UNIT - III . Numerical Integration

Numerical integration is used to obtain approximate answers for definite integrals that can not be solved analytically.

The process of evaluating a definite integral from a set of tabulated values of the integrand $f(x)$ which is not known explicitly is called numerical integration.

Newton - Cote's quadrature formula

Let $I = \int_a^b f(x)dx$ where $f(x)$ takes values $y_0, y_1, y_2, \ldots y_n$ for $x = x_0, x_1, x_2, \ldots x_n$

Let us devide the interval $[a, b]$ into n equal sub intervals of width h so that $x_0 = a$, $x_1 = x_0 + h$, $x_2 = x_0 + 2h$, $\ldots$ $x_n = x_0 + nh = b$.

Newton's forward difference formula is

$$y(x) = y(x_0 + ph) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0$$

where $\boxed{p = \frac{x - x_0}{h}}$

$$I = \int_a^b f(x)dx$$

$$= \int_{x_0}^{x_n} f(x)dx = \int_{x_0}^{x_0+nh}\left[y_0 + p\Delta y_0 + \frac{p(p-1)\cancel{(p-2)}}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \cdots\right]dx$$

Since $\boxed{x = x_0 + ph}$ $\boxed{dx = h\,dp}$ and p varies from 0 to n

hence above integral becomes

$$I = h\int_0^n \left[y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \cdots\right]dx$$

Since $x = x_0 + ph$ $dx = h\,dp$

and p varies from 0 to n hence above integral becomes

$$I = h \int_0^n \left[y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \cdots \right] dp$$

$$= h\left[p y_0 + \frac{p^2}{2}\Delta y_0 + \frac{1}{2}\left(\frac{p^3}{3} - \frac{p^2}{2}\right)\Delta^2 y_0 + \frac{1}{6}\left(\frac{p^4}{4} - p^3 + p^2\right)\Delta^3 y_0 + \cdots \right]_0^n$$

$$= h\left[n y_0 + \frac{n^2}{2}\Delta y_0 + \frac{1}{2}\left(\frac{n^3}{3} - \frac{n^2}{2}\right)\Delta^2 y_0 + \frac{1}{6}\left(\frac{n^4}{4} - n^3 + n^2\right)\Delta^3 y_0 + \cdots \right]$$

$$\int_{x_0}^{x_0+nh} f(x)\,dx = h\left[n y_0 + \frac{n^2}{2}\Delta y_0 + \frac{1}{2}\left(\frac{n^3}{3} - \frac{n^2}{2}\right)\Delta^2 y_0 + \cdots \right]$$

$$\boxed{\int_{x_0}^{x_0+nh} f(x)\,dx = h\left[n y_0 + \frac{n^2}{2}\Delta y_0 + \frac{1}{2}\left(\frac{n^3}{3} - \frac{n^2}{2}\right)\Delta^2 y_0 + \cdots \right]}$$

This is called Newton - Cotes quadrature formula.

It is called general quadrature formula from which we can derive various special formulae giving different values of n.

Trapezoidal Rule:

put $n=1$, in the quadrature formula. Then all differences higher than the first will become zero.

$$\int_{x_0}^{x_1} f(x)\,dx = \int_{x_0}^{x_0+h} f(x)\,dx = h\left[y_0 + \frac{1}{2}\Delta y_0\right] = h\left[y_0 + \frac{1}{2}(y_1 - y_0)\right]$$

$$= \frac{h}{2}(y_0 + y_1)$$

Similarly $\displaystyle\int_{x_0}^{x_2} f(x)\,dx = \int_{x_0+h}^{x_0+2h} f(x)\,dx = h\left[y_1 + \frac{1}{2}\Delta y_1\right]$

$$= h\left[y_1 + \frac{1}{2}(y_2 - y_1)\right]$$

$$= \frac{h}{2}(y_1 + y_2)$$

$$\int_{x_2}^{x_3} f(x)\,dx = \int_{x_0+2h}^{x_0+3h} f(x)\,dx = \frac{h}{2}(y_2 + y_3)$$

$$\cdots \int_{x_{n-1}}^{x_n} f(x)\,dx = \int_{x_0+(n-1)h}^{x_0+nh} f(x)\,dx = \frac{h}{2}(y_{n-1} + y_n)$$

Adding these n integrals we obtain

$$\int_{x_0}^{x_n} f(x)dx = \int_{x_0}^{x_0+h} f(x)dx + \int_{x_0+h}^{x_0+2h} f(x)\,dx + \cdots + \int_{x_{n-1}}^{x_n} f(x)\,dx$$

$$= \left[\frac{h}{2}(y_0+y_1) + \frac{h}{2}(y_1+y_2) + \cdots + \frac{h}{2}(y_{n-1}+y_n)\right]$$

$$\int_{x_0}^{x_n} f(x)dx = \frac{h}{2}\left[(y_0+y_n) + 2(y_1+y_2+\cdots+y_{n-1})\right]$$

$$\int_{x_0}^{x_n} f(x)dx = \frac{h}{2}\left[\text{Sum of first and last ordinates} + 2(\text{Sum of the remaining ordinates})\right]$$

This is known as the Trapezoidal rule.

