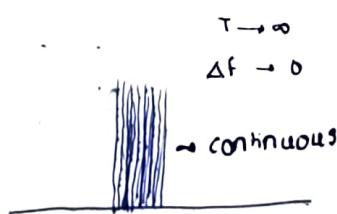
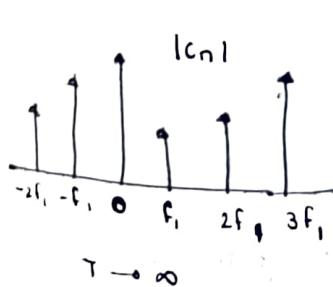


Unit-4 : Fourier Transform

Fourier series can be generalised only for periodic signals. This limitation is overcome through Fourier transform which is applicable for both periodic and aperiodic signals. And an aperiodic signal is considered as one which is periodic with an infinite period.

As the period increases, the fundamental frequency decreases, which makes aperiodic signal close in frequency and the representation in the form of linear combination takes the form of integral rather than summation. This is called Fourier transform.



Discrete Spectrum in Fourier series and continuous spectrum in Fourier transform.

Derivation of Fourier transform from Fourier series

$$C_n = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t) e^{-jn\omega_0 t} dt$$

$$T \rightarrow \infty \quad n\omega_0 \rightarrow \omega$$

$$T \cdot C_n = \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t) e^{-jn\omega_0 t} dt$$

$$x(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

Inverse Fourier transform:

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} x(\omega) e^{j\omega t} d\omega$$

$$\begin{aligned} \text{Proof:- } x(t) &= \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t} \\ &= \frac{1}{T \cdot \omega_0} \sum_{n=-\infty}^{\infty} T \cdot C_n e^{jn\omega_0 t} \Delta\omega_0 \\ &= \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} (T \cdot C_n) e^{jn\omega_0 t} \Delta\omega_0 \end{aligned}$$

$$T \rightarrow \infty, \quad T \cdot C_n = X(\omega), \quad n\omega_0 = \omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} x(\omega) e^{j\omega t} d\omega,$$

Convergence of Fourier transform:

i) Absolutely integrable

$$\int_{-\infty}^{\infty} |x(t)| dt < \infty$$

- ii) signal must have finite no. of discontinuities and finite no. of minimas and maximas.
- iii) signals must be energy signals or stable signal.
- iv) we can calculate Fourier transforms for power Signals approximation to energy signals.

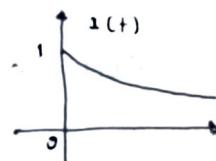
Fourier transform of basic signals:

$$i) x(t) = e^{-at} u(t)$$

$$X(\omega) = \int_{-\infty}^{\infty} e^{-at} u(t) e^{-j\omega t} dt$$

$$= \int_0^{\infty} e^{-at} e^{-j\omega t} dt$$

$$= \left[\frac{e^{-(a+j\omega)t}}{-a-j\omega} \right]_0^{\infty} = \frac{1}{a+j\omega}$$

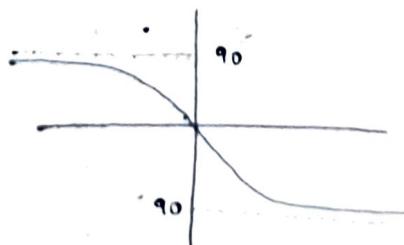
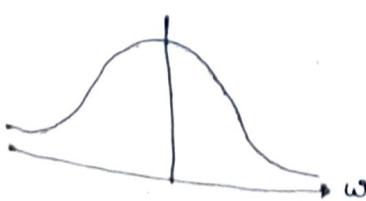


$$e^{-at} u(t) \xleftrightarrow{F.T} \frac{1}{a+j\omega}$$

Magnitude plot and phase plot:

$$X(\omega) = |x(\omega)| \angle x(\omega)$$

$$|x(\omega)| = \frac{1}{\sqrt{a^2 + \omega^2}} \quad \text{and} \quad \angle x(\omega) = -\tan^{-1}\left(\frac{\omega}{a}\right)$$



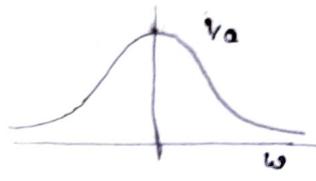
$$2) x(t) = e^{at} u(-t)$$

$$X(\omega) = \int_{-\infty}^0 e^{at} e^{-j\omega t} dt = \left[\frac{e^{(a-j\omega)t}}{a-j\omega} \right]_0^{-\infty}$$

$$= \frac{1}{a-j\omega} \left\{ \frac{1}{a-j\omega} \times \frac{a+j\omega}{a+j\omega} = \frac{a+j\omega}{a^2 + \omega^2} \right\}$$

$$|x(\omega)| = |x(\omega)| \angle x(\omega)$$

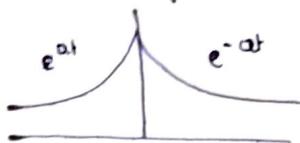
$$|x(\omega)| = \frac{1}{\sqrt{a^2 + \omega^2}}$$



$$\angle x(\omega) = \tan^{-1}\left(\frac{\omega}{a}\right)$$



$$3) x(t) = e^{-|at|}$$



$$|x| = \begin{cases} x & x > 0 \\ -x & x < 0 \end{cases}$$

$$x(\omega) = \int_0^\infty e^{-at} e^{-j\omega t} dt = \left[\frac{e^{-(a+j\omega)t}}{-(a+j\omega)} \right]_0^\infty = \frac{1}{a+j\omega}$$

$$x(\omega) = \int_{-\infty}^0 e^{at} e^{-j\omega t} dt = \left[\frac{e^{(a-j\omega)t}}{(a-j\omega)} \right]_{-\infty}^0 = \frac{1}{a-j\omega}$$

$$\therefore \frac{1}{a+j\omega} + \frac{1}{a-j\omega} = \frac{2a}{a^2 + \omega^2}$$

④ Fourier transform of $\delta(t)$.

$$x(\omega) = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt$$

$$\therefore = \int_{-\infty}^{\infty} \delta(t - t_0) e^{-j\omega_0 t} dt = e^{-j\omega_0 (0)} = 1$$

④ Find Fourier transform of $\delta(t)$

$$X(\omega) = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega_0 t} dt$$

$$= \int_{-\infty}^{\infty} \delta(t-t_0) e^{-j\omega_0 t} dt = e^{-j\omega_0 t_0} = 1$$

