UNIT - V

Models — Inequalities & Generating Functions

(a) Chebchev's inequality

(b) Cauchy Schwartz inequality

(a) Moment Generating Function M.G.F

(b) Characteristic Function Ch.F

(c) Cumulant Generating Function C.G.F

(d) Probability Generating Function P.G.F

Chebychev's inequality

If X is a random variable with mean μ and variance σ^2 then

$$P(|X-\mu| \geq \epsilon) \leq \frac{\sigma^2}{\epsilon^2} \quad (\epsilon > 0)$$

Proof:

let X be a continuous random variable with p.d.f $f(x)$ then by the def of variance

$$\sigma^2 = E(X - E(X))^2$$

$$= E(X-\mu)^2 \qquad \boxed{E(X) = \int_{-\infty}^{\infty} x f(x)\, dx}$$

$$= \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\, dx$$

$$= \int_{-\infty}^{\mu-\epsilon} (x-\mu)^2 f(x)\, dx + \int_{\mu-\epsilon}^{\mu+\epsilon} (x-\mu)^2 f(x)\, dx$$

$$+ \int_{\mu+\epsilon}^{\infty} (x-\mu)^2 f(x)\, dx$$

$$\geq \int_{-\infty}^{\mu-\epsilon} (x-\mu)^2 f(x)\,dx + \int_{\mu+\epsilon}^{\infty} (x-\mu)^2 f(x)\,dx \rightarrow ①$$

let us consider $x-\mu=\epsilon$

Then ① becomes

$$\sigma^2 \geq \int_{-\infty}^{\mu-\epsilon} \epsilon^2 f(x)\,dx + \int_{\mu+\epsilon}^{\infty} \epsilon^2 f(x)\,dx$$

$$= \epsilon^2 \left[\int_{-\infty}^{\mu-\epsilon} f(x)\,dx + \int_{\mu+\epsilon}^{\infty} f(x)\,dx \right]$$

$$= \epsilon^2 \left[P(X \leq \mu-\epsilon) + P(\mu+\epsilon \leq X) \right]$$

$$= \epsilon^2 P(\mu+\epsilon \leq X \leq \mu-\epsilon)$$

$$= \epsilon^2 P(\epsilon \leq X-\mu \leq -\epsilon)$$

$$= \epsilon^2 P(|X-\mu| \geq \epsilon)$$

$$\sigma^2 \geq \epsilon^2 P(|X-\mu| \geq \epsilon)$$

$$\Rightarrow \frac{\sigma^2}{\epsilon^2} \geq P(|X-\mu| \geq \epsilon)$$

$$\Rightarrow \boxed{P(|X-\mu| \geq \epsilon) \leq \frac{\sigma^2}{\epsilon^2}}$$

Hence the proof

Cauchy - Schwartz inequality

For any two random variables X and Y; $(E(XY))^2 \leq E(X^2)\,E(Y^2)$

Proof:

Consider $E(X-tY)^2 \geq 0$ for any real number t

$$E[X^2 - 2XYt + t^2Y^2] \geq 0$$

$$\Rightarrow E(X^2) - 2t\,E(XY) + t^2 E(Y^2) \geq 0$$

i.e $t^2 E(Y^2) - 2t\,E(XY) + E(X^2) \geq 0$

which is quadratic eqn in 't'

It has a complex roots when

$b^2 - 4ac \leq 0$

Here $a = E(Y^2)$, $b = -2E(XY)$

$c = E(X^2)$

$$\therefore\ 4\left(E(XY)\right)^2 - 4\,E(X^2)\,E(Y^2) \le 0$$

$$\boxed{\left(E(XY)\right)^2 \le E(X^2)\,E(Y^2)}$$

Hence the theorem

Generating Functions

Application: By using Generating Functions we find Mean $E(x)$, $E(x^2)$, $E(x^3)$, --- variance σ^2, S.D σ of Binomial, Poisson, Normal, Negative Binomial, Geometric, Hyper Geometric distributions

Types of Generating Functions:

Generating Functions classifieds 4 types

1. Moment Generating Function M.G.F $M_x(t) = E[e^{tx}]$
2. Characteristic Function Ch.F $\Phi_x(t) = E[e^{itx}]$
3. Cumulant Generating Function C.G.F $K_x(t) = \ln \Phi_x(t)$
4. Probability Generating Function P.G.F $G_x(t) = E[t^x]$