<u>Note</u>: Trapezoidal rule can be applied to any number of subintervals odd or even.

$\boxed{\text{Simpson's } \tfrac{1}{3}^{rd} \text{ Rule:}}$

put $n=2$, in Newton's Cote's quadrature formula.

Since x takes one of three values x_0, x_1 or x_2 all the differences of third and higher order become zero.

$$\int_{x_0}^{x_2} f(x)dx = h\left[2y_0 + \frac{4}{2}\Delta y_0 + \frac{1}{2}\left(\frac{8}{3}-\frac{4}{2}\right)\Delta^2 y_0\right]$$

$$= h\left[2y_0 + 2(y_1-y_0) + \frac{1}{2}\left(\frac{2}{3}\right)\Delta^2 y_0\right]$$

$$= h\left[2y_0 + 2(y_1-y_0) + \frac{1}{3}(E^2-2E+1)\,y_0\right]$$

$$= h\left[2y_1 + \frac{1}{3}(y_2-2y_1+y_0)\right]$$

$$= h\left[\frac{y_2}{3} + \frac{4}{3}y_1 + \frac{y_0}{3}\right]$$

Thus $\displaystyle\int_{x_0}^{x_2} f(x)dx = \frac{h}{3}\left[y_0+4y_1+y_2\right]$

Similarly $\displaystyle\int_{x_2}^{x_4} f(x)dx = \frac{h}{3}\left[y_2+4y_3+y_4\right]$

Finally, $\displaystyle\int_{x_{n-2}}^{x_n} f(x)dx = \frac{h}{3}\left[y_{n-2}+4y_{n-1}+y_n\right]$, n being even

Adding all these integrals, we have when n is even

$$\int_{x_0}^{x_n} f(x)dx = \int_{x_0}^{x_2} f(x)dx + \int_{x_2}^{x_4} f(x)dx + \dots + \int_{x_{n-2}}^{x_n} f(x)dx$$

$$= \frac{h}{3}\left[(y_0+4y_1+y_2)+(y_2+4y_3+y_4)+\dots+(y_{n-2}+4y_{n-1}+y_n)\right]$$

$$\int_{x_0}^{x_n} f(x)dx = \frac{h}{3}\left[(y_0+y_n)+4(y_1+y_3+y_5+\dots)+2(y_2+y_4+y_6+\dots)\right]$$

$$\Rightarrow \int_{x_0}^{x_n} f(x)dx = \frac{h}{3}\Big[(\text{Sum of the first and last ordinates})$$
$$+ 4(\text{Sum of the odd ordinates})$$
$$+ 2(\text{Sum of the remaining even ordinates})\Big]$$

This is known as the Simpson's $\frac{1}{3}$rd rule or simply Simpson's rule, and the rule requires the given interval must be divided into an even number of equal subintervals of width "h".

Simpson's $\frac{3}{8}$th Rule:

$$\int_{x_0}^{x_n} f(x)dx = \frac{3h}{8}\left[(y_0+y_n)+3(y_1+y_2+y_4+\dots)+2(y_3+y_6+y_9+\dots)\right]$$

This formula is known as the Simpson's $\frac{3}{8}$th rule. This rule is employed by dividing the interval into n equal subintervals when n is a multiple of 3.

1) Given that

x	4	4.2	4.4	4.6	4.8	5	5.2
$y = \log x$	1.3863	1.4351	1.4816	1.5261	1.5686	1.6094	1.6487

Evaluate $\int_4^{5.2} \log x \, dx$ by (a) Trapezoidal rule (b) Simpson's 1/3 rule (c) Simpson's 3/8 rule.

Solution: Given $h = 0.2$, $y_0 = 1.3861$, $y_1 = 1.4351$, $y_2 = 1.4816$, $y_3 = 1.5261$, $y_4 = 1.5686$, $y_5 = 1.6094$, $y_6 = 1.6487$

(a) By the Trapezoidal rule, we have

$$\int_4^{5.2} \log x \, dx = \frac{h}{2}\left[(y_0+y_6) + 2(y_1+y_2+y_3+y_4+y_5)\right]$$

$$= \frac{0.2}{2}\left[(1.3861+1.6487) + 2(1.4351+1.4816+1.5261+1.5686+1.6094)\right]$$

$$= \frac{0.2}{2}\left[3.035 + 2(7.6208)\right]$$

$$= 0.1\left[18.2766\right] = \boxed{1.82766}$$

(b) Simpson's $\frac{1}{3}$rd rule, we have

$$\int_4^{5.2} \log x \, dx = \frac{h}{3}\left[(y_0+y_6) + 4(y_1+y_3+y_5) + 2(y_4+y_6)\right]$$

$$= \frac{0.2}{3}\left[3.035 + 4(4.5706) + 2(3.0502)\right]$$

$$= \boxed{1.82785}$$

(c) By the Simpson's 3/8th rule, we have

$$\int_4^{5.2} \log x \, dx = \frac{3h}{8}\left[(y_0+y_6) + 3(y_1+y_2+y_4+y_5) + 2y_3\right]$$

$$= \frac{0.6}{8}\left[3.035 + 3(6.0947) + 2(1.5261)\right]$$

$$= \boxed{1.82785}$$

(2) By using Simpson's $\frac{3}{8}$ th rule, evaluate $\int_0^1 \sqrt{1+x^4}\,dx$

