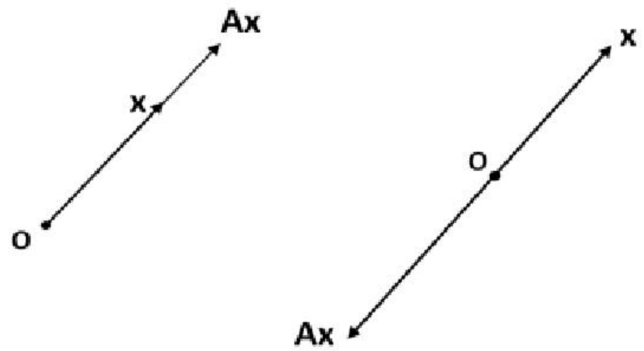
1.10

Eigenvalues and Eigenvectors

Definition: Let A be an $n \times n$ matrix. A scalar λ is called an eigenavlue of A if there exists a nonzero vector X in $\mathbb{R}^n$ such that $AX = \lambda X$. The vector X is called an eigenvector corresponding to λ.

Let us look at the geometrical significance of an eigenvector that corresponds to a nonzero eigenvalue. The vector AX is in the same or opposite direction as X , depending on the sign of λ . See the figure below. An eigenvector of A is thus a vector whose direction is unchanged or reversed when multiplied by A.



Computation of Eigenvalues and Eigenvectors

Let A be an $n \times n$ matrix with eigenvalue λ and corresponding eigenvector X. Thus $AX - \lambda X$. This equation may be rewritten as $AX - \lambda X = 0$, giving $(A - \lambda I_n)X = 0$.

This matrix equation represents a system of homogeneous linear equations having matrix of coefficients $(A - \lambda I_n)X = 0$

is a solution to this system. However, eigenvectors have been defined to be nonzero vectors. Further, nonzero solutions to this system of equations can only exist if the matrix of coefficients is singular, $|A - \lambda I_n| = 0$. Hence, solving the equation $|A - \lambda I_n| = 0$ for λ leads to all the eigenvalues of A.

On expanding the determinant $|A - \lambda I_n|$, we get a polynomial in λ. This polynomial is called the characteristic polynomial of A. The equation $|A - \lambda I_n| = 0$ is called the characteristic equation of A.

The eigenvalues are then substituted back into the equation $(A - \lambda I_n)X = 0$ to find the corresponding eigenvectors.

We are always interested in knowing whether sets of vectors form subspaces!

Theorem: Let A be an $n \times n$ matrix and λ an eigenvalue of A. The set of all eigenvectors corresponding to λ, together with the zero vectors, is a subspace of $\mathbb{R}^n$. This subspace is called the eigenspace of λ.

Proof: In order to show that the eigenspace is a subspace, we have to show that it is closed under vector addition and scalar multiplication.

Let X_1 and X_2 be two vectors in the eigenspace of λ and let c be a scalar.

Then $AX_1 = \lambda X_1$ and $AX_2 = \lambda X_2$.

Hence, $AX_1 + AX_2 = \lambda X_1 + \lambda X_2$

$A(X_1 + X_2) = \lambda(X_1 + X_2)$.

Thus $X_1 + X_2$ is a vector in the eigenspace of λ. The eigenspace is closed under addition.

Further, since $AX_1 = \lambda X_1$,

$cAX_1 = c\lambda X_1$

$A(cX_1) = \lambda(cX_1)$.

Therefore cX_1 is a vector in the eigenspace of λ. The eigenspace is closed under scalar multiplication.

Thus the eigenspce is a subspace.

Properties of Eigen Values

Property 1: The sum of the eigenvalues of a square matrix A is the sum of the diagonal elements (trace) of A.

Property 2: The product of the eigenvalues is $|A|$.

