09-06-2023.

5. GRAPH THEORY

5.1

* Graph Theory :-

→ It plays an important role in several areas of 'computer science', such as 'switching theory' and 'logical design', 'artificial intelligence', 'formal languages', 'computer graphics', 'Operating systems', 'compiler writing' & 'information organizatio & retrieval'.

→ Graph theory grew out of an interesting physical problem, the celebrated "konigsberg Bridge Puzzle".

→ The swiss mathematician, 'Leonhard Euler' was "father of graph theory" who solved the puzzle in 1736.

* Graph :-

→ A 'graph' G = (V, E) consists of a non empty set 'V', called the 'set of nodes' [or points or vertices] of the graph and 'E' [set of edges], which is a subset of the set of ordered or unordered pairs of element of 'V'.

→ We shall assume that, both 'V' and 'E' sets of a graph are finite.

→ If 'e ∈ E', then 'e' is an 'edge', 'e' is an 'ordered pair (u, v)' or an 'unordered pair {u, v}', where u, v ∈ V and 'e' connects or joins the nodes 'u' & 'v'.

→ A 'pair of nodes' are said to be 'adjacent', if they are connected by an 'edge'.

* **Directed, undirected and Mixed graphs:-**

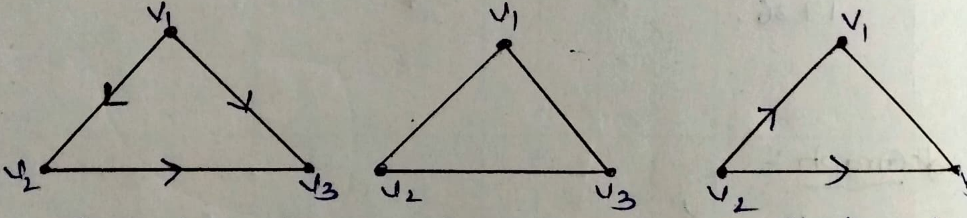
→ In a graph "G=(V, E)", an edge which is an ordered pair of "V×V" is called a "Directed edge".

→ A graph in which every edge is directed is called a "Directed graph" (or) "Digraph".

→ An edge which is an unordered pair {u,v}, u,v ∈ V is called an "undirected edge".

→ A graph in which every edge is undirected is called an "Undirected graph".

→ If some edges are directed & some are undirected in graph, then the graph is called a "Mixed graph".



Directed graph   undirected graph   Mixed graph

→ Let $G=(V, E)$ be a graph and $e \in E$ be a directed edge (u,v).

Here u = "initial node"
v = "terminal node"

→ An edge $e \in E$ which joins the nodes u & v, whether it be directed or undirected is said to be "incident" the nodes u & v.

→ An edge of a graph which joins a node to itself is called a "loop".

* Distinct :-

→ In the case of directed edges, the two possible edges (u,v) and (v,u) between a pair of nodes u,v which are opposite in direction are considered 'Distinct.'

* Parallel edges :-

→ Two or more distinct edges between a pair of vertices are called 'parallel edges.'

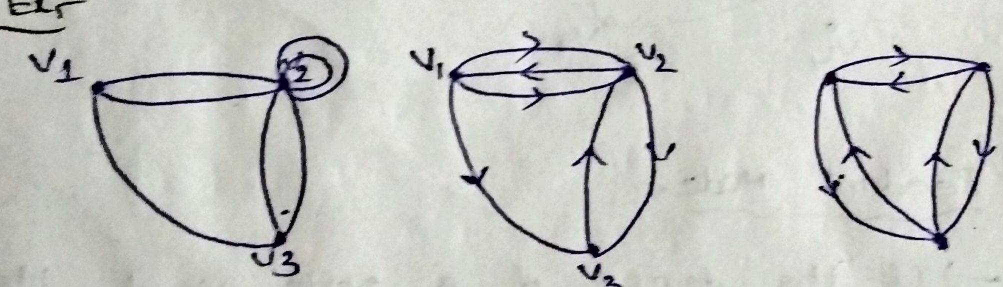
* Multigraph :-

→ Any graph which contains parallel edges is called a 'multigraph.'

* simple :-

→ A graph is said to be simple if there is no more than one edge between a pair of nodes.

Ex :-



G_1 G_2 G_3

Here G_1, G_2, are all ~~simple~~ Multi graphs.
G_3 is a simple graph.

* multiplicity :-

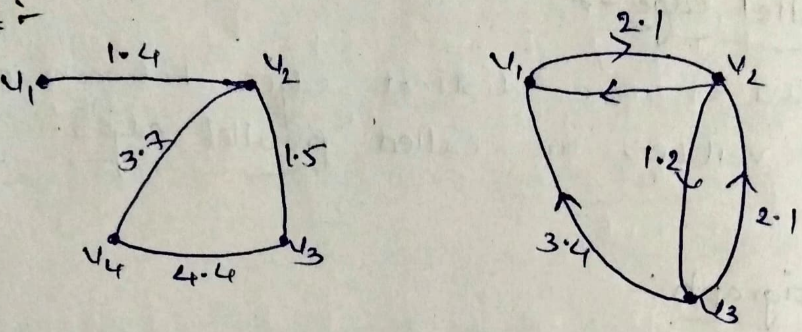
→ If 'e' denotes the edge joining two nodes 'u,' 'v' and there are 'n' parallel edges between 'u,v' then we say that the 'multiplicity' of 'e' is 'n.'

* Weight :-

→ We may also consider the multiplicity as a 'weight' assigned to an edge.

→ A graph in which weights are assigned to every edge is called a 'weighted graph'.

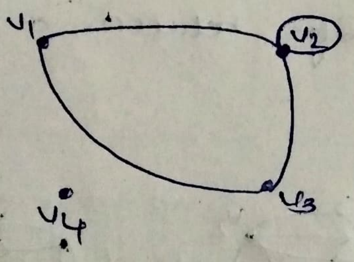
Ex :-



* Isolated Node :-

→ A node in a graph is said to be an 'isolated node' if it is not adjacent to any node.

Ex :-



Here v_4 is Isolated Node

* Pendant Node :-

→ If the degree of a Node is 1 then it is called pendant Node.

(or)

→ If the Incident edges to a node is only one then it is called pendant Node.

* Degree of a vertex in undirected graph:-

→ Let $G = G(V, E)$ is a graph, $v \in V$.

Notation:- $\deg_G(v)$ (or) $d_G(v)$.

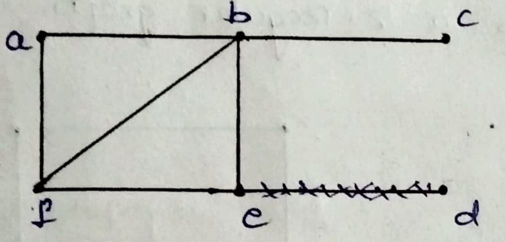
→ The No. of edges incident to v with self loop counted twice. is called 'degree'

→ Degree is a 'non-negative integers'

* Exercise:

1] Find the No. of vertices, edges & degree of given undirected graphs and also find isolated & pendant vertices

(i)



A) No. of vertices = 6

No. of edges = 6

Vertex	degree
a	2
b	4
c	1
d	0
e	2
f	3

Odd vertices :- c, f

Even vertices :- a, b, d, e.

Isolated Node/vertice :- d [Degree 0]

Pendant vertice/Node :- c [Degree 1]

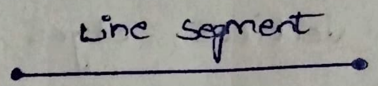
* Regular Graph :-

→ A Graph in which all vertices are of equa degree, that graph is called "Regular Graph"

→ If the degree of every vertex in a graph "G" is 'n' then that graph is called n-Regular gra

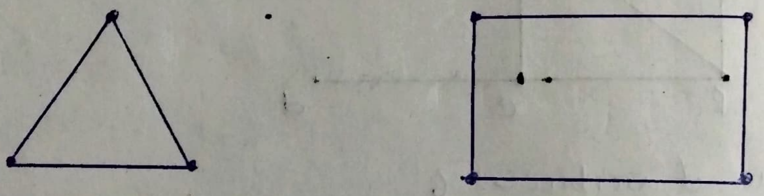
Examples :-

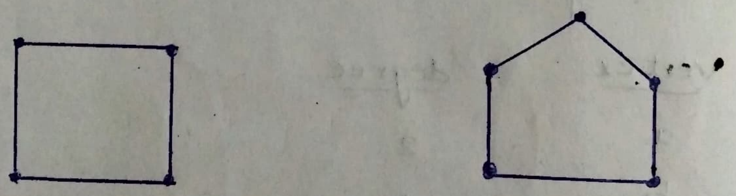
1-Regular Graph :-



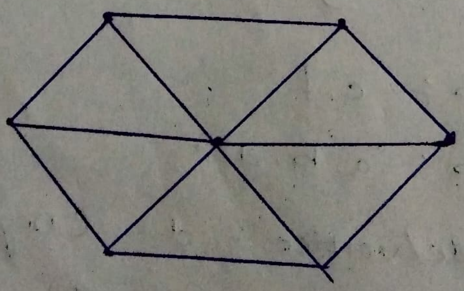
2-Regular Graph :-

Every polygon is a 2-Regular graph.





3-Regular graph :-



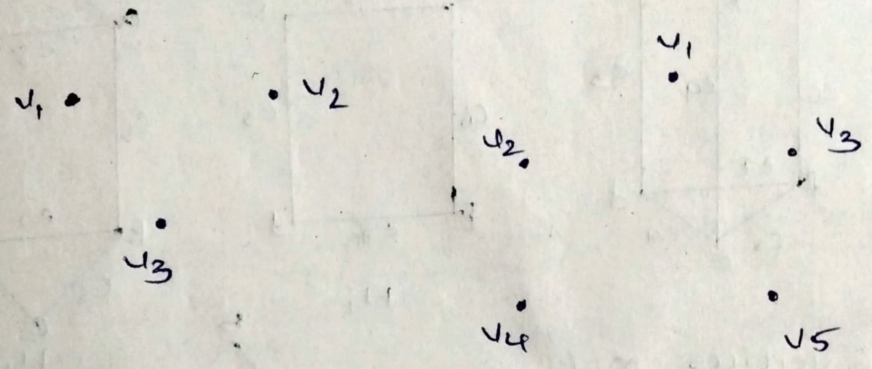
* NULL Graph :-

→ A graph without edges is called a 'Null graph'.

→ In Null graph, every vertex is an 'Isolated Node'.

→ Null graph is a '0-regular graph.'

Ex :-



* Sub graph :-

→ Let G be a graph with two finite sets '$G = G(V, E)$' & H be a graph with two finite sets '$H = H(V', E')$'.

Notation :- $H \subseteq G$

→ When '$V' \subseteq V$' and '$E' \subseteq E$' then '$H \subseteq G$'.

Note points :-

1. Every vertex in 'H' is also a vertex in 'G'.
2. Every 'edge' in 'H' is also an 'edge' in 'G'.
3. Every edge of 'H' has same end points in 'H' as in 'G'.

→ By Removing certain vertices & edges we can get a subgraph.

* **No. of Subgraphs:**

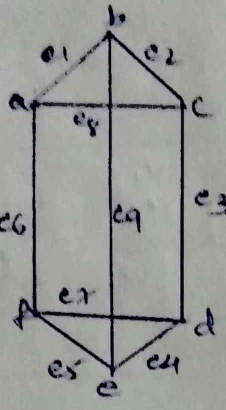
→ Let $G = G(V, E)$ be a graph.

$|V| = m$

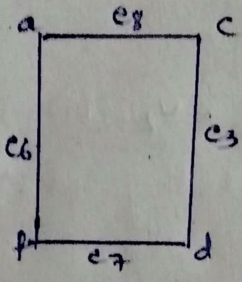
$|E| = n$

∴ Total no. of subgraphs $= \underline{2^n(2^m - 1)}$.

