2.8

Computing Integrals Dependent on a Parameter

Consider an integral dependent on the parameter α:

$$I(\alpha) = \int_a^b f(x,\alpha)dx$$

We state without proof that if a function $f(x,\alpha)$ is continuous with respect to x over an interval $[a,b]$ and with respect to α over an interval $[\alpha_1,\alpha_2]$, then the function

$$I(\alpha) = \int_a^b f(x,\alpha)dx$$

is a continuous function on $[\alpha_1,\alpha_2]$. Consequently, the function $I(\alpha)$ may be integrated with respect to α over the interval $[\alpha_1,\alpha_2]$

$$\int_{\alpha_1}^{\alpha_2} I(\alpha)\, d\alpha = \int_{\alpha_1}^{\alpha_2} \left[\int_a^b f(x,\alpha)dx\right] d\alpha$$

The expression on the right is an iterated integral of the function $f(x,\alpha)$ over a rectangle situated in the plane $X\alpha$. We can change the order of integration in this integral:

$$\int_{\alpha_1}^{\alpha_2} \left[\int_a^b f(x,\alpha)dx\right] d\alpha = \int_a^b \left[\int_{\alpha_1}^{\alpha_2} f(x,\alpha)d\alpha\right] dx$$

This formula shows that for integration of an integral dependent on a parameter, it is sufficient to integrate the element of integration with respect to the parameter . This formula is also useful when computing definite integral

Example: Compute the integral

$$\int_{0}^{\infty} \frac{e^{-ax} - e^{-bx}}{x} \, dx \; (a > 0, b > 0)$$

This integral is not expressible in terms of the elementary function .To evaluate it , we consider another integral that may be readily computed:

$$\int_{0}^{\infty} e^{-\alpha x} \, dx = \frac{1}{\alpha} (\alpha > 0)$$

Integrating this equation between the limits $\alpha_1 = a$ and $\alpha_2 = b$, we get

$$\int_{a}^{b} \left[\int_{0}^{\infty} e^{-\alpha x} \, dx \right] d\alpha = \int_{a}^{b} \frac{d\alpha}{\alpha} = \ln \frac{b}{a}$$

changing the order of integration in the first integral, we rewrite this equation in the following form:

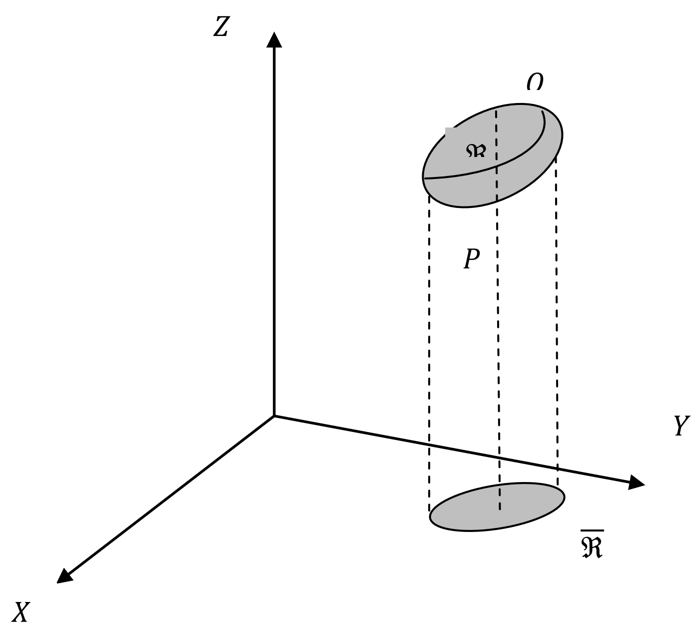
$$\int_{0}^{\infty} \left[\int_{a}^{b} e^{-\alpha x} \, d\alpha \right] dx = \ln \frac{b}{a}$$

whence, computing the inner integral, we get

$$\int_0^\infty \frac{e^{-ax} - e^{-bx}}{x}\, dx = \ln\frac{b}{a}.$$

Triple Integrals

The definition of a double integral can be extended to three-dimensions. For this purpose, let us consider a region $\Re$, in three dimensional spaces, bounded by a closed surface S whose Cartesian equation is of the form $\varphi(x,y,z) = 0$. Let $\overline{\Re}$ be the projection of $\Re$ on the $XY-$ plane. Then, as a point (x,y,z) varies over $\Re$, the corresponding point (x,y) varies over $\overline{\Re}$. Consider a line parallel to the $Z-$ axis and suppose this line cuts S at two points P and Q, P being below Q. Let z_1 and z_2 be the $Z-$ coordinates of P and Q respectively. Since P and Q lie on S, whose Cartesian equation is of the form $\varphi(x,y,z) = 0, z_1$ and z_2 are appropriate functions of $x, y; i.e. z_1 = z_1(x,y)$ and $z_2 = z_2(x,y)$.



Now, consider a function $f(x,y,z)$ defined over $\Re$ and S. On the line PQ, the coordinates (x,y) are held fixed so that $f(x,y,z)$ is a function of z only. Suppose we integrate this function with respect to z from $z_1(x,y)$ to $z_2(x,y)$. The resulting integral is a function of (x,y); let us denote it by $\psi(x,y)$. Thus,

$$\psi(x,y) = \int_{z_1(x,y)}^{z_2(x,y)} f(x,y,z)\,dz \tag{1}$$

Now, suppose we take the double integral of $\psi(x,y)$ over the plane region $\overline{\Re}$, and denote it by I. The integral I has following representation:

$$I = \int_a^b \left\{\int_{y_1(x)}^{y_2(x)} \psi(x,y)\,dy\right\} dx \tag{2}$$

Substituting for $\psi(x,y)$ from (1) in (2), we get

$$I = \int_a^b \left\{\int_{y_1(x)}^{y_2(x)} \left[\int_{z_1(x,y)}^{z_2(x,y)} f(x,y,z)\,dz\right] dy\right\} dx \tag{3}$$

