1.5

BASES AND DIMENSION

DEFINITION: A finite set of vectors $\{\overrightarrow{v_1}, \ldots\ldots, \overrightarrow{v_m}\}$ is called a basis for a vector space V, if the set spans V and is linearly independent.

Intuitively, a basis is an efficient set for characterizing a vector space, in that any vector can be expressed as a linear combination of the basis vectors, and the basis vectors are independent of one another.

Example: The set of 'n' vectors $\{(1,0, \ldots ,0), (0,1,0, \ldots ,0), \ldots, (0, \ldots ,0,1)\}$ is a basis for $\mathbb{R}^n$. This basis is called the standard basis for $\mathbb{R}^n$.

THEOREM: Let $\{\overrightarrow{v_1}, \overrightarrow{v_2}, \ldots, \overrightarrow{v_n}\}$ be a basis be a basis for a vector space V. If $\{\overrightarrow{\omega_1}, \overrightarrow{\omega_2}, \ldots, \overrightarrow{\omega_m}\}$ is a set of more than n vectors in V, then this set is linearly dependent.

Proof: We examine the identity $c_1\overrightarrow{\omega_1} + \cdots . + c_m\overrightarrow{\omega_m} = 0$ .....(1)

We will show that values of $c_1, \ldots\ldots, c_{m,}$ not all zono, exist, satisfying this identity and proving that the vectors are linearly dependent.

Since the set $\{\overrightarrow{v_1}, \overrightarrow{v_2}, \ldots, \overrightarrow{v_n}\}$ is a basis for V, each of the vectors $\{\overrightarrow{\omega_1}, \overrightarrow{\omega_2}, \ldots, \overrightarrow{\omega_m}\}$ can be expressed as a linear combination of $\overrightarrow{v_1}, \overrightarrow{v_2}, \ldots\ldots, \overrightarrow{v_n}$

Let $$\omega_1 = a_{11}\overrightarrow{v_1} + a_{12}\overrightarrow{v_2} + \cdots . + a_{1n}\overrightarrow{v_n}$$

$$\omega_2 = a_{21}\overrightarrow{v_1} + a_{22}\overrightarrow{v_2} + \cdots . + a_{2n}\overrightarrow{v_n}$$

.

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$$\omega_m = a_{m1}\overrightarrow{v_1} + a_{m2}\overrightarrow{v_2} + \cdots . + a_{mn}\overrightarrow{v_n}$$

Substituting these values in (1) we get

$$c_1(a_{11}\overrightarrow{v_1} + a_{12}\overrightarrow{v_2} + \cdots . + a_{1n}\overrightarrow{v_n}) + \cdots + c_m(a_{m1}\overrightarrow{v_1} + a_{m2}\overrightarrow{v_2} + \cdots . + a_{mn}\overrightarrow{v_n}) = 0$$

Rearranging, we get

$$(c_1a_{11} + c_2a_{21} + \cdots + c_m a_{m1})\overrightarrow{v_1} + \cdots + (c_1a_{1n} + c_2a_{2n} + \cdots + c_m a_{mn})\overrightarrow{v_n} = 0$$

Since $\overrightarrow{v_1}, \overrightarrow{v_2}, \dots, \overrightarrow{v_n}$ are linearly independent, we get

$$a_{11}c_1 + a_{21}c_2 + \cdots + a_{m1}c_m = 0$$

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$$a_{1n}c_1 + a_{2n}c_2 + \cdots + a_{mn}c_m = 0$$

Thus finding $c's$ that satisfy equation (1) reduces to finding solutions to this system of $'n'$ equations in $'m'$ variables. Since $m > n$, the number of variables is greater than the number of equations. We know that such a system of homogeneous equations has many solutions.

Therefore, there are non-zero values of $c's$ that satisfy equation(1). Thus the set $\{\overrightarrow{\omega_1}, \overrightarrow{\omega_2}, \dots, \overrightarrow{\omega_m}\}$ is linearly dependent.

THEOREM: Any two bases for a vector space V consist of the same number of vectors.

Proof: Let $\{\overrightarrow{v_1}, \overrightarrow{v_2}, \dots, \overrightarrow{v_n}\}$ and $\{\overrightarrow{\omega_1}, \overrightarrow{\omega_2}, \dots, \overrightarrow{\omega_m}\}$ be two bases for V . If we interpret $\{\overrightarrow{v_1}, \overrightarrow{v_2}, \dots, \overrightarrow{v_n}\}$ as a basis for V and $\{\overrightarrow{\omega_1}, \overrightarrow{\omega_2}, \dots, \overrightarrow{\omega_m}\}$ as a set of linearly independent vectors in V, than the previous theorem tells us that $m \leq n$. conversely, if we interpret $\{\overrightarrow{\omega_1}, \overrightarrow{\omega_2}, \dots, \overrightarrow{\omega_m}\}$ as a basis for V and $\{\overrightarrow{v_1}, \overrightarrow{v_2}, \dots, \overrightarrow{v_n}\}$ as a set of linearly independent vectors in V, then $n \leq m$. Thus $n = m$, proving that both the bases consists of same number of vectors.