Ex:-

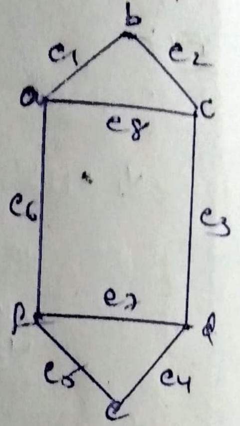


subgraph



H_1

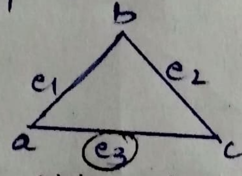
subgraph.



H_2

6 vertices, $m = 6$

9 edges, $n = 9$.



Not subgraph.

* **operations on Graphs:-**

① **Union of Graphs:-**

→ Let consider two graphs, $G_1 = (V_1, E_1)$ and $G_2 = (V_2,$

Now, $G_1 \cup G_2 = G = (V, E)$.

Here $V = V_1 \cup V_2$

$E = E_1 \cup E_2$

② **Intersection:-**

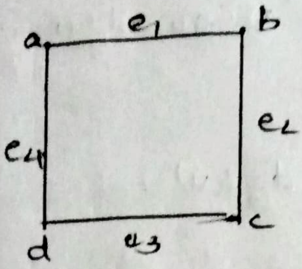
$G_1 = (V_1, E_1)$ & $G_2 = (V_2, E_2)$

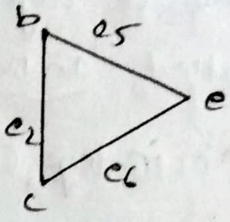
∴ $G_1 \cap G_2 = G = (V, E)$

Here $V = V_1 \cap V_2$

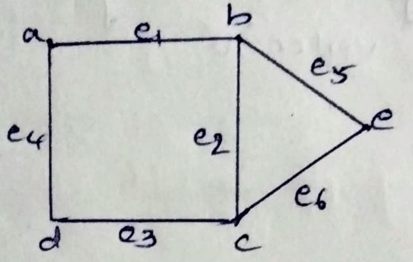
$E = E_1 \cap E_2$

* **Example :-**





* **Union of Graphs :-** $G_1 \cup G_2$



* **Relation b/w simple digraph & Binary relation :-**

→ Every digraph can generates a Binary relation.

→ Every Binary relation can be represented as a Digraph.

→ Let 'G=G(V,E)' be a 'simple directed graph' then '$E \subseteq V \times V$' is a binary relation on 'V'.

→ The converse need not be true because they may have loops.

Ex :- $V = \{1, 2, 3\}$ $E = \{(1,1), (1,2), (2,3), (3,2)\}$

* **Degree of a vertex in Directed graph :-**

Out Degree :-

→ Let $G = G(V, E)$ be a digraph and $v \in V$.

Notation :- $outdeg_G(v)$ or $d^+_G(v)$.

→ The No. of edges with v as initial vertex is called 'out Degree'.

In Degree:-

→ The no. of edges with 'v' as terminal vertex is called 'In Degree'.

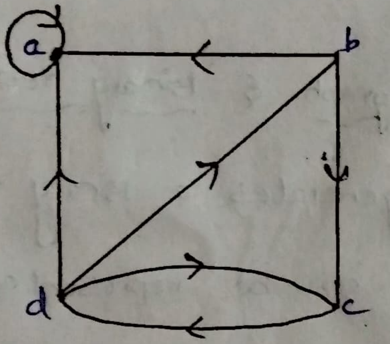
Notation:- $indeg_G(v)$ (or) $d_G^-(v)$.

Degree (or) Total Degree:-

→ The sum of outdegree and indegree is called 'Total degree' of a vertex 'v'.

*** Exercise:-**

4 ⓐ



A) No. of vertices = 4

No. of Edges = 7

vertex (v)	$d_G^+(v)$	$d_G^-(v)$	$d(v)$
a	1	3	4
b	2	1	3
c	1	2	3
d	3	1	4
	7	7	

*** Theorem:- 1**

→ Let $G = G(V,E)$ be a digraph then

$$\boxed{\sum_{v \in V} d_G^+(v) = \sum_{v \in V} d_G^-(v) = |E|}$$

* Degree sequence of a Graph :-

→ It is the sequence of degrees of vertices of the graph is non increasing order.

→ $G = G(V, E)$ with n - vertices.

Let $v_1, v_2, \text{———} v_n$ are vertices of graph G.

$$\begin{matrix} v_1, & v_2, & \text{———} & v_n \\ \downarrow & \downarrow & & \downarrow \\ d_1, & d_2, & \text{———} & d_n \end{matrix}$$

$d_1, d_2, \text{———} d_n$ are degrees of vertices $v_1, v_2 \text{———} v_n$.

→ Minimum Degree,

$$\delta(G) = \text{Min}\{d(v_i) / v_i \in V\}$$

→ maximum Degree,

$$\Delta(G) = \text{Max}\{d(v_i) / v_i \in V\}.$$

→ In our content, Degree sequence is Non-Increasing order.

$$\Delta(G) \geq d_n \geq d_{n-1} \geq \text{———} \geq d_2 \geq d_1 \geq \delta(G).$$

* Graphic sequence :- (or) (Graphic) :-

→ The sequence $d_1, d_2, \text{———}, d_n$ is called a 'graphic' if it is a degree sequence of a 'simple graph'.

(or)

The Degree sequence of a simple graph is called 'Graphic'

* Handshaking Theorem :-

→ Let $G = G(V, E)$ be an undirected graph. with E edges.

→ Then Handshaking Theorem states that "The sum of degrees of all vertices is equal to twice the no. of edges."

$$\sum_{v \in V} d(v) = 2e \quad \text{[even no]}.$$

→ Every undirected graph should follows this Theorem.

Applications :-

① It tells about the validity of degree sequence

② It tells the relation b/w No. of vertices, edges. and its degrees.

* Exercise :-

3] 4) ③ Let G be a graph with 15 vertices & each of degree 5.

Then sum of degrees of each vertice in G

$$= \sum_{v \in V} d(v) = 15 \times 5 = 75 \text{ [Not even]}.$$

∴ It doesnot following Handshaking Theorem.

Thus such graph doesnot exist.

* Theorem-3 :-

→ In an undirected graph,

(i) The no. of odd vertices is even.

(ii) The no. of even vertices may be even or odd.

* **Exercise**

5 (b) 5, 4, 3, 2, 1.

A) No. of odd vertices = 3

It is not even

$\therefore$ It is not valid degree sequence.

* **Theorem-4 :-**

→ The Maximum no. of edges of a simple graph with n-vertices is '$^nC_2 = \frac{n(n-1)}{2}$'.

→ Thus, the no. of edges of a simple graph with n-vertices cannot exceed '$^nC_2$'.

Ex:- 5, 4, 3, 1

$5+4+3+1 = 2e$

$\Rightarrow e = \frac{13}{2}$.

Not possible.

* **Theorem-5 :-**

→ The maximum degree of any vertex of a simple graph with n-vertices is $n-1$.

→ Thus the maximum degree of any vertex of a simple graph with n-vertices cannot exceed '$n-1$'

Ex:- 5, 4, 3, 2

No. of degrees = 4

Max degree = $n-1 = 4-1 = 3$.

Here max degree given is 5.

$\therefore$ simple graph doesnot possible.

* Theorem-6 :-

→ There is no simple graph has all degrees of vertices are distinct i.e, in graphic atleast one number should be repeated.

* Theorem-7 :-

→ Let 'G' be a graph with 'v' vertices & 'E' edges.

→ Let 'M' be maximum degree & 'm' be minimum degree of vertices. of Graph 'G'.

→ Then show that

(i) $\frac{2e}{v} \geq m$

(ii) $\frac{2e}{v} \leq M$.

Proof :-

Let $d_1, d_2, \_\_\_ d_v$ are degrees of vertices.

Handshaking Theorem,

$d_1 + d_2 + \_\_\_ + d_v = 2e$

(i) $m + m + \_\_\_ + m \leq d_1 + d_2 + \_\_\_ + d_v = 2e$

$mv \leq 2e$ $\qquad m = \delta(v)$

$m \leq \frac{2e}{v}$

(ii) $M + M + \_\_\_ + M \geq d_1 + d_2 + d_3 \_\_\_ + d_v = 2e$

$Mv \geq 2e$

$M \geq \frac{2e}{v}$

* Exercise

5] (c) 4, 4, 3, 2, 1

A) By Handshaking Theorem,

sum of degrees $= 14$

$\Rightarrow 14 = 2e$

No. of edges $= 7$.

$\Rightarrow 14 = 2(e)$

$\Rightarrow \frac{14}{2} = e$

$\therefore$ Edges $= 7$.

$nC_2 = \frac{n(n-1)}{2} = \frac{5(4)}{2} = 10$

v_1	v_2	v_3	v_4	v_5
4	4	3	2	1

$v_1 \longleftrightarrow v_5$

$v_2 \longleftrightarrow v_5$

Degree of v_5 is 1.

Not possible.

clearly, No. of terms $= 5$.

We have to take five vertices v_1, v_2, v_3, v_4, v_5.

Let us ~~ther~~ assume there exists a simple graph with given sequence.

Let 4, 4, 3, 2, 1 are degrees of vertices v_1, v_2, v_3, v_4, v_5.

since $d(v_1) = d(v_2) = 4$.

so v_1 & v_2 are adjacent with v_5.

Then degree of v_5 atleast greater or equal to 2

which is contradiction to degree of $v_5 = 1$.

so such simple graph doesnot exists.

$\therefore$ The given degree sequence is not a graphic.

* Problem 2

Is there a simple graph with degree sequence

1, 1, 3, 3, 3, 4, 6, 7

A) Let degrees of vertices are...

$v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8$

1, 1, 3, 3, 3, 4, 6, 7

$d(v_1) = d(v_2) = 1$

v_1, v_2 are adjacent to v_8 only.

The Maximum degree of v_7 is 5. [self loop Not possible]

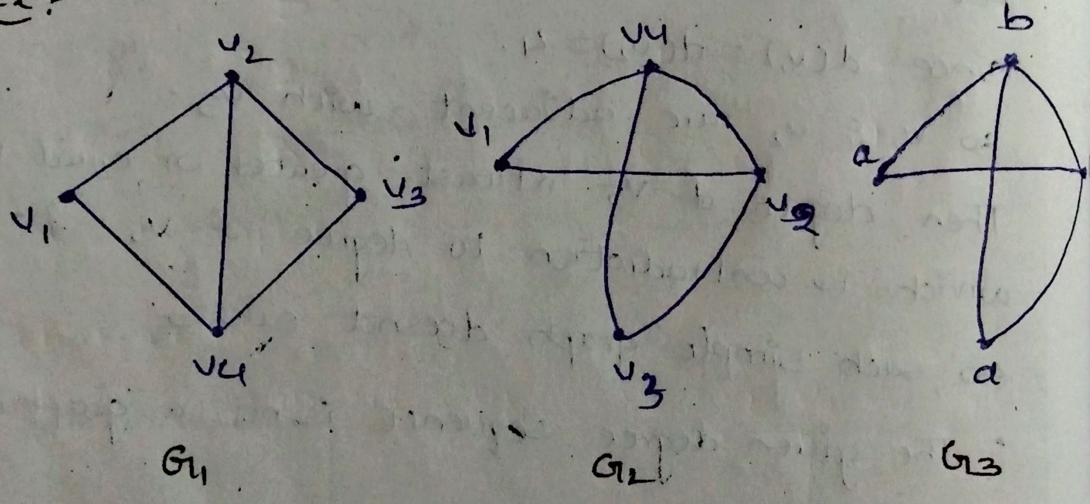
∴ The sequence is not valid.

