15-04-2023

3. SETS, RELATIONS AND FUNCTIONS.

Module - 3.1

* **Set:-**

→ A 'set' is a well-defined collection of objects.

→ The objects of sets are called Elements.

→ A collection is 'well-defined' if there is no ambiguity in determining whether or not a given object belongs to the collection.

→ Father of set theory was 'Georg cantor'.

→ sets are denoted by capital letters & their elements by lower case letters.

* **Ex-1:-** The set V of all vowels in the English alphabets is given by $V = \{a, e, i, o, u\}$.

→ This is 'Lisiting method.'

* **set builder form:-**

Ex-2:- Let M be the set of all months of year with exactly 30 days. Then M in set builder notation is

$M = \{x ; x$ is a month of year with exactly 30 days$\}$.

M = {April, June, september, november}.

* **Note:-**

$N = \{1, 2, 3, \ldots\}$, set of natural numbers

$Z = \{\ldots -2, -1, 0, 1, 2, \ldots\}$, set of integers.

$Q = \{\frac{p}{q} \mid p, q \in Z, q \neq 0\}$, set of rational numbers

R : set of Real number.

* Types of sets:-

① **Equality of sets:-**

→ Two sets A and B are equal if they have same elements.

Ex: $A = \{1, 3, 5\}$, $B = \{3, 5, 1\}$

$A = B$.

② **Universal set:-** [U (or) μ].

→ The largest set which is a superset of subsets is called 'universal set'.

③ **Equivalent set:-**

→ If two sets have same cardinal number [No. of elements] then they are called 'Equivalent set'.

④ **Finite set:-**

→ If no. of elements in a set are countable then it is called 'Finite set'.

⑤ **Infinite set:-**

→ If no. of elements in a set are not countable then it is called 'Infinite set'.

⑥ **Null set:-** [ϕ].

→ If a set has no elements then it is called 'Null set'.

⑦ **Disjoint sets:-**

→ If two sets have no common elements then they are called 'Disjoint sets'.

⑧ **Powerset:-**

→ The set of all subsets of A is called Powerset of A. i.e, P(A).

* **Index set:-**

→ Let $I = \{i_1, i_2, \ldots\}$. If A be a family of sets $A = \{A_{i_1}, A_{i_2}, \ldots\}$ such that for any $i_j \in I$ there exists a set $A_{i_j} \in A$ and $A_{i_j} = A_{i_k}$ iff $i_j = i_k$ then 'A' is called Indexed set, 'I' is Index set & any subscript i_j in A_{i_j} is called an Index.

* **set operations:-**

* **Union & Intersection:-**

Note:-

Let A or I (Index set) $= \{1, 2, 3, 4, \ldots, n\}$

$A = \{A_1, A_2, A_3, \ldots A_n\}$

Here A is family of Index set.

$A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_n = \bigcup_{i=1}^{n} A_i$ → Finite

$A_1 \cup A_2 \cup A_3 \cup \ldots \cup \ldots = \bigcup_{i=1}^{\infty} A_i$

↓ Arbitrary family of union set

* **Union:-**

$A \cup B \cup C = \{x / x \in A \text{ or } x \in B \text{ (or) } x \in C\}$

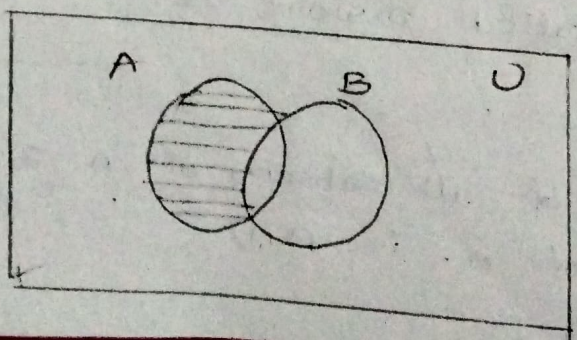
* **Intersection:-**

$A \cap B \cap C = \{x / x \in A \text{ and } x \in B \text{ and } x \in C\}$

* **Difference:-**

'A−B' → complement of B w.r.to A.

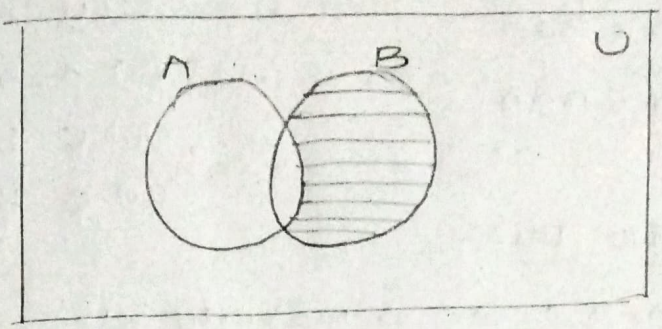
$A - B = \{x / x \in A \text{ and } x \notin B\}$.



$B-A = \{x \mid x \in B \text{ and } x \notin A\}$

↓

complement of ~~B~~ 'A' w.r.to 'B'.

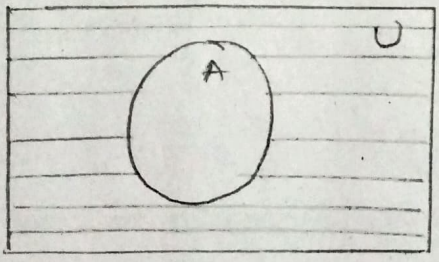


$\therefore \boxed{(A-B) \cap (B-A) = \phi}$

* complement of a set :-

$A' = U - A$ = complement of A w.r.to 'U'.

$A' = \{x \mid x \in U \text{ and } x \notin A\}$.



$A \cup A' = U$

$A \cap A' = \phi$

A, A' are Disjoint sets and Exastic sets.

* symmetric Difference :- [Δ (or) ⊕], [Boolean sum]

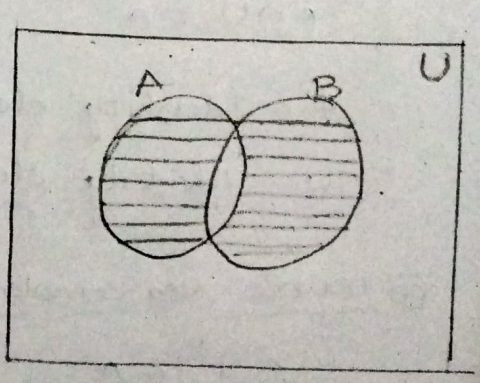
$A \oplus B = \{x \mid x \in A \text{ (or) } x \in B \text{ and } x \notin A \cap B\}$

$= \{x \mid x \in A \cup B \text{ and } x \notin A \cap B\}$

$\boxed{A \oplus B = (A \cup B) - (A \cap B)}$

(or)

$\boxed{A \oplus B = (A-B) \cup (B-A)}$



* Set Theory Identities:-

① **Commutative Law:-**

$A \cup B = B \cup A$

$A \cap B = B \cap A$.

Dual of Identity

(i) $\cup$ by $\cap$

(ii) $\cap$ by $\cup$

(iii) U by ϕ

(iv) ϕ by U.

② **Associative Law:-**

→ Let A, B, C are 3 Non empty sets:-

$A \cup (B \cup C) = (A \cup B) \cup C$

$A \cap (B \cap C) = (A \cap B) \cap C$.

③ **Distributive Law:-**

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

④ **Idempotent:-**

$A \cup A = A$

$A \cap A = A$.

⑤ **Inverse Law:-**

$A \cup A' = U$

$A \cap A' = \phi$

⑥ **Identity Law:-**

$A \cup \phi = A$

$A \cap U = A$.

ϕ = Identity element w.r.to 'Union'

U = Identity element w.r.to Intersection.

⑦ **Double ~~Nos~~ complement Law:-**

$(A')' = A$.

⑧ Demorgan's Law:-

$(A\cup B)' = U - (A\cup B)$

$= \{x \mid x\in U \text{ and } x\notin A\cup B\}$

$= \{x \mid x\in U \text{ and } x\notin A \text{ and } x\notin B\}$.

$\therefore (A\cup B)' = A'\cap B'$

$(A\cap B)' = A'\cup B'$.

⑨ Domination Laws:-

$A\cup U = U$; $A\subseteq U$

$A\cap \phi = \phi$; $\phi\subseteq A$.

⑩ Absorption Law:-

$A\cup(A\cap B) = A$

$A\cap(A\cup B) = A$

*P-1:-

Prove or disprove: For any sets $A, B \subseteq U$.

(i) $P(A\cup B) = P(A)\cup P(B)$

A) Let $A=\{a\}$, $B=\{b\}$

$A\cup B = \{a,b\}$. $\Rightarrow P(A\cup B) = \{a,b\}$

$P(A) = \{\phi, \{a\}\}$, $P(B) = \{\phi, \{b\}\}$

$P(A)\cup P(B) = \{\{\phi, \{1\}\}\cup\{\phi, \{2\}\}\} = \{\phi, \{1\}, \{2\}\}$.

$\therefore P(A\cup B) \neq P(A)\cup P(B)$.

(ii) $P(A\cap B) = P(A)\cap P(B)$.

A) Let $A=\{a\}$, $B=\{b\}$.

$A\cap B = \{\phi\}$. (or)

$\therefore P(A\cap B) = P(A)\cap P(B)$.

Let $x\in P(A\cap B)$

$\Rightarrow$ ~~$x\subseteq P(A\cap B)$~~

$\Rightarrow x\subseteq (A\cap B)$

$\Rightarrow x\subseteq A$ and $x\subseteq B$.

$\Rightarrow x\in P(A)$ and $x\in P(B)$

$\Rightarrow x\in P(A)\cap P(B)$

* P-2:-

Show that for any sets, A, B, C and D.

$(A\cap B) \cup (B\cap((C\cap D)\cup(C\cap D')))) = B\cap(A\cup C)$.

A) $(A\cap B) \cup (B\cap((C\cap D)\cup(C\cap D')))$

$\Rightarrow (A\cap B) \cup (B\cap(C\cap(D\cup D')))$. [∵ Distributive]

$\Rightarrow (A\cap B) \cup (B\cap(C\cap U))$

$\Rightarrow (A\cap B) \cup (B\cap C)$

$\Rightarrow (B\cap A) \cup (B\cap C)$ [∵ commutative]

$\Rightarrow B\cap(A\cup C)$. [Distributive]

Hence proved.

* P-3:-

Show that for any sets A and B, $A-(A\cap B)=A-B$

A) $A-(A\cap B) = A\cap(A\cap B)'$ [∵ $A-B = A\cap B'$]

$= A\cap(A'\cup B')$ [∵ Demorgan's]

$= (A\cap A')\cup(A\cap B')$

$= \phi \cup (A\cap B')$

$= A\cap B'$

$= A-B$.

* P-4:-

Show that for any sets A, B and C.

$(((A\cup B)\cap C)'\cup B')' = B\cap C$.

A) $(((A\cup B)\cap C)'\cup B') = (((A\cup B)\cap C)')'\cap(B')'$ [De morgan]

$= ((A\cup B)\cap C)\cap B$. [Double Negative]

$= (A\cup B)\cap(C\cap B)$ [Associative]

$= (A\cup B)\cap(B\cap C)$ [commutative]

$= ((A\cup B)\cap B)\cap C$

$= (B\cap C)$ [Absorption]

* P-5:-

simplify, using laws of set theory.

(i) $A \cap (B-A)$.

A) $A \cap (B-A) = A \cap (B \cap A')$ [$\because A-B = A \cap B'$]

$= A \cap (A' \cap B)$ [$\because$ commutative]

$= (A \cap A') \cap B$ [$\because$ Associative].

$= \phi \cap B$

$= \phi$

(ii) $(A \cap B) \cup (A \cap B \cap C' \cap D) \cup (A' \cap B)$.

A) $= (A \cap B) \cup (A \cap B \cap C' \cap D) \cup (A' \cap B)$

$= (A \cap B) \cup ((A \cap B) \cap (C' \cap D)) \cup (A' \cap B)$

$= (A \cap B) \cup (A' \cap B)$ [$\because$ Absorption]

$= (A \cup A') \cap B$

$= U \cap B$

$= B$.

(iii) $(A-B) \cup (A \cap B)$.

A) $(A-B) \cup (A \cap B) = (A \cap B') \cup (A \cap B)$

$= A \cap (B' \cup B) = A \cap U = A$.

(iv) $A' \cup B' \cup (A \cap B \cap C')$.

A) $A' \cup B' \cup (A \cap B \cap C') = (A \cap B)' \cup (A \cap B \cap C')$ [$\because$ Demorgan]

$= ((A \cap B)' \cup (A \cap B)) \cap ((A \cap B)' \cup C')$ [$\because$ Distributive]

$= U \cap (A' \cup B' \cup C')$ [$\because$ Demorgans]

$= (A' \cup B' \cup C')$.

(v) $A' \cup (A \cap B') \cup (A \cap B \cap C') \cup (A \cap B \cap C \cap D') \cup ___$

A) consider $A' \cup (A \cap B') = (A' \cup A) \cap (A' \cup B')$

$= U \cap (A' \cup B')$

$= (A \cap B)' \longrightarrow (1)$ [∵ demorgan]

consider $A' \cup (A \cap B') \cup (A \cap B \cap C')$.

$= (A \cap B)' \cup (A \cap B \cap C')$. [by (1)]

$= ((A \cap B)' \cup (A \cap B)) \cap ((A \cap B)' \cup C')$ [∵ Distributive

$= U \cap ((A \cap B)' \cup C')$

$= (A \cap B)' \cup C')$

$= (A \cap B \cap C)' \longrightarrow (2)$ [∵ Demorgan].

Consider $A' \cup (A \cap B') \cup (A \cap B \cap C') \cup (A \cap B \cap C \cap D')$

$= (A \cap B \cap C)' \cup (A \cap B \cap C \cap D')$ [by (2)]

$= (A \cap B \cap C)' \cup ((A \cap B \cap C) \cap D')$

$= ((A \cap B \cap C)' \cup (A \cap B \cap C)) \cap ((A \cap B \cap C)' \cup D')$ [Distributive

$= U \cap ((A \cap B \cap C)' \cup D')$

$= ((A \cap B \cap C)' \cup D')$

$= ((A \cap B \cap C) \cap D)'$ [∵ Demorgan

$= (A \cap B \cap C \cap D)' \longrightarrow (3)$

By observing (1), (2) & (3).

$A' \cup (A \cap B') \cup (A \cap B \cap C') \cup (A \cap B \cap C \cap D') \cup _____$

$= (A \cap B \cap C \cap D \cap ____)'$

* P-6:-

For any sets A and B.

(i) $A \oplus B = B \oplus A$.

A) $A \oplus B = (A-B) \cup (B-A)$

$= (B-A) \cup (A-B)$. [∵ commutative]

$= B \oplus A$

(ii) $A \oplus \phi = A$

A) $A \oplus \phi = (A' \cap \phi) \cup (A \cap \phi')$

$\cancel{= (A' \cup A \cap \phi') \cap (\phi \cup A \cap \phi')}$

$= \phi \cup (A \cap U) = \phi \cup A = A$.

(iii) $A \oplus A = \phi$.

A) $A \oplus A = (A' \cap A) \cup (A \cap A') = \phi \cup \phi = \phi$.

(iv) $A \oplus A' = U$.

A) $A \oplus A' = (A \cap (A')') \cup (A' \cap A')$

$= (A \cap A) \cup (A' \cap A')$

$= (A \cap A) \cup (A' \cap A') = A \cup A' = U$.

(v) $A \oplus U = A'$

A) $A \oplus U = (A \cap U') \cup (A' \cap U)$

$= (A \cap \phi) \cup A'$

$= \phi \cup A' = A'$.

* P-7:-

For any sets, A, B and C.

$A \oplus (B \oplus C) = (A \oplus B) \oplus C$ [$\oplus$ is associative].

A) $A \oplus (B \oplus C) = (A' \cap (B \oplus C)) \cup (A \cap (B \oplus C)')$

Now, $A' \cap (B \oplus C) = A' \cap ((B' \cap C) \cup (B \cap C'))$

$= (A' \cap B' \cap C) \cup (A' \cap B \cap C')$.

Now, $A \cap (B \oplus C)' = A \cap ((B' \cap C) \cup (B \cap C'))'$

$= A \cap ((B' \cap C)' \cap (B \cap C')')$ [De morgan

$= A \cap ((B \cup C') \cap (B' \cup C))$

$= A \cap ((B \cap B') \cup (B \cap C) \cup (C' \cap B') \cup (C' \cap$

$= A \cap (\phi \cup (B \cap C) \cup (C' \cap B') \cup \phi)$

$= A \cap ((B \cap C) \cup (C' \cap B'))$

$= (A \cap B \cap C) \cup (A \cap B' \cap C')$.

$\therefore A \oplus (B \oplus C) = (A' \cap B' \cap C) \cup (A' \cap B \cap C') \cup (A \cap B \cap C)$
$\cup (A \cap B' \cap C')$.

Now, $(A \oplus B) \oplus C = C \oplus (A \oplus B)$ [$\because$ commutative]

$= (C' \cap A' \cap B) \cup (C' \cap A \cap B') \cup (C \cap A' \cap B') \cup (C \cap A \cap B)$.

$= (A' \cap B' \cap C) \cup (A' \cap B \cap C') \cup (A \cap B' \cap C') \cup (A \cap B \cap C)$

$\therefore A \oplus (B \oplus C) = (A \oplus B) \oplus C$.

* P-81

show that $(A \oplus B)' = A \oplus B'$.

A) $(A \oplus B)' = ((A - B) \cup (B - A))'$

$= ((A \cap B') \cup (B \cap A'))'$

$= (A \cap B')' \cap (B \cap A')'$ [De morgan]

$= (A' \cup B) \cap (B' \cup A)$ [Double Negation]

$= (A' \cap B') \cup (B \cap B') \cup (A' \cap A) \cup (B \cap A)$ [Distrib

$= (A \cap B) \cup (A' \cap B') \cup \phi \cup \phi$

$= (A \cap (B')') \cup (A' \cap B') = A \oplus B'$.

Exercise

1] Prove that $[(A \cup (B \cap C)) \cap (A' \cup (B \cap C))] \cap (B' \cup C') = \phi$.

A) consider L.H.S.

$\equiv [(A \cup (B \cap C)) \cap (A' \cup (B \cap C))] \cap (B' \cup C')$

$\equiv [(A \cap A') \cup (B \cap C)] \cap (B \cap C)'$ [∵ Distribution, Demorgan].

$\equiv [\phi \cup (B \cap C)] \cap (B \cap C)'$

$\equiv (B \cap C) \cap (B \cap C)'$

$\equiv \phi$

2] show that $(A \cap B) \cup [B \cap ((C \cap D) \cup (C \cap D'))] = B \cap (A \cup C)$.

A) consider L.H.S

$\equiv (A \cap B) \cup [B \cap ((C \cap D) \cup (C \cap D'))]$.

$\equiv (A \cap B) \cup [B \cap (C \cap (D \cup D'))]$.

$\equiv (A \cap B) \cup [B \cap (C \cap U)]$

$\equiv (A \cap B) \cup [B \cap C)$

$\equiv (A \cap B) \cup (B \cap C)$

$\equiv (B \cap A) \cup (B \cap C) \equiv B \cap (A \cup C)$.

3] If $A \cup B = A \cup C$ and $A \cap B = A \cap C$, then prove $B = C$.

A) Given $A \cup B = A \cup C$ and $A \cap B = A \cap C$.

$A \cap B = A \cap C$ —— (1)

$A \cup B = A \cup C$ —— (2)

claim: $B = C$.

Consider B, $\Rightarrow B = B \cap (B \cup A)$ [Absorption]

$= B \cap (A \cup B)$ [commutative]

$= B \cap (A \cup C)$ [by (2)]

$= (B \cap A) \cup (B \cap C)$ [Distributive]

$= (A \cap B) \cup (B \cap C)$ [commutative]

$= (A \cap C) \cup (B \cap C)$ [by (1)]

$= (A \cup B) \cap C$ [Distributive]

$= (A \cup C) \cap C$ by (2) $= C$

4] Prove the following.

ⓐ $A \cup (B-A) = A \cup B$.

A) $A \cup (B-A) = A \cup (B \cap A')$ [$\because A-B = A \cap B'$]

$= (A \cup B) \cap (A \cup A')$ [$\because$ distributive]

$= (A \cup B) \cap U$

$= (A \cup B)$.

ⓑ $(A \cup B) \cap (A \cup C) = A \cup (B' \cup C')'$.

A) $(A \cup B) \cap (A \cup C) = A \cup (B' \cup C')'$.

consider R.H.S,

$A \cup (B' \cup C')' = A \cup ((B')' \cap (C')')$

$= A \cup (B \cap C)$

$= (A \cup B) \cap (A \cup C)$.

ⓒ $[C \cap (A \cup B)] \cup [(A \cup B) \cap C'] = A \cup B$.

A) $[C \cap (A \cup B)] \cup [(A \cup B) \cap C']$.

$\Rightarrow [(A \cup B) \cap C] \cup [(A \cup B) \cap C']$ [$\because$ commutative]

$\Rightarrow (A \cup B) \cap [C \cup C']$ [$\because$ distributive].

$\Rightarrow (A \cup B) \cap U$

$\Rightarrow A \cup B$.

ⓓ $(A \cup C) \cap [(A \cap B) \cup (C' \cap B')] = A \cap B$.

A) L.H.S.

$\Rightarrow (A \cup C) \cap [(A \cap B) \cup (C' \cap B')]$

$\Rightarrow (A \cup C) \cap ((A \cup C') \cap B)$

$\Rightarrow ((A \cup C) \cap (A \cup C')) \cap B$

$\Rightarrow (A \cup (C \cap C')) \cap B$

$\Rightarrow (A \cup \phi) \cap B$

$\Rightarrow A \cap B$

$\Rightarrow$ R.H.S

5) Prove the following Identities.

(a) $A-(B-C) = (A-B) \cup (A \cap C)$.

A) L.H.S, $A-(B-C) = A-(B \cap C')$

$= A \cap (B \cap C')'$

$= A \cap (B' \cup C)$

$= (A \cap B') \cup (A \cap C)$

$= (A-B) \cup (A \cap C)$.

$=$ R.H.S

(b) $A \cup (B-C) = (A \cup B) - (C-A)$.

A) $A \cup (B-C) = A \cup (B \cap C')$

$= (A \cup B) \cap (A \cup C')$

$= (A \cup B) \cap (A' \cap C)'$

$= A \cup B \cap (C \cap A')'$

$= A \cup B \cap (C-A)'$

$= (A \cup B) - (C-A)$.

(c) $A \cap (B-C) = (A \cap B) - (A \cap C)$.

A) consider R.H.S

$= (A \cap B) - (A \cap C) = (A \cap B) \cap (A \cap C)'$

$= (A \cap B) \cap (A' \cup C')$

$= (A \cap B \cap A') \cup (A \cap B \cap C')$.

