FREQUENCY DOMAIN ANALYSIS

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Contents:

1) Frequency response of a system
2) Nyquist stability criterion
3) Bode Plot
4) Gain margin and phase margin

What is frequency response of a system?

- The frequency response of a system is defined as the steady-state response of the system to a sinusoidal input signal.
- The sinusoid is a unique input signal, and the resulting output signal for a linear system, as well as signals throughout the system, is sinusoidal in the steady-state.

Time Domain Analysis Vs Frequency Domain Analysis

1) Variable frequency, sinusoidal signal generators are readily available and precision measuring instruments are available for measurement of magnitude and phase angle. The time response for a step input is more difficult to measure with accuracy.
2) It is easier to obtain the transfer function of a system by a simple frequency domain test. Obtaining transfer function from the step response is more tedious.
3) If the system has large time constants, it makes more time to reach steady state at each frequency of the sinusoidal input. Hence time domain method is preferred over frequency domain method in such systems.
4) In order to do a frequency response test on a system, the system has to be isolated and the sinusoidal signal has to be applied to the system. This may not be possible in systems which can not be interrupted. In such cases, a step or an impulse signal may be given to the system to find its transfer function. Hence for systems which cannot be interrupted, time domain method is more suitable.

Time Domain Analysis Vs Frequency Domain Analysis

5) The design of a controller is easily done in the frequency domain method than in time domain method.

6) The effect of noise signals can be assessed easily in frequency domain rather than time domain.

7) The most important advantage of frequency domain analysis is the ability to obtain the relative stability of feedback control systems. The RH criterion is essentially a time domain method which determines the absolute stability of a system and the determination of relative stability is cumbersome. Nyquist criterion will not only give stability but also relative stability of the system without actually finding the roots of the characteristic equation.

❖ Since the time response and frequency response of a system are related through Fourier transform, the time domain response can be easily obtained from the frequency response. The correlation between time frequency response can be easily established so that the time domain performance measures can be obtained from the frequency domain specifications and vice versa.

Nyquist stability criterion:

- ✓ Frequency domain technique.
- ✓ Relates the location of closed loop poles of the system with the frequency response of the open loop system (stability of the closed loop system from open loop system).
- ✓ Is a graphical technique.
- ✓ Does not require the exact determination of the closed loop poles.
- ✓ Frequency response can be obtained by subjecting the system to a sinusoidal input of constant amplitude and variable frequency and measuring the amplitude and phase angle of the output.
- ✓ It is based on a theory due to Cauchy ”the principle of argument” in the complex variable theory.

Nyquist diagram or Nyquist plot or Polar plot

We are going to determine the

stability of the closed-loop system **from** the **open-loop system** features

(i.e. the graphical representation of the open-loop frequency response $F(j\omega)$)

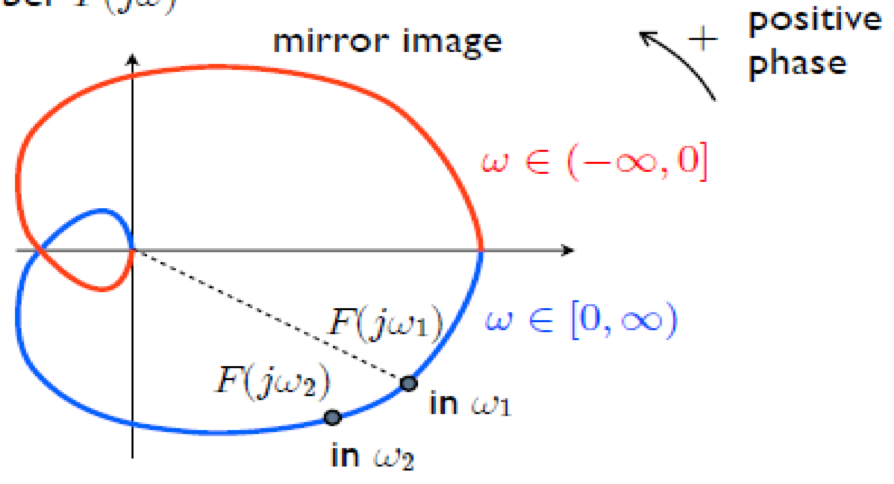
Nyquist diagram: (closed) polar plot of $F(j\omega)$ with $\omega \in (-\infty, \infty)$

we plot the magnitude and phase on the same plot using the frequency as a parameter, that is we use the polar form for the complex number $F(j\omega)$

being $F(s)$ a rational fraction

$$\boxed{F(-j\omega) = F^*(j\omega)}$$

and therefore the plot for negative angular frequencies ω is the **symmetric** wrt the real axis of the one obtained for positive ω



- ✓ Nyquist plots display both amplitude and phase angle on a single plot.
- ✓ Nyquist plot is a plot of the transfer function, G(s) with s = j**ω**. That means you want to plot G(j **ω**). G(j **ω**) is a complex number for any angular frequency, **ω**, so the plot is a plot of complex numbers.
- ✓ The complex number, G(j **ω**), depends upon frequency, so frequency will be a parameter if you plot the imaginary part of G(j **ω**) against the real part of G(j **ω).**
- ✓ To sketch the polar plot of $G(j\omega)$ for the entire range of frequency ω, i.e., from 0 to infinity, there are four key points that usually need to be known:

1) The start of plot where $\omega = 0$,
2) The end of plot where $\omega = \infty$,
3) Where the plot crosses the real axis, i.e., $\mathrm{Im}(G(j\omega)) = 0$, and
4) Where the plot crosses the imaginary axis, i.e., $\mathrm{Re}(G(j\omega)) = 0$.

Ex-1: Polar Plot of Integrator

Consider a first order system, $G(s) = \frac{1}{s}$

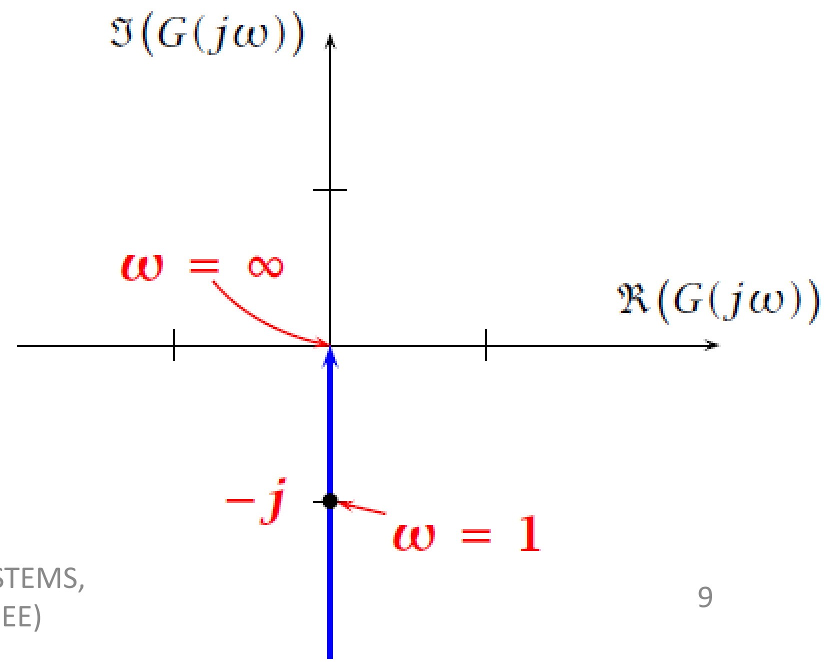
Representing $G(s)$ in the frequency response form $G(j\omega)$ by replacing $s = j\omega$:

$$G(j\omega) = \frac{1}{j\omega} = \frac{-j}{\omega}$$

The **magnitude** of $G(j\omega)$, i.e., $|G(j\omega)|$, is obtained as; $|G(j\omega)| = 1/\omega$

The **phase** of $G(j\omega)$, denoted by, ϕ, is obtained as; $\angle G(j\omega) = -90^\circ$

Polar Plot



Ex-2: Polar Plot of First Order System

Consider a first order system $G(s) = \dfrac{1}{1 + sT}$ where T is the time constant.

Representing $G(s)$ in the frequency response form $G(j\omega)$ by replacing **s = jω**:

$$G(j\omega) = \frac{1}{1 + j\omega T}$$

The **magnitude** of $G(j\omega)$, i.e., $|G(j\omega)|$, is obtained as;

$$|G(j\omega)| = \frac{1}{\sqrt{1 + \omega^2 T^2}}$$

The **phase** of $G(j\omega)$, denoted by, ϕ , is obtained as;

$$\angle G(j\omega) = -\arctan(\omega T)$$

The start of plot where ω = 0

$$|G(j\omega)| = \frac{1}{\sqrt{1+0}} = 1 , \qquad \phi = \tan^{-1}\left(\frac{0}{1}\right) = 0$$

The end of plot where ω = ∞

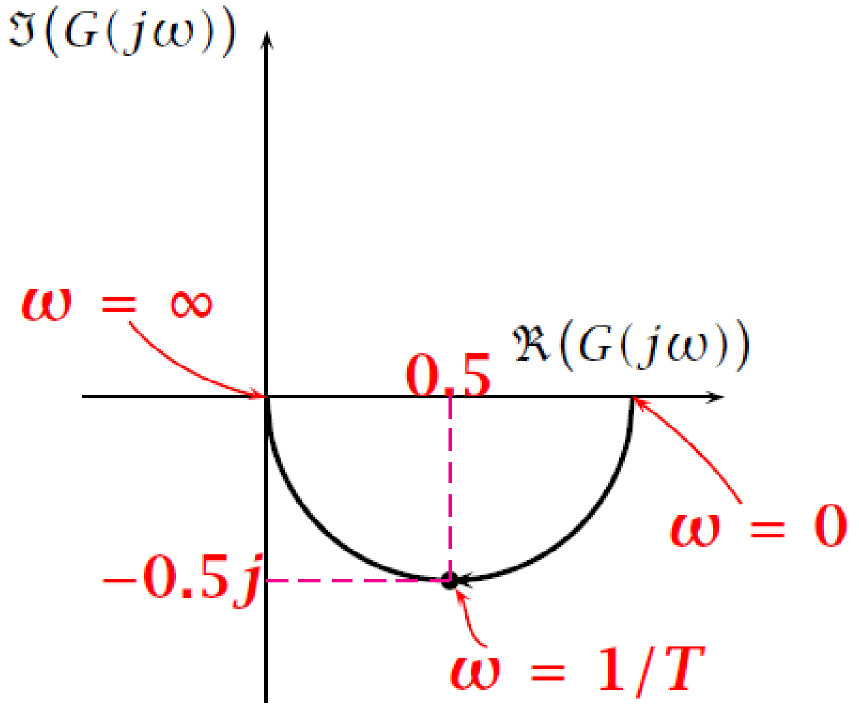
$$|G(j\omega)| = \frac{1}{\sqrt{1+\infty}} = 0 , \qquad \phi = \tan^{-1}\left(\frac{-\infty}{1}\right) = -90^{\circ}$$

The mid part of plot where ω = 1/T

$$|G(j\omega)| = \frac{1}{\sqrt{1+1}} = \frac{1}{\sqrt{2}} , \qquad \phi = \tan^{-1}\left(\frac{-1}{1}\right) = -45^{\circ}$$

	$\lvert G(j\omega)\rvert$	$\angle G(j\omega)$
$\omega = 0$	1	0
$\omega = \frac{1}{T}$	$\frac{1}{\sqrt{2}}$	$-45°$
$\omega = \to \infty$	0	$-90°$

Polar Plot



Ex-3: Polar Plot of Second Order System

Consider a second order system $G(s) = \dfrac{1}{(1+sT_1)(1+sT_2)}$ where T is the time constant.

Representing *G*(*s*) in the frequency response form *G*(*j*ω) by replacing **s = jω**:

$$G(j\omega) = \frac{1}{(1+j\omega T_1)(1+j\omega T_2)}$$

The **magnitude** of *G*(*j*ω), i.e., | *G*(*j*ω) |, is obtained as;

$$|G(j\omega)| = \frac{1}{\sqrt{1+\omega^2 T_1^2}\sqrt{1+\omega^2 T_2^2}}$$

The **phase** of *G*(*j*ω), denoted by, ϕ , is obtained as;

$$\angle G(j\omega) = -\arctan(\omega T_1) - \arctan(\omega T_2)$$

The start of plot where ω = 0

$$|G(j\omega)| = \frac{1}{\sqrt{1+0}\sqrt{1+0}} = 1$$

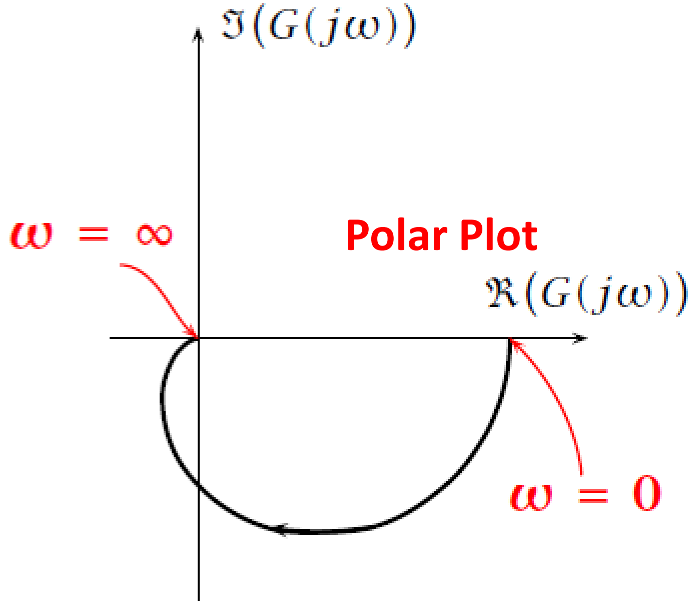
$$\angle G(j\omega) = -\tan^{-1}(0) - \tan^{-1}(0) = 0°$$

The end of plot where ω = ∞

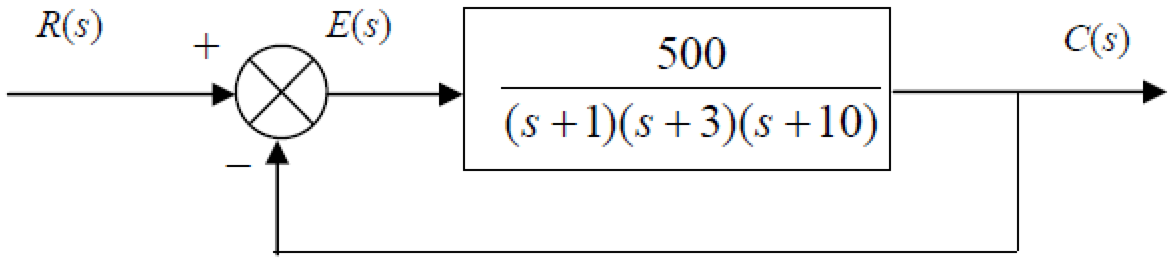
$$|G(j\omega)| = \frac{1}{\sqrt{\infty}\sqrt{\infty}} = 0$$

$$\angle G(j\omega) = -\tan^{-1}(\infty) - \tan^{-1}(\infty) = -90° - 90° = -180°$$

	$\lvert G(j\omega)\rvert$	$\angle G(j\omega)$
$\omega = 0$	1	0
$\omega = \rightarrow \infty$	0	$-180°$



Ex-4: Sketch the Nyquist diagram for the system shown in the following figure



The open-loop transfer function: $G(s)H(s) = \dfrac{500}{(s+1)(s+3)(s+10)}$.

Replacing s with $j\omega$ yields the frequency response of $G(s)H(s)$, i.e.,

$$G(j\omega)H(j\omega) = \frac{500}{(j\omega+1)(j\omega+3)(j\omega+10)} = \frac{500}{(-14\omega^2+30)+j(43\omega-\omega^3)}$$

$$= 500\frac{(-14\omega^2+30)-j(43\omega-\omega^3)}{(-14\omega^2+30)^2+(43\omega-\omega^3)^2} \longrightarrow \text{(a)}$$

Magnitude response:

$$\left|G(j\omega)H(j\omega)\right| = \sqrt{\mathrm{Re}^2 + \mathrm{Im}^2} = \frac{500}{\sqrt{(-14\omega^2+30)^2 + (43\omega-\omega^3)^2}}$$

Phase response:

$$\angle G(j\omega)H(j\omega) = \tan^{-1}\left(\frac{\mathrm{Im}}{\mathrm{Re}}\right) = \tan^{-1}\left(\frac{-\left(43\omega-\omega^3\right)}{-14\omega^2+30}\right)$$

Now that we have expressions for the magnitude and phase of the frequency response, we can sketch the polar plot using the **4 key points**.

