Vector Calculus

UNIT-IV

① Definitions
② Model wise problems
③ Objective Questions
included

Gradient $\nabla\phi$

(i) Find grad ϕ

(ii) Find unit normal vector $\bar{e}$

(iii) Directional Derivative

Surface to vector

Surface to two points

Surface to surface

Surface to curve

(iv) Angle between

Surface to surface

Surface to two points

If Angle is 90° Find Constants a & b

(v) Application

Problems on Application

Divergence $\nabla \cdot f$

(i) Find div f (or) $\nabla \cdot \bar{f}$

(ii) prove that $\bar{f}$ is solenoidal

$$\nabla \cdot \bar{f} = 0$$

(iii) Find constant c if $\bar{f}$ is solenoidal

(iv) If $\bar{f}$ = grad ϕ then find div $\bar{f}$

(v) Application

problems on Application

curl $\bar{f}$ $\nabla \times \bar{f}$

(i) Find curl $\bar{f}$ or $\nabla \times \bar{f}$

(ii) prove that $\bar{f}$ is irrotational

$$\nabla \times \bar{f} = 0$$

(iii) Find constants a, b, c if $\bar{f}$ is irrotational

(iv) If $\bar{f}$ = grad ϕ then find curl $\bar{f}$

(v) Application

problems on Application

misslenious problems on Laplace operator

$$\nabla^2 \phi$$

Line integral

$$\int_C \bar{F} \cdot d\bar{r}$$

Surface integral

$$\int_S \bar{F} \cdot \bar{n} \, dS$$

Volume integral

$$\int_V \bar{F} \, dV$$

Gauss Divergence Theorem

$$\int_S \bar{F} \cdot \bar{n} \, dS = \int_V \nabla \cdot F \, dV$$

Green's Theorem

$$\int_C M dx + N dy = \iint_S M dx + N dy$$

Stoke's Theorem

$$\int_C \bar{F} \cdot d\bar{r} = \int_S \text{curl} \, \bar{F} \cdot \bar{n} \, dS$$

Scalar point function and vector point function :

Consider a region in three dimensional space, to each point $P(x,y,z)$ suppose we associate a unique real number called scalar say ϕ, this $\phi(x,y,z)$ is called scalar potential function defined on the region. Similarly if to each point $P(x,y,z)$ we associate a unique vector $\bar{f}(x,y,z)$, $\bar{f}$ is called a vector point function

Vector Differential operator :

The vector differential operator ∇ read as del ; is defined as

$$\nabla = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}$$

we will define now some quantities known as gradient, divergence, curl involving this operator ∇

we must note that this operator has no meaning by itself unless it operates on some function suitably

Gradient of a Scalar potential function

let $\phi(x,y,z)$ be a scalar point function of position defined in some region space. Then the vector function

$i \frac{\partial \phi}{\partial x} + j \frac{\partial \phi}{\partial y} + k \frac{\partial \phi}{\partial z}$ is known as the gradient of ϕ and it is denoted by grad ϕ or $\nabla \phi$

$$\nabla \phi = i \frac{\partial \phi}{\partial x} + j \frac{\partial \phi}{\partial y} + k \frac{\partial \phi}{\partial z}$$

Direction Derivative :

The Directional Derivative of a scalar point function ϕ at a point $P(x,y,z)$ in the Direction of a unit vector $\bar{e}$ is equal to $\bar{e} \cdot \text{grad}\, \phi = \bar{e} \cdot \nabla \phi$

The physical interpretation of $\nabla \phi$

The gradient of a scalar function $\phi(x,y,z)$ at a point $P(x,y,z)$ is a vector along the normal to the level surface $\phi(x,y,z) = c$ at P and is in increasing direction

Its magnitude is equal to the greatest rate of increase of ϕ. Greatest value of Directional Derivative of ϕ at a point P $= |\text{grad}\,\phi|$ at that point P

Divergence of a vector:

Let $\bar{f}$ is any continuously differentiable vector point function. Then

$i \cdot \frac{\partial \bar{f}}{\partial x} + j \cdot \frac{\partial \bar{f}}{\partial y} + k \cdot \frac{\partial \bar{f}}{\partial z}$ is called divergence of $\bar{f}$; and is written as div $\bar{f}$

$$\text{div}\,\bar{f} = i \cdot \frac{\partial \bar{f}}{\partial x} + j \cdot \frac{\partial \bar{f}}{\partial y} + k \cdot \frac{\partial \bar{f}}{\partial z}$$

i.e $\bar{f} = f_1 x + f_2 y + f_3 z$

$$\text{div}\,f = \nabla \cdot f = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

div f is a scalar point function

Solenoidal vector:

A vector point function $\bar{f}$ is said to solenoidal if div $\bar{f} = 0$

Curl of a vector

Let $\bar{f}$ is continuously differentiable vector point function, then the vector function defined by $i \times \frac{\partial f}{\partial x} + j \times \frac{\partial f}{\partial y} + \bar{k} \times \frac{\partial f}{\partial z}$ is called curl $\bar{f}$ and is denoted by curl $\bar{f}$ (or) $\nabla \times \bar{f}$

$$\text{curl } \bar{f} = \nabla \times \bar{f} = i \times \frac{\partial f}{\partial x} + j \times \frac{\partial f}{\partial y} + k \times \frac{\partial f}{\partial y}$$

$$= \sum i \times \frac{\partial f}{\partial x}$$

Matrix representation of curl $\bar{f}$

If $\bar{f} = f_1 i + f_2 j + f_3 \bar{k}$

$$\text{curl } \bar{f} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

Irrotational vector:

A vector $\bar{f}$ is said to be irrotational if curl $\bar{f} = 0$

Laplacian operator: ∇^2

$$\nabla^2 \phi = \nabla \cdot \nabla \phi$$

$$= \nabla \cdot \left[i \frac{\partial \phi}{\partial x} + j \frac{\partial \phi}{\partial y} + k \frac{\partial \phi}{\partial z} \right]$$

$$= \left[i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z} \right] \cdot \left[i \frac{\partial \phi}{\partial x} + j \frac{\partial \phi}{\partial y} + k \frac{\partial \phi}{\partial z} \right]$$

$$= \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}$$

$$\therefore \nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}$$

$$\therefore \nabla^2 \phi = \nabla \cdot \nabla \phi$$

$$= \text{Div grad } \phi$$

Vector Identities

① $\nabla\cdot(\phi\bar{a}) = \nabla\phi\cdot\bar{a} + \phi(\nabla\cdot\bar{a})$

② $\nabla\times(\phi\bar{a}) = \nabla\phi\times\bar{a} + \phi(\nabla\times\bar{a})$

③ $\nabla(\bar{A}\cdot\bar{B}) = (\bar{B}\cdot\nabla)\bar{A} + (\bar{A}\cdot\nabla)\bar{B}$
$+ \bar{B}\times(\nabla\times\bar{A}) + \bar{A}\times(\nabla\times\bar{B})$

④ $\nabla\cdot(\bar{a}\times\bar{b}) = \bar{b}\cdot(\nabla\times\bar{a}) - \bar{a}\cdot(\nabla\times\bar{b})$

⑤ $\nabla\times(\bar{a}\times\bar{b}) = (\nabla\cdot\bar{b})\bar{a} - (\nabla\cdot\bar{a})\bar{b}$
$+ (\bar{b}\cdot\nabla)\bar{a} - (\bar{a}\cdot\nabla)\bar{b}$

⑥ curl grad $\phi = 0$ (or) $\nabla\times\nabla\phi = 0$

⑦ Div curl $f = 0$ (or) $\nabla\cdot(\nabla\times f) = 0$

⑧ div $(f\nabla g) = f\nabla^2 g + \nabla f\cdot\nabla g$

⑨ $\nabla\times(\nabla\times\bar{a}) = \nabla(\nabla\cdot\bar{a}) - \nabla^2\bar{a}$

Line integral

Let $\bar{F}$ represents the force vector acting on a particle moving along an arc AB. Then the work done by $\bar{F}$ during displacement from A to B is given by the line integral $\int_A^B \bar{F} \cdot d\bar{r}$

If the force $\bar{F}$ is conservative $\bar{F} = \nabla \phi$ then the work done is independent of the path. In this case curl $\bar{F} = 0$

If $\bar{F}$ is conservative force field then

$$\nabla \times \bar{F} = 0$$

$\therefore$ A conservative force field is irrotational

Surface integral:

let $\bar{F}(r) = F_1 i + F_2 j + F_3 \bar{k}$ where

F_1, F_2, F_3 are continuous and differen

functions of x, y, z

Then $\int \bar{F}\cdot\bar{n}\, dS = \iint F_1\, dy\, dz + \iint F_2\, dz\, dx$
$+ \iint F_3\, dx\, dy$

Note: let R_1 be the projection of S on xy-plane Then

$$\int_S \bar{F}\cdot\bar{n}\, dS = \iint_{R_1} \frac{\bar{F}\cdot\bar{n}\, dx\, dy}{|\bar{n}\cdot\bar{k}|}$$

lly $$\int_S \bar{F}\cdot\bar{n}\, dS = \iint_{R_2} \frac{\bar{F}\cdot\bar{n}\, dy\, dz}{|\bar{n}\cdot i|}$$

$$\int_S \bar{F}\cdot\bar{n}\, dS = \iint_{R_3} \frac{\bar{F}\cdot\bar{n}\, dx\, dz}{|\bar{n}\cdot j|}$$

where R_2, R_3 are the projection of S on yz, zx planes respectively

Volume integrals

let $\bar{F}(r) = F_1 i + F_2 j + F_3 \bar{k}$ where F_1, F_2, F_3 are functions of x, y, z. we know that $dV = dx\,dy\,dz$. The volume integral is given by

$$\int_V \bar{F}\, dV = \iiint (F_1 i + F_2 j + F_3 \bar{k})\, dx\,dy\,dz$$

$$= i \iiint F_1\, dx\,dy\,dz + j \iiint F_2\, dx\,dy\,dz$$

$$+ k \iiint F_3\, dx\,dy\,dz$$

Gauss Divergence Theorem:

let S be a closed surface enclosing a volume V. If $\bar{F}$ is a continuously differentiable vector point function

Then $$\int_V \operatorname{div} \bar{F}\, dV = \int_S \bar{F} \cdot \bar{n}\, dS$$

where $\bar{n}$ is outward drawn normal vector at any point of S

Green's Theorem:

If S is a closed region in xy-plane bounded by a simple closed curve C and if M and N are continuous functions of x and y having continuous derivatives in R then

$$\oint_C M\,dx + N\,dy = \iint_S \left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right] dx\,dy$$

where C is the traversed in the positive direction

Stoke's Theorem:

let S be a open surface bounded by a closed, non intersecting curve C, if $\bar{F}$ is any differentiable vector point function then $\oint_C \bar{F}\cdot d\bar{r} = \int_S \text{curl}\,\bar{F}\cdot\bar{n}\,dS$ where C is the traversed in the positive direction, and n is unit outward drawn normal at any point of the surface.

(I) Directional Derivative of surface to vector :

(1) Find the D.D of $2xy + z^2$ at $(1,-1,2)$ in the direction of $i + 2j + 3\bar{k}$

Sol: given $\phi = 2xy + z^2$

$$\boxed{\begin{array}{l} (1)\ \nabla\phi|_p \\ (2)\ \bar{e} = \frac{\nabla f}{|\nabla f|} \\ (3)\ \bar{e} \cdot \nabla\phi|_p \end{array}}$$

Now $\nabla\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$

$\nabla\phi = \hat{i}(2y) + \hat{j}(2x) + k(2z)$

$\nabla\phi|_{P(1,-1,2)} = -2\bar{i} + 2\bar{j} + 4\bar{k} \rightarrow (1)$

Since $\bar{f} = i + 2j + 3\bar{k}$

$$\bar{e} = \frac{f}{|f|} = \frac{i + 2j + 3\bar{k}}{\sqrt{1 + 2^2 + 3^2}}$$

$$\bar{e} = \frac{1}{\sqrt{14}}(1i + 2j + 3\bar{k}) \rightarrow (2)$$

Hence The D.D of $\phi = 2xy + z^2$ at $(1,-1,2)$

in The direction of $i + 2j + 3k = \bar{e} \cdot \nabla\phi$

$$\bar{e} \cdot \nabla\phi = \frac{1}{\sqrt{14}}(1i + 2j + 3k) \cdot (-2i + 2j + 4k)$$

$$= \frac{(-2) + (4) + (12)}{\sqrt{14}}$$

$$= \frac{14}{\sqrt{14}}$$

$$\boxed{\bar{e} \cdot \nabla\phi = \sqrt{14}}$$

(II) Directional Derivative of surface to two points

Here

(1) Find the Directional Derivative of the function $f = x^2 - y^2 + 2z^2$ at the point $P(1,2,3)$ in the direction of the line PQ where $Q(5,0,4)$

$$\boxed{\begin{array}{l} (1)\ \nabla f|_p \\ (2)\ \bar{e} \\ (3)\ \bar{e} \cdot \nabla f|_p \end{array}}$$

Sol. given $f = x^2 - y^2 + 2z^2$

Now $\nabla f = i \dfrac{\partial f}{\partial x} + j \dfrac{\partial f}{\partial y} + k \dfrac{\partial f}{\partial z}$

$\nabla f = i(2x) + j(-2y) + k(4z)$

$\nabla f|_{P(1,2,3)} = 2\bar{i} - 4\bar{j} + 12\bar{k} \rightarrow (1)$

Here $\overline{OP} = 1\bar{i} + 2\bar{j} + 3\bar{k}$

$\overline{OQ} = 5\bar{i} + 0\bar{j} + 4\bar{k}$

$\overline{PQ} = \overline{OQ} - \overline{OP}$

$\overline{PQ} = 4\bar{i} - 2\bar{j} + \bar{k}$

Hence $\bar{e} = \frac{\overline{PQ}}{|PQ|}$

$$= \frac{4\bar{i} - 2\bar{j} + \bar{k}}{\sqrt{16+4+1}}$$

$$\bar{e} = \frac{1}{\sqrt{21}}(4\bar{i} - 2\bar{j} + \bar{k}) \rightarrow (2)$$

Hence The D.D $= \bar{e} \cdot \nabla f \big|_p$

$$= \frac{1}{\sqrt{21}}(4\bar{i} - 2\bar{j} + \bar{k}) \cdot (2i - 4j + 12k)$$

$$= \frac{8 + 8 + 12}{\sqrt{21}}$$

$$\boxed{\bar{e} \cdot \nabla\phi = \frac{28}{\sqrt{21}}}$$

Directional Derivative of surface to surface

① Find the D.D of $\phi(x,y,z) = x^2yz + 4xz^2$ at the point $(1,-2,-1)$ in the direction of the normal to the surface $f(x,y,z) = x\log z - y^2$ at $(-1,2,1)$

① $\nabla\phi|_P$
② $\bar{e}$
③ $\bar{e}\cdot\nabla\phi_P$

Sol given $\phi = x^2yz + 4xz^2$

Now $\nabla\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$

$\nabla\phi = i(2xyz + 4z^2) + j(x^2z) + k(x^2y + 8xz)$

$\nabla\phi|_{P(1,-2,-1)} = 8i - j - 10\bar{k} \rightarrow ①$

given $f = x\log z - y^2$

Now $\nabla f = i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} + k\frac{\partial f}{\partial z}$

$\nabla f = i(\log z) + j(-2y) + k\, x\cdot\frac{1}{z}$

$$\nabla f \Big|_{(-1,2,1)} = i(\log 1) + j(-2(2)) + k\left(\frac{-1}{1}\right)$$

$$= 0i - 4j - k$$

Now unit vector along $0i - 4j - \bar{k}$ is

$$\bar{e} = \frac{0i - 4j - \bar{k}}{\sqrt{16+1}}$$

$$\bar{e} = \frac{1}{\sqrt{17}}(0i - 4j - \bar{k}) \rightarrow (2)$$

Hence $\boxed{D \cdot D = \bar{e} \cdot \nabla \phi|_p}$

$$= \frac{1}{\sqrt{17}}(0i - 4j - \bar{k}) \cdot (8i - j - 10\bar{k})$$

$$= \frac{4+10}{\sqrt{17}} = \frac{14}{\sqrt{17}}$$

$$\boxed{\bar{e} \cdot \nabla \phi = \frac{14}{\sqrt{17}}}$$

(iv) Directional Derivative of surface to curve

① Find the D.D of the function $xy^2+yz^2+zx^2$ along the tangent to curve $x=t, y=t^2, z=t^3$ at the point $(1,1,1)$

