UNIT - III

Multiple Integrals

(a) Double Integrals

(b) Double Integrals in Polar. Coordinates

(c) Change of Variables from Cartesian to Polar - Coordinates

(d) Change of order of Integration

(e) Triple Integrals

(f) Area Enclosed by a curve

(g) Volume as a Double Integral

(h) Volume as a Triple Integral

(i) Change of Variable in Triple Integral

Assignment problems
objective problems } included.

Formulas

Chapter 7 Class 12

Integration **Formula Sheet**

by teachoo.com

Basic Formulae

1. $\int x^n \, dx = \frac{x^{n+1}}{n+1} + C, n \neq -1.$ *Particularly* , $\int dx = x + c$

2. $\int \cos x \; dx = \sin x + C$

3. $\int \sin x \; dx = -\cos x + C$

4. $\int \sec^2 x \, dx = \tan x + c$

5. $\int \text{cosec}^2 x \, dx = -\cot x + c$

6. $\int \sec x \tan x \, dx = \sec x + c$

7. $\int \text{cosec}\, x \cot x \, dx = -\text{cosec}\, x + c$

8. $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + c$

9. $\int \frac{dx}{\sqrt{1-x^2}} = -\cos^{-1} x + c$

10. $\int \frac{dx}{1+x^2} = \tan^{-1} x + c$

Questions in **Ex 7.2** and **Ex 7.3**

11. $\int \frac{dx}{1+x^2} = -\cot^{-1} x + c$

12. $\int e^x \, dx = e^x + c$

$\int e^{ax} dx = \frac{e^{ax}}{a} + c$

13. $\int a^x dx = \frac{a^x}{\log a} + c$

14. $\int \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1} x + c$

15. $\int \frac{dx}{x\sqrt{x^2-1}} = -\text{cosec}^{-1} x + c$

16. $\int \frac{1}{x} dx = \log |x| + c$

17. $\int \tan x \, dx = \log |\sec x| + c$

18. $\int \cot x \, dx = \log |\sin x| + c$

19. $\int \sec x \, dx = \log |\sec x + \tan x| + c$

20. $\int \text{cosec}\, x \, dx = \log |\text{cosec}\, x - \cot x| + c$

$\boxed{a = 1}$

Integrals of some special functions

1. $\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \log \left| \frac{x - a}{x + a} \right| + c$

2. $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \log \left| \frac{a + x}{a - x} \right| + c$

3. $\int \frac{dx}{x^2 + a^2} = \frac{1}{a} \tan^{-1} \frac{x}{a} + c$

4. $\int \frac{dx}{\sqrt{x^2 - a^2}} = \log \left| x + \sqrt{x^2 - a^2} \right| + C$

5. $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \frac{x}{a} + c$

$\int \frac{1}{\sqrt{1 - x^2}} dx = \sin^{-1} x + c$

6. $\int \frac{dx}{\sqrt{x^2 + a^2}} = \log \left| x + \sqrt{x^2 + a^2} \right| + C$

Questions in **Ex 7.4**

Integrals by partial fractions

1. $\dfrac{px+q}{(x-a)(x-b)} = \dfrac{A}{x-a} + \dfrac{B}{x-b}$, $a \neq$ b

2. $\dfrac{px+q}{(x-a)^2} = \dfrac{A}{x-a} + \dfrac{B}{(x-a)^2}$

3. $\dfrac{px^2+qx+r}{(x-a)(x-b)(x-c)} = \dfrac{A}{x-a} + \dfrac{B}{x-b} + \dfrac{C}{x-c}$

4. $\dfrac{px^2+qx+r}{(x-a)^2(x-b)} = \dfrac{A}{x-a} + \dfrac{B}{(x-a)^2} + \dfrac{C}{x-b}$

5. $\dfrac{px^2+qx+r}{(x-a)(x^2+bx+c)} = \dfrac{A}{x-a} + \dfrac{Bx+C}{x^2+bx+c}$

Where x^2+ bx + c can not be factorised further.

Questions in **Ex 7.5**

Integration by parts

1. $\int f(x)\, g(x)\ dx = f(x) \int g(x)\, dx - \int \left(f'(x) \int g(x)\, dx\right) dx$

To decide first function. We use

I → Inverse *(Example $\sin^{-1} x$)*

L → Log *(Example $\log x$)*

A → Algebra *(Example x^2, x^3)*

T → Trignometry *(Example $\sin^2 x$)*

E → Exponential *(Example e^x)*

Questions in **Ex 7.6**

2. $\int e^x\ [f(x) + f'(x)]$ dx = $\int e^x$ f(x) dx + C

Other Special Integrals

1. $\int \sqrt{\boldsymbol{x^2 - a^2}}\, \boldsymbol{dx} = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\log\left|x + \sqrt{x^2 - a^2}\right| + C$

2. $\int \sqrt{\boldsymbol{x^2 + a^2}}\, \boldsymbol{dx} = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2}\log\left|x + \sqrt{x^2 + a^2}\right| + C$

3. $\int \sqrt{\boldsymbol{a^2 - x^2}}\, \boldsymbol{dx} = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{1}\frac{x}{a} + C$

Questions in
Ex 7.7

Integral of the form

$$\int (px+q)\sqrt{ax^2+bx+c}\,dx$$

We solve this using a specific method.

1. First we write

$$\text{px + q = A}\left(\frac{d\left(\sqrt{ax^2+bx+c}\right)}{dx}\right)+\text{B}$$

2. Then we find A and B
3. Our equation becomes two seperate identities and then we solve.

Some examples are

- $(x+3)\sqrt{3-4x-x^2}$ - View solution
- $x\sqrt{1+x-x^2}\,dx$ – View Solution

Questions in **Ex 7.7**

Area as a sum

$$\int f(x)\,dx$$

$$= (b-a)\lim_{n\to\infty}\frac{1}{n}\left(f(a)+f(a+h)+f(a+2h)\ldots+\right.$$

Some other results

(1) $\int_0^{\infty} e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$

(2)

Questions in
Ex 7.8

Properties of definite integration

P_0 : $\int_a^b f(x)dx = \int_a^b f(t)dt$

P_1 : $\int_a^b f(x)dx = -\int_b^a f(x)dx$. In particular, $\int_a^a f(x)dx = 0$

P_2 : $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$

P_3 : $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx.$

P_4 : $\int_0^a f(x)dx = \int_0^a f(a-x)dx$

Questions in **Ex 7.11**

P_5 : $\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_0^a f(2a-x)dx$

P_6 : $\int_0^{2a} f(x) = \begin{cases} 2\int_0^a f(x)dx, & if\ f(2a-x) = f(x) \\ 0, & if\ f(2a-x) = -f(x) \end{cases}$

P_6 : $\int_{-a}^{a} f(x) = \begin{cases} 2\int_0^a f(x)dx, & if\ f(-x) = f(x) \\ 0, & if\ f(-x) = -f(x) \end{cases}$

Problems on Double integrals

① Evaluate $\int_0^2 \int_0^x y \, dx \, dy$

Sol Given $\int_{x=0}^{2} \int_{y=0}^{x} y \, dx \, dy$

$$\int_{x=0}^{2} \left[\int_{y=0}^{x} y \, dy \right] dx$$

$$\int_{x=0}^{2} \left[\frac{y^2}{2} \right]_0^x dx$$

$$\int_{x=0}^{2} \left[\frac{x^2}{2} - 0 \right] dx$$

$$= \int_{x=0}^{2} \frac{x^2}{2} \, dx$$

$$= \frac{1}{2} \left[\frac{x^3}{3} \right]_0^2$$

$$= \frac{1}{2}\left[\frac{2^3}{3} - 0\right]$$

$$= \frac{4}{3}$$

$$\boxed{\int_0^2 \int_0^x y \, dx \, dy = \frac{4}{3}}$$

② Evaluate $\int_{y=0}^{2} \int_{x=0}^{3} xy \, dx \, dy$

Sol: Given $\int_{y=0}^{2} \int_{x=0}^{3} x y \, dx \, dy$

$$\int_{y=0}^{2} \left[\int_{x=0}^{3} x \, dx\right] y \, dy$$

$$= \int_{y=0}^{2} \left[\frac{x^2}{2}\right]_0^3 y \, dy$$

$$= \int_{y=0}^{2} \left[\frac{3^2}{2} - 0 \right] y \, dy$$

$$= \frac{9}{2} \int_{y=0}^{2} y \, dy$$

$$= \frac{9}{2} \left[\frac{y^2}{2} \right]_0^2$$

$$= \frac{9}{2} \left[\frac{2^2}{2} - 0 \right]$$

$$= 9$$

$$\boxed{\int_{y=0}^{2} \int_{x=0}^{3} xy \, dx \, dy = 9}$$

③ Evaluate $\int_{0}^{1}\int_{x}^{\sqrt{x}}(x^2+y^2)\,dx\,dy$

Sol

given $\int_{x=0}^{1}\int_{y=x}^{\sqrt{x}}(x^2+y^2)\,dx\,dy$

$$\int_{x=0}^{1}\left[\int_{y=x}^{\sqrt{x}}(x^2+y^2)\,dy\right]dx$$

$$\int_{x=0}^{1}\left[x^2y+\frac{y^3}{3}\right]_{x}^{\sqrt{x}}dx$$

$$\int_{x=0}^{1}\left[\left(x^2\sqrt{x}+\frac{(\sqrt{x})^3}{3}\right)-\left(x^2\cdot x+\frac{x^3}{3}\right)\right]dx$$

$$=\int_{x=0}^{1}\left[\left(x^{5/2}+\frac{x^{3/2}}{3}\right)-\left(x^3+\frac{x^3}{3}\right)\right]dx$$

$$\int x^n\,dx=\frac{x^{n+1}}{n+1}$$

$$= \left[\left(\frac{x^{\frac{5}{2}+1}}{\frac{5}{2}+1} + \frac{1}{3} \cdot \frac{x^{\frac{3}{2}+1}}{\frac{3}{2}+1} \right) - \left(\frac{x^4}{4} + \frac{1}{3} \cdot \frac{x^4}{4} \right) \right]_0^1$$

$$= \left(\frac{2}{7} + \frac{1}{3} \cdot \frac{2}{5} \right) - \left(\frac{1}{4} + \frac{1}{3} \cdot \frac{1}{4} \right)$$

$$= \frac{2}{7} + \frac{2}{15} - \frac{1}{4} - \frac{1}{12}$$

$$= \frac{3}{35}$$

$$\boxed{\int_0^1 \int_x^{\sqrt{x}} (x^2 + y^2)\, dx\, dy = \frac{3}{35}}$$

④ Evaluate $\int_0^5 \int_0^{x^2} x(x^2+y^2)\, dx\, dy$

sol given $\int_{x=0}^{5} \int_{y=0}^{x^2} [x^3 + xy^2]\, dx\, dy$

$$\int_{x=0}^{5} \left[\int_{y=0}^{x^2} [x^3 + xy^2]\, dy \right] dx$$

$$\int_{x=0}^{5} \left[x^3 y + x \cdot \frac{y^3}{3} \right]_0^{x^2} dx$$

$$\int_{x=0}^{5} \left[x^3 \cdot x^2 + \frac{x}{3} \cdot (x^2)^3 \right] dx$$

$$\int_{x=0}^{5} \left[x^5 + \frac{x^7}{3} \right] dx$$

$$= \left[\frac{x^6}{6} + \frac{1}{3} \frac{x^8}{8} \right]_0^5$$

$$= \left[\frac{5^6}{6} + \frac{1}{24}\cdot 5^8\right]$$

$$= \frac{29(5^6)}{24}$$

$$\boxed{\int_0^5 \int_0^{x^2} x(x^2+y^2)\,dx\,dy = \frac{29\times 5^6}{24}}$$

(5) Evaluate $\int_0^4 \int_0^{x^2} e^{y/x}\,dx\,dy$

Sol given $\int_{x=0}^4 \int_{y=0}^{x^2} e^{y/x}\,dx\,dy$

$$\int_{x=0}^4 \left[\int_{y=0}^{x^2} e^{y/x}\,dy\right] dx$$

$$\boxed{\int e^{ay}\,dy = \frac{e^{ay}}{a} \qquad a = \frac{1}{x}}$$

$$\int_{x=0}^{4} \left[\frac{e^{\frac{y}{x}}}{\frac{1}{x}}\right]_0^{x^2} dx$$

$$\int_{x=0}^{4} \left[x e^{\frac{y}{x}}\right]_0^{x^2} dx$$

$$\int_{x=0}^{4} \left[x e^{x^2/x} - x \cdot e^{0}\right] dx$$

$$\int_{x=0}^{4} \left[x e^{x} - x\right] dx$$

$$\int_{x=0}^{4} x e^{x} \, dx - \int_{x=0}^{4} x \, dx$$

$$\left[x e^{x} - 1 e^{x} - \frac{x^2}{2}\right]_0^{4}$$

$$= \left(4e^{4} - e^{4} - \frac{16}{2}\right) - (0 - 1 - 0)$$

$$= 3e^{4} - 7$$

$$\boxed{\int_0^4 \int_0^{x^2} e^{y/x}\,dx\,dy = 3e^4 - 7}$$

Problems

① $$\int_0^1 \int_0^{x^2} e^{y/x}\,dx\,dy$$

② $$\int_0^1 \int_x^{\sqrt{x}} x^2 y^2 (x+y)\,dx\,dy$$

Ans

$$\int_0^1 \int_0^{x^2} e^{y/x}\,dx\,dy = \frac{1}{2}$$

$$\int_0^1 \int_x^{\sqrt{x}} x^2 y^2 (x+y)\,dx\,dy = \frac{18}{660}$$

⑥ Evaluate $\int_0^1 \int_0^1 \frac{dx\ dy}{\sqrt{(1-x^2)(1-y^2)}}$

Sol: Given $\int_{x=0}^1 \int_{y=0}^1 \frac{dx\ dy}{\sqrt{(1-x^2)(1-y^2)}}$

$$\int_{x=0}^1 \left[\int_{y=0}^1 \frac{1}{\sqrt{1-y^2}}\, dy\right] \frac{1}{\sqrt{1-x^2}}\, dx$$

$$\int_{x=0}^1 \left[\sin^{-1} y\right]_0^1 \frac{1}{\sqrt{1-x^2}}\, dx$$

$$= \int_{x=0}^1 \left[\sin^{-1} 1 - \sin^{-1} 0\right] \frac{1}{\sqrt{1-x^2}}\, dx$$

$$\boxed{\sin \frac{\pi}{2} = 1 \ ; \ \sin 0 = 0}$$

$$= \int_{x=0}^1 \left[\sin^{-1} \sin \frac{\pi}{2} - \sin^{-1} \sin 0\right] \frac{1}{\sqrt{1-x^2}}\, dx$$

$$= \frac{\pi}{2} \int_{x=0}^{1} \frac{1}{\sqrt{1-x^2}}\, dx$$

$$= \frac{\pi}{2} \cdot \left[\sin^{-1} x\right]_0^1$$

$$= \frac{\pi}{2} \left[\sin^{-1} 1 - \sin^{-1} 0\right]$$

$$= \frac{\pi}{2} \left[\sin^{-1} \sin\frac{\pi}{2} - \sin^{-1} \sin 0\right]$$

$$= \frac{\pi}{2} \cdot \frac{\pi}{2}$$

$$= \frac{\pi^2}{4}$$

$$\boxed{\int_0^1 \int_0^1 \frac{dx\, dy}{\sqrt{(1-x^2)(1-y^2)}} = \frac{\pi^2}{4}}$$