⑤ Find Fourier transform of $A \text{rect}(t/T)$

$$X(\omega) = \int_{-T/2}^{T/2} A e^{-j\omega t} dt$$


$$= A \left[\frac{e^{-j\omega t}}{-j\omega} \right] \Big|_{-T/2}^{T/2}$$

$$= \frac{-A}{j\omega} \left[e^{-j\omega(T/2)} - e^{j\omega(T/2)} \right]$$

$$= \frac{A}{j\omega} \left[e^{j\omega(T/2)} - e^{-j\omega(T/2)} \right] \quad \left\{ \frac{e^{j\theta} - e^{-j\theta}}{2j} = \sin \theta \right\}$$

$$= \frac{2A}{\omega} \left[\sin \left(\frac{\omega T}{2} \right) \right]$$

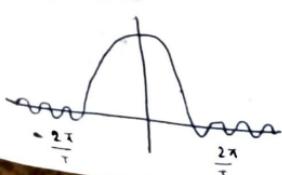
$$\left\{ \begin{array}{l} \text{Sinc} x = \frac{\sin x}{\pi x} \\ \text{Sa}(x) = \frac{\sin x}{x} \end{array} \right\}$$

$$= \frac{T}{2} (2A) \left[\frac{\sin(\omega T/2)}{\omega T/2} \right]$$

$$= AT \text{sinc} \left(\frac{\omega T}{2} \right) \text{ or } AT \left[\frac{\sin(2\pi f T/2)}{2\pi f T/2} \right]$$

$$= \text{sinc}(fT)$$

$A \text{rect}(t/T) \rightarrow AT \text{sinc}(\omega T/2) \text{ or } AT \sin(f\pi)$
magnitude



$$B \cdot \omega = \frac{4\pi}{T}$$

Inverse Fourier transform of $\delta(\omega)$

$$⑦ \quad x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \delta(\omega) e^{j\omega t} d\omega = \frac{1}{2\pi} [e^0] = \frac{1}{2\pi}$$

$$\delta(\omega) \xrightarrow[\text{F.T.}]{\text{I.F.T.}} \frac{1}{2\pi}$$

$$1 \xrightarrow{\text{F.T.}} 2\pi \delta(\omega)$$

$$2\pi \delta(\omega) \xleftarrow{\text{F.T.}} 1$$

$$⑧ \quad x(t) = \begin{cases} e^{-at} u(t) & 0 < t < \infty \\ -e^{at} u(-t) & -\infty < t < 0 \end{cases}$$

$$x(t) = \int_0^{\infty} e^{-at} - e^{-j\omega t} dt$$

$$= \left[\frac{e^{-(a+j\omega)t}}{-(a+j\omega)} \right]_0^{\infty} = \frac{1}{a+j\omega}$$

$$x(t) = \int_{-\infty}^0 e^{-at} e^{-j\omega t} dt = - \left[\frac{e^{(a-j\omega)t}}{a-j\omega} \right]_0^{\infty}$$

$$= \frac{-1}{a-j\omega}$$

$$x(t) = \frac{1}{a+j\omega} - \frac{1}{a-j\omega} = \frac{-2j\omega}{a^2 + \omega^2}$$

$$\underset{a \rightarrow 0}{\text{Lt}} \quad x(t) = \underset{a \rightarrow 0}{\text{Lt}} e^{-at} u(t) = 1 \quad \left. \begin{array}{l} \\ \end{array} \right\} \text{sgn}(t)$$

$$\underset{a \rightarrow 0}{\text{Lt}} \quad x(t) = \underset{a \rightarrow 0}{\text{Lt}} -e^{-at} u(-t) = -1$$

$$x(\omega) |_{\text{sgn}(t)} = \underset{a \rightarrow 0}{\text{Lt}} x(\omega)$$

$$= \underset{a \rightarrow 0}{\text{Lt}} \frac{-2j\omega}{a^2 + \omega^2} = -\frac{2j}{\omega} = \frac{2}{j\omega}$$

$$\text{sgn}(t) \xleftarrow{\text{F.T}} \frac{2}{j\omega}$$

Fourier transform of $u(t)$.

$$\text{sgn}(t) = 2u(t) - 1$$

$$u(t) = \frac{1}{2} [1 + \text{sgn}(t)]$$

$$= \frac{1}{2} + \frac{1}{2} \text{sgn}(t)$$

$$= \pi \delta(\omega) + \frac{1}{2} \frac{2}{j\omega}$$

$$= \pi \delta(\omega) = \frac{1}{j\omega}$$

$$u(t) \xrightarrow{\text{F.T}} \frac{1}{j\omega} + \pi \delta(\omega)$$

Area under property:

$$x(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \quad \{\omega = 0\}$$

$$x(0) = \int_{-\infty}^{\infty} x(t) (1) dt$$

$$x(0) = \int_{-\infty}^{\infty} x(t) dt$$

In F.T the area under the graph is at $\omega = 0$

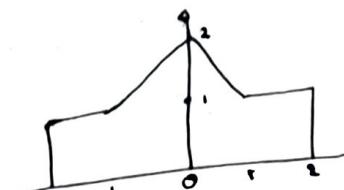
$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} x(\omega) e^{j\omega t} d\omega \quad t \rightarrow 0$$

$$x(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} x(\omega) d\omega$$

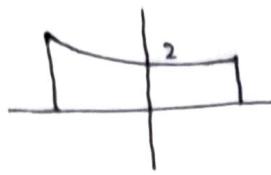
$$2\pi x(0) = \int_{-\infty}^{\infty} x(\omega) d\omega$$

