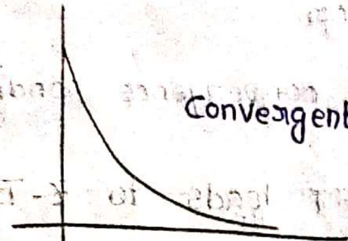
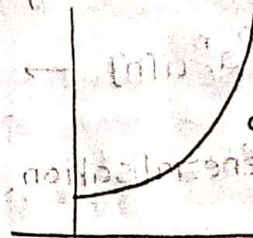
# UNIT - 05    Z - TRANSFORM

→ Z - Transform in discrete time is analogus to Laplace transform in continuous time.

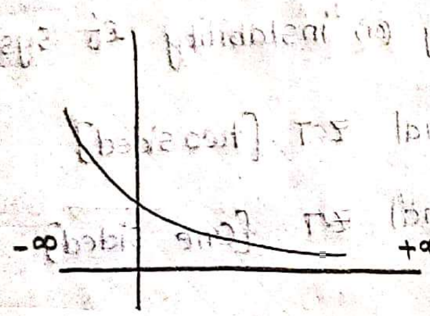
→ F.T does not exist for not absolutely summable signals.



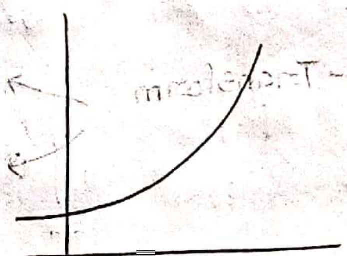

Finite, Abs summable




Not abs summable



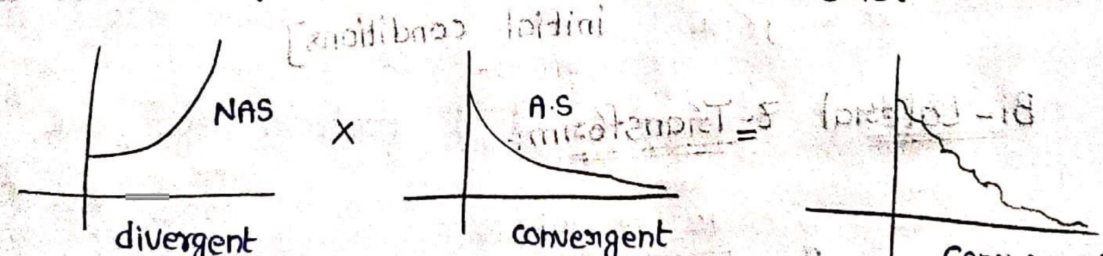

Not absolutely Summable



Not abs. summable

Absolutely summable $\Longrightarrow$ DTFT excists

Not Absolutely Summable $\Longrightarrow$ DTFT doesn't exist




→ Inputs : $a^n u[n]$ ; $a > 1$

$a^{-n}$ ; $-\infty < n < \infty$

$nu[n]$ and other signals which are not absolutely summable.

→ Similarly , DTFT does not exist for signals that are not absolutely summable.

* Generalising the DTFT, so that the signal $x[n]$ is expressed as the sum of complex exponentials, $z^n$ where $z = re^{j\omega}$

* This is equivalent to multiplying the signal by an exponential convergent factor.

  Ex:- $r^{-n} a^n u[n] \rightarrow$ Satisfies convergence condition for $r > a$

* This generalisation of DTFT leads to Z-Transform

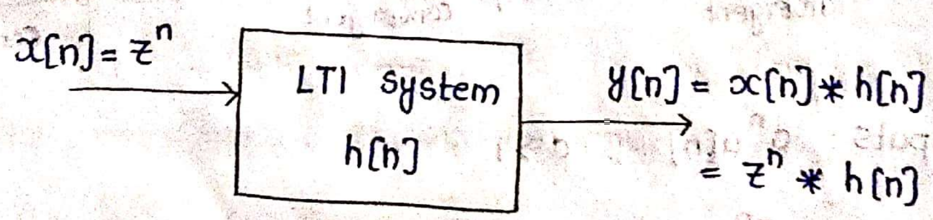
* Z-Transform $\longrightarrow$ Analysis on unstable systems [Stability or instability of systems]

