2.2

Bivariate random variable

In the real life situations more than one variable effects the outcome of a random experiment. For example, consider an electronic system consisting of two components. Suppose the system will fail if both the components fail. The probability distribution of the life of the system depends jointly on the probability distributions of lives of the components. Knowing the probability distributions of lives of the components will not provide us the enough information. What we need is the probability distribution of the simultaneous behavior of lives of the components. A pair of random variables is known as a **bivariate random variable.** The individual random variables in the pair may be related.

Bivariate random variable: let S be the sample space associated with a random experiment. Let $\boldsymbol{R}$ be the real line. If (X, Y): $S \longrightarrow R \times R$, $i.e., (X,Y)(\omega) = \big(X(\omega), Y(\omega)\big)$ $\forall\ \omega \in S$, then the pair (X, Y) is known as a bivariate random variable.

Note:

1. If X and Y are both discrete random variables, then (X, Y) is a bivariate discrete random variable.
2. If X and Y are both continuous random variables, then (X, Y) is a bivariate continuous random variable.

Joint probability mass function: Let (X, Y) be a bivariate discrete random variable, which takes the values (x_i, y_j) for $i = 1,2,3, \dots, m$ and $j = 1,2,3, \dots, n$. Let

$$p(x_i, y_j) = P(X = x_i, Y = y_j)\ \forall\ i \text{ and } j$$

Then $p(x_i, y_j) \geq 0\ \forall\ i$ and j and $\sum_{i=1}^{m} \sum_{j=1}^{n} p\left(x_i, y_j\right) = 1$. The function $p(x, y)$ is known as **joint probability mass function** (j.p.m.f) of (X, Y).

Marginal probability mass functions: Let (X, Y) be a bivariate discrete random variable with joint probability mass function given by (x, y) . The marginal probability mass function of X and Y are given by

$$p_1(x_i) = \sum_{j=1}^{n} p\left(x_i, y_j\right) \text{ for } i = 1,2,3, \dots, m \text{ and}$$

$$p_2(y_j) = \sum_{i=1}^{m} p\left(x_i, y_j\right) \text{ for } i = 1,2,3, \dots, n$$

respectively.

Note: X and Y are independent if and only if $p(x_i, y_j) = p_1(x_i)p_2(x_j)\ \forall\ (i, j)$

Conditional probability mass functions: Let (X, Y) be a bivariate discrete random variable with joint probability mass function given by $p(x, y)$. The conditional probability mass function of X given $y = y_j$ and the conditional probability mass function of Y given $x = x_i$ are given by

$$p_{1|2}(x_i|\ y_j) = \frac{p(x_i, y_j)}{p_2(y_j)} \text{ for } i = 1,2,3, \dots, m \text{ and}$$

$$p_{2|1}(\ y_j|\ x_i) = \frac{p(x_i, y_j)}{p_1(x_i)}\ \ j = 1,2,3, \dots, n$$

respectively.

Example 1: A fair coin is tossed three times. Let X be a random variable that takes the value 0 if the first toss is a tail and the value 1 if the first toss is a head and Y be a random variable that defines the total number of heads in the three tosses. Then

i. **Determine the joint, marginal and conditional mass functions of X and Y.**
ii. **Are X and Y independent?**

Solution:

i. The sample space and values of X and Y are given in the following table:

Out comes in sample space	Value of X	Value of Y
HHH	1	3
HHT	1	2
HTH	1	2
HTT	1	1
THH	0	2
THT	0	1
TTH	0	1
TTT	0	0

