2.5

Continuous Probability Distributions

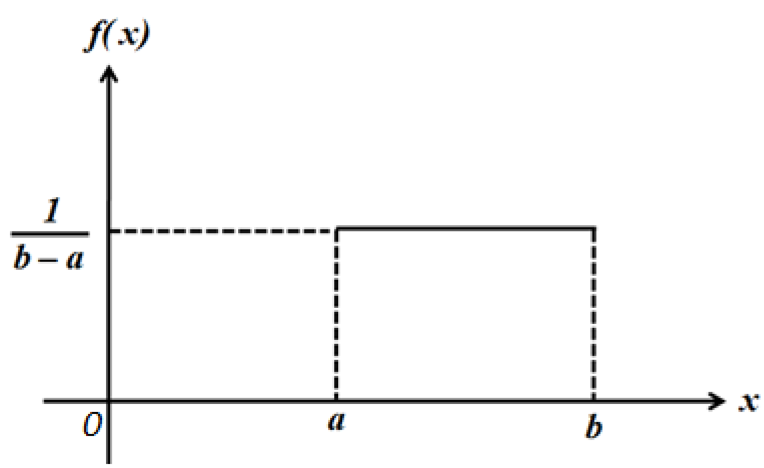
The **continuous probability distributions** are used in a number of applications in *engineering*. For example in *error analysis*, given a set of data or probability distribution, it is possible to estimate the probability that a measurement (temperature, pressure, flow rate) will fall within a desire range, and hence determine how reliable an instrument or piece of equipment is. Also, one can calibrate an instrument (ex. Temperature sensor) from the manufacturer on a regular basis and use a probability distribution to see of the variance in the instruments' measurements increases or decreases over time.

Uniform Distribution

A continuous random variable (c. r. v.) X is said to have a uniform distribution over the interval $[a, b]$ if its p. d. f. is given by

$$f(x) = \begin{cases} \dfrac{1}{b-a}, & a \leq x \leq b \\ 0, & otherwise \end{cases}$$

Notation: $\boldsymbol{X \sim U}(a, b)$. Read as X *follows uniform distribution with parameters* a *and* b. It is used to model events that are equally likely to occur at any time within a given time interval. The plot of p. d. f. is given below:



The cumulative distribution function (c. d. f.) of X is given by

$$F(x) = p(X \leq x) = \begin{cases} 0 \quad , & x < a \\ \dfrac{x-a}{b-a} \quad , & a \leq x \leq b \\ 1 \quad , & x \geq b \end{cases}$$

The mean of X is given by

$$\mu = E(X) = \int_{-\infty}^{\infty} x\, f(x)\, dx = \int_{a}^{b} x\, f(x)\, dx = \int_{a}^{b} \frac{x}{b-a}\, dx$$

$$= \left[\frac{x^2}{2(b-a)}\right]_a^b = \frac{b^2-a^2}{2(b-a)} = \frac{(b-a)(b+a)}{2(b-a)} = \frac{b+a}{2} \implies \mu = \frac{b+a}{2}$$

Now, $E(X^2) = \int_{-\infty}^{\infty} x^2\, f(x)\, dx = \int_{a}^{b} x^2\, f(x)\, dx = \int_{a}^{b} \frac{x^2}{b-a}\, dx$

$$= \left[\frac{x^3}{3(b-a)}\right]_a^b = \frac{b^3-a^3}{3(b-a)} = \frac{(b-a)(b^2+ab+a^2)}{3(b-a)} = \frac{b^2+ab+a^2}{3}$$

$$\implies E(X^2) = \frac{b^2+ab+a^2}{3}$$

Thus, the variance of X is given by $\sigma^2 = E(X^2) - \left(E(X)\right)^2$

$$= \frac{b^2+ab+a^2}{3} - \frac{b^2+2ab+a^2}{4}$$

$$= \frac{b^2-2ab+a^2}{12}$$

$$\implies \sigma^2 = \frac{(b-a)^2}{12}$$

Example 1: The time that a professor takes to grade a paper is uniformly distributed between 5 *minutes* and 10 *minutes*. Find the mean and variance of the time the professor takes to grade a paper.

Solution: Let X denotes the time the professor takes to grade a paper. Then $X \sim U(5, 10)$.

$$\mu = E(X) = \frac{10+5}{2} = 7.5 \text{ and } \sigma^2 = V(X) = \frac{(10-5)^2}{12} = \frac{25}{12}\ (minutes)^2$$

Normal Distribution

The normal distribution was first discovered by **De – Movire** and **Laplace** as the limiting form of Binomial distribution. Through a historical error it was credited to Gauss who first made reference to it as the distribution of errors in Astromy. Gauss used the normal curve to describe theory of accidental errors of measurements involved in the calculation of orbits of heavenly bodies.