* Z-Transform $\rightarrow$ Bi-lateral Z.T [two sided]
  $\rightarrow$ Unilateral Z.T [One sided]

* Bi-lateral Z.T [nature of system characteristics such as stability, causality, freq. response]

* Uni-lateral Z.T [Solving difference equations with initial conditions]

**Bi-Lateral Z-Transform:-**




Consider applying a complex exponential input $x[n] = z^n$ to an LTI system with impulse response $h[n]$.

System o/p $y[n] = h[n] * x[n]$

$$y[n] = \sum_{k=-\infty}^{\infty} h[k] \cdot x[n-k]$$

$$= \sum_{k=-\infty}^{\infty} h[k] \cdot z^{n-k}$$

$$= z^n \underbrace{\sum_{k=-\infty}^{\infty} h[k] \cdot z^{-k}}_{=H[z]}$$

$$y[n] = H[z] \cdot z^n$$

where, $H[z] = \sum_{k=-\infty}^{\infty} h[k] z^{-k}$

Equivalently, $H[z] = \sum_{n=-\infty}^{\infty} h[n] z^{-n}$

↓

② ← Transfer function / System function of the LTI system.

Note:- A signal for which the system output is a constant times the input is referred to as an eigen function of the system and the amplitude factor is referred to as the system Eigen value.

$z^n \Rightarrow$ Eigen function of LTI system

$H[z] \Rightarrow$ Eigen value.

* Z.T $H[z] = \sum_{n=-\infty}^{\infty} h[n] z^{-n}$ Bi-lateral

$H[z] = \sum_{n=0}^{\infty} h[n] z^{-n}$ Uni-lateral

} ①

Inverse Z-Transform:-

Substituting $z = re^{j\omega}$ in $H[z]$ equation and using 'n' as the variable of summation.

$$H[re^{j\omega}] = \sum_{n=-\infty}^{\infty} h[n] \, (re^{j\omega})^{-n}$$

$$= \sum_{n=-\infty}^{\infty} \left[h[n] \, (r^{-n})\right] e^{-j\omega n}$$

$$H[re^{j\omega}] = \text{DTFT}\left\{h[n] r^{-n}\right\}$$

$\therefore$ The inverse DTFT of $H[re^{j\omega}]$ must be $\{h[n] r^{-n}\}$

$$\therefore \quad h[n] r^{-n} = \frac{1}{2\pi} \int_{2\pi} H[re^{j\omega}] \, e^{j\omega n} \, d\omega$$

$$h[n] = r^{n} \, \frac{1}{2\pi} \int_{2\pi} H[re^{j\omega}] \, e^{j\omega n} \, d\omega$$

$$h[n] = \frac{1}{2\pi} \int_{2\pi} H[re^{j\omega}] \, (re^{j\omega})^{n} \, d\omega \longrightarrow ②$$

A change of variables is performed by letting

$$z = re^{j\omega}$$

$$dz = jre^{j\omega} \, d\omega = jz \, d\omega$$

$$dz = jz \, d\omega$$

$$\Rightarrow d\omega = \frac{1}{j} z^{-1} dz$$

$\omega \longrightarrow 0$ to $2\pi$ interval

which corresponds to one traversal around the circle $|z| = r$.

$$\therefore \quad \boxed{h[n] = \frac{1}{2\pi j} \oint_{c} H[z] \, z^{n-1} \, dz}$$

Inverse of Z.T $\longrightarrow ⑧$

'C' $\Rightarrow$ closed contour in the region of convergence [ROC] of $H[z]$ in anti-clockwise direction.

