UNIT $\underline{VI}$

Numerical Integration & Solution of Ordinary Differential Equations

Numerical Integration

(a) Trapezoidal Rule

(b) Simpson's $\frac{1}{3}$rd Rule

(c) Simpson's $\frac{3}{8}$th Rule

Solution of O.D.E

(a) Euler's Method.

(b) R-K Method

Numerical integration

A definite integral is of the form

$\int_a^b f(x)\,dx$ represents the area enclosed under the curve $y = f(x)$ between the limits $x = a$ and $x = b$

Now we find the solutions of above integrals by using Numerical integration methods

(i) Trapezoidal Rule

(ii) Simpson's $\frac{1}{3}$rd Rule

(iii) Simpson's $\frac{3}{8}$th Rule

$y = f(x)$

x	x_0	x_1	x_2	- - -	x_n
y	y_0	y_1	y_2	- - -	y_n

Trapezoidal Rule

$$\int_a^b f(x)\,dx = \frac{h}{2}\left[(y_0+y_n)+2(y_1+y_2+\cdots+y_{n-1})\right]$$

$$\int_a^b f(x)\,dx = \frac{h}{2}\left[(\text{sum of first and last ordinates})+2(\text{sum of remaining ordinates})\right]$$

where $\boxed{h=\frac{b-a}{n}}$

Simpson's $\frac{1}{3}$rd Rule

$$\int_a^b f(x)\,dx = \frac{h}{3}\left[(y_0+y_n)+2(y_2+y_4+y_6+\cdots+y_{n-2})+4(y_1+y_3+y_5+\cdots+y_{n-1})\right]$$

$\boxed{h=\frac{b-a}{n}}$

$$\int_a^b f(x)\,dx = \frac{h}{3}\left[(\text{sum of first and last ordinates})+2(\text{even } y\text{'s})+4(\text{odd } y\text{'s})\right]$$

Simpson's $\frac{3}{8}$th Rule

$$\int_a^b f(x)\,dx = \frac{3h}{8}\Big[(y_0+y_n)+2(y_3+y_6+y_9+\cdots+y_n)+3(y_1+y_2+y_4+y_5+y_7+\cdots+y_{n-1})\Big]$$

$$\int_a^b f(x)\,dx = \frac{3h}{8}\Big[(\text{sum of first and last ordinate})+2(\text{sum of multiple of 3 of } y\text{'s})+3(\text{remaing } y\text{'s})\Big]$$

where $\boxed{h = \frac{b-a}{n}}$

① Evaluate $\int_0^1 x^3 \, dx$ with five ($n=5$) subintervals by using

(i) Trapezoidal Rule

(ii) Simpson's $\frac{1}{3}$rd Rule

(iii) Simpson's $\frac{3}{8}$th Rule

Sol Given $\int_0^1 x^3 \, dx$

Here $a=0$, $b=1$, $y=f(x)=x^3$

Here $n=5$

$$h = \frac{b-a}{n}$$

$$h = \frac{1-0}{5} = \frac{1}{5} = 0.2$$

$$\boxed{h=0.2}$$

The value of x and y are Tabulated below

	$x_0 = a$	x_1 x_0+h	x_2 x_1+h	x_3 x_2+h	x_4 x_3+h	$x_5 = b$ x_4+h
x	0	0.2	0.4	0.6	0.8	1
$y = x^3$	0 y_0	0.008 y_1	0.064 y_2	0.216 y_3	0.512 y_4	1 y_5