Definition: A c. r. v. X is said to have a **normal distribution** with parameters μ and σ^2 if its p. d. f. is given by

$$\boldsymbol{f(x)} = \frac{\mathbf{1}}{\boldsymbol{\sigma\sqrt{2\pi}}} \boldsymbol{exp}\left\{-\frac{\mathbf{1}}{\mathbf{2}}\left(\frac{\boldsymbol{x-\mu}}{\boldsymbol{\sigma}}\right)^{\mathbf{2}}\right\}, -\infty < x < \infty, -\infty < \mu < \infty, \sigma > 0$$

The c. d. f. of X is given by

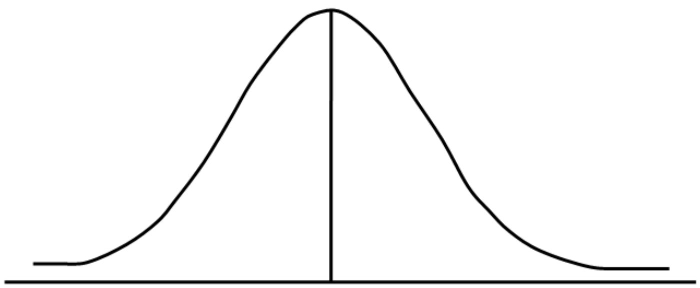
$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(t)\,dt = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{x} exp\left\{-\frac{1}{2}\left(\frac{t-\mu}{\sigma}\right)^2\right\} dt$$

Notation: $X \sim N(\mu, \sigma^2)$. *Read as X follows normal distribution with parameters μ and σ^2*.

Note:

1. The graph of $f(x)$ is famous ***bell – shaped*** curve and is symmetric about the line $X = \mu$. The top of the bell is directly above μ. For large values of σ, the curve tends

to flatten out and for small values of σ, it has a sharp peak. The curve of $f(x)$ is given below.



Normal probability curve

2. Whenever the random variable is continuous and the probabilities of it are increasing and then decreasing, in such cases we can think of using normal distribution.

Real life examples:

1) The heights of students.
2) The weights of students.
3) The diameters of bolts manufactured.
4) The lives of electrical bulbs manufactured.

3. Note that $\int_{-\infty}^{\infty} f(x)\,dx = 1$

Standard Normal distribution

If $X \sim N(\mu, \sigma^2)$ then $Z = \frac{X-\mu}{\sigma}$ is known as **standard normal distribution** with mean $E(Z) = 0$, with variance $V(Z) = 1$ and we write $Z \sim N(0,1)$.Its p. d. f. is given by

$$\boldsymbol{g(z) = \frac{1}{\sqrt{2\pi}}.\, exp\left(-\frac{1}{2}z^2\right)}, -\infty < z < \infty$$

and its c. d. f. is given by

$$\Phi(z) = P(Z \le z) = \int_{-\infty}^{x} g(t)\,dt = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} exp\left\{-\frac{1}{2}t^2\right\} dt$$

Area Property of Normal Distribution

If $X \sim N(\mu, \sigma^2)$, then $P(\mu < X < x_1) = \int_{\mu}^{x} f(x)\,dx = \frac{1}{\sigma\sqrt{2\pi}} \int_{\mu}^{x} exp\left\{-\left(\frac{x-\mu}{\sigma}\right)^2\right\} dx$

Let $Z = \frac{X-\mu}{\sigma}$. Then $X - \mu = \sigma Z$.

If $X = \mu$, then $Z = 0$. If $X = x_1$, then $Z = \frac{X-\mu}{\sigma} = z_1$ (say).

$$\therefore P(\mu < X < x_1) = P(0 < Z < z_1) = \int_{0}^{z_1} g(z)dz = \frac{1}{\sqrt{2\pi}} \int_{0}^{z_1} \exp\left\{-\frac{1}{2}z^2\right\} dz$$

where $g(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$ is the p. d. f. of standard normal variate. The definite integral $\int_{0}^{z_1} g(z)\,dz$ is known as **normal probability integral** and gives the area under standard normal curve between the ordinates at $z = 0$ and $z = z_1$. These areas have been tabulated for different values of z_1 at intervals of 0.01 in the table given at the ***end of the module.***

In particular, the probability that the random variable X lies in the interval $(\mu - \sigma, \mu + \sigma)$ is given by

