UNIT – I Oscillations

Periodic motion: A motion which repeats itself after equal (regular) intervals of time is called periodic motion. Ex:- Spin of earth, motion of a satellite around a planet, vibrations of atoms in molecules etc.

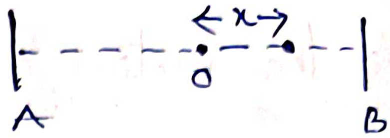
Oscillatory motion: when a body moves back and forth repeatedly about the mean position (or) fixed point, then it is said to be oscillatory motion (or) vibratory motion. Ex:- motion of prongs of a tuning fork, motion of simple pendulum, vertical oscillations of a loaded spring etc.

Basic definitions in oscillatory motion:

(a) Periodic time: Time taken to complete one oscillation is called periodic time (or) time period.

(b) Frequency: The no. of oscillations per second is known as frequency. It is the reciprocal of periodic time. ie $n = 1/T$ cycles/sec.

(c) Displacement: The distance of a particle in any direction from equilibrium position at any instant is called displacement of particle at that instant.



(d) Amplitude: The maximum displacement taken by a body from equilibrium position is known as amplitude (A).

(e) Phase: The position and direction of motion of a particle at any instant is known as phase (ϕ) at that instant. (or) The state of a particle in terms of the angle is called phase.

(f) Restoring force: The periodic force acting on a body so as to bring it to its mean (or) equilibrium position is called restoring force.

Simple Harmonic motion: when a body moves back and forth about mean position and the acceleration of a body is proportional to displacement in opposite direction. Such a motion of a body is said to be Simple harmonic motion. The body executing SHM is called Simple harmonic Oscillator.

(PTO)

Characteristics of SHM:

(a) The motion is periodic

(b) The motion is to and fro about the fixed point.

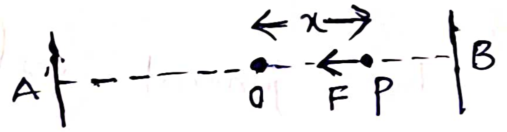
(c) Restoring force is proportional to displacement in opposite direction.

Equation of motion of a Simple Harmonic Oscillator

Consider a particle(P) of mass (m) executing SHM about equilibrium position 'O' along X-axis as shown in fig. So, the restoring force acting on 'P' is given by $F \propto -x$ (or) $F = -kx$ — ①

Here 'x' is displacement of 'P' from 'O'; $k \rightarrow$ force constant

(or) $k \rightarrow$ force/unit displacement



But according to Newton's II nd law of motion,

$F = ma \Rightarrow F = m\frac{d^2x}{dt^2}$ — ② $\left(\because a = \frac{d^2x}{dt^2}\right)$

From ① & ②,

$$m\frac{d^2x}{dt^2} = -kx$$

$\Rightarrow m\frac{d^2x}{dt^2} + kx = 0$ (or) $\boxed{\frac{d^2x}{dt^2} + \frac{k}{m}x = 0}$ — ③

$\frac{d^2x}{dt^2} + \omega^2 x = 0$ — ③ where $\omega^2 = k/m$

Equation ③ is known as D.E of a simple harmonic Oscillator.

Solution for D.E of a Simple Harmonic Oscillator

We have the differential equation as

$$\frac{d^2x}{dt^2} + \omega^2 x = 0$$

Let the solution be in the form of $x = ce^{\alpha t}$ —①

where $c, \alpha \rightarrow$ are arbitrary constants.

on differentiating ①, $\frac{dx}{dt} = c\alpha e^{\alpha t}$, $\frac{d^2x}{dt^2} = c\alpha^2 e^{\alpha t}$

put these values in diff. equation $\frac{d^2x}{dt^2} + \omega^2 x = 0$

we get, $c\alpha^2 e^{\alpha t} + \omega^2 c e^{\alpha t} = 0$

$$\Rightarrow ce^{\alpha t}(\alpha^2 + \omega^2) = 0$$

Here $ce^{\alpha t} \neq 0$, hence $\alpha^2 + \omega^2 = 0 \Rightarrow \alpha^2 = -\omega^2$

$\alpha = \pm\sqrt{-\omega^2} \Rightarrow \alpha = \pm i\omega \quad (\because i^2 = -1)$

So, the general solution is given by $x = c_1 e^{+i\omega t} + c_2 e^{-i\omega t}$

where $c_1, c_2 \rightarrow$ are arbitrary constants.

ie $x = c_1[\cos(\omega t) + i\sin(\omega t)] + c_2[\cos(\omega t) - i\sin(\omega t)]$

$$x = (c_1 + c_2)\cos\omega t + i(c_1 - c_2)\sin\omega t$$

Let $c_1 + c_2 = A\sin\phi$, $i(c_1 - c_2) = A\cos\phi$,

A and ϕ are new constants.

$$x = A\sin\phi\cos\omega t + A\cos\phi\sin\omega t$$

$$\boxed{x = A\sin(\omega t + \phi)}$$

This is the solution of D.E of a Simple harmonic Oscillator.

