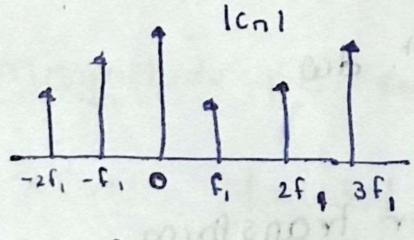
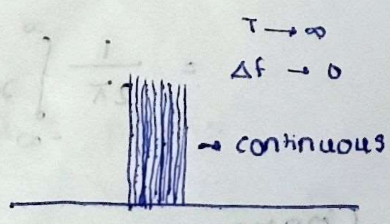
Unit-4 :- Forier Transform

Forier series can be generalised only for periodic signals. This limitation is overcome through forier transform which is applicable for both periodic and aperiodic, signals. And an aperiodic signal is considered as one which is periodic with an infinite period.

As the period increases the fundamental frequency decreases, which makes aperiodic sig-nal close in frequency, and the representation in the form of linear combination takes the form of integral rather than summation. This is called forier transform.





Discrete spectrum in forier series and continuo-us spectrum in forier transform.

* Derivation of forier transform from forier series

$$C_n = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\, e^{-jn\omega_0 t}\, dt$$

$T \to \infty \qquad n\omega_0 \to \omega$

$$T \cdot C_n = \int_{-T/2}^{T/2} x(t)\, e^{-jn\omega_0 t}\, dt$$

$$\boxed{x(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-jn\omega_0 t}\, dt}$$

Inverse forier transform:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} x(\omega)\, e^{j\omega t}\, d\omega$$

proof:- $x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$

$$= \frac{1}{T\cdot\omega_0}\sum_{n=-\infty}^{\infty} T C_n e^{jn\omega_0 t}\, \Delta\omega_0$$

$$= \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} (T\cdot C_n)\, e^{jn\omega_0 t}\, \Delta\omega_0$$

$T \to \infty$, $T C_n = X(\omega)$, $n\omega_0 = \omega$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} x(\omega)\, e^{j\omega t}\, d\omega$$

Convergence of forier transform:

1) Absolutely integrable

$$\int_{-\infty}^{\infty} |x(t)|\, dt < \infty$$

2) Signal must have finite no. of discontinuties and finite no. of minimas and maximas.

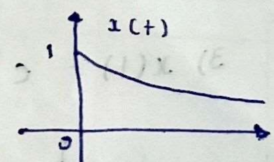
3) Signals must be energy signals on stable signal.

4) we can calculate forier transforms for power Signals approximation to energy signals.

Forier transform of basic signals:-

1) $x(t) = e^{-at} u(t)$

$$X(\omega) = \int_{-\infty}^{\infty} e^{-at} u(t) e^{-j\omega t} dt$$



$$= \int_{0}^{\infty} e^{-at} e^{-j\omega t} dt$$

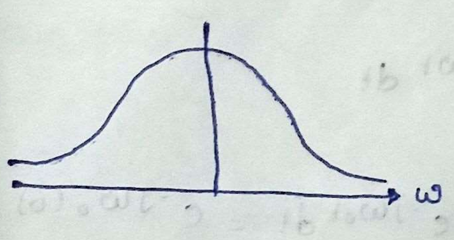
$$= \left[\frac{e^{-(a+j\omega)t}}{-(a+j\omega)t} \right]_0^{\infty} = \frac{1}{a+j\omega}$$

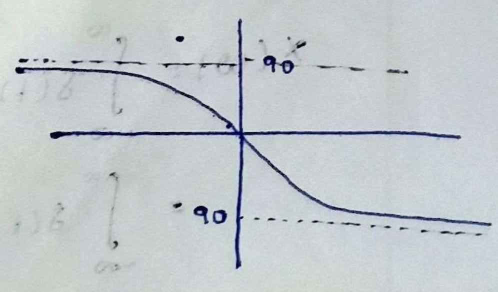
$$e^{-at} u(t) \xleftrightarrow{F.T} \frac{1}{a+j\omega}$$

Magnitude plot and phase plot:-

$X(\omega) = |x(\omega)| \angle x(\omega)$

$|x(\omega)| = \frac{1}{\sqrt{a^2+\omega^2}}$ & $\angle x(\omega) = -\tan^{-1}\left(\frac{\omega}{a}\right)$





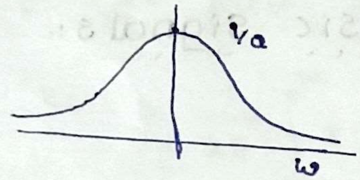
2) $x(t) = e^{at} u(-t)$

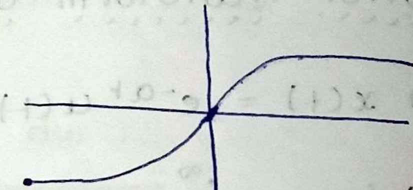
$$X(\omega) = \int_{-\infty}^{0} e^{at} e^{-j\omega t}\, dt = \left[\frac{e^{(a-j\omega)t}}{a-j\omega} \right]_{-\infty}^{0}$$

$$= \frac{1}{a-j\omega} \quad \left\{ \frac{1}{a-j\omega} \times \frac{a+j\omega}{a+j\omega} = \frac{a+j\omega}{a^2+\omega^2} \right\}$$

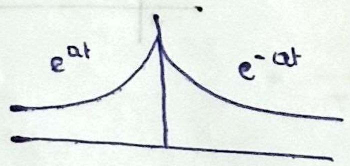
$$X(\omega) = |X(\omega)| \angle X(\omega)$$

$$|X(\omega)| = \frac{1}{\sqrt{a^2+\omega^2}} \qquad \angle X(\omega) = \tan^{-1}\left(\frac{\omega}{a}\right)$$





3) $x(t) = e^{-a|t|}$



$$|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

$$X(\omega) = \int_{0}^{\infty} e^{-at} e^{-j\omega t}\, dt = \left[\frac{e^{-(a+j\omega)t}}{-(a+j\omega)} \right]_{0}^{\infty} = \frac{1}{a+j\omega}$$

$$X(\omega) = \int_{-\infty}^{0} e^{at} e^{-j\omega t}\, dt = \left[\frac{e^{(a-j\omega)t}}{(a-j\omega)} \right]_{-\infty}^{0} = \frac{1}{a-j\omega}$$

$$\rightarrow \frac{1}{a+j\omega} + \frac{1}{a-j\omega} = \frac{2a}{a^2+\omega^2}$$

④ Forier transform of $\delta(t)$.

$$X(\omega) = \int_{-\infty}^{\infty} \delta(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} \delta(t-t_0)\, e^{-j\omega_0 t}\, dt = e^{-j\omega_0 (0)} = 1$$

