16-11-2022

2. FUNCTIONS OF SEVERAL VARIABLES.

* Topics :- [M.L]

→ Function

→ Limit of Function

→ continuity

→ Differentiation

→ Partial Differentiation.

→ Taylor series.

→ Applications.

(Degree of Freedom)

* 'n'- Dimensional Euclidian space :-

→ 'Euclid' - Father of 'Geometry'.

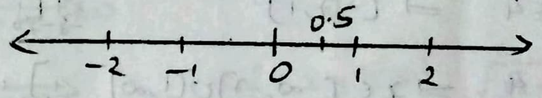
→ Let 'n∈N,' ℝ be the set of 'real numbers'

$\mathbb{R}^n = \mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\_\_\_\_\times\mathbb{R} = \{(x_1, x_2, x_3, \_\_, x_n); \forall\, x \in \mathbb{R},$

$1 \le i \le n\}$. is known as "n-Dimensional Euclidian space".

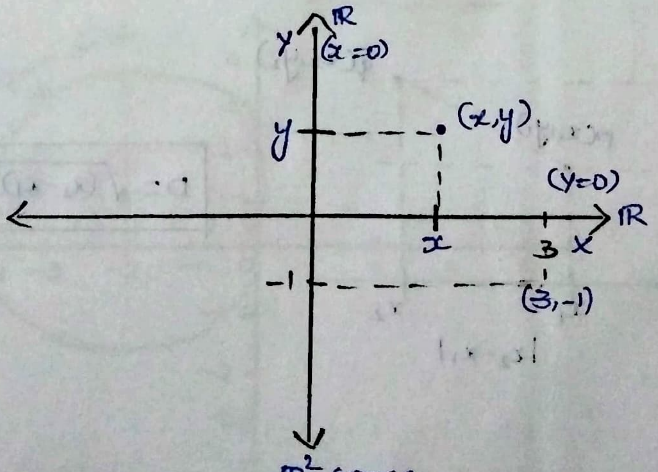
Ex :-

1-d E.space :- $\mathbb{R} = \{x; x \in \mathbb{R}\}$ is 1-d E.space



2-d E.space :-

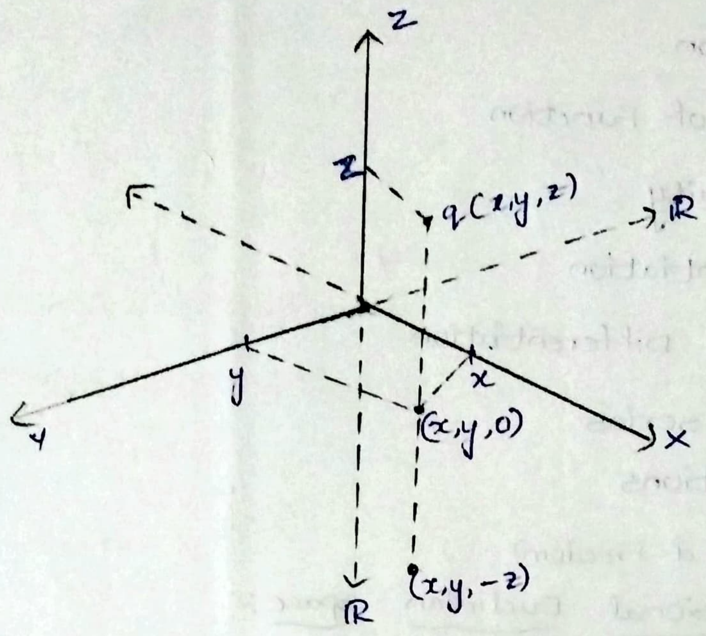
$\mathbb{R}^2 = \mathbb{R}\times\mathbb{R} = \{(x,y); \forall\, x, y \in \mathbb{R}\}$ is 2-d E.S.



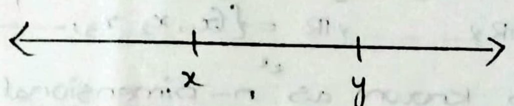
$\mathbb{R}^2$ space.

3-dimensional E.S :-

→ $\mathbb{R}\times\mathbb{R}\times\mathbb{R} = \mathbb{R}^3 = \{(x,y,z);\ \forall\, x,y,z \in \mathbb{R}\}$ is 3-d E.S.

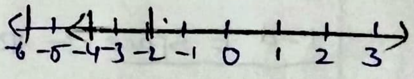


* Distance b/w two points in 1-d E.S :-

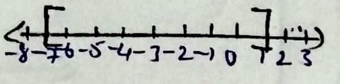


$\Rightarrow \boxed{D(x,y) = |x-y|}$

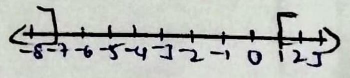
Ex :- ① $|x+4| = 2 \Rightarrow x = -2 \,/\, x = -6$



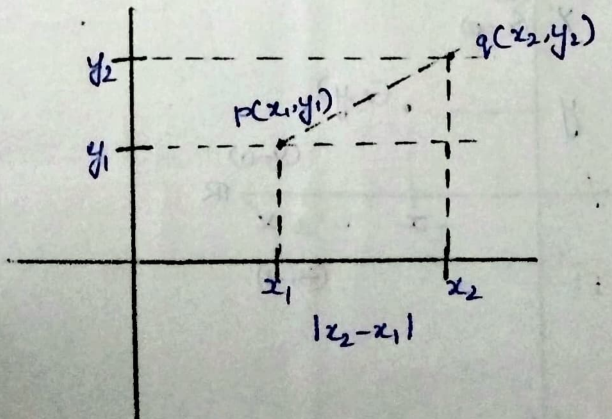
② $|x+3| \le 4 \Rightarrow [-7, 1]$



③ $|x+3| \ge 4 \Rightarrow x \in [-\infty, -7] \cup [1, \infty]$



* Distance b/w two points in 2-d E.S :-



$$\boxed{D = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}}$$

* Neighbourhood of a point :-

→ Let $a = (a_1, a_2, \ldots, a_n) \in \mathbb{R}^n$, $\delta (>0) \in \mathbb{R}$.

the δ - Neighbourhood of a point $a = (a_1, a_2 \ldots a_n) \in \mathbb{R}^n$ is denoted and defined as

$$\boxed{N_\delta(a) = \{ x = (x_1, x_2, \ldots x_n) \in \mathbb{R}^n ;\ d(x,a) < \delta \}}$$

Ex:-

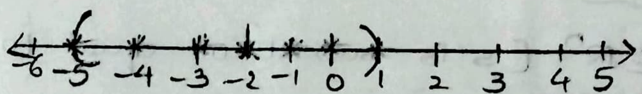
① $-2 \in \mathbb{R}$, $\delta = 3$

then find $N_\delta(-2) = \{ ? \}$

$\Rightarrow N_\delta(-2) = \{ x \in \mathbb{R} ;\ d(x, -2) < 3 \}$.

$= \{ x \in \mathbb{R} ;\ |x+2| < 3 \}$

$= (-5, 1)$ - open interval.



② $P(-2, 1)$, $\delta = +3$, find $N_\delta(P) = ?$

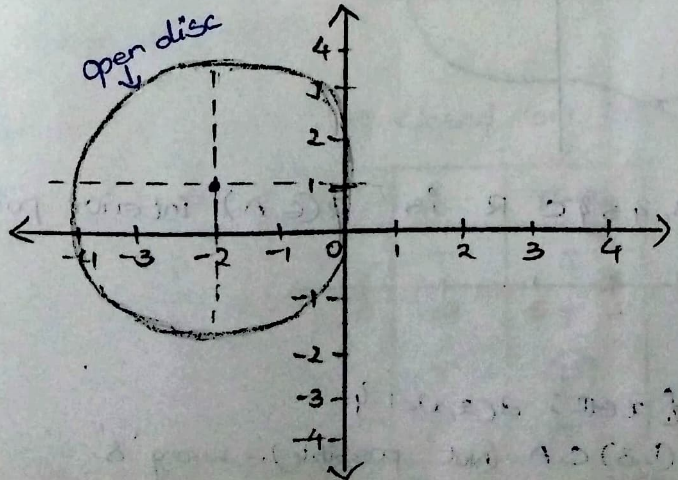
$\Rightarrow P(-2, 1) \in \mathbb{R}^2$.

$\Rightarrow N_\delta(P) = N_{\delta=3}((-2,1)) = \{ (x,y) \in \mathbb{R}^2 ;\ d(x, p) < 3 \}$

$= \{ (x,y) \in \mathbb{R}^2 ;\ \sqrt{(x+2)^2 + (y-1)^2} < 3 \}$

$= \{ (x,y) \in \mathbb{R}^2 ;\ (x+2)^2 + (y-1)^2 < 9 \}$.

= open Disc



③ $p(1,2,3) \in \mathbb{R}^3$, $\delta = 4$, then find $N_\delta(p) = ?$

$\Rightarrow N_{\delta=4}((1,2,3)) = \{(x,y,z) \in \mathbb{R}^3 ;\ (x-1)^2 + (y-2)^2 + (z-3)^2 < 4^2\}$

= open sphere.

* Deleted δ - Neighbourhood of a point :-

$\rightarrow$ Let $a = (a_1, a_2, ---, a_n) \in \mathbb{R}^n$, $\delta > 0$.

The Deleted δ - Neighbourhood of a point $a = (a_1, a_2, --, a_n)$ is denoted and defined as,

$$\boxed{N^*_\delta(a) = \{(x_1, x_2, ---, x_n) \in \mathbb{R}^n ;\ d(x,a) < \delta\} - \{(a_1, a_2, --, a_n)\}}$$

i.e, $\boxed{N^*_\delta(a) = N_\delta(a) - \{a\}}$

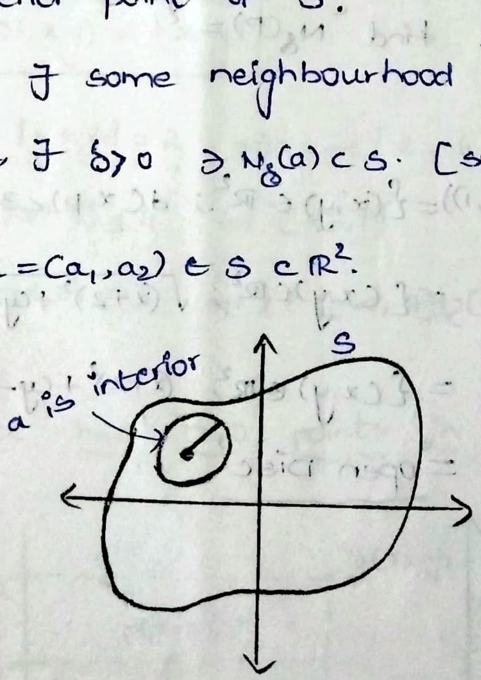
* Interior point :-

$\rightarrow$ Let $S \subseteq \mathbb{R}^n$ [S is subset or equal to $\mathbb{R}^n$], and $a = (a_1, a_2, ---, a_n) \in S$ [Belongs to S] is said to be Interior point of S.

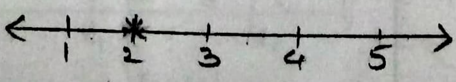
if $\exists$ some neighbourhood of a is in S.

i.e, $\exists\ \delta > 0 \ni N_\delta(a) \subset S$. [subset].

Ex:- $a = (a_1, a_2) \in S \subset \mathbb{R}^2$.



Ex:- $A = \{1, 2, 3, 4, 5\} \subset \mathbb{R}$ is $2 (\in A)$ interior point of A?



Let $\delta = 1$

$\Rightarrow N_1(2) = \{x \in \mathbb{R} ;\ d(2,x) < 1\}$

$= (1, 3) \subset A$ (Not possible) — wrong δ.

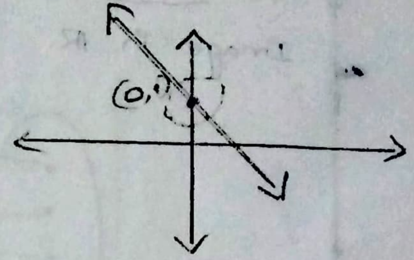
Ex :-

Let $A = (1, 2)$ $\rightarrow$ 1.5 is a interior point

Ex :-

Let $S = \{ (x,y) \in \mathbb{R}^2,\ x+y = 1 \}$

$\rightarrow$ Is $(0,1)$ Interior point of S?



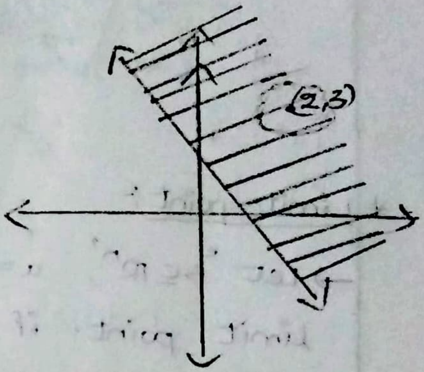
$(0,1)$ Not a interior point.

Ex :-

Let $S = \{ (x,y) \in \mathbb{R}^2,\ x+y \geq 1 \}$

$\rightarrow$ Is $(2,3)$ interior point of S?

Yes.



* Open set :-

$\rightarrow$ Every open interval is a openset.

$\rightarrow$ Every δ - Neighbourhood of a point is always in openset.

* closed set :-

$\rightarrow$ Let $A = \{1, 2, 3, 4, 5\} \in \mathbb{R}$.