Trapezoidal Rule:

$$\int_a^b f(x)\,dx = \frac{h}{2}\left[(y_0+y_n)+2(y_1+y_2+\cdots y_{n-1})\right]$$

$$\int_0^1 x^3\,dx = \frac{h}{2}\left[(y_0+y_5)+2(y_1+y_2+y_3+y_4)\right]$$

$$= \frac{0.2}{2}\left[(0+1)+2(0.008+0.064+0.216+0.512)\right]$$

$$= 0.26$$

$$\boxed{\int_0^1 x^3\,dx = 0.26}$$

Simpson's $\frac{1}{3}$rd Rule

$$\int_a^b f(x)\,dx = \frac{h}{3}\left[(y_0+y_n) + 2(y_2+y_4+\cdots) + 4(y_1+y_3+y_5+\cdots)\right]$$

$$\int_0^1 x^3\,dx = \frac{h}{3}\left[(y_0+y_5) + 2(y_2+y_4) + 4(y_1+y_3)\right]$$

$$= \frac{0.2}{3}\left[(0+1) + 2(0.064+0.512) + 4(0.008+0.216)\right]$$

$$\boxed{\int_0^1 x^3\,dx = 0.2032}$$

Simpson's $\frac{3}{8}$th Rule

$$\int_a^b f(x)\,dx = \frac{3h}{8}\left[(y_0+y_n) + 2(y_3+y_6+\cdots) + 3(y_1+y_2+y_4+\cdots)\right]$$

$$\int_0^1 x^3\,dx = \frac{3h}{8}\left[(y_0+y_5) + 2(y_3) + 3(y_1+y_2+y_4)\right]$$

$$= \frac{3(0.2)}{8}\Big[(0+1) + 2(0.216) + 3(0.008 + 0.064 + 0.512)\Big]$$

$$\boxed{\int_0^1 x^3\,dx = 0.2388}$$

<u>Exact method</u>

$$\int_0^1 x^3\,dx = \left[\frac{x^4}{4}\right]_0^1$$

$$= \frac{1}{4}$$

$$= 0.25$$

$$\boxed{\int_0^1 x^3\,dx = 0.25}$$

	$y = \int_0^1 x^3\,dx$
exact	0.25
trapezoidal	0.26
Simpson $\frac{1}{3}$rd	0.2032
Simpson $\frac{3}{8}$th	0.2388

② Evaluate $\int_0^2 e^{-x^2} dx$ by Taking $h=0.25$

Sol given $\int_0^2 e^{-x^2} dx$

Here $a=0$, $b=2$, $y=f(x)=e^{-x^2}$

$h=0.25$

$$h=\frac{b-a}{n} \Rightarrow n=\frac{b-a}{h}$$

$$=\frac{2-0}{0.25}=8$$

$$\boxed{n=8}$$

x	x_0 a 0	x_1 x_0+h 0.25	x_2 x_1+h 0.5	x_3 x_2+h 0.75	x_4 x_3+h 1	x_5 x_4+h 1.25	x_6 x_5+h 1.5	x_7 x_6+h 1.75	x_8 x_7+h 2
y e^{-x^2}	1 y_0	0.9394 y_1	0.7788 y_2	0.56978 y_3	0.36781 y_4	0.20961 y_5	0.10539 y_6	0.0467 y_7	0.0183 y_8

<u>Trapezoidal Rule</u>:

$$\int_a^b f(x)dx = \frac{h}{2}\left[(y_0+y_n)+2(y_1+y_2+\cdots)\right]$$

$$\int_0^2 e^{-x^2} dx = \frac{h}{2}\left[(y_0+y_8)+2(y_1+y_2+\cdots+y_7)\right]$$

$$= \frac{0.25}{2}\left[(1+0.0183)+2(0.9394+0.7788 + 0.5697+0.367+0.2096 + 0.1053+0.046)\right]$$

$$= 0.8816$$

$$\boxed{\int_0^2 e^{-x^2}\,dx = 0.8816}$$

Simpson's $\frac{1}{3}$ Rule

$$\int_a^b f(x)\,dx = \frac{h}{3}\left[(y_0+y_n)+2(y_2+y_4+\cdots) + 4(y_1+y_3+\cdots)\right]$$

$$\int_0^2 e^{-x^2}\,dx = \frac{h}{3}\left[(y_0+y_8)+2(y_2+y_4+y_6) + 4(y_1+y_3+y_5+y_7)\right]$$

