State Space Models For Electri
Networks

- Concept of state
- State equations
- Equivalent source method
- State space model and evaluation of state tra
  matrix
- Application to electrical networks

- Transfer function of a systems is used for analysis and desig
  control systems. It is also called as classical approach or techn

- This is a useful representation if the system is
  - Linear and time invariant
  - Having a single input and single output (SISO)
  - Having zero initial conditions only.

- But transfer function representation is not useful for,
    - Systems with initial conditions
    - Nonlinear systems
    - Time varying systems and
    - Multiple input multiple output (MIMO) systems
    - It does not throw any light on the variatio internal variables (Sometimes this informatio necessary because, some internal variables ma out of bounds, even though the output ren within the desired limits).

To overcome all these disadvantages of transfer function represe
the state space representation of the system was evolved. This is c
modern approach.

- This representation forms the basis for the development of
  control systems.
- This representation contains the information about some of the
  variables along with the output variable and is useful for analy
  design using digital computer.

- It is suitable for representing
  - Linear and nonlinear systems.
  - Time invariant and time varying systems.
  - SISO and MIMO systems.
- State variable analysis is a way of analyzing the dyn circuits in order to obtain its transfer function.
- Matrix notations and Laplace transformations are extens used in state variable analysis.

Concept of State:

STATE:

The concept of state is related to minimum set of variables. variables are called as state variables.

Now we can define the **state and state variables** *The minimum number of variables required to be known at time along with the input for $t \geq t_0$, to completely determine the behav system for $t > t_0$, are known as the state variables of the syste state of the system at any time 't' is given by the values of these va at time 't'.*

- **State vector:** If n variables represent behavior of a given s then n State variables can be considered as n components of vector x. Such vector is called as state vector.

- **State space:** The n dimensional space whose co-ordinat consist of x_1 axis, x_2 axis,….., x_n axis is called as state space

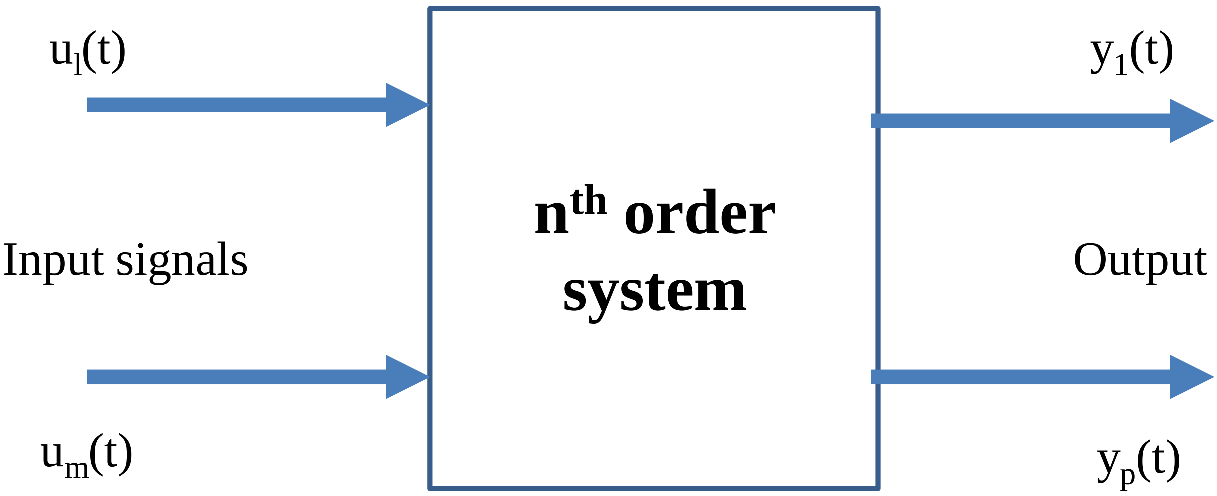
- **State equations:** It is the set of n simultaneous firs differential equation with n variables.

- **Output equation:** It is the linear combination of state va and inputs.

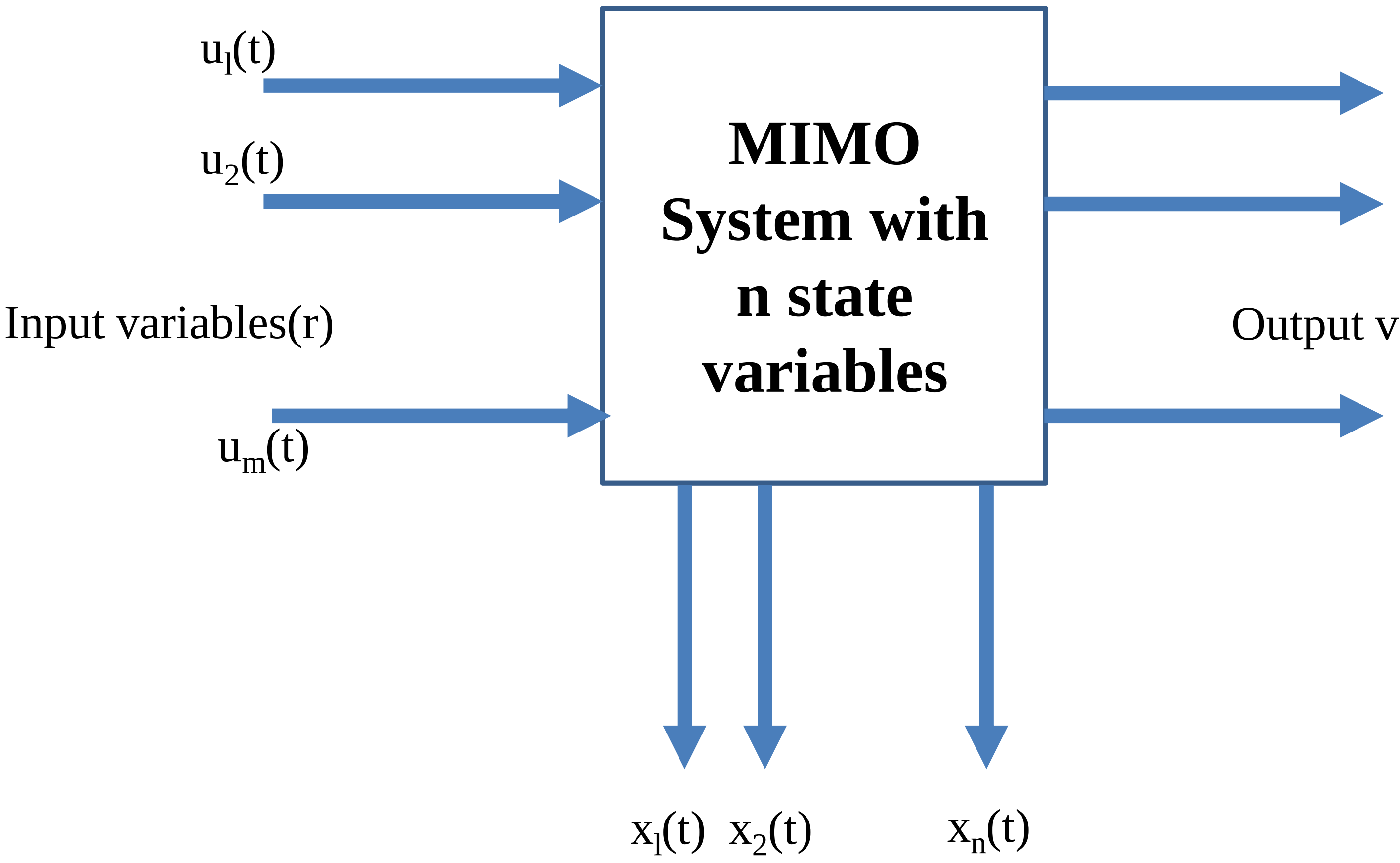
If the dynamic behavior of a system can be described by an *nth* order differential equation, we require *n* initial conditions of the system and hence, a minimum of *n* state variables are required known at $t = t_0$ completely determine the behavior of the syste a given input.

