6.4

Markov chains

Markov Chain: If, for all n,

$P\{X_n = a_n | X_{n-1} = a_{n-1}, X_{n-2} = a_{n-2}, \dots, X_0 = a_0\} = P\{X_n = a_n | X_{n-1} = a_{n-1}\}$, then the process $\{X_n\}, n = 0,1, \dots, \dots$ is called a **Markov chain** and $(a_0, a_1, a_2, \dots, a_n, \dots)$ are called the **states** of the Markov chain. The conditional probability $P\{X_n = a_j | X_{n-1} = a_i\}$ is called the **one-step transition probability** from state a_i to state a_j at the n^{th} step (trial) and is denoted by $\boldsymbol{p_{ij}(n-1, n)}$. If the one-step transition probability does not depend on the step $i.e.$, $p_{ij}(n-1, n) = p_{ij}(m-1, m)$ the Markov chain is called a **homogeneous Markov chain** or the chain is said to have **stationary transition probabilities**. The use of the word *stationary* does not imply a stationary random sequence.

When the Markov chain is homogeneous, the one-step transition probability is denoted by p_{ij}. The matrix $P = \{p_{ij}\}$ is called (one-step) **transition probability matrix**,(t.p.m in short)

Note: The t.p.m of a Markov chain is a stochastic matrix, since $p_{ij} \geq 0$ and $\sum_j p_{ij} = 1$ for all i, $i.e.$, the sum of all the elements of any row of the t.p.m is 1. This is obvious because the transition from state a_i to any one of the states (including a_i itself) is a certain event.

The conditional probability that the process is in state a_j at step n, given that it was in state a_i at step 0, $i.e.$, $P\{X_n = a_j | X_0 = a_i\}$ is called the ***n*-step transition probability** and denoted by $p_{ij}^{(n)}$.

Note: ${p_{ij}}^{(1)} = p_{ij}$.

Let us consider an example in which we explain how the t.p.m is formed for a Markov chain. Assume that a man is at an integral point of the x-axis between the origin and the point $x = 3$. He takes a unit step either to the right with probability

0.7 or to the left with probability 0.3, unless he is at the origin when he takes a step to the right to reach $x = 1$ or he is at the point $x = 3$, when he takes a step to the left to reach $x = 2$. The chain is called **Random walk with reflecting barriers**.

The t.p.m is given below:

$$\begin{array}{cc} & \textit{States of } X_n \\ & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \textit{States of } X_{n-1} \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0.3 & 0 & 0.7 & 0 \\ 0 & 0.3 & 0 & 0.7 \\ 0 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

Note: p_{23} = the element in the 2nd row, 3rd column of this t.p.m is 0.7. This means that, if the process is at state 2 at step $(n-1)$, the probability that it moves to state 3 at step n *is* 0.7, where n is any positive integer.

Definition: If the probability that the process is in state a_i is $p_i (i = 1,2, \dots, k)$ at any arbitrary step, then the row vector $p = (p_1, p_2, \dots, p_k)$ is called the **probability distribution** of the process at that time. In particular, $p^{(0)} = \{p_1{}^{(0)}, p_2{}^{(0)}, \dots, p_k{}^{(0)}\}$ is the initial probability distribution.

Remark: The transition probability matrix together with the initial probability distribution completely specifies a Markov chain $\{X_n\}$.

In the example given above, let us assume that the initial probability distribution of the chain is $p^{(0)} = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)$. $i.e.$, $P\{X_0 = i\} = \frac{1}{4}$, $i = 0,1,2,3$.

