2.6

Countable and uncountable sets

We are all familiar with a useful application of the set of natural numbers in counting. This property of natural numbers is used in measuring the *size* of a set and in comparing the *sizes* of any two sets. In the process of counting we establish a one to one correspondence between the objects to be counted and the set of natural numbers $\{1, 2, 3, \dots, n\}$. From this correspondence we say that the number of objects is n.The following is a generalization of this concept:

Equivalent Sets

Two sets A and B are said to be ***equivalent*** or ***equipotent*** or to have the ***same cardinality*** or ***similar*** written as $A \sim B$, if and only if there is a one- to- one correspondence between the sets A and B. A one -to -one correspondence can be established by showing a **bijective** mapping $f: A \longrightarrow B$.

Example 1: $\boldsymbol{N}$, the set of natural numbers and $\boldsymbol{N_2} = \{\mathbf{2}, \mathbf{4}, \mathbf{6}, \dots\}$, the set of even natural numbers are equivalent.

Solution: Define a map $f: \boldsymbol{N} \longrightarrow N_2$ by $f(n) = 2n$. It is easy to see that f is bijective. Thus $\boldsymbol{N} \sim N_2$.

Note: Notice that $N_2 \subset \boldsymbol{N}$

Example 2: Let $\boldsymbol{P}$ be the set of all possible real numbers and $\boldsymbol{S}$ be the subset of $\boldsymbol{P}$ given by $\boldsymbol{S} = \{\boldsymbol{x} | \boldsymbol{x} \in \boldsymbol{P} \land (\mathbf{0} < x < 1)\}$, then show that $\boldsymbol{S} \sim \boldsymbol{P}$.

Solution: Define a mapping $f: P \longrightarrow S$ by $f(x) = \frac{x}{1+x}$ for $x \in P$. Clearly, the range of f is in S. Now,

$$f(x_1) = f(x_2) \Longrightarrow \frac{x_1}{1 + x_1} = \frac{x_2}{1 + x_2} \Longrightarrow x_1 + x_1 x_2 = x_2 + x_1 x_2 \Longrightarrow x_1 = x_2$$

This shows that f is injective. Further, for any $y \in S$ there exists a $\frac{y}{1-y} \in P$, such that $f\left(\frac{y}{1-y}\right) = \frac{\frac{y}{1-y}}{1+\frac{y}{1-y}} = y$, $i.e.$, every element in S has a pre-image. Thus f is surjective. Therefore, f is bijective and $S \sim P$

Example 3: Any two closed intervals $[\boldsymbol{a}, \boldsymbol{b}]$, $\boldsymbol{a} < b$ and $[\boldsymbol{c}, \boldsymbol{d}]$, $\boldsymbol{c} < d$ are equivalent. (See P1 for solution)

Note: It is easy to see that the equivalence of sets (equipotence of sets) is an equivalence relation on a family of sets and hence partitions the family of sets into equivalence classes.(See P3)

Let F be a family of sets and let $\sim$ denotes the relation of equivalence (equipotence) on F. The equivalence classes of F under the relation $\sim$ are called **cardinal numbers**. For any set $A \in F$, the equivalence class to which A belongs is denoted by $[\boldsymbol{A}]$ or $\boldsymbol{card}(\boldsymbol{A})$ and is called the ***cardinal number* of $\boldsymbol{A}$**. For $A, B \in F, [A] = [B] \Leftrightarrow A \sim B$.

We shall first start with the empty set and denote its cardinal number by 0. For the time being, denote the cardinal number of a set A by $k(A)$, so that $k(\emptyset) = 0$. If $p \notin A$, then the cardinal number of $A \cup \{p\}$ $i.e.$, $k(A \cup \{p\})$ can be written as $k(A) + 1$. We can build sets starting with null set and building successive unions such that the cardinalities of these sets can be represented by zero and the natural numbers.

For example, Let $A_1 = \{a\}$, $A_2 = \{a, b\}$, $A_3 = \{a, b, c\}$, ... Then we have,
$A_1 = \emptyset \cup \{a\}$ and $k(A_1) = k(\emptyset) + 1 = 0 + 1 = 1$.

$A_2 = \{a, b\} = A_1 \cup \{b\}$, and $k(A_2) = k(A_1) + 1 = 1 + 1 = 2$

$A_3 = \{a, b, c\} = A_2 \cup \{c\}$, and $k(A_3) = k(A_2) + 1 = 2 + 1 = 3$

and so on .In this way the cardinal number of a set containing n elements can be denoted by the natural number n.

Remark: It is not possible to represent the cardinality of every set by a natural number, because there are sets which cannot be built by successive unions, as was done above.