(a) Displacement:

The displacement of a particle at any instant executing SHM is given by $x = A\sin(\omega t + \phi)$

(b) Velocity:

The Velocity of oscillating particle is given by

$$V = \frac{dx}{dt} = \frac{d}{dt}\left[A\sin(\omega t+\phi)\right] = A\omega\cos(\omega t+\phi)$$

$$\Rightarrow V = A\omega\sqrt{1-\sin^2(\omega t+\phi)} = A\omega\sqrt{1-\left(\frac{x}{A}\right)^2} \quad \left(\because \sin(\omega t+\phi) = \frac{x}{A}\right)$$

$$\therefore V = A\omega\frac{\sqrt{A^2-x^2}}{A} \Rightarrow \boxed{V = \omega\sqrt{A^2-x^2}}$$

At mean position, ie $x=0$, $V = A\omega$ (Maximum)

At extreme position, ie $x = A$, $V = 0$ (minimum)

(c) Acceleration:

The acceleration of oscillating particle is given by

$$a = \frac{dv}{dt} = \frac{d}{dt}\left[A\omega\cos(\omega t+\phi)\right]$$

$$= -A\omega^2\sin(\omega t+\phi)$$

$$\boxed{a = -\omega^2 x} \quad (\because x = A\sin(\omega t+\phi))$$

At mean position, $x=0$, $a=0$

At extreme position, $x=A$, $a = -\omega^2 A$.

periodic time: It is given by $T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{K/m}}$

$\therefore \quad T = 2\pi\sqrt{M/K} \qquad (\because \omega^2 = K/m)$

(e) frequency: It is given by $n = 1/T = \frac{\omega}{2\pi}$

$\therefore \quad n = \frac{1}{2\pi}\sqrt{\frac{K}{m}}$

(f) phase: The angle $(\omega t + \phi)$ is called the phase of vibration.

(g) Epoch: The value of phase at $t=0$ is called epoch. Here 'ϕ' is called epoch.

Energy of Simple Harmonic Oscillator:

A simple harmonic oscillator possesses K.E as well as P.E. Thus $E = K.E + P.E$

To find K.E, Consider $x = A\sin(\omega t + \phi)$

then $\frac{dx}{dt} = v = A\omega\cos(\omega t + \phi)$

$K.E = \frac{1}{2}mv^2 \Rightarrow K.E = \frac{1}{2}mA^2\omega^2\cos^2(\omega t + \phi)$

$\therefore \; K.E = \frac{1}{2}m\omega^2A^2\cos^2(\omega t + \phi)$ — ①

(PTO)

To find P.E, Consider restoring force $\boxed{F = -Kx}$ — (2)

But the force in terms of P.E (U) is given by

$$\boxed{F = -\frac{du}{dx}} \quad -(3)$$

From (3) and (4), $\frac{du}{dx} = Kx$

on integration, we get $U = \frac{1}{2}Kx^2 + c$, $c \rightarrow$ integration const.

At $x=0$, $P.E(U) = 0$, hence $c=0$

$\therefore$ P.E, $U = \frac{1}{2}Kx^2 = \frac{1}{2}KA^2\sin^2(\omega t+\phi)$

$\therefore$ $U = P.E = \frac{1}{2}m\omega^2A^2\sin^2(\omega t+\phi)$ $\left(\because \omega^2 = K/m\right)$ — (4)

Now, total energy $E = K.E + P.E$

$\therefore$ $E = \frac{1}{2}m\omega^2A^2\cos^2(\omega t+\phi) + \frac{1}{2}m\omega^2A^2\sin^2(\omega t+\phi)$

$\therefore$ $E = \frac{1}{2}m\omega^2A^2\left[\cos^2(\omega t+\phi) + \sin^2(\omega t+\phi)\right]$

$\therefore$ $\boxed{E = \frac{1}{2}m\omega^2A^2}$ (or) $\boxed{E = \frac{1}{2}mA^2(2\pi n)^2}$.

$\therefore$ T.E of the oscillator is (i) directly proportional to square of Amplitude (A^2)

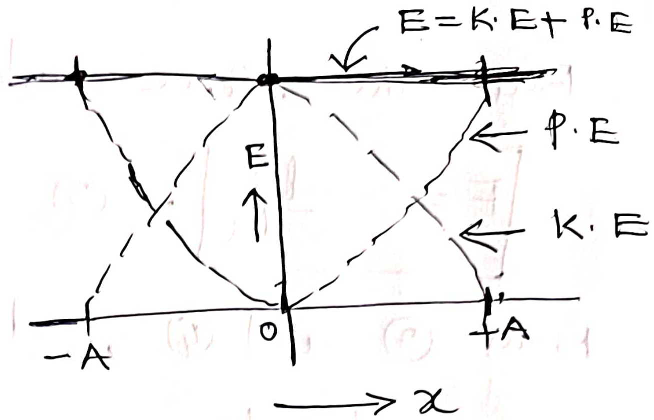
(ii) directly proportional to square of frequency (n^2)

(iii) inversely proportional to square of time period (T^2)

The distribution of energy with displacement is as shown.

Here K.E ↑ses and P.E ↓ses and vice-versa.

But the total energy remains constant. It can be seen that P.E is minimum at mean position and K.E is maximum at mean position. At extreme positions, K.E is minimum and P.E is maximum.



Damped Harmonic Oscillator:

In free oscillations, the energy remains constant i.e, these oscillations continues indefinitely. However, in real practise, the amplitude of oscillations of a body gradually decreases due to experiences of damping force like friction and resistance of media.

The oscillations whose amplitude goes on decreasing due to the presence of resistive forces like friction, resistance of media are called damped oscillations and the body is called damped oscillator.

Equation of a damped harmonic Oscillator

Consider a damped harmonic oscillator in which two forces are acted on it as follows.

(i) Restoring force is proportional to displacement (x) in opposite direction. i.e $F_r \propto -x$

$\therefore$ $F_r = -kx$, where $k \rightarrow$ force constant.

(ii) Frictional force (F_f) is proportional to velocity but in opposite direction. i.e $F_f \propto -\frac{dx}{dt}$

There fore, $F_f = -r\frac{dx}{dt}$, $r \rightarrow$ frictional force per unit velocity.