Then we have, for the example given above,

$$P\{X_1 = 2 | X_0 = 1\} = 0.7; \; P\{X_2 = 1 | X_1 = 2\} = 0.3$$

$$P\{X_2 = 1, X_1 = 2 | X_0 = 1\} = P\{X_2 = 1 | X_1 = 2\} P\{X_1 = 2 | X_0 = 1\}$$

$$= 0.3 \times 0.7 = 0.21 \qquad \dots (1)$$

$$P\{X_2 = 1, X_1 = 2, X_0 = 1\} = P\{X_0 = 1\} \, P\{X_2 = 1, X_1 = 2 | X_0 = 1\}$$

$$= \frac{1}{4} \times 0.21 = 0.0525 \text{ [by (1)]} \qquad \ldots (2)$$

$P\{X_3 = 3, X_2 = 1, X_1 = 2, X_0 = 1\}$

$$= P\{X_2 = 1, X_1 = 2, X_0 = 1\} \times P\{X_3 = 3 | X_2 = 1, X_1 = 2, X_0 = 1\}$$

$$= 0.0525 \times P\{X_3 = 3 | X_2 = 1\} \text{ (Markov property) [by (2)]}$$

$$= 0.0525 \times 0 = 0$$

Chapman-Kolmogorov Theorem:

If P is the t.p.m of a homogeneous Markov chain, then the n-step t.p.m $P^{(n)}$ is equal to P^n. $i.e.$, $\left[p_{ij}{}^{(n)}\right] = \left[p_{ij}\right]^n$.

Proof: $p_{ij}{}^{(2)} = P\{X_2 = j | X_0 = i\}$, since the chain is homogeneous.

The state j can be reached from the state i in 2 steps through some intermediate state k.

Now $p_{ij}{}^{(2)} = P\{X_2 = j | X_0 = i\} = P\{X_2 = j, X_1 = k | X_0 = i\}$

$$= P\{X_2 = j | X_1 = k, X_0 = i\} P\{X_1 = k | X_0 = i\}$$

$$= p_{kj}{}^{(1)} p_{ik}{}^{(1)}$$

$$= p_{ik} p_{kj}$$

Since the transition from state i to state j in 2 steps can take place through any one of the intermediate states, k can assume the values 1,2,3, ... The transitions through various intermediate states are mutually exclusive.

Hence $p_{ij}{}^{(2)} = \sum_k p_{ik}\, p_{kj}$

$i.e.$, the ij-th element of 2 step t.p.m $=$ the ij-th element of the product of the 2 one-step t.p.m's

$i.e.$, $P^{(2)} = P^2$

Now $p_{ij}{}^{(3)} = P\{X_3 = j | X_0 = i\}$

$$= \sum_k P\{X_3 = j | X_2 = k\}\, P\{X_2 = k | X_0 = i\}$$

$$= \sum_k p_{kj} p_{ik}{}^{(2)}$$

$$= \sum_k p_{ik}{}^{(2)} p_{kj}$$

Similarly $p_{ij}{}^{(3)} = \sum_k p_{ik}\, p_{kj}{}^{(2)}$

i.e., $P^{(3)} = P^2 P = P P^2 = P^3$

Proceeding further in a similar way, we get

$$P^{(n)} = P^n$$

For example, consider the problem of Random walk with reflecting barriers discussed above, for which the t.p.m is

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0.3 & 0 & 0.7 & 0 \\ 0 & 0.3 & 0 & 0.7 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$P^2 = \begin{pmatrix} 0.3 & 0 & 0.7 & 0 \\ 0 & 0.51 & 0 & 0.49 \\ 0.09 & 0 & 0.91 & 0 \\ 0 & 0.3 & 0 & 0.7 \end{pmatrix}$$

From this matrix, we see that $P_{11}{}^{(2)} = 0.51$. This is so, because

$$P_{11}{}^{(2)} = \sum_{k=0}^{3} P_{1k}P_{k1} = P_{10}P_{01} + P_{11}P_{11} + P_{12}P_{21} + P_{13}P_{31}$$

$$= (0.3)(1) + (0)(0) + (0.7)(0.3) + (0)(0) = 0.51$$

Definition: A stochastic matrix P is said to be a **regular matrix**, if all the entries of P^m(for some positive integer m) are positive. A homogeneous Markov chain is said to be regular if its t.p.m is regular.

