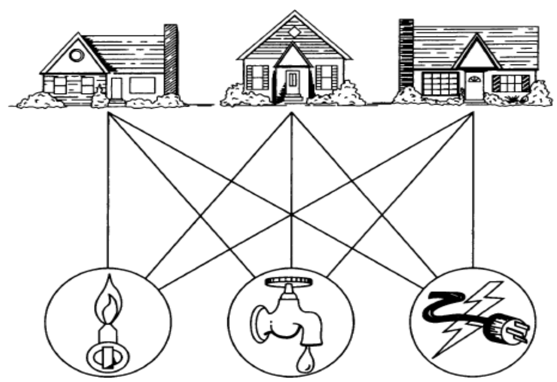
3.5

PLANAR GRAPHS

Three houses and three utilities problem:

Consider the problem of joining three houses to each of three separate utilities as shown below: **Is it possible to join these houses and utilities so that none of the connections cross?**



Three houses and three utilities

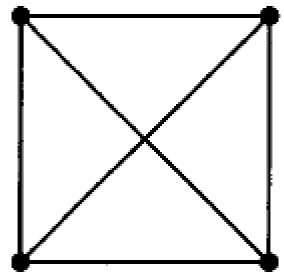
This problem can be modeled using the complete bipartite graph $K_{3,3}$. The above question is rephrased as: **Can $K_{3,3}$ be drawn in the plane so that no two of its edges cross?**

In this module we study the question of whether a graph can be drawn in the plane without edges crossing.

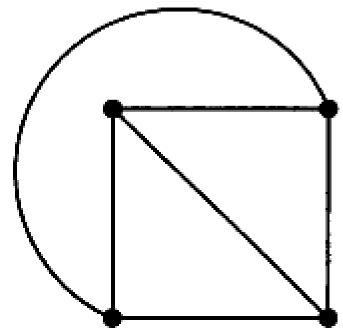
Planar graph: A graph is said to be **planar** if it can be drawn in the plane without any edges crossing. Such a drawing is called a **planar representation** of the graph.

A graph may be planar even if it is usually drawn with crossings, because it may be possible to draw it in a different way without crossings.

For example, K_4 drawn as below is planar, because it can be drawn without crossings as shown below.

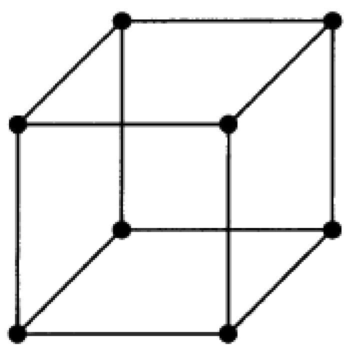


K_4 with a crossing

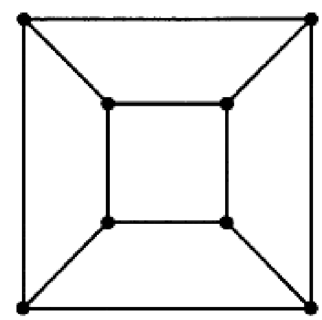


Planar representation of K_4

Example 1: Q_3 is planar

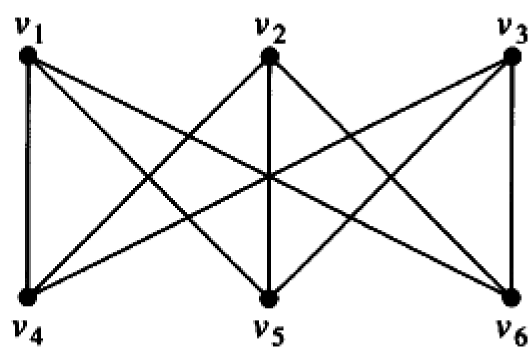


Q_3



Planar representation of Q_3

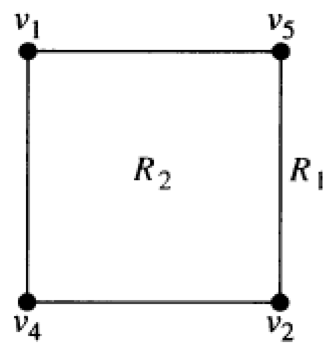
Example 2: $K_{3,3}$ is not planar.



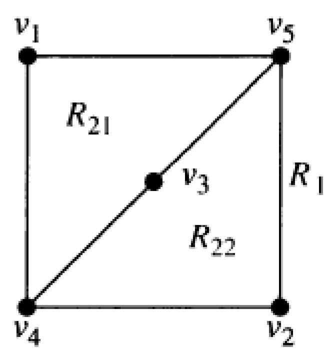
Solution: First notice that any attempt to draw $K_{3,3}$ in the plane with no edges crossing is failed.