$= ((A \cap A') \cap B) \cup (A \cap B \cap C')$

$= (\phi \cap B) \cup (A \cap B \cap C')$

$= \phi \cup (A \cap B \cap C')$

$= (A \cap B \cap C')$

$= A \cap (B-C)$

$=$ L.H.S

Hence proved.

(d) $(A-B)-C = A-(B\cup C) = (A-C)-B$.

A) consider $(A-B)-C = (A\cap B')\cap C'$

$= A\cap(B'\cap C')$

$= A\cap(B\cup C)'$

$= A-(B\cup C) \longrightarrow (1)$

Consider $(A-C)-B = (A-C)\cap B'$

$= (A\cap C')\cap B'$

$= A\cap(C'\cap B')$

$= A\cap(B'\cap C')$

$= A-(B\cup C) \longrightarrow (2)$

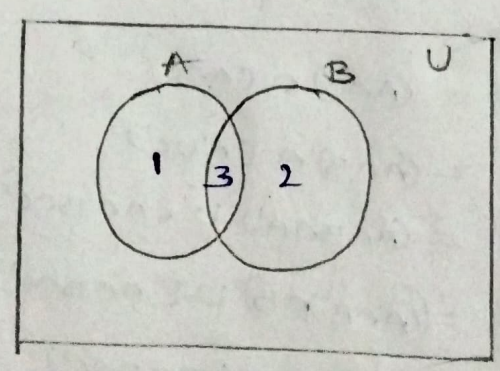
From (1) & (2)

$\therefore (A-B)-C = A-(B\cup C) = (A-C)-B$.

5] Write the following sets as disjoint union:

(a) $A\cup B'$

A)

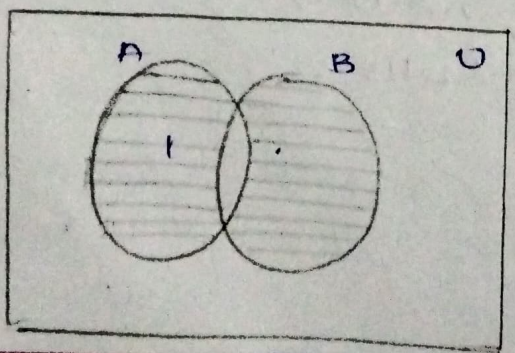


$A\cup B = (A-B)\cup(B-A)\cup(A\cap B)$.

Here, $A-B$, $B-A$, $A\cap B$ all are Disjoint sets

(b) $A\oplus B'$

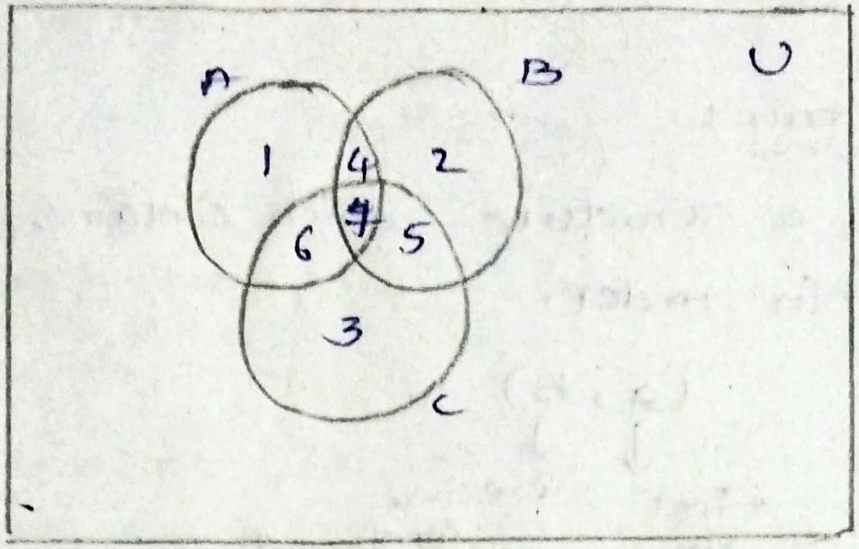
A)



$A\oplus B = (A-B)\cup(B-A)$

(c) $A \cup B \cup C$.

A)



$$A \cup B \cup C = [A - (B \cup C)] \cup [B - (A \cup C)] \cup [C - (A \cup B)].$$

$$\cup [(A \cap B) - (A \cap B \cap C)] \cup [(B \cap C) - (A \cap B \cap C)]$$

$$\cup [(A \cap C) - (A \cap B \cap C)] \cup [(A \cap B \cap C)].$$

*

<u>Second way</u> :-

$$A \cup B \cup C = (A \cap B' \cap C') \cup (A' \cap B \cap C') \cup (A' \cap B' \cap C) \cup$$

$$(A \cap B \cap C') \cup (A' \cap B \cap C) \cup (A \cap B' \cap C) \cup$$

$$(A \cap B \cap C).$$

Module - 3.R :- RELATIONS.

* Ordered pair :-

→ It is a structure which contains two elements fixed in order.

$$(a, b)$$

a → First component, b → second component.

→ If its corresponding components are equal then two ordered pairs are equal.

$$(a,b) = (c,d)$$

$\iff a = c$ and $b = d$.

otherwise $(a,b) \neq (c,d)$.

* Cartesian product :-

→ It is denoted by $A \times B$. (read as A cross B)

→ $A \times B$ is a set which contains all ordered pairs of A and B.

$$A \times B = \{(a,b) \mid a \in A,\ b \in B\}.$$

Ex:-

Let $A = \{1, 2\}$, $B = \{a, b\}$.

$A \times B = \{(1,a), (1,b), (2,a), (2,b)\}$

$B \times A = \{(a,1), (b,1), (a,2), (b,2)\}$.

A	B	Ordered pair
1	a	(1,a)
	b	(1,b)
2	a	(2,a)
	b	(2,b)

Note:-

① $\boxed{A \times B \neq B \times A}$

② $|A \times B| = 4$ [cardinality]

$|B \times A| = 4$

$\therefore \boxed{|A \times B| = |B \times A|}$

③ $|A \times B| = 4 = 2 \times 2 = |A| \cdot |B|$

$\therefore \boxed{|A \times B| = |A| \cdot |B|}$

④ Cartesian product of Three sets:-

→ It is denoted by '$A \times B \times C$'.

→ It contains all ordered pairs. from 3 sets.

$$A \times B \times C = \{(a, b, c)\}$$

$$\therefore A \times B \times C = \{\underbrace{(a,b,c)}_{\text{3 tuple}} \mid a \in A, b \in B, c \in C\}.$$

Note:-

⑤ $A_1 \times A_2 \times ___ \times A_n = \{\underbrace{(a_1, a_2, ___ a_n)}_{\text{n-tuple}} \mid a_i \in A_i, 1 \leq i \leq n\}$

⑥ $A \times A = \{(a,b) \mid a, b \in A\}$

$\therefore \boxed{A \times A = A^2}$

⑦ $A \times A \times A = A^3 = \{(a,b,c) \mid a, b, c\}$.

⑧ $A \times A \times A ___ \times A = A^n = \{(a_1, a_2, ___, a_n) \mid a_i \in A\}$.

⑨ $|A_1 \times A_2 \times ___ \times A_n| = |A_1| \cdot |A_2| \cdot ___ |A_n|$.

⑩ $A \times (B \times C) \neq (A \times B) \times C$.

(ii) cartesian product doesnot follow Associative Law.

$$\boxed{A \times (B \times C) \neq (A \times B) \times C}$$

consider R.H.S

$$(A \times B) \times C = \{(x, c) / x \in A \times B, c \in C\}.$$

$$= \{((a,b), c) / x = (a,b) \in A \times B, c \in C\}.$$

Here $((a,b),c)$ stands for (a,b,c).

$\therefore (A \times B) \times C = A \times B \times C$.

order triplet.

Consider L.H.S

$$A \times (B \times C) = \{(a, x) / x \in B \times C, a \in A\}$$

$$= \{a, (b,c) / a \in A, b \in B, c \in C\}.$$

$$= (a, (b,c)) \neq (a,b,c).$$

ordered pair.

$\therefore$ cartesian product doesnot follow. Associative law

* Relation (or) Binary Relation :-

→ 'Relation' is a subset of Cartesian product

→ Let A, B are two non-empty sets, if $R \subseteq A \times B$ then R is called a 'Relation from A to B'.

(or)

→ 'Relation' is a set which contains ordered pairs in which first element comes from A and second element comes from B and the set is relation from A to B.

$$R = \{(a,b) / a \in A, b \in B\}.$$

→ If $R \subseteq A \times A$;

'R is relation from A to A' (or) 'on A'.

→ $A_1, A_2, _____, A_n$.

$R \subseteq A_1 \times A_2 \times _____ \times A_n$.

∴ R is a n-ary Relation. [n-terms].

→ If "$(a,b) \in R$" then "aRb" i.e "a is related to b" (or) "a is in relation R to b".

→ No. of subsets of A is "$2^{|A|}$".

No. of subsets of $A \times B$ is "$2^{|A \times B|} = 2^{|A| \cdot |B|}$".

* P-1 :-

Write down all relations from $A = \{1, 2\}$ to $B = \{a, b\}$.

A) Cardinality of A is 2, B is 2.

No. of subsets $= 2^{|A \times B|} = 2^{|A| \cdot |B|} = 2^{2 \times 2} = 2^4 = 16$.

Now, $A \times B = \{(1,a), (1,b), (2,a), (2,b)\}$.

Relations are as follows:-

$R_1 = \phi$ [void relation] (Trivial relation).

$R_2 = \{(1,a)\}$

$R_3 = \{(1,b)\}$

$R_4 = \{(2,a)\}$

$R_5 = \{(2,b)\}$

$R_6 = \{(1,a), (1,b)\}$

$R_7 = \{(1,a), (2,a)\}$

$R_8 = \{(1,a), (2,b)\}$.

$R_9 = \{(1,b), (2,a)\}$.

$R_{10} = \{(1,b), (2,b)\}$.

$R_{11} = \{(2,a), (2,b)\}$

$R_{12} = \{(1,a), (1,b), (2,a)\}$.

$R_{13} = \{(1,a), (1,b), (2,b)\}$

$R_{14} = \{(1,a), (2,a), (2,b)\}$

$R_{15} = \{(1,b), (2,a), (2,b)\}$

$R_{16} = \{(1,a), (1,b), (2,a), (2,b)\}$.

(or)

$R_{16} = A \times B$. [Trivial relation (or) Universal relation]

* Exercise :-

7) (a) How many relations are there on the set $\{a,b,c,d\}$

A) $A = \{a,b,c,d\}$.

$|A| = 4$.

$\therefore$ Relations $= 2^{|A \times A|} = 2^{|A| \cdot |A|} = 2^{4 \cdot 4} = 2^{16}$.

(b) How many relations are there on the set $\{a,b,c,d\}$ that contain the pair (a,a)?

A) $\therefore \frac{2^{16}}{2} = 2^{15}$.

* Special Kinds of Relations :-

→ Let A, B are two non-empty sets.

① Void Relation :-

→ clearly 'ϕ' is a relation from 'A' to 'B' and 'ϕ' is also a relation on 'A'. That relation 'ϕ' is called void Relation.

$$\boxed{A \times \phi = \phi = \phi \times A}$$

② Universal Relation :-

→ We have '$A \times B$' is a relation from 'A' to 'B' and '$A \times A$' is a relation on 'A'.

Then these two relations '$A \times B$' and '$A \times A$' are universal relations.

③ Identity Relation :-

→ Let 'R' is a relation on 'A' and that relation 'R' contains all identical pairs. is called 'Identity Relation' (I_A).

$R = \{(a,a) / a \in A\}$.

Ex:- $A = \{1, 2, 3\}$

$R = \{(1,1), (2,2), (3,3)\}$.

→ Another name for Identical relation is 'Diagonal relation'.

④ converse of a relation:- (or) Inverse Relation:-

→ Let 'R' be given relation from A to B and

$R = \{(a,b) / a \in A, b \in B\}$.

R^{-1} (or) $\bar{R} = \{(b,a) / b \in B, a \in A\}$.

→ R^{-1} is a relation from B to A.

Ex:- $R_7 = \{(1,a), (2,b)\}$

$\bar{R_7} = \{(a,1), (a,2)\}$

⑤ complement of a relation:-

→ It is denoted by R' (or) R^c.

→ complement of given relation is equal to universal relation − given relation.

$\therefore R' = A \times B - R$.

Ex:- $R_7' = A \times B - R_7$

$= \{(1,b), (2,b)\}$.

* operations:-

① union:-

$R \cup S = \{(a,b) / (a,b) \in R \vee (a,b) \in S\}$.

② Intersections

$R \cap S = \{(a,b) | (a,b) \in R \wedge (a,b) \in S\}$

③ Difference:-

$R - S = \{(a,b) / (a,b) \in R \wedge (a,b) \notin S\}. = R \cap S'$.

④ Boolean sum:- $R \oplus S = (R \cup S) - (R \cap S)$.

* Theorem statements:-

(1) $A \times (B \cup C) = (A \times B) \cup (A \times C)$.

(2) $A \times (B \cap C) = (A \times B) \cap (A \times C)$

(3) $(A \cup B) \times C = (A \times C) \cup (B \times C)$.

(4) $(A \cap B) \times C = (A \times C) \cap (B \times C)$.

(5) $(A \times (B - C)) = (A \times B) - (A \times C)$

(6). $A \times (B - C)$

* Theorem:-

→ Let 'R' be a relation from 'A' to 'B'.

(i) $(\bar{R}) = R$ (or) $(R^{-1})^{-1} = R$

→ Let 'R' and 'S' be a relation from A to B.

(ii) If $R = S$ iff $\bar{R} = \bar{S}$ (or) $R^{-1} = S^{-1}$.

(iii) $R \subseteq S$ iff $R^{-1} \subseteq S^{-1}$ (or) $\bar{R} \subseteq \bar{S}$.

(iv) $\overline{R \cup S} = \bar{R} \cup \bar{S}$.

(v) $\overline{R \cap S} = \bar{R} \cap \bar{S}$.

* Domain and Range of a Relation:-

→ Let 'R' be a relation from A to B.

$R = \{(a,b) / a \in A, b \in B\}$.

→ $Dom(R)$ (or) $D(R) = \{a \in A / (a,b) \in R\}$.

$Range(R)$ (or) $R(R) = \{b \in B / (a,b) \in R\}$.

$\therefore D(R) \subseteq A$, $R(R) \subseteq B$.

$R_5 = \{(2,b)\}$

$R_8 = \{(1,a), (2,b)\}$

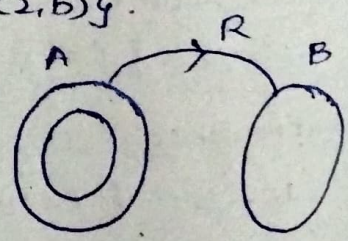
$R_{16} = A \times B = \{(1,a), (1,b), (2,a), (2,b)\}$.

Here, $D(R_5) = \{2\} \subseteq A$.

$R(R_5) = \{b\} \subseteq B$.

$R(R_8) = \{a,b\} \subseteq B$.

$D(R_8) = \{1,2\} \subseteq A$.



→ Domain and Range of Identity Relation is same set.

$A = \{1, 2, 3\}$

Ex:- $I_A = \{(1,1), (2,2), (3,3)\}$.

$\therefore D(I_A) = R(I_A) = A$.

→ Domain and Range of void Relation is ϕ.

→ Relation is based on given property.

Ex:- $A = \{1, 2\}$, $B = \{1, 2, 3\}$.

$R = \leq$ less than or equal.

$a \leq b \Rightarrow aRb. \Rightarrow (a,b) \in R$.

$\therefore R = \{(1,1), (1,2), (1,3), (2,2), (2,3)\}$.

* Exercise:-

1] List the ordered pairs in the Relation R from $A = \{0,1,2,3,4\}$ to $B = \{0,1,2,3\}$. where $(a,b) \in R$ if and only if

(a) $a = b$.

A) aR_1b iff $a = b$.

Then $R_1 = \{(0,0), (1,1), (2,2), (3,3)\}$.

Here R_1 is a equality Relation from A to B.

(b) $a+b = 4$.

A) aR_2b iff $a+b = 4$.

Then $R_2 = \{(1,3), (2,2), (3,1), (4,0)\}$.

(c) $a > b$.

A) i.e, aR_3b iff $a > b$

Then $R_3 = \{(1,0), (2,1), (3,2), (4,0), (3,1), (3,0), (2,0), (4,1), (4,2); (4,3)\}$.

(d) $a|b$.

A) i.e aR_4b iff $a|b$ $\boxed{\frac{b}{a} = a|b.}$

Then $R_4 = \{(1,0), (1,1), (1,2), (1,3), (2,0), (2,2), (3,0), (3,3), (4,0)\}$.

(e) $\gcd(a,b) = 1$.

A) That is $a R_5 b$ (or) $a, b \in R_5$ iff $G.C.D(a,b) = 1$.

Then $R_5 = \{(0,1), (1,0), (1,1), (1,2), (1,3), (2,1), (2,3), (3,1), (3,2), (4,1), (4,3)\}$.

(f) $\text{lcm}(a,b) = 2$.

A) That is $a R_6 b$ iff $\text{l.c.m}(a,b) = 2$.

Then $R_6 = \{(1,2), (2,1), (2,2)\}$

*

Relation R	aRb	converse Relation $\bar{R}$ or R^{-1}	$bR^{-1}a$ $(b,a) \in R^{-1}$
$\leq$	$a \leq b$	$\geq$	$b \geq a$
$\geq$	$a \geq b$	$\leq$	$b \leq a$
$=$	$a = b$	$=$	$b = a$
$<$	$a < b$	$>$	$b > a$
$>$	$a > b$	$<$	$b < a$
$\subseteq$	$A \subseteq B$	$\supseteq$	$B \supseteq A$
$\supseteq$	$A \supseteq B$	$\subseteq$	$B \subseteq A$
$\subset$	$A \subset B$	$\supset$	$B \supset A$
$\supset$	$A \supset B$.	$\subset$	$B \subset A$

* <u>Composition (or) product of Relations</u> :- $[R \circ S]$.

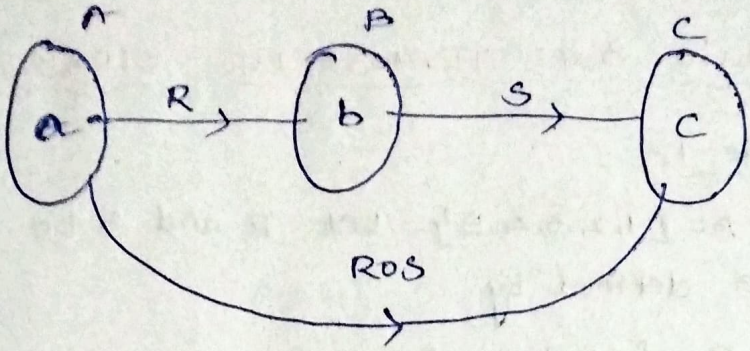
→ Let A, B, C are three non-empty sets.

$R : A \to B$

$S : B \to C$ $\qquad R \circ S : A \to C$

Now, $R \circ S = \{(a,c) / a \in A, c \in C, (a,b) \in R$ and $(b,c) \in S\}$.

If $(a,b) \in R \wedge (b,c) \in S \Longrightarrow (a,c) \in R \circ S$.



Note :-

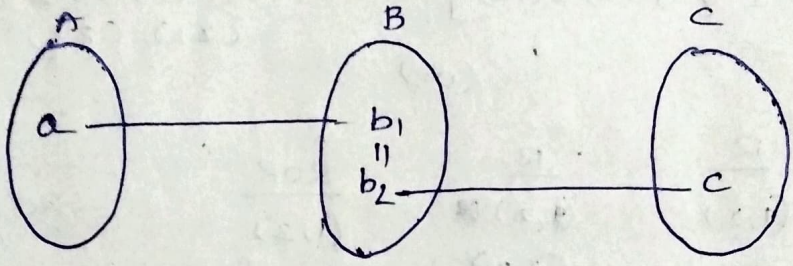
→ If $R(R) \cap D(S) \neq \phi$ then $ROS \neq \phi$.

→ If $R(R) \cap D(S) = \phi$ then $ROS = \phi$.

→ $Dom(ROS) \subseteq D(R)$.

$Range(ROS) \subseteq R(S)$.

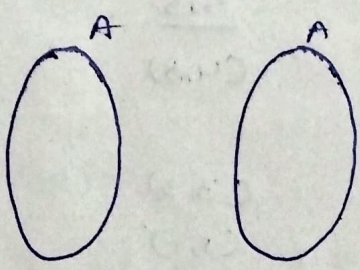
→



$b_1 = b_2 \longrightarrow ROS = \{(a, c)\}$

$b_1 \neq b_2$.

→



R on A $\quad A \to \cancel{A}$

R on A $\quad \cancel{A} \to A$.

$\therefore$ ROR on A.

ROR on A.

→ 11'y ROR on $A \to \cancel{A}$

R on $\cancel{A} \to A$

$\therefore (ROR)OR$ on A

→ P, S, R

$Po(SOR) = (POS)OR$.

→ $P = S = R$

$= R^3$.

$RO(ROR) = (ROR)OR = ROROR$

* Properties of Relation :-

① Reflexive Relation :- having Identical pairs.

→ $\forall a \ (a \in A \rightarrow a R a)$

(or)

$\forall a \ (a \in A \rightarrow (a,a) \in R)$

(or)

If $a \in A$ then aRa (or) $(a,a) \in R$.

→ 'Every Identity relation' is a 'Reflexive relation'.

→ 'ϕ' is not a reflexive relation.

→ Every 'universal relation' is a Reflexive relation.

→ '$=$', '$\leq$', '$\geq$' also reflexive relations over on 'N', 'Z', 'Q', 'R'.

→ Set Inclusion/subset '$\subseteq$' is also a Reflexive relation over powerset. 'P(U)'.

→ '$\parallel$' is a reflexive relation over straightlines but '$\perp$' is not.

② Irreflexive relation :-

→ If R is a reflexive relation on A iff $I_A \subseteq R$.

→ $\forall a \ (a \in R \rightarrow a \not R a ; (a,a) \notin R)$ is a Irreflexive relation.

→ A Relation doesnot contain any Identical pair then it is called 'Irreflexive relation.'

→ R is an irreflexive relation iff R doesnot contain any identical pair.

$I_A \subseteq R^c$.

→ $<, >$ are Irreflexive. and ϕ also.

③ Non-Reflexive relation:-

→ If R is neither Reflexive nor irreflexive then it is Non-reflexive relation.

Ex:- $A = \{1, 2, 3\}$

$R = \{(1,1), (2,3), (3,2)\}$

→ $<, >$ are Irreflexive.