$\downarrow$

closed set [Not open set]

Vacuus proof :-

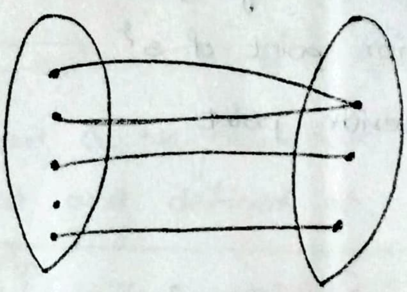
$\rightarrow$ If sun rise in the west then the triangle has 4 sides.

$\rightarrow$ ϕ is open (or) closed (or) Both [clopen].

p	q	$p \rightarrow q$
T	T	T
F	F	T
F	T	F
T	F	T

* Function :- [Multivariable - single valued Function].

→ Let '$\mathbb{R}^n$' be a 'n-dimensional euclidian space'. '$D \subseteq \mathbb{R}^n$' 'f' is mapping from '$D \subseteq \mathbb{R}^n$' to 'R' is said to be 'function' if $\forall\ (x_1, x_2, x_3 ---- x_n) \in D$ has unique Image in $\mathbb{R}$.



* Limit point :-

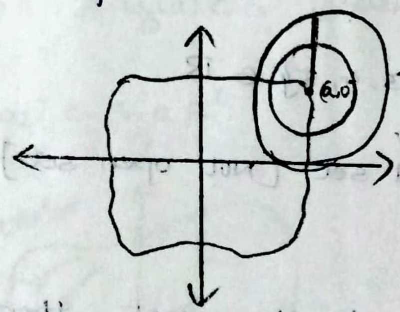
→ Let '$S \subseteq \mathbb{R}^n$', $a = (a_1, a_2, --- a_n) \in \mathbb{R}^n$ is said to be limit point. if for every '$\delta > 0$',

$\Rightarrow \boxed{N_\delta^*(a) \cap S \neq \phi \text{ of } S.}$

Ex:-

$S = (2,3) \in \mathbb{R}$ is

→ Every limit point is an Interior point.



Ex:-

→ Give example of $S \subset \mathbb{R}$, $a \in S$ which neighbour limit nor interior points?

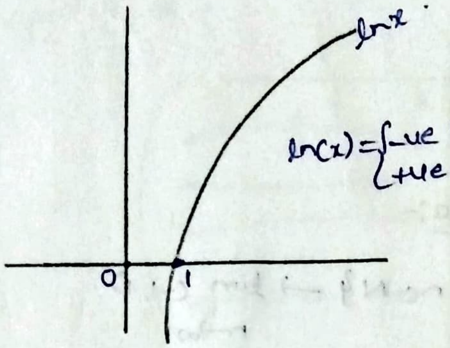
$\{1,2\} \in \mathbb{R}$, $a \in 1$, $\delta = 0.1 \cap S \neq \phi$.

→ Give example of set $S \subset \mathbb{R}$, $a \in S$ which neighbour limit nor interior point & boundary.

* $f(x,y) = e^{x^2+y^2} = e^{x^2} \cdot e^{y^2}$.

Domain: $\mathbb{R} \cdot \mathbb{R} = \mathbb{R}^2$.

Range: $[1, \infty)$



$\ln(x) = \begin{cases} -ve & x \in (0,1) \\ +ve & x \in [1,\infty) \end{cases}$

Domain

$x \in \mathbb{R}$.

$x \longrightarrow x^2 \longrightarrow e^{x^2}$

$y \in \mathbb{R}$.

$y \longrightarrow y^2 \longrightarrow e^{y^2}$

Range

$f(x,y) = e^{x^2+y^2} = 0$

$\Rightarrow x^2+y^2 = \ln(0)$.

$f(x,y) = e^{x^2+y^2} = -1$

$\Rightarrow x^2+y^2 = \ln(-1)$.

$f(x,y) = e^{x^2+y^2} = 0.1$

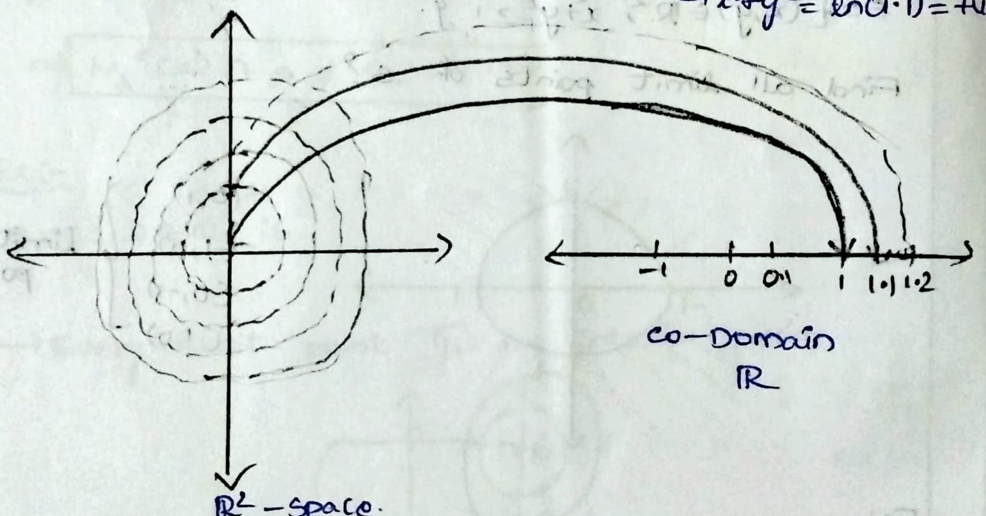
$\Rightarrow x^2+y^2 = \ln(0.1) = -ve$.

$f(x,y) = e^{x^2+y^2} = 1$

$\Rightarrow x^2+y^2 = \ln(1) = 0$.

$f(x,y) = e^{x^2+y^2} = 1.1$

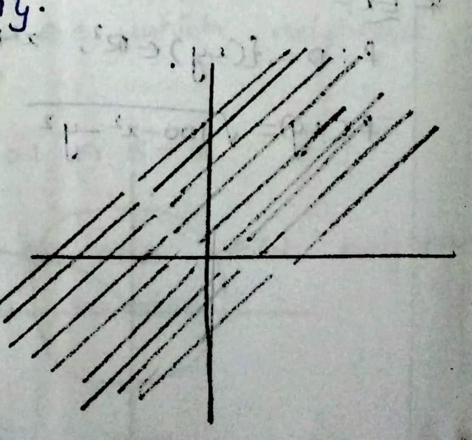
$\Rightarrow x^2+y^2 = \ln(1.1) = +ve$



* $f(x,y) = \dfrac{x^2}{x+y-1}$.

A) Domain: $\mathbb{R}^2 - \{(x,y) \in \mathbb{R}^2 ; x+y \neq 1\}$.

Range:



Let $\frac{x^2}{x+y-1} = -1$

$\Rightarrow x^2 = -x-y+1$

$\Rightarrow x^2+x+y-1=0.$

Let $\frac{x^2}{x+y-1} = -2$

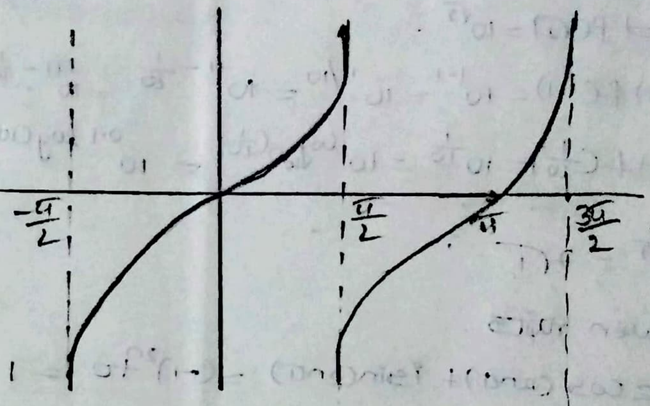
$\Rightarrow x^2 = -2x-2y+2$

$\Rightarrow x^2 = -2(x+y-1)$

$\Rightarrow x^2+2x+2y+2=0.$

* $f(x,y) = \tan^{-1}\left(\frac{y}{x}\right).$

A) <u>Graph of $\tan x$;</u>

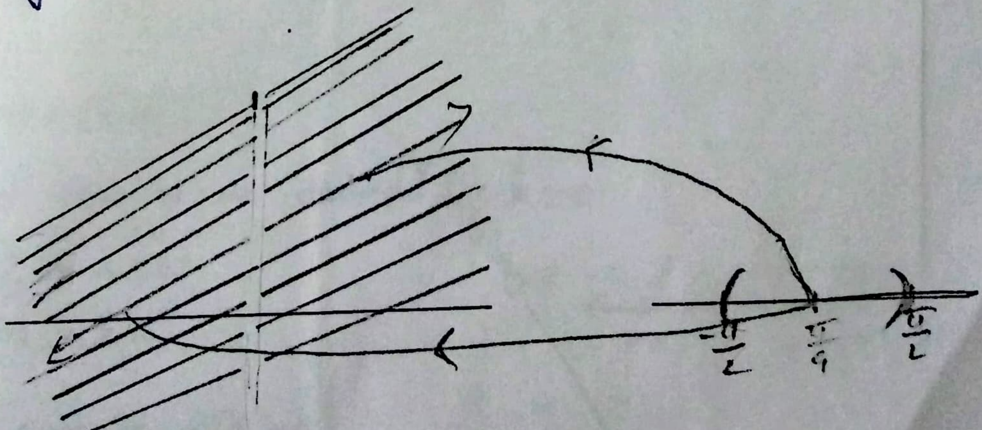


Domain of $\tan x$: $\mathbb{R} - \left\{\frac{(2n+1)\pi}{2} ; n \in \mathbb{Z}\right\}$

Range : $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right).$

Domain of $\tan^{-1}\left(\frac{y}{x}\right)$: ($\mathbb{R}$ − y axis)

Range : $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.



* **Level curve :-**

→ Let $f(x,y)=k$ is called Level curve where $k \in$ Range of (f).

* **Ex :-** $z = f(x,y) = x^2+y^2$

Domain: $\mathbb{R}^2$

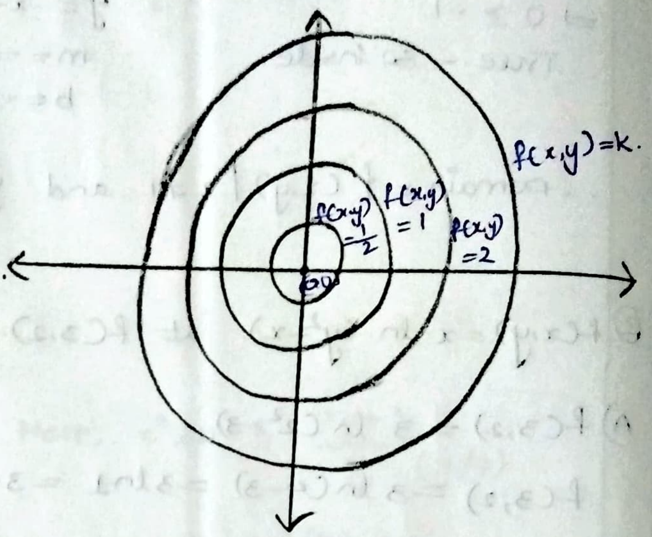
Range : $[0,\infty)$

$\Rightarrow$ Let $k=0$

$\Rightarrow x^2+y^2=0$

$\Rightarrow (x,y)=(0,0)$.

$\Rightarrow x=0$ & $y=0$.



Let $k=\frac{1}{2}$

$\Rightarrow x^2+y^2=\frac{1}{2}$

$\Rightarrow x^2+y^2=(\frac{1}{\sqrt{2}})^2$.

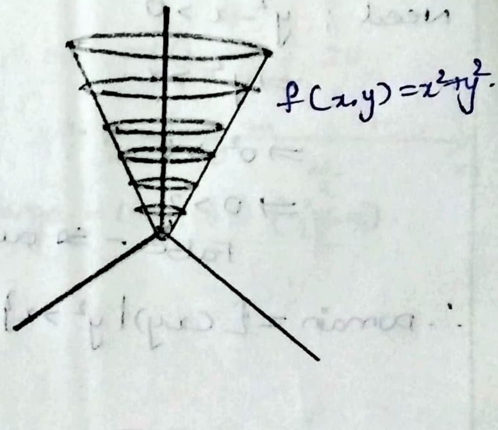
Let $k=1$

$\Rightarrow f(x,y)=1$

$\Rightarrow x^2+y^2=1$

Let $k=2$

$\Rightarrow x^2+y^2=(\sqrt{2})^2$.



* **Ex-2 :-** $f(x,y)=\sqrt{100-x^2+y^2}$.

A) Given, $f(x,y)=\sqrt{100-x^2+y^2}$

Domain: ~~$(0,\infty)$~~ closed Disk - $[x^2+y^2 \le 100]$ (or) $\{(x,y)\in\mathbb{R}^2; x^2+y^2\le 100\}$.

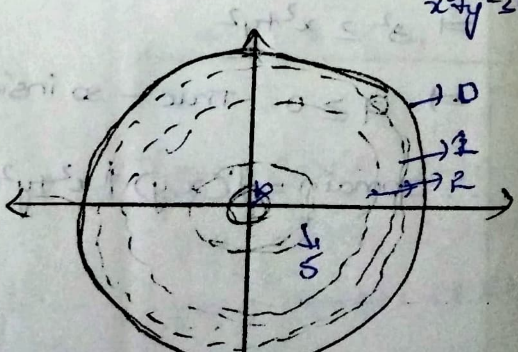
Range: $[0,10]$.