* **Graph Isomorphism:-**

→ 'Iso' means 'similar' & 'morphism' means 'forms'

→ 'Isomorphism' means similar form. It is a greek word.

Ex:-



Here $G_1 = G_2$

$G_1 \neq G_3$ [But similar].

→ Let $G_1 = (V_1, E_1)$ & $G_2 = (V_2, E_2)$.

Isomorphism, '$G_1 \cong G_2$'

→ It is said to be Isomorphism, when

$\exists\ f : V_1 \longrightarrow V_2$

(i) f is Bijective

(ii) $\forall \{a,b\} \in \hat{E}_1$ iff $\{f(a), f(b)\} \in E_2$ [undirected]

$\forall (a,b) \in E_1$ iff $(f(a), f(b)) \in E_2$ [Digraph]

where $a, b \in V_1$, $f(a), f(b) \in V_2$.

→ On the other hand, G_1 & G_2 are said to be Isomorphic there is a one-one correspondance b/w $V_1 \Longleftrightarrow V_2$ and $E_1 \longleftrightarrow E_2$ and also

Incidency is preserving [stable]

Adjacency is preserving [stable].

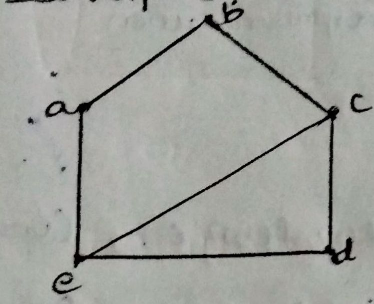
* Isomorphism Invariants / Necessary conditions for two graphs be Isomorphic :-

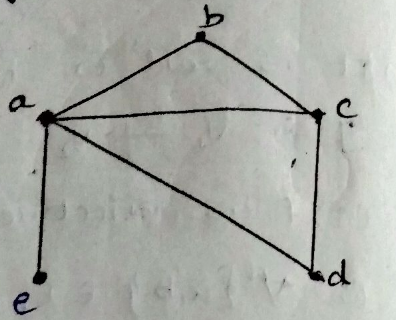
1. Both graphs have same number of vertices.
2. Both graphs have same no. of Edges.
3. Both graphs have same degree sequences.
4. If Graph 'G_1' have a loop at 'v' then 'G_2' have loop at '$f(v)$'.
5. For Digraphs, Both have
   (i) Same out degrees.
   (ii) Same In degrees.

→ We need to follow these conditions to show that two graphs are Not Isomorphic. [whether the conditions fail].

* P-5 :-

show that the following graphs are Not Isomorphic ?





A) No. of vertices in $G_1 = 5$

No. of edges in $G_1 = 6$

No. of vertices in $G_2 = 5$

No. of edges in $G_2 = 6$

Graph G_1

vertex	degree
a	2
b	2
c	3
d	2
e	3

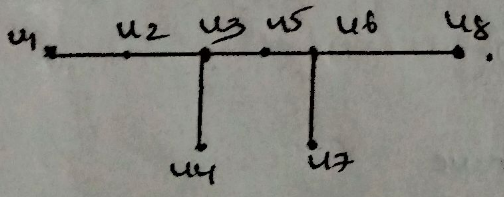
Graph G_2

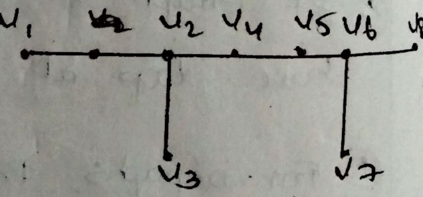
vertex	degree
a	~~3~~ 4
b	2
c	3
d	2
e	1

Not Isomorphic.

6) Determine

d)





A) Graph G

vertex	degree
u_1	1
u_2	2
u_3	3
u_4	1
u_5	2
u_6	3
u_7	1
u_8	1

Graph H

Vertex	degree
v_1	1
v_2	3
v_3	1
v_4	2
v_5	2
v_6	3
v_7	1
v_8	1

Degree sequence - 1 :- 3 3 2 2 1 1 1 1

Degree sequence - 2 :- 3 3 2 2 1 1 1 1

$u_5 \xrightarrow{f} v_4$

(or)

$u_5 \xrightarrow[f]{} v_5$

It is not possible.

∴ These two graphs are not Isomorphic.

since degree of u_5 is 2 in graph 'G'.

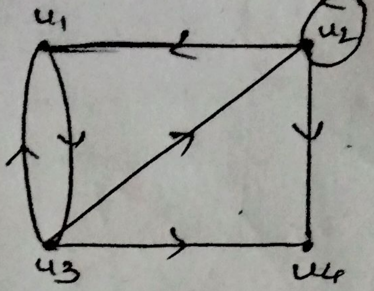
The corresponding vertices for u_5 in graph H are either v_4 (or) v_5.

Since the adjacent vertices of u_5 in graph 'G' are of degree 3 whereas the adjacent vertices of v_4 and v_5 in graph 'H' are degree 3 & 2. so that Incidency preserving fails. so that so that the given two graphs are not Isomorphic.

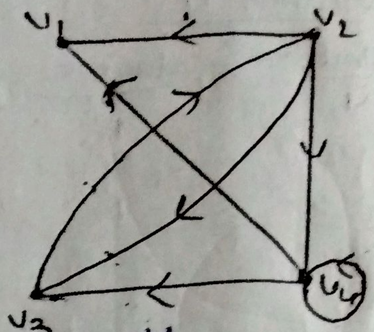
For u_3, degree is 3

$u_3 \begin{cases} v_2 \\ v_6 \end{cases}$

7] ⓐ



G.



H

A) No. of vertices in G = 4

No. of edges in G = 8

No. of vertices in H = 4

No. of edges in H = 8

$|V_1| = 4 = |V_2|$

$|E_1| = 8 = |E_2|$

Graph G

vertex	out degree	Indegree	Total degree
u1	1	2	3
u2	3.	2	5
u3	3	1	4
u4	0	2	2
	3, 3, 1, 0	2, 2, 2, 1	

Graph H

vertex	out degree	Indegree	Total degree
v1	0	2	2
v2	3	1	4
v3	1	2	3
v4	3	2	5.
	3, 3, 1, 0	2, 2, 2, 1	

Now, mappings [The preserving mappings]

$u_4 \xrightarrow{f} v_1$ i.e $f(u_4) = v_1$

$u_1 \xrightarrow{f} v_3$ i.e $f(u_1) = v_3$

$u_2 \xrightarrow{f} v_4$ i.e $f(u_2) = v_4$

$u_3 \xrightarrow{f} v_2$ i.e $f(u_3) = v_2$.

path for G :- $u_1 \to u_3 \to u_2 \to u_4$.

$$\begin{array}{cccc} u_1 \to & u_3 \to & u_2 \to & u_4 \\ f\downarrow & f\downarrow & f\downarrow & f\downarrow \\ v_3 \to & v_2 \to & v_4 \to & v_1 \end{array}$$

Path for H :- $v_3 \to v_2 \to v_4 \to v_1$.

(i) clearly, it is Bijective.

~~The prese~~

(ii) Adjacent edges.

$(u_1, u_3) \in E_1 \Rightarrow (f(u_1), f(u_3)) = (v_3, v_2)$

$(u_3, u_1) \in E_2 \Rightarrow (f(u_3), f(u_1)) = (v_2, v_3)$

$(u_2, u_1) \in E_3 \Rightarrow (f(u_2), f(u_1)) = (v_4, v_3)$

$(u_2, u_4) \in E_4 \Rightarrow (f(u_2), f(u_4)) = (v_4, v_1)$

$(u_3, u_4) \in E_5 \Rightarrow (f(u_3), f(u_4)) = (v_2, v_1)$

$(u_3, u_2) \in E_6 \Rightarrow (f(u_3), f(u_2)) = (v_2, v_4)$

$(u_2, u_2) \in E_7 \Rightarrow (f(u_2), f(u_2)) = (v_4, v_4)$.

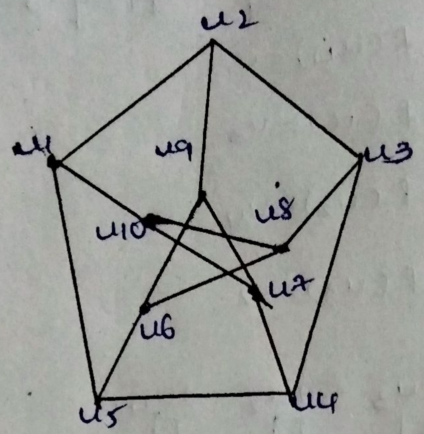
$(u_2, u_2) \in E_8 \Rightarrow f(u_2, f(u_2)) = (v_4, v_4)$.

$\therefore$ All corresponding edges are available.

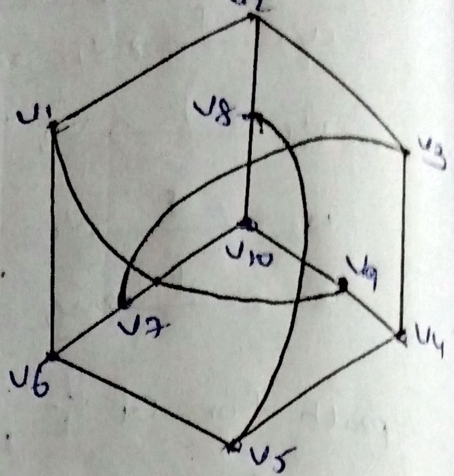
From (i) & (ii)

The two graphs are Isomorphic.

6] ⓔ



G



H

A) $|V_1| = |V_2| = 10$

$|E_1| = |E_2| = 15$

Graph G

vertex	degree
u_1	3
u_2	3
u_3	3
u_4	3
u_5	3
u_6	3
u_7	3
u_8	3
u_9	3
u_{10}	3

Graph H

vertex	degree
v_1	3
v_2	3
v_3	3
v_4	3
v_5	3
v_6	3
v_7	3
v_8	3
v_9	3
v_{10}	3

Path for G :- $u_1 \rightarrow u_{10} \rightarrow u_8 \rightarrow u_3 \rightarrow u_2 - u_9 - u_7$

$u_6 - u_5 - u_4$

Path for H :-

$u_1 - v_9 - v_{10} - v_8 - v_2 - v_3 - v_4 - v_5 - v_6 - v_7$

correspondence.

$u_1 \xrightarrow{f} v_1$

$u_2 \xrightarrow{f} v_2$

$u_3 \xrightarrow{f} v_8$

$u_4 \xrightarrow{f} v_5$

$u_5 \xrightarrow{f} v_6$ $\quad\therefore$ It is bijective

$u_6 \xrightarrow{f} u_7$

$u_7 \xrightarrow{f} v_4$

$u_8 \xrightarrow{f} v_{10}$

$u_9 \xrightarrow{f} v_3$

$u_{10} \xrightarrow{f} v_9$

Edges:-

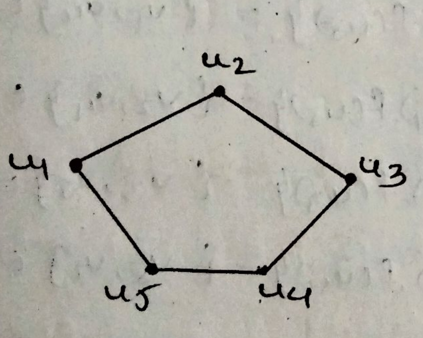
$\{u_1, u_2\} \in E_1 \Rightarrow \{f(u_1), f(u_2)\} \in E_2 = \{v_1, v_2\}$.

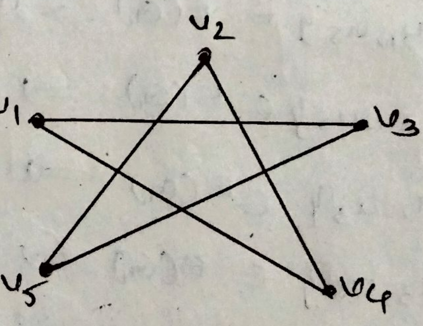
$\{u_9, u_7\} \in E_2 \Rightarrow \{f(u_9), f(u_7)\} = \{v_3, v_4\}$.

similarly for 15 edges corresponding edges are exist.

