* System of linear equations :-

① $x+y=0$
$2x-y=0.$ $\Big\}$ $(0,0)$

$$\begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

$\Rightarrow AX = 0.$

Here $|A| \neq 0$., so A^{-1} exist.

$\Rightarrow A^{-1}AX = A^{-1}0.$

$\Rightarrow IX = 0$

$\therefore X = 0.$

$X=0, Y.$

$AY=0$

$A^{-1}A \cdot Y = A^{-1}0$

$\Rightarrow I \cdot Y = 0$

$\therefore Y = 0.$

This is "Homogeneous linear system". since, zero is always a solution.

② $x+y=1$
$2x-y=1$

$$\begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

$AX = b.$

$x = \frac{2}{3}$ & $y = \frac{1}{3}$..

This is "Non-Homogeneous linear system".

③ $x+y+z=5$
$2x-y+z=6$
$x-3y+2z=1$

$Ax=b$

$[A\ b]$ – Augmented matrix.

A – coefficient matrix

Homogeneous.

(4) (i) $x+y-z=0 \longrightarrow (1)$

$2x-y+z=0 \longrightarrow (2)$

$$\begin{bmatrix} 1 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

$(1)+(2) \Rightarrow 3x=0 \Rightarrow x=0.$

$\Rightarrow x+y-z=0$

$0+y-z=0$

$y=z.$ (or) $-y+z=0.$

$\therefore$ solution set is $(0,y,y)$ (or) $(0,z,z)$.

$S=\{(0,z,z) \mid z \in \mathbb{R}\}.$

(ii) Non-Homogeneous

$x+y-z=1$

$2x-y+z=2.$

$$\begin{bmatrix} 1 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

$\therefore$ solution set is $(1,z,z)$

$\Rightarrow (1,z,z) = (0,z,z)+(1,0,0).$

$\therefore$ solution set for "Non-Homogeneous" system is Homogeneous system & other solution.

$\left.\begin{array}{l} x=1 \\ y-z=0 \end{array}\right\}$ $\begin{array}{l} x+0y+0z=1 \\ 0x+y-z=0. \end{array}$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$A=\begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix} \xrightarrow[R_3-3R_1]{R_2-R_1} \begin{bmatrix} R \\ R_2-R_1 \\ R_3-3R_1 \end{bmatrix}.$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & -1 & 1 \\ 2 & -1 & 1 & 2 \end{bmatrix} \xrightarrow{R_2=R_2-R_1} \begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & -3 & 3 & 0 \end{bmatrix} \xrightarrow{\frac{R_2}{-3}} \begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & +1 & -1 & 0 \end{bmatrix}.$$

pivot element

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \end{bmatrix} \xleftarrow{R_1=R_1-R_2}$$

③ ⑤ $x-y+z+w=1$

$2x-y+3z-w=5$

$x+y+2z+3w=1$

$$[A:b] = \begin{bmatrix} 1 & -1 & 1 & 1 & 1 \\ 2 & -1 & 3 & -1 & 5 \\ 1 & 1 & 2 & 3 & 1 \end{bmatrix}$$

$R_2 = R_2 - 2R_1$

$R_3 = R_3 - R_1$

$$\simeq \begin{bmatrix} 1 & -1 & 1 & 1 & 1 \\ 0 & 1 & 1 & -3 & 3 \\ 0 & 2 & 1 & 2 & 0 \end{bmatrix}$$

$R_2 = R_1 + R_2$

$R_3 = R_3 - 2R_2$

$$\simeq \begin{bmatrix} 1 & 0 & 2 & -2 & 4 \\ 0 & 1 & 1 & -3 & 3 \\ 0 & 0 & -1 & 8 & -6 \end{bmatrix}$$

$R_1 = R_1 + 2R_3$

$R_2 = R_2 + R_3$, $R_3 = -R_3$

$$\simeq \begin{bmatrix} 1 & 0 & 0 & 14 & -8 \\ 0 & 1 & 0 & 5 & -3 \\ 0 & 0 & 1 & -8 & 6 \end{bmatrix}$$

∴ This is reduced Echelon form.

