3.2

DIVERGENCE, CURL

In continuation of our previous module, first we study the concept below. Then we discuss divergence and curl of a vector field.

Lagrange Multiplier

Here we want to find the point (x, y) that give the extreme (maximum or mimimum value) of a function $f(x, y)$ subject to the constraint $g(x, y) = d$, where d is a constant.

This will occur only when the gradients ∇f and ∇g (directional derivatives) are orthogonal to the given curve (surface) $g(x, y) = d$. Thus ∇f and ∇g are parallel; and hence there must is a constant λ such that $\nabla f = \lambda \nabla g$.

The Greek letter λ(lamda) introduced above is called a Lagrange multiplier. The condition $\nabla f = \lambda \nabla g$ together with the original constraint yield three $(n + 1)$ equations in the unknowns x, y and λ:

$$f_x(x, y) = \lambda\, g_x(x, y),\ f_y(x, y) = \lambda g_y(x, y),\ g(x, y) = d.$$

Solutions of the system x and y give the candidates for the extreme of $f(x, y)$ subject to the constraint $g(x, y) = d$.

Example: Minimize the function $f(x, y) = x^2 + 2y^2$. Subject to the constraint $g(x, y) = 2x + y = 9$.

Solution: Using the condition that $\nabla f = \lambda \nabla g$ and the constraint, we obtain the three equations

$$2x = 2\lambda, 4y = \lambda, 2x + y = 9.$$

Eliminating λ from the first two equations, we obtain $x = 4y$. This and $2x + y = 9$ gives $9y = 9$. Thus we obtain the solution $y = 1$ and $x = 4$. Thus $f(4,1) = 16 + 2 = 18$ is the minimum value of f subject to the constraint $2x + y = 9$.

Remark: The greatest rate of change of $\emptyset$, i.e. the maximum directional derivative, takes place in the direction of, and has the magnitude of, the vector $\nabla\emptyset$.

Example: Let $\emptyset = x^2y^3z^6$

a) In what direction from the point $P(1,1,1)$ is the directional derivative of $\emptyset$ a maximum?
b) What is the magnitude of this maximum?

Solution: $\nabla\emptyset = \nabla(x^2y^3z^6) = 2xy^3z^6\boldsymbol{i} + 3x^2y^2z^6\boldsymbol{j} + 6x^2y^3z^5\boldsymbol{k}$. Then $\nabla\emptyset(1,1,1) = 2\boldsymbol{i} + 3\boldsymbol{j} + 6\boldsymbol{k}$. Then,

a) The directional derivative is a maximum in the direction $\nabla\emptyset(1,1,1) = 2\boldsymbol{i} + 3\boldsymbol{j} + 6\boldsymbol{k}$.
b) The magnitude of this maximum is $|\nabla\emptyset(1,1,1)| = \sqrt{(2)^2 + (3)^2 + (6)^2} = 7$.

In the previous module, we have defined the gradient operator which, when operated on a scalar field, produces a vector field. We shall now discuss two other important vector operations.

Divergence of vector field $\boldsymbol{v}$

Divergence of $\boldsymbol{v}$, denoted by div$\boldsymbol{v}$, is defined as the scalar field

$$\text{div}\boldsymbol{v} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}$$

We observe that div$\boldsymbol{v}$ can also be written in terms of the gradient operator as

$$\text{div}\boldsymbol{v} = \nabla.\boldsymbol{v} = \left(\boldsymbol{i}\frac{\partial}{\partial x} + \boldsymbol{j}\frac{\partial}{\partial y} + \boldsymbol{k}\frac{\partial}{\partial z}\right).(v_1\boldsymbol{i} + v_2\boldsymbol{j} + v_3\boldsymbol{k})$$

$$= \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}.$$

Note that $\nabla.\boldsymbol{v}$ is just a notation for div$\boldsymbol{v}$ and it is not a scalar product in the usual sense, since $\nabla.\boldsymbol{v} \neq \boldsymbol{v}.\nabla$. In fact

$$\boldsymbol{v}.\nabla = v_1\frac{\partial}{\partial x} + v_2\frac{\partial}{\partial y} + v_3\frac{\partial}{\partial z}$$

is a scalar operator.

Example: Suppose $\boldsymbol{A} = x^2z^2\boldsymbol{i} - 2y^2z^2\boldsymbol{j} + xy^2z\boldsymbol{k}$. Find $\nabla.\boldsymbol{A}$ (or div$\boldsymbol{A}$) at the point $P(1,-1,1)$.

$$\nabla.\boldsymbol{A} = \left(\frac{\partial}{\partial x}\boldsymbol{i} + \frac{\partial}{\partial y}\boldsymbol{j} + \frac{\partial}{\partial z}\boldsymbol{k}\right).(x^2z^2\boldsymbol{i} - 2y^2z^2\boldsymbol{j} + xy^2z\boldsymbol{k})$$

$$= \frac{\partial}{\partial x}(x^2z^2) + \frac{\partial}{\partial x}(-2y^2z^2) + \frac{\partial y}{\partial x}(xy^2z)$$

$$= 2xz^2 - 4yz^2 + xy^2.$$

At the point $P(1,-1,1)$,

$$\nabla.\boldsymbol{A} = 2(1)(1)^2 - 4(-1)(1)^2 + (1)(-1)^2 = 7.$$

Curl of vector field $\boldsymbol{v}$

curl of a vector field$\boldsymbol{v}$, denoted by curl$\boldsymbol{v}$, is defined as the vector field

$$\text{curl}\boldsymbol{v} = \left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right)\boldsymbol{i} + \left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}\right)\boldsymbol{j} + \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right)\boldsymbol{k}$$