$\therefore$ It is Isomorphic.

6) (a)





A) Here $V(G_1) = \{u_1, u_2, u_3, u_4, u_5\}$

$V(G_2) = \{v_1, v_2, u_3, v_4, v_5\}$

so $|V(G_1)| = |V(G_2)| = 5$.

$|E(G_1)| = |E(G_2)| = 5$.

Graph G_1

vertex	degree
u_1	2
u_2	2
u_3	2
u_4	2
u_5	2

Graph G_2

vertex	degree
v_1	2
v_2	2
v_3	2
v_4	2
v_5	2

Path for G_1 :- $u_1 \to u_2 \to u_3 \to u_4 \to u_5$

Path for G_2 :- $v_1 \to v_3 \to v_5 \to v_2 \to v_4$

Now, $f(u_1) = v_1$

$f(u_2) = v_3$

$f(u_3) = v_5$

$f(u_4) = v_2$

$f(u_5) = v_4$

clearly 'f' is Bijective.

Edges:-

$\{u_1, u_2\} \in E(G_1) \Rightarrow \{f(u_1), f(u_2)\} = \{v_1, v_3\} \in E(G_2)$

$\{u_1, u_5\} \in E(G_1) \Rightarrow \{f(u_1), f(u_5)\} = \{v_1, v_4\} \in E(G_2)$

$\{u_4, u_5\} \in E(G_1) \Rightarrow \{f(u_4), f(u_5)\} = \{v_2, v_4\} \in E(G_2)$

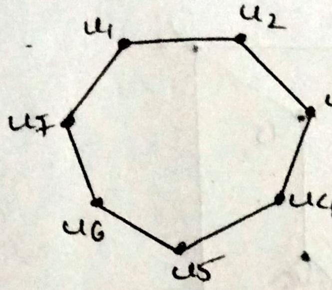
$\{u_2, u_3\} \in E(G_1) \Rightarrow \{f(u_2), f(u_3)\} = \{v_3, v_5\} \in E(G_2)$

$\{u_3, u_4\} \in E(G_1) \Rightarrow \{f(u_3), f(u_4)\} = \{v_5, v_2\} \in E(G_2)$

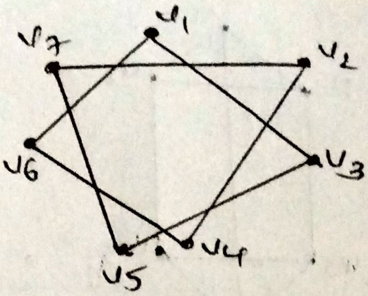
Hence 'f' preserves adjacency of vertices

$\therefore$ G_1 and G_2 are Isomorphic

(b)



G



H

A) $|V_1| = |V_2| = 7$

$|E_1| = |E_2| = 7$

Graph G

vertex	degree
u_1	2
u_2	2
u_3	2
u_4	2
u_5	2
u_6	2
u_7	2

Graph H

vertex	degree
v_1	2
v_2	2
v_3	2
v_4	2
v_5	2
v_6	2
v_7	2

Path for G : $u_1 \to u_2 \to u_3 \to u_4 \to u_5 \to u_6 \to u_7$

$\quad\;\;\updownarrow \quad \updownarrow \quad \updownarrow \quad \updownarrow \quad \updownarrow \quad \updownarrow \quad \updownarrow$

Path for H : $v_1 \to v_3 \to v_5 \to v_7 \to v_2 \to v_4 \to v_6$

Now $f(u_1) = v_1$ $\quad f(u_4) = v_7$ $\quad f(u_7) = v_6.$

$f(u_2) = v_3$ $\quad f(u_5) = v_2$

$f(u_3) = v_5$ $\quad f(u_6) = v_4$

clearly 'f' is Bijective.

$\therefore$ It is Isomorphic.

Edges:

$\{u_1, u_2\} \in E(G) \Rightarrow \{f(u_1), f(u_2)\} \cancel{\in E(H)} = \{v_1, v_3\} \in E(H)$

$\{u_2, u_3\} \in E(G) \Rightarrow \{f(u_2), f(u_3)\} \cancel{\in E(H)} = \{v_3, v_5\} \in E(H)$

$\{u_3, u_4\} \in E(G) \Rightarrow \{f(u_3), f(u_4)\} \cancel{\in E(H)} = \{v_5, v_7\} \in E(H)$

$\{u_4, u_5\} \in E(G) \Rightarrow \{f(u_4), f(u_5)\} \cancel{\in E(H)} = \{v_7, v_2\} \in E(H)$

$\{u_5, u_6\} \in E(G) \Rightarrow \{f(u_5), f(u_6)\} \cancel{\in E(H)} = \{v_2, v_4\} \in E(H)$

$\{u_6, u_7\} \in E(G) \Rightarrow \{f(u_6), f(u_7)\} \cancel{\in E(H)} = \{v_4, v_6\} \in E(H)$

$\{u_7, u_1\} \in E(G) \Rightarrow \{f(u_7), f(u_1)\} \cancel{\in E(H)} = \{v_6, v_1\} \in E(H)$

10-06-2023.

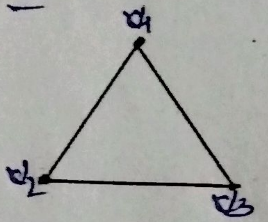
5.2 :- SPECIAL TYPES OF GRAPHS & GRAPH REPRESENTATIONS

① **Cycle (C_n) :-**

→ C_n means a graph cycle with n vertices. $n \geq 3$.

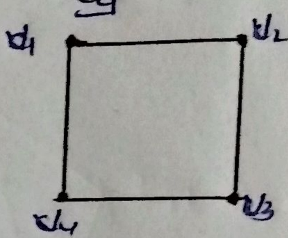
→ It consists of n vertices $v_1, v_2, \text{---} v_n$ and edges are $\{v_1, v_2\}$, $\{v_2, v_3\}$ ⟶ $\{v_{n-1}, v_n\}$, $\{v_n, v_1\}$.

Ex :- C_3



2- regular graph

C_4

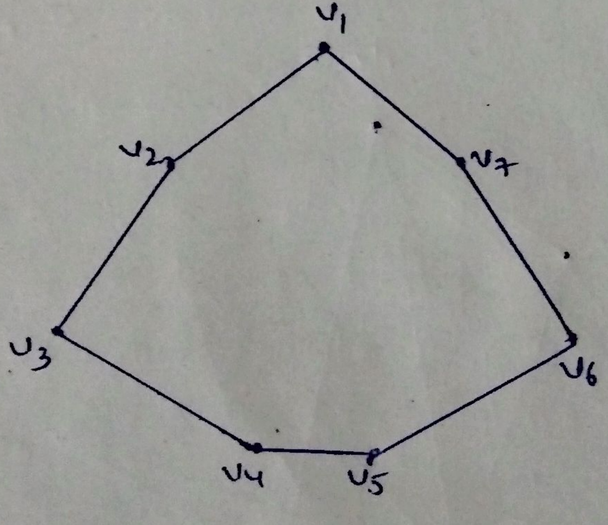


2- regular graph.

* **Exercise :-**

3] ⓐ C_7 :-

A)



* **Note :-**

① No. of vertices of cycle $= n$.

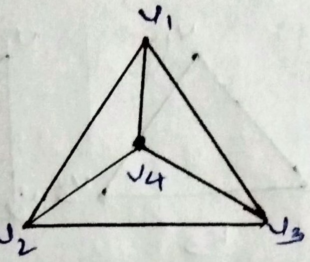
② C_n is 2-regular graph [Degree of each vertex =

③ No. of edges $= n$.

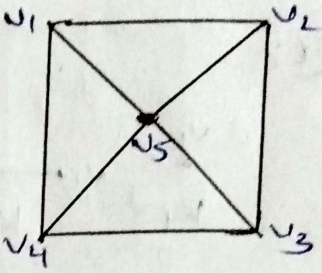
② **Wheel (W_n, $n \geq 3$) :-**

→ It is also a 'simple graph' & obtained by adding additional vertex interior to cycle & connects it to each vertex of cycle.

Ex:- W_3

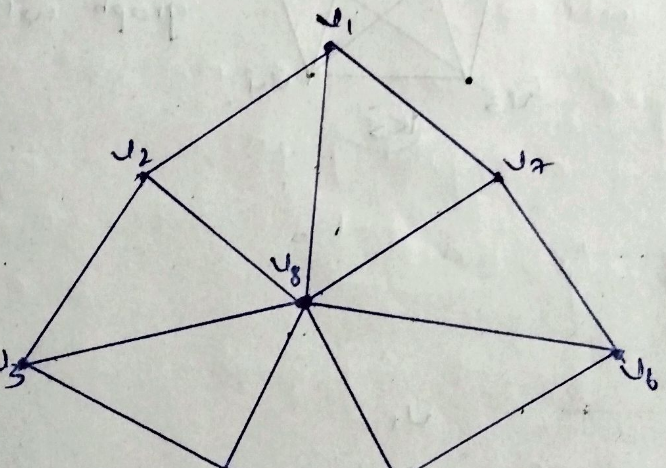


W_4



*** Exercise :-**

2] ⓔ W_7 :-



*** Note points :-**

① No. of vertices = $n+1$

② No. of edges = $2n$. By H.S.T $\left[\begin{array}{l} 3n+n = 2e \\ 4n = 2e \end{array} \quad \therefore e = 2n \right]$.

③ Degree of each vertex except additional vertex is 3.

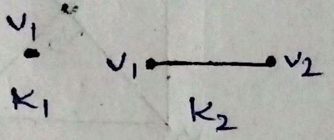
④ Degree of additional vertex = n.

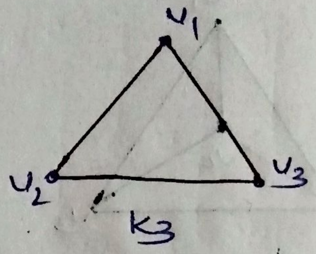
③ complete graph :- (K_n)

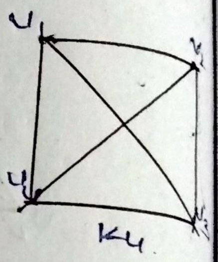
→ A graph is said to be complete graph that contains exactly one edge b/w each pair of distinct vertices.

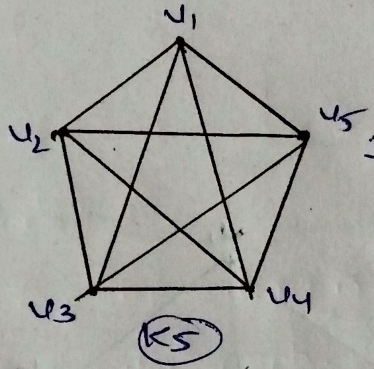
→ It is also a simple graph.