⑤ Find forier transform of $\delta(t)$

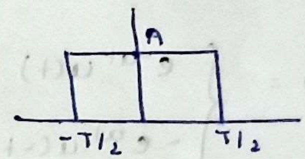
$$X(\omega) = \int_{-\infty}^{\infty} \delta(t)\, e^{-j\omega_0 t}\, dt$$

$$= \int_{-\infty}^{\infty} \delta(t-t_0)\, e^{-j\omega_0 t}\, dt = e^{-j\omega_0 (0)} = 1$$

⑤ Find forier transform of $A\,\mathrm{rect}\,(t/T)$

$$X(\omega) = \int_{-T/2}^{T/2} A\, e^{-j\omega t}\, dt$$

$x(\omega)$



$$= A\left[\frac{e^{-j\omega t}}{-j\omega}\right]_{-T/2}^{T/2}$$

$$\left\{ \mathrm{Sinc}\,x = \frac{\sin \pi x}{\pi x} \qquad Sa(x) = \frac{\sin x}{x} \right\}$$

$$= \frac{-A}{j\omega}\left[e^{-j\omega(T/2)} - e^{j\omega(T/2)}\right]$$

$$= \frac{A}{j\omega}\left[e^{j\omega(T/2)} - e^{-j\omega(T/2)}\right] \qquad \left\{ \frac{e^{j\theta} - e^{-j\theta}}{2j} = \sin\theta \right\}$$

$$= \frac{2A}{\omega}\left[\sin\left[\frac{\omega T}{2}\right]\right]$$

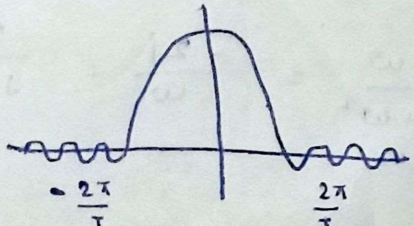
$$= \frac{T}{2}(2A)\left[\frac{\sin(\omega T/2)}{\omega (T/2)}\right]$$

$$= AT\, Sa\left(\frac{\omega T}{2}\right) \quad \text{or} \quad AT\left[\frac{\sin(2\pi f \frac{T}{2})}{2\pi f T/2}\right]$$

$$= \mathrm{Sinc}(fT)$$

* $A\,\mathrm{rect}(t/T) \longrightarrow AT\, Sa(\omega T/2)$ or $AT \sin(f\pi)$

magnitude



$$B.\omega = \frac{4\pi}{T}$$

⑦ Inverse forier transform of $\delta(\omega)$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\omega)\, e^{j\omega t}\, d\omega = \frac{1}{2\pi}[e^0] = \frac{1}{2\pi}$$

$$\delta(\omega) \underset{FT}{\overset{IFT}{\rightleftarrows}} \frac{1}{2\pi}$$

$$\boxed{1 \xrightarrow{F.T} 2\pi\,\delta(\omega)}$$

$$2\pi\,\delta(\omega) \xleftarrow{FT} 1$$

⑧ $$x(t) = \begin{cases} e^{-at}u(t) & 0 < t < \infty \\ -e^{at}u(-t) & -\infty < t < 0 \end{cases}$$

$$X(\omega) = \int_0^{\infty} e^{-at}\, e^{-j\omega t}\, dt$$
$x(t)$

$$= \left[\frac{e^{-(a+j\omega)t}}{-(a+j\omega)}\right]_0^{\infty} = \frac{1}{a+j\omega}$$

$$X(\omega) = \int_{-\infty}^{0} e^{-at}\, e^{-j\omega t}\, dt = -\left[\frac{e^{(a-j\omega)t}}{a-j\omega}\right]_{-\infty}^{0}$$
$x(t)$

$$= \frac{-1}{a-j\omega}$$

$$X(\omega) = \frac{1}{a+j\omega} - \frac{1}{a-j\omega} = \frac{-2j\omega}{a^2+\omega^2}$$
$x(t)$

$$\left.\begin{aligned} \lim_{a\to 0} x(t) &= \lim_{a\to 0} e^{-at}u(t) = 1 \\ \lim_{a\to 0} x(t) &= \lim_{a\to 0} -e^{-at}u(-t) = -1 \end{aligned}\right\} sgn(t)$$

$$X(\omega)\big|_{sgn(t)} = \lim_{a\to 0} X(\omega)$$

$$= \lim_{a\to 0} \frac{-2j\omega}{a^2+\omega^2} = \frac{-2j}{\omega} = \frac{2}{j\omega}$$

$$sgn(t) \xleftrightarrow{F.T} \frac{2}{j\omega}$$

* Forier transform of $u(t)$.

$$sgn(t) = 2u(t) - 1$$

$$u(t) = \frac{1}{2}[1 + sgn(t)]$$

$$= \frac{1}{2} + \frac{1}{2} sgn(t)$$

$$= \pi\delta(\omega) + \frac{1}{2}\frac{2}{j\omega}$$

$$= \pi\delta(\omega) = \frac{1}{j\omega}$$

$$\boxed{u(t) \xrightarrow{F.T} \frac{1}{j\omega} + \pi\delta(\omega)}$$

Area under property :-

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \qquad \{\omega = 0\}$$

$$X(0) = \int_{-\infty}^{\infty} x(t)(1) dt$$

$$X(0) = \int_{-\infty}^{\infty} x(t) dt$$

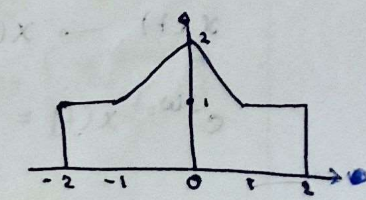
In F.T the area under the graph is at $x = 0$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega \qquad t \to 0$$

$$x(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega) d\omega$$

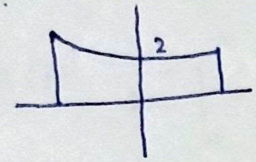
$$\boxed{2\pi x(0) = \int_{-\infty}^{\infty} X(\omega) d\omega}$$