① $\nabla\phi|_P$
② $\bar{e}$
③ $\bar{e}\cdot\bar{\nabla}\phi$

Sol. given $\phi = xy^2+yz^2+zx^2$

now $\nabla\phi = i\frac{\partial\phi}{\partial x}+j\frac{\partial\phi}{\partial y}+k\frac{\partial\phi}{\partial z}$

$\nabla\phi = i\left[y^2+2xz\right]+j\left[2xy+z^2\right]+\bar{k}\left[2yz+x^2\right]$

$\nabla\phi\Big|_{P(1,1,1)} = 3i+3j+3\bar{k} \rightarrow ①$

given curve $x=t, y=t^2, z=t^3$

let $\bar{r} = xi+yj+z\bar{k}$

$\bar{r} = ti+t^2j+t^3\bar{k}$

$\frac{d\bar{r}}{dt} = 1i+2tj+3t^2\bar{k}$

Here $x=t, y=t^2, z=t^3$
$1=t, 1=t^2, 1=t^3$

$\boxed{t=1}$

Hence $\left.\frac{d\bar{r}}{dt}\right|_{t=1} = 1\hat{i} + 2\hat{j} + 3\bar{k}$

Hence $\bar{e} = \frac{1\hat{i} + 2\hat{j} + 3\bar{k}}{\sqrt{9+4+1}}$

$$\bar{e} = \frac{1}{\sqrt{14}}(1\hat{i} + 2\hat{j} + 3\bar{k}) \rightarrow (2)$$

Hence $\boxed{D \cdot D = \bar{e} \cdot \nabla\phi}$

$$= \frac{1}{\sqrt{14}}(1\hat{i} + 2\hat{j} + 3k) \cdot (3\hat{i} + 3\hat{j} + 3k)$$

$$= \frac{3+6+9}{\sqrt{14}} = \frac{18}{\sqrt{14}}$$

$$\boxed{\bar{e} \cdot \nabla\phi = \frac{18}{\sqrt{14}}}$$

① Find the unit normal vector to the surface $x^2+y^2+2z^2=26$ at the point $(2,2,3)$

sol given surface $\phi = x^2+y^2+2z^2-26$

Now $\nabla\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$

$\nabla\phi = i(2x)+j(2y)+k(4z)$

$\nabla\phi|_{P(2,2,3)} = 4\bar{i}+4\bar{j}+12\bar{k} = \bar{f}$ (say)

Hence $\bar{e} = \frac{\bar{f}}{|f|} = \frac{4\bar{i}+4\bar{j}+12\bar{k}}{\sqrt{16+16+144}}$

$= \frac{4\bar{i}+4\bar{j}+12\bar{k}}{\sqrt{176}}$

$$\boxed{\bar{e} = \frac{\bar{i}+\bar{j}+3\bar{k}}{\sqrt{11}}}$$

Angle between to the normals

① Angle between surface to surface

② Angle between surface to two points

$$\cos\theta = \frac{n_1 \cdot n_2}{|n_1|\,|n_2|}$$

Angle between surface to surface

① Find the angle between the surfaces $x^2+y^2+z^2=29$ and $x^2+y^2+z^2+4x-6y-8z-47=0$

at the point $(4,-3,2)$

se given $f = x^2+y^2+z^2-29$

$$\nabla f = i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} + k\frac{\partial f}{\partial z}$$

$$\nabla f = i(2x) + j(2y) + k(2z)$$

$$\nabla f\Big|_{P(4,-3,2)} = 8\bar{i} - 6\bar{j} + 4\bar{k} = n_1 \text{ (say)}$$

Now $|n_1| = \sqrt{64+36+16}$

$= \sqrt{116}$

given $g = x^2+y^2+z^2+4x-6y-8z-47$

Now $\nabla g = i\frac{\partial g}{\partial x} + j\frac{\partial g}{\partial y} + k\frac{\partial g}{\partial z}$

$\nabla g = i(2x+4) + j(2y-6) + k(2z-8)$

$\nabla g\Big|_{P(4,-3,2)} = 12i - 12j - 4k = n_2$ (say)

$|n_2| = \sqrt{144+144+16} = \sqrt{304}$

let θ be the Angle between the surfaces Then

$$\cos\theta = \frac{n_1 \cdot n_2}{|n_1|\,|n_2|}$$

$$= \frac{(8i-6j+4k)\cdot(12i-12j-4k)}{\sqrt{116}\,\sqrt{304}}$$

$$= \frac{96 + 72 - 16}{\sqrt{116}\ \sqrt{304}}$$

$$\cos\theta = \frac{152}{\sqrt{116}\ \sqrt{304}}$$

$$\boxed{\theta = \cos^{-1}\left(\frac{152}{\sqrt{116}\ \sqrt{304}}\right)}$$

(or)

$$\theta = \cos^{-1}\left(\sqrt{\frac{19}{29}}\right)$$

Angle between surface to two points

① Find the angle between the normals to the surface $x^2 = yz$ at the points $(1,1,1)$ and $(2,4,1)$

Sol: given $\phi = x^2 - yz$

Now $\nabla\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$

$\nabla\phi = i(2x) + j(-z) + k(-y) \rightarrow ①$

$\nabla\phi\Big|_{P(1,1,1)} = 2i - j - \bar{k} = n_1$ (say)

$|n_1| = \sqrt{4+1+1} = \sqrt{6}$

$\nabla\phi\Big|_{Q(2,4,1)} = 4i - j - 4\bar{k} = n_2$ (say)

$|n_2| = \sqrt{16+1+16} = \sqrt{33}$

let θ be be the angle between two normals

$$\cos\theta = \frac{n_1 \cdot n_2}{|n_1|\,|n_2|}$$

$$= \frac{(2i - j - k)\cdot(4i - j - 4k)}{\sqrt{6}\,\sqrt{33}}$$

$$= \frac{8 + 1 + 4}{\sqrt{6}\,\sqrt{33}} = \frac{13}{\sqrt{198}}$$

Hence $\boxed{\theta = \cos^{-1}\left(\frac{13}{\sqrt{198}}\right)}$

Application

$$* \quad \nabla[f(r)] = \frac{f'(r)}{r}\,\bar{r} \quad \text{where } \bar{r} = xi + yj + zk$$

*

Sol Since $\bar{r} = xi + yj + z\bar{k}$

$$|r| = \sqrt{x^2 + y^2 + z^2}.$$

$$r^2 = x^2 + y^2 + z^2$$

Diff partially wrt x, y, z; we get

$$r^2 = x^2 + y^2 + z^2$$

$$2r\frac{\partial r}{\partial x} = 2x$$

$$\frac{\partial r}{\partial x} = \frac{2x}{2r} = \frac{x}{r}$$

$$\frac{\partial r}{\partial x} = \frac{x}{r}$$

lly $\dfrac{\partial r}{\partial y} = \dfrac{y}{r}$

$$\frac{\partial r}{\partial z} = \frac{z}{r}$$

Consider $\nabla f(r) =$

$$= \left(i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z} \right) f(r)$$

$$= i \frac{\partial}{\partial x} f(r) + j \frac{\partial}{\partial y} f(r) + k \frac{\partial}{\partial z} f(r)$$

$$= \sum i \frac{\partial}{\partial x} f(r)$$

$$= \sum i \, f'(r) \frac{\partial r}{\partial x}$$

$$= \sum i \, f'(r) \cdot \frac{x}{r}$$

$$= i \, f'(r) \frac{x}{r} + j \, f'(r) \frac{y}{r} + k \, f'(r) \frac{z}{r}$$

$$= \frac{f'(r)}{r} \left[x i + y j + z k \right]$$

*

$$\boxed{\nabla f(r) = \frac{f'(r)}{r} \cdot \bar{r}}$$ *

*

① Solve $\nabla \log r = \frac{1}{r^2} \bar{r}$

Sol. We know that

$$\nabla f(r) = \frac{f'(r)}{r} \cdot \bar{r}$$

Here $f(r) = \log r$

$$f'(r) = \frac{1}{r}$$

Hence $\boxed{\nabla \log r = \frac{1}{r^2} \cdot \bar{r}}$

② Solve $\nabla r^n = n r^{n-2} \cdot \bar{r}$

Sol We know that

$$\nabla f(r) = \frac{f'(r)}{r} \cdot \bar{r}$$

Here $f(r) = r^n$

$$f'(r) = n r^{n-1}$$

$$\nabla r^n = n\frac{r^{n-1}}{r}\cdot \bar{r}$$

$$\boxed{\nabla r^n = n \cdot r^{n-2}\, \bar{r}}$$

Evaluate $\nabla\left(\frac{1}{r}\right)$; $\nabla\left(\frac{1}{r^2}\right)$

① If $\nabla\phi = yz\,\bar{i} + zx\,\bar{j} + xy\,\bar{k}$, find ϕ

so Given $\nabla\phi = \bar{i}\, yz + \bar{j}\, zx + \bar{k}\, xy$

since $\nabla\phi = \bar{i}\,\frac{\partial\phi}{\partial x} + \bar{j}\,\frac{\partial\phi}{\partial y} + \bar{k}\,\frac{\partial\phi}{\partial z}$

$$\frac{\partial\phi}{\partial x} = yz \Rightarrow d\phi = yz\,dx$$

$$\int 1\, d\phi = yz\int 1\, dx$$

$$\boxed{\phi = xyz + c}$$

$$\frac{\partial\phi}{\partial y} = zx \Rightarrow d\phi = xz\, dy$$

$$\int 1\, d\phi = xz\int 1\, dy$$

$$\boxed{\phi = xyz + c}$$

$$\frac{\partial \phi}{\partial z} = xy \Rightarrow d\phi = xy \, dz$$

$$\int 1 \, d\phi = xy \int 1 \, dz$$

$$\boxed{\phi = xyz + c}$$

Hence $\boxed{\phi = xyz + c}$

Divergence :

Let $\bar{f}$ be any continuously differentiable vector point function Then

$\bar{i} \cdot \frac{\partial \bar{f}}{\partial x} + \bar{j} \cdot \frac{\partial \bar{f}}{\partial y} + \bar{k} \cdot \frac{\partial \bar{f}}{\partial z}$ called the Divergence of $\bar{f}$

Hence $\text{Div } \bar{f} = \nabla \cdot \bar{f}$

$$= \left(\bar{i} \frac{\partial}{\partial x} + \bar{j} \frac{\partial}{\partial y} + \bar{k} \frac{\partial}{\partial z} \right) \cdot \bar{f}$$

let $\bar{f} = f_1 i + f_2 j + f_3 \bar{k}$

$\text{Div } f = \nabla \cdot f$

$$= \left(i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z} \right) \cdot \left(f_1 \bar{i} + f_2 \bar{j} + f_3 \bar{k} \right)$$

$$= \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

Solenoidal vector:

A vector point function $\bar{f}$ is said to be solenoidal if $\text{Div } \bar{f} = 0$

① If $\bar{f} = xy^2\,\bar{i} + 2x^2 z\,\bar{j} - 3yz^2\,\bar{k}$

Then find $\operatorname{div}\bar{f}$ at $(1,-1,1)$

Sol: Given $\bar{f} = xy^2\,\bar{i} + 2x^2 z\,\bar{j} - 3yz^2\,\bar{k}$

Now $\operatorname{Div} f = \nabla \cdot f$

$$\operatorname{Div}\bar{f} = y^2 - 6yz$$

$$\operatorname{Div}\bar{f}\Big|_{(1,-1,1)} = 1 - 6(-1)(1) = 7$$

② If $\bar{f} = 3y^4 z^2\,\bar{i} + z^3 x^2\,\bar{j} - 3x^2 y^2\,\bar{k}$

Then find $\operatorname{div}\bar{f}$

Sol: Given $\bar{f} = 3y^4 z^2\,\bar{i} + z^3 x^2\,\bar{j} - 3x^2 y^2\,\bar{k}$

Now $\operatorname{Div} f = \nabla \cdot f$

$$\operatorname{Div}\bar{f} = 0$$

$\Rightarrow$ $\bar{f}$ is solenoidal

③ Find Div $\bar{f}$ where $\bar{f} = \text{grad}(x^3 + y^3 + z^3 - 3xyz)$

Sol given $\bar{f} = \text{grad}(x^3 + y^3 + z^3 - 3xyz)$

Now $\text{grad}\,\phi = \nabla\phi = \hat{i}\dfrac{\partial\phi}{\partial x} + \hat{j}\dfrac{\partial\phi}{\partial y} + k\dfrac{\partial\phi}{\partial z}$

$$\text{grad}\,\phi = \hat{i}(3x^2 - 3yz) + \hat{j}(3y^2 - 3xz) + \bar{k}(3z^2 - 3xy)$$

Hence

$$\bar{f} = (3x^2 - 3yz)\,\bar{i} + (3y^2 - 3xz)\,\bar{j} + (3z^2 - 3xy)\,\bar{k}$$

Now $\text{Div}\,\bar{f} = \nabla \cdot f$

$$= 6x + 6y + 6z \quad = 6(x + y + z)$$

Hence $\boxed{\text{Div}\,\bar{f} = \nabla \cdot \bar{f} = 6(x + y + z)}$

(4) If $\bar{f} = (x+3y)\bar{i} + (y-2z)\bar{j} + (x+pz)\bar{k}$ is solenoidal : Then find p

Sol given $\bar{f} = (x+3y)\bar{i} + (y-2z)\bar{j} + (x+pz)\bar{k}$

Since $\bar{f}$ is solenoidal $\Leftrightarrow$ Div $\bar{f} = 0$

$$1+1+p = 0$$

$$\boxed{p = -2}$$

Application;

Prove That $\text{Div}(r^n \bar{r}) = (n+3) r^n$

Hence show That $\frac{\bar{r}}{r^3}$ is solenoidal

(or)

Prove That $r^n \bar{r}$ is solenoidal if $n = -3$

(or)

Find Div $\bar{f}$ where $\bar{f} = r^n \bar{r}$; Find n if it is solenoidal

Sol We prove That

$$\operatorname{Div}(r^n \bar{r}) = (n+3) r^n$$

Since $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

$$|r| = \sqrt{x^2+y^2+z^2}$$

$$r^2 = x^2+y^2+z^2$$

Here $\frac{\partial r}{\partial x} = \frac{x}{r}$, $\frac{\partial r}{\partial y} = \frac{y}{r}$, $\frac{\partial r}{\partial z} = \frac{z}{r}$

Consider $\operatorname{Div}(r^n \bar{r})$

$$\operatorname{Div}\ r^n (x\bar{i} + y\bar{j} + z\bar{k})$$

$$\operatorname{Div}(x r^n i + y r^n j + z r^n k)$$

$$= \frac{\partial}{\partial x}(x r^n) + \frac{\partial}{\partial y}(y r^n) + \frac{\partial}{\partial z}(z r^n)$$

$$= \sum \frac{\partial}{\partial x}(x r^n)$$

$$+$$

$$= \sum \left[r^n + x\, n\, r^{n-1} \cdot \frac{\partial r}{\partial x} \right]$$

$$= \Sigma \left[r^n + n \cdot x \cdot r^{n-1} \cdot \frac{x}{r} \right]$$

$$= \Sigma \left[r^n + n \cdot r^{n-2} \cdot x^2 \right]$$

$$= r^n + n\, r^{n-2} \cdot x^2 + r^n + n \cdot r^{n-2} \cdot y^2 + r^n + n \cdot r^{n-2} \cdot z^2$$

$$= 3r^n + n\, r^{n-2} \left[x^2 + y^2 + z^2 \right]$$

$$= 3r^n + n\, r^{n-2} \cdot r^2$$

$$= 3r^n + n\, r^n$$

$$= (n+3)\, r^n$$

Hence $\boxed{\operatorname{Div}(r^n \bar{r}) = (n+3)\, r^n}$

If $n = -3$

Div $r^n \bar{r} = 0$

$\Leftrightarrow$ $r^n \bar{r}$ is solenoidal if $n = -3$

$\Leftrightarrow$ $r^{-3} \bar{r}$ is solenoidal

$\Leftrightarrow$ $\dfrac{\bar{r}}{r^3}$ is solenoidal

$$\boxed{\text{Evaluate } \nabla \cdot \frac{\bar{r}}{r^3} = 0}$$

① Evaluate $\nabla \cdot \dfrac{\bar{r}}{r^2}$

Sol $\nabla \cdot r^n \bar{r} = (n+3)\, r^n$

given $\nabla \cdot r^{-2} \bar{r}$

Here $n = -2$

Hence $\nabla \cdot \dfrac{\bar{r}}{r^2} = 1 \cdot r^{-2} = \dfrac{1}{r^2}$

② Evaluate Div $\frac{\bar{r}}{r^6}$, Div $\frac{\bar{r}}{r}$

Curl:

let $\bar{f}$ is continuously differentiable vector point function, Then the vector function defined by

$i \times \frac{\partial \bar{f}}{\partial x} + j \times \frac{\partial \bar{f}}{\partial y} + \bar{k} \times \frac{\partial \bar{f}}{\partial z}$ is called curl $\bar{f}$ and it is denoted by $\nabla \times \bar{f}$

$$\nabla \times \bar{f} = \text{curl } f = i \times \frac{\partial \bar{f}}{\partial x} + j \times \frac{\partial \bar{f}}{\partial y} + k \times \frac{\partial \bar{f}}{\partial z}$$

matrix representation for curl f

$$\text{curl } \bar{f} = \begin{vmatrix} \bar{i} & \bar{j} & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

where $\bar{f} = f_1 \bar{i} + f_2 \bar{j} + f_3 \bar{k}$

irrotational: If curl $\bar{f} = 0 \Leftrightarrow \bar{f}$ is irrotational.