Solution: Let $f(x) = \sqrt{1+x^4}$

Given $x_0 = 0$, $x_n = 1$ and take $n = 6$

$$h = \frac{b-a}{n} = \frac{1-0}{6} = \frac{1}{6}.$$

x	0	1/6	2/6	3/6	4/6	5/6	6/6 = 1
y	1 y_0	1.003 y_1	1.0062 y_2	1.0307 y_3	1.0943 y_4	1.2175 y_5	1.4142 y_6

Simpson's $\frac{3}{8}$ th rule is given by

$$\int_{x_0}^{x_n} f(x).dx = \frac{3h}{8}\left[(y_0+y_n) + 3(y_1+y_2+y_4+y_5) + 2(y_3)\right]$$

$$= \frac{3}{8}\cdot\frac{1}{6}\left[(1+1.4142) + 3(1.003 + 1.0062 + (1.0943) + 1.2175) + 2(1.0307)\right]$$

$$= \frac{1}{16}\left[2.4142 + 3(4.3183) + 2(1.0307)\right]$$

$$= \frac{1}{16}\left[17.4305\right] = 1.0894$$

(3) Evaluate $\int_0^6 \frac{dx}{(1+x^2)}$ using (1) Trapezoidal rule (2) Simpson's $\frac{1}{3}$ rule (3) Simpson's $\frac{3}{8}$ rule.

Solution: We divide the interval (0,6) into six equal parts. Here $f(x)\,dx = \frac{dx}{1+x^2}$ i.e. $\boxed{f(x) = \frac{1}{1+x^2}}$

The values of $f(x)$ are given below.

x	0	1	2	3	4	5	6
y	1 y_0	0.5 y_1	0.2 y_2	0.1 y_3	0.058824 y_4	0.03846 y_5	0.027027 y_6

(1) Trapezoidal rule:

$$\int_{x_0}^{x_n} f(x)dx = \frac{h}{2}\left[(y_0+y_n)+2(y_1+y_2+y_3+\cdots)\right]$$

$$\int_0^6 \frac{dx}{(1+x^2)} = \frac{h}{2}\left[(y_0+y_6)+2(y_1+y_2+y_3+y_4+y_5)\right]$$

$$= \frac{1}{2}\left[(1+0.027027)+2(0.5+0.2+0.1+0.058824+0.03846)\right]$$

$$= (0.5)\left[1.027027+2(0.897284)\right]$$

$$= (0.5)\left[2.821595\right] = \underline{1.4108}$$

(2) Simpson's $\frac{1}{3}$rd rule:

$$\int_{x_0}^{x_n} f(x)dx = \frac{h}{3}\left[(y_0+y_n)+4(y_1+y_3+y_5+\cdots)+2(y_2+y_4+y_6+\cdots)\right]$$

$$= \frac{1}{3}\left[(1+0.027027)+4(0.5+0.1+0.03846)+2(0.2+0.058824)\right]$$

$$= \frac{1}{3}\left[1.027027+4(0.63846)+2(0.258824)\right]$$

$$= \frac{1}{3}(4.0986) = \underline{1.3662}$$

(3) Simpson's $\frac{3}{8}$th rule:

$$\int_{x_0}^{x_n} f(x)dx = \frac{3h}{8}\left[(y_0+y_n)+3(y_1+y_2+y_4+y_5+y_7+\cdots)+2(y_3+y_6+\cdots)\right]$$

$$\int_0^6 \frac{dx}{(1+x^2)} = \frac{3}{8}\left[(1+0.027027)+3(0.5+0.2+0.058824+0.03846)+0.2\right]$$

$$= 0.375\left[1.027027+2.3918+0.2\right]$$

$$= \underline{1.3571}$$

(4) Evaluate $\int_0^{\pi} \frac{\sin x}{x} dx$ using (i) Trapezoidal rule (ii) Simpson's $\frac{1}{3}$rd rule (iii) Simpson's $\frac{3}{8}$th rule.

Solution: All the formulae are applicable if the number of subintervals is a multiple of 6.

So, we divide the $(0, \pi)$ into 6 equal parts

$$h = \frac{(\pi - 0)}{6} = \frac{\pi}{6}$$

x	0	$\pi/6$	$2\pi/6$	$3\pi/6$	$4\pi/6$	$5\pi/6$	$\frac{6\pi}{6} = \pi$
y	1	0.9549	0.8269	0.6366	0.4134	0.1909	0