$x + 14w = -8$

$\Rightarrow x = -8 - 14w.$

$y = -3 - 5w$

$z = 6 + 8w.$

∴ The solution set for given system is,

$S = \{(-8-14w, -3-5w, 6+8w, w) / w \in \mathbb{R}\}.$

$\Rightarrow S = \{(-8,-3,6,0) + (-14w, -5w, 8w, w) / w \in \mathbb{R}\}.$

Non-Homogeneous · Homogeneous · [Infinite sols].

∴ Rank of A = 3.

* Rank :-

→ Number of non-zero rows in a reduced echelon form is called 'Rank'.

(6) $x+y+z=1$

$2x-y+z=3$

$4x+y+3z=6$.

Augmented matrix.

$$\Rightarrow [A:b] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & -1 & 1 & 3 \\ 4 & 1 & 3 & 6 \end{bmatrix} \quad \begin{matrix} R_2 = R_2 - 2R_1 \\ R_3 = R_3 - 4R_1 \end{matrix}$$

$$\simeq \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -3 & -1 & 1 \\ 0 & -3 & -1 & 2 \end{bmatrix} \quad R_3 = R_3 - R_2$$

$$\simeq \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -3 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Here Last row is 0 0 0 1.

$\Rightarrow (0)x + (0)y + (0)z = 1$

$\Rightarrow 0 = 1$

But $0 \neq 1$ [Never].

It is impossible [$0 = 1$]

∴ No solution set for given system.

Rank of A is 2.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -3 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

Rank of $[A:b]$ is 3.

* Note :-

→ Rank of $[A:b] \geq$ Rank of A.

* Ex:-

$$\begin{bmatrix} 1 & 1 & 0 \\ 2 & -1 & 0 \\ 3 & 1 & 0 \end{bmatrix} \simeq \begin{bmatrix} 1 & 1 & 0 \\ 0 & -3 & 0 \\ 0 & -2 & 0 \end{bmatrix} \begin{matrix} \\ \leftarrow R_2 - 2R_1 \\ R_3 - 3R_1 \end{matrix}$$

$$\simeq \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \leftarrow R_3 - \frac{2}{3}R_2$$

$$\simeq \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \leftarrow R_1 - R_2.$$

$\Rightarrow (3,1) = \alpha(1,1) + \beta(2,-1)$

$\alpha + 2\beta = 3$

$\alpha - \beta = 1 \Rightarrow \alpha = \beta + 1$

$\Rightarrow \alpha + 2\beta = 3$

$1 + 3\beta = 3 \Rightarrow \beta = \frac{2}{3}$. & $\alpha = \frac{5}{3}$.

$x + y = 0$
$2x - y = 0$ }. $\therefore \frac{5}{3}(x+y) = \frac{5}{3}(0)$
$3x + y = 0$.

$+ \frac{2}{3}(2x - y) = \frac{2}{3}(0)$.

$\frac{\cancel{3}9x}{\cancel{3}} - y = 0 \quad \therefore 3x + y = 0$

* Note:-

① $\text{Rank}(A) \neq \text{Rank}[A:b] \longrightarrow$ No solution.

② $\text{Rank}(A) = \text{Rank}[A:b] =$ No. of variables $=$ No. of columns of A.
$\rightarrow$ Unique soln.

③ $\text{Rank}(A) = \text{Rank}[A:b] <$ no. of variables $\longrightarrow$ Infinite solutions

① $\rightarrow$ Inconsistent.

② & ③ $\rightarrow$ consistent.

* **Note:-**

1. $Ax=b$; No. of variables = No. of equations
2. $|A| \neq 0$; unique solution.
3. $|A| = 0$; → Infinitely many solutions. → No solution.

* **Eigen values and Eigen vectors.**

→ $T(v) = \lambda v$, $\lambda \in F$, then 'v' is called 'Eigen vector' and 'λ' is 'Eigen value'

$\Rightarrow T(v) = \lambda v$

~~$Ax=0$~~

$\Rightarrow T(v) = \lambda I(v)$.

$\Rightarrow [T - \lambda I](v) = 0$.

$(A - \lambda I)x = 0$. ⟶ ① (In matrix form).