We state below two theorems without proof:

1. **If $\boldsymbol{p = \{p_i\}}$ is the state probability distribution of the process at an arbitrary time, then that after one step is $\boldsymbol{pP}$, where $\boldsymbol{P}$ is the t.p.m of the chain and that after $\boldsymbol{n}$ steps in $\boldsymbol{pP^n}$.**
2. **If a homogeneous Markov chain is regular, then every sequence of state probability distributions approaches a unique fixed probability distribution called the *stationary (state) distribution* or *steady-state distribution* of the Markov chain.**
   That is, $\boldsymbol{\lim_{n \to \infty} \{p^{(n)}\} = \pi}$, where the state probability distribution at step $\boldsymbol{n}$, $\boldsymbol{p^{(n)} = \left(p_1{}^{(n)}, p_2{}^{(n)}, \ldots, p_k{}^{(n)}\right)}$ $\boldsymbol{i.e., p^{(n)} = p^{(0)}P^n}$ and the stationary distribution $\boldsymbol{\pi = (\pi_1, \pi_2, \ldots, \pi_k)}$ are row vectors.
3. Moreover, if P is the t.p.m of the regular chain, then $\pi P = \pi$(π is a row vector). Using this property of π, it can be found out, as in the worked examples given below:

Classification of States of a Markov Chain

If $P_{ij}{}^{(n)} > 0$ for some n and for all i and j, then every state can be reached from every other state. When this condition is satisfied, the Markov chain is said to be **irreducible**. The t.p.m of an irreducible chain is an irreducible matrix. Otherwise, the chain is said to be **nonirreducible or reducible.**

State i of a Markov chain is called a **return state**, if $P_{ii}{}^{(n)} > 0$ for some $n > 1$.

The period d_i of a return state i is defined as the greatest common divisor of all m such $p_{ii}{}^{(m)} > 0$, i.e., $d_i = GCD\{m: p_{ii}{}^{(m)} > 0\}$. State i is said to be **periodic with period d_i** if $d_i > 1$ and **aperiodic** if $d_i = 1$.

Obviously state i is aperiodic if $p_{ii} \neq 0$. The probability that the chain returns to state i, having started from state i, for the first time at the nth step (or after n transitions) is denoted by $f_{ii}{}^{(n)}$ and called the **first return time probability** or the **recurrence time probability**. $\{n, f_{ii}{}^{(n)}\}, n = 1,2,3, \ldots$, is the distribution of recurrence times of the state i.

If $F_{ii} = \sum_{n=1}^{\infty} f_{ii}{}^{(n)} = 1$, the return to state i is certain.

$\mu_{ii} = \sum_{n=1}^{\infty} n f_{ii}{}^{(n)}$ is called the **mean recurrence time** of the state i.

A state i is said to be persistent or recurrent if the return to state i is certain, i.e., if $F_{ii} = 1$. The state i is said to be transient if the return to state i is uncertain, i.e., if $F_{ii} < 1$. The state i is said to be nonnull persistent if its mean recurrence time μ_{ii} is finite and null persistent, if $\mu_{ii} = \infty$.

A nonnull persistent and aperiodic state is called ergodic.

We give below two theorems, without proof, which will be helpful to classify the states of a Markov chain.

1. If a Markov chain is irreducible, all its states are of the same type. They are all transient, all null persistent or all nonnull persistent. All its states are either aperiodic or periodic with the same period.
2. If a Markov chain is finite irreducible, all its states are nonnull persistent.

Example 1: The transition probability matrix of a Markov chain $\{X_n\}$, $n = 1, 2, 3, \ldots$, having 3 states 1, 2 and 3 is

$$\boldsymbol{P} = \begin{pmatrix} \mathbf{0.1} & \mathbf{0.5} & \mathbf{0.4} \\ \mathbf{0.6} & \mathbf{0.2} & \mathbf{0.2} \\ \mathbf{0.3} & \mathbf{0.4} & \mathbf{0.3} \end{pmatrix}$$

and the initial distribution is $\boldsymbol{p}^{(0)} = (\mathbf{0.7}, \mathbf{0.2}, \mathbf{0.1})$.

Find (i) $P\{X_2 = 3\}$ and (ii) $P\{X_3 = 2, X_2 = 3, X_1 = 3, X_0 = 2\}$.