The net force acting in damped oscillator is given by $F = F_r + F_f$

$$F = -kx - r\frac{dx}{dt} \quad \text{--- (1)}$$

but we know that $\boxed{F = ma = m\frac{d^2x}{dt^2}}$

put this value in (1), we get

$$\Rightarrow m\frac{d^2x}{dt^2} = -kx - r\frac{dx}{dt}$$

$$\frac{d^2x}{dt^2} + \frac{r}{m}\frac{dx}{dt} + \frac{k}{m}x = 0$$

$2b \rightarrow$ damping coefficient

let $r/m = 2b$, $\frac{k}{m} = \omega^2$

Then, the equation becomes, $\boxed{\frac{d^2x}{dt^2} + 2b\frac{dx}{dt} + \omega^2 x = 0}$ --- (2)

The differential equation of a damped Oscillator is given by $\frac{d^2x}{dt^2} + 2b\frac{dx}{dt} + w^2x = 0$ — ②

Let the solution can be assumed as $x = Ae^{\alpha t}$ — ③

then $\frac{dx}{dt} = A\alpha e^{\alpha t}$, $\frac{d^2x}{dt^2} = A\alpha^2 e^{\alpha t}$

put these values in ②, we get

$A\alpha^2 e^{\alpha t} + 2b \cdot A\alpha e^{\alpha t} + w^2 \cdot Ae^{\alpha t} = 0$

$Ae^{\alpha t}\left[\alpha^2 + 2b\alpha + w^2\right] = 0$

Here $Ae^{\alpha t} \neq 0$ and hence $\alpha^2 + 2b\alpha + w^2 = 0$ — ④

This is a quadratic equation $(ax^2 + bx + c = 0)$

∴ The two roots are $\alpha = -b \pm \sqrt{b^2 - w^2}$

i.e $\alpha = -b + \sqrt{b^2 - w^2}$ and $\alpha = -b - \sqrt{b^2 - w^2}$.

∴ The solution can be written as $x = A_1 e^{\left(-b + \sqrt{b^2 - w^2}\right)t} + A_2 e^{\left(-b - \sqrt{b^2 - w^2}\right)t}$

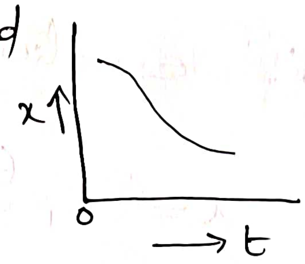
i.e $$\boxed{x = A_1 e^{\left(-b + \sqrt{b^2 - w^2}\right)t} + A_2 e^{\left(-b - \sqrt{b^2 - w^2}\right)t}} \quad —⑤$$

Special Cases: Case (a): over damped (or) dead beat

When $b^2 > w^2$, then $\sqrt{b^2 - w^2}$ is real in ⑤. So, the powers in equation ⑤ are both negative. Thus, the displacement 'x' consists of two terms, both dying off exponentially to zero without performing oscillations as shown. This type of motion is called over damped.

EX:- Pendulum moving in a thick oil.

The overdamping motion is also called as dead beat motion.



Case: b: Critical damping

When $b^2 = \omega^2$, then the solution does not satisfies the differential equation. Hence $\sqrt{b^2-\omega^2} \neq 0$

$\therefore$ let $\sqrt{b^2-\omega^2} = h \to 0$

Then the solution becomes $x = A_1 e^{(-b+h)t} + A_2 e^{(-b-h)t}$

$$\therefore x = e^{-bt}\left[A_1 e^{ht} + A_2 e^{-ht}\right]$$

$$x = e^{-bt}\left[A_1 (1+ht+\cdots) + A_2(1-ht+\cdots)\right]$$

$$x = e^{-bt}\left[(A_1+A_2) + h(A_1-A_2)t + \cdots\right]$$

$$\therefore x = e^{-bt}[P + qt] \quad -(6)$$ where $P = A_1 + A_2$, $q = h(A_1 - A_2)$

From this equation (6), it is clear that as 't' increases, the factor $(P+qt)$ increases but the factor e^{-bt} decreases. Hence the particle acquire equilibrium position much rapidly (in a short interval).

This motion is called critical damped motion.

Ex:- Pointer instruments such as Voltmeter, ammeter in which pointer moves to correct position and comes to rest without any oscillation in minimum time.

Case-(C): When $b^2 < \omega^2 \Rightarrow \sqrt{b^2-\omega^2}$ becomes imaginary

Let $\sqrt{b^2-\omega^2} = \pm i\sqrt{\omega^2-b^2} = \pm i\beta$, where $\beta = \sqrt{\omega^2-b^2}$

$i = \sqrt{-1}$

$\therefore$ equation (5) becomes

$$x = A_1 e^{(-b+i\beta)t} + A_2 e^{(-b-i\beta)t}$$

$$x = e^{-bt}\left[A_1 e^{i\beta t} + A_2 e^{-i\beta t}\right]$$

$$\therefore x = e^{-bt}\left[A_1(\cos i\beta t + i\sin\beta t) + A_2(\cos\beta t - i\sin\beta t)\right]$$

$$= e^{-bt}\left[(A_1+A_2)\cos\beta t + i(A_1-A_2)\sin\beta t\right]$$

let $A_1 + A_2 = A\sin\phi$, $i(A_1-A_2) = A\cos\phi$

$$\therefore x = e^{-bt}\left[A\sin\phi\cdot\cos\beta t + A\cos\phi\cdot\sin\beta t\right]$$

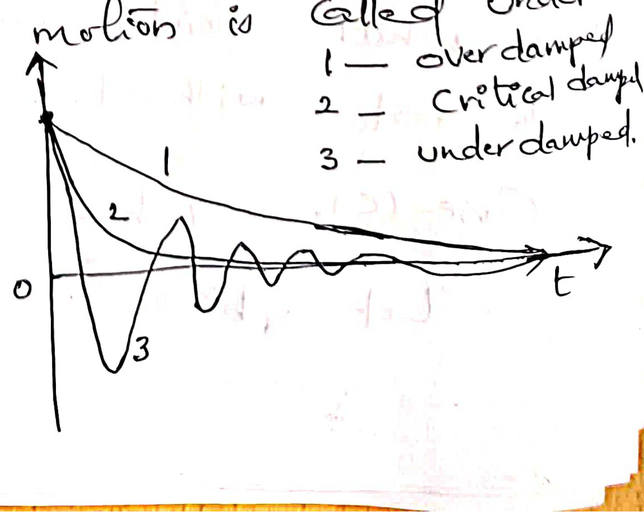
$$x = e^{-bt}\left[A\sin(\beta t+\phi)\right] = Ae^{-bt}\cdot\sin\left[\left[\sqrt{\omega^2-b^2}\right]t+\phi\right]$$

i.e $$\boxed{x = Ae^{-bt}\sin\left[\left(\sqrt{\omega^2-b^2}\right)t+\phi\right]}$$