Let $k=0$, $\Rightarrow f(x,y)=0$

$\Rightarrow \sqrt{100-x^2+y^2}=0$

$\Rightarrow x^2+y^2=100$

$\Rightarrow x^2+y^2=(10)^2$.



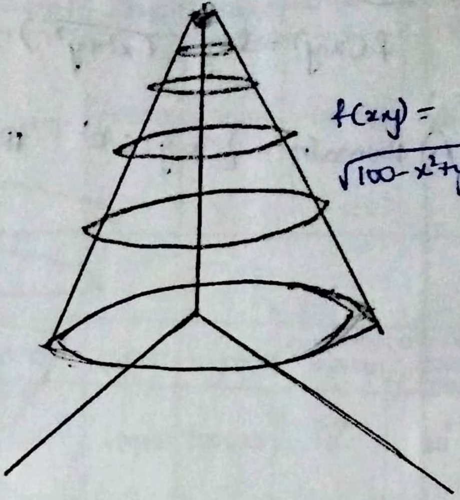
Let $k=1$ $\Rightarrow \sqrt{100-x^2+y^2}=1$

$\Rightarrow x^2+y^2=99$.

Let $k=2 \Rightarrow f(x,y)=2$

$\Rightarrow \sqrt{100-x^2+y^2}=2$

$\Rightarrow x^2+y^2=96$



$f(x,y)=\sqrt{100-x^2+y^2}$

Let $k=5 \Rightarrow f(x,y)=5$

$\Rightarrow \sqrt{100-x^2+y^2}=5$

$\Rightarrow x^2+y^2=75.$

Let $k=10 \Rightarrow f(x,y)=10$

$\Rightarrow \sqrt{100-x^2+y^2}=10$

$\Rightarrow x^2+y^2=0.$

* Ex-3 :-

$f(x,y)=\sin(x+y).$

A) Given, $f(x,y)=\sin(x+y).$

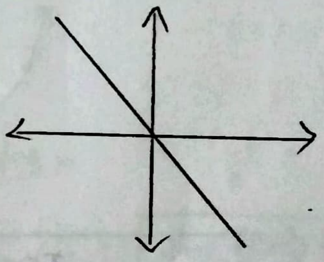
Domain $=\mathbb{R}^2$

Range $=[-1,1]$.

$f(x,y)=x+y$:-

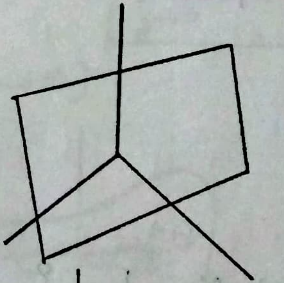
Domain $=\mathbb{R}$

Range $=\mathbb{R}$.



Level curves are 'Lines'.

Surface is 'plane'.

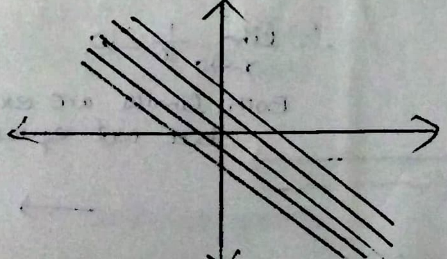


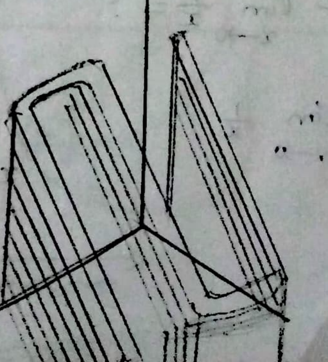
$\Rightarrow z=x+y$

$\Rightarrow x+y-z=0.$

Let $k=0 \Rightarrow f(x,y)=0$

$\Rightarrow \sin(x+y)=0$

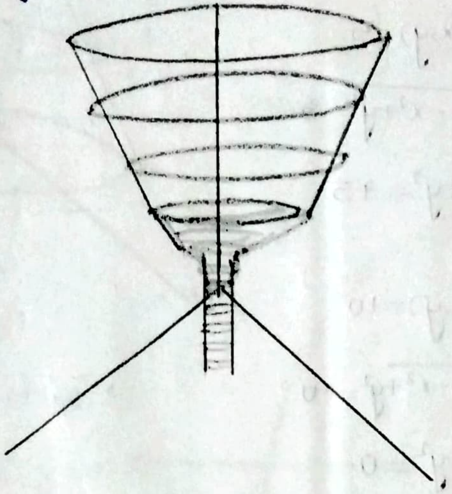




* Ex-4 :-

$f(x,y) = \log(\sqrt{x^2+y^2})$.

A) Domain :- $\{(x,y) \in R^2 : R - 0\}$.



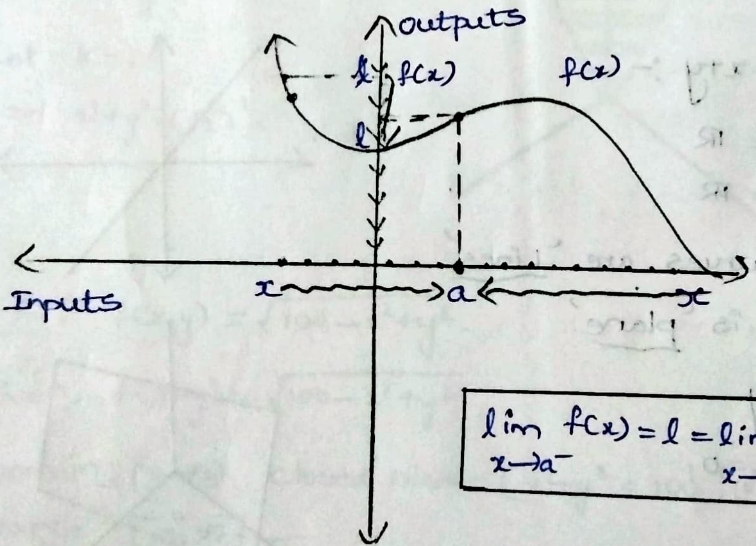
* **Limit of a function :-**

→ $\lim_{x \to a} f(x) = l$.

if 'x' tends to 'a' then 'f(x)' tends to 'l'.

if 'x' goes to 'a' then 'f(x)' goes to 'a'.

$x \to a$ then $f(x) \to l$.

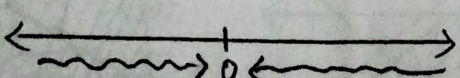


$$\boxed{\lim_{x \to a^-} f(x) = l = \lim_{x \to a^+} f(x).}$$

* Ex :- $\lim_{x \to 0} \frac{1}{x} = ?$

A) $\lim_{x \to 0} \frac{1}{x}$.

$x \to 0$



$\therefore \lim_{x \to 0} \frac{1}{x} = \infty$.

Both limits are exist but Not equal.

→

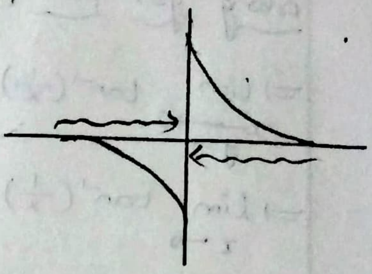
$\lim_{x \to 0^-}$

x	-0.5	-0.4	-0.3	-0.2	-0.1	-0.01	-0.001	-0.0000	-0.00000	-0.00000001
$f(x)=\frac{1}{x}$	-2	-2.5	-3.33	-5	-10	-100	-1000	-10000	-100000	-10^7.

$\lim_{x \to 0^+}$

x	0.5	0.4	0.3	0.2	0.1	0.01	0.001	0.0000	0.00000000001	0.000000000000000001
$f(x)=\frac{1}{x}$	2	2.5	3.33	5	10	100	1000	10000	10^{10}	10^{17}

* Ex :- $\lim_{\substack{\text{time} \to 2026 \\ \text{(or)} \\ \text{year}}}$ ♀(k) $= \infty$.



* Ex :- $\lim_{x \to 0} \tan^{-1}\left(\frac{1}{x}\right) = ?$

$\Rightarrow \lim_{x \to 0^-} \tan^{-1}\left(\frac{1}{x}\right) = \frac{1}{-0} = \infty = \frac{-\pi}{2} = -1.57$.

$\Rightarrow \lim_{x \to 0^+} \tan^{-1}\left(\frac{1}{x}\right) = \frac{1}{0} = \infty = \frac{\pi}{2} = 1.57$.

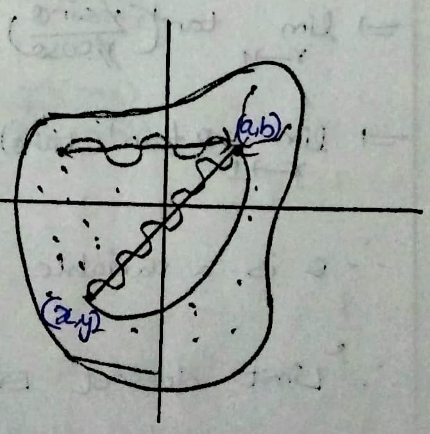
$\therefore \lim_{x \to 0} \tan^{-1}\left(\frac{1}{x}\right) = 1.57$.

* Limit of two variable functions :-

$\rightarrow \lim_{\boxed{x \to a}} f(x) = \boxed{l}$ [1-variable].
input output

(or). $\boxed{\lim_{\substack{x \to a \\ y \to b}} f(x,y) = L}$

$\rightarrow \lim_{\boxed{(x,y) \to (a,b)}} f(x,y) = \boxed{L}$ [$\because f : D \subseteq \mathbb{R}^2 \to \mathbb{R}$].
input output



$\lim_{(x,y) \xrightarrow[\text{path - I}]{\text{along}} (a,b)} f(x,y) = L$

$\lim_{(x,y) \xrightarrow[\text{path - II}]{\text{along}} (a,b)} f(x,y) = L$.

⋮

Infinite paths.

* $\lim_{(x,y)\to(0,0)} \tan^{-1}\left(\frac{y}{x}\right)$.

A) $\lim_{x \xrightarrow[y=0]{along} 0} \tan^{-1}\left(\frac{0}{x}\right) = \tan^{-1} 0 = 0$.

Along $y=x^2$ path :-

$\lim_{(x,y) \xrightarrow[y=x^2]{} (0,0)} \tan^{-1}\left(\frac{y}{x}\right)$.

$\Rightarrow \lim_{x\to 0} \tan^{-1}\left(\frac{x^2}{x}\right)$.

$\Rightarrow \lim_{x\to 0} \tan(x) = 0$.

Along $y=1$ path :-

$\Rightarrow \lim_{x \xrightarrow[y=1]{} 0} \tan^{-1}\left(\frac{1}{x}\right)$

$\Rightarrow \lim_{x\to 0} \tan^{-1}\left(\frac{1}{x}\right)$.

$x\to 0^-$ $\tan\frac{1}{x} \to -\infty$ $\Rightarrow \lim_{x\to 0^-} \tan^{-1}\left(\frac{1}{x}\right) = -\frac{\pi}{2}$

$x\to 0^+$ $\tan\frac{1}{x} \to \infty$ $\Rightarrow \lim_{x\to 0^+} \tan^{-1}\left(\frac{1}{x}\right) = \frac{\pi}{2}$.

Mathematically [using polar coordinates] :-

$x = r\cos\theta$

$y = r\sin\theta$

$x^2+y^2 = r^2 \Rightarrow r = \sqrt{x^2+y^2}$.

Along $y=1$ path [i.e, (0,1)].

$\Rightarrow (x,y) \to (0,1)$ then $r = \sqrt{0^2+1^2} = 1 \Rightarrow r \to 1$.

$\Rightarrow \lim_{r\to 1} \tan^{-1}\left(\frac{\cancel{r}\sin\theta}{\cancel{r}\cos\theta}\right)$

$\Rightarrow \lim_{r\to 1} \tan^{-1}(\tan\theta) = \lim_{r\to 1} \theta = \theta$.

$\because$ θ is a variable containing many values.

$\therefore$ 'Limit doesnot Exist'

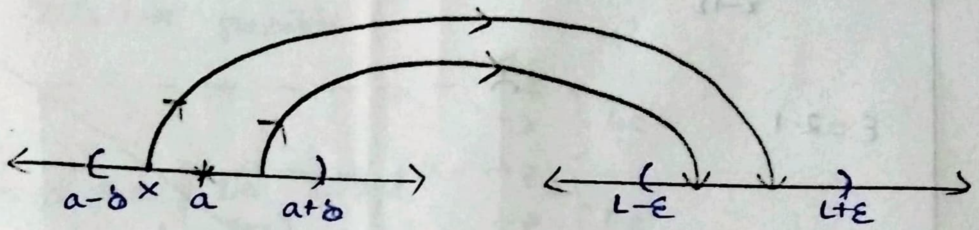
* Epsilon Delta Definition:-

→ $\lim_{x \to a} f(x) = L$,

For given '$\epsilon > 0$', there exists '$\delta > 0$' such that if $x \in N_\delta^*(a)$ then $f(x) \in N_\epsilon(L)$.