① Area under the graph

$$\int_{-\infty}^{\infty} x(\omega) d\omega = 2\pi x(0) = 2x(2) = 4\lambda$$



Area under time domain



$$\int_{-\infty}^{\infty} x(t) dt = x(0) = 2$$

Properties:

i) Linearity:

$$x(t) \rightarrow x(\omega)$$

$$y(t) \rightarrow y(\omega)$$

$$ax(t) + by(t) \rightarrow ax(\omega) + by(\omega)$$

ii) Time reversal

$$x(t) \rightleftharpoons x(\omega)$$

$$x(-t) \rightarrow x(-\omega)$$

iii) Time shifting

$$x(t) \rightarrow x(\omega)$$

$$x(t-t_0) \rightarrow x(\omega) e^{-j\omega t_0} \quad e^{-j\omega t_0}$$

$$x(\omega) = \int_{-\infty}^{\infty} x(t-t_0) e^{-j\omega_0 t} dt$$

$$= \int x(\tau) e^{-j\omega_0(\tau+t)} d\tau$$

$$= e^{-j\omega_0 t_0} \int_{-\infty}^{\infty} x(\tau) e^{-j\omega_0 \tau} d\tau$$

$$= e^{-j\omega_0 t_0} x(\omega)$$

$$\begin{cases} t-t_0 = \tau \\ d\tau = dt \end{cases}$$

iv) Frequency shifting

$$x(t) \rightarrow x(\omega)$$

$$e^{j\omega_0 t} x(t) = x(\omega - \omega_0)$$

$$x(\omega) = \int_{-\infty}^{\infty} e^{j\omega t} \cdot e^{-j\omega t} (x(t)) dt$$

$$x(\omega - \omega_0) = \int_{-\infty}^{\infty} e^{-j\omega_0 t} (x(t)) dt \cdot e^{j\omega_0 t}$$

$$= x(\omega) e^{j\omega_0 t}$$

$$\therefore = x(\omega - \omega_0)$$

Find Fourier transform of $\cos \omega_0 t$ & $\sin \omega_0 t$

$$x(t) = \cos \omega_0 t$$

$$\cos \omega_0 t = \frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}$$

$$\begin{aligned} &= \frac{1}{2} \left[\int_{-\infty}^{\infty} [e^{j\omega_0 t} + e^{-j\omega_0 t}] e^{-j\omega_0 t} dt \right] \\ &= \frac{1}{2} \left[\int_{-\infty}^{\infty} e^{j\omega_0 t} \cdot e^{-j\omega_0 t} + \int_{-\infty}^{\infty} e^{-j\omega_0 t} \cdot e^{-j\omega_0 t} dt \right] \\ &= \frac{1}{2} \int_{-\infty}^{\infty} e^{-(\omega - \omega_0)jt} dt + \frac{1}{2} \int_{-\infty}^{\infty} e^{-(\omega + \omega_0)jt} dt \end{aligned}$$

$$x(t) = \sin \omega_0 t$$

$$FT(\cos \omega_0 t) = FT\left(\frac{e^{j\omega_0 t}}{2}\right) + FT\left(\frac{e^{-j\omega_0 t}}{2}\right)$$

$$= \frac{1}{2} [2\pi \delta(\omega - \omega_0)] + \frac{1}{2} [2\pi \delta(\omega + \omega_0)]$$

$$FT(\cos \omega_0 t) = \pi [\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$$

$$2\pi \delta(\omega - \omega_0) \xleftrightarrow{IFT} e^{j\omega_0 t}$$

$$\sin \omega_0 t = \frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j}$$

$$FT(\sin \omega_0 t) = FT\left[\frac{e^{j\omega_0 t}}{2j}\right] - FT\left[\frac{e^{-j\omega_0 t}}{2j}\right]$$

Similarly, we get after solving $\frac{\pi}{j} [\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$

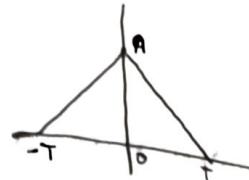
$$\sin \omega_0 t \rightarrow \frac{\pi}{j} [\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$$

$$\cos \omega_0 t \rightarrow \pi [\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$$

(d) $A \text{Tri}(\frac{t}{\tau})$

Line equation

$(-\tau, 0)$ $(0, A)$



$$x(t) = \left[\frac{A-0}{0+\tau} \right] (t+\tau)$$

$(\tau, 0)$ $(0, A)$

$$x(t) = \frac{A}{\tau} (t+\tau)$$

$$x(t) = \frac{A}{\tau} (t+\tau)$$

$$x(\omega) = \int_{-\tau}^0 \frac{A}{\tau} (t+\tau) e^{-j\omega t} dt + \int_0^\tau \frac{-A}{\tau} (t-\tau) e^{-j\omega t} dt$$

$$= \int_{-\tau}^0 \left[\frac{At e^{-j\omega t}}{\tau} + A e^{-j\omega t} \right] dt + \int_0^\tau \left[\frac{-At e^{-j\omega t}}{\tau} + A e^{-j\omega t} \right] dt$$

$$= \frac{A}{\tau} \left[\frac{-te^{-j\omega t}}{j\omega} + \int \frac{e^{-j\omega t}}{j\omega} dt \right] + A \int_{-\tau}^0 e^{-j\omega t} dt$$

$$= \frac{A}{\tau} \left[\frac{-te^{-j\omega t}}{j\omega} - \frac{e^{-j\omega t}}{(j\omega)^2} \right] \Big|_{-\tau}^0 - A \left[\frac{e^{-j\omega t}}{j\omega} \right] \Big|_{-\tau}^0$$

$$= \frac{A}{\tau} \left[0 - \frac{1}{(j\omega)^2} - \frac{-Te^{j\omega\tau}}{j\omega} + \frac{e^{j\omega\tau}}{(j\omega)^2} \right] - \frac{A}{j\omega} + \frac{e^{j\omega\tau}}{j\omega}$$

$$= \frac{-A}{(j\omega)^2 \tau} - \frac{A e^{j\omega\tau}}{j\omega} + \frac{A e^{j\omega\tau}}{\tau(j\omega)^2} - \frac{A}{j\omega} + \frac{A e^{j\omega\tau}}{j\omega}$$

$$= \frac{-A}{(j\omega)^2 \tau} + \frac{A e^{j\omega\tau}}{\tau(j\omega)^2} - \frac{A}{j\omega} \quad \left\{ A^2 \tau^2 \sin(\tau) \right\}$$