Solution:

$$P^{(2)} = P^2 = \begin{pmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{pmatrix}\begin{pmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{pmatrix} = \begin{pmatrix} 0.43 & 0.31 & 0.26 \\ 0.24 & 0.42 & 0.34 \\ 0.36 & 0.35 & 0.29 \end{pmatrix}$$

(i) $P\{X_2 = 3\} = \sum_{i=1}^{3} P\{X_2 = 3 \mid X_0 = i\} P\{X_0 = i\}$

$$\begin{aligned} &= p_{13}^{(2)} P(X_0 = 1) + p_{23}^{(2)} P(X_0 = 2) + p_{33}^{(2)} P(X_0 = 3) \\ &= 0.26 \times 0.7 + 0.34 \times 0.2 + 0.29 \times 0.1 \\ &= 0.182 + 0.068 + 0.029 \\ &= 0.279 \end{aligned}$$

(ii) $P\{X_1 = 3 | X_0 = 2\} = p_{23} = 0.2$ ... (1)

$P\{X_1 = 3, X_0 = 2\} = P\{X_1 = 3 | X_0 = 2\} \times P\{X_0 = 2\}$

$= 0.2 \times 0.2 = 0.04$ [by (1)] ... (2)

$P\{X_2 = 3, X_1 = 3, X_0 = 2\}$

$= P\{X_2 = 3 | X_1 = 3, X_0 = 2\} \times P\{X_1 = 3, X_0 = 2\}$

$= P\{X_2 = 3 | X_1 = 3\} \times P\{X_1 = 3, X_0 = 2\}$ (by Markov property)

$= 0.3 \times 0.04$ [by (2)]

$= 0.012$ ... (3)

$P\{X_3 = 2, X_2 = 3, X_1 = 3, X_0 = 2\}$

$= P\{X_3 = 2|X_2 = 3, X_1 = 3, X_0 = 2\} \times P\{X_2 = 3, X_1 = 3, X_0 = 2\}$

$= P\{X_3 = 2|\ X_2 = 3\} \times P\{\ X_2 = 3, X_1 = 3, X_0 = 2\}$ (by Markov property)

$= 0.4 \times 0.012$ [by (3)]

$= 0.0048$

Example 2: A fair dice is tossed repeatedly. If X_n denotes the maximum of the numbers occurring in the first n tosses, find the transition probability matrix P of the Markov chain $\{X_n\}$. Find also P^2 and $P(X_2 = 6)$

Solution: State space: $\{1,2,3,4,5,6\}$

The t.p.m is formed using the following analysis.

Let $X_n =$ the maximum of the numbers occurring in the first n trials $=$ 3, say

Then $X_{n+1} = 3$, if the $(n+1)$th trial results in 1, 2 or 3

$= 4$, if the $(n+1)$th trial results in 4

$= 5$, if the $(n+1)$th trial results in 5

$= 6$, if the $(n+1)$th trial results in 6

$\therefore P\{X_{n+1} = 3|X_n = 3\} = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{3}{6} = \frac{1}{2}$

$P\{X_{n+1} = i|X_n = 3\} = \frac{1}{6}$, when $i = 4,5,6$

Therefore, the transition probability matrix of the chain is

$$P = \begin{pmatrix} \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & \frac{2}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & 0 & \frac{3}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & 0 & 0 & \frac{4}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & 0 & 0 & 0 & \frac{5}{6} & \frac{1}{6} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$P^2 = \frac{1}{36} \begin{pmatrix} 1 & 3 & 5 & 7 & 9 & 11 \\ 0 & 4 & 5 & 7 & 9 & 11 \\ 0 & 0 & 9 & 7 & 9 & 11 \\ 0 & 0 & 0 & 16 & 9 & 11 \\ 0 & 0 & 0 & 0 & 25 & 11 \\ 0 & 0 & 0 & 0 & 0 & 36 \end{pmatrix}$$

Initial state probability distribution is $p^{(0)} = \left(\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}\right)$ since all the values $1, 2, \dots, 6$ are equally likely.