DEFINITION: If a vector space V has a basis consisting of $'n'$ vectors, than the dimension of V is said to be n. we write $\dim(v)$ for dimension of V.

EXAMPLE: The set of $'n'$ vectors $\{(1,0,\dots,0),\dots,(0,\dots 0,1)\}$ forms a basis (the stranded basis) for $\mathbb{R}^n$. Thus the dimension of $\mathbb{R}^n$ is $'n'$.

Note that we have defined a basis for a vector space to be a finite set of vectors that spans the space and is linearly independent. Such a set does not exist for all vector spaces. When such a finite set exists, we say that the vector space is finite dimensional. If such a finite set does not exist, we say that the vector space is infinite dimensional.

THEOREM: Let $\{\overrightarrow{v_1}, \overrightarrow{v_2}, \dots, \overrightarrow{v_n}\}$ be a basis for a vector space V. Then each vector in V can be expressed uniquely as a linear combination of these vectors.

Proof: Let 'v' be a vector in V. Since $\{\overrightarrow{v_1}, \overrightarrow{v_2}, \dots, \overrightarrow{v_n}\}$ is a basis, we can express v as a linear combination of these vectors.

Suppose we can write

$v = a_1\overrightarrow{v_1} + a_2\overrightarrow{v_2} + \cdots + a_n\overrightarrow{v_n}$ and

$v = b_1\overrightarrow{v_1} + b_2\overrightarrow{v_2} + \cdots + b_n\overrightarrow{v_n}$ then

$a_1\overrightarrow{v_1} + a_2\overrightarrow{v_2} + \cdots + a_n\overrightarrow{v_n} = b_1\overrightarrow{v_1} + b_2\overrightarrow{v_2} + \cdots + b_n\overrightarrow{v_n}$

$\Rightarrow (a_1 - b_1)\overrightarrow{v_1} + (a_2 - b_2)\overrightarrow{v_2} + \cdots + (a_n - b_n)\overrightarrow{v_n} = 0$

Since $\{\overrightarrow{v_1}, \overrightarrow{v_2}, \dots, \overrightarrow{v_n}\}$ is a basis, the vectors $\overrightarrow{v_1}, \overrightarrow{v_2}, \dots, \overrightarrow{v_n}$ are linearly independent. Thus $(a_1 - b_1) = 0, \dots, (a_n - b_n) = 0$ implying that $a_1 = b_1, \dots, a_n = b_n$

Therefore there is only one way of expressing v as a linear combination of the basis.

Lemma. Let S be a linearly independent subset of a vector space V. Suppose β is a vector in V which is not in the subspace spanned by S. Then the set obtained by adjoining β to S is linearly independent.

Proof: Suppose $\alpha_1, \dots, \alpha_m$ are distinct vectors in S and that $c_1\alpha_1 + \cdots + c_m\alpha_m + b\beta = 0$.

Then $b = 0$; for otherwise,

$$\beta = \left(-\frac{c_1}{b}\right)\alpha_1 + \cdots + \left(-\frac{c_m}{b}\right)\alpha_m$$

and β is in the subspace spanned by S. Thus $c_1\alpha_1 + \cdots + c_m\alpha_m = 0$, and since S is a linearly independent set each $c_i = 0$.

Theorem: If W is a subspace of finite-dimensional vector space V, every linearly independent subset of W is finite and is part of a basis for W.

Proof: Suppose S_0 is a linearly independent subset of W. If S is a linearly independent subset of W containing S_0, then S is also a linearly independent subset of V; since V is finite-dimensional, S contains no more than dim V elements.

We extend S_0, to a basis for W, as follows. If S_0 spans W, then S_0 is basis for W and we are done. If S_0 does not span W, we use the preceding lemma to find a vector β_1 in W such that the set $S_1 = S_0 \cup \{\beta_1\}$ is independent. If S_1 spans W, fine. If not, apply the lemma to obtain a vector β_2 in W such that $S_2 = S_1 \cup \{\beta_2\}$ is independent. If we continue in this way, then (in not more than dim V steps) we reach a set

$$S_m = S_0 \cup \{\beta_1, \dots, \beta_m\}$$

Which is a basis for W.

Suppose that a vector space is known to be a dimension n. The following theorem tells us that we do not have to check both linear independence and spanning conditions to see if a given set is a basis.

