3.3

LINE INTEGRALS

Let C be a simple curve. Let the parametric representation of C be written as

$$x = x(t),\ y = y(t),\ z = z(t),\ a \leq t \leq b.$$

Therefore, the position vector of a point on the curve C can be written as

$$\boldsymbol{r}(t) = x(t)\boldsymbol{i} + (t)\boldsymbol{j} + z(t)\boldsymbol{k}, \qquad a \leq t \leq b$$

Line integral with respect to arc length

Let C be a simple smooth curve whose parametric representation is as given in above equations. Let $f(x,\ y,\ z)$ be continuous on C. Then, we define the line integral of f over C with respect to the arc length s by

$$\int_C f(x, y, z)ds$$

$$= \int_a^b f[x(t), y(t), z(t)]\sqrt{x'(t)^2 + y'(t)^2 + z(t)^2}dt$$

since $ds = \frac{ds}{dt}dt = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\ dt.$

This result is true since the arc length along the curve from an initial point $(x(a), y(a), z(a))$ to any point $(x(t), y(t), z(t))$ is given by

$$s(t)=\int_a^t \sqrt{\left(\frac{dx}{d\eta}\right)^2+\left(\frac{dy}{d\eta}\right)^2+\left(\frac{dz}{d\eta}\right)^2}\,d\eta$$

and ds is as given above.

The initial point of C is given by $(x(a), y(a), z(a))$ and the terminal point of C is given by $\big(x(b), y(b), z(b)\big)$.

Example: Evaluate $\int_C \left(x^2+yz\right)ds$, where C is the curve defined by $x=4y, z=3$ from $(2, 1/2, 3)$ to $(4, 1, 3)$.

Solution:

Let $x=t$. Then $y=t/4$ and $z=3$. Therefore, the curve C is represented by $x=t, y=t/4, z=3,\ 2\le t\le 4$.

We have $ds=\sqrt{17}/4$.

Hence, $\int_C \left(x^2+yz\right)ds=\frac{\sqrt{17}}{4}\int_2^4\left(t^2+\frac{3}{4}t\right)dt$

$$=\frac{\sqrt{17}}{4}\left[\frac{1}{3}t^3+\frac{3}{8}t^2\right]_2^4$$

$$=\frac{139\sqrt{17}}{24}.$$

Line integral of vector fields

Let C be a smooth curve whose parametric representation is as given above equations. Let

$$\boldsymbol{V}(x,y,z)=v_1(x,y,z)\boldsymbol{i}+v_2(x,y,z)\boldsymbol{j}+v_3(x,y,z)\boldsymbol{k}$$

be a vector field that is continuous on C. Then, the line integral of $\boldsymbol{V}$ over C is defined by

$$\int_C \boldsymbol{V}.d\boldsymbol{r} = \int_C v_1 dx + v_2 dy + v_3 dz$$

$$= \int_a^b \boldsymbol{V}[x(t), y(t), z(t)].\frac{d\boldsymbol{r}}{dt}dt.$$

If $\boldsymbol{V} = v_1(x, y, z)\boldsymbol{i}$, then above equation reduces to

$$\int_C \boldsymbol{V}.d\boldsymbol{r} = \int_C v_1 dx = \int_a^b v_1\,[x(t), y(t), z(t)].\frac{dx}{dt}dt$$

Similarly, if $\boldsymbol{V} = v_2(x, y, z)\boldsymbol{j}$ or $\boldsymbol{V} = v_3(x, y, z)\boldsymbol{k}$, we respectively obtain

$$\int_C \boldsymbol{V}.d\boldsymbol{r} = \int_C v_2 dy = \int_a^b v_2\,[x(t), y(t), z(t)].\frac{dy}{dt}dt$$

and $\int_C \boldsymbol{V}.d\boldsymbol{r} = \int_C v_3 dz = \int_a^b v_3\,[x(t), y(t), z(t)].\frac{dz}{dt}dt$

If the curve C is piecewise smooth containing the arcs $C_1, C_2, \dots, C_n$, then we write

$$\int_C \boldsymbol{V}.d\boldsymbol{r} = \int_{C_1} \boldsymbol{V}.d\boldsymbol{r} + \int_{C_2} \boldsymbol{V}.d\boldsymbol{r} + \dots + \int_{C_n} \boldsymbol{V}.d\boldsymbol{r}$$

Example: Evaluate the line integral of $\boldsymbol{V} = x^2\boldsymbol{i} - 2y\boldsymbol{j} + z^2\boldsymbol{k}$

Over the straight line path from $(-1,2,3)$ to $(2,3,5)$.