① Area under the graph



$$\int_{-\infty}^{\infty} X(\omega) d\omega = 2\pi x(0) = 2\pi(2) = 4\pi$$

Area under time domain



$$\int_{-\infty}^{\infty} x(t)\,dt = X(0) = 2$$

Properties :-

i) Linearity :-

$x(t) \rightarrow X(\omega)$

$y(t) \rightarrow Y(\omega)$

$ax(t) + by(t) \longrightarrow aX(\omega) + bY(\omega)$

(ii) Time reversal

$x(t) \xrightarrow{FT} X(\omega)$

$x(-t) \rightarrow X(-\omega)$

(iii) Time shifting

$x(t) \rightarrow X(\omega)$ $\qquad Y(\omega-\omega_0)$

$x(t-t_0) \rightarrow X(\omega)e^{-j\omega t_0}$ $\qquad e^{-j\omega t_0}$

$$X(\omega) = \int_{-\infty}^{\infty} x(t-t_0)\,e^{-j\omega_0 t}\,dt$$

$$= \int x(T)\,e^{-j\omega_0(T+t)}\,dT \qquad \left\{ \begin{array}{l} t-t_0 = T \\ dt = dT \end{array} \right.$$

$$= e^{-j\omega_0 t_0} \int_{-\infty}^{\infty} x(T)\,e^{-j\omega_0 T}\,dT$$

$$= e^{-j\omega_0 t_0}\,X(\omega)$$

(iv) Frequency shifting

$x(t) \rightarrow X(\omega)$

$e^{j\omega_0 t}x(t) = X(\omega-\omega_0)$

$$X(\omega) = \int_{-\infty}^{\infty} e^{j\omega_0 t} \cdot e^{-j\omega_0 t} (x(t))\, dt$$

$$X(\omega - \omega_0) = \int_{-\infty}^{\infty} e^{-j\omega t} (x(t))\, dt \; e^{j\omega_0 t}$$

$$= X(\omega) e^{j\omega_0 t}$$

$$\therefore \; = X(\omega - \omega_0)$$

Q Find forier transform of $\cos \omega_0 t$ & $\sin \omega_0 t$

A $x(t) = \cos \omega_0 t$

$$\cos \omega_0 t = \frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}$$

$$\cancel{= \frac{1}{2} \int_{-\infty}^{\infty} \left[e^{j\omega_0 t} + e^{-j\omega_0 t} \right] e^{-j\omega_0 t}\, dt}$$

$$\cancel{= \frac{1}{2} \left[\int_{-\infty}^{\infty} e^{j\omega_0 t} \cdot e^{-j\omega_0 t} + \int_{-\infty}^{\infty} e^{-j\omega_0 t} \cdot e^{-j\omega t}\, dt \right]}$$

$$\cancel{= \frac{1}{2} \int_{-\infty}^{\infty} e^{-(\omega - \omega_0) j t}\, dt + \frac{1}{2} \int_{-\infty}^{\infty} e^{-(\omega + \omega_0) j t}\, dt}$$

$\cancel{x(t) = \sin \omega_0 t,}$

$$F.T(\cos \omega_0 t) = FT\left(\frac{e^{j\omega_0 t}}{2} \right) + FT\left(\frac{e^{-j\omega_0 t}}{2} \right)$$

$$= \frac{1}{2} \left[2\pi \delta(\omega - \omega_0) \right] + \frac{1}{2} \left[2\pi \delta(\omega + \omega_0) \right]$$

$$F.T(\cos \omega_0 t) = \pi \left[\delta(\omega - \omega_0) + \delta(\omega + \omega_0) \right]$$

$$\boxed{2\pi \delta(\omega - \omega_0) \xleftrightarrow{IFT} e^{j\omega_0 t}}$$

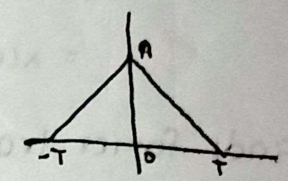
$$\sin \omega_0 t = \frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j}$$

$$FT(\sin \omega_0 t) = FT\left[\frac{e^{j\omega_0 t}}{2j} \right] - FT\left[\frac{e^{-j\omega_0 t}}{2j} \right]$$

Similarly, we get after solving $\frac{\pi}{j}[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)]$

* $\sin\omega_0 t \longrightarrow \frac{\pi}{j}[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)]$.
* $\cos\omega_0 t \longrightarrow \pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]$

6) $A\,Tri(t/T)$



Line equation

$(-T,0)\ (0,A)$

$(T,0)\quad(0,A)$

$\rightarrow x(t)-0 = \left[\frac{A-0}{0+T}\right](x+T)$

$x(t) = \frac{A}{-T}(t-T)$

$\rightarrow x(t) = \frac{A}{T}(t+T)$

$$x(\omega) = \int_{-T}^{0}\frac{A}{T}(t+T)e^{-j\omega t}dt + \int_{0}^{T}\frac{-A}{T}(t-T)e^{-j\omega t}dt$$

$$= \int_{-T}^{0}\left[\frac{Ate^{-j\omega t}}{T}+Ae^{-j\omega t}\right]dt + \int_{0}^{T}\left[\frac{-Ate^{-j\omega t}}{T}+Ae^{-j\omega t}\right]dt$$

$$= \frac{A}{T}\left[\frac{-te^{-j\omega t}}{j\omega}+\int\frac{e^{-j\omega t}}{j\omega}dt\right] + A\int_{-T}^{0}e^{-j\omega t}dt$$

$$= \frac{A}{T}\left[\frac{-te^{-j\omega t}}{j\omega}-\frac{e^{-j\omega t}}{(j\omega)^2}\right]_{-T}^{0} - A\left[\frac{e^{-j\omega t}}{j\omega}\right]_{-T}^{0}$$

$$= \frac{A}{T}\left[0-\frac{1}{(j\omega)^2}-\frac{Te^{j\omega T}}{j\omega}+\frac{e^{j\omega T}}{(j\omega)^2}\right]-\frac{A}{j\omega}+\frac{Ae^{j\omega T}}{j\omega}$$

$$= \frac{-A}{(j\omega)^2T}-\frac{Ae^{j\omega T}}{j\omega}+\frac{Ae^{j\omega T}}{T(j\omega)^2}-\frac{A}{j\omega}+\frac{Ae^{j\omega T}}{j\omega}$$

$$= \frac{-A}{(j\omega)^2T}+\frac{Ae^{j\omega t}}{T(j\omega)^2}-\frac{A}{j\omega}$$

