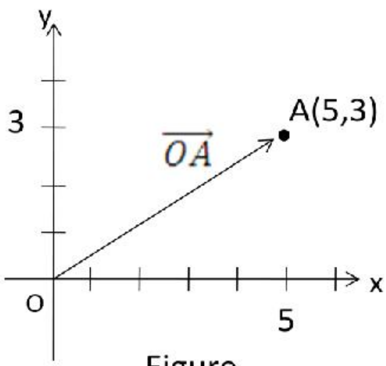
1.2

Vector Space $\mathbb{R}^n$

Introduction to vectors

The locations of points in a plane are usually discussed in terms of a coordinate system. For example, in Figure below, the location of each point in the plane can be described using a rectangular coordinate system. The point A is the point (5,3).

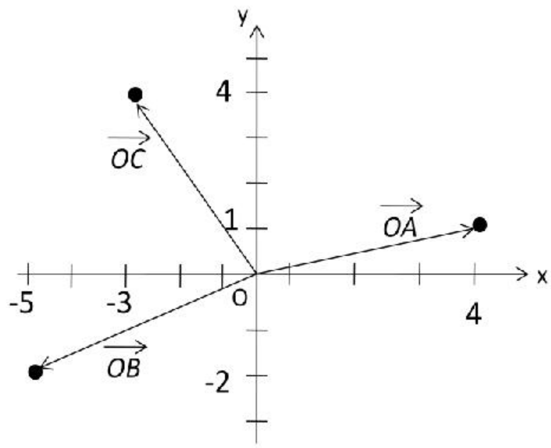


Figure

Furthermore, A is a certain distance in a certain direction from the origin (0,0). The distance and direction are characterized by the length and direction of the line segment from the origin O to A. We call such a directed line segment a **position vector** and denote it by $\overrightarrow{OA}$. O is called the initial point of $\overrightarrow{OA}$, and A is called the **terminal point**. There are thus two ways of interpreting (5,3); it defines the location of a point in a plane, and it also defines the position vector$\overrightarrow{OA}$.

EXAMPLE 1:

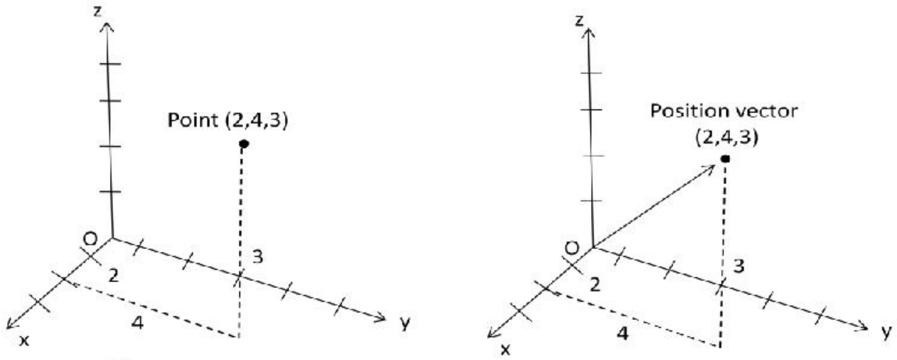
Sketch the position vectors $\overrightarrow{OA} = (4,1)$, $\overrightarrow{OB} = (-5,-2)$, and $\overrightarrow{OC} = (-3,4)$. See the figure below.



Figure

Denote the collection of all ordered pairs of real numbers by $\mathbb{R}^2$. Note the significance of "ordered" here; for example, (5,3) is not the same vector (3,5). The order is significant.

These concepts can be extended to arrays consisting of three real numbers, such as (2,4,3), which can be interpreted in two ways as the location of a point in three space relative to an *xyz* coordinate system, or as a position vector. These interpretations are illustrated in the figure below. We shall denote the set of all ordered triples of real numbers by $\mathbb{R}^3$.



Figure

We now generalize these concepts with the following definition.

DEFINITION: Let $(u_1, u_2, \ldots\ldots u_n)$ be a sequence of n real numbers. The set of all such sequences is called n-space and is denoted $\mathbb{R}^n$.

u_1 is the **first component** of $(u_1, u_2, \ldots\ldots u_n)$. u_2 is the **second component** and so on.

