STABILITY OF CONTROL SYSTEMS

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1) Concept of stability
2) Routh Hurwitz criterion
3) Root locus

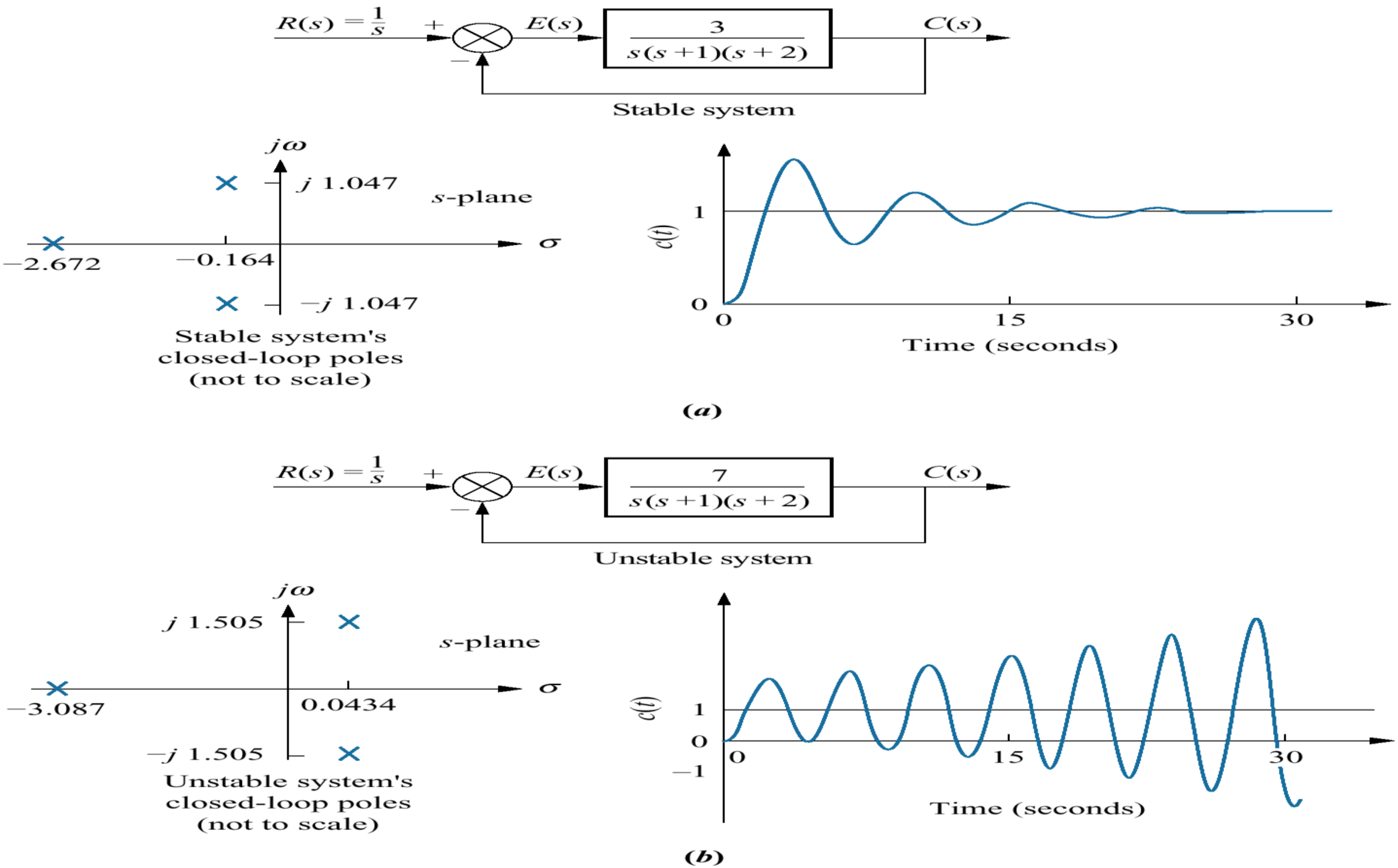
Concept of Stability:

- Not desirable that **a small change in the input**, **initial condition or parameters of the system produces a very large change in the response** of the system. If the response increases indefinitely with time, the system is said to be unstable.
- The roots of the characteristic equation D(S)=0 (i.e poles of the system) classified as

  1) Real roots
  2) Complex conjugate roots
  3) Roots at origin
  4) Purely imaginary roots

- The system is stable if the roots are negative, or have negative real parts if they are complex i.e Left Half of S-plane. (response is bounded)
- The system is unstable if the roots are positive, or have positive real parts if they are complex i.e Right Half of S-plane. (response is unbounded)
- The system is marginally or limited stable if the roots are purely imaginary.

Concept of Stability:



Necessary conditions for Stability:

- The characteristic polynomial is

$$D(s) = a_0 s^n + a_1 s^{n-1} + \text{L} + a_{n-1} s + a_n = 0$$

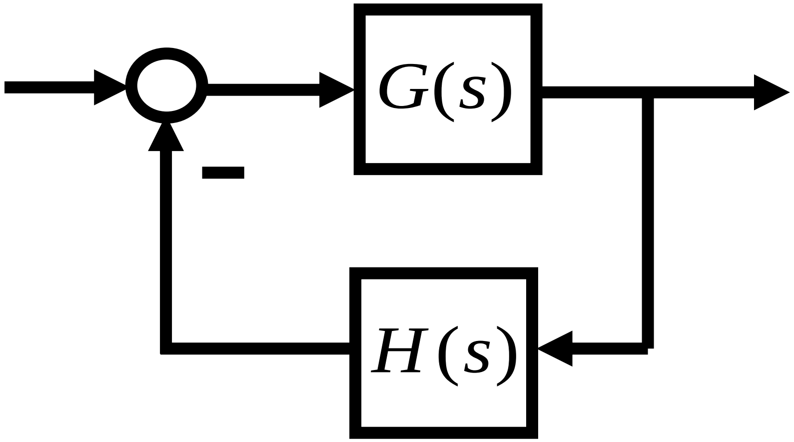
1. The characteristic polynomial must have all coefficients real and positive
2. None of the coefficients of the polynomial should be zero except the constant a_n
3. None of the powers of S between the highest and lowest powers of S should be missing. However, all odd powers of S or all even powers of S may be missing.

✓ These are only necessary conditions but not sufficient.

✓ All stable systems must satisfy these conditions but systems satisfying all these conditions need not be stable.

Routh Hurwitz stability criterion

Consider the following typical closed-loop system:



$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}$$

which can be written as

$$\frac{C(s)}{R(s)} = \frac{b_0 s^m + b_1 s^{m-1} + \cdots + b_m}{s^n + a_1 s^{n-1} + \cdots + a_n} = \frac{N(s)}{D(s)}$$

where a_i's and b_i's are constants and $m \leq n$.

Routh's stability criterion enables us to determine the number of closed-loop poles that lie in the right-half plane without having to factor the denominator polynomial.

1). Write the polynomial in *s* in the following form:

$$D(s) = a_0 s^n + a_1 s^{n-1} + \mathrm{L} \ + a_{n-1} s + a_n = 0$$

where we assume that $a_n \neq 0$; that is, any zero root has been removed.

2) Ascertain that all the coefficients are positive (or negative). Otherwise, the system is unstable. If we are interested in only the stability, there is no need to follow the procedure further.