Proof: Let $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$

The eigen values are got from the characteristic equation

$$\left|A - \lambda I\right| = \begin{vmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda \end{vmatrix} = 0$$

Let $\left|A-\lambda I\right| = a_0\lambda^n + a_1\lambda^{n-1} + \cdots + a_n$. . . ..(1)

Setting $\lambda = 0$ we get $\left|A\right| = a_n$ ....(2)

Also explaining $\left|A-\lambda I\right|$ by first row we get

$\left|A-\lambda I\right| = (-1)^n\lambda^n + (-1)^{n-1}(a11 + a22 + \cdots + a_{nn})\lambda^{n-1} + \cdots$ ....(3)

Comparing (1) and (3) we get

$$a_0 = (-1)^n; a_1 = (-1)^{n-1}(a_{11} + a_{22} + \cdots + a_{nn}); \quad \text{...............(4)}$$

If $\lambda_1, \lambda_2, \cdots \lambda_n$ are the eigenvalues ((i.e) the roots of the characteristic equation) then

Sum of the eigenvalues = sum of the roots

$$= \lambda_1 + \lambda_2 + \cdots + \lambda_n$$

$$= -\frac{a_1}{a_0} = a_{11} + a_{22} + \cdots + a_{nn} \text{ (Using 4)}$$

$$= \text{trace of } A.$$

Product of the eigenvalues = product of the roots

$$= \lambda_1, \lambda_2, \cdots \lambda_n$$

$$= (-1)^n \frac{a_n}{a_0} = \frac{(-1)^n a_n}{(-1)^n}$$

$$= a_n$$

$$= |A| \text{ (from (2))}$$

Property 3: The eigen values of A and its transpose A^T are the same.

Proof: It is enough if we prove that A and A^T have the same characteristic polynomial. Since for any square matrix $M, |M| = |M^T|$ we have,

$$|A - \lambda I| = |(A - \lambda I)^T| = |A^T - (\lambda I)^T| = |A^T - \lambda I|$$

Hence the result.

Property 4: If λ is an eigen value of a non singular matrix A then $\frac{1}{\lambda}$ is an eigen value of A^{-1}.

Proof : Let X be an eigen vector corresponding to λ.

Then $AX = \lambda X$. Since A is non singular A^{-1} exists.

$\therefore A^{-1}(AX) = A^{-1}(\lambda X)$

$$IX = \lambda A^{-1} X$$

$\therefore\ A^{-1}X = \left(\frac{1}{\lambda}\right)X.$

$\therefore \frac{1}{\lambda}$ is an eigen value of A^{-1}.

Corollary: If $\lambda_1, \lambda_2, \dots, \lambda_n$ are the eigen values of a non singular matrix A then $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \dots, \frac{1}{\lambda_n}$ are the eigen values of A^{-1}.

Property 5: If λ is an eigen value of A then $k\lambda$ is an eigen value of kA where k is a scalar.

Proof: Let X be an eigen vector corresponding to λ.

Then $AX = \lambda X$.

Now, $(kA)X = k(AX)$

$= k(\lambda X)$ (by (1))

$= (k\lambda)X$.

$\therefore k\lambda$ is an eigen value of kA.

Property 6: If λ is an eigen value of A then λ^k is an eigen value of A^k where k is any positive integer.

Proof: Let X be an eigen vector corresponding to λ.

Then $AX = \lambda X$.

Now, $A^2X = (AA)X = A(AX)$

$= A(\lambda X)$ (by (1))

$= \lambda(AX)$

$= \lambda(\lambda X)$ (by (1))

$= \lambda^2 X$.

$\therefore \lambda^2$ is an eigen value of A^2.

Proceeding like this we can prove that λ^k is an eigen value of A^k for any positive integer.

Corollary: If $\lambda_1, \lambda_2, \dots, \lambda_n$ are eigen values of A then $\lambda_1^k, \lambda_2^k, \dots, \lambda_n^k$ are given values of A^k for any positive integer.

CAYLEY-HAMILTON THEOREM:

Statement: Every square matrix satisfies its own characteristic equation.

Proof: Let A be a $n \times n$ square matrix. Then characteristic equation of A is $|A - \lambda I| = 0$,

Let $|A - \lambda I| = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_0$.

Cofactor of any element of $A - \lambda I$ is a polynomial of degree at most equal to $n - 1$.

Hence, adj $(A - \lambda I)$ may be expressed as a matrix polynomial in λ.