Ex :-







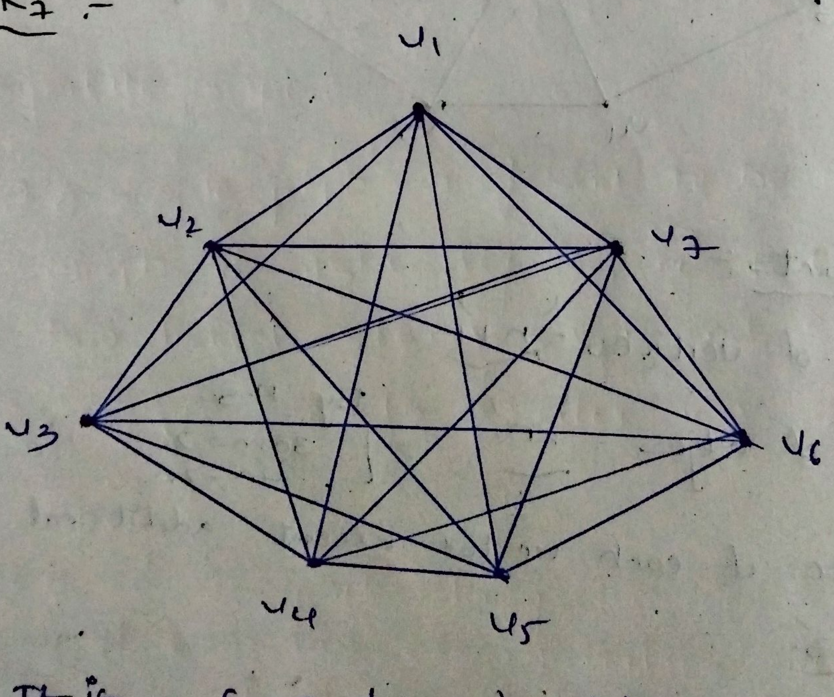


It is a 4-regular graph with 5 sides

* Exercise :-

2] ⓐ K_7 :-

A)



It is a 6-regular graph with 7 vertices.

*Note :-

① No. of vertices $\Rightarrow n$

② No. of edges $= {}^{n}C_{2} = \frac{n(n-1)}{2}$

$$n(n-1) = 2e$$

$$e = \frac{n(n-1)}{2} = {}^{n}C_{2}$$

③ The degree of each vertex $= n-1$.
So it is a $(n-1)$ regular graph.

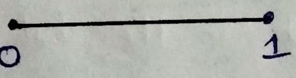
④ n-cubes (or) Hyper cubes :- $[Q_n]$ $(n \geq 1)$.

→ 'Q_n' is a Hyper cube with 2^n vertices. We can represent each vertex as a 'bit string' of length "n".

Q_1

$n=1$

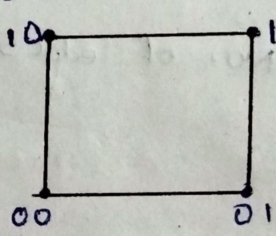
$2^n = 2^1 = 2$ vertices.



Q_2

$n=2$

$2^n = 2^2 = 4$ vertices.



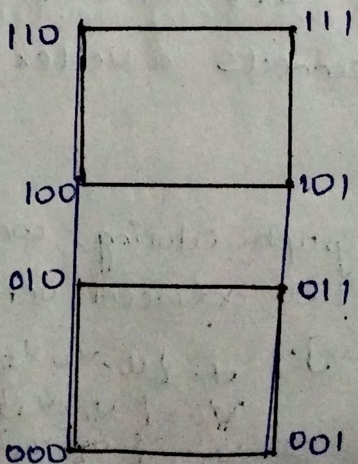
00
01
10
11

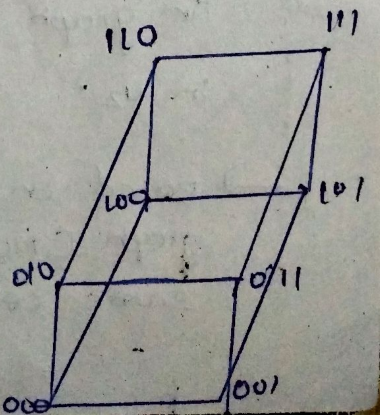
Q_3

$Q_{n+1} = Q_{2+1} = Q_3$.

$n=3$

$2^n = 2^3 = 8$ vertices.

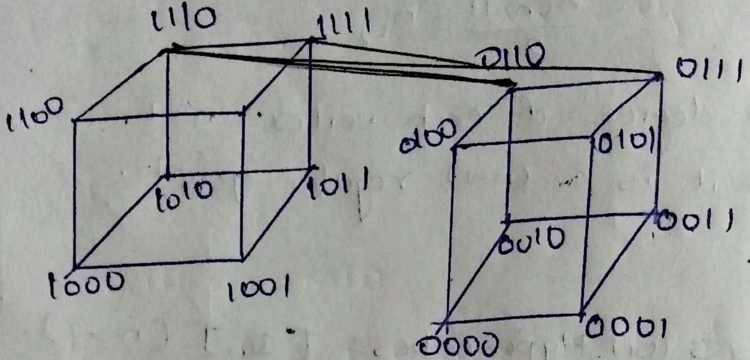




* Exercise

2) (f) Q_4 :-

A) $Q_4 = Q_{3+1}$



* Note :-

① No. of vertices $= 2^n$

② The degree of each vertex $= n$. i.e, Q_n is a n-regular graph.

③ No. of edges $= n \cdot 2^{n-1}$

By H.S.T

$2^n \cdot n = 2e$

$\Rightarrow e = \frac{2^n \cdot n}{2} = n 2^{n-1}$

⑤ Bipartite Graph :-

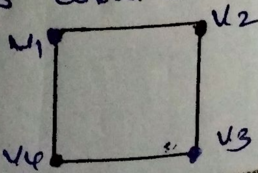
→ A simple graph 'G' is said to be Bipartite Graph, if its vertex can be partitioned into two disjoint sets. V_1 & V_2 and every edge in graph connects a vertex in V_1 & a vertex in V_2.

→ Based on graph coloring, we can define Bipartite graph [No two adjacent vertices contains same colour].

$V = \{v_1, v_2, v_3, v_4\}$

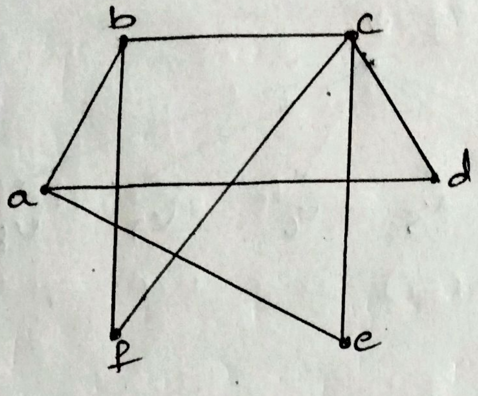
$V_1 = \{v_1, v_3\}$

$V_2 = \{v_2, v_4\}$.

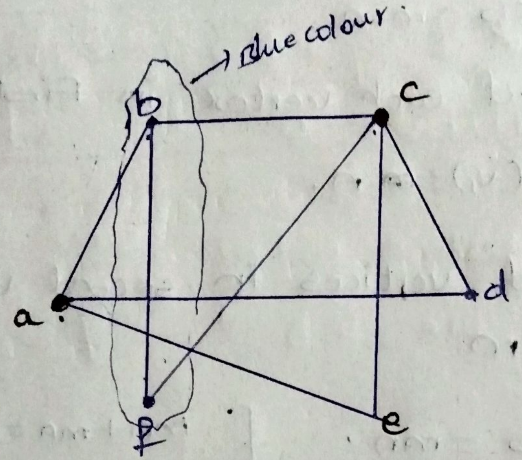


* Exercise :-

1) ⓒ



A)



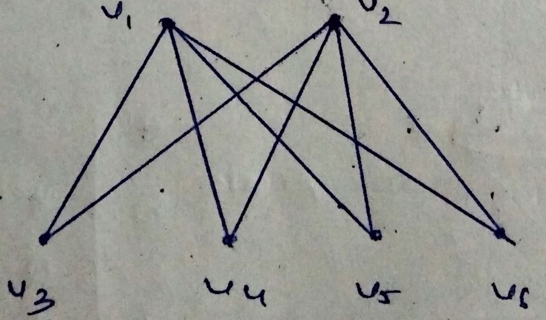
Not a Bipartite graph.

⑥ Complete Bipartite graph :- $(K_{m,n})$.

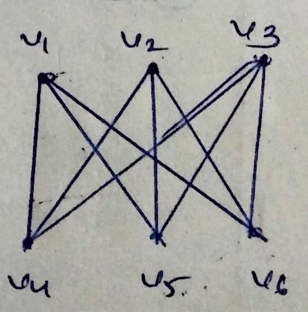
→ ① Vertex is partitioned into two subsets V_1 & V_2 of m and n vertices respectively.

② Every vertex in V_1 connects to Every vertex in V_2.

Ex :- $K_{2,4}$. m=2, n=4



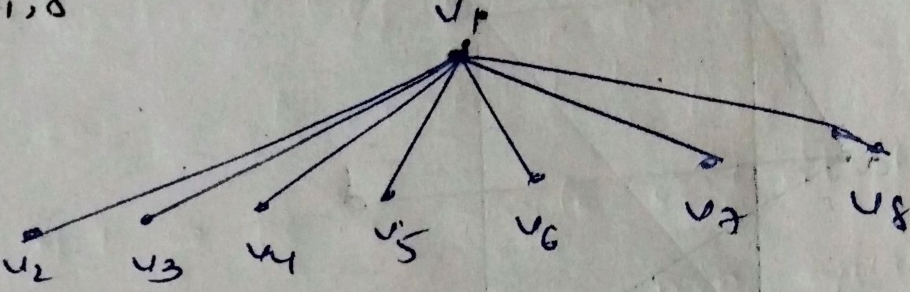
$K_{3,3}$.



* Exercise :-

2] (b) $K_{1,8}$

A)



* Note points :-

(1) No. of vertices $= m+n$.

(2) The degree of each vertex in First vertex set is. $d(v_1) = n$.

The degree of vertices in second vertex set is $d(v_2) = m$

(3) No. of edges $= mn$, $\left[\begin{array}{l} mn + mn = 2e \\ 2mn = 2e \quad \therefore e = mn \end{array} \right]$.

* Representations of graphs :-

→ We can represent any graph in the following 3-ways:

(1) Based on Adjacency List.

(2) Based on Adjacency Matrix

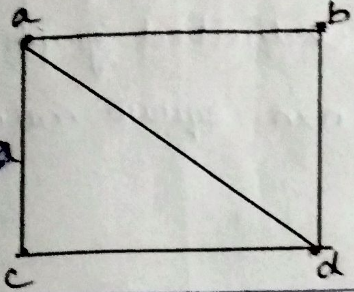
(3) Based on Incidency Matrix.

(1) Based on adjacency list :-

(a) For Undirected Graph :-

* Exercise

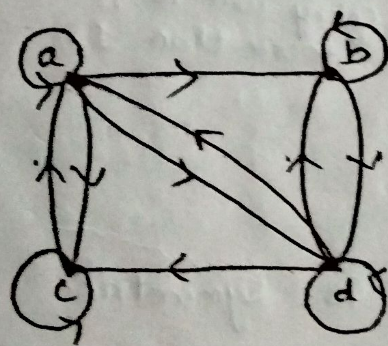
3(c)



Adjacency List of undirected graph.	
vertex	Adjacent vertices
a	b, c, d
b	a, d
c	a, d
d	a, b, c

(b) For Digraphs :-

3(b)



Adjacency list of Directed graph	
Initial vertex	Terminal vertex
a	a, b, c, d
b	b, d
c	a, c.
d	a, b, c, d.

(2) Adjacency Matrix :-

(a) For undirected graph :-

(i) For simple undirected graph :-

→ Let G be a simple undirected graph, $|V|=n$

(Notation :-) $A(G)$ is a $n \times n$ square matrix.

→ $A(G) = [a_{ij}]$.

$$a_{ij} = \begin{cases} 1, \text{ No. of edges b/w } \{v_i, v_j\} \\ \text{otherwise '0'} \end{cases}$$

→ It is a boolean matrix, since it is simple so entries are 0 & 1 only. [Not > 1].

(ii) For Multi undirected graph :-

→ It is not a boolean matrix.

→ For Multi & Pseudo entry will be more than 1.