3) If all coefficients are positive, arrange the coefficients of the polynomial in rows and columns according to the following pattern (called *Routh Array*):

$$D(s) = a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n = 0$$

Routh Array:

$$
\begin{array}{llllll}
s^n & a_0 & a_2 & a_4 & a_6 & \cdots \\
s^{n-1} & a_1 & a_3 & a_5 & a_7 & \cdots \\
s^{n-2} & b_1 & b_2 & b_3 & b_4 & \cdots \\
s^{n-3} & c_1 & c_2 & c_3 & c_4 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \cdots \\
s^0 & g_1 = a_n & & & &
\end{array}
$$

$$D(s) = a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n = 0$$

Routh Array:

$$\begin{array}{cccccc}
s^n & a_0 & a_2 & a_4 & a_6 & \cdots \\
s^{n-1} & a_1 & a_3 & a_5 & a_7 & \cdots \\
s^{n-2} & b_1 & b_2 & b_3 & b_4 & \cdots \\
s^{n-3} & c_1 & c_2 & c_3 & c_4 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \cdots \\
s^0 & g_1 = a_n & & & &
\end{array}$$

$$b_1 = \frac{a_1 a_2 - a_0 a_3}{a_1}$$

$$D(s) = a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n = 0$$

Routh Array:

$$\begin{array}{cccccc}
s^n & a_0 & a_2 & a_4 & a_6 & \cdots \\
s^{n-1} & a_1 & a_3 & a_5 & a_7 & \cdots \\
s^{n-2} & b_1 & b_2 & b_3 & b_4 & \cdots \\
s^{n-3} & c_1 & c_2 & c_3 & c_4 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \cdots \\
s^0 & g_1 = a_n & & & &
\end{array}$$

$$b_2 = \frac{a_1 a_4 - a_0 a_5}{a_1}$$

$$D(s) = a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n = 0$$

Routh Array:

$$\begin{array}{llllll}
s^n & a_0 & a_2 & a_4 & a_6 & \cdots \\
s^{n-1} & a_1 & a_3 & a_5 & a_7 & \cdots \\
s^{n-2} & b_1 & b_2 & b_3 & b_4 & \cdots \\
s^{n-3} & c_1 & c_2 & c_3 & c_4 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \cdots \\
s^0 & g_1 = a_n & & & &
\end{array}$$

$$b_3 = \frac{a_1 a_6 - a_0 a_7}{a_1}$$

The evaluation of b_i's is continue until the remaining ones are all zero.

The same pattern of cross-multiplying the coefficients of the two previous rows is followed in evaluating c_i's, d_i's and so on:

Routh Array:

$$
\begin{array}{llllll}
s^{n} & a_0 & a_2 & a_4 & a_6 & \mathrm{L} \\
s^{n-1} & a_1 & a_3 & a_5 & a_7 & \mathrm{L} \\
s^{n-2} & b_1 & b_2 & b_3 & b_4 & \mathrm{L} \\
s^{n-3} & c_1 & c_2 & c_3 & c_4 & \mathrm{L} \\
\mathrm{M} & \mathrm{M} & \mathrm{M} & \mathrm{M} & \mathrm{M} & \mathrm{L} \\
s^{0} & c_{1,n+1} = a_n & & & &
\end{array}
$$

$$c_1 = \frac{b_1 a_3 - a_1 b_2}{b_1}$$

$$D(s) = a_0 s^n + a_1 s^{n-1} + \mathrm{L} + a_{n-1} s + a_n = 0$$

Routh Array:

$$\begin{array}{llllll}
s^n & a_0 & a_2 & a_4 & a_6 & \mathrm{L} \\
s^{n-1} & a_1 & a_3 & a_5 & a_7 & \mathrm{L} \\
s^{n-2} & b_1 & b_2 & b_3 & b_4 & \mathrm{L} \\
s^{n-3} & c_1 & c_2 & c_3 & c_4 & \mathrm{L} \\
\mathrm{M} & \mathrm{M} & \mathrm{M} & \mathrm{M} & \mathrm{M} & \mathrm{L} \\
s^0 & g_1 = a_n & & & &
\end{array}$$

$$c_2 = \frac{b_1 a_5 - a_1 b_3}{b_1}$$

$$D(s) = a_0 s^n + a_1 s^{n-1} + \mathrm{L} + a_{n-1} s + a_n = 0$$

Routh Array:

$$\begin{array}{cccccc}
s^n & a_0 & a_2 & a_4 & a_6 & \mathrm{L} \\
s^{n-1} & a_1 & a_3 & a_5 & a_7 & \mathrm{L} \\
s^{n-2} & b_1 & b_2 & b_3 & b_4 & \mathrm{L} \\
s^{n-3} & c_1 & c_2 & c_3 & c_4 & \mathrm{L} \\
\mathrm{M} & \mathrm{M} & \mathrm{M} & \mathrm{M} & \mathrm{M} & \mathrm{L} \\
s^0 & g_1 = a_n & & & &
\end{array}$$

$$c_3 = \frac{b_1 a_7 - a_1 b_4}{b_1}$$

This process is continued until the *n*th row has been completed:

Routh Array:

$$
\begin{array}{c|c|ccccc}
s^{n} & a_0 & a_2 & a_4 & a_6 & \mathrm{L} \\
s^{n-1} & a_1 & a_3 & a_5 & a_7 & \mathrm{L} \\
s^{n-2} & b_1 & b_2 & b_3 & b_4 & \mathrm{L} \\
s^{n-3} & c_1 & c_2 & c_3 & c_4 & \mathrm{L} \\
\mathrm{M} & \mathrm{M} & \mathrm{M} & \mathrm{M} & \mathrm{M} & \\
s^{2} & e_1 & e_2 & & & \mathrm{L} \\
s^{1} & f_1 & & & & \\
s^{0} & g_1 = a_n & & & &
\end{array}
$$

Routh's stability criterion

1. The system is stable if and only if (1) all the coefficients of $D(s)$ are positive and (2) all the terms in the first column of the array have positive signs.
2. The number of roots of the D(s) with *positive* real parts is equal to the number of changes in sign of the coefficients of the first column of the array.

Example. Consider the following polynomial

$$s^4 + 2s^3 + 3s^2 + 4s + 5 = 0$$

Determine the number of unstable roots.

Example. The characteristic polynomial of a second-order system is

$$D(s) = a_0 s^2 + a_1 s + a_2$$

The Routh array is written as

$$\begin{array}{lll} s^2 & a_0 & a_2 \\ s^1 & a_1 & \\ s^0 & a_2 & \end{array}$$

Therefore, the requirement for a stable second-order system is simply that all the coefficients be positive or all the coefficients be negative.

Example. The characteristic polynomial of a third-order system is

$$D(s) = a_0 s^3 + a_1 s^2 + a_2 s + a_3$$

The Routh array is

$$\begin{array}{lll} s^3 & a_0 & a_2 \\ s^2 & a_1 & a_3 \\ s^1 & b_1 & \\ s^0 & c_1 & \end{array}$$

where

$$b_1 = \frac{a_1 a_2 - a_0 a_3}{a_1} \qquad c_1 = \frac{b_1 a_3}{b_1} = a_3$$

For third-order system to be stable, it is necessary and sufficient that the coefficients be positive and

$$a_1 a_2 - a_0 a_3 > 0$$

Case 1: There is a zero in the first column, but some other elements of *the row* containing the zero in the first column are nonzero.