① Find curl $\bar{f}$ where

$$\bar{f} = xy^2\,\bar{i} + 2x^2yz\,\bar{j} - 3yz^2\,\bar{k}$$

at the point $(1,-1,1)$

sol. given

$$\bar{f} = xy^2 i + 2x^2yz j - 3yz^2 k$$

Now

$$\text{curl } f = \nabla \times f = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy^2 & 2x^2yz & -3yz^2 \end{vmatrix}$$

$$= i\left[\frac{\partial}{\partial y}(-3yz^2) - \frac{\partial}{\partial z}(2x^2yz)\right]$$

$$- j\left[\frac{\partial}{\partial x}(-3yz^2) - \frac{\partial}{\partial z}(xy^2)\right]$$

$$+ k\left[\frac{\partial}{\partial x}(2x^2yz) - \frac{\partial}{\partial y}(xy^2)\right]$$

$$= i\left[-3z^2 - 2x^2y\right] - j\left[0 - 0\right]$$

$$+ k\left[4xyz - 2xy\right]$$

Now $\text{curl}\,\bar{f}\Big|_{P(1,-1,1)} = i[-3+2] + 0j + k[-4+2]$

$$\boxed{\text{curl}\,\bar{f} = -i + 0j - 2k}$$

(2) Find $\text{curl}\,\bar{f}$ where $\bar{f} = \text{grad}(x^3+y^3+z^3-3xyz)$

Sol: Given $\bar{f} = \text{grad}(x^3+y^3+z^3-3xyz)$

Here $\phi = x^3+y^3+z^3-3xyz$

Now $\text{grad}\,\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$

$\text{grad}\,\phi = i[3x^2-3yz] + j[3y^2-3xz] + k[3z^2-3xy]$

Hence

$\bar{f} = (3x^2-3yz)i + (3y^2-3xz)j + (3z^2-3xy)k$

Now

$$\operatorname{curl} \bar{f} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x^2-3yz & 3y^2-3xz & 3z^2-3xy \end{vmatrix}$$

$$= i\left[-3x+3x\right] - j\left[-3y+3y\right] + k\left[-3z+3z\right]$$

$$\operatorname{curl} \bar{f} = 0i + 0j + 0\bar{k}$$

$$\boxed{\operatorname{curl} \bar{f} = \bar{0}}$$

$\Leftrightarrow$ $\boxed{\bar{f} \text{ is irrotational.}}$

<u>Note:</u> Here $\bar{f} = \operatorname{grad} \phi$

$\Rightarrow \operatorname{curl} \bar{f} = \operatorname{curl} \operatorname{grad} \phi = \bar{0}$

$\Rightarrow \boxed{\operatorname{curl} \operatorname{grad} \phi = 0}$

$\Leftrightarrow$ $\operatorname{grad} \phi$ is irrotational

③ Find constants a, b, c if the vector

$$\bar{f} = (2x+3y+az)\,\bar{i} + (bx+2y+3z)\,\bar{k} + (2x+cy+3z)\,\bar{k} \text{ is irrotational}$$

sol: given $\bar{f}$ is irrotational $\Leftrightarrow$ curl $\bar{f} = 0$

$$\begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2x+3y+az & bx+2y+3z & 2x+cy+3z \end{vmatrix} = 0$$

$$i[c-3] - j[2-a] + \bar{k}[b-3] = 0$$

$$(c-3)\,i + (a-2)\,j + (b-3)\,\bar{k} = 0i + 0j + 0\bar{k}$$

$c-3=0$ $\quad$ $a-2=0$ $\quad$ $b-3=0$

$\boxed{c=3}$ $\quad$ $\boxed{a=2}$ $\quad$ $\boxed{b=3}$

④* Find constants a, b, c so that the vector $\bar{A} = (x+2y+az)\bar{i} + (bx-3y-z)\bar{j} + (4x+cy+2z)\bar{k}$ is irrotational. Also find ϕ such that $\bar{A} = \nabla\phi$

Sol Since $\bar{A}$ is irrotational $\Leftrightarrow$ curl $\bar{A} = 0$

$$\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+2y+az & bx-3y-z & 4x+cy+2z \end{vmatrix} = \bar{0}$$

$$i[c+1] - j[4-a] + k[b-2] = \bar{0}$$

$c+1=0$ $\quad$ $a-4=0$ $\quad$ $b-2=0$

$\boxed{c=-1}$ $\quad$ $\boxed{a=4}$ $\quad$ $\boxed{b=2}$

Hence

$$\bar{A} = (x+2y+4z)i + (2x-3y-z)j + (4x-y+2z)\bar{k}$$

Since $\boxed{\bar{A} = \nabla \phi}$

Since $\nabla \phi = i \frac{\partial \phi}{\partial x} + j \frac{\partial \phi}{\partial y} + k \frac{\partial \phi}{\partial z}$

Here $\frac{\partial \phi}{\partial x} = x + 2y + 4z$ [i coeff com]

$$d\phi = (x + 2y + 4z)\, dx$$

$$\int 1\, d\phi = \int (x + 2y + 4z)\, dx$$

$$\phi = \frac{x^2}{2} + 2xy + 4xz + c \rightarrow (1)$$

Here $\frac{\partial \phi}{\partial y} = 2x - 3y - z$ (j coeff compt)

$$d\phi = (2x - 3y - z)\, dy$$

$$\int 1\, d\phi = \int (2x - 3y - z)\, dy$$

$$\phi = 2xy - \frac{3y^2}{2} - yz + c \rightarrow (2)$$

Here $\frac{\partial \phi}{\partial z} = 4x - y + 2z$

$$d\varphi = (4x - y + 2z)\,dz$$

$$\int 1\, d\varphi = \int (4x - y + 2z)\,dz$$

$$\varphi = 4xz - yz + z^2 + c \rightarrow (3)$$

Hence required ϕ is

$$\boxed{\phi = \frac{x^2}{2} - \frac{3y^2}{2} + z^2 + 2xy - yz + 4xz + c}$$

Application

prove that curl $r^n \bar{r} = 0$

sol since $\bar{r} = xi + yj + z\bar{k}$

$$|r| = \sqrt{x^2 + y^2 + z^2}$$

$$r^2 = x^2 + y^2 + z^2$$

Here $\frac{\partial r}{\partial x} = \frac{x}{r}$, $\frac{\partial r}{\partial y} = \frac{y}{z}$, $\frac{\partial r}{\partial z} = \frac{z}{r}$

Here $r^n \bar{r} = r^n [xi + yj + z\bar{k}]$

$$r^n \bar{r} = x r^n i + y r^n j + z r^n \bar{k}$$

Now

$$\text{curl } r^n \bar{r} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x r^n & y r^n & z r^n \end{vmatrix}$$

$$= i\left[\frac{\partial}{\partial y} z r^n - \frac{\partial}{\partial z} y r^n\right] - j\left[\frac{\partial}{\partial x} z r^n - \frac{\partial}{\partial z} x r^n\right]$$

$$+ k\left[\frac{\partial}{\partial x} y r^n - \frac{\partial}{\partial y} x r^n\right]$$

$$= i\left[z\, n r^{n-1} \cdot \frac{y}{r} - y\, n r^{n-1} \frac{z}{r}\right]$$

$$- j\left[z\, n r^{n-1} \cdot \frac{x}{r} - x \cdot n r^{n-1} \cdot \frac{z}{r}\right]$$

$$+ k\left[y\, n r^{n-1} \cdot \frac{x}{r} - x \cdot n r^{n-1} \cdot \frac{y}{r}\right]$$

$= i \left[yz\, n r^{n-2} - yz\, n r^{n-2} \right]$

$+ j \left[xz\, n r^{n-2} - xz\, n r^{n-2} \right]$

$+ k \left[xy\, n r^{n-2} - xy\, n r^{n-2} \right]$

$= 0i + 0j + 0\bar{k}$

$\Rightarrow \boxed{\text{curl } r^n \bar{r} = 0}$

$\Leftrightarrow$ $r^n \bar{r}$ is irrotational

Note

Evaluate curl $\dfrac{\bar{r}}{r^2}$; curl $r^2 \bar{r}$

$= 0$ $= 0$

Laplacian operator ∇^2

$$\nabla^2 = \nabla \cdot \nabla$$

$$\nabla^2 \phi = \nabla \cdot \nabla \phi$$

$$= \nabla \cdot \left[i \frac{\partial \phi}{\partial x} + j \frac{\partial \phi}{\partial y} + k \frac{\partial \phi}{\partial z} \right]$$

$$= \left[i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z} \right] \cdot \left[i \frac{\partial \phi}{\partial x} + j \frac{\partial \phi}{\partial y} + k \frac{\partial \phi}{\partial z} \right]$$

$$= \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}$$

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}$$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$$\boxed{\begin{aligned} \nabla^2 \phi &= \nabla \cdot \underline{\nabla \phi} \\ &= \mathrm{Div}(\mathrm{grad}\,\phi) \end{aligned}}$$

① Prove That $\operatorname{Div}(\operatorname{grad} r^m) = m(m+1) r^{m-2}$

(or) $\nabla^2 r^m = m(m+1) r^{m-2}$

Sol: Since $\bar{r} = xi + yj + z\bar{k}$

$$|r| = \sqrt{x^2+y^2+z^2}$$

$$r^2 = x^2+y^2+z^2$$

$$\frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r}$$

consider

$$\nabla^2 r^m = \frac{\partial^2 r^m}{\partial x^2} + \frac{\partial^2 r^m}{\partial y^2} + \frac{\partial^2 r^m}{\partial z^2}$$

$$= \Sigma \frac{\partial^2}{\partial x^2} r^m$$

consider $\frac{\partial^2}{\partial x^2} r^m = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial x} r^m\right]$

$$= \frac{\partial}{\partial x} m r^{m-1} \cdot \frac{x}{r}$$

$$= m \frac{\partial}{\partial x} r^{m-2} \cdot x$$

$$\boxed{d(uv) = vu' + uv'}$$

$$= m\left[(m-2)\, r^{m-3}\, \frac{x}{r} \cdot x + 1 \cdot r^{m-2}\right]$$

$$\frac{\partial^2}{\partial x^2} r^m = m(m-2)\, r^{m-4} \cdot x^2 + m\, r^{m-2}$$

$$||^{ly} \quad \frac{\partial^2}{\partial y^2} r^m = m(m-2)\, r^{m-4} \cdot y^2 + m\, r^{m-2}$$

$$\frac{\partial^2}{\partial z^2} r^m = m(m-2)\, r^{m-4} \cdot z^2 + m\, r^{m-2}$$

Hence

$$\nabla^2 r^m = m(m-2)\, r^{m-4}\left[x^2 + y^2 + z^2\right] + 3m\, r^{m-2}$$

$$= m(m-2)\, r^{m-4} \cdot r^2 + 3m\, r^{m-2}$$

$$= m(m-2)\, r^{m-2} + 3m\, r^{m-2}$$

$$= m\, r^{m-2}\left[m - 2 + 3\right]$$

$$= m(m+1) r^{m-2}$$

$$\boxed{\nabla^2 r^m = m(m+1) r^{m-2}} \quad \otimes$$

Note:

① If $m = -1$ Then $\nabla^2 \left(\frac{1}{r}\right) = 0$

$$\nabla \cdot \left(\nabla \frac{1}{r}\right) = 0$$

$\Rightarrow$ $\nabla \frac{1}{r}$ is solenoidal

② If $m = 2$ Find $\nabla^2 r^2 = 6$

② Show that

$$\nabla^2 f(r) = \frac{d^2 f}{dr^2} + \frac{2}{r}\frac{df}{dr} = f''(r) + \frac{2}{r} f'(r)$$

Sol: consider $\nabla^2 f(r)$

$$\nabla^2 f(r) = \frac{\partial^2}{\partial x^2} f(r) + \frac{\partial^2}{\partial y^2} f(r) + \frac{\partial^2}{\partial z^2} f(r)$$

$$= \Sigma \frac{\partial^2}{\partial x^2} f(r)$$

consider $\frac{\partial^2}{\partial x^2} f(r) = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial x} f(r)\right]$

$$= \frac{\partial}{\partial x}\left[f'(r)\frac{x}{r}\right] = \frac{\partial}{\partial x}\left[\frac{x f'(r)}{r}\right]$$

$$d\left(\frac{uv}{w}\right) = \frac{w\, d(uv) - uv\, d(w)}{w^2}$$

$d(uv) = u'v + uv'$

$u = x$ $\quad d(x \cdot f'(r))$

$v = f'(r)$ $\quad = x f''(r) \frac{x}{r} + f'(r)$

$w = r$

$$= \frac{r\left[x f''(r)\frac{x}{r} + f'(r)\right] - x f'(r)\frac{x}{r}}{r^2}$$

$$= \frac{x^2 f''(r) + r f'(r) - \frac{x^2}{r} f'(r)}{r^2}$$

$$\frac{\partial^2 f(r)}{\partial x^2} = \frac{x^2 f''(r)}{r^2} + \frac{1}{r} f'(r) - \frac{x^2}{r^3} f'(r)$$

||ly

$$\frac{\partial^2}{\partial y^2} f(r) = \frac{y^2 f''(r)}{r^2} + \frac{1}{r} f'(r) - \frac{y^2}{r^3} f'(r)$$

$$\frac{\partial^2}{\partial z^2} f(r) = \frac{z^2 f''(r)}{r^2} + \frac{1}{r} f'(r) - \frac{z^2}{r^3} f'(r)$$

$$\nabla^2 f(r) = \frac{f''(r)}{r^2}\left[x^2+y^2+z^2\right] + \frac{3}{r} f'(r)$$

$$- \frac{f'(r)}{r^3}\left[x^2+y^2+z^2\right]$$

since $x^2+y^2+z^2 = r^2$

$$\nabla^2 f(r) = f''(r) + \frac{3}{r} f'(r) - \frac{1}{r} f'(r)$$

$$\boxed{\nabla^2 f(r) = f''(r) + \frac{2}{r} f'(r)} \quad (*)$$

① Evaluate $\nabla^2\left(\frac{1}{r}\right)$

Sol given $\nabla^2\left(\frac{1}{r}\right)$

we know that

$$\nabla^2 f(r) = f''(r) + \frac{2}{r} f'(r)$$

Here $f(r) = \frac{1}{r}$ $\quad \boxed{d(r^n) = n r^{n-1}}$

$$f'(r) = -\frac{1}{r^2}$$

$$f''(r) = \frac{2}{r^3}$$

hence

$$\nabla^2\left(\frac{1}{r}\right) = \frac{2}{r^3} + \frac{2}{r}\left(-\frac{1}{r^2}\right)$$

$$= \frac{2}{r^3} - \frac{2}{r^3}$$

$$\boxed{\nabla^2\left(\frac{1}{r}\right) = 0}$$

note: $\nabla \cdot \nabla\left(\frac{1}{r}\right) = 0$

$\text{Div } f = \nabla \cdot f = 0 \iff f$ is solenoidal

$\nabla\left(\frac{1}{r}\right)$ is solenoidal

Evaluate (i) $\nabla^2(\log r)$

(ii) $\nabla^2 r^m$

① If ϕ satisfies Laplacian equation show that $\nabla\phi$ is both solenoidal and irrotational