Solution:

The parametric representation of the straight line is given by $\boldsymbol{r}(t) = (-\boldsymbol{i} + 2\boldsymbol{j} + 3\boldsymbol{k}) + t(3\boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k})$

$$= (-1 + 3t)\boldsymbol{i} + (2 + t)\boldsymbol{j} + (3 + 2t)\boldsymbol{k},\ 0 \le t \le 1.$$

[If $\boldsymbol{a}$ and $\boldsymbol{b}$ are the two points, then the parametric representation of the line joining them is $\boldsymbol{r} = \boldsymbol{a} + t(\boldsymbol{b} - \boldsymbol{a})$]. Therefore, $\frac{d\boldsymbol{r}}{dt} = 3\boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k}$ and

$$\int_C \boldsymbol{V}.d\boldsymbol{r} = \int_C \boldsymbol{V}.\frac{d\boldsymbol{r}}{dt}dt = \int_0^1 [3(-1+3t)^2 - 2(2+t) + 2(3+2t)^2]dt$$

$$= \int_0^1 (17 + 4t + 35t^2)\, dt$$

$$= \left[17t + 2t^2 + \tfrac{35}{3}t^3\right]_0^1 = \tfrac{92}{3}.$$

Line integral of scalar fields

Let C be a smooth curve whose parametric representation is as given in above equations. Let $\mathrm{f}(x,y,z)$, $g(x,y,z)$ and $h(x,y,z)$ be scalar fields which are continuous at points over C. Then, we define a line integral as

$$\int_C f(x,y,z)dx + g(x,y,z)dz + h(x,y,z)dz$$
$$= \int_a^b \left[f\big(x(t),y(t),z(t)\big)\frac{dx}{dt} + g\big(x(t),y(t),z(t)\big)\frac{dy}{dt}\right.$$
$$\left. + h\big(x(t),y(t),z(t)\big)\frac{dz}{dt}\right]dt$$

This line integral does not contain any vector field, but involves three scalar fields. However, if we define $\boldsymbol{V} = f(x,y,z)\boldsymbol{i} + g(x,y,z)\boldsymbol{j} + h(x,y,z)\boldsymbol{k}$, and

$d\boldsymbol{r} = \boldsymbol{i}dx + \boldsymbol{j}dy + \boldsymbol{k}dz$, then the line integral is same as the line integral

If C is a closed curve, then we usually write $\int_C \boldsymbol{V}.d\boldsymbol{r} = \oint_C \boldsymbol{V}.d\boldsymbol{r}$.

Example: Evaluate $\int_C (x-y)dx - x^2dy + (y+z)dz$ Where C is $x^2 = 4y, z = x, 0 \leq x \leq 2$.

Solution: We parametrise C as $x = t$, $y = {t^2}/_4$, $z = t$, $0 \leq t \leq 2$. Therefore,

$$\int_C (x+y)dx - x^2dy + (y+z)dz = \int_0^2 \left[\left(t + \frac{t^2}{4}\right) - t^2\left(\frac{t}{2}\right) + \left(\frac{t^2}{4} + t\right)\right] dt$$

$$= \int_0^2 \left(2t + \frac{t^2}{2} - \frac{t^3}{2}\right) dt$$

$$= \left(t^2 + \frac{t^2}{6} - \frac{t^3}{8}\right)_0^2 = \frac{10}{3}.$$

Application of line integrals

Mass of a string

If the mass per unit length of the string C is $f(x,y)$, then the total mass of the string is the line integral of $f(x,y)$ over C with respect to arc length of the string s.

i.e., mass $= \int_C f(x,y)ds$.