$\{A^2T^2\,\mathrm{sinc}(tT)\}$

$$= \int_0^T \frac{-A}{T}(t-T)e^{-j\omega t}\,dt$$

$$= -\int_0^T \left[\frac{Ate^{-j\omega t}}{T} - Ae^{-j\omega t}\right]dt$$

$$= -\left[\frac{A}{T}\left[\frac{te^{-j\omega t}}{-j\omega} + \int \frac{e^{-j\omega t}}{j\omega}\,dt\right] - A\left[\frac{e^{-j\omega t}}{-j\omega}\right]\right]_0^T$$

$$= \left[\frac{Ate^{-j\omega t}}{j\omega T} + \frac{Ae^{-j\omega t}}{T(j\omega)^2} - \frac{Ae^{-j\omega t}}{j\omega}\right]_0^T$$

$$= \frac{Ae^{-j\omega t}}{j\omega} + \frac{Ae^{-j\omega T}}{T(j\omega)^2} - \frac{Ae^{-j\omega T}}{j\omega} - \frac{A(1)}{T(j\omega)^2} + \frac{A}{j\omega}$$

$$= \frac{Ae^{-j\omega T}}{T(j\omega)^2} - \frac{A}{T(j\omega)^2} + \frac{A}{j\omega}$$

$$X(\omega) = \frac{-2A}{(j\omega)^2 T} + \frac{A}{T(j\omega)^2}\left[e^{j\omega t} + e^{-j\omega t}\right]$$

$$= \frac{-2A}{(j\omega)^2 T} + \frac{A}{T(j\omega)^2}(2\cos\omega t)$$

$$= \frac{2A}{T(j\omega)^2}[\cos\omega t - 1]$$

$$= \frac{2A}{T\left(j^2 \frac{4\pi^2}{T^2}\right)}[\cos\omega t - 1]$$

$$= \frac{-A}{2\pi^2}[\cos\omega t - 1]$$

Q Find the inverse forier transform of $\delta(\omega - \omega_0)$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} x(\omega)e^{j\omega t}\,d\omega$$

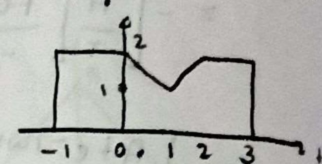
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\omega - \omega_0)e^{j\omega t}\,d\omega$$

$$\frac{1}{2\pi}\left[e^{j\omega_0 t}\right] = \frac{e^{j\omega_0 t}}{2\pi}$$

$$\delta(\omega-\omega_0) \xleftrightarrow{IFT} \frac{1}{2\pi} e^{j\omega_0 t}$$

2) Find the value of $X(0)$ for the following signal

$x(t)$



$\rightarrow$ $X(\omega) = \int_{-\infty}^{\infty} x(t)\,dt$ i.e area

$= 4(2) - \frac{1}{2}(2)(1)$

$= 8 - 1$

$= 7$

Duality property :-

$x(t) \xrightarrow{F} X(\omega)$

$X(t) \rightarrow 2\pi\, x(-\omega)$ $\quad x(-t) \quad x(-\omega)$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega$$

$$x(-t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{-j\omega t}\, d\omega$$

$$2\pi\, x(-t) = \int_{-\infty}^{\infty} X(\omega)\, e^{-j\omega t}\, d\omega$$

$$2\pi\, x(-\omega) = \int_{-\infty}^{\infty} X(t)\, e^{-j\omega t}\, dt$$

$$2\pi\, x(-\omega) = X(t)$$

1) Find the forier transform of $\frac{1}{4-jt}$

A. $e^{at}u(-t) \rightarrow \frac{1}{a-j\omega}$

$$x(t) \longrightarrow X(\omega)$$

$$\frac{1}{a-jt} \longrightarrow 2\pi X(-\omega)$$

$$\longrightarrow 2\pi e^{-a\omega} u(\omega)$$

$$\frac{1}{4-jt} \longrightarrow 2\pi e^{-4\omega} u(\omega)$$

2) Find the inverse forier transform of $X(\omega) = \frac{\omega^2+9}{\omega^2+12}$.

A $\frac{\omega^2+12}{\omega^2+9} = 1 + \frac{3}{\omega^2+9}$

$$X(\omega) = 1 + \frac{3}{\omega^2+9}$$

$$= \text{IFT}\{1\} + \text{IFT}\left\{\frac{3}{\omega^2+9}\right\}$$

$$\left[e^{-a|t|} \longrightarrow \frac{2a}{a^2+\omega^2}\right]$$

$$= 2\pi\delta(t) + \text{IFT}\left\{\frac{3}{\omega^2+9}\right\}$$

$$= 2\pi\delta(t) + \text{IFT}\left\{\frac{2(3)}{2(\omega^2+3^2)}\right\}$$

$$= 2\pi\delta(t) + \frac{1}{2}e^{-3|t|}$$

3) Find the forier transform of $\frac{1}{t^2+16}$,

A $x(t) = \frac{1}{t^2+4^2}$

$$e^{-a|t|} \longrightarrow \frac{2a}{a^2+\omega^2}$$

$$e^{-4|t|} \longrightarrow \frac{8}{\omega^2+16}$$

$$\frac{1}{8}e^{-4|t|} \longrightarrow \frac{1}{\omega^2+16}$$

$$\frac{1}{t^2+16} \longrightarrow 2\pi \frac{1}{8} e^{-4|-\omega|} = \frac{\pi}{4} e^{-4|\omega|}$$

5) Time scaling property:

$$x(t) \longrightarrow X(\omega)$$

$$x(at) \longrightarrow \frac{1}{|a|} X\left(\frac{\omega}{a}\right)$$

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty} x(at) e^{-j\omega t} dt$$

$at = \lambda$ $\quad = \int_{-\infty}^{\infty} x(\lambda) e^{-j\omega\left(\frac{\lambda}{a}\right)} \cdot \frac{d\lambda}{a}$

$a\,dt = d\lambda$

$dt = \frac{d\lambda}{a}$ $\quad = \frac{1}{a}\int_{-\infty}^{\infty} x(\lambda) e^{\frac{-j\omega\lambda}{a}} d\lambda = \frac{1}{|a|} x\left(\frac{\omega}{a}\right)$

6) Differentiation property

$$x(t) \longrightarrow X(\omega)$$

$$\frac{d}{dt} x(t) \longrightarrow j\omega X(\omega)$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$$

$$\frac{d}{dt} x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} x(\omega) \frac{d}{dt} e^{j\omega t} d\omega$$

$$\frac{d}{dt} x(t) = \frac{j\omega}{2\pi}\int_{-\infty}^{\infty} x(\omega) e^{j\omega t} d\omega$$

$$= j\omega \; FT\{x(t)\}$$

$$FT\left\{\frac{d}{dt} x(t)\right\} = j\omega X(\omega)$$

7) <u>Conjugate property</u>:

$$x(t) \longrightarrow X(\omega)$$

$$a-ib$$
$$a-ib$$

$$x^*(t) \longrightarrow X^*(-\omega) \qquad X^*(-j\omega)$$

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$X(-\omega) = \int_{-\infty}^{\infty} x(t)\, e^{+j\omega t}\, dt$$

$$X^*(-\omega) = \left[\int_{-\infty}^{\infty} x(t)\, e^{j\omega t}\, dt\right]^*$$

$$= \int_{-\infty}^{\infty} x^*(t)\, e^{-j\omega t}\, dt$$

$$= FT\left[x^*(t)\right]$$

8) <u>Integration property</u>:

