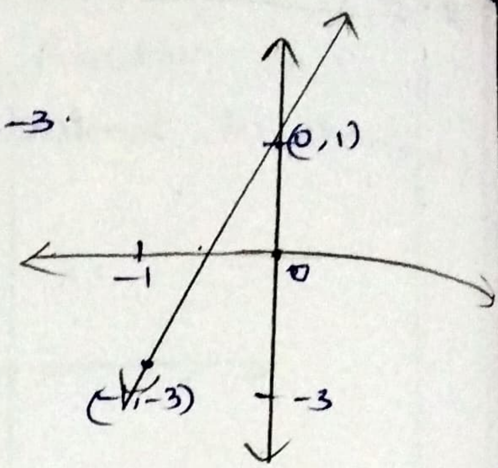
$f(x)$ has atmost '1' Negative real root.

$\therefore f(x)$ has '1' '-ve' real root.

$\rightarrow f(x) = 5x^3 - x + 1$

$f(0) = 5(0)^3 - 0 + 1 = 1$

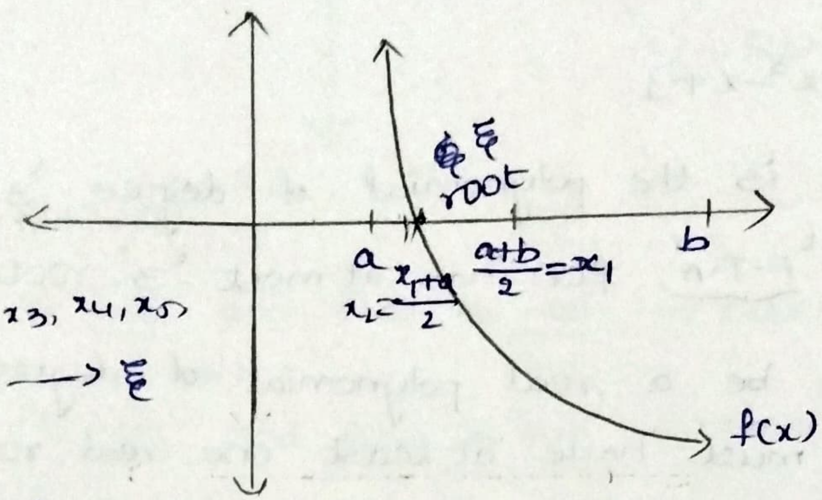
$f(-1) = 5(-1)^3 - (-1) + 1 = -5 + 1 + 1 = -3$.



By M.V.T,

$\therefore f(0) \cdot f(-1) < 0$

$\therefore f(x)$ has root in the interval $[-1, 0]$.



$x_1, x_2, x_3, x_4, x_5,$

$x_6, \ldots \longrightarrow \xi$

$f(a) \cdot f(b) < 0 \Rightarrow f(x)$ has root

i.e, $f(\xi) = 0$ where $\xi \in [a, b]$.

$\Rightarrow [a, b] = \underset{A}{[a, x_1]} + \underset{B}{[x_1, b]}$.

Case-1: ~~Now,~~ $f(a) \cdot f(x_1) < 0$

$\Rightarrow f(x)$ has root in $[a, x_1]$.

Case-2: $f(a) \cdot f(x_1) > 0$

$\Rightarrow f(x)$ has doesnot has root in $[a, x_1]$.

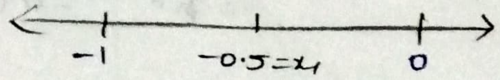
Now, $x_2 = \frac{a + x_1}{2}$.

$\therefore$ like these, we will take many iterations.

Now, $f(x) = 5x^3 - x + 1$

$f(x)$ has root in $[-1, 0]$
$\quad a, b$

First Iteration:



$\Rightarrow x_1 = \frac{a+b}{2} = \frac{-1+0}{2} = -0.5$

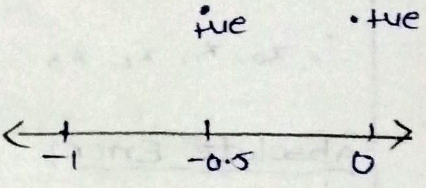
Second Iteration:

$f(x) = 5x^3 - x + 1$

$f(-1) = 5(-1)^3 - (-1) + 1 = -3$

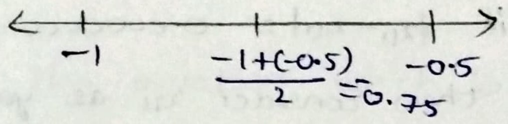
$f(-0.5) = +0.875$

$f(0) = 1$



-ve

$\therefore f(x)$ has root in $[-1, -0.5]$.



