# 5.2

## Auto correlation, Cross correlation Functions and Ergodicity

### Autocorrelation Function and its Properties

**Definition:** If the process $\{X(t)\}$ is stationary either in the *strict sense* or in the *wide sense*, then $E\{X(t)X(t-\tau)\}$ is a function of $\tau$, denoted by $R_{xx}(\tau)$ or $R_x(\tau)$ or $R(\tau)$. This function $R(\tau)$ is called the **autocorrelation function** of the process $\{X(t)\}$.

**Properties of $\boldsymbol{R(\tau)}$**

**1. $\boldsymbol{R(\tau)}$ is an even function of $\boldsymbol{\tau}$**

*Proof*: We have $R(\tau) = E\{X(t)X(t-\tau)\}$

$$R(-\tau) = E\{X(t)X(t+\tau)\} = E\{X(t+\tau)X(t)\} = R(\tau)$$

Therefore, $R(\tau)$ is an even function of $\tau$.

**2. $\boldsymbol{R(\tau)}$ is maximum at $\boldsymbol{\tau = 0}$** $i.e.$, $|R(\tau)| \leq R(0)$

*Proof:* By Cauchy-Schwarz inequality, we have

$$\{E(XY)\}^2 \leq E(X^2)E(Y^2)$$

Put $X = X(t)$ and $Y = X(t-\tau)$

Then $[E\{X(t)X(t-\tau)\}]^2 \leq E\{X^2(t)\}E\{X^2(t-\tau)\}$

$i.e.$, $\{R(\tau)\}^2 \leq [R(0)]^2$

Taking square-root on both sides

$$|R(\tau)| \leq R(0) \text{ [since } R(0) = E\{X^2(t)\} \text{ is positive]}$$

**3. If the autocorrelation function $R(\tau)$ of a real stationary process $\{X(t)\}$ is continuous at $\tau = 0$, then it is continuous at every other point.**

*Proof:* $E[\{X(t) - X(t-\tau)\}^2]$

$$= E\{X^2(t)\} + E\{X^2(t-\tau)\} - 2E\{X(t).X(t-\tau)\}$$

$$= R(0) + R(0) - 2R(\tau) = 2[R(0) - R(\tau)] \qquad \ldots(1)$$

Therefore, $E[\{X(t) - X(t-\tau)\}^2] = 2[R(0) - R(\tau)]$

Since $R(\tau)$ is continuous at $\tau = 0$, $\lim_{\tau \to 0} R(\tau) = R(0)$

Now, $\lim_{\tau \to 0} E[\{X(t) - X(t-\tau)\}^2] = \lim_{\tau \to 0} 2[R(0) - R(\tau)] = 0$

Thus, $\lim_{\tau \to 0} \{X(t) - X(t-\tau)\} = 0$

Therefore, $\lim_{\tau \to 0} \{X(t-\tau)\} = X(t)$

$i.e.$, $X(t)$ is continuous for all $t$ $\qquad \ldots(2)$

Consider $R(\tau + h) - R(\tau)$

$$= E[X(t).X\{t - (\tau + h)\}] - E[X(t).X(t-\tau)]$$

$$= E\big[X(t)\big\{X\{t - \tau - h\}\big\} - X(t-\tau)\big] \qquad \ldots (3)$$

Now, $\lim_{h \to 0} [X\{(t-\tau) - h\} - X\{(t-\tau)\}] = 0$, by (2)

$\therefore \lim_{h \to 0} \{R.S.of\ (3)\} = 0$

$\therefore \lim_{h \to 0} \{L.S.of\ (3)\} = 0$

$i.e.$, $\lim_{h \to 0} \{R(\tau + h)\} = R(\tau)$, $i.e.$, $R(\tau)$ is continuous for all $\tau$.

4. **If $R(\tau)$ is the autocorrelation function of a stationary process $\{X(t)\}$ with no periodic component, then $\lim_{\tau \to \infty} R(\tau) = \mu_x^2$, provided the limit exists.**

*Proof*: We have, $R(\tau) = E\{X(t)X(t-\tau)\}$.

When $\tau$ is very large, $X(t)$ and $X(t-\tau)$ are two sample functions (members) of the process $\{X(t)\}$ observed at a very long interval of time.

Therefore, $X(t)$ and $X(t-\tau)$ tend to become independent [$X(t)$ and $X(t-\tau)$ may be dependent, when $X(t)$ contains a periodic component, which is not true].