This equation represents SHM with amplitude Ae^{-bt} and time period $T = \frac{2\pi}{\beta} = \frac{2\pi}{\sqrt{\omega^2-b^2}}$

Depending upon the damping Coefficient 'b', the amplitude of oscillation decreases by a factor e^{-bt} and $\sin\left[\left(\sqrt{\omega^2-b^2}\right)t+\phi\right]$ varies between -1 to $+1$. Such a type of motion is called under damped motion.

Ex: Motion of pendulum in air, electric oscillations in LCR circuit.



Energy and power dissipation in damped Oscillator

The Mechanical energy of damped harmonic oscillator is given by $E = K.E + P.E$ — (1)

W.K.T $x = A e^{-bt} \cdot \sin(\beta t + \phi)$ for damped oscillator

where $\beta = \sqrt{\omega^2 - b^2}$

$$\frac{dx}{dt} = A(-b)e^{-bt}\sin(\beta t + \phi) + \beta\cos(\beta t + \phi) \cdot A e^{-bt}$$

$\left(\because \frac{d}{dt}(uv) = u\frac{dv}{dt} + v\frac{du}{dt}\right)$

As $b <<< \omega$, then $-bAe^{-bt}\sin(\beta t + \phi)$ can be neglected. Hence $\frac{dx}{dt} = \beta A e^{-bt}\cos(\beta t + \phi)$ — (2)

Here $K.E = \frac{1}{2}mv^2 = \frac{1}{2}m\left(\frac{dx}{dt}\right)^2$

$\therefore K.E = \frac{1}{2}m\beta^2 A^2 e^{-2bt}\cos^2(\beta t + \phi)$ — (3)

Here $P.E = \frac{1}{2}Kx^2 = \frac{1}{2}K\left[A e^{-bt} \cdot \sin(\beta t + \phi)\right]^2$

$\therefore P.E = \frac{1}{2}KA^2 e^{-2bt} \cdot \sin^2(\beta t + \phi)$ — (4)

put (3) and (4) in (1), we get

$$E = \frac{1}{2}m\beta^2 A^2 e^{-2bt}\cos^2(\beta t + \phi) + \frac{1}{2}KA^2 e^{-2bt} \cdot \sin^2(\beta t + \phi)$$

$$E = \frac{1}{2}KA^2 e^{-2bt}\cos^2(\beta t + \phi) + \frac{1}{2}KA^2 e^{-2bt}\sin^2(\beta t + \phi)$$

$\left(\because \beta = \sqrt{\omega^2 - b^2} \simeq \omega \;\&\; \omega^2 = K/m\right)$

$$\therefore E = \frac{1}{2}KA^2 e^{-2bt}\left[\cos^2(\beta t + \phi) + \sin^2(\beta t + \phi)\right]$$

$$\therefore \boxed{E = \frac{1}{2}KA^2 e^{-2bt}} \text{ — (5)}$$

This equation shows that energy of oscillator decreases with time.

power dissipation of damped harmonic oscillator is given by $P = -\frac{dE}{dt} = -\frac{d}{dt}\left[\frac{1}{2}KA^2 e^{-2bt}\right]$

$$\therefore \quad P = -\frac{1}{2}KA^2(-2b)e^{-2bt}$$

$$\therefore \quad P = 2b\left[\frac{1}{2}KA^2 \cdot e^{-2bt}\right] = 2b \cdot E$$

$$\therefore \quad \boxed{P = 2bE}$$

The rate of which the energy is lost is defined as power dissipation.

Forced Oscillations:

The oscillations in which a body vibrates with a frequency other than its natural frequency under the action of external periodic force are called Forced oscillations.

Ex:- Vibrations of bridge under the influence of marching soldiers, Vibrations of a tuning fork when subjected to periodic force of sound waves.

The body which executes forced oscillations is known as Forced oscillator.

Consider a forced oscillator in which 3-forces are acted upon the particle.

(i) Restoring force is proportional to displacement in opposite direction. $F_{\gamma} = -Kx$

(ii) Frictional force proportional to velocity in opp. direction

$$F_f \propto -dx/dt \Rightarrow F_f = -r\, dx/dt$$

(iii) external periodic force and is given by

$F_e = F \sin pt$, where $F \rightarrow$ max. value of force.

$p \rightarrow$ angular frequency.

∴ The total force acting on the particle is given by $F_T = F_{r} + F_f + F_e$ —①

But $F_T = ma = m \frac{d^2x}{dt^2}$.

∴ equation becomes $m \frac{d^2x}{dt^2} = -Kx - r \frac{dx}{dt} + F \sin pt$

∴ $\frac{d^2x}{dt^2} + \frac{r}{m} \frac{dx}{dt} + \frac{K}{m} x = \frac{F}{m} \sin pt$.

Let $\frac{r}{m} = 2b$, $\frac{K}{m} = \omega^2$, $\frac{F}{m} = f$

Then the equation becomes, $\frac{d^2x}{dt^2} + 2b \frac{dx}{dx} + \omega^2 x = f \sin pt$

ie $\boxed{\frac{d^2x}{dt^2} + 2b \frac{dx}{dt} + \omega^2 x = f \sin pt}$ —② is called Differential equation of forced oscillator.