Point 1: The start of plot where ω = 0

$$|G(0)H(0)| = \frac{500}{\sqrt{(30)^2}} = 16.67 \qquad \angle G(0)H(0) = \tan^{-1}\frac{0}{30} = 0^\circ$$

Point 2: The end of plot where ω = ∞

$$|G(\infty)H(\infty)| = \frac{500}{\sqrt{\infty}} = 0 \qquad \angle G(\infty)H(\infty) = \tan^{-1}\frac{\infty^3}{30} = -3 \times 90° = -270°$$

Point 3: Where the plot crosses the real axis, i.e., **Im(*G*(*j*ω)) = 0**

Take the imaginary part of equation (a), and put equal to zero, to get the value of frequency ω at the interception of real axis.

$$\frac{-(43\omega - \omega^3)}{(-14\omega^2 + 30)^2 + (43\omega - \omega^3)^2} = 0 \Rightarrow \begin{cases} 43\omega - \omega^3 = 0 \quad \Rightarrow \quad \omega = 0 \quad \text{and} \quad \omega = 6.56\,\text{rad/s} \\ \omega = \infty \end{cases}$$

Point 4: Where the plot crosses the imaginary axis, Re(*G*(*j*ω)) = 0

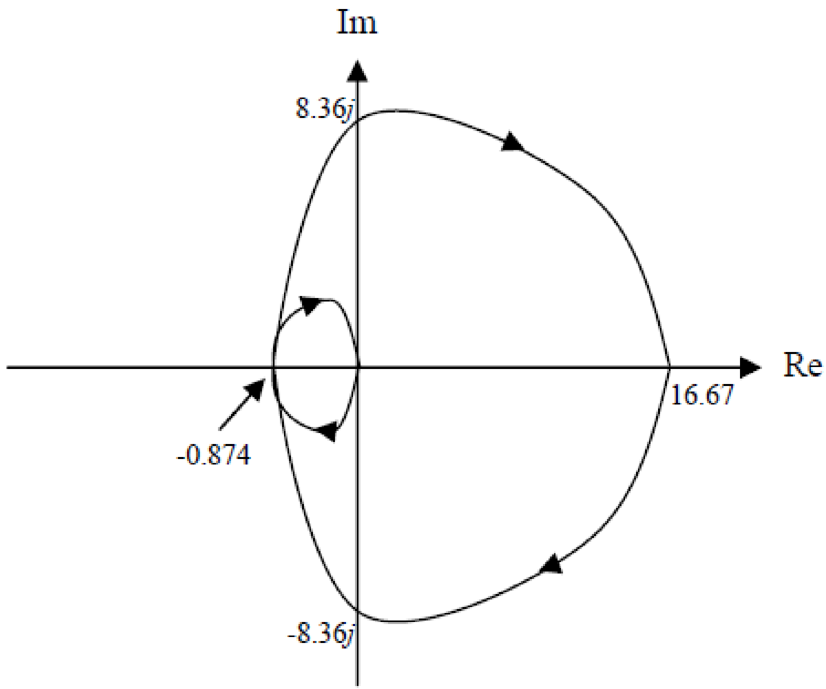
Take the real part of equation (a), and put equal to zero, to get the value of frequency ω at the interception of imaginary axis.

$$\frac{-14\omega^2+30}{(-14\omega^2+30)^2+(43\omega-\omega^3)^2}=0 \Rightarrow \begin{cases} -14\omega^2+30=0 & \Rightarrow \quad \omega=1.46 \text{ rad/s} \\ \omega=\infty \end{cases}$$

Key Points of the polar plot:

	$\lvert GH \rvert$	$\angle GH$
$\omega=0$	16.67	0
$\omega=\infty$	0	-270°
Cross Re: $\omega=0$ $\omega=\infty$ $\omega=6.56$ rad/s	See above 0.874	See above -180°
Cross Re: $\omega=0$ $\omega=\infty$ $\omega=1.46$ rad/s	See above 8.36	See above -90°

Nyquist diagram



Ex-5: Sketch the polar plot for the following transfer function.

$$G(s) = \frac{10}{s(s+1)(s+5)}$$

Representing *G*(*s*) in the frequency response form *G*(*j*ω) by replacing **s = jω**, and then Multiply both numerator and denominator by the conjugate of denominator.

$$G(j\omega) = \frac{10}{j\omega(j\omega+1)(j\omega+5)}$$

$$= \frac{10}{(-\omega^2 + j\omega)(j\omega+5)}$$

$$= \frac{10}{(-j\omega^3 - 5\omega^2 - \omega^2 + 5j\omega)}$$

$$= \frac{10}{-6\omega^2 + j(5\omega-\omega^3)} * \frac{-6\omega^2 - j(5\omega-\omega^3)}{-6\omega^2 - j(5\omega-\omega^3)}$$

$$G(j\omega) = \frac{-60\omega^2 - j10\omega(5-\omega^2)}{36\omega^4 - (5\omega-\omega^3)^2} \longrightarrow \text{(a)}$$

Point 1: The start of plot where ω = 0

At frequency ω = 0, we only observe the most significant terms that take the effect.

Magnitude at ω = 0:

$$G(j\omega)\Big|_{\omega=0} = \frac{10}{5j\omega}\Bigg|_{\omega=0} = \frac{2}{j\omega}\Bigg|_{\omega=0} = \infty$$

Phase at ω = 0:

$$\angle G(j\omega)\Big|_{\omega=0} = \lim_{\omega\to 0} \angle \frac{2}{j\omega} = -90^\circ$$

Point 2: The end of plot where ω = ∞

At frequency ω = ∞, we shall look at the most significant term that takes effect when the frequency approaches infinity.

Magnitude at ω = ∞:

$$\left|G(j\omega)\right|_{\omega\to\infty} = \lim_{\omega\to\infty}\left|\frac{10}{(j\omega^3)}\right| = \lim_{\omega\to\infty}\frac{10}{\omega^3} = 0$$

Phase at ω = ∞:

$$\angle G(j\omega)\,|_{\omega\to\infty} = \angle \lim_{\omega\to\infty}\left[\frac{10}{(j\omega)^3}\right] = -270^\circ$$

Point 3: Where the plot crosses the real axis, i.e., **Im(*G*(*j*ω)) = 0**

Take the imaginary part of equation (a), and put equal to zero, to get the value of frequency ω at the interception of real axis.

$$\Rightarrow -\frac{10\omega(5-\omega^2)}{36\omega^4+(5\omega-\omega^3)^2}=0$$

$$\Rightarrow 10(5-\omega^2)=0$$

$$\Rightarrow \omega^2=5$$

$$\Rightarrow \omega=\sqrt{5}$$

Therefore, the intersection point between the polar plot and the real axis, when $\boldsymbol{\omega} = \sqrt{\mathbf{5}}$, is located at;

$$G(j\omega)|_{\omega=\sqrt{5}}=-\frac{1}{3}$$

Point 4: Where the plot crosses the imaginary axis, Re($G(j\omega)$) = 0

Take the real part of equation (a), and put equal to zero, to get the value of frequency ω at the interception of imaginary axis.

$$\Rightarrow \quad -\frac{60\omega^2}{36\omega^4 + (5\omega - \omega^3)^2} = 0$$

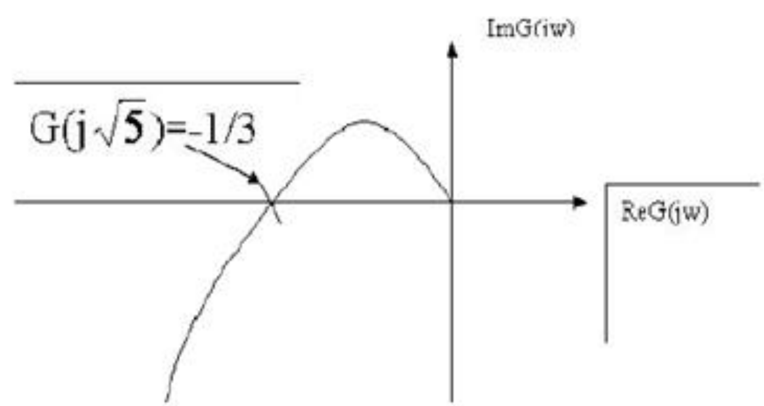
$$\Rightarrow \quad \omega = \infty$$

Therefore, the intersection point between the polar plot and the imaginary axis is when $\omega = \infty$ is located at;

$$G(j\infty) = 0$$

	|G(jω)|	∠G(jω)
ω = 0	∞	-90°
ω = ∞	0	-270°
Cross Re: $\omega = \sqrt{5}$	$-\frac{1}{3}$	
Cross Img: $\omega = \infty$	0	

Polar Plot



Ex-6: Sketch the polar plot for the following transfer function.

$$GH(s) = \frac{1}{s^4(s+p)} \qquad p > 0$$

Representing $G(s)H(s)$ in the frequency response form $G(j\omega)H(j\omega)$ by replacing **s = jω**:

$$GH(j\omega) = \frac{1}{j^4\omega^4(j\omega + p)}$$

The **magnitude** of $GH(j\omega)$ i.e., $|GH(j\omega)|$, is obtained as;

$$|GH(j\omega)| = \frac{1}{\omega^4\sqrt{\omega^2 + p^2}}$$

The **phase** of $GH(j\omega)$ denoted by, φ, is obtained as;

$$\varphi = \angle GH(0) = -\tan^{-1}\left(\frac{\omega}{p}\right)$$

Point 1: **The start of plot where ω = 0**

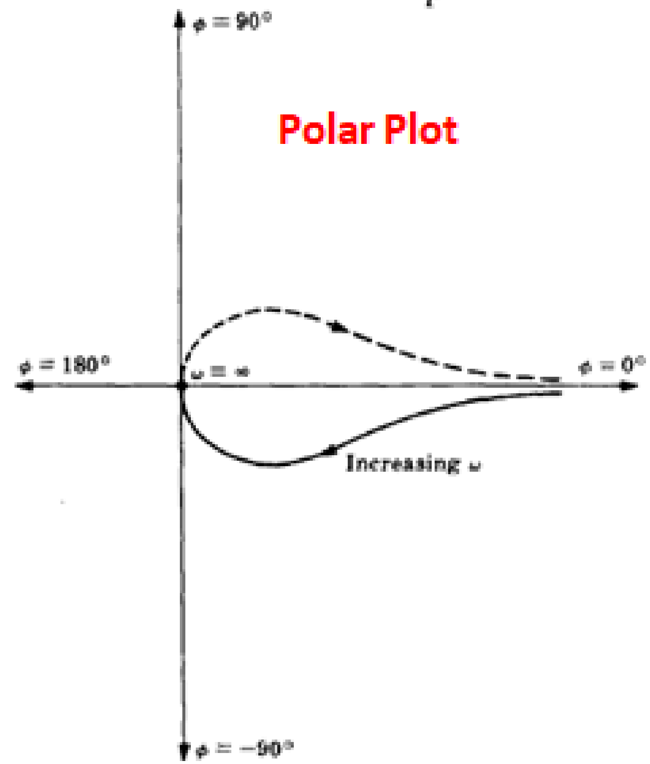
$$|GH(0)| = \frac{1}{0\sqrt{0 + p^2}} = \infty$$

$$\varphi = \angle GH(0) = -\tan^{-1}\frac{0}{p} = 0^{\circ}$$

Point 2: **The end of plot where ω = ∞**

$$|GH(\infty)| = \frac{1}{\infty\sqrt{\infty + p^2}} = 0$$

$$\varphi = \angle GH(\infty) = -\tan^{-1}\frac{\infty}{p} = -90^{\circ}$$



	|GH(jω)|	∠GH(jω)
ω = 0	∞	0°
ω = ∞	0	-90°

Nyquist criterion for stability:

P= Number of open loop poles in RHS

Z= Number of closed loop poles in RHS

N= Number of encirclements of (-1,j0) point by the Nyquist contour in the counter clockwise direction

= P-Z

- ✓ **If the closed loop system is stable, then Z=0**

  Thus, N=P

  The function G(S)H(S) encircles (-1,j0) in counter clockwise direction, as many times as number of poles of G(S)H(S) in the right half of s-plane.

- ✓ **If open loop system is stable, there are no poles of G(S)H(S) in the RHS and hence P=0**

  Thus N=0

Ex-7: Sketch the polar plot for the following transfer function and check the stability of the system.

$$GH(s) = \frac{1}{s(s+1)}$$

There is one pole at the origin.

Representing $G(s)H(s)$ in the frequency response form $G(j\omega)H(j\omega)$ by replacing **s = jω**:

$$GH(j\omega) = \frac{1}{j\omega(j\omega+1)}$$

The **magnitude** of $GH(j\omega)$ i.e., $|GH(j\omega)|$, is obtained as;

$$|GH(j\omega)| = \frac{1}{\omega\sqrt{\omega^2+1}}$$

The **phase** of $GH(j\omega)$ denoted by, φ, is obtained as;

$$\varphi = \angle GH(0) = -90^\circ - \tan^{-1}\left(\frac{\omega}{1}\right)$$

Point 1: **The start of plot where ω = 0**

$$|GH(0)| = \frac{1}{0\sqrt{0+1}} = \infty$$

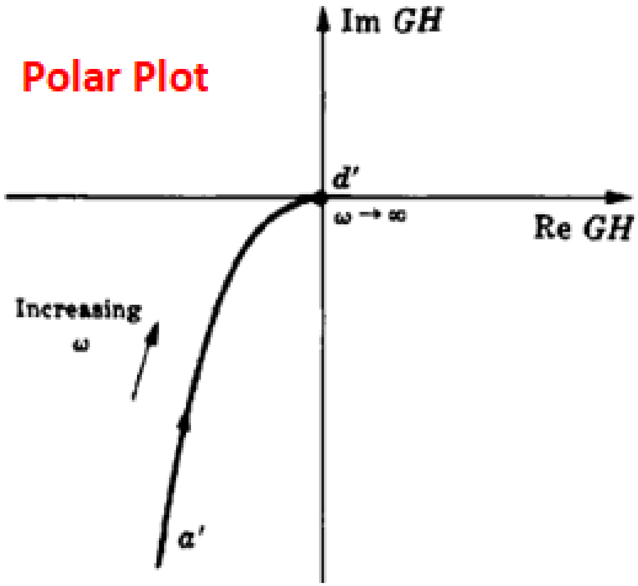
$$\varphi = \angle GH(0) = -90^\circ - \tan^{-1}\frac{0}{1} = -90^\circ$$

Point 2: **The end of plot where ω = ∞**

$$|GH(\infty)| = \frac{1}{\infty\sqrt{\infty+1}} = 0$$

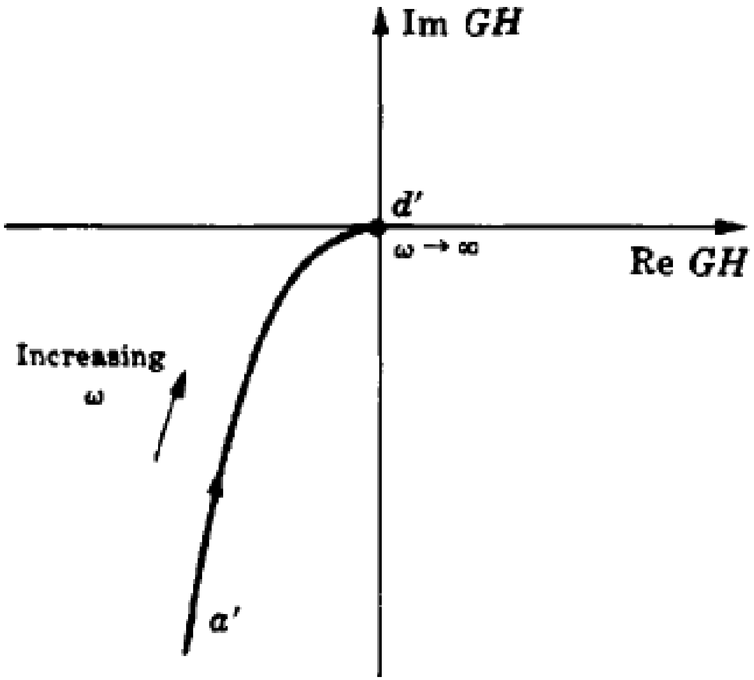
$$\varphi = \angle GH(\infty) = -90 - \tan^{-1}\frac{\infty}{p} = -90^\circ - 90^\circ = -180^\circ$$