We observe that curl$\boldsymbol{v}$ can also be written in terms of the gradient operator as

$$\text{curl}\boldsymbol{v} = \nabla \times \boldsymbol{v} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial x} \\ v_1 & v_2 & v_3 \end{vmatrix}$$

Note that $\nabla \times \boldsymbol{v}$ is just a notation for curl$\boldsymbol{v}$ and it is not a vector product in the usual sense, since $\nabla \times \boldsymbol{v} \neq -\boldsymbol{v} \times \nabla$

Sometimes, curl$\boldsymbol{v}$ is written as

$$\text{curl}\boldsymbol{v} = \sum \left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right)\boldsymbol{i}$$

Where denotes summation obtained by the cyclic rotation of the unit vectors $\boldsymbol{i},\boldsymbol{j},\boldsymbol{k}$, the components v_1, v_2, v_3 and the independent variables x, y, z respectively.

Example: Suppose $\boldsymbol{A} = x^2z^2\boldsymbol{i} - 2y^2z^2\boldsymbol{j} + xy^2z\boldsymbol{k}$. Find $\nabla \times \boldsymbol{A}$(or curl$\boldsymbol{A}$) at the point $P(1, -1, 1)$.

$$\nabla \times \boldsymbol{A} = \left(\frac{\partial}{\partial x}\boldsymbol{i} + \frac{\partial}{\partial y}\boldsymbol{j} + \frac{\partial}{\partial z}\boldsymbol{k}\right) \times (x^2z^2\boldsymbol{i} - 2y^2z^2\boldsymbol{j} + xy^2z\boldsymbol{k})$$

$$= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2z^2 & -2y^2z^2 & xy^2z \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}(xy^2z) - \frac{\partial}{\partial z}(-2y^2z^2)\right]\boldsymbol{i} - \left[\frac{\partial}{\partial x}(xy^2z) - \frac{\partial}{\partial z}(x^2z^2)\right]\boldsymbol{j} + \left[\frac{\partial}{\partial x}(-2y^2z^2) - \frac{\partial}{\partial y}(x^2z^2)\right]\boldsymbol{k}$$

$$= (2xyz + 4y^2z)\boldsymbol{i} - (y^2z - 2x^2z)\boldsymbol{j} + 0\boldsymbol{k}$$

at the point $P(1,-1,1), \nabla \times \boldsymbol{A} = 2\boldsymbol{i} + \boldsymbol{j}$.

There are two fundamental relations between the gradient, divergence and curl vectors. We prove these.

Curl of gradient: Let f be a differentiable scalar field. Then $\mathrm{Curl}(\mathrm{grad}\ f) = 0$ or $\nabla \times (\nabla f) = 0$

Proof: From the definition, we have

$$\nabla \times (\nabla f) = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix}$$

$$= \boldsymbol{i}\left(\frac{\partial^2 f}{\partial y\partial z} - \frac{\partial^2 f}{\partial y\partial z}\right) - \boldsymbol{j}\left(\frac{\partial^2 f}{\partial x\partial z} - \frac{\partial^2 f}{\partial x\partial z}\right) + \boldsymbol{k}\left(\frac{\partial^2 f}{\partial x\partial y} - \frac{\partial^2 f}{\partial x\partial y}\right) = 0.$$

Divergence of Curl: Let $\boldsymbol{V}$ be a differentiable vector field. Then $\mathrm{div}(\mathrm{curl}\boldsymbol{V}) = 0$ or $\nabla.(\nabla \times \boldsymbol{V}) = 0$.

Proof: From the definition, we have for $\boldsymbol{V} = v_1\boldsymbol{i} + v_2\boldsymbol{j} + v_3\boldsymbol{k}$

$$\nabla \times (\nabla \times V) = \left(\boldsymbol{i}\frac{\partial}{\partial x} + \boldsymbol{j}\frac{\partial}{\partial y} + \boldsymbol{k}\frac{\partial}{\partial z}\right).\left[\boldsymbol{i}\left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right) + \boldsymbol{j}\left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}\right) + \boldsymbol{k}\left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right)\right]$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right) = 0.$$

Formulas Involving ∇

The following propositions give many of the properties of the del operator ∇.

