4.3

Weak Law of Large Numbers

Let $\{X_n\}$ be a sequence of r.vs and let $\overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$ be the mean of first n r.vs. The weak laws deal with *limits of probabilities involving* $\overline{X}_n$. The strong laws deal with *probabilities involving limits of* $\overline{X}_n$.

Definition of Weak Law of Large Numbers

A sequence $\{X_n\}$ of r.vs is said to satisfy the ***Weak Law of Large Numbers (WLLN)*** if

$$\lim_{n \to \infty} P\left[\left|\frac{S_n}{n} - E\left(\frac{S_n}{n}\right)\right| < \epsilon\right] = 1$$

for any $\epsilon > 0$, where $S_n = \sum_{i=1}^{n} X_i$, $i.e.$, $\frac{S_n}{n} \xrightarrow{P} E\left(\frac{S_n}{n}\right)$

Theorem1: Let $\{X_n\}$ be a sequence of r.vs and let $S_n = X_1 + \cdots + X_n$ with $B_n = V(S_n) < \infty$. If $\frac{B_n}{n^2} \to 0$ as $n \to \infty$, then for any $\epsilon > 0$,

$$\lim_{n \to \infty} P\left[\left|\frac{S_n}{n} - E\left(\frac{S_n}{n}\right)\right| < \epsilon\right] = 1$$

$i.e.$, $\{X_n\}$ satisfies WLLN.

Proof: On applying Chebychev's inequality to the variable $\frac{S_n}{n}$, we have

$$P\left[\left|\frac{S_n}{n} - E\left(\frac{S_n}{n}\right)\right| \geq \epsilon\right] \leq \frac{V\left(\frac{S_n}{n}\right)}{\epsilon^2} = \frac{V(S_n)}{n^2\epsilon^2} = \frac{B_n}{n^2\epsilon^2} \to 0$$

as $n \to \infty$. Thus,

$$\lim_{n \to \infty} P\left[\left|\frac{S_n}{n} - E\left(\frac{S_n}{n}\right)\right| \geq \epsilon\right] = 0 \Rightarrow \lim_{n \to \infty} P\left[\left|\frac{S_n}{n} - E\left(\frac{S_n}{n}\right)\right| < \epsilon\right] = 1$$

$\Rightarrow \{X_n\}$ satisfies WLLN.

Corollary 1: Let $\{X_n\}$ be a sequence of r.vs, $\overline{X_n} = \frac{S_n}{n}$ and $\mu = E\left(\frac{S_n}{n}\right)$.

If $\frac{B_n}{n^2} \longrightarrow 0$ as $n \longrightarrow \infty$, then

$$\lim_{n\longrightarrow\infty} P[\overline{X_n} \leq k] = \begin{cases} 0 & ,if \;\; k < \mu \\ 1 & ,if \;\; k > \mu \end{cases}$$

Proof: Since WLLN holds for $\{X_n\}$, we have

$$\lim_{n\longrightarrow\infty} P[|\overline{X_n} - \mu| < \epsilon] = 1 \Longrightarrow \lim_{n\longrightarrow\infty} P[|\overline{X_n} - \mu| \geq \epsilon] = 0 \qquad \ldots(1)$$

Since $\{\overline{X_n} \leq \mu - \epsilon\} \subset \{|\overline{X_n} - \mu| \geq \epsilon\}$, we have

$$P(\overline{X_n} \leq \mu - \epsilon) \leq P(|\overline{X_n} - \mu| \geq \epsilon)$$

$$\Longrightarrow \lim_{n\longrightarrow\infty} P(\overline{X_n} \leq \mu - \epsilon) \leq \lim_{n\longrightarrow\infty} P(|\overline{X_n} - \mu| \geq \epsilon)$$

$$\Longrightarrow \lim_{n\longrightarrow\infty} P(\overline{X_n} \leq \mu - \epsilon) = 0$$

$\Longrightarrow \lim_{n\longrightarrow\infty} P(\overline{X_n} \leq k) = 0$, where $k = \mu - \epsilon$, $i.e., k < \mu$ since $\epsilon > 0$

$$\Longrightarrow \lim_{n\longrightarrow\infty} P(\overline{X_n} \leq k) = 0 \;\; if \; k < \mu$$

