2.2. Relations

Ordered pairs and $\boldsymbol{n}$ −tuples

We first introduce the concepts of ordered pairs and ordered n tuples:
The following is an intuitive definition of an ordered pair.

An ***order pair*** consists of two objects given in a fixed order. The ordering of the two objects is important. We shall denote an ordered pair by (x, y).

A familiar example of an ordered pair is the representation of points in the 2-dimentional Cartesian plane.

The ***equality*** of two ordered pairs (x, y) and (u, v) is defined by

$$\boldsymbol{(x, y) = (u, v) \Leftrightarrow \big((x = u) \wedge (y = v)\big)}$$

The concept of an ordered pair can be extended to define an ordered triple, and, more generally, an n-tuple.

An ***ordered triple*** is an ordered pair whose first member is itself an ordered pair. Thus, an ordered triple can be written as $\big((x, y), z\big)$.

We can derive at the equality of two ordered triples from the definition of the equality of two ordered pairs.

$$\big((x, y), z\big) = \big((u, v), w\big) \text{ iff } (x, y) = (u, v) \wedge \ z = w$$

But $(x, y) = (u, v) \Leftrightarrow \big((x = u) \wedge (y = v)\big)$,

Therefore, $\big((x, y), z\big) = \big((u, v), w\big) \Leftrightarrow \big((x = u) \wedge (y = v) \wedge (z = w)\big)$

From the above definition of equality of two ordered triples we may write an ordered triple as (x, y, z) with an understanding that $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$ ***stands for*** $\big((\boldsymbol{x}, \boldsymbol{y}), \boldsymbol{z}\big)$.

Continuing in this way, an ***ordered n-tuple*** is defined as an ordered pair whose first member is an ordered $(n-1)$-tuple. We write an ordered n-tuple as $\big((x_1, x_2, \dots, x_{n-1}), x_n\big)$.

Further,$\big((x_1, x_2, \dots, x_{n-1}), x_n\big) = \big((u_1, u_2, \dots, u_{n-1}), u_n\big)$

$\Leftrightarrow \big((x_1 = u_1) \wedge (x_2 = u_2) \wedge \dots \wedge (x_n = u_n)\big)$

Therefore, an ordered n-tuple will be written as $(\boldsymbol{x_1}, \boldsymbol{x_2}, \dots, \boldsymbol{x_n})$.

Cartesian product

Let A and B be any two sets. The ***Cartesian product*** of A and B (in this order) is written as $A \times B$ and is defined as the set of all ordered pairs such that the first and second members of the ordered pair are respectively the elements of A and B. That is,

$$\boldsymbol{A} \times \boldsymbol{B} = \{(\boldsymbol{x}, \boldsymbol{y}) | (\boldsymbol{x} \in \boldsymbol{A}) \wedge (\boldsymbol{x} \in \boldsymbol{B})\}$$

Note:

i. If any one of A, B is the empty set $\emptyset$, then $A \times B = \emptyset = B \times A$
ii. In general, $A \times B \neq B \times A$.
iii. If A and B are finite sets with m and n elements respectively, then

$$\boldsymbol{n}(\boldsymbol{A} \times \boldsymbol{B}) = \boldsymbol{mn}$$

iv. $(A \times B) \times C \neq A \times (B \times C)$.

From the definition

$$(A \times B) \times C = \big\{\big((a, b), c\big) | \big((a, b) \in A \times B\big) \wedge (c \in C)\big\}$$

$$= \{(a, b, c) | (a \in A) \wedge (b \in B) \wedge (c \in C)\}$$

and $A \times (B \times C) = \big\{\big(a, (b, c)\big) | (a \in A) \wedge \big((b, c) \in B \times C\big)\big\}$

Note that $\big(a, (b, c)\big)$ is not an ordered triple, since $A \times (B \times C)$ is the set of all ordered pairs, where the first member of the ordered pair is an element of A and the second member is an ordered pair from $B \times C$.Therefore

$$(A \times B) \times C \neq A \times (B \times C)$$

A Cartesian product satisfies the following distributive properties:

Theorem 1: For any three sets $\boldsymbol{A}, \boldsymbol{B}$ and $\boldsymbol{C}$

i. $\boldsymbol{A} \times (\boldsymbol{B} \cup \boldsymbol{C}) = (\boldsymbol{A} \times \boldsymbol{B}) \cup (\boldsymbol{A} \times \boldsymbol{C})$
ii. $\boldsymbol{A} \times (\boldsymbol{B} \cap \boldsymbol{C}) = (\boldsymbol{A} \times \boldsymbol{B}) \cap (\boldsymbol{A} \times \boldsymbol{C})$