Formula: $\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x$

⑦ Evaluate $\int_0^a \int_0^{\sqrt{a^2-y^2}} \sqrt{a^2-x^2-y^2}\,dx\,dy$

Sy Given $\int_{y=0}^{a} \int_{x=0}^{\sqrt{a^2-y^2}} \sqrt{a^2-x^2-y^2}\,dx\,dy$

$$\int_{y=0}^{a} \left[\int_{x=0}^{\sqrt{a^2-y^2}} \sqrt{(a^2-y^2)-x^2}\,dx \right] dy$$

$$\int_{y=0}^{a} \left[\int_{x=0}^{\sqrt{a^2-y^2}} \sqrt{\left(\sqrt{a^2-y^2}\right)^2-x^2}\,dx \right] dy$$

$$\int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + c$$

$a = \sqrt{a^2-y^2}$

$$\int_{y=0}^{a} \left[\frac{x}{2}\sqrt{a^2-y^2-x^2} + \frac{a^2-y^2}{2}\sin^{-1}\frac{x}{\sqrt{a^2-y^2}} \right]_{x=0}^{\sqrt{a^2-y^2}} dy$$

$$\int_{y=0}^{a} \left[\frac{\sqrt{a^2-y^2}}{2} \sqrt{a^2-y^2-a^2+y^2} + \frac{a^2-y^2}{2} \sin^{-1} \frac{\sqrt{a^2-y^2}}{\sqrt{a^2-y^2}} \right] dy$$

$$\int_{y=0}^{a} \frac{a^2-y^2}{2} \sin^{-1} 1 \; dy$$

$$\int_{y=0}^{a} \frac{a^2-y^2}{2} \sin^{-1} \sin \frac{\pi}{2} \; dy$$

$$\frac{\pi}{4} \int_{y=0}^{a} \left[a^2 - y^2 \right] dy$$

$$= \frac{\pi}{4} \left[a^2 y - \frac{y^3}{3} \right]_0^a$$

$$= \frac{\pi}{4} \left\{ a^3 - \frac{a^3}{3} \right\}$$

$$= \frac{\pi}{4} \frac{2a^3}{3}$$

$$= \frac{\pi a^3}{6}$$

$$\boxed{\int_0^a \int_0^{\sqrt{a^2-y^2}} \sqrt{a^2-y^2-x^2}\, dx\, dy = \frac{\pi}{6} a^3}$$

Formula: $\int \sqrt{a^2-x^2}\, dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}$

(8) Evaluate $\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dx\, dy}{1+x^2+y^2}$

Sol: given $\int_{x=0}^{1} \int_{y=0}^{\sqrt{1+x^2}} \frac{dx\, dy}{1+x^2+y^2}$

$$= \int_{x=0}^{1} \left[\int_{y=0}^{\sqrt{1+x^2}} \frac{dy}{\left(\sqrt{(1+x^2)}\right)^2 + y^2} \right] dx$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$$

$$a = \sqrt{1+x^2}$$

$$= \int_{x=0}^{1} \left[\frac{1}{\sqrt{1+x^2}} \tan^{-1}\frac{y}{\sqrt{1+x^2}}\right]_0^{\sqrt{1+x^2}} dx$$

$$= \int_{x=0}^{1} \left[\frac{1}{\sqrt{1+x^2}} \tan^{-1}\frac{\sqrt{1+x^2}}{\sqrt{1+x^2}}\right] dx$$

$$= \int_{x=0}^{1} \frac{1}{\sqrt{1+x^2}} \tan^{-1}(1)\, dx$$

$$= \int_{x=0}^{1} \frac{1}{\sqrt{1+x^2}} \tan^{-1}\tan\frac{\pi}{4}\, dx$$

$$= \frac{\pi}{4} \int_{x=0}^{1} \frac{1}{\sqrt{1+x^2}}\, dx$$

$$= \frac{\pi}{4}\left[\log\left(x+\sqrt{1+x^2}\right)\right]_0^1$$

$$= \frac{\pi}{4}\left[\log(1+\sqrt{2}) - \log(1)\right]$$

$$= \frac{\pi}{4}\log(1+\sqrt{2})$$

$$\boxed{\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dx\ dy}{1+x^2+y^2} = \frac{\pi}{4}\log(1+\sqrt{2})}$$

Formulas:

① $\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}$

② $\int \frac{1}{\sqrt{a^2+x^2}}\,dx = \log\left(x+\sqrt{a^2+x^2}\right)$

(9) Evaluate $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\, dx\, dy$

Sol. given $\int_{x=0}^{\infty} \int_{y=0}^{\infty} e^{-(x^2+y^2)}\, dx\, dy$

$$e^{-(x^2+y^2)} = e^{-x^2-y^2} = e^{-x^2} \cdot e^{-y^2}$$

$$\int_{x=0}^{\infty} \int_{y=0}^{\infty} e^{-x^2} \cdot e^{-y^2}\, dx\, dy$$

$$\int_{x=0}^{\infty} \left[\int_{y=0}^{\infty} e^{-y^2}\, dy \right] e^{-x^2}\, dx$$

$$\int_{x=0}^{\infty} \frac{\sqrt{\pi}}{2} e^{-x^2}\, dx$$

$$\frac{\sqrt{\pi}}{2} \qquad \int_{x=0}^{\infty} e^{-x^2}\, dx$$

$$\frac{\sqrt{\pi}}{2} \cdot \frac{\sqrt{\pi}}{2} = \frac{\pi}{4}$$

$$\boxed{\int_0^{\infty}\int_0^{\infty} e^{-(x^2+y^2)}\, dx\, dy = \frac{\pi}{4}}$$

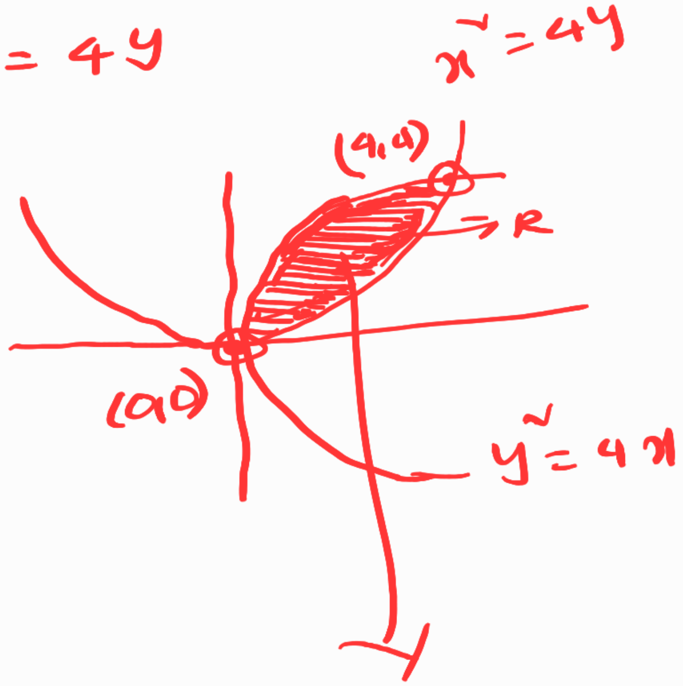
Formula : $\displaystyle\int_0^{\infty} e^{-x^2}\, dx = \frac{\sqrt{\pi}}{2}$

<u>Problems</u>

① $\displaystyle\int_0^{4}\int_{y^2/4}^{y} \frac{y}{x^2+y^2}\, dx\, dy$

② $\displaystyle\int_0^{1}\int_{x}^{\sqrt{2-x^2}} \frac{dx\, dy}{\sqrt{x^2+y^2}}$

(10) Evaluate $\iint_R y \, dx \, dy$ where R is the region bounded by the parabolas $y^2 = 4x$ and $x^2 = 4y$



Sol. given parabolas

$y^2 = 4x$

$x^2 = 4y \Rightarrow y = \frac{x^2}{4}$

Hence $\left(\frac{x^2}{4}\right)^2 = 4x$

$$\frac{x^4}{16} = 4x$$

$x^4 = 64x$

$x^4 - 64x = 0$

$x(x^3 - 64) = 0$

$x = 0$ and $x^3 - 64 = 0$

$x^3 = 64$

$x = 4$

Hence $\boxed{x = 0 \text{ to } 4}$

Since $y^2 = 4x \Rightarrow y = 2\sqrt{x}$

$x^2 = 4y \Rightarrow y = \frac{x^2}{4}$

Hence $\boxed{y = 2\sqrt{x} \text{ to } \frac{x^2}{4}}$

Hence $\iint_R y \, dx \, dy = \int_{x=0}^{4} \int_{y=2\sqrt{x}}^{x^2/4} y \, dx \, dy$

$$= \int_{x=0}^{4} \left[\int_{y=2\sqrt{x}}^{x^2/4} y \, dy \right] dx$$

$$= \int_{x=0}^{4} \left[\frac{y^2}{2} \right]_{2\sqrt{x}}^{x^2/4} dx$$

$$= \frac{1}{2} \int_{x=0}^{4} \left[\left(\frac{x^2}{4} \right)^2 - (2\sqrt{x})^2 \right] dx$$

$$= \frac{1}{2} \int_{x=0}^{4} \left[\frac{x^4}{16} - 4x \right] dx$$

$$= \frac{1}{2}\left[\frac{x^5}{5\times 16} - 4\cdot\frac{x^2}{2}\right]_0^4$$

$$= \frac{1}{2}\left[\frac{4^5}{5\times 16} - 4\cdot\frac{4^2}{2}\right]$$

$$= \frac{1}{2}\left[\frac{64}{5} - 32\right]$$

$$= -\frac{48}{5}$$

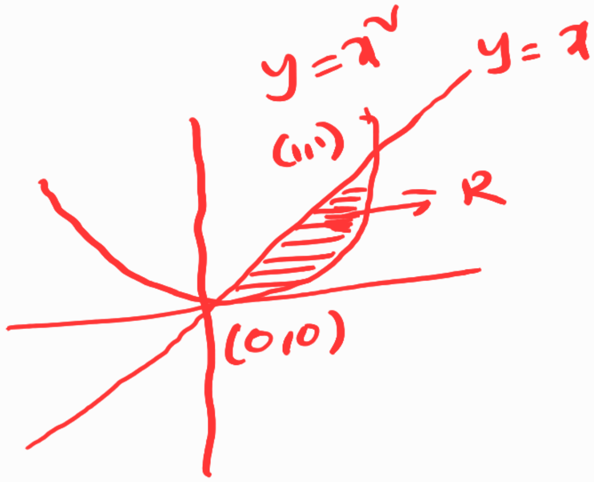
$$\iint_R y\,dx\,dy = \left|-\frac{48}{5}\right|$$

$$\boxed{\iint_R y\,dx\,dy = \frac{48}{5}}$$

(11) Evaluate $\iint_R xy(x+y)\,dx\,dy$ where

R is the region bounded by $y = x^2$

and $y = x$



Sol:

given $y = x^2$

$y = x$

Hence $x = x^2$

$x^2 - x = 0$

$x(x-1) = 0$

$x = 0$ or $x = 1$

$\boxed{x = 0 \text{ to } 1}$

Since $y = x^2$

$y = x$

$\boxed{y = x^2 \text{ to } x}$

Given $\iint_R xy(x+y)\,dx\,dy$

$$\int_{x=0}^{1}\int_{y=x^2}^{x}(x^2y+xy^2)\,dx\,dy$$

$$=\int_{x=0}^{1}\left[\int_{y=x^2}^{x}[x^2y+xy^2]\,dy\right]dx$$

$$=\int_{x=0}^{1}\left[x^2\frac{y^2}{2}+x\frac{y^3}{3}\right]_{x^2}^{x}dx$$

$$=\int_{x=0}^{1}\left[\left(x^2\cdot\frac{x^2}{2}+x\cdot\frac{x^3}{3}\right)-\left(x^2\frac{x^4}{2}+x\cdot\frac{x^6}{3}\right)\right]dx$$

$$=\int_{x=0}^{1}\left[\left(\frac{x^4}{2}+\frac{x^4}{3}\right)-\left(\frac{x^6}{2}+\frac{x^7}{3}\right)\right]dx$$

$$= \left[\left(\frac{x^5}{10} + \frac{x^5}{15}\right) - \left(\frac{x^7}{14} + \frac{x^8}{24}\right)\right]_0^1$$