Polar Plot

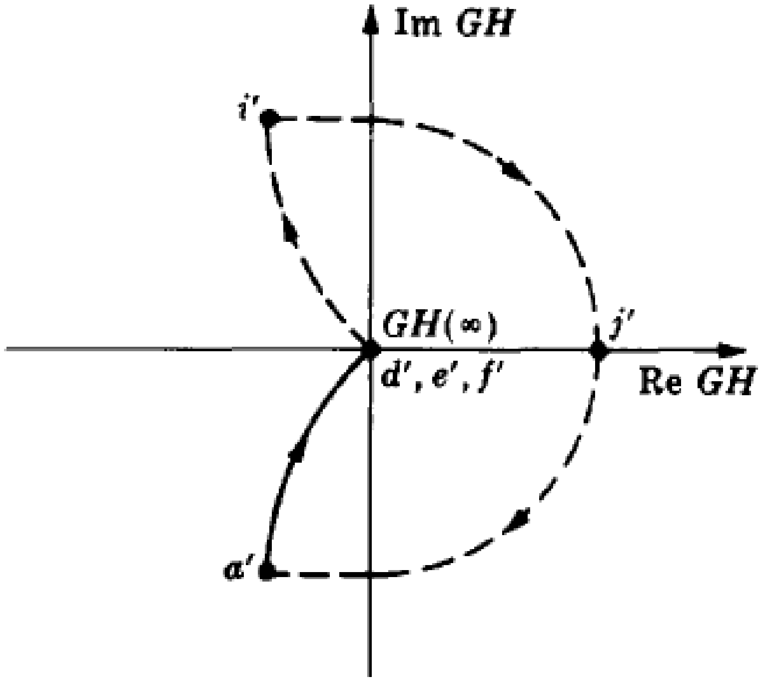


	|GH(jω)|	∠GH(jω)
ω = 0	∞	-90°
ω = ∞	0	-180°

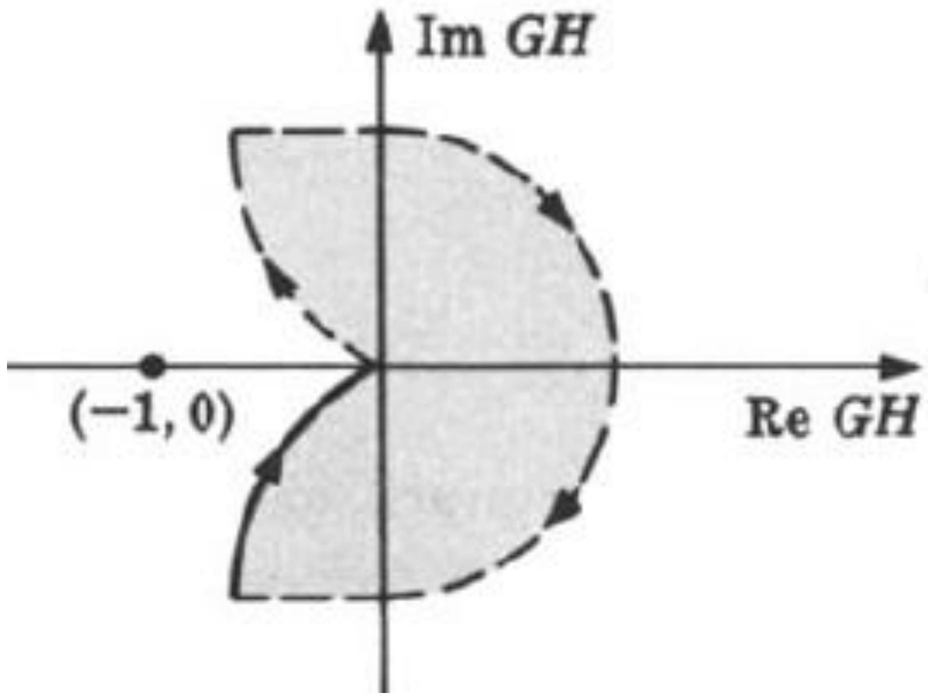
Nyquist or Polar Plot



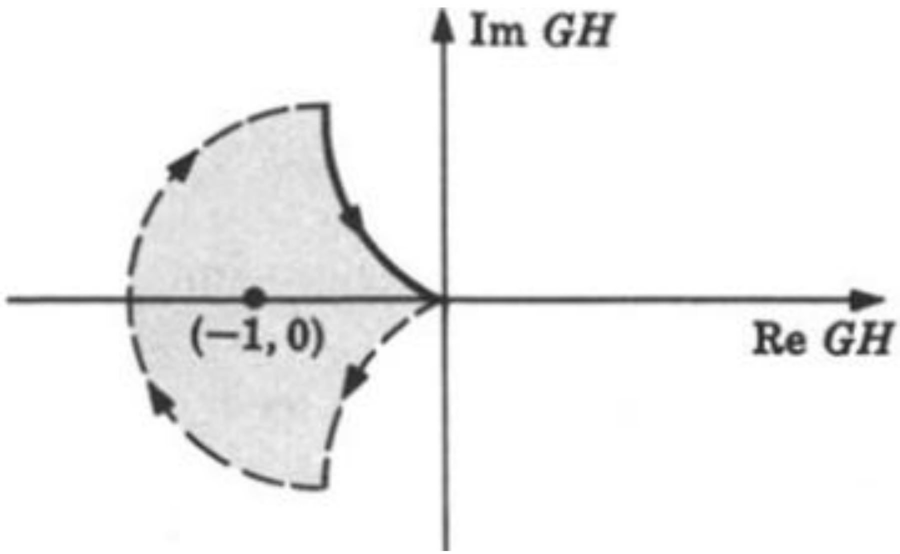
Nyquist Diagram



- The region to the right of the contour has been shaded.
- Clearly, the **(-1,0)** point is not in the shaded region; therefore it is not enclosed by the contour and so $N = 0$
- The poles of $GH(s)$ are at **s = 0** and **s = -1**, neither of which are in the right-hand-plane hence $P = 0$. Thus **Z=P-*N* = 0**, which means there are no closed loop poles in RHS and **the system is absolutely stable.**



➢**Ex-8:** The Nyquist Stability Plot for ***GH(s)* = 1/s(s-1)** is given in the figure below.

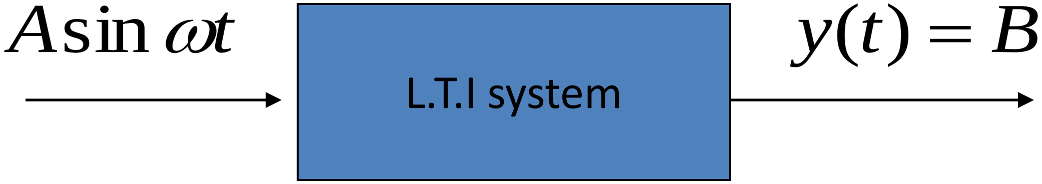


- The region to the right of the contour has been shaded and the **(-1,0)** point is enclosed in clockwise direction then ***N* = -1 (N +ve for counter clockwise direction)**.
- The poles of ***GH*** are at **s = 0** and **s = +1**, the latter pole being in the RHS. Hence ***P*** **= 1**.
- ***N ≠ P*** indicates that the **system is unstable.**
- N=P-Z
- Z=P-N= ***1 + 1 = 2,***
- Therefore the poles (2) of the closed-loop transfer function lie in the right-hand S-plane .

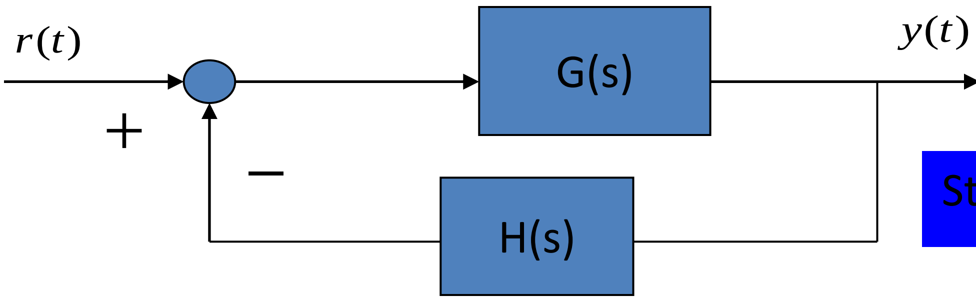
Bode Plot

- Bode Plot is a (semi log) plot of the transfer function magnitude and phase angle as a function of frequency.
- Representation of the sinusoidal transfer function G(jw) in db i.e 20log |G(jw)| is plotted against logw. Similarly angle of G(jw) is plotted against logw.

$$r(t) = A\sin\omega t \qquad y(t) = B\sin(\omega t + \phi)$$



Magnitude: $\dfrac{B}{A}$ Phase: ϕ



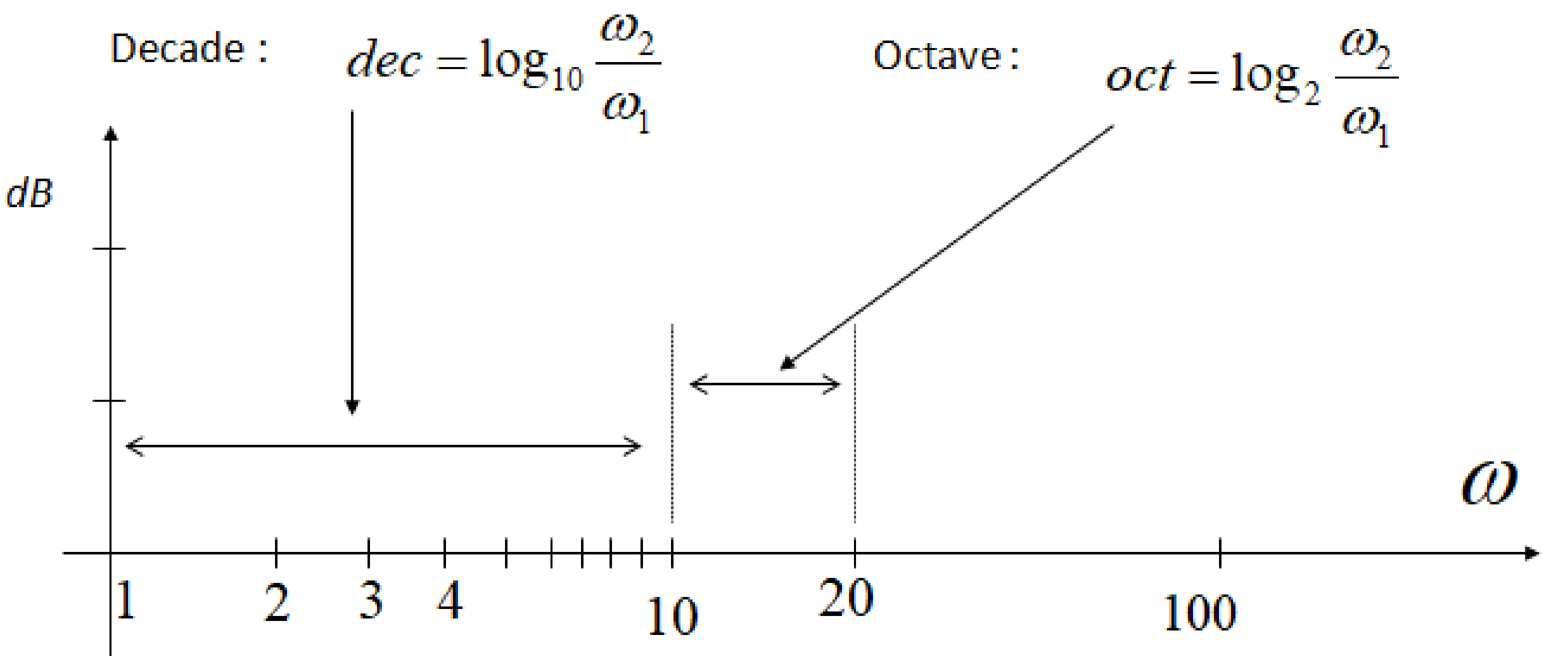
Steady state response

$$\frac{Y(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)} \qquad s = \sigma + j\omega \Rightarrow s = j\omega$$

Magnitude: $\dfrac{|G(j\omega)|}{|1+G(j\omega)H(j\omega)|}$

Phase: $\dfrac{\angle G(j\omega)}{\angle[1+G(j\omega)H(j\omega)]}$

Logarithmic coordinates



- The gain magnitude is many times expressed in terms of decibels (dB)

  dB = 20 $\log_{10}$ A

  where A is the amplitude or gain
  - a *decade* is defined as any 10-to-1 frequency range
  - an *octave* is any 2-to-1 frequency range

  20 dB/decade = 6 dB/octave

Bode plot construction

$$\frac{Y(s)}{R(s)} = \frac{k(s+z_1)(s+z_2)\cdots}{(s+p_1)(s+p_2)(s^2+as+b)\cdots}$$

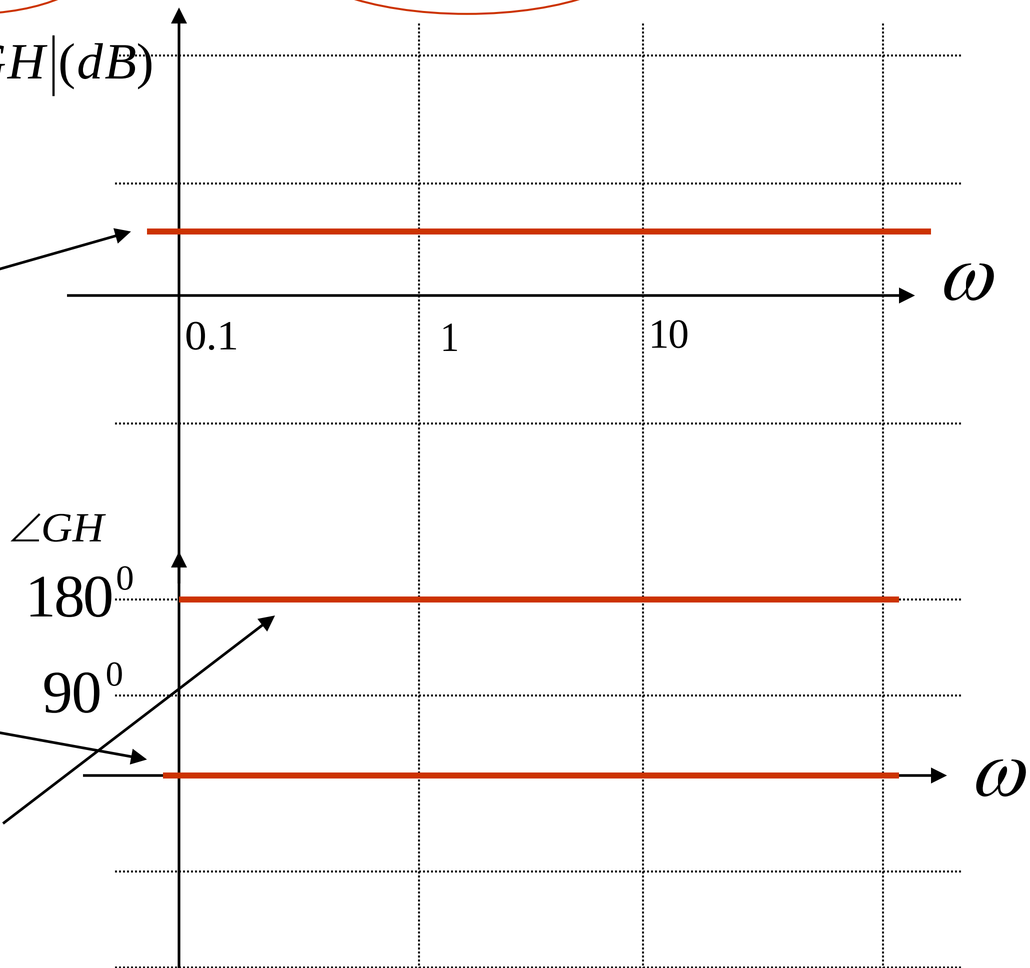
Case *I : k*

Magnitude:

$$|k|_{dB} = 20\log|k|(dB)$$

Phase:

$$\angle k = \begin{cases} 0^o & ,k > 0 \\ 180^o & ,k < 0 \end{cases}$$



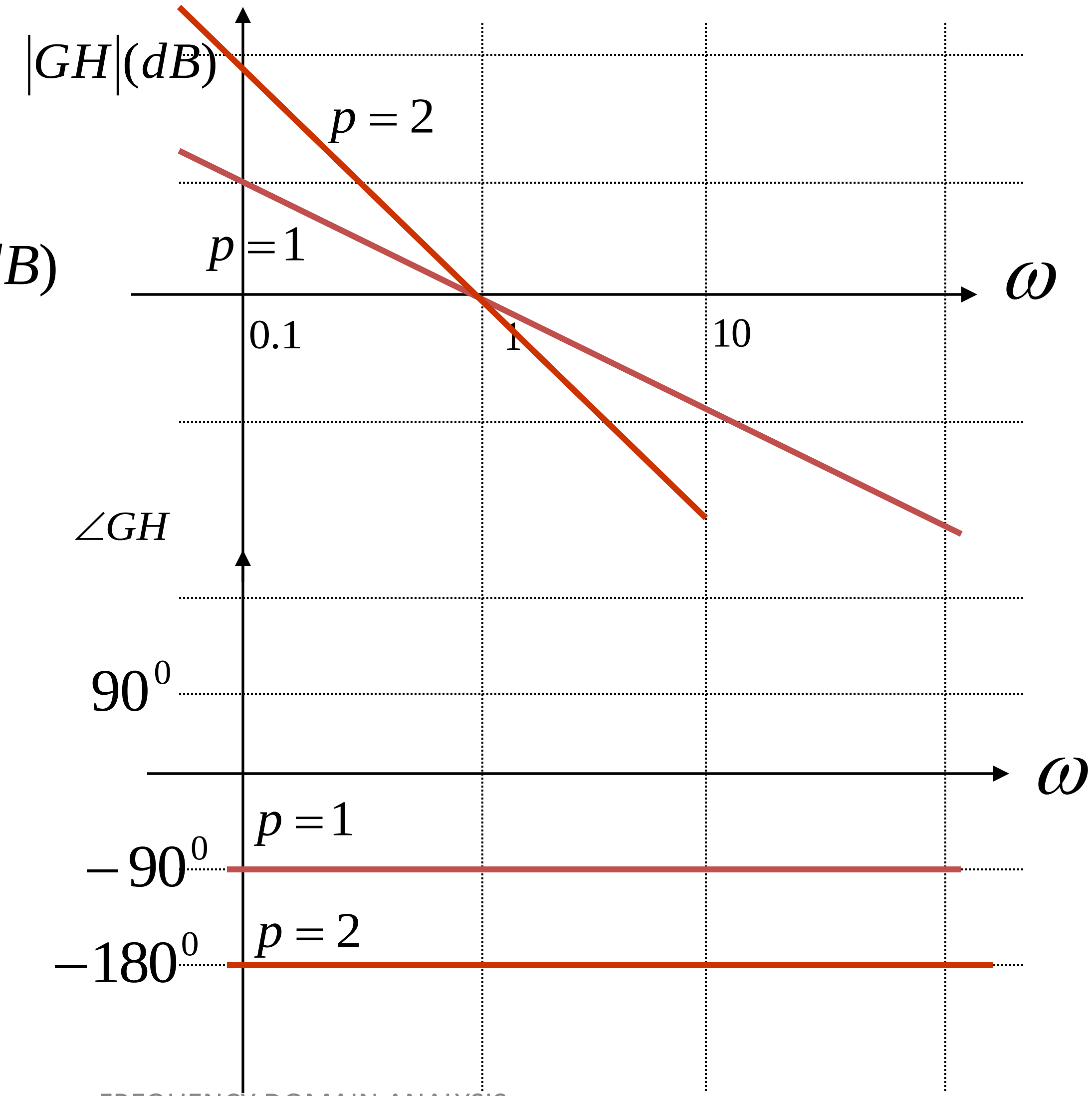
Case *II* : $\dfrac{1}{s^p}$

Magnitude:

$$\left|\frac{1}{(j\omega)^p}\right|_{dB} = -20\,p\log\omega(dB)$$

Phase:

$$\angle\frac{1}{(j\omega)^p} = (-90^o)\times p$$



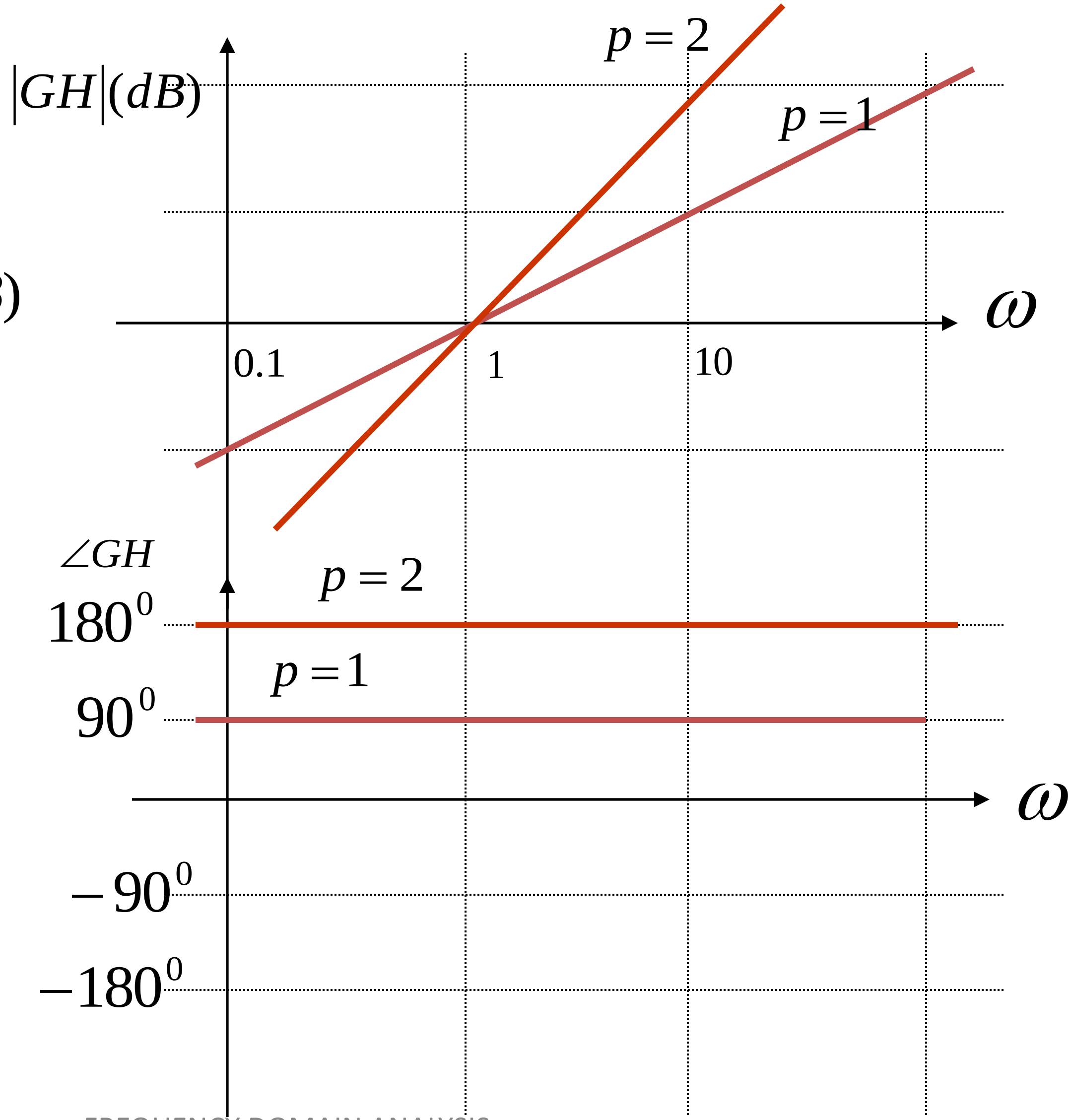
Case *III* : s^p

Magnitude:

$$\left|(j\omega)^p\right|_{dB} = 20\,p\log\omega(dB)$$

Phase:

$$\angle(j\omega)^p = (90^o)\times p$$



Case IV : $\dfrac{a}{(s+a)}$ or $(\dfrac{1}{a}s+1)^{-1}$

$$a = 1$$

Magnitude:

$$\left|(1+j\frac{\omega}{a})^{-1}\right|_{dB} = -20\log\sqrt{1+(\frac{\omega}{a})^2}$$

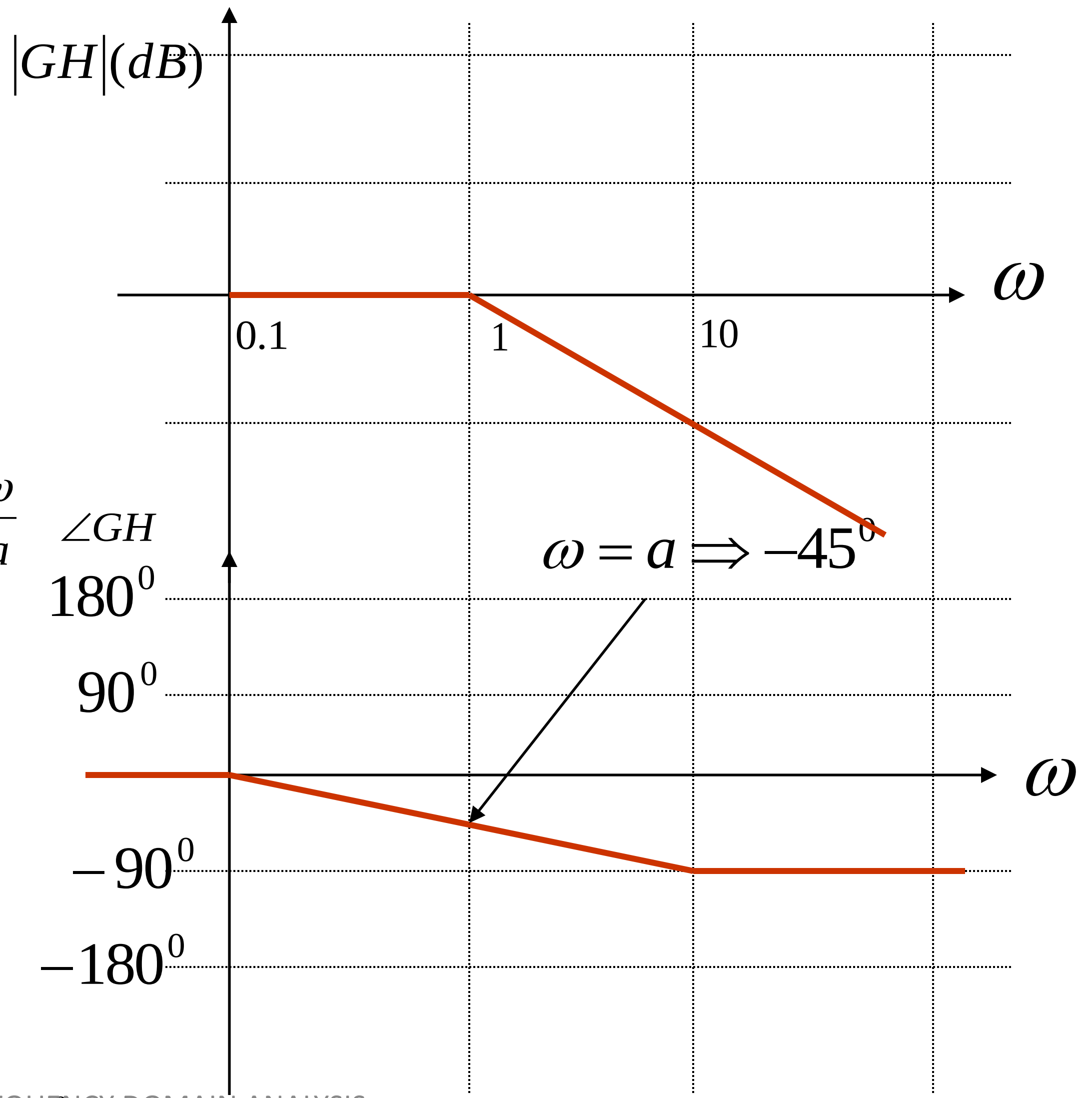
$$= -10\log[1+(\frac{\omega}{a})^2]$$

$$\omega \pi\pi\ a \Rightarrow \frac{\omega}{a} \approx 0 \Rightarrow dB = -10\log 1 = 0$$

$$\omega \phi\phi\ a \Rightarrow 1+j\frac{\omega}{a} \approx \frac{\omega}{a} \Rightarrow dB \approx -20\log\frac{\omega}{a}$$

$$dB = -[20\log\omega - 20\log a]$$

$$\omega = a \Rightarrow 1+j1 \Rightarrow dB = -10\log 2 = -3.01$$



Phase:

$$\angle(1+j\frac{\omega}{a}) = 0^0 - \tan^{-1}\frac{\omega}{a}$$

$$\omega \pi\pi\ a \Rightarrow \frac{\omega}{a} \approx 0 \Rightarrow \angle GH \approx \tan^{-1}0 = 0^o$$

$$\omega \phi\phi\ a \Rightarrow \frac{\omega}{a} \approx \infty \Rightarrow \angle GH \approx -\tan^{-1}\infty = -90^0$$

Case V : $\dfrac{(s+a)}{a}$ *or* $(\dfrac{1}{a}s+1)$

$$a=1$$

Magnitude:

$$\left|(1+j\frac{\omega}{a})\right|_{dB} = 20\log\sqrt{1+(\frac{\omega}{a})^2}$$

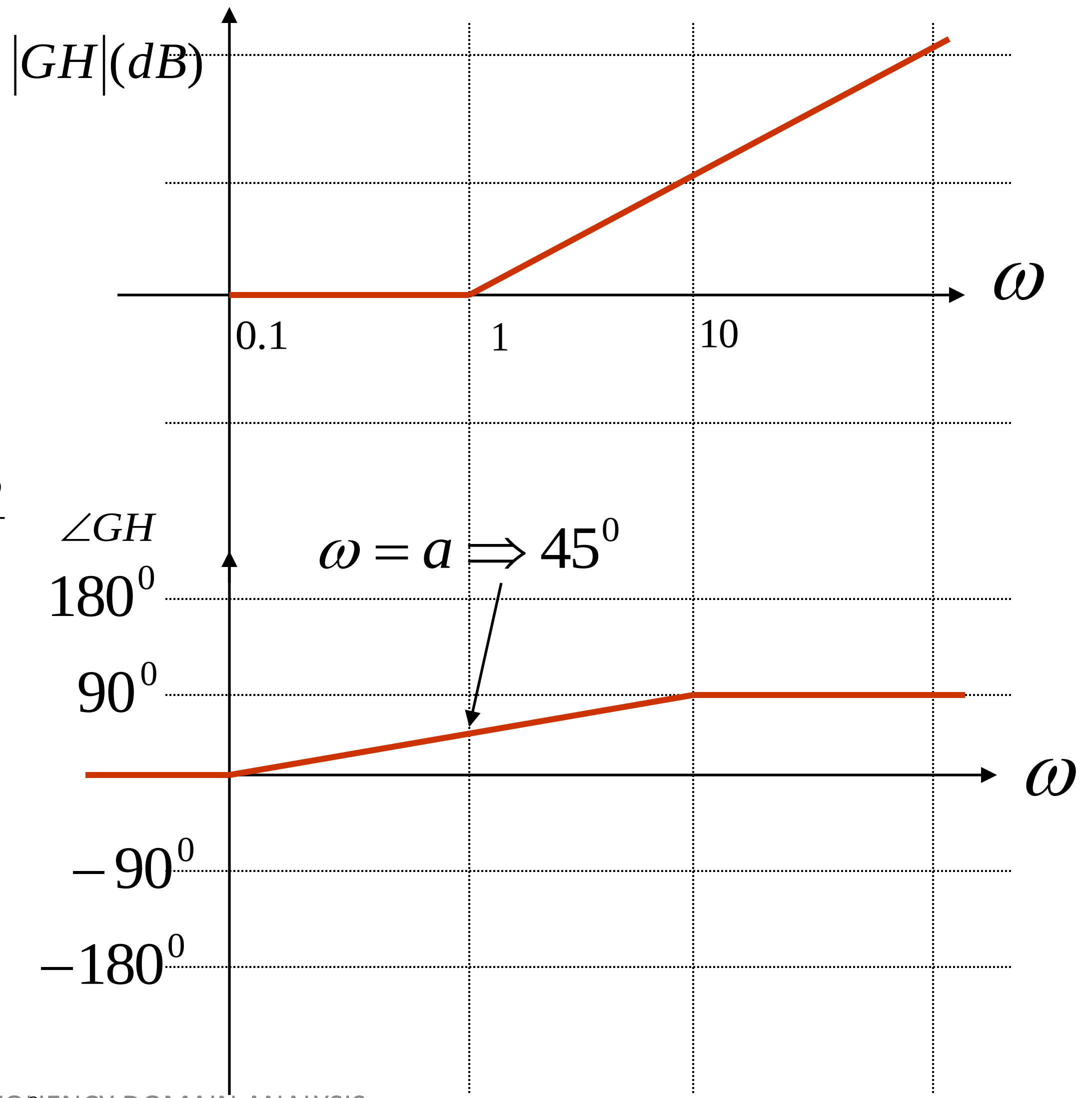
$$=10\log[1+(\frac{\omega}{a})^2]$$

$$\omega \pi\pi a \Rightarrow \frac{\omega}{a}\approx 0 \Rightarrow dB = 10\log 1 = 0$$