④ Symmetric relation:-

→ $\forall a \forall b \, (aRb \rightarrow bRa)$

(or)

$\forall a \forall b \, ((a,b) \in R \rightarrow (b,a) \in R)$

(or)

If aRb then bRa $\forall a, b \in A$.

Ex: $A = \{1, 2, 3\}$

$R = \{(1,2), (1,3), (2,1), (2,3), (3,2), (3,1)\}$.

consider, $R = \{(1,2), (1,3), (2,1), (2,3), (3,2), (2,2)\}$ ×

Symmetric relations are $(1,2)$, $(2,1)$
$(2,3)$, $(3,2)$

→ By vaccous proof, 'ϕ' is a symmetrical relation.

→ $\perp$ on $A = \{$set of straight lines$\}$. is symmetric relation
$a \perp b$. then $b \perp a$.

→ Equality also a symmetric relation.
$a = b$ then $b = a$.

→ If $R = R^{-1}$ then symmetric.

→ 'Brother' is symmetric over all males.
'Brother relation' is not symmetric over human-beings

→ Identity & universal relations are symmetric & reflexive.

⑤ Asymmetric Relation:-

$\forall a \forall b \; (aRb \rightarrow b \not R a)$.

(or)

If $(a,b) \in R$ then $(b,a) \notin R \;\; \forall a, b \in A$.

→ If a relation containing atleast one identical pair then that is not Asymmetric Relation.

→ R is a Asymmetric Relation on A iff $R \cap R^{-1} = \phi$.

⑥ Antisymmetric Relation :-

→ Let $a, b \in A$,

$\forall a \forall b$, If $(aRb) \wedge (bRa) \longrightarrow (a=b)$

(or)

$(a,b) \in R \wedge (b,a) \in R \longrightarrow a=b$.

(or)

If $(a,b) \in R$ and $(b,a) \in R$ then $a=b$.

Note:-

① Not symmetric doesrot mean Asymmetric or Antisymmetric.

② Identital relation is both symmetrical & antisymmetric

→ Ex:- $A = \{1, 2, 3\}$

I_A (or) $\Delta_A = \{(1,1), (2,2), (3,3)\}$.

③ Every universal relation is not an Antisymmetric relation.

④ $>, <, \leq, \geq, =$ are Antisymmetric Relations.

⑤ $\subseteq, \subset$ are Antisymmetric.

⑥ '$\perp$' is not an Antisymmetric. and '$\parallel$' also not an Antisymmetric.

Ex:-

'Father relation' over Humanbeings is not Antisymmetric.

7) Transitive Relation :-

→ Let $a, b, c \in A$.

$\forall a \forall b \forall c \; ((aRb) \wedge (bRc) \rightarrow (aRc))$

① I_A (or) Δ_A [Identical pair] is Transitive Relation.

Ex:- $(1,1) \wedge (1,1) \rightarrow (1,1)$

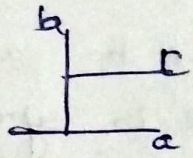
② '=' is a Transitive Relation.

③ $\leq, \geq, <, >$ all are transitive.

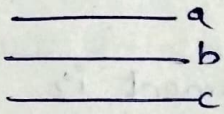
④ $\subseteq, \subset$ is also a transitive relation over power set.

⑤ $\perp$ also not transitive.

Ex:- $a \perp b$, $b \perp c$ $a \not\perp c$



⑥ $\parallel$ is Transitive.



⑦ Universal relation is Transitive

⑧ ϕ is Transitive relation.

8) Compatibility Relation :-

→ A Relation which is Both Reflexive and symmetric is called Compatibility Relation.

9) Equivalence Relation :-

→ A Relation R is said to be Equivalence Relation if it is Reflexive, symmetric & Transitive.

10) Partial order Relation :-

→ A Relation R is said to be partial order relation if R is reflexive, antisymmetric and Transitive.

* conclusions:-

① 'I_A (or) 'Δ_A' is an equivalence relation on A. and I_A is smallest equivalence relation on A.

② 'Universal Relation' ($A \times A$) is also an equivalence relation on A. & ($A \times A$) is Largest equivalence relation on A.

③ 'I_A' is both 'equivalence' & 'partial ordering Relation'

④ 'ϕ' is neither 'equivalence' nor 'partial Order Relation'. because ϕ is not reflexive.

* P-7:-

What is wrong with the following argument?
Let A be a set with R as a relation on A. If R is symmetric and transitive, then R is reflexive?

A) Given Argument is,

If R is symmetric and transitive then R is reflexive.

Let $A = \{1, 2, 3\}$.

$R_1 = \{(1,2), (2,1), (1,1), (2,2), (3,3)\}$.

$R_2 = \{(1,2), (2,1), (1,1), (2,2)\}$

clearly, R_1 and R_2 are both symmetric and Transitive.

But R_1 is reflexive whereas R_2 is not reflexive.

Here Hypothesis is True
conclusion is False.

∴ The given Argument is False.

* congruent modulo 'm' :-

→ Let 'm' be a positive integer, $m \in Z^+$ and $m > 1$.

Let $x, y \in Z$.

If x is said to be congruent to y module 'm' is denoted by.

$$\boxed{x \equiv y \pmod{m}}$$

if $m \mid x-y$ then $x \equiv y \pmod{m}$. i.e, $x-y$ = Integral multiple of m.

Ex:-

If $7 \mid x-y$ then $x \equiv y \pmod{7}$

If $13 \mid x-y$ then $x \equiv y \pmod{13}$

If $5 \mid a-b$ then $a \equiv b \pmod{5}$.

If $5 \mid 3-8$ then $3 \equiv 8 \pmod{5}$.

* Example-2 :-

The relation R on, the set of integers, defined by $R = \{(x,y) \mid x \in Z \wedge y \in Z \wedge (x \equiv y \pmod{m})\}$ is equivalent rel^n.

A) $R = \{(x,y) \mid x \in Z \wedge y \in Z \wedge x \equiv y \pmod{m}\}$ $m \in Z^+$ and $m > 1$

That $(x,y) \in R$ iff $x \equiv y \pmod{m}$ (or) xRy iff $x \equiv y \pmod{m}$

claim:- To prove that R is equivalent

(i) To prove that R is reflexive.

Let for any $x \in Z$,

we have $x - x = 0$ and $m \mid 0$

$\Rightarrow m \mid x - x$

$\Rightarrow x \equiv x \pmod{m}$ $\Rightarrow xRx \ \forall x \in Z$.

$\therefore$ R is reflexive.

(ii) To prove R is symmetric

Let $(x,y) \in Z$.

Let xRy

$\Rightarrow x \equiv y \pmod{m}$

$\Rightarrow m \mid x-y$

$\Rightarrow x-y = mk, \ k \in Z$.

$\Rightarrow y-x = -mk = m(-k)$

$\Rightarrow y-x = mn$, $n = -k \in Z$ $\Rightarrow m \mid y-x$

$\Rightarrow y \equiv x \bmod (m)$ $\Rightarrow yRx$.

(iii) To prove R is Transitive.

Let $x, y, z \in Z$

consider xRy and yRz $\forall x,y,z \in Z$

$x \equiv y \pmod{m}$ & $y \equiv z \pmod{m}$

$\Rightarrow m \mid x-y$ & $\Rightarrow m \mid y-z$

$\Rightarrow x-y = mp$ & $\Rightarrow y-z = mq$ $(p,q \in Z)$

$\Rightarrow x-y+y-z = mp+mq = m(p+q)$

$\Rightarrow x-z = mn$ $(n \in Z)$.

$\Rightarrow m \mid x-z$ $\Rightarrow x \equiv z \bmod (m)$

$\therefore$ It is Transitive.

* P-2 :-

Let $A = \{1,2,3\}$ and $B = \{1,2,3,4\}$

If $R_1 = \{(1,1),(2,2),(3,3)\}$ and $R_2 = \{(1,1),(1,2),(1,3),(1,4)\}$

then find $R_1 \cup R_2$, $R_1 \cap R_2$, R_1', $R_2 - R_1$?

A) $R_1 \cup R_2 = \{(x,y) \mid (x,y) \in R_1 \vee (x,y) \in R_2\}$.

$= \{(1,1),(2,2),(3,3),(1,2),(1,3),(1,4)\}$.

$R_1 \cap R_2 = \{(x,y) \mid (x,y) \in R_1 \wedge (x,y) \in R_2\}$

$= \{(1,1)\}$.

$R_1' = (A \times B) - R_1$

$= \{(1,2)\ (1,3),(2,1),(2,3),(3,1),(3,2),(3,4),(2,4),(1,4)\}$

$R_2 - R_1 = \{(x,y) \mid (x,y) \in R_2 \wedge (x,y) \notin R_1\}$. (or) $R_2 \cap R_1'$

$= \{(1,2),(1,3),(1,4)\}$.

* P-3 :-

Let $A = \{1,2,3,\ldots,7\}$ and

$R = \{(x,y) \mid x \in A \wedge y \in A \wedge (x \equiv y \pmod 3)\}$

$S = \{(x,y) \mid x \in A \wedge y \in A \wedge (x \equiv y \pmod 4)\}$

Find $R \cap S$, $R \cup S$, $R - S$?

A) $R = \{(1,1),(1,4),(1,7),(2,2),(2,5),(3,3),(3,6),(4,1),(4,4),(4,7),(5,2),(5,5),(6,3),(6,6),(7,1),(7,4),(7,7)\}$.

$S = \{(1,5),(1,1),(2,2),(2,6),(3,3),(3,7),(4,4),(5,1),(5,5),(6,2),(6,6),(7,3),(7,7)\}$.

$R \cap S = \{(1,1),(2,2),(3,3),(4,4),(5,5),(6,6),(7,7)\}$

$R \cup S = \{(1,1),(1,4),(1,5),(1,7),(2,2),(2,5),(2,6),(3,3),(3,6),(3,7),(4,1),(4,4),(4,7),(5,1),(5,2),(5,5),(6,2),(6,3),(6,6),(7,1),(7,3),(7,4),(7,7)\}$.

$R - S = \{(1,4),(1,7),(2,5),(3,6),(4,1),(4,7),(5,2),(6,3),(7,1),(7,4)\}$.

$S - R = \{(1,5),(2,6),(3,7),(5,1),(6,2),(7,3)\}$.

* Exercise:

2] Let $A=\{a,b,c\}$ and $B=\{1,2,3,4\}$.

If $R=\{(a,2),(a,4),(b,2),(b,3),(c,1),(c,4)\}$

$S=\{(a,3),(b,1),(b,2),(b,4),(c,4),(c,1)\}$.

Then find $R\cap S$, $R\cup S$, $R-S$, $S-R$?

A) $R\cap S=\{(x,y)\mid (x,y)\in R \wedge (x,y)\in S\}$.

$=\{(b,2),(c,1),(c,4)\}$.

$R\cup S=\{(x,y)\mid (x,y)\in R \vee (x,y)\in S\}$

$=\{(a,2),(a,3),(a,4),(b,1),(b,2),(b,3),(b,4),(c,1),(c,4)\}$.

$R-S=\{(x,y)\mid (x,y)\in R \wedge (x,y)\notin S\}$

$=\{(a,2),(a,4),(b,3)\}$

$S-R=\{(x,y)\mid (x,y)\in S \wedge (x,y)\notin R\}$

$=\{(a,3),(b,1),(b,4)\}$.

3] Let $A=\{1,2,3,4\}$ and $B=\{a,b,c\}$.

If $R=\{(1,a),(2,b),(3,c),(4,a)\}$ and

$S=\{(1,b),(2,c),(3,a),(4,a)\}$.

Then find $R\cup S$, $R\cap S$, $R-S$, $S-R$?

A) $R\cup S=\{(1,a),(1,b),(2,b),(2,c),(3,a),(3,c),(4,a)\}$

$R\cap S=\{(4,a)\}$.

$R-S=\{(1,a),(2,b),(3,c)\}$.

$S-R=\{(1,b),(2,c),(3,a)\}$.

4] Let $A=\{2,3,5,6,7\}$ and $B=\{3,4,10,12,14,15\}$. Let R and S be relations from A to B defined by.

For all $a\in A$, $b\in B$.

(i) aRb iff $a|b$ (ii) aSb iff $a\geq b$.

then find $R\cup S$ and $R\cap S$?

A) $R = \{ \cancel{(3,3)}, (2,4), (2,10), (2,12), (2,14), (3,3), (3,12), (3,15), (5,10), (5,15), (6,12), (7,14) \}$.

$S = \{ (3,3), (5,4), (5,3), (6,3), (6,4), (7,3), (7,4) \}$.

$R \cup S = \{ (2,4), (2,10), (2,12), (2,14), (3,3), (3,12), (3,15), (5,10), (5,15), (6,12), (7,14) \}$.

$R \cap S = \{ (3,3) \}$.

5) consider the following relation on Z:

$R_1 = \{ (a,b) \mid a \leq b \}$.

$R_2 = \{ (a,b) \mid a > b \}$

$R_3 = \{ (a,b) \mid a = b \text{ or } a = -b \}$

$R_4 = \{ (a,b) \mid a = b \}$.

$R_5 = \{ (a,b) \mid a = b + 1 \}$

$R_6 = \{ (a,b) \mid a + b \leq 3 \}$

Decide which of these are reflexive, symmetric, antisymmetric, transitive?

A)

Relation	Reflexive	Symmetric	Anti-symmetric	Transitive	Equivalence	Partial Order
R_1	✓	✗	✓	✓	✗	✓
R_2	✗	✗	✓	✓	✗	✗
R_3	✓	✓	✗	✓	✓	✗
R_4	✓	✓	✓	✓	✓	✓
R_5	✗	✗	✓	✗	✗	✗
R_6	✗	✓	✗	✗	✗	✗

R4-04-2023

MODULE - 3.3 :- EQUIVALENCE RELATIONS AND COMPATIBILITY RELATIONS

* Boolean Matrix (or) (0, 1) Matrix :-

→ A matrix with entries are bits (0 or 1) is called "Boolean Matrix".

Ex:- $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}_{2\times 3}$

→ The arithmetic operations over bits are boolean operations.

$\vee \longrightarrow$ Join (or) sum

$\wedge \longrightarrow$ meet (or) product

① Meet ($\wedge$) :-

$$b_1 \wedge b_2 = \begin{cases} 1, & \text{if } b_1 = b_2 = 1 \\ 0, & \text{otherwise} \end{cases}$$

(or)

$$b_1 \wedge b_2 = \text{Min}\{b_1, b_2\}.$$

Ex:- $1 \wedge 1 = 1$

$1 \wedge 0 = 0$

$0 \wedge 1 = 0$

$0 \wedge 0 = 0$

② Join ($\vee$) :-

$$b_1 \vee b_2 = \begin{cases} 1, & \text{if } b_1 = 1 \text{ or } b_2 = 1 \\ 0, & \text{otherwise} \end{cases}$$

(or)

$$b_1 \vee b_2 = \text{Max}(b_1, b_2).$$

Ex:- $1 \vee 1 = 1$

$1 \vee 0 = 1$

$0 \vee 1 = 1$

$0 \vee 0 = 0.$

* Meet and Join of Boolean Matrices:-

$A = [a_{ij}]_{m\times n}$; $B = [b_{ij}]_{m\times n}$

$A \vee B = [a_{ij} \vee b_{ij}]_{m\times n}$

$A \wedge B = [a_{ij} \wedge b_{ij}]_{m\times n}$

* Example-2 :- [Module-4]

Let $M = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ and $N = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$ then find $M \vee N$ and $M \wedge N$?

A) $M \vee N = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}_{2\times 3} \vee \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}_{2\times 3} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}_{2\times 3}$

$M \wedge N = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}_{2\times 3} \wedge \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}_{2\times 3} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}_{2\times 3}$

* Boolean product:-

→ It behaves like Matrix Multiplication.

$A = [a_{ij}]_{m\times k}$, $b = [b_{ij}]_{k\times n}$

→ $A \circ B$ (or) $A \odot B$ is the Notation.

$A \circ B = [c_{ij}]_{m\times n}$

$[c_{ij}] = (a_{i1} \wedge b_{1j}) \vee (a_{i2} \wedge b_{2j}) \vee \text{——} (a_{ik} \wedge b_{kj})$.

* Ex-3:-

Let $P = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ and $Q = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 0 \end{bmatrix}$, then find $P \circ Q$ and $Q \circ P$, if it is defined?

A) $P \circ Q = \begin{bmatrix} (1\wedge 1) \vee (0\wedge 1) \vee (1\wedge 0) & (1\wedge 0) \vee (0\wedge 1) \vee (1\wedge 0) \\ (0\wedge 1) \vee (1\wedge 1) \vee (0\wedge 0) & (0\wedge 0) \vee (1\wedge 1) \vee (0\wedge 0) \end{bmatrix}$

$= \begin{bmatrix} 1\vee 0\vee 0 & 0\vee 0\vee 0 \\ 0\vee 1\vee 0 & 0\vee 1\vee 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}_{2\times 2}$.

$$Q \circ P = \begin{bmatrix} (1\wedge 1)\vee(0\wedge 0) & (1\wedge 0)\vee(0\wedge 1) & (1\wedge 1)\vee(0\wedge 0) \\ (1\wedge 1)\vee(0\wedge 0) & (1\wedge 0)\vee(1\wedge 1) & (1\wedge 1)\vee(1\wedge 0) \\ (0\wedge 1)\vee(0\wedge 0) & (0\wedge 0)\vee(0\wedge 1) & (0\wedge 1)\vee(0\wedge 0) \end{bmatrix}$$

$$= \begin{bmatrix} 1\vee 0 & 0\vee 0 & 1\vee 0 \\ 1\vee 0 & 0\vee 1 & 1\vee 0 \\ 0\vee 0 & 0\vee 0 & 0\vee 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}_{3\times 3}$$

* Boolean power of a Boolean Matrix:-

→ Let A is square matrix & Boolean matrix of order 'n'. Then $A \circ A \circ A \circ A _ _ \circ A$ [n-factors] $= A^n$ (or) $A^{[n]}$, $n \in Z^+$.

∴ A^n is called Boolean power of boolean matrix.

* Properties:-

① $A^0 = I_n$

② $A^1 = A$

③ $A^n = A^{n-1} \circ A$.

* Ex-4:-

If $M = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$, find M^2, M^3?

A) $M^2 = M \circ M = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \circ \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

$M^3 = M^2 \circ M = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \circ \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.

* Less than property:-

→ Let $A = [a_{ij}]_{m\times n}$, $B = [b_{ij}]_{m\times n}$

$\boxed{A \le B \text{ iff } a_{ij} \le b_{ij}}$

*Ex: $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$.

A) $A \le B = \begin{bmatrix} 1 \le 1 & 0 \le 1 & 0 \le 0 \\ 0 \le 0 & 1 \le 1 & 0 \le 0 \end{bmatrix} = A \le B$.

* Relation as a Matrix:

→ Let A be a set containing 'm' elements, B be a set containing 'n' elements.

$A \to m$ $\quad$ $B \to n$ $\quad$ $\therefore |A \times B| = mn$.

$|A| = m$ $\quad$ $|B| = n$

→ Let R from A to B, $R \subseteq A \times B$; $m \times n$.

→ If R: A to B

$$\boxed{M_R = [m_{ij}]_{m \times n}}$$

$M_R \to$ matrix of Relation 'R' $(m \times n)$. and it is a boolean matrix.

$$M_R = \begin{bmatrix} m_{11}, & m_{12} & \text{———} & m_{1n} \\ m_{21} & m_{22} & \text{———} & m_{2n} \\ \vdots & \vdots & & \vdots \\ m_{m1} & m_{m2} & \text{———} & m_{mn} \end{bmatrix}$$

*Ex: $A = \{1, 2, 3\}$, $B = \{a, b\}$. & $R = \{(1,a), (2,b), (3,a)\}$ find M_R?

A) $|A| = 3$, $|B| = 2$.

$R = \{(1,a), (2,b), (3,a)\}$.

$$\therefore M_R = \begin{array}{c|cc} A & \multicolumn{2}{c}{B} \\ & a & b \\ \hline 1 & (1,a) & (1,b) \\ 2 & (2,a) & (2,b) \\ 3 & (3,a) & (3,b) \end{array} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

$(1,a) \in R$ so 1.

$(1,b) \notin R$ so 0

$(2,a) \in R$ so 1.

* Ex: $A = \{1, 2, 3\}$. Find M_R?

A) $R = \{(1,2), (2,2), (3,2)\}$.

$$M_R = \begin{bmatrix} (1,1) & (1,2) & (1,3) \\ (2,1) & (2,2) & (2,3) \\ (3,1) & (3,2) & (3,3) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

* Note :-

① If given all diagonal elements are '1' then the relation is reflexive.

② If all the diagonal elements are '0' then the relation is Irreflexive.

③ If some of diagonal elements are '1' and some are '0'. then the relation is Non-reflexive.

④ If the given relation is a symmetric relation then it is symmetric matrix $[M_R = M_R^T]$

~~⑤ If the~~

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & & \\ 0 & & \\ 1 & & \end{bmatrix}$$

Symmetric

⑤ If $(a_i, b_j) \in R$ then $(b_i, a_j) \notin R$.

If $(b_j, a_i) \in R$ then $(a_i, b_j) \notin R$.

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & & \\ 0 & & \\ 0 & & \end{bmatrix}$$

i.e, If $m_{ij} = 1$ then $m_{ji} = 0$.

(or)

If $m_{ij} = 0$ (or) $m_{ji} = 0$.

} Antisymmetric.

⑥ If R is transitive relation on A,

if $\boxed{M_R^2 \leq M_R}$

(or)

$\boxed{M_R^2 + M_R = M_R}$

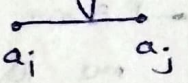
* Relation as a Digraph:-

→ We can show relation pictorally using graph that graph is called 'Digraph'.

→ Every graph contains two components
① 'vertices'/'Nodes'.
② 'Edges'.

→ Every element is a 'vertice' of a graph.

→ connecting two vertices is called a Edge.



→ Graphs are two types
① 'undirected Graph'.
② 'Directed graph / Digraph'.

→ It doesn't matter where we draw vertices. but it is important to draw edge correctly.

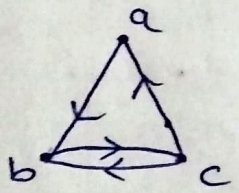
→ Edge need not be a straight line, it maybe a curve.

* Ex-2:-

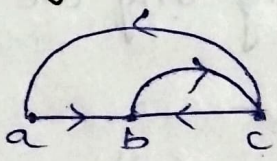
Let $A = \{a, b, c\}$

(ii) $R_2 = \{(a,b), (c,a), (c,b), (b,c)\}$.

A)

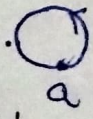


(or)



* Loop:-

→ An arc from a vertex to itself.
(a,a)



* Isolated Node:-

→ Isolated Node is a node which doesnot pass any edge through it.

