3.1

Preliminaries

Learning objectives:

- To defined a real valued function of n independent variables.
- To study the level curves of a function of two variables and level surfaces of a function of three variables.
- To defined the domain and range of functions of two and three variables.
- To study the interior ,boundary and boundedness of a given region

AND

- To practice the related problems

Functions of several variables

In earlier classes we have studied the calculus of functions of a single real variable. In day – to – day real – world applications we notice that a quantity under investigation depends on two or more independent variables. For example (i) the temperature (T) on earth's surface depends on its latitude (x) and longitude (y) $(ii)$the volume of a rectangular parallelepiped depends on its length(l), breadth (b) and height(h). Therefore, we need to extend the basic ideas of the calculus of functions of a single real variable to functions of several variables. The rules of this calculus broadly remain the same as the calculus of functions of a single real variable and this calculus is rich in generalization and elegance. The derivatives are more varies and interesting, since the variables can interact in different ways. Their integrals lead to a greater variety of applications. Various studies in probability, statistics, fluid dynamics, electricity and many more lead in natural ways to functions of more than one variable.

$\boldsymbol{n}$– Dimensional Euclidean space

Let $\boldsymbol{R}$ be the set of al real numbers. Then

$$\boldsymbol{R}^2 = \boldsymbol{R} \times \boldsymbol{R} = \{\boldsymbol{x} = (x_1, x_2) |\, x_i \in \boldsymbol{R}\,, i = 1, 2\}$$

i.e., $\boldsymbol{R}^2$ is the set of all ***ordered pairs*** whose components are from $\boldsymbol{R}$.

$$\boldsymbol{R}^3 = \boldsymbol{R} \times \boldsymbol{R} \times \boldsymbol{R} = \{\boldsymbol{x} = (x_1, x_2, x_3) |\ x_i \in \boldsymbol{R}, i = 1, 2, 3\}$$

i.e., $\boldsymbol{R}^3$ is the set of all ***ordered triplets*** whose components are from $\boldsymbol{R}$.

For a natural number n, we have

$$\boldsymbol{R}^n = \underbrace{\boldsymbol{R} \times \boldsymbol{R} \times \ldots \times \boldsymbol{R}}_{(n\ times)} = \{\boldsymbol{x} = (x_1, x_2, \ldots, x_n) | x_i \in \boldsymbol{R}, i = 1, 2, \ldots, n\}$$

The elements of $\boldsymbol{R}^n$ are called ordered ***n – tuplets***.

(The elements of $\boldsymbol{R}^n$ are called **points** or **vectors**, especially when $\boldsymbol{n} > 1$)

$\boldsymbol{R}^1 = \boldsymbol{R}$ is the **Real line**

$\boldsymbol{R}^2$ is the ***plane*** (or the ***complex plane***) and

$\boldsymbol{R}^3$ is the $\boldsymbol{3}$ ***– dimensional Euclidean space.***

Further, $\boldsymbol{R}^n$ is called the $\boldsymbol{n}$ ***– dimensional Euclidean space.***

Distance between two points:

Let $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$, $\boldsymbol{y} = (y_1, y_2, \ldots, y_n) \in \boldsymbol{R}^n$

Let $P(\boldsymbol{x})$ and $Q(\boldsymbol{y})$ be any two points in $\boldsymbol{R}^n$

Then the distance between P and Q is denoted by $d(P, Q)$ or $d(\boldsymbol{x}, \boldsymbol{y})$ and is defined as

$$d(P, Q) = d(\boldsymbol{x}, \boldsymbol{y}) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_n - y_n)^2}$$

Neighborhood of a point

Let $\boldsymbol{a} = (a_1, a_2, \dots, a_n)$ be a point P in $\boldsymbol{R^n}$ and $\delta > 0$.

The $\boldsymbol{\delta}$ – ***neighborhood*** of the point $P(\boldsymbol{a})$ is denoted by $N_\delta(\boldsymbol{a})$ or $N_\delta(P)$ and is defined as

$$N_\delta(\mathrm{P}) = N_\delta(\boldsymbol{a}) = \{\boldsymbol{x} = (x_1, x_2, \dots, x_n) \in \boldsymbol{R^n} \mid d(\boldsymbol{x}, \boldsymbol{a}) < \delta\}$$

$$= \left\{ (x_1, x_2, x_3, \dots, x_n) \in \boldsymbol{R^n} \mid \sqrt{\sum_{i=1}^{n} (x_i - a_i)^2} < \delta \right\}$$

The **deleted $\boldsymbol{\delta}$ – neighborhood** of the point P($\boldsymbol{a}$) is denoted by

$N^*_\delta(\boldsymbol{a})$ or $N^*_\delta(P)$ and

$$N^*_\delta(P) = N^*_\delta(\boldsymbol{a}) = \left\{ (x_1, x_2, \dots, x_n) \in \boldsymbol{R^n} \mid 0 < \sqrt{\sum_{i=1}^{n} (x_i - a_i)^2} < \delta \right\}$$

Function of $\boldsymbol{n}$ Independent variables

Let D be a set of n - tuplets of real numbers, $i.e.$, $D \subseteq \boldsymbol{R^n}$. A ***real valued function $\boldsymbol{f}$ on $\boldsymbol{D}$*** is a rule that assigns a unique real number w to each element $\boldsymbol{x} = (x_1, x_2, \dots, x_n) \in D$.

Then we write $f: D \longrightarrow \boldsymbol{R}$ defined by

$$f(x_1, x_2, \dots, x_n) = w \text{ for all } (x_1, x_2, \dots, x_n) \in D$$

The set D for whose elements, f is defined is called the ***domain*** of f and the set of w – values taken on by f is called the ***range*** of f.

$i.e.$, $\{f(x_1, x_2, \dots, x_n) \mid (x_1, x_2, \dots, x_n) \in D\}$ is the range of f.
The symbol w is the ***dependent variable*** of f and f is said to be a function of n ***independent variables*** $x_1, x_2, \dots, x_n$.
(We call the $x_j, 1 \le j \le n$, the ***input variables*** of f and w, the ***output variable*** of f).

As usual, we evaluate functions defined by formulas by substituting the values of the independent variables in the formula and calculate the corresponding value of the dependent variable.

In defining a function of more than one variable, we follow the usual practice of excluding inputs that lead to complex numbers or division by zero.

Functions of two variables:

If f is a function of two independent variables, we usually denote the independent variables by x and y and the dependent variable by z.

We often write $z = f(x, y)$ to make explicit the value taken on by f at the general point (x, y). A function of two variables is a function whose domain is a subset of $\boldsymbol{R}^2$ (*i.e.,* a region in xy −plane) and whose range is a subset of $\boldsymbol{R}$

Not all functions are given by explicit formulas.

Example 1:

Find the domain and range of $f(x, y) = \sqrt{9 - x^2 - y^2}$

Solution: The domain of f is

$D = \{(x, y) \in \boldsymbol{R^2} |\ 9 - x^2 - y^2 \geq 0\} = \{(x, y) |\ x^2 + y^2 \leq 9\}$

It is the disk with center $(0, 0)$ and radius 3. The range of f is
$\left\{z \in \boldsymbol{R} |\ z = \sqrt{9 - x^2 - y^2}, (x, y) \in D\right\}$

Since z is a positive square root; $z \geq 0$. Notice that
$9 - x^2 - y^2 \leq 9 \implies \sqrt{9 - x^2 - y^2} \leq 3.$ Therefore, the range is $\{z |\ 0 \leq z \leq 3\} = [0, 3]$

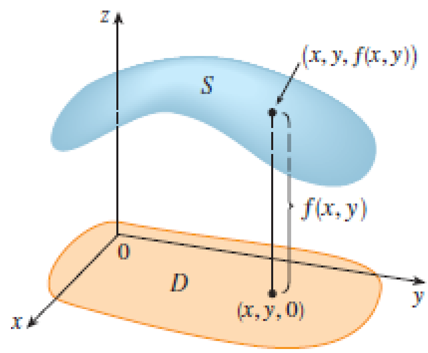
Graph:

A way of visualizing the behavior of a function of two variables is to consider its graph.

If f is a function of two variables with domain D, then the graph of f is the set of all points (x, y, z) in $\boldsymbol{R^3}$ such that

$$z = f(x, y) \text{ and } (x, y) \in D$$

Just as the graph of a function f of one variable is a curve C with equation $y = f(x)$, ***the graph of a function f of two variables is a surface S with equation $z = f(x, y)$***. We can visualize the graph S of f as lying directly above or below its domain in the xy – plane.



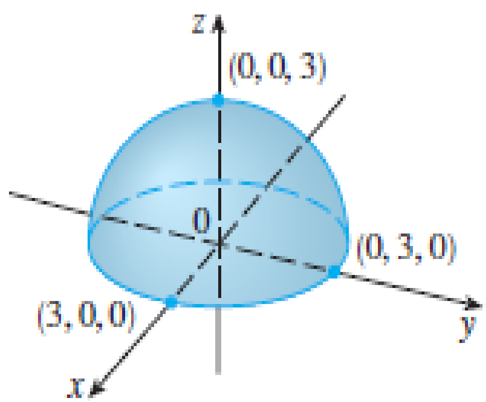
Example 2:

Sketch the graph of $f(x, y) = \sqrt{9 - x^2 - y^2}$

Solution: The graph has equation $z = \sqrt{9 - x^2 - y^2}$. Squaring

$z^2 = 9 - x^2 - y^2$ or $x^2 + y^2 + z^2 = 9$

It is the equation of the sphere with center at the origin and radius 3. Since $z \geq 0$, the graph of f is the top of their sphere, i.e., the upper hemisphere of the sphere $x^2 + y^2 + z^2 = 9$.



Note: An entire sphere can't be represented by a single function of x and y. The lower hemisphere is represented by the function $h(x, y) = -\sqrt{9 - x^2 - y^2}$.

Level Curves:

A method of visualizing functions is a contour map on which points of constant elevation are joined to form ***Contour curves*** or ***level curves***

The ***level curves*** of a function f of two variables are the curves with equations $f(x, y) = k$, where k is a constant in the range of f.

A level curve $f(x, y) = k$ is the set of all points in the domain of f at which f takes on a given value k. That is, it shows where the graph of f has height k.

Example 3:

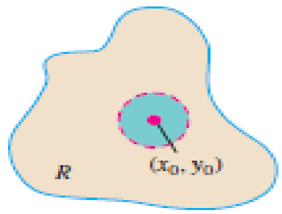
Sketch the level curves of the function

$f(x, y) = \sqrt{9 - x^2 - y^2}$, for $k = 0, 1, 2, 3$.

Solution: The level curves are $\sqrt{9 - x^2 - y^2} = k$ or $x^2 + y^2 = 9 - k^2$. This is a family of concentric circles, with center $(0,0)$ and radius $\sqrt{9 - k^2} \leq 3$. The four particular level curves with $k = 0, 1, 2$ and 3 are $x^2 + y^2 = 9, x^2 + y^2 = 8$, $x^2 + y^2 = 5$ and $x^2 + y^2 = 0$

Interior point

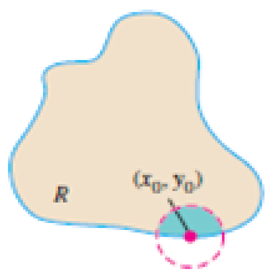
Let R be a region (set) in $\boldsymbol{R}^2$. A point P(x_0, y_0) in R is an **interior point** of R if it is the center of a disk of positive radius that lies entirely in R .



(a) Interior point

Boundary point

A point P(x_0, y_0) is a ***boundary point*** of R if every disk centered at P contains points that lie outside of R as well as points that are in R (The boundary point itself need not belong to R)

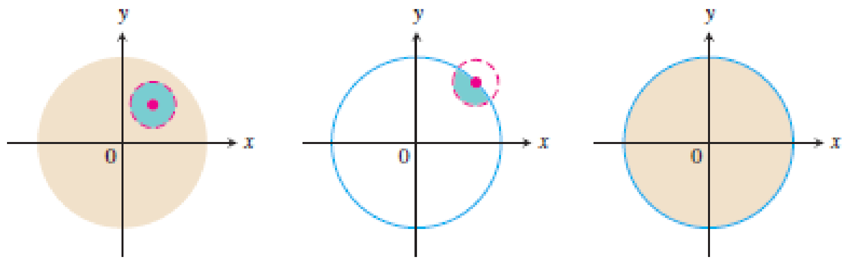


(b) Boundary point

Open and closed regions

A region R is said to be **open** if every point of R is an interior point of R.

A region R is said to be **closed** if R contains all its boundary points.



$\{(x, y) \mid x^2 + y^2 < 1\}$
Open unit disk.
Every point an interior point.

$\{(x, y) \mid x^2 + y^2 = 1\}$
Boundary of unit disk. (The unit circle.)

$\{(x, y) \mid x^2 + y^2 \le 1\}$
Closed unit disk.
Contains all boundary points.

Some regions in the plane are neither open nor closed. If you start with the open disk and add to it some but not all of its boundary points, then the resulting set is neither open nor closed.

Bounded and unbounded regions in the plane

A region in a plane is **bounded** if it lies inside a disk of fixed radius. A region is **unbounded** if it is not bounded.

Some examples of bounded sets in the plane:

Line segments, triangles, interiors of triangles, rectangles, circles and disks.

Some examples of unbounded sets in the plane:

Lines, coordinate axes, quadrants, half planes and the plane itself.

Functions of three variables

If f is a function of three independent variables, we usually call

the independent variables x, y and z and the domain is a region in the 3 – dimensional Euclidean space.

Example 4:

Find the domain of f if $f(x, y\ , z) = ln(z - y) + xy\ sin\ z$

Solution: The expression for $f(x, y, z)$ is defined as long as $z - y > 0$. Therefore, the domain D of f is

$$D = \{(x, y, z) \in R^3 : z > y\}$$

This is a half space consisting of all points that lie above the plane $z = y$.

It is very difficult to visualize a function f of three variables by its graph, since that would lie in a four-dimensional space. However, we do gain some insight into f by examining its **level surfaces**, which are the surfaces with equations $f(x, y, z) = k$, where k is a constant in the range of f. If the point (x, y, z) moves along a level surface, the value of $f(x, y, z)$remains fixed.

Example 5:

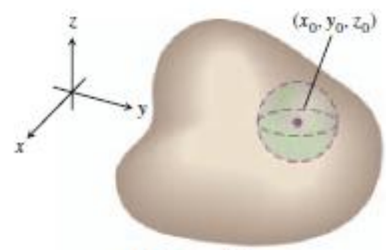
Find the level surfaces of the function $f(x, y, z) = x^2 + y^2 + z^2$

Solution: The level surfaces are $x^2 + y^2 + z^2 = k$, where $k \geq 0$.

These are a family of concentric sphere with radius $\sqrt{k}$. Thus, as (x, y, z) varies over any sphere with center $(0,0,0)$, the value of $f(x, y, z)$ remains fixed.

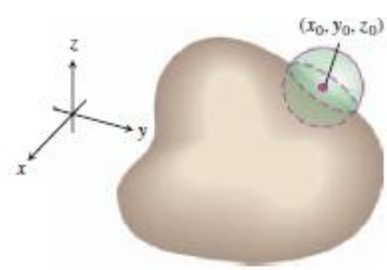
The definitions of interior, boundary, open, closed, bounded and unbounded for regions in space are similar to those for regions in the plane. To accommodate the extra dimension, we use solid balls instant of disks.

Let R be a region (set) in $\boldsymbol{R^3}$**.** A point $P(x_0, y_{0,} z_0)$ in R is an interior point of R if it is the center of a solid ball that lies entirely in R



(a) Interior point

A point $P(x_0, y_{0,} z_0)$ is ***boundary point*** of R if every sphere centered at P encloses points that lie outside of R as well as points that lie inside of R.



(b) Boundary point

The set of interior points of R is the ***interior of*** R. The set of boundary points of R is *the* ***boundary of*** R**.** A region R is said to be ***open*** if every point of R is an interior point of R. A region R is said to be ***closed*** of R contains all its boundary points.

Some examples of open sets in space:

The interior of a sphere, the half space $z > 0$, the first octant and the space itself.

Some examples of closed sets in space:

Lines, planes, the closed half space $z \geq 0$, the first octant together with its bounding planes and space itself.

A solid sphere with part of its boundary removed or a solid cube with a missing face, edge or a corner point would be neither open nor closed.

Linear function

A function of the form $f(x, y) = ax + by + c$ is called a ***linear function***. The graph of such a function has the equation $z = ax + by + c$ or $ax + by - z + c = 0$ and it is a plane. The linear functions of two variables play a central role in multi-variable calculus.

P1.

Find the domain, range and level curves for following functions:

(a) $\boldsymbol{f(x,y) = \sqrt{y - x^2}}$

(b) $\boldsymbol{g(x,y) = \frac{1}{xy}}$

Solution:

(a)

i. Notice that the domain D of the function $f(x,y)$ is

$$D = \{(x,y) \in \boldsymbol{R^2} \mid y - x^2 \geq 0\} = \{(x,y) \in \boldsymbol{R^2} \mid y \geq x^2\}$$

ii. The range of $f(x,y)$ is

$$\left\{z \in \boldsymbol{R} \mid z = \sqrt{y - x^2},\ \ (x,y) \in D\right\}$$

Since z is a positive square root; $z \geq 0$. Therefore,

Range of $f = \{z \in \boldsymbol{R} \mid z \geq 0\} = [0,\infty)$

iii. The level curves of the function $f(x,y)$ are the curves with equations $f(x,y) = k$, where k is a constant in the range of f.

i.e., $\sqrt{y - x^2} = k \Rightarrow y - x^2 = k^2 \Rightarrow y = x^2 + k^2$,

which is a parabola shifted k^2 units upwards.

(b)

i. Notice that the domain D of the function $g(x, y)$ is

$$D = \{(x, y) \in \boldsymbol{R^2} |\ xy \neq 0\}$$

That is, the whole of $\boldsymbol{R^2}$ except the points on x and y axes

ii. The range of $g(x, y)$ is

$$\left\{z \in \boldsymbol{R} \mid z = \frac{1}{xy},\quad (x, y) \in D\right\} = (-\infty, 0) \cup (0, \infty)$$

iii. The level curves of the function $g(x, y)$ are the curves with equations $g(x, y) = k$, where k is a constant in the range of g.

i.e., $\dfrac{1}{xy} = k \Longrightarrow xy = \dfrac{1}{k}$, which are Hyperbolas.

P2.

If $f(x,y) = y - x$ then find the boundary of the domain of f. Determine if the domain of f is an open region or closed region and decide if the domain of f is bounded or unbounded.

Solution:

Notice that $f(x,y) = y - x$ is defined for all $(x,y) \in \boldsymbol{R^2}$. Therefore, the domain of f is the whole of $\boldsymbol{R^2}$.

i. Let P be an arbitrary point in $D = \boldsymbol{R^2}$. Then for any $\delta > 0$, we have $N_\delta(P) \subset \boldsymbol{R^2}$. That is, P is an interior point of $\boldsymbol{R^2}$. Since P is arbitrary, every point of $D\left(= \boldsymbol{R^2}\right)$ is an interior point of $D\left(= \boldsymbol{R^2}\right)$. Thus, $\boldsymbol{R^2}$ is open.

ii. Since every point of $D\left(= \boldsymbol{R^2}\right)$ is an interior point and there are no points outside of $\boldsymbol{R^2}$, there are no boundary points.

Therefore, $b(D) = \phi$, where $b(D)$ denotes the boundary of D. Clearly, $b(D) \subset \boldsymbol{R^2} \Rightarrow \boldsymbol{R^2}$ is closed.

iii. $\boldsymbol{R^2}$ is unbounded, since we cannot enclose $\boldsymbol{R^2}$ in a disk of finite radius.

Note: $\boldsymbol{R^2}$ is both open and closed and it is unbounded.

P3.

If $f(x,y) = \frac{y}{x^2}$ then find the boundary of the domain of f. Determine if the domain of f is an open region or closed region and decide if the domain of f is bounded or unbounded.

Solution:

Notice that the domain D of the function $f(x,y)$ is

$$D = \{(x,y) \in \boldsymbol{R}^2 |\ (x,y) \neq (0,y)\}$$

That is, the domain is the whole of $\boldsymbol{R}^2$ except the points on the y −axis.

i. Let P be an arbitrary point on the y −axis. Then every disk centered at P contains the points (x,y) that lie outside of D (i.e., the points on the diameter of the disk along the y −axis) as well as the points (x,y) that lie in D. Thus, P is a boundary point of D. Since P is an arbitrary point on the y −axis, every point on the y −axis is a boundary point of D. Notice that $b(D)$ is not a subset of D. Therefore, D is not closed.

ii. Notice that every point in D is an interior point. Therefore, D is open.

iii. D is unbounded, because we cannot enclose D in a disk of finite radius.

P4:

Find an equation for the level curve of the function

$$f(x,y)=\int_x^y \frac{dt}{1+t^2}$$ **that passes through the point** $\left(-\sqrt{2},\sqrt{2}\right)$**.**

Solution:

Given $f(x,y)=\int_x^y \frac{dt}{1+t^2}=\left[tan^{-1}t\right]_x^y=tan^{-1}y-tan^{-1}x$

The level curves of a function $f(x,y)$ of two variables are the curves with equations $f(x,y)=k$ where k is a constant in the range of f

$$\text{i.e., } \tan^{-1}y-\tan^{-1}x=k$$

Since the level curve of $f(x,y)$ passes through the point $\left(-\sqrt{2},\sqrt{2}\right)$,

$$\tan^{-1}\left(\sqrt{2}\right)-\tan^{-1}\left(-\sqrt{2}\right)=k \Longrightarrow 2\tan^{-1}\left(\sqrt{2}\right)=k$$

Therefore, the required level curve is

$$\tan^{-1}y-\tan^{-1}x=2\tan^{-1}\left(\sqrt{2}\right)$$

IP1.

Find the domain, range and level curves for following functions:

(a) $\boldsymbol{f(x,y,z) = \sqrt{x^2+y^2+z^2}}$

(b) $\boldsymbol{g(x,y,z) = xylnz}$

Solution:

(a)

i. Notice that the domain D of the function $f(x,y,z)$ is

$$D = \{(x,y,z) \in \boldsymbol{R^3} |\, x^2+y^2+z^2 \geq 0\}$$

That is, the domain is the entire space $\boldsymbol{R^3}$.

ii. The range of $f(x,y,z)$ is

$$\left\{w \in \boldsymbol{R} \mid w = \sqrt{x^2+y^2+z^2},\ \ (x,y,z) \in D\right\}$$

Since w is a positive square root, $w \geq 0$. Therefore,

Range of $f = [0,\infty)$

iii. The level surfaces of the function $f(x,y,z)$ are the surfaces with equations $f(x,y,z) = k$, where k is a constant in the range of f.

i.e., $\sqrt{x^2+y^2+z^2} = k \Longrightarrow x^2+y^2+z^2 = k^2$, which is a sphere of radius k centered at the origin.

(b)

i. Notice that the domain D of the function $g(x, y, z)$ is

$$D = \{(x, y, z) \in \boldsymbol{R^3} |\ z > 0\}$$

That is, the domain is the upper half space $z > 0$

ii. The range of $g(x, y)$ is

$$\{w \in \boldsymbol{R} \mid z = xylnz,\ \ (x, y, z) \in D\} = (-\infty,\ \infty)$$

iii. The level surfaces of the function $g(x, y, z)$ are the surfaces with equations $g(x, y, z) = k$, where k is a constant in the range of g,

$$\text{i.e., } xylnz = k \Longrightarrow z = e^{\left(\frac{k}{xy}\right)}\ .$$

IP2.

If $\boldsymbol{g(x,y)=\sqrt{y-x}}$ then find the boundary of the domain of $\boldsymbol{g}$. Determine if the domain of $\boldsymbol{g}$ is an open region or closed region and decide if the domain of $\boldsymbol{g}$ is bounded or unbounded.

Solution:

Notice that the domain D of the function $g(x,y)$ is

$$D=\{(x,y)\in \boldsymbol{R}^2 \mid y-x\geq 0\}=\{(x,y)\in \boldsymbol{R}^2 \mid y\geq x\}$$

i. Let P be an arbitrary point on the line $y=x$. Then every disk centered at P contains the points (x,y) that lie outside of D (i.e., $y<x$) as well as the points (x,y) that lie in D (i.e., $y>x$). Thus, P is a boundary point of D. Since P is an arbitrary point on the line $y=x$, every point on the line $y=x$ is a boundary point of D. Thus,

$$b(D)=\{(x,y)\in \boldsymbol{R}^2 \mid y=x\}=\{(x,x)\in \boldsymbol{R}^2 \mid x\in \boldsymbol{R}\}$$

ii. Notice that $b(D)\subset D$. Therefore, D is closed.

iii. D is unbounded, because we cannot enclose D in a disk of finite radius.

IP3.

If $\boldsymbol{f(x,y)=ln\left(x^2+y^2\right)}$ then find the boundary of the domain of $\boldsymbol{f}$. Determine if the domain of $\boldsymbol{f}$ is an open region or closed region and decide if the domain of $\boldsymbol{f}$ is bounded or unbounded.

Solution:

Notice that the domain D of the function $f(x,y)$ is

$$D=\{(x,y)\in \boldsymbol{R}^2 \mid (x,y)\neq(0,0)\}=\boldsymbol{R}^2-\{(0,0)\}$$

i. Let P be an arbitrary point in the domain $D=\boldsymbol{R}^2-\{(0,0)\}$. There exists a $\delta>0$ such that $N_\delta(P)\subset D$. Thus, every point in the domain D is an interior point of D. Therefore, D is open.

ii. Let P be the point $(0,0)$. Then every disk centered at P contains points that lie outside of D (namely $P(0,0)$ itself) as well as points inside D. Therefore, $(0,0)$ is the only boundary point of D

iii. D is unbounded, because we cannot enclose D in a disk of finite radius.

IP4:

Find the equation for the level surface of the function

$$f(x,y,z)=\sum_{n=0}^{\infty}\frac{(x+y)^n}{n!\,z^n}$$ **passes through the point $(ln2, ln4, 3)$.**

Solution:

Given $f(x,y,z)=\sum_{n=0}^{\infty}\frac{(x+y)^n}{n!\,z^n}=e^{\left(\frac{x+y}{z}\right)}$

The level surfaces of a function $f(x,y,z)$ of the variables are the surfaces with equations $f(x,y,z)=k$, where k is a constant in the range of f

$$i.e.,\ e^{\left(\frac{x+y}{z}\right)}=k \Rightarrow \frac{x+y}{z}=ln\,k$$

Since the level surfaces of $f(x,y,z)$ passes through the point $(ln2, ln4, 3)$,

$$\frac{ln2+ln4}{3}=lnk \Longrightarrow ln8=3lnk \Longrightarrow ln8=lnk^3 \Longrightarrow k=2$$

Therefore, the required level surface is $\frac{x+y}{z}=ln2$

3.1. Preliminaries

Exercises

I. **Find the domain ,range and level curves(surfaces) of the following functions**

a. $z = sin\ xy$

b. $f(x, y) = 4x^2 + 9y^2$

c. $f(x, y) = xy$

d. $f(x, y) = 100 - x^2 - y^2$

e. $w = \frac{1}{x^2+y^2+z^2}$

II. **Find the boundary of the domain of f. Determine if the domain of f is an open region or closed region and decide if the domain of f is bounded or unbounded.**

a. $f(x, y) = x^2 - y^2$

b. $f(x, y) = \frac{1}{\sqrt{16-x^2-y^2}}$

c. $f(x, y) = e^{-(x^2+y^2)}$

d. $f(x, y) = tan\left(\frac{y}{x}\right)$

III. **Find an equation for the level curve of the following function $f(x,y)$ that passes through the given point.**

a. $f(x,y)=\sqrt{x^2-1}$, $(1,0)$

b. $f(x,y)=\sum_{n=0}^{\infty}\left(\frac{x}{y}\right)^n$, $(1,2)$

IV. **Find an equation for the level surfaces of the following function $f(x,y,z)$ that passes through the given point.**

a. $f(x,y,z)=\sqrt{x-y}-\ln z$, $(3,-1,1)$

b. $g(x,y,z)=\ln(x^2+y+z^2)$ $(-1,2,1)$

c. $g(x,y,z)=\int_x^y \frac{d\theta}{\sqrt{1-\theta^2}}+\int_{\sqrt{2}}^{z}\frac{dt}{t\sqrt{t^2-1}}$ $\left(0,\frac{1}{2},2\right)$

Answers:

I.

a. Domain(D): Entire plane

Range : $[-1,1]$

Level curve is $xy=\sin^{-1}k$

b. Domain(D): Entire plane

Range : $[0, \infty)$

Level curve: $4x^2 + y^2 = k$

c. Domain : Entire plane $\left(\boldsymbol{R^2}\right)$

Range: $(-\infty, \infty)$

Level curve: $xy = k$

d. Domain: Entire plane$(\boldsymbol{R^2})$

Range: $[0,100]$

Level curves: $x^2 + y^2 = 100 - k^2$

e. Domain :$(x, y, z) \neq (0,0,0)$

Range: $(0, \infty)$

Level surface: $x^2 + y^2 + z^2 = \frac{1}{k}$

II.

a. Domain : all points in xy plane

No boundary points

Both open and closed

Unbounded

b. Domain :All points(x, y) satisfying $x^2 + y^2 < 16$

Boundary is the circle $x^2 + y^2 = 16$

Open

Bounded

c. Domain :entire xy −plane
No boundary points
Both open and closed
Unbounded

d. Domain : entire xy −plane except $x = 0$
Boundary is the line $x = 0$
Open
Unbounded

III.

a. $x = 1$ or $x = -1$
b. $y = 2x$

IV.

a. $\sqrt{x - y} - lnz = 2$
b. $x^2 + y^2 + z^2 = 4$

$$sin^{-1}y - sin^{-1}x + sec^{-1}z = \frac{\pi}{2}$$

3.2

Limits and Continuity

Learning objectives:

- To define limits of function of two and three variables
- To define the continuity of a function of two and three variables at a point
  AND

To practice the related problems

Limits and Continuity

In this module we discuss limits and continuity for functions of two variables. The definition of the limit of a function of two variables is similar to the definition of the limit of a function of a single real variable, but with a difference.

Limits

If the values $f(x, y)$ lie arbitrarily close to a fixed real number L for all points (x, y) sufficiently close to a point (x_0, y_0), we say that f approaches the limit L as (x, y) approaches (x_0, y_0). Notice that, if (x_0, y_0) lies in the interior of the domain of f, then (x, y) can approach (x_0, y_0) from any direction.

Limit of a function of two variables

Let $D \subseteq \boldsymbol{R^2}$ and (x_0, y_0) be a point in D. Let $z = f(x, y)$ be a function (of two variables) defined on D except possibly at (x_0, y_0).

If for every given real number $\varepsilon > 0$, however small, there exists a real number $\delta > 0$ such that

$$|f(x, y) - L| < \varepsilon, \text{ for all } (x, y) \in {N_\delta}^*(P)$$

i.e., $|f(x, y) - L| < \varepsilon$, whenever $0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta$

then the finite real number L is called the ***limit of the function*** $\boldsymbol{f(x,y)}$ **as** $\boldsymbol{(x,y) \to (x_0,y_0)}$.

Symbolically, we write it as

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$$

Note:

(1) The definition of limit says that the distance between $f(x,y)$ and L becomes arbitrary small whenever the distance from (x,y) to (x_0,y_0) is made sufficiently small (but not 0).

(2) The definition of limit applies to boundary points (x_0,y_0) as well as interior points of the domain of f. ***The only requirement is that the point*** $\boldsymbol{(x,y)}$ ***remain in the domain at all times while reaching*** $\boldsymbol{(x_0,y_0)}$.

(3) The δ in the definition depends on ε and the point (x_0,y_0), in general.

(4) $\displaystyle \lim_{(x,y)\to(x_0,y_0)} f(x,y)$, ***if it exists is unique***.

(5) Since $(x,y) \to (x_0,y_0)$ in the two-dimensional plane in ${N_\delta}^*(P)$, there are infinite number of paths joining (x,y) to (x_0,y_0) in the δ- nbd of P. Since the limit is unique, the limit is same along all paths (inside the δ- nbd of P). That is the limit is independent of the path. Thus, the limit of a function cannot be obtained by approaching the point $P(x_0,y_0)$ along a particular path and finding the limit of

the function. If the limit depends on a path, then the limit does not exist.

Two – Path Test for Nonexistence of a limit

If a function $f(x, y)$ has different limits along different paths as (x, y) approaches (x_0, y_0), then $\lim\limits_{(x,y)\to(x_0,y_0)} f(x, y)$ does not exist.

(6) If $x = r\cos\theta$, $y = r\sin\theta$ (where $r^2 = x^2 + y^2$ and $\theta = \tan^{-1}\frac{y}{x}$), then the definition of the limit $\lim\limits_{(x,y)\to(0,0)} f(x, y) = L$ reduces to

$$|f(r\cos\theta, r\sin\theta) - L| < \varepsilon, \text{ whenever } r < \delta,$$

independent of θ.

(7) It is easy to see that

$$\lim_{(x,y)\to(x_0,y_0)} x = x_0 \quad ; \quad \lim_{(x,y)\to(x_0,y_0)} y = y_0$$

and $\lim\limits_{(x,y)\to(x_0,y_0)} k = k$ (for any constant $k \in \boldsymbol{R}$)

Theorem 1: Properties of Limits of Functions of Two Variables

If L, M and k are real numbers and

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = L \text{ and } \lim_{(x,y)\to(x_0,y_0)} g(x, y) = M$$

then the following rules hold:

1. ***Sum rule:*** $\lim\limits_{(x,y)\to(x_0,y_0)} [f(x, y) + g(x, y)] = L + M$

2. ***Difference Rule:*** $\lim\limits_{(x,y)\to(x_0,y_0)} [f(x,y) - g(x,y)] = L - M$

3. ***Product Rule:*** $\lim\limits_{(x,y)\to(x_0,y_0)} [f(x,y).g(x,y)] = L.M$
4. ***Constant Multiple Rule:*** $\lim\limits_{(x,y)\to(x_0,y_0)} \big(kf(x,y)\big) = kL$

   where k is any real constant.

5. ***Quotient Rule:*** $\lim\limits_{(x,y)\to(x_0,y_0)} \frac{f(x,y)}{g(x,y)} = \frac{L}{M}$, $M \neq 0$

6. ***Power Rule:*** If r and s are integers with no common factors and $s \neq 0$, then $\lim\limits_{(x,y)\to(x_0,y_0)} \left(f(x,y)\right)^{r/s} = L^{r/s}$, provided $L^{r/s}$ is a real number. (If s is even, then we assume $L > 0$).

Example 1: Applying the Limit definition

Find $\lim\limits_{(x,y)\to(0,0)} \frac{4xy^2}{x^2+y^2}$ if it exists.

Solution: Notice that the function always has value 0 along the line $x = 0$, when $y \neq 0$ and the function has value 0 along the line $y = 0$, when $x \neq 0$. This shows that the limit of the function as $(x, y) \to (0,0)$, if exists must be 0. To verify this we apply $\varepsilon - \delta$ definition of the limit.

For every given $\varepsilon > 0$, we have to find a $\delta > 0$ such that

$$\left|\frac{4xy^2}{x^2+y^2} - 0\right| < \varepsilon, \text{ whenever } 0 < \sqrt{x^2+y^2} < \delta$$

$$\text{or} \quad \frac{4|x|y^2}{x^2+y^2} < \varepsilon, \text{ whenever } 0 < \sqrt{x^2+y^2} < \delta$$

Now, $\frac{4|x|y^2}{x^2+y^2} \leq 4|x|$ since $y^2 \leq x^2 + y^2$

$$\leq 4\sqrt{x^2+y^2}\text{ , since } |x| \leq \sqrt{x^2+y^2}$$

$$< \varepsilon \text{ whenever } 0 < \sqrt{x^2+y^2} < \frac{\varepsilon}{4}$$

Thus, for each $\varepsilon > 0$ $\exists$ a $\delta = \frac{\varepsilon}{4}$ such that

$$\left|\frac{4xy^2}{x^2+y^2} - 0\right| < \varepsilon \text{ whenever } 0 < \sqrt{x^2+y^2} < \delta$$

Therefore, $\lim\limits_{(x,y)\to(0,0)} \frac{4xy^2}{x^2+y^2} = 0$

When we apply Theorem1 to polynomials and rational functions, we obtain the useful result that the limits of these functions as $(x, y) \longrightarrow (x_0, y_0)$can be calculated ***by evaluating the functions at*** $(\boldsymbol{x_0}, \boldsymbol{y_0})$. The only requirement is that the rational functions be defined at (x_0, y_0).

Example 2:

Find the limits

$(a)\ \lim\limits_{(x,y)\to(0,0)} \frac{3x^2 - y^2 + 5}{x^2 + y^2 + 2}$ $\qquad$ $(b)\ \lim\limits_{(x,y)\to(3,4)} \sqrt{x^2 + y^2 - 1}$

Solution:

(a) We have $\lim\limits_{(x,y)\to(0,0)} x = 0, \lim\limits_{(x,y)\to(0,0)} 2 = 2$

$$\Rightarrow \lim_{(x,y)\to(0,0)} x^2 = \lim_{(x,y)\to(0,0)} x. \lim_{(x,y)\to(0,0)} x = 0.0 = 0$$

Similarly $\lim\limits_{(x,y)\to(0,0)} y^2 = 0.$

Thus, $\lim\limits_{(x,y)\to(0,0)} (x^2 + y^2 + 2)$

$$= \lim_{(x,y)\to(0,0)} x^2 + \lim_{(x,y)\to(0,0)} y^2 + \lim_{(x,y)\to(0,0)} 2 = 0 + 0 + 2 \neq 0$$

and, $\lim\limits_{(x,y)\to(0,0)} (3x^2 - y^2 + 5)$

$$= 3\lim_{(x,y)\to(0,0)} x^2 - \lim_{(x,y)\to(0,0)} y^2 + \lim_{(x,y)\to(0,0)} 5 = 5$$

Therefore, $\lim\limits_{(x,y)\to(0,0)} \dfrac{3x^2 - y^2 + 5}{x^2 + y^2 + 2} = \dfrac{\lim\limits_{(x,y)\to(0,0)} (3x^2 - y^2 + 5)}{\lim\limits_{(x,y)\to(0,0)} (x^2 + y^2 + 2)} = \dfrac{5}{2}$

(b) We have $\lim\limits_{(x,y)\to(3,4)} x = 3, \lim\limits_{(x,y)\to(3,4)} y = 4, \lim\limits_{(x,y)\to(3,4)} 1 = 1$

$$\Rightarrow \lim_{(x,y)\to(3,4)} x^2 = 9, \lim_{(x,y)\to(3,4)} y^2 = 16$$

Thus, $\lim\limits_{(x,y)\to(3,4)} x^2 + y^2 - 1 = 9 + 16 - 1 = 24$

Therefore, $\lim\limits_{(x,y)\to(3,4)} \sqrt{x^2 + y^2 - 1} = \sqrt{24} = 2\sqrt{6}$

Example 3: Find $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2 - xy}{\sqrt{x} - \sqrt{y}}$

Solution: It is a rational function and notice that the denominator $\sqrt{x} - \sqrt{y} \to 0$ as $(x, y) \to (0,0)$. Therefore, we cannot use quotient rule.

Now, $\dfrac{x^2 - xy}{\sqrt{x} - \sqrt{y}} = \dfrac{x(x-y)(\sqrt{x} + \sqrt{y})}{(\sqrt{x} - \sqrt{y})(\sqrt{x} + \sqrt{y})}$

$$= \frac{x(x-y)(\sqrt{x} + \sqrt{y})}{x-y} = x(\sqrt{x} + \sqrt{y})$$

We can cancel $(x - y)$ because the points on the line $y = x$ are not in the domain of the function. So,

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - xy}{\sqrt{x} - \sqrt{y}} = \lim_{(x,y)\to(0,0)} x\left(\sqrt{x} + \sqrt{y}\right)$$

$$= 0(\sqrt{0} + \sqrt{0}) = 0 \qquad (\because it\ is\ a\ polynomial)$$

Example 4: Applying the Two-Path test

Show that the function $f(x,y) = \frac{2x^2y}{x^4+y^2}$ has no limit as $(x,y) \to (0,0)$

Solution: The limit cannot be found by the use of quotient rule because the denominator is 0 as $(x,y) \longrightarrow (0,0)$. We examine the value of f along curves that end at $(0,0)$, say $y = kx^2, x \neq 0$.

Then $f(x,y)|_{y=kx^2} = \frac{2x^2(kx^2)}{x^4+k^2x^4} = \frac{2k}{1+k^2}$

$$\therefore \lim_{\substack{(x,y)\to(0,0) \\ along\ y=kx^2}} f(x,y) = \frac{2k}{1+k^2}$$

Thus, $f(x,y)$have different limits along different paths as $(x,y) \to (0,0)$. By Two-Path Test, f has no limit as $(x,y) \to (0,0)$.

Continuity:

As in the case of functions of a single real variable, continuity of functions of two or more variables is defined in terms of limits.

Continuous functions of two variables

A function $f(x,y)$ is said to be ***continuous at the point*** $\boldsymbol{P(x_0,y_0)}$ if

(i) f is defined at (x_0,y_0)

(ii) $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ exists

(iii) $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0)$

Let D be the domain of the function f and $S \subseteq D$. We say that ***f* *is continuous*** **in *S*** if f is continuous at every point of S.

That is, $f(x,y)$ is continuous at $P(x_0,y_0)$ if for every given $\epsilon > 0$, there exists a $\delta > 0$ such that

$|f(x,y) - f(x_0,y_0)| < \epsilon$, whenever $(x,y) \in N_\delta(P)$

i.e., $|f(x,y) - f(x_0,y_0)| < \epsilon$, whenever $\sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta$

We say that a function is ***discontinuous*** at a point P if it is not continuous at P.

Note(1): In the definition of continuity,

$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0)$ holds for all paths approaching the point $P(x_0,y_0)$. Therefore, if the continuity of a function at a point is to be proved, we cannot choose a path and find the limit. However, if we want to show that a function is discontinuous, it is enough to choose a path and show that the limit does not exist.

Removable discontinuity

If $f(x_0,y_0)$is defined and $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$ exists,

$L \neq f(x_0,y_0)$ then the point (x_0,y_0) is said to be a point of ***removable discontinuity***.

In this case we redefine the function at (x_0, y_0) as $f(x_0, y_0) = L$, so that the redefined function is continuous at (x_0, y_0).

A consequence of theorem 1 is that the algebraic combinations of continuous functions are continuous at every point at which all the functions involved are continuous. That is,

The sums, differences, products, constant multiples, quotients and powers of continuous functions are continuous where they are defined.

In particular, polynomials and rational functions of two (or more) variables are continuous at every point at which they are defined.

A continuous function has the following properties:

P1: A continuous function in a closed and bounded domain D attains once its maximum value M and its minimum value m at some point inside or on the boundary of D.

P2: For any number μ that satisfies $m < \mu < M$, there exist a point (x_0, y_0) in D such that $f(x_0, y_0) = \mu$.

P3: A continuous function, in a closed and bounded domain D that attains both positive and negative values will have the value 0 at some point in D.

Example 5: show that

$$f(x,y) = \begin{cases} \frac{2xy}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}.$$

is continuous at every point except at the origin.

Solution: Notice that the domain of $f(x,y)$is $\boldsymbol{R^2}$.