Note: If x is Discrete R.V Then $E(x) = \sum_{0}^{\infty} x\, P(x)$

If x is Continuous R.V Then $E(x) = \int_{-\infty}^{\infty} x\, f(x)\, dx$

Here $P(x)$ = Probability mass function : P.m.f

$f(x)$ = Probability density function : P.d.f

1. Moment Generating Function: M.G.F

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$$

Here $z = tx$

$$\boxed{M_x(t) = E[e^{tx}]}$$

$$= E\left[1 + \frac{tx}{1!} + \frac{t^2x^2}{2!} + \frac{t^3x^3}{3!} + \cdots\right]$$

$$= E(1) + \frac{t}{1!}E(x) + \frac{t^2}{2!}E(x^2) + \frac{t^3}{3!}E(x^3) + \cdots$$

$$= 1 + \frac{t}{1!}E(x) + \frac{t^2}{2!}E(x^2) + \frac{t^3}{3!}E(x^3) + \cdots$$

$$= \sum_{r=0}^{\infty} \frac{t^r}{r!} E(x^r)$$

Hence $M_x(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} E(x^r)$

To find mean, variance using M.G.F

Since $M_X(t) = E[e^{tX}]$

Differentiation w.r.t 't' on both sides and Then put $t=0$

we get
$$\left[\frac{d}{dt} M_X(t)\right]_{t=0} = E(X) = \mu = \text{Mean}$$

Again Differentiate w.r.t 't' on both sides and Then put $t=0$

we get
$$\left[\frac{d^2}{dt^2} M_X(t)\right]_{t=0} = E(X^2)$$

Now variance $\sigma^2 = E(X^2) - [E(X)]^2$

Problem① If $X \sim B(n, p)$ Then find its M.G.F and hence find its mean and variance

Sol: Given $X \sim B(n, p)$

For Binomial Distribution its P.m.f is given by $P(x) = n_{c_x} \cdot p^x \cdot q^{n-x}$

Here $x = 0, 1, 2, \ldots n$, $p+q=1$, $q=1-p$

For Binomial Distribution $B(n, p)$
$P(x) = n_{c_x} \cdot p^x \cdot q^{n-x}$
$\mu = E(X) = np$ ✓
$\sigma^2 = npq$ ✓

Now its M.G.F is given by

$$M_X(t) = E[e^{tX}]$$
$$= \sum e^{tx} P(x)$$
$$= \sum e^{tx} \cdot n_{c_x} \cdot p^x \cdot q^{n-x}$$
$$= \sum n_{c_x} \cdot (pe^t)^x \cdot q^{n-x}$$

Since $E(X) = \sum x \, P(x)$

$M_x(t) = (pe^t + q)^n$ $\boxed{\sum nc_r \cdot p^r \cdot q^{n-r} = (p+q)^n}$

Hence M.G.F of Binomial Distribution

$$\boxed{M_x(t) = (q + pe^t)^n}$$

To find mean and variance using M.G.F

Since $M_x(t) = (q + pe^t)^n$

Now Diff w.r.t 't' on both sides

$$\frac{d}{dt} M_x(t) = n(q + pe^t)^{n-1} \cdot p \cdot e^t$$

$$= np\, e^t (q + pe^t)^{n-1} \longrightarrow (1)$$

Now $\left[\frac{d}{dt} M_x(t)\right]_{t=0} = np\, e^0 (q + pe^0)^{n-1}$ $\boxed{p+q=1}$

Hence $\boxed{\mu = E(x) = np = \text{mean}}$

Again Diff w.r.t 't' (1) on both sides

$$\frac{d^2}{dt^2} M_x(t) = np\left[e^t (q + pe^t)^{n-1} + e^t (n-1)(q + pe^t)^{n-2} pe^t\right]$$

Now $\left[\frac{d^2}{dt^2} M_x(t)\right]_{t=0} = np\left[e^0 (q + pe^0)^{n-1} + e^0 (n-1)(q + pe^0)^{n-2} pe^0\right]$

$$= np[1 + p(n-1)] = np[1 + np - p]$$ $1-p=q$

$$= np[np + q] = n^2p^2 + npq$$

Hence $\boxed{E(x^2) = n^2p^2 + npq}$

Hence variance $\sigma^2 = E(x^2) - [E(x)]^2$

$$\therefore \sigma^2 = n^2p^2 + npq - n^2p^2$$

$$\boxed{\sigma^2 = npq}$$

Note : For a Binomial Distribution mean always greater than variance

$$\mu > \sigma^2$$

Problem ②: If $X \sim P(\lambda)$ Then find its m.G.F and hence find its mean and variance