$$\omega \phi\phi a \Rightarrow 1+j\frac{\omega}{a}\approx\frac{\omega}{a} \Rightarrow dB \approx 20\log\frac{\omega}{a}$$

$$dB = 20\log\omega - 20\log a$$

$$\omega = a \Rightarrow 1+j1 \Rightarrow dB = 10\log 2 = 3.01$$

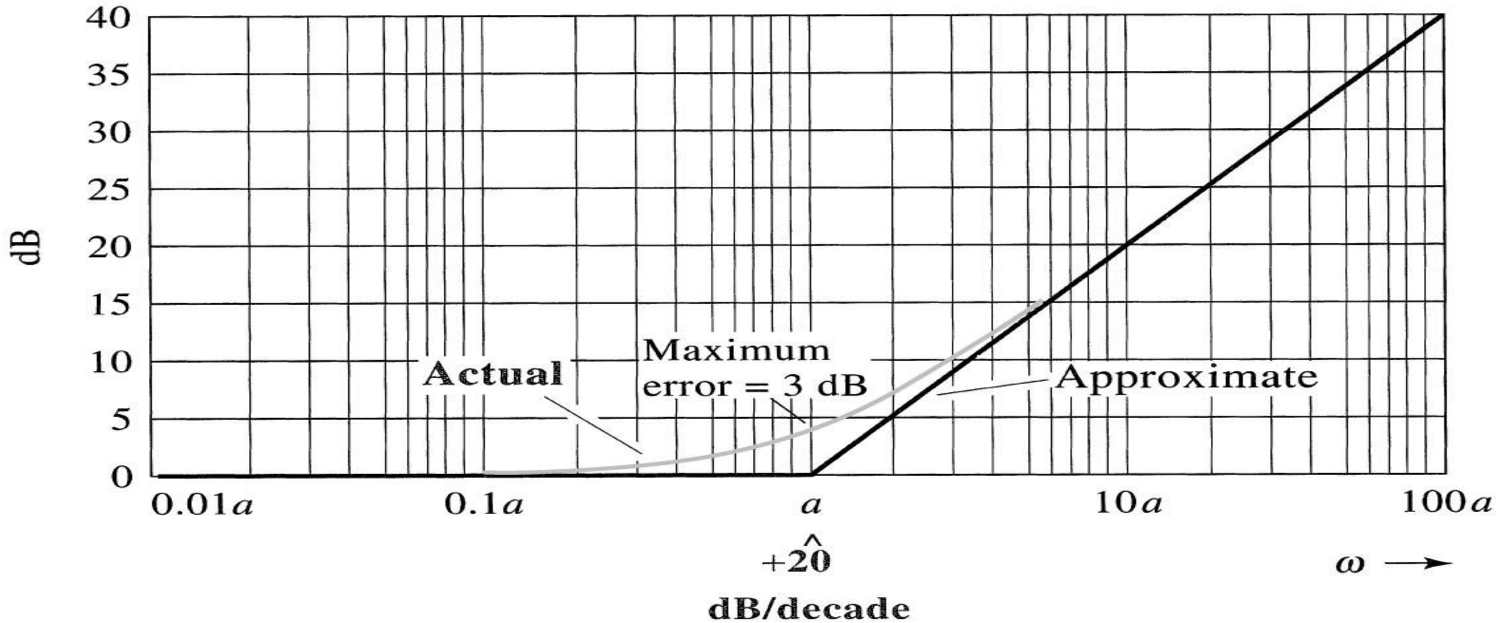


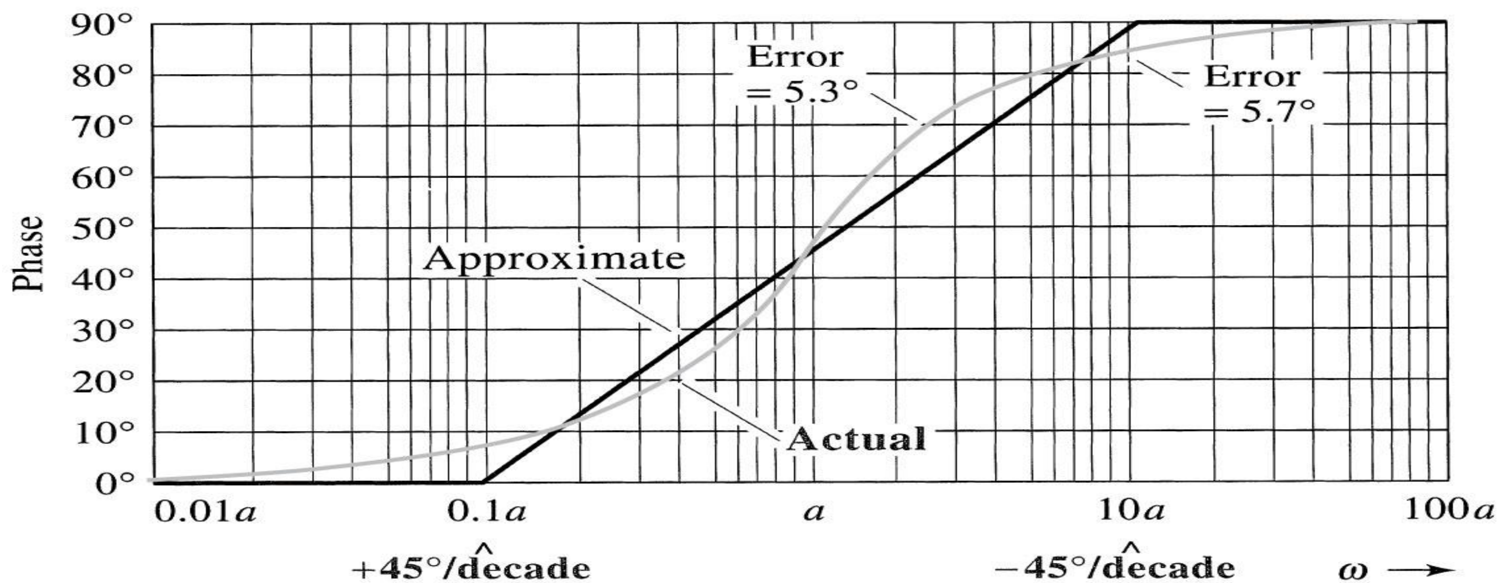
Phase:

$$\angle(1+j\frac{\omega}{a}) = \tan^{-1}\frac{\omega}{a}$$

$$\omega \pi\pi a \Rightarrow \frac{\omega}{a}\approx 0 \Rightarrow \angle GH \approx \tan^{-1}0 = 0^o$$

$$\omega \phi\phi a \Rightarrow \frac{\omega}{a}\approx \infty \Rightarrow \angle GH \approx \tan^{-1}\infty = 90^o$$


40
35
30
25
20
15
10
5
0
dB
Maximum
error = 3 dB
Actual
Approximate
0.01a
0.1a
a
10a
100a
+20
dB/decade
ω →

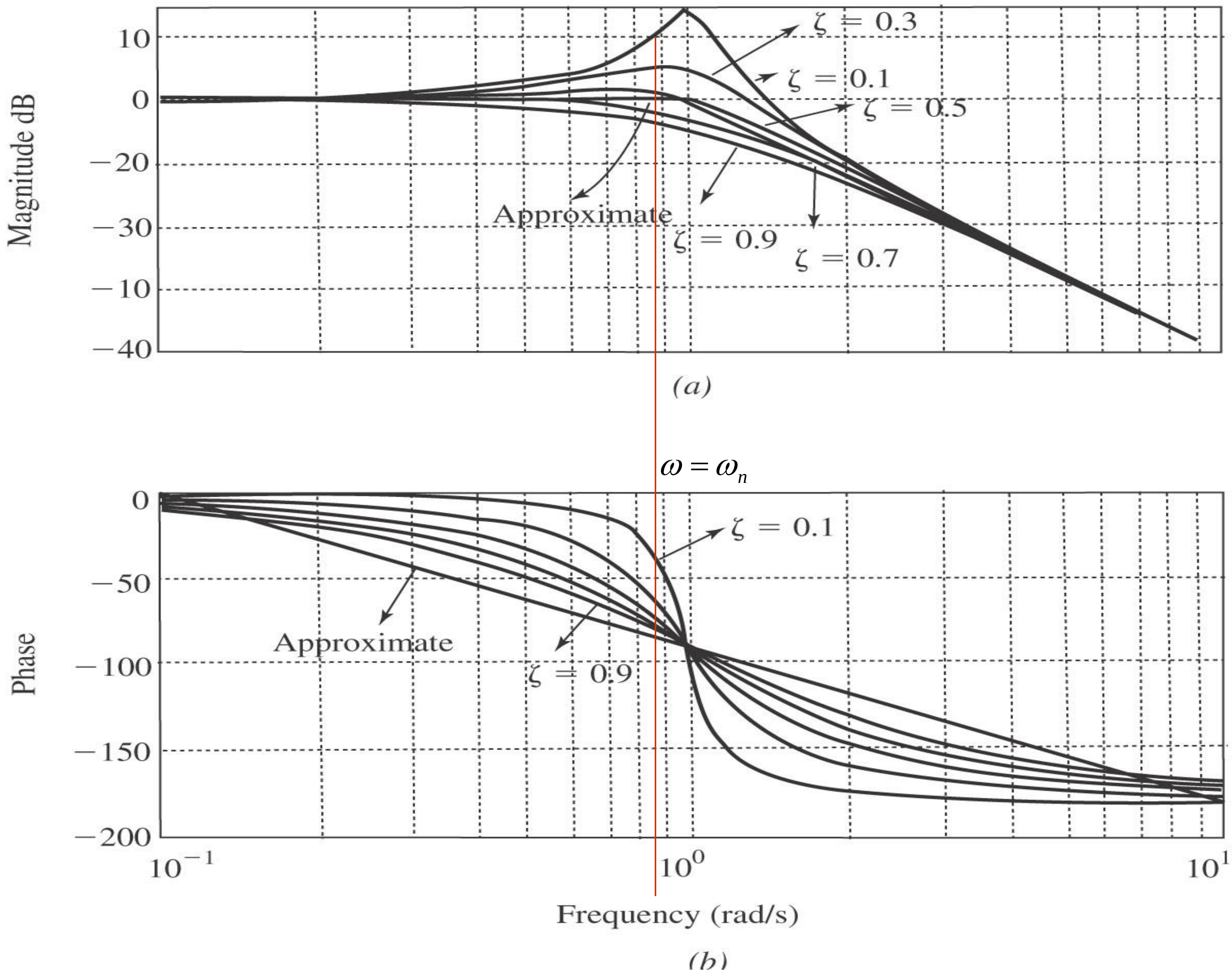

90°
80°
70°
60°
50°
40°
30°
20°
10°
0°
Phase
Error
= 5.3°
Error
= 5.7°
Approximate
Actual
0.01a
0.1a
a
10a
100a
+45°/decade
−45°/decade
ω →

Case *VI* : $T(s) = \dfrac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$

$$T(j\omega) = \frac{\omega_n^2}{(\omega_n{}^2 - \omega^2) + 2j\xi\omega_n\omega} \qquad \angle T(j\omega) = -\tan^{-1}\frac{2\xi\omega\,\omega_n}{(\omega_n{}^2 - \omega^2)}$$

$$T(j\omega) = \frac{1}{(1 - (\frac{\omega}{\omega_n})^2) + j2\xi\frac{\omega}{\omega_n}} \qquad \angle T(j\omega) = -\tan^{-1}\frac{2\xi\frac{\omega}{\omega_n}}{1 - (\frac{\omega}{\omega_n})^2}$$

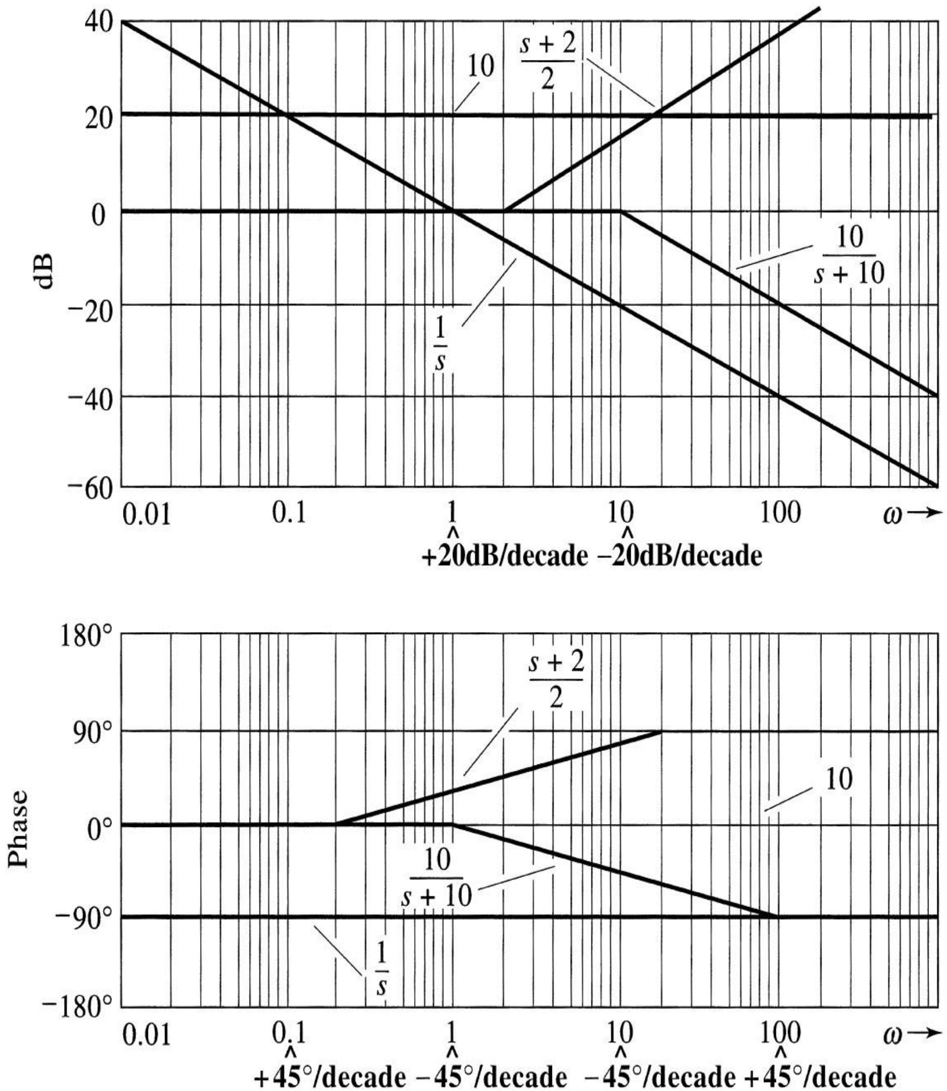
$$\left|T(j\omega)\right| = \begin{cases} 0 & , \frac{\omega}{\omega_n} \pi\pi\, 1 \\ -20\log(2\xi) & , \frac{\omega}{\omega_n} = 1 \\ -40\log(\frac{\omega}{\omega_n}) & , \frac{\omega}{\omega_n} \phi\phi\, 1 \end{cases} \qquad \angle T(j\omega) = \begin{cases} 0^0 & , \frac{\omega}{\omega_n} \pi\pi\, 1 \\ -90^0 & , \frac{\omega}{\omega_n} = 1 \\ -180^o & , \frac{\omega}{\omega_n} \phi\phi\, 1 \end{cases}$$