* Note:-

→ All Identical pairs have loops.

* Types of Relations using Digraph:-

① **Reflexive:-**

→ All Identical pairs have loops (or) Every Identical nodes have loops.

② **Irreflexive:-**

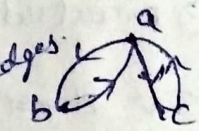
→ If any Identical vertice have no loop.

③ **Non reflexive:-**

→ some Identical nodes have loops & some have no loops.

④ **symmetric:-**

① some vertices have no edges / 2 edges



② some contains same edge with different direction

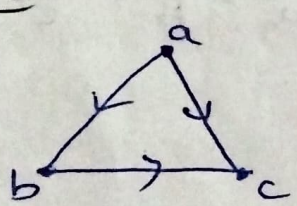
⑤ **Artisymmetric:-**

① some vertices have no edges.

(or)

② presenting only one edge.

⑥ **Transitive:-**



$a \to b$

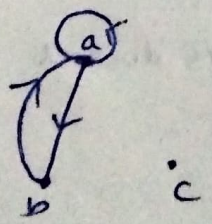
$b \to c$

$a \to c$.

* **Ex-2:-**

(iv) $R_4 = \{(a,a), (a,b), (b,a)\}$.

A)

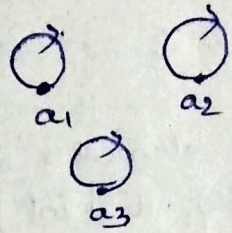


c is Isolated Node here.

* P-1 :-

Determine the property of relations given by following graphs and write corresponding relation matrices?

A) (ii)

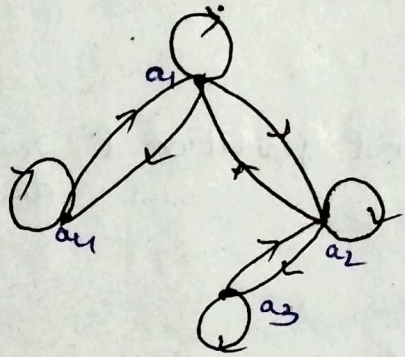


$$M_{R_2} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times 3}$$

All ~~I~~diagonal matrice entries are 1. [Identical].

∴ It is Equivalence.

(iii)



A) $$M_R = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}_{4\times 4}.$$

∴ ~~It is equivalence~~

* Covering and partition of set 'A'.

→ Let A is Non-empty set and

$A_1, A_2, ——, A_m$ are all subsets of A

→ Let $C = \{A_1, A_2, ——, A_m\}$.

we say C [family of sets] is covering of A, if

$$A = \bigcup_{i=1}^{m} A_i = A_1 \cup A_2 \cup A_3 \cup —— \cup A_m.$$

Partition :-

→ Let $P = \{A_1, A_2, \ldots, A_m\}$;

· P is partition of A;

(i) $A = \bigcup_{i=1}^{m} A_i$ [covering of A].

(ii) $A_i \cap A_j = \phi$ [Any two distinct elements are disi
$(i \neq j)$ [Mutually disjoint].

* Ex :-

$A = \{a, b, c\}$

$A_1 = \{a, b\}$ Find A is covering and partition?

$A_2 = \{b, c\}$.

$A_3 = \{c, a\}$.

A) $A_1 \cup A_2 \cup A_3 = A$.

∴ It is covering. But not partition [∵ Not disjoint].

* Blocks of partition :-

→ If P is partition of A.

$P = \{A_1, A_2, \ldots, A_m\}$. then $A_1, A_2, \ldots, A_m$ are called 'Blocks of partition'.

* No. of partitions of Non-empty set with 'n' element

→ The total no. of distinct partitions of A with K (Blocks / components) are equals to $S(n, k)$ [sterling number of second kind] is

$$\boxed{S(n,k) = \frac{1}{k!} \sum_{i=0}^{k} (-1)^i \binom{k}{i} (k-i)^n.}$$

$\binom{k}{i} = {}^kC_i$, n = No. of elements.

K = required blocks.

Note :- $\boxed{S(n, 1) = 1}$ $\boxed{S(n, n) = 1}$.

* Some Important partitions of a universal set :-

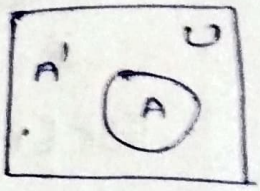
~~case-1~~ :-

→ Let 'U' be the universal set and 'n' be the no. of subsets of 'U' that you have to consider.

case-1 :- (n = 1)

→ Let 'A' be a non empty subset of 'U'.

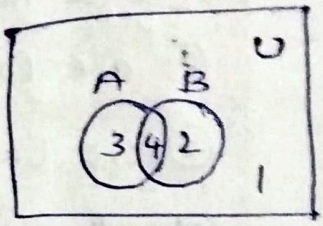
Then $P = \{A, A'\}$ is a partition of 'U'.



Here no. of blocks $= 2^n = 2^1 = 2$.

case-2 :- (n = 2).

→ Let consider two non empty subsets of 'U'. say A, B.



Here $I_0 = 1 = A' \cap B'$
$I_1 = 2 = A' \cap B$
$I_2 = 3 = A \cap B'$
$I_3 = 4 = A \cap B$

} Complete Intersections (or) Minterms

$I = I_0, I_1, I_2, I_3$

$\therefore P = \{I_0, I_1, I_2, I_3\}$

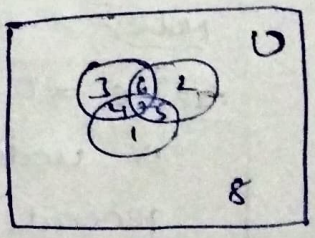
$\therefore$ No. of blocks $= 2^2 = 4$.

case-3 :- (n = 3).

→ Let A, B, C are three non empty subsets of 'U'

$P = \{I_0, I_1, \text{———}, I_{2^3-1}\}$

$P = \{I_0, I_1 \text{———}, I_7\}$.



$\therefore$ No. of blocks $= 2^3 = 8$.

Case-4 :- For (n = m) / case-n :-

→ $P = \{I_0, I_1, \text{———} I_{2^n-1}\}$

$\therefore$ No. of blocks $= 2^n$.

* Equivalence class :-

→ Let R be a equivalence on A and $a \in A$, then equivalence class is denoted by

$[a]_R$ (or) $\bar{a}$ = equivalence class of 'a' under 'R'.

→ $[a]_R = \{ b \in A / a R b \}$ - The set of elements of A which are equivalent to 'a'.

Ex:- $[x]_R$: R-equivalent to x.

$[b]_R$: R-equivalent to b

* Properties of Equivalent class :-

→ Let R be an equivalence relation on a set 'A' and $a, b \in A$.

① $a \in [a]_R$.

② $b \in [a]_R \Rightarrow [a]_R = [b]_R$

③ ~~If~~ aRb iff $[a]_R = [b]_R$.

④ $a \not R b$ iff $[a]_R \neq [b]_R$:

⑤ $[a]_R = [b]_R$ (or) $[a]_R \cap [b]_R = \phi$

Either two equivalent classes are Identical (or) Disjoint.

⑥ The union of all disjoint equivalent classes is equals to 'A'.

Note:-

① The set of all disjoint equivalent classes on 'A' under relation (equivalence relation) are generates a 'unique partition of A.'

Here each equivalent class is a 'Block.'

② ~~Every element~~

The Equivalent class is a set in which every pair of elements related to each other under relation [equivalence].

Ex:- $[a]_R = \{a, b, c, d\}$.

$[a]_R = [b]_R$

$[a]_R = [b]_R = [c]_R$.

③ Finding equivalence Relation on 'A' when partition of 'A' is known.

→ Let 'A' be a non empty set and $P = \{A_1, A_2, \ldots, A_m\}$ is a partition of 'A' with 'm' blocks.

$A_1 \subseteq A$

$A_1 \times A_1 \subseteq A$.

Then,

(i) R_1 is a relation on A_1 : $R_1 \subseteq A_1 \times A_1 \subseteq A \times A$ and hence on A [R_1 is a universal Relation].

(ii) R_1 is universal relation on A_1 and R_1 is a just Relation on A.

(iii) R_2 is an universal relation on A_2 and R_2 is a Relation on A.

(iv) R_m is an universal relation on A_m and R_m is Relation on A.

→ Let $R = R_1 \cup R_2 \cup \ldots \cup R_m$ is a relation on A

Then we can prove easily that,

$$\boxed{R = (A_1 \times A_1) \cup (A_2 \times A_2) \cup \ldots \cup (A_m \times A_m)}$$

where $A_1, A_2, \ldots, A_m$ are blocks of 'P'.

④ Thus we can establish a one-one correspondence b/w equivalence relation on A and partition of A.

⑤ Thus, the no. of different equivalence relations on A is same as no. of distinct partitions of A.

⑥ The no. of distinct equivalence relations on A is equals to no. of distinct partitions of A. $P(n)$.

* $$\boxed{P(n) = \sum_{j=0}^{n-1} C(n-1, j)\, P(n-j-1)}$$ $C(n-1, j) = {}^{n-1}C_j$

→ This is called 'Bell – Formula'.

→ It is used to find no. of partitions and no. of equivalent classes.

If $n=1$, $P(n) = P(1) = 1$.

If $n=2$, $P(n) = P(2) = 2$.

If $n=3$, $P(n) = P(3) = 5$

If $n=4$, $P(n) = P(4) = 15$

If $n=5$, $P(n) = P(5) = 52$.

* (7) Let R be a equivalence relation on a set A with 'n' elements. and 'm' disjoint equivalent classes of 'A'. then in this case the Digraph of equivalence relation is a disconducted graph with m-components.

* Quotient set :-

→ Let R be an equivalent relation on set A. The set of all collections of equivalence classes on A under relation R is called ~~Coeffi~~ 'Quotient set'.

→ It is denoted by 'A/R'. (or) 'A modulo R' (or) 'A mod R'.

→ $$\boxed{A/R = \{[a]_R / a \in A\}}$$

* P-3 :-

Let $A = \{1, 2, 3, ---, 7\}$ and $R = \{(x,y) \mid x \equiv y \pmod 3\}$. show that R is an equivalence relation through its graph. Find Quotient set A/R?

A) $R = \{(1,4), (1,7), (2,5), (3,6), (3,3), (4,7), (4,1), (5,2), (6,3), (7,4), (1,1), (2,2), (3,3), (4,4), (5,5), (6,6), (7,7)\}$.

clearly R is an equivalence relation on A.

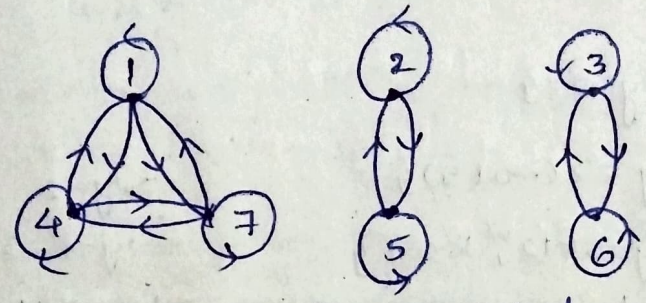
$[1]_R = \{1, 4, 7\} = [4]_R = [7]_R$ [By 2nd property].

$[2]_R = \{2, 5\} = [5]_R$.

$[3]_R = \{3, 6\} = [6]_R$

Hence $A/R = \{[1]_R, [2]_R, [3]_R\} = \{[1]_R, [2]_R, [3]_R\}$.

Digraph :-



Each subgraph is connected graph and the Entire graph is Disconnected.

* Example-3 :-

If R is the equivalence relation "Congruence modulo 5" on Z, then find equivalence classes generated by the elements of Z and the quotient set Z/R?

A) We know that the R, congruent modulo 5 on Z is an equivalence relation on Z and

$Z = \{0, \pm 1, \pm 2, \pm 3, ---\}$.

The equivalent class of elements of Z are as follows

$[0]_R = \{y \in Z \mid y R 0\}$

$= \{y \in Z \mid y \equiv 0 \pmod 5\}$

$= \{y \in Z \mid y = 5k,\ k \in Z\}$

$= \{y = 5k \mid k \in Z\}$.

$5 \mid y - 0$

$\Rightarrow y = y - 0 = 5k$.

$y = 5(-2) = -10$

$y = 5(-1) = -5$

$y = 5(1) = 5$

$y = 5(2) = 10$

$\therefore [0]_R = \{---, -10, -5, 0, 5, 10 ---\}$.

$[1]_R = \{y \in Z \mid y R 1\}$.

$= \{y \in Z \mid y \equiv 1 \pmod 5\}$

$= \{y \in Z \mid y = 5k+1, k \in Z\}$

$= \{y = 5k+1 \mid k \in Z\}$

$5 \mid y - 1$

$\Rightarrow y - 1 = 5k$

$\Rightarrow y = 5k+1$

$\therefore [1]_R = \{---, -9, -4, 1, 6, 11, ---\}$

$[2]_R = \{y \in Z \mid y R 2\}$

$= \{y \in Z \mid y \equiv 2 \pmod 5\}$

$= \{y \in Z \mid y = 5k+2,\ k \in Z\}$

$= \{y = 5k+2 \mid k \in Z\}$.

$5 \mid y - 2$

$\Rightarrow y - 2 = 5k$

$\Rightarrow y = 5k+2$.

$\therefore [2]_R = \{----, -8, -3, 2, 7, 12, ---\}$.

$[3]_R = \{y \in Z \mid y R 3\}$

$= \{y \in Z \mid y \equiv 3 \pmod 5\}$

$= \{y \in Z \mid y = 5k+3,\ k \in Z\}$

$= \{y = 5k+3 \mid k \in Z\}$

$\therefore [3]_R = \{---, -7, -2, 3, 8, 13 ---\}$

$[4]_R = \{y \in Z \mid y R 4\}$

$= \{y \in Z \mid y \equiv 4 \pmod 5\}$

$= \{y \in Z \mid y = 5k+4, k \in Z\} = \{y \in 5k+4 \mid k \in Z\}$

$\therefore [4]_R = \{---, -6, -1, 4, 9, 14, ---\}$.

$\therefore \frac{Z}{R} = \{ [0]_R, [1]_R, [2]_R, [3]_R, [4]_R \}$.

$\therefore$ Z_m = The set of equivalence class of congruent module, m is $\boxed{y = mk + x}$

$Z_m = \{ \bar{0}, \bar{1}, _ _ , (\overline{m-1}) \}$

Exercise

1] Show that the relation of logical equivalence on the set of all compound propositions is an equivalence relation. What are the equivalence classes of F and of T?

A) Let 'S' be set of all compound propositions.

Let 'R' be the relation on 'S' defined as follows

$a R b$ iff $a \equiv b$ $\forall a, b \in S$.

That is. $a R b$ iff a and b have same truth values in their truth table $\longrightarrow$ ①

claim :- "To prove that R is equivalence relation."

(i) To prove that R is Reflexive :-

→ We know that 'a' has the same truth table as 'a' $\forall$ 'a' $\in S$.

Then $a \equiv a$ $\forall a \in S$

$\therefore a R a$ $\forall a \in S$.

$\therefore$ R is Reflexive on S.

(ii) To prove that R is symmetric :-

→ Let $a, b \in S$ & suppose $a R b$ then 'a' and 'b' have same truth values in their truth table.

→ Then we conclude that 'b' and 'a' have same truth values

$\therefore b R a$

Hence R is symmetric on S.

(iii) To prove that 'R' is Transitive:

Let $a, b, c \in S$

→ suppose $a R b$ and $b R c$.

→ 'a' and 'b' have same truthvalues & b and 'c' have same truthvalues in their truth values.

∴ 'a' and 'c' have same truthvalues in their truthtable

Hence $a R c$.

∴ R is transitive relation on S.

∴ From (i), (ii), (iii) R is 'equivalence relation' on S.

Now,

We know that $T_0, F_0 \in S$.

$T_0 = p \vee \sim p$

Now, $[F_0]_R = \{a \in S \mid a R F_0\}$

$= \{a \in S \mid a \equiv F_0\}$

$= \{a \in S \mid a \text{ is a contradiction}\}$.

∴ $[F_0]_R \equiv$ The set of all contradictions in S.

Now, $[T_0]_R = \{a \in S \mid a R T_0\}$

$= \{a \in S \mid a \equiv T_0\}$

$= \{a \in S \mid a \text{ is a Tautology}\}$

∴ $[T_0]_R =$ The set of all tautologies in S.

* P-4:-

obtain the equivalence relation defined by the partition $P = \{A_1, A_2, A_3\}$ of the set $A = \{1, 2, 3, 4, 5, 6\}$, where $A_1 = \{1, 2, 3\}$, $A_2 = \{4, 5\}$ and $A_3 = \{6\}$?

A) Given, $A = \{1, 2, 3, 4, 5, 6\}$.

$P = \{A_1, A_2, A_3\}$ be partition of 'A'.

$A_1 = \{1, 2, 3\}$

$A_2 = \{4, 5\}$

$A_3 = \{6\}$.

Now, $A_1 \times A_1 = \{(1,1); (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3)\}$

$A_2 \times A_2 = \{(4,4), (4,5), (5,4), (5,5)\}$

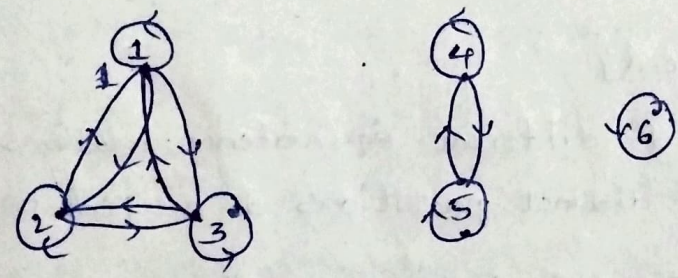
$A_3 \times A_3 = \{(6,6)\}$.

The equivalence class on A is given by

$R = (A_1 \times A_1) \cup (A_2 \times A_2) \cup (A_3 \times A_3)$

$\therefore R = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3),$
$(4,4), (4,5), (5,4), (5,5), (6,6)\}$

Digraph of R :-



* Ex:-

Determine the no. of different equivalence relations on a set with four elements by listing them?

A) Recurrence formula :-

$1 \le K \le n$.

$$S(n,k) = \frac{1}{k!} \sum_{i=0}^{k} (-1)^i \binom{k}{i} (k-i)^n$$

$$P(n) = \sum_{j=0}^{n-1} c(n-1, j)\, P(n-j-1).$$

$$= S(n,1) + S(n,2) + ____ + S(n,n).$$

Note:- $S(n,1) = 1$, $S(n,n) = 1$ } always.

$$P(4) = \sum_{j=0}^{3} {}^3C_j \, P(3-j)$$

$$= {}^3C_0 \, P(3) + {}^3C_1 \, P(2) + {}^3C_2 \, P(1) + {}^3C_3 \, P(0).$$

$$= P(3) + 3(P(2)) + 3\,P(1) + P(0)$$

$$= 5 + 3\cdot(2) + 3(1) + 1$$

$$= 5 + 6 + 3 + 1 = 15.$$

(or)

$$P(4) = S(4,1) + S(4,2) + S(4,3) + S(4,4)$$

$$= 1 + 7 + 6 + 1 = 15$$

Now, $S(4,2) = \frac{1}{2} \sum_{i=0}^{2} (-1)^i \, {}^2C_i \, (2-i)^4$

$$= \frac{1}{2}\{16 + (-1)(2) + 0\}$$

$$= \frac{1}{2}\{14\} = 7.$$

Similarly, $S(4,3) = 6$.

<u>Solution:-</u>

Let $A = \{a, b, c, d\}$

We have no. of different equivalence relations on A = no. of distinct partitions of $A = P(n) = P(4) = 15$.

Those partitions are listed as follows:-

$$P(4) = \underset{1}{S(4,1)} + \underset{7}{S(4,2)} + \underset{6}{S(4,3)} + \underset{1}{S(4,4)}$$

① $P_1 = \{a, b, c, d\}$.

② $P_2 = \{\{a\}, \{b, c, d\}\}$.

③ $P_3 = \{\{b\}, \{a, c, d\}\}$.

④ $P_4 = \{\{c\}, \{a, b, d\}\}$

⑤ $P_5 = \{\{d\}, \{a, b, c\}\}$

⑥ $P_6 = \{\{a, b\}, \{c, d\}\}$

⑦ $P_7 = \{\{a, c\}, \{b, d\}\}$

⑧ $P_8 = \{\{a, d\}, \{b, c\}\}$

(9) $P_9 = \{ \{a\}, \{b\}, \{c,d\} \}$

(10) $P_{10} = \{ \{a\}, \{c\}, \{b,d\} \}$

(11) $P_{11} = \{ \{a,b\}, \{c\}, \{d\} \}$

(12) $P_{12} = \{ \{a,c\}, \{b\}, \{d\} \}$

(13) $P_{13} = \{ \{a,d\}, \{b\}, \{c\} \}$

(14) $P_{14} = \{ \{a\}, \{d\}, \{b,c\} \}$

(15) $P_{15} = \{ \{a\}, \{b\}, \{c\}, \{d\} \}$.

* Compatibility Relations :-

→ A Relation R on a set A is said to be a compatibility relation if it is 'reflexive' & 'symmetric'.

→ If R is an equivalent Relation ~~then~~ and $a, b \in A$ then aRb, [a,b equivalent]

→ If R is a compatibility on R then aRb is compatible.

→ Let R be a relation on A (equivalence relation)

Let M be

(i) $M \subseteq A$

(ii) $\forall x, y \in M \Rightarrow xRy$.

→ Let R be a Relation on A [compatibility class and Block is $A \to n \Rightarrow 2^n$.

* Example-6 :-

Let $A = \{$ball, bed, dog, let, egg$\}$ and Let R be a relation on X given by for $x, y \in A$, xRy iff x and y contain some common letter?

A) The Digraph and Matrix of a compatibility relation are as follows:

Let $A = \{x_1, x_2, x_3, x_4, x_5\}$

Here x_1 = ball

x_2 = bed

x_3 = dog

x_4 = let

x_5 = egg.

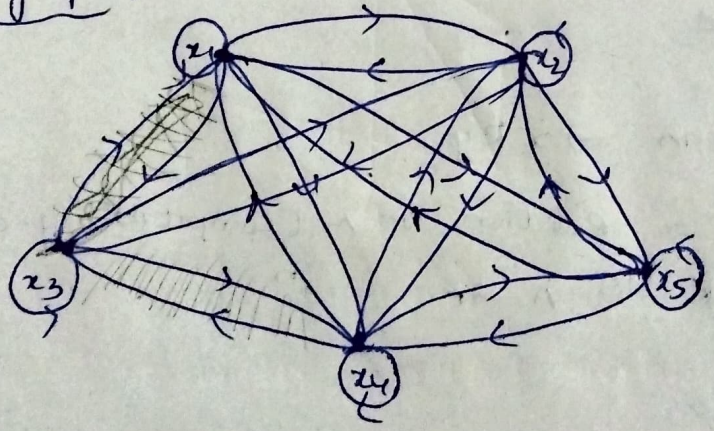
Let R be a relation on A defined as follows.