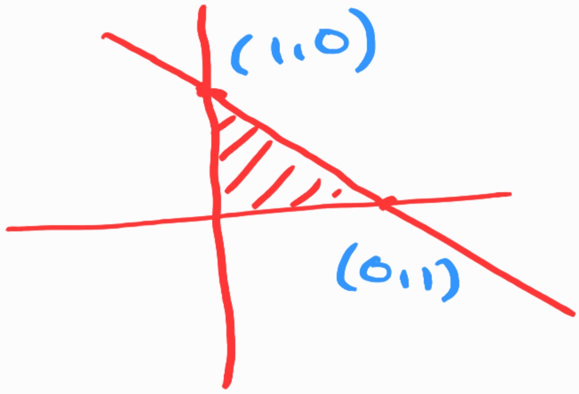
$$= \left(\frac{1}{10} + \frac{1}{15}\right) - \left(\frac{1}{14} + \frac{1}{24}\right)$$

$$= \frac{9}{168} = \frac{3}{56}$$

Hence

$$\boxed{\iint_R xy(x+y)\,dx\,dy = \frac{3}{56}}$$

(12) Evaluate $\iint_R (x^2+y^2)\,dx\,dy$ in the positive quadrant which $x+y \le 1$



So. given

$x + y \le 1$

$\boxed{x = 0 \text{ to } 1}$

$x + y = 1$

$y = 1 - x$

$\boxed{y = 0 \text{ to } 1 - x}$

Hence $\iint_R (x^2+y^2)\,dx\,dy$

$$= \int_{x=0}^{1} \int_{y=0}^{1-x} (x^2+y^2)\,dx\,dy$$

$$= \int_{x=0}^{1} \left[\int_{y=0}^{1-x} (x^2+y^2)\,dy \right] dx$$

$$= \int_{x=0}^{1} \left[x^2 y + \frac{y^3}{3}\right]_0^{1-x} dx$$

$$= \int_{x=0}^{1} \left[x^2 (1-x) + \frac{(1-x)^3}{3}\right] dx$$

$$= \int_{x=0}^{1} \left[x^2 - x^3 + \frac{(1-x)^3}{3}\right] dx$$

$$= \left[\frac{x^3}{3} - \frac{x^4}{4} + \frac{1}{3}\frac{(1-x)^4}{(-4)}\right]_0^1$$

$$= \left(\frac{1}{3} - \frac{1}{4} + 0\right) - \left(0 - 0 - \frac{1}{12}\right)$$

$$= \frac{1}{3} - \frac{1}{4} + \frac{1}{12} = \frac{1}{6}$$

$$\boxed{\iint_R (x^2 + y^2)\, dx\, dy = \frac{1}{6}}$$

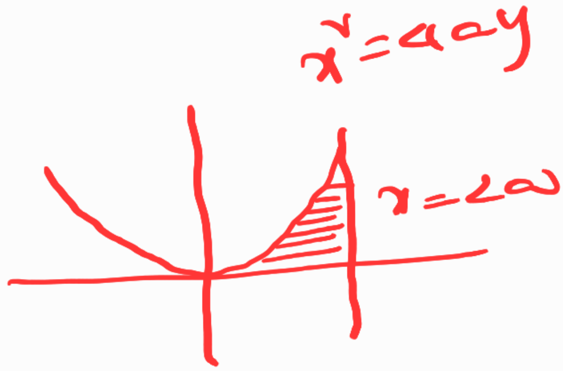
(13) Evaluate $\iint_R xy \, dx \, dy$ where

R is region bounded by x-axis

and $x = 2a$ and the curve $x^2 = 4ay$

$x = 0$ to $2a$

$y = 0$ to $\frac{x^2}{4a}$



(14) $\iint_R (x^2+y^2) \, dx \, dy$ where R is

The region bounded by the

ellipse $\boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1}$

Some special Functions

β - function

* $$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

$$\frac{1}{2}\beta(m,n) = \int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$$

Γ - function

* $$\Gamma m = \int_0^{\infty} e^{-x} x^{m-1}\,dx$$

$$\Gamma m = \frac{\Gamma m+1}{m}$$

* $$\Gamma m = (m-1)!$$

* $$\Gamma \tfrac{1}{2} = \sqrt{\pi}$$

β, Γ relation : $$\beta(m,n) = \frac{\Gamma m \, \Gamma n}{\Gamma m+n}$$

Example

① $$\int_0^{\pi/2} \sin^3\theta \cos^5\theta \, d\theta$$

$$\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta = \frac{1}{2} \beta(m,n)$$

Compare

$2m-1 = 3$; $2n-1 = 5$

$m = 2$ $\quad$ $n = 3$

$$\int_0^{\pi/2} \sin^3\theta \cos^5\theta \, d\theta = \frac{1}{2} \beta(2,3)$$

$$\beta(m,n) = \frac{\Gamma m \, \Gamma n}{\Gamma m+n}$$

$$= \frac{1}{2} \frac{\Gamma 2 \, \Gamma 3}{\Gamma 5}$$

$$= \frac{1}{2} \frac{1! \times 2!}{4!}$$

$$= \frac{1}{24}$$

② $\int_0^{\pi/2} \sin^2\theta \cos^4\theta \, d\theta = \frac{\pi}{32}$

$$\int_0^{\pi/2} \sin^{2m-1}\theta \cdot \cos^{2n-1}\theta \, d\theta = \frac{1}{2}\beta(m,n)$$

$2m-1 = 2 \qquad 2n-1 = 4$

$m = \frac{3}{2} \qquad n = \frac{5}{2}$

$$\int_0^{\pi/2} \sin^2\theta \cos^4\theta \, d\theta = \frac{1}{2}\beta\left(\frac{3}{2}, \frac{5}{2}\right)$$

$$\beta(m,n) = \frac{\Gamma m \, \Gamma n}{\Gamma m+n}$$

$$= \frac{1}{2} \frac{\Gamma\frac{3}{2} \, \Gamma\frac{5}{2}}{\Gamma 4} = \frac{1}{2} \frac{\Gamma\frac{3}{2} \, \Gamma\frac{5}{2}}{3!}$$

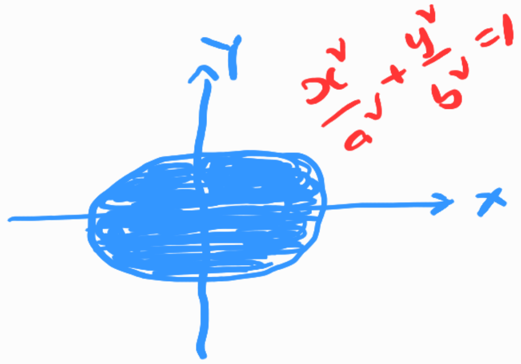
$$= \frac{1}{12} \Gamma\frac{3}{2} \, \Gamma\frac{5}{2}$$

$$= \frac{1}{12} \frac{\sqrt{\pi}}{2} \frac{3\sqrt{\pi}}{4} = \frac{\pi}{32}$$

(14) Evaluate $\iint_R x^2 + y^2 \, dx\, dy$ where

R is the area bounded by the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$



Sol Given ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

as $y \to 0$

$$\frac{x^2}{a^2} = 1 \Rightarrow x^2 = a^2 \Rightarrow x = \pm a$$

$$-a \leq x \leq a \; ;$$

since $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2}$$

$$\frac{y^2}{b^2} = \frac{1}{a^2}\left[a^2 - x^2\right]$$

$$y^2 = \frac{b^2}{a^2}\left[a^2 - x^2\right]$$

$$y = \sqrt{\frac{b^2}{a^2}(a^2 - x^2)}$$

$$y = \pm \frac{b}{a}\sqrt{a^2 - x^2}$$

$$-\frac{b}{a}\sqrt{a^2 - x^2} \le y \le \frac{b}{a}\sqrt{a^2 - x^2}$$

consider $\displaystyle\iint_R (x^2 + y^2)\, dx\, dy$

$$= \int_{-a}^{a} \int_{-\frac{b}{a}\sqrt{a^2-x^2}}^{\frac{b}{a}\sqrt{a^2-x^2}} (x^2 + y^2)\, dx\, dy$$

$$= 2\int_{-a}^{a} \left[\int_{0}^{\frac{b}{a}\sqrt{a^2-x^2}} (x^2 + y^2)\, dy \right] dx$$

$$= 2 \int_{-a}^{a} \left[\frac{x^2 y}{1} + \frac{y^3}{3} \right]_0^{\frac{b}{a}\sqrt{a^2-x^2}} dx$$

$$= 2\int_{-a}^{a}\left[x^2\,\frac{b}{a}\sqrt{a^2-x^2}+\frac{1}{3}\,\frac{b^3}{a^3}\left(a^2-x^2\right)^{3/2}\right]dx$$

$$= 4\int_{0}^{a}\left[x^2\,\frac{b}{a}\sqrt{a^2-x^2}+\frac{1}{3}\,\frac{b^3}{a^3}\left(a^2-x^2\right)^{3/2}\right]dx$$

put $x = a\sin\theta$; $x^2 = a^2\sin^2\theta$

$dx = a\cos\theta\, d\theta$

$$\boxed{\begin{array}{l} x = 0 \text{ to } a \\ \theta = 0 \text{ to } \frac{\pi}{2} \end{array}}$$

$$a^2 - x^2 = a^2 - a^2\sin^2\theta$$

$$= a^2(1-\sin^2\theta)$$

$$a^2 - x^2 = a^2\cos^2\theta$$

$$\sqrt{a^2-x^2} = a\cos\theta$$

$$(a^2-x^2)^{3/2} = (a^2\cos^2\theta)^{3/2}$$

$$= a^3\cos^3\theta$$

$$= 4\int_{\theta=0}^{\pi/2}\left[a^2\sin^2\theta\,\frac{b}{a}\,a\cos\theta+\frac{1}{3}\,\frac{b^3}{a^3}\,a^3\cos^3\theta\right]\cdot a\cos\theta\, d\theta$$

$$= 4 \int_{\theta=0}^{\frac{\pi}{2}} \left[a^3 b \sin^2\theta \cos^2\theta + \frac{b^3 a}{3} \cos^4\theta \right] d\theta$$

$$= 4 \left[a^3 b \int_{\theta=0}^{\pi/2} \sin^2\theta \cos^2\theta \, d\theta \right.$$

$$\left. + \frac{b^3 a}{3} \int_{0}^{\pi/2} \cos^4\theta \, d\theta \right]$$

Consider

$$\int_{\theta=0}^{\pi/2} \sin^2\theta \cos^2\theta \, d\theta = \frac{1}{2} \beta(m,n)$$

$$2m-1 = 2 \quad ; \quad 2n-1 = 2$$

$$m = \frac{3}{2} \qquad n = \frac{3}{2}$$

$$= \frac{1}{2} \beta(3/2, 3/2)$$

$$= \frac{1}{2} \frac{\Gamma\frac{3}{2} \, \Gamma 3/2}{\Gamma 3}$$

$$= \frac{1}{4} \Gamma\frac{3}{2} \, \Gamma\frac{3}{2}$$

$$\boxed{\Gamma\frac{3}{2} = \frac{1}{2} \times \Gamma\frac{1}{2} = \frac{\sqrt{\pi}}{2}}$$

$$= \frac{1}{4} \cdot \frac{\sqrt{\pi}}{2} \cdot \frac{\sqrt{\pi}}{2}$$

$$= \frac{\pi}{16}$$

consider $\displaystyle\int_0^{\pi/2} \cos^4\theta \, d\theta = \frac{1}{2} B(m,n)$

$2m-1 = 0 \quad ; \quad 2n-1 = 4$

$m = \frac{1}{2} \qquad n = \frac{5}{2}$

$$= \frac{1}{2} B\left(\frac{1}{2}, \frac{5}{2}\right)$$

$$= \frac{1}{2} \frac{\Gamma\frac{1}{2} \, \Gamma\frac{5}{2}}{\Gamma 3}$$

$$= \frac{1}{4} \sqrt{\pi} \; \frac{3}{2} \times \frac{1}{2} \times \Gamma\frac{1}{2}$$

$$= \frac{3\pi}{16}$$

Hence

$$4\left[a^3 b \cdot \frac{\pi}{16} + \frac{b^3 a}{\cancel{3}} \frac{\cancel{3}\pi}{16}\right]$$

$$\frac{4\pi}{16}\left[a^3 b + a b^3\right]$$

$$\frac{\pi}{4} ab(a^2+b^2)$$

$$\boxed{\iint_R (x^2+y^2)\,dx\,dy = \frac{\pi}{4} ab(a^2+b^2)}$$

(15) Evaluate $\iint_R (x+y)\,dx\,dy$ where

R is the region in the positive quadrant bounded by the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

Take
$$\begin{cases} 0 \le x \le a \\ 0 \le y \le \frac{b}{a}\sqrt{a^2-x^2} \end{cases}$$

Double integrals in polar coordinates

$P(r, \theta)$

① Evaluate $\int_0^{\pi} \int_0^{a\sin\theta} r \, dr \, d\theta$

Sol: given $\int_{\theta=0}^{\pi} \left[\int_{r=0}^{a\sin\theta} r \, dr \right] d\theta$

$$= \int_{\theta=0}^{\pi} \left[\frac{r^2}{2} \right]_0^{a\sin\theta} d\theta$$

$$= \frac{1}{2} \int_{\theta=0}^{\pi} \left[a^2 \sin^2\theta \right] d\theta$$

$$= \frac{a^2}{2} \int_{\theta=0}^{\pi} \sin^2\theta \, d\theta$$

$$\cos 2\theta = 1 - 2\sin^2\theta$$

$$\sin^2\theta = \frac{1 - \cos 2\theta}{2}$$

$$= \frac{a^2}{2} \int_{\theta=0}^{\pi} \frac{1-\cos 2\theta}{2} \, d\theta$$

$$= \frac{a^2}{4} \left[\theta - \frac{\sin 2\theta}{2} \right]_0^{\pi}$$

$$= \frac{a^2}{4} \left[\left(\pi - \frac{\sin 2\pi}{2} \right) - \left(0 - \frac{\sin 0}{2} \right) \right]$$

$$= \frac{a^2}{4} \left[(\pi - 0) - (0 - 0) \right]$$

$$= \frac{a^2}{4} \pi$$

$$\boxed{\int_0^{\pi} \int_0^{a \sin\theta} r \, dr \, d\theta = \frac{a^2}{4} \pi}$$

