6.2

Gaussian Process

Many random phenomena in physical problems including *noise* are well approximated by a special class of random process, namely Gaussian random process. A number of processes such as the Wiener process and the shot noise process can be approximated, as per central limit theorem, by a Gaussian process. Moreover the output of a linear system in which the input is a weighted sum of a large number of independent samples of a random process tends to approach a Gaussian process. Gaussian processes play an important role in the theory and analysis of random phenomena, because they are good approximations to the observations and multivariate Gaussian distributions are analytically simple.

One of the most important uses of the Gaussian process is to model and analyse the effects of thermal noise in electronic circuits used in communication systems. Individual circuits contain resistors, inductors and capacitors as well as semiconductor devices. The resistors and semiconductor elements contain charged particles (free electrons) subjected to random motion due to thermal energy. The random motion of charged particles causes fluctuations in the current waveforms or information bearing signals that flow through these components. These fluctuations are called **thermal noise**, which are of sufficient strength to disturb a weak signal and to make the recognition of signals a difficult task. Models of thermal noise are used to identify and minimize the effects of noise in signal recognition.

Gaussian Process: A real valued random process $\{X(t)\}$ is called a **Gaussian process** or **normal process**, if the random variables $X(t_1), X(t_2), \dots, X(t_n)$ are jointly normal for every $n = 1,2,\dots$ and for any set of t_i's.

The n^{th} order density of a Gaussian process is given by

$$f(x_1, x_2, \dots, x_n; t_1, t_2, \dots, t_n) = \frac{1}{(2\pi)^{\frac{n}{2}}|\Lambda|^{\frac{1}{2}}} exp\left[-\frac{1}{2|\Lambda|}\sum_{i=1}^{n}\sum_{j=1}^{n}|\Lambda|_{ij}(x_i - \mu_i)(x_j - \mu_j)\right]$$

where $\mu_i = E\{X(t_i)\}$ and Λ is the n^{th} order square matrix (λ_{ij}), where $\lambda_{ij} = C\{X(t_i), X(t_j)\}$ and $|\Lambda|_{ij}$ = cofactor of λ_{ij} in $|\Lambda|$ ... (1)

Note: *Gaussian process is completely specified by the first and second order moments, viz., means and covariances (variances).*

Note: When we consider the first order density of a Gaussian process,

$$\Lambda = (\lambda_{11}) = [cov(X(t_1), X(t_1))] = [Var\{X(t_1)\}] = \sigma_1^2$$

$\therefore |\Lambda| = \sigma_1^2$ and $|\Lambda|_{11} = 1$

$$\therefore f(x_1, t_1) = \frac{1}{\sigma_1\sqrt{2\pi}} exp\left\{-\frac{(x_1-\mu_1)^2}{2\sigma_1^2}\right\}$$

Note: When we consider the second order density of a Gaussian process,

$$\Lambda = \begin{pmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{pmatrix} = \begin{pmatrix} \sigma_1^2 & r_{12}\sigma_1\sigma_2 \\ r_{21}\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}$$

$\therefore |\Lambda| = \sigma_1^2\sigma_2^2(1-r^2)$, where $r_{12} = r_{21} = r$

$|\Lambda|_{11} = \sigma_2^2, |\Lambda|_{12} = -r\sigma_1\sigma_2, |\Lambda|_{21} = -r\sigma_1\sigma_2,\ |\Lambda|_{22} = \sigma_1^2$

$\therefore f(x_1, x_2; t_1, t_2)$

$$= \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-r^2}} exp\left[-\frac{1}{2\sigma_1^2\sigma_2^2(1-r^2)}\{\sigma_2^2(x_1-\mu_1)^2 - 2r\sigma_1\sigma_2(x_1-\mu_1)(x_2-\mu_2) + \sigma_1^2(x_2-\mu_2)^2\}\right]$$

$$i.e., f(x_1, x_2; t_1, t_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-r^2}} exp\left[-\frac{1}{2(1-r^2)}\left\{\frac{(x_1-\mu_1)^2}{\sigma_1^2} - \frac{2r(x_1-\mu_1)(x_2-\mu_2)}{\sigma_1\sigma_2} + \frac{(x_2-\mu_2)^2}{\sigma_2^2}\right\}\right]$$

Properties

1. If a Gaussian process is wide sense stationary, it is also strict sense stationary. (**See P1** for proof).
2. If the member functions of a Gaussian process are uncorrelated, then they are independent. (**See P2** for proof).

3. If the input $\{X(t)\}$ of a linear system is a Gaussian process, the output will also be a Gaussian process.

Example 1: If $\{X(t)\}$ is a Gaussian process with $\mu(t) = 10$ and $C(t_1, t_2) = 16e^{-|t_1 - t_2|}$, find the probability that

(i) $X(10) \leq 8$ and (ii) $|X(10) - X(6)| \leq 4.$

Solution: Since $\{X(t)\}$ is a Gaussian process, any member of the process is a normal r.v.