Further, $P(\overline{X_n} \leq \mu + \epsilon) + P(|\overline{X_n} - \mu| > \epsilon) \geq 1$, since the region is larger than sample space covered.

$$\Longrightarrow \lim_{n\longrightarrow\infty} P(\overline{X_n} \leq \mu + \epsilon) \geq 1 \;\; \left(\because \lim_{n\longrightarrow\infty} P(|\overline{X_n} - \mu| > \epsilon) = 0\right)$$

$$\Longrightarrow \lim_{n\longrightarrow\infty} P(\overline{X_n} \leq \mu + \epsilon) = 1$$

$\Longrightarrow \lim_{n\longrightarrow\infty} P(\overline{X_n} \leq k) = 1$ where $k = \mu + \epsilon \; i.e., k > \mu$ since $\epsilon > 0$

$\Longrightarrow \lim_{n\longrightarrow\infty} P(\overline{X_n} \leq k) = 1$ if $k > \mu$

Thus, $$\lim_{n\longrightarrow\infty} P(\overline{X_n} \leq k) = \begin{cases} 0\,, & k < \mu \\ 1\,, & k > \mu \end{cases}$$

Variations of the WLLN

The following are some special cases of Theorem1 which are stated without proof.

Theorem 2: (Bernoulli's WLLN)

Let $\{X_n\}$ be a sequence of Bernoulli trials with probability of success equal to p. If S_n is the number of successes in n trials, then

$$\lim_{n\to\infty} P\left[\left|\frac{S_n - np}{n}\right| < \epsilon\right] = 1, \ \ \forall\, \epsilon > 0$$

Theorem 3: (Khinchine's WLLN)

Let $\{X_n\}$ be a sequence of i.i.d.r.vs with $E(X_i) = \mu < \infty$, $i = 1,2, \ldots$, then the WLLNs holds i.e.,

$$\lim_{n\to\infty} P\left[\left|\frac{S_n}{n} - \mu\right| > \epsilon\right] = 0$$

Theorem 4: (Bernstein's WLLN)

Let $\{X_n\}$ be a sequence of random variables for which $var(X_n) = {\sigma_n}^2 < k, \ \forall\, i,$ where k is independent of n. If $\sigma_{ij} = cov(X_i, X_j) \longrightarrow 0$ as $|i - j| \longrightarrow \infty$ (*Asymptotic uncorrelatedness*) then the WLLN holds.

Example 1: Let $\{X_n\}$ be i.i.d.r.vs with mean μ and variance σ^2, if

$$\frac{{X_1}^2 + {X_2}^2 + \cdots + {X_n}^2}{n} \xrightarrow{P} c$$

as $n \longrightarrow \infty$ for some constant $c (0 \le c < \infty)$, then find c.

Solution: Here $E(X_i) = \mu$ and $V(X_i) = \sigma^2 \ \forall\, i$.

Let $S_n = {X_1}^2 + {X_2}^2 + \cdots + {X_n}^2$. Then

$E(S_n) = nE({X_1}^2)$ $\qquad$ ($\because$ Xs are i.i.d.r.vs)

$$= n\left[V(X_1) + \left(E(X_1)\right)^2\right]$$

$$\Longrightarrow E(S_n) = n(\sigma^2 + \mu^2)$$

Since $E(X^2) = V(X) + \left(E(X)\right)^2 = \sigma^2 + \mu^2$ exists for each X^2 in S_n, by Khinchine's WLLN, we have

$$\frac{S_n}{n} = \frac{{X_1}^2+{X_2}^2+\cdots+{X_n}^2}{n} \quad E\left({X_1}^2\right) = \mu^2 + \sigma^2$$

Thus, $c = \mu^2 + \sigma^2$.

Example 2: If the i.i.d. r.vs $X_k(k = 1, 2, \ldots)$ assume the value $2^{r-2\,ln\,r}$ with probability $\frac{1}{2^r}$, examine if the WLLN holds for the sequence $\{X_k\}$.