$$\begin{aligned} P\{X_2 = 6\} &= \sum_{i=1}^{6} P\{X_2 = 6 \mid X_0 = i\} \times P\{X_0 = i\} \\ &= \frac{1}{6} \sum_{i=1}^{6} P_{i6}^{(2)} \\ &= \frac{1}{6} \times \frac{1}{36} (11 + 11 + 11 + 11 + 11 + 36) \\ &= \frac{91}{216} \end{aligned}$$

Example 3: A man either drives a car or catches a train to go to office each day. He never goes 2 days in a row by train but if he drives one day, then the next day he is just as likely to drive again as he is to travel by train. Now suppose that on the first day of the week, the man tossed a fair dice and drove to work if any only if a 6 appeared. Find (i) the probability that he takes a train on the third day and (ii) the probability that he drives to work in the long run.

Solution: The travel pattern is a Markov chain, with state space = (train, car)

The t.p.m of the chain is

$$P = \begin{array}{c c} & \begin{array}{cc} T & C \end{array} \\ \begin{array}{c} T \\ C \end{array} & \begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \end{array}$$

The initial state probability distribution is $p^{(1)} = \left(\frac{5}{6}, \frac{1}{6}\right)$,
since P(travelling by car) = P(getting 6 in the toss of the dice) $= \frac{1}{6}$

and P(travelling by train) $= \frac{5}{6}$

$$p^{(2)} = p^{(1)}P = \left(\frac{5}{6}, \frac{1}{6}\right)\begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = \left(\frac{1}{12}, \frac{11}{12}\right)$$

$$p^{(3)} = p^{(2)}P = \left(\frac{1}{12}, \frac{11}{12}\right)\begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = \left(\frac{11}{24}, \frac{13}{24}\right)$$

$\therefore$ P(the man travels by train on the third day) $= \frac{11}{24}$

Let $\pi = (\pi_1, \pi_2)$ be the limiting form of the state probability distribution or stationary state distribution of the Markov chain.

By the property of $\pi, \pi P = \pi$

$$i.e., (\pi_1, \pi_2)\begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = (\pi_1, \pi_2)$$

$i.e., \frac{1}{2}\pi_2 = \pi_1$ ... (1)

and $\pi_1 + \frac{1}{2}\pi_2 = \pi_2 \Rightarrow \pi_1 = \frac{1}{2}\pi_2$ .... (2)

Equations (1) and (2) are one and the same.

Therefore, consider (1) or (2) with $\pi_1 + \pi_2 = 1$, since π is a probability distribution.

Solving, $\pi_1 = \frac{1}{3}$ and $\pi_2 = \frac{2}{3}$

$\therefore P\{\text{the man travels by car in the long run}\} = \frac{2}{3}$.

Example 4: Consider a communication system which transmits the 2 digits 0 and 1 through several stages. Let $X_n (n \geq 1)$ be the digit leaving the n^{th} stage of the system and X_0 be the digit entering the first stage (or leaving the 0^{th} stage). At each stage there is a constant probability q that the digit which enters will be transmitted unchanged (i. $e.$, the digit will remain unchanged when it leaves) and the probability P otherwise ($i. e.$, the digit changes when it leaves), where $p + q = 1$. Write down the t.p.m P of the homogeneous two-state Markov chain $\{X_n\}$. Find P^m, P^∞ and the conditional probability that the digit entering the first stage is 0, given that the digit leaving the mth stage is 0. Assume that the initial state probability distribution is $p^{(0)} = (a, 1-a)$.