A reason for this is given below:

In any planar representation of $K_{3,3}$ the vertices v_1 and v_2 must be connected to both the vertices v_4 and v_5 . These four edges form a closed curve that splits the plane into two regions R_1 and R_2 as shown below:



Now, the vertex v_3 is in either R_1 or R_2 .Suppose that v_3 is in R_2.The edges between v_3 , v_4 and v_3 , v_5 separate R_2 into two subregions R_{21} and R_{22} as shown below



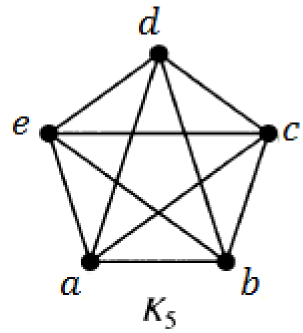
If v_6 is in R_1 , then the edge between v_6 and v_3 cannot be drawn without a crossing. If v_6 is in R_{21} , then the edge between v_6 and v_2 can not be drawn without a crossing. If v_6 is in R_{22} , then the edge between v_6 and v_1 can not be drawn without a crossing. This shows that $K_{3,3}$ can not be drawn without a crossing in this case. We obtain the same conclusion if v_3 is in R_1.

Thus, $K_{3,3}$ is not planar:

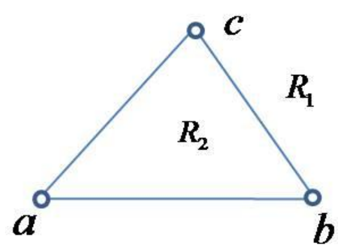
From the above result, we have: **The three houses and three utilities cannot be connected in the plane without a crossing.**

Example 3: Show that K_5 is nonplanar

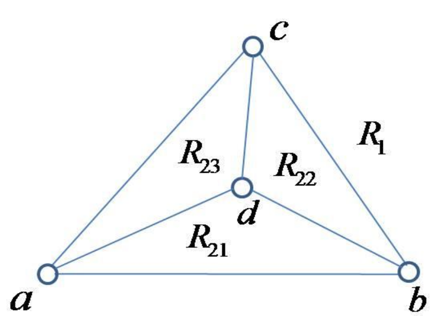
Solution: We have K_5



Solution: First notice that any attempt to draw K_5 in the plane with no edges crossing is failed. Assume that it has a planar representation. First, a triangle is formed by the planar representation of the subgraph of K_5 consisting of the edges connecting a, b and c; and this triangle splits the plane into two regions R_1 and R_2.



Now, the vertex d is in either R_1 or R_2. Suppose that the vertex d is in R_2 ($i.e.$, the inside of the closed curve) then the edges between d, a ; d, b and d, c separate R_2 into three subregions as shown below:



Note that there is no way to place the vertex e without forcing a crossing. If e is in R_1 , then the edge between e and d cannot be drawn without a crossing. If e is in R_{21} , then the edge between e and c cannot be drawn without a crossing. If e is in R_{22} , then the edge between e and a can not be drawn without a crossing. If e is in R_{23} , then the edge between e and b can not be drawn without a crossing.

This shows that K_5 cannot be drawn without a crossing in this case. We obtain the same conclusion if the vertex d is in R_1.

Thus, K_5 is nonplanar.

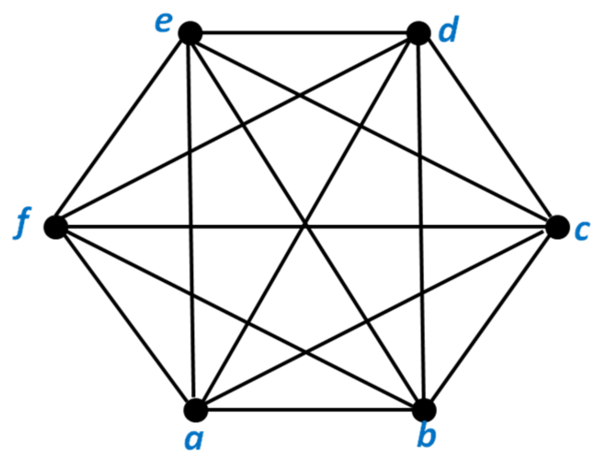
Planarity of graphs play an important role in the design of electronic circuits. We can model an electronic circuit with a graph by representing components of the electronic circuit by vertices and connections between them by edges. If the graph representing electronic circuit is planar, then we can print it on a single board with no connections crossing. If the graph is not planar then we must turn to more expensive options. For example, we can partition the vertices in the graph representing the electronic circuit into planar subgraphs. We then construct the circuit using multiple layers.