Solution:

$$E\left(X_k\right) = \sum_{r=1}^{\infty} 2^{r-2\,ln\,r} \cdot \frac{1}{2^r} = \sum_{r=1}^{\infty}\left(2^{-2}\right)^{ln\,r} = \sum_{r=1}^{\infty}\left(\frac{1}{4}\right)^{ln\,r}$$

$$= \sum_{r=1}^{\infty}(r)^{ln\left(\frac{1}{4}\right)} \left(\because a^{ln\,n} = n^{ln\,a}\right)$$

$$= \sum_{r=1}^{\infty}\left(\frac{1}{r}\right)^{ln\,4} \quad converges\ since\ ln\,4 = 1.39 > 1 \quad \left(\sum_{n=1}^{\infty}\frac{1}{n^p}\ converges\ if\ \ p > 1\right)$$

$$Thus\ \ E\left(X_k\right) < \infty$$

Since $\{X_k\}$ are i.i.d.r.vs with $E(X_k) < \infty$, the WLLN holds for the sequence, by Khinchine's theorem.

Example 3: Let $\{X_n\}$ be a sequence of i.i.d $U(0, 1)$ r.vs. For the geometric mean $G_n = (X_1.X_2.\ldots.X_n)^{\frac{1}{n}}$, show that $G_n \xrightarrow{P} c$ where c is some constant. Find c.

Solution: Let $Y = -\,ln\,X$ where $X \sim U(0,1)$. The c.d.f. of Y is given by

$$F_Y(y) = P(Y \le y) = P(-\ln X \le y) = P(X \ge e^{-y}) = \int_{e^{-y}}^{1} 1\,dx = 1 - e^{-y}$$

$\Longrightarrow F_Y(y) = 1 - e^{-y}$ and the p.d.f of Y is given by

$$f_Y(y) = \frac{d}{dx}\left(F_Y(y)\right) = e^{-y} \text{ for } y > 0.$$

Then $E(Y) = V(Y) = 1$.

Thus, the sequence $\{Y_n\}$ is i.i.d with finite mean $E(Y_n) = 1$. Hence, by Khinchine's WLLN

$$\sum_{i=1}^{n} \frac{Y_i}{n} \xrightarrow{P} E(Y_1) = 1 \qquad \dots(1)$$

$$But \ \ln G_n = \sum_{i=1}^{n} \ln \frac{X_i}{n} = -\sum_{i=1}^{n} \frac{Y_i}{n}$$

$$\Longrightarrow \sum_{i=1}^{n} \frac{Y_i}{n} = -\ln G_n \qquad \dots(2)$$

From (1) and (2), we have

$$-\ln G_n \xrightarrow{P} 1 \quad i.e., \quad G_n \xrightarrow{P} e^{-1}$$

Thus, $c = \frac{1}{e}$.

Example 4: Let X_i can have only two values i^{α} and $-i^{\alpha}$ with equal probabilities. If $\{X_i\}$ is a sequence of independent r.vs, then show that WLLN holds if $\alpha < \frac{1}{2}$.

Solution: Here $E(X_i) = i^{\alpha}\frac{1}{2} - i^{\alpha}\frac{1}{2} = 0$ and

$$V(X_i) = E\left({X_i}^2\right) = i^{2\alpha}\frac{1}{2} + i^{2\alpha}\frac{1}{2} = i^{2\alpha}$$

Let $S_n = \sum_{k=1}^{n} X_k$. Then

$$B_n = V(S_n) = \sum_{i=1}^{n} V(X_i) \qquad (\because X_i\text{s are independent})$$

$$= \sum_{i=1}^{n} i^{2\alpha} = 1^{2\alpha} + 2^{2\alpha} + \cdots + n^{2\alpha}$$

$$= \int_0^n x^{2\alpha} dx \quad (\text{Euler} - \text{Maclaurion formula})$$

$\Longrightarrow B_n = \frac{n^{2\alpha+1}}{2\alpha+1} \Longrightarrow \frac{B_n}{n^2} = \frac{n^{2\alpha+1}}{2\alpha+1} \longrightarrow 0$ as $n \longrightarrow \infty$ if $\alpha < \frac{1}{2}$

Thus, $\frac{B_n}{n^2} \longrightarrow 0$ as $n \longrightarrow \infty$ when $\alpha < \frac{1}{2}$

Therefore, $\{X_n\}$ holds WLLN when $\alpha < \frac{1}{2}$.