The following is the definition of a finite set:

Finite and infinite set: A set is said to be ***finite***, if its cardinal number is a natural number. A set is said to be ***infinite*** if it is not finite.

Denumerable set: A set A is said to be **denumerable**, if $\sim \boldsymbol{N}$ **,**$i.e.$, A is equivalent to the set of natural numbers. The cardinality of a denumerable set is denoted by the symbol $\mathcal{N}_0$ (read as *aleph null* or *aleph not*).The use of $k(A)$ is restricted to denote only the cardinality of a finite set A

Countable and uncountable sets: A set is said to be ***countable***, if it is either finite or denumerable. A set is said to be ***nondenumerable*** or ***uncountable*** if it is infinite and not denumerable.

Note: An important difference between a finite and an infinite set is that no proper subset of a finite set can be equivalent to itself, because a one-to-one correspondence between such sets is not possible.

Theorem 1: An infinite subset of a denumerable set is also denumerable.

Proof: Let A be a given denumerable set. Let S be any infinite subset of A . Since A is denumerable, $A \sim \boldsymbol{N}$ **,** that is , there exists a one-to-one correspondence between A and $\boldsymbol{N}$ **.** Let $f\colon \boldsymbol{N} \longrightarrow A$ be the bijective function, therefore, the elements of A can be arranged as $f(1), f(2), f(3)$, …. . Now, delete from this list those elements which are not in S. The remaining elements in the list are precisely the elements of S and they are infinite since S is infinite. Denote these elements by $f(i_1), f(i_2), f(i_3)$, …. . Define a function $g\colon \boldsymbol{N} \longrightarrow S$ by $g(n) = f(i_n)$. Now $g(n_1) = g(n_2) \Longrightarrow f\left(i_{n_1}\right) = f\left(i_{n_2}\right) \Longrightarrow i_{n_1} = i_{n_2}$
(since f is injective)$\Longrightarrow$ $n_1 = n_2$. This shows that g is injective.

Let x be any element of S. Then $x = f(i_k)$ for some $i_k \in \boldsymbol{N}$ and there exists a $k \in \boldsymbol{N}$ such that $g(i_k) = f(i_k) = x$. This shows that g is surjective. Thus $g: \boldsymbol{N} \longrightarrow S$ is bijective and hence S is denumerable. Hence the theorem.

Enumeration: A sequence which is used to establish one-to-one correspondence with the elements of a set S is called an *enumeration*.

Example 4: Show that $\boldsymbol{Z}$, the set of integers is denumerable.

Proof: Define a function $f: \boldsymbol{N} \longrightarrow \boldsymbol{Z}$ by

$$f(n) = \begin{cases} \dfrac{n}{2}, & if\ n\ is\ even \\ -\left(\dfrac{n-1}{2}\right), & if\ n\ is\ odd \end{cases}$$

This function enumerates the elements of $\boldsymbol{Z}$ as shown below:

$$f\ \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & \cdots\cdots \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \cdots\cdots \\ 0 & 1 & -1 & 2 & -2 & 3 & -3 & \cdots\cdots \end{array}$$

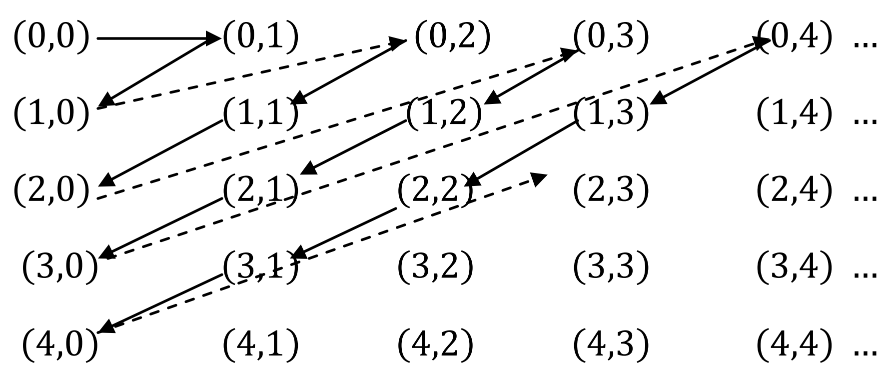
f is injective: Let $n_1, n_2 \in \boldsymbol{N}$. If n_1 is even and n_2 is odd then $f(n_1) = f(n_2) \Longrightarrow \frac{n_1}{2} = -\left(\frac{n_2-1}{2}\right) \Longrightarrow n_1 + n_2 = 1$ and this is not possible. Therefore n_1, n_2 are both even or both odd. Then $f(n_1) = f(n_2) \Longrightarrow n_1 = n_2$. This shows that f is injective.