EXAMPLE 2: $\mathbb{R}^4$ is the collection of all sets of four ordered real numbers. For example, $(1,2,3,4)$ and $(-1, \frac{3}{4}, 0,5)$ are elements of $\mathbb{R}^4$.

$\mathbb{R}^5$ is the collection of all sets of five ordered real numbers. For example, $\left(-1,2,0,\frac{7}{8},9\right)$ is in this collection.

DEFINITION: Let $u = (u_1, \ldots\ldots u_n)$ and $v = (v_1, \ldots\ldots v_n)$ be two elements of $\mathbb{R}^n$. We say that u and v are equal if $u_1 = v_1, \ldots u_n = v_n$. Thus two elements of $\mathbb{R}^n$ are equal if their corresponding components are equal.

Let us now develop the algebraic structure for $\mathbb{R}^n$.

DEFINITION: Let $u = (u_1, \ldots\ldots u_n)$ and $v = (v_1, \ldots\ldots v_n)$ be elements of $\mathbb{R}^n$ and let c be a scalar. Addition and scalar multiplication are performed as follows.

Addition: $u + v = (u_1 + v_1, \ldots u_n + v_n)$

Scalar multiplication: $cu = (cu_1, \ldots\ldots cu_n)$

To add two elements of $\mathbb{R}^n$, we add corresponding components. To multiply an element of $\mathbb{R}^n$ by a scalar, we multiply every component by that scalar. Observe that the resulting elements are in $\mathbb{R}^n$. We say that $\mathbb{R}^n$ is closed under addition and under scalar multiplication.

$\mathbb{R}^n$ with operations of component wise addition and scalar multiplication is an example of a **vector space**, and its elements are called **vectors**.

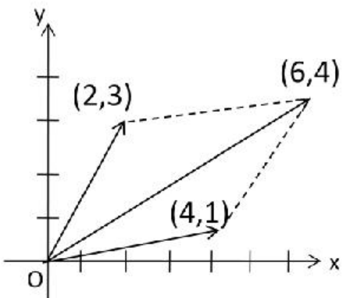
We shall henceforth in this course interpret $\mathbb{R}^n$ to be a vector space.

We now give example to illustrate geometrical interpretations of these vectors and their operations.

EXAMPLE 3: This example gives a geometrical interpretation of vector addition. Consider the sum of the vectors $(4,1)$ and $(2,3)$. We get

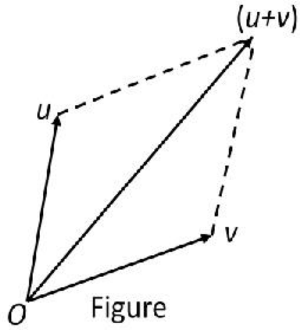
$$(4,1) + (2,3) = (6,4)$$

In the figure below, we interpret these vectors as position vectors. Construct the parallelogram having the vectors $(4,1)$ and $(2,3)$ as adjacent sides. The vector $(6,4)$, the sum, will be the diagonal of the parallelogram.



Figure

In general, if u and v are vectors in the same vector space, then $u + v$ is the diagonal of the parallelogram defined by u and v. See the figure below. This way of visualizing vector addition is useful in all vector spaces.

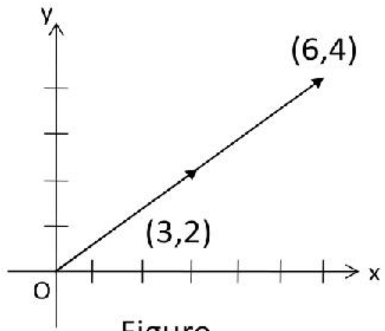


Figure

EXAMPLE 4: This example gives a geometrical interpretation of scalar multiplication. Consider the scalar multiple of the vectors (3,2) by 2. We get

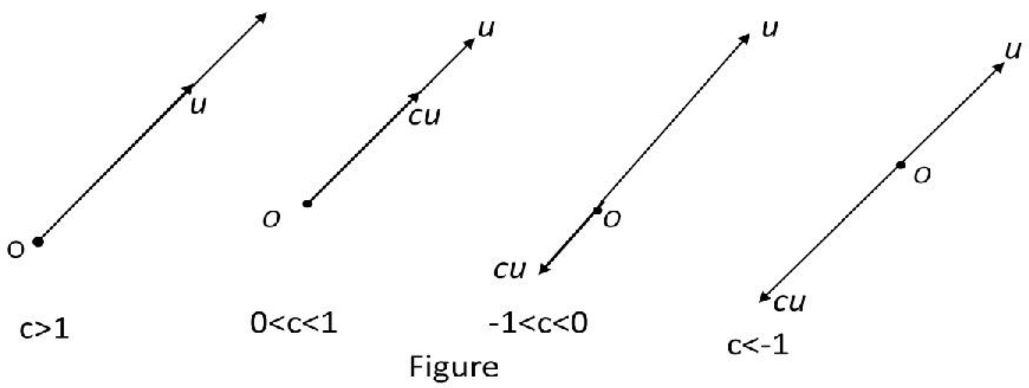
$$2(3,2) = (6,4)$$

Observe in the figure below that (6,4) is a vector in the same direction as (3,2) and 2 times it in length.



Figure

The direction will depend upon the sign of the scalar. The general result is as follows. Let u be a vector and c a scalar. The direction of cu will be the same as the direction of u if $c > 0$, the opposite direction to u if $c < 0$. The length of cu is $|c|$ times the length of u. See the figure below.



Figure

Zero vector:

The vector $(0,0,\dots,0)$ having n zero components, is called the **zero vector** of $\mathbb{R}^{\boldsymbol{n}}$ and is denoted by 0. For example, $(0,0,0)$ is zero vector of $\mathbb{R}^3$. We shall find that zero vectors play a central role in the development of vector spaces.

Negative Vector:

The vector $(-1)u$ is written $-\boldsymbol{u}$ and is called the **negative** of $\boldsymbol{u}$. It is a vector that has the same magnitude as $\boldsymbol{u}$, but lies in the opposite direction to $\boldsymbol{u}$.

Subtraction:

Subtraction is performed on elements of $\mathbb{R}^{\boldsymbol{n}}$ by subtracting corresponding components. For example, in $\mathbb{R}^3$,

$$(5,3,-6)-(2,1,3)=(3,2,-9)$$

Observe that this is equivalent to

$$(5,3,-6)+(-1)(2,1,3)=(3,2,-9)$$

Thus subtraction is not new operation on $\mathbb{R}^n$, but a combination of addition and scalar multiplication by -1. We have only two independent operations on $\mathbb{R}^n$, namely addition and scalar multiplication.

We now discuss some of the properties of vector addition and scalar multiplication. The properties are similar to those of matrices.

THEOREM:

Let $\boldsymbol{u}, \boldsymbol{v}$ and $\boldsymbol{w}$ be vectors in $\mathbb{R}^n$ and let c and d be scalars.

(a) $\boldsymbol{u} + \boldsymbol{v} = \boldsymbol{v} + \boldsymbol{u}$

(b) $\boldsymbol{u} + (\boldsymbol{v} + \boldsymbol{w}) = (\boldsymbol{u} + \boldsymbol{v}) + \boldsymbol{w}$

(c) $\boldsymbol{u} + \boldsymbol{0} = \boldsymbol{0} + \boldsymbol{u} = \boldsymbol{u}$

(d) $\boldsymbol{u} + (-\boldsymbol{u}) = \boldsymbol{0}$

(e) $\boldsymbol{c}\,(\boldsymbol{u} + \boldsymbol{v}) = \boldsymbol{cu} + \boldsymbol{cv}$

(f) $(\boldsymbol{c} + \boldsymbol{d})\,\boldsymbol{u} = \boldsymbol{cu} + \boldsymbol{du}$

(g) $\boldsymbol{c}\,(\boldsymbol{du}) = (\boldsymbol{cd})\boldsymbol{u}$

(h) $\boldsymbol{1u} = \boldsymbol{u}$

These results are verified by writing the vectors in terms of components and using the definitions of vector addition and scalar multiplication, and the properties of real numbers. We give the proofs of (a) and (e).

