1.1 Sequences

Learning objectives:

- To define an infinite sequence of real numbers and to discuss its convergence and divergence.
- To define the boundedness of a sequence and to prove that every convergent sequence is bounded.
- To define Non-decreasing and Non-increasing sequences and to discuss their convergence.

  And
- To practice the related problems.

Sequences

An infinite sequence of real numbers is a list of real numbers

$$a_1, a_2, a_3, \dots, \dots, \dots, a_n, \dots, \dots \qquad \text{------}(1)$$

in a definite order and formed according to a definite rule. Each number in the infinite sequence is called a ***term*** and $\boldsymbol{a_n}$ is called the $\boldsymbol{n^{th}}$ ***term.*** The suffix n is called the ***index*** of a_n and it indicates where a_n occurs in the list.

We can think the infinite sequence (1) as a function that sends a natural number n to a real number a_n (the n^{th} term of (1)) .

Thus we have a formal definition of an infinite sequence.

Infinite sequence:

*An **infinite sequence** of real numbers is a function*

$$\boldsymbol{a}: \boldsymbol{N} \to \boldsymbol{R}$$

The infinite sequence defined by the above function is denoted by $\{a_n\}$ or $\{a_n\}_{n=1}^{\infty}$, where $a_n = a(n)$, $n \in \boldsymbol{N}$.

From now onwards, a sequence means an infinite sequence of real numbers.

Sequences are described by writing rules that specify their terms, such as

$$a_n = \sqrt{n}\,,\ b_n = \begin{cases} \frac{1}{n} & if\ n\ is\ odd \\ -\frac{1}{n} & if\ n\ is\ even \end{cases}\ ,\ c_n = \frac{n-1}{n},\ d_n = (-1)^{n+1}$$

or by listing its terms (in the order as in (1))

$$\{a_n\}_{n=1}^{\infty}: \quad \sqrt{1}, \sqrt{2}, \sqrt{3}, \dots, \sqrt{n}\,, \dots$$

$\{b_n\}_{n=1}^{\infty}$: $\quad 1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \dots, (-1)^{n+1}\frac{1}{n}, \dots$

$\{c_n\}_{n=1}^{\infty}$: $\quad 0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots, \frac{n-1}{n}, \dots$

$\{d_n\}_{n=1}^{\infty}$: $\quad 1, -1, 1, -1, \dots, (-1)^{n+1}, \dots$

*A sequence $\{a_n\}$ is said to be a **constant sequence** if $a_n = k$, $\forall\, n \in \boldsymbol{N}$, where k is a constant.*

Example 1: Finding Terms of a sequence

If the formula for the n^{th} term a_n of a sequence $\{a_n\}$ is $a_n = 2 + (-1)^n$, then find the terms a_1, a_2, a_3, a_4 and a_5.

Solution: Given $a_n = 2 + (-1)^n$, $n \in N$. Then

$a_1 = 2 + (-1) = 1$; $a_2 = 2 + (-1)^2 = 3$; $a_3 = 2 + (-1)^3 = 1$

$a_4 = 2 + (-1)^4 = 3$; $a_5 = 2 + (-1)^5 = 1$.

The sequence $\{a_n\}$ is

$$1, 3, 1, 3, 1, 3, \dots\dots\dots\dots\dots$$

Example 2: Finding a Sequence's Formula

Find a formula for the n^{th} term of the sequence

$$0, 3, 8, 15, 24, \dots\dots\dots..$$

Solution: Notice that

$a_1 = 1 - 1$, $a_2 = 2^2 - 1$, $a_3 = 3^2 - 1$, $a_4 = 4^2 - 1$, $\dots\dots\dots$

Therefore, $a_n = n^2 - 1$, $n \in \boldsymbol{N}$.

Recursive Definitions

Sequences are often defined ***recursively*** by giving the value(s) of the initial term or terms, and a rule, called **recursion formula** for calculating any later term from terms that precede it.

Example 3: Sequences constructed Recursively

(a) The statements $a_1 = 1$ and $a_n = a_{n-1} + 1, n \geq 2$ define the sequence $1,2,3, \dots, n, \dots$ of natural numbers.

(b) The statements $a_1 = 1$ and $a_n = n \cdot a_{n-1}, n \geq 2$ define the sequence $1,2,6,24, \dots, n!, \dots$ of factorials.

(c) The statements $a_1 = 1$, $a_2 = 1$ and $a_{n+1} = a_n + a_{n-1}$, $n \geq 2$ define the sequence $1,1,2,3,5,8,13 \dots, \dots$ of **Fibonacci numbers.**

Convergence and divergence

Sometimes the numbers in a sequence approach a single value as the index n increases.

In the sequence $\left\{\frac{1}{n}\right\}$, the terms approach 0 as n gets large, and in the sequence $\left\{\frac{n-1}{n}\right\}$: $0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots, 1 - \frac{1}{n}, \dots$ the terms approach 1 as n increases. On the other hand sequences like $\left\{\sqrt{n}\right\}$: $\sqrt{1}, \sqrt{2}, \sqrt{3}, \dots, \sqrt{n}, \dots$ have terms larger than any number as n increases and sequences like $\{(-1)^{n+1}\}$: $1, -1, 1, -1, \dots, (-1)^{n+1}, \dots$ bounce back and forth between 1 and -1, never approach a single value.

The following definition captures the meaning of having a sequence converge to a limiting value.

Definitions:

The sequence of real numbers $\{a_n\}$ ***converges*** *to a number* L *if for every* $\epsilon > 0$ *there exists a natural number* N *such that*

$$|a_n - L| < \epsilon \text{ for all } n \geq N$$

We say that $\{a_n\}$ ***diverges*** *if no such number* L *exists.*

If $\{a_n\}$ *converges to* L *then we write* $\lim_{n\to\infty} a_n = L$ *or* $a_n \to L$ *as* $n \to \infty$ *and call* L *the* ***limit*** *of the sequence* $\{a_n\}$.

A sequence is said to be ***convergent*** *if it converges.*

A sequence is said to be ***divergent*** *if it diverges.*

The above definition says that if we go far enough out in the sequence $\{a_n\}$, by taking the index n larger than some value N, the difference between a_n and the limit L of the sequence becomes less than the preselected number $\epsilon > 0$.

Note:

1. The natural number N depends on ϵ.
2. $|a_n - L| < \epsilon$, for all $n \geq N \Leftrightarrow L - \epsilon < a_n < L + \epsilon$, for all $n \geq N$

$$\Leftrightarrow a_n \in (L - \epsilon, L + \epsilon), \text{ for all } n \geq N$$

3. The limit of a convergent sequence is unique.

Example 4: Applying the Definition

Prove that

i. The sequence $\left\{\frac{1}{n}\right\}$ converges to 0.

ii. Every constant sequence $\{a_n\} = \{k\}$ converges to k, where k is any constant.

Solution:

i. Let $\epsilon > 0$ be given. We have to show that for this $\epsilon > 0$ there exists a natural number N such that

$$|a_n - 0| = \left|\frac{1}{n} - 0\right| < \epsilon \text{ for all } n \geq N.$$

This holds if $\frac{1}{n} < \epsilon \;\; i.e., n > \frac{1}{\epsilon}$. Let N be any natural number greater than $\frac{1}{\epsilon}$ $i.e., N > \frac{1}{\epsilon}$.

Then for all $n \geq N$, we have $\frac{1}{n} \leq \frac{1}{N} < \epsilon$. Thus for each given $\epsilon > 0$, $\exists$ a natural number N $\left(> \frac{1}{\epsilon}\right)$ such that

$$\left|\frac{1}{n} - 0\right| = \frac{1}{n} \leq \frac{1}{N} < \epsilon \quad \text{for all } n \geq N.$$

This proves $\lim\limits_{n\to\infty} \frac{1}{n} = 0$ and therefore the sequence $\left\{\frac{1}{n}\right\}$ converges to 0.

ii. Let $\epsilon > 0$ be given. We have to show that for this $\epsilon > 0$ there exists a natural number N such that

$$|a_n - k| = |k - k| < \epsilon \text{ for all } n \geq N.$$

Since $k - k = 0$, this holds for any natural number N.

This proves $\lim\limits_{n\to\infty} k = k$ for any constant sequence $\{a_n\} = \{k\}$ converges to k.

Subsequence:

*If $\{a_n\}_{n=1}^{\infty}$ is a sequence and $\{n_k\}_{k=1}^{\infty}$ is a sequence of natural numbers such that $n_1 < n_2 < n_3 < \cdots$, then the sequence $\{a_{n_k}\}_{k=1}^{\infty}$ is called a **subsequence** of $\{a_n\}_{n=1}^{\infty}$.*

The following is a property of convergent sequences.

Theorem 1:

The sequence $\{a_n\}$ converges to L if and only if every subsequence of $\{a_n\}$ converges to L.

The following is a useful result.

Theorem 2:

If the subsequences $\{a_{2n-1}\}_{n=1}^{\infty}$ and $\{a_{2n}\}_{n=1}^{\infty}$ of a sequence $\{a_n\}$ converges to the same limit L, then the sequence $\{a_n\}$ converges to L.

The following is a way to prove that a given sequence is divergent.

Theorem 3:

If two subsequences of a sequence $\{a_n\}$ have different limits $L_1 \neq L_2$, then $\{a_n\}$ diverges.

Divergent Sequences

Example 5:

Prove that the sequence $\{a_n\} = \{(-1)^{n+1}\}$ diverges.

Solution:

Notice that the subsequence $\{a_{2n-1}\}_{n=1}^{\infty}$: 1,1,1, ... converges to 1 and the subsequence $\{a_{2n}\}_{n=1}^{\infty}$: $-1, -1, -1, \ldots$ converges to -1. Thus, the given sequence has two convergent subsequences converging to two different limits. Therefore, $\{a_n\} = \{(-1)^{n+1}\}$ diverges by Theorem 3.

We say that a sequence $\{a_n\}$ ***diverges to infinity*** *if for every number* M *there is a natural number* N *such that* $a_n > M$ *for all* $n \geq N$. *If this holds then write*

$$\lim_{n\to\infty} a_n = \infty \quad \textit{or} \quad a_n \to \infty \textit{ as } n \to \infty.$$

We say that a sequence $\{a_n\}$ ***diverges to negative infinity*** *if for every number* m *there is a natural number* N *such that* $a_n < m$ *for all* $n \geq N$ *and we write*

$$\lim_{n\to\infty} a_n = -\infty \quad \textit{or} \quad a_n \to -\infty \textit{ as } n \to \infty.$$

Example 6:

Show that the sequence $\{\sqrt{n}\}$ diverges to infinity.

Solution:

Let M be any given number. We have to show that ∃ a natural number N such that $a_n = \sqrt{n} > M$, for all $n \geq N$. This holds if

$n > M^2$. Let N be any natural number greater than M^2, $i.e., N > M^2$. Then for all $n \geq N$, we have

$$a_n = \sqrt{n} \geq \sqrt{N} > M$$

This shows that the sequence $\{\sqrt{n}\}$ diverges to ∞ and we write $\lim_{n\to\infty} \sqrt{n} = \infty$.

Note: A Sequence may diverge with out diverging to ∞ or $-\infty$. Example: $\{(-1)^{n+1}\}$.

Boundedness of a Sequence

*A sequence $\{a_n\}$ of real numbers is **bounded above** if there exists a number M such that $a_n \leq M$ for all n. The number M is an **upper bound** for $\{a_n\}$.*

*A sequence $\{a_n\}$ of real numbers is **bounded below** if there exists a number m such that $m \leq a_n$ for all n. The number m is a **lower bound** for $\{a_n\}$.*

*A sequence $\{a_n\}$ of real numbers is **bounded** if it is both bounded above and bounded below, i.e., if there exist numbers m and M such that $m \leq a_n \leq M$ for all n.*

*A number M is the **least upper bound (lub or supremum)** for $\{a_n\}$ if M is an upper bound for $\{a_n\}$ and no number less than M is an upper bound for $\{a_n\}$.*

*A number m is the **greatest lower bound (glb or infimum)** for $\{a_n\}$ if m is a lower bound for $\{a_n\}$ and no number greater than m is a lower bound for $\{a_n\}$.*

Note:

1. If m and M are lower and upper bounds of a sequence then $m \leq M$.
2. If M is an upper bound of a sequence then any number greater than M is also an upper bound of the sequence.
3. If m is a lower bound of a sequence then any number less than m is also a lower bound of the sequence.
4. Least upper bound of a sequence, if exists is unique.
5. Greatest lower bound of a sequence, if exists is unique.

Example 7:

(i) The sequence $\{n\}$: $1,2,3, \dots, n, \dots$ is bounded below by 1 and 1 is its greatest lower bound. Since the terms are becoming larger and larger, the sequence not bounded above. Thus, this sequence is not bounded.

(ii) The sequence $\left\{\frac{n}{n+1}\right\}$: $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots, \frac{n}{n+1}, \dots$ is bounded above by 1 and bounded below by $\frac{1}{2}$. Thus, the sequence is bounded.

We want to show that 1 is the lub of the sequence.

Let $0 < M < 1$. We show that M is not an upper bound of the sequence. If M is an upper bound then $\frac{n}{n+1} \leq M$ for all n, $i.e.$,

$n \leq \frac{M}{1-M}$ for all n. This is not possible. This shows that no number less than 1 is an upper bound of the sequence. Thus, 1 is the lub of this sequence. Further, $\frac{1}{2}$ is the glb of this sequence.

(iii) The sequence $\{(-1)^{n+1}\}$: $1, -1, 1, -1, \ldots$ is bounded below by -1 and bounded above by 1. Therefore, it is a bounded sequence. Further, -1 and 1 are the glb and lub of the sequence respectively.

Theorem 4:

Every convergent sequence is bounded.

Proof: Let $\{a_n\}$ be a sequence of real numbers converging to L. $i.e.$, $\lim_{n\to\infty} a_n = L$. For $\epsilon = 1$, $\exists$ a natural number N such that $|a_n - L| < 1$ for all $n \geq N$, $i.e.$, $L - 1 < a_n < L + 1$ for all $n \geq N$. Let $m = min\,\{a_1, a_2, a_3, \ldots, a_{N-1}, L - 1\}$ and $M = max\{a_1, a_2, a_3, \ldots, a_{N-1}, L + 1\}$. Then we have $m \leq a_n \leq M$ for $n \in \boldsymbol{N}$. Thus $\{a_n\}$ is bounded. Hence the theorem.

Note: The converse of the above theorem is not true, $i.e.$, a bounded sequence need not be convergent.

For example the sequence $\{(-1)^{n+1}\}$ is bounded but not convergent.

Non-decreasing and Non-increasing Sequences

Let $\{a_n\}$ be a sequence of real numbers.

*$\{a_n\}$ is said to be a **non-decreasing sequence** if $a_n \leq a_{n+1}$ for all $n \in \boldsymbol{N}$.*

*$\{a_n\}$ is said to be a **non-increasing sequence** if $a_n \geq a_{n+1}$ for all $n \in \boldsymbol{N}$.*

*A sequence of real numbers is said to be a **monotonic sequence** if it is either non-decreasing or non-increasing.*

The sequences $\{n\}$: $1,2,3,\dots,n,\dots$ and $\left\{\frac{n}{n+1}\right\}$: $\frac{1}{2},\frac{2}{3},\frac{3}{4},\dots,\frac{n}{n+1},\dots$ are non-decreasing sequences.

The sequences $\left\{\frac{1}{n}\right\}$: $1,\frac{1}{2},\frac{1}{3},\dots,\frac{1}{n},\dots$ and $\left\{\frac{n+1}{n}\right\}$: $2,\frac{3}{2},\frac{4}{3},\dots,\frac{n+1}{n},\dots$ are non-increasing sequences.

It may be noted that every constant sequence is both non-increasing and non-decreasing.

If the sequence $\{a_n\}$ is non-decreasing, then $a_n \leq a_{n+1}$ for all $n \in \boldsymbol{N}$ and so $a_1 \leq a_n$ for all $n \in \boldsymbol{N}$. This shows that

- ***Every non-decreasing sequence $\{a_n\}$ is bounded below and a_1 is the greatest lower bound for $\{a_n\}$.***

Completeness property of real numbers

A non-decreasing sequence of real numbers that is bounded above always has a least upper bound.

Theorem 5: The Non-decreasing Sequence Theorem

A non-decreasing sequence of real numbers converges if and only if it is bounded above.

Proof: Let $\{a_n\}$ be a non-decreasing sequence of real numbers. If $\{a_n\}$ is convergent then it is bounded (by Theorem 6). Therefore it is bounded above.

Suppose that $\{a_n\}$ is bounded above. Then by the completeness property of real numbers, $\{a_n\}$ has a least upper bound M. Let $\epsilon > 0$ be given. Now $M - \epsilon$ is not an upper bound of $\{a_n\}$. Therefore ∃ a natural number N such that $a_N > M - \epsilon$. Since $\{a_n\}$ is non-decreasing, $M - \epsilon < a_N \leq a_n$ for all $n \geq N$. Further $a_n < M < M + \epsilon$ for all n. Thus

$$M - \epsilon < a_n < M + \epsilon$$

for $n \geq N$. This proves for each $\epsilon > 0$, ∃ a natural number N such that

$$|a_n - M| < \epsilon \quad \text{for all } n \geq N$$

Therefore, $\{a_n\}$ is convent and converges to its least upper bound. Hence the theorem.

Theorem 5 implies that a non-decreasing sequence converges if it is bounded above. Further, it converges to its least upper bound.

Example 8: The sequence $\left\{\frac{n}{n+1}\right\}$: $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots, \frac{n}{n+1}, \ldots$ converges to 1 since it is a non-decreasing sequence which is bounded above and 1 is the least upper bound of the sequence.

The following is evident:

Theorem 6:

If a non-decreasing sequence of real numbers converges then it converges to its least upper bound.

The following theorem proves that a non-decreasing sequence diverges to infinity if it is not bounded above.

Theorem 7:

If a non-decreasing sequence of real numbers is not bounded above then it diverges to infinity.

Proof: Let $\{a_n\}$ be a non-decreasing sequence of real numbers. $i.e.,$ $a_n \leq a_{n+1}$ for all n. Let $M > 0$ be any given number. Since $\{a_n\}$ is not bounded above ∃ a natural number N such that $a_N > M$. Thus

$$a_n \geq a_N > M \quad \text{for all } n \geq N$$

This proves that the sequence $\{a_n\}$ diverges to infinity.

Example 9: The sequence $\{n\}$: $1,2,3, \ldots, n, \ldots$ diverges to infinity since it is non-decreasing sequence and not bounded above.

The following are the corresponding results for non-increasing sequences of real numbers

Theorem 8:

A non-increasing sequence $\{a_n\}$ of real numbers

(i) ***is always bounded above and a_1 is its lub.***

(ii) ***has a glb if it bounded below.***

(Completeness Property of real numbers)

(iii) ***converges if and only if it is bounded below.***

(The Non-increasing Sequence Theorem)

(iv) ***converges to its glb if it is bounded below.***

(v) ***diverges to minus infinity if it is not bounded below.***

Combining the Non-decreasing Sequence Theorem and Non-increasing Sequence Theorem we get

Theorem 9:

A monotonic sequence of real numbers is convergent if and only if it is bounded.

P1:

$(\boldsymbol{a})$ If the formula for $\boldsymbol{n^{th}}$ term $\boldsymbol{a_n}$ of a sequence $\{\boldsymbol{a_n}\}$ is $\boldsymbol{a_n = \frac{3(-1)^n}{n!}}$, then find the first five terms of the sequence.

$(\boldsymbol{b})$ Find a formula for the $\boldsymbol{n^{th}}$ term of the sequence

$$\mathbf{1}, -\frac{\mathbf{1}}{\mathbf{4}}, \frac{\mathbf{1}}{\mathbf{9}}, -\frac{\mathbf{1}}{\mathbf{16}}, \frac{\mathbf{1}}{\mathbf{25}}, \dots, \dots$$

Solution:

$(\boldsymbol{a})$ Given $a_n = \frac{3(-1)^n}{n!}$ for all $n \in \boldsymbol{N}.$ Then the first five terms of the sequence are

$$a_1 = \frac{3(-1)^1}{1!} = -3 \;,\; a_2 = \frac{3(-1)^2}{2!} = \frac{3}{2}, \quad a_3 = \frac{3(-1)^3}{3!} = -\frac{1}{2},$$

$$a_4 = \frac{3(-1)^4}{4!} = \frac{1}{8} \;, \quad a_5 = \frac{3(-1)^5}{5!} = -\frac{1}{40}$$

Therefore, the sequence $\{a_n\}$ is

$$-3, \frac{3}{2}, -\frac{1}{2}, \frac{1}{8}, -\frac{1}{40}, \dots, \dots$$

$(\boldsymbol{b})$ Given sequence is $1, -\frac{1}{4}, \frac{1}{9}, -\frac{1}{16}, \frac{1}{25}, \dots, \dots$

Notice that

$$a_1 = 1 \,,\, a_2 = -\frac{1}{2^2} \,,\, a_3 = \frac{1}{3^2} \,,\, a_4 = -\frac{1}{4^2} \,,\, a_5 = \frac{1}{5^2} \,, \dots\dots\dots$$

Therefore, $a_n = \frac{(-1)^{n-1}}{n^2}$ for all $n \in \boldsymbol{N}.$

P2:

Show that the sequence $\{\sqrt{n+1}-\sqrt{n}\}$ converges to 0.

Solution:

Let $a_n = \sqrt{n+1} - \sqrt{n}$.

Now, $\left|\sqrt{n+1}-\sqrt{n}\right| = \sqrt{n+1} - \sqrt{n} \times \frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}}$

$$= \frac{1}{\sqrt{n+1}+\sqrt{n}} < \frac{1}{\sqrt{n}+\sqrt{n}} = \frac{1}{2\sqrt{n}}$$

Let $\epsilon > 0$ be given.

Therefore, $|a_n - 0| < \frac{1}{2\sqrt{n}} < \epsilon$ whenever $n > \frac{1}{4\epsilon^2}$.

Let N be a natural number greater than $\frac{1}{4\epsilon^2}$ $i.e.,$ $N > \frac{1}{4\epsilon^2}$.

Now, for all $n \geq N$, we have $\frac{1}{2\sqrt{n}} \leq \frac{1}{2\sqrt{N}} < \epsilon$.

Thus for each given $\epsilon > 0$, ∃ a natural number N $\left(> \frac{1}{4\epsilon^2}\right)$ such that

$$|a_n - 0| < \frac{1}{2\sqrt{n}} \leq \frac{1}{2\sqrt{N}} < \epsilon \quad \text{for all } n \geq N.$$

This proves $\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(\sqrt{n+1}-\sqrt{n}\right) = 0$ and therefore the sequence $\{\sqrt{n+1}-\sqrt{n}\}$ converges to 0.

P3:

Prove that the sequence $\{a_n\}$ is convergent, where

$$\boldsymbol{a_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}}.$$

Proof:

Given $a_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}, n \in \boldsymbol{N}$

$\therefore \quad a_{n+1} - a_n = \frac{1}{(n+1)!} > 0$ for all $n \in \boldsymbol{N}$

$\Longrightarrow \quad a_n < a_{n+1}$ for all $n \in \boldsymbol{N}$

Thus, the sequence $\{a_n\}$ is a non-decreasing sequence.

Now, $a_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}, n \in \boldsymbol{N}$

$$< 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^{n-1}} = 1 + \left(\frac{1 - \frac{1}{2^n}}{1 - \frac{1}{2}}\right)$$

$$= 1 + 2 - \frac{1}{2^{n-1}} = 3 - \frac{1}{2^{n-1}} \leq 3$$

$\Longrightarrow \quad a_n \leq 3$ for all $n \in \boldsymbol{N}$

Thus, the sequence $\{a_n\}$ is bounded above by 3.

Therefore, by non-decreasing theorem for sequences, the sequence $\{a_n\}$ is convergent.

P4:

$(\boldsymbol{a})$ The limit of a convergent sequence is unique.

$(\boldsymbol{b})$ Prove that a sequence $\{\boldsymbol{a_n}\}$ converges to $\mathbf{0}$ if and only if the sequence of absolute values $\{|\boldsymbol{a_n}|\}$ converges to $\mathbf{0}$.

Proof:

$(\boldsymbol{a})$ Let $\{a_n\}$ be a convergent sequence converging to L and M. We intend to prove that $L = M$.

Let $\epsilon > 0$ be arbitrary. Then by definition there exist natural numbers N_1 and N_2 such that

$$|a_n - L| < \frac{\epsilon}{2} \text{ for all } n \geq N_1 \quad \text{and} \quad |a_n - M| < \frac{\epsilon}{2} \text{ for all } n \geq N_2$$

Let $N = max\{N_1, N_2\}$. Then

$$|a_n - L| < \frac{\epsilon}{2} \quad \text{and} \quad |a_n - M| < \frac{\epsilon}{2} \text{ for all } n \geq N$$

Now,

$$|L - M| = |(L - a_n) + (a_n - M)| \leq |L - a_n| + |a_n - M|$$
$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Notice that $|L - M|$ is non-negative. The only non-negative number less than every positive number ϵ is 0. Therefore $|L - M| = 0$. Thus $L = M$. Hence the result.

$(\boldsymbol{b})$ $\{a_n\}$ converges to $0 \Leftrightarrow$ for each $\epsilon > 0$ there exists a natural number N such that $|a_n - 0| < \epsilon$ for all $n \geq N \Leftrightarrow$ $|a_n| < \epsilon$ for all $n \geq N \Leftrightarrow$ $\big||a_n|\big| < \epsilon$ for all $n \geq N$ $\Leftrightarrow$ $\big||a_n| - 0\big| < \epsilon$ for all $n \geq N \Leftrightarrow \{|a_n|\}$ converges to 0.

IP1:

$(\boldsymbol{a})$ **If $\boldsymbol{a_1 = -2}$ and $\boldsymbol{a_{n+1} = \frac{na_n}{n+1}}$, then write the first ten terms of the sequence $\{\boldsymbol{a_n}\}$.**

$(\boldsymbol{b})$ **Find a formula for the $\boldsymbol{n^{th}}$ term of the sequence**

$$\frac{3}{5}, \frac{4}{25}, \frac{5}{125}, \frac{6}{625}, \frac{7}{3125}, \dots, \dots.$$

Solution:

$(\boldsymbol{a})$ Given $a_1 = -2$, $a_{n+1} = \frac{na_n}{n+1}$ for all $n \in \boldsymbol{N}$. Then the first ten terms of the sequence are

$$a_1 = -2 \;,\; a_2 = \frac{a_1}{2} = -1 \;,\; a_3 = \frac{2a_2}{3} = -\frac{2}{3},\; a_4 = \frac{3a_3}{4} = -\frac{1}{2}$$

$$a_5 = \frac{4a_4}{5} = -\frac{2}{5}, \quad a_6 = \frac{5a_5}{6} = -\frac{1}{3}, \quad a_7 = \frac{6a_6}{7} = -\frac{2}{7},$$

$$a_8 = \frac{7a_7}{8} = -\frac{1}{4}, \quad a_9 = \frac{8a_8}{9} = -\frac{2}{9}, \quad a_{10} = \frac{9a_9}{10} = -\frac{1}{5}$$

Therefore, the sequence $\{a_n\}$ is

$$-2, -1, -\frac{2}{3}, -\frac{1}{2}, -\frac{2}{5}, -\frac{1}{3}, \dots, \dots.$$

$(\boldsymbol{b})$ Given sequence is $\frac{3}{5}, \frac{4}{25}, \frac{5}{125}, \frac{6}{625}, \frac{7}{3125}, \dots, \dots.$

Notice that $a_1 = \frac{3}{5}, a_2 = \frac{4}{5^2}, a_3 = \frac{5}{5^3}, a_4 = \frac{6}{5^4}, \dots\dots\dots$

$$\Longrightarrow \quad a_1 = \frac{1+2}{5}, a_2 = \frac{2+2}{5^2}, a_3 = \frac{3+2}{5^3}, a_4 = \frac{4+2}{5^4}, \dots\dots\dots$$

Therefore, $a_n = \frac{n+2}{5^n}$ for all $n \in \boldsymbol{N}$.

IP2:

Prove that the sequence $\{\sqrt[n]{n}\}$ is converges to 1.

Proof:

Notice that $\sqrt[n]{n} > 1, n \in \boldsymbol{N}$, for if not $\sqrt[n]{n} < 1 \ \Rightarrow n < 1$, which is not true.

Let $a_n = \sqrt[n]{n} = 1 + b_n$, where $b_n \geq 0$. We first prove that $\{b_n\}$ converges to 0. Let $\epsilon > 0$ be given.

Now, $\sqrt[n]{n} = 1 + b_n$

$$\Rightarrow \quad n = (1 + b_n)^n = 1 + nb_n + \frac{n(n-1)}{2!} {b_n}^2 + \cdots + {b_n}^n$$

(By binomial theorem for positive integral index)

$$\Rightarrow \quad n > \frac{n(n-1)}{2} {b_n}^2 \qquad (\because \ b_n \geq 0, \forall \ n \in \boldsymbol{N})$$

$$\Rightarrow \quad {b_n}^2 < \frac{2}{n-1} \text{ for all } n \geq 2 \Rightarrow |b_n| < \sqrt{\frac{2}{n-1}} \quad \text{for all } n \geq 2$$

Therefore, $|b_n - 0| < \sqrt{\frac{2}{n-1}} < \epsilon$ whenever $n > 1 + \frac{2}{\epsilon^2}$.

Let N be any natural number greater than $1 + \frac{2}{\epsilon^2}\, i.e., N > 1 + \frac{2}{\epsilon^2}$

$i.e.,$ $\sqrt{\frac{2}{N-1}} < \epsilon.$

Now, for all $n \geq N$, we have $\sqrt{\frac{2}{n-1}} \leq \sqrt{\frac{2}{N-1}} < \epsilon$. Thus for each given $\epsilon > 0$, ∃ a natural number $N > \left(1 + \frac{2}{\epsilon^2}\right)$ such that

$$|b_n - 0| < \sqrt{\frac{2}{n-1}} \leq \sqrt{\frac{2}{N-1}} < \epsilon \text{ for all } n \geq N.$$

This proves $\{b_n\}$ converges to 0.

Now, for each $\epsilon > 0$, ∃ a natural number $N > \left(1 + \frac{2}{\epsilon^2}\right)$ such that

$$\left|\sqrt[n]{n} - 1\right| = |b_n| < \epsilon \qquad \text{for all } n \geq N.$$

This proves $\left\{\sqrt[n]{n}\right\}$ converges to 1.

IP3:

Prove that the sequence $\{\boldsymbol{a_n}\}$ is convergent, where

$$\boldsymbol{a_n = \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \cdots + \frac{1}{n+n}} \text{ .}$$

Proof:

Given $a_n = \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \cdots + \frac{1}{n+n}$, $\quad n \in \boldsymbol{N}$

$$a_{n+1} = \frac{1}{n+2} + \frac{1}{n+3} + \cdots + \frac{1}{n+n} + \frac{1}{2n+1} + \frac{1}{2n+2}$$

$$\therefore \quad a_{n+1} - a_n = \frac{1}{2n+1} + \frac{1}{2n+2} - \frac{1}{n+1}$$

$$= \frac{1}{(2n+1)(2n+2)} > 0 \quad \text{for all } n \in \boldsymbol{N}$$

$\Longrightarrow \quad a_n < a_{n+1}$ for all $n \in \boldsymbol{N}$

Thus, the sequence $\{a_n\}$ is non-decreasing sequence.

Now, $\quad a_n = \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \cdots + \frac{1}{n+n}$, $\quad n \in \boldsymbol{N}$

$\Longrightarrow \qquad a_n < \frac{1}{n} + \frac{1}{n} + \cdots + \frac{1}{n} = n.\frac{1}{n} = 1$, $\qquad n \in \boldsymbol{N}$

$\Longrightarrow \qquad a_n < 1$ for all $n \in \boldsymbol{N}$

Thus, the sequence $\{a_n\}$ is bounded above.

Therefore, by non-decreasing theorem for sequences, the sequence $\{a_n\}$ is convergent.

IP4:

$(\boldsymbol{a})$ Prove that a sequence of real numbers can not have two different least upper bounds.

$(\boldsymbol{b})$ Prove that if $\{a_n\}$ is a convergent sequence, then to every positive number ϵ there corresponds an integer N such that for all n and m.

$$\boldsymbol{n \geq N \quad \text{and} \quad m \geq N \quad \Longrightarrow \quad |a_n - a_m| < \epsilon}$$

Proof:

$(\boldsymbol{a})$ Let $\{a_n\}$ be a sequence of real numbers. Suppose M_1 and M_2 are least upper bounds of the sequence. We intend to prove that $M_1 = M_2$.

Since M_1 is a least upper bound and M_2 is an upper bound of the sequence, $M_1 \leq M_2$. Interchanging the roles of $M_1$and M_2, $M_2 \leq M_1$. Since both the inequalities hold, $M_1 = M_2$. Thus, a sequence of real numbers can not have two different least upper bounds.