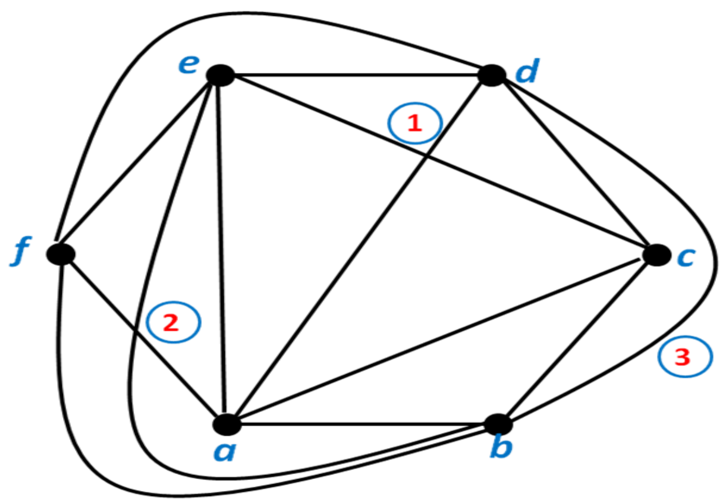
We can construct the electronic circuit using insulated wires whenever connections cross. In this case, drawing the graph with the fewest possible crossings is important.

Crossing number: The **Crossing number** of a simple graph is the minimum number of crossings that can occur when the graph is drawn in the plane where no three arcs representing the edges are permitted to cross the same point.

Example 4: Find the crossing number of K_6.

Solution: We have K_6



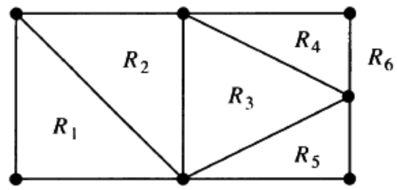


The minimum number of crossings that can occur when the graph is drawn in the plane is 3. Therefore, the crossing number of K_6 is 3.

Euler's Formula

A planar representation of a graph splits the plane into **regions**, including an unbounded region.

For example, the planar representation of the graph shown below splits the plane into six regions.



The region R_6 is the unbounded region, because it is not bounded by the edges.

Euler found a relation among the number of regions, the number of vertices, and the number of edges of a planar graph.

Theorem Euler's formula: Let G be any connected planar graph with v vertices and e edges. Let r be the number of regions in a planar representation of G. Then

$$\boldsymbol{v - e + r = 2}$$

Example 5: Suppose that a connected planar graph has 20 vertices, each of degree 3. Into how many regions does a representation of this planar graph split the plane?

Solution: The planar graph G has 20 vertices, each of degree 3. The sum of the degrees of the vertices is $20 \cdot 3 = 60$. By Handshaking theorem we have $2e = 60$, *i.e.,* $e = 30$. By Euler's formula we have, $v - e + r = 2$.
Therefore, $r = e - v + 2 = 30 - 20 + 2 = 12$.

Euler's formula can be used to prove some inequalities that must be satisfied by planar graphs.

Corollary1: Let G be a connected planar simple graph with no loops. If G has e edges and v vertices, $v \geq 3$, then

$$\boldsymbol{e \leq 3v - 6}$$

Corollary 2: If G is a connected planar simple graph with no loops then G has a vertex of degree not exceeding five.

Proof: If G has one or two vertices, then the result is true. If G has atleast three vertices then by Corollary 1, $e \leq 3v - 6$, so $2e \leq 6v - 12$. Assume the contrary.

That is, assume that the degree of every vertex is at least six. By Handshaking Theorem, $2e = \sum_{x \in V} deg\, x \geq 6v$. This contradicts the inequality $2e \leq 6v - 12$.

Therefore, our assumption is false. Thus, there is a vertex of degree not exceeding five.

Corollary3: Let G be a connected planar simple graph with no circuits of length 3 ($i.e.$, every region is bounded by four or more edges). If G has e edges and v vertices, $v \geq 3$, then

$$e \leq 2v - 4$$

Example 6: K_5 is nonplanar.

Solution: Notice that K_5 is a connected simple graph with no loops. It has $v = 5$ vertices and $e = 10$ edges. Assume that K_5 is planar then by Corollary 1, $e \leq 3v - 6$, *i.e.,* $10 \leq 3 \cdot 5 - 6 = 9$ - a contradiction. Thus, K_5 is nonplanar.

Example 7: $K_{3,3}$ is nonplanar.

Solution: First note that $K_{3,3}$ is a connected simple graph with $v = 6$ vertices and $e = 9$ edges. Note that it has no loops and it satisfies the inequality $e \leq 3v - 6$. This does not imply $K_{3,3}$ is nonplanar. Observe that $K_{3,3}$ has no circuit of length 3. Now assume that $K_{3,3}$ is nonplanar. By Corollary 3, $e \leq 2v - 4$, $i.e.$, $9 \leq 2 \cdot 6 - 4 = 8$, a contradiction. Thus $K_{3,3}$ is nonplanar.