$\boldsymbol{u} + \boldsymbol{v} = \boldsymbol{v} + \boldsymbol{u}$:

Let $u = (u_1, \ldots\ldots u_n)$ and $v = (v_1, \ldots\ldots v_n)$ Then

$$u + v = (u_1, \ldots\ldots u_n) + (v_1, \ldots\ldots v_n)$$

$$= (u_1 + v_1, \ldots u_n + v_n)$$

$$= (v_1 + u_1, \ldots v_n + u_n)$$

$$= v + u$$

$\boldsymbol{c}(\boldsymbol{u} + \boldsymbol{v}) = \boldsymbol{cu} + \boldsymbol{cv}$:

$$c(\boldsymbol{u} + \boldsymbol{v}) = c(u_1, \ldots\ldots u_n) + (v_1, \ldots\ldots v_n)$$

$$= c((u_1 + v_1, \ldots u_n + v_n))$$

$$= \big(c(u_1 + v_1), \dots, c(u_n + v_n)\big)$$

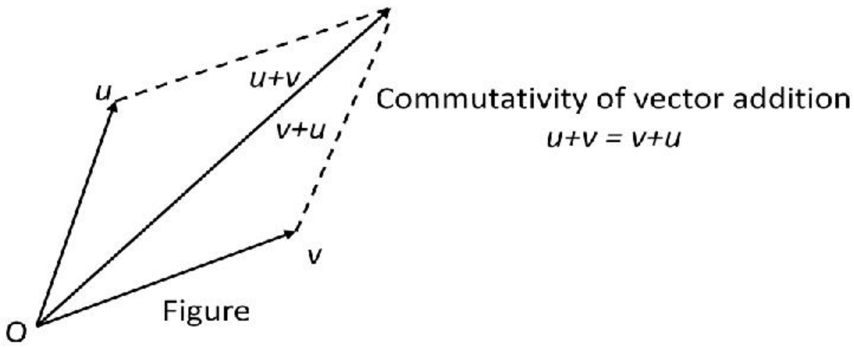
$$= (cu_1 + cv_1, \dots cu_n + cv_n)$$

$$= (cu_1, \dots\dots, cu_n) + (cv_1, \dots\dots, cv_n)$$

$$= c(u_1, \dots\dots, u_n) + c(v_1, \dots\dots, v_n)$$

$$= cu + cv$$

Some of the above properties can be illustrated geometrically. The commutative property of vector addition is illustrated in the figure below. Note that we get the same diagonal to the parallelogram whether we add the vectors in the order $u + v$ or in the order $v + u$. One implication of part (b) above is that we can write certain algebraic expressions involving vectors, without parentheses.



Figure

General Vector Space

In this section, we generalize the concept of the vector space $\mathbb{R}^n$. We examine the underlying algebraic structure of $\mathbb{R}^n$. Any set with this structure has the same mathematical properties and will be called a vector space. The results that were developed for the vector space $\mathbb{R}^n$ will also apply such sets. We will, for example, find that certain spaces of functions have the same mathematical properties as the vector space $\mathbb{R}^n$. Similarly the scalar set has the algebraic structure of the real number set and will be called a field. Precise definitions are as follows.

Definition: Let F be a set having at least two elements 0_F and 1_F $(0_F \neq 1_F)$ together two operations '$.$' (multiplication) and '$+$' (addition). A field $(F, +, .)$ is a triplet satisfying the following axioms:

For any three elements $a, b, c \in F$.

1) Addition and multiplication are closed

$a + b \in F$ and $ab \in F$.

2) Addition and multiplication are associative:

$(a + b) + c = a + (b + c)$,

$(ab)c = a(bc)$

3) Addition and multiplication are commutative:

$a + b = b + a, \ ab = ba$

4) The multiplicative operation distribute over addition:

$a\,(b + c) = ab + ac$

5) 0_F is the additive identity:

$0_F + a = a + 0_F = a$

6) 1_F is the multiplicative identity:

$$1_F\, a = a\, 1_F = a$$

7) Every element has additive inverse:

$$\exists -a \in F, a + (-a) = (-a) + a = 0_F$$

8) Every non-zero element has multiplicative inverse:

If $a \neq 0_F$, then $\exists \quad a^{-1} \in F$ such that $aa^{-1} = a^{-1}a = 1_F$

Example: $(\mathbb{Q}, +, .)$, $(\mathbb{R}, +, .)$ and $(\mathbb{C}, +, .)$ are all fields.

$(\mathbb{Z}, +, .)$ is not a field because every non-zero element except -1 and 1 has no multiplicative inverse.

Definition: A *vector space* $V(F)$ over a field 'F' is a non-empty set whose elements are called vectors, possessing two operations '$+$' (vector addition), and '$.$' (scalar multiplication) which satisfy the following axioms:

For $\vec{a}, \vec{b}$ and $\vec{c} \in V(F)$ and $\alpha, \beta \in F$.