Let $(x_0, y_0) \neq (0,0)$ be an arbitrary point.

$$\lim_{(x,y)\to(x_0,y_0)} 2xy = 2 \lim_{(x,y)\to(x_0,y_0)} x. \lim_{(x,y)\to(x_0,y_0)} y = 2x_0y_0 \textit{ and}$$

$$\lim_{(x,y)\to(x_0,y_0)} (x^2 + y^2)$$

$$= \lim_{(x,y)\to(x_0,y_0)} x^2 + \lim_{(x,y)\to(x_0,y_0)} y^2 \quad = x_0^2 + y_0^2 \neq 0.$$

Therefore, $\lim_{(x,y)\to(x_0,y_0)} \frac{2xy}{x^2+y^2} = \frac{\lim_{(x,y)\to(x_0,y_0)} 2xy}{\lim_{(x,y)\to(x_0,y_0)} (x^2+y^2)}$

$$= \frac{2x_0y_0}{x_0^2+y_0^2} = f(x_0, y_0)$$

Thus, f is continuous at $(x_0, y_0) \neq (0,0)$. Since it is an arbitrary point, it is continuous at every point of $\boldsymbol{R^2} - \{(0,0)\}$.

Continuity at $(\boldsymbol{0}, \boldsymbol{0})$:The function f is defined at $(0,0)$.We show that f has no limit as $(x,y) \to (0,0)$ along $y = mx$, $x \neq 0$. Then

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \left[f(x,y)|_{y=mx} \right]$$

$$= \lim_{(x,y)\to(0,0)} \frac{2mx^2}{x^2+m^2x^2} = \frac{2m}{1+m^2}$$

The limit changes with m. By Two- Path Test $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ does not exist. Thus, $f(x,y)$ is discontinuous at $(0,0)$. Therefore, $f(x,y)$ is continuous at every point except at the origin. That is, $f(x,y)$ is continuous in $\boldsymbol{R^2} - \{(0,0)\}$.

Continuity of Composites:

If f is continuous at (x_0, y_0) and g is a single variable function continuous at $f(x_0, y_0)$, then the composite function $h = gof$ defined by $h(x,y) = g(f(x,y))$ is continuous at (x_0, y_0).

Example 6: Show that the function e^{x-y} is continuous at every point $(x,y) \in \boldsymbol{R^2}$ and find $\lim_{(x,y)\to(0,ln\,2)} e^{x-y}$.

Solution: Let $f(x,y) = x - y$. First we note that f is continuous at every point (x,y) of $\boldsymbol{R^2}$(because it is a polynomial).

Let $g(t) = e^t$. Further, g is continuous at every point $t \in \boldsymbol{R}$ and so g is continuous at $f(x,y)$. Now, the composite function $h = gof$ defined by $h(x,y) = g\big(f(x,y)\big) = g(x-y) = e^{x-y}$ is continuous at every point $(x,y) \in \boldsymbol{R^2}$. Thus, $h(x,y) = e^{x-y}$ is continuous at $(0, ln2)$. Therefore,

$$\lim_{(x,y)\to(0,ln2)} h(x,y) = h(0,ln2)$$

$$\Rightarrow \lim_{(x,y)\to(0,ln2)} e^{x-y} = e^{0-ln2} = e^{ln\left(\frac{1}{2}\right)} = \frac{1}{2}$$

Functions of more than two variables

The definitions of limit and continuity for functions of two variables and the conclusions about limits and continuity for sums, difference, products, quotients, powers and composites all extend to functions of three or more variables.

Example 7: Find $\lim_{(x,y,z)\to(1,-1,-1)} \frac{2xy+yz}{x^2+z^2}$

Solution: The given function is a rational function and $x^2 + z^2 \neq 0$ at$(1,-1,-1)$. Therefore,

$$\lim_{(x,y,z)\to(1,-1,-1)} \frac{2xy+yz}{x^2+z^2} = \frac{2(1)(-1)+(-1)(-1)}{1^2+-1^2} = -\frac{1}{2}$$

Example 8:

a) At what points (x,y,z)in space is the function $h(x,y,z) = xy\, sin\frac{1}{z}$ continuous?

b) Find $\lim_{(x,y,z)\to(1,1,\frac{2}{\pi})} h(x,y,z)$

Solution: Let $f(x,y,z) = \frac{1}{z}$. It is continuous at every point$(x,y,z) \in \boldsymbol{R^3}, z \neq 0$ (Since it is a rational function)

Let $g(t) = sint$. Clearly, $g(t)$ is continuous at every $t \in \boldsymbol{R}$ and so, is continuous at $f(x, y, z)$, where $(x, y, z) \in \boldsymbol{R^3}, z \neq 0$

Now, the composite function $h_1 = gof$ defined by $h_1(x, y, z) = g\big(f(x, y, z)\big) = g\left(\frac{1}{z}\right) = sin\frac{1}{z}$ is continuous at every point $(x, y, z) \in \boldsymbol{R^3}, z \neq 0$.

Let $h_2(x, y, z) = xy$.It is continuous at every point $(x, y, z) \in \boldsymbol{R^3}$. Now, the product $h(x, y, z) = h_2(x, y, z).h_1(x, y, z) = xysin\frac{1}{z}$ is continuous at every point $(x, y, z) \in \boldsymbol{R^3}, z \neq 0$.

(b) Since h is continuous at $(1,1,\frac{2}{\pi})$

$$\lim_{(x,y,z)\to(1,1,\frac{2}{\pi})} h(x, y, z) = h\left(1,1,\frac{2}{\pi}\right) = 1.1.sin\frac{\pi}{2} = 1.$$

P1:

(a) Find $\lim\limits_{(x,y)\to(0,0)}\left(\dfrac{xy}{\sqrt{x^2+y^2}}\right)$**, if it exists.**

(b) Find $\lim\limits_{\substack{(x,y)\to(2,\,-4)\\ y\neq-4,\,x\neq x^2}}\left(\dfrac{y+4}{x^2y-xy+4x^2-4x}\right)$**, if it exists**

Solution:

(a) Notice that the given function always has value 0 along the line $x=0$, when $y\neq 0$ and the function has value 0 along the line $y=0$, when $x\neq 0$.

This shows that the limit of the function as $(x,y)\to(0{,}0)$, if exists must be 0. To verify this we apply $\varepsilon-\delta$ definition of the limit.

For every given $\varepsilon>0$, we have to find a $\delta>0$ such that

$$\left|\frac{xy}{\sqrt{x^2+y^2}}-0\right|<\varepsilon,\text{ whenever } 0<\sqrt{x^2+y^2}<\delta$$

Now, $\left|\dfrac{xy}{\sqrt{x^2+y^2}}-0\right|=\dfrac{|xy|}{\sqrt{x^2+y^2}}\leq\dfrac{1}{2}\dfrac{(x^2+y^2)}{\sqrt{x^2+y^2}}$, since $|xy|\leq\dfrac{x^2+y^2}{2}$

$$\leq\frac{1}{2}\sqrt{x^2+y^2}$$

$$<\varepsilon\text{ , whenever } 0<\sqrt{x^2+y^2}<2\varepsilon$$

Thus, for each $\varepsilon>0\ \exists$ a $\delta=2\varepsilon>0$ such that

$$\left|\frac{xy}{\sqrt{x^2+y^2}} - 0\right| < \varepsilon, \text{ whenever } 0 < \sqrt{x^2+y^2} < \delta$$

Therefore, $\lim_{(x,y)\to(0,0)} \left(\frac{xy}{\sqrt{x^2+y^2}}\right) = 0$

(b)

$$\lim_{\substack{(x,y)\to(2,-4)\\ y\neq -4, x\neq x^2}} \frac{(y+4)}{x^2y - xy + 4x^2 - 4x}$$

$$= \lim_{\substack{(x,y)\to(2,-4)\\ y\neq -4,\ x\neq x^2}} \frac{(y+4)}{x^2(y+4) - x(y+4)}$$

$$= \lim_{\substack{(x,y)\to(2,-4)\\ y\neq -4, x\neq x^2}} \frac{(y+4)}{x(x-1)(y+4)}$$

$$= \lim_{\substack{(x,y)\to(2,-4)\\ x\neq x^2}} \frac{1}{x(x-1)} = \frac{1}{2(2-1)} = \frac{1}{2}$$

P2.

a) **Find** $\lim\limits_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$ **, if it exists**

b) **Find** $\lim\limits_{(x,y)\to(0,0)} \frac{x+\sqrt{y}}{x^2+y}$**, if it exists**

Solution:

The limit of the given function cannot be found by the use of quotient rule because the denominator is 0 as $(x, y) \to (0,0)$.

We examine the value of f along the curves that end at $(0,0)$,

say $y = kx, x \neq 0$

Then $f(x, y)|_{y=kx} = \frac{x.kx}{x^2+k^2x^2} = \frac{k}{1+k^2}$

$\therefore \lim\limits_{(x,y)\to(0,0)} f(x, y) = \frac{k}{1+k^2}$

Thus, $f(x, y)$ have different limits along different paths as $(x, y) \to (0,0)$.

By Two -path Test, f has no limit as $(x, y) \to (0,0)$.

Hence, the limit of the given function does not exist.

(b)

The limit of the given function cannot be found by the use of quotient rule because the denominator is 0 as $(x, y) \to (0,0)$.

We examine the value of f along the curves that end at $(0,0)$, say $y = kx^2, x \neq 0$

Then $f(x,y)|_{y=kx^2} = \frac{x+\sqrt{kx^2}}{x^2+kx^2} = \frac{1+\sqrt{k}}{(1+k)x}$

$$\therefore \lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{1+\sqrt{k}}{(1+k)x} = \infty$$

Since the limit is not finite, the limit does not exist.

P3.

Discuss the continuity of the following functions:

(a) $\boldsymbol{f(x,y) = \begin{cases} \frac{2x^4+3y^4}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) \neq (0,0) \end{cases}}$ **, at the point** $\boldsymbol{(0,0)}$

(b) $\boldsymbol{h(x,y) = \cos\left(\frac{x^2+y^2}{x+y+1}\right)}$**, at the point** $\boldsymbol{(0,0)}$

Solution:

(a) The given function is defined for all $(x,y) \in \boldsymbol{R^2}$

$f(x,y) = 2x^2 + 3y^2 - \frac{5x^2y^2}{x^2+y^2}$ (by actual division) and

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \left[2x^2 + 3y^2 - \frac{5x^2y^2}{x^2+y^2}\right]$$

$$= -5 \lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^2+y^2}$$

Notice that the function $\frac{x^2y^2}{x^2+y^2}$ always has value 0 about the line $x = 0$, when $y \neq 0$ and this function has value 0 about the line $y = 0$, when $x \neq 0$. This shows that the limit of the function as $(x,y) \to (0,0)$, if exists must be 0. To verify this we apply $\varepsilon - \delta$ definition of the limit.

For any given $\varepsilon > 0$, we have to find a $\delta > 0$ such that

$$\left|\frac{x^2y^2}{x^2+y^2} - 0\right| < \varepsilon, \text{ whenever } 0 < x^2 + y^2 < \delta$$

Now, $\left|\frac{x^2y^2}{x^2+y^2} - 0\right| = \frac{x^2y^2}{x^2+y^2} \leq \frac{1}{4}\frac{(x^2+y^2)^2}{x^2+y^2}$ (since $|xy| \leq \frac{x^2+y^2}{2}$)

$$= \frac{1}{4}(x^2 + y^2) < \varepsilon, \text{ whenever } x^2 + y^2 < 4\varepsilon$$

For each $\varepsilon > 0$ $\exists$ a $\delta \leq 4\varepsilon$ such that $\left|\frac{x^2y^2}{x^2+y^2} - 0\right| < \varepsilon$, whenever $0 < x^2 + y^2 < \delta$

$$\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^2+y^2} = 0$$

Therefore,

$$\lim_{(x,y)\to(0,0)} f(x,y) = -5 \lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^2+y^2} = -5(0) = 0 = f(0,0)$$

$\therefore \lim_{(x,y)\to(0,0)} f(x,y) = f(0,0) = 0$

Thus, $f(x,y)$ is a continuous function at the point $(0,0)$

(b) Let $f(x,y) = \frac{x^2+y^2}{x+y+1}$

Now, $f(x,y)$ is a rational function, so it is continuous every where in $\boldsymbol{R^2}$ except at the points on the line $x + y = -1$

Notice that $(0,0)$ is not point on the line $x + y = -1$.

Therefore, $f(x,y)$ is continuous at $(0,0)$ and $f(0,0) = 0$

Let $g(t) = \cos t$ and $g(t)$ is continuous at $0 = f(0,0)$

Now, the composite function $h = gof$ defined by

$h(x, y) = g(f(x, y)) = g\left(\frac{x^2+y^2}{x+y+1}\right) = \cos\left(\frac{x^2+y^2}{x+y+1}\right)$ is continuous at $(0,0)$.

Therefore,

$$\lim_{(x,y)\to(0,0)} h(x, y) = h(0,0) = \cos\left(\frac{0^2+0^2}{0+0+1}\right) = cos0 = 1$$

P4

Discuss the continuity of the function

$$f(x,y)=\begin{cases}\dfrac{xy}{\sqrt{x^2+y^2}} & ,\ (x,y)\neq(0,0)\\ 4 & ,\ (x,y)=(0,0)\end{cases}$$

Solution:

In $P1$ problem we have seen that $\lim\limits_{(x,y)\to(0,0)}\left(\dfrac{xy}{\sqrt{x^2+y^2}}\right)=0$

Notice that $f(0,0)$ is defined and

$$\lim_{(x,y)\to(0,0)} f(x,y)=0\neq f(0,0)$$

$\Longrightarrow f$ is discontinuous at $(0,0)$.

This discontinuity is called Removable discontinuity.

Now, define $f(x,y)=0$ at $(x,y)=(0,0)$. Then f is continuous at $(0,0)$.

IP1:

(a) Show that $\lim\limits_{(x,y,z)\to(0,0,0)}\left[\dfrac{xy+xz+yz}{\sqrt{x^2+y^2+z^2}}\right]=0$

(b) Find $\lim\limits_{(x,y,z)\to(0,0,0)}\dfrac{xy^2z^2}{x^4+y^4+z^8}$ **if it exists.**

Solution:

(a) Given, $f(x,y,z)=\frac{xy+xz+yz}{\sqrt{x^2+y^2+z^2}}$

Show that $f(x,y,z)\to(0,0,0)$ as $(x,y,z)\to(0,0,0)$

For this, given $\varepsilon>0$, we have to find a $\delta>0$ such that

$$\left|\frac{xy+xz+yz}{\sqrt{x^2+y^2+z^2}}-0\right|<\varepsilon, \text{ whenever } 0<\sqrt{x^2+y^2+z^2}<\delta$$

Now, $\left|\frac{xy+xz+yz}{\sqrt{x^2+y^2+z^2}}-0\right|=\frac{|xy|+|xz|+|yz|}{\sqrt{x^2+y^2+z^2}}\leq\frac{1}{2}\left[\frac{x^2+y^2+x^2+z^2+y^2+z^2}{\sqrt{x^2+y^2+z^2}}\right]$

Since $|xy|\leq\frac{x^2+y^2}{2}$, $|yz|\leq\frac{y^2+z^2}{2}$, $|xz|\leq\frac{x^2+z^2}{2}$

$$\leq\frac{1}{2}\left[\frac{2(x^2+y^2+z^2)}{\sqrt{x^2+y^2+z^2}}\right]$$

$$<\sqrt{x^2+y^2+z^2}<\varepsilon\text{ , whenever } 0<\sqrt{x^2+y^2+z^2}<\varepsilon$$

Thus, for each $\varepsilon>0$ $\exists$ a $\delta\leq\varepsilon$ such that

$$\left|\frac{xy+xz+yz}{\sqrt{x^2+y^2+z^2}}-0\right|<\varepsilon, \text{ whenever } 0<\sqrt{x^2+y^2+z^2}<\delta$$

Therefore, $\lim\limits_{(x,y,z)\to(0,0,0)}\left[\dfrac{xy+xz+yz}{\sqrt{x^2+y^2+z^2}}\right]=0$

(b) Notice that the denominator of the given function approaches to 0 as $(x, y, z) \to (0, 0, 0)$.

To evaluate the limit

$$\lim_{(x,y,z)\to(0,0,0)} \frac{xy^2z^2}{x^4+y^4+z^8}$$

Choose the paths $z = \sqrt{x}$, $y = mx$, we get

$$\lim_{\substack{(x,y,z)\to(0,0,0)\\ \text{along } z=\sqrt{x},y=mx}} \frac{xy^2z^2}{x^4+y^4+z^8} = \lim_{(x,y,z)\to(0,0,0)} \frac{x(mx)^2\left(\sqrt{x}\right)^2}{x^4+(mx)^4+\left(\sqrt{x}\right)^8}$$

$$= \lim_{(x,y,z)\to(0,0,0)} \frac{x^4m^2}{x^4(2+m^4)} = \lim_{(x,y,z)\to(0,0,0)} \frac{m^2}{(2+m^4)}$$

Thus, $f(x, y, z)$ have different limits along different paths as $(x, y, z) \to (0, 0, 0)$.

By Two-Path Test, f has no limit as $(x, y, z) \to (0, 0, 0)$.

IP2.

(a) **Find** $\displaystyle \lim_{(x,y)\to(0,0)} \frac{x^3y}{x^6+y^2}$ **, if it exists**

(b) **Find** $\displaystyle \lim_{(x,y)\to(0,0)} \frac{xy}{|xy|}$ **, if it exists**

Solution:

The limit of the given function cannot be found by the use of quotient rule because the denominator is 0 as $(x, y) \to (0,0)$.

We examine the value of f along the curves that end at $(0,0)$,

say $y = kx^3, x \neq 0$

Then $f(x, y)|_{y=kx^3} = \frac{x^3kx^3}{x^6+k^2x^6} = \frac{k}{1+k^2}$

$\therefore \displaystyle \lim_{(x,y)\to(0,0)} f(x, y) = \lim_{(x,y)\to(0,0)} \frac{k}{1+k^2}$

Thus, $f(x, y)$ have different limits along different paths as $(x, y) \to (0,0)$. By Two-Path Test, f has no limits as $(x, y) \to (0,0)$.

Hence, the limit of the given function does not exist.

(b)

The limit of the given function cannot be found by the use of quotient rule because the denominator is 0 as $(x, y) \to (0,0)$.

We examine the value of f along the curves that end at $(0,0)$, say $y = kx^2, x \neq 0$

Then $f(x,y)|_{y=kx^2} = \frac{x\ kx}{|x\ kx|} = \frac{k}{|k|}$

$$\therefore \lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{k}{|k|} = \begin{cases} 1 & if \quad k > 0 \\ -1 & if \quad k < 0 \end{cases}$$

Thus, $f(x,y)$ have different limits along the different points as $(x,y) \to (0,0)$. By Two-Path Test, f has no limit as $(x,y) \to (0,0)$.

Hence, the limit of the given function does not exist.

IP3.

Discuss the continuity of the following functions at the given point

(a) $\boldsymbol{f(x,y) = \begin{cases} \frac{2x(x^2-y^2)}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) \neq (0,0) \end{cases} \quad ; (0,0)}$

(b) $\boldsymbol{h(x,y) = \ln(1 + x^2y^2) \quad ; (1,1)}$

Solution:

(a) Let $x = rcos\theta, y = rsin\theta.$

Then, $r = \sqrt{x^2 + y^2} \neq 0$.We have

$$|f(x,y) - f(0,0)| = \left|\frac{2x(x^2-y^2)}{x^2+y^2} - 0\right|$$

$$= \left|\frac{2r^3(cos^2\theta - sin^2\theta)cos\theta}{r^2(cos^2\theta + sin^2\theta)}\right| = |2r\, cos2\theta\, cos\theta| \leq 2r < \varepsilon,$$

whenever $r = \sqrt{x^2 + y^2} < \frac{\varepsilon}{2}$

For any $\varepsilon > 0$ ∃ a $\delta < \frac{\varepsilon}{2}$, we find that $|f(x,y) - f(0,0)| < \varepsilon$, whenever $0 < \sqrt{x^2 + y^2} < \delta$

Therefore, $\lim\limits_{(x,y)\to(0,0)} f(x,y) = f(0,0)$

Hence, $f(x,y)$ is continuous at $(0,0)$.

(b) Let $f(x, y) = 1 + x^2y^2$

Now, $f(x, y)$ is a polynomial. So it is continuous everywhere in $\boldsymbol{R^2}$**.** Thus, $f(x, y)$ is continuous at $(1, 1)$ and $f(1, 1) = 2$.

Let $g(t) = \ln t$, $t > 0$ and $g(t)$ is continuous at $2 = f(1, 1)$

Now, the composite function $h = gof$ defined by

$$h(x, y) = g\big(f(x, y)\big) = g(1 + x^2y^2) = \ln(1 + x^2y^2)$$

is continuous at $(1,1)$.

Therefore,

$$\lim_{(x,y)\to(1,1)} h(x, y) = h(1,1) = \ln(1 + (1)^2(1)^2) = ln2$$

IP4.

Discuss the continuity of the function

$$\boldsymbol{f(x,y,z) = \ln xyz}; \quad \boldsymbol{xyz} > 0 \quad ; (\mathbf{1},\mathbf{1},\mathbf{1})$$

Solution:

Given $f(x,y,z) = \ln xyz$

Let $h(x,y,z) = xyz$. First we note that h is continuous at every point (x,y,z) of $\boldsymbol{R}^3$(because it is a polynomial). So $h(x,y,z)$ is continuous at $(1,1,1)$ and $h(1,1,1) = 1.1.1 = 1$

Let $g(t) = \ln t\,, t > 0$. Further, g is continuous at $1 = h(1,1,1)$ Now, the composite function $f = g\ o\ h$ defined by $f(x,y,z) = g\big(h(x,y,z)\big) = g(xyz) = \ln(xyz)$ is continuous at $(1,1,1)$. Therefore,

$$\lim_{(x,y,z)\to(0,0,0)} f(x,y,z) = f(1,1,1) = \ln(1.1.1) = ln1 = 0$$

$$\Longrightarrow \lim_{(x,y,z)\to(0,0,0)} \ln xyz = 0$$

3.2. Limits and Continuity

Exercises:

I. Use the $\delta-\varepsilon$ approach, establish the following limits.

1. $\lim\limits_{(x,y)\to(0,0)} \dfrac{x+y}{x^2+y^2+1}=0$

2. $\lim\limits_{(x,y)\to(0,0)} \left[y+x\cos\left(\dfrac{1}{y}\right)\right]=0$

3. $\lim\limits_{(x,y)\to(0,0)} \left(x^2+y^2\right)\sin\dfrac{1}{xy}=0$

Determine the following limits if they exists

4. $\lim\limits_{(x,y)\to(0,0)} \dfrac{x}{\sqrt{x^2+y^2}}$

5. $\lim\limits_{(x,y)\to(0,1)} \dfrac{(y-1)\tan^2 x}{x^2\left(y^2-1\right)}$

6. $\lim\limits_{(x,y,z)\to(0,0,0)} \dfrac{xy+z}{x+y+z^2}$

7. $\lim\limits_{(x,y,z)\to(0,0,0)} \dfrac{x(x+y+z)}{x^2+y^2+z^2}$

Answers:

4. Limit does not exist.
5. $\frac{1}{2}$
6. Limit does not exist.

7. Limit does not exist.

II. Find

i. (a) $\lim_{(x,y)\to(1,1)} \frac{x^2-2xy+y^2}{x-y}, \ x \neq y$ (b) $\lim_{(x,y)\to(2,2)} \frac{x+y-4}{\sqrt{x+y}-2}$

ii. $\lim_{(x,y)\to(2,0)} \frac{\sqrt{2x-y}-2}{2x-y-4}$, $2x - y \neq 4$

iii. $\lim_{(x,y)\to(0,0)} \frac{x-y+2\sqrt{x}+2\sqrt{y}}{\sqrt{x}-\sqrt{y}}, \ x \neq y$

Answers:

i. (a) 0 (b) 4

ii. $\frac{1}{4}$

iii. 2

III.

Discuss the continuity of the following functions at the given points.

a) $f(x,y) = \begin{cases} \frac{(x-y)^2}{x^2+y^2} & , \quad (x,y) \neq (0,0) \\ 0 & , \quad (x,y) = (0,0) \end{cases}$ at $(0,0)$.

b) $f(x,y) = \begin{cases} \frac{x^2+y^2}{\tan xy} & , \quad (x,y) \neq (0,0) \\ 0 & , \quad (x,y) = (0,0) \end{cases}$ at $(0,0)$.

c) $f(x,y) = \begin{cases} \frac{xy(x-y)}{x^2+y^2} & , \quad (x,y) \neq (0,0) \\ 0 & , \quad (x,y) = (0,0) \end{cases}$ at $(0,0)$.

d) $f(x,y,z) = \begin{cases} \frac{xyz}{x^2+y^2+z^2} & , \quad (x,y,z) \neq (0,0,0) \\ 0 & , \quad (x,y,z) = (0,0,0) \end{cases}$ at $(0,0,0)$.

Answers:

a) Discontinuous.
b) Discontinuous.
c) Continuous.
d) Continuous.

3.3

Partial Derivatives

Learning objectives:

* To define partial derivatives.
* To calculate partial derivatives.
* To discuss the second and higher order partial derivatives.
* To state mixed derivative theorem.
  AND
* To practice the related problems.

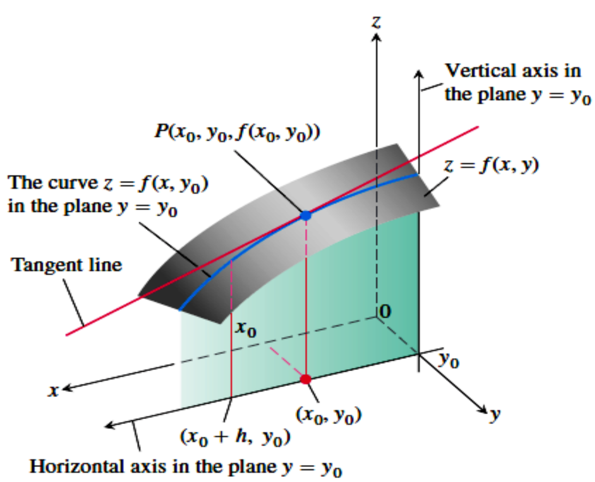
Partial Derivatives

The calculus of several variables is basically single variable calculus applied to severable variables one at a time. If we hold all but one of the independent variables of a function constant and differentiate w.r.t. that variable, then we get a partial derivative.

In this module we introduce the concept of partial derivatives and calculate them by applying the rules for differentiating function of a single real variable.

Partial derivative of a function of two variables

If (x_0, y_0) is a point in the domain of a function $f(x, y)$, then the vertical plane $y = y_0$ cuts the surface $z = f(x, y)$ in the curve $z = f(x, y_0)$. This curve is the graph of the function $z = f(x, y_0)$ in the plane$y = y_0$. The horizontal coordinate is x, the vertical coordinate is z and the y- value is held constant at y_0. Therefore, y is not a variable.



We define the partial derivative of f w.r.t x at the point (x_0, y_0) as the ordinary derivative of $f(x, y_0)$ w.r.t x at the point $x = x_0$. To distinguish partial derivative from ordinary derivative we use the symbol ∂ rather than the d previously used.

Partial derivative w.r.t x

The partial derivative of $f(x,y)$ w.r.t x at the point (x_0, y_0) is

$$\left.\frac{\partial f}{\partial x}\right|_{x=x_0} = \lim_{h\to 0} \frac{f(x_0+h, y_0)-f(x_0,y_0)}{h}$$

provided the limit exists.

Note:

(1) **Notation:** The partial derivative of f or z w.r.t. x is denoted by $\frac{\partial f}{\partial x}$, f_x , $\frac{\partial z}{\partial x}$ or z_x .
The partial derivative of f or z w.r.t x at (x_0, y_0) is denoted by $\left(\frac{\partial f}{\partial x}\right)_{(x_0,y_0)}$, $f_x(x_0, y_0)$, $\left.\frac{\partial z}{\partial x}\right|_{(x_0,y_0)}$ or $z_x(x_0, y_0)$.

(2) An equivalent expression for the partial derivative

$$\left.\frac{d}{dx} f(x, y_0)\right|_{x=x_0}$$

That is, the slope of the curve $z = f(x, y_0)$ at the point $P\big(x_0, y_0, f(x_0, y_0)\big)$ in the plane $y = y_0$ is $\left(\frac{df}{dx}\right)_{(x_0,y_0)}$, the value of the partial derivative of f w.r.t. x at (x_0, y_0). The tangent line to the curve $z = f(x, y_0)$ at $P(x_0, y_0)$ is the line in the plane $y = y_0$ that passes through P with this slope.

(3) The partial derivative $\frac{\partial f}{\partial x}$ at (x_0, y_0) gives the rate of change of f w.r.t. x when y is held fixed at y_0. This is the rate of change of f in the direction of the positive x-axis at (x_0, y_0) (i.e., in the direction of vector $\boldsymbol{i}$).

The definition of the partial derivative of $f(x, y)$ w.r.t. y at the point (x_0, y_0) is similar to the definition of the partial derivative of f w.r.t x. We hold x fixed at the value x_0 and take the ordinary derivative of $f(x_0, y)$ w.r.t. y at y_0.

Partial derivative w.r.t y

The partial derivative of $f(x, y)$ w.r.t y at the point (x_0, y_0) is

$$\left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)} = \left.\frac{d}{dx} f(x_0, y)\right|_{y=y_0} = \lim_{h\to 0} \frac{f(x_0\,, y_0+h)-f(x_0,y_0)}{h}$$

provided the limit exists.

Note:

(1) **Notation:** The partial derivative of f or z w.r.t. y is denoted by $\frac{\partial f}{\partial y}$, f_y , $\frac{\partial z}{\partial y}$ or z_y .

The partial derivative of f or z w.r.t y at (x_0, y_0) is denoted by $\left(\frac{\partial f}{\partial y}\right)_{(x_0,y_0)}$, $f_y(x_0, y_0)$, $\left.\frac{\partial z}{\partial y}\right|_{(x_0,y_0)}$ or $z_y(x_0, y_0)$.

(2) The slope of the curve $z = f(x_0, y)$ at the point $P\big(x_0, y_0, f(x_0, y_0)\big)$ in the vertical plane $x = x_0$ is $\left(\frac{\partial f}{\partial y}\right)_{(x_0,y_0)}$, the value of f w.r.t. y at (x_0, y_0). The tangent line to the curve $z = f(x_0, y)$ at $P(x_0, y_0)$ is the line in the plane $x = x_0$ that passes through P with this slope.

(3) The partial derivative $\frac{\partial f}{\partial y}$ at (x_0, y_0) gives the rate of change of f w.r.t. y when x is held fixed at x_0. This is the rate of change of f in the direction of the positive y-axis at (x_0, y_0) (i.e., in the direction of the vector $\boldsymbol{j}$).

Example 1:

Find the first order partial derivatives of $f(x, y) = ye^{-x}$ at the point (x, y) from the first principles.

Solution: We have,

$$\frac{\partial f}{\partial x} = \lim_{h\to 0} \frac{f(x+h\,,y)-f(x,y)}{h} = \lim_{h\to 0} \frac{ye^{-x-h}-ye^{-x}}{h}$$

$$= -ye^{-x} \lim_{h\to 0} \frac{1-e^{-h}}{h} = -ye^{-x} \lim_{h\to 0} -(-e^{-h}) = -ye^{-x}$$

$$\frac{\partial f}{\partial y} = \lim_{h\to 0} \frac{f(x\,,y+h)-f(x,y)}{h} = \lim_{h\to 0} \frac{(y+h)e^{-x}-ye^{-x}}{h} = e^{-x}$$

The definition of $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ give us two different ways of differentiating f at a point w.r.t.x in the usual way while treating y as constant and w.r.t. y in the usual way while treating x as constant.

Example 2: Finding partial derivatives at a point.

Find $\left(\frac{\partial f}{\partial x}\right)_{(4,-5)}$ and $\left(\frac{\partial f}{\partial y}\right)_{(4,-5)}$ if $f(x,y) = x^2 + 3xy + y - 1$.

Solution:

To find $\frac{\partial f}{\partial x}$, we treat y as a constant and differentiate w.r.t.x.

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x^2 + 3xy + y - 1) = 2x + 3 \cdot 1 \cdot y + 0 - 0 = 2x + 3y$$

Now, $\left(\frac{\partial f}{\partial x}\right)_{(4,-5)} = 2(4) + 3(-5) = -7$

To find $\frac{\partial f}{\partial y}$, we treat x as a constant and differentiate w.r.t.y.

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x^2 + 3xy + y - 1) = 0 + 3x \cdot 1 + 1 - 0 = 3x + 1$$

and $\left(\frac{\partial f}{\partial y}\right)_{(4,-5)} = 3(4) + 1 = 13$

Example 3: Finding Partial derivative of a function

Find $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ if $f(x,y) = x\cos xy$.

Solution: $\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x\cos xy) = x \cdot \frac{\partial}{\partial x}(\cos xy) + \cos xy \cdot \frac{\partial}{\partial x}(x)$

$$= -x\sin xy \frac{\partial}{\partial x}(xy) + \cos xy$$

$$= -x\sin xy \cdot y \cdot 1 + \cos xy = -xy\sin xy + \cos xy$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x\cos xy)$$

$$= x\frac{\partial}{\partial y}(\cos xy) = -x\sin xy\frac{\partial}{\partial y}(xy) = -x^2\sin xy$$

Example 4:

Find f_x , f_y if $f(x,y) = \frac{2y}{y+\cos\ x}$.

Solution: We treat f as a quotient. Treating y as constant, we obtain

$$f_x = \frac{\partial}{\partial x}\left(\frac{2y}{y+\cos\ x}\right) = \frac{(y+\cos\ x)\cdot\frac{\partial}{\partial x}(2y) - 2y\cdot\frac{\partial}{\partial x}(y+\cos\ x)}{(y+\cos\ x)^2}$$

$$= \frac{(y+\cos\ x)(0) - 2y(-\sin\ x)}{(y+\cos\ x)^2} = \frac{2y\sin\ x}{(y+\cos\ x)^2}$$

Treating x as constant, we obtain

$$f_y = \frac{\partial}{\partial y}\left(\frac{2y}{y+\cos\ x}\right) = \frac{(y+\cos\ x)\cdot\frac{\partial}{\partial y}(2y) - 2y\cdot\frac{\partial}{\partial y}(y+\cos\ x)}{(y+\cos\ x)^2}$$

$$= \frac{(y+\cos\ x)2 - 2y(1)}{(y+\cos\ x)^2} = \frac{2\cos\ x}{(y+\cos\ x)^2}$$

Implicit partial differentiation works for partial derivatives the way it works for ordinary derivatives.

Example 5: Implicit partial differentiation

Find $\frac{\partial z}{\partial x}$ if the equation $yz - \ln z = x + y$ defines z as a function of two independent variables x and y.

Solution: We differentiate both sides of the equation w.r.t.x, treating y as constant and treating z as a differentiable function of x:

$$\frac{\partial}{\partial x}(yz - \ln z) = \frac{\partial}{\partial x}(x + y) \Rightarrow \frac{\partial}{\partial x}(yz) - \frac{\partial}{\partial z}(\ln z) = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial x}$$

$$\Rightarrow y\frac{\partial z}{\partial x} - \frac{1}{z}\frac{\partial z}{\partial x} = 1 + 0 \Rightarrow \left(y - \frac{1}{z}\right)\frac{\partial z}{\partial x} = 1 \Rightarrow \frac{\partial z}{\partial x} = \frac{z}{yz - 1}$$

Example 6: Finding the slope of a surface in the y-direction.

The plane $x = 1$ intersects the paraboloid $z = x^2 + y^2$ in a parabola. Find the slope of the tangent to the parabola at $(1,2,5)$.

Solution: The slope is the value of $\frac{\partial z}{\partial y}$ at $(1,2)$.

$$\left(\frac{\partial z}{\partial y}\right)_{(1,2)} = \frac{\partial}{\partial y}(x^2 + y^2)\Big|_{(1,2)} = 2y|_{(1,2)} = 2(2) = 4$$

Functions of more than two variables

The definitions of the partial derivatives of more than two independent variables are like the definitions for functions of two variables. They are ordinary derivatives w.r.t one variable, taken while the other independent variables are held constant.

Example 7: A function of three variables

If x, y, z are independent variables and $f(x, y, z) = x \ \sin(y + 3z)$, then find $\frac{\partial f}{\partial z}$.

Solution: Given $f(x, y, z) = x \ \sin(y + 3z)$.

$$\frac{\partial f}{\partial z} = \frac{\partial}{\partial z}(x \ \sin(y + 3z)) = x \frac{\partial}{\partial z}(\sin(y + 3z))$$

$$= x \ \cos(y + 3z) \ \frac{\partial}{\partial z}(y + 3z) = 3x \cos(y + 3z).$$

Partial derivatives and continuity

A function $f(x, y)$can have partial derivates w.r.t both x and y at a point without the function being continuous there. Note that this is different from functions of a single variable, where the existence of a *derivative implies continuity.*

Example 8: Partial derivatives exist, but f is discontinuous

Show that the function $f(x, y) = \begin{cases} \frac{xy}{x^2+2y^2} & , (x, y) \neq (0,0) \\ 0 & , \ (x, y) = (0,0) \end{cases}$

is not continuous at $(0,0)$ but its partial derivatives f_x and f_y exist at $(0,0)$.

Solution: Choose the path $y = mx$, and let $(x, y) \to (0,0)$ along $y = mx, x \neq 0$.

$$f(x,y)|_{y=mx} = \frac{mx^2}{x^2(1+m^2)} = \frac{m}{1+m^2}$$

Now, $\lim\limits_{\substack{(x,y)\to(0,0) \\ along\ \ y=mx}} f(x,y) = \frac{m}{1+m^2}$

Since the limit depends on m,By Two-Path Test $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ does not exist.

Therefore, $f(x,y)$ is discontinuous at $(0,0)$.

Now, $f_x\,(0,0) = \lim\limits_{h\to0} \frac{f(0+h,0)-f(0,0)}{h} = \lim\limits_{h\to0} \frac{0-0}{h} = 0$

$$f_y\,(0,0) = \lim_{h\to0} \frac{f(0,0+h)-f(0,0)}{h} = lim_{h\to0} \frac{0-0}{h} = 0$$

Thus, the partial derivatives f_x and f_y exist at $(0,0)$, but the $f(x,y)$is discontinuous at $(0,0)$.

The following is a sufficient condition for continuity:

Theorem 1: Sufficient condition for continuity

A sufficient condition for a function $f(x,y)$ to be continuous at a point (x_0, y_0) is that one of its first order partial derivatives

exists and is bounded in a neighborhood of (x_0, y_0)***and that the other exists at*** (x_0, y_0)***.***

Second order partial derivatives

If we differentiate a function $f(x, y)$ twice, then we produce its second-order derivatives. These derivatives are usually denoted by $\frac{\partial^2 f}{\partial x^2}$ or f_{xx} ; or $\frac{\partial^2 f}{\partial y^2}$ or f_{yy} ; $\frac{\partial^2 f}{\partial x \partial y}$ or f_{yx} ; $\frac{\partial^2 f}{\partial y \partial x}$ or f_{xy}

The defining equations are

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right); \ \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right); \ f_{yx} = \left(f_y\right)_x; \dots \dots \dots$$

The derivatives f_{xy} and f_{yx} are called **mixed derivatives**

Example 9: Finding the second-order partial derivates

If$f(x, y) = x\ cosy + ye^x$,then find $\frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial y \partial x}, \frac{\partial^2 f}{\partial y^2}$ and $\frac{\partial^2 f}{\partial x \partial y}$

Solution:

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(xcos\ y + ye^x) = \cos y + ye^x$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = -\sin y + e^x$$

$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = ye^x$ and

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(xcos\ y + ye^x) = -x\ sin\ y + e^x$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = -\sin y + e^x$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = -x\cos y$$

Notice that the *mixed derivatives* are equal.

Example 10:

For the function $f(x,y) = \begin{cases} \frac{xy(2x^2-3y^2)}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$

Show that $f_{xy}(0,0) \neq f_{yx}(0,0)$.

Solution:

$$f_x(0,0) = \lim_{h\to 0} \frac{f(0+h,0)-f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0$$

$$f_x(0,y) = \lim_{h\to 0} \frac{f(0+h,y)-f(0,y)}{h} = \lim_{h\to 0} \frac{\frac{hy(2h^2-3y^2)}{h^2+y^2}-0}{h}$$

$$= \lim_{h\to 0} \frac{y(2h^2-3y^2)}{h^2+y^2} = -\frac{3y^3}{y^2} = -3y$$

$$f_y(0,0) = \lim_{h\to 0} \frac{f(0,0+h)-f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0$$

$$f_y(x,0) = \lim_{h\to 0} \frac{f(x,0+h)-f(x,0)}{h} = \lim_{h\to 0} \frac{\frac{xh(2x^2-3h^2)}{x^2+h^2}-0}{h}$$

$$= \lim_{h\to 0} \frac{x(2x^2-3h^2)}{x^2+h^2} = \frac{2x^3}{x^2} = 2x$$

Now,

$$f_{xy}(0,0) = \left[\frac{\partial}{\partial y} f_x\right]_{(0,0)} = \lim_{h\to 0} \frac{f_x(0,0+h) - f_x(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{-3h - 0}{h} = -3$$

$$f_{yx}(0,0) = \left[\frac{\partial}{\partial x} f_y\right]_{(0,0)} = \lim_{h\to 0} \frac{f_y(0+h,0) - f_y(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{2h - 0}{h} = 2$$

Thus, $f_{xy}(0,0) \neq f_{yx}(0,0)$

Theorem 2: The mixed derivatives theorem (Clairaut's Theorem)

If $f(x, y)$ and its partial derivatives f_x, f_y, f_{xy} and f_{yx} are defined throughout an open region containing a point (a, b), then

$$\boldsymbol{f_{xy}(a,b) = f_{yx}(a,b)}$$

(i.e., the order of differentiation is immaterial).

Partial derivatives of still higher order

We deal mostly with first and second order partial derivatives, because these derivatives occur most frequently is applications. There is no limit to how many times we can differentiate a function as long as the derivatives involved exist. We get the third and fourth order derivatives denoted by symbols like

$$\frac{\partial^3 f}{\partial x \partial y^2} = f_{yyx}\,; \quad \frac{\partial^4 f}{\partial x^2 \partial y^2} = f_{yyxx}\,; \; \dots \dots$$

As in the case of second-order derivatives, the order of differentiation is immaterial as long as all the derivatives through the order in question are continuous.

Laplace's equations:

If U is a function of two variables x and , then the partial differential equation

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = 0$$

is called ***Laplace's equation in two variables or Two-dimensional Laplace's equation.***

If V is a function of three variables x, y and z then the partial differential equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

is called ***Laplace's equation in three variables or Three-dimensional Laplace's equation***

Example 11:

Show that the function $f(x, y) = ln\sqrt{x^2 + y^2}$ satisfies the Laplace's Equation.

Solution:

$$\frac{\partial f}{\partial x} = \frac{1}{\sqrt{x^2+y^2}} \frac{1}{2}(x^2 + y^2)^{-\frac{1}{2}} \ (2x) = \frac{x}{x^2+y^2} \ ; \ \ \frac{\partial f}{\partial y} = \frac{y}{x^2+y^2}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{(x^2+y^2)(1)-x.2x}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{(x^2+y^2)(1)-y.2y}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2}$$

$$\therefore \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$

Thus, $f(x, y)$ satisfies the Laplace's equation.

P1.

Find the first order partial derivatives of the following functions

a. $\boldsymbol{f(x,y) = Sin(2x+3y)}$

b. $\boldsymbol{f(x,y) = 4+2x-3y-xy^2}$ **at the point** $\boldsymbol{(-2,1)}$

from the first principle.

