2.5

Solutions of Homogeneous Linear Differential Equations with Constant Coefficients

Learning objectives:

* To study the methods of finding solutions of homogeneous linear differential equations with constant coefficients

AND

* To practice the related problems.

A linear (ordinary) differential equation of order n, with ***variable coefficients*** is of the form

$$\boldsymbol{a_0(x)\frac{d^ny}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = r(x)}$$

or

$$\boldsymbol{a_0(x)y^{(n)}(x) + a_1(x)y^{(n-1)}(x) + \cdots + a_{n-1}(x)y'(x) + a_n(x)y(x) = r(x)} \text{ --- (1)}$$

where y is the dependent variable and x is the independent variable and $a_i(x), i = 0,1,2, \dots, n$ are functions defined on an interval $\boldsymbol{I}$, $a_0(x) \neq 0$. If $a_i(x), i = 0,1,2, \dots, n$ are constants, then (1) is a n^{th} order linear differential equation with ***constant coefficients.***

If $r(x) = 0$, then (1) is called a ***homogeneous equation,*** otherwise it is called a ***non-homogeneous equation***.

A second order homogeneous linear differential equation is of the form $\boldsymbol{a_0(x)y'' + a_1(x)y' + a_2(x)y = 0}$ and a non-homogeneous second order linear differential equation is of the form $\boldsymbol{a_0(x)y'' + a_1(x)y' + a_2(x)y = r(x)\ , r(x) \neq 0}$.

This module is devoted for the study of the solutions of homogeneous linear differential equations.

Solutions of linear differential equations

If $y(x)$ is a solution of (1), then it must identically satisfy the equation. Therefore, $y(x)$ must be continuously differentiable $(n-1)$ times and $y^{(n)}(x)$ must be continuous on $\boldsymbol{I}$.

The following is an important result related to the uniqueness of solutions.

Theorem 1:

If the functions $\boldsymbol{a_i(x), i = 0, 1, 2, \dots, n}$ and $\boldsymbol{r(x)}$ are continuous over an interval $\boldsymbol{I}$, then there exists a unique solution to the IVP

$$\boldsymbol{a_0 y^{(n)} + a_1 y^{(n-1)} + a_2 y^{(n-2)} + \cdots + a_{n-1} y' + a_n y = r(x)} \quad \text{---- } \textbf{\textit{(1)}}$$

$$\boldsymbol{y(x_0) = c_0, y'(x_0) = c_1, \dots, y^{(n-1)}(x_0) = c_{n-1}}$$

where $\boldsymbol{x_0 \in I}$ and $\boldsymbol{c_0, c_1, \dots, c_{n-1}}$ are given constants.

This theorem guarantees that there exists a unique solution if the conditions are satisfied but does not give a procedure to find the solution.

If the conditions given in Theorem1 are satisfied, then the equation (1) is said to be **normal** on $\boldsymbol{I}$. The conditions given in Theorem1 are necessary and sufficient for the differential equation (1) to be normal.

Note(1): If $a_i(x), i = 0,1,2, \dots, n$ are continuous on $\boldsymbol{I}$, then the trivial solution $y = 0$ on $\boldsymbol{I}$ is the only solution of the

homogeneous IVP

$$a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + a_2(x)y^{(n-2)} + \cdots + a_{n-1}(x)y' + a_n(x)y = 0$$

$$y(x_0) = y'(x_0) = \cdots = y^{(n-1)}(x_0) = 0$$

Linear combination of functions:

Let $f_1(x), f_2(x), \dots, f_n(x)$ be n functions defined on an interval $\boldsymbol{I}$. Then

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x)$$

is called a linear combination of functions $f_i(x), i = 1,2, \dots, n$, where $c_1, c_2, \dots, c_n$ are constants.

Solutions of homogeneous linear differential equations

The following theorem asserts that the ***superposition principle*** or ***linearity principle*** holds in the case of solutions of homogeneous linear differential equations.

Theorem 2:

If $y_i(x), i = 1, 2, \dots, m$ are m solutions of homogeneous linear differential equation (of order n)

$$\boldsymbol{a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + a_2(x)y^{(n-2)} + \cdots + a_{n-1}(x)y' + a_n(x)y = 0 \dots (2)}$$

on an interval I then a linear combination of solutions $c_1y_1 + c_2y_2 + \cdots + c_my_m$, where $c_1, c_2, \dots, c_m$ are constants, is also a solution of (2) on I.

Note(2): Superposition principle does not hold for non-homogeneous or non-linear differential equations.

Linear independence and dependence

The functions $f_i(x), i = 1,2, \dots, n$ are said to *be **linearly independent*** on $\boldsymbol{I}$ (where they are defined) if the equation
$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0 \Rightarrow c_1 = c_2 = \cdots = c_n = 0$

The functions $f_i(x), i = 1,2, \dots, n$ are said to *be **linearly dependent*** on $\boldsymbol{I}$ if there exist constants $c_1, c_2, \dots, c_n$ not all of them zero such that $c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0$.

Note(3): If $f_i(x), i = 1,2, \dots, n$ are linearly dependent on $\boldsymbol{I}$, then one or more functions can be expressed as a linear combination of the remaining functions.

For example if $c_1 \neq 0$, then

$$f_1(x) = -\frac{1}{c_1}[c_2 f_2(x) + c_3 f_3(x) + \cdots + c_n f_n(x)]$$

Conversely, if any function say $f_k(x)$ can be expressed as a linear combination of the remaining functions, then the functions $f_i(x), i = 1,2, \dots, n$ are linearly dependent.

Wronskian

A very elegant procedure to test the linear independence and dependence of a given set of functions is the application of *Wronskians*.

Wronskian: The Wronskian of n functions $f_1(x), f_2(x), \dots, f_n(x)$ is denoted by $\boldsymbol{W(f_1, f_2, \dots, f_n)}$ and is a determinant defined by

$$W\left(f_1, f_2, \dots, f_n\right) = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \cdots & \cdots & \cdots & \cdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix} = W(x)$$

The Wronskian of the n functions $f_1, f_2, \dots, f_n$ exist if all the functions $f_1, f_2, \dots, f_n$ are differentiable $(n-1)$ times on the interval $\boldsymbol{I}$. If any one or more functions are not differential $(n-1)$times, then their Wronskian does not exist.

The following is the result for testing the linear independence and dependence of the solutions of homogeneous linear differential equation (2) of order n.

Theorem3:

Let the coefficients $a_i(x), i = 0, 1, 2, \dots, n$ in the homogeneous linear differential equation of order n

$$a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_n y = 0 \text{ ----- (2)}$$

be continuous on an interval I and $y_1(x), y_2(x), \dots, y_n(x)$ be n solutions of (2). Then

(i) $y_1(x), y_2(x), \dots, y_n(x)$ are linearly independent on I if and only if $W(x) = W(y_1, y_2, \dots, y_n) \neq 0$ for all $x \in I$.

(ii) ***If $W(x_0) = 0$ for some $x_0 \in I$, then $W(x) = 0$ for all $x \in I$ and $y_1(x), y_2(x), \dots, y_n(x)$ are linearly dependent on I.***

The following theorem gives the general solution of (2).

Theorem4:

If the coefficients $a_i(x), i = 0, 1, 2, \dots, n$ ***in the homogeneous linear differential equation of order*** n

$$a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_n y = 0 \text{ ----- (2)}$$

are continuous on an interval I***, then the equation (2) has*** n ***linearly independent solutions***

If $y_1(x), y_2(x), \dots, y_n(x)$ ***are*** n ***linearly independent solutions of (2), then the general solution of (2) is given by***

$$y = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n,$$

where $c_1, c_2, \dots, c_n$ ***are arbitrary constants.***

Fundamental solutions and basis:

The n linearly independent solutions of (2) are called the ***fundamental solutions*** of (2) on I. This set of fundamental solutions is called a ***basis*** of (2).

Example 1: Show that e^x, e^{2x}, e^{3x} are the fundamental solutions of $y''' - 6y'' + 11y' - 6y = 0$ on any interval I.

Solution: Let $y_1 = e^x$. Then $y_1' = y_1'' = y_1''' = e^x$ and
$y_1''' - 6y_1'' + 11y_1' - 6y_1 = e^x - 6e^x + 11e^x - 6e^x = 0.$

Thus, e^x is a solution. Similarly, e^{2x} and e^{3x} are solutions.

Now, $W(x) = W(e^x, e^{2x}, e^{3x}) = \begin{vmatrix} e^x & e^{2x} & e^{3x} \\ e^x & 2e^{2x} & 3e^{3x} \\ e^x & 4e^{2x} & 9e^{3x} \end{vmatrix} = e^{6x} \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{vmatrix} = 2e^{2x}$

Note that, $W(x) = 2e^{2x} \neq 0$ on any interval. Hence, the solutions are linearly independent and so $\{e^x, e^{2x}, e^{3x}\}$ is a fundamental set of solutions (on any interval $\boldsymbol{I}$) of the given differential equation.

Differential operator $\boldsymbol{D}$

Sometimes it is convenient to write the given differential equation in a simple form using the differential operator $D = \frac{d}{dx}$. The differential operators $D^2, D^3, \dots, D^n$ respectively denote $\frac{d^2}{dx^2}, \frac{d^3}{dx^3}, \dots, \frac{d^n}{dx^n}$. The differential operators when applied on the function y of x yield

$$Dy = \frac{dy}{dx}, D^2y = \frac{d^2y}{dx^2}, \dots, D^ny = \frac{d^ny}{dx^n}$$

We now define the operator L by

$$L = a_0(x)\frac{d^n}{dx^n} + a_1(x)\frac{d^{n-1}}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{d}{dx} + a_n(x)$$
$$= a_0(x)D^n + a_1(x)D^{n-1} + \cdots + a_{n-1}(x)D + a_n(x),$$

which is a polynomial in D, so that

$$Ly = a_0(x)\frac{d^ny}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y$$
$$= a_0(x)D^ny + a_1(x)D^{n-1}y + \cdots + a_{n-1}(x)Dy + a_n(x)y$$

$$= [a_0(x)D^n + a_1(x)D^{n-1} + \cdots + a_{n-1}(x)D + a_n(x)]y = P(D)y$$

The n^{th} order homogeneous linear differential equation (2) can now be written as $L(y) = 0$, where

$$L = P(D) = a_0(x)D^n + a_1(x)D^{n-1} + \cdots + a_{n-1}(x)D + a_n(x)$$

For example, the differential equation $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = 0$ can be written as $L(y) = 0$, where the operator L is given by $L = P(D) = D^2 + 2D + 2$.

Solutions of Homogeneous Linear Differential Equations with Constant Coefficients

We now discuss the methods of solving homogeneous linear differential equations with constant coefficients.

Consider the n^{th} order homogeneous linear differential equation

$\boldsymbol{a_0 y^{(n)} + a_1 y^{(n-1)} + a_2 y^{(n-2)} + \cdots + a_{n-1} y' + a_n y = 0}$ ……(3), where $a_0, a_1, a_2, \dots, a_{n-1}, a_n$ are real constants.

In the differential operator notation, we write the equation (3) as

$\boldsymbol{L(y) = P(D)y = (a_0 D^n + a_1 D^{n-1} + a_2 D^{n-2} + \cdots + a_{n-1} D + a_n)y = 0}$ ---(4)

General solution of (4)

If $y_1(x), y_2(x), \dots, y_n(x)$ are n linearly independent solutions of

(4), then $y = c_1y_1 + c_2y_2 + \cdots + c_ny_n$ is the general solution of (4), where $c_1, c_2, \dots, c_n$ are arbitrary constants.

We now evolve methods to obtain n linearly independent solutions of (4).

Auxiliary Equation

We have noticed that the solution of the first order equation $y' - my = 0$, where m is a constant, is $y = e^{mx} + c$. Therefore, it gives an idea to try for a solution of the form $y = e^{mx}$ for (4), where m is an unknown constant to be determined. Note that $D^k e^{mx} = m^k e^{mx}$ and

$$P(D)e^{mx} = (a_0D^n + a_1D^{n-1} + a_2D^{n-2} + \cdots + a_{n-1}D + a_n)e^{mx}$$

$$= a_0m^n e^{mx} + a_1m^{n-1}e^{mx} + a_2m^{n-2}e^{mx} + \cdots + a_{n-1}me^{mx} + a_n e^{mx}$$

$$= (a_0m^n + a_1m^{n-1} + a_2m^{n-2} + \cdots + a_{n-1}m + a_n)e^{mx}$$

$$= P(m)e^{mx}$$

Now, $y = e^{mx}$ is a solution of (4), if $L(e^{mx}) = 0$, i.e., $P(m)e^{mx} = 0$. Since $e^{mx} \neq 0$, we obtain $P(m) = 0$, i.e.,

$$\boldsymbol{a_0m^n + a_1m^{n-1} + a_2m^{n-2} + \cdots + a_{n-1}m + a_n = 0} \text{ ----- (5)}$$

This is an algebraic equation in m of degree n (is equal to the order of the differential equation (3)). It is called the ***Auxiliary equation (A.E)*** or the ***characteristic equation*** of the homogenous equation (3). This equation has n roots and the roots of this equation are called the ***characteristic roots.*** Then, the following three cases arise:

Case (i): All the roots of the A.E are real and distinct

Case (ii): All the roots of the A.E are real and all or some are equal.

Case (iii): All or some roots of A.E are complex

We now consider following cases.

Case (i): Real and distinct roots

Let $m_1, m_2, \ldots, m_n$ be the roots of the A.E (5).Suppose that all the roots are real and distinct. Then we have n solutions

$$y_1 = e^{m_1 x}, y_2 = e^{m_2 x}, \ldots, y_n = e^{m_n x} \text{ and}$$

$$W\left(y_1, y_2, \ldots, y_n\right) = \begin{vmatrix} e^{m_1 x} & e^{m_2 x} & \ldots & e^{m_n x} \\ m_1 e^{m_1 x} & m_2 e^{m_2 x} & \ldots & m_n e^{m_n x} \\ m_1^{\,2} e^{m_1 x} & m_2^{\,2} e^{m_2 x} & \ldots & m_n^{\,2} e^{m_n x} \\ \ldots. & \ldots. & \ldots & \ldots \\ m_1^{\,n-1} e^{m_1 x} & m_2^{\,n-1} e^{m_2 x} & \ldots & m_n^{\,n-1} e^{m_n x} \end{vmatrix} = W(x)$$

$$= e^{(m_1 + m_2 + \ldots + m_n)x} \begin{vmatrix} 1 & 1 & \ldots & 1 \\ m_1 & m_2 & \ldots & m_n \\ m_1^{\,2} & m_2^{\,2} & \ldots & m_n^{\,2} \\ \ldots. & \ldots. & \ldots & \ldots \\ m_1^{\,n-1} & m_2^{\,n-1} & \ldots & m_n^{\,n-1} \end{vmatrix}$$

The determinate on the right is called ***Vandermonde determinant*** and it can be a shown that it equals $(-1)^{\frac{n(n-1)}{2}}V$ where V is the product of all factors $m_j - m_k$ with $j < k(\leq n)$.

(For example, when $n = 3$, we get

$$-V = -(m_1 - m_2)(m_1 - m_3)(m_2 - m_3))$$

Thus, $W(x) \neq 0$ for all x $(since\ m_1, m_2, \dots, m_n\ are\ distinct)$. Therefore, $y_1 = e^{m_1 x}, y_2 = e^{m_2 x}, \dots, y_n = e^{m_n x}$ are linearly independent and they are fundamental solutions of (4). The set of solutions $\{e^{m_1 x}, e^{m_2 x}, \dots, e^{m_n x}\}$ is a basis and the general solution of (4) is given by

$$\boldsymbol{y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \cdots + c_n e^{m_n x}},$$

where $c_1, c_2, \dots, c_n$ are arbitrary constants.

Example 2: Solve the differential equation

$$4y^{iv} - 12y''' - y'' - 27y' - 18y = 0$$

Solution: The given differential equation is 4^{th} order homogeneous linear differential equation.

The equation in the operator notation is

$$(4D^4 - 12D^3 - D^2 + 27D - 18)y = 0.$$

i.e., $P(D)y = 0, where\ P(D) = 4D^4 - 12D^3 - D^2 + 27D - 18$

The A.E. is $P(m) = 0, i.e., 4m^4 - 12m^3 - m^2 + 27m - 18 = 0$.

Notice that $P(1) = 4 - 12 - 1 + 27 - 18 = 0$. Therefore, $m = 1$ is a root and $m - 1$ is factor. Further,

$$(m-1)(4m^3 - 8m^2 - 9m + 18) = 0$$

$$\Rightarrow (m-1)[4m^2(m-2) - 9(m-2)] = 0$$

$$\Rightarrow (m-1)(m-2)(4m^2 - 9) = 0 \Rightarrow m = 1, 2, \frac{3}{2}, \frac{-3}{2}$$

Notice that the characteristic roots are all distinct. Therefore, the general solution is given by

$$y = c_1 e^x + c_2 e^{2x} + c_3 e^{\frac{3}{2}x} + c_4 e^{-\frac{3}{2}x},$$

where c_1, c_2, c_3 and c_4 are arbitrary constants

Case (ii): Real and multiple roots

Suppose that the A.E has some multiple roots. That is, if $m = m_1$ is repeated, say r times and the remaining $n - r$ roots, say $m_2, m_3, \ldots, m_{n-r+1}$ are real and distinct, then r linearly independent solutions corresponding to the multiple root

$m = m_1$ are given by

$$\boldsymbol{e^{m_1 x}, x e^{m_1 x}, x^2 e^{m_1 x}, \ldots, x^{r-1} e^{m_1 x},}$$

since the Wronskian of these solutions is not zero. Further, $e^{m_1 x}, x e^{m_1 x}, x^2 e^{m_1 x}, \ldots, x^{r-1} e^{m_1 x}, e^{m_2 x}, e^{m_3 x}, \ldots, e^{m_{n-r+1} x}$

are all linearly independent. The general solution is given by
$$y(x) = (c_0 + c_1x + c_2x^2 + \cdots + c_{r-1}x^{r-1})e^{m_1x} + c_re^{m_2x}$$
$$+c_{r+1}e^{m_3x} + \cdots + c_{n-1}e^{m_{n-r+1}x}$$

Example 3: Solve $y^{iv} - 5y^{'''} + 9y^{''} - 7y^{'} + 2$.

Solution: The given equation is a homogeneous linear differential equation of order 4. The given equation in differential operator notation is

$(D^4 - 5D^3 + 9D^2 - 7D + 2)y = 0$, i.e., $P(D)y = 0$, where $P(D) = D^4 - 5D^3 + 9D^2 - 7D + 2$. The A.E is $P(m) = 0$, i.e., $m^4 - 5m^3 + 9m^2 - 7m + 2 = 0$. Notice that $P(2) = 0$. Therefore, $m - 2$ is a factor and

$$(m-2)(m^3 - 3m^2 + 3m - 1) = 0 \Rightarrow (m-2)(m-1)^3 = 0$$

$\Rightarrow m = 2, 1$(repeated thrice).

Corresponding to the root $m = 1$(repeated thrice), the linearly independent solutions are e^x, xe^x, x^2e^x. The general solution is given by

$$y(x) = (a + bx + cx^2)e^x + de^{2x},$$

where a, b, c and d are arbitrary constants.

Case (iii)(a): Simple complex roots

Since the coefficients of the A.E are real, complex roots occur in conjugate pairs. That is, if $p + iq$ is a root then $p - iq$ is also a root. In this case, the linearly independent solutions are $\boldsymbol{e^{px}\cos qx}$ *and* $\boldsymbol{e^{px}\sin qx}$. If the A.E has r complex conjugate pairs of roots $p_k \pm iq_k$, $k = 1,2, \dots, r$, then the corresponding $2r$ linearly independent solutions are

$e^{p_1x}\cos q_1x$, $e^{p_1x}\sin q_1x$; $e^{p_2x}\cos q_2x$, $e^{p_2x}\sin q_2x$; … ;

$e^{p_rx}\cos q_rx$ and $e^{p_rx}\sin q_rx$.

The general solution can be written as usual depending on the nature of the roots.

Note: The part of the general solution corresponding to $\boldsymbol{p \pm iq}$ is $\boldsymbol{e^{px}(a\cos qx + b\sin q\,x)}$.

Example 4: Solve $y^{iv} - 2y''' + 2y'' - 2y' + y = 0$

Solution: The given equation is a 4th order homogeneous linear differential equation with constant coefficients. The given equation in the differential operator notation is

$(D^4 - 2D^3 + 2D^2 - 2D + 1)y = 0$, i.e., $P(D)y = 0$ where $P(D) = D^4 - 2D^3 + 2D^2 - 2D + 1$. The A.E is $P(m) = 0$,

i.e., $m^4 - 2m^3 + 2m^2 - 2m + 1 = 0$. Notice that $P(1) = 0$. Therefore, $m - 1$ is a factor and $(m-1)(m^3 - m^2 + m - 1) = 0$

$\Rightarrow (m-1)(m-1)(m^2+1) = 0 \Rightarrow m = \pm i, 1$(twice)

The linearly independent solutions are $e^{0x}\cos x$, $e^{0x}\sin x$, e^x, xe^x. The general solution is given by

$y(x) = a\cos x + b\sin x + (c + dx)e^{x}$, where a, b, c and d are arbitrary constants

Case (iii)(b): Multiple complex roots

This case is a combination of two earlier cases of real multiple roots and simple complex roots. Now if $p + iq$ is a multiple root multiplicity r, then $p - iq$ is also a multiple root of the same multiplicity r. The corresponding $2r$ linearly independent solutions are

$$e^{px}\cos qx\ , xe^{px}\cos qx\ , x^{2}e^{px}\cos qx\ , \dots , x^{r-1}e^{px}\cos qx$$

$$e^{px}\sin qx\ , xe^{px}\sin qx\ , x^{2}e^{px}\sin qx\ , \dots , x^{r-1}e^{px}\sin qx$$

Note: The part of the general solution corresponding to these multiple complex roots $p \pm iq$ of multiplicity r is

$$\boldsymbol{e^{px}[\left(c_0 + c_1x + c_2x^2 + \cdots + c_{r-1}x^{r-1}\right)\cos qx + \left(d_0 + d_1x + d_2x^2 + \cdots + d_{r-1}x^{r-1}\right)\sin qx]}$$

Example 5: Solve $(D^2 + 2D + 2)^2 y = 0$.

Solution: The given equation is a 4th order homogeneous linear differential equation $P(D)y = 0$ with constant coefficients,

where $P(D) = (D^2 + 2D + 2)^2$

The A.E is $P(m) = 0 \Rightarrow (m^2 + 2m + 2)^2 = 0$

$\Rightarrow m^2 + 2m + 2 = 0$(twice)$\Rightarrow m = -1 \pm i$ (twice)

The linearly independent solutions are

$$e^{-x}\cos x\,,x\,e^{-x}\cos x\,,e^{-x}\sin x\,,xe^{-x}\sin x$$

The general solution is given by

$$y(x)=e^{-x}[(a+bx)\cos x+(c+dx)\sin x]$$

where a,b,c,d are arbitrary constants.

P1:

Solve $\frac{d^3y}{dx^3} - 9\frac{d^2y}{dx^2} + 23\frac{dy}{dx} - 15y = 0.$

Solution:

The given differential equation is a 3rd order homogeneous linear differential equation with constant coefficients. The given equation in operator notation is $(D^3 - 9D^2 + 23D - 15)y = 0$, i.e., $P(D)y = 0$ where $P(D) = D^3 - 9D^2 + 23D - 15$.

The Auxiliary Equation (A.E) is $P(m) = 0$

$$\text{i.e., } m^3 - 9m^2 + 23m - 15 = 0$$

Notice that $P(1) = 1 - 9 + 23 - 15 = 0$.

Therefore, $m = 1$ is a root and $(m - 1)$ is a factor.

Further, $(m - 1)(m^2 - 8m + 15) = 0$

$$\Rightarrow (m - 1)(m - 3)(m - 5) = 0 \Rightarrow m = 1,3,5$$

Notice that the characteristic roots are all distinct. Therefore, the general solution of the given differential equation is

$$y = ae^x + be^{3x} + ce^{5x}$$

where a, b, c are arbitrary constants.

P2:

Solve the differential equation

$$\left(\boldsymbol{D^4 - 2D^3 - 3D^2 + 4D + 4}\right)\boldsymbol{y = 0.}$$

Solution:

The given differential equation is

$$(D^4 - 2D^3 - 3D^2 + 4D + 4)y = 0$$

i.e., $P(D)y = 0$, where $P(D) = D^4 - 2D^3 - 3D^2 + 4D + 4$.

The A.E is $P(m) = 0$, i.e., $m^4 - 2m^3 - 3m^2 + 4m + 4 = 0$.

Notice that $P(-1) = 1 + 2 - 3 - 4 + 4 = 0$.

Therefore, $m = -1$ is a root and $(m + 1)$ is a factor.

Further, $(m + 1)(m^3 - 3m^2 + 4) = 0$

$$\Rightarrow (m + 1)(m + 1)(m^2 - 4m + 4) = 0$$

$$\Rightarrow (m + 1)^2(m - 2)^2 = 0 \Rightarrow m = -1, -1, 2, 2$$

The linearly independent solutions are $e^{-x}, xe^{-x}, e^{2x}, xe^{2x}$.

The general solution is given by

$$y(x) = (a + bx)e^{-x} + (c + dx)e^{2x}$$

where a, b, c and d are arbitrary constants.

P3:

Solve the differential equation $(D^6 - 1)y = 0$.

Solution:

The given differential equation is $(D^6 - 1)y = 0$ ------- (1)

i.e., $P(D)y = 0$, where $P(D) = D^6 - 1$.

The A.E is $P(m) = 0$, i.e., $m^6 - 1 = 0$.

Now, $(m^2)^3 - (1)^3 = 0$

$\Rightarrow (m^2 - 1)(m^4 + m^2 + 1) = 0$

$\Rightarrow (m^2 - 1)\left((m^4 + 2m^2 + 1) - m^2\right) = 0$

$\Rightarrow (m - 1)(m + 1)(m^2 + m + 1)(m^2 - m + 1) = 0$

$\Rightarrow m = 1, -1, \ \frac{-1 \pm \sqrt{1-4}}{2}, \ \frac{1 \pm \sqrt{1-4}}{2}$

$\Rightarrow m = 1, -1, \ -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}, \ \frac{1}{2} \pm i\frac{\sqrt{3}}{2}$

Notice that the characteristic roots are distinct real and complex. Therefore, the general solution of (1) is given by

$$y = ae^{x} + be^{-x} + e^{-\frac{x}{2}}\left[c\cos\left(\frac{x\sqrt{3}}{2}\right) + d\sin\left(\frac{x\sqrt{3}}{2}\right)\right] + e^{\frac{x}{2}}\left[f\cos\left(\frac{x\sqrt{3}}{2}\right) + g\sin\left(\frac{x\sqrt{3}}{2}\right)\right],$$

where a, b, c, d, f, g are arbitrary constants.

P4.