$$\int_{-\infty}^{t} x(t)\,dt = \frac{X(\omega)}{j\omega} + \pi X(0)\,\delta(\omega)$$

$$x(t) * u(t) = \int_{-\infty}^{\infty} x(\tau)\, u(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{t} x(\tau)\, d\tau \qquad \begin{array}{l} t-\tau > 0 \\ t > \tau \end{array}$$

$$\int_{-\infty}^{t} x(\tau)\, d\tau = x(t) * u(t)$$

$$FT\left\{\int_{-\infty}^{t} x(\tau)\, d\tau\right\} = FT\left\{x(t) * u(t)\right\}$$

$$= X(\omega) \cdot FT\left\{u(t)\right\}$$

$$= X(\omega)\left[\frac{1}{j\omega} + \pi\delta(\omega)\right]$$

$$= \frac{X(\omega)}{j\omega} + \pi X(\omega)\,\delta(\omega)$$

From product property $= \frac{X(\omega)}{j\omega} + \pi X(0)\delta(\omega)$

9) Convolution property :-

$x(t) \longrightarrow X(\omega)$

$y(t) \longrightarrow Y(\omega)$

$x(t) * y(t) \longrightarrow X(\omega)Y(\omega)$

$$x(t) * y(t) = \int_{-\infty}^{\infty} x(T)\, y(t-T)\, dT$$

$$FT\{x(t) * y(t)\} = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x(T) y(t-T) e^{-j\omega t} dT\, dt \right]$$

$t - T = \lambda$ $\rightarrow dt = d\lambda$

$t = \lambda + T$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x(T)\, y(\lambda)\, e^{-j\omega(\lambda + T)}\, d\lambda \cdot dT$$

$$= \int_{-\infty}^{\infty} x(T) e^{-j\omega T} dT \int_{-\infty}^{\infty} y(\lambda) e^{-j\omega\lambda} d\lambda$$

$= X(\omega) Y(\omega)$

10) Derivative in frequency domain or multiplication with 't' in time domain.

$x(t) \rightarrow X(\omega)$

$t \cdot x(t) \longrightarrow j \frac{d}{d\omega} X(\omega)$

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$\frac{d}{d\omega} X(\omega) = \int_{-\infty}^{\infty} x(t) \frac{d}{d\omega} e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} x(t)\,(-jt)\, e^{-j\omega t}\, dt$$

$$= -j\int_{-\infty}^{\infty} t \cdot x(t)\, e^{-j\omega t} dt = \frac{1}{j}\, FT\{t \cdot x(t)\}$$

$$= j \frac{d}{d\omega} X(\omega) = t \cdot x(t).$$

ii) Convolution in frequency domain.

$x(t) \to X(\omega)$

$y(t) \to Y(\omega)$

$$x(t) \cdot y(t) \to \frac{1}{2\pi}[X(\omega) * Y(\omega)]$$

$$FT\{x(t) \cdot y(t)\} = \int_{-\infty}^{\infty} x(t)\, y(t)\, e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty} \frac{1}{2\pi} \int X(\omega')\, e^{j\omega' t} d\omega'\; y(t)\, e^{-j\omega t} dt$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} X(\omega')\, y(t)\, e^{j(\omega' - \omega) t} dt\, d\omega'$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega') \int_{-\infty}^{\infty} y(t)\, e^{-j(\omega - \omega') t} dt\, d\omega'$$

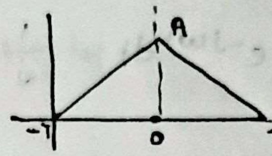
$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega')\, Y(\omega - \omega')\, d\omega'$$

$$= \frac{1}{2\pi}[X(\omega) * Y(\omega)]$$

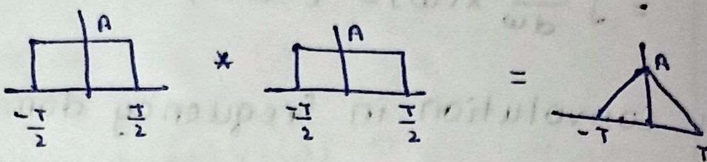
2 Parsvels theorem

$$\int_{-\infty}^{\infty} |x(t)|^2 dt \to \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 d\omega$$

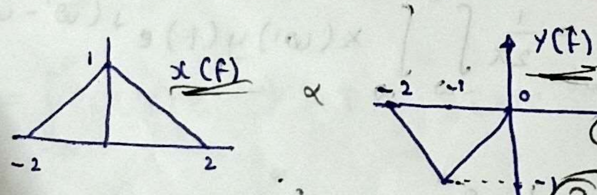
Q. Find the forier transform of $A\,tri(t/T)$ using properties



A. we have



2) Let $y(t)$ and $x(t)$ F.T of $X(f)$, $Y(f)$ as shown in fig. Then find $y(f)$ in terms of $x(f)$



① → $X(f)$ → $y(t)$

② → $Y(f)$ → $x(t)$

$y(t) \rightarrow x(f)$

$x(t) \rightarrow y(f)$

$x(at) \rightarrow \frac{1}{|a|} X(\omega)$

$x(t-t_0) \rightarrow e^{-j\omega t_0} X(\omega)$

$-x(2(t+1)) \rightarrow y(t)$

$Y(f) \rightarrow -\frac{1}{2} X(f/2)\, e^{j2\pi f \cdot 1}$

3) Find IFT of $X(3f+2)$ is where FT of $x(t)$ is $x(f)$

$x(t) \longrightarrow X(f)$

$x(3f+2) \longrightarrow$

$x\left(3\left(f+\frac{2}{3}\right)\right) \longrightarrow \frac{1}{3}x\left(\frac{t}{3}\right)e^{j2\pi f\, 2/3}$

$= \frac{1}{3}x(t/3)\, e^{j4\pi f/3}$

Find the forier transform of $x(5t-3)$ if $x(t) \xrightarrow{FT} X(\omega)$

$x\left(5\left(t-\frac{3}{5}\right)\right) = \frac{1}{5}x(t/5)e^{-j2\pi f\left(\frac{3}{5}\right)}$