Sol: Given $X \sim P(\lambda)$

For Poisson Distribution its p.m.f is given by $P(x) = \dfrac{e^{-\lambda}.\lambda^x}{x!}$

where $x = 0,1,2,\cdots$; $\lambda > 0$

For Poisson Distribution $P(\lambda)$

$P(x) = \dfrac{e^{-\lambda}.\lambda^x}{x!}$

mean $\mu = \lambda$

variance $\sigma^2 = \lambda$

Now its m.G.F is given by

$$M_X(t) = E\left[e^{tX}\right]$$

$$= \Sigma e^{tx} P(x)$$

$$= \Sigma e^{tx} \frac{e^{-\lambda}.\lambda^x}{x!}$$

$$= e^{-\lambda} \Sigma \frac{(\lambda e^t)^x}{x!}$$

$$= e^{-\lambda} e^{\lambda e^t}$$

$$\boxed{M_X(t) = e^{\lambda(e^t - 1)}}$$

$\Sigma \dfrac{z^x}{x!} = e^z$

Here $z = \lambda e^t$

which is the required Moment Generating function of Poisson Distr

To find mean and variance of Poisson Distribution using M.G.F

Since $M_X(t) = e^{\lambda(e^t - 1)}$

$$= e^{\lambda e^t - \lambda}$$

$$M_X(t) = e^{-\lambda} \cdot e^{\lambda e^t}$$

Now Diff w.r.t 't' on both sides

$$\frac{d}{dt} M_X(t) = e^{-\lambda} \cdot e^{\lambda e^t} \cdot \lambda \cdot e^t$$

$$= \lambda \cdot e^{-\lambda} \cdot e^{\lambda e^t} \cdot e^t \rightarrow (1)$$

Now $\left[\frac{d}{dt} M_X(t)\right]_{t=0} = \lambda$

Hence $\boxed{\text{mean} = E(X) = \mu = \lambda}$

Now Again Diff w.r.t 't' (1) on both sides

$$\frac{d^2}{dt^2}[M_X(t)] = \lambda e^{-\lambda}\left[\lambda e^t e^{\lambda e^t} \cdot e^t + e^{\lambda e^t} e^t\right]$$

Now $\left[\frac{d^2}{dt^2} M_X(t)\right]_{t=0} = \lambda e^{-\lambda}\left[\lambda e^{\lambda} + e^{\lambda}\right] = \lambda^2 + \lambda$

Hence $E(X^2) = \lambda^2 + \lambda$

Hence variance $\sigma^2 = E(X^2) - [E(X)]^2$

$$= \lambda^2 + \lambda - \lambda^2$$

$$\boxed{\text{Variance} = \sigma^2 = \lambda}$$

② Characteristic Function Ch.F

In some cases M.G.F does not exist. Then we use Ch.F

$$\boxed{\phi_X(t) = E[e^{itx}]}$$

$$= E\left[1 + (it)x + \frac{(it)^2}{2!}x^2 + \frac{(it)^3}{3!}x^3 + \cdots\right]$$

$$= E(1) + it\,E(x) + \frac{(it)^2}{2!}E(x^2) + \frac{(it)^3}{3!}E(x^3) + \cdots$$

$$= 1 + it\,E(X) + \frac{(it)^2}{2!}E(x^2) + \frac{(it)^3}{3!}E(x^3) + \cdots$$

$$= \sum_{r=0}^{\infty} \frac{(it)^r}{r!} E(x^r)$$

Hence $\phi_X(t) = \sum_{r=0}^{\infty} \frac{(it)^r}{r!} E(x^r)$

To find mean, variance using Ch.F

Since $\phi_X(t) = E[e^{itx}]$

Diff w.r.t 't' on both sides and then put $t=0$

we get $\left[\frac{d}{dt}\phi_X(t)\right]_{t=0} = i\,E(X)$

$$E(X) = \frac{1}{i}\left[\frac{d}{dt}\phi_X(t)\right]_{t=0} = (-i)\left[\frac{d}{dt}\phi_X(t)\right]_{t=0}$$

Hence $E(X) = (-i)\left[\frac{d}{dt}\phi_X(t)\right]_{t=0}$ = mean = μ

$|||^{ly}$ $E(X^2) = (-i)^2\left[\frac{d^2}{dt^2}\phi_X(t)\right]_{t=0}$

$$E(X^r) = (-i)^r\left[\frac{d^r}{dt^r}\phi_X(t)\right]_{t=0}$$

Variance $\sigma^2 = E(X^2) - [E(X)]^2$

Problem ③ If $X \sim B(n; p)$ then find its ch.F and hence find its mean and variance

