UNIT-I

Vector Space

Models

(a) vector space

(b) sub space

(c) linear combination; linear span

(d) linear Independent & Dependent

(e) Bases Dimension

(f) linear Transformation

(g) Rank, Nullity Theorem

(i) Matrix linear Transformation

Problems on vector space - model wise

objective questions on vector space

* Group $(G, +)$ Abelian group

closure law : $\forall a, b \in G \quad a+b \in G$

Asso law : $\forall a, b, c \in G$

$$a+(b+c) = (a+b)+c$$

Identi law : $\forall a \in G \quad \exists e \in G \ni$

$$a+e = e+a = a$$

Inverse law : $\forall a \in G \quad \exists b \in G \ni$

$$a+b = b+a = e$$

b is inverse of a

$$\boxed{b = -a}$$

Commutal law: $\forall a, b \in G$

$$a+b = b+a$$

* Ring $(R, +, \cdot)$

(i) $(R, +)$ Abelian group

(ii) $(R, \cdot)$ Semi group $\begin{cases} \to \text{closure law} \\ \to \text{Asso law} \end{cases}$ $(\cdot)$

(iii) Distributive laws holds

$$a \cdot (b+c) = a \cdot b + a \cdot c$$

$$(b+c) \cdot a = b \cdot a + c \cdot a$$

* Field $(F, +, \cdot)$

(i) $(F, +, \cdot)$ Ring

(ii) unit element exist

(iii) commutative laws holds in '$\cdot$'

(iv) non zero elements have multiplicative inverse

* Vector space $V(F)$

let 'V' be a non empty set whose elements are called vectors

let F be any set whose elements are called scalars; and also $(F, +, \cdot)$ is field.

Then $V(F)$ is called vector space

(i) $(V, +)$ Abelian group

(ii) External composition exists

i.e $\forall \alpha \in V, \forall a \in F \Rightarrow a \cdot \alpha \in V$

(iii) $a(\alpha + \beta) = a \cdot \alpha + a \cdot \beta$

$(a+b)\alpha = a \cdot \alpha + b \cdot \alpha$

$(ab)(\alpha) = a(b\alpha)$

$1 \cdot \alpha = \alpha$

where $\alpha, \beta \in V$, $a, b \in F$; $1 \in F$

NOTE

vector: $\alpha, \beta, \gamma, \delta, \omega \in V$

$$\alpha = (a_1\ a_2\ \cdots\ a_n)$$

$$\beta = (b_1\ b_2\ \cdots\ b_n)$$

$$\gamma = (c_1\ c_2\ \cdots\ c_n)$$

$$\delta = (d_1\ d_2\ \cdots\ d_n)$$

* $\bar{0} \in V$ zero vector *

$$\bar{0} = (0, 0, 0, \cdots 0)$$

Scalar $a, b, c, d \sim \in F$

zero scalar $= 0 \in F$

Note

① $$a \cdot \alpha = a(a_1\ a_2\ \cdots\ a_n)$$
$$= (aa_1, aa_2, \cdots, aa_n)$$

② $$\alpha + \beta = (a_1\ a_2\ \cdots\ a_n) + (b_1\ b_2\ \cdots\ b_n)$$
$$= (a_1 + b_1, a_2 + b_2, \cdots a_n + b_n)$$

③ $$(\alpha + \beta) + \gamma = ((a_1 + b_1) + c_1, (a_2 + b_2) + c_2, \cdots (a_n + b_n) + c_n)$$

④ $$a \cdot \bar{0} = a(0, 0, \cdots 0) = (0, 0 \cdots 0) = \bar{0}$$

$$a \cdot \bar{0} = \bar{0}$$

(5) $0 \cdot \alpha = 0(a_1, a_2 \cdots a_n)$

$$= (0, 0, \cdots 0)$$

$$0 \cdot \alpha = \bar{0}$$

(6) $a(-\alpha) = -(a\alpha)$

(7) $a\alpha = \bar{0}$ $\quad a = 0$ (or) $\alpha = \bar{0}$

(8) $a(\alpha - \beta) = a\alpha - a\beta$

(9) $(-a)(-\alpha) = a\alpha$

① The set C_n of all n-tuples of complex numbers with addition of the external composition and scalar multiplication of complex numbers by complex numbers is a vector space over the field of complex numbers with the following definition

(i) If $\alpha, \beta \in \mathbb{C}_n$ and

$\alpha = (a_1\ a_2 \ldots a_n)$, $\beta = (b_1\ b_2 \ldots b_n)$

$\forall\ a_i, b_i \in \mathbb{C}$

$\alpha + \beta = (a_1 + b_1, a_2 + b_2, \ldots a_n + b_n) \in \mathbb{C}_n$

(ii) $x\alpha = (xa_1, xa_2, \ldots xa_n)$

$\forall\ x \in \mathbb{C}$

Sol. $\underline{\mathbb{C}_n(\mathbb{C}) \text{ is vector space}}$

(i) $(\mathbb{C}_n, +)$ Abelian group

(ii) External composition exists

$\alpha \in \mathbb{C}_n$, $x \in \mathbb{C}$ $x \cdot \alpha \in \mathbb{C}_n$

(iii) $x(\alpha+\beta) = x\alpha + x\beta$

$$(x+y)\alpha = x\alpha + y\alpha$$

$$(xy)(\alpha) = x(y\alpha)$$

$$1\cdot\alpha = \alpha$$

$$\forall\ \alpha, \beta \in \mathbb{C}^n, \quad x, y \in \mathbb{C} \quad 1 \in \mathbb{C}$$

(i) $(\mathbb{C}^n, +)$ Abelian group

(i) closure law: $\alpha + \beta \in \mathbb{C}^n$

since $\forall\ \alpha, \beta \in \mathbb{C}^n$

we have $\boxed{\alpha + \beta \in \mathbb{C}^n}$

(ii) Associative law : $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$

$$\forall\ \alpha, \beta, \gamma \in \mathbb{C}^n$$

$$\alpha + \beta = (a_1 + b_1, a_2 + b_2 \ldots a_n + b_n)$$

$$\gamma = (c_1, c_2, \ldots c_n)$$

$$(\alpha + \beta) + \gamma = ((a_1 + b_1) + c_1, (a_2 + b_2) + c_2, \ldots (a_n + b_n) + c_n)$$

$$= \big((a_1 + (b_1 + c_1), a_2 + (b_2 + c_2), \ldots\ldots a_n + (b_n + c_n)\big)$$

$$= (a_1, a_2, \ldots a_n) + (b_1 + c_1, b_2 + c_2, \ldots b_n + c_n)$$

$$= \alpha + (\beta + \gamma)$$

$$\boxed{(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)}$$

(iii) Identity: $\alpha + \bar{0} = \bar{0} + \alpha = \alpha$

$$\forall \alpha = (a_1\ a_2 \ldots a_n) \in \mathbb{C}^n$$

$$\bar{0} = (0, 0, \ldots 0) \in \mathbb{C}^n$$

$$\alpha + \bar{0} = (a_1 + 0,\ a_2 + 0, \ldots a_n + 0)$$

$$= (a_1, a_2, \ldots a_n) = \alpha$$

$\alpha + \bar{0} = \alpha$ lly $\bar{0} + \alpha = \alpha$

$$\therefore \boxed{\alpha + \bar{0} = \bar{0} + \alpha = \alpha}$$

$\bar{0}$ is identity vector in $\mathbb{C}_n$

(iv) inverse: $\alpha + (-\alpha) = (-\alpha) + \alpha = \bar{0}$

$\forall \alpha = (a_1\ a_2\ --\ a_n) \in \mathbb{C}_n$

$-\alpha = (-a_1, -a_2, --\ -a_n) \in \mathbb{C}_n$

$\alpha + (-\alpha) = (a_1 - a_1, a_2 - a_2, \dots a_n - a_n)$

$= (0, 0, --\ 0) = \bar{0}$

$\alpha + (-\alpha) = \bar{0}$

lly $(-\alpha) + \alpha = \bar{0}$

$$\boxed{\alpha + (-\alpha) = (-\alpha) + \alpha = \bar{0}}$$

$-\alpha$ is additive inverse of α

(v) commutative law: $\alpha + \beta = \beta + \alpha$

$\alpha = (a_1\ a_2\ --\ a_n) \in \mathbb{C}_n$

$\beta = (b_1\ b_2\ --\ b_n) \in \mathbb{C}_n$

$$\alpha + \beta = (a_1 + b_1, a_2 + b_2, \ldots a_n + b_n)$$

$$= (b_1 + a_1, b_2 + a_2, \ldots b_n + a_n)$$

$$= \beta + \gamma$$

$$\boxed{\alpha + \beta = \beta + \gamma}$$

$(\mathbb{C}_n, +)$ is an Abelian group.

② External composition exists

Since $\forall x \in \mathbb{C}, \alpha \in \mathbb{C}_n$

$$x\alpha = x(a_1\ a_2 \ldots a_n)$$

$$= (xa_1\ xa_2 \ldots xa_n) \in \mathbb{C}_n$$

$$\boxed{x\alpha \in \mathbb{C}_n}$$

(3)

(1) $x(\alpha+\beta) = x\alpha + x\beta$

$x(\alpha+\beta) = x(a_1+b_1, a_2+b_2, \dots a_n+b_n)$

$= (x(a_1+b_1), x(a_2+b_2), \dots x(a_n+b_n))$

$= (xa_1+xb_1, xa_2+xb_2, \dots xa_n+xb_n)$

$= (xa_1, xa_2, \dots xa_n) + (xb_1\ xb_2 \dots xb_n)$

$= x(a_1\ a_2 \dots a_n) + x(b_1\ b_2 \dots b_n)$

$= x\alpha + x\beta$

$$\boxed{x(\alpha+\beta) = x\alpha + x\beta}$$

(2) $xy(\alpha) = x(y\alpha)$

$xy(\alpha) = xy(a_1\ a_2 \dots a_n)$

$= x(ya_1\ ya_2 \dots ya_n)$

$$\boxed{xy(\alpha) = x(y\alpha)}$$

③ $(x+y)\alpha = x\alpha + y\alpha$

$(x+y)\alpha = (x+y)(a_1\ a_2 \dots a_n)$

$= ((x+y)a_1,\ (x+y)a_2, \dots (x+y)a_n)$

$= (xa_1 + ya_1,\ xa_2 + ya_2, \dots xa_n + ya_n)$

$= (xa_1,\ xa_2 \dots xa_n) + (ya_1\ ya_2 \dots ya_n)$

$= x(a_1\ a_2 \dots a_n) + y(a_1\ a_2 \dots a_n)$

$= x\alpha + y\alpha$

$\boxed{(x+y)\alpha = x\alpha + y\alpha}$

④ $1\cdot\alpha = \alpha\cdot 1 = \alpha$

$1\cdot\alpha = 1(a_1\ a_2 \dots a_n)$

$= (a_1\ a_2 \dots a_n) = \alpha$

lly $\alpha\cdot 1 = \alpha$

$\therefore\ \boxed{1\cdot\alpha = \alpha\cdot 1 = \alpha}$

Hence $C_n(C)$ is a vector space.

1. Vector Space

   problem ✓

2. Sub space ✓

   problem ✓

3. Linear Combination

   problems ✓

4. Linearly independent, dependents

   problems ✓

5. Basis Dimension

   problems ✓

6. Linear Transformation

   problems ✓

7. matrix linear transformation

   problems ✓

② Prove that the set of all real valued continuous functions defined on open interval $(0,1)$ is a vector space over the field of real numbers, w.r.t. the operations of addition and scalar multiplication defined as

$$\rightarrow \boxed{\begin{aligned} (f+g)(x) &= f(x) + g(x) \\ (af)(x) &= a f(x) \end{aligned}}\ : \ a \text{ is real } \in \mathbb{R}$$

with $0 < x < 1$

Sol. $\underline{V(\mathbb{R}) \text{ vector space}}$

✓ $f(x), g(x), h(x) : 0(x) \in V$

$x \in \mathbb{R}$

✓ ① $(V,+)$ Abelian group

✓ ② External composition exist

✓ ③ $(a+b) f(x) = a f(x) + b f(x)$

$a(f(x) + g(x)) = a f(x) + a g(x)$

$(ab) f(x) = a(b f(x))$

$1 \cdot f(x) = f(x)$

① $(V, +)$ Abelian group

$f(x), g(x), h(x) \in V$

(i) closure law: $\forall f(x), g(x) \in V$

$$f(x) + g(x) \in V$$

$$\boxed{f+g \in V}$$

(2) Associative law $\forall f, g, h \in V$

$$((f+g)+h)(x) = (f+g)(x) + h(x)$$

$$= [f(x) + g(x)] + h(x)$$

$$= f(x) + [g(x) + h(x)]$$

$$= (f + (g+h))(x)$$

$$\boxed{(f+g)+h = f+(g+h)}$$

(3) identity law $\forall f \in V, \bar{0} \in V$

$$(f+\bar{0})(x) = f(x) + \bar{0}(x)$$

$$= f(x)$$

$$(f+\bar{0})(x) = f(x)$$

$$f+\bar{o}=f$$

||ly $\bar{o}+f=f$

$$\boxed{f+\bar{o}=\bar{o}+f=f}$$

$\bar{o}$ is identity vector in V

(4) inverse law: $\forall f \in \bar{V} ; -f \in \bar{V}$

$$(f+(-f))(x) = f(x) + (-f)(x)$$
$$= f(x) - f(x)$$
$$= 0(x)$$

$$f+(-f)=\bar{o}$$

||ly $-f+f=\bar{o}$

$$\boxed{f+(-f)=(-f)+f=\bar{o}}$$

$-f$ is inverse of f

(5) Commutative law:

$$f(x), g(x) \in V$$

$$(f+g)(x) = f(x) + g(x)$$

$$= g(x) + f(x)$$

$$= (g+f)(x)$$

$$(f+g)(x) = (g+f)(x)$$

$$\boxed{f+g = g+f}$$

$\therefore$ $(V, +)$ is an abelian group

② <u>External composition exist</u>

$$f(x) \in V, \; a \in \mathbb{R}$$

$$(af)(x) = a f(x) \in V$$

$\Rightarrow$ External composition exist.

