1.6. Tautological Implications and Rules of Inference

We have already noted that $p \wedge q, q \wedge p$ are equivalent and $p \vee q, q \vee p$ are equivalent. That is,

$$p \wedge q \equiv q \wedge p \text{ and } p \vee q \equiv q \vee p$$

Also, $p \longleftrightarrow q \equiv (p \longrightarrow q) \wedge (q \longrightarrow p) \equiv (q \longrightarrow p) \wedge (p \longrightarrow q) \equiv q \longleftrightarrow p$

On the other hand $p \longrightarrow q \not\equiv q \longrightarrow p$ ($i.e.$, $p \longrightarrow q$ is not equivalent to $q \longrightarrow p$), since $p \longrightarrow q, q \longrightarrow p$ are F, T respectively when p is T and q is F.

Tautological implication

Let a and b be *wffs*. We say that ***a* tautologically imply *b*** if and only if ***a* ⟶ *b* is a tautology.** *If* a tautologically imply b**,** then we write as $\boldsymbol{a \Longrightarrow b}$ and it is read as ***a* tautologically implies *b*** or simply ***a* implies *b***.

Note that $\Longrightarrow$ is not a connective and $a \Longrightarrow b$ is not proposition formula.

To show that $a \Longrightarrow b$, it is sufficient to show that the assignment of the truth value T to the antecedent *a leads to the truth value T for the* consequent .This procedure guarantees that the conditional proposition $a \longrightarrow b$ is a tautology.

Another method to show $a \Longrightarrow b$, is to assume that the consequent *b has the truth value F and then show that this assumption leads to the truth value F for the* antecedent *a.*

The following tautological implications have important applications:

(1) $p \wedge q \Longrightarrow p$; $p \wedge q \Longrightarrow q$
If the antecedent $p \wedge q$ is T then both p and q are T.Consequently the consequent is also T.Hence the result.

(2) $p \Longrightarrow p \vee q$; $q \Longrightarrow p \vee q$
If the antecedent p is true then the consequent $p \vee q$ is T for whatever truth value of q. This proves $p \Longrightarrow p \vee q$. Similarly the other follows.

(3) $(p \longrightarrow q) \wedge (q \longrightarrow r) \Longrightarrow p \longrightarrow r$

Suppose that the consequent $p \longrightarrow r$ is F. Then p is T and r is F. Now q can have truth value T or F. If q is T then $p \longrightarrow q$ is T and $q \longrightarrow r$ is F and consequently the antecedent is F. On the other hand if q is F then $p \longrightarrow q$ is F and $q \longrightarrow r$ is T and consequently, the antecedent is F. This proves the result.

(4) $(p \longrightarrow q) \wedge (r \longrightarrow s) \wedge (p \vee r) \Longrightarrow (q \vee s)$

Suppose that antecedent is true then $p \longrightarrow q,\ r \longrightarrow s$ and $p \vee r$ are true. Then q and s must be true. Therefore, the consequent $q \vee s$ is true. Thus the implication is valid.

(5) $(p \longrightarrow q) \wedge (r \longrightarrow s) \wedge (\sim q \vee \sim s) \Longrightarrow (\sim p \vee \sim r)$

Suppose that the consequent is false then both p and r are true. For any combination of truth values of q and s the antecedent is F. Thus, the implication is valid.

Theorem 1: *$a \Leftrightarrow b$ if and only if $a \Longrightarrow b$ and $b \Longrightarrow a$.*

$$\boldsymbol{i.e.}, a \equiv b \text{ if and only if } a \Longrightarrow b \text{ and } b \Longrightarrow a.$$

Proof: We have $a \longleftrightarrow b$ is equivalent to $(a \longrightarrow b) \wedge (b \longrightarrow a)$

$$i.e., a \longleftrightarrow b \Leftrightarrow (a \longrightarrow b) \wedge (b \longrightarrow a).$$

Suppose that $a \Longrightarrow b$ and $b \Longrightarrow a$. Then $a \longrightarrow b$ is a tautology and $b \longrightarrow a$ is a tautology. Therefore, $(a \longrightarrow b) \wedge (b \longrightarrow a)$ is a tautology (since the conjunction of two tautologies is a tautology).Thus, $a \longleftrightarrow b$ is a tautology. Hence a, b are equivalent $i.e.$, $a \Leftrightarrow b$.

Conversely, suppose that $a \Leftrightarrow b$. Therefore, $a \longleftrightarrow b$ is a tautology. Since $a \longleftrightarrow b \Leftrightarrow (a \longrightarrow b) \wedge (b \longrightarrow a)$, $(a \longrightarrow b) \wedge (b \longrightarrow a)$ is a tautology. If any one of $a \longrightarrow b, b \longrightarrow a$ is not a tautology then their conjugation is not a tautology. Thus, both $a \longrightarrow b$ and $b \longrightarrow a$ are tautologies. This proves $a \Longrightarrow b$ and $b \Longrightarrow a$.
Hence the result.

The following are some important facts about tautological implication and equivalence:

- $*$ If a formula is equivalent to a tautology, then it must be a tautology ($i.e.$, if $a \Leftrightarrow T_0$, then a must be a tautology)
- $*$ If a formula is implied by a tautology, then it is a tautology ($i.e.$, if $T_0 \Longrightarrow a$, then a is a tautology)
- $*$ Equivalence of formulas is transitive ($i.e.$, if $a \Leftrightarrow b$ and $b \Leftrightarrow c$ then $a \Leftrightarrow c$).
- $*$ Tautological implication of formulas is also transitive ($i.e.$, if $a \Longrightarrow b$ and $b \Longrightarrow c$ then $a \Longrightarrow c$).