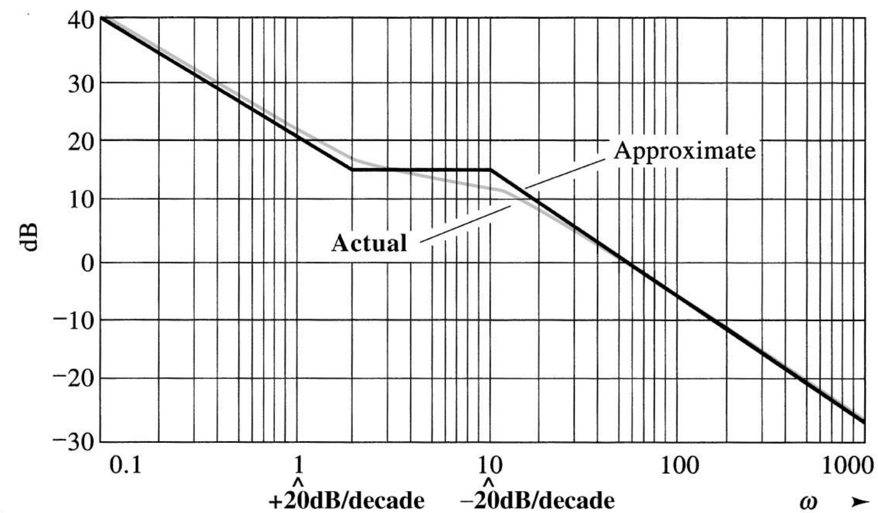
Magnitude dB
10
0
−20
−30
−10
−40
ζ = 0.3
ζ = 0.1
ζ = 0.5
Approximate
ζ = 0.9
ζ = 0.7
(a)
$\omega = \omega_n$
Phase
0
−50
−100
−150
−200
ζ = 0.1
Approximate
ζ = 0.9
10^{-1}
10^{0}
10^{1}
Frequency (rad/s)
(b)

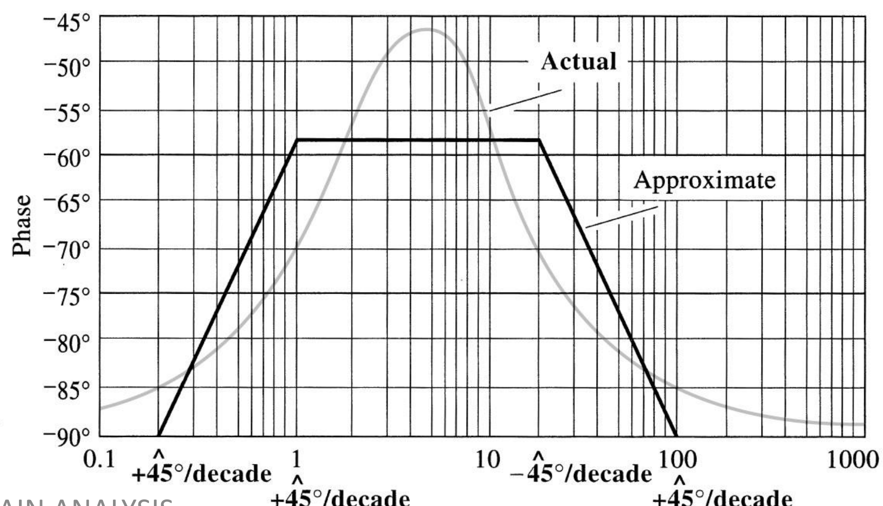
Example :

$$T(s) = \frac{50(s+2)}{s(s+10)} \Rightarrow T(s) = 10\left(\frac{1}{s}\right)\left(\frac{s+2}{2}\right)\left(\frac{10}{s+10}\right)$$



(a)





Ex-1: Draw the bode plot for the transfer function $H(s) = \dfrac{100}{s+30}$

Step 1: Rewrite the transfer function in proper form.

Make both the lowest order term in the numerator and denominator unity. The numerator is an order 0 polynomial, the denominator is order 1.

$$H(s) = \frac{100}{30} \frac{1}{\frac{s}{30}+1} = 3.3 \frac{1}{\frac{s}{30}+1}$$

Step 2: Separate the transfer function into its constituent parts.

The transfer function has 2 components:

- A constant of 3.3
- A pole at s=-30

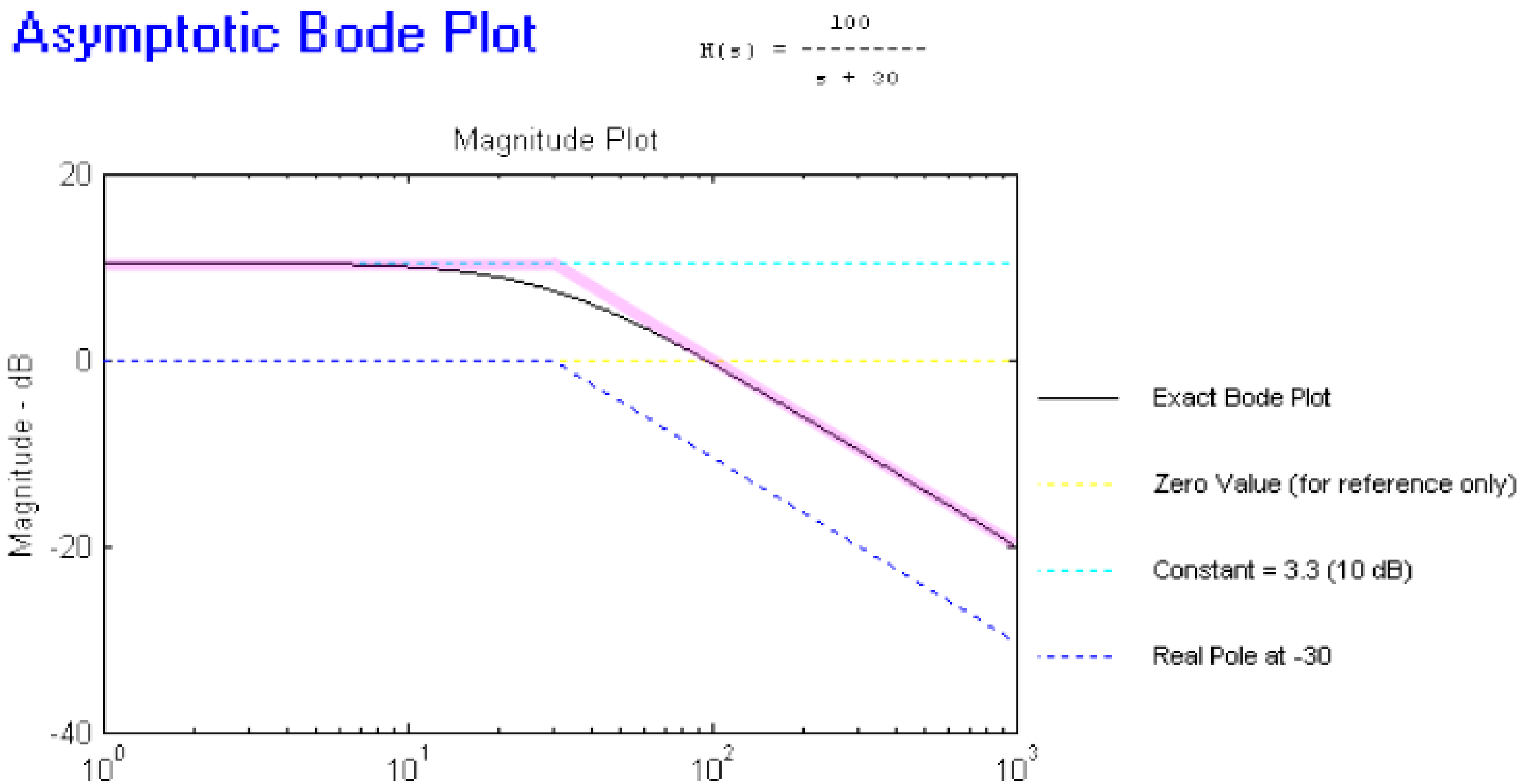
Step 3: Draw the Bode diagram for each part.

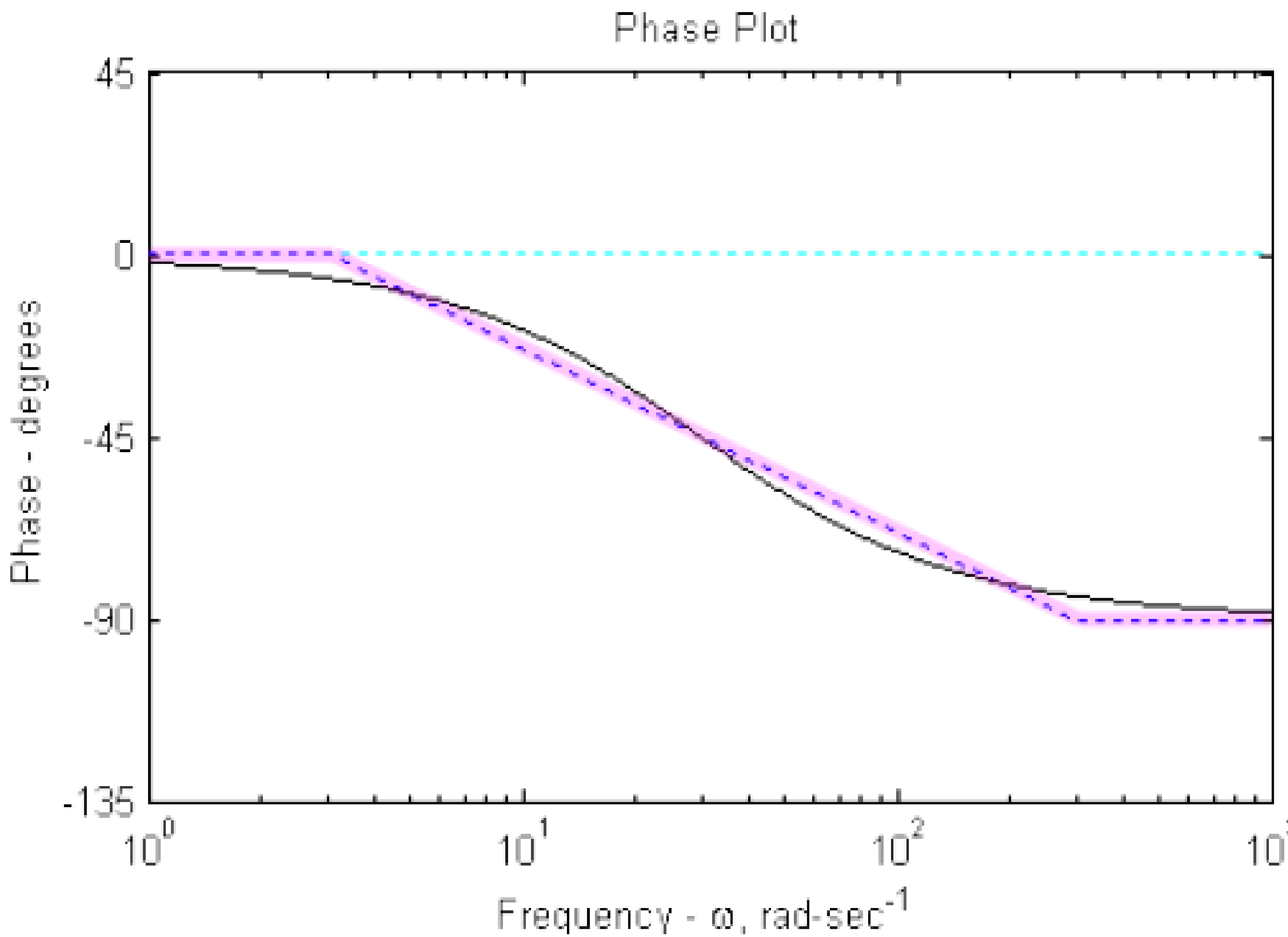
This is done in the diagram below.

- The constant is the cyan line (A quantity of 3.3 is equal to 10.4 dB). The phase is constant at 0 degrees.
- The pole at 30 rad/sec is the blue line. It is 0 dB up to the break frequency, then drops off with a slope of -20 dB/dec. The phase is 0 degrees up to 1/10 the break frequency (3 rad/sec) then drops linearly down to -90 degrees at 10 times the break frequency (300 rad/sec).

Step 4: Draw the overall Bode diagram by adding up the results from step 3.

The overall asymptotic plot is the translucent pink line, the exact response is the black line.




Phase Plot
45
0
-45
-90
-135
Phase - degrees
Frequency - ω, rad-sec-1

Ex-2: $$H(s) = 100\frac{(s+1)}{(s+10)(s+100)}$$

Step 1: Rewrite the transfer function in proper form.

Make both the lowest order term in the numerator and denominator unity. The numerator is an order 1 polynomial, the denominator is order 2.

$$H(s) = \frac{100}{10 \cdot 100}\frac{\frac{s}{1}+1}{\left(\frac{s}{10}+1\right)\left(\frac{s}{100}+1\right)} = 0.1\frac{\frac{s}{1}+1}{\left(\frac{s}{10}+1\right)\left(\frac{s}{100}+1\right)}$$

Step 2: Separate the transfer function into its constituent parts.

The transfer function has 4 components:

- A constant of 0.1
- A pole at s=-10
- A pole at s=-100
- A zero at s=-1

Step 3: Draw the Bode diagram for each part.

This is done in the diagram below.

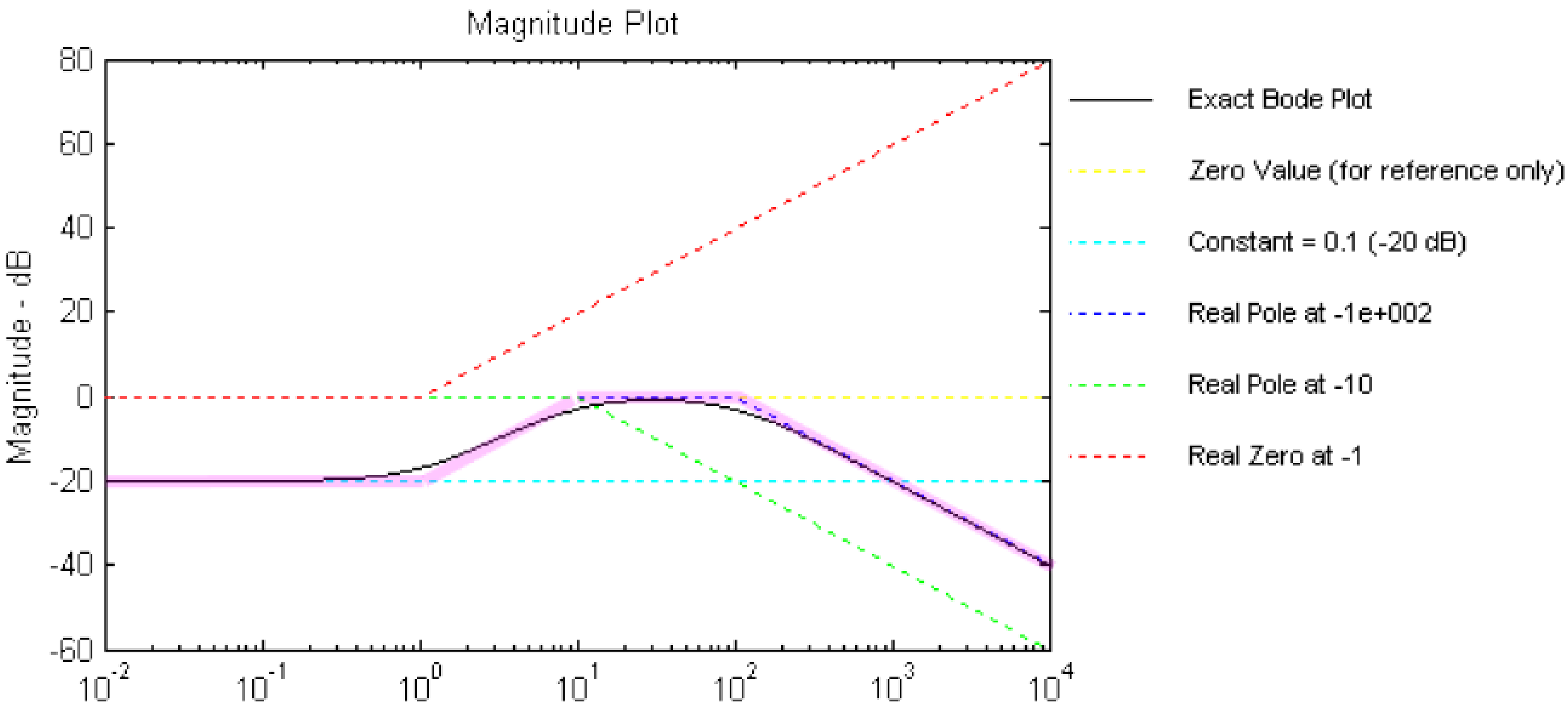
- The constant is the cyan line (A quantity of 0.1 is equal to -20 dB). The phase is constant at 0 degrees.
- The pole at 10 rad/sec is the green line. It is 0 dB up to the break frequency, then drops off with a slope of -20 dB/dec. The phase is 0 degrees up to 1/10 the break frequency (1 rad/sec) then drops linearly down to -90 degrees at 10 times the break frequency (100 rad/sec).
- The pole at 100 rad/sec is the blue line. It is 0 dB up to the break frequency, then drops off with a slope of -20 dB/dec. The phase is 0 degrees up to 1/10 the break frequency (10 rad/sec) then drops linearly down to -90 degrees at 10 times the break frequency (1000 rad/sec).
- The zero at 1 rad/sec is the red line. It is 0 dB up to the break frequency, then rises at 20 dB/dec. The phase is 0 degrees up to 1/10 the break frequency (0.1 rad/sec) then rises linearly to 90 degrees at 10 times the break frequency (10 rad/sec).