③ ① $a(f+g) = af + ag$

$$a(f+g)(x) = a\left[f(x) + g(x)\right]$$

$$= af(x) + ag(x)$$

$$= (af + ag)(x)$$

$$a(f+g)(x) = (af + ag)(x)$$

$$\boxed{a(f+g) = af + ag}$$

② $(a+b)f = af + bf$

$$\left[(a+b)f\right](x) = (a+b)f(x)$$

$$= af(x) + bf(x)$$

$$= \left[af + bf\right](x)$$

$$\left[(a+b)f\right](x) = \left[af + bf\right](x)$$

$$\boxed{(a+b)f = af + bf}$$

③ $\underline{(ab)(f) = a(bf)}$

$$(ab)(f)(x) = (ab)\, f(x)$$

$$= a\left[b\, f(x)\right]$$

$$(ab)(f)(x) = a\,[bf](x)$$

$$\boxed{(ab)(f) = a(bf)}$$

④ $\underline{1 \cdot f = f \cdot 1 = 1}$

$1 \in R,\ f \in V$

$$(1 \cdot f)(x) = 1 \cdot f(x)$$

$$= f(x)$$

$$(1 \cdot f)(x) = f(x)$$

$1 \cdot f = f$ lly $f \cdot 1 = f$

$$\boxed{1 \cdot f = f \cdot 1 = f}$$

Hence $V(R)$ is a vector space.

③** V is the set of all $m \times n$ matrices with real entries R is the field of real numbers. Addition of matrices is the internal composition and multiplication of matrix by a real number is an external composition in then show that $V(\mathbb{R})$ is a vector space

So. $V(\mathbb{R})$ vector space

✓ ① $(V, +)$ Abelian group

② External composition exists

③ $x\{A+B\} = x[A] + x[B]$

$(x+y)[A] = x\{A\} + y[A]$

$(xy)[A] = x[y(A)]$

$1 \cdot [A] = [A]$

$A = [a_{ij}] \quad B = \{b_{ij}\} \quad C = [c_{ij}]$

$O = [o_{ij}]$

① Associative law

$$[A+B] + [C] = [A] + [B+C]$$

$$[A+B] + [C] = [a_{ij} + b_{ij}] + [c_{ij}]$$

$$= [a_{ij}] + [b_{ij} + c_{ij}]$$

$$= A + [B+C]$$

Subspace

✓ let $V(F)$ be a vector space and let W is a said to be subspace of $V(F)$ if W it self a vector space over F with same operation of vector addition and scalar multiplication.

(or)

* let $V(F)$ be a vector space. A non empty set $W \subseteq V$. The necessary and sufficient condition for W to be a subspace of V is

$a, b \in F$; $\alpha, \beta \in V$

$$\Rightarrow \boxed{a\alpha + b\beta \in W}$$

$$\boxed{V(F) \quad V\text{-}S}$$

V vectors $\alpha, \beta \in V$

F scalars $a, b \in F$

$$\boxed{a\alpha + b\beta \in W}$$

$\Rightarrow W$ is subspace of V

Problem

① The set W of ordered triods $(x, y, 0)$ where $x, y \in F$ Then W is a subspace of $V_3(F)$

Sol. let $\alpha, \beta \in W$

$\alpha = (x_1, y_1, 0)$ where $x_1, y_1 \in F$

$\beta = (x_2, y_2, 0)$ where $x_2, y_2 \in F$

let $a, b \in F$

Now $a\alpha + b\beta = a(x_1, y_1, 0)$
$+ b(x_2, y_2, 0)$

$= (ax_1, ay_1, 0) + (bx_2, by_2, 0)$

$= (ax_1 + bx_2, ay_1 + by_2, 0)$

where $\left.\begin{array}{l} ax_1 + bx_2 \\ ay_1 + by_2 \end{array}\right\} \in F$

$\therefore a\alpha + b\beta \in W$

$\therefore W$ is subspace of V

② let $\underline{p, q, r}$ be the fixed elements of a field F, show that the set W of all triads (x, y, z) of elements of F such that $px + qy + rz = 0$ is a vector subspace of $V_3(F)$

sol let $\alpha, \beta \in W$

$\Rightarrow \alpha = (x_1, y_1, z_1) \Rightarrow px_1 + qy_1 + rz_1 = 0$

$\beta = (x_2, y_2, z_2) \Rightarrow px_2 + qy_2 + rz_2 = 0$

let $a, b \in F$

Now $\underline{a\alpha + b\beta} = a(x_1, y_1, z_1) + b(x_2, y_2, z_2)$

$= (ax_1, ay_1, az_1) + (bx_2, by_2, bz_2)$

$= (\underset{x}{ax_1 + bx_2}, \underset{y}{ay_1 + by_2}, \underset{z}{az_1 + bz_2})$

Now $\underline{px + qy + rz}$

$p[ax_1 + bx_2] + q[ay_1 + by_2]$

$+ r[az_1 + bz_2]$

$= pax_1 + pbx_2 + qay_1 + qby_2$

$+ raz_1 + rbz_2$

$= a[px_1 + qy_1 + rz_1]$

$+ b[px_2 + qy_2 + rz_2]$

$= a(0) + b(0)$

$= 0$

$\therefore \quad a\alpha + b\beta \in W$

$\therefore \quad \boxed{W \text{ is subspace of } V}$

③ let R be the field of Real numbers and $W = \{(x, y, z) \mid x, y, z \text{ are rational no}\}$ is W is a subspace of $V_3(R)$

Sol: $V_3(\mathbb{R})$ is a vector space

Given $W = \{(x, y, z) \mid x, y, z \text{ are rational nos}\}$

let $\alpha = (2, 3, 5) \in W$

$\beta = (1, 2, 7) \in W$

$\sqrt{7}, \sqrt{11} \in \mathbb{R}$

$a\alpha + b\beta = \sqrt{7}(2, 3, 5) + \sqrt{11}(1, 2, 7)$

$= (2\sqrt{7}, 3\sqrt{7}, 5\sqrt{7}) + (1\sqrt{11}, 2\sqrt{11}, 7\sqrt{11})$

$= (2\sqrt{7} + \sqrt{11}, 3\sqrt{7} + 2\sqrt{11}, 5\sqrt{7} + 7\sqrt{11})$

$\notin W$

$\boxed{a\alpha + b\beta \notin W}$

$\Rightarrow$ W is not a subspace of $V_3(\mathbb{R})$

Properties of subspace :

① The intersection of any two subspaces W_1 and W_2 of vector space $V(F)$ is also a subspace; lly $\bigcap_{i=1}^{n} W_i$ also subspace

② The union of two subspaces of $V(F)$ may not be subspace of $V(F)$

③ The union of two subspaces is a subspace $\Leftrightarrow$ one is contained in the other

$W_1 \cup W_2$ is subspace $\Leftrightarrow$ $W_1 \subseteq W_2$ (or) $W_2 \subseteq W_1$

④ If W_1 and W_2 are any two subspaces of vector space then $W_1 + W_2$ is a subspace of $V(F)$

$W_1 \cap W_2$ ✓ $W_1 + W_2$ ✓	$W_1 \cup W_2$ ✗ ✓

Linear Combination of vectors:

let $V(F)$ be a vector space. If $\alpha_1 \alpha_2 \cdots \alpha_n \in V$ Then any vector $\gamma = a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n$ where $a_1 a_2 \cdots a_n \in F$ is called a linear combination of The vectors $\alpha_1 \alpha_2 \cdots \alpha_n$ where γ is a vector belonging to $V(F)$

Linear Span of a Set:

let S be a non empty subset of a vector space $V(F)$. The linear Span of S is the set of all linear combination of all possible finite subset of S –

$$L(S) = \left\{ \gamma \mid \gamma = \sum_{i=1}^{n} a_i\alpha_i \quad \begin{array}{l} a_i \in F \\ \alpha_i \in S \end{array} \right\}$$

① Express the vector $\underline{\alpha} = (1, -2, 5)$ as a linear combination of the vectors $e_1 = (1,1,1)$, $e_2 = (1,2,3)$, $e_3 = (2,-1,1)$

Sol. given $\alpha = (1,-2,5)$

$e_1 = (1,1,1)$, $e_2 = (1,2,3)$, $e_3 = (2,-1,1)$

α is l.c of e_1, e_2, e_3

$$\alpha = x e_1 + y e_2 + z e_3$$

$$\left.\begin{aligned}(1,-2,5) = x(1,1,1) + y(1,2,3) \\ + z(2,-1,1)\end{aligned}\right\}$$

$$\left.\begin{aligned} x + y + 2z &= 1 \\ x + 2y - z &= -2 \\ x + 3y + z &= 5 \end{aligned}\right\}$$

Non-Homogeneous system of eqn $AX = B$

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & -1 \\ 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 5 \end{bmatrix}$$

$\boxed{r = n}$ unknown

$[A \mid B]$

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -6 \\ 3 \\ 2 \end{bmatrix}$$

$(1,-2,5) = -6(1,1,1) + 3(1,2,3)$

$+ 2(2,-1,1)$

② Show that the vector $\alpha = (2,-5,3)$ in R^3 cannot be expressed as l.c of the vectors

$e_1 = (1,-3,2)$, $e_2 = (2,-4,-1)$

$e_3 = (1,-5,7)$

Sol: $\boxed{\alpha = x e_1 + y e_2 + z e_3}$

$(2,-5,3) = x(1,-3,2) + y(2,-4,-1)$

$+ z(1,-5,7)$

$$\left.\begin{aligned} x + 2y + z &= 2 \\ -3x - 4y - 5z &= -5 \\ 2x - y + 7z &= 3 \end{aligned}\right\}$$

Non Homo Sys

$\boxed{AX = B}$

$2 \times \frac{1}{4} - \frac{3}{4} + \frac{7}{4}$

$$\begin{bmatrix} 1 & 2 & 1 \\ -3 & -4 & -5 \\ 2 & -1 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -5 \\ 3 \end{bmatrix}$$

$$[A \mid B] \sim \begin{bmatrix} 1 & 2 & 1 & 2 \\ -3 & -4 & -5 & -5 \\ 2 & -1 & 7 & 3 \end{bmatrix}$$

$R_2 \to R_2 + 3R_1, \quad R_3 \to R_3 - 2R_1$

$$\sim \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 2 & -2 & 1 \\ 0 & -3 & 5 & -1 \end{bmatrix}$$

$R_3 \to 2R_3 + 3R_2$

$-2 + 3$

$$\sim \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 2 & -2 & 1 \\ 0 & 0 & 4 & 1 \end{bmatrix}$$

$$\boxed{z = \frac{1}{4}}$$

$$\left.\begin{aligned} x + 2y + z &= 2 \\ 2y - 2z &= 1 \\ 4z &= 1 \end{aligned}\right\} \quad 2y - \frac{1}{2} = 1$$

$$\boxed{y = \frac{3}{4}}$$

$$x + 2 \times \frac{3}{4} + \frac{1}{4} = 2$$

$$\boxed{x = \frac{1}{4}}$$

$$(2, -5, 3) = \frac{1}{4}(1, -3, 2) + \frac{3}{4}(2, -4, -1)$$

$$+ \frac{1}{4}(1, -5, 7)$$

Linear Span $L(S)$

S is non empty finite subset of $V(F)$

$$L(S) = \left\{ \gamma \;\middle|\; \gamma = \sum_{i=1}^{n} a_i \alpha_i ;\ \begin{matrix} a_i \in F \\ \alpha_i \in S \end{matrix} \right\}$$

Prove that $L(S)$ is subspace of $V(F)$

Proof. $L(S)$ which is linear span of S

let $\alpha, \beta \in L(S)$

$$\alpha = \sum_{i=1}^{n} a_i \alpha_i , \quad \beta = \sum_{i=1}^{n} b_i \beta_i$$

$$\alpha = a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n \alpha_n$$

$$\beta = b_1\beta_1 + b_2\beta_2 + \cdots + b_n\beta_n$$

where a_is, b_is $\in F$

$\alpha_1 \alpha_2 \ldots \alpha_n, \beta_1 \beta_2 - \beta_n \in S$

Now

$$a\alpha + b\beta = a(a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n)$$

$$+ b(b_1\beta_1 + b_2\beta_2 + \cdots + b_n\beta_n)$$

$$= aa_1\alpha_1 + aa_2\alpha_2 + \cdots + aa_n\alpha_n$$

$$+ bb_1\beta_1 + bb_2\beta_2 + \cdots + bb_n\beta_n$$

$a\alpha + b\beta$ is l.c of $\alpha_1, \alpha_2 - - \alpha_n, \beta_1 - - \beta_n$

$\therefore a\alpha + b\beta \in L(S)$

$\Rightarrow L(S)$ is subspace of $V(F)$

Linearly independent:

$S = \{\alpha_1, \alpha_2 - - \alpha_n\}$ is called l.I set

If $\exists$ scalars $a_1, a_2 - a_n \in F \ni$

$a_1\alpha_1 + a_2\alpha_2 + - + a_n\alpha_n = \bar{0}$

here $a_1 = 0, a_2 = 0, - - a_n = 0$

all a_i's are zero's

Then S is l.I

Linearly Dependent:

$S = \{\underbrace{\alpha_1, \alpha_2 - - \alpha_n}\}$ is called l.D

If $\exists$ scalar $a_1, a_2 - a_n \in F \ni$

$a_1\alpha_1 + a_2\alpha_2 + - - + a_n\alpha_n = 0$

not all a_i's are zero's

① Show that the system of vectors $(1,2,0), (0,3,1), (-1,0,1)$ of $V_3(\mathbb{Q})$ is L.I where $\mathbb{Q}$ is the field of rational no's

Sol: Let $S = \{(1,2,0), (0,3,1), (-1,0,1)\}$
$\qquad\qquad\qquad\quad \alpha \qquad\quad \beta \qquad\quad \gamma$

S is L.I

$$x\alpha + y\beta + z\gamma = 0$$ where $x, y, z \in \mathbb{Q}$

$$x(1,2,0) + y(0,3,1) + z(-1,0,1) = 0$$

$$\left.\begin{aligned} x + 0y - z &= 0 \\ 2x + 3y + 0z &= 0 \\ 0x + y + z &= 0 \end{aligned}\right\}$$ Homogene $\boxed{AX = 0}$

$$\begin{bmatrix} 1 & 0 & -1 \\ 2 & 3 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Consider $\begin{bmatrix} 1 & 0 & -1 \\ 2 & 3 & 0 \\ 0 & 1 & 1 \end{bmatrix}$

$R_2 \to R_2 - 2R_1$,

$$\sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 3 & 2 \\ 0 & 1 & 1 \end{bmatrix}$$