Trapezoidal rule is given by

$$\int_{x_0}^{x_n} f(x)dx = \frac{h}{2}\left[(y_0+y_n) + 2(y_1+y_2+y_3+y_4+\cdots)\right]$$

$$\int_0^{\pi} \frac{\sin x}{x} dx = \frac{\pi}{12}\left[(1+0)+2(0.9549+0.8269+0.6366+0.4134+0.1909)\right]$$

$$= \frac{\pi}{12}\left[1 + 6.0454\right]$$

$$= (0.2618)(7.0454)$$

$$= 1.84448$$

Simpson's $\frac{1}{3}$rd rule is given by

$$\int_{x_0}^{x_n} f(x).dx = \frac{h}{3}\left[(y_0+y_n) + 4(y_1+y_3+y_5+\cdots) + 2(y_2+y_4+y_6+\cdots)\right]$$

$$= \frac{\pi}{18}\left[(1+0) + 4(0.9549+0.6366+0.19090) + 2(0.8269+0.4134)\right]$$

$$= \frac{\pi}{18}\left[1 + 7.1296 + 2.4806\right]$$

$$= \frac{\pi}{18}(10.6102) = 1.8518$$

Simpson's $\frac{3}{8}$th rule is given by

$$\int_{x_0}^{x_n} f(x)dx = \frac{3h}{8}\Big[(y_0+y_n)+3(y_1+y_2+y_4+y_5+\cdots) + 2(y_3+y_6+y_9+\cdots)\Big]$$

$$\int_0^{\pi}\left(\frac{\sin x}{x}\right)dx = \frac{3}{8}\cdot\left(\frac{\pi}{6}\right)\Big[(1+0)+3(0.9549+0.8269+0.4134+0.19090)+2(0.6366)\Big]$$

$$= \frac{\pi}{16}\Big[1+7.1583+1.2732\Big]$$

$$= \frac{\pi}{16}(9.4315) = 1.8515$$

(5) Use Simpson's $(1/3)^{rd}$ rule to find $\int_0^{0.6} e^{-x^2}dx$ by taking seven ordinates.

Solution: Let $f(x) = e^{-x^2}$. Here $x_0 = 0$, $x_n = 0.6$.
Divide the interval $(0, 0.6)$ into six equal parts.

$$h = \frac{x_n - x_0}{n} = \frac{(0.6-0)}{6} = \frac{0.6}{6} = 0.1$$

The values of $f(x)$ are given below.

x	0	0.1	0.2	0.3	0.4	0.5	0.6
$y=f(x)$	1 y_0	0.9900 y_1	0.9608 y_2	0.9139 y_3	0.8521 y_4	0.7788 y_5	0.6977 y_6

By Simpson's $\frac{1}{3}$rd rule, we have

$$\int_{x_0}^{x_n} f(x)dx = \frac{h}{3}\Big[(y_0+y_6)+4(y_1+y_3+y_5)+2(y_2+y_4)\Big]$$

$$= \frac{0.1}{3}\Big[(1+0.6977)+4(0.9900+0.9139+0.7788) + 2(0.9608+0.8521)\Big]$$

$$= \frac{0.1}{3}\Big[(1.6977)+4(2.6827)+2(1.8129)\Big]$$

$$= \frac{0.1}{3}\Big[1.6977+10.7308+3.6258\Big] = \frac{(16.0543)(0.1)}{3}$$

$$= 0.5351$$

(1) Evaluate $\int_0^1 \sqrt{(1+x^3)}\, dx$ taking $h=0.1$, using

(i) Trapezoidal rule Ans: 1.1123

(ii) Simpson's $\frac{1}{3}$rd rule Ans: 1.1114.

(2) Calculate an approximate value of $\int_0^{\pi/2} \sin x\, dx$ by

(a) Trapezoidal rule 0.9979

(b) Simpson's rule using 11 ordinates Ans: 1

(3) Evaluate $\int_0^{\pi/2} e^{\sin x}\, dx$ correct to four decimal places by Simpson's $\frac{3}{8}$ rule.

Ans: 3.1043

4) Evaluate $\int_0^{\pi/2} \sqrt{\sin\theta}\, d\theta$ using (i) Simpson's $1/3$ rule 1.1872

ii) Simpson's $3/8$ rule Ans: 1.18491

(5) Evaluate $\int_0^1 \frac{x}{(1+x^2)}\, dx$ using the Simpson's $(3/8)$th dividing the interval into 3 equal parts

Ans: 0.34807

(6) A rocket is launched from the ground. Its acceleration measured every 5 seconds is tabulated below. Find the velocity and the position of rocket at $t = 40$ seconds. Use trapezoidal rule as well as Simpson's rule.

t	0	5	10	15	20	25	30	35	40
a(t)	40	45.25	48.50	51.25	54.35	59.48	61.5	64.3	68.7

Let S be the distance traveled by the rocket with velocity $v(t)$. and acceleration $a(t)$ at time t

$a = \frac{dv}{dt}$. $v = \int_0^{40} a(t)\, dt$ $h=5$,

Trapezoidal: 2194.9

Simpson's $1/3$ rule: 2197.5

Numerical Solution of Ordinary Differential Equations

Numerical methods for the solution of first order differential equations of the form :

$$\frac{dy}{dx} = f(x,y), \text{ given } y(x_0) = y_0$$

Here the values of y are calculated in short steps for equal intervals of x and are therefore, termed as step by step methods.