Solve the differential equation $\left(\boldsymbol{D^2+1}\right)^2\left(\boldsymbol{D^2+D+1}\right)\boldsymbol{y=0}$

Solution:

The given differential equation is

$$(D^2+1)^2(D^2+D+1)y=0 \quad .... (1)$$

i.e., $P(D)y=0\ wher\ P(D)=(D^2+1)^2(D^2+D+1)$

The A.E is $P(m)=0,$ i.e., $(m^2+1)^2(m^2+m+1)=0$

$\implies (m^2+1)^2=0; m^2+m+1=0$

$\implies m=0\pm i\ (twice)\ ;\ m=\frac{-1\pm\sqrt{1-4}}{2}=-\frac{1}{2}\pm i\left(\frac{\sqrt{3}}{2}\right)$

Notice that the characteristic roots are multiple complex roots. The general solution of (1) is given by

$$y=(a+bx)cosx+(c+dx)sinx$$

$$+e^{-\frac{x}{2}}\left[fcos\left(\frac{x\sqrt{3}}{2}\right)+gsin\left(\frac{x\sqrt{3}}{2}\right)\right],$$

where a,b,c,d,f,g are arbitrary constants.

IP1:

Solve IVP: $\boldsymbol{\frac{d^2y}{dx^2} + 4\frac{dy}{dx} - 12y = 0,\ y(0) = 0,\ y'(0) = 1.}$

Solution:

The given differential equation is a second order homogeneous linear differential equation with constant coefficients. The given equation in operator notation is

$$(D^2 + 4D - 12)y = 0$$

i.e., $P(D)y = 0$, where $P(D) = D^2 + 4D - 12$

The A.E is $P(m) = 0$, i.e., $m^2 + 4m - 12 = 0$

$$\Rightarrow (m-2)(m+6) = 0 \Rightarrow m = 2, -6$$

Therefore, the linearly independent solutions are e^{2x} and e^{-6x}.

The general solution is given by $y(x) = ae^{2x} + be^{-6x}$ where a, b are arbitrary constants.

Now, $y'(x) = 2ae^{2x} - 6be^{-6x}$

Substituting the initial conditions, we get

$$y(0) = 0 \Rightarrow a + b = 0;\ y'(0) = 1 \Rightarrow 2a - 6b = 0$$

Solving, we get $a = \frac{1}{8}\ , b = -\frac{1}{8}$

The solution of the given IVP is $y(x) = \frac{1}{8}e^{2x} - \frac{1}{8}e^{-6x}$

IP2:

Solve IVP $\left(\boldsymbol{D^3 + 6D^2 + 12D + 8}\right)\boldsymbol{y = 0}$, **given that** $\boldsymbol{x = 0, y = 1, y' = -2}$ **and** $\boldsymbol{y'' = 2}$**.**

Solution:

The given differential equation is

$$(D^3 + 6D^2 + 12D + 8)y = 0$$

i.e., $P(D)y = 0$, where $P(D) = D^3 + 6D^2 + 12D + 8$.

The A.E is $P(m) = 0$, i.e., $m^3 + 6m^2 + 12m + 8 = 0$.

Notice that $P(-2) = -8 + 24 - 24 + 8 = 0$.

Therefore, $m = -2$ is a root and $(m + 2)$ is a factor.

Further, $(m + 2)(m^2 + 4m + 4) = 0$

$$\Rightarrow (m + 2)(m + 2)^2 = 0 \Rightarrow (m + 2)^3 = 0$$

$$\Rightarrow m = -2, -2, -2$$

The linearly independent solutions are e^{-2x}, xe^{-2x} and x^2e^{-2x}.

The general solution is $y(x) = (a + bx + cx^2)e^{-2x}$.

Now,

$$y'(x) = -2e^{-2x}(a + bx + cx^2) + e^{-2x}(b + 2cx)$$

$$\Rightarrow y'(x) = -2y(x) + e^{-2x}(b + 2cx)$$

$$\Rightarrow y''(x) = -2y'(x) - 2e^{-2x}(b + 2cx) + e^{-2x}(2c)$$

$$\Rightarrow y''(x) = -2y'(x) - 2\big(y'(x) + 2y(x)\big) + 2ce^{-2x}$$

$$\Rightarrow y''(x) = -4y'(x) - 4y(x) + 2ce^{-2x}$$

Substituting the given initial conditions, we get

$$y(0) = 1 \Rightarrow a = 1$$

$$y'(0) = -2 \Rightarrow -2 = -2(1) + b \Rightarrow b = 0$$

$$y''(0) = 2 \Rightarrow 2 = -4(-2) - 4(1) + 2c \Rightarrow c = -1$$

Therefore, the solution of the given IVP is

$$y(x) = (1 - x^2)e^{-2x}$$

IP3:

Solve the differential equation $\boldsymbol{(D^3-14D+8)y=0}$.

Solution:

The given differential equation is $(D^3-14D+8)y=0$ ----- (1)

i.e., $P(D)y=0$, where $P(D)=D^3-14D+8$.

The A.E is $P(m)=0$, i.e., $m^3-14m+8=0$.

Notice that $P(-4)=0$. Therefore, $m=-4$ is a root and $(m+4)$ is a factor. Further,

$$(m+4)(m^2-4m+2)=0 \Rightarrow m=-4, 2\pm\sqrt{2}$$

Therefore, notice that the characteristic roots are real and irrational.

Therefore, the general solution of (1) is given by

$$y=ae^{-4x}+e^{2x}\left[be^{\sqrt{2}x}+ce^{-\sqrt{2}x}\right],$$

where a, b, c are arbitrary constants.

Note:

The general solution of (1) is given by

$$y=ae^{-4x}+e^{2x}\left[b\cos h\left(x\sqrt{2}\right)+c\sin h\left(x\sqrt{2}\right)\right],$$

where a, b, c are arbitrary constants.

IP4:

Solve IVP $\boldsymbol{y'' - 4y' + 5y = 0, y(0) = 2\,;\; y'(0) = -1.}$

Solution:

The given differential equation is a second order homogeneous linear differential equation and in operator notation is

$$(D^2 - 4D + 5)y = 0$$

i.e., $P(D)y = 0$, where $P(D) = D^2 - 4D + 5$.

The A.E is $P(m) = 0$, i.e., $m^2 - 4m + 5 = 0$.

$$\Rightarrow m = \frac{4 \pm \sqrt{16-20}}{2} = 2 \pm i$$

Therefore, the linearly independent solutions are $e^{2x}\cos x$ and $e^{2x}\sin x$. The general solution is given by

$$y(x) = e^{2x}(a\cos x + b\sin x)$$

where a and b are arbitrary constants. Now,

$$y'(x) = 2e^{2x}(a\cos x + b\sin x) + e^{2x}(-a\sin x + b\cos x)$$

$$\Rightarrow y'(x) = e^{2x}[(2a\cos x - a\sin x) + (2b\sin x + b\cos x)]$$

Substituting the given initial conditions, we get

$$y(0) = 2 \Rightarrow a = 2;\; y'(0) = -1 \Rightarrow 2a + b = -1 \Rightarrow b = -5$$

Therefore, the solution of the given IVP is

$$y(x) = e^{2x}(2\cos x - 5\sin x)$$

2.5. Solutions of Homogeneous Linear Differential Equations with Constant Coefficients

EXERCISES

I. **Find the general solutions of the following differential equations.**

1) $2y''' + y'' - 13y' + 6y = 0$
2) $y^{'v} - 13y'' + 36y = 0$
3) $y^{'v} + y''' - 4y'' - 4y' = 0$
4) $144y^{'v} - 25y'' + y = 0$
5) $y''' + 4y'' + 5y' + 2y = 0$
6) $27y''' - 27y'' + 9y' - y = 0$
7) $y^{'v} - 3y''' + 3y'' - y' = 0$
8) $9y^{'v} - 66y''' + 157y'' - 132y' + 36y = 0$
9) $y''' - 2y'' + 4y' - 8y = 0$
10) $y''' - 7y'' + 19y' - 13y = 0$
11) $y^{'v} + y''' + 14y'' + 16y' - 32y = 0$
12) $y^{'v} + 2y''' - 9y'' - 10y' + 50y = 0$
13) $y^{'v} + 2y'' + y = 0$
14) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 5y = 0$
15) $(D^3 + 6D^2 + 12D + 8)y = o$
16) $(D^4 + 4D^3 - 5D^2 - 36D - 36)y = 0$
17) $(D^2 + D + 1)^2(D - 2)y = 0$

Answers:

1) $ae^{x/2} + be^{2x} + ce^{-3x}$
2) $ae^{2x} + be^{-2x} + ce^{3x} + de^{-3x}$
3) $a + be^{2x} + ce^{-2x} + de^{-x}$
4) $ae^{x/3} + be^{-x/3} + ce^{x/4} + de^{-x/4}$
5) $ae^{-2x} + (bx + c)e^{-x}$
6) $(a + bx + cx^2)e^{x/3}$
7) $a + (bx^2 + cx + d)e^x$
8) $(ax + b)e^{3x} + (cx + d)e^{2x/3}$
9) $ae^{2x} + bcos2x + csin2x$
10) $ae^x + e^{3x}(bcos2x + csin2x)$
11) $ae^x + be^{-2x} + c\ cos4x + dsin4x$
12) $(a + bx)cosx + (c + dx)sinx$
13) $y = e^{-x}(acos4x + bsin4x)$
14) $y = (a + bx + cx^2)e^{-2x}$
15) $y = ae^{-3x} + be^{3x} + (c_3 + c_4x)e^{-2x}$
16) $y = ae^{2x} + e^{\frac{-x}{2}}(b + cx)cos\left(\frac{x\sqrt{3}}{2}\right) + (d + fx)sin\left(\frac{x\sqrt{3}}{2}\right)$

II. Solve the following initial value problems.

a. $y''' - 2y'' - 5y' + 6y = 0, y(0) = 0, y'(0) = 0, y''(0) = 1$
b. $y''' - 5y'' + 7y' - 3y = 0, y(0) = 1, y'(0) = 0, y''(0) = -5$
c. $y^{iv} + y'' = 0, y(0) = 1, y'(0) = 2, y''(0) = -1, y'''(0) = -1$
d. $y''' + y'' - 2y = 0, y(0) = 2, y'(0) = 2, y''(0) = -3$

e. $(D^3 - 2D^2 - 5D + 6)y = 0 \; when \; y(0) = 1$

$$y'(0) = -7, y''(0) = -1$$

Answers

a. $\frac{(3e^{3x}+2e^{-2x}-5e^{x})}{30}$

b. $(2 + x)e^{x} - e^{3x}$

c. $x + cosx + sinx$

d. $e^{x} + e^{-x}(cosx + 2sinx)$

e. $y = 2e^{-2x} - e^{3x}$

2.6.

Solutions of Non-homogeneous Linear Differential Equations with Constant Coefficients (Part-I)

Learning objectives:

- To find the general solution of non-homogeneous linear differential equations with constant coefficients.
- To discuss three methods of finding particular integrals.
  AND
- To practice the related problems.

In the last module, we have studied the methods for finding general and particular solutions of homogeneous linear differential equations with constant coefficients. In this module we discuss the methods for finding the general solution of a non-homogeneous linear differential equation with constant coefficients, when the general solution of the corresponding homogeneous linear differential equation is known.

We first prove a general Theorem

Theorem1:

If $y_p(x)$ be any particular solution of a non-homogeneous linear differential equation

$$L(y) = r(x) \qquad \dots \qquad \dots \qquad (1)$$

where $L = P(D) = a_0(x)D^n + a_1(x)D^{n-1} + \cdots + a_{n-1}(x)D + a_n(x)$,

$a_0(x) \neq 0, r(x) \neq 0$ and $\{y_1(x), y_2(x), \dots, y_n(x)\}$ is a basis and $y_c(x) = c_1y_1(x) + c_2y_2(x) + \cdots + c_ny_n(x)$ (where $c_1, c_2, \dots, c_n$ are arbitrary constants) is a general solution of the corresponding homogeneous equation

$$L(y) = 0 \qquad \dots \qquad \dots \qquad (2)$$

then $y = y_c + y_p$ is a general solution of (1).

Proof: Given $y_p(x)$ is particular solution of (1) $\Rightarrow L(y_p) = r(x)$.

Given $\{y_1(x), y_2(x), \dots, y_n(x)\}$ is a basis and

$$y_c = c_1y_1(x) + c_2y_2(x) + \cdots + c_ny_n(x)$$

where $c_1, c_2, \dots, c_n$ are arbitrary constants, is a general solution of the homogeneous equation corresponding to (1).That is $L(y_c) = 0$. Now, consider $y(x) = y_c(x) + y_p(x)$. Then,

$$L(y) = P(D)y$$

$$= [a_0(x)D^n + a_1(x)D^{n-1} + \cdots + a_{n-1}(x)D + a_n(x)](y_c + y_p)$$

$$= a_0(x)D^n(y_c + y_p) + a_{n-1}(x)D^{n-1}(y_c + y_p) + \cdots$$

$$+a_{n-1}(x)D(y_c + y_p) + a_n(x)(y_c + y_p)$$

$$= a_0(x)\left(y_c{}^{(n)} + y_p{}^{(n)}\right) + a_{n-1}(x)\left(y_c{}^{(n-1)} + y_p{}^{(n-1)}\right) + \cdots$$

$$+a_{n-1}(x)\left(y_c{}' + y_p{}'\right) + a_n(x)(y_c + y_p)$$

$$= a_0(x)y_c{}^{(n)} + a_{n-1}(x)y_c{}^{(n-1)} + \cdots + a_{n-1}(x)y_c{}' + a_n(x)y_c$$

$$+a_0(x)y_p{}^{(n)} + a_{n-1}(x)y_p{}^{(n-1)} + \cdots + a_{n-1}(x)y_p{}' + a_n(x)y_p$$

$$= P(D)y_c + P(D)y_p = L(y_c) + L(y_p)$$

$$= r(x) + 0 = r(x)$$

Thus, $y(x) = y_c(x) + y_p(x)$ is a solution of (1). It is a general solution of (1), since it contains n arbitrary constants $c_1, c_2, \dots, c_n$.

Hence the theorem.

From the above theorem, it is clear that the general solution of a non-homogeneous linear equation (1) is the sum of following two parts:

(i) $y_c(x) = c_1y_1(x) + c_2y_2(x) + \cdots + c_ny_n(x)$, a general solution of the corresponding homogeneous equation (2) (where $c_1, c_2, \dots, c_n$ are arbitrary constants). This solution is called the ***complimentary function*** (C.F) of (1).

(ii) $y_p(x)$, a particular solution of (1). This solution is also called ***particular integral*** (P.I).

Note: Consider the non-homogeneous equation

$$L(y) = r_1(x) + r_2(x) + \cdots + r_m(x) \quad \dots \qquad \dots \qquad (3)$$

If $y_{p_i}(x)$ is a particular solution of $L(y) = r_i(x), i = 1,2, \dots, m$ then $y_{p_1}(x) + y_{p_2}(x) + \cdots + y_{p_m}(x)$ is a particular solution of (3).

The methods of finding $y_c(x)$ have been discussed in module 2.5. We now derive methods of finding particular integrals of non-homogeneous equations with constant coefficients.

Methods for Finding Particular Integrals

Consider the n^{th} order non-homogeneous linear equation with constant coefficients.

$$L(y) = P(D)y = r(x) \quad \dots \qquad \dots \qquad (4)$$

where $P(D) = a_0 D^n + a_1 D^{n-1} + a_2 D^{n-2} + \cdots + a_{n-1} D + a_n;$

$a_0, a_1, \dots, a_{n-1}, a_n$ are constants and D is the differential operator $\frac{d}{dx}$. Since D is a differential operator, its inverse $D^{-1}\left(=\frac{1}{D}\right)$ defines the integral operator and $\frac{1}{D} f(x) = \int f(x)dx$. Further, $D^{-1}\left(D\big(f(x)\big)\right) = f(x)$, where $f(x)$ is a differentiable function.

Particular integral of (4)

Operating (4) on both sides by $[P(D)]^{-1} = \frac{1}{P(D)}$ on both sides we get, $\frac{1}{P(D)}[P(D)y] = \frac{1}{P(D)}r(x) \Longrightarrow y = \frac{1}{P(D)}r(x)$. Notice that (4) is satisfied if we take $y = \frac{1}{P(D)}r(x)$. Thus,

$$\textbf{Particular integral (P.I.)} = \frac{\boldsymbol{1}}{\boldsymbol{P(D)}}\boldsymbol{r(x)}.$$

It may be noted that the P.I. contains no arbitrary constants. We now develop methods for finding particular integrals.

Note (1): If $\boldsymbol{\alpha}$ is a constant, then a particular value of $\frac{\boldsymbol{1}}{\boldsymbol{D-\alpha}}\boldsymbol{r(x)}$ is $\boldsymbol{e^{\alpha x}\int r(x)e^{-\alpha x}dx}$.

This follows as shown below:

Let $z = \frac{1}{D-\alpha}r(x)$. Operating on both sides by $D - \alpha$, we get

$$(D-\alpha)z = (D-\alpha)\left[\frac{1}{D-\alpha}r(x)\right] \Longrightarrow (D-\alpha)z = r(x)$$

$$\Longrightarrow \frac{dz}{dx} - \alpha z = r(x)$$

This is a first order linear differential equation in z.

$$I.F. = e^{\int -\alpha\, dx} = e^{-\alpha x}.$$

Its solution is given by $ze^{-\alpha x} = \int r(x)e^{-\alpha x}\, dx$

Since we are concerned about a particular value of z, the constant of integration is omitted. Therefore,

$z = e^{\alpha x}\int r(x)e^{-\alpha x}dx$. Thus, $\frac{\boldsymbol{1}}{\boldsymbol{D-\alpha}}\boldsymbol{r(x)} = \boldsymbol{e^{\alpha x}\int r(x)e^{-\alpha x}dx}$.

Note (2): If α and β are constants then

$$\frac{1}{(D-\beta)(D-\alpha)}r(x) = \frac{1}{D-\beta}\left[\frac{1}{D-\alpha}r(x)\right] = \frac{1}{D-\beta}e^{\alpha x}\int r(x)e^{-\alpha x}\,dx$$

$$= e^{\beta x}\int[e^{\alpha x}\int r(x)e^{-\alpha x}\,dx]e^{-\beta x}\,dx.$$

Note (3): If α is a constant and r is a positive integer, then

$$\boldsymbol{\frac{1}{(D-\alpha)^r}e^{\alpha x} = \frac{x^r}{r!}e^{\alpha x}}$$

$$\frac{1}{(D-\alpha)^r}e^{\alpha x} = \frac{1}{(D-\alpha)^{r-1}}e^{\alpha x}\int e^{\alpha x}e^{-\alpha x}\,dx = \frac{1}{(D-\alpha)^{r-1}}xe^{\alpha x}$$

$$= \frac{1}{(D-\alpha)^{r-2}}e^{\alpha x}\int(xe^{\alpha x})e^{-\alpha x}\,dx = \frac{1}{(D-\alpha)^{r-2}}\frac{x^2}{2}e^{\alpha x}$$

$$= \frac{1}{(D-\alpha)^{r-3}}e^{\alpha x}\int\left(\frac{x^2}{2}e^{\alpha x}\right)e^{-\alpha x} = \frac{1}{(D-\alpha)^{r-3}}\frac{x^3}{3!}e^{\alpha x}$$

$$= \cdots = \frac{x^r}{r!}e^{\alpha x}.$$

Method 1: P.I. of $L(y) = P(D)y = r(x)$ when $\frac{1}{P(D)}$ is expressed as partial fractions

Suppose that $P(D) = (D-\alpha_1)(D-\alpha_2)\dots(D-\alpha_n)$ and $\frac{1}{P(D)} = \frac{A_1}{D-\alpha_1} + \frac{A_2}{D-\alpha_2} + \cdots + \frac{A_n}{D-\alpha_n}$, where $\alpha_1, \alpha_2, \dots, \alpha_n$ and $A_1, A_2, \dots A_n$ are constants. Then

$$P.I. = \frac{1}{P(D)}r(x) = \left[\frac{A_1}{D-\alpha_1} + \frac{A_2}{D-\alpha_2} + \cdots \frac{A_n}{D-\alpha_n}\right]r(x)$$

$$= A_1\frac{1}{D-\alpha_1}r(x) + A_2\frac{1}{D-\alpha_2}r(x) + \cdots + A_n\frac{1}{D-\alpha_n}r(x)$$

$$= A_1e^{\alpha_1 x}\int r(x)\,e^{-\alpha_1(x)}dx + A_2e^{\alpha_2 x}\int r(x)e^{-\alpha_2(x)}dx$$

$$+\cdots+A_n e^{\alpha_n x}\int r(x)e^{-\alpha_n(x)}dx$$

Note: This method is a general method and it will be more useful if $r(x)$ is of the form $tan\ ax, sec\ ax, cot\ ax$ and $csc\ ax$.

Example 1: Solve $(D^2+a^2)y=\sec ax$.

Solution: The given equation is $P(D)y=\sec ax$, where $P(D)=D^2+a^2$. The A.E is $P(m)=0 \Rightarrow m^2+a^2=0 \Rightarrow m=\pm ai$.

Therefore, the C.F. is $y_c=c_1\cos ax+c_2\sin ax$. A particular integral y_p is obtained as follows.

$$y_p=\frac{1}{D^2+a^2}\sec ax=\frac{1}{2ai}\left(\frac{1}{D-ai}-\frac{1}{D+ai}\right)\sec ax$$

Now,

$$\frac{1}{D-ai}\sec ax=e^{iax}\int e^{-iax}\sec ax\,dx=e^{iax}\int\frac{\cos ax-i\sin ax}{\cos ax}dx$$

$$=e^{iax}\int(1-i\tan ax)dx=e^{iax}\left(x+\frac{i}{a}ln\,cos\,ax\right)$$

Replacing i by $-i$ in the above, we get

$$\frac{1}{D+ai}\sec ax=e^{-iax}\left(x-\frac{i}{a}ln\,cos\,ax\right)$$

Now,

$$y_p=\frac{1}{2ai}\left[e^{iax}\left(x+\frac{i}{a}ln\,cos\,ax\right)-e^{-iax}\left(x-\frac{i}{a}ln\,cos\,ax\right)\right]$$

$$=\frac{1}{a}\frac{x\left(e^{iax}-e^{-iax}\right)}{2i}+\frac{1}{a^2}\frac{e^{iax}+e^{-iax}}{2}ln\,cos\,ax$$

$$=\frac{x}{a}\sin ax+\frac{1}{a^2}\cos ax\,ln\,cos\,ax$$

The general solution is given by $y=C.F+P.I=y_c+y_p$

$$y = c_1 \cos ax + c_2 \sin ax + \frac{x}{a}\sin ax + \frac{1}{a^2}\cos ax \ln \cos ax\ ,$$

where c_1 and c_2 are arbitrary constants.

Method 2: P.I of (4) if $r(x) = e^{ax}$, where a is a constant.

We have two cases to deal with.

Case (i): Let $P(a) \neq 0$.

We have, $P(D)e^{ax} = P(a)e^{ax}$ (already seen in module 2.5)

Now, operating by $\frac{1}{P(D)}$ on both sides, we get

$$e^{ax} = \frac{1}{P(D)} \cdot P(a)e^{ax} = P(a)\frac{1}{P(D)}e^{ax} \Rightarrow \frac{1}{P(D)}e^{ax} = \frac{e^{ax}}{P(a)}$$

Thus, P.I. of (4) when $r(x) = e^{ax}$ is

$$\textbf{P.I.} = \frac{1}{P(D)}e^{ax} = \frac{e^{ax}}{P(a)},\ \textbf{when } P(a) \neq 0.$$

Case (ii): Let $P(a) = 0$.

Then $D - a$ is factor of $P(D)$. If a is multiple root of multiplicity r, then $P(D) = (D-a)^r \emptyset(D)$, where $\emptyset(a) \neq 0$. Now,

$$\text{P.I} = \frac{1}{P(D)}e^{ax} = \frac{1}{(D-a)^r \emptyset(D)}e^{ax} = \frac{1}{(D-a)^r}\left[\frac{1}{\emptyset(D)}e^{ax}\right]$$

$$= \frac{1}{(D-a)^r} \cdot \frac{1}{\emptyset(a)}e^{ax} \quad \text{(by case (i), since } \emptyset(a) \neq 0)$$

$$= \frac{1}{\emptyset(a)}\left[\frac{1}{(D-a)^r}e^{ax}\right] = \frac{1}{\emptyset(a)} \cdot \frac{x^r}{r!}e^{ax} \quad \text{(by Note (3) above)}$$

Thus, **if $P(D) = (D-a)^r\emptyset(D)$, where $\emptyset(a) \neq 0$, then**

$$\textbf{P.I.} = \frac{1}{P(D)}e^{ax} = \frac{1}{\emptyset(a)}\frac{x^r}{r!}e^{ax}$$

Example 2: Solve $\frac{d^3y}{dx^3} - 4\frac{d^2y}{dx^2} + 5\frac{dy}{dx} - 2y = \cos hx$

Solution: The given equation is a non-homogeneous third order linear differential equation with constant coefficients. The given equation in differential operator notation is
$P(D)\, y = coshx$, where $P(D) = D^3 - 4D^2 + 5D - 2$

The A.E. is $P(m) = 0$, i.e., $m^3 - 4m^2 + 5m - 2 = 0$. Notice that $P(1) = 0 \Rightarrow m - 1$ is a factor of $P(m) = 0$.

Now, $P(m) = (m-1)(m^2 - 3m + 2) = (m-1)^2(m-2)$.

The characteristic roots are $m = 1\ (twice), 2$. Notice that 1 is a root of multiplicity 2 and $P(-1) \neq 0$. The complimentary function is given by $y_c = (a + bx)e^x + c\, e^{2x}$

The P.I. is given by

$$y_p = \frac{1}{P(D)} cos\ hx = \frac{1}{(D-1)^2\,(D-2)} \frac{e^x + e^{-x}}{2}$$

$$= \frac{1}{2}\left[\frac{1}{(D-1)^2\,(D-2)} e^x + \frac{1}{(D-1)^2\,(D-2)} e^{-x}\right]$$

$$= \frac{1}{2}\left[\frac{1}{(D-1)^2\,(1-2)} e^x + \frac{1}{(-1-1)^2\,(-1-2)} e^{-x}\right]$$

$$= \frac{1}{2}\left[-\frac{x^2}{2!} e^x - \frac{e^{-x}}{12}\right] \text{(using case(i) and case(ii))}$$

$$= -\frac{1}{24}\left(6x^2 e^x + e^{-x}\right)$$

The general solution is $y = C.F + P.I = y_c + y_p$

i.e., $y = (a + bx)e^x + ce^{2x} - \frac{1}{24}(6x^2 e^x + e^{-x})$, where a, b, c are arbitrary constants.

Method3: P.I. of (4) if $r(x) = \sin ax$ or $\cos ax$, where a is a constant.

Let $P(D)y = \sin ax$. Operating both sides by $\frac{1}{P(D)}$, we get $y = \frac{1}{P(D)}\sin ax$. Thus P.I. $= \frac{1}{P(D)}\sin ax$.

Case (i): Let $P(D)$ be a polynomial in even powers of D, i.e., D^2 only. Then $P(D) = Q(D^2)$.