Suppose $a \Longrightarrow b$ and $b \Longrightarrow c$. Then $a \longrightarrow b$ and $b \longrightarrow c$ is a tautology. Therefore $(a \longrightarrow b) \wedge (b \longrightarrow c) \Longrightarrow (a \longrightarrow c)$ and thus $(a \longrightarrow c)$ is implied by a tautology. This proves $a \longrightarrow c$ is a tautology and hence $a \Longrightarrow c$.Thus if $a \Longrightarrow b$ and $b \Longrightarrow c$ then $a \Longrightarrow c$.This proves the tautological implication is transitive.

Note:

(1) If $a \Leftrightarrow b$ and $b \Longrightarrow c$ then $a \Longrightarrow c$
(2) If $a \Longrightarrow b$ and $b \Leftrightarrow c$ then $a \Longrightarrow c$
(3) If $a \Longrightarrow b$ and $a \Longrightarrow c$ then $a \Longrightarrow b \wedge c$.(If a is true then b is true and c is true. Thus, $b \wedge c$ is true. This shows that $a \longrightarrow (b \wedge c)$ is a tautology . Hence $a \Longrightarrow b \wedge c$).

The following are some more tautological implications which have important applications:

(6) $\sim p \Longrightarrow p \longrightarrow q$; $q \Longrightarrow p \longrightarrow q$

We have $p \Longrightarrow p \vee q, i.e., p \longrightarrow (p \vee q)$ is a tautology. Substituting $\sim p$ for p we get $\sim p \longrightarrow (\sim p \vee q)$ and it is a tautology (since the substitution instance of a tautology is also a tautology).Thus $\sim p \Longrightarrow (\sim p \vee q)$.But $\sim p \vee q \equiv p \longrightarrow q$. Therefore, $\sim p \Longrightarrow p \longrightarrow q$. The other result follows from the facts $q \Longrightarrow \sim p \vee q$ and $\sim p \vee q \equiv p \longrightarrow q$.

(7) $\sim(p \longrightarrow q) \Rightarrow p$; $\sim(p \longrightarrow q) \Rightarrow \sim q$

$\sim(p \longrightarrow q) \Leftrightarrow \sim(\sim p \vee q) \Leftrightarrow \sim(\sim p) \wedge \sim q \Leftrightarrow p \wedge \sim q \Rightarrow p$

and $\sim(p \longrightarrow q) \Leftrightarrow p \wedge \sim q \Rightarrow \sim q$

Thus $\sim(p \longrightarrow q) \Rightarrow p$ and $\sim(p \longrightarrow q) \Rightarrow \sim q$

(8) $\sim p \wedge (p \vee q) \Rightarrow q$

$\sim p \wedge (p \vee q) \Leftrightarrow (\sim p \wedge p) \vee (\sim p \wedge q) \Leftrightarrow F_0 \vee (\sim p \wedge q) \Leftrightarrow (\sim p \wedge q) \Rightarrow q$

Thus, $\sim p \wedge (p \vee q) \Rightarrow q$

(9) $p \wedge (p \longrightarrow q) \Rightarrow q$; $\sim q \wedge (p \longrightarrow q) \Rightarrow \sim p$

$p \wedge (p \longrightarrow q) \Leftrightarrow p \wedge (\sim p \vee q) \Leftrightarrow (p \wedge \sim p) \vee (p \wedge q) \Leftrightarrow F_0 \vee (p \wedge q) \Leftrightarrow p \wedge q \Rightarrow q.$

Thus, $p \wedge (p \longrightarrow q) \Rightarrow q$.

$\sim q \wedge (p \longrightarrow q) \Leftrightarrow (\sim q) \wedge (\sim p \vee q) \Leftrightarrow (\sim q \wedge \sim p) \vee (\sim q \wedge q) \Leftrightarrow$
$(\sim p \wedge \sim q) \vee F_0 \Leftrightarrow \sim p \wedge \sim q_0 \Rightarrow \sim p.$

Thus $\sim q \wedge (p \longrightarrow q) \Rightarrow \sim p$.

Let $p_1, p_2, \dots, p_m$ be formulas. These formulas jointly imply a particular formula q ($i.e., p_1, p_2, \dots, p_m \Rightarrow q$) means $p_1 \wedge p_2 \wedge \dots \wedge p_m \Rightarrow q$.

The theory of Inference for propositional calculus

The logic is mainly to provide rules of inference or principles of reasoning. The theory related to the rules of inference is known as the ***inference theory*** since it is concerned with the inferring of a conclusion from certain premises.

The process of deriving a conclusion from a set of premises by using the accepted rules of reasoning is known as a ***deduction*** or ***formal proof***.

In a formal proof every rule of inference that is used at any stage in the derivation is acknowledged. In mathematics we are concerned with the conclusion that is obtained by the rules of logic. This conclusion, called a ***theorem*** ,can be inferred from a set of premises, called the ***axioms*** *of the theory* and the truth value plays no part in the theory.

In any argument, a conclusion is admitted to be true provided that (i) the premises (assumptions, axioms, hypotheses) are accepted as true and (ii) the reasoning used in deriving the conclusion from the premises following a certain accepted rules of logical inference. Such an argument is called *sound*. In any argument we are always concerned with its soundness.

In logic we concentrate on the study of rules of inference by which conclusions are derived from premises. Any conclusion that is derived by following these rules is called a ***valid conclusion*** and the argument is called a ***valid argument***.

In logic we are concerned with the validity of the argument but not (necessarily) its soundness.

Let a and b be two wffs. We say that **b logically follows from a** or **b is a valid conclusion (consequence) of the premise a** iff a tautologically implies b, $(i.e., a \Longrightarrow b)$.Further, we say that a conclusion c logically follows from a set of premises $\{h_1, h_2, \dots, h_m\}$ iff $h_1 \wedge h_2 \wedge \dots \wedge h_m \Longrightarrow c$.

