1.1

System of Linear Equations

When mathematics is used to solve a problem it often becomes necessary to find a solution to a so-called system of linear equations. Historically, linear algebra developed from studying method for solving such equations. This module introduces methods for solving system of linear equations and includes examples of problems that reduce to solving such equations. The techniques of this module will be used throughout the remainder of the course.

Definition: A linear equation is an algebraic equation in which each term is either a constant or the product of a constant and the first power of a single variable.

Example: $x + 3y = 9$.

The graph of this equation is a straight line in the $x - y$ plane.

Note: Linear equations can have one or more variables.

Consider a system of two linear equations

$$x + 3y = 9$$

$$-2x + y = -4$$

A pair of values of x and y that satisfy both the equations is called a solution. It can be seen by substitution that $x = 3, y = 2$ is a solution to this system. A solution to such a system will be a point at which the graphs of the two equations intersect. The following examples illustrate that three possibilities can arise for such system of equations. There can be a unique solution, no solution, or many solutions.

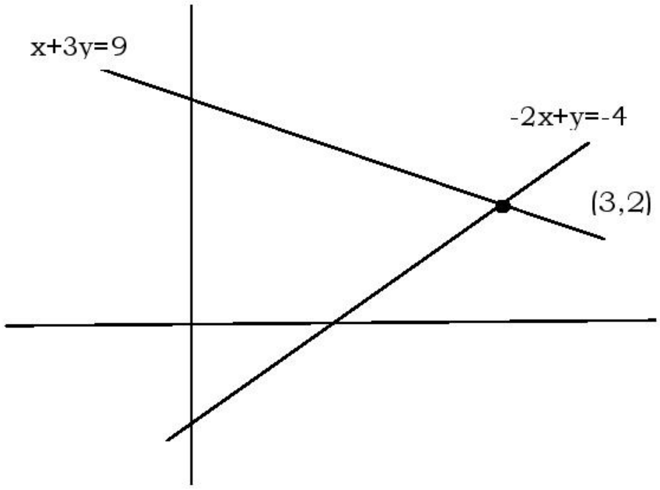
Unique Solution

$$x + 3y = 9$$

$$-2x + y = -4$$

Lines intersect at (3,2).

Unique solution, $x = 3, y = 2$.

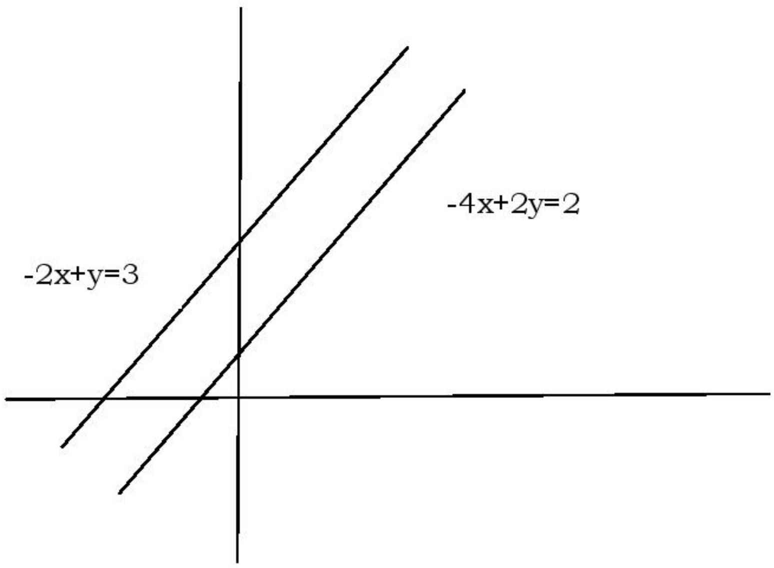


No Solution

$$-2x + y = 3$$

$$-4x + 2y = 2$$

Lines are parallel. No point of intersection.