$$= \frac{0.25}{3}\left[(1+0.0183)+2(0.7788 + 0.36788+0.10558) + 4(0.9394+0.569781+0.2096 + 0.0467)\right]$$

$$= 0.882028$$

$$\boxed{\int_0^2 e^{-x^2}\,dx = 0.8820}$$

Simpson's $\frac{3}{8}$th Rule

$$\int_a^b f(x)\,dx = \frac{3h}{8}\left[(y_0 + y_n) + 2(y_3 + y_6 + y_9 + \cdots) + 3(y_1 + y_2 + y_4 + \cdots)\right]$$

$$\int_0^2 e^{-x^2}\,dx = \frac{3h}{8}\left[(y_0 + y_8) + 2(y_3 + y_6) + 3(y_1 + y_2 + y_4 + y_5 + y_7)\right]$$

$$= \frac{3(0.25)}{8}\left[(1 + 0.0183) + 2(0.569778 + 0.1053) + 3(0.9394 + 0.7788 + 0.36788 + 0.2096 + 0.046)\right]$$

$$= 0.88061$$

$$\boxed{\int_0^2 e^{-x^2}\,dx = 0.88061}$$

$\int_0^2 e^{-x^2}\,dx$	
Trapezoidal	0 8816
Simpson $\frac{1}{3}$rd	0 8820
Simpson $\frac{3}{8}$th	0.8806

③ Evaluate $\int_0^{\pi} t \sin t \, dt$

$\boxed{y = f(t)}$

Sol: given $\int_0^{\pi} t \sin t \, dt$

$a = 0$; $b = \pi$, $y = f(t) = t \sin t$

Take $n = 4$

$h = \frac{b-a}{n} = \frac{\pi - 0}{4} = \frac{\pi}{4}$ $\boxed{h = \frac{\pi}{4}}$

t	$a = t$ 0	t_1 $\frac{\pi}{4}$	t_2 $\frac{\pi}{2}$	t_3 $\frac{3\pi}{4}$	$t_4 = b$ π
$y = t \sin t$	0 y_0	0.5553 y_1	1.5707 y_2	1.6660 y_3	0 y_4

Trapezoidal Rule

$$\int_0^{\pi} t \sin t \, dt = \frac{h}{2}\left[(y_0 + y_4) + 2(y_1 + y_2 + y_3)\right]$$

$$= \frac{\pi}{8}\left[(0+0) + 2(0553 + 1.5707 + 1.6660)\right]$$

$$\boxed{\int_0^{\pi} t \sin t \, dt = 2.978}$$

Simpson's $\frac{1}{3}$rd Rule :

$$\int_0^{\pi} t \sin t \, dt = \frac{h}{3}\left[(y_0+y_4)+2(y_2)+4(y_1+y_3)\right]$$

$$= \frac{\pi}{12}\left[(0+0)+2(1.5702)+4(0.553+1.666)\right]$$

$$\boxed{\int_0^{\pi} t \sin t \, dt = 3.1485}$$

Simpson's $\frac{3}{8}$th Rule

$$\int_0^{\pi} t \sin t \, dt = \frac{3h}{8}\left[(y_0+y_4)+2(y_3)+3(y_1+y_2)\right]$$

$$= \frac{3\pi}{32}\left[(0+0)+2(1.666)+3(0.553+1.5702)\right]$$

$$\boxed{\int_0^{\pi} t \sin t \, dt = 2.8598}$$

$\int_0^{\pi} t \sin t \, dt$	
Trapezoidal	2.978
Simpson's $\frac{1}{3}$rd	3.1485
Simpson's $\frac{3}{8}$th	2.8598

① Evaluate $\int_0^6 \frac{1}{1+x^2} dx$ by Taking 6 equal parts

② Evaluate $\int_0^1 \sqrt{1+x^3} \, dx$ by Taking $h = 0.1$

③ Evaluate $\int_0^{\pi} \sin x \, dx$ by Taking $n = 10$

④ Evaluate $\int_0^4 e^x dx$

⑤ If $f(0) = 1$, $f(\frac{\pi}{6}) = 1.6487$, $f(\frac{\pi}{3}) = 2.3632$, $f(\frac{\pi}{2}) = 2.7182$