Sol: Given $X \sim B(n; p)$

For Binomial Distribution its p.m.f is given by $P(x) = nc_x \cdot p^x \cdot q^{n-x}$

Here $x = 0, 1, 2, \ldots n$, $p + q = 1$

$q = 1 - p$

For Binomial Distn
$P(x) = nc_x \cdot p^x \cdot q^{n-x}$
$E(X) = np$
$\sigma^2 = npq$

Now its ch.F is given by

$$\phi_X(t) = E\left[e^{itX}\right]$$

$$= \Sigma e^{itx} P(x)$$

$$= \Sigma e^{itx} nc_x \cdot p^x \cdot q^{n-x}$$

$$= \Sigma nc_x (pe^{it})^x \cdot q^{n-x}$$

$$= (pe^{it} + q)^n$$

since
$E(X) = \Sigma x \, P(x)$
Here $x = e^{itx}$

$\Sigma nc_x \cdot p^x \cdot q^{n-x} = (p+q)^n$
Here $p = pe^{it}$
$q = q$

Hence $\boxed{\phi_X(t) = (q + pe^{it})^n}$ is ch.F of Binomial Distn

To find mean and variance using ch.F

Since $\phi_X(t) = (q + pe^{it})^n$

$E(X^r) = (-i)^r \dfrac{d^r}{dt^r} \phi_X(t) \Big|_{t=0}$

Now Diff wrt 't' on both sides

$$\frac{d}{dt} \phi_X(t) = n (q + pe^{it})^n \cdot p \cdot i \cdot e^{it}$$

$$= npi \, e^{it} (q + pe^{it})^{n-1} \rightarrow ①$$

Now $\left[\dfrac{d}{dt} \phi_X(t)\right]_{t=0} = npi$

Hence $E(X) = (-i)^1 \left[\frac{d}{dt} \phi_X(t)\right]_{t=0}$

$$= (-i)(npi) = np$$

Hence $\boxed{\text{mean} = E(X) = \mu = np}$

Again Diff w.r.t 't' (1) on both sides

$$\frac{d^2}{dt^2} \phi_X(t) = npi\left[i e^{it} (q + pe^{it})^{n-1} + e^{it} (n-1)(q+pe^{it})^{n-2} \, i \cdot p \cdot e^{it}\right]$$

Now $\left[\frac{d^2}{dt^2} \phi_X(t)\right]_{t=0} = npi\left[i + (n-1)ip\right]$

$$= -np\left[1 + (n-1)p\right]$$

$$= -np\left[1 + np - p\right] \qquad (1-p=q)$$

$$= -np(q+np)$$

Hence $E(X^2) = (-i)^2 \left[\frac{d^2}{dt^2} \phi_X(t)\right]_{t=0}$

$$= (-i)^2 (-np)(q+np)$$

$$= np(q+np) = npq + n^2p^2$$

$$\boxed{E(X^2) = npq + n^2p^2}$$

Hence variance $\sigma^2 = E(X^2) - [E(X)]^2$

$$= npq + \cancel{n^2p^2} - \cancel{n^2p^2}$$

$$\boxed{\sigma^2 = npq}$$

Problem 4: If $X \sim P(\lambda)$, Then find The characteristic Function and hence find its mean and variance

Sol. given $X \sim P(\lambda)$

For poisson Distribution its p.m.f is given by $P(x) = \frac{e^{-\lambda} \cdot \lambda^{x}}{x!}$

$x = 0, 1, 2, \ldots, \lambda > 0$

For poisson Distribu $P(x) = \frac{e^{-\lambda} \cdot \lambda^{x}}{x!}$

mean $\mu = \lambda$

variance $\sigma^2 = \lambda$

Now its ch.f is given by

$$\phi_X(t) = E\left[e^{itx}\right]$$

$$= \sum e^{itx} P(x)$$

$$= \sum e^{itx} \frac{e^{-\lambda} \cdot \lambda^{x}}{x!}$$

$$= e^{-\lambda} \sum \frac{(\lambda e^{it})^{x}}{x!}$$

$$\sum \frac{z^x}{x!} = e^z$$

Here $z = \lambda e^{it}$

$$= e^{-\lambda} \cdot e^{\lambda e^{it}}$$

$$= e^{\lambda(e^{it}-1)}$$

Hence $\boxed{\phi_X(t) = e^{\lambda(e^{it}-1)}}$

which is The required ch.F of poisson Distn.