$R_3 \to 3R_3 - R_2$

$$\sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 3 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 3 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x + 0y - z = 0$$

$$3y + 2z = 0$$

$$z = 0$$

$$y = 0 \quad \therefore \quad x = 0$$

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x\alpha + y\beta + z\gamma = 0$$

$$x = 0,\ y = 0,\ z = 0$$

$S = \{\alpha, \beta, \gamma\}$ is L.I

---

② If the system of vectors $(1,3,2), (1,-7,-8), (2,1,-1)$ of $V_3(R)$ is L.I or L.D

Sol. $S = \{(1,3,2), (1,-7,-8), (2,1,-1)\}$

Consider
$$\begin{vmatrix} 1 & 1 & 2 \\ 3 & -7 & 1 \\ 2 & -8 & -1 \end{vmatrix} = 0$$

If
$|A| = 0$
S is L.D
$|A| \neq 0$
S is L.I

$\Rightarrow$ S is L.D set

(or)

$$x\alpha + y\beta + z\gamma = 0$$

$$x(1,3,2) + y(1,-7,-8) + z(2,1,-1) = 0$$

$$\left.\begin{aligned} x + y + 2z &= 0 \\ 3x - 7y + z &= 0 \\ 2x - 8y - z &= 0 \end{aligned}\right\} \text{ Homo } \quad AX = 0$$

$$\underbrace{\begin{bmatrix} 1 & 1 & 2 \\ 3 & -7 & 1 \\ 2 & -8 & -1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}}_{0}$$

Consider $\begin{bmatrix} 1 & 1 & 2 \\ 3 & -7 & 1 \\ 2 & -8 & -1 \end{bmatrix}$

$R_2 \to R_2 - 3R_1, \quad R_3 \to R_3 - 2R_1$

$$\sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & -10 & -5 \\ 0 & -10 & -5 \end{bmatrix}$$

$R_3 \to R_3 - R_2, \quad R_2 \to \dfrac{R_2}{-5}$

$$\sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x + y + 2z = 0$$

$$2y + z = 0$$

$$z = K$$

$$y = -\frac{k}{2}$$

$$x - \frac{k}{2} + 2k = 0$$

$$x = \frac{-3k}{2}$$

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -\frac{3k}{2} \\ -\frac{k}{2} \\ k \end{bmatrix} = -\frac{k}{2} \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix}$$

vector space $V(F)$ $\left|\begin{array}{l} ① (V,+)\ AG \\ ② ECE \\ ③\ \_\_\_ \end{array}\right.$

Subspace $W \subseteq V(F)$

$\downarrow$ VS

$\boxed{a\alpha + b\beta \in W}$

linear combination $\alpha \in V(F)$

$$\alpha = a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n$$

linear span: $L(S) = \left\{ \alpha \;\middle|\; \alpha = \sum_{i=1}^{n} a_i\alpha_i \quad a_1 \ldots a_n \in F \quad \alpha_1, \alpha_2 \ldots \alpha_n \in S \right\}$

linearly indep $S = \{\alpha_1, \alpha_2, \ldots \alpha_n\}$

Depende $a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n = 0$

$a_1 = a_2 = \cdots = a_n = 0$

$a_n \neq 0$

Basis $\boxed{S\ \text{l.I}}$

Dimension $\downarrow$ $L(S)$

$\alpha.T$

M.L.T

Prove that the four vectors $\alpha = (1,0,0)$, $\beta = (0,1,0)$, $\gamma = (0,0,1)$, $\delta = (1,1,1)$ in $V_3(\mathbb{C})$ form L.D set but any three of them are L.I

Sol. $S = \{\underset{\alpha}{(1,0,0)}, \underset{\beta}{(0,1,0)}, \underset{\gamma}{(0,0,1)}, \underset{\delta}{(1,1,1)}\}$

$$a(1,0,0) + b(0,1,0) + c(0,0,1) + d(1,1,1) = 0$$

$$\left.\begin{aligned} a + 0b + 0c + d &= 0 \\ 0a + 1b + 0c + 1d &= 0 \\ 0a + 0b + 1c + 1d &= 0 \end{aligned}\right\} \quad \begin{aligned} a + d &= 0 \\ b + d &= 0 \\ c + d &= 0 \end{aligned}$$

$a = -d, \; b = -d, \; c = -d$

let $d = K$

$a = -K, \; b = -K, \; c = -K$

$$(-1)(1,0,0) + (-1)(0,1,0) + (-1)(0,0,1) + (1)(1,1,1) = 0$$

Here $a \neq 0, \; b \neq 0, \; c \neq 0, \; d \neq 0$

S is L.D

$\therefore$

Consider any three $\{\beta, \gamma, \delta\}$

now $\begin{vmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix} = 1$

$0[-1] + 0(1] + 1[1]$

Det $\neq 0$

$\therefore \{\beta, \gamma, \delta\}$ is L.I

lly $\{\alpha, \beta, \gamma\}$, $\{\alpha, \gamma, \delta\}$

are all L.I

$\therefore$ any three of them are L.I

① If α, β, γ are L.I of $V_3(R)$ then show that $\alpha+\beta, \beta+\gamma, \gamma+\alpha$ are L.I

① Let V be the vector space of 2×3 matrices over R; show that the vectors $\underline{A, B, C}$ form L.I set where

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 3 & -2 & 4 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 1 & -3 \\ -2 & 0 & 5 \end{bmatrix}$$

$$C = \begin{bmatrix} 4 & -1 & 2 \\ 1 & -2 & -3 \end{bmatrix}$$

Sol. Let $S = \{A, B, C\}$

we prove

If $\boxed{xA + yB + zC = 0}$

$x=0,\ y=0,\ z=0$

Then S is L.I

$$x\begin{bmatrix}2 & 1 & -1\\ 3 & -2 & 4\end{bmatrix} + y\begin{bmatrix}1 & 1 & -3\\ -2 & 0 & 5\end{bmatrix}$$

$$+\ z\begin{bmatrix}4 & -1 & 2\\ 1 & -2 & -3\end{bmatrix} = \begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$

$$\begin{bmatrix}2x+y+4z & x+y-z & -x-3y+2z\\ 3x-2y+z & -2x+0y-2z & 4x+5y-3z\end{bmatrix} = \begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$

$$\left.\begin{aligned}2x+y+4z&=0\\ x+y-z&=0\end{aligned}\right\}1 \qquad 3x-2y+z=0 \checkmark \quad 2$$

$$-x-3y+2z=0 \checkmark \qquad \left.\begin{aligned}-2x+0y-2z&=0\\ 4x+5y-3z&=0\end{aligned}\right\}3$$

$$\begin{aligned}2x+\cancel{y}+4z&=0\\ \underset{-}{x}\ \underset{-}{+}\ \cancel{y}\ \underset{+}{-}\ z&=0\\ \hline x+0y+5z&=0 \rightarrow 1\end{aligned}$$

$$\begin{aligned}-x-3y+\cancel{2z}&=0\\ \underset{-}{6x}\ \underset{+}{-}\ 4y\ \underset{-}{+}\ \cancel{2z}&=0\\ \hline -7x+y+0z&=0 \rightarrow 2\end{aligned}$$

$$\begin{array}{r} -\cancel{4x} + 0y - 4z = 0 \\ \cancel{4x} + 5y - 3z = 0 \\ \hline 0x + 5y - 7z = 0 \end{array} \rightarrow (3)$$

$$\left.\begin{array}{l} x + 0y + 5z = 0 \\ -7x + y + 0z = 0 \\ 0x + 5y - 7z = 0 \end{array}\right\}$$

$$\boxed{AX = 0}$$

$$\begin{bmatrix} 1 & 0 & 5 \\ -7 & 1 & 0 \\ 0 & 5 & -7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \rightarrow R_2 + 7R_1$

$$\begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 35 \\ 0 & 5 & -7 \end{bmatrix}$$

$R_3 \to R_3 - 5R_2$

$$\sim \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 35 \\ 0 & 0 & -182 \end{bmatrix}$$

$$\begin{pmatrix} 1 & 0 & 5 \\ 0 & 1 & 35 \\ 0 & 0 & -182 \end{pmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} x + 0y + 5z &= 0 \\ y + 35z &= 0 \\ -182z &= 0 \end{aligned}\right\}$$

$z = 0 \quad y = 0 \quad x = 0$

Hence $x = 0 \quad y = 0 \quad z = 0$

Hence $S = \{A, B, C\}$ is L.I

(2) prove that the vectors (x_1, y_1) and (x_2, y_2) of $V_3(F)$ are l.D if $\boxed{x_1 y_2 - x_2 y_1 = 0}$

Sol: let $S = \{ (x_1, y_1), (x_2, y_2) \}$
$\qquad\qquad\qquad\alpha, \quad \beta$

$$a\alpha + b\beta = 0$$

$$a(x_1, y_1) + b(x_2, y_2) = (0, 0)$$

$$\left.\begin{aligned} ax_1 + bx_2 &= 0 \\ ay_1 + by_2 &= 0 \end{aligned}\right\} AX = 0$$

$$\underbrace{\begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} a \\ b \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}_{O}$$

$|A| \neq 0$ S is l.I

$|A| = 0$ S is l.D

$$A = \begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix}$$

$$|A| = x_1 y_2 - x_2 y_1 = 0$$

$$|A| = 0$$

S is l.D

$$\boxed{x = 3,\ y = 1,\ z = -2}$$

$$x\alpha + y\beta + z\gamma = 0$$

$$x = y = z \neq 0$$

$S = \{\alpha, \beta, \gamma\}$ is L.D

③ S is L.I (or) L.D

ⓐ $S = \{(1,-2,1)\ (2,1,-1)\ (7,-4,1)\}$

ⓑ $S = \{(-1,2,1), (3,0,-1), (-5,4,3)\}$

Vector space: $V(F)$ ① $(V,+)$ A.G

② E.C.E

③ $a(\alpha+\beta) = a\alpha + a\beta$

$(a+b)\alpha = a\alpha + b\alpha$

$(ab)\alpha = a(b\alpha)$

$1\cdot\alpha = \alpha$

$a, b, 1 \in F$

$\alpha, \beta \in V$

Subspace: $W \subseteq V$: W itself a V.S

$a\alpha + b\beta \in W$

Linear combination: $S = \{\alpha_1, \alpha_2 \dots \alpha_n\}$

$\alpha \in V$

$\alpha = a_1\alpha_1 + a_2\alpha_2 + \dots + a_n\alpha_n$

$\rightarrow \boxed{\alpha = \sum_{i=1}^{n} a_i\alpha_i}$

Linear Span: $L(S) = \{\alpha \mid \alpha = \sum_{i=1}^{n} a_i\alpha_i\}$

Linearly independ: $S = \{\alpha_1, \alpha_2 - \alpha_n\}$ l.i

$a_1\alpha_1 + a_2\alpha_2 + \dots + a_n\alpha_n = 0$

$a_1 = 0 \quad a_2 = 0 \dots a_n = 0$

Linearly Depend: not all zero's S L.D

① If α, β, γ are L.I vectors of $V(R)$ show that $\alpha+\beta, \beta+\gamma, \gamma+\alpha$ are also L.I

Sol Given α, β, γ are L I

$S = \{\alpha, \beta, \gamma\}$ is L.I set

$a\alpha + b\beta + c\gamma = 0$

$\Rightarrow a = 0, b = 0, c = 0$

Now $T = \{\alpha+\beta, \beta+\gamma, \gamma+\alpha\}$

consider

$a(\alpha+\beta) + b(\beta+\gamma) + c(\gamma+\alpha) = 0$

$(a+c)\alpha + (a+b)\beta + (b+c)\gamma = 0$

Since α, β, γ are L.I

$a+c=0$; $a+b=0$, $b+c=0$

$a=-c \Rightarrow b-c=0$

$$\begin{aligned} b+c &= 0 \\ b-c &= 0 \\ \hline b &= 0 \end{aligned}$$

$\Rightarrow c=0 \Rightarrow a=0$

$a(\alpha+\beta) + b(\beta+\gamma) + c(\gamma+\alpha) = 0$

$\Leftarrow$ $a=0, b=0, c=0$

$\Leftrightarrow$ $\{\alpha+\beta, \beta+\gamma, \gamma+\alpha\}$ is L.I

② prove that the set $(1, i)$ is L.D in the vector space $\mathbb{C}(\mathbb{C})$ but it is L.I in the vector space $\mathbb{C}(\mathbb{R})$

Pf: Given $S=\{1, i\}$ in $\mathbb{C}(\mathbb{C})$

$a(1)+b(i)=0$

$a=1,\ b=i \in \mathbb{C}$ $\quad a, b \in \mathbb{C}$

$a \neq 0, b \neq 0$

$\Leftrightarrow$ w.r.t the vector space $\mathbb{C}(\mathbb{C})$

S is L.D

Consider $S=\{1, i\}$ in $\mathbb{C}(\mathbb{R})$

$\boxed{a(1)+b(i)=0}$ $\quad a, b \in \mathbb{R}$

$a=0, b=0$

$\Rightarrow$ $S=\{1, i\}$ is L.I in $\mathbb{C}(\mathbb{R})$

(3) Let $F(x)$ be the vector space of all polynomials over the field F. Show that the infinite set

$S = \{1, x, x^2, \cdots\}$ is L.I

Se

$S = \{1, x, x^2, \cdots\}$

$a_1(1) + a_2(x) + a_3(x^2) + \cdots = 0(1) + 0(x) + 0(x^2) + \cdots$

$a_1 = 0, a_2 = 0, a_3 = 0, a_4 = 0$

Hence S is L.I

Basis :

A subset S of a vector space $V(F)$ is said to be the Basis of V if

(i) S is L.I

(ii) $L(S) = V$

① Show that the vector $(1,1,2), (1,2,5),$ $(5,3,4)$ of $R^3(R)$ do not form a basis set of $R^3(R)$

Sol. Let $S = \{(1,1,2), (1,2,5), (5,3,4)\}$

writing the vectors as rows of a Matrix

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 5 \\ 5 & 3 & 4 \end{bmatrix}$$