$\therefore \lim_{\tau \to \infty} R(\tau) = E\{X(t)\}.E\{X(t-\tau)\} = \mu_x^2$ [since $E\{X(t)\}$ is a constant]

$i.e., \mu_x = \sqrt{\lim_{\tau \to \infty} R(\tau)}$

**Cross-Correlation Function and its Properties**

**Definition:** If the process $\{X(t)\}$ and $\{Y(t)\}$ are jointly wide-sense stationary, then $E\{X(t)Y(t-\tau)\}$ is a function of $\tau$, denoted by $R_{xy}(\tau)$. This function $R_{xy}(\tau)$ is called the **cross-correlation function** of the processes $\{X(t)\}$ and $\{Y(t)\}$.

We give below the properties of $R_{xy}(\tau)$ without proof. Proofs of these properties are left as exercises to the reader.

**Properties**

1. $R_{yx}(\tau) = R_{xy}(-\tau)$
2. $|R_{xy}(\tau)| \leq \sqrt{R_{xx}(0)R_{yy}(0)}$
   This means that the maximum of $R_{xy}(\tau)$ can occur anywhere, but it cannot exceed $\sqrt{R_{xx}(0) \times R_{yy}(0)}$.
3. $|R_{xy}(\tau)| \leq \frac{1}{2}\{R_{xx}(0) + R_{yy}(0)\}$
4. If the processes $\{X(t)\}$ and $\{Y(t)\}$ are orthogonal, then $R_{xy}(\tau) = 0$
5. If the processes $\{X(t)\}$ and $\{Y(t)\}$ are independent, then $R_{xy}(\tau) = \mu_x\mu_y$

**Ergodicity**

When we wish to take a measurement of a variable quantity in the laboratory, we usually obtain multiple measurements of the variable and average them to reduce measurement errors. If the value of the variable being measured is constant and errors are due to disturbances (noise) or due to the instability of the measuring instrument, then averaging is, in fact, a valid and useful technique. *Time averaging* is an extension of this concept, which is used in the estimation of various statistics of stochastic processes.

We normally use ensemble averages (or statistical averages) such as the *mean* and ***autocorrelation function*** for characterizing stochastic processes. To estimate ensemble averages, one has to compute a weighted average over all the member functions of the stochastic process.

For example, the ensemble mean of a discrete stochastic process $\{X(t)\}$ is computed by the formula $\mu_x = \sum x_i p_i$ . If we have access only to a single sample function of the process, then we use its time-average to estimate the ensemble averages of the process.

**Definition:** If $\{X(t)\}$ is a stochastic process, then $\frac{1}{2T}\int_{-T}^{T} X(t)\,dt$ is called the **time-average** of $\{X(t)\}$ over $(-T, T)$ and denoted by $\overline{\boldsymbol{X}_{\boldsymbol{T}}}$ .

In general, *ensemble averages* and *time averages* are not equal except for a very special class of stochastic processes called **Ergodic processes**. *The concept of ergodicity deals with the equality of time averages and ensemble averages.*

**Definition:** A stochastic process $\{X(t)\}$ is said to be ***ergodic***, if its ensemble averages are equal to appropriate time averages.

This definition implies that, with probability 1, any ensemble average of $\{X(t)\}$ can be determined from a single sample function of $\{X(t)\}$.

**Note:** Ergodicity is a stronger condition than stationarity and hence *all stochastic processes that are stationary are not ergodic*. Moreover, ergodicity is usually defined with respect to one or more ensemble averages (such as mean and autocorrelation function) as discussed below and a process may be ergodic with respect to one ensemble average but not others.

**Mean-Ergodic Process:** If the stochastic process $\{X(t)\}$ has a constant mean $E\{X(t)\} = \mu$ and $\overline{X_T} = \frac{1}{2T}\int_{-T}^{T} X(t)\,dt \to \mu \text{ as } T \to \infty$, then $\{X(t)\}$ is said to be **mean-ergodic**.

**Mean-Ergodic Theorem**

**If $\{X(t)\}$ is a stochastic process with constant mean $\mu$ and if $\overline{X_T} = \frac{1}{2T}\int_{-T}^{T} X(t)\,dt$, then $\{X(t)\}$ is mean-ergodic (or ergodic in the mean), provided**

$$\lim_{T \to \infty} \{Var\overline{X_T}\} = 0.$$

*Proof*: $\overline{X_T} = \frac{1}{2T}\int_{-T}^{T} X(t)\,dt$

$$\therefore E\left(\overline{X_T}\right) = \frac{1}{2T}\int_{-T}^{T} E\{X(t)\}\,dt = \mu \qquad \text{... (1)}$$