$x R y$ iff x and y contains some common letters $\forall x, y \in A$.

clearly R be a compatibility relation on A [since it is reflexive & symmetric].

Then $R = \{(x_1, x_1), (x_2, x_2), (x_3, x_3), (x_4, x_4), (x_5, x_5), (x_1, x_2), (x_2, x_5), (x_1, x_4), (x_2, x_3), (x_4, x_1), (x_3, x_2), (x_2, x_4), (x_4, x_2), (x_3, x_5), (x_5, x_3), (x_4, x_5), (x_5, x_4), (x_5, x_2)\}$.

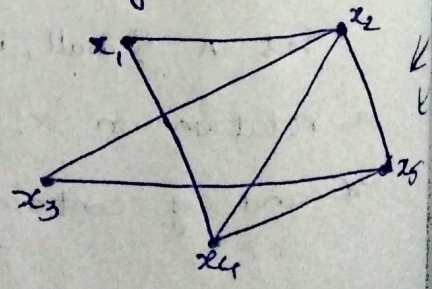
Digraph :-



→ The simplified graph of compatibility relation is as follows:-

① Delete all loops.

② Draw an undirected edge.



The matrix M_R of compatibility Relation is as follows

$$M_R = \begin{array}{c} \cancel{x_1} \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{array} \begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$
$$\begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & \cancel{x_5} \end{array}$$

~~We can~~

It is enough to write lower triangular matrix of M_R to represent compatibility relation R as follows.

$$M_R = \begin{array}{c} x_2 \\ x_3 \\ x_4 \\ x_5 \end{array} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$
$$\begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array}$$

$\therefore M_R =$

x_2	1			
x_3	0	1		
x_4	1	1	0	
x_5	1	1	1	1
	x_1	x_2	x_3	x_4

The compatibility classes of set A as follows.

$\{x_1\}, \{x_2\}, \{x_3\}, \{x_4\}, \{x_5\}$ [set of vertices]

$\{x_1, x_2\}, \{x_1, x_4\}, \{x_2, x_3\}, \{x_2, x_4\}, \{x_2, x_5\}, \{x_3, x_5\}, \{x_4, x_5\}$ [vertices of line segments].

$\{x_1, x_2, x_4\}, \{x_2, x_3, x_5\}, \{x_2, x_4, x_5\}$ [vertices of complete polygons].

* Maximum/Maximal compatibility Block/class :-

~~→ If 'M' is said to be~~

→ Let R be a compatibility Relation on A [$A \neq \phi$]. If a subset 'M' of 'A' is said to be Maximum compatibility Block, if 'M' doesnot properly contained in any other compatibility class of 'A'. i.e, 'M' is said to MCB if

(i) M is compatibility class of 'A'.

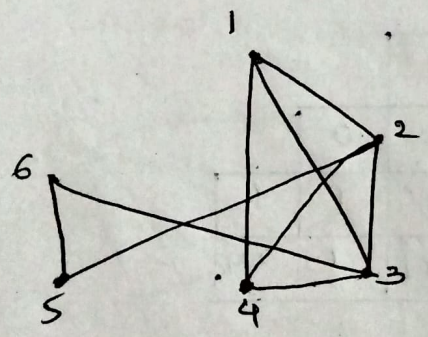
(ii) $M \not\subset N$, for any compatibility class N of N.

→ In the previous example, the MCBs are $\{x_1, x_2, x_4\}$, $\{x_2, x_3, x_5\}$, $\{x_2, x_4, x_5\}$.

Finding MCBs of a SET under compatibility Relation :-

* P-8 :-

The graph of a compatibility relation is given below. Write its relation matrix & find its MCBs?



A) (i) Through simplified Graph :-

Step-1 :- Draw the simplified Graph. [∵ Already given].

Step-2 :- Pick the largest complete polygon from simplified graph.

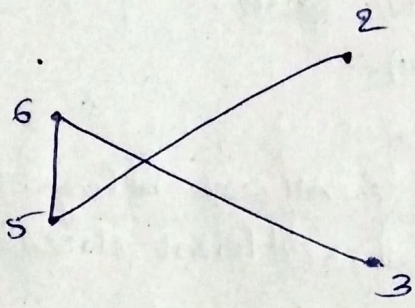
→ The vertices of that largest complete polygons are 1, 2, 3, 4.

Here we notice that the largest complete polygon is formed by vertices 1, 2, 3, 4.

→ The set of these vertices i.e, {1, 2, 3, 4} is a M.C.B.

Step-3:- Neglect the internal polygon in largest polygon

Step-4:- Pick the complete largest polygon (or) line segment [which ever present]. in the complement of largest polygon obtained in step-2.



→ Thus {3, 6}, {2, 5}, {5, 6} are M.C.B's.

Step-5:- Repeat this process until we get M.C.B's

Step-6:- consider if Isolated vertices are present.

Hence all M.C.Bs are {3, 6}, {2, 5}, {5, 6}, {1, 2, 3, 4}.

Method-2:- Through a matrix of compatibility

→ A vertice in which row and columns contain zero then that is Isolated vertice. [Remove that vertice]

Step-1:- We can write relations column-wise from Right most to left.

Step-2:- Determination of compatibility classes that those be elements belong to.

Step-3:- Repeat step ① & ② until we get all elements required.

→

2	1				
3	1	1			
4	1	1	1		
5	0	1	0	0	
6	0	0	1	0	1
	1	2	3	4	5

Step-1:- (5,6), ~~(3,6), (3,4)~~

(3,6), (3,4).

Step-2:- (5,6)

(3,4).

(3,6)

} All are belongs to different classes

Same procedure again

Step-1:- (5,6)

(3,4)

(3,6)

(2,3), (2,4), (2,5)

Step-2:- (5,6)

(3,4), (2,3), (2,4).

(3,6)

(2,5)

Again Repeat the process.

Step-1:- (5,6)

(3,4), (2,3), (2,4)

(3,6)

(2,5)

(1,2), (1,3), (1,4).

Step-2:- (5,6)

(3,4), (2,3), (2,4), (1,2), (1,3), (1,4).

(3,6).

(2,5).

∴ compatibility classes are {5,6}, {1,2,3,4}, {3,6}, {2,5}

The MCBs are (or)

Module - 3:3

*** Ex-1:-**

If $A = \{1,2,3,4\}$, $B = \{1,4,6,8,9\}$ and a relation R from A to B is defined by: for $a, b \in A$, aRb iff $a|b$. Write R and the matrix of R?

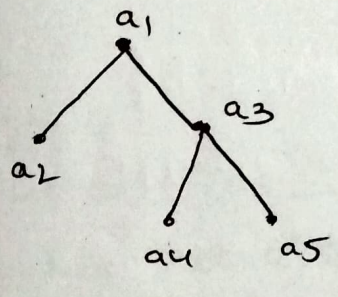
A) $R = \{(1,1), (1,4), (1,6), (1,8), (1,9), (2,4), (2,6), (2,8), (3,6), (3,9), (4,4), (4,8)\}$.

$$M_R = \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{bmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$
$$\begin{array}{ccccc} 1 & 4 & 6 & 8 & 9 \end{array}$$

*** P-1:-**

Determine the property of relations given by the following graphs and also write the corresponding relation matrices.

(i)

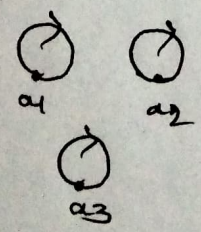


A) $R_1 = \{(a_1,a_2), (a_1,a_3), (a_3,a_4), (a_3,a_5)\}$.

$$MR_1 = \begin{array}{c} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{array} \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$
$$\begin{array}{ccccc} a_1 & a_2 & a_3 & a_4 & a_5 \end{array}$$

Irreflexive.
Antisymmetric
Not transitive.

(ii)

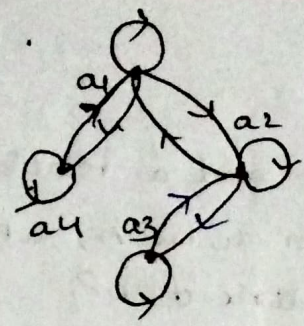


A) $R_2 = \{(a_1,a_1), (a_2,a_2), (a_3,a_3)\}$

$$MR_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Reflexive, symmetric, Transitive.

(iii)

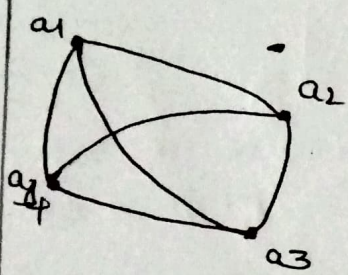


d) $R_3 = \{(a_1,a_1), (a_1,a_2), (a_1,a_4), (a_2,a_2), (a_2,a_1), (a_2,a_3), (a_3,a_3), (a_3,a_2), (a_4,a_1), (a_4,a_4)\}$

$$M_{R_3} = \begin{array}{c} a_1 \\ a_2 \\ a_3 \\ a_4 \end{array} \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$
$$\begin{array}{cccc} a_1 & a_2 & a_3 & a_4 \end{array}$$

Reflexive, symmetric, Transitive;

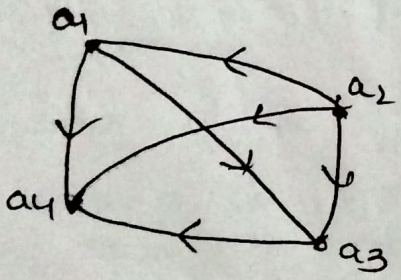
(iv)



d) $R_4 = \{(a_1,a_2), (a_1,a_3), (a_1,a_4), (a_2,a_1), (a_2,a_3), (a_2,a_4), (a_3,a_1), (a_3,a_2), (a_3,a_4), (a_4,a_1), (a_4,a_2), (a_4,a_4)\}$

$$M_{R_4} = \begin{array}{c} a_1 \\ a_2 \\ a_3 \\ a_4 \end{array} \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$
$$\begin{array}{cccc} a_1 & a_2 & a_3 & a_4 \end{array}$$

(v)



d) $R_4 = \{(a_1,a_3), (a_1,a_4), (a_2,a_1), (a_2,a_3), (a_3,a_4), (a_2,a_4)\}$

$$M_{R_4} = \begin{array}{c} a_1 \\ a_2 \\ a_3 \\ a_4 \end{array} \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$
$$\begin{array}{cccc} a_1 & a_2 & a_3 & a_4 \end{array}$$

~~Not sy~~

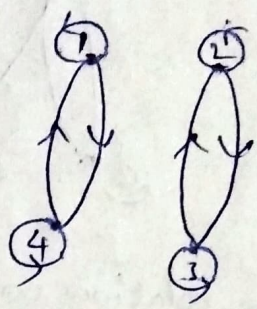
Irreflexive, Antisymmetric, transitive.

*P-2:

Let $A = \{1, 2, 3, 4\}$ and $R = \{(1,1), (1,4), (2,2), (2,3), (3,2), (3,3), (4,1), (4,4)\}$.

show that R is an equivalence relation through its relation matrix and graph? Find the quotient set A/R for the example?

A) $M_R = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 1 & 0 & 0 & 1 \\ 2 & 0 & 1 & 1 & 0 \\ 3 & 0 & 1 & 1 & 0 \\ 4 & 1 & 0 & 0 & 1 \end{array}$



R is symmetric, Reflexive thus Transitive.

$\therefore$ R is Equivalence relation.

$[1]_R = \{y \mid y \in A \ \& \ 1Ry\} = \{1, 4\}$.

$[2]_R = \{y \mid y \in A \ \& \ 2Ry\} = \{2, 3\}$

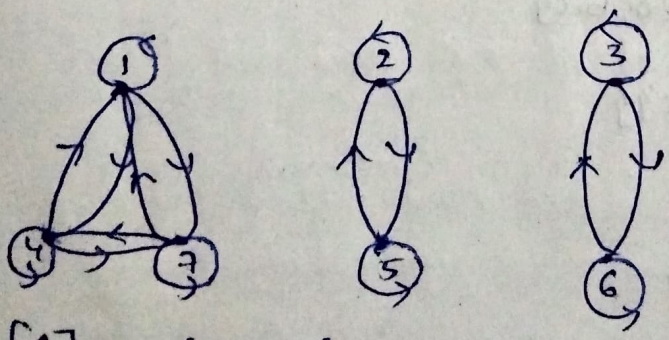
$\therefore A/R = \{[1]_R, [2]_R\} = \{\{1,4\}, \{2,3\}\}$.

*P-3:

Let $A = \{1, 2, 3, ___, 7\}$ and $R = \{(x,y) \mid x \equiv y \pmod 3\}$.

show that R is equivalence through its graph & Find A/R?

A) $R = \{(1,1), (1,4), (1,7), (2,5), (2,2), (3,3), (3,6), (4,4), (4,7), (4,1), (5,2), (5,5), (6,3), (6,6), (7,1), (7,4), (7,7)\}$.



R is reflexive, symmetric & Transitive

$\therefore$ R is equivalence.

$[1]_R = \{1, 4, 7\}$

$[2]_R = \{2, 5\}$

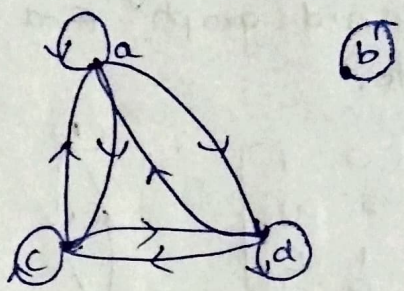
$[3]_R = \{3, 6\}$

$\therefore A/R = \{[1]_R, [2]_R, [3]_R\}$.

* P-5:

Determine whether the relation R represented by the following digraph is an equivalence relation. If R is an equivalence relation find its A/R?

A) Given,



Reflexive,
Symmetric,
Transitive
∴ Equivalence.

Let,

R be relation on $A = \{a, b, c, d\}$

$R = \{(a,a), (a,b), (a,d), (b,b), (c,c), (c,a), (c,d), (d,d), (d,a), (c,d), (d,c)\}$.

$$M_R = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

$[a]_R = \{a, b, c, d\}$.

$[b]_R = \{b\}$.

$\therefore A/R = \{[a]_R, [b]_R\}$

$= \{\{a, c, d\}, \{b\}\}$.

* P-6:

Determine the no. of equivalence relations on a set with three elements?

A) $P(3) = 5$, $A = \{a, b, c\}$

$P_1 = \{\{a\}, \{b\}, \{c\}\}$

$P_2 = \{\{a\}, \{b, c\}\}$

$P_3 = \{\{b\}, \{a, c\}\}$

$P_4 = \{\{c\}, \{a, b\}\}$

$P_5 = \{\{a, b, c\}\}$

Exercise.

8] suppose that A is a nonempty set and f is a function that has A as its domain. Let R be relation on A consisting of all ordered pairs (x,y) such that $f(x) = f(y)$.

ⓐ show that R is equivalence relation on A?

ⓑ What are the equivalence classes of R?

A) Let A is a non-empty set and f is a function that has domain as 'A'.

Let R be the relation on 'A' and it is defined as follows;-

$(x,y) \in R$ i.e $xRy \iff f(x) = f(y) \longrightarrow (1)$
$\forall\, x,y \in A$

a]

ⓐ Reflexive :-

Let $x \in A$

since $f(x) = f(y)$ for any $x \in A$ then xRx for any $x \in A$ [by (1)].

$\therefore R$ is reflexive on A.

ⓑ symmetric :-

Let $x,y \in A$.

Suppose $(x,y) \in R$ i.e. xRy.

Then $f(x) = f(y)$.

$\Rightarrow f(y) = f(x)$.

$\Rightarrow yRx$ i.e, $(y,x) \in R$.

$\therefore R$ is symmetric on 'A'.

ⓒ Transitive :-

Let $(x,y)\, z \in A$.

suppose $(x,y) \in R$ and $(y,z) \in R$ i.e xRy and yRz.

$\longrightarrow$

By (1), $f(x)=f(y)$ and $f(y)=f(z)$

$\Rightarrow f(x)=f(z)$

$\Rightarrow (x,z)\in R$ (or) $x R z$ [$\because$ by (1)]

$\therefore$ R is transitive on 'A'.

Hence R is an equivalence relation on 'A'.

b) Let $x \in A$.

$[x]_R = \{y \in A \mid f(x)=f(y)\}$

$= \{y \in A \mid a = f(y)\}$

where $a=f(x) \in$ Range of f.

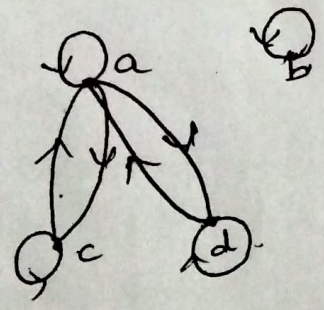
$= \{y \in A \mid y = f^{-1}(a)\}$

$= \{f^{-1}(a) \mid a$ is an element of range of f$\}$.

$\therefore$ The equivalence classes of Relation R is $\{f^{-1}(a) \mid a$ is an element in range of f$\}$.

3) Determine whether the relation with directed graphs show below is an equivalence relation.

(i)



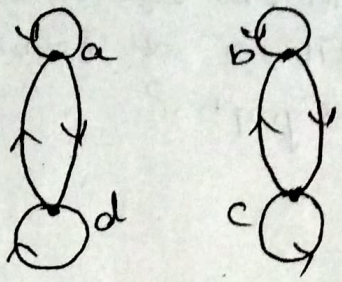
A) Here, every Identical node has loops, so it is Reflexive.

Here, every ~~Ide~~ vertice has either no edge or two edges, so it is symmetric.

But It is not Transitive.

Hence it is not equivalence relation.

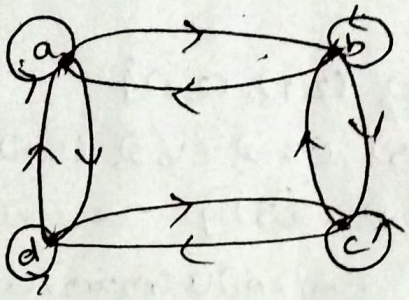
(ii)



A) It is Reflexive, symmetric & Transitive.

∴ It is equivalence Relation.

(iii)



A) It is reflexive, symmetric & Transitive.

∴ It is equivalence Relation.

5) Which of these collections of subsets of partitions of $\{1,2,3,4,5,6\}$?

(a) $\{1,2\}, \{2,3,4\}, \{4,5,6\}$...

A) Not disjoint sets

∴ Not a partition.

(b) $\{1\}, \{2,3,6\}, \{4\}, \{5\}$.

A) It is partition.

(c) $\{2,4,6\}, \{1,3,5\}$

A) It is partition.

(d) $\{1,4,5\}, \{2,6\}$.

A) Not a covering.

∴ Not a partition.

5) List the ordered pairs in the equivalence relations produced by these partitions of $\{0,1,2,3,4,5\}$. Draw the corresponding digraphs?

(a) $\{0\}, \{1,2\}, \{3,4,5\}$.

A) Let $A = \{0,1,2,3,4,5\}$

$A_1 = \{0\}$

$A_2 = \{1,2\}$

$A_3 = \{3,4,5\}$

Now, $A_1 \times A_1 = \{(0,0)\}$.

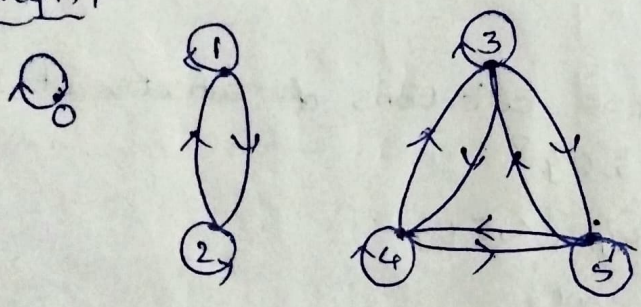
$A_2 \times A_2 = \{(1,1), (1,2), (2,1), (2,2)\}$.

$A_3 \times A_3 = \{(3,4), (3,5), (3,3), (4,3), (4,4), (4,5), (5,3), (5,4), (5,5)\}$.

$\therefore$ Equivalence Relation $= (A_1 \times A_1) \cup (A_2 \times A_2) \cup (A_3 \times A_3)$

$= \{(1,1), (1,2), (2,1), (2,2), (3,3), (3,4), (3,5), (4,3), (4,4), (4,5), (5,3), (5,4), (5,5)\}$.

Digraph :-



(b) $\{0,1\}, \{2,3\}, \{4,5\}$.

A) Let $A = \{0,1,2,3,4,5\}$

$A_1 = \{0,1\}$

$A_2 = \{2,3\}$

$A_3 = \{4,5\}$

Now, $A_1 \times A_1 = \{(0,0), (0,1), (1,0), (1,1)\}$

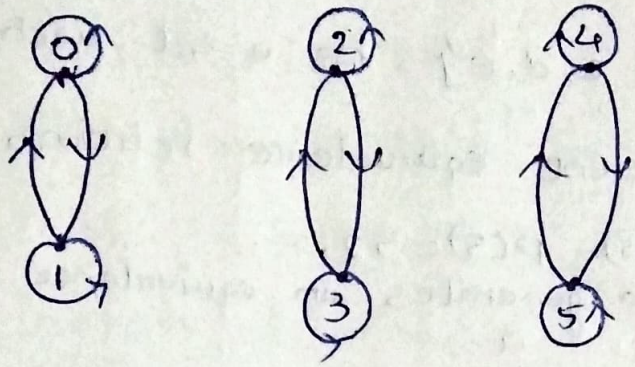
$A_2 \times A_2 = \{(2,2), (2,3), (3,2), (3,3)\}$

$A_3 \times A_3 = \{(4,4), (4,5), (5,4), (5,5)\}$.

$\therefore$ Equivalence Relation $= (A_1 \times A_1) \cup (A_2 \times A_2) \cup (A_3 \times A_3)$

$R = \{(0,0), (0,1), (1,0), (1,1), (2,2), (2,3), (3,2), (3,3), (4,4), (4,5), (5,4), (5,5)\}$.

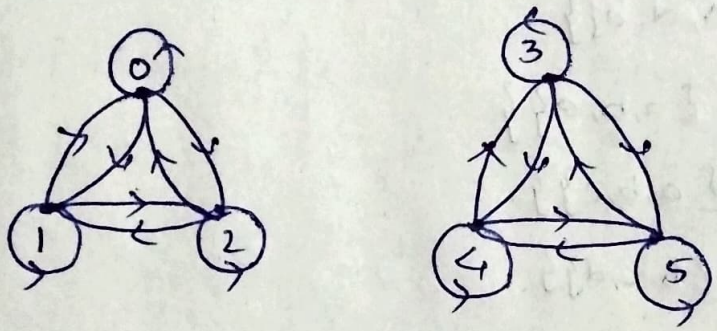
Digraph :-



(c) $\{0,1,2\}, \{3,4,5\}$.