Now
$$\begin{vmatrix} 1 & 1 & 2 \\ 1 & 2 & 5 \\ 5 & 3 & 4 \end{vmatrix} = 1[-7] - 1[-21] + 2[-7]$$

$$= -7 + 21 - 14$$

$$= -21 + 21$$

$$= 0$$

$|A| = 0$

$\Rightarrow$ S is L.D

$\Rightarrow$ S can not be form a basis of $R^3(R)$

$=$

② Show that the vectors $(1,0,0), (1,1,0), (1,1,1)$ form a basis of $R^3(R)$

sol. Let $S = \{(1,0,0), (1,1,0), (1,1,1)\}$

Now it can be written in Matrix form

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$|A| = 1[1] - 0[1-0] + 0[1-1] = 1$$

$$|A| = 1 \neq 0$$

$\therefore$ S is L.I ✓

$\therefore$ S forms a basis of $R^3(R)$

$$\boxed{L(S) = R^3(R)}$$

③ ** Show that the set (1,0,0), (1,1,0), (1,1,1) basis of $C^3(C)$ Hence find the coordinates of the vector $(3+4i, 6i, 3+7i)$ in $C^3(C)$

Sol. Let $S = \{(1,0,0), (1,1,0), (1,1,1)\}$

Now it can be written in Matrix form

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

Now it can be reduced to E. F

$R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

L. I

$|A| \neq 0$

$\rho(A) = n$

$a\alpha + b\beta + c\gamma = 0$

$a = 0$ $b = 0$ $c = 0$

Here $\rho(A) = 3$ = no of unknowns

$$\boxed{r = n}$$ ✓

$\Rightarrow S$ is L.I

$\therefore$ S forms a basis of $\mathbb{C}^3(\mathbb{C})$

Consider

$$(3+4i, 6i, 3+7i) = a(1,0,0) + b(1,1,0) + c(1,1,1)$$

$$a + b + c = 3+4i$$
$$0a + b + c = 6i$$
$$0a + 0b + c = 3+7i$$

Here $\boxed{c = 3+7i}$

$$b + 3+7i = 6i$$

$$\boxed{b = -3-i}$$

$$a + (-3-i) + (3+7i) = 3+4i$$

$$a + 6i = 3+4i$$

$$\boxed{a = 3-2i}$$

Hence the coordinates of $(3+4i, 6i, 3+7i)$

are $(3-2i, -3-i, 3+7i)$

Ex: ① Show that the vectors $(0,1,-1), (1,1,0), (1,0,2)$ form a Basis of $V_3(C)$ and also find the coordinates of $(1,0,-1)$

* vector space: ① $(V,+)$ Abelian group
$V(F)$ ② External comp ... $a\alpha \in V$
(or) V
③ $a(\alpha+\beta) = a\alpha + a\beta$
$(a+b)\alpha = a\alpha + b\alpha$
$(ab)\alpha = a(b\alpha)$
$1\alpha = \alpha$

* Subspace: $W \subseteq V$ with itself a V.S
$a\alpha + b\beta \in W$

* linear combination: $\alpha \in V$
$\alpha = a_1\alpha_1 + a_2\alpha_2 + \dots + a_n\alpha_n$
$\alpha = \sum_{i=1}^{n} a_i\alpha_i$ $a_i \in F$
$\alpha_i \in V$

* Linear span: $L(S) = \{\underline{\alpha} \mid \alpha = \sum a_i\alpha_i\}$
where $S = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$

* linearly independent: $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$ L.I
$a_1\alpha_1 + a_2\alpha_2 + \dots + a_n\alpha_n = 0$
$a_1 = 0$ $a_2 = 0$ $\dots$ $a_n = 0$
all a_i's are zero's

* linearly Dependent: " " "
not all a_i's are zero's

* Basis : $S = \{\alpha_1 \alpha_2 \dots \alpha_n\}$

① S L.I

② $L(S) = V$

* Dimension : S is basis of V(F)

The no of elements in S is called

Dimension of V(F)

* If S has (Basis) finite no. of elements Then

V(F) is Finite Dimensional vector space

* V(F) be a finite dimensional V.S. Then

any two basis have same no. of elements

* Every set of (n+1) or more vectors

in an n dimensional V-S is L.D

* Let V(F) be a finite dimensional V-S of

dimension n. Then any set of n L.I

vectors in V forms a basis of V

* Let V(F) be a finite dimensional

V.S with dimension n. and W be The

sub space of V. Then W is a finite

dimensional vector space with

$\boxed{\dim W \leq n}$ $\boxed{\dim W \leq \dim V}$

** let W_1 and W_2 are two subspaces of a finite dimensional vector space $V(F)$ then

$$\dim(W_1 + W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2)$$

* let W be a subspace of finite dimensional vector space $V(F)$ then

$$\dim\left(\frac{V}{W}\right) = \dim V - \dim W$$

(1) If W is the subspace of $V_4(R)$ generated by the vectors $(1,-2,5,-3)$ $(2,3,1,-4)$ and $(3,8,-3,-5)$ find the basis of W and its dimension

Sol: Arranging the given vectors as rows of a matrix

$$A = \begin{bmatrix} 1 & -2 & 5 & -3 \\ 2 & 3 & 1 & -4 \\ 3 & 8 & -3 & -5 \end{bmatrix}$$

Reduce to Echelon form

$R_2 \to R_2 - 2R_1,\ R_3 \to R_3 - 3R_1$

$$\sim \begin{bmatrix} 1 & -2 & 5 & -3 \\ 0 & 7 & -9 & 2 \\ 0 & 14 & -18 & 4 \end{bmatrix}$$

$R_3 \to R_3 - 2R_2$

$$\sim \begin{bmatrix} 1 & -2 & 5 & -3 \\ 0 & 7 & -9 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Hence two non zero rows $(1,-2,5,-3),(0,7,-9,2)$ form the basis of least l.I set and hence basis of W

$$\therefore \boxed{\dim W = 2}$$

2) V is the v.s generated by the polynomials

$\alpha = x^3 + 2x^2 - 2x + 1$, $(1, 2, -2, 1)$

$\beta = x^3 + 3x^2 - x + 4$ $(1, 3, -1, 4)$

$\gamma = 2x^3 + x^2 - 7x - 7$ $(2, 1, -7, -7)$

Find the basis of V and its Dimension

Sol. Now V is the polynomial space generated by $\{\alpha, \beta, \gamma\}$

Forming the Matrix A with these coordinates

$$A = \begin{bmatrix} 1 & 2 & -2 & 1 \\ 1 & 3 & -1 & 4 \\ 2 & 1 & -7 & -7 \end{bmatrix}$$

Reduce to Echelon form

$R_2 \to R_2 - R_1$, $R_3 \to R_3 - 2R_1$

$$\sim \begin{bmatrix} 1 & 2 & -2 & 1 \\ 0 & 1 & 1 & 3 \\ 0 & -3 & -3 & -9 \end{bmatrix}$$

$R_3 \to R_3 + 3R_2$

$$\sim \begin{bmatrix} 1 & 2 & -2 & 1 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Hence the two non zero rows

$(1,2,-2,1), (0,1,1,3)$ form the

basis of V

$\therefore \boxed{\dim V = 2}$

③ V is the vector space of polynomials

over $\mathbb{R}$; W_1 and W_2 are subspaces

generated by

$$W_1 = \{x^3 + x^2 - 1,\ x^3 + 2x^2 + 3x,\ 2x^3 + 3x^2 + 3x - 1\}$$

$$W_2 = \{x^3 + 2x^2 + 2x - 2,\ 2x^3 + 3x^2 + 2x - 3,\ x^3 + 3x^2 + 4x - 3\}$$

find

(i) $\dim(W_1 + W_2)$ (ii) $\dim(W_1 \cap W_2)$

Sol. The coordinates of the polynomials

w.r.t to basis $(x^3, x^2, x, 1)$ are

respectively

$$W_1 = \{(1,1,0,-1), (1,2,3,0), (2,3,3,-1)\}$$

$$W_2 = \{(1,2,2,-2), (2,3,2,-3), (1,3,4,-3)\}$$

(1) For W_1 we find Basis & Dimension

$$A = \begin{bmatrix} 1 & 1 & 0 & -1 \\ 1 & 2 & 3 & 0 \\ 2 & 3 & 3 & -1 \end{bmatrix}$$

Reduce to Echelon form

$R_2 \to R_2 - R_1, \quad R_3 \to R_3 - 2R_1$

$$\sim \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 1 & 3 & 1 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\sim \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Hence two non zero rows

$(1, 1, 0, -1), (0, 1, 3, 1)$ forms the

basis of W_1

$$\boxed{\text{Dim } W_1 = 2}$$

(2) For W_2 we find basis & Dimension

$$A = \begin{bmatrix} 1 & 2 & 2 & -2 \\ 2 & 3 & 2 & -3 \\ 1 & 3 & 4 & -3 \end{bmatrix}$$

Reduce to Echelon form

$R_2 \to R_2 - 2R_1, \quad R_3 \to R_3 - R_1$

$$\sim \begin{bmatrix} 1 & 2 & 2 & -2 \\ 0 & -1 & -2 & 1 \\ 0 & 1 & 2 & -1 \end{bmatrix}$$

$R_3 \to R_3 + R_2$

$$\sim \begin{bmatrix} 1 & 2 & 2 & -2 \\ 0 & -1 & -2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Here two non zero rows

$(1,2,2,-2), (0,-1,-2,1)$

form the basis of W_2

$$\boxed{\text{Dim } W_2 = 2}$$

(i) Dim $W_1 + W_2$

Now the subspace $W_1 + W_2$ is by all the six vectors

Hence arranging them in rows of a matrix and reduce to Echelon form

$$A = \begin{bmatrix} 1 & 1 & 0 & -1 \\ 1 & 2 & 3 & 0 \\ 2 & 3 & 3 & -1 \\ 1 & 2 & 2 & -2 \\ 2 & 3 & 2 & -3 \\ 1 & 3 & 4 & -3 \end{bmatrix}$$

Reduce to Echelon form

$R_2 \to R_2 - R_1$, $R_3 \to R_3 - 2R_1$, $R_4 \to R_4 - R_1$

$R_5 \to R_5 - 2R_1$, $R_6 \to R_6 - R_1$

$$\sim \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 1 & 3 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 1 & 2 & -1 \\ 0 & 2 & 4 & -2 \end{bmatrix}$$

$R_3 \to R_3 - R_2,\ R_4 \to R_4 - R_2,\ R_5 \to R_5 - R_2$

$R_6 \to R_6 - 2R_2$

$$\sim \begin{array}{cc} \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & -2 & -4 \end{bmatrix} & \begin{matrix} R_1 \\ R_2 \\ R_3 \\ R_4 \\ R_5 \\ R_6 \end{matrix} \end{array}$$

$R_5 \to R_5 - R_4,\ R_6 \to R_6 - 2R_4$

$$\sim \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Here the non zero rows

$\{ (1,1,0,-1), (0,1,3,1), (0,0,-1,-2) \}$

forms the basis of $W_1 + W_2$

$$\boxed{\dim (W_1 + W_2) = 3}$$

(ii) <u>$\dim (W_1 \cap W_2)$</u>

we know that

$\dim (W_1 + W_2) = \dim W_1 + \dim W_2 - \dim (W_1 \cap W_2)$

$3 = 2 + 2 - \dim (W_1 \cap W_2)$

$$\boxed{\dim (W_1 \cap W_2) = 1}$$

Linear Transformation

① Homomorphism :-

let U and V two vector space over the same field F : Then the mapping $f : U \to V$ is called Homomorphism from U into V if

(i) $f(\alpha + \beta) = f(\alpha) + f(\beta)$ $\forall \alpha, \beta \in U$

(ii) $f(a\alpha) = a f(\alpha)$ $a \in F$

② If f is onto function $f(U) = V$

③ If f is 1-1 If $\alpha = \beta \Rightarrow f(\alpha) = f(\beta)$

④ $f : U \to V$
① f hom
② f onto
③ f 1-1

f is isomorphism

$$\boxed{U \cong V}$$

Linear Transformation:

Let $U(F)$ and $V(F)$ be two vector spaces. Then the function

$T: U \to V$ is called linear transformation of U into V if

$$T(a\alpha + b\beta) = a\,T(\alpha) + b\,T(\beta)$$

$\forall\, a, b \in F$

$\alpha, \beta \in U$

Problem

① The mapping $T: V_3(R) \to V_2(R)$ is defined by $T(x, y, z) = (x-y, x-z)$. Show that T is a linear transformation.

Sol: given $T: V_3 \to V_2$ defined by

$$\boxed{T(x, y, z) = (x-y, x-z)}$$

Now we show that T is linear transformation ie we prove

$$\underline{T(a\alpha + b\beta) = aT(\alpha) + bT(\beta)}$$

let $\left.\begin{array}{l}\alpha = (x_1, y_1, z_1) \\ \beta = (x_2, y_2, z_2)\end{array}\right\} \in V_3(R)$

$a, b \in F$

$$T[a\alpha + b\beta] = T\{a(x_1, y_1, z_1) + b(x_2, y_2, z_2)]$$

$$= T[(ax_1, ay_1, az_1) + (bx_2, by_2, bz_2)]$$

$$= T[ax_1 + bx_2,\ ay_1 + by_2,\ az_1 + bz_2]$$

$$= (ax_1 + bx_2 - (ay_1 + by_2),\ ax_1 + bx_2 - (az_1 + bz_2))$$

$$= [a(x_1 - y_1) + b(x_2 - y_2),\ a(x_1 - z_1) + b(x_2 - z_2)]$$

$= (a(x_1 - y_1), a(x_1 - z_1)$

$+ b(x_2 - y_2), b(x_2 - z_2))$

$= a(x_1 - y_1, x_1 - z_1) + b(x_2 - y_2, x_2 - z_2)$

$= a\,T(x_1, y_1, z_1) + b\,T(x_2, y_2, z_2)$

$= a\,T(\alpha) + b\,T(\beta)$

$$\boxed{T(a\alpha + b\beta) = a\,T(\alpha) + b\,T(\beta)}$$

$\therefore$ Hence T is linear transformation

Linear Transformation

$T : U \rightarrow V$ is a Mapping

$\forall\ \alpha, \beta \in U$

$T(\alpha), T(\beta) \in V$

$* \quad \boxed{T(a\alpha + b\beta) = a\,T(\alpha) + b\,T(\beta)} \quad *$

T is called linear Transfor

$a, b \in F$

① The mapping $T : V_3(R) \longrightarrow V_1(R)$

is defined by $T(a, b, c) = a^2 + b^2 + c^2$

can T be a linear Transformation

Sol. let $\left.\begin{array}{l} \alpha = (a_1\ b_1\ c_1) \\ \beta = (a_2\ b_2\ c_2) \end{array}\right\} \in V_3(R)$

$T(\alpha) = T(a_1, b_1, c_1) = a_1^2 + b_1^2 + c_1^2$

$T(\beta) = T(a_2, b_2, c_2) = a_2^2 + b_2^2 + c_2^2$

Now we prove That $T : V_3 \rightarrow V_1$

is linear Transformation

i.e we show that

$$T(a\alpha + b\beta) = aT(\alpha) + bT(\beta)$$

LHS

$$T(a\alpha + b\beta) = T\left[a(a_1\ b_1\ c_1) + b(a_2\ b_2\ c_2)\right]$$

$$= T\left[(aa_1, ab_1, ac_1) + (ba_2, bb_2, bc_2)\right]$$

$$= T\left[aa_1 + ba_2,\ ab_1 + bb_2,\ ac_1 + bc_2\right]$$

$$T(a, b, c) = a^2 + b^2 + c^2$$

$$= (aa_1 + ba_2)^2 + (ab_1 + bb_2)^2 + (ac_1 + bc_2)^2 \rightarrow (1)$$

RHS:

$$aT(\alpha) + bT(\beta)$$

$$= a(a_1^2 + b_1^2 + c_1^2) + b(a_2^2 + b_2^2 + c_2^2)$$

$$= aa_1^2 + ab_1^2 + ac_1^2 + ba_2^2 + bb_2^2 + bc_2^2 \rightarrow (2)$$

Here we observe that

$$T(a\alpha + b\beta) \neq aT(\alpha) + bT(\beta)$$

$\therefore$ T is not a linear transformation

(3) $T : R^2 \to R^2$ defined by

$$T(a, b) = (2a + 3b, 3a - 4b)$$

is a linear transformation.