THEROM: Let V be a vector space of dimension n.

a) If $S = \{\overrightarrow{v_1}, \overrightarrow{v_2}, \dots, \overrightarrow{v_n}\}$ is a set of n linearly independent vectors in V, then S is a basis for V.

b) If $S = \{\overrightarrow{v_1}, \overrightarrow{v_2}, \dots, \overrightarrow{v_n}\}$ is a set of n vectors that spans V, then S is a basis for V.

Proof: (a) part is clear from the above theorem and the fact that every basis of V contains n number of elements.

(b) It is enough to show that S is linearly independent.

Let $\{\vec{u}, \overrightarrow{u_2}, \dots, \overrightarrow{u_n}\}$ be a basis of V. If we give a proof similar to the first theorem of this material and by using the fact that a homogeneous system of linear equations with equal number of variables and equations will have unique solution, we can prove that $\{\overrightarrow{v_1}, \overrightarrow{v_2}, \dots, \overrightarrow{v_n}\}$ is linearly independent.

THEOREM: If W_1 and W_2 are finite-dimensional subspaces of a vector space V, then $W_1 + W_2$is finite-dimensional and

$$\dim W_1 + \dim W_2 = \dim (W_1 \cap W_2) + \dim(W_1 + W_2)$$

Proof. By Theorem 5 and its corollaries, $W_1 \cap W_2$ has a finite basis $\{\alpha_1, \dots, \alpha_k\}$ which is part of a basis

$$\{\alpha_1, \dots, \alpha_k, \quad \beta_1, \dots, \beta_m\} \text{ for } W_1$$

and part of basis

$$\{\alpha_1, \dots, \alpha_k, \ \gamma_1, \dots, \gamma_n\} \text{ for } W_2.$$

The subspace $W_1 + W_2$ is spanned by the vectors

$\alpha_1, \dots, \alpha_k, \quad \beta_1, \dots, \beta_m, \quad \gamma_1, \dots, \gamma_n$ and these vectors form an independent set. For suppose

$$\sum x_i \alpha_i + \sum y_j \beta_j + \sum z_r \gamma_r = 0.$$

Then

$$-\sum z_r\, \gamma_r = \sum x_i\, \alpha_i + \sum y_j\, \beta_j$$

which shows that $\sum z_r\, \gamma_r$ belongs to W_1 . As $\sum z_r\, \gamma_r$ Also belongs to W_2 it follows that

$$\sum z_r\, \gamma_r = \sum c_i\, \alpha_i$$

for certain scalars $c_1, \ldots, c_k$. Because the set

$$\{\alpha_1, \ldots, \alpha_k, \quad \gamma_1, \ldots, \gamma_n\}$$

is independent, each of the scalars $z_r = 0$. Thus,

$$\sum x_i\, \alpha_i + \sum y_j\, \beta_j = 0$$

and since

$$\{\alpha_1, \ldots, \alpha_k, \quad \beta_1, \ldots, \beta_m\}$$

is also an independent set, each $x_i = 0$ and each $y_j = 0$. Thus $\{\alpha_1, \ldots, \alpha_k, \quad \beta_1, \ldots, \beta_m, \quad \gamma_1, \ldots, \gamma_n\}$

is a basis for $W_1 + W_2$. Finally

$$\begin{aligned}\dim W_1 + \dim W_2 &= (k + m) + (k + n)\\ &= k + (m + k + n)\\ &= \dim(W_1 \cap W_2) + \dim(W_1 + W_2).\end{aligned}$$

Problem 1: Show that the set $\{(1,0,-1),(1,1,1),(1,2,4)\}$ is a basis for $\mathbb{R}^3$.

Solution: Let us first show that the set spans $\mathbb{R}^3$.

Let (x_1, x_2, x_3) be an arbitrary element of $\mathbb{R}^3$.

We try to find scalars a_1, a_2, a_3such that $(x_1, x_2, x_3) = a_1(1,0,-1) + a_2(1,1,1) + a_3(1,2,4)$. This identity leads to the system of equations.

$$a_1 + a_2 + a_3 = x_1$$

$$a_2 + 2a_3 = x_2$$

$$-a_1 + a_2 + 4a_3 = x_3$$

This system of equations has the solution

$$a_1 = 2x_1 - 3x_2 + x_3$$

$$a_2 = -2x_1 + 5x_2 - 2x_3$$

$$a_3 = x_1 - 2x_2 + x_3$$

Thus the set spans the space. We now show that the set is linearly independent.

Consider the identity

$$b_1(1,0,-1) + b_2(1,1,1) + b_3(1,2,4) = (0,0,0)$$

This identity leads to the system of equations.

$$b_1 + b_2 + b_3 = 0$$

$$b_2 + 2b_3 = 0$$

$$-b_1 + b_2 + 4b_3 = 0$$

This system has the unique solution $b_1 = 0, b_2 = 0,$ and $b_3 = 0$. Thus the set is linearly independent.