Validity using truth tables

Given a set of premises and a conclusion, we can determine whether the conclusion logically follows from the given premises by constructing truth tables as follows:

Let $p_1, p_2, \dots, p_n$ be all atomic variables appearing in the premises $h_1, h_2, \dots, h_m$ and the conclusion c. For all possible combinations of truth values of $p_1, p_2, \dots, p_n$ find the truth values of $h_1, h_2, \dots, h_m$ and c and enter them in the truth table. If for every row in which $h_1, h_2, \dots, h_m$ have the value T, the conclusion c also has value T, then

$$h_1 \wedge h_2 \wedge \dots \wedge h_m \Longrightarrow c$$

Alternatively if for every row in which c has value F, at least one of $h_1, h_2, \dots, h_m$ has value F, then

$$h_1 \wedge h_2 \wedge \ldots \wedge h_m \Longrightarrow c$$

This method of determining the validity of a conclusion from a given set of premises is called ***Truth table technique***.

Example 1: Determine whether the conclusion c follows logically from the premises h_1, h_2 ... by truth table technique

a) $\boldsymbol{h_1 \colon p \longrightarrow q, h_2 \colon \sim p, c \colon q}$
b) $\boldsymbol{h_1 \colon \sim p \vee q, h_2 \colon \sim(q \wedge \sim r), h_3 \colon \sim r\ , c \colon \sim p}$

Solution:

a) We first construct the truth tables.

p	q	$h_1 \colon p \longrightarrow q$	$h_2 \colon \sim p$	$c \colon q$
T	T	T	F	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	F

Notice that in Table, h_1 and h_2 are true in the third and fourth rows but the conclusion $c \colon q$ is true only in the third row and not in the fourth row. Therefore, the conclusion is not valid.

b) We first construct the truth table

p	q	r	$h_1 \colon \sim p \vee q$	$h_2 \colon \sim(q \wedge \sim r)$	$h_3 \colon \sim r$	$c \colon \sim p$
T	T	T	T	T	F	F
T	T	F	T	F	T	F
T	F	T	F	T	F	F
T	F	F	F	T	T	F
F	T	T	T	T	F	T
F	T	F	T	F	T	T
F	F	T	T	T	F	T
F	F	F	T	T	T	T

Observe that the last row is the only row in which h_1, h_2 and h_3 have truth value T. The conclusion c also has the truth value T in that row. Therefore, c logically follows from the given premises h_1, h_2 and h_3.

Note: It is possible to determine in a finite number of steps whether a conclusion follows from a given set of premises through the truth table technique. However, this method becomes tedious when the number of atomic variables present in all the premises and conclusion is large.

Rules of Inference

In the process of derivation to demonstrate a particular formula is a consequence of a given set of premises, we take the following two rules of inference called **rule P** and **rule T**

Rule **P:** A premise that may be introduced at any point in the derivation.

Rule **T**: A formula s may be introduced in a derivation ,if s follows logically from any one or more of the preceding formulas in the derivation.

The following is the list of important implications and equivalences that will be referred frequently. These are not independent of one another.

Table 1 Implications

$p \Rightarrow p \vee q\ ; q \Rightarrow p \vee q$ $\sim p \Rightarrow p \longrightarrow q\ ; q \Rightarrow p \longrightarrow q$	Disjunctive Addition
$p \wedge q \Rightarrow p\ ; p \wedge q \Rightarrow q$ $\sim(p \longrightarrow q) \Rightarrow p\ ; \sim(p \longrightarrow q) \Rightarrow \sim q$	Simplification
$p, q \Rightarrow p \wedge q$	Conjunctive Addition
$\sim p, p \vee q \Rightarrow q$	Disjunctive Syllogism
$p, p \longrightarrow q \Rightarrow q$	Modus ponens^*
$\sim q, p \longrightarrow q \Rightarrow \sim p$	Modus tollens^{**}
$p \longrightarrow q, q \longrightarrow r \Rightarrow p \longrightarrow r$	Hypothetical Syllogism
$p \longrightarrow q\ , r \longrightarrow s, p \vee r \Rightarrow q \vee s$	Constructive Dilemma
$p \longrightarrow q\ , r \longrightarrow s, \sim q \vee \sim s \Rightarrow \sim p \vee \sim r$	Destructive Dilemma

$*$ Latin meaning: Method of affirming.

$**$ Latin meaning: Method of denying.

Table 2 Equivalences

$p \vee p \equiv p \;\; ; \;\; p \wedge p \equiv p$	Idempotent laws
$p \vee q \equiv q \vee p \;\; ; \;\; p \wedge q \equiv q \wedge p$	Commutative laws
$p \wedge (p \vee q) \equiv p \,; \;\; p \vee (p \wedge q) \equiv p$	Absorption laws
$\sim(\sim p) \equiv p$	Double negation law
$\sim(p \vee q) \equiv \sim p \wedge \sim q \;\; ; \sim(p \wedge q) \equiv \sim p \vee \sim q$	Demorgan's laws
$p \vee (q \vee r) \equiv (p \vee q) \vee r \,; \, p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r$	Associative laws
$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r) \,; \, p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$	Distributive laws
$p \wedge T_0 \equiv p \;\; ; p \vee F_0 \equiv p$	Identity laws
$p \vee T_0 \equiv T_0 \;\; ; p \wedge F_0 \equiv F_0$	Domination laws
$p \vee \sim p \equiv T_0 \;\; ; p \wedge \sim p \equiv F_0$	Inverse laws
$p \longrightarrow q \equiv \sim p \vee q$	
$\sim(p \longrightarrow q) \equiv p \wedge \sim q$	
$p \longrightarrow q \equiv \sim q \longrightarrow \sim p$	
$p \longrightarrow (q \longrightarrow r) \equiv (p \wedge q) \longrightarrow r$	
$p \leftrightarrow q \equiv (p \longrightarrow q) \wedge (q \longrightarrow p)$	
$p \leftrightarrow q \equiv (p \wedge q) \vee (\sim p \wedge \sim q)$	
$\sim(p \leftrightarrow q) \equiv p \leftrightarrow \sim q$	

Example 2: Show that s is a valid inference from the premises:

