2.3

Equivalence relations and Compatibility relations

We first discuss **two** methods of representing relations on a set. One method uses zero-one matrices and the other method uses directed graphs.

Relation Matrix of a Relation

Let A and B be finite sets with m and n elements respectively. Let the elements of A and B be ordered in certain order say $A = \{a_1, a_2, \dots, a_m\}$ and $B = \{b_1, b_2, \dots, b_n\}$. A relation R from A to B can be represented by an $m \times n$ matrix called the **relation matrix of R**, denoted by $\boldsymbol{M_R}$ with entries $0's$ and $1's$.($i.e.$, bits) and $M_R = (m_{ij})_{m\times n}$ is defined as

$$\boldsymbol{m_{ij}} = \begin{cases} \mathbf{1} & \boldsymbol{if} \quad \boldsymbol{a_i R\ b_j} \\ \mathbf{0} & \boldsymbol{if} \quad \boldsymbol{a_i \not{R}\ b_j} \end{cases}$$

Example 1: If $A = \{1, 2, 3, 4\}$, $B = \{1, 4, 6, 8, 9\}$ and a relation R from A to B is defined by: for $a, b \in A$, $a\ R\ b$ iff $a|b$.Write R and the matrix of R.

Solution: We have, $R = \{(a, b)|\ a \in A\ \wedge b \in B\ \wedge\ (a|b)\}$

$R\ = \{(1,1), (1,4), (1,6), (1,8), (1,9), (2,4), (2,6), (2,8), (3,6), (3,9), (4,4), (4,8)\}$

and $M_R = (m_{ij})_{4\times5}$ where $m_{ij} = \begin{cases} 1 & if \quad a_i \mid b_j \\ 0 & if \quad a_i \nmid b_j \end{cases}$

Therefore, $M_R = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}$

Note: The matrix of a relation from A to B depends on the ordering of the elements of A and B.

Conversely, given two sets A and B with $|A| = m$ and $|B| = n$, an $m \times n$ matrix whose entries are zeros and ones determines a relation .

Consider the matrix

$$M = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

Since M is a 3×4 matrix, we let $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2, b_3, b_4\}$ and we write down the corresponding relation R from A to B as follows

$$(a_i,\ b_j) \in R \text{ iff } m_{ij} = 1.$$

Therefore, $R = \{(a_1, a_1),\ (a_1, b_4), (a_2, b_2), (a_2, b_3), (a_3, b_1), (a_3, b_3)\}$

Throughout this module we consider the relations on a set A, $i.e.$,The relations form A to itself.

A relation matrix M_R of a relation R on a set A reflects some of the properties of R.

a) If R is reflexive, then all the diagonal entries of M_R must be 1.
b) If R is symmetric, then M_R is a symmetric matrix.
c) If R is antisymmetric, then M_R is such that $m_{ij} = 0$ whenever $m_{ji} = 1$ for $i \neq j$.

Graph of a relation:

A relation can also be represented pictorially by drawing its graph. Although we shall introduce some of the concepts of graph theory which are discussed in a sub sequent unit, here we shall use graph only as a tool to represent relations.

Let R be a relation on a set $A = \{a_1, a_2, \dots, a_m\}$ (we order the elements of A in a certain order and keep the ordering fixed throughout the discussion). The elements of A are represented by points (or small circles) called **nodes** (or **vertices**).

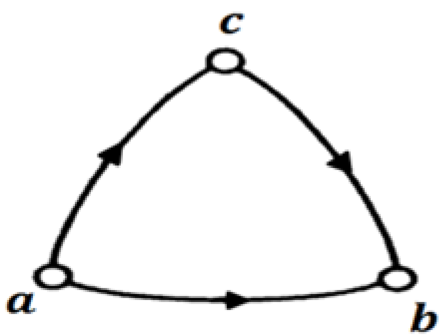
If $a_i\ R\ a_j$, $i.e.$, $(a_i, a_j) \in R$, then we connect the nodes a_i and a_j by means of an arc ,called an **edge**, and put an arrow mark on the arc form a_i to a_j. When all nodes corresponding to all the ordered pairs in R are connected by arcs with proper arrows, we get a **graph** (**directed graph** and **digraph**) of the relation R.

If $a_i \; R \; a_j$ and $a_j \; R \; a_i$, then we draw to arcs a_i to a_j and a_j to a_i. For the sake of simplicity, we may replace two arcs by one arc with arrows pointing in both directions. If $a_i R \; a_i$, we get an arc which starts from a_i and returns to a_i. Such an arc is called a **loop**.

A node $a \in A$ is said to be an **isolated node**, if $a \not{R} x$ and $x \not{R} \; a$, $for \; all \; x \in A, \; x \neq a$

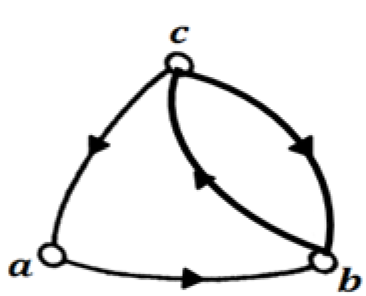
Example 2: Let $A = \{a, b, c\}$

(i) Let $R_1 = \{(a,b),(a,c),(c,b)\}$. Then its digraph and M_{R_1} are given by

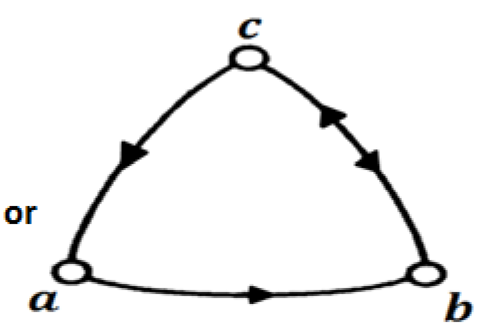


$$M_{R_1} = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

(ii) Let $R_2 = \{(a,b),(c,a),(c,b),(b,c)\}$. Then

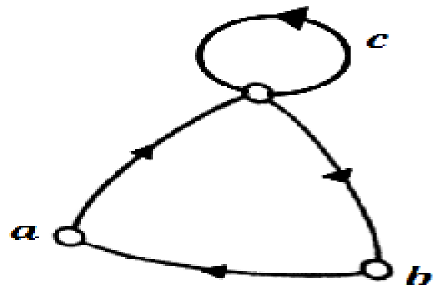


or



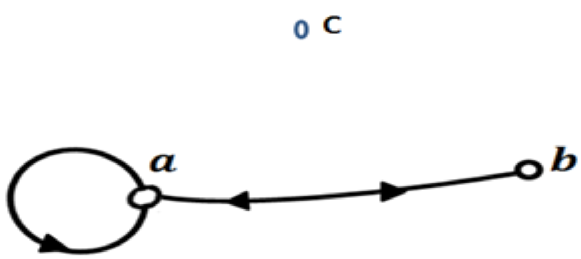
$$M_{R'_2} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