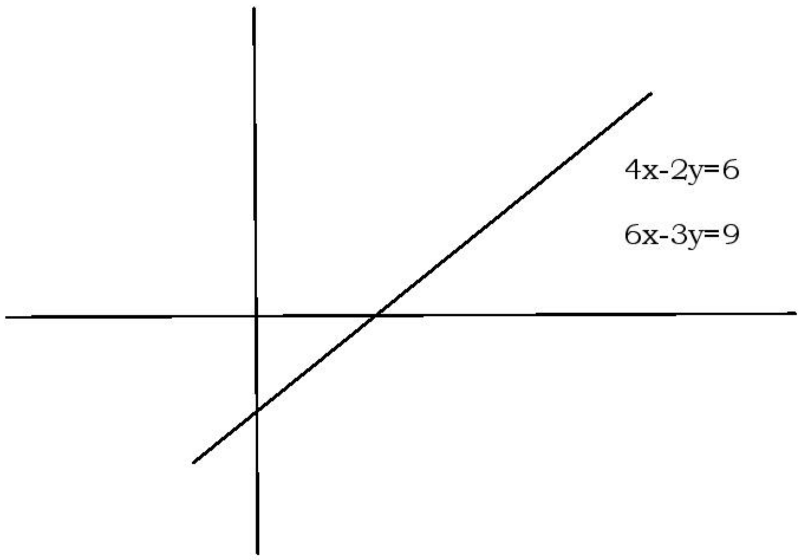


Many Solutions

$$4x - 2y = 6$$

$$6x - 3y = 9$$

Both equations have the same graph. Any point of the graph is a solution.



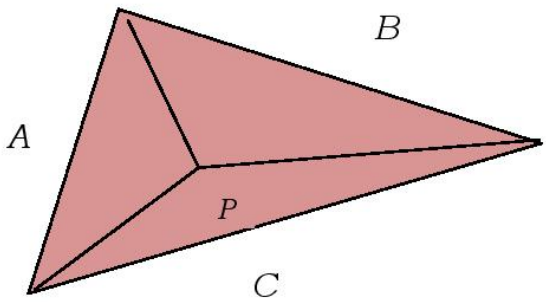
Our aim in this module is to analyze larger system of linear equation. The following is an example of a system of three linear equations.

$$x_1 + x_2 + x_3 = 2$$

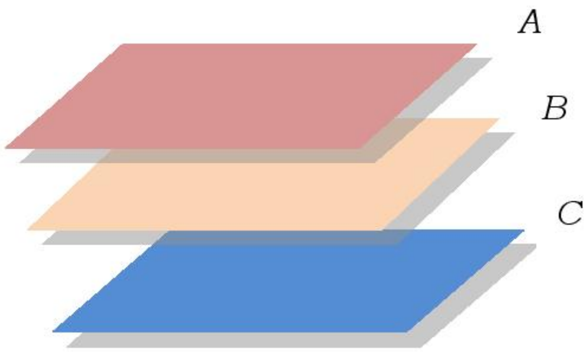
$$2x_1 + 3x_2 + x_3 = 3$$

$$x_1 - x_2 - 2x_3 = -6$$

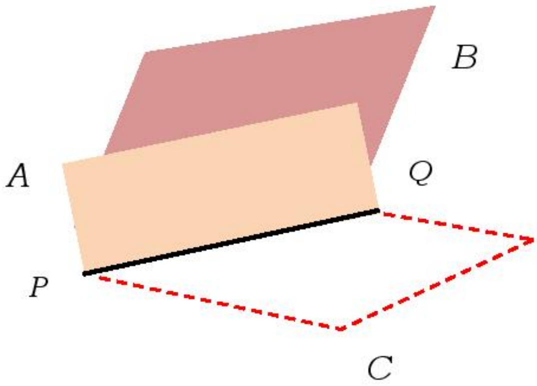
A linear equation in three variables corresponds to a plane in three-dimensional space. Solutions will be points that lie on all three planes. As for systems of two equations there can be a unique solution, no solution, or many solutions.



Three planes A, B and C intersects at a single point P.
P corresponds to a unique solution.



Planes A, B and C have no points in common.
There is no solution.



Three planes A, B and C intersects in a line PQ.
Any point on the line is a solution

As the number of variables increases, a geometrical interpretation of such a system of equations becomes increasingly complex. Each equation will represent a space embedded in a larger space. Solutions will be points that lie on all the embedded space. A geometrical approach to visualizing solutions becomes impractical. We have to rely solely on algebraic methods. We introduce a method for solving systems, of linear equations called Gauss-Jordan elimination.