Euler and Runge-Kutta (R.K) methods are used for computing y over a wide range of x. values

Here the initial condition is $y(x_0) = y_0$, specified at the point x_0. Such problems in which all the initial conditions are given at the initial point only are called initial value problem

Euler's method

The formula is

$$y_{n+1} = y_n + h\, f(x_n, y_n)$$

when $\frac{dy}{dx} = f(x,y)$, $y(x_0) = y_0$

and $x_i = x_{i-1} + h$

$n=0$, $y_1 = y_0 + h\, f(x_0, y_0)$

$n=1$, $y_2 = y_1 + h \cdot f(x_1, y_1)$

$n=2$, $y_3 = y_2 + h\, f(x_2, y_2)$

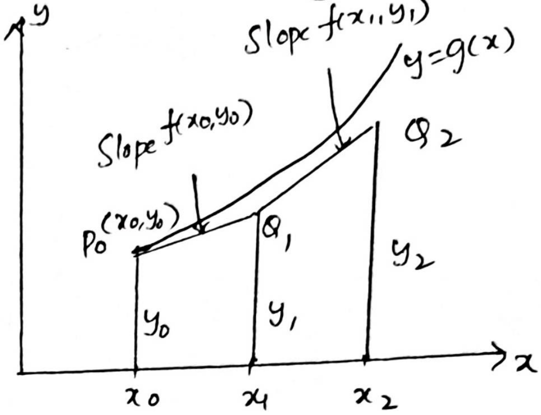
In general, we obtain a recursive relation as

$$y_{n+1} = y_n + h\, f(x_n, y_n)$$

Euler's method

In this method, we determine the change Δy is y corresponding to small increase in the argument x

Consider the differential equation



$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0 \quad \text{--- (1)}$$

Let $y = g(x)$ be the solution of (1)

Let $x_0, x_1, x_2, \ldots$ be equidistant values of x.

In this method, we use the property that in a small interval, a curve is nearly a straight line. Thus at the point (x_0, y_0) we approximate the curve by the tangent at the point (x_0, y_0)

The equation of the tangent at $P_0(x_0, y_0)$ is

$$y - y_0 = \left(\frac{dy}{dx}\right)_{P_0} (x - x_0)$$

$$= f(x_0, y_0)(x - x_0)$$

$$y = y_0 + f(x_0, y_0)(x - x_0)$$

$$y_1 = y_0 + f(x_0, y_0)(x_1 - x_0) \quad (\because \text{The curve is approximated by the tangent in the interval } (x_0, x_1))$$

$$y_1 = y_0 + h \cdot f(x_0, y_0)$$

$\therefore Q_1$ is (x_1, y_1)

Similarly approximating the curve in the interval (x_1, x_2) by a line through $Q_1(x_1, y_1)$ with slope $f(x_1, y_1)$ we get

$$y_2 = y_1 + f(x_1, y_1)(x_2 - x_1)$$

$$y_2 = y_1 + h\, f(x_1, y_1)$$

In general it can be shown that $\boxed{y_{n+1} = y_n + h f(x_n, y_n)}$

1. Using Euler method find an approximate value of y corresponding to $x = 1$; given that $\frac{dy}{dx} = x + y$ and $y = 0$ when $x = 0$. Choosing step length 0.2

Sol: By Euler's Method

$$y_{n+1} = y_n + h\, f(x_n, y_n) \quad -①$$

for $n = 0$, in ① $\quad y_1 = y_0 + h\, f(x_0, y_0)$

$f(x, y) = x + y \qquad y(0) = 0$

$x_0 = 0, \quad y_0 = 0$

$f(x_0, y_0) = x_0 + y_0$

$= 0 + 0 = 0.$

$\therefore y_1 = y_0 + (0.2)\, f(x_0, y_0) = 0 + (0.2) \cdot 0 = 0 \qquad \boxed{y_1 = 0}$

for $n = 1$, in ① $\quad y_2 = y_1 + h\, f(x_1, y_1)$

$x_1 = x_0 + h$

$f(x_1, y_1) = x_1 + y_1 \qquad = 0 + (0.2) = 0.2$

$= 0.2 + 0 \qquad y_1 = 0$

$f(x_1, y_1) = 0.2$

$y_2 = 0 + (0.2) \cdot (0.2) = 0.04 \qquad \boxed{y_2 = y(0.4) = 0.04}$

for $n = 2$, in ① $\quad y_3 = y_2 + h \cdot f(x_2, y_2)$

$x_2 = x_0 + 2h$

$= 0 + 2(0.2) = 0.4$

$f(x_2, y_2) = x_2 + y_2$

$= 0.4 + 0.04$

$= 0.44$

$y_3 = (0.04) + (0.2)\, f(x_2, y_2)$

$= (0.04) + (0.2)(0.44) \qquad \boxed{y_3 = y(0.6) = 0.128}$

$= .04 + 0.088 = 0.128$

$=$

for $n = 3$, in ① $\quad y_4 = y_3 + h \cdot f(x_3, y_3)$

$x_3 = x_0 + 3h$

$= 0 + 3(0.2) = 0.6$

$f(x_3, y_3) = x_3 + y_3$

$= 0.6 + 0.128 = 0.728$

$y_4 = 0.128 + (0.2) \cdot f(0.6, 0.128) = 0.128 + (0.2) \cdot (0.728)$

$= 0.128 + 0.1456 = 0.2736 \qquad \boxed{y_4 = y(0.8) = 0.2736}$

for $n = 4$, in ① $\quad y_5 = x_4 + y_4 \qquad x_4 = x_0 + 4h = 0 + 4(0.2) = 0.8$