$(\boldsymbol{b})$ Let L be the limit of the convergent sequence $\{a_n\}$. Then for each $\epsilon > 0$ there is a natural number N such that for all n and m, we have

$$n \geq N \Longrightarrow \quad |a_n - L| < \frac{\epsilon}{2} \text{ and } m \geq N \Longrightarrow \quad |a_m - L| < \frac{\epsilon}{2}$$

Now, $n \geq N$ and $m \geq N \Longrightarrow$

$$\begin{aligned} |a_n - a_m| &= |(a_n - L) + (L - a_m)| \\ &\leq |a_n - L| + |a_m - L| < \frac{\epsilon}{2} + \frac{\epsilon}{2} < \epsilon \end{aligned}$$

Hence the result.

1.1 Sequences

Exercises:

1. Find the first five terms of the sequence $\{a_n\}$

$(a)\ a_n = \frac{1-n}{n^2}$ $(b)\ a_n = \frac{1}{n!}$ $(c)\ a_n = \frac{(-1)^n}{2n-1}$

$(d)\ a_n = \frac{2^n}{2^{n+1}}$ $(e)\ a_n = \frac{2^n-1}{2^n}$ $(f)\ a_n = \frac{n+1}{3n-1}$

2. Write the first ten terms of the sequence $\{a_n\}$

$(a)\ a_1 = 1\ ,\ a_{n+1} = a_n + \frac{1}{2^n}$

$(b)\ a_1 = 1\ ,\ a_{n+1} = \frac{a_n}{n+1}$

$(c)\ a_1 = 2\ ,\ a_{n+1} = \frac{(-1)^n a_n}{2}$

$(d)\ a_1 = 3\ ,\ a_{n+1} = 2a_n - 1$

$(e)\ a_1 = 4\ ,\ a_{n+1} = \frac{a_n}{a_n - 1}$

$(f)\ a_1 = a_2 = 1\ ,\ a_{n+2} = a_{n+1} + a_n$

$(g)\ a_1 = 2, a_2 = -1\ ,\ a_{n+2} = \frac{a_{n+1}}{a_n}$

3. Find the formula for n^{th} term of the sequence

(a) The sequence $1, -1, 1, -1, \dots \dots ..$

(b) The sequence $-1, 1, -1, 1 \dots \dots ..$

(c) The sequence $1, -4, 9, -16, 25, \dots \dots ..$

(d) The sequence $-3, -2, -1, 0, 1 \dots \dots ..$

(e) The sequence $1, 5, 9, 13, 17, \dots \dots \dots$

(f) The sequence $2, 6, 10, 14, 18, \dots \dots.$

(g) The sequence $1, 0, 1, 0, 1 \dots \dots ..$

4. Show that the sequence $\{a_n\}$, where $a_n = \frac{n^2+1}{2n^2+5}$, for all $n \in \boldsymbol{N}$ converges to $\frac{1}{2}$.

5. Show that the sequence $\{a_n\}$, where $a_n = \frac{3n-1}{4n+5}$, for all $n \in \boldsymbol{N}$ converges to $\frac{3}{4}$.

6. Show that the sequence $\{a_n\}$, where $a_n = \frac{2n-7}{3n+2}$, for all $n \in \boldsymbol{N}$ converges to $\frac{2}{3}$.

7. Show that the following sequences diverge to ∞.

$(a)\ a_n = n^3 \qquad (b)\ a_n = \sqrt{n+1} \qquad (c)\ a_n = \frac{n}{\sqrt{n+1}}$

8. Show that the following are non-decreasing sequences and discuss their convergence.

$(a)\ a_n = \frac{3n+1}{n+1} \quad (b)\ a_n = \frac{(2n+3)!}{(n+1)!} \quad (c)\ a_n = 2 - \frac{2}{n} - \frac{1}{2^n}$

9. Show that the following are non-increasing sequences and discuss their convergence.

$(a)\ a_n = \frac{1+\sqrt{2n}}{\sqrt{n}} \quad (b)\ a_n = \frac{1-4^n}{2^n} \quad (c)\ a_n = \frac{4^{n+1}+3^n}{4^n}$

1.2

Limits of Sequences

Learning objectives:

- To state the rules for the algebra of convergent sequences.
- To study the sequence versions of
  - Sandwich Theorem
  - Continuous Function Theorem
  - L'Hopital's Rule
- To prove some commonly occurring limits.
  And
- To practice the related problems.

Calculating Limits of Sequences

Computing limits of sequences through the formal definition by calculating $\epsilon's$ and $N's$ would be a formidable task. Instead we define a few basic results on limits and then use them to compute the limits of many sequences. The following theorem deals with the algebra of convergent sequences.

Theorem 1:

If $\{a_n\}$ and $\{b_n\}$ are convergent sequences of real numbers converging to A and B respectively, then

1. ***Sum Rule:*** $\lim\limits_{n\to\infty}(a_n + b_n) = A + B$
2. ***Difference Rule:*** $\lim\limits_{n\to\infty}(a_n - b_n) = A - B$
3. ***Product Rule:*** $\lim\limits_{n\to\infty}(a_n \cdot b_n) = A \cdot B$
4. ***Constant Multiple Rule:*** $\lim\limits_{n\to\infty}(k \cdot a_n) = k \cdot A$ ***, where k is any constant***
5. ***Quotient Rule:*** $\lim\limits_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \frac{A}{B}$***, if $B \neq 0$.***

Example 1:

$$(i)\ \lim_{n\to\infty}\left(\frac{n-1}{n}\right) = \lim_{n\to\infty}\left(1 - \frac{1}{n}\right) = \lim_{n\to\infty} 1 - \lim_{n\to\infty}\frac{1}{n}$$

(By Difference Rule)

$$= 1 - 0 = 1$$

(ii) $\lim\limits_{n\to\infty} \frac{2}{3n^2} = \frac{2}{3} \lim\limits_{n\to\infty} \frac{1}{n} \cdot \lim\limits_{n\to\infty} \frac{1}{n}$

(By Constant Multiple and Product Rules)

$= 0$

(iii) $\lim\limits_{n\to\infty} \frac{4+2n^6}{n^6-7} = \lim\limits_{n\to\infty} \frac{\frac{4}{n^6}+2}{1-\frac{7}{n^6}} = \frac{\lim\limits_{n\to\infty}\left(\frac{4}{n^6}+2\right)}{\lim\limits_{n\to\infty}\left(1-\frac{7}{n^6}\right)}$ (By Quotient Rule)

$= \frac{4\lim\limits_{n\to\infty} \frac{1}{n^6} + \lim\limits_{n\to\infty} 2}{\lim\limits_{n\to\infty} 1 - 7\lim\limits_{n\to\infty} \frac{1}{n^6}}$ (By Sum, Difference and Constant Multiple Rules)

$= \frac{0+2}{1-0} = 2$

Remark:

Part 1 of Theorem 1 does not say that each of the sequences $\{a_n\}$ and $\{b_n\}$ are convergent if their sum $\{a_n + b_n\}$ is convergent. For example $\{a_n\} = \{(-1)^n\}$ and $\{b_n\} = \{(-1)^{n+1}\}$ both diverge, but their sum $\{a_n + b_n\}$: $0,0,0, \dots, \dots$ converges to 0.

The following is a consequence of Theorem 1:

Corollary:

Every nonzero multiple of a divergent sequence diverges.

$i.e.$, if $\{a_n\}$ is a divergent sequence then for every constant $k \neq 0$ the sequence $\{ka_n\}$ is also divergent.

Proof: Assume the contrary. That is, $\{ca_n\}$ converges for some number $c \neq 0$. Now by Constant Multiple Rule the sequence $\left\{\frac{1}{c}.ca_n\right\} = \{a_n\}$ converges – a contradiction. Hence the result.

Sequence Versions of Limits of Functions

Since sequences are functions with domain $\boldsymbol{N}$, the set of natural numbers, the theorems on limits of functions (studied in the earlier classes) have versions for sequences.

The following theorem is the sequence version of Sandwich Theorem.

Theorem 2: The Sandwich Theorem for Sequences

Let $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ be sequences of real numbers and $a_n \le b_n \le c_n$ for all n beyond some index N. If $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} c_n = L$, then $\lim\limits_{n\to\infty} b_n = L$.

The following is a consequence of the Sandwich Theorem for Sequences

Corollary:

Let $\{b_n\}$ and $\{c_n\}$ be sequences of real numbers. If $|b_n| \le c_n$ and $\lim\limits_{n\to\infty} c_n = 0$, then $\lim\limits_{n\to\infty} b_n = 0$.

Proof follows from the fact $-c_n \le b_n \le c_n$ and Sandwich Theorem.

Example 2: Applying the Sandwich Theorem

Prove that the sequences $\left\{\frac{1}{2^n}\right\}$, $\left\{(-1)^n \frac{1}{n}\right\}$ and $\left\{\frac{\cos n}{n^2}\right\}$ all converge to 0.

(i) Notice that $0 \leq \frac{1}{2^n} \leq \frac{1}{n}$, for all $n \in \boldsymbol{N}$ and $\lim_{n\to\infty} 0 = \lim_{n\to\infty} \frac{1}{n} = 0$.

Therefore, $\lim_{n\to\infty} \frac{1}{2^n} = 0$ by the Sandwich Theorem for Sequence.

Thus, $\left\{\frac{1}{2^n}\right\}$ converges to 0.

(ii) Notice that $\left|(-1)^n \frac{1}{n}\right| \leq \frac{1}{n}$, for all $n \in \boldsymbol{N}$.

Since $\lim_{n\to\infty} \frac{1}{n} = 0$, the result follows by the Corollary to Sandwich Theorem for Sequences.

(iii) Notice that $\left|\frac{\cos n}{n^2}\right| \leq \frac{1}{n^2}$ for all $n \in \boldsymbol{N}$. The result follows as in (ii).

Theorem 3: The Continuous Function Theorem for Sequences

Let $\{a_n\}$ be a sequence of real numbers and let f be a real valued function defined at all a_n. If $\{a_n\}$ converges to L and f is continuous at L, then the sequence $\{f(a_n)\}$ converges to $f(L)$.

Example 3:

Show that $\left\{\sqrt{\frac{n+1}{n}}\right\}$ converges to 1.

Solution: First note that $L = \lim_{n\to\infty} \frac{n+1}{n} = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right) = 1$.

Let $f(x) = \sqrt{x}$. It is defined at all the terms of the sequence $\left\{\frac{n+1}{n}\right\}$ and is continuous at L. By the above theorem

$$\lim_{n\to\infty} \sqrt{\frac{n+1}{n}} = \lim_{n\to\infty} f\left(\frac{n+1}{n}\right) = f(L) = f(1) = \sqrt{1} = 1$$

Example 4:

Show that $\left\{2^{\frac{1}{n}}\right\}$ converges to 1.

Solution: The sequence $\left\{\frac{1}{n}\right\}$ converges to $L = 0$. Let $f(x) = 2^x$.

It is defined at all the terms of the sequence $\left\{\frac{1}{n}\right\}$ and it is continuous at L. By the above theorem

$$\lim_{n\to\infty} 2^{\frac{1}{n}} = \lim_{n\to\infty} f\left(\frac{1}{n}\right) = f(L) = f(0) = 2^0 = 1$$

Using L'Hopital's Rule

The following theorem enables us to use L'Hopital's Rule to find the limits of some sequences. It formalizes the connection between $\lim_{n\to\infty} a_n$ and $\lim_{x\to\infty} f(x)$.

Theorem 4:

If $\boldsymbol{f(x)}$ is a differentiable function defined for all $\boldsymbol{x \geq n_0}$ $(\in N)$ and $\boldsymbol{\{a_n\}}$ is a sequence of real numbers such that $\boldsymbol{a_n = f(n)}$ for $\boldsymbol{n \geq n_0}$, then

$$\lim_{x\to\infty} f(x) = L \quad \Longrightarrow \quad \lim_{n\to\infty} a_n = L$$

Example 5: Applying L'Hopital's Rule

Show that the sequence $\left\{\frac{\ln n}{n}\right\}$ converges to 0.

Solution: Let $f(x) = \frac{\ln x}{x}$, $a_n = \frac{\ln n}{n}$. Clearly $f(x)$ is defined and is differentiable for all $x \geq 1$, and $a_n = f(n)$ for all $n \geq 1$. By the above theorem $\lim_{n\to\infty} \frac{ln\, n}{n}$ is equal to $\lim_{x\to\infty} f(x)$, if the latter exists. Now,

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{ln\, x}{x} \qquad \left(\frac{\infty}{\infty} \text{ form}\right)$$

$$= \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = 0 \qquad \text{(By L'Hopital's Rule)}$$

Therefore, $\lim_{n\to\infty} \frac{ln\, n}{n} = 0$ and the sequence $\left\{\frac{ln\, n}{n}\right\}$ is convergent and converges to 0.

Example 6: Applying L'Hopital's Rule to determine convergence.

Does the sequence whose n^{th} term is $a_n = \left(\frac{n+1}{n-1}\right)^n$ converge? If so, find $\lim_{n\to\infty} a_n$.

Solution: We notice that the $\lim_{n\to\infty} a_n$ leads to the indeterminate form 1^∞. Now, $ln\, a_n = n\, ln\left(\frac{n+1}{n-1}\right)$ and

$$\lim_{n\to\infty} ln\, a_n = \lim_{n\to\infty} n\, ln\left(\frac{n+1}{n-1}\right) \qquad (\infty.0 \text{ form})$$

$$= \lim_{n\to\infty} \frac{ln\left(\frac{n+1}{n-1}\right)}{\frac{1}{n}} \qquad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{n\to\infty} \frac{\frac{n-1}{n+1} . \frac{d}{dn}\left(\frac{n+1}{n-1}\right)}{-\frac{1}{n^2}} \qquad \text{(By L'Hopital's Rule)}$$

$$= \lim_{n\to\infty} \frac{\frac{2}{n^2-1}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{2n^2}{n^2-1} = 2$$

Thus, $a_n = e^{ln\, a_n} \to e^2$ as $n \to \infty$ and so the sequence $\{a_n\}$ converge to e^2.

Some commonly occurring limits

Theorem 5:

The following six sequences converge to the limits listed below:

1. $\lim\limits_{n\to\infty} \frac{ln\, n}{n} = 0$
2. $\lim\limits_{n\to\infty} \sqrt[n]{n} = 1$
3. $\lim\limits_{n\to\infty} x^{\frac{1}{n}} = 1,\ x > 0$
4. $\lim\limits_{n\to\infty} x^n = 0,\ |x| < 1$
5. $\lim\limits_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x$, ***for any x***
6. $\lim\limits_{n\to\infty} \frac{x^n}{n!} = 0$, ***for any x***

Proof:

1. See Example 5
2. $\lim\limits_{n\to\infty} n^{\frac{1}{n}} = \lim\limits_{n\to\infty} e^{\frac{1}{n} ln\, n} = e^0 = 1$

(Using Formula 1 and Theorem 3)

3. $\lim\limits_{n\to\infty} x^{\frac{1}{n}} = \lim\limits_{n\to\infty} e^{\frac{1}{n} ln\, x} = e^0 = 1$

(Using $\lim\limits_{n\to\infty} \frac{1}{n} = 0$ and Theorem 3)

4. Let $\epsilon > 0$ be given. By Formula 3 $\lim_{n\to\infty} \epsilon^{\frac{1}{n}} = 1$. Since $|x| < 1$, there exists a natural number N such that $\epsilon^{\frac{1}{N}} > |x|$. Now, $|x^N| = |x|^N < \epsilon$. Therefore, $|x^n - 0| = |x|^n < \epsilon$ for all $n \geq N$. Thus $\lim_{n\to\infty} x^n = 0$, where $|x| < 1$.

5. Let $a_n = \left(1 + \frac{x}{n}\right)^n$. Then $ln\, a_n = n\, ln\left(1 + \frac{x}{n}\right)$

Now, $\lim_{n\to\infty} n\, ln\left(1 + \frac{x}{n}\right)$ $\quad(\infty . 0 \text{ form})$

$$= \lim_{n\to\infty} \frac{ln\left(1+\frac{x}{n}\right)}{\frac{1}{n}} \qquad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{n\to\infty} \frac{\frac{1}{1+\frac{x}{n}}\left(\frac{-x}{n^2}\right)}{\frac{-1}{n^2}} \qquad \text{(By L'Hopital's Rule)}$$

$$= \lim_{n\to\infty} \frac{x}{1+\frac{x}{n}} = x$$

By Theorem 4 with $f(x) = e^x$ we conclude that

$$a_n = e^{ln\, a_n} \to e^x \text{ as } n \to \infty.$$

6. For any number x, we have

$$-\frac{|x|^n}{n!} \leq \frac{x^n}{n!} \leq \frac{|x|^n}{n!} \qquad \text{--------}(i)$$

We first prove $\lim_{n\to\infty} \frac{|x|^n}{n!} = 0$. Choose a natural number $M > |x|$.

Then $\frac{|x|}{M} < 1$ and $\lim_{n\to\infty} \left(\frac{|x|}{M}\right)^n = 0$ (By Formula 4). Now, for $n > M$.

$$\frac{|x|^n}{n!} = \frac{|x|^n}{1\cdot 2\cdot 3 \cdots M(M+1)(M+2)\cdots n} \leq \frac{|x|^n}{M!\, M^{n-M}} = \frac{M^M}{M!}\left(\frac{|x|}{M}\right)^n$$

Thus $0 \le \frac{|x|^n}{n!} \le \frac{M^M}{M!}\left(\frac{|x|}{M}\right)^n$, for $n > M$. As $n \to \infty$, by Sandwich Theorem $\lim_{n\to\infty} \frac{|x|^n}{n!} = 0$. The result now follows from (i) by Sandwich Theorem for Sequences. Hence the theorem.

Example 7:

Show that the sequences

$(a)\ \left\{\frac{(\ln n)^2}{n}\right\}$ $\qquad (b)\ \left\{\sqrt[n]{4^n \cdot n}\right\}$ $\qquad (c)\ \left\{\sqrt[n]{2^{1+3n}}\right\}$

$(d)\ \left\{\left(\frac{1}{3}\right)^n + \frac{1}{\sqrt{2}^n}\right\}$ $\qquad (e)\ \left\{\left(\frac{n+1}{n+2}\right)^n\right\}$ $\qquad (f)\ \left\{\frac{3^n \cdot 6^n}{2^{-n}.n!}\right\}$

converge and find their limits.

Solution:

$(a)\ \lim_{n\to\infty} \frac{(ln\, n)^2}{n} = \lim_{n\to\infty} \frac{2\, ln\, n \left(\frac{1}{n}\right)}{1}$ ($\frac{\infty}{\infty}$ form, by L'Hopital's Rule)

$= 2\lim_{n\to\infty} \frac{ln\, n}{n} = 2(0) = 0$ (By Formula 1)

$(b)\ \lim_{n\to\infty} \sqrt[n]{4^n \cdot n} = \lim_{n\to\infty} \left(4 \cdot n^{\frac{1}{n}}\right) = 4(1) = 4$ (By Formula 2)

$(c)\ \lim_{n\to\infty} \sqrt[n]{2^{1+3n}} = \lim_{n\to\infty} 2^{\frac{1}{n}+3} = \lim_{n\to\infty} \left(8 \cdot 2^{\frac{1}{n}}\right) = 8(1) = 8$

(By Formula 3 with $= 2$)

(d) We first note that $\lim_{n\to\infty} \left(\frac{1}{3}\right)^n = 0$ (By Formula 4 with $x = \frac{1}{3}$)

and $\lim_{n\to\infty} \frac{1}{\sqrt{2}^n} = \lim_{n\to\infty} \left(\frac{1}{\sqrt{2}}\right)^n = 0$ by (By Formula 4 with $x = \frac{1}{\sqrt{2}}$).

Now,

$$\lim_{n\to\infty} \left[\left(\frac{1}{3}\right)^n + \frac{1}{\sqrt{2}^n}\right] = \lim_{n\to\infty} \left(\frac{1}{3}\right)^n + \lim_{n\to\infty} \frac{1}{\sqrt{2}^n} = 0 \text{ (By Theorem 1)}$$

$(e)\ \lim_{n\to\infty}\left(\frac{n+1}{n+2}\right)^n = \lim_{n\to\infty}\left(\frac{1+\frac{1}{n}}{1+\frac{2}{n}}\right)^n = \frac{\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n}{\lim_{n\to\infty}\left(1+\frac{2}{n}\right)^n}$ (By Theorem 1)

$= \frac{e}{e^2} = \frac{1}{e}$ (By Formula 5)

$(f)\ \lim_{n\to\infty}\frac{3^n \cdot 6^n}{2^{-n}.n!} = \lim_{n\to\infty}\frac{36^n}{n!} = 0$ (By Formula 6 with $x = 36$)

P1:

Determine whether the sequences $\{a_n\}$ converge or diverge. If converges, then find the limit.

$(\boldsymbol{a})\ \boldsymbol{a_n = \frac{1-5n^4}{n^4+8n^3}}$ $\quad\quad$ $(\boldsymbol{b})\ \boldsymbol{a_n = \frac{1-n^3}{70-4n^2}}$

Solution:

$(\boldsymbol{a})$ The given sequence is $\left\{\frac{1-5n^4}{n^4+8n^3}\right\}$.

Now, $\lim_{n\to\infty} \frac{1-5n^4}{n^4+8n^3} = \lim_{n\to\infty} \frac{\left(\frac{1}{n^4}-5\right)}{\left(1+\frac{8}{n}\right)} = \frac{\lim_{n\to\infty}\frac{1}{n^4} - \lim_{n\to\infty} 5}{\lim_{n\to\infty} 1 + 8\cdot \lim_{n\to\infty}\frac{1}{n}}$

(By Quotient, Sum, Difference and Constant Multiple Rules)

$$= \frac{0-5}{1+0} = -5$$

Therefore, the sequence $\left\{\frac{1-5n^4}{n^4+8n^3}\right\}$ converges to -5.

$(\boldsymbol{b})$ The given sequence is $\left\{\frac{1-n^3}{70-4n^2}\right\}$.

Now, $\lim_{n\to\infty} \frac{1-n^3}{70-4n^2} = \lim_{n\to\infty} \frac{\left(\frac{1}{n^2}-n\right)}{\left(\frac{70}{n^2}-4\right)} = \frac{\lim_{n\to\infty}\frac{1}{n^2} - \lim_{n\to\infty} n}{70\cdot \lim_{n\to\infty}\frac{1}{n^2} - \lim_{n\to\infty} 4}$

(By quotient, Difference and Constant Multiple Rules)

$$= \frac{0-\infty}{0-4} = \infty$$

Therefore, the given sequence $\left\{\frac{1-n^3}{70-4n^2}\right\}$ diverges.

P2:

Prove that the sequence $\left\{\frac{n!}{n^n}\right\}$ converges to 0.

Solution:

The given sequence is $\left\{\frac{n!}{n^n}\right\}$. Notice that $\frac{n!}{n^n} \geq 0$ for all $n \in \boldsymbol{N}$ and

$$\frac{n!}{n^n} = \frac{1\cdot 2\cdot 3\cdots\cdots\cdots(n-2)(n-1)n}{n\cdot n\cdot n\cdots\cdots n} \leq \frac{1}{n}$$

Thus, $\qquad 0 \leq \frac{n!}{n^n} \leq \frac{1}{n}$

We have, $\lim_{n\to\infty} 0 = \lim_{n\to\infty} \frac{1}{n} = 0$. By Sandwich Theorem for Sequences, $\lim_{n\to\infty} \frac{n!}{n^n} = 0$. Thus, the sequence $\left\{\frac{n!}{n^n}\right\}$ converges to 0.

P3:

Does the sequence whose $\boldsymbol{n^{th}}$ term is $\boldsymbol{a_n = \left(\frac{1}{n}\right)^{\frac{1}{\ln n}}}$ converge? If so, find $\boldsymbol{\lim_{n\to\infty} a_n}$.

Solution:

We notice that the $\lim_{n\to\infty} a_n$ leads to the indeterminate form 0^0.

Now, $\ln a_n = \ln\left(\frac{1}{n}\right)^{\frac{1}{\ln n}} = \frac{1}{\ln n}\ln\left(\frac{1}{n}\right) = \frac{1}{\ln n}(-\ln n) = -1$ and $\lim_{n\to\infty} \ln a_n = -1$.

Thus, $a_n = e^{\ln a_n} \to e^{-1}$ as $n \to \infty$. Therefore, the sequence $\{a_n\}$ converges to e^{-1}.

P4:

Show that the sequences

$$(\boldsymbol{a})\ \left\{\left(\frac{\boldsymbol{3n+1}}{\boldsymbol{3n-1}}\right)^{\boldsymbol{n}}\right\}\quad (\boldsymbol{b})\ \left\{\frac{\boldsymbol{n+ln\,n}}{\boldsymbol{n}}\right\}\quad (\boldsymbol{c})\ \left\{\left(\frac{\boldsymbol{3}}{\boldsymbol{n}}\right)^{\frac{\boldsymbol{1}}{\boldsymbol{n}}}\right\}\quad (\boldsymbol{d})\ \left\{\frac{(\boldsymbol{-3})^{\boldsymbol{n-1}}}{\boldsymbol{2^{3n}}}\right\}$$

converge and find their limits.

Solution:

$(\boldsymbol{a})$ The given sequence is $\left\{\left(\frac{3n+1}{3n-1}\right)^{n}\right\}$.

Now, $$\lim_{n\to\infty}\left(\frac{3n+1}{3n-1}\right)^{n} = \lim_{n\to\infty}\left(\frac{1+\frac{1}{3n}}{1-\frac{1}{3n}}\right)^{n}$$

$$= \frac{\lim_{n\to\infty}\left(1+\frac{\frac{1}{3}}{n}\right)^{n}}{\lim_{n\to\infty}\left(1+\frac{-\frac{1}{3}}{n}\right)^{n}} \quad \text{(By Quotient Rule)}$$

$$= \frac{e^{\frac{1}{3}}}{e^{-\frac{1}{3}}} = e^{\frac{2}{3}} \qquad \left(\because \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^{n} = e^{x}\right)$$

Therefore, the sequence $\left\{\left(\frac{3n+1}{3n-1}\right)^{n}\right\}$ converges to $e^{\frac{2}{3}}$.

$(\boldsymbol{b})$ The given sequence is $\left\{\frac{n+ln\,n}{n}\right\}$.

Now, $$\lim_{n\to\infty}\frac{n+ln\,n}{n} = \lim_{n\to\infty}\left(1+\frac{ln\,n}{n}\right)$$

$$= \lim_{n\to\infty}1 + \lim_{n\to\infty}\frac{ln\,n}{n} \quad \text{(By Sum Rule)}$$

$$= 1 + 0 = 1 \qquad \left(\because \lim_{n\to\infty} \frac{ln\, n}{n} = 0\right)$$

Therefore, the sequence $\left\{\frac{n+ln\, n}{n}\right\}$ converges to 1.

$(\boldsymbol{c})$ The given sequence is $\left\{\left(\frac{3}{n}\right)^{\frac{1}{n}}\right\}$.

Now, $\lim_{n\to\infty} \left(\frac{3}{n}\right)^{\frac{1}{n}} = \lim_{n\to\infty} \frac{3^{\frac{1}{n}}}{n^{\frac{1}{n}}} = \frac{\lim_{n\to\infty} 3^{\frac{1}{n}}}{\lim_{n\to\infty} n^{\frac{1}{n}}}$ (By Quotient Rule)

$$= \frac{1}{1} = 1 \left(\because \lim_{n\to\infty} \sqrt[n]{n} = 1, \lim_{n\to\infty} x^{\frac{1}{n}} = 1 \text{ if } x > 0\right)$$

Therefore, the sequence $\left\{\left(\frac{3}{n}\right)^{\frac{1}{n}}\right\}$ converges to 1.

$(\boldsymbol{d})$ The given sequence is $\left\{\frac{(-3)^{n-1}}{2^{3n}}\right\}$.

Now, $\lim_{n\to\infty} \frac{(-3)^{n-1}}{2^{3n}} = \lim_{n\to\infty} \frac{(-3)^{n}(-3)^{-1}}{8^{n}}$

$$= \frac{-1}{3} \lim_{n\to\infty} \left(\frac{-3}{8}\right)^{n} = 0 \left(\because \lim_{n\to\infty} x^{n} = 0 \text{ if } |x| < 1\right)$$

IP1:

Determine whether the sequences $\{a_n\}$ converge or diverge. If converges, then find the limit.

$(\boldsymbol{a})\ \boldsymbol{a_n = \sqrt{n^2+n+1} - \sqrt{n^2+1}}$ $(\boldsymbol{b})\ \boldsymbol{a_n = \sqrt{n} - \sqrt{n^2-1}}$

Solution:

$(\boldsymbol{a})$ The given sequence is $\left\{\sqrt{n^2+n+1} - \sqrt{n^2+1}\right\}$.

Now, $\lim\limits_{n\to\infty}\left(\sqrt{n^2+n+1} - \sqrt{n^2+1}\right)$

$$= \lim_{n\to\infty}\left[\left(\sqrt{n^2+n+1} - \sqrt{n^2+1}\right) \times \frac{\sqrt{n^2+n+1}+\sqrt{n^2+1}}{\sqrt{n^2+n+1}+\sqrt{n^2+1}}\right]$$

$$= \lim_{n\to\infty}\frac{n}{n\left(\sqrt{1+\frac{1}{n}+\frac{1}{n^2}}+\sqrt{1+\frac{1}{n^2}}\right)} = \lim_{n\to\infty}\frac{1}{\sqrt{1+\frac{1}{n}+\frac{1}{n^2}}+\sqrt{1+\frac{1}{n^2}}}$$

$$= \frac{1}{\lim\limits_{n\to\infty}\sqrt{1+\frac{1}{n}+\frac{1}{n^2}}+\lim\limits_{n\to\infty}\sqrt{1+\frac{1}{n^2}}}$$ (By Sum and Quotient Rules)

$$= \frac{1}{2}$$ (By Continuous Function Theorem for Sequences)

Therefore, the given sequence converges to $\frac{1}{2}$

$(\boldsymbol{b})$ The given sequence is $\left\{\sqrt{n} - \sqrt{n^2-1}\right\}$.

Now, $\lim\limits_{n\to\infty}\left(\sqrt{n} - \sqrt{n^2-1}\right)$

$$= \lim_{n\to\infty}\left[\left(\sqrt{n} - \sqrt{n^2-1}\right) \times \frac{\sqrt{n}+\sqrt{n^2-1}}{\sqrt{n}+\sqrt{n^2-1}}\right]$$

$$= \lim_{n\to\infty} \frac{n-n^2+1}{\sqrt{n}+\sqrt{n^2-1}}$$

$$= \lim_{n\to\infty} \frac{1-n+\frac{1}{n}}{\sqrt{\frac{1}{n}}+\sqrt{1-\frac{1}{n^2}}} = \frac{\lim_{n\to\infty} 1-\lim_{n\to\infty} n+\lim_{n\to\infty}\frac{1}{n}}{\lim_{n\to\infty}\sqrt{\frac{1}{n}}+\lim_{n\to\infty}\sqrt{1-\frac{1}{n^2}}}$$

(By Sum, Difference, Quotient Rules and Continuous Function Theorem for Sequences)

$$= -\infty$$

Therefore, the given sequence diverges.

IP2:

Prove that the sequence $\left\{\frac{\sin^2 n}{2^n}\right\}$ converges to 0.

Solution:

The given sequence is $\left\{\frac{\sin^2 n}{2^n}\right\}$. Notice that $0 \leq \frac{\sin^2 n}{2^n} \leq \frac{1}{2^n}$.

We have, $\lim_{n\to\infty} 0 = \lim_{n\to\infty} \frac{1}{2^n} = 0$. By Sandwich Theorem for Sequences, $\lim_{n\to\infty} \frac{\sin^2 n}{2^n} = 0$. Thus, the sequence $\left\{\frac{\sin^2 n}{2^n}\right\}$ converges to 0.

IP3:

Does the sequence whose $\boldsymbol{n^{th}}$ term is $\boldsymbol{a_n = (1+n^2)^{e^{-n}}}$ converge? If so, find $\boldsymbol{\lim_{n\to\infty} a_n}$.

Solution:

We notice that the $\lim_{n\to\infty} a_n$ leads to the indeterminate form ∞^0.

Now, $ln\, a_n = ln(1+n^2)^{e^{-n}} = e^{-n}\, ln(1+n^2) = \frac{ln(1+n^2)}{e^n}$

and $\lim_{n\to\infty} ln\, a_n = \lim_{n\to\infty} \frac{ln(1+n^2)}{e^n} \qquad \left(\frac{\infty}{\infty} \text{ form}\right)$

$$= \lim_{n\to\infty} \frac{\frac{1}{1+n^2}(2n)}{e^n} \qquad \text{(By L'Hopital's Rule)}$$

$$= \lim_{n\to\infty} \frac{2n}{(1+n^2)e^n} \qquad \left(\frac{\infty}{\infty} \text{ form}\right)$$

$$= \lim_{n\to\infty} \frac{2}{2ne^n+(1+n^2)e^n} \qquad \text{(By L'Hopital's Rule)}$$

$$= 0$$

Thus, $a_n = e^{ln\, a_n} \to e^0 = 1$ as $n \to \infty$. Therefore, the sequence $\{a_n\}$ converges to 1.