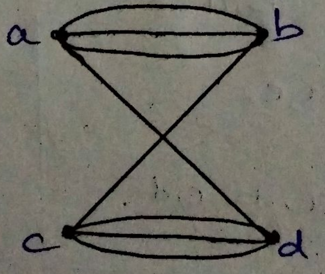
Note :-

(1) The Adjacency matrix is symmetric.

(2) All diagonal entries are zero in simple undirected graph.

* Exercise :-

5



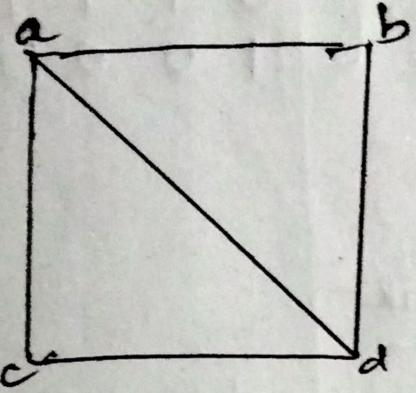
$$A(G) = \begin{array}{c|cccc} & a & b & c & d \\ \hline a & 0 & 3 & 0 & 1 \\ b & 3 & 0 & 1 & 0 \\ c & 0 & 1 & 0 & 3 \\ d & 1 & 0 & 3 & 0 \end{array}$$

observations :-

① sum of all entries in either upper or lower triangle is equal to No. of edges.

② sum of row entries is equal to degree [Not for pseudograph]

3 ⓐ



A)
$$A(G) = \begin{array}{c|c} & \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}_{4\times4} \begin{array}{l} \rightarrow 3 \\ \rightarrow 2 \\ \rightarrow 2 \\ \rightarrow 3 \end{array} \end{array} \Bigg\} \text{degrees}$$

Edges = 5.

ⓑ For Digraphs :-

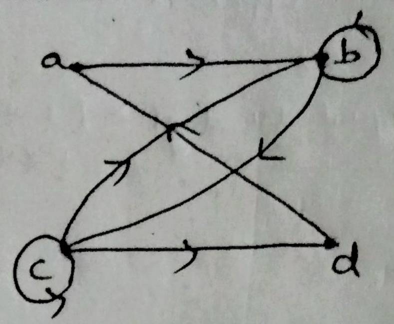
$\rightarrow a_{ij} = \begin{cases} \text{no. of edges b/w } (v_i, v_j) \\ \text{otherwise '0'.} \end{cases}$

Here $a_{ij} \neq a_{ji}$.

$\rightarrow$ Need not be symmetric.

* Exercise - 5(c) :-

5] (c)



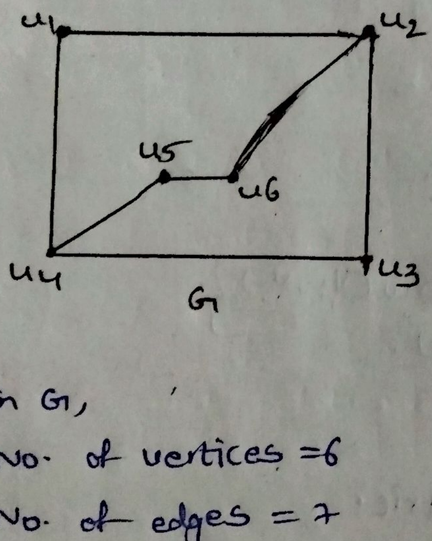
$$A(G) = \begin{array}{c|cccc} & a & b & c & d \\ \hline a & 0 & 1 & 0 & 0 \\ b & 0 & 1 & 1 & 0 \\ c & 0 & 1 & 1 & 1 \\ d & 1 & 0 & 0 & 0 \end{array}$$

* Conclusions on Adjacency Matrix :-

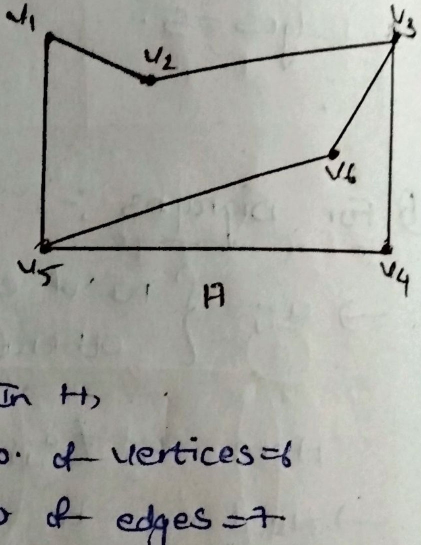
(1) We can check whether two graphs are Isomorphic or not.

Example-13 :-

Determine G & H are Isomorphic or not.



G



H

A) In G,

No. of vertices = 6

No. of edges = 7

In H,

No. of vertices = 6

No of edges = 7

G

vertices	degree
u_1	2
u_2	3
u_3	2
u_4	3
u_5	2
u_6	2

Degree sequence is 3, 3, 2, 2, 2, 2.

H

vertices	degree
v_1	2
v_2	2
v_3	3
v_4	2
v_5	3
v_6	2

Degree sequence is 3, 3, 2, 2, 2, 2.

Simple path :-

→ It is a path in which any edge doesnot occurs more than one time.

Now simple path for Graphs G, & H.

$$\begin{array}{ccccccccccc}
2 & & 3 & & 2 & & 3 & & 2 & & 2 \\
u_1 & - & u_2 & - & u_3 & - & u_4 & - & u_5 & - & u_6. \\
| & & | & & | & & | & & | & & | \\
v_1 & - & v_2 & - & v_3 & - & v_4 & - & v_5 & - & v_6. \\
2 & & 2 & & 3 & & 2 & & 3 & & 2
\end{array}$$

This path is not suitable.

Now,

$$\begin{array}{ccccccccccc}
2 & & 3 & & 2 & & 3 & & 2 & & 2 \\
u_1 & - & u_2 & - & u_3 & - & u_4 & - & u_5 & - & u_6 \\
| & & | & & | & & | & & | & & | \\
v_4 & - & v_3 & - & v_6 & - & v_5 & - & v_1 & - & v_2 \\
2 & & 3 & & 2 & & 3 & & 2 & & 2
\end{array}$$

$u_1 \xrightarrow{f} v_4$; $f(u_1) = v_4$

$u_2 \xrightarrow{f} v_3$; $f(u_2) = v_3$

$u_3 \xrightarrow{f} v_6$; $f(u_3) = v_6$

$u_4 \xrightarrow{f} v_5$; $f(u_4) = v_5$

$u_5 \xrightarrow{f} v_1$; $f(u_5) = v_1$

$u_6 \xrightarrow{f} v_2$; $f(u_6) = v_2$

∴ It is Bijective.

Adjacency matrix of 'G' with ordering $u_1, u_2, u_3, u_4, u_5, u_6$.

$$A_G = \begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{array}
\begin{array}{c}
\begin{array}{cccccc} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \end{array} \\
\begin{bmatrix}
0 & 1 & 0 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 0
\end{bmatrix}_{6 \times 6}
\end{array}$$

Adjacency Matrix of Graph 'H' with ordering $v_4, v_3, v_6, v_5, v_1, v_2$.

$$A_H = \begin{array}{c} \\ v_4 \\ v_3 \\ v_6 \\ v_5 \\ v_1 \\ v_2 \end{array} \begin{array}{c} \begin{array}{cccccc} v_4 & v_3 & v_6 & v_5 & v_1 & v_2 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

∴ A_G and A_H are equal.

∴ The graphs 'G' & 'H' are Isomorphic.

[since A_G & A_H are same & 'G' & H are Bijecti

* Requirements to draw a graph :-

1. 'Undirected' / 'Directed'
2. 'No. of vertices'
3. 'No. of Edges'
4. 'Adjacency Relationship' (or) 'Incidency Relationship'.

* Note :-

1. If Adjaceny matrix is symmetric then it is undirected graph otherwise ∴ Digraph.
2. Based on order of matrix, we can find no. of vertices in a graph.
3. No. of edges.

Undirected	Directed
→ The sum of all entries in upper (or) lower Triangular Matrix of Adjacency matrix is equal to no. of edges.	→ The sum of all entries of Adjacency matrix is equal to no. of edges.

④ If the corresponding entry is zero then there is no Adjacency Relationship.

* **Example-14:-**

Are the simple graphs with following adjacency matrices isomorphic?

ⓑ
$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

A) ① Since two matrices are symmetric, so undirected graph.

② No. of vertices in G = 4 [∵ 4x4 Matrix].
No. of vertices in H = 4

③ No. of edges in G = 3. [sum of entries in upper/lower triangular matrices].
No. of edges in H = 4.

∴ It is not Isomorphic.

ⓒ
$$\begin{array}{cc} G & H \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

A) ① Directed graphs [∵ Not symmetric]

② No. of vertices,
In G = 4
In H = 4

③ No. of edges,
In G = 9
In H = 5 [sum of entries = no. of edges].

∴ Not Isomorphic

(a) $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

A) $A_G = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$; $A_H = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

consider A_G,

$$A_G = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}_{R_1 \longleftrightarrow R_3}$$

$$\Rightarrow A_G = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}_{C_1 \longleftrightarrow C_3}$$

$$\Rightarrow A_G = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$\Rightarrow A_G = A_H$.

$\therefore$ Graphs G & H are Isomorphic.

(3) Incidency Matrix :- [M/D].

(a) undirected graph

If $G = G(V, E)$

$|V| = n$

$|E| = m$.

$M = [m_{ij}]_{n \times m}$.

$$m_{ij} = \begin{cases} 1, & \text{if the edge } e_j \text{ is incident with } v_i. \\ 0, & \text{otherwise.} \end{cases}$$

always Boolean matrix.

(b) Directed graph

If $G = G(V, E)$

$|V| = n$

$|E| = m$

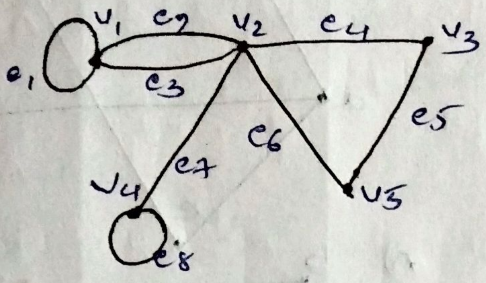
$M = [m_{ij}]_{n \times m}$.

$$m_{ij} = \begin{cases} 1, & \text{if } e_j \text{ is incident out the vertex } v_i \\ -1, & \text{if } e_j \text{ is incident in vertex } v_j \\ 0, & \text{otherwise} \end{cases}$$

$v_i \xrightarrow{e_j} v_j$

*** Example-12 :-**

Represent the following multigraph with an Incidence matrix.



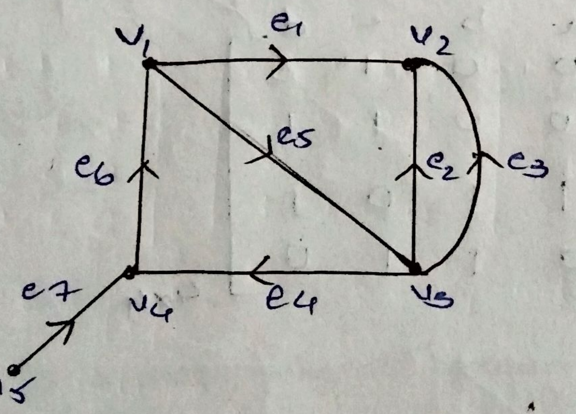
A) Here $|V| = 5$

$|E| = 8$

$M = 5 \times 8$.

$$M = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{cccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}_{5 \times 8} \end{array}$$

* Find the Incidence matrix of digraph shown below.



A) $|V| = 5$, $|E| = 7$, $M = 5 \times 7$ matrix

$$M = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{ccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & -1 & 0 \\ -1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}_{5 \times 7} \end{array}$$

5.3 - CONNECTIVITY

12-06-2023

* Path:-

→ Path is a sequence of edges that starts and ends with vertices and travels along edges.