$y_5 = 0.2736 + (0.2)\, f(0.8, 0.2736) = 0.2736 + (0.2) \cdot (0.8 + 0.2736)$

$= 0.2736 + (0.2)(1.0736) = 0.4883$

$\boxed{y_5 = y(x_5) = y(1) = 0.4883}$

2. Use Euler's method to approximate y when $x = 0.1$, given that $\frac{dy}{dx} = \frac{(y-x)}{(y+x)}$ $y(0) = 1$ by taking $h = 0.02$

Solution: ~~Here~~ $f(x,y) = \frac{(y-x)}{(y+x)}$, $y(0) = 1$

Given initial condition $y(0) = 1$, $x_0 = 0$, $y_0 = 1$.

Euler's method, we know that $y_{n+1} = y_n + h f(x_n, y_n)$

for $n=0$, $y_1 = y_0 + h \cdot f(x_0, y_0)$

$$= 1 + (0.02) \frac{(y_0 - x_0)}{(y_0 + x_0)} = 1 + (0.02) \frac{(1-0)}{(1+0)} = 1.02$$

$y_1 = y(x_1) = y(0.02) = 1.02$ $\boxed{y_1 = 1.02}$

for $n=1$, $y_2 = y_1 + h \cdot f(x_1, y_1)$ $x_1 = x_0 + h = 0 + 0.02 = 0.02$

$$= 1.02 + (0.02) \frac{(y_1 - x_1)}{(y_1 + x_1)} = 1.02 + (0.02) \frac{(1.02 - 0.02)}{(1.02 + 0.02)}$$

$$= 1.02 + (0.02) \cdot \frac{1}{(1.04)}$$

$y_2 = 1.0392$ $\boxed{y(x_2) = y(0.04) = 1.0392}$

for $n=2$, $y_3 = y_2 + h f(x_2, y_2)$ $x_2 = x_0 + 2h = 0 + 0.04 = 0.04$

$$= 1.0392 + (0.02) \frac{(y_2 - x_2)}{(y_2 + x_2)}$$

$$= 1.0392 + (0.02) \frac{(1.0392 - 0.04)}{(1.0392 + 0.04)}$$

$$= 1.0392 + (0.02) \frac{(0.9992)}{(1.0792)}$$

$y_3 = 1.0577$. $\boxed{y(0.06) = 1.0577}$

for $n=3$, $y_4 = y_3 + h f(x_3, y_3)$ $x_3 = x_0 + 3h = 0 + 0.06 = 0.06$

$$= 1.0577 + (0.02) \frac{(y_3 - x_3)}{(y_3 + x_3)}$$

$$= 1.0577 + (0.02) \frac{(1.0577 - 0.06)}{(1.0577 + 0.06)}$$

$y_4 = 1.0756$

$\boxed{y(x_4) = y(0.08) = 1.0756}$.

for $n=4$, $y_5 = y_4 + h f(x_4, y_4)$ $\quad x_4 = x_0 + 4h = 0 + 0.08 = 0.08$

$$= 1.0756 + (0.02)\frac{(y_4 - x_4)}{(y_4 + x_4)}$$

$$= 1.0756 + (0.02)\frac{(1.0756 - 0.08)}{(1.0756 + 0.08)}$$

$$= 1.0756 + 0.01721 = 1.09281$$

$y_5 = y(x_5) = y(0.1) = 1.09281.$

$\therefore \boxed{y(0.1) = 0.09281}$

3. using Euler's method find $y(0.6)$ of $y' = 1 - 2xy$ given that $y(0) = 0$. by taking $h = 0.2$

Sol: Here $f(x,y) = 1 - 2xy$

$y(0) = 0$, $h = 0.2$ $\qquad y(x_0) = y_0$

$y(0) = 0 \Rightarrow \boxed{x_0 = 0}$ $\boxed{y_0 = 0}$

for $n=0$, $y_1 = y_0 + h f(x_0, y_0)$ $\qquad f(x_0, y_0) = 1 - 2x_0 y_0$

$= 0.0 + (0.2) f(0,0)$ $\qquad f(0,0) = 1 - 2(0)(0)$

$= 1 - 0 = 1$

$= 0.0 + (0.2)(1) = 0.2$ $\qquad \boxed{y_1 = 0.2}$

for $n=1$, $y_2 = y_1 + h f(x_1, y_1)$ $\qquad f(0.2, 0.2)$

$= 0.2 + (0.2) f(0.2, 0.2)$ $\qquad = 1 - 2(0.2)(0.2)$

$= 0.2 + (0.2)(0.92)$ $\qquad = (1 - 0.08) = 0.92$

$= 0.384$ $\qquad \boxed{y_2 = 0.384}$

for $n=2$ $\quad y_3 = y_2 + h f(x_2, y_2)$

$= 0.384 + 0.2 f(0.2, 0.384)$

$= 0.384 + 0.2(1 - 2(0.2)(0.384))$

$= 0.384 + 0.2(1 - 0.3072)$

$= 0.384 + 0.13856 = 0.52256$

$x_3 = x_2 + h = 0.4 + 0.2 = 0.6$ we stop here

$\boxed{y(x_3) = y(0.6) = 0.52256}$

Fourth order Runge-Kutta Method (R.K Method)

It is one of the most widely used methods and is particularly suitable in cases when the computation of higher derivatives is complicated.