PROPOSITION: Suppose $\boldsymbol{A}$ and $\boldsymbol{B}$ are differentiable vector functions, and $\emptyset$ and ψ are differentiable scalar functions of position (x, y, z). Then the following laws hold.

i. $\nabla(\emptyset + \psi) = \nabla\phi + \nabla\psi$ or $\text{grad}(\emptyset + \psi) = \text{grad}\phi + \text{grad}\psi$
ii. $\nabla.(\boldsymbol{A} + \boldsymbol{B}) = \nabla.\boldsymbol{A} + \nabla.\boldsymbol{B}$ or $\text{div}(\boldsymbol{A} + \boldsymbol{B}) = \text{div}\boldsymbol{A} + \text{div}\boldsymbol{B}$
iii. $\nabla \times (\boldsymbol{A} + \boldsymbol{B}) = \nabla \times \boldsymbol{A} + \nabla \times \boldsymbol{B}$ or $\text{curl}(\boldsymbol{A} + \boldsymbol{B}) = \text{curl}\boldsymbol{A} + \text{curl}\boldsymbol{B}$
iv. $\nabla.(\emptyset\boldsymbol{A}) = (\nabla\emptyset).\boldsymbol{A} + \emptyset(\nabla.\boldsymbol{A})$
v. $\nabla \times (\emptyset\boldsymbol{A}) = (\nabla\emptyset).\boldsymbol{A} + \emptyset(\nabla \times \boldsymbol{A})$
vi. $\nabla \times (\boldsymbol{A} \times \boldsymbol{B}) = (\boldsymbol{B}.\nabla)\boldsymbol{A} - \boldsymbol{B}(\nabla.\boldsymbol{A}) - (\boldsymbol{A}.\nabla)\boldsymbol{B} + \boldsymbol{A}(\nabla.\boldsymbol{B})$
vii. $\nabla(\boldsymbol{A}.\boldsymbol{B}) = (\boldsymbol{B}.\nabla)\boldsymbol{A} + (\boldsymbol{A}.\nabla)\boldsymbol{B} + \boldsymbol{B} \times (\nabla \times \boldsymbol{A}) + \boldsymbol{A} \times (\nabla \times \boldsymbol{B})$

PROPOSITION: Suppose $\emptyset$ and $\boldsymbol{A}$ are differentiable scalar and vector functions, respectively, and both have continuous second partial derivatives. Then the following laws hold.

i. $\nabla.(\nabla\emptyset) = \nabla^2\emptyset = \frac{\partial^2\emptyset}{\partial x^2} + \frac{\partial^2\emptyset}{\partial y^2} + \frac{\partial^2\emptyset}{\partial z^2}$, where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is called the Laplacian operation.
ii. $\nabla \times (\nabla\emptyset) = 0$. The curl of the gradient of $\emptyset$ is zero.
iii. $\nabla.(\nabla \times \boldsymbol{A}) = 0$. The divergence of the curl of $\boldsymbol{A}$ is zero.
iv. $\nabla \times (\nabla \times \boldsymbol{A}) = \nabla(\nabla.\boldsymbol{A}) - \nabla^2\boldsymbol{A}$.

Remark

Let $\boldsymbol{v}$ denote the velocity of a fluid in a medium. If $\text{div}(\boldsymbol{v}) = 0$, then the fluid is said to be incompressible. In electromagnetic theory, if $\operatorname{div}(\boldsymbol{v}) = 0$, then the vector field $\boldsymbol{v}$ is said to be solenoidal.

Remark

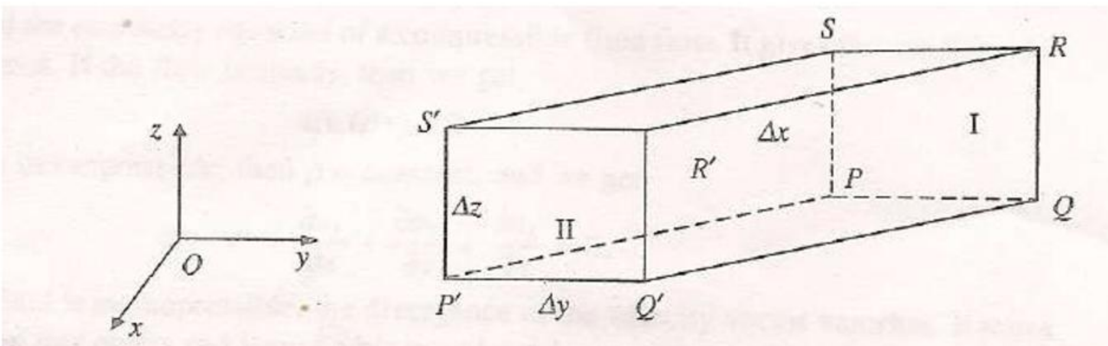
Some books use the word rotation in place of curl, that is, $\operatorname{curl}(\boldsymbol{v})$ is written as $\operatorname{rot}(\boldsymbol{v})$. In fluid mechanics, $\operatorname{curl}(\boldsymbol{v})$ where $\boldsymbol{v}$ is the velocity, measure the vorticity of the fluid. If $\operatorname{curl}(\boldsymbol{v}) = 0$, then $\boldsymbol{v}$ is said to be an irritation field.

Physical interpretation of divergence

We shall present an interpretation in fluid mechanics. Consider the flow of a compressible fluid of density $\rho(x, y, z, t)$, (density is mass per unit volume) and velocity

$$\boldsymbol{v}(x, y, z, t) = v_1(x, y, z, t)\boldsymbol{i} + v_2(x, y, z, t)\boldsymbol{j} + v_3(x, y, z, t)\boldsymbol{k}.$$

Therefore, the density and velocity vary from point to point and also with respect to time. Consider an infinitesimal volume element (parallelepiped of sides $\Delta x, \Delta y, \Delta z$) placed in the fluid as given in figure. The fluid enters the elemental volume through the faces and goes out from the other faces.