Let adj $(A - \lambda I) = B_{n-1}\lambda^{n-1} + B_{n-2}\lambda^{n-2} + \cdots + B_1\lambda + B_0$,

Where $B_0, \ldots, B_{n-1}$ are all matrices of order n and whose elements are functions of the elements of the matrix A. Now, we have

$$(A - \lambda I)\mathrm{adj}(A - \lambda I) = |A - \lambda I| I.$$

i.e.,

$$(A - \lambda I)(B_{n-1}\lambda^{n-1} + \cdots + B_0) = (-1)^n [\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_0] I.$$

Comparing the coefficients of like powers of λ on both sides,

$$-B_{n-1} = (-1)^n I.$$

$$AB_{n-1} - B_{n-2} = (-1)^n a_{n-1} I.$$

$$AB_{n-2} - B_{n-3} = (-1)^n a_{n-2} I.$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$AB_1 - B_0 = (-1)^n a_1 I.$$

$$AB_0 = (-1)^n a_0 I.$$

Pre-multiplying the first by A^n, the second by $A^{n-1}, \dots$ and adding the above relations, we get

$$0 = (-1)^n [A^n + a_{n-1}A^{n-1} + \dots + a_1 A + a_0 I]$$

$$\Rightarrow (-1)^n [A^n + a_{n-1}A^{n-1} + \dots + a_1 A + a_0 I] = 0$$

Hence A satisfies its characteristic equation.

Remark: Determination of A^{-1} using Cayley-Hemilton theorem.

A satisfies its characteristic equation

i.e $(-1)^n [A^n + a_{n-1}A^{n-1} + \dots + a_1 A + a_0 I] = 0$

$$\Rightarrow A^n + a_{n-1}A^{n-1} + \dots + a_1 A + a_0 I = 0$$

$\Rightarrow A^{-1}[A^n + a_{n-1}A^{n-1} + \dots + a_1 A + a_0 I]$ if A is non-singular

$$\Rightarrow a_0 A^{-1} = -A^{n-1} - a_{n-1}A^{n-2} - \dots - a_1 I$$

$$\therefore A^{-1} = -\frac{1}{a_0}[A^{n-1} + a_{n-1}A^{n-2} + \dots + a_1 I].$$

Applications

Eigenvalues and eigenvectors are extremely important tools in applying mathematics. They are used in many branches of engineering and the natural and social sciences. Now we see use of eigenvectors in long term behavior of the population distribution.

It is estimated that the number of people living in cities in the United States during 2000 is 58 million. The number of people living in the surrounding suburbs is 142 million. Let us represent this information by the matrix $X_0 = \begin{bmatrix} 58 \\ 142 \end{bmatrix}$.

Consider the population flow from cities to suburbs. During 2000, the probability of a person staying in the city was 0.96. Thus the probability of moving to the suburbs was 0.04 (assuming that all those who moved went to the suburbs). Consider now the reverse population flow, from suburbia to city. The probability of a person moving to the city was 0.01; the probability of remaining in suburbia was 0.99. These probabilities can be written as the elements of a stochastic matrix P:

$$\begin{array}{cc} & \text{(from)} \qquad\qquad \text{(to)} \\ & \begin{array}{cc} \text{City} & \text{Suburb} \end{array} \\ P = & \begin{bmatrix} 0.96 & 0.01 \\ 0.04 & 0.99 \end{bmatrix} \begin{array}{l} \text{city} \\ \text{Suburb} \end{array} \end{array}$$

The probability of moving from location A to location B is given by the element in column A and row B. In this context, the stochastic matrix is called a **matrix of transition probabilities.**

Now consider the population distribution in 2001, one year later:

City population in 2001 = People who remained from 2000 + People who moved in the from the suburbs

$$= (0.96 \times 58) + (0.01 \times 142)$$

$$= 57.1 \text{ million}$$

Suburban population in 2001 = People who moved in from the city +People who stayed from 2000

$$= (0.04 \times 58) + (0.99 \times 142)$$

$$= 142.9 \text{ million.}$$

Note that we can arrive at these numbers using matrix multiplication:

$$\begin{bmatrix} 0.96 & 0.01 \\ 0.04 & 0.99 \end{bmatrix} \begin{bmatrix} 58 \\ 142 \end{bmatrix} = \begin{bmatrix} 57.1 \\ 142.9 \end{bmatrix}$$

Using 2000 as the base year, let X_1 be the population in 2002, one year later. We can write $X_1 = PX_0$.

Assume that the population flow represented by the matrix P is unchanged over the years. The population distribution X_2 after 2 years is given by

$$X_2 = PX_1$$

After 3 years the population distribution is given by

$$X_3 = PX_2$$

After n years we get

$$X_n = PX_{n-1}$$

The predictions of this model (to four decimal places) are

$$X_0 = \begin{bmatrix} 58 \\ 142 \end{bmatrix} \begin{array}{l} \text{City} \\ \text{Suburb} \end{array} \qquad X_1 = \begin{bmatrix} 57.1 \\ 142.9 \end{bmatrix} \qquad X_2 = \begin{bmatrix} 56.245 \\ 143.755 \end{bmatrix}$$

$$X_3 = \begin{bmatrix} 55.4327 \\ 144.5672 \end{bmatrix} \qquad X_4 = \begin{bmatrix} 54.6611 \\ 145.3389 \end{bmatrix}$$

and so on.