Sol: given ϕ satisfies Laplacian Eqn

$$\nabla^2\phi = 0$$

$$\nabla \cdot \nabla\phi = 0$$

$$\left[\text{Div } f = \nabla \cdot f = 0 \Leftrightarrow f \text{ solenoidal} \right]$$

$$\Leftrightarrow \boxed{\nabla\phi \text{ is solenoidal}}$$

also curl grad $\phi = 0$ ⊛

grad ϕ irrotational

$$\boxed{\nabla\phi \text{ is irrotational}}$$

② prove that curl grad $\phi = 0$

$\nabla \times \text{grad}\,\phi = 0$

sol: we prove that $\nabla \times (\nabla \phi) = 0$

$$\nabla \phi = i \frac{\partial \phi}{\partial x} + j \frac{\partial \phi}{\partial y} + k \frac{\partial \phi}{\partial z}$$

Now

curl grad ϕ

$$= \nabla \times \nabla \phi = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial z} \end{vmatrix}$$

$$= i \left[\frac{\partial^2 \phi}{\partial y \partial z} - \frac{\partial^2 \phi}{\partial z \partial y} \right] - j \left[\frac{\partial^2 \phi}{\partial x \partial z} - \frac{\partial^2 \phi}{\partial z \partial x} \right]$$

$$+ k \left[\frac{\partial^2 \phi}{\partial x \partial y} - \frac{\partial^2 \phi}{\partial y \partial x} \right]$$

$$= 0$$

curl grad $\phi = 0$

$\boxed{\begin{array}{l} \text{curl } f = 0 \\ f \text{ irrotational} \end{array}}$

$\Leftrightarrow$ grad ϕ irrotational

$\nabla\phi$ irrotational

* *
*

Find the values of a and b so that the surfaces $ax^2 - byz = (a+2)x$ and $4x^2y + z^3 = 4$ may intersect orthogonally at the point $(1, -1, 2)$

(or)

Find the constants a and b so that the surface $ax^2 - byz = (a+2)x$ will be orthogonal to the surface $4x^2y + z^3 = 4$ at the point $(1, -1, 2)$

Sol: let the given surfaces be

$$f(x, y, z) = ax^2 - byz - (a+2)x$$

$$g(x, y, z) = 4x^2y + z^3 - 4$$

f and g intersect at $P(1, -1, 2)$

$f|_P = 0$ and $g|_P = 0$

$$g(1,-1,2) = 4(1)(-1) + 8 - 4$$

$$= -4 + 8 - 4$$

$$= 0$$

$$f(1,-1,2) = a(1) - b(-1)(2) - (a+2)(1) = 0$$

$$a + 2b - a - 2 = 0$$

$$2b - 2 = 0$$

$$\boxed{b = 1}$$

Hence

$$f(x,y,z) = ax^2 - yz - (a+2)x$$

Now $\nabla f = i(2ax - a - 2) + j(-z) + k(-y)$

$$\nabla f\Big|_{P(1,-1,2)} = (a-2)\bar{i} - 2\bar{j} + \bar{k} = n_1$$

Here

$$g(x,y,z) = 4x^2y + z^3 - 4$$

Now $\nabla g = i(8xy) + j(4x^2) + k(3z^2)$

$$\nabla g\Big|_{P(1,-1,2)} = -8\bar{i} + 4\bar{j} + 12\bar{k} = n_2$$

Since f and g are orthogonal at $(1,-1,2)$

Hence $\theta = \frac{\pi}{2}$

Angle between two normals n_1, n_2 is

$$\boxed{\cos\theta = \frac{n_1 \cdot n_2}{|n_1||n_2|}}$$

Here $\theta = \frac{\pi}{2}$ $\quad \cos\frac{\pi}{2} = 0$

$$\Rightarrow \boxed{n_1 \cdot n_2 = 0}$$

$$\therefore \left[(a-2)i - 2j + k\right] \cdot \left[-8i + 4j + 12k\right] = 0$$

$$-8(a-2) - 8 + 12 = 0$$

$$-8(a-2) = -4$$

$$a - 2 = \frac{1}{2}$$

$$\boxed{a = \frac{5}{2}}$$

Hence $\boxed{a = \frac{5}{2}, \quad b = 1}$

line integral

surface integral

volume integral

line integral:

The integral $\int_C \bar{F} \cdot d\bar{r}$ taken along the curve C is called line integral of $\bar{F}$ along C

Note: If $\int_C \bar{F} \cdot d\bar{r} = 0$ then the field $\bar{F}$ is called conservative. i.e no work is done and the energy is conserved.

line integral: ① Evaluate $\int \bar{F} \cdot d\bar{r}$

② work done by force

③ conservative force field

① Find the work done by the force

$$\bar{F} = (3x^2 - 6yz)\,i + (2y + 3xz)\,j + (1 - 4xyz^2)\,k$$

in moving particle from the point $(0,0,0)$ to the point $(1,1,1)$ along the curve

$c:\ x = t,\ y = t^2,\ z = t^3$

Sol. Let $\bar{r} = x i + y j + z k$

$$d\bar{r} = dx\, i + dy\, j + dz\, k$$

Now $\bar{F} = (3x^2 - 6yz)\,i + (2y + 3xz)\,j$

$$+ (1 - 4xyz^2)\,k$$

Now $\bar{F} \cdot d\bar{r} = (3x^2 - 6yz)\,dx + (2y + 3xz)\,dy$

$$+ (1 - 4xyz^2)\,dz$$

Given curve $c:\ x = t,\ y = t^2,\ z = t^3$

$$dx = dt,\quad dy = 2t\,dt$$

$$dz = 3t^2\,dt$$

Hence

$$\bar{F} \cdot d\bar{r} = (3t^2 - 6t^5)\,dt + (2t^2 + 3t^4)\,2t\,dt$$

$$+ (1 - 4t^9)\,3t^2\,dt$$

$$= \left[3t^2 - \cancel{6t^5} + 4t^3 + \cancel{6t^5} + 3t^2 - 12t^{11}\right] dt$$

$$\bar{F} \cdot d\bar{r} = \left[-12t^{11} + 4t^3 + 6t^2\right] dt$$

The end points of curve are

$(0,0,0)$ and $(1,1,1)$

Here $t = 0$ to 1

Hence

$$\int_{(0,0,0)}^{(1,1,1)} \bar{F} \cdot d\bar{r} = \int_{t=0}^{1} \left[-12t^{11} + 4t^3 + 6t^2\right] dt$$

$$= \left[-12\frac{t^{12}}{12} + 4\frac{t^4}{4} + 6\frac{t^3}{3}\right]_0^1$$

$$= \left[-t^{12} + t^4 + 2t^3\right]_0^1$$

$$= -1 + 1 + 2 = 2$$

$$\boxed{\int_C \bar{F} \cdot d\bar{r} = 2}$$

② Find the work done in moving a particle in the force field

$\bar{F} = 3x^2 \bar{i} + j + z\bar{k}$ along the st line from $(0,0,0)$ to $(2,1,3)$

Sol: let $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

$d\bar{r} = dx\, i + dy\, j + dz\, \bar{k}$

Now $\bar{F} \cdot d\bar{r} = 3x^2\, dx + 1\, dy + z\, dz$

Given curve C: along the st line OP from $O(0,0,0)$ to $P(2,1,3)$

Equation of OP is $\dfrac{x-0}{2} = \dfrac{y-0}{1} = \dfrac{z-0}{3} = t$ (say)

$x = 2t, \quad y = t, \quad z = 3t$

$dx = 2dt, \quad dy = dt, \quad dz = 3dt$

from $O\,(0,0,0) \rightarrow t = 0$
$\; x, y, z$

$P\,(2,1,3) \rightarrow t = 1$
x, y, z

Hence t varies from 0 to 1

Hence $\bar{F}\cdot d\bar{r}$ along the curve C:

$$\bar{F}\cdot d\bar{r} = 3(4t^2)\,2\,dt + 1\,dt + 3t\cdot 3\,dt$$

$$= [24t^2 + 1 + 9t]\,dt$$

Work done in moving a particle in the force field

$$\int_C \bar{F}\cdot d\bar{r} = \int_{t=0}^{1} [24t^2 + 1 + 9t]\,dt$$

$$= \left[24\frac{t^3}{3} + t + 9\frac{t^2}{2}\right]_0^1$$

$$= \frac{24}{3} + 1 + \frac{9}{2}$$

$$= \frac{27}{2}$$

$$\boxed{\int_C \bar{F}\cdot d\bar{r} = \frac{27}{2}}$$

Examples:

① Find the work done in moving a particle in the force field

$\bar{F} = 3x^2\,\bar{i} + (2xz - y)\bar{j} + z\bar{k}$

along the st line from $(0,0,0)$ to $(2,1,3)$

② If $\bar{F} = xy\,\bar{i} - z\bar{j} + x^2\,\bar{k}$ and C is the curve $x = t^2,\ y = 2t,\ z = t^3$ from $t = 0$ to $t = 1$ Evaluate $\int_C \bar{F} \cdot d\bar{r}$

③ Find the work done by the force $\bar{F} = z\bar{i} + x\bar{j} + y\bar{k}$ when it moves a particle along the arc of the curve $\bar{r} = \cos t\,\bar{i} + \sin t\,\bar{j} - t\,\bar{k}$ from $t = 0$ to 2π

① If $\bar{F} = 3xy\,\bar{i} - y^2\,\bar{j}$ Evaluate $\int_C \bar{F}\cdot d\bar{r}$ where C is the curve $y = 2x^2$ in the xy-plane from $(0,0)$ to $(1,2)$

Sol. let $\bar{r} = x\,i + y\,j$

$d\bar{r} = dx\,i + dy\,j$

Now $\bar{F}\cdot d\bar{r} = 3xy\,dx - y^2\,dy$

Given curve C: $y = 2x^2$ in the xy plane from $(0,0)$ to $(1,2)$

Here $y = 2x^2$

$dy = 4x\,dx$

x vary from 0 to 1

Hence $\bar{F}\cdot d\bar{r}$ along the curve C:

$$\bar{F}\cdot d\bar{r} = 3x(2x^2)\,dx - 4x^4 \cdot 4x\,dx$$

$$= [6x^3 - 16x^5]\,dx$$

$$\int_C \bar{F}\cdot d\bar{r} = \int_{x=0}^{1} 6x^3 - 16x^5 \, dx$$

$$= \left[6\frac{x^4}{4} - 16\frac{x^6}{6} \right]_0^1$$

$$= \frac{6}{4} - \frac{16}{6}$$

$$= -\frac{7}{6}$$

$$\boxed{\int_C \bar{F}\cdot d\bar{r} = \frac{7}{6}}$$

(2) If $\bar{F} = (5xy - 6x^2)\bar{i} + (2y - 4x)\bar{j}$

Evaluate $\int_C \bar{F}\cdot d\bar{r}$ along the curve C in xy-plane $y = x^3$ from $(1,1)$ to $(2,8)$

① Find the circulation of

$$\bar{F} = (2x - y + 2z)\bar{i} + (x + y - z)\bar{j} + (3x - 2y - 5z)\bar{k}$$

along the circle $x^2 + y^2 = 4$ in the xy-plane

Sol: let $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

$$d\bar{r} = dx\,\bar{i} + dy\,\bar{j} + dz\,\bar{k}$$

on the xy-plane $z = 0 \Rightarrow dz = 0$

$$\boxed{\bar{F} \cdot d\bar{r} = (2x - y)\,dx + (x + y)\,dy}$$

Given curve c: along the circle

$$\boxed{x^2 + y^2 = 4}$$

Now it can be changed to polar-coord

$x = 2\cos\theta$, $y = 2\sin\theta$

$dx = -2\sin\theta\,d\theta$, $dy = 2\cos\theta\,d\theta$

θ varies from 0 to 2π

Hence $\bar{F} \cdot d\bar{r}$ along curve c is

$$\bar{F}\cdot d\bar{r} = \left[2(2\cos\theta) - 2\sin\theta\right]\left[-2\sin\theta\right]d\theta$$

$$+ \left[2\cos\theta + 2\sin\theta\right]\left[2\cos\theta\right]d\theta$$

$$= \left[-8\sin\theta\cos\theta + 4\sin^2\theta + 4\cos^2\theta + 4\sin\theta\cos\theta\right]d\theta$$

$$= \left[4 - 4\sin\theta\cos\theta\right]d\theta$$

$$= \left[4 - 2\cdot 2\sin\theta\cos\theta\right]d\theta$$

$$= \left[4 - 2\sin 2\theta\right]d\theta$$

Hence

$$\int_C \bar{F}\cdot d\bar{r} = \int_{\theta=0}^{2\pi}\left[4 - 2\sin 2\theta\right]d\theta$$

$$= \left[4\theta + 2\frac{\cos 2\theta}{2}\right]_0^{2\pi}$$

$$= \left[4\theta + \cos 2\theta\right]_0^{2\pi}$$

$$= (8\pi + \cos 4\pi) - (1)$$

$$= 8\pi + 1 - 1$$

$$= 8\pi$$

$$\boxed{\int_C \bar{F} \cdot d\bar{r} = 8\pi}$$

(2) Find the work done by

$$\bar{F} = (2x - y - z)\bar{i} + (x + y - z)\bar{j} + (3x - 2y - 5z)\bar{k}$$

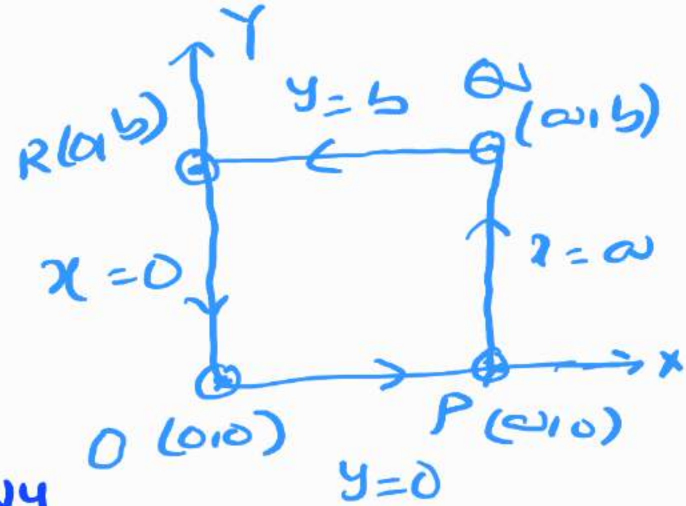
along the curve C in the xy-plane given by $x^2 + y^2 = 9$, $z = 0$

① If $\bar{F} = (x^2+y^2)\bar{i} - 2xy\,\bar{j}$ Evaluate $\int_C \bar{F}\cdot d\bar{r}$

where c is the curve is the rectangle

in xy-plane bounded by $y=0, y=b,$

$x=0, x=a$



Sol: let $\bar{r} = xi + yj$

$d\bar{r} = dx\,i + dy\,j$

$$\bar{F}\cdot d\bar{r} = (x^2+y^2)dx + (-2xy)\,dy$$

Given curve c: is the rectangle in xy-plane bounded by $y=0, y=b, x=0, x=a$

$$\int_C \bar{F}\cdot d\bar{r} = \int_{OPQR} \bar{F}\cdot d\bar{r}$$

$$= \int_{OP} \bar{F}\cdot d\bar{r} + \int_{PQ} \bar{F}\cdot d\bar{r} + \int_{QR} \bar{F}\cdot d\bar{r} + \int_{RO} \bar{F}\cdot d\bar{r}$$

OP	PQ	QR	RO
$y=0$	$x=a$	$y=b$	$x=0$
$dy=0$	$dx=0$	$dy=0$	$dx=0$
integral of x	integ of y	integ of x	integral y
$x=0$ to a	$y=0$ to b	$x=a$ to 0	$y=b$ to a