Note:-

For an arbitrary signal $x[n]$, the z-transform is defined as

$$z\{x[n]\} = X[z] = \sum_{n=-\infty}^{\infty} x[n].\, z^{-n} \longrightarrow ④$$

Inverse z.T of $X[z]$ is defined as

$$x[n] = \frac{1}{2\pi j}\oint_C X[z]\, z^{n-1}\, dz \longrightarrow ⑤$$

* The transform relationship between $x[n]$ & $X[z]$ as

$$x[n] \longleftrightarrow X[z]$$

Region of Convergence: The range till z-transform exists.

Relation b/w z-T & DTFT:

Discrete time signal $x[n]$

$$z\{x[n]\} = X[z] = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$$

Substituting $z = re^{j\omega}$ in the above equation,

$$z\{x[n]\} = \sum_{n=-\infty}^{\infty} x[n]\,[re^{j\omega}]^{-n}$$

$$= \sum_{n=-\infty}^{\infty} [x[n]\, r^{-n}]\, e^{-j\omega n}$$

$$z\{x[n]\} = \mathcal{F}\{x[n]\, r^{-n}\}$$

$\therefore$ The z.T of $x[n]$ is the DTFT of $x[n]r^{-n}$

* If $r=1$

$$z\{x[n]\} = \mathcal{F}\{x[n]\} \text{ for } r=1.$$

The Z.T reduces to the DTFT when $|z| = 1$.

$$\left[ \text{i.e } z = 1 \cdot e^{j\omega} \right]$$

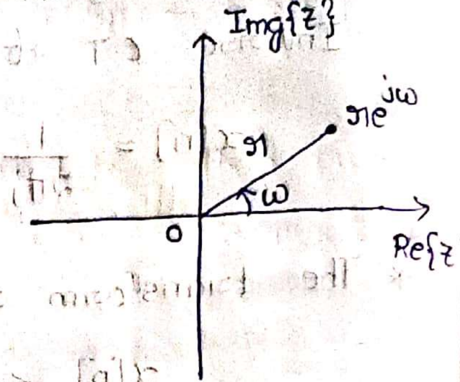
## Z-plane:-

Complex number $z \Rightarrow$ Complex plane

$$z = re^{j\omega}$$

where, $r =$ distance from origin

$\omega =$ Angle from real axis



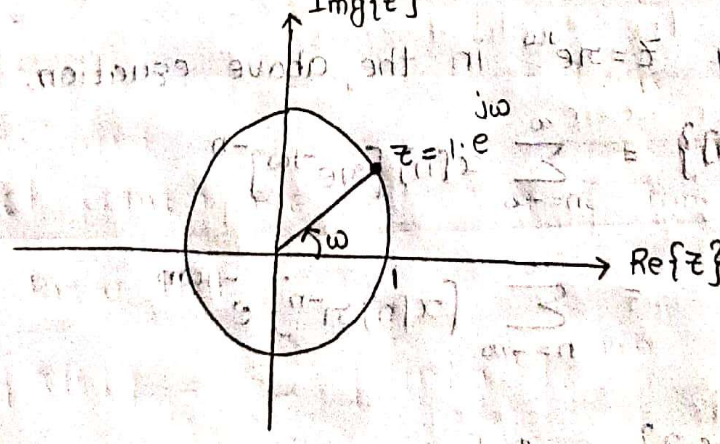

Z-plane

If $r = 1 \Rightarrow$ Z.T = DTFT

$$X[e^{j\omega}] = X[z]\Big|_{z=e^{j\omega}}$$

**Note:-** The Z.T reduces to the DTFT on the contour in the complex z-plane corresponding to a circle with a radius of unity.

Z-plane




The DTFT corresponds to the z-transform evaluated on the unit circle.

## Poles and zeroes :-

General form of Z-transform is the ratio of two polynomials in 'z'.

$$H[z] = \frac{b_m z^m + b_{m-1} z^{m-1} + \cdots + b_1 z + b_0}{a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0}$$

In terms of factors of the polynomials in the numerator and denominator and to write the transfer function in terms of those factors.

$$H[z] = \frac{N[z]}{D[z]} = \frac{K(z-z_1)(z-z_2)(z-z_3)\cdots(z-z_{m-1})(z-z_m)}{(z-P_1)(z-P_2)\cdots(z-P_{n-1})(z-P_n)} \quad \to ⑥$$

'o' $\Leftarrow z \Rightarrow$ zeroes

'x' $\Leftarrow P \Rightarrow$ Poles

<u>Note:-</u> Where the numerator & denominator polynomials $N[z]$ and $D[z]$ have real coefficients and

$$K = \frac{b_m}{a_n}$$

* $z_i$'s are the roots of the equation $N[z] = 0$ and are defined as "<u>zeroes</u>".