② Evaluate $\int_{0}^{\pi/2}\int_{a(1-\cos\theta)}^{a} r^2\, dr\, d\theta$

Sol Given $\int_{\theta=0}^{\pi/2}\left[\int_{r=a(1-\cos\theta)}^{a} r^2\, dr\right] d\theta$

$$= \int_{\theta=0}^{\pi/2}\left[\frac{r^3}{3}\right]_{a(1-\cos\theta)}^{a} d\theta$$

$$= \frac{1}{3}\int_{\theta=0}^{\pi/2}\left[a^3 - a^3(1-\cos\theta)^3\right] d\theta$$

$$= \frac{a^3}{3}\int_{\theta=0}^{\pi/2}\left[1-(1-\cos\theta)^3\right] d\theta$$

$$= \frac{a^3}{3}\int_{\theta=0}^{\pi/2}\left[1-\left[1-3\cos\theta+3\cos^2\theta-\cos^3\theta\right]\right] d\theta$$

$$= \frac{a^3}{3} \int_{\theta=0}^{\pi/2} \left[3\cos\theta - 3\cos^2\theta + \cos^3\theta \right] d\theta$$

$$= \frac{a^3}{3} \left[3\int_{\theta=0}^{\pi/2} \cos\theta \, d\theta - 3\int_{0}^{\pi/2} \cos^2\theta \, d\theta + \int_{0}^{\pi/2} \cos^3\theta \, d\theta \right]$$

Consider $\int_{0}^{\pi/2} \cos^2\theta \, d\theta = \frac{1}{2} B(m,n)$

$2m-1 = 0, \quad 2n-1 = 2$

$m = \frac{1}{2}, \quad n = 3/2$

$= \frac{1}{2} B\left(\frac{1}{2}, \frac{3}{2}\right)$

$= \frac{1}{2} \frac{\Gamma\frac{1}{2}\,\Gamma\frac{3}{2}}{\Gamma 2}$

$= \frac{1}{2} \sqrt{\pi} \; \frac{1}{2} \Gamma\frac{1}{2}$

$= \frac{\pi}{4}$

Consider $\int_0^{\pi/2} \cos^3\theta \, d\theta = \frac{1}{2} \beta(m,n)$

$2m-1=0$; $2n-1=3$

$m=\frac{1}{2}$ $\quad n=2$

$$= \frac{1}{2} \beta\left(\frac{1}{2}, 2\right)$$

$$= \frac{1}{2} \frac{\Gamma\frac{1}{2} \, \Gamma 2}{\Gamma\frac{5}{2}}$$

$$= \frac{\sqrt{\pi}}{2} \frac{1}{\frac{3}{2} \times \frac{1}{2} \times \Gamma\frac{1}{2}}$$

$$= \frac{\cancel{\sqrt{\pi}}}{2} \frac{1}{\frac{3}{4} \times \cancel{\sqrt{\pi}}}$$

$$= \frac{2}{3}$$

Consider $\displaystyle\int_0^{\pi/2} \cos\theta \, d\theta$

$$= \left[\sin\theta\right]_0^{\pi/2}$$

$$= \sin\frac{\pi}{2} - \sin 0$$

$$= 1$$

Hence

$$= \frac{a^3}{3}\left[3(1) - 3\left(\frac{\pi}{4}\right) + \frac{2}{3}\right]$$

$$= \frac{a^3}{3}\left[3 - \frac{3\pi}{4} + \frac{2}{3}\right]$$

$$= \frac{a^3}{36}(44 - 9\pi)$$

$$\boxed{\int_0^{\pi/2} \int_{a(1-\cos\theta)}^{a} r^2 \, dr \, d\theta = \frac{a^3}{36}(44 - 9\pi)}$$

③ Evaluate $\iint_R r\sin\theta \, dr\, d\theta$ where

R is the cardioid $r = a(1+\cos\theta)$

above the initial line.

Sol. Given cardioid

$$r = a(1+\cos\theta)$$

Here $r = 0$ to $a(1+\cos\theta)$

$\theta = 0$ to π

Hence $\iint_R r\sin\theta \, dr\, d\theta$

$$= \int_{\theta=0}^{\pi} \int_{r=0}^{a(1+\cos\theta)} r\sin\theta \, dr\, d\theta$$

$$= \int_{\theta=0}^{\pi} \left[\int_{r=0}^{a(1+\cos\theta)} r \, dr \right] \sin\theta \, d\theta$$

$$= \int_{\theta=0}^{\pi} \left[\frac{r^2}{2}\right]_0^{a(1+\cos\theta)} \sin\theta \, d\theta$$

$$= \frac{1}{2} \int_{\theta=0}^{\pi} \left[a^2(1+\cos\theta)^2 \sin\theta\right] d\theta$$

$$= \frac{a^2}{2} \int_{\theta=0}^{\pi} \left[1+\cos^2\theta + 2\cos\theta\right] \sin\theta \, d\theta$$

$$= \frac{a^2}{2} \int_{\theta=0}^{\pi} \left[\sin\theta + \sin\theta\cos^2\theta + \sin 2\theta\right] d\theta$$

$$= \frac{a^2}{2} \left[\int_{\theta=0}^{\pi} \sin\theta \, d\theta + \int_{\theta=0}^{\pi} \sin\theta\cos^2\theta \, d\theta + \int_{\theta=0}^{\pi} \sin 2\theta \, d\theta \right]$$

consider $\displaystyle\int_{\theta=0}^{\pi} \sin\theta \cos^2\theta \, d\theta$

$$\cos\theta = t$$

$$-\sin\theta \, d\theta = dt$$

$$\sin\theta \, d\theta = (-dt)$$

$$\theta = 0 \Rightarrow t = 1$$

$$\theta = \pi \Rightarrow t = -1$$

$$\int_{1}^{-1} t^2 \, (-dt)$$

$$(-2) \int_{0}^{1} t^2 \, (-dt)$$

$$= 2 \int_0^1 t^2 \, dt = 2 \left[\frac{t^3}{3} \right]_0^1$$

$$= \frac{2}{3}$$

$$= \frac{a^2}{2}\left[\left[-\cos\theta\right]_0^{\pi} + \frac{2}{3} + \left[-\frac{\cos 2\theta}{2}\right]_0^{\pi}\right]$$

$$= \frac{a^2}{2}\left[(-\cos\pi + \cos 0) + \frac{2}{3} + \frac{1}{2}\left[-\cos 2\pi + \cos 0\right]\right]$$

$$= \frac{a^2}{2}\left[(1+1) + \frac{2}{3} + \frac{1}{2}(1-1)\right]$$

$$= \frac{a^2}{2}\left[2 + \frac{2}{3}\right]$$

$$= \frac{4}{3}a^2$$

$$\boxed{\iint r\sin\theta \, dr\, d\theta = \frac{4}{3}a^2}$$

(4) Evaluate $\iint\limits_R r^3\, dr\, d\theta$ where R is the region bounded by the circles $r = 2\sin\theta$ and $r = 4\sin\theta$

Sol: given circles

$r = 2\sin\theta$ and $r = 4\sin\theta$

$r = 2\sin\theta$ to $4\sin\theta$

$\theta = 0$ to π

Hence $\iint\limits_R r^3\, dr\, d\theta$

$$= \int_{\theta=0}^{\pi} \left[\int_{r=2\sin\theta}^{4\sin\theta} r^3\, dr \right] d\theta$$

$$= \int_{\theta=0}^{\pi} \left[\frac{r^4}{4} \right]_{2\sin\theta}^{4\sin\theta} d\theta$$

$$= \frac{1}{4}\int_0^{\pi} \left[(4\sin\theta)^4 - (2\sin\theta)^4\right] d\theta$$

$$= \frac{1}{4}\int_0^{\pi} \left[256\sin^4\theta - 16\sin^4\theta\right] d\theta$$

$$= 60\int_0^{\pi} \sin^4\theta \, d\theta$$

$$= 60\times 2\int_0^{\pi/2} \sin^4\theta \, d\theta$$

$$\int_0^{\pi/2} \sin^4\theta \, d\theta = \frac{1}{2} B(m,n)$$

$$2m-1 = 4 \qquad 2n-1 = 0$$

$$m = \frac{5}{2} \qquad n = \frac{1}{2}$$

$$= \frac{1}{2} B\left(\frac{5}{2}, \frac{1}{2}\right)$$

$$= \frac{1}{2} \frac{\Gamma\frac{5}{2}\,\Gamma\frac{1}{2}}{\Gamma 3}$$

$$= \frac{1}{4} \quad \frac{3}{2} \times \frac{1}{2} \times \sqrt{\pi}\ \sqrt{\pi}$$

$$= \frac{3\pi}{16}$$

$$= \overset{15}{\cancel{60}} \times \cancel{2} \times \frac{3\pi}{\cancel{16}\ \cancel{4}\ 2}$$

$$= \frac{45}{2}\pi$$

$$\boxed{\iint r^3\, dr\, d\theta = \frac{45}{2}\pi}$$

(5) Evaluate $\iint_R r^2 \sin\theta \, dr\, d\theta$ where

R is the semicircle $r = 2a\cos\theta$

above the initial line

Change the variables from Cartesian to polar-coordinates

In this case the given integral is of the form $\iint F(x,y)\,dx\,dy$

Here $x = r\cos\theta$; $y = r\sin\theta$

and Jacobian $J = \dfrac{\partial(x,y)}{\partial(r,\theta)}$

$$= \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} \end{vmatrix}$$

$$= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta$$

$$= r(\cos^2\theta + \sin^2\theta)$$

$$\boxed{J = r}$$

$$\iint_{(x,y)} F(x,y)\, dx\, dy$$

$$= \iint F(r\cos\theta, r\sin\theta)\, |J|\, dr\, d\theta$$

$$= \iint_{(r,\theta)} F(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

Note:

1. $x = r\cos\theta$, $y = r\sin\theta$

2. $dx\, dy = r\, dr\, d\theta$

3. x, y limits

$\downarrow$

r, θ limits

① Evaluate the following integral by transforming into polar co-ordinates

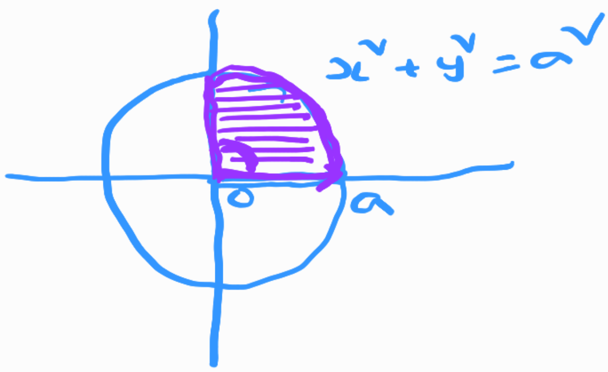
$$\int_0^a \int_0^{\sqrt{a^2-x^2}} y\sqrt{x^2+y^2}\,dx\,dy$$

Sol: given integral

$$\int_{x=0}^{a} \int_{y=0}^{\sqrt{a^2-x^2}} y\sqrt{x^2+y^2}\,dx\,dy$$

$y=0$; $y=\sqrt{a^2-x^2}$; $x=0$: $x=a$

$y=0$, $x^2+y^2=a^2$ $x=0$, $x=a$



changing to polar co-ordinates

$x = r\cos\theta$, $y = r\sin\theta$

Here $r = 0$ to a

$\theta = 0$ to $\frac{\pi}{2}$

given $y\sqrt{x^2+y^2} = r\sin\theta\sqrt{r^2\cos^2\theta + r^2\sin^2\theta}$

$= r^2\sin\theta$

$dx\,dy = r\,dr\,d\theta$

Hence

$$\int_{x=0}^{a}\int_{y=0}^{\sqrt{a^2-x^2}} y\sqrt{x^2+y^2}\,dx\,dy$$

$$= \int_{r=0}^{a}\int_{\theta=0}^{\pi/2} r^2\sin\theta\; r\,dr\,d\theta$$

$$= \left[\int_{r=0}^{a} r^3\,dr\right]\left[\int_{\theta=0}^{\pi/2}\sin\theta\,d\theta\right]$$

$$= \left[\frac{r^4}{4}\right]_0^a \left[-\cos\theta\right]_0^{\pi/2}$$

$$= \frac{a^4}{4}\left[-\cos\frac{\pi}{2} + \cos 0\right]$$

$$= \frac{a^4}{4}$$

$$\boxed{\int_0^a \int_0^{\sqrt{a^2-x^2}} y\sqrt{x^2+y^2}\, dx\, dy = \frac{a^4}{4}}$$

② By changing into polar co-ordinates

Evaluate $\displaystyle\int_0^1 \int_x^{\sqrt{2-x^2}} \frac{x}{x^2+y^2}\, dx\, dy$

Sol

Here $x=0$; $x=1$; $y=x$: $y=\sqrt{2-x^2}$

$x=0$; $x=1$; $\boxed{y=x}$: $\boxed{x^2+y^2=2}$

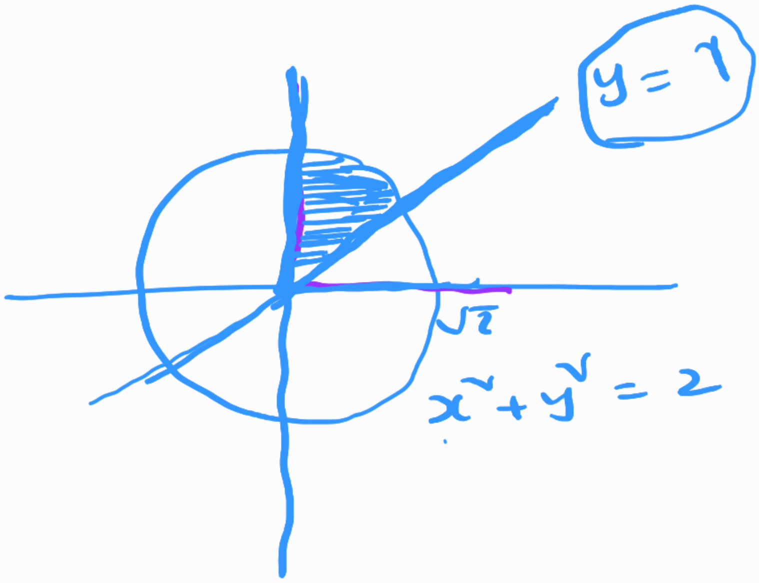
To change the given double integral into polar coordinates of