Along OP : $y=0 \Rightarrow dy=0$

$\bar{F}\cdot d\bar{r} = x^2\,dx$

x vary from 0 to a

$$\int_{OP} \bar{F}\cdot d\bar{r} = \int_{x=0}^{a} x^2\,dx = \frac{a^3}{3}$$

Along PQ : $x=a \Rightarrow dx=0$

$\bar{F}\cdot d\bar{r} = -2ay\,dy$

y vary from 0 to b

$$\int_{PQ} \bar{F}\cdot d\bar{r} = \int_{y=0}^{b} (-2ay)\,dy$$

$$= (-2a)\left[\frac{y^2}{2}\right]_0^b = -ab^2$$

Along QR $y=b \Rightarrow dy=0$

$\bar{F}\cdot d\bar{r} = (x^2+b^2)\,dx$

x vary from a to a

$$\int_{QR} \bar{F}\cdot d\bar{r} = \int_{x=a}^{0} (x^2+b^2)\,dx$$

$$= \left[\frac{x^3}{3} + b^2 x\right]_a^0$$

$$= 0 - \left(\frac{a^3}{3} + ab^2\right)$$

$$= -\frac{a^3}{3} - ab^2$$

Along RO $\quad x=0 \Rightarrow dx=0$

$$\bar{F}\cdot d\bar{r} = 0$$

$$\int_{RO} \bar{F}\cdot d\bar{r} = \int_b^0 0\,dy = 0$$

Hence

$$\int_C \bar{F}\cdot d\bar{r} = \int_{OPQR} \bar{F}\cdot d\bar{r}$$

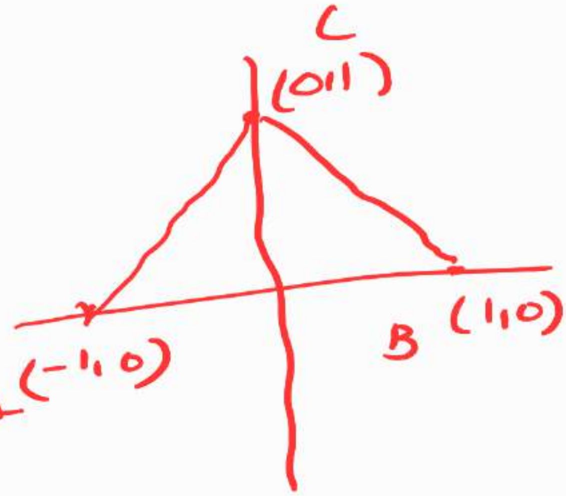
$$= \int_{OP} \bar{F}\cdot d\bar{r} + \int_{PQ} \bar{F}\cdot d\bar{r} + \int_{QR} \bar{F}\cdot d\bar{r} + \int_{RO} \bar{F}\cdot d\bar{r}$$

$$= \cancel{\frac{a^3}{3}} - ab^2 - \cancel{\frac{a^3}{3}} - ab^2$$

$$= -2ab^2$$

$$\boxed{\int_C \bar{F}\cdot d\bar{r} = 2ab^2}$$

(2) Compute the line integral $\int_C y^2\,dx - x^2\,dy$ round the triangle whose vertices are $(1,0), (0,1), (-1,0)$ in the xy plane



AB: $y = 0$

BC: $x + y = 1$

AC: $y = x + 1$

$$\int_C \bar{F}\cdot d\bar{r} = \int_C y^2\,dx - x^2\,dy$$

$$\int_{\Delta ABC} = \int_{AB} + \int_{BC} + \int_{CA}$$

① If $\bar{F} = (4xy - 3x^2z^2)\bar{i} + 2x^2\bar{j} - 2x^3z\bar{k}$,

prove that $\int_C \bar{F} \cdot d\bar{r}$ i.e. work done is independent of the curve joining two points

sol $\boxed{\text{curl } \bar{F} = 0}$ work done is independent

① prove that the force field given by

$$\bar{F} = 2xyz^3\,\bar{i} + x^2z^3\,\bar{j} + 3x^2yz^2\,\bar{k}$$

is conservative, find the work done by moving a particle from $(1,-1,2)$ to $(3,2,-1)$ in the force field.

Sol. given $\bar{F} = 2xyz^3\,i + x^2z^3\,j + 3x^2yz^2\,\bar{k}$

now

$$\text{curl}\,\bar{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xyz^3 & x^2z^3 & 3x^2yz^2 \end{vmatrix}$$

$$= i\left[3x^2z^2 - 3x^2z^2\right] - j\left[6xyz^2 - 6xyz^2\right] + k\left[2xz^3 - 2xz^3\right]$$

$$\boxed{\text{curl}\,\bar{F} = 0}$$

$\bar{F}$ is conservative force field.

$\Leftrightarrow$ There exist a scalar potential function such that $\bar{F} = \nabla \phi$

since

$$2xyz^3\, i + x^2 z^3\, j + 3x^2 y z^2\, \bar{k}$$

$$= i \frac{\partial \phi}{\partial x} + j \frac{\partial \phi}{\partial y} + k \frac{\partial \phi}{\partial z}$$

here

$$\frac{\partial \phi}{\partial x} = 2xyz^3$$

$$\int 1\, d\phi = \int 2xyz^3\, dx$$

$$\boxed{\phi = x^2 y z^3 + c}$$

$$\frac{\partial \phi}{\partial y} = x^2 z^3$$

$$\int 1\, d\phi = \int x^2 z^3\, dy$$

$$\boxed{\phi = x^2 y z^3 + c}$$

$$\frac{\partial \phi}{\partial z} = 3x^2 y z^2$$

$$\int 1 \, d\phi = \int 3x^2 y z^2 \, dz$$

$$\boxed{\phi = x^2 y z^3 + c}$$

Hence $\phi = x^2 y z^3$

Hence $$\int_C \bar{F} \cdot d\bar{r} = \int_{(1,-1,2)}^{(3,2,-1)} d\phi$$

$$= \left[\phi\right]_{(1,-1,2)}^{(3,2,-1)}$$

$$= \left[x^2 y z^3\right]_{(1,-1,2)}^{(3,2,-1)}$$

$$= \left[9 \cdot 2 \cdot (-1)\right] - \left[1 \cdot (-1)(8)\right]$$

$$= -18 + 8 = -10$$

$$\boxed{\int_C \bar{F} \cdot d\bar{r} = 10}$$

Surface integral

The surface integral of a vector point function $\bar{F}$ express the normal flux through a surface. If $\bar{F}$ represents the velocity vector of a fluid then the surface integral $\int_S \bar{F}\cdot\bar{n}\,dS$ over a closed surface S represents the rate of flow of fluid through the surface.

Note ① let R_1 be the projection of S on xy-plane

Then $$\int_S \bar{F}\cdot\bar{n}\,dS = \iint_{R_1} \bar{F}\cdot\bar{n}\,\frac{dx\,dy}{|\bar{n}\cdot\bar{k}|}$$

② let R_2 be the projection of S on yz-plane

Then $$\int_S \bar{F}\cdot\bar{n}\,dS = \iint_{R_2} \bar{F}\cdot\bar{n}\,\frac{dy\,dz}{|\bar{n}\cdot\bar{i}|}$$

③ let R_3 be the projection of S on zx-plane

Then $$\int_S \bar{F}\cdot\bar{n}\,dS = \iint_{R_3} \bar{F}\cdot\bar{n}\,\frac{dz\,dx}{|n\cdot j|}$$

① Evaluate $\int_S \bar{F}\cdot\bar{n}\,ds$ where $\bar{F} = z\bar{i} + x\bar{j} - 3y^2z\,k$ and S is the surface $x^2+y^2=16$ include the first octant between $z=0$ and $z=5$

① $\nabla\phi$

② $\bar{n} = \frac{\nabla\phi}{|\nabla\phi|}$

③ $\bar{F}\cdot\bar{n}$

④ $\bar{n}\cdot i$ or j or k

⑤ $\int_S \bar{F}\cdot\bar{n}\,ds$

Sol given surface

$$\phi = x^2+y^2-16$$

Now $\nabla\phi = i\,2x + j\,2y$

$$|\nabla\phi| = \sqrt{4x^2+4y^2}$$

$$= 2\sqrt{x^2+y^2}$$

unit normal $\bar{n} = \frac{\nabla\phi}{|\nabla\phi|}$

$$= \frac{2(xi+yj)}{2\sqrt{x^2+y^2}} = \frac{xi+yj}{\sqrt{x^2+y^2}}$$

Since $x^2+y^2=16$

$$\Rightarrow \bar{n} = \frac{xi+yj}{4}$$

Now $\bar{F}\cdot\bar{n} = (zi+xj-3y^2zk)\cdot\frac{xi+yj}{4}$

$$= \frac{xz+xy}{4} = \frac{x(y+z)}{4}$$

let R be the projection of S on YZ-plane

$$\int_S \bar{F}\cdot\bar{n}\, dS = \iint_{y\,z} \bar{F}\cdot\bar{n}\, \frac{dy\, dz}{|\bar{n}\cdot i|}$$

Now $\bar{n}\cdot i = \frac{1}{4}(x i + y j)\cdot i = \frac{x}{4}$

$$|\bar{n}\cdot i| = \frac{x}{4}$$

Since R be the projection of S on yz-plane

For the surface $x^2+y^2=16$ in yz plane $x=0$

$$\Rightarrow y = 4$$

Hence y varies from 0 to 4

z vary from 0 to 5 [given]

$$\int_S \bar{F}\cdot\bar{n}\, dS = \int_{y=0}^{4}\int_{z=0}^{5} \bar{F}\cdot\bar{n}\, \frac{dy\, dz}{|n\cdot i|}$$

$$= \int_{y=0}^{4}\int_{z=0}^{5} \cancel{\frac{x}{4}}(y+z)\cdot \frac{dy\, dz}{\cancel{\left(\frac{x}{4}\right)}}$$

$$= \int_{y=0}^{4} \int_{z=0}^{5} (y+z)\, dz\, dy$$

$$= \int_{y=0}^{4} \left[yz + \frac{z^2}{2} \right]_0^5 dy$$

$$= \int_{y=0}^{4} \left[5y + \frac{25}{2} \right] dy$$

$$= \left[\frac{5y^2}{2} + \frac{25}{2} y \right]_0^4$$

$$= 40 + 50 = 90$$

$$\boxed{\int_S \bar{F} \cdot \bar{n}\, dS = 90}$$

(2) Evaluate $\int_S \bar{F} \cdot \bar{n}\, dS$ if $\bar{F} = yz\bar{i} + 2y^2\bar{j} + xz^2\bar{k}$ and S is the surface of the cylinder $x^2 + y^2 = 9$ contained in the first octant between the planes $z=0$ and $z=2$

③ Evaluate $\int_S \bar{F}\cdot\bar{n}\, dS$ where $\bar{F} = 12x^2y\,\bar{i} - 3yz\,\bar{j} + 2z\,\bar{k}$

and S is the portion of the plane $x+y+z=1$ included in the first octant

Sol: given surface $\phi = x+y+z-1$

Now $\nabla\phi = i(1) + j(1) + k(1)$

$$\nabla\phi = 1i + 1j + 1k$$

$$|\nabla\phi| = \sqrt{3}$$

unit normal vector $\bar{n} = \dfrac{\nabla\phi}{|\nabla\phi|} = \dfrac{i+j+k}{\sqrt{3}}$

Now $\bar{F}\cdot\bar{n} = (12x^2y\,\bar{i} - 3yz\,\bar{j} + 2z\,k)\cdot\dfrac{i+j+k}{\sqrt{3}}$

$$\bar{F}\cdot\bar{n} = \frac{1}{\sqrt{3}}(12x^2y - 3yz + 2z)$$

let R be the projection of S on xy-plane

$$\int_S \bar{F}\cdot\bar{n}\, dS = \iint_{xy} \bar{F}\cdot\bar{n}\,\frac{dx\,dy}{|\bar{n}\cdot\bar{k}|}$$

so $\bar{n}\cdot\bar{k} = \dfrac{1}{\sqrt{3}}(i+j+k)\cdot k = \dfrac{1}{\sqrt{3}}$

$$|\bar{n}\cdot\bar{k}| = \frac{1}{\sqrt{3}}$$

Since R be the projection of S on xy-plane

given surface $x+y+z=1$ For xy-plane $z=0$

$$\Rightarrow \quad x+y=1$$

$\Rightarrow$ y varies from 0 to $1-x$

x varies from 0 to 1

and also in $\bar{F}\cdot\bar{n}$ replace z by $1-x-y$

$$\therefore \bar{F}\cdot\bar{n} = \frac{1}{\sqrt{3}}\left(12x^2y-3y(1-x-y)+2(1-x-y)\right)$$

$$= \frac{1}{\sqrt{3}}\left[12x^2y+3xy+3y^2-2x-5y+2\right]$$

Hence
$$\int_S \bar{F}\cdot\bar{n}\,dS = \int_{x=0}^{1}\int_{y=0}^{1-x} \bar{F}\cdot\bar{n}\,\frac{dx\cdot dy}{|\bar{n}\cdot\bar{k}|}$$

$$= \int_{x=0}^{1}\int_{y=0}^{1-x} \cancel{\frac{1}{\sqrt{3}}}\left[12x^2y+3xy+3y^2-2x-5y+2\right]\frac{dx\,dy}{\cancel{\left(\frac{1}{\sqrt{3}}\right)}}$$

$$= \int_{x=0}^{1}\int_{y=0}^{1-x}\left[12x^2y+3xy+3y^2-2x-5y+2\right]dy\,dx$$

$$= \int_{x=0}^{1} \left[12x^2 \frac{y^2}{2} + 3x\frac{y^2}{2} + 3\frac{y^3}{3} - 2xy - 5\frac{y^2}{2} + 2y \right]_0^{1-x} dx$$

$$= \int_{x=0}^{1} \left\{ \left[6x^2 + \frac{3x}{2} - \frac{5}{2} \right] y^2 + y^3 - 2x(y) + 2y \right\}_0^{1-x} x$$

$$= \int_{x=0}^{1} \left\{ \left(6x^2 + \frac{3x}{2} - \frac{5}{2} \right)(1-x)^2 + (1-x)^3 - 2x(1-x) + 2(1-x) \right\} dx$$

on simplefying we get

$$= \int_{x=0}^{1} \left[x^3 + 11x^2 - x - 8 \right] dx$$

$$= \frac{1}{2} \left[\frac{x^4}{4} + 11\frac{x^3}{3} - \frac{x^2}{2} - 8x \right]_0^1$$

$$= -\frac{55}{24}$$

$$\boxed{\int_S \bar{F} \cdot \bar{n} \, dS = -\frac{55}{24}}$$

④ Evaluate $\int_S \bar{F}\cdot\bar{n}\,dS$ where $\bar{F} = 18z\bar{i} - 12\bar{j} + 3y\bar{k}$ and S is the part of the surface of the plane $2x+3y+6z=12$ located in the first octant.