* $P_i$'s are the roots of the equation $D[z] = 0$ and are defined as "<u>poles</u>".

In equ ⑥, the factor in the numerator & denominator are written so that when $z = z_0$, the numerator $N[z] = 0$ and the transfer function vanishes.

$$\lim_{z \to z_0} H[z] = 0$$

Similarly when $z = P_i$ the denominator $D[z] = 0$ and the transfer function becomes infinity / unbounded.

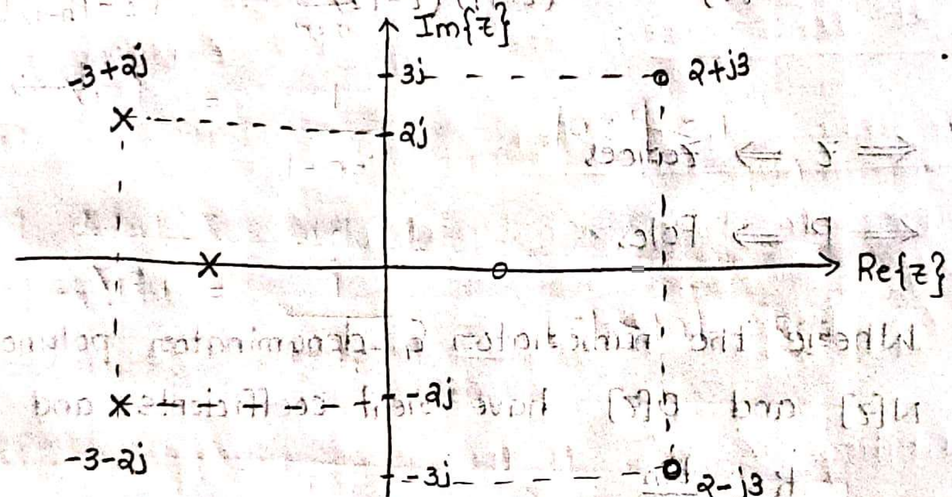
$$\lim_{z \to P_i} H[z] = \infty.$$

**Note:-** All ~~of~~ the coefficients of polynomial $N[z]$ and $D[z]$ are real.

↳ Poles and zeroes must be either purely real or appear in complex conjugate pairs

⇒ Zeroes $z = 1,\ 2 \pm j3$

Poles $P = -2,\ -3 \pm j2$




**Region of Convergence [ROC] for Z-Transform:**

We know that,

$$z\{x[n]\} = \sum_{n=-\infty}^{\infty} [x[n] r^{-n}] e^{-j\omega n}$$

Z.T is gaurateed to converge if $x[n] r^{-n}$ is absolutely summable.

i.e, $$\sum_{n=-\infty}^{\infty} |x[n] r^{-n}| < \infty$$

This gaurantees that $X[z]$ is finite,

$\because \ Z\{x[n]\} = X[z] = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$

Substituting $z = re^{j\omega}$ in above equation,

$$X[z] = \sum_{n=-\infty}^{\infty} x[n] \left(re^{j\omega}\right)^{-n}$$

$$X[z] = \sum_{n=-\infty}^{\infty} \left[x[n] r^{-n}\right] e^{-j\omega n}$$

$$|X[z]| = \left| \sum_{n=-\infty}^{\infty} \left[x[n] r^{-n}\right] e^{-j\omega n} \right|$$

$$|X[z]| \leq \sum_{n=-\infty}^{\infty} \left|x[n] r^{-n}\right| \underbrace{\left|e^{-j\omega n}\right|}_{=1}$$

$$|X[z]| \leq \sum_{n=-\infty}^{\infty} \left|x[n] r^{-n}\right|$$

So if $\sum_{n=-\infty}^{\infty} \left|x[n] r^{-n}\right| < \infty$ then $|X[z]| < \infty$

* The range of values of $|z| = r$ for which the z.T converges is termed as "Roc".