Let the solution be of the form $x = A \sin(pt - \theta)$ —③

$\frac{dx}{dt} = Ap \cos(pt - \theta)$, $\frac{d^2x}{dt^2} = -Ap^2 \sin(pt - \theta)$.

put these values in equation ②, we get

$$-Ap^2 \sin(pt-\theta) + 2b \cdot Ap \cos(pt-\theta) + \omega^2 A \sin(pt-\theta) = f \sin[(pt-\theta)+\theta]$$

$$A(\omega^2 - p^2) \sin(pt-\theta) + 2bAp \cos(pt-\theta) = f \sin(pt-\theta).\cos\theta + f \cos(pt-\theta).\sin\theta$$

Compare the coefficients of $\sin(pt-\theta)$ and $\cos(pt-\theta)$

$$A(\omega^2 - p^2) = f\cos\theta \quad \text{--- (4)}$$

$$2bAp = f\sin\theta \quad \text{--- (5)}$$

$$(4)^2 + (5)^2 \Rightarrow f^2(\cos^2\theta + \sin^2\theta) = 4b^2A^2p^2 + A^2(\omega^2 - p^2)^2$$

$$= A^2\left[4b^2p^2 + (\omega^2 - p^2)^2\right]$$

$$\Rightarrow A^2 = \frac{f^2}{4b^2p^2 + (\omega^2 - p^2)^2}$$

$$\therefore A = \frac{f}{\sqrt{4b^2p^2 + (\omega^2 - p^2)^2}} = \frac{f}{\sqrt{(\omega^2 - p^2)^2 + 4b^2p^2}} \quad \text{--- (6)}$$

$$\frac{(5)}{(4)} \Rightarrow \frac{f\sin\theta}{f\cos\theta} = \frac{2bAp}{A(\omega^2 - p^2)}$$

$$\tan\theta = \frac{2b\cancel{A}p}{\cancel{A}(\omega^2 - p^2)} = \frac{2bp}{\omega^2 - p^2}$$

$$\therefore \tan\theta = \frac{2bp}{\omega^2 - p^2} \Rightarrow \theta = \tan^{-1}\left(\frac{2bp}{\omega^2 - p^2}\right) \quad \text{--- (7)}$$

put (6) in (3), we get $x = \dfrac{f}{\sqrt{\omega^2 - p^2 + 4b^2p^2}}\sin(pt - \theta)$ --- (8)

ie $x = \dfrac{f}{\sqrt{\omega^2 - p^2 + 4b^2p^2}}\sin(pt - \theta)$ is the solution

of D.E of a forced oscillator.

(PTO)

Special Cases:

Case (a): if $p << \omega$

i.e if the frequency of applied force is very less than the natural frequency.

WKT $$A = \frac{f}{\sqrt{(\omega^2-p^2)^2+4b^2p^2}}$$

$\because$ $p << \omega$, then $4p^2b^2$ term can be neglected.

Hence, $$A = \frac{f}{\omega^2} = \frac{F}{m\omega^2} \quad (\because \omega^2-p^2 \simeq \omega^2)$$

$\therefore$ Amplitude of oscillation depends on $\frac{F}{m}$ and is independent of external force frequency p.

$$\theta = \tan^{-1}\left(\frac{2bp}{\omega^2}\right) = \tan^{-1}(0) = 0 \quad (\because p<<<\omega)$$

Force and displacement are in the same phase.

Case-b: if $p = \omega$,

if if the frequency of force = natural frequency

then $$A = \frac{f}{2bp} = \frac{F}{2bm \cdot p}$$

Here the amplitude of oscillations is maximum since there is no square for p in the denominator

$$\theta = \tan^{-1}\left(\frac{2bp}{0}\right) = \tan^{-1}(\infty) = \pi/2$$

displacement lags behind the force by a phase $\pi/2$

Case: (c): when $p >> \omega$, ie at very high frequency

$$A = \frac{F}{p^2} = \frac{f}{mp^2} \quad (\because f = F/m)$$

$\therefore$ Amplitude depends on mass and driving frequency

$$\theta = \tan^{-1}\left(\frac{2bp}{-p^2}\right) = \tan^{-1}\left(-\frac{2b}{p}\right) = \tan^{-1}(0) = \pi$$

Resonance: If the frequency of applied periodic force (p) matches (equal) with the natural frequency (w) of oscillator, then the amplitude of oscillations will be maximum and maximum transfer of energy occurs from applied force to oscillator. This condition is called Resonance.

Amplitude Resonance: The amplitude of forced oscillations changes with the frequency of applied force and becomes maximum at a particular frequency. This phenomenon is known as Amplitude Resonance.

Condition for Amplitude Resonance:

We have $$A = \frac{f}{\sqrt{(w^2-p^2)^2 + 4b^2p^2}}$$

The amplitude A is maximum, when the denominator is minimum.

Thus $$\frac{d}{dp}\left[(w^2-p^2)^2 + 4p^2b^2\right] = 0$$

$$\therefore\ 2(w^2-p^2)\times(-2p) + 4\times 2p \cdot b^2 = 0.$$

(PTO)

$$\Rightarrow \quad 2(\omega^2-p^2)(-2p) + 4b^2(2p) = 0$$

$$\cancel{4}(\omega^2-p^2)\cancel{p} = \cancel{4}b^2 \times 2\cancel{p}$$

$$\therefore \quad \omega^2 - p^2 = 2b^2 \Rightarrow p^2 = \omega^2 - 2b^2$$

$$\text{or} \quad p = \sqrt{\omega^2 - 2b^2} \quad \text{--- (3)}$$

This is the condition of Amplitude Resonance.