A) Equivalence Relation = $\{(0,1), (0,2), (0,0), (1,0), (1,1), (1,2), (2,0), (2,1), (2,2), (3,3), (3,4), (3,5), (4,3), (4,4), (4,5), (5,3), (5,4), (5,5)\}$.

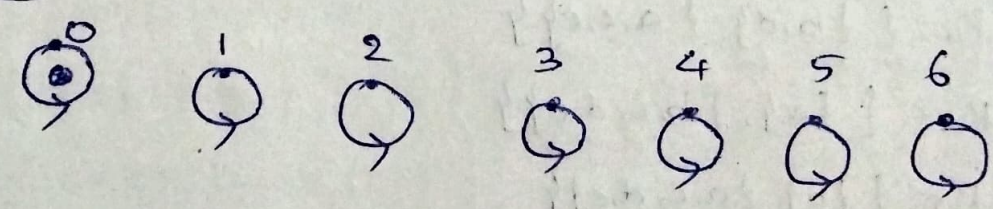
Digraph :-



(d) $\{0\}, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}$.

A) Equivalence Relation = $\{(0,0), (1,1), (2,2), (3,3), (4,4), (5,5), (6,6)\}$.

Digraph :-

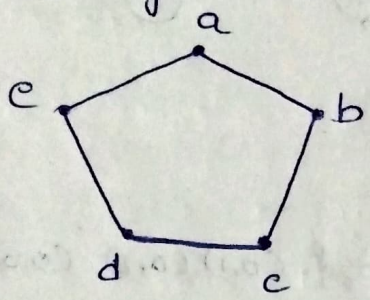


7] Determine the number of different equivalence relations on a set with five elements by listing them?

A) Let $A = \{a, b, c, d, e\}$ be a set with 5 elements

No. of different equivalence relations on a set $A = P(n) = P(5) = 52$.

Every partition generates an equivalence relation.



$P_1 = \{\{a\}, \{b\}, \{c\}, \{d\}, \{e\}\}$

$P_2 = \{\{a,b\}, \{c,d,e\}\}$

$P_3 = \{\{b,c\}, \{a,d,e\}\}$

$P_4 = \{\{c,d\}, \{a,b,e\}\}$

$P_5 = \{\{d,e\}, \{a,b,c\}\}$

$P_6 = \{\{a,e\}, \{b,c,d\}\}$

$P_7 = \{\{c,e\}, \{a,b,d\}\}$

$P_8 = \{\{a,d\}, \{b,c,e\}\}$

$P_9 = \{\{b,e\}, \{a,c,d\}\}$

$P_{10} = \{\{a,c\}, \{b,d,e\}\}$

$P_{11} = \{\{b,d\}, \{a,c,e\}\}$

$P_{12} = \{\{a\}, \{b,c,d,e\}\}$

$P_{13} = \{\{b\}, \{a,c,d,e\}\}$

$P_{14} = \{\{c\}, \{a,b,d,e\}\}$

$P_{15} = \{\{d\}, \{a,b,c,e\}\}$

$P_{16} = \{\{e\}, \{a,b,c,d\}\}$

$P_{17} = \{\{a,b\}, \{c\}, \{d\}, \{e\}\}$

$P_{18} = \{\{b,c\}, \{a\}, \{d\}, \{e\}\}$

$P_{19} = \{\{c,d\}, \{a\}, \{b\}, \{e\}\}$

$P_{20} = \{\{d,e\}, \{a\}, \{b\}, \{c\}\}$

$P_{21} = \{\{a,e\}, \{b\}, \{c\}, \{d\}\}$

$P_{22} = \{\{c,e\}, \{a\}, \{b\}, \{d\}\}$

$P_{23} = \{\{a,d\}, \{b\}, \{c\}, \{e\}\}$

$P_{24} = \{\{b,e\}, \{a\}, \{c\}, \{d\}\}$

$P_{25} = \{\{a,c\}, \{b\}, \{d\}, \{e\}\}$

$P_{26} = \{\{b,d\}, \{a\}, \{c\}, \{e\}\}$

$P_{27} = \{\{a\}, \{b\}, \{c,d,e\}\}$

$P_{28} = \{\{b\}, \{c\}, \{a,d,e\}\}$

$P_{29} = \{\{c\}, \{d\}, \{a,b,e\}\}$

$P_{30} = \{\{d\}, \{e\}, \{a,b,c\}\}$

$P_{31} = \{\{b\}, \{e\}, \{a,c,d\}\}$

$P_{32} = \{\{a\}, \{e\}, \{b,d,e\}\}$

$P_{33} = \{\{b\}, \{d\}, \{a,c,e\}\}$

$P_{34} = \{\{c\}, \{e\}, \{a,b,d\}\}$

$P_{35} = \{\{a\}, \{e\}, \{b,c,d\}\}$

$P_{36} = \{\{a\}, \{d\}, \{b,c,e\}\}$

$P_{37} = \{\{a,e\}, \{b,c\}, \{d\}\}$

$P_{38} = \{\{a,b\}, \{c,d\}, \{e\}\}$

$P_{39} = \{\{d,e\}, \{b,c\}, \{a\}\}$

$P_{40} = \{\{a,e\}, \{c,d\}, \{b\}\}$

$P_{41} = \{\{a,b\}, \{d,e\}, \{c\}\}$

$P_{42} = \{\{a,c\}, \{b,e\}, \{d\}\}$

$P_{43} = \{\{a,c\}, \{b,d\}, \{e\}\}$

$P_{44} = \{\{b,d\}, \{c,e\}, \{a\}\}$

$P_{45} = \{\{a,d\}, \{c,e\}, \{b\}\}$

$P_{46} = \{\{a,d\}, \{b,e\}, \{c\}\}$

$P_{47} = \{\{a,b\}, \{c,e\}, \{d\}\}$

$P_{48} = \{\{a,d\}, \{b,c\}, \{e\}\}$

$P_{49} = \{\{b,e\}, \{c,d\}, \{a\}\}$

$P_{50} = \{\{a,c\}, \{d,e\}, \{b\}\}$

$P_{51} = \{\{a,e\}, \{b,d\}, \{c\}\}$

$P_{52} = \{a, b, c, d\}$.

Module - 3.4 :- TRANSITIVE CLOSURE

* Example - 1 :-

Let $A = \{1, 2, 3, 4, 5\}$. Let R and S be relations on A defined by.

$R = \{(1,2), (2,2), (3,4)\}$

$S = \{(4,2), (2,5), (3,1), (1,3)\}$.

Find ROR, SoS, RoS, SoR, Ro(SoR), (RoS)oR and RoRoR?

A) ① ROR :-

$ROR = \{(1,2), (2,2), (3,4)\} \circ \{(1,2), (2,2), (3,1), (1,3)\}$

$= \{(1,2), (2,2)\}$

(1,2), (1,2)
(2,2), (2,2)

(or)

R	R	ROR
(1,2)	(1,2)	(1,2)
(2,2)	(2,2)	(2,2)
(3,4)		

② SoS :-

S	S	SoS
(4,2)	(2,5)	(4,5)
(2,5)		
(3,1)	(1,3)	(3,3)
(1,3)	(3,1)	(1,1)

③ RoS :-

R	S	RoS
(1,2)	(2,5)	(1,5)
(2,2)	(2,5)	(2,5)
(3,4)	(4,2)	(3,2)

④ SOR :-

S	R	SOR
(4,2)	(2,2)	(4,2)
(2,5)		
(3,1)	(1,2)	(3,2)
(1,3)	(3,4)	(1,4)

⑤ RO(SOR) :-

R	SOR	RO(SOR)
(1,2)		
(2,2)		
(3,4)	(4,2)	(3,2).

⑥ (ROS)OR :-

ROS	R	(ROS)OR
(1,5)		
(2,5)		
(3,2)	(2,2)	(3,2).

⑦ ROROR :-

ROROR = (ROR)OR

ROR	R	ROROR
(1,2)	(2,2)	(1,2)
(2,2)	(2,2)	(2,2).

∴ ROROR = (ROR)OR = RO(ROR).

* Important properties :

→ Let R, S on A & M_R, M_S are Matrices.

$R \cup S$ and $R \cap S$ on A

$\downarrow$ $\qquad$ $\downarrow$

$M_{R \cup S}$ $\qquad$ $M_{R \cap S}$

① $M_{R \cup S} = M_R \vee M_S$.

② $M_{R \cap S} = M_R \wedge M_S$.

③ $M_{\bar{R}}$ (or) $M_{R^{-1}} = (M_R)^T$ (or) M_R'.

④ $A \to m$ elements, $B \to n$ elements, $C \to p$ elements

(i) R from A to B : $M_R \to mn$ elements

(ii) S from B to C : $M_S \to n \times p$ elements.

(iii) $R \circ S$ from A to C : $M_{R \circ S} \to m \times p$ elements.

(iv) $M_{R \circ S} = M_R \circ M_S$. [Boolean product].

⑤ Let R relation on A,

Then $R^2 = R \circ R$ is also a relation on A.

Now, $M_{R^2} = M_{R \circ R} = M_R \circ M_R = (M_R)^2$.

Also, $R^3 = R \circ R \circ R$ is also relation on A.

Now, $\Rightarrow M_{R^3} = M_{R \circ R \circ R} = M_R \circ M_R \circ M_R = (M_R)^3$.

Thus, $M_{R \circ R \circ R \circ \cdots R} = M_{R^n} = (M_R)^n \ \forall \ n \in Z^+$.

⑥ $\overline{R \circ S} = \bar{S} \circ \bar{R}$.

Then $M_{\overline{R \circ S}} = M_{\bar{S} \circ \bar{R}} = M_{\bar{S}} \circ M_{\bar{R}}$.

$\therefore M_{\overline{R \circ S}} = M_{\bar{S}} \circ M_{\bar{R}}$.

* Example-6 :-

If $M_R = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ and $M_S = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}$.

Then show that $M_{\overline{R \circ S}} = M_{\overline{S}} \circ M_{\overline{R}}$?

A) consider R.H.S $M_{\overline{S}} \circ M_{\overline{R}}$

$$M_{\overline{R}} = (M_R)^T = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$M_{\overline{S}} = (M_S)^T = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$M_{\overline{S}} \circ M_{\overline{R}} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}_{5\times 3} \circ \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}_{3\times 3}$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

L.H.S :-

$M_{R \circ S} = M_R \circ M_S$

$$= \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}_{3\times3} \circ \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}_{3\times5}$$

$$M_{R \circ S} = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$M_{\overline{R \circ S}} = (M_{R \circ S})^T = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$\therefore$ LHS = RHS

Hence proved.

*P-4:-

Two equivalence relations R and S are given by the relation matrices M_R and M_S, where

$$M_R = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad M_S = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

(a) Show that $R \circ S$ is not an equivalence relation.

(b) obtain equivalence relation R_1 and R_2 on the set $A = \{1, 2, 3\}$ such that $R_1 \circ R_2$ is an equivalence relation?

A) (a) Given R and S are relations.

. Then $R \circ S$ is also a relation.

$$M_{R \circ S} = M_R \circ M_S = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Here $M_{R \circ S}$ is not a symmetric & reflexive

$\therefore$ It is not an equivalence relation.

(b) Let A be a set. contains 3 elements.

Let $R_1 = I_A$ and $R_2 = A \times A$.

$$M_{R_1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times 3},$$

$$M_{R_2} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}_{3\times 3}$$

Then $R_1 \circ R_2$ is also a relation on A.

$$M_{R_1 \circ R_2} = M_{R_1} \circ M_{R_2}$$

$$\Rightarrow M_{R_1 \circ R_2} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}_{3\times 3} = M_{R_2}$$

$$\Rightarrow M_{R_1 \circ R_2} = M_{R_2}$$

Here M_{R_2} is equivalence relation.

$\therefore$ $R_1 \circ R_2$ is also equivalence relation.

* Reflexive closure :-

→ Let R be a relation on A.

$R^{(r)}$: The reflexive closure of Relation R.

→ Let $A = \{1, 2, 3\}$.

Let, $R = \{(1,1), (2,3), (1,3)\}$ → Not reflexive.

① $R_1 = \{(1,1), (2,2), (3,3)\}$

② $R_2 = \{(1,1), (2,2), (3,2), (2,3)\}$

③ $R_3 = \{(1,1), (2,2), (3,3), (1,3)\}$

④ $R_1^* = \{(1,1), (2,2), (3,3), (2,3), (1,3)\}$

⑤ $R_2^* = \{(1,1), (2,2), (3,3), (2,3), (1,3), (2,1), (3,1)\}$

⑥ $R_3^* = \{(1,1), (2,2), (3,3), (2,3), (1,3), (2,1) (3,2)\}$

(i) $R^{(r)}$ is a Reflexive relation

(ii) $R^{(r)}$ contains R i.e, $R \subseteq R^{(r)}$

(iii) $R^{(r)}$ is a smallest subset to R.

→ By satisfying the above 3-properties, the relation becomes Reflexive closure.

$$\boxed{R^{(r)} = R \cup I_A.}$$

Note:-

→ If given relation is reflexive then corresponding reflexive ~~clous~~ closure is itself.

$$\boxed{R \cup I_A = R.}$$

* Symmetric closure :- $(R^{(s)})$

→ Let R is relation on A.

→ If $R^{(s)}$ satisfies the following properties

(i) $R^{(s)}$ is symmetric

(ii) $R \subseteq R^{(s)}$

(iii) $R^{(s)}$ is smallest

* Transitive closure :- (R^{+}) / (R^{*})

→ Let R be a given relation on A.

R^{+} = Transitive clauser of a R on A.

(i) R^{+} is a transitive relation

(ii) $R \subseteq R^{+}$

(iii) R^{+} is smallest.

→ A be a set with 'n' elements. & 'R' on A.

Then $\boxed{R^{+} = R \cup R^{2} \cup ___ \cup R^{n}.}$ [n = cardinality of A]

and Matrix of transitive closure is $M_{R^{+}} = M_{R \cup R^{2} \cup __ \cup R^{n}}$

$M_{R^+} = M_{R \cup R^2 \cup __ R^n} = M_R \vee M_{R^2} \vee ____ M_{R^n}$

$\therefore M_{R^+} = M_R \vee (M_R)^2 \vee (M_R)^3 ____ \vee (M_R)^n.$

<u>Note:-</u>

→ If we don't know the property [Reflexive, symmetric or Transitive] then it is just a 'closure'.

* <u>P-5:-</u>

Find the matrix of transitive ~~clous~~ closure of the relation R with

$$M_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}_{3\times 3}.$$

$\therefore M_{R^+} = M_R \vee (M_R)^2 \vee (M_R)^3.$

Now, $(M_R)^2 = M_R \circ M_R = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$

$(M_R)^3 = (M_R)^2 \circ M_R = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} = M_{R^2}$

$\therefore M_{R^+} = M_R \vee (M_R)^2 \vee (M_R)^3$

08-05-2023

Module -3.5 :- POSETS

* Partial ordering relation :- ('$\leq$')

→ Let R be a relation on A said to be partial ordering relation if it satisfies.

(i) Reflexive

(ii) Antisymmetric

(iii) Transitive.

* Poset :-

→ A set say 'A' together with partial ordering relation on it is called 'Poset'.

Ex :-

① '$\leq$' less than or equal to '$(A, \leq)$'

② '$\geq$' greater than or equal to '$(A, \geq)$'

③ divides is partial ordering on set A. '$(A, |)$'

④ Let A be a set, P(A) is powerset then Relation '$\subseteq$' is partial ordering, so '$(P(A), \subseteq)$'.

Notation :-

→ Poset Notation is '(A, R)'. (or) '$(A, \preceq)$'.

where 'A' is a 'set'

'R' is partial ordering Relation.

Note :-

→ Every partial ordering relation is denoted by '$\preceq$'. [Read as less than or equal to].

i.e, '$\preceq$' may be $\leq, \geq, =, \subseteq, |$ ____ ____.

⊕ In a set 'A',

$$\boxed{a \preceq a \quad \forall a \in A}.$$

② Let $(A, \preceq)$ be a poset and $a, b \in R$ $(a \neq b)$

→ Every distinct pair of elements in a poset may or maynot be related by '$\preceq$'.

Ex: $A = \{1, 2, 3, 4, 5, 6\}$

$(A, \preceq)$: Here $\preceq$ is '|'.

$2, 3 \in A$

$2 \nmid 3$ i.e $2 \nmid 3$.

* Precedes and succeds :-

→ Let $(A, \preceq)$ be a poset and $a \preceq b$ for $a, b \in A$ then we can say as follows.

'a preceds b'

(or)

'b succeds a'

(or)

'a is less than or equal to b'

(or)

'b is greater than or equal to a'

→ Here a is called 'predecessor' and b is called 'successor.'

* Strictly preceds and strictly succeds :-

→ Let '$(A, \preceq)$' be a poset and $a, b \in A$.

If $a \preceq b$ and $a \neq b$ then $a \prec b$ and we say that 'a is strictly preceds b' ~~and~~ (or) 'b is strictly succeds a'. (or) 'a is less than b' (or) 'b is greater than a'.

Note :-

→ If $a \prec b$ then $b \nprec a$. [For identical pairs only two sides true].

→ If $a \prec b$ then $b \nprec a$ if $b \prec a$ then $a = b$ is not possible.

* Comparability in a poset :-

→ Let $(A, \leq)$ be a poset.

If $a, b \in A$ are said to be 'comparable' if either '$a \leq b$' (or) '$b \leq a$'.

→ If $a, b \in A$ are said to be 'Incomparable' if neither '$a \leq b$' nor '$b \leq a$'.

Note :-

→ Every distinct pairs of poset may or maynot be comparable.

* Totally ordered set (or) Linearly ordered set (or) chain

→ If a poset containing elements have relation for every pair then it is called Totally ordered set.

→ The Relation b/w them is 'Totally ordered relation'.

→ Here Every distinct pairs of poset ~~may or~~ have must 'comparable'.

Ex :- ① $(N, \leq)$

5, 97 16, 2

$5 \leq 97$. $16 \leq 2$. [Here '$\leq$' is '$\geq$']

② $A = \{1, 2, 3, 4, 6\}$

$\leq$: $|$

Not a Total ordered set.

* Antichain :-

→ No of two distinct pairs/elements of poset are comparable is called 'Antichain'.

Exercise

1] Let (S, R) be a poset. show that $(S, \bar{R})$ is also a poset, where $\bar{R}$ is the inverse of R. The poset $(S, \bar{R})$ is called the dual of (S, R)?

A) Proof:-

- Let (S, R) be a poset.
- Then R is a partial ordering relation on S.

Thus R is Reflexive, Antisymmetric and transitive on S.

claim:- To prove that $(S, \bar{R})$ is also a poset.

It is enough to Prove that $\bar{R} = R^{-1}$ is also a partial ordering on S.

(i) Reflexivity:-

since R is reflexive on S.

$(x, x) \in R \quad \forall x \in S$.

$(x, x) \in \bar{R} \quad \forall x \in S$.

$\therefore \bar{R}$ is Reflexive on S.

(ii) Anti symmetric:-

Let $x, y \in S$.

suppose $(x, y) \in \bar{R}$ and $(y, x) \in \bar{R}$.

$\Rightarrow (y, x) \in R$ and $(x, y) \in R$.

since R is Antisymmetric and $(x, y), (y, x) \in R \Rightarrow x = y$.

Thus $\bar{R}$ is also antisymmetric on S.

(iii) Transitivity:-

Let $x, y, z \in S$

suppose $(x, y) \in \bar{R}$ and $(y, z) \in \bar{R}$

Then $(y, x) \in R$ and $(z, y) \in R$

since R is Transitive and $(z, y), (y, x) \in R \Rightarrow (z, x) \in R$

Then $(x, z) \in \bar{R}$.

$\therefore$ It is Transitivity $[\bar{R}]$ on S.

From (i), (ii), (iii),

$\bar{R}$ is 'partial ordering Relation' on S.

$\therefore$ '$(S, \bar{R})$' is also a 'poset'.

Note :-

→ The poset '$(S, \bar{R})$' is called as "Dual of a poset (S, R)".

2) Find the duals of the following posets.

ⓐ $(\{0,1,2\}, \leq)$

ⓑ $(Z, \geq)$

ⓒ $(P(Z), \supseteq)$

ⓓ $(Z^+, |)$

A) ⓐ We know that, '$\geq$' is the converse relation of a relation '$\leq$'

Then $(\{0,1,2\}, \geq)$ is also a poset.

$\therefore$ $(\{0,1,2\}, \geq)$ is dual of given poset.

ⓑ $(Z, \geq)$

converse relation of '$\geq$' is '$\leq$'

$(Z, \leq)$ is also a poset

$\therefore$ $(Z, \leq)$ is dual of $(Z, \geq)$.

ⓒ $(P(Z), \supseteq)$

converse relation of '$\supseteq$' is '$\subseteq$'

$(P(Z), \subseteq)$ is also a poset and dual of $(P(Z), \supseteq)$

ⓓ $(Z^+, |)$

converse relation of '$|$' is 'multiple of'.

Thus the dual of '$(Z^+, |)$' is '$(Z^+$, multiple of$)$'.

3) Which of the following pairs of elements are comparable in the poset $(Z^+, |)$?

(a) 5, 15

A) $5 | 15$, so It is comparable.

(b) 6, 9

A) $6 \nmid 9$, so It is not comparable.

(c) 8, 16

A) $8 | 16$, so It is comparable.

(d) 7, 7.

A) $7 | 7$, so It is comparable.

4) Find two incomparable elements in the following posets?

(a) $(P(\{0,1,2\}), \subseteq)$.

A) $A = \{\phi, \{0\}, \{1\}, \{2\}, \{0,1\}, \{1,2\}, \{0,2\}, \{0,1,2\}\}$

Here $\{0\} \not\subseteq \{1\}$

$\{1\} \not\subseteq \{2\}$

$\{0,1\} \not\subseteq \{1,2\}$

$\therefore$ It is Incomparable.

(b) $(\{1,2,4,6,8\}, |)$.

A) Here, $4 \nmid 6$. (4 doesnot divides 6)

and also $6 \nmid 8$.

$\therefore$ It is Incomparable.

* Immediate predecer and Immediate successor

→ Let $(A, \leq)$ be a poset and $a, b \in A$.

if ① $a < b$

② There is no element $c \in A$ such that $a < c < b$.

→ Then we can say 'a' is immediate predecer and 'b' is immediate successor. when it follows above two conditions.

Notation :- '$a \lessdot b$'

Ex:- $A = \{1, 2, 3, 4, 6, 8\}$

clearly $A \subseteq N$.

Then $(A, |)$ is a poset.