Gauss-Jordan Elimination:

The general form of system of linear equations is

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$

$$.$$

$$.$$

$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m$$

The matrix representation of the above system of linear equations is

$$\begin{bmatrix} a_{11} & a_{12} \ldots\ldots & a_{1n} \\ a_{21} & a_{22} \ldots\ldots & a_{2n} \\ a_{m1} & a_{m2} \ldots\ldots & a_{mn} \end{bmatrix}_{m\times n} \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ x_n \end{bmatrix}_{n\times 1} = \begin{bmatrix} b_1 \\ b_2 \\ . \\ . \\ . \\ b_m \end{bmatrix}_{m\times 1}$$

i.e., $AX = B$

A is called coefficient matrix.

The matrix $[A \quad B] = \begin{bmatrix} a_{11} & a_{12} \ldots\ldots & a_{1n} & b_1 \\ a_{21} & a_{22} \ldots\ldots & a_{2n} & b_2 \\ a_{m1} & a_{m2} \ldots\ldots & a_{mn} & b_m \end{bmatrix}$

is the augmented matrix of the system of linear equations.

Definition: A matrix is in reduced echelon form if

1. Any row consisting entirely of zeros is grouped at the bottom of the matrix.
2. The first non-zero element of each row is 1. This element is called leading coefficient or pivot.
3. The leading coefficient of each row after the first is positioned to the right of the leading coefficient of the previous row.
4. All other elements in a column that contains a leading coefficient are zero.

Note: If we give relaxation to the condition that leading coefficient is 1 and to the point 4, then the form of the matrix is called echelon form.

Example: The following matrices are all in reduced echelon form.

$$\begin{bmatrix}1 & 0 & 8\\0 & 1 & 2\\0 & 0 & 0\end{bmatrix}, \begin{bmatrix}1 & 0 & 0 & 7\\0 & 1 & 0 & 3\\0 & 0 & 1 & 9\end{bmatrix}, \begin{bmatrix}1 & 4 & 0 & 0\\0 & 0 & 1 & 0\\0 & 0 & 0 & 1\end{bmatrix}, \begin{bmatrix}1 & 2 & 3 & 0\\0 & 0 & 0 & 1\\0 & 0 & 0 & 0\end{bmatrix}$$

The following matrixes are not in reduced echelon form.

$$\begin{bmatrix}1 & 2 & 0 & 4\\0 & 0 & 0 & 0\\0 & 0 & 1 & 3\end{bmatrix}$$

Row of zeros not at bottom of matrix

$$\begin{bmatrix}1 & 2 & 0 & 3 & 0\\0 & 0 & 3 & 4 & 0\\0 & 0 & 0 & 0 & 1\end{bmatrix}$$

First non-zero element is row 2 is not 1.

$$\begin{bmatrix}1 & 0 & 0 & 2\\0 & 0 & 1 & 4\\0 & 1 & 0 & 3\end{bmatrix}$$

Leading 1 in row 3 not to the right of leading 1 in row 2.

$$\begin{bmatrix}1 & 7 & 0 & 8\\0 & 1 & 0 & 3\\0 & 0 & 1 & 2\\0 & 0 & 0 & 0\end{bmatrix}$$

Non zero element above leading 1 in row 2

There are usually many sequences of row operation that can be used to transform a given matrix to reduced echelon form they all, however, lead to the same reduced echelon form. We say that the reduced echelon form of a matrix is unique.

Gauss-Jordan Elimination:

1. Write down the augmented matrix of the system of linear equations.
2. Derive the reduced echelon form of the augmented matrix using elementary row operations.
3. Write down the system of equations corresponding to the reduced echelon form. This system gives the solution.