$\boldsymbol{p \longrightarrow q}$, $\quad \boldsymbol{p \longrightarrow r}, \sim(\boldsymbol{q \wedge r})$ **and** $\boldsymbol{s \vee p}$

Solution:

(1)	$p \longrightarrow q$	**P**
(2)	$p \longrightarrow r$	**P**
(3)	$(p \longrightarrow q) \wedge (p \longrightarrow r)$	**T**; $(1), (2)$, Conjunctive Addition
(4)	$\sim(q \wedge r)$	**P**
(5)	$\sim q \vee \sim r$	**T**; (4), De Morgan's law
(6)	$\sim p \vee \sim p$	**T**; (3),(5), Destructive dilemma
(7)	$\sim p$	**T**; (6), Idempotent law
(8)	$s \vee p$	**P**
(9)	s	**T**; $(7), (8)$, Disjunctive syllogism

Thus, s is a valid inference from the given premises.

Notation: The numbers in the first column designate the line of derivation and the formula. In the second column $\mathbf{P}$ or $\mathbf{T}$ represent the rule of inference, followed by a comment showing from which formulas and tautology that particular formula has been obtained.

Example 3: "If there was a ball game, then travelling was difficult. If they arrived on time, then travelling was not difficult. They arrived on time. Therefore, there was no ball game". Show that these statements constitute a valid argument.

Solution: We first symbolize the given statements and then use the method of derivation.

Let p: There was a ball game.

q: Travelling was difficult.

r: They arrived on time.

Required to show that the conclusion $\sim p$ follows from the premises $p \longrightarrow q, r \longrightarrow \sim q$ and r. The derivation is given below

(1) r	$\mathbf{P}$
(2) $r \longrightarrow \sim q$	$\mathbf{P}$
(3) $\sim q$	$\mathbf{T}$; (1), (2), Modus ponens
(4) $p \longrightarrow q$	$\mathbf{P}$
(5) $\sim p$	$\mathbf{T}$; (3), (4), Modus tollens

Thus, the argument is valid.

Third Inference Rule

The following is the basis for the third inference rule

Theorem 2: ***If*** $\boldsymbol{p_1, p_2, \dots, p_m}$ ***and*** $\boldsymbol{p}$ ***imply*** $\boldsymbol{q}$***, then*** $\boldsymbol{p_1, p_2, \dots, p_m}$ ***imply*** $\boldsymbol{p \longrightarrow q}$***.***
i.e., if $\boldsymbol{(p_1 \wedge p_2 \wedge \dots \wedge p_m \wedge p) \Longrightarrow q}$ ***then*** $\boldsymbol{(p_1 \wedge p_2 \wedge \dots \wedge p_m) \Longrightarrow (p \longrightarrow q)}$***.***

Proof: We have $p_1, p_2, \dots, p_m$ and p imply q.

$$i.e., (p_1 \wedge p_2 \wedge \dots \wedge p_m \wedge p) \Longrightarrow q.$$

$i.e.$, $(p_1 \wedge p_2 \wedge \dots \wedge p_m \wedge p) \longrightarrow q$ is a tautology.

We have the following result on equivalences:

$$(p_1 \wedge p_2) \longrightarrow p_3 \Leftrightarrow p_1 \longrightarrow (p_2 \longrightarrow p_3)$$

Therefore $(p_1 \wedge p_2 \wedge \dots \wedge p_m \wedge p) \longrightarrow q \Leftrightarrow (p_1 \wedge p_2 \wedge \dots \wedge p_m) \longrightarrow (p \longrightarrow q)$.

Since $(p_1 \wedge p_2 \wedge \dots \wedge p_m \wedge p) \longrightarrow q$ is tautology, $(p_1 \wedge p_2 \wedge \dots \wedge p_m) \longrightarrow (p \longrightarrow q)$ is a tautology. Thus $(p_1 \wedge p_2 \wedge \dots \wedge p_m) \;\Longrightarrow\; (p \longrightarrow q)$.
Hence the theorem.

The third inference rule is known **as Rule CP** or R**ule of conditional proof.**

Rule CP: If we can derive s from r and a set of premises ,then we can derive $r \longrightarrow s$ from the set of premises alone.

The Rule **CP** is also known as the ***deduction theorem*** and is generally used if the conclusion is of the form $r \longrightarrow s$. In such cases, r is taken as an additional premise and s is derived from the given premises and r.

Example 4: Derive the following ,using rule CP:

$$\boldsymbol{p \longrightarrow (q \longrightarrow r), q \longrightarrow (r \longrightarrow s) \Rightarrow p \longrightarrow (q \longrightarrow s)}$$

Solution: We introduce p as an additional premise and show that $q \longrightarrow s$ follows.

(1) $p \longrightarrow (q \longrightarrow r)$	**P**
(2) p	**P** (additional Premise)
(3) $q \longrightarrow r$	**T** ; (1), (2), Modus ponens
(4) $\sim q \vee r$	**T** ; (3), $\alpha \longrightarrow \beta \Leftrightarrow \sim\alpha \vee \beta$
(5) $\sim q \vee r \vee s$	**T** ; (4), Disjunctive Addition
(6) $\sim q \vee s \vee r$	**T** ; (5), commutativity
(7) $q \longrightarrow (r \longrightarrow s)$	**P**
(8) $\sim q \vee (\sim r \vee s)$	**T** ; (7), $\alpha \longrightarrow \beta \Leftrightarrow \sim\alpha \vee \beta$

(9) $\sim q \vee s \vee \sim r$ **T** ; (8), Commutativity

(10) $(\sim q \vee s \vee r) \wedge (\sim q \vee s \vee \sim r)$ **T** ; (6), (9), Conjunctive Addition

(11) $(\sim q \vee s) \vee (r \wedge \sim r)$ **T** ; (10), Distributive law

(12) $\sim q \vee s$ **T** ; (11), $r \wedge \sim r = F_0$ and $a \vee F_0 \equiv a$

(13) $q \longrightarrow s$ **T** ; $\alpha \longrightarrow \beta \Leftrightarrow \sim \alpha \vee \beta$

(14) $p \longrightarrow (q \longrightarrow s)$ **CP**

Thus, $p \longrightarrow (q \longrightarrow s)$ follows from the given premises.

