4.4

Recurrence Relations

Many counting problems cannot be solved using the methods discussed in the modules 4.1-4.3. One such problem is: How many bit strings of length n do not contain two consecutive zeros? To solve this problem, let a_n be the number of such strings of length n. An argument can be given that shows $a_{n+1} = a_n + a_{n-1}$. This equation is called a *recurrence relation* and the initial conditions $a_1 = 1$ and $a_2 = 3$ to determine the sequence $\{a_n\}$. Further, an explicit formula can be found for a_n from the equation relating the terms of the sequence. A similar technique can be used to solve many different types of counting problems.

Recurrence relation: let $\{a_n\}$ be a sequence. A ***recurrence relation*** for the sequence $\{a_n\}$ is an equation that expresses a_n in terms of one or more of the previous terms of the sequence, namely $a_0, a_1, a_2, \ldots, a_{n-1}$ for all integers n with $n \geq n_0$, where n_0 is a nonnegative integer.

A sequence is called a ***solution*** of a recurrence relation if its terms satisfy the recurrence relation.

Note: A recurrence relation together with initial condition provide a recursive definition of the sequence.

Example 1: The number of bacteria in a colony doubles every hour. If a colony begins with five bacteria, obtain a recurrence relation for the number of bacteria in n hours.

Solution: Let a_n be the number of bacteria at the end of n hours. Because the number of bacteria doubles every hour, the relation $a_n = 2a_{n-1}$ holds whenever n is a positive integer. It is given the initial condition $a_0 = 5$. The recurrence relation is: $a_n = 2a_{n-1}$ for all positive integers n, and $a_0 = 5$

Example 2: Determine (a) whether the sequence $\{a_n\}$, where $a_n = 3n$ for every nonnegative integer n is a solution of the recurrence relation $a_n = 2a_{n-1} - a_{n-2}$ for $n = 2, 3, 4, \ldots$

(b) Show the sequence $\{a_n\}$, where $a_n = 2^n$ for every nonnegative integer n is a not a solution of the recurrence relation $a_n = 2a_{n-1} - a_{n-2}$ for $n = 2, 3, 4, \ldots$

Solution: The given recurrence relation $a_n = 2a_{n-1} - a_{n-2}$ for $n = 2, 3, 4, \ldots$

a) Suppose that $a_n = 3n$ for every nonnegative integer.
For $n \geq 2$, we have

$$2a_{n-1} - a_{n-2} = 2[3(n-1)] - 3(n-2) = 3n = a_n$$

Thus, $a_n = 3n$ is a solution of the recurrence relation.

b) Suppose that $a_n = 2^n$ for every nonnegative integer.
For $n \geq 2$, we have

$$2a_{n-1} - a_{n-2} = 2.2^n - 2^{n-2} = 2^{n-2}(2^2 - 1) = 3.2^{n-2} \neq a_n$$

Thus, $a_n = 2^n$ is not a solution of the recurrence relation

Note: If $a_n = 5$ for every nonnegative integer n, then for $n \geq 2$, we have

$$2a_{n-1} - a_{n-2} = 2.5 - 5 = 5 = a_n$$

Thus, $\{a_n\}$, where $a_n = 5$ is a solution of the recurrence relation.

We can use recurrence relations to model a wide variety of problems.

Example 3: Compound Interest

Suppose that a person deposits $Rs. 10,000$ in a savings account at a bank yielding 11% per year with interest compounded annually. How much will be in the account after 30 years?

Solution: To solve the problem, let P_n denote the amount in the account after n years. Note that the amount in the account after n years equals the amount in the account after $n-1$ years plus the interest for the n^{th}year. Thus, we see that the sequence $\{P_n\}$ satisfies the recurrence relation

$$P_n = P_{n-1} + \frac{P_{n-1} \times 1 \times 11}{100}$$

$$= P_{n-1} + 0.11P_{n-1} = (1.11)P_{n-1}$$

The initial condition is $P_0 = 10{,}000$.

We use an iterative approach to find a formula for P_n. Note that

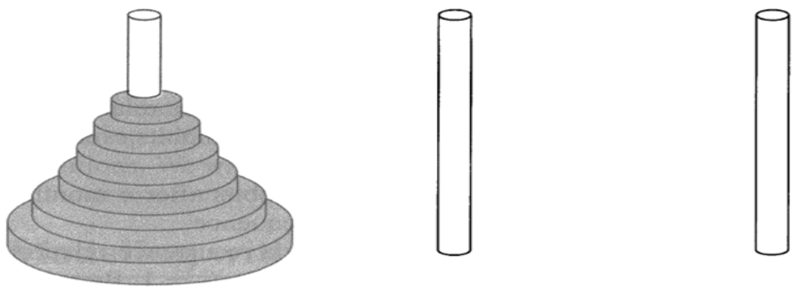
$$
\begin{aligned}
P_1 &= (1.11)P_0 \\
P_2 &= (1.11)P_1 = (1.11)^2 P_0 \\
P_3 &= (1.11)P_2 = (1.11)^3 P_0 \\
&\dots \qquad \dots \qquad \dots \\
&\dots \qquad \dots \qquad \dots \\
P_n &= (1.11)P_{n-1} = (1.11)^n P_0
\end{aligned}
$$

Using initial condition $P_0 = 10{,}000$, the formula $P_n = (1.11)^n 10{,}000$ is obtained.

We can use mathematical induction to establish the validity of the formula.
Putting $n = 30$, we obtain $P_{30} = (1.11)^{30} 10{,}000 = Rs.\,2{,}28{,}92{,}297$

Example 4: The Tower of Hanoi

A popular puzzle of the late nineteenth century invented by the French Mathematician Edourd Lucas, called the ***Tower of Hanoi***, consists of three pegs mounted on a board together with disks of decreasing sizes from bottom to top. Initially these disks are placed on peg 1 in order of size, with the largest on the bottom as shown below



Peg 1 Peg 2 Peg 3

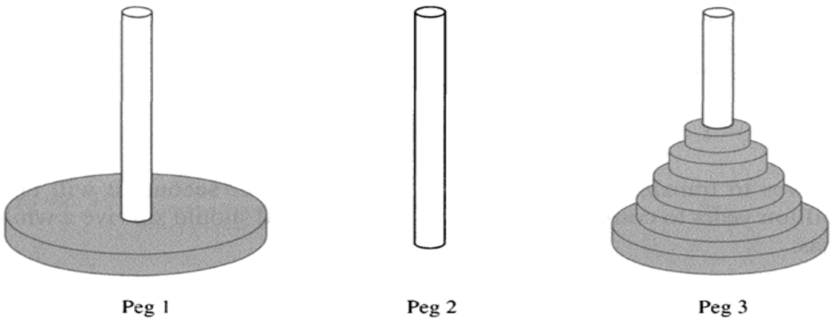
Rules: The rules of the puzzle allows disks to be moved one at a time from one peg to another as long as a disk is never placed on top of a smaller disk.

Goal: The goal of the puzzle is to have all the disks on the second peg in order of size, with the largest on the bottom.

Let H_n denote the number of moves needed to solve Tower of Hanoi problem with n disks.

Set up a recurrence relation for the sequence $\{H_n\}$.

Solution: Begin with n disks on peg 1 (as in the above figure). We can transfer the top $n-1$ disks, following the rules of the puzzle, to peg 3 using H_{n-1} moves as in the figure shown below:



We keep the largest disk fixed during these moves. Then, we use one move to transfer the largest disk to peg 2. We can transfer the $n-1$ disks on peg 3 to peg 1 using H_{n-1} additional moves, placing them on top of the largest disk, which always stays fixed on the bottom of peg 2. Further, it is easy to see that puzzle cannot be solved using fewer steps. Thus,

$$H_n = H_{n-1} + 1 + H_{n-1}$$

i.e., $H_n = 2H_{n-1} + 1$

The initial condition is $H_1 = 1$, because one disk can be transferred from peg 1 to peg 2, according to the rules of the puzzle, in one move. Therefore, the recurrence relation is

$$H_n = 2H_{n-1} + 1, \text{ for } n = 2, 3, \ldots, \text{ with the initial condition } H_1 = 1$$

Solution of the recurrence relation:

We can use an iterative approach to solve the recurrence relation. For $n \geq 2$,

$$H_n = 2H_{n-1} + 1$$

$$= 2(2\,H_{n-2} + 1) + 1 = 2^2\,H_{n-2} + 2 + 1$$

$$= 2^2(2\,H_{n-3} + 1) + 2 + 1 = 2^3\,H_{n-3} + 2^2 + 2 + 1$$

$$\dots \quad \dots \quad \dots \quad \dots$$

$$\dots \quad \dots \quad \dots \quad \dots$$

$$= 2^{n-1}H_1 + 2^{n-2} + 2^{n-3} + \dots + 2 + 1$$

$$= 2^{n-1} + 2^{n-2} + 2^{n-3} + \dots + 2 + 1 \qquad \text{(Since } H_1 = 1\text{)}$$

$$= \frac{2^n - 1}{2 - 1} = 2^n - 1$$

Example 5: Codeword Enumeration

A computer system considers a string of decimal digits a valid codeword if it contains an even number of 0 digits. (For instance, 1230407869 is valid, where as 120987045608 is not valid). If a_n is the number of valid n- digit code words then find a recurrence relation for a_n.

Solution: Note that there are ten one - digit strings, and only one, namely 0 is not valid. Therefore, $a_1 = 9$. Let $n \geq 2$.

We can obtain a valid string with n digits from a string with $n - 1$ digits in two ways.

(i) A string of n digits can be obtained by appending a digit other than 0 to a valid string of $n - 1$ digits. This appending can be done in 9 ways. Therefore, a valid string of n digits can be obtained in this way in $9a_{n-1}$ ways.

(ii) Note that there are 10^{n-1} strings of length $n - 1$ and in this a_{n-1} are valid. Thus, we have $10^{n-1} - a_{n-1}$ invalid strings of length $n - 1$. Now, a valid string of n digits can be obtained by appending a 0 to an invalid string of length $n - 1$.

Thus, the number of valid strings of n digits obtained in this way is $10^{n-1} - a_{n-1}$.

Because all valid strings of length n are obtained in one of these two ways,

$$a_n = 9\, a_{n-1} + (10^{n-1} - a_{n-1})$$

i.e., $\quad a_n = 8\, a_{n-1} + 10^{n-1}$

The recurrence relation of $\{a_n\}$ is $a_n = 8\, a_{n-1} + \ 10^{n-1}$ for $n \geq 2$ and $a_1 = 9$

Example 6: Solve the recurrence relation $\boldsymbol{a_n = c\, a_{n-1} + f(n)}$ for $\boldsymbol{n \geq 1}$ by iteration.

Solution: For $n \geq 1,\ a_n = c\, a_{n-1} + f(n)$

$$= c\big(c\, a_{n-2} + f(n-1)\big) + f(n) = c^2 a_{n-2} + c\, f(n-1) + f(n)$$

$$= c^2\, \big(c\, a_{n-3} + f(n-2)\big) + c\, f(n-1) + f(n)$$

$$= c^3\, a_{n-3} + c^2\, f(n-2) + c\, f(n-1) + f(n)$$

$$\cdots \qquad \cdots\cdot \qquad \cdots\cdot$$

$$= c^n\, a_{n-n} + c^{n-1}\, f\big(n-(n-1)\big) + c^{n-2}\, f\big(n-\ (n-2)\big) + \cdots + c\, f(n-1) + f(n)$$

$$= c^n\, a_0 + c^{n-1}\, f(1) + c^{n-2}\, f(2) + \cdots + c\, f\, (n-1) + f(n)$$

$$= c^n\, a_0 + \sum_{k=1}^{n} c^{n-k}\ f(k)$$

$$\therefore\ a_n = c^n\, a_0 + \sum_{k=1}^{n} c^{n-k}\ f(k)$$

Example 7: Show that $\boldsymbol{\{a_n\}}$ defined by $\boldsymbol{a_n = 4.2^n + 7.3^n}$ is a solution of the recurrence relation $\boldsymbol{a_n - 5\, a_{n-1} + \ 6\, a_{n-2} = 0}$

Solution: We have, $\ a_n = 4.2^n + 7\,.3^n$

$$a_{n-1} = 4.2^{n-1} + 7.3^{n-1}$$

$$a_{n-2} = 4.2^{n-2} + 7\,.3^{n-2}$$

Then $\ a_n - 5a_{n-1} + 6a_{n-2}$

$$= 4.2^n + 7.3^n - 5(4.2^{n-1} + 7.3^{n-1}) + 6(4.2^{n-2} + 7.3^{n-2})$$

$$= 4.2^{n-2}(2^2 - 5.2 + 6) + 7.3^{n-2}\,(3^2 - 5.3 + 6) = 0$$

Thus, $a_n = 4.2^n + 7.3^n$ is a solution of $a_n - 5a_{n-1} + 6 = 0$

Example 8: Find a recurrence relation for the sequence $\{a_n\}$ given by

$a_n = A.2^n + B(-3)^n$, where A, B are arbitrary constants.

Solution: The recurrence relation is obtained by eliminating A and B.

We have $a_n = A.2^n + B(-3)^n$ ... (1)

$a_{n-1} = A.2^{n-1} + B(-3)^{n-1}$ ... (2)

$a_{n-2} = A.2^{n-2} + B(-3)^{n-2}$ ... (3)

Multiplying (2) by 2 and subtracting from (1), we get

$$a_n - 2a_{n-1} = B(-3)^n - 2B(-3)^{n-1} = B(-3)^{n-1}(-3-2) = -5B(-3)^{n-1}$$

i.e., $a_n - 2a_{n-1} = -5B(-3)^{n-1}$ ... (4)

Replacing n by $n-1$ we get

$a_{n-1} - 2a_{n-2} = -5B(-3)^{n-2}$ ... (5)

From (4) we have $a_n - 2a_{n-1} = (-3)(-5B\,(-3)^{n-2})$

$= (-3)\,(a_{n-1} - 2\,a_{n-2})$ (from (5))

i.e., $a_n + 5a_{n-1} - 6a_{n-2} = 0$ is the required recurrence relation.

Example 9: A person climbs a stair case by climbing either (i) two steps in a single stride or (ii) only one step in a single stride. Find a recurrence relation for the number of ways climbing n stairs.

Solution: Let a_n denote the number of ways of climbing n stairs. In reaching n steps, the person can climb either one step or two steps in his last stride. For these two choices, the number of ways are a_{n-1} and a_{n-2} respectively.

Therefore, $a_n = a_{n-1} + a_{n-2}$

Example 10: A factory makes custom sports cars at an increasing rate. In the first month only one car is made, in the second month two cars are made, and so on, with $\boldsymbol{n}$ cars made in the $\boldsymbol{n^{th}}$ month.

a) Set up a recurrence relation for the number of cars produced in the first $\boldsymbol{n}$ months by this factory.

b) Find an explicit formula for the number of cars produced in the first $\boldsymbol{n}$ months by this factory.

Solution:

(a) Let a_n be the total number of cars produced in n months. Initially $a_0 = 0$
The number of cars produced in the n^{th} month is $a_n - a_{n-1}$ and it is n.
Therefore, $a_n - a_{n-1} = n$

i.e., $a_n = a_{n-1} + n, \;\; a_0 = 0$ is the recurrence relation.

(b) Now, $a_n = a_{n-1} + n$

$$= a_{n-2} + (n-1) + n$$
$$= a_{n-3} + (n-2) + (n-1) + n$$
$$\dots \quad \dots \quad \dots$$
$$= a_0 + \big(n - (n-1)\big) + \cdots + (n-2) + (n-1) + n$$
$$= a_0 + 1 + 2 + \cdots + n = \frac{n\,(n+1)}{2} \qquad (\because \;\; a_0 = 0)$$

4.4. Recurrence Relations

EXERCISES:

1. Is the sequence $\{a_n\}$ a solution of the recurrence relation $a_n = 8a_{n-1} - 16a_{n-2}$ if
   a) $a_n = 0$? b) $a_n = 1$? c) $a_n = 2^n$?

   d) $a_n = 4^n$? e) $a_n = n4^n$? e) $a_n = 2.4^n + 3\,n\,4^n$?

   f) $a_n = (-4^n)$? g) $a_n = n^2 4^n$?

2. Show that the sequence $\{a_n\}$ is a solution of the recurrence relation
   $a_n = a_{n-1} + 2a_{n-2} + 2n - 9$ if

   a) $a_n = -n + 2$ b) $a_n = 5(-1)^n - n + 2$

   c) $a_n = 3(-1)^n + 2^n - n + 2$ d) $a_n = 7.2^n - n + 2$

3. Find the solution to each of these recurrence relations and initial conditions. Use an iterative approach such as that used in example 5.
   a) $a_n = 3\,a_{n-1},\ \ a_0 = 2$ b) $a_n = a_{n-1} + 2,\ \ a_0 = 3$
   c) $a_n = a_{n-1} + n,\ \ a_0 = 1$ d) $a_n = a_{n-1} + 2n,\ \ 3a_0 = 4$
   e) $a_n = 2a_{n-1} - 1,\ \ a_0 = 1$ f) $a_n = 3a_{n-1} + 1,\ \ a_0 = 1$
   g) $a_n = na_{n-1},\ \ a_0 = 5$ h) $a_n = 2n\,a_{n-1},\ \ a_0 = 1$

4. A person deposits $Rs. 1000$ in an account that yields 9% interest compounded annually.
   a) Set up a recurrence relation for the amount in the account at the end of n years.
   b) Find an explicit formula for the amount in the account at the end of n years
   c) How much money will the account contain after 100 years?

5.

a) Find a recurrence relation for the number of permutations of a set with n elements.
b) Use this recurrence relation to find the number of permutations of a set with n elements using iteration.