It is a standard practice to denote these *n* **state variables** by $x_l($ $x_2(t)$ ..... $x_n(t)$ and *m* **inputs** by $u_l(t)$, $u_2(t)$ ..... $u_m(t)$ and *p* **outpu** $y_I(t)$, $y_2(t)$ ..... $y_p(t)$.

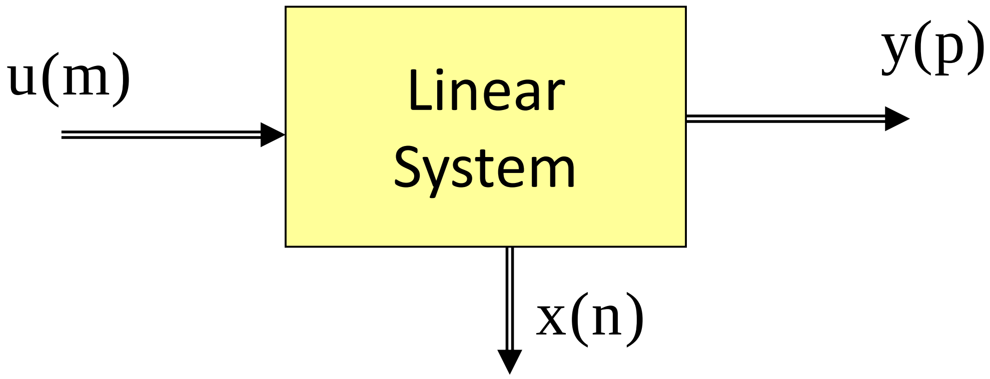
System block diagram



State space representation



State variable technique



$$u(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ u_i(t) \\ u_m(t) \end{bmatrix} \qquad x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_i(t) \\ x_n(t) \end{bmatrix} \qquad y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_i(t) \\ y_p(t) \end{bmatrix}$$

Input vector State vector Output vect

The state representation can be arranged in the form of n fir differential equations, as shown in the next slide.

For linear systems, the derivatives of state variables can be express
combinations of the state variables and inputs.

$$\dot{x}_1 = a_{11} x_1 + a_{12} x_2 + \ldots + a_{1n} x_n + b_{11} u_1 + b_{12} u_2 + \ldots + b$$

$$\dot{x}_2 = a_{21} x_1 + a_{22} x_2 + \ldots + a_{2n} x_n + b_{21} u_1 + b_{22} u_2 + \ldots + b$$

$$\vdots$$

$$\dot{x}_n = a_{n1} x_1 + a_{n2} x_2 + \ldots + a_{nn} x_n + b_{n1} u_1 + b_{n2} u_2 + \ldots + b$$

where a_{ij} and b_{ij} are constants.

Eqns. (1) can be written in a matrix form as,

$$\dot{X}(t) = A X(t) + BU(t) \qquad \text{.......(2)}$$

where X(t) is a n × 1 state vecotr

A is a n × n constant system matrix

B is a n × m constant input matrix

U(t) is a m × 1 input vector.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{bmatrix}; \; B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1m} \\ b_{21} & b_{22} & \cdots & b_{2m} \\ \vdots & \vdots & & \\ b_{n1} & b_{n2} & \ldots & b_{nm} \end{bmatrix}$$

Similarly, the outputs can also be expressed as linear combinations of state variables

$$y_1(t) = c_{11}\,x_1 + c_{12}\,x_2 \dots + c_{1n}\,x_n + d_{11}\,u_1 + d_{12}\,u_2 + \dots + $$

$$\vdots$$

$$y_p(t) = c_{p1}\,x_1 + c_{p2}\,x_2 \dots + c_{pn}\,x_n + d_{p1}\,u_1 + d_{p2}\,u_2 + \dots + d_{pm}$$

The above equation, eq(3), can be written as:

$$Y(t) = C\,X(t) + D\,U(t) \qquad \text{......(4)}$$

Where Y (t) is a p × 1 output vector

C is a p × n output matrix

D is a p × m transmission matrix

$$C = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ \vdots & & & \vdots \\ c_{p1} & c_{p2} & \cdots & c_{pn} \end{bmatrix}; \; D = \begin{bmatrix} d_{11} & d_{12} \\ \vdots & \\ d_{p1} & d_{p2} \end{bmatrix}$$

The complete state model for linear systems is given by the two equations, eq (2) and eq (4), as,

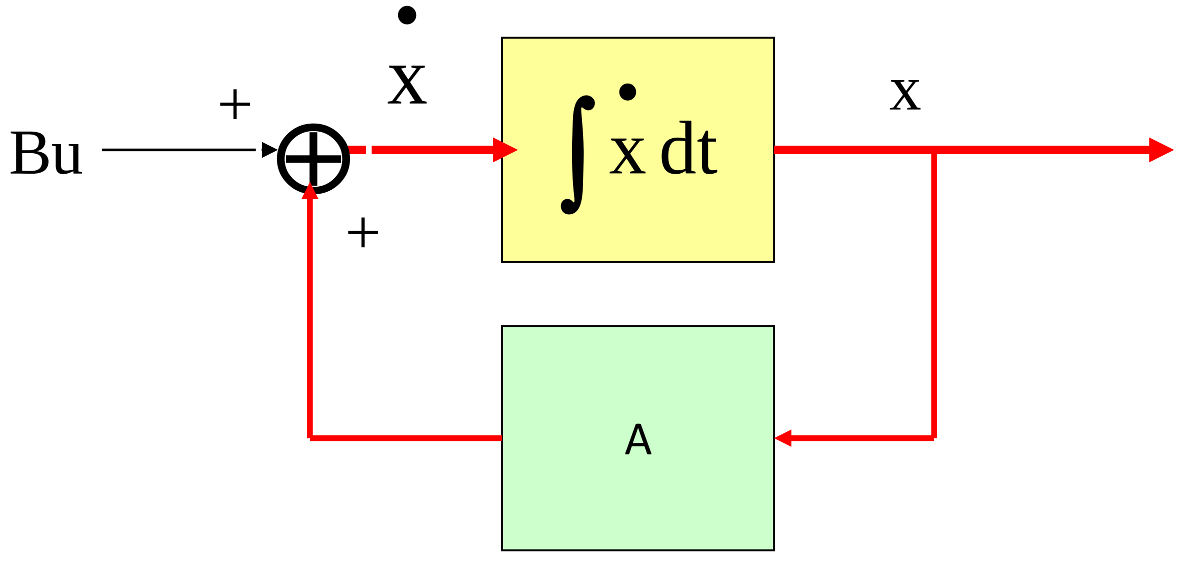
$$\boxed{\begin{aligned} \dot{x} &= \frac{dx}{dt} = AX(t) + BU(t) \\ Y(t) &= CX(t) + DU(t) \end{aligned}}$$