Evidently, the integration covers the whole of the region $\Re$, and the integral is called the volume integral of $f(x,y,z)$ over the region $\Re$. It is denoted by $\int_V f(x,y,z)dV$; here, V stands for the volume of $\Re$. Expression (3) shown that this volume integral is represented in terms of three repeated integrals. For this reason a volume integral is also called a Triple integral and is denoted by

$$\iiint_{\Re} f(x,y,z)dxdydz \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \ (*)$$

Thus,

$$\int_V f(x,y,z) = \iiint_{\Re} f(x,y,z)dxdydz$$

$$= \int_a^b \int_{y_1(x)}^{y_2(x)} \int_{z_1(x,y)}^{z_2(x,y)} f(x,y,z)\, dzdydx \qquad (4)$$

Here, it is understood that, in the right hand side, the first integral sign is with respect to x, the second integral sign is with respect to y and the third integral sign is with respect to z. Thus, while evaluating the repeated integrals in (4), we have to first integrate $f(x,y,z)$ with respect to z from z_1 to z_2 , keeping (x,y) fixed. Then we have to integrate the resulting function with respect to y from y_1 to y_2, keeping x fixed. Finally, we have to integrate the resulting function with respect to x from a to b . A similar procedure is adopted when the integrals with respect to x,y,z appear in other orders.

Change of Variables

While evaluating triple integrals computational work can often be reduced by changing the variables x,y,z to some other appropriate variables u,v,w which are related to x,y,z and which are such that the Jacobian

$$J \equiv \frac{\partial(x,y,z)}{\partial(u,v,w)} \neq 0.$$

It can be proved that (we omit the proof)

$$\iiint_{\Re} f(x,y,z) = \iiint_{\bar{\Re}} \phi(u,v,w) Jdudvdw \qquad (1)$$

Here, $\mathfrak{R}$ is the region in which (x, y, z) vary and $\bar{\mathfrak{R}}$ is the corresponding region in which (u, v, w) vary, and

$$\psi(u, v, w) = f\{x\,(u, v, w), y(u, v, w), z(u, v, w)\}$$

Once the triple integral with respect to (x, y, z) is changed to a triple integral with respect to (u, v, w) by using formula (1), the latter integral may be evaluated by expressing it in terms of repeated integrals with appropriate limits of integration.

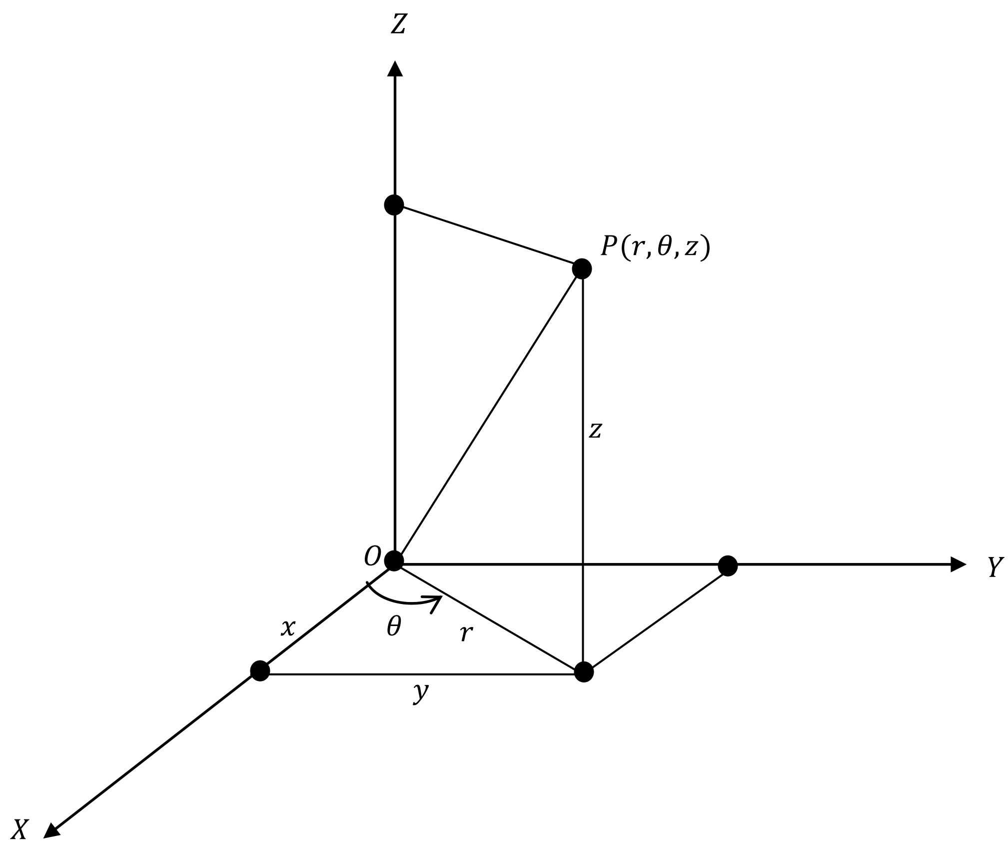
Two standard examples of (u, v, w) which are widely used in the evaluation of triple integrals are the cylindrical polar coordinates (r, θ, z) and the spherical polar coordinates (r, θ, ϕ). Let us specialize the expression (1) for these two cases.

Triple Integral in Cylindrical Polar Coordinates

Suppose (x, y, z) are related to three variables (r, θ, z) through the relations

$$x = r\cos\theta, \quad y = r\,\sin\theta\,, \quad z = z \tag{2}$$

Then (r, θ, z) are called cylindrical polar coordinates. These coordinates are depicted in following figure.