Volume integrals

let V be the volume bounded by a surface $\bar{r}$; let $\bar{F}(\bar{r})$ be a vector point function defined over V. The volume integral of $\bar{F}(\bar{r})$ in the region V is denoted by $\int_V \bar{F}(\bar{r})\,dV$ (or) $\int_V \bar{F}\,dV$

Cartesian form: let $\bar{F}(\bar{r}) = F_1\bar{i} + F_2\bar{j} + F_3\bar{k}$ where F_1, F_2, F_3 are functions of x, y, z, we know that $dV = dx\,dy\,dz$, The volume integral is given by $\quad V = xyz \therefore dV = dx\,dy\,dz$

$$\int_V \bar{F}\,dV = \iiint F_1 i + F_2 j + F_3 k \; dx\,dy\,dz$$

$$= i \iiint F_1\,dx\,dy\,dz + j\iiint F_2\,dx\,dy\,dz + \bar{k}\iiint F_3\,dx\,dy\,dz$$

① If $\bar{F} = 2xz\bar{i} - x\bar{j} + y^2\bar{k}$ Evaluate

$\int_V \bar{F}\, dv$ where v is the region bounded

by the surface $x=0, x=2, y=0, y=6,$

$z = x^2, z = 4$

10 Given $\bar{F} = 2xz\bar{i} - x\bar{j} + y^2\bar{k}$

The volume integral is

$$\int_V \bar{F}\, dv = \iiint 2xz i - xj + y^2 \bar{k}\; dx\, dy\, dz$$

$$= \bar{i} \int_{x=0}^{2} \int_{y=0}^{6} \int_{z=x^2}^{4} 2xz \, dx\, dy\, dz$$

$$- \bar{j} \int_{x=0}^{2} \int_{y=0}^{6} \int_{z=x^2}^{4} x \, dx\, dy\, dz$$

$$+ \bar{k} \int_{x=0}^{2} \int_{y=0}^{6} \int_{z=x^2}^{4} y^2 \, dx\, dy\, dz$$

$$= i \int_{x=0}^{2} \int_{y=0}^{6} \left[x z^2\right]_{x^2}^{4} \, dx \, dy$$

$$- j \int_{x=0}^{2} \int_{y=0}^{6} \left[x z\right]_{x^2}^{4} \, dx \, dy$$

$$+ \bar{k} \int_{x=0}^{2} \int_{y=0}^{6} y^2 \left(z\right]_{x^2}^{4} \, dx \, dy$$

$$= i \int_{x=0}^{2} \int_{y=0}^{6} x \left[16 - x^4\right] dx \, dy$$

$$- j \int_{x=0}^{2} \int_{y=0}^{6} x \left[4 - x^2\right] dx \, dy$$

$$+ \bar{k} \int_{x=0}^{2} \int_{y=0}^{6} y^2 \left[4 - x^2\right] dx \, dy$$

$$= i \int_{x=0}^{2} \left[16x - x^5\right] \left[y\right]_0^6 dx - j \int_{x=0}^{2} \left[4x - x^3\right] \left[y\right]_0^6 dx$$

$$+ \bar{k} \int_{x=0}^{2} \left[4 - x^2\right) \left[\frac{y^3}{3}\right]_0^6 dx$$

$$= 6i\left[16\frac{x^2}{2} - \frac{x^6}{6}\right]_0^2 - 6j\left[4\frac{x^2}{2} - \frac{x^4}{4}\right]_0^2$$

$$+ \bar{k}\,\frac{6^3}{3}\left[4x - \frac{x^3}{3}\right]_0^2$$

$$= 128i - 24j - 384\bar{k}$$

Hence $\boxed{\int_V \bar{F}\cdot dV = 128i - 24j - 384\bar{k}}$

② If $\bar{F} = (2x^2 - 3z)i - 2xyj - 4x\bar{k}$

Then evaluate (i) $\int_V \nabla\cdot\bar{F}\,dV$ (ii) $\int_V \nabla\times\bar{F}\,dV$

where V is the closed region bounded

by $x=0$, $y=0$, $z=0$, $2x+2y+z=4$

Sol: given $\bar{F} = \underset{F_1}{(2x^2 - 3z)}i \underset{F_2}{- 2xy}j \underset{F_3}{- 4x}\bar{k}$

(i) $\nabla\cdot F = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$

$$= 4x - 2x = 2x$$

$$\nabla \cdot F = 2x$$

Now $\int_V \nabla \cdot F \, dV = \iiint 2x \, dx \, dy \, dz$

Here the limits are

$z = 0$ to $4 - 2x - 2y$

$y = 0$ to $2 - x$

$x = 0$ to 2

$$= \int_{x=0}^{2} \int_{y=0}^{2-x} \int_{z=0}^{4-2x-2y} 2x \, dx \, dy \, dz$$

$$= \int_{x=0}^{2} \int_{y=0}^{2-x} 2x \left[z\right]_0^{4-2x-2y} dx \, dy$$

$$= \int_{x=0}^{2} \int_{y=0}^{2-x} 2x \left[4 - 2x - 2y\right] dy \, dx$$

$$= \int_{x=0}^{2} 2x \left[(4-2x)y - y^2\right]_0^{2-x} dx$$

$$= \int_{x=0}^{2} 2x \left[2(2-x)(2-x) - (2-x)^2 \right] dx$$

$$= \int_{x=0}^{2} 2x (2-x)^2 \, dx$$

$$= \int_{x=0}^{2} 2x \left[4 + x^2 - 4x \right] dx$$

$$= \int_{x=0}^{2} \left[8x + 2x^3 - 8x^2 \right] dx$$

$$= \left[8\frac{x^2}{2} + 2\frac{x^4}{4} - 8\frac{x^3}{3} \right]_0^2 dx$$

$$= \frac{8}{3}$$

$$\boxed{\int_V \nabla \cdot \vec{F} \, dV = \frac{8}{3}}$$

(ii)

$$\text{Now } \nabla \times \bar{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x^2-3z & -2xy & -4x \end{vmatrix}$$

$$= j - 2y\bar{k}$$

$$\text{Now } \int_V \nabla \times \bar{F}\, dV = \iiint \nabla \times \bar{F}\, dx\, dy\, dz$$

$$= \int_{x=0}^{2} \int_{y=0}^{2-x} \int_{z=0}^{4-2x-2y} [j - 2y\bar{k}]\, dx\, dy\, dz$$

$$= \int_{x=0}^{2} \int_{y=0}^{2-x} [j - 2y\bar{k}]\, [z]_0^{4-2x-2y}\, dy\, dx$$

$$= \int_{x=0}^{2} \int_{y=0}^{2-x} [j - 2y\bar{k}]\, [4-2x-2y]\, dy\, dx$$

$$= \int_{x=0}^{2} \int_{y=0}^{2-x} \left\{ \hat{j} \left[4 - 2x - 2y \right] - \bar{k} \left[(4 - 2x) 2y - 4y^2 \right] \right\} dy\, dx$$

$$= \int_{x=0}^{2} \hat{j} \left[(4-2x) y - y^2 \right]_0^{2-x} - \bar{k} \left[(4-2x) y^2 - 4 \frac{y^3}{3} \right]_0^{2-x} dx$$

$$= \int_{x=0}^{2} \hat{j} \left[2(2-x)(2-x) - (2-x)^2 \right]$$

$$- \bar{k} \left[2(2-x)(2-x)^2 - \frac{4}{3}(2-x)^3 \right] dx$$

$$= \int_{x=0}^{2} \left\{ (2-x)^2 \hat{j} - \bar{k} \frac{2}{3} (2-x)^3 \right\} dx$$

$$= \left[\hat{j} \frac{(2-x)^3}{-3} - \frac{2\bar{k}}{3} \frac{(2-x)^4}{-4} \right]_0^2$$

$$= 0 - \left(-\frac{8}{3} \hat{j} - \frac{2\bar{k}}{3} \frac{16}{(-4)} \right)$$

$$= \frac{8}{3} (\hat{j} - \bar{k})$$

$$\boxed{\int_V \nabla \times \bar{F} \, dV = \frac{8}{3} (\hat{j} - \bar{k})}$$

Vector integral Theorems

Gauss Divergence Theorem

Green's Theorem

Stokes Theorem

Gauss Divergence Theorem:

let S be a closed surface enclosing a volume V, If $\bar{F}$ is a continuously differentiable vector point Function, Then

$$\int_V \operatorname{div} \bar{F} \, dV = \int_S \bar{F} \cdot \bar{n} \, dS$$

where $\bar{n}$ is outward drawn normal vector at any point of S

Note: Gauss Divergence Theorem is useful only for closed surfaces.

① verify Gauss Divergence Theorem for $\bar{F} = x^2\bar{i} + y^2\bar{j} + z^2\bar{k}$ over the surface S of the solid cut off by the plane $x+y+z=a$ in the first octant

Sol. By Gauss Divergence Theorem

$$\int_S \bar{F}\cdot\bar{n}\, dS = \int_V \operatorname{div} F\, dV$$

(I) $\int_S \bar{F}\cdot\bar{n}\, dS$

since $\phi = x+y+z-a$

$\nabla\phi = i + j + k$

$|\nabla\phi| = \sqrt{3}$

unit normal $\bar{n} = \dfrac{\nabla\phi}{|\nabla\phi|} = \dfrac{1}{\sqrt{3}}(\bar{i}+\bar{j}+\bar{k})$

let R be the projection of S on xy-plane

$$\therefore \int_S \bar{F}\cdot\bar{n}\, dS = \iint_{xy} \bar{F}\cdot\bar{n}\,\frac{dx\,dy}{|\bar{n}\cdot\bar{k}|}$$

Hence the eqn of the given plane

will be $x + y = a$

y varies from 0 to $a - x$

x varies from 0 to a

$$\bar{n} \cdot \bar{k} = \frac{1}{\sqrt{3}}(i + j + \bar{k}) \cdot \bar{k} = \frac{1}{\sqrt{3}}$$

$$|n \cdot \bar{k}| = \frac{1}{\sqrt{3}}$$

$$\bar{F} \cdot \bar{n} = \frac{x^2 + y^2 + z^2}{\sqrt{3}} \qquad [z = a - x - y]$$

$$= \frac{x^2 + y^2 + (a - x - y)^2}{\sqrt{3}}$$

$$\frac{\bar{F} \cdot \bar{n}}{|\bar{n} \cdot \bar{k}|} = x^2 + y^2 + (a - x - y)^2$$

$$\int_S \bar{F} \cdot \bar{n} \, dS = \int_{x=0}^{a} \int_{y=0}^{a-x} [x^2 + y^2 + (a - x - y)^2] \, dy \, dx$$

$$= \int_{x=0}^{a} \int_{y=0}^{a-x} \left[x^2+y^2+a^2+x^2+y^2-2ax+2xy -2ay\right] dx\, dy$$

$$= \int_{x=0}^{a} \int_{y=0}^{a-x} \left[2x^2+2y^2-2ax-2ay +2xy+a^2\right] dy\, dx$$

$$= \int_{x=0}^{a} \left[2x^2y+\frac{2}{3}y^3-2axy-ay^2+xy^2 +a^2y\right]_0^{a-x} dx$$

$$= \int_{x=0}^{a} \left[2x^2(a-x)+\frac{2}{3}(a-x)^3-2ax(a-x) -a(a-x)^2+x(a-x)^2+a^2(a-x)\right] dx$$

$$= \int_{x=0}^{a} \left[-\frac{5}{3}x^3+3ax^2-2a^2x+\frac{2}{3}a^3\right] dx$$

$$= \left[-\frac{5}{3}\frac{x^4}{4}+3a\frac{x^3}{3}-2a^2\frac{x^2}{2}+\frac{2}{3}a^3x\right]_0^a$$

$$= -\frac{5}{3}\frac{a^4}{4}+a\cdot a^3-a^2a^2+\frac{2}{3}a^3a$$

$$= \frac{a^4}{4}$$

$$\boxed{\int_S \bar{F}\cdot\bar{n}\, dS = \frac{a^4}{4}}$$

(ii) $\int_V \operatorname{div} \bar{F}\, dV$

Given $\bar{F} = x^2 i + y^2 j + z^2 \bar{k}$

Now $\operatorname{div} F = \nabla \cdot F$

$= 2x + 2y + 2z$

$= 2(x+y+z)$

Now $\int_V \operatorname{div} F\, dV = \iiint \operatorname{div} F\, dx\, dy\, dz$

given plane $x+y+z=a$

z varies from 0 to $a-x-y$

y varies from 0 to $a-x$

x varies from 0 to a

$$= 2\int_{x=0}^{a}\int_{y=0}^{a-x}\int_{z=0}^{a-x-y}(x+y+z)\,dx\,dy\,dz$$

$$= 2\int_{x=0}^{a}\int_{y=0}^{a-x}\left[(x+y)z+\frac{z^2}{2}\right]_0^{a-x-y}dy\,dx$$

$$= 2\int_{x=0}^{a}\int_{y=0}^{a-x}\left\{z\left[x+y+\frac{z}{2}\right]\right\}_0^{a-x-y}dy\,dx$$

$$= 2\int_{x=0}^{a}\int_{y=0}^{a-x}(a-x-y)\left[x+y+\frac{a-x-y}{2}\right]dy\,dx$$

$$= \int_{x=0}^{a}\int_{y=0}^{a-x}(a-x-y)\,(a+x+y)\,dy\,dx$$

$$= \int_{x=0}^{a}\int_{y=0}^{a-x}\left[a-(x+y)\right]\left[a+(x+y)\right]dy\,dx$$

$$= \int_{x=0}^{a} \int_{y=0}^{a-x} a^2 - (x+y)^2 \, dy \, dx$$

$$= \int_{x=0}^{a} \int_{y=0}^{a-x} \left[a^2 - x^2 - y^2 - 2xy\right] dy \, dx$$

$$= \int_{x=0}^{a} \left[a^2 y - x^2 y - \frac{y^3}{3} - x y^2\right]_0^{a-x} dx$$

$$= \int_{x=0}^{a} \left[a^2 (a-x) - x^2 (a-x) - \frac{(a-x)^3}{3} - x (a-x)^2\right] dx$$

$$= \int_{x=0}^{a} (a-x) \left[a^2 - x^2 - \frac{(a-x)^2}{3} - x(a-x)\right] dx$$

$$= \int_{x=0}^{a} (a-x)^2 \left[a + \cancel{x} - \frac{a-x}{3} - \cancel{x}\right] dx$$

$$= \int_{x=0}^{a} \left[a^2 + x^2 - 2ax\right] \left[\frac{2a+x}{3}\right] dx$$

$$\int_{x=0}^{a}\left[2a^3 + a^2 x + 2a x^2 + x^3 - 4a^2 x - 2a x^2\right] dx$$

$$= \left[2a^3 x + a^2 \frac{x^2}{2} + 2a\frac{x^3}{3} + \frac{x^4}{4} - 4a^2\frac{x^2}{2} - 2a\frac{x^3}{3}\right]_0^a$$

$$= \cancel{2a^4} + \frac{a^4}{2} + \cancel{\frac{2}{3}a^4} + \frac{a^4}{4} - \cancel{2a^4} - \cancel{\frac{2a^4}{3}}$$

$$= \frac{a^4}{4}$$

② Using Gauss Divergence Theorem Evaluate $\iint_S (y z^2 i + z x^2 j + 2z^2 k)\, dS$ where S is the closed surface bounded by the xy-plane and the upper half of the sphere $x^2 + y^2 + z^2 = a^2$ above this plane

Sol Gauss Divergence Theorem states that

$$\int_S \bar{F}\cdot\bar{n}\, dS = \int_V \operatorname{Div} F\, dV$$

Here $\bar{F} = yz^2 i + zx^2 j + 2z^2 k$

Here $\text{Div } F = 4z$

$\int \text{Div } F \, dV = \iiint 4z \, dx \, dy \, dz$

Here spherical coordinates

$x = r\sin\theta\cos\phi, \; y = r\sin\theta\sin\phi,$

$z = r\cos\theta$

Here $dx \, dy \, dz = r^2 \, dr \, d\theta \, d\phi$

r varies from 0 to a

θ varies from 0 to π

ϕ varies from 0 to 2π

$$\int_{r=0}^{a} \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} 4 r\cos\theta \; r^2 \, dr \, d\theta \, d\phi$$

On integrating w r t ϕ, θ, r

we get the Answer $= 0$

Hence

$$\boxed{\int_S \bar{F} \cdot \bar{n} \, dS = \int_V \text{Div } F \, dV = 0}$$

③ Evaluate $\int_S \bar{F} \cdot \bar{n}\, dS$ if

$\bar{F} = xy\,\bar{i} + z^2\,\bar{j} + 2yz\,\bar{k}$ over the tetrahedron bounded by $x=0$, $y=0$, $z=0$ and the plane $x+y+z=1$

Sol

By Gauss Divergence The

$$\int_S \bar{F} \cdot \bar{n}\, dS = \int_V \operatorname{div} F\, dV$$

since $F = xy\,\bar{i} + z^2\,\bar{j} + 2yz\,\bar{k}$

$\operatorname{div} F = 3y$

given plane $\boxed{x+y+z=1}$

z vary from 0 to $1-x-y$

y vary from 0 to $1-x$

x vary from 0 to 1

$$\int_V \operatorname{div} \vec{F} \, dV = \int_{x=0}^{1} \int_{y=0}^{1-x} \int_{z=0}^{1-x-y} 3y \, dx \, dy \, dz$$

$$= \int_{x=0}^{1} \int_{y=0}^{1-x} 3y \left[z\right]_0^{1-x-y} dy \, dx$$

$$= \int_{x=0}^{1} \int_{y=0}^{1-x} 3y\,(1-x-y) \, dy \, dx$$

$$= 3 \int_{x=0}^{1} \int_{y=0}^{1-x} \left[(1-x)y - y^2\right] dy \, dx$$

$$= 3 \int_{x=0}^{1} \left[(1-x)\frac{y^2}{2} - \frac{y^3}{3}\right]_0^{1-x} dx$$

$$= 3 \int_{x=0}^{1} (1-x)\frac{(1-x)^2}{2} - \frac{(1-x)^3}{3} \, dx$$

$$= 3 \int_{x=0}^{1} \frac{(1-x)^3}{2} - \frac{(1-x)^3}{3} \, dx$$

$$= 3 \int_{x=0}^{1} \frac{(1-x)^3}{6} \, dx$$

$$= \frac{1}{2} \left[\frac{(1-x)^4}{-4} \right]_0^1$$

$$= -\frac{1}{8} \left[(1-x)^4 \right]_0^1$$

$$= -\frac{1}{8} \left[0 - 1 \right] = \frac{1}{8}$$

$$\boxed{\int_V \operatorname{div} F \, dV = \int_S \bar{F} \cdot \bar{n} \, dS = \frac{1}{8}}$$

Green's Theorem :