put (3) in (1), we get $A = \dfrac{f}{\sqrt{(\omega^2-p^2)^2 + 4b^2p^2}}$

$$\Rightarrow A_{max} = \frac{f}{\sqrt{(\omega^2-\omega^2+2b^2)^2 + 4b^2(\omega^2-2b^2)}} = \frac{f}{\sqrt{4b^4 + 4b^2(\omega^2-2b^2)}}$$

$$\therefore A_{max} = \frac{f}{\sqrt{4b^2(b^2+\omega^2-2b^2)}} = \frac{f}{\sqrt{4b^2(\omega^2-b^2)}}$$

$$\therefore A_{max} = \frac{f}{2b\sqrt{\omega^2-b^2}} = \frac{f}{2b\sqrt{p^2+2b^2-b^2}}$$

$$\left(\because p^2 = \omega^2 - 2b^2 \Rightarrow \omega^2 = p^2 + 2b^2\right)$$

$$\therefore A_{max} = \frac{f}{2b\sqrt{p^2+b^2}}$$

for low damping $b \ll p$, hence $A_{max} = \dfrac{f}{2b \times p}$

b^2 can be neglected compared to p^2

$\therefore A_{max} = \dfrac{f}{2bp}$ which shows that

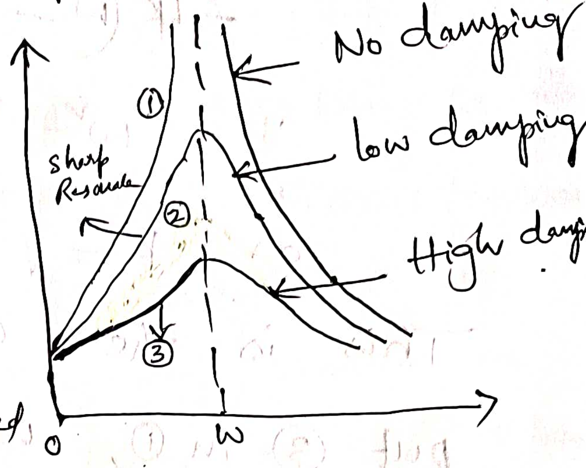
for $b \to 0$, ie if the damping is zero, then

$A_{max} \to \infty$.

The variation of amplitude with forced frequency at different amounts of damping is as shown.

In fig, curve ① denotes Amplitude of vibration when there is no damping ($b=0$).

So, Amplitude becomes '∞' at $p=\omega$ which is never attained in practise.



Due to frictional resistance, slight damping is always present as shown by curve ② and ③

Q - factor (or) Quality factor :

It is defined as 2π times the ratio of energy stored in the system to energy lost per period.

ie $$Q = 2\pi \times \left[\frac{\text{Energy stored}}{\text{Energy lost per period}}\right] = 2\pi \frac{E}{P \times T}$$

where $P \rightarrow$ power dissipated, $T \rightarrow$ time period.

$$\therefore Q = 2\pi \frac{E}{\left(\frac{E}{\tau}\right) \times T} \qquad \left(\because P = \frac{E}{\tau}, \text{ where } \tau = \frac{1}{2b}\right)$$

$$\therefore Q = \frac{2\pi}{T} \cdot \tau = \omega \cdot \tau$$

Quality factor measures the Quality of resonating circuit.

Velocity Amplitude and Velocity Resonance:

The displacement 'x' of a forced Oscillator is

$$x = \frac{f}{\sqrt{(\omega^2-p^2)^2+4b^2p^2}} \sin(pt-\theta)$$

Velocity $v = \frac{dx}{dt}$

$$\therefore v = \frac{dx}{dt} = \frac{fp}{\sqrt{(\omega^2-p^2)^2+4b^2p^2}} \cos(pt-\theta)$$

Here, the Velocity will be maximum when

$\cos(pt-\theta) = 1$

This maximum value of Velocity is Called Velocity Amplitude. It Can be denoted by V_{max}.

$$\therefore V_{max} = \frac{fp}{\sqrt{(\omega^2-p^2)^2+4b^2p^2}} = \frac{f}{\sqrt{\left(\frac{\omega^2-p^2}{p}\right)^2+4b^2}}$$

This Velocity Amplitude will be maximum when the denominator is minimum.

i.e $\left(\frac{\omega^2-p^2}{p}\right)^2 = 0 \Rightarrow \omega^2-p^2=0$ (or) $\boxed{\omega=p}$.

$\therefore$ $\boxed{\omega=p}$ is the Condition for Velocity Resonance

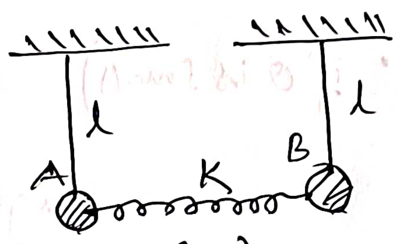
i.e At this frequency Velocity is maximum.

When the frequency of applied force = natural frequency of oscillator, Velocity Amplitude is maximum.

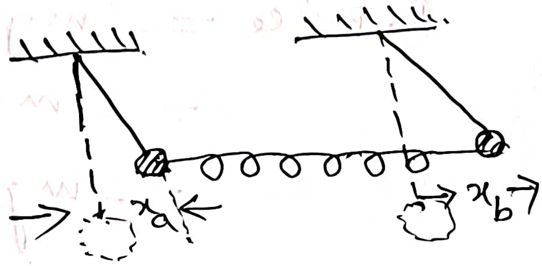
Coupled Oscillators

Consider two identical simple pendulums A and B each of mass 'm' and length 'l' coupled by a spring of force constant 'K' as shown.

When the pendulum 'B' is pulled aside while the pendulum 'A' is held fixed and then both are released. The amplitude of oscillation 'B' continuously decreases while that of 'A' increases continuously. Soon after, the amplitudes of A and B are equal. Further, the amplitude of 'B' continues to decrease till it becomes zero while that of 'A' continues to increase till it becomes equal to originally given to 'B'. This process is not stopped at this point. The motion of 'A' is now transferred back to 'B'. Now the process continues. In this way, the energy is transferred between B and A and continues to shuttle back and forth between B and A.