In this case, the zero is replaced by a very small positive number ε and the rest of the array is evaluated then.

Example.

$$D(s) = s^5 + 2s^4 + 2s^3 + 4s^2 + 11s + 10$$

$$\begin{array}{llll}
s^5 & 1 & 2 & 11 \\
s^4 & 2 & 4 & 10 \\
s^3 & 0 \approx e & 6 & \\
s^2 & c_1 & 10 & \\
s & d_1 & & \\
s^0 & 10 & &
\end{array}$$

where $e \to 0$

$$c_1 = \frac{4e - 12}{e} < 0$$

$$d_1 = \frac{6c - 10e}{c_1} \to 6 > 0$$

There are two changes in sign, that is, two poles lie in the right-half s-plane.

Case 2: There is a zero in the first column, and the other elements of the row containing the zero are also zero.

For example,

$$D(s) = s^6 + s^5 - 2s^4 - 3s^3 - 7s^2 - 4s - 4 = 0$$

$$\begin{array}{lllll}
s^6 & 1 & -2 & -7 & -4 \\
s^5 & 1 & -3 & -4 & \\
s^4 & 1 & -3 & -4 & \\
s^3 & 0 & 0 & &
\end{array}$$

To solve the problem, from s^4 row we obtain an auxiliary polynomial:

$$P(s) = s^4 - 3s^2 - 4$$

which indicates that there are two roots on the $j\omega$ axis and the system is in the marginal stability case. Then, consider

$$dP(s)/ds = 4s^3 - 6s$$

Let the term in s^3 row be replaced by $4s^3 - 6s$. The array becomes

$$\begin{array}{lllll}
s^6 & 1 & -2 & -7 & -4 \\
s^5 & 1 & -3 & -4 & \\
s^4 & 1 & -3 & -4 & \\
s^3 & 4 & 6 & & \\
s^2 & -1.5 & -4 & & \\
s^1 & -16.7 & 0 & & \\
s^0 & -4 & & &
\end{array}$$

There is one change in sign of the first column of the array. Hence, only one pole lies in the right-half s-plane.

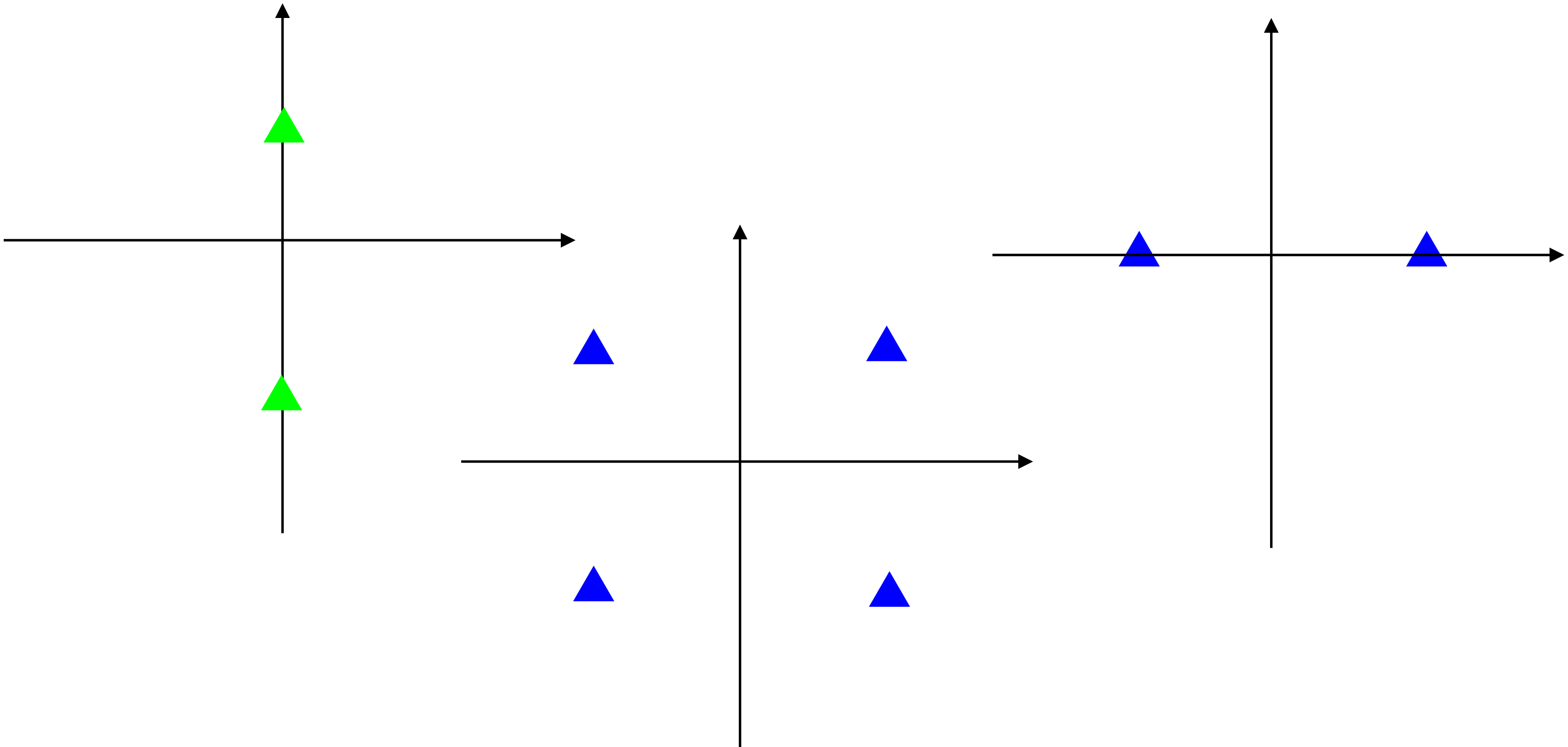
Example.

$$D(s) = s^3 + 2s^2 + s + 2$$

$$\begin{array}{lll} s^3 & 1 & 1 \\ s^2 & 2 & 2 \\ s^1 & 0 \approx e & \\ s^0 & 2 & \end{array}$$

Question: Is the system stable or unstable?
In this case, with ε>0, the signs of the first column are the same, which indicates that there is a pair of imaginary roots. The system is in the marginal stability

In general, the terms of a row are all zero indicate that there are roots of equal magnitude lying radially opposite in the *s*-plane.