Let x be any arbitrary element in $\boldsymbol{Z}$. Then x is either positive or nonpositive. If x is positive then there is an even number $y = 2x$ such that $f(y) = \frac{y}{2} = x$. If x is nonpositive then there is an odd natural number $y = -2x + 1$ such that $f(y) = -\left(\frac{y-1}{2}\right) = x$. In any case, each element $x \in \boldsymbol{Z}$ has a pre-image under f, showing f is surjective.

Thus, f is bijective and $\boldsymbol{Z}$ is denumerable.

Example 5: Let $\boldsymbol{W} = \{0,1,2,3,\dots\} = \boldsymbol{N} \cup \{0\}$.Show that $\boldsymbol{W} \times \boldsymbol{W}$ is denumerable.

Solution: We have $\boldsymbol{W} \times \boldsymbol{W} = \{(m,n) | m,n \in W\}$, write the elements of $\boldsymbol{W} \times \boldsymbol{W}$ as shown below:



Now, arrange the elements of $\boldsymbol{W} \times \boldsymbol{W}$ in the order shown by the arrows, namely,

$(0,0), (0,1), (1,0), (0,2), (1,1), (2,0), (0,3), (1,2), (2,1), (3,0), (0,4), (1,3),$

$(2,2), (3,1), (4,0), \dots, \dots$

Define a mapping $f: \boldsymbol{W} \times \boldsymbol{W} \longrightarrow \boldsymbol{N}$ by

$$\boldsymbol{f(m,n) = \frac{1}{2}(m+n+1)(m+n) + m + 1}$$

Notice that f maps $(0,0), (0,1), (1,0), (0,2), (1,1), (2,0), \dots$ onto $1,2,3,4,5,6, \dots$

respectively. Further f is bijective. Thus, $\boldsymbol{W} \times \boldsymbol{W}$ is denumerable.

Example 6: Show that $\boldsymbol{N} \times \boldsymbol{N}$ is denumerable.

Solution: Notice that $\boldsymbol{N} \times \boldsymbol{N} \subset \boldsymbol{W} \times \boldsymbol{W}$ and $\boldsymbol{N} \times \boldsymbol{N}$ is an infinite set. Since $\boldsymbol{W} \times \boldsymbol{W}$ is denumerable, by Theorem 1, $\boldsymbol{N} \times \boldsymbol{N}$ is also denumerable.

Example 7: Show that the set $\boldsymbol{Q}^+$ of positive rational numbers is denumerable.

Solution: Obtain a subset S of $\boldsymbol{N} \times \boldsymbol{N}$ by deleting all ordered pairs (m,n) in which m and n are not relatively prime ($i.e$, m and n have a common factor greater than 1). Note that S contains at least $(1,1), (2,1), (3,1), (4,1), \dots$ .Therefore S is infinite. Since $\boldsymbol{N} \times \boldsymbol{N}$ is denumerable, and S is an infinite subset of $\boldsymbol{N} \times \boldsymbol{N}$, S is also denumerable(by Theorem 1).

We have $Q^+ = \left\{\frac{m}{n} | m, n \in \boldsymbol{N}, m \text{ and } n \text{ are relatively prime}\right\}$

Define a map $f: Q^+ \longrightarrow S$ by $f\left(\frac{m}{n}\right) = (m, n)$

Now, $f\left(\frac{m_1}{n_1}\right) = f\left(\frac{m_2}{n_2}\right) \Longrightarrow (m_1, n_1) = (m_2, n_2) \Longrightarrow m_1 = m_2, n_1 = n_2$

$$\Longrightarrow \frac{m_1}{n_1} = \frac{m_2}{n_2}$$

This shows f is injective. For any $(m, n) \in S$,(where m, n are relatively prime) there is an $\frac{m}{n} \in Q^+$ such that $f\left(\frac{m}{n}\right) = (m, n)$. This shows that f is surjective. Thus $f: Q^+ \longrightarrow S$ is bijective and $Q^+ \sim S$. Since S is denumerable $S \sim N$. Thus $Q^+ \sim S$, $S \sim \boldsymbol{N}$. Therefore $Q^+ \sim \boldsymbol{N}$ (since $\sim$ is transitive).This proves that Q^+ is denumerable.

Lemma 1: If $A_1, A_2, \ldots$ are a countable number of finite sets then $S = \bigcup_i A_i$ is countable.