Consider the differential equation $y' = f(x, y)$ with the initial condition $y(x_0) = y_0$. Let h be the interval between equidistant values of x then the first increment in y is computed from the formula

$$\left.\begin{aligned} K_1 &= h \cdot f(x_0, y_0) \\ K_2 &= h\, f\left(x_0 + \frac{h}{2},\ y_0 + \frac{K_1}{2}\right) \\ K_3 &= h\, f\left(x_0 + \frac{h}{2},\ y_0 + \frac{K_2}{2}\right) \\ K_4 &= h \cdot f(x_0 + h,\ y_0 + K_3) \end{aligned}\right\} \quad \text{——— (1)}$$

$$\Delta y = \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4).$$

Taken in the given order: $y_1 = y_0 + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$

Then $x_1 = x_0 + h$, and $y_1 = y_0 + \Delta y$.

In a similar manner, the increment in y for the second interval is computed by means of the formulae

$$K_1 = h\, f(x_1, y_1)$$

$$K_2 = h\, f\left(x_1 + \frac{h}{2},\ y_1 + \frac{K_1}{2}\right)$$

$$K_3 = h\, f\left(x_1 + \frac{h}{2},\ y_1 + \frac{K_2}{2}\right)$$

$$K_4 = h\, f(x_1 + h,\ y_1 + K_3)$$

$$\Delta y = \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$$

and similarly for the next intervals.

This method is also termed as Runge-Kutta's method.

or $y_2 = y_1 + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$

1. use Runge-Kutta method of 4th order to find $y(0.1)$, $y(0.2)$ given that $\frac{dy}{dx} = y - x$; $y(0) = 2$

Sol: Here $f(x,y) = y - x$, $y(0) = 2$

ie $x_0 = 0$, $y_0 = 2$ Take $h = 0.1$

By Runge's Kutta Method

$$K_1 = h f(x_0, y_0)$$
$$= (0.1) f(0,2) = (0.1)(2) = 0.2$$

$$K_2 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}\right)$$
$$= (0.1) f(0.05, 2 + 0.1)$$
$$= (0.1)(2.1 - 0.05) = (0.1)(2.05) = 0.205$$

$$K_3 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}\right)$$
$$= 0.1 f(0.05, 2 + 0.1025)$$
$$= 0.1 f(0.05, 2.1025) = (0.1)(2.1025 - 0.05)$$
$$= (0.1)(2.0525) = 0.20525$$

$$K_4 = h f(x_0 + h, y_0 + K_3)$$
$$= (0.1) f(0.1, 2 + 0.20525)$$
$$= (0.1) f(0.1, 2.20525)$$
$$= (0.1)(2.10525) = 0.210525$$

Hence $K = \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$

$$= \frac{1}{6}\left[0.2 + 0.205 + 0.20525 + 0.210525\right]$$

$$= \frac{1}{6}\left[1.23105\right]$$

$$= 0.205175$$

$$\Rightarrow y_1 = y_0 + K = 2 + 0.205175$$
$$y_1 = 2.20517$$

Similarly $y_2 = y_1 + K$

$x_1 = x_0 + h = 0.1, \quad y_1 = 2.20517$. and $K_1 = h f(x_1, y_1)$

$K_2 = h f(x_1 + \frac{h}{2}, y_1 + \frac{K_1}{2})$

$K_3 = h f(x_1 + \frac{h}{2}, y_1 + \frac{K}{2})$

$K_4 = h f(x_1 + h, y_1 + K_3)$

$\therefore K = \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4) \Rightarrow y_2 = y_1 + K \Rightarrow$

$\boxed{y(0.2) = 2.42139}$

(2) using Runge's. Kutta method to obtain y when $x = 1.1$ given that $y = 1.2$ when $x = 1$ and y satisfies the equation $\frac{dy}{dx} = 3x + y^2$

<u>Solution:</u> Here $f(x,y) = 3x + y^2$; $x_0 = 1$, $y_0 = 1.2$, $h = 0.1$

$y_1 = y(x_0 + h) = y(1 + 0.1) = y(1.1)$ we need $y_1 = y_0 + K$.

To find K, By Runge's Kutta method. we have

$K_1 = h f(x_0, y_0)$

$= (0.1) f(1, 1.2) = (0.1)(3(1) + (1.2)^2) = 0.444$

$K_2 = h f(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2})$

$= (0.1) f(1.05, 1.4222)$

$= (0.1)(3(1.05) + (1.4222)^2)$

$= 0.5172$

$K_3 = h f(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2})$

$= 0.1 f(1.05, 1.2 + 0.2586) = 0.1 f(1.05, 1.4586)$

$= (0.1)[3(1.05) + (1.4586)^2]$

$= (0.1)[5.27751] = 0.527751$

$K_4 = h\, f(x_0+h,\ y_0+K_3)$

$= (0.1)\, f(1.01,\ 1.2+0.5277)$

$= (0.1)\, f(1.01,\ 1.7277)$

$= (0.1)\left[3(1.01) + (1.7277)^2\right]$

$= 0.6015$

$K = \frac{1}{6}\left[K_1 + 2K_2 + 2K_3 + K_4\right]$

$= \frac{1}{6}\left[0.444 + 2(0.5172) + 2(0.5277) + 0.6015\right]$

$= \frac{1}{6}\left[0.444 + 1.0344 + 1.0555 + 0.6015\right]$

$= \frac{1}{6}(3.1354) = 0.5225$

Hence $y_1 = y_0 + K = 1.2 + 0.5225 = 1.7225$.