(iii) Let $R_3 = \left\{(a, c), (c, c), \left(c, b\right), (b, a)\right\}$



$$M_{R_3} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

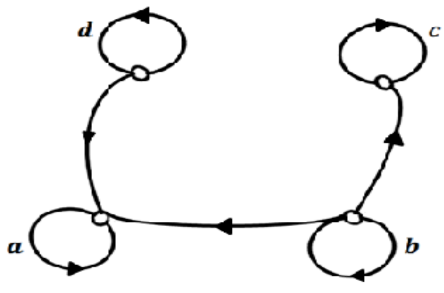
(iv) Let $R_4 = \{(a, a), (a, b), (b, a)\}$



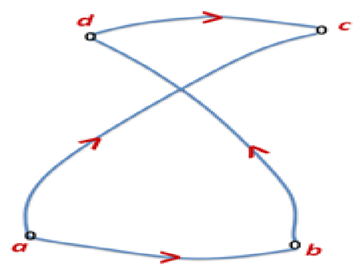
$$M_{R_4} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

From the graph of a relation it is possible to observe some of its properties

If a relation is reflexive, then there must be loops at each node. On the other hand, if the relation is irreflexive, then there is no loop at any node .



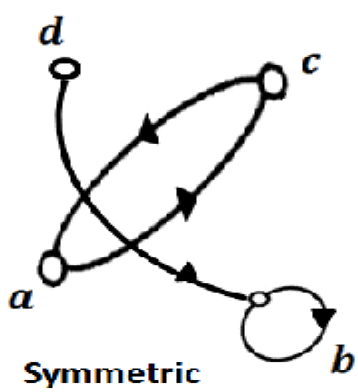
Reflexive relation



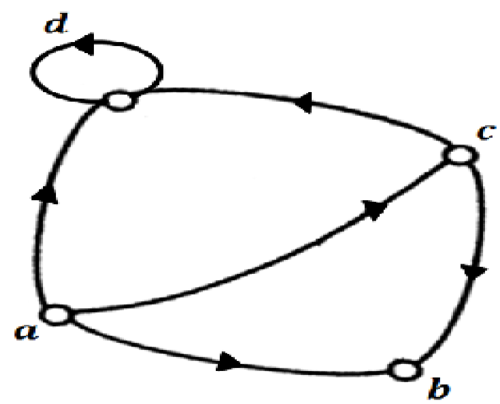
Irreflexive relation

If a relation is symmetric then there must be an arc form one node b to another node a whenever there is an arc from a to b.

If a relation is antisymmetric then an arc from one node b to another node a $(a \neq b)$ does not exist, whenever there is an arc form a to b.

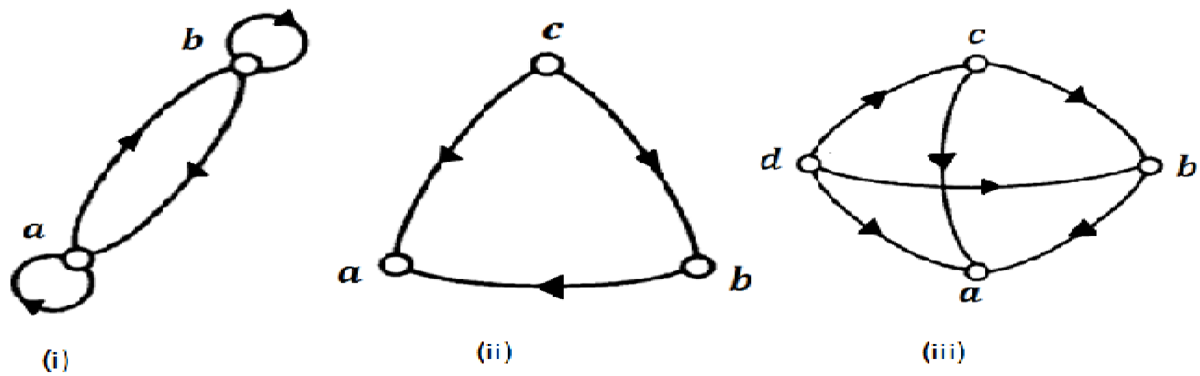


Symmetric



Antisymmetric relation

If a relation is transitive, then the situation is not so simple. In any case, its digraph must have circuits of the type shown in the following figure



Transitive

Note:

i) When the number of elements in a set A over which a relation R is defined is large, say greater than or equal to 5 or 6, both the graphical and the matrix representations of the relation become unwieldy. However, the matrix representation can be considered, since the computing machines can handle matrices well.

ii) It is easy to determine from the matrix of the relation whether the relation is reflexive or symmetric. But it is not always easy to determine from the matrix whether the relation is transitive.

iii) The entries of the relation matrix are denoted by T and F instead of 1 and 0 respectively in order to conserve storage (note that in FORTRAN only 1 byte is needed for each logical entry, but at least 2 bytes are required for an integer entry).

Partition and Covering of a set

Let A be a given set and $C = \{A_1, A_2, \dots, A_m\}$, where each $A_i, i = 1, 2, 3, \dots, m$ is a subset of A. We say that C is a ***covering*** of A if $A = \bigcup_{i=1}^{m} A_i$ and the sets $A_1, A_2, \dots, A_m$ are said to ***cover*** A.

We say that C is a **partition** of A if $A_i \cap A_j = \emptyset$, when $i \neq j$ and $A = \bigcup_{i=1}^{m} A_i$

That is C is a partition of A, if C is a covering of A and the elements of C, which are subsets of A are mutually disjoint. If C is a partition of A, then the sets $A_1, A_2, \dots, A_m$ of C are called the ***blocks*** of the partition.

For example, Let $A = \{a, b, c\}$ and consider the following collection of subsets of A.

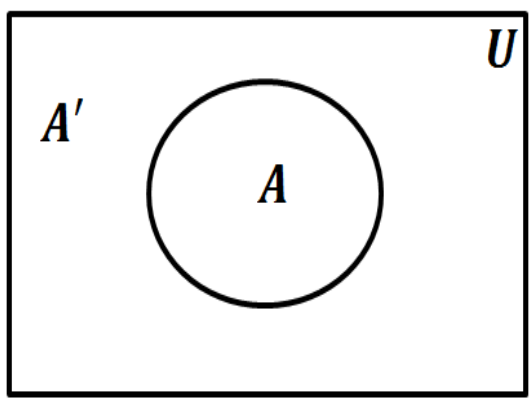
$$C_1 = \{\{a, b\}, \{b, c\}\}, C_2 = \{\{a\}, \{b, c\}\}$$

$$C_3 = \{\{a\}, \{b\}, \{c\}\}, C_4 = \{\{a\}, \{a, b\}, \{a, c\}\}$$

The sets C_1 and C_4 are coverings of A and are not partitions of A.The sets C_2 and C_3 are partitions of A.