→ Here 'δ' is a function of 'ϵ'
i.e, 'δ' depends on the 'ϵ'

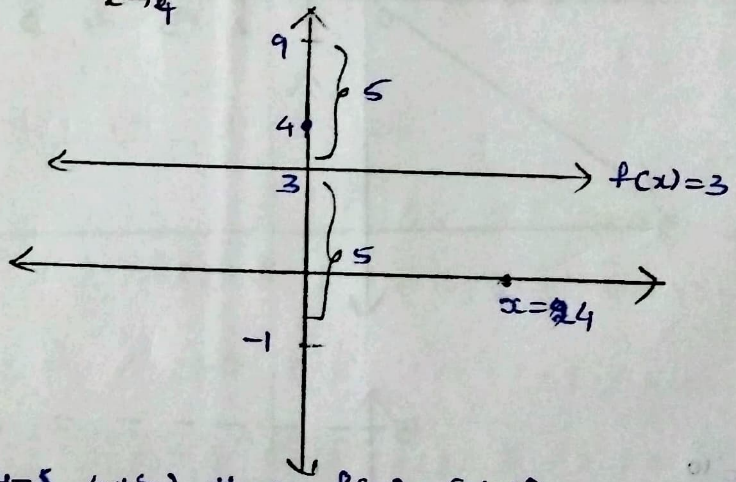
→ Inputs & outputs.
if $x \in (a-\delta, a+\delta) - \{a\}$ then $f(x) \in (L-\epsilon, L+\epsilon)$.



Ex:-

Prove that $\lim_{x \to 4} 3 \neq 4$.

sol:-



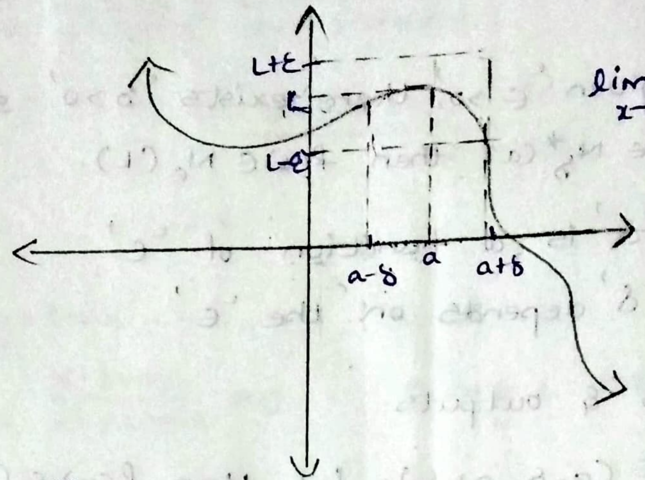
$\epsilon = 5$:
if $x \in (4-\delta, 4+\delta)$ then $f(x) = (-1, 9)$.

$\epsilon = 5$ & $\delta = 0.1$
if $x \in (3.9, 4.1) - \{4\} \rightarrow 3 \in (-1, 9)$.

$\epsilon = 4$ & $\delta = 0.1$
if $x \in (4-\delta, 4+\delta)$ then $f(x) = (0, 8)$
if $x \in (4-\delta, 4+\delta) - \{4\} \Rightarrow f(x) \in (0, 8)$.
$\epsilon = 4$ & $\delta = 0.001$
$\therefore$ It's not possible.

* Prove that $\lim_{x \to 2} x \neq 5$?

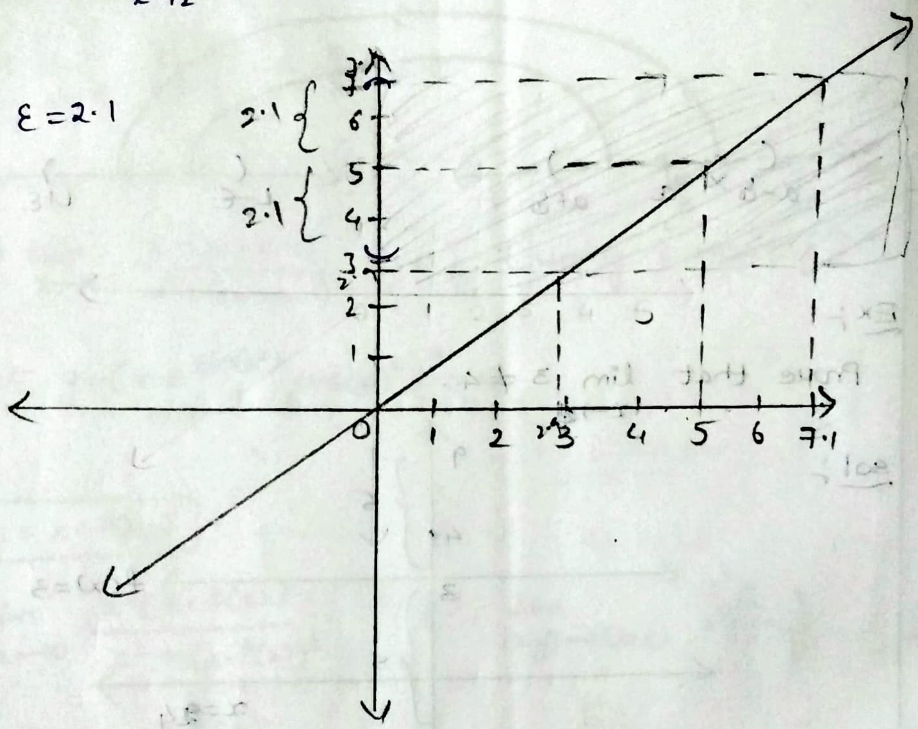
A)



$\lim_{x \to a} f(x) = l$.

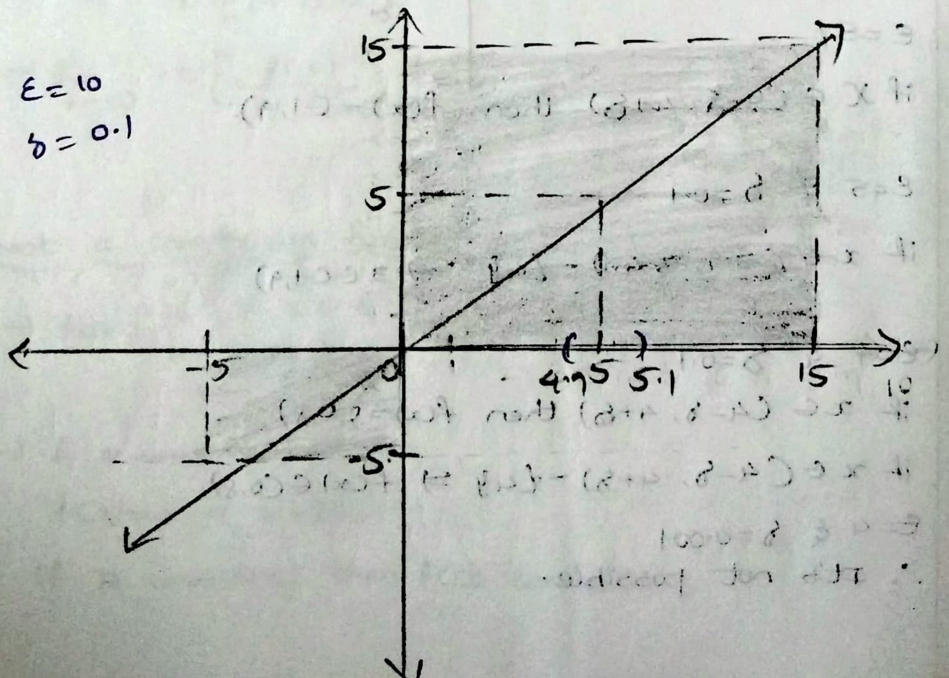
Given $\lim_{x \to 2} x \neq 5$.

$\varepsilon = 2.1$



$\varepsilon = 10$

$\delta = 0.1$



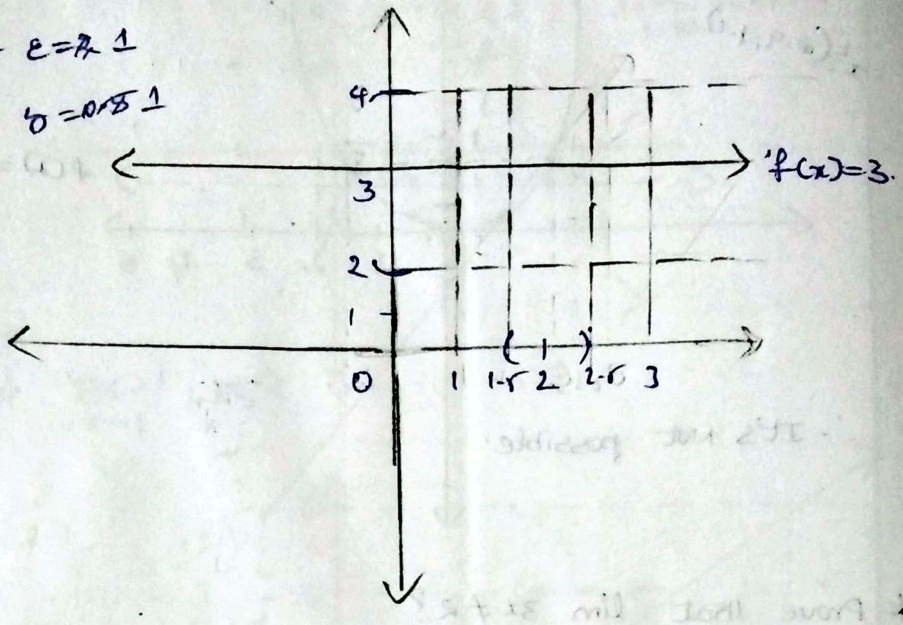
* Show that by using $\varepsilon - \delta$ definition.

$$\lim_{x \to 2} 3 = 3.$$

A)

Let $\varepsilon = \cancel{2}\ 1$

$\delta = \cancel{0.5}\ 1$



For given $\varepsilon > 0$, $\delta = ?$

such that

if $x \in (2-\delta, 2+\delta) - \{2\}$ then $\delta \in (3-\varepsilon, 3+\varepsilon)$.

if $x \in$.

Here,

for given $\varepsilon > 0$, $\delta = \varepsilon$.

since it is constant function.

if $x \in (2-1, 2+1) - \{2\}$ then $\delta = (3-1, 3+1)$

$\Rightarrow$ $x \in (1, 3) - \{2\}$ then $\delta = (2, 4)$

* Epsilon - Delta Definition :-

Given $\varepsilon > 0$, $\exists\ \delta > 0$, $\ni$

if $x \in N_\delta^*(a)$ then $f(x) \in N_\varepsilon(L)$.

$\Rightarrow$ if $0 < |x-a| < \delta$ then $|f(x) - L| < \varepsilon$.

$\Rightarrow$ if $x \in (a-\delta, a+\delta) - \{a\}$, then $f(x) \in (L-\varepsilon, L+\varepsilon)$.

* show that $\lim_{x\to 0} \frac{\sin x}{x} = 1$.

A) For every '$\epsilon>0$', $\exists$ '$\delta>0$' $\ni$ if '$x \in N^*_\delta(0)$' then '$f(x) \in N_\epsilon(1)$'.

i.e, If '$x \in (-\delta, \delta) - \{0\}$' then '$f(x) \in (1-\epsilon, 1+\epsilon)$'.

i.e, if '$0<|x|<\delta$' then '$|f(x)-1|<\epsilon$'.

we known that,

$|\sin x| \le |x| \longrightarrow ①$

$|x| \le |\tan x| \longrightarrow ②$

$N_\delta(a) = \{x \in \mathbb{R}; d(x,a)<\delta\}$
$= \{x \in \mathbb{R}; |x-a|<\delta\}$
$= \{x \in \mathbb{R}; 0<|x-a|<\delta\}$

From ②

$\Rightarrow |x| \le \left|\frac{\sin x}{\cos x}\right|$.

$\Rightarrow |\cos x| \le \left|\frac{\sin x}{x}\right|$

$\Rightarrow |\cos x| \le \left|\frac{\sin x}{x}\right| \le 1$ [$\because$ we know $|\cos x| \le 1$].

$\Rightarrow |\cos x - 1| \le \left|\frac{\sin x}{x} - 1\right| \le 1-1$

$\Rightarrow |\cos x - 1| \le \left|\frac{\sin x}{x} - 1\right| \le 0$

$\Rightarrow |1-\cos x| \ge \left|\frac{\sin x}{x} - 1\right|$.

$\Rightarrow \left|\frac{\sin x}{x} - 1\right| \le \left|\frac{1+\cos x}{1+\cos x} \times 1-\cos x\right|$

$\Rightarrow \left|\frac{\sin x}{x} - 1\right| \le \left|\frac{1-\cos^2 x}{1+\cos x}\right|$

$\Rightarrow \left|\frac{\sin x}{x} - 1\right| \le \left|\frac{\sin^2 x}{1+\cos x}\right|$ [$\because$ Identity].

$\Rightarrow \left|\frac{\sin x}{x} - 1\right| \le \left|\frac{\sin^2 x}{1+\cos x}\right| \le |\sin^2 x|$ [$\because \left|\frac{4}{2}\right| \le |4|$].

$\Rightarrow \left|\frac{\sin x}{x} - 1\right| \le \left|\frac{\sin^2 x}{1+\cos x}\right| \le |\sin^2 x| \le |x^2|$ [$\because$ From ①]

$\Rightarrow \left|\frac{\sin x}{x} - 1\right| \le \delta^2 < \epsilon$.