$$P(\mu - \sigma < X < \mu + \sigma) = P(-1 < Z < 1)$$

$$= \int_{-1}^{1} g(z)\,dz$$

$$= 2\int_{0}^{1} g(z)\,dz \qquad \text{(by symmetry)}$$

$$= 2 \times 0.3413 \qquad \text{(from table)}$$

$$\Rightarrow P(\mu - \sigma < X < \mu + \sigma) = 0.6826$$

Similarly, $P(\mu - 2\sigma < X < \mu + 2\sigma) = P(-2 < Z < 2)$

$$= 2 \times P(0 < Z < 2) = 2 \times 0.4772 \text{(see table)}$$

$$\Rightarrow P(\mu - 2\sigma < X < \mu + 2\sigma) = 0.9544$$

and $P(\mu - 3\sigma < X < \mu + 3\sigma) = P(-3 < Z < 3)$

$$= 2 \times P(0 < Z < 3)$$

$$= 2 \times 0.49865 \qquad \text{(see table)}$$

$$\Rightarrow P(\mu - 3\sigma < X < \mu + 3\sigma) = 0.9973$$

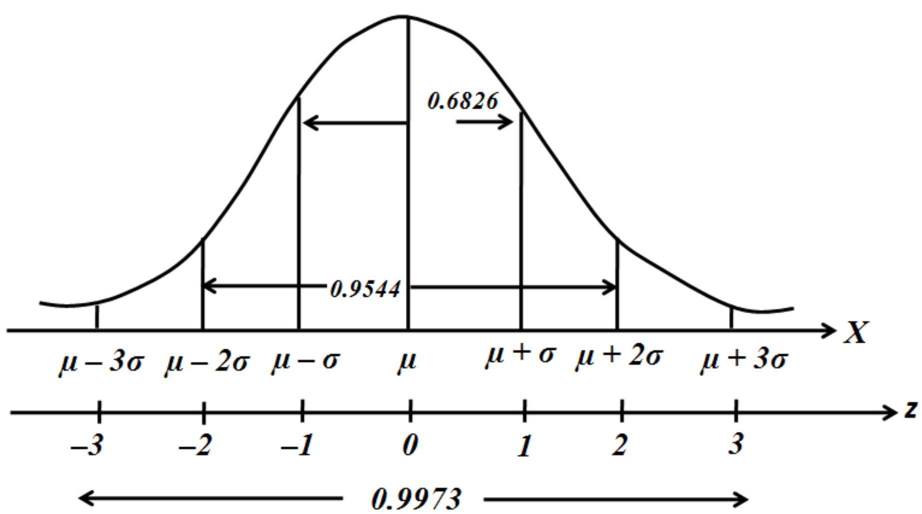
Thus, the probability that a normal variate X lies outside the range $\mu \pm 3\sigma$ is given by

$$P(|x - \mu > 3\sigma|) = P(|Z| > 3) = 1 - P(|Z| \leq 3)$$

$$= 1 - P(-3 \leq Z \leq 3) = 1 - 0.9973 = 0.0027$$

Thus, in all probability, we should expect a normal variate to lie within the range $\mu \pm 3\sigma$, though theoretically, it may range from $-\infty$ to ∞.

The probabilities computed above are exhibited in the following figure.



Note: The Gamma function defined below is used to evaluate mean and variance of the normal distribution.

$$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx \text{ for } n > 0$$

$$\Gamma(n+1) = n\,\Gamma(n)$$

$$\Gamma(n+1) = n!\text{, where } n \text{ is a positive integer.}$$

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

Mean of Normal distribution

$$E(X) = \int_{-\infty}^{\infty} xf(x)\,dx = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} x e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

Let $z = \frac{x-\mu}{\sigma}$.Then $x = \mu + \sigma z$,$dx = \sigma dz$ and

$$E(X) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (\mu + \sigma z) e^{-\frac{1}{2}z^2} dz$$

$$= \mu.\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}z^2} dz + \frac{\sigma}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z e^{-\frac{1}{2}z^2} dz = \mu \times 1 + 0$$

Note that the integral in first term is 1 since total probability is one and the integral in the second term in zero since the integral is an odd function.

Therefore, Mean $= E(X) = \mu$

Variance of Normal distribution

$$V(X) = E\big(X - E(X)\big)^2 = E(X-\mu)^2 \qquad (\because E(X) = \mu)$$

$$= \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} (x-\mu)^2 e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

Let $\frac{x-\mu}{\sigma} = z$. Then $x - \mu = \sigma z$, $dx = \sigma dz$ and

$$V(X) = \frac{\sigma^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z^2 e^{-\frac{1}{2}z^2} dz = \frac{2\sigma^2}{\sqrt{2\pi}} \int_{0}^{\infty} z^2 e^{-\frac{1}{2}z^2} dz$$

(Since the integrand is an even function)

Let $\frac{1}{2}z^2 = t \implies z = \sqrt{2t}$ and $dz = \frac{dt}{\sqrt{2t}}$. Then

$$V(X) = \frac{2\sigma^2}{\sqrt{2\pi}} \int_{0}^{\infty} 2t.e^{-t} \frac{dt}{\sqrt{2t}} = \frac{2\sigma^2}{\sqrt{\pi}} \int_{0}^{\infty} e^{-t}.t^{\frac{3}{2}-1} dt$$

$$= \frac{2\sigma^2}{\sqrt{\pi}} \Gamma\left(\frac{3}{2}\right) \qquad \text{(Gamma function)}$$

$$= \frac{2\sigma^2}{\sqrt{\pi}}.\frac{1}{2}.\Gamma\left(\frac{1}{2}\right) = \frac{\sigma^2}{\sqrt{\pi}}.\sqrt{\pi} = \sigma^2$$

$$\implies V(X) = \sigma^2$$

Note: Standard deviation $= \sqrt{V(X)} = \sqrt{\sigma^2} = \sigma$

Example 2: If X is normally distributed with mean 12 and standard deviation , then

(a) Find the probabilities of the following :