In the above examples, the characteristic polynomial can be factored as:

$$(s+2)(s-2)(s+j)(s-j)(s+1+j\sqrt{3})(s+1-j\sqrt{3})/4$$

Example. Consider the following equation:

$$D(s) = s^5 + 2s^4 + 24s^3 + 48s^2 - 25s - 50 = 0$$

$$\begin{array}{llll} s^5 & 1 & 24 & -25 \\ s^4 & 2 & 48 & -50 \\ s^3 & 0 & 0 & \end{array}$$

The auxiliary polynomial is then formed from the coefficients of the second row:

$$P(s) = 2s^4 + 48s^2 - 50$$

Hence,

$$dP(s)/ds = 8s^3 + 96s$$

Let the term in s^3 row be replaced by $4s^3+96s$. The array becomes

$$\begin{array}{llll} s^5 & 1 & 24 & -25 \\ s^4 & 2 & 48 & -50 \\ s^3 & 8 & 96 & \\ s^2 & 24 & -50 & \\ s & 112.7 & & \\ s^0 & -50 & & \end{array}$$

The system has one pole in the right-half s-plane and two poles on the $j\omega$ axis. In fact,

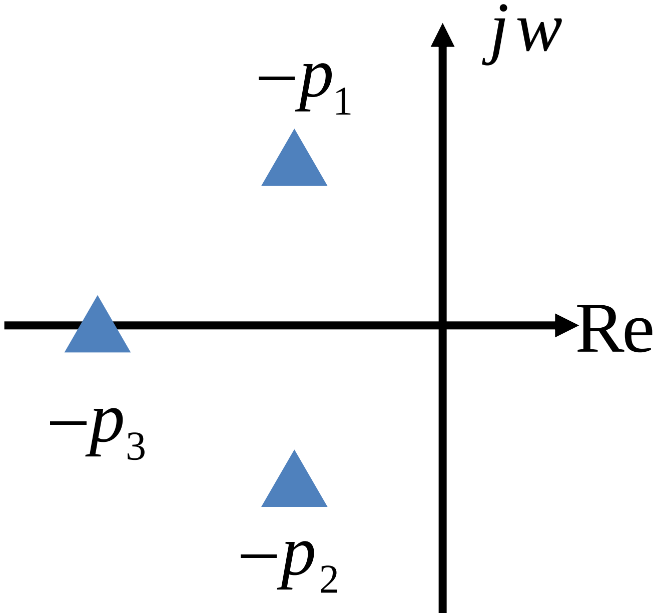
$$D(s) = (s+1)(s-1)(s+j5)(s-j5)(s+2)$$

Relative Stability Analysis

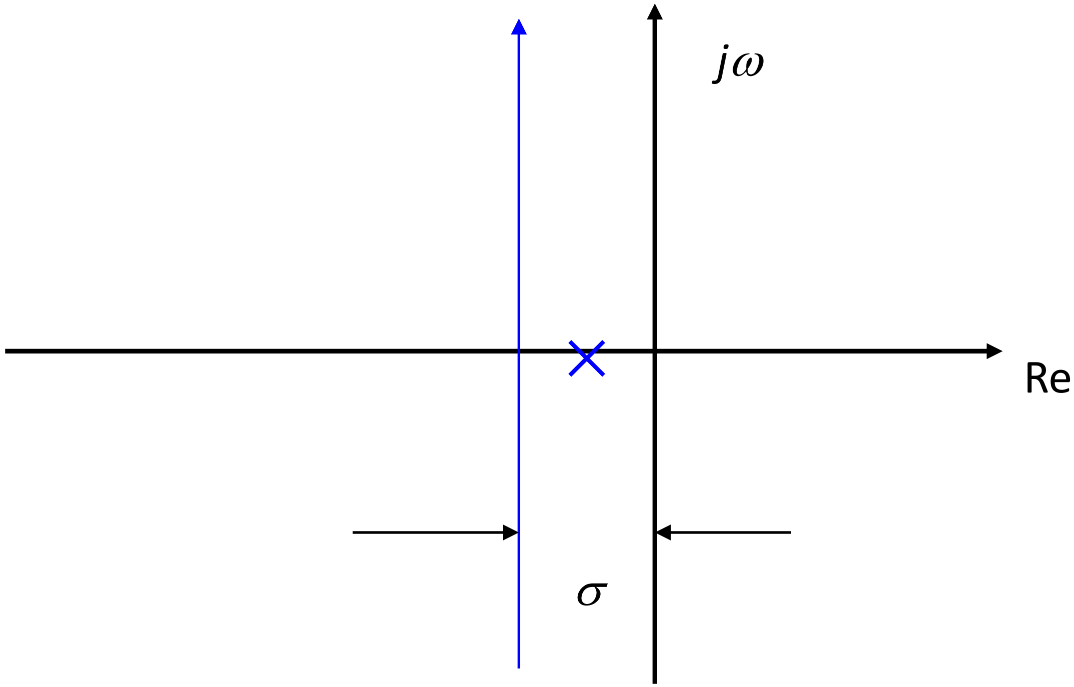
- The R-H can determine whether a system is stable or not(known as absolute stability).
- If a system has some roots with a very small negative real part i.e the roots are very near to the jw-axis, the roots may cross the threshold and enter unstable region due to some environment conditions the parameter values have changed.
- Systems with more-ve real parts of the dominant poles (poles nearest to the jw-axis) are relatively more stable than systems with less –ve real parts of the dominant poles.
- R-H gives us absolute stability of the system only. However we can modify the procedure to determine how far the dominant pole is from the jw-axis by shifting the jw-axis to the left by a small amount.

Relative Stability Analysis

The relative stability of a system can be defined as the property that is measured by the relative real part of each root or pair of roots.



The root $-p_3$ is relatively more stable than the roots $-p_1$ and $-p_2$.

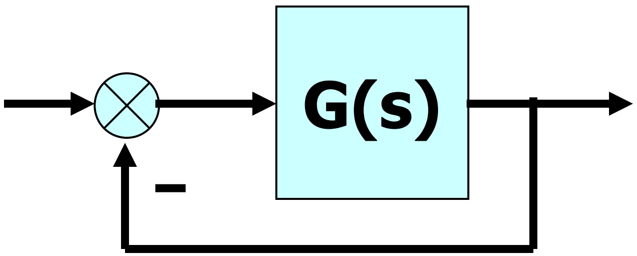


One useful approach for examining relative stability is to shift the *s*-plane axis as shown above. Then

$$s = \hat{s} - s \;\Rightarrow \hat{s} = s + s$$

where σ>0 implies that to make all the poles lie in the left-half $\hat{s}$-plane, more restrictions must be made.

Example (Axis shift). Consider a unity-feedback system:



$$G(s) = \frac{K}{s(0.1s+1)(0.25s+1)}$$

Determine the range of *K* for which the system is stable.

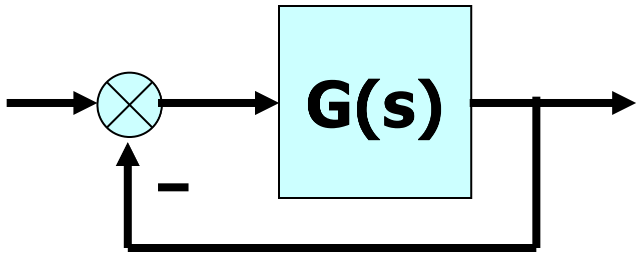
Solution: The closed-loop characteristic equation is

$$0.025s^3 + 0.35s^2 + s + K = 0$$

1) $K > 0$

2) $0.35 - 0.025K > 0 \quad \Rightarrow \quad K < 14$

The range of *K* for system to be stable is $0 < K < 14$



$$G(s) = \frac{K}{s(0.1s+1)(0.25s+1)}$$

Now, if it is required that the real part of all roots be less than –1, what is the range of *K* for system to be stable?