The above method of derivation provides a partial solution to the decision problem, because if an argument is valid, then it is possible to show by this method that the argument is valid. On the other hand, if an argument is not valid, then it is difficult to decide after a finite number of steps that this is so.

Example 5: If A works hard, then either B or C will enjoy themselves. If B enjoys himself, then A will not work hard. If D enjoys himself, then C will not. Therefore if A works hard, D will not enjoy himself. Show that these statements constitute a valid argument.

Solution: Let a: A works hard

b: B will enjoy himself

c: C will enjoy himself

d: D will enjoy himself

We have to show that

$a \longrightarrow \sim d$ follows from $a \longrightarrow b \vee c$, $b \longrightarrow \sim a$ and $d \longrightarrow \sim c$

Since the conclusion is in the form of a conditional $a \longrightarrow \sim d$, include a as an additional premise and show that $\sim d$ follows logically from all the premises including a. The result follows by the rule **CP.** The following is the derivation.

(1) a Assumed premise

(2) $a \longrightarrow b \vee c$ **P**

(3) $b \vee c$ **T** ; (1), (2), Modus ponens

(4) $b \longrightarrow \sim a$ **P**

(5)	$a \longrightarrow \sim b$	**T** ; (4), $b \longrightarrow \sim a \equiv \sim(\sim a) \longrightarrow \sim b \equiv a \longrightarrow \sim b$
(6)	$\sim b$	**T** ; (1), (5), Modus ponens.
(7)	$(\sim b) \wedge (b \vee c)$	**T** ; (3), (6), Conjunctive Addition
(8)	$\sim b \wedge c$	**T** ; (7), Distributive law
(9)	c	**T,**Simplification
(10)	$d \longrightarrow \sim c$	**P**
(11)	$\sim d$	**T** ; (9), (10), Modus tollens
(12)	$a \longrightarrow \sim d$	**CP**

Consistency of Premises and Indirect Method of Proof

A set of formulas $h_1, h_2, \ldots, h_m$ is said to be ***consistent*** if their conjunction has truth value T for *some* assignment of the truth values to the atomic variables appearing in $h_1, h_2, \ldots, h_m$.

If for *every* assignment of the truth values to the atomic variables, at least one of the formulas $h_1, h_2, \ldots, h_m$ is false (so that their conjunction is identically false) then the formulas $h_1, h_2, \ldots, h_m$ are called ***inconsistent****.*

Alternatively, a set of formulas $h_1, h_2, \ldots, h_m$ is inconsistent if their conjunction implies a contradiction, $i.e.$, $h_1 \wedge h_2 \wedge \ldots \wedge h_m \Rightarrow r \wedge \sim r$, where r is any formula.

The notion of inconsistency is used in a procedure called ***proof by contradiction*** *or* ***reduction absurdum*** *or* ***indirect method of proof.***

To show that c follows logically from the premises $h_1, h_2, \ldots, h_m$,we assume that c is false and consider $\sim c$ as an additional premise. If the new set of premises ($i.e.$, $h_1, h_2, \ldots, h_m$ and $\sim c$) is in consistent, then the assumption $\sim c$ is true does not hold simultaneously with $h_1, h_2, \ldots, h_m$ being true. Therefore c is true whenever $h_1, h_2, \ldots, h_m$ are true. Thus, c follows logically from the premises $h_1, h_2, \ldots, h_m$.

Example 6: Show that the following premises are inconsistent

1. **If Jack misses many classes through illness, then he fails high school.**
2. **If Jack fails high school, then he is uneducated**
3. **If Jack reads a lot of books, then he is not educated**
4. **Jack misses many classes through illness and reads a lot of books.**

Solution: We first symbolize the given statements and then use the method of derivation.

Let m: Jack misses many classes

f: Jack fails high school

r: Jack reads a lot of books

u: Jack is uneducated

The given premises are

$m \longrightarrow f,\ \ f \longrightarrow u,\ \ r \longrightarrow \sim u$ and $m \wedge r$

Derivation:

(1)	$m \longrightarrow f$	**P**
(2)	$f \longrightarrow u$	**P**
(3)	$m \longrightarrow u$	**T**; (1), (2), Hypothetical syllogism
(4)	$r \longrightarrow \sim u$	**P**
(5)	$u \longrightarrow \sim r$	**T**; (4), equivalence of contrapositive
(6)	$m \longrightarrow \sim r$	**T**; (3), (5), Hypothetical syllogism
(7)	$\sim m \vee \sim r$	**T**; (6), $p \longrightarrow q \equiv \sim p \vee q$
(8)	$\sim(m \wedge r)$	**T**; (7), DeMorgan's law
(9)	$m \wedge r$	**P**
(10)	$(m \wedge r) \wedge \sim(m \wedge r)$	**T**; (8), (9) Conjunctive Addition
(11)	F_0	**T**; (10), Contradiction.

Thus, the given premises are inconsistent.

Example 7 :Using indirect method ,show that

$$\sim(\boldsymbol{p} \longrightarrow \boldsymbol{q}) \longrightarrow \sim(\boldsymbol{r} \vee \boldsymbol{s}),\ \left((\boldsymbol{q} \longrightarrow \boldsymbol{p}) \vee \sim\boldsymbol{r}\right),\ \boldsymbol{r} \Rightarrow \boldsymbol{p} \leftrightarrow \boldsymbol{q}$$

Solution: We introduce $\sim(p \leftrightarrow q)$ as an additional premise and to show that the given premises and additional premise leads to a contradiction.