Sol. Let $\left.\begin{aligned} \alpha &= (a_1, b_1) \\ \beta &= (a_2, b_2) \end{aligned}\right\} \in R^2$

$$T(\alpha) = T(a_1, b_1) = (2a_1 + 3b_1, 3a_1 - 4b_1)$$

$$T(\beta) = T(a_2, b_2) = (2a_2 + 3b_2, 3a_2 - 4b_2)$$

Now

$$T(a\alpha + b\beta) = T\left(a(a_1, b_1) + b(a_2, b_2)\right)$$

$$= T\left[(aa_1, ab_1) + (ba_2, bb_2)\right]$$

$$= T\left[\underbrace{aa_1 + ba_2}_{a}, \underbrace{ab_1 + bb_2}_{b}\right]$$

$$T(a, b) = (2a + 3b,\ 3a - 4b)$$

$$= \left[2(aa_1 + ba_2) + 3(ab_1 + bb_2),\right.$$

$$\left. 3(aa_1 + ba_2) - 4(ab_1 + bb_2)\right] \longrightarrow (1)$$

Now

$$aT(\alpha) + bT(\beta)$$

$$= a(2a_1 + 3b_1,\ 3a_1 - 4b_1)$$

$$+ b(2a_2 + 3b_2,\ 3a_2 - 4b_2)$$

$$= (2aa_1 + 3ab_1,\ 3aa_1 - 4ab_1)$$

$$+ (2ba_2 + 3bb_2,\ 3ba_2 - 4bb_2)$$

$$= (2aa_1 + 3ab_1 + 2ba_2 + 3bb_2,$$

$$3aa_1 - 4ab_1 + 3ba_2 - 4bb_2)$$

$$= \left[2(a a_1 + b a_2) + 3(a b_1 + b b_2), \right.$$
$$\left. 3(a a_1 + b a_2) - 4(a b_1 + b b_2) \right] \rightarrow (2)$$

Hence from (1) & (2)

$$\boxed{T(a\alpha + b\beta) = a\,T(\alpha) + b\,T(\beta)}$$

$\therefore$ T is linear Transformation

(4) $T : R^3 \rightarrow R^3$ defined by

$$T(x, y, z) = (x+1, y, z)$$

(5) $T : V_1 \rightarrow V_3$ defined by

$$T(x) = (x, 2x, 3x)$$

are linear transformation (or) not!

① * Find $T(x, y, z)$ where $T: \mathbb{R}^3 \to \mathbb{R}$

is defined by $T(1,1,1) = 3$,

$T(0,1,-2) = 1$, $T(0,0,1) = -2$

Sol (i) let $S = \{(1,1,1), (0,1,-2), (0,0,1)\}$

let $a(1,1,1) + b(0,1,-2) + c(0,0,1) = 0$

$a = 0$; $a + b = 0$ $a - 2b + c = 0$

$\therefore$ $a = 0$, $b = 0$, $c = 0$

$\therefore$ S is L.I

(ii) let $(x, y, z) \in \mathbb{R}^3$

$$\boxed{(x, y, z) = a(1,1,1) + b(0,1,-2) + c(0,0,1)}$$

$a = x$, $a + b = y$, $a - 2b + c = z$

$\boxed{a = x}$, $b = y - a$ $c = z - a + 2b$

$\boxed{b = y - x}$ $= z - x + 2(y - x)$

$\boxed{c = z + 2y - 3x}$

Here

$$(x, y, z) = a(1,1,1) + b(0,1,-2) + c(0,0,1)$$

$$T(x, y, z) = T\left[a(1,1,1) + b(0,1,-2) + c(0,0,1)\right]$$

$$= a\,T(1,1,1) + b\,T(0,1,-2) + c\,T(0,0,1)$$

$$= x(3) + (y-x)(1) + (z+2y-3x)(-2)$$

$$= 3x + y - x - 2z - 4y + 6x$$

$$= 8x - 3y - 2z$$

$$\boxed{T(x, y, z) = 8x - 3y - 2z}$$

② $T : V_3 \to V_3$ such that

$T(0,1,2) = (3,1,2)$,

$T(1,1,1) = (2,2,2)$ Then find

the linear transformation

Sol: (i) let $S = \{(0,1,2), (1,1,1)\}$

Now $a(0,1,2) + b(1,1,1) = 0$

$b = 0, \quad a + b = 0, \quad 2a + b = 0$

$a = 0, \; b = 0$

$\therefore$ S is L.I

(ii) let $(x, y, z) \in V_3$

$$\boxed{(x, y, z) = a(0,1,2) + b(1,1,1)}$$

$x = b, \quad y = a + b, \quad z = 2a + b$

$\boxed{b = x}$ $\quad$ $\boxed{a = y - x}$

NOW

$$T(x,y,z) = T\left[a(0,1,2) + b(1,1,1)\right]$$

$$= a\,T(0,1,2) + b\,T(1,1,1)$$

$$= (y-x)(3,1,2) + x(2,2,2)$$

$$= \left[3(y-x), (y-x), 2(y-x)\right]$$

$$+ \left[2x, 2x, 2x\right]$$

$$= \left[3(y-x)+2x,\ y-x+2x,\ 2(y-x)+2x\right]$$

$$= \left[3y-x,\ y-x,\ 2y\right]$$

$$\boxed{T(x,y,z) = (3y-x,\ y-x,\ 2y)}$$

Example

(1) Find a Linear Transformation

$T : R^2 \to R^2$ such That

$T(1,0) = (1,1)$

$T(0,1) = (-1,2)$

(2) $T : R^2 \to R \to$

$T(2,3) = (4,5)$

$T(1,0) = (0,0)$

* $T: U \to V$ L.T

$T(a\alpha + b\beta) = aT(\alpha) + bT(\beta)$

$\alpha, \beta \in U$

$T(\alpha) : T(\beta) \in V$

$a, b \in F$

* $\boxed{T: R^3 \to R^2}$

$T(x, y, z) = (x-y, y-z)$

$T(\alpha) = \beta \in R^2$

$\alpha \in R^3$

$\alpha = (x, y, z) \in R^3$

$T(\alpha) = \beta = (x_1\ x_2) \in R^2$

standard basis $V_4(F)$

$e_1 = (1, 0, 0, 0)$

$e_2 = (0, 1, 0, 0)$

$e_3 = (0, 0, 1, 0)$

$e_4 = (0, 0, 0, 1)$

(4) $V_3(R)$

Standard Basis $\to$ L.I
$\to L(S) = V$

$S = \{(1,0,0), (0,1,0), (0,0,1)\}$
$\quad e_1 \quad e_2 \quad e_3$

no. of elements in $S = 3$

$$\boxed{\dim V_3(R) = 3}$$

Sum of two linear Transformations

$T_1 : U \rightarrow V \qquad T_2 : U \rightarrow V$

$$(T_1 + T_2)(\alpha) = T_1(\alpha) + T_2(\alpha)$$

Scalar multiplication

$T : U \rightarrow V \qquad \exists\, a \in F$

$$(aT)(\alpha) = a\,T(\alpha)$$

① Let $T : R^3 \rightarrow R^2$ and $H : R^3 \rightarrow R^2$

be defined by

$T(x, y, z) = (3x, y + z)$

$H(x, y, z) = (2x - z, y)$

compute

(i) $T + H$ (ii) $4T - 5H$

(iii) TH (iv) HT

(i)

$$(T+H)(x,y,z) = T(x,y,z) + H(x,y,z)$$

$$= (3x, y+z) + (2x-z, y)$$

$$= (3x+2x-z, y+z+y)$$

$$= (5x-z, 2y+z)$$

(ii)

$$(4T-5H)(x,y,z) = 4T(x,y,z) - 5H(x,y,z)$$

$$= 4(3x, y+z) - 5(2x-z, y)$$

$$= (12x, 4y+4z) - (10x-5z, 5y)$$

$$= (12x-10x+5z, 4y+4z-5y)$$

$$= (2x+5z, -y+4z)$$

(iii) $TH(x, y, z) = T[H(x, y, z)]$

$= T[2x - z, y]$

$\therefore$ which is undefine

$\therefore$ TH does not exists

Ily HT does not exists

Range and Null Space of Linear Transformation

Range : $T: U \to V$ be a l.T

Range of T is denoted by $R(T)$
and is defined by

$$R(T) = \{ T(\alpha) : \alpha \in U \}$$

since $T(\alpha) \in V$

Here $\boxed{R(T) \subseteq V}$

$\dim R(T) = \text{Rank of } T = \rho(T)$

Null space : $T: U \to V$ be a l.T

Null space of T is denoted by $N(T)$
and is defined by

$$N(T) = \{ T(\alpha) = \bar{0} ; \alpha \in U \}$$

Here $\boxed{N(T) \subseteq U}$

$\dim N(T) = \text{Nullity of } T = \nu(T)$

* Rank, Nullity Theorem

let $U(F)$ and $V(F)$ be two V.S and $T : U \to V$ be a L.T, let U be finite dimensional Then

Rank of T + Nullity of T = dim U

$$\rho(T) + \nu(T) = \dim U$$

$\boxed{T : \underline{U} \longrightarrow V}$ $\quad \underline{T(x, y, z) = (x+y, z)}$

$\boxed{\text{Dim } U}$ $\to$ Range $T \to R(T)$

$+$

$\to$ Nul $T \to \nu(T)$

Range of T

$\boxed{T : \underline{U} \to V}$

$R(T) = \{T(\alpha) \mid \alpha \in U\}$

$\text{Dim } R(T) = \rho(T)$

Null space of T

$\boxed{T : U \to V}$

$N(T) = \{\alpha \in U \mid T(\alpha) = 0\}$

$\text{Dim } N(T) = \nu(T)$

$$\boxed{\rho(T) + \nu(T) = \dim U}$$

If $T: V_4(R) \longrightarrow V_3(R)$ is a linear transformation defined by

$$T(a,b,c,d) = (a-b+c+d,\ a+2c-d,\ a+b+3c-3d)$$

for $a,b,c,d \in R$:

Then verify $\rho(T) + \nu(T) = \dim V_4(R)$

Sol. consider

$$S = \{(1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1)\}$$

given

$$T(a,b,c,d) = (a-b+c+d,\ a+2c-d,\ a+b+3c-3d)$$

$$T(1,0,0,0) = (1,1,1)$$

$$T(0,1,0,0) = (-1,0,1)$$

$$T(0,0,1,0) = (1,2,3)$$

$$T(0,0,0,1) = (1,-1,-3)$$

Sol

$S_1 = \{(1,1,1), (-1,0,1), (1,2,3), (1,-1,-3)\}$

Now its matrix form is

$$\begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & 2 & 3 \\ 1 & -1 & -3 \end{bmatrix}$$

Now it can be reduce to E.F

$R_2 \to R_2 + R_1$, $R_3 \to R_3 - R_1$,

$R_4 \to R_4 - R_1$

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & -2 & -4 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$R_4 \to R_4 + 2R_2$

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Here the non zero rows of

vectors $\{(1,1,1), (0,1,2)\}$

forms the basis of $R(T)$

$$\boxed{\operatorname{Dim} R(T) = \rho(T) = 2}$$

Now we find the Basis of Null Space of T

$$N(T) = \{ U \in V_4(R) \mid T(U) = 0 \}$$

$$T(a, b, c, d) = \hat{0}$$

$$(a - b + c + d,\ a + 2c - d,\ a + b + 3c - 3d) = 0$$

$$\therefore \left. \begin{aligned} a - b + c + d &= 0 \\ a + 2c - d &= 0 \\ a + b + 3c - 3d &= 0 \end{aligned} \right\}$$

Now its matrix form

$$\begin{bmatrix} 1 & -1 & 1 & 1 \\ 1 & 0 & 2 & -1 \\ 1 & 1 & 3 & -3 \end{bmatrix}$$

$R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$

$$\sim \begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 1 & 1 & -2 \\ 0 & 2 & 2 & -4 \end{bmatrix}$$

$R_3 \to R_3 - 2R_2$

$$\sim \begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

no. of columns − no. of non zero rows

$= 4 - 2$

$= 2$

$\therefore \quad \mathrm{Dim}\{N(T)\} = 2$

$$\boxed{\nu(T) = 2}$$

Hence $\quad \rho(T) + \nu(T) = \mathrm{Dim}\, V_4(R)$

$$2 + 2 = 4$$

Hence The theorem verified.