Ex:- $\epsilon=1$, $\delta \le 1$
$\epsilon=100$, $\delta \le 10$

$\Rightarrow \boxed{\delta^2 \le \epsilon}$ $\therefore \boxed{\delta \le \sqrt{\epsilon}}$.

$\therefore$ "$\lim_{x\to 0} \frac{\sin x}{x} = 1$"

$\therefore$ For every $\epsilon>0$, $\delta \le \sqrt{\epsilon}$ then if $0<|x|<\delta$ then $\left|\frac{\sin x}{x} - 1\right| < \epsilon$.

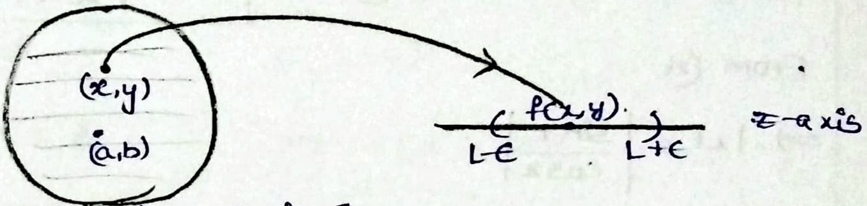
* Epsilon-Delta definition for two variables :-

$$\lim_{(x,y)\to(a,b)} f(x,y) = L.$$

→ (x,y) are inputs.

→ For every $\epsilon > 0$, there exist $\delta > 0$ such that if $(x,y) \in N^*_\delta(a,b)$ then $f(x,y) \in N_\epsilon(L)$.

i.e, $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$ then $|f(x,y) - L| < \epsilon$.



xy plane [Disk].

* Epsilon-Delta Definition for 'n' variables :-

$$\lim_{(x_1, x_2, \ldots x_n)} f(x_1, x_2, \ldots x_n) = L.$$

→ For every given $\epsilon > 0$, there exist $\delta > 0$ such that if $(x_1, x_2, \ldots x_n) \in N^*_\delta(a_1, a_2, \ldots a_n)$ then $f(x_1, x_2, \ldots x_n) \in N_\epsilon(L)$.

i.e, if $0 < \sqrt{(x_1-a_1)^2 + (x_2-a_2)^2 + \ldots + (x_n-a_n)^2} < \delta$ then $|f(x_1, x_2, \ldots x_n) - L| < \epsilon$.

* Find $\displaystyle\lim_{(x,y)\to(0,0)} \frac{2xy^2}{x^2+y^2}$?

A) Along $y=x$ path :-

$$\Rightarrow \lim_{(x,y)\to(0,0)} \frac{2xy^2}{x^2+y^2} = \lim_{(x,y)\xrightarrow[y=x]{\text{along}}} \frac{2xy^2}{x^2+y^2}.$$

$$= \lim_{x\to 0} \frac{2x \cdot x^2}{x^2+x^2}$$

$$= \lim_{x\to 0} \frac{\cancel{2}x \cdot \cancel{x^2}}{\cancel{2}\cancel{x^2}} = \lim_{x\to 0} x = 0.$$

Along $y=2x$ path :-

$$\lim_{(x,y)\xrightarrow[y=2x]{\text{along}}(0,0)} \frac{2xy^2}{x^2+y^2} = \lim_{x\to 0} \frac{2x(2x)^2}{x^2+(2x)^2}$$

$$= \lim_{x\to 0} \frac{2x\cdot 4x^2}{x^2+4x^2}$$

$$= \lim_{x\to 0} \frac{\cancel{x^2}[2x\cdot 4]}{\cancel{x^2}[1+4]}.$$

$$= \lim_{x\to 0} \frac{8x}{5} = 0$$

Along $y=mx$ path :-

$$\lim_{(x,y)\xrightarrow[y=mx]{\text{along}}(0,0)} \frac{2xy^2}{x^2+y^2} = \lim_{x\to 0} \frac{2x(mx)^2}{x^2+(mx)^2}$$

$$= \lim_{x\to 0} \frac{2x\cdot m^2x^2}{x^2+m^2x^2} = \lim_{x\to 0} \frac{\cancel{x^2}[2xm^2]}{\cancel{x^2}[1+m^2]}$$

$$= \lim_{x\to 0} x\cdot\left[\frac{2m^2}{1+m^2}\right] = 0$$

Along $y=x^k\cdot g(x)$ path :-

$$\lim_{(x,y)\to(0,0)} \frac{2xy^2}{x^2+y^2} = \lim_{x\to 0} \frac{2x(x^{2k}\cdot g(x)^2)}{x^2+x^{2k}\cdot g(x)^2} = 0$$

We can suspect that "$\lim_{(x,y)\to(0,0)} \frac{2xy^2}{x^2+y^2} = 0$"

$$\Rightarrow \lim_{(x,y)\to(0,0)} \frac{2xy^2}{x^2+y^2} = 0.$$

if $0<\sqrt{x^2+y^2}<\delta$ then $\left|\frac{2xy^2}{x^2+y^2}-0\right|<\epsilon$.

$$\Rightarrow \left|\frac{2xy^2}{x^2+y^2}\right| < \epsilon \longrightarrow (1)$$

$$\left|\frac{2xy^2}{x^2+y^2}\right| = |2x|\cdot\left|\frac{y^2}{x^2+y^2}\right|$$

$$\leq |2x|\cdot|1|$$

$$\leq 2|x|$$

$$\leq 2|\sqrt{x^2}|$$

$$\leq 2\left|\sqrt{x^2+y^2}\right|$$

$$\leq 2\delta$$

$$\leq 2\delta \leq \epsilon \quad \therefore \delta \leq \frac{\epsilon}{2}.$$

Ex:-

$2=2$

$2\leq 2+1$

$\frac{2}{3}\leq 1$

* $\lim_{(x,y)\to(0,0)} x \sin\left(\frac{1}{y}\right)$.

Find the limit?

A) $\epsilon > 0$, $\delta > 0$.

$0 < \sqrt{(x-0)^2 + (y-0)^2} < \delta$

$\Rightarrow 0 < \sqrt{x^2+y^2} < \delta \longrightarrow (1)$

Now $|f(x,y) - L| < \epsilon$

$\Rightarrow \left| x \cdot \sin\left(\frac{1}{y}\right) - L \right| < \epsilon$

We need to find 'L' value.

Along 'x=0' path:-

$$\lim_{(x,y) \xrightarrow[x=0]{\text{Along}} (0,0)} x \cdot \sin\left(\frac{1}{y}\right) = \lim_{x\to 0} x \cdot \sin\left(\frac{1}{y}\right)$$

$$= 0 \cdot \sin\left(\frac{1}{y}\right)$$

$$= 0$$

Along 'y=x' path:-

$$\lim_{(x,y) \xrightarrow[y=x]{\text{Along}} (0,0)} x \cdot \sin\left(\frac{1}{y}\right) = \lim_{x\to 0} x \cdot \sin\left(\frac{1}{y}\right)$$

$$= \lim_{x\to 0} x \cdot \sin\left(\frac{1}{x}\right). \quad [\because y = x].$$

$$= \lim_{x\to 0} \frac{1}{\frac{1}{x}} \sin\left(\frac{1}{x}\right)$$

$$= \lim_{x\to 0} \frac{\sin\left(\frac{1}{x}\right)}{\left(\frac{1}{x}\right)} \quad \left[\begin{array}{l} x\to 0 \text{ then} \\ \frac{1}{x} \to \infty \end{array}\right]$$

$$= \lim_{\frac{1}{x}\to\infty} \frac{\sin\left(\frac{1}{x}\right)}{\left(\frac{1}{x}\right)}$$

(or)

$$= \lim_{x\to 0} x \cdot \sin\left(\frac{1}{x}\right)$$

$$= 0.$$

$\because$ since $\sin\left(\frac{1}{x}\right)$ is a bounded value, so

$$\lim_{x\to 0} x \cdot \sin\left(\frac{1}{x}\right) = \lim 0 \cdot \sin\left(\frac{1}{x}\right) = 0.$$

Along the path $y = mx$

$$\lim_{(x,y) \xrightarrow[y=mx]{\text{Along}} (0,0)} x \cdot \sin\left(\frac{1}{y}\right) = \lim_{x \to 0} x \cdot \sin\left(\frac{1}{mx}\right) = 0.$$

$\therefore$ We suspected that $\lim_{(x,y) \to (0,0)} x \sin\left(\frac{1}{y}\right) = 0.$

For given $\varepsilon > 0$, $\exists$ $\delta > 0$, such that if $(x,y) \in N_\delta^*(0,0)$

then $f(x,y) \in N_\varepsilon L$.

$\Rightarrow 0 < \sqrt{x^2+y^2} < \delta$ then $\left|x \sin\left(\frac{1}{y}\right) - 0\right| < \varepsilon$.

Now

$\Rightarrow \left|x \sin\left(\frac{1}{y}\right)\right| < \varepsilon \longrightarrow (1)$

$$\Rightarrow \left|x \sin\left(\frac{1}{y}\right)\right| = |x| \left|\sin\left(\frac{1}{y}\right)\right|$$

$$\leq |x| \underbrace{\left|\sin\left(\frac{1}{y}\right)\right| < 1}_{}$$

$$\leq |x|$$

$$\leq |\sqrt{x^2}|$$

$$\leq \left|\sqrt{x^2+y^2}\right|.$$

$$\leq \delta \leq \varepsilon.$$

$\therefore$ For given $\varepsilon > 0$, choose $\underline{\delta \leq \varepsilon}$.

* Find $\lim\limits_{(x,y)\to(0,0)} \frac{2x^3}{x^2+y^2}$

A) Along $x=0$ path : $\frac{2(0)}{y^2} = 0$

Along $y=0$ path : $2x = 2(0) = 0$

Along $y=mx$ path : $\lim\limits_{x\to 0} \frac{2x^3}{x^2+m^2x^2} = \frac{2x}{1+m^2} = 0$

Along $y=x^k \cdot g(x)$ path : $\lim\limits_{x\to 0} \frac{2x^3}{x^2+(x^k)^2} = 0$

We suspected that $\lim\limits_{(x,y)\to(0,0)} \frac{2x^3}{x^2+y^2} = 0$.

i.e, $0 < \sqrt{x^2+y^2} < \delta$ and $\left| \frac{2x^3}{x^2+y^2} - 0 \right| < \varepsilon$.

$\Rightarrow \left| \frac{2x^3}{x^2+y^2} \right| < \varepsilon \longrightarrow (1)$.

$\Rightarrow |2x| \left| \frac{x^2}{x^2+y^2} \right|$

$\Rightarrow |2x| \underbrace{\left| \frac{x^2}{x^2+y^2} \right| \leq 1}$

$\Rightarrow |2x|$

$\Rightarrow 2|x| \Rightarrow 2|\sqrt{x^2}| \Rightarrow 2|\sqrt{x^2+y^2}|$

$\Rightarrow 2|\sqrt{x^2+y^2}| \leq 2\delta$

$\Rightarrow 2\delta \leq \varepsilon$

$\Rightarrow \underline{\delta \leq \frac{\varepsilon}{2}}$

* Find $\lim_{(x,y)\to(0,0)} (x^2+y^2)\sin\left(\frac{1}{x^2+y^2}\right)$.

A) Along $y=0$ path :- '0'.

Along $x=0$ path :- $y^2\sin\left(\frac{1}{y^2}\right)$

Along $y=mx$ path :- $(x^2+m^2x^2)\sin\left(\frac{1}{x^2+m^2x^2}\right)$

:- sin $x^2(1+m^2)\sin\left(\frac{1}{x^2+m^2x^2}\right)$.

:- $(1+m^2)\lim_{x\to 0} x^2\sin\left(\frac{1}{x^2+m^2x^2}\right) = '0'$.

We suspected $\lim_{(x,y)\to(0,0)} (x^2+y^2)\sin\left(\frac{1}{x^2+y^2}\right)=0$.

$0<\sqrt{x^2+y^2}<\delta$ & $\left|(x^2+y^2)\sin\left(\frac{1}{x^2+y^2}\right)-0\right|<\varepsilon$.

$\Rightarrow \left|(x^2+y^2)\sin\left(\frac{1}{x^2+y^2}\right)\right|<\varepsilon \longrightarrow (1)$

$\Rightarrow |x^2+y^2|\,\left|\sin\left(\frac{1}{x^2+y^2}\right)\right|$.

$\leq |x^2+y^2|$ $\left[\underbrace{\left|\sin\left(\frac{1}{x^2+y^2}\right)\right|}\leq 1\right]$.

$\leq |x^2+y^2|$.

$\leq \left|\sqrt{(x^2+y^2)}^{\,2}\right| \leq \delta^2$

$\leq \delta^2 \leq \varepsilon$.

$\leq$ '$\delta \leq \sqrt{\varepsilon}$'

$\therefore$ '$\delta \leq \sqrt{\varepsilon}$'

* continuity :-

→ Let $f(x)$ be a function is said to be continuous at $x=a$,

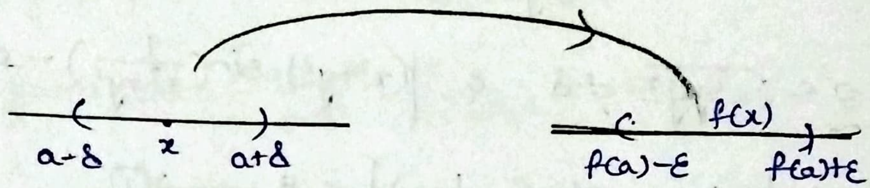
if (i) '$f(x)$' is defined at '$x=a$'.