Problem 1: Solve the following system of linear equations.

$$x_1 - 2x_2 + 4x_3 = 12$$

$$2x_1 - x_2 + 5x_3 = 18$$

$$-x_1 + 3x_2 - 3x_3 = -8$$

Solution:

Step 1:

Start with the augmented matrix and use the first row to create zeros in the first column (This corresponds to using the first equation to eliminate x_1 from the second and third equations).

$$\begin{bmatrix} 1 & -2 & 4 & 12 \\ 2 & -1 & 5 & 18 \\ -1 & 3 & -3 & -8 \end{bmatrix} \begin{matrix} \approx \\ R2+(-2)R1 \\ R3+R1 \end{matrix} \begin{bmatrix} 1 & -2 & 4 & 12 \\ 0 & 3 & -3 & -6 \\ 0 & 1 & 1 & 4 \end{bmatrix}$$

Step 2:

Next, multiply row 2 by $\frac{1}{3}$ to make the (3,3) element 1. (This corresponds to making the coefficient of x_2 in the second equation 1.)

$$\begin{matrix} \approx \\ \left(\frac{1}{3}\right)R2 \end{matrix} \begin{bmatrix} 1 & -2 & 4 & 12 \\ 0 & 1 & -1 & -2 \\ 0 & 1 & 1 & 4 \end{bmatrix}$$

Step 3:

Create zeros in the second column as follows. (This corresponds to using the second equation to eliminate x_2 from the first and third equations.)

$$\begin{array}{c} \approx \\ R1+(2)R2 \\ R3+(-1)R2 \end{array} \begin{bmatrix} 1 & 0 & 2 & 8 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 2 & 6 \end{bmatrix}$$

Step 4:

Multiply row 3 by $\frac{1}{2}$. (This corresponds to making the coefficient of in the third equation 1.)

$$\begin{array}{c} \approx \\ \left(\frac{1}{2}\right)R3 \end{array} \begin{bmatrix} 1 & 0 & 2 & 8 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Finally, create zeros in the third column. (This corresponds to using the third equation to eliminate x_3 from the first and second equations.)

$$\begin{array}{c} \approx \\ R1+(-2)R3 \\ R2+R3 \end{array} \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

This matrix corresponds to the system

$$x_1 \quad = 2$$

$$x_2 = 1$$

$$x_3 = 3$$

The solution is $x_1 = 2,\ x_2 = 1,\ x_3 = 3$.

Problem 2: Solve, if possible, the system of equations.

$$2x_3 - 2x_4 = 2$$

$$3x_1 + 3x_2 - 3x_3 + 9x_4 = 12$$

$$4x_1 + 4x_2 - 2x_3 + 11x_4 = 12$$

Solution:

Step 1:

We interchange rows, if necessary, to bring a nonzero element to the top of the first nonzero column. This nonzero element is called a pivot.

Pivot

$$\underset{R1 \leftrightarrow R2}{\approx} \begin{bmatrix} 3 & 3 & -3 & 9 & 12 \\ 0 & 0 & 2 & -2 & 2 \\ 4 & 4 & -2 & 11 & 12 \end{bmatrix}$$

Step 2:

Create a 1 in the pivot location by multiplying the pivot row by $\frac{1}{pivot}$

$$\underset{\left(\frac{1}{3}\right)R1}{\approx} \begin{bmatrix} 1 & 1 & -1 & 3 & 4 \\ 0 & 0 & 2 & -2 & 2 \\ 4 & 4 & -2 & 11 & 12 \end{bmatrix}$$

Step 3:

Create zeros elsewhere in the pivot column by adding suitable multiple of the pivot row to all other rows of the matrix

$$\underset{R3+(-4)R1}{\approx} \begin{bmatrix} 1 & 1 & -1 & 3 & 4 \\ 0 & 0 & 2 & -2 & 2 \\ 0 & 0 & 2 & -1 & -4 \end{bmatrix}$$

Step 4:

Cover the pivot row and all rows above it. Repeat steps 1 and 2 for the remaining sub Matrix. Repeat step 3 for the whole matrix. Continue until the reduced echelon from is reached.