Therefore $\{(1,0,-1),(1,1,1),(1,2,4)\}$ forms a basis for $\mathbb{R}^3$.

Problem 2: Prove that the set $\{(1,3,-1),(2,1,0),(4,2,1)\}$ is a basis for $\mathbb{R}^3$

Solution: The dimension of $\mathbb{R}^3$ is three. Thus a basis of $\mathbb{R}^3$consists of three vectors. We have the correct number of vectors for a basis.

Normally, we would have to show that this set is linearly independent and that it spans $\mathbb{R}^3$

Since $\mathbb{R}^3$ is finite dimensional vector space, we need to check only one of these two conditions. Let us check for linear independence. We get $c_1(1,3,-1)+c_2(2,1,0)+c_3(4,2,1)=(0,0,0)$. This identity leads to the system of equations.

$$c_1+2c_2+4c_3=0$$

$$3c_1+c_2+2c_3=0$$

$$-c_1+4c_3=0$$

This system has unique solution $c_1=0, c_2=0, c_3=0$

Thus the vectors are linearly independent. The set $\{(1,3,-1),(2,1,0),(4,2,1)\}$ is therefore a basis for $\mathbb{R}^3$.

Problem 3: State (with a brief explanation) whether the following statements are true or false.

(a) The vectors (1, 2), (-1, 3), (5, 2) are linearly dependent in $\mathbb{R}^2$.

(b) The vectors (1, 0, 0), (0, 2, 0), (1, 2, 0) span $\mathbb{R}^3$.

(c) {(1, 0, 2), (0, 1, -3)} is a basis for the subspace of $\mathbb{R}^3$ consisting of vectors of the form (a, b, 2a-3b).

(d) Any set of two vectors can be used to generate a two-dimensional subspace of $\mathbb{R}^3$.

Solution:

(a) True: The dimension of $\mathbb{R}^2$ is two. Thus any three vectors are linearly dependent.

(b) False: The three vectors are linearly dependent. Thus they cannot span a three-dimensional space.

(c) True: The vectors span the subspace since

$$(a, b, 2a-3b) = a(1, 0, 2) + b(0, 1, -3)$$

The vectors are also linearly independent since they are not collinear.

(d) False: The two vectors must be linearly independent.

Exercise

1. Prove that the subspace of $\mathbb{R}^3$ generated by the vectors $(-1,2,1), (2,-1,0)$, and $(1,4,3)$ is a two dimensional subspace of $\mathbb{R}^3$ and give a basis for this subspace.
2. Find a basis for $\mathbb{R}^3$ that includes the vectors $(1,1,1)$ and $(1,0,-2)$.
3. Determine a basis for each of the following subspaces of $\mathbb{R}^3$. Give the dimension of each subspace.

a) The set of vectors of the form (a, a, b).

b) The set of vectors of the form $(a, b, a+b)$

c) The set of vectors of the form (a, b, c), where $a+b+c = 0$.

4. Which of the following sets of vectors are bases for $\mathbb{R}^2$?

   (a) $\{(3,1),(2,1)\}$ (b) $\{(1,-3),(-2,6)\}$

5. Which of the following sets are bases for $\mathbb{R}^3$?

   (a) $\{(1,-1,2),(2,0,1),(3,0,0)\}$

   (b) $\{(2,1,0),(-1,1,1),(3,3,1)\}$

6. Prove that the vector (1, 2, -1) lies in the two dimensional subspace of $\mathbb{R}^3$ generated by the vectors (1, 3, 1) and (1, 4, 3).
7. Let $\{v_1, v_2\}$ be a basis for a vector space V. Show that the set of vectors $\{u_1, u_2\}$, where $u_1 = v_1 + v_2, u_2 = v_1 - v_2$, is also a basis for V.
8. Let V be a vector space of dimension n. Prove that no set of n - 1 vectors can span V.
9. Let V be a vector space, and let W be a subspace of V. If dim (V) = n and dim (W) = m, prove that m ≤ n.

Answers

1. $\{(-1,2,1),(2,-1,0)\}$ is a basis.
2. $\{(1,1,1),(1,0,-2),(1,0,0)\}$.
3. a) Basis $= \{(1,1,0),(0,0,1)\}$, dimension $= 2$.
   b) Basis $= \{(1,0,1),(0,1,1)\}$, dimension $= 2$.
   c) Basis $= \{(1,0,-1),(0,1,-1)\}$, dimension $= 2$.
4. (a) Basis (b) Not a basis
5. (a) Basis (b) Not a basis
6. $(1,2,-1) = 2(1,3,1) - (1,4,3)$.