3.5

GREEN'S THEOREM

In unit 2, we have defined the definite integral and discussed methods for evaluating it. The vector field theory provides three important theorems. One of the theorems is the Green's theorem, which provides a relationship between a double integral over a region R and the line integral over the closed curve C bounding R. Green's theorem is also called the first fundamental theorem of integral vector calculus.

Theorem (Green's Theorem):

Let C be a piecewise smooth simple closed curve bounding a region R. If $f, g, \frac{\partial f}{\partial y}$ and $\frac{\partial g}{\partial x}$ are continuous functions on R, then

$$\oint_C f(x,y)dx + g(x,y)dy = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dxdy$$

the integration being carried in the positive direction (counter clockwise direction) of C.

Proof: we shall prove Green's theorem for a particular case of the region R.

Let the region R be simultaneously expressed in the following form

$R: u_1(x) \leq y \leq u_2(x), \qquad a \leq x \leq b$ (Figure 1)……… (1)

$R: v_1(y) \leq x \leq v_2(y), \qquad c \leq y \leq d$ (Figure 2)………. (2)

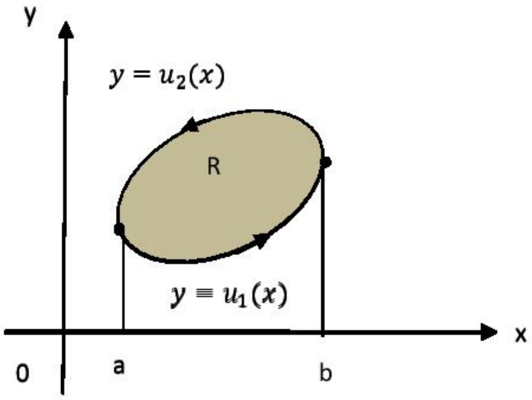


Figure 1

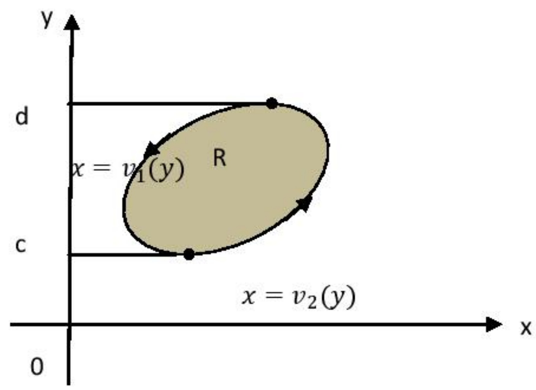


Figure 2

Using figure 2 we obtain

$$\iint_R \frac{\partial g}{\partial x} dxdy = \int_c^d \left[\int_{v_1(y)}^{v_2(y)} \frac{\partial g}{\partial x} dx\right] dy = \int_c^d [g(v_2(y), y) - g(v_1(y), y)]dy$$

$$= \int_c^d g(v_2(y), y)dy + \int_d^c g(v_1(y), y)\, dy = \oint_c g(x, y)dy$$

The integration being carried in the counter clockwise direction

Using figure 1, we obtain

$$\iint_R \frac{\partial f}{\partial y} dxdy = \int_a^b \left[\int_{u_1(x)}^{u_2(x)} \frac{\partial f}{\partial y} dy\right] dx = \int_a^b [f(x, u_2(x)) - f(x, u_1(x)]dx$$

$$= \int_a^b f\big(x, u_2(x)\big)dx + \int_b^a f\big(x, u_1(x)\big)dx = -\oint_c f(x, y)dx$$

the integration being carried in the counter clockwise direction.

Therefore,

$$\iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dxdy = \oint_c f(x, y)dx + g(x, y)dy.$$

The proof can be extended to more general region R. The region R is decomposed into a finite number of sub regions $R_1, R_2, \dots \dots, R_n$ such that each region can be expressed in both the forms given in Equations (1) and (2) (figures (a) and (b)).