…… State equation

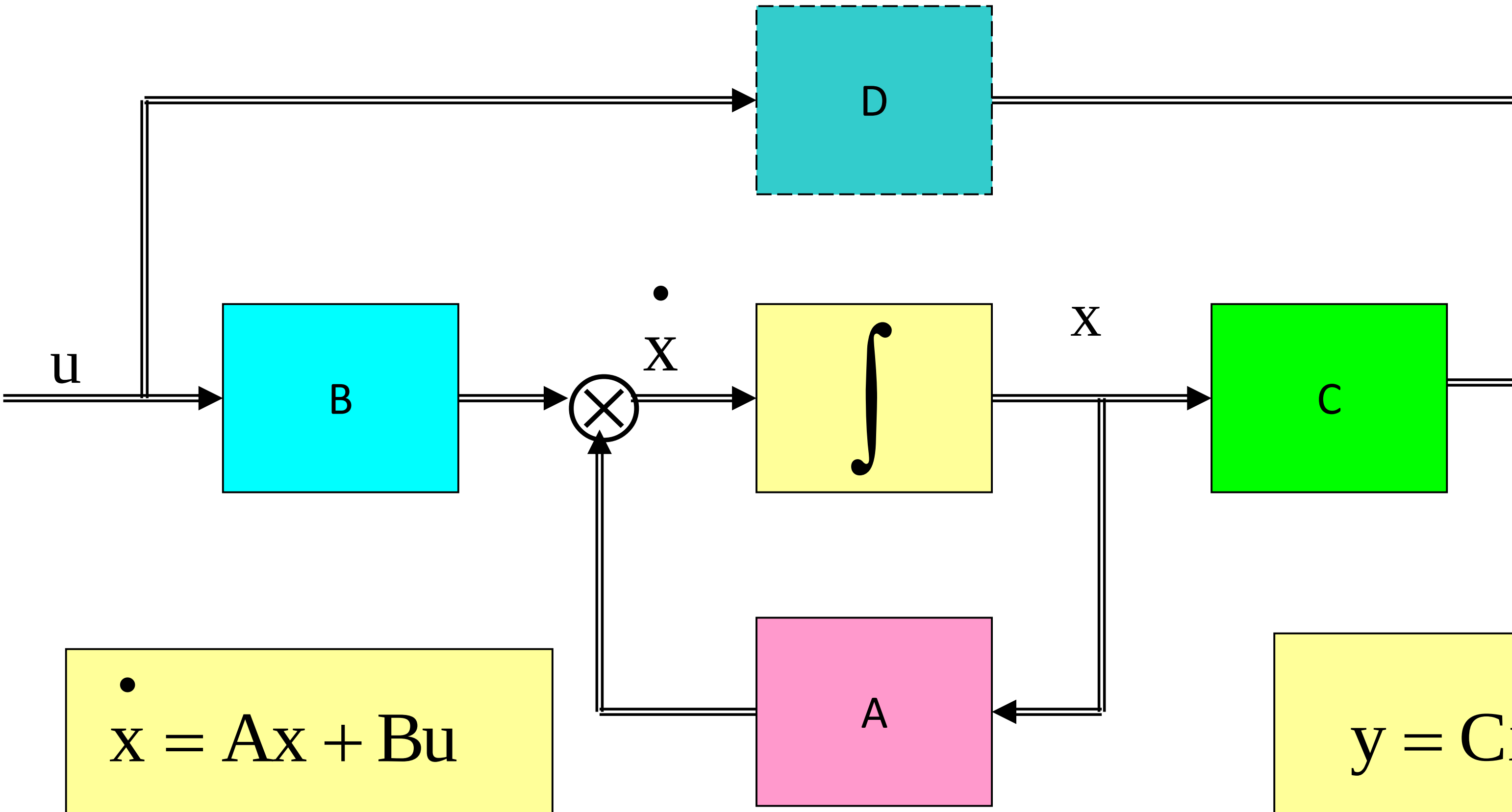
……. Output equati

Block diagram representation of state model of a linear multi-input-m
systems

Representation of $\dot{x} = Ax + Bu$



Block diagram representation of the state model of an MIM



$$\dot{x} = Ax + Bu$$

$$y = C$$

A= System Matrix(n $\times$ n)
B= Input Matrix (n $\times$ m)
x= State Vector (n $\times$ 1)
u= Input Vector (m $\times$ 1)

C= Output Matrix (p $\times$ n)
D= Direct Transmission Mat
y= Output Vector (p $\times$ 1)

The system is described by *n* first order differential equations in these state

$$\frac{dx_1}{dt} = \dot{x}_1 = f_1\,(x_1, x_2 \ldots x_n; u_1, u_2 \ldots u_m, t)$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$\frac{dx_n}{dt} = \dot{x}_n = f_n\,(x_1, x_2 \ldots x_n; u_1, u_2 \ldots u_m, t)$$

The functions $f_1, f_2 \ldots f_n$ may be time varying or time invariant and linear or nonli

Using vector notation to represent the states, their derivatives, and inp

$$X(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}; \; \dot{X}(t) = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \vdots \\ \dot{x}_n(t) \end{bmatrix}; \; U(t) =$$

where X(t) is known as state vector and U(t) is known as inpu

State Transition Matrix:

Consider a homogeneous(unforced) state equation,

$$\dot{x}(t) = Ax(t)$$

The solution of this equation can be found out using Laplace trai

$$sX(s) - x(0) = AX(s)$$

$$(sI - A)X(s) = x(0)$$

$$X(s) = (sI - A)^{-1}x(0)$$

Inverse Laplace transform of X(S) gives the required solution, i

$$\Rightarrow \; x(t) = e^{At}\, x(0)$$

Let $x(0) = x_0$

Therefore the solution is $x(t) = e^{At}x(0)$.

- This solution of homogeneous state equation shows that the init x_0 at t=0, is driven to a state x(t) at time t.

