$$x = 1 - \int_0^t t\,dt = 1 - \frac{t^2}{2}, \quad y = \int_0^t \left(1 - \frac{t^2}{2}\right) dt = t - \frac{t^3}{6}$$

$$x = 1 - \int_0^t \left(t - \frac{t^3}{6}\right) dt = 1 - \frac{t^2}{2} + \frac{t^4}{24}$$

$$y = \int_0^t \left(t - \frac{t^2}{2} + \frac{t^4}{24}\right) dt = t - \frac{t^3}{6} + \frac{t^5}{120},$$ **Ans.**

EXERCISE 52.2

Using Picard's method, solve the following:

1. $\dfrac{dy}{dx} = x + y^2$, given $y(0) = 0$. (*RGPV., Bhopal, June 2008*)

   Ans. $y = \dfrac{1}{2}x^2 + \dfrac{1}{20}x^5 + \dfrac{1}{160}x^8 + \dfrac{1}{4400}x^{11}$

2. Apply Picard's iteration method to find approximate solutions to the initial value problem
   $y' = 1 + y^2,\ y(0) = 0$

3. $\dfrac{dy}{dx} = x - y$, given $y(0) = 1$ and find $y(0.2)$ to five places of decimals.

   (*RGPV., Bhopal, June 2001, 2000*) **Ans.** $y = 1 - x + x^2 - \dfrac{x^3}{3} + \dfrac{x^4}{12} - \dfrac{x^5}{60} + \dfrac{x^6}{720}$, 0.83746

4. $\dfrac{dy}{dx} = y + x$, given $y(0) = 1$, find $y(1)$, **Ans.** $y = 1 + x + x^2 - \dfrac{x^3}{3} + \dfrac{x^4}{12} - \dfrac{x^5}{60} + \dfrac{x^6}{120}$ $+3.434$.

5. $\dfrac{dy}{dx} = x^2 + y^2$ ' for $y(0) = 0$, find $y(0.4)$. **Ans.** 0.0214.

6. $\dfrac{dy}{dx} = 2y - z,\ \dfrac{dz}{dx} = y + 2z$ given $y(0) = 0,\ z(0) = 1$

   Ans. $y = x + 2x^2 + \dfrac{13}{6}x^3 + \dfrac{5}{3}x^4 + \ldots,\ z = 1 + 2x + \dfrac{5}{2}x^2 + \dfrac{7}{3}x^3 + \dfrac{41}{40}x^4 + \ldots$

7. Use Picard's method to approximate y when x = 0.1, given that y = 1, when x = 0 and $\dfrac{dy}{dx} = \dfrac{y - x}{y + x}$. (*RGPV., Bhopal, III Sem. June 2003*) **Ans.** y = 1.0906.

52.5 EULER'S METHOD

This is purely numerical method for solving the first order differential equations. This is an elementary method and which will demonstrate the procedure underlying these methods. This method should not be used for practical solution.

Consider the differential equation

$$\frac{dy}{dx} = f(x, y) \qquad \ldots(1)$$

Let $y = \phi(x)$ be the solution of (1). ...(2)

Let $(x_0, y_0), (x_1, y_1) \ldots (x_n, y_n), (x_n + 1, y_{n-1})$ be the points on the curve of (2). $x_0, x_1, \ldots x_n, x_{n+1} \ldots$ are equispaced at equal interval h.

$$y_{n-1} = \phi(x_{n+1}) \qquad [(x_{n-1}, y_{n+1}) \text{ lies on (2).}]$$

$$= \phi(x_n + h) \qquad (x_{n+1} = x_n + h)$$

$$= \phi(x_n) - h\,f(x_n) - \frac{1}{2}h^2\,\phi''(x_n) + \ldots \qquad \ldots(3)$$

$$= \phi(x_n) - h\phi'(x_n) \qquad (h \text{ is very small})$$

$$= \phi(x_n) + hf(x_n, y_n) \qquad \left[\text{since}\, \frac{dy}{dx} = f(x, y)\right]$$

$$y_{n+1} = y_n + hf(x_n, y_n) \qquad [\text{since } y_n = \phi(x_n) \text{ from (2)}] \quad ...(4)$$

This formula (4) can be used to find y_{n+1}, where y_n is known.

On substituting the value of y_0, $(n = 0)$ in (4) we get y_1,

Similarly putting the value of $(n = 1)$ in (4), we obtain y_2 and so on.

Note. Since we have neglected $1/2\, h^2 \phi''(x_n)$ and higher powers of h from formula (4) there will be a larger error in y_{n+1}. Therefore it is not used in practical problems.

Geometrically

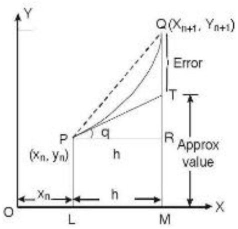
Let $y = \phi(x)$ be a solution curve PQ. The ordinate of P *i.e.* y_n is known.