Here $1 \lessdot 2$; $1|2$; $1 R 2$

$1 \lessdot 3$. $1|3$

× $1 \lessdot 4$ $1|4$ $2|4$ $1|2$

$3 \lessdot 6$

$2 \lessdot 6$

$4 \lessdot 8$

× $2 \lessdot 8$

* Hasse diagrams :-

→ Representing posets in the form of diagrams are called 'Hasse diagrams'.

→ It contains

① vertices

② Edges

→ In a Hasse diagram we can draw Edges based on 'Immediate' predecer' & 'successor'.

→ Every vertex is represented by '•'

→ Draw edges in upward direction always.

* Exercise :-

Q) Draw the Hasse diagram for divisibility on the set.

(a) $\{1, 2, 3, 4, 5, 6, 7, 8\}$.

A) Let $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$

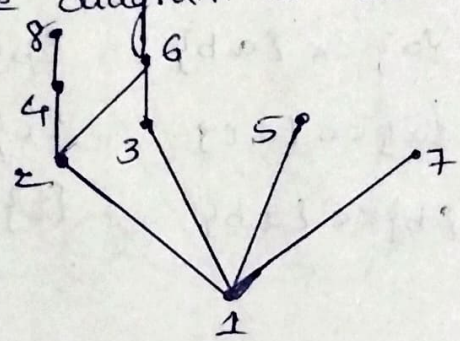
clearly $A \subseteq N$

Then $(A, |)$ is a poset.

Immediate predecesors & successors are follows:-

$1 \prec 2$	$2 \prec 4$
$1 \prec 3$	$2 \prec 6$
$1 \prec 5$	$3 \prec 6$
$1 \prec 7$	$4 \prec 8$

The Hasse diagram is as follows.

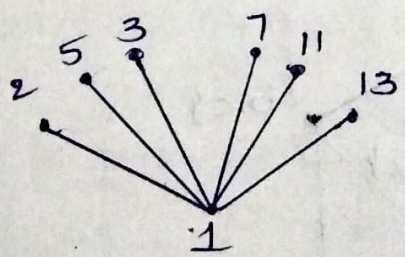


(b) $\{1, 2, 3, 5, 7, 11, 13\}$.

A) $A = \{1, 2, 3, 5, 7, 11, 13\}$

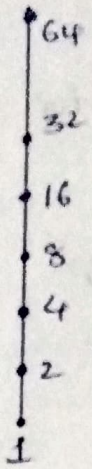
$A \subseteq N$

$\therefore (A, |)$ is a poset.



(d) $\{1, 2, 4, 8, 16, 32, 64\}$

A)



* Example-8 :-

Draw Hasse diagram of poset $(P(A), \subseteq)$, where $A = \{a, b, c\}$.

A) Here, $P(A) = \{\phi, \{a\}, \{b\}, \{c\}, \{a,b\}, \{b,c\}, \{a,c\}, A\}$.

$\phi << \{a\}$ $\quad$ $\{a\} << \{a,b\}$ $\quad$ $\{b\} << \{b,c\}$

$\phi << \{b\}$ $\quad$ $\{a\} << \{a,c\}$ $\quad$ $\{c\} << \{a,c\}$

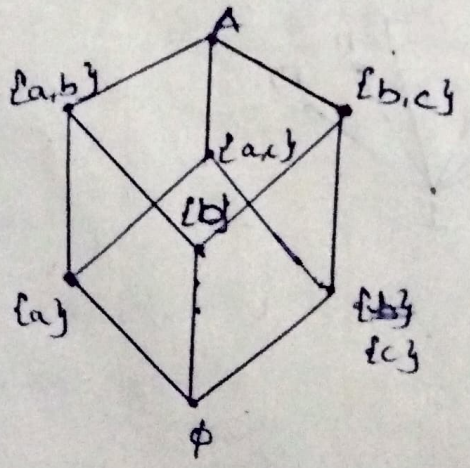
$\phi << \{c\}$ $\quad$ $\{b\} << \{a,b\}$ $\quad$ $\{c\} << \{b,c\}$

$\{a,b\} << A$

$\{a,c\} << A$

$\{b,c\} << A$

Hasse diagram :-



Method-2:

8) Q.3: $\{(a,b) \mid a \text{ divides } b\}$

Draw the Hasse diagram for ~~less than or equal~~ to relation defined on set $A = \{1,2,3,4\}$ $\{1,2,3,4,6,8,12\}$.

i) Let $A = \{1,2,3,4,6,8,12\} \subseteq N$.

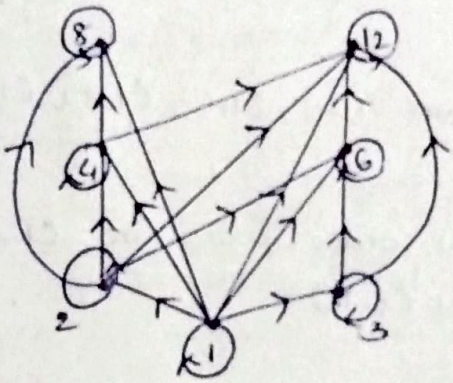
$R = \{(a,b) \mid a,b \in A;\ a|b\}$

Clearly (A, R) be a poset.

Then $R = \{(1,1), (1,2), (1,3), (1,4), (1,6), (1,8), (1,12), (2,2), (2,4), (2,6), (2,8), (2,12), (3,3), (3,6), (3,12), (4,4), (4,8), (4,12), (6,6), (6,12), (8,8), (12,12)\}$.

Step-1:

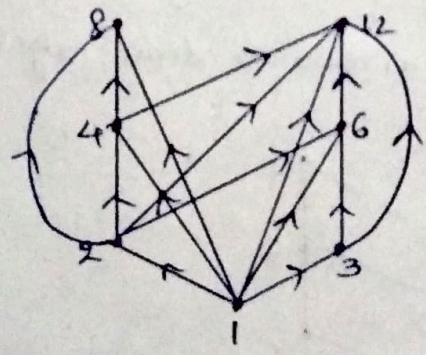
Draw diagraph of R.



Step-2:

Delete all loops [Identical pairs]. Then

$(1,2), (1,3), (1,4), (1,6), (1,8), (1,12), (2,4), (2,6), (2,8), (2,12), (3,6), (3,12), (4,8), (4,12), (6,12)$.



Step-3:-

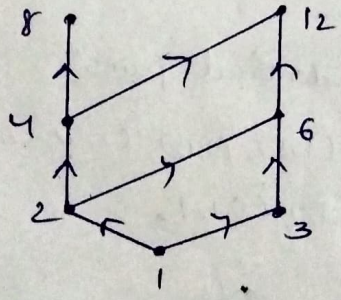
Delete all edges implied by transitivity.

Edges	Transitivity	Deleted Edges
(1,2), (2,4)	(1,4) →	(1,4)
(1,2), (2,6)	→	(1,6)
(1,2), (2,8)	→	(1,8)
(1,2), (2,12)	→	(1,12)
(1,3), (3,6)	→	(1,6)
(1,3), (3,12)	→	(1,12)
(2,4), (4,8)	→	(2,8)
(2,4), (4,12)	→	(2,12)
(2,6), (6,12)	→	(2,12)
(3,6), (6,12)	→	(3,12)

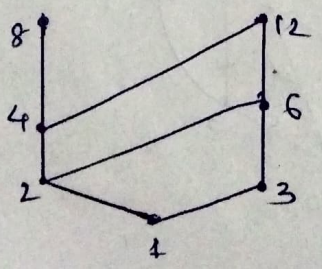
Deleted edges are (1,4), (1,6), (1,8), (1,12), (2,8), (2,12), (3,12).

Then rest of the order pairs are (1,2), (1,3), (2,4), (4,6), (3,6), (4,8), (4,12), (6,12).

Modified graph:-



Step-4:- omit all arrows & draw edges upwards.



* Conclusions:-

① Hasse diagram is not a graph.

② Relation is always in upward direction only.

③ Hasse diagram allows a g. path towards upwards from a vertex [bottom vertex] through a vertex. [Intermediate vertex].

* Extremal elements of a Poset:-

→ There are 4 extremal elements of a poset.

① Maximal element:-

→ Let $(A, \leq)$ be a poset. & $a \in A$.

'a' is said to be a maximal element in $(A, \leq)$ if there is no element $b \in A$ such that '$a < b$'.

② Minimal element:-

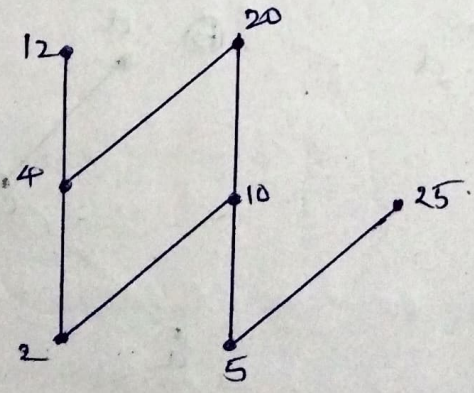
→ Let $(A, \leq)$ be a poset & $a \in A$

'a' is said to be minimal element in $(A, \leq)$ if there is no element $b \in A$ ~~se~~ such that '$b < a$'.

* P-4:-

which elements of poset $(A, \mid)$ are maximal & minimal, defined on set $A = \{2, 4, 5, 10, 12, 20, 25\}$.

Given,



Maximal elements:- 12, 20, 25

Minimal elements:- 2, 5.

Conclusion :-

① For Totally ordering set contains no Maximal elements [It is infinite]

② Every poset contains may or maynot a Maximal element.

③ Greatest element :- / Last element / unit element.

→ Let $(A, \leq)$ be a poset $a \in A$ and a is said to be a greatest element in a poset $(A, \leq)$ if $x \leq a \ \forall \ x \in A$. i.e, Every element in A preceds 'a'.

→ It is denoted by '1'.

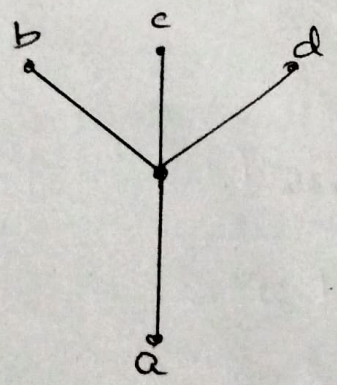
④ Last element :- / First element / Bottom element :-

→ Let $(A, \leq)$ is a poset & $a \in A$ and a is said to be a least element if $a \leq x \ \forall \ x \in A$. i.e, every element in A succeds 'a'.

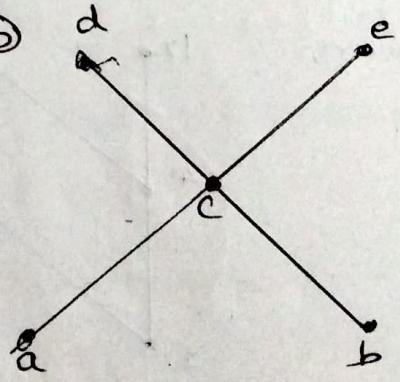
* P.5 :-

Determine whether posets represented by each of following Hasse diagrams have a greatest element & least element ?

ⓐ



ⓑ



A) Greatest :- No.
Least :- a

No Greatest & No Least

* Algorithm for Topological Sorting:

Step-1:- Draw a Hasse diagram of a given poset $(A, \leq)$.

Step-2:- Select a minimal element and put it in the list

Step-3:- Delete the minimal element from A.

Step-4:- Repeats step 2 & 3 until all elements of A and appeared in the list.

* Exercise :-

Q] Find a compatible total order for the divisibility relation on the set. $\{1, 2, 3, 6, 8, 12, 24, 36\}$

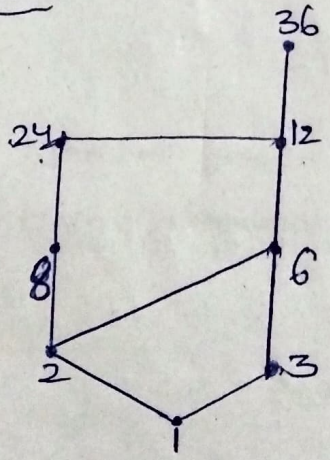
A) Given, $A = \{1, 2, 3, 6, 8, 12, 24, 36\}$.

clearly $A \subseteq N$.

Then $(A, |)$ is a poset.

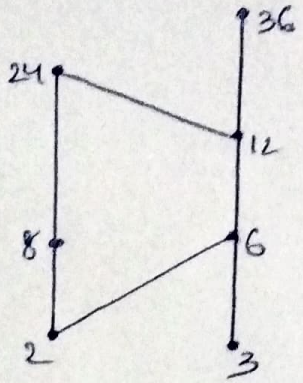
Hasse diagram:-

Step-1:-



Step-2:- $\{1\}$

Step-3:- Delete 1 from A and rest of elements are 2, 3, 6, 8, 12, 24, 36.



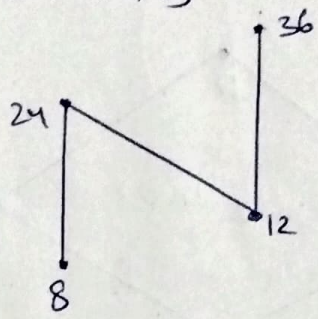
Repeat step-2 :- ~~{3}~~ {1, 3}.

Repeat step-3 :- 2, 6, 12, 8, 24, 36.

Repeat step-2 :- ~~{2}~~ {1, 3, 2}

Repeat step-3 :- {1, 3, 2} The rest are 6, 8, 12, 24, 36.

Repeat step-2 :- {1, 3, 2, 6}



Repeat step-3 :- Rest of elements are 8, 12, 24, 36.

Repeat step-2 :- {1, 3, 2, 6, 8}

Repeat step-3 :- Rest of elements are 12, 24, 36

Repeat step-2 :- {1, 3, 2, 6, 8, 12}

Repeat step-3 :- Rest of elements are 24, 36.

Repeat step-2 :- {1, 3, 2, 6, 8, 12, 24}

Repeat step-3 :- Rest of elements are 36. [Delete 24].

Repeat step-2 :- {1, 3, 2, 6, 8, 12, 24, 36}.

$\therefore 1 < 3 < 2 < 6 < 8 < 12 < 24 < 36.$

* Lexicographic ordering:-

→ Let R be a set of any Number set [Z, N, Q, R] and the Lexicographic ordering is always defined on n-fold product [n=2].

→ consider R×R (R, ≤) total ordering set and Lexico-graphic ordering (≤) on R×R.

→ Lexico graphing order ~~stasf~~ satisfies the following

$(x_1, y_1) \leq (x_2, y_2)$ iff $\begin{cases} \text{(i) } x_1 < x_2 \\ \quad \text{(or)} \\ \text{(ii) } x_1 = x_2 \text{ and } y_1 \leq y_2. \end{cases}$

Note:-

$A = \{a, b, c, d, ____, z\}$

we define a ≤ on A.

$a \leq b \leq c \leq d \leq ____ \leq z$

Then ≤ is a totally ordered relation on A.

That is (A, ≤) is a totally ordered set.

→ Let 's' be string of length 'n' can be considered as a n-tuple.

class ⟶ (c, l, a, s, s)

BUS ⟶ (B, u, s)

Let,

→ s_1 is string of length p [s_1 is p-tuple]

s_2 is string of length q [s_2 is q-tuple].

Ex: (1) discreet discrete

e < t

∴ discreet ≤ discrete (lexico graphic order)

3.6 — FUNCTIONS

* Function :-

→ A Relation [$R \subseteq A \times B$] is said to be a 'function' if it satisfies the following.

(i) "Domain $(R) = A$."

(ii) $\forall\, a \in A$. there is an unique element $b \in A$ $\ni (a,b) \in R$.

Ex:-

(1) $A = \{1, 2, 3\}$, $B = \{a, b\}$.

$R_1 = \{(1,a), (2,b)\}$ not a function

$R_2 = \{(1,a), (1,b), (2,a)\}$ not a function.

Notation :- $f : A \to B$ (or) $A \xrightarrow{f} B$.

synonyms :- Function, Mapping, Transformation.

Ex:-

$f = \{(1,a), (2,b), (3,a)\} \subseteq A \times B$.

$D_f = \{1, 2, 3\} = A$.

∴ It is a function.

* Representation of Functions :-

(1) In set form :-

$A = \{1, 2, 3\}$, $B = \{a, b\}$

$f = \{(1,a), (2,b), (3,a)\}$

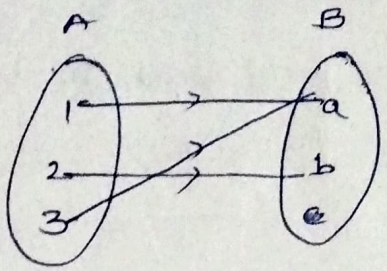
$f \subseteq A \times B$

$D_f = \{1, 2, 3\} = A$

∴ f is a function.

② In pictorial form :-

→ only suitable for finite sets.

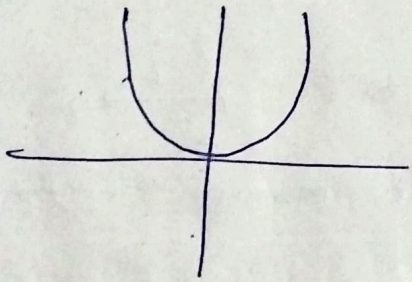


③ In matrix form :-

→ Every function will be represented in the form of matrix.

④ In Graph :-

$f(x) = x^2$



⑤ In Explicit form :-

Ex :- $f(x) = x+1 \quad \forall \; x \in A$

* Note :-

→ Every function contains '3 components'.

① Domain, codomain & Range :-

$f : A \to B$

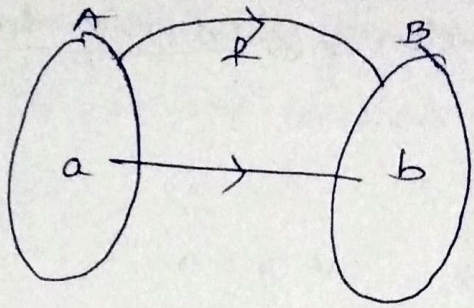
$A =$ 'Df'

$B =$ 'co-domain of f'

'Rf' = The set of all images of elements of A
$= f(A)$.

$\therefore Rf = \{ b \in B / \exists \; a \in A \ni f(a) = b \}$.

Note:-



$(a,b) \in f$ (or) $a f b$ (or) $\boxed{f(a)=b}$

Here 'a' = 'Argument' / 'preimage of b'.

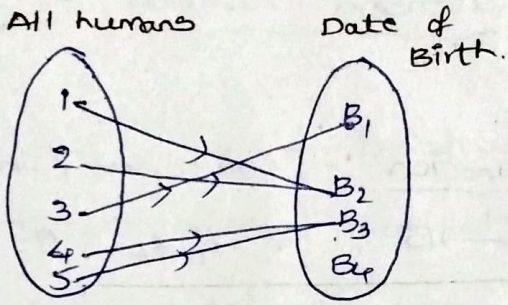
'b' = 'Image of a'. (or) value of 'a' under 'f'.

Note:-

→ Range set is always subset to co-domain

$$\boxed{R_f \subseteq B}$$

Ex:-



* Types of functions:-

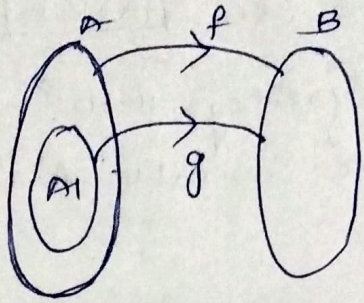
1. Restriction function
2. Extension function
3. One-one function
4. On-to function
5. In-to function
6. Bijection
7. Identity function.
8. Inverse function
9. composition function

① Restrictive function & ② Extension function :-

$f: A \to B$

$A_1 \subseteq A$

$g: A_1 \to B \quad \forall x \in A$



$\boxed{f(x) = g(x) \ \forall x \in A}$ $\quad \therefore \boxed{g = f/A_1}$

$\therefore$ 'g' is a restriction function of 'f' to 'A_1'

$\therefore$ 'f' is a extension function of 'g'

③ one-one function :- [one to one / Injective].

$f: A \to B \quad \forall x_1, x_2 \in A$

$\boxed{\text{If } \underset{p}{x_1 \neq x_2} \text{ then } \underset{q}{f(x_1) \neq f(x_2)}}$

Ex:- $f = \{(1,a), (2,a), (3,b)\}$ (Not one-one).

$\to$ It is in the form of $p \to q$.

$\to$ contrapositive statement is $\sim q \to \sim p$.

$\boxed{\text{If } f(x_1) = f(x_2) \text{ then } x_1 = x_2.}$

* conclusions :-

$f: A \to B$

Let $|A| = m$, $|B| = n$

① Total no. of functions from A to B is B^A (or) $|B|^{|A|}$ (or) n^m

② If $f: A \to B$ is one-one, then $|A| \leq |B|$ $[m \leq n]$.

③ If '$|A| > |B|$' then we can't define one-one function.

④ The no. of one-one functions from A to B is

'$^nP_m = \frac{n!}{(n-m)!}$' $(m \le n)$.

④ on-to function :- [surjective, surjection] & ⑤ Into :-

→ If '$R_f \subset B$' then 'f is into'.

→ If '$R_f = B$' then 'f is onto'. [Range = co-domain]

$\forall$ 'b ∈ B' (co-domain) $\exists$ 'a ∈ A' (co-main) ∋ 'f(a) = b'.

→ If atleast one element in B doesnot contain pre-image then it is 'into function'.

Conclusions :-

Let $f: A \to B$, $|A| = m$, $|B| = n$.

① If $f: A \to B$ is on-to, then $|A| \ge |B|$ $(m \ge n)$.

② The total no. of on-to functions from A to B is

$\boxed{n!\, S(m,n)}$.

$$S(n,k) = \frac{1}{k!} \sum_{i=0}^{k} (-1)^i \binom{k}{i} (k-i)^n.$$

$$\Rightarrow S(m,n) = \frac{1}{n!} \sum_{i=0}^{n} (-1)^i \binom{n}{i} (n-i)^m.$$

$$\therefore \boxed{n!\, S(m,n) = \sum_{i=0}^{n} (-1)^i \binom{n}{i} (n-i)^m}$$

⑥ Bijection :- [correspondance].

'$f: A \to B$'

(i) it must be one to one

(ii) it must be on-to.

① If f is bijective then $|A| = |B|$. [$|A| = n$, $|B| = n$]

② Total no. of bijective functions are '$n!$'.

Here '$|A| = |B| = n$.'

③ If A is 'Non-empty set & '$|A| = n$'

$F_A = \{ f / f : A \to A \text{ is bijection} \}$

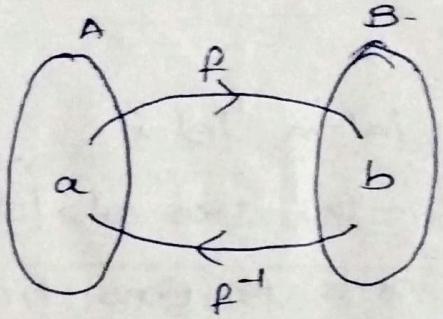
$\therefore F_A$ = The set of all bijections on A.