(a)



Def:- If two oscillators are connected in such a way that exchange of energy transfer takes place between them and the motion of all oscillators depends upon each other, such a system is called coupled oscillators.

Equation of motion of a coupled system:

Consider a coupled system in which they are connected a spring. Now, let the coupled system is slightly disturbed from equilibrium. Then the two pendulums begin to oscillate. Let x_a, x_b be the displacements of two bobs respectively. We say that the spring will be stretched when $x_b > x_a$ and compressed when $x_a > x_b$.

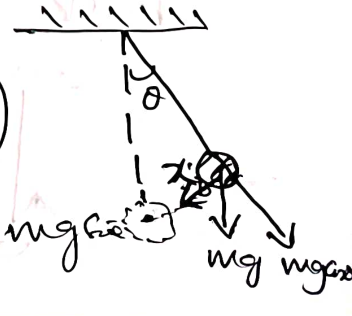
Considering the bob, there are two forces acting on it:

(i) Restoring force due to gravity.

$$R.\ Force = -mg \sin\theta$$

$$= -mg\theta \quad (\because \text{ if } \theta \text{ is small})$$

$$= -mg \frac{x_b}{l}$$



(ii) Return force due to stretching of string.

ie $-K(x_b - x_a)$, $K \rightarrow$ force constant

The net force acting on pendulums A and B,

For pendulum 'A': $m\frac{d^2x_a}{dt^2} = -mg\frac{x_a}{l} - K(x_a - x_b)$

ie $\frac{d^2x_a}{dt^2} = -\frac{\cancel{m}g}{\cancel{m}}\frac{x_a}{l} + \frac{K}{m}(x_b - x_a)$

$\therefore \frac{d^2x_a}{dt^2} = -\frac{g}{l}x_a + \frac{K}{m}(x_b - x_a)$

let $\omega_0^2 = g/l \Rightarrow \omega_0 = \sqrt{g/l}$,

$$\therefore \boxed{\frac{d^2x_a}{dt^2} = -\omega_0^2 x_a + \frac{K}{m}(x_b - x_a) \quad \text{--- (1)}}$$

For Pendulum 'B': $m\frac{d^2x_b}{dt^2} = -mg\frac{x_b}{l} - K(x_b - x_a)$

$\Rightarrow \frac{d^2x_b}{dt^2} = -\frac{g}{l}x_b - \frac{K}{m}(x_b - x_a)$

$$\therefore \boxed{\frac{d^2x_b}{dt^2} = -\omega_0^2 x_b - \frac{K}{m}(x_b - x_a) \quad \text{--- (2)}}$$

To find the effect of coupling on each pendulum, these (1) and (2) must be solved for x_a and x_b.

$\therefore$ Adding (1) and (2), we get

$$\frac{d^2(x_a+x_b)}{dt^2} = -\omega_0^2(x_a+x_b)$$

Let $x_a + x_b = X$

$$\therefore \boxed{\frac{d^2X}{dt^2} + \omega_0^2 X = 0} \quad \text{--- (3)}$$

Now, (2) − (1) $\Rightarrow$ $\frac{d^2}{dt^2}(x_b - x_a) = -\omega_0^2(x_b - x_a) - \frac{2k}{m}(x_b - x_a)$

$$\therefore \frac{d^2}{dt^2}(x_b - x_a) + \omega_0^2(x_b - x_a) + \frac{2k}{m}(x_b - x_a) = 0$$

Let $x_b - x_a = X'$

$$\frac{d^2X'}{dt^2} + \omega_0^2 X' + \frac{2k}{m}X' = 0 \Rightarrow \boxed{\frac{d^2X'}{dt^2} + \left(\omega_0^2 + \frac{2k}{m}\right)X' = 0} \quad \text{--- (4)}$$

Equations (3) and (4) are familiar equations of simple harmonic oscillations.

Cases.

Case (i) :- If $x_a = x_b$, then equation (4) vanished.

Hence, the motion is completely described by equation (3) $\quad (\because X = x_a + x_b,\ X' = x_b - x_a)$

Now, the angular frequency of oscillation is given by $\omega_1 = \omega_0 = \sqrt{g/l}$.

This frequency is same as either pendulum oscillates in isolation. ie effect of spring is absent

$\therefore$ Both the pendulums are always in phase.

$\therefore$ The spring has natural length through-out the motion. This is the first normal mode (in phase mode).

Case-(2) :- if $x_a = -x_b$, equation ③ is vanished

$\therefore$ The motion is completely described by equation ④.

$\therefore$ The angular frequency is given by ω_2.

$$\therefore \quad \omega_2 = \left(\omega_0^2 + \frac{2k}{m}\right)^{1/2}$$

So, $\omega_2 > \omega_1$ ie the frequency of oscillation of coupled system > the natural frequency of the pendulum when they are seperate.

In this case the spring is either compressed (or) extended. Thus the pendulums of the coupled system swing always out of phase. This is the second normal mode.

Discuss the motion of a Coupled System and derive total energy of a Coupled System.

The motion of a coupled system will be a combination of two normal modes. ie (i) in phase mode. (ii) out of phase mode.

For in phase mode: We have,

$$\boxed{\frac{d^2x}{dt^2} + \omega_1^2 x = 0}, \text{ where } x = x_a + x_b.$$

This is a SHM equation. Let the solution can be taken as $x = x_0 \sin\omega_1 t$, but $x = x_a + x_b$.

$\therefore$ $x_a + x_b = x_0 \sin\omega_1 t$

Let $x_a = x_b = a$ (maximum values of x_a and x_b)

Then $\boxed{x_a + x_b = 2a \sin\omega_1 t}$ $(\because x_0 = 2a)$ —(1)

For out of phase: We have

$$\frac{d^2x'}{dt^2} + \omega_2^2 x' = 0, \text{ where } x' = x_b - x_a.$$

This is also S.H.M equation.