The face $PQRS$ is denoted as I and the face $P'Q'R'S'$ is denoted as II. Let us now compute the loss of fluid as it flows through the element in time Δt. We assume the following (volume of the fluid flowing through an element of surface area Δs in time Δt) $\approx$ (component of fluid velocity normal to the surface$\times$area of surface $\Delta S \times \Delta t$).

The area of face I is $\Delta y \Delta z$ and the direction of the normal is $-\boldsymbol{i}$. Therefore, the mass of the fluid entering through the face I in time Δt, is approximately equal to

$$-(\rho v_1)(x, y, z, t)\Delta y\, \Delta z\, \Delta t.$$

The area of the face II is $\Delta y\, \Delta z$ and the direction of the normal is $\boldsymbol{i}$. Hence, the mass of the fluid leaving this face in time Δt is approximately equal to

$$(\rho v_1)(x + \Delta x, y, z, t)\Delta y\, \Delta z\, \Delta t.$$

Therefore, the approximate loss of mass as the fluid flows through the faces, perpendicular to the YZ- plane, is

$$[(\rho v_1)(x + \Delta x, y, z, t) - (\rho v_1)(x, y, z, t)]\Delta y\, \Delta z\, \Delta t.$$

Similarly, the approximate losses of mass through other faces of the elemental volume $\Delta V (= \Delta x\, \Delta y\, \Delta z)$, are

$$[(\rho v_2)(x, y + \Delta y, z, t) - (\rho v_2)(x, y, z, t)]\Delta x\, \Delta z\, \Delta t$$

and $$[(\rho v_3)(x , y, z + \Delta z, t) - (\rho v_3)(x, y, z, t)]\Delta y\, \Delta x\, \Delta t$$

Therefore, adding equations, we obtain the total loss of mass of the fluid during the time Δt as

$$\left[\left\{\frac{1}{\Delta x}\{(\rho v_1)(x+\Delta x,y,z,t)-(\rho v_1)(x,y,z,t\}+\frac{1}{\Delta y}\{(\rho v_2)(x,y+\Delta y,z,t)-(\rho v_2)(x,y,z,t)\}+\frac{1}{\Delta z}\{(\rho v_3)(x,y,z+\Delta z)-(\rho v_3)(x,y,z,t)\}\right\}\right]\Delta V\Delta t \qquad (1)$$

This loss of mass is due to the rate of change of density with respect to time and hence is equal to $-\frac{\partial\rho}{\partial t}\Delta V\Delta t$. (2)

Equate the expression in equations (1) and (2), let $\Delta x \to 0, \Delta y \to 0, \Delta z \to 0, \Delta t \to 0$ and divide the resulting equation by $\Delta V\Delta t$. In the limit, we get

$$\frac{\partial}{\partial x}(\rho v_1)+\frac{\partial}{\partial y}(\rho v_2)+\frac{\partial}{\partial z}(\rho v_3)=-\frac{\partial\rho}{\partial t}$$

$$\frac{\partial\rho}{\partial t}+\mathrm{div}(\rho\boldsymbol{v})=0$$

or

This equation is called the continuity equation of a compressible fluid flow. It gives the condition for the conservation of mass. If the flow is steady, then we get

$$\mathrm{div}(\rho\boldsymbol{v})=0$$

Further, if the fluid is incompressible, then $\rho=$ constant, and we get

$$\mathrm{div}(\boldsymbol{v})=\frac{\partial v_1}{\partial x}+\frac{\partial v_2}{\partial y}+\frac{\partial v_3}{\partial z}=0$$

Therefore, when the fluid is incompressible, the divergence of the velocity vector vanishes. It states that the amount of fluid that enters and leaves a given volume is same; there is no loss in the mass of the fluid.

Physical interpretation of curl

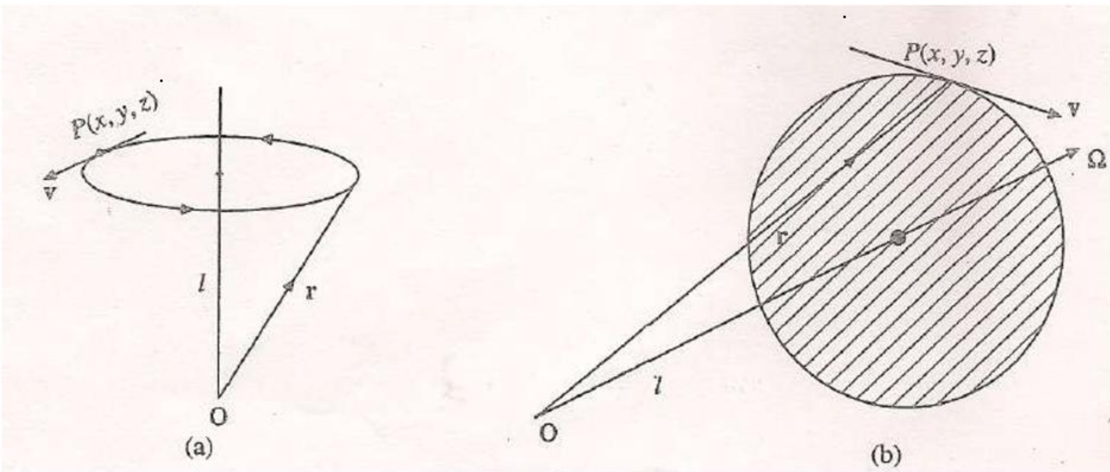
Let a rigid body rotate with the uniform angular velocity $\boldsymbol{\Omega} = a\boldsymbol{i} + b\boldsymbol{j} + c\boldsymbol{k}$, about an axis l through the origin O. Let the position vector of any point $P(x, y, z)$ on the rotating body be $\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$. The tangential (linear) velocity $\boldsymbol{v}$ of the point $P(x, y, z)$ is given by