- Since this transition in state is carried out by the matrix exponen e^{At}, e^{At} is known as the **state transition matrix (STM)** and is d by φ(t).

$$\boxed{\boldsymbol{\varphi(t)=e^{At}}}$$

Summary :

- State Transition matrix

$$\boxed{\varphi(t)=e^{At}}$$

$$\boxed{\Phi(t) = L^{-1}[(sI - A)^{-1}]}$$

Significance of STM:

- Since state transition matrix satisfies the homogeneous state equations, it represents the free response of the system.
- In other words, it governs the response that is excited by the in conditions only.
- As the STM is dependent on A only, it is therefore, sometimes referred to as STM of A.
- As the name implies, the STM describes the change of state fr initial time t=0 to any time t, when the inputs are zero.

Properties

$$\Phi(t) \stackrel{\Delta}{=} e^{At} = L^{-1}[(sI - A$$

1. $\Phi(0) = I$
2. $\Phi^{-1}(t) = \Phi(-t)$
3. $x(0) = \Phi(-t)x(t)$
4. $\Phi(t_2 - t_1)\Phi(t_1 - t_0) = \Phi(t_2$
5. $\Phi(t)^k = \Phi(kt)$

State space representation of electrical networks:

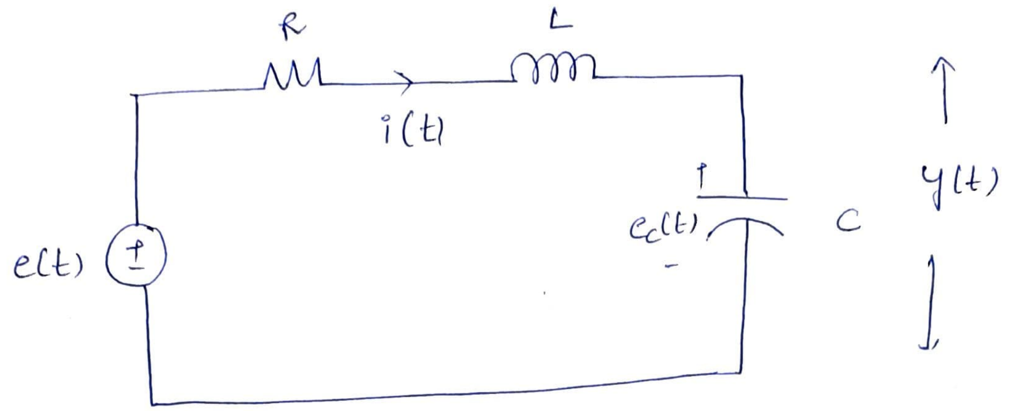
- The desired output information is usually the voltage and currents associated with various elements of the network.

- This information at any time 't' can be obtained if the initial voltage across the capacitor $e_c(t_0)$ and initial current through the inductor $i_L(t_0)$ are known in addition to the values of the input $e(t)$ applied for $t > t_0$.

- The voltage across the capacitor and current through inductor thus constitute a set of characterizing variables of the circuit. The values of these variables at any time 't' describe the state of the network at that time. These variables are therefore called as state variables of the circuit.

{ The initial state of the circuit is given by $e_c(t_0)$ & $i_L(t_0)$, and the state of the circuit at any time t is given by $e_c(t)$ and $i_L(t)$. }

Q1) Obtain the state space representation of the given series RLC circuit. ~~Obta~~ Output is $y(t)$. & input is $e(t)$.



Sol) State model equations are:

① $$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = [A]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + [B]\begin{bmatrix} U_1 \\ U_2 \end{bmatrix}$$

Here, the state variables are, $e_c(t)$ and $i_L(t)$ and input is $e(t)$.

(As $i_L(t) = i(t)$)

$$\therefore \begin{bmatrix} \dot{e}_c(t) \\ \dot{i}_L(t) \end{bmatrix} = \begin{bmatrix} A \end{bmatrix}\begin{bmatrix} e_c(t) \\ i_L(t) \end{bmatrix} + \begin{bmatrix} B \end{bmatrix}\begin{bmatrix} e(t) \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \dot{e}_c(t) \\ \dot{i}_L(t) \end{bmatrix} = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}\begin{bmatrix} e_c(t) \\ i_L(t) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}\begin{bmatrix} e(t) \end{bmatrix}$$

$$\Rightarrow \dot{e}_c(t) = A_1 e_c(t) + A_2 i_L(t) + B_1 e(t) \longrightarrow ①$$

and $$\dot{i}_L(t) = A_3 e_c(t) + A_4 \dot{e}_L(t) + B_2 e(t). \longrightarrow ②$$

(2)

$y(t) = C\,x(t) + D\,u(t)$

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} C_1 & C_2 \\ C_3 & C_4 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} D_1 \\ D_2 \end{bmatrix} \begin{bmatrix} U_1(t) \\ U_2(t) \end{bmatrix}$$

Here only one output is present i.e., $y(t)$

and only one input, i.e., $u_1(t) = e(t)$

$$\therefore \begin{bmatrix} y_1(t) \end{bmatrix} = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} D_1 \end{bmatrix} \begin{bmatrix} e(t) \end{bmatrix}$$

$$\Rightarrow y_1(t) = C_1\,x_1(t) + C_2\,x_2(t) + D_1\,e(t) \longrightarrow (3)$$

$$\Rightarrow y(t) = C_1\,e_C(t) + C_2\,i_L(t) + D_1\,e(t) \longrightarrow (3)$$

So, apply KVL or KCL (or mesh (or) nodal or any other technique

and frame equations in the form of eq (1), (2) and (3). }

from the circuit,

apply KVL in the loop,

$$e(t) = i_L(t).R + L\frac{di_L(t)}{dt} + e_C(t)$$

$$\Rightarrow e(t) = i(t)R + L\frac{di(t)}{dt} + e_C(t)$$

$$\Rightarrow e(t) = R.i(t) + L\,\dot{i}(t) + e_C(t)$$

$$\Rightarrow \dot{i}(t) = \frac{e(t) - R\,i(t) - e_c(t)}{L}$$

$$\Rightarrow \dot{i}(t) = \frac{-1}{L}(e_c(t)) - \frac{R}{L}(i(t)) + \frac{1}{L}e(t). \rightarrow (4)$$

and, $e_c(t) = \frac{1}{c}\int i(t)\,dt$

Apply differentiation,

$$\frac{d\,e_c(t)}{dt} = \frac{1}{c}\,i(t)$$

$$\Rightarrow \dot{e}_c(t) = \frac{1}{C}\,i(t).$$

$$\dot{e}_c(t) = 0(e_c(t)) + \frac{1}{C}(i(t)) + 0(e(t)) \rightarrow (5)$$

Comparing eq (1) and (2) with eq (5) and (4) respectively,

to write the state equation.

$$\therefore \begin{bmatrix} \dot{e}_c(t) \\ \dot{i}_L(t) \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{C} \\ -\frac{1}{L} & -\frac{R}{L} \end{bmatrix} \begin{bmatrix} e_c(t) \\ i_L(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix} [e(t)]$$

from the circuit,

$$y(t) = e_c(t)$$

$$\Rightarrow y(t) = e_c(t) + 0\,(i_L(t)) + 0\,(e(t)) \longrightarrow (6)$$

Comparing eq (3) and eq (6), to write the output equation,

$$[y(t)] = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} e_c(t) \\ i_L(t) \end{bmatrix} + [0][e(t)]$$