Solution: State space $= (0,1)$;

$$P \equiv \text{State of } X_n \begin{array}{c} \\ \\ 0 \\ 1 \end{array} \begin{array}{c} \text{State of } X_{n+1} \\ \begin{array}{cc} 0 & 1 \end{array} \\ \begin{pmatrix} q & p \\ p & q \end{pmatrix} \end{array}$$

Now $P^2 = \begin{pmatrix} q & p \\ p & q \end{pmatrix} \begin{pmatrix} q & p \\ p & q \end{pmatrix}$

$$= \begin{pmatrix} p^2 + q^2 & 2pq \\ 2pq & p^2 + q^2 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2}[(q+p)^2 + (q-p)^2] & \frac{1}{2}[(q+p)^2 - (q-p)^2] \\ \frac{1}{2}[(q+p)^2 - (q-p)^2] & \frac{1}{2}[(q+p)^2 + (q-p)^2] \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2} + \frac{1}{2}r^2 & \frac{1}{2} - \frac{1}{2}r^2 \\ \frac{1}{2} - \frac{1}{2}r^2 & \frac{1}{2} + \frac{1}{2}r^2 \end{pmatrix}, \text{ where } q - p = r$$

$$P^3 = \begin{pmatrix} \frac{1}{2} + \frac{1}{2}r^3 & \frac{1}{2} - \frac{1}{2}r^3 \\ \frac{1}{2} - \frac{1}{2}r^3 & \frac{1}{2} + \frac{1}{2}r^3 \end{pmatrix}$$

The values of P^2 and P^3 make us guess that

$$P^m = \begin{pmatrix} \frac{1}{2} + \frac{1}{2}r^m & \frac{1}{2} - \frac{1}{2}r^m \\ \frac{1}{2} - \frac{1}{2}r^m & \frac{1}{2} + \frac{1}{2}r^m \end{pmatrix}$$

It is correct as can be proved by induction as follows:

$$P^{m+1} = \begin{pmatrix} q & p \\ p & q \end{pmatrix} \begin{pmatrix} \frac{1}{2} + \frac{1}{2}r^m & \frac{1}{2} - \frac{1}{2}r^m \\ \frac{1}{2} - \frac{1}{2}r^m & \frac{1}{2} + \frac{1}{2}r^m \end{pmatrix}$$

$$= \begin{bmatrix} \frac{q}{2} + \frac{q}{2}r^m + \frac{p}{2} - \frac{p}{2}r^m & \frac{q}{2} - \frac{q}{2}r^m + \frac{p}{2} + \frac{p}{2} + \frac{p}{2}r^m \\ \frac{p}{2} + \frac{p}{2}r^m + \frac{q}{2} - \frac{q}{2}r^m & \frac{p}{2} - \frac{p}{2}r^m + \frac{q}{2} + \frac{q}{2}r^m \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} + \frac{1}{2}r^{m+1} & \frac{1}{2} - \frac{1}{2}r^{m+1} \\ \frac{1}{2} - \frac{1}{2}r^{m+1} & \frac{1}{2} + \frac{1}{2}r^{m+1} \end{bmatrix}$$

$$\therefore \quad P^m = \begin{bmatrix} \frac{1}{2} + \frac{1}{2}r^m & \frac{1}{2} - \frac{1}{2}r^m \\ \frac{1}{2} - \frac{1}{2}r^m & \frac{1}{2} + \frac{1}{2}r^m \end{bmatrix}, \text{ where } m \text{ is a positive integer} \geq 1$$

$P^{\infty} = \lim_{m \to \infty} (P^m) = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$ since $|r| < 1$

Now $P\{X_m = 0\,, X_0 = 0\} = P\{X_m = 0 | X_0 = 0\} \times P\{X_0 = 0\} = aP_{00}^{(m)}$

and $P\{X_m = 0\,, X_0 = 1\} = bp_{10}^{(m)} b = 1 - a$

Now $P\{X_0 = 0 |\, X_m = 0\} = \frac{p\{X_0=0\}\, P\{X_m=0|X_0=0\}}{P\{X_0=0\}\, P_{00}^{(m)} + p\{X_0=1\} P_{10}^{(m)}}$

$$= \frac{a\left\{\frac{1}{2}+\frac{1}{2}r^m\right\}}{a\left\{\frac{1}{2}+\frac{1}{2}r^m\right\}+b\left\{\frac{1}{2}-\frac{1}{2}r^m\right\}} = \frac{a(1+r^m)}{1+(a-b)r^m},$$ where $b = 1 - a$