Some partitions of the Universal set U

First consider a subset A of U. The subsets A and A' generate a partition of U, since $A \cap A' = \emptyset$ and $A \cup A' = U$.

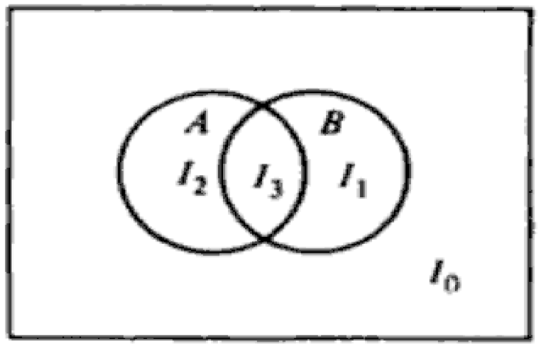


Let A and B be any two subsets of U.Consider the subsets

$$I_0 = A' \cap B', I_1 = A' \cap B, I_2 = A \cap B' \text{ and } I_3 = A \cap B$$

The sets I_0, I_1, I_2 and I_3 are called ***complete intersections*** or the ***minterms*** generated by the subsets A and B. Note that $I_i \cap I_j = \emptyset, i \neq j$ and $U = \bigcup_{i=0}^{3} I_i$.

Therefore,$P = \{I_0, I_1, I_2, I_3\}$ is a partition of U. The complete intersections or the minterms generated by A and B are the blocks of a partition of U.

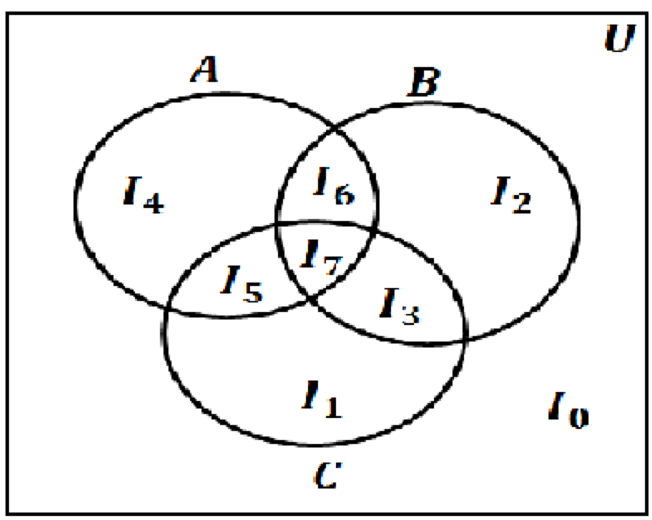


Let A, B and C be three subsets of U and let the 2^3 minterms denoted by I_i, $i = 0,1,2,\dots,7$

$I_0 = A' \cap B' \cap C'$, $\quad I_1 = A' \cap B' \cap C$, $\quad I_2 = A' \cap B \cap C'$, $\quad I_3 = A' \cap B \cap C$,
$I_4 = A \cap B' \cap C'$, $\quad I_5 = A \cap B' \cap C$, $\quad I_6 = A \cap B \cap C'$, $\quad I_7 = A \cap B \cap C$

The subscript of I shows indirectly the minterm under consideration as in the propositional calculus. Note that $I_i \cap I_j = \phi$, $i \neq j$ and $\boldsymbol{U} = \bigcup_{\boldsymbol{i}=0}^{7} \boldsymbol{I_i}$

Therefore $P = \{I_0, I_1, I_2, \dots, I_7\}$ is a partition of U. The minterms generated by A, B and C are the blocks of a partition of U.



In general, if $A_1, A_2, \dots, A_n$ are n subsets of the universal set U, then the complete intersections or minterms generated by these n subsets are denoted by

$I_0, I_1, I_2, \dots, I_{2^n-1}$. These are mutually disjoint and $U = \bigcup_{i=0}^{2^n-1} I_i$

Note: Note the similarity between the minterms defined here and those given in the propositional calculus.

Equivalence relation:

A relation R on a set A is called an **equivalence relation** if it is reflexive, symmetric and transitive.

Equivalence class: Let R be an equivalence relation on a set A. For any $x \in A$, the set $[x]_R \subseteq X$ defined by

$$[\boldsymbol{x}]_{\boldsymbol{R}} = \{\boldsymbol{y} | \boldsymbol{y} \in \boldsymbol{X} \wedge \boldsymbol{xRy}\}$$

is called an ***R*- equivalence class** generated by $x \in X$.That is, the set $[x]_R$ consists of all the ***R*-relatives** of x in the set A.
Some times $[x]_R$ is also written as x/R.

Properties of equivalence classes

1) $\boldsymbol{x \in [x]_R}$ **, for all** $\boldsymbol{x \in R}$.
   For any $x \in A$, we have xRx (since R is reflexive).Therefore $x \in [x]_R$,$\forall x \in A$.
   That is **each element of A generates an R-equivalence class $[x]_R$ which is nonempty.**
2) $\boldsymbol{[x]_R = [y]_R}$ **iff** $\boldsymbol{xRy}$.
   Suppose $[x]_R = [y]_R$.Clearly, from above $y \in [y]_R$. Therefore, $y \in [x]_R$ and this means xRy.
   Conversely, suppose xRy. If $z \in [y]_R$ then yRz. Now xRy and yRz imply xRz (since R is transitive). Thus, $z \in [x]_R$.Therefore $[y]_R \subseteq [x]_R$ when xRy.
   Further $xRy \Longrightarrow yRx$ (since R is symmetric).Therefore $[y]_R \subseteq [x]_R$.
   Thus , if xRy then $[x]_R = [y]_R$.

3) **If $\boldsymbol{x} \not{R}\, \boldsymbol{y}$ then $[\boldsymbol{x}]_{\boldsymbol{R}} \cap [\boldsymbol{y}]_{\boldsymbol{R}} = \boldsymbol{\phi}$.**
   Assume the contrary, $i.e.$, $[x]_R$ and $[y]_R$ are not disjoint. Therefore, there is at least one element $z \in [x]_R$ and $z \in [y]_R$; i.e., xRz and yRz. Since R is transitive we have xRy. This is a contradiction. The result now follows.