Solution: We list the elements of A_1 ,then we list the elements of A_2 which do not belong to A_1, then we list the elements of A_3 which have not been listed in $A_1 \cup A_2$, and so on. We can always list the elements of each set, since each set is finite. Now define sets $B_1, B_2, B_3, \ldots$ as given below:

$$B_1 = A_1 \text{ and for } k \geq 2 \text{ , } B_k = A_k - (A_1 \cup A_2 \cup \ldots \cup A_{k-1})$$

Then, the sets B_i are disjoint and

$$S = \bigcup_i A_i = \bigcup_i B_i$$

Let $b_{i1}, b_{i2}, \ldots, b_{im_i}$ be all the elements of B_i. Then $S = \{b_{ij}\}$.

Now, we have two possibilities: either S is finite or infinite.

If S is infinite define a map $f: S \longrightarrow \boldsymbol{N}$ by

$$f\left(b_{ij}\right) = m_1 + m_2 + \cdots + m_{i-1} + j$$

It may be seen that f is bijective. Thus S is denumerable.

Therefore, S is either finite or denumerable, $i.e.$, S is countable.

Theorem 2: A countable union of countable sets is countable.

Proof: Suppose $A_1, A_2, \ldots$ are countable number of sets. Let $S = \bigcup_i A_i$.

If $A_1, A_2, \ldots$ are finite sets then S is countable (by the above Lemma 1…).

Suppose that the sets $A_1, A_2, \ldots$ are countable number of denumerable sets. Since A_i is denumerable, its elements can be indexed as

$$a_{i1}, a_{i2}, a_{i3}, \ldots$$

Now define sets $B_2, B_3, B_4, \ldots$ as follows:

$$B_k = \left\{a_{ij} \mid i + j = k\right\}$$

Note that $B_2 = \{a_{11}\},\ B_3 = \{a_{12}, a_{21}\},\ B_4 = \{a_{13}, a_{22}, a_{31}\}, \ldots$. Observe that each B_k is finite and

$$S = \bigcup_i A_i = \bigcup_k B_k$$

By Lemma1, $\bigcup_k B_k$ is countable. Therefore, $S = \bigcup_i A_i$ is countable.

In any case countable union of countable sets is countable.

Example 8: The set $\boldsymbol{Q}$ of rational numbers is denumerable.

Solution: It is known that Q^+, the set of positive rational numbers is denumerable. Similarly Q^-, the set of negative rational numbers is also denumerable. Thus, $\boldsymbol{Q} = Q^- \cup \{0\} \cup Q^+$ is also denumerable (Since countable union of countable sets is countable).

Example 9: The open interval $(\mathbf{0},\mathbf{1})$ is equivalent to $\boldsymbol{R}$

Solution: Define a map $f: (0,1) \longrightarrow \boldsymbol{R}$ by

$$f(x) = \begin{cases} \dfrac{1}{2x} - 1 & , \quad 0 < x \leq \dfrac{1}{2} \\ \dfrac{1}{2(x-1)} + 1 & , \quad \dfrac{1}{2} \leq x < 1 \end{cases}$$

It may be seen that f is bijective. Therefore $(0,1)\sim\boldsymbol{R}$.

Cantor's diagonal argument

We now show that the set of real numbers lying between 0 and 1 is not denumerable (i.e., uncountable). The proof is based on cantor's diagonal argument and the indirect method. We assume that the set $(0,1)$ is denumerable and then show that an element of the set is different from all those enumerated exists, showing that the enumeration is not exhaustive and hence arriving at a contradiction. The arrangement is called ***diagonal*** because to obtain this particular element, we move along the diagonal of an array.

The diagonal argument is used frequently in the theory of automata and other logical investigations.

This method of proof can be used for showing the nondenumerabilty of other sets also.

Theorem 3: The open interval $(\mathbf{0},\mathbf{1})$ is nondenumerable.

Solution: Assume that $(0,1)$ is denumerable. Then the elements of S can be arranged in an infinite sequence $s_1, s_2, s_3, \ldots$. It is known that every positive real number less than 1 can be expressed as $s = 0.y_1y_2y_3 \ldots$ where $y_i \in \{0,1,2,\ldots,9\}$ and s has an infinite number of nonzero $y_i\,'s$ .This statement is true because the real numbers such as 0.5 and 0.312 can be written as $0.49999\ldots$ and $0.311999\ldots$ respectively. In the light of this fact , we can write the elements $s_1, s_2, s_3, \ldots.$ as