$T : U \to V$ L.T

same field ✓

$$\boxed{T(a\alpha + b\beta) = aT(\alpha) + bT(\beta)}$$

$T : U \to V$

$a, b \in F$

$T(U) \subseteq V$

$\alpha, \beta \in U$

✓ $T(\alpha), T(\beta) \in V$

$\alpha \in U$

$T(a\alpha + b\beta) \in V$

$T(\alpha) \in V$

Range of T

$R(T)$

$\alpha_1 \in U \quad T(\alpha_1) \in V$

$\alpha_2 \in U \quad T(\alpha_2) \in V$

$\alpha_3 \in U \quad T(\alpha_3) \in V$

$\vdots$

$\alpha_n \in U \quad T(\alpha_n) \in V$

$R(T) = \{T(\alpha) \mid \alpha \in U\}$

Dim $R(T)$ = Rank of T

$= \rho(T)$

Null space of T

$N(T)$

$\alpha \in U$

$T(\alpha) = \hat{0}$

$\alpha_1 \in U$

✓ $T(\alpha_1) = \hat{0}$

$\alpha_2 \in U$

$T(\alpha_2) = \hat{0}$

$N(T) = \{T(\alpha) = \hat{0} \mid \alpha \in U\}$

Dim $N(T)$ = Nullity of T

$= \nu(T)$

Rank nullity Theorem $T : U \to V$

$$\boxed{\rho(T) + \nu(T) = \dim U}$$

① Verify Rank nullity Theorem for the L.T $T : R^2 \Rightarrow R^3$ defined by

$$T(x,y) = (x+y,\ x-y,\ y)$$

Sol: given $T : R^2 \Rightarrow R^3$ defined by

$$T(x,y) = (x+y,\ x-y,\ y)$$

(i)

let $S = \{(1,0), (0,1)\}$ is a standard basis for R^2

Now T on S

$$T(1,0) = (1,1,0)$$

$$T(0,1) = (1,-1,1)$$

Here $S_1 = \{(1,1,0), (1,-1,1)\}$

Now we verify S_1 is L.I or not

$$S_1 \sim \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix}$$

$R_2 \to R_2 - R_1$

$$\sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

Here the two non zero rows

$\{(1,1,0), (0,-2,1)\}$ forms a

Basis of $R(T)$

Hence Dim $R(T)$ = Rank of $T = \rho(T) = 2$

$$\boxed{\rho(T) = 2}$$

(ii) Now we find Dimension of Null space of T

$\alpha \in N(T) \Rightarrow T(\alpha) = \hat{0}$

$\alpha = (x, y)$

$T(x, y) = \hat{0}$

$(x+y, x-y, y) = \hat{0} = (0,0,0)$

$$\left.\begin{aligned} x+y &= 0 \\ x-y &= 0 \\ y &= 0 \end{aligned}\right\} \text{ Now it can be written in Matrix form}$$

$$\sim \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 0 & 1 \end{bmatrix}$$

$R_2 \to R_1$ $\sim \begin{bmatrix} 1 & 1 \\ 0 & -2 \\ 0 & 1 \end{bmatrix}$

$R_3 \to 2R_3 + R_2$

$$\sim \begin{bmatrix} 1 & 1 \\ 0 & -2 \\ 0 & 0 \end{bmatrix}$$

Here the no. of columns - no of non zero rows $= 2 - 2 = 0$

Hence Dimension of $N(T) = \nu(T) = 0$

$$\boxed{\nu(T) = 0}$$

Hence $\rho(T) + \nu(T) = 2 + 0 = 2 = \text{Dim } R^2$

Hence the theorem verified

② Verify Rank, Nullity Theorem for the L.T $T: R^3 \rightarrow R^3$ is defined by

$T(x, y, z) = (x+2y-z, y+z, x+y-2z)$

Sol. given $T: R^3 \rightarrow R^3$ is defined by

$T(x, y, z) = (x+2y-z, y+z, x+y-2z)$

(i) Find the Dimension of Range of T

let $S = \{(1,0,0), (0,1,0), (0,0,1)\}$

be the Standard Basis for R^3

Now T on S

$T(1,0,0) = (1,0,1)$

$T(0,1,0) = (2,1,1)$

$T(0,0,1) = (-1,1,-2)$

Here $S_1 = \{(1,0,1), (2,1,1), (-1,1,-2)\}$

$$S_1 \sim \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ -1 & 1 & -2 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1, \; R_3 \to R_3 + R_1$

$$\sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Here the non zero rows

$\{(1,0,1), (0,1,-1)\}$ forms the

Basis of Range of T

$\therefore$ Dim Range of T $= 2 =$ Rank of T

$$\boxed{\rho(T) = 2}$$

(ii) Now we find the Dimension of N(T)

$\alpha \in N(T) \Rightarrow T(\alpha) = \hat{0}$

$\alpha = (x, y, z) \Rightarrow T(x, y, z) = \hat{0}$

$(x+2y-z,\ y+z,\ x+y-2z) = \hat{0} = (0,0,0)$

$$\left.\begin{aligned} x+2y-z &= 0 \\ y+z &= 0 \\ x+y-2z &= 0 \end{aligned}\right\}$$

Now it can be written in Matrix form

$$\sim \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & -2 \end{bmatrix}$$

$R_3 \to R_3 - R_1$

$$\sim \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \end{bmatrix}$$

$R_3 \to R_3 + R_2$

$$\sim \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Here $\vartheta(T)$ = no. of columns − no. of non zero rows in E.F

$= 3 - 2 = 1$

$$\boxed{\vartheta(T) = 1}$$

Hence By Rank, Nullity Theorem

$\rho(T) + \nu(T) = 2 + 1 = 3 = \text{Dim } \mathbb{R}^3$

Hence The Theorem verified

Verify Rank, Nullity Theorem

① $T : \mathbb{R}^3 \to \mathbb{R}^3$

$T(x, y, z) = (x - y + 2z, 2x + y - z, -x - 2y)$

② $T : V_3 \to V_2$

$T(a, b, c) = (a, b)$

Matrix of Linear Transformation

$\boxed{T: U \to V}$ be a L.T

F be the same field of U & V

let $B_1 = \{\alpha_1\ \alpha_2\ \alpha_3\}$ ordered Basis of U

$B_2 = \{\beta_1, \beta_2\ \beta_3\}$ ordered Basis of V

$\alpha \in U \Rightarrow T(\alpha) \in V$

$T(\alpha)$ can be expressed as L.C of elements of ordered Basis B_2

$$\left.\begin{aligned}
T(\alpha_1) &= a_1\beta_1 + b_1\beta_2 + c_1\beta_3 \\
T(\alpha_2) &= a_2\beta_1 + b_2\beta_2 + c_2\beta_3 \\
T(\alpha_3) &= a_3\beta_1 + b_3\beta_2 + c_3\beta_3
\end{aligned}\right\}$$

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} = [T : B_1, B_2]$$

① let $T: R^3 \to R^2$ be the linear transformation defined by

$T(x,y,z) = (3x+2y-4z,\ x-5y+3z)$

find the matrix of T relative to the bases $B_1 = \{(1,1,1),(1,1,0),(1,0,0)\}$

$B_2 = \{(1,3),(2,5)\}$ $[T : B_1, B_2]$

Sol: given

$$B_1 = \{\underset{\alpha_1}{(1,1,1)},\ \underset{\alpha_2}{(1,1,0)},\ \underset{\alpha_3}{(1,0,0)}\}$$

$T(x,y,z) = (3x+2y-4z,\ x-5y+3z)$

$T(1,1,1) = (1,-1)$

$T(1,1,0) = (5,-4)$

$T(1,0,0) = (3,1)$

since $B_2 = \{(1,3),(2,5)\}$

let $(a,b) \in R^2$

$(a,b) = p(1,3) + q(2,5)$

$$1p + 2q = a$$

$$3p + 5q = b$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & a \\ 3 & 5 & b \end{bmatrix}$$

$$R_2 \to R_2 - 3R_1$$

$$\begin{bmatrix} 1 & 2 & a \\ 0 & -1 & b-3a \end{bmatrix}$$

$$p + 2q = a$$

$$-q = b - 3a$$

$$\boxed{q = 3a - b}$$

$$p + 2(3a - b) = a$$

$$\boxed{p = -5a + 2b}$$

$$(a,b) = (-5a+2b)(1,3) + (3a-b)(2,5)$$

$$T(1,1,1) = (1,-1) = -7(1,3) + 4(2,5)$$

$$T(1,1,0) = (5,-4) = -33(1,3) + 19(2,5)$$

$$T(1,0,0) = (3,1) = -13(1,3) + 8(2,5)$$

Hence the matrix of L.T relative to the order Basis B_1 and B_2

$$[T : B_1, B_2] = \begin{bmatrix} -7 & -33 & -13 \\ 4 & 19 & 8 \end{bmatrix}$$

② Let $T : R^3 \to R^2$ be a linear Transform defined by

$T(x, y, z) = (2x + y - z, 3x - 2y + 4z)$

obtain the matrix of T relative to the bases

$B_1 = \{(1,1,1), (1,1,0), (1,0,0)\}$

$B_2 = \{(1,3), (1,4)\}$

$[T : B_1, B_2]$

Sol. given $B_2 = \{(1,3), (1,4)\}$

let $(a, b) \in R^2$

$(a, b) = p(1,3) + q(1,4)$

$p + q = a$

$3p + 4q = b$

$\boxed{AX = B}$

$$\underbrace{\begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} p \\ q \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} a \\ b \end{bmatrix}}_{B}$$

$\sim [A \mid B]$

$$\begin{bmatrix} 1 & 1 & a \\ 3 & 4 & b \end{bmatrix}$$

$R_2 \to R_2 - 3R_1$

$$\sim \begin{bmatrix} 1 & 1 & a \\ 0 & 1 & b-3a \end{bmatrix}$$

$p + q = a$

$$\boxed{q = b - 3a}$$

$p + b - 3a = a$

$$\boxed{p = 4a - b}$$

Hence

$$\boxed{(a, b) = (4a - b)(1, 3) + (b - 3a)(1, 4)}$$

since

$B_1 = \{(1,1,1), (1,1,0), (1,0,0)\}$

$T(x, y, z) = (2x + y - z,\ 3x - 2y + 4z)$

$T(1,1,1) = (2,5) = 3(1,3) + (-1)(1,4)$

$T(1,1,0) = (3,1) = 11(1,3) + (-8)(1,4)$

$T(1,0,0) = (2,3) = 5(1,3) + (-3)(1,4)$

$$\boxed{\begin{aligned} T(\alpha_1) &= a_1 \beta_1 + a_2 \beta_2 \\ T(\alpha_2) &= b_1 \beta_1 + b_2 \beta_2 \\ T(\alpha_3) &= c_1 \beta_1 + c_2 \beta_2 \\ \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{bmatrix} &= T : B_1 : B_2 \end{aligned}}$$

$$[T : B_1 ; B_2] = \begin{bmatrix} 3 & 11 & 5 \\ -1 & -8 & -3 \end{bmatrix}$$

---

① let $T : V_2 \to V_3$ be defined by

$T(x, y) = (x+y, 2x-y, 7y)$

Find $[T : B_1, B_2]$

where $B_1 = \{(1,0), (0,1)\}$

$B_2 = \{(1,0,0), (0,1,0), (0,0,1)\}$

(2) let $T: \mathbb{R}^3 \to \mathbb{R}^2$ be defined by

$$T(x, y, z) = (x + y, 2z - x)$$

Find $[T : B_1, B_2]$

where $B_1 = \{(1, 0, -1), (1, 1, 1), (1, 0, 0)\}$

$B_2 = \{(0, 1), (1, 0)\}$

$T : U \to V$

T is invertible or not

singular

$T(\alpha) = \hat{0}$

$\alpha \neq \hat{0}$

$\downarrow$

T^{-1} does not exist

non singular

$T(\alpha) = \hat{0}$

$\alpha = \hat{0}$

$\downarrow$

T is invertible

① let T be a linear operator on $V_3(R)$ defined by

$$T(x, y, z) = (3x + z, -2x + y, -x + 2y + z)$$

prove that T is invertible.

Sol let $S = \{(1,0,0), (0,1,0), (0,0,1)\}$ be the standard basis of V_3

Now T on S

$$T(1,0,0) = (3, -2, -1)$$

$$T(0,1,0) = (0, 1, 2)$$

$$T(0,0,1) = (1, 0, 1)$$

Now its coeffect Matrix is

$$P = \begin{bmatrix} 3 & 0 & 1 \\ -2 & 1 & 0 \\ -1 & 2 & 1 \end{bmatrix}$$

Now $|P| = 3[1] + 0 + 1[-4 + 1]$

$$= 3 - 3 = 0$$

P is singular

$\Rightarrow T$ is singular

$\Rightarrow T^{-1}$ does not exist

vector space: $V(F)$

① $(V, +)$ Abelian group

② External composition exist

$\alpha \in V,\ a \in F$

$a\alpha \in V$

③ $a(\alpha+\beta) = a\alpha + a\beta$

$(a+b)\alpha = a\alpha + b\alpha$

$(ab)\alpha = a(b\alpha)$

$1 \cdot \alpha = \alpha \cdot 1 = \alpha$

$a, b, 1 \in F,\ \alpha, \beta \in V$

Subspace: $W \subseteq V \rightarrow W$ itself a V. S

(or) $a\alpha + b\beta \in W \quad \forall a, b \in F$
$\alpha, \beta \in V$

linear Combination: $\forall \alpha \in V\ \exists \alpha_1, \alpha_2 \ldots \alpha_n \in V$

$a_1, a_2 \ldots a_n \in F$

$\alpha = a_1\alpha_1 + a_2\alpha_2 + \ldots + a_n\alpha_n$

$$\alpha = \sum_{i=1}^{n} a_i\alpha_i$$

$\alpha_1, \alpha_2 \ldots \alpha_n \in S$

linear Span: $L(S) = \left\{ \alpha \mid \alpha = \sum_{i=1}^{n} a_i\alpha_i \right\}$

$\alpha_1, \alpha_2 \ldots \alpha_n \in S$

$a_1, a_2 \ldots a_n \in F$

Linearly independent: $\{\alpha_1, \alpha_2 \cdots \alpha_n\}$ is L.I

$\exists\ a_1, a_2 \cdots a_n \ni$

$a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n = 0$

$\Rightarrow a_1 = a_2 = \cdots = a_n = 0$

all a_i's are zero's

Linearly Dependent: not all a_i's are zero's

Bases: S is bases ① S is L.I

② $L(S) = V$

Dimension: no. of elements in Bases

Finite Dimensional V.S: no. of elements in Bases are finite then The V.S is F.D.V.S

Linear Transformation: $T: U \rightarrow V$

$T(a\alpha + b\beta) = a\,T(\alpha) + b\,T(\beta)$

$a, b \in F,$

$\alpha, \beta \in U$

Range of T: $R(T) = \{T(\alpha) \mid \alpha \in U\}$

Dimension $R(T)$ = rank of $T = \rho(T)$

Null space of T: $N(T) = \{\alpha \in U \mid T(\alpha) = \hat{0}\}$

Dimension $N(T)$ = Nullity of $T = \nu(T)$

Rank Nullity Theorem: $T : U \to V$

$$\rho(T) + \nu(T) = \text{Dim } U$$

Matrix linear Transformation: $T : U \to V$

B_1 Basis of U

B_2 Basis of V

$$[T : B_1, B_2] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

let W_1 and W_2 are any two subspace of $V(F)$

Then

$$\dim(W_1 + W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2)$$

① vector Space:

Show that the set of all Triads

(x_1, x_2, x_3) where x_1, x_2, x_3 are

real numbers forms a V.S over

field of real numbers w.r.t the

vector addition, scalar multiplication

(i) $(x_1, x_2, x_3) + (y_1, y_2, y_3)$

$= (x_1 + y_1, x_2 + y_2, x_3 + y_3)$

(ii) $c(x_1, x_2, x_3) = (cx_1, cx_2, cx_3)$

② Subspace

prove that the set of solutions

(x, y, z) of the Eqn $x + y + 2z = 0$ is

a sub space of $R^3(R)$

③ linear span (or) linear combination

In the vector space $R^3(R)$ let $\alpha = (1,2,1)$

$\beta = (3,1,5)$, $\gamma = (3,-4,5)$ are forms

a l.c of $(1,-2,5)$?