Let $Q(D^2) = b_0(D^2)^m + b_1(D^2)^{m-1} + \cdots + b_{m-1}D^2 + b_m$, where b_i , $i = 0,1,2, \dots , m$ are constants, we have

$D(\sin ax) = a\cos ax$, $D^2(\sin ax) = -a^2 \sin ax$,

$D^3(\sin ax) = -a^3 \cos ax$, $D^4(\sin ax) = (-a^2)^2 \sin ax$

… … , $(D^2)^k(\sin ax) = (-a^2)^k \sin ax$

… …

Now, $Q(D^2)\sin ax = Q(-a^2)\sin ax$,

Sub case (a) of case(i): Let $Q(-a^2) \neq 0$.

Operating $Q(D^2)\sin ax = Q(-a^2)\sin ax$ by $\frac{1}{Q(D^2)}$, we get

$$\sin ax = \frac{1}{Q(D^2)} \cdot Q(-a^2)\sin ax \Rightarrow \frac{\sin ax}{Q(-a^2)} = \frac{1}{Q(D^2)}\sin ax$$

Thus,

$$\textbf{P.I.} = \frac{1}{P(D)}\sin ax = \frac{1}{Q(D^2)}\sin ax = \frac{\sin ax}{Q(-a^2)}, \text{ when } Q(-a^2) \neq 0$$

Similarly, if $r(x) = \cos ax$, then

$$\textbf{P.I.} = \frac{1}{P(D)}\cos ax = \frac{1}{Q(D^2)}\cos ax = \frac{\cos ax}{Q(-a^2)}, \text{ when } Q(-a^2) \neq 0$$

Sub case (b) of case(i): Let $Q(-a^2) = 0$.

Then $D^2 + a^2$ is a factor of $Q(D^2)$.

Let $P(D) = (D^2 + a^2)R(D^2)$, where $R(-a^2) \neq 0$.

Now, P.I. $= \frac{1}{P(D)} \sin ax = \frac{1}{(D^2+a^2)R(D^2)} \sin ax$

$$= \frac{1}{D^2+a^2}\left[\frac{1}{R(D^2)} \sin ax\right] = \frac{1}{D^2+a^2} \cdot \frac{\sin\ ax}{R(-a^2)}$$

(Since $R(-a^2) \neq 0$ and using the above subcase)

$$= \frac{1}{R(-a^2)} \cdot \frac{1}{D^2+a^2} \sin ax$$

Evaluation of $\boldsymbol{\frac{1}{D^2+a^2} \sin ax}$

$\frac{1}{D^2+a^2} \sin ax =$ Imaginary part of $\frac{1}{D^2+a^2} e^{iax}$

$=$ I.P of $\frac{1}{(D-ia)(D+ia)} e^{iax}$ $=$ I.P of $\frac{1}{(D-ia)} \cdot \frac{1}{2ia} e^{iax}$

$=$ I.P of $\frac{(-i)}{2a} \cdot xe^{iax}$ $=$ I.P of $\frac{(-i)}{2a} x(\cos ax + i \sin ax) = -\frac{x}{2a} \cos ax$

Thus, $\boldsymbol{\frac{1}{D^2+a^2} \sin ax = -\frac{x}{2a} \cos ax}$

Therefore, P.I. $= \frac{1}{P(D)} \sin ax = \frac{1}{R(-a^2)} \cdot \frac{(-x)}{2a} \cos ax$

By a similar argument we can show that

$$\boldsymbol{\frac{1}{D^2+a^2} \cos ax = \frac{x}{2a} \sin ax}$$

If $r(x) = \cos ax$, then P.I. $= \frac{1}{P(D)} \cos ax = \frac{1}{R(-a^2)} \cdot \frac{x}{2a} \sin ax$

Case (ii): Let $P(D)$ contains both even and odd powers of D.

Then $P(D) = Q_1(D^2) + DQ_2(D^2)$ and

P.I. $= \frac{1}{P(D)} \sin ax = \frac{1}{Q_1(D^2)+DQ_2(D^2)} \sin ax$

$$= \frac{1}{Q_1(-a^2)+DQ_2(-a^2)} \sin ax \text{ (use case (i)(a))}$$

$$= \frac{1}{p+qD} \sin ax, \text{ where } p = Q_1(-a^2), q = Q_2(-a^2)$$

$$= \frac{p-qD}{p^2-q^2D^2} \sin ax = \frac{1}{p^2-q^2(-a^2)} (p - qD) \sin ax$$

$$= \frac{p \sin ax - qa \cos ax}{p^2+q^2a^2}$$

Similarly, P.I $= \frac{1}{P(D)} \cos ax = \frac{p \cos ax + q\, a \sin ax}{p^2+q^2a^2}$

Example 3: Solve $(D^2 - 4D + 3)y = \sin 3x \cos 2x$.

Solution: The given equation is a 2nd order non-homogeneous linear differential equation with constant coefficients. It is of the form $P(D)y = \sin 3x \cos 2x$, where $P(D) = D^2 - 4D + 3$

The A.E is $P(m) = 0 \Rightarrow (m-1)(m-3) = 0 \Rightarrow m = 1,3$.

Therefore, C.F $= y_c = ae^x + be^{3x}$.

P.I $= y_p = \frac{1}{P(D)} \sin 3x \cos 2x = \frac{1}{2} \frac{1}{D^2-4D+3} (\sin 5x + \sin x)$

$$= \frac{1}{2}\left[\frac{1}{D^2-4D+3} \sin 5x + \frac{1}{D^2-4D+3} \sin x\right]$$

$$= \frac{1}{2}\left[\frac{1}{-5^2-4D+3} \sin 5x + \frac{1}{-1^2-4D+3} \sin x\right]$$

$$= \frac{1}{2}\left[-\frac{1}{4D+22} \sin 5x + \frac{1}{2-4D} \sin x\right]$$

$$= \frac{1}{4}\left[\frac{1}{1-2D}\sin x - \frac{1}{2D+11}\sin 5x\right]$$

$$= \frac{1}{4}\left[\frac{1+2D}{1-4D^2}\sin x - \frac{2D-11}{4D^2-121}\sin 5x\right]$$

$$= \frac{1}{4}\left[\frac{1+2D}{1-4(-1^2)}\sin x - \frac{2D-11}{4(-5^2)-121}\sin 5x\right]$$

$$= \frac{1}{4}\left[\frac{1}{5}(\sin x + 2\cos x) + \frac{1}{221}(10\cos 5x - 11\sin 5x)\right]$$

$$= \frac{1}{20}(\sin x + 2\cos x) + \frac{1}{884}(10\cos 5x - 11\sin 5x)$$

The general solution is given by

$$y = C.F + P.I = y_c + y_p$$

$$= ae^x + be^{3x} + \frac{1}{20}(\sin x + 2\cos x) + \frac{1}{884}(10\cos 5x - 11\sin 5x)$$

where a, b are arbitrary constants.

Example 4: Solve $\frac{d^4y}{dx^4} - 2\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = \sin x + e^x$

Solution: The given equation is a 4$^{\text{th}}$ order non-homogeneous linear differential equation with constant coefficients. The equation in the differential operator notation is $P(D)y = \sin x + e^x$, where $P(D) = D^4 - 2D^3 + 2D^2 - 2D + 1$.

The A.E is $P(m) = 0 \Rightarrow m^4 - 2m^3 + 2m^2 - 2m + 1 = 0$

Notice that $P(1) = 0 \Rightarrow (m-1)(m^3 - m^2 + m - 1) = 0$

$\Rightarrow (m-1)(m-1)(m^2+1) = 0 \Rightarrow m = 1$(twice), $\pm i$

Therefore, C.F $= y_c = (a + bx)e^x + c\cos x + d\sin x$

$$\text{P.I} = y_p = \frac{1}{(D-1)^2(D^2+1)}(\sin x + e^x)$$

$$= \frac{1}{(D-1)^2(D^2+1)}\sin x + \frac{1}{(D-1)^2(D^2+1)}e^x$$

$$= \frac{(D+1)^2}{(D^2-1)^2(D^2+1)}\sin x + \frac{1}{(D-1)^2(1^2+1)}e^x$$

$$= \frac{D^2+2D+1}{(-1^2-1)^2(D^2+1)}\sin x + \frac{1}{2}\frac{1}{(D-1)^2}e^x$$

$$= \frac{1}{4}\frac{1}{D^2+1}(-\sin x + 2\cos x + \sin x) + \frac{1}{2}\frac{x^2}{2!}e^x$$

$$= \frac{1}{2}\frac{1}{D^2+1}\cos x + \frac{x^2e^x}{4}$$

$$= \frac{1}{2}\cdot\frac{x}{2}\sin x + \frac{x^2e^x}{4} \quad (\because \frac{1}{D^2+a^2}\cos ax = \frac{x}{2a}\sin ax)$$

$$= \frac{x}{4}(\sin x + xe^x)$$

The general solution is

$$y = y_c + y_p = (a+bx)e^x + c\cos x + d\sin x + \frac{x}{4}(\sin x + xe^x),$$

where a, b, c, d are arbitrary constants.

P1:

Solve $\frac{d^2y}{dx^2} + y = \csc^2 x$

Solution:

The given equation in the operator notation is

$$(D^2 + 1)y = \csc^2 x \qquad \text{------ (1)}$$

The given equation is in the form of $P(D)y = \csc^2 x$, where $P(D) = D^2 + 1$.

The A.E is $P(m) = 0$, i.e., $m^2 + 1 = 0 \Rightarrow m = \pm i$.

Therefore, the C.F. is $y_c = a\cos x + b\sin x$.

The particular integral y_p is obtained as follows:

$$y_p = \frac{1}{D^2+1}\csc^2 x = \frac{1}{(D-i)(D+i)}\csc^2 x = \frac{1}{2i}\left[\frac{1}{D-i} - \frac{1}{D+i}\right]\csc^2 x$$

Now, $\frac{1}{D-i}\csc^2 x = e^{ix}\int e^{-ix}\csc^2 x\, dx = e^{ix}\int \frac{\cos x - i\sin x}{\sin^2 x}dx$

$$= e^{ix}\int(\cot x\csc x - i\csc x)dx$$

$$= e^{ix}(-\csc x - i\ln|\csc x - \cot x|)$$

Replacing i by $-i$ in the above, we get

$$\frac{1}{D+i}\csc^2 x = e^{-ix}(-\csc x + i\ln|\csc x - \cot x|)$$

Now,

$$y_p = \frac{1}{2i}\left[e^{ix}(-\csc x - i\ln|\csc x - \cot x|)\right]$$

$$-\frac{1}{2i}\left[e^{-ix}(-\csc x + i\ln|\csc x - \cot x|)\right]$$

$$= \left[\frac{e^{ix} - e^{-ix}}{2i}\right](-\csc x) - \left[\frac{e^{ix} + e^{-ix}}{2}\right]\ln|\csc x - \cot x|$$

$$= -\sin x \csc x - \cos x \ln|\csc x - \cot x|$$

$$= -1 - \cos x \ln|\csc x - \cot x|$$

∴ The general solution is given by $y = y_c + y_p$

$y = (a\cos x + b\sin x) - \cos x \ln|\csc x - \cot x| - 1$, where a, b are arbitrary constants.

P2:

Solve $(\boldsymbol{D}-\mathbf{1})^{4}\boldsymbol{y}=(\boldsymbol{e}^{\boldsymbol{x}}+\mathbf{1})^{2}$**.**

Solution:

The given equation is a 4th order non-homogeneous linear differential equation with constant coefficients.

The given equation of the form $P(D)y=(e^{x}+1)^{2}$, where $P(D)=(D-1)^{4}$. The A.E is $P(m)=0$ i.e.,$(m-1)^{4}=0$

$\Rightarrow m=1$(repeated 4 times).

Therefore, the C.F, $y_{c}=(a+bx+cx^{2}+dx^{3})e^{x}$.

The particular integral y_{p} is obtained as follows.

$$y_{p}=\frac{1}{(D-1)^{4}}(e^{x}+1)^{2}=\frac{1}{(D-1)^{4}}(e^{2x}+2e^{x}+1)$$

$$=\frac{1}{(D-1)^{4}}e^{2x}+\frac{1}{(D-1)^{4}}2e^{x}+\frac{1}{(D-1)^{4}}e^{0\cdot x}$$

$$=\frac{1}{(2-1)^{4}}e^{2x}+\frac{2x^{4}}{4!}e^{x}+\frac{1}{(0-1)^{4}}e^{0\cdot x}=e^{2x}+\frac{x^{4}}{4!}2e^{x}+1$$

$\therefore$ The general solution is given by

$$y=y_{c}+y_{p}=(a+bx+cx^{2}+dx^{3})e^{x}+e^{2x}+\frac{x^{4}}{4!}2e^{x}+1$$

where a,b,c,d are arbitrary constants.

P3:

Solve $(\boldsymbol{D^3 + a^2 D})\boldsymbol{y} = \mathbf{sin}\,\boldsymbol{ax}$.

Solution:

The given equation is a 3rd order non-homogeneous linear differential equation with constant coefficients.

The given equation is of the form $P(D)y = \sin ax$, where $P(D) = D^3 + a^2 D$. The A.E is $P(m) = 0$ i.e.,$m^3 + a^2 m = 0$

$\Rightarrow m(m^2 + a^2) = 0 \Rightarrow m = 0, \pm ia$

Therefore, the C.F, $y_c = c_1 + c_2 \cos ax + c_3 \sin ax$

The particular integral y_p is obtained as follows.

$$y_p = \frac{1}{D^3+a^2D} \sin ax = \frac{1}{D^2+a^2}\frac{1}{D} \sin ax = \frac{1}{D^2+a^2} \int \sin ax\, dx$$

$$= \frac{1}{D^2+a^2}\left(-\frac{1}{a}\cos ax\right) = -\left(\frac{1}{a}\right)\left(\frac{x}{2a}\right)\sin ax = -\left(\frac{x}{2a^2}\right)\sin ax$$

∴ The general solution is given by

$$y = y_c + y_p = c_1 + c_2 \cos ax + c_3 \sin ax - \left(\frac{x}{2a^2}\right)\sin ax,$$

where c_1, c_2, c_3 are arbitrary constants.

P4:

Solve $\boldsymbol{\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 9y = 40\sin 5x.}$

Solution:

The given equation is a 2nd order non-homogeneous linear differential equation with constant coefficients.

The given equation $(D^2 - 8D + 9)y = 40\sin 5x$ is of the form $P(D)y = 40\sin 5x$, where $P(D) = D^2 - 8D + 9$.

The A.E is $P(m) = 0$ i.e.,$m^2 - 8m + 9 = 0$

$$\Rightarrow m = \frac{8 \pm \sqrt{64-36}}{2} = \frac{8 \pm 2\sqrt{7}}{2} = 4 \pm \sqrt{7}$$

$\therefore$ C.F $= y_c = e^{4x}\left(c_1 \cosh x\sqrt{7} + c_2 \sinh x\sqrt{7}\right)$

P.I $= y_p = \frac{1}{D^2-8D+9} 40\sin 5x = 40\frac{1}{-5^2-8D+9}\sin 5x$

$$= 40\frac{1}{-8(D+2)}\sin 5x = -5(D-2)\frac{1}{(D-2)(D+2)}\sin 5x$$

$$= -5(D-2)\frac{1}{D^2-4}\sin 5x = -5(D-2)\frac{1}{-5^2-4}\sin 5x$$

$$= \left(\frac{5}{29}\right)(D\sin 5x - 2\sin 5x) = \left(\frac{5}{29}\right)(5\cos 5x - 2\sin 5x)$$

$\therefore$ The general solution is given by $y = y_c + y_p$

$$y = e^{4x}\left(c_1 \cosh x\sqrt{7} + c_2 \sinh x\sqrt{7}\right) + \left(\frac{5}{29}\right)(5\cos 5x - 2\sin 5x)$$

where c_1, c_2 are arbitrary constants.

IP1:

Solve $\boldsymbol{(D^2 - 9D + 18)y = e^{e^{-3x}}}$**.**

Solution:

The given equation is a non-homogeneous 2nd order linear differential with constant coefficients. The given equation is in the form of $P(D)y = e^{e^{-3x}}$, where $P(D) = D^2 - 9D + 18$.

The A.E is $P(m) = 0$ i.e.,$m^2 - 9m + 18 = 0$

$$\Rightarrow (m-6)(m-3) = 0 \Rightarrow m = 6,3$$

Therefore, the C.F. is $y_c = ae^{6x} + be^{3x}$.

The particular integral y_p is obtained as follows.

$$y_p = \frac{1}{D^2-9D+18} e^{e^{-3x}} = \frac{1}{(D-6)(D-3)} e^{e^{-3x}}$$

$$= \frac{-1}{3}\left[\frac{1}{D-3} - \frac{1}{D-6}\right] e^{e^{-3x}} = \frac{-1}{3}\left[\frac{1}{D-3} e^{e^{-3x}} - \frac{1}{D-6} e^{e^{-3x}}\right]$$

Now, $\frac{1}{D-3} e^{e^{-3x}} = e^{3x} \int e^{e^{-3x}} e^{-3x} dx$

Put $e^{-3x} = t \Rightarrow e^{-3x} dx = -\frac{dt}{3}$

$$= -\frac{e^{3x}}{3} \int e^t dt = -\frac{e^{3x}}{3} e^{e^{-3x}}$$

$\frac{1}{D-6} e^{e^{-3x}} = e^{6x} \int e^{e^{-3x}} e^{-6x} dx$

Put $e^{-3x} = t \Rightarrow e^{-3x} dx = -\frac{dt}{3}$

$$= -\frac{e^{6x}}{3}\int e^t t dt = -\frac{e^{6x}}{3}e^t(t-1)$$

$$= -\frac{e^{6x}}{3}e^{e^{-3x}}(e^{-3x}-1)$$

$$= -\frac{e^{e^{-3x}}}{3}(e^{3x}-e^{6x})$$

Now,

$$y_p = -\frac{1}{3}\left[-\frac{e^{3x}}{3}e^{e^{-3x}} + \frac{e^{e^{-3x}}}{3}(e^{3x}-e^{6x})\right]$$

$$= -\frac{e^{e^{-3x}}}{9}[-e^{3x}+e^{3x}-e^{6x}] = \frac{e^{6x}}{9}e^{e^{-3x}}$$

∴ The general solution is given by

$y = y_c + y_p = ae^{6x} + be^{3x} + \frac{e^{6x}}{9}e^{e^{-3x}}$, where a, b are arbitrary constants.

IP2:

Solve $\left(D^4 - 2D^3 + 5D^2 - 8D + 4\right)y = e^x$.

Solution:

The given equation is a 4th order non-homogeneous linear differential equation with constant coefficients.

The given equation is of the form $P(D)y = e^x$, where $P(D) = D^4 - 2D^3 + 5D^2 - 8D + 4$.

The A.E is $P(m) = 0$ i.e.,$m^4 - 2m^3 + 5m^2 - 8m + 4 = 0$

Notice that $P(1) = 0$. Therefore, $m = 1$ is a root and $(m-1)$ is a factor.

$\Rightarrow (m-1)[m^3 - m^2 + 4m - 4] = 0$

$\Rightarrow (m-1)[m^2(m-1) + 4(m-1)] = 0$

$\Rightarrow (m-1)^2(m^2+4) = 0 \Rightarrow m = 1$(twice), $m = \pm 2i$

Therefore, the C.F, $y_c = (a + bx)e^x + (c\cos 2x + d\sin 2x)$.

The particular integral y_p is obtained as follows.

$$y_p = \frac{1}{(D-1)^2(D^2+4)} e^x = \frac{1}{(D-1)^2}\frac{1}{5} e^x = \frac{1}{5}\frac{x^2}{2!} e^x = \frac{x^2}{10} e^x$$

$\therefore$ The general solution is given by $y = y_c + y_p$

$y = (a + bx)e^x + (c\cos 2x + d\sin 2x) + \frac{x^2}{10} e^x$, where a, b, c, d are arbitrary constants.

IP3:

Solve $\frac{d^4y}{dx^4} - a^4y = \sin ax.$

Solution:

The given equation is a 4th order non-homogeneous linear differential equation with constant coefficients.

The given equation $(D^4 - a^4)y = \sin ax$ is of the form $P(D)y = \sin ax$, where $P(D) = D^4 - a^4$.

The A.E is $P(m) = 0$ i.e.,$m^4 - a^4 = 0$

$\Rightarrow (m^2 + a^2)(m^2 - a^2) \Rightarrow m = \pm a, \pm ia$

$\therefore$ C.F $= y_c = c_1e^{ax} + c_2e^{-ax} + c_3 \cos ax + c_4 \sin ax$

P.I. $= y_p = \frac{1}{(D^2+a^2)(D^2-a^2)} \sin ax = \frac{1}{(D^2+a^2)(-a^2-a^2)} \sin ax$

$= -\frac{1}{2a^2}\frac{1}{D^2+a^2} \sin ax = -\frac{1}{2a^2} \times \left(-\frac{x}{2a} \cos ax\right) = \frac{x}{4a^3} \cos ax$

$\therefore$ The general solution is given by $y = y_c + y_p$

$y = c_1e^{ax} + c_2e^{-ax} + c_3 \cos ax + c_4 \sin ax + \frac{x}{4a^3} \cos ax$

where c_1, c_2, c_3, c_4 are arbitrary constants.

IP4:

Solve $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 10y + 37\sin 3x = 0$, and find the value of y when $x = \frac{\pi}{2}$ if it is given that $y = 3$ and $\frac{dy}{dx} = 0$ when $x = 0$.

Solution:

The given equation is a 2nd order non-homogeneous linear differential equation with constant coefficients.

The given equation $(D^2 + 2D + 10)y = -37\sin 3x$ is of the form $P(D)y = -37\sin 3x$, where $P(D) = D^2 + 2D + 10$.

The A.E is $P(m) = 0$ i.e.,$m^2 + 2m + 10 = 0$

$$\Rightarrow m = \frac{-2 \pm \sqrt{4-40}}{2} = \frac{-2 \pm 6i}{2} = -1 \pm 3i$$

$\therefore$ C.F $= y_c = e^{-x}(c_1\cos 3x + c_2 \sin 3x)$

$$\text{P.I} = y_p = \frac{1}{D^2+2D+10}(-37\sin 3x) = -37\frac{1}{-3^2+2D+10}\sin 3x$$

$$= -37\frac{1}{2D+1}\sin 3x = -37(2D-1)\frac{1}{(2D-1)(2D+1)}\sin 3x$$

$$= -37(2D-1)\frac{1}{4D^2-1}\sin 3x$$

$$= -37(2D-1)\frac{1}{4(-3^2)-1}\sin 3x$$

$$= (2D-1)\sin 3x = 6\cos 3x - \sin 3x$$

$\therefore$ The general solution is given by $y = y_c + y_p$

$$y = e^{-x}(c_1 \cos 3x + c_2 \sin 3x) + 6\cos 3x - \sin 3x \qquad \text{... (1)}$$

where c_1, c_2 are arbitrary constants.

Differentiating both sides of (1) w.r.t. x, we have

$$\frac{dy}{dx} = e^{-x}(-3c_1 \sin 3x + 3c_2 \cos 3x)$$

$$-e^{-x}(c_1 \cos 3x + c_2 \sin 3x) - 18\sin 3x - 3\cos 3x \text{... (2)}$$

It is given that $y = 3, \frac{dy}{dx} = 0$ when $x = 0$. So (1) and (2) give

$3 = c_1 + 6$ and $0 = 3c_2 - c_1 - 3 \Rightarrow c_1 = -3$ and $c_2 = 0$

From (1), $y = -3e^{-x}\cos 3x + 6\cos 3x - \sin 3x$ ... (3)

Putting $x = \frac{\pi}{2}$ in (3), the corresponding value of y is given by

$$y\left(\frac{\pi}{2}\right) = -3e^{-\frac{\pi}{2}}\cos\left(\frac{3\pi}{2}\right) + 6\cos\left(\frac{3\pi}{2}\right) - \sin\left(\frac{3\pi}{2}\right)$$

$$= -3e^{-\frac{\pi}{2}}(0) + 6(0) - (-1) = 1$$

2.6.

Solutions of Non-homogeneous Linear Differential Equations with Constant Coefficients (Part-I)

EXERCISES

1. **Solve the following differential equations.**
   A. $(D^2 + a^2)y = \cot ax$
   B. $(D^2 + a^2)y = \csc a\, x$
   C. $\frac{d^2y}{dx^2} + y = sec^2x$
   D. $(D^2 + 4)y = tan2x$
   E. $(D^2 + 4)y = 4tan2x$
   F. $(D^2 - 5D + 6)y = xe^{4x}$

Answers

A. $y = c_1 \cos ax + c_2\, sinax + \frac{1}{a^2} sinax.\, ln \tan\left(\frac{ax}{2}\right)$

B. $y = c_1 cosax + c_2 sinax + \frac{1}{a^2} sinax.\, lnsinax - \frac{x}{a} cosax$

C. $y = acosx + bsinx + sinx\ ln|secx + tanx| - 1$

D. $y = acos2x + bsin2x - \frac{1}{4} cos2x\ lntan\left(\frac{\pi}{4} + x\right)$

E. $y = acos2x + bsin2x - cos2x\ lntan\left(\frac{\pi}{4} + x\right)$

F. $y = ae^{2x} + be^{3x} + \frac{e^{4x}}{4}(2x - 3)$

2. Solve the following differential equations

A. $(D^2 - 3D + 2)y = e^{3x}$
B. $(D + 2)(D - 1)^3 y = e^x$
C. $(D - 1)^2(D^2 + 1)^2 y = e^x$
D. $(D^2 + D - 2)y = e^x$
E. $(D - 1)(D^2 - 2D + 2)y = e^x$
F. $(D^3 - D)y = e^x + e^{-x}$
G. $(D^4 + D^3 + D^2 - D - 2)y = e^x$
H. $(D^2 - 3D + 2)y = e^x + e^{2x}$
I. $(D^3 - 1)y = (e^x + 1)^2$

Answers:

A. $y = ae^x + be^x + \frac{e^{3x}}{2}$

B. $y = ae^{-2x} + (b + cx + dx^2)e^x + \frac{x^3 e^x}{18}$

C. $y = (a + bx)e^x + (c + dx)cosx + (f + gx)sinx + \frac{x^2 e^x}{8}$

D. $y = ae^{-2x} + be^x + \frac{xe^x}{3}$

E. $y = e^x(a + bcosx + csinx + x)$

F. $y = a + be^x + ce^{-x} + \frac{x(e^x + e^{-x})}{2}$

G. $y = ae^x + be^{-x} + e^{-\frac{x}{2}}\left(ccos\left(\frac{\sqrt{7}x}{2}\right) + dsin\left(\frac{\sqrt{7}x}{2}\right)\right) + \frac{x}{8}e^x$

H. $y = ae^x + be^{2x} - xe^x + xe^{2x}$

I. $y = ae^{x} + e^{-\frac{x}{2}}\left(b\cos\left(\frac{\sqrt{3}x}{2}\right) + c\sin\left(\frac{\sqrt{3}x}{2}\right)\right) + \frac{e^{2x}}{7} + \frac{2xe^{x}}{7} - 1$

3. Solve the following the differential equations:

i. $(D^2 + 1)y = cos2x$

ii. $(D^2 + 9)y = cos4x$

iii. $(D^2 - 3D + 2)y = sin3x$

iv. $(D^2 + 4)y = 8cos2x$, given that $y(0) = 0$ and $y^{'}(0) = 0$

v. $(D^2 + 4)y = sin2x$, given that $y(0) = 0$ and $y^{'}(0) = 2$

vi. $(D^2 - 4)y = 2cos^2x$

vii. $(D^2 - 3D + 2)y = cos3x$

viii. $(D^2 + 4)y = e^x + sin2x + cos2x$

ix. $y^{''} - 2y^{'} + 2y = e^x + cosx$

x. $y^{''} + 4y^{'} + 4y = 4cosx + 3sinx, y(0) = 1, y^{'}(0) = 0$

xi. $(D^2 + 1)y = sinx.sin2x$

Answers:

i. $y = acosx + bsinx - \frac{\cos 2x}{3}$

ii. $y = acos3x + bsin3x - \frac{\cos 4x}{7}$

iii. $y = ae^{x} + be^{2x} + \frac{1}{130}(9cos3x - 7sin3x)$

iv. $y = 2x\ sin2x$

v. $y = \frac{1}{8}(9sin2x - 2xcos2x)$

vi. $y = ae^{2x} + be^{-2x} - \frac{1}{4} - \frac{\cos 2x}{8}$

vii. $y = ae^{x} + be^{2x} - \frac{1}{130}(9sin3x + 7cos3x)$

viii. $y = acos2x + bsin2x + \frac{e^{x}}{5} - \frac{x}{4}(cos2x - sin2x)$

ix. $y = e^{x}(acosx + bsinx) + e^{x} + \frac{cosx - 2sinx}{5}$

x. $y = -xe^{-2x} + sinx$

xi. $y = acosx + bsinx + \frac{xsinx}{2} + \frac{cos3x}{16}$

2.7

Solutions of Non-homogeneous Linear Differential Equations with Constant Coefficients (Part-2)

Learning objectives:

- To find the particular integrals and to write general solutions of the given differential equations.

  AND

- To practice the related problems.

Consider an n^{th} order non-homogeneous linear differential equation with constant coefficients

$$L(y) = P(D)y = r(x) \qquad \dots \qquad \dots \qquad \dots \quad (4)$$

where $P(D) = a_0 D^n + a_1 D^{n-1} + \cdots + a_{n-1} D + a_n$; $a_0, a_1, \dots, a_n$ are constants, $a_0 \neq 0$ and $r(x) \neq 0$. In this module we discuss the methods of finding particular integrals when $r(x)$ is of the forms (i) x^k, where k is a positive integer (ii) $e^{ax}V$ and (iii) xV, where a is a constant and V is a function of x.

Method 4: P.I. of (4) when $r(x) = x^k$, where k is a positive integer

$$\text{P.I.} = \frac{1}{P(D)} x^k$$

To evaluate this, reduce $\frac{1}{P(D)}$ to the form $\frac{1}{1+\emptyset(D)}$ by taking out the lowest degree term from $\frac{1}{P(D)}$. Now write $\frac{1}{f(D)}$ as $[1 + \emptyset(D)]^{-1}$, expand it in ascending integral powers of D (regarding D as a number) by Binomial theorem up to the term containing D^k and operate each term of the expansion on x^k.

The above method is applicable when $r(x)$ is a polynomial degree k.

Example 5: Solve $(D^4 + D^2 + 16)y = 16x^2 + 256.$

Solution: The given equation is a 4th order non-homogeneous linear differential equation with constant coefficients.

It is of the form $P(D)y = r(x)$, where $P(D) = D^4 + D^2 + 16$ and $r(x) = 16x^2 + 256$. The A.E is given by $P(m) = 0$

i.e.,$m^4 + m^2 + 16 = 0 \Rightarrow (m^2 + 4)^2 - \left(\sqrt{7}m\right)^2 = 0$

$$\Rightarrow \left(m^2 + \sqrt{7}m + 4\right)\left(m^2 - \sqrt{7}m + 4\right) = 0 \Rightarrow m = \frac{-\sqrt{7}\pm 3i}{2}, \frac{\sqrt{7}\pm 3i}{2}$$

The C.F. is given by

$$y_c = e^{-\frac{\sqrt{7}}{2}x}\left(a\cos\frac{3x}{2} + b\sin\frac{3x}{2}\right) + e^{\frac{\sqrt{7}}{2}x}\left(c\cos\frac{3x}{2} + d\sin\frac{3x}{2}\right).$$

The P.I. is

$$y_p = \frac{1}{D^4+D^2+16}(16x^2 + 256) = \frac{1}{16\left(1+\frac{D^2+D^4}{16}\right)}(16x^2 + 256)$$

$$= \left[1 + \frac{D^2+D^4}{16}\right]^{-1}(x^2 + 16)$$

$$= \left[1 - \frac{D^2+D^4}{16} + \left(\frac{D^2+D^4}{16}\right) - \cdots\right](x^2 + 16)$$

$$= x^2 + 16 - \frac{D^2}{16}(x^2 + 16) - \frac{D^4}{16}(x^2 + 16)$$

$$= x^2 + 16 - \frac{1}{16}\cdot 2 = x^2 + \frac{127}{8}$$

$\therefore$ The general solution is given by $y = C.F. + P.I. = y_c + y_p$

$$y = e^{-\frac{\sqrt{7}}{2}x}\left(a\cos\frac{3x}{2} + b\sin\frac{3x}{2}\right) + e^{\frac{\sqrt{7}}{2}x}\left(c\cos\frac{3x}{2} + d\sin\frac{3x}{2}\right) + x^2 + \frac{127}{8},$$

where a, b, c, d are arbitrary constants.

Example 6: Solve $(D^4 - 2D^3 + D^2)y = x^3$.

Solution: The A.E is $m^4 - 2m^3 + m^2 = 0 \Rightarrow m^2(m-1)^2 = 0$
$\Rightarrow m = 0$ (twice), 1(twice).

The C.F. is given by $y_c = (a + bx)e^{0\cdot x} + (c + dx)e^x$

$$\text{i.e., } y_c = a + bx + (c + dx)e^x.$$

Now, P.I. $= y_p = \frac{1}{D^2(D-1)^2}x^3 = \frac{1}{D^2}[1 - D]^{-2}x^3$

$= \frac{1}{D^2}(1 + 2D + 3D^2 + 4D^3 + \cdots)x^3$ (recall Binomial Theorem)

$= \frac{1}{D^2}(x^3 + 6x^2 + 18x + 24) \qquad \left(\because D = \frac{d}{dx}\right)$

$= \frac{1}{D}\int(x^3 + 6x^2 + 18x + 24)dx$

$= \frac{1}{D}\left(\frac{x^4}{4} + 2x^3 + 9x^2 + 24x\right) = \frac{x^5}{20} + \frac{x^4}{2} + 3x^3 + 12x^2$

The general solution is given by $y = C.F. + P.I. = y_c + y_p$

i.e., $y = a + bx + (c + dx)e^x + \frac{x^5}{20} + \frac{x^4}{2} + 3x^3 + 12x^2$,

where a, b, c, d are arbitrary constants.

Method 5: P.I. of (4) when $r(x) = e^{ax}V$, where a is a constant and V is a function of x

We have, P.I. $= \frac{1}{P(D)} e^{ax} V$.Let X be a function of x.

By successive differentiation, we notice that

$$D(e^{ax}X) = e^{ax}DX + ae^{ax}X = e^{ax}(D+a)X$$

$$D^2(e^{ax}X) = D\big(D(e^{ax}X)\big) = D(e^{ax}(D+a)X)$$

$$= e^{ax}(D+a)\big((D+a)X\big) = e^{ax}(D+a)^2X$$

$$D^3(e^{ax}X) = D\big(D^2(e^{ax}X)\big) = D(e^{ax}(D+a)^2X)$$

$$= e^{ax}(D+a)((D+a)^2X) = e^{ax}(D+a)^3X$$

... ... For a positive integer n

$D^n(e^{ax}X) = e^{ax}(D+a)^nX$. Now,

$$P(D)(e^{ax}X) = (a_0D^n + a_1D^{n-1} + \cdots + a_{n-1}D + a_n)(e^{ax}X)$$

$$= a_0D^n(e^{ax}X) + a_1D^{n-1}(e^{ax}X) + \cdots + a_{n-1}D(e^{ax}X) + a_ne^{ax}X$$

$$= a_0e^{ax}(D+a)^nX + a_1e^{ax}(D+a)^{n-1}X + \cdots + a_{n-1}e^{ax}(D+a)X + a_ne^{ax}X$$

$$= e^{ax}[a_0(D+a)^n + a_1(D+a)^{n-1} + \cdots + a_{n-1}(D+a) + a_n]X$$

$$= e^{ax}P(D+a)X$$

Thus, $P(D)(e^{ax}X) = e^{ax}P(D+a)X$ ------- (i)

Let $P(D+a)X = V$. Notice that V is a function of x (since X is a function of x). Operating on both sides by $\frac{1}{P(D+a)}$, we get

$X = \frac{1}{P(D+a)}V$. Now from (i), $(D)\left(e^{ax}\frac{1}{P(D+a)}V\right) = e^{ax}V$.

Operating on both sides by $\frac{1}{P(D)}$, we get

$$\textbf{P.I.} = \frac{\mathbf{1}}{\boldsymbol{P(D)}}\boldsymbol{e^{ax}V} = \boldsymbol{e^{ax}}\frac{\mathbf{1}}{\boldsymbol{P(D+a)}}\boldsymbol{V}$$

Example 7: Solve $(D^3 - 3D - 2)y = 540x^3e^{-x}$.

Solution: The A.E is $m^3 - 3m - 2 = 0$

$\Rightarrow (m+1)^2(m-2) = 0 \Rightarrow m = -1$ (twice), 2.

The C.F. is given by $y_c = (a+bx)e^{-x} + ce^{2x}$

P.I. $= y_p = \frac{1}{D^3-3D-2}540x^3e^{-x} = 540e^{-x}\frac{1}{(D-1)^3-3(D-1)-2}x^3$

$= 540e^{-x}\frac{1}{D^3-3D^2}x^3 = 540e^{-x}\frac{1}{(-3D^2)\left(1-\frac{D}{3}\right)}x^3$

$= -180e^{-x}\frac{1}{D^2}\left(1-\frac{D}{3}\right)^{-1}x^3 = -180e^{-x}\frac{1}{D^2}\left(1+\frac{D}{3}+\frac{D^2}{9}+\frac{D^3}{27}+\cdots\right)x^3$

$= -180e^{-x}\frac{1}{D^2}\left(x^3+x^2+\frac{2}{3}x+\frac{2}{9}\right) = -180e^{-x}\frac{1}{D}\int\left(x^3+x^2+\frac{2}{3}x+\frac{2}{9}\right)dx$

$= -180e^{-x}\frac{1}{D}\left(\frac{x^4}{4}+\frac{x^3}{3}+\frac{x^2}{3}+\frac{2x}{9}\right) = -180e^{-x}\left(\frac{x^5}{20}+\frac{x^4}{12}+\frac{x^3}{9}+\frac{x^2}{9}\right)$

$= -e^{-x}(9x^5 + 15x^4 + 20x^3 + 20x^2)$

The general solution is given by $y = C.F. + P.I. = y_c + y_p$

i.e., $y = (a+bx)e^{-x} + ce^{2x} - e^{-x}(9x^5 + 15x^4 + 20x^3 + 20x^2)$ where a, b, c are arbitrary constants.

Example 8: Solve $(D^2 - 1)y = \cosh x \cos x$.

Solution: The A.E. is $m^2 - 1 = 0 \Rightarrow m = \pm 1$.

C.F.: $y_c = ae^x + be^{-x}$

P.I. $= y_p = \frac{1}{D^2-1} \cosh x \cos x = \frac{1}{D^2-1}\left(\frac{e^x+e^{-x}}{2}\right) \cos x$

$$= \frac{1}{2}\frac{1}{D^2-1} e^x \cos x + \frac{1}{2}\frac{1}{D^2-1} e^{-x} \cos x = y_{p_1} + y_{p_2}$$

where $y_{p_1} = \frac{1}{2}\frac{1}{D^2-1} e^x \cos x = \frac{e^x}{2}\frac{1}{(D+1)^2-1} \cos x$

$$= \frac{e^x}{2}\frac{1}{D^2+2D} \cos x = \frac{e^x}{2}\frac{1}{-1^2+2D} \cos x$$

$$= \frac{e^x}{2}\frac{2D+1}{4D^2-1} \cos x = \frac{e^x}{2}\frac{2D+1}{4(-1^2)-1} \cos x$$

$$= -\frac{e^x}{10}(2D \cos x + \cos x) = -\frac{e^x}{10}(\cos x - 2 \sin x)$$

and $y_{p_2} = \frac{1}{2}\frac{1}{D^2-1} e^{-x} \cos x = \frac{e^{-x}}{2}\frac{1}{(D-1)^2-1} \cos x$

$$= \frac{e^{-x}}{2}\frac{1}{D^2-2D} \cos x = \frac{e^{-x}}{2}\frac{1}{-1^2-2D} \cos x$$

$$= -\frac{e^{-x}}{2}\frac{2D-1}{4D^2-1} \cos x = -\frac{e^{-x}}{2}\frac{2D-1}{4(-1^2)-1} \cos x$$

$$= \frac{e^{-x}}{10}(2D \cos x - \cos x) = -\frac{e^{-x}}{10}(\cos x + 2 \sin x)$$

The general solution is given by $y = C.F. + P.I. = y_c + y_p$

i.e., $y = ae^x + be^{-x} - \frac{e^x}{10}(\cos x - 2\sin x) - \frac{e^{-x}}{10}(\cos x +$ $2\sin x)$ where a, b are arbitrary constants.

or $y = ae^x + be^{-x} - \frac{1}{5}\cos x\left(\frac{e^x+e^{-x}}{2}\right) + \frac{2}{5}\sin x\left(\frac{e^x-e^{-x}}{2}\right)$

$$= ae^x + be^{-x} - \frac{1}{5}\cosh x\cos x + \frac{2}{5}\sinh x\sin x$$

Method 6: P.I. of (4) where $r(x) = xV$, where V is a function of x

$$\text{P.I.} = \frac{1}{P(D)}xV$$

Let u and v be functions of x. Leibniz theorem for the n^{th} derivative of the product uv says that

$$D^n(uv) = D^n u\cdot v + n_{C_1}D^{n-1}u\cdot Dv + n_{C_2}D^{n-2}u\cdot D^2v + \cdots + n_{C_n}u\cdot D^n v$$

Let U be a function of x, then

$$D^n(xU) = D^n(Ux) = D^nU\cdot x + n_{C_1}D^{n-1}U\cdot 1$$

$$= xD^nU + nD^{n-1}U = (xD^n + nD^{n-1})U$$

Now,

$$P(D)(xU) = (a_0D^n + a_1D^{n-1} + a_2D^{n-2} + \cdots + a_{n-1}D + a_n)xU$$

$$= a_0D^n(xU) + a_1D^{n-1}(xU) + a_2D^{n-2}(xU) + \cdots + a_{n-1}D(xU) + a_nxU$$

$$= a_0(xD^n + nD^{n-1})U + a_1(xD^{n-1} + (n-1)D^{n-2})U$$

$$+a_2(xD^{n-2} + (n-2)D^{n-3})U + \cdots + a_{n-1}(xD+1)U + a_nxU$$

$$= x(a_0D^n + a_1D^{n-1} + a_2D^{n-2} + \cdots + a_{n-1}D + a_n)U$$

$$+(a_0nD^{n-1} + a_1(n-1)D^{n-2} + a_2(n-2)D^{n-3} + \cdots + a_{n-1})U$$

$$= xP(D)U + P'(D)U$$

Thus, $P(D)xU = xP(D)U + P'(D)U$ (Here $P'(D)$ is written with an agreement that $P'(D)$ is the derivative of $P(D)$ w.r.t. D).

Put $P(D)U = V$. Therefore, $U = \frac{1}{P(D)}V$. From the above, we get,

$$P(D)\left(x \cdot \frac{1}{P(D)}V\right) = xV + P'(D)\frac{1}{P(D)}V$$

Operating on both sides by $\left(P(D)\right)^{-1} = \frac{1}{P(D)}$, we get

$$x \cdot \frac{1}{P(D)}V = \frac{1}{P(D)}(xV) + \frac{1}{P(D)}\left(P'(D)\frac{1}{P(D)}V\right)$$

$$\Rightarrow \frac{1}{P(D)}(xV) = x\frac{1}{P(D)}V - \frac{1}{P(D)}P'(D)\frac{1}{P(D)}V = \left(x - \frac{1}{P(D)}P'(D)\right)\frac{1}{P(D)}V$$

Thus, **P.I.**$= \boldsymbol{\frac{1}{P(D)}xV = \left(x - \frac{1}{P(D)}P'(D)\right)\frac{1}{P(D)}V}$

Example 9: Solve $(D^2 + 2D + 1)y = xcosx$

Solution: The A.E. is $m^2 + 2m + 1 = 0 \Longrightarrow m = -1(twice)$

The C.F is $y = (a + bx)e^{-x}$

$$P.I:\ y_p = \frac{1}{D^2 + 2D + 1}xcosx$$

$$= \left(x - \frac{1}{D^2+2D+1}(D^2+2D+1)'\right)\frac{1}{D^2+2D+1}cosx$$

$$= \left(x - \frac{1}{(D+1)(D+1)}2(D+1)\right)\frac{1}{-1^2+2D+1}cosx$$

$$= \left(x - \frac{2}{D+1}\right)\frac{1}{2D}cosx$$

$$= \left(\frac{x}{2} - \frac{1}{D+1}\right)sinx \quad \left(\because \ \frac{1}{D}\ cosx = \int \cos x\, dx = sinx\right)$$

$$= \frac{x}{2}\sin x - \frac{1}{D+1}\ sinx = \frac{x}{2}\sin x - \frac{D-1}{D^2-1}\sin x$$

$$= \frac{x}{2}\sin x \ - \frac{D-1}{-1^2-1}\sin x = \frac{x}{2}\sin x \ + \frac{1}{2}(D\sin x - \sin x)$$

$$= \frac{x}{2}\sin x \ + \frac{1}{2}(\cos x - \sin x)$$

The general solution is $y = \ y_c + y_p$

i.e., $y = \ (a+bx)e^{-x} + \frac{x}{2}\sin x + \ \frac{1}{2}(\cos x - \sin x)$,

where a, b are arbitrary constants.

Example 10: Solve $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = xe^x sinx$

Solution: The given differential equation in D notation is

$P(D)y = xe^x sinx, \ where\ P(D) = D^2 + 3D + 2$

$A.E$: $P(m) = 0 \Longrightarrow m = -1, -2$

$C.F$: $y_c = ae^{-x} + be^{-2x}$

$$P.I:\ y_p = \frac{1}{D^2+3D+2} xe^x sinx$$

$$= e^x \frac{1}{(D+1)^2+3(D+1)+2} xsinx = e^x \frac{1}{D^2+5D+6} xsinx$$

$$= e^x \left[x - \frac{1}{D^2+5D+6}(D^2+5D+6)'\right] \frac{1}{D^2+5D+6} \sin x$$

$$= e^x \left[x.\frac{1}{D^2+5D+6} sinx - (2D+5)\frac{1}{(D^2+5D+6)^2} \sin x\right]$$

$$= e^x \left[x.\frac{1}{-1^2+5D+6} sinx - (2D+5)\frac{1}{(-1^2+5D+6)^2} \sin x\right]$$

$$= e^x \left[\frac{x}{5}.\frac{1}{D+1} sinx - \frac{1}{25}(2D+5)\frac{1}{D^2+2D+1} \sin x\right]$$

$$= e^x \left[\frac{x}{5}.\frac{D-1}{D^2-1} sinx - \frac{1}{25}(2D+5)\frac{1}{-1^2+2D+1} \sin x\right]$$

$$= e^x \left[\frac{x}{5}.\frac{D-1}{-1^2-1} sinx - \frac{1}{25}(2D+5).\frac{1}{2D} \sin x\right]$$

$$= e^x \left[\frac{-x}{10}.(D\ sinx - sinx) + \frac{1}{50}(2D+5)\cos x\right]$$

$$\left(\because\ \frac{1}{D}\sin x = -\cos x\right)$$

$$= e^x \left[-\frac{x}{10}(cos\ x - sinx) + \frac{1}{50}(-2\ sinx + 5\cos x)\right]$$

$$= \frac{e^x}{10}\left[x(\ sinx - cosx) + \frac{1}{5}(5\cos x - 2\sin x)\right]$$

The general solution is given by: $y = y_c + y_p$

$$y = ae^{-x} + be^{-2x} + \frac{e^x}{10}\left[x.(\ sinx - cosx) + \frac{1}{5}(5\cos x - 2\sin x)\right]$$

where a, b are arbitrary constants.

Example 11: Solve $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 3x^2e^{2x}\, sin\, 2x$

Solution: The given equation in D notation is

$P(D)y = 3x^2e^{2x}\sin 2x$, where $P(D) = D^2 - 4D + 4$,
$A.E:\ P(m) = 0 \Longrightarrow\ \ m = 2\ (twice)$

$C.F:\ y_c = (a + bx)e^{2x}$

$$P.I:\ y_p = \frac{1}{(D-2)^2}.3x^2e^{2x}sinx$$

$$= 3.e^{2x}.\frac{1}{(D+2-2)^2}.x^2\sin 2x$$

$$= 3.e^{2x}.\frac{1}{D^2}x^2\sin 2x$$

$$= 3.e^{2x}.I.P\ of\ \frac{1}{D^2}e^{2ix}x^2$$

$$= 3\ e^{2x}.I.P\ of\ e^{2ix}\frac{1}{(D+2i)^2}\ x^2$$

$$= 3\ e^{2x}.I.P\ of\ e^{2ix}\frac{1}{\left[2i\left(1+\frac{D}{2i}\right)\right]^2}\ x^2$$

$$= 3\ e^{2x}.I.P\ of\ \ e^{2ix}\left(-\frac{1}{4}\right)\left(1-\frac{iD}{2}\right)^{-2}x^2$$

$$= \ -\frac{3}{4}\ e^{2x}.I.P\ of\ \ e^{2ix}\left[1+2\left(\frac{iD}{2}\right)+3\left(\frac{iD}{2}\right)^2+\cdots\right]x^2$$

(By Binomial theorem)

$$= -\frac{3}{4}\ e^{2x}\ .I.P\ of\ \ e^{2ix}\left[x^2+2ix-\frac{3}{2}\right]$$

$$= -\frac{3}{4}\, e^{2x}. I.P\ of\ (cos2x + i\ sin2x)\left[\left(x^2 - \frac{3}{2}\right) + 2ix\right]$$

$$= -\frac{3}{4}\, e^{2x}\left[2x\cos 2x + \left(x^2 - \frac{3}{2}\right) sin2x\right]$$

$$= -\frac{3}{8}\, e^{2x}\,[4x\cos 2x + (2x^2 - 3)sin2x]$$

The general solution is: $y = y_c + y_p$

i.e., $y = (a + bx)e^{2x} - \frac{3}{8}\, e^{2x}[4x\ cos2x + (2x^2 - 3)sin\ 2x]$

where a, b are arbitrary constants.

P1.

Solve $\left(\boldsymbol{D^3+3D^2+2D}\right)\boldsymbol{y}=\boldsymbol{x^2}$

Solution:

The given equation is of the form $P(D)y=r(x)$, where $P(D)=D^3+3D^2+2D \; and \; r(x)=x^2$

The A.E. is given by $P(m)=0$

i.e., $m^3+3m^2+2m=0 \Longrightarrow m(m^2+3m+2)=0$

$\Longrightarrow m(m+1)(m+2)=0 \Longrightarrow m=0,-1,-2$

∴ The C.F is $y_c=a+be^{-x}+ce^{-2x}$.

P.I.= $y_p=\frac{1}{D^3+3D^2+2D}x^2=\frac{1}{2D\left[1+\frac{3D}{2}+\frac{D^2}{2}\right]}x^2$

$$=\frac{1}{2D}\left[1+\left(\frac{3D}{2}+\frac{D^2}{2}\right)\right]^{-1}x^2$$

$$=\frac{1}{2D}\left[1-\left(\frac{3D}{2}+\frac{D^2}{2}\right)+\left(\frac{3D}{2}+\frac{D^2}{2}\right)^2\ldots\right]x^2$$

$$=\frac{1}{2D}\left[1-\left(\frac{3D}{2}+\frac{D^2}{2}\right)+\frac{9D^2}{4}+\cdots\right]x^2$$

$$=\frac{1}{2D}\left[1-\frac{3D}{2}+\frac{7D^2}{4}+\cdots\right]x^2$$

$$=\frac{1}{2D}\left[x^2-\frac{3}{2}(2x)+\frac{7}{4}(2)\right]$$

$$= \frac{1}{2}\int\left(x^2 - 3x + \frac{7}{2}\right)dx = \frac{x^3}{6} - \frac{3x^2}{4} + \frac{7x}{4}$$

$\therefore$The general solution is $y = y_c + y_p$

$$y = a + be^{-x} + ce^{-2x} + \frac{x^3}{6} - \frac{3x^2}{4} + \frac{7x}{4},$$

where a, b, c are arbitrary constants.

P2.

Solve $\boldsymbol{(D^2 - 4D + 3)y = 2xe^{3x} + 3e^x \cos 2x}$

Solution:

The given equation is of the form $P(D)y = r(x)$, where $P(D) = (D^2 - 4D + 3)$ and $r(x) = 2xe^{3x} + 3e^x \cos 2x$

The A.E. is given by $P(m) = 0$

i.e., $m^2 - 4m + 3 = 0 \Longrightarrow m = 1, 3$

The C.F. is $y_c = ae^x + be^{3x}$

P.I. $= y_p = \frac{1}{D^2-4D+3}(2xe^{3x} + 3e^x \cos 2x)$

$$= 2\frac{1}{D^2-4D+3}xe^{3x} + 3\frac{1}{D^2-4D+3}e^x \cos 2x$$

$$= 2e^{3x}\frac{1}{(D+3)^2-4(D+3)+3}x + 3e^x\frac{1}{(D+1)^2-4(D+1)+3}\cos 2x$$

$$= 2e^{3x}\frac{1}{D^2+2D}x + 3e^x\frac{1}{D^2-2D}\cos 2x$$

$$= 2e^{3x}\frac{1}{2D\left(1+\frac{D}{2}\right)}x + 3e^x\frac{1}{-2^2-2D}\cos 2x$$

$$= e^{3x}\frac{1}{D}\left(1+\frac{D}{2}\right)^{-1}x - \frac{3}{2}e^x\frac{1}{D+2}\cos 2x$$

$$= e^{3x}\frac{1}{D}\left(1-\frac{D}{2}+\frac{D^2}{4}\dots.\right)x - \frac{3e^x}{2}\frac{D-2}{D^2-4}\cos 2x$$

$$= e^{3x}\left(\frac{1}{D}-\frac{1}{2}+\frac{D}{4}\dots.\right)x - \frac{3e^x}{2}\frac{D-2}{-2^2-4}\cos 2x$$

$$= e^{3x}\left(\frac{x^2}{2} - \frac{x}{2} + \frac{1}{4}\right) + \frac{3e^x}{16}(-2\ sin2x - 2\ cos2x)$$

$$= e^{3x}\left(\frac{x^2-x}{2}\right) + \frac{e^{3x}}{4} - \frac{3}{8}e^x(\ sin2x +\ cos2x)$$

The general solution of (1) is $y = y_c + y_p$

$$y = ae^x + be^{3x} + e^{3x}\left(\frac{x^2-x}{2}\right) + \frac{e^{3x}}{4} - \frac{3}{8}e^x(\ sin2x +\ cos2x),$$

where a, b are arbitrary constants.

P3.

Solve $(\boldsymbol{D^2+1})\boldsymbol{y} = \boldsymbol{x^2 sin2x}$

Solution:

The given equation is of the form $P(D)y = r(x)$, where $P(D) = (D^2+1)$ and $r(x) = x^2 sin2x$

The A.E is given by $P(m) = 0$, i.e., $m^2 + 1 = 0 \Longrightarrow m = \pm i$

The C.F is $y_c = acosx + bsinx$

P.I.$= y_p = \frac{1}{D^2+1}(x^2 sin2x)$

$= \text{I.P. of } \frac{1}{D^2+1} x^2 e^{2ix}$

$= \text{I.P. of } e^{2ix} \frac{1}{(D+2i)^2+1} x^2 = \text{I.P. of } e^{2ix} \frac{1}{D^2+4iD-3} x^2$

$= (-1/3) \text{ I.P. of } e^{2ix} \frac{1}{1-\frac{D^2+4iD}{3}} x^2$

$= (-1/3) \text{ I.P. of } e^{2ix} (1 - \frac{D^2+4iD}{3})^{-1} x^2$

$= (-1/3) \text{ I.P. of } e^{2ix} (1 + \frac{D^2+4iD}{3} + \frac{-16D^2}{9} + \cdots) x^2$

$= (-1/3) \text{ I.P. of } e^{2ix} \left[x^2 + \frac{2}{3} + \frac{4i}{3}(2x) - \frac{16}{9}(2)\right]$

$= (-1/3) \text{ I.P. of } (\cos 2x + i \sin 2x) \left[\left(x^2 - \frac{26}{9}\right) + \frac{8ix}{3}\right]$

$= (-1/3) \left[\frac{8x}{3} \cos 2x + \left(x^2 - \frac{26}{9}\right) \sin 2x\right]$

$\therefore$The general solution of (1) is $y = y_c + y_p$

i.e. $y = a\cos x + b\sin x - \frac{8}{9}x\ \cos 2x - \frac{1}{3}\left(x^2 - \frac{26}{9}\right) sin2x,$

where a, b are constants.

P4:

Solve $\left(\boldsymbol{D^2-4D+4}\right)\boldsymbol{y}=\boldsymbol{8x^2e^{2x}\sin 2x}$.

Solution:

The given equation is of the form $P(D)y=r(x)$, where $P(D)=D^2-4D+4$ and $r(x)=8x^2e^{2x}\sin 2x$.

The A.E is given by $P(m)=0$

i.e., $m^2-4m+4=0 \Rightarrow (m-2)^2=0 \Rightarrow m=2$ (twice)

The C.F. is given by $y_c=(a+bx)e^{2x}$

P.I. $=y_p=\frac{1}{(D-2)^2}8x^2e^{2x}\sin 2x=8e^{2x}\frac{1}{(D+2-2)^2}x^2\sin 2x$

$$=8e^{2x}\frac{1}{D^2}x^2\sin 2x=8e^{2x}\frac{1}{D}\int x^2\sin 2x\,dx$$

$$=8e^{2x}\frac{1}{D}\left[x^2\left(-\frac{\cos 2x}{2}\right)-\int(2x)\left(-\frac{\cos 2x}{2}\right)dx\right],$$

$$(integration\ by\ parts\ )$$

$$=8e^{2x}\frac{1}{D}\left[-\frac{1}{2}x^2\cos 2x+\int x\cos 2x\,dx\right]$$

$$=8e^{2x}\frac{1}{D}\left[-\frac{1}{2}x^2\cos 2x+x\left(\frac{\sin 2x}{2}\right)-\int 1\left(\frac{\sin 2x}{2}\right)dx\right]$$

$$=8e^{2x}\frac{1}{D}\left[-\frac{1}{2}x^2\cos 2x+\frac{1}{2}x\sin 2x+\frac{1}{4}\cos 2x\right]$$

$$=8e^{2x}\int\left(-\frac{1}{2}x^2\cos 2x+\frac{1}{2}x\sin 2x+\frac{1}{4}\cos 2x\right)dx$$

$$= 8e^{2x}\left[-\frac{1}{2}\int x^2\cos 2x\,dx + \frac{1}{2}\int x\sin 2x\,dx + \frac{1}{4}\int \cos 2x\,dx\right]$$

$$= 8e^{2x}\left[\begin{array}{c}-\frac{1}{2}\left\{x^2\left(\frac{1}{2}\sin 2x\right) - \int 2x\left(\frac{1}{2}\sin 2x\right)dx\right\}\\ +\frac{1}{2}\int x\sin 2x\,dx + \frac{1}{8}\sin 2x\end{array}\right]$$

$$= 8e^{2x}\left[-\frac{1}{4}x^2\sin 2x + \frac{1}{2}\int x\sin 2x\,dx + \frac{1}{2}\int x\sin 2x\,dx + \frac{1}{8}\sin 2x\right]$$

$$= 8e^{2x}\left[-\frac{1}{4}x^2\sin 2x + \int x\sin 2x\,dx + \frac{1}{8}\sin 2x\right]$$

$$= 8e^{2x}\left[-\frac{1}{4}x^2\sin 2x + x\left(-\frac{1}{2}\cos 2x\right) - \int 1\left(-\frac{1}{2}\cos 2x\right)dx + \frac{1}{8}\sin 2x\right]$$

$$= 8e^{2x}\left[-\frac{1}{4}x^2\sin 2x - \frac{1}{2}x\cos 2x + \frac{1}{4}\sin 2x + \frac{1}{8}\sin 2x\right]$$

$$= 8e^{2x}\left[-\frac{1}{4}x^2\sin 2x - \frac{1}{2}x\cos 2x + \frac{3}{8}\sin 2x\right]$$

The general solution is given by $y = C.F. + P.I. = y_c + y_p$

$$y = (a + bx)e^{2x} + 8e^{2x}\left[-\frac{1}{4}x^2\sin 2x - \frac{1}{2}x\cos 2x + \frac{3}{8}\sin 2x\right]$$

where a, b are arbitrary constants.

IP1.

Solve $\boldsymbol{D^2(D^2+4)y = 320(x^3+2x^2)}$

Solution:

The given equation is of the form $P(D)y = r(x)$, where $P(D) = D^2(D^2+4)$ and $r(x) = 320(x^3+2x^2)$

The A.E. is given by $P(m) = 0$

i.e.,$m^2(m^2+4) = 0 \Longrightarrow m = 0(twice), \pm 2i$

$\therefore$The C.F is $y_c = (a+bx)e^{0.x} + (c\cos 2x + d\sin 2x)$

$\Longrightarrow y_c = (a+bx) + c\cos 2x + d\sin 2x$

Now, P.I.= $y_p = \frac{1}{D^2(D^2+4)} 320(x^3+2x^2)$

$$= \frac{1}{4D^2\left(1+\frac{D^2}{4}\right)} 320(x^3+2x^2) = \frac{80}{D^2}\left(1+\frac{D^2}{4}\right)^{-1}(x^3+2x^2)$$

$$= \frac{80}{D^2}\left[1 - \frac{D^2}{4} + \frac{D^4}{16} - \frac{D^6}{64} + \cdots\right](x^3+2x^2)$$

$$= 80\left[\frac{1}{D^2} - \frac{1}{4} + \frac{D^2}{16} - \frac{D^4}{16} + \cdots\right](x^3+2x^2)$$

$$= 80\left[\frac{1}{D^2}(x^3+2x^2) - \frac{1}{4}(x^3+2x^2) + \frac{D^2}{16}(x^3+2x^2) + \cdots\right]$$

$$= 80\left[\int\left(\int (x^3+2x^2)\,dx\right)dx - \frac{1}{4}(x^3+2x^2) + \frac{6x+4}{16}\right]$$

$$= 80\left[\frac{x^5}{20} + \frac{x^4}{6} - \frac{x^3}{4} - \frac{x^2}{2} + \frac{3x+2}{8}\right]$$

$$y_p = 4x^5 + \frac{40x^4}{3} - 20x^3 - 40x^2 + 30x + 20$$

∴The general solution is $y = y_c + y_p$

$$y = a + bx + c\cos 2x + d\sin 2x + 4x^5 + \frac{40}{3}x^4$$
$$-20x^3 - 40x^2 + 30x + 20,$$

where a, b, c, d are arbitrary constant.

IP2:

Solve $\left(D^2 - 5D + 6\right)y = e^{4x}\left(x^2 + 9\right)$.

Solution:

The given equation is of the form $P(D)y = r(x)$, where
$P(D) = D^2 - 5D + 6$ and $r(x) = e^{4x}(x^2 + 9)$.

The A.E is given by $P(m) = 0$

i.e., $m^2 - 5m + 6 = 0 \Rightarrow (m-2)(m-3) = 0 \Rightarrow m = 2, 3$

The C.F. is given by $y_c = ae^{2x} + be^{3x}$

P.I. $= y_p = \frac{1}{D^2-5D+6} e^{4x}(x^2 + 9)$

$$= e^{4x} \frac{1}{(D+4)^2-5(D+4)+6}(x^2+9) = e^{4x}\frac{1}{D^2+3D+2}(x^2+9)$$

$$= \frac{e^{4x}}{2}\frac{1}{1+\frac{D^2}{2}+\frac{3D}{2}}(x^2+9) = \frac{1}{2}e^{4x}\left\{1+\left(\frac{D^2}{2}+\frac{3D}{2}\right)\right\}^{-1}(x^2+9)$$

$$= \frac{1}{2}e^{4x}\left\{1-\left(\frac{D^2}{2}+\frac{3D}{2}\right)+\left(\frac{D^2}{2}+\frac{3D}{2}\right)^2-\cdots\right\}(x^2+9)$$

$$= \frac{1}{2}e^{4x}\left\{1-\frac{D^2}{2}-\frac{3D}{2}+\frac{9D^2}{4}+\cdots\right\}(x^2+9)$$

$$= \frac{1}{2}e^{4x}\left\{(x^2+9)-\frac{3}{2}D(x^2+9)+\frac{7}{4}D^2(x^2+9)\right\}$$

$$= \frac{1}{2}e^{4x}\left\{x^2+9-\frac{3}{2}(2x)+\frac{7}{4}(2)\right\}$$

$$= \frac{1}{2} e^{4x} \left(x^2 - 3x + \frac{25}{2} \right) = \frac{1}{4} e^{4x} (2x^2 - 6x + 25)$$

The general solution is given by $y = C.F. + P.I. = y_c + y_p$

i.e., $y = ae^{2x} + be^{3x} + \frac{1}{4} e^{4x} (2x^2 - 6x + 25)$, where a, b are arbitrary constants.

IP3:

Solve $(D^4 + 2D^2 + 1)y = x^2 \cos x$.

Solution:

The given equation is of the form $P(D)y = r(x)$, where $P(D) = D^4 + 2D^2 + 1$ and $r(x) = x^2 \cos x$.

The A.E is given by $P(m) = 0$ i.e., $m^4 + 2m^2 + 1 \ = 0$

$$\Rightarrow (m^2 + 1)^2 = 0 \Rightarrow m = \pm i \ (twice)$$

The C.F. is given by $y_c = (a + bx)\cos x + (c + dx)\sin x$

P.I. $= y_p = \frac{1}{(D^2+1)^2} x^2 \cos x$

$= \text{R.P of } \frac{1}{(D^2+1)^2} x^2 e^{ix}$

$= \text{R.P of } e^{ix} \frac{1}{\{(D+i)^2+1\}^2} x^2$

$= \text{R.P of } e^{ix} \frac{1}{(D^2+2Di)^2} x^2$

$= \text{R.P of } e^{ix} \frac{1}{(2Di)^2\left(1+\frac{D}{2i}\right)^2} x^2$

$= \text{R.P of } \frac{e^{ix}}{-4D^2}\left(1 + \frac{D}{2i}\right)^{-2} x^2$

$= \text{R.P of } \frac{e^{ix}}{-4D^2}\left(1 - \frac{iD}{2}\right)^{-2} x^2$

$$= \text{R.P of } \frac{e^{ix}}{-4D^2}\left(1 + iD + \frac{3i^2D^2}{4} + \cdots\right)x^2$$

$$= \text{R.P of } \frac{e^{ix}}{-4D^2}\left(1 + iD - \frac{3D^2}{4}\right)x^2$$

$$= \text{R.P of } \frac{e^{ix}}{-4D^2}\left(x^2 + 2ix - \frac{3}{4}\cdot 2\right)$$

$$= \text{R.P of } \frac{e^{ix}}{-4}\frac{1}{D}\left(\frac{x^3}{3} + ix^2 - \frac{3}{2}x\right)$$

$$= \text{R.P of } \frac{e^{ix}}{-4}\left(\frac{x^4}{12} + \frac{ix^3}{3} - \frac{3x^2}{4}\right)$$

$$= \text{R.P of } \left(-\frac{1}{4}\right)(\cos x + i\sin x)\left(\frac{x^4}{12} - \frac{3x^2}{4} + \frac{ix^3}{3}\right)$$

$$= -\frac{1}{4}\left[\left(\frac{x^4}{12} - \frac{3x^2}{4}\right)\cos x - \frac{x^3}{3}\sin x\right]$$

The general solution is given by $y = C.F. + P.I. = y_c + y_p$

$$y = (a + bx)\cos x + (c + dx)\sin x - \frac{1}{4}\left[\left(\frac{x^4}{12} - \frac{3x^2}{4}\right)\cos x - \frac{x^3}{3}\sin x\right]$$

where a, b, c, d are arbitrary constants.

IP4:

Solve $\boldsymbol{(D^5 - D)y = 12e^x + 8\sin x - 2x.}$

Solution:

The given equation is of the form $P(D)y = r(x)$, where $P(D) = D^5 - D$ and $r(x) = 12e^x + 8\sin x - 2x$.

The A.E is given by $P(m) = 0$ i.e., $m^5 - m = 0$

$\Rightarrow m(m^2 - 1)(m^2 + 1) = 0 \Rightarrow m = 0, \pm 1, \pm i$

The C.F. is given by $y_c = a + be^x + ce^{-x} + d\cos x + e\sin x$

P.I. $= y_p = \frac{1}{D(D^2-1)(D^2+1)}(12e^x + 8\sin x - 2x)$

$= 12\frac{1}{D(D^2-1)(D^2+1)}e^x + 8\frac{1}{D(D^2-1)(D^2+1)}\sin x - 2\frac{1}{D(D^2-1)(D^2+1)}x$

$= 12\frac{1}{(D-1)D(D+1)(D^2+1)}e^x + 8\frac{1}{(D^2+1)D(D^2-1)}\sin x + 2\frac{1}{D(1-D^2)(1+D^2)}x$

$= 12\frac{1}{(D-1)1(1+1)(1+1)}e^x + 8\frac{1}{(D^2+1)D(-1^2-1)}\sin x$

$+2\frac{1}{D}(1 - D^2)^{-1}(1 + D^2)^{-1}x$

$= 3\frac{1}{D-1}e^x - 4\frac{1}{(D^2+1)}\left[\frac{1}{D}\sin x\right] + 2\frac{1}{D}(1 + D^2 + \cdots)(1 - D^2 + \cdots)x$

$= 3\frac{x}{1!}e^x + 4\frac{1}{D^2+1}\cos x + 2\frac{1}{D}(1 + D^2 - D^2 + \cdots)x$

$= 3xe^x + 4\left(\frac{x}{2\times 1}\sin x\right) + 2\frac{1}{D}x = 3xe^x + 2x\sin x + x^2$

The general solution is given by $y = C.F. + P.I. = y_c + y_p$

$$y = a + be^{x} + ce^{-x} + d\cos x + e\sin x + 3xe^{x}$$

$$+2x\sin x + x^2 ,$$

where a, b, c, d, e are arbitrary constants.

2.7

Solutions of Non-homogeneous Linear Differential Equations with Constant Coefficients (Part-2)

EXERCISES:

I. Solve the following differential equations:

a) $(D^4 - D^2)y = 2$

b) $(D^2 + D)y = x^2 + 2x + 4$

c) $(D^4 - a^4)y = x^4$

d) $(D^3 + 8)y = x^4 + 2x + 1$

e) $(D^2 + 2D + 2)y = x^2$

f) $(D^2 - 4D + 4)y = x^2$

g) $(D^3 + 3D^2 + 2D)y = x^2$

h) $(D^4 - 2D^3 + D^2)y = x$

i) Solve the IVP: $\frac{d^2y}{dx^2} = a + bx + cx^2$, given that $y(0) = d$, $y'(0) = 0$

j) $(D^3 - D^2 - D - 2)y = x.$

k) $(D^3 - D^2 - 6D)y = x^2 + 1$

Answers

a) $y = c_1 + c_2x + c_3e^x + c_4e^x - x^2$

b) $y = a + be^{-x} + \frac{x^3}{3} + 4x$

c) $y = c_1e^{ax} + c_2e^{-ax} + c_3 \cos ax + c_4 \sin ax - \left(\frac{1}{a^4}\right)\left(x^4 + \frac{24}{a^4}\right)$

d) $y = c_1e^{-2x} + e^x\left(c_2 \cos x\sqrt{3} + c_3 \sin x\sqrt{3}\right) + (x^4 - x + 1)/8.$

e) $y = (c_1cosx + c_2sinx)e^{-x} + (x^2 - 2x + 1)/2$

f) $y = (c_1 + c_2x)e^{2x} + (2x^2 + 4x + 3)/4$

g) $y = c_1 + c_2e^{-x} + c_2e^{-2x} + \left(\frac{x^3}{6}\right) - \left(\frac{3x^2}{4}\right) + \left(\frac{7x}{4}\right).$

h) $y = c_1 + c_2x + c_2x + (c_3 + c_4x)e^x - \left(\frac{x^3}{6}\right) + x^2$

i) $y = e^{-x\sqrt{\frac{7}{2}}}\left(c_1 \cos\frac{3x}{2} + c_2 \sin\frac{3x}{2}\right)$

$+e^{x\sqrt{\frac{7}{2}}}\left(c_3 \cos\frac{3x}{2} + c_4 \sin\frac{3x}{2}\right) + x^2 + \frac{127}{8}$

j) $y = d + \left(\frac{1}{2}\right) ax^2 + \left(\frac{1}{6}\right) bx^3 + \left(\frac{1}{12}\right) cx^4$

k) $y = c_1e^{2x} + e^{\frac{-x}{2}}\left\{c_3 \cos\left(x\sqrt{\frac{3}{2}}\right) + c_4 \sin\left(x\sqrt{\frac{3}{2}}\right)\right\} - \left(\frac{1}{8}\right)(4x - x^2).$

l) $y = c_1 + c_2e^{3x} + c_3e^{-2x} - \left(\frac{1}{18}\right)\left(x^3 - \frac{x^2}{2} + \frac{25x}{6}\right)$

II. Solve the following differential equations:

A. $(D^2 - 2D + 1)y = x^2e^{3x}$

B. $(D^2 - 2D + 1)y = x^2e^x$

C. $(D^2 - 2D + 1)y = xe^x$

D. $(D^3 - 3D - 2)y = 540x^3e^{-x}$

E. $(D^2 + 3D + 2)y = e^{2x} \sin x$

F. $(D^4 - 1)y = e^x \cos x$

G. $(D^3 - D^2 + 3D + 5)y = e^x \cos x$

Answers:

A. $y = (c_1 + c_2x)e^x + \left(\frac{1}{8}\right)e^{3x}(2x^2 - 4x + 3)$

B. $y = (c_1 + c_2x)e^x + \left(\frac{1}{12}\right)x^4e^x$

C. $y = (c_1 + c_2x)e^x + \left(\frac{1}{6}\right)x^3e^x$

D. $y = c_1e^{2x} + (c_2 + c_3x)e^{-x} - e^{-x}(9x^5 + 15x^4 + 20x^3 + 20x^2)$

E. $y = c_1e^{-2x} + c_2e^{-x} - \left(\frac{1}{170}\right)e^{2x}(11\sin x - 7\cos x)$

F. $y = c_1e^x + c_2e^{-x} + c_3\cos x + c_4\sin x - \left(\frac{1}{5}\right)e^x\cos x$

G. $y = c_1e^{-x} + e^x(c_2\cos 2x + c_3\sin 2x) + \left(\frac{1}{34}\right) \times e^x(3\sin x + 5\cos x)$

III. Solve the following differential equations:

1. $(D^2 - 4D + 4)y = x^2 + e^x + cos2x$
2. $(D^2 - 1)y = xe^x + cos^2x$
3. $(D^4 - 4D^2 - 5)y = e^x(x + cosx)$
4. $\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} - 4\frac{dy}{dx} - 2y = e^x + cosx$
5. $(D^2 + a^2)y = sinax + xe^{2x}$
6. $(D^2 - 6D + 8)y = (e^{2x} - 1)^2 + sin3x$

Answers:

1. $y = (c_1 + c_2x)e^{2x} + \frac{1}{8}(2x^2 + 4x + 3) - \frac{1}{8}sin2x$
2. $y = c_1e^x + c_2e^{-x} + \frac{1}{4}e^x(x^2 - x) - \frac{1}{2} - \frac{1}{10}cos2x$
3. $y = c_1coshx\sqrt{5} + c_1sinhx\sqrt{5} + c_3cosx + c_4sinx$
$$- \frac{e^x}{16}(2x - 1)$$

4. $y = e^{x}(c_1 + c_2 cosx + c_3 sinx) + xe^{x} + \left(\frac{3sinx + cosx}{10}\right)$

5. $y = c_1 cosx + c_2 sinax - \left(\frac{x}{2a}\right) cosax$

$$+e^{2x}(4 + a^2)^{-2}(4x + xa^2 - 4)$$

6. $y = c_1 e^{2x} + c_2 e^{4x} + \left(\frac{x}{2}\right) e^{4x}$

$$+xe^{2x} + \frac{1}{8} + \frac{1}{325}(18cos3x - sin3x)$$

2.8

Method of Variation of Parameters

Learning objectives:

* To learn the Method of Variation of Parameters to solve 2^{nd} and 3^{rd} order non-homogeneous linear differential equations

  AND

* To solve the related problems.

Method of Variation of Parameters

Consider the second-order non-homogeneous linear differential equation

$$a_0(x)y'' + a_1(x)y' + a_2(x)y = r(x)\,, a_0(x) \neq 0 \ldots (1)$$

Let the equation (1) be normal on an interval $\boldsymbol{I}$. Here, we discuss a general method of solving equation (1), called the method of ***Variation of Parameters***. This method can always be used to find a particular integral whenever the complementary function of (1) is known.

Consider first the general solution of the corresponding homogeneous equation

$$a_0(x)y'' + a_1(x)y' + a_2(x)y = 0\,, a_0(x) \neq 0 \ \ldots (2)$$

Let $y_1(x)$ and $y_2(x)$ be two linearly independent solutions of (2). The complementary function of (1) is given by

$$y_c(x) = Ay_1(x) + By_2(x),$$

where A and B are arbitrary constants.

The concept behind this method is to vary the parameters A and$B$. That is, we assume A and B to be functions of x and determine $A(x)$ and $B(x)$ such that

$$y(x) = A(x)y_1(x) + B(x)y_2(x) \quad ...(3)$$

is the general solution of (1). Clearly, we need two equations to determine $A(x)$ and $B(x)$. Now,

$$y' = A'y_1 + Ay_1' + B'y_2 + By_2'$$
$$= (A'y_1 + B'y_2) + (Ay_1' + By_2')$$

Choose $A(x), B(x)$ such that $A'y_1 + B'y_2 = 0$.

Then, $y' = Ay_1' + By_2'$ and

$$y'' = (A'y_1' + B'y_2') + Ay_1'' + By_2''$$

Substituting y, y', y'' in (1), we get

$$a_0(x)[(A'y_1' + B'y_2') + Ay_1'' + By_2'']$$
$$+a_1(x)[A\,y_1' + By_2'] + a_2(x)[Ay_1 + By_2] = r(x)$$

i.e., $a_0(x)[A'y_1' + B'y_2'] + A[a_0(x)y_1'' + a_1(x)y_1' + a_2(x)y_1]$

$$+B[a_0(x)y_2'' + a_1(x)y_2' + a_2(x)y_2] = r(x)$$

Since y_1, y_2 are solutions of the homogeneous equation (2), the above equation becomes,

$$a_0(x)[A'y_1' + B'y_2'] = r(x)$$
$$\text{i.e., } A'y_1' + B'y_2' = g(x)$$

where $g(x) = \frac{r(x)}{a_0(x)}$ is continuous on $\boldsymbol{I}$, since $a_0(x) \neq 0$ on $\boldsymbol{I}$.

Thus $A(x)$ and $B(x)$ are chosen such that

$A^{'}y_1 + B^{'}y_2 = 0$ and $A^{'}y_1{'} + B^{'}y_2{'} = g(x)$ … … … (4)

where $g(x) = \frac{r(x)}{a_0(x)}$

Solving the above equations, we obtain

$$A^{'} = -\frac{g(x)y_2}{y_1y_2{'} - y_2y_1{'}} \quad , \; B^{'} = \frac{g(x)y_1}{y_1y_2{'} - y_2y_1{'}}$$

Notice that $y_1y_2{'} - y_2y_1{'} = W(y_1, y_2) = W(x)$. Further, $W(x) \neq 0$, since y_1, y_2 are linearly independent of solutions of (2). Thus,

$$A^{'} = -\frac{g(x)y_2}{W(x)} \quad , \; B^{'} = \frac{g(x)y_1}{W(x)}$$

Integrating, we obtain

$A(x) = -\int \frac{g(x)y_2}{W(x)} dx + c_1$ and $B(x) = \int \frac{g(x)y_1}{W(x)} dx + c_2$ --- (5)

where c_1, c_2 are arbitrary constants . The general solution of (1) is given by

$$y = A(x)y_1(x) + B(x)y_2(x)$$

$$= c_1y_1(x) + cy_2(x) + \left(-\int \frac{g(x)y_2}{W(x)} dx\right) y_1(x) + \left(\int \frac{g(x)y_1}{W(x)} dx\right) y_2(x)$$

If we do not add arbitrary constant while carrying out integration in (5),then we get the P.I as

$$y_p(x) = A(x)y_1 + B(x)y_2$$

Note:

(1) The method of variation of parameters is applicable for non-homogeneous normal linear equations with constant coefficients and variable coefficients.

(2) The method does not depend on the form of $r(x)$.

Example 1: Solve: $y'' + 16y = 32\sec 2x$.

Solution: We apply the method of variation of parameters to obtain a general solution.

A.E.: $m^2 + 16 = 0 \Rightarrow m = \pm 4i$

The linearly independent solutions of the homogeneous equation are $y_1 = \cos 4x$, $y_2 = \sin 4x$

C.F.: $y_c(x) = Ay_1 + By_2 = A\cos 4x + B\sin 4x$

By the method of variation of parameters, we write the general solution

$$y(x) = A(x)\cos 4x + B(x)\sin 4x,$$

where $A(x) = -\int \frac{g(x)y_2(x)}{W(x)}dx + c_1$

$$B(x) = \int \frac{g(x)y_1(x)}{W(x)}dx + c_2,\ g(x) = \frac{r(x)}{a_0(x)} = 32\sec x$$

and

$$W(x) = W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos 4x & \sin 4x \\ -4\sin 4x & 4\cos 4x \end{vmatrix} = 4$$

Therefore,

$$A(x) = -\frac{1}{4}\int 32 \sec 2x \sin 4x \, dx + c_1$$

$$= -16\int \sin 2x \, dx + c_1 = 8\cos 2x + c_1$$

$$B(x) = \frac{1}{4}\int 32 \sec 2x \cos 4x \, dx + c_2 = 8\int \frac{2\cos^2 2x - 1}{\cos 2x} dx + c_2$$

$$= 8\int (2\cos 2x - \sec 2x) dx + c_2$$

$$= 8\sin 2x - 4\ln|\sec 2x + \tan 2x| + c_2$$

The general solution is

$$y(x) = A(x)\cos 4x + B(x)\sin 4x$$

$$= c_1 \cos 4x + c_2 \sin 4x + 8\cos 2x \cos 4x$$

$$+8\sin 2x \sin 4x - 4\sin 4x \ln|\sec 2x + \tan 2x|$$

$$= c_1 \cos 4x + c_2 \sin 4x + 8\cos 2x$$

$$-4\sin 4x \ln|\sec 2x + \tan 2x|,$$

where c_1 and c_2 are arbitrary constants.

Example 2: Solve: $y'' - 2y' + y = xe^x \ln x \ , x > 0.$

Solution: We apply the method of variation of parameters to obtain a general solution.

A.E.: $m^2 - 2m + 1 = 0 \Rightarrow m = 1$(twice)

The linearly independent solutions of the homogeneous

equation are $y_1 = e^x$, $y_2 = xe^x$

C.F.: $y_c(x) = Ay_1 + By_2 = Ae^x + Bxe^x$

By the method of variation of parameters, we write the general solution

$$y(x) = A(x)e^x + B(x)xe^x$$

where $A(x) = -\int \frac{g(x)y_2(x)}{W(x)} dx + c_1$

$$B(x) = \int \frac{g(x)y_1(x)}{W(x)} dx + c_2 \ , \quad g(x) = \frac{r(x)}{a_0(x)} = xe^x \ln x$$

and

$$W(x) = W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^x & xe^x \\ e^x & (x+1)e^x \end{vmatrix} = e^{2x}$$

Therefore,

$$A(x) = -\int \frac{xe^x \ln x \cdot xe^x}{e^{2x}} dx + c_1 = -\int x^2 \ln x \, dx + c_1$$

$$= -\left[\ln x \cdot \frac{x^3}{3} - \int \frac{1}{x} \cdot \frac{x^3}{3} dx\right] + c_1 = -\frac{x^3}{3} \ln x + \frac{x^3}{9} + c_1$$

$$B(x) = \int \frac{xe^x \ln x \cdot e^x}{e^{2x}} dx + c_2 = \int x \ln x \, dx + c_2$$

$$= \ln x \cdot \frac{x^2}{2} - \int \frac{1}{x} \cdot \frac{x^2}{2} dx + c_2 = \frac{x^2}{2} \ln x - \frac{x^2}{4} + c_2$$

The general solution is

$$y(x) = A(x)e^x + B(x)xe^x$$

$$= \left(-\frac{x^3}{3}\ln x + \frac{x^3}{9} + c_1\right)e^x + \left(\frac{x^2}{2}\ln x - \frac{x^2}{4} + c_2\right)xe^x$$

$$= (c_1 + c_2 x)e^x + \left[\left(\frac{1}{2} - \frac{1}{3}\right)\ln x + \left(\frac{1}{9} - \frac{1}{4}\right)\right]x^3 e^x$$

$$= (c_1 + c_2 x)e^x + \left(\frac{1}{6}\ln x - \frac{5}{36}\right)x^3 e^x$$

where c_1 and c_2 are arbitrary constants.

Example 3: Find a P.I. and a general solution of

$$(x-1)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = (x-1)^2\ , x \neq 1$$

if $y = e^x$ and $y = x$ are the solutions of the homogeneous equation corresponding to the above equation.

Solution:

Given that, $y_1 = e^x$ and $y_2 = x$ are the solutions of the homogeneous equation corresponding to the given equation.

Now, $W(x) = W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^x & x \\ e^x & 1 \end{vmatrix}$

$$= e^x(1-x) \neq 0$$

Thus, y_1, y_2 are linearly independent.

C.F.: $y_c(x) = c_1 e^x + c_2 x$

P.I.: $y_p(x) = A(x)y_1 + B(x)y_2$

where $A(x) = -\int \frac{g(x)y_2(x)}{W(x)} dx$, $B(x) = \int \frac{g(x)y_1(x)}{W(x)} dx$.

Here $g(x) = \frac{r(x)}{a_0(x)} = \frac{(x-1)^2}{x-1} = x - 1$

Now, $A(x) = -\int \frac{(x-1)\cdot x}{e^x(1-x)} dx + c_1 = \int xe^{-x} dx = -(1+x)e^{-x}$

$$B(x) = \int \frac{(x-1)e^x}{e^x(1-x)} dx = -x$$

Therefore, $y_p(x) = A(x)y_1 + B(x)y_2$

$$= -(1+x)e^{-x}e^x + (-x)x = -(1+x+x^2)$$

The general solution is $y = y_c + y_p$

i.e., $y = c_1 e^x + c_2 x - (1 + x + x^2)$

where c_1 and c_2 are arbitrary constants.

Extension to higher order equations:

The method of variation of parameters can be extended to non-homogeneous (normal) linear differential equations of any order. At each differentiation step, we set the part containing the derivatives of unknown function to zero, until we arrive the final substitution step.

For example, consider the third order normal non-homogeneous linear differential equation

$$a_0(x)y''' + a_1(x)y'' + a_2(x)y' + a_3(x)y = r(x)\ , a_0(x) \neq 0 \quad \text{--- (1)}$$

Let the C.F of (1) be

$$y(x) = Ay_1(x) + By_2(x) + Cy_3(x)$$

where A, B, C are arbitrary constants and y_1, y_2, y_3 are linearly independent solutions of the corresponding homogeneous equation

$$a_0(x)y''' + a_1(x)y'' + a_2(x)y' + a_3(x)y = 0 \quad \text{--- (2)}$$

We assume the general solution as

$$y(x) = A(x)y_1(x) + B(x)y_2(x) + C(x)y_3(x) \quad \text{--- (3)}$$

where $A(x), B(x), C(x)$ are to be determined.

Following the procedure discussed earlier, we obtain the required equations for determining $A(x), B(x)$ and $C(x)$ as

$$\left.\begin{array}{r} A'y_1 + B'y_2 + C'y_3 = 0 \\ A'y_1' + B'y_2' + C'y_3' = 0 \\ A'y_1'' + B'y_2'' + C'y_3'' = \frac{r(x)}{a_0(x)} = g(x) \end{array}\right\} \quad \text{------ (4)}$$

Notice that $W(y_1, y_2, y_3) \neq 0$, since y_1, y_2, y_3 are linearly independent solutions of (2). We now determine $A(x), B(x)$ and $C(x)$ and substitute in (3), we get the general solution.

Example 4: Solve: $y''' + 4y' = \sec 2x$.

Solution: We apply the method of variation of parameters to obtain a general solution.

A.E.: $m^3 + 4m = 0 \Rightarrow m(m^2 + 2^2) = 0 \Rightarrow m = 0, \pm 2i$

The linearly independent solutions of the homogeneous

equation are $y_1 = 1\,, y_2 = \cos 2x\ \,, y_3 = \sin 2x$ and

$$W(x) = W(y_1, y_2, y_3) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}$$

$$= \begin{vmatrix} 1 & \cos 2x & \sin 2x \\ 0 & -2\sin 2x & 2\cos 2x \\ 0 & -4\cos 2x & -4\sin 2x \end{vmatrix} = 8$$

C.F.: $y_c(x) = Ay_1 + By_2 + Cy_3 = A + B\cos 2x + C\sin 2x$

By the method of variation of parameters, we assume the general solution as

$$y(x) = A(x)y_1 + B(x)y_2 + C(x)y_3$$

$$= A(x) + B(x)\cos 2x + C(x)\sin 2x$$

The equations for determining $A(x), B(x)$ and $C(x)$ are:

$$A'(x)y_1 + B'(x)y_2 + C'(x)y_3 = 0$$
$$A'(x)y_1' + B'(x)y_2' + C'(x)y_3' = 0$$
$$A'(x)y_1'' + B'(x)y_2'' + C'(x)y_3'' = \frac{r(x)}{a_0(x)} = g(x)$$

i.e., $A'(x) + B'(x)\cos 2x + C'(x)\sin 2x = 0$

$$-2B'(x)\cos 2x + 2C'(x)\cos 2x = 0$$

$$-4B'(x)\sin 2x - 4C'(x)\sin 2x = \sec 2x$$

Now, W_i is same as $W(x)$ with the i^{th} column replaced by the column $\begin{bmatrix} 0 \\ 0 \\ secx \end{bmatrix}$ for $i = 1, 2, 3$.

$$W_1(x) = \begin{vmatrix} 0 & \cos 2x & \sin 2x \\ 0 & -2\sin 2x & 2\cos 2x \\ \sec 2x & -4\cos 2x & -4\sin 2x \end{vmatrix}$$

$$= \sec 2x\,(2\cos^2 2x + 2\sin^2 2x) = 2\sec 2x$$

$$W_2(x) = \begin{vmatrix} 1 & 0 & \sin 2x \\ 0 & 0 & 2\cos 2x \\ 0 & \sec 2x & -4\sin 2x \end{vmatrix} = -2\cos 2x \sec 2x = -2$$

$$W_3(x) = \begin{vmatrix} 1 & \cos 2x & 0 \\ 0 & -2\sin 2x & 0 \\ 0 & -4\cos 2x & \sec 2x \end{vmatrix} = -2\sin 2x \sec 2x$$

$$= -2\tan 2x$$

By Cramer's Rule,

$$A'(x) = \frac{W_1(x)}{W(x)}\ , B'(x) = \frac{W_2(x)}{W(x)}\ , C'(x) = \frac{W_3(x)}{W(x)}$$

i.e., $A'(x) = \frac{1}{4}\sec 2x\,, B'(x) = \frac{1}{8}(-2) = -\frac{1}{4}\ , C'(x) = -\frac{1}{4}\tan 2x$

Integrating, we get

$$A(x) = \frac{1}{4}\int \sec 2x\,dx + c_1 = \frac{1}{8}\ln|\sec 2x + \tan 2x| + c_1$$

$$B(x) = -\frac{x}{4} + c_2$$

$$C(x) = -\frac{1}{4}\int \tan 2x \, dx + c_3 = \frac{1}{8}\ln|\cos 2x| + c_3$$

The general solution is given by

$$y = A(x) + B(x)\cos 2x + C(x)\sin 2x$$

$$= c_1 + c_2 \cos 2x + c_3 \sin 2x + \frac{1}{8}\ln|\sec 2x + \tan 2x|$$

$$-\frac{x}{4}\cos 2x + \frac{1}{8}\sin 2x \ln|\cos 2x|$$

where c_1, c_2, c_3 are arbitrary constants.

P1:

Solve $(D^2-1)y = e^{-x}\sin(e^{-x}) + \cos(e^{-x})$ by the Method of Variation of Parameters.

Solution:

The given differential equation is of the form $P(D)y = r(x)$, where $P(D) = D^2 - 1$ and $r(x) = e^{-x}\sin(e^{-x}) + \cos(e^{-x})$

The A.E is given by $P(m) = 0$, i.e., $m^2 - 1 = 0 \Longrightarrow m = \pm 1$

The linearly independent solutions of the corresponding homogeneous equation of the given equation are $y_1 = e^x$

, $y_2 = e^{-x}$.

∴ The C.F: $y_c(x) = Ay_1 + By_2 = Ae^x + Be^{-x}$

By the method of variation of parameters, we assume the general solution as $y(x) = A(x)e^x + B(x)e^{-x}$

where $A(x) = -\int \frac{g(x)y_2(x)}{W(x)}dx + c_1$

$$B(x) = \int \frac{g(x)y_1(x)}{W(x)}dx + c_2$$

$$g(x) = \frac{r(x)}{a_0(x)} = e^{-x}\sin(e^{-x}) + \cos(e^{-x})$$

and

$$W(x) = W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = -2 \neq 0$$

Therefore, $A(x) = \frac{1}{2}\int [e^{-x}\sin(e^{-x}) + \cos(e^{-x})]\, e^{-x}\, dx + c_1$

$$A(x) = \frac{1}{2}\int e^{-2x}\sin(e^{-x})\, dx + \frac{1}{2}\int e^{-x}\cos(e^{-x})\, dx + c_1$$

Put $e^{-x} = t \Longrightarrow -e^{-x}dx = dt$

$$= -\frac{1}{2}\int t\sin t\, dt - \frac{1}{2}\int \cos t\, dt + c_1$$

$$= -\frac{1}{2}\left[-t\cos t + \int \cos t\, dt\right] - \frac{1}{2}\sin t + c_1$$

$$= -\frac{1}{2}\left[-t\, cos\, t + \sin t\right] - \frac{1}{2}\sin t + c_1$$

$$= \frac{t}{2}\cos t - \sin t + c_1 = \frac{e^{-x}}{2}\cos(e^{-x}) - \sin(e^{-x}) + c_1$$

$$B(x) = -\frac{1}{2}\int [e^{-x}\sin(e^{-x}) + \cos(e^{-x})]\, e^{x}dx + c_2$$

$$= -\frac{1}{2}\int \sin(e^{-x})\, dx - \frac{1}{2}\int e^{x}\cos(e^{-x})\, dx + c_2$$

$$= -\frac{1}{2}\int \sin(e^{-x})\, dx - \frac{1}{2}[e^{x}\cos(e^{-x}) - \int e^{x}(-\sin(e^{-x}))(-e^{-x})dx] + c_2$$

$$= -\frac{1}{2}\int \sin(e^{-x})\, dx - \frac{1}{2}e^{x}\cos(e^{-x}) + \frac{1}{2}\int \sin(e^{-x})\, dx + c_2$$

$$= -\frac{1}{2}e^{x}\cos(e^{-x}) + c_2$$

The general solution is $y(x) = A(x)e^{x} + B(x)e^{-x}$

i.e., $y(x) = c_1e^{x} + c_2e^{-x} + \frac{1}{2}\cos(e^{-x}) - e^{x}\sin(e^{-x}) - \frac{1}{2}\cos(e^{-x})$

$$= c_1e^{x} + c_2e^{-x} - e^{x}\sin(e^{-x}),$$

where c_1, c_2 are arbitrary constants

P2:

Solve $(D^3 + D)y = \sec x$ by the Method of Variation of Parameters.

Solution:

The given differential equation is of the form $P(D)y = r(x)$, where $P(D) = D^3 + D$ and $r(x) = \sec x$

The A.E. is given by $P(m) = 0$

i.e., $m^3 + m = 0 \Rightarrow m(m^2 + 1) = 0 \Rightarrow m = 0, \pm i$

The linearly independent solutions of the corresponding homogeneous equation of the given equation are

$$y_1 = 1\,, y_2 = \cos x\ , y_3 = \sin x$$

$$W(x) = W(y_1, y_2, y_3) = \begin{vmatrix} 1 & \cos x & \sin x \\ 0 & -\sin x & \cos x \\ 0 & -\cos x & -\sin x \end{vmatrix} = 1$$

$\therefore$ The C.F.: $y_c(x) = Ay_1 + By_2 + Cy_3 = A + B\cos x + C\sin x$

By the method of variation of parameters, we assume the general solution as

$$y(x) = A(x) + B(x)\cos x + C(x)\sin x$$

The equations for obtaining of $A(x), B(x)$ and $C(x)$ are

$$A^{'}(x)y_1 + B^{'}(x)y_2 + C^{'}(x)y_3 = 0$$

$A^{'}(x)y_1{}' + B^{'}(x)y_2{}' + C^{'}(x)y_3{}' = 0$

$A^{'}(x)y_1{}'' + B^{'}(x)y_2{}'' + C^{'}(x)y_3{}'' = g(x)$, where $g(x) = \frac{a_0(x)}{r(x)}$

$\Rightarrow A^{'}(x) + B^{'}(x)\cos x + C^{'}(x)\sin x = 0$

$\Rightarrow \qquad -B^{'}(x)\sin x + C^{'}(x)\cos x = 0$

$\Rightarrow \qquad -B^{'}(x)\cos x - C^{'}(x)\sin x = \sec x$

Now, W_i is same as $W(x)$ with the i^{th} column replaced by the column $\begin{bmatrix} 0 \\ 0 \\ \sec x \end{bmatrix}$ for $i = 1, 2, 3$.

$$W_1(x) = \begin{vmatrix} 0 & \cos x & \sin x \\ 0 & -\sin x & \cos x \\ \sec x & -\cos x & -\sin x \end{vmatrix} = secx$$

$$W_2(x) = \begin{vmatrix} 1 & 0 & \sin x \\ 0 & 0 & \cos x \\ 0 & \sec x & -\sin x \end{vmatrix} = 1(0-1) = -1$$

$$W_3(x) = \begin{vmatrix} 1 & \cos x & 0 \\ 0 & -\sin x & 0 \\ 0 & -\cos x & \sec x \end{vmatrix} = -\tan x$$

By Cramer's rule,

$A^{'}(x) = \frac{W_1(x)}{W(x)} = \sec x$, $B^{'}(x) = \frac{W_2(x)}{W(x)} = -1$,

$C^{'}(x) = \frac{W_3(x)}{W(x)} = -\tan x$

Integrating, we get

$$A(x) = \int \sec x \, dx + c_1 = \ln|\sec x + \tan x| + c_1$$

$$B(x) = -\int dx + c_2 = -x + c_2$$

$$C(x) = -\int \tan x \, dx + c_3 = -\ln|\sec x| + c_3 = \ln|\cos x| + c_3$$

Therefore, the general solution is

$$y = A(x) + B(x)\cos x + C(x)\sin x$$

$$= [\ln|\sec x + \tan x| + c_1] + \cos x\,[-x + c_2] + \sin x\,[\ln|\cos x| + c_3]$$

$$= c_1 + c_2\cos x + c_3\sin x + \ln|\sec x + \tan x| - x\cos x + \sin x\ln|\cos x|$$

where c_1, c_2, c_3 are arbitrary constants.

P3.

Solve $\left(D^2 - 3D + 2\right)y = \frac{e^x}{1+e^x}$ by the Method of Variation of Parameters.

Solution:

The given differential equation is of the form $P(D)y = r(x)$, where $P(D) = D^2 - 3D + 2$ and $r(x) = \frac{e^x}{1+e^x}$

The A.E is given by $P(m) = 0$,

i.e., $m^2 - 3m + 2 = 0 \implies m = 1, 2$

The linearly independent solutions of the corresponding homogeneous equation of the given equation are
$y_1 = e^x \,, y_2 = e^{2x}$

The C.F is $y_c = Ay_1 + By_2 = Ae^x + Be^{2x}$

By the method of variation of parameters, we assume the general solution as

$$y(x) = A(x)y_1 + B(x)y_2 = A(x)e^x + B(x)e^{2x}$$

where $\quad A(x) = -\int \frac{g(x)y_2(x)}{W(x)}dx + c_1$

$B(x) = \int \frac{g(x)y_1(x)}{W(x)}dx + c_2 \;,\; g(x) = \frac{r(x)}{a_0(x)} = \frac{e^x}{1+e^x}$

and $\quad W(x) = W(y_1, y_2) = \begin{vmatrix} e^x & e^{2x} \\ e^x & 2e^{2x} \end{vmatrix} = e^{3x} \neq 0.$

Therefore,

$$A(x) = -\int \frac{e^x}{1+e^x} . \frac{e^{2x}}{e^{3x}} dx + c_1 = -\int \frac{1}{1+e^x}\ dx + c_1$$

$$= \int \frac{-e^{-x}}{e^{-x}+1} dx + c_1 = ln|e^{-x} + 1| + c_1$$

$$B(x) = \int \frac{e^x}{1+e^x} . \frac{e^x}{e^{3x}}\ \ dx + c_2 = \int \frac{1}{(1+e^x)e^x}\ \ dx + c_2$$

put $e^x = u \implies e^x dx = du \implies dx = \frac{du}{u}$

$$= \int \frac{1}{u^2(1+u)} du + c_2 = \int \left(\frac{1}{u^2} - \frac{1}{u} + \frac{1}{1+u}\right) du + c_2$$

$$= \frac{-1}{u} - \ln|u| + ln|1 + u| + c_2 = -e^{-x} - x + ln|1 + e^x| + c_2$$

The general solution is $\quad y(x) = A(x)e^x + B(x)e^{2x}$

$$y = [ln|e^{-x} + 1| + c_1]e^x + [-e^{-x} - x + ln|1 + e^x| + c_2]e^{2x}$$

$$= c_1e^x + c_2e^{2x} - e^x - xe^{2x} + e^x\ ln|e^x + 1| + e^{2x}ln|1 + e^x|$$

where c_1, c_2 are arbitrary constants.

P4.

If $y = x$ and $y = xe^{2x}$ are solutions of the homogenous equation corresponding to

$$x^2 \frac{d^2y}{dx^2} - 2x(1+x)\frac{dy}{dx} + 2(x+1) = x^3$$

Then find the general solution of the above differential equation.

Solution:

Given that $y_1 = x$ and $y_2 = xe^{2x}$ are the solutions of the corresponding homogenous equation to given equation.

Now, $W(x) = W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x & xe^{2x} \\ 1 & 2xe^{2x} + e^{2x} \end{vmatrix}$

$$= x(2xe^{2x} + e^{2x}) - xe^{2x}$$

$$= 2x^2e^{2x} + xe^{2x} - xe^{2x} = 2x^2e^{2x} \neq 0$$

Thus, $y_1, \; y_2$ are linearly independent.

The C.F is $y_c = Ax + B\, xe^{2x}$

By the Method of variation of parameters, we assume the general solution as

$$y(x) = A(x)\, x + B(x)\, xe^{2x}$$

where $\quad A(x) = -\int \frac{g(x)y_2(x)}{W(x)} dx + c_1$

$$B(x) = \int \frac{g(x)y_1(x)}{W(x)}dx + c_2 \quad ; \quad g(x) = \frac{x^3}{x^2} = x$$

$$A(x) = -\int \frac{x.xe^{2x}}{2x^2e^{2x}}dx + c_1 = -\int \frac{1}{2}dx + c_1 = -\frac{x}{2} + c_1$$

$$B(x) = \int \frac{x.x}{2x^2e^{2x}}dx + c_2 = \frac{1}{2}\int e^{-2x}dx + c_2 = -\frac{1}{4}e^{-2x} + c_2$$

Therefore, the general solution is

$$y = A(x)\, x + B(x)\, xe^{2x}$$

$$y(x) = \left(-\frac{x}{2} + c_1\right)x + \left(-\frac{1}{4}e^{-2x} + c_2\right)xe^{2x}$$

$$= c_1x + c_2xe^{2x} - \frac{x}{4} - \frac{x^2}{2} \quad ,$$

where c_1, c_2 are arbitrary constants.

IP1:

Solve $y^{''} - y = \frac{2}{1+e^x}$ by the Method of Variation of Parameters.

Solution:

The given differential equation in the operator notation is $(D^2 - 1)y = \frac{2}{1+e^x}$, which is of the form $P(D)y = r(x)$, where $P(D) = D^2 - 1$ and $r(x) = \frac{2}{1+e^x}$

The A.E is given by $P(m) = 0$, i.e., $m^2 - 1 = 0 \Longrightarrow m = \pm 1$

The linearly independent solutions of the homogeneous equation of the given equation are $y_1 = e^x$, $y_2 = e^{-x}$.

$\therefore$ The C.F.: $y_c(x) = Ae^x + Be^{-x}$

By the method of variation of parameters, we assume the general solution as $y(x) = A(x)e^x + B(x)e^{-x}$

where $A(x) = -\int \frac{g(x)y_2(x)}{W(x)} dx + c_1$

$B(x) = \int \frac{g(x)y_1(x)}{W(x)} d(x) + c_2$, $g(x) = \frac{r(x)}{a_0(x)} = \frac{2}{1+e^x}$

and $W(x) = W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = -2 \neq 0$

Therefore,

$$A(x) = \frac{1}{2}\int \frac{2}{1+e^x} e^{-x} dx + c_1 = \int \frac{dx}{(e^x+1)e^x} + c_1$$

Put $e^x = u \Longrightarrow e^x dx = du \Longrightarrow dx = \frac{du}{u}$

$$= \int \frac{du}{u^2(1+u)} + c_1 = \int \left(\frac{1}{u^2} - \frac{1}{u} + \frac{1}{u+1}\right) du + c_1$$

$$= -\frac{1}{u} - \ln u + \ln|1+u| + c_1 = -e^{-x} - x + \ln|1+e^x| + c_1$$

$$B(x) = -\frac{1}{2}\int \frac{2}{1+e^x} e^x dx + c_2$$

$$= -\int \frac{e^x}{1+e^x} dx + c_2 = -\ln|1+e^x| + c_2$$

The general solution is $y(x) = A(x)e^x + B(x)e^{-x}$

i.e., $y(x) = e^x[-e^{-x} - x + \ln|1+e^x| + c_1] + e^{-x}[-\ln|1+e^x| + c_2]$

$$= c_1 e^x + c_2 e^{-x} - 1 - xe^x + (e^x - e^{-x})\ln|1+e^x|,$$

where c_1, c_2 are arbitrary constants.

IP2.

Solve $y''' - 6y'' + 11y' - 6y = e^{2x}$ by the Method of Variation of Parameters.