Solution:

a. We have

$$\frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h,y)-f(x,y)}{h} = \lim_{h \to 0} \frac{Sin\ (2(x+h)+3y)-\sin(2x+3y)}{h}$$

$$= \lim_{h \to 0} \frac{2\cos\left(\frac{2x+2h+3y+2x+3y}{2}\right).Sin\left(\frac{2x+2h+3y-2x-3y}{2}\right)}{h}$$

$$= 2\lim_{h \to 0} \frac{cos\ (2x+3y+h)\sin\ h}{h}$$

$$= 2\lim_{h \to 0} \cos(2x+3y+h) \lim_{h \to 0} \frac{\sin h}{h} = 2\cos(2x+3y)$$

$$\frac{\partial f}{\partial y} = \lim_{h \to 0} \frac{f(x,y+h)-f(x,y)}{h} = \lim_{h \to 0} \frac{sin\ (2x+3(y+h))-\sin\ (2x+3y)}{h}$$

$$= \lim_{h \to 0} \frac{2\ cos\left(\frac{2x+3y+3h+2x+3y}{2}\right).\ sin\left(\frac{2x+3y+3h-2x-3y}{2}\right)}{h}$$

$$= \lim_{h \to 0} \frac{2\ cos\left(2x+3y+\frac{3h}{2}\right).\ sin\ \left(\frac{3h}{2}\right)}{h}$$

$$= 3\lim_{h\to 0} \cos\left(2x + 3y + \frac{3h}{2}\right)\frac{sin\ (3h/2)}{(3h/2)}$$

$$= 3\lim_{h\to 0} cos\left(2x + 3y + \frac{3h}{2}\right)\lim_{h\to 0}\frac{sin\ (3h/2)}{(3h/2)} = 3\cos(2x + 3y)$$

b. $f_x(x,y) = \lim_{h\to 0}\frac{f(x+h,\ y)-f(x,y)}{h}$

$$= \lim_{h\to 0}\frac{4+2(x+h)-3y-(x+h)y^2-\left[4+2x-3y-xy^2\right]}{h}$$

$$= \lim_{h\to 0}\frac{\left(4+2x-3y-xy^2\right)+\left(2h-hy^2\right)-\left[4+2x-3y-xy^2\right]}{h}$$

$$= \lim_{h\to 0}\frac{h\left(2-y^2\right)}{h} = 2 - y^2$$

$f_x(-2,1) = 2 - (1)^2 = 1$

$f_x(x,y) = \lim_{h\to 0}\frac{f(x,y+h)-f(x,y)}{h}$

$$= \lim_{h\to 0}\frac{4+2x-3(y+h)-x-(y+h)^2-\left[4+2x-3y-xy^2\right]}{h}$$

$$= \lim_{h\to 0}\frac{\left(4+2x-3y-xy^2\right)+\left(-3h-h^2x-2hxy\right)-\left[4+2x-3y-xy^2\right]}{h}$$

$$= \lim_{h\to 0}\frac{h\ (-3-hx-2xy)}{h}$$

$$= \lim_{h\to 0}(-3 - h - 2xy) = -3 - 2xy$$

$f_x(-2,1) = -3 - 2(1)(-2) = 1$

P2:

If $z(x+y) = x^2 + y^2$, show that

$$\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 = 4\left(1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)$$

Solution: Given $z = \frac{x^2+y^2}{x+y}$

Differentiating z partially with respect to x,

$$\frac{\partial z}{\partial x} = \frac{(x+y)(2x)-(x^2+y^2)(1)}{(x+y)^2} = \frac{2x^2+2xy-x^2-y^2}{(x+y)^2} = \frac{x^2+2xy-y^2}{(x+y)^2}$$

Differentiating z partially with respect to y,

$$\frac{\partial z}{\partial y} = \frac{(x+y)(2y)-(x^2+y^2)(1)}{(x+y)^2} = \frac{-x^2+2xy+y^2}{(x+y)^2}$$

Now $\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} = \frac{x^2+2xy-y^2}{(x+y)^2} - \left[\frac{-x^2+2xy+y^2}{(x+y)^2}\right]$

$$\Rightarrow \left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right) = \frac{2x^2-2y^2}{(x+y)^2} = \frac{2(x+y)(x-y)}{(x+y)^2} = \frac{2(x-y)}{(x+y)}$$

$$\Rightarrow \left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 = \frac{4(x-y)^2}{(x+y)^2} \qquad \text{------} \qquad (1)$$

Now $1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} = 1 - \left[\frac{x^2+2xy-y^2}{(x+y)^2} + \frac{y^2-x^2+2xy}{(x+y)^2}\right]$

$$= 1 - \frac{4xy}{(x+y)^2} = \frac{(x+y)^2-4xy}{(x+y)^2} = \frac{(x-y)^2}{(x+y)^2}$$

$$\Rightarrow 4\left(1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right) = \frac{4(x-y)^2}{(x+y)^2} \qquad \text{------} \qquad (2)$$

From (1) and (2), we have

$$\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 = 4\left(1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)$$

P3:

If $\boldsymbol{U} = \log(x^3 + y^3 + z^3 - 3xyz)$, prove that

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 U = \frac{-9}{(x+y+z)^2}$$

Solution: Given that $U = \log(x^3 + y^3 + z^3 - 3xyz)$

$$\therefore \frac{\partial U}{\partial x} = \frac{3x^2 - 3yz}{x^3+y^3+z^3-3xyz} \quad ; \quad \frac{\partial U}{\partial y} = \frac{3y^2 - 3xz}{x^3+y^3+z^3-3xyz}$$

$$\frac{\partial U}{\partial z} = \frac{3z^2 - 3xy}{x^3+y^3+z^3-3xyz}$$

$$\therefore \frac{\partial U}{\partial x} + \frac{\partial U}{\partial y} + \frac{\partial U}{\partial z} = \frac{3(x^2+y^2+z^2-xy-yz-zx)}{x^3+y^3+z^3-3xyz}$$

$$= \frac{3(x^2+y^2+z^2-xy-yz-zx)}{(x+y+z)(x^2+y^2+z^2-xy-yz-zx)}$$

$$= \frac{3}{x+y+z} \qquad \text{------ (1)}$$

Now,

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 U = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right) . \left(\frac{\partial U}{\partial x} + \frac{\partial U}{\partial y} + \frac{\partial U}{\partial z}\right)$$

$$= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right) . \left(\frac{3}{x+y+z}\right) \qquad \text{[from (1)]}$$

$$= \frac{\partial}{\partial x}\left(\frac{3}{x+y+z}\right) + \frac{\partial}{\partial y}\left(\frac{3}{x+y+z}\right) + \frac{\partial}{\partial z}\left(\frac{3}{x+y+z}\right)$$

$$= -\frac{3}{(x+y+z)^2} - \frac{3}{(x+y+z)^2} - \frac{3}{(x+y+z)^2} = -\frac{9}{(x+y+z)^2}$$

P4:

Find all the second order partial derivatives of the function $\boldsymbol{f(x,y) = \ln(x^2+y^2) + \tan^{-1}\left(\frac{y}{x}\right), (x,y) \neq (0,0).}$

Solution: We have

$$f_x(x,y) = \frac{2x}{x^2+y^2} + \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(-\frac{y}{x^2}\right) = \frac{2x-y}{x^2+y^2}$$

$$f_y(x,y) = \frac{2y}{x^2+y^2} + \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(\frac{1}{x}\right) = \frac{2y+x}{x^2+y^2}$$

$$f_{xy}(x,y) = \frac{\partial}{\partial y}(f_x) = \frac{\partial}{\partial y}\left(\frac{2x-y}{x^2+y^2}\right)$$

$$= \frac{(x^2+y^2)(-1)-(2x-y)(2y)}{(x^2+y^2)^2} = \frac{y^2-x^2-4xy}{(x^2+y^2)^2}$$

$$f_{yx}(x,y) = \frac{\partial}{\partial x}(f_y) = \frac{\partial}{\partial x}\left(\frac{2y+x}{x^2+y^2}\right)$$

$$= \frac{(x^2+y^2)(1)-(2y+x)(2x)}{(x^2+y^2)^2} = \frac{y^2-x^2-4xy}{(x^2+y^2)^2}$$

$$f_{xx}(x,y) = \frac{\partial}{\partial x}(f_x) = \frac{\partial}{\partial x}\left(\frac{2x-y}{x^2+y^2}\right)$$

$$= \frac{(x^2+y^2)(2)-(2x-y)(2x)}{(x^2+y^2)^2} = \frac{2y^2-2x^2+2xy}{(x^2+y^2)^2}$$

$$f_{yy}(x,y) = \frac{\partial}{\partial y}(f_y) = \frac{\partial}{\partial x}\left(\frac{2y+x}{x^2+y^2}\right)$$

$$= \frac{(x^2+y^2)(2)-(2y+x)(2y)}{(x^2+y^2)^2} = \frac{2x^2-2y^2-2xy}{(x^2+y^2)^2}$$

Notice that $f_{xy}(0,0) = f_{yx}(0,0)$.

IP1.

Find the first order partial derivatives of the following functions

(a) $\boldsymbol{f(x,y) = x^2 + y^2 + x}$**, at the point** $\boldsymbol{(x,y)}$

(b) $\boldsymbol{f(x,y) = 1 - x + y - 3x^2y}$**,** $\boldsymbol{\frac{\partial f}{\partial x}}$ **and** $\boldsymbol{\frac{\partial f}{\partial y}}$ **at** $\boldsymbol{(1,2)}$

from the first principle.

Solution:

a) We have

$$\frac{\partial f}{\partial x} = \lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h}$$

$$= \lim_{h\to 0} \frac{[(x+h)^2+y^2+(x+h)]-[x^2+y^2+x]}{h} = \lim_{h\to 0} \frac{(2x+1)h+h^2}{h}$$

$$= \lim_{h\to 0}(2x+1+h) = 2x+1$$

$$\frac{\partial f}{\partial y} = \lim_{h\to 0} \frac{f(x,y+h)-f(x,y)}{h} = \lim_{h\to 0} \frac{[x^2+(y+h)^2+x]-[x^2+y^2+x]}{h}$$

$$= \lim_{h\to 0} \frac{(h^2+2hy)}{h} = \lim_{h\to 0} \frac{h(h+2y)}{h} = 2y$$

b) We have

$$f_x(x,y) = \lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h}$$

$$= \lim_{h\to 0} \frac{\left[1-(x+h)+y-3(x+h)^2y\right]-(1-x+y-3x^2y)}{h}$$

$$= \lim_{h\to 0} \frac{1}{h}[-h-3h^2y-6xhxy]$$

$$= \lim_{h\to 0}[-1-3hy-6xy] = -1-6xy$$

$$f_x(1,2) = -1-6(1)(2) = -13$$

$$f_y(x,y) = \lim_{h\to 0} \frac{f(x,y+h)-f(x,y)}{h}$$

$$= \lim_{h\to 0} \frac{1-x+(y+h)-3x^2(y+h)-\left(1-x+y-3x^2\right)}{h}$$

$$= \lim_{h\to 0} \frac{1}{h}[h-3x^2h] = 1-3x^2$$

$$f_y(1,2) = 1-3(1) = -2$$

IP2.

If $\boldsymbol{z = log(\tan x + \tan y)}$, show that

$$\boldsymbol{(\sin 2x)z_x + (\sin 2y)z_y = 2}$$

Solution:

We have

$$z_x = \frac{\partial z}{\partial x} = \frac{\partial}{\partial x}[log(\tan x + \tan y)] = \frac{\sec^2 x}{\tan x + \tan y}$$

$$z_y = \frac{\partial z}{\partial y} = \frac{\partial}{\partial y}[log(\tan x + \tan y)] = \frac{\sec^2 y}{\tan x + \tan y}$$

Therefore, $(\sin 2x)z_x + (\sin 2y)z_y$

$$= \frac{2sinx\ \ cos\ x\ \ sec^2 x}{tan\ x + tan\ y} + \frac{2siny\ \ cos\ y\ \ sec^2 y}{tan\ x + tan\ y}$$

$$= \frac{2\tan x}{\tan x + \tan y} + \frac{2\tan y}{\tan x + \tan y} = 2$$

IP3:

If $\boldsymbol{U} = \log(\boldsymbol{x^2} + \boldsymbol{y^2} + \boldsymbol{z^2})$, then find $\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2}$.

Solution: Given that $U = \log(x^2 + y^2 + z^2)$

$$\therefore \frac{\partial U}{\partial x} = \frac{2x}{x^2+y^2+z^2}$$

$$\frac{\partial^2 U}{\partial x^2} = \frac{(x^2+y^2+z^2)2-(2x)(2x)}{(x^2+y^2+z^2)^2} = \frac{2y^2+2z^2-2x^2}{(x^2+y^2+z^2)^2}$$

Similarly,

$$\frac{\partial^2 U}{\partial y^2} = \frac{2z^2+2x^2-2y^2}{(x^2+y^2+z^2)^2} \quad \text{and} \quad \frac{\partial^2 U}{\partial z^2} = \frac{2x^2+2y^2-2z^2}{(x^2+y^2+z^2)^2}$$

$$\therefore \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} = \frac{2(x^2+y^2+z^2)}{(x^2+y^2+z^2)^2} = \frac{2}{x^2+y^2+z^2}$$

IP2:

Show that $f_{xy}(0,0) \neq f_{yx}(0,0)$ for the function

$$f(x,y) = \begin{cases} \dfrac{xy^3}{x+y^2} & , \quad (x,y) \neq (0,0) \\ 0 & , \quad (x,y) = (0,0) \end{cases}$$

Solution: We have

$$f_x(0,0) = \lim_{h\to 0} \frac{f(0+h\,,0)-f(0,0)}{h} = \lim_{h\to 0} \frac{f(h\,,0)-f(0,0)}{h} = 0$$

$$f_y(0,0) = \lim_{h\to 0} \frac{f(0,0+h)-f(0,0)}{h} = \lim_{h\to 0} \frac{f(0,h)-f(0,0)}{h} = 0$$

$$f_x(0,y) = \lim_{h\to 0} \frac{f(0+h,y)-f(0,y)}{h} = \lim_{h\to 0} \frac{f(h,y)-f(0,y)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{y^3 h}{(h+y^2)} - 0}{h} = \lim_{h\to 0} \frac{y^3}{(h+y^2)} = \frac{y^3}{y^2} = y$$

$$f_y(x,0) = \lim_{h\to 0} \frac{f(x,0+h)-f(x,0)}{h} = \lim_{h\to 0} \frac{f(x,h)-f(x,0)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{xh^3}{(x+h^2)} - 0}{h} = \lim_{h\to 0} \frac{xh^3}{(x+h^2)} = 0$$

$$f_{xy}(0,0) = \left[\frac{\partial}{\partial x}(f_y)\right]_{(0,0)}$$

$$= \lim_{h\to 0} \frac{f_y(0+h,0)-f_y(0,0)}{h} = \lim_{h\to 0} \frac{f_y(h,0)-f_y(0,0)}{h} = 0$$

$$f_{yx}(0,0) = \left[\frac{\partial}{\partial y}(f_x)\right]_{(0,0)} = \lim_{h\to 0} \frac{f_x(0,0+h)-f_x(0,0)}{h}$$

$$= \lim_{h \to 0} \frac{f_x(0,h) - f_x(0,0)}{h} = \lim_{h \to 0} \frac{h - 0}{h} = 1$$

Notice that $f_{xy}(0,0) \neq f_{yx}(0,0)$.

3.3. Partial Derivatives

EXCERCISES

I. Compute the partial derivatives of the following functions at the specified points.

a. $f(x,y) = x^4 - x^2y^2 + y^4$; $(-1,1)$

b. $f(x,y) = \frac{x}{\sqrt{x^2+y^2}}$; $(6,7)$

c. $f(x,y) = \frac{x+y}{xy-1}$; $(1,2)$

d. $f(x,y) = \left(x^3 + \frac{y}{2}\right)^{2/3}$; $(-1,4)$

e. $f(x,y) = 5xy - 7x^2 - y^2 + 3x - 6y + 2$; $(-1,1)$

Answers:

a. $f_x = -2; f_y = 2$

b. $f_x = \frac{49}{(85)^{3/2}}; f_y = \frac{42}{(85)^{3/2}}$

c. $f_x = -5; f_y = -2$

d. $f_x = 2; f_y = \frac{1}{3}$

e. $f_x = 22; f_y = -13$

II. Find f_x and f_y

a. $f(x,y) = e^{xy}\, lny$

b. $f(x,y) = sin^2(x-3y)$

c. $f(x,y) = cos^2(3x-y^2)$

d. $f(x,y) = \tan^{-1}\left(\frac{y}{x}\right)$

e. $f(x,y) = \ln(x+y)$

Answers:

a. $f_x = ye^{xy}\, lny; f_y = xe^{xy}\, lny + \frac{e^{xy}}{y}$

b. $f_x = 2\sin(x-3y)\cos(x-3y)$;

$f_y = -6\sin(x-3y)\cos(x-3y)$

c. $f_x = -6\cos(3x-y^2)\sin(3x-y^2)$; $f_y = 4y\cos(3x-y^2)\sin(3x-y^2)$

d. $f_x = -\frac{y}{x^2+y^2}$; $f_y = \frac{x}{x^2+y^2}$

e. $f_x = \frac{1}{x+y}$; $f_y = \frac{1}{x+y}$

III. Find $\boldsymbol{f_x}$, $\boldsymbol{f_y}$ $\boldsymbol{and}$ $\boldsymbol{f_z}$

a. $f(x,y,z) = 1 + xy^2 - 2z^2$

b. $f(x,y,z) = \sin^{-1}(xyz)$

c. $f(x,y,z) = \sec^{-1}(x + y + z)$

d. $f(x,y,z) = yz\ln(xy)$

e. $f(x,y,z) = \tanh(x + y + 2z)$

f. $f(x,y,z) = \sinh(xy - y^2)$

Answers:

a. $f_x = 1 + y^2; f_y = 2xy, f_z = -4z$

b. $f_x = \frac{yz}{\sqrt{1-x^2y^2z^2}}; f_y = \frac{xz}{\sqrt{1-x^2y^2z^2}}, f_z = \frac{xy}{\sqrt{1-x^2y^2z^2}}$

c. $f_x = \frac{1}{|x+yz|\sqrt{(x+yz)^2-1}}; f_y = \frac{z}{|x+yz|\sqrt{(x+yz)^2-1}},$

$f_z = \frac{y}{|x+yz|\sqrt{(x+yz)^2-1}}$

d. $f_x = \frac{yz}{x}; f_y = zln(xy) + z, f_z = yln(xy)$

e. $f_x = sech^2(x + 2y + 3z); f_y = 2sech^2(x + 2y + 3z),$

$f_z = 3sech^2(x + 2y + 3z)$

f. $f_x = y\cosh(xy - z^2);\ f_y = x\cosh(xy - z^2);$

$f_z = -2z\cosh(xy - z^2);$

IV. Find all the second order partial derivatives of the following functions.

a) $f(x, y) = x + y + xy$
b) $f(x, y) = \sin xy$
c) $g(x, y) = x^2y + \cos y + y \sin x$
d) $h(x, y) = xe^4 + y + 1$
e) $r(x, y) = \ln(x + y)$
f) $s(x, y) = \tan^{-1} \frac{y}{x}$

V. Find $\boldsymbol{w_{xy}}, \boldsymbol{w_{yx}}$ and verify that $\boldsymbol{w_{xy} = w_{yx}}$

a) $w = \ln(2x + 3y)$
b) $w = e^x + x \ln y + y \ln x$
c) $w = xy^2 + x^2y^2 + x^3y^4$
d) $w = x \sin y + y \sin x + xy$

VI. Show that each of the following function satisfies the Laplace's equation:

a) $f(x, y, z) = x^2 + y^2 - 2z^2$

b) $f(x, y, z) = 2z^3 - 3(x^2 - y^2)z$

c) $f(x, y, z) = e^{-2y} cos2x$

d) $f(x, y, z) = (x^2 + y^2 + z^2)^{-\frac{1}{2}}$

e) $f(x, y, z) = e^{3x+4y}\ cos5z$

3.4

Homogeneous Functions and Euler's Theorem

Learning objectives:

* To define a homogeneous function of two and three variables.
* To state and prove Euler's Theorem for functions of two variables.

  AND
* To practice the related problems.

In this module we define a homogeneous function of two and three variables and prove Euler's theorem for homogeneous functions of two variables.

Homogeneous function of two variables

A function $f(x, y)$ is said to be a ***homogeneous function of degree n in x and y*** if it can be written in any one of the following forms:

(i) $f(\lambda x, \lambda y) = \lambda^n f(x, y)\ , \lambda > 0$

(ii) $f(x, y) = x^n g\left(\frac{y}{x}\right)$

(iii) $f(x, y) = y^n h\left(\frac{x}{y}\right)$

Homogeneous function of three variables

A function $f(x, y, z)$ is said to be a ***homogeneous function of degree n in x, y and z*** if it can be written in any one of the following forms:

(i) $f(\lambda x, \lambda y, \lambda z) = \lambda^n f(x, y, z)$

(ii) $f(x, y, z) = x^n g\left(\frac{y}{x}, \frac{z}{x}\right)$

(iii) $f(x, y, z) = y^n u\left(\frac{x}{y}, \frac{z}{y}\right)$

(iv) $f(x, y, z) = z^n v\left(\frac{x}{z}, \frac{y}{z}\right)$

The degree of homogeneity n can be an integer or any real number.

Example 1: Find the degree of homogeneity of the functions

(i) $f(x,y) = \frac{1}{x+y}$

(ii) $g(x,y,z) = \frac{\sqrt{x}}{\sqrt{x^2+y^2+z^2}}$

Solution:

(i) For any $\lambda > 0$, we have $f(\lambda x, \lambda y) = \frac{1}{\lambda x+\lambda y} = \lambda^{-1}\frac{1}{x+y}$

$$= \lambda^{-1} f(x,y)$$

Therefore, the degree of homogeneity of $f(x,y)$ is -1.

(ii) For any $\lambda > 0$, we have

$$g(\lambda x, \lambda y, \lambda z) = \frac{\sqrt{\lambda x}}{\sqrt{(\lambda x)^2+(\lambda y)^2+(\lambda z)^2}} = \frac{\sqrt{\lambda}}{\lambda}\frac{\sqrt{x}}{\sqrt{x^2+y^2+z^2}}$$

$$= \lambda^{-\frac{1}{2}} g(x,y,z)$$

Therefore, the degree of homogeneity of $g(x,y,z)$ is $-\frac{1}{2}$.

The following is an important theorem concerning homogeneous functions.

Theorem 1: Euler's Theorem

Let $f(x,y)$ be a homogeneous function of degree n in x and y.

(i) If the first order partial derivatives of f exist, then

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf \qquad \dots (1)$$

for all (x,y) in the domain of f.

(ii) If the first and second order partial derivatives of f are continuous, then

$$\boldsymbol{x^2 \frac{\partial^2 f}{\partial x^2} + 2xy \frac{\partial^2 f}{\partial x \partial y} + y^2 \frac{\partial^2 f}{\partial y^2} = n(n-1)f} \quad \textit{\textbf{... (2)}}$$

for all (x, y) in the domain of f.

Proof: Since $f(x, y)$ is a homogeneous function of degree n in x and y, we can write $f(x, y) = x^n g\left(\frac{y}{x}\right)$, for all (x, y) in the domain of f.

Differentiating partially w.r.t. x and y, we obtain,

$$\frac{\partial f}{\partial x} = nx^{n-1} g\left(\frac{y}{x}\right) + x^n g'\left(\frac{y}{x}\right)\left(-\frac{y}{x^2}\right)$$

$$= nx^{n-1} g\left(\frac{y}{x}\right) - yx^{n-2} g'\left(\frac{y}{x}\right)$$

$$\frac{\partial f}{\partial y} = x^n g'\left(\frac{y}{x}\right)\left(\frac{1}{x}\right) = x^{n-1} g'\left(\frac{y}{x}\right)$$

Now, $x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = nx^n g\left(\frac{y}{x}\right) - yx^{n-1} g'\left(\frac{y}{x}\right) + yx^{n-1} g'\left(\frac{y}{x}\right)$

$$= nx^n g\left(\frac{y}{x}\right) = nf$$

This proves the first part of the theorem.

We have, $x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = nf$ ... (1)

Differentiating (1) partially w.r.t. x and y, we obtain

$$x\frac{\partial^2 f}{\partial x^2}+\frac{\partial f}{\partial x}+y\frac{\partial^2 f}{\partial x\partial y}=n\frac{\partial f}{\partial x}$$

$$\Rightarrow x\frac{\partial^2 f}{\partial x^2}+y\frac{\partial^2 f}{\partial x\partial y}=(n-1)\frac{\partial f}{\partial x} \quad \ldots \ (3)$$

$$x\frac{\partial^2 f}{\partial y\partial x}+\frac{\partial f}{\partial y}+y\frac{\partial^2 f}{\partial y^2}=n\frac{\partial f}{\partial y}$$

$$\Rightarrow x\frac{\partial^2 f}{\partial y\partial x}+y\frac{\partial^2 f}{\partial y^2}=(n-1)\frac{\partial f}{\partial y} \quad \ldots \ (4)$$

Multiplying (3) by x, (4) by y and adding, we obtain

$$x^2\frac{\partial^2 f}{\partial x^2}+xy\left(\frac{\partial^2 f}{\partial x\partial y}+\frac{\partial^2 f}{\partial y\partial x}\right)+y^2\frac{\partial^2 f}{\partial y^2}=(n-1)\left(x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}\right)f$$

i.e., $x^2\frac{\partial^2 f}{\partial x^2}+xy\left(\frac{\partial^2 f}{\partial x\partial y}+\frac{\partial^2 f}{\partial y\partial x}\right)+y^2\frac{\partial^2 f}{\partial y^2}=n(n-1)f$ ------- (5)

(since $x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=nf$)

Since the first and second order partial derivatives are continuous in the domain f, by mixed derivative theorem $\frac{\partial^2 f}{\partial x\partial y}=\frac{\partial^2 f}{\partial y\partial x}$ at all points in the domain of f.

Therefore, (5) reduces to

$$x^2\frac{\partial^2 f}{\partial x^2}+2xy\frac{\partial^2 f}{\partial x\partial y}+y^2\frac{\partial^2 f}{\partial y^2}=n(n-1)f$$

for all (x, y) in the domain of f.

Thus, the theorem is proved.

Note:

If $f(x, y)$ is a homogeneous function of degree n, then $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are homogeneous functions of degree $n - 1$ in x and y.

Proof: We have $f(x, y) = x^n g\left(\frac{y}{x}\right)$. Then

$$\frac{\partial f}{\partial x} = nx^{n-1}g\left(\frac{y}{x}\right) - yx^{n-2}g'\left(\frac{y}{x}\right) = x^{n-1}\left[ng\left(\frac{y}{x}\right) - \frac{y}{x}g'\left(\frac{y}{x}\right)\right]$$

$$= x^{n-1}\emptyset\left(\frac{y}{x}\right), \text{ where } \emptyset\left(\frac{y}{x}\right) = ng\left(\frac{y}{x}\right) - \frac{y}{x}g'\left(\frac{y}{x}\right)$$

and $\frac{\partial f}{\partial y} = x^{n-1}g'\left(\frac{y}{x}\right)$

Thus, $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are homogeneous functions of degree $n - 1$ in x and y.

Example 2: If $u(x, y) = \sin^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$, $0 < x, y < 1$, then prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}\tan u$.

Solution: For all x, y: $0 < x, y < 1$, we have $\frac{x+y}{\sqrt{x}+\sqrt{y}} < 1$.

Therefore, $u(x, y)$ is defined. The given function can be written as $V = \sin u = \frac{x+y}{\sqrt{x}+\sqrt{y}} = \frac{x\left(1+\frac{y}{x}\right)}{\sqrt{x}\left(1+\sqrt{\frac{y}{x}}\right)} = x^{\frac{1}{2}}\frac{1+\frac{y}{x}}{1+\sqrt{\frac{y}{x}}} = x^{\frac{1}{2}}\, g\left(\frac{y}{x}\right)$

$\Rightarrow V$ is a homogeneous function of degree $\frac{1}{2}$. By Euler's theorem, we have

$$x\frac{\partial V}{\partial x} + y\frac{\partial V}{\partial y} = \frac{1}{2}V$$

$$\Rightarrow x\frac{\partial}{\partial x}(\sin u) + y\frac{\partial}{\partial y}(\sin u) = \frac{1}{2}\sin u$$

$$\Rightarrow x\cos u\frac{\partial u}{\partial x} + y\cos u\frac{\partial u}{\partial y} = \frac{1}{2}\sin u$$

i.e., $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}\tan u$

Example 3: If $V(x, y)$ is a homogeneous function of degree n and $u = \ln V$, then $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n$.

Solution: Given $V(x, y)$ is a homogeneous function of degree n and $u = \ln V$, i.e., $V = e^u$. Clearly, the first order partial derivatives of V exist and $\frac{\partial V}{\partial x} = e^u\frac{\partial u}{\partial x}$, $\frac{\partial V}{\partial y} = e^u\frac{\partial V}{\partial y}$

Since V is a homogeneous function of degree n in x, y, by Euler's theorem

$$x\frac{\partial V}{\partial x} + y\frac{\partial V}{\partial y} = nV \Rightarrow xe^u\frac{\partial u}{\partial x} + ye^u\frac{\partial u}{\partial y} = n$$

i.e., $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n$

Example 4: If $u(x,y) = x^2 \tan^{-1}\left(\frac{y}{x}\right) - y^2 \tan^{-1}\left(\frac{x}{y}\right)$ $, x > 0,$

$y > 0$, then evaluate $x^2 u_{xx} + 2xyu_{xy} + y^2 u_{yy}$.

Solution: We have for all $\lambda > 0$,

$$u(\lambda x, \lambda y) = \lambda^2 x^2 \tan^{-1}\left(\frac{\lambda y}{\lambda x}\right) - \lambda^2 y^2 \tan^{-1}\left(\frac{\lambda x}{\lambda y}\right)$$

$$= \lambda^2\left(x^2 \tan^{-1}\left(\frac{y}{x}\right) - y^2 \tan^{-1}\left(\frac{x}{y}\right)\right) = \lambda^2 u(x,y)$$

Therefore, $u(x,y)$ is a homogeneous function of degree $n = 2$.

Note that

$$u_x = 2x \tan^{-1}\left(\frac{y}{x}\right) + x^2 \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(-\frac{y}{x^2}\right) - y^2 \frac{1}{1+\left(\frac{x}{y}\right)^2}\left(\frac{1}{y}\right)$$

$$= 2x \tan^{-1}\left(\frac{y}{x}\right) - y$$

$$u_y = x^2 \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(\frac{1}{x}\right) - 2y \tan^{-1}\left(\frac{x}{y}\right) - y^2 \frac{1}{1+\left(\frac{x}{y}\right)^2}\left(-\frac{x}{y^2}\right)$$

$$= -2y \tan^{-1}\left(\frac{x}{y}\right) + x$$

Now, $u_{xy} = (u_x)_y = 2x \frac{1}{1+\left(\frac{y}{x}\right)^2}\frac{1}{x} - 1 = \frac{2x^2}{x^2+y^2} - 1 = \frac{x^2-y^2}{x^2+y^2}$

$$u_{yx} = \left(u_y\right)_x = -2y \frac{1}{1+\left(\frac{x}{y}\right)^2}\frac{1}{y} + 1 = -\frac{2y^2}{x^2+y^2} + 1 = \frac{x^2-y^2}{x^2+y^2}$$

Notice that $u_{xy} = u_{yx}$ at all points (x,y) in the domain of u.

By Euler's Theorem, we have

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = n(n-1)u = 2(2-1)u = 2u$$

Example 5: Let $u(x, y) = \frac{x^3+y^3}{x+y}$, $(x, y) \neq (0,0)$. Then evaluate $x \frac{\partial^2 u}{\partial x^2} + y \frac{\partial^2 u}{\partial x \partial y} - \frac{\partial u}{\partial x}$.

Solution: We have,

$$u(x, y) = \frac{x^3+y^3}{x+y} = \frac{x^3\left(1+\left(\frac{y}{x}\right)^3\right)}{x\left(1+\frac{y}{x}\right)} = x^2 \frac{1+\left(\frac{y}{x}\right)^2}{1+\frac{y}{x}} = x^2 g\left(\frac{y}{x}\right)$$

$\Rightarrow u(x, y)$ is a homogeneous function of degree 2.

By Euler's Theorem,

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 2u$$

Differentiating partially w.r.t. x, we obtain

$$\frac{\partial u}{\partial x} + x \frac{\partial^2 u}{\partial x^2} + y \frac{\partial^2 u}{\partial x \partial y} = 2 \frac{\partial u}{\partial x}$$

i.e., $x \frac{\partial^2 u}{\partial x^2} + y \frac{\partial^2 u}{\partial x \partial y} - \frac{\partial u}{\partial x} = 0$

Example 6: Let $f(x, y)$ and $g(x, y)$ be two homogeneous functions of degree m and n respectively, where $m \neq 0$. If $h = f + g$ and $x \frac{\partial h}{\partial x} + y \frac{\partial h}{\partial y} = 0$, then show that $f = kg$ for some scalar k.

Solution: Given f and g are homogeneous functions of degree m and n respectively. By Euler's theorem,

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = mf$$

$$x\frac{\partial g}{\partial x} + y\frac{\partial g}{\partial y} = ng$$

Adding, we obtain

$$x\left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial x}\right) + y\left(\frac{\partial f}{\partial y} + \frac{\partial g}{\partial y}\right) = mf + ng$$

$$\Rightarrow x\frac{\partial}{\partial x}(f+g) + y\frac{\partial}{\partial y}(f+g) = mf + ng$$

i.e., $x\dfrac{\partial h}{\partial x} + y\dfrac{\partial h}{\partial y} = 0$

Given $x\dfrac{\partial h}{\partial x} + y\dfrac{\partial h}{\partial y} = 0$

$\Rightarrow mf + ng = 0$

$\Rightarrow f = \left(-\dfrac{n}{m}\right)g = kg,$ where $k = -\dfrac{n}{m}$, $(m \neq 0)$.

Euler's theorem for homogeneous functions of three variables

If $f(x, y, z)$ is a homogeneous function of degree n in x, y and z and first order partial derivatives of f exist, then

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} = nf$$

for all (x, y, z) in the domain of f.

Example 7: If $w = \sin^{-1} u \ , u = \frac{x^2+y^2+z^2}{x+y+z}$, then

$$x \frac{\partial w}{\partial x} + y \frac{\partial w}{\partial y} + z \frac{\partial w}{\partial z} = \tan w$$

Solution: Given, $w = \sin^{-1} u \ , u = \frac{x^2+y^2+z^2}{x+y+z}$

$\Rightarrow u = \sin w$ and u is a homogeneous function of degree $n = 1$. By Euler's theorem

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z} = nu = \sin w$$

i.e., $x \cos w \frac{\partial w}{\partial x} + y \cos w \frac{\partial w}{\partial y} + z \cos w \frac{\partial w}{\partial z} = \sin w$

i.e., $x \frac{\partial w}{\partial x} + y \frac{\partial w}{\partial y} + z \frac{\partial w}{\partial z} = \tan w$

P1.

Find the degree of homogeneity of the function

$$\boldsymbol{f(x,y) = \frac{x^{\frac{1}{3}}+y^{\frac{1}{3}}}{\sqrt{x}+\sqrt{y}}}$$

Solution:

The function $f(x, y)$ can be written as

$$f(x,y) = \frac{x^{\frac{1}{3}}\left(1+\left(\frac{y}{x}\right)^{\frac{1}{3}}\right)}{\sqrt{x}\left(1+\sqrt{\frac{y}{x}}\right)}$$

$$= x^{-\frac{1}{6}}\ f(x,y)$$

Therefore, the given function is a homogeneous function of degree $-\frac{1}{6}$

P2.

If $\boldsymbol{u = \sin^{-1}\left(\frac{x^3+y^3+z^3}{ax+by+cz}\right)}$ then prove that

$$\boldsymbol{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 2\tan u.}$$

Solution:

The given function can be written as

$V = \sin u = \frac{x^3+y^3+z^3}{ax+by+cz}$

Let $f(x,y,z) = \frac{x^3+y^3+z^3}{ax+by+cz}$

Now, $f(\lambda x, \lambda y, \lambda z) = \frac{(\lambda x)^3+(\lambda y)^3+(\lambda z)^3}{\lambda ax+\lambda by+\lambda cz} = \lambda^2\left(\frac{x^3+y^3+z^3}{ax+by+cz}\right)$

$$= \lambda^2 f(x,y,z)$$

$\Longrightarrow V$ is a homogeneous function of degree 2.

By Euler's Theory, we have

$$x\frac{\partial V}{\partial x} + y\frac{\partial V}{\partial y} + z\frac{\partial V}{\partial z} = 2V$$

$$\Longrightarrow x\frac{\partial}{\partial x}(\sin u) + y\frac{\partial}{\partial y}(\sin u) + z\frac{\partial}{\partial z}(\sin u) = 2\sin u$$

$$\Longrightarrow x\cos u\frac{\partial u}{\partial x} + y\cos u\frac{\partial u}{\partial y} + z\cos u\frac{\partial u}{\partial z} = 2\sin u$$

$$\Longrightarrow x\frac{\partial u}{\partial y} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial y} = 2\tan u$$

Hence proved

P3.

If $\boldsymbol{u(x,y) = \cos^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)}$**,** $\boldsymbol{0 < x, y < 1}$**, then prove that**

$$\boldsymbol{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = -\frac{1}{2}cotu}$$

Solution:

For all $x, y;\ 0 < x, y < 1$, we have $\frac{x+y}{\sqrt{x}+\sqrt{y}} < 1$

Therefore, $u(x,y)$is defined. The given function can be written as

$$V = cosu = \frac{x+y}{\sqrt{x}+\sqrt{y}} = \frac{x\left(1+\frac{y}{x}\right)}{\sqrt{x}\left(1+\sqrt{\frac{y}{x}}\right)} = x^{\frac{1}{2}}\left[\frac{1+\left(\frac{y}{x}\right)}{1+\sqrt{\frac{y}{x}}}\right] = x^{\frac{1}{2}}\, g\left(\frac{y}{x}\right)$$

$\Longrightarrow V$ is a homogeneous function of degree $\frac{1}{2}$.

By Euler's theorem, we have $x\frac{\partial V}{\partial x} + y\frac{\partial V}{\partial y} = \frac{1}{2}V$

$$\Longrightarrow x\frac{\partial}{\partial x}(cosu) + y\frac{\partial}{\partial y}(cosu) = \frac{1}{2}cosu$$

$$\Longrightarrow x(-sinu)\frac{\partial u}{\partial x} + y(-sinu)\frac{\partial u}{\partial y} = \frac{1}{2}cosu$$

$$\Longrightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = -\frac{1}{2}cotu$$

Hence proved

P4.

If $\boldsymbol{u = \tan^{-1}\frac{x^3+y^3}{x-y}, x \neq y}$ then show that

$$\boldsymbol{x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = sin4u - sin2u}$$

Solution:

The given function can be written as

$$V = tanu = \frac{x^3+y^3}{x-y} = x^2\frac{\left(1+\left(\frac{y}{x}\right)^3\right)}{1-\left(\frac{y}{x}\right)} = x^2\, g\left(\frac{y}{x}\right)$$

$\Longrightarrow V$ is a homogeneous function of degree 2.

By Euler's theorem, we have

$$x\frac{\partial V}{\partial x} + y\frac{\partial V}{\partial y} = 2V$$

$$\Longrightarrow x\frac{\partial}{\partial x}(tanu) + y\frac{\partial}{\partial y}(tanu) = 2tanu$$

$$\Longrightarrow x\, sec^2u\,\frac{\partial u}{\partial x} + y\, sec^2u\,\frac{\partial u}{\partial y} = 2tanu$$

$$\Longrightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{2tanu}{sec^2u} = sin2u \quad \ldots (1)$$

Differentiating (1) partially w.r.t. x, we get

$$x\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} + y\frac{\partial^2 u}{\partial x \partial y} = 2cos2u\frac{\partial u}{\partial x} \qquad \ldots (2)$$

Multiplying (2) by x on both sides, we get

$$x^2\frac{\partial^2 u}{\partial x^2} + x\frac{\partial u}{\partial x} + xy\frac{\partial^2 u}{\partial x \partial y} = 2x cos2u\frac{\partial u}{\partial x} \quad \text{... (3)}$$

Differentiating (1) partially w.r.t. y, we get

$$x\frac{\partial^2 u}{\partial y \partial x} + y\frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial y} = 2cos2u\frac{\partial u}{\partial y}$$

Multiplying (3) by y on both sides, we get

$$xy\frac{\partial^2 u}{\partial y \partial x} + y^2\frac{\partial^2 u}{\partial y^2} + y\frac{\partial u}{\partial y} = 2y\ cos2u\frac{\partial u}{\partial y} \qquad \text{... (4)}$$

Note that $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$ and adding (3) and (4), we get

$$x^2\frac{\partial^2 u}{\partial x^2} + y^2\frac{\partial^2 u}{\partial y^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$$

$$= \left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right)(2cos2u)$$

$$\Longrightarrow x^2\frac{\partial^2 u}{\partial x^2} + y^2\frac{\partial^2 u}{\partial y^2} + 2xy\frac{\partial^2 u}{\partial y \partial x} + sin2u$$

$$= (sin2u)(2cos2u)$$

$$\Longrightarrow x^2\frac{\partial^2 u}{\partial x^2} + y^2\frac{\partial^2 u}{\partial y^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} = 2sin2u\ cos2u - sin2u$$

$$= sin4u - sin2u$$

$$\therefore \quad x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = sin4u - sin2u$$

Hence proved

IP1.

Find the degree of homogeneity of the function

$$\boldsymbol{f(x,y,z) = \left(\frac{x}{z}\right)^2 + \left(\frac{y}{z}\right)^2 + \frac{x}{y}}$$

Solution:

Given function is $f(x,y,z) = \left(\frac{x}{z}\right)^2 + \left(\frac{y}{z}\right)^2 + \frac{x}{y}$

For any $\lambda > 0$, we have

$$\begin{aligned} f(\lambda x, \lambda y, \lambda z) &= \left(\frac{\lambda x}{\lambda z}\right)^2 + \left(\frac{\lambda y}{\lambda z}\right)^2 + \frac{\lambda x}{\lambda y} \\ &= \lambda^0 \left[\left(\frac{x}{z}\right)^2 + \left(\frac{y}{z}\right)^2 + \frac{x}{y}\right] \\ &= \lambda^0 f(x,y,z) \end{aligned}$$

Therefore, the given function is a homogeneous function of degree 0.

IP2.

If $\boldsymbol{u = \sec^{-1}\left[\frac{x+2y+3z}{\sqrt{x^8+y^8+z^8}}\right]}$, then find $\boldsymbol{x\frac{\partial u}{\partial x} + x\frac{\partial u}{\partial y} + x\frac{\partial u}{\partial z}}$

Solution:

The given function can be written as

$$V = \sec u = \left[\frac{x+2y+3z}{\sqrt{x^8+y^8+z^8}}\right]$$

Let $f(x,y,z) = \left[\frac{x+2y+3z}{\sqrt{x^8+y^8+z^8}}\right]$

Now, $f(\lambda x, \lambda y, \lambda z) = \left[\frac{\lambda x+2\lambda y+3\lambda z}{\sqrt{\lambda^8 x^8+\lambda^8 y^8+\lambda^8 z^8}}\right] = \lambda^{-3}\left[\frac{x+2y+3z}{\sqrt{x^8+y^8+z^8}}\right]$

$$= \lambda^{-3} f(x,y,z)$$

$\Longrightarrow V$ is a homogenous function of degree -3.

By Euler's Theorem, we have

$$x\frac{\partial V}{\partial x} + y\frac{\partial V}{\partial y} + z\frac{\partial V}{\partial z} = -3V$$

$$\Longrightarrow x\frac{\partial}{\partial x}(\sec u\,) + y\frac{\partial}{\partial y}(\sec u\,) + z\frac{\partial}{\partial z}(\sec u\,) = -3\sec u$$

$$\Longrightarrow \sec u \tan u\left[\,x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}\right] = -3\sec u$$

$$\Longrightarrow\ x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = -3\cot u$$

IP3.

If $\boldsymbol{u(x,y) = \tan^{-1}\frac{x^3+y^3}{x+y}}$ **then prove that**

$$\boldsymbol{xu_x + yu_y = sin2u}.$$

Solution:

The given function can be written as

$$V = tanu = \frac{x^3+y^3}{x+y} = \frac{x^3\left(1+\left(\frac{y}{x}\right)^3\right)}{x\left(1+\frac{y}{x}\right)} = x^2\frac{\left(1+\left(\frac{y}{x}\right)^3\right)}{\left(1+\frac{y}{x}\right)} = x^2 g(x,y)$$

$\Longrightarrow V$ is a homogeneous function of degree 2.

By Euler's theorem, we have

$$x\frac{\partial V}{\partial x} + y\frac{\partial V}{\partial y} = 2V$$

$$\Longrightarrow x\frac{\partial}{\partial x}(tanu) + y\frac{\partial}{\partial y}(tanu) = 2\ tanu$$

$$\Longrightarrow x\ sec^2u\frac{\partial u}{\partial x} + y\ sec^2u\frac{\partial u}{\partial y} = 2\ tanu$$

$$\Longrightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2\ tanu\ cos^2u$$

$$\Longrightarrow xu_x + yu_y = sin2u$$

Hence proved

IP4.

If $\boldsymbol{u} = \tan^{-1}\left(\frac{y^2}{x}\right)$ show that

$$\boldsymbol{x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial y\partial x} + y^2\frac{\partial^2 u}{\partial y^2} = -sin2u\, sin^2 u}$$

Solution:

The given function can be written as

$$V = tanu = \frac{y^2}{x} = x\left(\frac{y^2}{x^2}\right) = x\, g\left(\frac{y}{x}\right)$$

$\Longrightarrow V$ is a homogeneous function degree 1.

By Euler's theorem, we have

$$x\frac{\partial V}{\partial x} + y\frac{\partial V}{\partial y} = V$$

$$\Longrightarrow x\frac{\partial}{\partial x}(tanu) + y\frac{\partial}{\partial y}(tanu) = (tan\, u)$$

$$\Longrightarrow xsec^2u\frac{\partial u}{\partial x} + ysec^2u\frac{\partial u}{\partial y} = tan\, u$$

$$\Longrightarrow x\ \frac{\partial u}{\partial x} +\ y\frac{\partial u}{\partial y} = \frac{tan\, u}{sec^2u} = sin2u$$

$$\Longrightarrow x\ \frac{\partial u}{\partial x} +\ y\frac{\partial u}{\partial y} = \frac{1}{2}sin2u \quad \dots (1)$$

Differentiating (1) partially w.r.t x, we get

$$x\ \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} +\ y\frac{\partial^2 u}{\partial x\partial y} = \frac{1}{2}(2cos2u)\frac{\partial u}{\partial x} = cos2u\frac{\partial u}{\partial x}$$

Multiplying on both sides with x, we get

$$x^2 \frac{\partial^2 u}{\partial x^2} + x\frac{\partial u}{\partial x} + xy\frac{\partial^2 u}{\partial x \partial y} = xcos2u\frac{\partial u}{\partial x} \qquad \ldots (2)$$

Differentiating (1) partially with respect to y , we get

$$x\frac{\partial^2 u}{\partial y \partial x} + y\frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial y} = \frac{1}{2}(2cos2u)\frac{\partial u}{\partial y} = cos2u\frac{\partial u}{\partial y}$$

Multiplying on both sides with y, we get

$$xy\frac{\partial^2 u}{\partial y \partial x} + y^2\frac{\partial^2 u}{\partial y^2} + y\frac{\partial u}{\partial y} = ycos2u\frac{\partial u}{\partial y} \qquad \ldots (3)$$

$(2) + (3)$ gives

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = (cos2u - 1)\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right)$$

$$= (cos2u - 1)\left(\frac{1}{2}sin2u\right) \qquad \text{[From (1)]}$$

$$= \frac{1}{2}sin2u[-2\, sin^2 u] = -sin2u\, sin^2 u$$

Hence proved

3.4. Homogeneous Functions and Euler's Theorem

Exercises:

1. If $u = \sqrt{y^2 - x^2}\sin^{-1}\left(\frac{y}{x}\right)$ then show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = u$

2. If $u = \sin^{-1}\left[\frac{x^2+y^2}{x+y}\right]$ then show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = tanu$

3. If $u = \tan^{-1}\left(\frac{x^3+y^3}{x+y}\right)$ then show that

   a) $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = sin2u$

   b) $x^2u_{xx} + 2xyu_{xy} + y^2u_{yy} = (1 - 4sin^2u)sin2u$

4. If $u = x^4y^2\sin^{-1}\left(\frac{y}{x}\right) + lnx - lny$ then show that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 6x^4y^2\sin^{-1}\left(\frac{y}{x}\right)$$

5. If $u = x^3 + y^3 + z^3 + 3xyz$ then prove that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 3u$$

6. If $u = (x^2 + y^2 + z^2)^{-\frac{1}{2}}$ then prove that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = -u$$

7. If $u = \tan^{-1}\left[\frac{x^2+y^2}{x+y}\right]$ then prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}sin2u$

8. If $u = \sec^{-1}\left[\frac{x^2+y^2}{x+y}\right]$ then prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = cotu$

9. If $u = \sin^{-1}\left[\frac{x^{\frac{1}{3}}+y^{\frac{1}{3}}}{x+y}\right]^{\frac{1}{2}}$ then show that

$$x^2u_{xx} + 2xyu_{xy} + y^2u_{yy} = \frac{1}{144}tanu(sec^2u + 12)sin2u$$

3.5

Differentiability

Learning objectives:

- To state and prove Increment theorem for functions of two variables.
- To define the differentiability of a function of two variables at a point.
- To prove Differentiability implies Continuity.
  AND
- To practice the related problems.

We recall the following result from the differential calculus of functions of a single variable.

Change in $\boldsymbol{y = f(x)}$ near $\boldsymbol{x = x_0}$:

If $f(x)$ is differentiable at $x = x_0$, then the change in the value f that results from changing x_0 to $x_0 + \Delta x$ is given by an equation of the form

$$\boldsymbol{\Delta y = f(x_0 + \Delta x) - f(x_0) = f'(x_0)\Delta x + \varepsilon\, \Delta x}$$

in which $\varepsilon \to 0$ as $\Delta x \to 0$.

For functions of two variables, the analogous property becomes the definition of differentiability. The following theorem tells us when to expect the property hold.

Theorem1: The increment theorem for functions of two variables

Suppose that the first order partial derivatives of $z = f(x, y)$ are defined throughout an open region R containing the point $(x_0\,, y_0)$ and f_x and f_y are continuous at $(x_0\,, y_0)$. Then the change

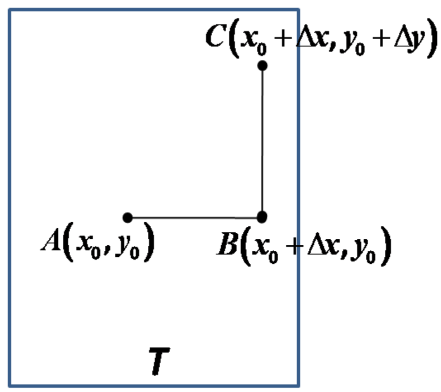
$$\Delta z = f(x_0 + \Delta x\,, y_0 + \Delta y) - f(x_0\,, y_0)$$

in the value of f that results from moving from $(x_0\,, y_0)$ to another point $(x_0 + \Delta x\,, y_0 + \Delta y)$ in R satisfies the equation of the form

$$\boldsymbol{\Delta z = f_x(x_0\,, y_0)\Delta x + f_y(x_0\,, y_0)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y}$$

in which each of $\varepsilon_1, \varepsilon_2 \to 0$ as both $\Delta x, \Delta y \to 0$.

Proof: Let T be a rectangle centered at $A(x_0\,, y_0)$, lying entirely in R. Let Δx and Δy be choosen so small such that the line segment joining A to $B(x_0 + \Delta x\,, y_0)$ and the line segment joining B to $C(x_0 + \Delta x\,, y_0 + \Delta y)$ lie in the interior of T.



Let $\Delta z_1 = f(x_0 + \Delta x\,, y_0) - f(x_0\,, y_0)$ be the change in the value of f that results from moving from A to B.

Let $\Delta z_2 = f(x_0 + \Delta x\,, y_0 + \Delta y) - f(x_0 + \Delta x\,, y_0)$ be the change in the value of f that results from moving from B to C.

Then the change in the value of f that results from moving from A to C is given by

$$\begin{aligned}
\Delta z &= f(x_0 + \Delta x\,, y_0 + \Delta y) - f(x_0\,, y_0) \\
&= f(x_0 + \Delta x\,, y_0 + \Delta y) - f(x_0 + \Delta x\,, y_0) \\
&\qquad\qquad + f(x_0 + \Delta x\,, y_0) - f(x_0\,, y_0) \\
&= \Delta z_2 + \Delta z_1
\end{aligned}$$

Computation of Δz_1

Let $F(x) = f(x\,, y_0)$, for x in the closed interval joining x_0 to $x_0 + \Delta x$. Since the partial derivatives of f are defined throughout R, $F(x)$ is a differentiable (and hence continuous) function of one variable x and

$$F'(x) = f_x(x, y_0)$$

By the Lagrange's mean value theorem for single real variable, there is a c between x_0 and $x_0 + \Delta x$ such that,

$$\frac{F(x_0+\Delta x)-F(x_0)}{x_0+\Delta x-x_0} = F'(c)$$

i.e., $F(x_0 + \Delta x) - F(x_0) = F'(c)\Delta x$

i.e., $f(x_0 + \Delta x\,, y_0) - f(x_0\,, y_0) = f_x(c\,, y_0)\Delta x$

i.e., $\Delta z_1 = F_x(c\,, y_0)\Delta x$

Similarly, $G(y) = f(x_0 + \Delta x\,, y)$ is a differentiable (and hence continuous) function of one variable y on the closed y-interval joining y_0 and $y_0 + \Delta y$, with derivative,

$$G'(y) = f_y(x_0 + \Delta x\,, y).$$

Hence, by Lagrange's mean value theorem there is a d between y_0 and $y_0 + \Delta y$ such that

$$\frac{G(y_0+\Delta y)-G(y_0)}{y_0+\Delta y-y_0} = G'(d)$$

i.e., $G(y_0 + \Delta y) - G(y_0) = G'(d)\Delta y$

i.e., $f(x_0 + \Delta x\,, y_0 + \Delta y) - f(x_0 + \Delta x\,, y_0) = f_y(x_0 + \Delta x\,, d)\Delta y$

i.e., $\Delta z_2 = f_y(x_0 + \Delta x\,, d)\Delta y$

Notice that $c \to x_0$ and $d \to y_0$ as both $\Delta x \to 0$ and $\Delta y \to 0$. Since f_x and f_y are continuous at $(x_0\,, y_0)$, $f_x(c, y_0) \to f_x(x_0\,, y_0)$ and $f_y(x_0 + \Delta x\,, d) \to f_y(x_0\,, y_0)$ as both Δx and $\Delta y \to 0$. Therefore,

$$f_x(c\,, y_0) = f_x(x_0\,, y_0) + \varepsilon_1$$

$$f_y(x_0 + \Delta x\,, d) = f_y(x_0\,, y_0) + \varepsilon_2$$

where both ε_1 and $\varepsilon_2 \to 0$ as both Δx and $\Delta y \to 0$. Thus,

$$\begin{aligned}
\Delta z &= \Delta z_1 + \Delta z_2 \\
&= f_x(c\,, y_0)\Delta x + f_y(x_0 + \Delta x\,, d)\Delta y \\
&= [f_x(x_0\,, y_0) + \varepsilon_1]\Delta x + \left[f_y(x_0\,, y_0) + \varepsilon_2\right]\Delta y \\
&= f_x(x_0\,, y_0)\Delta x + f_y(x_0\,, y_0)\Delta y + \varepsilon_1\Delta x + \varepsilon_2\Delta y
\end{aligned}$$

where both ε_1 and $\varepsilon_2 \to 0$ as both Δx and $\Delta y \to 0$.

Hence the theorem

Definition: Differentiable function

*A function $z = f(x, y)$ is said to be **differentiable at** $(\boldsymbol{x_0}\,, \boldsymbol{y_0})$ if $f_x(x_0\,, y_0)$ and $f_y(x_0\,, y_0)$ exist and Δz satisfies an equation of* *the form*

$$\Delta \boldsymbol{z} = \boldsymbol{f_x(x_0\,, y_0)}\Delta \boldsymbol{x} + \boldsymbol{f_y(x_0\,, y_0)}\Delta \boldsymbol{y} + \boldsymbol{\varepsilon_1}\Delta \boldsymbol{x} + \boldsymbol{\varepsilon_2}\Delta \boldsymbol{y} \; \ldots \; (1)$$

in which each of $\varepsilon_1, \varepsilon_2 \to 0$ *as both* $\Delta x\,, \Delta y \to 0$

We say that f *is* ***differentiable*** *if it is differentiable at every point of its domain*.

Note:

(1) $\Delta z = f(x_0 + \Delta x\,, y_0 + \Delta y) - f(x_0\,, y_0)$ is called the ***total increment*** in z corresponding to the increments Δx in x and Δy in y.

(2) The first part $f_x \Delta x + f_y \Delta y$ in (1) which is linear in Δx and Δy is called ***total differential*** of z at the point $(x_0\,, y_0)$ and is denoted by dz or df. That is

$$dz = f_x \Delta x + f_y \Delta y \text{ or } \boldsymbol{dz = f_x dx + f_y dy}$$

In the light of this definition (1) takes the form

$$\Delta \boldsymbol{z} = \boldsymbol{dz} + \boldsymbol{\varepsilon_1}\Delta \boldsymbol{x} + \boldsymbol{\varepsilon_2}\Delta \boldsymbol{y} \qquad \ldots (2)$$

(3) The second part $\varepsilon_1 \Delta x + \varepsilon_2 \Delta y$ is the infinitesimal nonlinear part and is of higher order relative to $\Delta x, \Delta y$ or $\Delta \rho = \sqrt{(\Delta x)^2 + (\Delta y)^2}$. Note that $(\Delta x, \Delta y) \to (0,0)$ implies $\Delta \rho \to 0$. Equation (2) can be written as

$$\frac{\Delta \boldsymbol{z} - \boldsymbol{dz}}{\Delta \boldsymbol{\rho}} = \boldsymbol{\varepsilon_1}\left(\frac{\Delta \boldsymbol{x}}{\Delta \boldsymbol{\rho}}\right) + \boldsymbol{\varepsilon_2}\left(\frac{\Delta \boldsymbol{y}}{\partial \boldsymbol{\rho}}\right) \qquad \ldots (3)$$

If $f(x, y)$ if differentiable, then both $\varepsilon_1, \varepsilon_2 \to 0$ as both

$\Delta x, \Delta y \to 0$, i.e., as $\Delta\rho \to 0$. Now taking the limit as $\Delta\rho \to 0$ in equation (3), we obtain

$$\lim_{\Delta\rho\to 0} \frac{\Delta z - dz}{\Delta\rho} = \lim_{\Delta\rho\to 0} \left[\varepsilon_1\left(\frac{\Delta x}{\Delta\rho}\right) + \varepsilon_2\left(\frac{\Delta y}{\partial\rho}\right)\right] = 0$$

Since $\left|\frac{\Delta x}{\Delta\rho}\right| \leq 1$ and $\left|\frac{\Delta y}{\Delta\rho}\right| \leq 1$.

Thus, to test the differentiability of $f(x, y)$ at a point (x_0, y_0), we can use one of the following two ways:

(i) Show that $\boldsymbol{\lim_{\Delta\rho\to 0} \frac{\Delta z - dz}{\Delta\rho} = 0}$

(ii) Find the expressions $\varepsilon_1(\Delta x, \Delta y), \varepsilon_2(\Delta x, \Delta y)$ from equation (2) and then show that $\varepsilon_1 \to 0$ and $\varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0,0)$, i.e., $\Delta\rho \to 0$.

In the light of the definition of differentiability, we have the following corollary of Theorem1, that a function is differentiable if its first order partial derivatives are continuous.

Corollary of Theorem1: Continuity of partial derivatives implies differentiability

If the first order partial derivatives f_x and f_y of a function $f(x, y)$ are continuous at a point (x_0, y_0), then $z = f(x, y)$ is differentiable at (x_0, y_0).

Notice that the continuity of the first order partial derivatives f_x and f_y at a point (x_0, y_0) is a sufficient condition for the differentiability at (x_0, y_0).

Note that the conditions of this corollary can be relaxed. ***It is sufficient that of the first order partial derivatives is continuous at $(\boldsymbol{x_0}, \boldsymbol{y_0})$ and the other exists at $(\boldsymbol{x_0}, \boldsymbol{y_0})$.***

The following theorem assures that a function of two variables is continuous at every point where it is differentiable.

Theorem 2:

If $(\boldsymbol{x_0}, \boldsymbol{y_0})$ is a point in the domain of the function $\boldsymbol{z = f(x, y)}$ such that one of the partial derivatives $\boldsymbol{f_x}$ and $\boldsymbol{f_y}$ is continuous at $(\boldsymbol{x_0}, \boldsymbol{y_0})$ and the other exists at $(\boldsymbol{x_0}, \boldsymbol{y_0})$ then f is differentiable at $(\boldsymbol{x_0}, \boldsymbol{y_0})$

Theorem 3: Differentiability implies continuity

If a function $\boldsymbol{f(x, y)}$ is differentiable at $(\boldsymbol{x_0}, \boldsymbol{y_0})$, then $\boldsymbol{f}$ is continuous at $(\boldsymbol{x_0}, \boldsymbol{y_0})$.

Proof: Given that the function $z = f(x, y)$ is differentiable at (x_0, y_0).

That is by definition, $\Delta z = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0)$ satisfies the equation

$f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0)$

$$= f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

where both $\varepsilon_1, \varepsilon_2 \to 0$ as both $\Delta x, \Delta y \to 0$. Now,

$$\lim_{(\Delta x, \Delta y) \to (0,0)} [f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0)] = 0$$

$$\Rightarrow \lim_{(\Delta x, \Delta y) \to (0,0)} f(x_0 + \Delta x, y_0 + \Delta y) = f(x_0, y_0)$$

$\Rightarrow f(x, y)$ is continuous at (x_0, y_0)

Hence the result

Example 1: Find Δz and the total differential of $z = f(x, y) = x^2 y - 3y$.

Solution: At any point $(x, y) \in R^2$, we have

$$\Delta z = f(x + \Delta x\,, y + \Delta y) - f(x, y)$$

$$= [(x + \Delta x)^2 (y + \Delta y) - 3(y + \Delta y)] - [x^2 y - 3y]$$

$$= 2xy\Delta x + (x^2 - 3)\Delta y + (y\Delta x + 2\Delta y)\Delta x + (\Delta x)^2 \Delta y$$

Note that $\frac{\partial f}{\partial x} = 2x\,, \frac{\partial f}{\partial y} = x^2 - 3$

Therefore the total differential is

$$dz = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = 2xy dx + (x^2 - 3) dy$$

Note:

At any point $(x, y) \in \boldsymbol{R^2}$,

$$\Delta z = f_x(x,y)\Delta x + f_y(x,y)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

Where both $\varepsilon_1 = y\Delta x + 2\Delta y$, $\varepsilon_2 = (\Delta x)^2 \to 0$ when Δx , $\Delta y \to 0$. This shows that $z = f(x,y)$ is differentiable at every point $(x,y) \in \boldsymbol{R^2}$.

Example 2:

Show that the function

$$f(x,y) = \begin{cases} xy\dfrac{x^2 - y^2}{x^2 + y^2} , & (x,y) \neq (0,0) \\ 0 , & (x,y) = (0,0) \end{cases}$$

is differentiable at the origin.

Solution:

We first calculate the first order partial derivatives of f at the origin.

$$f_x(0,0) = \lim_{h \to 0} \frac{f(0+h,0) - f(0,0)}{h} = \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h \to 0} \frac{0-0}{h} = 0$$

$$f_y(0,0) = \lim_{h \to 0} \frac{f(0,0+h) - f(0,0)}{h} = \lim_{h \to 0} \frac{f(0,h) - f(0,0)}{h} = \lim_{h \to 0} \frac{0-0}{h} = 0$$

We have $\Delta z = f(\Delta x, \Delta y) - f(0,0) = dz + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$

where $dz = f_x(0,0)\Delta x + f_y(0,0)\Delta y = 0$

Now, $\Delta z = \Delta x\, \Delta y \frac{(\Delta x)^2 - (\Delta y)^2}{(\Delta x)^2 + (\Delta y)^2} - 0 = \Delta x\, \Delta y \frac{(\Delta x)^2 - (\Delta y)^2}{(\Delta x)^2 + (\Delta y)^2}$

and $\frac{\Delta z - dz}{\Delta \rho} = \Delta x \, \Delta y \frac{(\Delta x)^2 - (\Delta y)^2}{((\Delta x)^2 + (\Delta y)^2)^{3/2}}$, where $\Delta \rho = \sqrt{(\Delta x)^2 + (\Delta y)^2}$

Let $\Delta x = r\, cos\theta$, $\Delta y = r\, sin\theta$. Therefore,

$$\frac{\Delta z - dz}{\Delta \rho} = r\, cos\theta\, sin\theta . cos2\theta$$

and $\left|\frac{\Delta z - dz}{\Delta \rho}\right| = r\, |sin\theta\, cos\theta . cos2\theta| \leq r$

As $(\Delta x, \Delta y) \to (0,0)$; $\Delta \rho = r \to 0$. This shows that
$\lim\limits_{\Delta \rho \to 0} \frac{\Delta z - dz}{\Delta \rho} = 0.$

$\Rightarrow$ $f(x, y)$ is differentiable at $(0,0)$.

Aliter:

$$\frac{\Delta z - dz}{\Delta \rho} = \frac{1}{\Delta \rho} \frac{(\Delta x)^3 \Delta y - \Delta x (\Delta y)^3}{(\Delta x)^2 + (\Delta y)^2}$$

$$= -\frac{(\Delta y)^3}{(\Delta x)^2 + (\Delta y)^2} \left(\frac{\Delta x}{\Delta \rho}\right) + \frac{(\Delta x)^3}{(\Delta x)^2 + (\Delta y)^2} \left(\frac{\Delta y}{\Delta \rho}\right) = \varepsilon_1 \frac{\Delta x}{\Delta \rho} + \varepsilon_2 \frac{\Delta y}{\Delta \rho}$$

where $\varepsilon_1 = -\frac{(\Delta y)^3}{(\Delta x)^2 + (\Delta y)^2}$, $\varepsilon_2 = \frac{(\Delta x)^3}{(\Delta x)^2 + (\Delta y)^2}$

Notice that

$|\varepsilon_1| = \left| -\frac{(\Delta y)^3}{(\Delta x)^2 + (\Delta y)^2} \right| \leq |\Delta y| \leq \sqrt{(\Delta x)^2 + (\Delta y)^2}$ and

$|\varepsilon_2| = \left| \frac{(\Delta x)^3}{(\Delta x)^2 + (\Delta y)^2} \right| \leq |\Delta x| \leq \sqrt{(\Delta x)^2 + (\Delta y)^2}$

Clearly $\varepsilon_1, \varepsilon_2 \to 0$ as both Δx and $\Delta y \to 0$

Thus $f(x, y)$ is differentiable at $(0,0)$

Example 3:

Show that the function

$$f(x,y)=\begin{cases} \dfrac{xy}{x^2+y^2}, & (x,y)\neq(0,0) \\ 0, & (x,y)=(0,0) \end{cases}$$

is differentiable at the origin.

Solution: We first test its continuity at $(0,0)$.

Let $(x, y) \to (0,0)$ along the path $y = mx,\ x \neq 0$

Then $f(x, y)|_{y=mx} = \frac{mx^2}{(1+m^2)x^2} = \frac{m}{1+m^2}$

Now, $\lim\limits_{\substack{(x,y)\to(0,0) \\ along\ y=mx}} f(x, y) = \frac{m}{1+m^2}$

and the limit does not exist by Two-Path Test. This shows that $f(x, y)$ is discontinuous at $(0, 0)$. Therefore $f(x, y)$ is not differentiable at $(0, 0)$

(Assume that f is differentiable at $(0, 0) \Longrightarrow f$ is continuous at $(0, 0)$ – a contradiction, since it is discontinuous at $(0, 0)$. Therefore, our assumption is wrong. Thus f is not differentiable at $(0, 0)$)

P1.

Find the total differential of the function $z = \tan^{-1}\left(\frac{x}{y}\right)$, $(x, y) \neq (0, 0)$.

Solution:

If $z = f(x, y)$ is a function of two variables x and y then the total differential is

$$dz = f_x\,dx + f_y\,dy$$

Given that $z = \tan^{-1}\left(\frac{x}{y}\right) = f(x, y)$

Now, $f_x = \frac{1}{1+\left(\frac{x}{y}\right)^2}\left(\frac{1}{y}\right) = \frac{y}{x^2+y^2}$

$$f_y = \frac{1}{1+\left(\frac{x}{y}\right)^2}\left(-\frac{x}{y^2}\right) = -\frac{x}{x^2+y^2}$$

Therefore, the total differential is

$$dz = f_x\,dx + f_y\,dy = \frac{y}{x^2+y^2}\,dx - \frac{x}{x^2+y^2}\,dy = \frac{(ydx-xdy)}{x^2+y^2}$$

P2.

Show that function

$$f(x,y)=\begin{cases}\dfrac{x^3+2y^3}{x^2+y^2}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$$

(i) is continuous at $(0,0)$

(ii) possess partial derivatives $f_x(0,0)$ and $f_y(0,0)$

(iii) is not differentiable at $(0,0)$

Solution:

Continuity at $(0,0)$:

let $x = rcos\theta$ and $y = rsin\theta$, we have

$$|f(x,y) - f(0,0)| = \left|\frac{r^3(cos^3\theta+2sin^3\theta)}{r^2}\right|$$

$$\leq r[|cos^3\theta| + 2|sin^3\theta|]$$

$$\leq 3r = 3\sqrt{x^2+y^2} < \varepsilon,$$

whenever $\sqrt{x^2+y^2} < \frac{\varepsilon}{3}$

Therefore, $\lim\limits_{(x,y)\to(0,0)} f(x,y) = 0 = f(0,0)$

Hence, $f(x,y)$ is continuous at $(0,0)$.

Partial derivatives at $(\mathbf{0}, \mathbf{0})$:

$$f_x(0,0) = \lim_{h \to 0} \frac{f(0+h,0) - f(0,0)}{h} = \lim_{h \to 0} \frac{h-0}{h} = 1$$

$$f_y(0,0) = \lim_{h \to 0} \frac{f(0,0+h) - f(0,0)}{h} = \lim_{h \to 0} \frac{2h-0}{h} = 2$$

Therefore, the partial derivatives exists at $(0,0)$

Differentiability at $(\mathbf{0}, \mathbf{0})$:

We have $dz = f_x(0,0)\,\Delta x + f_y(0,0)\,\Delta y = \Delta x + 2\Delta y$

$$\Delta z = f(\Delta x, \Delta y) - f(0,0) = \frac{(\Delta x)^3 + 2(\Delta y)^3}{(\Delta x)^2 + (\Delta y)^2}$$

Therefore,

$$\frac{\Delta z - dz}{\Delta \rho} = \frac{1}{\Delta \rho}\left[\frac{(\Delta x)^3 + 2(\Delta y)^3}{(\Delta x)^2 + (\Delta y)^2} - (\Delta x + 2\Delta y)\right] = \frac{1}{\Delta \rho}\left[-\frac{\Delta x . \Delta y(\Delta y + 2\Delta x)}{(\Delta x)^2 + (\Delta y)^2}\right]$$

$$= \left(-\frac{(\Delta y)^2}{(\Delta x)^2 + (\Delta y)^2}\right)\left(\frac{\Delta x}{\Delta \rho}\right) + \left(\frac{2(\Delta x)^2}{(\Delta x)^2 + (\Delta y)^2}\right)\left(\frac{\Delta y}{\Delta \rho}\right)$$

$$= \varepsilon_1\left(\frac{\Delta x}{\Delta \rho}\right) + \varepsilon_2\left(\frac{\Delta y}{\Delta \rho}\right),$$

where $\varepsilon_1 = -\frac{(\Delta y)^2}{(\Delta x)^2 + (\Delta y)^2}$, $\varepsilon_2 = \frac{2(\Delta x)^2}{(\Delta x)^2 + (\Delta y)^2}$

Clearly, both $\lim_{(\Delta x, \Delta y) \to (0,0)} \varepsilon_1$, $\lim_{(\Delta x, \Delta y) \to (0,0)} \varepsilon_2$ does not exist (by Two Path-Test).

This shows that $f(x,y)$ is not differentiable at $(0,0)$.

P3.

Show that the function $\boldsymbol{f(x,y)=\sqrt{x^2+y^2}}$ is not differentiable at $(\mathbf{0},\mathbf{0})$.

Solution:

Partial derivatives at $(\mathbf{0},\mathbf{0})$:

$$f_x(0,0)=\lim_{h\to 0}\frac{f(0+h,0)-f(0,0)}{h}=\lim_{h\to 0}\frac{h-0}{h}=1$$

$$f_y(0,0)=\lim_{h\to 0}\frac{f(0,0+h)-f(0,0)}{h}=\lim_{h\to 0}\frac{h-0}{h}=1$$

Therefore, the partial derivatives exists at $(0,0)$

Differentiability at $(\mathbf{0},\mathbf{0})$:

We have $dz=\ f_x(0,0)\,\Delta x+\ f_y(0,0)\,\Delta y=\Delta x+\Delta y$

$$\Delta z=f(0+\Delta x,\Delta y)-f(0,0)=\sqrt{(\Delta x)^2+(\Delta y)^2}$$

Therefore,

$$\frac{\Delta z-dz}{\Delta\rho}=\frac{1}{\Delta\rho}\left[\sqrt{(\Delta x)^2+(\Delta y)^2}-\Delta x-\Delta y\right]=\left[1-\frac{\Delta x+\Delta y}{\sqrt{(\Delta x)^2+(\Delta y)^2}}\right]$$

Notice that $\displaystyle\lim_{(\Delta x,\Delta y)\to(0,0)}\frac{\Delta x+\Delta y}{\sqrt{(\Delta x)^2+(\Delta y)^2}}$ does not exists.

Hence, $\displaystyle\lim_{\Delta\rho\to 0}\left[\frac{\Delta z-dz}{\Delta\rho}\right]$ does not exist (by Two- Path Test)

Therefore, $f(x,y)$ is not differentiable at $(0,0)$.

P4.

Show that function

$$f(x,y)=\begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}}, & (x,y)\neq(0,0) \\ 0 \quad , & (x,y)=(0,0) \end{cases}$$

is not differentiable at $(\mathbf{0},\mathbf{0})$

Solution:

Partial derivatives at $(\mathbf{0},\mathbf{0})$:

$$f_x(0,0)=\lim_{h\to 0}\frac{f(0+h,0)-f(0,0)}{h}=\lim_{h\to 0}\frac{0-0}{h}=0$$

$$f_y(0,0)=\lim_{h\to 0}\frac{f(0,0+h)-f(0,0)}{h}=\lim_{h\to 0}\frac{0-0}{h}=0$$

Therefore, the partial derivatives exists at $(0,0)$

Differentiability at $(\mathbf{0},\mathbf{0})$:

We have $dz = f_x(0,0)\,\Delta x + f_y(0,0)\,\Delta y = 0$

$$\Delta z = f(\Delta x,\Delta y) - f(0,0) = \frac{\Delta x.\Delta y}{\sqrt{(\Delta x)^2+(\Delta y)^2}}$$

Therefore,

$$\frac{\Delta z - dz}{\Delta\rho} = \frac{1}{\Delta\rho}\left[\frac{\Delta x.\Delta y}{\sqrt{(\Delta x)^2+(\Delta y)^2}}\right] = \left[\frac{\Delta x.\Delta y}{(\Delta x)^2+(\Delta y)^2}\right]$$

Notice that, $\lim_{(\Delta x,\Delta y)\to(0,0)}\left[\frac{\Delta x.\Delta y}{(\Delta x)^2+(\Delta y)^2}\right]$ does not exists (by Two – Path Test).

Therefore, $\lim_{\Delta\rho\to 0}\left[\frac{\Delta z-dz}{\Delta\rho}\right]$ does not exist.

This shows that $f(x,y)$ is not differentiable at $(0,0)$.

IP1.

Find the total differential of the function $\boldsymbol{z = \left(xz + \frac{x}{z}\right)^{y}}$**,** $\boldsymbol{z \neq 0}$**.**

Solution:

If $z = f(x, y, z)$ is a function of three variables x, y and z then the total differential is

$$dz = f_x \, dx + f_y \, dy + f_z \, dz$$

Given that $z = \left(xz + \frac{x}{z}\right)^{y} = f(x, y, z)$

Now, $f_x = y\left(xz + \frac{x}{z}\right)^{y-1}\left(z + \frac{1}{z}\right)$; $f_y = \left(xz + \frac{x}{z}\right)^{y} ln\left(xz + \frac{x}{z}\right)$

$$f_z = y\left(xz + \frac{x}{z}\right)^{y-1}\left(x - \frac{x}{z^2}\right)$$

Therefore, the total differential is

$$dz = f_x \, dx + f_y \, dy + f_z \, dz$$

$$= y\left(xz + \frac{x}{z}\right)^{y-1}\left(z + \frac{1}{z}\right) dx + \left(xz + \frac{x}{z}\right)^{y} ln\left(xz + \frac{x}{z}\right) dy$$
$$+ y\left(xz + \frac{x}{z}\right)^{y-1}\left(x - \frac{x}{z^2}\right)$$

$$= \left(xz + \frac{x}{z}\right)^{y-1}\left[y\left(z + \frac{1}{z}\right) dx + xy\left(1 - \frac{1}{z^2}\right) dz\right]$$
$$+ \left(xz + \frac{x}{z}\right)^{y} ln\left(xz + \frac{x}{z}\right) dy$$

IP2.

Show that function

$$f(x,y)=\begin{cases}\dfrac{x^2-y^2}{x-y}, & (x,y)\neq(1,-1)\\ 0 \quad\quad\quad , & (x,y)=(1,-1)\end{cases}$$

is continuous and differentiable at $(1\,,-1)$.

Solution:

Continuity at $(1,-1)$:

We have

$$\lim_{(x,y)\to(1,-1)} f(x,y) = \lim_{(x,y)\to(1,-1)} \frac{x^2-y^2}{x-y}$$

$$= \lim_{(x,y)\to(1,-1)} (x+y) = 0$$

and $f(1,-1) = 0$

Therefore, $\lim_{(x,y)\to(1,-1)} f(x,y) = 0 = f(1,-1)$

Hence, $f(x,y)$ is continuous at $(1,-1)$.

Partial derivatives at $(1,-1)$:

$$f_x(1,-1) = \lim_{h\to 0} \frac{f(1+h,-1)-f(1,-1)}{h}$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{(1+h)^2-1}{(1+h)+1}\right] = \lim_{h\to 0} \frac{2+h}{2+h} = 1$$

$$f_y(1,-1) = \lim_{h\to 0} \frac{f(1,-1+h)-f(1,-1)}{h}$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{1-(-1+h)^2}{1-(-1+h)}\right] = \lim_{h\to 0} \frac{2-h}{2-h} = 1$$

Therefore, the partial derivatives exists at $(1,-1)$

Differentiability at $(\mathbf{1},-\mathbf{1})$:

Now, we have

$$f_x(x,y) = \frac{(x-y)(2x)-\left(x^2-y^2\right)(1)}{(x-y)^2} = \frac{x^2-2xy+y^2}{(x-y)^2} = \frac{(x-y)^2}{(x-y)^2},$$

$$(x,y) \neq (1,-1)$$

and $f_x(x,y) = 1$ when $(x,y) = (1,-1)$

Since $\lim_{(x,y)\to(1,-1)} f_x(x,y) = \lim_{(x,y)\to(1,-1)} \frac{(x-y)^2}{(x-y)^2} = 1 = f_x(1,-1)$, the partial derivative f_x is continuous at $(1,-1)$.

Also $f_y(1,-1)$ is continuous at $(1,-1)$.

Hence, $f(x,y)$ is differentiable at $(1,-1)$ (by relaxed conditions).

IP3.

Show that function

$$f(x,y)=\begin{cases}\dfrac{x^3-y^3}{x^2+y^2}, & (x,y)\neq(0,0)\\ 0 \quad , & (x,y)=(0,0)\end{cases}$$

(i) is continuous at $(\mathbf{0}\,,\mathbf{0})$

(ii) possess partial derivatives $f_x(\mathbf{0},\mathbf{0})$ and $f_y(\mathbf{0},\mathbf{0})$

(iii) is not differentiable at $(\mathbf{0},\mathbf{0})$

Solution:

Continuity at $(\mathbf{0},\mathbf{0})$:

let $x = rcos\theta$ and $y = rsin\theta$, we have

$$|f(x,y)-f(0,0)| = \left|\frac{r^3(cos^3\theta-sin^3\theta)}{r^2(cos^2\theta-sin^2\theta)}\right| = r|cos^3\theta - sin^3\theta|$$

$$\leq r[|cos^3\theta| + |sin^3\theta|]$$

$$\leq 2r = 2\sqrt{x^2+y^2} < \varepsilon$$

whenever $\sqrt{x^2+y^2} < \frac{\varepsilon}{2}$.

Therefore, $\lim_{(x,y)\to(0,0)} f(x,y) = f(0,0)$

Hence, $f(x,y)$ is continuous at $(0,0)$.

Partial derivatives at $(0, 0)$:

$$f_x(0,0) = \lim_{h\to 0} \frac{f(0+h,0)-f(0,0)}{h} = \lim_{h\to 0} \frac{h-0}{h} = 1$$

$$f_y(0,0) = \lim_{h\to 0} \frac{f(0,0+h)-f(0,0)}{h} = \lim_{h\to 0} \frac{-h-0}{h} = -1$$

Therefore, the partial derivatives exists at $(0, 0)$

Differentiability at $(0, 0)$:

We have $dz = f_x(0,0)\,\Delta x + f_y(0,0)\,\Delta y = \Delta x - \Delta y$

$$\Delta z = f(\Delta x, \Delta y) - f(0,0) = \frac{(\Delta x)^3-(\Delta y)^3}{(\Delta x)^2+(\Delta y)^2}$$

Therefore,

$$\frac{\Delta z - dz}{\Delta\rho} = \frac{1}{\Delta\rho}\left[\frac{(\Delta x)^3-(\Delta y)^3}{(\Delta x)^2+(\Delta y)^2} - \Delta x + \Delta y\right] = \left[\frac{\Delta x.\Delta y(\Delta x-\Delta y)}{[(\Delta x)^2+(\Delta y)^2]^{\frac{3}{2}}}\right]$$

$$\lim_{(x,y)\to(0,0)}\left[\frac{\Delta z - dz}{\Delta\rho}\right] = \lim_{(x,y)\to(0,0)}\left[\frac{\Delta x.\Delta y(\Delta x-\Delta y)}{[(\Delta x)^2+(\Delta y)^2]^{\frac{3}{2}}}\right]$$

Notice that $\lim_{(\Delta x,\Delta y)\to(0,0)}\left[\frac{\Delta x.\Delta y(\Delta x-\Delta y)}{[(\Delta x)^2+(\Delta y)^2]^{\frac{3}{2}}}\right]$ does not exist (by Two Path-Test).

This shows that $f(x, y)$ is not differentiable at $(0, 0)$.

IP4.

Discuss the continuity and differentiability of the function

$$f(x,y)=\begin{cases}\dfrac{xy^2}{x^2+y^2}, & (x,y)\neq(0,0)\\ 0 \quad, & (x,y)=(0,0)\end{cases}$$

at $(0,0)$.

Solution:

Continuity at $(0,0)$:

We have

$$|f(x,y)-f(0,0)| = \left|\frac{xy^2}{x^2+y^2}\right| = |x| \le \sqrt{x^2+y^2} < \varepsilon,$$

whenever $\sqrt{x^2+y^2} < \varepsilon$

Therefore, $\lim\limits_{(x,y)\to(0,0)} f(x,y) = f(0,0)$

Hence, $f(x,y)$ is continuous at $(0,0)$.

Partial derivatives at $(0,0)$:

$$f_x(0,0) = \lim_{h\to 0}\frac{f(0+h,0)-f(0,0)}{h} = \lim_{h\to 0}\frac{0-0}{h} = 0$$

$$f_y(0,0) = \lim_{h\to 0}\frac{f(0,0+h)-f(0,0)}{h} = \lim_{h\to 0}\frac{0-0}{h} = 0$$

Therefore, the partial derivatives exists at $(0,0)$

Differentiability at $(0,0)$:

We have $dz = f_x(0,0)\,\Delta x + f_y(0,0)\,\Delta y = 0$

$$\Delta z = f(\Delta x, \Delta y) - f(0,0) = \frac{\Delta x.(\Delta y)^2}{(\Delta x)^2+(\Delta y)^2}$$

Therefore,

$$\frac{\Delta z - dz}{\Delta \rho} = \frac{1}{\Delta \rho}\left[\frac{\Delta x.(\Delta y)^2}{(\Delta x)^2+(\Delta y)^2}\right] = \left[\frac{\Delta x.(\Delta y)^2}{[(\Delta x)^2+(\Delta y)^2]^{\frac{3}{2}}}\right]$$

Notice that, $\lim\limits_{(\Delta x,\Delta y)\to(0,0)}\left[\frac{\Delta x.(\Delta y)^2}{[(\Delta x)^2+(\Delta y)^2]^{\frac{3}{2}}}\right]$ does not exists (by Two- Path-Test).

Therefore, $\lim\limits_{\Delta\rho\to 0}\left[\frac{\Delta z - dz}{\Delta \rho}\right]$ does not exist.

This shows that $f(x,y)$ is not differentiable at $(0,0)$.

3.5

Differentiability

EXERCISES:

1. Discuss the continuity and differentiability of the following functions.

a) $f(x,y)=\begin{cases} \dfrac{x^3-y^3}{x^2+y^2} & , \quad (x,y)\neq(0,0) \\ 0 & , \quad (x,y)=(0,0) \end{cases}$

b) $f(x,y)=\begin{cases} \dfrac{x^2y}{x^4+y^2} & , \quad (x,y)\neq(0,0) \\ 0 & , \quad (x,y)=(0,0) \end{cases}$

3.6.

Derivatives of composite functions and implicit functions (Chain Rule)

Learning objectives:

- To discuss the chain rule for functions two and three independent variables.
- To discuss two and three variable Implicit differentiation

  AND

- To practice the related problems

Derivatives of composite functions and implicit functions (Chain Rule)

The chain rule for functions of a single real variable states the following:

If $y = f(x)$ is a differentiable function of x and $x = g(t)$ is a differentiable function of t, then $y = f\big(g(t)\big)$ is a composite differentiable function of t and its derivative $\frac{dy}{dt}$ is given by

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

For functions of two or more variables the chain rule has several forms. The form depends on the number of variables involved but it works like the chain rule as above once we account for the presence of additional variables.

Functions of two variables

Theorem 1: ***Chain rule for functions of two independent variables***

If $\boldsymbol{z = f(x,y)}$ has continuous partial derivatives $\boldsymbol{f_x}$ and $\boldsymbol{f_y}$ and if $\boldsymbol{x = \phi(t), y = \psi(t)}$ are differentiable functions of t, then the composite function $\boldsymbol{z = f\big(\phi(t), \psi(t)\big)}$ is a differentiable function of t and

$$\boldsymbol{\frac{df}{dt} = f_x\big(\phi(t), \psi(t)\big)x'(t) + f_y\big(\phi(t), \psi(t)\big)y'(t)}$$

or $\quad \frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$

Proof: Let $\Delta x, \Delta y$ and Δz be the increments that result from changing t from t_0 to $t_0 + \Delta t$. Let P be the point $\left(x(t_0), y(t_0)\right)$. Since f_x and f_y are continuous at P; f is differentiable at P. Therefore,

$$\Delta z = \left(\frac{\partial z}{\partial x}\right)_{P_0} \Delta x + \left(\frac{\partial z}{\partial y}\right)_{P_0} \Delta y + \varepsilon_1 \frac{\Delta x}{\Delta t} + \varepsilon_2 \frac{\Delta y}{\Delta t}\text{, where}$$

$t_1, t_2 \to 0$ as both $\Delta x, \Delta y \to 0$. Dividing throughout by Δt, we obtain

$$\frac{\Delta z}{\Delta \mathrm{t}} = \left(\frac{\partial z}{\partial x}\right)_{P_0} \frac{\Delta x}{\Delta t} + \left(\frac{\partial z}{\partial y}\right)_{P_0} \frac{\Delta y}{\Delta t} + \varepsilon_1 \frac{\Delta x}{\Delta t} + \varepsilon_2 \frac{\Delta y}{\Delta t} \quad \text{... (1)}$$

Since x and y are differential functions of t, $\lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{\mathrm{d}x}{\mathrm{d}t}$ and $\lim\limits_{\Delta t \to 0} \frac{\Delta y}{\Delta t} = \frac{\mathrm{d}y}{\mathrm{d}t}$. Further, both Δx and $\Delta y \to 0$ as $\Delta t \to 0$. Therefore both ε_1 and $\varepsilon_2 \to 0$ as $\Delta t \to 0$. Thus, as $\Delta t \to 0$ the right hand side of (1) exists and so $\lim\limits_{\Delta t \to 0} \frac{\Delta z}{\Delta t}$ exists and it is $\frac{dz}{dt}$. Therefore,

$$\lim_{\Delta t \to 0} \frac{\Delta z}{\Delta t} = \left(\frac{\partial z}{\partial x}\right)_{P_0} \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} + \left(\frac{\partial z}{\partial y}\right)_{P_0} \lim_{\Delta t \to 0} \frac{\Delta y}{\Delta t}$$

$$+ \lim_{\Delta t \to 0} \varepsilon_1 \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} + \lim_{\Delta t \to 0} \varepsilon_2 \lim_{\Delta t \to 0} \frac{\Delta y}{\Delta t}$$

i.e., $\frac{\mathrm{d}z}{\mathrm{d}t} = \left(\frac{\partial z}{\partial x}\right)_{P_0} \left(\frac{dx}{dt}\right) + \left(\frac{\partial z}{\partial y}\right)_{P_0} \left(\frac{dy}{dt}\right)$

i.e., $\frac{\mathrm{d}z}{\mathrm{d}t} = \left(\frac{\partial f}{\partial x}\right)_{P_0} \left(\frac{dx}{dt}\right) + \left(\frac{\partial f}{\partial y}\right)_{P_0} \left(\frac{dy}{dt}\right)$

Hence the theorem

Note:

(1) A more precise notation for the above formula is

$$\left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)_{t=t_0} = \left(\frac{\partial f}{\partial x}\right)_{(x_0,y_0)} \left(\frac{dx}{dt}\right)_{t=t_0} + \left(\frac{\partial f}{\partial y}\right)_{(x_0,y_0)} \left(\frac{dy}{dt}\right)_{t=t_0}$$

where $(x_0, y_0) = \big(x(t_0), y(t_0)\big)$.

(2) In the above theorem the variables x, y are referred to ***intermediate variables*** and theorem is about ***the chain rule for two intermediate variables and one independent variable.***

The following is the chain rule for one independent variable and three intermediate variables.

Theorem 2: Chain rule for functions of three independent variables

If $w = f(x, y, z)$ is differentiable and x, y, z are differentiable functions of t, then w is a differential function of t and

$$\boldsymbol{\frac{dw}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}}$$

Example 1: Find $\frac{df}{dt}$ at $t = 0$, where

$f(x,y) = x\cos y + e^x \sin y,\ x = t^2 + 1,\ y = t^3 + t.$

Solution: We have, $x = t^2 + 1,\ y = t^3 + t$

Then $(x_0, y_0) = \big(x(0), y(0)\big) = (1,0)$ and

$\frac{dx}{dt} = 2t,\ \frac{dy}{dt} = 3t^2 + 1,\ \left(\frac{dx}{dt}\right)_{t=0} = 0,\ \left(\frac{dy}{dt}\right)_{t=0} = 1$. Further

$$\frac{\partial f}{\partial x} = \cos y + e^x \sin y\ ,\ \frac{\partial f}{\partial y} = -x\sin y + e^x \cos y$$

and $\left(\frac{\partial f}{\partial x}\right)_{(x_0,y_0)} = (\cos y + e^x \sin y)|_{(1,0)} = 0$

$$\left(\frac{\partial f}{\partial y}\right)_{(x_0,y_0)} = (-x\sin y + e^x \cos y)|_{(1,0)} = e$$

Now, by chain rule,

$$\left(\frac{\mathrm{d}f}{\mathrm{d}t}\right)_{t=0} = \left(\frac{\partial f}{\partial x}\right)_{(x_0,y_0)}\left(\frac{dx}{dt}\right)_{t=0} + \left(\frac{\partial f}{\partial y}\right)_{(x_0,y_0)}\left(\frac{dy}{dt}\right)_{t=0} = e$$

Example 2: Find $\frac{dw}{dt}$ at $t = 0$, where $w = xy + z$, $x = \cos t$, $y = \sin t$, $z = t$.

Solution: We have, by chain rule

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

$$= (y)(-\sin t) + (x)(\cos t) + (1)(1)$$

$$= (\sin t)(-\sin t) + (\cos t)(\cos t) + 1 = 1 + \cos 2t$$

Now, $\left(\frac{\mathrm{d}w}{\mathrm{d}t}\right)_{t=0} = 1 + \cos 0 = 2.$

A physical interpretation of change along a curve

If $w = T(x, y, z)$ is the temperature at each point (x, y, z) along the curve C with parametric equations $x = x(t), y = y(t)$, $z = z(t)$, then the composite function $w = T\big(x(t), y(t), z(t)\big)$ represents the temperature relative to t along the curve C. The derivative $\frac{dw}{dt}$ gives the instantaneous rate of change of temperature along C. (The curve C with parametric equations $x = \cos t\,, y = \sin t\,, z = t$ is a *helix*).

Let $z = f(x, y)$ and x, y be functions of two independent variables u and v, say $x = \phi(u, v), y = \psi(u, v)$. Then $z = f\big(\phi(u, v), \psi(u, v)\big)$ is a composite function of two independent variables u and v. The following is the chain rule that computes the partial derivatives of z w.r.t. u and v.

Theorem 3: Chain rule for two independent variables and two intermediate variables

Suppose that $z = f(x, y), x = \phi(u, v), y = \psi(u, v)$. If all the three functions are differentiable, then z has partial derivatives w.r.t. u and v, given by

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$$

$$\boldsymbol{\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}}$$

Example 3: Compute $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$ in terms of u and v if

$z = x^2 + y^2, x = u - v, y = u + v.$

Solution:

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} = (2x)(1) + 2y(1) = 2(x + y) = 4u$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}.\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}.\frac{\partial y}{\partial v} = (2x)(-1) + (2y)(1) = 2(y - x) = 4v$$

Example 4: If $z = f(x, y), x = e^{2u} + e^{-2v}, y = e^{-2u} + e^{2v}$

then show that $\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = 2\left[x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}\right]$

Solution: By chain rule we have

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}.\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}.\frac{\partial y}{\partial u} = 2e^{2u}\frac{\partial z}{\partial x} - 2e^{-2u}\frac{\partial z}{\partial y}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}.\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}.\frac{\partial y}{\partial v} = -2e^{-2v}\frac{\partial z}{\partial x} + 2e^{2v}\frac{\partial z}{\partial y}$$

$$\Rightarrow \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = 2(e^{2u} + e^{-2v})\frac{\partial z}{\partial x} - 2(e^{-2u} + e^{2v})\frac{\partial z}{\partial y}$$

$$= 2\left[x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}\right]$$

Note:

Chain rule for two independent variables and one intermediate variable:

Suppose that $y = f(x)$ and $x = \emptyset(u, v)$. If both are differentiable, then y has partial derivatives w.r.t u and v and they are given by

$$\frac{\partial y}{\partial u} = \frac{dy}{dx} \cdot \frac{\partial x}{\partial u}$$

$$\frac{\partial y}{\partial v} = \frac{dy}{dx} \cdot \frac{\partial x}{\partial v}$$

Theorem 4: Chain rule for two independent variables and three intermediate variables

Suppose that $\boldsymbol{w = f(x, y, z), x = \emptyset(u, v), y = \psi(u, v)}$,

$\boldsymbol{z = \chi(u, v)}$. ***If all four functions are differentiable then*** $\boldsymbol{w}$ ***has partial derivatives w.r.t*** $\boldsymbol{u}$ ***and*** $\boldsymbol{v}$ ***and they are given by***

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial u} + \frac{\partial w}{\partial z} \cdot \frac{\partial z}{\partial u}$$

$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial v} + \frac{\partial w}{\partial z} \cdot \frac{\partial z}{\partial v}$$

Example 5: Express $\frac{\partial w}{\partial u}$ and $\frac{\partial w}{\partial v}$ in terms of u and v if
$w = x + 2y + z^2,\ x = \frac{u}{v},\ \ y = u^2 + lnv,\ z = 2u.$

Solution:

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial u} + \frac{\partial w}{\partial z} \cdot \frac{\partial z}{\partial u}$$

$$= (1)\frac{1}{v} + (2)(2u) + (2z)(2)$$

$$= \frac{1}{v} + 4u + (4u)(2) = \frac{1}{v} + 12u$$

$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x} . \frac{\partial x}{\partial v} + \frac{\partial w}{\partial y} . \frac{\partial y}{\partial v} + \frac{\partial w}{\partial z} . \frac{\partial z}{\partial v}$$

$$= (1)\left(-\frac{u}{v^2}\right) + (2)\left(\frac{1}{v}\right) + (2z)(0) = \frac{2}{v} - \frac{u}{v^2}$$

Implicit Differentiation

Theorem 4: A formula

Suppose that $F(x, y)$ is differentiable and the equation $F(x, y) = 0$ defines y implicitly as a differentiable function of x. Then at any point where $F_y \neq 0$,

$$\frac{dy}{dx} = -\frac{F_x}{F_y}$$

Proof: Let $w = F(x, y) = 0$, Then $\frac{dw}{dx} = 0$.

By chain rule,

$$0 = \frac{dw}{dx} = \frac{\partial F}{\partial x} . \frac{dx}{dx} + \frac{\partial F}{\partial y} . \frac{dy}{dx}$$

$\Rightarrow F_x + F_y \frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{F_x}{F_y}$ (since $F_y \neq 0$)

Hence the result

Note: Under the conditions of the above theorem, we have

$$\frac{d^2y}{dx^2} = -\frac{F_{xx}(F_y)^2 - 2F_{yx}F_xF_y + F_{yy}(F_x)^2}{(F_y)^3}$$

Example 6:

Find $\frac{dy}{dx}$ when $f(x,y) = \ln(x^2 + y^2) + tan^{-1}\left(\frac{y}{x}\right) = 0.$

Solution: By implicit differentiation, we have

$$\frac{dy}{dx} = -\frac{f_x}{f_y}$$

Now, $f_x = \frac{2x}{x^2+y^2} + \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(-\frac{y}{x^2}\right) = \frac{2x}{x^2+y^2} - \frac{y}{x^2+y^2} = \frac{2x-y}{x^2+y^2}$

$$f_y = \frac{2y}{x^2+y^2} + \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(\frac{1}{x}\right) = \frac{2y}{x^2+y^2} + \frac{x}{x^2+y^2} = \frac{2y+x}{x^2+y^2}$$

Therefore, $\frac{dy}{dx} = -\frac{f_x}{f_y} = -\frac{2x-y}{2y+x} = \frac{y-2x}{2y+x}, y \neq -\frac{x}{2}$

Three- variable implicit differentiation

If the equation $F(x,y,z) = 0$ determines z implicitly as a function x and y, then at the points where $f_z \neq 0$,

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} \quad \text{and} \quad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}.$$

Example 7:

Find $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$ at (π, π, π), when

$$sin(x + y) + sin(y + z) + sin(z + x) = 0.$$

Solution:

Let $F(x,y,z) = sin(x+y) + sin(y+z) + sin(z+x) = 0.$

Now, $F_x(x,y,z) = cos(x+y) + cos(z+x)$

$F_y(x,y,z) = cos(x+y) + cos(y+z)$

$F_z(x,y,z) = cos(y+z) + cos(z+x)$

and $\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{cos\,(x+y)+cos\,(z+x)}{cos\,(y+z)+cos\,(z+x)}$

$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{cos\,(x+y)+cos\,(y+z)}{cos\,(y+z)+cos\,(z+x)}$

Therefore, $\left(\frac{\partial z}{\partial x}\right)_{(\pi,\pi,\pi)} = -\frac{1+1}{1+1} = -1$

$\left(\frac{\partial z}{\partial y}\right)_{(\pi,\pi,\pi)} = -\frac{1+1}{1+1} = -1$

P1:

If $\boldsymbol{f(x,y,z) = x^3 + xz^2 + y^3 + xyz, x = e^t, y = cost, z = t^3}$ then find $\boldsymbol{\frac{df}{dt}}$ at $\boldsymbol{t = 0}$

Solution:

Given $x = e^t, y = cost, z = t^3$

At $t = 0$:

$$x = 1,\, y = cos(0) = 1, z = 0$$

Now,

$$\left.\frac{dx}{dt}\right|_{t=0} = e^t|_{t=0} = 1 \quad ; \quad \left.\frac{dy}{dt}\right|_{t=0} = -sint|_{t=0} = 0$$

$$\left.\frac{dz}{dt}\right|_{t=0} = 3t^2|_{t=0} = 0$$

$$\frac{\partial f}{\partial x} = 3x^2 + z^2 + yz\Big|_{(1,1,0)} = 3 \quad ; \quad \frac{\partial f}{\partial y} = 3y^2 + xz|_{(1,1,0)} = 3,$$

$$\frac{\partial f}{\partial z} = 2xz + xy|_{(1,1,0)} = 1$$

Therefore by chain rule, at $t = 0$

$$\frac{df}{dt} = \frac{\partial f}{\partial x}.\frac{dx}{dt} + \frac{\partial f}{\partial y}.\frac{dy}{dt} + \frac{\partial f}{\partial z}.\frac{dz}{dt} = 3(1) + 3(0) + 1(0) = 3$$

P2:

If $\boldsymbol{f(x,y) = x^3 - xy + y^3}$, $\boldsymbol{x = r\cos\theta}$, $\boldsymbol{y = r\sin\theta}$, then find $\boldsymbol{\frac{\partial f}{\partial r}, \frac{\partial f}{\partial \theta}}$.

Solution:

Given, $f(x,y) = x^3 - xy + y^3, x = r\cos\theta\,, y = r\sin\theta$

$$\frac{\partial x}{\partial r} = \cos\theta \;\; ; \; \frac{\partial x}{\partial \theta} = -r\sin\theta$$

$$\frac{\partial y}{\partial r} = \sin\theta \;\; ; \; \frac{\partial y}{\partial \theta} = r\cos\theta$$

Now, $\frac{\partial f}{\partial x} = 3x^2 - y\,;\, \frac{\partial f}{\partial y} = -x + 3y^2$

Therefore by chain rule, we have

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r}$$

$$= (3x^2 - y)\cos\theta + (3y^2 - x)\sin\theta$$

$$= 3r^3cos^3\theta - 2rcos\theta.sin\theta + 3r^2sin^3\theta$$

$$\frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta}$$

$$= (3x^2 - y)(-r\sin\theta) + (3y^2 - x)(r\cos\theta)$$

$$= -3r^3cos^2\theta.sin\theta + r^2sin^2\theta + 3r^3sin^2\theta.cos\theta - r^2cos^2\theta$$

P3:

If $x^y = y^x$, then show that $\frac{dy}{dx} = \frac{y}{x}\frac{(y - x\ln y)}{(x - y\ln x)}$.

Solution:

Let $f(x, y) = x^y - y^x = 0$

Notice that f is an implicit function of x and y.

We have $\frac{dy}{dx} = -\frac{f_x}{f_y}$

Now, $f_x = yx^{y-1} - y^x \ln y$

$f_y = x^y \ln x - xy^{x-1}$

Therefore,

$$\frac{dy}{dx} = -\frac{yx^{y-1} - y^x \ln y}{x^y \ln x - xy^{x-1}} = -\frac{yx^{y-1} - x^y \ln y}{y^x \ln x - xy^{x-1}} \quad (\because x^y = y^x)$$

$$= \frac{x^y\left(\ln y - \frac{y}{x}\right)}{y^x\left(\ln x - \frac{x}{y}\right)} = \frac{\frac{1}{x}(x\ln y - y)}{\frac{1}{y}(y\ln x - x)} = \frac{y}{x}\left[\frac{x\ln y - y}{y\ln x - x}\right]$$

Hence proved

P4:

If $\boldsymbol{u = f(x-y, y-z, z-x)}$ then prove that

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$$

Solution:

Given $u = f(x-y, y-z, z-x)$

Put $x - y = X, \quad y - z = Y, \quad z - x = Z$

Therefore, $u = f(X, Y, Z)$, where X, Y, Z are functions of x, y, z

Thus, u is a composite function of x, y, z.

Now, $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial X} \cdot \frac{\partial X}{\partial x} + \frac{\partial u}{\partial Y} \cdot \frac{\partial Y}{\partial x} + \frac{\partial u}{\partial Z} \cdot \frac{\partial Z}{\partial x}$

$$= \frac{\partial u}{\partial X}(1) + \frac{\partial u}{\partial Y}(0) + \frac{\partial u}{\partial Z}(-1) = \frac{\partial u}{\partial X} - \frac{\partial u}{\partial Z}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial X} \cdot \frac{\partial X}{\partial y} + \frac{\partial u}{\partial Y} \cdot \frac{\partial Y}{\partial y} + \frac{\partial u}{\partial Z} \cdot \frac{\partial Z}{\partial y}$$

$$= \frac{\partial u}{\partial X}(-1) + \frac{\partial u}{\partial Y}(1) + \frac{\partial u}{\partial Z}(0) = -\frac{\partial u}{\partial X} + \frac{\partial u}{\partial Y}$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial X} \cdot \frac{\partial X}{\partial z} + \frac{\partial u}{\partial Y} \cdot \frac{\partial Y}{\partial z} + \frac{\partial u}{\partial Z} \cdot \frac{\partial Z}{\partial z}$$

$$= \frac{\partial u}{\partial X}(0) + \frac{\partial u}{\partial Y}(-1) + \frac{\partial u}{\partial Z}(1) = -\frac{\partial u}{\partial Y} + \frac{\partial u}{\partial Z}$$

Therefore,

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{\partial u}{\partial X} - \frac{\partial u}{\partial Z} - \frac{\partial u}{\partial X} + \frac{\partial u}{\partial Y} - \frac{\partial u}{\partial Y} + \frac{\partial u}{\partial Z} = 0$$

IP1:

If $\boldsymbol{w = x^2 + y^2, x = cost + sint, y = cost - sint}$ then find $\boldsymbol{\frac{dw}{dt}}$ at $\boldsymbol{t = 0}$

Solution:

Given $w = x^2 + y^2,\ x = cost + sint,\ y = cost - sint$

At $t = 0$,

$x = cos(0) + sin(0) = 1$

$y = cos(0) - sin(0) = 1$

Now,

$$\left.\frac{dx}{dt}\right|_{t=0} = -sint + cost|_{t=0} = -sin(0) + cos(0) = 1$$

$$\left.\frac{dy}{dt}\right|_{t=0} = -sint - cost|_{t=0} = -sin(0) - cos(0) = -1$$

$$\frac{\partial w}{\partial x} = 2x|_{(1,1)} = 2$$

$$\frac{\partial w}{\partial y} = 2y|_{(1,1)} = 2$$

Therefore by chain rule, at $t = 0$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}.\frac{dx}{dt} + \frac{\partial w}{\partial y}.\frac{dy}{dt} = 2(1) + 2(-1) = 0$$

IP2:

If $\boldsymbol{f(x,y,z) = z\sin\left(\frac{y}{x}\right)}$, where $\boldsymbol{x = 3u^2 + 2v\,,\, y = 4u - 2v^3}$, $\boldsymbol{z = 2u^2 - 3v^2}$, then find $\boldsymbol{\frac{\partial f}{\partial u}\,,\frac{\partial f}{\partial v}}$.

Solution:

Given, $f(x,y,z) = z\sin\left(\frac{y}{x}\right)$,

$$x = 3u^2 + 2v\,,\, y = 4u - 2v^3,\, z = 2u^2 - 3v^2$$

$$\frac{\partial x}{\partial u} = 6u\,;\ \frac{\partial x}{\partial v} = 2$$

$$\frac{\partial y}{\partial u} = 4 \quad ;\ \frac{\partial y}{\partial v} = -6v^2$$

$$\frac{\partial z}{\partial u} = 4u \quad ;\ \frac{\partial z}{\partial v} = -6v$$

Now,

$$\frac{\partial f}{\partial x} = z\cos\left(\frac{y}{x}\right)\left(-\frac{y}{x^2}\right) = -\frac{yz}{x^2}\cos\left(\frac{y}{x}\right)$$

$$\frac{\partial f}{\partial y} = z\cos\left(\frac{y}{x}\right)\frac{1}{x} = \frac{z}{x}\cos\left(\frac{y}{x}\right)$$

$$\frac{\partial f}{\partial z} = \sin\left(\frac{y}{x}\right)$$

Therefore by chain rule,

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial u}$$

$$= \left(-\frac{yz}{x^2}\right)\cos\left(\frac{y}{x}\right)(6u) + \frac{4z}{x}\cos\left(\frac{y}{x}\right) + \sin\left(\frac{y}{x}\right)(4u)$$

$$= \left(-\frac{6u(4u-2v^3)(2u^2-3v^2)}{(3u^2+2v)^2} + \frac{4(2u^2-3v^2)}{3u^2+2v}\right)\cos\left(\frac{4u-2v^3}{3u^2+2v}\right)$$

$$+4u\sin\left(\frac{4u-2v^3}{3u^2+2v}\right)$$

$$\frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial v}$$

$$= \left(-\frac{2yz}{x^2}\right)\cos\left(\frac{y}{x}\right) + \frac{z}{x}\cos\left(\frac{y}{x}\right)(-6v^2) + \sin\left(\frac{y}{x}\right)(-6v)$$

$$= \left(-\frac{2(4u-2v^3)(2u^2-3v^2)}{(3u^2+2v)^2} - \frac{6v^2(2u^2-3v^2)}{3u^2+2v}\right)\cos\left(\frac{4u-2v^3}{3u^2+2v}\right)$$

$$-6v\sin\left(\frac{4u-2v^3}{3u^2+2v}\right)$$

IP3:

Find $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$ at $(1,1,1)$ where $z^3 - xy + yz + y^3 - 2 = 0$.

Solution:

Let, $F(x,y,z) = z^3 - xy + yz + y^3 - 2 = 0$

Notice that F is an implicit function.

Now, $F_x = -y$

$F_y = -x + z + 3y^2$

$F_z = 3z^2 + y$

and $\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{(-y)}{3z^2+y} = \frac{y}{3z^2+y}$

$\Rightarrow \left(\frac{\partial z}{\partial x}\right)_{(1,1,1)} = \frac{1}{4}$

$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{-x+z+3y^2}{3z^2+y} = \frac{x-z-3y^2}{3z^2+y}$

$\Rightarrow \left(\frac{\partial z}{\partial y}\right)_{(1,1,1)} = -\frac{3}{4}$

IP4:

If $\boldsymbol{x = u + v + w}$, $\boldsymbol{y = uv + vw + wu}$, $\boldsymbol{z = uvw}$ and F is a function of $\boldsymbol{x, y, z}$ then show that

$$\boldsymbol{u\frac{\partial F}{\partial u} + v\frac{\partial F}{\partial v} + w\frac{\partial F}{\partial W} = x\frac{\partial F}{\partial x} + 2y\frac{\partial F}{\partial y} + 3z\frac{\partial F}{\partial z}}$$

Solution:

Given $x = u + v + w$, $y = uv + vw + wu$, $z = uvw$

Since x, y, z are the functions of u, v, w, F is a composite function of u, v, w.

Now, $\frac{\partial F}{\partial u} = \frac{\partial F}{\partial x}.\frac{\partial x}{\partial u} + \frac{\partial F}{\partial y}.\frac{\partial y}{\partial u} + \frac{\partial F}{\partial z}.\frac{\partial z}{\partial u}$

$$= \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}(v + w) + \frac{\partial F}{\partial z}(vw)$$

$$\frac{\partial F}{\partial v} = \frac{\partial F}{\partial x}.\frac{\partial x}{\partial v} + \frac{\partial F}{\partial y}.\frac{\partial y}{\partial v} + \frac{\partial F}{\partial z}.\frac{\partial z}{\partial v}$$

$$= \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}(u + w) + \frac{\partial F}{\partial z}(uw)$$

$$\frac{\partial F}{\partial w} = \frac{\partial F}{\partial x}.\frac{\partial x}{\partial w} + \frac{\partial F}{\partial y}.\frac{\partial y}{\partial w} + \frac{\partial F}{\partial z}.\frac{\partial z}{\partial w}$$

$$= \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}(v + u) + \frac{\partial F}{\partial z}(uv)$$

Therefore,

$$u\frac{\partial F}{\partial u}+v\frac{\partial F}{\partial v}+w\frac{\partial F}{\partial W}$$

$$=u\left[\frac{\partial F}{\partial x}+\frac{\partial F}{\partial y}(v+w)+\frac{\partial F}{\partial z}(vw)\right]$$

$$+v\left[\frac{\partial F}{\partial x}+\frac{\partial F}{\partial y}(u+w)+\frac{\partial F}{\partial z}(uw)\right]$$

$$+w\left[\frac{\partial F}{\partial x}+\frac{\partial F}{\partial y}(v+u)+\frac{\partial F}{\partial z}(uv)\right]$$

$$=(u+v+w)\frac{\partial F}{\partial x}+2(uv+vw+wu)\frac{\partial F}{\partial y}+3\,uvw\;\frac{\partial F}{\partial z}$$

$$=x\frac{\partial F}{\partial x}+2y\frac{\partial F}{\partial y}+3z\frac{\partial F}{\partial z}$$

Hence proved

3.6.

Derivatives of Composite functions and Implicit functions (Chain Rule)

EXERCISES:

I.

a) Express $\frac{dw}{dt}$ as a function of t, using the chain rule and evaluate at the given value of t .

1) $w = x^2 + y^2,\ x = cost, y = sint \quad ; t = \pi$

2) $w = x^2 + y^2,\ x = \frac{t^2-1}{t}, y = \frac{t}{t^2+1} \quad ; t = 1$

3) $w = \frac{x}{z} + \frac{y}{z}\ ,\quad x = cos^2t\ , y = sin^2t, z = \frac{1}{t} \quad ; t = 3$

4) $w = ln(x^2 + y^2 + z^2), x = cost, y = sint, z = 4\sqrt{t}\ ; t = 3$

5) $w = 2ye^x - lnz, x = ln(t^2 + 1), y = tan^{-1}t, z = e^t; t = 1$

6) $w = z - sinxy, x = t, y = lnt, z = e^{t-1}\ ;\quad t = 1$

7) $w = x^2 + y^2 + z^2, x = cost, y = \ln(t + 1)\ , z = e^t;\quad t = 0$

Answers:

1) $\left(\frac{dw}{dt}\right)_{t=\pi} = 0$

2) $\left(\frac{dw}{dt}\right)_{t=1} = 0$

3) $\left(\frac{dw}{dt}\right)_{t=3} = 1$

4) $\left(\frac{dw}{dt}\right)_{t=3} = \frac{16}{49}$

5) $\left(\frac{dw}{dt}\right)_{t=1} = \pi + 1$

6) $\left(\frac{dw}{dt}\right)_{t=1} = 0$

7) $\left(\frac{dw}{dt}\right)_{t=0} = 2$

II. Express $\frac{dw}{dt}$ as a function of t, using the chain rule

a) $w = e^x \sin(y + 2z)\,, x = t, y = \frac{1}{t}\,, z = t^2$

b) $w = xy + yz + zx, x = t^2, y = te^t, z = te^{-t}$

Answers:

a) $\frac{dw}{dt} = e^x \left\{\sin(y + 2z) + \left(\frac{4t^3 - 1}{t^2}\right) \cos(y + 2z)\right\}$

b) $\frac{dw}{dt} = 2(y + z)t + (x + z)(t + 1)e^t + (x + y)(1 - t)e^{-t}$

III. Express $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$ as functions of u and v by using the chain rule and evaluate them at the given point (u,v)

i) $z = 4e^{x} lny\,,\ x = \ln(ucosv)\,, y = usinv\,;\ \ (u,v) = \left(2,\frac{\pi}{4}\right)$

ii) $z = tan^{-1}\left(\frac{x}{y}\right), x = ucosv, y = usinv;\ (u,v) = \left(\frac{1}{3},\frac{\pi}{6}\right)$

iii)$z = xy + yz + zx,\ x = u + v, y = u - v, z = uv;$

$$(u,v) = \left(\frac{1}{2},1\right)$$

iv)If $z = ln(x^2 + y^2 + z^2), x = ue^{v}sinu, x = ue^{v}cosu,$

$z = ue^{v}\,; (u,v) = (-2,0)$

Answers:

i) $\left(\frac{\partial z}{\partial u}\right)_{\left(2,\frac{\pi}{4}\right)} = \sqrt{2}(ln2 + 2)\quad;\quad \left(\frac{\partial z}{\partial v}\right)_{\left(2,\frac{\pi}{4}\right)} = \sqrt{2}(4 - 2ln2)$

ii) $\left(\frac{\partial z}{\partial u}\right)_{\left(\frac{1}{3},\frac{\pi}{6}\right)} = 0\quad;\quad \left(\frac{\partial z}{\partial v}\right)_{\left(\frac{1}{3},\frac{\pi}{6}\right)} = -1$

iii)$\left(\frac{\partial z}{\partial u}\right)_{\left(\frac{1}{2},1\right)} = 3\quad;\quad \left(\frac{\partial z}{\partial v}\right)_{\left(\frac{1}{2},1\right)} = -\frac{3}{2}$

iv)$\left(\frac{\partial z}{\partial u}\right)_{(-2,0)} = -1\quad;\quad \left(\frac{\partial z}{\partial v}\right)_{(-2,0)} = 2$

IV.

1) If $z = f(x, y), x = rcosh\theta, y = rsinh\theta$ then show that

$$\left(\frac{\partial z}{\partial x}\right)^2 - \left(\frac{\partial z}{\partial y}\right)^2 = \left(\frac{\partial z}{\partial r}\right)^2 - \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2$$

2) If $z = f(x, y), x = ucos\alpha - vsin\alpha, y = usin\alpha - vcos\alpha,$

where α is constant then show that

$$\left(\frac{\partial f}{\partial u}\right)^2 + \left(\frac{\partial f}{\partial v}\right)^2 = \frac{\partial f}{\partial x} + \left(\frac{\partial f}{\partial y}\right)^2$$

3) If $z = ln(u^2 + v), u = e^{x+y^2}, v = x + y^2$ then show that

$$2y\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} = 0$$

4) If $z = \sqrt{x^2 + y^2 + z^2}, x = ucosv, y = usinv, z = uv,$ then

show that $u\frac{\partial z}{\partial u} - v\frac{\partial z}{\partial v} = \frac{u}{\sqrt{1+v^2}}$

V. Assuming that the equations define y as a differentiable function of x , find the value of $\frac{dy}{dx}$ at the given point.

1. $x^3 - 2y^2 + xy = 0 \ ; \ (1, 1)$
2. $xy + y^2 - 3x - 3 = 0 \ ; \ (-1, 1)$
3. $x^2 + xy + y^2 - 7 = 0 \ ; \ (1, 2)$
4. $xe^y + sinxy + y - ln2 = 0 \ ; \ (0, ln2)$

Answers:

1. $\left(\frac{dy}{dx}\right)_{(1,1)} = \frac{4}{3}$
2. $\left(\frac{dy}{dx}\right)_{(-1,1)} = 2$
3. $\left(\frac{dy}{dx}\right)_{(1,2)} = -\frac{4}{5}$
4. $\left(\frac{dy}{dx}\right)_{(0,ln2)} = -(ln2 + 2)$

VI. Assuming that the equations define z as a differentiable function of x and y, find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ at the given point.

1. $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} - 1 = 0$; $(2, 3, 6)$
2. $xe^y + ye^z + 2lnx - 2 - 3ln2 = 0$; $(1, ln2, ln3)$

Answers:

1. $\left(\frac{\partial z}{\partial y}\right)_{(2,3,6)} = -4$
2. $\left(\frac{\partial z}{\partial y}\right)_{(1,ln2,ln3)} = -\frac{5}{3ln2}$

3.7

JACOBIANS

Learning objectives:

- To define the Jacobian of a coordinate transformation.
- To study properties of Jacobians.

  AND

- To practice the related problems.

Jacobians arise naturally when there is a coordinate transformation. This concept is named after the German mathematician Carl Gustav Jacob Jacobi (1804-1851), who made significant contributions to mechanics, partial differential equations and calculus of variations.

Jacobian

The Jacobian determinant or Jacobian of the coordinate transformation $x = \phi(u, v), y = \psi(u, v)$ *is*

$$J(u, v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

It is also denoted by $J(u, v) = \frac{\partial(x,y)}{\partial(u,v)}$

Suppose that a region R in the uv-plane is transformed into the region R' in the xy-plane by equations of the form $x = \phi(u, v)$, $y = \psi(u, v)$. If ϕ and ψ are differentiable functions in R, then the transformation is one-to-one, and the Jacobian of the transformation $\frac{\partial(x,y)}{\partial(u,v)} \neq 0$. It measures how much the transformation is expanding or contracting the area around a point in R as R is transformed into R'.

The Jacobian of the coordinate transformation $x = \phi(u, v, w)$, $y = \psi(u, v, w)$, $z = \chi(u, v, w)$ *is*

$$J(u, v, w) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \frac{\partial(x,y,z)}{\partial(u,v,w)}$$

Suppose that a region G in uvw-space is transformed one-to-one into the region D in xyz-space by differentiable equations of the form $x = \phi(u, v, w)$, $y = \psi(u, v, w)$, $z = \chi(u, v, w)$.

The Jacobian of the transformation $\frac{\partial(x,y,z)}{\partial(u,v,w)}$ measures how much the volume near a point in G is being expanded or contracted by the transformation from uvw-space to the xyz-space.

Theorem 1: Chain rule for Jacobians

Suppose that $\boldsymbol{x = f(u, v)}$, $\boldsymbol{y = g(u, v)}$ ***where*** $\boldsymbol{u = \phi(r, s)}$, $\boldsymbol{v = \psi(r, s)}$. ***If all four functions are differentiable, then***

$$\boldsymbol{\frac{\partial(x,y)}{\partial(r,s)} = \frac{\partial(x,y)}{\partial(u,v)} \cdot \frac{\partial(u,v)}{\partial(r,s)}}$$

Proof: We have,

$$x = f(u, v), u = \phi(r, s) \text{ and } v = \psi(r, s)$$

and all are differentiable. Therefore, by chain rule for two intermediate variables and two independent variables, we have

$$\frac{\partial x}{\partial r} = \frac{\partial x}{\partial u}\frac{\partial u}{\partial r} + \frac{\partial x}{\partial v}\frac{\partial v}{\partial r}$$

$$\frac{\partial x}{\partial s} = \frac{\partial x}{\partial u}\frac{\partial u}{\partial s} + \frac{\partial x}{\partial v}\frac{\partial v}{\partial s}$$

Similarly, since $y = g(u, v), u = \phi(r, s), v = \psi(r, s)$ are differentiable,

$$\frac{\partial y}{\partial r} = \frac{\partial y}{\partial u}\frac{\partial u}{\partial r} + \frac{\partial y}{\partial v}\frac{\partial v}{\partial r}$$

$$\frac{\partial y}{\partial s} = \frac{\partial y}{\partial u}\frac{\partial u}{\partial s} + \frac{\partial y}{\partial v}\frac{\partial v}{\partial s}$$

Now, $\frac{\partial(x,y)}{\partial(r,s)} = \begin{vmatrix} x_r & x_s \\ y_r & y_s \end{vmatrix} = \begin{vmatrix} x_u u_r + x_v v_r & x_u u_s + x_v v_s \\ y_u u_r + y_v v_r & y_u u_s + y_v v_s \end{vmatrix}$

$$= \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} \begin{vmatrix} u_r & u_s \\ v_r & v_s \end{vmatrix}$$

(by the multiplication of determinants)

$$= \frac{\partial(x,y)}{\partial(u,v)} \cdot \frac{\partial(u,v)}{\partial(r,s)}$$

Hence the result

Theorem 2:

If $x = f(u, v)$ and $y = g(u, v)$ are differentiable (in a region R) then

$$\frac{\partial(x,y)}{\partial(u,v)} \cdot \frac{\partial(u,v)}{\partial(x,y)} = 1$$

Proof: Since x, y are differentiable functions of u and v, the transformation is 1-1, and the Jacobian of transformation $\frac{\partial(x,y)}{\partial(u,v)} \neq 0$. Since the transformation is 1-1, we solve for u, v in terms of x and y and we obtain the inverse transformation $u = \phi(x, y)$, $v = \psi(x, y)$. Further, these are also differentiable. Thus we have, $x = f(u, v)$, $y = g(u, v)$ where $u = \phi(x, y)$, $v = \psi(x, y)$ where all the four functions are differentiable: This context is precisely same as theorem1 with $r = x, s = y$.

Thus from theorem1,

$$\frac{\partial(x,y)}{\partial(u,v)} \cdot \frac{\partial(u,v)}{\partial(x,y)} = \frac{\partial(x,y)}{\partial(x,y)} = \begin{vmatrix} x_x & x_y \\ y_x & y_y \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$

Hence the result

Some coordinate transformations and their Jacobians

(1) Polar Coordinates to Cartesian coordinate

$$\boldsymbol{x = r\cos\theta, y = r\sin\theta,}$$

$$\frac{\partial x}{\partial r} = \cos\theta \ , \frac{\partial x}{\partial \theta} = -r\sin\theta \ , \frac{\partial y}{\partial r} = \sin\theta \ , \frac{\partial y}{\partial \theta} = r\cos\theta$$

$$\therefore \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} x_r & x_\theta \\ y_r & y_\theta \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$$

(2) Cylindrical coordinates to Cartesian coordinates

$$\boldsymbol{x = r\cos\theta, y = r\sin\theta, z = z}$$

$$x_r = \cos\theta \ , x_\theta = -r\sin\theta \ , x_z = 0,$$
$$y_r = \sin\theta \ , y_\theta = r\cos\theta \ , y_z = 0,$$
$$z_r = 0 \ , z_\theta = 0 \ , z_z = 1.$$
$$\therefore \frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \begin{vmatrix} x_r & x_\theta & x_z \\ y_r & y_\theta & y_z \\ z_r & z_\theta & z_z \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r$$

(3) Spherical polar coordinates, to Cartesian coordinates

$$\boldsymbol{x = r\sin\theta\cos\phi, y = r\sin\theta\sin\phi, z = r\cos\theta}$$

$$x_r = \sin\theta\cos\phi \ , x_\theta = r\cos\theta\cos\phi \ , x_\phi = -r\sin\theta\sin\phi,$$
$$y_r = \sin\theta\sin\phi \ , y_\theta = r\cos\theta\sin\phi \ , y_\phi = r\sin\theta\cos\phi,$$
$$z_r = \cos\theta \ , z_\theta = -r\sin\theta \ , z_\phi = 0.$$

$$\therefore \frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{vmatrix} x_r & x_\theta & x_\phi \\ y_r & y_\theta & y_\phi \\ z_r & z_\theta & z_\phi \end{vmatrix}$$

$$= \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$$= r^2\sin\theta$$

Example 1: If $u = x^2 - y^2$, $v = 2xy$ and $x = r\cos\theta$,

$y = r\sin\theta$ then find $\frac{\partial(u,v)}{\partial(r,\theta)}$

Solution: By chain rule for Jacobians, we have,

$$\frac{\partial(u,v)}{\partial(r,\theta)} = \frac{\partial(u,v)}{\partial(x,y)}\frac{\partial(x,y)}{\partial(r,\theta)}$$

Now, $\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} 2x & -2y \\ 2y & 2x \end{vmatrix} = 4(x^2 + y^2)$

$\frac{\partial(x,y)}{\partial(r,\theta)} = r$ (we have already seen)

Therefore, $\frac{\partial(u,v)}{\partial(r,\theta)} = 4(x^2 + y^2) \cdot r = 4r^3$

Theorem 3:

Suppose that $f(x, y)$ is a function of two (independent) variables x, y and x, y are functions of two independent variables u and v given by $x = \phi(u, v)$, $y = \psi(u, v)$. If all the three functions are differentiable, then

$$\frac{\partial f}{\partial x} = \frac{1}{J}\frac{\partial(f,y)}{\partial(u,v)} \text{ and } \frac{\partial f}{\partial y} = \frac{1}{J}\frac{\partial(x,f)}{\partial(u,v)} = -\frac{\partial(f,x)}{\partial(u,v)} \text{ where } J = \frac{\partial(x,y)}{\partial(u,v)}$$

Proof: Since x, y are differentiable functions, by chain rule,

$$\frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} = \frac{\partial f}{\partial u}$$

$$\frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v} = \frac{\partial f}{\partial v}$$

Since, $J = \frac{\partial(x,y)}{\partial(u,v)} \neq 0$; we solve for $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

By Cramer's rule $\frac{\partial f}{\partial x} = \frac{\begin{vmatrix} f_u & y_u \\ f_v & y_v \end{vmatrix}}{\begin{vmatrix} x_u & y_u \\ x_v & y_v \end{vmatrix}} = \frac{\begin{vmatrix} f_u & f_v \\ y_u & y_v \end{vmatrix}}{\begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}} = \frac{1}{J}\left[\frac{\partial(f,y)}{\partial(u,v)}\right]$

$$\frac{\partial f}{\partial y} = \frac{\begin{vmatrix} x_u & f_u \\ x_v & f_v \end{vmatrix}}{\begin{vmatrix} x_u & y_u \\ x_v & y_v \end{vmatrix}} = \frac{\begin{vmatrix} x_u & x_v \\ f_u & f_v \end{vmatrix}}{\begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}} = \frac{1}{J}\left[\frac{\partial(x,f)}{\partial(u,v)}\right] = -\frac{1}{J}\frac{\partial(f,x)}{\partial(u,v)}$$

The following is the corresponding result for three intermediate variables and three independent variables.

Theorem 4:

Suppose that $f(x,y,z)$ is a function of three (independent) variables x,y,z and x,y,z are functions of three independent variables u,v and w given by $x = \phi(u,v,w)$, $y = \psi(u,v,w)$, $z = \chi(u,v,w)$. If all the four functions are differentiable, then

$$\frac{\partial f}{\partial x} = \frac{1}{J}\frac{\partial(f,y,z)}{\partial(u,v,w)}$$

$$\frac{\partial f}{\partial y} = \frac{1}{J}\frac{\partial(x,f,z)}{\partial(u,v,w)} = -\frac{1}{J}\frac{\partial(f,x,z)}{\partial(u,v,w)}$$

$$\frac{\partial f}{\partial z} = \frac{1}{J}\frac{\partial(x,y,f)}{\partial(u,v,w)} = \frac{1}{J}\frac{\partial(f,x,y)}{\partial(u,v,w)} \quad , \quad \textit{where } J = \frac{\partial(x,y,z)}{\partial(u,v,w)}$$

Example 2: If $z = f(x,y), x = r\cos\theta \ , y = r\sin\theta$, then

$$\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 = \left(\frac{\partial f}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial f}{\partial \theta}\right)^2$$

Solution: The Jacobian of the transformation is

$$J = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$$

By Theorem 3 we have $\frac{\partial f}{\partial x} = \frac{1}{J}\frac{\partial(f,y)}{\partial(r,\theta)}$ and $\frac{\partial f}{\partial y} = \frac{1}{J}\frac{\partial(x,f)}{\partial(r,\theta)}$

Now $\frac{\partial(f,y)}{\partial(r,\theta)} = \begin{vmatrix} f_r & f_\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos\theta\frac{\partial f}{\partial r} - \sin\theta\frac{\partial f}{\partial\theta}$

$$\frac{\partial(x,f)}{\partial(r,\theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ f_r & f_\theta \end{vmatrix} = \cos\theta\frac{\partial f}{\partial\theta} + r\sin\theta\frac{\partial f}{\partial r}$$

Therefore,

$$\frac{\partial f}{\partial x} = \frac{1}{r}\left(r\,cos\,\theta\frac{\partial f}{\partial r} - sin\,\theta\frac{\partial f}{\partial\theta}\right) = cos\,\theta\frac{\partial f}{\partial r} - \frac{sin\,\theta}{r}\frac{\partial f}{\partial\theta}$$

$$\frac{\partial f}{\partial y} = \frac{1}{r}\left(r\,sin\,\theta\frac{\partial f}{\partial r} + cos\,\theta\frac{\partial f}{\partial\theta}\right) = sin\,\theta\frac{\partial f}{\partial r} + \frac{cos\,\theta}{r}\frac{\partial f}{\partial\theta}$$

Squaring and adding, we get

$$\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 = \left(\frac{\partial f}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial f}{\partial\theta}\right)^2$$

Example 3: If $w = f(x,y,z)$ and $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$, then show that

Solution: The Jacobian of the transformation is

$$J = \frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$$= r^2\sin\theta$$

Now,

$$\frac{\partial(f,y,z)}{\partial(r,\theta,\phi)} = \begin{vmatrix} f_r & f_\theta & f_\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$$= r^2 \sin^2 \theta \cos \phi \frac{\partial f}{\partial r} + r \sin \theta \cos \theta \cos \phi \frac{\partial f}{\partial \theta} - r \sin \phi \frac{\partial f}{\partial \phi}$$

$$\frac{\partial(x,f,z)}{\partial(r,\theta,\phi)} = \begin{vmatrix} \sin \theta \cos \phi & r \cos \theta \cos \phi & -r \sin \theta \sin \phi \\ f_r & f_\theta & f_\phi \\ \cos \theta & -r \sin \theta & 0 \end{vmatrix}$$

$$= - \begin{vmatrix} f_r & f_\theta & f_\phi \\ \sin \theta \cos \phi & r \cos \theta \cos \phi & -r \sin \theta \sin \phi \\ \cos \theta & -r \sin \theta & 0 \end{vmatrix}$$

$$= - \left[-r^2 \sin^2 \theta \sin \phi \frac{\partial f}{\partial r} - r \sin \theta \cos \theta \sin \phi \frac{\partial f}{\partial \theta} - r \cos \phi \frac{\partial f}{\partial \phi} \right]$$

$$= r^2 \sin^2 \theta \sin \phi \frac{\partial f}{\partial r} + r \sin \theta \cos \theta \sin \phi \frac{\partial f}{\partial \theta} + r \cos \phi \frac{\partial f}{\partial \phi}$$

$$\frac{\partial(x,y,f)}{\partial(r,\theta,f)} = \frac{\partial(f,x,y)}{\partial(r,\theta,\emptyset)} = \begin{vmatrix} f_r & f_\theta & f_\emptyset \\ sin\theta cos\emptyset & rcos\theta cos\emptyset & -rsin\theta sin\emptyset \\ sin\theta sin\emptyset & rcos\theta sin\emptyset & rsin\theta cos\emptyset \end{vmatrix}$$

$$= r^2 sin\theta cos\theta \frac{\partial f}{\partial r} - rsin^2\theta \frac{\partial f}{\partial \theta}$$

By Theorem 4,

$$\frac{\partial f}{\partial x} = \frac{1}{J} . \frac{\partial(f,x,y)}{\partial(r,\theta,\emptyset)} = sin\theta cos\emptyset \frac{\partial f}{\partial r} + \frac{cos\theta sin\emptyset}{r} \frac{\partial f}{\partial \theta} - \frac{sin\emptyset}{rsin\theta} \frac{\partial f}{\partial \emptyset}$$

$$\frac{\partial f}{\partial y} = \frac{1}{J} . \frac{\partial(x,f,z)}{\partial(r,\theta,\emptyset)} = sin\theta sin\emptyset \frac{\partial f}{\partial r} + \frac{cos\theta sin\emptyset}{r} \frac{\partial f}{\partial \theta} + \frac{cos\emptyset}{rsin\theta} \frac{\partial f}{\partial \emptyset}$$

$$\frac{\partial f}{\partial z} = \frac{1}{J} . \frac{\partial(x,y,f)}{\partial(r,\theta,\emptyset)} = cos\theta \frac{\partial f}{\partial r} - \frac{sin\theta}{r} \frac{\partial f}{\partial \theta}$$

Squaring and adding we obtain.

$$\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2+\left(\frac{\partial f}{\partial z}\right)^2=\left(\frac{\partial f}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial f}{\partial \theta}\right)^2+\frac{1}{r^2 sin^2\theta}\left(\frac{\partial f}{\partial \emptyset}\right)^2$$

Functional Dependence

Theorem 5:

Let $u=f(x,y), v=g(x,y)$ be differentiable functions in some region R.

A necessary and sufficient condition for a functional relation between u and v of the form $\emptyset(u,v)=0$ is that the jacobian $\frac{\partial(u,v)}{\partial(x,y)}$ vanishes in R. That is $\frac{\partial(u,v)}{\partial(x,y)}=0$ identically in R .

Example 6: If $u=\frac{x+y}{1-xy}$, $v=tan^{-1}x+tan^{-1}y$ then show that they are functionally related. Find the relationship.

Solution:

$$\frac{\partial(u,v)}{\partial(x,y)}=\begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}=\begin{vmatrix} \frac{1+y^2}{(1-xy)^2} & \frac{1+x^2}{(1-xy)^2} \\ \frac{1}{1+x^2} & \frac{1}{1+y^2} \end{vmatrix}$$

$$=\frac{1}{(1-xy)^2}-\frac{1}{(1-xy)^2}=0 \text{ , if } xy\neq 1$$

By Theorem 5, there must be a functional relationship between u and v.

$$v=Tan^{-1}x+Tan^{-1}y$$

$$\Rightarrow tanv = tan(Tan^{-1}x + Tan^{-1}y) = \frac{x+y}{1-xy} = u.$$

The relationship is $u = tanv$.

Example 7: Show that the functions $u = x + y - z$,

$v = x - y + z, W = x^2 + y^2 + z^{\,2} - 2yz$ are not independent. Find the functional relationship among them.

Solution:

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix} = \begin{vmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ 2x & 2y-2z & 2z-2y \end{vmatrix}$$

$$\underset{=}{\boldsymbol{C_3 = C_3 + C_2}} \begin{vmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 2x & 2y-2z & 0 \end{vmatrix} = 0$$

Since the Jacobian vanishes, there must be a functional relation among them. Now,

$u + v = 2x, u - v = 2(y - z)$and

$(u + v)^2 + (u - v)^2 = 4x^2 + 4(y^2 + z^{\,2} - 2yz) = 4w$, is the required relationship.

P1.

If $u = x + \frac{y^2}{x}, v = \frac{y^2}{x}$ then find $J = \frac{\partial(u,v)}{\partial(x,y)}$ and $J^* = \frac{\partial(x,y)}{\partial(u,v)}$ and verify that $J.J^* = 1$.

Solution:

Given that $u = x + \frac{y^2}{x}, v = \frac{y^2}{x}$

$$J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$$

$$= \begin{vmatrix} 1 - \frac{y^2}{x^2} & \frac{2y}{x} \\ -\frac{y^2}{x^2} & \frac{2y}{x} \end{vmatrix} = \frac{2y}{x} - \frac{2y^3}{x^3} + \frac{2y^3}{x^3} = \frac{2y}{x}$$

Solving for x, y in terms of u and v, we have

$$u = x + \frac{y^2}{x} = x + v \implies x = u - v, v = \frac{y^2}{x}$$

$$y^2 = vx, y^2 = v(u-v) \implies y = \sqrt{v(u-v)}$$

Now, $J^* = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ \frac{1}{2}\frac{v}{\sqrt{v(u-v)}} & \frac{1}{2}\frac{(u-2v)}{\sqrt{v(u-v)}} \end{vmatrix}$

$$= \frac{1}{2}\frac{1}{\sqrt{v(u-v)}}[u - 2v + v] = \frac{1}{2}\frac{u-v}{\sqrt{v(u-v)}} = \frac{x}{2y}$$

Verification: $J.J^* = \frac{2y}{x}.\frac{1}{2}\frac{x}{y} = 1$

P2.

If $y_1 = \frac{x_2x_3}{x_1}, y_2 = \frac{x_3x_1}{x_2}, y_3 = \frac{x_1x_2}{x_3}$, show that the Jacobian of y_1, y_2, y_3 with respect to x_1, x_2, x_3 is 4.

Solution:

We have $\quad \frac{\partial y_1}{\partial x_1} = -\frac{x_2x_3}{x_1^2}, \quad \frac{\partial y_1}{\partial x_2} = \frac{x_3}{x_1}, \quad \frac{\partial y_1}{\partial x_3} = \frac{x_2}{x_1}$

$$\frac{\partial y_2}{\partial x_1} = \frac{x_3}{x_2}, \quad \frac{\partial y_2}{\partial x_2} = -\frac{x_3x_1}{x_2^2}, \quad \frac{\partial y_2}{\partial x_3} = \frac{x_1}{x_2}$$

and $\quad \frac{\partial y_3}{\partial x_1} = \frac{x_2}{x_3}, \quad \frac{\partial y_3}{\partial x_2} = \frac{x_1}{x_3}, \quad \frac{\partial y_3}{\partial x_3} = -\frac{x_1x_2}{x_3^2}$

$$\therefore \frac{\partial(y_1,y_2,y_3)}{\partial(x_1,x_2,x_3)} = \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \frac{\partial y_1}{\partial x_3} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \frac{\partial y_2}{\partial x_3} \\ \frac{\partial y_3}{\partial x_1} & \frac{\partial y_3}{\partial x_2} & \frac{\partial y_3}{\partial x_3} \end{vmatrix} = \begin{vmatrix} -\frac{x_2x_3}{x_1^2} & \frac{x_3}{x_1} & \frac{x_2}{x_1} \\ \frac{x_3}{x_2} & -\frac{x_3x_1}{x_2^2} & \frac{x_1}{x_2} \\ \frac{x_2}{x_3} & \frac{x_1}{x_3} & -\frac{x_1x_2}{x_3^2} \end{vmatrix}$$

$$= \frac{1}{x_1^2x_2^2x_3^3} \begin{vmatrix} -x_2x_3 & x_3x_1 & x_1x_2 \\ x_2x_3 & -x_3x_1 & x_1x_2 \\ x_2x_3 & x_3x_1 & -x_1x_2 \end{vmatrix}$$

$$= \frac{x_1^2x_2^2x_3^3}{x_1^2x_2^2x_3^3} \begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

$$= -1(1-1) - 1(-1-1) + 1(1+1)$$

$$= 0 + 2 + 2 = 4$$

P3.

If $\boldsymbol{u = 2axy, v = a(x^2 - y^2)}$,where $\boldsymbol{x = rcos\theta, y = rsin\theta}$ then find $\boldsymbol{\frac{\partial(u,v)}{\partial(r,\theta)}}$.

Solution:

Given that $u = 2axy, v = a(x^2 - y^2)$,where $x = rcos\theta$, $y = rsin\theta$

Since u, v are functions of x and y which are themselves functions of r, θ,

By using chain rule for Jacobians, we have

$$\frac{\partial(u,v)}{\partial(r,\theta)} = \frac{\partial(u,v)}{\partial(x,y)} \cdot \frac{\partial(x,y)}{\partial(r,\theta)}$$

Given $u = 2axy, v = a(x^2 - y^2)$

$\Longrightarrow u_x = 2ay, u_y = 2ax$ and $v_x = 2ax, v_y = -2ay$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 2ay & 2ax \\ 2ax & -2ay \end{vmatrix}$$

$$= -4a^2(y^2 + x^2) = -4a^2r^2$$

Also, $\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} cos\theta & -rsin\theta \\ sin\theta & rcos\theta \end{vmatrix} = r$

Hence, $\frac{\partial(u,v)}{\partial(r,\theta)} = \frac{\partial(u,v)}{\partial(x,y)} \cdot \frac{\partial(x,y)}{\partial(r,\theta)} = (-4a^2r^2)r = -4a^2r^3$.

P4.

If $u = x\sqrt{1-y^2} + y\sqrt{1-x^2}, v = \sin^{-1} x + \sin^{-1} y,$ then show that u, v are functionally related and find the relationship.

Solution:

Given that $u = x\sqrt{1-y^2} + y\sqrt{1-x^2}, v = \sin^{-1} x + \sin^{-1} y$

Now, $\frac{\partial u}{\partial x} = \sqrt{1-y^2} - \frac{xy}{\sqrt{1-x^2}}, \quad \frac{\partial u}{\partial y} = -\frac{xy}{\sqrt{1-y^2}} + \sqrt{1-x^2}$

and $\frac{\partial v}{\partial x} = \frac{1}{\sqrt{1-x^2}}, \quad \frac{\partial v}{\partial y} = \frac{1}{\sqrt{1-y^2}}$

$$\therefore \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} \sqrt{1-y^2} - \frac{xy}{\sqrt{1-x^2}} & \sqrt{1-x^2} - \frac{xy}{\sqrt{1-y^2}} \\ \frac{1}{\sqrt{1-x^2}} & \frac{1}{\sqrt{1-y^2}} \end{vmatrix}$$

$$= 1 - \frac{xy}{\sqrt{[(1-x^2)(1-y^2)]}} - 1 + \frac{xy}{\sqrt{[(1-x^2)(1-y^2)]}} = 0$$

Hence, u and v are functionally related i.e., they are not independent.

We have

$$v = \sin^{-1} x + \sin^{-1} y = \sin^{-1}\left[x\sqrt{1-y^2} + y\sqrt{1-x^2}\right]$$

i.e., $u = \sin v$, which is the required relationship between u and v.

IP1.

If $x = e^u cosv, y = e^u sinv$ then find $J = \frac{\partial(u,v)}{\partial(x,y)}$ and $J^* = \frac{\partial(x,y)}{\partial(u,v)}$ and verify that $J.J^* = 1$.

Solution:

Given that $x = e^u cosv, \; y = e^u sinv$

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} e^u cosv & -e^u sinv \\ e^u sinv & e^u cosv \end{vmatrix}$$

$$= e^{2u}(cos^2 v + sin^2 v) = e^{2u}$$

Solving for u, v in terms of x, y, we get

$$tanv = \frac{e^u sinv}{e^u cosv} = \frac{y}{x} \implies v = Tan^{-1}\frac{y}{x}$$

$$x^2 + y^2 = e^{2u}(cos^2 v + sin^2 v) = e^{2u}$$

$$\therefore \; u = \frac{1}{2} ln(x^2 + y^2)$$

Now, $J^* = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{x}{x^2+y^2} & \frac{y}{x^2+y^2} \\ -\frac{y}{x^2+y^2} & \frac{x}{x^2+y^2} \end{vmatrix}$

$$= \frac{x^2+y^2}{(x^2+y^2)^2} = \frac{1}{x^2+y^2} = \frac{1}{e^{2u}}$$

Verification: $J.J^* = e^{2u}\frac{1}{e^{2u}} = 1.$

IP2.

If $\boldsymbol{u = \frac{2yz}{x}, v = \frac{3zx}{y}, w = \frac{4xy}{z}}$ **then find** $\boldsymbol{\frac{\partial(x,y,z)}{\partial(u,v,w)}}$ **.**

Solution:

Given that $u = \frac{2yz}{x}, v = \frac{3zx}{y}, w = \frac{4xy}{z}$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{vmatrix}$$

$$= \begin{vmatrix} \frac{-2yz}{x^2} & \frac{2z}{x} & \frac{2y}{x} \\ \frac{3z}{y} & -\frac{3zx}{y^2} & \frac{3x}{y} \\ \frac{4y}{z} & \frac{4x}{z} & -\frac{4xy}{z^2} \end{vmatrix}$$

$$= -\frac{2yz}{x^2}\left[\frac{12x^2yz}{y^2z^2} - \frac{12x^2}{yz}\right] - \frac{2z}{x}\left[-\frac{12xyz}{yz^2} - \frac{12xy}{yz}\right] + \frac{2y}{x}\left[\frac{12xz}{yz} + \frac{12xyz}{zy^2}\right]$$

$$= 0 + 48 + 48 = 96$$

IP3.

If $= \sqrt{vw}, y = \sqrt{wu}, z = \sqrt{uv}$, where $u = rsin\theta.cos\emptyset$,

$v = rsin\theta.sin\emptyset, w = rcos\theta$, then find $\frac{\partial(x,y,z)}{\partial(r,\theta,\emptyset)}$

Solution:

Given that $x = \sqrt{vw}, y = \sqrt{wu}, z = \sqrt{uv}$, where
$u = rsin\theta.cos\emptyset, v = rsin\theta.sin\emptyset, w = rcos\theta$

Since x, y, z are functions of u, v, w which are the functions in $r, \theta, \emptyset$.

By using chain rule for Jacobians, we have

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\emptyset)} = \frac{\partial(x,y,z)}{\partial(u,v,w)} \frac{\partial(u,v,w)}{\partial(r,\theta,\emptyset)}$$

Therefore,

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} x_u & x_v & x_w \\ y_u & y_v & x_w \\ z_u & z_v & z_w \end{vmatrix}$$

$$= \begin{vmatrix} 0 & \frac{1}{2}\sqrt{\frac{w}{v}} & \frac{1}{2}\sqrt{\frac{v}{w}} \\ \frac{1}{2}\sqrt{\frac{w}{u}} & 0 & \frac{1}{2}\sqrt{\frac{u}{w}} \\ \frac{1}{2}\sqrt{\frac{v}{u}} & \frac{1}{2}\sqrt{\frac{u}{v}} & 0 \end{vmatrix}$$

$$= \frac{1}{8}\left[\sqrt{\frac{w}{v}\frac{v}{u}\frac{u}{v}} + \sqrt{\frac{v}{w}\frac{w}{u}\frac{u}{v}}\right] = \frac{2}{8} = \frac{1}{4}$$

We know that the Jacobian for spherical coordinates is

$$\frac{\partial(u,v,w)}{\partial(r,\theta,\emptyset)} = r^2 sin\theta$$

Thus, $\frac{\partial(x,y,z)}{\partial(r,\theta,\emptyset)} = \frac{\partial(x,y,z)}{\partial(u,v,w)} \cdot \frac{\partial(u,v,w)}{\partial(r,\theta,\emptyset)} = \frac{1}{4} r^2 sin\theta$

IP4:

Determine the functions $\boldsymbol{u = x^2 e^{-y}\cosh z, v = x^2 e^{-y}\sinh z,}$ $\boldsymbol{w = 3x^4 e^{-2y}}$ are functionally dependent. If so, find the relationship between them.

Solution:

Given that $u = x^2 e^{-y}\cosh z$, $v = x^2 e^{-y}\sinh z$, $w = 3x^4 e^{-2y}$

Now, $J\left(\frac{u,v,w}{x,y,z}\right) = \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix}$

$$= \begin{vmatrix} 2xe^{-y}\cosh z & -x^2e^{-y}\cosh z & x^2e^{-y}\sinh z \\ 2xe^{-y}\sinh z & -x^2e^{-y}\sinh z & x^2e^{-y}\cosh z \\ 12x^3e^{-2y} & -6x^4e^{-2y} & 0 \end{vmatrix}$$

$$= 12x^7e^{-4y}(\cosh^2 z - \sinh^2 z) - 12x^7e^{-4y}(\cosh^2 z - \sinh^2 z)$$

$$= 0$$

$\therefore u, v, w$ are functionally dependent.

We have

$$3u^2 - 3v^2 = 3(x^4e^{-2y}\cosh^2 z - x^4e^{-2y}\sinh^2 z)$$

$$= 3x^4e^{-2y} = w$$

That is, $3u^2 - 3v^2 = w$, is the required relationship between u and v.

3.7. Jacobians

EXERCISES:

I. Find the Jacobian $\frac{\partial(u,v)}{\partial(x,y)}$ when

a) $u = 3x + 5y, v = 4x - 3y$

b) $x + y = u,\ \ v = uv$

c) $u = \frac{(x+y)}{(1-xy)},\ \ v = Tan^{-1}x + Tan^{-1}y$

Answers:

a) -29

b) $(x + y)^{-1}$

c) 0

II. Find $\frac{\partial(u,v)}{\partial(r,\theta)}$ if

1. $u = x^2 - 2y^2, v = 2x^2 - y^2$ and $x = rcos\theta, y = rsin\theta$
2. $u = 2xy, v = x^2 - y^2$ and $r = cos\theta, y = rsin\theta$
3. $u = x^2 + y^2, v = y, x = rcos\theta, y = rsin\theta$

Answers:

1. $6r^3sin2\theta$
2. $4r^3$
3. $2xr$

III.

1) Find $\frac{\partial(u,v)}{\partial(r,\theta)}$ if $u = 2xy, v = x^2 - y^2$ and

$r = cos\theta, y = rsin\theta$

2) If $X = u^2v, Y = uv^2$ and $u = x^2 - y^2, v = xy$ then find

$\frac{\partial(X,Y)}{\partial(x,y)}$

3) If $u = x^2, v = siny, w = e^{-3z}$ then find $\frac{\partial(u,v,w)}{\partial(x,y,z)}$

4) If $u = x + y + z, uv = y + z, uvw = z$ then find $\frac{\partial(x,y,z)}{\partial(u,v,w)}$

5) If $u = xyz, v = xy + yz + zx\ , w = x + y + z$ then find

$\frac{\partial(x,y,z)}{\partial(u,v,w)}$

6) If $x = rcos\theta, y = rsin\theta$,evaluate $\frac{\partial(r,\theta)}{\partial(x,y)}, \frac{\partial(x,y)}{\partial(r,\theta)}$ and prove

that $\frac{\partial(r,\theta)}{\partial(x,y)} \frac{\partial(x,y)}{\partial(r,\theta)} = 1$

Answers:

1) $4r^3$
2) $6x^2y^2(x^2 + y^2)(x^2 - y^2)^2$
3) $-6e^{-3z}x\ cosy$
4) u^2v
5) $(x - y)(y - z)(z - x)$

IV. Determine whether the following functions are functionally dependent or not. If so, find the functional relation between them.

a) $u = \frac{x^2-y^2}{x^2+y^2}, v = \frac{2xy}{x^2+y^2}$

b) $u = sinx + siny,\ \ v = \sin(x + y)$

c) $u = \frac{x-y}{x+y}, v = \frac{xy}{(x-y)^2}$

d) $u = xe^y sinz, y = xe^y cosz, w = x^2e^{2y}$

e) $u = x + y + z, v = x^3 + y^3 + z^3 - 3xyz,$
$w = x^2 + y^2 + z^2 - xy - yz - zx$

Answers:

a) Dependent , $u^2 + v^2 = 1$

b) Independent

c) Dependent, $u^2 = 1 + 4v$

d) Dependent, $u^2 + v^2 = w$

e) Dependent, $uw = v$

3.8

Taylor's Expansion

Learning objectives:

- To state and prove Taylor's Theorem for functions of two variables
- To compute the maximum absolute error for the linear and quadratic approximations of $f(x, y)$ about a point

  AND

- To practice the related problems.

Taylor's Expansion

We have already derived the Taylor's theorem for functions of a single real variable.

If $f(x)$ has continuous derivatives upto $(n+1)^{\text{th}}$ order in some interval containing $x = a$, then

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \cdots + \frac{(x-a)^n}{n!}f^{(n)}(a) + R_n(x)$$

where $R_n(x)$ is the remainder term given by

$$R_n(x) = \frac{(x-a)^{n+1}}{(n+1)!}f^{(n+1)}(c) = \frac{(x-a)^{n+1}}{(n+1)!}f^{(n+1)}[a+\theta(x-a)],$$

$a < c < x\,, 0 < \theta < 1.$

Theorem1: Taylor's Theorem for functions of two variables

Let $f(x,y)$ be a function of two variables x and y defined in some domain D in $\boldsymbol{R^2}$. If f has continuous partial derivatives upto $(n+1)^{\text{th}}$ order in some neighborhood of a point (x_0, y_0) in D, then for some point (x_0+h, y_0+k) in this neighborhood, we have

$$\boldsymbol{f(x_0+h, y_0+k) = f(x_0, y_0) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)f(x_0, y_0)}$$

$$\boldsymbol{+\frac{1}{2!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f(x_0, y_0) + \cdots}$$

$$+\frac{\mathbf{1}}{\boldsymbol{n}!}\left(\boldsymbol{h}\frac{\boldsymbol{\partial}}{\boldsymbol{\partial x}}+\boldsymbol{k}\frac{\boldsymbol{\partial}}{\boldsymbol{\partial y}}\right)^{\boldsymbol{n}}\boldsymbol{f}(\boldsymbol{x_0},\boldsymbol{y_0})+\boldsymbol{R_n} \quad \dots (\mathbf{1})$$

where R_n is the remainder term given by

$$\boldsymbol{R_n}=\frac{\mathbf{1}}{(\boldsymbol{n}+\mathbf{1})!}\left(\boldsymbol{h}\frac{\boldsymbol{\partial}}{\boldsymbol{\partial x}}+\boldsymbol{k}\frac{\boldsymbol{\partial}}{\boldsymbol{\partial y}}\right)^{\boldsymbol{n}+\mathbf{1}}\boldsymbol{f}(\boldsymbol{x_0}+\boldsymbol{\theta h},\boldsymbol{y_0}+\boldsymbol{\theta k})\,,\mathbf{0}<\theta<1$$

$$\dots (\mathbf{2})$$

Proof: Let $x = x_0 + th\,, y = y_0 + tk$, where the parameter t takes the values in the interval $[0,1]$. That is $(x,y) = (x_0,y_0)$ when $t = 0$ and $(x,y) = (x_0+h, y_0+h)$ when $t = 1$.

Define a function $\phi(t) = f(x,y) = f(x_0+th, y_0+th)$,

$t \in [0,1]$. Now, $\frac{dx}{dt} = h, \frac{dy}{dt} = k$ and by chain rule, we obtain

$$\phi'(t) = \frac{\partial f}{\partial x}\cdot\frac{dx}{dt}+\frac{\partial f}{\partial y}\cdot\frac{dy}{dt} = h\frac{\partial f}{\partial x}+k\frac{\partial f}{\partial y} = \left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)f$$

$$\phi''(t) = \frac{\partial}{\partial x}\left(h\frac{\partial f}{\partial x}+k\frac{\partial f}{\partial y}\right)\frac{dx}{dt}+\frac{\partial}{\partial y}\left(h\frac{\partial f}{\partial x}+k\frac{\partial f}{\partial y}\right)\frac{dy}{dt}$$

$$= h^2\frac{\partial^2 f}{\partial x^2}+hk\frac{\partial^2 f}{\partial x\partial y}+hk\frac{\partial^2 f}{\partial y\partial x}+k^2\frac{\partial^2 f}{\partial y^2}$$

$$= h^2\frac{\partial^2 f}{\partial x^2}+2hk\frac{\partial^2 f}{\partial x\partial y}+k^2\frac{\partial^2 f}{\partial y^2}$$

(**by the continuity of the partial derivatives upto $(\boldsymbol{n}+\mathbf{1})^{\text{th}}$ order**)

$$= \left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^2 f$$

Continuing in this way,

$$\phi^{(n)}(t) = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n f \text{ and } \phi^{(n+1)}(t) = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^{n+1} f$$

By Maclaurin's theorem,

$$\phi(t) = \phi(0) + t\phi'(0) + \frac{t^2}{2!}\phi''(0) + \cdots + \frac{t^n}{n!}\phi^{(n)}(0)$$
$$+ \frac{t^{n+1}}{(n+1)!}\phi^{(n+1)}(\theta t) \text{ where } 0 < \theta < 1$$

Put $t = 1$, we obtain

$$\phi(1) = \phi(0) + \phi'(0) + \frac{1}{2!}\phi''(0) + \cdots + \frac{1}{n!}\phi^{(n)}(0)$$
$$+ \frac{1}{(n+1)!}\phi^{(n+1)}(\theta) \qquad \dots \dots \dots \dots \dots \dots \dots \text{ (3)}$$

where $\phi(0) = f(x_0, y_0)$, $\phi(1) = f(x_0 + h, y_0 + h)$

$$\phi^{(i)}(0) = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^i f(x_0, y_0) \text{ , } i = 1,2, \dots, n$$

and $\phi^{(n+1)}(\theta) = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^{n+1} f(x_0 + \theta h, y_0 + \theta k),\ 0 < \theta < 1$

Putting the above expressions in (3), we obtain

$$f(x_0 + h, y_0 + k) = f(x_0, y_0) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) f(x_0, y_0)$$
$$+ \frac{1}{2!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f(x_0, y_0) + \cdots$$

$$+\frac{1}{n!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^n f(x_0,y_0)+R_n \ \dots (1)$$

where $R_n=\frac{1}{(n+1)!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^{n+1} f(x_0+\theta h,y_0+\theta k)\,,0<\theta<1$

$$\dots\dots (2)$$

Hence the theorem

Note 1: Substituting $x=x_0+h\,,y=y_0+h$. Then $h=x-x_0$ and $k=y-y_0$ and Taylor's theorem becomes

$$\boldsymbol{f(x,y)=f(x_0,y_0)+\left[(x-x_0)\frac{\partial}{\partial x}+(y-y_0)\frac{\partial}{\partial y}\right]f(x_0,y_0)}$$

$$\boldsymbol{+\frac{1}{2!}\left[(x-x_0)\frac{\partial}{\partial x}+(y-y_0)\frac{\partial}{\partial y}\right]^2 f(x_0,y_0)}$$

$$\boldsymbol{+\frac{1}{n!}\left[(x-x_0)\frac{\partial}{\partial x}+(y-y_0)\frac{\partial}{\partial y}\right]^n f(x_0,y_0)+R_n \dots (4)}$$

Where$\boldsymbol{R_n=\frac{1}{(n+1)!}\left[(x-x_0)\frac{\partial}{\partial x}+(y-y_0)\frac{\partial}{\partial y}\right]^{n+1} f(\xi,\eta),}$
$\boldsymbol{\xi=(1-\theta)x_0+\theta x,\ \eta=(1-\theta)y_0+\theta y,\ 0}<\theta<1$ **.... (5)**

Note 2: For $n=1$, we get the ***linear polynomial approximation to*** $\boldsymbol{f(x,y)}$ as

$$f(x,y)\approx f(x_0,y_0)+(x-x_0)f_x+(y-y_0)f_y$$

where the partial derivatives are evaluated at (x_0,y_0).

Note 3: For $n=2$, we get the ***second degree (quadratic) polynomial approximation to*** $\boldsymbol{f(x,y)}$ as

$$f(x,y) \approx f(x_0,y_0) + (x-x_0)f_x + (y-y_0)f_y$$
$$+\frac{1}{2!}\left[(x-x_0)^2 f_{xx} + 2(x-x_0)(y-y_0)f_{xy} + (y-y_0)^2 f_{yy}\right]$$

where the partial derivatives are evaluated at (x_0, y_0).

Note 4: For $n = 3$, we get the ***third degree (cubic) polynomial approximation to $f(x,y)$*** as

$$f(x,y) \approx f(x_0,y_0) + (x-x_0)f_x + (y-y_0)f_y$$
$$+\frac{1}{2!}\left[(x-x_0)^2 f_{xx} + 2(x-x_0)(y-y_0)f_{xy} + (y-y_0)^2 f_{yy}\right]$$
$$+\frac{1}{3!}[(x-x_0)^3 f_{xxx} + 3(x-x_0)^2(y-y_0)f_{xxy} +$$
$$3(x-x_0)(y-y_0)^2 f_{xyy} + (y-y_0)^3 f_{yyy}]$$

where the partial derivatives are evaluated at (x_0, y_0).

Note 5: If we set $(x_0, y_0) = (0,0)$ in (4), we obtain the **Maclaurin's theorem for functions of two variables** as

$$f(\boldsymbol{x},\boldsymbol{y}) = \boldsymbol{f(0,0)} + \left(\boldsymbol{x}\frac{\boldsymbol{\partial}}{\boldsymbol{\partial x}} + \boldsymbol{y}\frac{\boldsymbol{\partial}}{\boldsymbol{\partial y}}\right)\boldsymbol{f(0,0)} + \frac{\mathbf{1}}{2!}\left(\boldsymbol{x}\frac{\boldsymbol{\partial}}{\boldsymbol{\partial x}} + \boldsymbol{y}\frac{\boldsymbol{\partial}}{\boldsymbol{\partial y}}\right)^{\mathbf{2}}\boldsymbol{f(0,0)} + \cdots$$
$$+\frac{\mathbf{1}}{\boldsymbol{n}!}\left(\boldsymbol{x}\frac{\boldsymbol{\partial}}{\boldsymbol{\partial x}} + \boldsymbol{y}\frac{\boldsymbol{\partial}}{\boldsymbol{\partial y}}\right)^{\boldsymbol{n}}\boldsymbol{f(0,0)} + \boldsymbol{R_n} \quad \ldots \ \mathbf{(6)}$$

where $\boldsymbol{R_n} = \frac{\mathbf{1}}{(\boldsymbol{n}+\mathbf{1})!}\left(\boldsymbol{x}\frac{\boldsymbol{\partial}}{\boldsymbol{\partial x}} + \boldsymbol{y}\frac{\boldsymbol{\partial}}{\boldsymbol{\partial y}}\right)^{\boldsymbol{n}+\mathbf{1}}\boldsymbol{f(\theta x, \theta y)}, \mathbf{0} < \theta < 1$ **..** (7)

Note 6: When $\lim_{n\to\infty} R_n = 0$, we obtain the ***Taylor's series*** *or* ***Taylor's expansion of $f(x,y)$ about the point (x_0, y_0)*** from (1) and the ***Maclaurin's series expansion of $\boldsymbol{f(x, y)}$ about the origin*** $(\mathbf{0}, \mathbf{0})$ from (6).

Note 7: Taylor's theorem can be extended to functions of more than two variables.

Estimation of error

Since the value of θ or the point (ξ, η) in the error term R_n (given in (5)) is not known, the evaluation the error term exactly, is not possible. However, we can find a bound of the error term in a given rectangular region

$$R: |x - x_0| < \delta_1 , |y - y_0| < \delta_2$$

While estimating, we assume that all the partial derivatives of the required order are continuous throughout this rectangular region R.

Estimation of error term for linear approximation

For $n = 1$, i.e., for linear approximation, the error term is given by

$$R_1 = \frac{1}{2!}\left[(x - x_0)^2 f_{xx} + 2(x - x_0)(y - y_0) f_{xy} + (y - y_0)^2 f_{yy}\right]$$

where the partial derivatives f_{xx} , f_{xy} , f_{yy} are evaluated at the point $(\xi, \eta) = \left(x_0 + \theta(x - x_0) , y_0 + \theta(y - y_0)\right), 0 < \theta < 1.$

Now,

$$|R_1| \le \frac{1}{2}\left[|x - x_0|^2|f_{xx}| + 2|x - x_0||y - y_0|\left|f_{xy}\right| + |y - y_0|^2\left|f_{yy}\right|\right]$$

$$\le \frac{B}{2}[|x - x_0|^2 + 2|x - x_0||y - y_0| + |y - y_0|^2]$$

where $B = max\{|f_{xx}|, \left|f_{xy}\right|, \left|f_{yy}\right|\}$ for all $(x, y) \in R$

$$= \frac{B}{2}[|x - x_0| + |y - y_0|]^2 \le \frac{B}{2}[\delta_1 + \delta_2]^2$$

This value of $|\ R_1\ |$ is called the ***maximum absolute error*** in the linear approximation of $f(x, y)$ about the point (x_0, y_0).

Estimation of error term for quadratic approximation

For $n = 2$, i.e., for quadratic approximation, the error term is given by

$$R_2 = \frac{1}{3!}[(x - x_0)^3 f_{xxx} + 3(x - x_0)^2(y - y_0)f_{xxy}$$

$$+3(x - x_0)(y - y_0)^2 f_{xyy} + (y - y_0)^3 f_{yyy}]$$

where the partial derivatives are evaluated at the point $(\xi, \eta) = \left(x_0 + \theta(x - x_0)\ , y_0 + \theta(y - y_0)\right), 0 < \theta < 1.$

Now,

$$|R_2| \le \frac{1}{6}\begin{bmatrix} |x - x_0|^3|f_{xxx}| + 3|x - x_0|^2|y - y_0|\left|f_{xxy}\right| \\ +3|x - x_0||y - y_0|^2\left|f_{xyy}\right| + |y - y_0|^3\left|f_{yyy}\right| \end{bmatrix}$$

$$\le \frac{B}{6}[|x - x_0|^3 + 3|x - x_0|^2|y - y_0| + 3|x - x_0||y - y_0|^2 + |y - y_0|^3]$$

where $B = max\{|f_{xxx}|, |f_{xxy}|, |f_{xyy}|, |f_{yyy}|\}$ for all $(x, y) \in R$

$$= \frac{B}{6}[|x - x_0| + |y - y_0|]^3 \leq \frac{B}{6}[\delta_1 + \delta_2]^3$$

This value of $|R_2|$ is the maximum absolute error in the quadratic approximation of $f(x, y)$ about the point (x_0, y_0).

Example 1: Expand $f(x, y) = 21 + x - 20y + 4x^2 + xy + 6y^2$ in Taylor series of maximum order about the point $(-1,2)$.

Solution: Notice that all the third order partial derivatives of $f(x, y)$ are zero, the maximum order of the Taylor series expansion of $f(x, y)$ about the point $(-1,2)$ is two. We have,

$$f(x, y) = f(-1,2) + \left[(x + 1)\frac{\partial}{\partial x} + (y - 2)\frac{\partial}{\partial y}\right] f(-1,2)$$

$$+\frac{1}{2!}\left[(x + 1)\frac{\partial}{\partial x} + (y - 2)\frac{\partial}{\partial y}\right]^2 f(-1,2)$$

$$= f(-1,2) + (x + 1)f_x(1, -2) + (y - 2)f_y(-1,2)$$

$$+\frac{1}{2!}[(x + 1)^2 f_{xx}(-1,2) + 2(x + 1)(y - 2)f_{xy}(-1,2)$$

$$+(y - 2)^2 f_{yy}(-1,2)]$$

Now, $f(-1,2) = 6$, $f_x(x, y) = 1 + 8x + y \Rightarrow f_x(-1,2) = -5$

$f_y(x, y) = -20 + x + 12y \Rightarrow f_y(-1,2) = 3$

$f_{xx}(x, y) = 8\,, f_{xy}(x, y) = 1\,, f_{yy}(x, y) = 12$

$\Rightarrow f_{xx}(-1,2) = 8\,, f_{xy}(-1,2) = 1\,, f_{yy}(-1,2) = 12$

$$\therefore f(x,y) = 6 + (x+1)(-5) + (y-2)(3)$$
$$+\frac{1}{2}[(x+1)^2(8) + 2(x+1)(y-2)(1) + (y-2)^2(12)]$$
$$= 6 - 5(x+1) + 3(y-2) + 4(x+1)^2 + (x+1)(y-2) + 6(y-2)^2$$

Example 2: Expand e^{xy} at $(1,1)$

Solution:

$f(x,y) = e^{xy}$	;	$f(1,1) = e$
$f_x(x,y) = ye^{xy}$	;	$f_x(1,1) = e$
$f_{xx}(x,y) = y^2e^{xy}$	;	$f_{xx}(1,1) = e$
$f_{xxy}(x,y) = (xy^2 + 2y)e^{xy}$	;	$f_{xxy}(1,1) = 3e$
$f_{xxx}(x,y) = y^3e^{xy}$	;	$f_{xxx}(1,1) = e$
$f_{xy}(x,y) = (1 + xy)e^{xy}$	;	$f_{xy}(1,1) = 2e$
$f_y(x,y) = xe^{xy}$	;	$f_y(1,1) = e$
$f_{yy}(x,y) = x^2e^{xy}$	;	$f_{yy}(1,1) = e$
$f_{yyy}(x,y) = x^3e^{xy}$	;	$f_{yyy}(1,1) = e$
$f_{yyx}(x,y) = (2x + yx^2)e^{xy}$	;	$f_{yyx}(1,1) = 3e$

By Taylors expansion,

$$f(x,y) = e^{xy}$$

$$= f(1,1) + \left[(x-1)f_x(1,1) + (y-1)f_y(x-1)\right]$$

$$+\frac{1}{2!}\left[(x-1)^2 f_{xx}(1,1) + 2(x-1)(y-1)f_{xy}(1,1) + (y-1)^2 f_{yy}(1,1)\right]$$

$$+\frac{1}{3!}\left[(x-1)^3 f_{xxx}(1,1) + 3\,(x-1)^2(y-1)f_{xxy}(1,1) + \right.$$
$$\left. 3(x-1)(y-1)^2 f_{xyy}(1,1) + (y-1)^3 f_{yyy}(1,1)\right] + \cdots$$

$$= e + [(x-1)e + (y-1)e]$$

$$+\frac{1}{2!}[(x-1)^2 e + 2(x-1)(y-1)2e + (y-1)^2 e]$$

$$+\frac{1}{3!}[(x-1)^3 e + 3\,(x-1)^2(y-1)3e + 3(x-1)(y-1)^2 3e + (y-1)^3 e]$$

$$= e\left[x + y - 1 + \frac{1}{2!}\{(x-1)^2 + 4(x-1)(y-1) + (y-1)^2\}\right]$$

$$+\frac{1}{3!}\{(x-1)^3 + 9\,(x-1)^2(y-1) + 9(x-1)(y-1)^2 + (y-1)^3\} + \cdots$$

Example 3: Find the linear and quadratic Taylor series polynomial approximations to the function

$$f(x,y) = \ 2x^3 + 3y^3 - 4x^2 y$$

about the point $(1,2)$. Obtain the maximum absolute error in the region $|x-1| < 0.01$ and $|y-2| < 0.1$

Solution: We have

$f(x,y) = 2x^3 + 3y^3 - 4x^2 y$; $f(1,2) = 18$

$f_x(x,y) = 6x^2 - 8xy$; $f_x(1,2) = -10$

$f_y(x,y) = 9y^2 - 4x^2$; $f_y(1,2) = 32$

$f_{xx}(x,y) = 12x - 8y \quad ; \quad f_{xx}(1,2) = -4$

$f_{xy}(x,y) = -8x \quad ; \quad f_{xy}(1,2) = -8$

$f_{yy}(x,y) = 18y \quad ; \quad f_{yy}(1,2) = 36$

$f_{xxx}(x,y) = 12$,$f_{xxy}(x,y) = -8$, $f_{xyy}(x,y) = 0$,$f_{yyy}(x,y) = 18$

The linear approximation is given by

$$f(x,y) \approx f(1,2) + \left[(x-1)f_x(1,2) + (y-2)f_y(1,2)\right]$$

$$= 18 + (x-1)(-10) + (y-2)(32)$$

$$= 18 - 10(x-1) + 32(y-2)$$

Maximum absolute error in the linear approximation is given by

$|R_1| \le \frac{B}{2}(\delta_1 + \delta_2)^2$, where $\delta_1 = 0.01,\ \delta_2 = 0.1$

$\Longrightarrow\ |R_1| \le \frac{B}{2}(0.11)^2 = 0.00605\, B$, where

$B = max\{|f_{xx}|, |f_{xy}|, |f_{yy}|\}$ in R: $|x-1| < 0.01\ |y-1| < 0.1$

$$max|f_{xx}| = max|12x - 8y|$$
$$= max|12(x-1) - 8(y-2) - 4|$$
$$\le max\ [12|x-1| + 8|y-2| + 4]$$
$$= 0.12 + 0.8 + 4 = 4.92$$

$$max|f_{xy}| = max|-8x| = max\ |8(x-1) + 8|$$
$$\le max\ [8|x-1| + 8] = 8.08$$

$max|f_{yy}| = max|18y| = max\ |18(y-2)+36|$

$\leq max\ \ [18|y-2|+36] = 37.8$

$\therefore B = max\{4.92, 8.08, 37.8\} = 37.8$

$\therefore |R_1| \leq 0.00605\ \times 37.8 \approx 0.23$

The quadratic approximation is given by

$f(x,y) = f(1,2) + \left[(x-1)f_x(1,2) + (y-2)f_y(1,2)\right]$

$+\frac{1}{2!}\left[(x+1)^2 f_{xx}(1,2) + 2(x-1)(y-\ 2)f_{xy}(1,2) + (y-2)^2 f_{yy}(1,2)\right]$

$= 18 - 10(x-1) + 32(y-2)$

$+\dfrac{1}{2!}[(x-1)^2(-4) + 2(x-1)(y-\ 2)(-8) + (y-2)^2(36)]$

$= 18 - 10(x-1) + 32(y-2) - 2(x-1)^2 - 8(x-1)(y-2) + 18(y-2)^2$

The maximum absolute error in the quadratic approximation is

$|R_2| \leq \frac{B}{6}(\delta_1 + \delta_2)^3 = \frac{B}{2}(0.11)^3 = \frac{B}{6}(0.001331),$

where $B = max\left\{|f_{xxx}|, |f_{xxy}|, |f_{xyy}|, |f_{yyy}|\right\}$ in R

$= max[12, 8, 0, 18] = 18$

$\therefore |R_2| \leq \dfrac{18}{6}(0.001331) \approx 0.04$

Note: The error decreases as the order of the approximation increases.

P1.

Use Taylor's formula for $f(x, y) = e^x \cos y$ at the origin to find Quadratic and Cubic polynomial approximations of f near the origin.

Solution:

Given that $f(x, y) = e^x \cos y$

$$\Rightarrow f_x = e^x \cos y\,, f_y = -e^x \sin y\,, f_{xx} = e^x \cos y,$$

$$f_{xy} = -e^x \sin y\,, f_{yy} = -e^x \cos y, f_{xxx} = e^x \cos y,$$

$$f_{xxy} = -e^x \sin y, f_{xyy} = -e^x \cos y\,, f_{yyy} = e^x \sin y$$

Now, the Quadratic polynomial approximation is given by

$$f(x, y) \approx f(0,0) + xf_x(0,0) + yf_y(0,0)$$

$$+\tfrac{1}{2}\left[x^2 f_{xx}(0,0) + 2xyf_{xy}(0,0) + y^2 f_{yy}(0,0)\right]$$

$$= 1 + x.1 + y.0 + \tfrac{1}{2}[x^2.1 + 2xy.0 + y^2.(-1)]$$

$$= 1 + x + \tfrac{1}{2}(x^2 - y^2)$$

The Cubic polynomial approximation is given by

$$f(x, y) \approx \text{ quadratic}$$

$$+\tfrac{1}{6}\left[x^3 f_{xxx}(0,0) + 3x^2 y f_{xxy}(0,0) + 3xy^2 f_{xyy}(0,0) + y^3 f_{yyy}(0,0)\right]$$

$$= 1 + x + \tfrac{1}{2}(x^2 - y^2) + \tfrac{1}{6}[x^3(1) + 3x^2 y(0) + 3xy^2(-1) + y^3(0)]$$

$$= 1 + x + \tfrac{1}{2}(x^2 - y^2) + \tfrac{1}{6}(x^3 - 3xy^2)$$

P2.

Use Taylor's theorem to expand $\boldsymbol{f(x, y) = x^2 + xy + y^2}$in powers of $\boldsymbol{(x-1)}$ and $\boldsymbol{(y-2)}$.

Solution:

Notice that all the third order partial derivatives of $f(x, y)$ are zero, the maximum order of the Taylor series expansion of $f(x, y)$ about the point $(1,2)$ is two. We have,

$$f(x, y) = f(1, 2) + \left[(x-1)\frac{\partial}{\partial x} + (y-2)\frac{\partial}{\partial y}\right] f(1, 2)$$

$$+\frac{1}{2!}\left[(x-1)\frac{\partial}{\partial x} + (y-2)\frac{\partial}{\partial y}\right]^2 f(1, 2)$$

$$= f(1,2) + (x-1)f_x(1, 2) + (y-2)f_y(1, 2)$$

$$+\frac{1}{2!}[(x-1)^2 f_{xx}(1, 2) + 2(x-1)(y-2)f_{xy}(1, 2)$$

$$+(y-2)^2 f_{yy}(1, 2)]$$

Now,

$$f_x = 2x + y, f_y = x + 2y, f_{xy} = 1, f_{yy} = 2 = f_{xx}$$

$$f(1,2) = 7, f_x(1,2) = 4, f_y(1,2) = 5$$

$$f(x, y) = 7 + 4(x-1) + 5(y-2)$$

$$+\frac{1}{2!}[2(x-1)^2 + 2(x-1)(y-2) + 2(y-2)^2]$$

P3.

Expand x^y in powers of $(x-1)$ and $(y-1)$ upto the third degree terms.

Solution:

Given $f(x,y) = x^y$. We expand $f(x,y)$ about the point $(1,1)$

$f(x,y) = x^y$; $f(1,1) = 1$

$f_x(x,y) = yx^{y-1}$; $f_x(1,1) = 1$

$f_y(x,y) = x^y logx,$; $f_y(1,1) = 0$

$f_{xx}(x,y) = y(y-1)x^{y-2}$; $f_{xx}(1,1) = 0$

$f_{xy}(x,y) = x^{y-1} + yx^{y-1}logx$; $f_{xy}(1,1) = 1$

$f_{yy}(x,y) = x^y(logx)^2$; $f_{yy}(1,1) = 0$

$f_{xxx}(x,y) = y(y-1)(y-2)x^{y-3}$; $f_{xxx}(1,1) = 0$

$f_{xxy}(x,y) = (2y-1)x^{y-2} + y(y-1)x^{y-2}(logx)$

; $f_{xxy}(1,1) = 1$

$f_{xyy}(x,y) = yx^{y-1}(logx)^2 + x^y.2(logx).\frac{1}{x},$

; $f_{xyy}(1,1) = 0$

$f_{yyy}(x,y) = x^y(logx)^3$; $f_{yyy}(1,1) = 0$

By Taylor's expansion, we have

$$f(x,y) = f(1,1) + \left[(x-1)f_x(1,1) + (y-1)\ \ f_y(1,1)\right]$$

$$+\tfrac{1}{2!}\left[(x-1)^2 f_{xx}(1,1) + 2(x-1)(y-1)f_{xy}(1,1) + (y-1)^2 f_{yy}(1,1)\right]$$

$$+\tfrac{1}{3!}[(x-1)^3 f_{xxx}(1,1) + 3(x-1)^2(y-1)f_{xxy}(1,1)$$

$$+3(x-1)(y-1)^2 f_{xyy}(1,1) + (y-1)^3 f_{yyy}(1,1)]$$

$$= 1 + [(x-1).0 + (y-1).0] + \frac{1}{2!}[(x-1)^2.0 + 2(x-1)(y-1)1 + (y-1)^2.0]$$

$$+\tfrac{1}{3!}[(x-1)^3.1 + 3(x-1)^2(y-1).1$$

$$+3(x-1)(y-1)^2.0 + (y-1)^3.0]$$

$$= x + (x-1)(y-1) + \tfrac{1}{2}(x-1)^2(y-1)$$

P4.

Use Taylor's formula to find a quadratic approximation of $\boldsymbol{f(x,y) = cosx.cosy}$ at the origin. Estimate the error in the approximation if $|\boldsymbol{x}| \leq \boldsymbol{0.1}$ and $|\boldsymbol{y}| \leq \boldsymbol{0.1}$.

Solution:

Given $f(x,y) = \cos x \cos y$

$\Longrightarrow f_x = -\sin x \cos y\,, f_y = -\cos x \sin y$

$f_{xx} = -\cos x \cos y\,, f_{xy} = \sin x \sin y\,, f_{yy} = -\cos x \cos y$

By Taylor's expansion, we have

$$f(x,y) \approx f(0,0) + xf_x(0,0) + yf_y(0,0)$$

$$+\frac{1}{2}\left[x^2 f_{xx}(0,0) + 2xyf_{xy}(0,0) + y^2 f_{yy}(0,0)\right]$$

$$= 1 + x.0 + y.0 + \frac{1}{2}[x^2.(-1) + 2xy.0 + y^2.(-1)]$$

$= 1 - \frac{x^2}{2} - \frac{y^2}{2}$, is the quadratic approximation.

Since all partial derivatives of f are products of sines and cosines, the absolute value of third order derivatives is less than or equal to 1.

Thus, $B = 1$ in R: $|x| \leq 0.1, |y| \leq 0.1$

$$|R_2| \leq \frac{B}{6}(\delta_1 + \delta_2)^3 = \frac{1}{6}(0.1 + 0.1)^3 = 0.0013$$

IP1.

Use Taylor's formula for $\boldsymbol{f(x,y) = e^x \ln(1+y)}$ at the origin to find Quadratic and Cubic polynomial approximations of $\boldsymbol{f}$ near the origin

Solution:

Given that $f(x,y) = e^x \ln(1+y)$

$$\Longrightarrow f_x = e^x \ln(1+y)\,, f_y = \frac{e^x}{1+y}\,, f_{xx} = e^x \ln(1+y)\,, f_{xy} = \frac{e^x}{1+y},$$

$$f_{yy} = -\frac{e^x}{(1+y)^2}\,, f_{xxx} = e^x \ln(1+y)\,, f_{xxy} = \frac{e^x}{1+y},$$

$$f_{xyy} = -\frac{e^x}{(1+y)^2}, f_{yyy} = \frac{2e^x}{(1+y)^3}$$

Now, the Quadratic polynomial approximation is given by

$$f(x,y) \approx f(0,0) + xf_x(0,0) + yf_y(0,0)$$

$$+\frac{1}{2}\left[x^2 f_{xx}(0,0) + 2xyf_{xy}(0,0) + y^2 f_{yy}(0,0)\right]$$

$$= 0 + x.0 + y.1 + \frac{1}{2}[x^2.0 + 2xy.1 + y^2.(-1)]$$

$$= y + \frac{1}{2}(2xy - y^2)$$

The Cubic polynomial approximation is given by

$$f(x,y) \approx \text{quadratic}$$

$$+\frac{1}{6}\left[x^3 f_{xxx}(0,0) + 3x^2 y f_{xxy}(0,0) + 3xy^2 f_{xyy}(0,0) + y^3 f_{yyy}(0,0)\right]$$

$$= y + \frac{1}{2}(2xy - y^2) + \frac{1}{6}[x^3.0 + 3x^2y.1 + 3xy^2.(-1) + y^3.2]$$

$$= y + \frac{1}{2}(2xy - y^2) + \frac{1}{6}(3x^2y - 3xy^2 + 2y^3)$$

IP2.

Expand $x^2y + 3y - 2$in powers of $x - 1$ and $y + 2$.

Solution:

Notice that all the third order partial derivatives of $f(x, y)$ are zero, the maximum order of the Taylor series expansion of $f(x, y)$ about the point $(1, -2)$ is two. We have,

$$f(x, y) = f(1, -2) + \left[(x-1)\frac{\partial}{\partial x} + (y+2)\frac{\partial}{\partial y}\right] f(-1,2)$$

$$+\frac{1}{2!}\left[(x-1)\frac{\partial}{\partial x} + (y+2)\frac{\partial}{\partial y}\right]^2 f(-1,2)$$

$$= f(1, -2) + (x-1)f_x(1, -2) + (y-2)f_y(1, -2)$$

$$+\frac{1}{2!}[(x-1)^2 f_{xx}(1, -2) + 2(x-1)(y+2)f_{xy}(1, -2)$$

$$+(y+2)^2 f_{yy}(1, -2)]$$

Now,

$f(x, y) = x^2y + 3y - 2 \quad ; \quad f(1, -2) = -10$

$f_x(x, y) = 2xy \quad ; \quad f_x(1, -2) = -4$

$f_y(x, y) = x^2 + 3 \quad ; \quad f_y(1, -2) = 4$

$f_{xx}(x, y) = 2y \quad ; \quad f_{xx}(1, -2) = -4$

$f_{xy}(x, y) = 2x \quad ; \quad f_{xy}(1, -2) = 2$

$f_{yy}(x,y) = 0 \qquad ; \quad f_{yy}(1,-2) = 0$

$f_{xxx}(x,y) = 0 = f_{yyy}(x,y) \; ; \quad f_{xyy}(1,-2) = 2 = f_{xxy}(1,-2)$

$\therefore f(x,y) = x^2y + 3y - 2$

$= -10 - 4(x-1) + 4(y+2)$

$+\frac{1}{2}[-4(x-1)^2 + 4(x-1)(y+2)] + \frac{1}{3!}[3(x-1)^2(y+2)(2) + 0]$

$= -10 - 4(x-1) + 4(y+2) - 2(x-1)^2$

$+2(x-1)(y+2) + (x-1)^2(y+2)$

IP3.

If $\boldsymbol{f(x,y) = tan^{-1}(xy)}$, compute an approximate value of $\boldsymbol{f(0,9,-1.2)}$.

Solution:

Given $f(x,y) = tan^{-1}(xy)$. We expand $f(x,y)$ about the point $(1,-1)$.

$$f(x,y) = tan^{-1}(xy) \quad ; \quad f(1,-1) = -\frac{\pi}{4}$$

$$f_x(x,y) = \frac{y}{1+x^2y^2} \quad ; \quad f_x(1,-1) = -\frac{1}{2}$$

$$f_y(x,y) = \frac{x}{1+x^2y^2} \quad ; \quad f_y(1,-1) = \frac{1}{2}$$

$$f_{xx}(x,y) = -\frac{2xy}{(1+x^2y^2)^2} \quad ; \quad f_{xx}(1,-1) = \frac{1}{2}$$

$$f_{xy}(x,y) = \frac{1-x^2y^2}{(1+x^2y^2)^2} \quad ; \quad f_{xy} = 0$$

$$f_{yy}(x,y) = \frac{-2x^2y^2}{1+x^2y^2} \quad ; \quad f_{yy}(1,-1) = \frac{1}{2}$$

By Taylor's expansion, we have

$$f(0{,}9,-1.2) = f(1-0.1,-1-0.2)$$

$$= f(1,-1) + \left[(-0.1)\frac{\partial f}{\partial x} + (-0.2)\frac{\partial f}{\partial y}\right]_{(1,-1)}$$

$$+\frac{1}{2!}\left[(-0.1)^2\frac{\partial^2 f}{\partial x^2} + 2(-0.1)(-0.2)\frac{\partial^2 f}{\partial x \partial y} + (-0.2)^2\frac{\partial^2 f}{\partial y^2}\right]_{(1,-1)}$$

$$= -\frac{\pi}{4} + \left[(-0.1)\left(-\frac{1}{2}\right) + (-0.2)\left(\frac{1}{2}\right)\right]$$

$$+ \frac{1}{2!}\left[(-0.1)^2 . \frac{1}{2} + 2(-0.1)(-0.2). 0 + (-0.2)^2 . \frac{1}{2}\right]$$

$$= -\frac{\pi}{4} + 0.05 - 0.1 + \frac{1}{2}(0.005 + 0.02)$$

$$= -\frac{\pi}{4} 0.05 - 0.1 + 0.0125 = -0.823$$

IP4.

Use Taylor's formula to find a quadratic approximation of $\boldsymbol{f(x,y) = e^x siny}$ at the origin. Estimate the error in the approximation if $|\boldsymbol{x}| \le \mathbf{0.1}$ and $|\boldsymbol{y}| \le \mathbf{0.1}$.

Solution:

Given $f(x,y) = e^x \sin y$

$$\Longrightarrow f_x = e^x \sin y\,, f_y = e^x \cos y, f_{xx} = e^x \sin y,$$

$$f_{xy} = e^x \cos y\,, f_{yy} = -e^x \sin y$$

By Taylor's expansion, we have

$$f(x,y) \approx f(0,0) + xf_x(0,0) + yf_y(0,0)$$

$$+\frac{1}{2}\left[x^2 f_{xx}(0,0) + 2xyf_{xy}(0,0) + y^2 f_{yy}(0,0)\right]$$

$$= 0 + x.0 + y.1 + \frac{1}{2}(x^2.0 + 2xy.1 + y^2.0) = y + xy,$$

is the quadratic approximation.

Now,

$f_{xxx} = e^x \sin y\,, f_{xxy} = e^x \cos y\,, f_{xyy} = -e^x \sin y\,, and\ f_{yyy} = -e^x \cos y.$

Since $|x| \le 0.1, |e^x \sin y| \le |e^{0.1} \sin 0.1| \approx 0.11$ and

$$|e^x \cos y| \le |e^{0.1} \cos 0.1| \approx 1.11.$$

Thus, $B = max\{|f_{xxx}|, |f_{xxy}|, |f_{xyy}|, |f_{yyy}|\}$ in $R = 1.11$

$$|R_2| \le \frac{B}{6}(\delta_1 + \delta_2)^3 = \frac{1.11}{6}(0.1 + 0.1)^3 = 0.0014$$

3.8. Taylor's Expansion

EXERCISES:

I. Use Taylor's formula for *f*(*x*, *y*) at the origin to find quadratic and cubic approximations of *f* near the origin.

a) $f(x,y) = xe^y$

b) $f(x,y) = ysinx$

c) $y = sinx.cosy$

d) $f(x,y) = \ln(2x + y + 1)$

e) $f(x,y) = sin(x^2 + y^2)$

f) $f(x,y) = cos(x^2 + y^2)$

g) $f(x,y) = \frac{1}{1-x-y}$

h) $f(x,y) = \frac{1}{1-x-y+xy}$

Answers:

a) Quadratic approximation: $f(x,y) = x + xy$
Cubic approximation: $f(x,y) = x + xy + \frac{1}{2}xy^2$

b) Quadratic approximation: $f(x,y) = xy$
Cubic approximation: $f(x,y) = xy$

c) Quadratic approximation: $f(x,y) = x$
Cubic approximation: $f(x,y) = x - \frac{1}{6}(x^3 + 3xy^2)$

d) Quadratic approximation: $f(x,y) = (2x+y) - \frac{1}{2}(2x+y)^2$

Cubic approximation:

$$f(x,y) = (2x+y) - \frac{1}{2}(2x+y)^2 + \frac{1}{3}(2x+y)^3$$

e) Quadratic approximation: $f(x,y) = x^2 + y^2$

Cubic approximation: $f(x,y) = x^2 + y^2$

f) Quadratic approximation: $f(x,y) = 1$

Cubic approximation: $f(x,y) = 1$

g) Quadratic approximation: $f(x,y) = 1 + (x+y) + (x+y)^2$

Cubic approximation:

$$f(x,y) = 1 + (x+y) + (x+y)^2 + (x+y)^3$$

h) Quadratic approximation:

$$f(x,y) = 1 + x + y + x^2 + xy + y^2$$

Cubic approximation:

$$f(x,y) = 1 + x + y + x^2 + xy + y^2$$
$$+x^3 + xy^2 + yx^2 + y^3$$

II.

1. The function $f(x,y) = x^2 - xy + y^2$ is approximated by a first degree Taylor's polynomial about the point $(2,3)$. Find a square $|x-2| < \delta, |y-3| < \delta$ with center at $(2,3)$ such that the error of approximation is less than or equal to 0.1 in magnitude for all points within this square.

2. If $f(x,y) = tan^{-1}xy$, find an approximation value of $f(1.1, 0.8)$ using the Taylor's series (i)linear approximation (ii) Quadratic approximation

3. Obtain the Taylor's series expansion of the maximum order for the function $f(x,y) = x^2 + 3y^2 - 9x - 9y + 26$ about the point $(2,2)$

4. Obtain the Taylor's linear approximation to the function $f(x,y) = 2x^2 + y^2 - xy + 3x - 4y + 1$ about the point $(-1, 1)$. Find the maximum error in the region $|x+1| < 0.1$, $|y-1| < 0.1$.

5. Obtain the Taylor's quadratic approximation to the function $f(x,y) = \sqrt{x+y}$ about the point $(1, 3)$. Find the maximum error in the region $|x-1| < 0.2$, $|y-3| < 0.1$.

6. Expand $f(x,y,z) = \sqrt{x^2 + y^2 + z^2}$ in Taylor's series upto first order terms about the point $(2, 2, 1)$. Find the maximum error in the region $|x-2| < 0.2$, $|y-2| < 0.1$, $|z-1| < 0.1$

7. Expand $f(x,y,z) = \sqrt{xy + yz + zx}$ in Taylor's series upto first order terms about the point $\left(1, 3, \frac{3}{2}\right)$. Find the maximum error in the region $|x-2| < 0.2$, $|y-3| < 0.1$, $\left|z - \frac{3}{2}\right| < 0.1$

Answers:

1. $\delta \approx 0.1581$
2. Linear approximation: $f(1.1, 0.8) = 0.7354$
   Quadratic approximation: $f(1.1, 0.8) = 0.7229$
3. $f(x,y) = 6 - 5(x-2) + 3(y-2) + (x-2)^2 + 3(y-2)^2$
4. $f(x,y) = -2 - 2(x-1) - (y-1); B = 4;\ |E| \le 0.08$
5. $f(x,y) = 2 + \frac{1}{4}[(x-1) + (y-3)]$
   $-\frac{1}{64}[(x-1)^2 + 2(x-1)(y-3) + (y-3)^2]$
   $B = 0.0142, |E| \le 0.64 \times 10^{-4}$

6. $f(x,y,z) = 3 + \frac{2}{3}[(x-2) + (y-2) + (z-1)]; B = 0.3872$
   $|E| \le 0.017$

7. $f(x,y,z) = 3 + \frac{3}{4}(x-1) + \frac{5}{12}(y-3) + \frac{2}{3}\left(z - \frac{3}{2}\right);$
   $B = 0.3985;\ |E| \le 0.0179$

3.9

Maxima and Minima

Learning objectives:

- To define local extrema of functions of two variables
- To state second derivative test for local extreme values.

AND

- To practice the related problems

Maxima and Minima

Continuous functions of two variables attain extreme values on closed and bounded domains. In this module the search for these extreme values is narrowed to the behaviour of the first order partial derivatives of the function. A function of two variables can assume extreme values only at the boundary points or interior points of its domain, where both of its first order partial derivatives are zero or where one or both of its first order partial derivatives fails to exist. However, the vanishing of the first order partial derivatives at an interior point of the domain does not always signal the presence of an extreme value.

Local maximum and local minimum

Let $f(x, y)$ be a function defined on a region R containing the point (a, b).

i) We say that $f(a, b)$ is a ***local*** (***relative***) ***maximum*** value of f if $f(a, b) \geq f(x, y)$ for all domain points (x, y) in an open disk centered at (a, b).

ii) We say that $f(a, b)$ is a ***local*** (***relative***) ***minimum*** value of f if $f(a, b) \leq f(x, y)$ for all domain points (x, y) in an open disk centered at (a, b).

The points at which local maximum/minimum values of the function occur are also called ***points of extrema*** and the local maximum/minimum values taken together are called ***the extreme values*** of the function.

As with functions of a single real variable, the key to identifying the local extrema is a *first derivative test*.

Theorem1: First derivative test for local extreme values

If $\boldsymbol{f(x,y)}$ has local maximum or local minimum value at an interior point $\boldsymbol{(a,b)}$ of its domain and if the first order partial derivatives exist there, then $\boldsymbol{f_x(a,b)=0}$ and $\boldsymbol{f_y(a,b)=0}$.

Proof: Suppose that the first order partial derivatives exist at the point (a,b). If f has a local extremum at (a,b), then the function $g(x)=f(x,b)$ has a local extremum at $x=a$. By the first derivative test for local extremum of a function a single variable, $g^{'}(a)=0$, i.e., $0=g^{'}(a)=\frac{d}{dx}f(x,b)\Big|_{x=a}=f_x(a,b)$

A similar argument proves $f_y(a,b)=0$. Hence the theorem

Critical point

*An interior point of the domain of a function $f(x,y)$ where both f_x and f_y are zero or where one or both of f_x and f_y do not exist is a **critical (stationary) point** of f.*

Note (1): By Theorem1, the only points where a function $f(x, y)$ can have extreme values are critical points and boundary points.

Note (2): As with differentiable functions of a single variable, *not every critical point gives rise to a local extremum*. A differentiable function of a single variable might have a point of inflection. A differentiable function of two variables might have a *saddle point.*

Saddle point

*A differentiable function $f(x, y)$ has a **saddle point** at a critical point (a, b) if every open disk centered at (a, b) conains domain points (x_1, y_1) and (x_2, y_2) such that $f(x_1, y_1) > f(a, b)$ and $f(x_2, y_2) < f(a, b)$. The corresponding point $\left(a, b, f(a, b)\right)$ on the surface $z = f(x, y)$ is called a **saddle point** of the surface.*

Example1: Find the local extreme values of $f(x, y) = x^2 + y^2$.

Solution: The domain of f is $\boldsymbol{R}^2$. Notice that there are no boundary points. Further, the first order partial derivatives $f_x = 2x$, $f_y = 2y$ exist everywhere. Thus, the local extreme values can occur only where

$$f_x = 2x = 0 \text{ and } f_y = 2y = 0$$

i.e., only at the origin $(0,0)$. At the origin f is 0 and it is the local minimum value since, f is never negative.

Example2: **Identifying a saddle point**

Find the local extreme values, if any, of $f(x, y) = y^2 - x^2$.

Solution: The domain of f is $\boldsymbol{R}^2$(and so there are no boundary points). Further, the first order partial derivatives $f_x = -2x$, $f_y = 2y$ exist everywhere. Thus, the local extreme values can occur only at the origin $(0,0)$ and $f(0,0) = 0$. Notice that

$f(x, 0) = -x^2 < 0$ along the positive x-axis and

$f(0, y) = y^2 > 0$ along the positive y-axis

This shows that every open disk centered at $(0,0)$ contains points $(c, 0)$ and $(d, 0)$, $c \neq 0, d \neq 0$ such that

$$f(c, 0) < f(0,0) \text{ and } f(d, 0) > f(0,0).$$

Thus, the function f has the saddle point at $(0,0)$.The function has no local extreme values (although $(0,0)$ is a critical point of f).

Theorem2: **Second derivative test for local extreme values**

Suppose that $f(x, y)$ and its first and second order partial derivatives are continuous throughout a disk centered at (a, b) and that

$$\boldsymbol{f_x(a, b) = 0 = f_y(a, b)}$$

Then

(i) ***f has a local maximum at (a, b) if***

$$f_{xx} < 0 \textbf{\textit{ and }} f_{xx} f_{yy} - f_{xy}^{\ 2} > 0 \textbf{\textit{ at }} (a, b)$$

(ii) ***f has a local minimum at (a, b) if***

$$f_{xx} > 0 \textbf{\textit{ and }} f_{xx} f_{yy} - f_{xy}^{\ 2} > 0 \textbf{\textit{ at }} (a, b)$$

(iii) ***f has a saddle point at (a, b) if $f_{xx} f_{yy} - f_{xy}^{\ 2} < 0$ at (a, b)***

(iv) ***The test is inconclusive at (a, b) if $f_{xx} f_{yy} - f_{xy}^{\ 2} = 0$ at (a, b)***

In case (iv) we have to find some other way to determine the behavior of f at (a, b).

The expression $f_{xx} f_{yy} - f_{xy}^{\ 2}$ is called the **discriminant** or Hessian of f. It is easier to remember in determinant form:

$$f_{xx} f_{yy} - f_{xy}^{\ 2} = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix}$$

Example 3: Find the local extreme values of the function $f(x, y) = xy - x^2 - y^2 - 2x - 2y + 4$

Solution:

The domain of $f(x, y)$ is R^2 and it has no boundary points. The critical points are given by

$$f_x = y - 2x - 2 = 0, \qquad f_y = x - 2y - 2 = 0$$

i.e., $(x, y) = (-2, -2)$. Therefore, $(-2, -2)$ is the only point where f may have extreme value. Now,

$f_{xx} = -2, \qquad f_{xy} = 1, \qquad f_{yy} = -2$

The discriminant of f at $(-2,-2)$

$$f_{xx}\ f_{yy} - {f_{xy}}^2 = (-2)(-2) - 1 = 3$$

We see that $f_{xx} < 0$ and $f_{xx}\ f_{yy} - {f_{xy}}^2 > 0$ at $(-2,-2)$.

Therefore, by the second derivative test for local extreme values, f has a local maximum at $(-2,-2)$. The local maximum values is $f(-2,-2) = 8$

Example 4: Find the local extreme values of $f(x,y) = xy$

Solution: Since $f(x,y)$ is differentiable every where, it can assume extreme values only at the critical points. That is, the points where $f_x = y = 0$ and $f_y = x = 0$. This shows that $(0,0)$ is the only point where f might have an extreme value. We see that

$$f_{xx} = 0, \quad f_{yy} = 0 \ \text{ and } \ f_{xy} = 1$$

and the discriminant $f_{xx}\ f_{yy} - {f_{xy}}^2 = -1 < 0\ at\ (0,0)$ and this shows that $f(x,y)$ has a saddle point at $(0,0)$ and $f(x,y)$ has no local extreme values.

Example 5: Find the local extreme values of the function

$$f(x,y) = 2(x^2 - y^2) - x^4 + y^4$$

Solution: We have

$f_x = 4x - 4x^3 = 4x(1 - x^2) = 0 \quad or \;\; x = 0, \; \pm 1$

$f_y = -4y + 4y^3 = -4y(1 - y^2) = 0 \quad or \;\; y = 0, \pm 1$

The critical points of $f(x, y)$ are $(0{,}0)$, $(0, \pm 1)$, $(\pm 1, 0)$, $(\pm 1, \pm 1)$

We see that $f_{xx} = 4 - 12x^2$, $f_{xy} = 0$, $f_{yy} = -4 + 12y^2$ and the discriminant is $f_{xx} \;\; f_{yy} - {f_{xy}}^2 = -16(1 - 3x^2)(1 - 3y^2)$

(i) At $(0, \pm 1)$, we have $f_{xx} \;\; f_{yy} - {f_{xy}}^2 = 32 > 0$ and

$f_{xx} = 4 > 0$

Therefore, by the second derivative test, f has local minimum at $(0, \pm 1)$ and the local minimum value is $f(0, \pm 1) = -1$

(ii) At $(\pm 1, 0)$, we have $f_{xx} \;\; f_{yy} - {f_{xy}}^2 = 32 > 0$ and

$f_{xx} = -8 < 0$

Therefore, by the second derivative test, f has local maximum at $(\pm 1, 0)$ and local maximum values is $f\,(\pm 1, 0) = 1$

(iii) At $(0, 0)$, we have $f_{xx} \;\; f_{yy} - {f_{xy}}^2 = -16 < 0$

and at $(\pm 1, \pm 1\,)$, we have $f_{xx} \;\; f_{yy} - {f_{xy}}^2 = -64 < 0$

Therefore, by the second derivative test, f has saddle points at $(0, 0), (\pm 1, \pm 1\,)$.

Thus, f has no local extrema at $(0, 0)$, $(\pm 1, \pm 1\,)$.

Absolute maxima and minima on closed and bounded regions:

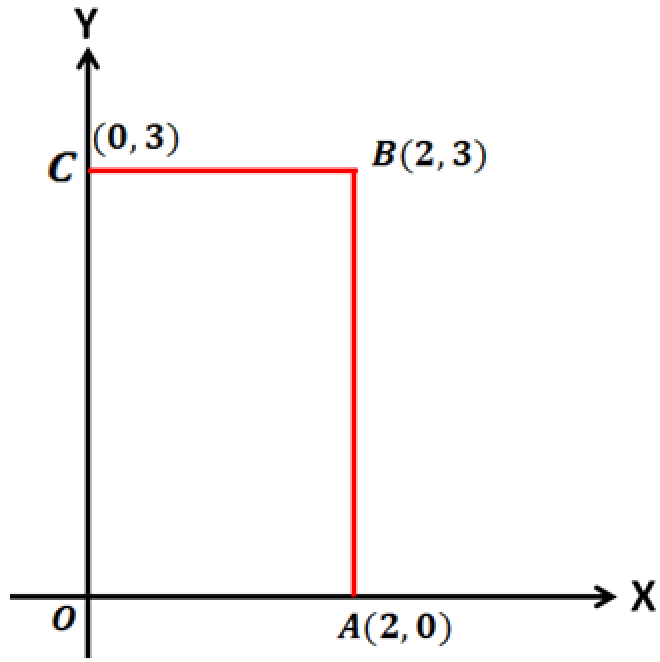
Let $f(x,y)$ be a continuous function on a closed and bounded region R. The function $f(x,y)$ may also attain its minimum or maximum values on the boundary of the region R. The largest and the smallest values attained by $f(x,y)$ over the entire region including the boundary are called **absolute (global) maximum** and **absolute (global) minimum values** respectively.

We organize the search for the absolute extrema of $f(x,y)$ on R into three steps.

1. *List the interior points of R*(i.e. the critical points of f) where f may have local extrema and evaluate *f* at these points.
2. *List boundary points of R* where f has local extrema and evaluate *f* at these points.
3. Look through the lists for the maximum and minimum values of f. These will be the absolute maximum and absolute minimum of f on *R*.

Example 6: Find the absolute maximum and minimum values of $f(x,y) = 4x^2 + 9y^2 - 8x - 12y + 4$ over the rectangle in the first quadrant by the lines $x = 2, y = 3$ and the coordinate axes.

Solution: The function can attain maximum/minimum values at the critical points or on the boundary of the rectangle $OABC$.



(a) Interior points:

We have $f_x = 8x - 8 = 0, f_y = 18y - 12 = 0$. The critical point is $\left(1, \frac{2}{3}\right)$. Now, $f_{xx} = 8, f_{xy} = 0, f_{yy} = 18$

The discriminant at the point $\left(1, \frac{2}{3}\right)$ is $f_{xx}\ f_{yy} - {f_{xy}}^2 = 144 > 0$ and $f_{xx}\left(1, \frac{2}{3}\right) = 8 > 0$. By second derivative test f has a local minimum at $\left(1, \frac{2}{3}\right)$ .The local minimum value is $f\left(1, \frac{2}{3}\right) = -4$.

(b) Boundary points:

i) On the boundary line $OA, y = 0$ and

$f(x,y) = f(x,0) = g(x) = 4x^2 - 8x + 4$, a function of x on the closed interval $0 \le x \le 2$.

Setting $g'(x) = 0$, we get $8x - 8 = 0$, i.e., $x = 1$. Now, $g''(x) = 8 > 0$ $(at\ x = 1)$. By second derivative test (for functions of one variable), g has minimum at $x = 1$. The minimum value is $g(1) = f(1,0) = 0$. The extreme values of g may occur at the end points

$$x = 0, \text{ where } g(0) = f(0,0) = 4$$

$$x = 2, \text{ where } g(2) = f(2,0) = 4$$

ii) On the boundary line AB , $x = 2$ and $f(x,y) = f(2,y) = h(y) = 9y^2 - 12y + 4$, a function of y on the closed interval $0 \le y \le 3$.

Setting $h'(y) = 0$, we get $18y - 12 = 0$, i.e., $y = \frac{2}{3}$.

Now, $h''(y) = 18 > 0$ (at $y = \frac{2}{3}$) . By second derivative test (for functions of one variable), h has minimum at $y = \frac{2}{3}$. The minimum value is $h\left(\frac{2}{3}\right) = f\left(2, \frac{2}{3}\right) = 0$. The extreme values of h may occur at the end points

$$y = 0, \text{ where } h(0) = f(2,0) = 4$$

$$y = 3, \text{ where } h(3) = f(2,3) = 49$$

iii) On the boundary line CB , $y = 3$ and

$f(x, y) = f(x, 3) = 4x^2 - 8x + 49 = g(x)$ (say) a function of x on the closed interval $0 \leq x \leq 2$.

Setting $g^{'}(x) = 0$, we get $8x - 8 = 0$, i.e., $x = 1$. Now, $g^{''}(x) = 8 > 0$ (at $x = 1$). By second derivative test (for functions of one variable), g has minimum at $x = 1$. The minimum value is $g(1) = f(3,1) = 45$. The extreme values of g may occur at the end points

$$x = 0, \text{ where } g(0) = f(0, 3) = 49$$

$$x = 3, \text{ where } g(2) = f(2, 3) = 49 = h(3)$$

iv)On the boundary line $OC, x = 0$ and

$f(x, y) = f(0, y) = 9y^2 - 12y + 4,$

which is the same case as for $x = 2$

Summary:

We list all the extreme values:

$$-4, 1, 4, 0, 4, 49, 45, 49, 49$$

The absolute maximum value is 49, and it occurs at the points $(0, 3)$ and $(2, 3)$. The absolute minimum value is -4 and it occurs at $\left(1, \frac{2}{3}\right)$.

P1:

Find the relative maxima and minima of

$$\boldsymbol{f(x,y) = x^3 + y^3 - 3x - 12y + 20.}$$

Solution:

Given $f(x) = x^3 + y^3 - 3x - 12y + 20$

Now, $f_x = 3x^2 - 3 = 0 \Longrightarrow x = \pm 1$ and

$f_y = 3y^2 - 12 = 0 \Longrightarrow x = \pm 2$

The critical points are $P(1,2), Q(-1,2), R(1,-2), S(-1,-2)$

$f_{xx} = 6x,\ f_{yy} = 6y,\ f_{xy} = 0$. Then $\Delta = f_{xx}\, f_{yy} - f_{xy}^2 = 36xy$

At $P(1,2), \Delta > 0$ and $f_{xx} > 0$. By second derivative test, f has a local minimum at the point $P(1,2)$. The local minimum value is $f(1,2) = 2$

At $Q(-1,2), \Delta < 0$ and so Q is a saddle point.

At $R(1,-2), \Delta < 0$ and so R is a saddle point.

At $S(-1,-2), \Delta > 0$ and $f_{xx} < 0$. By second derivative test, f has a local maximum at $S(-1,-2)$. The local maximum value is $f(-1,-2) = 38$.

P2.

Discuss the local maxima and local minima of the function

$$\boldsymbol{f(x,y) = x^3y^2(1-x-y)}$$

Solution:

We have $f_x = 3x^2y^2 - 4x^3y^2 - 3x^2y^3$;

$f_y = 2x^3y - 2x^4y - 3x^3y^2$ and

$r = f_{xx} = 6xy^2 - 12x^2y^2 - 6xy^3 = 6xy^2(1-2x-3y)$

$s = f_{xy} = 6x^2y - 8x^3y - 9x^2y^2 = x^2y(6-8x-9y)$

$t = f_{yy} = 2x^3 - 2x^4 - 6x^3y = 2x^3(1-x-3y)$

When $f_x = 0$, $f_y = 0$, we have

$x^2y^2(3-4x-3y) = 0$ ..... (1)

$x^3y(2-2x-3y) = 0$ ..... (2)

Solving (1) and (2), the stationary points are $\left(\frac{1}{2},\frac{1}{3}\right)$, $(0,0)$. Now,

$$rt - s^2 = x^4y^2[12(1-2x-y)(1-x-3y) - (6-8x-9y)^2]$$

At the point $\left(\frac{1}{2},\frac{1}{3}\right)$, we have

$$rt - s^2 = \left(\frac{1}{16}\right)\frac{1}{9}\left[12\left(1-1-\frac{1}{3}\right)\left(1-\frac{1}{2}-1\right) - (6-4-3)^2\right]$$

$$= \frac{1}{14} > 0$$

Also, $r = 6\left(\frac{1}{2}.\frac{1}{9} - \frac{2}{4}.\frac{1}{9} - \frac{1}{2}.\frac{1}{27}\right) = -\frac{1}{9} < 0$

Hence, $f(x, y)$ has local maximum at $\left(\frac{1}{2}, \frac{1}{3}\right)$ (by second derivative test).

The local maximum value is $f\left(\frac{1}{2}, \frac{1}{3}\right) = \frac{1}{8}.\frac{1}{9}\left(1 - \frac{1}{2} - \frac{1}{3}\right) = \frac{1}{432}$

At $(0,0)$, $rt - s^2 = 0$. Therefore, we need further investigation.

For the points along the line $y = x$, $f(x, y) = x^5(1 - 2x)$ which is positive for $x = 0.1$ and negative for $x = -0.1$, $i.e.$, in the neighborhood of $(0,0)$, there are the points where $f(x, y) > f(0,0)$ and there are points where $f(x, y) < f(0,0)$.

Hence $f(0,0)$ is not an extreme value.

P3:

Find the local maximum and local minimum values of the function $\boldsymbol{f(x,y) = x^3 + y^3 - 3axy}$.

Solution:

Given that $f(x,y) = x^3 + y^3 - 3axy$

$$\frac{\partial f}{\partial x} = 3x^2 - 3ay \text{ and } \frac{\partial f}{\partial y} = 3y^2 - 3ax$$

For stationary points $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$, which gives $x^2 = ay$ and $y^2 = ax$.

Thus, the stationary points are $(0,0)$ and (a,a)

Now, $r = \frac{\partial^2 f}{\partial x^2} = 6x$, $s = \frac{\partial^2 f}{\partial x \partial y} = -3a$ and $t = \frac{\partial^2 f}{\partial y^2} = 6y$.

At $(0,0)$, $rt - s^2 = 0 - 9a^2 < 0$. Therefore, the point $(0,0)$ is a saddle point.

At $(0,0)$, $f\,(x,y)$ is neither maximum nor minimum

At (a,a) , we have $rt - s^2 = 36\,a^2 - 9a^2 = 27\,a^2\ (> 0)$

Now the sign of r decides the existence of maxima or minima at the point (a,a), since $r = 6a$ at (a,a).

If $a > 0$ then $r > 0$ and so f has local minimum at (a,a).

The minimum value of f is $a^3 + a^3 - 3a^3 = -a^3$

If $a < 0$ then $r < 0$ and so f has local maximum at (a, a).

The maximum value of f is $a^3 + a^3 - 3a^3 = -a^3$.

P4.

Find the absolute maximum and minimum values of the function $\boldsymbol{f(x,y) = 3x^2 + y^2 - x}$ over the region $\boldsymbol{2x^2 + y^2 \leq 1}$

Solution:

The given function is $f(x,y) = 3x^2 + y^2 - x$.

The function can attain maximum/minimum at the critical points or on the boundary of the region $2x^2 + y^2 \leq 1$.

(a) Interior points

We have $f_x = 6x - 1 = 0$ and $f_y = 2y = 0$. Therefore, the critical point is $(x,y) = \left(\frac{1}{6}, 0\right)$.

Now, $r = f_{xx} = 6 > 0$, $s = f_{xy} = 0, t = f_{yy} = 2$,

and $rt - s^2 = 12 > 0$. Therefore, $\left(\frac{1}{6}, 0\right)$ is point of local minimum (by second derivative test).The minimum value at this point is $f\left(\frac{1}{6}, 0\right) = -\frac{1}{12}$

(b) Boundary points:

On the boundary, we have $y^2 = 1 - 2x^2$, $-\frac{1}{\sqrt{2}} \leq x \leq \frac{1}{\sqrt{2}}$.

Substituting in $f(x,y)$, we obtain

$f(x,y) = 3x^2 + 1 - 2x^2 - x = 1 - x - x^2 = g(x)$, which is a function of one variable. Setting $g'(x) = 0$, we get

$g'(x) = 2x - 1 = 0$ or $x = \frac{1}{2}$. Also $g''(x) = 2 > 0$

(Therefore, by second derivative test for functions of one variable, f has a minimum at $x = \frac{1}{2}$)

For $x = \frac{1}{2}$, we get $y^2 = 1 - 2x^2 = \frac{1}{2}$ or $y = \pm\frac{1}{\sqrt{2}}$. Hence, the points $\left(\frac{1}{2}\ , \pm\frac{1}{\sqrt{2}}\right)$ are points of local minimum. The minimum value is $f\left(\frac{1}{2}\ , \pm\frac{1}{\sqrt{2}}\right) = \frac{3}{4}$.

The extreme values of f may occur at the vertices $\left(\pm\frac{1}{\sqrt{2}}\ , 0\right)$,

$(0, \pm 1)$. We have

$f\left(\frac{1}{\sqrt{2}}\ , 0\right) = \frac{3-\sqrt{2}}{2}, f\left(-\frac{1}{\sqrt{2}}\ , 0\right) = \frac{3+\sqrt{2}}{2}\ , f(0, \pm 1) = 1.$

Therefore, the given function has absolute minimum value $-\frac{1}{12}$ at $\left(\frac{1}{6}, 0\right)$ and absolute maximum value $\frac{3+\sqrt{2}}{2}$ at $\left(-\frac{1}{\sqrt{2}}\ , 0\right)$.

IP1:

A rectangular box, open at the top, is to have a volume of 32 cubic feet. What must be the dimensions so that total surface is a minimum?

Solution:

If x, y and z are the edges of the rectangular box, then

i) Volume of box $= \mathrm{V} = xyz = 32$
ii) Surface area of box $= S = xy + 2yz + 2xz$

or, since $z = \frac{32}{xy}$ from (i), $S = xy + \frac{64}{x} + \frac{64}{y}$

$\frac{\partial S}{\partial x} = y - \frac{64}{x^2} = 0 \Rightarrow x^2 y = 64$ ...... (iii)

$\frac{\partial S}{\partial y} = x - \frac{64}{y^2} = 0 \Rightarrow xy^2 = 64$ ...... (iv)

Dividing equations (iii) and (iv), we obtain $y = x$ so that $x^3 = 64$ or $x = y = 4$ and $z = 2$.

For $x = y = 4$, $\Delta = S_{xx}\, S_{yy} - S_{xy}^2 = \left(\frac{128}{x^3}\right)\left(\frac{128}{y^3}\right) - 1 > 0$

and $S_{xx} = \frac{128}{x^3} > 0$. By second derivative test, S is minimum.

Hence, when $x = y = 4$, the dimensions $x = 4\,ft$, $= 4\,ft$,

$z = 2\,ft$ gives the minimum surface of the rectangular box.

IP2.

Find the minimum value of $x^2+y^2+z^2$ given that $xyz=a^3$

Solution:

Let $f(x,y,z)=x^2+y^2+z^2$ ...... (1)

Given that $xyz=a^3 \Longrightarrow z=\frac{a^3}{xy}$

Substituting in (1), we get

$$f=x^2+y^2+\frac{a^6}{x^2y^2}$$

We have $\frac{\partial f}{\partial x}=2x-\frac{2a^6}{x^3y^2}$ and $\frac{\partial f}{\partial y}=2y-\frac{2a^6}{x^2y^3}$

$\frac{\partial f}{\partial x}=0$ and $\frac{\partial f}{\partial y}=0 \Longrightarrow x=y=a$

$r=\frac{\partial^2 f}{\partial x^2}=2+\frac{6a^6}{x^3y^3}$, $s=\frac{\partial^2 f}{\partial x\partial y}=\frac{4a^6}{x^3y^3}$, $t=\frac{\partial^2 f}{\partial y^2}=2+\frac{6a^6}{x^2y^4}$

At $(a,a), rt-s^2=64-16=48>0$ also $r>0$

Therefore, f is minimum at (a,a) (by second derivative test).

The minimum value is $=a^2+a^2+\frac{a^6}{a^2.a^2}=3a^2$

IP3:

Examine local minimum and local maximum values of the function $\boldsymbol{f(x,y) = sinx + siny + sin(x+y)}$

Solution:

Given that $f(x,y) = sinx + siny + sin(x+y)$

$f_x = cosx + cos(x+y), f_y = cosy + cos(x+y)$

$r = f_{xx} = -sinx - sin(x+y),$

$s = f_{xy} = -sin(x+y)$

$t = f_{yy} = -siny - sin(x+y)$

Now, $f_x = 0$ and $f_y = 0$

$\Rightarrow \quad cosx + cos(x+y) = 0 \qquad \text{....... (1)}$

$\quad cosy + cos(x+y) = 0 \qquad \text{....... (2)}$

Subtracting (2) from (1), we get

$cosx - cosy = 0 \Rightarrow cosx = cosy \Rightarrow x = y$

From (1), $cosx + cos2x = 0$

$\Rightarrow cos2x = -cosx = cos(\pi - x) \Rightarrow 2x = \pi - x \Rightarrow x = \frac{\pi}{3}$

$\therefore x = y = \frac{\pi}{3}$ is a stationary point

At $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$, $r = -\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2} = -\frac{\sqrt{3}}{2} < 0,\ s = \frac{\sqrt{3}}{2},\ t = -\sqrt{3}$

$\therefore \quad rt - s^2 = 3 - \frac{3}{4} = \frac{9}{4} > 0$, also $r < 0$

$\therefore f(x, y)$ has local maximum at $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$ (by second derivative test).

Local maximum value $= f\ \left(\frac{\pi}{3}, \frac{\pi}{3}\right) = sin\frac{\pi}{3} + sin\frac{\pi}{3} + sin\frac{2\pi}{3}$

$$= \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{2}$$

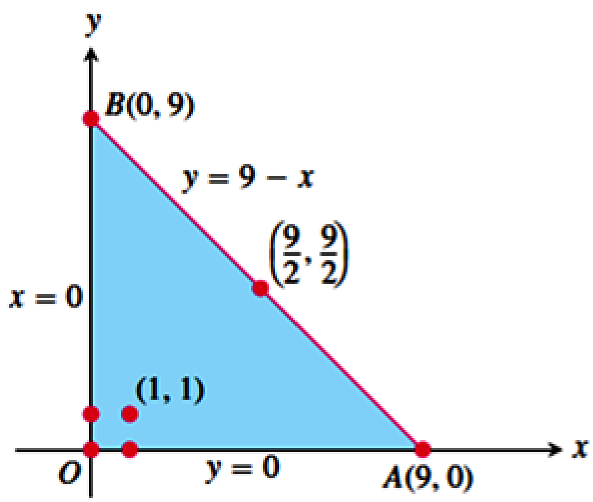
IP4.

Find the absolute maximum and minimum values of the function $f(x,y) = 2 + 2x + 2y - x^2 - y^2$ on the triangular region in the first quadrant bounded by the lines $x = 0, y = 0$,

$y = 9 - x$

Solution:

Since f is differentiable, the only points where f can assume these values are points inside the triangle where $f_x = 0, f_y = 0$ and points on the boundary.



(a) Interior points :

We have $f_x = 2 - 2x = 0$, $f_y = 2 - 2y = 0$, yielding the critical point $(x, y) = (1,1)$. Now, $r = f_{xx} = -2, s = f_{xy} = 0$,

$t = f_{yy} = -2.$

At $(1,1)$, $rt - s^2 = 4 > 0$ and $r > 0$. Therefore, f attains local minimum at $(1,1)$ and the local minimum value is $f(1,1) = 4$

(b) Boundary points:

We take the triangle one side at a time:

i) On the segment OA, $y = 0$.

The function $f(x,y) = f(x,0) = 2 + 2x - x^2$, may now be regarded as a function of x defined on the closed interval

$0 \le x \le 9$. Its extreme values may occur at the end points

$$x = 0 \quad \text{where } f(0,0) = 2$$
$$x = 9 \quad \text{where } f(9,0) = 2 + 18 - 81 = -61$$

and at the interior points where $f'(x,0) = 2 - 2x = 0$. The only interior point where $f'(x,0) = 0$ is $x = 1$, where $f''(x,0) = -2 < 0$. Therefore, f has local maximum at $(1,0)$ and the value is $f(1,0) = 3$

ii) On the segment OB, $x = 0$ and

$$f(x,y) = f(0,y) = 2 + 2y - y^2$$

We know from the symmetry of f in x and y from the analysis we just carried out that the candidates on this segments are

$$f(0,0) = 2, \qquad f(0,9) = -61, f(0,1) = 3$$

iii)We have already accounted for the values of f at the end points of AB, so we need only look at the interior points of AB . with $y = 9 - x$, we have

$$f(x, y) = 2 + 2x + 2(9 - x) - x^2 - (9 - x)^2$$
$$= -61 + 18x - 2x^2$$

Setting $f^{'}(x, 9 - x) = 18 - 4x = 0$ gives $x = \frac{18}{4} = \frac{9}{2}$ and $f^{''}(x, 9 - x) = -4 < 0$. The local maximum is attained at $x = \frac{9}{2}$ and at this value of x,

$$y = 9 - \frac{9}{2} = \frac{9}{2} \text{ and } f(x, y) = f\left(\frac{9}{2}, \frac{9}{2}\right) = -\frac{41}{4}$$

Summary:

We list all the extreme values:

$$4\ , 2\ , -61\ , 3\ , -\frac{41}{2}\ .$$

The absolute maximum is 4, which f assumes at $(1,1)$. The absolute minimum is -61 which f assumes at $(0,9)$ and $(9,0)$

3.9. Maxima and Minima

EXERCISES:

I. Find all the local maxima, local minima, and saddle points of the following functions:

a) $f(x,y) = x^2 + xy + 3x + 2y + 5$

b) $f(x,y) = y^2 + xy - 2x - 2y + 2$

c) $f(x,y) = 9x^3 + \frac{y^3}{3} - 4xy$

d) $f(x,y) = 8x^3 + y^3 + 6xy$

e) $f(x,y) = 2x^3 + 2y^3 - 9x^2 + 3y^2 - 12y$

f) $f(x,y) = 4xy - x^4 - y^4$

g) $f(x,y) = \frac{1}{x^2+y^2-1}$

h) $f(x,y) = \frac{1}{x} + xy + \frac{1}{y}$

i) $f(x,y) = y\ sinx$

j) $f(x,y) = e^{2x}\ cosy$

Answers:

a) Critical point: $(-2,1)$ Saddle point: $(-2,1)$

b) Critical point: $(-2,2)$ Saddle point: $(-2,2)$

c) Critical points: $(0,0), \left(\frac{4}{9}, \frac{4}{3}\right)$ Saddle point: $(0,0)$

Local minimum: $f\left(\frac{4}{9}, \frac{4}{3}\right) = -\frac{64}{81}$

d) Critical points: $(0,0), \left(-\frac{1}{2}, -1\right)$ Saddle point: $(0,0)$

Local maximum: $f\left(-\frac{1}{2}, -1\right) = 1$

e) Critical points: $(0,0), (0,1), (3,-2), (3,1)$

Saddle points: $(0,1), (3,-2)$

Local maximum: $f(0,-2) = 20$

Local minimum: $f(3,1) = -34$

f) Critical points: $(0,0), (1,1), (-1,-1)$ Saddle point: $(0,0)$

Local maximum: $f(-1,-1) = 2 = f(1,1)$

g) Critical point: $(0,0)$

Local maximum: $f(0,0) = -1$

h) Critical point: $(1,1)$

Local minimum: $f(1,1) = 3$

i) Critical point: $(n\pi, 0), n \in \boldsymbol{Z}$ Saddle point: $(n\pi, 0), n \in \boldsymbol{Z}$

j) No critical points No saddle points

No Extrema.

II. Find the absolute maxima and minima of the functions on the given domains.

A) $f(x,y) = 2x^2 - 4x + y^2 - 4y + 1$ on the closed triangular plate bounded by the lines $x = 0, y = 2, y = 2x$ in the first quadrant .

B) $D(x,y) = x^2 - xy + y^2 + 1$ on the closed triangular plate in the first quadrant bounded by the lines $x = 0, y = 4, y = x$

C) $f(x,y) = x^2 + y^2$ on the closed triangular plate bounded by the lines $x = 0, y = 0, y + 2x = 2$ in the first quadrant

D) $T(x,y) = x^2 + xy + y^2 - 6x$ on the rectangular plate $0 \leq x \leq 5, -3 \leq y \leq 0$

E) $T(x,y) = x^2 + xy + y^2 - 6x + 2$ on the rectangular plate $0 \leq x \leq 5, -3 \leq y \leq 0$

F) $f(x,y) = 48xy - 32x^3 - 24y^2$ on the rectangular plate $0 \leq x \leq 1, 0 \leq y \leq 1$

Answers:

A) Absolute maximum at $(0,0)$: 1
Absolute minimum at $(1,2)$: -5

B) Absolute maximum at $(0,4), (4,4)$: 17
Absolute minimum at $(0,0)$: 1

C) Absolute maximum at $(0,2)$: 4
Absolute minimum at $(0,0)$: 0

D) Absolute maximum at $(5,3)$: 19

Absolute minimum at $(4,-2)$: -12

E) Absolute maximum at $(0,-3)$: 11

Absolute minimum at $(4,-2)$: -10

F) Absolute maximum at $\left(\frac{1}{2},\frac{1}{2}\right)$: 2

Absolute minimum at $(1,0)$: -32

3.10

Lagrange's Method of Multipliers

Learning objectives:

- To find the extremum of a function subject to given set of constraints by the method of Lagrange's Multipliers.

AND

- To practice the related problems

Lagrange's Method of Multipliers

In many practical problems, it is required to find the maximum or minimum of a function $f(x_1, x_2, \dots, x_n)$, where the variables are not independent but are connected by one or more constraints of the form

$$\phi_i(x_1, x_2, \dots, x_n) = 0\ , i = 1, 2, \dots, k,$$

where generally $n > k$. In this module we discuss the Lagrange's method of multipliers to find the solution of such problems.

Lagrange method of multipliers

Suppose that $f(x_1, x_2, \dots, x_n)$ and $\phi_i(x_1, x_2, \dots, x_n)$, $i = 1,2, \dots, k$ are differentiable. To find the maximum/minimum of the function $f(x_1, x_2, \dots, x_n)$ subject to the constraints

$$\phi_i(x_1, x_2, \dots, x_n), \quad i = 1,2, \dots, k \qquad \dots (1)$$

We construct an auxiliary function of the form:

$$F(x_1, x_2, \dots, x_n, \lambda_1, \lambda_2, \dots, \lambda_k) = f(x_1, x_2, \dots, x_n) + \sum_{i=1}^{k} \lambda_i \phi_i(x_1, x_2, \dots, x_n) \qquad \dots (2)$$

where λ_i's are undetermined parameters known as ***Lagrange's multipliers.***

To determine the critical (stationary) points of F, we have the necessary conditions

$$\frac{\partial F}{\partial x_1} = 0 \, , \frac{\partial F}{\partial x_2} = 0 \, , \dots , \frac{\partial F}{\partial x_n} = 0$$

which give the equations

$$\frac{\partial f}{\partial x_j} + \sum_{i=1}^{k} \lambda_i \frac{\partial \phi_i}{\partial x_j} = 0 \, , j = 1,2, \dots , n \qquad \dots (3)$$

The equations (1) and (3) give $n + k$ equations in $n + k$ unknowns $x_1, x_2, \dots , x_n, \lambda_1, \lambda_2, \dots , \lambda_k$. Solving these equations we obtain the required critical points $(x_1, x_2, \dots , x_n)$ at which the function f has an extremum.

Note:

Further investigation is needed to determine the exact nature of these points.

Example 1: Find the maximum and minimum values of the function $f(x, y) = 3x + 4y$ on the circle $x^2 + y^2 = 1$.

Solution: We want the extreme values of $f(x, y) = 3x + 4y$ subject to the constraint $\phi(x, y) = x^2 + y^2 - 1$.

Consider the auxiliary function

$F(x, y, \lambda) = f(x, y) + \lambda\phi(x, y) = 3x + 4y + \lambda(x^2 + y^2 - 1)$
where λ is Lagrange's multiplier. The necessary conditions for extremum is

$$\frac{\partial F}{\partial x} = 3 + 2x\lambda = 0$$

$$\frac{\partial F}{\partial y} = 4 + 2y\lambda = 0$$

The above equations imply

$$x = -\frac{3}{2\lambda}, y = -\frac{2}{\lambda}$$

Using the constraint $x^2 + y^2 = 1$, we obtain

$$\frac{9}{4\lambda^2} + \frac{4}{\lambda^2} = 1 \Rightarrow 4\lambda^2 = 25 \Rightarrow \lambda = \pm\frac{5}{2}$$

Thus, $= -\frac{3}{2\lambda} = \pm\frac{3}{5}, y = -\frac{2}{\lambda} = \pm\frac{4}{5}$. Note that

$f(x, y) = 5$, when $(x, y) = \left(\frac{3}{5}, \frac{4}{5}\right)$

$f(x, y) = -\frac{7}{5}$, when $(x, y) = \left(\frac{3}{5}, -\frac{4}{5}\right)$

$f(x, y) = \frac{7}{5}$, when $(x, y) = \left(-\frac{3}{5}, \frac{4}{5}\right)$ and

$f(x, y) = -5$, when $(x, y) = \left(-\frac{3}{5}, -\frac{4}{5}\right)$

The maximum and minimum values of $f(x, y) = 3x + 4y$ on the circle $x^2 + y^2 = 1$ are 5 and -5 respectively, since we want to find extreme values of $f(x, y)$ gives that (x, y) also lie on the circle $x^2 + y^2 = 1$.

Example 2: Find the minimum value of $x^2 + y^2 + z^2$ subject to the condition $xyz = a^3$.

Solution: We have $f(x, y, z) = x^2 + y^2 + z^2$ and $\phi(x, y, z) = xyz - a^3 = 0$. Consider the auxiliary function

$$F(x, y, z, \lambda) = f(x, y, z) + \lambda\phi(x, y, z)$$

$$= x^2 + y^2 + z^2 + \lambda(xyz - a^3)$$

where λ is Lagrange's multiplier. The necessary conditions for extremum is

$$\frac{\partial F}{\partial x} = 2x + \lambda yz = 0$$

$$\frac{\partial F}{\partial y} = 2y + \lambda xz = 0$$

$$\frac{\partial F}{\partial z} = 2z + \lambda xy = 0$$

The above equations imply

$$\lambda xyz = -2x^2 = -2y^2 = -2z^2 \Rightarrow x^2 = y^2 = z^2.$$

Using the constraint $xyz = a^3$, we obtain the solutions as $(a, a, a), (a, -a, -a), (-a, a, -a)$ and $(-a, -a, a)$

At each of these points the value of $f(x, y, z)$ is $x^2 + y^2 + z^2 = 3a^2$.

For any x, y, z, we have

The AM (arithmetic mean) of x^2, y^2, z^2 is $\frac{x^2+y^2+z^2}{3}$

The GM (geometric mean) of x^2, y^2, z^2 is $(x^2y^2z^2)^{\frac{1}{3}}$

Since $AM \geq GM$, we obtain $\frac{x^2+y^2+z^2}{3} \geq (xyz)^{\frac{2}{3}}$

Under the given constraint, we have $\frac{x^2+y^2+z^2}{3} \geq (a^3)^{\frac{2}{3}} = a^2$.

Thus, $x^2 + y^2 + z^2 \geq 3a^2$ and so the constrained minimum of $f(x, y, z)$ is $3a^2$.

Example 3: Find the point on the curve of intersection of the surface $4z = x^2 + y^2$ and the plane $2x + 3y = 6$ nearest to the origin.

Solution: We want the extreme values of
$f(x, y, z) = x^2 + y^2 + z^2$ subject to the conditions

$$\phi_1(x, y, z) = x^2 + y^2 - 4z = 0$$

and $\phi_2(x, y, z) = 2x + 3y - 6 = 0$

Consider the auxiliary function

$$F(x, y, z, \lambda_1, \lambda_2) = f(x, y, z) + \lambda_1\phi_1(x, y, z) + \lambda_2\phi_2(x, y, z)$$

$$= x^2 + y^2 + z^2 + \lambda_1(x^2 + y^2 - 4z) + \lambda_2(2x + 3y - 6)$$

where λ_1 and λ_2 are Lagrange's multipliers. The necessary conditions for extremum is

$$\frac{\partial F}{\partial x} = 2x + 2\lambda_1 x + 2\lambda_2 = 0$$

$$\frac{\partial F}{\partial y} = 2y + 2\lambda_1 y + 3\lambda_2 = 0$$

$$\frac{\partial F}{\partial z} = 2z - 4\lambda_1 = 0$$

From the first two relations above, we have

$$2(1+\lambda_1)x = -2\lambda_2\,, 2(1+\lambda_1)y = -3\lambda_2$$

On division, we get $3x = 2y$. Substituting in $\phi_2(x,y,z) = 0$, we get $x = \frac{12}{13}\,, y = \frac{18}{13}$. Substituting these values in $\phi_1(x,y,z) = 0$, we get $z = \frac{9}{13}$. Therefore, the point $P\left(\frac{12}{13},\frac{18}{13},\frac{9}{13}\right)$ lies on the surfaces $\phi_1(x,y,z) = 0$ and the plane $\phi_2(x,y,z) = 0$ and is either nearest or farthest from the origin.

Choose another point $Q(0,2,1)$ that lies on both.

Notice that $OP = \sqrt{\frac{(12)^2+(18)^2+(9)^2}{(13)^2}} = \frac{\sqrt{549}}{13} \approx 1.8$,

$OQ = \sqrt{0^2+2^2+1^2} = \sqrt{5}$ and $OP < OQ$. Thus, the point $P\left(\frac{12}{13},\frac{18}{13},\frac{9}{13}\right)$ is nearest to the origin.

P1:

Find the minimum and maximum value of $\boldsymbol{x+y+z}$ such that $\boldsymbol{xyz = 1}$

Solution:

Let $f(x, y, z) = x + y + z$ and $\phi(x, y, z) = xyz - 1 = 0$ consider the auxiliary function

$$F\ (x, y, z, \lambda) = f\ (x, y, z) + \lambda\ \phi(x, y, z)$$

$$= (x + y + z) + \lambda\ (xyz - 1),$$

where λ is Lagrange's multiplier. The necessary conditions for extremum is

$$\frac{\partial F}{\partial x} = 1 + \lambda yz = 0$$

$$\frac{\partial F}{\partial y} = 1 + \lambda xz = 0$$

$$\frac{\partial F}{\partial z} = 1 + \lambda xy = 0$$

The above equation imply

$$\lambda yz = -1, \qquad \lambda xz = -1, \qquad \lambda xy = -1$$

$$\Longrightarrow \lambda xyz = -x = -y = -z \Longrightarrow x = y = z$$

Using the constraint $xyz = 1$, we obtain the solution as $(1, 1, 1)$.

The value of $f\ (x, y, z) = x + y + z = 1 + 1 + 1 = 3$ at the point $(1, 1, 1)$, which is a local minimum, since, if we consider another point $\left(\frac{3}{4}, \frac{3}{4}, \frac{16}{9}\right)$ close to $(1, 1, 1)$ on the surface $xyz = 1$, then $x + y + z = \frac{59}{18} > 3$.

Therefore, the point $(1, 1, 1)$ gives minima by comparison.

P2.

Find the shortest distance from the origin to the hyperbola $\boldsymbol{x^2 + 8xy + 7y^2 = 225, z = 0}$

Solution:

Let $f(x, y) = x^2 + y^2$ is the square of the distance from the origin to any point in the xy −plane, which is to be minimized with the constraint $\phi(x, y) = x^2 + 8xy + 7y^2 - 225 = 0.$

Consider the auxiliary function

$$F(x, y, \lambda) = f(x, y) + \lambda\phi(x, y)$$

$$= x^2 + y^2 + \lambda(x^2 + 8xy + 7y^2 - 225),$$

where λ is the Lagrange's multiplier. The necessary condition for extremum is

$$\frac{\partial F}{\partial x} = 0 \Longrightarrow 2x + 2\lambda x + 8\lambda y = 0 \Longrightarrow (1 + \lambda)x + 4\lambda y = 0 \quad \ldots (1)$$

$$\frac{\partial F}{\partial y} = 0 \Longrightarrow 2y + 8\lambda x + 14\lambda y = 0 \Longrightarrow 4\lambda x + (1 + 7\lambda)y = 0 \quad \ldots (2)$$

Since $(x, y) \neq 0$, we must have

$$\begin{vmatrix} 1 + \lambda & 4\lambda \\ 4\lambda & 1 + 7\lambda \end{vmatrix} = 0$$

i.e., $1 + 8\lambda - 9\lambda^2 = 0 \Longrightarrow \lambda = 1, -\frac{1}{9}$

Case (i):

If $\lambda = 1$ then from (1), we get $x = -2y$ and substituting in $\phi(x, y) = 0$, we obtain $y^2 = -45$, which has no real solutions.

Case (ii):

If $\lambda = -\frac{1}{9}$ then from (1), we get $y = 2x$ and substituting in $\phi(x, y) = 0$, we obtain $x^2 = 5$. Then $x^2 = 5$, $y^2 = 4x^2 = 20$ and $x^2 + y^2 = 25$. Thus, the shortest distance is $\sqrt{25} = 5$.

P3:

A wire of length b is cut into two parts which are bent in the form of a square and a circle respectively. Find the least value of the sum of the areas so found.

Solution:

Let x and y be two parts into which the given wire is cut so that $x + y = b$. Suppose the piece of wire of length x is bent into a square so that each side is $\frac{x}{4}$ and thus the area of the square is $\frac{x}{4} \cdot \frac{x}{4} = \frac{x^2}{16}$.

Suppose the wire of length y is bent into a circle with the perimeter y. So, the area of this circle so formed is

$$\pi(radius)^2 = \pi\left(\frac{y}{2\pi}\right)^2 = \frac{\pi y^2}{4\pi^2} = \frac{y^2}{4\pi}$$

Thus, to find the minimum of the sum of the two areas subject to the constraint that sum is b. So the auxiliary equation is

$$F(x, y, \lambda) = \left(\frac{x^2}{16} + \frac{y^2}{4\pi}\right) + \lambda(x + y - b)$$

$$F_x = \frac{x}{8} + \lambda = 0,\ F_y = \frac{y}{2\pi} + \lambda = 0$$

Solving $x = -8\lambda$, $y = -2\pi\lambda$

Substituting these values in the constraint

$x + y = -8\lambda - 2\pi\lambda = b$

$$\therefore \lambda = -\frac{b}{8+2\pi}$$

Thus $= -8\lambda = \frac{8b}{8+2\pi}$, $y = -2\pi\lambda = \frac{2\pi b}{8+2\pi}$

The least value of the sum of the areas of the square and circle is

$$f(x,y) = \frac{x^2}{16} + \frac{y^2}{4\pi}\bigg|_{(x,y)} = \frac{64b^2}{16(8+2\pi)^2} + \frac{4\pi^2 b^2}{(8+2\pi)^2}$$

$$= \frac{b^2(\pi+4)}{4(\pi+4)^2} = \frac{b^2}{4(\pi+4)}$$

P4.

Maximum on line of intersection:

Find the maximum value of $\boldsymbol{W = xyz}$ on the line of intersection of two planes $\boldsymbol{x + y + z = 40}$ and $\boldsymbol{x + y - z = 0}$

Solution:

We want to find the maximum value of $f\ (x, y, z) = xyz$ subject to the constraints $\phi_1(x, y, z) = x + y + z - 40 = 0$,

$\phi_2(x, y, z) = x + y - z = 0$

Now, the auxiliary function is

$$F(x, y, z, \lambda_1, \lambda_2) = f(x, y, z) + \lambda_1 \phi_1(x, y, z) + \lambda_2 \phi_2(x, y, z)$$

$$= xyz + \lambda_1(x + y + z - 40) + \lambda_2(x + y - z),$$

where λ_1, λ_2 are Lagrange's multipliers. The necessary conditions for extremum is

$$\frac{\partial F}{\partial x} = 0 \quad \Rightarrow yz + \lambda_1 + \lambda_2 = 0 \qquad \text{……… (1)}$$

$$\frac{\partial F}{\partial y} = 0 \quad \Rightarrow xz + \lambda_1 + \lambda_2 = 0 \qquad \text{……… (2)}$$

$$\frac{\partial F}{\partial z} = 0 \quad \Rightarrow xy + \lambda_1 - \lambda_2 = 0 \qquad \text{……… (3)}$$

Now, (1) and (2) imply

$$yz = xz \Rightarrow \ z(y - x) = 0 \Rightarrow z = 0 \quad or \ \ y = x$$

Case (i):

If $z = 0$ then from the two constraints, we get $x + y = 40$ and $x + y = 0$. Notice that these two equations are inconsistent. Thus, it has no solution.

Case (2):

If $y = x$ then from the two constraints, we get $2x + z = 40$ and $2x - z = 0 \quad \Rightarrow \; z = 20, \; x = 10$

Since $y = x$, we get $y = 10$.

Therefore, the maximum value of xyz is

$$10 \times 10 \times 20 = 2000.$$

IP1:

Find the point on the plane $x + 2y + 3z = 6$ nearest to the origin?

Solution:

Let $f(x, y, z) = x^2 + y^2 + z^2$ is the square of the distance of the point $P(x, y, z)$ from the origin, which is to be minimized with the constraint $\phi(x, y, z) = x + 2y + 3z - 6 = 0.$

Consider the auxiliary function

$$F(x, y, z, \lambda) = f(x, y, z) + \lambda\, \phi(x, y, z)$$

$$= (x^2 + y^2 + z^2) + \lambda(x + 2y + 3z - 6\,),$$

where λ is Lagrange's multiplier. The necessary condition for extremum is

$$\frac{\partial F}{\partial x} = 2x + \lambda = 0$$

$$\frac{\partial F}{\partial y} = 2y + 2\lambda = 0$$

$$\frac{\partial F}{\partial z} = 2z + 3\lambda = 0$$

The above equations imply

$$x = -\frac{\lambda}{2},\ y = -\lambda,\ z = -\frac{3\lambda}{2}$$

Substituting these in $\phi(x, y, z) = 0 \implies -\frac{\lambda}{2} - 2\lambda - \frac{9\lambda}{2} = 1$

$$\Longrightarrow -\lambda - 4\lambda - 9\lambda = 12 \Longrightarrow \lambda = -\frac{12}{14} = -\frac{6}{7}$$

Substituting the λ value in the above relation, we get

$$x = -\frac{\lambda}{2} = \frac{+6}{14} = \frac{3}{7} \;\; ; \;\; y = -\lambda = \frac{6}{7} \;\; ; \;\; z = -\frac{3}{2}\left(-\frac{6}{7}\right) = \frac{9}{7}$$

The point $P\left(\frac{3}{7}, \frac{6}{7}. \frac{9}{7}\right)$ is either nearest or farthest to the origin.

Now, consider another point $Q(0,0,2)$, which is also on the plane. The distance $\mathrm{OQ} = 2$ and $OP = \sqrt{\frac{9}{49} + \frac{36}{49} + \frac{81}{49}} = \sqrt{\frac{18}{7}} < 2.$ Hence $OP < OQ$.

Therefore, the point $P\left(\frac{3}{7}, \frac{6}{7}. \frac{9}{7}\right)$ of the plane is nearest to the origin.

IP2:

Find the maximum and minimum distances from the origin to the curve $3x^2 + 4xy + 6y^2 = 140$.

Solution:

Let $f(x, y) = x^2 + y^2$ is the square of the distance from the origin to any point in the xy −plane, which is to be minimized with the constraint $\phi(x, y) = 3x^2 + 4xy + 6y^2 - 140 = 0$.

Consider the auxiliary function

$$F(x, y, \lambda) = f(x, y) + \lambda\, \phi(x, y)$$

$$= (x^2 + y^2) + \lambda(3x^2 + 4xy + 6y^2 - 140),$$

where λ is the Lagrange's multiplier. The necessary condition for extremum is

$$F_x = 2x + \lambda(6x + 4y) = 0$$

$$F_y = 2y + \lambda(12y + 4x) = 0$$

The above equations imply

$$\lambda = -\frac{x}{3x+2y} = -\frac{y}{6y+2x}$$

$$-\lambda = \frac{x^2}{3x^2+2xy} = \frac{y^2}{6y^2+2xy} = \frac{x^2+y^2}{3x^2+4xy+6y^2}$$

$$\therefore -\lambda = \frac{f}{140}$$

Substituting λ in $F_x = 0$ and $F_y = 0$, we get

$$(140 - 3f)x - 2fy = 0$$

$$-2fx + (140 - 6f)y = 0$$

This system has non-trivial solution if

$$\begin{vmatrix} 140 - 3f & -2f \\ -2f & 140 - 6f \end{vmatrix} = 0$$

i.e., $(140 - 3f)(140 - 6f) - 4f^2 = 0$

$$14f^2 - 1260f + 140^2 = 0$$

$$f^2 - 90f + 1400 = 0$$

$$(f - 70)(f - 20) = 0$$

$\therefore f = 70,20$

Thus the maximum and minimum distances are $\sqrt{70}$, $\sqrt{20}$.

IP3:

Find the dimensions of a rectangular box of maximum capacity whose surface area is given when (a) box is open at the top (b) box is closed.

Solution:

Let x, y, z be the dimensions of the rectangular box so that its volume V is

$$V = xyz \qquad \dots (1)$$

The total surface area of the box is

$$nxy + 2yz + 2zx = S = \text{given constant} \qquad \dots (2)$$

Here $n = 1$, if the box is open at the top

$n = 2$, if the box is closed (on all sides)

The constrained maximum problem is to maximize V subject to constraint (2).

So the auxiliary function is

$$F(x, y, z) = xyz + \lambda(nxy + 2yz + 2zx - S) \qquad \dots (3)$$

$$F_x = yz + \lambda(ny + 2z) = 0 \qquad \dots (4)$$

$$F_y = xz + \lambda(nx + 2z) = 0 \qquad \dots (5)$$

$$F_z = xy + \lambda(2y + 2x) = 0 \qquad \dots (6)$$

Multiplying (4), (5), (6) by x, y, z respectively and adding, we get

$$3xyz + 2\lambda(nxy + 2yz + 2zx) = 0$$

or $3 \cdot V + 2\lambda \cdot S = 0$ using (1) and (2)

$$\Longrightarrow \lambda = -\frac{3V}{2S} \qquad \dots (7)$$

Substituting value of λ from (7) in (4), (5), (6)

$$yz - \frac{3V}{2S}(ny + 2z) = 0 \text{ or } yz - \frac{3xyz}{2S}(ny + 2z) = 0$$

$$\text{i.e., } nxy + 2xz = \frac{2S}{3} \qquad \dots (8)$$

Similarly $\quad nxy + 2yz = \frac{2S}{3} \qquad \dots (9)$

$$2yz + 2zx = \frac{2S}{3} \qquad \dots (10)$$

(8) – (9) $\Longrightarrow x = y \qquad \dots (11)$

(9) – (10) $\Longrightarrow ny = 2z \qquad \dots (12)$

Substituting (11) and (12) in the given constraint (2)

$$n \cdot x \cdot x + 4x\frac{nx}{2} = S \Longrightarrow 3nx^2 = S \Longrightarrow x^2 = \frac{S}{3n}$$

a. When box is open : $n = 1$

$$\therefore x^2 = \frac{S}{3} \text{ or } x = \sqrt{\frac{S}{3}}$$

The dimensions of the open top box are

$$x = y = \sqrt{\frac{S}{3}}\ , z = \frac{1}{2}\sqrt{\frac{S}{3}}$$

b. When the box is closed: $n = 2$

$$x^2 = \frac{S}{6} \text{ or } x = \sqrt{\frac{S}{6}}\ , x = y = z$$

The dimensions are $x = y =, z = \sqrt{\frac{S}{6}}$

IP4.

Minimum distance to the origin:

Find the point closest to the origin on the curve of the intersection of the plane $2y + 4z = 5$ and the cone

$\boldsymbol{z^2 = 4x^2 + 4y^2}$.

Solution:

Let $f(x, y, z) = x^2 + y^2 + z^2$ be the square of the distance from the origin.

We want to minimize $f(x, y, z)$ subject to the constraints $\phi_1(x, y, z) = 2y + 4z - 5 = 0$ and

$\phi_2(x, y, z) = 4x^2 + 4y^2 - z^2 = 0$

The auxiliary function is

$$F(x, y, z, \lambda_1, \lambda_2) = f(x, y, z) + \lambda_1\phi_1(x, y, z) + \lambda_2\phi_2(x, y, z)$$

$$= (x^2 + y^2 + z^2) + \lambda_1(2y + 4z - 5) + \lambda_2(4x^2 + 4y^2 - z^2),$$

where λ_1, λ_2 are the Lagrange's multipliers. The necessary conditions for extremum is

$$\frac{\partial F}{\partial x} = 0 \Longrightarrow 2x + 8x\lambda_2 = 0 \Longrightarrow 2x = -8x\lambda_2 \quad \text{... (1)}$$

$$\frac{\partial F}{\partial y} = 0 \Longrightarrow 2y + 2\lambda_1 + 8y\lambda_2 = 0 \Longrightarrow 2y = -2\lambda_1 - 8y\lambda_2 \quad \text{... (2)}$$

$\frac{\partial F}{\partial z} = 0 \Longrightarrow 2z + 4\lambda_1 - 2z\lambda_2 = 0 \Longrightarrow 2z = -4\lambda_1 + 2z\lambda_2$ ... (3)

Now, (1)$\Longrightarrow 2x(1 + 4\lambda_2) = 0 \Longrightarrow \lambda_2 = -\frac{1}{4}$ or $x = 0$

Case (i):

If $\lambda_2 = -\frac{1}{4}$ then from (2) we get

$$2y = -2\lambda_1 - 8y\left(-\frac{1}{4}\right) = -2\lambda_1 + 2y$$

$\Longrightarrow y = -\lambda_1 + y \Longrightarrow \lambda_1 = 0$

Now, (3) gives $2z = -4(0) + 2z\left(-\frac{1}{4}\right) \Longrightarrow z = 0$

The constraint $\phi_1(x, y, z) = 0$ now becomes

$$2y + 4(0) = 5 \Longrightarrow y = \frac{5}{2}$$

and the constraint $\phi_2(x, y, z) = 0$ will become

$4x^2 + 4\left(\frac{5}{2}\right)^2 - 0 = 0 \Longrightarrow x^2 = -\left(\frac{5}{2}\right)^2$, which is not possible

Case (ii):

If $x = 0$ then $4x^2 + 4y^2 - z^2 = 0 \Longrightarrow 4y^2 - z^2 = 0 \Longrightarrow z = \pm 2y$

Now, $2y + 4z - 5 = 0$

$$\Longrightarrow y = \frac{1}{2} \text{ when } z = 2y \quad ; \quad y = -\frac{5}{6} \text{ when } z = -2y$$

Thus, $(x, y, z) = \left(0, \frac{1}{2}, 1\right)$and $\left(0, -\frac{5}{6}, \frac{5}{3}\right)$

Now, $f\left(0, \frac{1}{2}, 1\right) = (0)^2 + \left(\frac{1}{2}\right)^2 + (1)^2 = \frac{5}{4}$ and

$$f\left(0, -\frac{5}{6}, \frac{5}{3}\right) = (0)^2 + \left(-\frac{5}{6}\right)^2 + \left(\frac{5}{3}\right)^2 = \frac{125}{36}$$

Clearly, $f\left(0, \frac{1}{2}, 1\right) < f\left(0, -\frac{5}{6}, \frac{5}{3}\right)$

Thus, the point $\left(0, \frac{1}{2}, 1\right)$ is closest to the origin and the shortest distance is $\sqrt{f\left(0, \frac{1}{2}, 1\right)} = \frac{\sqrt{5}}{2}$

3.10 Lagrange's Method of Multipliers

EXERCISES:

I.

a) Find the point on the plane $x - 2y - 3z = 6$ nearest to the origin.

b) Find the point on the curve of inter section of the surface $4z = x^2 + y^2$ and the plane $x - y = 4$ nearest to the origin.

c) Find the volume of the greatest rectangular parallelepiped that can be inscribed in the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0$

d) Find a point on the plane $3x + 2y + z - 12 = 0$ which is nearest to the origin.

e) Find the maximum value of $u = x^2y^3z^4$ if $2x + 3y + 4z = a$.

f) Find the maximum and minimum distances of the point $(3,4,12)$ from the sphere $x^2 + y^2 + z^2 = 1$

g) Show that the rectangular solid of maximum volume that can be inscribed in a sphere is a cube.

Answers:

a) Nearest point $\left(\frac{3}{7}, \frac{6}{7}, -\frac{9}{7}\right)$

b) $(2,2,2)$

c) Volume $V = \frac{8abc}{3\sqrt{3}}$ cubic units

d) $\left(\frac{18}{7}, \frac{12}{7}, \frac{6}{7}\right)$

e) $\left(\frac{a}{9}\right)^9$

f) Maximum $= 14$, minimum $= 12$

g) (HINT: Assume that $2x, 2y, 2z$ be the length, breadth, height of the rectangular solid whose volume is $V = 8xyz$; R be the radius of the sphere so that $x^2 + y^2 + z^2 = R^2$)

II.

1. Find the points on the ellipse $x^2 + 2y^2 = 1$ where $f(x, y) = xy$ as its extreme values.
2. Find the extreme values of $f(x, y) = xy$ subject to the constraint $g(x, y) = x^2 + y^2 - 10 = 0$
3. Find the point on the plane $x + 2y + 3z = 13$ closest to the point $(1,1,1)$
4. Find the point on the sphere $x^2 + y^2 + z^2 = 4$ farthest from the point $(1, -1, 1)$

Answers:

1. Extreme values : $\pm\frac{\sqrt{2}}{2}$; 2. Extreme values : ± 5

3. $\left(\frac{3}{2}, 2, \frac{5}{2}\right)$; 4. $\left(\frac{-2}{\sqrt{3}}, \frac{2}{3}, \frac{-2}{\sqrt{3}}\right)$

III. Extreme values subject to two constraints

a) Maximize the function $f(x, y, z) = x^2 + 2y - z^2$ subject to the constraints $2x - y = 0$ and $y + z = 0$

b) Minimize the function $f(x, y, z) = x^2 + y^2 + z^2$ subject to the constraints $x + 2y + 3z = 6$ and $x + 3y + 9z = 9$.

c) **Minimum distance to the origin:** Find the point closest to the origin on the line intersection of the planes $y + 2z = 12$ and $x + y = 6$.

Answers:

a) The minimum value is $f\left(\frac{81}{59}, \frac{123}{59}, \frac{9}{59}\right) = \frac{369}{59}$

b) $(2, 4, 4)$

c) The maximum value is $f\left(\frac{2}{3}, \frac{4}{3}, -\frac{4}{3}\right) = \frac{4}{3}$