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To find mean and variance of Poisson Distribution using ch.f

Since $\phi_x(t) = e^{\lambda(e^{it}-1)}$

$= e^{\lambda e^{it} - \lambda} = e^{-\lambda} \cdot e^{\lambda e^{it}}$

$\phi_x(t) = e^{-\lambda} \cdot e^{\lambda e^{it}}$

Now Diff w.r.t 't' on both sides

$\frac{d}{dt}\phi_x(t) = e^{-\lambda} \cdot e^{\lambda e^{it}} \cdot \lambda \cdot i \cdot e^{it}$

$= \lambda \cdot i \cdot e^{-\lambda} \cdot e^{\lambda e^{it}} \cdot e^{it} \rightarrow (1)$

Now $\left[\frac{d}{dt}\phi_x(t)\right]_{t=0} = \lambda \cdot i \cdot 1 = \lambda i$

Hence $E(x^r) = (-i)^r \left[\frac{d^r}{dt^r}\phi_x(t)\right]_{t=0}$

for $r=1$

$E(x) = (-i)\left[\frac{d}{dt}\phi_x(t)\right]_{t=0}$

$= (-i)(\lambda i) = \lambda$

Hence $\boxed{\text{mean} = E(x) = \lambda}$

Again Diff (1) w.r.t 't' on both sides

$\frac{d^2}{dt^2}[\phi_x(t)] = \lambda i e^{-\lambda}\left[\lambda \cdot i \cdot e^{it} e^{\lambda e^{it}} \cdot e^{it} + e^{\lambda e^{it}} i \cdot e^{it}\right]$

$\left[\frac{d^2}{dt^2}\phi_x(t)\right]_{t=0} = \lambda \cdot i \cdot e^{-\lambda}\left[\lambda \cdot i \cdot e^{\lambda} + i \cdot e^{\lambda}\right]$

$= \lambda i e^{-\lambda}\, i e^{\lambda}(\lambda+1) = -\lambda(\lambda+1) =$

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$$\left[\frac{d^2}{dt^2}\phi_x(t)\right]_{t=0} = -\lambda^2 - \lambda$$

Now $\quad E(x^2) = (-i)^2\left[\frac{d^2}{dt^2}\phi_x(t)\right]_{t=0}$

$$= (-i)^2(-\lambda^2 - \lambda)$$

Hence $\boxed{E(x^2) = \lambda^2 + \lambda}$

Hence variance $\sigma^2 = E(x^2) - [E(x)]^2$

$$= \lambda^2 + \lambda - \lambda^2$$

$$\boxed{\sigma^2 = \lambda}$$

③ Cumulant Generating Function C.G.F: $K_X(t)$

$$\boxed{K_X(t) = \ln[\phi_X(t)]}$$

Since $\phi_X(t) = E[e^{itx}]$

$$= E\left[1 + \frac{itx}{1!} + \frac{(itx)^2}{2!} + \frac{(itx)^3}{3!} + \cdots\right]$$

$$= E(1) + \frac{it}{1!}E(x) + \frac{(it)^2}{2!}E(x^2) + \cdots$$

$$\phi_X(t) = 1 + \frac{it}{1!}E(x) + \frac{(it)^2}{2!}E(x^2) + \cdots$$

Now $K_X(t) = \ln \phi_X(t)$

$$= \ln\left[1 + \frac{it}{1!}E(x) + \frac{(it)^2}{2!}E(x^2) + \cdots\right]$$

Note: $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$

Here $x = \frac{it}{1!}E(x) + \frac{(it)^2}{2!}E(x^2) + \cdots$

$$K_X(t) = \left[\frac{it}{1!}E(x) + \frac{(it)^2}{2!}E(x^2) + \cdots\right] - \frac{1}{2}\left[\frac{it}{1!}E(x) + \frac{(it)^2}{2!}E(x^2) + \cdots\right]^2$$

$$+ \frac{1}{3}\left[\frac{it}{1!}E(x)\left(\frac{it}{2!}\right)^2 E(x^2) + \frac{(it)^3}{3!}E(x^3) + \cdots\right]^3 - \frac{1}{4}\cdots$$

which is the form of C.G.F

To find mean and variance using C.G.F

Since $K_x(t) = \ln \phi_x(t)$

Now diff w.r.t 't' on both sides and then put $t=0$

continuously we get

$$\text{mean} = E(X) = (-i)^1 \left[\frac{d}{dt} K_x(t)\right]_{t=0}$$

$$\text{variance} = \sigma^2 = (-i)^2 \left[\frac{d^2}{dt^2} K_x(t)\right]_{t=0}$$