Work Done by a Force

Let $\boldsymbol{V}(x,y,z) = v_1(x,y,z)\boldsymbol{i} + v_2(x,y,z)\boldsymbol{j} + v_3(x,y,z)\boldsymbol{k}$ be a vector function defined and continuous at every point on C. Then, the integral of the tangential component of $\boldsymbol{V}$ along the curve C from a point P to the point Q is given by

$$\int_Q^P \boldsymbol{V}.\,d\boldsymbol{r} = \int_{C^*} \boldsymbol{V}.\,d\boldsymbol{r} = \int_{C^*} v_{\boldsymbol{1}}dx + v_2\,dy + v_3dz.$$

Where C^* is the part of C, whose initial and terminal points are P and Q.

Let now $\boldsymbol{V} = \boldsymbol{F}$, a variable force acting on a particle which moves along a curve C. Then, the work W done by the force $\boldsymbol{F}$ in displacing the particle from the point P to the point Q along the curve C is given by

$$W = \int_P^Q \boldsymbol{F}.d\boldsymbol{r} = \int_{C^*} \boldsymbol{F}.d\boldsymbol{r},$$

where C^* is the part of C, whose initial and terminal points are P and Q.

Example: Find the work done by the force $\boldsymbol{F} = -xy\,\boldsymbol{i} + y^2\boldsymbol{j} + z\boldsymbol{k}$ in moving a particle over the circular path $x^2 + y^2 = 4, z = 0$ from $(2,0,0)$ to $(0,2,0)$.

Solution: The parametric representation of the given curve is $x = 2\cos t$, $y = 2\sin t$, $z = 0$, $0 \le t \le \pi/2$. Therefore, work done W is given by

$$W = \int_C \boldsymbol{F}.d\boldsymbol{r} = \int_C -xy\,dx + y^2 dy + z\,dz$$

$$= \int_0^{\frac{\pi}{2}} [-4\sin t\cos t\,(-2\sin t) + 4\sin^2 t\,(2\cos t)]\,dt$$

$$= 16\int_0^{\frac{\pi}{2}} \sin^2 t\cos t\,dt$$

$$= 16\left[\frac{1}{3}\sin^3 t\right]_0^{\frac{\pi}{2}} = \frac{16}{3}.$$

Line Integrals independent of the path

We have seen that the value of $\int_c \boldsymbol{F}.d\boldsymbol{r}$ or $\int_c f\,dx + g\,dy + h\,dz$ depends not only on the end points P and Q of the curve C but also on the path of C. We shall now discuss the conditions under which the line integral is independent of the path of integration, that is, it depends only on the end points P and Q of the curve C.

Let $\emptyset(x, y, z)$ be a differentiable scalar function. The differential of $\emptyset(x, y, z)$ is defined by

$$d\emptyset = \frac{\partial\emptyset}{\partial x}dx + \frac{\partial\emptyset}{\partial y}dy + \frac{\partial\emptyset}{\partial z}dz = (\text{grad}\emptyset).\,d\boldsymbol{r}$$

Therefore, a differential expression

$f(x, y, z)dx + g(x, y, z)dy + h(x, y, z)dz$ is an exact differential if there exists a scalar function $\emptyset(x, y, z)$ such that

$$d\emptyset = f(x, y, z)dx + g(x, y, z)dy + h(x, y, z)dz$$

We now present the result (with out proof) on the independence of the path of a line integral.

Theorem: Let C be a curve in a simply connected domain D in space. Let f, g and h be continuous functions having continuous first partial derivatives in D. Then $\int_C f\,dx + g\,dy + h\,dz$ is independent of path C if and only if the integrand is an exact differential in D.

We now state the conditions for testing the path independence.

Theorem: Let C be a curve in a simply connected domain D in space. Let f, g and h be continous functions having continuous first partial derivatives in D. Then $\int_C f\,dx + g\,dy + h\,dz$ is independent of path C if and only if

$$\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}, \frac{\partial f}{\partial z} = \frac{\partial h}{\partial x}, \text{ and } \frac{\partial g}{\partial z} = \frac{\partial h}{\partial y}.$$

Remark

If we define $\boldsymbol{F} = f\boldsymbol{i} + g\boldsymbol{j} + h\boldsymbol{k}$ then we can write

$$\int_C f dx + g dy + h dz = \int_C \boldsymbol{F} \,.\, d\boldsymbol{r}.$$

If the line integral is path independent, then $\boldsymbol{F} = \text{grad}(\emptyset)$. Hence, curl $(\boldsymbol{F}) = \text{curl}(\text{grad}\,\emptyset) = 0$. We say that the given vector field $\boldsymbol{F}$ is a gradient field and the function $\emptyset$ is called the potential function for $\boldsymbol{F}$. Therefore, in a gradient force field, the work done by force $\boldsymbol{F}$ in moving a particle from a position P to a position Q is independent of the path of integration, that is, it is same for all paths. Such a force field is also called a conservative field, that is

Total energy =Kinetic energy+potential energy =constant

Remark

In two dimensions, the conditions for testing the path independence of $\int_C f dx + g\,dy$ reduce to $\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}$.