④ Linearly Independent Dependent

Show That The vectors $(2,1,4),(1,-1,2)$
$(3,1,-2)$ are L.I $\to |A|$?
$\to a, b, c$?

⑤ Bases

Show That The set of vectors
$(1,2,1)$, $(2,1,0)$, $(1,-1,2)$ forms a
Bases of $V_3(F)$

*
⑥ Find The coordinates of α w.r.t
The Bases $\{x, y, z\}$ where

$\alpha = (4,5,6)$
$x = (1,1,1)$ $y = (-1,1,1)$, $z = (1,0,-1)$

**
⑦ let W_1 and W_2 be The subspace of
R^4 generated by
$\{(1,1,0,-1), (1,2,3,0), (2,3,3,-1)\}$
and $\{(1,2,2,-2), (2,3,2,-3), (1,3,4,-3)\}$
Find (i) Dim W_1 (ii) Dim W_2 (iii) Dim$(W_1 + W_2)$
(iv) Dim$(W_1 \cap W_2)$

⑧ linear Transformation

let $T: R^2 \to R^3$ is defined by

$T(x,y) = (2x, 4x-y, 2x+3y)$

is linear transformation?

⑨ let $T: R^3 \to R^2$, $H: R^2 \to R^2$ are two linear transformations defined by

$T(x,y,z) = (3x, 4y-z)$

$H(x,y) = (-x, y)$

Then find

(i) $T+H$ (ii) $T-H$ (iii) $3T-4H$

(iv) TH (v) HT

**

⑩ verify Rank, Nullity theorem

$T: R^3 \to R^3$ be defined by

$T(x,y,z) = (x+y-2z, 2x-y-z, x+2y)$

(11) Matrix linear Transformation

let $T: V_4 \to V_3$ is defined by

$$T(x, y, z, w) = (7x + 2y + 11z - 8w,$$
$$11x + 7y + 22z - 19w,$$
$$13x + 8y + 30z - 23w)$$

Then Find $[T : B_1, B_2]$ where

$$B_1 = \{(1,1,1,2), (1,-1,0,0), (0,0,1,1), (0,1,0,0)\}$$

$$B_2 = \{(1,2,3), (1,-1,1), (2,1,1)\}$$

(1)

Objective Type Questions

Matrices; Vector Spaces; Characteristic Roots, Characteristic Vectors of a Square Matrix; Real Quadratic Forms

Multiple Choice Questions

1. The inverse of an orthogonal matrix is

   (*a*) a unit matrix (*b*) orthogonal (*c*) hermitian

2. The adjoint of the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is

   (*a*) $\begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$ (*b*) $\begin{bmatrix} d & c \\ b & a \end{bmatrix}$ (*c*) $\begin{bmatrix} d & b \\ c & a \end{bmatrix}$

3. The necessary and sufficient conditon for the square matrix A to have an inverse is

   (*a*) $|A|=0$ (*b*) $|A|\neq 0$ (*c*) $|A|=1$

4. The rank of the matrix $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & -6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix}$ is

   (*a*) 0 (*b*) ≥ 4 (*c*) < 4

5. If $A=\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$, then $A^2-2A+2I=$

   (*a*) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ (*b*) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (*c*) $\begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix}$

6. If A and B are $n \times n$ matrices then det (AB)

   (*a*) $\leq (\det A)(\det B)$ (*b*) $= (\det A)(\det B)$ (*c*) $= \det A + \det B$

7. If A is a square matrix, then adj $(AB)=$

   (*a*) (adj A) (adj B) (*b*) (adj B) (adj A) (*c*) (adj A) + (adj B)

8. The system of equations

   $2x+3y+4z+1=0,\ 2x+6y+8z+3=0,\ x+y+z+1=0$ has

   (*a*) no solution (*b*) a unique solution

   (*c*) infinitely many solutions

9. The matrix $\begin{bmatrix} 2 & 1 & 0 \\ 3 & 0 & -1 \\ 0 & -3 & -2 \end{bmatrix}$ is

2 (a) symmetric (b) skew-symmetric
(c) neither symmetric nor skew-symmetric

10. The matrix $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 3 & 0 & 0 \end{bmatrix}$ has
(a) no inverse (b) inverse (c) two inverse

11. The matrix $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 4 & 5 & 6 \end{bmatrix}$ has
(a) an inverse (b) no inverse (c) more than one inverse

12. If A is a $n \times n$ matrix and $X = \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix}$ and if A^{-1} exists, then the system of equations $AX = 0$ has
(a) a unique solution (b) no solution
(c) more than one solution

13. The matrix $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ has
(a) an inverse (b) two inverses (c) no inverse

14. A is singular, then $AB = AC$
(a) implies $B = C$ (b) does not necessarily imply $B = C$
(c) implies $B \neq C$

15. If $A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 3 & 1 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 2 \\ -3 & -6 \end{bmatrix}$, $D = \begin{bmatrix} 0 & 2 \\ 0 & -1 \end{bmatrix}$, then
(a) $AB = CD = 0$ (b) $AB = 0$ and $CD \neq 0$ (c) $AB \neq 0$ and $CD = 0$

16. If A is a square matrix and A' is its transpose, then $A' - A$ is
(a) neither symmetric nor skew-symmetric
(b) skew-symmetric (c) symmetric

3 17. The equations $2x - y + 3z = 0, 3x + y + 5z = 13,$ $4x + 3y + 7z = 26$ have

(*a*) only one solution (*b*) unique solution (*c*) infinite solutions

18. The inverse of the matrix $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ is

(*a*) $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ (*b*) $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ (*c*) $\begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

19. If $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 1 \end{bmatrix}$, then $A\,A^T$ is

(*a*) symmetric (*b*) skew-symmetric (*c*) orthogonal

20. Interchange of a pair of rows of a matrix

(*a*) does not alter the rank

(*b*) does alter the rank

(*c*) make it zero

21. The det of every skew-symmetric matrix is

(*a*) odd (*b*) even (*c*) zero

22. The rank of the matrix $\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$ is

(*a*) 0 (*b*) 1 (*c*) 2

23. The equations $2x - y - z = 0, x + y + z = 0, 3x + 6y + 8z = 0$ have

(*a*) unique solution (*b*) infinitely many solutions

(*c*) no solution

24. The system of linear equations $x + y + z = 0, 2x + y - z = 0$ and $3x + 2y = 0$ has

(*a*) no solution (*b*) a unique solution

(*c*) an infinite number of soultions

25. $[x \quad y]\begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} =$

(*a*) $5x^2 + 3xy + 2y^2$ (*b*) $5x^2 + 6xy + 2y^2$ (*c*) $2x^2 + 6xy + 5y^2$

26. The adjoint of the matrix $\begin{bmatrix} 2 & -3 \\ 4 & -1 \end{bmatrix}$ is

(a) $\begin{bmatrix} -1 & -4 \\ 3 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} -1 & -3 \\ -3 & 2 \end{bmatrix}$ (c) $\begin{bmatrix} -1 & 4 \\ -3 & 2 \end{bmatrix}$

27. The matrix $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ is

(a) singular (b) symmetric (c) skew-symmetric

28. The rank of the matrix $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ is

(a) 1 (b) 2 (c) 3

29. The rank of the matrix $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ is

(a) 1 (b) 2 (c) 3

30. The rank of the matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & i \end{bmatrix}$ is

(a) 1 (b) i (c) 3

31. Every superset of a linearly dependent set is

(a) linealy independent (b) linearly dependent (c) none

32. Every subset of a linearly independent set is

(a) linearly independent (b) linearly dependent (c) none

33. S is a subset of vector space V over F. If S contains in distinct vectors, then S generates a subspace of dimension

(a) $> n$ (b) $< n$ (c) $\leq n$

34. The charecteristic roots of the matrix $\begin{bmatrix} 5 & 1 \\ 1 & 5 \end{bmatrix}$ are

(a) 4, 6 (b) 4, 4 (c) 5, 5

35. The set M of all 2×2 matrices over the field of reals R is a vector space over R under the usual operations. The dimension of M over R is

(a) 1 (b) 2 (c) 4

36. A finite system of vectors $x_1, x_2, \ldots\ldots, x_n$ of vector space $V(x)$ is said to be linearly independent if every relation of the form $a_1 x_1 + a_1 x_2 + \ldots + a_m x_m = 0, a_i \in k, 1 \le i \le m \Rightarrow$
   (a) $a_i = 0$ for every $i, 1 \le i \le m$
   (b) $a_i = 0$ for at least one $a_i, 1 \le i \le m$
   (c) no $a_i = 0$

37. Let S span a vector space M. Let B be a set of linearly independent members of M. Then, the number of members of $S =$
   (a) the numbers of members of B
   (b) greater than the number of members of B
   (c) at least the number of members of B

38. If T is a linear transformation, then the range of T and $\{X : T(X = 0)\}$
   (a) are vector spaces
   (b) only range of T is a vector space
   (c) only $\{X : T(X) = 0\}$ is a vector space

39. The set $\{(1, 0, 0); (0, 1, 0); (0, 0, 1)\}$ of vectors in R^3 is a
   (a) linearly dependent set
   (b) basis for R^3
   (c) both (a) and (b) are false

40. The set of vectors (1, 2, 1), (0, 1, 0), (3, 4, 3) in R^3 is
   (a) a linearly dependent set
   (b) a basis for R^3
   (c) not a basis for R^3.

41. If W is a proper subspace of a finite dimensional vector space V, then
   (a) $\dim V < \dim W$
   (b) $\dim V = \dim W$
   (c) $\dim V > \dim W$

42. If in a vector space V, A is a set of vector containing zero vector, then A is
   (a) linearly independent
   (b) a basis of V
   (c) linearly dependent

43. A subspace of R^3 is
   (a) $\{(x_1, x_2, x_3) \mid x_1 + x_2 + x_3 = 2\}$
   (b) $\{(x_1, x_2, x_3) \mid 2x_1 = x_2 + x_3\}$
   (c) $\{(x_1, x_2, x_3) \mid x_1 x_2 = 0\}$

6 44. For the linear transformation $T : R^3 \to R^2$ defined by $T(a, b, c) = (a+b+c, a-b-c)$, an element in the null space of T is

(a) $(1, -1, 0)$ (b) $(0, 1, 1)$ (c) $(0, 1, -1)$

45. If A is a linear transformation such that $A^3 - A^2 + A = 0$, then $A^{-1} =$

(a) $A^2 - A$ (b) $A - A^2$ (c) does not exist

46. $T : R^2 \to R^2$ is defined by $T(x, y) = (4x - 2y, 2x + y)$. Its matrix relative to the basis $\{(1, 0), (0, 1)\}$ is given by

(a) $\begin{bmatrix} 4 & -2 \\ 2 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 4 & 2 \\ -2 & 1 \end{bmatrix}$

(c) more information is needed

47. If the quadratic form $ax^2 + 2hxy + by^2$ is written as $X' A X$, then $A =$

(a) $\begin{bmatrix} a & b \\ h & b \end{bmatrix}$ (b) $\begin{bmatrix} a & h \\ b & h \end{bmatrix}$ (c) $\begin{bmatrix} a & h \\ h & b \end{bmatrix}$

48. The eigen values of the matrix are $\begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix}$ are

(a) $0, 3$ (b) $1, 2$ (c) $-1, -2$

49. The quadratic form $3x^2 + 4y^2 - z^2$ is

(a) positive definite (b) negative definite (c) indefinite

50. Every n-dimentional vector space $V(F)$ is — to $V_n(F)$.

(a) isomorphic (b) homomorphic only

(c) not the first two

51. If T is an invertible linear transformation on a vector space $V(F)$, then $TT^{-1} =$

(a) $\dfrac{T}{T^{-1}}$ (b) $T^{-1}T$ (c) $T.(TT^{-1})$

52. The union of two subspaces of $V(F)$

(a) need not be subspace of $V(F)$

(b) is a subspace of $V(F)$

(c) is greater than $V(F)$

53. If W is a proper subspace of a finite dimension vector space V, then
(a) $\dim V < \dim W$ (b) $\dim V = \dim W$ (c) $\dim V > \dim W$

54. If e_1, e_2, e_3 are L.I. vectors in a vector space, then $e_1 + e_2$; $e_2 + e_3$; $e_3 + e_1$ are
(a) linearly independent (b) linearly dependent
(c) none of the two

55. The rank of the quadratic form $x^2 + y^2 + z^2 - t^2$ is
(a) 4 (b) 3 (c) 2

56. If S and T are subsets of a vector space $V(F)$, then $S \subseteq L(T) \Rightarrow$
(a) $L(S) \subseteq L(T)$ (b) $L(S) \supset L(T)$ (c) $L(S) = L(T)$

57. If T_1 and T_2 are linear transformations on V, their product T_1T_2 is
(a) also a linear transformation
(b) not a linear transformation
(c) none of the two