Note:-

① $Ax=0$ has Non-zero solution if $|A|=0$.

Finding Eigen values:-

Now ①

$\Rightarrow |A - \lambda I| = 0$.

Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}$ $\lambda I = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$

$\rightarrow (A - \lambda I) = \begin{bmatrix} 1-\lambda & 2-0 \\ 3-0 & 0-\lambda \end{bmatrix}$

$\Rightarrow (A - \lambda I) = \begin{bmatrix} 1-\lambda & 2 \\ 3 & -\lambda \end{bmatrix}$

$\Rightarrow \det(A - \lambda I) = (1-\lambda)(-\lambda) - (2)(3)$

$= -\lambda + \lambda^2 - 6$

$= \lambda^2 - \lambda - 6$ $= (\lambda+2)(\lambda-3)$ ∴ $\lambda = -2, 3$

∴ Eigen values are -2, 3.

* Characteristic Equation:-

→ 'det $(A-\lambda I)=0$' is called 'characteristic equation'

Note:-

→ Roots of the characteristic equation are 'Eigen values' of 'A'. $(\lambda_1, \lambda_2, \ldots \lambda_n)$.

Finding Eigen vectors:-

$\lambda=-2$:-

$\Rightarrow (A-\lambda I)x=0$

$\Rightarrow [A-(-2)I]x=0$ [∵ -2 is one Eigen value)

$\Rightarrow [A+2I]x=0.$

$$\begin{bmatrix} 3 & 2 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$ ∵ $\begin{bmatrix} 1-\lambda & 2 \\ 3 & -\lambda \end{bmatrix}$

$3x+2y=0$

$y=\frac{-3x}{2}.$

∴ Eigen span $=\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} x \\ \frac{-3x}{2} \end{bmatrix}$, $x \in \mathbb{R}$.

For $\lambda=-2$, the Eigen vector is $v=\begin{bmatrix} 2 \\ -3 \end{bmatrix}$.

$\lambda=3$:-

$\Rightarrow [A-\lambda I]x=0$

$\Rightarrow [A-3I]x=0.$

$$\begin{bmatrix} 1-(+3) & 2 \\ 3 & -(+3) \end{bmatrix}=\begin{bmatrix} -2 & 2 \\ 3 & -3 \end{bmatrix}.$$

$$\begin{bmatrix} -2 & 2 \\ 3 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$-2x+2y=0$ $\Rightarrow 2x-2y=0 \Rightarrow x=y.$

$3x-3y=0$. $\Rightarrow y=+x$

* Characteristic Equation:-

→ 'det $(A-\lambda I)=0$' is called "characteristic equation"

Note:-

→ Roots of the characteristic equation are 'Eigen' values' of 'A'. $(\lambda_1, \lambda_2, \ldots \lambda_n)$.

Finding Eigen vectors:-

$\lambda=-2$:-

$\Rightarrow [A-\lambda I]\, x=0$

$\Rightarrow [A-(-2)I]\, x=0$ [∵ -2 is one Eigen value)

$\Rightarrow [A+2I]\, x=0$.

$$\begin{bmatrix} 3 & 2 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$ ∵ $\begin{bmatrix} 1-\lambda & 2 \\ 3 & -\lambda \end{bmatrix}$

$3x+2y=0$

$y=\frac{-3x}{2}$.

∴ Eigen span $=\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} x \\ \frac{-3x}{2} \end{bmatrix}$, $x \in \mathbb{R}$.

For $\lambda=-2$, the Eigen vector is $v=\begin{bmatrix} 2 \\ -3 \end{bmatrix}$.

$\lambda=3$:-

$\Rightarrow [A-\lambda I]\, x=0$

$\Rightarrow [A-3I]\, x=0$.

$$\begin{bmatrix} 1-(+3) & 2 \\ 3 & -(+3) \end{bmatrix}=\begin{bmatrix} -2 & 2 \\ 3 & -3 \end{bmatrix}.$$

$$\begin{bmatrix} -2 & 2 \\ 3 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow 2x-2y=0 \Rightarrow x=y$.