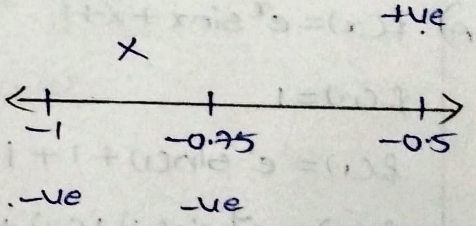
$\Rightarrow x_2 = \frac{-1 + (-0.5)}{2} = -0.75$

Third Iteration:

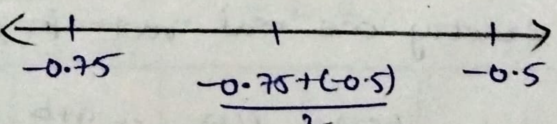
$f(-1) = -3$ $\qquad [-1, -0.75] + [0.75, -0.5]$.

$f(-0.75) = -0.3593$

$f(0.5) = 0.875$



$\therefore f(x)$ has root in $[-0.75, -0.5]$.



$\Rightarrow x_3 = \frac{-0.75 + (-0.5)}{2} = -0.625$.

$\therefore$ Third iteration, $x_3 = -0.625$.

Fourth iteration:

$[-0.75, -0.625]$, $[-0.625, -0.5]$

+ve ✓ ; +ve ✗

-0.75 ; -0.625 ; -0.5

$-ve$.

$f(-0.75) = -0.3593$

$f(-0.625) = 0.404$

$f(-0.5) = 0.875$

$$x_4 = \frac{-0.75 + (-0.625)}{2} = \underline{-0.6875}.$$

$\therefore x_0, x_1, x_2, x_3 \text{ ---- } x_i, x_{i+1}, x_{i+2}, \text{ -------- }$

Absolute Error:-

$|x_{i+1} - x_i| \approx 0.000001$ ----

$\boxed{x_{i+1}} \approx \xi$

$|x_2 - x_1|$
$|x_3 - x_2|$
$|x_n - x_{n-1}|$

Ex:-

x_1
x_2
x_3
x_4
|
|
$\boxed{x_{11}}$.

if $|x_{11} - x_{10}| \approx 0.0000000001$

then consider x_{11} as your root $- \xi$.

* Find the root of function $e^x \sin x + x + 1 = f(x)$?

A) $f(x) = e^x \sin x + x + 1$

$f(0) = 1$

$f(1) = e^1 \sin(1) + 1 + 1 = +ve$

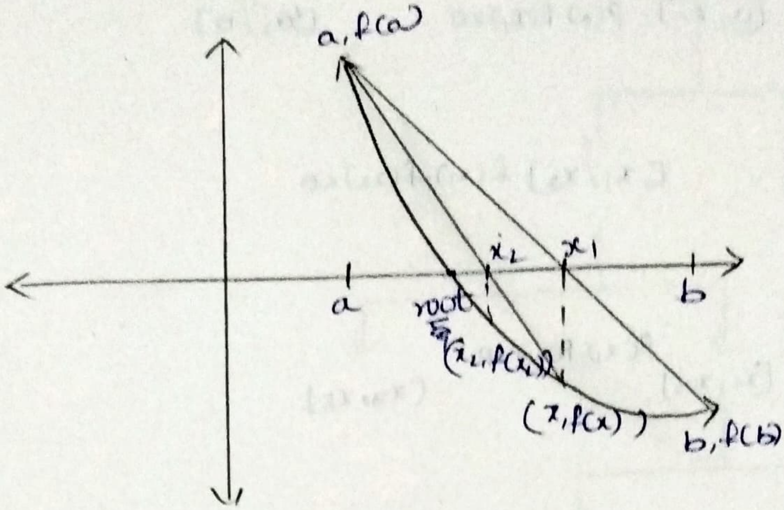
$f(-1) = e^{-1} \sin(-1) + (-1) + 1 = -ve$.