Proof: We prove the first and the second can be proved on similar lines.

i. $A \times (B \cup C) = \{(x,y) | (x \in A) \wedge (y \in B \cup C)\}$

$$= \{(x,y) | (x \in A) \wedge ((y \in B) \vee (y \in C))\}$$

$$= \{(x,y) | ((x \in A) \wedge (y \in B)) \vee ((x \in A) \wedge (y \in C))\}$$

(by the distributivity of the predicate calculus)

$$= \{(x,y) | ((x,y) \in A \times B) \vee ((x,y) \in A \times C)\}$$

$$= (A \times B) \cup (A \times C)$$

We now define the Cartesian product of any finite number of sets.

Let $A = \{A_i\}_{i \in I_n}$ be an indexed set and $I_n = \{1,2, \dots, n\}$. We denote the Cartesian product of the sets $A_1, A_2, \dots, A_n$ by

$$\bigtimes_{i \in I_n} A_i = A_1 \times A_2 \times A_3 \dots\dots \times A_n$$

and it is defined by

$$\bigtimes_{i \in I_1} A_i = A_1 \text{ and } \bigtimes_{i \in I_m} A_i = \left(\bigtimes_{i \in I_{m-1}} A_i \right) \times A_m \quad for\ m = 2,3,4, \dots\dots, n$$

By the above definition

$$A_1 \times A_2 \times A_3 = (A_1 \times A_2) \times A_3$$

$$A_1 \times A_2 \times A_3 \times A_4 = (A_1 \times A_2 \times A_3) \times A_4$$

$$= \big((A_1 \times A_2) \times A_3\big) \times A_4$$

This definition of Cartesian product of n sets is related to the definition of n-tuples in the following way:

$$A_1 \times A_2 \times \ldots \times A_n = \{(x_1, x_2, \ldots, x_n) | (x_1 \in A_1) \wedge (x_2 \in A_2) \wedge \ldots \wedge (x_n \in A_n)\}$$

Note:

i. We write $A \times A$ by $\boldsymbol{A^2}$, $A \times A \times A$ by $\boldsymbol{A^3}$ and so on.
ii. If $A_1, A_2, A_3, \ldots., A_n$ are finite sets ,then

$$n(A_1 \times A_2 \times \ldots \times A_n) = n(A_1).n(A_2)\ldots.n(A_n)$$

Relations

The concept of a relation is a basic concept in everyday life as well as in mathematics. We have already come across various relations. Familiar examples in arithmetic are relations such as 'greater than', 'less than' or that of 'equality between two real numbers'. Similar examples exist for relations among more than two objects. In this module we only consider relations, called binary relations, between a pair of objects. We note that a relation between two objects can be defined by listing the two objects as an ordered pair. A set of all such ordered pairs, in each of which the first member has some definite relationship to the second, describes a particular relationship.

Relation: Any set of ordered pairs defines a ***binary relation***.

We call a binary relation simply a relation. If R is a relation and $(x, y) \in R$ then we sometimes write $\boldsymbol{xRy}$ and it is read as "**$\boldsymbol{x}$ is in relation $\boldsymbol{R}$ to $\boldsymbol{y}$**" or "**$\boldsymbol{x}$ is related to $\boldsymbol{y}$ by the relation $\boldsymbol{R}$**".

Let A and B be any two sets. A subset S of the Cartesian product $A \times B$ is said to be a **binary relation from $\boldsymbol{A}$ to $\boldsymbol{B}$**.

Let

$$D(S) = \{x \in A | (\exists y)\big((x, y) \in S\big)\}$$

$i.e.$, $D(S)$ is the set of objects $x \in A$ such that for some $y \in B$, $(x, y) \in S$ and $D(S)$ is called the ***domain of S***. Clearly $D(S) \subseteq A$.

Let

$$R(S) = \{y \in B | (\exists x)((x, y) \in S)\}$$

$i.e.$, $R(S)$ is the set of all objects $y \in B$ such that for some $x \in A$, $(x, y) \in S$ and $R(S)$ is called the ***range of S.*** Clearly, $R(S) \subseteq B$.

Note that $A \times B \subseteq A \times B$ and $\emptyset \subseteq A \times B$. Therefore $A \times B$ itself defines a relation from A to B called the ***Universal relation*** from A to B, while the empty set defines a relation called a ***void relation*** from A to B.

Note: If A and B are finite sets with m and n elements respectively, then the number of relations from A to B is 2^{mn}.