Solution:

The given differential in the operator notation is

$(D^3 - 6D^2 + 11D - 6)y = e^{2x}$, which is of the form $P(D)y = r(x)$, where $P(D) = D^3 - 6D^2 + 11D - 6$ and $r(x) = e^{2x}$. The A.E. is given by $P(m) = 0$

i.e., $m^3 - 6m^2 + 11m - 6 = 0 \Longrightarrow m = 1, 2, 3$

The linearly independent solutions of the corresponding homogeneous equation of the given equation are

$$y_1 = e^x, y_2 = e^{2x}, \; y_3 = e^{3x}$$

$\therefore$The C.F $y_c(x) = Ay_1 + By_2 + Cy_3 = Ae^x + Be^{2x} + Ce^{3x}$

$$W(x) = W(y_1, y_2, y_3) = \begin{vmatrix} e^x & e^{2x} & e^{3x} \\ e^x & 2e^{2x} & 3e^{3x} \\ e^x & 4e^{2x} & 9e^{3x} \end{vmatrix} = 2e^{6x} \neq 0$$

By the method of variation of parameters, we assume the general solution as

$$y(x) = A(x)e^x + B(x)e^{2x} + C(x)e^{3x}$$

The equations for obtaining $A(x), B(x), C(x)$ are

$A'(x)y_1 + B'(x)y_2 + C'(x)y_3 = 0$

$A'(x)y_1' + B'(x)y_2' + C'(x)y_3' = 0$

$A'(x)y_1'' + B'(x)y_2'' + C'(x)y_3'' = g(x)$, where $g(x) = \frac{a_0(x)}{r(x)}$

$\Longrightarrow A'(x)e^x + B'(x)e^{2x} + C'(x)e^{3x} = 0$

$\Longrightarrow A'(x)e^x + 2B'(x)e^{2x} + 3C'(x)e^{3x} = 0$

$\Longrightarrow A'(x)e^x + 4B'(x)e^{2x} + 9C'(x)e^{3x} = e^{2x}$

Now, W_i is same as $W(x)$ with the i^{th} column replaced by the column $\begin{bmatrix} 0 \\ 0 \\ e^{2x} \end{bmatrix}$ for $i = 1, 2, 3.$

$$W_1(x) = \begin{vmatrix} 0 & e^{2x} & e^{3x} \\ 0 & 2e^{2x} & 3e^{3x} \\ e^{2x} & 4e^{2x} & 9e^{3x} \end{vmatrix} = e^{7x}$$

$$W_2(x) = \begin{vmatrix} e^x & 0 & e^{3x} \\ e^x & 0 & 3e^{3x} \\ e^x & e^{2x} & 9e^{3x} \end{vmatrix} = -2e^{6x}$$

$$W_3(x) = \begin{vmatrix} e^x & e^{2x} & 0 \\ e^x & 2e^{2x} & 0 \\ e^x & 4e^{2x} & e^{2x} \end{vmatrix} = e^{5x}$$

By Cramer's rule,

$A'(x) = \frac{W_1(x)}{W(x)} = \frac{e^x}{2}$; $B'(x) = \frac{W_2(x)}{W(x)} = -1$; $C'(x) = \frac{W_3(x)}{W(x)} = \frac{e^{-x}}{2}$

Integrating, we get

$$A(x) = \int \frac{e^x}{2} dx + c_1 = \frac{e^x}{2} + c_1 \,;\, B(x) = -\int dx + c_2 = -x + c_2;$$

$$C(x) = \int \frac{e^{-x}}{2} dx + c_3 = \frac{-e^{-x}}{2} + c_3$$

The general solution is

$$y(x) = A(x)e^x + B(x)e^{2x} + C(x)e^{3x}$$

$$= \left(\frac{e^x}{2} + c_1\right)e^x + (-x + c_2)e^{2x} + \left(\frac{-e^{-x}}{2} + c_3\right)e^{3x}$$

$$= c_1 e^x + c_2 e^{2x} + c_3 e^{3x} - xe^{2x},$$

where c_1, c_2, c_3 are the arbitrary constants.

IP3.

Solve $\left(D^2 - 4D + 3\right)y = \frac{e^x}{1+e^x}$ by the Method of Variation of Parameters

Solution:

The given differential equation is of the form $p(D)y = r(x)$, where $P(D) = D^2 - 4D + 3$ and $(x) = \frac{e^x}{1+e^x}$.

The A.E is given by $P(m) = 0$

i.e., $m^2 - 4m + 3 = 0 \implies m = 1, 3$

The linearly independent solutions of the corresponding homogeneous equation are $y_1 = e^x$, $y_2 = e^{3x}$

The C.F is $y_c = Ay_1 + By_2 = Ae^x + Be^{3x}$

By the method of variation of parameters, we assume the general solution as

$$y(x) = A(x)e^x + B(x)e^{3x}$$

where $\qquad A(x) = -\int \frac{g(x)y_2(x)}{w(x)}dx + c_1$

$B(x) = \int \frac{g(x)y_1(x)}{w(x)}dx + c_2$, $g(x) = \frac{r(x)}{a_0(x)} = \frac{e^x}{1+e^x}$

and $W(x) = W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^x & e^{3x} \\ e^x & 3e^{3x} \end{vmatrix} = 2e^{4x} \neq 0$

Therefore,

$$A(x) = -\int \frac{e^x}{1+e^x}.\frac{e^{3x}}{2e^{4x}}\ dx + c_1 = \frac{1}{2}\int \frac{(-e^{-x})}{e^{-x}+1} dx + c_1 = \frac{ln|e^{-x}+1|}{2} + c_1$$

$$B(x) = \int \frac{e^x}{1+e^x}.\frac{e^x}{2e^{4x}}\ dx + c_2 = \frac{1}{2}\int \frac{1}{e^{2x}(1+e^x)}\ dx + c_2$$

$$= \frac{1}{2}\int \frac{e^{-2x}}{(1+e^x)}\ dx + c_2 = \frac{1}{2}\int \frac{e^{-3x}}{e^{-x}+1}\ dx + c_2$$

Put $e^{-x} = t \Longrightarrow -e^{-x}dx = dt \Longrightarrow e^{-x}dx = -dt$

$$= -\frac{1}{2}\int \frac{t^2}{t+1} dt + c_2 == -\frac{1}{2}\int \frac{t^2-1+1}{1+t} dt + c_2$$

$$= -\frac{1}{2}\int (t-1)dt - \frac{1}{2}\int \frac{1}{1+t} dt + c_2$$

$$= -\frac{1}{2}\left[\frac{t^2}{2} - t + ln|1+t|\right] + c_2$$

$$= -\frac{1}{2}\left[\frac{e^{-2x}}{2} - e^{-x} + ln|1+e^{-x}|\right] + c_2$$

$$= \frac{-e^{-2x}}{4} + \frac{e^{-x}}{2} - \frac{ln|1+e^{-x}|}{2} + c_2$$

The general solution $y(x) = A(x)e^x + B(x)e^{3x}$

$$y(x) = \left[\frac{1}{2} ln|1+e^{-x}| + c_1\right]e^x + \left[\frac{-e^{-2x}}{4} + \frac{e^{-x}}{2} - \frac{ln|1+e^{-x}|}{2} + c_2\right]e^{3x}$$

$$= c_1e^x + c_2e^{3x} + \frac{e^x}{2} ln|1+e^{-x}| - \frac{e^{3x}}{2} ln|1+e^{-x}| - \frac{e^x}{4} + \frac{e^{2x}}{2}$$

$$= c_1e^x + c_2e^{3x} + \left(\frac{e^x - e^{3x}}{2}\right) ln\left|1+e^{-x}\right| + \frac{e^{2x}}{2}$$

where c_1, c_2 are arbitrary constants.

IP4.

If $y_1 = x^2$ and $y_2 = \frac{1}{x^2}$ are solutions of the homogenous equation corresponding to

$$x^2y'' + xy' - 4y = x^2 ln|x|, \quad x > 0$$

Then find the general solution of the above differential equation.

Solution:

Given that $y_1 = x^2$ and $y_2 = \frac{1}{x^2}$ are the solutions of the corresponding homogenous equation to the given equation.

Now, $W(x) = W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x^2 & \frac{1}{x^2} \\ 2x & -\frac{2}{x^3} \end{vmatrix}$

$$= x^2\left(-\frac{2}{x^3}\right) - 2x\left(\frac{1}{x^2}\right) = -\frac{2}{x} - \frac{2}{x} = -\frac{4}{x} \neq 0$$

Thus, $y_1, \;\; y_2$ are linearly independent.

The C.F: $y_c = Ax^2 + B\left(\frac{1}{x^2}\right)$

By Method of variation of parameters, we assume the general solution as

$$y(x) = A(x).x^2 + B(x).\left(\frac{1}{x^2}\right)$$

where $\quad A(x) = -\int \frac{g(x)y_2(x)}{w(x)} dx + c_1$

$$B(x) = \int \frac{g(x)y_1(x)}{w(x)} dx + c_2 \;\; ; \;\; g(x) = \frac{x^2 ln|x|}{x^2} = ln|x|, \;\; x > 0$$

$$A(x) = \int \frac{\ln|x|.\left(1/x^2\right)}{\left(-4/x\right)} dx + c_1 = \frac{1}{4}\int \ln|x|\left(\frac{1}{x}\right) dx + c_1 = \frac{1}{8}(ln|x|)^2 + c_1$$

$$B(x) = \int \frac{\ln|x|.x^2}{\left(-4/x\right)} dx + c_2 = -\frac{1}{4}\int \ln|x|.x^3 dx + c_2$$

$$= -\frac{1}{4}\left[\ln|x|.\frac{x^4}{4} - \int \frac{1}{x}.\frac{x^4}{4} dx\right] + c_2 = -\frac{1}{4}\left[\frac{x^4 \ln|x|}{4} - \frac{x^4}{16}\right] + c_2$$

$$= \frac{x^4}{64}\left[4\ln|x| - 1\right] + c_2$$

Therefore, the general solution is

$$y(x) = A(x)\, x^2 + B(x)\, \left(\frac{1}{x^2}\right)$$

$$= \left[\frac{1}{8}(ln|x|)^2 + c_1\right] x^2 + \left[\frac{x^4}{64}(1 - 4ln|x|) + c_2\right]\frac{1}{x^2}$$

$$= c_1 x^2 + c_2\left(\frac{1}{x^2}\right) + \frac{x^2}{64}\left[8(ln|x|)^2 - 4ln|x| + 1\right]$$

where c_1, c_2 are arbitrary constants.

2.8. Method of Variation of Parameters

EXERCISES:

I. Solve the following differential equations:

1. $y'' - 2y' + y = xe^x lnx, \;\; x > 0$
2. $y'' + 4y = 4sec^2 2x$
3. $y'' + 4y = 4csc^2 2x$
4. $(D^2 - 2D + 2)y = e^x tanx$
5. $y'' + y = \frac{1}{1+sinx}$
6. $y''' + y' = cscx$
7. $y'' - 2y' + y = \frac{e^x}{x^3}$
8. $(D^2 + 1)y = cscx.cotx$
9. $(D^2 - 3D + 2)y = cos(e^{-x})$

Answers:

1. $y = c_1 e^x + c_2 xe^x + \frac{x^3 e^x}{6} ln|x| - \frac{5}{36} x^3 e^x$
2. $y = c_1 cos2x + c_2 sin2x + sin2x \, ln|sec2x + tan2x| - 1$
3. $y = c_1 cos2x + c_2 sin2x - cos2x \, ln|tanx| - 1$
4. $y = e^x(c_1 cosx + c_2 sinx) + e^x cosx \, ln|sec2x + tan2x|$
5. $y = c_1 cosx + c_2 sinx - 1 + sinx - xcosx + sinx \, ln|1 + sinx|$
6. $y = c_1 + c_2 cosx + c_3 sinx - ln|cscx + cotx| - cosxln|sinx| - xsinx$

7. $y = (c_1 + c_2 x)e^x - \frac{e^x}{2x}$

8. $y = c_1 cosx + c_2 sinx - xsinx - cosx\ ln|sinx|$

9. $y = c_1 e^x + c_2 e^{2x} - e^{2x}\ cos(e^{-x})$

II. **In the following problems, using the method of variation of parameters and the given linearly independent solutions , find a particular integral and the general solution:**

a) $x^2 y'' + xy' - y = x^3\ ;\ \ y_1 = x, y_2 = \frac{1}{x}$

b) $x^2 y'' - xy' + y = \frac{1}{x^4}\ ;\ \ y_1 = x, y_2 = x\ ln|x|$

c) $x^2 y'' - 2xy' + 2y = x^3 + x\ ;\ \ y_1 = x, y_2 = x^2$

d) $x^2 y'' + xy' - y = x\ ;\ \ y_1 = x, y_2 = \frac{1}{x}$

e) $(1 - x)y'' + xy' - y = 2(1 - x)^2 e^{-x}, 0 < x < 1;$

$y_1 = x, y_2 = e^x$

Answers:

a) $y = c_1 x + c_2\left(\frac{1}{x}\right) + \frac{x^3}{8}$

b) $y = c_1 x + c_2 x\ ln|x| + \frac{1}{25x^4}$

c) $y = c_1 x + c_2 x^2 + \frac{x^3}{2} - x(1 + ln|x|)$

d) $y = c_1 x + c_2\left(\frac{1}{x}\right) + \frac{x}{2} ln|x|$

e) $y = c_1 x + c_2 e^x + \left(\frac{1}{2} - x\right) e^{-x}$

2.9

Solution of Euler-Cauchy equation

Learning objectives:

* To learn the method of solving Euler – Cauchy equation
* To study the equations reducible to Euler – Cauchy form and their solutions

  AND

* To practice the related problems

Solution of Euler-Cauchy equation

A linear differential equation of the form

$a_0x^n y^{(n)} + a_1 x^{n-1} y^{(n-1)} + \cdots + a_{n-1}xy' + a_n y = r(x), x \neq 0$ ... (1)
where $a_0 \neq 0, a_1, a_2, \dots, a_{n-1}, a_n$ are constants, is called as ***Euler-Cauchy equation***.

The equation (1) in operator notation is

$(a_0x^n D^n + a_1x^{n-1}D^{n-1} + \cdots + a_{n-1}xD + a_n)y = r(x)$ ... (2)
where $= \frac{d}{dx}$.

Method of solving Euler-Cauchy equation

We change the independent variable from x to t by the transformation $x = e^t$ or $t = \ln x \ , x > 0$.

Then $\frac{dt}{dx} = \frac{1}{x}$. Let θ be the differential operator w.r.t. t, i.e.,$\theta = \frac{d}{dt}$. Now, $Dy = \frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{1}{x}\theta y \Rightarrow xDy = \theta y$

$$D^2y = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dt}\right) = -\frac{1}{x^2}\frac{dy}{dt} + \frac{1}{x}\frac{d}{dx}\left(\frac{dy}{dt}\right)$$

$$= -\frac{1}{x^2}\frac{dy}{dt} + \frac{1}{x}\frac{d}{dt}\left(\frac{dy}{dt}\right)\frac{dt}{dx} = -\frac{1}{x^2}\frac{dy}{dt} + \frac{1}{x^2}\frac{d^2y}{dt^2}$$

$$\Rightarrow x^2D^2y = \frac{d^2y}{dt^2} - \frac{dy}{dt} = \theta(\theta - 1)y$$

By induction, we can prove that

$$x^n D^n y = \theta(\theta - 1)(\theta - 2) \dots (\theta - n + 1)y$$

Substituting in (2), we obtain

$$[a_0\theta(\theta - 1)(\theta - 2) \dots (\theta - n + 1)$$

$$+a_1\theta(\theta - 1)(\theta - 2) \dots (\theta - n + 2)$$

$$+ \cdots + a_{n-1}\theta + a_n]y = r(e^t) \dots (3)$$

This is a linear differential equation with constant coefficients and t is the independent variable. The methods described in the earlier modules can be applied to find its solution. The solution of the given equation is obtained by replacing t by $\ln x$.

Note:

The case $x < 0$ can also be considered by taking the transformation as

$$|x| = e^t \text{ or } t = \ln|x|$$

Example 1: Solve $x^3y''' - 3xy' + 3y = 16x + 9x^2 \ln x , x > 0$

Solution: The given equation is an Euler-Cauchy equation. The equation is $(x^3D^3 - 3xD + 3)y = 16x + 9x^2 \ln x$, where $D = \frac{d}{dx}$. Changing the independent variable from x to t by the transformation $x = e^t$ or $t = \ln x$, we get

$x^3D^3y = \theta(\theta - 1)(\theta - 2)y,\ x^2D^2y = \theta(\theta - 1)y,\ xDy = \theta y,$

where $\theta = \frac{d}{dt}$.

The equation now becomes,

$$[\theta(\theta-1)(\theta-2)-3\theta+3]y=16e^{t}+9te^{2t}$$

$$\Rightarrow (\theta^3-3\theta^2-\theta+3)y=16e^{t}+9te^{2t}$$

It is a 3rd order non-homogeneous linear differential equation with constant coefficients.

The A.E.: $m^3-3m^2-m+3=0 \Rightarrow (m-3)(m^2-1)=0$

$$\Rightarrow (m-1)(m+1)(m-3)=0$$

The characteristic roots are: $\pm 1, 3$.

The complementary function is given by

$$y_c(t)=ae^{t}+be^{-t}+ce^{3t}$$

A particular integral is given by:

$$y_p(t)=\frac{1}{\theta^3-3\theta^2-\theta+3}(16e^{t}+9te^{2t})=y_{p_1}+y_{p_2}$$

$$y_{p_1}=16\frac{1}{(\theta-1)(\theta+1)(\theta-3)}e^{t}=16\frac{1}{(\theta-1)(1+1)(1-3)}e^{t}=-4te^{t}$$

$$y_{p_2}=9\frac{1}{\theta^3-3\theta^2-\theta+3}te^{2t}=9e^{2t}\frac{1}{(\theta+2)^3-3(\theta+2)^2-(\theta+2)+3}t$$

$$=9e^{2t}\frac{1}{\theta^3-3\theta^2-\theta-3}t=9e^{2t}\frac{1}{(-3)\left(1+\frac{\theta+3\theta^2-\theta^3}{3}\right)}t$$

$$=-3e^{2t}\left(1+\frac{\theta+3\theta^2-\theta^3}{3}\right)^{-1}t=-3e^{2t}\left(1-\frac{\theta+3\theta^2-\theta^3}{3}+\cdots\right)t$$

$$= -3e^{2t}\left(t - \frac{1}{3}\right) = (1 - 3t)e^{2t}$$

Therefore, $y_p = (1 - 3t)e^{2t} - 4te^t$

The general solution is: $y(t) = y_c(t) + y_p(t)$

i.e., $y(t) = ae^t + be^{-t} + ce^{3t} + (1 - 3t)e^{2t} - 4te^t$

Substituting $x = e^t$ or $= lnx$, we get the general solution of the given equation as

$$y(x) = ax + \frac{b}{x} + cx^3 + (1 - 3lnx)x^2 - 4xlnx,$$

where a, b and c are arbitrary constants.

Example 2:

Solve $(x^2D^2 + xD - 1)y = x^2e^x \, , x > 0 \, , D = \frac{d}{dx}$

Solution: The given equation is an Euler- Cauchy equation. Changing the independent variable form x to t by the transformation $x = e^t$ or $t = lnx$, we get

$x^2D^2 = \theta(\theta - 1) \, , xD = \theta$ where $\theta = \frac{d}{dt}$

The equation now becomes

$$[\theta(\theta - 1) + \theta - 1]\, y = e^{2t}.e^{e^t} \Longrightarrow (\theta^2 - 1)y = e^{2t}.e^{e^t}$$

It is a non – homogeneous linear differential equation with constant coefficients.

A.E. : $m^2 - 1 = 0 \Rightarrow m = \pm 1$.

The characteristic roots are ± 1. The complementary function is given by $y_c(t) = ae^t + be^{-t}$. A particular integral is given by:

$$y_p(t) = \frac{1}{(\theta+1)(\theta-1)} e^{2t}.\, e^{e^t} = e^{2t} \frac{1}{(\theta+2+1)(\theta+2-1)} e^{e^t}$$

$$= e^{2t} \frac{1}{\theta+3} \left[\frac{1}{\theta+1} e^{e^t}\right] = \frac{e^{2t}}{\theta+3} \left[e^{-t} \int e^{e^t}.e^t\, dt\right]$$

$$\left(\because \frac{1}{\theta-\alpha} r(t) = e^{\alpha t} \int r(t)\, .e^{-\alpha t} dt\right)$$

$$= e^{2t} \frac{1}{\theta+3} [\, e^{-t} . \int e^u du] \quad \text{(where } u = e^t, du = e^t dt \text{)}$$

$$= e^{2t} \frac{1}{\theta+3} \left(e^{-t}\, e^{e^t}\right) = e^{2t}\left[e^{-3t} \int e^{-t}\, e^{e^t}.\, e^{3t} dt\right]$$

$$= e^{-t}\left[\int e^t e^{e^t} e^t dt\right] = e^{-t} \int u\, e^u du,$$

where $u = e^t, du = e^t dt$

$$= e^{-t}(u-1)e^u = e^{-t}(e^t - 1)e^{e^t} = (1 - e^{-t})\, e^{e^t}$$

The general solution is $y(t) = y_c(t) + y_p(t)$.

i.e., $y(t) = ae^t + be^{-t} + (1 - e^{-t})e^{e^t}$

Substituting $x = e^t$ or $t = lnx$, we get the general solution of the given differential equation as

$$y(x) = ax + \frac{b}{x} + \left(1 - \frac{1}{x}\right) e^x$$

where a and b are arbitrary constants.

Example 3:

Solve $(x^2D^2 - 3xD + 1)y = \frac{lnx \ \sin(lnx)+1}{x}, x > 0, D = \frac{d}{dx}$

Solution: The given equation is an Euler-Cauchy equation. We change the independent variable from x to t by the transformation $x = e^t$ or $t = lnx$

Then $x^2D^2 = \theta(\theta - 1), xD = \theta$ where $\theta = \frac{d}{dt}$

The equation now becomes

$$[\theta(\theta - 1) - 3\theta + 1]y = e^{-t}(t\sin t + 1)$$

$$\Rightarrow (\theta^2 - 4\theta + 1)y = e^{-t} + e^{-t}t\sin t$$

It is a second order non-homogeneous linear differential equation with constant coefficients.

A.E. : $m^2 - 4m + 1 = 0 \Rightarrow m = \frac{-4\pm\sqrt{16-4}}{2} = 2 \pm \sqrt{3}$

C.F. : $y_c(t) = e^{2t}\left(ae^{\sqrt{3}\,t} + be^{-\sqrt{3}\,t}\right)$

or $y_c(t) = e^{2t}\left(a\cosh\sqrt{3}\,t + b\sinh\sqrt{3}\,t\right)$

P.I. : $y_p(t) = \frac{1}{\theta^2-4\theta+1}(e^{-t} + e^{-t}t\sin t) = y_{p_1} + y_{p_2}$

where $y_{p_1} = \frac{1}{\theta^2-4\theta+1}e^{-t} = \frac{1}{(-1)^2+4(-1)+1}e^{-t} = \frac{e^{-t}}{6}$

$$y_{p_2} = \frac{1}{\theta^2-4\theta+1} e^{-t} t \ \sin t = e^{-t} \frac{1}{(\theta-1)^2-4(\theta-1)+1} t \sin t$$

$$= e^{-t} \frac{1}{\theta^2-6\theta+6} t \sin t$$

$$= e^{-t} \left[t - \frac{1}{\theta^2-6\theta+6} (\theta^2 - 6\theta + 6)'\right] \frac{1}{\theta^2-6\theta+6} \sin t$$

$$= e^{-t} \left[t \ \frac{1}{\theta^2-6\theta+6} \sin t - 2(\theta - 3) \frac{1}{(\theta^2-6\theta+6)^2} \sin t\right]$$

$$= e^{-t} \left[t \ \frac{1}{-1^2-6\theta+6} \sin t - 2(\theta - 3) \frac{1}{(-1^2-6\theta+6)^2} \sin t\right]$$

$$= e^{-t} \left[t \ \frac{1}{5-6\theta} \sin t - 2(\theta - 3) \frac{1}{25-60\theta+36\theta^2} \sin t\right]$$

$$= e^{-t} \left[t \ \ \frac{5+6\theta}{25-36\theta^2} \sin t - 2(\theta - 3) \frac{1}{25-60\theta+36(-1)^2} \sin t\right]$$

$$= \ e^{-t} \left[\frac{t}{25-36(-1)^2} (5 \sin t + 6 \cos t) + 2(\theta - 3) \frac{1}{60\theta+11} \sin t\right]$$

$$= \ e^{-t} \left[\frac{t}{61} (5 \sin t + 6 \cos t) + 2(\theta - 3) \frac{(60\theta-11)}{3600\theta^2-121} \sin t\right]$$

$$= \ e^{-t} \left[\frac{t}{61} (5 \sin t + 6 \cos t) + \frac{120\theta^2-382\theta+66}{3600(-1)^2-121} \sin t\right]$$

$$= e^{-t} \left[\frac{t}{61} (5 \sin t + 6 \cos t) - \frac{120(-\sin t)-382 \cos t+66 \sin t}{3721}\right]$$

$$= e^{-t} \left[\frac{t}{61} (5 \sin t + 6 \cos t) + \frac{54 \sin t+382 \cos t}{3721}\right]$$

The general solution is $y(t) = y_c(t) + y_{p_1} + y_{p_2}$

i.e. , $y(t) = e^{2t}\left(ae^{\sqrt{3}\,t} + be^{-\sqrt{3}\,t}\right) + \frac{e^{-t}}{6}$

$$+e^{-t}\left[\frac{t}{61}(5\sin t + 6\cos t) + \frac{54\sin\,t + 382\cos\,t}{3721}\right]$$

The general solution of the given equation is

$$y(x) = x^2\left(ax^{\sqrt{3}} + bx^{-\sqrt{3}}\right) + \frac{1}{6x}$$

$$+\frac{1}{x}\left[\frac{lnx}{61}(5\sin(lnx) + 6\cos(lnx)) + \frac{1}{3721}(54\sin(lnx) + 382\cos(lnx)\right],$$

where a and b are arbitrary constants.

Equations Reducible to Euler-Cauchy Form

A linear differential equation of the form

$$a_0(a+bx)^n y^{(n)} + a_1(a+bx)^{n-1}y^{(n-1)} + \cdots$$

$$\ldots + a_{n-1}(a+bx)y' + a_n y = r(x) \quad \ldots(3)$$

where $a, b, a_0 \neq 0, a_1, a_2, \ldots, a_{n-1}, a_n$ are constants, is called a ***Legendre's linear equation***.

Equation (3) in the operator notation is

$$[a_0(a+bx)^n D^n + a_1(a+bx)^{n-1}D^{n-1} + \cdots$$

$$\ldots + a_{n-1}(a+bx)D + a_n y = r(x) \ldots (4)$$

where $= \frac{d}{dx}$.

Method of solving (4)

Substitute $a + bx = e^t$ or $t = \ln(a + bx)$.

Then $\frac{dt}{dx} = \frac{b}{a+bx}$. Let $= \frac{d}{dt}$. Now, $Dy = \frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{b}{a+bx} \frac{dy}{dt}$

$$\Rightarrow (a + bx)Dy = b\frac{dy}{dt} = b\theta y$$

$$D^2y = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{b}{a+bx}\frac{dy}{dt}\right)$$

$$= -\frac{b^2}{(a+bx)^2}\frac{dy}{dt} + \frac{b}{a+bx}\frac{d}{dt}\left(\frac{dy}{dt}\right)\frac{dt}{dx}$$

$$= \frac{b^2}{(a+bx)^2}\left[\frac{d^2y}{dt^2} - \frac{dy}{dt}\right] = \frac{b^2}{(a+bx)^2}\theta(\theta - 1)y$$

$$\Rightarrow (a + bx)^2D^2y = b^2\theta(\theta - 1)y$$

Similarly, $(a + bx)^3D^3y = b^3\theta(\theta - 1)(\theta - 2)y$

Continuing in this way, we prove

$$(a + bx)^nD^ny = b^n\theta(\theta - 1)(\theta - 2)\ldots(\theta - n + 1)y$$

The equation (2) now becomes,

$$[a_0b^n\theta(\theta - 1)(\theta - 2)\ldots(\theta - n + 1)$$
$$+a_1b^{n-1}\theta(\theta - 1)(\theta - 2)\ldots(\theta - n + 2) + \cdots$$
$$\ldots + a_{n-1}b\theta + a_n]y = r\left(\frac{e^t - a}{b}\right) = s(t) \text{ (say)}$$

This is a linear differential equation with constant coefficients, whose solution can be found by the methods studied in earlier modules. After obtaining the solution replace t by $\ln(a+bx)$.