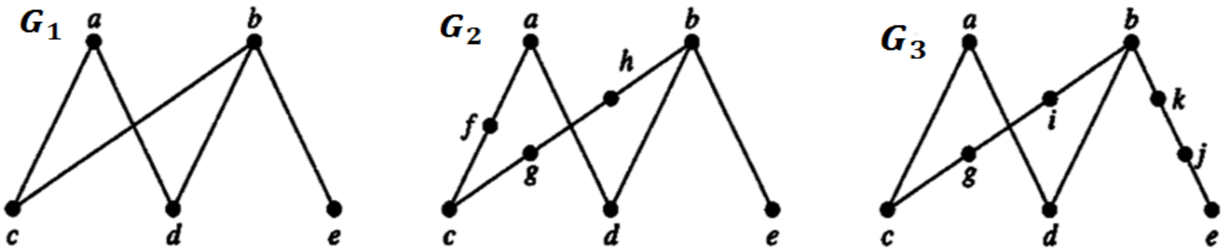
Kuratowski's Theorem

It is known that $K_{3,3}$ and K_5 are nonplanar. Clearly, a graph is not planar if it contains either of these two graphs $K_{3,3}$, K_5 as a subgraph.

Elementary subdivision: If a graph is planar, so will be any graph obtained by removing an edge $\{u, v\}$ and adding a new vertex w together with edges $\{w, u\}$ and $\{w, v\}$. Such an operation is called an ***elementary subdivision.***

Homeomorphic graphs: Two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are said to ***homeomorphic*** if they can be obtained from the same graph by a sequence of elementary subdivisions.

Example 8: Show that the following graphs G_1, G_2 and G_3 are all homeomorphic.



Solution: We show that all the three graphs can be obtained from G_1 by elementary subdivisions.

First note that G_1 can be obtained from itself by an empty sequence of elementary subdivisions.

We obtain G_2 from G_1 by the following sequence of elementary subdivisions:

(i) Remove the edge $\{a, c\}$, add the vertex f, and add the edges $\{f, a\}$ and $\{f, c\}$
(ii) Remove the edge $\{b, c\}$, add the vertex g, and add the edges $\{g, b\}$ and $\{g, c\}$
(iii) Remove the edge $\{b, g\}$, add the vertex h, and add the edges $\{h, b\}$ and $\{b, g\}$

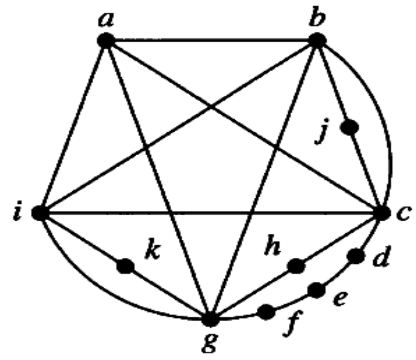
Similarly, we obtain G_3 from G_1 by a sequence of elementary subdivisions.

The Polish mathematician Kazimierz Kuratowski proved the following theorem in 1930. This theorem characterizes planar graphs using the concept of graph homeomorphism.

Theorem (Kuratowski): A graph is nonplanar if and only if it contains a subgraph homeomorphic to $K_{3,3}$ or K_5.

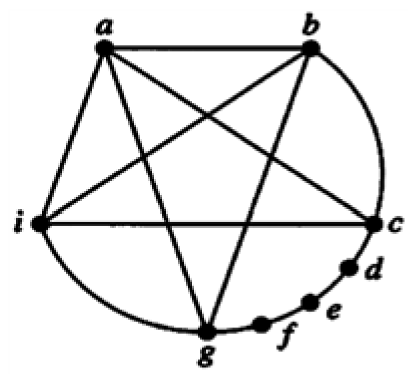
It is clear that a graph containing a subgraph homeomorphic to K_5 or $K_{3,3}$ is nonplanar. However, the proof of the converse, namely that every nonplanar graph contains a subgraph homeomorphic to $K_{3,3}$ or K_5 is complicated and it is out of scope of this course.

Example 9: Determine whether the following graph G is planar.