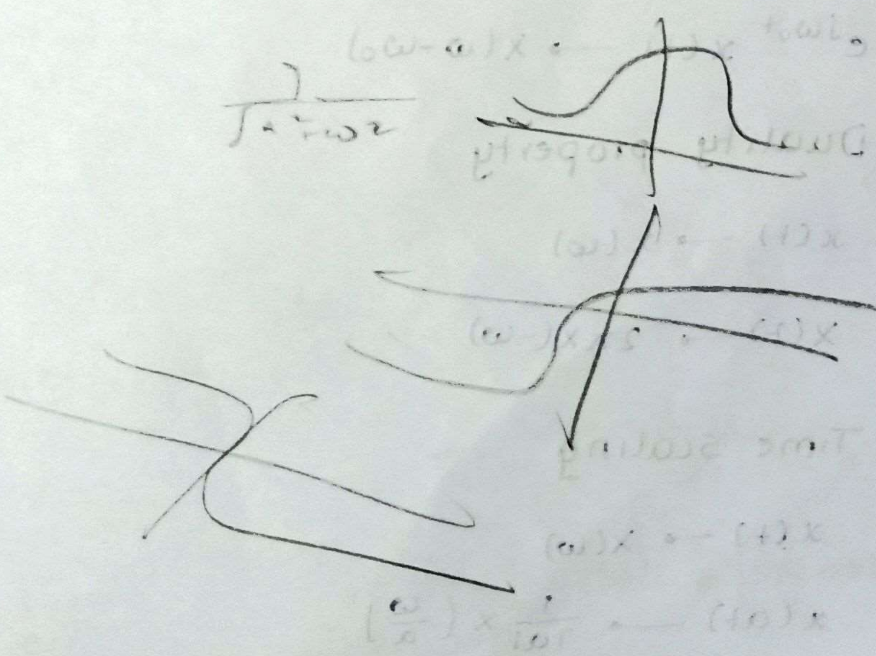
$= \frac{1}{5}x(t/5)e^{-j\frac{6\pi f}{5}}$

$= \frac{1}{5}X(\omega/5)\, e^{-j3\omega/5}$

Q The forier transform of signal $h(t)$ is $H(\omega)$

$H(\omega) = (2\cos\omega)\left(\frac{\sin 2\omega}{\omega}\right)$ find the value of $h(0)$

$x(t) \longrightarrow X(\omega)$



All properties

1) Area under property :

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} \, dt$$

2) Linearity

$x(t) \rightarrow X(\omega)$

$y(t) \rightarrow Y(\omega)$

$a x(t) + b y(t) \rightarrow a X(\omega) + b Y(\omega)$

3) Time reversal

$x(t) \rightarrow X(\omega)$

$x(-t) \rightarrow X(-\omega)$

4) Time shifting

$x(t) \rightarrow X(\omega)$

$x(t-t_0) \rightarrow X(\omega) e^{-j\omega t_0}$

5) Frequency shifting

$e^{j\omega_0 t} x(t) \rightarrow X(\omega - \omega_0)$

6) Duality property

$x(t) \rightarrow X(\omega)$

$X(t) \rightarrow 2\pi x(-\omega)$

7) Time scaling

$x(t) \rightarrow X(\omega)$

$x(at) \rightarrow \frac{1}{|a|} X\left(\frac{\omega}{a}\right)$

8) Differentiation

$x(t) \rightarrow X(\omega)$

$\frac{d}{dt}(x(t)) \rightarrow j\omega X(\omega)$

9) Conjugate property.

$x(t) \rightarrow X(\omega) \implies x^*(t) = X^*(-\omega)$

$y(t) \rightarrow Y(\omega)$

10) Integration

$$\int_{-\infty}^{t} x(t)\,dt = \frac{X(\omega)}{j\omega} + \pi X(0)\,\delta(\omega)$$

11) Convolution property

$x(t) \rightarrow X(\omega)$

$y(t) \rightarrow Y(\omega)$

$x(t) * y(t) \rightarrow X(\omega) \cdot Y(\omega)$

12) Convolution in frequency domain

$x(t) \rightarrow X(\omega)$

$y(t) \rightarrow Y(\omega)$

$x(t) \cdot y(t) \rightarrow \frac{1}{2\pi}[X(\omega) * Y(\omega)]$

13) Parsevel's theorem

$$\int_{-\infty}^{\infty} |x(t)|^2 dt \rightarrow \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2 d\omega$$

14) Derivative in frequency domain or multiplication with 't'

$t \cdot x(t) \longrightarrow j \frac{d}{d\omega} X(\omega)$

Problems

1) $e^{4t} u(-t+3)$

$e^{at} u(-t) \longleftrightarrow \frac{1}{a - j\omega}$

$e^{4t} u(-t) \longrightarrow \frac{1}{4 - j\omega}$

$x(t - t_0) \longrightarrow e^{-j\omega t_0} X(\omega)$

$x(t-3) \longrightarrow e^{-j\omega 3} X(\omega)$

$e^{4(t-3)} u[-(t-3)] \longrightarrow e^{-j\omega 3} \frac{1}{4 - j\omega}$

$e^{4t} u(-t+3)$

$e^{4(t-3+3)} u(-t+3)$

$e^{12} e^{4(t-3)} u(-t+3)$

$= e^{12 - j\omega 3} \frac{1}{4 - j\omega}$

2) Forier transform of $1/t$

$\text{sgn}(t) \longrightarrow \frac{2}{j\omega}$

$x(t) \longrightarrow X(\omega)$ duality

$X(t) \longrightarrow 2\pi x(-\omega)$

$\frac{2}{jt} \longrightarrow 2\pi \, \text{sgn}(-\omega)$

$\frac{1}{t} \longrightarrow j\pi \, \text{sgn}(-\omega)$

Find the F.T of signal $x(t)$ is given by

$$X(\omega) = \frac{1}{2-\omega^2+3j\omega} \cdot \text{ Find } x(t)$$

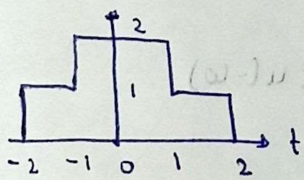
A $= \dfrac{1}{(j\omega)^2+3j\omega+2} = \dfrac{1}{(j\omega)^2+j\omega+2j\omega+2}$

$$= \frac{1}{j\omega(1+j\omega)+2(1+j\omega)} = \frac{1}{(j\omega+1)(j\omega+2)}$$

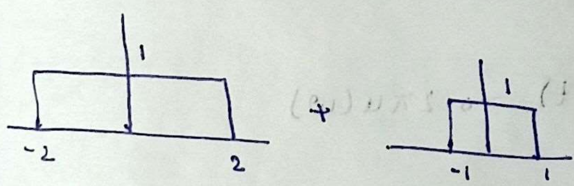
$$= \frac{1}{j\omega+1} - \frac{1}{j\omega+2} \quad \left\{ e^{-at}u(t) \rightarrow \frac{1}{a+j\omega} \right\}$$

$$= e^{-t}u(t) - e^{-2t}u(t)$$

4) Find the forier transform of signal shown in fig.