Here X takes the values 0 and 1 and Y takes the values 0 ,1,2 and 3. Then the j.p.m.f of (x, y) is computed as below:

$$p(0,0) = P(X = 0, Y = 0) = P(\{TTT\}) = \frac{1}{8}$$

$$p(0,1) = P(X = 0, Y = 1) = P(\{THT, TTH\}) = \frac{2}{8} = \frac{1}{4}$$

$$p(0,2) = P(X = 0, Y = 2) = P(\{THH\}) = \frac{1}{8}$$

$$p(0,3) = P(X = 0, Y = 3) = 0$$

$$p(1,0) = P(X = 1, Y = 0) = 0$$

$$p(1,1) = P(X = 1, Y = 1) = P(\{HTT\}) = \frac{1}{8}$$

$$p(1,2) = P(X = 1, Y = 2) = P(\{\ HTH, HHT\}) = \frac{2}{8} = \frac{1}{4}$$

$$p(1,3) = P(X = 1, Y = 3) = P(\{HHH\}) = \frac{1}{8}$$

The m.p.m.f of X is given by

$$p_1(0) = p(0,0) + p(0,1) + p(0,2) + p(0,3) = \frac{1}{8} + \frac{2}{8} + \frac{1}{8} + 0 = \frac{1}{2}$$ and

$$p_1(1) = p(1,0) + \ p(1,1) + p(1,2) + p(1,3) = 0 + \frac{1}{8} + \frac{2}{8} + \frac{1}{8} = \frac{1}{2}$$

The m.p.m.f of Y is given by

$$p_2(0) = p(0,0) + p(1,0) = \frac{1}{8} + 0 = \frac{1}{8}$$

$$p_2(1) = p(0,1) + p(1,1) = \frac{2}{8} + \frac{1}{8} = \frac{3}{8}$$

$$p_2(2) = p(0,2) + p(1,2) = \frac{1}{8} + \frac{2}{8} = \frac{3}{8}$$

$$p_2(3) = p(0,3) + p(1,3) = 0 + \frac{1}{8} = \frac{1}{8}$$

The conditional p.m.f of X given Y is computed below:

$$p_{1|2}\ (0|0) = \frac{p(0,0)}{p_2(0)} = \frac{\frac{1}{8}}{\frac{1}{8}} = 1\ , \qquad p_{1|2}\ (1|0) = \frac{p(1,0)}{p_2(0)} = \frac{0}{\frac{1}{8}} = 0$$

$$p_{1|2}\ (0|1) = \frac{p(0,1)}{p_2(1)} = \frac{\frac{2}{8}}{\frac{3}{8}} = \frac{2}{3}, \qquad p_{1|2}\ (1|1) = \frac{p(1,1)}{p_2(1)} = \frac{\frac{1}{8}}{\frac{3}{8}} = \frac{1}{3}$$

$$p_{1|2}\ (0|2) = \frac{p(0,2)}{p_2(2)} = \frac{\frac{1}{8}}{\frac{3}{8}} = \frac{1}{3}, \qquad p_{1|2}\ (1|2) = \frac{p(1,2)}{p_2(2)} = \frac{\frac{2}{8}}{\frac{3}{8}} = \frac{2}{3}$$

$$p_{1|2}\ (0|3) = \frac{p(0,3)}{p_2(3)} = \frac{0}{\frac{1}{8}} = 0\ , \qquad p_{1|2}\ (1|3) = \frac{p(1,3)}{p_2(3)} = \frac{\frac{1}{8}}{\frac{1}{8}} = 1$$

The conditional p.m.f. of Y given X is computed as below:

$$p_{2|1}\ (0|0) = \frac{P(0,0)}{P_1(0)} = \frac{\frac{1}{8}}{\frac{1}{2}} = \frac{1}{4}, \qquad P_{2|1}\ (1|0) = \frac{p(0,1)}{p_1(0)} = \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2}$$

$$p_{2|1}\ (2|0) = \frac{p(0,2)}{p_1(0)} = \frac{\frac{1}{8}}{\frac{1}{2}} = \frac{1}{4}, \qquad p_{2|1}\ (3|0) = \frac{p(0,3)}{p_1(0)} = \frac{0}{\frac{1}{2}} = 0$$

$$p_{2|1}\ (0|1) = \frac{p(1,0)}{p_1(1)} = \frac{0}{\frac{1}{2}} = 0, \qquad p_{2|1}\ (1|1) = \frac{p(1,1)}{p_1(1)} = \frac{\frac{1}{8}}{\frac{1}{2}} = \frac{1}{4}$$

$p_{2|1}\ (2|1) = \frac{p(1,2)}{p_1(1)} = \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2}$, $p_{2|1}\ (3|1) = \frac{p(1,3)}{p_1(1)} = \frac{\frac{1}{8}}{\frac{1}{2}} = \frac{1}{4}$

ii. Here $p(0,0) = \frac{1}{8}$, $p_1(0) = \frac{1}{2}$ and $p_2(0) = \frac{1}{8}$

Since $p(0,0) \neq p_1(0)p_2(0)$, X and Y are not independent .