$-2x+2y=0$

$3x\mp 3y=0$. $\Rightarrow y=+x$

$\lambda = 3$:-

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}.$$

$$[A - 3I]\, X = 0$$

$$\begin{bmatrix} -2 & 2 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow -2x + 2y = 0 \Rightarrow x - y = 0 \Rightarrow x = y.$

$3x - 3y = 0 \Rightarrow x - y = 0 \Rightarrow x = y.$

$\therefore$ Eigen space $= \begin{bmatrix} x \\ x \end{bmatrix}$, $x \in \mathbb{R}$.

For $\lambda = 3$

$\therefore v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$

* Problem :-

$$A = \begin{bmatrix} \overset{+}{1} & \overset{-}{2} & \overset{+}{1} \\ 3 & 2 & -1 \\ 4 & 4 & 0 \end{bmatrix}.$$ Find Eigen values & vectors?

A) $[A - \lambda I]\, X = 0.$

For E. values, $|A - \lambda I| = 0.$

$$\Rightarrow A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 2 & -1 \\ 4 & 4 & 0 \end{bmatrix}, \quad \lambda I = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$\Rightarrow |A - \lambda I| = \begin{vmatrix} \overset{+}{1-\lambda} & \overset{-}{2} & \overset{+}{1} \\ 3 & 2-\lambda & -1 \\ 4 & 4 & -\lambda \end{vmatrix}$$

$$= (1-\lambda)\left[(2-\lambda)(-\lambda) - (-4)\right] - 2\left[-3\lambda + 4\right]$$

For $A_{2\times 2}$:- $\lambda^2 - \text{Trace}(A) \cdot \lambda + \det(A) = 0.$

For $A_{3\times 3}$:- $\lambda^3 - \text{Trace}(A) \cdot \lambda^2 + (A_{11} + A_{22} + A_{33})\lambda - \det(A) = 0.$

characteristic eq^n :-

$\lambda^3 - \text{trace}(A) \cdot \lambda^2 + (A_{11} + A_{22} + A_{33})\lambda - \det(A) = 0.$

$\Rightarrow$ Trace $(A) = 1+2+0 = 3.$

$A_{11} = [(2\times 0) - (-4)] = 4$

$A_{22} = [(1\times 0) - 4] = -4$

$A_{33} = [(1\times 2) - (2\times 3)] = -4$

$|A| = 0$ [$\because$ Third row is = First row + sec row].

$|A| = 1(0+4) - 2(0+4) + 1(12-8)$

$= 4 + 8 + 4 = 8 - 8 = 0.$

$\therefore \lambda^3 - 3\cdot\lambda^2 + (4-4-4)\cdot\lambda - 0 = 0$

$\Rightarrow \lambda^3 - 3\lambda^2 - 4\lambda = 0$

$\Rightarrow \lambda(\lambda^2 - 3\lambda - 4) = 0$

$\Rightarrow \lambda(\lambda-4)(\lambda+1) = 0$

$\therefore \lambda_1 = 0, \quad \lambda_2 = 4, \quad \lambda_3 = -1$ [Eigen values].

$\lambda_1 = 0$:-

$[A - 0I]\,X = 0$

$$\begin{bmatrix} 1 & 2 & 1 \\ 3 & 2 & -1 \\ 4 & 4 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0.$$

$x + 2y + z = 0$ $\Rightarrow x - 2x + z = 0 \Rightarrow z = x$

$3x + 2y - z = 0$ $\Rightarrow 3x - 2x - x = 0 \Rightarrow x = 0,$

$4x + 4y = 0 \Rightarrow x + y = 0 \Rightarrow y = -x \Rightarrow x = -y.$

(1) $\Rightarrow -y + 2y + z = 0. \Rightarrow y + z = 0 \Rightarrow z = -y.$

$\therefore$ E.Vector $= \begin{bmatrix} -y \\ y \\ -y \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}.$

$$\begin{bmatrix} 1 & 2 & 1 \\ 3 & 2 & -1 \\ 4 & 4 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

$\downarrow$

$R_2 = R_2 - 3R_1$

$R_3 = R_3 - 4R_1$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & -4 & -4 \\ 0 & -4 & -4 \end{bmatrix}$$