This follows from the fact that every subset of $A \times B$ is a relation and there are 2^{mn} subsets of $\times B$.

Relation on A: A relation from A to itself is called *a **relation on A.*** That is, R is a relation on A iff $\boldsymbol{R \subseteq A \times A}$.

Throughout this module a relation means a relation on a set A.

If R and S are relations on a set A then $R \cap S$, $R \cup S$, $R - S$ and R' define relations on A in the following way:

$$\boldsymbol{R \cap S = \{(x, y) | (x, y) \in R \wedge (x, y) \in S\}}$$

$$\boldsymbol{R \cup S = \{(x, y) | (x, y) \in R \vee (x, y) \in S\}}$$

$$\boldsymbol{R - S = \{(x, y) | (x, y) \in R \wedge (x, y) \notin S\}}$$

$$\boldsymbol{R' = \{(x, y) | (x, y) \notin R\} = A \times A - R}$$

Example 1: Let $A = \{1, 2, 3, 4\}$. If R and S are relations on a set A defined by

$$\boldsymbol{R = \{(x, y) | x \in A \wedge y \in A \wedge (x \equiv y (mod\ 2))\}}$$

$$\boldsymbol{S = \{(x, y) | x \in A \wedge y \in A \wedge (x \leq y)\}}$$

then find $\boldsymbol{R} \cap \boldsymbol{S}$, $\boldsymbol{R} \cup \boldsymbol{S}$, $\boldsymbol{R} - \boldsymbol{S}$ and $\boldsymbol{S}'$.

Solution: We have

$$R = \{(x,y) | x \in A \wedge y \in A \wedge ((x-y) \text{ } is \text{ } an \text{ } integral \text{ } multiple \text{ } of \text{ } 2)\}$$

$$= \{(1,3), (2,4), (3,1), (4,2)\}$$

$$S = \{(x,y) | x \in A \wedge y \in A \wedge (x \leq y)\}$$

$$= \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,3), (3,4), (4,4)\}$$

Now,

i. $R \cap S = \{(x,y) | (x,y) \in R \wedge (x,y) \in S\} = \{(1,3), (2,4)\}$

ii. $R \cup S = \{(x,y) | (x,y) \in R \vee (x,y) \in S\}$

$$R \cup S = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,1), (3,3), (3,4), (4,2), (4,4)\}$$

iii. $R - S = R \cap S' = \{(3,1), (4,2)\}$

iv. $S' = \{(x,y) | (x,y) \notin S\} = \{(x,y) | (x,y) \in A \times A \wedge (x,y) \notin S\}$

$$= \{(2,1), (3,1), (3,2), (4,1), (4,2), (4,3)\}$$

Properties of binary relations on a set

Reflexive relation: A (binary) relation R on a set A is ***reflexive*** if for every $x \in A$, xRx, $i.e.$, $(x,x) \in R$ or

R is reflexive on $A \Leftrightarrow \forall x (x \in A \longrightarrow xRx)$

Examples:

(i) The relation $\leq$ is reflexive on $\boldsymbol{R}$,the set of real numbers, since $x \leq x$, $\forall x \in \boldsymbol{R}$.

(ii) The relation inclusion ($\subseteq$) is reflexive in the family of all subsets of a universal set.

(iii) The relation equality of sets is also reflexive.

The relation $<$ is not reflexive on $\boldsymbol{R}$ and the relation proper inclusion ($\subset$) is not reflexive on the family of subsets of a universal set.

Symmetric relation: A relation R on a set A is ***symmetric*** if, for every $x, y \in A$. yRx whenever xRy, $i.e.$, $(y, x) \in R$ whenever $(x, y) \in R$.
That is R is symmetric $\Leftrightarrow \forall x\ \forall y\big(x \in A \wedge y \in A \wedge (xRy \longrightarrow yRx)\big)$

The relation $\leq$ and $<$ are not symmetric on $\boldsymbol{R}$, while the relation of equality is symmetric. The relation of similarity is both reflexive and symmetric on the set of triangles in a plane.

(The relation of being a brother is not symmetric on the set of all people. However, in the set of all males it is symmetric).

Transitive relation: A relation R on a set A is ***transitive*** if for every x, y and $z \in A$

$$xRz \text{ whenever } xRy \text{ and } yRz$$

$$i.e., (x, z) \in R \text{ whenever } (x, y) \in R \text{ and } (y, z) \in R.$$

That is,
R is transitive $\Leftrightarrow \forall x\ \forall y\ \forall z\big(x \in A \wedge y \in A \wedge z \in A \wedge (xRy \wedge yRz \longrightarrow xRz)\big)$

The relations $\leq$, $<$ and $=$ are transitive on $\boldsymbol{R}$.