From expressions (2), we find that

$$J = \frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{vmatrix} = \begin{vmatrix} \cos\theta & -R\ \sin\theta & 0 \\ \sin\theta & R\cos\theta & 0 \\ 1 & 0 & 0 \end{vmatrix} = r \qquad (3)$$

Hence, for $(u, v, w) = (r, \theta, z)$, expression (1) becomes

$$\iiint_{\Re} f(x, y, z) = \iiint_{\Re} \varphi(r, \theta, z) r dr d\theta dz \qquad (4)$$

Here $\overline{\Re}$ is the region in which (r, θ, z) vary as (x, y, z) vary in $\Re$, and $\varphi(r, \theta, z) = f(x, y, z)$ with x, y, z given by (2). Observe that when (x, y, z) are changed to (r, θ, z) and $dxdydz$ changes to $rdrd\theta dz$.

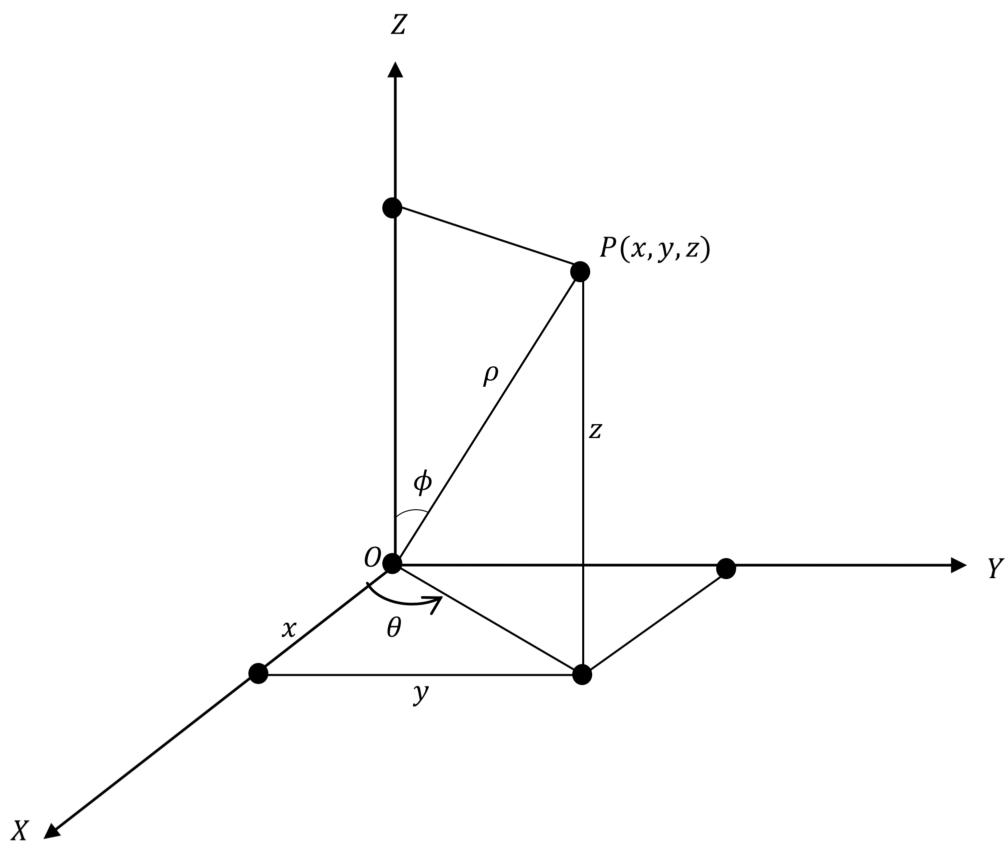
The formula (4) is particularly useful when the region $\Re$ is bounded by a cylindrical surface.

Triple Integral in Spherical Polar Coordinates

Suppose (x, y, z) are related to three variables $(\rho, \theta, \emptyset)$ through the relations

$$x = \rho\ \sin\emptyset \cos\theta,\ \ y = \rho\ \sin\emptyset \sin\theta,\ \ z = \rho\ \cos\emptyset \tag{5}$$

Then $(\rho, \theta, \emptyset)$ are called spherical polar coordinates. These coordinates are depicted in following figure.



From expressions (5), we find that

$$J = \frac{\partial(x,y,z)}{\partial(\rho,\theta,\emptyset)} = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \emptyset} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \emptyset} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \emptyset} \end{vmatrix}$$

$$= \begin{vmatrix} \sin\emptyset\cos\theta & -\rho\ \sin\phi\sin\theta & \rho\ \cos\ \theta\cos\ \emptyset \\ \sin\theta\sin\emptyset & \rho\ \cos\theta\sin\emptyset & \rho\ \sin\theta\cos\emptyset \\ \cos\phi & 0 & -\rho\ \sin\phi \end{vmatrix}$$

$$= \rho^2 \sin\phi. \tag{6}$$

Hence, for $(u, v, w) = (\rho, \theta, \emptyset)$, expression (1) becomes

$$\iiint_{\Re} f(x,y,z)dxdydz = \iiint_{\bar{\Re}} \phi(r,\theta,\phi)\rho^2 \sin\phi d\rho d\theta dz \tag{7}$$

Here $\bar{\Re}$ is the region in which $(\rho, \theta, \emptyset)$ vary as (x, y, z) vary in $\Re$, and $\varphi(\rho, \theta, \emptyset) = f(x, y, z)$ wirh x, y, z given by (5). Observe that when (x, y, z) are changed to $(\rho, \theta, \emptyset)$ and $dx\,dy\,dz$ changes to $\rho^2 \sin\phi\ d\rho d\theta d\emptyset$.

The formula (7) is particularly useful when the region $\Re$ is bounded by a spherical surface.

Computation of Volume:

Let us recall expression $(*)$. In the particular case where $f(x, y, z) \equiv 1$, this expression becomes

$$\int_V dV \equiv \iiint_{\Re} dx\,dy\,dz = \int_a^b \int_{y_1(x)}^{y_2(x)} \int_{Z_1(x,y)}^{Z_2(x,y)} dz\,dy\,dx \tag{1}$$

The integral $\int_V dV$ represents the volume V of the region $\Re$. Thus, expression (1) may be used to compute V.