IP4:

Show that the sequences

$(\boldsymbol{a})\ \left\{\left(\frac{\boldsymbol{n}}{\boldsymbol{n}+\mathbf{1}}\right)^{\boldsymbol{n}}\right\}\quad (\boldsymbol{b})\ \left\{\frac{\mathbf{5}^{\mathbf{2}\boldsymbol{n}}}{\boldsymbol{n}!\ \mathbf{9}^{\boldsymbol{n}}}\right\}\quad (\boldsymbol{c})\ \left\{\frac{\left(\frac{\mathbf{10}}{\mathbf{11}}\right)^{\boldsymbol{n}}}{\left(\frac{\mathbf{9}}{\mathbf{10}}\right)^{\boldsymbol{n}}+\left(\frac{\mathbf{11}}{\mathbf{12}}\right)^{\boldsymbol{n}}}\right\}$ **converge and find their limits.**

Solution:

$(\boldsymbol{a})$ The given sequence is $\left\{\left(\frac{n}{n+1}\right)^n\right\}$.

$$\text{Now, } \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n = \lim_{n\to\infty}\left(\frac{1}{1+\frac{1}{n}}\right)^n$$

$$= \lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{n}\right)^n} = \frac{1}{e}\quad \left(\because\ \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x\right)$$

Therefore, the sequence $\left\{\left(\frac{n}{n+1}\right)^n\right\}$ converges to $\frac{1}{e}$.

$(\boldsymbol{b})$ The given sequence is $\left\{\frac{5^{2n}}{n!\ 9^n}\right\}$.

$$\text{Now, } \lim_{n\to\infty}\frac{5^{2n}}{n!\ 9^n} = \lim_{n\to\infty}\frac{25^n}{n!\ 9^n} = \lim_{n\to\infty}\frac{\left(\frac{25}{9}\right)^n}{n!} = 0\quad \left(\because\ \lim_{n\to\infty}\frac{x^n}{n!} = 0\right)$$

Therefore, the sequence $\left\{\frac{5^{2n}}{n!\ 9^n}\right\}$ converges to 0.

$(\boldsymbol{c})$ The given sequence is $\left\{\frac{\left(\frac{10}{11}\right)^n}{\left(\frac{9}{10}\right)^n+\left(\frac{11}{12}\right)^n}\right\}$.

Now, $\lim\limits_{n\to\infty} \frac{\left(\frac{10}{11}\right)^n}{\left(\frac{9}{10}\right)^n+\left(\frac{11}{12}\right)^n}$ $\left(\frac{0}{0}\text{ form}\right)$ $\left(\because \lim\limits_{n\to\infty} x^n = 0 \text{ if } |x| < 1\right)$

$$= \lim_{n\to\infty} \frac{\left(\frac{12}{11}\right)^n\left(\frac{10}{11}\right)^n}{\left(\frac{12}{11}\right)^n\left(\frac{9}{10}\right)^n+1} = \lim_{n\to\infty} \frac{\left(\frac{120}{121}\right)^n}{\left(\frac{108}{110}\right)^n+1} = \frac{\lim\limits_{n\to\infty}\left(\frac{120}{121}\right)^n}{\lim\limits_{n\to\infty}\left(\frac{108}{110}\right)^n+1}$$

(By Sum and Quotient Rules)

$$= \frac{0}{0+1} \qquad \left(\because \lim_{n\to\infty} x^n = 0 \text{ if } |x| < 1\right)$$

$$= 0$$

Therefore, the sequence $\left\{\frac{\left(\frac{10}{11}\right)^n}{\left(\frac{9}{10}\right)^n+\left(\frac{11}{12}\right)^n}\right\}$ converges to 0.

1.2 Limits of Sequences

Exercises:

1. Which of the following sequences converge, and which diverge? Find the limit of each convergent sequence.

$(a)\ \left\{\frac{n+(-1)^n}{n}\right\}$ $\quad (b)\ \left\{\frac{1-2n}{1+2n}\right\}$ $\quad (c)\ \left\{\frac{2n+1}{1-3\sqrt{n}}\right\}$

$(d)\ \left\{\frac{n+3}{n^2+5n+6}\right\}$ $\quad (e)\ \{1+(-1)^n\}$ $\quad (f)\ \left\{(-1)^n\left(1-\frac{1}{n}\right)\right\}$

$(g)\ \left\{\left(\frac{n-1}{2n}\right)\left(1-\frac{1}{n}\right)\right\}$ $\quad (h)\ \left\{\left(2-\frac{1}{2^n}\right)\left(3+\frac{1}{2^n}\right)\right\}$

$(i)\ \left\{\frac{(-1)^{n+1}}{2n-1}\right\}$ $\quad (j)\ \left\{\sqrt{\frac{2n}{n+1}}\right\}$ $\quad (k)\ \left\{\sin\left(\frac{\pi}{2}+\frac{1}{n}\right)\right\}$

$(l)\ \{n\pi\cos n\pi\}$ $\quad (m)\ \left\{n-\sqrt{n^2-n}\right\}$ $\quad (n)\ \left\{\frac{1}{\sqrt{n^2-1}-\sqrt{n^2+n}}\right\}$

$(n)\ \{\tanh n\}$ $\quad (o)\ \{\sinh(ln\, n)\}$

2. Which of the following sequences converge, and which diverge? Find the limit of each convergent sequence.

$(a)\ \left\{\frac{\sin n}{n}\right\}$ $\quad (b)\ \left\{\frac{\cos^2 n}{2^n}\right\}$ $\quad (c)\ \left\{\frac{\sin 2n}{1+\sqrt{n}}\right\}$

3. Which of the following sequences converge, and which diverge? Find the limit of each convergent sequence.

$(a)\ \left\{\frac{n}{2^n}\right\}$ $\quad (b)\ \left\{\frac{3^n}{n^3}\right\}$ $\quad (c)\ \left\{\frac{\ln(n+1)}{\sqrt{n}}\right\}$ $\quad (d)\ \left\{\frac{\ln n}{\ln 2n}\right\}$

$(e)\ \left\{\left(\frac{x^n}{2n+1}\right)^{\frac{1}{n}}\right\}, x > 0$ $(f)\ \left(1-\frac{1}{n^2}\right)^n$ $(g)\ \left\{\frac{n^2}{2n-1}\sin\frac{1}{n}\right\}$

$(f)\ \left\{n\left(1-\cos\frac{1}{n}\right)\right\}$ $(g)\ \left\{\sqrt[n]{n^2+n}\right\}$ $(h)\ \left\{\frac{(\ln n)^5}{\sqrt{n}}\right\}$

4. Which of the following sequences converge, and which diverge? Find the limit of each convergent sequence.

$(a)\ \left\{8^{\frac{1}{n}}\right\}$ $(b)\ \left\{(0.03)^{\frac{1}{n}}\right\}$ $(c)\ \left\{\left(1+\frac{7}{n}\right)^n\right\}$ $(d)\ \left\{\left(1-\frac{1}{n}\right)^n\right\}$

$(e)\ \left\{\sqrt[n]{10n}\right\}$ $(f)\ \left\{\sqrt[n]{n^2}\right\}$ $(g)\ \left\{(n+4)^{\frac{1}{n+4}}\right\}$ $(h)\ \left\{\sqrt[3]{3^{1+2n}}\right\}$

$(i)\ \left\{\frac{(-4)^n}{n!}\right\}$ $(j)\ \left\{\frac{n!}{10^{6n}}\right\}$ $(k)\ \left\{\frac{n!}{2^n 3^n}\right\}$

1.3 Infinite Series

Learning objectives:

- To define the convergence and divergence of an infinite series of real numbers.
- To discuss the convergence and divergence of Geometric Series of real numbers.
- To state the n^{th} term test for divergence.
- To prove the sum, difference and constant multiple rules for convergent series.
  And
- To practice the related problems.

Infinite Series

In this module we study the infinite series of real numbers and their convergence and divergence. We revisit the geometric series and discuss its convergence and divergence. To the end the n^{th} term test for divergence of infinite series is given.

An infinite series of real numbers is the sum of an infinite sequence of real numbers.
If $\{a_n\}_{n=1}^{\infty}$ *is an infinite sequence of real numbers, then the expression*

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

is called an ***infinite series****.*
It is written in sigma notation as

$$\sum_{n=1}^{\infty} \boldsymbol{a_n} \quad \text{or} \quad \text{simply} \quad \sum \boldsymbol{a_n}$$

and a_n *is called the* $\boldsymbol{n^{th}}$ ***term*** *of the infinite series.*
From here after a series means an infinite series of real numbers.
The sequence $\{s_n\}$ *defined by*

$$s_n = a_1 + a_2 + a_3 + \cdots + a_n = \sum_{k=1}^{n} a_k \quad , \quad n \in \boldsymbol{N}$$

is called the ***sequence of*** $\boldsymbol{n^{th}}$ ***partial sums****, where* s_n *is called the* $\boldsymbol{n^{th}}$ ***partial*** *sum of the series.*

We say that the series $\sum_{n=1}^{\infty} a_n$ ***converges*** *if the sequence of* n^{th} *partial sums* $\{s_n\}_{n=1}^{\infty}$ *converges.*
If $\{s_n\}$ *converges to a limit* l*, then we say that the series* $\sum a_n$ *is* ***convergent*** *and* $\sum \boldsymbol{a_n}$ ***converges to*** $\boldsymbol{l}$*. Further,* l *is called the* ***sum*** *of the series and we write* $\sum \boldsymbol{a_n} = \boldsymbol{l}$*.*
We say that the series $\sum a_n$ ***diverges*** *if the sequence of* n^{th} *partial sums of the series does not converge.*

Geometric Series:

Geometric Series are series of the form

$$a + ar + ar^2 + \cdots + ar^{n-1} + \cdots = \sum_{n=1}^{\infty} ar^{n-1}$$

in which a and r are the fixed real numbers and $a \neq 0$. The series can also be written as $\sum_{n=0}^{\infty} ar^n$.

Theorem 1:

The geometric series

$$\boldsymbol{a + ar + ar^2 + \cdots + ar^{n-1} + \cdots}$$

$(\boldsymbol{i})$ ***converges to*** $\frac{\boldsymbol{a}}{\boldsymbol{1-r}}$ ***if*** $|\boldsymbol{r}| < 1$

$\boldsymbol{i.e.}$**,** $$\sum_{n=1}^{\infty} \boldsymbol{ar^{n-1}} = \frac{\boldsymbol{a}}{\boldsymbol{1-r}} \qquad \textit{\textbf{if}} \ |\boldsymbol{r}| < 1.$$

$(\boldsymbol{ii})$ ***diverges if*** $|\boldsymbol{r}| \geq \boldsymbol{1}$**.**

Proof: Let $\{s_n\}$ be the n^{th} partial sum of the geometric series $\sum_{n=1}^{\infty} ar^{n-1}$, *i.e.*, $\quad s_n = a + ar + ar^2 + \cdots + ar^{n-1}$

If $r = 1$, then $s_n = na$ and the series diverges, since $\lim_{n\to\infty} s_n = \pm\infty$, depending on the sign of a. If $r = -1$, then

$$s_n = \begin{cases} a & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$$

and the series diverges, since the n^{th} partial sums alternate between a and 0.

If $|r| \neq 1$ then we determine the convergence and divergence of the series in the following way. We have,

$$s_n - rs_n = a - ar^n$$

$$\Longrightarrow \quad s_n = \frac{a(1-r^n)}{1-r}, \quad r \neq 1 .$$

Now, $\lim_{n\to\infty} s_n = \frac{a}{1-r} - \frac{1}{1-r}\lim_{n\to\infty} r^n$

If $|r| < 1$ then $\lim_{n\to\infty} r^n = 0$ (By Theorem 5 of Module 1.2) and $\lim_{n\to\infty} s_n = \frac{a}{1-r}$. Thus, $\sum_{n=1}^{\infty} ar^{n-1} = \dfrac{a}{1-r}$ if $|r| < 1$.

If $|r| > 1$ then $r^n \to \infty$ as $n \to \infty$ and the sequence $\{s_n\}$ diverges. Thus, $\sum_{n=1}^{\infty} ar^{n-1}$ diverges if $|r| \geq 1$.

Hence the theorem.

Note: The formula $\frac{a}{1-r}$ for the sum of a geometric series applies only when the summation index begins with $n = 1$ in the expression $\sum_{n=1}^{\infty} ar^{n-1}$ or with the index $n = 0$ if we write the series as $\sum_{n=0}^{\infty} ar^{n}$.

Example 1: Index starts with $n = 1$

The series $\sum_{n=1}^{\infty} \frac{1}{9}\left(\frac{1}{3}\right)^{n-1} = \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \cdots$ is a geometric series with $a = \frac{1}{9}$ and $r = \frac{1}{3}$. It is convergent (since $|r| < 1$) and converges to $\frac{a}{1-r} = \frac{\frac{1}{9}}{1-\frac{1}{3}} = \frac{1}{6}$. Thus, $\sum_{n=1}^{\infty} \frac{1}{9}\left(\frac{1}{3}\right)^{n-1} = \frac{1}{6}$.

Example 2: Index starts with $n = 0$

The geometric series with $a = \frac{2}{5}$ and $r = -\frac{1}{3}$ is

$$\frac{2}{5} - \frac{2}{5\cdot 3} + \frac{2}{5\cdot 9} - \frac{2}{5\cdot 27} + \cdots = \sum_{n=0}^{\infty} \frac{2}{5}\left(-\frac{1}{3}\right)^{n}$$

It converges to $\frac{a}{1-r} = \frac{\frac{2}{5}}{1-\left(-\frac{1}{3}\right)} = \frac{3}{10}$ (since $|r| < 1$) and

$$\sum_{n=0}^{\infty} \frac{2}{5}\left(-\frac{1}{3}\right)^{n} = \frac{3}{10}.$$

Example 3: A Bouncing Ball

A ball is dropped from a height of a meters above a flat surface. Each time the ball hits the surface after falling a distance h, it rebounds a distance rh, where r is positive number less than 1. Find the total distance the ball travels up and down if $a = 6\ m$ and $r = \frac{2}{3}$.

Solution: Let s be the total distance travelled by the ball. First the ball hits the surface after falling a distance $a\ mts$. Then it rebounds a distance ar, falls a distance ar and the total distance is $2ar$. Since it has fallen a distance ar, it rebounds a distance ar^2, falls a distance ar^2 and there by the total distance is $2ar^2$. The argument continues. Therefore,

$$s = a + \underbrace{2ar + 2ar^2 + 2ar^3 + \cdots}_{Geometric\ series} = a + \frac{2ar}{1-r} = \frac{a(1+r)}{1-r}$$

If $a = 6$ and $r = \frac{2}{3}$ then the total distance the ball travels up and down is given by

$$s = \frac{6\left(1+\frac{2}{3}\right)}{1-\frac{2}{3}} = 30\ mts.$$

Example 4: Repeating Decimals

Express the repeating decimal $5.232323\cdots$ $(i.e.,\ 5.\overline{23}\ )$ as a rational number.

Solution:

$$5.\overline{23} = 5.232323\cdots$$

$$= 5 + \frac{23}{100} + \frac{23}{(100)^2} + \frac{23}{(100)^3} + \cdots$$

$$= 5 + \frac{23}{100}\underbrace{\left(1 + \frac{1}{100} + \frac{1}{(100)^2} + \cdots\right)}_{\text{G.S with } a=1, r=\frac{1}{100}<1}$$

$$= 5 + \frac{23}{100} \cdot \frac{a}{1-r} = 5 + \frac{23}{100} \cdot \frac{1}{1-\frac{1}{100}} = 5 + \frac{23}{99} = \frac{518}{99}.$$

A series $\sum a_n$ is said to a ***telescoping series*** *if the terms of s_n, (the n^{th} partial sum) other than first and last cancel out in pairs.*

Example 5: A non-geometric but Telescoping series

Find the sum of the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$.

Solution: Notice that

$$\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1} \quad \text{(By partial fraction decomposition)}$$

Therefore, the n^{th} partial sum

$$s_n = \sum_{k=1}^{n} \frac{1}{k(k+1)} = \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1}\right)$$

$$= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

Cancelling the adjacent terms of opposite sign.

$$= 1 - \frac{1}{n+1}$$

Now, $\lim_{n\to\infty} s_n = 1$. Thus, the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converges and its sum is 1. Therefore, $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$.

Divergent Series

A reason for a series failing to converge is that its terms do not become small.

Example 6: Partial Sums Outgrow Any Number

(i) The series $\sum n^2 = 1 + 4 + 9 + \cdots + n^2 + \cdots$ diverges.

Notice that $s_n = 1 + 4 + 9 + \cdots + n^2 > n^2$ for all $n \in \boldsymbol{N}, n > 1$.

Let $M > 0$ be given. Choose a natural number $N > \sqrt{M}$. Then

$$n \geq N \implies s_n > n^2 \geq N^2 > M$$

Thus, $\{s_n\}$ diverges and $\sum n^2$ diverges.

(ii) The series $\sum \frac{n+1}{n} = \frac{2}{1} + \frac{3}{2} + \frac{4}{3} + \cdots + \frac{n+1}{n} + \cdots$ diverges.

Notice that each term of the series is greater than 1 and so $s_n = \frac{2}{1} + \frac{3}{2} + \frac{4}{3} + \cdots + \frac{n+1}{n} > n$ for all $n \in \boldsymbol{N}$. Let $M > 0$ be given. Choose a natural number $N > M$. Then

$$n \geq N \implies s_n > n \geq N > M$$

Thus $\{s_n\}$ diverges and $\sum \frac{n+1}{n}$ diverges.

Theorem 2:

If $\sum_{n=1}^{\infty} \boldsymbol{a_n}$ ***converges, then*** $\lim_{n\to\infty} \boldsymbol{a_n} = \mathbf{0}$.

Proof: Given $\sum_{n=1}^{\infty} a_n$ converges. Let s be the sum of the series.

Then $\lim_{n\to\infty} s_n = s$, where s_n is the n^{th} partial sum of the series.
Further, $\lim_{n\to\infty} s_{n-1} = s$. Now,

$$a_n = s_n - s_{n-1} \Longrightarrow \lim_{n\to\infty} a_n = \lim_{n\to\infty} (s_n - s_{n-1})$$
$$= \lim_{n\to\infty} s_n - \lim_{n\to\infty} s_{n-1} = s - s = 0$$

Hence the theorem.

Corollary 1: The $\boldsymbol{n}^{\text{th}}$ Term Test for Divergence

$\sum_{n=1}^{\infty} \boldsymbol{a_n}$ ***diverges if*** $\lim_{n\to\infty} \boldsymbol{a_n} \neq \mathbf{0}$ ***or fails to exist.***

Proof: We have a series $\sum_{n=1}^{\infty} a_n$. Given $\lim_{n\to\infty} a_n \neq 0$ or fails to exist. Assume $\sum_{n=1}^{\infty} a_n$ is convergent. Then by the above theorem $\lim_{n\to\infty} a_n$ exists and is equal to zero. This contradicts our hypothesis. The result now follows.

Example 7: Applying the $\boldsymbol{n}^{\text{th}}$ Term Test

(i) $\sum_{n=1}^{\infty} n^2$ diverges, since $\lim_{n\to\infty} n^2$ does not exist.

(ii) $\sum_{n=1}^{\infty} \frac{n+1}{n}$ diverges, since $\lim_{n\to\infty} \frac{n+1}{n} = 1 \neq 0$.

(iii) $\sum_{n=1}^{\infty} (-1)^{n+1}$ diverges, since $\lim_{n\to\infty} (-1)^{n+1}$ does not exist.

(iv) $\sum_{n=1}^{\infty} \frac{-n}{2n+5}$ diverges, since $\lim_{n\to\infty} \frac{-n}{2n+5} = -\frac{1}{2} \neq 0$.

Note: Theorem 2 does not say that $\sum_{n=1}^{\infty} a_n$ converges if $\lim_{n\to\infty} a_n = 0$. It is possible for a series to diverge when $a_n \to 0$ as $n \to \infty$.

Example 8: $a_n \to 0$ as $n \to \infty$ but the series $\sum a_n$ diverges.

Does the series $\sum_{n=1}^{\infty} ln\left(\frac{n}{n+1}\right)$ converge or diverge?

Solution:

Consider the series $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} ln\left(\frac{n}{n+1}\right)$, where $a_n = ln\left(\frac{n}{n+1}\right)$.

Note that

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} ln\left(\frac{n}{n+1}\right) = \lim_{n\to\infty} ln\left(1 - \frac{1}{n+1}\right) = ln\, 1 = 0$ but the series diverges.

Let s_n be the n^{th} partial sum of the series $\sum a_n$. Then

$$s_n = \sum_{k=1}^{n} ln\left(\frac{k}{k+1}\right) = \sum_{k=1}^{n}\left[ln\,k - ln(k+1)\right]$$
$$= (ln\,1 - ln\,2) + (ln\,2 - ln\,3) + (ln\,3 - ln\,4) + \cdots + (ln(n-1) - ln\,n) + (ln\,n - ln(n+1))$$
$$= -\,ln(n+1)$$

and $\lim_{n\to\infty} s_n = -\lim_{n\to\infty} ln(n+1) = -\infty$. Thus, the sequence of partial sums of the series diverges. Therefore, the series $\sum a_n$ diverges.

Combining Series:

If $\sum a_n$ and $\sum b_n$ are two given series then we can add them term by term, subtract them term by term or multiply them by constant to make new series. If the given series are convergent then the series obtained as above are also convergent.

Theorem 3:

If $\sum a_n = A$ and $\sum b_n = B$ are convergent series, then

1. ***Sum Rule:*** $\quad \sum(a_n + b_n) = A + B$
2. ***Difference Rule:*** $\quad \sum(a_n - b_n) = A - B$
3. ***Constant Multiple Rule:*** $\quad \sum ka_n = kA$

(where k is any constant)

Proof: Let A_n and B_n be the n^{th} partial sums of the series $\sum a_n$ and $\sum b_n$ respectively. Since $\sum a_n = A$ and $\sum b_n = B$; $\lim_{n\to\infty} A_n = A$ and $\lim_{n\to\infty} B_n = B$. Let s_n be the n^{th} partial sum of the series $\sum(a_n + b_n)$. Then,

$$s_n = \sum_{k=1}^{n}(a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k = A_n + B_n$$

and $\lim_{n\to\infty} s_n = \lim_{n\to\infty}(A_n + B_n) = \lim_{n\to\infty} A_n + \lim_{n\to\infty} B_n$

(By Sum Rule for Sequences)

$= A + B$

Thus, the series $\sum(a_n + b_n)$ is convergent and converges to $A + B$. Therefore, $\sum(a_n + b_n) = A + B$.

Similarly, the difference rule for the convergent series can be proved.

Let t_n be the n^{th} partial sum of the series $\sum ka_n$. Then

$t_n = ka_1 + ka_2 + \cdots + ka_n = k(a_1 + a_2 + \cdots + a_n) = kA_n$

and $\lim_{n\to\infty} t_n = \lim_{n\to\infty} kA_n = k\lim_{n\to\infty} A_n$

(By Constant Multiple Rule for Sequences)

$= kA$

Thus, $\sum ka_n = kA$.

Hence the theorem.

Corollary 2: ***Every non-zero constant multiple of a divergent series is divergent.***

Corollary 3: ***If one of the series $\sum a_n$ and $\sum b_n$ converges and the other diverges then both the series $\sum(a_n + b_n)$ and $\sum(a_n - b_n)$ diverge.***

Note: $\sum(a_n + b_n)$ can converge when both $\sum a_n$ and $\sum b_n$ both diverge.

For example, $\sum a_n = 1 + 1 + 1 + \cdots + \cdots$ and $\sum b_n = (-1) + (-1) + (-1) + \cdots + \cdots$ both diverge, where as $\sum(a_n + b_n) = 0 + 0 + 0 + \cdots + \cdots$ converges to 0.

Example 9:

Find the sum of the series

$(i)\ \sum_{n=1}^{\infty} \frac{3^{n-1}-1}{6^{n-1}} \qquad (ii)\ \sum_{n=1}^{\infty} \frac{4}{2^n}$

Solution:

$$(i)\ \sum_{n=1}^{\infty} \frac{3^{n-1}-1}{6^{n-1}} = \sum_{n=1}^{\infty} \left(\frac{1}{2^{n-1}} - \frac{1}{6^{n-1}} \right)$$

Notice that $\sum_{n=1}^{\infty} \frac{1}{2^{n-1}}$ and $\sum_{n=1}^{\infty} \frac{1}{6^{n-1}}$ are convergent (since they are geometric series with $|r| < 1$)

$$= \sum_{n=1}^{\infty} \frac{1}{2^{n-1}} - \sum_{n=1}^{\infty} \frac{1}{6^{n-1}} \quad \text{(By Difference Rule for Series)}$$

$$= \frac{1}{1-\frac{1}{2}} - \frac{1}{1-\frac{1}{6}}$$

(Geometric Series with $a = 1$ and $r = \frac{1}{2}, \frac{1}{6}$)

$$= 2 - \frac{6}{5} = \frac{4}{5}$$

$$(ii)\ \sum_{n=0}^{\infty} \frac{4}{2^n} = 4 \sum_{n=0}^{\infty} \frac{1}{2^n} \quad \text{(By Constant Multiple Rule for Series)}$$

$$= 4 \cdot \frac{1}{1-\frac{1}{2}} \qquad \text{(Geometric Series with } a = 1, r = \tfrac{1}{2}\text{)}$$

$$= 8$$

Adding or Deleting Terms

Addition and deletion of a finite number of terms from a series will not alter its convergence or divergence. However, the addition or deletion of a finite number of terms from a convergent series will change its sum.

Notice that for any $k > 1$, $\sum\limits_{n=1}^{\infty} a_n = a_1 + a_2 + \cdots + a_{k-1} + \sum\limits_{n=k}^{\infty} a_n$

If $\sum\limits_{n=1}^{\infty} a_n$ converges to s (say) then $\sum\limits_{n=k}^{\infty} a_n$ also converges for any $k > 1$ and it converges to $s - \left(\sum\limits_{n=1}^{k-1} a_n\right)$. Conversely if $\sum\limits_{n=k}^{\infty} a_n$ converges to t (say) for any $k > 1$ then $\sum\limits_{n=1}^{\infty} a_n$ also converges and it converges to $t + \sum\limits_{n=1}^{k-1} a_n$.

Reindexing

Preserving the order of the terms of a series, we can reindex it without altering its convergence.

To raise the starting value of the index h units we replace the index n in the formula for a_n by $n - h$:

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1+h}^{\infty} a_{n-h}$$

To lower starting value of index h units we replace the index n in the formula for a_n by $n+h$:

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1-h}^{\infty} a_{n+h}$$

Note: The partial sum of the series remains same no matter what indexing we choose.

Example 10: Reindexing Geometric Series

$$\sum_{n=1}^{\infty} \frac{1}{2^{n-1}} = 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{1}{2^n}$$

(By lowering the starting value of the index by -1 units)

$$= \sum_{n=5}^{\infty} \frac{1}{2^{n-5}}$$

(By raising the starting value of the index by 4 units)

P1:

Find the sum of the series $\sum_{n=0}^{\infty}\left(\frac{1}{2^n}+\frac{(-1)^n}{5^n}\right)$.

Solution:

The given series is $\sum_{n=0}^{\infty}\left(\frac{1}{2^n}+\frac{(-1)^n}{5^n}\right)$.

$$\sum_{n=0}^{\infty}\left(\frac{1}{2^n}+\frac{(-1)^n}{5^n}\right)=\sum_{n=0}^{\infty}\frac{1}{2^n}+\sum_{n=0}^{\infty}\frac{(-1)^n}{5^n} \qquad \text{(By Sum Rule for Series)}$$

$$=\sum_{n=0}^{\infty}\frac{1}{2^n}+\sum_{n=0}^{\infty}\left(\frac{-1}{5}\right)^n$$

$$=\frac{1}{1-\frac{1}{2}}+\frac{1}{1+\frac{1}{5}}$$

(Geometric Series with $a=1$ and $r=\frac{1}{2},\frac{-1}{3}$)

$$=2+\frac{5}{6}=\frac{17}{6}$$

Thus, the series $\sum_{n=0}^{\infty}\left(\frac{1}{2^n}+\frac{(-1)^n}{5^n}\right)$ converges and its sum is $\frac{17}{6}$.

Therefore, $\sum_{n=0}^{\infty}\left(\frac{1}{2^n}+\frac{(-1)^n}{5^n}\right)=\frac{17}{6}$.

P2:

Express the number $1.24\overline{123}$ as a rational number.

Solution:

$$
\begin{aligned}
1.24\overline{123} &= 1.24123123\cdots \\
&= 1.24 + \frac{123}{10^5} + \frac{123}{10^8} + \frac{123}{10^{11}} + \cdots \\
&= 1.24 + \frac{123}{10^5}\left(1 + \frac{1}{10^3} + \frac{1}{10^5} + \cdots\right) \\
&= 1.24 + \frac{123}{10^5}\left(\frac{1}{1-\frac{1}{10^3}}\right)
\end{aligned}
$$

(Geometric Series with $a = 1$ and $r = \frac{1}{10^3}$)

$$
\begin{aligned}
&= 1.24 + \frac{123}{10^5}\left(\frac{10^3}{999}\right) \\
&= \frac{124}{100} + \frac{123}{99900} = \frac{123999}{99900} = \frac{41333}{33300}
\end{aligned}
$$

P3:

Does the series $\sum_{n=1}^{\infty}\left(1-\frac{1}{n}\right)^n$ converge or diverge.

Solution:

The given series is $\sum_{n=1}^{\infty}\left(1-\frac{1}{n}\right)^n$. Let $a_n = \left(1-\frac{1}{n}\right)^n$.

$$\text{Now,} \quad \lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(1-\frac{1}{n}\right)^n$$

$$= \lim_{n\to\infty}\left(1+\frac{(-1)}{n}\right)^n$$

$$= e^{-1} \neq 0 \qquad \left(\because \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x\right)$$

$\therefore$ By the n^{th} term test for divergence, the series $\sum_{n=1}^{\infty}\left(1-\frac{1}{n}\right)^n$ diverges.

P4:

Find the values of x for which the series $\sum_{n=0}^{\infty} 3\left(\frac{x-1}{2}\right)^n$ converges. Also, find the sum of the series for those values of x.

Solution:

The given series $\sum_{n=0}^{\infty} 3\left(\frac{x-1}{2}\right)^n$ is a geometric series with $a = 3$ and $r = \frac{x-1}{2}$.

The series converges if $|r| < 1$

$$\Longrightarrow \quad \left|\frac{x-1}{2}\right| < 1 \quad \Longrightarrow \quad -1 < \frac{x-1}{2} < 1$$

$$\Longrightarrow -1 < x < 3$$

Therefore, the series converges to

$$\frac{a}{1-r} = \frac{3}{1-\left(\frac{x-1}{2}\right)} = \frac{6}{3-x} \quad \text{when} \quad -1 < x < 3.$$

IP1:

Find the sum of the series $\sum_{n=1}^{\infty}\left(\frac{2n+1}{n^2(n+1)^2}\right)$.

Solution:

Notice that

$$\frac{2k+1}{k^2(k+1)^2} = \frac{1}{k^2} - \frac{1}{(k+1)^2} \quad \text{(By partial fraction decomposition)}$$

Therefore, the n^{th} partial sum

$$s_n = \sum_{k=1}^{n}\frac{2k+1}{k^2(k+1)^2} = \sum_{k=1}^{n}\left(\frac{1}{k^2} - \frac{1}{(k+1)^2}\right)$$

$$= \left(1 - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{9}\right) + \left(\frac{1}{9} - \frac{1}{16}\right) + \cdots + \left(\frac{1}{n^2} - \frac{1}{(n+1)^2}\right)$$

Cancelling the adjacent terms of opposite sign.

$$= 1 - \frac{1}{(n+1)^2}$$

Now, $\lim_{n\to\infty} s_n = 1$. Thus, the series $\sum_{n=1}^{\infty}\frac{2n+1}{n^2(n+1)^2}$ converges and its sum is 1. Therefore, $\sum_{n=1}^{\infty}\frac{2n+1}{n^2(n+1)^2} = 1$.

IP2:

A ball is dropped from a height of $4\ m$. Each time it strikes the pavement after falling from a height of h *meters*, it rebounds to a height $0.75h$ *meters*. Find the total distance the ball travels up and down.

Solution:

A ball is dropped from a height of a meters. Each time the ball hits the surface after falling a distance h *meters*, it rebounds a distance rh *meters*. Then the total distance(s) the ball travels up and down is given by

$$s = \frac{a(1+r)}{1-r}$$

Here $a = 4\ m$ and $r = 0.75$. Therefore, the total distance ball travels up and down is

$$s = \frac{4(1+0.75)}{1-0.75} = \frac{4(1.75)}{0.25} = 28\ m.$$

IP3:

Does the series $\sum_{n=1}^{\infty} n\tan\frac{1}{n}$ converge or diverge.