$x = r\cos\theta$, $y = r\sin\theta$

$$x^2+y^2=2 \Rightarrow r^2\cos^2\theta + r^2\sin^2\theta = 2$$
$$r^2 = 2$$
$$r = \sqrt{2}$$

r : varies from 0 to $\sqrt{2}$

θ varies from $\pi/4$ to $\frac{\pi}{2}$



$$\frac{x}{x^2+y^2} = \frac{r\cos\theta}{r^2\cos^2\theta + r^2\sin^2\theta} = \frac{1}{r}\cos\theta$$

$$dx\,dy = r\,dr\,d\theta$$

Hence
$$\int_{x=0}^{1}\int_{y=x}^{\sqrt{2-x^2}} \frac{x}{x^2+y^2}\,dx\,dy$$

$$= \int_{r=0}^{\sqrt{2}}\int_{\theta=\frac{\pi}{4}}^{\pi/2} \frac{1}{r}\cos\theta \cdot r\,dr\,d\theta$$

$$= \left[\int_{y=0}^{\sqrt{2}} 1 \, dy \right] \left[\int_{\theta=\frac{\pi}{4}}^{\pi/2} \cos\theta \, d\theta \right]$$

$$= \left[y \right]_0^{\sqrt{2}} \left[\sin\theta \right]_{\frac{\pi}{4}}^{\pi/2}$$

$$= \sqrt{2} \left[1 - \frac{1}{\sqrt{2}} \right]$$

$$= \sqrt{2} - 1$$

$$\boxed{\int_0^1 \int_x^{\sqrt{2-x^2}} \frac{x}{x^2+y^2} \, dx \, dy = \sqrt{2} - 1}$$

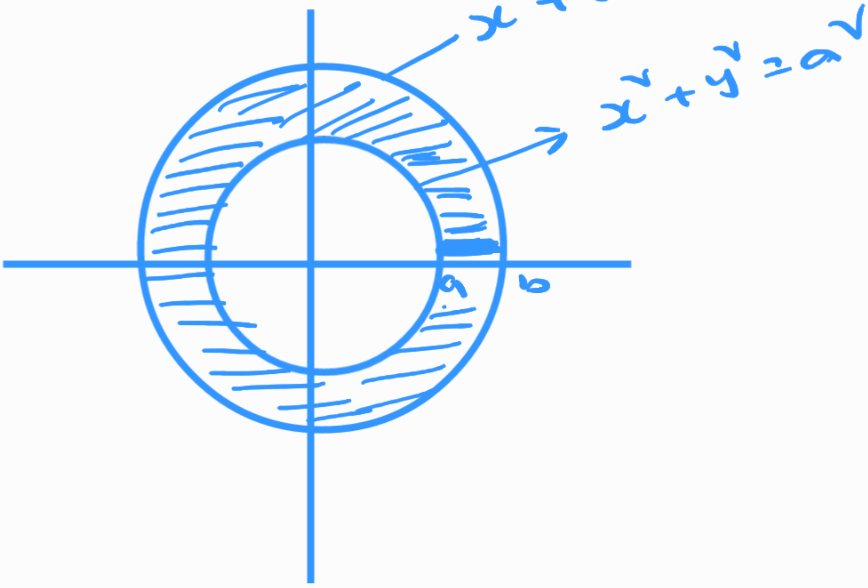
③ By changing into polar coordinates

Evaluate $\iint \frac{x^2 y^2}{x^2+y^2}\,dx\,dy$ over

The annular region between the

circles $x^2+y^2=a^2$; $x^2+y^2=b^2$ $(b>a)$

sol



Changing into polar coordinates

$x = r\cos\theta$, $y = r\sin\theta$

$dx\,dy = r\,dr\,d\theta$

$x^2+y^2=a^2 \Rightarrow r^2\cos^2\theta + r^2\sin^2\theta = a^2$

$\boxed{r=a}$

$x^2+y^2=b^2 \Rightarrow r^2\cos^2\theta + r^2\sin^2\theta = b^2$

$\boxed{r=b}$

r varies from a to b

θ vary from 0 to 2π

Now $\dfrac{x^2 y^2}{x^2+y^2} = \dfrac{r^2\cos^2\theta \; r^2 \sin^2\theta}{r^2}$

$= r^2 \cos^2\theta \sin^2\theta$

$= r^2 \; \dfrac{1}{4} \left[2\sin\theta\cos\theta\right]^2$

$= r^2 \; \dfrac{1}{4} \; \sin^2 2\theta$

$$\boxed{\sin^2 2\theta = \frac{1-\cos 4\theta}{2}}$$

$= \dfrac{r^2}{4} \; \dfrac{1-\cos 4\theta}{2}$

$= \dfrac{r^2}{8} \left[1-\cos 4\theta\right]$

$$\iint \frac{x^2 y^2}{x^2+y^2}\,dx\,dy$$

$$= \int_{r=a}^{b}\int_{\theta=0}^{2\pi} \frac{r^2}{8}\left[1-\cos 4\theta\right] r\,dr\,d\theta$$

$$= \frac{1}{8}\left[\int_{a}^{b} r^3\,dr\right]\left[\int_{\theta=0}^{2\pi} 1-\cos 4\theta\,d\theta\right]$$

$$= \frac{1}{8}\left[\frac{r^4}{4}\right]_a^b \left[\theta - \frac{\sin 4\theta}{4}\right]_0^{2\pi}$$

$$= \frac{1}{8}\left[\frac{b^4-a^4}{4}\right]\left[(2\pi-0)-(0-0)\right]$$

$$= \frac{\pi}{16}\left(b^4-a^4\right)$$

$$\boxed{\iint \frac{x^2 y^2}{x^2+y^2}\,dx\,dy = \frac{\pi}{16}\left(b^4-a^4\right)}$$

④ Evaluate $\int_0^2 \int_0^{\sqrt{2x-x^2}} (x^2+y^2)\, dx\, dy$

by changing into polar co-ordinates

Sol Here $x=0$ to $x=2$

$y=0$ to $y=\sqrt{2x-x^2}$

$y^2 = 2x - x^2$

$x^2+y^2 = 2x$

let $x = r\cos\theta$, $y = r\sin\theta$

$x^2+y^2 = 2x$

$r^2\cos^2\theta + r^2\sin^2\theta = 2r\cos\theta$

$r^2 = 2r\cos\theta$

$\boxed{r = 2\cos\theta}$

r varies from 0 to $2\cos\theta$

θ varies from 0 to $\frac{\pi}{2}$

Here $x^2+y^2 = r^2$

$dx\,dy = r\,dr\,d\theta$

$$\int_0^2 \int_0^{\sqrt{2x-x^2}} (x^2+y^2)\,dx\,dy$$

$$= \int_{\theta=0}^{\pi/2} \int_{r=0}^{2\cos\theta} r^2\, r\,dr\,d\theta$$

$$= \int_{\theta=0}^{\pi/2} \left[\frac{r^4}{4}\right]_0^{2\cos\theta} d\theta$$

$$= \int_{\theta=0}^{\pi/2} \frac{2^4\cos^4\theta}{4}\,d\theta$$

$$= 4\int_{\theta=0}^{\pi/2} \cos^4\theta\,d\theta \qquad \boxed{B\text{-}\Gamma \text{ function}}$$

$$= 4\cdot \frac{3}{2}\times\frac{1}{2}\times\frac{\pi}{2} = \frac{3\pi}{2}$$

$$\int_0^2 \int_0^{\sqrt{2x-x^2}} (x^2+y^2)\,dx\,dy = \frac{3\pi}{2}$$

Change of order of integration

$$\int_{x=a}^{b} \int_{y=f_1(x)}^{f_2(x)} f(x,y)\,dx\,dy$$

By changing the order of integration the above integral becomes

$$\int_{y=a}^{b} \int_{x=f_1(y)}^{f_2(y)} f(x,y)\,dx\,dy$$

① Change the order of integration and Evaluate

$$\int_0^{4a} \int_{x^2/4a}^{2\sqrt{ax}} dy\, dx$$

Sol: given $\displaystyle\int_{x=0}^{4a} \int_{y=\frac{x^2}{4a}}^{2\sqrt{ax}} dy\, dx$

$x = 0$ to $x = 4a$

$y = \dfrac{x^2}{4a}$ to $y = 2\sqrt{ax}$

$x = 2\sqrt{ay}$ to $x = \dfrac{y^2}{4a}$

for $x = 0$ we have $y = 0$

$x = 4a$ we have $y = 4a$

$y = 0$ to $4a$

Hence $\displaystyle\int_{x=0}^{4a} \int_{y=\frac{x^2}{4a}}^{2\sqrt{ax}} dy\, dx$

$$= \int_{y=0}^{4a} \int_{x=2\sqrt{ay}}^{y^2/4a} dx\, dy$$

$$= \int_{y=0}^{4a} \left[x\right]_{2\sqrt{ay}}^{y^2/4a} dy$$

$$= \int_{y=0}^{4a} \left[\frac{y^2}{4a} - 2\sqrt{ay}\right] dy$$

$$= \int_{y=0}^{4a} \left[\frac{1}{4a} \cdot y^2 - 2\sqrt{a} \cdot y^{\frac{1}{2}}\right] dy$$

$$= \left[\frac{1}{4a} \frac{y^3}{3} - 2\sqrt{a}\, y^{\frac{3}{2}} \cdot \frac{2}{3}\right]_0^{4a}$$

$$= \frac{1}{12a}(4a)^3 - \frac{4\sqrt{a}}{3} \cdot (4a)^{3/2}$$

$$= \frac{1}{12a} \cdot 64a^3 - \frac{32}{3}a^2$$

$$= \frac{64}{12}a^2 - \frac{32}{3}a^2$$

$$= \left| -\frac{16}{3}a^2 \right|$$

$$= \frac{16}{3}a^2$$

$$\boxed{\int_0^{4a} \int_{x^2/4a}^{2\sqrt{ax}} dx\, dy = \frac{16}{3}a^2}$$

② Change the order of integration and Evaluate $\int_{0}^{a} \int_{\frac{x}{a}}^{\sqrt{\frac{x}{a}}} (x^2+y^2)\, dx\, dy$

Sol given $\int_{x=0}^{a} \int_{y=\frac{x}{a}}^{\sqrt{\frac{x}{a}}} (x^2+y^2)\, dx\, dy$

Here $x=0$ to $x=a$

$y=\frac{x}{a}$ to $y=\sqrt{\frac{x}{a}}$

$x=ay$ to $x=ay^2$

for $x=a \Rightarrow y=0$

$x=a \Rightarrow y=1$

$y=0$ to $y=1$

Hence

$$\int_{x=0}^{a} \int_{y=\frac{x}{a}}^{\sqrt{\frac{x}{a}}} (x^2+y^2)\,dx\,dy$$

$$= \int_{y=0}^{1} \int_{x=ay}^{ay^2} (x^2+y^2)\,dx\,dy$$

$$= \int_{y=0}^{1} \left[\frac{x^3}{3} + xy^2\right]_{ay}^{ay^2} dy$$

$$= \int_{y=0}^{1} \left[\left(\frac{(ay^2)^3}{3} + ay^2\,y^2\right) - \left(\frac{(ay)^3}{3} + ayy^2\right)\right] dy$$

$$= \int_{y=0}^{1} \left[\left(\frac{a^3}{3}\cdot y^6 + ay^4\right) - \left(\frac{a^3}{3}\cdot y^3 + ay^3\right)\right] dy$$

$$= \left[\left(\frac{a^3}{3}\,\frac{y^7}{7} + a\,\frac{y^5}{5}\right) - \left(\frac{a^3}{3}\,\frac{y^4}{4} + a\,\frac{y^4}{4}\right)\right]_0^1$$

$$= \frac{a^3}{21} + \frac{a}{5} - \frac{a^3}{12} - \frac{a}{4}$$

③ By changing the order of the integration evaluate $\int_0^1 \int_1^{2-x} xy\,dx\,dy$

Sol given $\int_{x=0}^{1} \int_{y=1}^{2-x} xy \; dx \; dy$

Here $x = 0$ and $x = 1$

$y = 1$ and $\boxed{y = 2 - x}$

at $x = 0$; $y = 2$

at $x = 1$: $y = 1$

Hence y varies from 1 to 2

Here x vary from $x=0$ to $2-y$

$$\int_{x=0}^{1} \int_{y=1}^{2-x} xy \, dx \, dy$$

$$= \int_{y=1}^{2} \int_{x=0}^{2-y} x y \, dx \, dy$$

$$= \int_{y=1}^{2} \left[\frac{x^2}{2}\right]_0^{2-y} y \, dy$$

$$= \int_{y=1}^{2} \frac{(2-y)^2}{2} y \, dy$$

$$= \frac{1}{2} \int_{y=1}^{2} [4+y^2-4y] \, y \, dy$$

$$= \frac{1}{2} \int_{y=1}^{2} \left[4y + y^3 - 4y^2\right] dx$$

$$= \frac{1}{2} \left[4\frac{y^2}{2} + \frac{y^4}{4} - 4\frac{y^3}{3}\right]_1^2$$

$$= \frac{21}{24} = \frac{7}{8}$$

$$\boxed{\int_0^1 \int_1^{2-x} xy \, dx \, dy = \frac{7}{8}}$$

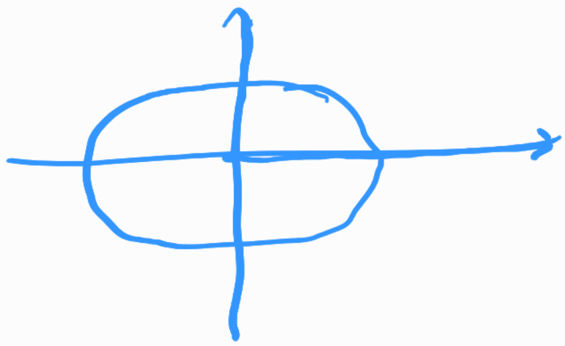
④ change the order of integration and Evaluate $\int_0^b \int_0^{\frac{a}{b}\sqrt{b^2-y^2}} xy \, dx \, dy$