$$\boldsymbol{v} = \boldsymbol{\Omega} \times \boldsymbol{r} = (a\boldsymbol{i} + b\boldsymbol{j} + c\boldsymbol{k}) \times (x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k})$$

$$= (bz - cy)\boldsymbol{i} + (cx - az)\boldsymbol{j} + (ay - bx)\boldsymbol{k}.$$

Now, $\text{curl}\boldsymbol{v} = \nabla \times \boldsymbol{v} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ bz - cy & cx - az & ay - bx \end{vmatrix}$

$$= 2(a\boldsymbol{i} + b\boldsymbol{j} + c\boldsymbol{k}) = 2\boldsymbol{\Omega}.$$



Therefore, the angular velocity of the point $P(x, y, z)$ is given by $\mathbf{\Omega} = \left(\text{curl}\,\frac{\mathbf{v}}{2}\right)$. Hence, the angular velocity of a uniformly body is equal to one-half of the curl of the linear velocity. Because of this interpretation, the notation *rotation* or *rot* curl is also used

Remark

A force field $\boldsymbol{F}$ is said to be conservative if it is derivable from a potential function f, that is $\boldsymbol{F} = \text{grad}f$. Then, $\text{curl}(\boldsymbol{F}) = \text{curl}(\text{grad}f) = 0$. Therefore, if $\boldsymbol{F}$ is conservative then $\text{curl}(\boldsymbol{F}) = 0$ and there exists a scalar potential function f such that $\boldsymbol{F} = \text{grad}f$.

Problem 1: Suppose $\emptyset(x, y, z) = 3x^2y - y^2z^2$. Find $\nabla\emptyset$(or grad $\emptyset$) at the point $(1, -2, -1)$.

Solution:

$$\nabla\emptyset = \left(\frac{\partial}{\partial x}\boldsymbol{i} + \frac{\partial}{\partial y}\boldsymbol{j} + \frac{\partial}{\partial z}\boldsymbol{k}\right)(3x^2y - y^3z^2)$$

$$= \boldsymbol{i}\frac{\partial}{\partial x}(3x^2y - y^3z^2) + \boldsymbol{j}\frac{\partial}{\partial y}(3x^2y - y^2z^2) + \boldsymbol{k}\frac{\partial}{\partial z}(3x^2y - y^2z^2)$$

$$= 6xy\boldsymbol{i} + (3x^2 - 3y^2z^2)\boldsymbol{j} - 2y^3z\boldsymbol{k}$$

$$= 6(1)(-2)\boldsymbol{i} + \{3(1)^2 - 3(-2)^2(-1)^2\}\boldsymbol{j} - 2(-2)^3(-1)\boldsymbol{k}$$

$$= -12\boldsymbol{i} - 9\boldsymbol{j} - 16\boldsymbol{k}.$$

Problem 2: Find $\nabla\emptyset$ if

a) $\emptyset = ln\,|\boldsymbol{r}|$
b) $\emptyset = \frac{1}{r}$

Solution:

a) $\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$. Then $|\boldsymbol{r}| = \sqrt{x^2 + y^2 + z^2}$ and

$$\emptyset = ln|\boldsymbol{r}| = \frac{1}{2}\, ln(x^2 + y^2 + z^2)$$

$$\nabla\emptyset = \frac{1}{2}\nabla\, ln\,(x^2 + y^2 + z^2)$$

$$= \frac{1}{2}\left\{\boldsymbol{i}\frac{\partial}{\partial x}\, ln\,(x^2 + y^2 + z^2) + \boldsymbol{j}\frac{\partial}{\partial y}\, ln\,(x^2 + y^2 + z^2) + k\frac{\partial}{\partial z}\, ln\,(x^2 + y^2 + z^2)\right\}$$

$$= \frac{1}{2}\left\{i\,\frac{2x}{x^2+y^2+z^2} + j\,\frac{2y}{x^2+y^2+z^2} + k\,\frac{2z}{x^2+y^2+z^2}\right\}$$

$$= \frac{x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k}}{x^2+y^2+z^2} = \frac{\boldsymbol{r}}{r^2}.$$

b) $$\nabla\emptyset = \nabla\left(\frac{1}{r}\right) = \nabla\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right) = \nabla\left\{(x^2 + y^2 + z^2)^{-1/2}\right\}$$

$$= \boldsymbol{i}\frac{\partial}{\partial x}(x^2 + y^2 + z^2)^{-1/2} + \boldsymbol{j}\frac{\partial}{\partial y}(x^2 + y^2 + z^2)^{-1/2} + \boldsymbol{k}\frac{\partial}{\partial z}(x^2 + y^2 + z^2)^{-1/2}$$

$$= \boldsymbol{i}\left\{-\frac{1}{2}(x^2 + y^2 + z^2)^{-3/2}2x\right\} + \boldsymbol{j}\left\{-\frac{1}{2}(x^2 + y^2 + z^2)^{-3/2}2y\right\} + \boldsymbol{k}\left\{-\frac{1}{2}(x^2 + y^2 + z^2)^{-3/2}2z\right\}$$

$$= \frac{-x\boldsymbol{i}-y\boldsymbol{j}-z\boldsymbol{k}}{(x^2+y^2+z^2)^{3/2}} = -\frac{\boldsymbol{r}}{r^3}.$$