If S is a closed region in xy plane bounded by a simple closed curve C and if M and N are continuous functions of x and y having continuous derivatives in R Then

$$\oint_C M\,dx + N\,dy = \iint_S \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

where C is traversed in the positive Direction

① verify Green's Theorem for

$\int_C (xy + y^2)\,dx + x^2\,dy$ where C is bounded by $y = x$ and $y = x^2$

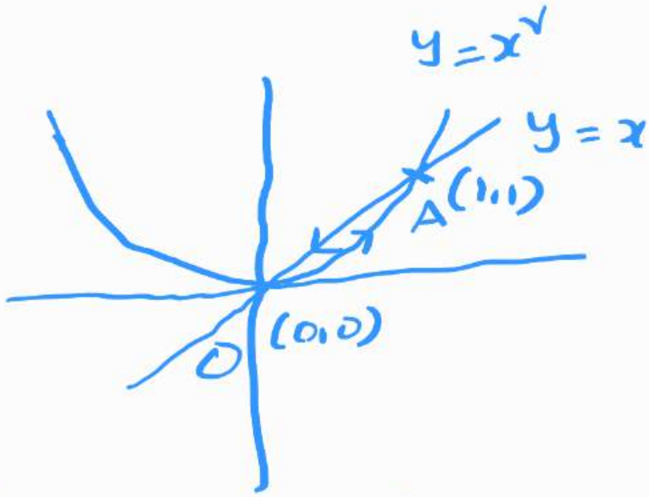
sol: By Green's Theorem

$$\int_C M\,dx + N\,dy = \iint_S \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

Given curves

$y = x$

$y = x^2$



Here $M = \underline{xy + y^2}$; $N = x^2$

$$\int_C M\,dx + N\,dy = \underset{\substack{y=x^2 \\ C_1}}{\int} M\,dx + N\,dy + \underset{\substack{y=x \\ C_2}}{\int} M\,dx + N\,dy$$

For C_1: $y = x^2$

$dy = 2x\,dx$

x varies from 0 to 1

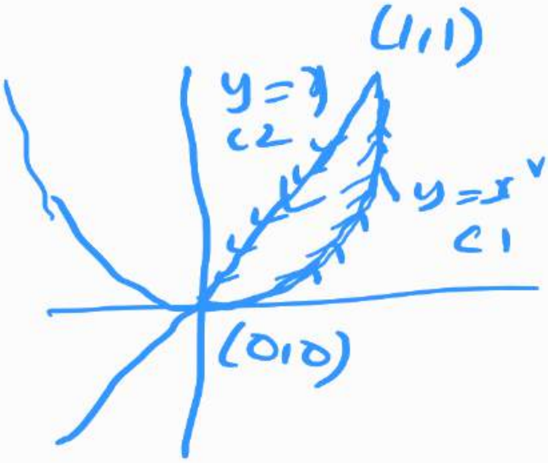
$$\int_{y=x^2} M\,dx + N\,dy = \int_{x=0}^{1} [x(x^2) + x^4]\,dx + x^2(2x)\,dx$$

$$= \int_{x=0}^{1} [x^3 + x^4 + 2x^3]\,dx$$

$$= \int_{x=0}^{1} [3x^3 + x^4]\,dx$$

$$= \left[3\frac{x^4}{4} + \frac{x^5}{5} \right]_0^1$$

$$= \frac{19}{20}$$



For C_2: $y = x$

$dy = dx$

x vary from 1 to 0

$$\int_{y=x} M\,dx + N\,dy = \int_{x=1}^{0} \left[x(x) + x^2\right] dx + x^2\,dx$$

$$= \int_{x=1}^{0} 3x^2\,dx$$

$$= \left[3\frac{x^3}{3}\right]_1^0$$

$$= 0 - 1$$

$$= -1$$

Hence

$$\oint_C M\,dx + N\,dy = \int_{C_1} M\,dx + N\,dy + \int_{C_2} M\,dx + N\,dy$$

$$= \frac{19}{20} - 1$$

$$\boxed{\oint_c M\,dx + N\,dy = -\frac{1}{20}}$$

Since $M = xy + y^2 \quad N = x^2$

$$\frac{\partial M}{\partial y} = x + 2y \qquad \frac{\partial N}{\partial x} = 2x$$

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 2x - x - 2y = x - 2y$$

$$\iint_S \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \int_{x=0}^{1} \int_{y=x^2}^{y=x} (x - 2y)\,dy\,dx$$

$$= \int_{x=0}^{1} \left[xy - \frac{2y^2}{2}\right]_{y=x^2}^{x} dx$$

$$= \int_{x=0}^{1} \left[(x^2 - x^2) - (x^3 - x^4)\right] dx$$

$$= \int_{x=0}^{1} x^4 - x^3 \, dx$$

$$= \left[\frac{x^5}{5} - \frac{x^4}{4}\right]_0^1$$

$$= \frac{1}{5} - \frac{1}{4}$$

$$= -\frac{1}{20}$$

$$\boxed{\iint \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy = -\frac{1}{20}}$$

Hence

$$\boxed{\int_C M\,dx + N\,dy = \iint_S \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy = -\frac{1}{20}}$$

Stoke's Theorem:

let S be a open surface bounded by a closed, non intersecting curve C. If $\bar{F}$ is any differentiable vector point function Then

$$\oint_C \bar{F} \cdot d\bar{r} = \int_S \text{curl } \bar{F} \cdot \bar{n} \, dS$$

where c is Traversed in The positive direction and $\bar{n}$ is unit outward drawn normal at any point of The surface.

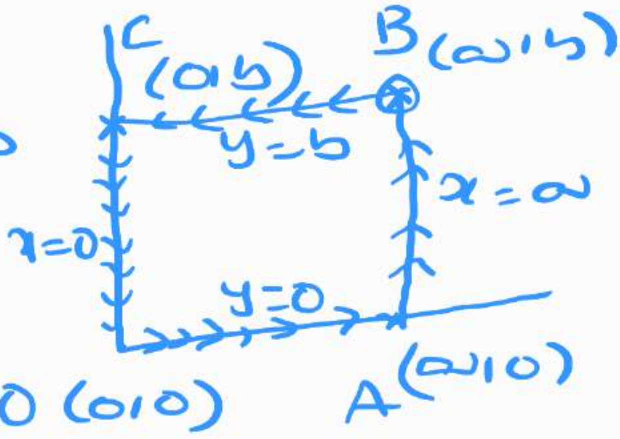
① verify stoke's Theorem For $\bar{F} = (x^2 - y^2)\bar{i} + 2xy\bar{j}$ over The box bounded by The planes $x = 0, x = a, y = 0, y = b$

Sol By stoke's Theorem

$$\int_C \bar{F} \cdot d\bar{r} = \int_S \text{curl } \bar{F} \cdot \bar{n} \, dS$$

given planes

$x=0,\ x=a,\ y=0,\ y=b$



we $\bar{r} = x\hat{i} + y\hat{j}$

$d\bar{r} = dx\,\hat{i} + dy\,\hat{j}$

$$\bar{F}\cdot d\bar{r} = (x^2-y^2)\,dx + 2xy\,dy$$

$$\int_C \bar{F}\cdot d\bar{r} = \int_{OA} \bar{F}\cdot d\bar{r} + \int_{AB} \bar{F}\cdot d\bar{r} + \int_{BC} \bar{F}\cdot d\bar{r}$$

$$+ \int_{CO} \bar{F}\cdot d\bar{r}$$

<u>Along OA line</u> $\quad y=0 \quad \therefore dy=0$

$\bar{F}\cdot d\bar{r} = x^2\,dx$

x vary from 0 to a

$$\int_{OA} \bar{F}\cdot d\bar{r} = \int_{x=0}^{a} x^2\,dx = \frac{a^3}{3}$$

Along AB line $\quad x = a$

$dx = 0$

$$\bar{F} \cdot d\bar{r} = 2ay \, dy$$

y vary from 0 to b

$$\int_{AB} \bar{F} \cdot d\bar{r} = \int_{y=0}^{b} 2ay \, dy = ab^2$$

Along BC line $\quad y = b$

$dy = 0$

$$\bar{F} \cdot d\bar{r} = (x^2 - b^2) \, dx$$

x vary from a to 0

$$\int_{BC} \bar{F} \cdot d\bar{r} = \int_{x=a}^{0} (x^2 - b^2) \, dx$$

$$= \left[\frac{x^3}{3} - b^2 x \right]_a^0$$

$$= 0 - \left(\frac{a^3}{3} - ab^2 \right)$$

$$= ab^2 - \frac{a^3}{3}$$

Along CO - Line $\quad x = 0$

$dx = 0$

$$\bar{F} \cdot d\bar{r} = 0$$

y varies from b to 0

$$\int_{CO} \bar{F} \cdot d\bar{r} = \int_{y=b}^{0} 0 \, dy = 0$$

$$\int_{C} \bar{F} \cdot d\bar{r} = \int_{OA} \bar{F} \cdot d\bar{r} + \int_{AB} \bar{F} \cdot d\bar{r} + \int_{BC} \bar{F} \cdot d\bar{r} + \int_{CO} \bar{F} \cdot d\bar{r}$$

$$= \frac{a^3}{3} + ab^2 + ab^2 - \frac{a^3}{3} + 0$$

$$= 2ab^2$$

$$\boxed{\int_{C} \bar{F} \cdot d\bar{r} = 2ab^2}$$

Since $\bar{F} = (x^2 - y^2)i + 2xy\,j + 0\bar{k}$

$$\operatorname{curl} \bar{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 - y^2 & \underline{2xy} & 0 \end{vmatrix}$$

$$= i[0] - j[0] + k[2y + 2y]$$

$$= 4y\,\bar{k}$$

$$\operatorname{curl} \bar{F} = 4y\,\bar{k}$$

$$\int_S \operatorname{curl} \bar{F} \cdot \bar{n}\, ds = \int_S 4y\,\bar{k} \cdot \bar{n}\, ds$$

$$= 4 \int_S y\; \bar{n} \cdot \bar{k}\, ds$$

*

* let R be The projection of S on xy-plane

$$dS = \frac{dx\,dy}{|\bar{n} \cdot \bar{k}|}$$

$$\boxed{dx\,dy = \bar{n} \cdot \bar{k}\, dS}$$

$$= 4 \iint_{x\ y} y\, dx\, dy$$

$$= 4 \int_{x=0}^{a} \int_{y=0}^{b} y\, dy\, dx$$

$$= 4 \int_{x=0}^{a} \left[\frac{y^2}{2}\right]_0^b dx$$

$$= 4 \frac{b^2}{2} [x]_0^a$$

$$= 4\frac{ab^2}{2}$$

$$= 2ab^2$$

$$\boxed{\int_S \operatorname{curl} \bar{F} \cdot \bar{n} \, dS = 2ab^2}$$

Hence

$$\boxed{\int_C \bar{F} \cdot d\bar{r} = \int_S \operatorname{curl} \bar{F} \cdot \bar{n} \, dS = 2ab^2}$$

gradiant $\nabla\phi$ (or) grad ϕ

Divergence $\nabla\cdot\bar{f}$ (or) Div $\bar{f}$

curl $\nabla\times\bar{f}$

Line integral $\int_C \bar{F}\cdot d\bar{r}$

surface integral $\int_S \bar{F}\cdot\bar{n}\, dS$

Volume integral $\int_V F\, dV$

Gauss Divergence: $\int_S \bar{F}\cdot\bar{n}\, dS = \int_V \mathrm{div}\, F\, dV$

Green's: $\oint_C M\,dx + N\,dy = \iint_S \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy$

Stokes $\int_C \bar{F}\cdot d\bar{r} = \int_S \mathrm{curl}\, \bar{F}\cdot\bar{n}\, dS$

Problems on gradient

(1) Find grad f where

(i) $f = x^3 + y^3 + 3xyz$

(ii) $f = x^2y + y^2x + z^2$

(iii) $f = x^3 - y^3 + x^2z$

at the point $(1, 1, -2)$.

(2) If $\phi = 2xz^4 - x^2y$, find $|\nabla\phi|$ at the point $(2, -2, -1)$

(3) Find the directional derivative of

(i) $\phi = x^2 - 2y^2 + 4z^2$ at $(1, 1, -1)$ in the direction of $2\bar{i} + \bar{j} - \bar{k}$.

(ii) $\phi = xy + yz + zx$, at A in the direction of $\overline{AB}$ where $A = (1, 2, -1)$, $B = (1, 2, 3)$.

(iii) $\phi = xyz$ at $(1, 1, 1)$ in the direction of the vector $\bar{i} + \bar{j} + \bar{k}$.

(iv) Find the directional derivative of $\phi(x, y, z) = xy^2 + yz^2$ at the point $(2, -1, 1)$ in the direction of $i + 2j + 2k$. [JNTU 1999 S]

(v) Find the directional derivative of the scalar point function $\phi(x, y, z) = 4xy^2 + 2x^2yz$ at the point $A(1, 2, 3)$ in the direction of the line AB where $B = (5, 0, 4)$. [JNTU 2007S (Set No. 1)]

(4) Find the maximum value of the directional derivative of $\phi = x^2yz$ at $(1, 4, 1)$.

(5) Find the angle between the surfaces $xy^2z = 3x + z^2$ and $3x^2 - y^2 + 2z = 1$ at $(1, -2, 1)$.

(6) Find the scalar point function whose gradient is $2xyz\bar{i} + x^2z\bar{j} + x^2y\bar{k}$.

(7) Show that grad $r = \frac{\bar{r}}{|\bar{r}|}$.

(8) Find a unit normal to the surface $x^3 + y^3 + 3xyz = 3$ at the point $(1, 2, -1)$.

ANSWERS

(1) (i) $(3x^2 + 3yz)\bar{i} + (3y^2 + 3xy)\bar{j} + 3xyk$; (ii) $(2xy + y^2)i + (x^2 + 2xy)\bar{j} + 2z\bar{k}$ (iii) $-\bar{i} - 3\bar{j} + \bar{k}$

(2) $2\sqrt{93}$ (3) (i) $\frac{8}{\sqrt{6}}$ (ii) 3 (iii) $\sqrt{3}$ (iv) -3 (v) $\frac{120}{\sqrt{21}}$ (4) 9

(5) $\cos^{-1}\left(\frac{3}{7\sqrt{6}}\right)$ (6) x^2yz + costant (8) $\frac{-\bar{i} + 3\bar{j} + 2\bar{k}}{\sqrt{14}}$

Problems on Divergence

(1) Show that (*i*) $3y^4z^2\bar{i}+z^3x^2\bar{j}-3x^2y^2\bar{k}$

(*ii*) $(x+3y)\bar{i}+(y-2z)\bar{j}+(x-2z)\bar{k}$ [JNTU 1998]

is solenoidal.

(2) If $\phi=2x^3y^2z^4$, show that div (grad ϕ) $=12xy^2z^4+4x^3z^4+24x^3y^2z^2$

(3) Prove that $\text{div}(\bar{r}\times\bar{a})=0$ where $\bar{a}$ is a constant vector.

(4) Prove that div $\left(\frac{\bar{r}}{r}\right)=\frac{2}{r}$ where $\bar{r}=x\bar{i}+y\bar{j}+z\bar{k}$.

Problems Curl

(1) If $\bar{f}=e^{x+y+z}(\bar{i}+\bar{j}+\bar{k})$ find curl $\bar{f}$.

(2) Prove that $\bar{f}=(y+z)\bar{i}+(z+x)\bar{j}+(x+y)\bar{k}$ is irrotational.

(3) Prove that $\nabla.(\bar{a}\times\bar{f})=-\bar{a}.\text{ curl }\bar{f}$ where $\bar{a}$ is a constant vector.

(4) Prove that $\text{curl}(\bar{a}\times\bar{r})=2\bar{a}$ where $\bar{a}$ is a constant vector. [JNTU 1996, 2006S(Set No.1)]

(5) If $\bar{f}=x^2y\bar{i}-2xz\bar{j}+2yz\bar{k}$ find (*i*) curl $\bar{f}$ (*ii*) curl curl $\bar{f}$.

(6) If $\bar{a}=(x+y+1)\bar{i}+\bar{j}-(x+y)\bar{k}$, then show that $\bar{a}.\text{ curl }\bar{a}=0$.

[JNTU 1995, JNTU (H) June 2010 (Set No. 4)]

Problems on Line Integral

in

(1) Find $\int_c \bar{F}.d\bar{r}$ where $\bar{F}=x^2y^2\bar{i}+y\bar{j}$ and the curve $y^2=4x$ in the *xy*-plane from (0,0) to (4,4).