Therefore, $X(10)$ is a normal r.v with mean $\mu(10) = 10$ and variance $C(10,10) = 16.$

i. $P\{X(10) \leq 8\} = P\left\{\frac{X(10)-10}{4} \leq -0.5\right\}$

$= P\{Z \leq -0.5\}$ (where Z is the standard normal r.v)

$= 0.5 - P\{0 \leq Z \leq 0.5\}$

$= 0.5 - 0.1915$ (from normal tables)

$= 0.3085$

ii. $X(10) - X(6)$ is also a normal r.v with mean

$\mu(10) - \mu(6) = 10 - 10 = 0.$

$$Var\{X(10) - X(6)\} = Var\{X(10)\} + Var\{X(6)\} - 2Cov\{X(10), X(6)\}$$

$$= C(10,10) + C(6,6) - 2C(10,6)$$

$$= 16 + 16 - 2.16e^{-4}$$

$$= 31.4139$$

$$P\{X(10) - X(6) \leq 4\} = P\left\{\frac{|X(10)-X(6)|}{5.6048} \leq \frac{4}{5.6048}\right\}$$

$$= P\{|Z| \leq 0.7137\}$$

$$= 2 \times 0.2611$$

$$= 0.5222$$

Example 2: The process $\{X(t)\}$ is normal with $\mu_t = 0$ and $R_x(\tau) = 4e^{-3|\tau|}$. Find a memoryless system $g(x)$ such that the first order density $f_y(y)$ of the resulting output $Y(t) = g\{X(t)\}$ is uniform in the interval $(6, 9)$.

Solution: Since $\{X(t)\}$ is a normal process, a sample function $X(t)$ follows a normal distribution with mean 0 and variance given by $R_x(0) = 4$.

$$\therefore f_X(x) = \frac{1}{2\sqrt{2\pi}} e^{-\frac{x^2}{8}}, -\infty < x < \infty$$

Now $Y(t)$ is to be uniform in (6,9)

$$\therefore f_Y(y) = \frac{1}{3}, 6 < y < 9$$

Therefore, the distribution function of Y is given by

$$F_Y(y) = \int_6^y f_Y(y)dy = \frac{1}{3}(y-6) \qquad .....(1)$$

Now $F_Y(y) = P\{Y(t) \leq y\} = P\{g[X(t)] \leq y\}$

$$= P\{X(t) \leq g^{-1}(y)\}$$

$$= P\{X(t) \leq x\} \qquad \text{[since } y = g(x)\text{]}$$

$$= F_X(x)$$

But, from (1), $F_Y\{g(x)\} = \frac{1}{3}\{g(x) - 6\}$

$$\therefore \frac{1}{3}\{g(x) - 6\} = F_X(x)$$

$$\therefore g(x) = 6 + 3F_X(x) = 6 + 3\int_{-\infty}^{x} \frac{1}{2\sqrt{2\pi}} e^{-\frac{x^2}{8}} dx$$

Example 3: It is given that $R_x(\tau) = e^{-|\tau|}$ for a certain stationary Gaussian random process $\{X(t)\}$. Find the joint p.d.f of the r.vs $X(t), X(t+1), X(t+2)$.

Solution: Let us denote the given r.vs by $X(t_1), X(t_2), X(t_3)$.

The joint p.d.f of $\{X(t_1), X(t_2), X(t_3)\}$ is given by

$$f(x_1, x_2, x_3; t_1, t_2, t_3) = \frac{1}{(2\pi)^{\frac{3}{2}}|\Lambda|^{\frac{1}{2}}} exp\left[-\frac{1}{2|\Lambda|}\sum_{i=1}^{3}\sum_{j=1}^{3}|\Lambda|_{ij}\left(x_i - \mu_i\right)\left(x_j - \mu_j\right)\right]$$

where $\mu_i = E\{X(t_i)\}$ and Λ is the third order square matrix $\left(\lambda_{ij}\right)$, where $\lambda_{ij} = C\{X(t_i), X(t_j)\}$ and $|\Lambda|_{ij}$ = cofactor of λ_{ij} in $|\Lambda|$.

$$E\{X(t)\} = \sqrt{\lim_{\tau \to \infty} R_x(\tau)} = \sqrt{\lim_{\tau \to \infty} e^{-|\tau|}} = 0$$

$$\therefore \lambda_{ij} = C\{X(t_i)X(t_j)\} = R(t_i - t_j) \quad (\because \text{ it is stationary})$$

$$\therefore \lambda_{11} = R(0) = 1, \lambda_{12} = R(1) = e^{-1} \text{ etc.} \qquad \text{(Compute!)}$$

$$\therefore \Lambda = \begin{pmatrix} 1 & \frac{1}{e} & \frac{1}{e^2} \\ \frac{1}{e} & 1 & \frac{1}{e} \\ \frac{1}{e^2} & \frac{1}{e} & 1 \end{pmatrix} \text{ and } |\Lambda| = \left(1 - \frac{1}{e^2}\right)^2$$

$$|\Lambda|_{11} = 1 - \frac{1}{e^2}, |\Lambda|_{12} = -\frac{1}{e} + \frac{1}{e^3}, |\Lambda|_{13} = 0 \text{ etc.} \qquad \text{(do it !)}$$

Therefore, the required joint p.d.f is given by

$$= \frac{1}{(2\pi)^{\frac{3}{2}}\left(1 - \frac{1}{e^2}\right)} exp\left[-\frac{1}{2\left(1 - \frac{1}{e^2}\right)^2}\left\{\left(1 - \frac{1}{e^2}\right)x_1^2 - \frac{2}{e}\left(1 - \frac{1}{e^2}\right)x_1x_2 + \left(1 - \frac{1}{e^4}\right)x_2^2 - \frac{2}{e}\left(1 - \frac{1}{e^2}\right)x_2x_3 + \left(1 - \frac{1}{e^2}\right)x_3^2\right\}\right]$$

$i.e.,$

$$= \frac{1}{(2\pi)^{\frac{3}{2}}\left(1-\frac{1}{e^2}\right)} exp\left[-\frac{1}{2\left(1-\frac{1}{e^2}\right)}\left\{x_1^2 - \frac{2}{e}x_1x_2 + \left(1+\frac{1}{e^2}\right){x_2}^2 - \frac{2}{e}x_2x_3 + x_3^2\right\}\right]$$