Solution:

The given series is $\sum_{n=1}^{\infty} n\tan\frac{1}{n}$. Let $a_n = n\tan\frac{1}{n}$.

Now, $\lim_{n\to\infty} a_n = \lim_{n\to\infty} n\tan\frac{1}{n}$

$$= \lim_{n\to\infty} \frac{\tan\left(\frac{1}{n}\right)}{\left(\frac{1}{n}\right)} \qquad \left(\because \frac{0}{0} \text{ form}\right)$$

$$= \lim_{n\to\infty} \frac{\left(-\frac{1}{n^2}\right)\sec^2\left(\frac{1}{n}\right)}{\left(-\frac{1}{n^2}\right)} \qquad \text{(By L'Hopital's Rule)}$$

$$= \lim_{n\to\infty} \sec^2\left(\frac{1}{n}\right) = \sec^2 0 = 1 \neq 0$$

$\therefore$ By the n^{th} term test for divergence, the series $\sum_{n=1}^{\infty} n\tan\frac{1}{n}$ diverges.

IP4:

Find the values of x for which the series $\sum_{n=0}^{\infty} (\ln x)^n$ converges.

Also, find the sum of the series for those values of x.

Solution:

The given series $\sum_{n=0}^{\infty} (\ln x)^n$ is a geometric series with $a = 1$ and

$r = \ln x$.

The given series converges if $|r| < 1$

$$\Rightarrow \quad |\ln x| < 1 \quad \Rightarrow \quad -1 < \ln x < 1$$

$$\Rightarrow e^{-1} < x < e$$

Therefore, the series converges to $\frac{1}{1-\ln x}$ when $e^{-1} < x < e$.

1.3 Infinite Series

Exercises:

1. Find the sum of the following series.

$(a)\ \sum_{n=0}^{\infty} \frac{(-1)^n}{4^n}$ $(b)\ \sum_{n=2}^{\infty} \frac{1}{4^n}$ $(c)\ \sum_{n=1}^{\infty} \frac{7}{4^n}$ $(d)\ \sum_{n=0}^{\infty} (-1)^n \frac{5}{4^n}$

$(e)\ \sum_{n=0}^{\infty} \left(\frac{5}{2^n} + \frac{1}{3^n}\right)$ $(f)\ \sum_{n=0}^{\infty} \left(\frac{5}{2^n} - \frac{1}{3^n}\right)$ $(g)\ \sum_{n=0}^{\infty} \left(\frac{2^{n+1}}{5^n}\right)$

2. Find the sum of the following series.

$(a)\ \sum_{n=1}^{\infty} \frac{4}{(4n-3)(4n+1)}$ $(b)\ \sum_{n=1}^{\infty} \frac{6}{(2n-1)(2n+1)}$

$(c)\ \sum_{n=1}^{\infty} \frac{40n}{(2n-1)^2(2n+1)^2}$ $(d)\ \sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}\right)$

$(e)\ \sum_{n=1}^{\infty} \left(\frac{1}{2^{\frac{1}{n}}} - \frac{1}{2^{\frac{1}{(n+1)}}}\right)$ $(f)\ \sum_{n=1}^{\infty} \left(\frac{1}{ln(n+2)} - \frac{1}{ln\,(n+1)}\right)$

$(g)\ \sum_{n=1}^{\infty} \left(tan^{-1}(n) - tan^{-1}(n+1)\right)$

3. Express each of the following numbers as a rational number.

$(i)\ 0.\overline{23} = 0.232323\cdots$ $(ii)\ 0.\overline{234} = 0.234234\cdots$

$(iii)\ 0.\overline{7} = 0.777\cdots$ $(iv)\ 0.\bar{d} = 0.ddd\cdots$

$(v)\ 0.0\overline{6} = 0.0666\cdots$ $(vi)\ 1.\overline{414} = 1.414414\cdots$

$(vii)\ 3.\overline{142817} = 3.142817142817\cdots$

4. Find which of the following series converge and which diverge? If a series converges, then find its sum.

$(a)\ \sum_{n=0}^{\infty}\left(\frac{1}{\sqrt{2}}\right)^n$ $\quad (b)\ \sum_{n=0}^{\infty}\left(\sqrt{2}\right)^n$ $\quad (c)\ \sum_{n=1}^{\infty}(-1)^n\frac{3}{2^n}$

$(d)\ \sum_{n=1}^{\infty}(-1)^n n$ $\quad (e)\ \sum_{n=0}^{\infty}\cos n\pi$ $\quad (f)\ \sum_{n=0}^{\infty}\frac{\cos n\pi}{5^n}$

$(g)\ \sum_{n=0}^{\infty}e^{-2n}$ $\quad (h)\ \sum_{n=1}^{\infty}\ln\frac{1}{n}$ $\quad (i)\ \sum_{n=1}^{\infty}\frac{2}{10^n}$

$(j)\ \sum_{n=0}^{\infty}\left(\frac{1}{x^n}\right),\ |x|>1$ $\quad (k)\ \sum_{n=0}^{\infty}\frac{2^n-1}{3^n}$ $\quad (l)\ \sum_{n=0}^{\infty}\frac{n!}{1000^n}$

$(m)\ \sum_{n=1}^{\infty}\frac{n^n}{n!}$ $\quad (n)\ \sum_{n=1}^{\infty}\ln\left(\frac{n}{n+1}\right)$ $\quad (o)\ \sum_{n=1}^{\infty}\ln\left(\frac{n}{2n+1}\right)$

$(p)\ \sum_{n=0}^{\infty}\left(\frac{e}{\pi}\right)^n$ $\quad (q)\ \sum_{n=0}^{\infty}\frac{e^{n\pi}}{\pi^{ne}}$

5. Find the values of x for which the following series converges. Also, find the sum of the series (as a function of x) for those values of x.

$(a)\ \sum_{n=0}^{\infty}(-1)^n x^n$ $\quad (b)\ \sum_{n=0}^{\infty}(-1)^n x^{2n}$ $\quad (c)\ \sum_{n=0}^{\infty}\frac{(-1)^n}{2}\left(\frac{1}{3+\sin x}\right)^n$

$(d)\ \sum_{n=0}^{\infty}2^n x^n$ $\quad (e)\ \sum_{n=0}^{\infty}(-1)^n x^{-2n}$ $\quad (f)\ \sum_{n=0}^{\infty}(-1)^n(x+1)^n$

$(g)\ \sum_{n=0}^{\infty}\left(\frac{-1}{2}\right)^n(x-3)^n$ $\quad (h)\ \sum_{n=0}^{\infty}\sin^n x$

1.4

Integral Test

Learning objectives:

- To state and prove integral test for the convergence and divergence of series of positive real numbers.
- To study convergence and divergence of $\sum \frac{1}{n^p}$, the p-series. And
- To practice the related problems.

Integral Test

In this module we consider series with non negative terms. We state and prove integral test for the series positive real numbers and we discuss the convergence and divergence of p-series.

Series of non-negative real numbers

Let $\sum\limits_{n=1}^{\infty} a_n$ be an infinite series with $a_n \geq 0$ for all $n \in \boldsymbol{N}$. Then each partial sum is greater than or equal to its predecessor, $i.e.$, $s_n \leq s_{n+1}$, since $s_{n+1} = s_n + a_n$ for all n. Thus,

$$s_1 \leq s_2 \leq s_3 \leq \cdots \leq s_n \leq s_{n+1} \leq \cdots$$

$i.e.$, $\{s_n\}_{n=1}^{\infty}$ is a non-decreasing sequence.

By Non-decreasing Sequence Theorem, $\{s_n\}$ converges if and only if it is bounded above. Thus the series $\sum\limits_{n=1}^{\infty} a_n$, $a_n \geq 0 \;\; \forall n$, converges if and only if the sequence of its partial sums $\{s_n\}$ is bounded above. Thus we have the following result.

Theorem 1:

A series $\sum\limits_{n=1}^{\infty} a_n$ of non negative terms converges if and only if its sequence of partial sums is bounded above.

The Harmonic Series

The series $\sum_{n=1}^{\infty}\frac{1}{n}=1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}+\cdots$ *is called the* ***harmonic series.***

Theorem 2:

The harmonic series is divergent.

Solution:

We have the harmonic series $\sum_{n=1}^{\infty}\frac{1}{n}$. It is an infinite series with non-negative terms. We will show that the sequence of n^{th} partial sums is not bounded above. We group the terms of the series in the following way:

$$1+\frac{1}{2}+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)+\left(\frac{1}{9}+\frac{1}{10}+\cdots+\frac{1}{16}\right)+\cdots$$

Notice that the sum of the first two terms is $\frac{3}{2}$. The sum of the next two terms $\frac{1}{3}+\frac{1}{4}>\frac{1}{4}+\frac{1}{4}=\frac{1}{2}$.

The sum of the next four terms

$$\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}>\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}=\frac{1}{2}$$

and so on. In general, the sum of 2^n terms ending with $\frac{1}{2^{n+1}}$ is greater than $\frac{2^n}{2^{n+1}}=\frac{1}{2}$. Now, notice that, if $n=2^k$, then the partial sum $s_n>\frac{k}{2}$.

Assume that $\{s_n\}$ is bounded above by M (say). Then for $n>2^{2M}$ we have

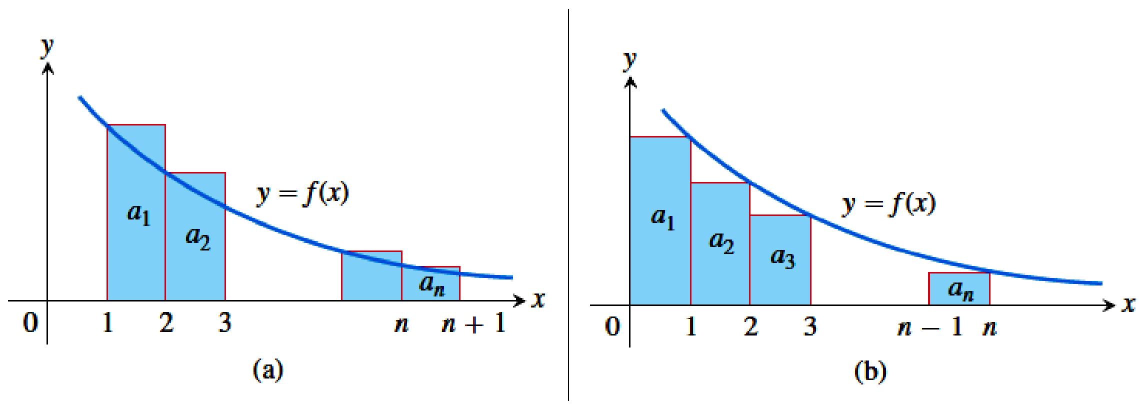
$$s_n > s_{2^{2M}} > \frac{2M}{2} = M$$

a contradiction. Thus, $\{s_n\}$ is not bounded above. By Theorem 1 the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

Theorem 3: The Integral Test

Let $\{a_n\}$ be a sequence of positive terms. Suppose $a_n = f(n)$, where f is a continuous, positive valued decreasing function of x for $x \geq N$, where N is a natural number. Then the series $\sum_{n=N}^{\infty} a_n$ and $\int_N^{\infty} f(x)dx$ both converge or both diverge.

Proof:



We establish the test for the case $N = 1$. The proof for general N is similar.

Under the given conditions for f, we notice that the rectangles in fig (a) have areas $a_1, a_2, \dots, a_n$ and their sum enclose more area than the area under the curve $y = f(x)$ from $x = 1$ to $x = n + 1$.

Therefore, $\int\limits_{1}^{n+1} f(x)\,dx \le a_1 + a_2 + \cdots + a_n$

In fig (b) the rectangles are faced to the left instead of right.

Then notice that

$$a_2 + a_3 + \cdots + a_n \le \int\limits_{1}^{n} f(x)\,dx$$

Now, $a_1 + a_2 + a_3 + \cdots + a_n \le a_1 + \int\limits_{1}^{n} f(x)\,dx$

Combining these results, we get

$$\int\limits_{1}^{n+1} f(x)\,dx \le a_1 + a_2 + \cdots + a_n \le a_1 + \int\limits_{1}^{n} f(x)\,dx$$

These inequalities hold for each n and continue to hold as $n \to \infty$.

If $\int\limits_{1}^{\infty} f(x)\,dx$ is finite, $i.e.$, $\int\limits_{1}^{\infty} f(x)\,dx$ converge, then the right hand inequality shows that $\sum\limits_{n=1}^{\infty} a_n$ is finite, $i.e.$, $\sum\limits_{n=1}^{\infty} a_n$ converge.

If $\int\limits_{1}^{\infty} f(x)\,dx$ is not finite, $i.e.$, $\int\limits_{1}^{\infty} f(x)\,dx$ diverge, then the left hand inequality shows that $\sum\limits_{n=1}^{\infty} a_n$ is infinite, $i.e.$, $\sum\limits_{n=1}^{\infty} a_n$ diverge.

Hence the series and integral are both converge or diverge.

Thus, the Integral Test is proved.

p*-series:** *The series* $\sum\limits_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} + \cdots$ *where* p *is a (real) constant is called a* ***p-series*.*

The p-series with $p = 1$ is the harmonic series.

Theorem 4:

The p-series

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} + \cdots$$

where p is a (real) constant, converges if $p > 1$ and diverges if $p \leq 1$.

Proof: If $p > 1$ then $f(x) = \frac{1}{x^p}$ is a continuous, positive valued and decreasing function for $x \geq 1$.

Now, $a_n = \frac{1}{n^p} = f(n)$, $n \in \boldsymbol{N}$ and

$$\int_1^{\infty} f(x)\,dx = \int_1^{\infty} \frac{1}{x^p}\,dx = \lim_{b\to\infty} \int_1^{b} x^{-p}\,dx = \lim_{b\to\infty} \left[\frac{x^{-p+1}}{-p+1} \right]_1^b$$

$$= \frac{1}{1-p} \cdot \lim_{b\to\infty} \left(\frac{1}{b^{p-1}} - 1 \right) = \frac{1}{1-p} \quad (\because p > 1)$$

Thus, $\int_1^{\infty} f(x)\,dx$ converges. By integral test the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges.

(**Note:** The sum of the p-series is not $\frac{1}{p-1}$. The series converges, but the value to which it converges is not known)

If $p < 1$ then $1 - p > 0$ and

$$\int_1^{\infty} f(x)\,dx = \int_1^{\infty} \frac{1}{x^p}\,dx = \frac{1}{1-p} \cdot \lim_{b\to\infty} \left(b^{1-p} - 1 \right) = \infty$$

Thus, $\int_1^{\infty} f(x)\,dx$ diverges. By integral test the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges.

If $p = 1$, then we have divergent harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots$$

Hence the theorem.

Example 1: A Convergent Series

Prove that the series $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ converges.

Solution: Let $f(x) = \frac{1}{x^2+1}$. Note that $f(x)$ is positive, continuous and decreasing for $x \geq 1$ and

$$\int_1^{\infty} \frac{1}{x^2+1}\,dx = \lim_{b\to\infty}\left[\tan^{-1} x\right]_1^b = \lim_{b\to\infty}\left[\tan^{-1} b - \tan^{-1} 1\right]$$

$$= \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

Thus, $\int_1^{\infty} f(x)\,dx$ converges. By integral test $\sum_{n=1}^{\infty} f(x) = \sum_{n=1}^{\infty} \frac{1}{n^2+1}$ also converges.

Note: $\frac{\pi}{4}$ is not the sum of the series.

Logarithmic p-series:

The series

$$\sum_{n=2}^{\infty}\frac{1}{n(\ln n)^p}=\frac{1}{2(\ln 2)^p}+\frac{1}{3(\ln 3)^p}+\frac{1}{4(\ln 4)^p}+\cdots+\frac{1}{n(\ln n)^p}+\cdots$$

*where p is a real constant is called a **Logarithmic p-series.***

Theorem 5:

The logarithmic p-series

$$\sum_{n=2}^{\infty}\frac{1}{n(\ln n)^p}=\frac{1}{2(\ln 2)^p}+\frac{1}{3(\ln 3)^p}+\frac{1}{4(\ln 4)^p}+\cdots+\frac{1}{n(\ln n)^p}+\cdots$$

where p is a real constant, converges if $p > 1$ and diverges if $p \leq 1$.

Proof:

Let $f(x) = \frac{1}{x(\ln x)^p}$. Note that $f(x)$ is positive, continuous and decreasing for $x \geq 2$ and

$$\int_2^{\infty} f(x)dx=\int_2^{\infty}\frac{1}{x(\ln x)^p}\,dx=\int_{\ln 2}^{\infty}\frac{1}{u^p}\,du \quad \text{where } u = \ln x$$

$$=\lim_{b\to\infty}\left[\frac{u^{-p+1}}{-p+1}\right]_{\ln 2}^{b}=\frac{1}{1-p}\lim_{b\to\infty}\left[b^{-p+1}-(\ln 2)^{-p+1}\right]$$

$$=\begin{cases} \frac{1}{1-p}(\ln 2)^{-p+1} & \text{if } p > 1 \\ \infty & \text{if } p < 1 \end{cases}$$

For $p = 1$: $\int_2^{\infty}\frac{1}{x\ln x}\,dx=\lim_{b\to\infty}\left[\ln(\ln x)\right]_2^b=\lim_{b\to\infty}\left[\ln(\ln b)-\ln(\ln 2)\right]=\infty$

Thus, $\int_{2}^{\infty} f(x)dx$ converges if $p > 1$ and diverges if $p \leq 1$.

By integral test $\sum_{n=2}^{\infty} f(n) = \sum_{n=2}^{\infty} \frac{1}{n(ln n)^p}$ also converges if $p > 1$ and diverges if $p \leq 1$. Hence the theorem.

P1:

Use integral test to determine the convergence of the series $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$.

Solution:

Let $f(x) = \frac{x}{x^2+1}$. Note that $f(x)$ is positive, continuous and decreasing for $x \geq 1$ and

$$\int_1^{\infty} f(x)\,dx = \int_1^{\infty} \frac{x}{x^2+1}\,dx = \lim_{b\to\infty} \int_1^{b} \frac{x}{x^2+1}\,dx = \lim_{b\to\infty} \left[\frac{1}{2}\ln(x^2+1)\right]_1^b$$

$$= \lim_{b\to\infty} \left[\frac{1}{2}\ln(b^2+1) - \frac{1}{2}\ln 2\right] = \infty$$

Thus, $\int_1^{\infty} f(x)\,dx$ diverges. By integral test $\sum_{n=1}^{\infty} f(n) = \sum_{n=1}^{\infty} \frac{n}{n^2+1}$ also diverges.

P2:

Use integral test to determine the convergence of the series $\sum_{n=1}^{\infty} \frac{e^n}{e^{2n}+1}$**.**

Solution:

Let $f(x) = \frac{e^x}{e^{2x}+1}$. Notice that $f(x)$ is positive, continuous and decreasing for $x \geq 1$ and

$$\int_1^{\infty} f(x)\,dx = \int_1^{\infty} \frac{e^x}{e^{2x}+1}\,dx = \int_e^{\infty} \frac{1}{u^2+1}\,du$$

(where $u = e^x \Longrightarrow du = e^x dx$; $x = 1 \Longrightarrow u = e; x = \infty \Longrightarrow u = \infty$)

$$= \lim_{b\to\infty} \int_e^{b} \frac{1}{u^2+1}\,du = \lim_{b\to\infty} \left[\tan^{-1} u \right]_e^b$$

$$= \lim_{b\to\infty} \left[\tan^{-1} b - \tan^{-1} e \right] = \frac{\pi}{2} - \tan^{-1} e$$

Thus, $\int_1^{\infty} f(x)\,dx$ converges. By integral test $\sum_{n=1}^{\infty} f(n) = \sum_{n=1}^{\infty} \frac{e^n}{e^{2n}+1}$ also converges.

P3:

Which of the following series converge and which diverge

$$(\boldsymbol{i})\sum_{n=2}^{\infty}\frac{\mathbf{1}}{\boldsymbol{n\,ln\,n}}\quad(\boldsymbol{ii})\sum_{n=2}^{\infty}\frac{\mathbf{1}}{\boldsymbol{n(ln\,n)^{1.01}}}\quad(\boldsymbol{iii})\sum_{n=2}^{\infty}\frac{\mathbf{1}}{\boldsymbol{n\,ln\,n^{3}}}\quad(\boldsymbol{iv})\sum_{n=2}^{\infty}\frac{\mathbf{1}}{\boldsymbol{n(ln\,n)^{3}}}$$

Solution:

The logarithmic p-series

$$\sum_{n=2}^{\infty}\frac{1}{n(ln\,n)^p}=\frac{1}{2(ln\,2)^p}+\frac{1}{3(ln\,3)^p}+\frac{1}{4(ln\,4)^p}+\cdots+\frac{1}{n(ln\,n)^p}+\cdots$$

where p is a real constant, converges if $p>1$ and diverges if $p\le 1$.

$(\boldsymbol{i})$ The given series $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n\,ln\,n}$ is the Logarithmic p-series for $p=1$. Therefore, it diverges.

$(\boldsymbol{ii})$ The given series $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n(ln\,n)^{1.01}}$ is the Logarithmic p-series for $p=1.01>1$. Therefore, it converges.

$(\boldsymbol{iii})$ The given series $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n\,ln\,n^3}=\frac{1}{3}\sum_{n=2}^{\infty}\frac{1}{n\,ln\,n}$ is the Logarithmic p-series for $p=1$. Therefore, it diverges.

$(\boldsymbol{iv})$ The given series $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n(ln\,n)^3}$ is the Logarithmic p-series for $p=3>1$. Therefore, it converges.

P4:

Determine whether the series is convergent or divergent

$$(\boldsymbol{a})\ \sum_{n=1}^{\infty} \frac{2}{n^{0.85}} \qquad (\boldsymbol{b})\ \sum_{n=1}^{\infty} \left(n^{-1.4} + 3n^{-1.2}\right)$$

Solution:

(a) The given series $\sum_{n=1}^{\infty} \frac{2}{n^{0.85}}$ is divergent by p-series test because $p = 0.85 < 1$.

(b) The given series is $\sum_{n=1}^{\infty} \left(n^{-1.4} + 3n^{-1.2}\right)$.

Now, $\sum_{n=1}^{\infty} \left(n^{-1.4} + 3n^{-1.2}\right) = \sum_{n=1}^{\infty} n^{-1.4} + 3\sum_{n=1}^{\infty} n^{-1.2}$

$$= \sum_{n=1}^{\infty} \frac{1}{n^{1.4}} + 3\sum_{n=1}^{\infty} \frac{1}{n^{1.2}}$$

Both series $\sum_{n=1}^{\infty} \frac{1}{n^{1.4}}$ and $\sum_{n=1}^{\infty} \frac{1}{n^{1.2}}$ are convergent by p-series test because $p = 1.4, 1.2 > 1$ and sum of the convergent series is also convergent. Thus, the series $\sum_{n=1}^{\infty} \left(n^{-1.4} + 3n^{-1.2}\right)$ is convergent.

IP1:

Use integral test to determine the convergence of the series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}\left(\sqrt{n}+1\right)}$.

Solution:

Let $f(x)=\frac{1}{\sqrt{x}(\sqrt{x}+1)}$. Note that $f(x)$ is positive, continuous and decreasing for $x\geq 1$ and

$$\int_{1}^{\infty} f(x)\,dx=\int_{1}^{\infty}\frac{1}{\sqrt{x}\left(\sqrt{x}+1\right)}\,dx$$

$$=\lim_{b\to\infty}\int_{1}^{b}\frac{1}{\sqrt{x}\left(\sqrt{x}+1\right)}\,dx=\lim_{b\to\infty}2\int_{2}^{\sqrt{b}+1}\frac{du}{u}$$

(where $u=\sqrt{x}+1 \Longrightarrow du=\frac{1}{2\sqrt{x}}dx$)

$$=\lim_{b\to\infty}2\left[\ln\left(\sqrt{b}+1\right)-\ln 2\right]=\infty$$

Thus, $\int_{1}^{\infty} f(x)\,dx$ diverges. By integral test $\displaystyle\sum_{n=1}^{\infty} f(n)=\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}\left(\sqrt{n}+1\right)}$ also diverges.

IP2:

Use integral test to determine the convergence of the series $\sum_{n=1}^{\infty} ne^{-n}$.

Solution:

Let $f(x) = xe^{-x}$. Notice that $f(x)$ is positive, continuous and decreasing for $x \geq 1$ and

$$\int_1^{\infty} f(x)\,dx = \int_1^{\infty} xe^{-x}\ dx = \lim_{b\to\infty}\int_1^{b} xe^{-x}\ dx$$

$$= \lim_{b\to\infty}\left[-xe^{-x} - e^{-x}\right]_1^b$$

$$= \lim_{b\to\infty}\left[-be^{-b} - e^{-b} + 2e^{-1}\right]$$

$$= 2e^{-1} - \lim_{b\to\infty}\frac{b}{e^b}$$

$$= 2e^{-1} - \lim_{b\to\infty}\frac{1}{e^b} \qquad \text{(By L'Hopital's Rule)}$$

$$= 2e^{-1}$$

Thus, $\int_1^{\infty} f(x)\,dx$ converges. By integral test $\sum_{n=1}^{\infty} f(n) = \sum_{n=1}^{\infty} ne^{-n}$ also converges.

IP3:

Use integral test to determine the convergence of the series

$$\sum_{n=1}^{\infty} \frac{1}{n\left(1+(\ln n)^2\right)}.$$

Solution:

Let $f(x) = \frac{1}{x(1+(\ln x)^2)}$. Notice that $f(x)$ is positive, continuous and decreasing for $x \geq 1$ and

$$\int_1^{\infty} f(x)dx = \int_1^{\infty} \frac{1}{x\left(1+(\ln x)^2\right)} dx = \int_0^{\infty} \frac{du}{1+u^2}$$

(where $u = \ln x \Longrightarrow du = \frac{1}{x} dx; x = 1 \Longrightarrow u = 0; x = \infty \Longrightarrow u = \infty$)

$$= \lim_{b\to\infty} \int_0^b \frac{du}{1+u^2} = \lim_{b\to\infty} \left[\tan^{-1} u\right]_0^b = \lim_{b\to\infty} \tan^{-1} b = \frac{\pi}{2}$$

Thus, $\int_1^{\infty} f(x)dx$ converges. By integral test

$\sum_{n=1}^{\infty} f(n) = \sum_{n=1}^{\infty} \frac{1}{n\left(1+(\ln n)^2\right)}$ also converges.

IP4:

Determine whether the series is convergent or divergent

$(\boldsymbol{a})\ \mathbf{1+\frac{1}{8}+\frac{1}{27}+\frac{1}{64}+\frac{1}{125}+\cdots}$

$(\boldsymbol{b})\ \mathbf{1+\frac{1}{2\sqrt{2}}+\frac{1}{3\sqrt{3}}+\frac{1}{4\sqrt{4}}+\frac{1}{5\sqrt{5}}+\cdots}$

Solution:

(a) The given series

$1+\frac{1}{8}+\frac{1}{27}+\frac{1}{64}+\frac{1}{125}+\cdots=\sum_{n=1}^{\infty}\frac{1}{n^3}$ is convergent because it is a p-series with $p=3>1$.

(b) The given series

$1+\frac{1}{2\sqrt{2}}+\frac{1}{3\sqrt{3}}+\frac{1}{4\sqrt{4}}+\frac{1}{5\sqrt{5}}+\cdots=\sum_{n=1}^{\infty}\frac{1}{n\sqrt{n}}=\sum_{n=1}^{\infty}\frac{1}{n^{3/2}}$ is convergent because it is a p-series with $p=\frac{3}{2}>1$.

1.4 Integral Test

Exercises:

1. Determine which of the following series converge and which diverge?

$(a)\ \sum_{n=1}^{\infty} \frac{5}{n+1}$ $(b)\ \sum_{n=1}^{\infty} -\frac{2}{n+1}$ $(c)\ \sum_{n=1}^{\infty} \frac{\ln n}{n}$ $(d)\ \sum_{n=1}^{\infty} \frac{1}{2n-1}$

$(e)\ \sum_{n=1}^{\infty} \frac{\ln n}{\sqrt{n}}$ $(f)\ \sum_{n=1}^{\infty} \frac{\left(\frac{1}{n}\right)}{(\ln n)\sqrt{(\ln n)^2-1}}$ $(g)\ \sum_{n=1}^{\infty} \frac{2}{1+e^n}$

$(h)\ \sum_{n=1}^{\infty} \frac{8\tan^{-1} n}{1+n^2}$ $(i)\ \sum_{n=1}^{\infty} \operatorname{sech} n$ $(j)\ \sum_{n=1}^{\infty} \operatorname{sech}^2 n$

$(k)\ \sum_{n=1}^{\infty} \frac{n^2}{n^3+1}$ $(l)\ \sum_{n=1}^{\infty} \frac{2n^3}{n^4+3}$ $(m)\ \sum_{n=2}^{\infty} \frac{1}{n\sqrt{n^2-1}}$

2. Determine which of the following series converge and which diverge?

$(a)\ \sum_{n=1}^{\infty} \frac{3}{\sqrt{n}}$ $(b)\ \sum_{n=1}^{\infty} \frac{-2}{n\sqrt{n}}$ $(c)\ \sum_{n=1}^{\infty} \frac{-8}{n}$ $(d)\ \sum_{n=1}^{\infty} \frac{3}{n^3}$

1.5

Comparison Tests

Learning objectives:

- To state and prove the comparison and limit comparison tests for the convergence of the series of real numbers. And
- To practice the related problems.

Comparison Tests

We have learnt how to determine the convergence of geometric series, p-series and a few others. We can test the convergence of many more series by comparing their terms to those of a series whose convergence is known.

Theorem 1: The Comparison Test

Let $\sum \boldsymbol{a_n}$ be a series with non negative terms.

$(\boldsymbol{i})$ $\sum \boldsymbol{a_n}$ converges if there is a convergent series $\sum \boldsymbol{c_n}$ with $\boldsymbol{a_n} \leq \boldsymbol{c_n}$ for all $\boldsymbol{n} \geq \boldsymbol{N}$, for some natural number $\boldsymbol{N}$.

$(\boldsymbol{ii})$ $\sum \boldsymbol{a_n}$ diverges if there is a divergent series of non negative terms $\sum \boldsymbol{d_n}$ with $\boldsymbol{a_n} \geq \boldsymbol{d_n}$ for all $\boldsymbol{n} \geq \boldsymbol{N}$, for some natural number $\boldsymbol{N}$.

Proof:

We have the series $\sum a_n$ with non negative terms.

Let $\{s_n\}$ be the sequence of n^{th} partial sums of the series.

Clearly, it is a non-decreasing sequence.

$(\boldsymbol{i})$ Given that there is a convergent series $\sum c_n$ with $a_n \leq c_n$ for all $n \geq N$, for some natural number N. Thus, $\sum\limits_{n=1}^{\infty} c_n$ is finite and $\sum\limits_{n=N+1}^{\infty} c_n$ is also finite. Then

$$M = a_1 + a_2 + \cdots + a_N + \sum_{n=N+1}^{\infty} c_n$$

is a finite real number and $s_n \leq M$ for all $n \in \boldsymbol{N}$. Thus, $\{s_n\}$ is bounded above. Therefore, $\sum a_n$ converges by Theorem 1 of Module 1.4.

$(\boldsymbol{ii})$ Given that there is a divergent series $\sum d_n$ of non negative terms with $a_n \geq d_n$ for all $n \geq N$, for some natural number N. Let $\{t_n\}$ be the sequence of n^{th}partial sums of $\sum d_n$. Then $\{t_n\}$ is a non-decreasing sequence (since each d_n is non-negative). Required to prove that $\sum a_n$ diverges. In the light of Theorem 1 of Module 1.4, it is sufficient to prove that $\{s_n\}$, the sequence of partial sums of $\sum a_n$ is not bounded above.
Assume that $\{s_n\}$ is bounded above. Then by Theorem 1 of Module 1.4, $\sum a_n$ is convergent. Therefore, $\sum a_n$ is finite, and so $\sum_{n=N+1}^{\infty} a_n$ is also finite. Then

$$M^* = d_1 + d_2 + \cdots + d_N + \sum_{n=N+1}^{\infty} a_n$$

is finite. Clearly $t_n \leq M^*$, for all $n \in \boldsymbol{N}$. By Theorem 1 of Module 1.4, $\sum d_n$ is convergent. – a contradiction.
The result now follows. Hence the result.

Example1: Applying the Comparison Test

Discuss the convergence and divergence of the series

$(i)\ \sum_{n=0}^{\infty} \frac{1}{n!}$ $(ii) \sum \frac{7}{7n-2}$ $(iii)\ 5 + \frac{2}{3} + 1 + \frac{1}{2+\sqrt{1}} + \frac{1}{4+\sqrt{2}} + \frac{1}{8+\sqrt{3}} + \cdots$

Solution:

(i) We have the series

$$\sum_{n=0}^{\infty}\frac{1}{n!}=1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\cdots+\cdots$$

Notice that the terms are all positive and are less than or equal to the corresponding terms of the series

$$1+\sum_{n=0}^{\infty}\frac{1}{2^n}=1+1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots+\cdots$$

Further, $1+\sum_{n=0}^{\infty}\frac{1}{2^n}$ is a geometric series and converges to $1+\frac{1}{1-\frac{1}{2}}=3$. Thus, by comparison test the series $\sum_{n=0}^{\infty}\frac{1}{n!}$ converges.