Then Evaluate $\int_0^{\pi/2} f(x)\,dx$

⑥ If $y_0 = 0$, $y_1 = 10$, $y_2 = 18$, $y_3 = 25$

Then Evaluate $\int_0^3 f(x)\,dx$

⑦ The velocity v (m/sec) of a particle at distance S (m) from a point on its path is given by the following table

d 20

S	0	10	20	30	40	50	60
v	47	58	64	65	61	52	38

Estimate the time taken to travel 60 meters by using

(i) S. $1/3^{rd}$ rule

(ii) S. $3/8^{th}$ rule

Solution of ordinary Differential Equations:

Suppose we wish to solve the equation

$$\frac{dy}{dx} = f(x, y)$$ subject to condition

that $y(x_0) = y_0$

The solution of this D.E by

using (i) Euler's method

(ii) R-K method.

Euler's method:

$$y_{n+1} = y_n + h\, f(x_n, y_n)$$

$n = 0, 1, 2, \ldots$

where $h = x_i - x_{i-1}$

① Using Euler's method solve for y at $x = 2$: from $\frac{dy}{dx} = 3x^2 + 1$

$y(1) = 2$ by taking step size

$h = 0.5$

Sol. given $\frac{dy}{dx} = f(x, y) = 3x^2 + 1$; $h = 0.5$

with initial conditions $y(1) = 2$ ($x_0 = 1$, $y_0 = 2$)

	x_0	$x_0 + h$ x_1	$x_1 + h$ x_2
x	1	1.5	2
y	y_0 2	y_1 4	y_2 7.875

Euler's Formula

$$\boxed{y_{n+1} = y_n + h\,f(x_n, y_n)}$$

Put $n = 0$

$$\boxed{y_1 = y_0 + h\,f(x_0, y_0)}$$

Here $y_0 = 2$; $x_0 = 1$

$h = 0.5$

$$f(x_0, y_0) = 3x_0^2 + 1$$
$$= 3(1) + 1$$
$$= 4$$

$$y_1 = 2 + (0.5)(4)$$

$$\boxed{y_1 = 4}$$

Euler's formula

$$\boxed{y_{n+1} = y_n + h f(x_n, y_n)}$$

put $n=1$

$$\boxed{y_2 = y_1 + h f(x_1, y_1)}$$

$y_1 = 4, \quad x_1 = 1.5,$

$h = 0.5$

$$f(x_1, y_1) = 3x_1^2 + 1$$

$$= 3(1.5)^2 + 1$$

$$= 6.75 + 1$$

$$= 7.75$$

$$y_2 = 4 + (0.5)(7.75)$$

$$\boxed{y_2 = 7.875}$$

② Solve $\frac{dy}{dx} = x + y$ with $y(0) = 1$

compute $y(0.1)$ by using

Euler's method

Sol given $\frac{dy}{dx} = f(x, y) = x + y$

$y(0) = 1 \Rightarrow x_0 = 0;\ y_0 = 1$

$h = 0.1$

	x_0	x_1
x	0	0.1
y	1	1.1
	y_0	y_1

Euler's Formula

$$y_{n+1} = y_n + h\, f(x_n, y_n)$$

put $n = 0$

$$y_1 = y_0 + h f(x_0, y_0)$$

$y_0 = 1, \; h = 0.1,$

$$f(x_0, y_0) = x_0 + y_0$$

$$= 0 + 1$$

$$= 1$$

$$y_1 = 1 + (0.1)(1)$$

$$= 1 + 0.1$$

$$\boxed{y_1 = 1.1}$$

$$\boxed{y(0.1) = 1.1}$$

③ Given $\frac{dy}{dx} = y' = x^2 - y$; $y(0) = 1$

compute $y(0.1)$ by using Euler's method.