By Tchebycheff's inequality

$$P\{|\overline{X_T} - E(\overline{X_T})| \le \epsilon\} \ge 1 - \frac{Var(\overline{X_T})}{\epsilon^2} \qquad \text{... (2)}$$

Taking limits as $T \longrightarrow \infty$ and using (1) we get

$$P\left\{\left|\lim_{T \to \infty}\{\overline{X_T}\} - \mu\right| \le \epsilon\right\} \ge 1 - \frac{\lim_{T \to \infty} Var\overline{X_T}}{\epsilon^2}$$

Therefore, When $\lim_{T \to \infty} Var\overline{X_T} = 0$ , (2) becomes

$$P\left\{\left|\lim_{T \to \infty}(\overline{X_T}) - \mu\right| \le \epsilon\right\} \ge 1$$

$i.e.,\ \lim_{T \to \infty}(\overline{X_T}) = E\{X(t)\}$ with probability 1.

**Note:** This theorem provides a sufficient condition for the mean-ergodicity of a stochastic process. That is, to prove the mean-ergodicity of $\{X(t)\}$, it is enough to prove $\lim_{T \longrightarrow \infty} Var(\overline{X_T}) = 0$.

**Correlation Ergodic Process**

The stationary process $\{X(t)\}$ is said to be **correlation ergodic** (or ergodic in the correlation), if the process $\{Y(t)\}$ is mean-ergodic, where $Y(t) = X(t+\tau)X(t)$.

That is, the stationary process $\{X(t)\}$ is correlation ergodic, if

$$\overline{Y_T} = \frac{1}{2T}\int_{-T}^{T} X(t+\tau)X(t)dt \text{ tends to } E\{X(t+\tau)X(t)\} = R(\tau) \text{ as } T \longrightarrow \infty.$$

**Distribution Ergodic Process**

If $\{X(t)\}$ is a stationary process and if $\{Y(t)\}$ is another process such that

$$Y(t) = \begin{cases} 1 & if \quad X(t) \le x \\ 0 & if \quad X(t) > x \end{cases}$$

then $\{X(t)\}$ is said to be **distribution-ergodic**, if $\{Y(t)\}$ is mean-ergodic. That is, the stationary process $\{X(t)\}$ is distribution ergodic, if

$$\overline{Y_T} = \frac{1}{2T}\int_{-T}^{T} Y(t)dt \to E\{Y(t)\} \; as \; T \to \infty.$$

We note that

$$E\{Y(t)\} = 1 \times P\{X(t) \le x\} + 0 \times P\{X(t) > x\} = F_X(x)$$

Thus the stationary process $\{X(t)\}$ is distribution-ergodic, if

$$\frac{1}{2T}\int_{-T}^{T} Y(t)dt \to F_X(x) \; as \; T \to \infty.$$

**Example1: Given that the autocorrelation function for a stationary ergodic process with no periodic component is**

$$\boldsymbol{R_{xx}(\tau) = 25 + \frac{4}{1+6\tau^2}}$$

**Find the mean value and variance of the process $\{X(t)\}$.**

**Solution:** Here $R_{xx}(\tau) = R(\tau)$. By the property of autocorrelation function,

$$\mu_x^2 = \lim_{\tau \to \infty} R_{xx}(\tau) = 25 \quad \text{(Property 4 of } R(\tau)\text{ )}$$

Therefore, $\mu_x = 5$

$$E\{X^2(t)\} = R_{xx}(0) = 25 + 4 = 29$$

Therefore, $Var\{X(t)\} = E\{X^2(t)\} - (E\{X(t)\})^2 = 29 - 25 = 4$

**Example2: Express the autocorrelation function of the process $\{X'(t)\}$ in terms of the autocorrelation function of the process $\{X(t)\}$.**

**Solution:** Consider

$$R_{xx'}(t_1, t_2) = \text{cross correlation function of } X(t_1) \text{ and } X'(t_2)$$

$$= E\{X(t_1)X'(t_2)\}$$

$$= E\left[X(t_1)\left\{\frac{X(t_2+h)-X(t_2)}{h}\right\}\right] \text{ as } h \to 0$$

$$= \lim_{h \to 0}\left[\frac{R_{xx}(t_1,t_2+h)-R_{xx}(t_1,t_2)}{h}\right]$$

$$= \frac{\partial}{\partial t_2} R_{xx}(t_1, t_2) \qquad \dots (1)$$

Similarly, $R_{x'x'}(t_1, t_2) = \frac{\partial}{\partial t_1} R_{xx'}(t_1, t_2)$ ... (2)

Using (1) in (2),

$$R_{x'x'}(t_1, t_2) = \frac{\partial^2}{\partial t_1 \partial t_2} R_{xx}(t_1, t_2) \qquad \dots . (3)$$

If $\{X(t)\}$ is a stationary process, we put $t_1 - t_2 = \tau$. From (1), (2) and (3), then we get

$$R_{xx'}(\tau) = -\frac{\partial}{\partial\tau} R_{xx}(\tau)$$

$$R_{x'x'}(\tau) = \frac{\partial}{\partial\tau} R_{xx'}(\tau) \text{ and}$$

$$R_{x'x'}(\tau) = -\frac{\partial^2}{\partial\tau^2} R_{xx}(\tau)$$

**Example3: Prove that the stochastic process $\{X(t)\}$ with constant mean is mean-ergodic, if $\lim_{T\to\infty}\left[\frac{1}{4T^2}\int_{-T}^{T}\int_{-T}^{T} C(t_1,t_2)\,dt_1 dt_2\right] = 0$.**

**Solution**: By the mean-ergodic theorem, the condition for the mean-ergodicity of the process $\{X(t)\}$ is

$\lim_{T\to\infty}\{Var(\overline{X_T})\} = 0$ , where

$$\overline{X_T} = \frac{1}{2T}\int_{-T}^{T} X(t)\,dt \text{ and } E\left(\overline{X_T}\right) = E\{X(t)\} \qquad \text{(Since the mean is constant)}$$

Now $\overline{X_T}^2 = \frac{1}{4T^2}\int_{-T}^{T}\int_{-T}^{T} X(t_1)X(t_2)\,dt_1 dt_2$

$$\therefore E\left\{\overline{X_T}^2\right\} = \frac{1}{4T^2}\int_{-T}^{T}\int_{-T}^{T} R(t_1,t_2)\,dt_1 dt_2$$

$$\therefore Var\left(\overline{X_T}\right) = E\left\{\overline{X_T}^2\right\} - \left(E\left(\overline{X_T}\right)\right)^2$$

$$= \frac{1}{4T^2}\int_{-T}^{T}\int_{-T}^{T}\left[R(t_1,t_2) - E\{X(t_1)\}E\{X(t_2)\}\right] dt_1 dt_2$$

$$= \frac{1}{4T^2}\int_{-T}^{T}\int_{-T}^{T} C(t_1,t_2)\,dt_1 dt_2 \qquad \text{...(1)}$$

Therefore, the condition $\lim_{T\to\infty}\{Var(\overline{X_T})\} = 0$ is equivalent to the condition

$$\lim_{T\to\infty}\left[\frac{1}{4T^2}\int_{-T}^{T}\int_{-T}^{T}C(t_1,t_2)dt_1dt_2\right]=0$$

Hence the result .

**Example 4: If $\overline{X_T}$ is the time – average of a stationary stochastic process $\{X(t)\}$ over $(-T,T)$, prove that $Var\left(\overline{X}_T\right)=\frac{1}{T}\int_0^{2T}C(\tau)\left[1-\frac{|\tau|}{2T}\right]d\tau$ and hence prove that the sufficient condition for the mean – ergodicity of the process $\{X(t)\}$ is**

**(i)** $\lim_{T\to\infty}\frac{1}{T}\int_0^{2T}C(\tau)\left[1-\frac{|\tau|}{2T}\right]d\tau=0$ **and**

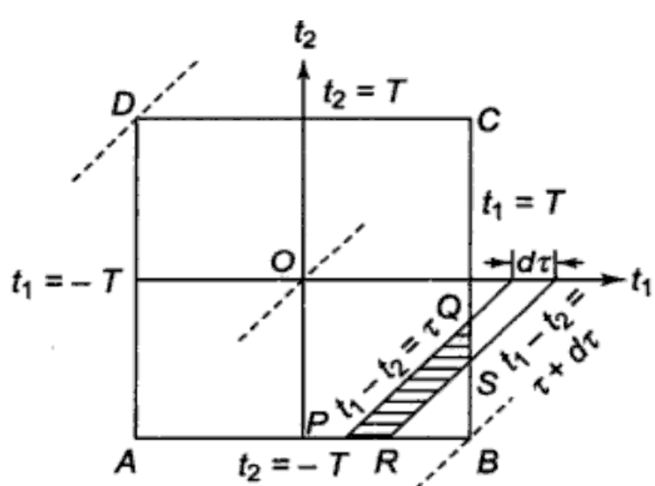
**(ii)** $\int_{-\infty}^{\infty}|C(\tau)|d\tau<\infty$

***Solution:*** Step (1) of the Example 3 gives

$$Var\left(\overline{X}_T\right)=\frac{1}{4T^2}\int_{-T}^{T}\int_{-T}^{T}C(t_1,t_2)\,dt_1dt_2 \qquad ...(1)$$

We shall convert the double integral (1) into a single definite integral with respect to the variable $\tau = t_1 - t_2$ as explained below:

The double integral (1) is evaluated over the area of the square bounded by $t_1 = \pm T$ and $t_2 = \pm T$ as shown in the figure.



We divide the area of the square $ABCD$ into a number of strips parallel to the line $t_1 - t_2 = 0$. Let a typical strip be $PQRS$, where $PQ$ is given by $t_1 - t_2 = \tau$ and $RS$ is given by $t_1 - t_2 = \tau + d\tau$.

When $PQRS$ is at the initial position $D$, $t_1 - t_2 = -2T$, $i.e.$, the initial value of $\tau = -2T$.

When $PQRS$ is at finial position , $t_1 - t_2 = 2T$, $i.e.$, final value of $\tau = 2T$. Hence to cover the given area $ABCD$, $\tau$ has to vary from $-2T$ to $2T$. Since $d\tau$ is very small, $C(t_1 - t_2) = C(\tau)$ can be assumed to be a constant in the strip $PQRS$.

Now $dt_1\, dt_2$ = element area in the $t_1 t_2$ - plane

$$= \text{area of the small strip } PQRS \qquad \dots (2)$$

$t_1$ co-ordinate of $P$ is obtained by solving the equations $t_1 - t_2 = \tau$ and $t_2 = -T$.

Thus $(t_1)_P = \tau - T$ .

$\therefore PB(= BQ) = T - (\tau - T) = 2T - \tau$ if $\tau > 0$

$= 2T + \tau$ if $\tau < 0$

When $\tau > 0$,

Area of $PQRS$ = Area of $\Delta\, PBQ$ − Area of $\Delta\, RSB$

$$= \frac{1}{2}(2T - \tau)^2 - \frac{1}{2}(2T - \tau - d\tau)^2$$

$$= (2T - \tau)d\tau, \text{ omitting } (d\tau)^2 \qquad \dots (3)$$

From (2) and (3),

$$dt_1 dt_2 = \{2T - |\tau|\}d\tau \qquad \dots (4)$$

Using (4) in (1),

$$Var\left(\bar{X}_T\right) = \frac{1}{2T}\int_{-2T}^{2T} C(\tau)\left\{1 - \frac{|\tau|}{2T}\right\}d\tau$$

i.e., $Var\left(\bar{X}_T\right)=\frac{1}{T}\int_0^{2T} C(\tau)\left\{1-\frac{|\tau|}{2T}\right\}d\tau$ (since the integral is even) ....(5)

(i) The sufficient condition for mean – ergodicity of a stationary process $\{X(t)\}$ can also be stated as

$$\lim_{T\to\infty}\left[\frac{1}{T}\int_0^{2T} C(\tau)\left\{1-\frac{|\tau|}{2T}\right\}d\tau\right]=0$$

(ii) The sufficient condition for mean ergodicity of $\{X(t)\}$ can also given as

$$\lim_{T\to\infty}\left[\frac{1}{2T}\int_{-2T}^{2T} C(\tau)\left\{1-\frac{|\tau|}{2T}\right\}d\tau\right]=0 \quad \text{... (6)}$$

Since $\tau$ varies from $-2T$ to $2T$, $|\tau| \leq 2T$.

$\therefore \quad 1-\frac{|\tau|}{2T} \leq 1$

*Thus,* $\frac{1}{2T}\int_{-2T}^{2T} C(\tau)\left\{1-\frac{|\tau|}{2T}\right\}d\tau \leq \frac{1}{2T}\int_{-2T}^{2T}|C(\tau)|d\tau$

(6) will be true, only if $\lim_{T\to\infty}\frac{1}{2T}\int_{-2T}^{2T}|C(\tau)|d\tau=0$

$i.e., if \int_{-\infty}^{\infty}|C(\tau)|d\tau$ is finite.

$i.e., if \int_{-\infty}^{\infty}|C(\tau)|d\tau<\infty$

Therefore, a sufficient condition for mean ergodicity of the stationary process $\{X(t)\}$ can also be stated as $\int_{-\infty}^{\infty}|C(\tau)|d\tau<\infty$