$$s_1 = 0.a_{11}a_{12}a_{13} \dots a_{1n} \dots$$

$$s_2 = 0.a_{21}a_{22}a_{23} \dots a_{2n} \dots$$

$$s_3 = 0.a_{31}a_{32}a_{33} \dots a_{3n} \dots$$

$$s_4 = 0.a_{41}a_{42}a_{43} \dots a_{4n} \dots \quad \text{and so on...}$$

Construct a real number $r = 0.b_1b_2b_3 \dots b_n \dots$,where $b_j = 1$ if $a_{jj} \neq 1$ and $b_j = 2$ if $a_{jj} = 1$ for $j = 1,2,3, \dots$.Notice that r is not equal to any of the numbers $s_1, s_2, s_3, \dots$., because r differs from s_1 in the first position, from s_2 in the second position and so on. This shows that $r \notin (0,1)$ which is a contradiction . Thus the open interval $(0,1)$ is nondenumerable

Corollary: The set $\boldsymbol{R}$ of real numbers is nondenumerable.

Proof: The proof is by contradiction. Assume that $\boldsymbol{R}$ is denumerable. Note that $(0,1) \subset \boldsymbol{R}$ and $(0,1)$ is infinite. By Theorem 1, $(0,1)$ is denumerable. This is a contradiction. Thus $\boldsymbol{R}$ is nondenumerable.

Cardinality of $\boldsymbol{R}$: The cardinality of $\boldsymbol{R}$ is denoted by $\boldsymbol{c}$ and is called the **power of continuum**. That is, $\boldsymbol{card}(\boldsymbol{R}) = \boldsymbol{c}$

Note:

1. The result that the nondenumerabilty of the set of real numbers in the open interval $(0,1)$ is true for the set of real numbers in any open interval (a, b) with $a < b$.
2. The cardinality of all these sets (a, b),$a < b$ which are mutually equivalent and equivalent to $\boldsymbol{R}$, is denoted by $\boldsymbol{c}$
3. The sets $\boldsymbol{R}$,$\boldsymbol{R}^2$,$\boldsymbol{R}^3$, … are all nondenumerable and have the cardinality c

Example 10: Prove that the set of irrational numbers is nondenumerable.

Solution: It is known that $\boldsymbol{R} = \boldsymbol{Q} \cup X$, where X is the set of irrational numbers. To prove that X is nondenumerable. Assume the contrary, $i.e.$, X is denumerable. It is known that $\boldsymbol{Q}$ is denumerable. Therefore $\boldsymbol{Q} \cup X$ is also denumerable, $i.e.$, $\boldsymbol{R}$ is denumerable, a contradiction. Therefore X is nondenumerable.

We now introduce an ordering relation on the family of subsets of the universal set and a corresponding ordering on the set of cardinal numbers.

If A and B are sets such that A is equivalent to a subset of B, then we say that A is ***dominated*** by B or A **precedes** B and write $A \preccurlyeq B$.

If α and β denote the cardinal numbers of the sets A and B respectively and if $A \preccurlyeq B$, then we say that $\boldsymbol{\alpha}$ ***is less than or equal to*** $\boldsymbol{\beta}$. Symbolically,

$$\boldsymbol{A \preccurlyeq B \iff \alpha \leq \beta}$$

Note: The choice of the term *less than or equal to* express the relation on the set of cardinal numbers is based on the fact that for finite sets $A \preccurlyeq B$ implies the natural number representing $k(A)$ is less than or equal to the natural number representing $k(B)$.

From the definition, it is clear that the relation $\preccurlyeq$ and $\leq$ are both reflexive and transitive. The **Schroder – Bernstein theorem** states the following:

If $\boldsymbol{A}$ and $\boldsymbol{B}$ are sets such that $\boldsymbol{A \preccurlyeq B}$ and $\boldsymbol{B \preccurlyeq A}$ then $\boldsymbol{A \sim B}$.

Associated with the relations $\preccurlyeq$ and $\leq$, we have the relations $\prec$ and $<$ given by

$$A \prec B \iff A \preccurlyeq B \text{ and } A \nsim B.$$

$$\alpha < \beta \iff \alpha \leq \beta \text{ and } \alpha \neq \beta.$$

Since $\boldsymbol{N}$ is a proper subset of real numbers, we have $\boldsymbol{N} \preccurlyeq \boldsymbol{R}$. Further $\boldsymbol{N} \nsim \boldsymbol{R}$, since $\boldsymbol{N}$ is denumerable and $\boldsymbol{R}$ is nondenumerable. Therefore, $\boldsymbol{N} \preccurlyeq \boldsymbol{R}$ and $\boldsymbol{N} \nsim \boldsymbol{R} \Longrightarrow \boldsymbol{N} \prec \boldsymbol{R}$. From this it follows that $card\ \boldsymbol{N} < card\ \boldsymbol{R}$, i.e., $\mathcal{N}_0 < \boldsymbol{c}$.