$\therefore f(x)$ has root in $[-1, 0]$.

$\therefore f(x)$ has exactly one real root in $[-1, 0]$.

Iteration No.	a	b	f(a)	f(b)	$c = \frac{a+b}{2}$	f(c)	update.
1	-1	0	-0.309	1	-0.5	0.209	b=c
2	-1	-0.5	-0.309	0.209	-0.75	-0.071	a=c
3	-0.75	-0.5	-0.071	0.209	-0.625	0.0618	b=c
4	-0.75	0.625	-0.071	0.0618	-0.6875	-0.006	a=c

* Regular Falsi Method :-



I^{st} Iteration :-

→ By M.V.T. f(x) has a root in [a, b].

step-1 :-

→ Find the line equation which is passing through (a, f(a)), (b, f(b)).

$$\boxed{(y - f(a)) = \frac{f(b) - f(a)}{b-a} \cdot (x-a).}$$

step-2 :-

→ Find the value of x, where y=0

$$\Rightarrow 0 - f(a) = \frac{f(b) - f(a)}{b-a} (x-a)$$

$$\Rightarrow \frac{-f(a) \cdot [b-a]}{f(b) - f(a)} = x - a$$

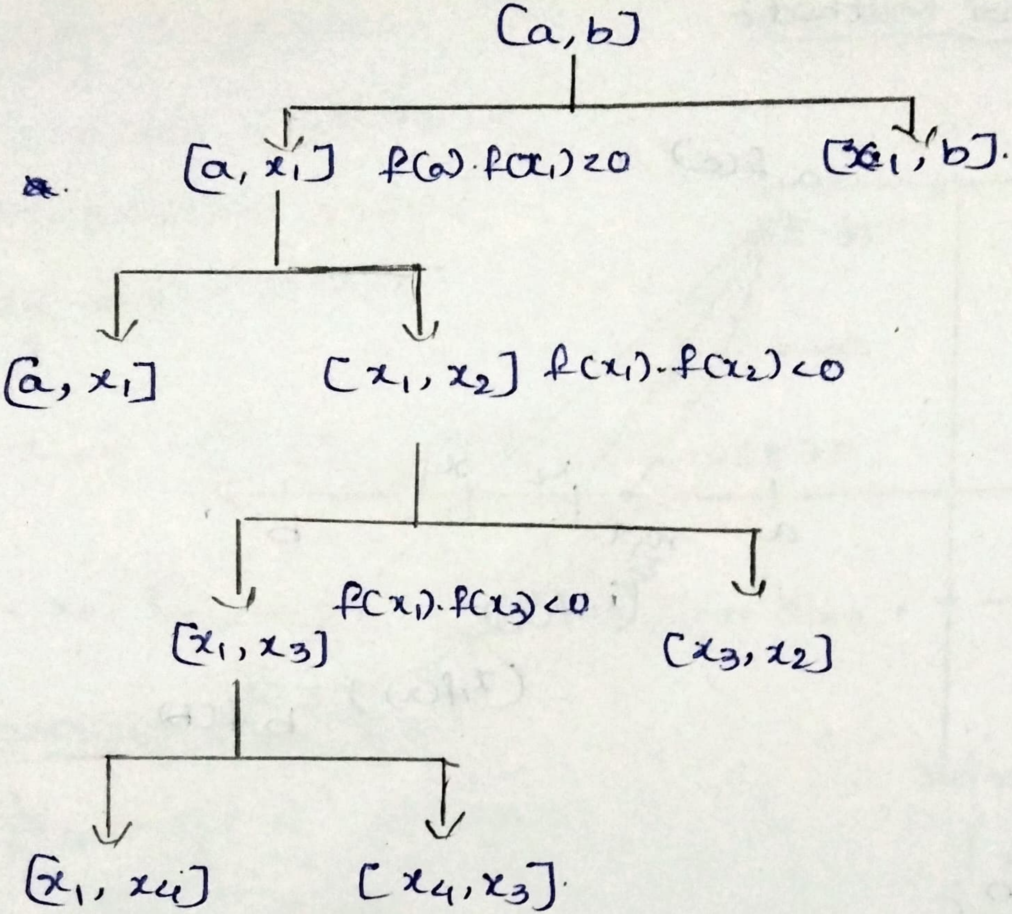
$$\Rightarrow \frac{a \cdot f(a) - b(fa)}{f(b) - f(a)} = x - a.$$