Pivot

$$\begin{bmatrix} 1 & 1 & -1 & 3 & 4 \\ 0 & 0 & 2 & -2 & 2 \\ 0 & 0 & 2 & -1 & -4 \end{bmatrix}$$

$$\underset{\left(\frac{1}{2}\right)R2}{\approx} \begin{bmatrix} 1 & 1 & -1 & 3 & 4 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 2 & -1 & -4 \end{bmatrix}$$

$$\underset{\substack{R1+R2 \\ R3+(-2)R2}}{\approx} \begin{bmatrix} 1 & 1 & 0 & 2 & 5 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 & -6 \end{bmatrix}$$

Pivot

$$\begin{array}{c} \approx \\ R1+(-2)R3 \\ R2+R3 \end{array} \quad \begin{bmatrix} 1 & 1 & 0 & 0 & 17 \\ 0 & 0 & 1 & 0 & -5 \\ 0 & 0 & 0 & 1 & -6 \end{bmatrix}$$

This matrix is the reduced echelon form of the given matrix. The corresponding system of equations is

$$x_1 + x_2 = 17$$

$$x_3 = -5$$

$$x_4 = -6$$

Let us assign an arbitrary value r to x_1. The general solution to the system of equation is

$x_1 = r, x_2 = 17 - r, x_3 = -5$, and $x_4 = -6$.

As r ranges over the set of real numbers we get many solutions.

Problem 3: solve, if possible, the system of equations

$$3x_1 - 3x_2 + 3x_3 = 9$$

$$2x_1 - x_2 + 4x_3 = 7$$

$$3x_1 - 5x_2 - x_3 = 7$$

Solution: Start with the augmented matrix and follow the Gauss-Jordan algorithm. Pivots and leading 1s are circled

$$\begin{bmatrix} 3 & -3 & 3 & 9 \\ 2 & -1 & 4 & 7 \\ 3 & -5 & -1 & 7 \end{bmatrix} \underset{\left(\frac{1}{3}\right)R1}{\approx} \begin{bmatrix} 1 & -1 & 1 & 3 \\ 2 & -1 & 4 & 7 \\ 3 & -5 & -1 & 7 \end{bmatrix}$$

$$\underset{\substack{R2+(-2)R1 \\ R3+(-3)R1}}{\approx} \begin{bmatrix} 1 & -1 & 1 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & -2 & -4 & -2 \end{bmatrix} \underset{\substack{R1+R2 \\ R3+2R2}}{\approx} \begin{bmatrix} 1 & 0 & 3 & 4 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We have arrived at the reduced echelon form. The corresponding system of equations is

$$x_1 + 3x_3 = 4$$

$$x_2 + 2x_3 = 1$$

There are many values of x_1, x_2 and x_3 that satisfy these equations. This is a system of equations that has many solutions. x_1 is called the leading variable of the first equation and x_2 is the leading variable of the second equation. To express these many solutions, we write the leading variables in each equation in terms of the remaining variables. We get

$$x_1 = -3x_3 + 4$$

$$x_2 = -2x_3 + 1$$

Let us assign the arbitrary value r to x_3. The **general solution** to the system is

$$x_1 = -3r + 4, x_2 = -2r + 1, \ x_3 = r$$

As r ranges over the set of real numbers we get many solutions. r is called a parameter. We can get specific solutions by giving r different values. For example,

$r = 1$ gives $x_1 = 1, \ x_2 = -1, x_3 = 1$

r = -2 gives $x_1 = 10, \ x_2 = -5, x_3 = -2$.

Problem 4: Solve, if possible, the system of equations.

$$x_1 - x_2 + 2x_3 = 3$$

$$2x_1 - 2x_2 + 5x_3 = 4$$

$$x_1 + 2x_2 - x_3 = -3$$

$$2x_2 + 2x_3 = 1$$

Solution: Starting with the augmented matrix we get

$$\begin{bmatrix} 1 & -1 & 2 & 3 \\ 2 & -2 & 5 & 4 \\ 1 & 2 & -1 & -3 \\ 0 & 2 & 2 & 1 \end{bmatrix} \underset{\substack{R2+(-2)R1 \\ R3+(-1)R1}}{\approx} \begin{bmatrix} 1 & -1 & 2 & 3 \\ 0 & 0 & 1 & -2 \\ 0 & 3 & -3 & -6 \\ 0 & 2 & 2 & 1 \end{bmatrix}$$

$$\underset{R2 \leftrightarrow R3}{\approx} \begin{bmatrix} 1 & -1 & 2 & 3 \\ 0 & 3 & -3 & -6 \\ 0 & 0 & 1 & -2 \\ 0 & 2 & 2 & 1 \end{bmatrix}$$

$$\underset{\left(\frac{1}{3}\right)R2}{\approx} \begin{bmatrix} 1 & -1 & 2 & 3 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 1 & -2 \\ 0 & 2 & 2 & 1 \end{bmatrix}$$

$$\underset{\substack{R1+R2 \\ R4+(-2)R2}}{\approx} \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 4 & 5 \end{bmatrix}$$

$$\underset{\substack{R1+(-1)R3 \\ R2+R3 \\ R4+(-4)R3}}{\approx} \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 13 \end{bmatrix}$$

$$\underset{\left(\frac{1}{13}\right)R4}{\approx} \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