(i) $X \geq 20$

(ii) $X \leq 20$ and

(iii) $0 \leq X \leq 12$

(b) Find x when $P(X > x) = 0.24$

(c) Find x_1 and x_2 when $P(x_1 < X < x_2) = 0.5$ and $P(X > x_2) = 0.25$

Solution:

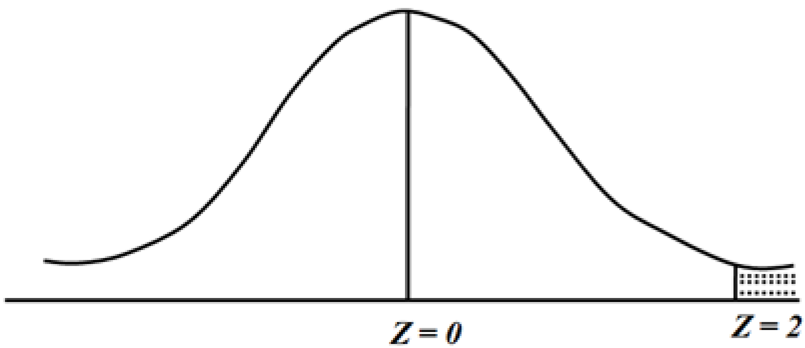
(a) it is given that $\mu = 12$ and $\sigma = 4$ $i.e.$, $X \sim N(12, 16)$

(i) Let $Z = \frac{X-12}{4}$. then $P(X \geq 20) = P\left(\frac{X-12}{4} \geq \frac{20-12}{4}\right)$

$= P(Z \geq 2) = 0.5 - P(0 \leq Z \leq 2)$

$= 0.5 - 0.4772$ (from table)

$= 0.0228$



(ii) $P(X \leq 20) = 1 - P(X \geq 20) = 1 - 0.0228 = 0.9722$

(iii) $P(0 \leq X \leq 12) = P\left(\frac{0-12}{4} \leq \frac{X-12}{4} \leq \frac{12-12}{4}\right)$

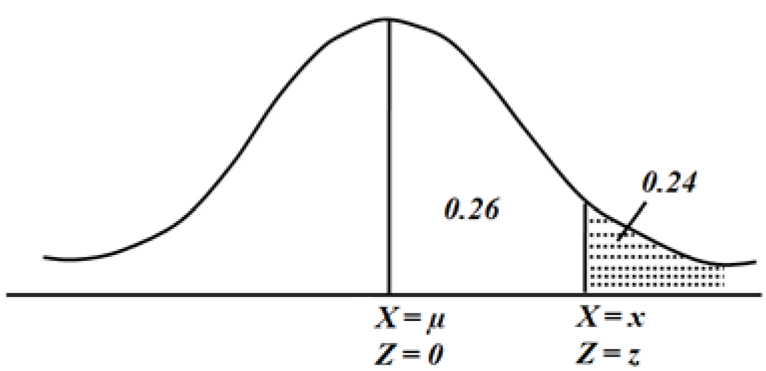
$= P(-3 \leq Z \leq 0)$

$= P(0 \leq Z \leq 3)$ (by symmetry)

$= 0.4986$ (from table)

(b) $P(X > x) = 0.24$

$\Rightarrow P\left(\frac{X-12}{4} > \frac{x-12}{4}\right) = 0.24 \Rightarrow P(Z > z) = 0.24$, where $z = \frac{x-12}{4}$



$\Rightarrow P(0 < Z < z) = 0.5 - 0.24 - 0.26$

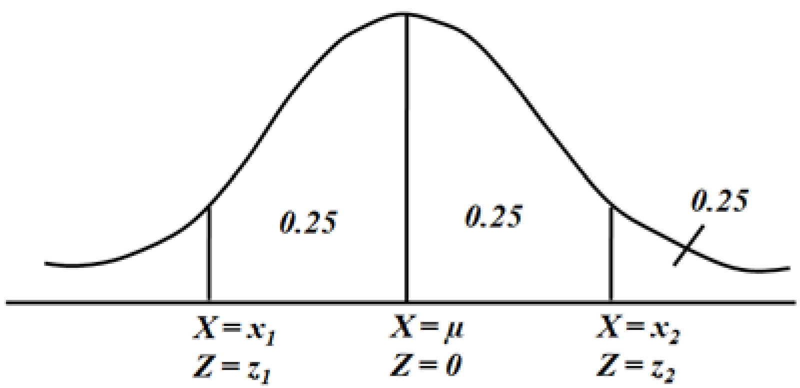
∴ From normal tables, corresponding to probability 0.26, value of $z = 0.71$(approximately)

Hence $0.71 = z = \frac{x-12}{4} \Longrightarrow x = 0.71 \times 4 + 12 = 14.84$

(c) We are given that $P(x_1 < X < x_2) = 0.5$ and $(X > x_2) = 0.25$

$\Longrightarrow P\left(\frac{x_1-12}{4} < \frac{X-12}{4} < \frac{x_2-12}{4}\right) = 0.5$ and $P\left(\frac{X-12}{4} > \frac{x_2-12}{4}\right) = 0.25$