$$\Rightarrow x = a + \frac{a \cdot f(a) - b \cdot f(a)}{f(b) - f(a)}.$$

$$\Rightarrow x = \frac{a(f(b)) - \cancel{a f(a)} + \cancel{a f(a)} - b \cdot f(a)}{f(b) - f(a)}$$

$$\therefore \boxed{x = \frac{a \cdot f(b) - b \cdot f(a)}{f(b) - f(a)}}$$

*



[a, b]
[a, x1] f(a)·f(x1)<0
[x1, b].
[a, x1]
[x1, x2] f(x1)·f(x2)<0
f(x1)·f(x3)<0
[x1, x3]
[x3, x2]
[x1, x4]
[x4, x3].

* Fixed point - Iteration method :-

Ex :-

$x^2-5x+6=0$

Roots are 2, 3 roots.

$\frac{-b \pm \sqrt{b^2-4ac}}{2a}$, by using

Now, $x^2-5x+6=0$

$\Rightarrow x^2+6=5x$

$\Rightarrow x = \frac{x^2+6}{5}$ → Iteration function.

$\therefore \boxed{x_{n+1} = \frac{x_n^2+6}{5}}$ → (I) Iteration scheme

If $n=0 \Rightarrow x_1 = \frac{x_0^2+6}{5}$

If $n=1 \Rightarrow x_2 = \frac{x_1^2+6}{5}$

If $n=2 \Rightarrow x_3 = \frac{x_2^2+6}{5}$

$\vdots$

$x_0, x_1, x_2, x_3, x_4, \ldots x_n, x_{n+1}$ ------ converges to some root.

Here x_1 depends on x_0 ---- x_{n+1} depends on x_n.

$\rightarrow x_0 \in (3, \infty)$

$x_{n+1} = \frac{x_n^2 + 6}{5} \rightarrow \infty$

$x_0 \in (-\infty, -3)$.

$\rightarrow x^2 - 5x + 6$

$x = \frac{x^2+6}{5}$

$\boxed{x_{n+1} = \frac{x_n^2+6}{5}}$

$\times$

$f(x) = 0$

$\Rightarrow x - f(x) = 0$

$\Rightarrow x = f(x)$

$\Rightarrow x_{n+1} = f(x_n)$.

$x^2 = 6 - 5x$

$x = \sqrt{6-5x} = f(x)$

$\boxed{x_{n+1} = \sqrt{6-5x}}$.

$\rightarrow x^2 - 5x + 6 = 0$

$x^2 + 6 = 5x$

$x^2 = 5x - 6$

$x = \sqrt{5x-6}$.

$\boxed{x_{n+1} = \sqrt{5x_n - 6}}$.

$x_0 \in (2, \infty)$.