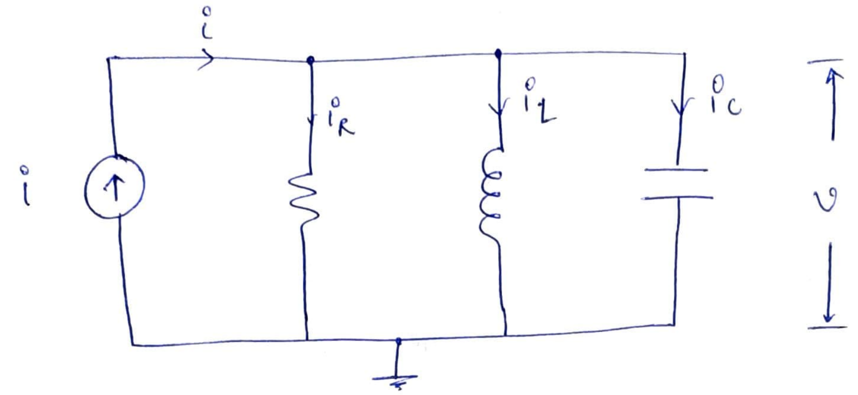
$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} e_c(t) \\ i_L(t) \end{bmatrix} \cancel{+}$$

∴ The state model equations are,

$$\begin{bmatrix} \dot{e}_c(t) \\ \dot{i}_L(t) \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{C} \\ -\frac{1}{L} & -\frac{R}{L} \end{bmatrix} \begin{bmatrix} e_c(t) \\ i_L(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix} e(t)$$

and $$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} e_c(t) \\ i_L(t) \end{bmatrix}$$

Q2) Write the state variable formulation of the parallel RLC network shown below. The current through the inductor and voltage across the capacitor are the output variables.



Sol) Here the state variables are i_L and v_c ie v.

& output variables are also given as i_L and v.

{ from state model equations,

① $$\begin{bmatrix}\dot{i}_L\\ \dot{V}_c\end{bmatrix} = \begin{bmatrix}A_1 & A_2\\ A_3 & A_4\end{bmatrix}\begin{bmatrix}i_L\\ V_c\end{bmatrix} + \begin{bmatrix}B_1\\ B_2\end{bmatrix}[i]$$

$$\Rightarrow \dot{i}_L = A_1 i_L + A_2 V_c + B_1 i \longrightarrow ①$$

$$\text{and } \dot{V}_c = A_3 i_L + A_4 V_c + B_2 i \longrightarrow ②$$

② $y : Y(t) = C\,X(t) + D\,U(t)$

$$\begin{bmatrix}i_L\\ V_c\end{bmatrix} = \begin{bmatrix}C_1 & C_2\\ C_3 & C_4\end{bmatrix}\begin{bmatrix}i_L\\ V_c\end{bmatrix} + \begin{bmatrix}D_1\\ D_2\end{bmatrix}[i]$$

$$\Rightarrow \dot{i}_L = C_1 i_L + C_2 V_C + D_1 i \longrightarrow (3)$$

and $$\dot{v}_C = C_3 i_L + C_4 V_C + D_2 i \longrightarrow (4)$$

Apply KCL ~~in~~ at top node,

$$i = i_R + i_L + i_C$$

$$\Rightarrow i = \frac{v(t)}{R} + i_L + C\frac{dv_C(t)}{dt}$$

$$\Rightarrow i = \frac{V_C(t)}{R} + i_L + C\,\dot{v}_C(t)$$

$$\Rightarrow \dot{v}_C(t) = \frac{1}{C}\left(i - i_L - \frac{V_C}{R}\right)$$

$$\Rightarrow \dot{v}_C(t) = -\frac{1}{C}(i_L) + \left(\frac{-1}{RC}\right)(V_C) + \frac{i}{C} \longrightarrow (5)$$

for inductor,

$$v_L(t) = L\frac{di_L(t)}{dt}$$

$$\Rightarrow V_C(t) = L\frac{di_L(t)}{dt}$$

$$\Rightarrow V_C(t) = L\,\dot{i}_L(t)$$

$$\Rightarrow \dot{i}_L(t) = \frac{1}{L}V_C(t)$$

$$\Rightarrow \dot{i}_L = 0(i_L) + \left(\frac{1}{L}\right)V_C + 0(i). \longrightarrow (6)$$

Comparing eq 5 and 6 with 1 and 2,

$$\begin{bmatrix} \dot{i}_L \\ \dot{V}_c \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{L} \\ -\frac{1}{C} & -\frac{1}{RC} \end{bmatrix} \begin{bmatrix} i_L \\ V_c \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{C} \end{bmatrix} i$$

and output is current through inductor and voltage across capacitor. so, it can be written as,

$$\begin{bmatrix} i_L \\ V_c \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} i_L \\ V_c \end{bmatrix} + [0]\, i$$

$$\Rightarrow \begin{bmatrix} i_L \\ V_c \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} i_L \\ V_c \end{bmatrix}$$

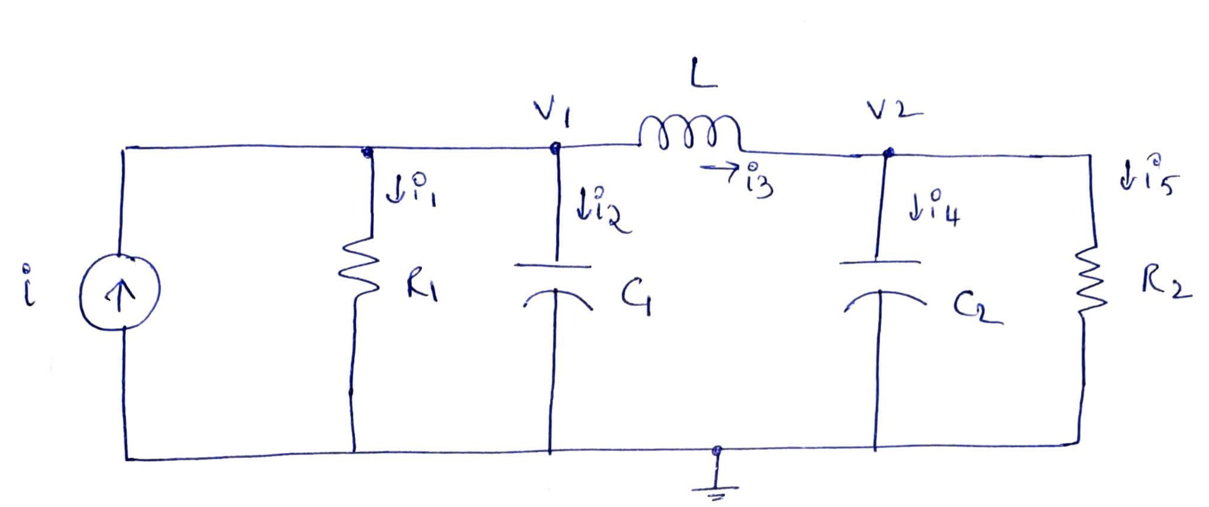
∴ State model equations are,

$$\begin{bmatrix} \dot{i}_L \\ \dot{V}_c \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{L} \\ -\frac{1}{C} & -\frac{1}{RC} \end{bmatrix} \begin{bmatrix} i_L \\ V_c \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{C} \end{bmatrix} i$$

and $$\begin{bmatrix} i_L \\ V_c \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} i_L \\ V_c \end{bmatrix}$$

Q3) Obtain the state model for the below network. 'i' is input

& output is i_5 and v_2.