$\Longrightarrow P(z_1 < Z < z_2) = 0.5$ and $P(Z > z_2) = 0.25$, where $z_1 = \frac{x_1-12}{4}$ and $z_2 = \frac{x_2-12}{4}$



By symmetry of normal curve, $z_1 = -z_2$. Find z_2 such that $P(0 < Z < z_2) = 0.25$

Corresponding to probability 0.25 from the normal table, we have $z_2 = 0.67$ approximately. Thus

$\frac{x_2-12}{4} = 0.67 \Longrightarrow x_2 = 12 + 4 \times 0.67 = 14.68$

Similarly, $z_1 = -z_2 \Longrightarrow \frac{x_2-12}{4} = -0.67 \Longrightarrow x_1 = 12 - 4 \times 0.67 = 9.32$

Example 3: The local authorities in a certain city install $10,000$ electric lamps in the streets of the city. If these lamps have an average life of $1,000$ burning hours with a standard deviation of 200 hours, assuming normality, what number of lamps might be expected to fail

(i) **in the first 800 and 1200 burning hours?**
(ii) **between 800 and 1200 burning hours?**

After what period of burning hours would you expect that

(a) **10% of the lamps would fail?**
(b) **10% of the lamps would be still burning?**

Solution:

Let X denote the life of a bulb in burning hours. Here $\mu = 1000,\ \sigma = 200$ and $X \sim N(1000, 40000)$

(i) Find $P(X < 800) = P\left(\frac{X-1000}{200} < \frac{800-1000}{200}\right)$

$= P(Z < -1)$, where $Z = \frac{X-1000}{200} \sim N(0,1)$

$= P(Z > 1) = 0.5 - P(0 < Z < 1)$

$= 0.5 - 0.3413 = 0.1587$

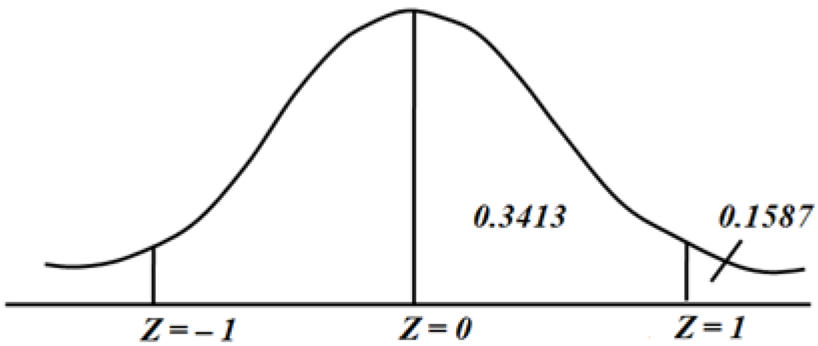
$\therefore$ Out of $10{,}000$ bulbs, number of bulbs which fail in the first 800 hours is

$10{,}000 \times 0.1587 = 1{,}587.$

(ii) Find $P(800 < X < 1200) = P\left(\frac{800-1000}{200} < \frac{X-1000}{200} < \frac{1200-1000}{200}\right)$

$= P(-1 < Z < 1) = 2.P(0 < Z < 1)$

$= 2 \times 0.3413 = 0.6826$



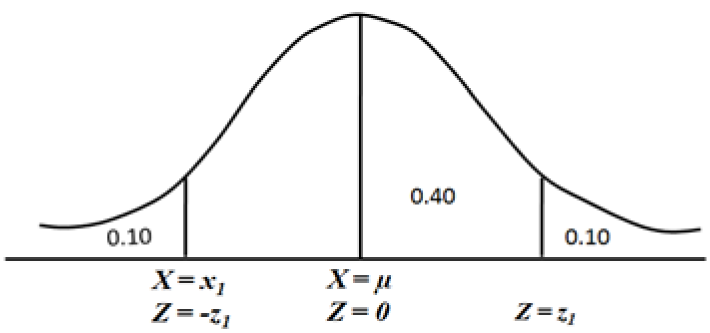
Hence, the expected number of bulbs with life between 800 and 1200 hours of burning life is $10{,}000 \times 0.6826 = 6{,}826$.

(a) Let 10% of the bulbs fail after x_1 hours of burning life. Then we have to find x_1 such that

$$P(X < x_1) = 0.10 \Rightarrow P\left(\frac{X-1000}{200} < \frac{x_1-1000}{200}\right) = 0.10$$
$$\Rightarrow P(Z < -z_1) = 0.10\text{ , where } z_1 = -\left(\frac{x_1-1000}{200}\right)$$
$$\Rightarrow P(Z > z_1) = 0.10$$
$$\Rightarrow P(0 < Z < z_1) = 0.5 - 0.10 = 0.40$$

From table corresponding to probability 0.40, we have

$$z_1 = 1.28 \Rightarrow -\left(\frac{x_1-1000}{200}\right) = 1.28$$
$$\Rightarrow x_1 = 1000 - 1.28 \times 200 = 1000 - 256 = 744.$$



Thus, after 744 hours of burning life, 10% of the bulbs will fail.