Theorem 1: Every equivalence relation $\boldsymbol{R}$ on a set generates a unique partition of the set. The blocks of this partition correspond to the $\boldsymbol{R}$-equivalence classes.

Proof: Let R be any equivalence relation on a set A. It is known from the properties of the equivalence classes, each element $x \in A$ generates an R- equivalence class which is non empty and any two R- equivalence classes are either disjoint or equal. Thus, the family of R-equivalence classes generated by the elements of A defines a partition P of A.That is the R-equivalence class are the blocks of this partition P. Such a partition P is unique because an R-equivalence class of any element of A is unique. Hence the theorem.

Quotient set: We shall denote the family of equivalence classes of R on A by A/R, (also written as $\boldsymbol{A}$ **modulo** $\boldsymbol{R}$ or $\boldsymbol{A}$ **mod** $\boldsymbol{R}$ (inshort)) called the **quotient set of $\boldsymbol{A}$ by $\boldsymbol{R}$**.

Two special equivalence relations:

Consider two special equivalence relations on a set A.

i) The relation $R_1 = A \times A$, *i.e.,* the universal relation on A. Note that every element of A is in R_1-relation to all the elements of A.In this case the quotient set of A by R_1 is the set $\{A\}$.
ii) The relation $R_2 = \{(a,a)|a \in A\}$, i.e., the equality relation or **identity relation** on A. It is an equivalence relation on A and $[a] = \{a\}$ for each $a \in A$ the quotient set of A by R_2 consists of all singleton subsets of A. Note that R_2 generates the largest partition of A.

Example 3: If R is the equivalence relation "*congruence modulo* 5" on Z, then find equivalence classes generated by the elements of Z and the quotient set $\boldsymbol{Z/R}$.

Solution:

$$[0]_R = \{y | y \in \boldsymbol{Z} \wedge 0Ry\} = \{y \in \boldsymbol{Z} \,| y \equiv 0(mod\ 5)\}$$

$$= \{y \in \boldsymbol{Z} \,| y = 5k\ , k \in Z\} = \{\dots, -10, -5, 0, 5, 10, \dots\}$$

$$[1]_R = \{y \in \boldsymbol{Z} \,| yR1\} = \{y \in Z | y \equiv 1(mod\ 5)\}$$

$$= \{y \in \boldsymbol{Z} | y - 1 \text{ is divisible by } 5\} = \{y \in \boldsymbol{Z} \,| y - 1 = 5k\ , k \in \boldsymbol{Z}\}$$

$$= \{y \in \boldsymbol{Z} \,| y = 5k + 1, k \in \boldsymbol{Z}\}$$

$$= \{\dots, -9, -4, 1, 6, 11, \dots\}$$

$$[2]_R = \{y \in \boldsymbol{Z} | y \equiv 2(mod\ 5)\} = \{y \in \boldsymbol{Z} \,| y = 5k + 2, k \in \boldsymbol{Z}\}$$

$$= \{\dots, -8, -3, 2, 7, 12, \dots\}$$

$$[3]_R = \{y \in \boldsymbol{Z} | y \equiv 3(mod\ 5)\} = \{5k + 3 \mid k \in \boldsymbol{Z}\}$$

$$= \{\dots, -7, -2, 3, 8, 13, \dots\}$$

$$[4]_R = \{y \in \boldsymbol{Z} | y \equiv 4(mod\ 5)\} = \{5k + 4 \mid k \in \boldsymbol{Z}\}$$

$$= \{\dots, -6, -1, 4, 9, 14, \dots\}$$

Therefore, the quotient $\boldsymbol{Z}/R$ is a family of an equivalence classes of R on $\boldsymbol{Z}$.
Thus, $\boldsymbol{Z}/R = \{[0]_R, [1]_R, [2]_R, [3]_R, [4]_R\}$

In a similar manner we can find the equivalence classes generated by an equivalence relation "*congruence relation modulo* m" on $\boldsymbol{Z}$, where $m \in \boldsymbol{N}, m > 1$.

Example 4: Let S be the set of all well-formed formulas in n proposition variables and let R be the relation given by

$$\boldsymbol{R} = \{(\boldsymbol{a}, \boldsymbol{b}) | \boldsymbol{a} \in \boldsymbol{S} \wedge \boldsymbol{b} \in \boldsymbol{S} \wedge (\boldsymbol{a} \Leftrightarrow \boldsymbol{b})\}$$

Prove that R is an equivalence relation on S

Solution: For any $a, b \in S$, a is equivalent to b (written as $a \Leftrightarrow b$ or $a \equiv b$) if a and b have the same truth values in their truth tables, $i.e.$, $a \leftrightarrow b$ is a tautology.

Let R be the propositional equivalence. Let $a, b, c \in S$. Clearly R is reflexive. If aRb, i.e., a and b have the same truth values in their truth tables then b and a also have the same truth values in their truth tables, i.e., bRa. Thus, R is symmetric.

If aRb and bRc, i.e., a, b have the same entries in their truth tables and b, c have the same entries in their truth tables, then a and c also have the same entries in their truth tables, $i.e.$, aRc. Thus, R is transitive.

Therefore, R is an equivalence relation on S and R partitions S into mutually disjoint equivalence classes.

Note that the set S of all well-formed formulas in n propositional variables is an infinite set.

Observe that there are 2^n rows in the truth table for any formula in n variables. Since each row can have any one of the truth values T or F, we have 2^{2^n} possible truth tables. Every formula in n variables will have one of these 2^{2^n} truth tables. All those formulas which have one of these truth tables are equivalent to each other and are in one R-equivalence class. Since there are 2^{2^n} distinct truth tables, there are 2^{2^n} R-equivalence classes generated by the elements of S.

Converse of Theorem 1

So far we have considered the partition of a set A generated by an equivalence relation on A. We now show that the converse of Theorem 1 is also true. That is, if we start with a definite partition P of a given set A, then we can define an equivalence relation R with $A/R = P$.

Theorem 2: If P is any partition of a set A then there is an equivalence relation R on A whose equivalence classes are the blocks of the partition P.

Proof: Let $P = \{A_1, A_2, \dots, A_m\}$ be a partition of A. That is $A_i \cap A_j = \phi, i \neq j$ and $\bigcup_{i=1}^{m} A_i = A$. For each $x \in A$ there is a block say A_1 of P such that $x \in A_1$. Evidently x does not belong to any other block of P.