Problem(5): If $X \sim B(n, p)$ then find its C.G.F and hence find its mean and variance.

$$\boxed{\begin{array}{l} X \sim B(n,p) \\ P(x) = nc_x \, p^x q^{n-x} \\ \text{mean} = np \\ \text{variance} = npq \end{array}}$$

Sol: Since $X \sim B(n, p)$ its p.m.f is given by $P(x) = nc_x \cdot p^x \cdot q^{n-x}$

Now its ch.F is given by

$$\phi_x(t) = E\left[e^{itx}\right]$$

$$\boxed{E(x) = \sum x \, P(x)}$$

$$= \sum e^{itx} \, P(x)$$

$$= \sum e^{itx} \, nc_x \cdot p^x \cdot q^{n-x}$$

$$= \sum nc_x \cdot (pe^{it})^x \cdot q^{n-x}$$

$$\boxed{\sum nc_x \, p^x q^{n-x} = (p+q)^n}$$

$$\boxed{\phi_x(t) = (pe^{it} + q)^n}$$

Now its C.G.F is given by

$$\boxed{K_x(t) = \ln \phi_x(t)} \quad \therefore K_x(t) = \ln (pe^{it}+q)^n$$

Since $K_X(t) = \ln (pe^{it}+q)^n$

$$\boxed{K_X(t) = n \ln (pe^{it}+q)}$$

which is the required C.G.F for Binomial Distri

To find mean and variance using C.G.F

Since $K_X(t) = n \ln (pe^{it}+q)$

we know that mean $= E(X) = (-i)^1 \left[\frac{d}{dt} K_X(t)\right]_{t=0}$

$$\therefore \frac{d}{dt} K_X(t) = n \frac{1}{pe^{it}+q} p.i.e^{it} \longrightarrow (1)$$

Now $\left[\frac{d}{dt} K_X(t)\right]_{t=0} = npi$

Hence mean $= E(X) = (-i)(npi)$

$$\boxed{E(X) = np}$$

we know that variance $= \sigma^2 = (-i)^2 \left[\frac{d^2}{dt^2} K_X(t)\right]_{t=0}$

$\therefore$ Diff (1) w.r.t 't' on both sides d(u/v)

$$\frac{d^2}{dt^2} K_X(t) = npi \left[\frac{(pe^{it}+q)\, ie^{it} - e^{it}\, p.i.e^{it}}{(pe^{it}+q)^2}\right]$$

Now $\left[\frac{d^2}{dt^2} K_X(t)\right]_{t=0} = npi\,(i-ip) = npi^2(1-p)$ $\boxed{1-p=q}$

$= -npq$

Hence Variance $= \sigma^2 = (-i)^2(-npq)$

$$\boxed{\sigma^2 = npq}$$

⑥ Problem:
If $X \sim P(\lambda)$ Then find its C.G.F and hence find its Mean and Variance

$X \sim P(\lambda)$

p.m.f

$P(x) = \frac{e^{-\lambda} \cdot \lambda^x}{x!}$

Mean $= \lambda$

Variance $= \lambda$

Sol: Given $X \sim P(\lambda)$ Now its p.m.f is given by $P(\lambda) = \frac{e^{-\lambda} \cdot \lambda^x}{x!}$

Now its ch.f is given by

$$\boxed{\phi_X(t) = e^{\lambda(e^{it}-1)}}$$

[See Problem (4) in Page no 9]

Hence $K_X(t) = \ln \phi_X(t)$

$\therefore K_X(t) = \ln e^{\lambda(e^{it}-1)}$

$$\boxed{K_X(t) = \lambda(e^{it}-1)}$$

Now we find mean and variance

Since $K_X(t) = \lambda(e^{it}-1)$

mean

Since mean $= E(X) = (-i)^1 \left[\frac{d}{dt} K_X(t)\right]_{t=0}$

$\therefore \frac{d}{dt} K_X(t) = \lambda \cdot i \cdot e^{it} \longrightarrow ①$ $\Rightarrow \left[\frac{d}{dt} K_X(t)\right]_{t=0} = \lambda i$

