3.4

SURFACE INTEGRALS

Consider a surface S whose equation is $f(x, y, z) = c$. We may also write the equation as $f(x, y, z) = 0$. We have shown in last modules that

$$\boldsymbol{n} = \text{grad}(f) \text{ and } \hat{\boldsymbol{n}} = \frac{\text{grad}\,(f)}{|\text{grad}\,(f)|}$$

are the normal and unit normal vectors respectively to the surface S. We assume that $f(x, y, z)$ has continuous first order partial derivatives at each point (x, y, z) in its domain and at least one of them is not equal to zero. Then, a unique normal exists at each point of the surface S. We then say that S is a smooth surface. A piecewise smooth surface consists of a number of surfaces each of which is a smooth surface. For example, the surface of a sphere is a smooth surface while the surfaces of a closed cylinder or a cube are piecewise smooth surfaces.

We have discussed the parametric representation of a surface in previous modules. If u, v are two independent paramenters taking values in a region R in the $uv-$ plane and if we write $x = x(u, v), y = y(u, v), z = z(u, v)$, then the parametric representation of the surface S can be written as

$$\boldsymbol{r} = \boldsymbol{r}(u, v) = x(u, v)\boldsymbol{i} + y(u, v)\boldsymbol{j} + z(u, v)\boldsymbol{k}$$

If the equation of the surface is in the form $z = f(x, y)$, then the parametric representation of the surface S can be written as

$$\boldsymbol{r}(u, v) = u\boldsymbol{i} + v\boldsymbol{j} + f(u, v)\boldsymbol{k}$$

Where $u = x$ and $v = y$, or simply as

$$\boldsymbol{r}(x, y) = x\boldsymbol{i} + y\boldsymbol{j} + f(x, y)\boldsymbol{k}.$$

Consider now a surface S whose parametric representation is $\boldsymbol{r} = \boldsymbol{r}(u, v)$. Let C be a curve on S. Then, the parametric representation of the curve C can be written as

$$u = f(t), v = g(t), t \text{ a real parameter}$$

that is, $$\boldsymbol{r} = \boldsymbol{r}(t) = \boldsymbol{r}[u(t), v(t)] = \boldsymbol{r}[f(t), g(t)]$$

We assume that $f(t)$ and $g(t)$ have continuous first order derivatives with respect to t. Then, the tangent vector to C for any value of the parameter t is given by

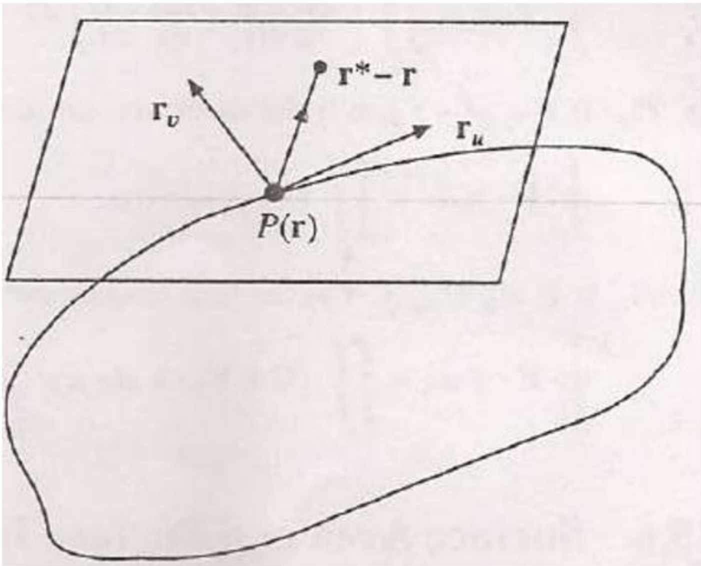
$$\frac{d\boldsymbol{r}}{dt} = \frac{\partial \boldsymbol{r}}{\partial u}\frac{du}{dt} + \frac{\partial \boldsymbol{r}}{\partial v}\frac{dv}{dt} = \boldsymbol{r}_u\frac{du}{dt} + \boldsymbol{r}_v\frac{dv}{dt}.$$

Now, since u, v are independent parameters, $\partial\boldsymbol{r}/\partial u$ and $\partial\boldsymbol{r}/\partial v$ are independent vectors and hence they determine a plane. Consider the point $P(\boldsymbol{r})$ on the surface S. At this point, we have two independent vectors $\boldsymbol{r}_u$ and $\boldsymbol{r}_v$. Let $\boldsymbol{r}^*$ be the position vector of any point in the tangent plane see in figure. Then, the vector $\boldsymbol{r}^* - \boldsymbol{r}$ can be written as a linear combination of the vectors $\boldsymbol{r}_u$ and $\boldsymbol{r}_v$. Alternately, the vectors $\boldsymbol{r}^* - \boldsymbol{r}$, $\boldsymbol{r}_u$ and $\boldsymbol{r}_v$ are coplanar. Therefore, the equation of the tangent plane at P is given by

$$(\boldsymbol{r}^* - \boldsymbol{r}).(\boldsymbol{r}_u \times \boldsymbol{r}_v) = [\boldsymbol{r}^* - \boldsymbol{r} \;\; \boldsymbol{r}_u \; \boldsymbol{r}_v] = 0$$

Where [...] is the scalar triple product.

Which is the equation of the tangent plane at a point P on the surface S.



Tangent plane

Now,

$$\frac{\partial \boldsymbol{r}}{\partial u} \times \frac{\partial \boldsymbol{r}}{\partial v} = \boldsymbol{r}_u \times \boldsymbol{r}_v$$

is a vector perpendicular to both the vectors $\frac{\partial \boldsymbol{r}}{\partial u}$ and $\frac{\partial \boldsymbol{r}}{\partial v}$ and hence it is the normal vector to the tangent plane, that is normal vector to the surface S at a point P. This given the representation of the normal vector in terms of the parametric representation while we have earlier shown that grad(f) is the normal vector to the surface S. Since S is a smooth surface, a unique normal exists. This implies that

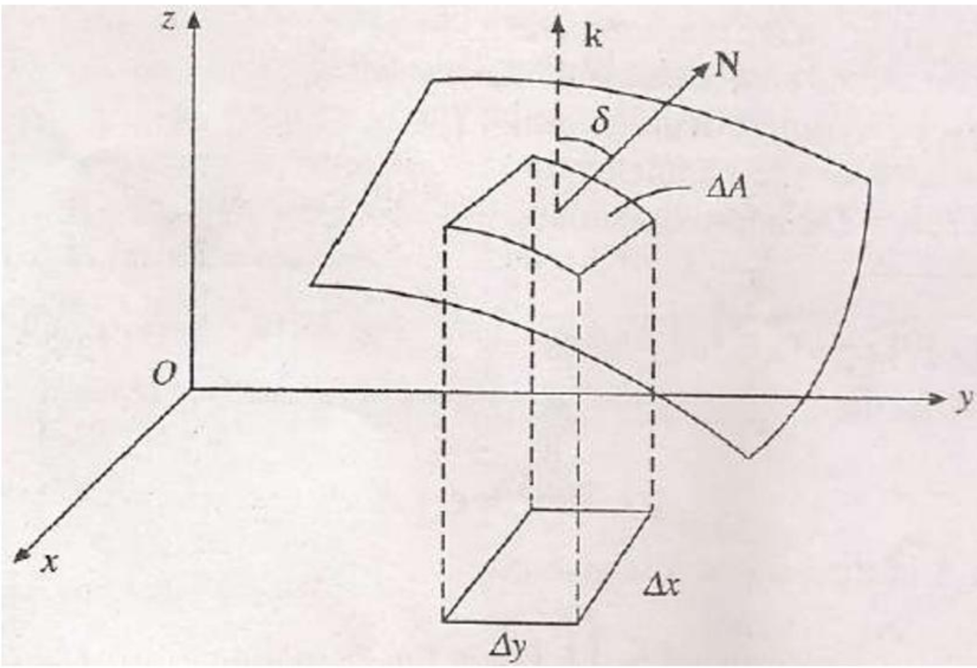
$\boldsymbol{r}_u \times \boldsymbol{r}_v \neq 0$. The unit normal vector at a point P on the surface S is given by

$$\widehat{\boldsymbol{n}} = \frac{\boldsymbol{r}_u \times \boldsymbol{r}_v}{|\boldsymbol{r}_u \times \boldsymbol{r}_v|}.$$

We now derive an alternate form for the element surface area ΔA using projections, which is of use in evaluating surface integrals.

(a). Let $z = f(x, y)$ be the equation of the surface. Let the element area ΔA be projected on the XY $-$plane. Let $\boldsymbol{n}$ be the normal vector to the element area. Now,

$$\boldsymbol{n}.\boldsymbol{k} = (\boldsymbol{r}_u \times \boldsymbol{r}_v).\boldsymbol{k} = (-f_u \boldsymbol{i} - f_v \boldsymbol{j} + \boldsymbol{k}).\boldsymbol{k} = 1.$$



Projection of element surface area

Let $\delta(< \pi/2)$ be the angle which the normal vector $\boldsymbol{n}$ makes with the positive direction of $Z-$axis. Then, using above equation, we obtain

$$\boldsymbol{n}.\boldsymbol{k} = |\boldsymbol{n}| \cos\delta = 1.$$

Therefore,

$$\sec\delta = |\boldsymbol{n}| = |\boldsymbol{r}_u \times \boldsymbol{r}_v| = \sqrt{1 + f_x^2 + f_y^2}, \qquad dA = (\sec\delta) dxdy$$

and

$$A = \iint_{R^*} \sqrt{1 + f_x^2 + f_y^2} dx\, dy.$$

Similarly, we have the following cases.

(b). Equation of the surface: $x = g(y, z)$. We project S on the $YZ-$plane. Then

$$dA = (\sec\alpha) dy\, dz, \qquad (\alpha < \pi/2)$$

Where α is the angle which the normal vector $\boldsymbol{n}$ makes with the positive direction of $X-$axis. We have

$$A = \iint_{R^*} \sqrt{1 + g_y^2 + g_z^2}\, dy\, dz$$

Where R^* is the projection of S on the $YZ-$plane.

(c). Equation of the surface: $y = h(x, z)$. We project S on the $XZ-$ plane. Then

$$dA = (\sec\beta) dx\, dz, \qquad (\beta < \pi/2)$$

Where β is the angle which the normal vector $\boldsymbol{n}$ makes with the positive direction of Y −axis. We have

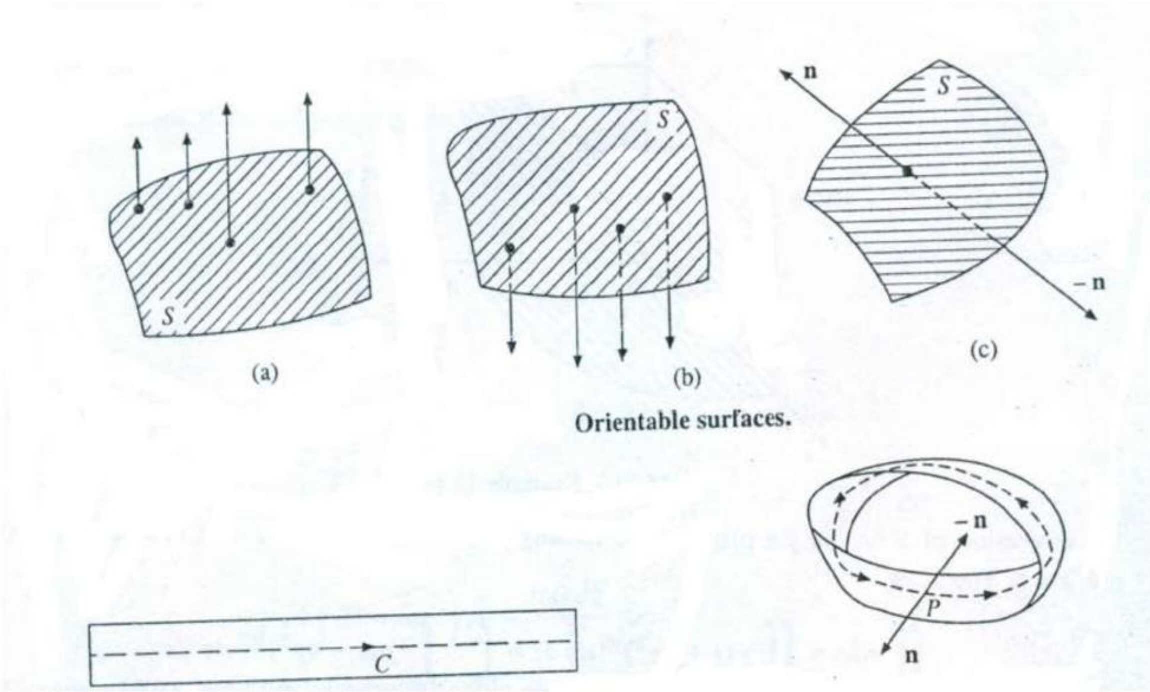
$$A = \iint_{R^*} \sqrt{1 + h_x^2 + h_z^2}\, dx\, dz,$$

Where R^* is the projection of S the XZ −palne,

For example, consider the lateral surface S of an open cylinder $x^2 + y^2 = a^2, 0 < z < k$. We can project the surface on the YZ or on the XZ − planes. The projections are rectangular regions on the corresponding planes. We cannot project the surfaces S on the XY −plane.

Orientable surfaces: Before we try to evaluate a surface integral of a vector field we need the concept of an orientable surface. Without mathematical rigour, we may say that S is an orientable surface if it has two sides (which may be painted in two different colours). A surface which has only one side is not an orientable surface. A famous example of a non-orientable surface is the möbius strip. To construct a Möbius strip, we can cut out a long strip of paper, give one of the ends a half twist and then attach the ends.

A smooth surface S is said to be orientable if there exists a continuous unit normal vector field $\boldsymbol{n}$ defined at each point (x, y, z) on the surface. We then say that the vector field $\boldsymbol{n}(x, y, z)$ is the orientation of S. An Orientation surface has two orientations since a unit normal to a surface S at (x, y, z) can be $\boldsymbol{n}(x, y, z)$ or $-\boldsymbol{n}(x, y, z)$. We usually call them, outward and inward normal's. They are also sometimes called upward and downward orientations. The Möbius strip is not an orientable surface, since if a unit normal vector $\boldsymbol{n}$ starts at a point P_0 on the surface and moves once round the curve C, then the resulting normal vector points in the opposite direction. Most of the surfaces that we encounter are oreintable surfaces.



Orientable surfaces.

Strip of paper　　　　　　　Möbius strip

Let $\boldsymbol{V}(x, y, z) = v_1(x, y, z)\boldsymbol{i} + v_2(x, y, z)\boldsymbol{j} + v_3(x, y, z)\boldsymbol{k}$ be a vector field Then,

$$\iint_S \boldsymbol{V}.\boldsymbol{n}\ dA$$

where $\boldsymbol{n}$ is the unit normal vector to the given surface (for smooth surfaces only one normal vector will exists, namely gradient).

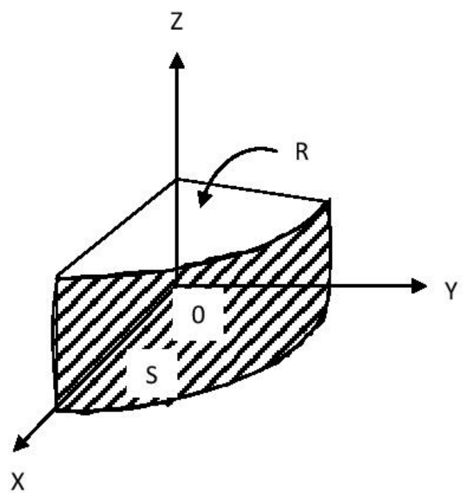
If the projection of the given surface is taken on XY −plane then $dA = \frac{dx\, dy}{\boldsymbol{n}.\boldsymbol{k}}$.

If the projection of the given surface is taken on YZ −plane then $dA = \frac{dy\, dz}{\boldsymbol{n}.\boldsymbol{i}}$.

If the projection of the given surface is taken on XZ −plane then $dA = \frac{dx\, dz}{\boldsymbol{n}.\boldsymbol{j}}$.

Example: Evaluate the surface integral $\iint_S \boldsymbol{F}.\boldsymbol{n}\, dA$ Where $\boldsymbol{F} = z^2\boldsymbol{i} + xy\,\boldsymbol{j} - y^2\boldsymbol{k}$ and S is the portion of the surface of the cylinder $x^2 + y^2 = 36, 0 \le z \le 4$ included in the first octant.

Solution: Let $f(x, y, z) = x^2 + y^2 - 36 = 0$ be the surface.



Then grad$f = 2x\,\boldsymbol{i} + 2y\,\boldsymbol{j}$, $\boldsymbol{n} = \frac{\text{grad } f}{|\text{grad } f|} = \frac{2(x\boldsymbol{i}+y\boldsymbol{j})}{\sqrt{4((x^2+y^2)}} = \frac{1}{6}(x\boldsymbol{i} + y\boldsymbol{j})$.

The projection of S on XY − plane cannot be considered. Project S on the YZ − plane. The projection is a rectangle with sides of lengths 6 and 4. We have

$$dA = \frac{dydz}{\boldsymbol{n}.\boldsymbol{i}} = \frac{dydz}{x/6}.$$

Therefore,

$$\iint_S \boldsymbol{F}.\boldsymbol{n}\, dA = \iint_S \frac{1}{6}(z^2x + xy^2)dA = \int_{z=0}^{4}\int_{y=0}^{6} \frac{1}{6}\, x(y^2+z^2)\frac{dydz}{x/6}$$

$$= \int_0^4 \left[\int_0^6 (y^2+z^2)\, dy\right] dz = \int_0^4 \left(\frac{y^3}{3} + yz^2\right)_0^6 dz$$

$$= \int_0^4 (72 + 6z^2)\, dz = [72\, z + 2z^3]_0^4 = 416.$$

Applications of Surface Integrals

We now present some of the important applications of the surface integrals.

Mass of a surface Let $\rho(x,y,z)$ denote the density of a surface S at any point or mass per unit surface area. Then, the mass m of the surface is given by

$$m = \iint_S \rho(x,y,z)dA$$

Moment of Inertia Let $\rho(x,y,z)$ denote the density of a surface S at a point. Then, the moment of inertia I of the mass m with respect to a given axis l is defined by the surface integral

$$I = \iint_S \rho(x,y,z)d^2dA$$

Where d is the distance of the point (x,y,z) from the reference axis l. If the surface is homogeneous, then

$\rho(x, y, z) =$ constant and, $\rho(x, y, z) = m/A$ Where A is the surface area of S. Then,

$$I = \frac{m}{A} \iint_S d^2 dA.$$

Flux of a vector field V through a surface S

Let $\boldsymbol{V}(x, y, z) = v_1(x, y, z)\boldsymbol{i} + v_2(x, y, z)\boldsymbol{j} + v_3(x, y, z)\boldsymbol{k}$ be a vector field representing the velocity field of a fluid. The flux of the velocity vector field V through the area ΔA is approximated by $(V.n)\Delta A$, where n is a unit vector normal to the surface. The total volume of the fluid flowing through S per unit time is called flux of V through S. It is given by

$$\text{flux} = \iint_S (\boldsymbol{V}.\boldsymbol{n}) dA$$

The other surface integrals are

$$\iint_S \phi dS, \iint_S \phi n dS, \iint_S A \times dS$$

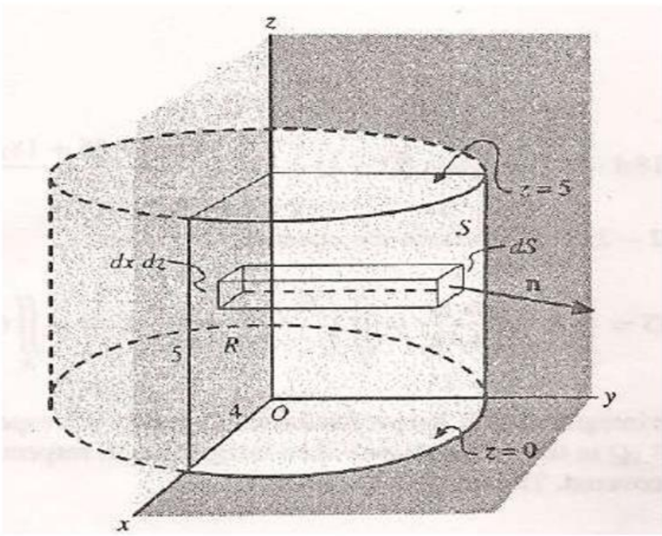
Where $\emptyset$ is a scalar function.

Problem 1: Evaluate $\int\int_S \boldsymbol{A}.\boldsymbol{n}ds$ where $\boldsymbol{A} = z\boldsymbol{i} + x\boldsymbol{j} - 3y^2z\boldsymbol{k}$ and S is the surface of the cylinder $x^2 + y^2 = 16$ included in the first octant between $z = 0$ and $z = 5$.

Solution:

Project S on the XZ −plane as in following figure and call the projection R. Note that the projection of S on the XY −plane cannot be used here. Then

$$\int\int_S \boldsymbol{A}.\boldsymbol{n}ds = \int\int_R \boldsymbol{A}.\boldsymbol{n}\frac{dxdz}{|\boldsymbol{n}.\boldsymbol{j}|}$$



A normal to $x^2 + y^2 = 16$ is $\nabla(x^2 + y^2) = 2x\boldsymbol{i} + 2y\boldsymbol{j}$. Thus as shown in following figure the unit normal to S is

$$\boldsymbol{n} = \frac{2x\boldsymbol{i} + 2y\boldsymbol{j}}{\sqrt{(2x)^2 + (2y)^2}} = \frac{x\boldsymbol{i} + y\boldsymbol{j}}{4}$$

Since $x^2 + y^2 = 16$ on S.

$$\boldsymbol{A}.\boldsymbol{n} = (z\boldsymbol{i} + x\boldsymbol{j} - 3y^2z\boldsymbol{k}).\left(\frac{x\boldsymbol{i} + y\boldsymbol{j}}{4}\right) = \frac{1}{4}(xz + xy)$$

$$\boldsymbol{n}.\boldsymbol{j} = \frac{x\boldsymbol{i} + y\boldsymbol{j}}{4}.\boldsymbol{j} = \frac{y}{4}$$

Then the surface integral equals

$$\int\int_R \frac{xz+xy}{y} dxdz = \int_{z=0}^{5}\int_{x=0}^{4}\left(\frac{xz}{\sqrt{16-x^2}} + x\right) dx\, dz$$

$$= \int_{z=0}^{5}(4z+8)\, dz = 90.$$

Problem 2: Evaluate $\int\int_S \emptyset \boldsymbol{n} ds$ where $\emptyset = \frac{3}{8}xyz$ and S is the surface of above problem.

Solution:

We have $\int\int_S \emptyset \boldsymbol{n} ds = \int\int_R \emptyset \boldsymbol{n} \frac{dxdz}{|\boldsymbol{n}.\boldsymbol{j}|}$

Using $\boldsymbol{n} = x\boldsymbol{i} + y\boldsymbol{j}/4$, $\boldsymbol{n}.\boldsymbol{j} = y / 4$ as in above problem, this last integral becomes

$$\int\int_S \frac{3}{8} xz(x\boldsymbol{i} + y\boldsymbol{j})dxdz = \frac{3}{8}\int_{z=0}^{5}\int_{x=0}^{4}(x^2 z\boldsymbol{i} + xz\sqrt{16 - x^2\boldsymbol{j}})dx\, dz$$

$$= \frac{3}{8}\int_{z=0}^{5}(\frac{64}{3}z\boldsymbol{i} + \frac{64}{3}z\boldsymbol{j})dz = 100\boldsymbol{i} + 100\boldsymbol{j}.$$

Problem 3: Le$\boldsymbol{F} = 4xz\boldsymbol{i} - y^2\boldsymbol{j} + yz\boldsymbol{k}$. Evaluate$\int\int_S \boldsymbol{F}.\boldsymbol{n}ds$ where S is the surface of the cube bounded by $x = 0, x = 1, y = 0, y = 1, z = 0, z = 1$. (see above figure)

Solution:

Face *DEFG:* $\boldsymbol{n} = \boldsymbol{i}, x = 1$. Then

$$\int\int_{DEFG} \boldsymbol{F}.\boldsymbol{n}ds = \int_0^1\int_0^1 (4xz\boldsymbol{i} - y^2\boldsymbol{j} + yz\boldsymbol{k}).\boldsymbol{i}\ dy\, dz$$

$$= \int_0^1\int_0^1 4z\ dy\, dz = 2.$$

Face *ABCO:* $\boldsymbol{n} = -\boldsymbol{i}, x = 0$. Then

$$\int\int_{ABCO} \boldsymbol{F}.\boldsymbol{n}ds = \int_0^1\int_0^1 (-y^2\boldsymbol{j} + yz\boldsymbol{k}).(-\boldsymbol{i})\ dy\, dz = 0.$$

Face *ABEF:* $\boldsymbol{n} = \boldsymbol{j}, y = 1$. Then

$$\int\int_{ABEF} \boldsymbol{F}.\boldsymbol{n}ds = \int_0^1\int_0^1 (4xz\boldsymbol{i} - \boldsymbol{j} + z\boldsymbol{k}).\boldsymbol{j}\ dx\, dz$$

$$= \int_0^1\int_0^1 -dx\, dz = -1.$$

Face *OGDC:* $\boldsymbol{n} = -\boldsymbol{j}, y = 0$. Then

$$\int\int_{OGDC} \boldsymbol{F}.\boldsymbol{n}ds = \int_0^1\int_0^1 (4xz\boldsymbol{i}).(-\boldsymbol{j})\ dx\, dz = 0.$$

Face *BCDE:* $\boldsymbol{n} = \boldsymbol{k}, z = 1$. Then

$$\int\int_{BCDE} \boldsymbol{F}.\boldsymbol{n}ds = \int_0^1\int_0^1 (4x\boldsymbol{i} - y^2\boldsymbol{j} + y\boldsymbol{k}).\boldsymbol{k}\ dx\, dy$$

$$= \int_0^1\int_0^1 y dx\, dy = \frac{1}{2}.$$

Face AFGO: $\boldsymbol{n} = -\boldsymbol{k}, z = 0$. Then

$$\int\int_{AFGO} \boldsymbol{F}.\boldsymbol{n}ds = \int_0^1\int_0^1 (-y^2\boldsymbol{j}).(-\boldsymbol{k})dx\, dy = 0.$$

Adding all faces, then we get

$$\int\int_S \boldsymbol{F}.\boldsymbol{n}ds = 2 + 0 + (-1) + 0 + \frac{1}{2} + 0 = \frac{3}{2}.$$

Problem 4: Find the mass of the surface of the cone $z = 2 + \sqrt{x^2 + y^2}$, $2 \leq z \leq 7$, in the first octant, if the density $\rho(x, y, z)$ at any point of the surface is proportional to its distance from the XY −plane.

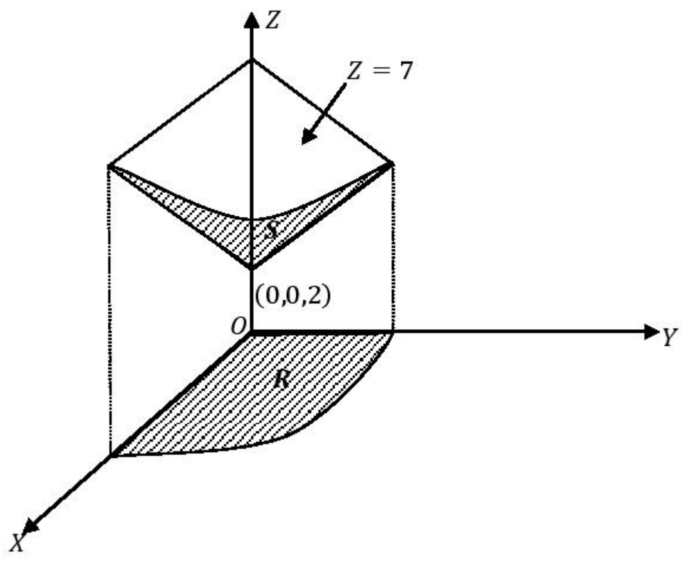
Solution:

The density is given by $\rho(x, y, z) = cz$, c constant. We have

$$z = f(x, y) = 2 + \sqrt{x^2 + y^2},\ f_x = \frac{x}{\sqrt{x^2+y^2}},\ f_y = \frac{y}{\sqrt{x^2+y^2}}$$

$$dA = \sqrt{1 + f_x^2 + f_y^2}\,dxdy = \sqrt{1 + \frac{x^2}{x^2+y^2} + \frac{y^2}{x^2+y^2}}\,dxdy = \sqrt{2}\,dxdy.$$

The projection of S on the XY − plane is given by figure $R: x^2 + y^2 = 25$, in the first quadrant.



Therefore, mass of the surface is given by

$$m = \iint_s czdA = \iint_R c\left[2 + \sqrt{x^2 + y^2}\right]\sqrt{2}\ dxdy$$

$$= c\sqrt{2}\iint_R\left[2 + \sqrt{x^2 + y^2}\right]dxdy.$$

Substituting $x = r\cos\theta,\ \ y = r\sin\theta, \quad 0 \leq \theta \leq \frac{\pi}{2}$, we obtain

$$m = c\sqrt{2}\int_0^5 \int_0^{\frac{\pi}{2}}(2 + r)rdrd\theta = c\sqrt{2}\int_0^{\frac{\pi}{2}}\left(r^2 + \frac{r^3}{3}\right)_0^5 d\theta$$

$$= c\sqrt{2}\left(25 + \frac{125}{3}\right)\frac{\pi}{2} = \frac{100\sqrt{2}}{3}\pi c.$$

Problem 5: Find the moment of inertia of the homogenous cylindrical lamina $x^2 + y^2 = a^2$, $0 \le z \le b$ of mass m about the $Z-$axis.

Solution: Since the surface is homogenous, the density $\rho(x, y, z)$ =constant and the moment of the inertia is

$$I = \frac{m}{A} \iint_S d^2 dA$$

Where A =surface are of cylinder= $2\pi ab$,

d =distance of any point on S from Z –axis= $\sqrt{x^2 + y^2}$.

Now, the parametric form of the surface S is

$$r(u, v) = acosu\boldsymbol{i} + a \sin u\boldsymbol{j} + v\boldsymbol{k}$$

We have $\boldsymbol{r}_u = -a\sin u\boldsymbol{i} + a\cos u\boldsymbol{j}$, $\boldsymbol{r}_v = \boldsymbol{k}$, $\boldsymbol{r}_u^2 = a^2$, $\boldsymbol{r}_u^2 = 1$, $\boldsymbol{r}_u.\boldsymbol{r}_v = 0$.

Therefore,

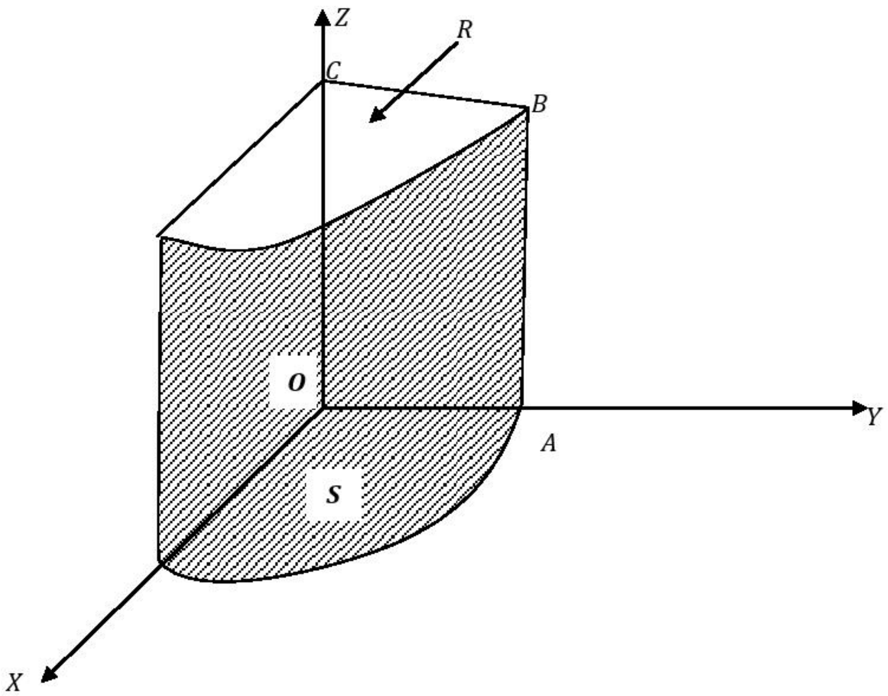
$$I = \frac{m}{2\pi ab} \iint_R (a^2).\sqrt{a^2} dudv = \frac{ma^2}{2\pi b} \int_0^b \int_0^{2\pi} dudv = ma^2.$$

Problem 6: Evaluate the integral $\iint_S y dA$ where S is the portion of the cylinder $x = 6 - y^2$ in the first octant bounded by the planes $x = 0$, $y = 0, z = 0$ and $z = 8$.

Solution:

The equation of the surface is in the form $x = h(y,z)$. Therefore, $h(y,z) = 6 - y^2$. We have

$$h_y = -2y, h_z = 0, \ \ dA = \left(1 + h_y^2 + h_z^2\right)^{\frac{1}{2}} dydz = (1 + 4y^2)^{\frac{1}{2}} dydz$$



The projection of S on the YZ −plane is the rectangle $OABC$ with sides $y = 0$, $y = \sqrt{6}$, $z = 0$ and $z = 8$. Therefore,

$$\iint_S y dA = \iint_R y(1 + 4y^2)^{\frac{1}{2}} dydz = \int_0^{\sqrt{6}} \int_0^8 y(1 + 4y^2)^{\frac{1}{2}} dydz$$

$$= 8\left[\frac{\left(1+4y^2\right)^{\frac{3}{2}}}{8\left(\frac{3}{2}\right)}\right]_0^{\sqrt{6}} = \frac{2}{3}\left[(25)^{\frac{3}{2}} - 1\right] = \frac{248}{3}.$$

Problem 7: Suppose $\boldsymbol{F} = y\boldsymbol{i} + (x - 2xz)\boldsymbol{j} - xy\boldsymbol{k}$. Evaluate $\int\int_S (\nabla \times \boldsymbol{F}).\boldsymbol{n}ds$ where S is the surface of the sphere $x^2 + y^2 + z^2 = a^2$ above XY −plane.

Solution:

$$\boldsymbol{\nabla} \times \boldsymbol{F} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y & x - 2xz & -xy \end{vmatrix} = x\boldsymbol{i} + y\boldsymbol{j} - 2z\boldsymbol{k}$$

A normal to $x^2 + y^2 + z^2 = a^2$ is

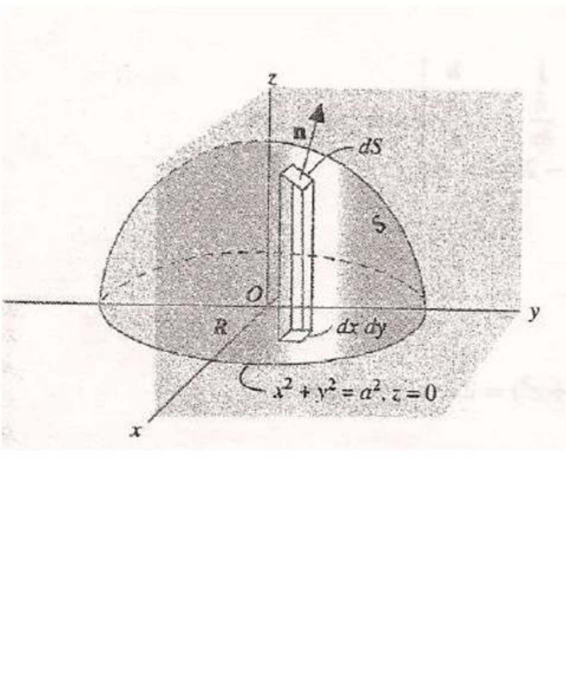
$$\boldsymbol{\nabla}(x^2 + y^2 + z^2) = 2x\boldsymbol{i} + 2y\boldsymbol{j} + 2z\boldsymbol{k}$$