<u>Note</u>:- Roc consists of those values of '$r$' for which the DTFT of $x[n] r^{-n}$ converges.

⊛ z-T exists for some signals that do not have a DTFT But by limiting to certain range of values of '$r$' we may ensure that $x[n] r^{-n}$ is absolutely summable, even though $x[n]$ is not absolutely summable by itself.

⊛ Roc provides the information about whether $x[n]$ is Fourier transformable or not.

$\because$ DTFT is obtained from bilateral z-transform by setting "$r = 1$" the Roc in this case is a unit circle.

∴ If the ROC for $X[z]$ includes the unit circle, $x[n]$ is fourier transformable.

(Pb) $x[n] = a^n u[n]$

$$z\{x[n]\} = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} a^n u[n] z^{-n}$$

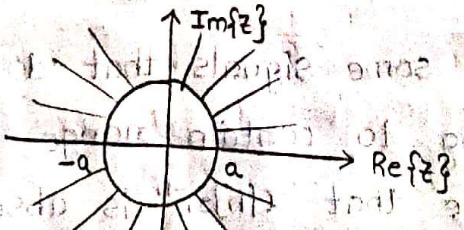
$$= \sum_{n=0}^{\infty} a^n z^{-n}$$

$$= \sum_{n=0}^{\infty} (az^{-1})^n$$

$$X[z] = \frac{1}{1-az^{-1}}; \quad |az^{-1}| < 1$$

$$\boxed{X[z] = \frac{1}{1-az^{-1}}; \quad |z| > |a| \;: \text{ROC}}$$

$$a^n u[n] \xleftrightarrow{\text{Z-T}} \frac{1}{1-az^{-1}} \quad |z| > |a|$$




(Pb) $x[n] = -a^n u[-n-1]$

$$z\{x[n]\} = X[z] = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

$$u[n] = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$$

$$u[-n-1] = \begin{cases} 1 & (-n-1) \geq 0 \Rightarrow n \leq -1 \\ 0 & (-n-1) < 0 \Rightarrow n > -1 \end{cases}$$

$$z\{x[n]\} = X[z] = \sum_{n=-\infty}^{\infty} -a^n \underbrace{u[-n-1]}_{\hookrightarrow -\infty \text{ to } -1} \cdot z^{-n}$$

$$= -\sum_{n=-\infty}^{-1} a^n z^{-n}$$

$$= -\sum_{n=-\infty}^{-1} (az^{-1})^n$$

$$\sum_{n=-1}^{-\infty} (az^{-1})^n = -\sum_{n=1}^{\infty} (a^{-n} z^{n})$$

$$X[z] = -\sum_{n=1}^{\infty} (a^{-1}z)^n$$

$$= -\frac{a^{-1}z}{1-a^{-1}z} \qquad \left( \because \sum_{n=1}^{\infty} \alpha^n = \frac{\alpha}{1-\alpha}, |\alpha| < 1 \right)$$