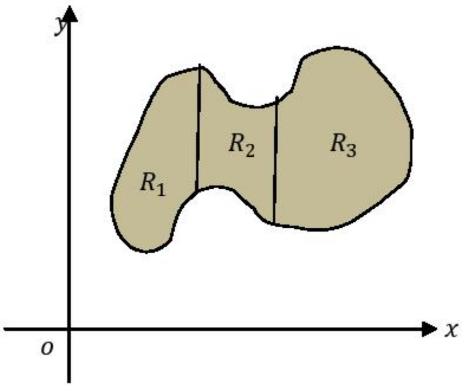


Figure (a)

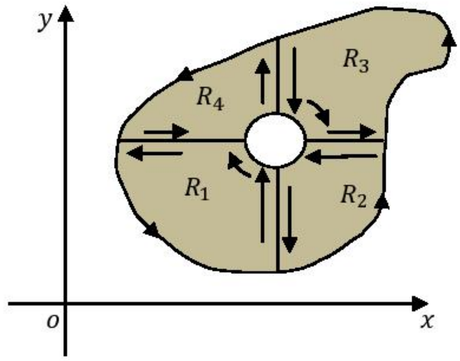
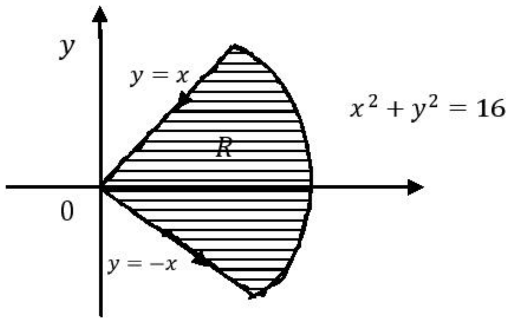


Figure (b)

Example: Find the work done by the force $\boldsymbol{F} = (x^2 - y^3)\boldsymbol{i} + (x + y)\boldsymbol{j}$ in moving a particle along the closed path C contacting the curves $x + y = 0, x^2 + y^2 = 16$ and $y = x$ in the first and fourth quadrant

Solution: The work done by the force is given by

$$W = \oint_C \boldsymbol{F}.d\boldsymbol{r} = \oint_C (x^2 - y^3)dx + (x + y)dy$$



The closed path C bounds the region R as given in figure. Using the Green's theorem, we obtain

$$\oint_C (x^2 - y^3)dx + (x + y)dy = \iint_R (1 + 3y^2))dxdy.$$

It is convenient to use polar coordinates to evaluate the integral. The region R is given by

$$R: x = rcos\ \theta, y = r\sin\theta\,, 0 \le r \le 4, -\frac{\pi}{4} \le \theta \le \frac{\pi}{4}.$$

Therefore,

$$\iint_R (1 + 3y^2))dxdy = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left[\int_0^4 (1 + 3r^2\sin^2\theta)r\ dr\right] d\theta$$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left[\frac{r^2}{2} + \frac{3}{4}r^4 sin^2\theta\right]_0^4 d\theta$$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (8 + 192sin^2\theta)d\theta$$

$$= 2\int_0^{\frac{\pi}{4}} [8 + 96(1 - \cos 2\theta)]d\theta$$

$$= 2[104\theta - 48\sin 2\theta]_0^{\pi/4} = 52\pi - 96.$$

Sometime we use Green's theorem to replace a double integral by a line integral. In particular, there is a formula for the area of a region bounded by a curve C, in terms of a line integral around C.

Area of Region

If C is a curve that bounds a region R, then the area of R is

$$A = \frac{1}{2}\int_C xdy - ydx$$

Proof: Let $f = -y$ and $g = x$; then by Green's theorem,

$$\frac{1}{2}\int_C xdy - ydx = \frac{1}{2}\iint_R \left[\frac{\partial x}{\partial x} - \frac{\partial(-y)}{\partial y}\right] dxdy = \iint_R dxdy,$$

which is the area of R.

Example: Verify the above area formula if R is the disk $x^2 + y^2 \leq r^2$.