$$\begin{aligned}
&= \int_0^T -A \frac{1}{T} (t-T) e^{-j\omega t} dt \\
&= - \int_0^T \left[\frac{A t e^{-j\omega t}}{T} - A e^{-j\omega t} \right] dt \\
&= - \left[\frac{A}{T} \left[\frac{t e^{-j\omega t}}{-j\omega} + \int \frac{e^{-j\omega t}}{j\omega} dt \right] - A \left[\frac{e^{-j\omega t}}{-j\omega} \right] \right]_0^T \\
&= \left[\frac{A t e^{-j\omega t}}{j\omega T} + \frac{A e^{-j\omega t}}{T(j\omega)^2} - \frac{A e^{-j\omega T}}{j\omega} \right]_0^T \\
&= \frac{A e^{-j\omega T}}{j\omega} + \frac{A e^{-j\omega T}}{T(j\omega)^2} - \frac{A e^{-j\omega T}}{j\omega} - \frac{A(1)}{T(j\omega)^2} + \frac{A}{j\omega} \\
&= \frac{A e^{-j\omega T}}{T(j\omega)^2} - \frac{A}{T(j\omega)^2} + \frac{A}{j\omega} \\
x(\omega) &= \frac{-2A}{(j\omega)^2 T} + \frac{A}{T(j\omega)^2} [e^{j\omega t} + e^{-j\omega t}] \\
&= \frac{-2A}{(j\omega)^2 T} + \frac{A}{T(j\omega)^2} (2 \cos \omega t) \\
&= \frac{2A}{T(j\omega)} [\cos \omega t - 1] \\
&= \frac{2A}{T(j\omega \frac{4\pi}{T^2})} [\cos \omega t - 1] \\
&= \frac{-A}{2\pi^2} [\cos \omega t - 1]
\end{aligned}$$

Find the inverse Fourier transform of $\delta(\omega - \omega_0)$

$$\begin{aligned}
x(t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} x(\omega) e^{j\omega t} d\omega \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \delta(\omega - \omega_0) e^{j\omega t} d\omega
\end{aligned}$$

$$\frac{1}{2\pi} [e^{j\omega_0 t}] = \frac{e^{j\omega_0 t}}{2\pi}$$

$$\delta(\omega - \omega_0) \xleftarrow{\text{IFT}} \frac{1}{2\pi} e^{j\omega_0 t}$$

2) Find the value of $x(0)$ for the following sig.

$$x(t)$$

$$\rightarrow x(\omega) = \int_{-\infty}^{\infty} x(t) dt \text{ i.e area}$$

$$= 4(2) - \frac{1}{2}(2)(1)$$

$$= 8 - 1$$

$$= 7,$$



Duality property :

$$x(t) \xrightarrow{\quad} x(\omega)$$

$$x(t) \xrightarrow{\quad} 2\pi x(-\omega)$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} x(\omega) e^{j\omega t} d\omega$$

$$x(-t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} x(\omega) e^{-j\omega t} d\omega$$

$$2\pi x(-t) = \int_{-\infty}^{\infty} x(\omega) e^{-j\omega t} d\omega$$

$$2\pi x(-\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$$2\pi x(-\omega) = x(t)$$

1) Find the Fourier Transform of $\frac{1}{4-jt}$

$$e^{at} u(-t) \xrightarrow{\quad} \frac{1}{a-j\omega}$$

$$x(t) \longrightarrow x(\omega)$$

$$\frac{1}{a-jt} \longrightarrow 2\pi x(-\omega)$$

$$\longrightarrow 2\pi e^{-\omega t} u(\omega)$$

$$\frac{1}{4-jt} \longrightarrow 2\pi e^{-4\omega t} u(\omega)$$

Find the inverse Fourier transform of $x(\omega) = \frac{\omega^2 + 12}{\omega^2 + 9}$

$$\frac{\omega^2 + 12}{\omega^2 + 9} = 1 + \frac{3}{\omega^2 + 9}$$

$$x(\omega) = 1 + \frac{3}{\omega^2 + 9}$$

$$= \text{IFT}\{1\} + \text{IFT}\left\{\frac{3}{\omega^2 + 9}\right\} \quad \left[e^{-\alpha|t|} \rightarrow \frac{2\alpha}{\alpha^2 + \omega^2} \right]$$

$$= 2\pi \delta(t) + \text{IFT}\left\{\frac{3}{\omega^2 + 9}\right\}$$

$$= 2\pi \delta(t) + \text{IFT}\left\{\frac{2(3)}{2(\omega^2 + 9^2)}\right\}$$

$$= 2\pi \delta(t) + \frac{1}{2} e^{-3|t|}$$

Find the Fourier transform of $\frac{1}{t^2 + 16}$

$$x(t) = \frac{1}{t^2 + 4^2}$$

$$e^{-\alpha|t|} \longrightarrow \frac{2\alpha}{\alpha^2 + \omega^2}$$

$$e^{-4|t|} \longrightarrow \frac{8}{\omega^2 + 16}$$

$$\frac{1}{8} e^{-4|t|} \longrightarrow \frac{1}{\omega^2 + 16}$$

$$\frac{1}{t^2 + 16} \rightarrow 2\pi \frac{1}{8} e^{-4|-\omega|} = \frac{\pi}{4} e^{-4|\omega|}$$

5) Time scaling property:

$$x(t) \rightarrow x(\omega)$$

$$x(at) \rightarrow \frac{1}{|a|} X\left(\frac{\omega}{a}\right)$$

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty} x(at) e^{-j\omega t} dt$$

$$at = \lambda \quad = \int_{-\infty}^{\infty} x(\lambda) e^{-j\omega\left(\frac{\lambda}{a}\right)} \cdot \frac{d\lambda}{a}$$

$$adt = d\lambda$$

$$dt = \frac{d\lambda}{a} \quad = \frac{1}{a} \int_{-\infty}^{\infty} x(\lambda) e^{-j\omega\lambda} d\lambda = \frac{1}{|a|} x\left(\frac{\omega}{a}\right)$$

6) Differentiation property

$$x(t) \rightarrow x(\omega)$$

$$\frac{d}{dt} x(t) \rightarrow j\omega x(\omega)$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$$

$$\frac{d}{dt} x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} x(\omega) \frac{d}{dt} e^{j\omega t} d\omega$$

$$\frac{d}{dt} x(t) = \frac{j\omega}{2\pi} \int_{-\infty}^{\infty} x(\omega) e^{j\omega t} d\omega$$

$$= j\omega \text{ FT}\{x(t)\}$$

$$\text{FT}\left\{ \frac{d}{dt} x(t) \right\} = j\omega X(\omega)$$

7) Conjugate property:

$$x(t) \longrightarrow x(\omega)$$

$$x^*(t) \longrightarrow x^*(-\omega) \quad x^*(-\omega)$$

$$x(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$$x(-\omega) = \int_{-\infty}^{\infty} x(t) e^{+j\omega t} dt$$

$$x^*(-\omega) = \left[\int_{-\infty}^{\infty} x(t) e^{j\omega t} dt \right]^*$$

$$= \int_{-\infty}^{\infty} x^*(t) e^{-j\omega t} dt$$

$$= FT[x^*(t)]$$

8) Integration property:

$$\int_{-\infty}^t x(t) dt = \frac{x(\omega)}{j\omega} + \pi x(0) \delta(\omega)$$

$$x(t) * u(t) = \int_{-\infty}^{\infty} x(\tau) u(t-\tau) d\tau$$

$$= \int_{-\infty}^t x(\tau) d\tau \quad t - \tau > 0 \\ t > \tau$$

$$\int_{-\infty}^t x(\tau) d\tau = x(t) * u(t)$$

$$FT \left\{ \int_{-\infty}^t x(\tau) d\tau \right\} = FT \{ x(t) * u(t) \}$$

$$= x(\omega) \cdot FT \{ u(t) \}$$

$$= x(\omega) \left[\frac{1}{j\omega} + \pi \delta(\omega) \right]$$

$$= \frac{x(\omega)}{j\omega} + \pi x(\omega) \delta(\omega)$$

From product property = $\frac{x(\omega)}{j\omega} + \pi x(0)\delta(\omega)$

Q) Convolution property:

$$\underline{x(t) \rightarrow x(\omega)}$$

$$\underline{y(t) \rightarrow Y(\omega)}$$

$$x(t) * y(t) \rightarrow \underline{x(\omega)Y(\omega)}$$

$$x(t) * y(t) = \int_{-\infty}^{\infty} x(\tau) y(t-\tau) d\tau$$

$$FT\{x(t) * y(t)\} = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x(\tau) y(t-\tau) e^{-j\omega t} d\tau \right] dt$$

$$t - \tau = \lambda \quad \rightarrow dt = d\lambda$$

$$t = \lambda + \tau$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x(\tau) y(\lambda) e^{-j\omega(\lambda+\tau)} d\lambda d\tau$$

$$= \int_{-\infty}^{\infty} x(\tau) e^{-j\omega\tau} d\tau \int_{-\infty}^{\infty} y(\lambda) e^{-j\omega\lambda} d\lambda$$

$$= x(\omega)Y(\omega)$$

Q) Derivative in frequency domain or multiplication
with 't' in time domain.

$$x(t) \rightarrow x(\omega)$$

$$t \cdot x(t) \rightarrow j \frac{d}{d\omega} x(\omega)$$

$$x(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$$\frac{d}{d\omega} x(\omega) = \int_{-\infty}^{\infty} x(t) \frac{d}{d\omega} e^{-j\omega t} dt$$

$$\begin{aligned}
 &= \int_{-\infty}^{\infty} x(t) (-jt) e^{-j\omega t} dt \\
 &= -j \int_{-\infty}^{\infty} t \cdot x(t) e^{-j\omega t} dt = \frac{1}{j} \text{FT}\{t \cdot x(t)\} \\
 &= j \frac{d}{d\omega} x(\omega) = t \cdot x(t)
 \end{aligned}$$

ii) Convolution in frequency domain.

$$x(t) \rightarrow x(\omega)$$

$$y(t) \rightarrow Y(\omega)$$

$$x(t) \cdot y(t) \rightarrow \frac{1}{2\pi} [x(\omega) * Y(\omega)]$$

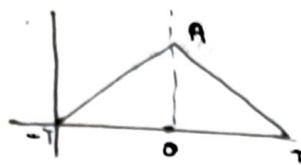
$$\begin{aligned}
 \text{FT}\{x(t) \cdot y(t)\} &= \int_{-\infty}^{\infty} x(t) y(t) e^{-j\omega t} dt \\
 &= \int_{-\infty}^{\infty} \frac{1}{2\pi} \int x(\omega') e^{j\omega' t} d\omega' y(t) e^{-j\omega t} dt \\
 &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x(\omega') y(t) e^{j(\omega' - \omega)t} dt d\omega' \\
 &= \frac{1}{2\pi} \int_{-\infty}^{\infty} x(\omega') \int_{-\infty}^{\infty} y(t) e^{-j(\omega - \omega')t} dt d\omega' \\
 &= \frac{1}{2\pi} \int_{-\infty}^{\infty} x(\omega) Y(\omega) d\omega
 \end{aligned}$$

$$= \frac{1}{2\pi} [x(\omega) * Y(\omega)]$$

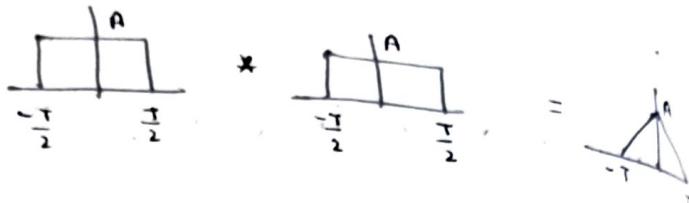
i) Parseval's theorem

$$\int_{-\infty}^{\infty} |x(t)|^2 dt \rightarrow \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 d\omega$$

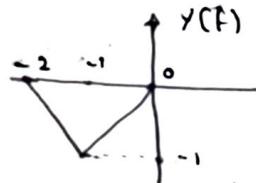
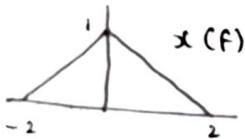
Find the Fourier transform of $A \text{tri}(t/T)$ using properties



We have



Let $y(t)$ and $x(t)$ F.S of $x(f)$, $y(f)$ as shown in fig. Then, find $y(f)$ in terms of $x(f)$



$$y(t) \rightarrow x(f)$$

$$x(t) \rightarrow y(f)$$

$$-x(2(t+1)) \rightarrow y(t)$$

$$\left\{ \begin{array}{l} x(\omega t) \rightarrow \frac{1}{|\omega|} x(\omega) \\ x(t-t_0) \rightarrow e^{-j\omega t_0} x(\omega) \end{array} \right.$$

$$y(f) \rightarrow -\frac{1}{2} x(f_{1/2}) e^{j2\pi f_{1/2}}$$

Find IFT of $x(3f+2)$ is where FT of $x(t)$ is $x(f)$

$$x(t) \rightarrow X(f)$$

$$x(3f+2) \rightarrow$$

$$x\left(3\left(f+\frac{2}{3}\right)\right) \rightarrow \frac{1}{3}x\left(\frac{t}{3}\right)e^{j2\pi f t/3}$$
$$= \frac{1}{3}x\left(\frac{t/3}{s}\right)e^{j4\pi f t/3}$$