If (x, y, z) are changed to (u, v, w), we obtain the following expression for the volume.

$$\int_V dV \equiv \iiint_{\Re} dx\,dy\,dz = \int_{\overline{\Re}} J\,dudvdw \tag{2}$$

Taking $(u, v, w) = (r, \theta, z)$ in the above expression, we obtain the following expression for volume in terms of cylindrical polar coordinates

$$\int_V dV \equiv \iiint_{\Re} r\,dr\,d\theta dz \tag{3}$$

Similarly, we obtain the following expression for volume in terms of spherical polar coordinates $(\rho, \theta, \emptyset)$;

$$\int_V dV \equiv \iiint_{\Re} \rho^2 \sin\phi\,d\rho\,d\theta d\emptyset \tag{4}$$

Observe that $rdrd\theta dz$ is the volume element in cylindrical polar coordinates (r, θ, z) and $\rho^2 \sin\phi\,d\rho\,d\theta\,d\emptyset$ is the volume element in spherical polar coordinates $(\rho, \theta, \emptyset)$.

Problem 1: Evaluate the following integrals:

i. $\int_0^1 \int_{y^2}^1 \int_0^{1-x} x\, dzdxdy$

ii. $\int_0^4 \int_0^{2\sqrt{z}} \int_0^{\sqrt{4z-x^2}} dydxdz$

Solution: we have

i. $$\int_0^1 \int_{y^2}^1 \int_0^{1-x} x\, dzdxdy = \int_{y=0}^1 \left\{ \int_{x=y^2}^1 \left(\int_{z=0}^{1-x} x\, dz \right) dx \right\} dy$$
$$= \int_{y=0}^1 \left\{ \int_{x=y^2}^1 x[z]_0^{1-x}\, dx \right\} dy$$
$$= \int_{y=0}^1 \left\{ \int_{x=y^2}^1 x(1-x)\, dx \right\} dy$$
$$= \int_0^1 \left\{ \left[\frac{x^2}{2} - \frac{x^3}{3} \right]_{y^2}^1 \right\} dy$$
$$= \int_0^1 \left\{ \frac{1}{2}(1-y^4) - \frac{1}{3}(1-y^6) \right\} dy$$
$$= \int_0^1 \left(\frac{1}{6} - \frac{1}{2}y^4 + \frac{1}{3}y^6 \right) dy$$
$$= \frac{1}{6} - \frac{1}{2}.\frac{1}{5} + \frac{1}{3}.\frac{1}{7} = \frac{1}{35}.$$

ii. $$\int_0^4 \int_0^{2\sqrt{z}} \int_0^{\sqrt{4z-x^2}} xdydxdz = \int_{z=0}^4 \left\{ \int_{x=0}^{2\sqrt{z}} \left(\int_{y=0}^{\sqrt{4z-x^2}} dy \right) dx \right\} dz$$
$$= \int_{z=0}^4 \left\{ \int_{x=0}^{2\sqrt{z}} \left([y]_0^{\sqrt{4z-x^2}} \right) dx \right\} dz$$
$$= \int_{z=0}^4 \left\{ \int_{x=0}^{2\sqrt{z}} \sqrt{4z-x^2}\, dx \right\} dz$$
$$= \int_0^4 \left\{ \int_0^t \left(\sqrt{t^2-x^2} \right) dx \right\} dz,$$

where $t = 2\sqrt{z}$

$$= \int_0^4 \left\{ \int_0^{\frac{\pi}{2}} (t\cos\theta)(t\cos\theta)\, d\theta \right\} dz,$$

where $x = t\sin\theta$

$$= \int_0^4 \left\{ t^2 \int_0^{\frac{\pi}{2}} \cos^2 \theta \, d\theta \right\} dz$$

$$= \int_0^4 t^2 \left(\frac{1}{2} . \frac{\pi}{2} \right) dz = \frac{\pi}{4} \int_0^4 4z \, dz,$$

Using $t = 2\sqrt{z}$.

$$= \pi . \frac{4^2}{2} = 8\pi.$$

Problem 2: Evaluate the following integrals

i. $\int_{-1}^{1}\int_{0}^{z}\int_{x-z}^{x+z}(x+y+z)\,dy\,dx\,dz$

ii. $\int_{-c}^{c}\int_{-b}^{b}\int_{-a}^{a}(x^2+y^2+z^2)\,dx\,dy\,dz$

Solution: we have

$\int_{-1}^{1}\int_{0}^{z}\int_{x-z}^{x+z}(x+y+z)\,dy\,dx\,dz$

$$= \int_{-1}^{1}\left\{\int_{0}^{z}\left[\int_{x-z}^{x+z}(x+y+z)dy\right]dx\right\}dz$$

$$= \int_{-1}^{1}\left\{\int_{0}^{z}\left[(x+z)\int_{x-z}^{x+z}dy + \int_{x-z}^{x+z}ydy\right]dx\right\}dz$$

$$= \int_{-1}^{1}\left\{\int_{0}^{z}\left[(x+z)\{(x+z)-(x-z)\} + \frac{1}{2}\{(x+z)^2 - (x-z)^2\}\right]dx\right\}dz$$

$$= \int_{-1}^{1}\left\{\int_{0}^{z}[(x+z)(2z)+2zx]dx\right\}dz = \int_{-1}^{1}\left\{\int_{0}^{z}(4zx+2z^2)dx\right\}dz$$

$$= \int_{-1}^{1}\left\{4z\int_{0}^{z}x\,dx + 2z^2\int_{0}^{z}dx\right\}dz$$

$$= \int_{-1}^{1}\{4z(z^2/2)+2z^2(z)\}\,dz = \int_{-1}^{1}4z^3\,dz = 0.$$

ii. $\int_{-c}^{c}\int_{-b}^{b}\int_{-a}^{a}(x^2+y^2+z^2)\,dxdydz$

$$= \int_{-c}^{c}\left\{\int_{-b}^{b}\left[\int_{-a}^{a}(x^2+y^2+z^2)dx\right]dy\right\}dz$$

$$= \int_{-c}^{c}\left\{\int_{-b}^{b}\left[\frac{1}{3}(2a^3)+y^2(2a)+z^2(2a)\right]dy\right\}dz$$

$$= \int_{-c}^{c} \left\{ \frac{2}{3} a^3 (2b) + 2a \left(\frac{2b^3}{3} \right) + 2az^2 (2b) \right\} dz$$
$$= \frac{4a^3 b}{3} (2c) + \frac{4ab^3}{3} (2c) + 4ab \left(\frac{2c^3}{3} \right) = \frac{8}{3} abc(a^2 + b^2 + c^2).$$