Hence $E(X) = (-i)\, \lambda \cdot i$

Hence $\boxed{E(X) = \lambda = \text{mean}}$

Variance

Since variance $\sigma^2 = (-i)^2 \left[\frac{d^2}{dt^2} K_X(t)\right]_{t=0}$

Now Diff (1) w.r.t 't'

$\frac{d^2}{dt^2} K_X(t) = \lambda \cdot i \cdot i \cdot e^{it} = -\lambda e^{it}$

$\Rightarrow \left[\frac{d^2}{dt^2} K_X(t)\right]_{t=0} = -\lambda$

Hence variance $= (-i)^2(-\lambda) = \lambda$

$$\boxed{\text{variance} = \lambda = \sigma^2}$$

(9) Probability Generating Function P.G.F : $G_X(t)$

$$\boxed{G_X(t) = E[t^X]}$$

$$G_X(t) = \sum t^x P(x)$$

To find mean and variance using P.G.F

Since $G_X(t) = E[t^X]$

Now Diff w.r.t 't' on both sides and then put $t=1$

$$\text{Mean} = E(X) = \left[\frac{d}{dt} G_X(t)\right]_{t=1} = G_X^{(1)}(1)$$

$$\text{Here} \quad E(X^2) = \left[\frac{d^2}{dt^2} G_X(t)\right]_{t=1} + \left[\frac{d}{dt} G_X(t)\right]_{t=1} = G_X^{(2)}(1) + G_X^{(1)}(1)$$

$$\text{Hence variance } \sigma^2 = E(X^2) - [E(X)]^2$$

$$\therefore \sigma^2 = G_X^{(2)}(1) + G_X^{(1)}(1) - \left[G_X^{(1)}(1)\right]^2$$

Problem(7): If $X \sim B(n, p)$ then find its P.G.F and hence find its mean and variance

$$\boxed{\begin{gathered} X \sim B(n, p) \\ P(x) = nC_x\, p^x q^{n-x} \\ \mu = np \\ \sigma^2 = npq \end{gathered}}$$

Sol: Since $X \sim B(n, p)$ its p.m.f is given by $P(x) = nC_x \cdot p^x \cdot q^{n-x}$

Now its P.G.F is given by

$$G_X(t) = E[t^X]$$

$$= \sum t^x P(x)$$

$$= \sum t^x\, nC_x \cdot p^x \cdot q^{n-x}$$

$= \sum nc_x \cdot (pt)^x \cdot q^{n-x}$

$\boxed{\sum nc_x \cdot p^x \cdot q^{n-x} = (p+q)^n}$

$$\boxed{G_X(t) = (pt+q)^n}$$

which is the required P.G.F of Binomial Distr

<u>To find mean and variance using P.G.F</u>

Since $G_X(t) = (pt+q)^n$

Since $E(X) = G_X'(1)$

$\therefore \frac{d}{dt} G_X(t) = n(pt+q)^{n-1} \cdot p \longrightarrow (1)$

$G_X'(1) = \left[\frac{d}{dt} G_X(t)\right]_{t=1} = n(p+q)^{n-1} \cdot p$

Hence $\boxed{G_X'(1) = E(X) = np}$

Since $E(X^2) = G_X^{(2)}(1) + G_X^{(1)}(1)$

Now Diff (1) w.r.t 't'

$\frac{d^2}{dt^2} G_X(t) = np\,(n-1)(pt+q)^{n-2} \cdot p$

$= n(n-1)p^2 (pt+q)^{n-2}$

$\left[\frac{d^2}{dt^2} G_X(t)\right]_{t=1} = G_X^{(2)}(1) = n(n-1)p^2 = n^2p^2 - np^2$

Hence $E(X^2) = \underbrace{n^2p^2 - np^2}_{G_X^2(1)} + \underbrace{np}_{G_X^{(1)}(1)}$

Hence variance $= \sigma^2 = E(X^2) - [E(X)]^2$

$= \cancel{n^2p^2} - np^2 + np - \cancel{n^2p^2}$

$= np - np^2 = np(1-p) = npq$

Hence $\boxed{\text{variance} = npq}$

(8) Problem: If $X \sim P(\lambda)$ Then find its P.G.F and hence find its mean and variance

Sol: Since $X \sim P(\lambda)$ Now its p.m.f is given by $P(x) = \dfrac{e^{-\lambda} \cdot \lambda^{x}}{x!}$