1) Vector addition and scalar multiplication are closed:

$$\vec{a} + \vec{b} \in V(F), \alpha\vec{a} \in V(F).$$

2) Commutativity:

$$\vec{a} + \vec{b} = \vec{b} + \vec{a}$$

3) Associativity:

$$\left(\vec{a} + \vec{b}\right) + \vec{c} = \vec{a} + \left(\vec{b} + \vec{c}\right)$$

4) Existence of an additive identity:

$\exists\ \vec{0} \in V(F)$ such that $\vec{a} + \vec{0} = \vec{0} + \vec{a} = \vec{a}$

5) Existence of additive inverse:

$\exists\ -\vec{a} \in V(F)$ such that $\vec{a} + (-\vec{a}) = (-\vec{a}) + (\vec{a}) = \vec{0}$

6) Distributive Laws:

$$\alpha\left(\vec{a} + \vec{b}\right) = \alpha\vec{a} + \alpha\vec{b}$$

7) $1_F\vec{a} = \vec{a}$

8) $(\alpha\beta)\,\vec{a} = \alpha(\beta\vec{a})$.

Note: Throughout this course we use the field of real numbers as scalar set. We may refer a vector space simply

Example:

1) The set of all 2×2 matrices with entries real numbers is a vector space over the field of real numbers under usual addition and scalar multiplication of matrices.

2) The set of all functions having real numbers as their domain is a vector space over the field of real numbers under the following operations.

$$(\boldsymbol{f} + \boldsymbol{g})(\boldsymbol{x}) = \boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{g}(\boldsymbol{x})$$

$$(\boldsymbol{cf})(\boldsymbol{x}) = \boldsymbol{c}.\,[\boldsymbol{f}(\boldsymbol{x})]$$

For all functions f and g and scalar c.

We now give a theorem that contains useful properties of vectors. These are properties that were immediately apparent for $\mathbb{R}^n$ and were taken almost for granted. They are not, however, so apparent for all vector spaces.

Theorem: Let 'V' be a vector space, $\vec{v}$a vector in $V, \vec{0}$ the zero vector of V, 'c' a scalar, and 0 the zero scalar. Then

(a) $0\vec{v} = \vec{0}$

(b) $c\vec{v} = 0$

(c) $(-1)\,\vec{v} = -\vec{v}$

(d) If $c\vec{v} = 0$, then either $c = 0$ or $\vec{v} = 0$

Proof: (a) $0\vec{v} + 0\vec{v} = (0+0)\vec{v} = 0\vec{v}$

Add the negative of $0\vec{v}$ namely $-0\vec{v}$ to both sides of this equation.

$(0\vec{v} + 0\vec{v}) + (-0\vec{v}) = 0\vec{v} + (-0\vec{v})$

$\Rightarrow 0\vec{v} + \left[\left(0\vec{v} + (-0\vec{v})\right)\right] = \vec{0}$

$\Rightarrow 0\vec{v} + \vec{0} = \vec{0}$

$\Rightarrow 0\vec{v} = \vec{0}$

(b) $c\vec{0} = c\left(\vec{0} + \vec{0}\right)$

$\Rightarrow c\vec{0} + c\vec{0}$

$\Rightarrow c\vec{0} = \vec{0}$.

(c)$(-1)\,\vec{v} + \vec{v} = (-1)\,\vec{v} + 1\vec{v}$

$= [(-1) + 1]\vec{v}$

$= 0\vec{v} = 0$

Thus (-1) $\vec{v}$ is the additive inverse of $\vec{v}$.

i.e (-1) $\vec{v} = -\vec{v}$

(d) Assume that $c \neq 0_F$.

Then $\exists\ c^{-1}$ such that $c^{-1}c = 1_F$

$c\vec{v} = \vec{0} \Rightarrow c^{-1}(c\vec{v}) = c^{-1}\vec{0}$

$\Rightarrow c^{-1}(c\vec{v}) = \vec{0}$

$\Rightarrow 1_F\vec{v} = \vec{0}$

$\Rightarrow \vec{v} = \vec{0}$.

SUBSPACES

Definition: Let $V(F)$ be a vector space. A non-empty subset $U \subseteq V$ which is also a vector space under the inherited operations of V is called a vector subspace of V.