$x + 2y + z = 0$

$-4y - 4z = 0$

$y + z = 0$

$z = -y$

$x + 2y - y = 0$

$x + y = 0 \quad \Rightarrow x = -y.$

$\therefore$ Eigen vector, $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -y \\ y \\ -y \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}.$

Note :

① $\lambda_1 + \lambda_2 + \lambda_3 \text{ ------ } + \lambda_n = \text{Trace}(A)$

② $\lambda_1 \cdot \lambda_2 \cdot \lambda_3 \text{ ------ } \cdot \lambda_n = \text{Det}(A).$

$$\rightarrow \begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix}$$

$\lambda_1 = 4 \rightarrow \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$\lambda_2 = 0 \rightarrow \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

$\lambda_3 = 0 \rightarrow \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}.$

$\lambda^3 - 4\lambda^2 = 0$

$\Rightarrow \lambda^2(\lambda - 4) = 0$

$\Rightarrow \lambda = 0, 4, 0.$

$AX = 0$

$\Rightarrow x + 2y + z = 0$

$\Rightarrow z = -x - 2y.$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ -x-2y \end{bmatrix} = x\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + y\begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}.$$

* Properties of Eigen values & vectors:

(1) $AX = \lambda X$

$A^2X = A(AX) = A(\lambda X) = \lambda(AX) = \lambda(\lambda X) = \lambda^2 X.$

$A^3X = \lambda^3 X.$

$A^4X = \lambda^4 X.$

$\vdots$

$A^n x = \lambda^n x.$

(2) $[A^5 - A + I]X = A^5X - AX + IX$

$= \lambda^5 X - \lambda X + X$

$= (\lambda^5 - \lambda + 1)X$

$\downarrow$

$P(A) - P(\lambda)$

(3) $A_{4\times4}$ $\cdot 0, -1, 3, 2.$

$A^2 + A - 3I.$

$\Rightarrow \lambda = 0, \quad 0^2 + 0 - 3 = -3$

$\lambda = -1, \quad (-1)^2 + (-1) - 3 = -3$

$\lambda = 3, \quad 3^2 + 3 - 3 = 9$

$\lambda = 2, \quad 2^2 + 2 - 3 = 3.$

$\therefore$ Eigen values of $A^2 + A - 3I$ are $-3, -3, 9, 3.$

(4) $\lambda_1 + \lambda_2 + \lambda_3$ ——— $+ \lambda_n$ = Trace(A).

(5) $\lambda_1 \cdot \lambda_2 \cdot \lambda_3$ ——— $\cdot \lambda_n$ = Det(A).

(6) $\det(A - 3I) = 0$

$\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 = 0$ $\Rightarrow \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = -3$

$\lambda_5 = 3.$

* Cayley - Hamilton Theorem:-

→ Every square matrix satisfies its characteristic ~~square matrix~~ equation

Ex:-

$$A = \begin{bmatrix} 1 & 3 \\ 3 & -1 \end{bmatrix}.$$

$\lambda^2 - tr(A)\lambda + det(A) = 0$

$\Rightarrow \lambda^2 - 0 + (-1-9) = 0$

$\Rightarrow \lambda^2 - 10 = 0.$

$\therefore A^2 - 10I = 0$

$$A^2 = \begin{bmatrix} 1 & 3 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}$$

$$\therefore \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix} - \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix} = 0.$$

* Ex:-

$$A = \begin{array}{c} \begin{array}{ccc} + & - & + \end{array} \\ \begin{bmatrix} 3 & 2 & 0 \\ 1 & -1 & 1 \\ -2 & 1 & 1 \end{bmatrix} \end{array}.$$

given ~~Find~~ (1) $A^5 = \alpha A^2 + \beta A + \gamma I.$
Find α, β, γ? & (2) $A^{-1} = aA^2 + bA + cI.$
Find $a + 2b + c$?