for $|a^{-1}z| < 1$

$|z| < |a|$

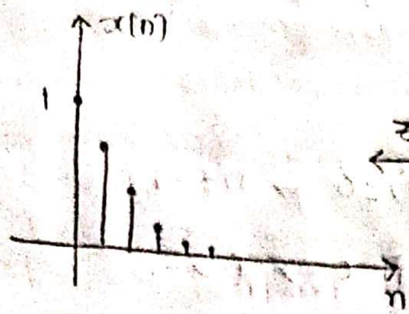
$$= \frac{-z/a}{1-z/a}$$

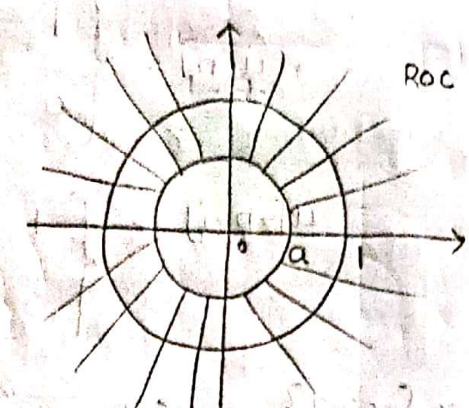
$$X[z] = \frac{-z}{a-z} \quad ; \quad |z| < |a| \Rightarrow X[z] = \frac{z}{z-a} \; ; |z| < |a|$$

$$= \frac{1}{1-az^{-1}}$$

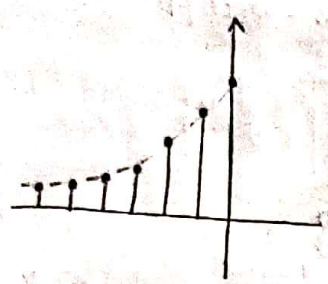
$$\boxed{\begin{array}{lll} a^n u[n] \longleftrightarrow \dfrac{1}{1-az^{-1}} & & |z| > |a| \\[2ex] -a^n u[-n-1] \longleftrightarrow \dfrac{z}{z-a} = \dfrac{1}{1-az^{-1}} & & |z| < |a| \end{array}}$$

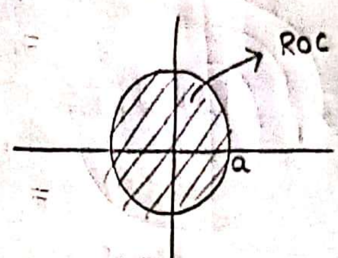
* $x[n] = a^n u[n] \quad 0 < a < 1$





* $-a^n u[-n-1] \xleftrightarrow{z.T} \dfrac{1}{1-az^{-1}} \; ; \; |z| < |a|$





(Pb) Determine the z-Transform $x[n] = \left(\frac{1}{2}\right)^n u[n] + 2^n u[n]$ and depict Roc and location of poles & zeroes in the z-plane.

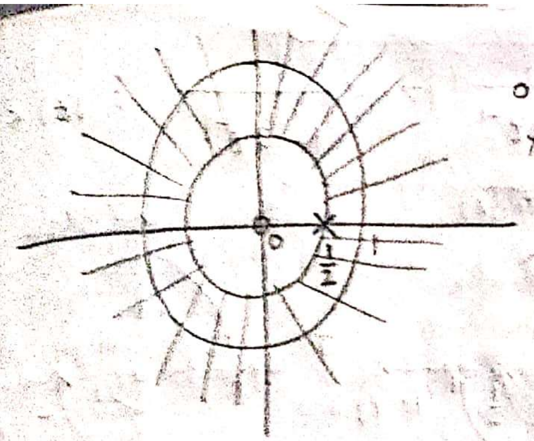
Ans:- $z\{x[n]\} = z\left\{\left(\frac{1}{2}\right)^n u[n]\right\} + z\{2^n u[n]\}$

By using $a^n u[n] \longleftrightarrow \dfrac{1}{1-az^{-1}} \quad |z| > |a|$

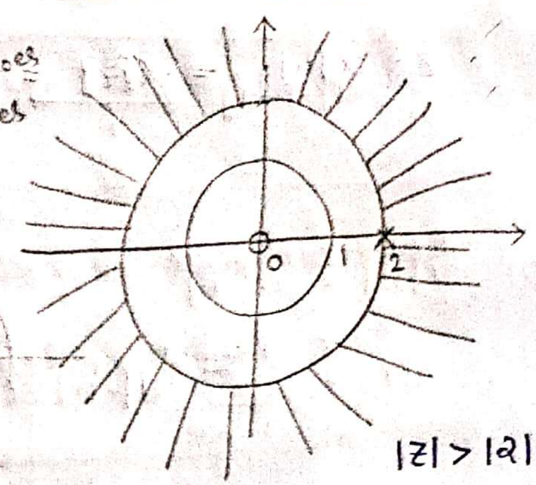
① $\left(\frac{1}{2}\right)^n u[n] \longleftrightarrow \dfrac{1}{1-\frac{1}{2}z^{-1}} \quad |z| > \left|\frac{1}{2}\right|$