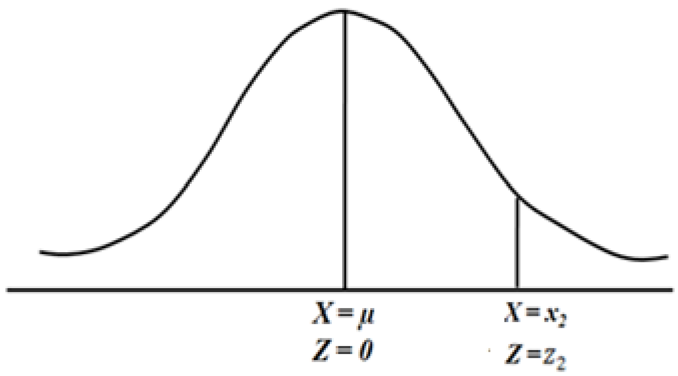
(b) Let 10% of the bulbs be still burning after x_2 hours of burning life. Then we have

$$P(X > x_2) = 0.10 \Rightarrow P\left(\frac{X-1000}{200} > \frac{x_2-1000}{200}\right) = 0.10$$

$\Rightarrow P(Z > z_2) = 0.10$,where $z_2 = \frac{x_2 - 1000}{200}$

From normal tables, $z_2 = 1.28$ and hence

$$\frac{x_2 - 1000}{200} = 1.28 \Rightarrow x_2 = 1000 + 1.28 \times 200 = 1000 + 256 = 1256$$



Hence, after 1,256 hours of burning life, 10% of the bulbs will be still burning.

De Moivre-Laplace Theorem (Normal Approximation to Binomial Distribution)

Let $X \sim B(n, p)$. Then its p.m.f. is given by $p(x) = \binom{n}{x} p^x q^{n-x}$ for $x = 0,1,2, \dots, n$. The mean and variance of X are given by $\mu = np$ and $\sigma^2 = npq$ respectively. Now,

$P(k_1 \le X \le k_2) = \sum_{x=k_1}^{k_2} \binom{n}{x} p^x q^{n-k}$ for some non-negative integers k_1 and k_2 such that $k_1 < k_2$. Since the binomial coefficient $\binom{n}{x}$ grows quite rapidly with n, it is very difficult to compute $P(k_1 \le X \le k_2)$ for large n. In this context, normal approximation to binomial distribution is extremely useful.

Let $Z = \frac{X - \mu}{\sigma} = \frac{X - np}{\sqrt{npq}}$. If n is large with neither p nor q close to zero, the binomial distribution can be approximated by the standard normal distribution. Thus,

$$\lim_{n \to \infty} P(k_1 \le X \le k_2) = \lim_{n \to \infty} P\left(\frac{k_1 - np}{\sqrt{npq}} \le Z \le \frac{k_2 - np}{\sqrt{npq}}\right) = \frac{1}{\sqrt{2\pi}} \int_{z_1}^{z_2} e^{-\frac{1}{2}z^2} dz$$

$$\text{where } z_1 = \frac{k_1 - np}{\sqrt{npq}} \text{ and } z_2 = \frac{k_2 - np}{\sqrt{npq}}$$

This is a very good approximation when both np and npq are greater than 5.

Example 4: A coin is tossed 10 times. Find the probability of getting between 4 and 7 heads inclusive using the (a) binomial distribution and (b) the normal approximation to the binomial distribution.

Solution:

(a) Let X denote the number of heads in 10 tosses. Then $X \sim B\left(10, \frac{1}{2}\right)$ and $\mu = np = 5$ and $\sigma^2 = npq = 2.5$ and

$$P(4 \le X \le 7) = \sum_{x=4}^{7} p(x) = \sum_{x=4}^{7} \binom{n}{x} \left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{10-x} = \sum_{x=4}^{7} \binom{n}{x} \left(\frac{1}{2}\right)^{10}$$

$$= \frac{\binom{10}{4} + \binom{10}{5} + \binom{10}{6} + \binom{10}{7}}{1024} = \frac{792}{1024} = 0.7734$$

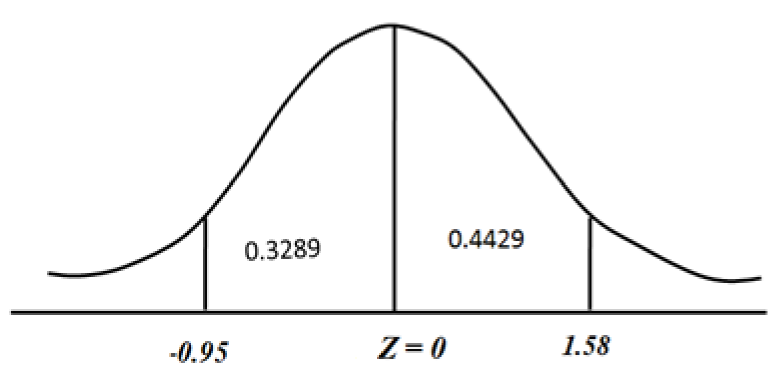
(b) The discrete binomial probability distribution is approximated to continuous normal probability distribution. The integers 4,5,6,7 lie in the interval (3.5 to 7.5). Thus,

$$P(4 \le X \le 7) = P(3.5 \le X \le 7.5) = P\left(\frac{3.5-5}{\sqrt{25}} \le Z \le \frac{7.5-5}{\sqrt{2.5}}\right)$$

$$= P(-0.95 \le Z \le 1.58) = P(-0.95 \le Z \le 0) + P(0 \le Z \le 1.58)$$

$$= P(0 \le Z \le 0.95) + P(0 \le Z \le 1.58)$$

$$= 0.3289 + 0.4429 = 0.7718$$



Exponential distribution: A c.r.v. X is said to follow **exponential distribution** with parameter λ if its p.m.f. is given by