$f(x) = \sqrt{5x-6}$

Domain: $5x - 6 > 0$

$5x > 6$

$x > \frac{6}{5}$ / 1.2

$\rightarrow x^2 - 5x + 6 = 0$

$x^2 - 5x = -6$

$x(x-5) = -6$

$x = \frac{-6}{x-5}$

$x = \frac{-6}{x} + 5$

$\boxed{x_{n+1} = \frac{-6}{x_n} + 5}$

converges to 3

$\rightarrow x^2 - 5x + 6 = 0$

$x^2 - 5x = -6$

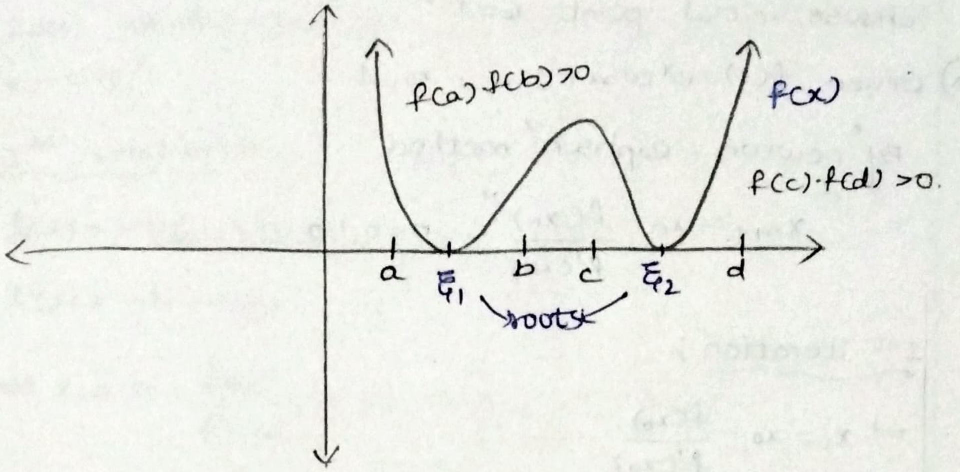
$x(x-5) = -6$

$x = \frac{-6}{-(5-x)} = \frac{6}{5-x}$

$\therefore \boxed{x_{n+1} = \frac{6}{5-x_n}}$.

converges to 2.

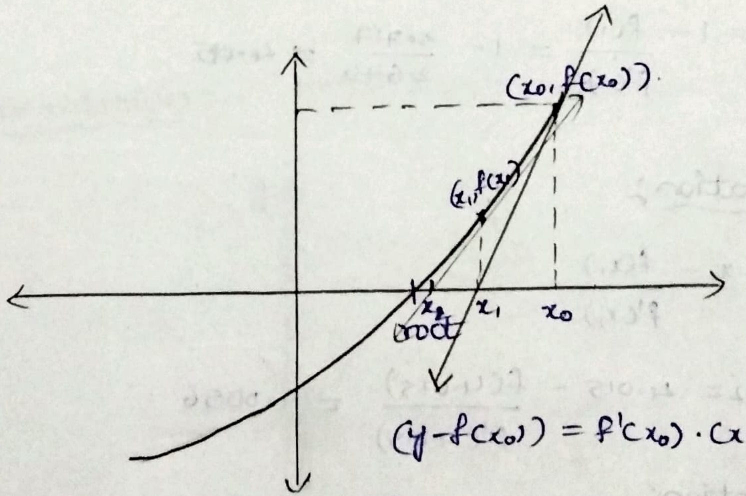
* Newton – Raphson Method :-



→ For using bisection method we have to choose (a,b) such that $f(a)\cdot f(b) < 0$.

→ Bisection method [converse] & Regular falsi method fails sometimes as $f(a)\cdot f(b) > 0$.

→ To overcome those fails, Newton – Raphson method will be useful.



$$(y - f(x_0)) = f'(x_0)\cdot(x - x_0).$$

$$\Rightarrow 0 - f(x_0) = f'(x_0)\cdot(x_1 x_0)$$

$$\Rightarrow (x_1 - x_0) = \frac{-f(x_0)}{f'(x_0)}.$$

First iteration :- $\Rightarrow \boxed{x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}}$, $f'(x_0) \neq 0$.

Second iteration. $\Rightarrow \boxed{x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}}$.

$$\therefore \boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}, \quad f'(x_n) \neq 0.$$

* Ex:-

Find the root of the function $f(x) = e^x \cos x + 1$, choose initial point $x_0 = 1$?

A) Given $f(x) = e^x \cos x + 1$, $x_0 = 1$.

By newton-Raphson method,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, 2, \text{----}$$

1st iteration:-

$$\Rightarrow x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Now, $f(x) = e^x \cos x + 1$

$$f'(x) = e^x(-\sin x) + e^x \cos x = -e^x \sin x + e^x \cos x$$
$$= e^x(\cos x - \sin x).$$

$$f(1) = e^1 \cos(1) + 1 = 3.7178$$

$$f'(1) = e^1(\cos(1) - \sin(1)) = 2.670$$

$$\therefore x_1 = 1 - \frac{f(1)}{f'(1)} = 1 - \frac{3.717}{2.670} = 4.015.$$

2nd Iteration:-

$$\Rightarrow x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}.$$

Now, $x_2 = 4.015 - \frac{f(4.015)}{f'(4.015)} = 9.0056$

3rd Iteration:-

$$\Rightarrow x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$\Rightarrow x_3 = 9.0056 - \frac{f(9.0056)}{f'(9.0056)} = 8.313.$$

4th Iteration:-

$$\Rightarrow x_4 = x_3 - \frac{f(x_3)}{f'(x_3)}$$

$$\Rightarrow x_4 = 8.313 - \frac{f(8.313)}{f'(8.313)} = 7.982$$

* Ex :-

Find the root of the function, $f(x) = -4x + \cos x + 2$, from Newton-Raphson method, choose initial point, $x_0 = 0.5$?

