Unit-5

Queuing models

Service rate (μ)

$\boxed{S(t) = \lambda e^{-\lambda t}}$ $\boxed{\text{Server}}$

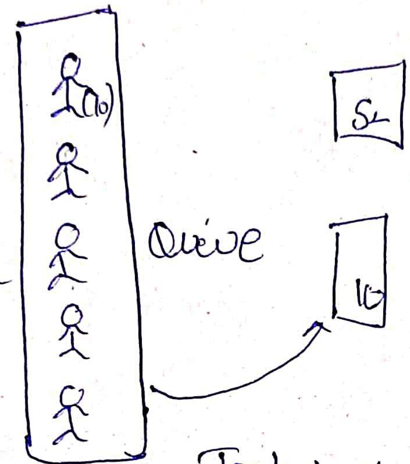
exponential distribution

Inter-arrival rate (λ)

(λ or δ)

arrival rate

(per hr/min)



Jockying

(Jumping from large Queue to Small Queue)

General Discipline

FCFS $\rightarrow$ GD

LCFS

SIRO

Queuing System

$\phi = \frac{50}{25} = 2 > 1$

$\mu = 25$ cust/hr

$\lambda = 50$ cust/hr

$\mu < \lambda$

(inefficient)

Some customers will be left over

$\boxed{\mu > \lambda}$

* utilization factor $(\phi) = \frac{\lambda}{\mu} < 1$

$\mu = 60/hr$

$\lambda = 50/hr$

$\phi = \frac{50}{60}$

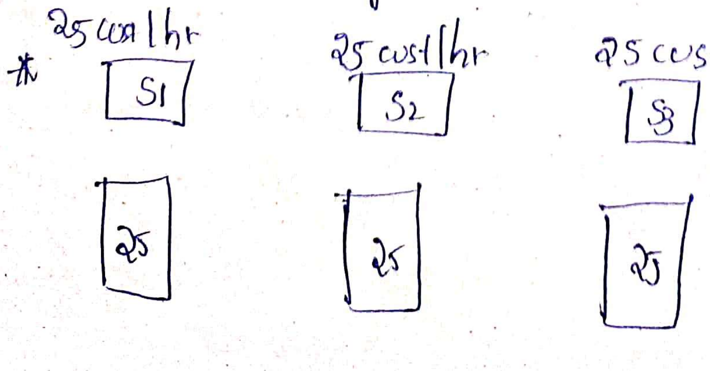
$= 0.8 < 1$

* Group of people - Bulk

* Reneging → already present in queue you will leave, because it will be late

* Balking → By seeing the length of Queue you don't join it

*



→ It will take only 1hr,

If we have more service counters, it will take less time

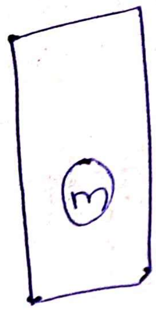
① Single Server → 3 models

② Multiple server → not in symbols

$\boxed{\text{pure Birth}}$ $\boxed{\text{pure death}}$

P1 $\boxed{\text{Server}}$

P#? $\boxed{\text{Server}}$



n=0
1
2
3
4
5
n-1

only departure
No arrival
n o. of departure
$\boxed{0 < n < N}$

N
N-1
N-2
N=50
n=12
$\boxed{N-n}$

At t=0
arrival

Interarrival in
rate=h t=T
is very small

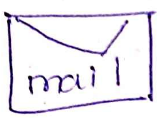
N | C (finite)

No departure
① single $\boxed{\infty/\infty}$

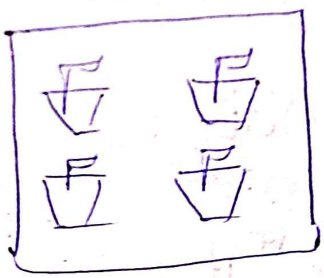
At
finit $\boxed{N/\infty}$ - infinit

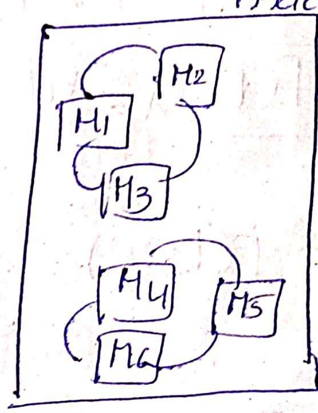
limited
customers

factory only ± maching



$\boxed{M/m/1:GD/\infty/\infty}$





Kendall's notation

P|Q|R : X/Y/Z

M|M|1 GD/∞/∞

N/∞

N/C(N)