Example 2 : The j.p.m.f. of (X, Y) is given by

$$p(x,y)=\begin{cases} k(2x+y) & \text{for } x=1,2;\ y=1,2 \\ 0 & \text{otherwise} \end{cases}$$

where k is a constant

a. **Find the value of k.**
b. **Find marginal and conditional p.m.fs.**
c. **Are X and Y independent.**

Solution:

a. Since $p(x, y)$ is a j.p.m.f, $\sum_{x=1}^{2}\sum_{y=1}^{2} p(x,y)=1$

$$\sum_{x=1}^{2}\sum_{y=1}^{2} p(x,y)=k\sum_{x=1}^{2}\sum_{y=1}^{2}(2x+y)=k(3+4+5+6)=18k=1.$$

Thus $k = \frac{1}{18}$

b. The m.p.m.f. of X is given by

$$p_1(x)=\sum_{y=1}^{2} p(x,y)=\frac{1}{18}\sum_{y=1}^{2} 2x+y=\frac{1}{18}\left[(2x+1)+(2x+2)\right]=\frac{4x+3}{18}$$

Thus, $p_1(x) = \frac{4x+3}{18}$ for $x = 1,2$.

The m.p.m.f of Y is given by

$$p_2(y)=\sum_{x=1}^{2} p(x,y)=\frac{1}{18}\sum_{x=1}^{2} 2x+y=\frac{1}{18}\left[(2+y)+(4+y)\right]=\frac{2y+6}{18}=\frac{y+3}{9}$$

Thus, $p_2(y)=\frac{y+3}{9}$ for $y=1,2$

The c.p.m.f. of X given Y is given by

$$p_{1|2}(x|y)=\frac{p(x,y)}{p_2(y)}=\frac{\frac{1}{18}(2x+y)}{\frac{1}{18}(2y+6)}=\frac{2x+y}{2y+6}$$

Thus, $p_{1|2}(x|y)=\frac{2x+y}{2y+6}$ for $x=1,2$

The c.p.m.f. of Y given X is given by

$$p_{2|1}(y|x)=\frac{p(x,y)}{p_1(x)}=\frac{\frac{1}{18}(2x+y)}{\frac{1}{18}(4x+3)}=\frac{2x+y}{4x+3}$$

Thus, $p_{2|1}(y|x)=\frac{2x+y}{4x+3}$ for $y=1,2$

c. Note that $p_1(x).p_2(y)=\frac{4x+3}{18}.\frac{y+3}{9}\neq p(x,y)$.
   Thus, X and Y are not independent.

Joint probability density function: Let (X,Y) be a bivariate continuous random variable. Let

$$P(a\leq x\leq b, c\leq y\leq d)=\int_a^b\int_c^d f(x,y)dx\,dy$$

for some real numbers a, b, c, d such that $a < b$ and $c < d$. Then

i. $f(x, y) \geq 0 \ \forall (x, y)$ and

ii. $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y) dx\, dy = 1$

and the function $f(x, y)$ is known as the **joint probability density function** of the bivariate continuous random variable (X, Y).

Marginal probability density function: Let (X, Y) be a bivariate continuous random variable with j.p.d.f. $f(x, y)$. The marginal probability density functions of X and Y are given by

$$f_1(x) = \int_{-\infty}^{\infty} f(x,y) dy \quad \text{and} \quad f_2(y) = \int_{-\infty}^{\infty} f(x,y) dx$$

respectively.