$e_1, e_2, e_3, \ldots, e_n$

$\{x_0, x_1\}, \{x_1, x_2\}, \{x_2, x_3\}, \ldots \{x_{n-1}, x_n\}$. [Undirected graph]

$x_0, x_1, x_2, x_3, \ldots, x_{n-1}, x_n$ ← undirected

$(x_0, x_1), (x_1, x_2), (x_2, x_3), \ldots, (x_{n-1}, x_n)$ [Digraph].

$x_0, x_1, x_2, \ldots, x_{n-1}, x_n$. Directed path.

Length of path:-

→ The No. of edges appearing in a path is called Length of the path.

Note:-

→ Path allows repeated edges.

* Circuit:-

→ Circuit is a path that starts and ends with same path.

* Simple Path:-

→ It is a path doesnot appear same edge more than once.

→ Edge repeatations are not allowed.

→ But allows the repeated vertices.

*

	Repeated edge	Repeated vertex	starts & ends with same vertex
Path	Allowed	Allowed	Not Allowed
simple path	Not	Allowed	Not Allowed
circuit	Allowed	Allowed	~~Not~~ Allowed
simple circuit	Not	Allowed	~~Not~~ Allowed

* Paths and Isomorphisms :-

→ Paths & circuits can help to determine whether two graphs are Isomorphic.

→ Based on simple circuit,

If $G \cong H$, then Both have simple circuits of length k $(k \geq 3)$ [k is natural no].

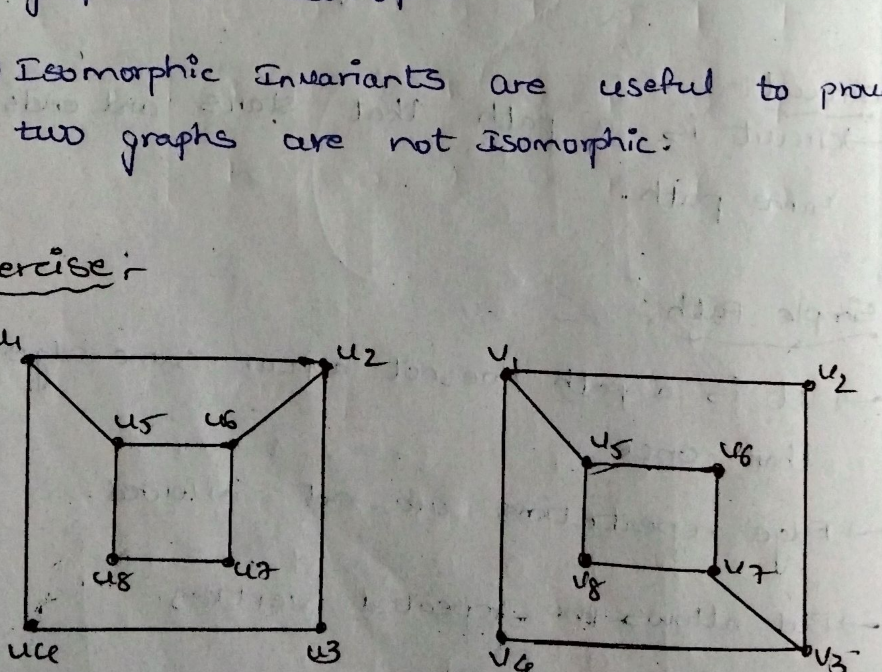
Note :-

→ Adjaceny Matrix, is useful to prove two graphs are Isomorphic.

→ Isomorphic Invariants are useful to prove two graphs are not Isomorphic.

* Exercise :-

4] (a)



G H

A) Path for Graph G :- $u_1 \to u_5 \to u_8 \to u_7 \to u_6 \to u_2 \to u_3 \to u_4 \to u_1$

The length of simple circuit is 8.

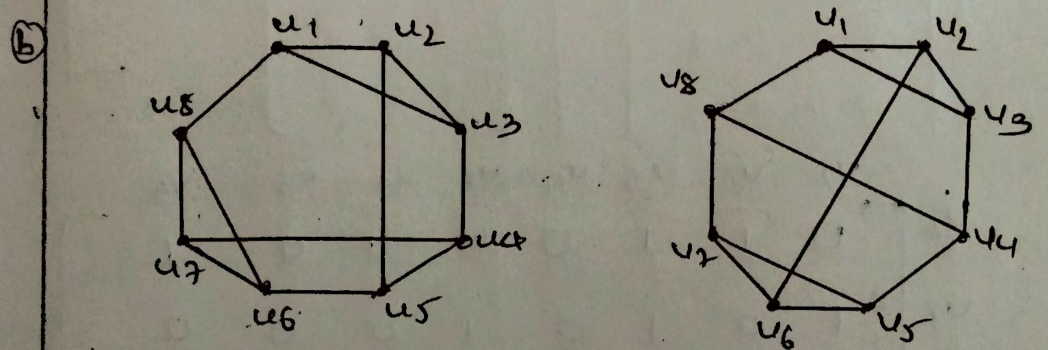
Path for H :- $v_1 \to v_5 \to v_6 \to v_7 \to v_3 \to v_2 \to v_1$

The length of simple circuit is 6.

Based on 4th Isomorphic Invariant, lengths are different.

$\therefore$ G & H are Not Isomorphic.

(b)



G H.

A) $|V_1| = |V_2| = 8$

$|E_1| = |E_2| = 12$

Degree of all vertices in G & H = 3.

Simple path for G :- $u_1 \to u_2 \to u_3 \to u_4 \to u_5 \to u_6 \to u_7 \to u_8$

Simple path for H :- $v_2 \to v_1 \to v_3 \to v_4 \to v_8 \to v_7 \to v_5 \to v_6$

Now, $f(u_1) = v_2$ $f(u_5) = v_8$

$f(u_2) = v_1$ $f(u_6) = v_7$

$f(u_3) = v_3$. $f(u_7) = v_5$

$f(u_4) = v_4$ $f(u_8) = v_6$.

$\therefore$ f is Bijective.

$$A_G = \begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \end{array} \begin{array}{c} \begin{array}{cccccccc} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 & u_7 & u_8 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

$$A_H = \begin{array}{c} \\ u_2 \\ v_1 \\ u_3 \\ u_4 \\ v_8 \\ u_7 \\ u_5 \\ v_6 \end{array} \begin{array}{c} \begin{array}{cccccccc} u_2 & v_1 & u_3 & u_4 & u_8 & u_7 & v_5 & v_6 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

Here $A_G = A_H$.

$\therefore$ G & H are Isomorphic.

* counting no. of paths b/w two vertices of a graph:-

Theorem:-

→ Let $G = G(V, E)$ be a graph with $|V| = n$. Let $v_1, v_2, \ldots, v_n$ are vertices of G.

The no. of paths of length k b/w vertices $v_i, v_j = (i, j)$ entry in the matrix A^k. [$A = A_G = A$·Matrix of G]

The no. of paths of length 2 b/w $v_i, v_j = (i, j)$ in A^2

The no. of paths of length 3 b/w $v_i, v_j = (i, j)$ in A^3.

* Exercise

5] Find the no. of paths of length n between two different vertices in K_4 if n is

A) Algorithm:-

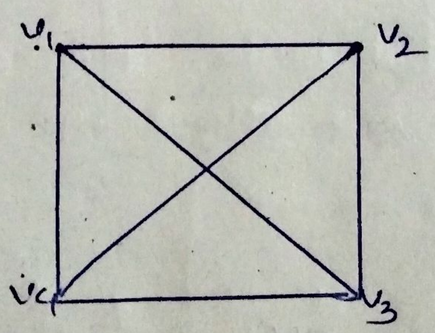
step-1:- Draw the Graph 'G'

step-2:- write Adjacency matrix 'A_G' = A.

step-3:- Find the length k & A^k matrix

step-4:- write (i, j) entry in A^k.

(b) K_4:-



The Adjacency matrix with ordering v_1, v_2, v_3, v_4 is as follows:

$$A = \begin{array}{c|c} & \begin{matrix} v_1 & v_2 & v_3 & v_4 \end{matrix} \\ \begin{matrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{matrix} & \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

Let us take $K=4$ [length $=4$]

$\Rightarrow A^4 = A^3 \cdot A$

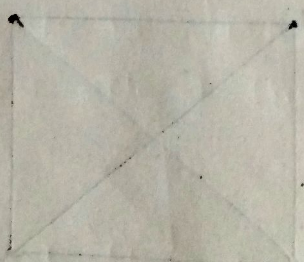
$$\Rightarrow A^4 = \begin{bmatrix} 6 & 7 & 7 & 7 \\ 7 & 6 & 7 & 7 \\ 7 & 7 & 6 & 7 \\ 7 & 7 & 7 & 6 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

$$\therefore A^4 = \begin{bmatrix} 21 & 20 & 20 & 20 \\ 20 & 21 & 20 & 20 \\ 20 & 20 & 21 & 20 \\ 20 & 20 & 20 & 21 \end{bmatrix}$$

$\therefore$ The no. of paths in any two different vertic[es]
[Non-diagonal] are 20.

(a) 2.

A) $$A^2 = \begin{bmatrix} 3 & 2 & 2 & 2 \\ 2 & 3 & 2 & 2 \\ 2 & 2 & 3 & 2 \\ 2 & 2 & 2 & 3 \end{bmatrix}$$



$\therefore$ No. of paths in two different vertices $= 2$

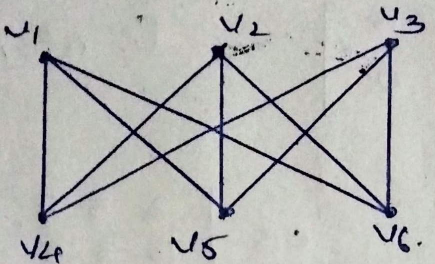
No. of paths in two same vertices $= 3$

No. of circuits $= 3$.

* Exercise :

(b)(i)

(a)



Adjacent vertices = $\{u_1, u_2, u_3\}$, $\{u_4, u_5, u_6\}$.

Non-Adjacent vertices are vertices in V_1 & vertices in V_2.

$$A = \begin{array}{c|c} & \begin{matrix} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \end{matrix} \\ \begin{matrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{matrix} & \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{bmatrix} \end{array} = \begin{bmatrix} 0 & B \\ B & 0 \end{bmatrix}$$

where $B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

Now, $A^2 = \begin{bmatrix} 0 & B \\ B & 0 \end{bmatrix} \begin{bmatrix} 0 & B \\ B & 0 \end{bmatrix} = \begin{bmatrix} B^2 & 0 \\ 0 & B^2 \end{bmatrix}$

$$\Rightarrow A^2 = \begin{array}{c|c} & \begin{matrix} \quad V_1 \quad & \quad V_2 \quad \end{matrix} \\ \begin{matrix} \\ V_1 \\ \\ \\ V_2 \\ \\ \end{matrix} & \begin{bmatrix} 3 & 3 & 3 & 0 & 0 & 0 \\ 3 & 3 & 3 & 0 & 0 & 0 \\ 3 & 3 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 3 & 3 \\ 0 & 0 & 0 & 3 & 3 & 3 \\ 0 & 0 & 0 & 3 & 3 & 3 \end{bmatrix} \end{array} \qquad \because B^2 = \begin{bmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{bmatrix}$$

(a) R :-

No paths in Adjacency vertices

3 paths in Non-adjacency vertices.