Lsys > Lq

Wsys > Wq

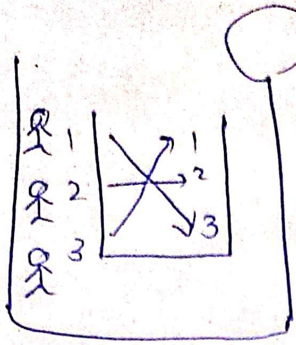
R → no. of servers (single server)

P → arrival rate (polynomial)

Q → service rate (FCFS LCFS)

X → service discipline

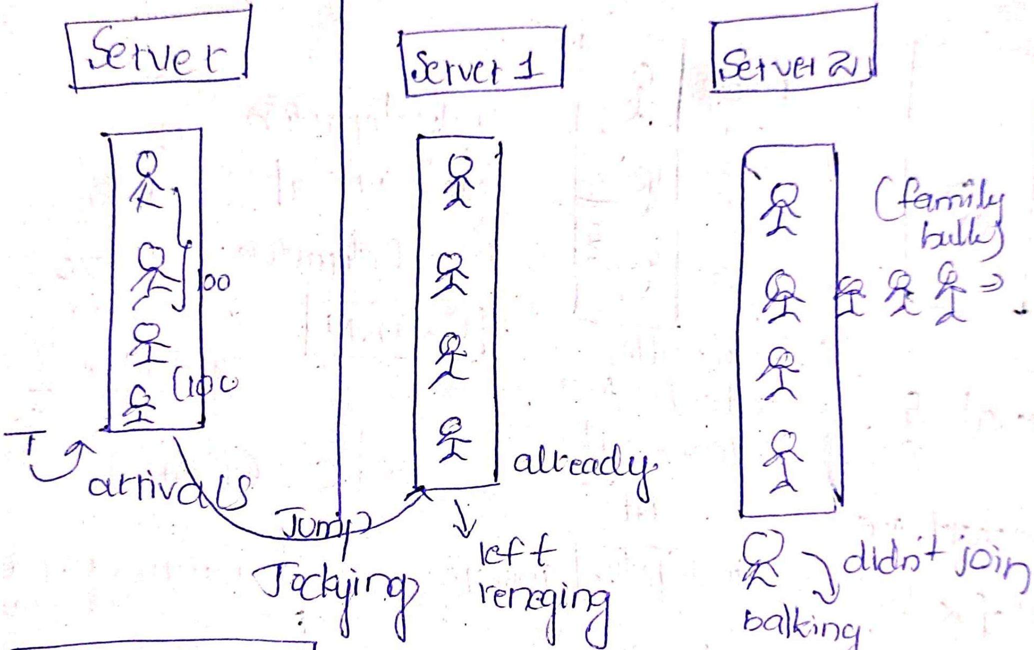
Y →
Z → } depending on model



Service counter.