3. Using Runge-Kutta method of order four, compute $y(0.2)$ in steps of 0.1. if $\frac{dy}{dx} = x+y^2$; given that $y=1$, when $x=0$.

Sol: Given $f(x,y) = x+y^2$ $x_0=0,\ y_0=1$.

Take $h=0.1$ By Runge's Kutta method

$K_1 = h\, f(x_0, y_0) = 0.1\, f(0,1)$

$= 0.1\,(0+1) = 0.1$

$K_2 = h.f\left(x_0+\frac{h}{2},\ y_0+\frac{K_1}{2}\right)$

$= (0.1)\, f(0.05,\ 1+0.05)$

$= (0.1)\, f(0.05,\ 1.05) = (0.1)\left[0.05 + (1.05)^2\right]$

$= (0.1)(1.1525)$

$= 0.11525$

$$K_3 = h f\left(x_0+\frac{h}{2}, y_0+\frac{K_2}{2}\right)$$

$$=(0.1)\ f\ (0.05\ ,1+0.057625)$$

$$=(0.1)\ [0.05+1.11855]$$

$$=(0.1)\ (1.1685) = 0.11685$$

$$K_4 = h\ f(x_0+\frac{h}{2},\ y_0+K_3)$$

$$=(0.1)\ f(0.1,\ 1.11685)$$

$$=(0.1)\ (1.3473) = 0.13473$$

$$K = \frac{1}{6}\left[K_1+2K_2+2K_3+K_4\right]$$

$$= \frac{1}{6}(0.1+2(0.1152)+2(0.1168)+0.1347)$$

$$= \frac{1}{6}(0.6989) = 0.1164$$

$$= 0.1165$$

$$\therefore y_1 = y_0+K = 1+0.1165 = 1.1165$$

To find $y(0.2)$ we find $y_2 = y_1+K$

$$x_1 = x_0+h = 0+0.1 = 0.1$$

$$y_1 = 1.1165,\quad h=0.1$$

By Runge's Kutta method

$$K_1 = h f(x_1,y_1) = (0.1)\ f\ (0.1,\ 1.1165)$$

$$= (0.1)\cdot(0.1+(1.1165)^2)$$

$$= (0.1)\ (1.3465)$$

$$= 0.13465$$

$$K_2 = h f(x_1+\frac{h}{2},\ y+\frac{K_1}{2}) = 0.1 f\left(0.1+\frac{0.1}{2},\ 1.1165+0.67328\right)$$

$$= 0.1\ f\ (0.15,\ 1.18382)$$

$$= 0.1\ (0.15+(1.18382)^2)$$

$$= 0.155144$$

$$K_3 = hf\left(x_1 + \frac{h}{2},\ y_1 + \frac{K_2}{2}\right)$$

$$= 0.1\, f(0.15,\ 1.1165 + 0.077757)$$

$$= 0.1\, f(0.15,\ 1.194072)$$

$$= 0.1\,(0.15 + (1.194072)^2)$$

$$= (0.1)(0.15 + 1.4258)$$

$$= 0..157580$$

$$K_4 = h\, f(x_1 + h,\ y_1 + K_3)$$

$$= (0.1)\, f(0.2,\ 1.1165 + 0.157580)$$

$$= (0.1)\, f(0.2,\ 1.27408)$$

$$= (0.1)(0.2 + 1.6232)$$

$$= (0.1)(1.823279)$$

$$= 0.1823279$$

$$K = \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$$

$$= \frac{1}{6}\left(0.1346 + 2(0.1551) + 2(0.1575) + (0.1823)\right)$$

$$= \frac{1}{6}(0.1346 + 0.3102\,8 + 0.3151 + 0.1823)$$

$$= \frac{1}{6}(0.9424) = 0.1570$$

$$y_2 = y_1 + K = 1.1165 + 0.1570$$

$$= 1.2735$$

$$\therefore y(0.2) = 1.2735.$$

H.W

(1) using Runge-Kutta method find $y(0.2)$

given $\frac{dy}{dx} = 3x + \frac{y}{2}$

$y(0) = 1$ Taking $h = 0.1$

Ans $y_1 = y(0.1) = 1.0665$

$y_2 = y(0.2) = 1.1672$

(2) Given $\frac{dy}{dx} = \frac{(y^2 - x^2)}{(y^2 + x^2)}$ $x_0 = 0$, $y_0 = 1$.

$h = 0.2$ find y_1 and y_2 by using Runge-Kutta method

Ans: $y(0.2) = 1.19599$

$y(0.4) = 1.3751$