Sol Given $\int_{y=0}^{b} \int_{x=0}^{\frac{a}{b}\sqrt{b^2-y^2}} xy \, dx \, dy$

Here $y=0$ and $y=b$

$x=0$ and $x=\frac{a}{b}\sqrt{b^2-y^2}$

$x^2=\frac{a^2}{b^2}(b^2-y^2)$



$x^2b^2=a^2b^2-a^2y^2$

$x^2b^2+a^2y^2=a^2b^2$

Divide with a^2b^2

$$\boxed{\frac{x^2}{a^2}+\frac{y^2}{b^2}=1}$$

If $y=0$ $\quad x=a$

$y=b$ $\quad x=0$

Hence x varies from 0 to a

y varies from 0 to $\frac{b}{a}\sqrt{a^2-x^2}$

$$\int_{y=0}^{b}\int_{x=0}^{\frac{a}{b}\sqrt{b^2-y^2}} xy \, dx \, dy$$

$$= \int_{x=0}^{a}\int_{y=0}^{\frac{b}{a}\sqrt{a^2-x^2}} xy \, dx \, dy$$

$$= \int_{x=0}^{a} x\left[\frac{y^2}{2}\right]_0^{\frac{b}{a}\sqrt{a^2-x^2}} dx$$

$$= \frac{1}{2}\int_{x=0}^{a} x \, \frac{b^2}{a^2}(a^2-x^2) \, dx$$

$$= \frac{b^2}{2a^2} \int_{x=0}^{a} \left[a^2 x - x^3\right] dx$$

$$= \frac{b^2}{2a^2} \left[a^2 \frac{x^2}{2} - \frac{x^4}{4}\right]_0^a$$

$$= \frac{b^2}{2a^2} \left[\frac{a^4}{2} - \frac{a^4}{4}\right]$$

$$= \frac{b^2}{2a^2} \frac{a^4}{4}$$

$$= \frac{a^2 b^2}{8}$$

$$\boxed{\int_0^b \int_0^{\frac{a}{b}\sqrt{b^2-y^2}} xy \, dx \, dy = \frac{a^2 b^2}{8}}$$

⑤ By changing the order of integration

Evaluate $\int_0^3 \int_1^{\sqrt{4-y}} (x+y)\,dx\,dy$

Sol Given $\int_{y=0}^{3} \int_{x=1}^{\sqrt{4-y}} (x+y)\,dx\,dy$

Here $y=0$ and $y=3$

$x=1$ and $\boxed{x=\sqrt{4-y}}$

$x^2 = 4-y$

$y = 4-x^2$

If $y=0$ $x=2$

$y=3$ $x=1$

$\begin{cases} x \text{ varies from } 1 \text{ to } 2 \\ y \text{ varies from } 0 \text{ to } 4-x^2 \end{cases}$

$\boxed{\text{Ans: } \frac{241}{60}}$

(6) By changing The order of integration Evaluate $\int_0^1 \int_0^{\sqrt{1-x^2}} y^2 \, dx \, dy$

Is given $\int_{x=0}^{1} \int_{y=0}^{\sqrt{1-x^2}} y^2 \, dx \, dy$

Here $x=0$ and $x=1$

$y=0$ and $y=\sqrt{1-x^2}$

$y^2 = 1-x^2$

$x^2+y^2=1$

$x=\sqrt{1-x^2}$

If $x=0$ $y=1$

$x=1$ $y=0$

y varies from 0 to 1

x vary from 0 to $\sqrt{1-y^2}$

$\boxed{\text{Ans: } \frac{\pi}{16}}$

(5) By changing the order of integration Evaluate

(i) $$\int_0^a \int_x^a (x^2+y^2)\, dx\, dy$$

(ii) $$\int_0^a \int_{a-x}^{\sqrt{a^2-x^2}} y\, dx\, dy$$

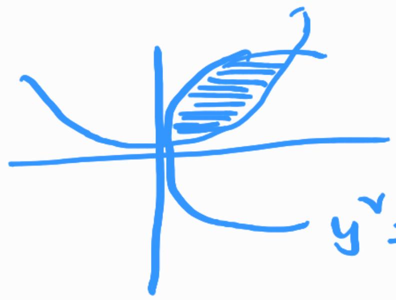
Area Enclosed by a plane curve

① Find the Area of the region bounded by the parabolas $y^2 = 4ax$;
$x^2 = 4ay$

Sol Given parabolas $y^2 = 4ax$
$x^2 = 4ay$

Here $y^2 = 4ax$

$x^2 = 4ay \Rightarrow y = \frac{x^2}{4a}$



$\frac{x^4}{16a^2} = 4ax$

$x^4 = 64a^3x$

$x^4 - 64a^3x = 0$

$x(x^3 - 64a^3) = 0$

$x = 0 \quad ; \quad x = 4a$

x vary from 0 to $4a$

y vary from $2\sqrt{ax}$ to $\frac{x^2}{4a}$

Hence Area $A = \iint_R dx\,dy$

$$= \int_{x=0}^{4a} \int_{y=2\sqrt{ax}}^{x^2/4a} 1 \, dy \, dx$$

$$= \int_{x=0}^{4a} \left[y \right]_{2\sqrt{ax}}^{x^2/4a} dx$$

$$= \int_{x=0}^{4a} \left[\frac{x^2}{4a} - 2\sqrt{ax} \right] dx$$

$$= \left[\frac{x^3}{12a} - 2\sqrt{a}\,\frac{2}{3} \cdot x^{3/2} \right]_0^{4a}$$

$$= \frac{(4a)^3}{12a} - \frac{4\sqrt{a}}{3} (4a)^{3/2}$$

$$= \frac{64}{12} \cdot a^2 - \frac{32}{3} a^2 = \frac{16a^2}{3}$$

② Find by double integral the area enclosed by the curves $y = 2-x$ and

$y^2 = 2(2-x)$

Sol Given curves $y = 2-x$

$$y^2 = 2(2-x)$$

$$\Rightarrow (2-x)^2 = 2(2-x)$$

$$\Rightarrow (2-x)^2 - 2(2-x) = 0$$

$$(2-x)\left[2-x-2\right] = 0$$

$$2-x = 0 \quad \text{and} \quad x = 0$$

$$x = 2 \quad \text{and} \quad x = 0$$

x varies from 0 to 2

y varies from $2-x$ to $\sqrt{2(2-x)}$

Hence Area $A = \iint_R dx\, dy$

$$= \int_{x=0}^{2} \int_{y=2-x}^{\sqrt{2(2-x)}} 1 \, dy \, dx$$

$$= \int_{x=0}^{2} [y]_{2-x}^{\sqrt{2(2-x)}} \, dx$$

$$= \int_{x=0}^{2} \left[\sqrt{2(2-x)} - (2-x)\right] dx$$

$$= \int_{x=0}^{2} \left[\sqrt{2}\,(2-x)^{\frac{1}{2}} - (2-x)\right] dx$$

$$= \left[\sqrt{2}\,\frac{(2-x)^{\frac{1}{2}+1}}{(\frac{1}{2}+1)(-1)} - \frac{(2-x)^2}{2(-1)}\right]_0^2$$

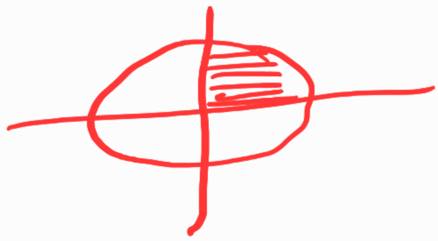
$$= 0 - \left(-\sqrt{2}\,\frac{2^{3/2}}{\frac{3}{2}} + \frac{2^2}{2}\right)$$

$$= \frac{2}{3} \cdot \sqrt{2} \cdot 2^{3/2} - 2 = \frac{2}{3}$$

$$\boxed{A = \frac{2}{3}}$$

③ Find the area of a plane in the form of a first quadrant of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Sol Given ellipse



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

x varies from 0 to a

y vary from 0 to $\frac{b}{a}\sqrt{a^2-x^2}$

Hence Area $A = \iint_R dx\, dy$

$$= \int_{x=0}^{a} \int_{y=0}^{\frac{b}{a}\sqrt{a^2-x^2}} 1 \, dy \, dx$$

$$= \int_{x=0}^{a} [y]_0^{\frac{b}{a}\sqrt{a^2-x^2}} \, dx$$

$$= \int_{x=0}^{a} \frac{b}{a}\sqrt{a^2-x^2} \, dx$$

$$= \frac{b}{a} \int_{x=0}^{a} \sqrt{a^2 - x^2} \, dx$$

$$= \frac{b}{a} \left[\frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} \right]_0^a$$

$$= \frac{b}{a} \left[\frac{a^2}{2} \sin^{-1}(1) \right]$$

$$= \frac{b}{a} \frac{a^2}{2} \sin^{-1} \sin \frac{\pi}{2}$$

$$= \frac{ba}{2} \cdot \frac{\pi}{2}$$

$$= \frac{\pi ab}{4}$$

$$\boxed{A = \frac{\pi ab}{4}}$$

④ Find by double integral : The area bounded by the curve $y = x^3$ and $y = x$

Sol given curve $y = x^3$;

$y = x$

$$\Rightarrow \quad x = x^3$$

$$x^3 - x = 0$$

$$x(x^2 - 1) = 0$$

$$x = 0 \qquad x^2 = 1$$

$$x = 0 \qquad x = \pm 1$$

x vary from 0 to 1

y vary from x^3 to x

Hence Area $A = \iint_R dx\, dy$

$$= \int_{x=0}^{1} \int_{y=x^3}^{x} 1 \, dy \, dx$$

$$= \int_{x=0}^{1} [y]_{x^3}^{x} \, dx$$

$$= \int_{x=0}^{1} [x - x^3] \, dx$$

$$= \left[\frac{x^2}{2} - \frac{x^4}{4}\right]_0^1$$

$$= \frac{1}{2} - \frac{1}{4}$$

$$= \frac{1}{4}$$

$$\boxed{A = \frac{1}{4}}$$

(5) Find by double integral the area lying between the parabolas $y^2 = ax$ and $x^2 = ay$

Triple integrals

$$\int_x \int_y \int_z f(x,y,z)\, dx\, dy\, dz$$

$$\Leftarrow \int_{x}\int_{y=f(x_1)}^{f(x_2)} \int_{z=f_1(x,y)}^{f_2(x,y)} dx\, dy\, dz$$

const

① Evaluate $\int_0^1 \int_1^2 \int_2^3 xyz\, dx\, dy\, dz$

Sol given $\int_{x=0}^{1} \int_{y=1}^{2} \int_{z=2}^{3} xyz\, dx\, dy\, dz$

$$= \int_{x=0}^{1} \int_{y=1}^{2} xy \left[\frac{z^2}{2}\right]_2^3 dy\, dx$$

$$= \int_{x=0}^{1} x \int_{y=1}^{2} y \left(\frac{9}{2} - \frac{4}{2}\right) dy\, dx$$

$$= \frac{5}{2} \int_{x=0}^{1} x \left[\frac{y^2}{2}\right]_1^2 dx$$

$$= \frac{5}{2} \int_{x=0}^{1} x \left(\frac{4}{2} - \frac{1}{2}\right) dx$$

$$= \frac{15}{4} \int_{x=0}^{1} x \, dx$$

$$= \frac{15}{4} \left[\frac{x^2}{2}\right]_0^1$$

$$= \frac{15}{8}$$

$$\boxed{\int_{x=0}^{1} \int_{y=1}^{2} \int_{z=2}^{3} xyz \, dx \, dy \, dz = \frac{15}{8}}$$

② Evaluate $\displaystyle\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} xyz \, dx \, dy \, dz$

Sol given $\displaystyle\int_{x=0}^{1} \int_{y=0}^{\sqrt{1-x^2}} \int_{z=0}^{\sqrt{1-x^2-y^2}} xyz \, dx \, dy \, dz$

$$= \int_{x=0}^{1} \int_{y=0}^{\sqrt{1-x^2}} xy \left[\frac{z^2}{2}\right]_0^{\sqrt{1-x^2-y^2}} dx\, dy$$

$$= \int_{x=0}^{1} \int_{y=0}^{\sqrt{1-x^2}} xy \, \frac{\left(\sqrt{1-x^2-y^2}\right)^2}{2} dx\, dy$$

$$= \int_{x=0}^{1} \int_{y=0}^{\sqrt{1-x^2}} xy \, \frac{(1-x^2-y^2)}{2} dy\, dx$$

$$= \frac{1}{2} \int_{x=0}^{1} \int_{y=0}^{\sqrt{1-x^2}} \left[xy - x^3y - xy^3\right] dy\, dx$$

$$= \frac{1}{2} \int_{x=0}^{1} \left[x\frac{y^2}{2} - x^3\frac{y^2}{2} - x\frac{y^4}{4}\right]_0^{\sqrt{1-x^2}} dx$$

$$= \frac{1}{2} \int_{x=0}^{1} \left[\frac{x}{2}(1-x^2) - \frac{x^3}{2}(1-x^2) - \frac{x}{4}(1-x^2)^2\right] dx$$

$$= \frac{1}{2}\int_0^1 \left[\frac{x}{2} - \frac{x^3}{2} - \frac{x^3}{2} + \frac{x^5}{2} - \frac{x}{4} - \frac{x^5}{4} + \frac{2x^3}{4}\right] dx$$

$$= \frac{1}{2}\left[\frac{x^2}{4} - \frac{x^4}{8} - \frac{x^4}{8} + \frac{x^6}{12} - \frac{x^2}{8} - \frac{x^6}{24} + 2\,\frac{x^4}{16}\right]_0^1$$

$$= \frac{1}{2}\left\{\frac{1}{4} - \frac{1}{8} - \frac{1}{8} + \frac{1}{12} - \cancel{\frac{1}{8}} - \frac{1}{24} + \cancel{\frac{1}{8}}\right\}$$

$$= \frac{1}{48}$$

$$\boxed{\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} xyz \; dx\,dy\,dz = \frac{1}{48}}$$