Last come first service

Serricerate



$$\boxed{f(t) = \lambda e^{-\lambda t}}$$

λ = arrival rate

$$* \quad P(t \le T) = \int_0^T f(t)\,dt$$

$$= \int_0^T \lambda e^{-\lambda t}\,dt$$

$$= \left[\frac{\lambda e^{-\lambda t}}{-\lambda}\right]_0^T$$

$$= -e^{-\lambda T} + e^{-\lambda(0)}$$

$P(t>T) = 1 - P(t \le T)$

$= 1 - 1 + e^{-\lambda T}$

$= e^{-\lambda T}$

Service rate is Departure rate

(v) Pure birthe model

$P_0(t) = e^{-\lambda t}$

$P_0(h) = e^{-\lambda h}$

$P_0(h) = 1 - \lambda h$

$P_1(h) = 1 - P_0(h)$

$= \lambda h$

$P_n(t+h) = P_n(t)(1-\lambda h) + P_{n-1}(t)\lambda h \,;\, n>0$

$n=0 \quad \boxed{P_0(t+h) = P_0(t)(1-\lambda h)t}$

$P_n'(t) = \lim_{h \to 0} \dfrac{P_n(t+h) - P_n(t)}{h}$

$= \dfrac{P_n(t) - P_n(t)\lambda + P_{n-1} - P_n(t)}{h}$

$P_n(t) = \lambda(P_{n-1}(t) - P_n(t))$

$P_0'(t) = -\lambda P_0(t)$

$= -\lambda e^{-\lambda t}$

$P_0(t) = \lambda t e^{-\lambda t}$

$$\boxed{P_n(t) = \frac{(\lambda t)^n e^{-\lambda t}}{n!}} \qquad n = 0, 1, 2 \dots n$$

poison

② pure death model

* No arrival rate

* $P_1(t) = e^{-\mu t}$

$P_n(t)(1-\mu h) + P_{n+1}(t) = P_n(t+h)$

$$P'_n(t) = \lim_{h \to 0} \frac{P_n(t+h) - P_n(t)}{h}$$

$$= \frac{-P_n(t)\mu h + \mu h P_{n+1}(t)}{h}$$

$$= \mu[P_{n+1}(t) - P_n(t)]$$

$P'_0(t) = \mu P_1(t)$

$$\boxed{P_n(t) = \frac{(\mu t)^{N-n} e^{-\mu t}}{(N-n)!}}$$ → poison

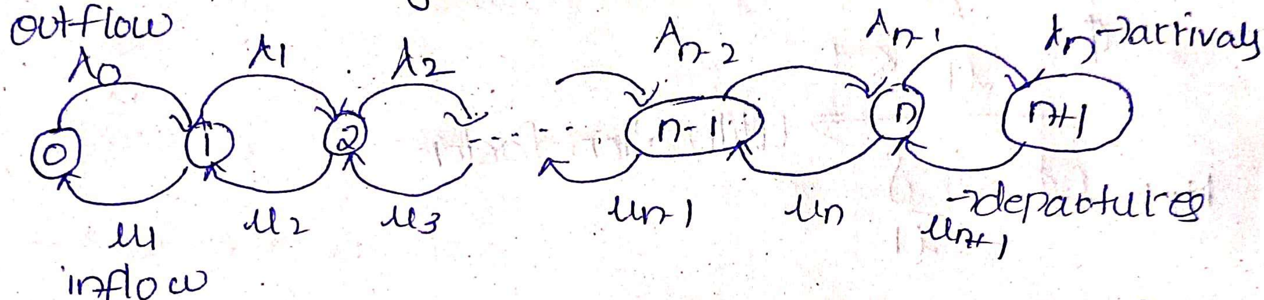
* Departure rate = poison rate
* Difference b/w pure birth and pure death
* explain queuing model
* Real time q.s
* Reneging, Balking

combining arrivals & departures in the real time queuing system

λ_n = arrival rate

μ_n = departure rate

P_n = probability of having 'n' no. of customers in steady state system



Expected rate of inflows = Expected rate of outflows

State '0'

$\mu_1 P_1 = \lambda_0 P_0$.

$$P_1 = \left[\frac{\lambda_0}{\mu_1}\right] P_0$$

State '1'

$$\lambda_0 P_0 + \mu_2 P_2 = (\lambda_1 + \mu_1) P_1$$

$$\lambda_0 P_0 + \mu_2 P_2 = (\lambda_1 + \mu_1)\left(\frac{\lambda_0}{\mu_1}\right) P_0$$

$$= \frac{\lambda_1 \lambda_0}{\mu_1} P_0 + \frac{\cancel{\mu_1}\lambda_0}{\cancel{\mu_1}} P_0$$

$$= \frac{\lambda_1 \lambda_0}{\mu_1} P_0 + \lambda_0 P_0$$

$$P_2 = \left(\frac{\lambda_1 \lambda_0}{\mu_1 \mu_2}\right) P_0$$

State 'n' :-

$$\lambda_{n-1} P_{n-1} + \mu_{n+1} P_{n+1} = (\lambda_n + \mu_n) P_n$$

$$P_n = P_0 \left[\frac{\lambda_0 \lambda_1 \lambda_2 \cdots \lambda_{n-1}}{\mu_1 \mu_2 \mu_3 \cdots \mu_n} \right]$$