Note: X and Y are independent if and only if $f(x, y) = f_1(x). f_2(y)$

Conditional probability density functions: Let (X, Y) be a bivariate continuous random variable with j.p.d.f. $f(x, y)$. Let $f_1(x)$ and $f_2(y)$ be the m.p.d.fs of X and Y respectively.The conditional probability density function of X given Y and the conditional probability density function of Y given $X$are given by

$$f_{1|2}\ (x|y) = \frac{f(x,y)}{f_2(y)} \text{ and}$$

$$f_{2|1}\ (y|x) = \frac{f(x, y)}{f_1(x)}$$

respectively.

Cumulative distribution function: The cumulative distribution of a bivariate random variable (X, Y) is defined by

$$F(x, y) = P(X \leq x, Y \leq y) \text{ and}$$

$$F(x,y)=\begin{cases}\sum\limits_{t\le x}\sum\limits_{s\le y} p(t,s) \ \ if\ (X,Y) is\, a\ \ d.r.v\ with\ j.p.m.f.\,p(x,y)\\ \int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} f(t,s)dt\,ds\ if\ \ (X,Y) is\, a\ \ c.r.v\ with\ j.p.d.f.\,f(x,y)\end{cases}$$

Properties of cumulative distribution function

1. $0 \le F(x,y) \le 1$
2. $F(\infty,\infty) = 1\ , F(-\infty,-\infty) = 0$
3. $P(a < X \le b, Y \le d) = F(b,d) - F(a,d)$ and
   $P(\ X \le b, c < Y \le d) = F(b,d) - F(b,c)$
4. $P(a < X \le b\ , c < Y \le d) = F(b,d) - F(a,d) - F(b,c) + F(a,c)$

Marginal cumulative distribution function: Let (X,Y) be a bivariate random variable with c.d.f. $F(x,y)$.The marginal cumulative distribution functions of X and Y are given by $F_1(\ x) = F(x,\infty)$ and $F_2(\ y) = F(\infty,y)$ respectively.

Note:

If (X,Y) is a bivariate continuous random variable with c.d.f. $F(x,y)$, then its j.p.d.f. is given by

$$f(x,y) = \frac{\partial^2 F(x,y)}{\partial x\ \partial y}$$

Example 3: The j.p.d.f. of (X,Y) is given by

$$f(x,y)=\begin{cases} e^{-(x+y)} & for\ 0<x<\infty, 0<y<\infty \\ 0 & otherwise\end{cases}$$

a. **Find marginal p.d.fs of X and Y.**
b. **Are X and Y independent?**

Solution:

a. The m.p.d.f of X is given by

$$f_1(x)=\int\limits_0^{\infty} f(x,y)dy=\int\limits_0^{\infty} e^{-(x+y)}dy=e^{-x}\int\limits_0^{\infty} e^{-y}dy=e^{-x}.1=e^{-x}$$

$$=\begin{cases} e^{-x} \text{ for } x \geq 0 \\ 0 \quad \text{otherwise} \end{cases}$$

The m.p.d.f. of Y is given by

$$f_2(y) = \int_0^{\infty} f(x,y)dx = \int_0^{\infty} e^{-(x+y)}dx = e^{-y}\int_0^{\infty} e^{-x}dx = e^{-y}.1 = e^{-y}$$

$$=\begin{cases} e^{-y} \text{ for } y \geq 0 \\ 0 \quad \text{otherwise} \end{cases}$$

b. Since $f_1(x).f_2(y) = e^{-x}.e^{-y} = e^{-(x+y)} = f(x,y)$, X and Y are independent.