We now take all the elements of $A_1 \times A_1$ as members of a relation R. Clearly, every element of A that is in A_1 is R-related to every other element of A_1. Further, no other element of A which is not in A_1 is related to the elements of A_1. Similarly, we take other blocks A_i , $i = 2,3, \dots, m$ and take all the elements of $A_i \times A_i$ as member of R. Then

$$R = (A_1 \times A_1) \cup (A_2 \times A_2) \cup \dots \cup (A_m \times A_m)$$

It is straight forward to check R is an equivalence relation on A and for any $a_i \in A_i$, $i = 1,2, \dots, m$. We see

$$[a_i]_R = \{y \in A | yRa_i\} = A_i \text{ and}$$

$$A/R = \{[a_i]_R | i = 1,2, \dots, m\} = \{A_1, A_2, \dots, A_m\} = P$$

Hence the theorem.

Theorem 1 and Theorem 2 together proves the following result:

An equivalence relation on a set generates a partition of the set and conversely

Example 5: Let $A = \{a, b, c, d, e\}$ and let $P = \big\{\{a, b\}, \{c\}, \{d, e\}\big\}$. Obtain the equivalence relation defined by P.

Solution: Let R be the equivalence relation defined by the partition $P = \{A_1, A_2, \dots, A_m\}$. Then $R = (A_1 \times A_1) \cup (A_2 \times A_2) \cup \dots \cup (A_m \times A_m)$.

The required equivalence relation is

$$R = \big\{\{a, b\} \times \{a, b\}\big\} \cup \big\{\{c\} \times \{c\}\big\} \cup \big\{\{d, e\} \times \{d, e\}\big\}$$

$= \{(a,a),(a,b),(b,a),(b,b),(c,c),(d,d),(d,e),(e,d),(e,e)\}$

Compatibility Relations

A relation R on a set A is said to be a ***Compatibility relation*** if it is reflexive and symmetric.

Let R be a compatibility relation on A and $x, y \in A$, we say that x, y are **compatible** if xRy. A compatibility relation is sometimes denoted by $\approx$.

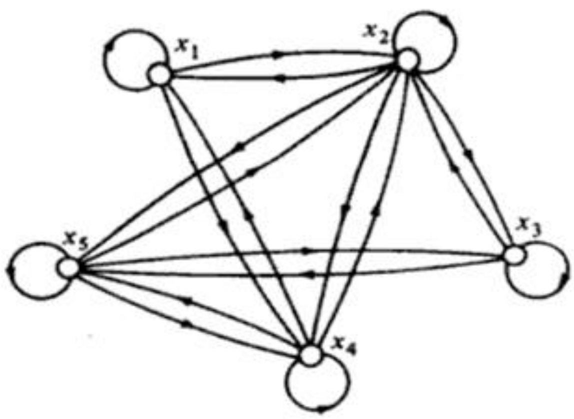
Compatibility relations are useful in solving certain minimization problems of switching theory, particularly for incompletely specified minimization problems.

Evidently all equivalence relations are compatibility relations. We are concerned with those compatibility relations which are not equivalent relations.

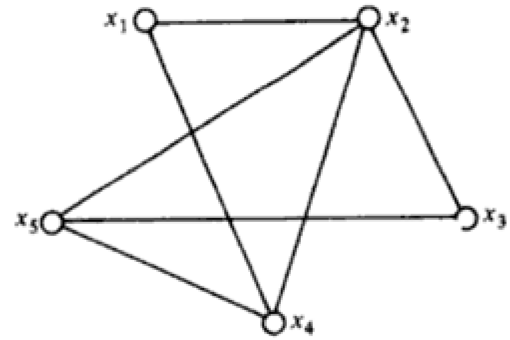
Example 6: Let $A = \{ball, bed, dog, let, egg\}$ and let R be a relation on X given by for $x, y \in A$, xRy iff x and y contain some common letter.

Clearly R is reflexive. If xRy $i.e.$, x and y contain some common letter, then y and x contain the same common letter $i.e.$, yRx. This shows that R is symmetric. Thus, R is a compatibility relation on A. Note that *ball* $\approx$ *bed* and *bed* $\approx$ *egg* but *ball* $\not\approx$ *egg*. This shows that $\approx$ is not transitive.

Thus, R is a compatibility relation. Denote "ball", "bed" , "dog", "let" and "egg" by x_1, x_2, x_3, x_4 and x_5 respectively. The graph of the compatibility relation $\approx$ on A is given below.



(i)



(ii) Simplified graph of compatibility relation

Since $\approx$ is a compatibility relation, it is not necessary to draw the loops at each node nor is it necessary to draw both the arcs xRy and yRx. Thus ,we can simplify the graph of $\approx$ on A as shown in the figure (ii)

The relation matrix of a compatibility relation is symmetric and has diagonal elements unity. It is therefore, sufficient to give only the elements of the lower triangular part of the relational matrix in such a case. For this compatibility relation, the relation matrix is given below.

x_2	1			
x_3	0	1		
x_4	1	1	0	
x_5	0	1	1	1
	x_1	x_2	x_3	x_4

Note that the elements in each of the sets $\{x_1, x_2, x_4\}$ and $\{x_2, x_3, x_5\}$ are mutually compatible. Notice that the union of these two sets is A. Therefore, $\{\{x_1, x_2, x_4\}, \{x_2, x_3, x_5\}\}$ is a covering of A. It may be seen that the elements of the set $\{x_2, x_4, x_5\}$ are also mutually compatible.

An equivalence relation on a set A defines a partition of A into equivalence classes and a compatibility relation does not necessarily define a partition. However, a compatibility relation on A defines a covering of the set A.

Maximal compatibility block

Let A be a set and $\approx$ a compatibility relation on A. A subset $M \subseteq A$ is called a ***maximal compatibility block*** if any element of M is compatible to every other element of M and no element of $X - M$ is compatible to all the elements of M. From figure (ii) that the subsets $\{x_1, x_2, x_4\}, \{x_2, x_3, x_5\}$ and $\{x_2, x_4, x_5\}$ are maximal compatible blocks.

There are two procedures to find the maximal compatibility block.

Procedure 1

To find the maximal compatibility blocks corresponding to a compatibility relation R on a set A, first draw a simplified graph of R and pick from this graph the **largest complete polygons**. By a largest complete polygon we mean a polygon in which any vertex is connected to every other vertex. The set of elements of A which are the vertices of these largest complete polygons are the maximal compatibility blocks of R. In addition to these largest complete polygons, any element of A which is related only to itself forms a maximal compatibility block. Also, any two elements which are compatible to one another but not compatible to any other elements is a maximal compatibility block.

Procedure 2

This procedure is to find maximal compatibility block from the table of the relation matrix.

First delete all the elements which are only compatible to themselves and obtain the simplified table of the relation matrix, because they are in a maximal compatible block by themselves and are in no other compatibility block. Such blocks are listed at the end.