If the sequence $X_0, X_1, X_2, \ldots$ converges to some fixed vector X, where $PX = X$. The population movement would then be in a steady-state with the total city population and total suburban population remaining constant thereafter. We then write

$$X_0, X_1, X_2, \ldots \to X$$

Since such a vector X satisfies $PX = X$, it would be an eigenvector of P corresponding the eigenvalue 1. Knowledge of the existence and value of such a vector would give us information about the long term behavior of the population distribution.

Problem 1: Find the eigenvalue and eigenvectors of the matrix $A = \begin{bmatrix} -4 & -6 \\ 3 & 5 \end{bmatrix}$.

Solution: Let us first derive the characteristic polynomial of A. We get

$$A - \lambda I_2 = \begin{bmatrix} -4 & -6 \\ 3 & 5 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -4-\lambda & -6 \\ 3 & 5-\lambda \end{bmatrix}$$

Note that the matrix $A - \lambda I_2$ is obtained by subtracting λ from the diagonal elements of A. The characteristic polynomial of A is $|A - \lambda I_2| = (-4-\lambda)(5-\lambda) + 18 = \lambda^2 + \lambda - 2$.

We now solve the characteristic equation of A.

$$\lambda^2 + \lambda - 2 = 0$$

$$(\lambda - 2)(\lambda + 1) = 0$$

$$\lambda = 2 \text{ or } -1.$$

The eigenvalues of A are 2 and -1.

The corresponding eigenvectors are found by using these values of λ in the equation$(A - \lambda I_2)X = 0$. There are many eigenvectors corresponding to each eigenvalue.

$\lambda = 2$: We solve the equation $(A - 2I_2)X = 0$ for x. The matrix $(A - 2I_2)$is obtained by subtracting 2 from the diagonal elements of A. We get

$$\begin{bmatrix} -6 & -6 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

This leads to the system of equations

$$-6x_1 - 6x_2 = 0$$

$$3x_1 + 3x_2 = 0$$

giving $x_1 = -x_2$. The solutions to this system of equations are $x_1 = -r$, $x_2 = r$, where r is a scalar. Thus the eigenvectors of A corresponding to $\lambda = 2$ are nonzero vectors of the form

$$r\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$\boldsymbol{\lambda = -1}$: We solve the equation $(A + 1I_2)X = 0$for X. The matrix $(A + 1I_2)$is obtained by adding 1 to the diagonal elements of A. We get

$$\begin{bmatrix} -3 & -6 \\ 3 & 6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

This leads to the system of equations

$$-3x_1 - 6x_2 = 0$$

$$3x_1 + 3x_2 = 0$$

Thus $x_1 = -2x_2$. The solutions to these equations are $x_1 = -2s, x_2 = s$, where s is a scalar. Thus the eigenvectors of A corresponding to $\lambda = -1$ are nonzero vectors of the form $s\begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

Problem 2: Find the eigenvalues and eigenvectors of the matrix $\begin{bmatrix} 5 & 4 & 2 \\ 4 & 5 & 2 \\ 2 & 2 & 2 \end{bmatrix}$.

Solution: The matrix $A - \lambda I_3$is obtained by subtracting λ from the diagonal elements of A. Thus

$$A - \lambda I_3 = \begin{bmatrix} 5-\lambda & 4 & 2 \\ 4 & 5-\lambda & 2 \\ 2 & 2 & 2-\lambda \end{bmatrix}.$$

The characteristic polynomial of A is$|A - \lambda I_3|$. Using row and column operations to simplify determinants, we get

$$|A - \lambda I_3| = \begin{bmatrix} 5-\lambda & 4 & 2 \\ 4 & 5-\lambda & 2 \\ 2 & 2 & 2-\lambda \end{bmatrix}$$

$$= \begin{bmatrix} 1-\lambda & -1+\lambda & 0 \\ 4 & 5-\lambda & 2 \\ 2 & 2 & 2-\lambda \end{bmatrix}$$

$$= \begin{bmatrix} 1-\lambda & 0 & 0 \\ 4 & 9-\lambda & 2 \\ 2 & 4 & 2-\lambda \end{bmatrix}$$

$$= (1-\lambda)[(9-\lambda)(2-\lambda) - 8] = (1-\lambda)[\lambda^2 - 11\lambda + 10]$$

$$= (1-\lambda)(\lambda - 10)(\lambda - 1) = -(\lambda - 10)(\lambda - 1)^2$$

We now solve the characteristic equation of A:

$$= -(\lambda - 10)(\lambda - 1)^2 = 0$$

$$\lambda = 10 \text{ or } 1$$

The eigenvalues of A are 10 and 1.