The solution of this equation is $x' = x_0' \sin\omega_2 t$

but $x' = x_b - x_a$

$\therefore$ $\boxed{x_b - x_a = 2a \sin\omega_2 t}$ —(2) $\left(\because \text{max. value of } x_a = -a \text{ \& } x_b = a \;\; \therefore a - (-a) = 2a\right)$

The displacement of the bob 'B' is given by 'x_b'.

It Can be obtained by adding ① & ②

$$2x_b = 2a \sin \omega_1 t + 2a \sin \omega_2 t$$

$$\Rightarrow \cancel{2} x_b = \cancel{2} a \left[\sin \omega_1 t + \sin \omega_2 t\right]$$

$$x_b = 2a \sin\left(\frac{\omega_1+\omega_2}{2}\right)t \cdot \cos\left(\frac{\omega_1-\omega_2}{2}\right)t$$

$$\Rightarrow x_b = 2a \sin\left(\frac{\omega_1+\omega_2}{2}\right)t \cdot \cos\left(\frac{\omega_2-\omega_1}{2}\right)t \quad —③$$

Similarly,

The displacement of the bob 'A' is given by 'x_a'

'x_a' Can be obtained by Subtracting ② from ①.

ie ① − ②, we get $2x_a = 2a\left[\sin \omega_1 t - \sin \omega_2 t\right]$

$$\therefore x_a = 2a \sin\left(\frac{\omega_1-\omega_2}{2}\right)t \cdot \cos\left(\frac{\omega_1+\omega_2}{2}\right)t$$

$$\Rightarrow x_a = -2a \sin\left(\frac{\omega_2-\omega_1}{2}\right)t \cdot \cos\left(\frac{\omega_1+\omega_2}{2}\right)t \quad —④$$

Let $\frac{\omega_2-\omega_1}{2} = \omega_m$, $\frac{\omega_1+\omega_2}{2} = \omega_{av}$.

Then equations ③ & ④ becomes

$$x_a = -2a \sin \omega_m t \cdot \cos \omega_{av} t \quad \text{and}$$

$$x_b = 2a \sin \omega_{av} \cancel{t} \cdot \cos \omega_m t$$

Let the amplitudes be 'A', 'B' in the displacements of x_a and x_b. Here, let $A = 2a \sin \omega_m t$

$B = 2a \sin \omega_{av} t$.

$\therefore \quad x_a = A \cos \omega_{av} t, \quad x_b = B \sin \omega_{av} t$

$$\frac{dx_a}{dt} = -A\omega_{av} \sin \omega_{av} t, \quad \frac{dx_b}{dt} = B\omega_{av} \cos \omega_{av} t.$$

In general, for a S.H. motion, $V_{max} = A\omega$

Hence $V_{max} = \left(\frac{dx_a}{dt}\right)_{max} = -A\omega_{av}$ and

$$V_{max} = \left(\frac{dx_b}{dt}\right)_{max} = B \cdot \omega_{av}$$

The above two equations are maximum Velocities for the bobs A and B.

let 'E_A' be the K.E of the bob 'A', $E_A = \frac{1}{2} m (V_{max})^2_{for\ A}$

$$\therefore \quad E_A = \frac{1}{2} m A^2 \omega_{av}^2 = \frac{1}{2} m \times 4a^2 \sin^2 \omega_m t . \omega_{av}^2$$

lly K.E of bob B, $E_B = \frac{1}{2} m B^2 \omega_{av}^2 = \frac{1}{2} m 4a^2 \cos^2 \omega t \, \omega_{av}^2$

Total energy $E = E_A + E_B$

$$\therefore \quad E = \frac{1}{2} m 4a^2 \sin^2 \omega_m t . \omega_{av}^2 + \frac{1}{2} m 4a^2 \cos^2 \omega_m t . \omega_{av}^2$$

$$\therefore E_A + E_B = \frac{1}{2} m 4a^2 \omega_{av}^2 \left[\sin^2 \omega_m t + \cos^2 \omega_m t\right]$$

$$\therefore \boxed{E = E_A + E_B = \frac{1}{2} m 4a^2 \omega_{av}^2} \text{ --- (5)} \quad \text{let } E = 2ma^2\omega_{av}^2 = E_0$$

Consider $E_B - E_A = \frac{1}{2} m 4a^2 \omega_{av}^2 \left[\cos^2 \omega_m t - \sin^2 \omega_m t\right]$

$$E_B - E_A = 2ma^2\omega_{av}^2 \left[\cos 2\omega_m t\right]$$

$$\therefore \quad E_B - E_A = E_0 \left[\cos 2\omega_m t\right] \text{ --- (6)}$$

(5)+(6) $\Rightarrow \quad 2E_B = E_0 + E_0 \left[\cos 2\omega_m t\right]$

$$\therefore \quad E_B = \frac{1}{2} E_0 \left[1 + \cos 2\omega_m t \right]$$

$$\Rightarrow \quad E_B = \frac{E_0}{2} \left[1 + \cos \cancel{2} \left(\frac{\omega_2 - \omega_1}{\cancel{2}} \right) t \right]$$

(5) − (6) $\Rightarrow \quad 2E_A = E_0 - E_0 \cos 2\omega_m t$

$$2E_A = E_0 \left[1 - \cos 2\omega_m t \right]$$

$$\Rightarrow \quad E_A = \frac{E_0}{2} \left[1 - \cos \cancel{2} \left(\frac{\omega_2 - \omega_1}{\cancel{2}} \right) t \right]$$

$$\therefore \quad \boxed{\begin{aligned} E_A &= \frac{E_0}{2} \left[1 - \cos (\omega_2 - \omega_1) t \right] \\ E_B &= \frac{E_0}{2} \left[1 + \cos (\omega_2 - \omega_1) t \right] \end{aligned}}$$

This shows that total energy is constant, flows back and forth between the pende