Step 1: Start in the right most column of the table and proceed to the left until a column containing at least one nonzero entry is encountered. List all the compatible pairs represented by the entries in that column.

Step 2: Proceed left to the next column that contains at least one nonzero entry.

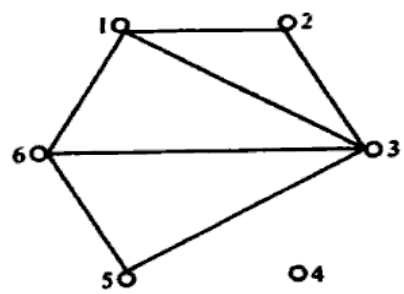
a) If any element is compatible to all the members of some previously defined compatibility class, then add this element to that class.
b) If a member is compatible to only some members of a previously defined class, then form a new class which includes all the elements that are compatible.

Next, list all the compatible pairs not included in any previously defined class.

Step 3: Repeat step 2 until all the columns are considered.

The final sets of compatibility classes including those which are isolated elements constitute the maximal compatibility classes (blocks).

Example 7: The graph of a compatibility relation is given below. Write its relation matrix and find its maximal compatibility block.



Solution: We have $A = \{1,2,3,4,5,6\}$ and Its relation matrix is

2	1				
3	1	1			
4	0	0	0		
5	0	0	1	0	
6	1	0	1	0	1
	1	2	3	4	5

Procedure 1

The largest complete polygons are $\{1,2,3\}, \{1,3,6\}, \{3,5,6\}$ and these are maximal compatibility blocks. $\{4\}$ is also a m.c.b. Note that 4 is an isolated node.

Procedure 2

We delete all the elements which are only compatible to themselves. That is we denote the element 4, because 4 is only compatible to itself. The node 4 is an isolated node. Now, the simplified table of the relation matrix is

2	1			
3	1	1		
5	0	0	1	
6	1	0	1	1
	1	2	3	5

Step 1: $(5,6)$

$(3,6), (3,5)$

Step 2: $(5,6), (3,5), (3,6)$ 2(a) Since 3 is compatible with all the elements of the previously defined class $i.e., \{5,6\}$.

$(5,6), (3,5), (3,6)$

Step 2:

$(2,3)$

$(5,6), (3,5), (3,6)$

$(2,3)$

Step 2:

$(1,2), (1,3), (1,6)$

$(5,6), (3,5), (3,6)$

$(2,3), (1,2), (1,3)$

$(1,6)$

$(5,6), (3,5), (3,6)$

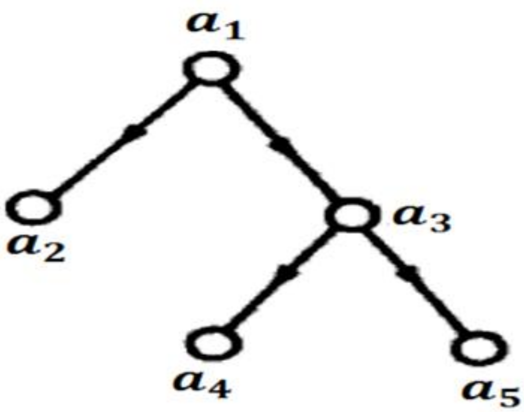
$(2,3), (1,2), (1,3)$

$(1,2), (1,3), (1,6)$ 2(b)

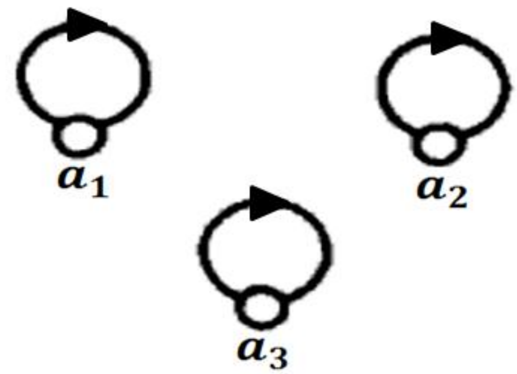
The m.c.bs are $\{3,5,6\}$, $\{1,2,3\}$, $\{1,3,6\}$ and $\{4\}$

P1:

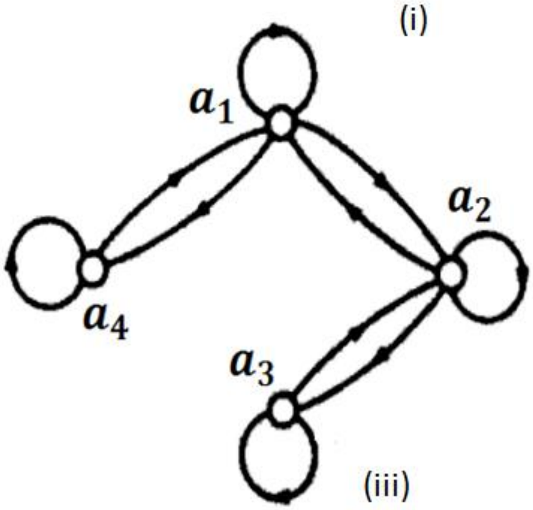
Determine the property of the relations given by the following graphs and also write the corresponding relation matrices.



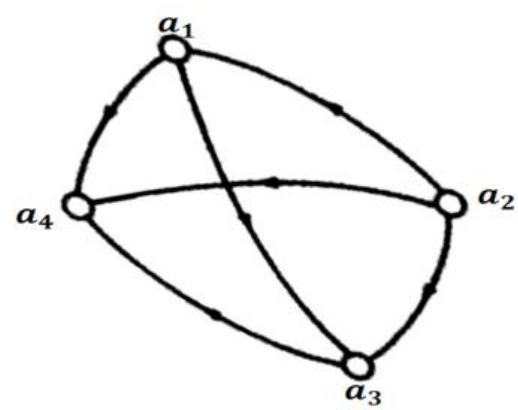
(i)



(ii)



(iii)



(iv)

Solution:

i) The relation corresponding to the graph given by figure (i) is $R_1 = \{(a_1, a_2), (a_1, a_3), (a_3, a_4), (a_3, a_5)\}$ on $A = \{a_1, a_2, a_3, a_4, a_5\}$ and its corresponding relation matrix is

$$M_{R_1} = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

It is irreflexive, antisymmetric and not transitive

ii) The relation corresponding to the graph given by figure (ii) is
$R_2 = \{(a_1, a_1), (a_2, a_2), (a_3, a_3)\}$ on $A = \{a_1, a_2, a_3\}$ and its corresponding relation matrix is

$$M_{R_2} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

It is reflexive, symmetric and transitive.

iii) The relation corresponding to the graph given by figure (iii) is

$$R_3 = \left\{ \begin{matrix} (a_1, a_1), (a_1, a_2), (a_1, a_4), (a_2, a_1), (a_2, a_2), (a_2, a_3), (a_3, a_2), (a_3, a_3), (a_4, a_1) \\ , (a_4, a_4) \end{matrix} \right\}$$

on $A = \{a_1, a_2, a_3, a_4\}$ and its corresponding relation matrix is

$$M_{R_3} = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

It is reflexive, symmetric and not transitive.

iv) The relation corresponding to the graph given by figure (iv) is
$R_4 = \{(a_1, a_3), (a_1, a_4), (a_2, a_1), (a_2, a_3), (a_2, a_4), (a_3, a_4)\}$
on $A = \{a_1, a_2, a_3, a_4\}$ and its corresponding relation matrix is

$$M_{R_4} = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

It is irreflexive, antisymmetric and transitive.