For a given set, can we find another set whose cardinality greater than that of the given set? For finite sets, such a construction is easy. For infinite sets, a theorem due to Cantor shows the existence of such sets.

Theorem 4 (Cantor): Power Set Theorem

For any set A, $A \prec 2^{A}$ where 2^{A} is the power set of A. If α is the cardinality of A and 2^{α} denotes the cardinality of 2^{A}, then $\alpha < 2^{\alpha}$.

(See P8 for proof)

Remark: The cardinality of 2^{N} is c. That is $\mathbf{2^{\mathcal{N}_0} = c}$.

P1:

Any two closed intervals $[\boldsymbol{a}, \boldsymbol{b}]$, $\boldsymbol{a} < b$ and $[\boldsymbol{c}, \boldsymbol{d}]$, $\boldsymbol{c} < d$ are equivalent.

Solution:

Define a map f: $[a, b] \longrightarrow [c, d]$ by

$$f(x) = c + \frac{d-c}{b-a}(x-a)$$

Notice that $f(a) = c$, $f(b) = d$

Let $\alpha, \beta \in [a, b]$ such that $f(\alpha) = f(\beta)$. Then

$$c + \frac{d-c}{b-a}(\alpha - a) = c + \frac{d-c}{b-a}(\beta - a)$$

$\Longrightarrow$ $\alpha - a = \beta - a$ $\Longrightarrow$ $\alpha = \beta$. This proves f is injective.

Let β be any element in $[c, d]$.

Do we have a α such that $f(\alpha) = \beta$? If so then $c + \frac{d-c}{b-a}(\alpha - a) = \beta$,

i.e., $\alpha = a + \frac{b-a}{d-c}(\beta - c)$ (solve for α)

Clearly $a \leq \alpha \leq b$. Thus β has a pre-image. This shows that f is surjective.

Therefore, $[a, b] \sim [c, d]$. Thus, any two closed intervals are equivalent.

P2:

If A and B are sets with the same cardinality and; C and D are sets with the same cardinality, then $A \times C$ and $B \times D$ have the same cardinality.

Solution:

Given that A and B have the same cardinality and; C and D have the same cardinality. That is, $A \sim B$ and $C \sim D$.

Let f and g be the bijections from A to B and from C to D respectively, *i.e.,* $f: A \longrightarrow B$ and $g: C \longrightarrow D$ are bijections. Required to prove that $A \times C \sim B \times D$

Define a map $h: A \times C \longrightarrow B \times D$ by $h(a, c) = \big(f(a), g(c)\big)$.

h is injective:

$h(a_1, c_1) = h(a_2, c_2) \Longrightarrow \big(f(a_1), g(c_1)\big) = \big(f(a_2), g(c_2)\big)$

$\Longrightarrow f(a_1) = f(a_2)$ and $g(c_1) = g(c_2)$

$\Longrightarrow a_1 = a_2$ and $c_1 = c_2$ (since f, g are injective)

$\Longrightarrow (a_1, c_1) = (a_2, c_2)$

Thus, h is injenctive.

h is surjunctive:

Let (b, d) be any element of $B \times D$. That is, b and d are arbitrary elements of B and D respectively. Since f is surjective, there exists an element $x \in A$ such that $f(x) = b$. Similarly, there exists an element $y \in C$ such that $g(y) = d$.Thus, there is an element $(x, y) \in A \times C$ such that

$$h(x, y) = \big(f(x), g(y)\big) = (b, d)$$

This shows that h is surjective. Therefore, $h : A \times C \longrightarrow B \times D$ is bijective.

Thus, $A \times C \sim B \times D$ and so $A \times C$ and $B \times D$ have the same cardinality.

P3:

The relation of equivalence (or equipotence) of sets on any family of sets is an equivalence relation.

Solution:

Let F be a family of sets and let $\sim$ denotes the relation of equivalence of sets on F.

Recall that, for any $A, B \in F$, $A \sim B$ iff there is a bijection between A and B.

(i) *$\sim$ is reflexive:*

Let A be any sets in F. We have an identify function $i : A \longrightarrow A$ defined by $i(x) = x$, $\forall x \in A$. It is known that the identity function is bijective. Thus, $A \sim A$, for every $A \in F$. Therefore, the relation $\sim$ is reflexive.

(ii) *$\sim$ is symmetric:*

Let $A, B \in F$ and $A \sim B$.

$A \sim B \Longrightarrow$ There exists a bijective function $f : A \longrightarrow B$. It is known that the inverse of a bijective function exists and it is bijective.