(1)	r	$\mathbf{P}$
(2)	$(q \longrightarrow p) \vee \sim r$	$\mathbf{P}$
(3)	$q \longrightarrow p$	$\mathbf{T}$; $(1), (2)$, Disjunctive syllogism
(4)	$\sim(p \leftrightarrow q)$	$\mathbf{P}$ (Additional premise)
(5)	$\sim(p \longrightarrow q) \vee \sim(q \longrightarrow p)$	$\mathbf{T}$; $(4), \sim(p \leftrightarrow q) \equiv \sim\left((p \longrightarrow q) \wedge (q \longrightarrow p)\right)$ $\equiv \sim(p \longrightarrow q) \vee \sim(q \longrightarrow p)$
(6)	$\sim(p \longrightarrow q)$	$\mathbf{T}$; $(3), (5)$, Disjunctive syllogism
(7)	$\sim(p \longrightarrow q) \longrightarrow \sim(r \vee s)$	$\mathbf{P}$
(8)	$\sim(r \vee s)$	$\mathbf{T}$; $(6), (7)$,Modus Ponens
(9)	$\sim r \wedge \sim s$	$\mathbf{T}$; (8), De Morgan's law
(10)	$r \wedge (\sim r \wedge \sim s)$	$\mathbf{T}$; $(1), (9)$, Conjunctive Addition
(11)	F_0	$\mathbf{T}$; (10) Contradiction.

Thus , $(p \leftrightarrow q)$ follows from the given premises.

Note: See annexure for some more solved problems.

1.6. Tautological Implications and Rules of Inference

Annexure:

1. Show that r is tautologically implied by the premises $p \longrightarrow q, q \longrightarrow r$ and p

Solution:

(1) p Rule $\mathbf{P}$
(2) $p \longrightarrow q$ Rule $\mathbf{P}$
(3) q Rule $\mathbf{T}$; (1), (2) and modus ponens
(4) $q \longrightarrow r$ Rule $\mathbf{P}$
(5) r Rule $\mathbf{T}$; (3), (4) and modus ponens

Thus, r is a valid inference from the premises $p \longrightarrow q, q \longrightarrow r$ and p

2.

a. q is a valid inference from the premises: $p \longrightarrow q$, $p \vee q$ and $\sim q$.
b. $r \wedge (p \vee q)$ is a valid inference from the premises

$p \vee q, q \longrightarrow r, p \longrightarrow m$ and $\sim m$

a. *Solution:*

(1) $p \longrightarrow q$ $\mathbf{P}$
(2) $\sim q$ $\mathbf{P}$
(3) $\sim p$ $\mathbf{T}$; (1), (2), Modus tollens
(4) $p \vee q$ $\mathbf{P}$
(5) q $\mathbf{T}$; (3), (4), disjunctive syllogism.

Thus, q is a valid inference from the given premises.

b. *Solution:*

(1) $p \longrightarrow m$ $\mathbf{P}$
(2) $\sim m$ $\mathbf{P}$
(3) $\sim p$ $\mathbf{T}$; (1), (2), modus tollens
(4) $p \vee q$ $\mathbf{P}$
(5) q $\mathbf{T}$; (3), (4), disjunctive syllogism
(6) $q \longrightarrow r$ $\mathbf{P}$
(7) r $\mathbf{T}$; (5), (6), modus ponens
(8) $r \wedge (p \vee q)$ $\mathbf{T}$; (4), (7), Conjunctive addition

Thus, $r \wedge (p \vee q)$ is a valid conclusion that can be drawn from the given set of premises .

3. Show that $\sim(\boldsymbol{p} \wedge \boldsymbol{q})$ follows from $\sim\boldsymbol{p} \wedge \boldsymbol{q}$

Solution: We introduce $\sim\left(\sim(p \wedge q)\right)$ as an additional premise and show that this leads to a contradiction.

Derivation:

(1)	$\sim\left(\sim(p \wedge q)\right)$	$\mathbf{P}$ (assumed)
(2)	$p \wedge q$	$\mathbf{T}$; (1), double negation law
(3)	p	$\mathbf{T}$; (2), simplification
(4)	$\sim p \wedge \sim q$	$\mathbf{P}$
(5)	$\sim p$	$\mathbf{T}$; (4), simplification
(6)	$p \wedge \sim p$	$\mathbf{T}$; (3), (5), conjunctive addition.
(7)	F_0	$\mathbf{T}$; (6) contradition

P1:

Determine whether the conclusion c follows logically from the premises h_1, h_2 by truth table technique

a) $h_1: \sim p, h_2: p \leftrightarrow q, c: \sim(p \wedge q)$
b) $h_1: p \longrightarrow (q \longrightarrow r), h_2: r, c: p$

Solution:

a) We first construct the truth table

p	q	$h_1: \sim p$	$h_2: p \leftrightarrow q$	$c: \sim(p \wedge q)$
T	T	F	T	F
T	F	F	F	T
F	T	T	F	T
F	F	T	T	T

Observe that the fourth row is the only row in which both h_1 and h_2 have the truth value T. The conclusion $c: \sim(p \wedge q)$ also has the truth value T in that row. Therefore, the conclusion is valid

b) We first construct the truth table

p	q	r	$q \longrightarrow r$	$h_1: p \longrightarrow (q \longrightarrow r)$	$h_2: r$	$c: p$
T	T	T	T	T	T	T
T	T	F	F	F	F	T
T	F	T	T	T	T	T
T	F	F	T	T	F	T
F	T	T	T	T	T	F
F	T	F	F	T	F	F
F	F	T	T	T	T	F
F	F	F	T	T	F	F

Observe that h_1 and h_2 are true in the first, third, fifth and seventh rows, but the conclusion c is true only first and third row and not in the fifth and seventh rows. Therefore, the conclusion does not logically follow from the given premises.