P2:

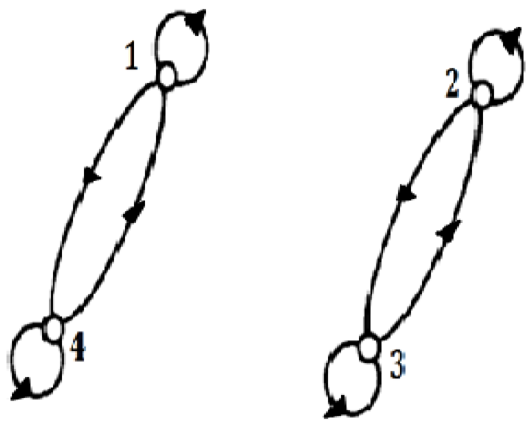
Let $A = \{1, 2, 3, 4\}$ and

$$\boldsymbol{R = \{(1,1), (1,4), (2,2), (2,3), (3,2), (3,3), (4,1), (4,4)\}.}$$

Show that R is an equivalence relation through its relation matrix and graph.
Find the quotient set A/R for the example

Solution:

$$M_R = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$



Note that the diagonal entries of M_R are all 1, and M_R is a symmetric matrix. Therefore, R is reflexive and symmetric. Further, it is transitive. Thus, R is an equivalence relation. The equivalence classes are

$$[1]_R = \{y | y \in A \ \wedge 1Ry\} = \{1,4\}$$

$$[2]_R = \{y | y \in A \ \wedge 2Ry\} = \{2,3\}$$

Therefore , $A/R = \{[1]_R, [2]_R\} = \{\{1,4\}, \{2,3\}\}$

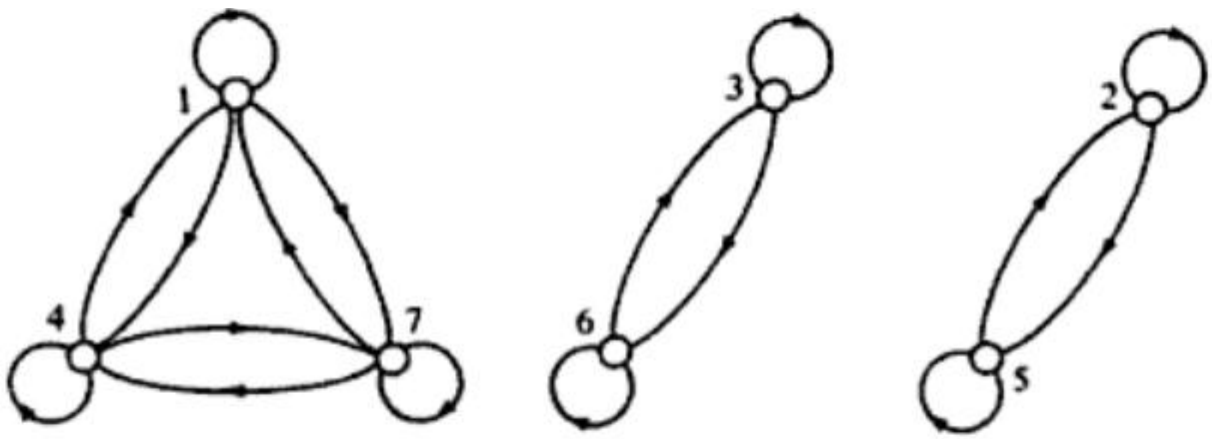
P3:

Let $\boldsymbol{A} = \{\mathbf{1}, \mathbf{2}, \mathbf{3}, \dots, \mathbf{7}\}$ and $\boldsymbol{R} = \{(\boldsymbol{x}, \boldsymbol{y}) | \boldsymbol{x} \equiv \boldsymbol{y}(\boldsymbol{mod}\ \mathbf{3})\}$.

Show that R is an equivalence relation through its graph. Find the quotient set A/R for the example.

Solution: We have

$$R = \left\{ \begin{matrix} (1,1), (1,4), (1,7), (2,2), (2,5), (3,3), (3,6), (4,1), (4,4), (4,7), (5,2), (5,5), (6,3), (6,6), \\ (7,1), (7,4), (7,7) \end{matrix} \right\}$$



Clearly R is reflexive and symmetric. Further R is transitive. Thus R is an equivalence relation. The equivalence classes are

$$[1]_R = \{y | y \in A\ \wedge 1Ry\} = \{1,4,7\}$$

$$[2]_R = \{y | y \in A\ \wedge 2Ry\} = \{2,5\}$$

$$[3]_R = \{y | y \in A\ \wedge 3Ry\} = \{3,6\}$$

Therefore , $A/R = \{[1]_R, [2]_R, [3]_R\} = \{\{1,4,7\}, \{2,5\}, \{3,6\}\}$

P4:

Obtain the equivalence relation defined by the partition $\boldsymbol{P = \{A_1, A_2, A_3\}}$ of the set $\boldsymbol{A = \{1, 2, 3, 4, 5, 6\}}$, where $\boldsymbol{A_1 = \{1, 2, 3\}}$, $\boldsymbol{A_2 = \{4, 5\}}$ and $\boldsymbol{A_3 = \{6\}}$

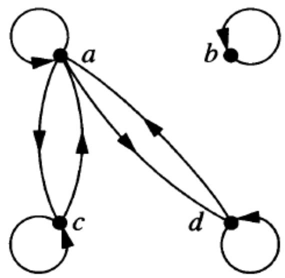
Solution:

Let R be the equivalence relation defined by the partition $P = \{A_1, A_2, A_3\}$ of A.
Then $R = (A_1 \times A_1) \cup (A_2 \times A_2) \times (A_3 \times A_3)$

$$R = \left\{ \begin{matrix} (1,1),(1,2),(1,3),(2,1),(2,2),(2,3),(3,1),(3,2), \\ (3,3),(4,4),(4,5),(5,4),(5,5),(6,6) \end{matrix} \right\}$$