The closed-loop characteristic equation is

$$0.025s^3 + 0.35s^2 + s + K = 0$$

Let $s = \hat{s} - 1$, then

$$\hat{s}^3 + 11\hat{s}^2 + 15\hat{s} + (40K - 27) = 0$$

$$a_i > 0, \quad K > 0.675$$

$$11 \times 15 - (40K - 27) > 0$$

$$K < 4.8$$

The range of *K* is $0.675 < K < 4.8$

Root Locus:

- Transient response of a system plays an important in the design of a control system.
- The transient response is determined by the location of poles of the closed loop system. Usually the loop gain (K) of the system is adjustable and the value of this gain determines the location of poles of the closed loop system.
- The locus of these roots as one parameter of the system, usually the gain, is varied over a wide range (0 to ∞), is known as the root locus plot of the system.
- The adjustment of the system gain enables the designer to place the poles at the desired locations. If this is not possible a compensator or a controller has to be designed.

Properties of Root Locus:

1. The root locus is symmetrical about the real axis
2. There are n root locus branches each starting from an open loop pole for K = 0. m of these branches terminate on m open loop zeros. The remaining $n - m$ branches go to zero at infinity.
3. The $n - m$ branches going to zeros at infinity, do so along asymptotes making angles

$$\phi = \frac{(2k \pm 1)180}{n - m} \; ; \qquad k = 0, 1, 2 \ldots. (n - m - 1)$$

with the real axis.

Note : For different values of $(n - m)$ the angles of asymptotes are fixed. For example if

(i) $n - m = 1$ $\phi = 180^0$

(ii) $n - m = 2$ $\phi = 90, -90$

(iii) $n - m = 3$ $\phi = 60, 180, -60$

(iv) $n - m = 4$ $\phi = 45, 135, -135, -45$ and so on.

4. The asymptotes meet the real axis at

$$\sigma_a = \frac{\Sigma \text{ real parts of poles} - \Sigma \text{ real parts of zeros}}{n - m}$$

5. Segments of real axis are parts of root locus if the total number of real poles and zeros together to their right is odd.

6. Breakaway or Breakin points.

   These are points in s-plane where multiple closed loop poles occur. These are the roots of the equation,

$$\frac{dK}{ds} = 0$$

   Only those roots which satisfy the angle criterion also, are the breakaway or breakin points. If r root locus branches break away at a point on real axis, the breakaway directions are given by $\pm \frac{180^0}{r}$.

7. The angle of departure of the root locus at a complex pole is given by,

$$\phi_p = \pm (2k + 1)\ 180 + \phi$$

   where ϕ is the net angle contributed by all other open loop zeros and poles at this pole.

   Similarly the angle of arrival at a complex zero is given by

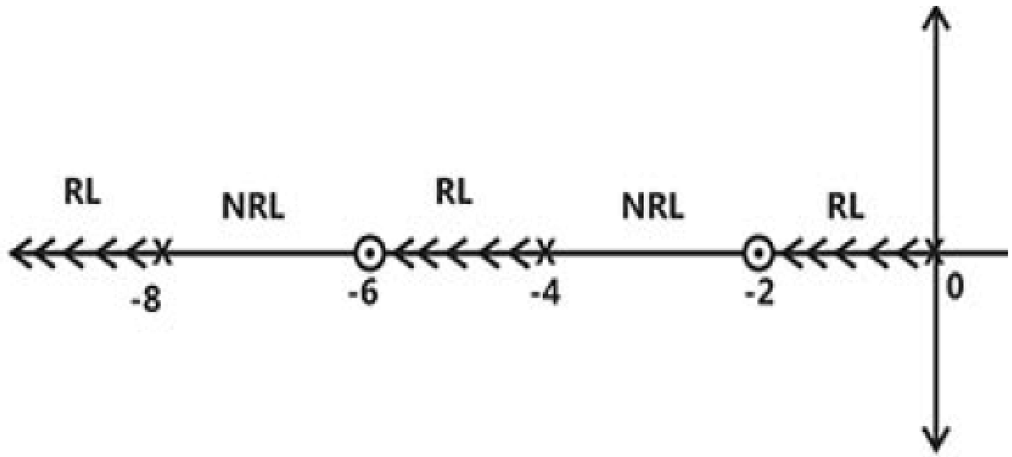
$$\phi_z = \pm (2k + 1)\ 180 - \phi$$

   where ϕ is the net angle contributed by all other open loop poles and zeros at this zero.

8. The cross over point of the root locus on the imaginary axis is obtained by using Routh Hurwitz criterion.

Part of Root Locus or Not

Ex: $G(s) = \dfrac{k(s+2)(s+6)}{s(s+4)(s+8)}$



$P = 3 \;;\; Z = 2 \;;\; P - Z = 1$

NRL – Not Root Locus; RL – Root Locus

Angle of asymptotes

$$G(s)H(s) = \frac{K(s+1)}{s(s+4)\left(s^2+2s+1\right)}$$

$$P = 4 \;;\; Z = 1 \;;\; P - Z = 3$$

$$\theta_1 = \frac{[2(0)+1]}{3} \times 180^0 = 60^0$$

$$\theta_2 = \frac{[2(1)+1]}{3} \times 180^0 = 180^0$$

$$\theta_3 = \frac{[2(2)+1]}{3} \times 180^0 = 300^0$$

Centroid

$$\frac{k(s+1)}{s(s+2)(s+3)}$$

Poles = 0, -2, -3 ; zeroes = -1

$$\text{centroid} = \frac{(0-2-3)-(-1)}{3-1} = [-2,0]$$

Break Points

They are those points where multiple roots of the characteristics equation occur.

Procedure to find Break Points:

1. Construct 1 + G(s)H(s) = 0
2. Write k in terms of s
3. Find $\frac{dk}{ds} = 0$
4. The roots of $\frac{dk}{ds} = 0$ give the locations of break-away points.
5. To test the validity of breakaway points substitute in step – 2.

If k > 0, it means a valid breakaway point.

Breakaway Points

If the break point lies between two successive poles then it is termed as a Breakaway Point.

Predictions

1) The branches of root locus either approach or leave the breakaway points at an angle of

$$\pm\frac{180^0}{n}$$

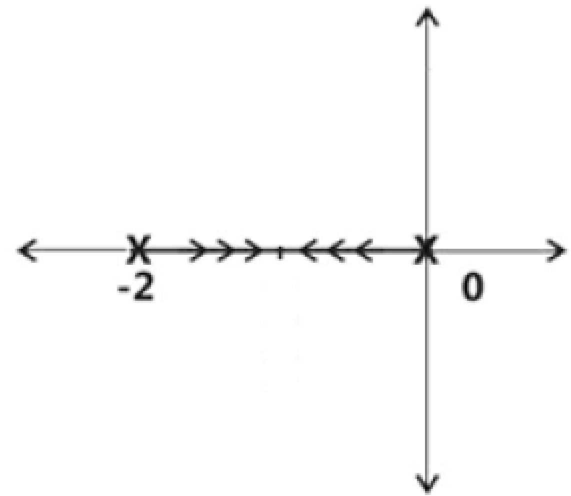
Where n = no. of branches approaching or leaving breakaway point.

2) The complex conjugate path for the branches of root locus approaching or leaving or breakaway points is a circle.