A) 1^{st} iteration :-

$f(x) = -4x + \cos x + 2$

$f'(x) = -4 - \sin x$.

$\Rightarrow x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)} = 0.5 - \dfrac{f(0.5)}{f'(0.5)} = 0.89$

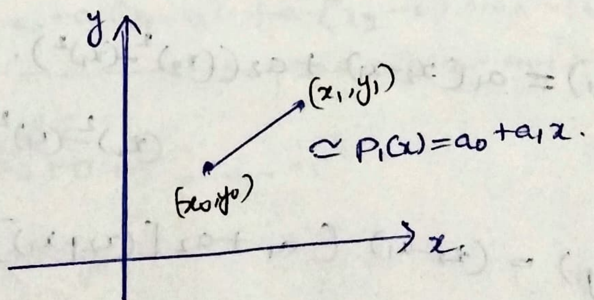
2^{nd} iteration :-

$\Rightarrow x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)} = 0.89 - \dfrac{f(0.89)}{f'(0.89)} = 0.7$

3^{rd} iteration :-

$\Rightarrow x_3 = x_2 - \dfrac{f(x_2)}{f'(x_2)} = 0.7 - \dfrac{f(0.7)}{f'(0.7)} = 0.693$

* Interpolation :-



$P_1(x_0) = a_0 + a_1 x_0 = y_0$

$P_2(x_1) = a_0 + a_1 x_1 = y_1$

$\Rightarrow \cancel{a_0} + a_1 x_0 = y_0$

$\cancel{a_0} + a_1 x_1 = y_1$

$a_1(x_0 - x_1) = y_0 - y_1$

$a_1 = \dfrac{(y_0 - y_1)}{(x_0 - x_1)}$

Now, $a_0 + \left(\dfrac{y_0 - y_1}{x_0 - x_1}\right) \cdot x_0 = y_0$

$\Rightarrow a_0 = y_0 - \dfrac{(y_0 - y_1)}{(x_0 - x_1)} \cdot x_0$

$\therefore P_1(x) = y = y_0 - \left(\dfrac{y_0 - y_1}{x_0 - x_1}\right) \cdot x_0 + \left(\dfrac{y_0 - y_1}{x_0 - x_1}\right) \cdot x$.

$$\Rightarrow \boxed{y - y_0 = \left(\dfrac{y_0 - y_1}{x_0 - x_1}\right) \cdot (x - x_0)}.$$

General form.

* Newton-Forward Interpolation:-

→ $(x_0,y_0), (x_1,y_1), (x_2,y_2)$ ———— (x_n,y_n) are $n+1$ data points such that '$x_{i+1}-x_i=h$' $\forall$ $0\le i\le n$.

Newton-Forward Difference:-

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
x_0	y_0	$y_1-y_0=\Delta y_0$	$y_2-2y_1+y_0=\Delta^2 y_0$	$y_3-3y_2+3y_1-y_0$
x_1	y_1	$y_2-y_1=\Delta y_1$	$y_3-2y_2+y_1=\Delta^2 y_1$	$y_4-3y_3+3y_2-y_1$
x_2	y_2	$y_3-y_2=\Delta y_2$	$y_4-2y_3+y_2=\Delta^2 y_2$	$y_5-3y_4+3y_3-y_2$
x_3	y_3	$y_4-y_3=\Delta y_3$	$y_5-2y_4+y_3=\Delta^2 y_3$	
x_4	y_4	$y_5-y_4=\Delta y_4$		
x_5	y_5			

	$\Delta^4 y$	$\Delta^5 y$
$\Delta^3 y_0$	$\Delta^3 y_1-\Delta^3 y_0=\Delta^4 y_0$	$\Delta^5 y_0$.
$\Delta^3 y_1$	$\Delta^3 y_2-\Delta^3 y_1=\Delta^4 y_1$	
$\Delta^3 y_2$		

Newton-consideration polynomial of degree 'n':-

$P_n(x)=f(x)=a_0+a_1(x-x_0)+a_2(x-x_0)(x-x_1)+a_3(x-x_0)(x-x_1)(x-x_2)+$ ————

———— $+a_n(x-x_0)\dots(x-x_{n-1})$.