* Find the Fourier transform of $x(5t - 3)$ if

$$x\left(5\left(t-\frac{3}{5}\right)\right) = \frac{1}{5}x\left(\frac{t}{s}\right)e^{-j2\pi f\left(\frac{3}{5}\right)}$$
$$\xrightarrow{x(t) \xrightarrow{\text{FT}} X(\omega)}$$
$$= \frac{1}{5}x\left(\frac{t}{s}\right)e^{-j\frac{6\pi f}{5}}$$
$$= \frac{1}{5}x\left(\frac{\omega}{s}\right)e^{-j3\omega/s}$$

* The Fourier transform of signal $h(t)$ is $H(\omega)$

$$H(\omega) = (2\cos\omega)(\underline{\sin\frac{2\omega}{\omega}}) \text{ find the value of } h(0)$$

All properties

1) Area under property :

$$x(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

2) Linearity

$$x(t) \rightarrow x(\omega)$$

$$y(t) \rightarrow Y(\omega)$$

$$ax(t) + by(t) \rightarrow ax(\omega) + bY(\omega)$$

3) Time reversal

$$x(t) \rightarrow x(-\omega)$$

$$x(-t) \rightarrow x(-\omega)$$

4) Time shifting

$$x(t) \rightarrow x(\omega)$$

$$x(t-t_0) \rightarrow x(\omega) e^{-j\omega t_0}$$

5) Frequency shifting

$$e^{j\omega_0 t} x(t) \rightarrow x(\omega - \omega_0)$$

6) Duality property

$$x(t) \rightarrow x(\omega)$$

$$x(t) \rightarrow 2\pi x(-\omega)$$

7) Time scaling

$$x(t) \rightarrow x(\omega)$$

$$x(at) \rightarrow \frac{1}{|a|} x\left(\frac{\omega}{a}\right)$$

8) Differentiation

$$x(t) \rightarrow X(\omega)$$

$$\frac{d}{dt}(x(t)) \rightarrow j\omega X(\omega)$$

9) Conjugate property.

$$x(t) \rightarrow X(\omega) \quad \Rightarrow \quad x^*(t) = X^*(-\omega)$$

$$y(t) \rightarrow Y(\omega)$$

10) Integration

$$\int_{-\infty}^t x(t) dt = \frac{x(\omega)}{j\omega} + \pi x(0) \delta(\omega)$$

11) Convolution property

$$x(t) \rightarrow X(\omega)$$

$$y(t) \rightarrow Y(\omega)$$

$$x(t) * y(t) \rightarrow X(\omega) \cdot Y(\omega)$$

12) Convolution in frequency domain

$$x(t) \rightarrow X(\omega)$$

$$y(t) \rightarrow Y(\omega)$$

$$x(t) \cdot y(t) \rightarrow \frac{1}{2\pi} [X(\omega) * Y(\omega)]$$

13) Parseval's theorem

$$\int_{-\infty}^{\infty} |x(t)|^2 dt \rightarrow \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 d\omega$$

14) Derivative in frequency domain or multiplication with t

$$t \cdot x(t) \rightarrow j \frac{d}{d\omega} X(\omega)$$

Problems

1) $e^{4t} u(-t+3)$

$$e^{at} u(-t) \rightarrow \frac{1}{a-j\omega}$$

$$e^{4t} u(-t) \rightarrow \frac{1}{4-j\omega}$$

$$x(t-t_0) \rightarrow e^{-j\omega t_0} x(\omega)$$

$$x(t-3) \rightarrow e^{-j\omega 3} x(\omega)$$

$$e^{4(t-3)} u[-(t-3)] \rightarrow e^{-j\omega 3} \frac{1}{4-j\omega}$$

$$e^{4t} u(-t+3)$$

$$e^{4(t-3+3)} u(-t+3)$$

$$e^{12} e^{4(t-3)} u(-t+3)$$

$$= e^{12-j\omega 3} \frac{1}{4-j\omega}$$

2) Fourier transform of γ_t

$$\text{sgn}(t) \rightarrow \frac{2}{j\omega}$$

$$x(t) \rightarrow X(\omega) \quad \text{duality}$$

$$x(t) \rightarrow 2\pi x(-\omega)$$

$$\frac{2}{jt} \rightarrow 2\pi \text{sgn}(-\omega)$$

$$\frac{1}{t} \rightarrow j\pi \text{sgn}(-\omega)$$

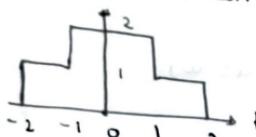
3) Find the F.T of signal $x(t)$ is given by

$$X(\omega) = \frac{1}{2 - \omega^2 + 3j\omega} \quad \text{Find } x(t)$$

$$\begin{aligned} X(\omega) &= \frac{1}{(\omega)^2 + 3j\omega + 2} = \frac{1}{(\omega)^2 + j\omega + 2j\omega + 2} \\ &= \frac{1}{j\omega(1+j\omega) + 2(1+j\omega)} = \frac{1}{(j\omega+1)(j\omega+2)} \\ &= \frac{1}{j\omega+1} - \frac{1}{j\omega+2} \quad \left\{ e^{-at} u(t) \xrightarrow{\text{FT}} \frac{1}{a+j\omega} \right\} \\ &= e^{-t} u(t) - e^{-2t} u(t) \end{aligned}$$

4) Find the Fourier transform of signal shown in

Fig.



$$x(t) = \begin{cases} 1 & -2 \leq t < -1 \\ 0 & -1 \leq t < 0 \\ 2 & 0 \leq t < 1 \\ 1 & 1 \leq t \leq 2 \\ 0 & \text{elsewhere} \end{cases}$$

$$= 1 \operatorname{rect}(t/4) + 1 \operatorname{rect}(t/2)$$

$$= 4 \operatorname{sinc}(\frac{\omega t}{2}) + 2 \operatorname{sinc}(\frac{\omega t}{2})$$

$$= 4 \operatorname{sinc}(2\omega) + 2 \operatorname{sinc}(\omega) \quad \text{or } \operatorname{sinc}(4t) + \operatorname{sinc}(2t)$$