3) Whenever there are 2 adjacently placed poles on the real axis with the section of real axis between there as part of RL, then there exists a breakaway point between the adjacently placed poles.

$$G(s) = \frac{k}{s(s+2)}$$

P = 2, Z = 0, P – Z = 2



Section of real axis lying on Root Locus

$$\theta_1 = \frac{(2(0)+1)}{2} \times 180 = 90^0; \;\; \theta_2 = \left(\frac{2(1)+}{2}\right) 180^0 = 270^0$$

$$\text{centroid} = \frac{0+(-2)-0}{2} = -1$$

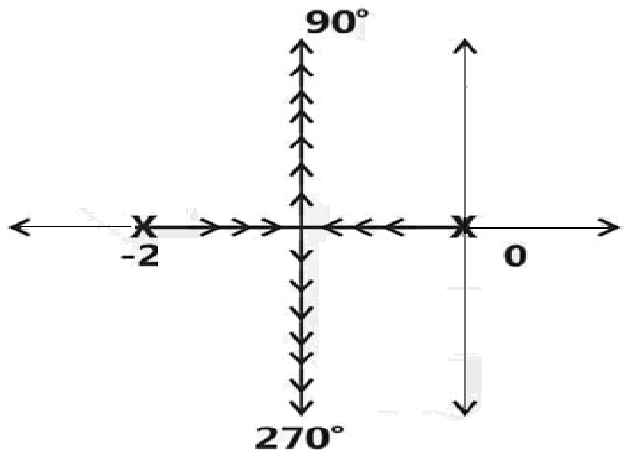
Breakaway points

$$s^2 + 2s + k = 0 \;\; => \;\; k = \left(-s^2 - 2s\right)$$

$$\frac{dk}{ds} = -2s - 2 = 0 \qquad => s = -1$$

From the third prediction about Breakaway points we know that there must be a breakaway point between s=0 and s=-2. Hence s= -1 is a valid breakaway point.

Since centroid and breakaway point coincide the branches of root locus will leave breakaway point along the asymptotes.



Break – in points

The points where two branches of roots locus converge are called as break in points. A Break-in Point lies between two successive zeroes.

To differentiate between break in & break away points.

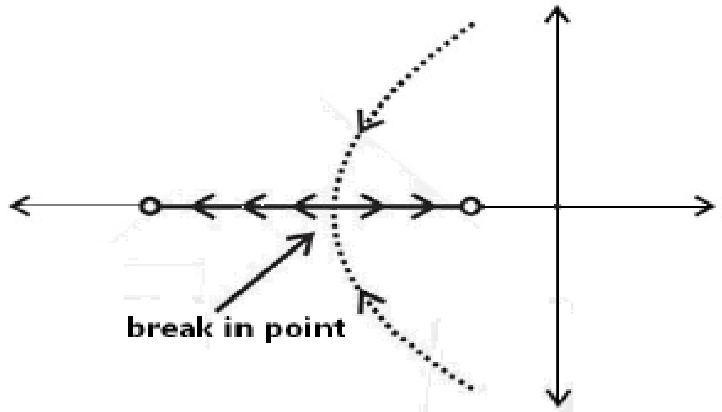
$$\frac{d^2k}{ds^2} > 0 \text{ For break in points}$$

$$\frac{d^2k}{ds^2} < 0 \text{ For break-away points}$$

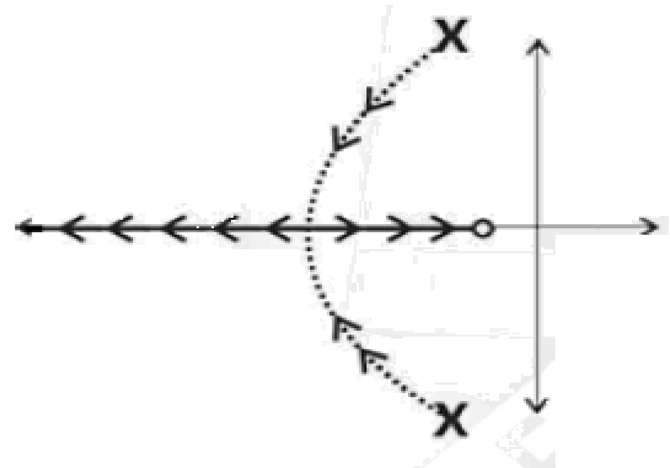
Otherwise by observation a breakaway point lies between two poles and a break-in point lies between two zeroes.

Predictions about Break-in Points:

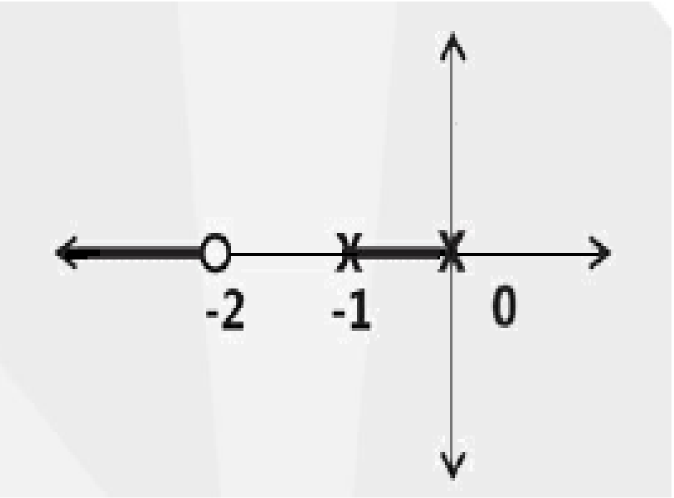
1. Whenever there are 2 zeroes on real axis & the portion of real axis between 2 zeroes lies on root locus then there is a break in points between 2 zeroes.



2. Whenever there exists a zero on real axis & real axis on left is on root locus & P > Z, then there will be a break in to left of zero.



Example-1: $G(s)=\dfrac{K(s+2)}{s(s+1)}$



$$P=2,\ Z=1,\ P-Z=1$$

Angle of Aymptotes $\theta=180^0$

$$\text{centroid}=\frac{0+(-1)-(-2)}{1}=1$$

Break away points

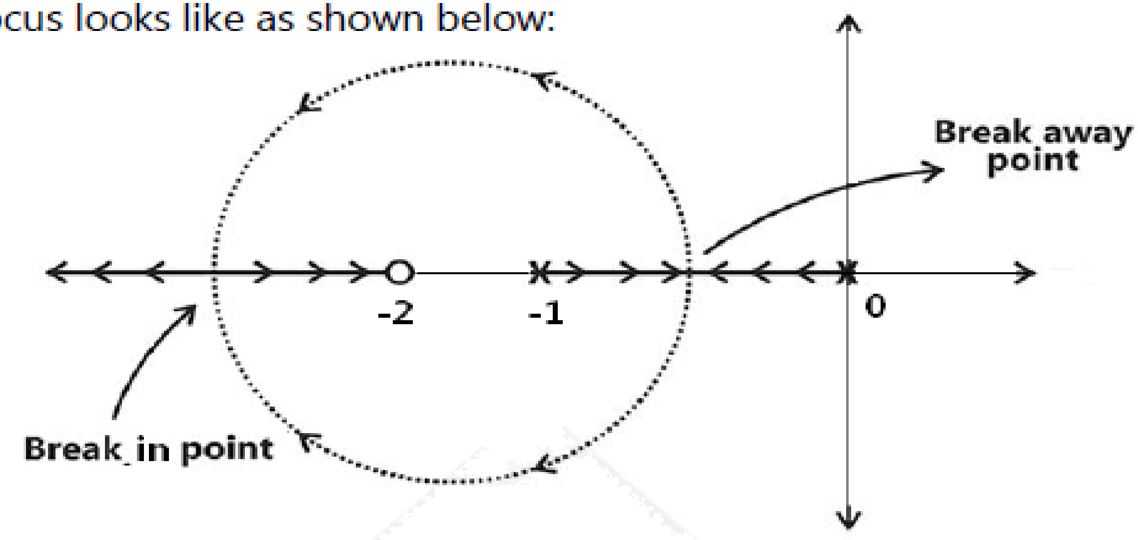
$$s(s+1)+K(s+2)=0 \Rightarrow K=\frac{-s^2-s}{(s+2)}$$