Now we have to find the ordinate y_{n+1} of any point Q.

$$y_{n-1} = MQ = MR + RQ = PL + RT + TQ \qquad (TQ = \text{Error})$$

$$= y_n + h\tan\theta = y_n + h\left(\frac{dy}{dx}\right) = y_n + hf(x_n, y_n)$$

Example 8. *Using Euler's method find an approximate value of y corresponding to x = 2, given that* $\frac{dy}{dx} = x + 2y$ *and* $y = 1$ *when* $x = 1$.



Solution. $f(x, y) = x + 2y$

$$y_{n-1} = y_n + hf(x_n, y_n) = y_n + 0.1\,(x + 2y)$$

Method: In column 3 we record the value of $x + 2y$ and in column 4 we enter the sum of the value of y and the product of 0.1 with the value of $x + 2y$. This value entered in 4th column is transfered to second column for the next calculation.

x	y	$x + 2y = \frac{dy}{dx}$ old	$y + 0.1\left(\frac{dy}{dx}\right)$ = new y
1.0	1.00	3.00	1.0 + 0.1 (3) = 1.30
1.1	1.3	3.70	1.3 + 0.1 (3.7)= 1.67
1.2	1.67	4.54	1.67 + 0.1 (4.54) = 2.12
1.3	2.12	5.54	2.12 + 0.1 (5.54) = 2.67
1.4	2.67	6.74	2.67 + 0.1 (6.74) = 3.34
1.5	3.34	8.18	3.34 + 0.1 (8.18) = 4.16
1.6	4.16	9.92	4.16 + 0.1 (9.92) =5.15
1.7	5.15	12.00	5.15 + 0.1 (12.0) =6.35
1.8	6.35	14.50	6.35 + 0.1 (14.50) =7.80
1.9	7.80	17.50	7.80 + 0.1 (17.50) = 9.55
2.0	9.55		

Thus the required approximate value of $y = 9.55$ **Ans.**

EXERCISE 52.3

1. Using Euler's method, find an approximate value of corresponding to $x = 1$, given that $\frac{dy}{dx} = x + y$ and $y = 1$ when $x = 0$. **Ans.** 3.18

2. Using Euler's method, find an approximate value of y corresponding to $x = 1.4$, given $\frac{dy}{dx} = xy^{1/2}$ and $y = 1$ when $x = 1$. **Ans.** 1.49857.

3. Using Euler's method, find an approximate value of y corresponding to $x = 1.6$, given $\frac{dy}{dx} = y^2 - \frac{y}{x}$ and $y = 1$ when $x = 1$. **Ans.** 1.1351

4. Using Euler's method to solve the differential equation in six steps

   $\frac{dy}{dx} = x + y;$ $y(0) = 0$ choosing h = 0.2. (*RGPV., Bhopal, III Sem. Dec. 2003*) **Ans.** $y = 0.785984$

52.6 EULER'S MODIFIED FORMULA

In equation (3) of Art 52.14 the expansion of y_{n+1} is

$$y_{n-1} = y_n + hf(x_n, y_n) + \frac{1}{2}h^2\phi''(x_n, y_n) + \frac{1}{\underline{|3}}h^3\,\phi'''+(x_n, y_n) + \ldots \qquad \ldots(1)$$

In Euler's formula we omit $\frac{1}{2}h^2\phi''(x_n, y_n)$ and higher powers of h.

The error due to this omission is called **Truncation error.**

Now a formula is derived with small error.

Differentiating (1) w.r.t. x we get

$$\left(\frac{dy}{dx}\right)_{n+1} = \left(\frac{dy}{dx}\right)_n + hf'\,(x_n, y_n) + \frac{1}{2}h^2\phi'''(x_n, y_n) + \ldots$$

$$\therefore \qquad f(x_{n-1}, y_{n+1}) = f(x_n, y_n)\ hf'(x_n, y_n)' + \frac{1}{2}h^2\phi'''(x_n, y_n) + \ldots$$

$$= f(x_n, y_n) + h\phi'''(x_n, y_n) + \frac{1}{2}h^2\phi'''(x_n, y_n) + \ldots \qquad \ldots(2)$$

Multiplying (2) by $\frac{h}{2}$ and subtracting from (1) we get

$$y_{n-1} - \frac{1}{2}hf(x_{n-1}, y_{n-1}) = y_n + \frac{h}{2}f(x_n, y_n) - \frac{h^3}{12}\phi'''(x_n, y_n)$$

Neglecting terms containing h^3 and higher powers, we obtain

$$y_{n+1} = y_n + h\left[\frac{f(x_n, y_n) + f(x_{n+1}, y_{n+1})}{2}\right] \qquad \ldots(3)$$

Equation (3) is the Euler's modified formula.

But $f(x_{n-1}, y_{n+1})$ which occurs on the right hand side of equation (3), cannot be calculated since y_{n+1} is unknown. So first we calculate y_{n+1} from Euler's first formula.

$$y_{n+1} = y_n + hf(x_n y_n)$$

Thus for each stage we use the following two formulae.

$$y_{n+1} = y_n + hf(x_n y_n)$$

$$y_{n+1} = y_n + \frac{h}{2}f(x_n y_n)\ (y_{n+1}, y_{n+1})$$