Q3) Ans:

$$\begin{bmatrix} \dot{v}_1 \\ \dot{v}_2 \\ \dot{i}_3 \end{bmatrix} = \begin{bmatrix} -1/R_1C_1 & 0 & -1/C_1 \\ 0 & -1/R_2C_2 & 1/C_2 \\ 1/L & -1/L & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ i_3 \end{bmatrix} + \begin{bmatrix} 1/C_1 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} i_5 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 & 1/R_2 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ i_3 \end{bmatrix}$$

Q4) Obtain the state model for the above problem, when,

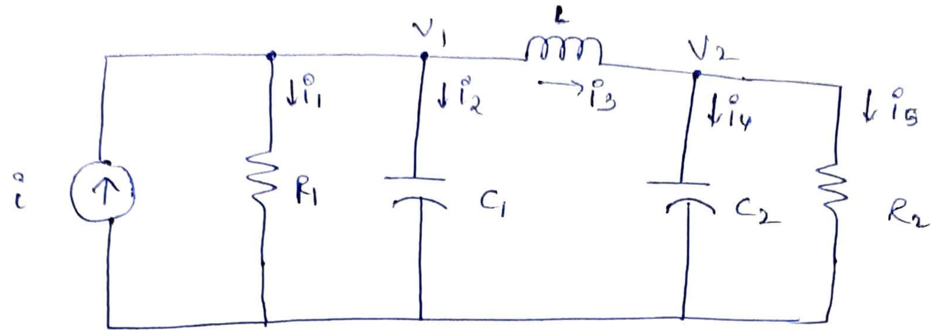
$R_1 = R_2 = 1\,\Omega$, $C_1 = C_2 = 1F$ and $L = 1H$

4Q Ans:

$$\begin{bmatrix} \dot{v}_1 \\ \dot{v}_2 \\ \dot{i}_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & -1 \\ 0 & -1 & 1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ i_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \dot{i}$$

$$\begin{bmatrix} i_5 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ i_3 \end{bmatrix}$$

Q3) Obtain the state model of the below network.



Input is i & output is i_5 and v_2

Sol) The behavior of the network at any time t can be determined if the initial current through inductor and initial voltages across the capacitors together with the input i applied for $t>0$ are known.

$\therefore$ i_3, v_1, v_2 are selected as state variable.

- To obtain state equations, express the first derivatives of the state variables in terms of input variable & state variables.

Apply KCL at node 1,

$$i = i_1 + i_2 + i_3$$

$$i = \frac{V_1}{R_1} + C_1\frac{dv_1}{dt} + i_3$$

$$\Rightarrow i = \frac{V_1}{R_1} + C_1\dot{v}_1 + i_3 \Rightarrow \dot{v}_1 = \frac{1}{C_1}\left(-\frac{V_1}{R_1} - i_3 + i\right) \rightarrow (1)$$

Apply KCL at node 2,

$i_3 = i_4 + i_5$

$$\Rightarrow i_3 = C_2 \frac{dV_2}{dt} + \frac{V_2}{R_2}$$

$$\Rightarrow i_3 = C_2 \dot{V}_2 + \frac{V_2}{R_2}$$

$$\Rightarrow \dot{V}_2 = \frac{1}{C_2}\left(-\frac{V_2}{R_2} + i_3\right) \rightarrow ②$$

Apply KVL in loop containing inductor,

$$L\frac{di_3}{dt} + V_2 - V_1 = 0$$

$$\Rightarrow \dot{i}_3 = \frac{1}{L}(V_1 - V_2) \rightarrow ③$$

Outputs are i_5 and v_2

(To obtain, Output equation, i_5 and v_2 must be written in terms of input & state variables) ie i & i_3, V_1, V_2

$$i_5 = \frac{V_2}{R_2} \rightarrow ④$$

and V_2 is another output

∴ The complete state model can be written with the help of eq ①, ②, ③ and ④, ⑤.

$$\begin{bmatrix} \dot{V}_1 \\ \dot{V}_2 \\ \dot{i}_3 \end{bmatrix} = \begin{bmatrix} -1/R_1C_1 & 0 & -1/C_1 \\ 0 & -1/R_2C_2 & 1/C_2 \\ \frac{1}{L} & -\frac{1}{L} & 0 \end{bmatrix}\begin{bmatrix} V_1 \\ V_2 \\ i_3 \end{bmatrix} + \begin{bmatrix} 1/C_1 \\ 0 \\ 0 \end{bmatrix} i$$

$$\& \begin{bmatrix} i_5 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{R_2} & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ i_3 \end{bmatrix}$$

Q4) Obtain the state model for the above problem when,

$R_1 = R_2 = 1\,\Omega$, $C_1 = C_2 = 1F$ and $L = 1H$

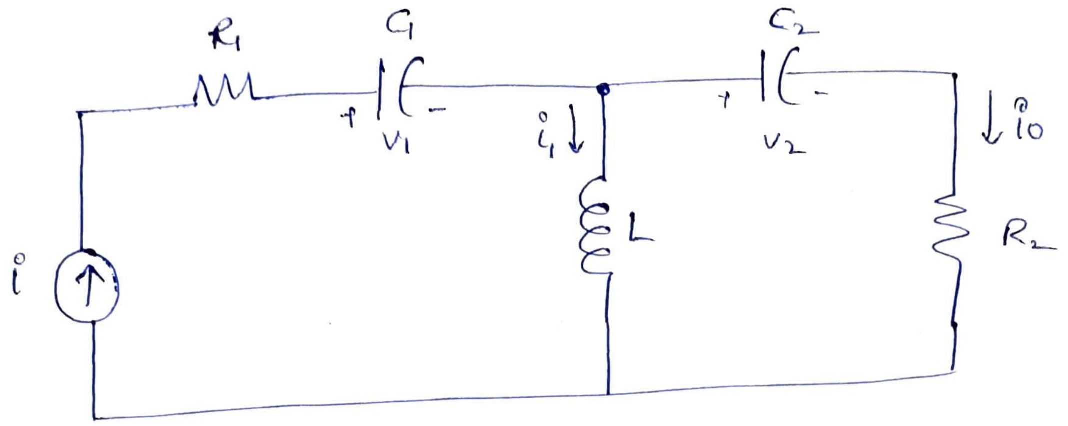
Sol)
$$\begin{bmatrix} \dot{v}_1 \\ \dot{v}_2 \\ \dot{i}_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & -1 \\ 0 & -1 & 1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ i_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} i$$

and
$$\begin{bmatrix} i_5 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ i_3 \end{bmatrix}$$

//

Q5) Obtain the dynamic equations of the network shown below.