P5:

Determine whether the relation $\boldsymbol{R}$ represented by the following digraph is an equivalence relation. If $\boldsymbol{R}$ is an equivalence relation then find its quotient set



Solution:

Let R be the relation on $A = \{a, b, c, d\}$ represented by the given graph and M_R be its relation matrix. Then

$$R = \{(a, a), (a, c), (a, d), (b, b), (c, a), (c, c), (d, a), (d, d)\}$$

$$M_R = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

From the diagraph, we see that there is a loop at every node. Therefore R is reflexive. From M_R we see that $(M_R)^T = M_R$. Therefore M_R is a symmetric matrix and thus R is a symmetric relation. Further, R is symmetric. Therefore, R is an equivalence relation. The equivalence classes are

$$[a]_R = \{x \in A | x\ R\ a\} = \{a, b, c, d\}$$

$$[b]_R = \{x \in A | x\ R\ b\} = \{b\}$$

Therefore the quotient set is

$$A/R = \{[a]_R, [b]_R\} = \{\{a, c, d\}, \{b\}\}$$

P6:

Determine the number of equivalence relations on a set with three elements.

Solution:

It is known that every equivalence relation on a set generates a unique partition of the set. Conversely if P is any partition on a set then there is an equivalence relation on the set.

That is an equivalence relation on a set generates a partition of the set and conversely.

From this it follows that the number of equivalence relation on a finite set A is the number of partitions of A:

Let $A = \{a, b, c\}$. We now list all partitions of A:

$P_1 = \{\{a\}, \{b\}, \{c\}\}$

$P_2 = \{\{a\}, \{b, c\}\}$

$P_3 = \{\{b\}, \{a, c\}\}$

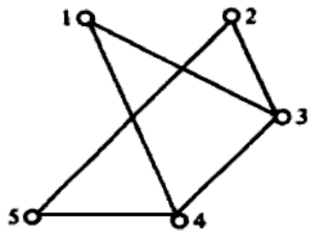
$P_4 = \{\{c\}, \{a, b\}\}$

$P_5 = \{\{a, b, c\}\}$

Thus there are five equivalence relations on a set with three elements.

P7:

The graph of a compatibility relation is given below. Write its relation matrix and find its maximal compatibility blocks.



Solution: We have $A = \{1,2,3,4,5\}$

The relation matrix is

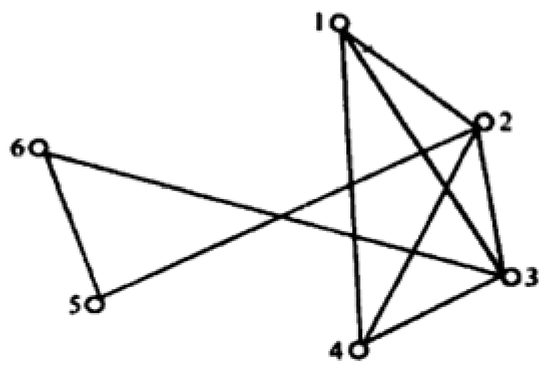
2	0			
3	1	1		
4	1	0	1	
5	0	1	0	1
	1	2	3	4

The largest complete polygon is $\{1,3,4\}$. Therefore it is maximal compatibility block. The other m.c.bs are $\{2,3\}, \{4,5\}, \{2,5\}$

Obtain the m.c.bs by procedure 2 (Do it !)

P8:

The graph of a compatibility relation is given below. Write its relation matrix and find its maximal compatibility blocks.



Solution: We have $A = \{1,2,3,4,5,6\}$

The relation matrix is

2	1				
3	1	1			
4	1	1	1		
5	0	1	0	0	
6	0	0	1	0	1
	1	2	3	4	5

The largest complete polygon is $\{1,2,3,4\}$ and it is a maximal compatibility block. The other m.c.bs are $\{2,5\}, \{3,6\}, \{5,6\}$

Obtain the m.c.bs by procedure 2 (Do it !)

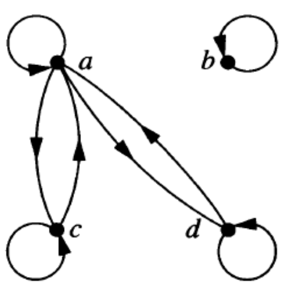
2.3. Equivalence relations and Compatibility relations

Exercise:

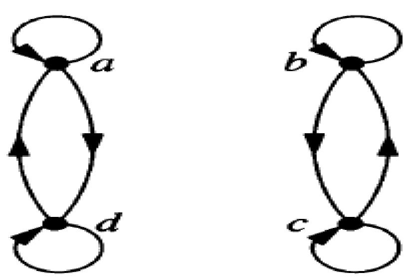
1. Show that the relation of logical equivalence on the set of all compound propositions is an equivalence relation. What are the equivalence classes of F and of T?
2. Suppose that A is a nonempty set, and f is a function that has A as its domain. Let R be the relation on A consisting of all ordered pairs (x, y) such that $f(x) = f(y)$.
    a. Show that R is an equivalence relation on A.
    b. What are the equivalence classes of R?

3. Determine whether the relation with the directed graphs shown below is an equivalence relation.

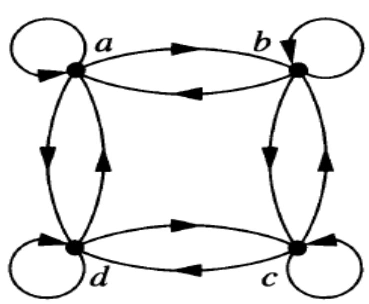
i)



ii)



iii)



4. Determine whether the relations represented by these zero-one matrices are equivalence relations.
   a. $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$
   b. $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$
   c. $\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

5. Which of these collections of subsets are partitions of $\{1,2,3,4,5,6\}$?
   a. $\{1,2\}, \{2,3,4\}, \{4,5,6\}$
   b. $\{1\}, \{2,3,6\}, \{4\}, \{5\}$
   c. $\{2,4,6\}, \{1,3,5\}$
   d. $\{1,4,5\}, \{2,6\}$

6. List the ordered pairs in the equivalence relations produced by these partitions of $\{0,1,2,3,4,5\}$. Draw the corresponding digraphs.
   a. $\{0\}, \{1,2\}, \{3,4,5\}$
   b. $\{0,1\}, \{2,3\}, \{4,5\}$
   c. $\{0,1,2\}, \{3,4,5\}$
   d. $\{0\}, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}$

7. Determine the number of different equivalence relations on a set with five elements by listing them.