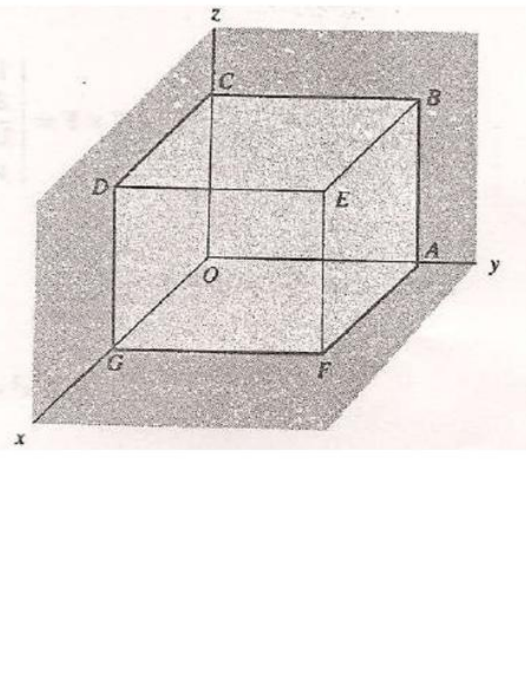
Then the unit normal $\boldsymbol{n}$ of following figure is given by

$$\boldsymbol{n} = \frac{2x\boldsymbol{i} + 2y\boldsymbol{j} + 2z\boldsymbol{k}}{\sqrt{4x^2 + 4y^2 + 4z^2}} = \frac{x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}}{a}$$

Since $x^2 + y^2 + z^2 = a^2$.

The projection of S on the XY −plane is the region R bounded by the circle $x^2 + y^2 = a^2, z = 0$ (see following figure).





Then

$$\int\int_S (\nabla \times \boldsymbol{F}).\boldsymbol{n}ds = \int\int_R (\nabla \times \boldsymbol{F}).\boldsymbol{n}\frac{dxdy}{|n.k|}$$

$$= \int\int_S (x\boldsymbol{i} + y\boldsymbol{j} - 2z\boldsymbol{k}).(\frac{x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k}}{a})\frac{dxdy}{z/a}$$

$$= \int_{x=-a}^{a}\int_{y=-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \frac{3(x^2+y^2)-2a^2}{\sqrt{a^2-x^2-y^2}}\,dy\,dx$$

Using the fact that $z = \sqrt{a^2 - x^2 - y^2}$. To evaluate the double integral, transform to polar coordinates $(\rho, \emptyset)$ where $x = \rho\cos\emptyset,\ y = \rho\sin\emptyset,\ and\ dy\,dx$ is replaced by $\rho\,d\rho\,d\emptyset$. The double integral becomes

$$\int_{\emptyset=0}^{2\pi}\int_{\rho=0}^{a}\frac{3\rho^2-2a^2}{\sqrt{a^2-\rho^2}}\rho\,d\rho\,d\emptyset = \int_{\emptyset=0}^{2\pi}\int_{\rho=0}^{a}\frac{3(\rho^2-a^2)+a^2}{\sqrt{a^2-\rho^2}}\rho\,d\rho\,d\emptyset$$

$$= \int_{\emptyset=0}^{2\pi}\int_{\rho=0}^{a}(-3\rho\sqrt{a^2-\rho^2} + \frac{a^2\rho}{\sqrt{a^2-\rho^2}})d\rho\,d\emptyset$$

$$= \int_{\emptyset=0}^{2\pi}\left[(a^2-\rho^2)^{3/2} - a^2\sqrt{a^2-\rho^2}\Big|_{\rho=0}^{a}\right]d\emptyset$$

$$= \int_{\emptyset=0}^{2\pi}(a^3 - a^3)\,d\emptyset = 0.$$

Exercise

1. Evaluate $\iint_S \boldsymbol{F}.\boldsymbol{n}dA$ if $\boldsymbol{F} = yz\boldsymbol{i} + 2y^2\boldsymbol{j} + xz^2\boldsymbol{k}$ and S is the surface of the cylinder $x^2 + y^2 = 9$ contained in the first octant between the planes $z = 0$ and $z = 2$.
2. Evaluate $\iint_S \boldsymbol{F}.\boldsymbol{n}\, dA$ where $\boldsymbol{F} = 12x^2y\boldsymbol{i} - 3yz\boldsymbol{j} + 2z\boldsymbol{k}$ and S is the portion of the plane $x + y + z = 1$ included in the first octant.
3. Evaluate the surface integral $\iint_S \boldsymbol{F}.\boldsymbol{n}\, dA$ where $\boldsymbol{F} = 6z\boldsymbol{i} + 6\boldsymbol{j} + 3y\boldsymbol{k}$ and S is the portion of the plane $2x + 3y + 4z = 12$, which is in the first octant.
4. Find the mass of the surface of the $2x + 3y + 4z = 12$, in the first octant. Density at $\rho(x, y, z)$ is directly proportional to the square of its distance from the XZ −plane.
5. Evaluate the integral $\iint_S 6xyz\, dA$ where S is the portion of the plane $x + y + z = 1$ in the first octant.
6. Let $\boldsymbol{F} = 4xz\boldsymbol{i} - y^2\boldsymbol{j} + yz\boldsymbol{k}$. Evaluate $\int\int_S \boldsymbol{F}.\boldsymbol{n}dA$ where S is the closed unit cube bounded by the planes $x = 0, x = a, y = 0, y = a, z = 0, z = a$.
7. If $\emptyset = \frac{3}{8}xyz$, find $\iint_S \emptyset\boldsymbol{n}\, dA$ where S is the surface of the cylinder $x^2 + y^2 = 16$ included in the first octant between $z = 0$ and $z = 5$.

Answers

1. 81
2. $\frac{49}{120}$
3. 138
4. $8\sqrt{29}c$, c is a constant
5. $\frac{\sqrt{3}}{20}$
6. $\frac{3a^4}{4}$
7. $100(\boldsymbol{i} + \boldsymbol{j})$