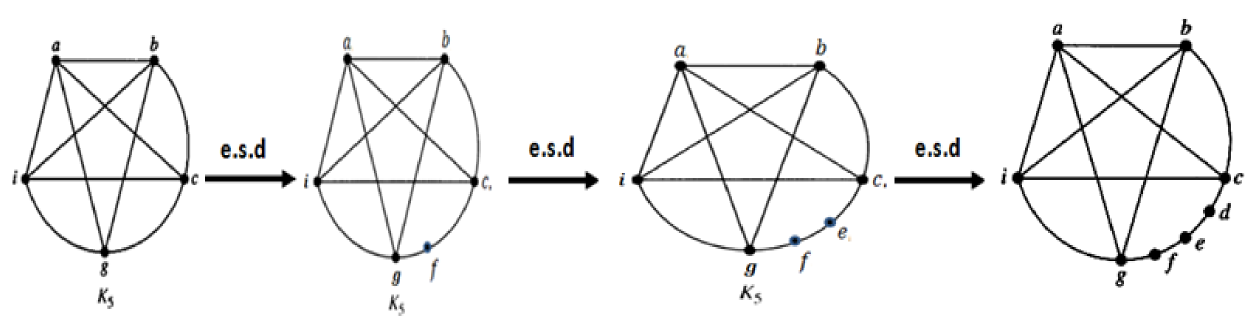
G

Solution: Let H be the subgraph of G obtained by deleting the vertices h, j and k and all edges incident with them.



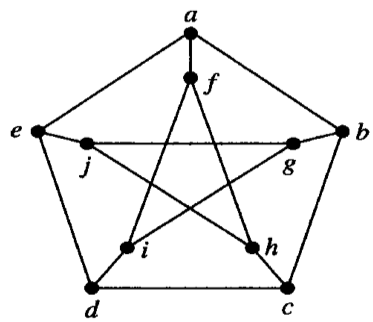
H

Now consider K_5 with vertices a, b, c, g and i. We obtain H from K_5 by the following sequence of elementary subdivisions (e.s.d) as shown below.



Thus, H is homeomorphic to K_5. This shows, that G has a subgraph homeomorphic to K_5. Therefore, by Kuratowski theorem G is nonplanar.

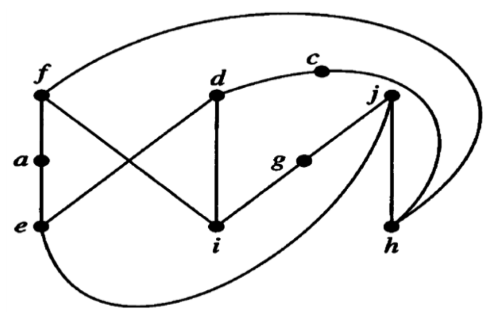
Petersen Graph: The Danish mathematician Julis Petersen studied this graph in 1891. It is often used to illustrate various theoretical properties of graphs.



$Petersen\ Graph$

Example 10: Is Petersen graph, shown above, planar?

Solution: Let H be the subgraph of the Petersen graph obtained by deleting the vertex b and all the edges incident with b.



H

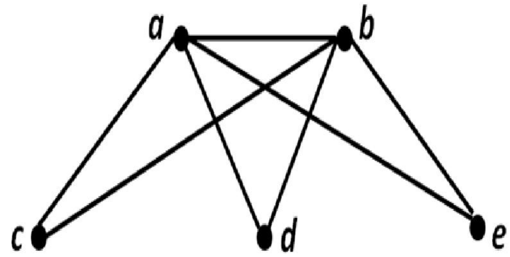
Now, consider $K_{3,3}$ with vertex sets $\{f, d, j\}$ and $\{e, i, h\}$, ($i.e.$, a bipartition).

We obtain H from $K_{3,3}$ by the following sequence of elementary subdivisions:
i. delete $\{d, h\}$ and add the vertex c and add edges $\{c, h\}$ and $\{c, d\}$:
ii. delete $\{e, f\}$ and add the vertex a and add edges adding $\{a, e\}$ and $\{a, f\}$ and
iii. delete $\{i, j\}$ and add the vertex g and add edges $\{g, i\}$ and $\{g, j\}$

Thus, H is homeomorphic to $K_{3,3}$. This shows that the Petersen graph has a subgraph homeomorphic to $K_{3,3}$. Therefore, by Kuratowski theorem, the Petersen graph is nonplanar.

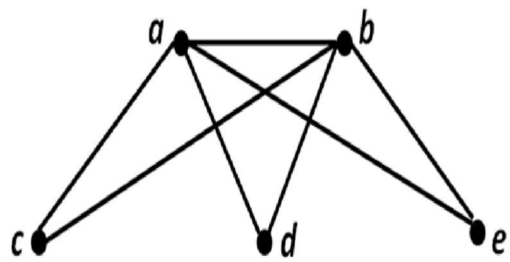
P1.