③ Evaluate $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \frac{1}{\sqrt{1-x^2-y^2-z^2}}\, dx\, dy\, dz$

Sol Given $\int_{x=0}^{1} \int_{y=0}^{\sqrt{1-x^2}} \int_{z=0}^{\sqrt{1-x^2-y^2}} \frac{1}{\sqrt{1-x^2-y^2-z^2}}\, dz\, dy\, dx$

$$\boxed{\int \frac{1}{\sqrt{a^2-z^2}}\, dz = \sin^{-1} \frac{z}{a}}$$

consider

$$\int \frac{1}{\sqrt{\left(\sqrt{1-x^2-y^2}\right)^2 - z^2}}\, dz = \sin^{-1} \frac{z}{\sqrt{1-x^2-y^2}}$$

$$\int_{x=0}^{1} \int_{y=0}^{\sqrt{1-x^2}} \left[\sin^{-1} \frac{z}{\sqrt{1-x^2-y^2}}\right]_{z=0}^{\sqrt{1-x^2-y^2}} dy\, dx$$

$$\int_{x=0}^{1} \int_{y=0}^{\sqrt{1-x^2}} \left[\sin^{-1} \frac{\sqrt{1-x^2-y^2}}{\sqrt{1-x^2-y^2}} \right] dy\, dx$$

$$\int_{x=0}^{1} \int_{y=0}^{\sqrt{1-x^2}} \sin^{-1}(1)\, dy\, dx$$

$$\boxed{\sin^{-1}(1) = \sin^{-1} \sin\frac{\pi}{2} = \frac{\pi}{2}}$$

$$\frac{\pi}{2} \int_{x=0}^{1} \int_{y=0}^{\sqrt{1-x^2}} 1\, dy \quad dx$$

$$\frac{\pi}{2} \int_{x=0}^{1} [y]_0^{\sqrt{1-x^2}} \, dx$$

$$\frac{\pi}{2} \int_{x=0}^{1} \sqrt{1-x^2} \quad dx$$

$$\frac{\pi}{2} \left[\frac{x}{2} \sqrt{1-x^2} + \frac{1}{2} \sin^{-1} x \right]_0^1$$

$$\frac{\pi}{2}\left[\frac{1}{2}\sin^{-1} 1\right]$$

$$= \frac{\pi}{2}\cdot\frac{1}{2}\cdot\frac{\pi}{2}$$

$$= \frac{\pi^2}{8}$$

$$\boxed{\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \frac{1}{\sqrt{1-x^2-y^2-z^2}}\, dx\, dy\, dz = \frac{\pi^2}{8}}$$

**

(4) Evaluate $\iiint_V (xy+yz+zx)\, dx\, dy\, dz$ where V is the region of space bounded by $x=0, x=1, y=0, y=2, z=0, z=3$

So given $\int_{x=0}^{1} \int_{y=0}^{2} \int_{z=0}^{3} (xy+yz+zx)\, dx\, dy\, dz$

$$= \int_{x=0}^{1} \int_{y=0}^{2} \left[xyz + y\frac{z^2}{2} + x\frac{z^2}{2} \right]_0^3 dy\, dx$$

$$= \int_{x=0}^{1} \int_{y=0}^{2} \left[3xy + \frac{9}{2}y + \frac{9}{2}x \right] dy\, dx$$

$$= \int_{x=0}^{1} \left[3x\frac{y^2}{2} + \frac{9}{2}\frac{y^2}{2} + \frac{9}{2}xy \right]_0^2 dx$$

$$= \int_{x=0}^{1} \left[6x + 9 + 9x \right] dx$$

$$= \int_{x=0}^{1} \left[15x + 9 \right] dx$$

$$= \left[15\frac{x^2}{2} + 9x \right]_0^1$$

$$= \left(\frac{15}{2} + 9 \right)$$

$$= \frac{33}{2}$$

$$\boxed{\iiint\limits_V (xy + yz + zx)\, dx\, dy\, dz = \frac{33}{2}}$$

⑤ Evaluate $\int_{-1}^{1}\int_{0}^{z}\int_{x-z}^{x+z}(x+y+z)\,dx\,dy\,dz$

Sol given $\int_{z=-1}^{1}\int_{x=0}^{z}\int_{y=x-z}^{x+z}(x+y+z)\,dx\,dy\,dz$

$$\int_{z=-1}^{1}\int_{x=0}^{z}\left[xy+\frac{y^2}{2}+yz\right]_{x-z}^{x+z}dx\,dz$$

$$\int_{z=-1}^{1}\int_{x=0}^{z}\Big\{x\left[(x+z)-(x-z)\right]$$
$$+\frac{1}{2}\left[(x+z)^2-(x-z)^2\right]$$
$$+z\left[(x+z)-(x-z)\right]\Big\}dx\,dz$$

$$\int_{z=-1}^{1}\int_{x=0}^{z}\left[2zx+2xz+2z^2\right]dx\,dz$$

$$\int_{z=-1}^{1}\int_{x=0}^{z}(4xz+2z^2)\,dx\,dz$$

$$\int_{z=-1}^{1} \left[4z\,\frac{x^2}{2} + 2z^2 x\right]_0^{z} dz$$

$$\int_{z=-1}^{1} \left[2z\,z^2 + 2z^2 z\right] dz$$

$$4\int_{z=-1}^{1} z^3 \, dz$$

$$4\left[\frac{z^4}{4}\right]_{-1}^{1}$$

$$= 1-1$$

$$= 0$$

$$\boxed{\int_{-1}^{1}\int_{0}^{z}\int_{x-z}^{x+z} (x+y+z)\,dx\,dy\,dz = 0}$$

⑥ Evaluate $\displaystyle\int_0^1 \int_0^{1-z} \int_0^{1-y-z} xyz \, dx \, dy \, dz$

⑦ Evaluate $\displaystyle\int_0^1 \int_0^{1-x} \int_0^{1-x-y} dx \, dy \, dz$

⑧ ** Evaluate $\displaystyle\int_0^1 \int_0^{1-x} \int_0^{1-x-y} e^x \, dx \, dy \, dz$

Se given $\displaystyle\int_{x=0}^{1} \int_{y=0}^{1-x} \int_{z=0}^{1-x-y} e^x \, dx \, dy \, dz$

$$= \int_{x=0}^{1} e^x \int_{y=0}^{1-x} \int_{z=0}^{1-x-y} 1 \, dz \, dy \, dx$$

$$= \int_{x=0}^{1} e^x \int_{y=0}^{1-x} \left[z\right]_0^{1-x-y} dy \, dx$$

$$= \int_{x=0}^{1} e^x \int_{y=0}^{1-x} \left[\underbrace{1-x}-y\right] dy \, dx$$

$$= \int_{x=0}^{1} e^{x} \left[(1-x)y - \frac{y^2}{2}\right]_0^{1-x} dx$$

$$= \int_{x=0}^{1} e^{x} \left[(1-x)(1-x) - \frac{(1-x)^2}{2}\right] dx$$

$$= \int_{x=0}^{1} e^{x} \, \frac{1}{2}(1-x)^2 \, dx$$

$$= \frac{1}{2} \int_{x=0}^{1} e^{x} (1-x)^2 \, dx$$

$$\frac{1}{2}\left[(1-x)^2 e^{x} + 2(1-x)e^{x} + 2e^{x}\right]_0^1$$

$$\frac{1}{2}\left[2e - (1+2+2)\right]$$

$$\frac{1}{2}\left[2e-5\right]$$

$$\boxed{\int_0^1 \int_0^{1-x} \int_0^{1-x-y} e^{x} \, dx \, dy \, dz = \frac{1}{2}(2e-5)}$$

Volume of a Double integral

let $z = f(x,y)$ be a surface above the xy-plane

The volume of the surface between xy plane and the given surface $z = f(x,y)$ is given by

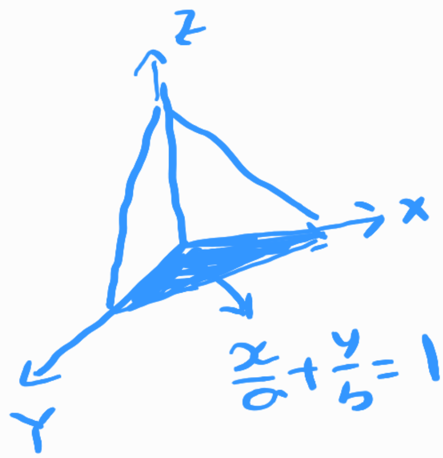
$$\boxed{\iint_S z \, dx \, dy}$$

(or) $\iint_S f(x,y) \, dx \, dy$ integrated over the region S

① Using double integration, find the volume of the tetrahedron bounded by the coordinate planes and the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$

Sol Given $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1 \rightarrow ①$

$\Rightarrow \quad z=c\left(1-\frac{x}{a}-\frac{y}{b}\right)$



In xy-plane $z=0$

put $z=0$ in (1)

$\frac{x}{a}+\frac{y}{b}=1 \qquad \boxed{y=b\left(1-\frac{x}{a}\right)}$

Here $y: 0$ to $b\left(1-\frac{x}{a}\right)$

$x: 0$ to a $\quad (y=0)$

Hence required volume $\displaystyle\iint_{x\,y} z \, dx \, dy$

$$= \int_{x=0}^{a} \int_{y=0}^{b(1-\frac{x}{a})} c\left(1-\frac{x}{a}-\frac{y}{b}\right) dx\, dy$$

$$= c \int_{x=0}^{a} \int_{y=0}^{b(1-\frac{x}{a})} \left(1-\frac{x}{a}-\frac{y}{b}\right) dy\, dx$$

$$\int (a+bx)\, dx = \frac{(a+bx)^2}{2 \cdot b}$$

$$= c \int_{x=0}^{a} \left[\frac{\left(1-\frac{x}{a}-\frac{y}{b}\right)^2}{2\cdot\left(-\frac{1}{b}\right)} \right]_{y=0}^{b(1-\frac{x}{a})} dx$$

$$= -\frac{bc}{2} \int_{x=0}^{a} \left\{ \left[1-\frac{x}{a}-\frac{1}{b}\, b\left(1-\frac{x}{a}\right)\right]^2 - \left(1-\frac{x}{a}\right)^2 \right\} dx$$

$$= \frac{bc}{2} \int_{x=0}^{a} \left(1-\frac{x}{a}\right)^2 dx$$

$$= \frac{bc}{2}\left[\frac{\left(1-\frac{x}{a}\right)^3}{3\left(-\frac{1}{a}\right)}\right]_0^a$$

$$= -\frac{abc}{6}\left[\left(1-\frac{x}{a}\right)^3\right]_0^a$$

$$= -\frac{abc}{6}\left[0-(1)\right]$$

$$= \frac{abc}{6}$$

(2) Find the volume bounded by the cylinders $x^2+y^2=4$; $y+z=4$, $z=0$

Sol Eqn of the cylinder is $x^2+y^2=4$ and $y+z=4$; $z=0$

Here $z=4-y$

$x^2+y^2=4$ represents circle

$x = r\cos\theta, \quad y = r\sin\theta$

$$r^2\cos^2\theta + r^2\sin^2\theta = 4$$

$$r^2 = 4$$

$$r = 2$$

r varies from 0 to 2

θ varies from 0 to 2π

$dx\,dy = r\,dr\,d\theta$

Since $z = 4 - y$

$$z = 4 - r\sin\theta$$

Hence required volume

$$= \iint z\,dx\,dy$$

$$= \int_{r=0}^{2} \int_{\theta=0}^{2\pi} (4 - r\sin\theta)\, r\,dr\,d\theta$$

$$= \int_{r=0}^{2} r\left[4\theta + r\cos\theta\right]_0^{2\pi} dr$$

$$= \int_{r=0}^{2} r\left[\left(8\pi + r\cos 2\pi\right) - r\right] dr$$

$$= 8\pi \int_{r=0}^{2} r \, dr$$

$$= 8\pi \left[\frac{r^2}{2}\right]_0^2$$

$$= \underline{16\pi}$$

③ Find the volume bounded by the xy plane. The cylinder $x^2+y^2=1$ and the plane $2x+3y+4z=12$

Sol given cylinder $x^2+y^2=1$ and

The plane $2x+3y+4z=12$

Here $z = \frac{1}{4}\left[12-2x-3y\right]$

Given circle $x^2+y^2=1$

$x = r\cos\theta$, $y = r\sin\theta$

$r^2\cos^2\theta + r^2\sin^2\theta = 1$

$r^2 = 1$

$r = 1$

r varies from 0 to 1

θ varies from 0 to 2π

Since $z = \frac{1}{4}[12 - 2x - 3y]$

$$z = \frac{1}{4}[12 - 2r\cos\theta - 3r\sin\theta]$$

$dx\,dy = r\,dr\,d\theta$

Hence required volume $= \iint_V z\,dx\,dy$

$$= \int_{r=0}^{1} \int_{\theta=0}^{2\pi} \frac{1}{4}[12 - 2r\cos\theta - 3r\sin\theta]\, r\,dr\,d\theta$$

$$= \frac{1}{4}\int_{\theta=0}^{2\pi} \int_{r=0}^{1} [12r - 2r^2\cos\theta - 3r^2\sin\theta]\,dr\,d\theta$$

$$= \frac{1}{4}\int_{\theta=0}^{2\pi} \left[12\frac{r^2}{2} - 2\frac{r^3}{3}\cos\theta - 3\frac{r^3}{3}\sin\theta\right]_0^1 d\theta$$

$$= \frac{1}{4} \int_{\theta=0}^{2\pi} \left[6 - \frac{2}{3}\cos\theta - \sin\theta \right] d\theta$$

$$= \frac{1}{4} \left[6\theta - \frac{2}{3}\sin\theta + \cos\theta \right]_0^{2\pi}$$

$$= \frac{1}{4} \left[\left(12\pi - \frac{2}{3}(0) + 1\right) - \left(0 - \frac{2}{3}(0) + 1\right) \right]$$

$$= \frac{1}{4} \left[12\pi + 1 - 1 \right]$$

$$= \underline{\underline{3\pi}}$$

④ Find the double integral, the volume of the solid bounded by $z = 0$, $x^2 + y^2 = 1$ and $x + y + z = 3$

Volume as a Triple integral

Suppose a three dimensional solid is cut into rectangular parallelpiped by drawing planes parallel to the coordinate planes

The volume of the solid is

$$\iiint_V dV = \iiint_V dx\,dy\,dz$$

where the integration is carried over the entire volume.