A



$$1\,\mathrm{rect}(t/4) + 1\,\mathrm{rect}(t/2)$$

$$4\,Sa\left(\frac{\omega t}{2}\right) + 2\,Sa\left(\frac{\omega t}{2}\right)$$

$$4\,Sa\left(\frac{\omega 4}{2}\right) + 2\,Sa\left(\frac{\omega 2}{2}\right)$$

$$= 4Sa(2\omega) + 2Sa(\omega) \text{ or } \mathrm{sinc}(4t) + \mathrm{sinc}(2t)$$

5) Find the F.T. $t\{e^{-at}u(t)\}$

A $t \cdot x(t) \rightarrow j\dfrac{d}{d\omega}X(\omega)$

$$e^{-at}u(t) \rightarrow \frac{1}{a+j\omega}$$

$t \cdot e^{-at} u(t) \longrightarrow j \frac{d}{d\omega}\left(\frac{1}{a+j\omega}\right)$

$t \cdot e^{-at} u(t) \longrightarrow j \frac{-1}{(a+j\omega)^2} j$

$\longrightarrow j^2 \frac{-1}{(a+j\omega)^2}$

$\longrightarrow \frac{1}{(a+j\omega)^2}$

$t \cdot e^{-at} u(t) \longrightarrow \frac{1}{(a+j\omega)^2}$

6) Find inverse F.T of $u(\omega)$

$u(t) \longrightarrow \frac{1}{j\omega} + \pi\delta(\omega)$

$\frac{1}{jt} + \pi\delta(t) \longrightarrow 2\pi u(-\omega)$

$x(-t) \longrightarrow x(-\omega)$

$\frac{-1}{jt} + \pi\delta(-t) \longrightarrow 2\pi u(\omega)$

$\frac{j}{2\pi t} + \frac{1}{2}\delta(t) \longleftrightarrow u(\omega)$

$u(\omega) \longrightarrow \frac{j}{2\pi t} + \frac{1}{2}\delta(t)$

7) Find the F.T of $\frac{4t}{(1+t^2)^2}$

A: $e^{-|t|} \longrightarrow \frac{2}{\omega^2+1}$

$t \cdot e^{-|t|} \longrightarrow j \frac{d}{d\omega}\left(\frac{2}{\omega^2+1}\right)$

$\longrightarrow \frac{j(-2 \cdot 2\omega)}{(1+\omega^2)^2}$

$t \cdot e^{|t|} \longrightarrow \frac{-4j\omega}{(\omega^2+1)^2}$

$$\left\{ \begin{array}{l} x(t) \longrightarrow X(\omega) \\ X(t) \longrightarrow 2\pi x(-\omega) \end{array} \right\}$$

$$\frac{-4jt}{(t^2+1)^2} \longrightarrow 2\pi(-\omega)e^{-|\omega|}$$

$$\frac{4jt}{(t^2+1)^2} \rightarrow 2\pi\omega e^{-|\omega|}.$$

$$\frac{4t}{(t^2+1)^2} \rightarrow -j2\pi\omega e^{-|\omega|}$$

8) F.T of $x(t)$ is $2\,rect\,(0.25\,f)$. Then find F.T of $x(t)\cos 2\pi t$.

A: $x(t) \longrightarrow 2\,rect\,(0.25\,f)$

$Sa\left(\frac{\omega T}{2}\right)$

$AT\left(\frac{\sin\frac{\omega T}{2}}{\frac{\omega T}{2}}\right)$

$x(t)\cos 2\pi t \longrightarrow \pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)$

$\mathrm{sinc}(FT)$

x

9) Find F.T of rect (t) * $\cos\pi t$

$$Sa\left(\frac{\omega}{2}\right) \cdot \pi\left[\delta(\omega-\pi)+\delta(\omega+\pi)\right]$$

* Forier transform of periodic signals

$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$$

$$F.T\{x(t)\} = \sum_{n=-\infty}^{\infty} C_n FT\{e^{jn\omega_0 t}\}$$

$$= \sum_{n=-\infty}^{\infty} C_n 2\pi\delta(\omega-\omega_0 n)$$

$$X(\omega) = 2\pi\sum_{n=-\infty}^{\infty} C_n\delta(\omega-\omega_0 n)$$

Thus forier transform of periodic Signal having impulse train having strength $2\pi C_n$ logcated at $\omega = n\omega_0$

1) Find the F.T of $x(t) = \sum_{n=-\infty}^{\infty} \delta(at-nT)$

A: $$C_n = \frac{1}{T}\int_{-T/2}^{T/2} x(t) e^{-jn\omega_0 t} dt$$

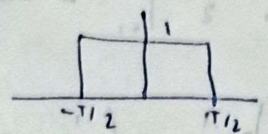
$$= \frac{1}{T}\int \delta(t) e^{jn\omega_0 t} dt$$

$$C_n = \frac{1}{T}$$

$$X(\omega) = \frac{2\pi}{T}\sum_{n=-\infty}^{\infty} \delta(\omega-n\omega_0)$$

Q Find F.T of $\left\{ \frac{\sin at}{\pi t} \right\}$

A: $\left\{ \frac{\sin at}{\pi t} \right\}$



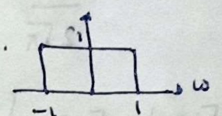
$\text{rect}(t/\tau) \longrightarrow \frac{T\sin(\omega T/2)}{\omega T/2}$

$\rightarrow \frac{T \sin(t\, T/2)}{t\, T/2} \longrightarrow 2\pi\, \text{rect}(\omega/T)$

$\rightarrow \frac{\sin(t T/2)}{\pi t} \longrightarrow \text{rect}(\omega/T)$ $\quad a = T/2$

$T = 2a$

$\rightarrow \frac{\sin at}{\pi t} \longrightarrow \text{rect}(\omega/2a)$



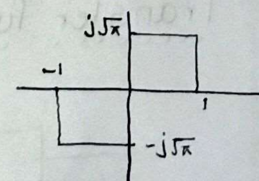
Q - Find I.F.T of $\text{rect}\left(\frac{\omega - 10}{2\pi}\right)$

$\frac{\sin at}{\pi t} \underset{I.F.T}{\overset{FT}{\rightleftharpoons}} \text{rect}(\omega/2a)$

$\text{rect}\left(\frac{\omega - 10}{2\pi}\right) \xrightarrow{IFT} \frac{\sin at}{\pi t}$ $\quad a = \pi$

$\text{rect}\ \frac{\omega - 10}{2\pi} \xrightarrow{I.F.T} e^{j10t} \cdot \frac{\sin \pi t}{\pi t}$

For the spectrum is shown in figure, find $\frac{d}{dt}x(t)$ at $t = 0$



A: $\frac{d}{dt}(x(t))\Big|_{t=0}$

$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega$

$\frac{d}{dt}x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega\, (j\omega)$

$\frac{d}{dt}x(t)\Big|_{t=0} = \frac{1}{2\pi} \cancel{X(\omega)} \int_{-\infty}^{\infty} j\omega\, e^{0}\, d\omega\, (X(\omega))$