$\Longrightarrow f^{-1} : B \longrightarrow A$ exists and f^{-1} is bijective $\Longrightarrow B \sim A$

Thus, $A \sim B \Longrightarrow B \sim A$. Therefore, the relation $\sim$ is symmetric.

(iii) *$\sim$ is transitive:*

Let $A, B, C \in F, A \sim B$ and $B \sim C$.

$A \sim B$ and $B \sim C \Longrightarrow$ There exist bijections $f : A \longrightarrow B$ and $g : B \longrightarrow C$.

$\Longrightarrow g \circ f : A \longrightarrow C$ is bijective (∵The composition of two bijective functions is also bijective)

$\Longrightarrow A \sim C$

Therefore, the relation $\sim$ is transitive.

Thus, the relation $\sim$ (equivalence of sets) defined on F is an equivalence relation.

P4:

Determine whether each of these sets is countable or uncountable. For those that are countable, exhibit a one-to-one correspondence between the set of natural numbers and the set of

a) **the negative integers**
b) **the odd integers**
c) **the real numbers between 0 and $\frac{1}{2}$**

Solution:

a) We have $\boldsymbol{Z}^- = \{\dots, -4, -3, -2, -1\}$. Clearly, $\boldsymbol{Z}^-$ infinite.
Define a map $f: \boldsymbol{N} \longrightarrow \boldsymbol{Z}^-$ by $f(n) = -n,\ n \in \boldsymbol{N}$.
It is injective and surjective (show!)

Thus, $\boldsymbol{Z}^- \sim \boldsymbol{N}$. Thus, $\boldsymbol{Z}^-$ is denumerable. Therefore, $\boldsymbol{Z}^-$ is countable.

b) We have $A = \{\dots, -5, -3, -1, 1, 3, 5, \dots\} = B \cup C$, where $= \{1, 3, 5, \dots\}$, the set of positive odd integers and $C = \{\dots, -5, -3, -1\}$, the set of negative odd integers. Required to show that A is countable.

Clearly, both B and C are infinite sets. Define a map $f: \boldsymbol{N} \longrightarrow B$ by $f(n) = 2n - 1$. It is bijective (show!). Thus, $\boldsymbol{N} \sim B$. Therefore, $B \sim \boldsymbol{N}$ (why?), B is denumerable and B is countable.

Notice that $B \sim C$. Therefore, $C \sim B$ and $B \sim \boldsymbol{N} \Longrightarrow C \sim \boldsymbol{N}$ (why?) .This shows that C is countable.

Since B and C are countable; $A = B \cup C$ is also countable (*because countable union of countable sets is countable*).

c) We have $\left(0, \frac{1}{2}\right)$, $i.e.$, $\left\{x \mid x \in \boldsymbol{R} \wedge \left(0 < x < \frac{1}{2}\right)\right\}$. It is uncountable, because the set of real numbers in any open interval (a, b) with $a < b$, is uncountable.

P5:

Determine whether each of these sets is countable or uncountable. For those that are countable, exhibit a one-to-one correspondence between the set of natural numbers and the set of

a) **the integers that are multiples of 7**
b) **the integers not divisible by 3**
c) **all rational numbers that cannot be written with denominators less than 4**

Solution:

a) We have $\boldsymbol{Z_7}$, $i.e$, the set of integers that are multiples of 7

$$= \{\dots, -21, -14, -7, 0, 7, 14, 21, \dots\}$$

Clearly $\boldsymbol{Z_7}$ is infinite.

First note that $\boldsymbol{Z} \sim \boldsymbol{Z_7}$. Define a map $f : \boldsymbol{Z} \longrightarrow \boldsymbol{Z_7}$ by $f(n) = 7n, n \in \boldsymbol{Z}$. It is easy to see that f is bijective. Therefore, $\boldsymbol{Z} \sim \boldsymbol{Z_7}$

Now, $\boldsymbol{Z_7} \sim \boldsymbol{Z}$ and $\boldsymbol{Z} \sim \boldsymbol{N}$ ($\because Z$ is denumerable) $\Longrightarrow \boldsymbol{Z_7} \sim \boldsymbol{N}$ ($\because \sim$ is transitive)