$$\frac{dk}{ds}=0=\frac{(s+2)(-2s-1)+\left(s^2+s\right)(1)}{(s+2)^2}=0$$

$$s^2+4s+2=0 \Rightarrow s=\left(-2\pm\sqrt{2}\right)=-0.6,-3.4$$

Since, s =-0.6Lie between two consecutive poles it is a Break away point and since s = -3.4 lies between a zero and infinity it is a break-in point.

So, the root locus looks like as shown below:



Intersection of Root Locus with jw (Imaginary) Axis

Roots of auxiliary equation A(s) at $k = k_{max}$ (i.e. Maximum value of Gain K for which the closed loop system is Stable) from Routh Array gives the intersection of Root locus with imaginary axis.

The value of $k = k_{max}$ is obtained by equating the coefficient of s^1 in the Routh Array to zero.

Ex-2: Find the intersection of root locus with the imaginary axis

$$G(s) = \frac{k}{s(s+2)(s+4)}$$

Characteristic equation of the system is

$$s^3 + 6s^2 + 8s + k = 0$$

Routh Array:

$$\begin{array}{lll} s^3 & 1 & 8 \\ s^2 & 6 & K \\ s^1 & \dfrac{48-k}{6} & \\ s^0 & K & \end{array}$$

For stability $\frac{48-k}{6} > 0 = k < 48 \ \& \ k > 0$

$\therefore$ range $0 < k < 48$

At $k = k_{max} = 48$

$$A(s) = 6s^2 + k = 0 \ \Rightarrow \ 6s^2 + 48 = 0$$

$$s = \pm j\sqrt{8} = \pm j2.82$$

Shortcut method

If the Transfer Function is of the type

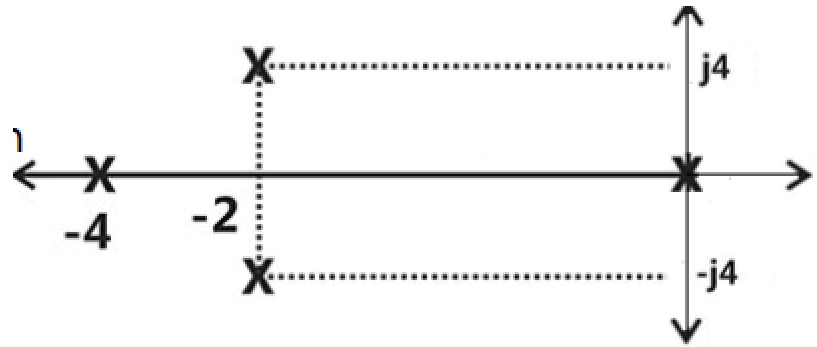
$$G(s) = \frac{K}{s(s+a)(s+b)}$$

Intersection of RL with jω axis is given by $s = \pm j\sqrt{ab}$

Ex-3:

$$G(s)H(s)=\frac{k}{s(s+4)\left(s^2+4s+20\right)}$$

$P = 4, Z = 0, P - Z = 4$



$$\theta_1=45^0, \theta_2=135^0, \theta_3=225^0, \ \theta_4=315^0$$

$$\text{centroid}=\frac{0+(-2)+(-2)+(-4)-0}{4}=-2$$

Breakaway points

$$s^4+8s^3+36s^2+80s+K=0$$

$$K=-s^4-8s^3-36s^2-80s$$

$$\frac{dk}{ds}=0=4s^3+24s^2+72s+80=0 \Rightarrow s=-2, \ -2\pm j2.45$$

Routh Array

s^4	1	36	K
s^3	8	80	0
s^2	26	K	
s^1	$\dfrac{2080-8k}{26}$		
s^0	K		

For stability $\dfrac{2080-8k}{26} > 0 \Rightarrow k < 260$ & $K > 0$

$\therefore$ range $0 < k < 260$

For k=260 A.E.= $A(s) = 26s^2 + k = 0$

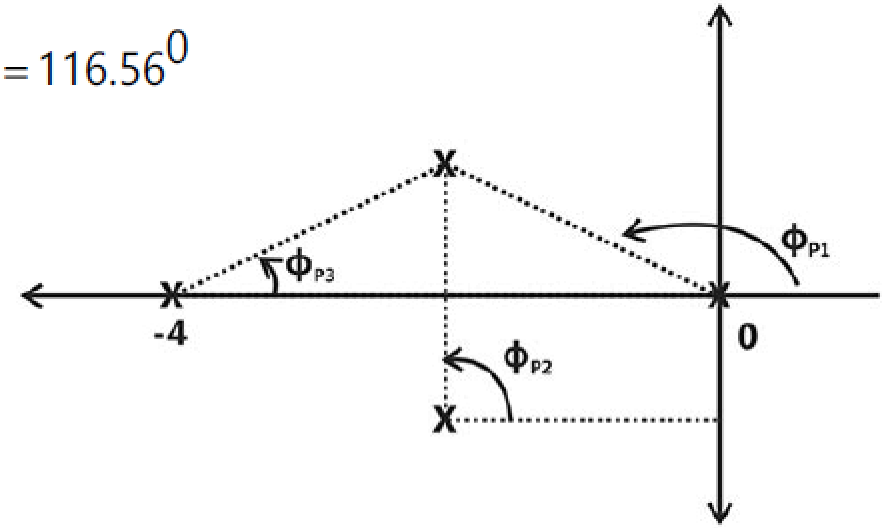
$26s^2 + 260 = 0 \;\; => \;\; s = \pm j3.16$