5) Find the F.T. of $\{e^{-at} u(t)\}$

$$+ x(t) \rightarrow j \frac{d}{d\omega} X(\omega)$$

$$e^{-at} u(t) \rightarrow \frac{1}{a+j\omega}$$

$$t \cdot e^{-at} u(t) \rightarrow j \frac{d}{d\omega} \left(\frac{1}{a+j\omega} \right)$$

$$\begin{aligned} t \cdot e^{-at} u(t) &\rightarrow j \frac{-1}{(a+j\omega)^2} j \\ &\rightarrow j^2 \frac{-1}{(a+j\omega)^2} \\ &\rightarrow \frac{1}{(a+j\omega)^2} \end{aligned}$$

$$t \cdot e^{-at} u(t) \rightarrow \frac{1}{(a+j\omega)^2}$$

6) Find inverse F.T of $u(\omega)$

$$u(t) \rightarrow \frac{1}{j\omega} + \pi \delta(\omega)$$

$$\frac{1}{jt} + \pi \delta(t) \rightarrow 2\pi u(-\omega)$$

$$x(-t) \rightarrow x(-\omega)$$

$$\frac{-1}{jt} + \pi \delta(-t) \rightarrow 2\pi u(\omega)$$

$$\frac{j}{2\pi t} + \frac{1}{2} \delta(t) \leftrightarrow u(\omega)$$

$$u(\omega) \rightarrow \frac{j}{2\pi t} + \frac{1}{2} \delta(t)$$

7) Find the F.T of $\frac{4t}{(1+t^2)^2}$

$$e^{-it\tau} \rightarrow \frac{2}{\omega_{+1}}$$

$$t \cdot e^{-it\tau} \rightarrow j \frac{d}{d\omega} \left(\frac{2}{\omega_{+1}} \right)$$

$$\rightarrow \frac{j(-2\omega)}{(1+\omega^2)^2}$$

$$t \cdot e^{jt} \longrightarrow \frac{-4j\omega}{(\omega^2+1)^2}$$

$$\left\{ \begin{array}{l} x(+)\longrightarrow x(\omega) \\ x(+)\longrightarrow 2\pi x(-\omega) \end{array} \right\}$$

$$\frac{-4jt}{(t^2+1)^2} \longrightarrow 2\pi(-\omega)e^{-j\omega t}$$

$$\frac{4jt}{(t^2+1)^2} \longrightarrow 2\pi\omega e^{-j\omega t}$$

$$\frac{4t}{(t^2+1)^2} \longrightarrow -j2\pi\omega e^{-j\omega t}$$

8) F.T of $x(+)$ is $2\text{rect}(0.25f)$. Then find F.T of

$$x(+)\cos 2\pi f$$

$$x(+)\longrightarrow 2\text{rect}(0.25f)$$

$$x(+)\cos 2\pi f \longrightarrow \pi [\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$$

a) Find F.T of $\text{rect}(t) * \cos \pi t$

$$\text{Sa}\left(\frac{\omega}{2}\right) \cdot \pi [\delta(\omega - \pi) + \delta(\omega + \pi)]$$

* Fourier transform of periodic signals

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}$$

$$\text{F.T}\{x(t)\} = \sum_{n=-\infty}^{\infty} c_n \text{F.T}\{e^{jn\omega_0 t}\}$$
$$= \sum_{n=-\infty}^{\infty} c_n 2\pi \delta(\omega - \omega_0 n)$$

$$X(\omega) = 2\pi \sum_{n=-\infty}^{\infty} c_n \delta(\omega - \omega_0 n)$$

Thus Fourier transform of periodic signal having

impulse train having strength $2\pi c_n$ located

at $\omega = n\omega_0$

b) Find the F.T of $x(t) = \sum_{n=-\infty}^{\infty} \delta(at - nT)$

$$c_n = \frac{1}{T} \int_{-T/2}^{T/2} x(t) e^{-jn\omega_0 t} dt$$

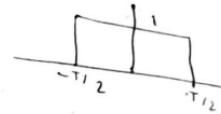
$$= \frac{1}{T} \int \delta(t) e^{jn\omega_0 t} dt$$

$$c_n = \frac{1}{T}$$

$$X(\omega) = \frac{2\pi}{T} \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_0)$$

Q Find FT of $\left\{ \frac{\sin at}{\pi t} \right\}$

$$\left\{ \frac{\sin at}{\pi t} \right\}$$



$$\text{rect}(t/T)$$

$$\xrightarrow{\quad \frac{T \sin(t/T)}{t/T} \quad}$$

$$\xrightarrow{\quad 2\pi \text{rect}(\omega/T) \quad}$$

$$\xrightarrow{\quad \frac{T \sin(\omega T/2)}{\omega T/2} \quad}$$

$$\xrightarrow{\quad \frac{\sin(t/T)}{\pi t} \quad}$$

$$\xrightarrow{\quad \text{rect}(\omega/T) \quad}$$

$$a = T/2$$

$$\xrightarrow{\quad \frac{\sin at}{\pi t} \quad}$$

$$\xrightarrow{\quad \text{rect}(\omega/2a) \quad}$$

$$T = 2a$$



Q Find I.F.T of $\text{rect}\left(\frac{\omega - 10}{2\pi}\right)$

$$\frac{\sin at}{\pi t} \xrightarrow[\text{I.F.T.}]{\quad} \text{rect}\left(\frac{\omega}{2a}\right)$$

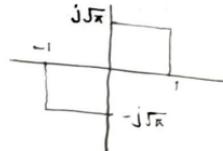
$$\text{rect}\left(\frac{\omega - 10}{2\pi}\right) \xrightarrow{\text{I.F.T.}} \frac{\sin at}{\pi t} \quad a = \pi$$

$$\text{Rect} \frac{\omega - 10}{2\pi} \xrightarrow{\text{I.F.T.}} e^{j10t} \cdot \frac{\sin \pi t}{\pi t}$$