The relations $\subseteq$, $\subset$ and equality are transitive on the family of subsets of a universal set. The relation similarity of triangles in a plane is transitive, while the relation of being a mother is not on the set of people.

Irreflexive relation: A relation R on a set A is ***Irreflexive*** if, for every $x \in A, (x, x) \notin R. i.e.$ $x \not\!\!R\, x \ \ \forall x \in A$

The relation $<$ on $\boldsymbol{R}$ is Irreflexive because for no $x \in \boldsymbol{R}$ do we have $x < x$. The relation of proper inclusion on the set of all non-empty subsets of a universal set is irreflexive.

Let R, S and T be relations on $A = \{1,2,3\}$ given by

$$R = \{(1,2), (2,3), (3,2)\}$$

$$S = \{(1,1), (1,2), (3,2), (2,3), (3,3)\}$$

$$T = \{(1,1), (2,1), (2,2), (3,2), (3,3)\}$$

Here R is irreflexive, S is not reflexive and T is reflexive.

Antisymmetric relation: A relation R on a set A is ***antisymmetric*** if, for every $x, y \in A$,

$$x = y \text{ whenever } xRy \text{ and } yRx.$$

$$i.e., x = y \text{ whenever } (x, y) \in R \text{ and } (y, x) \in R.$$

That is

$$\forall x\, \forall y\, \forall z \left(x \in A \land y \in A \land \left(xRy \land yRx \longrightarrow (x = y) \right) \right)$$

Let X be the collection of subsets of a universal set. For $A, B \in X$, $A \subseteq B$ and $B \subseteq A \Longrightarrow A = B$.Thus, inclusion is an antisymmetric relation on X.

Also the relation of proper inclusion on X is antisymmetric.

Note: A relation R on A is antisymmetric if there are no pairs of distinct elements a, b such that aRb and bRa.

Equivalence relation: A relation R on a set A is called an **equivalence relation** if it is reflexive, symmetric and transitive.

The following are some examples of equivalence relations:

i. Equality of numbers on the set of real numbers
ii. Equality of subsets of a universal set.
iii. Similarity of triangles on the set of triangles
iv. Relation of lines being parallel on a set of lines in a plane
v. Relation of living in the same town on the set of people living in Andhra Pradesh
vi. Relation of propositions being equivalent in the set of propositions.

Congruence modulo m: Let $m > 1$ be a given natural number. For any $x, y \in Z$, $x \equiv y(mod\ m)$, read as "x is congruent to y modulo m", if $x - y$ is divisible by m. The relation is called congruence modulo m

Example 2: The relation R on , the set of integers , defined by

$$\boldsymbol{R = \{(x, y) | x \in Z \wedge y \in Z \wedge (x \equiv y(mod\ m))\}}$$

is an equivalence relation.

Solution:

i. For any $x \in Z$,$x - x$ is divisible by m. Therefore, $x \equiv x(mod\ m)\ \forall x \in Z$. Thus R is reflexive.

ii. For any $x, y \in Z$, if $x \equiv y(mod\ m)$ then $x - y$ is divisible by m. Therefore $y - x$ is also divisible by m . Thus ,$x \equiv y(mod\ m) \Longrightarrow y \equiv x(mod\ m)$.This shows that R is symmetric

iii. For $x, y, z \in Z$, if $x \equiv y(mod\ m)$ and $y \equiv z(mod\ m)$, then $x - y$ and $y - z$ are divisible by m. Therefore,$x - z = (x - y) + (y - z)$ is also divisible by m $i.e.$, $x \equiv z(mod\ m)$

Thus, ***the relation congruence modulo m is an equivalence relation on the set Z of integers.***

Partial order: A relation R on a set A is called a **partial order relation** or **partial ordering relation** or **partial order** iff R is reflexive, anti symmetric and transitive.

The following are some examples of partial order relations:

i. The relation $\leq$ is a partial ordering on $\boldsymbol{R}$
ii. The relation $\geq$ is a partial ordering on $\boldsymbol{R}$
iii. Let A be any set. The relation of inclusion($\subseteq$) is a partial ordering on $P(A)$
iv. The relation "divides" is a partial ordering on the set of positive integers

Note:

(i) If R_1 and R_2 are equivalence relations on A, then $R_1 \cap R_2$ is also an equivalence relation A.