Example 4:

$$[(3x+2)^2D^2+3(3x+2)D-36]y=3x^2+4x+1$$

Solution: Put $3x+2=e^t$ or $t=\ln(3x+2)$.

Then $(3x+2)Dy=3\theta y$, $(3x+2)D^2y=3^2\theta(\theta-1)y$, where $\theta=\frac{d}{dt}$. The equation now becomes,

$$[3^2\theta(\theta-1)+3\cdot 3\theta-36]y=3\left(\frac{e^t-2}{3}\right)^2+4\left(\frac{e^t-2}{3}\right)+1$$

$$\Rightarrow 9(\theta^2-4)y=\frac{e^{2t}-4e^t+4+4e^t-8+3}{3}=\frac{e^{2t}-1}{3}$$

i.e., $(\theta^2-4)y=\frac{1}{27}(e^{2t}-1)$

It is a 2nd order non-homogeneous linear differential equation with constant coefficients.

A.E.: $m^2-4=0\Rightarrow m=\pm 2$

C.F.: $y_c(t)=ae^{2t}+be^{-2t}$

P.I.: $y_p(t)=\frac{1}{\theta^2-4}\cdot\frac{1}{27}(e^{2t}-1)=\frac{1}{27}\left[\frac{1}{(\theta-2)(\theta+2)}e^{2t}-\frac{1}{\theta^2-4}e^{0t}\right]$

$$=\frac{1}{27}\left[\frac{1}{4}te^{2t}-\frac{1}{0^2-4}\right]=\frac{1}{108}(te^{2t}+1)$$

The general solution is given by $y(t) = y_c(t) + y_p(t)$

i.e., $y(t) = ae^{2t} + be^{-2t} + \frac{1}{108}(te^{2t} + 1)$

The general solution of the given equation is

$$y(x) = a(3x+2)^2 + \frac{b}{(3x+2)^2} + \frac{1}{108}[(3x+2)^2 \ln(3x+2) + 1]$$

where a, b are arbitrary constants.

Example 5:

$$[(x+1)^4D^3 + 2(x+1)^3D^2 - (x+1)^2D + (x+1)]y = \frac{1}{x+1}, D = \frac{d}{dx}$$

Solution: Dividing both sides of $(x+1)$, we obtain

$$[(x+1)^3D^3 + 2(x+1)^2D^2 - (x+1)D + 1]y = \frac{1}{(x+1)^2}$$

Put $x + 1 = e^t$ or $t = \ln(x+1)$.

Then $(x+1)^3D^3y = \theta(\theta-1)(\theta-2)y$,

$(x+1)^2D^2y = \theta(\theta-1)y$ and $(x+1)Dy = \theta y$.

The given equation now becomes,

$$[\theta(\theta-1)(\theta-2) + 2\theta(\theta-1) - \theta + 1]y = e^{-2t}$$

$$\Rightarrow (\theta^3 - \theta^2 - \theta + 1)y = e^{-2t}$$

It is a 3rd order non-homogeneous linear differential equation with constant coefficients.

A.E.: $m^3 - m^2 - m + 1 = 0 \Rightarrow m^2(m-1) - 1(m-1) = 0$

$$\Rightarrow (m-1)^2(m+1) = 0 \Rightarrow m = 1(\text{twice}), -1$$

C.F.: $y_c(t) = (a+bt)e^t + ce^{-t}$

P.I.: $y_p(t) = \frac{1}{(\theta-1)^2(\theta+1)} e^{-2t} = \frac{1}{(-2-1)^2(-2+1)} e^{-2t} = -\frac{e^{-2t}}{9}$

The general solution is given by $y(t) = y_c(t) + y_p(t)$

i.e., $y(t) = (a+bt)e^t + ce^{-t} - \frac{e^{-2t}}{9}$

The general solution of the given equation is

$$y(x) = [a + b\ln(x+1)](x+1) + \frac{c}{x+1} - \frac{1}{9(x+1)^2},$$ where

a, b, c are arbitrary constants.

P1:

Solve $\left(\boldsymbol{x^2D^2+3xD+1}\right)\boldsymbol{y}=\frac{\mathbf{1}}{(\mathbf{1}-\boldsymbol{x})^2}$ **,** $\boldsymbol{x}>0$

Solution:

The given equation is an Euler-Cauchy equation. Changing the independent variable x to t by the transformation $x=e^t$ or $t=\ln x$.

Then $x^2D^2=\theta(\theta-1), xD=\theta$, where $\theta=\frac{d}{dt}$.

The equation now becomes,

$$[\theta(\theta-1)-3\theta+1]y=\frac{1}{(1-e^t)^2}\Rightarrow(\theta^2+2\theta+1)y=\frac{1}{(1-e^t)^2}$$

It is a 2nd order non-homogeneous linear differential equation with constant coefficients.

The A.E. is given by $P(m)=0$

i.e., $m^2+2m+1=0\Rightarrow m=-1$(twice)

The C.F. is $y_c(t)=(c_1+c_2t)e^{-t}$

Now, P.I. $=y_p(t)=\frac{1}{\theta^2+2\theta+1}\cdot\frac{1}{(1-e^t)^2}=\frac{1}{(\theta+1)^2}\frac{1}{(1-e^t)^2}$

$$=\frac{1}{(\theta+1)}\left[\frac{1}{(\theta+1)}\frac{1}{(1-e^t)^2}\right]=\frac{1}{(\theta+1)}\left[e^{-t}\int\frac{e^t}{(1-e^t)^2}dt\right]$$

$$=\frac{1}{(\theta+1)}\left[e^{-t}\frac{1}{1-e^t}\right]=e^{-t}\int\frac{e^{-t}}{1-e^t}e^t dt$$

$$= e^{-t}\int \frac{1}{1-e^{t}}\,dt = -e^{-t}\int \frac{(-e^{-t})}{e^{-t}-1}\,dt = -e^{-t}\ln|e^{-t}-1|$$

$\therefore$ The general solution is given by

$$y(t) = y_c(t) + y_p(t)$$

i.e., $y(t) = (c_1 + c_2 t)e^{-t} - e^{-t}\ln|e^{-t}-1|$

Substituting $x = e^{t}$ or $t = \ln x$, we get the general solution of the given differential equation as

$$y(x) = (c_1 + c_2 \ln x)\left(\frac{1}{x}\right) - \frac{1}{x}\ln\left|\frac{1-x}{x}\right|,$$

where a, b are arbitrary constants.

P2:

Solve:

$$\left(\boldsymbol{x^4D^4 + 6x^3D^3 + 9x^2D^2 + 3xD + 1}\right)\boldsymbol{y} = (\boldsymbol{1 + lnx})^{\boldsymbol{2}}, \boldsymbol{x} > 0$$

Solution:

The given equation is an Euler – Cauchy equation. Changing the independent variable form x to t by the transformation $x = e^t$ or $t = ln\,|x|,\ x > 0,$ we get

$$x^4D^4 = \theta(\theta - 1)(\theta - 2)(\theta - 3)\,, x^3D^3 = \theta(\theta - 1)(\theta - 2)$$

$$x^2D^2 = \theta(\theta - 1), xD = \theta, \text{where } \theta = \frac{d}{dt}$$

The equation now becomes

$$(\theta^4 + 2\theta^2 + 1)y = (1 + t)^2$$

It is a 4th order non-homogeneous linear differential equation with constant coefficients.

The A.E is given by $P(m) = 0$

i.e., $m^4 - 2m^2 + 1 = 0 \Longrightarrow (m^2 + 1)^2 = 0 \Longrightarrow m = \pm i\ (twice)$

The C.F. is $\quad y_c(t) = (c_1 + c_2t)cost + (c_3 + c_4t)sint$

Now, $P.I. = y_p(t) = \frac{1}{\theta^4 + 2\theta^2 + 1}(1 + t)^2$

$$= \frac{1}{(\theta^2+1)^2}[1 + t^2 + 2t] = (1 + \theta^2)^{-2}(1 + t^2 + 2t)$$

$$= (1 - 2\theta^2 + \cdots)(1 + t^2 + 2t)$$

$$= 1 + t^2 + 2t - 4 = t^2 + 2t - 3$$

∴ The general solution is

$$y(t) = y_c(t) + y_p(t)$$

i.e., $y(t) = (c_1 + c_2 t)cost + (c_3 + c_4 t)sint + t^2 + 2t - 3$

Substituting $x = e^t \;\; or \; t = \; ln\, x$, we get the general solution of the given differential equation as

$$y = (c_1 + c_2\, ln\, x)cos(ln\, x) + (c_3 + c_4\, ln\, x)sin(ln\, x)$$
$$+(ln\, x)^2 + 2ln\, x - 3$$

where c_1, c_2, c_3, c_4 are arbitrary constants.

P3.

Solve $\left(\boldsymbol{x^2 D^2 - 3xD + 5}\right)\boldsymbol{y} = \boldsymbol{x^2 \, sin(ln\, x)} \ \ , \boldsymbol{x} > 0$

Solution:

The given equation is an Euler-Cauchy equation. Changing the independent variable form x to t by the transformation
$x = e^t \; or \; t = \ln x$, we get
$x^2 D^2 = \theta(\theta - 1),\; xD = \theta,$ where $\theta = \frac{d}{dt}$
The equation now becomes
$[\theta(\theta - 1) - 3\theta + 5]y = e^{2t} \sin t \Longrightarrow (\theta^2 - 4\theta + 5)y = e^{2t} \sin t$

It is a 2nd order non-homogeneous linear differential equation with constant coefficients.

The A.E is $P(m) = 0,$ i.e.,$m^2 - 4m + 5 = 0 \Longrightarrow m = 2 \pm i$

The C.F is $y_c(t) = e^{2t}(\,c_1 \cos t + c_2 \sin t)$

Now, P.I.= $y_p(t) = \frac{1}{\theta^2 - 4\theta + 5} e^{2t} \sin t$

$$= e^{2t} \frac{1}{(\theta + 2)^2 - 4(\theta + 2) + 5} \sin t = e^{2t} \frac{1}{\theta^2 + 1} \; sint$$

$$= e^{2t}\left(-\frac{t}{2}\cos t\right) = -\frac{te^{2t}}{2}\cos t$$

$\therefore$The general solution is $y(t) = y_c(t) + y_p(t)$.

i.e., $y(t) = e^{2t}(c_1 \cos t + c_2 \sin t) - \frac{te^{2t}}{2}\cos t$

Substituting $x = e^t or\ t = \ln x$, we get the general solution of the given differential equation as

$$y(x) = x^2(c_1 \cos(\ln x) + c_2 \sin(\ln x)) - \frac{x^2}{2} \log x \cos(\ln x) \ ,$$

where c_1, c_2 are arbitrary constants.

P4:

Solve:

$$[(2x-1)^2D^2+(2x-1)D-2]y=8x^2-2x+3,\ \ x>\frac{1}{2}$$

Solution:

The given differential equation is a Legendre linear equation

Put $2x-1=e^t$ or $t=\ln(2x-1)$

Then $(2x-1)Dy=2\theta y$, $(2x-1)^2D^2y=2^2\theta(\theta-1)y$,

where $\theta=\frac{d}{dt}$. The equation now becomes

$$[2^2\theta(\theta-1)+2\theta y-2]y=8\left(\frac{e^t+1}{2}\right)^2-2\left(\frac{e^t+1}{2}\right)+3$$

$$[2\theta^2-\theta-1]y=e^{2t}+\frac{3}{2}e^t+2$$

It is a 2nd order non – homogeneous linear differential equation with constant coefficients.

The A.E is given by $P(m)\ =\ 0$

$$i.e.,2m^2-m-1=0\ \ \Rightarrow m=1,-\frac{1}{2}$$

The C.F. is $\quad y_c(t)=c_1e^t+c_2e^{-\frac{t}{2}}$

Now, $P.I=y_p(t)=\frac{1}{2\theta^2-\theta-1}e^{2t}+\frac{1}{2\theta^2-\theta-1}\frac{3}{2}e^t+\frac{1}{2\theta^2-\theta-1}2e^{0.t}$

$$=\frac{1}{2(4)-2-1}e^{2t}+\frac{3}{2}\frac{1}{(2\theta+1)(\theta-1)}e^t+\frac{1}{2(0)^2-(0)-1}2e^{0.t}$$

$$= \frac{1}{5}e^{2t} + \frac{3}{2}\frac{1}{3(\theta-1)}e^{t} - 2 = \frac{1}{5}e^{2t} + \frac{t}{2}e^{t} - 2$$

∴ The general solution is given by

$$y(t) = y_c(t) + y_p(t)$$

$$y(t) = c_1 e^{t} + c_2 e^{-\frac{t}{2}} + \frac{1}{5}e^{2t} + \frac{t}{2}e^{t} - 2$$

Hence the general solution of the given differential equation is

$$y(x) = c_1(2x-1) + c_2(2x-1)^{-\frac{1}{2}} + \frac{1}{5}(2x-1)^2$$

$$+ \frac{1}{2}(2x-1)\ln(2x-1) - 2,$$

where c_1, c_2 are arbitrary constants.

IP1:

Solve $\left(\boldsymbol{x^2 D^2 - 2xD + 2}\right)\boldsymbol{y} = \boldsymbol{x + x^2 \ln x + x^3}\ , \boldsymbol{x} > 0$

Solution:

The given differential equation is an Euler-Cauchy equation. Changing the independent variable from x to t by the transformation $x = e^t$ or $t = \ln x$, we get

$$x^2 D^2 = \theta(\theta - 1),\ \ xD = \theta, \text{where} = \frac{d}{dt}\,.$$

The equation now becomes,

$$[\theta(\theta - 1) - 2\theta + 2]y = e^t + e^{2t}\,t + e^{3t}$$

$$\Rightarrow (\theta^2 - 3\theta + 2)y = e^t + te^{2t} + e^{3t}$$

It is a 2nd order non-homogeneous linear differential equation with constant coefficients.

The A.E. is given by $(m) = 0$,

i.e., $m^2 - 3m + 2 = 0 \Rightarrow m = 1, 2$

The C.F. is $y_c(t) = ae^t + be^{2t}$

Now, P.I. $= y_p(t) = \frac{1}{\theta^2 - 3\theta + 2}(e^t + te^{2t} + e^{3t})$

$$= \frac{1}{\theta^2 - 3\theta + 2}e^t + \frac{1}{\theta^2 - 3\theta + 2}te^{2t} + \frac{1}{\theta^2 - 3\theta + 2}e^{3t}$$

$$= y_{p_1} + y_{p_2} + y_{p_3}$$

$$y_{p_1} = \frac{1}{\theta^2-3\theta+2}e^t = \frac{1}{(\theta-1)(\theta-2)}e^t = -\frac{1}{\theta-1}e^t = -te^t$$

$$y_{p_2} = \frac{1}{\theta^2-3\theta+2}te^{2t} = e^{2t}\frac{1}{(\theta+2)^2-3(\theta+2)+2}t = e^{2t}\frac{1}{\theta^2+\theta}t$$

$$= e^{2t}\frac{1}{\theta(\theta+1)}t = e^{2t}\frac{1}{\theta}(1+\theta)^{-1}t$$

$$= e^{2t}\frac{1}{\theta}[1-\theta+\cdots]t = e^{2t}\frac{1}{\theta}(t-1)$$

$$= e^{2t}\int(t-1)dt = e^{2t}\left[\frac{t^2}{2}-t\right]$$

$$y_{p_3} = \frac{1}{\theta^2-3\theta+2}e^{3t} = \frac{e^{3t}}{3^2-3(3)+2} = \frac{e^{3t}}{2}$$

$$\therefore y_p(t) = -te^t + e^{2t}\left[\frac{t^2}{2}-t\right] + \frac{e^{3t}}{2}$$

$\therefore$ The general solution is $y(t) = y_c(t) + y_p(t)$

i.e., $y(t) = ae^t - be^{2t} - te^t + e^{2t}\left[\frac{t^2}{2}-t\right] + \frac{e^{3t}}{2}$

Substituting $x = e^t$ or $t = \ln x$, we get the general solution of the given differential equation as

$y(x) = ax - bx^2 - x\ln x + \frac{x^2}{2}[(\ln x)^2 - 2\ln x] + \frac{x^3}{2}$, where a, b are arbitrary constants.

IP2.

Solve $\left(x^3D^3 + 3x^2D^2 + xD + 8\right)y = 65\cos(\ln x), x > 0$

Solution:

The given equation is an Euler-Cauchy equation. Changing the independent variable form x to t by the transformation
$x = e^t \; or \; t = \ln x$, we get
$x^3D^3 = \theta(\theta-1)(\theta-2),\; x^2D^2 = \theta(\theta-1)$
$xD = \theta$ where $\theta = \frac{d}{dt}$.
The equation now becomes
$[\theta(\theta-1)(\theta-2) + 3\theta(\theta-1) + \theta + 8]y = 65\cos t$

$\Longrightarrow (\theta^3 + 8)y = 65\cos t$

It is a 3rd order non-homogeneous linear differential equation with constant coefficients.

The A.E is $P(m) = 0$ i.e.,$m^3 + 8 = 0$

Notice that $P(-2) = 0$, therefore$(m+2)$is a root

$\Longrightarrow (m+2)(m^2 - 2m + 4) = 0 \Longrightarrow m = -2, 1 \pm i\sqrt{3}$

The C.F is $y_c(t) = c_1e^{-2t} + e^t\left(c_2\cos\left(t\sqrt{3}\right) + c_3\sin\left(t\sqrt{3}\right)\right)$

Now, P.I.$= y_p(t) = \frac{1}{\theta^3+8}65\cos t = \frac{1}{\theta(\theta^2)+8}65\cos t$

$$= \frac{65}{\theta(-1^2)+8}\cos t = 65\frac{8+\theta}{(8+\theta)(8-\theta)}\cos t$$

$$= 65\frac{8+\theta}{64-\theta^2}\cos t = \frac{65(8+\theta)}{(64+1)}\cos t$$

$$= 8\cos t - \sin t$$

∴ The general solution is

$$y(t) = y_c(t) + y_p(t)$$

i.e., $y(t) = c_1 e^{-2t} + e^t[c_2\cos\left(t\sqrt{3}\right) + c_3\sin\left(t\sqrt{3}\right)]$

$$+8\cos t - \sin t$$

Substituting $x = e^t or\ t = \ln x$, we get the general solution of the differential equation is

$$y(x) = c_1\left(\frac{1}{x^2}\right) + x\left[c_2\,cos\left(\sqrt{3}\,ln\,x\right) + c_3\,sin\left(\sqrt{3}\,ln\,x\right)\right]$$

$$+8\,cos(ln\,x) - sin(ln\,x)$$

where c_1, c_2, c_3 are arbitrary constants.

IP3.

Solve the $\left(\boldsymbol{x^2D^2+xD+1}\right)\boldsymbol{y}=\boldsymbol{ln\,x\,.\,sin(ln\,x)}\,,\boldsymbol{x}>0$

Solution:

The given equation is an Euler-Cauchy equation. Changing the independent variable form x to t by the transformation
$x=e^t\ or\ t=\ln x$, we get
$x^2D^2=\theta(\theta-1),\ \ xD=\theta$, where $\theta=\frac{d}{dt}$
The equation now becomes
$[\theta(\theta-1)+\theta+1]y=t\sin t\Longrightarrow(\theta^2+1)y=t\sin t$

It is a 2nd order non-homogeneous linear differential equation with constant coefficients.

The A.E is $P(m)=0,$ i.e.,$m^2+1=0\Longrightarrow m=\pm i$

The C.F is $y_c(t)=c_1\cos t+c_2\sin t$

Now, P.I.$=y_p(t)=\frac{1}{\theta^2+1}t\sin t$

$=$I.P. of $\frac{1}{\theta^2+1}te^{it}$ $=$I.P of $e^{it}\frac{1}{(\theta+i)^2+1}t$

$=$I.P. of $e^{it}\frac{1}{\theta^2+2i\theta}t$ $=$I.P. of $e^{it}\frac{1}{2i\theta\left(1+\frac{\theta}{2i}\right)}t$

$=$I.P. of $e^{it}\frac{1}{2i\theta}\left[1+\frac{\theta}{2i}\right]^{-1}t$

$=$I.P. of $\frac{e^{it}}{2i}\frac{1}{\theta}\left[1-\frac{\theta}{2i}+\cdots\right]t$ $=$I.P. of $\frac{e^{it}}{2i}\frac{1}{\theta}\left[t-\frac{1}{2i}\right]$

$=$I.P. of $\frac{e^{it}}{2i}\left[\frac{t^2}{2}-\frac{t}{2i}\right]$ $=$I.P. of $\frac{1}{4}(\cos t + i\sin t)(-it^2+t)$

$=\frac{1}{4}(\text{t} \sin t - t^2\cos t)$

$\therefore$ The general solution is $y(t) = y_c(t) + y_p(t)$

i.e., $y(t) = c_1\cos t + c_2\sin t + \frac{1}{4}(t\sin t - t^2\cos t)$

Substituting $x = e^t or\ t = \ln x$, we get the general solution of the given differential equation as

$$y(x) = c_1\cos(\ln x) + c_2\sin(\ln x) + \frac{1}{4}[\ln x\sin(\ln x) - (\ln x)^2\cos(\ln x)]$$

where c_1, c_2 are arbitrary constants.

IP4:

Solve: $[(2x+1)^2D^2 - 6(2x+1)D + 16]y = 8(2x+1)^2$

Solution:

The given differential equation is a Legendre linear equation

Put $2x + 1 = e^t$ or $t = \ln(2x+1)$

Then $(2x+1)Dy = 2\theta y$, $(2x+1)^2D^2y = 2^2\theta(\theta-1)y$,

where $\theta = \frac{d}{dt}$. The equation now becomes

$$[2^2\theta(\theta-1) - 12\theta y + 16]y = 8e^{2t}$$

$$[\theta(\theta-1) - 3\theta + 4]y = 2e^{2t} \Rightarrow (\theta^2 - 4\theta + 4)y = 2e^{2t}$$

It is a 2nd order non – homogeneous linear differential equation with constant coefficients.

The A.E is given by $P(m) = 0$

i.e., $m^2 - 4m + 4 = 0 \quad \Rightarrow m = 2\ (twice)$

The C.F. is $\quad y_c(t) = (c_1 + c_2t)e^{2t}$

Now, $P.I = y_p(t) = \frac{1}{\theta^2 - 4\theta + 4} 2e^{2t} = \frac{1}{(\theta-2)^2} 2e^{2t}$

$$= 2.\frac{t^2}{2!}e^{2t} = t^2e^{2t}$$

∴ The general solution is

$y(t) = (c_1 + c_2t)e^{2t} + t^2e^{2t}$

Hence the general solution of the given differential equation is

$$y(x) = [c_1 + c_2 \; ln\,(1 + 2x)](1 + 2x)^2 + (1 + 2x)^2[ln\,|1 + 2x|]^2,$$

where c_1, c_2 are arbitrary constants.

2.9. Solution of Euler-Cauchy equation

EXERCISES

I. Solve the following differential equations:

1. $x^2y'' + xy' - 4y = 0$
2. $4x^2y'' + y = 2\sin(lnx)$
3. $4x^2y'' + 16xy' + 9y = 19\cos(lnx) + 22\sin(lnx)$
4. $x^2y'' + y = 3x^2$
5. $x^2y'' - 2y = x^2 + \frac{1}{x}$
6. $x^2y'' + 2xy = lnx$
7. $x^2y'' + 5xy' + 4y = x\, lnx$
8. $x^3y''' + 2x^2y'' + 2y = 10\left(x + \frac{1}{x}\right)$
9. $x^3y''' + 2xy' - 2y = x^2 lnx + 3x$
10. $x^4y^{iv} + 6x^3y''' + 4x^2y'' - 2xy' - 4y = 2\cos(lnx)$
11. $x^3y''' + 9x^2y'' + 18xy' + 6y = 0$
12. $x^3y''' - 3x^2y'' + 7xy' - 8y = 3x^3 + 8x$
13. $x^4y^{iv} + 6x^3y''' + 4x^2y'' - 2xy' + y = 0$
14. $4x^4y^{iv} + 16x^3y''' - x^2y'' + 9xy' - 9y = 14x^2 + 1$
15. $(x+1)^2\frac{d^2y}{dx^2} - 3(x+1)\frac{dy}{dx} + 4y = x^2 + x + 1$
16. $(x+1)^2\frac{d^2y}{dx^2} + (x+1)\frac{dy}{dx} = (2x+3)(2x+4)$
17. $(2x+1)^2\frac{d^2y}{dx^2} - 2(2x+1)\frac{dy}{dx} - 12 = 6x$
18. $(x+1)^2\frac{d^2y}{dx^2} + (x+1)\frac{dy}{dx} + y = 4\cos(lnx + 1)$

Answers:

1. $y = Ax^2 + B\left(\frac{1}{x^2}\right)$
2. $y = (A + B\ lnx)x^{\frac{1}{2}} + 4\cos(lnx) - 3\sin(lnx)$
3. $y = (A + B\ lnx)x^{-\frac{3}{2}} - \cos(lnx) + 2\sin(lnx)$
4. $y = x^{\frac{1}{2}}\left[c_1 cos\left(\frac{\sqrt{3}}{2}lnx\right) + c_2 sin\left(\frac{\sqrt{3}}{2}lnx\right)\right] + x^2$
5. $y = c_1x^2 + c_2\left(\frac{1}{x}\right) + \frac{1}{3}\left(x^2 + \frac{1}{x}\right) lnx$
6. $y = c_1x^2 + c_2\left(\frac{1}{x}\right) + \frac{1}{2}(lnx)^2 - lnx$
7. $y = (A + B\ lnx)x^2 + \frac{x}{27}(3\ lnx - 2)$
8. $y = c_1\left(\frac{1}{x}\right) + x[c_2 cos(lnx) + c_3 sin(lnx)] + 5x + \frac{2\ lnx}{x}$
9. $y = x[c_1 + c_2 cos(lnx) + c_3 sin(lnx)] + \frac{x^2}{2}(lnx - 2) + 3x\ lnx$
10. $y = c_1x^2 + c_2x^{-2} + c_3\cos(lnx) + c_4\sin(lnx) - \frac{1}{5}lnx\sin(lnx)$
11. $y = A\left(\frac{1}{x}\right) + B\left(\frac{1}{x^2}\right) + C\left(\frac{1}{x^3}\right)$
12. $y = (A + B\ lnx + C(lnx)^2)x^2 + 3x^3 - 8x$
13. $y = Ax^2 + B\left(\frac{1}{x^2}\right) + C\ cos(lnx) + D\sin(lnx)$
14. $y = Ax^{\frac{3}{2}} + Bx^{-\frac{3}{2}} + (C + D\ lnx)x + 2x^2 - \frac{1}{9}$
15. $y = (c_1 + c_2\ln(x + 1))(x + 1)^2 + \frac{[ln(x+1)]^2}{2}(x + 1)^2 - (x + 1) + \frac{1}{4}$
16. $y = (c_1 + c_2\ln(x + 1))(x + 1)^2 + 6(x + 1) + \big(ln(x + 1)\big)^2$
17. $y = c_1(2x + 1)^2 + c_2(2x + 1)^{-1} - \frac{3}{16}(2x + 1) + \frac{1}{4}$
18. $y = c_1\cos(\ln(1 + x)) + c_2\sin(\ln(1 + x))$
$+2\ln(x + 1)\sin(\ln(1 + x))$