$\therefore$ '$|F_A| = n!$'.

④ Inverse function:-

$f : A \to B$ is bijection.

$f^{-1} : B \to A$ exists.



$(a, b) \in f$

$(b, a) \in f^{-1}$

If $a \xrightarrow{f} b$ then $b \xrightarrow{f^{-1}} a$.

$$\boxed{\text{If } f(a) = b \iff f^{-1}(b) = a.}$$

⑧ Identity function:-

→ A function which maps every element to itself.

$f(x) = x$.

$I_A : A \to A$ $\qquad x \xrightarrow{f} x$

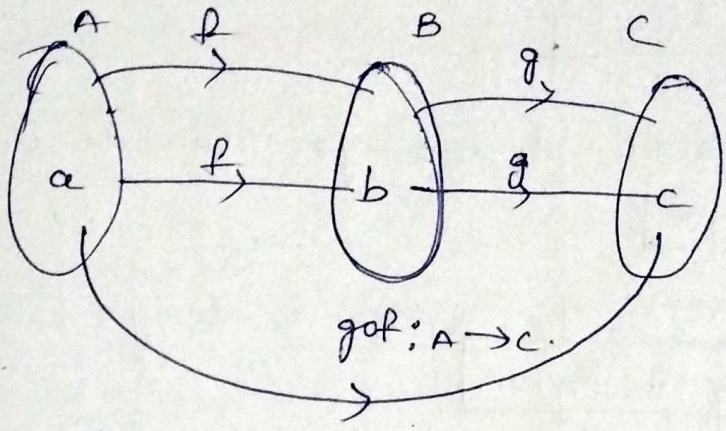
$$\boxed{I_A(x) = x \quad \forall\, x \in A}$$

→ Inverse of Identity function is itself

$$\boxed{I_A^{-1} = I_A.}$$

⑨ composition of function :-

$f: A \to B$ & $g: B \to C$.



$a \xrightarrow{f} b \xrightarrow{g} c \Rightarrow a \xrightarrow{gof} c.$

$(a,b) \in f \wedge (b,c) \in g$ then $(a,c) \in gof$.

Here Domain = A

Co-domain = C.

Conclusions :-

① If $f: A \to B$, $g: C \to D$.

(a) gof exists iff $\boxed{R_f \subseteq D_g}$.

If $R_f \not\subseteq D_g$ then gof Doesnot exists.

(b) fog exists. iff $\boxed{R_g \subseteq D_f}$.

If $R_g \not\subseteq D_f$ then fog doesnot exists.

*** Important points over composition mapping :-**

① Let $f: A \to B$; $g: B \to A$.

$f \circ I_A = f$. and $I_B \circ f = f$.

② $fog : B \to B$

$fog = I_B$; $gof : A \to A$

If $fog = I_B$. If $gof = I_A$

$g = f^{-1}$ $g = f^{-1}$.

③ $f: A \to B$ is bijection

$f^{-1}: B \to A$ is also a bijection.

$$\boxed{(f^{-1})^{-1} = f.}$$

④ composition of mappings neednot be commutative

$$\boxed{fog \neq gof}$$

⑤ composition of mappings is always associative.

$$\boxed{fo(goh) = (fog)oh.}$$

* Binary operations:-

→ Let A be a set. and Binary operation is a function.

$f: A \times A \longrightarrow A$ [Binary operation]

$f: A^2 \longrightarrow A$

$$\boxed{f(a,b) = a f b.}$$

→ $f: A \times A \times A \longrightarrow A$ [Ternary operation].

→ $f: A^n \longrightarrow A$ [n-ary operation].

→ No. of Binary operations, $|A| = n$.

$|A| = |A \times A| = |A|^2 = n^2$.

* Domain:-

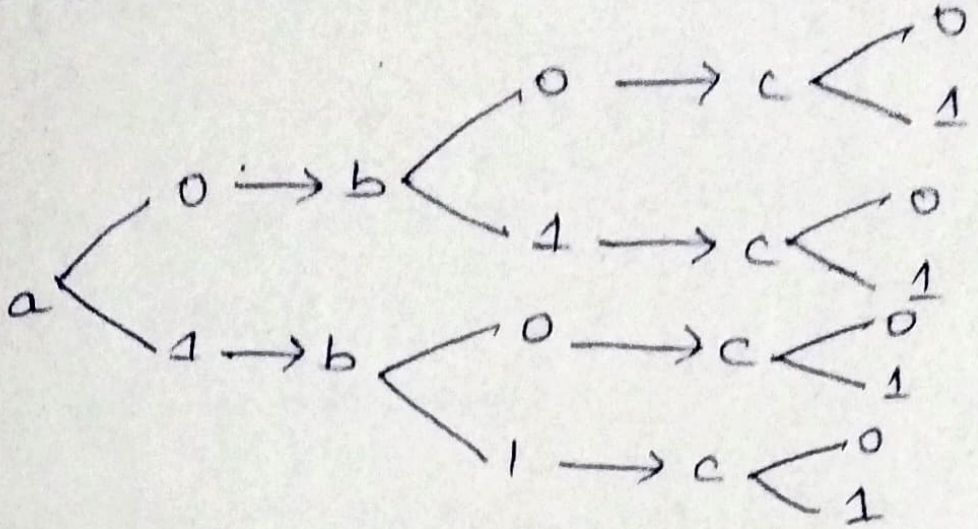
→ The set where the given function is well defined is called 'Domain'.

Ex:- $f(x) = \sqrt{x}$; $x \in R^+$.

* P-1:

List all possible functions from $A = \{a, b, c\}$ and $B = \{0, 1\}$ and indicate in each case whether the function is one to one & is onto?

A) No. of functions from A to B is $|B|^{|A|} = 2^3 = 8$.



$|A| = 3$

$|B| = 2$

$|A| > |B|$

$\therefore$ Not one-one.

$f_0 = \{(a,0), (b,0), (c,0)\}$, into

$f_1 = \{(a,0), (b,0), (c,1)\}$, onto

$f_2 = \{(a,0), (b,1), (c,0)\}$, onto

$f_3 = \{(a,0), (b,1), (c,1)\}$, onto

$f_4 = \{(a,1), (b,0), (c,0)\}$, onto

$f_5 = \{(a,1), (b,0), (c,1)\}$, onto

$f_6 = \{(a,1), (b,1), (c,0)\}$, onto

$f_7 = \{(a,1), (b,1), (c,1)\}$, into.

* **Example:-**

Let $f: R \to R$ be given by $f(x) = -x^2$ and $g: R^+ \to R^+$ be given by $g(x) = \sqrt{x}$, where R^+ is the set of non-negative real numbers. Find fog. Is gof defined?

A) $f: R \to R$

$g: R^+ \to R^+$

We have, gof exists if $R_f \subseteq D_g$

Here $R \not\subseteq R^+$.

$\therefore$ gof is not exists.

Now, fog ; $R_g \subseteq D_f$ i.e, $R^+ \subseteq R$

$\therefore$ fog is exists.

$\Rightarrow fog: R^+ \to R \quad \forall x \in R^+$

$\Rightarrow fog(x) = f[g(x)]$

$= f[\sqrt{x}]$

$= -(\sqrt{x})^2 = -x$

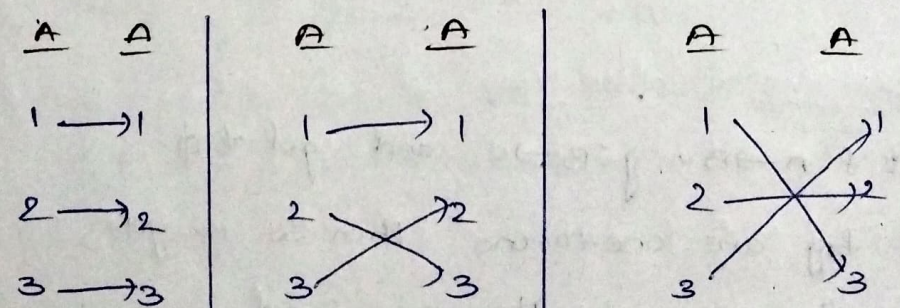
* **Example:-**

Let $A = \{1, 2, 3\}$. Find all elements of F_A and find the inverse of each element?

A) $F_A = \{f / f: A \to A$ is bijections$\}$

$|F_A| = 3$

No. of Bijection functions, $|F_A|! = n! = 6$.



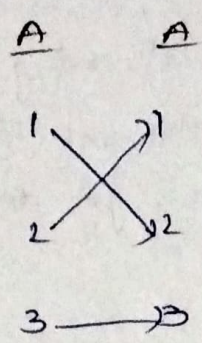
$f_0 = I_A$

f_1

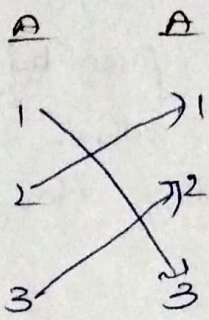
$f_1 = \{(1,1), (2,3), (3,2)\}$

f_2

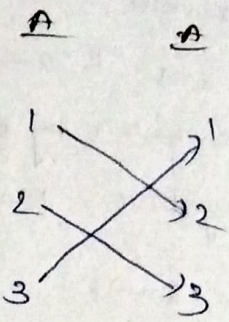
$f_2 = \{(1,3), (2,2), (3,1)\}$



$f_3 = \{(1,2),(2,1)$
$(3,3)\}$



$f_4 = \{(1,3),(2,1),(3,2)\}$



$f_5 = \{(1,2),(2,3),(3,1)\}$

$\therefore F_A = \{f_0, f_1, f_2, f_3, f_4, f_5\}$.

① $f_0^{-1} = f_0$

$f_0^{-1} = I_A^{-1} = I_A = f_0$.

② $f \circ g = I$

$g = f^{-1}$

$f \circ f_1 = I_A$

$f_1^{-1} = f_1$

$f_1 \circ f_1 : A \to A$

$(f_1 \circ f_1)(1) = f_1[f_1(1)] = f_1(1) = 1 = I_A(1)$.

$(f_1 \circ f_1)(2) = f_1[f_1(2)] = f_1(3) = 2 = I_A(2)$.

$(f_1 \circ f_1)(3) = f_1[f_1(3)] = f_1(2) = 3 = I_A(3)$.

$\therefore f_1^{-1} = f_1$

clearly,

③ $f_2^{-1} = f_2$

④ $f_3^{-1} = f_3$

⑤ $f_4^{-1} = f_4$

⑥ $f_5^{-1} = f_5$.

* P-2 :-

Let $f: A \to B$, $g: B \to C$ and $gof \neq \phi$.

(i) if f, g are one-to-one then so is gof.

*** (ii) if f, g are onto then so is gof

(iii) if f, g are bijective then so is gof.

A) Given, $f: A \to B$, $g: B \to C$.

Then $gof: A \to C$ is also a mapping.

(i) suppose f and g are one to one.

<u>claim</u>: To prove that $gof: A \to C$ is one-one.

Let $x_1, x_2 \in A$.

Suppose $gof(x_1) = gof(x_2)$.

$\Rightarrow g(f(x_1)) = g(f(x_2))$.

$\Rightarrow f(x_1) = f(x_2)$

$\Rightarrow x_1 = x_2$

$f(x_1) = f(x_2)$
$\Rightarrow x_1 = x_2$
$g(x_1) = g(x_2)$
$\Rightarrow x_1 = x_2$

$\therefore gof: A \to C$ is one to one.

(ii) suppose f and g are onto.

<u>claim</u>: To prove that $gof: A \to C$ is onto.

Let $c \in C$.

Since, $g: B \to C$ is onto and $c \in C$. then

$\exists\, b \in B \ni g(b) = c$. $\longrightarrow$ (1)

Since, $f: A \to B$ is onto and $b \in B$ then $\exists\, a \in A$

$\ni f(a) = b \longrightarrow$ (2).

From (1) $c = g(b)$

from (2) $\Rightarrow c = g(f(a))$

$\Rightarrow c = (gof)(a)$

Thus $(gof)(a) = c$.

Hence $gof: A \to C$ is onto.

(iii) suppose f and g are Bijections.

$\Rightarrow f$ and g are one-one & onto.

By (1) & (2)

$\therefore gof$ is a bijection.

* P-3:

show that there exists a one-to-one & onto mapping from $A \times B$ to $B \times A$. Is it also onto?

Sol: Let A, B are two non-empty sets.

claim: $\exists$ a one-one & onto [Bijection] from $A \times B$ to $B \times A$.

Now define a function $f: A \times B \to B \times A$ by

$f(a,b) = (b,a) \ \forall \ (a,b) \in A \times B \longrightarrow (1)$

(i) one-one:

Let (a_1, b_1) ; $(a_2, b_2) \in A \times B$.

suppose $f(a_1, b_1) = f(a_2, b_2)$.

By using (1) $\Rightarrow (b_1, a_1) = (b_2, a_2)$.

$\therefore b_1 = b_2$ and $a_1 = a_2$

$\therefore (a_1, b_1) = (a_2, b_2) \rightarrow$ one to one.

(ii) onto:

Let $(b,a) \in B \times A$ then $b \in B$, $a \in A$.

$\Rightarrow (a,b) \in A \times B$.

By (1), we have $f(a,b) = (b,a)$.

$\therefore f$ is onto

Hence f is Bijective.

* characteristic function of a set:

$\rightarrow$ Let A be a subset of U [$\mho$].

$P = \{A, A'\}$ i.e. $A \cup A' = U$ & $A \cap A' = \phi$.

Let for any $x \in U$, $P = \{A, A'\}$ is the partition of U.

Either $x \in A$ or $x \in A'$.

→ characteristic function is denoted by 'ψ' [psi].

→ It is denoted by 'ψ_A' [characteristic function of A]

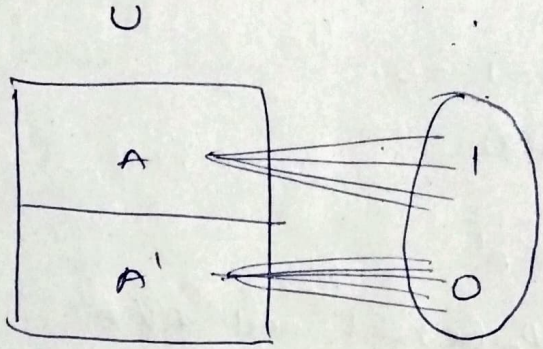
Domain :- U

Co-domain :- $\{0, 1\}$.

~~Range :-~~

→ It is defined by $\boxed{\psi_A(x) = \begin{cases} 1, & \text{if } x \in A \\ 0, & \text{if } x \in A' \text{ (or) } x \notin A \end{cases}}$

→ characteristic function 'always onto' but not 'one-to-one.'

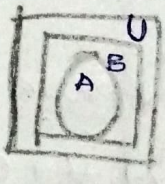


* Properties :- Let A & B are 2 subsets of U.

① $\psi_A(x) = 0, \forall x \in U \iff A = \phi$.

② $\psi_A(x) = 1, \forall x \in U \iff A = U$.

③ $\psi_A(x) \le \psi_B(x) \quad \forall x \in U \iff A \subseteq B$.



④ $\psi_A(x) = \psi_B(x) \quad \forall x \in U \iff A = B$.

⑤ $\psi_{A'}(x) = 1 - \psi_A(x), \quad \forall x \in U$.

⑥ $\psi_{A \cap B}(x) = \psi_A(x) \cdot \psi_B(x) \quad \forall x \in U$.

⑦ $\psi_{A \cup B}(x) = \psi_A(x) + \psi_B(x) \quad \forall x \in U$

⑧ $\psi_{A-B}(x) = \psi_A(x) - \psi_{A \cap B}(x)$.

* P-6:-

Let A and B be subsets of a universal set U.

Then.

(i) $\psi_{A'}(x) = 1 - \psi_A(x)$, for all $x \in U$..

A) Let for any $x \in U$

then either $x \in A$ or $x \in A'$

Case (i):- If $x \in A$

Then $\psi_A(x) = 1$ and $x \notin A'$

$\Rightarrow \psi_{A'}(x) = 0$.

Consider R.H.S:

$\Rightarrow 1 - \psi_A(x) = 1 - 1 = 0$.

$\therefore \psi_{A'}(x) = 1 - \psi_A(x)$.

Case-ii:- If $x \in A'$

Then L.H.S $= \psi_{A'}(x) = 1$ and $x \notin A$.

$\Rightarrow \psi_A(x) = 0$

R.H.S, $1 - \psi_A(x) = 1 - 0 = 1$

$\therefore$ "$\psi_{A'}(x) = 1 - \psi_A(x)$."

* Prove that $\psi_{A \cap B}(x) = \psi_A(x) \cdot \psi_B(x)$ $\forall$ $x \in U$.

A) $\forall x \in U$.

Then either $x \in A \cap B$ (or) $x \in (A \cap B)'$

Case-i:- Suppose $x \in A \cap B$

L.H.S $= \psi_{A \cap B}(x) = 1$.

~~R.H.S~~, Since $x \in A \cap B \Rightarrow x \in A$ and $x \in B$.

$\Rightarrow \psi_A(x) = 1$ & $\psi_B(x) = 1$

R.H.S $= \psi_A(x) \cdot \psi_B(x) = 1$

$\therefore$ L.H.S = R.H.S.

U

case ii :- suppose $x \in (A \cap B)'$.

$\Rightarrow x \notin A \cap B$

L.H.S, $\psi_{A \cap B} = 0$.

since $x \in (A \cap B)' = A' \cup B'$

either $x \in A'$ (or) $x \in B'$.

$\Rightarrow$ either $x \notin A$ (or) $x \notin B$

$\Rightarrow$ either $\psi_A(x) = 0$ (or) $\psi_B(x) = 0$.

$\therefore$ R.H.S $= \psi_A(x) \cdot \psi_B(x) = 0$.

$\therefore$ L.H.S = R.H.S

* Prove that $\psi_{A-B}(x) = \psi_A(x) - \psi_{A \cap B}(x)$

A) $\psi_{A-B}(x) = \psi_{A \cap B'}(x) \ \forall \ x \in U$.

$= \psi_A(x) \cdot \psi_{B'}(x) \cdot \forall \ x \in U$ [property-6]

$= \psi_A(x) \cdot [1 - \psi_B(x)] \ \forall \ x \in U$ [property-5].

$= \psi_A(x) - \psi_A(x) \cdot \psi_B(x) \ \forall \ x \in U$.

$= \psi_A(x) - \psi_{A \cap B}(x) \ \forall \ x \in U$.

$\therefore \psi_{A-B}(x) = \psi_A(x) - \psi_{A \cap B}(x)$.

* Prove that $(A')' = A$ by using characteristic function?

A) $\forall \ x \in U$, we have $\psi_{(A')'}(x) = 1 - \psi_{A'}(x)$ [prop-5]

$= 1 - [1 - \psi_A(x)]$ [prop-5]

$\psi_{(A')'}(x) = \psi_A(x) \ \forall \ x \in U$.

$\therefore (A')' = A$ [By property-4].

* Prove that $A\cap(B\cup C) = A\cap B \cup A\cap C$ by using characteristic function?

A) $\forall x \in U$,

Consider $\psi_{A\cap(B\cup C)}(x) = \psi_A(x)\cdot\psi_{B\cup C}(x)$. [By property]

$= \psi_A(x)\cdot[\psi_B(x)+\psi_C(x) - \psi_{B\cap C}(x)]$ [By prop-3]

$= \psi_A(x)\cdot\psi_B(x) + \psi_A(x)\cdot\psi_C(x) - \psi_A(x)\cdot\psi_{B\cap C}(x)$.

$A\cap B\cap C \equiv$

$\Rightarrow (A\cap B)\cap(A\cap C)$

$\Rightarrow A\cap A\cap B\cap C$

$\Rightarrow A\cap B\cap C$.

$= \psi_{A\cap B}(x) + \psi_{A\cap C}(x) - \psi_{A\cap B\cap C}(x)$ [By prop-5]

$= \psi_{A\cap B}(x) + \psi_{A\cap C}(x) - \psi_{(A\cap B)\cap(A\cap C)}(x)$

$= \psi_{(A\cap B)\cup(A\cap C)}(x)$ [By prop-7]

$\therefore \psi_{A\cap(B\cup C)}(x) = \psi_{(A\cap B)\cup(A\cap C)}(x) \quad \forall x \in U$.

$\therefore A\cap(B\cup C) = (A\cap B)\cup(A\cap C)$.

= * =

* Note:-

$\rightarrow U = \{a, b, c\}$

$P(U) = \{\phi, \{a\}, \{b\}, \{c\}, \{a,b\}, \{b,c\}, \{a,b,c\}\}$.

Elements	$\{a\}$	$\{b\}$	$\{c\}$	$\{a,b\}$	U
a	$\psi_{\{a\}}(a)=1$	0	0	1	1
b	$\psi_{\{a\}}(b)=0$	1	0	1	1
c	$\psi_{\{a\}}(c)=0$	0	1	0	1

$\{a\} \rightarrow 100$

* P-6:-

Prove that $\psi_{A\oplus B}(x) = \psi_A(x) + \psi_B(x) - 2\psi_{A\cap B}(x)$ $\forall x \in U$.

A) $\forall x \in U$.

L.H.S $= \psi_{A\oplus B}(x)$

$\boxed{\psi_{A\cup B}(x) = \psi_A(x) + \psi_B(x)}$ $\forall x \in U$. when A & B are disjoint.

$\therefore A\oplus B = (A-B) \cup (B-A)$.

$$
\begin{aligned}
\text{L.H.S} &= \psi_{A\oplus B}(x) \\
&= \psi_{(A-B)\cup(B-A)}(x). \\
&= \psi_{A-B}(x) + \psi_{B-A}(x). \\
&= \psi_{A\cap B'}(x) + \psi_{B\cap A'}(x) \\
&= \psi_A(x)\cdot\psi_{B'}(x) + \psi_B(x)\cdot\psi_{A'}(x). \\
&= \psi_A(x)[1-\psi_B(x)] + \psi_B(x)[1-\psi_A(x)] \\
&= \psi_A(x) - \psi_A(x)\cdot\psi_B(x) + \psi_B(x) - \psi_A(x)\cdot\psi_B(x). \\
&= \psi_A(x) + \psi_B(x) - 2\psi_A(x)\cdot\psi_B(x) \\
&= \psi_A(x) + \psi_B(x) - 2\psi_{A\cap B}(x). \\
&= \text{R.H.S}.
\end{aligned}
$$

* Hashing Function:-

→ Division method [Mod functions] is one of the most important Hash function.

$$\boxed{a \pmod n = r}$$

$r \in \{0, 1, 2, \ldots, n-1\}$.