Example: Show that $\int_C (yz - 1)dx + (z + xy + z^2)dy + (y + xy + 2yz)dz$ is independent of the path of integration from $(1,2,2)$ to $(2,3,4)$. Evaluate the integral.

Solution: We have $f(x, y, z) = yz - 1$, $g(x, y, z) = z + xz + z^2$ and $h(x, y, z) = y + xy + 2yz$

Now, $\frac{\partial f}{\partial y} = z = \frac{\partial g}{\partial x}, \frac{\partial f}{\partial z} = y = \frac{\partial h}{\partial x}$, and $\frac{\partial g}{\partial z} = 1 + x + 2z = \frac{\partial h}{\partial y}$

The integral is independent of path of integration. Also, the integrand is an exact differential. Therefore, there exists a function $\phi(x, y, z)$ such that

$\frac{\partial \phi}{\partial x} = yz - 1$, $\frac{\partial \phi}{\partial y} = z + xz + z^2$, and $\frac{\partial \phi}{\partial z} = y + xy + 2yz$

Integrating the first equation with respect to x, we get

$$\phi(x, y, z) = xyz - x + h(y, z).$$

Substituting in the second equation, we get

$$\frac{\partial \phi}{\partial y} = z + xz + z^2 = xz + \frac{\partial h}{\partial y}(y, z), \text{ or } \frac{\partial h}{\partial y} = z + z^2.$$

Integrating, we get

$h(y, z) = yz + yz^2 + s(z)$, and $\phi(x, y, z) = xyz - x + yz + yz^2 + s(z)$

Substituting in the third equation, we get

$\frac{\partial \phi}{\partial z} = y + xy + 2yz + \frac{ds}{dz} = y + xy + 2yz$, or $\frac{ds}{dz} = 0$, or $s = k$, constant.

Therefore, $\phi(x, y, z) = xyz - x + yz + yz^2 + k$.

The value of the integral is

$$\int_c (yz - 1)dx + (z + xz + z^2)dy + (y + xy + 2yz)dz$$

$$= \int_{(-1,2)}^{(2,3)} d(xyz - x + yz + yz^2)$$

$$= [xyz - x + yz + yz^2]_{(1,2,2)}^{(2,3,4)} = 82 - 15 = 67.$$

Conservative Field

A vector $\boldsymbol{F}$ is called a conservative vector field if $\boldsymbol{F}$ can be written as $\boldsymbol{F} = \text{grad} f$, where f is a scalar potential (field). Then, the work done

$$W = \int_{C^*} \boldsymbol{F}.d\boldsymbol{r} = \int_{C^*} (\text{ grad } f).d\boldsymbol{r}$$

$$= \int_{C^*} \left(\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz \right) = \int_P^Q df = [f(x,y,z)]_P^Q.$$

Therefore, work done depends only the initial and terminal points of the curve C^*, that is the work done is independent of the path of integration. The units of work depend on the units $|\boldsymbol{F}|$ and on the units of distance.

Remark

The necessary and sufficient condition that a field $\boldsymbol{F}$ be conservative is that $\text{curl}\boldsymbol{F} = \boldsymbol{\nabla} \times \boldsymbol{F} = \boldsymbol{0}$.

Remark

If $\boldsymbol{F}$ is a conservative force field, the work done a long any simple closed path is zero.

Problem 1: Suppose $\boldsymbol{A} = (3x^2 + 6y)\boldsymbol{i} - 14yz\boldsymbol{j} + 20xz^2\boldsymbol{k}$. Evaluate $\int_C \boldsymbol{A}.d\boldsymbol{r}$ from $(0,0,0)$to $(1,1,1)$ along the following paths C:

(a) $x = t,\ y = t^2,\ z = t^3$.

(b) The straight lines form $(0,0,0)$to $(1,0,0)$, then to $(1,1,0)$, and then to $(1,1,1)$.

(c) The straight line joining $(0,0,0)$ and $(1,1,1)$.

Solution:

$$\int_C \boldsymbol{A}.d\boldsymbol{r} = \int_C [(3x^2 + 6y)\boldsymbol{i} - 14yz\boldsymbol{j} + 20xz^2\boldsymbol{k}].(dx\boldsymbol{i} + dy\boldsymbol{j} + dz\boldsymbol{k})$$

$$= \int_C (3x^2 + 6y)dx - 14yzdy + 20xz^2dz.$$

(a) If $x = t,\ y = t^2,\ z = t^3$, points $(0,0,0)$ and $(1,1,1)$ correspond to $t = 0$ and $t = 1$, respectively. Then

$$\int_C \boldsymbol{A}.d\boldsymbol{r} = \int_{t=0}^{1} (3t^2 + 6t^2)dt - 14(t^2)(t^3)d(t^3) + 20(t)(t^3)^2 d(t^3)$$

$$= \int_{t=0}^{1} 9t^2\,dt - 28t^6\,dt + 60t^9\,dt$$

$$= \int_{t=0}^{1} (9t^2 - 28t^6 + 60t^9)dt = 3t^3 - 4t^7 + 6t^{10}\,|_0^1 = 5.$$

Another method:

Along C, $\boldsymbol{A} = 9t^2\boldsymbol{i} - 14t^5\boldsymbol{j} + 20t^7\boldsymbol{k}$ and $\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k} = t\boldsymbol{i} + t^2\boldsymbol{j} + t^3\boldsymbol{k}$ and $d\boldsymbol{r} = (\boldsymbol{i} + 2t\boldsymbol{j} + 3t^2\boldsymbol{k})dt$.

$$\therefore \quad \int_C \boldsymbol{A}.d\boldsymbol{r} = \int_{t=0}^{1} (9t^2\boldsymbol{i} - 14t^5\boldsymbol{j} + 20t^7\boldsymbol{k}).(\boldsymbol{i} + 2t\boldsymbol{j} + 3t^2\boldsymbol{k})dt$$

$$= \int_0^1 (9t^2 - 28t^6 + 60t^9)\,dt = 5.$$

(b) Along the straight line from $(0,0,0)$ to $(1,0,0)$, $y=0$, $=0$, $dy=0, dz=0$ while x varies from 0 to 1. Then the integral over this part of the path is

$$\int_{x=0}^{1}(3x^2+6(0))dx-14(0)(0)(0)+20x(0)^2(0)=\int_{x=0}^{1}3x^2\ dx$$

$$=x^3|_0^1=1. \qquad \text{..................}(1)$$

Along the straight line from $(1,0,0)$to $(1,1,0)$, $x=1$, $z=0, dx=0, dz=0$ while y varies from 0 to 1. Then the integral over this part of the path is

$$\int_{y=0}^{1}(3(1)^2+6y)0-14y(0)dy+20(1)(0)^2 0=. \qquad \text{..................}(2)$$

Along the straight line from $(1,1,0)$to $(1,1,1)$, $x=1$, $y=1$, $dx=0$, $dy=0$ while z varies from 0 to 1. Then the integral over this part of the path is

$$\int_{z=0}^{1}(3(1)^2+6(1))0-14(1)z(0)+20(1)z^2dz$$

$$=\int_{z=0}^{1}20z^2dz=\left.\frac{20z^3}{3}\right|_0^1=\frac{20}{3}. \qquad \text{...............}(3)$$

Adding (1),(2) and (3), then

$$\int_C \boldsymbol{A}.\,d\boldsymbol{r}=1+0+\frac{20}{3}=\frac{23}{3}.$$

(c) The straight line joining $(0,0,0)$ and $(1,1,1)$ is given in parametric from by $x=t, y=t,\ z=t$. Then

$$\int_C \boldsymbol{A}.\,d\boldsymbol{r}=\int_{t=0}^{1}(3\,t^2+6t)dt-14(t)(t)dt+20(t)(t)^2dt$$

$$=\int_{t=0}^{t}(3\,t^2+6t-14t^2+20t^3)dt$$

$$=\int_{t=0}^{1}(6t-11t^2+20t^3)dt=\frac{13}{3}.$$

Problem 2: Find the total work done in moving a particle in the force field given by $\boldsymbol{F} = z\boldsymbol{i} + z\boldsymbol{j} + x\boldsymbol{k}$ along the helix C given by $x = \cos t$, $y = \sin t$, $z = t$ from $t = 0$ to $t = \pi/2$.

Solution:

Total work $= \int_C \boldsymbol{F}.d\boldsymbol{r} = \int_C (z\boldsymbol{i} + z\boldsymbol{j} + x\boldsymbol{k}).(dx\boldsymbol{i} + dy\boldsymbol{j} + dz\boldsymbol{k})$

$$= \int_C zdx + zdy + xdz$$

$$= \int_0^{\frac{\pi}{2}} (t\ \ d(\cos t) + t\, d(\sin t) + \cos t\ \ dt)$$

$$= \int_0^{\pi/2} (-t \sin t\ t)dt + \int_0^{\pi/2} (t+1)\cos t\ \ dt$$

Evaluating $\int_0^{\pi/2} (-t \sin t\ t)dt$ by parts we get

$$[t \cos t]_0^{\pi/2} - \int_0^{\pi/2} \cos t\, dt = 0 - [\sin r]_0^{\pi/2} = -1.$$

Evaluating $\int_0^{\pi/2} (t+1)\cos t\, dt$ by parts we get

$$[(t+1)\sin t]_0^{\pi/2} - \int_0^{\pi/2} \sin t\, dt = \frac{\pi}{2} + 1 + [\cos t]_0^{\pi/2} = \frac{\pi}{2}.$$

Thus the total work is $(\pi\ /\ 2) - 1$.

Problem 3: Suppose a force field is given by

$$\boldsymbol{F} = (2x - y + z)\boldsymbol{i} + (x + y - z^2)\boldsymbol{j} + (3x - 2y + 4z)\boldsymbol{k}$$

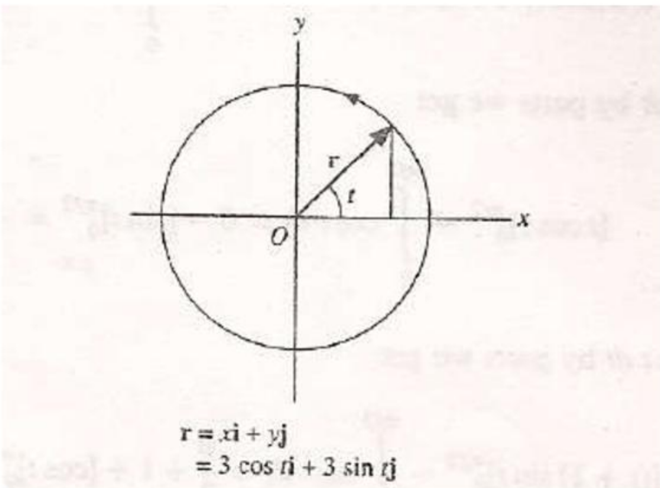
Find the work done in moving a particle once around a circle C in the XY-plane with its centre at the origin and a radius of 3.

Solution:

In the plane$z = 0$, $\boldsymbol{F} = (2x - y)\boldsymbol{i} + (x + y)\boldsymbol{j} + (3x - 2y)\boldsymbol{k}$ and $d\boldsymbol{r} = dx\boldsymbol{i} + dy\boldsymbol{j}$ so that the work done is

$$\int_C \boldsymbol{F}.d\boldsymbol{r} = \int_C [(2x - y)\boldsymbol{i} + (x + y)\boldsymbol{j} + (3x - 2y)\boldsymbol{k}].(dx\boldsymbol{i} + dy\boldsymbol{j})$$

$$= \int_C (2x - y)dx + (x + y)dy$$



Choose the parametric equations of the circle as $x = 3\cos y, y = 3\sin t$, where t varies from 0 to 2π (as in above figure). Then the line integral equals

$$\int_{t=0}^{2\pi} [2(3\cos t) - 3\sin t](-3\sin t)\,dt + (3\cos t + 3\sin t)(3\cos t)dt$$

$$= \int_0^{2\pi} (9 - 9\sin t\cos t)\,dt = 9t - \frac{9}{2}\sin^2 t|_0^{2\pi} = 18\pi\ .$$

In traversing C, we have chosen the counter clockwise direction indicated in the adjoining figure. We call this the positive direction, or say that C has been traversed in the positive sense. If C were traversed in the clockwise (negative) direction the value of the integral would be -18π.

Problem 4: Show that a necessary and sufficient condition that $F_1 dx + F_2 dy + F_3 dz$ be an exact differential is that $\boldsymbol{\nabla} \times \boldsymbol{F} = 0$, where $\boldsymbol{F} = F_1\boldsymbol{i} + F_2\boldsymbol{j} + F_3\boldsymbol{k}$.

Solution:

Suppose $F_1 dx + F_2 dy + F_3 dz = d\emptyset = \frac{\partial \emptyset}{\partial x} dx + \frac{\partial \emptyset}{\partial y} dy + \frac{\partial \emptyset}{\partial z} dz$

an exact differential. Then since x, y and z are independent variables,

$$F_1 = \frac{\partial \emptyset}{\partial x}, \quad F_2 = \frac{\partial \emptyset}{\partial y}, \quad F_3 = \frac{\partial \emptyset}{\partial z}$$

and so $\boldsymbol{F} = F_1\boldsymbol{i} + F_2\boldsymbol{j} + F_3\boldsymbol{k} = \left(\frac{\partial \emptyset}{\partial x}\right)\boldsymbol{i} + \left(\frac{\partial \emptyset}{\partial y}\right)\boldsymbol{j} + \left(\frac{\partial \emptyset}{\partial z}\right)\boldsymbol{k} = \nabla\emptyset$.

Thus $\nabla \times \boldsymbol{F} = \nabla \times \nabla\emptyset = 0$.

Conversely, if $\boldsymbol{\nabla} \times \boldsymbol{F} = 0$, then we have $\boldsymbol{F} = \boldsymbol{\nabla}\emptyset$ and so $\boldsymbol{F}.d\boldsymbol{r} = \boldsymbol{\nabla}\emptyset.d\boldsymbol{r}$, that is, $F_1 dx + F_2 dy + F_3 dz = d\emptyset$, an exact differential.

Problem 5: Suppose $\emptyset = 2xyz^2, \boldsymbol{F} = xy\boldsymbol{i} - z\boldsymbol{j} + x^2\boldsymbol{k}$ and C is the curve $x = t^2, y = 2t,\ z = t^3$ from $t = 0$ to $t = 1$. Evaluate the line integrals (a) $\int_C \emptyset d\boldsymbol{r}$ (b) $\int_C \boldsymbol{F} \times d\boldsymbol{r}$.

Solution:

(a) Along C, $\emptyset = 2xyz^2 = 2(t^2)(2t)(t^3)^2 = 4t^9$
$\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k} = t^2\boldsymbol{i} + 2t\boldsymbol{j} + t^3\boldsymbol{k}$, and
$d\boldsymbol{r} = (2t\boldsymbol{i} + 2\boldsymbol{j} + 3t^2\boldsymbol{k})dt.$

Then

$$\int_C \emptyset d\boldsymbol{r} = \int_{t=0}^{1} 4t^9 (2t\boldsymbol{i} + 2\boldsymbol{j} + 3t^2\boldsymbol{k})dt$$

$$= \boldsymbol{i}\int_0^1 8t^{10}\,dt + \boldsymbol{j}\int_0^1 8\,t^9\,dt + \boldsymbol{k}\int_0^1 12\,t^{11}\,dt$$

$$= \frac{8}{11}\boldsymbol{i} + \frac{4}{5}\boldsymbol{j} + \boldsymbol{k}.$$

(b) Along C, we have $\boldsymbol{F} = xy\boldsymbol{i} - z\boldsymbol{j} + x^2\boldsymbol{k} = 2t^3\boldsymbol{i} - t^3\boldsymbol{j} + t^4\boldsymbol{k}$

Then $\boldsymbol{F} \times d\boldsymbol{r} = (2t^3\boldsymbol{i} - t^3\boldsymbol{j} + t^4\boldsymbol{k}) \times (2t\boldsymbol{i} - t^3\boldsymbol{j} + t^4\boldsymbol{k})dt$

$$= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 2t^3 & -t^3 & t^4 \\ 2t & 2 & 3t^2 \end{vmatrix} dt$$

$$= [(-3t^5 - 2t^4)\boldsymbol{i} + (2t^5 - 6t^5)\boldsymbol{j} + (4t^3 + 2t^4)\boldsymbol{k}]dt$$

and

$$\int_C \boldsymbol{F} \times d\boldsymbol{r} = \boldsymbol{i}\int_0^1 (-3t^5 - 2t^4)dt + \boldsymbol{j}\int_0^1 (-4t^5)dt + \boldsymbol{k}\int_0^1 (4t^3 + 2t^4)dt$$

$$= -\frac{9}{10}\boldsymbol{i} - \frac{2}{3}\boldsymbol{j} + \frac{7}{5}\boldsymbol{k}.$$

Exercise

1. Find the mass of the string C defined by $x = 3\cos t, y = 3\sin t,\ 0 \le t \le \pi / 2$, where the density function of C is $x^2 y$.
2. Evaluate the line integral of $\boldsymbol{V} = xy\boldsymbol{i} + y^2\boldsymbol{j} + e^z\boldsymbol{k}$ over the curve C whose parametric representation is given by $x = t^2, y = 2t, z = t, 0 \le t \le 1$.
3. Show that $\int_C \frac{xdx + ydy}{\sqrt{x^2+y^2}}$ is independent of any path of integration which does not pass through the origin. Find the value of the integral from the point $P(-1,2)$ to the point $Q(2,3)$.
4. Suppose $\boldsymbol{F} = -3x^2\boldsymbol{i} + 5xy\boldsymbol{j}$. Evaluate $\int_C \boldsymbol{F}.d\boldsymbol{r}$, where C is the curve in the XY-plane, $y = 2x^2$, form $(0,0)$ to $(1,2)$.
5. Show that in an irrotational field, the value of a line integral between two points A and B will be independent of the path of integration.
6. Find the work done by the force $\boldsymbol{F} = (3x^2 - 6yz)\boldsymbol{i} + (2y + 3xz)\boldsymbol{j} + (1 - 4xyz^2)\boldsymbol{k}$ in moving particle from the point (0,0,0) to the point (1,1,1) along the curve $C: x = t, y = t^2, z = t^3$.
7. If $\boldsymbol{F} = (4xy - 3x^2z^2)\boldsymbol{i} + 2x^2\boldsymbol{j} - 2x^3z\boldsymbol{k}$, prove that $\int_c \boldsymbol{F}.d\boldsymbol{r}$ is independent of the curve joining two points
8. If $\boldsymbol{F} = (x^2 + y^2)\boldsymbol{i} - 2xy\boldsymbol{j}$ evaluate $\oint_c \boldsymbol{F}.d\boldsymbol{r}$ where curve C is the rectangle in XY-plane bounded by $y = 0, y = b, x = 0, x = a$
9. Compute the line integral $\int(y^2dx - x^2dy)$ round the triangle whose vertices are $(1,0), (0,1), (-1,0)$ in the XY-plane.

10. If $\boldsymbol{F} = 2y\boldsymbol{i} - z\boldsymbol{j} + x\boldsymbol{k}$, evaluate $\int_c \boldsymbol{F} \times d\boldsymbol{r}$ along the curve $x = \cos t, y = \sin t, z = 2\cos t$ from $t = 0$ to $t = \pi/2$.

11. Prove that force given by $\boldsymbol{F} = 2xyz^3\boldsymbol{i} + x^2z^3\boldsymbol{j} + 3x^2yz^2\boldsymbol{k}$ is conservative .find the work done by moving a particle from $(1,-1,2)$ to $(3,2,-1)$ in this force field.

Answers

1. 27
2. $\frac{37}{15} + e$
3. $\sqrt{13} - \sqrt{5}$
4. 7
6. 2
8. $-2ab^2$
9. $-\frac{2}{3}$
10. $\boldsymbol{i}\left(2 - \frac{\pi}{4}\right) + \boldsymbol{j}\left(\pi - \frac{1}{2}\right)$
11. -10