The current through R_2 is the output required.



Sol) State variables are V_1, V_2, i_1

Input is i

Output is i_0

$\therefore$ for state equation, the first derivatives of the state variables

i.e, $\dot{V}_1$; $\dot{V}_2$ and $\dot{i}_1$ must be expressed in terms of state variables

ie $x_1(t) = V_1$, $x_2(t) = V_2$ and $x_3(t) = i_1$ and inputs ie $u(t) =$

and output $y = i_0$ must be expressed in terms of the same

state variables and input

Apply KCL at node 1,

$$i = i_1 + i_o$$

$$i = i_1 + C_2 \frac{dv_2}{dt}$$

$$i = i_1 + C_2 \dot{v}_2$$

$$\Rightarrow \dot{v}_2 = \frac{1}{C_2}\left(-i_1 + i\right) \longrightarrow (1)$$

from capacitor C_1,

$$i = C_1 \frac{dv_1}{dt}$$

$$\Rightarrow i = C_1 \dot{v}_1$$

$$\Rightarrow \dot{v}_1 = \frac{1}{C_1} i \longrightarrow (2)$$

Apply KVL in loop 2,

$$v_2 + i_o R_2 = L \frac{di_1}{dt}$$

$$\Rightarrow v_2 + \left(C_2 \frac{dv_2}{dt}\right) R_2 = L \dot{i}_1 \quad \left(\because i_o = C_2 \frac{dv_2}{dt}\right)$$

$$\Rightarrow v_2 + C_2 R_2 \dot{v}_2 = L \dot{i}_1$$

$$\Rightarrow v_2 + C_2 R_2 \left(\frac{1}{C_2}(-i_1 + i)\right) = L \dot{i}_1 \quad (\because \text{from } (1))$$

$$\Rightarrow v_2 + R_2(-i_1 + i) = L \dot{i}_1$$

$$\Rightarrow \dot{i}_1 = \frac{1}{L}\left(v_2 - R_2 i_1 + R_2 i\right) \longrightarrow (3)$$

from ①, ②, ③, state equation can be written as,

$$\begin{bmatrix} \dot{v}_1 \\ \dot{v}_2 \\ \dot{i}_1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1/C_2 \\ 0 & 1/L & -R_2/L \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ i_1 \end{bmatrix} + \begin{bmatrix} 1/C_1 \\ 1/C_2 \\ R_2/L \end{bmatrix} \dot{i}.$$

Output is i_0, $i_0 = C_2 \dfrac{dv_2}{dt}$

$$= C_2 \dot{v}_2$$

$$= C_2 \left(\frac{1}{C_2} \left(-i_1 + i \right) \right)$$

$$\boxed{i_0 = i - i_1}$$

Or simply apply KCL at node.

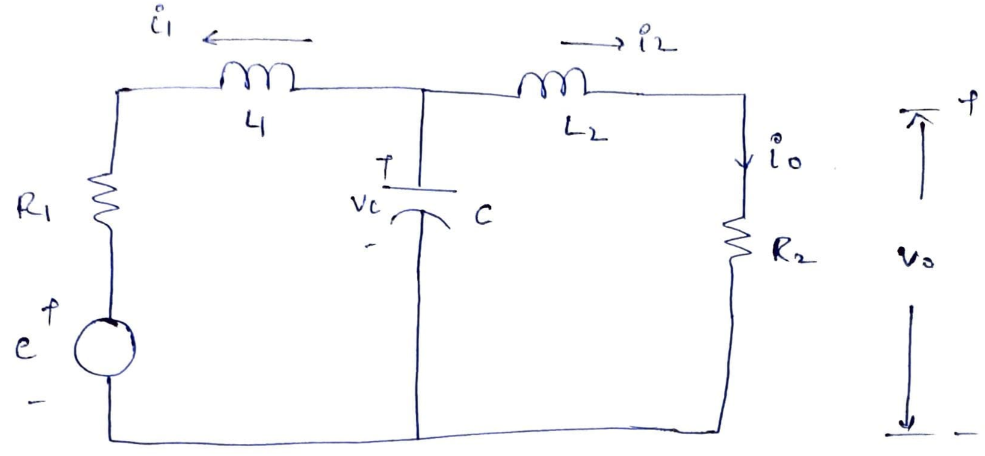
$$i = i_1 + i_0 \Rightarrow i_0 = i - i_1$$

∴ output equation is

$$i = \begin{bmatrix} 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ i_1 \end{bmatrix} + \begin{bmatrix} 1 \end{bmatrix} i$$

//.

Q6) Obtain the state space representation of RLC network shown.



Ans:

$$\begin{bmatrix} \dot{i}_1 \\ \dot{i}_2 \\ \dot{V}_c \end{bmatrix} = \begin{bmatrix} -R_1/L_1 & 0 & 1/L_1 \\ 0 & -R_2/L_2 & 1/L_2 \\ -1/C & -1/C & 0 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ V_C \end{bmatrix} + \begin{bmatrix} \frac{-1}{L_1} \\ 0 \\ 0 \end{bmatrix} e$$

$$\begin{bmatrix} i_o \\ V_o \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & R_2 & 0 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ V_C \end{bmatrix}$$

Numericals on state transition matrix:

Q1) Obtain the STM for the state model whose A matrix is given below,

(i) $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ ii $A = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$ iii, $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$

Sol) $\phi(t)$ is the state transition matrix

$\phi(t) = L^{-1}[(SI-A)^{-1}]$ (or) $\phi(t) = e^{At}$

i, $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

$$SI - A = S\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} S-1 & -1 \\ 0 & S-1 \end{bmatrix}$$

$$(SI-A)^{-1} = \frac{1}{(S-1)(S-1)-(-1)(0)} \begin{bmatrix} S-1 & 1 \\ 0 & S-1 \end{bmatrix}$$

$$\left\{ \because \text{If } M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad M^{-1} = \frac{1}{(ad-bc)}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \right\}$$

$$= \frac{1}{(S-1)^2} \begin{bmatrix} S-1 & 1 \\ 0 & S-1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{s-1}{(s-1)^2} & \frac{1}{(s-1)^2} \\ 0 & \frac{s-1}{(s-1)^2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{s-1} & \frac{1}{(s-1)^2} \\ 0 & \frac{1}{(s-1)} \end{bmatrix}$$

$$\therefore \quad \phi(t) = \begin{bmatrix} e^{t} & te^{t} \\ 0 & e^{t} \end{bmatrix}$$

ii $A = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$

$$\phi(t) = L^{-1}\left[(sI - A)^{-1}\right]$$

$$= L^{-1}\left\{\left[\begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix}\right]^{-1}\right\}$$