P2:

Show that $\boldsymbol{s \vee r}$ is tautologically implied by the premises

$$\boldsymbol{(p \vee q) \wedge (p \longrightarrow r) \wedge (q \longrightarrow s)}$$

Solution:

(1) $p \vee q$	**P**
(2) $\sim p \longrightarrow q$	**T** ; $\sim p \longrightarrow q \Leftrightarrow \sim(\sim p) \vee q \Leftrightarrow p \vee q$
(3) $q \longrightarrow s$	**P**
(4) $\sim p \longrightarrow s$	**T** ; (2), (3), Hypothetical syllogism
(5) $\sim s \longrightarrow p$	**T** ; (4), $\sim p \longrightarrow s \Leftrightarrow \sim s \longrightarrow \sim(\sim p) \Leftrightarrow \sim s \longrightarrow p$
(6) $p \longrightarrow r$	**P**
(7) $\sim s \longrightarrow r$	**T** ; (5), (6), Hypothetical syllogism
(8) $s \vee r$	**T** ; (7), $\sim s \longrightarrow r \Leftrightarrow \sim(\sim s) \vee r \Leftrightarrow s \vee r$

Thus, $s \vee r$ is tautologically implied by the given set of premises.

P3:

Show that $\boldsymbol{r \vee s}$ is a valid inference from the premises

$$\boldsymbol{c \vee d, (c \vee d) \longrightarrow \sim h, \sim h \longrightarrow (a \wedge \sim b)} \text{ and } \boldsymbol{(a \wedge \sim b) \longrightarrow r \vee s}\ .$$

Solution:

(1) $c \vee d \longrightarrow \sim h$	**P**
(2) $\sim h \longrightarrow (a \wedge \sim b)$	**P**
(3) $c \vee d \longrightarrow (a \wedge \sim b)$	**T** ; (1), (2) Hypothetical syllogism
(4) $(a \wedge \sim b) \longrightarrow r \vee s$	**P**
(5) $c \vee d \longrightarrow r \vee s$	**T** ; (3), (4) Hypothetical syllogism
(6) $c \vee d$	**P**
(7) $r \vee s$	**T** ; (5), (6) ,Modus ponens

Thus, $r \vee s$ follows logically from the given set of premises.

P4:

Derive the following, using CP rule:

$$\sim\boldsymbol{p} \vee \boldsymbol{q}, \quad \sim\boldsymbol{q} \vee \boldsymbol{r}, \boldsymbol{r} \longrightarrow \boldsymbol{s} \Longrightarrow \boldsymbol{p} \longrightarrow \boldsymbol{s}$$

Solution: we will introduce p as an additional premise and show that s follows.

(1) $\sim p \vee q$	$\mathbf{P}$
(2) $p \longrightarrow q$	$\mathbf{T}$; (1) and (2), $\alpha \longrightarrow \beta \equiv \sim\alpha \vee \beta$
(3) p	$\mathbf{P}$ (assumed premise)
(4) q	$\mathbf{T}$; (1), (2), (3), Modus ponens
(5) $\sim q \vee r$	$\mathbf{P}$
(6) r	$\mathbf{T}$; (4), (5), Disjunctive syllogism
(7) $r \longrightarrow s$	$\mathbf{P}$
(8) s	$\mathbf{T}$; (6), (7), Modus ponens
(9) $p \longrightarrow s$	$\mathbf{CP}$

Thus, $p \longrightarrow s$ follows from the given premises.

P5:

Show that $\boldsymbol{r \longrightarrow s}$ can be derived from the premises
$\boldsymbol{p \longrightarrow (q \longrightarrow s), \sim r \vee p}$ **and** $\boldsymbol{q}$

Solution: Instead of deriving $r \longrightarrow s$, we include r as an additional premise and show that s follows.

(1) $\sim r \vee p$	$\mathbf{P}$
(2) $r \longrightarrow p$	$\mathbf{T}$; (1), $\alpha \longrightarrow \beta \equiv \sim\alpha \vee \beta$
(3) r	$\mathbf{P}$; Assumed premise
(4) p	$\mathbf{T}$; (2), (3), Modus ponens
(5) $p \longrightarrow (q \longrightarrow s)$	$\mathbf{P}$
(6) $q \longrightarrow s$	$\mathbf{T}$; (4), (5), Modus ponens
(7) q	$\mathbf{P}$
(8) s	$\mathbf{T}$; (6), (7), Modus ponens
(9) $r \longrightarrow s$	$\mathbf{CP}$

P6:

Show that the following set of premises is inconsistent:

$$\boldsymbol{p \longrightarrow (q \longrightarrow r)\,,q \longrightarrow (r \longrightarrow s)\,,p \wedge q \wedge \sim s}$$

Solution:

(1)	$p \wedge q \wedge \sim s$	$\mathbf{P}$
(2)	p	$\mathbf{T}$; (1), simplification
(3)	$p \longrightarrow (q \longrightarrow r)$	$\mathbf{P}$
(4)	$q \longrightarrow r$	$\mathbf{T}$; $(2), (3)$, Modus ponens
(5)	q	$\mathbf{T}$; (1), simplification
(6)	r	$\mathbf{T}$; $(4), (5)$, Modus ponens
(7)	$q \longrightarrow (r \longrightarrow s)$	$\mathbf{P}$
(8)	$r \longrightarrow s$	$\mathbf{T}$; $(5), (7)$, Modus ponens
(9)	s	$\mathbf{T}$; $(6), (8)$, Modus ponens
(10)	$\sim s$	$\mathbf{T}$; (1), simplification
(11)	$s \wedge \sim s$	$\mathbf{T}$; $(9), (10)$, Conjunctive Addition
(12)	F_0	$\mathbf{T}$; (11), contradiction

Thus, the given premises are inconsistent.