Problem 3: Evaluate the following triple integrals:

i. $\iiint_{\mathfrak{R}} (x+y+z)dxdydz$

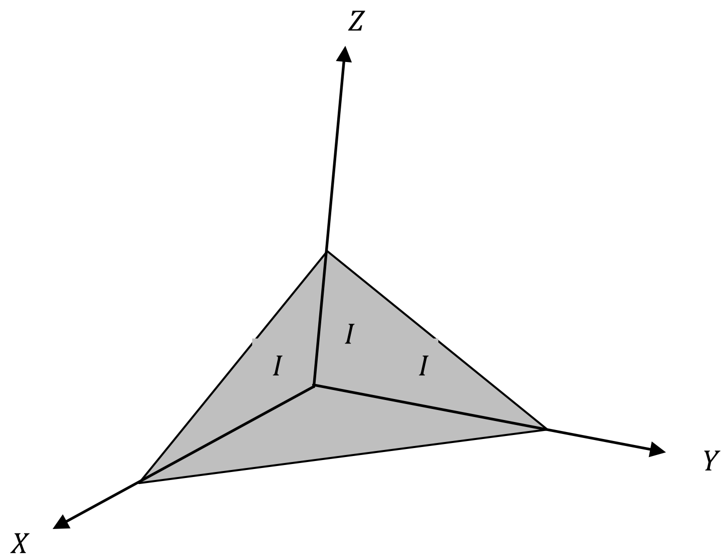
ii. $\iiint_{\mathfrak{R}} zdxdydz$

iii. $\iiint_{\mathfrak{R}} xyzdxdydz$

iv. $\iiint_{\mathfrak{R}} \frac{dxdydz}{(1+x+y+z)^3}$

Here, $\mathfrak{R}$ is the region bounded by the planes $x = 0, y = 0, z = 0$ and $x + y + z = 1$.

Solution:



The given region $\mathfrak{R}$ is shown in figure. In this region, z varies from a point in the $XY-$ plane to a point on the plane $x + y + z = 1$; that is z increases from 0 to $z = (1 - x - y)$. In the $xy-$ plane, y varies from a point on the $x-$ axis to a point on the line $x + y = 1$. That is, for $z = 0, y$ increases from 0 to $y = 1 - x$. For $y = 0, z = 0$ (that is, on the $X-$ axis), x varies from 0 to 1. Therefore:

i. $\iiint_{\Re}(x+y+z)dxdydz = \int_{x=0}^{1}\int_{y=0}^{1-x}\int_{z=0}^{1-x-y}(x+y+z)\,dz\,dy\,dx$

$$= \int_{x=0}^{1}\left\{\int_{y=0}^{1-x}\left[\int_{z=0}^{1-x-y}(x+y+z)\,dz\right]\,dy\right\}dx$$

$$= \int_{x=0}^{1}\left\{\int_{y=0}^{1-x}\left(\left[(x+y)z+\frac{z^2}{2}\right]_{z=0}^{1-x-y}\right)dy\right\}dx$$

$$= \int_{x=0}^{1}\left\{\int_{y=0}^{1-x}\left[(x+y)(1-x-y)+\frac{1}{2}(1-x-y)^2\right]dy\right\}dx$$

$$= \int_{x=0}^{1}\left\{\int_{y=0}^{1-x}\frac{1}{2}(1-x-y)(2x+2y+1-x-y)dy\right\}dx$$

$$= \int_{x=0}^{1}\left\{\int_{y=0}^{1-x}\frac{1}{2}[1-(x+y)^2]dy\right\}dx$$

$$= \frac{1}{2}\int_0^1\left\{\left[y-\frac{1}{3}(x+y)^3\right]_{y=0}^{1-x}\right\}dx = \frac{1}{2}\int_0^1\left\{(1-x)-\frac{1}{3}(1-\right.$$

$x3)dx$

$$= \frac{1}{6}\int_0^1(2-3x+x^3)dx = \frac{1}{6}\left(2-\frac{3}{2}+\frac{1}{4}\right) = \frac{1}{8}.$$

ii. $\iiint_{\Re} zdxdydz = \int_{x=0}^{1}\int_{y=0}^{1-x}\int_{z=0}^{1-x-y} z\,dz\,dy\,dx$

$$= \int_{x=0}^{1}\left\{\int_{y=0}^{1-x}\left[\int_{z=0}^{1-x-y} z\,dz\right]\,dy\right\}dx$$

$$= \int_{x=0}^{1}\left\{\int_{y=0}^{1-x}\frac{(1-x-y)^2}{2}dy\right\}dx$$

$$= \int_{x=0}^{1}\left\{\frac{1}{2}\left[-\frac{(1-x-y)^3}{3}\right]_{y=0}^{1-x}\right\}$$

$$= \frac{1}{6}\int_0^1(1-x)^3\,dx = -\frac{1}{6}.\left.\frac{(1-x)^4}{4}\right|_0^1 = \frac{1}{24}.$$

iii. $\iiint_{\Re} xyzdxdydz = \int_{x=0}^{1}\int_{y=0}^{1-x}\int_{z=0}^{1-x-y} xyz\;dz\,dy\,dx$

$$= \int_0^1 x\left\{\int_0^{1-x}\left[\int_0^{1-x-y} z\,dz\right]dy\right\}dx$$

$$= \int_0^1 x\left\{\int_0^{1-x} y\left[\frac{1}{2}(1-x-y)^2\right]dy\right\}dx$$