Example 4: The j.p.d.f. of (X, Y) is given by

$$f(x,y) = \begin{cases} xe^{-x(y+1)} & 0 < x < \infty, 0 < y < \infty \\ 0 & \text{otherwise} \end{cases}$$

a. **Determine the marginal and conditional p.d.fs**
b. **Are X and Y independent.**

Solution: The m.p.d.f of X is given by

$$f_1(x) = \int_0^{\infty} f(x,y)dy = \int_0^{\infty} xe^{-x(y+1)}dy = xe^{-x}\int_0^{\infty} e^{-xy}dy = xe^{-x}\left[\frac{e^{-xy}}{-x}\right]_{y=0}^{y=\infty} = e^{-x}$$

$$\Rightarrow f_1(x) = e^{-x} \text{ for } 0 < x < \infty$$

The m.p.d.f. of Y is given by

$$f_2(y) = \int_0^{\infty} f(x,y)dx = \int_0^{\infty} xe^{-x(y+1)}dx = \left[\frac{x.e^{-x(y+1)}}{y+1}\right]_0^{\infty} + \frac{1}{y+1}\int_0^{\infty} e^{-x(y+1)}dx$$

(using integration by parts)

$$= 0 - \frac{1}{(y+1)^2}\left[e^{-x(y+1)}\right]_0^\infty = \frac{1}{(y+1)^2}$$

$$\Rightarrow f_2(y) = \frac{1}{(y+1)^2} \quad for\ 0 < y < \infty$$

The conditional p.d.f of X given Y is given by

$$f_{1|2}(x|y) = \frac{f(x,y)}{f_2(y)} = \frac{x.e^{-x.(y+1)}}{\frac{1}{(y+1)^2}} = x(y+1)^2 e^{-x.(y+1)}$$

$\Longrightarrow f_{1|2}(x|y) = x(y+1)^2 e^{-x.(y+1)}$ for $0 < x < \infty$

The conditional p.d.f of Y given X is given by

$$f_{2|1}(x|y) = \frac{f(x,y)}{f_1(x)} = \frac{x.e^{-x.(y+1)}}{e^{-x}} = x.e^{-xy}$$

$\Longrightarrow f_{2|1}(x|y) = x.e^{-xy}$for $0 < x < \infty$

Note that $f_1(x).f_2(y) = e^{-x}.\frac{1}{(y+1)^2} \neq f(x,y)$. Hence, X and Y are not independent .

Example 5: The j.p.d.f of (X, Y) is given by $f(x, y) = kx^3y$ for $0 < x < 2,\ 0 < y < 1$.

a. **Find k**
b. **Find the m.p.d.fs of X and Y**
c. **Are X and Y independent.**

Solution:

a. We have $\int_0^2\int_0^1 f(x,y)dx\,dy = \int_0^2\int_0^1 kx^3y\,dx\,dy = k\int_0^2 x^3\left(\int_0^1 y\,dy\right)dx = k\int_0^2 x^3\frac{1}{2}dx$

$$= \frac{k}{2}\left[\frac{x^4}{4}\right]_0^2 = \frac{k}{8}\times 16 = 2k$$

Now , $\int_0^2\int_0^1 f(x,y)dx\,dy = 1 \Rightarrow 2k = 1 \Rightarrow k = \frac{1}{2}$

The j.p.d.f of (X, Y) is given by

$$f(x, y) = \frac{1}{2}x^3y \text{ for } 0 < x < 2,\ 0 < y < 1$$

The m.p.d.f of X is given by

$$f_1(x) = \int_0^1 f(x,y)dy = \frac{1}{2}x^3\int_0^1 y\,dy = \frac{1}{2}x^3\left[\frac{y^2}{2}\right]_0^1 = \frac{1}{4}x^3$$

$\Rightarrow f_1(x) = \frac{x^3}{4}$ for $0 < x < 2$.

The m.p.d.f. of Y is given by

$$f_2(y) = \int_0^2 f(x,y)dx = \frac{1}{2}y\int_0^2 x^3\,dx = \frac{1}{2}y\left[\frac{x^4}{4}\right]_0^2 = 2y$$

$\Rightarrow f_2(y) = 2y$ for $0 < y < 1$.

b. Note that $f_1(x)f_2(y) = \frac{x^3}{4}.2y = \frac{x^3y}{2} = f(x,y), \forall(x,y)$

Since $f_1(x)f_2(y) = f(x,y), \forall(x,y)$ X and Y are independent.