$\Longrightarrow \boldsymbol{Z_7}$ is denumerable. Therefore, $\boldsymbol{Z_7}$ is countable

b) We have $\boldsymbol{Z_3} = \{\dots, -9, -6, -3, 0, 3, 6, 9, \dots\}$, $\boldsymbol{Z} - \boldsymbol{Z_3}$ is the set of all integers not divisible by 3 and it is an infinite set. Note that Z is denumerable and $\boldsymbol{Z} - \boldsymbol{Z_3} \subset \boldsymbol{Z}$. Therefore, $\boldsymbol{Z} - \boldsymbol{Z_3}$ is denumerable (*by Theorem: Every infinite subset of a denumerable set is denumerable*). Thus $\boldsymbol{Z} - \boldsymbol{Z_3}$ is countable.

c) We have $\boldsymbol{Q}$, the set of rational numbers. Let A be the set of all rational numbers which are written with the denominator less than 4. That is

$$A = \left\{\frac{p}{1} \mid p \in \boldsymbol{Z}\right\} \cup \left\{\frac{p}{2} \mid p \in \boldsymbol{Z}\right\} \cup \left\{\frac{p}{3} \mid p \in \boldsymbol{Z}\right\}$$

Clearly, A is infinite. Now, $\boldsymbol{Q} - \boldsymbol{A}$ is the set of all rational numbers. That cannot be written with denominator less than 4. Note that $\boldsymbol{Q}$ is denumerable and $\boldsymbol{Q} - \boldsymbol{A} \subset \boldsymbol{Q}$. Therefore, $\boldsymbol{Q} - \boldsymbol{A}$ is denumerable (*by Theorem: Every infinite subset of a denumerable set is denumerable*). Thus, $\boldsymbol{Q} - \boldsymbol{A}$ is countable.

P6:

If $\boldsymbol{A}$ is uncountable and $\boldsymbol{A} \subseteq \boldsymbol{B}$ then $\boldsymbol{B}$ is uncountable.

Proof:
We have that A is an uncountable set and $A \subseteq B$. Required to prove that B is uncountable.

Now, A is uncountable $\Longrightarrow A$ is infinite and A is not denumerable. Since A is infinite, $A \subseteq B$; B must be an infinite set.

Assume that B is denumerable. Since A is infinite, $A \subseteq B$ and B is denumerable, A must be denumerable (*by Theorem: Every infinite subset of a denumerable set is denumerable*).This is a contradiction to the nondenumerabilty of A. Therefore, B is not denumerable. Thus, B is uncountable.

P7

If $\boldsymbol{A}$ is an uncountable set and $\boldsymbol{B}$ is a countable set then $\boldsymbol{A - B}$ is uncountable.

Solution:

We have that A is an uncountable set and B is a countable set. Required to prove that $A - B$ is uncountable.

Assume the contrary, $i.e.$, Assume that $A - B$ is countable. Then

$$A = (A - B) \cup B$$

is countable, because countable union of countable sets is countable. This is a contradiction to the uncountability of A. The result now follows.

P8:

Theorem (Cantor): Power Set Theorem
For any set A, $A \prec 2^A$ where 2^A is the power set of A. If α is the cardinality of A and 2^α denotes the cardinality of 2^A, then $\alpha < 2^\alpha$.

Solution:
Let $f: A \longrightarrow 2^A$ be defined by $f(a) = \{a\}$ for every $a \epsilon A$. If $f(a) = f(b)$ then $\{a\} = \{b\}$ and $a = b$. This shows that f is injective. Thus A is equivalent to a (proper) subset of 2^A. Therefore $A \preccurlyeq 2^A$.

To show that $A \prec 2^A$, we have to show that $A \nsim 2^A$, we show this by indirect method of proof.

Assume that $A \sim 2^A$. Let g: $A \longrightarrow 2^A$ be a bijection. For any $a \in A$, $g(a)$ is a subset of A (because $g(a)$ is an element of 2^A). We call a an *interior member* of A if $a \in g(a)$; otherwise a is called an *exterior member* of A.

Let B be the set of all exterior members of A, that is,

$$B = \{x | (x \in A) \wedge (x \notin g(x))\}$$

Clearly $B \subseteq A$. Therefore, $B \in 2^A$. Since g is onto, B has a pre-image say $b \in A$. That is, there exists an element $b \in A$ such that $g(b) = B$.

Now, we have two cases: either $b \in B$ or $b \notin B$.

(i) If $b \in B$ then $b \in A$ and $b \notin g(b)$. Since $g(b) = B$; $b \notin B$. This is a contradiction.
(ii) If $b \notin B$ then $b \in g(b)$. Since $g(b) = B$; $b \in B$. Again we arrive a contradiction.

This argument shows that $A \nsim 2^A$. Therefore $A \prec 2^A$.

If $ard(A) = \alpha$, then $card(2^A) = 2^\alpha$.Now, $\alpha < 2^\alpha$, since $A \prec 2^A$.

Hence the theorem.