The corresponding eigenvectors are found by using these values of λ in the equation $(A - \lambda I_3)X = 0$.

$\lambda = 10$: We get $(A - 10I_3)X = 0$

$$\begin{bmatrix} -5 & 4 & 2 \\ 4 & -5 & 2 \\ 2 & 2 & -8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

The solutions to this system of equations are $x_1 = 2r, x_2 = 2r$and $x_3 = r$, where r is a scalar. Thus the eigenspace of $\lambda = 10$ is the one-dimensional space of vectors of the form

$$r\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$$

$\lambda = 1$: Let $\lambda = 1$ in $(A - \lambda I_3)X = 0$. We get

$$(A - 1I_3)X = 0$$

$$\begin{bmatrix} 4 & 4 & 2 \\ 4 & 4 & 2 \\ 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

The solutions to this system of equations can be shown to be $x_1 = -s - t, x_2 = s$, and $x_3 = 2t$, where s and t are scalars. Thus the eigenspace of $\lambda = 1$ is the space of vectors of the form

$$\begin{bmatrix} -s - t \\ s \\ 2t \end{bmatrix}$$

Separating the parameters s and t, we can write

$$\begin{bmatrix} -s-t \\ s \\ 2t \end{bmatrix} = s\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$$

Thus the eigenspace of $\lambda = 1$ is a two-dimensional subspace of $\mathbb{R}^2$with basis

$$\left\{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}\right\}$$

If an eigenvalue occurs as a k times repeated root of the characteristic equation, we say that it is of multiplicity k. Thus $\lambda = 10$ has multiplicity 1, while $\lambda = 1$ has multiplicity 2 in this problem.

Problem 3: Let A be an $n \times n$ matrix A with eigenvalues $\lambda_1, \ldots\ldots \lambda_n$ and corresponding eigenvectors $X_1, \ldots\ldots X_n$. Prove that if $c \neq 0$, then the eigenvalues of cA are $c\lambda_1, \ldots\ldots c\lambda_n$ with corresponding eigenvectors $X_1, \ldots\ldots X_n$.

Solution: Let λ_i be one of the eigenvalues of A with corresponding eigenvector X_i. Then $AX_i = \lambda_i X_i$. Multiply both sides of this equation by c to get

$$cAX_i = c\lambda_i X_i.$$

Thus $c\lambda_i$ is an eigenvalue of cA with corresponding eigenvector X_i.

Problem 4: If the given values of $A = \begin{pmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{pmatrix}$ are $2, 2, 3$ find the given values of A^{-1} and A^2.

Solution: Since 0 is not an eigen value of A, A is a non singular matrix and hence A^{-1} exists.

Eigen values of A^{-1} are $\frac{1}{2}, \frac{1}{2}, \frac{1}{3}$ and eigen values of A^2 are $2^2, 2^2 3^2$.

Problem 5: Find the eigenvalues of A^5 when $A = \begin{pmatrix} 3 & 0 & 0 \\ 5 & 4 & 0 \\ 3 & 6 & 1 \end{pmatrix}$.

Solution: The characteristic equation of A is obviously $(3-\lambda)94-\lambda)(1-\lambda) = 0$.

Hence the eigen values of A are $3, 4, 1$.

$\therefore$ The eigen values of A^5 are $3^5, 4^5, 1^5$.

Problem 6: Find the sum and product of the eigen values of the matrix $\begin{pmatrix} 3 & -4 & 4 \\ 1 & -2 & 4 \\ 1 & -1 & 3 \end{pmatrix}$ without actually finding the eigen values.

Solution: Let $A = \begin{pmatrix} 3 & -4 & 4 \\ 1 & -2 & 4 \\ 1 & -1 & 3 \end{pmatrix}$

Sum of the eigen values = trace of $A = 3 + (-2) + 3 = 4$.

Product of the eigen values $= |A|$.