Solution: We know the area is $\pi\, r^2$. The preceding formula with $x = r\cos t\,, y = r\sin t\,, 0 \leq t \leq 2\pi$, gives

$$\begin{aligned} A &= \frac{1}{2}\int_C xdy - ydx \\ &= \frac{1}{2}\int_0^{2\pi}(r\cos t)(r\cos t)dt - (r\sin t)(-r\sin t)\,dt \\ &= \frac{1}{2}\int_0^{2\pi} r^2\,dt = \pi\, r^2. \end{aligned}$$

So the formula checks.

Example: Using the Green's theorem, show that

$$\oint_C \frac{\partial u}{\partial n} ds = \iint_R \nabla^2 u dx dy$$

where ∇^2 is the Laplace operator $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ and $\boldsymbol{n}$ is the unit outward normal to $\boldsymbol{C}$.

Solution: Let the position vector of a point on C, in terms of the arc length s be $\boldsymbol{r}(s) = x(s)\boldsymbol{i} + y(s)\boldsymbol{j}$. Then, the tangent vector to $\boldsymbol{C}$ is given by

$$\boldsymbol{T} = \frac{d\boldsymbol{r}}{ds} = \frac{dx}{ds}\boldsymbol{i} + \frac{dy}{ds}\boldsymbol{j}$$

and the normal vector $\boldsymbol{n}$ is given by (since $\boldsymbol{n}.\boldsymbol{T} = 0$)

$$\boldsymbol{n} = \frac{dy}{ds}\boldsymbol{i} - \frac{dx}{ds}\boldsymbol{j}.$$

Note that $\boldsymbol{n}$ is the unit normal vector. Now,

$$\oint_C \frac{\partial u}{\partial n} ds = \oint_C \nabla \mathrm{u}.\mathbf{n} ds$$

since $\frac{\partial u}{\partial n}$ is the directional derivative of u in the direction of $\boldsymbol{n}$. Therefore, using Green's theorem, we obtain

$$\oint_C \frac{\partial u}{\partial n} ds = \oint_C \left(\frac{\partial u}{\partial x}\frac{dy}{ds} - \frac{\partial u}{\partial y}\frac{dx}{ds}\right) ds = \oint_C \left(-\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy\right)$$

$$= \iint_R \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) dx dy = \iint_R \nabla^2 \mathrm{udxdy}.$$

Problem 1: Let C be the boundary of the square $[0,1] \times [0,1]$ oriented counterclockwise evaluate $\int_C (y^4 + x^3)dx + 2x^6 dy$.

Solution: We obtain a match with the line integral $\int_C f dx + g dy$ by choosing C to be the boundary of $D = [0,1] \times [0,1]$ and $f = y^4 + x^3, g = 2x^6$, then by Green's theorem,

$$\int_C (y^4 + x^3)dx + 2x^6 dy = \iint_D \left[\frac{\partial}{\partial x} 2x^6 - \frac{\partial}{\partial y}(y^4 + x^3)\right] dxdy$$

$$= \iint_D [12x^5 - 4y^3]dxdy$$

$$= \int_{\mathbf{0}}^{\mathbf{1}} \left[\int_{\mathbf{0}}^{\mathbf{1}} (12x^5 - 4y^3)\, dx\right] dy$$

$$= \int_{\mathbf{0}}^{\mathbf{1}} (2 - 4y^3)\, dy = 1.$$

Problem 2: Verify Green's theorem for $f(x, y) = x$ and $g(x, y) = xy$ where D is the unit disk $x^2 + y^2 \leq 1$.

Solution: We evaluate the integral on both side of Green's theorem directly. The boundary of D is the unit circle. It is parameterized in the positive sense by $x = \cos t\,, y = \sin t\,, 0 \leq t \leq 2\pi$, so

$$\int_{\partial D} f dx + g dy = \int_0^{2\pi} [(\cos t)(-\sin t) + \cos t \sin t \cos t]\, dt$$

$$= \left[\frac{\cos^2 t}{2}\right]_0^{2\pi} + \left[-\frac{\cos^3 t}{3}\right]_0^{2\pi} = 0.$$