$P_n(x)$ ———— (x_0,y_0):-

$\Rightarrow f(x_0)=a_0+0+0+0$ ———— $+0$.

$\Rightarrow a_0=f(x_0)=y_0$

$P_n(x)$ ———— (x_1,y_1):-

$\Rightarrow f(x_1)=a_0+a_1(x_1-x_0)+0+0$ ———— $+0$.

$\Rightarrow y_1=a_0+a_1(x_1-x_0)$.

$\Rightarrow y_1=y_0+a_1(x_1-x_0)$ $\Rightarrow a_1=\left(\frac{y_1-y_0}{x_1-x_0}\right)$ $\therefore \boxed{a_1=\frac{\Delta y_0}{h}}$

$P_n(x)$ ———— (x_2,y_2):-

$\Rightarrow$ $y_2=a_0+a_1(x_2-x_0)+a_2(x_2-x_0)(x_2-x_1)$.

$\Rightarrow y_2=y_0+\left(\frac{y_1-y_0}{x_1-x_0}\right)(2h)+a_2(2h)\cdot h$

$\Rightarrow y_2=y_0+2y_1-2y_0+a_2\cdot 2h^2$

$\Rightarrow y_2-2y_1+y_0=a_2\cdot 2h^2$ $\Rightarrow \boxed{a_2=\frac{\Delta^2 y_0}{2h^2}}$

$P_n(x)$ —— (x_3, y_3) :-

$\Rightarrow y_3 = a_0 + a_1(x - x_0) + a_2(x_2 - x_0)(x_2 - x_1) + a_3(x_3 - x_0)(x_3 - x_1)(x_3 - x_2) + 0 + \text{——} + 0.$

$$\therefore \boxed{a_3 = \frac{\Delta^3 y_0}{3!\,h^3}}$$

$$\therefore \boxed{a_n = \frac{\Delta^n y_0}{n!\,h^n}}$$

$$\therefore P_n(x) = f(x) = y_0 + \frac{\Delta y_0}{h}(x - x_0) + \frac{\Delta^2 y_0}{2h^2}(x - x_0)(x - x_1) + \frac{\Delta^3 y_0}{3!\,h^3}(x - x_0)(x - x_1)(x - x_2) \text{ ——— } + \frac{\Delta^n y_0}{n!\,h^n}(x - x_0)(x - x_1) \text{ ——— } (x - x_{n-1}).$$

*Problem :-

x	1	2	3	4	5
y	1	0	4	5	-1

Find a polynomial using these 5 data points

A)

x	y	Δy_0	$\Delta^2 y$	$\Delta^3 y_0$	$\Delta^4 y_0$
x_0 1	1	$-1 = \Delta y_0$	$5\ \Delta^2 y_0$	$-8 = \Delta^3 y_0$	$\boxed{4} = \Delta^4 y_0$
x_1 2	0	$4 = \Delta y_1$	$-3\ \Delta^2 y_1$	$-4 = \Delta^3 y_1$	
x_2 3	4	$1\ \Delta y_2$	$-7\ \Delta^2 y_2$		
x_3 4	5	$-6 = \Delta y_3$			
x_4 5	-1				

Here, $h = 1$.

$$\Rightarrow f(x) = 1 + (-1)(x-1) + \frac{5}{2!}(x-1)(x-2) + \frac{-8}{6}(x-1)(x-2)(x-3) + \frac{4}{4!}(x-1)(x-2)(x-3)(x-4).$$