Note: We see that 3 is an upper bound for the sequence of n^{th} partial sums of the series $\sum_{n=0}^{\infty}\frac{1}{n!}$. It does not mean that the series converges to 3. Infact the series $\sum_{n=0}^{\infty}\frac{1}{n!}$ converges to e.

(ii) Notice that $\frac{7}{7n-2}=\frac{1}{n-\frac{2}{7}}>\frac{1}{n}$ for all $n\in \boldsymbol{N}$ $i.e.$, the n^{th} term of the series $\sum\frac{7}{7n-2}$ is greater than the n^{th} term of the harmonic series $\sum\frac{1}{n}$. Since $\sum\frac{1}{n}$ is divergent, the series $\sum\frac{7}{7n-2}$ is also divergent by comparison test.

(iii) We have the series

$$5+\frac{2}{3}+1+\frac{1}{2+\sqrt{1}}+\frac{1}{4+\sqrt{2}}+\frac{1}{8+\sqrt{3}}+\cdots$$

Notice that the first three terms do not follow a pattern and so we ignore these first three terms. After deleting these terms we see that the series is $\sum\limits_{n=0}^{\infty}\dfrac{1}{2^n+\sqrt{n}}$. Observe that $\frac{1}{2^n+\sqrt{n}} \le \frac{1}{2^n}$ for $n = 0,1,2,3\cdots$ and $\sum\limits_{n=0}^{\infty}\dfrac{1}{2^n}$ is a convergent geometric series.

Therefore the series $\sum\limits_{n=0}^{\infty}\dfrac{1}{2^n+\sqrt{n}}$ is convergent by comparison test. The given series is convergent since the convergence and divergence of a series is not altered by adding or deleting a finite number of terms.

The Limit Comparison Test

The following comparison test is useful for series in which the n^{th} term is a rational function of n.

Theorem 2: Limit Comparison Test

Let $\sum a_n$ and $\sum b_n$ be series and $a_n > 0$, $b_n > 0$ for all $n \ge N^*$, for some natural number N^*.

(i) If $\lim\limits_{n\to\infty}\frac{a_n}{b_n} = c > 0$, then $\sum a_n$ and $\sum b_n$ both converge or both diverge.

(ii) If $\lim\limits_{n\to\infty}\frac{a_n}{b_n} = 0$ and $\sum b_n$ converges, then $\sum a_n$ converges.

(iii) If $\lim\limits_{n\to\infty}\frac{a_n}{b_n} = \infty$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.

Proof:

$(\boldsymbol{i})$ Suppose that $\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0$. Then for $\epsilon = \frac{c}{2}$, there exists a natural number $N(> N^*)$ such that $\left|\frac{a_n}{b_n} - c\right| < \frac{c}{2}$ for all $n \geq N$.

Thus, $n \geq N \implies -\frac{c}{2} < \frac{a_n}{b_n} - c < \frac{c}{2} \implies \left(\frac{c}{2}\right) b_n < a_n < \left(\frac{3c}{2}\right) b_n$

If $\sum b_n$ converges, then $\sum \left(\frac{3c}{2}\right) b_n$ converges and

$a_n < \left(\frac{3c}{2}\right) b_n$, $\forall\, n \geq N \implies \sum a_n$ converges by comparison test.

If $\sum b_n$ diverges, then $\sum \left(\frac{c}{2}\right) b_n$ diverges and

$\left(\frac{c}{2}\right) b_n < a_n$, $\forall\, n \geq N \implies \sum a_n$ diverges by comparison test.

This proves the first part of the theorem.

$(\boldsymbol{ii})$ Suppose that $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$. Then for $\epsilon = 1$, there exists a natural number $N(> N^*)$ such that

$$\left|\frac{a_n}{b_n} - 0\right| < 1 \quad \text{for all } n \geq N.$$

Thus, $n \geq N \implies -1 < \frac{a_n}{b_n} < 1 \implies a_n < b_n$

If $\sum b_n$ converges then $\sum a_n$ converges by comparison test.

$(\boldsymbol{iii})$ Suppose that $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$. Given $M = 1$, there is a natural number $N(> N^*)$ such that

$$\frac{a_n}{b_n} > 1, \qquad \text{for all } n \geq N$$

Thus, $n \geq N \implies a_n > b_n$. Now, $\sum a_n$ diverges whenever $\sum b_n$ diverges by comparison test.

Hence the theorem.

Example 2: Using the Limit Comparison Test

Discuss the convergence and divergence of the series

$(i)\ \frac{3}{4}+\frac{5}{9}+\frac{7}{16}+\frac{9}{25}+\cdots$

$(ii)\ 1+\frac{1}{3}+\frac{1}{7}+\frac{1}{15}+\cdots$

$(iii)\ \frac{1+\ln 4}{9}+\frac{1+\ln 9}{14}+\frac{1+\ln 256}{21}+\cdots$

Solution:

$(\boldsymbol{i})$ The given series is

$$\frac{3}{4}+\frac{5}{9}+\frac{7}{16}+\frac{9}{25}+\cdots=\sum_{n=1}^{\infty} a_n \quad \text{where } a_n=\frac{2n+1}{(n+1)^2},\ n\in \boldsymbol{N}$$

Clearly, $a_n>0\ \forall n$. Notice that for large values of n, a_n behave like $\frac{2n}{n^2}=\frac{2}{n}$ (since leading terms dominate for large values of n).

Now, take $b_n=\frac{1}{n}$, $n\in \boldsymbol{N}$ and consider the series $\sum b_n$. Observe that $\lim\limits_{n\to\infty}\frac{a_n}{b_n}=\lim\limits_{n\to\infty}\frac{n(2n+1)}{(n+1)^2}=\lim\limits_{n\to\infty}\frac{2n^2+n}{n^2+2n+1}=2>0.$

By part (i) of limit comparison test $\sum a_n$ and $\sum b_n$ both converge or both diverge. Since $\sum b_n=\sum\frac{1}{n}$ is divergent; $\sum a_n$ is also divergent.

$(\boldsymbol{ii})$ The given series is

$$1+\frac{1}{3}+\frac{1}{7}+\frac{1}{15}+\cdots=\sum_{n=1}^{\infty} a_n$$

where $a_n=\frac{1}{2^n-1}$, $n\in \boldsymbol{N}$. Clearly, $a_n>0\ \forall n$. Notice that for large values of n, a_n behaves like $\frac{1}{2^n}$. Now, take $b_n=\frac{1}{2^n}$, $n\in \boldsymbol{N}$ and consider the series $\sum b_n$. Observe that

$$\lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}\frac{2^n}{2^n-1}=\lim_{n\to\infty}\frac{1}{1-\frac{1}{2^n}}=1>0$$

By part(i) of limit comparison test $\sum a_n$ and $\sum b_n$ both converge or both diverge. Since $\sum b_n=\sum\frac{1}{2^n}$ is convergent; $\sum a_n$ is also convergent.

$(\boldsymbol{iii})$ The given series is

$$\frac{1+\ln 4}{9}+\frac{1+\ln 9}{14}+\frac{1+\ln 256}{21}+\cdots=\sum_{n=2}^{\infty}\frac{1+n\ln n}{n^2+5}=\sum_{n=2}^{\infty}a_n$$

where $a_n=\frac{1+n\ln n}{n^2+5}$, $n\geq 2$, $n\in\boldsymbol{N}$**.** Clearly $a_n>0$, $n\geq 2$, $n\in\boldsymbol{N}$

Notice that for large values of n, a_n behaves like $\frac{n\ln n}{n^2}=\frac{\ln n}{n}$.

Further, $\frac{\ln n}{n}>\frac{1}{n}$ for $n\geq 3$. Now, take $b_n=\frac{1}{n}$, $n\geq 3$, $n\in\boldsymbol{N}$ and consider the series $\sum_{n=3}^{\infty}b_n$. Notice that

$$\begin{aligned}\lim_{n\to\infty}\frac{a_n}{b_n}&=\lim_{n\to\infty}\frac{n(1+n\ln n)}{n^2+5}=\lim_{n\to\infty}\frac{n+n^2\ln n}{n^2+5}\left(\frac{\infty}{\infty}\text{ form}\right)\\&=\lim_{n\to\infty}\frac{1+n+2n\ln n}{2n}\quad\text{(by L'Hopital's Rule)}\\&=\lim_{n\to\infty}\frac{3+\ln n}{2}\qquad\text{(by L'Hopital's Rule)}\\&=\infty\end{aligned}$$

and $\sum_{n=3}^{\infty}b_n=\sum_{n=3}^{\infty}\frac{1}{n}$ diverges. By part (iii) of limit comparison test $\sum_{n=2}^{\infty}a_n$ is also diverges.

Note: $ln\, x$ grows slower than x^c for any positive constant c.

Notice that for any positive constant c,

$$\lim_{x\to\infty}\frac{ln\, x}{x^c}=\lim_{x\to\infty}\frac{\frac{1}{x}}{cx^{c-1}}=\frac{1}{c}\cdot\lim_{x\to\infty}\frac{1}{x^c}=0$$

Thus, $ln\, x$ grows slower than x^c for any positive constant c.

Example 3:

Does the series $\sum_{n=1}^{\infty}\frac{ln\, n}{n^{3/2}}$ converge?

Solution:

It is known that $\ln n$ grows slower than n^c for any positive constant c. Now,

$$\frac{\ln n}{n^{3/2}}<\frac{n^c}{n^{3/2}}=\frac{1}{n^{\left(\frac{3}{2}-c\right)}}$$

for sufficiently large n. Now, taking $a_n=\frac{\ln n}{n^{3/2}}$ and $b_n=\frac{1}{n^{\left(\frac{3}{2}-c\right)}}$,

we have

$$\lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}\frac{ln\, n}{n^c}=\lim_{n\to\infty}\frac{\frac{1}{n}}{cn^{c-1}}=\frac{1}{c}\cdot\lim_{n\to\infty}\frac{1}{n^c}=0$$

By part (ii) of limit comparison test $\sum a_n=\sum\frac{\ln n}{n^{3/2}}$ converges if $\sum b_n=\sum\frac{1}{n^{\left(\frac{3}{2}-c\right)}}$ converges. Clearly $\sum b_n=\sum\frac{1}{n^{\left(\frac{3}{2}-c\right)}}$ converges if $\frac{3}{2}-c>1,\ i.e.,c<\frac{1}{2}$. For any $0<c<\frac{1}{2}$, $\sum b_n$ converges.

Therefore, $\sum\frac{\ln n}{n^{3/2}}$ converges.

P1:

Test the convergence of the series $\sum_{n=1}^{\infty} \frac{\sin^2 n}{2^n}$.

Solution:

The given series is $\sum_{n=1}^{\infty} \frac{\sin^2 n}{2^n}$. Notice that $\frac{\sin^2 n}{2^n} \leq \frac{1}{2^n}$ for all $n \in \boldsymbol{N}$, $i.e.,$ the n^{th} term of the series $\sum_{n=1}^{\infty} \frac{\sin^2 n}{2^n}$ is less than or equal to the n^{th} term of the geometric series $\sum_{n=1}^{\infty} \frac{1}{2^n}$. Since $\sum_{n=1}^{\infty} \frac{1}{2^n}$ is convergent, the series $\sum_{n=1}^{\infty} \frac{\sin^2 n}{2^n}$ is also convergent by Comparison Test.

P2:

Test the convergence of the series $\sum\limits_{n=3}^{\infty} \dfrac{1}{ln(ln n)}$.

Solution:

The given series is $\sum\limits_{n=3}^{\infty} \dfrac{1}{ln(ln n)}$. Notice that

$$n > ln\, n \;\; \Rightarrow ln\, n > ln(ln\, n)$$

$$\Rightarrow \frac{1}{n} < \frac{1}{ln\, n} < \frac{1}{ln(ln\, n)} \;\; \Rightarrow \frac{1}{n} < \frac{1}{ln(ln\, n)} \text{ for all } n \geq 3,$$

$i.e.,$ the n^{th} term of the series $\sum\limits_{n=3}^{\infty} \dfrac{1}{ln(ln n)}$ is greater than the n^{th} term of the series $\sum\limits_{n=3}^{\infty} \dfrac{1}{n}$. Since $\sum\limits_{n=3}^{\infty} \dfrac{1}{n}$ is divergent, the series $\sum\limits_{n=3}^{\infty} \dfrac{1}{ln(ln n)}$ is also divergent by Comparison Test.

P3:

Does the series $\sum_{n=1}^{\infty} \frac{n}{1+n\sqrt{n+1}}$ converge or diverge.

Solution:

The given series is

$$\sum_{n=1}^{\infty} \frac{n}{1+n\sqrt{n+1}} = \sum_{n=1}^{\infty} a_n \text{ , where } a_n = \frac{n}{1+n\sqrt{n+1}}, n \in \boldsymbol{N}$$

Clearly, $a_n > 0\ \forall n$. Notice that for large values of n, a_n behaves like $\frac{n}{n\sqrt{n}} = \frac{1}{\sqrt{n}}$. Now, take $b_n = \frac{1}{\sqrt{n}}$, $n \in \boldsymbol{N}$ and consider the series $\sum_{n=1}^{\infty} b_n$. Observe that

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n\sqrt{n}}{1+n\sqrt{n+1}} = \lim_{n\to\infty} \left(\frac{1}{\frac{1}{n\sqrt{n}}+\sqrt{1+\frac{1}{n}}} \right) = 1 > 0.$$

By part (i) of Limit Comparison Test $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converge or both diverge. Since $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is divergent by p-series test; $\sum_{n=1}^{\infty} a_n$ is also divergent.

P4:

Test the convergence of the series $\sum_{n=1}^{\infty} \frac{1}{1+\ln n}$.

Solution:

The given series is

$$\sum_{n=1}^{\infty} \frac{1}{1+\ln n} = \sum_{n=1}^{\infty} a_n \text{ , where } a_n = \frac{1}{1+\ln n}, n \in \boldsymbol{N}$$

Clearly, $a_n > 0\ \forall n$. Notice that for large values of n, a_n behaves like $\frac{1}{n}$. Now, take $b_n = \frac{1}{n}$, $n \in \boldsymbol{N}$ and consider the series $\sum_{n=1}^{\infty} b_n$.

Observe that

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n}{1+\ln n} = \lim_{n\to\infty} \frac{1}{\left(\frac{1}{n}\right)} = \lim_{n\to\infty} n = \infty$$

and $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n}$ is divergent. By part (iii) of Limit Comparison Test $\sum_{n=1}^{\infty} a_n$ is also divergent.

IP1:

Test the convergence of the series $\sum_{n=1}^{\infty}\left(\frac{n}{3n+1}\right)^{n}$.

Solution:

The given series is $\sum_{n=1}^{\infty}\left(\frac{n}{3n+1}\right)^{n}$. Notice that

$$\left(\frac{n}{3n+1}\right)^{n} < \left(\frac{n}{3n}\right)^{n} = \left(\frac{1}{3}\right)^{n} \text{ for all } n \in \boldsymbol{N},$$

$i.e.,$ the n^{th} term of the series $\sum_{n=1}^{\infty}\left(\frac{n}{3n+1}\right)^{n}$ is less than the n^{th} term of the geometric series $\sum_{n=1}^{\infty}\frac{1}{3^{n}}$. Since $\sum_{n=1}^{\infty}\frac{1}{3^{n}}$ is convergent, the series $\sum_{n=1}^{\infty}\left(\frac{n}{3n+1}\right)^{n}$ is also convergent by Comparison Test.

IP2:

Test the convergence of the series $\sum_{n=1}^{\infty} \frac{n+2^n}{n^2 2^n}$.

Solution:

The given series is $\sum_{n=1}^{\infty} \frac{n+2^n}{n^2 2^n} = \sum_{n=1}^{\infty}\left(\frac{1}{n2^n}+\frac{1}{n^2}\right)$. Notice that

$$\frac{1}{n2^n}+\frac{1}{n^2} \leq \frac{1}{2^n}+\frac{1}{n^2} \text{ for all } n \in \boldsymbol{N},$$

$i.e.,$ the n^{th} term of the series $\sum_{n=1}^{\infty}\left(\frac{1}{n2^n}+\frac{1}{n^2}\right)$ is less than or equal to the n^{th} term of the series $\sum_{n=1}^{\infty}\left(\frac{1}{2^n}+\frac{1}{n^2}\right)$.

Notice that $\sum_{n=1}^{\infty}\left(\frac{1}{2^n}+\frac{1}{n^2}\right)$ is convergent, since $\sum_{n=1}^{\infty}\frac{1}{2^n}$ and $\sum_{n=1}^{\infty}\frac{1}{n^2}$ are convergent. Thus, the given series is convergent by Comparison Test.

IP3:

Test the convergence of the series

$$\frac{2^q}{1^p} + \frac{3^q}{2^p} + \frac{4^q}{3^p} + \cdots + \cdots$$

Solution:

The given series is

$$\frac{2^q}{1^p} + \frac{3^q}{2^p} + \frac{4^q}{3^p} + \cdots = \sum_{n=1}^{\infty} \frac{(n+1)^q}{n^p} = \sum_{n=1}^{\infty} a_n$$

where $a_n = \frac{(n+1)^q}{n^p} = \frac{\left(1+\frac{1}{n}\right)^q}{n^{p-q}}$, $n \in \boldsymbol{N}$. Clearly, $a_n > 0\ \forall n$.

Notice that for large values of n, a_n behaves like $\frac{1}{n^{p-q}}$.

Now, take $b_n = \frac{1}{n^{p-q}}$, $n \in \boldsymbol{N}$ and consider the series $\sum_{n=1}^{\infty} b_n$.

Observe that

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{\left(1+\frac{1}{n}\right)^q}{n^{p-q}}}{\frac{1}{n^{p-q}}} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^q = 1 > 0$$

By part(i) of Limit Comparison Test $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converge or both diverge. Clearly $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^{p-q}}$ is convergent if $p - q > 1$ and is divergent if $p - q \leq 1$.

Therefore, the given series is convergent if $p - q > 1$ and is divergent if $p - q \leq 1$.

IP4:

Test the convergence of the series $\sum_{n=1}^{\infty} \frac{1}{n^3}\left(\frac{n+2}{n+3}\right)^n$.

Solution:

The given series is

$$\sum_{n=1}^{\infty} \frac{1}{n^3}\left(\frac{n+2}{n+3}\right)^n = \sum_{n=1}^{\infty} a_n \text{ , where } a_n = \frac{1}{n^3}\left(\frac{n+2}{n+3}\right)^n , n \in \boldsymbol{N}.$$

Clearly, $a_n > 0 \ \forall n$. Notice that for large values of n, a_n behaves like $\frac{1}{n^3}$. Now, take $b_n = \frac{1}{n^3}$, $n \in \boldsymbol{N}$ and consider the series $\sum_{n=1}^{\infty} b_n$.

Observe that

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \left(\frac{n+2}{n+3}\right)^n = \lim_{n\to\infty} \frac{\left(1+\frac{2}{n}\right)^n}{\left(1+\frac{3}{n}\right)^n} = \frac{e^2}{e^3} = \frac{1}{e} > 0.$$

By part (i) of Limit Comparison Test $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converge or both diverge. Since $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^3}$ is convergent; $\sum_{n=1}^{\infty} a_n$ is also convergent.

1.5 Comparison Tests

Exercises:

1. Which of the following series converge and which diverge.

$(a)\ \sum_{n=1}^{\infty} \frac{1}{2\sqrt{n}+\sqrt[3]{n}}$ $(b)\ \sum_{n=1}^{\infty} \frac{3}{n+\sqrt{n}}$ $(c)\ \sum_{n=1}^{\infty} \frac{1+\cos n}{n^2}$

$(d)\ \sum_{n=1}^{\infty} \frac{n+1}{n^2\sqrt{n}}$ $(e)\ \sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3+2}}$ $(f)\ \sum_{n=2}^{\infty} \frac{1}{(\ln n)^2}$

$(g)\ \sum_{n=2}^{\infty} \frac{1}{n\sqrt{n^2-1}}$ $(h)\ \sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+1}$ $(i)\ \sum_{n=1}^{\infty} \frac{1-n}{n2^n}$

$(j)\ \sum_{n=1}^{\infty} \frac{1}{3^{n-1}+1}$ $(k)\ \sum_{n=1}^{\infty} \sin\frac{1}{n}$ $(l)\ \sum_{n=1}^{\infty} \frac{10n+1}{n(n+1)(n+2)}$

$(m)\ \sum_{n=3}^{\infty} \frac{5n^3-3n}{n^2(n-2)(n^2+5)}$ $(n)\ \sum_{n=1}^{\infty} \frac{\tan^{-1} n}{n^{1.1}}$ $(o)\ \sum_{n=1}^{\infty} \frac{\sec^{-1} n}{n^{1.3}}$

$(p)\ \sum_{n=1}^{\infty} \frac{\coth n}{n^2}$ $(q)\ \sum_{n=1}^{\infty} \frac{\tanh n}{n^2}$ $(r)\ \sum_{n=1}^{\infty} \frac{1}{n\sqrt[n]{n}}$

$(s)\ \sum_{n=1}^{\infty} \frac{n\sqrt[n]{n}}{n^2}$ $(t)\ \sum_{n=1}^{\infty} \frac{1}{1+2+3+\cdots+n}$

$(u)\ \sum_{n=1}^{\infty} \frac{1}{1^2+2^2+3^2+\cdots+n^2}$

1.6
The Ratio and Root Tests

Learning objectives:

- To learn the Ratio Test and the Root Test for the convergence and divergence of series of real numbers. And
- To practice the related problems.

The Ratio and Root Tests

This module is devoted for the study of two tests of convergence and divergence of series of real numbers. The tests under discussion are (i) The Ratio Test and (ii) The Root Test.

The Ratio Test measures the rate of growth (or decline) of a series $\sum a_n$ by examining the ratio $\frac{a_{n+1}}{a_n}$. For geometric series $\sum ar^n$, this rate is a constant $\left(\frac{ar^{n+1}}{ar^n} = r\right)$ and the series converges if and only if its ratio is less than 1 in absolute value. The Ratio Test is a powerful rule extending that result.

Theorem 1: The Ratio Test (D' Alembert's Test)

Let $\sum \boldsymbol{a_n}$ ***be a series with positive terms and suppose that*** $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = l$***. Then***

(i) ***The series*** **converges** ***if*** $l < 1$

(ii) ***The series*** **diverges** ***if*** $l > 1$ ***or*** l ***is infinite.***

(iii) ***The test is*** **inconclusive** ***if*** $l = 1$***.***

Proof:

We have a series $\sum a_n$, with $a_n > 0$ and $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = l$.

(i) Suppose $l < 1$ and r is a number such that $l < r < 1$. Then $\epsilon = r - l > 0$. For this $\epsilon > 0$, there is a natural number N such that

$$\frac{a_{n+1}}{a_n} < l + \epsilon = r \text{ , for all } n \geq N$$

$i.e.,$ $a_{N+1} < ra_N,$

$a_{N+2} < ra_{N+1} < r^2 a_N,$

$a_{N+3} < ra_{N+2} < r^2 a_{N+1} < r^3 a_N,$

$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$

$a_{N+m} < ra_{N+m-1} < r^m a_N,$

$\ldots \quad \ldots \quad \ldots.$

Now, consider the series $\sum c_n$, where $c_n = a_n$ for $n = 1,2, \ldots, N$ and $c_{N+1} = ra_N, c_{N+2} = r^2 a_N, \ldots, c_{N+m} = r^m a_N, \ldots$

Clearly, $a_n \leq c_n \; \forall \, n$ and

$$\sum_{n=1}^{\infty} c_n = a_1 + a_2 + \cdots + a_{N-1} + a_N(1 + r + r^2 + \cdots)$$

Since $r < 1$, the geometric series $1 + r + r^2 + \cdots$ converges and so $\sum c_n$ converges. By Comparison Test $\sum a_n$ converges.

Thus, $\sum a_n$ converges if $l < 1$.

(ii) Suppose $l > 1$ or l is infinite. Since $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = l > 1;$

$\frac{a_{n+1}}{a_n} > 1$ for some index N onwards, $i.e., a_{n+1} > a_n$, $\forall n \geq N.$

Thus, $\quad a_N < a_{N+1} < a_{N+2} < \cdots$

That is the terms of the series are growing and do not approach zero as $n \to \infty$. Therefore, by n^{th} Term Test the series diverges.

Thus, $\sum a_n$ diverges if $l > 1$ or l is infinite.

(iii) Suppose $l = 1$. Consider the series $\sum \frac{1}{n}$ and $\sum \frac{1}{n^2}$.

For $\sum \frac{1}{n}$, $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim\limits_{n\to\infty} \frac{n}{n+1} = 1$

For $\sum \frac{1}{n^2}$, $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim\limits_{n\to\infty} \left(\frac{n}{n+1}\right)^2 = 1$

In both the cases $l = 1$, yet the first series diverges where as the second series converges (by p-series Test). Therefore the Ratio Test is inconclusive if $l = 1$.

Hence the theorem

Note: The Ratio Test is effective when the terms of a series contain factorial expressions involving n or expressions raised to a power of n.

Example 1: Applying the Ratio Test

Investigate the convergence and divergence of the series

$$(i)\ \sum_{n=0}^{\infty} \frac{2^n + 5}{3^n} \qquad (ii)\ \sum_{n=1}^{\infty} \frac{2^n}{n!} \qquad (iii)\ \sum_{n=1}^{\infty} \frac{4^n (n!)^2}{(2n)!}$$

Solution:

(i) For the series $\sum\limits_{n=0}^{\infty} \frac{2^n + 5}{3^n}$,

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\left(\frac{2^{n+1}+5}{3^{n+1}}\right)}{\left(\frac{2^n+5}{3^n}\right)} = \frac{1}{3} \lim_{n\to\infty} \left(\frac{2+\frac{5}{2^n}}{1+\frac{5}{2^n}}\right) = \frac{2}{3} < 1$$

The series converges by Ratio Test, since $l = \frac{2}{3} < 1$.

Note: This does not mean that $\frac{2}{3}$ is the sum of the series.

Notice that $\displaystyle\sum_{n=0}^{\infty}\frac{2^n+5}{3^n}=\sum_{n=0}^{\infty}\left(\frac{2}{3}\right)^n+5\sum_{n=0}^{\infty}\frac{1}{3^n}=\frac{1}{1-\frac{2}{3}}+5\,\frac{1}{1-\frac{1}{3}}=\frac{21}{2}.$

(ii) For the series $\displaystyle\sum_{n=1}^{\infty}\frac{2^n}{n!}$

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{\frac{2^{n+1}}{(n+1)!}}{\frac{2^n}{n!}}=\lim_{n\to\infty}\frac{2}{n+1}=0$$

The series converges by Ratio Test, since $l=0<1$.

(iii) Let $a_n=\frac{4^n\,(n!)^2}{(2n)!}$. Then

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\left[\frac{4^{n+1}\cdot(n+1)!(n+1)!}{(2n+2)!}\cdot\frac{(2n)!}{4^n\cdot n!\cdot n!}\right]=\lim_{n\to\infty}\frac{2n+2}{2n+1}=1$$

Since $l=1$, the Ratio Test is inconclusive.

To decide the convergence or divergence of the series, we look for the other tests.

Notice that $\frac{a_{n+1}}{a_n}=\frac{2n+2}{2n+1}>1$ for all $n\in \boldsymbol{N}$**.** Thus, a_{n+1} is always greater than a_n. Further each term is greater than or equal to $a_1=2$. Therefore, the n^{th} term does not approach to 0 as $n\to\infty$. Thus, $\sum a_n$ diverges by the n^{th} Term Test.

Theorem 2: The Root Test (Cauchy's *n*th Root Test)

Let $\sum a_n$ be a series with $a_n\geq 0$ for $n\geq M$, for some natural number M and suppose that $\lim_{n\to\infty}\sqrt[n]{a_n}=l$. Then

$(\boldsymbol{i})$ **The series converges if $\boldsymbol{l} < 1$**

$(\boldsymbol{ii})$ **The series diverges if $\boldsymbol{l} > 1$ or $\boldsymbol{l}$ is infinite**

$(\boldsymbol{iii})$ **The test is inconclusive if $\boldsymbol{l} = \mathbf{1}$.**

Proof:

We have a series $\sum a_n$, with $a_n \geq 0$ for $n \geq M$, for some natural number M and $\lim_{n\to\infty} \sqrt[n]{a_n} = l$.

$(\boldsymbol{i})$ Suppose $l < 1$. Choose an $\epsilon > 0$ such that $l + \epsilon < 1$. For this ϵ, there exists a natural number $N \geq M$, such that

$$\left|\sqrt[n]{a_n} - l\right| < \epsilon, \quad \text{for all } n \geq N,$$

$$i.e., \quad \sqrt[n]{a_n} < l + \epsilon, \quad \text{for all } n \geq N$$

$$\Rightarrow a_n < (l+\epsilon)^n, \quad \text{for all } n \geq N.$$

Notice that $\sum_{n=N}^{\infty} (l+\epsilon)^n$ converges, since it is a geometric series with common ratio $l + \epsilon < 1$. By Comparison Test $\sum a_n$ converges.

(ii) Suppose $l > 1$ or l is infinite. There exists a natural number $N \geq M$ such that $\sqrt[n]{a_n} > 1$, for all $n \geq N$, $i.e.$, $a_n > 1$ for all $n \geq N$. Thus, the terms of the series do not converge to zero and therefore, $\sum a_n$ diverges by n^{th} Term Test.

(iii) Suppose $l = 1$. Consider the series $\sum \frac{1}{n}$ and $\sum \frac{1}{n^2}$. Notice that $\lim_{n\to\infty} \sqrt[n]{a_n} = 1$ in both the cases, but the first series diverges and the second converges. These two series show that the test is not conclusive when $l = 1$.

Hence the theorem

Example 3: Applying the Root Test

Which of the following series converges and which diverges?

$$(i)\ \sum_{n=1}^{\infty} \frac{n^2}{2^n} \quad (ii)\ \sum_{n=1}^{\infty} \frac{2^n}{n^2} \quad (iii)\ \sum_{n=1}^{\infty} \left(\frac{1}{1+n}\right)^n$$

Solution:

(i) For the series $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$, we have

$$l = \lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \frac{\left(\sqrt[n]{n}\right)^2}{2} = \frac{1}{2} < 1$$

The series converges by the Root Test, since $l < 1$.

(ii) For the series $\sum_{n=1}^{\infty} \frac{2^n}{n^2}$, we have

$$l = \lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \frac{2}{\left(\sqrt[n]{n}\right)^2} = 2 > 1$$

The series diverges by Root Test, since $l > 1$.

(iii) For the series $\sum_{n=1}^{\infty} \left(\frac{1}{1+n}\right)^n$, we have

$$l = \lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \frac{1}{1+n} = 0 < 1$$

The series converges by the Root Test, since $l < 1$.

Example 4:

Let $a_n = \begin{cases} \frac{n}{2^n} & \text{if } n \text{ is odd} \\ \frac{1}{2^n} & \text{if } n \text{ is even} \end{cases}$, Does the series $\sum a_n$ converge?

Solution:

We have $\sqrt[n]{a_n} = \begin{cases} \frac{\sqrt[n]{n}}{2} & \text{if } n \text{ is odd} \\ \frac{1}{2} & \text{if } n \text{ is even} \end{cases}$

Therefore, $\frac{1}{2} \le \sqrt[n]{a_n} \le \frac{\sqrt[n]{n}}{2}$ for all $n \in \boldsymbol{N}$.

Now, $l = \lim_{n\to\infty} \sqrt[n]{a_n} = \frac{1}{2}$ by Sandwich Theorem, since $\lim_{n\to\infty} \frac{1}{2} = \lim_{n\to\infty} \frac{\sqrt[n]{n}}{2} = \frac{1}{2}$. Thus, the series $\sum a_n$ converges by Root Test since $l < 1$.

Example 5:

Does the series $\sum_{n=1}^{\infty} \frac{(n+1)^n}{n^{n+1}}$ converge or diverge?

Solution:

Let $a_n = \frac{(n+1)^n}{n^{n+1}} = \frac{(n+1)^n}{n^n \cdot n}$. Then

$$l = \lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \frac{n+1}{n \cdot n^{\frac{1}{n}}} = \lim_{n\to\infty} \left(\frac{n+1}{n}\right) \cdot \lim_{n\to\infty} \frac{1}{n^{\frac{1}{n}}} = 1 \cdot 1 = 1$$

Since $l = 1$, the test is inconclusive from the Root test. To decide the convergence or divergence of the series we look for other tests.

Now, $a_n = \frac{1}{n}\left(\frac{n}{n+1}\right)^n$; $n \in \boldsymbol{N}$. Clearly $a_n > 0 \ \forall n$.