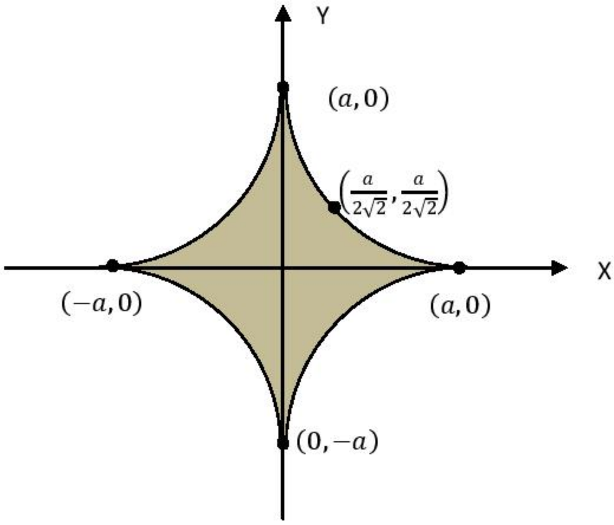
On the other hand,

$$\iint_D \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dxdy = \iint_D y dxdy$$

which is zero by symmetry (the integrals over the portions $y \geq 0$ and $y \leq 0$ cancel). Thus Green's theorem is verified in the case.

Problem 3: Compute the area of the region enclosed by the hypocycloid $x^{2/3} + y^{2/3} = a^{2/3}$ using the parameterization

$$x = a\cos^3\theta, \qquad y = a\sin^3\theta, \qquad 0 \le \theta \le 2\pi.$$



Solution: $A = \frac{1}{2}\int_C xdy - ydx$

$$= \frac{1}{2}\int_0^{2\pi} [(a\cos^3\theta)(3a\sin^2\theta\cos\theta) - (a\sin^3\theta)(-3a\cos^2\theta\sin\theta)]\,d\theta$$

$$= \frac{3}{2}a^2\int_0^{2\pi} (\sin^2\theta\cos^4\theta + \cos^2\theta\ \sin^4\theta)\,d\theta$$

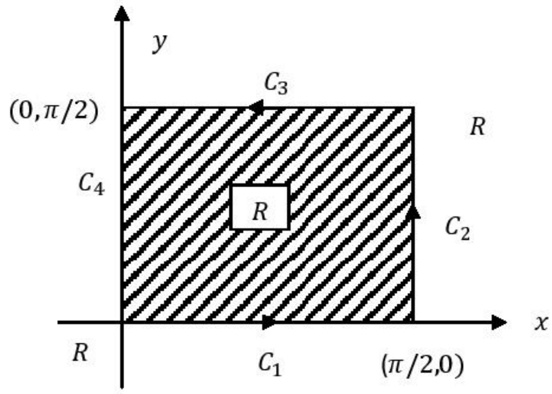
Using $\sin^2\theta + \cos^2\theta = 1$ and $2\sin\theta\cos\theta = \sin 2\theta$, one derives the trigonometric identity $\sin^2\theta\cos^4\theta + \cos^2\theta\sin^4\theta = \frac{1}{4}\sin^2 2\theta$, so we get

$$A = \frac{3}{8}a^2\int_0^{2\pi}\sin^2 2\theta\,d\theta = \frac{3}{8}a^2\int_0^{2\pi}\left(\frac{1-\cos 4\theta}{2}\right)d\theta$$

$$= \frac{3}{16}a^2\int_0^{2\pi}d\theta - \frac{3}{16}a^2\int_0^{2\pi}\cos 4\theta\,d\theta = \frac{3}{8}\pi a^2.$$

Problem 4: Verify the Green's theorem for $f(x,y) = e^{-x}\sin y, g(x,y) = e^{-x}\cos y$ and C is the square with vertices at $(0,0), \left(\frac{\pi}{2},0\right), \left(\frac{\pi}{2},\frac{\pi}{2}\right), \left(0,\frac{\pi}{2}\right)$.