For the spectrum $x(\omega)$ shown in figure, find $\frac{d}{dt} x(t)$

At $t=0$

$$A \quad \frac{d}{dt} (x(t)) \Big|_{t=0}$$



$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} x(\omega) e^{j\omega t} d\omega$$

$$\frac{d}{dt} x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} x(\omega) e^{j\omega t} d\omega \cdot (j\omega)$$

$$\frac{d}{dt} x(t) \Big|_{t=0} = \frac{1}{2\pi} X(\omega) \int_{-\infty}^{\infty} j\omega e^0 d\omega \quad (x(\omega))$$

$$\frac{d}{dt} x(t) \Big|_{t=0} = \frac{1}{2\pi} \left[\int_{-1}^0 -j\sqrt{\pi} j\omega d\omega + \int_0^1 j\sqrt{\pi} j\omega d\omega \right]$$

$$= \frac{5\pi}{2\pi} \left[\frac{\omega^2}{2} \Big|_{-1}^0 - \frac{\omega^2}{2} \Big|_0^1 \right]$$

$$= \frac{5\pi}{2\pi} \left[0 - \frac{1}{2} - \left(\frac{1}{2} - 0 \right) \right]$$

$$= \frac{1}{2\sqrt{\pi}} (-1)$$

$$= \frac{-1}{2\sqrt{\pi}}$$

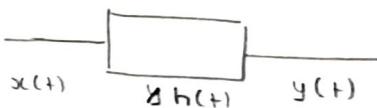
Q Find F.T. $\left\{ \frac{\sin t \cdot \sin t_{1/2}}{\pi t^2} \right\}$

$$\frac{\sin t}{\pi t} \rightarrow \pi \text{rect}\left(\frac{\omega}{2}\right)$$

$$\frac{\sin t_{1/2}}{\pi t_{1/2}} \rightarrow \text{rect}\left(\frac{\omega}{2 \cdot t_{1/2}}\right)$$

$$2 \cdot \frac{\sin t}{\pi t} \cdot \frac{\sin t_{1/2}}{\pi t} \rightarrow \frac{\pi}{2} \left[\pi \text{rect}\left(\frac{\omega}{2}\right) * \text{rect}\left(\frac{\omega}{2 \cdot t_{1/2}}\right) \right]$$

* Transfer function of an LTI system



$$y(t) = x(t) * h(t)$$

$$Y(\omega) = X(\omega) \cdot H(\omega)$$

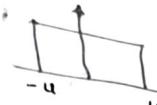
$$H(\omega) = \frac{Y(\omega)}{X(\omega)}$$

Q An LTI system is having impulse response $h(t)$
 $= \frac{\sin ut}{\pi t}$ for which the input applied

(i) $x(t) = \cos 2t + \sin gt$ find the output.

$$h(t) = \frac{\sin ut}{\pi t}$$

$$x(t) = \cos 2t + \sin gt$$

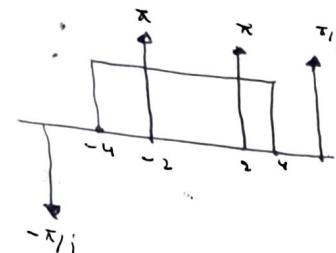
$$H(\omega) =$$


$$X(\omega) = \pi [\delta(\omega-2) + \delta(\omega+2)] + \frac{\pi}{j} [\delta(\omega-g) - \delta(\omega+g)]$$

$$Y(\omega) = H(\omega)x(\omega)$$

$$Y(\omega) = \pi [\delta(\omega-2) + \delta(\omega+2)]$$

$$y(t) = \cos 2t$$

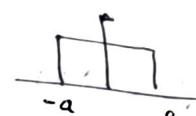


* Find energy in signal $x(t) = \frac{\sin at}{\pi t}$

$$x(t) = \frac{\sin at}{\pi t}$$

$$\text{energy signal} = \int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |x(\omega)|^2 d\omega$$

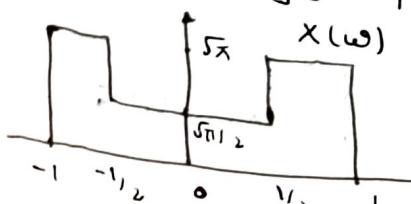
$$\frac{\sin at}{\pi t} \xrightarrow{F.T} \text{rect}\left(\frac{\omega}{2a}\right)$$



$$= \frac{j}{2\pi} \int_{-a}^{a} 1 d\omega = \frac{1}{2\pi} [a]_{-a}^a$$

$$= \frac{1}{2\pi} [a+a] = \frac{a}{\pi}$$

Find the energy spectrum as shown in figure



$$\begin{aligned}
 &= \frac{1}{2\pi} \int_{-\infty}^{\infty} |x(\omega)|^2 d\omega \\
 &= \frac{2}{2\pi} \int_0^{\pi} |x(\omega)|^2 d\omega \\
 &= \frac{1}{\pi} \int_0^{\pi} \left(\frac{\sqrt{\pi}}{2}\right)^2 d\omega + \int_{\pi}^{\infty} (\sqrt{\pi})^2 d\omega \\
 &= \frac{1}{\pi} \left[\frac{\pi^2}{4} \right] \frac{1}{2} + \pi \left[\frac{1}{2} \right] \left[\right] \\
 &= \frac{1}{\pi} \left[\frac{\pi}{8} + \frac{\pi}{2} \right] \\
 &= \frac{1}{8} + \frac{1}{2} = \boxed{\frac{5}{8}}
 \end{aligned}$$

Q Find the value of integral $\int_{-\infty}^{\infty} \text{sinc}^2(st) dt$

$$\begin{aligned}
 &\int_{-\infty}^{\infty} \text{sinc}^2(st) dt \quad \left\{ \frac{\sin s\pi t}{s\pi t} \right\} \\
 &= \frac{1}{2\pi} \int_{-s\pi}^{s\pi} \left(\frac{1}{s}\right)^2 d\omega \quad \left\{ \frac{1}{s} \cdot \frac{\sin s\pi t}{\pi t} \right\} \\
 &= \frac{1}{2\pi} \cdot \frac{1}{2s} (s\pi + s\pi) \quad \begin{array}{c} 1/s \\ \hline -s\pi \qquad \qquad s\pi \end{array} \\
 &= \frac{1}{s\pi} (10\pi) \\
 &= \frac{1}{s},
 \end{aligned}$$

Q Find the value of integral $\int_{-\infty}^{\infty} \frac{8}{(\omega^2+4)^2} d\omega$

$$\int_{-\infty}^{\infty} \frac{8}{(\omega^2+4)^2} d\omega$$

$$\int_{-\infty}^{\infty} |x(t)|^2 dt \rightarrow \frac{1}{2\pi} \int_{-\infty}^{\infty} |x(\omega)|^2 d\omega$$

$$2\pi \int_{-\infty}^{\infty} x(t)^2 dt = \int_{-\infty}^{\infty} |x(\omega)|^2 d\omega$$

$$e^{-\alpha t^2} \rightarrow \frac{2\alpha}{\omega^2 + \alpha^2}$$

$$|x(\omega)|^2 = \frac{8}{\omega^2 + 4\alpha^2} = \left(\frac{2\sqrt{2}}{\omega^2 + 4}\right)^2$$