Models

$$\phi = \frac{\delta}{\mu} < 1$$

$\rightarrow$ Utilization factor

M_1 : $\phi = \frac{\delta}{\mu}$

$$L_{sys} = \frac{\phi}{1-\phi} \text{ customers}$$

$$L_q = \frac{\phi^2}{1-\phi} \text{ customers}$$

$$W_{sys} = \frac{L_{sys}}{\delta} \text{ hr}$$

$$W_q = \frac{L_q}{\delta} \text{ hr}$$

$$P_n = (1-\phi)\phi^n$$

Idle time $= (1-\phi) = P_0 < 25\%$.

It is Justifiable, to have any changes

Model 1: M|M|1 : GD|∞|∞

(P1) The no. of arrivals of cars in the sheet is 45/hr. Service rate is 60 cars per hour. Find the average no. of cars in the system, queue; waiting time in system & queue probability of having n = 0, 15, 20 cars in the system

Assuming $N = \infty \rightarrow M_1$

Arrival rate $\lambda = 45$ cars/hr

Service rate $\mu = 60$ cars/hr

$$\phi = \frac{\lambda}{\mu} = \frac{45}{60} = 0.75$$

$$L_{sys} = \frac{\phi}{1-\phi} = \frac{0.75}{1-0.75} = \frac{0.75}{0.25} = 3 \text{ cars}$$

$$L_q = \frac{\phi^2}{1-\phi} = \frac{(0.75)^2}{1-0.75} = 2.25 \text{ Cars}$$

(or)

$$= (0.75)(3)$$

$$W_{sys} = \frac{L_{sys}}{\lambda} = \frac{3}{45} \text{ hour} = 0.66 \times 60 \times 60 \text{ Secs}$$

$$W_q = \frac{L_q}{\lambda} = \frac{2.25}{45} \text{ hr}$$

$$= 0.05 \times 3600 \text{ secs}$$

$$= 180 \text{ Secs}$$

$P_0 = (1-\phi)\phi^0 = 1-\phi = 1-0.75 = 0.25$

$P_{15} = (1-\phi)\phi^{15} = (1-0.75)(0.75)^{15} = 0.00334$

$P_{20} = (1-\phi)\phi^{20} = (1-0.75)(0.75)^{20} = 0.0007$

(P2) The no. of ships anchored for service in a heaven is 8 ships/week. The service rate of the server is 14 ships/week.

Find L_q, L_{sys}, W_q, W_{sys}, P_0, P_2

(Assuming $N=\infty$)

$\delta = 8$ ships/week

$\mu = 14$ ships/weak

$\phi = \frac{8}{\mu} = \frac{8}{14} = 0.571$

$L_{sys} = \frac{\phi}{1-\phi} = \frac{0.571}{1-0.571} = 1.331$ ships

$L_q = \frac{\phi^2}{1-\phi} = 0.760$ ships

$W_{sys} = \frac{L_{sys}}{\delta} = \frac{1.331}{8} = 0.166$ weeks

$= 0.166 \times 7$ days

$= 0.166 \times 7 \times 24 \times 60 \times 60$ sec

$P_0 = (1-\phi)\phi^0 = (1-0.5711) = 0.429$

$P_2 = (1-\phi)\phi^2 = (1-0.5711)(0.511)^2$

$= 0.139$

(13) The no. of packages of vehicles at toll gate in 1 an hour is 90. The service can serve upto 3600 vehicles/hr. But we have the avg to pass. So for cars a, b & c. Find whether the new gate replacements in cars 'b' & 'c' is Justifiable.

(a) $\delta = 90$ vehicles/hr

a) Old gate

Avg time to pass $= 36$ secs

$\mu = \frac{3600}{36}$ vehicles/hr

$= 100$ vehicles/hr.

$\phi = \frac{90}{100} = 0.9$

Idle time $= 1 - \phi = 0.1 = 10\% < 25\%$.

Justifiable

(b) New gate

$\mu = \frac{3600}{30}$ vehicles/hr

$= 120$ veh/hr

$\phi = \frac{90}{120} = 0.75$

Idle time $= 0.25 < 25\%$.

Not Justifiable

(c) $\mu = \frac{3600}{33}$

$= 109.09$ veh/hr

$\phi = \frac{90}{109.09}$ 0.825

Idle time $= 1 - \phi = 1 - 0.825 = 0.175 < 25\%$.

Justifiable

Model 2: M|M|1 : GD|N|∞ $\boxed{\text{Erlang}}$

$$\delta_{eff} = \delta(1 - P_N),\ \phi = \frac{\delta}{\mu}$$

$$P_N = \frac{(1-\phi)\phi^n}{(1-(\phi)^{n+1})},\ \phi \neq 1$$

$$= \frac{1}{N+1},\ \phi = 1\ (\delta = \mu)$$

$$L_{sys} = \frac{\phi(1-(N+1)\phi^N + N\phi^{N+1})}{(1-\phi)(1-\phi^{N+1})},\ \phi > 1$$

$$= N/2,\ \phi = 1 \quad \text{customers}$$

$$W_{sys} = \frac{L_{sys}}{\delta_{eff}}\ hr$$

$$W_q = \frac{L_q}{\delta_{eff}}\ hr$$

(P1) The arrival rate of car in a shed is 24 per hour. The service rendered per hour is for 20 cars. 4 cars are permitted in the system: fine the avg no. of cars present in waiting in the system queue & their waiting times. $\boxed{N=4}$

Sol:- δ or λ = 24 cars/hr

μ = 20 cars/hr

Model 12: N=4

$$\phi = \frac{\delta}{\mu} = \frac{24}{20} = 1.2.$$

$$\delta_{eff} = \delta(1 - P_N)$$

$$P_{N=4} = \frac{(1-\phi)\phi^{n}}{1-(\phi)^{n+1}} = \frac{(1-1.2)(1.2)^{4}}{1-(1.2)^{5}} = 0.275$$

$$\delta_{eff} = 4(1 - 0.2787)$$

$$= 17.311 \text{ cars/hr.}$$

$$L_{sys} = \frac{\phi[1-(N+1)\phi^{N} + N\phi^{N+1}]}{1-\phi[1-\phi^{N+1})}$$

$$= \frac{1.2[1-5(1.2)^{4} + 4(1.2)^{5}]}{1-1.2[1-(1.2)^{5}]}$$

$$= 2.36 \text{ cars.}$$

$$L_q = L_{sys} - \frac{\delta_{eff}}{} = 2.36 - \frac{17.311}{20} = 1.49 \text{ cars}$$

$$W_{sys} = \frac{L_{sys}}{\delta_{eff}} = \frac{2.36}{17.311} \text{ hr} = 0.136 \times 3600 \text{ secs}$$

$$W_q = \frac{L_q}{\delta_{eff}} = \frac{1.49}{17.311} \text{ hr} = 0.86 \times 3600 \text{ secs}$$

1 week = 7 days

1 day = 24 hrs

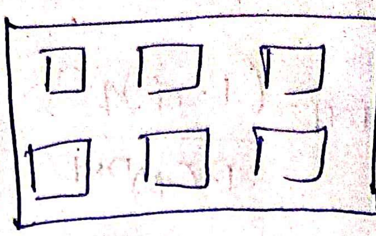
1 hr = 60 min

1 min = 60 secs

(P2) $\delta = 21$ tvsets/hr

$\mu = 25$ tvsets/hr

$N = 6$



Find L_{sys}, L_q, W_{sys}, W_q, P_N

$$\phi = \frac{\delta}{\mu} = \frac{21}{25} = 0.840$$

$$P_{N=6} = \frac{\overset{0.16}{(1-0.840)}\ \overset{0.351}{(0.840)}}{(1-(0.84)^7)} = \frac{0.056}{0.295} = 0.705$$

$$= 0.079$$

$$\delta_{eff} = 21(1-0.079) = 19.34 \text{ Tvsets/hr}$$

$$L_{sys} = \frac{0.84\left[1-(7)(0.84)^6 + 6(0.84)^7\right]}{1-0.84\left[1-(0.84)^7\right]}$$

$$= 2.31 \text{ TVsets}$$

$$L_q = 2.31 - \frac{19.34}{25} = 1.536 \text{ TV sets}$$

$$W_{sys} = \frac{2.31}{19.34} = 0.119 \text{ hr}$$

$$W_q = \frac{1.536}{19.34} = 0.079 \text{ hr}$$

Permutation

$$^nP_r = \frac{n!}{(n-r)!}$$

$$^5P_4 = \frac{5!}{1!} = 120$$

combination

$$^nC_{n-r} = {^nC_r} = \frac{n!}{n(n-r)!}$$

$$^5C_4 = \frac{5!}{4!\,1!}$$

$$= \frac{4! \times 5}{4!} = 5$$

$$^5C_2 = \frac{5 \times 4}{2 \times 1} = 10$$

$$^5C_1 = \frac{5}{1} = 5$$

Model 3: $M|M|1 : GD|N|(N)C$

Assumptions

$$\lambda_n = (N-n)\lambda \quad 0 \le n \le N$$
$$= 0 \quad n > N$$

$$\mu_n = n\mu \quad 0 \le n \le 1$$
$$= \mu \quad 1 \le n \le N$$
$$= 0 \quad n > N$$

$$\lambda_{eff} = \mu(1 - P_0) \text{ cost/hr}$$

$$P_0 = \left[\sum_{n=0}^{c} {^NC_n}\, \phi^n + \sum_{n=c+1}^{n} {^NC_n} \frac{n!\,\phi^n}{c!\,c^{n-c}} \right]^{-1}$$

$$P_n = {^NC_n}\, \phi^n P_0 \quad 0 \le n \le c$$

$$= {^NC_n} \frac{n!\,\phi^n}{c!\,c^{n-c}} P_0 \quad c+1 \le n \le N$$

$$L_{sys} = N - \frac{(1-P_0)}{\phi} \text{ customers}$$

$$L_q = N - \left(1 + \frac{1}{\phi}\right)(1-P_0) \text{ customers}$$

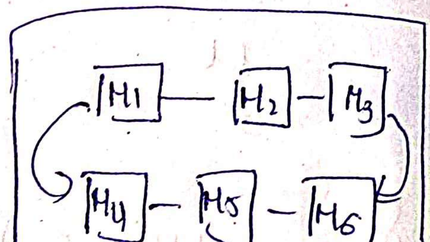
$$W_{sys} = \frac{L_{sys}}{\delta_{eff}} \text{ hr}$$

$$W_q = \frac{L_q}{\delta_{eff}} \text{ hr}$$

(P1) In case of maintenance & repairs for machinery in a factory a crew is assigned to 6 machinery (N=6)
The arrivals of machines in the factory is given by 2.5/hr. & Service rate is given by 1.5 machines/hr. Find L_q, L_S, W_q, W_s, P_0

λ (or) $\delta = 2.5$ machines/hr

$\mu = 1.5$ machines/hr



Models 3 : N=6 , C=1

$$\phi = \frac{\delta}{\mu} = \frac{25}{15} = 1.667$$

$$\delta_{eff} = \mu(1-P_0)$$

$$P_0 = \left[\sum_{n=0}^{1} 6C_n (1.667)^n + \sum_{n=2}^{6} 6C_n \frac{n!}{1!\,1^{n-1}} (1.66)\right.$$

$$= \left[1(1.667)^0 + 6(1.667)^1 + 15(2!)(1.667)^2 + 20(3!)(1.667)^3 + 15(4!) + (1.667)^4 + 6(5!)(1.667)^5 + 1(6!)(1.667) \right]$$

$$= 0.0000355$$

$$\delta_{eff} = 1.5(1 - 0.0000355) = 1.49 \text{ machines/hr}$$

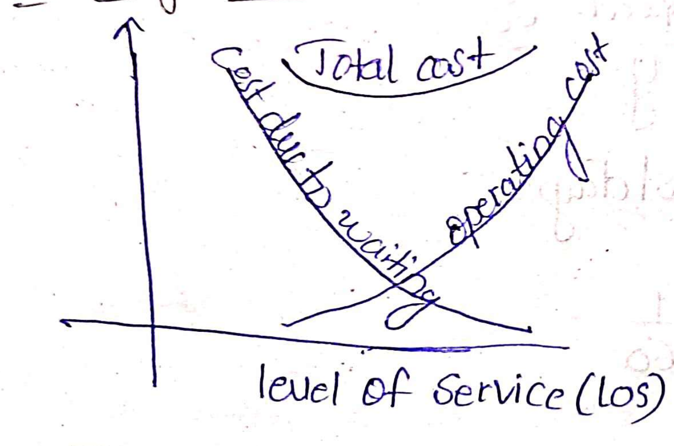
$$L_{sys} = \frac{6 - (1 - 0.0000355)}{1.667} = 5.001 \text{ machines}$$