P7:

Show that the following set of premises is inconsistent:

$$\boldsymbol{p \longrightarrow q\ , p \longrightarrow r\ , q \longrightarrow \sim r\ , p}$$

Solution:

(1) $p \longrightarrow q$	$\mathbf{P}$
(2) p	$\mathbf{P}$
(3) q	$\mathbf{T}$; $(1), (2)$, Modus ponens
(4) $p \longrightarrow r$	$\mathbf{P}$
(5) r	$\mathbf{T}$; $(2), (4)$, Modus ponens
(6) $q \wedge r$	$\mathbf{T}$; $(3), (5)$, Conjunctive Addition
(7) $q \longrightarrow \sim r$	$\mathbf{P}$
(8) $\sim q \vee \sim r$	$\mathbf{T}$; $(7), \alpha \longrightarrow \beta \Leftrightarrow \sim\alpha \vee \beta$
(9) $\sim(q \wedge r)$	$\mathbf{T}$; (8), DeMorgan's law
(10) $(q \wedge r) \wedge \sim(q \wedge r)$	$\mathbf{T}$; $(6), (9)$, Conjunctive addition
(11) F_0	$\mathbf{T}$; (10), Contradiction

Thus, the given premises are inconsistent.

Aliter:

(1) p	$\mathbf{P}$
(2) $p \longrightarrow q$	$\mathbf{P}$
(3) q	$\mathbf{T}$; $(1), (2)$, Modus ponens
(4) $q \longrightarrow \sim r$	$\mathbf{P}$
(5) $\sim r$	$\mathbf{T}$; $(3), (4)$, Modus ponens
(6) $p \longrightarrow r$	$\mathbf{P}$
(7) r	$\mathbf{T}$; $(1), (6)$, Modus ponens
(8) $r \wedge \sim r$	$\mathbf{T}$; $(5), (7)$, Conjunctive addition
(9) F_0	$\mathbf{T}$; (8), contradiction

Thus, the given premises are inconsistent.

P8:

Using indirect method show that

$$\boldsymbol{r \longrightarrow \sim q\ , r \vee s, s \longrightarrow \sim q, p \longrightarrow q \Rightarrow \sim p}$$

Solution: We introduce $\sim(\sim p) \equiv p$ as an additional premise and show the given premises and additional premise leads to a contradiction.

(1) $p \longrightarrow q$	$\mathbf{P}$
(2) p	$\mathbf{P}$ (additional premise $\sim(\sim p)$)
(3) q	$\mathbf{T}$; $(1), (2)$, Modus ponens
(4) $s \longrightarrow \sim q$	$\mathbf{P}$
(5) $\sim s$	$\mathbf{T}$; $(3), (4)$, Modus tollens
(6) $r \vee s$	$\mathbf{P}$
(7) r	$\mathbf{T}$; $(5), (6)$, Disjunctive Syllogism
(8) $r \longrightarrow \sim q$	$\mathbf{P}$
(9) $\sim q$	$\mathbf{T}$; $(7), (8)$, Modus ponens
(10) $q \wedge \sim q$	$\mathbf{T}$; $(3), (9)$, Conjunctive Addition
(11) F_0	$\mathbf{T}$; (10), contradiction

Thus, $\sim p$ follows from the given premises.

1.6. Tautological Implications and Rules of Inference

Exercise:

1. Show that the conclusions *c* follows from the premises h_1 and h_2 in the following cases:
   a. $h_1: \sim q \; ; h_2: p \longrightarrow q \quad , \quad c: \sim p$
   b. $h_1: p \longrightarrow q \; ; h_2: q \longrightarrow r \quad , \quad c: p \longrightarrow r$

2. Determine whether the conclusion c is valid in the following, when h_1 , h_2 ... are the premises.
   a. $h_1: p \vee q \quad ; \quad h_2: p \longrightarrow r \; ; h_3: q \longrightarrow r \quad , c: r$
   b. $h_1: \sim p \quad ; \quad h_2: p \vee q \quad , c: p \wedge q$

3. Show that the validity of the following arguments, for which the premises are given on the left and conclusion on the right.
   a. $p \longrightarrow q \, , (\sim q \vee r) \wedge \sim r \, , \sim(\sim p \wedge s) \quad : \quad \sim s$
   b. $b \wedge c \, , (b \leftrightarrow c) \longrightarrow (h \vee g) \quad : \quad g \vee h$

4. Derive the following, using rule **CP.**

   a. $p, p \longrightarrow \big(q \longrightarrow (r \wedge s)\big) \Longrightarrow q \longrightarrow s$
   b. $(p \vee q) \longrightarrow r \Longrightarrow (p \wedge q) \longrightarrow r$

5. Show that the following sets of premises are inconsistent.
   a. $a \longrightarrow (b \longrightarrow c) \, , d \longrightarrow (b \wedge \sim c) \, , a \wedge d$
   b. $p \longrightarrow q \, , q \vee r \longrightarrow s, s \longrightarrow \sim p, \; p \wedge \sim r$

6. Show the following using indirect method.
   a. $p \longrightarrow (q \longrightarrow \sim r) \, , \sim s \longrightarrow q \, , \sim t \wedge (p \vee t) \Longrightarrow (r \longrightarrow s)$
   b. $(p \vee q) \longrightarrow (r \wedge s) \, , r \longrightarrow \sim s \Longrightarrow \sim p$

7. Test the validity of the following arguments.

a. If Ram has completed B.E.Computer Science or MBA, then he is assured of a good job. If Ram is assured of a good job, he is happy. Ram is not happy. So Ram has not completed MBA

b. If US tightens visa restrictions, then the demand for BPO will increase. Either US tightens visa restrictions or some computer companies in India close down. The demand for BPO will not increase. Therefore some computer companies in India will close down.

c. If the advertisement is successful, then the sales of the product will go up. Either the advertisement is successful or the production of the product will be stopped. The sales of the product will not go up. Therefore the production of the product will be stopped.

d. If Ram is clever, then Prem is well-behaved. If Joe is good, then Sam is bad and prem is not well-behaved. If Lal is educated, then Joe is good or Ram is clever. Hence, if Lal is educated and Prem is not well-behaved ,then Sam is bad.