Problem 3: Find a unit normal to the surface $-x^2yz^2 + 2xy^2z = 1$ at the point $P(1,1,1)$.

Solution: Let $\emptyset = -x^2yz^2 + 2xy^2z$. $\nabla\emptyset$ is normal to the surface$-x^2yz^2 + 2xy^2z = 1$ at the point $P(1,1,1)$; hence $\frac{\nabla\emptyset_{(1,1,1)})}{|\nabla\emptyset_{(1,1,1)}|}$ will suffice.

$$\nabla\emptyset = (-2xyz^2 + 2y^2z)\boldsymbol{i} + (-x^2z^2 + 4xyz)\boldsymbol{j} + (-2x^2yz + 2xy^2)\boldsymbol{k}$$

Then $\nabla\emptyset_{(1,1,1)} = 3\boldsymbol{j}$. $\left|\nabla\emptyset_{(1,1,1)}\right| = |3\mathbf{j}| = 3|\mathbf{j}| = 3$.

Thus, at the point $P(1,1,1)$ $\frac{3\boldsymbol{j}}{3} = \boldsymbol{j}$ is a unit normal to $-x^2yz^2 + 2xy^2z = 1$.

Problem 4: Find an equation for the tangent plane to the surface $x^2yz - 4xyz^2 = -6$ at the point $P(1, 2, 1)$.

Solution:

$$\nabla(x^2yz - 4xyz^2)$$
$$= (2xyz - 4yz^2)\boldsymbol{i} + (x^2z - 4xz^2)\boldsymbol{j} + (x^2y - 8xyz)\boldsymbol{k}$$

Evaluating the gradient at the point $P(1, 2, 1)$, we get

$-4\boldsymbol{i} - 3\boldsymbol{j} - 14\boldsymbol{k}$. Then $4\boldsymbol{i} + 3\boldsymbol{j} + 14\boldsymbol{k}$ is normal to the surface at P. An equation of the plane with normal $\boldsymbol{N} = a\boldsymbol{i} + b\boldsymbol{j} + c\boldsymbol{k}$ has the form

$$ax + by + cz = k$$

Thus the equation has the form $4x + 3y + 14z = k$. substituting P in the equation, we get $k = 24$.

Thus the required equation is $4x + 3y + 14z = 24$.

Problem 5: Find the angle between the surfaces $z = x^2 + y^2$ and $z = \left(x - \frac{\sqrt{6}}{6}\right)^2 + \left(y - \frac{\sqrt{6}}{6}\right)^2$ at the point $P = \left(\frac{\sqrt{6}}{12}, \frac{\sqrt{6}}{12}, \frac{1}{12}\right)$.

Solution: The angle between the surfaces at the point is the angle between the normals to the surfaces at the point.

Let $\emptyset_1 = x^2 + y^2 - z$ and $\emptyset_2 = \left(x - \frac{\sqrt{6}}{6}\right)^2 + \left(y - \frac{\sqrt{6}}{6}\right)^2 - z$.

A normal to $z = x^2 + y^2$ is

$$\nabla\emptyset_1 = 2x\boldsymbol{i} + 2y\boldsymbol{j} - \boldsymbol{k} \text{ and } \nabla\emptyset_1(P) = \frac{\sqrt{6}}{6}\boldsymbol{i} + \frac{\sqrt{6}}{6}\boldsymbol{j} - \boldsymbol{k}.$$

A normal to $z = \left(x - \frac{\sqrt{6}}{6}\right)^2 + \left(y - \frac{\sqrt{6}}{6}\right)^2$ is

$$\nabla\emptyset_2 = 2\left(x - \frac{\sqrt{6}}{6}\right)\boldsymbol{i} + 2\left(y - \frac{\sqrt{6}}{6}\right)\boldsymbol{j} - \boldsymbol{k} \text{ and } \nabla\emptyset_2 = \frac{\sqrt{6}}{6}\boldsymbol{i} + \frac{\sqrt{6}}{6}\boldsymbol{j} - \boldsymbol{k}.$$

Now $(\nabla\emptyset_1(P)).(\nabla\emptyset_2(P)) = |(\nabla\emptyset_1(P))||(\nabla\emptyset_2(P))| \cos\Theta$ where θ is the required angle.