Draw the given planar graph without any crossings

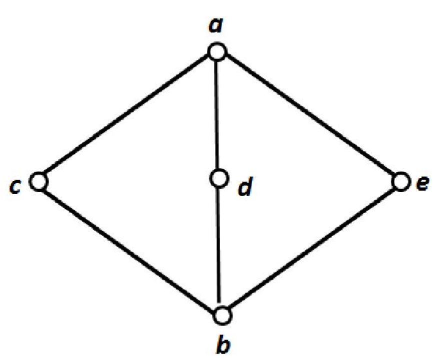


Solution:

The given planar graph is



It is $K_{2,3}$. Its planar representation is



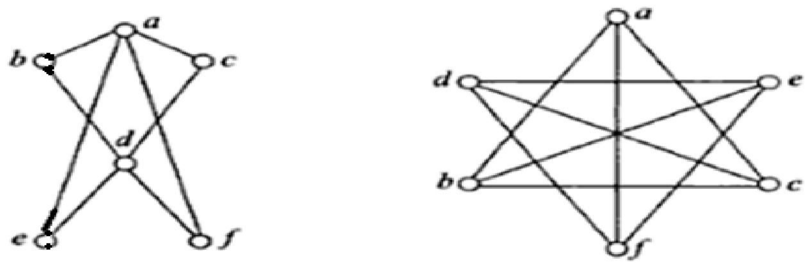
Remark:

(i) $K_{2,3}$ is planar

(ii) $K_{3,3}$ is nonplanar

P2.

Draw the given planar graph without any crossings.



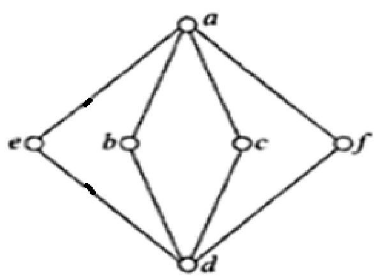
G_1 G_2

Solution:

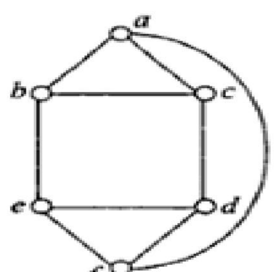
Notice that these are two simple circuits

a, c, d, b, a and a, f, d, e, a in G_1

The planar representation of G_1 can now be drawn as



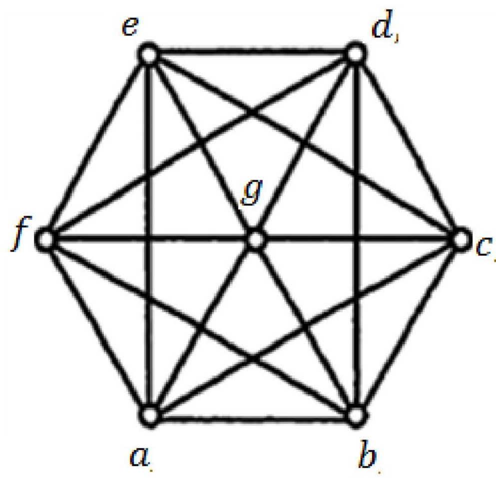
Notice that G_2 has a circuit a, b, c, d, e, f, a .The remaining edges can be drawn as shown below:



The planar representation of G_2

P3.

Show that the following graph is nonplanar.



Solution:

Notice that the graph is a connected simple graph with no loops and
$v = 7, e = 18$

Assume that the graph is planar. By Corollary 1,

$e \leq 3v - 6$, i.e., $18 \leq 3(7) - 6 = 15$, a contradiction

Therefore, the given graph is nonplanar.

P4.

Suppose that a connected planar graph has six vertices, each of degree four. Into how many regions is the plane divided by a planar representation of this graph.

Solution:

The planar graph G has 6 vertices, each of degree 4. The sum of the degrees of all the vertices is $6(4) = 24$.
By Hand shaking theorem, we have $2e = 24,\ i.e.,\quad e = 12$

By Euler's formula, we have

$v - e + r = 2,\quad i.e.,\ 6 - 12 + r = 2,\quad i.e.,\quad r = 8$

P5.

Which of these nonplanar graphs have the property that the removal of any vertex and all edges incident with that vertex produces a planar graph?

a) $\boldsymbol{K_5}$ **b)** $\boldsymbol{K_6}$ **C)** $\boldsymbol{K_{3,\ 3}}$ **D)** $\boldsymbol{K_{3,4}}$

Solution:

a) If we remove any vertex from K_5 and all edges incident with that vertex, then it produces K_4 and it is planar.

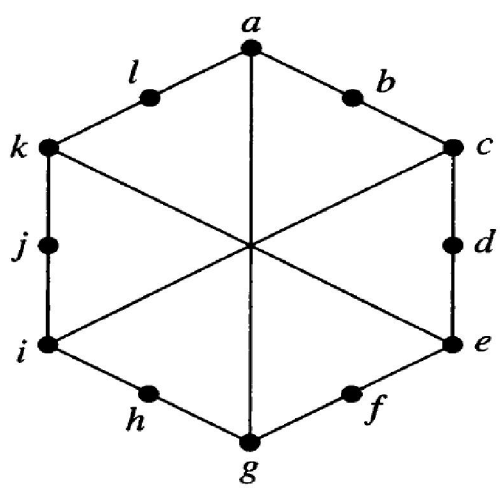
b) If we remove any vertex from K_6 and all edges incident with that vertex, then it produces K_5 and it is nonplanar.

c) If we remove any vertex from $K_{3,3}$ and all edges incident with that vertex, then it produces $K_{2,3}$ or $K_{3,2}$ and both are planar.

d) If we remove any vertex from $K_{3,4}$ and all edges incident with that vertex, then it produces $K_{3,3}$ or $K_{2,4}$ and both are nonplanar.