(ii) If R_1 and R_2 are partial orders on A, then $R_1 \cap R_2$ is also a partial order on A.

(iii) For any set A, $A \times A$ is an equivalence relation on A.

(iv) If $A = \{a_1, a_2, \dots a_n\}$ then the equality relation $R = \{(a_i, a_i) | i = 1, 2, \dots, n\}$ is the smallest equivalence relation on A.

(v) Let R be a relation on a set A . R is both an equivalence relation and a partial order on A if and only if R is the equality relation on A.

P1:

Write down all relations from $\boldsymbol{A} = \{\mathbf{1}, \mathbf{2}\}$ to $\boldsymbol{B} = \{\boldsymbol{a}, \boldsymbol{b}\}$.

Solution: We have $A = \{1,2\}$, $B = \{a, b\}$, and $n(A) = n(B) = 2$

Now $A \times B = \{(x, y)|x \in A \wedge y \in B\}$

$= \{(1, a), (1, b), (2, a), (2, b)\}$

The number of relations from a finite set A to a finite set B is $2^{n(A)\cdot n(B)} = 16$

We have to enlist all relations from A to B.

$R_1 : \phi$

$R_2 : \{(1, a)\}$

$R_3 : \{(1, b)\}$

$R_4 : \{(2, a)\}$

$R_5 : \{(2, b)\}$

$R_6 : \{(1, a), (1, b)\}$

$R_7 : \{(1, a), (2, a)\}$

$R_8 : \{(1, a), (2, b)\}$

$R_9 : \{(1, b), (2, a)\}$

$R_{10} : \{(1, b), (2, b)\}$

$R_{11} : \{(2, a), (2, b)\}$

$R_{12} : \{(1, a), (1, b), (2, a)\}$

$R_{13} : \{(1, a), (1, b), (2, b)\}$

$R_{14} : \{(1, a), (2, a), (2, b)\}$

$R_{15} : \{(1, b), (2, a), (2, b)\}$

$R_{16}: \{(1, a), (1, b), (2, a), (2, b)\}$

P2:

Let $\boldsymbol{A} = \{\mathbf{1}, \mathbf{2}, \mathbf{3}\}$ and $\boldsymbol{B} = \{\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}\}$.

If $\boldsymbol{R_1} = \{(\mathbf{1}, \mathbf{1}), (\mathbf{2}, \mathbf{2}), (\mathbf{3}, \mathbf{3})\}$ and $\boldsymbol{R_2} = \{(\mathbf{1}, \mathbf{1}), (\mathbf{1}, \mathbf{2}), (\mathbf{1}, \mathbf{3}), (\mathbf{1}, \mathbf{4})\}$, then find $\boldsymbol{R_1} \cup \boldsymbol{R_2}$, $\boldsymbol{R_1} \cap \boldsymbol{R_2}$, $\boldsymbol{R_1}'$, $\boldsymbol{R_2} - \boldsymbol{R_1}$.

Solution:

i) $R_1 \cup R_2 = \{(x, y) | (x, y) \in R_1 \vee (x, y) \in R_2\}$

$= \{(1,1), (1,2), (1,3), (1,4), (2,2), (3,3)\}$

ii) $R_1 \cap R_2 = \{(x, y) | (x, y) \in R_1 \wedge (x, y) \in R_2\}$

$= \{(1,1)\}$

iii) $R_1' = (A \times B) - R_1$
$= \{(1,2), (1,3), (1,4), (2,1), (2,3), (2,4), (3,1), (3,2), (3,4)\}$

iv) $R_2 - R_1 = R_2 \cap R_1' = \{(1,2), (1,3), (1,4)\}$

P3:

Let $\boldsymbol{A} = \{\mathbf{1}, \mathbf{2}, \mathbf{3}, \dots, \mathbf{7}\}$ **and**

$$\boldsymbol{R} = \{(\boldsymbol{x}, \boldsymbol{y}) | \boldsymbol{x} \in \boldsymbol{A} \wedge \boldsymbol{y} \in \boldsymbol{A} \wedge (\boldsymbol{x} \equiv \boldsymbol{y}(\boldsymbol{mod}\ \mathbf{3}))\}$$

$$\boldsymbol{S} = \{(\boldsymbol{x}, \boldsymbol{y}) | \boldsymbol{x} \in \boldsymbol{A} \wedge \boldsymbol{y} \in \boldsymbol{A} \wedge (\boldsymbol{x} \equiv \boldsymbol{y}(\boldsymbol{mod}\ \mathbf{4}))\}$$