* Newton-Backward Interpolation :-

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
x_0	y_0					
		$y_0 - y_1 = \Delta y_1$				
x_1	y_1		$\Delta^2 y_2$			
		$y_1 - y_2 = \Delta y_2$		$\Delta^3 y_3$		
x_2	y_2		$\Delta^2 y_3$		$\Delta^4 y_4$	
		$y_2 - y_3 = \Delta y_3$		$\Delta^3 y_4$		$\Delta^5 y_4$
x_3	y_3		$\Delta^2 y_4$		$\Delta^4 y_5$	
		$y_3 - y_4 = \Delta y_4$		$\Delta^3 y_5$		
x_4	y_4		$\Delta^2 y_5$			
		$y_4 - y_5 = \Delta y_5$				
x_5	y_5					

$$f(x) = y_n + \frac{\Delta y_n}{h}(x - x_n) + \frac{\Delta^2 y_n}{2!\,h^2}(x - x_n)(x - x_{n-1}) +$$

$$\frac{\Delta^3 y_n}{3!\,h^3}(x - x_n)(x - x_{n-1})(x - x_{n-2}) + \text{———}$$

$$+ \frac{\Delta^n y_n}{n!\,h^n}(x - x_n)(x - x_{n-1}) \text{———} (x - x_1).$$

* Problem :-

A)

x	y_0	Δy_0	$\Delta^2 y_0$	$\Delta^3 y_0$	$\Delta^4 y_0$
1	1				
		$1 = \Delta y_1$			
2	0		$5 = \Delta^2 y_2$		
		$-4 = \Delta y_2$		$8 = \Delta^3 y_3$	
3	4		$-3 = \Delta^2 y_3$		$\boxed{4} = \Delta^4 y_4$
		$-1 = \Delta y_3$		$+4 = \Delta^3 y_4$	
4	5		$-7 = \Delta^2 y_4$		
		$6 = \Delta y_4$			
5	-1				

* Legrange Interpolation:

→ Let $(x_0, y_0), (x_1, y_1)$ —— (x_n, y_n) are '$n+1$' data points.

Let legrange interpolation polynomial is

$$P_n(x) = f(x) = a_0 \cdot (x-x_1)(x-x_2) \text{——} (x-x_n) +$$
$$a_1 (x-x_0)(x-x_2) \text{——} (x-x_n) +$$
$$a_2 (x-x_0)(x-x_1)(x-x_3)(x-x_n) + \text{——} +$$
$$a_n (x-x_0)(x-x_1)(x-x_2) \text{——} (x-x_n).$$

(1) —— (x_0, y_0)

$$\Rightarrow f(x_0) = a_0 (x_0-x_1)(x_0-x_2)(x_0-x_3) \text{——} (x_0-x_n).$$

$$\Rightarrow y_0 = a_0 (x_0-x_1)(x_0-x_2)(x_0-x_3) \text{——} (x_0-x_n)$$

$$\therefore a_0 = \frac{y_0}{(x_0-x_1)(x_0-x_2)(x_0-x_3) \text{——} (x_0-x_n)}.$$

$$a_1 = \frac{y_1}{(x_1-x_0)(x_1-x_2)(x_1-x_3) \text{——} (x_1-x_n)}$$

$$\therefore a_n = \frac{y_n}{(x_n-x_0)(x_n-x_1) \text{——} (x_n-x_{n-1})}.$$

* problem :

x	$\sin x$
15	0.2588190
20	0.3420201
25	0.4226183
30	0.5
35	0.5735764
40	0.6427876.

A)

x	$\sin x$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
15	0.2588190					
		0.0832011				
20	0.3420201		-0.0026029			
		0.0805982		-0.0006136		
25	0.4226183		-0.0032165		0.0000248	
		0.0773817		-0.0005888		0.0000[illegible]
30	0.5		-0.0038053		0.0000289	
		0.0735764		-0.0005599		
35	0.5735764		-0.0043652			
		0.0692112				
40	0.6427876					

$$f(x) = y_0 + \frac{\Delta y_0}{h}(x-x_0) + \frac{\Delta^2 y_0}{2h^2}(x-x_0)(x-x_1) + \frac{\Delta^3 y_0}{6h^3}(x-x_0)(x-x_1)(x-x_2) + \frac{\Delta^4 y_0}{24h^4}(x-x_0)(x-x_1)(x-x_2)(x-x_3) + \frac{\Delta^5 y_0}{125h^5}(x-x_0)(x-x_1)(x-x_2)(x-x_3)(x-x_4).$$

$= (0.2588190) +$

* Note :

$\Delta y_0 = y_1 - y_0$

$\nabla y_1 = y_1 - y_0$.