UNIT-3-FOURIER SERIES:-

→ Any signal $f(t)$ can be represented in terms of sum of sinusoids as -

$$\boxed{f(t) = a_0 + \sum_{n=-\infty}^{\infty} a_n \cos(n\omega_0 t) + \sum_{n=-\infty}^{\infty} b_n \sin(n\omega_0 t)}$$

$$\omega_0 = \frac{2\pi}{T}$$

$\Rightarrow f(t) = a_0 + a_1 \cos\omega_0 t + a_2 \cos(2\omega_0 t) + \ldots + b_1 \sin(\omega_0 t) + b_2 \sin(2\omega_0 t) + \ldots$

where, $\boxed{a_0 = \frac{1}{T}\int_0^T f(t)\,dt}$

$$\boxed{a_n = \frac{2}{T}\int_0^T f(t)\cos(n\omega_0 t)\,dt}$$

$$\boxed{b_n = \frac{2}{T}\int_0^T f(t)\sin(n\omega_0 t)\,dt}$$

Formulae

→ $\int_0^T \cos(n\omega_0 t)\,dt = \int_0^T \sin(n\omega_0 t)\,dt = 0$,

→ $\int_0^T \cos(m\omega_0 t)\cos(n\omega_0 t)\,dt = \begin{cases} T/2, & m = n \\ 0, & m \neq n. \end{cases}$

→ $\int_0^T \sin(m\omega_0 t)\sin(n\omega_0 t)\,dt = \begin{cases} T/2, & m = n \\ 0, & m \neq n. \end{cases}$

→ $\int_0^T \cos(m\omega_0 t)\sin(n\omega_0 t)\,dt = 0.$

→ cos ω₀t and sin ω₀t are Orthogonal Signals.

→ For two signals, $f_1(t)$ & $f_2(t)$ to be Orthogonal,

$$\cancel{\int_{t_1}^{\infty} f_1(t)\, f_2(t)\, dt = 0} \qquad \boxed{\int_{t_1}^{t_2} f_1(t) \cdot f_2(t)\, dt = 0}$$

Proofs:

① Let,

$$f(t) = a_0 + \sum_{n=-\infty}^{\infty} a_n \cos(n\omega_0 t) + \sum_{n=-\infty}^{\infty} b_n \sin(n\omega_0 t)$$

$$\Rightarrow f(t) = a_0 + a_1 \cos\omega_0 t + a_2 \cos(2\omega_0 t) + \ldots + b_1 \sin\omega_0 t + b_2 \sin(2\omega_0 t) \ldots$$

$$\Rightarrow \int_0^T f(t)\, dt = \int_0^T a_0\, dt + \underbrace{\int_0^T a_1 \cos\omega_0 t\, dt}_{0} + \underbrace{\ldots}_{0} + \underbrace{\int_0^T b_1 \sin\omega_0 t\, dt}_{0} + \underbrace{\ldots}_{0}$$

$$\Rightarrow \int_0^T f(t)\, dt = a_0 T$$

$$\Rightarrow \boxed{a_0 = \frac{1}{T}\int_0^T f(t)\, dt.}$$

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② $f(t) = a_0 + a_1 \cos\omega_0 t + \ldots + a_n \cos n\omega_0 t + \ldots + b_1 \sin\omega_0 t + \ldots + b_n \sin n\omega_0 t$

Multiply both sides with 'cos nω₀t'.

$$\Rightarrow f(t)\cos(n\omega_0 t) = a_0 \cos(n\omega_0 t) + a_1 \cos\omega_0 t \cos n\omega_0 t + \ldots + a_n \cos^2 n\omega_0 t$$
$$+ b_1 \sin\omega_0 t \cos n\omega_0 t + \ldots + b_n \sin n\omega_0 t \cos n\omega_0 t.$$

Integrating on Both Sides,

$$\int_0^T f(t)\cos(n\omega_0 t)\, dt = \overbrace{\int_0^T a_0 \cos(n\omega_0 t)}^{=0} + \overbrace{\int_0^T a_1 \cos\omega_0 t \cos n\omega_0 t\, dt}^{=0} + \ldots^{=0=0}$$
$$+ \overbrace{\int_0^T a_n \cos^2 n\omega_0 t\, dt}^{\neq 0} + \ldots + \overbrace{\int_0^T b_1 \sin\omega_0 t \cos n\omega_0 t\, dt}^{=0} + \ldots + \overbrace{\int_0^T b_n \sin n\omega_0 t \cos n\omega_0 t}^{=0}$$

$$\Rightarrow \int_0^T f(t)\cos n\omega_0 t\,dt = \int_0^T a_n \cos^2 n\omega_0 t\,dt$$

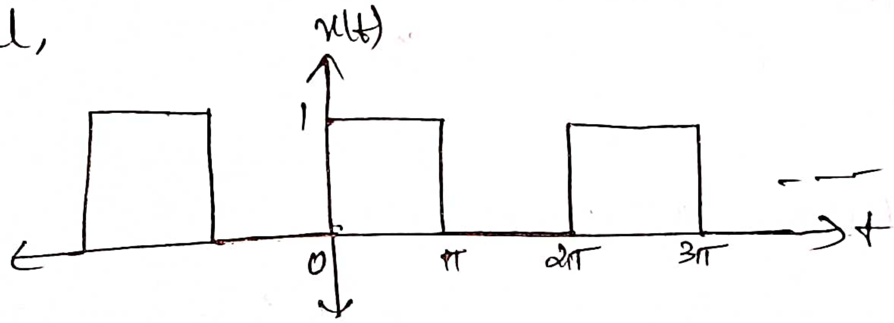
$$\int_0^T f(t)\cos n\omega_0 t\,dt = a_n(T/2)$$

$$\Rightarrow \boxed{a_n = \frac{2}{T}\int_0^T f(t)\cos(n\omega_0 t)\,dt}$$

③ Similarly, multiplying $f(t)$ with $\sin(n\omega_0 t)$, and the integrating gives us,

$$\boxed{b_n = \frac{2}{T}\int_0^T f(t)\sin(n\omega_0 t)\,dt}$$

① Find the trigonometric fourier series of the following signal,



Sol's

$$a_0 = \frac{1}{T}\int_0^T x(t)\,dt \quad \text{, Here, } T = 2\pi.$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x(t)\,dt$$

$$= \frac{1}{2\pi}\int_0^{\pi} 1\,dt + \frac{1}{2\pi}\int_{\pi}^{2\pi} 0\cdot dt$$

$$= \frac{1}{2\pi}(\pi - 0) + 0$$

$$\boxed{a_0 = \frac{1}{2}}$$

$$\Rightarrow a_n = \frac{2}{T}\int_0^T x(t)\cos(n\omega_0 t)\,dt$$

$$= \frac{2}{2\pi}\int_0^{\pi}\cos(n\omega_0 t)\,dt + \underbrace{\frac{2}{2\pi}\int_{\pi}^{2\pi}\cos(n\omega_0 t)\,dt}_{=0}$$

$$= \frac{1}{\pi}\left[\frac{\sin n\omega_0 t}{n\omega_0}\right]_0^{\pi} \cancel{+\frac{1}{\pi}\left[\frac{\sin(n\omega_0 t)}{n\omega_0}\right]_{\pi}^{2\pi}}$$

$$\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{2\pi}$$

$$\cancel{= \frac{1}{\pi}\cdot\frac{1}{n\omega_0}[0] + \frac{1}{\pi}[0]}$$

$$a_n = \frac{1}{\pi}\left[\frac{\sin(n\omega_0 t)}{n\omega_0}\right]_0^{\pi}$$

$$\Rightarrow a_n = \frac{1}{\pi}\left(\frac{\sin(n\pi\omega_0)}{n\omega_0}\right) = \frac{1}{\pi}\left(\frac{\sin(n\pi)}{n}\right)$$

$$\Rightarrow a_n = \frac{\sin(n\pi)}{n\pi} = 0.$$

$$\Rightarrow \boxed{a_n = 0}$$

$$\Rightarrow b_n = \frac{2}{T}\int_0^T x(t)\sin(n\omega_0 t)\,dt$$

$$\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{2\pi} = 1.$$

$$= \frac{1}{\pi}\int_0^{\pi}\sin(n\omega_0 t)\,dt$$

$$= \frac{1}{\pi}\left[-\frac{\cos(n\omega_0 t)}{n\omega_0}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{\cos(0)}{n} - \frac{\cos(n\pi)}{n}\right] = \frac{1}{\pi}\left[\frac{1}{n} - \frac{\cos(n\pi)}{n}\right]$$

$\Rightarrow b_n = \frac{1}{n\pi}\left[1 - \cos(n\pi)\right]$

If n is even, $\boxed{b_n = 0}$

If n is odd, $\boxed{b_n = \frac{2}{n\pi}}$

$$\boxed{b_n = \begin{cases} 0, & n \text{ is even} \\ \frac{2}{n\pi}, & n \text{ is odd.} \end{cases}}$$

$$\Rightarrow \boxed{x(t) = \frac{1}{2} + \frac{2}{\pi}\cos(\omega_0 t) + \frac{2}{3\pi}\cos(3\omega_0 t) + \frac{2}{5\pi}\cos(5\omega_0 t) + \dots}$$

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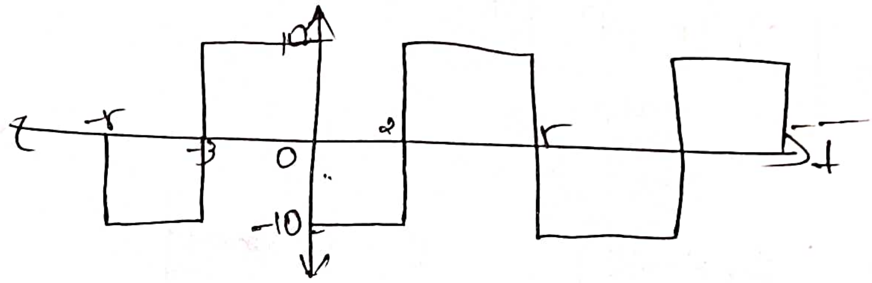
(2)



Sol:- $T = 5$

$$a_0 = \frac{1}{T}\int_0^T x(t)\,dt$$

$$= \frac{1}{5}\int_0^2 (-10)\,dt + \frac{1}{5}\int_2^5 (10)\,dt$$

$$= \frac{1}{5}(-20+0) + \frac{1}{5}(50-20)$$

$$= \cancel{\frac{-20}{5} + \frac{30}{5}} = -4 + 6$$

$$\Rightarrow \boxed{a_0 = 2}$$

$$a_n = \frac{2}{T}\int_0^T x(t)\cos(n\omega_0 t)\,dt \qquad \omega_0 = \frac{2\pi}{T} = \frac{2\pi}{5}$$

$$= \frac{2}{5}\int_0^5 x(t)\cos\left(\frac{2n\pi}{5}t\right)dt$$

$$= \frac{2}{5}\left[\int_0^2 -10\cos\left(\frac{2n\pi t}{5}\right)dt + \int_2^5 10\cos\left(\frac{2n\pi t}{5}\right)dt\right]$$

$$= \frac{2}{5}\left[\frac{-10}{\frac{2n\pi}{5}}\left[\sin\left(\frac{2n\pi t}{5}\right)\right]_0^2 + 10\times\frac{5}{2n\pi}\left[\sin\left(\frac{2n\pi t}{5}\right)\right]_2^5\right]$$

$$\cancel{= \frac{2}{5}\left[-\frac{25}{n\pi}\left[\sin\right] + \frac{25}{n\pi}\left[0\right]\right]}$$

$$\cancel{a_n = 0} \quad = \frac{2}{5}\left[-\frac{25}{n\pi}\sin\left(\frac{4n\pi}{5}\right) - \frac{25}{n\pi}\sin\left(\frac{4n\pi}{5}\right)\right]$$

$$= \frac{2}{5}\left[-\frac{50}{n\pi}\sin\left(\frac{4n\pi}{5}\right)\right]$$

$$\boxed{a_n = -\frac{20}{n\pi}\sin\left(\frac{4n\pi}{5}\right)}$$

$$b_n = \frac{2}{T}\int_0^T x(t)\sin(n\omega_0 t)\,dt \qquad \omega_0 = \frac{2\pi}{5}$$

$$= \frac{2}{5}\left[-10\int_0^2 \sin\left(\frac{2n\pi}{5}t\right)dt + 10\int_2^5 \sin\left(\frac{2n\pi}{5}t\right)dt\right]$$

$$= \frac{2}{5}\left[10\times\frac{5}{2n\pi}\left[\cos\left(\frac{4n\pi}{5}\right) - \cos 0\right] - 10\times\frac{5}{2n\pi}\left[1 - \cos\left(\frac{4n\pi}{5}\right)\right]\right]$$

$$= \frac{2}{5}\left[\frac{50}{n\pi}\cos\left(\frac{4n\pi}{5}\right)\right] = \boxed{\frac{20}{n\pi}\cos\left(\frac{4n\pi}{5}\right)}$$

$$\Rightarrow x(t) = 2 + \sum_{n=-\infty}^{\infty} \frac{-20}{n\pi}\sin\left(\frac{4n\pi}{5}\right)\cos(n\omega_0 t) + \sum_{n=-\infty}^{\infty} \frac{20}{n\pi}\left(\cos\left(\frac{4n\pi}{5}\right)\right)\sin(n\omega_0 t)$$

Even Symmetry:-

→ For a signal to be having even symmetry,

$\boxed{f(t) = f(-t)}$; here, $\boxed{b_n = 0}$; $a_n \neq 0$

Odd Symmetry:-

→ For a signal to be having odd symmetry,

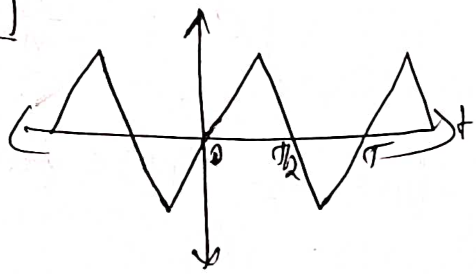
$\boxed{f(t) = -f(-t)}$;

→ Here, $b_n \neq 0$, $\boxed{a_n = 0}$

Half-Wave Symmetry:-

→ A signal is Half-Wave Symmetric, if,

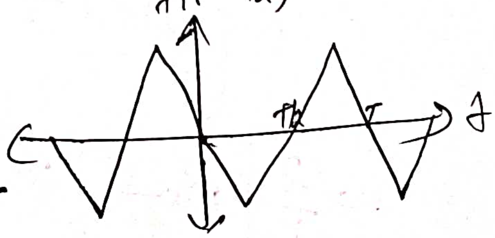
$\boxed{f(t) = -f(t - T/2)}$ Ex: $f(t) \rightarrow$



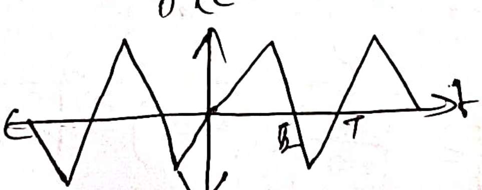
→ Here, $a_n = \begin{cases} 0, & n\text{-even} \\ \neq 0, & n\text{-odd} \end{cases}$

$b_n = \begin{cases} 0, & n\text{-even} \\ \neq 0, & n\text{-odd.} \end{cases}$

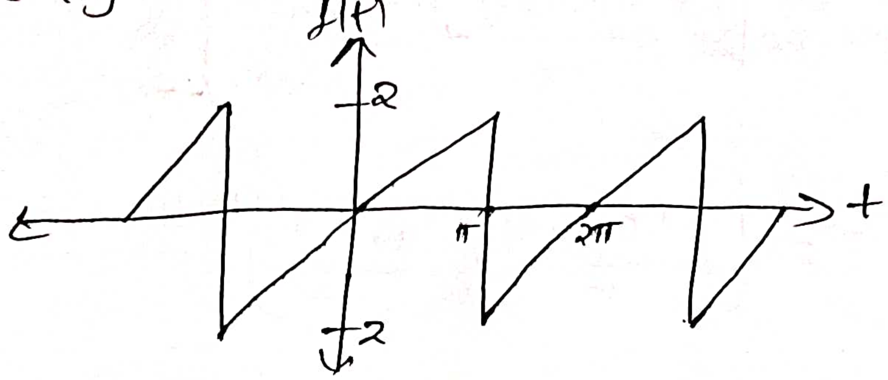
$f(t - T/2)$



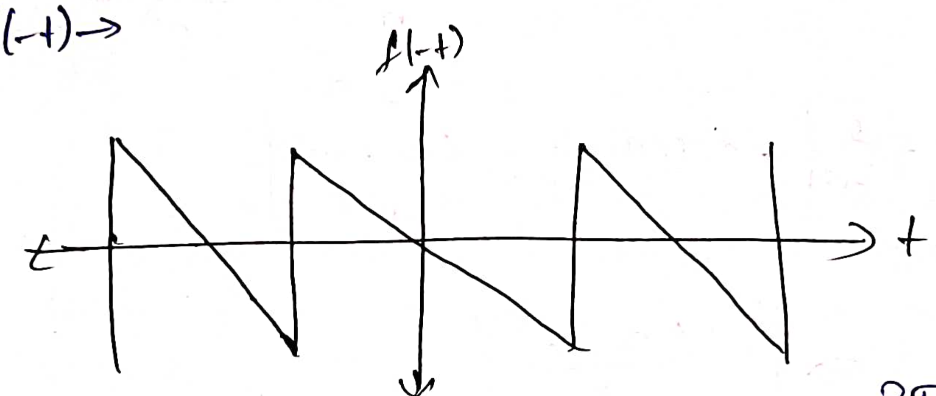
$-f(t - T/2)$



③ find the trigonometric fourier series of the following signal.



Sol: $f(-t) \rightarrow$



Clearly, it is odd symmetry $\qquad \omega_0 = \frac{2\pi}{2\pi} = 1$

$\Rightarrow a_n = 0.$

$$\Rightarrow a_0 = \frac{1}{T}\int_0^T f(t)\,dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,dt$$

$$\cancel{= \frac{1}{2\pi}\left[\int_0^{\pi}\frac{2t}{\pi}dt + \int_{\pi}^{2\pi}\frac{2t}{\pi}dt\right]}$$

$$\cancel{= \frac{1}{2\pi}\left[\frac{2}{\pi}[\pi] + \frac{2}{\pi}[\pi]\right] = \frac{1}{2\pi}\times 4 = \frac{2}{\pi}}$$

$$\cancel{\Rightarrow a_0 = 2/\pi}$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{2t}{\pi}dt$$

$$= \frac{1}{\pi^2}\left[\frac{t^2}{2}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi^2}\cdot\left(\frac{1}{2}\right)\left[\pi^2 - \pi^2\right]$$

$$\Rightarrow a_0 = 0.$$

$$\Rightarrow b_n = \frac{2}{T}\int_0^T f(t)\sin(n\omega_0 t)\,dt$$

$u = t,\ du = dt$
$dv = \sin(n\omega_0 t)\,dt$
$v = -\frac{\cos(n\omega_0 t)}{n\omega_0}$

$$= \frac{2}{2\pi}\int_{-\pi}^{\pi}\frac{2t}{\pi}\sin(n\omega_0 t)\,dt = \frac{2}{\pi^2}\int_{-\pi}^{\pi} t\sin(n\omega_0 t)\,dt$$

$$= \frac{2}{\pi^2}\left[-\frac{t}{n\omega_0}\cos(n\omega_0 t) - \int_{-\pi}^{\pi}\frac{-\cos(n\omega_0 t)}{n\omega_0}\,dt\right]$$

$$\Rightarrow b_n = \frac{2}{\pi^2}\left[\frac{-t\cos(n\omega_0 t)}{n\omega_0} + \frac{\sin(n\omega_0 t)}{n^2\omega_0^2}\right]_{-\pi}^{\pi}$$

$$= \frac{2}{\pi^2}\left[\frac{-t\cos(nt)}{n} + \frac{\sin(nt)}{n^2}\right]_{-\pi}^{\pi}$$

$$= \frac{2}{\pi^2}\left[\frac{-\pi\cos(n\pi)}{n} - \frac{\pi\cos(n\pi)}{n} + \frac{\sin(n\pi)}{n^2} - \frac{\sin(-n\pi)}{n^2}\right]$$

$$= \frac{2}{\pi^2}\left[\frac{-2\pi\cos(n\pi)}{n} + \frac{\overset{=0}{2\sin(n\pi)}}{n^2}\right]$$

$$= \frac{2}{\pi^2}\left[\frac{-2\pi\cos(n\pi)}{n}\right]$$

$$\Rightarrow b_n = -\frac{4}{n\pi}\cos(n\pi)$$

$$\Rightarrow b_n = -\frac{4}{n\pi}(-1)^n = \begin{cases} -4/n\pi, & n \text{ is even} \\ 4/n\pi, & n \text{ is odd.} \end{cases}$$

$$\Rightarrow \boxed{f(t) = \sum_{n=-\infty}^{\infty} -\frac{4}{n\pi}(-1)^n \sin(n\omega_0 t)}$$

Complex-Exponential Fourier Series:-

$\rightarrow$ Consider,

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t) + \sum_{n=1}^{\infty} b_n \sin(n\omega_0 t)$$

We know that, $$\cos(n\omega_0 t) = \frac{e^{jn\omega_0 t} + e^{-jn\omega_0 t}}{2}$$

and $$\sin(n\omega_0 t) = \frac{e^{jn\omega_0 t} - e^{-jn\omega_0 t}}{2j}$$

$$\Rightarrow f(t) = a_0 + \sum_{n=1}^{\infty} a_n \left(\frac{e^{jn\omega_0 t} + e^{-jn\omega_0 t}}{2} \right) + \sum_{n=1}^{\infty} b_n \left(\frac{e^{jn\omega_0 t} - e^{-jn\omega_0 t}}{2} \right)$$

$$f(t) = a_0 + \sum_{n=1}^{\infty} \left(\frac{a_n - jb_n}{2} \right) e^{jn\omega_0 t} + \sum_{n=1}^{\infty} \left(\frac{a_n + jb_n}{2} \right) e^{-jn\omega_0 t}$$

Let, $\boxed{\frac{a_n - jb_n}{2} = C_n}$

$$\Rightarrow \boxed{C_{-n} = \frac{a_n + jb_n}{2}}$$

Also, $\boxed{C_0 = a_0}$

$$\Rightarrow f(t) = C_0 + \sum_{n=1}^{\infty} C_n e^{jn\omega_0 t} + \sum_{n=1}^{\infty} C_{-n} e^{-jn\omega_0 t}$$

$$\Rightarrow \boxed{f(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}}$$

This is the complex-exponential fourier series,

where, $\boxed{C_n = \frac{1}{T} \int_{0}^{T} f(t)\, e^{-jn\omega_0 t}\, dt}$

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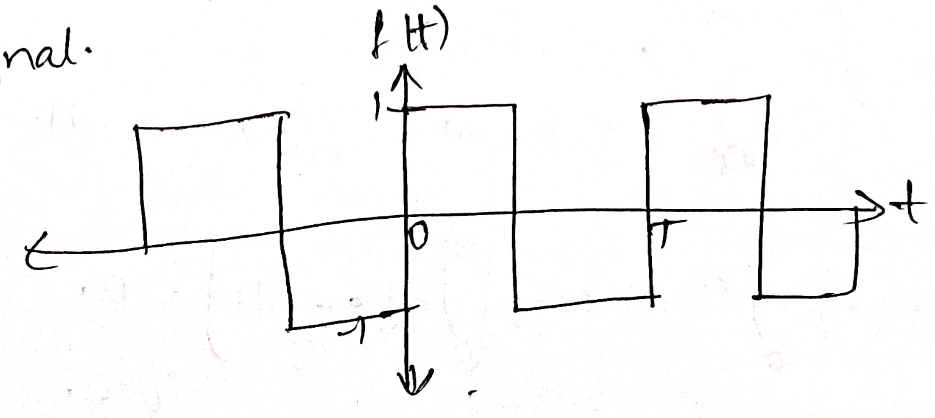
(1) find the complex-exponential fourier series for the following signal.



Soln $C_n = \frac{1}{T}\int_{0}^{T} f(t)\, e^{-jn\omega_0 t}\, dt$ $\qquad \omega_0 = \frac{2\pi}{T}$

$$= \frac{1}{T}\left[\int_0^{T/2} e^{-jn\omega_0 t}\, dt - \int_{T/2}^{T} e^{-jn\omega_0 t}\, dt\right]$$

$\omega_0 \times T/2 = \frac{2\pi}{T}\times\frac{T}{2} = \pi$

$$= \frac{1}{T}\left[\left[\frac{1 - e^{-jn\pi}}{jn\omega_0}\right] - \left[\frac{e^{-jn\pi} - e^{-j2n\pi}}{jn\omega_0}\right]\right]$$

$$= \frac{1}{T}\left[\frac{1}{jn\omega_0} - \frac{e^{-jn\pi}}{jn\omega_0} - \frac{e^{-jn\pi}}{jn\omega_0} + \frac{e^{-j2n\pi}}{jn\omega_0}\right]$$

$$= \frac{1}{T}\left[\frac{2}{jn\omega_0} - \frac{2e^{-jn\pi}}{jn\omega_0}\right]$$

$$= \frac{1}{T}\left[\frac{T}{jn\pi} - \frac{T\cos n\pi}{jn\pi}\right]$$

$$= \frac{1}{jn\pi}\left[1 - \cos n\pi\right]$$

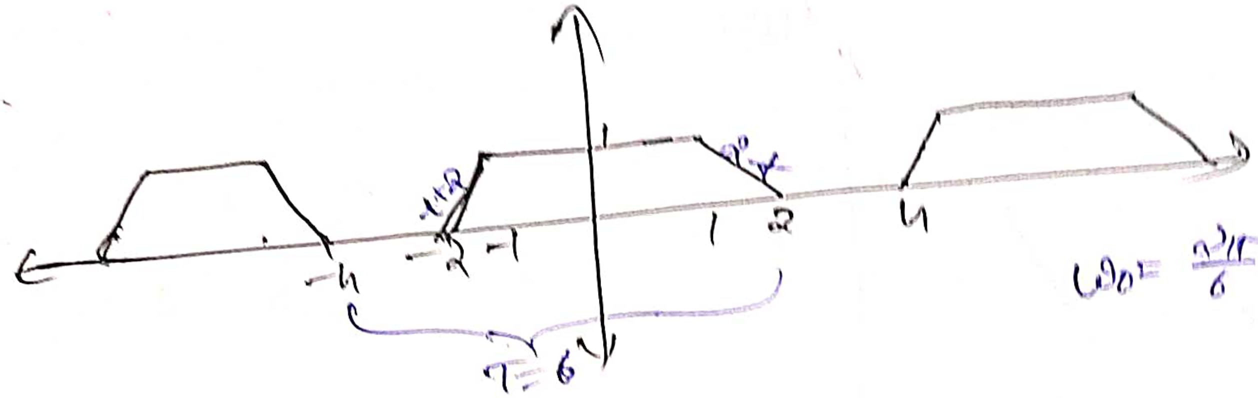
$$C_n = \frac{j}{n\pi}\left[(-1)^n - 1\right]$$

$$\Rightarrow C_n = \begin{cases} 0, & n \text{ is even} \\ \cancel{\frac{-2j}{n\pi}, n \text{ is odd}}\ \frac{-2j}{n\pi}, & n \text{ is odd.} \end{cases}$$

$$\Rightarrow C_0 = a_0 = \frac{1}{T}\int_0^T f(t)\,dt = \frac{1}{T}\left[\int_0^{T/2} dt - \int_{T/2}^{T} dt\right] = 0.$$

$\Rightarrow C_0 = 0.$

(2) Find the complex exponential fourier series of:-



$\omega_0 = \frac{2\pi}{6} = \pi/3.$

Sol^n: $C_n = \frac{1}{T}\int_0^T f(t)\, e^{-jn\omega_0 t}\, dt$

$$= \frac{1}{6}\left[\int_{-2}^{-1}(t+2)e^{-jn\omega_0 t}dt + \int_{-1}^{1} e^{-jn\omega_0 t}dt + \int_{1}^{2}(2-t)e^{-jn\omega_0 t}dt\right]$$

$$= \frac{1}{6}\left[\left(\frac{te^{-jn\omega_0 t}}{-jn\omega_0} - \frac{e^{-jn\omega_0 t}}{j^2n^2\omega_0^2} + \frac{2e^{-jn\omega_0 t}}{-jn\omega_0}\right)_{-2}^{-1} + \left(\frac{-e^{-jn\omega_0 t}}{jn\omega_0}\right)_{-1}^{1}\right.$$

$$\left. + \left(\frac{2e^{-jn\omega_0 t}}{-jn\omega_0} - \frac{te^{-jn\omega_0 t}}{-jn\omega_0} + \frac{e^{-jn\omega_0 t}}{j^2n^2\omega_0^2}\right)_{1}^{2}\right]$$

$$= \frac{1}{6}\left[\cancel{\frac{e^{jn\omega_0}}{jn\omega_0}} \mp \cancel{\frac{2e^{j2n\omega_0}}{jn\omega_0}} - \frac{e^{jn\omega_0}}{j^2n^2\omega_0^2} + \frac{e^{j2n\omega_0}}{j^2n^2\omega_0^2} \mp \cancel{\frac{2e^{jn\omega_0}}{jn\omega_0}} + \cancel{\frac{2e^{j2n\omega_0}}{jn\omega_0}} \right.$$

$$- \cancel{\frac{e^{-jn\omega_0}}{jn\omega_0}} + \cancel{\frac{e^{jn\omega_0}}{jn\omega_0}} - \cancel{\frac{2e^{-j2n\omega_0}}{jn\omega_0}} + \cancel{\frac{2e^{-jn\omega_0}}{jn\omega_0}} + \cancel{\frac{2e^{-j2n\omega_0}}{jn\omega_0}}$$

$$\left. - \cancel{\frac{e^{-jn\omega_0}}{jn\omega_0}} + \frac{e^{-j2n\omega_0}}{j^2n^2\omega_0^2} - \frac{e^{-jn\omega_0}}{j^2n^2\omega_0^2} \right]$$

$$= \frac{1}{6}\left[\cancel{\frac{4e^{jn\omega_0}}{jn\omega_0}} - \cancel{\frac{4e^{j2n\omega_0}}{jn\omega_0}} - \frac{e^{jn\omega_0}}{j^2n^2\omega_0^2} + \frac{e^{j2n\omega_0}}{j^2n^2\omega_0^2} + \frac{e^{-j2n\omega_0}}{j^2n^2\omega_0^2} - \frac{e^{-jn\omega_0}}{j^2n^2\omega_0^2} \right]$$

$$= \frac{1}{6\left(j^2 n^2 \pi^2/9\right)}\left[e^{j2n\pi/3} + e^{-j2n\pi/3} - e^{jn\pi/3} - e^{-jn\pi/3} \right]$$

$$= \frac{3}{j^2 2n^2\pi^2}\left[2\cos(2n\pi/3) - 2\cos(n\pi/3)\right]$$

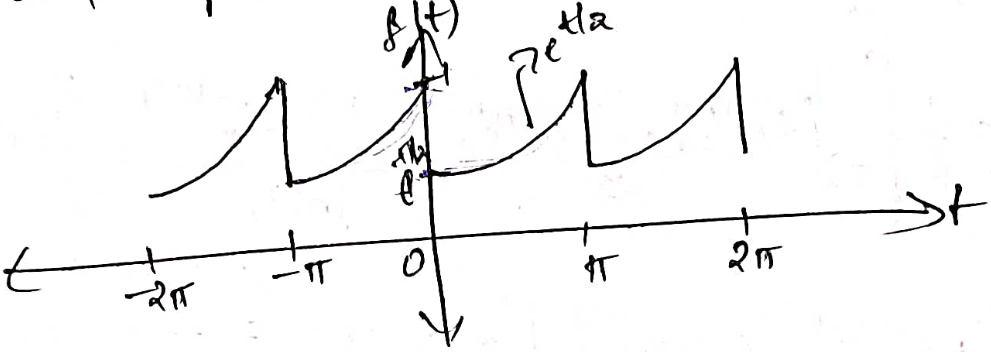
$$\Rightarrow C_n = \frac{3}{j^2 n^2\pi^2}\left[\cos(2n\pi/3) - \cos(n\pi/3)\right]$$

$$\Rightarrow C_n = \frac{3}{n^2\pi^2}\left[\cos(n\pi/3) - \cos(2n\pi/3)\right]$$

(3) Find the complex-exp. fourier series of the signal-

$T = \pi$

$\omega_0 = \frac{2\pi}{\pi} = 2.$



Sol:

$$C_n = \frac{1}{T}\int_0^T f(t)\, e^{-jn\omega_0 t}\, dt$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} e^{t/2} e^{-j2nt}\, dt$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} e^{t\left(\frac{1}{2}-j2n\right)}\, dt$$

$$= \frac{1}{\pi}\left(\frac{2}{1-j4n}\right)\left[e^{t(1/2-j2n)}\right]_{-\pi}^{0} = \frac{2}{\pi(1-j4n)}\left[-e^{-\pi/2+j2\pi n} + 1\right]$$

$$= \frac{2}{\pi(1-4jn)}\left[-e^{-\pi/2}\left(\cos(2\pi n)+j\sin(2\pi n)\right)+1\right]$$

$$= \frac{2}{\pi(1-4jn)}\left[-e^{-\pi/2}[1]+1\right] = \frac{2}{\pi(1-4jn)}\left(-e^{-\pi/2}+1\right)$$

$$= \frac{2(1+4jn)}{\pi(1+16n^2)}\left(-e^{-\pi/2}+1\right)$$

$$\Rightarrow C_n = \frac{2}{\pi(1-4jn)}\left(-e^{-\pi/2}+1\right) = \frac{2}{\pi(1-4jn)}\left(1-e^{-\pi/2}\right)$$

$$C_0 = \frac{1}{T}\int_0^T f(t)\,dt$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} e^{t/2}\,dt$$

$$= \frac{2}{\pi}\left[e^{t/2}\right]_{-\pi}^{0} = \frac{2}{\pi}\left(1-e^{-\pi/2}\right)$$

$$\Rightarrow \boxed{C_0 = \frac{2}{\pi}\left(1-e^{-\pi/2}\right)}; \quad \boxed{C_n = \frac{2}{\pi(1-4jn)}\left(1-e^{-\pi/2}\right)}$$

Convergence of Fourier Series:- (Dirichlet Conditions):-

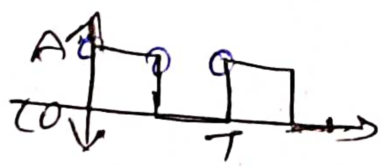
① Signal must be absolutely integrable over the range of time period,

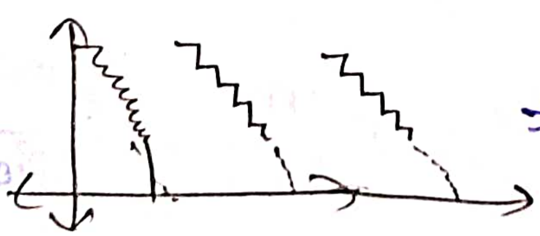
i.e; $\int_0^T |x(t)|\,dt < \infty$.

→ Eg: 'tant' is periodic signal, but can't be absolutely integrable, since it has infinite values over its time period.

→ Eg:  ⇒ Absolutely integrable.

② → Signals must have finite no. of discontinuities, over the range of time period.

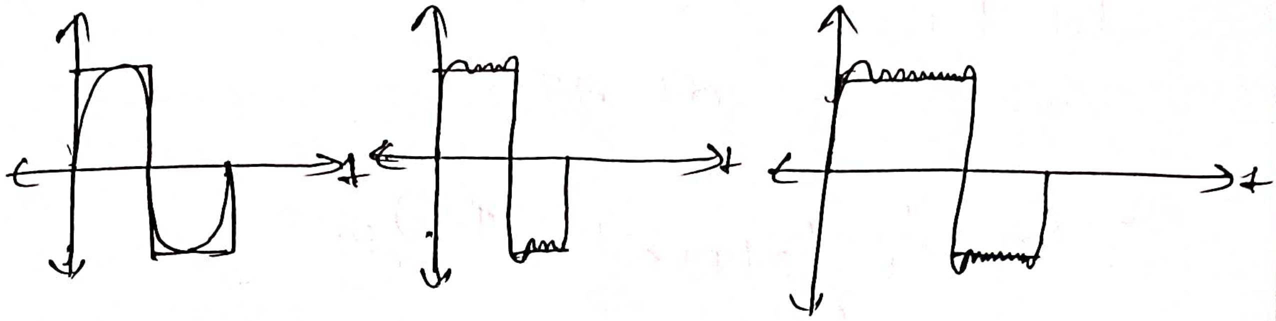
→ Eg:-  ⇒ Limited (3)

 ⇒ Unlimited.

→ Gibbs Phenomenon:-

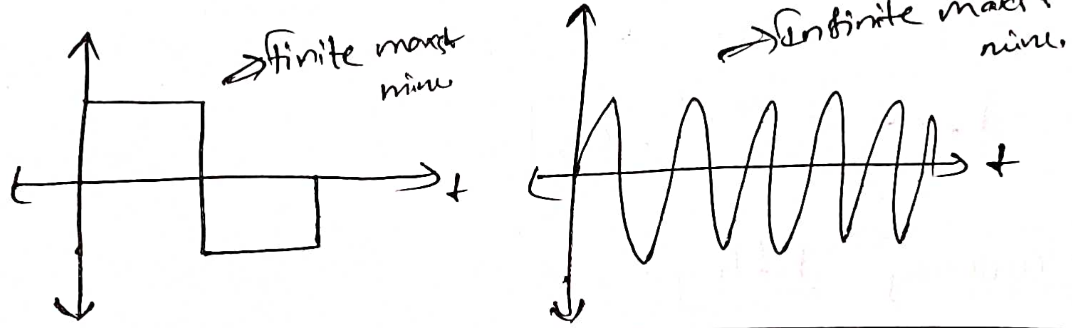
At a discontinuity, the Fourier Series representation of the function will overshoot 9% of its values. It'll never disappear even in the limit of infinite no. of terms.

→ n=1 n=3 n=32



③ Signal must have finite no. of maximas and minimas.

→ Ex:



Properties of Fourier Series :-

① **Linearity:-**

→ Consider two fourier Series,

$x(t) \xrightarrow{F.S.} C_n$ [x(t) fourier series coefficient is 'C_n']

$y(t) \xrightarrow{F.S.} D_n$

$$\Rightarrow \boxed{\alpha x(t) + \beta y(t) \xrightarrow{F.S.} \alpha C_n + \beta D_n}$$

② **Time Shifting:-**

→ $x(t) \xrightarrow{F.S.} C_n$

$$\rightarrow \boxed{x(t-t_0) \xrightarrow{F.S.} e^{-jn\omega_0 t_0} \cdot C_n}$$

→ Proof:- W.K.T., $C_n = \frac{1}{T} \int_0^T f(t) e^{-jn\omega_0 t} dt$

$\Rightarrow C_{n,t_0} = \frac{1}{T}\int_0^T x(t-t_0)\, e^{-jn\omega_0 t}\, dt$

Let $t - t_0 = \tau \Rightarrow t = \tau + t_0$

$dt = d\tau$

$\Rightarrow C_{n,t_0} = \frac{1}{T}\int_0^T x(\tau)\, e^{-jn\omega_0(\tau+t_0)}\, d\tau$

$= \frac{e^{-jn\omega_0(t_0)}}{T}\int_0^T x(\tau)\, e^{-jn\omega_0\tau}\, d\tau$

$\Rightarrow \boxed{C_{n,t_0} = e^{-jn\omega_0 t_0} \cdot C_n}$

③ Frequency Shifting:-

$\rightarrow \quad x(t) \xrightarrow{F.S.} C_n$

$\Rightarrow \boxed{e^{jm\omega_0 t}\, x(t) \xrightarrow{F.S.} C_{n-m}}$

$\rightarrow$ Proof:- $C_n = \frac{1}{T}\int_0^T e^{jm\omega_0 t}\, x(t)\, e^{-jn\omega_0 t}\, dt$

$= \frac{1}{T}\int_0^T x(t)\, e^{-j(n-m)\omega_0 t}\, dt$

$= C_{n-m}$

④ Differentiation property:-

$\rightarrow \quad x(t) \xrightarrow{F.S.} C_n$

$\Rightarrow \boxed{\frac{d}{dt}x(t) \xrightarrow{F.S.} (jn\omega_0)\, C_n}$

→ **Proof:** W.K.T,

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}$$

$$\Rightarrow \frac{d}{dt}x(t) = \sum_{n=-\infty}^{\infty} c_n \frac{d}{dt} e^{jn\omega_0 t}$$

$$\Rightarrow \frac{d}{dt}x(t) = \sum_{n=-\infty}^{\infty} c_n (jn\omega_0) e^{jn\omega_0 t}.$$

$$\Rightarrow \frac{d}{dt}(x(t)) \xrightarrow{f.s.} (jn\omega_0)\, c_n.$$

---

① Let $x(t)$ be a periodic signal with period 'T' and fourier series coefficient is 'c_n'. And $y(t) = x(t-t_0) + x(t+t_0)$ having fourier series cofficient 'd_n', where $d_n = 0 \ \forall$ n is odd. Find the value of 't_0'.

Sol: Given

$$x(t) \xrightarrow{f.s} c_n$$

$$y(t) = x(t-t_0) + x(t+t_0)$$

$$\Rightarrow y(t) \xrightarrow{f.s.} e^{-jn\omega_0 t_0} c_n + e^{jn\omega_0 t_0} c_n$$

$$\Rightarrow y(t) \xrightarrow{f.s} c_n\left(e^{-jn\omega_0 t_0} + e^{jn\omega_0 t_0}\right)$$

$$\omega_0 = \frac{2\pi}{T}$$

$$\Rightarrow d_n = c_n\left(e^{-jn\omega_0 t_0} + e^{jn\omega_0 t_0}\right)$$

$$\Rightarrow d_n = c_n\left(e^{-jn\frac{2\pi}{T}t_0} + e^{jn\frac{2\pi}{T}t_0}\right)$$

$$\therefore \boxed{t_0 = T/4}$$

When $\Rightarrow d_n = c_n\left(2\cos\left(\frac{2n\pi}{T}t_0\right)\right)$

When $t_0 = T/4$, $d_n = 2c_n \cos\left(\frac{2n\pi}{T}(T/4)\right) = 2c_n \cos(n\pi/2) = 0.$

(5) Time Reversal Property:-

$\rightarrow$ $x(t) \xrightarrow{f.s} C_n$

$\Rightarrow$ $\boxed{x(-t) \xrightarrow{f.s.} C_{-n}}$

$\rightarrow$ Proof:- $C_n = \frac{1}{T}\int_0^T x(-t)\, e^{-jn\omega_0 t}\, dt$

$\Rightarrow$ for $x(-t)$,

$$C_n = \frac{1}{T}\int_{-T/2}^{T/2} x(-t)\, e^{-jn\omega_0 t}\, dt$$

Let $-t = \tau \Rightarrow dt = -d\tau$

$\Rightarrow$ If $t = -T/2$, $\tau = T/2$,
$t = T/2$, $\tau = -T/2$.

$\Rightarrow$ $$C_n = \frac{1}{T}\int_{T/2}^{-T/2} x(\tau)\, e^{+jn\omega_0 \tau}\, -d\tau$$

$$= \frac{1}{T}\int_{-T/2}^{T/2} x(\tau)\, e^{jn\omega_0 \tau}\, d\tau$$

$$= \frac{1}{T}\int_{-T/2}^{T/2} x(\tau)\, e^{-j(-n)\omega_0 \tau}\, d\tau$$

$= C_{-n}$.

(6) Conjugation Property:-

$\rightarrow$ $x(t) \xrightarrow{f.s.} C_n$

$\Rightarrow$ $\boxed{x^*(t) \xrightarrow{f.s.} C^*_{-n}}$

→ **Proof:** $C_n = \frac{1}{T}\int_0^T x(t)e^{-jn\omega_0 t}\,dt$

For $x^*(t)$,

$$\cancel{C_n = \frac{1}{T}\int_0^T x^*(t)e^{-jn\omega_0 t}\,dt}$$

$$C_n = \frac{1}{T}\int_0^T x^*(t)\,e^{-j(-n)\omega_0 t}\,dt$$

$$\Rightarrow C^*_{-n} = \frac{1}{T}\int_0^T x^*(t)\,e^{-jn\omega_0 t}\,dt$$

⑦ **Time Scaling Property:-**

→ $\cancel{y(t)}\ x(t) \xrightarrow{F.S.} C_n$

$$\Rightarrow \boxed{x(at) \xrightarrow{F.S.} C_n}$$

→ **Proof:-**

$$x(t) = \sum_{n=\infty}^{\infty} C_n e^{jn\omega_0 t}$$

$$\Rightarrow x(at) = \sum_{n=\infty}^{\infty} C_n e^{jn(\omega_0 a)t}$$

⑧ **Integral Property:-**

$x(t) \xrightarrow{F.S.} C_n$

$$\boxed{\int_{-\infty}^{t} x(\tau)d\tau \longrightarrow \frac{C_n}{jn\omega_0}}\ ;\ C_0 = 0.$$

→ Let $y(t) = \int_{t_0}^{t} x(\tau)\,d\tau$, where,

$$x(\tau) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 \tau}$$

$$\Rightarrow y(t) = \sum_{n=-\infty}^{\infty} C_n \int_{t_0}^{t} e^{jn\omega_0 \tau}\,d\tau$$

$$= C_0\int_{t_0}^{t} d\tau + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} C_n \int_{t_0}^{t} e^{jn\omega_0 \tau}\,d\tau$$

$$\Rightarrow y(t) = C_0(t-t_0) + \sum_{\substack{n=-\infty \\ (n \neq 0)}}^{\infty} C_n \cdot \frac{e^{jn\omega_0 t} - e^{jn\omega_0 t_0}}{jn\omega_0}$$

Here, $C_0(t-t_0)$ is linearly increasing with t.

So, if $t \to \infty$, $C_0(t-t_0) \to \infty$, which is not possible for us to write fourier series for $y(t)$, and it is periodic.

So, consider $C_0(t-t_0) = 0$.

$\rightarrow$ $y(t)$ has a fourier series Representation (if it is periodic), only if $C_0 = 0$.

We have, ~~$C_n = \frac{1}{T}\int_0^T x(t) e^{-jn\omega_0 t} dt$~~

$$\Rightarrow C_0 = \frac{1}{T}\int_0^T x(t)\,dt$$

$\rightarrow$ Consider $y(t) = \int_{t_0}^{t} x(\tau)\,d\tau$, assuming $a_0 = 0$.

$$\Rightarrow y(t) = \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} a_n \cdot \frac{e^{jn\omega_0 t} - e^{jn\omega_0 t_0}}{jn\omega_0}$$

$$\Rightarrow y(t) = \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{a_n}{jn\omega_0} e^{jn\omega_0 t} - \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} \frac{a_n e^{jn\omega_0 t_0}}{jn\omega_0}$$

If $y(t) = \sum_{k=-\infty}^{\infty} d_n e^{jn\omega_0 t}$,

$$d_n = \begin{cases} \frac{a_n}{jn\omega_0}, & n \neq 0 \\ -\sum_{k \neq 0} \frac{c_n e^{jn\omega_0 t_0}}{jn\omega_0}, & n = 0. \end{cases}$$

$$\therefore \boxed{d_n = \begin{cases} \dfrac{a_n}{jn\omega_0}, & n \neq 0 \\ -\sum\limits_{k\neq 0} \dfrac{a_n \cdot e^{jn\omega_0 t_0}}{jn\omega_0}, & n = 0. \end{cases}}$$

→ If $y(t) = \int_{-\infty}^{t} x(\tau)d\tau$, we assume $b_0 = 0$.

(9) Parseval's Theorem (Power Theorem):-

→ $x(t) \xrightarrow{F.S.} C_n$.

Power of $x(t)$,

$$\boxed{\Rightarrow \frac{1}{T}\int_0^T |x(t)|^2 dt \xrightarrow{F.S.} \sum_{n=-\infty}^{\infty} |C_n|^2}$$

Proof:

→ We know that,

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + \sum_{n=1}^{\infty} b_n \sin n\omega_0 t$$

Now, Power of DC constant $= a_0^2$

Power of $a_n \cos n\omega_0 t = \dfrac{a_n^2}{2}$

Power of $b_n \sin n\omega_0 t = \dfrac{b_n^2}{2}$

We have, $C_n = \dfrac{a_n + jb_n}{2}$, $C_{-n} = \dfrac{a_n - jb_n}{2}$

$$\Rightarrow |C_n|^2 = \frac{a_n^2 + b_n^2}{4} \;;\; |C_{-n}|^2 = \frac{a_n^2 + b_n^2}{4}$$

$$\Rightarrow |C_n|^2 + |C_{-n}|^2 = \frac{a_n^2 + b_n^2}{2}$$

$$\therefore \boxed{\text{Power of } x(t)} = C_0^2 + \sum_{n=1}^{\infty} |C_n|^2 + |C_{-n}|^2 = \boxed{\sum_{n=-\infty}^{\infty} |C_n|^2}.$$

$$\Rightarrow \boxed{\text{Power of } x(t) = a_0^2 + \sum_{n=1}^{\infty} \left(\frac{a_n^2 + b_n^2}{2}\right)}$$

(10) <u>Multiplication Property:-</u>

$\rightarrow$ $x(t) \xrightarrow{F.S.} C_n$

$y(t) \xrightarrow{F.S.} B_m$

$$\Rightarrow \boxed{x(t) \cdot y(t) \xrightarrow{F.S.} C_n * B_m}$$

$\rightarrow$ <u>Proof:-</u>

$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$$

$$y(t) = \sum_{m=-\infty}^{\infty} B_m e^{jm\omega_0 t}$$

$$\Rightarrow x(t) \cdot y(t) = \left(\sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}\right) \cdot \left(\sum_{m=-\infty}^{\infty} B_m e^{jm\omega_0 t}\right)$$

~~Let m+n=k~~

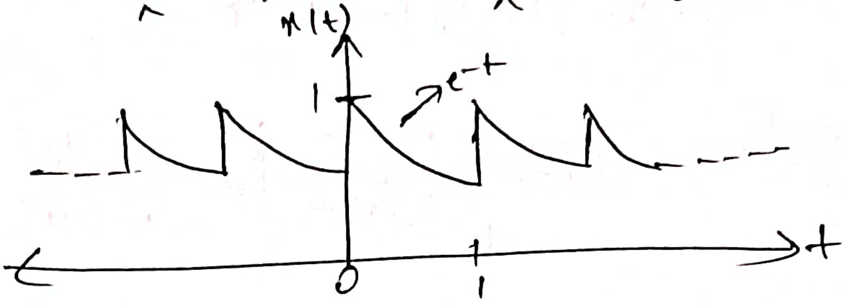
$$= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} C_n B_m e^{j(n+m)\omega_0 t}$$

Let $m+n=k \Rightarrow m=k-n$

$$\Rightarrow x(t) \cdot y(t) = \sum_{k=-\infty}^{\infty} \underbrace{\left(\sum_{n=-\infty}^{\infty} C_n B_{k-n}\right)}_{\rightarrow C_k * B_k} e^{jk\omega_0 t}$$

$$= \sum_{k=-\infty}^{\infty} (C_k * B_k)\, e^{jk\omega_0 t}$$

② (i) Find the exponential fourier series coefficient (EFS coeff) of signal $x(t)$.



(ii) Find the power in DC and first two harmonics of the signal

$T = 1,$

$\omega_0 = 2\pi$

Sol^n (i) $C_n = \frac{1}{T}\int_0^T f(t)\, e^{-jn\omega_0 t}\, dt$

$= \frac{1}{1}\int_0^1 e^{-t} e^{-j2n\pi t}\, dt$

$C_0 = \frac{1}{T}\int_0^T f(t)\, dt$

$= \int_0^1 e^{t(-1-j2n\pi)}\, dt$

$= \int_0^1 e^{-t}\, dt$

$= -\left[e^{-t}\right]_0^1$

$= \frac{1}{-1-j2n\pi}\left[e^{-t-j2n\pi t}\right]_0^1$

$\boxed{C_0 = 1 - e^{-1}}$

$= \frac{1}{-1-j2n\pi}\left[e^{-1-j2n\pi} - e^{0}\right]$

$= \frac{1}{-1-j2n\pi}\left[e^{-1} - 1\right]$

$\Rightarrow \cancel{C_n = \frac{e^{-1}}{-1-j2n\pi}}$

$\Rightarrow \boxed{C_n = \frac{1-e^{-1}}{1+j2n\pi}}$

$\Rightarrow \boxed{C_n = \frac{(1-1/e)(1-j2n\pi)}{1+4n^2\pi^2}}$

(ii) Since $C_n = \frac{1-e^{-1}}{1+j2n\pi}$ for $x(t)$.

$$P = \sum_{n=-2}^{2} |c_n|^2$$

$$= |c_{-2}|^2 + |c_{-1}|^2 + |c_0|^2 + |c_1|^2 + |c_2|^2$$

$$= \frac{(1-1/e)^2 \cancel{(1+16\pi^2)^2}}{\cancel{(1+16\pi^2)^2}\ \sqrt{1+16\pi^2}} + \frac{(1-1/e)^2 \cancel{(1+4\pi^2)}}{\cancel{1+4\pi^2}\ \sqrt{1+4\pi^2}} + (1-1/e)^2 + \left(\frac{1-1/e}{\sqrt{1+4\pi^2}}\right)^2 + \left(\frac{1-1/e}{\sqrt{1+16\pi^2}}\right)^2$$

$$= \cancel{\boxed{P = 8(1-e^{-1})}} = \cancel{\frac{2(1-e^{-1})}{\sqrt{1+16\pi^2}} + \frac{2(1-e^{-1})}{\sqrt{1+4\pi^2}} + 1 - 1/e}$$

$$= (1-1/e)^2 \left(\frac{2}{1+16\pi^2} + \frac{2}{1+4\pi^2} + 1\right) = 0.424\ W.$$

③ The EFS representation of signal $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$. The following information is given about $x(t)$ and a_k.

(i) $x(t)$ is real and even, having fundamental time period of 6.

(ii) The avg. value of $x(t)$ is 2. $\Rightarrow a_0 = 2$.

(iii) $a_k = \begin{cases} k, & \cancel{0} \le k \le 3 \\ 0, & k > 3 \end{cases}$

find the avg. power of the signal $x(t)$.

Sol:- $T = 6$,

Since $x(t)$ is even, and $a_k = k$ for $0 \le k \le 3$,

the signal $x(t)$ only exits for $k \in [-3, 3]$.

$\therefore$ Avg. power of $x(t) = \sum_{k=-3}^{3} |a_k|^2$

$= \sum_{k=-3}^{3} |k|^2$

$= |-3|^2 + |-2|^2 + |-1|^2 + |2|^2 + |1|^2 + |2|^2 + |3|^2$

$= 9 + 4 + 1 + \overset{4}{\cancel{0}} + 1 + 4 + 9$

$\boxed{\therefore \text{Avg. power} = 32\ W.}$

(4) Find the EFS coeffs of the signal-

$$x(t) = 2 + \cos\left(\frac{2\pi}{5}t\right) + 4\sin\left(\frac{5\pi}{3}t\right)$$

Sol: Given, $x(t) = 2 + \cos\left(\frac{2\pi}{3}t\right) + 4\sin\left(\frac{5\pi}{3}t\right)$

~~from this $a_0 = 2$.~~

$$x(t) = 2 + \frac{1}{2}\left(e^{j(2\pi/3)t} + e^{-j(2\pi/3)t}\right) + 2je^{-j(5\pi/3)t}$$
$$- 2je^{j(5\pi/3)t}$$

$$\Rightarrow x(t) = \underset{n=0}{2} + \underset{n=-2}{\frac{1}{2}e^{j2(2\pi/6)t}} + \underset{n=2}{\frac{1}{2}e^{-j2(2\pi/6)t}} \underset{n=5}{+ 2je^{-j(5)(2\pi/6)t}}$$
$$\underset{n=-5}{- 2je^{j(5)(2\pi/6)t}}$$

from this, $\omega_0 = \frac{2\pi}{6} = \pi/3 \Rightarrow \boxed{T = 6.}$

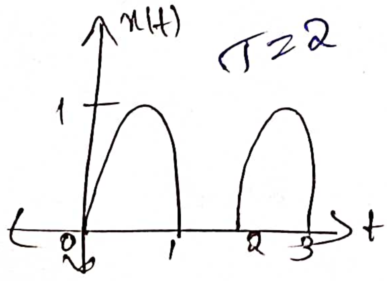
$\boxed{C_0 = 2,}$ $\boxed{C_2 = C_{-2} = \frac{1}{2},}$ $\boxed{C_5 = C^*_{-5} = -2j}$

---

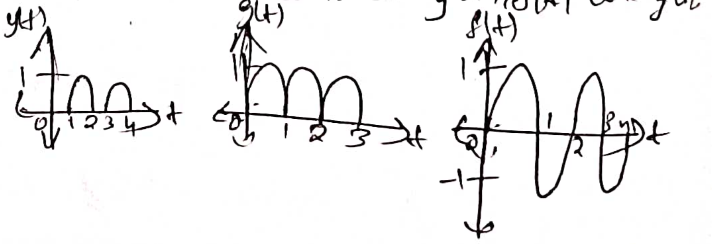
(5) The FS coeffs of signal $x(t)$ as shown in fig are

$C_0 = \frac{1}{\pi}$, $C_1 = -j(0.25)$, $C_n = \frac{1}{\pi(1-n^2)}$ (n is even)

$T = 2$



Find FS coeffs of $y(t)$, $f(t)$ and $g(t)$



Sol: For $y(t) = x(t-1)$,

$x(t) \rightarrow C_n$ $\qquad \omega_0 = \pi$

$x(t - t_0) \rightarrow e^{-jn\omega_0 t_0} C_n$

$x(t-1) \rightarrow e^{-jn\omega_0} C_n = e^{-jn\pi} C_n$

$= (-1)^n C_n.$

$\Rightarrow C_0 = \frac{1}{\pi}(-1)^0 = \frac{1}{\pi}$

$C_1 = j(0.25)$

$C_n = \frac{1}{\pi(1-n^2)}$ (n is even).

For $g(t) = x(t) + x(t-1)$

$$C_0 = \frac{1}{\pi} + \frac{1}{\pi} = \frac{2}{\pi}$$

$$C_1 = -j(0.25) + j(0.25) = 0.$$

$$C_n = \frac{1}{\pi(1-n^2)} + \frac{1}{\pi(1-n^2)} = \frac{2}{\pi(1-n^2)} \text{ (n is even)}$$

for $f(t) = x(t) - x(t-1)$

$$C_0 = \frac{1}{\pi} - \frac{1}{\pi} = 0.$$

$$C_1 = -j(0.25) - j(0.25) = -j(0.5)$$

$$C_n = \frac{1}{\pi(1-n^2)} - \frac{1}{\pi(1-n^2)} = 0.$$