Example: $\{\vec{0}\}$ and V are trivial vector subspaces of V.

Theorem: Let $V(F)$ be a vector space. Then $U \subseteq V$, $U \neq \emptyset$ is a subspace of V if and only if for all $\alpha \in F$ and $\vec{a}, \vec{b} \in U$ it is verified that $\vec{a} + \alpha\vec{b} \in U$.

Proof: Let $V(F)$ be a vector space and let U be a non-empty subset of V. If U is a subspace of V, then it is clear that $\vec{a} + \alpha\vec{b} \in U$ for all $\alpha \in F$and $\vec{a}, \vec{b} \in U$.

Conversely, suppose that U is non-empty subset of V and for all $\alpha \in F$ and $\vec{a}, \vec{b} \in U, \vec{a} + \alpha\vec{b} \in U$.

We prove that U is a subspace of V. That is, U is a vector space under the inherited operations of V.

1) Vector addition and scalar multiplication are closed, commutative and Associative. By taking $\alpha = -1$, we get $-\vec{v} \in U$ for every $\vec{v} \in U$

So $\vec{0} = \vec{v} - \vec{v} \in U$

All other properties hold as they hold in V.

Therefore 'U' is a vector space with the same binary operations as on V and hence U is a subspace of V.

Theorem: Let $X \subseteq V$, $Y \subseteq V$ be vector subspaces of a vector space $V(F)$. Then their intersection $X \cap Y$ is also a vector subspace of V.

Proof: Follows from the above Theorem.

Problem 1: Let $u = (-1,4,3,7)$ and $v = (-2,-3,1,0)$ be vectors in $\mathbb{R}^4$. Find $u + v$ and $3u$.

Solution: We get

$$u + v = (-1,4,3,7) + (-2,-3,1,0) = (-3,1,4,7)$$

$$3u = 3(-1,4,3,7) = (-3,12,9,21)$$

Note that the resulting vector under each operation is in the original vector space $\mathbb{R}^4$.

Problem 2: Let $u = (2,5,-3), v = (-4,1,9), w = (4,0,2)$. Determine the vector $2u - 3v + w$.

Solution: $2u - 3v + w = 2(2,5,-3) - 3(-4,1,9) + (4,0,2)$

$$= (4,10,-6) - (-12,3,27) + (4,0,2)$$

$$= (4 + 12 + 4, 10 - 3 + 0, -6 - 27 + 2)$$

$$= (20,7,-31).$$

Problem 3: Let $\mathbb{C}$ denote the complex numbers and $\mathbb{R}$ denote the real numbers. Is $\mathbb{C}$ a vector space over $\mathbb{R}$ under ordinary addition and multiplication? Is $\mathbb{R}$ a vector space over $\mathbb{C}$?

Solution: $\mathbb{C}$ is a vector space over $\mathbb{R}$ but $\mathbb{R}$ is not a vector space over $\mathbb{C}$ since $\mathbb{R}$ is not closed under scalar multiplication over $\mathbb{C}$.

Problem 4: Let $V(F)$ be a vector space, and let $U_1 \subseteq V$ and $U_2 \subseteq V$ be vector subspaces. Prove that if $U_1 \cup U_2$ is a vector subspace of V, then either $U_1 \subseteq U_2$ on $U_2 \subseteq U_1$

Solution: If $U_1 \subseteq U_2$on $U_2 \subseteq U_1$ then it is trivial that $U_1 \cup U_2$ is a subspace of V.

Suppose that $U_1 \nsubseteq U_2$on $U_2 \nsubseteq U_1$

So $\exists\ u_1 \in U_1$ and $U_2 \in U_1$ consider $u_1 + u_2$

Then $u_1 + u_2$ cannot be in U_1 and U_2.

(If $u_1 + u_2 \in U_1$, then$(u_1 + u_2) - u_1 = u_2 \in U_1$)

Therefore $U_1 \cup U_2$ is not closed with respect to vector addition. Hence $U_1 \cup U_2$ is not a subspace of V.

Problem 5: Prove that $X = \{(a, b, c, d) \in a - b - 3d = 0\}$ is a vector subspace of $\mathbb{R}^4$.