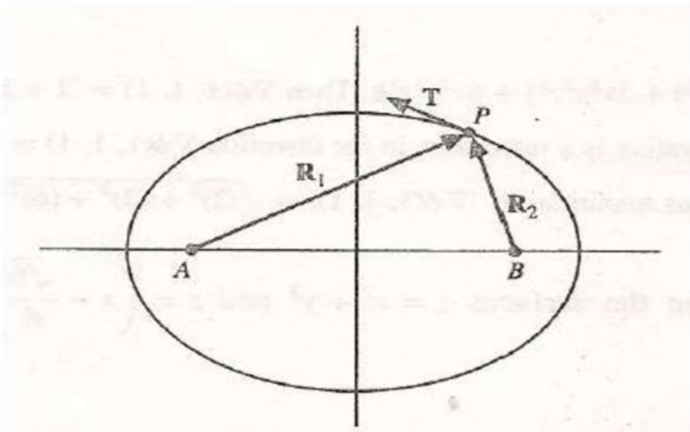
$$\left(\frac{\sqrt{6}}{6}\boldsymbol{i} + \frac{\sqrt{6}}{6}\boldsymbol{j} - \boldsymbol{k}\right).\left(-\frac{\sqrt{6}}{6}\boldsymbol{i} - \frac{\sqrt{6}}{6}\boldsymbol{j} - \boldsymbol{k}\right) = \left|\frac{\sqrt{6}}{6}\boldsymbol{i} + \frac{\sqrt{6}}{6}\boldsymbol{j} - \boldsymbol{k}\right|\left|-\frac{\sqrt{6}}{6}\boldsymbol{i} - \frac{\sqrt{6}}{6}\boldsymbol{j} - \boldsymbol{k}\right| \cos\theta$$

$$-\frac{1}{6} - \frac{1}{6} + 1 = \sqrt{\frac{1}{6} + \frac{1}{6} + 1}\,\sqrt{\frac{1}{6} + \frac{1}{6} + 1}\,\cos\theta \text{ and } \cos\theta = \frac{2/3}{4/3} = \frac{1}{2}.$$

Problem 6: Let P be any point on an ellipse whose foci are at points A and B. Prove that lines AP and BP make equal angles with the tangent to the ellipse at P.

Solution: Let $R_1 = AP$ and $R_2 = BP$ denote vectors drawn respectively from foci A and B to point P on the ellipse, and let T be a unit tangent to the ellipse at P. Since an ellipse is the locus of all points P, the sum of distances from two fixed points A and B is a constant p, it is seen that the equation of the ellipse is $R_1 + R_2 = p$.

$\nabla(R_1 + R_2)$ is a normal to the ellipse; hence $[\nabla(R_1 + R_2)].T = 0$ or $(\nabla R_2).\mathrm{T} = -(\nabla R_1).\mathrm{T}$.



Since ∇R_1 and ∇R_2 are unit vectors in direction R_1 and R_2 respectively, the cosine of the angle between ∇R_2 and T is equal to the cosine of the angle between ∇R_1 and $-T$; hence the angles themselves are equal.

The problem has a physical interpretation. Light rays (or sound waves) originating at focus A, for example, will be reflected from the ellipse to focus B.

Problem 7: Suppose $\boldsymbol{A} = x^2z^2\boldsymbol{i} - 2y^2z^2\boldsymbol{j} + xy^2z\boldsymbol{k}$. Find $\nabla.\boldsymbol{A}$ (or div$\boldsymbol{A}$) at the point $P(1,-1,1)$.

Solution: $\nabla.\boldsymbol{A} = \left(\frac{\partial}{\partial x}\boldsymbol{i} + \frac{\partial}{\partial y}\boldsymbol{j} + \frac{\partial}{\partial z}\boldsymbol{k}\right).(x^2z^2\boldsymbol{i} - 2y^2z^2\boldsymbol{j} + xy^2z\boldsymbol{k})$

$$= \frac{\partial}{\partial x}(x^2z^2) + \frac{\partial}{\partial y}(-2y^2z^2) + \frac{\partial}{\partial z}(xy^2z)$$

$$= 2xz^2 - 4yz^2 + xy^2$$

$$\nabla.\boldsymbol{A}_{(1,-1,1)} = 2(1)(1)^2 - 4(-1)(1)^2 + (1)(-1)^2 = 7.$$

Problem 8: Suppose $\boldsymbol{A} = x^2z^2\boldsymbol{i} - 2y^2z^2\boldsymbol{j} + xy^2z\boldsymbol{k}$. Find $\nabla \times \boldsymbol{A}$ (or curl$\boldsymbol{A}$) at the point $P = (1,-1,1)$.

Solution: $\nabla \times \boldsymbol{A} = \begin{bmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2z^2 & -2y^2z^2 & xy^2z \end{bmatrix}$

$$= \left[\frac{\partial}{\partial x}(xy^2z) - \frac{\partial}{\partial y}(-2y^2z^2)\right]\boldsymbol{i} - \left[\frac{\partial}{\partial x}(xy^2z) - \frac{\partial}{\partial y}(x^2z^2)\right]\boldsymbol{j} + \left[\frac{\partial}{\partial x}(-2y^2z^2) + \frac{\partial}{\partial y}(x^2z^2)\right]\boldsymbol{k}$$

$$= (2xyz + 4yz^2)\boldsymbol{i} - (y^2z - 2x^2z)\boldsymbol{j}$$

Thus $\nabla \times \boldsymbol{A}_{(\mathrm{P})} = 2\boldsymbol{i} + \boldsymbol{j}$.