④ Given $\frac{dy}{dx} = \frac{x^2 - y^2}{x^2 + y^2}$ with $y(1.1) = 2$

Compute $y(1.3)$ by using Euler's method

R.K Methods

First order RK method

$$y_1 = y_0 + h\, f(x_0, y_0)$$

Second order RK method

$$y_1 = y_0 + \frac{1}{2}(K_1 + K_2)$$

where $K_1 = h\, f(x_0, y_0)$

$K_2 = h\, f(x_0 + h, y_0 + K_1)$

Third order RK method

$$y_1 = y_0 + \frac{1}{6}(K_1 + 4K_2 + K_3)$$

where $K_1 = h f(x_0, y_0)$

$K_2 = h\, f(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2})$

$K_3 = h\, f(x_0 + h, y_0 + 2K_2 - K_1)$

RK Fourth order method

$$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

where $k_1 = h f(x_0, y_0)$

$k_2 = h f(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2})$

$k_3 = h f(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2})$

$k_4 = h f(x_0 + h, y_0 + k_3)$

Similarly we write the formula for y_2

$$y_2 = y_1 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$$

where $k_1 = h f(x_1, y_1)$

$k_2 = h f(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2})$

$k_3 = h f(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2})$

$k_4 = h f(x_1 + h, y_1 + k_3)$

① Solve $\frac{dy}{dx} = x+y$ with $y(0) = 1$

compute $y(0.1)$ by using

RK - Fourth order Method

given

	x_0	x_1
x	0	0.1
y	1	?
	y_0	y_1

$$\frac{dy}{dx} = f(x,y) = x+y$$

$x_0 = 0,\ y_0 = 1,\ h = 0.1$

$x_1 = 0.1$

RK Fourth order

$$y_1 = y_0 + \frac{1}{6}\left[k_1 + 2k_2 + 2k_3 + k_4\right]$$

where $k_1 = h\ f(x_0, y_0)$

Here $h = 0.1$

$$f(x_0, y_0) = x_0 + y_0$$

$$= 1$$

Hence $k_1 = (0.1)(1)$

$$\boxed{k_1 = 0.1}$$

where $k_2 = h\, f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$

Here $h = 0.1$

$$f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$= f\left(0 + \frac{0.1}{2}, 1 + \frac{0.1}{2}\right)$$

$$= f(0.05, 1.05)$$

$$= 0.05 + 1.05$$

$$= 1.1$$

Hence $k_2 = (0.1)(1.1)$

$$\boxed{k_2 = 0.11}$$

where $k_3 = h\, f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$

Here $h = 0.1$

$$f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = f\left(0 + \frac{0.1}{2}, 1 + \frac{0.11}{2}\right)$$

$$= f(0.05, 1+0.055)$$

$$= f(0.05, 1.055)$$

$$= 1.105$$

Hence $k_3 = (0.1)(1.105)$

$$\boxed{k_3 = 0.1105}$$

where $k_4 = h f(x_0+h, y_0+k_3)$

Here $h = 0.1$

$$f(x_0+h, y_0+k_3) = f(0+0.1, 1+0.1105)$$

$$= f(0.1, 1.1105)$$

$$= 1.2105$$

Hence $k_4 = (0.1)(1.2105)$

$$\boxed{k_4 = 0.12105}$$

Hence

$$y_1 = 1 + \frac{1}{6}\left[0.1 + 2(0.11) + 2(0.1105) + 0.12105\right]$$

$$\boxed{y_1 = 1.25}$$

② Solve $\frac{dy}{dx} = x^2 + y^2$ with $y(1) = 1.5$

compute $y(1.2)$ by using

RK - Fourth order method.

③ Solve the following using RK Fourth order method $y' = y - x$, $y(0) = 2$, $h = 0.2$ compute $y(0.2)$

④ Apply RK-Fourth order method to find $y(0.1)$ and $y(0.2)$ given $y' = xy + y^2$, $y(0) = 1$