Intersection of asymptotes with $j\omega$ – axis

$y = \tan 45 \times 2 = \pm j2$

Angle of departure

$$\phi_{p1} = 180 - \tan^{-1}\left[\frac{4-0}{0-(-2)}\right] = 180 - \tan^{-1} 2 = 116.56^0$$



$$\phi_{p2} = 90^0$$

$$\phi_{p3} = \tan^{-1}\left[\frac{4-0}{(-2)(-4)}\right] = 63.4^0$$

$\phi =$ Sum of Angles from zeroes - Sum of Angles from Poles

$$\phi = \Sigma\phi_z - \Sigma\phi_p$$

$$\phi_D = 180 + \phi = 180 - [116.6 + 90 + 63.4] = 180 - 270 = -90^0$$

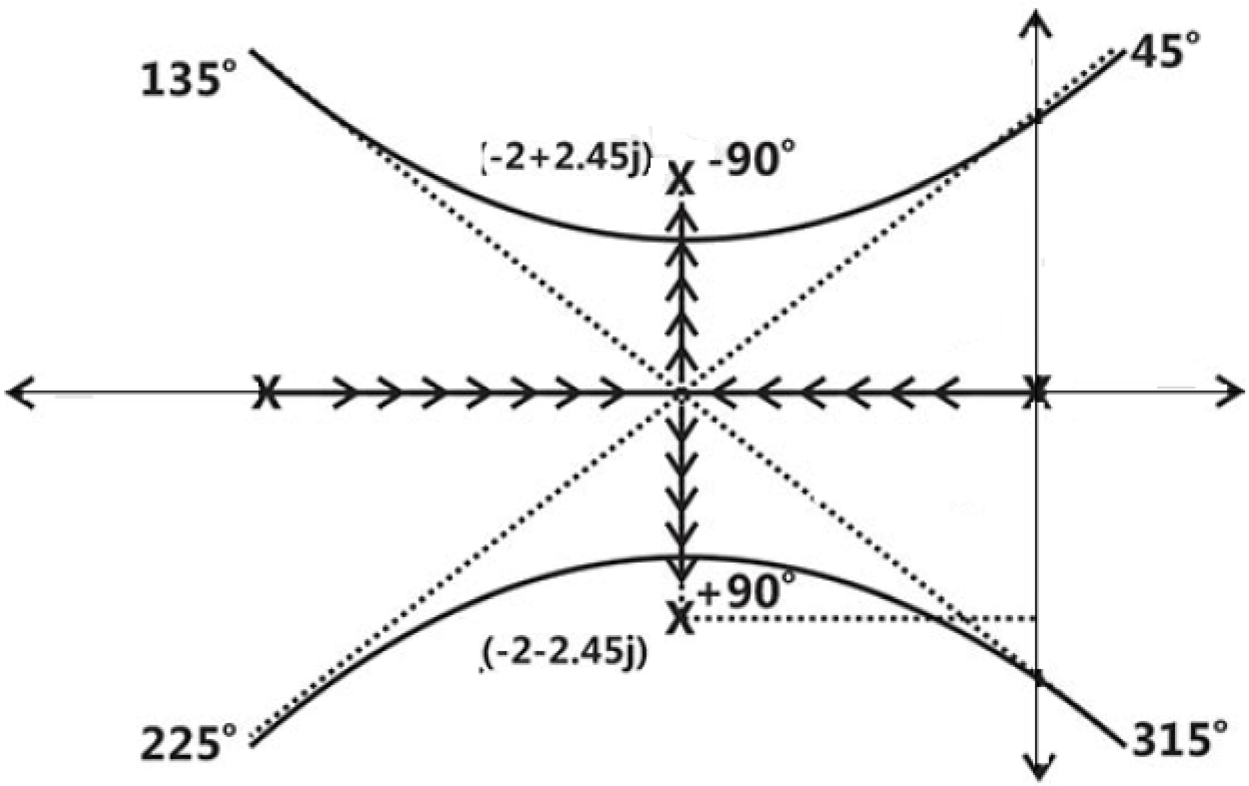
Else

$$G(s)H(s)\Big|_{s=-2+4j} = \frac{K}{s(s+4)(s+2+4j)(s+2-4j)} = \frac{K}{(-2+4j)(2+4j)(8j)}$$

$$\angle GH' = -\tan^{-1}\left(\frac{4}{-2}\right) - \tan^{-1}\left(\frac{4}{2}\right) - 90^0 = -\left(180 - \tan^{-1}\left(4/2\right)\right) - \tan^{-1}\left(4/2\right) - 90^0 = -270$$

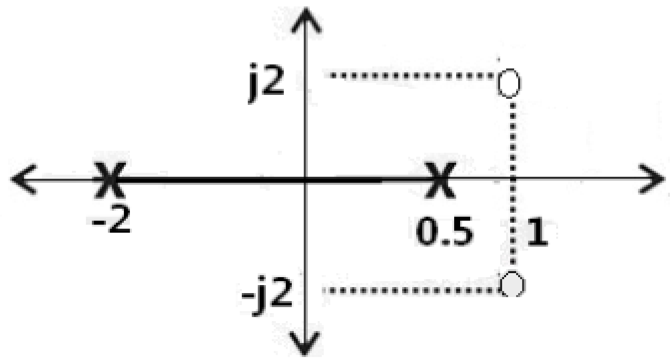
$$\phi_D = 180 - 270 = -90^0$$

So, the root locus of the system looks like as shown below:



Ex-4:

$$G(s)=\frac{k\left(s^2-2s+5\right)}{(s+2)(s-0.5)}$$



$P = 2, Z = 2, P - Z = 0$

As the number of poles and zeros are equal, there will be no asymptotes and hence angles

Breakaway points

Characteristic Equation of the system is

$$(s+2)(s-0.5)+k\left(s^2-2s+5\right)=0 \Rightarrow k=\frac{-\left(s^2+1.5s-1\right)}{\left(s^2-2s+5\right)}$$

$$\frac{dk}{ds}=0 \Rightarrow s=-0.4, 3.6$$

Here, since Breakaway point must lie between two consecutive poles so s = -0.4 is a valid Breakaway Pont whereas s = 3.6 is an invalid point.

$$s^2(1+k)+s(1.5-2k)+(5k-1)=0$$

Routh Array:

s^2	$1+k$	$5k-1$
s^1	$1.5-2k$	0
s^0	$5k-1$	0

For stability $1+k>0 \Rightarrow k>-1$ & $1.5-2k>0 \Rightarrow k<0.75$ & $5k-1>0 = k>0.2$

$\therefore$ range $0.2<k<0.75$

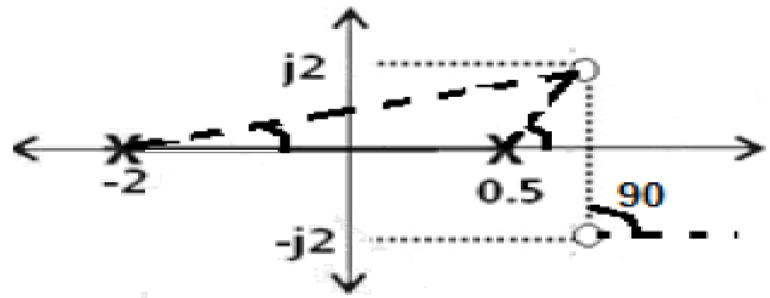
$$k=k_{max}=0.75$$

$$A.E.=A(s)=(1+k)s^2+(5k-1)=1.75s^2+(5\times 0.75-1)=0$$

$$s=\pm j1.25$$

Angle of arrival

$$G(s) = \frac{k(s-1+2j)(s-1-2j)}{(s+2)(s-0.5)}$$



ϕ_A at $s = (1+2j)$

$$\angle GH' = \frac{\angle(1+2j-1+2j)}{\angle(3+2j)\angle(0.5+2j)} = \tan^{-1}(4j) - \tan^{-1}(2/3) - \tan^{-1}(2/0.5)$$

$$\angle GH' = 90^0 - \tan^{-1}\left(\frac{2}{3}\right) - \tan^{-1}(4) = 90^0 - \left(76^0 + 34^0\right) = -20^0$$

$$\phi = 180^0 + 20^0 = 200^0$$

The Root Locus for the system looks like as shown below: