3|1|21 # Co-ordinate Transformation System UNIT-01

Vector:

→ It is defined as the physical quantity that has both magnitude and direction.

→ If the magnitude equals to one then it is called unit vector. It is represented by a lowercase alphabet with a "hat" circumflex, i.e "$\hat{u}$".

Ex: Linear momentum, Acceleration, Displacement, Momentum, Angular velocity, Force, Electric field.

→ Vector is represented by an arrow.

$$\overrightarrow{a}$$

Scalar:

→ It is defined as the physical quantity with magnitude and no direction.

Ex: Mass, Speed, Distance, Time, Area, Volume, Density, Temperature.

Vectors addition and Subtraction:

Let $\vec{u} = (u_1, u_2)$ and $\vec{v} = (v_1, v_2)$ be two vectors.

Then the sum of $\vec{u}$ and $\vec{v}$ is

$$\vec{u} + \vec{v} = (u_1 + v_1, u_2 + v_2)$$

The difference of $\vec{u}$ and $\vec{v}$ is

$$\vec{u} - \vec{v} = \vec{u} + (-\vec{v})$$

$$= (u_1 - v_1, u_2 - v_2)$$

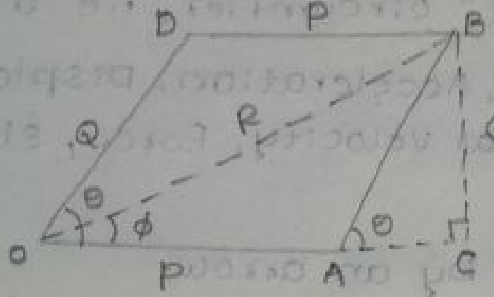
→ The sum of two or more vectors is called the resultant.

→ The resultant of two vectors can be found using either the parallelogram method or the triangle method.

Parallelogram method:

(parallelogram law of vector addition)

Let P and Q be two vectors acting simultaneously at a point and represented both in magnitude and direction by two adjacent sides OA and OD of a parallelogram OABD as in figure.



Let θ be the angle between P and Q and R be the resultant vector. Then according to parallelogram law of vector addition, diagonal OB represent the resultant of P and Q.

So, we have $R = P + Q$

Now expand A to C and draw BC ⊥er to OC.

From Δle OCB

$$OB^2 = OC^2 + BC^2$$

$$OB^2 = (OA + AC)^2 + BC^2 \rightarrow ①$$

In ΔABC

$$\cos\theta = \frac{AC}{AB}$$

$$\Rightarrow AC = AB\cos\theta \text{ (or)}$$

$$AC = OD\cos\theta$$

$$\Rightarrow \boxed{AC = Q\cos\theta}$$

Also

$$\sin\theta = \frac{BC}{AB}$$

$$\Rightarrow BC = AB\sin\theta \text{ (or)}$$

$$BC = OD\sin\theta$$

$$\Rightarrow \boxed{BC = Q\sin\theta}$$

Magnitude of resultant:

Substituting value of AC and BC in eq ①

$$OB^2 = (OA + AC)^2 + BC^2$$

$$R^2 = (P + Q\cos\theta)^2 + (Q\sin\theta)^2$$

$$R^2 = P^2 + 2PQ\cos\theta + Q^2\cos^2\theta + Q^2\sin^2\theta$$

$$R^2 = P^2 + 2PQ\cos\theta + Q^2(\cos^2\theta + \sin^2\theta)$$

$$\therefore \boxed{R^2 = P^2 + 2PQ\cos\theta + Q^2}$$

Direction:

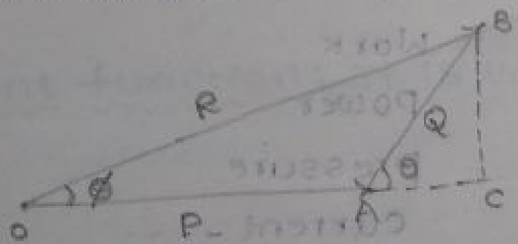
From Δle OCB

$$\tan\phi = \frac{Q\sin\theta}{P + Q\cos\theta}$$

$$\boxed{\phi = \tan^{-1}\left(\frac{Q\sin\theta}{P + Q\cos\theta}\right)}$$

Triangle law of vector addition:

Statement: If two vectors acting simultaneously on a body are represented both in magnitude and direction by two sides of a triangle taken in order, then the resultant is represented in magnitude and direction by the third side of the Δle taken in opposite direction.



By theorem, we have

$$\vec{R} = \vec{P} + \vec{Q}$$

from Δle OBC

$$OB^2 = OC^2 + BC^2$$

$$OB^2 = (OA + AC)^2 + BC^2 \rightarrow ①$$

In $\triangle$le ACB

$$\cos\theta = \frac{AC}{AB} \Rightarrow AC = AB\cos\theta$$
$$= Q\cos\theta$$

$$\sin\theta = \frac{BC}{AB} \Rightarrow BC = AB\sin\theta$$
$$= Q\sin\theta$$

Magnitude:

Substitute AC and BC in eq(1)

$$OB^2 = (OA + AC)^2 + BC^2$$

$$R^2 = (P + Q\cos\theta)^2 + (Q\sin\theta)^2$$

$$\boxed{R^2 = P^2 + 2PQ\cos\theta + Q^2}$$

Direction:

$$\tan\phi = \frac{BC}{OC} = \frac{BC}{OA + AC}$$

$$= \frac{Q\sin\theta}{P + Q\cos\theta}$$

$$\boxed{\phi = \tan^{-1}\left(\frac{Q\sin\theta}{P + Q\cos\theta}\right)}$$

4/1/21

Vectors	Scalars
Electric field ($\vec{E}$)	Mass
Magnetic field ($\vec{H}$)	Work
Poynting vector ($\vec{P}$)	power
$\vec{P} = \vec{E} \times \vec{H}$	pressure
	current ----
Electric force	
Velocity	
Acceleration....	

$\vec{A} \cdot \vec{B} = |\vec{A}||\vec{B}|\cos\theta$

$\vec{A} \times \vec{B} = |\vec{A}||\vec{B}|\sin\theta \cdot \hat{n} \rightarrow$ unit vector

→ Dot product of two vectors gives scalar

→ Cross product of two vectors gives vector.

Examples

$\phi_E = \vec{E} \cdot \vec{A}$

$\phi_B = \vec{B} \cdot \vec{A}$

$P = \vec{F} \cdot \vec{v}$

$W = \vec{F} \cdot \vec{S}$

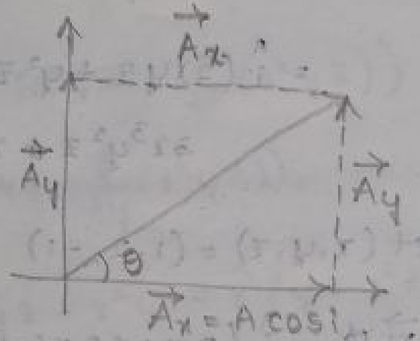
Torque $(\vec{T}) = \vec{r} \times \vec{F}$

Poynting vector $(\vec{P}) = \vec{E} \times \vec{H}$

$\vec{L} = \vec{r} \times \vec{P}$

$\vec{v} = \vec{\omega} \times \vec{r}$

Resolution of vectors: Resolution means one vector resolve with its two components.



$\vec{A} = \vec{A_x} + \vec{A_y}$

$= A\cos\theta\,\hat{i} + A\sin\theta\,\hat{j}$

$\cos\theta = \frac{A_x}{A}$ $\quad\quad$ $\sin\theta = \frac{A_y}{A}$

$A_x = A\cos\theta$ $\quad\quad$ $A_y = A\sin\theta$

$\vec{A_x} = A\cos\theta\,\hat{i}$ $\quad\quad$ $\vec{A_y} = A\sin\theta\,\hat{i}$

Vector point function: It is variable with respect to x.

Ex:

$\vec{F}(x,y,z) = (x^3y^2z + xy)\hat{i} + (x^3y^3z^3 + x^2y^2z^2)\hat{j} + xyz\,\hat{k}$

Scalar point function:

Ex:

$\phi(x,y,z) = x^2yz + x^3y^3z^3 + y + x^2y$

1\. Vector differential operator

$$\vec{\nabla} = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

*** Gradient of a scalar point function (gradient = change of a parameter)

$$\vec{\nabla}\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\phi$$

$$\vec{\nabla}\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

Let $\phi = x^2yz + xy^2z + x^3y^3z + xyz$

$$\vec{\nabla}\phi = \hat{i}\left(\frac{\partial}{\partial x}(x^2yz) + \hat{j}\frac{\partial}{\partial y}(xy^2z) + \frac{\partial}{\partial x}(x^3y^3z) + \frac{\partial}{\partial x}(xyz)\right)$$

$$\hat{j}\left(\frac{\partial}{\partial y}(x^2yz) + \frac{\partial}{\partial y}(xy^2z) + \frac{\partial}{\partial y}(x^3y^3z) + \frac{\partial}{\partial y}(xyz)\right)$$

$$\hat{k}\left(\frac{\partial}{\partial z}(x^2yz) + \frac{\partial}{\partial z}(xy^2z) + \frac{\partial}{\partial z}(x^3y^3z) + \frac{\partial}{\partial z}(xyz)\right)$$

$$= \hat{i}(2xyz + y^2z + 3x^2y^3z + yz) + \hat{j}(x^2z + 2xyz + 3x^3y^2z + xz) + \hat{k}(x^2y + xy^2 + x^3y^3 + xy)$$

Let $(x, y, z) = (1, 2, -1)$

$$\vec{\nabla}\phi = \hat{i}(2\times1\times2\times(-1) + 4\times(-1) + 3\times1\times8\times(-1) + 2\times(-1)) + \hat{j}(1\times(-1) + 2\times1\times2\times(-1) + 3\times1\times4\times(-1) + 1\times(-1)) + \hat{k}(1\times2 + 1\times4 + 1\times8 + 1\times2)$$

$$= \hat{i}(-4-4-24-2) + \hat{j}(-1+(-4)+(-12)-1) + \hat{k}(2+4+8+2)$$

$$= \hat{i}(-34) + \hat{j}(-18) + \hat{k}(16)$$

$$= -34\hat{i} - 18\hat{j} + 16\hat{k}$$

→ Gradient of a scalar point function gives us a vector point function.

2. Curl of a vector point function:

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$$ (curl = rotation)

$\vec{F}(x,y,z) = x^2yz\,\hat{i} + xy\,\hat{j} + x^3yz\,\hat{k}$

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \overset{+}{\hat{i}} & \overset{-}{\hat{j}} & \overset{+}{\hat{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2yz & xy & x^3yz \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial}{\partial y}(x^3yz) - \frac{\partial}{\partial z}(xy)\right) - \hat{j}\left(\frac{\partial}{\partial x}(x^3yz) - \frac{\partial}{\partial z}(x^2yz)\right)$$
$$+ \hat{k}\left(\frac{\partial}{\partial x}(xy) - \frac{\partial}{\partial y}(x^2yz)\right)$$

$$= \hat{i}(x^3z - xy) - \hat{j}(3x^2yz - x^2y) + \hat{k}(y - x^2z)$$

let $(x,y,z) = (1,-1,2)$

$$\vec{\nabla} \times \vec{F} = \hat{i}(1\times 2 - (-1)) - \hat{j}(3\times 1\times(-1)\times 2 - 1\times(-1)) + \hat{k}(-1 - 1\times 2)$$

$$= \hat{i}(3) - \hat{j}(-6+1) + \hat{k}(-1-2)$$

$$= 3\hat{i} + 5\hat{j} - 3\hat{k}$$

3. Divergence of a vector point function:

$$\vec{\nabla}\cdot\vec{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left(F_x\hat{i} + F_y\hat{j} + F_z\hat{k}\right)$$

$$\vec{\nabla}\cdot\vec{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

→ Divergence of a vector point function gives scalar point function.

→ Gradient of a vector point function doesn't exist.

$$\vec{F} = x\cdot y\cdot z\,\hat{i} + x^2yz\,\hat{j} + xy^3z\,\hat{k}$$

$$\vec{\nabla}\cdot\vec{F} = \frac{\partial}{\partial x}(xyz) + \frac{\partial}{\partial y}(x^2yz) + \frac{\partial}{\partial z}(xy^3z)$$

$$= yz + x^2z + xy^3$$

Problems

1. Find the gradient of $r = \sqrt{x^2+y^2+z^2}$

Sol:
$$\vec{\nabla} = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$$

$$r = \sqrt{x^2+y^2+z^2}$$

$$\vec{\nabla}\cdot r = \frac{\partial r}{\partial x}\hat{i} + \frac{\partial r}{\partial y}\hat{j} + \frac{\partial r}{\partial z}\hat{k}$$

$$= \frac{\partial}{\partial x}\left(\sqrt{x^2+y^2+z^2}\right)\hat{i} + \frac{\partial}{\partial y}\left(\sqrt{x^2+y^2+z^2}\right)\hat{j} + \frac{\partial}{\partial z}\left(\sqrt{x^2+y^2+z^2}\right)\hat{k}$$

$$= \frac{1}{\cancel{2}\sqrt{x^2+y^2+z^2}}\cdot\cancel{2}x\,\hat{i} + \frac{1}{\cancel{2}\sqrt{x^2+y^2+z^2}}\cdot\cancel{2}y\,\hat{j} + \frac{1}{\cancel{2}\sqrt{x^2+y^2+z^2}}\cdot\cancel{2}z\,\hat{k}$$

$$= \frac{x\hat{i} + y\hat{j} + z\hat{k}}{\sqrt{x^2+y^2+z^2}}$$

2. Find the gradients

(a) $f(x,y,z) = x^2+y^3+z^4$

Sol. $\vec{\nabla}.f(x,y,z) = \frac{\partial}{\partial x}(x^2+y^3+z^4)\hat{i} + \frac{\partial}{\partial y}(x^2+y^3+z^4)\hat{j} + \frac{\partial}{\partial z}(x^2+y^3+z^4)\hat{k}$

$= (2x+\overset{0}{y^3}+\overset{0}{z^4})\hat{i} + (\overset{0}{x^2}+3y^2+\overset{0}{z^4})\hat{j} + (\overset{0}{x^2}+\overset{0}{y^3}+4z^3)\hat{k}$

$= 2x\hat{i}+3y^2\hat{j}+4z^3\hat{k}$

(b) $f(x,y,z) = x^2y^3z^4$

Sol. $\vec{\nabla}.f(x,y,z) = \frac{\partial}{\partial x}(x^2y^3z^4)\hat{i} + \frac{\partial}{\partial y}(x^2y^3z^4)\hat{j} + \frac{\partial}{\partial z}(x^2y^3z^4)\hat{k}$

$= 2xy^3z^4\hat{i} + 3x^2y^2z^4\hat{j} + 4x^2y^3z^3\hat{k}$

(c) $f(x,y,z) = e^x \sin(y)\ln(z)$

Sol. $\vec{\nabla}.f(x,y,z) = \frac{\partial}{\partial x}(e^x\sin(y)\ln(z))\hat{i} + \frac{\partial}{\partial y}(e^x\sin(y)\ln(z))\hat{j} + \frac{\partial}{\partial z}(e^x\sin(y)\ln(z))\hat{k}$

$= e^x\sin(y)\ln(z)\hat{i} + e^x\cos(y)\ln z\,\hat{j} + e^x\sin(y)\times\frac{1}{z}\hat{k}$

3. $V_a = r = x\hat{i}+y\hat{j}+z\hat{k}$, $V_b = \hat{k}$ and $V_c = z\hat{k}$

Calculate their divergences.

Sol. $\nabla.V_a = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z)$

$= 1+1+1 = 3$

$\nabla.V_b = \frac{\partial}{\partial z}(0) = 0$

$\nabla.V_c = \frac{\partial}{\partial z}(z) = 1$

4. Calculate the divergence

(a) $V_a = x^2\hat{i} + 3xz^2\hat{j} - 2xy\hat{k}$ (or) $2xz\hat{k}$

Sol. $\nabla \cdot V_a = \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(3xz^2) - \frac{\partial}{\partial z}(2xy)$

$= 2x + 3xz^2 - 2xy$

(b) $V_b = xy\hat{i} + 2yz\hat{y} + 3zx\hat{k}$

Sol. $\nabla \cdot V_b = \frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial y}(2yz) + \frac{\partial}{\partial z}(3zx)$

$= y + 2z + 3x$

(c) $V_c = y^2\hat{i} + (2xy + z^2)\hat{j} + 2yz\hat{k}$

Sol. $\nabla \cdot V_c = \frac{\partial}{\partial x}(y^2) + \frac{\partial}{\partial y}(2xy + z^2) + \frac{\partial}{\partial z}(2yz)$

$= y^2 + 2x + z^2 + 2y$

5. Suppose the function sketched is $V_a = -y\hat{i} + x\hat{j}$, and $V_b = x\hat{j}$. Calculate their curls.

Sol.

$$\nabla \cdot V_a = \begin{vmatrix} \overset{+}{\hat{i}} & \overset{-}{\hat{j}} & \overset{+}{\hat{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & x & 0 \end{vmatrix}$$

$= \hat{i}\left(\frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}(x)\right) - \hat{j}\left(\frac{\partial}{\partial z}(0) - \frac{\partial}{\partial z}(-y)\right) + \hat{k}\left(\frac{\partial}{\partial z}(x) - \frac{\partial}{\partial y}(-y)\right)$

$= \hat{i}(+0) - \hat{j}(0) + \hat{k}(1+1)$

$= \cancel{(-x\hat{i} - y\hat{j})} + 2\hat{k}$

$$\nabla \cdot V_b = \begin{vmatrix} \overset{+}{i} & \overset{-}{j} & \overset{+}{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & x & 0 \end{vmatrix}$$

$= \hat{i}(0-0) - \hat{j}(0-0) + \hat{k}(1-0)$

$= (\cancel{-x\hat{i}}) + 1\hat{k} = \hat{k}$

$\rightarrow \frac{\partial}{\partial x}(x^2) = 2x$

$\rightarrow \frac{\partial}{\partial x}(x^2+y^2+z^2) = 2x+0+0$
$= 2x$

$\rightarrow \frac{\partial}{\partial y}(xy^3z) = 3y^2xz$

$\rightarrow \frac{\partial}{\partial z}(xyz) = xy$

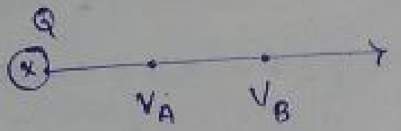
6|1|21

$\rightarrow \vec{\nabla} \times \vec{r} = 0$

$\rightarrow \vec{\nabla} \cdot \vec{r} = 3$

$\vec{E} = -\text{Grad } V$

→ Electric field is -ve gradient of potential.



$V_A > V_B$ (indicates -ve sign)

1. Divergence of gradient of a scalar point function:

$$\vec{\nabla} \cdot \vec{\nabla}\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot \hat{i}\left(\frac{\partial \phi}{\partial x}\right) + \hat{j}\left(\frac{\partial \phi}{\partial y}\right) + \hat{k}\left(\frac{\partial \phi}{\partial z}\right)$$

$$= \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}$$

$$= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\phi$$

$$\vec{\nabla} \cdot \vec{\nabla}\phi = \nabla^2 \phi$$

Where, $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$

↓

Laplacian operator

2. Curl of a gradient of a scalar point function:

$$\vec{\nabla} \times (\vec{\nabla}\phi) = 0$$

3. Divergence of a curl of a vector point function:

$$\vec{\nabla} (\vec{\nabla} \times \vec{F}) = 0$$

4. Curl of a curl of a vector point function:

** $\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})$

$$BAC = CAB$$

$$\Rightarrow \vec{\nabla} \times (\vec{\nabla} \times \vec{F}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{F}) \cdot \nabla^2 \vec{F}$$

Problems: $\rightarrow \vec{\nabla} \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\vec{\nabla} \times \vec{A}) - \vec{A} \cdot (\vec{\nabla} \times \vec{B})$

1. If $r^2 = x^2 + y^2 + z^2$ then show grad $r^n = n \cdot r^{n-2} \cdot \vec{r}$

Sol. Given

$$r^2 = x^2 + y^2 + z^2$$

$$r = \sqrt{x^2 + y^2 + z^2}$$

$$\Rightarrow r = (x^2 + y^2 + z^2)^{1/2}$$

$$\vec{\nabla} \cdot r^n = \vec{\nabla} (x^2 + y^2 + z^2)^{n/2}$$

$$= \hat{i} \frac{\partial}{\partial x} (x^2 + y^2 + z^2)^{n/2} + \hat{j} \frac{\partial}{\partial y} (x^2 + y^2 + z^2)^{n/2} + \hat{k} \frac{\partial}{\partial z} (x^2 + y^2 + z^2)^{n/2}$$

$$\frac{\partial}{\partial x} (x^2 + y^2 + z^2)^{n/2} = \frac{n}{2} (x^2 + y^2 + z^2)^{\frac{n-2}{2}} \cdot 2x$$

$$= nx (x^2 + y^2 + z^2)^{\frac{n}{2} - 1}$$

Similarly

$$\frac{\partial}{\partial y} (x^2 + y^2 + z^2)^{n/2} = ny (x^2 + y^2 + z^2)^{\frac{n}{2} - 1}$$

$$\frac{\partial}{\partial z} (x^2 + y^2 + z^2)^{n/2} = nz (x^2 + y^2 + z^2)^{\frac{n}{2} - 1}$$

then

$$\vec{\nabla} \cdot r^n = n (x^2 + y^2 + z^2)^{\frac{n}{2} - 1} [x\hat{i} + y\hat{j} + z\hat{k}]$$

$$= n \cdot r^{n-2} \cdot \vec{r}$$

Since

$$r = (x^2 + y^2 + z^2)^{1/2}$$

$$r^{n-2} = (x^2 + y^2 + z^2)^{\frac{n-2}{2}}$$

2. Curl $\vec{F}$, where $\vec{F} = $ grad $(2x^2 - 3y^2 + 4z^2)$

(A) $4x - 6y + 8z$

(B) $4x\hat{i} - 6y\hat{j} + 8z\hat{k}$

(C) 0

(D) 3

curl(grad $\vec{F}$)

$\vec{\nabla} \times \vec{\nabla} F = 0$

3. The value of 'λ'. So that the vector $\vec{u} = (x+3y)\hat{i} + (y-2z)\hat{j} + (x+\lambda z)\hat{k}$ is a Solenoid.

Sol. Condition to be a solenoid field is

divergence = 0

$\Rightarrow \vec{\nabla}\cdot\vec{F} = 0$ (then $\vec{F}$ is a solenoid field)

$$\vec{\nabla}\cdot\vec{u} = 0$$

$$\frac{\partial}{\partial x}(x+3y)\hat{i} + \frac{\partial}{\partial y}(y-2z)\hat{j} + \frac{\partial}{\partial z}(x+\lambda z)\hat{k} = 0$$

$$1 + 1 + \lambda = 0$$

$$\boxed{\lambda = -2}$$

4. $\vec{F} = (ax+3y+4z)\hat{i} + (x-2y+3z)\hat{j} + (3x+2y-z)\hat{k}$ is a solenoid then $a = ?$

Sol. $\vec{\nabla}\cdot\vec{F} = 0$

$$\frac{\partial}{\partial x}(ax+3y+4z)\hat{i} + \frac{\partial}{\partial y}(x-2y+3z)\hat{j} + \frac{\partial}{\partial z}(3x+2y-z)\hat{k} = 0$$

$$a + (-2) - 1 = 0$$

$$a - 3 = 0$$

$$\boxed{a = 3}$$

5. Value of n for which the vectors $r^n\vec{r}$ is solenoid.

(A) 3 (B) -3 (C) 2 (D) -2

Sol. $\vec{\nabla}(r^n.\vec{r}) = 0$

$$\vec{\nabla}\left((x^2+y^2+z^2)^{n/2}.(x\hat{i}+y\hat{j}+z\hat{k})\right) = 0$$

$$r^n.\vec{r} = (x^2+y^2+z^2)^{n/2}.(x\hat{i}+y\hat{j}+z\hat{k})$$

$$= x(x^2+y^2+z^2)^{n/2}\hat{i} + y(x^2+y^2+z^2)^{n/2}\hat{j} + z(x^2+y^2+z^2)^{n/2}\hat{k}$$

$\vec{\nabla} \cdot r^n \cdot \vec{r} = 0$

$$\Rightarrow \frac{\partial}{\partial x}\left[x(x^2+y^2+z^2)^{n/2}\right] + \frac{\partial}{\partial y}\left[y(x^2+y^2+z^2)^{n/2}\right] + \frac{\partial}{\partial z}\left[z(x^2+y^2+z^2)^{n/2}\right] = 0$$

then

$$\frac{\partial}{\partial x}\left[x(x^2+y^2+z^2)^{n/2}\right] = (x^2+y^2+z^2)^{n/2} + x \cdot \frac{n}{\cancel{2}}(x^2+y^2+z^2)^{\frac{n}{2}-1} \cdot \cancel{2}x$$

$$= (x^2+y^2+z^2)^{n/2} + n \cdot x^2 (x^2+y^2+z^2)^{\frac{n}{2}-1}$$

Similarly

$$\frac{\partial}{\partial y}\left[y(x^2+y^2+z^2)^{n/2}\right] = (x^2+y^2+z^2)^{n/2} + n \cdot y^2 (x^2+y^2+z^2)^{\frac{n}{2}-1}$$

$$\frac{\partial}{\partial z}\left[z(x^2+y^2+z^2)^{n/2}\right] = (x^2+y^2+z^2)^{n/2} + n \cdot z^2 (x^2+y^2+z^2)^{\frac{n}{2}-1}$$

$$\vec{\nabla} \cdot r^n \cdot \vec{r} = 3(x^2+y^2+z^2)^{n/2} + n(x^2+y^2+z^2)^{\frac{n}{2}-1}(x^2+y^2+z^2)$$

$$\Rightarrow \underbrace{(x^2+y^2+z^2)^{n/2}}_{\neq 0}(n+3) = 0$$

$$\Rightarrow n+3 = 0$$

$$\boxed{n = -3}$$

P_2. Show that $\vec{\nabla} \cdot \left(\frac{\vec{r}}{r^3}\right) = 0$

P_1. If $\phi = \log(x^2+y^2+z^2)$, then show that grad $\phi = \frac{2(x\hat{i}+y\hat{j}+z\hat{k})}{x^2+y^2+z^2}$

P_3. $\phi = \log r$ then find $\vec{\nabla}\phi$.

Solutions

P_1. Given

$$\phi = \log(x^2+y^2+z^2)$$

$$\vec{\nabla} = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$$

grad $\phi = \vec{\nabla}\phi$

$$\vec{\nabla}\phi = \frac{1}{x^2+y^2+z^2}\left[\frac{\partial}{\partial x}(x^2+y^2+z^2)\hat{i} + \frac{\partial}{\partial y}(x^2+y^2+z^2)\hat{j} + \frac{\partial}{\partial z}(x^2+y^2+z^2)\hat{k}\right]$$

$$\text{grad}\,\phi = \frac{2x\hat{i}+2y\hat{j}+2z\hat{k}}{x^2+y^2+z^2} = \frac{2(x\hat{i}+y\hat{j}+z\hat{k})}{x^2+y^2+z^2}$$

Hence proved

P3. Given

$$\phi = \log r$$

$$r = x^2+y^2+z^2$$

$$\vec{\nabla} = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$$

$$\vec{\nabla}\phi = \vec{\nabla}.\log(x^2+y^2+z^2)$$

$$= \frac{1}{x^2+y^2+z^2}(2x\hat{i} + 2y\hat{j} + 2z\hat{k})$$

$$= \frac{2(x\hat{i}+y\hat{j}+z\hat{k})}{x^2+y^2+z^2}$$

P2. Given $r^2 = x^2+y^2+z^2$

$$\vec{\nabla}.\left(\frac{\vec{r}}{r^3}\right) = 0 \Rightarrow \vec{\nabla}.(r^{-3}.\vec{r}) = 0$$

$$r^{-3}.\vec{r} = x(x^2+y^2+z^2)^{-3/2}\hat{i} + y(x^2+y^2+z^2)^{-3/2}\hat{j} + z(x^2+y^2+z^2)^{-3/2}\hat{k}$$

$$\vec{\nabla}.r^{-3}.\vec{r} = \frac{\partial}{\partial x}(x(x^2+y^2+z^2)^{-3/2})\hat{i} + \frac{\partial}{\partial y}(y(x^2+y^2+z^2)^{-3/2})\hat{j} + \frac{\partial}{\partial z}(z(x^2+y^2+z^2)^{-3/2})\hat{k}$$

$$= \frac{-3x}{\cancel{2}}(x^2+y^2+z^2)^{-5/2}.\cancel{2}x + (x^2+y^2+z^2)^{-3/2} +$$

$$\frac{-3y}{\cancel{2}}(x^2+y^2+z^2)^{-5/2}.\cancel{2}y + (x^2+y^2+z^2)^{-3/2} +$$

$$\frac{-3z}{\cancel{2}}(x^2+y^2+z^2)^{-5/2}.\cancel{2}z + (x^2+y^2+z^2)^{-3/2}$$

$$= -3x^2(x^2+y^2+z^2)^{-5/2} + (x^2+y^2+z^2)^{-3/2} -$$

$$3y^2(x^2+y^2+z^2)^{-5/2} + (x^2+y^2+z^2)^{-3/2} -$$

$$3z^2(x^2+y^2+z^2)^{-5/2} + (x^2+y^2+z^2)^{-3/2}$$

$$= -3(x^2+y^2+z^2)(x^2+y^2+z^2)^{-5/2} + 3(x^2+y^2+z^2)^{-3/2}$$

$$= -3(x^2+y^2+z^2)^{1-\frac{5}{2}} + 3(x^2+y^2+z^2)^{-3/2}$$

$$= -3(x^2+y^2+z^2)^{-3/2} + 3(x^2+y^2+z^2)^{-3/2} = 0$$

$$\therefore \vec{\nabla}\left(\frac{\vec{r}}{r^3}\right) = 0$$

7|1|21

1. Electric potential in a region of space is given by $V = 5x - 7x^2y + 8y^2 + 16yz - 5z$ volt where the distance in metre. Obtain an expression for the electric field intensity and y-component field at (2,4,-3) in the space.

sol. Given

$$V = 5x - 7x^2y + 8y^2 + 16yz - 5z$$

$$\vec{E} = -\text{grad}\,\vec{V}$$

$$\vec{E} = -\vec{\nabla}\vec{V}$$

$$= -\vec{\nabla}\cdot(5x - 7x^2y + 8y^2 + 16yz - 5z)$$

$$\vec{\nabla}\cdot\vec{V} = \frac{\partial}{\partial x}(5x - 7x^2y + 8y^2 + 16yz - 5z)\hat{i} + \frac{\partial}{\partial y}(5x - 7x^2y + 8y^2 + 16yz - 5z)\hat{j} + \frac{\partial}{\partial z}(5x - 7x^2y + 8y^2 + 16yz - 5z)\hat{k}$$

$$= (5 - 14xy)\hat{i} + (-7x^2 + 16y + 16z)\hat{j} + (16y - 5)\hat{k}$$

$$E = -\vec{\nabla}\cdot\vec{V}$$

$$E = (-5 + 14xy)\hat{i} + (7x^2 - 16y - 16z)\hat{j} + (5 - 16y)\hat{k}$$

Given point (2,4,-3)

$$E_y\big|_{(2,4,-3)} = 7\times 2^2 - 16\times 4 - 16\times(-3)$$

$$= 7\times 4 - 64 + 48$$

$$= 28 - 64 + 48$$

$$= 12 \text{ volts}$$

2. Find the value of constant c for which the vector $\vec{A} = \hat{i}(x+3y) + \hat{j}(y-2z) + \hat{k}(x+cz)$ is Solenoid.

Sol. $\vec{\nabla}\cdot\vec{A} = 0$

$$\Rightarrow \frac{\partial}{\partial x}(x+3y) + \frac{\partial}{\partial y}(y-2z) + \frac{\partial}{\partial z}(x+Cz) = 0$$

$$1+1+C = 0$$

$$\boxed{C = -2}$$

3. Prove that $\vec{\nabla}\cdot(\vec{A}\times\vec{r}) = \vec{r}\cdot(\vec{\nabla}\times\vec{A})$

$$\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$$

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

Sol. Given $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$$\vec{\nabla}\times\vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix}$$

$$= \hat{i}(0-0) - \hat{j}(0-0) + \hat{k}(0-0) = 0$$

$$\vec{\nabla}\cdot(\vec{A}\times\vec{r}) = \vec{r}\cdot(\vec{\nabla}\times\vec{A}) - \vec{A}\cdot\underbrace{(\vec{\nabla}\times\vec{r})}_{0}$$

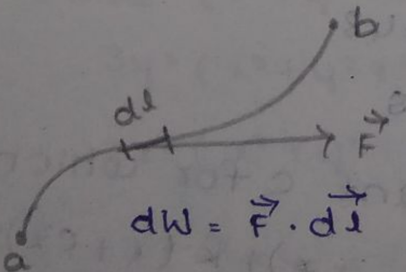
$$\vec{\nabla}\cdot(\vec{A}\times\vec{r}) = \vec{r}\cdot(\vec{\nabla}\times\vec{A})$$

Hence proved.

1. Line integral (or) path integral:

$$W = \int_a^b \vec{F}\cdot d\vec{l}$$

Small displacement



$$dW = \vec{F}\cdot d\vec{l}$$

$$\vec{E} = \frac{-dv}{dx}$$

$$dv = \vec{E}\cdot d\vec{r}$$

$$\Delta V = -\int \vec{E}\cdot d\vec{l}$$

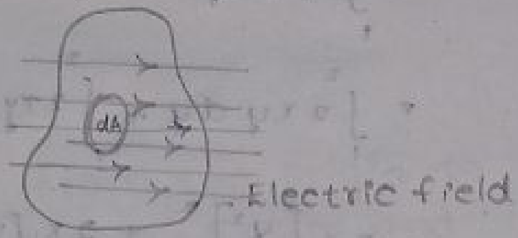
$\rightarrow$ path dependent

2. Surface Integral:

→ It is indicated by double integral.

→ Area vector is always ⊥er to area.

$$\iint_{l\,b=a} \vec{F}\cdot d\vec{A}$$



$$\vec{E}\cdot d\vec{A} = d\phi_E$$

$$\phi_E = \iint_S \vec{E}\cdot d\vec{A}$$

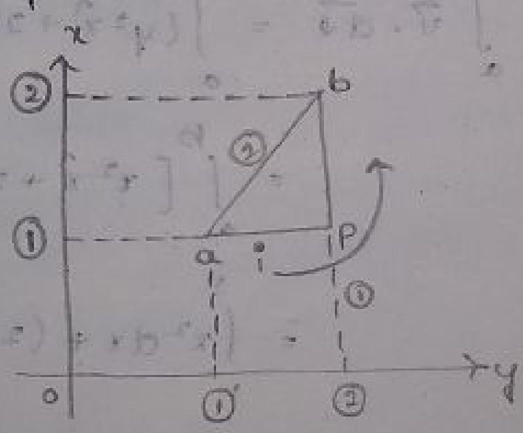
3. Volume Integral:

$$\iiint T.\,dv$$

any function × volume

problems:

1. Calculate the line integral of the function $v = y^2\hat{x} + 2x(y+1)\hat{y}$ from a point $a(1,1,0)$ to the point $b(2,2,0)$ along the paths as shown in the figure.



(In the direction of x y is const)

sol. from the figure.

a to p

$\vec{dl} = dx\,\hat{x}$

p to b

$\vec{dl} = dy\,\hat{y}$

Now,

$$\int_a^p \vec{v}\cdot dx\,\hat{x} = \int (y^2\hat{x} + 2x(y+1)\hat{y})\,dx\,\hat{x}$$

$$= \int_1^2 y^2\,dx = 1^2[x]_1^2 = 1[2-1]$$

$$= 2-1 = 1$$

$$\int_p^b \vec{v}\cdot\vec{dl} = \int_p^b (y^2\hat{x} + 2x(y+1)\hat{y})\cdot dy\,\hat{y}$$

$$= \int_p^b 2x(y+1)\,dy$$

$$= \int_1^2 2x(y+1)\,dy$$

$$= \int_1^2 2xy\,dy + \int_1^2 2xy\,dy$$

$$= 2x\left[\frac{y^2}{2}\right]_1^2 + 2x\,[y]_1^2$$

× $\boxed{= 2\left[\frac{4}{2}-\frac{1}{2}\right] + 2[2-1]}$ $= \cancel{2}\left[\frac{2}{\cancel{2}}\right](2^2-1) + 2(2)(2-1)$

$$= 2(4-1) + 4 = 6+4 = 10$$

For the path ①

$$\int_a^b \vec{v}\cdot d\vec{l} = \int_a^p \vec{v}\cdot d\vec{l} + \int_p^b \vec{v}\cdot d\vec{l} = 1+10 = 11$$

if $x = y$
$dx = dy$

For the path (2)

$$\int_a^b \vec{v}\cdot d\vec{l} = \int_a^b (y^2\hat{x} + 2x(y+1)\hat{y})\cdot(dx\,\hat{x} + dy\,\hat{y})$$

$$= \int_a^b \left[x^2\hat{x} + 2x(x+1)\hat{y}\right]\cdot\left[dx\,\hat{x} + dx\,\hat{y}\right]$$

$$= \int_1^2 x^2\,dx + (2x^2+2x)\,dx$$

$$= \int_1^2 (3x^2+2x)\,dx$$

$$= \cancel{3}\cdot\frac{x^3}{\cancel{3}}\Big]_1^2 + \cancel{2}\cdot\frac{x^2}{\cancel{2}}\Big]_1^2 = (8-1)+(4-1)$$

$$= 7+3 = 10$$

$$= 3\cdot\left[\frac{8}{8}-\frac{1}{8}\right] + [4-1] \quad ×$$

$$= 3\left[\frac{8-1}{8}\right] + 3$$

$$= 3\times\frac{7}{8} + 3 = \frac{21}{8} + 3 = \frac{21+24}{8}$$

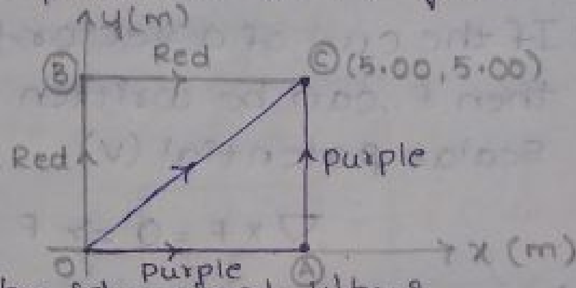
18/01/21

$$\left[W = \int \vec{F} \cdot d\vec{r} \right.$$

$$\vec{F} = mg(-\hat{j})$$

$$\left. O \to C = O \to A + A \to C \right]$$

2. A 4.00-kg particle moves from the origin to position Ⓒ, having coordinates $x = 5.00$ m and $y = 5.00$ m. One force of the particle is the gravitational force acting in the negative direction. Using equation, calculate the work done by the gravitational force on the particle as it goes from O to Ⓒ along
(a) the purple path
(b) the red path and
(c) the blue path
(d) Your results should all be identical. Why?



Sol. $W = \int \vec{F} \cdot d\vec{r}$

$\vec{F} = mg(-\hat{j})$

(a) $O \to C = O \to A + A \to C$

$O \to A$: $\vec{F} = mg(-\hat{j})$

$d\vec{r} = dx\,\hat{i}$

$$W_{O\to A} = \int mg(-\hat{j}) \cdot dx\,\hat{i} = 0$$

$$W_{A\to C} = \int_A^C mg(-\hat{j}) \cdot dy\,\hat{j} = -mg \int_0^5 dy = -mg(5)$$

$$= -4 \times 9.8 \times 5$$

$$= -196\text{ J}$$

(b) $W_{O\to B} = \int mg(-\hat{j}) \cdot dy\,\hat{j}$

$$= -mg[y]_0^5 = -4 \times 9.8 \times 5 = -196\text{ J}$$

$$W_{B\to C} = \int mg(-\hat{j}) \cdot dx\,\hat{i} = 0$$

(c) $W_{o\to c} = \int_0^c mg(-\hat{j})(dx\hat{i} + dy\hat{j})$

$= \int_0^c (0) + \int_0^c (-mg)dy$

$= -mg \times 5 = -196 J$

(d) Because the force is conservative force.
→ Work done by conservative force depends only on initial and final positions.

Surface integral $\iint \vec{F} \cdot d\vec{A}$

→ If the curl of a vector field (F) vanishes (everywhere) then F can be written as the gradient of a scalar potential (V).

$$\nabla \times F = 0 \Rightarrow F = -\nabla V$$

(The minus sign is purely conventional)

(a) $\nabla \times F = 0$ (everywhere)

(b) $\int_a^b F \cdot dl$ is independent of path for any given end points.

(c) $\oint F \cdot dl = 0$ for any closed loop

(d) F is the gradient of some scalar function:

$$F = -\nabla V$$

The potential is not unique - any constant can be added to V with impurity. Since this will not affect its gradient.

If the divergence of a vector field (F) vanishes (everywhere), then F can be expressed as the curl of a vector potential (A):

$\nabla \cdot F = 0 \Longleftrightarrow F = \nabla \times A$ → conclusion

Theorem-2:

Divergence-less (or solenoidal) fields

(a) $\nabla \cdot F = 0$ everywhere

(b) $\int F \cdot da$ is independent of surface, for any given boundary line

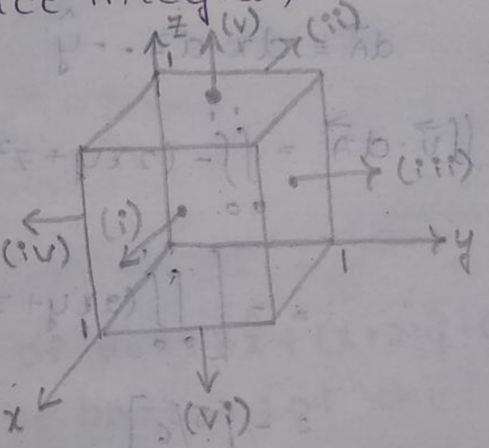
(c) $\oint F \cdot da = 0$ for any closed surface

(d) F is the curl of some vector function

$$F = \nabla \times A$$

3. $V = y^2 \hat{x} + (2xy + z^2)\hat{y} + (2yz)\hat{z}$

Calculate the $\iint \vec{V} \cdot d\vec{A}$ (surface integral)



Sol.

(i) $\iint \vec{V} \cdot d\vec{A}$

$$d\vec{A} = dy\,dz\,\hat{x}$$

$$\vec{V} \cdot d\vec{A} = y^2 \hat{x} \cdot dy\,dz\,\hat{x}$$

$$= y^2\,dy\,dz$$

$$\iint \vec{V} \cdot d\vec{A} = \int_0^1\int_0^1 y^2\,dy\,dz$$

$$= \int_0^1 dz \cdot \int_0^1 y^2\,dy = 1 \times \frac{y^3}{3} = 1/3$$

(ii) $\iint \vec{V} \cdot d\vec{A}$

$$d\vec{A} = dy \cdot dz \cdot -\hat{x}$$

$$\vec{V} = y^2 \hat{x}$$

$$\iint \vec{V} \cdot d\vec{A} = \int_0^1\int_0^1 -y^2\,dy\,dz = \frac{-1}{3}$$

(iii) $\iint \vec{V} \cdot d\vec{A} = \iint (2xy + z^2)\hat{y} \cdot dx \cdot dz\,\hat{y}$

$$= \iint (2xy + z^2)\,dx\,dz$$

$$= \int_0^1\int_0^1 (2xy + z^2)\, dx\, dz$$

$$= \int_0^1\int_0^1 2xy\, dx\, dz + \int_0^1\int_0^1 z^2\, dx\, dz \left[= \frac{4}{3}\right]$$

$$= \cancel{2}\left[\frac{x^2}{\cancel{2}} \cdot \frac{y^2}{2} \times z\right]_0^1 + \left[z \times \frac{z^3}{3}\right]_0^1$$

$$= 2\left[\frac{1}{2} \cdot \frac{1}{2} \cdot 1\right] + \left[1 \cdot \frac{1}{3}\right]$$

WRONG $= \cancel{2}\left[\frac{1}{\cancel{4}_2}\right] + \left[\frac{1}{3}\right] = \frac{1}{2} + \frac{1}{3} = \frac{3+2}{6} = 5/6$

$$= 2\left[\frac{x^2}{2}\right] \times z + \frac{z^3}{3} \times x$$

$$= \cancel{2} \times \frac{1}{\cancel{2}} \times 1 + \frac{1}{3} \times 1$$

$$= 1 + \frac{1}{3} = \frac{4}{3}$$

(iv) $\iint \vec{v} \cdot d\vec{A}$

$$d\vec{A} = dx \cdot dz \cdot -\hat{y}$$

$$\iint \vec{v} \cdot d\vec{A} = \int_0^1\int_0^1 -(2xy + z^2)\, \hat{y} \times dx \cdot dz\, \hat{y}$$

$$= -\left[\int_0^1\int_0^1 (2xy + z^2)\, dx\, dz\right]$$

$$= -\left[5/6\right]$$

(iv) $\frac{2}{3}$

(v) 1

(vi) -1

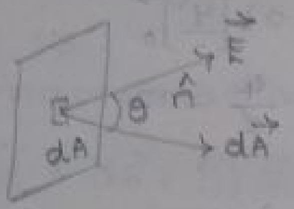
20|01|21

Line integral

Surface integral → $\iint S = \iint \vec{V} \cdot d\vec{A}$

Volume integral → $\iiint_V \vec{V} \cdot d\vec{r}$ (or) $\iiint \tau \cdot dv$

$d\vec{r}$ ↓ Elemental volume



$d\vec{A} = dA \cdot \hat{n}$ ↳ unit vector

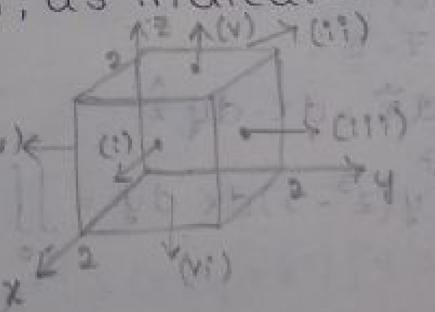
$d\phi_E = \vec{E} \cdot d\vec{A}$

$\phi_S = \iint d\phi_E = \iint_S \vec{E} \cdot d\vec{A}$

$\hat{n} = \frac{\vec{A}}{|\vec{A}|}$ | $d\vec{A} = dA\hat{n}$

$\hat{n} = \frac{d\vec{A}}{dA}$

Q. Calculate the surface integral of $v = 2xz\hat{x} + (x+2)\hat{y} + y(z^2-3)\hat{z}$ over five sides (excluding the bottom) of the cubical box (side 2) in figure. Let "upward and outward" be the +ve direction, as indicated by the arrows.



Sol. $\iint_S \vec{V} \cdot d\vec{A}$ for (i) along x-axis

$x = 2$

$d\vec{A} = dy \cdot dz\ \hat{x}$

$\iint 2xz\, dy\, dz = \iint 2(2) z\, dy\, dz$

$z \to 0 - 2$

$y \to 0 - 2$

$= 4 \int_0^2 dy \int_0^2 z\, dz$

$= 4(2-0)\left[\frac{z^2}{2}\right]_0^2$

$= 4 \times \cancel{2} \times \frac{1}{\cancel{2}} \times 4 = 16$

for (ii)

$x = 0$

$d\vec{A} = dy \cdot dz(-\hat{x})$

$\Rightarrow -2xz\,dy\,dz = 0$

for (iii)

$y = 2$

$x \to 0-2$

$z \to 0-2$

$d\vec{A} = dx \cdot dz \cdot \hat{y}$

$$\int_0^2\int_0^2 (x+2)\,dx\,dz = \int_0^2\int_0^2 x\,dx \cdot dz + \int_0^2\int_0^2 2 \cdot dx \cdot dz$$

$$= \left[\frac{x^2}{2}\right]_0^2 \cdot [z]_0^2 + 2 \cdot [x]_0^2 \cdot [z]_0^2$$

$$= \frac{4}{\cancel{2}} \cdot \cancel{2} + 2 \cdot 2 \cdot 2$$

$$= 4 + 8 = 12$$

for (iv)

~~$z = 0$~~ (-ve y-axis)

$d\vec{A} = dx \cdot dz(-\hat{y})$

result $= -12$

for (v)

$z = 2$

$d\vec{A} = dx \cdot dy\,\hat{z}$

$$\int_0^2\int_0^2 y(z^3 - 3)\,dx\,dy = \int_0^2\int_0^2 yz^3\,dx\,dy - 3\int_0^2\int_0^2 dx \cdot dy$$

$$= \frac{y^2}{2} \cdot \frac{z^4}{4} \cdot x - 3 \cdot x \cdot y \quad 8$$

$$= \frac{\cancel{4}}{\cancel{2}} \cdot \frac{16^{8}}{\cancel{4}} \cdot 2 - 3 \cdot 2 \cdot 2$$

$$= 16 - 12 = 4 \quad (20)$$

for (vi) $z = 0$

$d\vec{A} = dx \cdot dy(-\hat{z})$

$\vec{v} \cdot d\vec{A} = -y(z^3 - 3)\,dx \cdot dy$

$\vec{v} \cdot d\vec{A} = 3y\,dx\,dy$

$$= \int_0^2\int_0^2 3y\,dx\,dy$$

$$= 3\,(2-0)\left[\frac{y^2}{2}\right]_0^2$$

$$= 3 \cdot \cancel{2} \cdot \frac{4}{\cancel{2}} = 12$$

Total $= 16 + 0 + 12 + \cancel{12} - \cancel{12} + 4$

$= 32 \quad (48)$

Gauss Theorem: (Gauss divergence theorem)

Gauss theorem of divergence states that the surface integral of the normal component of vector 'A' taken over a closed surface 's' is equal to the volume integral of the divergence of vector 'A' over the volume 'v' enclosed by the surface 's'.

Its mathematical form is $\left[\begin{matrix} dv \\ d\tau \end{matrix}\right] \rightarrow$ elemental volume

$$\iint \vec{A}\cdot d\vec{s} = \iiint \vec{\nabla}\cdot\vec{A}\, dv \quad (\vec{A} \text{ or } \vec{v}) \quad (dv \text{ or } d\tau) \quad d\tau/dv = dx\cdot dy\cdot dz$$

$$= \iiint_V \operatorname{div}\vec{A}\cdot dv$$

Hence this theorem is the transformation between the surface and volume integrals.

$$\int_V (\vec{\nabla}\cdot\vec{v})\, d\tau = \oint_S \vec{v}\cdot d\vec{a}$$

As per the gauss theorem (problem 3)

$$\boxed{\iint_S \vec{v}\cdot d\vec{s} = \iiint_V (\vec{\nabla}\cdot\vec{v})\, d\tau}$$

$$\iint_S \vec{v}\cdot d\vec{s} = \iint_1 \vec{v}\cdot d\vec{s} + \iint_2 \vec{v}\cdot d\vec{s} + \iint_3 \vec{v}\cdot d\vec{s} + \iint_4 \vec{v}\cdot d\vec{s} + \iint_5 \vec{v}\cdot d\vec{s} + \iint_6 \vec{v}\cdot d\vec{s}$$

$$= \cancel{\frac{4}{3}} - \cancel{\frac{4}{3}} + \frac{4}{3} + \frac{2}{3} + \cancel{1} - \cancel{1}$$

$$= \frac{6}{3} = 2$$

$$\vec{\nabla}\cdot\vec{v} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left(y^2\hat{i} + (2xy+z^2)\hat{j} + 2yz\hat{k}\right)$$

$$= 2(x+y)$$

Now

$$\iiint 2(x+y)\, d\tau = \int_0^1\int_0^1\int_0^1 2(x+y)\, dx\cdot dy\cdot dz$$

$$= \int_0^1\int_0^1 2(x+y)\, dx\cdot dy \cdot \int_0^1 dz$$

$$= \iint_0^1 2x\, dx\, dy + \iint_0^1 2y\, dx\cdot dy$$

$$= \cancel{2}\left[\frac{x^2}{\cancel{2}}\right]_0^1 * [y]_0^1 + \cancel{2}\left[\frac{y^2}{\cancel{2}}\right]_0^1 \times [x]_0^1$$

$$= 1\cdot 1 + 1\cdot 1$$

$$= 1 + 1 = 2$$

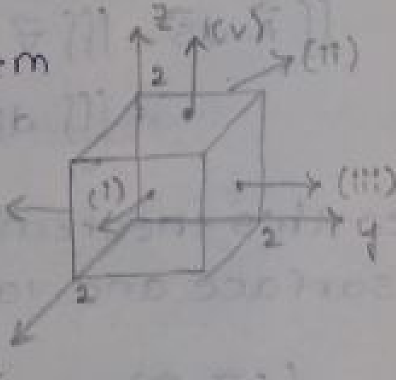
Hence gauss theorem is verified.

* $\oint_L \vec{A}\cdot d\vec{l} = \iint_S \text{curl}\, \vec{A}\cdot d\vec{S}$ (stoke's theorem)

21/01/21

1. Verify the Gauss divergence theorem

$$\vec{V} = xy\,\hat{x} + 2yz\,\hat{y} + 3zx\,\hat{z}$$



Sol. By gauss divergence theorem

$$\iint_S \vec{V}\cdot d\vec{S} = \iiint_V (\vec{\nabla}\cdot\vec{V})\, dv$$

Now,

$$\vec{\nabla}\cdot\vec{V} = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right)(xy\,\hat{x} + 2yz\,\hat{y} + 3zx\,\hat{z})$$

$$= \frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial y}(2yz) + \frac{\partial}{\partial z}(3zx)$$

$$= y + 2z + 3x$$

$$\iiint_V (\vec{\nabla}\cdot\vec{V})\, dv = \int_0^2\int_0^2\int_0^2 y\, dx\, dy\, dz + 2\int_0^2\int_0^2\int_0^2 dx\, dy\, dz + 3\int_0^2\int_0^2\int_0^2 x\, dx\, dy\, dz$$

$$= \left[\frac{y^2}{2}\right]_0^2 (2-0)(2-0) + 2\cdot\left[\frac{z^2}{2}\right]_0^2\cdot(2-0)\cdot(2-0) + 3\cdot\left[\frac{z^2}{2}\right]_0^2\cdot(2-0)\cdot(2-0)$$

$$= \frac{4}{\cancel{2}}\cdot\cancel{4}^{2} + \cancel{2}\cdot\frac{4}{\cancel{2}}\cdot 2\cdot 2 + 3\cdot\frac{4}{\cancel{2}}\cdot\cancel{2}\cdot 2$$

$$= 8 + 16 + 24 = 48$$

- Here surface integral should be calculated.

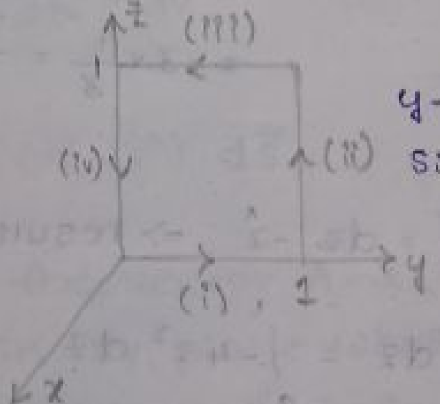
Stoke's Theorem: It states that the line integral of a vector field A around a closed curve is equal to the surface integral of the curl of vector A taken over the surface S surrounded by the closed curve. This theorem is the transformation between the line and surface integrals. For a vector field A stoke's theorem can be written as

$$\oint_L \vec{A}\cdot d\vec{r} = \oiint_S \text{curl}\,\vec{A}\cdot d\vec{S}$$

$$= \iint (\vec{\nabla}\times\vec{V})\cdot d\vec{S}$$

Problem:

1. Suppose $V = (2xz + 3y^2)\hat{y} + (4yz^2)\hat{z}$. Check stoke's theorem for the square surface shown in figure



y - z plane
side = 1 unit

Sol. By stoke's theorem

$$\oint_L \vec{A}\cdot d\vec{r} = \oiint_S \text{curl}\,\vec{A}\cdot d\vec{S}$$

first L.H.S

$$\oint \vec{V}\cdot d\vec{l} = \int_0^A \vec{V}\cdot d\vec{l} + \int_A^B \vec{V}\cdot d\vec{l} + \int_B^C \vec{V}\cdot d\vec{l} + \int_C^0 \vec{V}\cdot d\vec{l}$$

for (i)

$$\vec{V} = (2xz + 3y^2)\hat{y} + (4yz^2)\hat{z}, \quad d\vec{l} = dy\,\hat{y}$$

$$\int_0^A \vec{V}\cdot d\vec{l} = \int_0^1 \left((2xz + 3y^2)\hat{y} + 4yz^2\hat{z}\right)\cdot dy\,\hat{y}$$

$$= \int_0^1 (2\underset{\substack{\downarrow \\ x=0}}{x}z + 3y^2)\,dy = \int_0^1 (2z + 3y^2)\,dy$$

$$= \int_0^1 3y^2\,dy$$

$$= 3\left[\frac{y^3}{3}\right]_0^1 = \cancel{3}\times\frac{1}{\cancel{3}}$$

$$= 1$$

for (ii)

$$d\vec{l} = dz\,\hat{z}$$

$$\int_A^B \vec{v}\cdot d\vec{l} = \int_0^1 \left((2xz+3y^2)\hat{y} + (4yz^2)\hat{z}\right)\cdot dz\,\hat{z}$$

$$= \int_0^1 4yz^2\,dz$$

$y=0$

$$= \int_0^1 4z^2\,dz = 4\times\left[\frac{z^3}{3}\right]_0^1 = 4\times\frac{1}{3} = 4/3$$

for (iii)

$$d\vec{l} = dy\ -\hat{y}$$

$$\int_B^C -(2z+3y^2)\,dy = \int_0^1 -3y^2\,dy$$

$$= -\cancel{3}\times\frac{1}{\cancel{3}} = -1$$

for (iv)

$$d\vec{l} = dz\ -\hat{z} \Rightarrow \text{result} = 0$$

$$\left[\int_C^0 -4yz^2\,dz = \int_0^1 -4z^2\,dz = -4\times\left[\frac{1}{3}\right]\right]^{\times}$$

then total = (i) + (ii) + (iii) + (iv)

$$= \cancel{1} + \frac{4}{3} - \cancel{1} + 0 = 4/3$$

Now R.H.S

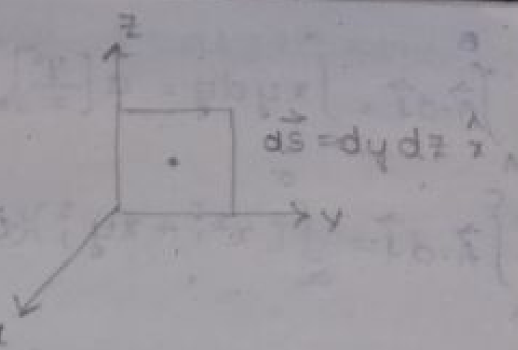
$$\vec{\nabla}\times\vec{v} = \begin{vmatrix} \overset{+}{\hat{x}} & \overset{-}{\hat{y}} & \overset{+}{\hat{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (4z^2 - 2x) & 0 & 4yz^2 \end{vmatrix}$$

$(2xz+3y^2)$

$\{\ \cancel{\hat{x}(4z^2 - (2x)}$

$$= \hat{x}\left(4z^2 - (2x+0)\right) - \hat{y}(0-0) + \hat{z}(2z+0-0)$$

$$= (4z^2 - 2x)\hat{x} + 2z\,\hat{z}$$



Now

$$\iiint((4z^2 - 2x)\hat{x} + 2z\,\hat{z})\,dy\,dz\,\hat{i}$$

$$= \int_0^1\int_0^1 (4z^2 - 2x)\,dy\,dz$$

$$= 4\left[\frac{z^3}{3}\right][y]_0^1 \cancel{(-2\times 0)}$$

$$= 4 \times \frac{1}{3} \times 1 = \frac{4}{3}$$

Hence stoke's theorem verified. $\boxed{\text{L.H.S} = \text{R.H.S}}$

27/1/21

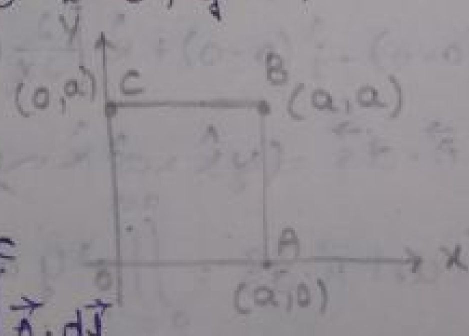
1. Gauss divergence theorem
2. Stokes theorem

$$\iint_S \vec{F}\cdot\vec{ds} = \iiint_V (\underbrace{\vec{\nabla}\cdot\vec{F}}_{\text{Scalar}})\,dv = \iiint_V \text{div}\,\vec{F}\,dv$$

$$\oint_C \vec{F}\cdot\vec{dl} = \iint_S \text{curl}\,\vec{F}\cdot\vec{ds} = \iint_S (\vec{\nabla}\times\vec{F})\cdot\vec{ds}$$

2. Verify stoke's theorem for the vector $\vec{A} = x(\hat{i}x + \hat{j}y)$ integrated round a square in xy plane whose sides are along the lines $x=0$, $y=0$, $x=a$, $y=a$.

Sol. Given vector



$$\vec{A} = x(\hat{i}x + \hat{j}y)$$

$$= x^2\hat{i} + xy\,\hat{j}$$

$$\oint_{\substack{\text{loop}\\ OABCO}} \vec{A}\cdot\vec{dl} = \int_O^A \vec{A}\cdot\vec{dl} + \int_A^B \vec{A}\cdot\vec{dl} + \int_B^C \vec{A}\cdot\vec{dl} + \int_C^O \vec{A}\cdot\vec{dl}$$

$\vec{dl} = dx\,\hat{i}$

Now,

$$\int_O^A \vec{A}\cdot\vec{dl} = \int_O^A (x^2\hat{i} + xy\,\hat{j})\cdot dx\,\hat{i} = \int_0^a x^2\,dx = \frac{a^3}{3} \rightarrow (1)$$

$$d\vec{l} = dy\,\hat{j}$$

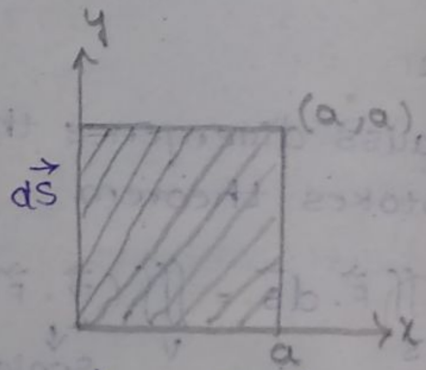
$$\int_A^B \vec{A}\cdot d\vec{l} = \int_0^a xy\,dy = a\left[\frac{y^2}{2}\right]_0^a = \frac{a^3}{2} \longrightarrow (2)$$

$$\int_B^C \vec{A}\cdot d\vec{l} = \int_a^0 (x^2\hat{i} + xy\,\hat{j})\cdot(-dx)\hat{i} = -\int_a^0 x^2\,dx = -\left[\frac{x^3}{3}\right]_a^0$$

$$= -\left[\frac{0^3 - a^3}{3}\right] = \frac{a^3}{3}$$

$$\int_C^0 (x^2\hat{i} + xy\,\hat{j})\cdot(-dy\,\hat{j}) = -\int_a^0 xy\,dy = 0 \quad (\because x = 0)$$

$$\oint_{\substack{\text{loop}\\ \text{OABCO}}} \vec{A}\cdot d\vec{l} = \frac{a^3}{3} + \frac{a^3}{2} + \frac{a^3}{3} = \frac{7a^3}{6}$$



Now, R.H.S

$$\iint \text{curl}\,\vec{A}\cdot d\vec{S}$$

$$d\vec{S} = a^2\hat{k}$$

$$d\vec{S} = |d\vec{S}|\,d\hat{S}$$

$$d\vec{S} = a^2\hat{z}$$

$$= dx\,dy\,\hat{k}$$

$$\text{curl}\,\vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & xy & 0 \end{vmatrix}$$

$$= \hat{i}(0-0) - \hat{j}(0-0) + \hat{k}\left[\frac{\partial}{\partial x}(xy) - \frac{\partial}{\partial y}(x^2)\right]$$

$$\text{curl}\,\vec{A}\cdot d\vec{S} = \cancel{(y\hat{k} \times a^2\hat{k} = a^2 y)}\; y\hat{k}\cdot dx\,dy\,\hat{k} = y\,dx\,dy$$

$$\iint \text{curl}\,\vec{A}\cdot d\vec{S} = \int_0^a\!\!\int_0^a a^2 y\,dx\,dy$$

$$= a\left[\frac{y^2}{2}\right]_0^a = a \times \frac{a^2}{2} = \frac{a^3}{2}$$

Stokes theorem is not verified.

3. Using Stoke's theorem prove that $\oint_c r \cdot dl = 0$ where r is position vector.

Sol. $\oint_c \vec{r} \cdot d\vec{l} = \oiint (\vec{\nabla} \times \vec{r}) \cdot d\vec{s}$

We know

$$\vec{\nabla} \times \vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = 0$$

$\therefore \oint \vec{r} \cdot d\vec{l} = 0$ Hence proved

4. If $F = ax\hat{i} + by\hat{j} + cz\hat{k}$ where a, b, c are constants, show that $\iint_S F \cdot ds\,\hat{n} = \frac{4}{3}\pi(a+b+c)$, Where S is the Surface area of a unit sphere.

Sol. Unit Sphere $= 1 \Rightarrow r$

$$\iint \vec{F} \cdot d\vec{s} = \iiint \operatorname{div} \vec{F}\, dv = \iiint (\vec{\nabla} \cdot \vec{F})\, dv$$

given $\vec{F} = ax\hat{i} + by\hat{j} + cz\hat{k}$

$$\vec{\nabla} \cdot \vec{F} = \left[\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right] \cdot \left[ax\hat{i} + by\hat{j} + cz\hat{k}\right]$$

$$= \frac{\partial}{\partial x}(ax) + \frac{\partial}{\partial y}(by) + \frac{\partial}{\partial z}(cz)$$

$$= a + b + c$$

$$\iiint (\vec{\nabla} \cdot \vec{F})\, dv = \iiint (a+b+c)\, dv$$

$$= (a+b+c) \iiint dv$$

$$= (a+b+c) \cdot \frac{4}{3}\pi r^3 \quad (\because r = 1)$$

$$= (a+b+c) = \frac{4}{3}\pi$$

Hence shown.

5. Use divergence theorem to show that $\iint_S F\cdot\hat{n}\,ds = \frac{12}{5}\pi r^5$ Where S is the surface of sphere of radius r and $F = x^3\hat{i} + y^3\hat{j} + z^3\hat{k}$.

Sol. Given

$$F = x^3\hat{i} + y^3\hat{j} + z^3\hat{k}$$

$$\vec{\nabla}\cdot\vec{F} = 3x^2 + 3y^2 + 3z^2$$

$$= 3(x^2+y^2+z^2)$$

$$\iiint(\vec{\nabla}\cdot\vec{F})\,dv = \iiint 3(x^2+y^2+z^2)\,dx\,dy\,dz$$

$$= 3r^2\iiint dv \quad (\because r^2 = x^2+y^2+z^2)$$

$$= 3r^2\left(\frac{4}{3}\pi r^3\right)$$

$$= 4\pi r^5$$

(OR)

→ According to spherical polar coordinates (r,θ,ϕ)

$$r^2 = x^2+y^2+z^2$$

$$dv = dx\,dy\,dz \Rightarrow dv = r^2\sin\theta\,dr\,d\theta\,d\phi$$

Also; $0\le r\le a$, $0\le\theta\le\pi$, and $0\le\phi\le 2\pi$

$$\Rightarrow \iint_S \vec{F}\cdot\hat{n}\,ds = 3\int_{r=0}^{a}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi}(r^2)\cdot r^2\sin\theta\,dr\,d\theta\,d\phi$$

$$= 3\int_{r=0}^{a} r^4\,dr \times \int_{\theta=0}^{\pi}\sin\theta\,d\theta \times \int_{\phi=0}^{2\pi} d\phi$$

$$= 3\times\left[\frac{r^5}{5}\right]_0^a \times [-\cos\theta]_0^{\pi} \times [\phi]_0^{2\pi}$$

$$= \frac{3a^5}{5}\times(-\cos\pi + 1)\times 2\pi$$

$$= \frac{3a^5}{5}\times 2\times 2\pi$$

$$= \frac{12}{5}\pi a^5$$

28/01/2021 (UNIT-02)

Spherical polar co-ordinates:

(r, θ, ϕ)

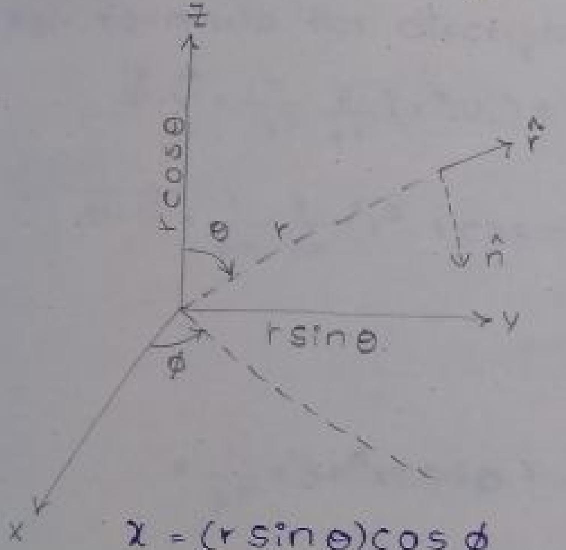
here r = radial vector

θ = polar angle

ϕ = Azimuthal angle

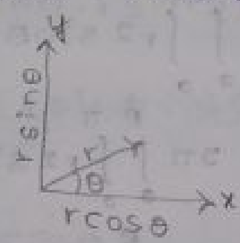
Polar coordinates (r, θ)

cartesian - (x, y, z)



The component of r in x-y plane is $r\sin\theta$

z-component = $r\cos\theta$



$x = (r\sin\theta)\cos\phi$

$y = (r\sin\theta)\sin\phi$

$z = r\cos\theta$

$(x, y, z) = (r\sin\theta\cos\phi,\ r\sin\theta\sin\phi,\ r\cos\theta)$

→ Any vector 'A' can be expressed in terms of them, in the usual way

$$A = A_r\hat{r} + A_\theta\hat{\theta} + A_\phi\hat{\phi}$$

here A_r = radial, A_θ = polar, A_ϕ = azimuthal $\Big\}$ components of A

In terms of the cartesian unit vectors

$$\hat{r} = \sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + \cos\theta\,\hat{z}$$

$$\hat{\theta} = \cos\theta\cos\phi\,\hat{x} + \cos\theta\sin\phi\,\hat{y} - \sin\theta\,\hat{z}$$

$$\hat{\phi} = -\sin\phi\,\hat{x} + \cos\phi\,\hat{y}$$

$d\tau = r^2 \sin\theta \, dr \, d\theta \, d\phi$

$d\tau = r^2 \sin\theta \, dr \, d\theta \, d\phi$

$V = \iiint d\tau$

$= \iiint r^2 \sin\theta \, dr \, d\theta \, d\phi$

$$\boxed{\begin{array}{l} r \to 0 - R \\ \theta \to 0 - \pi \\ \phi \to 0 - 2\pi \end{array}}$$

$$= \int_0^R \int_0^\pi \int_0^{2\pi} r^2 \sin\theta \, dr \, d\theta \, d\phi$$

$$= \int_0^R \int_0^\pi r^2 \sin\theta \, dr \, d\theta \int_0^{2\pi} d\phi$$

$$= 2\pi \int_0^R \int_0^\pi r^2 \sin\theta \, dr \, d\theta$$

$$= 2\pi \int_0^R r^2 [-\cos\theta]_0^\pi \, dr$$

$$= 2\pi \int_0^R -[\cos\pi - \cos 0] r^2 dr$$

$$= 2\pi \times 2 \int_0^R r^2 dr$$

$$= 4\pi \left[\frac{r^3}{3}\right]_0^R = \frac{4}{3}\pi R^3$$

→ The vector derivatives in spherical co-ordinates

Gradient: $(\vec{v} = v_r \hat{r} + v_\theta \hat{\theta} + v_\phi \hat{\phi})$

$$\nabla T = \frac{\partial T}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial T}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial T}{\partial \phi}\hat{\phi}$$

Divergence:

$$\nabla \cdot v = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta \, u_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\theta}{\partial\phi}$$

Curl:

$$\nabla \times v = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta \, v_\phi) - \frac{\partial v_\theta}{\partial\phi}\right]\hat{r} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial\phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\theta}$$

$$+ \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial\theta}\right]\hat{\phi}$$

Laplacian:

$$\nabla^2 T = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial T}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\cdot\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial T}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\cdot\frac{\partial^2 T}{\partial\phi^2}$$

Ex. Compute the divergence of function

$$\vec{V} = \overset{V_r}{r\cos\theta}\,\hat{r} + \overset{V_\theta}{r\sin\theta}\,\hat{\theta} + \overset{V_\phi}{r\sin\theta\cos\phi}\,\hat{\phi}$$

sol. formula for divergence

$$\vec{\nabla}\cdot\vec{V} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 V_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, V_\theta) + \frac{1}{r\sin\theta}\frac{\partial V_\phi}{\partial\phi}$$

$$\vec{\nabla}\cdot\vec{V} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\cdot r\cos\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\cdot r\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(r\sin\theta\cos\phi)$$

$$= \frac{1}{\cancel{r^2}}\times 3\cancel{r^2}\times\cos\theta + \frac{1}{r\cancel{\sin\theta}}\times\cancel{r}\times 2\cancel{\sin\theta}\cos\theta + \frac{1}{r\cancel{\sin\theta}}\times r\cancel{\sin\theta}(-\sin\phi)$$

$$= 3\cos\theta + 2\cos\theta - \sin\phi$$

$$= 5\cos\theta - \sin\phi$$

Cylindrical co-ordinates:

$x = s\cos\phi$ ($s \to$ distance from the z-axis)

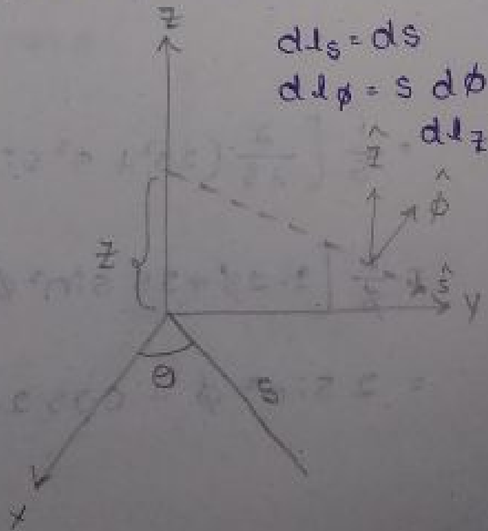
$y = s\sin\phi$

$z = z$

$dl_s = ds$

$dl_\phi = s\,d\phi$

$dl_z = dz$



The unit vectors are

$\hat{s} = \cos\phi\,\hat{x} + \sin\phi\,\hat{y}$

$\hat{\phi} = -\sin\phi\,\hat{x} + \cos\phi\,\hat{y}$

$\hat{z} = \hat{z}$

(s, ϕ, z)

$d\tau = s\,ds\cdot d\phi\cdot dz$

$s \to 0 - \infty$

$\phi \to 0 - 2\pi$

$z \to -\infty - \infty$

→ The vector derivatives in cylindrical coordinates are:

Gradient:

$$\nabla T = \frac{\partial T}{\partial s}\hat{s} + \frac{1}{s}\frac{\partial T}{\partial \phi}\hat{\phi} + \frac{\partial T}{\partial z}\hat{z}$$

Divergence:

$$\nabla \cdot V = \frac{1}{s}\frac{\partial}{\partial s}(s V_s) + \frac{1}{s}\frac{\partial V_\phi}{\partial \phi} + \frac{\partial V_z}{\partial z}$$

Curl:

$$\nabla \times V = \left(\frac{1}{s}\frac{\partial V_z}{\partial \phi} - \frac{\partial V_\phi}{\partial z}\right)\hat{s} + \left(\frac{\partial V_s}{\partial z} - \frac{\partial V_z}{\partial s}\right)\hat{\phi} + \frac{1}{s}\left(\frac{\partial}{\partial s}(s V_\phi) - \frac{\partial V_s}{\partial \phi}\right)\hat{z}$$

Laplacian:

$$\nabla^2 T = \frac{1}{s}\frac{\partial}{\partial s}\left(s \cdot \frac{\partial T}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 T}{\partial \phi^2} + \frac{\partial^2 T}{\partial z^2}$$

Ex. Compute the divergence of function

$$\vec{V} = \underbrace{s(2+\sin^2\phi)}_{V_s}\hat{s} + \underbrace{s\sin\phi\cos\phi}_{V_\phi}\hat{\phi} + \underbrace{3z}_{V_z}\hat{z}$$

Sol. Formula

$$\nabla \cdot V = \frac{1}{s}\frac{\partial}{\partial s}(s V_s) + \frac{1}{s}\frac{\partial V_\phi}{\partial \phi} + \frac{\partial V_z}{\partial z}$$

$$= \frac{1}{s}\cdot\frac{\partial}{\partial s} s\left[s(2+\sin^2\phi)\right] + \frac{1}{s}\frac{\partial}{\partial \phi}(s\sin\phi\cos\phi) + \frac{\partial}{\partial z}(3z)$$

$$= \frac{1}{s}\left[\frac{\partial}{\partial s}(2s^2 + s^2\sin^2\phi) + \frac{1}{s}\frac{\partial}{\partial \phi} s\left(\frac{1}{2}\sin 2\phi\right) + 3\right.$$

$$= \frac{1}{\cancel{s}}\left[2\cdot 2\cancel{s} + 2\underline{s}\sin^2\phi\right] + \frac{1}{\cancel{s}}\cdot\frac{\cancel{s}}{\cancel{2}}\cos 2\phi \times \cancel{2} + 3$$

$$= 2\sin^2\phi + \cos 2\phi + 7$$

29/01/2021

Gauss divergence theorem

$\iint \vec{F}\, d\vec{s} = \iiint \mathrm{div}\vec{F}\, d\tau$

$$\boxed{\iint_S \vec{F}\cdot d\vec{s} = \iiint_V \vec{\nabla}\cdot\vec{F}\, d\tau}$$

Stoke's theorem

$$\boxed{\oint_C \vec{F}\cdot d\vec{l} = \oiint_S \mathrm{curl}\,\vec{F}\cdot d\vec{s}}$$

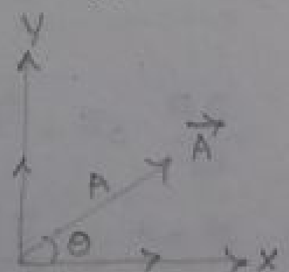
$\oint \vec{F}\cdot d\vec{l} = \oiint (\vec{\nabla}\times\vec{F})\, d\vec{s}$

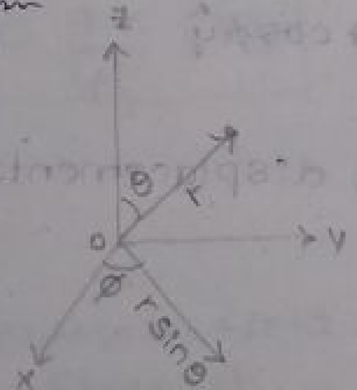
Co-ordinates:

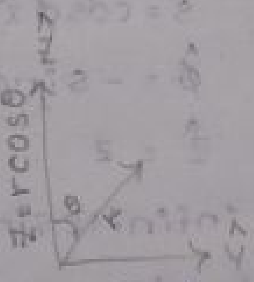
1. Polar coordinates (r,θ)
2. Spherical polar coordinates
3. Cylindrical coordinates

Spherical polar coordinates:

(r,θ,ϕ)







y-z plane

$\vec{A}_x = A\cos\theta\,\hat{x}$

$\vec{A}_y = A\sin\theta\,\hat{y}$

$x = r\sin\theta\cos\phi$

$y = r\sin\theta\sin\phi$

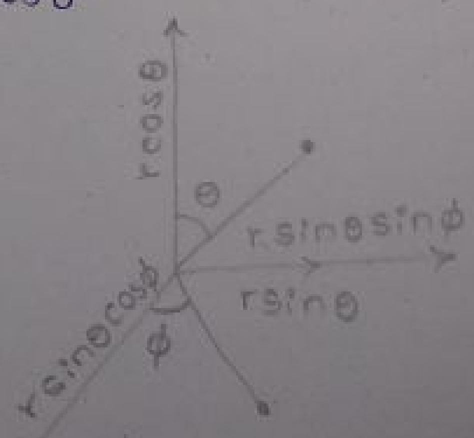
$z = r\cos\theta$

$(x,y,z) \Rightarrow (r,\theta,\phi)$

r, $\theta \rightarrow$ polar angle $(0-\pi)$

$\phi \rightarrow$ Azimuthal angle $(0-2\pi)$

$d\tau$ (or) $dv = r^2\sin\theta\, dr\, d\theta\, d\phi$



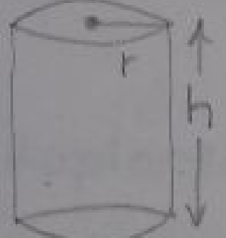
$$d\vec{r} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$$

$$\left.\begin{aligned} dl_r &= dr \\ dl_\theta &= r\,d\theta \\ dl_\phi &= r\sin\theta\,d\phi \end{aligned}\right\} \text{infinitesimal elements (displacements)}$$

$$d\tau = dl_r\,dl_\theta\,dl_\phi$$

$$\therefore\ d\tau = r^2 \sin\theta\,dr\,d\theta\,d\phi$$

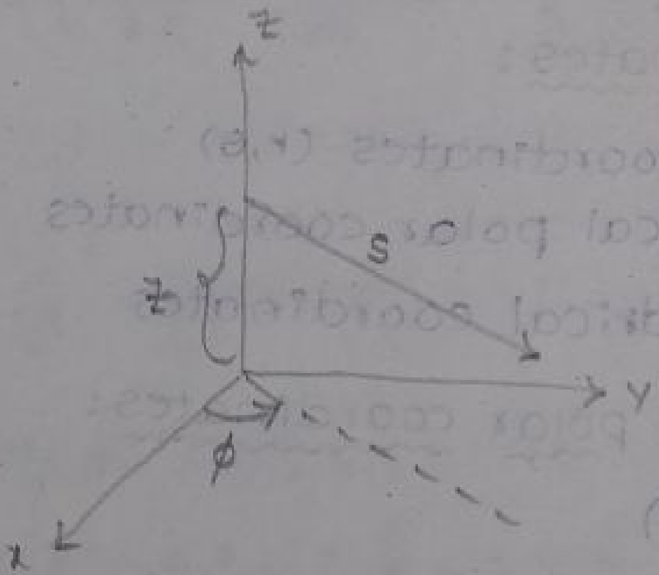
Cylindrical coordinates:



$x = s\cos\phi$

$y = s\sin\phi$

$z = z$



The unit vectors

$\hat{s} = \cos\phi\,\hat{x} + \sin\phi\,\hat{y}$

$\hat{\phi} = -\sin\phi\,\hat{x} + \cos\phi\,\hat{y}$

$\hat{z} = z$

The infinitesimal displacements are

$dl_s = ds$

$dl_\phi = s\,d\phi$

$dl_z = dz$

$d\tau = s\,ds\,d\phi\,dz$