② $2^n u[n] \longleftrightarrow \dfrac{1}{1-2z^{-1}} \quad |z| > |2|$

$$X[z] = \frac{1}{1-\frac{1}{2}z^{-1}} + \frac{1}{1-2z^{-1}} \; ; \; |z| > |2|$$




$|z| > \frac{1}{2}$




$|z| > |2|$

$$\frac{1}{1-\frac{1}{2}z^{-1}} \Rightarrow \frac{1}{1-\frac{1}{2z}} = \frac{z}{z-\frac{1}{2}}$$

$z \rightarrow N(z)$ zeroes

$z-\frac{1}{2} \rightarrow D(z)$ poles

$$\frac{1}{1-2z^{-1}} = \frac{z}{z-2}$$

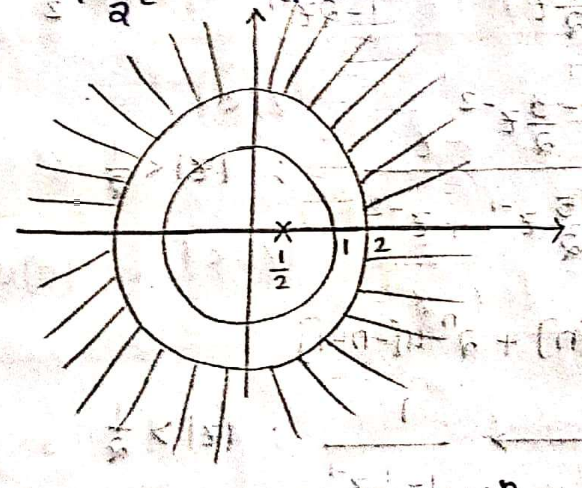
$z \rightarrow N(z)$ zeroes

$z-2 \rightarrow D(z)$ poles

$N(z)=0 \Rightarrow z=0$

$D(z)=0 \Rightarrow z=2$

$$X[z] = \frac{1}{1-\frac{1}{2}z^{-1}} + \frac{1}{1-2z^{-1}} \quad ; \; |z| > |2|$$




(Pb) Determine the Z.T of $x[n] = -\left(\frac{1}{2}\right)^n u[-n-1] + 2^n u[-n-1]$ ROC, location of poles & zeroes.

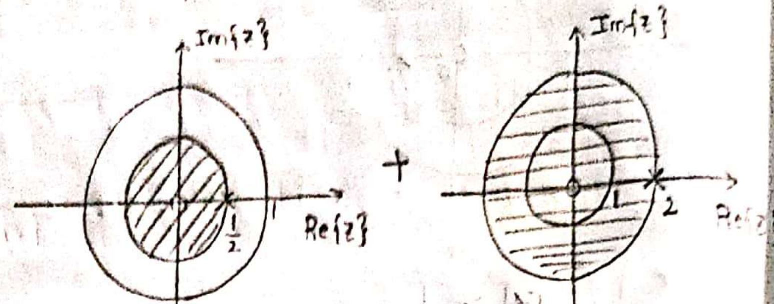
Ans:

$$-a^n u[-n-1] \xleftrightarrow{Z.T} \frac{1}{1-az^{-1}} \qquad |z| < |a|$$

$$-\left(\frac{1}{2}\right)^n u[-n-1] \xleftrightarrow{Z.T} \frac{1}{1-\frac{1}{2}z^{-1}} \qquad |z| < \left|\frac{1}{2}\right|$$