$$= \frac{1}{2}\int_0^1 x\left\{\int_0^{1-x}[(1-x)^2 y - 2(1-x)y^2 +\right.$$
$$\left. y^3]dy\right\}dx$$

$$= \frac{1}{2}\int_0^1 x\left\{\frac{1}{2}(1-x)^2(1-x)^2 - \frac{2}{3}(1-x)(1-x)^3 +\right.$$
$$\left.\frac{1}{4}(1-x)^4\right\}dx$$

$$= \frac{1}{24}\int_0^1 x\,(1-x)^4 dx$$

$$= \frac{1}{24}\left\{-x\,\frac{(1-x)^5}{5}\Big|_0^1 + \int_0^1 \frac{(1-x)^5}{5}dx\right\}$$

$$= \frac{1}{24}\left\{-\frac{(1-x)^6}{30}\right\}_0^1 = \frac{1}{24}\left(\frac{1}{30}\right) = \frac{1}{720}.$$

iv. $\iiint\limits_{\Re} \frac{dxdydz}{(1+x+y+z)^3}$

$$= \int_{x=0}^1\left\{\int_{y=0}^{1-x}\left[\int_{z=0}^{1-x-y}(1+x+y+z)^{-3}dz\right]dy\right\}dx$$
$$= \int_{x=0}^1\left\{\int_{y=0}^{1-x}\left[\frac{(1+x+y+z)^{-2}}{-2}\right]_{z=0}^{1-x-y} dy\right\}dx$$
$$= -\frac{1}{2}\int_{x=0}^1\left\{\int_{y=0}^{1-x}[\{(1+x+y+1-x-y)^{-2}\} -\right.$$
$$1+x+y-2dydx$$

$$= -\frac{1}{2}\int_{x=0}^1\left\{\int_{y=0}^{1-x}\left[\frac{1}{2^2} - (1+x+y)^{-2}\right]dy\right\}dx$$
$$= -\frac{1}{2}\int_{x=0}^1\left\{\left[\frac{1}{4}y + \frac{1}{(1+x+y)}\right]_{y=0}^{1-x}\right\}dx$$
$$= -\frac{1}{2}\int_{x=0}^1\left\{\frac{1}{4}(1-x) + \left(\frac{1}{2} - \frac{1}{1+x}\right)\right\}dx$$

$$= -\frac{1}{2}\left[-\frac{1}{4}.\frac{(1-x)^2}{2} + \frac{1}{2}x - \log(1+x)\right]_0^1$$
$$= -\frac{1}{2}\left\{\frac{1}{8} + \frac{1}{2} - \log 2 + \log 1\right\} = \frac{1}{2}\left(\log 2 - \frac{5}{8}\right).$$

Problem 4: Evaluate $\mathrm{I} = \int_0^{\pi/2}\int_0^{a\sin\theta}\int_0^{(a^2-r^2)/a} r\,dz\,dr\,d\theta$, given in cylindrical polar coordinates (r, θ, z).

Solution: By using the meaning of repeated integrals, we find that

$$I = \int_{\theta=0}^{\frac{\pi}{2}}\left[\int_{r=0}^{a\sin\theta}\left\{\int_{z=0}^{\frac{a^2-r^2}{a}} r\,dz\right\}dr\right]d\theta$$

$$= \int_{\theta=0}^{\pi/2}\left[\int_{r=0}^{a\sin\theta} r\left\{[z]_0^{(a^2-r^2)/a}\right\}dr\right]d\theta$$

$$= \int_{\theta=0}^{\frac{\pi}{2}}\left[\int_{r=0}^{a\sin\theta}\left(\frac{a^2-r^2}{a}\right)r\,dr\right]d\theta$$

$$= \int_0^{\pi/2}\left\{\left[a\frac{r^2}{2} - \frac{1}{a}\frac{r^4}{4}\right]_0^{a\sin\theta}\right\}d\theta$$

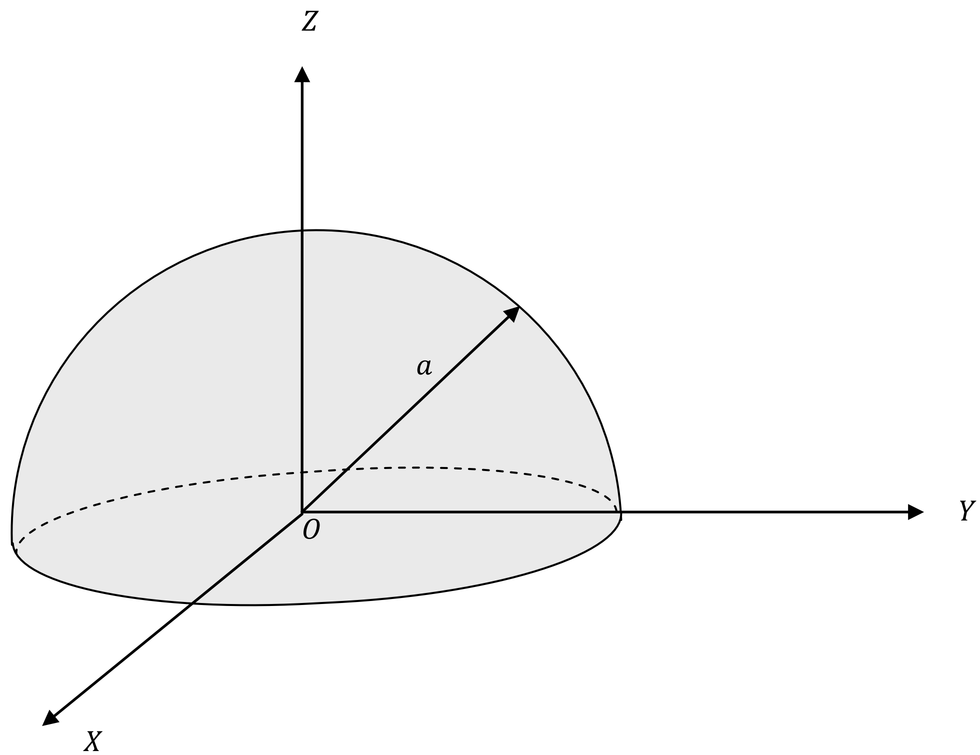
$$= \int_0^{\pi/2}\left\{\frac{a^3}{2}\sin^2\theta - \frac{a^3}{4}\sin^4\theta\right\}d\theta$$

$$= \frac{a^3}{2}\int_0^{\pi/2}\sin^2\theta\,d\theta - \frac{a^3}{4}\int_0^{\pi/2}\sin^4\theta\,d\theta$$

$$= \frac{a^3}{2}.\frac{1}{2}.\frac{\pi}{2} - \frac{a^3}{4}.\frac{3}{4}.\frac{1}{2}.\frac{\pi}{2} = \frac{5\pi}{64}a^3.$$

Problem 5: Find the value of $\iiint z\,dx\,dy\,dz$ over the hemisphere $x^2+y^2+z^2 \le a^2, z \ge 0$.

Solution:



Let $(\rho, \theta, \emptyset)$ be the spherical polar coordinates. In the given hemisphere (shown in above figure), ρ increases from 0 to a, ϕ increases from 0 to $\pi/2$, and θ increases form 0 to 2π, Therefore,

$$\iiint z\,dx\,dy\,dz = \int_{\rho=0}^{a}\int_{\phi=0}^{\pi/2}\int_{\theta=0}^{2\pi}(\rho\cos\phi)\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

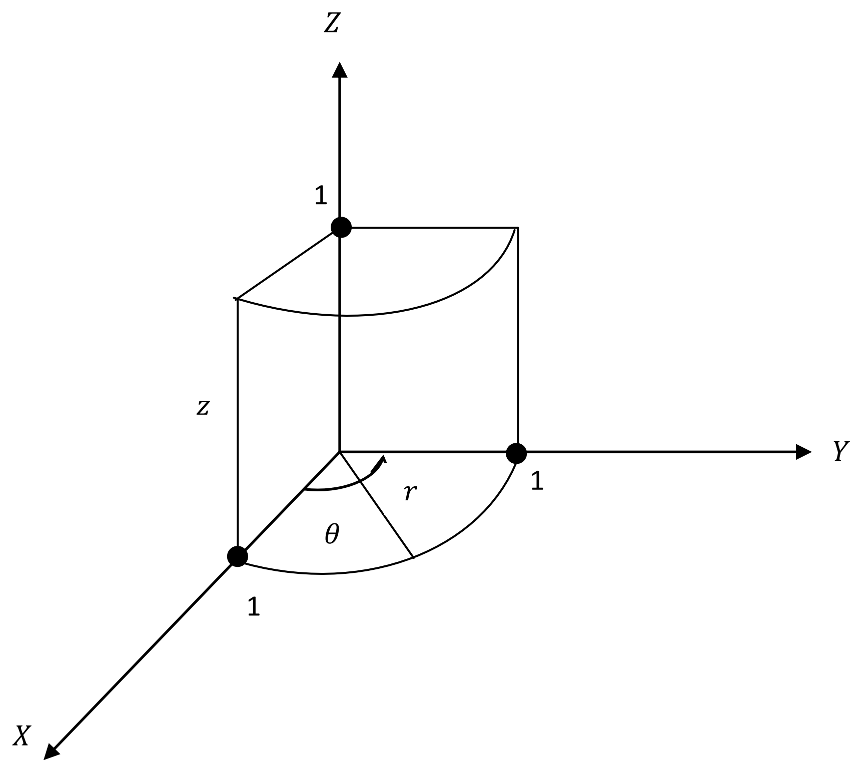
$$= \int_0^a \rho^3 d\rho \times \int_0^{\pi/2}\sin\phi\cos\phi\,d\phi \times \int_0^{2\pi} d\theta$$

$$= \frac{\pi a^4}{4}.$$

Problem 6: *If $\Re$ is the region bounded by the planes* $x = 0, y = 0, z = 1$ and the cylinder $x^2 + y^2 = 1$, evaluate the integral $\iiint_{\Re} xyzdxdydz$, by changing it to cylindrical polar coordinates.

Solution:

Let (r, θ, z) be cylindrical polar coordinates. In the given region (shown in following figure), r increases form 0 to $1, \theta$ increases form 0 to $\pi/2$ and z increases form 0 to 1. Therefore,



$$\iiint_{\Re} xyzdxdydz = \int_{r=0}^{1} \int_{\theta=0}^{\pi/2} \int_{z=0}^{1} (r \cos\theta)\,(r \sin\theta)\, z\,(r\, dr d\theta dz)$$

$$= \int_0^1 r^3 dr \times \int_0^{\frac{\pi}{2}} \sin\theta \cos\theta \, d\theta \times \int_0^1 z\, dz$$

$$= \frac{1}{4} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{16}.$$

Problem 7: Using triple integrals find the volume bounded by the surface $z^2 = a^2 - x^2$ and the planes $x = 0, y = 0$ and $y = b$.