Problem 9: A vector $\boldsymbol{v}$ is called irrotational if curl$\boldsymbol{v} = 0$

(a). Find constants a, b, and c such that

$\boldsymbol{v} = (-4x - 3y + az)\boldsymbol{i} + (bx + 3y + 5z)\boldsymbol{j} + (4x + cy + 3z)\boldsymbol{k}$ is irrotational .

(b). Show that $\boldsymbol{v}$ can be expressed as the gradient of a scalar function.

Solution:

(a). curl$\boldsymbol{v} = \nabla \times \boldsymbol{v}$

$$\nabla \times \boldsymbol{v} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ -4x - 3y + az & bx + 3y + 5z & 4x + cy + 3z \end{vmatrix}$$

$$= (c - 5)\boldsymbol{i} - (4 - a)\boldsymbol{j} + (b + 3)\boldsymbol{k}$$

This equals the zero vector when $a = 4, b = -3,$ and $c = 5.$

So $\boldsymbol{v} = (-4x - 3y + 4z)\boldsymbol{i} + (-3x + 3y + 5z)\boldsymbol{j} + (4x + 5y + 3z)\boldsymbol{k}$

(b).

Assume $\boldsymbol{v} = \nabla\emptyset = \frac{\partial\emptyset}{\partial x}\boldsymbol{i} + \frac{\partial\emptyset}{\partial y}\boldsymbol{j} + \frac{\partial\emptyset}{\partial z}\boldsymbol{k}$.

Then

$$\frac{\partial\emptyset}{\partial x} = -4x - 3y + 4z \qquad \text{.........(1)}$$

$$\frac{\partial\emptyset}{\partial y} = -3x + 3y + 5z \qquad \text{........(2)}$$

$$\frac{\partial\emptyset}{\partial z} = 4x + 5y + 3z \qquad \text{........(3)}$$

Integrating (1) partially with respect to x keeping y and z constant, we obtain

$$\emptyset = -2x^2 - 3xy + 4xz + f(y,z) \qquad \text{........(4)}$$

Where $f(y,z)$ is an arbitrary function of y and z.

Partially differentiate (4) with respect to y, we get

$$\frac{\partial \emptyset}{\partial y} = -3x + \frac{\partial f}{\partial y} \qquad \text{...........(5)}$$

From (2) and (5), we get $\frac{\partial f}{\partial y} = 3y + 5z$ and integrating partially with respect to y, we obtain

$$f(y,z) = \frac{3y^2}{2} + 5yz + h(z)$$

Where $h(z)$ is an arbitrary function in z.

Therefore $\emptyset = -2x^2 - 3xy + 4xz + \frac{3y^2}{2} + 5yz + h(z)$ ....(6)

Partially differentiate (7) with respect to z, we get

$$\frac{\partial \emptyset}{\partial z} = 4x + 5y + \frac{\partial h}{\partial z} \qquad \text{..........(7)}$$

From (3) and (7), we get $\frac{\partial h}{\partial z} = 3z$ and integrating partially with respect to z, we obtain

$$h(z) = \frac{3z^2}{2} + \text{constant}$$

Therefore $\emptyset = -2x^2 - 3xy + 4xz + \frac{3y^2}{2} + 5yz + \frac{3z^2}{2} + \text{constant}$

So that $\emptyset = -2x^2 + \frac{3}{2}y^2 + \frac{3}{2}z^2 - 3xy + 4xz + 5yz$ +constant.

Note that we can add any constant to $\emptyset$. In general, if $\nabla \times \boldsymbol{v} = 0$, then we can find $\emptyset$ so that $\boldsymbol{v} = \nabla\emptyset$.

A vector field $\boldsymbol{v}$; which can be obtained from a scalar field $\emptyset$, so that $\boldsymbol{v} = \nabla\emptyset$ is called a conservative vector field and $\emptyset$ is called the scalar potential. Note conversely that, if $\boldsymbol{v} = \nabla\emptyset$, then $\nabla \times \boldsymbol{v} = 0$.

EXERCISE

1. Suppose F and G are differentiable scalar functions of x, y and z. Prove
a) $\nabla(F+G) = \nabla F + \nabla G$.
b) $\nabla(FG) = F\nabla G + G\nabla F$.
2. Show that $\nabla r^n = nr^{n-1}\boldsymbol{r}$.
3. Show that $\nabla\emptyset$ is a vector perpendicular to the surface $\emptyset(x, y, z) = c$ where c is a constant.
4. Given $\emptyset = 6x^3y^2z$.
a) Find $\nabla.\nabla\emptyset$(or div grad $\emptyset$).
b) Show that $\nabla.\nabla\emptyset = \nabla^2\emptyset$ where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ denotes the Laplacian operator.
5. Prove that $\nabla^2\left(\frac{1}{r}\right) = 0$.
6. If $r = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$ and $r = |\boldsymbol{r}|$, show that div $\left(\frac{\boldsymbol{r}}{r^3}\right) = 0$.
7. Show that if $\emptyset(x, y, z)$ is any solution of Laplace's equation, then $\nabla\emptyset$ is a vector that is both solenoidal and irrotational.

ANSWERS

4. a) $36xy^2z + 12x^3z$