58. An element in R^3 in the span of $\{(1, 1, 1), (1, 0, 1)\}$ is
(a) $(0, 1, 0)$ (b) $(2, 2, 0)$ (c) $(1, 3, 4)$

59. In R^3 with $e_1 = (1, 0, 0)$, $e_2 = (1, 1, 0)$, $e_3 = (1, 1, 1)$ as a basis, the element $(1, 2, 3) =$
(a) $e_1 + 2e_2 + 3e_3$ (b) $e_1 + 2e_2 - 2e_3$ (c) $3e_3 - e_1 - e_2$

60. If A, B are invertible transformations, then $(AB)^{-1} =$
(a) $A^{-1}B^{-1}$ (b) $B^{-1}A^{-1}$ (c) BA^{-1}

61. $T : R^3 \to R^3$ is defined by
$T(x_1, x_2, x_3) = (x_1 + x_2 + x_3, -x_1 - x_2 - 4x_3, 2x_1 - x_3)$. The image of $(1, 1, 2)$ is
(a) $(3, 1, 0)$ (b) $(4, -10, 0)$ (c) $(3, -6, -1)$

62. If $T : V \to W$ is a linear transformation, then rank T + nullity T =
(a) $\dim V$ (b) $\dim W$ (c) $\dim (V + W)$

63. If $v_1, v_2, \ldots\ldots, v_m$ are linearly independent in an n dimensional vector space, then
(a) $m \leq n$ (b) $m = n$ (c) $n \leq m$

64. The rank of the Q.F. $2x^2 + y^2 + 4z^2 - 2v^2$ is
(*a*) 2 (*b*) 3 (*c*) 4

65. In the vector space of complex numbers C over the real number field R
(*a*) $\{1, i\}$ is a basis (*b*) $\{i, -i\}$ is a basis (*c*) $\{i\}$ is a basic

66. Let U, V be two finite dimensional vector spaces over the same field. If $T: U \to V$ is a linear transformation, then rank $T =$
(*a*) dimensional of Ker T
(*b*) dimension of U (*c*) dimension of $T(U)$

Fill up the Blanks

1. If I is the unit matrix of order n, then $|I| =$ ....
2. The product of two invertible matrices is necessarily ....
3. The rank of a 3×3 matrix whose elements are all 2 is ....
4. The rank of a matrix can be got by ....
5. The rank of a singular matrix of order 3 is ....
6. The rank of the product of two matrices each of order 3 is ....
7. The rank of a matrix nedeeded to the form $\begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix}$ is ....
8. If A is a non-zero 3×1 matrix and B is a non zero 1×3 matrix, the rank of AB is ....
9. A square matrix is non-singular if and only if its columns are ....
10. If A and B are non-singular if and only if its columns are ....
11. If A and B are matrices and if AB is defined then the rank of AB ....
12. If A is non-singular and if B is any matrix such that AB is defined, then the rank of AB ....
13. If S, A, B are $n \times n$ matrices and S is non-singular such that $B = S^{-1} A S$, then rank of A .... rank of B.
14. If A is a singular matrix then the system of equations $AX = 0$ has ....
15. If A and B are 3×4 matrices, then the rank $(A + B)$ is ....

a

16. If A and B are $n \times n$ matrices and if AB is non-singular, then both are ....

17. If the elements of any row of a determinant are multiplied by the cofactors of the corresponding elements of another row, the sum of the products so formed is ....

18. A solution of the equations $x - y + z = 0,\ 2x + y - z = 0$ is ....

19. The rank of the matrix $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 2 & 2 \end{bmatrix}$ is ....

20. The row rank of a matrix is equal to ....

21. The rank of the $n \times n$ unit matrix is ....

22. An elementary column operation on an $m \times n$ matrix amounts to .... by the corresponding elementary $n \times n$ matrix.

23. The determinant of a triangular matrix is equal to the .... its diagonal elements.

24. If a square matrix can be written as a product of elementary matrices, then ....

25. Two $m \times n$ matrices are equivalent if and only if they have ....

26. The product matrix of two elementary matrices is ....

27. If the determinant of a system of n non-homogeneous linear equations in n unkonwns is different from zero, then the equations have ....

28. If A^{-1} denotes the inverse of a matrix A, then $(AB)^{-1}$ is equal to ....

29. If A is $n \times n$ matrix and B is an $n \times m$ matrix, then $(A\ B)^T$ is .... matrix.

30. D is an $n \times m$ matrix, then $\begin{bmatrix} A & C \\ D & B \end{bmatrix}$ is .... matrix.

31. The system of non-homogeneous linear equations $AX = B$ is consistent if ....

32. If A is skew-symmetric, the diagonal elements of A are all ....

33. If $A^{-1} = A^T$, then A is ....

34. If A is a non-singular matrix and k is any positive integer, then $(A^k)^{-1} =$ ....

35. The determinant of an orthogonal matrix is ....

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36. The square matrix A is invertible if ....
37. The product of two orthogonal matrices is ....
38. Every square matrix A satisfies its own ....
39. If S is a 3×3 non-singular matrix, then its rank is ....
40. A sufficient condition for m non-homogeneous linear equations in n variables to have a unique solution is ....
41. If two vectors are L.D., then one of them is ....
42. The union of two subspaces is a subspace iff ....
43. A vector space is said to be finite dimensional iff ....
44. In a finite dimensional vector space with dim n, and L.I. set of n vectors is ....
45. The dimension of a subspace W of V is .... the dimension of V.
46. If W_1 and W_2 are two subspaces of a finite dimensional vector space $V(F)$, then dim $(W_1 + W_2) =$ ....
47. In a vector space, the null set of vectors is linearly ....
48. The vectors $L_1, L_2, \ldots\ldots, L_n$ are L.I. if ....
49. If V is a direct sum of two of its subspaces U and V, then every element of V can be written as ....
50. If λ_1, λ_2 are two distinct characteristic roots of a square matrix, then the corresponding characteristic vectors are ....
51. The determinant of a diagonal matrix A of order n is equal to .... the characteristic roots of A.
52. The kernel of a homomorphism of a vector space V onto a vector space W is a ....
53. If the characteristic roots of a non-singular $n \times n$ matrix A are $\lambda_1, \lambda_2, \ldots\ldots, \lambda_n$, then the characteristic roots of A^{-1} are ....
54. If A is a square matrix and $f(x)$ is any polynomial $a_0 + a_1 x^1 + a_2 x^2 + \ldots\ldots + a_n x^n$ such that $f(A) = 0$, then $f(x)$ is divisible by ....
55. If U and W are finite dimensional subspaces of a vector space, then dim $(U \cap W)$ ....
56. If U and W are finite dimensional subspaces of a vector space, then dim $(U + W)$ ....

57. Any base of a non-trivial vector space .... the zero vector.
58. If W is a kernel of a homomorphism of a vector space U onto a vector space V, then U/W is .... to V.
59. The vector space V of ordered pairs of complex numbers over the real field is of dimension ....
60. If V is a vector space of finite dimension n, then any subset of V of $(n+1)$ or more vectors is ....
61. V is the vector space of 3×3 symmetric matrices over k. The dimensionality of V is ....
62. $F : R^3 \to R^3$ is projection mapping into the XY-plane $F(x, y, z) = (x, y, 0)$. The kernel of F is ....
63. If W_1 and W_2 are subspaces of a vector space V over a field then $W_1 \cap W_2$ is a .... of V.
64. The dimension of the solution space of the homogeneous system of linear equations $AX = 0$ is .... where n is the number of unknowns and r is the rank of the coefficient matrix A.
65. The non-zero rows of a matrix in Echelon form are linearly ....
66. Let a vector space V of finite dimension n. Then any linearly independent set of V is ....
67. A set containing a single non-zero vector is linearly ....
68. If a vector space V has dimension n, then no set of $(n-1)$ elements can .... V.
69. If B is a subset of a set A of linearly independent vectors, then B is linearly ....
70. Any two basis of a finite dimensional vector space have ....
71. The minimal polynomial of a square matrix A divides the .... of A.
72. If V is a finite dimensional vector space and W is a subspace of V, then $\dim (V/W) =$ ....
73. If W is a subspace of a vector space U, then there is a homomorphism of U onto ....

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74. The rank of the product of two linear transformations S and T is less then or equal to the .... of the ranks of S and T.
75. If a vector space has dimension n, then any $(n+1)$ elements of it are ....
76. The index of a quadratic form is invariant under ....
77. In a finite dimensional vector space, every non-empty independent set of vectors if part of a ....
78. If α, β, γ are scalars not all zero and U, V, W are vectors such that $\alpha U + \beta V + \gamma W = 0$, then the set $\{U, V, W\}$ is ....
79. The minimal polynomial of a square matrix is .... its characteristic polynomial.
80. Any two finite dimensional vector spaces of the some dim are ....
81. The subspace of a vector space is also a ....
82. If the vector space V is the direct sum of the subspaces U and W, then dim V is .... dim U + dim W.
83. The set of vectors $(1, 0, 0), (0, 1, 0), (0, 0, 1)$ is a .... of R^3.
84. The characteristic equation of a sqaure matrix is satisfied by ....
85. The kernel of homomorphism of a vector space V onto a vector space W is .... of V.
86. The dimension of the vector space of complex numbers as a vector space over the field of real numbers is ....
87. The elements $x_1, x_2, \ldots\ldots, x_k$ are linearly dependent if there exist scalars $a_1, a_2, \ldots\ldots, a_k$ ....
88. $T : R^6 \to R^6$ is a linear transformation such that the rank of T is 4. The nullity of T is ....
89. The singnature of the Q.F. is ....
90. The eigen values of symmetric matrix are ....
91. If A is a linear trasnformation such that $A^2 - A + I = 0$, then A is ....
92. If a diagonal matrix A is idempotent, then its diagonal elements can be ....
93. If T is a homomorphism of $U(F)$ into $V(F)$. then $\alpha \in U$, $T(-\alpha) =$ ....

(B)

94. The matrices $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ and $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}$ have .... characteristic roots.

95. The number of elements in the basis of a finitely generated vector space is called ....

96. If W_1 and W_2 are two subspaces of a finite dimensional vector space $V(F)$, then $\dim W_1 + \dim W_2 =$ ....

97. The degree of the characteristic equation of a matrix is equal to ....

98. The rank of every skew-symmetric is ....

99. In a vector space V, the singleton set $\{V\}, V \neq 0$ is linearly ....

100. If u, v, w are vectors such that $u = 2v + 3w$, then the set $\{u, v, w\}$ is linearly ....

ANSWERS

Multiple Choice Questions

1. *b*	2. *a*	3. *b*	4. *c*	5. *c*	6. *b*	7. *b*	8. *b*	9. *c*	10. *b*
11. *b*	12. *a*	13. *c*	14. *b*	15. *c*	16. *b*	17. *c*	18. *a*	19. *a*	20. *a*
21. *c*	22. *b*	23. *b*	24. *c*	25. *b*	26. *a*	27. *c*	28. *c*	29. *b*	30. *c*
31. *b*	32. *a*	33. *c*	34. *a*	35. *c*	36. *a*	37. *c*	38. *a*	39. *b*	40. *a*
41. *c*	42. *c*	43. *b*	44. *c*	45. *a*	46. *a*	47. *c*	48. *b*	49. *c*	50. *a*
51. *b*	52. *a*	53. *c*	54. *a*	55. *a*	56. *a*	57. *a*	58. *a*	59. *c*	60. *b*
61. *b*	62. *a*	63. *a*	64. *c*	65. *a*	66. *c*.				

Filling up of Blanks

1. 1 2. invertible 3. 1 4. elementary transformations

5. ≤ 2 6. ≤ 3 7. 3 8. 1

9. L.I. 10. $\begin{bmatrix} A^{-1} & 0 \\ 0 & B^{-1} \end{bmatrix}$ 11. $\leq \min\{\text{rank}\, A, \text{rank}\, B\}$

12. = rank of B 13. = 14. a non-trivial solution

15. ≥ 0 16. non-singular 17. zero 18. $x = 0, y = 1, z = 1$

19. 2 20. its column rank 21. n 22. post multiplication

23. product of 24. it is non-singular

(14)

Objective Type Questions

25. the same rank
26. non-singular
27. a unique solution
28. $B^{-1}A^{-1}$
29. $m \times n$
30. an $(m+n) \times (m+n)$
31. $\rho(A) = \rho[AB]$
32. zero
33. orthogonal
34. $(A^{-1})^k$
35. ± 1
36. A is non-singular
37. orthogonal
38. characteristic equation
39. 3
40. $m = n$ where the coefficient matrix is non-singular
41. a scalar multiple of the other
42. one is contained in the other
43. it has a finite basis
44. a basis of the vector space
45. $\leq$
46. $\dim W_1 + \dim W_2 - \dim (W_1 \cap W_2)$
47. dependent
48. There exist scalars $a_1, a_2, \ldots\ldots, a_n$ so that

$a_1\alpha_1 + a_2\alpha_2 + \ldots\ldots + a_n\alpha_n = 0 \Rightarrow a_1 = 0, a_2 = 0, \ldots\ldots, a_n = 0.$

49. the sum of two unique elements in U and V.
50. L.I.
51. the product of
52. subspace of V
53. $1/\lambda_1, 1/\lambda_2, \ldots\ldots 1/\lambda_3$
54. the characteristic polynomial of A
55. $\leq \dim U$ and $\leq \dim W$
56. $\geq \dim U$ and $\geq \dim W$
57. does not contain
58. isomorphic
59. 2
60. L.D.
61. 6
62. $\{(0, 0, z)/z \in R\}$
63. subspace
64. $n - r$
65. independent
66. a part of its basis
67. independent
68. form a basis of
69. independent
70. the same numbers of its basis
67. independent
68. form a basis of
69. independent
70. the same number of elements
71. charactersitic polynomial
72. $\dim V - \dim W$
73. U/W
74. minimum
75. linearly dependent
76. a real non-singular transformation
77. basis
78. linearly independent
79. a divisor of
80. isomorphic

81. vector space 82. equal to

83. basis 84. it self

85. a subspace 86. 2

87. not all zero such that $a_1 x_1 + a_2 x_2 + \ldots\ldots + a_k x_k = 0$ 88. 2

89. $s-(r-s) = 2s-r$ $r-(s-r) = 2s-r$ where r is the rank of the Q.F. and s is the number of positive terms 90. not distinct

91. non-singular 92. 0, 1

93. $-T(\alpha)$ 94. the same

95. dimension of the vector space

96. $\dim(W_1 + W_2) + \dim(W_1 \cap W_2)$

97. the number of rows or columns of the matrix

98. greater than or equal to 2 99. independent

100. dependent.