Notice that for large values of n, a_n behave like $\frac{1}{n}$. Now, take

$b_n = \frac{1}{n}$, $n \in \boldsymbol{N}$ and consider the series $\sum b_n$. Observe that

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n = \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^n} = \frac{1}{e} > 0$$

By part (i) of Limit Comparison Test, $\sum a_n$ and $\sum b_n$ both converge or both diverge. Since $\sum b_n = \sum \frac{1}{n}$ is divergent; $\sum a_n$ is also divergent.

P1:

Test the convergence of the series $\sum_{n=1}^{\infty} \frac{(n+3)!}{3!\,n!\,3^n}$.

Solution:

Let $a_n = \frac{(n+3)!}{3!n!3^n}$. Then

$$l = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+4)!}{3!(n+1)!3^{n+1}} \cdot \frac{3!n!3^n}{(n+3)!}$$

$$= \lim_{n\to\infty} \frac{n+4}{3(n+1)} = \frac{1}{3}$$

The series converges by Ratio Test, since $l = \frac{1}{3} < 1$.

P2:

Test the convergence of the series $\sum_{n=1}^{\infty} \dfrac{2^n - 2}{2^n + 1}$.

Solution:

Let $a_n = \frac{2^n - 2}{2^n + 1}$. Then

$$l = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{2^{n+1}-2}{2^{n+1}+1} \cdot \frac{2^n+1}{2^n-2}$$

$$= \lim_{n\to\infty} \frac{2-\frac{2}{2^n}}{2+\frac{1}{2^n}} \cdot \frac{1+\frac{1}{2^n}}{1-\frac{2}{2^n}} = 1$$

Since $l = 1$, the Ratio Test is inconclusive. To decide the convergence or divergence of the series we look for the other tests.

Now, $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{2^n-2}{2^n+1} = \lim_{n\to\infty} \frac{1-\frac{2}{2^n}}{1+\frac{1}{2^n}} = 1 \neq 0$ and

$\sum a_n$ diverges by the n^{th} Term Test.

P3:

Test the convergence of the series $\sum_{n=1}^{\infty} \frac{n^{n^2}}{(n+1)^{n^2}}$.

Solution:

Let $a_n = \frac{n^{n^2}}{(n+1)^{n^2}}$. Then

$$l = \lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \frac{n^n}{(n+1)^n} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n$$

$$= \lim_{n\to\infty} \left(\frac{1}{1+\frac{1}{n}}\right)^n = \frac{1}{\lim\limits_{n\to\infty} \left(1+\frac{1}{n}\right)^n} = \frac{1}{e} < 1$$

The series converges by the Root Test, since $l < 1$.

P4:

Test the convergence of the series $\displaystyle\sum_{n=1}^{\infty}\left[\left(\frac{n+1}{n}\right)^{n+1}-\left(\frac{n+1}{n}\right)\right]^{-n}$.

Solution:

Let $a_n = \left[\left(\frac{n+1}{n}\right)^{n+1} - \left(\frac{n+1}{n}\right)\right]^{-n}$. Then

$$
\begin{aligned}
l &= \lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \left[\left(\frac{n+1}{n}\right)^{n+1} - \left(\frac{n+1}{n}\right)\right]^{-1} \\
&= \lim_{n\to\infty} \frac{1}{\left[\left(1+\frac{1}{n}\right)^{n+1} - \left(1+\frac{1}{n}\right)\right]} = \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)\left[\left(1+\frac{1}{n}\right)^{n} - 1\right]} \\
&= \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)} \cdot \lim_{n\to\infty} \frac{1}{\left[\left(1+\frac{1}{n}\right)^{n} - 1\right]} \\
&= \frac{1}{e-1} < 1 \qquad \left(\because \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x\right)
\end{aligned}
$$

The series converges by the Root Test, since $l < 1$.

IP1:

Test the convergence of the series $\displaystyle\sum_{n=1}^{\infty} \frac{1\cdot 3\cdot 5\cdots(2n-1)}{\left[2\cdot 4\cdot 6\cdots(2n)\right]\left(3^n+1\right)}$**.**

Solution:

Let $a_n = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{[2\cdot 4\cdot 6\cdots(2n)](3^n+1)}$. Then

$$l = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{1\cdot 3\cdot 5\cdots(2n+1)}{2\cdot 4\cdot 6\cdots 2n.(2n+2)(3^{n+1}+1)} \cdot \frac{2\cdot 4\cdot 6\cdots 2n.(3^n+1)}{1\cdot 3\cdot 5\cdots(2n-1)}$$

$$= \lim_{n\to\infty} \frac{(2n+1)}{(2n+2)} \cdot \frac{(3^n+1)}{(3^{n+1}+1)}$$

$$= \lim_{n\to\infty} \left(\frac{1+\frac{1}{2n}}{1+\frac{1}{n}}\right)\left(\frac{1+\frac{1}{3^n}}{3+\frac{1}{3^n}}\right)$$

$$= 1\cdot\frac{1}{3} = \frac{1}{3}$$

The series converges by Ratio Test, since $l = \frac{1}{3} < 1$.

IP2:

Test the convergence of the series

$$(\boldsymbol{i})\ \sum_{n=1}^{\infty} \boldsymbol{n!\,e^{-n}} \quad (\boldsymbol{ii})\ \sum_{n=1}^{\infty} \frac{\boldsymbol{n^n}}{\boldsymbol{n!}}$$

Solution:

(i) Let $a_n = n!\,e^{-n} = \frac{n!}{e^n}$. Then

$$l = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)!}{e^{n+1}} \cdot \frac{e^n}{n!} = \lim_{n\to\infty} \frac{n+1}{e} = \infty$$

The series diverges by Ratio Test, since l is infinite.

(ii) Let $a_n = \frac{n^n}{n!}$. Then

$$l = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)^{n+1}}{(n+1)!} \cdot \frac{n!}{n^n} = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^n$$

$$= \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e \qquad \left(\because \lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x\right)$$

The series diverges by Ratio Test, since $l = e > 1$.

IP3:

Test the convergence of the series $\sum_{n=1}^{\infty} \frac{3^n}{n^3 2^n}$.

Solution:

Let $a_n = \frac{3^n}{n^3 2^n}$. Then

$$l = \lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \left(\frac{3^n}{n^3 2^n}\right)^{\frac{1}{n}}$$

$$= \lim_{n\to\infty} \frac{3}{\left(n^{\frac{1}{n}}\right)^3 \cdot 2} = \frac{3}{2} \cdot \frac{1}{\lim_{n\to\infty} \left(n^{\frac{1}{n}}\right)^3} = \frac{3}{2} \qquad \left(\because \lim_{n\to\infty} n^{\frac{1}{n}} = 1\right)$$

The series diverges by the Root Test, since $l > 1$.

IP4:

Test the convergence of the series $\sum_{n=1}^{\infty} 3^{-n-(-1)^n}$.

Solution:

The given series is $\sum_{n=1}^{\infty} 3^{-n-(-1)^n}$.

Let $a_n = 3^{-n-(-1)^n} = \begin{cases} 3^{-n} \cdot 3^{-1}, & \text{if } n \text{ is even} \\ 3^{-n} \cdot 3 \quad , & \text{if } n \text{ is odd} \end{cases}$

Now, $\sqrt[n]{a_n} = \begin{cases} 3^{-1} \cdot 3^{-\frac{1}{n}} = \frac{1}{3} \cdot \frac{1}{3^{\frac{1}{n}}}, & \text{if } n \text{ is even} \\ 3^{-1} \cdot 3^{\frac{1}{n}} = \frac{1}{3} \cdot 3^{\frac{1}{n}} \quad , & \text{if } n \text{ is odd} \end{cases}$

Therefore, $l = \lim_{n\to\infty} \sqrt[n]{a_n} = \frac{1}{3}$ $\qquad \left(\because \lim_{n\to\infty} x^{\frac{1}{n}} = 1 \text{ if } x > 0\right)$

Thus, the series $\sum a_n$ converges by Root Test, since $l < 1$.

The Ratio and Root Tests

Exercises:

I. Which of the following series converge and which diverge.

(1) $\sum_{n=1}^{\infty} \frac{n^{\sqrt{2}}}{2^n}$ (2) $\sum_{n=1}^{\infty} n^2 e^{-n}$ (3) $\sum_{n=1}^{\infty} \frac{n!}{10^n}$

(4) $\sum_{n=1}^{\infty} \frac{n^{10}}{10^n}$ (5) $\sum_{n=1}^{\infty} \left(\frac{n-2}{n}\right)^n$ (6) $\sum_{n=1}^{\infty} \left(1-\frac{3}{n}\right)^n$

(7) $\sum_{n=1}^{\infty} \left(1-\frac{1}{3n}\right)^n$ (8) $\sum_{n=1}^{\infty} \frac{(\ln n)^n}{n^n}$ (9) $\sum_{n=1}^{\infty} \left(\frac{1}{n}-\frac{1}{n^2}\right)^n$

(10) $\sum_{n=1}^{\infty} \frac{n \ln n}{2^n}$ (11) $\sum_{n=1}^{\infty} \frac{(n+1)(n+2)}{n!}$ (12) $\sum_{n=1}^{\infty} e^{-n}\left(n^3\right)$

(13) $\sum_{n=1}^{\infty} \frac{n2^n(n+1)!}{3^n n!}$ (14) $\sum_{n=1}^{\infty} \frac{n!}{(2n+1)!}$ (15) $\sum_{n=1}^{\infty} \frac{n!}{n^n}$

(16) $\sum_{n=2}^{\infty} \frac{n}{(\ln n)^n}$ (17) $\sum_{n=2}^{\infty} \frac{n}{(\ln n)^{n/2}}$ (18) $\sum_{n=1}^{\infty} \frac{n! \ln n}{n(n+2)!}$

(19) $\sum_{n=1}^{\infty} \frac{(n!)^n}{\left(n^n\right)^2}$ (20) $\sum_{n=1}^{\infty} \frac{(n!)^n}{n^{\left(n^2\right)}}$ (21) $\sum_{n=1}^{\infty} \frac{n^n}{2^{\left(n^2\right)}}$

(22) $\sum_{n=1}^{\infty} \frac{n^n}{\left(2^n\right)^2}$ (23) $\sum_{n=1}^{\infty} \frac{1\cdot 3\cdots\cdot(2n-1)}{4^n 2^n n!}$ (24) $\sum_{n=1}^{\infty} \left(\frac{n+1}{2n+5}\right)^n$

(25) $\sum_{n=1}^{\infty} \frac{\left(1+\frac{1}{n}\right)^{2n}}{e^n}$ (26) $\sum_{n=1}^{\infty} \left(\sqrt[n]{n}-1\right)^n$

1.7

Alternating Series, Absolute and Conditional Convergence

Learning objectives:

- To state and prove the alternating series test for the convergence of alternate series.
- To discuss Absolute and Conditional convergence of series of real numbers.
- To discuss the rearrangement of absolute and conditionally convergent series.

  And
- To practice the related problems.

Alternating Series, Absolute and Conditional Convergence

The convergence tests so far studied apply only to series with positive or non-negative terms. In this module we learn how to deal with series whose terms are not necessarily positive or non-negative.

Alternating Series

A series in which the terms are alternately positive and negative is called an ***alternating series****.*

The following are alternating series:

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\cdots+\frac{(-1)^{n+1}}{n}+\cdots$$

$$-2+1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\cdots+\frac{(-1)^{n}4}{2^{n}}+\cdots$$

$$1-2+3-4+5-6+\cdots+(-1)^{n+1}n+\cdots$$

The following is a test for the convergence of the alternating series.

Theorem 1: The Alternating Series Test (Leibniz's Theorem)

The alternating series

$$\sum_{n=1}^{\infty}(-1)^{n+1}a_n = a_1 - a_2 + a_3 - a_4 + \cdots$$

converges if

$(\boldsymbol{i})$ $\boldsymbol{a_n} > 0$ ***for all*** $\boldsymbol{n} \in \boldsymbol{N}$

$(\boldsymbol{ii})$ $\boldsymbol{a_n} \geq \boldsymbol{a_{n+1}}$ ***for all n*** $\in$ ***N,***

$(\boldsymbol{iii})$ $\lim_{n\to\infty} \boldsymbol{a_n} = \boldsymbol{0}$.

Proof:

We prove that the sequence $\{s_n\}$ of n^{th} partial sums of $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ converges. Let n be an even natural number, say $n = 2m$, $m \in \boldsymbol{N}$. Then its n^{th} partial sum is

$$s_{2m} = (a_1 - a_2) + (a_3 - a_4) + \cdots + (a_{2m-1} - a_{2m})$$

Since $a_n \geq a_{n+1}$ for all $n \in \boldsymbol{N}$, s_{2m} is the sum of m non negative terms. Hence $s_{2m+2} \geq s_{2m}$ and the sequence $\{s_{2m}\}$ is non-decreasing. Further

$$s_{2m} = a_1 - (a_2 - a_3) - (a_4 - a_5) - \cdots - (a_{2m-2} - a_{2m-1}) - a_{2m}$$

shows that $s_{2m} \leq a_1$. Thus, $\{s_{2m}\}$ is a non-decreasing sequence and bounded above and it converges to its least upper bound, say l, $i.e.$,

$$\lim_{m\to\infty} s_{2m} = l$$

Let n be an odd natural number, say $n = 2m + 1$, $m \in \boldsymbol{N}$. Then $s_{2m+1} = s_{2m} + a_{2m+1}$. As $n \to \infty$, we have $a_n \to 0$ $\big(\text{by } (iii)\big)$, and so $\lim_{m\to\infty} a_{2m+1} = 0$.

Now, $\lim_{m\to\infty} s_{2m+1} = \lim_{m\to\infty} s_{2m} + \lim_{m\to\infty} a_{2m+1} = l$

The results $\lim_{m\to\infty} s_{2m} = l$, $\lim_{m\to\infty} s_{2m+1} = l$ together implies that $\lim_{n\to\infty} s_n = l$ (by Theorem 2 of Module 1.1). Thus, the given alternating series converges.

Hence the theorem

Note 1:

Since the convergence (and divergence) of a series is not altered by addition and deletion of finite terms, Theorem 1 remains true even if the condition (ii) is replaced by the following condition:

$(ii)'$ $\quad a_n \geq a_{n+1}$ for all $n \geq N$, for some natural number N.

Example 1:

The alternating harmonic series

$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$ converges.

Solution:

We have $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \sum_{n=1}^{\infty} (-1)^{n+1} a_n$ where $a_n = \frac{1}{n}$

Notice that (i) $a_n > 0 \quad \forall\, n$

(ii) $a_n \geq a_{n+1} \quad \forall\, n$ and

(iii) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0$

Therefore, by Leibnitz Theorem, the given alternating harmonic series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ is convergent.

Absolute convergence:

*A series $\sum a_n$ converges **absolutely** (is absolutely convergent), if the corresponding series of absolute values, $\sum |a_n|$ converges.*

Example 2:

The Geometric series $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots$ is absolutely convergent, since the corresponding series of absolute values

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$$

is convergent.

Example 3:

The alternating harmonic series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ is not absolutely convergent, since the corresponding series of absolute values

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

is the divergent harmonic series.

The following result proves that every absolutely convergent series is convergent.

Theorem 2: The Absolute Convergence Test

If $\sum_{n=1}^{\infty} a_n$ ***converges absolutely then*** $\sum_{n=1}^{\infty} a_n$ ***converges.***

(i.e., Absolute convergence $\Longrightarrow$ ***Convergence)***

(i.e., Every absolute convergent series converges)

Proof:

We have for each n, $-|a_n| \leq a_n \leq |a_n|$

Therefore, $0 \leq a_n + |a_n| \leq 2|a_n|$

Given $\sum_{n=1}^{\infty} a_n$ converges absolutely, i.e., $\sum_{n=1}^{\infty} |a_n|$ converges.

Now, $\sum_{n=1}^{\infty} 2a_n$ converges and so the non-negative series

$\sum_{n=1}^{\infty} \left(a_n + |a_n|\right)$ converges by Comparison Test. Further, by Difference Rule for convergent series $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \left(a_n + |a_n| - |a_n|\right)$ also converges. Hence the theorem

Note 2:

The converse of the above theorem is false.

For example, the alternating harmonic series converges but is not absolutely convergent.

Example 4: Applying the Absolute Convergent Test

Show that the series $(i)\ \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3}$ and $(ii)\ \sum_{n=1}^{\infty} \frac{\sin n}{n^2}$ are convergent.

Solution:

$(i)\ \sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{n^3} \right| = \sum_{n=1}^{\infty} \frac{1}{n^3}$ is convergent (by p-series Test).

Therefore, the series (i) is absolutely convergent and so (i) is convergent by the Absolute Convergence Test.

$(ii)\ \sum_{n=1}^{\infty} \left| \frac{\sin n}{n^2} \right| = \sum_{n=1}^{\infty} \frac{|\sin n|}{n^2}$

Notice that $\frac{|\sin n|}{n^2} \leq \frac{1}{n^2}$, since $|\sin n| \leq 1$ for all $n \in \boldsymbol{N}$

Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent (by p-series Test), $\sum_{n=1}^{\infty} \frac{|\sin n|}{n^2}$ is convergent by Comparison Test. Therefore, (ii) is absolutely convergent and so (ii) is convergent by the Absolute Convergence Test.

Conditional Convergence:

A series converges ***conditionally*** *if it converges but does not converge absolutely.*

In other words,

*A series $\sum a_n$ is **conditionally convergent** if $\sum_{n=1}^{\infty} a_n$ is convergent and $\sum_{n=1}^{\infty} |a_n|$ is divergent.*

For example the alternating harmonic series converges conditionally.

Theorem 3: Alternating p-series

The Alternating p-series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^p} = 1 - \frac{1}{2^p} + \frac{1}{3^p} - \frac{1}{4^p} + \dots$$

(i) converges if $p > 0$

(ii) converges absolutely if $p > 1$

(iii) converges conditionally if $0 < p \leq 1$

Proof:

Let $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^p} = \sum_{n=1}^{\infty} (-1)^{n+1} a_n$ where $a_n = \frac{1}{n^p}$

Let $p > 0$. Notice that $a_n > 0 \ \forall\, n$, $a_n \geq a_{n+1} \ \forall n$ and $\lim_{n\to\infty} a_n = 0$. Therefore, by Leibnitz Theorem the alternating p-series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^p}$ converges if $p > 0$.

Notice that $\sum_{n=1}^{\infty}\left|\frac{(-1)^{n+1}}{n^p}\right| = \sum_{n=1}^{\infty}\frac{1}{n^p}$ converges if $p > 1$ (by p-series Test) and diverges if $p \leq 1$. Therefore, the alternating p-series converges absolutely if $p > 1$.

Further, the alternating p-series converges and does not converge absolutely if $0 < p \leq 1$. Therefore, the alternating p-series converges conditionally if $0 < p \leq 1$.

Example 5:

(i) The series $1-\frac{1}{2^{3/2}}+\frac{1}{3^{3/2}}-\frac{1}{4^{3/2}}+\ldots=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^{3/2}}$ converges absolutely, since it is an alternating $p-$ series for $p = \frac{3}{2} > 1$. It is a convergent series (since absolute convergence $\Longrightarrow$ convergence).

(ii) The series $1-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{4}}+\ldots=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{\sqrt{n}}$ converges conditionally, since it is an alternating $p-$ series for $p = \frac{1}{2}$, $0 < p = \frac{1}{2} \leq 1$.

Any of the tests used for series with positive terms or non negative terms can be used to test the absolute convergence. The Ratio and Root Tests can also be used to test the convergence/divergence of series whose terms are not necessarily positive or non-negative.

Theorem 4: The Ratio Test

Let $\sum a_n$ be a series of real numbers with $a_n \neq 0, \forall n$ and suppose that $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = l$

Then

(i) ***The series converges absolutely (and therefore converges) if $l < 1$***
(ii) ***The series diverges if $l > 1$ or l is infinite***
(iii) ***The Test is inconclusive if $l = 1$***

Theorem 5: The Root Test

Let $\sum a_n$ be a series of real numbers and suppose that $\lim_{n\to\infty} \sqrt[n]{|a_n|} = l$.Then

(i) ***The series converges absolutely (and therefore converges) if $l < 1$***
(ii) ***The series diverges if $l > 1$ or l is infinite***
(iii) ***The Test is inconclusive if $l = 1$***

Example 6:

Test the series $\sum_{n=1}^{\infty} (-1)^n \frac{n^3}{3^n}$ for absolutely convergence

Solution:

We use the Ratio Test with $a_n = (-1)^n \frac{n^3}{3^n}$. Then

$$l = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(-1)^{n+1}\ \frac{(n+1)^3}{3^{n+1}}}{(-1)^n\ \frac{n^3}{3^n}}\right| = \frac{1}{3}\lim_{n\to\infty}\left(\frac{n+1}{n}\right)^3 = \frac{1}{3}$$

The given series $\sum_{n=1}^{\infty}(-1)^n \frac{n^3}{3^n}$ converges absolutely and therefore, converges by Ratio Test, since $l < 1$.

Example 7:

Test convergence and divergence of $\sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{n}{10}\right)^n$

Solution:

We use the Ratio Test with $a_n = (-1)^{n+1}\left(\frac{n}{10}\right)^n$. Then

$$l = \lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left|\left(\frac{n}{10}\right)^n\right|} = \lim_{n\to\infty}\frac{n}{10} = \infty$$

By the Root Test, the given series diverges, since l is infinite.

Rearranging Series:

*If $\sum_{n=1}^{\infty} a_n$ is a series and $b_1, b_2, b_3, \ldots b_n, \ldots$ is a rearrangement of the sequence $\{a_n\}$ then $\sum_{n=1}^{\infty} b_n$ is called a **rearranged series** of the given series $\sum_{n=1}^{\infty} a_n$.*

The following theorem asserts that any rearranged series of a given absolute convergent series is absolutely convergent and has the same sum as the given series.

Theorem 6: The Rearrangement Theorem for absolutely convergent series

If $\sum_{n=1}^{\infty} a_n$ converges absolutely and $\sum_{n=1}^{\infty} b_n$ is any rearranged series of $\sum_{n=1}^{\infty} a_n$ then $\sum_{n=1}^{\infty} b_n$ converges absolutely and $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} b_n$.

Example 8: Applying the Rearrangement Theorem

The series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \ldots + \frac{(-1)^{n+1}}{n^2} + \ldots$ converges absolutely (by Alternating p-series Test because $p = 2 > 1$).

A possible rearrangement of the terms of the series might start with a positive term, then two negative terms, then three positive terms, then four negative terms and so on. That is, after k terms of one sign, take $k+1$ terms of the opposite sign. This arranged series is

$$1 - \frac{1}{4} - \frac{1}{16} + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} - \frac{1}{36} - \frac{1}{64} - \frac{1}{100} - \frac{1}{144} + \cdots$$

By the Rearrangement Theorem for absolutely convergent series, this rearranged series is also absolutely convergent and converges to the same sum as the given series.

Note 3:
If we rearrange infinitely many terms of a conditionally convergent series, then we can get results that are far different from the sum of the original series.

To illustrate this fact let's consider the alternating harmonic series. It is conditional convergent and converges to $ln\,2$. Now,

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\frac{1}{8}+\cdots=ln\,2 \qquad \text{------ (1)}$$

Multiplying each term by $\frac{1}{2}$ we get

$$\frac{1}{2}-\frac{1}{4}+\frac{1}{6}-\frac{1}{8}+\cdots=\frac{1}{2}ln\,2$$

Inserting zeros between the terms of the series, we get

$$0+\frac{1}{2}+0-\frac{1}{4}+0+\frac{1}{6}+0-\frac{1}{8}+\cdots=\frac{1}{2}ln\,2 \qquad \text{------ (2)}$$

Adding the corresponding terms of (1) and (2) we get

$$1+0+\frac{1}{3}-\frac{1}{2}+\frac{1}{5}+0+\frac{1}{7}-\frac{1}{4}+\cdots=\frac{3}{2}ln\,2$$

$i.e.,$ $$1+\frac{1}{3}-\frac{1}{2}+\frac{1}{5}+\frac{1}{7}-\frac{1}{4}+\cdots=\frac{3}{2}ln\,2 \qquad \text{------ (3)}$$

Notice that the series (3) contains the same terms as in (1), but rearranged so that one negative term occur after each pair of positive terms. The sums of these series (1) and (3), however, are different.

Rearranging Alternating Harmonic Series

The conditionally convergent alternating harmonic series

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\frac{1}{8}+\frac{1}{9}-\frac{1}{10}+\frac{1}{11}-\ldots$$

can be rearranged to diverge or to reach any pre assigned sum.

Note 4:

Riemann proved the following:

Theorem 7:

If $\sum a_n$ is conditionally convergent series and r is any real number whatsoever, then there is a rearrangement of $\sum a_n$ that has the sum equal to r.

Note 5:

We must always add the terms of a conditionally convergent series in the given order.

P1:

Which of the alternating series converge and which diverge?

$$(\boldsymbol{a})\ \sum_{n=2}^{\infty}\frac{(-1)^{n+1}}{\ln n} \qquad (\boldsymbol{b})\ \sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sqrt{n}+1}{n+1}$$

Solution:

(a) We have $\sum_{n=2}^{\infty}\frac{(-1)^{n+1}}{\ln n}=\sum_{n=2}^{\infty}(-1)^{n+1}a_n$, where $a_n=\frac{1}{\ln n}$

It is an alternating series and $a_n>0$, for $n\geq 2$. Let $f(x)=\frac{1}{\ln x}$.

Note that $\ln x$ is an increasing function of x.

$\Longrightarrow \frac{1}{\ln x}$ is a decreasing function of x. Therefore, $a_n\geq a_{n+1}$ for $n\geq 2$ and $\lim_{n\to\infty}a_n=0$. By Leibniz's Theorem the given series converges.

(b) We have $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sqrt{n}+1}{n+1}=\sum_{n=1}^{\infty}(-1)^{n+1}a_n$, where $a_n=\frac{\sqrt{n}+1}{n+1}$

It is an alternating series and $a_n>0$, for all n.

To know its increasing, decreasing nature we use calculus:

Let $f(x)=\frac{\sqrt{x}+1}{x+1}$. Then $f'(x)=\frac{1-x-2\sqrt{x}}{2\sqrt{x}(x+1)^2}<0$ for $x\geq 1$.

$\Longrightarrow f(x)$ is decreasing for $x\geq 1$

$\Longrightarrow a_n=f(n)\geq f(n+1)=a_{n+1}$ for $n\geq 1$ and $\lim_{n\to\infty}a_n=0$.

Thus, by Leibniz's Theorem the given alternating series converges.

P2:

Does the series $\displaystyle\sum_{n=2}^{\infty}(-1)^{n+1}\frac{1}{n\ln n}$ **converge absolutely or conditionally?**

Solution:

We have $\displaystyle\sum_{n=2}^{\infty}(-1)^{n+1}\frac{1}{n\ln n}=\sum_{n=2}^{\infty}(-1)^{n+1}a_n$, where $a_n=\frac{1}{n\ln n}$.

It is an alternating series and $a_n>0$ for all $n\geq 2$.

Let $f(x)=\frac{1}{x\ln x}\implies f'(x)=-\frac{\ln x+1}{(x\ln x)^2}<0$, for all $x\geq 2$.

Therefore, $f(x)$ is decreasing for $x\geq 2$.

$\implies a_n=f(n)>f(n+1)=a_{n+1}$ for all $n\geq 2$ and $\displaystyle\lim_{n\to\infty}a_n=0$. By Leibniz's Theorem the given alternating series converges.

Now, $\displaystyle\sum_{n=2}^{\infty}\left|(-1)^{n+1}\frac{1}{n\ln n}\right|=\sum_{n=2}^{\infty}\frac{1}{n\ln n}$.

We try Integral Test for convergence or divergence.

$$\int_2^{\infty}f(x)dx=\lim_{b\to\infty}\int_2^b\frac{1}{x\ln x}\,dx=\lim_{b\to\infty}\int_2^b\frac{\left(\frac{1}{x}\right)}{\ln x}\,dx$$
$$=\lim_{b\to\infty}\left[\ln(\ln b)-\ln(\ln 2)\right]=\infty$$

$\implies$ By Integral Test $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n\ln n}$ diverges. $i.e.$, $\displaystyle\sum_{n=2}^{\infty}\left|(-1)^{n+1}\frac{1}{n\ln n}\right|$ diverges. Thus, the given series is conditionally convergent.

P3:

Show that the series $\sum_{n=1}^{\infty} \frac{(-2)^{n+1}}{n+5^n}$ absolutely convergent.

Solution:

We have $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} \cdot 2^{n+1}}{n+5^n} = \sum_{n=1}^{\infty} (-1)^{n+1} a_n$, where $a_n = \frac{2^{n+1}}{n+5^n}$.

It is an alternating series and $a_n > 0$ for all n.

Now, $\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1} \cdot 2^{n+1}}{n+5^n} \right| = \sum_{n=1}^{\infty} \frac{2^{n+1}}{n+5^n} = \sum_{n=1}^{\infty} a_n$.

Notice that $\frac{2 \cdot 2^n}{n+5^n} < 2\left(\frac{2}{5}\right)^n$ for all n. $i.e.$, the n^{th} term of the series $\sum_{n=1}^{\infty} \frac{2^{n+1}}{n+5^n}$ is less than the n^{th} term of the series $2\sum_{n=1}^{\infty} \left(\frac{2}{5}\right)^n$.

Since $2\sum_{n=1}^{\infty} \left(\frac{2}{5}\right)^n$ is convergent, the series $\sum_{n=1}^{\infty} \frac{2^{n+1}}{n+5^n}$ is also convergent by Comparison Test. Therefore, $\sum_{n=1}^{\infty} a_n$ is convergent.

Thus, the given series is absolutely convergent.

P4:

Show that the series $\sum\limits_{n=1}^{\infty} \frac{(-1)^n (n!)^2 3^n}{(2n+1)!}$ **is absolutely convergent.**

Solution:

We have $\sum\limits_{n=1}^{\infty} \frac{(-1)^n (n!)^2 3^n}{(2n+1)!} = \sum\limits_{n=1}^{\infty} a_n$, where $a_n = \frac{(-1)^n (n!)^2 3^n}{(2n+1)!}$.

$$l = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{\left((n+1)!\right)^2 3^{n+1}}{(2n+3)!} \cdot \frac{(2n+1)!}{(n!)^2 3^n}\right|$$

$$= \lim_{n\to\infty} \frac{3(n+1)^2}{(2n+3)(2n+2)} = 3 \lim_{n\to\infty} \frac{\left(1+\frac{1}{n}\right)^2}{\left(2+\frac{3}{n}\right)\left(2+\frac{2}{n}\right)} = \frac{3}{4} < 1$$

Therefore, the given series is absolutely convergent by Ratio Test, since $l < 1$.

IP1:

Which of the alternating series converge absolutely and which converge conditionally.

$(\boldsymbol{a})\ \sum_{n=1}^{\infty} \frac{\cos n\pi}{n^3}\quad (\boldsymbol{b})\ \sum_{n=1}^{\infty} \frac{\cos n\pi}{n^{3/4}}$

Solution:

(a) We have $\sum_{n=1}^{\infty} \frac{\cos n\pi}{n^3} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3} = (-1)\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3}$.

$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3}$ is an alternating p-series for $p = 3 > 1$. Therefore, it is absolutely convergent series. Thus, the series $\sum_{n=1}^{\infty} \frac{\cos n\pi}{n^3}$ is absolutely convergent.

(b) We have $\sum_{n=1}^{\infty} \frac{\cos n\pi}{n^{3/4}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^{3/4}} = (-1)\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^{3/4}}$.

$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^{3/4}}$ is an alternating p-series for $0 < p = \frac{3}{4} < 1$.

Therefore, it is conditionally convergent series. Thus, the series $\sum_{n=1}^{\infty} \frac{\cos n\pi}{n^{3/4}}$ is conditionally convergent.

IP2:

Show that the series $\sum_{n=1}^{\infty}(-1)^n \frac{\tan^{-1} n}{1+n^2}$ is absolutely convergent.

Solution:

We have $\sum_{n=1}^{\infty}(-1)^n \frac{\tan^{-1} n}{1+n^2} = \sum_{n=1}^{\infty}(-1)^n a_n$, where $a_n = \frac{\tan^{-1} n}{1+n^2}$.

It is an alternating series and $a_n > 0$, for all n.

Now, $\sum_{n=1}^{\infty}\left|(-1)^n \frac{\tan^{-1} n}{1+n^2}\right| = \sum_{n=1}^{\infty} \frac{\tan^{-1} n}{1+n^2}$.

Let $f(x) = \frac{\tan^{-1} x}{1+x^2}$. Notice that $f(x)$ is positive, continuous and decreasing for $x \geq 1$ and

$$\int_1^{\infty} f(x)dx = \int_1^{\infty} \frac{tan^{-1} x}{1+x^2}\, dx = \lim_{b\to\infty} \int_1^{b} \frac{tan^{-1} x}{1+x^2}\, dx$$

$$= \lim_{b\to\infty}\left[\frac{\left(tan^{-1} x\right)^2}{2}\right]_1^b = \frac{1}{2}\lim_{b\to\infty}\left[\left(tan^{-1} b\right)^2 - \left(tan^{-1} 1\right)^2\right]$$

$$= \frac{1}{2}\left[\left(\frac{\pi}{2}\right)^2 - \left(\frac{\pi}{4}\right)^2\right] = \frac{3\pi^2}{32}$$

Thus, $\int_1^{\infty} f(x)dx$ converges. By integral test $\sum_{n=1}^{\infty} f(n) = \sum_{n=1}^{\infty} \frac{tan^{-1} n}{1+n^2}$ also converges. Thus the given series is absolutely convergent.