(2) If $\bar{F}=3xy\bar{i}-5z\bar{j}+10x\bar{k}$ evaluate $\int_c \bar{F}.d\bar{r}$ along the curve $x=t^2+1, y=2t^2, z=t^3$ from $t=1$ to $t=2$.

(3) If $\bar{F}=y\bar{i}+z\bar{j}+x\bar{k}$, find the circulation of $\bar{F}$ round the curve *c* where *c* is the circle $x^2+y^2=1, z=0$.

(4) (i) If $\phi=x^2yz^3$, evaluate $\int_c \phi d\bar{r}$ along the curve $x=t, y=2t, z=3t$ from $t=0$ to $t=1$.

(ii) If $\phi=2xy^2z+x^2y$, evaluate $\int_c \phi d\bar{r}$ where *c* is the curve $x=t, y=t^2, z=t^3$ from $t=0$ to $t=1$.

[JNTU 2008S (Set No. 4)]

(5) Find the work done by the force $\bar{F}=(x^2-yz)\bar{i}+(y^2-zx)\bar{j}+(z^2-xy)\bar{k}$ in taking particle from (1, 1, 1) to (3, −5, 7).

(6) If $\overline{F}=(3x^2+6y)i-14yzj+20xz^2k$ evaluate the line integral $\int_c \overline{F}\cdot d\overline{r}$ from (0,0,0) to (1, 1, 1) along the path $x=t, y=t^2, z=t^3$. [JNTU 2007S (Set No. 2)]

(7) Show that $\overline{F}=(2xy+z^3)i+x^2j+3xz^2k$ is a conservative force field. Find the potential function and the work done by $\overline{F}$ in moving an object in this field from (1, −2, 1) to (3, 1, 4). [JNTU 2007S (Set No. 4)]

(8). Prove that the scalar field $\overline{F}=(x^2+xy^2)\,i+(y^2+x^2y)\,j$ is conservative and find the scalar potential (potential function). [JNTU 2008S (Set No. 1)]

(9). Show that $\overline{F}=(e^xz-2xy)\,\overline{i}-(x^2-1)\,\overline{j}+(e^x+z)\,\overline{k}$ is conservative field. Hence evaluate $\int_C \overline{F}.\,d\overline{r}$ where the end points of C are $(0,1,-1)$ and $(2,3,0)$ [JNTU (K) Nov. 2009S (Set No. 4)]

(10). Find the work done by a force $\overline{F}=(x^2-y^2+x)\,\overline{i}-(2xy+y)\,\overline{j}$ which moves a particle in xy - plane from $(0,0)$ to $(1,1)$ along the parabola $y^2=x$. [JNTU (K) May 2010 (Set No. 3)]

ANSWERS

(1) 264 (2) 303 (3) $-\pi$ (4) (i) $\frac{54}{7}(\overline{i}+2\overline{j}+3\overline{k})$ (5) $184\frac{2}{3}$

(6) 5 (7) x^2y+xz^3; 202 (8) $\frac{x^3}{3}+\frac{x^2y^2}{2}+\frac{z^3}{3}$

problems on surface integral

(1) If $\overline{F}=(x+y^2)\overline{i}-2x\overline{j}+2yz\overline{k}$ evaluate $\int_S \overline{F}.\overline{n}\,dS$ where S is of the surface of the plane $2x+y+2z=6$ in the first octant.

(2) If $\overline{F}=yzi+zx\overline{j}+xy\overline{k}$, evaluate $\int_S \overline{F}.\overline{n}dS$ over the surface $x^2+y^2+z^2=1$ in the first octant.

(3) If $\phi=\frac{3}{8}xyz$, find $\int_S \phi\overline{n}\,dS$ where S is the surface of the cylinder $x^2+y^2=16$ included in the first octant between $z=0$ and $z=5$. [**Hint:** Take projection on zx-plane]

(4) If $\overline{F}=2xy\overline{i}+yz^2\overline{j}+xz\overline{k}$ find the surface integral over the parallelopiped $x=0,\ y=0,\ z=0,\ x=2,\ y=1,\ z=3$.

(5) Evaluate $\int\int_s \overline{F}\cdot\overline{n}ds$ where $\overline{F}=zi+xj-3y^2zk$, where S is the surface of the cylinder $x^2+y^2=1$ in the first octant between $z=0$ and $z=2$. [JNTU 2008S (Set No. 2)]

ANSWERS

(1) 81 (2) $\frac{3}{8}$ (3) $100(\overline{i}+j)$ (4) 30 (5) 3.

problems on volume integral

(1) Evaluate $\iiint(2x+y)dv$ where V is the closed region bounded by the cylinder $z=4-x^2$, and planes $x=0,\ y=0,\ y=2,$ and $z=0$.

(2) If $\phi=45x^2y$ evaluate $\iiint_V \phi dv$ where V is the closed region bounded by the planes $4x+2y+z=8, y=0, z=0$.

(3) Evaluate $\int_V \overline{F}dv$ when $\overline{F}=x\overline{i}+y\overline{j}+z\overline{k}$ and V is the region bounded by $x=0, y=0, y=6, z=4, z=x^2$.

ANSWERS

(1) $\frac{80}{3}$ (2) 128 (3) $24\overline{i}+96\overline{j}+\frac{384}{5}\overline{k}$

Problems on Gauss Divergence

1. Apply Gauss Divergence theorem to compute $\iint_S \overline{F}.\overline{n}\, ds$ where $\overline{F} = x\overline{i} - y\overline{j} + z\overline{k}$ over the surface of the cylinder $x^2 + y^2 = a^2$ bounded by the planes. $z = 0, z = b$.

2. Evaluate $\iint_S \overline{F}.\overline{n}\, ds$ where $\overline{F} = (2x+z)\overline{i} + yz\overline{j} + z^2\overline{k}$ over the upper half of the sphere $x^2+y^2+z^2 = a^2$.

3. Verify the divergence theorem for $\overline{F} = (2x - z)\overline{i} + x^2\, y\overline{j} - xz^2\overline{k}$ in the region bounded by the planes $x = y = z = 0$ and $x = y = z = 1$.

4. Verify Gauss Divergence theorem for $\overline{F} = x^2\overline{i} + y^2\overline{j} + z^2\overline{k}$ over the surface S of the solid cut off by the plane. $x + y + z = a$ in the first octant.

5. Verify Gauss Divergence theorem for $\overline{F} = z\overline{i} + x\overline{j} - 3y^2 z\overline{k}$ and s is the surface $x^2 + y^2 = 16$ included in the first octant between $z = 0$ and $z = 5$.

6. Compute $\iint_S (ax^2 + by^2 + cz^2)\, ds$ over the surface of the sphere $x^2 + y^2 + z^2 = 1$.

ANSWERS

1. $\pi a^2 b$ 2. $\frac{\pi a^3}{12}(16 + 9a)$ 6. $\frac{4\pi}{3}(a+b+c)$.

Problems on Green's Theorem

EXERCISE 8(E)

1. Verify Green's theorem for $\oint_c (y - \sin x)\, dx + \cos x\, dy$ where c is the plane triangle enclosed by the lines $y = 0$, $y = \pi/2$, $y = \frac{2x}{\pi}$

2. Verify Green's theorem in a plane for $\oint_c (x^2 - 2xy)\, dx + (x^2 y + 3)\, dy$ where c is the boundary of region bounded by the parabola $y^2 = 8x$ and the line $x = 2$.

3. Verify Green's theorem in plane $\oint_c (2x - y^3) dx - xy\, dy$ where c is the boundary of the annulus enclosed by the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 9$.

4. Evaluate $\oint_c \sin y\, dx + x(1 + \cos y)\, dy$ over the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1; z = 0$.

5. Evaluate $\oint_c \overline{F}.d\overline{r}$ where $\overline{F} = (x - 2y)\,\overline{i} + x\overline{j}$ where c is the circle $x^2 + y^2 = 1$; $z = 0$ using Green's theorem.

6. Verify Green's theorem for $\oint_c (3x^2 + 2y)\, dx - (x + 3\cos y)\, dy$ where c is the curve bounded by the parallelogram with vertices (0, 0), (2, 0), (3, 1), (1, 1).

ANSWERS

4. πab. 5. 3π

Problems on Stoke's Theorem

1. Verify stoke's theorem for $\overline{F} = (x^2 + y^2)\overline{i} - 2xy\,\overline{j}$ taken round the Rectangle bounded the lines $x = \pm a; y = 0; y = b$. [2003, 2005]

2. Verify stoke's theorem for $\overline{F} = (2x - y)\overline{i} - yz^2\,\overline{j} - y^2 z\,\overline{k}$ over the half surface of the sphere $x^2 + y^2 + z^2 = 1$ bounded by the projection of the xy plane. [JNTU 2006, 07, 08, 09, 10]

3. Apply stoke's theorem to evaluate $\oint_c (y\, dx + z\, dy + x\, dz)$ where c is the curve of intersection of the sphere $x^2 + y^2 + z^2 = a^2$ and plane $x + z = a$.

4. Verify stoke's theorem for $\overline{F} = -y^3\,\overline{i} + x^3\overline{j}$ where s is the circular disc $x^2 + y^2 \leq 1; z = 0$. [JNTU 2007, 2008]

5. Evaluate $\oint_c (xy\, dx + xy^2\, dy)$ taken round the square with vertices (1, 0) (–1, 0) (0, 1) (0, –1). Using stoke's theorem.

ANSWERS

3. $\frac{\pi a^2}{\sqrt{2}}$ 5. 4/3

Objective Questions

OBJECTIVE TYPE QUESTIONS

(1) If $\phi = x^2 + y^2 + z^2 - 3xyz$ then curl (grad ϕ) =

(a) $\overline{0}$ (b) $6x + 6y + 6z$ (c) $x + y + z$ (d) None

(2) If $\phi = x^2 + y^2 + z^2 - 3xyz$ the div grad ϕ =

(a) 0 (b) $6x + 6y + 6z$ (c) $x+y+z$ (d) None

(3) If $\overline{r} = x\overline{i} + y\overline{j} + z\overline{k}$ then div $\overline{r}$ =

(a) 3 (b) $\overline{0}$ (c) 0 (d) None

(4) If $\overline{r} = x\overline{i} + y\overline{j} + z\overline{k}$ then curl $\overline{r}$ =

(a) $\overline{0}$ (b) $3x\overline{i}$ (c) $3y\overline{j}$ (d) $3z\overline{k}$

(5) If $\overline{a}$ is a constant vector then curl $(\overline{r} \times \overline{a})$ =

(a) $2\overline{a}$ (b) $-2\overline{a}$ (c) $\overline{a}$ (d) $-\overline{a}$

(6) If $\overline{r} = x\overline{i} + y\overline{j} + z\overline{k}$ then $\nabla \overline{r}$ =

(a) $\frac{\overline{r}}{r}$ (b) $\overline{r}$ (c) $x\overline{i}$ (d) $y\overline{j}$

(7) If $\overline{a}$ and $\overline{b}$ are irrotational vectors, then $\overline{a} \times \overline{b}$ is

(a) Solenoidal vector (b) Irrotational vector

(c) Free vector (d) None

(8) If $\phi = ax^2 + by^2 + cz^2$ satisfies Laplacian equation, then $a + b + c$ =

(a) 0 (b) 1 (c) 2 (d) None

(9) If $\overline{r} = x\overline{i} + y\overline{j} + z\overline{k}$ then $\nabla^2\left(\frac{1}{r}\right)$ =....

(a) 0 (b) $3x$ (c) $2x$ (d) $3(x+y+z)$

(10) If $\overline{F} = x(y+z)\overline{i} + y(z+x)\overline{j} + z(x+y)\overline{k}$ then div$\overline{F}$ =

(a) $(x+y+z)$ (b) $2(x+y+z)$

(c) $3(x+y+z)$ (d) None

(11) If $\overline{r} = x\overline{i} + y\overline{j} + z\overline{k}$ and if $(r^n\overline{r})$ is solenoidal then n =

(a) 3 (b) -3 (c) 1 (d) None

(12) If $\phi(x,y,z) = c$ is a surface then $\nabla\phi$ is

(a) Normal to $\phi = c$ (b) tangent to $\phi = c$ (c) binormal to $\phi = c$ (d) None

(13) If curl $\overline{f} = \overline{0}$ then $\overline{f}$ is

(a) Solenoidal vector (b) Constant vector

(c) Irrotational vector (d) Can not say

(14) If $\overline{f} = f_1(y,z)i + f_2(z,x)\overline{j} + f_3(x,y)\overline{k}$ then $\overline{f}$ is

(a) Irrotational (b) Solenoidal

(c) Both Solenoidal and Irrotational (d) None.

(15) If $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ then $\nabla^2(\log r) =$

(*a*) 0 (*b*) $1/r^2$ (*c*) $x + y + z$ (*d*) None

(16) If $u = x^2 + y^2 + z^2$ and $V = xi + yj + zk$ then $\nabla \cdot (uV) =$

(a) 0 (b) 2u (c) 3u (d) 5u

(17) If $div\,\bar{A} = 0$ then $\bar{A}$ is called

(a) Irrotational (b) Solenoidal (c) constant vector (d) none

(18) $\int_C \bar{F} \cdot d\bar{r}$ is independent of the path joining any two points if and only if it is ________

(19) Physical interpretation of $\nabla\phi$ is that ________

ANSWERS

1) a 2) d 3) a 4) a 5) b 6) a 7) a 8) a 9) a 10) b
11) c 12) a 13) a 14) b 15) b 16) d 17) b 18) Irrotational field
19) it gives maximum rate of change of ϕ.

OBJECTIVE TYPE QUESTIONS

(1) For any closed surface S, $\iint_S \operatorname{curl} \overline{F} \cdot \overline{n}\, dS =$

(a) 0 (b) $2\overline{F}$ (c) $\overline{n}$ (d) $\oint \overline{F}.d\overline{r}$

(2) If S is any closed surface enclosing a volume V and $\overline{F} = x\overline{i} + 2y\overline{j} + 3z\overline{k}$ then $\iint_S \overline{F} \cdot \overline{n}\, dS =$

(a) V (b) $3V$ (c) $6V$ (d) None

(3) If $\overline{r} = x\overline{i} + y\overline{j} + z\overline{k}$ then $\oint_c \overline{r}.d\overline{r} =$

(a) 0 (b) $\overline{r}$ (c) x (d) None

(4) $\int \overline{r} \times \overline{n} dS =$

(a) 0 (b) r (c) 1 (d) None

(5) $\int_s \overline{r}.\overline{n}\, dS =$

(a) V (b) $3V$ (c) $4V$ (d) None

(6) If $\overline{n}$ is the unit outward drawn normal to any closed surface then $\int_V \operatorname{div} \overline{n}\, dV =$

(a) S (b) $2S$ (c) $3S$ (d) None

(7) $\oint f\, \nabla f \cdot d\overline{r} =$

(a) f (b) $2f$ (c) 0 (d) None

(8) The value of the line integral $\int grad(x+y-z)d\overline{r}$ from $(0, 1, -1)$ to $(1, 2, 0)$ is

(a) -1 (b) 0 (c) 2 (d) 3

(9) A necessary and sufficient condition that the line integral $\int_c A \cdot dr = 0$ for every closed curve c is that

(a) div A = 0 (b) div A $\neq$ 0 (c) curl A = 0 (d) curl A $\neq$ 0

(10) If $\overline{F} = axi + byj + czk$ where a, b, c are constants then $\iint \overline{F} \cdot \overline{n} dS$ where S is the surface of the unit sphere is

(a) 0 (b) $\frac{4}{3}\pi(a+b+c)$ (c) $\frac{4}{3}\pi(a+b+c)^2$ (d) none

(11) $\int_V D \times \overline{F} dv =$ ______

(a) $\int_S \overline{n} \times \overline{F} ds$ (b) 0 (c) V (d) S

(12) $\int_V \phi \times dv =$ ________

(a) $\int \overline{n}\phi ds$ (b) 0 (c) V (d) ϕ

(13) $\int fog \cdot d\overline{r} =$ ________

(a) 0 (b) $\int_S (\nabla f \times \overline{F} Dg)$ (c) $\overline{r}$ (d) S

(14) $\iint_S xdydx + ydzdx + zdxdy$ where $S: x^2 + y^2 + z^2 = a^2$ as

(a) 4p (b) $\frac{4}{3}\pi a^3$ (v) $4\pi a^3$ (d) 4π

ANSWERS

(1) d (2) c (3) a (4) a (5) b (6) a (7) c (8) d (9) c (10) b

(11) a (12) a (13) b (14) c