① Evaluate $\iiint (x+y+z)\,dx\,dy\,dz$ over the tetrahedron bounded by the co-ordinate planes and the plane $x+y+z=1$

Sol. Given plane $x+y+z=1$

z varies from 0 to $1-x-y$

y varies from 0 to $1-x$

x varies from 0 to 1

$$\iiint_V (x+y+z)\,dx\,dy\,dz$$

$$= \int_{x=0}^{1} \int_{y=0}^{1-x} \int_{z=0}^{1-x-y} (x+y+z)\,dz\,dy\,dx$$

$$\boxed{\text{Ans} = \frac{1}{4}}$$

(2) Evaluate $\iiint\limits_R (x+y+z)\,dx\,dy\,dz$

where R is the region bounded by

the plane $x=0$, $x=1$, $y=0$, $y=1$,

$z=0$, $z=1$

$$\boxed{\text{Ans: } \frac{3}{2}}$$

Multiple integrals Assignment

Models :

1. Double integrals
2. Double integrals without limits
3. polar co-ordinates
4. polar co-ordinates without limits
5. change the variables from cartesian to polar coordin
6. change the order of the integration
7. Area enclosed by the curve
8. Triple integrals
9. Volume of a double integral

① Evaluate $\int_0^1 \int_{x^2}^{\sqrt{x}} (x^2+y^2)\, dx\, dy$

② Evaluate $\iint_R y\, dx\, dy$ where R is the region bounded by the parabolas $y^2 = 16x$; $x^2 = 16y$

③ Evaluate $\int_0^{\pi} \int_0^{a(1+\cos\theta)} r\, dr\, d\theta$

④ Show that $\iint_R r^2 \sin\theta\, dr\, d\theta = \frac{2a^2}{3}$ where R is the semi circle $r = 2a\cos\theta$ above the initial line

⑤ Evaluate $\int_{-a}^{a} \int_0^{\sqrt{a^2-x^2}} \sqrt{x^2+y^2}\, dx\, dy$ by changing into polar co-ordinates

⑥ By changing the order of integration Evaluate $\int_0^2 \int_1^{\sqrt{2-y}} (x+y)\, dx\, dy$

7. Find the Area Enclosed by the Parabolas $x^2 = y$ and $y^2 = x$

8. Evaluate $\int_0^{\pi/2} \int_0^{a\cos\theta} \int_0^{\sqrt{a^2 - r^2}} r \, dz \, dr \, d\theta$

9. Find the volume bounded by the xy-plane, the cylinder $x^2 + y^2 = 1$ and the plane $x + y + z = 3$

OBJECTIVE TYPE QUESTIONS

1. $\int_{-1}^{2}\int_{x^2}^{x+2} dy\, dx =$

(a) $\frac{9}{2}$ (b) $\frac{9}{4}$ (c) $\frac{3}{2}$ (d) none

2. $\int_0^1\int_0^1\int_0^1 e^{x+y+z}\, dx\, dy\, dz =$

(a) $(e-1)^2$ (b) $(e-1)$ (c) $(e-1)^3$ (d) none

3. An equivalent iterated integral with the order of integration reversed for $\int_0^1\int_1^{e^x} dy\, dx$ is

(a) $\int_0^1\int_1^{e^y} dx\, dy$ (b) $\int_1^e\int_{\log y}^1 dx\, dy$ (c) $\int_e^1\int_1^{\log y} dx\, dy$ (d) none

4. $\iint xy\,dx\,dy$ over the region bounded by x - axis, ordinate at $x = 2a$ and the parabola $x^2 = 4ay$ is

(a) $\frac{a^2}{3}$ (b) $\frac{a^2}{8}$ (c) $\frac{a^4}{3}$ (d) none

5. The limits of integration of $\iint (x^2 + y^2)\,dx\,dy$ over the domain bounded by $y = x^2$ and $y^2 = x$ are

(a) $x = 0$ to 1, $y = \sqrt{x}$ to x^2 (b) $x = 0$ to $1, y = 0$ to 1

(c) $x = y^2$ to $\sqrt{y}$, $y = 0$ to 1 (d) none

6. $\int_0^1\int_0^2\int_1^2 x^2\,yz\,dz\,dy\,dx =$

(a) 1/2 (b) 1/4 (c) 1 (d) 3/2

7. $\iint r^3\,dr\,d\theta$ over the region included between the circles $r = 2\sin\theta$ and $r = 4\sin\theta$ is

(a) $\int_0^{\pi}\int_{2\sin\theta}^{4\sin\theta} r^3\,dr\,d\theta$ (b) $\int_0^{\pi/2}\int_{2\sin\theta}^{4\sin\theta} r^3\,dr\,d\theta$ (c) $\int_{-\pi}^{\pi}\int_{2\sin\theta}^{4\sin\theta} r^3\,dr\,d\theta$ (d) none

8. $\iiint (x^2 + y^2 + z^2)\,dz\,dy\,dx$ where V is the volume of the cube bounded by the coordinate planes $x = y = z = a$ is

(a) $\frac{a^5}{5}$ (b) $\frac{a^5}{25}$ (c) a^5 (d) $\left(\frac{a}{5}\right)^5$

9. The value of the double integral $\int_0^3\int_0^2 (4 - y)^2\,dy\,dx$ is

(a) 16 (b) $\frac{16}{3}$ (c) $\frac{8}{3}$ (d) none

10. $\iint \frac{xy}{\sqrt{1 - y^2}}\,dx\,dy$ over the positive quadrant of the circle $x^2 + y^2 = 1$ is

(a) 1/6 (b) 2/3 (c) 5/6 (d) none

11. $\int_0^1\int_1^2 xy\,dy\,dx =$ ____________.

12. $\int_0^2\int_0^x (x + y)\,dx\,dy =$ ____________.

13. $\int_0^2\int_{y/2}^1 e^{x^2}\,dx\,dy =$ ____________.

14. $\int_{y=0}^{1}\int_{x=y^{3/2}}^{y^{2/3}} dx\,dy =$ ____________.

15. $\int_0^a\int_0^{\sqrt{a^2 - y^2}} dx\,dy =$ ____________.

16. $\int_0^a \int_0^{\sqrt{ay}} xy\, dx\, dy =$ ____________.

17. $\int_0^2 \int_1^3 \int_1^2 xy^2 z\, dz\, dy\, dx =$ ____________.

18. $\int_{-1}^{1} \int_{-2}^{2} \int_{-3}^{3} dx\, dy\, dz =$ ____________.

19. $\int_0^1 \int_0^{1-x} \int_0^{1-x^2-y^2} dz\, dy\, dx =$ ____________.

20. The iterated integral for $\int_{-1}^{1} \int_0^{\sqrt{1-x^2}} f(x,y)\, dy\, dx$ after changing the order of integration is ______.

21. The iterated integral for $\int_0^1 \int_{y=x^2}^{x} f(x,y)\, dy\, dx$ after changing the order of integration is ______.

22. The iterated integral for $\int_0^1 \int_x^{\sqrt{x}} f(x,y)\, dy\, dx$ after changing the order of integration is ______.

23. The iterated integral for $\int_0^1 \int_1^{e^x} dy\, dx$ after changing the order of integration is ____________.

24. The iterated integral for $\int_{-a}^{a} \int_0^{\sqrt{a^2-y^2}} x\, dx\, dy$ after changing the order of integration is ______.

25. $\int_1^0 \int_0^1 (x+y) dx\, dy =$

(a) 2 (b) -2 (c) -1 (d) 1

26. $\int_0^1 dx \int_0^x e^{y/x}\, dy =$

(a) $e-1$ (b) $\frac{1}{2}(e-1)$ (c) $\frac{1}{3}(e-1)$ (d) none

27. $\int_0^1 \int_0^x e^x\, dx\, dy =$

(a) 2 (b) -2 (c) 1 (d) -1

28. $\int_0^1 \int_x^{\sqrt{x}} xy \, dy \, dx =$

(a) $\frac{1}{8}$ (b) $\frac{1}{16}$ (c) $\frac{1}{24}$ (d) none

29. $\int_0^2 \int_0^{x^2} x(x^2+y^2) dx \, dy =$

(a) $\frac{32}{3}$ (b) $\frac{64}{3}$ (c) $\frac{84}{3}$ (d) none

30. $\int_0^{\pi} \int_0^{a\cos\theta} r \sin\theta \, dr \, d\theta =$

(a) $\frac{a^2}{2}$ (b) $\frac{a^2}{3}$ (c) $\frac{a^3}{3}$ (d) $\frac{a^3}{4}$

31. $\int_0^{\infty} \int_0^{\infty} e^{-(x^2+y^2)} dx \, dy =$

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{2}$ (c) 0 (d) $\frac{\pi}{6}$

32. On converting into polar coordinates,

$\int_0^{a} \int_0^{\sqrt{a^2-x^2}} (x^2+y^2) dy \, dx =$

(a) $\int_0^{a} \int_0^{\pi/2} r^2 \, dr \, d\theta$ (b) $\int_0^{a} \int_0^{\pi/2} r^3 \, dr \, d\theta$ (c) $\int_0^{a} \int_0^{\pi/4} r^3 \, dr \, d\theta$ (d) $\int_0^{a} \int_0^{\pi/4} r^2 \, dr \, d\theta$

33. In polar coordinates $\int_0^{\infty} \int_0^{\infty} e^{-(x^2+y^2)} dx \, dy =$

(a) $\int_0^{\pi/2} \int_0^{\infty} e^{-r^2} dr \, d\theta$ (b) $\int_0^{\pi/4} \int_0^{\infty} e^{-r} r \, dr \, d\theta$ (c) $\int_0^{\pi/2} \int_0^{\infty} e^{-r^2} r \, dr \, d\theta$ (d) $\int_0^{\pi/2} \int_0^{\infty} e^{-r} dr \, d\theta$

34. The value of $\iint_R xy \, dx \, dy =$______, where R is the region in the positive quadrant of the circle $x^2+y^2=a^2$.

(a) $\frac{a^2}{8}$ (b) $\frac{a^3}{8}$ (c) $\frac{a^4}{8}$ (d) $\frac{a^4}{2}$

35. The value of $\iint_R x^2y^3\,dx\,dy =$ ________, where R is the region bounded by the rectangle $0 \le x \le 1$ and $0 \le y \le 3$.

(a) $\frac{27}{4}$ (b) $\frac{27}{8}$ (c) $\frac{29}{4}$ (d) $\frac{29}{8}$

36. The iterated integral for $\int_0^\infty \int_x^\infty f(x,y)\,dx\,dy$ after changing the order of integration is

(a) $\int_0^\infty \int_0^y f(x,y)\,dx\,dy$ (b) $\int_0^\infty \int_0^x f(x,y)\,dx\,dy$ (c) $\int_0^\infty \int_y^\infty f(x,y)\,dx\,dy$ (d) $\int_x^\infty \int_0^\infty f(x,y)\,dx\,dy$

37. $\int_{-1}^{1} \int_{-2}^{2} \int_{-3}^{3} dx\,dy\,dz =$

(a) 12 (b) 24 (c) 48 (d) 36

38) $\int_0^1 \int_0^1 \int_0^1 xyz\,dx\,dy\,dz =$

(a) $\frac{1}{3}$ (b) $\frac{1}{5}$ (c) $\frac{1}{8}$ (d) $\frac{1}{12}$

39. The volume of the tetrahedron bounded by the surfaces $x=0, y=0, z=0$ and $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ is

(a) $\frac{abc}{2}$ (b) $\frac{abc}{4}$ (c) $\frac{abc}{6}$ (d) $\frac{abc}{3}$

40. The volume of the tetrahedron bounded by the coordinate planes and the plane $x+y+z=1$ is

(a) $\frac{1}{2}$ (b) $\frac{1}{3}$ (c) $\frac{1}{4}$ (d) $\frac{1}{6}$

ANSWERS

1) a 2) c 3) b 4) c 5) a 6) c 7) b

8) c 9) a 10) a 11) $\frac{3}{4}$ 12) 4 13) e–1 14) $\frac{1}{5}$

15) $\frac{\pi a^2}{4}$ 16) $\frac{a^4}{6}$ 17) 26 18) 48 19) $\frac{\pi a^2}{4}$ 20) $\int_0^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} f(x,y)\,dx\,dy$

21) $\int_0^1 dy \int_y^{\sqrt{y}} f(x,y)\,dx$ 22) $\int_0^1 dy \int_{y^2}^{y} f(x,y)\,dx$ 23) $\int_1^e \int_{\log y}^{1} dx\,dy$ 24) $\int_0^a \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-y^2}} x\,dy\,dx$

25) d 26) b 27) c 28) c 29) b 30) b

31) a 32) b 33) c 34) c 35) a 36) a

37) c 38) c 39) c 40) d