Find $\boldsymbol{R} \cap \boldsymbol{S}$ **,** $\boldsymbol{R} \cup \boldsymbol{S}$ **,** $\boldsymbol{R} - \boldsymbol{S}$**.**

Solution: We have

$$R = \left\{ \begin{matrix} (1,1), (1,4), (1,7), (2,2), (2,5), (3,3), (3,6), (4,1), (4,4), \\ (4,7), (5,2), (5,5), (6,3), (6,6), (7,1), (7,4), (7,7) \end{matrix} \right\}$$

$$S = \left\{ \begin{matrix} (1,1), (1,5), (2,2), (2,6), (3,3), (3,7), (4,4), \\ (5,1), (5,5), (6,2), (6,6), (7,3), (7,7) \end{matrix} \right\}$$

$$R \cap S = \{(1,1), (2,2), (3,3), (4,4), (5,5), (6,6), (7,7)\}$$

$$R \cup S = \left\{ \begin{matrix} (1,1), (1,4), (1,5), (1,7), (2,2), (2,5), (2,6), (3,3), (3,6), (3,7), (4,1), \\ (4,4), (4,7)(5,1), (5,2), (5,5), (6,2), (6,3), (6,6), (7,1), (7,3), (7,4), (7,7) \end{matrix} \right\}$$

$$R - S = \{(1,4), (1,7), (2,5), (3,6), (4,1), (4,7), (5,2), (6,3), (7,1), (7,4)\}$$

$$S - R = \{(1,5), (2,6), (3,7), (5,1), (6,2), (7,3)\}$$

P4:

The relation "divides" is a partial order relation on $\boldsymbol{N}$, the set of natural numbers.

Solution: We have $\boldsymbol{N}$ and let

$$R = \{(x, y)|x \in \boldsymbol{N} \wedge y \in \boldsymbol{N} \wedge (x|y)\}$$

(i) For any $x \in \boldsymbol{N}$, we have $x|x$. Thus xRx , $\forall x \in \boldsymbol{N}$. Therefore the relation R is reflexive.

(ii) If $x|y$ and $y|x$ then $y = kx$ and $x = ly$ for some $k, l \in \boldsymbol{N}$. Therefore, $x = klx$. Thus $kl = 1$, implying $k = l = 1$. This shows that $x = y$, proving R is antisymmetric.

(iii) If $x|y$ and $y|z$ then $y = kx$ and $z = ly$ for some $k, l \in \boldsymbol{N}$.Therefore $z = lkx$. This proves that $x|z$. Thus R is transitive.

Therefore, R is reflexive, anti symmetric and transitive. This proves R is a partial order relation on $\boldsymbol{N}$.

Note: The relation "divides" is not a partial order on $\boldsymbol{Z}$, the set of integers(why?)

P5:

If $\boldsymbol{R}$ is a relation on $\boldsymbol{Z} \times \boldsymbol{Z}$, where $(\boldsymbol{x}, \boldsymbol{y})\ \boldsymbol{R}\ (\boldsymbol{u}, \boldsymbol{v})$ if $\boldsymbol{x} \le \boldsymbol{u}$. Determine whether $\boldsymbol{R}$ is reflexive, symmetric, anti symmetric or transitive

Solution:

First note that $R \subseteq (\boldsymbol{Z} \times \boldsymbol{Z}) \times (\boldsymbol{Z} \times \boldsymbol{Z})$

(i) For any $x \in \boldsymbol{Z}$ we have $x \le x$
For any $(x, y) \in \boldsymbol{Z} \times \boldsymbol{Z}$, we have ,$(x, y)R(x, y)$, since $x \le x$
Thus, $(x, y)R\ (x, y)$, $\forall\ (x, y) \in Z \times Z$.
Therefore R is reflexive

(ii) If $x \le u$ then $(x, y)R\ (u, v)$, but $(u, v) \not R\ (x, y)$,since $u \ge x$ when $x \le u$
Therefore R is not symmetric.
If $(x, y)\ R\ (u, v)$ and $(u, v)\ R\ (x, y)$ then $x \le u$ and $u \le x$,
$i.e.$, $x = u$ with no restriction y and v.
Note that $(1, 2)\ R\ (1, -3)$ and $(1, -3)\ R\ (1, 2)$ without being equal. This shows that R is not anti symmetric. Thus R is neither symmetric nor anti symmetric.

Remark: Not symmetric does not mean anti symmetric.

(iii) $(x, y)\ R\ (u, v)$ and $(u, v)\ R\ (r, s) \implies x \le u$ and $u \le r$

$$\implies x \le r$$

$$\implies (x, y)\ R\ (r, s)$$

This, shows that R is transitive.