$\frac{d}{dt}x(t)\Big|_{t=0} = \frac{1}{2\pi}\left[\int_{-1}^{0} -j\sqrt{\pi}\, j\omega\, d\omega + \int_{0}^{1} j\sqrt{\pi}\, j\omega \cdot d\omega\right]$

$$= \frac{\sqrt{\pi}}{2\pi}\left[\frac{\omega^2}{2}\Big|_{-1}^{0} - \frac{\omega^2}{2}\Big|_{0}^{1} \right]$$

$$= \frac{\sqrt{\pi}}{2\pi}\left[0 - \frac{1}{2} - \left(\frac{1}{2} - 0\right) \right]$$

$$= \frac{1}{2\sqrt{\pi}}(-1)$$

$$= \frac{-1}{2\sqrt{\pi}}$$

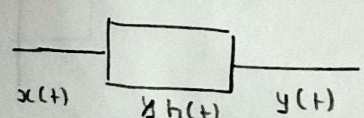
Q Find F.T $\left\{ \frac{\sin t \cdot \sin t/2}{\pi t^2} \right\}$

$$\frac{\sin t}{\pi t} \longrightarrow \pi \,\mathrm{rect}\left(\frac{\omega}{2}\right)$$

$$\frac{\sin t/2}{\pi t/2} \longrightarrow \mathrm{rect}\left(\frac{\omega}{2 \cdot 1/2}\right)$$

$$2 \cdot \frac{\sin t}{\pi t} \cdot \frac{\sin t/2}{\pi t} \longrightarrow \frac{\pi}{2}\left[\pi \,\mathrm{rect}\left(\frac{\omega}{2}\right) * \mathrm{rect}(\omega) \right]$$

* Transfer function of an LTI system.



$$y(t) = x(t) * h(t)$$

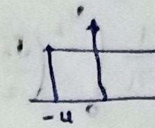
$$Y(\omega) = X(\omega) \cdot H(\omega)$$

$$\boxed{H(\omega) = \frac{Y(\omega)}{X(\omega)}}$$

An LTI system is having impulse response $h(t) = \frac{\sin 4t}{\pi t}$ for which the input applied

(i) $x(t) = \cos 2t + \sin 6t$ find the output.

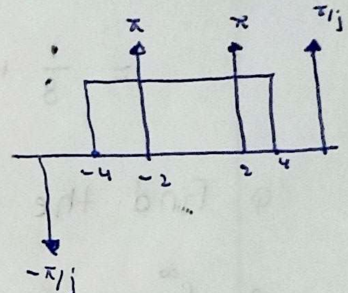
$$h(t) = \frac{\sin 4t}{\pi t} \qquad H(\omega) =$$



$$x(t) = \cos 2t + \sin 6t$$

$$X(\omega) = \pi\left[\delta(\omega-2) + \delta(\omega+2)\right] + \frac{\pi}{j}\left[\delta(\omega-6) - \delta(\omega+6)\right]$$

$$Y(\omega) = H(\omega)X(\omega)$$



$$Y(\omega) = \pi\left[\delta(\omega-2) + \delta(\omega+2)\right]$$

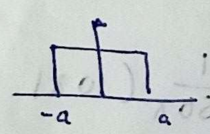
$$y(t) = \cos 2t$$

* Find energy in signal $x(t) = \frac{\sin at}{\pi t}$

A. $x(t) = \frac{\sin at}{\pi t}$

$$\text{energy signal} = \int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2 d\omega$$

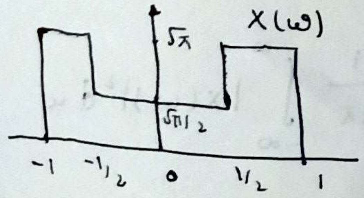
$$\frac{\sin at}{\pi t} \xrightarrow{F.T} \text{rect}\left(\frac{\omega}{2a}\right)$$



$$= \frac{1}{2\pi}\int_{-a}^{a} 1 \cdot d\omega = \frac{1}{2\pi}[\omega]_{-a}^{a}$$

$$= \frac{1}{2\pi}[a+a] = \frac{a}{\pi}$$

* Find the energy spectrum as shown in figure



$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} |x(\omega)|^2 \, d\omega$$

$$= \frac{2}{2\pi}\int_{0}^{1} |x(\omega)|^2 \, d\omega$$

$$= \frac{1}{\pi}\int_{0}^{1/2} \left(\frac{\sqrt{\pi}}{2}\right)^2 d\omega + \int_{1/2}^{0} (\sqrt{\pi})^2 \, d\omega$$

$$= \frac{1}{\pi}\left[\frac{\pi}{4}\right]\frac{1}{2} + \pi\left[\frac{1}{2}\right]\Big]$$

$$= \frac{1}{\pi}\left[\frac{\pi}{8} + \frac{\pi}{2}\right]$$

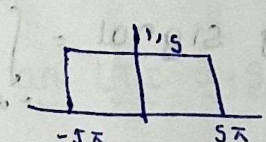
$$= \frac{1}{8} + \frac{1}{2} = \boxed{\frac{5}{8}}$$

Q Find the value of integral $\int_{-\infty}^{\infty} \text{sinc}^2(5t)\,dt$

A $\int_{-\infty}^{\infty} \text{sinc}^2(5t)\,dt \qquad \left\{ \frac{\sin 5\pi t}{5\pi t} \right\}$

$$= \frac{1}{2\pi}\int_{-5\pi}^{5\pi} (1/5)^2 \, d\omega \qquad \left\{ \frac{1}{5} \cdot \frac{\sin 5\pi t}{\pi t} \right\}$$

$$= \frac{1}{2\pi} \cdot \frac{1}{25} (5\pi + 5\pi)$$



$$= \frac{1}{50\pi}(10\pi)$$

$$= \frac{1}{5}$$

Q Find the value of integral $\int_{-\infty}^{\infty} \frac{8}{(\omega^2+4)^2}\,d\omega$

A $\int_{-\infty}^{\infty} \frac{8}{(\omega^2+4)^2}\,d\omega$

$$\int_{-\infty}^{\infty} |x(t)|^2 \, dt \longrightarrow \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2 \, d\omega$$

$$2\pi \int_{-\infty}^{\infty} x(t)^2 \, dt \longrightarrow \int_{-\infty}^{\infty} |X(\omega)|^2 d\omega$$

$$e^{-a|t|} \longrightarrow \frac{2a}{\omega^2 + a^2}$$

$$|X(\omega)|^2 = \frac{8}{\omega^2 + 4^2} = \left(\frac{2\sqrt{2}}{\omega^2 + 4}\right)^2$$