$$= L^{-1}\left\{\begin{pmatrix} s & -1 \\ 1 & s+2 \end{pmatrix}^{-1}\right\}$$

$$= L^{-1}\left\{ \frac{1}{s(s+2)-(-1)(1)} \begin{bmatrix} s+2 & 1 \\ -1 & s \end{bmatrix} \right\}$$

$$= L^{-1}\left\{ \frac{1}{s^2+2s+1} \begin{bmatrix} s+2 & 1 \\ -1 & s \end{bmatrix} \right\}$$

$$= L^{-1}\left\{ \frac{1}{(s+1)^2} \begin{bmatrix} s+2 & 1 \\ -1 & s \end{bmatrix} \right\}$$

$$= L^{-1}\begin{bmatrix} \frac{s+2}{(s+1)^2} & \frac{1}{(s+1)^2} \\ \frac{-1}{(s+1)^2} & \frac{s}{(s+1)^2} \end{bmatrix}$$

$$= L^{-1}\begin{bmatrix} \frac{s+1}{(s+1)^2} + \frac{1}{(s+1)^2} & \frac{1}{(s+1)^2} \\ \frac{-1}{(s+1)^2} & \frac{s+1}{(s+1)^2} - \frac{1}{(s+1)^2} \end{bmatrix}$$

$$= L^{-1}\begin{bmatrix} \frac{1}{(s+1)} + \frac{1}{(s+1)^2} & \frac{1}{(s+1)^2} \\ \frac{-1}{(s+1)^2} & \frac{1}{(s+1)} - \frac{1}{(s+1)^2} \end{bmatrix}$$

$$= \begin{bmatrix} e^{-t} + te^{-t} & te^{-t} \\ -te^{-t} & e^{-t} - te^{-t} \end{bmatrix} = e^{-t}\begin{bmatrix} 1+t & t \\ -t & 1-t \end{bmatrix}$$

iii) $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$

$$\phi(t) = L^{-1}\left[(SI-A)^{-1}\right]$$

$$= L^{-1}\left\{\left[\begin{pmatrix} S & 0 \\ 0 & S \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}\right]^{-1}\right\}$$

$$= L^{-1}\left\{\begin{bmatrix} S & -1 \\ 2 & S+3 \end{bmatrix}^{-1}\right\}$$

$$= L^{-1}\left\{\frac{1}{S(S+3)-(-1)(2)}\begin{bmatrix} S+3 & 1 \\ -2 & S \end{bmatrix}\right\}$$

$$= L^{-1}\left\{\frac{1}{S^2+3S+2}\begin{bmatrix} S+3 & 1 \\ -2 & S \end{bmatrix}\right\}$$

$$= L^{-1}\begin{bmatrix} \frac{S+3}{(S+1)(S+2)} & \frac{1}{(S+1)(S+2)} \\ \frac{-2}{(S+1)(S+2)} & \frac{S}{(S+1)(S+2)} \end{bmatrix}$$

$$= L^{-1}\begin{bmatrix} \frac{2}{S+1} - \frac{1}{S+2} & \frac{1}{S+1} + \frac{-1}{S+2} \\ \frac{-2}{S+1} + \frac{+2}{S+2} & \frac{-1}{S+1} + \frac{2}{S+2} \end{bmatrix}$$

$$\therefore \phi(t) = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix}$$

2(a) Obtain the time response (or) solution of the system described by,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \cancel{\begin{bmatrix} 1 \\ -1 \end{bmatrix}} u$$

with initial conditions or initial state vector as,

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Sol) Solution of a homogeneous state equation is,

given by $x(t) = \phi(t)\, x(0)$

where, $x(0)$, = initial state vector

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$\phi(t)$ is the state transition matrix,

$$\phi(t) = L^{-1}\left[(sI - A)^{-1}\right]$$

where, $A = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$

$$\phi(t) = L^{-1}\left[(SI - A)^{-1}\right]$$

$$= L^{-1}\left\{\left[\begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix}\right]^{-1}\right\}$$

$$= L^{-1}\left\{\begin{bmatrix} s & -1 \\ 1 & s+2 \end{bmatrix}^{-1}\right\}$$

$$= L^{-1}\left\{\frac{1}{s(s+2)-(-1)(1)}\begin{bmatrix} s+2 & 1 \\ -1 & s \end{bmatrix}\right\}$$

$$= L^{-1}\left\{\frac{1}{s^2+2s+1}\begin{bmatrix} s+2 & 1 \\ -1 & s \end{bmatrix}\right\}$$

$$= L^{-1}\begin{bmatrix} \frac{s+2}{(s+1)^2} & \frac{1}{(s+1)^2} \\ \frac{-1}{(s+1)^2} & \frac{s}{(s+1)^2} \end{bmatrix}$$

$$= L^{-1}\begin{bmatrix} \frac{s+1}{(s+1)^2} + \frac{1}{(s+1)^2} & \frac{1}{(s+1)^2} \\ \frac{-1}{(s+1)^2} & \frac{s+1}{(s+1)^2} - \frac{1}{(s+1)^2} \end{bmatrix}$$

$$= \begin{bmatrix} e^{-t} + te^{-t} & te^{-t} \\ -te^{-t} & e^{-t} - te^{-t} \end{bmatrix} = e^{-t}\begin{bmatrix} 1+t & t \\ -t & 1-t \end{bmatrix}$$

Now, the solution of homogeneous state equation is given by,

$$x(t) = \phi(t)\, x(0)$$

$$= e^{-t}\begin{bmatrix} 1+t & t \\ -t & 1-t \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= e^{-t}\begin{bmatrix} t \\ 1-t \end{bmatrix} = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

3Q) A linear time invariant system is characterized by the homogeneous state equation,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Compute the solution of the homogeneous equation assuming the initial state vector $x_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Sol) Solution, $x(t) = \phi(t)\, x(0)$

$$\phi(t) = L^{-1}\{(sI - A)^{-1}\} \quad ; \text{here } A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$= L^{-1}\left[\left\{\begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\right\}^{-1}\right]$$

$$= L^{-1}\left\{\begin{bmatrix} s-1 & 0 \\ -1 & s-1 \end{bmatrix}^{-1}\right\}$$

(Solution already done in Q

$$= L^{-1}\left\{\frac{1}{(s-1)^2-(0)(-1)}\begin{bmatrix} s-1 & 0 \\ +1 & (s-1) \end{bmatrix}\right\}$$

$$= L^{-1}\left\{\frac{1}{(s-1)^2}\begin{bmatrix} s-1 & 0 \\ 1 & s-1 \end{bmatrix}\right\}$$

$$= L^{-1}\left\{\begin{bmatrix} \frac{1}{s-1} & 0 \\ \frac{1}{(s-1)^2} & \frac{1}{(s-1)} \end{bmatrix}\right\}$$

$$= \begin{bmatrix} e^t & 0 \\ te^t & e^t \end{bmatrix}$$

$$\therefore x(t) = \phi(t)\, x(0)$$

$$= \begin{bmatrix} e^t & 0 \\ te^t & e^t \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\therefore \boxed{x(t) = \begin{bmatrix} e^t \\ te^t \end{bmatrix}}$$