Therefore, R is reflexive, transitive and neither symmetric nor anti symmetric.

P6:

Let $\boldsymbol{R_1}$ and $\boldsymbol{R_2}$ be relations on a set $\boldsymbol{A}$.

(a) **Prove or disprove that $\boldsymbol{R_1}$ and $\boldsymbol{R_2}$ reflexive $\Longrightarrow$ $\boldsymbol{R_1 \cap R_2}$ is reflexive.**

(b) **Answer part (a) when each occurrence of "reflexive" is replaced by (i) symmetric (ii) anti symmetric and (iii) transitive.**

Solution:

(a) R_1 and R_2 are reflexive $\Longrightarrow (x,x) \in R_1, \forall\, x \in A$ and $(x,x) \in R_2, \forall\, x \in A$

$$\Longrightarrow (x,x) \in R_1 \cap R_2\,,\ \forall\, x \in A$$

$$\Longrightarrow R_1 \cap R_2 \text{ is reflexive.}$$

(b)

(i) We have R_1 and R_2 are symmetric.

$(x,y) \in R_1 \cap R_2 \Longrightarrow (x,y) \in R_1$ and $(x,y) \in R_2$

$\Longrightarrow (y,x) \in R_1$ and $(y,x) \in R_2$

(Since R_1 and R_2 are symmetric)

$\Longrightarrow (y,x) \in R_1 \cap R_2$

Thus, $R_1 \cap R_2$ is symmetric whenever R_1 and R_2 are symmetric.

(ii) We have R_1 and R_2 are anti symmetric

$(x,y) \in R_1 \cap R_2$ and $(y,x) \in R_1 \cap R_2$

$\Longrightarrow \big((x,y) \in R_1$ and $(x,y) \in R_2\big)$ and $\big((y,x) \in R_1$ and $(y,x) \in R_2\big)$

$\Longrightarrow \big((x,y) \in R_1$ and $(y,x) \in R_1\big)$ and $\big((x,y) \in R_2$ and $(y,x) \in R_2\big)$

$\Longrightarrow (x = y)$ and $(y = x)$ (Since R_1 and R_2 are anti symmetric).

$\Longrightarrow x = y$

Thus, $R_1 \cap R_2$ is anti symmetric whenever R_1 and R_2 are anti symmetric.

(iii) We have R_1 and R_2 are transitive

$(x, y) \in R_1 \cap R_2$ and $(y, z) \in R_1 \cap R_2$

$\Longrightarrow \big((x, y) \in R_1 \text{and } (x, y) \in R_2\big)$ and $\big((y, z) \in R_1 \text{and } (y, z) \in R_2\big)$

$\Longrightarrow \big((x, y) \in R_1 \text{and } (y, z) \in R_1\big)$ and $\big((x, y) \in R_2 \text{and } (y, z) \in R_2\big)$

$\Longrightarrow (x, z) \in R_1$ and $(x, z) \in R_2$ (Since R_1 and R_2 are transitive)

$\Longrightarrow (x, z) \in R_1 \cap R_2$

Thus, $R_1 \cap R_2$ is transitive whenever R_1 and $R_2$are transitive.

P7:

What is wrong with the following argument?

Let $\boldsymbol{A}$ be a set with $\boldsymbol{R}$ a relation on $\boldsymbol{A}$. If $\boldsymbol{R}$ is symmetric and transitive, then $\boldsymbol{R}$ is reflexive

Solution:

Let $(x, y) \in R$ .By the symmetry $(y, x) \in R$. Now $(x, y) \in R$ and $(y, x) \in R$ will imply $(x, x) \in R$, by transitivity. Hence R is reflexive.

The conclusion $(x, x) \in R$ is true for $x \in A$ only. The relation R be reflexive we need $(x, x) \in R, \forall x \in A$

There may exist an element $a \in A$ such that for all $b \in A$, neither(a, b) nor $(b, a) \in R$. The argument is not correct.

P8:

Which of the following relations on $A = \{0, 1, 2, 3\}$ are equivalent relations/partial orders? Determine the properties of an equivalence/a partial order the others lack.