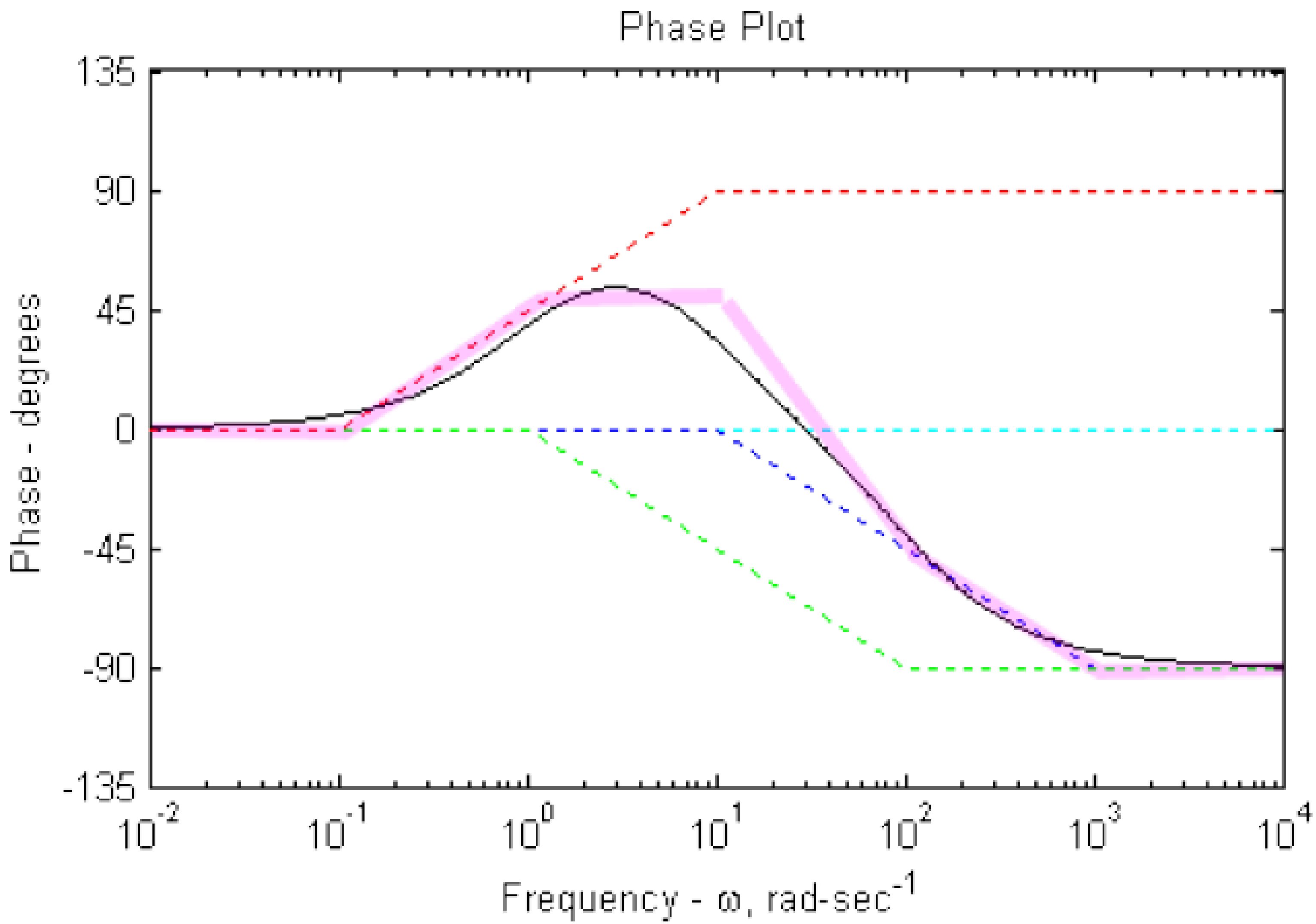
Step 4: Draw the overall Bode diagram by adding up the results from step 3.

The overall asymptotic plot is the translucent pink line, the exact response is the black line.

Asymptotic Bode Plot

$$H(s) = \frac{100\,s + 100}{s^2 + 110\,s + 1000}$$




Phase Plot
135
90
45
0
-45
-90
-135
Phase - degrees
10^{-2}
10^{-1}
10^{0}
10^{1}
10^{2}
10^{3}
10^{4}
Frequency - ω, rad-sec^{-1}

Ex-3: $H(s) = 10\frac{s+10}{s^2+3s}$

Step 1: Rewrite the transfer function in proper form.

Make both the lowest order term in the numerator and denominator unity. The numerator is an order 1 polynomial, the denominator is order 2.

$$H(s) = 10\frac{10}{3}\frac{\frac{s}{10}+1}{s\left(\frac{s}{3}+1\right)} = 33.3\frac{\frac{s}{10}+1}{s\left(\frac{s}{3}+1\right)}$$

Step 2: Separate the transfer function into its constituent parts.

The transfer function has 4 components:

- A constant of 33.3
- A pole at s=-3
- A pole at s=0
- A zero at s=-10

Step 3: Draw the Bode diagram for each part.

This is done in the diagram below.

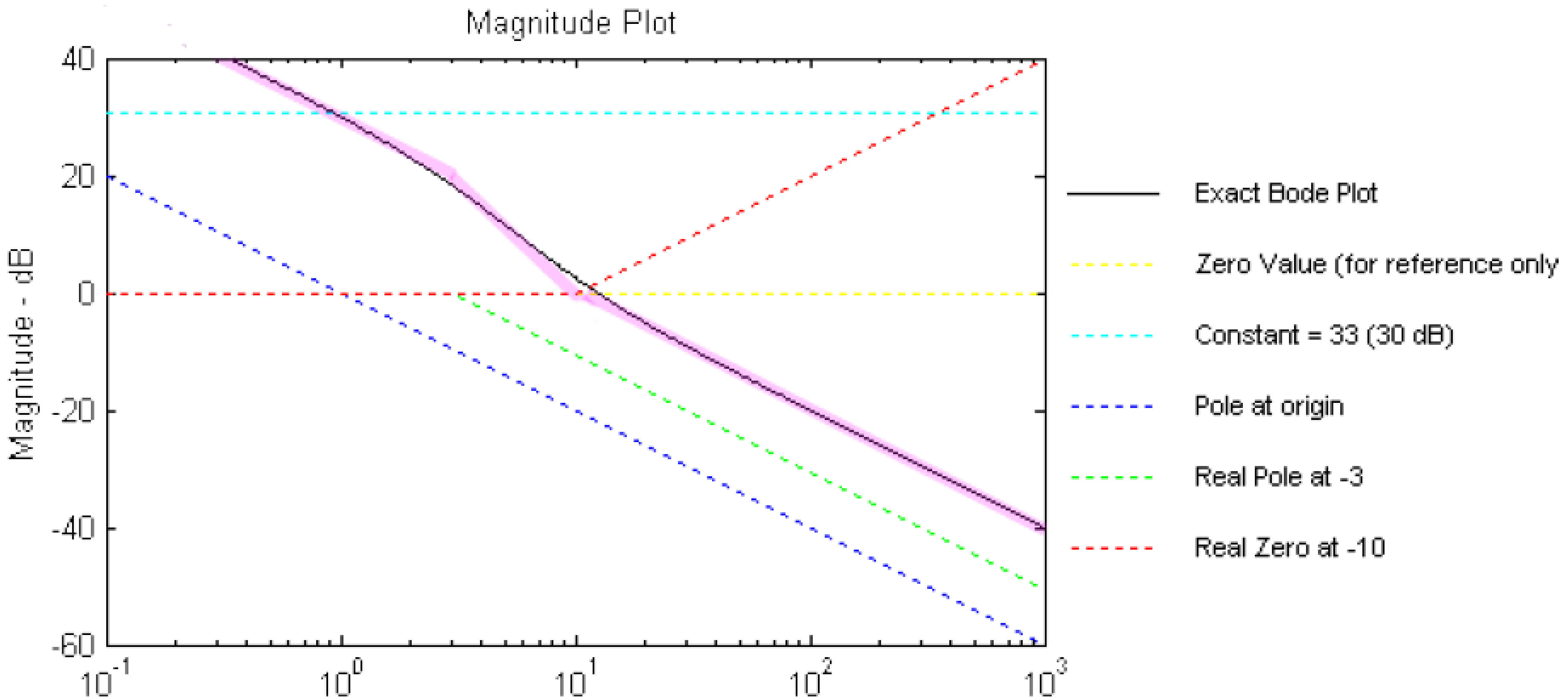
- The constant is the cyan line (A quantity of 33.3 is equal to 30 dB). The phase is constant at 0 degrees.
- The pole at 3 rad/sec is the green line. It is 0 dB up to the break frequency, then drops off with a slope of -20 dB/dec. The phase is 0 degrees up to 1/10 the break frequency (0.3 rad/sec) then drops linearly down to -90 degrees at 10 times the break frequency (30 rad/sec).
- The pole at the origin. It is a straight line with a slope of -20 dB/dec. It goes through 0 dB at 1 rad/sec. The phase is -90 degrees.
- The zero at 10 rad/sec is the red line. It is 0 dB up to the break frequency, then rises at 20 dB/dec. The phase is 0 degrees up to 1/10 the break frequency (1 rad/sec) then rises linearly to 90 degrees at 10 times the break frequency (100 rad/sec).

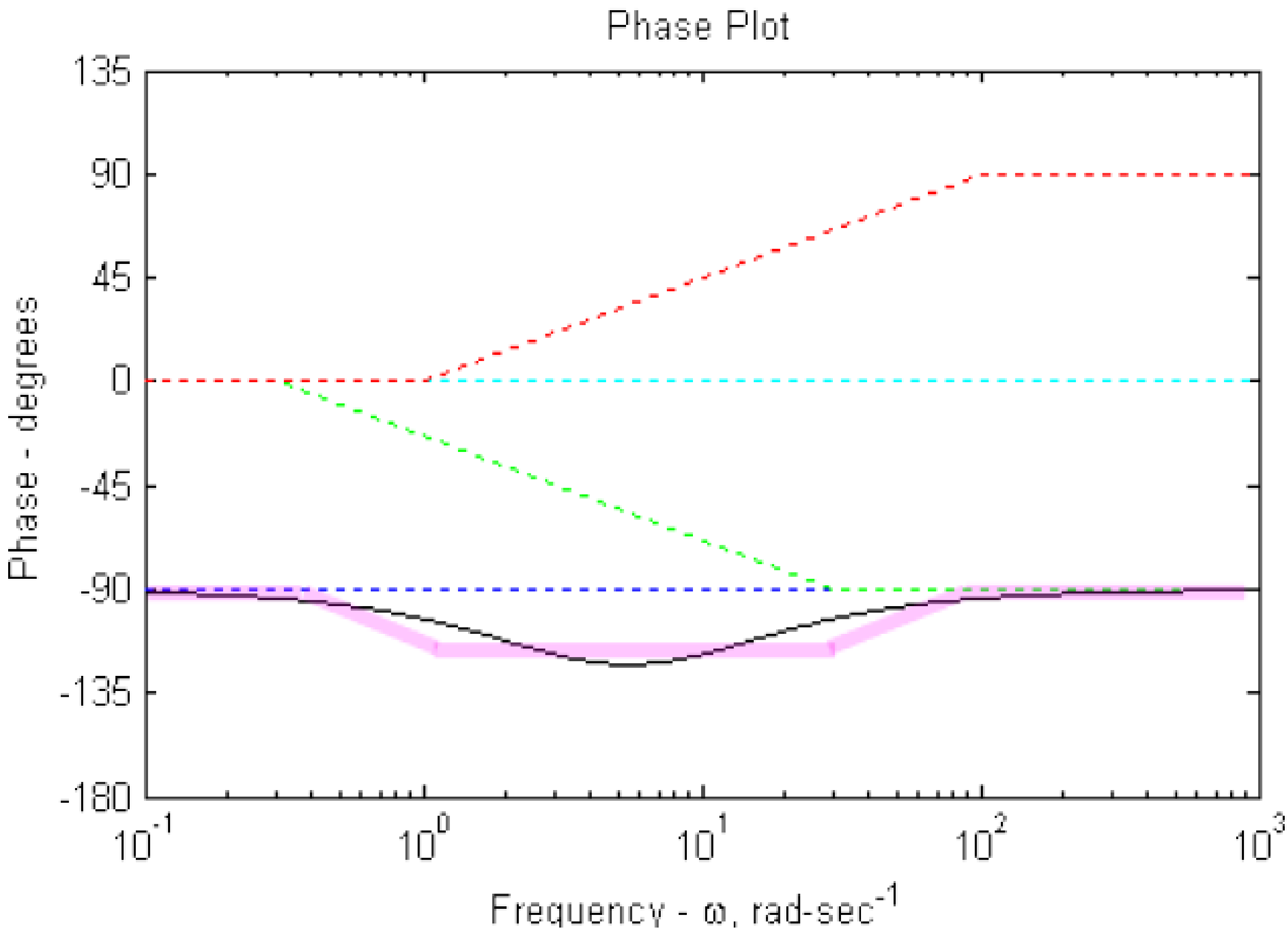
Step 4: Draw the overall Bode diagram by adding up the results from step 3.

The overall asymptotic plot is the translucent pink line, the exact response is the black line.

Asymptotic Bode Plot

```
            10 s + 100
H(s)  =  ------------
            s^2 + 3 s
```

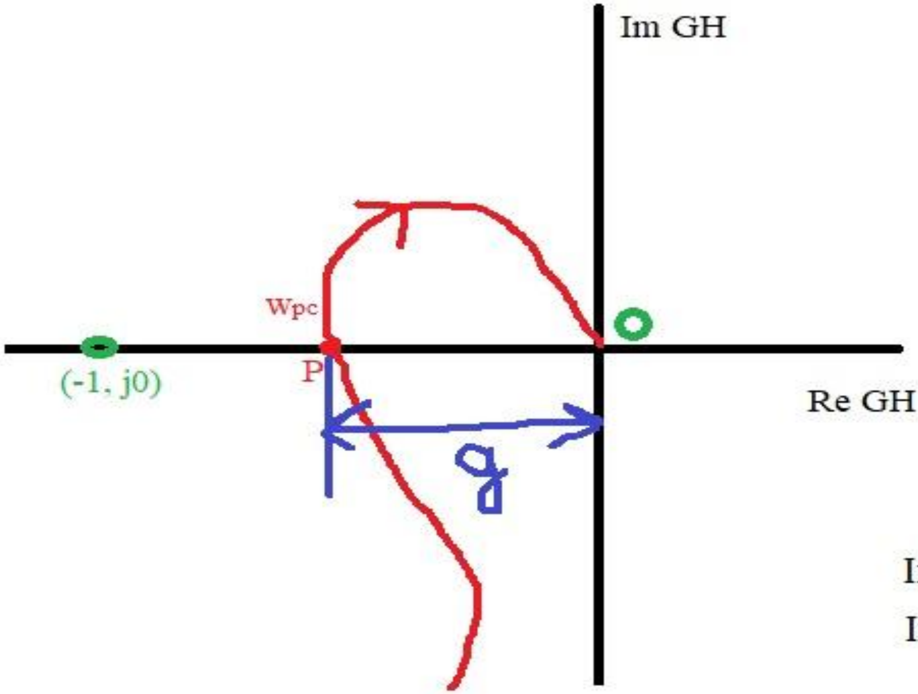




Phase Plot
135
90
45
0
-45
-90
-135
-180
Phase - degrees
10^-1
10^0
10^1
10^2
10^3
Frequency - ω, rad-sec^-1

Gain margin (GM):

- **The amount of increase in the gain(usually specified in db and is positive) that can be permitted before the system becomes unstable**.
- Is a measure of closeness of phase crossover point to (-1, j0) in GH plane.
- **Phase crossover frequency(w_{pc}):** The point where GH plot crosses the –ve real axis i.e at $w=w_{pc}$, phase angle changes from -180 to +180.
- If (-1, j0) point is to the right of phase crossover point |GH|> 1, system is unstable.
- If GH never cuts –ve axis, the system is forever stable.
- The GM is +ve for stable systems.
- If GM is –ve, the plot encloses the critical point and the system is unstable.
- The crossover point moves nearer to the critical point, as gain K is increased. For a particular value of K, |GH| becomes unity and the system will be on the verge of instability.