This matrix is still not in reduced echelon form; zeros still have to be created above the 1 in the last row. However, in such a situation, when the last nonzero row of a matrix is of the form (0 0 . . . 0 1), there is no need to proceed further. They system has no solution. To see this, let us write down the equation that corresponds to the last row of the matrix.

$$0x_1 + 0x_2 + 0x_3 = 1$$

This equation cannot be satisfied by any values of x_1, x_2 and x_3. Thus the system of equation has no solution.

Exercise:

1. Determine the matrix of coefficients and augmented matrix of each following systems of equations.

(a)
$$\begin{aligned} x_1 + 3x_2 &= 7 \\ 2x_1 - 5x_2 &= -3 \end{aligned}$$

(b)
$$\begin{aligned} -x_1 + 3x_2 - 5x_3 &= -3 \\ 2x_1 - 2x_2 + 4x_3 &= 8 \\ x_1 + 3x_2 \qquad\quad &= 6 \end{aligned}$$

(c)
$$\begin{aligned} 5x_1 + 2x_2 - 4x_3 &= 8 \\ 4x_2 + 3x_3 &= 0 \\ x_1 \qquad\quad - x_3 &= 6 \end{aligned}$$

2. Interpret the following matrices as augmented matrices of systems of equations. Write down each system of equations.

(a) $\begin{bmatrix} 7 & 9 & 8 \\ 6 & 4 & -3 \end{bmatrix}$

(b) $\begin{bmatrix} 8 & 7 & 5 & -1 \\ 4 & 6 & 2 & 4 \\ 9 & 3 & 7 & 6 \end{bmatrix}$

(c) $\begin{bmatrix} 0 & -2 & 4 \\ 5 & 7 & -3 \\ 6 & 0 & 8 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 2 & -1 & 6 \\ 0 & 1 & 4 & 5 \\ 0 & 0 & 1 & -2 \end{bmatrix}$

3. In the following exercises you are given a matrix followed by an elementary row operation. Determine each resulting matrix.

(a) $\begin{bmatrix} 2 & 6 & -4 & 0 \\ 1 & 2 & -3 & 6 \\ 8 & 3 & 2 & 5 \end{bmatrix} \quad \begin{matrix} \approx \\ \left(\frac{1}{2}\right) R1 \\ \end{matrix}$

(b) $\begin{bmatrix} 1 & 2 & 3 & -1 \\ -1 & 1 & 7 & 1 \\ 2 & -4 & 5 & -3 \end{bmatrix} \quad \begin{matrix} \approx \\ R2+R1 \\ R3+(-2)R1 \end{matrix}$

(c) $\begin{bmatrix} 1 & 0 & 4 & -3 \\ 0 & 1 & -3 & 2 \\ 0 & 0 & 1 & 5 \end{bmatrix} \quad \begin{matrix} \approx \\ R1+(-4)R3 \\ R2+3R3 \end{matrix}$

4. The following systems of equations all have unique solutions. Solve these systems using the method of Gauss-Jordan elimination with matrices.

(a)
$$\begin{aligned} 2x_2 + 4x_3 &= 8 \\ 2x_1 + 2x_2 &= -3 \\ x_1 + x_2 + x_3 &= 5 \end{aligned}$$

(b) $$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 14 \\ 2x_1 + 5x_2 + 8x_3 &= 36 \\ x_1 - x_2 &= -4 \end{aligned}$$

(c) $$\begin{aligned} 2x_1 + 2x_2 - 4x_3 &= 14 \\ 3x_1 + x_2 + x_3 &= 8 \\ 2x_1 - x_2 + 2x_3 &= -1 \end{aligned}$$

5. Determine the following matrices are in reduced echelon form. If a matrix is not in reduced echelon form, give a reason.