Now its P.G.F is given by

$$G_X(t) = E[t^X]$$

$$= \sum t^x \, P(x)$$

$$= \sum t^x \cdot \frac{e^{-\lambda} \cdot \lambda^x}{x!}$$

$$= e^{-\lambda} \sum \frac{(\lambda t)^x}{x!}$$

$$= e^{-\lambda} e^{\lambda t}$$

$$\boxed{G_X(t) = e^{\lambda(t-1)}}$$

which is required P.G.F of ~~Binomial~~ Poisson Distribution

$X \sim P(\lambda)$

$P(x) = \dfrac{e^{-\lambda} \cdot \lambda^x}{x!}$

mean $= \lambda$

Variance $= \lambda$

$\boxed{E(X) = \sum x \, P(x)}$

$\sum \dfrac{z^x}{x!} = e^z$

Here $z = \lambda t$

To find Mean and Variance using P.G.F

Since $G_X(t) = e^{\lambda(t-1)}$

$$G_X(t) = e^{\lambda t} \cdot e^{-\lambda}$$

Since $E(X) = G_X'(1)$

Now $\dfrac{d}{dt} G_X(t) = e^{-\lambda} \cdot \lambda \cdot e^{\lambda t} \rightarrow (1)$

$$G_X'(1) = \left[\frac{d}{dt} G_X(t)\right]_{t=1} = \lambda \cdot e^{-\lambda} \cdot e^{\lambda} = \lambda$$

Hence $\boxed{G_X'(1) = E(X) = \lambda}$

Since $E(X^2) = G_X^{(2)}(1) + G_X^{(1)}(1)$

Now Diff (1) wrt 't'

$$\frac{d^2}{dt^2} G_X(t) = e^{-\lambda} \cdot \lambda \cdot \lambda e^{\lambda t}$$

Now $G_X^{(2)}(1) = \left[\frac{d^2}{dt^2} G_X(t)\right]_{t=1} = e^{-\lambda} \cdot \lambda^2 \cdot e^{\lambda} = \lambda^2$

Hence $E(X^2) = \lambda^2 + \lambda$

Hence variance $\sigma^2 = E(X^2) - [E(X)]^2$

$$= \lambda^2 + \lambda - \lambda^2$$

$$\boxed{\sigma^2 = \lambda = \text{variance}}$$

Moment Generating Function

M.G.F

$$\boxed{M_X(t) = E[e^{tX}]}$$

Binomial Distribution

$$M_X(t) = (q+pe^t)^n$$

Poisson Distribution

$$M_X(t) = e^{\lambda(e^t-1)}$$

$$\text{Mean} = \left[\frac{d}{dt} M_X(t)\right]_{t=0} = E(X)$$

$$E(X^2) = \left[\frac{d^2}{dt^2} M_X(t)\right]_{t=0}$$

$$\sigma^2 = E(X^2) - [E(X)]^2$$

Characteristic Function

Ch.F

$$\boxed{\phi_X(t) = E[e^{itX}]}$$

Binomial Distribution

$$\phi_X(t) = (q+pe^{it})^n$$

Poisson Distribution

$$\phi_X(t) = e^{\lambda(e^{it}-1)}$$

$$\text{mean} = E(X) = (-i)^1\left[\frac{d}{dt} \phi_X(t)\right]_{t=0}$$

$$E(X^2) = (-i)^2\left[\frac{d^2}{dt^2} \phi_X(t)\right]_{t=0}$$

$$\sigma^2 = E(X^2) - [E(X)]^2$$

Cumulant Generating Function

C.G.F

$$\boxed{K_X(t) = \ln \phi_X(t)}$$

Binomial Distribution

$$K_X(t) = n\ln(q+pe^{it})$$

Poisson Distribution

$$K_X(t) = \lambda(e^{it}-1)$$

$$\text{mean} = E(X) = (-i)^1\left[\frac{d}{dt} K_X(t)\right]_{t=0}$$

$$\text{Variance} = \sigma^2 = (-i)^2\left[\frac{d^2}{dt^2} K_X(t)\right]_{t=0}$$

Probability Generating Function

P.G.F

$$\boxed{G_X(t) = E[t^X]}$$

Binomial Distribution

$$G_X(t) = (pt+q)^n$$

Poisson Distribution

$$G_X(t) = e^{\lambda(t-1)}$$

$$\text{Mean} = G_X'(1) = \left[\frac{d}{dt} G_X(t)\right]_{t=1}$$

$$E(X^2) = \left[\frac{d^2}{dt^2} G_X(t)\right]_{t=1} + \ldots = G_X^{2}(1) + G_X^{(1)}(1)$$

$$\text{Variance} = E(X^2) - [E(X)]^2$$