$R_1 = \{(0,0), (1,1), (2,2), (3,3)\}$

$R_2 = \{(0,0), (0,2), (2,0), (2,2), (2,3), (3,2), (3,3)\}$

$R_3 = \{(0,0), (1,1), (1,2), (2,1), (2,2), (3,3)\}$

$R_4 = \{(0,0), (1,1), (1,3), (2,2), (2,3), (3,1), (3,2), (3,3)\}$

$R_5 = \{(0,0), (0,1), (0,2), (1,0), (1,1), (1,2), (2,0), (2,2), (3,3)\}$

$R_6 = \{(0,0), (2,0), (2,2), (2,3), (3,2), (3,3)\}$

$R_7 = \{(0,0), (1,1), (1,2), (2,2), (3,3)\}$

$R_8 = \{(0,0), (1,1), (1,2), (1,3), (2,2), (2,3), (3,3)\}$

Solution:

Relation	Reflexive	Symmetric	Anti symmetric	transitive	Equivalence relation	Partial order
R_1	✓	✓	✓	✓	✓	✓
R_2	×	✓	×	×	×	×
R_3	✓	✓	×	✓	✓	×
R_4	✓	✓	×	×	×	×
R_5	✓	×	×	×	×	×
R_6	×	×	×	×	×	×
R_7	✓	×	✓	✓	×	✓
R_8	✓	×	✓	✓	×	✓

2.2. Relations

Exercises

1. List the ordered pairs in the relation R from $A = \{0,1,2,3,4\}$ to $B = \{0,1,2,3\}$, where $(a, b) \in R$ if and only if
    a) $a = b$
    b) $a + b = 4$
    c) $a > b$
    d) $a|b$
    e) $gcd(a, b) = 1$
    f) $lcm(a, b) = 2$

2. Let $A = \{a, b, c\}$ and $B = \{1,2,3,4\}$.
   If $R = \{(a, 2), (a, 4), (b, 2), (b, 3), (c, 1), (c, 4)\}$
   $S = \{(a, 3), (b, 1), (b, 2), (b, 4), (c, 4), (c, 1)\}$
   Then find $R \cap S$, $R \cup S$, $R - S$, $S - R$.

3. Let $A = \{1,2,3,4\}$ and $B = \{a, b, c\}$.
   If $R = \{(1, a), (2, b), (3, c), (4, a)\}$ and $S = \{(1, b), (2, c), (3, a), (4, a)\}$
   Then find $R \cup S$, $R \cap S$, $R - S$, $S - R$.

4. Let $A = \{2,3,5,6,7\}$ and $B = \{3,4,10,12,14,15\}$. Let R and S be relations from A to B defined by:
   For all $a \in A, b \in B$
    i. $a\ R\ b$ iff $a|b$
    ii. $a\ S\ b$ iff $a \geq b$
   then find $R \cup S$ and $R \cap S$.

5. Consider the following relation on Z:

$$R_1 = \{(a,b) | a \le b\}$$

$$R_2 = \{(a,b) | a > b\}$$

$$R_3 = \{(a,b) | a = b \text{ or } a = -b\}$$

$$R_4 = \{(a,b) | a = b\}$$

$$R_5 = \{(a,b) | a = b + 1\}$$

$$R_6 = \{(a,b) | a + b \le 3\}$$

Decide which of these are reflexive, symmetric, anti symmetric, transitive.

6. Consider the following relation on $A = \{1,2,3,4\}$.

$$R_1 = \{(1,1), (1,2), (2,1), (2,2), (3,4), (4,1), (4,4)\}$$

$$R_2 = \{(1,1), (1,2), (2,1)\}$$

$$R_3 = \{(1,1), (1,2), (1,4), (2,1), (2,2), (3,3), (4,1), (4,4)\}$$

$$R_4 = \{(2,1), (3,1), (3,2), (4,1), (4,2), (4,3)\}$$

$$R_5 = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,3), (3,4), (4,4)\}$$

$$R_6 = \{(3,4)\}$$

Which of these relations are reflexive, symmetric, anti symmetric, transitive?

7.

a) How many relations are there on the set $\{a, b, c, d\}$?

b) How many relations are there on the set $\{a, b, c, d\}$ that contain the pair (a, a)?

8. Let R be the relation on the set of ordered pairs of positive integers such that $\big((a, b), (c, d)\big) \in R$ if and only if $ad = bc$. Show that R is an equivalence relation.

9. Give an example of a relation in $\{1,2,3\}$ which is
    a) Reflexive but neither symmetric nor transitive
    b) Symmetric but neither reflexive nor transitive
    c) Transitive but neither reflexive nor symmetric
    d) Reflexive and symmetric but not transitive
    e) Reflexive and transitive but not symmetric
    f) Symmetric and transitive but not reflexive
    g) Reflexive, symmetric and transitive
    h) Not reflexive, not symmetric and not transitive