Solution: Here, z varies from $z = 0$ to $z = a^2 - x^2$. For $z = 0$, x varies form 0 to a, and y varies from 0 to b.Therefore, the required volume is

$$V = \int_{x=0}^{a} \int_{y=0}^{b} \int_{z=0}^{a^2-x^2} dz\, dy\, dx$$

$$= \int_{x=0}^{x} \left[\int_{y=0}^{b} \left\{ \int_{z=0}^{a^2-x^2} dz \right\} dy \right] dx$$

$$= \int_0^a \left[\int_0^b (a^2 - x^2) dy \right] dx$$

$$= \int_0^a (a^2 - x^2) \{ [y]_0^b \} dx$$

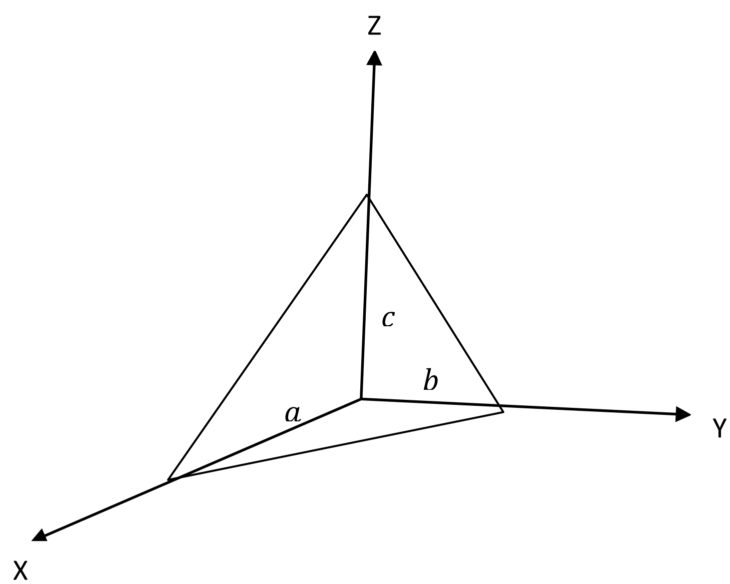
$$= \int_0^a b(a^2 - x^2) dx$$

$$= b \left[a^2 x - \frac{x^3}{3} \right]_0^a$$

$$= b \left(a^3 - \frac{a^3}{3} \right) = \frac{2}{3} b a^3.$$

Problem 8: Using triple integrals, find the volume of the tetrahedron bounded by the planes $x = 0, y = 0, z = 0$ and $(x/a) + (y/b) + (z/c) = 1$.

Solution:



The given tetrahedron is shown in above figure. In this tetrahedron, z varies from 0 to $c(1 - x/a - y/b)$. For $z = 0$, y varies from 0 to $b(1 - x/a)$. for $y = 0, z = 0$, x varies from 0 to a.Therefore, the required volume is

$$V = \int_{x=0}^{a} \int_{y=0}^{b\left(1-\frac{x}{a}\right)} \int_{z=0}^{c\left(1-\frac{x}{a}-\frac{y}{b}\right)} dz\, dy\, dx$$

$$= \int_{x=0}^{a} \left[\int_{y=0}^{b\left(1-\frac{x}{a}\right)} \left\{ c\left(1 - \frac{x}{a} - \frac{y}{b}\right) \right\} dy \right] dx$$

$$= c \int_{x=0}^{a} \left\{ \left[\left(1 - \frac{x}{a}\right) y - \frac{y^2}{2b} \right]_0^{b(1-x/a)} \right\} dx$$

$$= c \int_{x=0}^{a} \left\{ b\left(1 - \frac{x}{a}\right)^2 - \frac{1}{2b} . b^2 \left(1 - \frac{x}{a}\right)^2 \right\} dx$$

$$= \frac{cb}{2} \int_{x=0}^{a} \left(1 - \frac{x}{a}\right)^2 dx$$

$$= \frac{cb}{2} \left\{ \left[\frac{\left(1 - \frac{x}{a}\right)^3}{3\left(-\frac{1}{a}\right)} \right]_0^a \right\} = \frac{1}{6} abc.$$

Exercise

1. Compute the integral $\int_0^1 \frac{x^a - x^b}{\log x} dx \quad a > b > -1.$
2. Compute the integral $\int_0^\infty \left(\frac{e^{-ax} - e^{-bx}}{x}\right) \sin x \, dx$.
3. Evaluate the integral $I = \iiint\limits_{\mathfrak{R}} xy dx dy dz$ where $\mathfrak{R}$ is the positive octant of the sphere $x^2 + y^2 + z^2 = a^2$.
4. If $\mathfrak{R}$ is the region in the first octant bounded by the sphere $x^2 + y^2 + z^2 = a^2$, evaluate the integral $\iiint\limits_{\mathfrak{R}} (x + y + z) dx dy dz$ by changing it to spherical polar coordinates.
5. Using triple integrals, find the volume of the ellipsoid

$$\left(\frac{x^2}{a^2}\right) + \left(\frac{y^2}{b^2}\right) + \left(\frac{z^2}{c^2}\right) = 1.$$

6. Using triple integrals, find the volume of the sphere $x^2 + y^2 + z^2 = a^2$.
7. Compute $\iiint \frac{dx\ dy\ dz}{(x+y+z+1)^3}$ if the domain of integration is bounded by the coordinate planes and the plane $x + y + z = 1$.
8. Evaluate $\int_0^a \left[\int_0^x \left(\int_0^y xyz \ dz\right) dy\right] dx$.
9. Evaluate $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \frac{dx\ dy\ dz}{\sqrt{1-x^2-y^2-z^2}}$ by changing to spherical polar co-ordinates.
10. Evaluate $\iiint (x^2 + y^2)\, dx\, dy\, dz$ taken over the volume bounded by the XY-plane and the parabolaid $z = 9 - x^2 - y^2$ by using cylindrical polar coordinates.

11. Find the volume common to the cylinders $x^2 + y^2 = a^2$ and $x^2 + z^2 = a^2$ by triple integral.

12. Find the volume bounded by the XY-plane, the cylinder $x^2 + y^2 = 1$ and the plane $x + y + z = 3$ by triple integral.

Answers

1. $\log(\frac{b+1}{a+1})$
2. $\tan^{-1} b - \tan^{-1} a$
3. $\frac{1}{15} a^5$
4. $\frac{3\pi a^4}{16}$
5. $\frac{4\pi}{3} abc$
6. $\frac{4\pi}{3} a^3$
7. $\frac{\ln 2}{2} - \frac{5}{16}$
8. $\frac{a^6}{48}$.
9. $\frac{\pi^2}{8}$.
10. $\frac{243\pi}{2}$.
11. $\frac{16a^3}{3}$ Cubic units.
12. 3π Cubic units.