IP3:

Show that the series $\sum_{n=1}^{\infty}(-1)^n \operatorname{csch} n$ is absolutely convergent.

Solution:

We have $\sum_{n=1}^{\infty}(-1)^n \operatorname{csch} n = \sum_{n=1}^{\infty}(-1)^n a_n$, where $a_n = \operatorname{csch} n$.

It is an alternating series and $a_n > 0$ for all n.

Now $\sum_{n=1}^{\infty}\left|(-1)^n \operatorname{csch} n\right| = \sum_{n=1}^{\infty} a_n$, where $a_n = \frac{2}{e^n - e^{-n}}$.

Notice that for large values of n, a_n behaves like $\frac{1}{e^n}$. Now, take

$b_n = \frac{1}{e^n}$, $n \in \boldsymbol{N}$ and consider the series $\sum_{n=1}^{\infty} b_n$. Observe that

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{2e^n}{e^n - e^{-n}} = 2\lim_{n\to\infty}\frac{1}{1-\left(\frac{1}{e^{2n}}\right)} = 2 > 0$$

By part(i) of Limit Comparison Test $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converge or both diverge. Clearly $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty}\frac{1}{e^n}$ is convergent (as it is a geometric series with $r = \frac{1}{e} < 1$). Therefore, $\sum_{n=1}^{\infty} a_n$ is also convergent. Thus, the given series is absolutely convergent.

IP4:

Show that the series $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^n (n+1)^n}{(2n)^n}$ **is absolutely convergent.**

Solution:

We have $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^n (n+1)^n}{(2n)^n} = \sum\limits_{n=1}^{\infty} a_n$ where $a_n = \frac{(-1)^n (n+1)^n}{(2n)^n}$.

$$l = \lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \frac{n+1}{2n} = \frac{1}{2} < 1$$

Therefore, the given series is absolutely convergent by Root Test, since $l < 1$.

1.7 Alternating Series, Absolute and Conditional Convergence

Exercises:

1. Which of the following Alternating series is converges and which is diverges:

$(a)\ \sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{n}{10}\right)^{n}$ $\qquad (b)\ \sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{10^{n}}{n^{10}}\right)$

$(c)\ \sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{\ln n}{n}\right)$ $\qquad (d)\ \sum_{n=2}^{\infty}(-1)^{n+1}\left(\frac{\ln n}{\ln n^{2}}\right)$

$(e)\ \sum_{n=1}^{\infty}(-1)^{n+1}\frac{3\sqrt{n}+1}{\sqrt{n}+1}$

2. Which of the following series converge absolutely and which converge conditionally and which diverge:

$(a)\ \sum_{n=1}^{\infty}(-1)^{n+1}\frac{(0.1)^{n}}{n}$ $\qquad (b)\ \sum_{n=1}^{\infty}\frac{(-1)^{n}}{1+\sqrt{n}}$

$(c)\ \sum_{n=1}^{\infty}(-1)^{n+1}\frac{n}{n^{3}+1}$ $\qquad (d)\ \sum_{n=1}^{\infty}(-1)^{n+1}\frac{n!}{2^{n}}$

$(e)\ \sum_{n=1}^{\infty}(-1)^{n}\frac{\sin n}{n^{2}}$ $\qquad (f)\ \sum_{n=1}^{\infty}(-1)^{n+1}\frac{3+n}{5+n}$

$(g)\ \sum_{n=2}^{\infty}(-1)^{n}\frac{1}{\ln\left(n^{3}\right)}$ $\qquad (h)\ \sum_{n=1}^{\infty}(-1)^{n}\frac{1+n}{n^{2}}$

$(i)\ \sum_{n=1}^{\infty}(-1)^{n}n^{2}\left(\frac{2}{3}\right)^{n}$ $\qquad (j)\ \sum_{n=1}^{\infty}(-1)^{n+1}\sqrt[n]{10}$

$(k)\ \sum_{n=1}^{\infty} \frac{\cos n\pi}{n\sqrt{n}}$ $(l)\ \sum_{n=1}^{\infty} (-1)^n \frac{n}{n+1}$

$(m)\ \sum_{n=1}^{\infty} (-1)^n \frac{\ln n}{n-\ln n}$ $(n)\ \sum_{n=1}^{\infty} \frac{(-100)^n}{n!}$

$(o)\ \sum_{n=1}^{\infty} (-5)^{-n}$ $(p)\ \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2+2n+1}$

$(q)\ \sum_{n=2}^{\infty} (-1)^n \left(\frac{\ln n}{\ln(n^2)}\right)^n$ $(r)\ \sum_{n=1}^{\infty} (-1)^n \frac{(n+1)^2}{(2n)!}$

$(s)\ \sum_{n=1}^{\infty} (-1)^n \frac{(2n)!}{(2^n n!)n}$ $(t)\ \sum_{n=1}^{\infty} (-1)^n \left(\sqrt{n+\sqrt{n}}-\sqrt{n}\right)$

$(u)\ \sum_{n=1}^{\infty} (-1)^n \operatorname{sech} n$

1.8. Power Series

Learning objectives:

- To define the Power series and to prove the convergence theorem for Power series.
- To find the Interval of convergence and radius of convergence of a Power series.
- To state the Term by Term differentiation and integration theorem for Power series.

  AND
- To practice the related problems.

Power Series

Let a be given real number and x be a real variable. A ***power series in $x-a$*** *or a* ***power series centered at a*** *or a* ***power series about a*** *is a series of the form*

$$\sum_{n=0}^{\infty} a_n(x-a)^n = a_0 + a_1(x-a) + a_2(x-a)^2 + \ldots + a_n(x-a)^n + \ldots (1)$$

where a_n's are constants called ***coefficients*** *of the series.*

A special case

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots$$

is a power series about $x = 0$ by taking $a = 0$ in (1).

Example 1: A Geometric Series

The series

$$1 - \frac{1}{2}(x-2) + \frac{1}{4}(x-2)^2 + \cdots + \left(-\frac{1}{2}\right)^n (x-2)^n + \cdots$$

is a power series $\sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^n (x-2)^n$ about $x = 2$ and it is a geometric series with first term $a = 1$ and common ratio $r = -\frac{x-2}{2}$. The series converges to the sum $\frac{a}{1-r} = \frac{1}{1+\frac{x-2}{2}} = \frac{2}{x}$ if $|r| < 1$, i.e., $\left|\frac{x-2}{2}\right| < 1$ or $0 < x < 4$.

Hence $\sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)^{n}(x-2)^{n}=\frac{2}{x}, \qquad 0<x<4$

Example 2: Testing the Convergence Using the Ratio Test

For what values of x do the following power series converge?

(i) $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^{n}}{n}=x-\frac{x^{2}}{2}+\frac{x^{3}}{3}-\ldots$

(ii) $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^{2n-1}}{2n-1}=x-\frac{x^{3}}{3}+\frac{x^{5}}{5}-\ldots$

(iii) $\sum_{n=0}^{\infty}\frac{x^{n}}{n!}=1+x+\frac{x^{2}}{2!}+\frac{x^{3}}{3!}+\ldots$

(iv) $\sum_{n=0}^{\infty}n!x^{n}=1+x+2!x^{2}+3!x^{3}+\ldots$

Solution:

We apply the Ratio Test to the series $\sum a_n$ where a_n is the n^{th} term of the series under discussion.

$$(i)\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^n x^{n+1}}{n+1}\cdot\frac{n}{(-1)^{n-1}x^n}\right| = \lim_{n\to\infty}\frac{n}{n+1}|x| = |x|$$

By Ratio Test, the series converges absolutely for $|x|<1$ and it diverges for $|x|>1$.

When $x=1$, we get $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}$ the alternating harmonic series which is convergent.

When $x = -1$, we get $-1-\frac{1}{2}-\frac{1}{3}-\frac{1}{4}-\ldots=-\sum_{n=1}^{\infty}\frac{1}{n}$ the negative of the harmonic series, which is divergent.

Thus, the series (i)converges for $-1 < x \leq 1$ and diverges elsewhere.

$$(ii)\ \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^n x^{2n+1}}{2n+1}\cdot\frac{2n-1}{(-1)^{n-1}x^{2n-1}}\right|$$

$$= \lim_{n\to\infty}\frac{2n-1}{2n+1}x^2 = x^2$$

By Ratio Test, the series converges absolutely for $|x^2| < 1$, i.e., $|x| < 1$ and it diverges for $|x^2| > 1$, i.e., $|x| > 1$.

We need to test at $x^2 = 1$, i.e., $x = \pm 1$.

When $x = 1$, we get the alternating series

$$1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2n-1}$$

and it converges by Leibniz's Theorem.

When $x = -1$, we get the alternating series

$$-1+\frac{1}{3}-\frac{1}{5}+\frac{1}{7}-\ldots=\sum_{n=1}^{\infty}\frac{(-1)^{n}}{2n-1}$$

and it converges by Leibniz's Theorem.

Thus, the series (ii) converges for $-1 \leq x \leq 1$ and diverges elsewhere.

$$(iii)\ \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+1)!}\cdot\frac{n!}{x^n}\right|$$

$$= \lim_{n\to\infty}\frac{|x|}{n+1} = 0 \text{ for every } x$$

The series converges absolutely by Ratio Test for all $x \in \boldsymbol{R}$, i.e., the series converges everywhere on $\boldsymbol{R}$.

$$(iv)\ \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!\,x^{n+1}}{n!\,x^n}\right|$$

$$= \lim_{n\to\infty}(n+1)|x| = \infty\text{ , unless } x = 0$$

The series diverges by Ratio Test for all $x \in \boldsymbol{R}$, except $x = 0$.

Theorem 1: The Convergence Theorem for Power Series

If the power series $\sum_{n=0}^{\infty} a_n x^n$ converges for $x = c \neq 0$, then the series converges absolutely for all x with $|x| < |c|$. If the series diverges for $x = d$, then it diverges for all x with

$|x| > |d|$.

Proof:

(i) Suppose the series $\sum_{n=0}^{\infty} a_n x^n$ converges for $x = c \neq 0$, i.e.,

$\sum_{n=0}^{\infty} a_n c^n$ converges. Then $\lim_{n\to\infty} a_n c^n = 0$. Therefore, there exists a natural number N such that $|a_n c^n - 0| < 1$ for all $n \geq N$, i.e., $|a_n c^n| < 1$ for all $n \geq N$, i.e., $|a_n| < \frac{1}{|c|^n}$ for all $n \geq N$.

Let x be any real number such that $|x| < |c|$. Now, consider the series $\sum_{n=0}^{\infty} \left|a_n x^n\right|$. Notice that $|a_n x^n| < \left|\frac{x}{c}\right|^n$ for all $n \geq N$ and $\sum_{n=0}^{\infty} \left|\frac{x}{c}\right|^n$ is a convergent geometric series$\left(\text{since } \left|\frac{x}{c}\right| < 1\right)$. Thus, by Comparison Test $\sum_{n=0}^{\infty} \left|a_n x^n\right|$ is convergent and so $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely for all x with $|x| < |c|$.

(ii) Given that the series $\sum_{n=0}^{\infty} a_n x^n$ diverges at $x = d$.

Assume that the series converges at a value x_0 with $|x_0| > |d|$. Therefore, by the first part of the theorem, the series converges for all x with $|x| < |x_0|$. Since $|d| < |x_0|$, the series converges absolutely at d. This contradicts the hypothesis that the series diverges at d. Thus, the series diverges for all x with $|x| > |d|$ whenever it diverges at d.

Hence the theorem

For simplicity, we dealt the convergence of the series of the form $\sum_{n=0}^{\infty} a_n x^n$. For the series of the form $\sum a_n (x-a)^n$ we can replace $x-a$ by y and apply the results to the series $\sum a_n y^n$

The Radius of convergence of a Power Series

From the above examples and the convergence theorem for power series; we draw the following conclusions regarding the convergence of a power series $\sum a_n (x-a)^n$.

A power series $\sum a_n (x-a)^n$ behaves in one of the three possible ways. It might converge only at $x = a$ or converge everywhere or converge on some interval of radius R centered at $x = a$.

Corollary to Theorem 1:

The convergence of the series $\sum a_n (x-a)^n$ is described by one of the following three possibilities:

1. *There is a positive number R such that the series diverges for all x with $|x-a| > R$ but converges for all x with $|x-a| < R$. The series may or may not converge at either of the end points $x = a - R$ and $x = a + R$.*
2. *The series converges absolutely for every x $(R = \infty)$*
3. *The series converges at $x = a$ and diverges elsewhere $(R = 0)$*

*R is called the **Radius of convergence** of the power series and the interval of radius R centered at $x = a$ is called the **interval of convergence**.*

The interval of convergence may be open, closed or half-open depending on the given series.

At the points x with $|x - a| < R$, the series converges absolutely. If the series converges for all values of x, then we say that its radius of convergence is infinite. If it converges only at $x = a$, then we say that its radius of convergence is zero.

How to Test a Power Series for Convergence

1. *Use Ratio Test or Root Test to find the interval where the series converges absolutely. Normally, this is an open interval*
$$|x - a| < R \;\; or \;\; a - R < x < a + R$$
2. *If the radius of absolute convergence is finite, then test the convergence and divergence at end points (Use a Comparison Test, the Integral Test or the Alternating Series Test).*
3. *If the interval of absolute convergence is $a - R < x < a + R$, the series diverges for $|x - a| > R$ (it does not even converge conditionally) since the n^{th} term does not approach zero for those values of x.*

Term-by-Term Differentiation:

A theorem from advanced calculus says that a power series can be differentiated term by term at each interior point of its interval of convergence.

Theorem2: The Term-by-Term Differentiation Theorem

If $\sum a_n(x-a)^n$ converges for $a-R<x<a+R$ for some $R>0$, then it defines a function f such that

$$f(x)=\sum_{n=0}^{\infty} a_n(x-a)^n \quad , a-R<x<a+R$$

Such a function f has derivatives of all orders inside the interval of convergence. We can obtain the derivatives by differentiating the original series term by term:

$$f'(x)=\sum_{n=1}^{\infty} n a_n(x-a)^{n-1}, \quad f''(x)=\sum_{n=1}^{\infty} n(n-1)a_n(x-a)^{n-2}, \ldots$$

Each of these derived series converges at every interior point of the interval of convergence of the original series.

Example 3: Applying Term-by-Term Differentiation

Consider the series

$$\sum_{n=0}^{\infty} x^n = 1+x+x^2+x^3+\ldots+x^n+\ldots=\sum_{n=0}^{\infty} a_n \ \ where \ a_n = x^n$$

Now, $l = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = |x|$

By Ratio Test, the series converges if $|x| < 1$, diverges if $|x| > 1$ and the test is inconclusive if $|x| = 1$. Notice that the series diverges for $x = \pm 1$.

The interval of convergence is $-1 < x < 1$ and $R = 1$ is the radius of convergence. Further, its sum is $\frac{1}{1-x}$ if $|x| < 1$.

Therefore, the given series defines a function f, where

$$f(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad ,-1 < x < 1$$

By Term by Term Differentiation Theorem, f has derivatives of all orders inside the interval of convergence,

$$f'(x) = \frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n x^{n-1} \quad , -1 < x < 1$$

$$f''(x) = \frac{2}{(1-x)^3} = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} \quad , -1 < x < 1$$

Remark: (1) The Term by Term differentiation Theorem is valid for power series only and it is not valid for other kinds of series.

(2) From the Term by Term Differentiation Theorem, it is clear that within its interval of convergence the sum of a power series is a continuous function with derivatives of all orders.

Term-by-Term Integration:

Another theorem from advanced calculus says that a power series can be integrated term by term throughout its interval of convergence.

Theorem 3: The Term-by-Term Integration Theorem

If $\sum \boldsymbol{a_n(x-a)^n}$ ***converges for*** $\boldsymbol{a-R} < x < a+R$, ***for some*** $\boldsymbol{R} > 0$ ***and defines a function*** $\boldsymbol{f}$***:***

$$f(x)=\sum_{n=0}^{\infty} a_n(x-a)^n \quad , a-R<x<a+R, \quad \textbf{\textit{then}}$$

$\displaystyle\sum_{n=0}^{\infty} a_n \frac{(x-a)^{n+1}}{n+1}$ converges for $a-R<x<a+R$***, and***

$$\int f(x)\,dx=\sum_{n=0}^{\infty} a_n \frac{(x-a)^{n+1}}{n+1}+C \quad \textbf{\textit{for}}\ \boldsymbol{a-R}<x<a+R.$$

Example 4: A series for $\tan^{-1} x$, $-1 \le x \le 1$

Identify the function $f(x)=x-\frac{x^3}{3}+\frac{x^5}{5}-\cdots$, $-1 \le x \le 1$

Solution:

We have already seen that the given series

$$x-\frac{x^3}{3}+\frac{x^5}{5}-\ldots\ldots=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^{2n-1}}{2n-1}$$

converges for $-1 \le x \le 1$ and diverges elsewhere. By the Term by Term Differentiation Theorem

$$f(x)=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^{2n-1}}{2n-1} \quad , -1<x<1$$

can be differentiated term by term and

$$f'(x)=\sum_{n=1}^{\infty}(-1)^{n-1}x^{2n-2} \quad , -1<x<1$$

This is a geometric series with first term 1 and common ratio $-x^2$ and so,

$$f'(x) = \frac{1}{1-(-x^2)} = \frac{1}{1+x^2}$$

Now, integrate and we get

$$f(x) = \int \frac{dx}{1+x^2} = \tan^{-1} x + C$$

$$i.e., \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^{2n-1}}{2n-1}=\tan^{-1}x+C$$

Putting $x = 0$, we get $C = 0$. Thus,

$$f(x)=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^{2n-1}}{2n-1}=\tan^{-1}x \; , -1<x<1$$

Note (1): The above is also valid for $-1 \le x \le 1$, i.e., $|x| \le 1$.

Therefore, $f(x)=\sum_{n=1}^{\infty}(-1)^{n-1}\dfrac{x^{2n-1}}{2n-1}=\tan^{-1}x$, $-1\le x\le 1$

Note (2): Note that the original series in the above example converges at both end points of the original interval of convergence, but the term by term differentiation theorem can guarantee the convergence of the differentiated series only inside the interval.

Note (3): Putting $x = 1$ in the above, we get the following *Leibniz's formula.*

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots + \frac{(-1)^{n-1}}{2n-1} + \cdots$$

Example 5: A series for $ln(1 + x)$, $-1 < x \le 1$

Consider the series $\sum_{n=0}^{\infty}(-1)^n t^n = 1 - t + t^2 - t^3 + \ldots..$

By Ratio Test, the series converges absolutely for $|t| < 1$ i.e., $-1 < t < 1$. It diverges for $|t| > 1$. Further, the series diverges when $t = \pm 1$.

The given series is a convergent geometric series with first term 1 and common ratio $-t$ and its sum is $\frac{1}{1-(-t)} = \frac{1}{1+t}$. Thus,

$$\frac{1}{1+t} = \sum_{n=0}^{\infty}(-1)^n t^n , \qquad -1 < t < 1$$

By the Term by Term integration Theorem

$$\int_0^x \frac{1}{1+t} = \left[\sum_{n=0}^{\infty} (-1)^n \frac{t^{n+1}}{n+1} \right]_0^x, \qquad -1 < x < 1$$

$$\Rightarrow \log(1+x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}$$

$$i.e., \log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + ..., \quad -1 < x < 1$$

It can also be shown that the above series converges at $x = 1$ to $\ln 2$. (But this is not guaranteed by the theorem). Thus,

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + ..., \quad -1 < x \le 1$$

Multiplication of Power series:

Another theorem from advanced calculus states that absolutely convergent series can be multiplied the way we multiply polynomials.

Theorem 4: The Series Multiplication Theorem for Power Series

If $A(x) = \sum_{n=0}^{\infty} a_n x^n$ ***and*** $B(x) = \sum_{n=0}^{\infty} b_n x^n$ ***converge absolutely for*** $|x| < R$ ***and***

$\boldsymbol{c_n = a_0 b_n + a_1 b_{n-1} + \cdots + a_{n-1} b_1 + a_n b_0} = \sum_{k=0}^{\infty} a_k b_{n-k}$***.Then***

$\sum_{n=0}^{\infty} c_n x^n$ ***converges absolutely to $\boldsymbol{A(x)B(x)}$ for $|\boldsymbol{x}| < R$:***

$$\left(\sum_{n=0}^{\infty} a_n x^n\right) \cdot \left(\sum_{n=0}^{\infty} b_n x^n\right) = \left(\sum_{n=0}^{\infty} a_n x^n\right)$$

Example 6: Multiply the Geometric series

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \quad , \quad for \quad |x| < 1$$

by itself to get a power series for $\frac{1}{(1-x)^2}$, for $|x| < 1$

Solution:

Let

$$A(x) = \sum_{n=0}^{\infty} a_n x^n = 1 + x + x^2 + x^3 + \ldots + x^n + \ldots = \frac{1}{1-x}, \; for\,|x| < 1$$

$$B(x) = \sum_{n=0}^{\infty} b_n x^n = 1 + x + x^2 + x^3 + \ldots + x^n + \ldots = \frac{1}{1-x}, \; for\,|x| < 1$$

and $c_n = \underbrace{a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + \cdots + a_{n-1} b_1 + a_n b_0}_{(n+1) terms}$

$$= \underbrace{1 + 1 + 1 + \cdots + 1 + 1}_{(n+1)\ ones} = n + 1$$

Now, by the series Multiplication Theorem

$$A(x).B(x) = \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} (n+1)x^n$$

$$\Rightarrow \frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + (n+1)x^n + \cdots \ , \quad \text{for } |x| < 1$$

P1.

Find the radius and the interval of convergence of the series

$$\sum_{n=0}^{\infty}(-1)^n(4x+1)^n$$

For what values of x does the series converge absolutely; converge conditionally.

Solution:

We apply Ratio Test for $\sum a_n$, where $a_n = (-1)^n(4x+1)^n$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}(4x+1)^{n+1}}{(-1)^n(4x+1)^n}\right|$$

$$= \lim_{n\to\infty}|(-1)(4x+1)| = |(4x+1)|$$

By Ratio Test, the series converges absolutely for $|4x+1| < 1$

i.e., $-1 < 4x+1 < 1$, i.e., $-\frac{1}{2} < x < 0$. It diverges for

$|4x+1| > 1.$

We test the convergence at $x = -\frac{1}{2}, x = 0$

When $x = -\frac{1}{2}$, we have the series $\sum_{n=0}^{\infty}(-1)^n(-1)^n = \sum_{n=0}^{\infty}(1)^n$,

which is divergent.

When $x = 0$, we have the series $\sum_{n=0}^{\infty} (-1)^n (1)^n = \sum_{n=0}^{\infty} (-1)^n$, which is divergent. Therefore,

- The interval of absolute convergence is: $-\frac{1}{2} < x < 0$
- The interval of convergence is: $-\frac{1}{2} < x < 0$
- The radius of convergence is: $\frac{1}{4}$
- There are no values (for x) for which the series converges conditionally.

P2.

Find the radius and the interval of convergence of the series

$$\sum_{n=1}^{\infty} \frac{x^n}{\sqrt{n^2+3}}$$

For what values of x does the series converge absolutely; converge conditionally.

Solution:

We apply Ratio Test for $\sum a_n$, where $a_n = \frac{x^n}{\sqrt{n^2+3}}$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{x^{n+1}}{\sqrt{(n+1)^2+3}} \cdot \frac{\sqrt{n^2+3}}{x^n}\right|$$

$$= |x| \lim_{n\to\infty} \left|\sqrt{\frac{n^2+3}{n^2+2n+4}}\right| = |x|$$

By Ratio Test, the series converges absolutely for $|x| < 1$

i.e., $-1 < x < 1$. It diverges for $|x| > 1$.

We test the convergence at $x = -1$, $x = 1$

When $x = -1$, we have the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n^2+3}}$, which is an alternating series and converges by Leibniz's theorem. Further, $\sum_{n=1}^{\infty} \left|\frac{(-1)^n}{\sqrt{n^2+3}}\right| = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+3}}$ is divergent by Limit Comparison Test, by comparing with $\sum b_n$ where $b_n = \frac{1}{n}$. Thus, it converges conditionally for $x = -1$.

When $x = 1$, we have the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+3}}$, which is divergent.

Therefore,

- The interval of absolute convergence is: $-1 < x < 1$
- The interval of convergence is: $-1 \le x < 1$
- The radius of convergence is: 1
- The series converges conditionally at $x = -1$

P3.

Find the radius and the interval of convergence of the series

$$\sum_{n=1}^{\infty} (\ln n) x^n$$

For what values of x does the series converge absolutely; converge conditionally.

Solution:

We apply Ratio Test for $\sum a_n$, where $a_n = (ln\, n)\, x^n$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(ln\,(n+1))x^{n+1}}{(ln\ n)x^n}\right|$$

$$= |x| \lim_{n\to\infty} \left|\frac{ln\,(n+1)}{ln\ n}\right|$$

$$= |x| \lim_{n\to\infty} \left|\frac{1/n+1}{1/n}\right| = |x|$$

By Ratio Test, the series converges absolutely for $|x| < 1$

i.e., $-1 < x < 1$. It diverges for $|x| > 1$.

We test the convergence at $x = -1$, $x = 1$

When $x = -1$, we have the series $\sum_{n=1}^{\infty} (-1)^n \ln n$, which is divergent by nth - term Test, since $\lim_{n\to\infty} ln\, n \neq 0$.

When $x = 1$, we have the series $\sum_{n=1}^{\infty} \ln n$, which is divergent.

Therefore,

- The interval of absolute convergence is: $-1 < x < 1$
- The interval of convergence is: $-1 < x < 1$
- The radius of convergence is: 1
- There are no values for which the series converges conditionally.

P4.

Find the interval of convergence of the series $\sum_{n=1}^{\infty}(\ln x)^n$ and find the sum of the series as a function of x within this interval.

Solution:

We apply Ratio Test for $\sum a_n$, where $a_n = (ln\, x)^n$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(ln\ x)^{n+1}}{(ln\ x)^n}\right| = |ln\, x|$$

By Ratio Test, the series converges absolutely for $|ln\, x| < 1$

i.e., $-1 < ln\, x < 1$, i.e., $e^{-1} < x < e$ It diverges for $|ln\, x| > 1$.

We test the convergence at $x = e^{-1}$, $x = e$

When $x = e^{-1}$ or $x = e$, we obtain the series $\sum_{n=1}^{\infty}(1)^n$ and $\sum_{n=1}^{\infty}(-1)^n$ which are divergent.

Therefore, the interval of convergence is $e^{-1} < x < e$.

The series $\sum_{n=1}^{\infty}(\ln x)^n$ is a convergent geometric series when $e^{-1} < x < e$ and its sum is $\frac{1}{1-ln\ x}$.

Thus, $\sum_{n=1}^{\infty}(\ln x)^n = \dfrac{1}{1-\ln x}$ when $e^{-1} < x < e$

IP1.

Find the radius and the interval of convergence of the series

$$\sum_{n=1}^{\infty} \frac{(3x-2)^n}{n}$$

For what values of x does the series converge absolutely; converge conditionally.

Solution:

We apply Ratio Test for $\sum a_n$, where $a_n = \frac{(3x-2)^n}{n}$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(3x-2)^{n+1}}{n+1} \frac{n}{(3x-2)^n}\right|$$

$$= |3x-2| \lim_{n\to\infty} \left|\frac{n}{n+1}\right| = |3x-2|$$

By Ratio Test, the series converges absolutely for $|3x-2| < 1$

i.e., $-1 < 3x - 2 < 1$, i.e., $\frac{1}{3} < x < 1$. It diverges for

$|3x-2| > 1.$

We test the convergence at $x = \frac{1}{3}, x = 1$

When $x = \frac{1}{3}$, we have the series $\sum_{n=0}^{\infty} \frac{(-1)^n}{n}$, which is the alternating harmonic series and it is conditionally convergent.

When $x = 1$, we have the series $\sum_{n=0}^{\infty} \frac{1}{n}$, which is the divergent harmonic series. Therefore,

- The interval of absolute convergence is: $\frac{1}{3} < x < 1$
- The interval of convergence is: $\frac{1}{3} \leq x < 1$
- The radius of convergence is: $\frac{1}{3}$
- The series converges conditionally at $x = \frac{1}{3}$

IP2.

Find the radius and the interval of convergence of the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{\sqrt{n^2+3}}$$

For what values of x does the series converge absolutely; converge conditionally.

Solution:

We apply Ratio Test for $\sum a_n$, where $a_n = \frac{(-1)^n x^n}{\sqrt{n^2+3}}$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(-1)^{n+1} x^{n+1}}{\sqrt{(n+1)^2+3}} \cdot \frac{\sqrt{n^2+3}}{(-1)^n x^n}\right|$$

$$= |x| \lim_{n\to\infty} \left|\sqrt{\frac{n^2+3}{n^2+2n+4}}\right| = |x|$$

By Ratio Test, the series converges absolutely for $|x| < 1$

i.e., $-1 < x < 1$. It diverges for $|x| > 1$.

We test the convergence at $x = 1$, $x = -1$

When $x = 1$, we have the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n^2+3}}$, which is conditionally convergent.

When $x = -1$, we have the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+3}}$, which is divergent. Therefore,

- The interval of absolute convergence is: $-1 < x < 1$
- The interval of convergence is: $-1 < x \leq 1$
- The radius of convergence is: $\frac{1}{2}$
- The series converges conditionally at $x = 1$

IP3.

Find the radius and the interval of convergence of the series

$$\sum_{n=1}^{\infty} \frac{x^n}{n(\ln n)^2}$$

For what values of x does the series converge absolutely; converge conditionally.

Solution:

We apply Ratio Test for $\sum a_n$, where $a_n = \frac{x^n}{n(ln\ n)^2}$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{x^{n+1}}{(n+1)(ln\,(n+1))^2} . \frac{n(ln\ n)^2}{x^n}\right|$$

$$= |x| \lim_{n\to\infty} \frac{n}{n+1} . \left(\lim_{n\to\infty} \frac{ln\ n}{ln\,(n+1)}\right)^2$$

$$= |x| \lim_{n\to\infty} \left|\frac{1/n}{1/n+1}\right| = |x|$$

By Ratio Test, the series converges absolutely for $|x| < 1$

i.e., $-1 < x < 1$. It diverges for $|x| > 1$.

We test the convergence at $x = -1$, $x = 1$

When $x = -1$, we have the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n(\ln n)^2}$ and it converges absolutely (by logarithmic $p -$ series).

When $x = 1$, we have the series $\sum_{n=1}^{\infty} \frac{1}{n(\ln n)^2}$ and it converges (by logarithmic $p-$ series). Therefore,

- The interval of absolute convergence is: $-1 \leq x \leq 1$
- The interval of convergence is: $-1 \leq x \leq 1$
- The radius of convergence is: 1
- There are no values for which the series converges conditionally.

IP4.

Find the interval of convergence of the series $\sum_{n=1}^{\infty}\left(\frac{x^2+1}{3}\right)^n$ and find the sum of the series as a function of x within this interval.

Solution:

We apply Ratio Test for $\sum a_n$, where $a_n = \left(\frac{x^2+1}{3}\right)^n$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\left(\frac{x^2+1}{3}\right)^{n+1}\left(\frac{3}{x^2+1}\right)^n\right| = \left|\frac{x^2+1}{3}\right|$$

By Ratio Test, the series converges absolutely for $\left|\frac{x^2+1}{3}\right| < 1$

i.e., $\frac{x^2+1}{3} < 1\ (why?)$, i.e., $|x| < \sqrt{2}$, i.e., $-\sqrt{2} < x < \sqrt{2}$.

It diverges for $|x| > \sqrt{2}$.We test the convergence at $x = \pm\sqrt{2}$.

When $x = \pm\sqrt{2}$, we obtain the series $\sum_{n=1}^{\infty}(1)^n$ which is divergent

Thus, the series $\sum_{n=1}^{\infty}\left(\frac{x^2+1}{3}\right)^n$ is a convergent geometric series

when $-\sqrt{2} < x < \sqrt{2}$ and its sum is $\frac{1}{1-\frac{x^2+1}{3}} = \frac{3}{2-x^2}$

Therefore, $\sum_{n=1}^{\infty}\left(\frac{x^2+1}{3}\right)^n = \frac{3}{2-x^2}$ when $-\sqrt{2} < x < \sqrt{2}$.