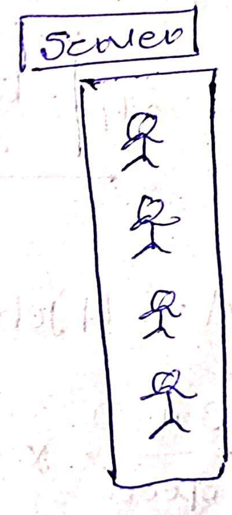
$$L_q = 6 - \left(1 + \frac{1}{1.667}\right)(1 - 0.0000355) = 4.4001 \text{ machines}$$

$$L_{sys} = \frac{5.001}{149} \text{ hr} = 3624 \text{ hr}$$

$$W_q = \frac{4.4001}{149} \text{ hr} = 2.45 \text{ hr}$$

Queuing costs





200x60
=12000
60min→60 people
@ 2 employ
2 cus/min
30 min
=>120 people/hr
↓
120 x 200
=24000
Penalty cost

ETC = EOC + EWC

↳ Operating ↓ reading.

$= C_1(x) + C_2 L_{sys}$

$= EOC + EWC = C_1(x) + C_2 L_{sys}$

(17) $ETC = EOC + EWIC = C_1(x) + C_2 L_{sys}$

In the process of purchasing a high Speed commercial co pet 4 models are given

Model	C_1 operating cost (\$/hr)	for $C_1 x$	Speed (sheets/min)
1	15 x 24	360	30
2	20 x 24	480	36
3	24 x 24	576	50
4	22 x 24	648	66

* Jobs arrival at kence according to a x poison distribution with a mean of 4 job/avg(24hr)

Job Size is random but average about 10,000 sh/job

The penalty cost per late delivery is \$ 80 jobs/day

Which to produce?

$$\mu = \frac{24 \Big/ \frac{10,000}{Speed} \times \frac{1}{60}}{y}$$

$\lambda = 4$ jobs/day

$$y = \frac{10,000}{Speed} \times \frac{1}{60}$$

$$= \frac{10,000}{30} \times \frac{1}{60} = 5556$$

$$= \frac{10,000}{20} \times \frac{1}{60} = 4629$$

$$= \frac{10000}{50} \times \frac{1}{60} = 333$$

$$= \frac{10000}{66} \times \frac{1}{60} = 2525$$

$$u = \frac{24}{y} = \frac{24}{5556} = 4.32$$

$$= \frac{24}{4629} = 5.184$$

$$= \frac{24}{333} = 7.207$$

$$= \frac{24}{2525} = 9.505$$

$$\phi = \frac{\lambda}{u} = \frac{4}{u} = \frac{4}{4.32} = 0.925$$

$$= \frac{4}{5.184} = 0.771$$

$$= \frac{4}{7.207} = 0.555$$

$$= \frac{4}{9.505} = 0.420$$

$$L_{sys} = \phi / 1-\phi \text{ jobs}$$

$$= \frac{0.925}{1-0.925} = 12.33$$

$$= \frac{0.771}{1-0.771} = 3.366$$

$$= \frac{0.555}{1-0.555} = 1.247$$

$$= \frac{0.420}{1-0.420} = 0.724$$

$EW_c = 80 \cdot Q_{sys}(\$)$

$80(12.33) = 986.4$

$80(3.365) = 269.2$

$80(1.247) = 99.76$

$80(0.724) = 57.9$

$ETC(\$)$

$360 + 986.4 = 1346.4$

$480 + 269.2 = 749.2$

$576 + 99.76 = 675.76$

$648 + 57.9 = 705.9$

Model 3 can be purchased