P6.

Determine whether the graph given below is homeomorphic to $K_{3,3}$



Solution:

We have $V = \{a, b, c, d, \ldots, j, k, l\}$. Let $v_1 = \{a, e, i\}$ and $v_2 = \{c, g, k\}$. Now, we have $K_{3,3}$ with vertex sets v_1 and v_2 ($i.e.$ a bipartition)

We obtain the given graph from $K_{3,3}$ by the following sequence of elementary sub divisions.

i) Remove $\{a, c\}$, add the vertex b and add edges $\{b, a\}$ and $\{b, c\}$

ii) Remove $\{e, c\}$, add the vertex d and add edges $\{d, e\}$ and $\{d, c\}$

iii) Remove $\{e, g\}$, add the vertex f and add edges $\{f, e\}$ and $\{f, g\}$

iv) Remove $\{i, g\}$, add the vertex h and add edges $\{h, i\}$ and $\{h, g\}$

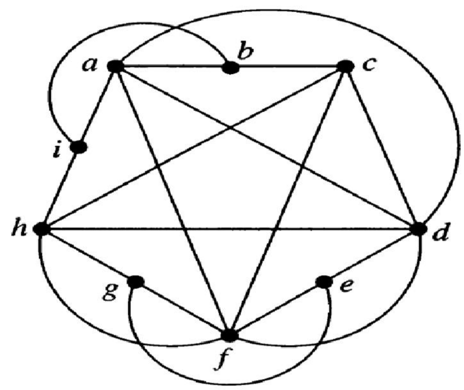
v) Remove $\{i, k\}$, add the vertex j and add edges $\{j, i\}$ and $\{j, k\}$

vi) Remove $\{a, k\}$, add the vertex l and add edges $\{l, a\}$ and $\{l, k\}$

Thus, the given graph is homeomorphic to $K_{3,3}$.

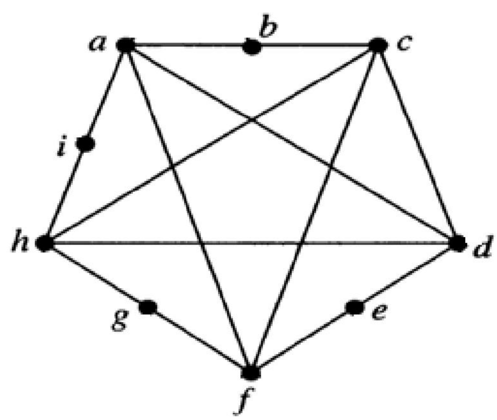
P7.

Use Kuratowski's theorem to determine whether the given graph is planar.



Solution:

Let H be the subgraph of the given graph by deleting one of the two edges between a, d; and delete the edges $\{b, i\}, \{f, d\}, \{e, g\}, \{f, h\}$.



Now consider K_5 with vertices a, c, d, f and h. We obtain H from K_5 by the following sequence of elementary subdivisions:

i) delete $\{a, c\}$, add the vertex b and add edges $\{b, a\}$ and $\{b, c\}$
ii) delete $\{a, h\}$, add the vertex i and add edges $\{i, a\}$ and $\{i, h\}$
iii) delete $\{h, f\}$, add the vertex g and add edges $\{g, h\}$ and $\{g, f\}$
iv) delete $\{f, d\}$, add the vertex e and add edges $\{e, f\}$ and $\{e, d\}$

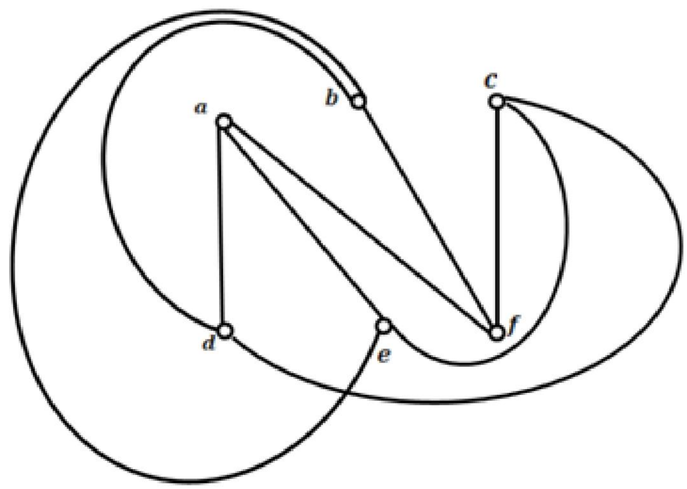
Thus, H is homeomorphic to K_5. This shows that the given graph has a subgraph homeomorphic to K_5. Therefore, by Kuratowski's theorem, the given graph is nonplanar.