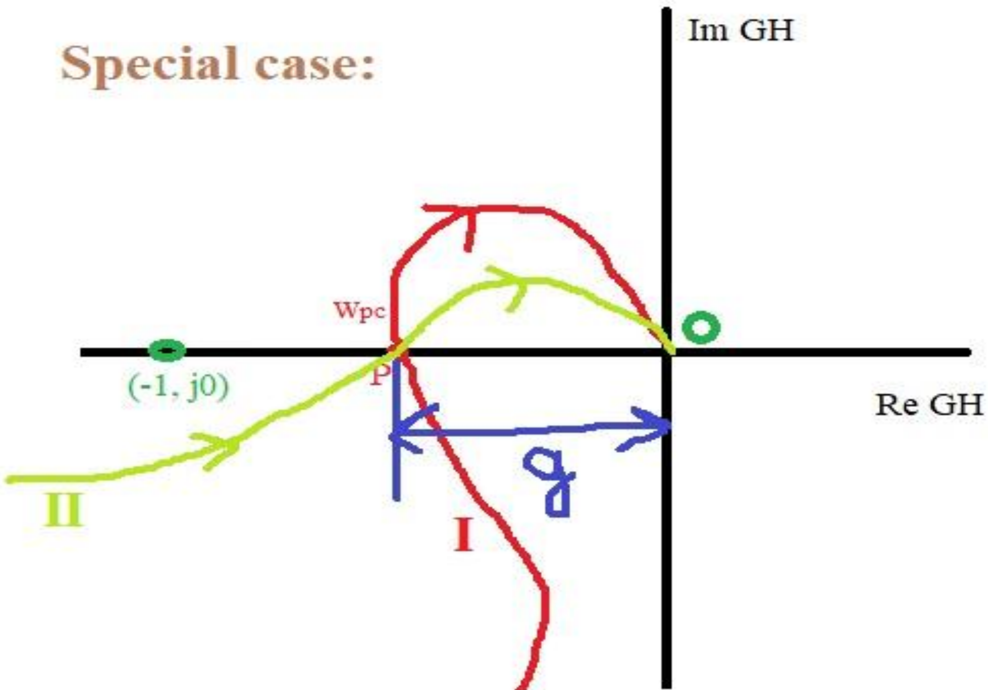
$$GM = \frac{1}{|GH|_{w=w_{pc}}} = \frac{1}{g}$$

$$\text{in db} = 20\log_{10}\frac{1}{g}$$

If g>1, the GM is -ve, unstable

If the gain is increased by 1/g, GH=1

Special case:



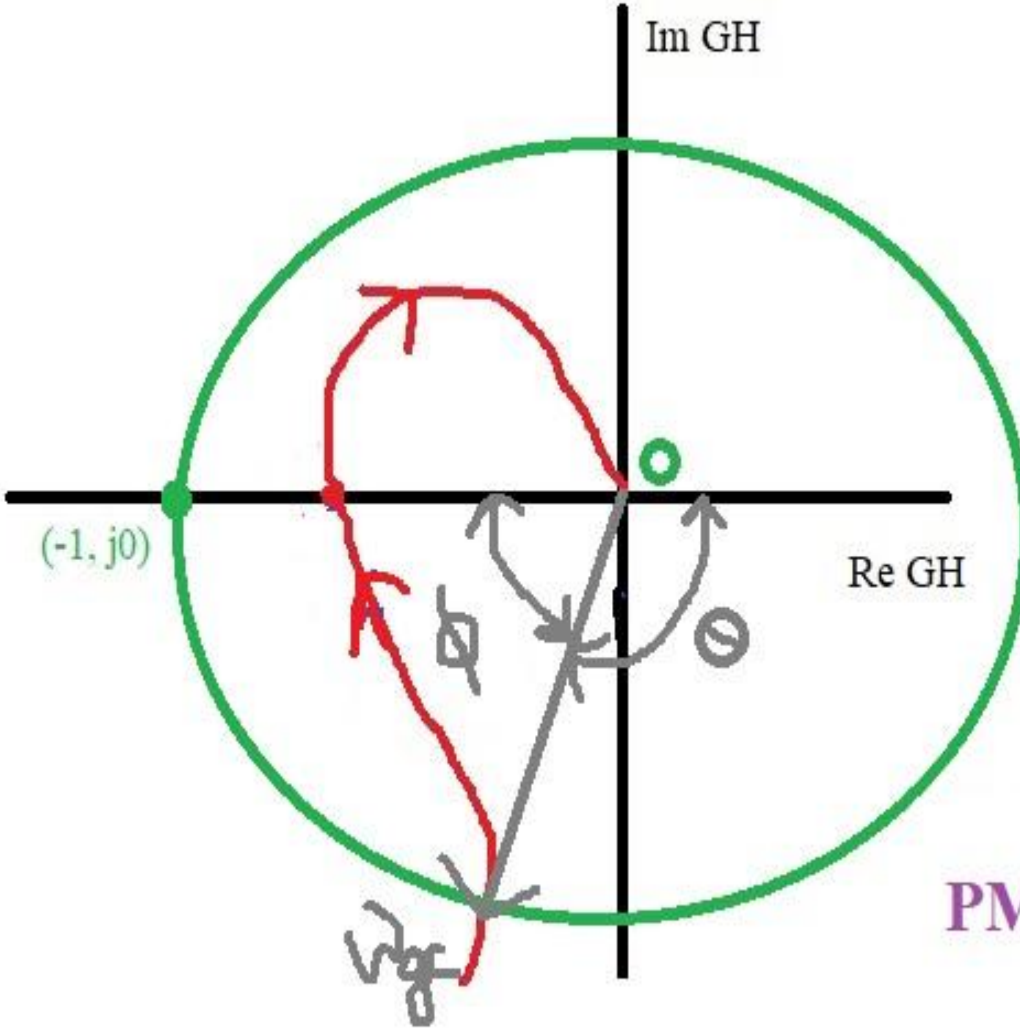
Both the systems I and II have same crossover point and hence same gain margin.

But II is nearer to critical point than I

I is relatively more stable than II

Phase margin (PM):

- **The amount of phase lag that can be introduced into the system at the gain cross over frequency to bring the system to the verge of instability.**
- Is measured positively from -180 line in the counter clockwise direction.
- **Gain crossover frequency(w_{gc}):** The frequency at which the GH plot crosses the unit circle i.e at $w=w_{gc}$, $|GH| = 1$ or 0db.
- The PM is +ve for stable systems.



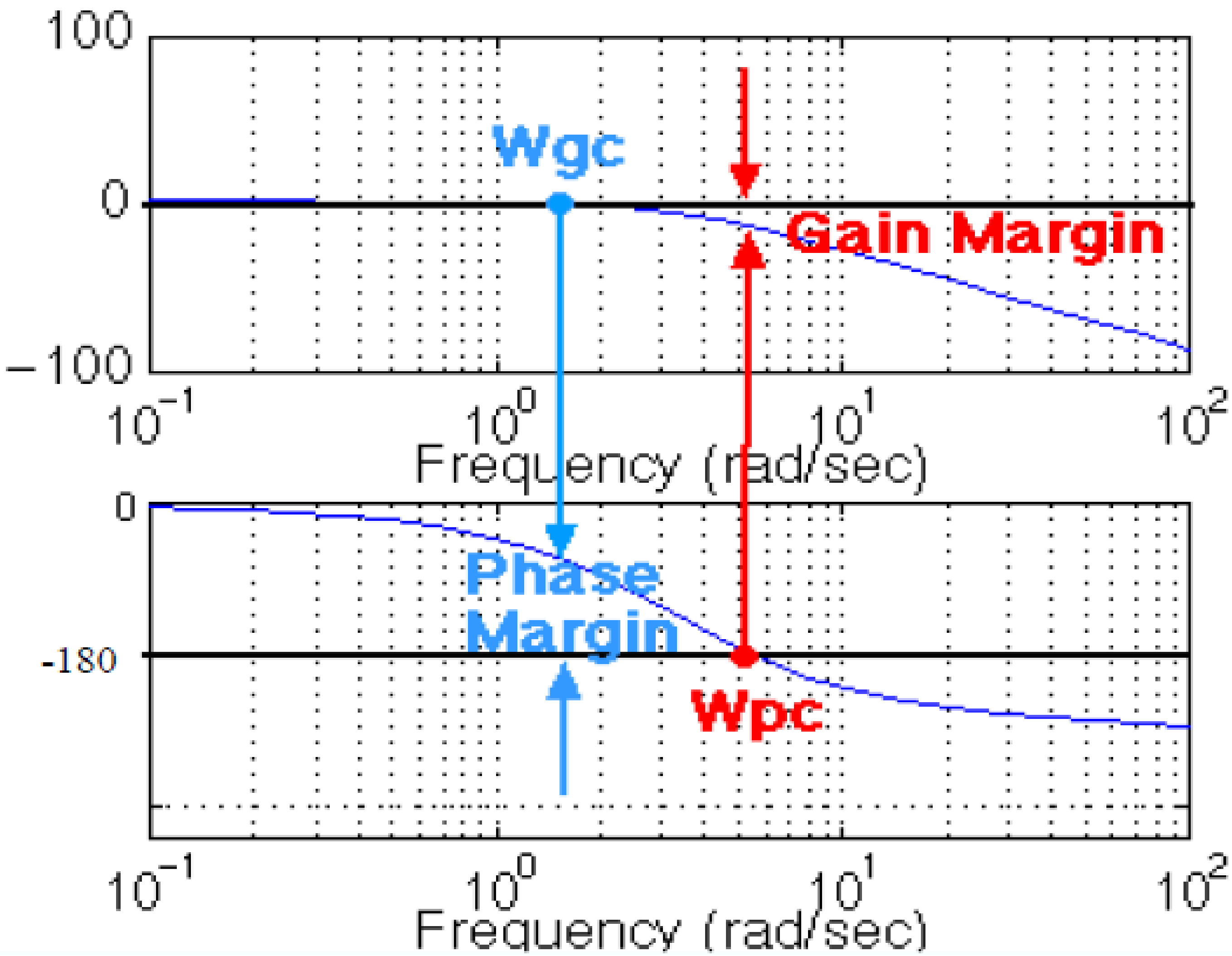
$\theta = \angle GH_{\omega=\omega_{gc}}$

If an additional phase lag of ϕ = 180-θ is added without changing the magnitude at this frequency, the plot will cross (-1, j0)

PM= $\angle GH_{\omega=\omega_{gc}}$ **+180**

If the plot is rotated by an angle equal to ϕ in the clockwise direction, the syetm becomes unstable.

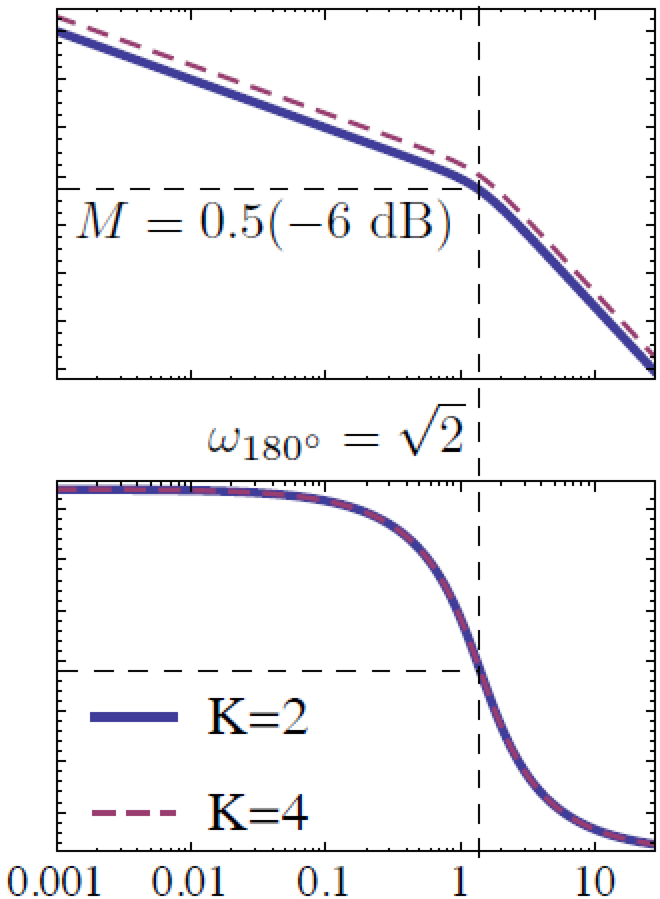
Gain and Phase Margin



- Systems with GM of about 5 to 10db and PM of around 30 to 40 degrees are reasonably stable systems.
- Good GM ensures a good PM also. But there may be systems with good GM but low PM and vice-versa.
- PM is the relative stability measure than GM.
- Usually PM is one of the frequency domain specifications in the design of a control system rather than GM.

Gain Margin

Our example: $G(s) = \dfrac{1}{s(s^2+2s+2)}$, $K = 2$ (stable)



Gain margin (GM) is the factor by which K can be multiplied before we get $M = 1$ when $\phi = 180^\circ$

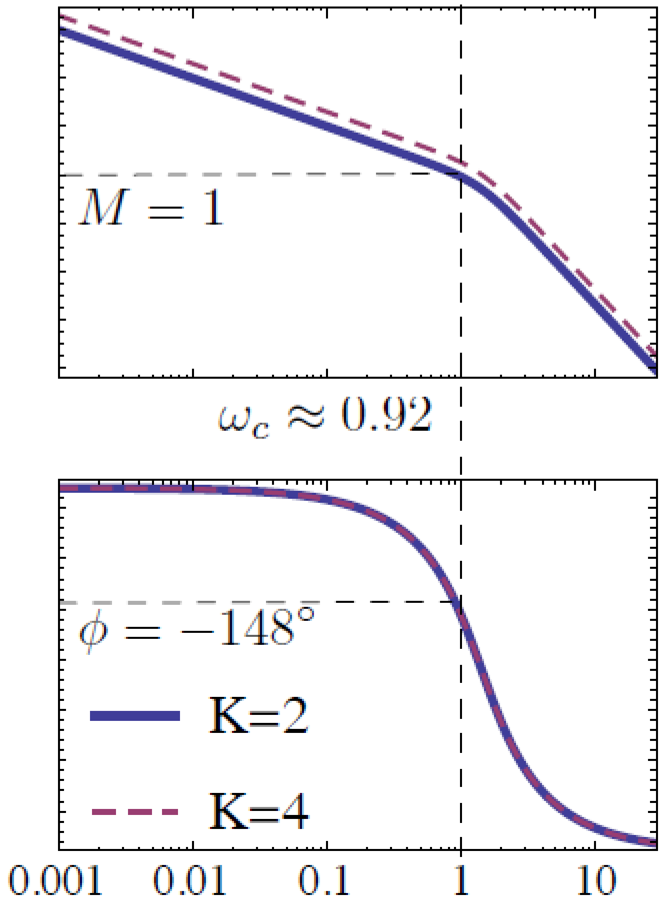
Since varying K doesn't change ω_{180°, to find GM we need to inspect M at $\omega = \omega_{180^\circ}$

In this example:

$$\text{at } \omega_{180^\circ} = \sqrt{2}$$
$$M = 0.5\,(-6\text{ dB}),$$
$$\text{so GM} = 2$$

Phase Margin

Our example: $G(s) = \dfrac{1}{s(s^2+2s+2)}$, $K = 2$ (stable)



Phase margin (PM) is the amount by which the phase at the crossover frequency ω_c differs from $180° \mod 360°$

To find PM, we need to inspect ϕ at $\omega = \omega_c$

In this example:

at $\omega_c \approx 0.92$

$\phi = -148°$,

so PM $= (-148°) - (-180°) = 32°$

(in practice, want PM $\geq 30°$)