Now, $|A| = \begin{vmatrix} 3 & -4 & 4 \\ 1 & -2 & 4 \\ 1 & -1 & 3 \end{vmatrix} = 3(-6+4) + 4(3-4) - 4(-1+2)$

$$= -6 - 4 - 4 = -14.$$

$\therefore$ Product of the eigen values$= -14$.

EXERCISE

1. Determine the characteristic polynomials, eigenvalues, and corresponding eigenspaces of the given 3×3 matrices.

$a.\ \begin{bmatrix} 3 & 2 & -2 \\ -3 & -1 & 3 \\ 1 & 2 & 0 \end{bmatrix}$ $\qquad b.\ \begin{bmatrix} 1 & -2 & 2 \\ -2 & 1 & 2 \\ -2 & 0 & 3 \end{bmatrix}$

$c.\ \begin{bmatrix} 15 & 7 & -7 \\ -1 & 1 & 1 \\ 13 & 7 & -5 \end{bmatrix}$

2. Determine the characteristic polynomials, eigenvalues, and corresponding eigenspaces of the matrix.

$$\begin{bmatrix} 4 & 2 & -2 & 2 \\ 1 & 3 & 1 & -1 \\ 0 & 0 & 2 & 0 \\ 1 & 1 & -3 & 5 \end{bmatrix}$$

3. Prove that if A is a diagonal matrix, then its eigenvalues are the diagonal elements.

4. Prove that A and A^T have the same eigenvalues.

5. Prove that $\lambda = 0$ is an eigenvalue of a matrix A if and only if A is singular.

6. Prove that if the eigenvalues of a matrix A are $\lambda_1, \ldots .. \lambda_n$ with corresponding eigenvectors $X_1, \ldots .. X_n$, then $\lambda_1^m, \ldots .. \lambda_n^m$ are eigenvalues of A^m with corresponding eigenvectors $X_1, \ldots .. X_n$.

7. Show that the following matrices satisfy their characteristic equations.

$(a). \begin{bmatrix} 0 & 2 \\ -1 & 3 \end{bmatrix}$ $(b). \begin{bmatrix} 8 & -10 \\ 5 & -7 \end{bmatrix}$ $(c). \begin{bmatrix} 6 & -8 \\ 4 & -6 \end{bmatrix}$ $(d). \begin{bmatrix} -1 & 5 \\ -10 & 14 \end{bmatrix}$

8. Verify Cayley-Hamiltion theorem for the matrix A and find its inverse, where $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$.

9. Find the characteristic equation of the matrix $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$ and hence find the matrix $A^7 - 5A^6 + 9A^5 - 13A^4 + 17A^3 - 21A^2 + 21A - 8I$.

10. Using Cayley-Haimaltion theorem find A^8 , if $A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$.

ANSWERS

1)

a. Characteristic polynomial $-\lambda^3 + 2\lambda^2 + \lambda - 2$; eigenvalues 1,-1,2; corresponding eigenspaces $r\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, s\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, t\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

b. Characteristic polynomial $(1-\lambda)^2(3-\lambda)$; eigen values 1,3; corresponding eigenspaces $r\begin{bmatrix}1\\1\\2\end{bmatrix}$, $s\begin{bmatrix}0\\1\\1\end{bmatrix}$

c. Characteristic polynomial $(1-\lambda)(2-\lambda)(8-\lambda)$; eigenvalues 1,2,8; corresponding eigenspaces $r\begin{bmatrix}1\\-1\\1\end{bmatrix}, s\begin{bmatrix}0\\1\\1\end{bmatrix}, t\begin{bmatrix}1\\0\\1\end{bmatrix}$

2) Characteristic polynomial $(2-\lambda)(2-\lambda)(4-\lambda)(6-\lambda)$; eigenvalues 2,4,6; corresponding eigenspaces $r\begin{bmatrix}1\\-1\\0\\0\end{bmatrix}+s\begin{bmatrix}0\\0\\1\\1\end{bmatrix},\ t\begin{bmatrix}0\\1\\0\\-1\end{bmatrix}, p\begin{bmatrix}1\\0\\0\\1\end{bmatrix}$

8) $\frac{1}{4}\begin{bmatrix}3 & 1 & -1\\1 & 3 & 1\\-1 & 1 & 3\end{bmatrix}$.

9) $\lambda^3-5\lambda^2+7\lambda-3=0,\ \ I$

10) $625I$