A) $\lambda^3 - tr(A)\lambda^2 + (A_{11} + A_{22} + A_{33})\lambda - det(A) = 0.$

$\Rightarrow \lambda^3 - 3\lambda^2 + (-2 + 3 + -5)\lambda - (-6 + 6 + 0) = 0$

$\Rightarrow \lambda^3 - 3\lambda^2 - 4\lambda + 12 = 0.$

A & A^T have equal characteristic eq^{n}s. since

$$\boxed{det(A - \lambda I) = det(A^T - \lambda I)}$$

From C-H Theorem,

$A^3 - 3A^2 - 4A + 12I = 0.$

→

$$A^3 = \begin{bmatrix} 3 & 2 & 0 \\ 1 & -1 & 1 \\ -2 & 1 & 1 \end{bmatrix}\begin{bmatrix} 3 & 2 & 0 \\ 1 & -1 & 1 \\ -2 & 1 & 1 \end{bmatrix}\begin{bmatrix} 3 & 2 & 0 \\ 1 & -1 & 1 \\ -2 & 1 & 1 \end{bmatrix} =$$

Given $A^3 - 3A^2 \mp 4A + 12I = 0 \longrightarrow (1)$

$\Rightarrow A^3 = 3A^2 + 4A - 12I$

$\Rightarrow A^4 = 3A^3 + 4A^2 - 12A \longrightarrow (2)$

From (1)

$\Rightarrow A^4 = 3(3A^2 + 4A \pm 12I) + 4A^2 - 12A$

$\Rightarrow A^4 = 9A^2 + 4A^2 + \cancel{12A} + 36I - \cancel{12A}$

$\Rightarrow A^4 = 13A^2 - 36I$

$\Rightarrow A^5 = 13A^3 - 36A$ [Multiply with A].

$\therefore A^5 = 13(3A^2 + 4A - 12I) - 36A$.

$A^5 = 39A^2 + 16A - 156I$.

$\Rightarrow \alpha = 39, \quad \beta = 16, \quad \gamma = -156$.

(2) $A^{-1} = aA^2 + bA + cI$, $\quad a + 2b + c = ?$

consider, char eqⁿ.

$\Rightarrow A^3 - 3A^2 - 4A + 12I = 0$.

$\Rightarrow A^{-1}(A^3 - 3A^2 - 4A + 12I) = 0$ [Mul with A^{-1}].

$\Rightarrow A^2 - 3A - 4I + 12A^{-1} = 0$.

$\Rightarrow 12A^{-1} = 4I + 3A - A^2$.

$\therefore A^{-1} = \frac{1}{12}[4I + 3A - A^2]$.

$A^{-1} = -\frac{1}{12}A^2 + \frac{1}{4}A + \frac{1}{3}I$

$\therefore a = -\frac{1}{12}$

$b = \frac{1}{4}$

$c = \frac{1}{3}$

(3) What is $A^8 - 3A^7 - 4A^6 + 11A^5 = ?$.

A) Char of A,

$\lambda^3 - 3\lambda^2 - 4\lambda + 12 = 0$

$\Rightarrow A^3 - 3A^2 - 4A + 12 = 0$.

$\Rightarrow A^5[A^3 - 3A^2 - 4A + 12] = A^5[0]$. [$\because$ multiply].

$\Rightarrow A^8 - 3A^7 - 4A^6 + 12A^5 = 0$.

$\Rightarrow A^8 - 3A^7 - 4A^6 + 11A^5 + A^5 = 0$

$\Rightarrow A^8 - 3A^7 - 4A^6 + 11A^5 = -A^5$.

$\therefore A^8 - 3A^7 - 4A^6 + 11A^5 = -[39A^2 + 16A - 156I]$.

$= -39A^2 - 16A + 156I$.

* Ex:-

$A = \begin{bmatrix} 3 & 2 \\ -4 & -3 \end{bmatrix}$, $A^{2023} = ?$

A) $\lambda^2 - tr(A)\lambda + det(A) = 0$

$\Rightarrow \lambda^2 - 0 + (-1) = 0$

$\Rightarrow \lambda^2 - 1 = 0$

$\Rightarrow A^2 - I = 0$.

$\Rightarrow A^2 = I$.

$\Rightarrow A^{2023} = A$.

(or)

$A^{2023} = A \cdot A^{2022}$

$= A \cdot (A^2)^{1011}$

$= A \cdot (I)^{1011}$ [$\because A^2 = I$]

$= A \cdot I$

$= A$.