(ii) '$\lim_{x \to a} f(x) = L$' [L is finite]

(iii) '$L = f(a)$'.

* ε-δ definition :-

→ For given '$\varepsilon > 0$', there exist '$\delta > 0$' such that if '$x \in N_\delta(a)$' then '$f(x) \in N_\varepsilon(f(a))$'.

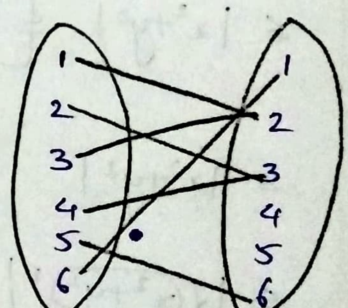


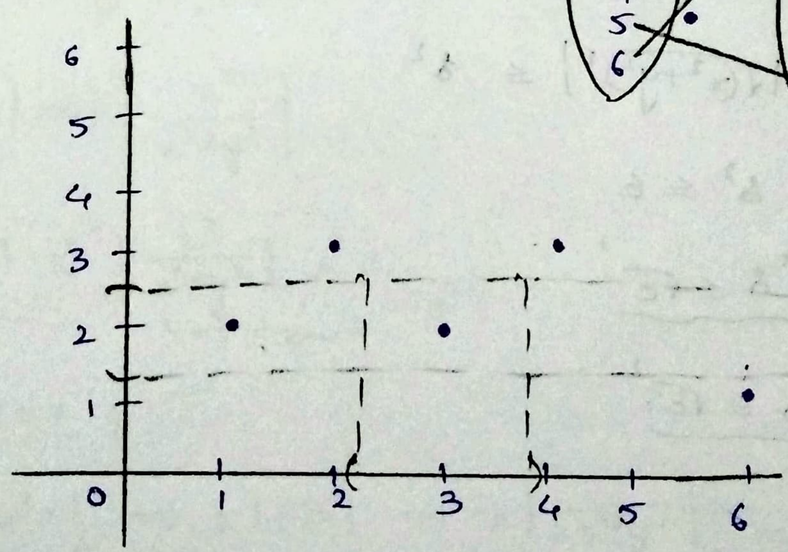
i.e, if '$|x-a| < \delta$' then '$|f(x) - f(a)| < \varepsilon$'.

* Example :-

$A = \{1, 2, 3, 4, 5, 6\}$.

$B = \{1, 2, 3, 4, 5, 6\}$





$\lim_{x \to 3} f(x) = \underline{\quad 2. \quad}$

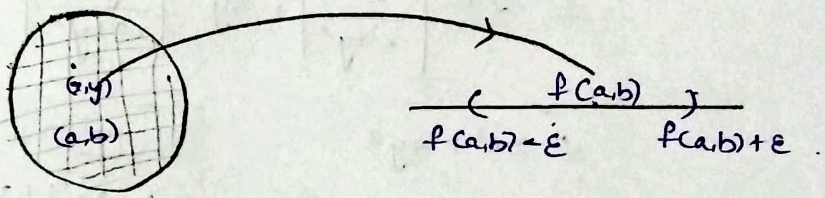
* **continuity for two variables :-**

→ Let $f(x,y)$ be a function, $D \subseteq \mathbb{R}^2 \to \mathbb{R}$ is said to be continuous at a point $(a,b) \in D$.

if $\boxed{\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b).}$

* **ε-δ definition :-**

→ For every given 'ε > 0', ∃ 'δ > 0', ∋ if $(x,y) \in N_\delta(a,b)$ then $f(x,y) \in N_\epsilon(f(a,b))$.



i.e, '$\sqrt{(x-a)^2 + (y-b)^2} < \delta$' then '$|f(x,y) - f(a,b)| < \epsilon$.'

* Discuss the continuity for $f(a,b) = x^2+y^2$ at $(0,0)$?

A) $f(a,b) = x^2+y^2$

At $(0,0)$

$\Rightarrow f(0,0) = 0^2+0^2 = 0.$

$\therefore \lim_{(x,y)\to(a,b)} x^2+y^2 = 0.$

$\Rightarrow \lim_{r\to 0} r^2 = 0.$ — ∴ It is continuous.

At $(1,1)$

$\Rightarrow f(1,1) = 1^2+1^2 = 2$

$\therefore \lim_{(x,y)\to(a,b)} x^2+y^2 = 2.$

$\therefore \lim_{r\to\sqrt{2}} r^2 = 2.$

∴ It is continuous.

∴ It is continuous at every point in $\mathbb{R}^2$ space.

* Discuss the continuity of the function,

$$f(x,y) = \begin{cases} \dfrac{2x^4+3y^4}{x^2+y^2} & \text{at } (x,y) \neq (0,0). \\ 0 & \text{at } (x,y) = (0,0). \end{cases}$$

A) At (0,0) ⊢

$$f(0,0) = \frac{2(0^4)+3(0)^4}{0^2+0^2} = \frac{0}{0} = \text{Not defined.}$$

$$\left|\frac{2x^4+3y^4}{x^2+y^2}\right| \leq 2\left|\frac{x^4}{x^2+y^2}\right| + 3\left|\frac{y^4}{x^2+y^2}\right|$$

$$\leq 2x^2\left|\frac{x^2}{x^2+y^2}\right| + 3y^2\left|\frac{y^2}{x^2+y^2}\right|$$

$$\leq 2x^2+3y^2$$

$$\leq 2x^2+3y^2+x^2$$

$$\leq 3x^2+3y^2$$

$$\leq 3(x^2+y^2).$$

$$\leq 3\delta^2 \leq \varepsilon$$

$$\therefore \delta \leq \sqrt{\frac{\varepsilon}{3}}.$$

By polar ⊢

$$\Rightarrow \frac{2x^4+3y^4}{x^2+y^2}$$

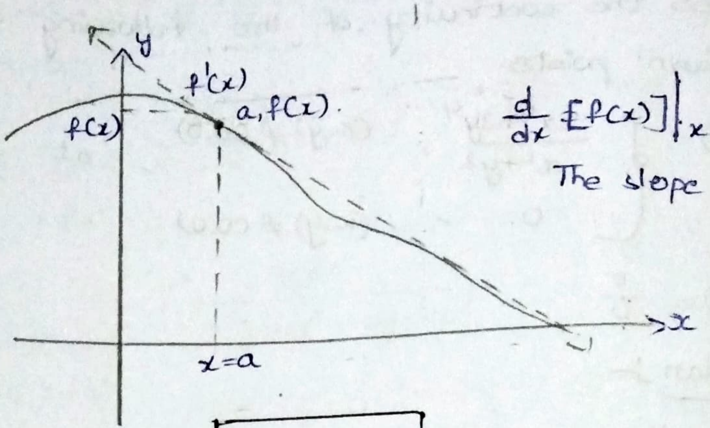
$$\Rightarrow \frac{2r^4\cos^4\theta+3r^4\sin^4\theta}{r^2}$$

$$\Rightarrow r^2[2\cos^4\theta+3\sin^4\theta]$$

$$\lim_{r\to 0} \; 0[\;\;] = 0.$$

∴ It is continuous function.

* Partial Derivatives :-



$\frac{d}{dx}[f(x)]\Big|_{x=a}$

The slope of .

Derivative of $\boxed{f(x) = f'(x)}$ — only in one direction

Definition :-

→ Let $f(x_1, x_2, x_3, \ldots x_n)$ be a n-variable function.

Partial derivative of $f(x_1, x_2 \ldots x_n)$ w.r.to x_i is the usual derivative of $f(x_1, x_2, \ldots x_n)$ w.r.to x_i. Remaining other variable x_j's $(j \neq i)$. keeping as a constant.

$$\boxed{\frac{\partial f}{\partial x_i} = f_{x_i} = \frac{d}{dx_i}[f(x_1, x_2, x_3, \ldots x_n)]\Big|_{x_j\text{'s are constant}}}$$

* Partial Derivatives for two variables:-

Let $f(x,y)$ be a function.

① partial derivative of $f(x,y)$ w.r.to "x" is the usual derivative of $f(x,y)$. keeping 'y' is constant

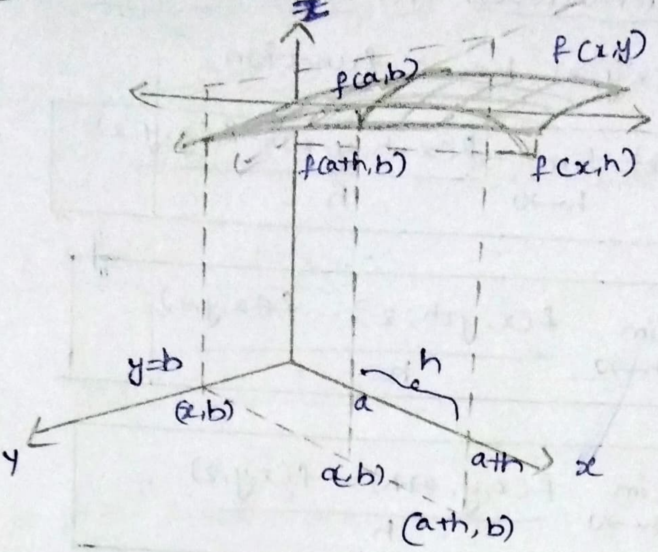
i.e,
$$\boxed{\frac{\partial f}{\partial x} = f_x = \frac{d[f(x,y)]}{dx}\Big|_{y \text{ is constant}}}$$

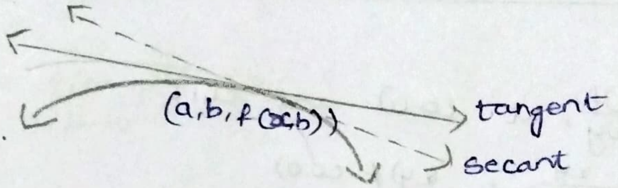
② Partial derivative of $f(x,y)$ w.r.to 'y' is the usual derivative of $f(x,y)$ keeping 'x' is constant

i.e,
$$\boxed{\frac{\partial f}{\partial y} = f_y = \frac{d[f(x,y)]}{dy}\Big|_{x \text{ is constant}}}$$

'f_x, f_y' are "first order partial derivatives."

* Case-1: At Y constant.



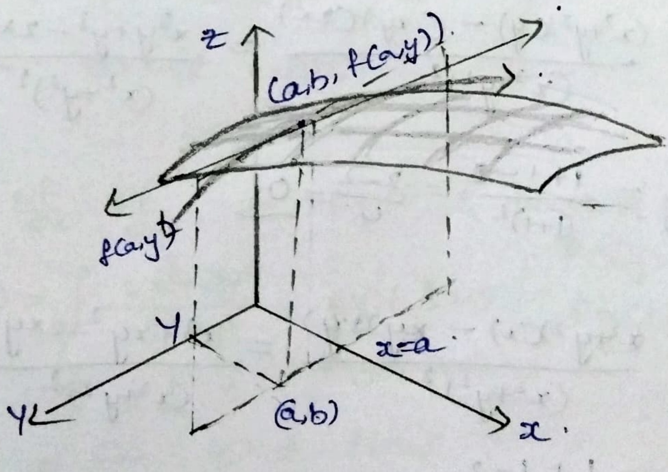


$$\left.\frac{\partial f}{\partial x}\right|_{(a,b)} = \frac{d}{dx}\left[f(x,b)\right].$$

slope of the tangent of f(a,b) which is passing through (a, b, f(a,b)) in x-direction.

$$\therefore \boxed{f_x(a,b) = \lim_{h \to 0} \frac{f(a+h,b) - f(a,b)}{h}.}$$

* case: x is constant



$$\left.\frac{\partial f}{\partial y}\right|_{(a,b)} = \frac{d}{dy}\left[f(a,y)\right].$$

The slope of the tangent line of f(x,y) which is passing through (a, b, f(x,y)) in y-direction.

$$\boxed{f_y(a,b) = \lim_{h \to 0} \frac{f(a,b+h) - f(a,b)}{h}}$$

* Partial derivatives for three variables:-

Let $f(x,y,z)$ be a function,

$$\boxed{\frac{\partial f}{\partial x}(x,y,z) = \lim_{h\to 0} \frac{f(x+h,y,z)-f(x,y,z)}{h}}$$

$$\boxed{\frac{\partial f}{\partial y} = \lim_{h\to 0} \frac{f(x,y+h,z)-f(x,y,z)}{h}}$$

$$\boxed{\frac{\partial f}{\partial z} = \lim_{h\to 0} \frac{f(x,y,z+h)-f(x,y,z)}{h}}$$

* Find $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$, at $(0,0)$ and $(1,1)$

$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y)=(0,0) \end{cases}$$

A) x^2+y^2, at $(1,1)$.

$\Rightarrow \frac{\partial f}{\partial x} = 2x$ & $\frac{\partial f}{\partial y} = 2y$.

$\Rightarrow \frac{\partial f}{\partial x}\Big|_{(1,1)} = 2$ & $\frac{\partial f}{\partial y}\Big|_{(1,1)} = 2$.

Now given problem

$$\Rightarrow \frac{\partial f}{\partial x} = \frac{(x^2+y^2)(y)-(xy)(2x)}{(x^2+y^2)^2} = \frac{x^2y+y^3-2x^2y}{(x^2+y^2)^2}$$

$$\Rightarrow \frac{\partial f}{\partial x}\Big|_{(1,1)} = \frac{1+1-2}{(1+1)^2} = \frac{2-2}{4} = 0.$$

$$\frac{\partial f}{\partial y} = \frac{(x^2+y^2)(x)-(xy)(2y)}{(x^2+y^2)^2} = \frac{x^3+xy^2-2xy^2}{(x^2+y^2)^2}$$

$$\frac{\partial f}{\partial y}\Big|_{(1,1)} = \frac{1+1-2}{2^2} = 0.$$

At $(0,0)$

$\frac{\partial f}{\partial x}\Big|_{(0,0)} = 0$, $\frac{\partial f}{\partial y}\Big|_{(0,0)} = 0$

* Partial Derivatives [second order].

$$
\begin{tikzcd}[column sep=small]
 & & & f(x,y) \arrow[dll, "x \text{ direction}"'] \arrow[drr, "y \text{ direction}"] & & & \\
 & \frac{\partial f}{\partial x} \arrow[dl] \arrow[dr] & & & & \frac{\partial f}{\partial y} \arrow[dl] \arrow[dr] & \\
\frac{\partial^2 f}{\partial x^2} & & \frac{\partial^2 f}{\partial y^2} & & \frac{\partial^2 f}{\partial x^2} & & \frac{\partial^2 f}{\partial y^2}
\end{tikzcd}
$$

$\therefore$ No. of partial derivatives exists $= 2^n$

* ## Homogeneous Function:-

→ Let $f(x,y)$ is said to be Homogeneous function of degree n, if $f(\lambda x, \lambda y) = \lambda^n f(x,y)$; where $\lambda \neq 0$, $n \in \mathbb{R}$.

Ex:- $f(x,y) = x+y$

$$f(\lambda x, \lambda y) = \lambda x + \lambda y = \lambda(x+y) = \lambda^1 f(x,y)$$

$\therefore f(x,y)$ is a Homogeneous function of degree '1'.

Ex:- $f(x,y) = \sin(x+y)$

$$f(\lambda x, \lambda y) = \sin(\lambda x + \lambda y) \neq \lambda \cdot \sin(x+y)$$

$\therefore$ Non Homogeneous function.

* ## Euler's First Theory:- / Theorem:-

→ Let '$f(x+y)$' is a homogeneous function of degree 'n' then $\boxed{x \cdot \frac{\partial f}{\partial x} + y \cdot \frac{\partial f}{\partial y} = n \cdot f(x,y).}$

Ex:- $f(x,y) = x+y$.

$f(x,y)$ is a Homogeneous fun of degree 1.

Here $n=1$.

$$\therefore x \cdot \frac{\partial f}{\partial x} + y \cdot \frac{\partial f}{\partial y} = (x)(1) + (y)(1) = 1 \cdot f(x,y).$$

* Euler's theorem for n variables:-

→ Let $f(x_1, x_2, \ldots, x_n)$ homogeneous function of degree 'n'

→ Then Euler's First theorem is

$$\boxed{x_1 \frac{\partial f}{\partial x_1} + x_2 \frac{\partial f}{\partial x_2} + \cdots + x_n \frac{\partial f}{\partial x_n} = n f(x_1, x_2, \ldots, x_n)}$$

→ Euler's second theorem is

$$\boxed{x^2 \cdot \frac{\partial^2 f}{\partial x^2} + 2xy \frac{\partial^2 f}{\partial x \partial y} + y^2 \frac{\partial^2 f}{\partial y^2} = n(n-1) \cdot f(x,y).}$$

* Euler's theorem for three variables:-

→ Let $f(x,y,z)$ be three variable function which is Homogeneous.

① $$\boxed{x \cdot \frac{\partial f}{\partial x} + y \cdot \frac{\partial f}{\partial y} + z \cdot \frac{\partial f}{\partial z} = n f(x,y,z)}$$

② $$\boxed{x^2 \frac{\partial^2 f}{\partial x^2} + y^2 \frac{\partial^2 f}{\partial y^2} + z^2 \frac{\partial^2 f}{\partial z^2} + 2xy \frac{\partial^2 f}{\partial x \partial y} + 2yz \frac{\partial^2 f}{\partial y \partial z} + 2zx \frac{\partial^2 f}{\partial z \partial x} = n(n-1) \cdot f(x,y,z).}$$

* Let $f(x,y)$, $g(x,y)$ be two functions with degree m, n respectively, $m \neq 0$.

if $h = f + g$ and $x \frac{\partial h}{\partial x} + y \frac{\partial h}{\partial y} = 0$

then show that $f(x,y) = K \cdot g(x,y)$, where 'K' be a real constant?

A) Given,

$$x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = m.f \longrightarrow ①$$

$$x \cdot \frac{\partial g}{\partial x} + y \frac{\partial g}{\partial y} = n.f \longrightarrow ②$$

given, $h(x,y) = f(x,y) + g(x,y)$.

$$\Rightarrow \frac{\partial h}{\partial x} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial x}$$

$$\Rightarrow \frac{\partial h}{\partial y} = \frac{\partial f}{\partial y} + \frac{\partial g}{\partial y}$$

given, $x \frac{\partial h}{\partial x} + y \frac{\partial h}{\partial y} = 0$.

$$\Rightarrow x \cdot \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial x} \right) + y \left(\frac{\partial f}{\partial y} + \frac{\partial g}{\partial y} \right) = 0.$$

$$-\left(x\cdot\frac{\partial f}{\partial x}+y\cdot\frac{\partial f}{\partial y}\right)+\left(x\cdot\frac{\partial g}{\partial x}+y\cdot\frac{\partial g}{\partial y}\right)=0.$$

By comparing with ① & ②

$$\Rightarrow m\cdot f(x,y)+n\cdot g(x,y)=0.$$

$$\Rightarrow f(x,y)=\frac{-n}{m}\cdot g(x,y)$$

$$\therefore \boxed{f(x,y)=k\cdot g(x,y)}$$

Hence proved.

* If $u(x,y)=\cos^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$. $0<x<y<1$ then prove that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{-1}{2}\cot u$?

A) Given $u(x,y)=\cos^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$

$$\Rightarrow \cos u=\frac{x+y}{\sqrt{x}+\sqrt{y}}$$

Let $\cos u=\frac{x+y}{\sqrt{x}+\sqrt{y}}=f$

$$\Rightarrow \frac{\partial f}{\partial x}=\frac{\partial}{\partial x}[\cos u]=-\sin u\,\frac{\partial u}{\partial x}$$

$$\Rightarrow \frac{\partial f}{\partial y}=\frac{\partial}{\partial x}[\cos u]=-\sin y\,\frac{\partial u}{\partial x}.$$

Now, $x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=nf$ [$\because$ First Theorem of Euler's].

Here, degree is $\frac{1}{2}$.

$$\frac{\lambda x+\lambda y}{\lambda^{1/2}x+\lambda^{1/2}y}=\lambda^{1/2}f(x,y)$$

$$\Rightarrow x\cdot\frac{\partial f}{\partial x}+y\cdot\frac{\partial f}{\partial y}=\frac{1}{2}f$$

$$\Rightarrow x\cdot\left[-\sin u\,\frac{\partial u}{\partial x}\right]+y\left[-\sin\frac{\partial u}{\partial y}\right]=\frac{1}{2}[\cos u]$$

$$\Rightarrow -\sin u\left[x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right]=\frac{1}{2}\cos u$$

$$\therefore x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=-\frac{1}{2}\cot u.$$

* If $w = \sin^{-1} u$, $u = \frac{x^2+y^2+z^2}{x+y+z}$, then prove that

$$x \frac{\partial w}{\partial x} + y \frac{\partial w}{\partial y} + z \frac{\partial w}{\partial z} = \tan w ?$$

A) Given $w = \sin^{-1} u$

$\Rightarrow \sin w = u$

Let $\sin w = u = f$.

Here degree is 1.

$$u = \frac{x^2+y^2+z^2}{x+y+z}$$

$$u(\lambda x, \lambda y, \lambda z) = \frac{\lambda^2 x^2 + \lambda^2 y^2 + \lambda^2 z^2}{\lambda x + \lambda y + \lambda z}$$

$$= \frac{\lambda^2 [x^2+y^2+z^2]}{\lambda [x+y+z]} = \lambda \cdot u f(x, y, z).$$

By Euler's first theorem,

$$x \cdot \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = n \cdot f$$

$$\Rightarrow \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}[\sin w] = \cos w \frac{\partial w}{\partial x}$$

$$\Rightarrow \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}[\sin w] = \cos w \frac{\partial w}{\partial y}.$$

$$\therefore x \cdot \frac{\partial f}{\partial x} + y \cdot \frac{\partial f}{\partial y} = n \cdot f.$$

$$\Rightarrow x \cdot \left[\cos w \frac{\partial w}{\partial x}\right] + y\left[\cos w \frac{\partial w}{\partial y}\right] = 1 \cdot \sin w$$

$$\Rightarrow \cos w \left[x \frac{\partial w}{\partial x} + y \frac{\partial w}{\partial y}\right] = \sin w$$

$$\therefore x \frac{\partial w}{\partial x} + y \frac{\partial w}{\partial y} = \frac{\sin w}{\cos w} = \tan w$$

Hence proved.

* Differentiability:-

→ Let $f(x)$ be a function, $f: D \subseteq \mathbb{R} \to \mathbb{R}$, is said to be differentiable at a point 'x' $[x \in D]$.

if when 'Δx' is the increment in 'x' then the increment of '$y = f(x)$' is 'Δy' can be written in the form.

$$\boxed{\Delta y = A \cdot \Delta x + \epsilon \cdot \Delta x}$$

Here, "$\Delta y = f(x+\Delta x) - f(x)$".

where 'A' is independent from 'Δx' & 'A' is 'unique'

and '$\epsilon \to 0$' as '$\Delta x \to 0$'

i.e., '$\lim_{\Delta x \to 0} \epsilon = 0$.'

* show that $f(x) = x^2$ is differentiable?

A) $\Delta y = f(x+\Delta x) - f(x)$

$= (x+\Delta x)^2 - x^2$

$= \cancel{x^2} + (\Delta x)^2 + 2x \cdot \Delta x - \cancel{x^2}$

$= 2x \cdot \Delta x + (\Delta x)^2$

$= 2x \cdot \Delta x + \Delta x \cdot \Delta x$

$\therefore \Delta y = A \cdot \Delta x + \epsilon \cdot \Delta x$

where $A = 2x$, $\epsilon = \Delta x$. and $\epsilon \to 0$ as $\Delta x \to 0$

i.e. $\lim_{\Delta x \to 0} = 0$.

$\therefore f(x)$ is differentiable.

* show that $f(x) = \sin x$ is differentiable?

A) $\Delta y = f(x+\Delta x) - f(x)$.

$= \sin(x+\Delta x) - \sin x$

$= \sin x \cos\Delta x + \cos x \cdot \sin\Delta x - \sin x$.

$= \sin x(\cos\Delta x - 1) + \cos x \cdot \sin\Delta x$.

$= \cos x \cdot \sin\Delta x + \sin x(\cos\Delta x - 1)$.

$\Delta y = \frac{\cos x \sin\Delta x}{\Delta x} + \frac{(\cos\Delta x - 1)}{\Delta x} \cdot \sin x \cdot \Delta x. = A \cdot \Delta x + \epsilon \Delta x.$

where $\varepsilon = \left(\frac{\cos\Delta x - 1}{\Delta x}\right) \cdot \sin x$. i.e, $\varepsilon \to 0$ as $\Delta x \to 0$

and $\lim_{\Delta x \to 0} \varepsilon = \lim_{\Delta x \to 0} \sin x \cdot \left(\frac{\cos\Delta x - 1}{\Delta x}\right)$

$= \sin x \lim_{\Delta x \to 0} \left(\frac{\cos\Delta x - 1}{\Delta x}\right)$

$= \sin x \, (0)$

$= 0$.

where $A = \cos x$.

$\because \lim_{\Delta x \to 0} \frac{\sin\Delta x}{\Delta x} = 1$.

$\therefore f(x)$ is Differentiable.

* **Differentiability :**

→ Let f(x) is said to be differentiable at x if when given Δx is increment of x then Δy is increment in y is written in the form of

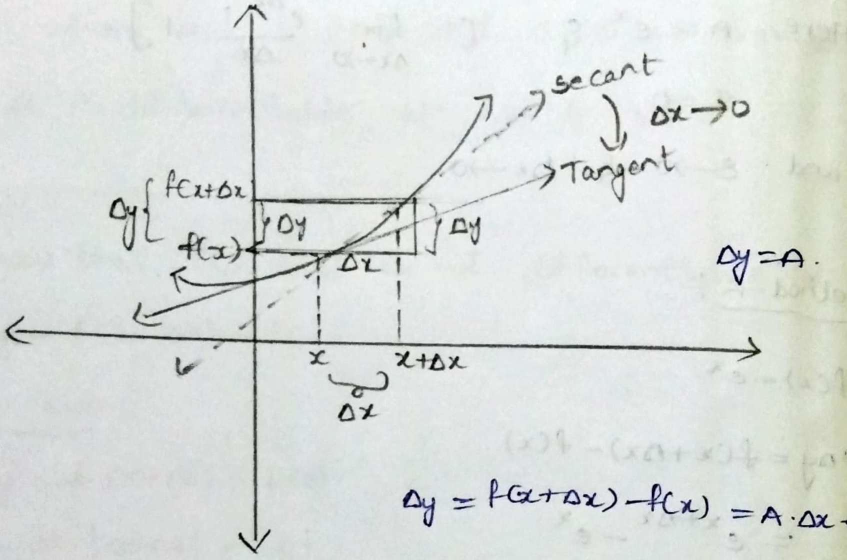
$$\boxed{\Delta y = A\,\Delta x + \epsilon\,\Delta x}$$

Here, A is independent from 'Δx' & 'unique'

Here, $A \cdot \Delta x$ = Total differentiability = 'dy'

$$\therefore \boxed{\Delta y = dy + \epsilon\,\Delta x} \quad , \quad 'dy = A \cdot \Delta x'$$

* **Graphically (or) Geometrical Representation :**



$$\Delta y = f(x+\Delta x) - f(x) = A \cdot \Delta x + \epsilon \cdot \Delta x$$

* **Differentiability Test :**

① $\boxed{\Delta y = A \cdot \Delta x + \epsilon\,\Delta x,}$ $\quad \epsilon \to 0$ as $\Delta x \to 0$.

② $\boxed{f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}}$ exist.

③ From ①

$\Rightarrow \Delta y = A \cdot \Delta x + \epsilon\,\Delta x$

$\Rightarrow \epsilon\,\Delta x = \Delta y - A \cdot \Delta x$.

$\Rightarrow \epsilon = \frac{\Delta y - A \cdot \Delta x}{\Delta x}$.

We know that, $\lim_{\Delta x \to 0} \epsilon = 0$.

$$\therefore \boxed{\lim_{\Delta x \to 0} \frac{\Delta y - A \cdot \Delta x}{\Delta x} = 0}$$

* Test the differentiability of $f(x) = 1 + \sqrt[3]{(x-1)^2}$ at $x=1$?

A) 2nd Rule :-

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$\Rightarrow \lim_{\Delta x \to 0} \frac{\cancel{1} + \sqrt[3]{(\cancel{x}+\Delta x-\cancel{1})^2} - \cancel{1} - \sqrt[3]{(x-1)^2}}{\Delta x}$$

$$\Rightarrow \lim_{\Delta x \to 0} \frac{((\Delta x)^2)^{1/3}}{\Delta x}$$

$$\Rightarrow \lim_{\Delta x \to 0} \frac{(\Delta x)^{2/3}}{\Delta x} = \lim_{\Delta x \to 0} \frac{1}{(\Delta x)^{1/3}} = \infty$$

$\therefore$ Limit doesnot exist

$\therefore$ Derivative doesnot exist.

* Tangents :-

By 3rd Rule :-

$$\Rightarrow \lim_{\Delta x \to 0} \frac{\Delta y - A \cdot \Delta x}{\Delta x}.$$

$$\Rightarrow \lim_{\Delta x \to 0} \frac{[A \cdot \Delta x + E \Delta x] - A \Delta x}{\Delta x} \quad \text{(or)}$$

$$\Rightarrow \lim_{\Delta x \to 0} \frac{f[(1+\Delta x)] - f(1)}{\Delta x}.$$

$$\Rightarrow \lim_{\Delta x \to 0} \frac{(1 + \sqrt[3]{(\Delta x)^2} - 1) - A \Delta x}{\Delta x}$$

$$\Rightarrow \lim_{\Delta x \to 0} \frac{(\Delta x)^{2/3} - A \cdot \Delta x}{\Delta x}$$

$$\Rightarrow \lim_{\Delta x \to 0} \frac{(\Delta x)^{2/3}}{\Delta x} - \lim_{\Delta x \to 0} A.$$

$$\Rightarrow \lim_{\Delta x \to 0} \frac{1}{(\Delta x)^{1/3}} - A.$$

$\therefore \lim_{\Delta x \to 0} \frac{1}{(\Delta x)^{1/3}}$ doesnot exist.

Since $\lim_{\Delta x \to 0} \frac{\Delta y - A \Delta x}{\Delta x} \neq 0.$

$\therefore f(x)$ is not differentiable at $x=1$.

* Differentiability for two variables :-

→ Let '$f(x,y)$' be a two-variable function is said to be differentiable at '(x,y)'.

if given increment in 'x' and 'y' are 'Δx', 'Δy' respectively, then increment in 'z' is 'Δz' can be written in the form of

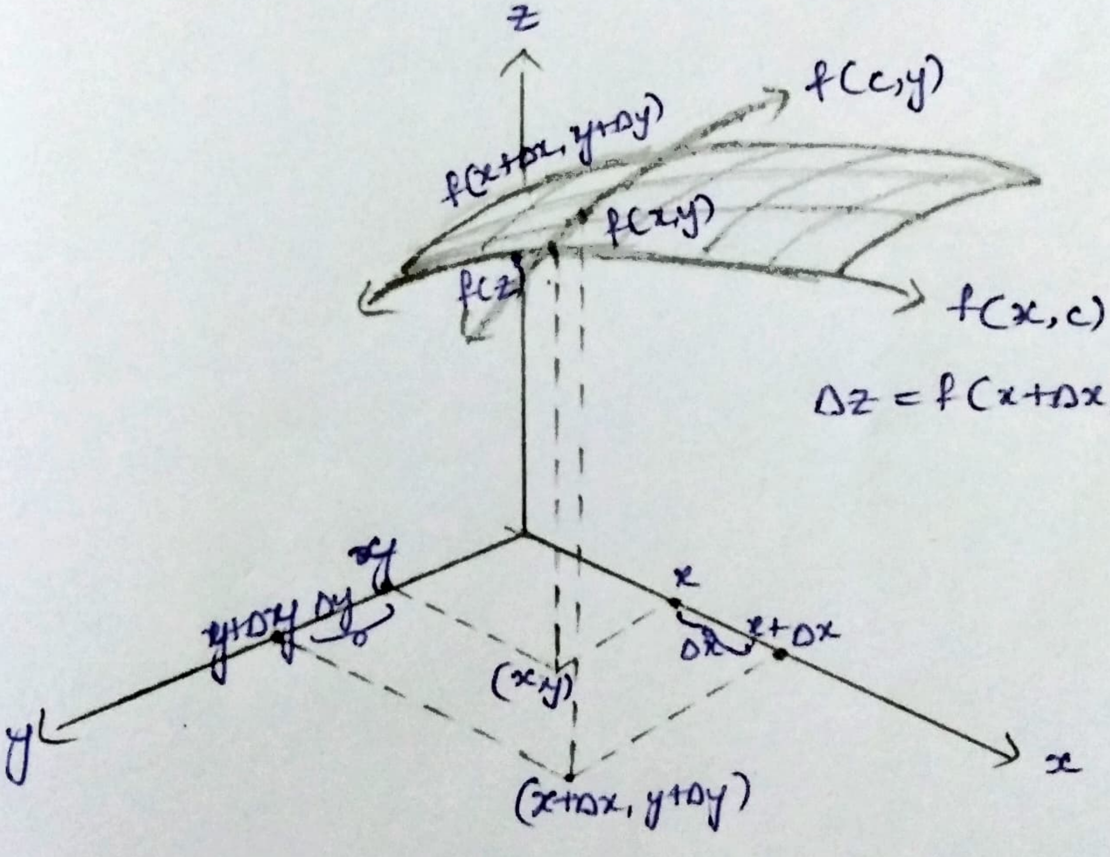
$$\boxed{\Delta z = A \cdot \Delta x + B \cdot \Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y}$$

where 'ε_1', 'ε_2' $\to 0$ as 'Δx', 'Δy' $\to 0$

& 'A', 'B' are independent from 'Δx' & 'Δy'

& 'A', 'B' are 'unique.'

* Geometrical Representation :-



$\Delta z = f(x+\Delta x, y+\Delta y) - f(x,y)$.

*** chain Rule :-**

→ Let '$y=f(x)$' be a 'differentiable function', '$x=\phi(t)$' is a 'differentiable function' then the composite function '$y=f(\phi(t))$' is also a 'differentiable function'.

$$\boxed{\frac{dy}{dt}=\frac{dy}{dx}\cdot\frac{dx}{dt}}$$

$$\begin{tikzcd}
y \arrow[d, "\frac{dy}{dx}"] \arrow[dd, bend right=60, "\frac{dy}{dx}\times\frac{dx}{dt}"'] \\
x \arrow[d, "\frac{dx}{dt}"] \\
t
\end{tikzcd}$$

Here, 'y' is 'dependent variable'

't' is 'Independent variable'

'x' is 'Intermediate variable'

*** chain Rule for two variables :-**

→ Let '$z=f(x,y)$' has continuous first order partial derivatives and '$x=\phi(t)$' and '$y=\psi(t)$' are two differentiable functions in 't'.

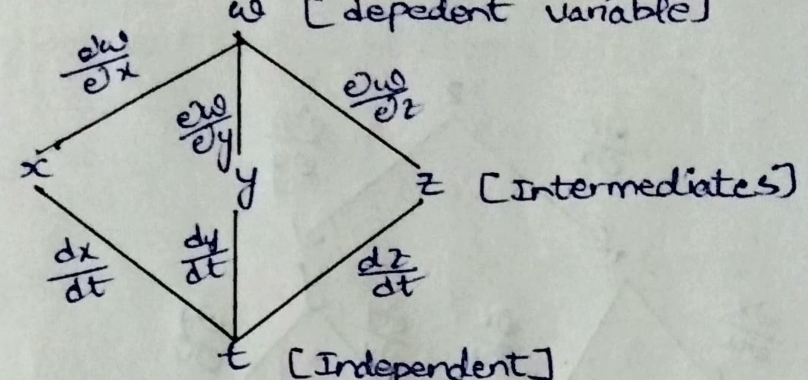
The composite function $\boxed{z=f(\phi(t);\ \psi(t))}$

$$\begin{tikzcd}
 & z \text{ (Dependent)} \arrow[dl, "\frac{\partial z}{\partial x}"'] \arrow[dr, "\frac{\partial z}{\partial y}"] & \\
x \arrow[dr, "\frac{dx}{dt}"'] & & y \text{ (Intermediate variables)} \arrow[dl, "\frac{dy}{dt}"] \\
 & t \text{ (Independent)} &
\end{tikzcd}$$

$$\therefore \boxed{\frac{dz}{dt}=\underbrace{\frac{\partial z}{\partial x}}_{f_x}\cdot\frac{dx}{dt}+\underbrace{\frac{\partial z}{\partial y}}_{f_y}\cdot\frac{dy}{dt}}$$

* Chain Rule for Three variables :-

→ Let $w = f(x, y, z)$ be a function which has F.O.P.D and $x = \phi(t)$, $y = \psi(t)$ and $z = \eta(t)$ then the composite function is $w = f(\phi(t), \psi(t), \eta(t))$.



$$\therefore \boxed{\frac{dw}{dt} = \frac{\partial w}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial w}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial w}{\partial z} \cdot \frac{dz}{dt}}$$

$$\Rightarrow \boxed{\frac{dw}{dt} = f_x \cdot x'(t) + f_y \cdot y'(t) + f_z \, z'(t)}$$

* Chain Rule for 'n' variables :-

→ Let $z = f(x_1, x_2, \_\_\_ x_n)$ be a n-variable function which has first order partial derivatives $f_{x_1}, f_{x_2}, \_\_\_ f_{x_n}$ exists.

Let $x_1 = \phi_1(t)$, $x_2 = \phi_2(t)$ — — $x_n = \phi_n(t)$ are differentiable functions.

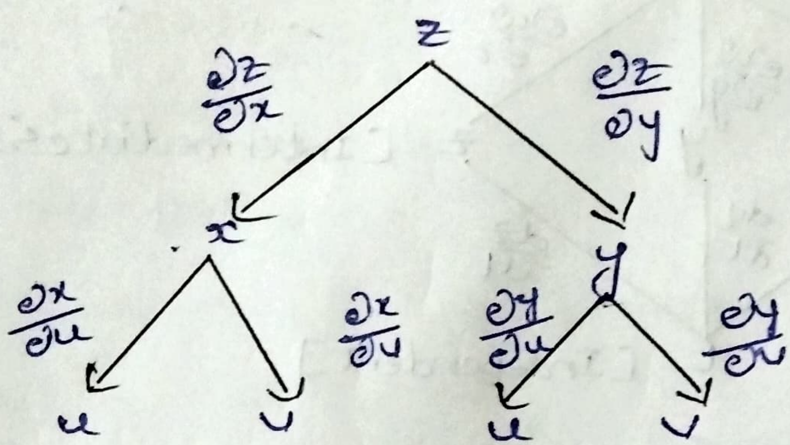
Then the derivative of $z = f(\phi_1(t), \phi_2(t), \phi_3(t), \_\_\_ \phi_n(t))$.

$$\therefore \boxed{\frac{dz}{dt} = \sum_{i=1}^{n} \frac{\partial z}{\partial x_i} \cdot \frac{\partial x_i}{dt}}$$

* chain rule for ~~three~~ two Independent & ~~three~~ two Intermediate variables

→ Let $z=f(x,y)$ has continuous first order partial derivatives and $x=\phi(u,v)$, $y=\psi(u,v)$ are two differentiable functions in u, v. then composite function is

$z=f(\phi(u,v), \psi(u,v))$.



$$\Rightarrow \boxed{\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}}$$

$$\boxed{\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}}$$

Note :-

→ For 'n' independent variables and 'm' intermediate variables, 'n' partial derivatives will form.