Solution: $(a_1, b_1, c_1, d_1) + \alpha(a_2, b_2, c_2, d_2) = (a_1 + \alpha a_2, b_1 + \alpha b_2, c_1 + \alpha c_2, d_1 + \alpha d_2, a_1 + \alpha a_2 - b_1 + \alpha b_2 - 3(d_1 + \alpha d_2))$

$$= a_1 - b_1 - 3\lambda_1 + \alpha(a_2 - b_2 - 3\lambda_2)$$

$$= 0 + \alpha(0) = 0.$$

Problem 6: Prove that $X = \{(a, 2a - 3b, 5b, a + 2b, a): a, b \in R\}$ is a vector subspace of $\mathbb{R}^5$.

Solution:

$$(a_1, 2a_1 - 3b_1, 5b_1, a_1 + 2b_1, a_1) \\ + \alpha(a_2, 2a_2 - 3b_2, 5b_2, a_2 + 2b_2, a_2)$$

$= (a_1 + \alpha a_2, 2(a_1 + \alpha a_2) - 3(b_1 + \alpha b_2), 5(b_1 + \alpha b_2), a_1 + \alpha a_2 + 2(b_1 + \alpha b_2), a_1 + \alpha a_2)$.

Exercise

1. Compute the following vector expressions for $u = (1,2)$, $v = (4,-1)$, and $w = (-3,5)$.

(a) $u + 3v$

(b) $2u + 3v - w$

(c) $-3u + 4v - 2w$

2. Prove that the set C^n with the operations of addition and scalar multiplication defined as follow is a vector space $(u_1, \dots, u_n) + (v_1, \dots, v_n) = (u_1 + v_1, \dots, u_n + v_n)$

$$c(u_1, \dots, u_n) = (cu_1, \dots, cu_n)$$

Determine $u + v$ and cu for the following vectors and scalars in C^2.

(a) $u = (2 - i, 3 + 4i), v = (5,1 + 3i), c = 3 - 2i.$

(b) $u = (1 + 5i, -2 - 3i), v = (2i, 3 - 2i), c = 4 + i.$

3. Let W be the set of vectors of the form (a, a^2, b). Show that W is not a subspace of $\mathbb{R}^3$.

4. Prove that the set U of 2×2 diagonal matrices is a subspace of the vector space M_{22} of 2×2 matrices.

5. Let P_n denote the set of real polynomial functions of degree $\leq n$. Prove that P_n is a vector space if addition and scalar multiplication are defined on polynomials in a point wise manner.

6. Let W be the set of vectors of the form $(a, a, a + 2)$. Show that W is not a subspace of $\mathbb{R}^3$.

7. Consider the sets of vectors of the following form. Prove that the sets are subspaces of $\mathbb{R}^3$.

(a) $(a, b, 0)$

(b) $(a, 2a, b)$

(c) $(a, a + b, 3a)$

8. Are the following sets subspaces of $\mathbb{R}^3$? The set of all vectors of the form (a, b, c) where

(a) $a + b + c = 0$

(b) $ab = 0$

(c) $ab = ac$.

9. Prove that the following sets are not subspaces of $\mathbb{R}^3$. The set of all vectors of the form

(a) $(a, a + 1, b)$

(b) $(a, b, a + b - 4)$.

10. Let U be the set of all vectors of the form (a, b, c) and V be the set of all vectors of the form $(a, a + b, c)$. Show that U and V are the same set. Is this set a subspace of $\mathbb{R}^3$?

Answers

1. (a) $(13, -1)$ (b) $(17, -4)$ (c) $(19, -20)$

2. (a)$(7 - i, 4 + 7i)$ (b) $(4 - 7i, 17 + 6i)$

3. W is not a subspace.

4. It is a vector space of matrices, embedded in M_{22}

5. Subspace

6. Not a subspace

7. (a) The set is the xy plane

   (b) The set is the plane given by the equation $y = 2x$.

   (c) The set is the plane given by the equation $z = 3x$.

8. (a) Subspace (b) Not a subspace (c) Not a subspace

9. (a) If $(a, a + 1, b) = (0,0,0)$ then $a = a + 1 = b = 0. a = a + 1$ is impossible. That is, the set does not contain the zero vector.

(b) If $(a, b, a + b - 4) = (0,0,0)$, then $a = b = 0$ and $-4 = 0$ Not possible. That is, the set does not contain the zero vector.

10. Yes.