(a) $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 2 & 5 & 6 \\ 0 & 1 & 3 & -7 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 0 & 0 & 3 & 2 \\ 0 & 2 & 0 & 6 & 1 \\ 0 & 0 & 1 & 2 & 3 \end{bmatrix}$

6. Determine the following matrices are in reduced echelon form. If a matrix is not in reduced echelon form, give a reason.

(a) $\begin{bmatrix} 1 & 0 & 3 & -2 \\ 0 & 0 & 1 & 8 \\ 0 & 1 & 4 & 9 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 5 & 0 & 2 & 0 \\ 0 & 0 & 1 & 9 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 & 2 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 7 \\ 0 & 0 & 0 & 1 & 3 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 0 & 0 & 5 & 3 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 1 & 2 & 3 & 7 \end{bmatrix}$

(e) $\begin{bmatrix} 1 & 5 & -3 & 0 & 7 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

7. Solve (if possible) each the following systems of equations using the method of Gauss-Jordan elimination.

(a)
$$\begin{aligned} x_1+2x_2-\ x_3-\ x_4 &= 0 \\ x_1+2x_2\qquad\quad +x_4 &= 4 \\ -x_1-2x_2+2x_3+4x_4 &= 5 \end{aligned}$$

(b)
$$\begin{aligned} x_2 - 3x_3 + x_4 &= 0 \\ x_1 + x_2 - x_3 - 4x_4 &= 0 \\ -2x_1 - 2x_2 + 2x_3 - 8x_4 &= 0 \end{aligned}$$

(c)
$$\begin{aligned} x_1 - x_2 - 2x_3 &= 7 \\ 2x_1 - 2x_2 + 2x_3 - 4x_4 &= 12 \\ -x_1 + x_2 - x_3 + 2x_4 &= -4 \\ -3x_1 + x_2 - 8x_3 - 10x_4 &= -29 \end{aligned}$$

Answers:

1. (a) $\begin{bmatrix} 1 & 3 \\ 2 & -5 \end{bmatrix}$ (b) $\begin{bmatrix} -1 & 3 & -5 \\ 2 & -2 & 4 \\ 1 & 3 & 0 \end{bmatrix}$ (c) $\begin{bmatrix} 5 & 2 & -4 \\ 0 & 4 & 3 \\ 1 & 0 & -1 \end{bmatrix}$

2. (a)
$$\begin{aligned} 7x_1 + 9x_2 &= 8 \\ 6x_1 + 4x_2 &= -3 \end{aligned}$$

(b)
$$\begin{aligned} 8x_1 + 7x_2 + 5x_3 &= -1 \\ 4x_1 + 6x_2 + 2x_3 &= 4 \\ 9x_1 + 3x_2 + 7x_3 &= 6 \end{aligned}$$

(c)
$$\begin{aligned} -2x_2 &= 4 \\ 5x_1 + 7x_2 &= -3 \end{aligned}$$

$$6x_1 \qquad\quad = 8$$

(d) $x_1 + 2x_2 - x_3 = 6$

$$x_2 + 4x_3 = 5$$

$$x_3 = -2$$

3. (a) $\begin{bmatrix} 1 & 3 & -2 & 0 \\ 1 & 2 & -3 & 6 \\ 8 & 3 & 2 & 5 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 3 & 10 & 0 \\ 0 & -8 & -1 & -1 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 0 & 0 & -23 \\ 0 & 1 & 0 & 17 \\ 0 & 0 & 1 & 5 \end{bmatrix}$

4. (a) $x_1 = 3, x_2 = 0, x_3 = 2$ (b) $x_1 = 0, x_2 = 4, x_3 = 2$

(c) $x_1 = 2, x_2 = 3, x_3 = -1$

5. (a) Yes (b) No (c) Yes (d) No

6. (a) No (b) Yes (c) No (d) No (e) Yes

7. (a) $x_1 = 3 - 2r, x_2 = r, x_3 = 2, and\ x_4 = 1$

(b) $x_1 = -2r - 3s, x_2 = 3r - s, x_3 = r, x_4 = s$

(c) No solution