P8:

Example: Show that the crossing number of $K_{3,3}$ is 1.

Solution:

First note that $K_{3,3}$ is nonplanar

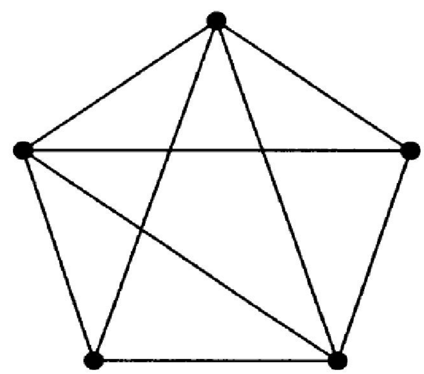


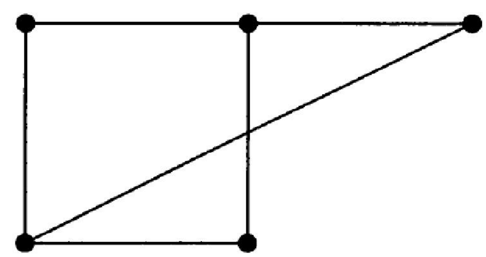
The minimum number of crossings that can occur when the graph is drawn in the plane is 1. Therefore, the crossing number of $K_{3,3}$ is 1.1

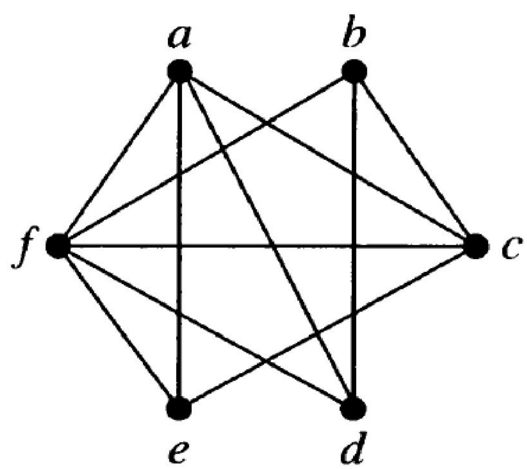
3.5 Planar Graphs

Exercises:

1. Draw the given planar graph without any crossings:





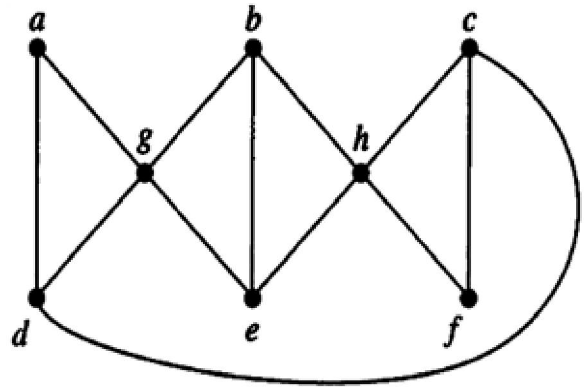


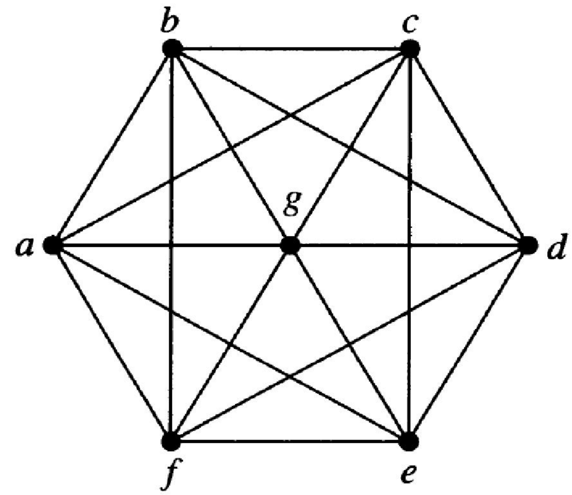
2.

a) Suppose that a connected planar graph has 8 vertices, each of degree 3. Into how many regions is the plane divided by a planar representation of this graph?

b) Suppose that a connected planar graph has 30 edges. If a planar representation of this graph divides the plane into 20 regions, how many vertices does this graph have?

3. Use Kuratowski's theorem to determine whether the given graphs is planar.





4. Find the crossing numbers of each of these nonplanar graphs.
    a) K_5 b) K_7 c) $K_{3,\ 4}$ d) $K_{4,4}$