EXERCISES

I. **Find the radius and the interval of convergence of the following series. For what values of x does the series converge absolutely; converge conditionally.**

$(a)\ \sum_{n=0}^{\infty}(x+5)^n \quad (b)\ \sum_{n=0}^{\infty}\frac{(x-2)^n}{10^n} \quad (c)\ \sum_{n=0}^{\infty}\frac{(x-2)^n}{10^n}$

$(d)\ \sum_{n=0}^{\infty}(2x)^n \quad (e)\ \sum_{n=0}^{\infty}\frac{nx^n}{n+2} \quad (f)\ \sum_{n=0}^{\infty}\frac{(x-1)^n}{\sqrt{n}}$

$(g)\ \sum_{n=0}^{\infty}\frac{x^n}{n\sqrt{n}.3^n} \quad (h)\ \sum_{n=0}^{\infty}\frac{3^n x^n}{n!} \quad (i)\ \sum_{n=0}^{\infty}\frac{x^{2n+1}}{n!}$

$(j)\ \sum_{n=0}^{\infty}\frac{(2x+3)^{2n+1}}{n!} \quad (k)\ \sum_{n=1}^{\infty}\frac{(-1)^{n+1}(x+2)^n}{n.2^n}$

$(l)\ \sum_{n=0}^{\infty}(-2)^n(n+1)(x-1)^n \quad (m)\ \sum_{n=2}^{\infty}\frac{x^n}{n\ln n} \quad (n)\ \sum_{n=0}^{\infty}\frac{(4x-5)^{2n+1}}{n^{3/2}}$

$(0)\ \sum_{n=0}^{\infty}\frac{(3x+1)^{n+1}}{2n+2}$

II. Find the interval of convergence of the following series and find the sum of the series as a function of x within this interval.

$$(a)\ \sum_{n=0}^{\infty}\frac{(x-1)^{2n}}{4n} \qquad (b)\ \sum_{n=0}^{\infty}\frac{(x+1)^{2n}}{9^n}$$

$$(c)\ \sum_{n=0}^{\infty}\left(\frac{\sqrt{x}}{2}-1\right)^n \qquad (d)\ \sum_{n=0}^{\infty}\left(\frac{x^2-1}{2}\right)^n$$

1.9

Taylor and Maclaurin Series

Learning objectives:

- To define the Taylor series and Maclaurin series generated by a function at a point.
- To define a Taylor polynomial of a given order generated by a function at a point.
- To state and prove the Taylor's Theorem.
- To define the Taylor's Formula and to state the Remainder Estimation Theorem.

  AND

To practice the related problems

We know from the Term by Term Differentiation theorem (Theorem 2 of module 1.8) that within its interval of convergence, the sum of a power series is a continuous function with derivatives of all orders. We ask the question about the other way around. That is, if a function $f(x)$ has derivatives of all orders on an interval I, can it be expressed as a power series on I ? If it can, what will its coefficients be?

Theorem 1:

If $f(x)$ is the sum of a power series $\sum_{n=0}^{\infty} a_n (x-a)^n$ with a positive radius of convergence then $a_n = \frac{f^{(n)}(a)}{n!}$, $n = 0, 1, 2, \ldots$ and $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$.

Proof:

We have $f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n$. By repeated Term by Term Differentiation within the interval of convergence I, we obtain

$$f'(x) = \sum_{n=1}^{\infty} n a_n (x-a)^{n-1}, \quad f''(x) = \sum_{n=2}^{\infty} n(n-1) a_n (x-a)^{n-2}, \ldots$$

and $f^{(n)}(x) = n!\ a_n +$ sum of the factors with $(x-a)$ as a factor.

Now, $f^{(n)}(a) = n!\, a_n$ and $a_n = \frac{f^{(n)}(a)}{n!}$ for all $n = 0,1,2,\ldots$

The result now follows. Hence the result

If we start with an arbitrary function f that has derivatives of all orders on an interval I centered at $x = a$ and use it to generate the series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + f''(a)(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Will the series converge to $f(x)$ at each x in the interior of I ? The answer is may be - for some functions it will, but for other functions it will not.

Taylor series and Maclaurin Series:

*Let f be a function with derivatives of all orders throughout some interval containing a as an interior point. Then the **Taylor series generated by f at $x = a$** is*

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \ldots$$

The Maclaurin Series generated by f is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}(x-a)^k = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \ldots + \frac{f^{(n)}(0)}{n!}x^n + \ldots$$

the Taylor series generated by f at $x = 0$

Example 1: Finding a Taylor series

Find the Taylor series generated by $f(x) = \frac{1}{x}$ at $a = 2$. Where (if any where) does the series converge to $\frac{1}{x}$?

Solution:

We have $f(x) = \frac{1}{x}$. Taking the derivatives, we get

$$f(x) = x^{-1}, \qquad f(2) = \frac{1}{2}$$

$$f'(x) = (-1)x^{-2}, \qquad f'(2) = \frac{-1}{2^2}$$

$$f''(x) = (-1)^2\ 2!\ x^{-3}, \qquad \frac{f''(2)}{2!} = \frac{(-1)^2}{2^3}$$

$$f'''(x) = (-1)^3\ 3!\ x^{-4}, \qquad \frac{f'''(2)}{3!} = \frac{(-1)^3}{2^4}$$

………… ……………

$$f^n(x) = (-1)^n\ n!\ x^{-(n+1)}, \qquad \frac{f^{(n)}(2)}{n!} = \frac{(-1)^n}{2^{n+1}}$$

The Taylor series generated by $f(x) = \frac{1}{x}$ at $a = 2$ is

$$f(2) + f'(2)(x-2) + \frac{f''(2)}{2!}(x-2)^2 + \cdots + \frac{f^{(n)}(x)}{n!}(x-2)^n + \cdots$$

$$= \frac{1}{2} - \frac{x-2}{2^2} + \frac{(x-2)^2}{2^3} - \cdots + (-1)^n \frac{(x-2)^n}{2^{n+1}} + \cdots$$

This is a geometric series with first term $\frac{1}{2}$ and common ratio $r = -\frac{x-2}{2}$. It converges absolutely for $|x - 2| < 2$, i.e., $0 < x < 4$ and its sum is

$$\frac{a}{1-r} = \frac{\frac{1}{2}}{1+\frac{x-2}{2}} = \frac{1}{x}$$

In this example the Taylor series generated by $f(x) = \frac{1}{x}$ at $a = 2$ converges to $\frac{1}{x}$ for $0 < x < 4$.

Taylor Polynomials:

The linearization of differentiable function f at a point a is the polynomial of degree one given by

$$P_1(x) = f(a) + f'(a)(x - a)$$

We used this linearization to approximate $f(x)$ at values of x near a.

If f has derivatives of higher order at $x = a$, then it has higher order polynomial approximations, one for each available derivative. These polynomials are called **Taylor Polynomials of f.**

Taylor Polynomial of Order n:

Let f be a function with derivatives of order k for $k = 1,2, \dots, N$ in some interval containing a as an interior point. Then, for any *integer n from 0 through N, the* ***Taylor Polynomial of Order n*** *generated by f at $x = a$ is the polynomial*

$$P_n(x) = f(a) + f^{'}(a)(x-a) + \frac{f^{''}(a)}{2!}(x-a)^2 + \cdots$$

$$\ldots + \frac{f^{(k)}(a)}{k!}(x-a)^k + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

It is called a Taylor polynomial of Order n rather than degree n because $f^{(n)}(a)$ may be zero.

Note:

Just as the linearization of f at $x = a$ provides the best approximation of f in the neighborhood of a, the higher order Taylor polynomials provide the best polynomial approximations of their respective degrees.

Example 2: Finding Taylor Polynomial for e^x

Find the Taylor series and Taylor polynomials generated by $f(x) = e^x$ at $x = 0$

Solution:

We have, $f(x) = e^x$, $f^{'}(x) = e^x$, $f^{''}(x) = e^x, \ldots f^{(n)}(x) = e^x$ and $f(0) = 1$, $f^{'}(0) = 1$, $f^{''}(0) = 1, \ldots, f^{(n)}(0) = 1, \ldots$

The Taylor series generated by $f(x) = e^x$ at $x = 0$ is

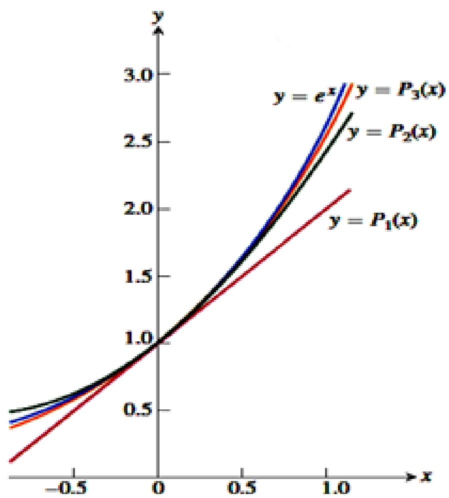
$$f(0) + f^{'}(0)x + \frac{f^{''}(0)}{2!}x^2 + \cdots + \frac{f^{n}(0)}{n!}x^n + \cdots$$

$$= 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

This is also Maclaurin series for e^x . (We will see later that the series converges to e^x at every x).

The Taylor polynomial of order n at $x = 0$ is

$$P_n(x) = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}$$



The graph of $f(x) = e^x$ and its Taylor polynomials
$P_1(x) = 1 + x$
$P_2(x) = 1 + x + (x^2/2!)$
$P_3(x) = 1 + x + (x^2/2!) + (x^3/3!)$.
Notice the very close agreement near the center $x = 0$

The above figure shows how well these polynomials approximate $f(x) = e^x$ near $x = 0$

Example3: A function f whose Taylor Series converges at every x but converges to $f(x)$ only at $x = 0$.

The function $f(x) = \begin{cases} 0 \ , & x = 0 \\ e^{-\frac{1}{x^2}} \ , & x \neq 0 \end{cases}$ has derivatives of all orders at $x = 0$ and that $f^{(n)}(0) = 0$ for all n. This Taylors series generated by f at $x = 0$ is

$$f(0) + f^{'}(0)x + \frac{f^{''}(0)}{2!}x^2 + \cdots + \frac{f^{n}(0)}{n!}x^n + \cdots$$

$$= 0 + 0.x + 0.x^2 + \cdots + 0.x^n + \cdots = 0 + 0 + 0 + \cdots + 0 + \cdots$$

The series converges for every x (its sum is zero) but converges to $f(x)$ only at $x = 0$.

We now answer the question:

When does the Taylor series generated by a function converge to its generating function?

We answer this question with the following theorem:

Theorem 2: Taylor's Theorem

If f and its first n derivatives $f^{'}, f^{''}, \dots, f^{(n)}$ are continuous on the closed interval between a and b, and f is differentiable on the open interval between a and b, then there exists a number c between a and b such that

$$\boldsymbol{f(b) = f(a) + f^{'}(a)(b-a) + \frac{f^{''}(a)}{2!}(b-a)^2 + \cdots}$$

$$\boldsymbol{+ \frac{f^{(n)}(a)}{n!}(b-a)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1}}$$

Proof:

We prove the Taylor's Theorem when $a < b$. The proof of $a > b$ is nearly same. The Taylor polynomial of order n generated by f at $x = a$ is

$$P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Notice that $P_n(x)$ and its first n derivatives match f and its first n derivatives at $x = a$.

Define $\emptyset_n(x) = P_n(x) + K(x-a)^{n+1}$, where K is a constant

This new function and its first n derivatives agree with f and its first n derivatives at $x = a$.

Choose K such that the curves $y = \emptyset_n(x)$ and $y = f(x)$ agree at $x = b$. That is $\emptyset_n(b) = f(b)$,

$$\text{i.e., } f(b) = P_n(b) + K(b-a)^{n+1} \text{ --------- } (1)$$

Now, the function

$$\begin{aligned} F(x) &= f(x) - \emptyset_n(x) \\ &= f(x) - f(a) - f'(a)(x-a) - \frac{f''(a)}{2!}(x-a)^2 - \cdots \\ &\quad - \frac{f^{(n)}(a)}{n!}(x-a)^n - K(x-a)^{n+1} \end{aligned}$$

measures the difference between the original function f and the approximating function $\emptyset_n$ for each x in $[a, b]$.

Notice that F and $F^{'}$ are continuous on $[a, b]$ and $F(a) = 0$, $F(b) = f(b) - \emptyset_n(b) = 0$. Thus, by Rolle's Theorem $F^{'}(c_1) = 0$ for some $c_1 \in (a, b)$.

Notice that $F^{'}$ and $F^{''}$ are continuous on $[a, c_1]$ and $F^{'}(a) = F^{'}(c_1) = 0$. By Rolle's Theorem $F^{''}(c_2) = 0$ for some $c_2 \in (a, c_1)$. Applying Rolle's Theorem successively to $F^{''}, F^{'''}, \dots, F^{(n-1)}$ implies the existence of c_3 in (a, c_2), such that $F^{'''}(c_3) = 0$; c_4 in (a, c_3) such that $F^{(4)}(c_4) = 0; \dots;$ c_n in (a, c_{n-1}) such that $F^{(n)}(c_n) = 0$.

Now notice that $F^{(n)}$ is continuous on $[a, c_n]$ and differentiable on (a, c_n) and $F^{(n)}(a) = F^{(n)}(c_n) = 0$. By Rolle's Theorem there is a number $c_{n+1} \in (a, c_n)$ such that $F^{(n+1)}(c_n) = 0$ where $F^{(n+1)}(x) = f^{(n+1)}(x) - (n+1)!\,K$. Therefore, $K = \frac{f^{(n+1)}(c)}{(n+1)!}$ for some $c = c_{n+1} \in (a, b)$. The equation (1) gives

$$f(b) = P_n(b) + \frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1}$$

This completed the proof of Taylor's Theorem.

Note:

(1) Taylor's Theorem is a generalization of Lagrange's Mean Value Theorem.

(2) When we apply Taylor's Theorem, we usually hold a fixed and treat b as an independent variable. The following is a version of Taylor's Theorem with this change.

Taylor's Formula:

If f has derivatives of all orders in an open interval $\boldsymbol{I}$ containing a, then for each natural number n and for each x in $\boldsymbol{I}$.

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots$$

$$+ \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x) \text{ ------ } (2)$$

where $R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$ for some c between a and x.

Note:

(1) If we state Taylor's theorem in this way, then it concludes that $f(x) = P_n(x) + R_n(x)$ for each $x \in \boldsymbol{I}$.

(2) The function $R_n(x)$ is determined by the value of $f^{(n+1)}$ at a point c, where c depends on a and x, and lies between a and x.

(3) For any value of n, the equation (2) gives both a polynomial approximation of f of order n and a formula for ***error*** involved in using the approximation over the interval $\boldsymbol{I}$.

Equation (2) is called ***Taylor's Formula***. The function $R_n(x)$ is called the ***remainder of order n*** or the ***error term*** for the approximation of f by $P_n(x)$ over $\boldsymbol{I}$. ***If $R_n(x) \to 0$ as $n \to \infty$ for all $x \in I$, then we say that the Taylor series generated by f at $x = a$ converges to f on I and we write***

$$f(x) = \sum_{k=0}^{\infty} \frac{f^k(a)}{n!}(x-a)^k$$

Remark:

The remainder mentioned in the Taylor's Formula is called Lagrange's form of Remainder. There are other forms of remainders like Cauchy's form of Remainder, Integral form of Remainder, etc.

Often we can estimate R_n without knowing the value of c.

Example 4: The Taylor Series for e^x

Show that the Taylor Series generated by $f(x) = e^x$ at $x = 0$ converges to $f(x)$ for every real value of x.

Solution:

The function $f(x) = e^x$ has derivatives of all orders in the interval $\boldsymbol{I} = (-\infty, \infty)$ and $f^n(0) = 1$ for all n.

By Taylor's formula, we have $f(x) = e^x = P_n(x) + R_n(x)$

where $P_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n$

$$= 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}$$

and $R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1} = \frac{e^c}{(n+1)!} x^{n+1}$ for some c between 0 and x.

Since e^x is an increasing function of x, e^c lies between $e^0 = 1$ and e^x. If x is negative, then c is negative and $e^c < 1$. If $x = 0$, then $R_n(x) = 0$. If x is positive, then c is positive and $e^c < e^x$.

Thus, $\quad |R_n(x)| = \frac{e^c |x|^{n+1}}{(n+1)!} \leq \frac{|x|^{n+1}}{(n+1)!}$ when $x \leq 0$

and $\quad |R_n(x)| < \frac{e^x x^{n+1}}{(n+1)!}$ when $x > 0$

Since $\lim_{n\to\infty} \frac{x^{n+1}}{(n+1)!} = 0$ for all x ; $\lim_{n\to\infty} R_n(x) = 0$

Thus, the Taylor series generated by e^x at $x = 0$ converges to e^x for every x and

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^k}{k!} + \ldots$$

Estimating the Remainder:

It is often possible to estimate $R_n(x)$. This method of estimation is so convenient that we state it as a theorem.

Theorem 3: The Remainder Estimation Theorem

If there is a positive constant M such that $\left|f^{(n+1)}(t)\right| \le M$ for all t between x and a, inclusive, then the remainder term $R_n(x)$ in Taylor's Theorem satisfies the inequality

$$|R_n(x)| \le \frac{M}{(n+1)!}|x-a|^{n+1}$$

If this condition holds for every n and other conditions of Taylor's Theorem are satisfied by f, then the series converges to $f(x)$.

Example 5: Finding a Taylor series by substitution

Find the Taylor series generated by e^{-5x} at $x = 0$

Solution:

We have $e^x = \sum\limits_{k=0}^{\infty} \frac{x^k}{k!}$ $,-\infty < x < \infty$ $\Rightarrow$It holds for $-\infty < -5x < \infty$.

Substituting $-5x$ for x, we get

$$e^{-5x} = \sum_{k=0}^{\infty} \frac{(-5x)^k}{k!} = \sum_{k=0}^{\infty} \frac{(-1)^k 5^k x^k}{k!}$$

$$= 1 - 5x + \frac{5^2}{2!}x^2 - \frac{5^3}{3!}x^3 + \dots$$

and it holds for $-\infty < -5x < \infty$ and this series converges for all x.

Example 6: Finding a Taylor series by Multiplication

Find the Taylor series for xe^x at $x = 0$

Solution:

We can find the Taylor series for xe^x by multiplying the Taylor series for e^x by x:

$$xe^x = x\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right)$$

$$= x + x^2 + \frac{x^3}{2!} + \frac{x^4}{3!} + \frac{x^5}{4!} + \cdots$$

valid for $-\infty < x < \infty$, since the series for e^x is valid for $-\infty < x < \infty$.

P1.

Find the Taylor series generated by $f(x) = \frac{1}{x^2}$ at $x = 1$

Solution:

Given $f(x) = \frac{1}{x^2}$, $a = 1$

The Taylor series generated by $f(x)$ at $x = a$ is $\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$

We have $f(x) = x^{-2}$, $f'(x) = -2x^3$, $f''(x) = 3!\,x^{-4}$,

$f'''(x) = -4!\,x^{-5}$.......$f^{(k)}(x) = (-1)^k (k+1)!\,x^{-k-2}$

Now, $\frac{f^{(k)}(1)}{k!} = \frac{(-1)^k (k+1)!}{k!} = (-1)^k (k+1)$

Therefore, the Taylor series generated by $\frac{1}{x^2}$ at $x = 1$ is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(1)}{k!}(x-1)^k = \sum_{k=0}^{\infty} (-1)^k (k+1)(x-1)^k$$

$$= 1 - 2(x-1) + 3(x-1)^2 - 4(x-1)^3 + ...$$

P2.

Find the Taylor polynomials of orders $0, 1, 2$ and 3 generated by $f(x) = ln(1 + x)$ at $x = 0$

Solution:

We have $f(x) = ln(1 + x), f'(x) = \frac{1}{1+x}, f''(x) = (-1)\frac{1}{(1+x)^2}$

$f'''(x) = \frac{(-1)^2 1.2}{(1+x)^3}$, $f'^{v}(x) = \frac{(-1)^3 1.2.3}{(1+x)^4}$, ... $f^{(n)}(x) = \frac{(-1)^{n-1}(n-1)!}{(1+x)^n}$

Now, $f(0) = 0, f'(0) = 1, f''(0) = -1$ $f'''(0) = 2!$,

$f'^{v}(0) = -3!, \ldots f^{(n)}(0) = (-1)^{n-1}(n-1)!$

Let $P_n(x)$ be the Taylor polynomial of order n generated by f at $x = a$. We have

$$P_n(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x - a)^n$$

Now, $P_0(x) = f(0) = 0$

$$P_1(x) = f(0) + f'(0)x = x$$

$$P_2(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 = x - \frac{x^2}{2!}$$

$$P_3(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f''(0)}{3!}x^3 = x - \frac{x^2}{2!} + \frac{x^3}{3!}$$

In general, for $n = 1, 2, 3 \ldots$

$$P_n(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n-1}\frac{x^n}{n}$$

P3.

(i) The Taylor Series for $\sin x$ at $x = 0$.

Show that the Taylor Series generated by $\sin x$ at $x = 0$ converges to $sinx$ for all x.

Solution:

We have, $f(x) = \sin x$, $f'(x) = \cos x$,

$$f''(x) = -\sin x \ , f'''(x) = -\cos x \ , \ldots..$$

……. $f^{(2k)}(x) = (-1)^k \sin x$, $f^{(2k+1)}(x) = (-1)^k \cos x$,

Now, $f^{(2k)}(0) = 0$ and $f^{(2k+1)}(0) = (-1)^k$.

Notice that the series has only odd powered terms and for $n = 2k + 1$, Taylor's Theorem gives

$$sin\, x = P_{2k+1}(x) + R_{2k+1}(x)$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + \frac{(-1)^k x^{2k+1}}{(2k+1)!} + R_{2k+1}(x)$$

where $R_{2k+1}(x) = \frac{f^{(2k+1)}(c)}{(2k+2)!} x^{2k+2} = \frac{(-1)^{k+1} \sin x}{(2k+2)!} x^{2k+2}$ for every c between 0 and x.

Now, $|R_{2k+1}(x)| \leq \frac{|x|^{2k+2}}{(2k+2)!}$, since $|\sin x| \leq 1$ (by Remainder Estimation Theorem) and

$\frac{|x|^{2k+2}}{(2k+2)!} \to 0$ as $k \to \infty$ for all x. Thus, $R_{2k+1}(x) \to 0$ for all k.

Therefore, the Maclaurin series for $\sin x$ converges to $\sin x$ for all x and

$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$$

(ii) Finding a Taylor series by Multiplication

Find the Taylor series for $xsinx$ at $x = 0$

Solution:

We can find the Taylor series for $xsinx$ by multiplying the Taylor series for $sinx$ by x:

$$xsinx = x\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right) = \left(x^2 - \frac{x^4}{3!} + \frac{x^6}{5!} - \frac{x^8}{7!} + \cdots\right)$$

valid for $-\infty < x < \infty$, since the series for $sinx$ is valid for $-\infty < x < \infty$.

P4.

Find the Maclaurin series for the function

$$\boldsymbol{f(x) = x^4 - 2x^3 - 5x + 4}$$

Solution:

Given $f(x) = x^4 - 2x^3 - 5x + 4$

The Maclaurin series for $f(x)$ is $\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$

We have $f(x) = x^4 - 2x^3 - 5x + 4$

$f'(x) = 4x^3 - 6x^2 - 5,\qquad f''(x) = -12x$

$f'''(x) = 24x\ ,\qquad f^{(4)}(x) = 24$

$f^{(k)}(x) = 0$ for $k \geq 5$

Now, $f(0) = 4,\ f'(0) = -5,\ f''(0) = 0,\ \ f'''(x) = -12$

$f^{(4)}(x) = 24, \ldots\ldots, f^{(k)}(x) = 0$ for $k \geq 5$

The Maclaurin series for $f(x) = x^4 - 2x^3 - 5x + 4$ is

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \cdots$$

$$= 4 - 5x + \frac{0}{2!}x^2 - \frac{12}{3!}x^3 + \frac{24}{4!}x^4 = x^4 - 2x^3 - 5x + 4$$

The Maclaurin series for $x^4 - 2x^3 - 5x + 4$ is itself.

IP1.

Find the Taylor series generated by $f(x) = 2^x$ at $x = 1$

Solution:

Given $f(x) = 2^x$, $a = 1$

The Taylor series generated by $f(x)$ at $x = a$ is $\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$

We have $f(x) = 2^x$, $f'(x) = 2^x ln2$, $f''(x) = 2^x (ln2)^2$,

$f'''(x) = 2^x (ln2)^3 ....f^{(k)}(x) = 2^x (ln2)^k$

Now, $\frac{f^{(k)}(1)}{k!} = \frac{2(ln2)^k}{k!}$

Therefore, the Taylor series generated by 2^x at $x = 1$ is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(1)}{k!}(x-1)^k = \sum_{k=0}^{\infty} \frac{2(\ln 2)^k}{k!}(x-1)^k$$

$$= 2 + 2\ln 2.(x-1) + \frac{2(\ln 2)^2}{2!}(x-1)^2 + \frac{2(\ln 2)^3}{3!}(x-1)^3 + ...$$

IP2.

Find the Taylor polynomials of orders $0, 1, 2$ and 3 generated by $\boldsymbol{f(x) = \sqrt{x}}$ at $\boldsymbol{x = 4}$

Solution:

We have $f(x) = \sqrt{x}$ $, f'(x) = \frac{1}{2}x^{-\frac{1}{2}}, f''(x) = \frac{1}{2}\left(-\frac{1}{2}\right)x^{-\frac{3}{2}}$

$f'''(x) = \frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)x^{-\frac{5}{2}}$

Now, $f(4) = 2, f'(4) = \frac{1}{4}, f''(4) = -\frac{1}{32}$, $f'''(4) = \frac{3}{256}$,

Let $P_n(x)$ be the Taylor polynomial of order n generated by f at $x = a$, we have

$$P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Now, $P_0(x) = f(4) = 2$

$$P_1(x) = f(4) + f'(4)(x-4) = 2 + \frac{1}{4}(x-4)$$

$$P_2(x) = f(4) + f'(4)x + \frac{f''(4)}{2!}(x-4)^2$$

$$= 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2\text{s}$$

$$P_3(x) = f(4) + f'(4)x + \frac{f''(4)}{2!}(x-4)^2 + \frac{f'''(4)}{3!}(x-4)^3$$

$$= 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2 + \frac{1}{512}(x-4)^3$$

IP3.

(i) The Taylor series for $cosx$ at $x = 0$

Show that the Taylor series generated by $cosx$ at $x = 0$ converges to $cosx$ for every value of x.

Solution:

Let $f(x) = cosx$. We have

$f(x) = cosx, f^{'}(x) = -sinx, f^{''}(x) = -cosx, f^{'''}(x) = sinx$
……$f^{(2n)}(x) = (-1)^n\ cosx, f^{(2n+1)}(x) = (-1)^{n+1}\ sinx$

Now, $f^{(2n)}(0) = (-1)^n$, $f^{(2n+1)}(0) = 0$

The Taylor series generated by $f(x) = cosx$ at $x = 0$ is

$$f(0) + f^{'}(0)x + \frac{f^{''}(0)}{2!}x^2 + \frac{f^{'''}(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots$$

$$= 1 + 0.x - \frac{x^2}{2!} + 0.x^3 + \frac{x^4}{4!} + \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots$$

The Taylor polynomial of order $2k$ for $cosx$ is

$$P_{2k}(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^k \frac{x^{2k}}{(2k)!}$$

The Taylor's formula for $f(x) = cosx$ with $n = 2k$ is

$$f(x) = cosx = P_{2k}(x) + R_{2k}(x)$$

where $R_{2k}(x) = \frac{f^{(2k+1)}(c)}{(2k+1)!}x^{2k+1} = \frac{(-1)^{k+1}sinc}{(2k+1)!}x^{2k+1}$ for some c between 0 and x.

Now, $|R_{2k}(x)| = \left|\frac{(-1)^{k+1}sinc}{(2k+1)!}x^{2k+1}\right| \leq \frac{|x|^{2k+1}}{(2k+1)!}$

For every value of x, $R_{2k}(x) \to 0$ as $k \to \infty$ (by the Remainder Estimation Theorem)

Therefore, the Taylor series for $cosx$ at $x = 0$ converges to $cosx$. Thus,

$$cosx = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \ldots + (-1)^k \frac{x^{2k}}{(2k)!} + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}$$

(ii) Finding a Taylor series by substitution

Find the Taylor series for $cos2x$ at $x = 0$

Solution:

We have $cosx = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}$, $-\infty < x < \infty$

$\Longrightarrow$It holds for $-\infty < 2x < \infty$.

Substituting $2x$ for x, we get

$$\cos 2x = \sum_{k=0}^{\infty} \frac{(-1)^k (2x)^{2k}}{(2k)!} = \sum_{k=0}^{\infty} \frac{(-1)^k 2^{2k} x^{2k}}{(2k)!}$$ and it holds for $-\infty < x < \infty$.

IP4.

Find the Taylor series generated by $f(x) = x^4 + x^2 + 1$ at $x = -2$

Solution:

Given $f(x) = x^4 + x^2 + 1$

The Taylor series generated by $f(x)$ at $x = a$ is $\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} (x-a)^k$

We have $f(x) = x^4 + x^2 + 1$

$f'(x) = 4x^3 + 2x,$ $\qquad f''(x) = 12x^2 + 2$

$f'''(x) = 24x$, $\qquad f^{(4)}(x) = 24$

$f^{(k)}(x) = 0$ for $k \geq 5$

Now, $f(-2) = 21,\ f'(-2) = -36,\ f''(-2) = 50,$

$f'''(-2) = -48, f^{(4)}(-2) = 24, \ldots\ldots, f^{(k)}(-2) = 0$ for $k \geq 5$

The Taylor series generated by $f(x) = x^4 + x^2 + 1$ at $x = -2$ is

$$f(-2) + f'(-2)x + \frac{f''(-2)}{2!}(x+2)^2 + \frac{f'''(-2)}{3!}(x+2)^3 + \frac{f^{(4)}(-2)}{4!}(x+2)^4 + \cdots$$

$$= 21 - 36(x+2) + \frac{50}{2!}(x+2)^2 - \frac{48}{3!}(x+2)^3 + \frac{24}{4!}(x+2)^4$$

$$= 21 - 36(x+2) + 25(x+2)^2 - 8(x+2)^3 + (x+2)^4$$

EXERCISES:

I. Find the Taylor Polynomials of order 0, 1, 2 and 3 generated by $\boldsymbol{f(x)}$ at $\boldsymbol{x = a}$:

a. $f(x) = lnx \quad ; \quad a = 1$

b. $f(x) = \frac{1}{x} \quad ; \quad a = 2$

c. $f(x) = \frac{1}{x+2} \quad ; \quad a = 0$

d. $f(x) = sinx \quad ; \quad a = \frac{\pi}{4}$

e. $f(x) = \sqrt{x+4} \quad ; \quad a = 0$

II. Find the Maclaurin series for the functions:

a. e^{-x}

b. $e^{x/2}$

c. $\frac{1}{1+x}$

d. $\frac{1}{1-x}$

e. $sin3x$

f. $coshx$

g. $sinhx$

III. Find the Taylor series generated by $\boldsymbol{f(x)}$ at $\boldsymbol{x = a}$:

a. $f(x) = 2x^3 + x^2 + 3x - 8 \quad ; \quad a = 1$

b. $f(x) = 3x^5 - x^4 + 2x^3 + x^2 - 2 \quad ; \quad a = -1$

c. $f(x) = \frac{x}{1-x} \quad ; \quad a = 0$

d. $f(x) = e^x \quad ; \quad a = 2$

IV. Use substitution to find the Taylor series at $x = 0$:

a. $e^{-x/2}$

b. $5\sin(-x)$

c. $sin\left(\frac{\pi x}{2}\right)$

d. $cos\sqrt{x+1}$

e. $cos\left(\frac{x^{3/2}}{\sqrt{2}}\right)$

V. Find the Taylor series at $x = 0$:

a. $x^2 sinx$

b. $\frac{x^2}{2} - 1 + cosx$

c. $xcos\pi x$

d. $x^2 cos(x^2)$

e. $sin^2 x$

f. $\frac{x^2}{1-2x}$

g. $\frac{1}{(1-x)^2}$

h. $\frac{2}{(1-x)^3}$

Frequently used Taylor series:

1. $\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \cdots = \sum_{n=0}^{\infty} x^n$, $|x| < 1$

2. $\frac{1}{1+x} = 1 - x + x^2 - \cdots + (-x)^n + \cdots = \sum_{n=0}^{\infty} (-1)^n x^n$, $|x| < 1$

3. $e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, $|x| < \infty$

4. $sinx = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

$|x| < \infty$,

5. $cosx = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$

$|x| < \infty$

6. $\ln(x+1) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n-1} \frac{x^n}{n} + \cdots$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}, \quad -1 < x \le 1$$

7. $ln\left(\frac{1+x}{1-x}\right) = 2tanh^{-1}x$

$$= 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots + \frac{x^{2n+1}}{2n+1} + \cdots\right) = 2\sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1}, \quad |x| < 1$$

8. $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + (-1)^n \frac{x^{2n+1}}{2n+1} + \cdots$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)}, \quad |x| \le 1$$