Solution:



We can write the line integral as

$$\oint_C f dx + g dy = \left[\int_{C_1} + \int_{C_{21}} + \int_{C_3} + \int_{C_4}\right](f dx + g dy)$$

Where C_1, C_2, C_3 and C_4 are the boundary lines as given in figure, we have along C_1: $y = 0, 0 \leq x \leq \frac{\pi}{2}$ and

$$\int_{C_1} e^{-x}(\sin y\, dx + \cos y\, dy) = 0.$$

Along $C_{2:}x = \frac{\pi}{2}, 0 \leq y \leq \frac{\pi}{2}$ and

$$\int_{C_2} e^{-x}(\sin y\, dx + \cos y\, dy) = \int_0^{\frac{\pi}{2}} e^{-\frac{\pi}{2}} \cos y\, dy = e^{-\frac{\pi}{2}}.$$

Along $C_{3:}y = \frac{\pi}{2}, \frac{\pi}{2} \leq x \leq 0$ and

$$\int_{C_3} e^{-x}(\sin y\, dx + \cos y\, dy) = \int_{\frac{\pi}{2}}^{0} e^{-x}\, dx = e^{-\frac{\pi}{2}} - 1.$$

Along $C_{4:}x = 0, \frac{\pi}{2} \leq y \leq 0$ and

$$\int_{C_4} e^{-x}\,(\sin y\, dx + \cos y\, dy) = \int_{\frac{\pi}{2}}^{0} \cos y\, dy = -1.$$

Therefore, $\oint_c f dx + g dy = e^{-\pi/2} + e^{-\pi/2} - 1 - 1 = 2\left(e^{-\frac{\pi}{2}} - 1\right)$.

Using Green's theorem, we obtain

$$\oint_c f dx + g dy = \iint_R (-2e^{-x}\cos y) dx dy$$

$$= \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} -2e^{-x} \cos y dx dy = 2\left(e^{-\frac{\pi}{2}} - 1\right).$$

Problem 5: Evaluate$\oint_c fdx + gdy$, where $f = y^2 - 7y$, $g = 2xy + 2x$ and C is $x^2 + y^2 = 1$.

Solution: By Green's theorem we know $\oint_c fdx + gdy = \int_R \int \left(\frac{\partial g}{\partial x} - \frac{\partial f}{dy}\right) dxdy$.

Therefore $\int_R \int \left(\frac{\partial g}{\partial x} - \frac{\partial f}{dy}\right) dxdy = \int_R \int [(2y + 2) - (2y - 7)]\, dxdy$

$$= 9 \int_R \int dxdy$$

$$= 9\pi.$$

since area of the circle is π.

Exercise

1. Let $\boldsymbol{F}(x, y) = y\boldsymbol{i} - x\boldsymbol{j}$ and let C be a circle of radius r traversed counterclockwise. Write the line integral $\int_C \boldsymbol{F} . \boldsymbol{dr}$ as a double integral using Green's theorem and evaluate.
2. Use Green's theorem to find the area bounded by the ellipse C: $x^2/\,a^2 \,+\, y^2/\,b^2 \,= 1$.
3. Evaluate $\oint_C (x^2 + y^2)dx + (y + 2x)dy$, where C is the boundary of the region in the first quadrant that is bounded by the curves $y^2 = x$ and $x^2 = y$.
4. Evaluate $\int_C (y - \sin x)\, dx + \cos x dy$, where C is the plane triangle enclosed by the lines $y = 0, x = \frac{\pi}{2}, y = \frac{2}{\pi} x$.
5. A vector field is given by $\boldsymbol{F} = \sin y \boldsymbol{i} + x(1 + \cos y)\boldsymbol{j}$. Evaluate the line integral over the circular path $x^2 + y^2 = a^2$, $z = 0$.
6. Verify Green's theorem for $\int_C \frac{y^3}{3} dx + (\frac{x^3}{3} + xy^2)dy$, where C is the boundary of the region between $y = x^2$ and $y = x$.
7. Verify Green's theorem in the plane for $\int_C \left(\frac{e^x}{3} - y\right) dx + \left(\frac{e^y}{3} + 2x\right) dy$, where C is the boundary of the ellipse $x^2 + 4y^2 = 4$.
8. Verify Green's theorem for $\int_C 2xy dx + (e^x + x^2)dy$, where C is the triangle with vertices at $(0,0), (1,0)$ and $(1,1)$.

Answers

1. $-2\pi r^2$
2. πab
3. $\frac{11}{30}$
4. $-(\frac{\pi}{4}+\frac{2}{\pi})$
5. πa^2