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & , \quad x \geq 0 \\ 0 & , \quad x < 0 \end{cases}$$

The c.d.f. is given by

$$F(x) = P(X \leq x) = \int_0^x f(t)\,dt = \lambda \int_0^x e^{-\lambda t} dt = \lambda \left[\frac{e^{-\lambda t}}{-\lambda} \right]_0^x = 1 - e^{-\lambda x}$$

$$\Rightarrow F(x) = 1 - e^{-\lambda x}$$

Notation: $X \sim E(\lambda)$. Read as X *follows exponential distribution with parameter* λ.

Real life examples of exponential distribution

1. The time taken to serve a customer at a petrol pump, railway booking counter or any other service facility.
2. The period of time for which an electronic component operates without any breakdown.
3. The time between two successive arrivals at any service facility.

Mean and Variance of exponential distribution

For $r \geq 1$, $E(X^r) = \int_0^\infty x^r f(x)\,dx = \lambda \int_0^\infty x^r e^{-\lambda x} dx$

Let $\lambda x = t$. Then t varies between 0 to ∞ and $dx = \frac{dt}{\lambda}$. Then

$$E(X^r) = \lambda \int_0^\infty \left(\frac{t}{\lambda} \right)^r e^{-t} \frac{dt}{\lambda} = \frac{1}{\lambda^r} \int_0^\infty e^{-t} t^{(r+1)-1} dt$$

$$\Rightarrow E(X^r) = \frac{\Gamma(r+1)}{\lambda^r} = \frac{r!}{\lambda^r} \text{(using Gamma function)}$$

Thus, mean $= \mu = E(X) = \frac{1}{\lambda}$ and $E(X^2) = \frac{2!}{\lambda^2} = \frac{2}{\lambda^2}$

Hence $\sigma^2 = V(X) = E(X^2) - (E(X))^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$

Therefore, $\mu = \frac{1}{\lambda}$ and $\sigma^2 = \frac{1}{\lambda^2}$.

Example 5: Assume that the length of phone calls made at a particular telephone booth is exponentially distributed with a mean of 3 minutes. If you arrive at the telephone booth just as Ramu was about to make a call, find the following:

a. **The probability that you will wait more than 5 minutes before Ramu is done with the call.**
b. **The probability that Ramu's call will last between 2 minutes and 6 minutes.**

Solution: Let X be a r.v. that denotes the length of calls made at the telephone booth. Since the mean length of calls $\frac{1}{\lambda} = 3$, the p.d.f. is given by

$$f(x) = \frac{1}{3} e^{-\frac{x}{3}}$$

a. $P(X > 5) = \int_5^\infty f(x)dx = \frac{1}{3}\int_5^\infty e^{-\frac{x}{3}}dx = \left[e^{-\frac{x}{3}}\right]_5^\infty = e^{-\frac{5}{3}}$

b. $P[2 \le X \le 6] = \int_2^6 f(x)dx = \frac{1}{3}\int_2^6 e^{-\frac{x}{3}}dx = \left[-e^{-\frac{x}{3}}\right]_2^6 = e^{-\frac{2}{3}} - e^{-2}$

Memory lessness property of exponential distribution

The exponential distribution is used extensively in reliability engineering to model the lifetimes of systems. Suppose the life X of an equipment is exponentially distributed with a mean of $\frac{1}{\lambda}$. Assume that the equipment has not failed by time t. We want to find the probability that $X \le t + s$ given that $X > t$ for some nonnegative additional time s.

Thus,

$$P(X \le s + t | X > t) = \frac{P(X \le s+t, X > t)}{P(X > t)} = \frac{P(t < X \le s+t)}{P(X > t)} = \frac{F(s+t) - F(t)}{1 - F(t)}$$

$$= \frac{(1 - e^{-\lambda(s+t)}) - (1 - e^{-\lambda t})}{e^{-\lambda t}} = \frac{e^{-\lambda t} - e^{-\lambda(s+t)}}{e^{-\lambda t}} = 1 - e^{-\lambda s} = F(s) = P(X \le s)$$

$$\Rightarrow P(X \le s + t | X > t) = P(X \le s)$$

This indicates that the process only remembers the present and not the past.

Example 6: In example 5, Ramu, who is using the phone at the telephone booth, had already talked for 2 minutes before you arrived. According to the memory lessness property of the exponential distribution, the mean time until Ramu is done with the call is still 3 minutes. The random variable forgets the length of time the call had lasted before you arrived.