A^3:-

$$A^3 = \begin{bmatrix} C & 0 \\ 0 & C \end{bmatrix}\begin{bmatrix} 0 & B \\ B & 0 \end{bmatrix} \qquad [\because A^3 = A^2 \cdot A]$$

$$A^3 = \begin{bmatrix} 0 & CB \\ CB & 0 \end{bmatrix}$$

$$CB = \begin{bmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 9 & 9 & 9 \\ 9 & 9 & 9 \\ 9 & 9 & 9 \end{bmatrix}$$

$$\therefore A^3 = \begin{array}{c c} & \begin{array}{cccccc} \quad V_1 & & & & V_2 & \end{array} \\ \begin{array}{c} \\ V_1 \\ \\ \\ V_2 \\ \\ \end{array} & \begin{bmatrix} 0 & 0 & 0 & 9 & 9 & 9 \\ 0 & 0 & 0 & 9 & 9 & 9 \\ 0 & 0 & 0 & 9 & 9 & 9 \\ 9 & 9 & 9 & 0 & 0 & 0 \\ 9 & 9 & 9 & 0 & 0 & 0 \\ 9 & 9 & 9 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

- Paths b/w Adjacency vertices $= 9$

Paths b/w Non-Adjacency vertices $= 0$.

* Connected graph:-

→ A graph 'G' is said to be 'connected' for every Distinct pair of vertices having path.

* Disconnected graph:-

→ A graph G is said to be 'Disconnected' when there is no path b/w atleast two vertices.

* Connected component:-

→ The connected component 'H' (say) of Graph 'G'

① 'H' is a subgraph of 'G'.

② Maximal graph 'G'.

③ 'H' is connected graph.

Maximal (or) longest Subgraph of G

* **Theorem:-**

→ For every Distinct pair of vertices there is a simple graph.

Note points:-

① Every Isolated vertex is also one of the connected graph.

② If 'G' is connected graph, then the whole graph is one-connected graph.

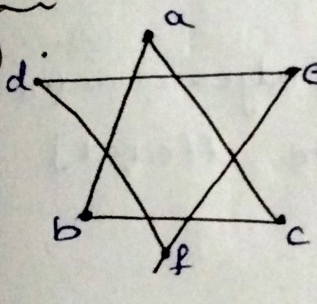
③ If 'G' is Disconnected graph then 'G' can be represented as union of Disjoint connected graph.

$$\overline{G = H_1 \cup H_2 \cup H_3 \cup \_\_\_\_}$$

Here H_1, H_2, H_3 ____ are disjoint connected graphs.

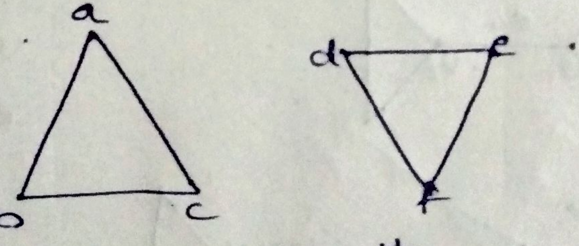
* **P-1:-**

(iii)



→ Here a to d is no connection etc ____

∴ It is Disconnected graph.



Here $G = \overline{H_1 \cup H_2}$.

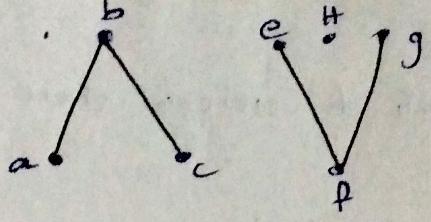
$H_1 = (V_1, E_1)$; $V_1 = \{a, b, c\}$

$E_1 = [\{a,b\}, \{b,c\}, \{c,a\}]$.

$H_2 = (V_2, E_2)$; $V_2 = \{d, e, f\}$

$E_2 = \{\{d,e\}, \{e,f\}, \{f,d\}\}$

(i)



A) $H_1 = (V_1, E_1)$; $V_1 = \{a, b, c\}$.

E_1 ;

$H_2 = (V_2, E_2)$; $V_2 = \{e, f, g\}$; E_2

$H_3 = (V_3, E_3)$; $V_3 = \{d\}$, $E_3 = \phi$.

Disconnected graph.

(ii) connected graph.

* Deletion of vertex;

Graph, G

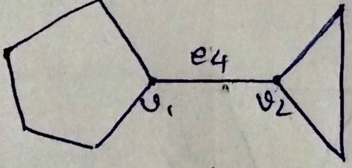
$\Rightarrow G - v = (V_1, E_1)$

$V_1 = V_1 - \{v\}$

E_1 = effects the graph [edges connected to deleted vertex are effected].

* Deletion of edge;

Eg

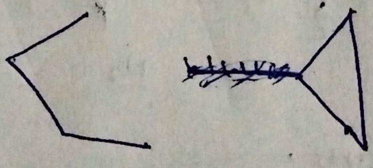


$G - e = (V_2, E_2)$

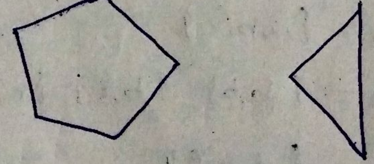
$V_2 = V$

$E_2 = E - \{e\}$

$G - v_1 \Rightarrow$



$G - e_4 \Rightarrow$



* connectedness in undirected graph:-

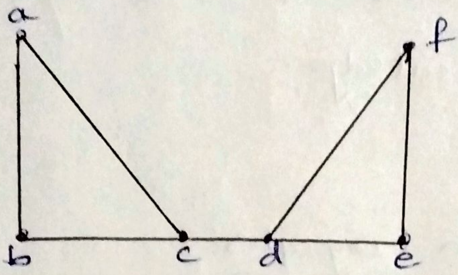
* cut vertex / cut point / articulation point :-

→ If 'G' is a connected graph & v be a vertex in 'G' then 'G-v' is a disconnected graph.

→ 'G-v' graph has more components than 'G'

* Ex:-

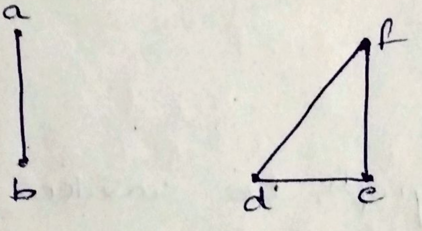
P-R :-



A) Given graph is connected [It has only one component]

cut vertices c, d :-

G-c :

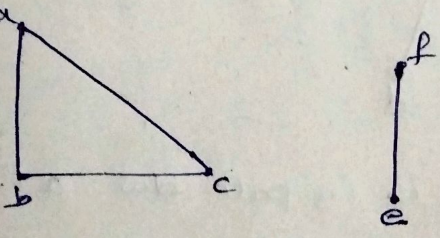


→ Disconnected graph

Here 'G-c' has 2 components

∴ 'c' is a cut vertex.

G-d :-



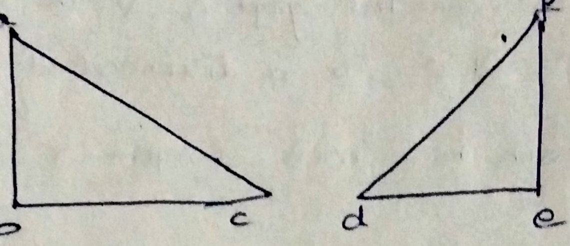
→ Disconnected graph.

∴ d is a cut-vertex

G-d has 2 components.

* cut Edge / Bridge :-

G - {c, d}



It has 2 components & it is Disconnected.

∴ {c, d} is a cut edge

* connectedness in Digraphs :-

① strongly connected :-

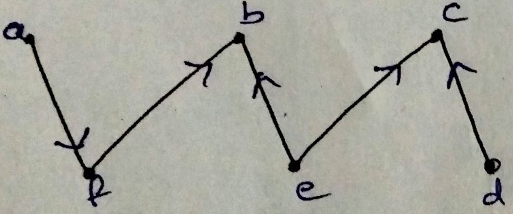
→ For every distinct pair of 2 vertices (a, b) (i.e.
and (a ≠ b) there is a path from a to b & b to a

② weakly connected :-

→ Let G be a digraph, we consider undirected
graph of G is connected.

* Exercise

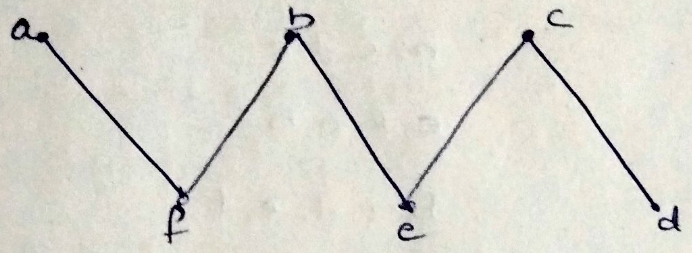
2] ⓐ



A) In this digraph, there is a path b/w a to b
but no path b/w b to a.

∴ Not strongly connected.

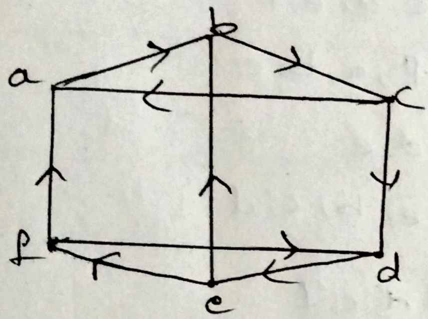
corresponding undirected graph of G is.



It is weakly connected.

ⓒ This path is Neither weakly nor strongly connected [∵ There is no path b/w a to b & b to a].

ⓑ



A)

verte to vertex	Path
a to b	a, b
b to a	b, c, a
a to c	a, b, c
c to a	c, a
a to d	a, b, c, d
d to a	d, e, f, a
a to e	a, b, c, d, e
e to a	e, f, a
a to f	a, b, c, d, e, f
f to a	f, a
b to c	b, c
c to b	c, a, b.
b to d	b, c, d

	Path
d to b	d, e, f, a, b.
b to e	b, c, d, e.
e to b	e, f, a, b.
b to f	b, c, d, e, f
f to b	f, a, b
c to e	c, d, e
e to c	e, f, a, b, c / e, b, c
c to d	c, d
d to c	d, e, b, c
c to f	c, d, e, f
f to c	f, a, b, c.
d to e	d, e
e to d	e, b, c, d
d to f	d, e, f
f to d	f, a, b, c, d
e to f	e, f
f to e.	f, a, b, c, d, e.

* strongly connected component:

→ It is need to satisfy the following.

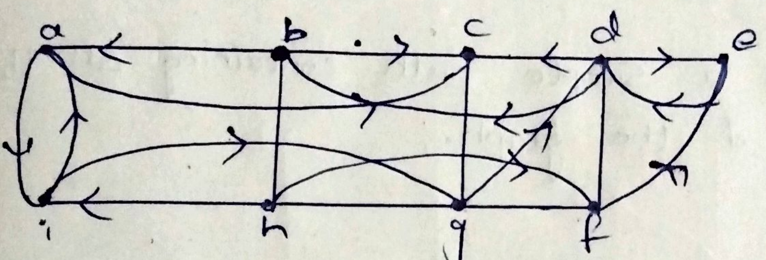
① It is a subgraph of 'G'

② Maximal graph.

③ strongly connected.

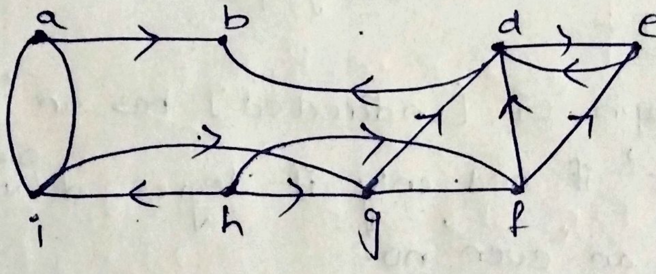
→ Every strongly connected

* Exercise:

③


A) There is no ~~gr~~ edge from c to any vertex.

so, $G - \{c\}$

⇒


Here,

① $G - \{c\}$ is strongly connected [Isolated].

② $H = (V_1, E_1)$

⇒ $V_1 = \{c\}$; $E_1 = \phi$.

Here Remaining graph is also strongly connected

∴ It has 2 components.

Thus given graph is strongly connected