Relationship between exponential and Poisson distributions

Let λ denote the mean number of arrivals per unit of time, say per second. Then the mean number of arrivals in t seconds is λt.

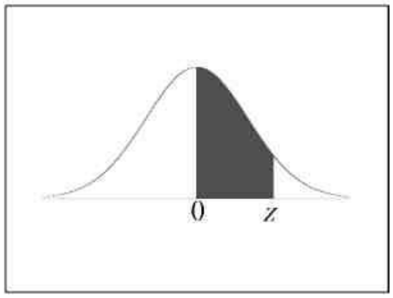
Let X denote the number of arrivals during an interval of t seconds.

Let Y denote the time between two successive arrivals.

If $X \sim P(\lambda t)$ i.e., $p(x) = P(X = x) = \frac{e^{-\lambda t}(\lambda t)^x}{x!}$ for $x = 0,1,2, \dots; \quad t \geq 0$, then

$Y \sim E(\lambda)$ i.e., $f(x) = \lambda e^{-\lambda t}$.

Standard Normal Distribution Table



z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.0000	.0040	.0080	.0120	.0160	.0199	.0239	.0279	.0319	.0359
0.1	.0398	.0438	.0478	.0517	.0557	.0596	.0636	.0675	.0714	.0753
0.2	.0793	.0832	.0871	.0910	.0948	.0987	.1026	.1064	.1103	.1141
0.3	.1179	.1217	.1255	.1293	.1331	.1368	.1406	.1443	.1480	.1517
0.4	.1554	.1591	.1628	.1664	.1700	.1736	.1772	.1808	.1844	.1879
0.5	.1915	.1950	.1985	.2019	.2054	.2088	.2123	.2157	.2190	.2224
0.6	.2257	.2291	.2324	.2357	.2389	.2422	.2454	.2486	.2517	.2549
0.7	.2580	.2611	.2642	.2673	.2704	.2734	.2764	.2794	.2823	.2852
0.8	.2881	.2910	.2939	.2967	.2995	.3023	.3051	.3078	.3106	.3133
0.9	.3159	.3186	.3212	.3238	.3264	.3289	.3315	.3340	.3365	.3389
1.0	.3413	.3438	.3461	.3485	.3508	.3531	.3554	.3577	.3599	.3621
1.1	. 3643	.3665	.3686	.3708	.3729	.3749	.3770	.3790	.3810	.3830
1.2	. 3849	.3869	.3888	.3907	.3925	.3944	.3962	.3980	.3997	.4015
1.3	.4032	.4049	.4066	.4082	.4099	.4115	.4131	.4147	.4162	.4177
1.4	.4192	.4207	.4222	.4236	.4251	.4265	.4279	.4292	.4306	.4319
1.5	.4 332	.4345	.4357	.4370	.4382	.4394	.4406	.4418	.4429	.4441
1.6	.4452	.4463	.4474	.4484	.4495	.4505	.4515	.4525	.4535	.4545
1.7	.4554	.4564	.4573	.4582	.4591	.4599	.4608	.4616	.4625	.4633
1.8	.4641	.4649	.4656	.4664	.4671	.4678	.4686	.4693	.4699	.4706
1.9	.4713	.4719	.4726	.4732	.4738	.4744	.4750	.4756	.4761	.4767
2.0	.4772	.4778	.4783	.4788	.4793	.4798	.4803	.4808	.4812	.4817
2.1	.4821	.4826	.4830	.4834	.4838	.4842	.4846	.4850	.4854	.4857
2.2	.4861	.4864	.4868	.4871	.4875	.4878	.4881	.4884	.4887	.4890
2.3	.4893	.4896	.4898	.4901	.4904	.4906	.4909	.4911	.4913	.4916
2.4	.4918	.4920	.4922	.4925	.4927	.4929	.4931	.4932	.4934	.4936
2.5	.4938	.4940	.4941	.4943	.4945	.4946	.4948	.4949	.4951	.4952

2.6	.4953	.4955	.4956	.4957	.4959	.4960	.4961	.4962	.4963	.4964
2.7	.4965	.4966	.4967	.4968	.4969	.4970	.4971	.4972	.4973	.4974
2.8	.4974	.4975	.4976	.4977	.4977	.4978	.4979	.4979	.4980	.4981
2.9	.4981	.4982	.4982	.4983	.4984	.4984	.4985	.4985	.4986	.4986
3.0	.4987	.4987	.4987	.4988	.4988	.4989	.4989	.4989	.4990	.4990
3.1	.4990	.4991	.4991	.4991	.4992	.4992	.4992	.4992	.4993	.4993
3.2	.4993	.4993	.4994	.4994	.4994	.4994	.4994	.4995	.4995	.4995
3.3	.4995	.4995	.4995	.4996	.4996	.4996	.4996	.4996	.4996	.4997
3.4	.4997	.4997	.4997	.4997	.4997	.4997	.4997	.4997	.4997	.4998
3.5	.4998	.4998	.4998	.4998	.4998	.4998	.4998	.4998	.4998	.4998