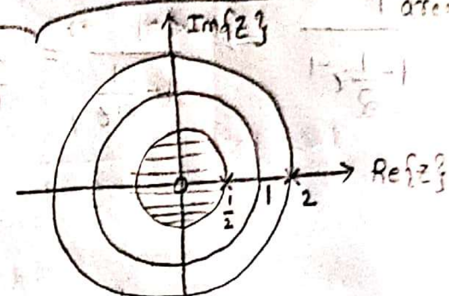
$$-(2)^n u[-n-1] \xleftrightarrow{Z.T} \frac{1}{1-2z^{-1}} \qquad |z| < 2$$

$$Z\{x[n]\} = X[z] = \frac{1}{1-\frac{1}{2}z^{-1}} - \frac{1}{1-2z^{-1}} \quad ; \ |z| < \frac{1}{2}$$




common area

$X[z]$ ROC :




$$X[z] = \frac{1}{1-\frac{1}{2}z^{-1}} - \frac{1}{1-2z^{-1}} \quad ; \ |z| < \frac{1}{2}$$

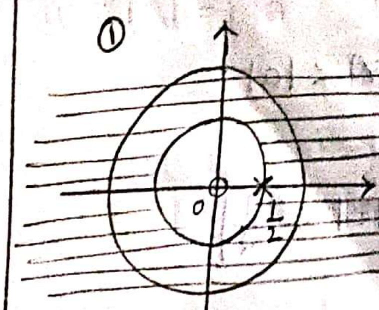
$$X[z] = \frac{-\frac{3}{2}z^{-2}}{1-\frac{5}{2}z^{-1}+z^{-2}} \quad ; \ |z| < \frac{1}{2}$$

(Pb) $x[n] = \left(\frac{1}{2}\right)^n u[n] + 2^n u[-n-1]$

Ans:- ① $\left(\frac{1}{2}\right)^n u[n] \longleftrightarrow \frac{1}{1-\frac{1}{2}z^{-1}} \quad ; \ |z| > \frac{1}{2}$

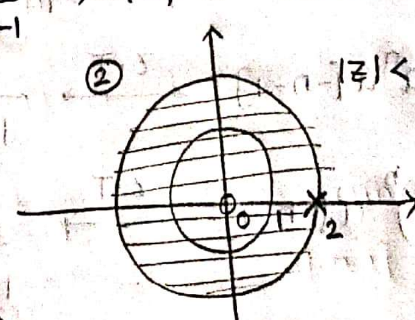
② $-2^n u[-n-1] \longleftrightarrow \frac{1}{1-2z^{-1}} \quad ; \ |z| < 2$

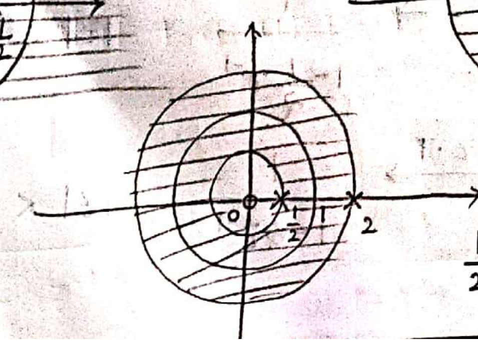
①




$|z| > \frac{1}{2}$

② $|z| < 2$




$\frac{1}{2} < |z| < 2$

(Pb) Find the z-transform of unit impulse function $x[n] = \delta[n]$.

Ans:- By definition,

$$z\{x[n]\} = X[z] = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} \delta[n] \cdot z^{-n} = z^{-n}\Big|_{n=0} = z^{0} = 1$$

$\hookrightarrow$ exists at origin "n=0"

$$X[z] = 1$$

$$\boxed{\delta[n] \xleftrightarrow{Z.T} 1}$$

ROC : Entire z-plane including $z=0$ & $z=\infty$

(Pb) $x[n] = u[n]$

$$X[z] = \sum_{n=-\infty}^{\infty} u[n]\, z^{-n} = \sum_{n=0}^{\infty} z^{-n} = \sum_{n=0}^{\infty} (z^{-1})^{n}$$

$$= \frac{1}{1-z^{-1}} \qquad |z^{-1}| < 1$$

$$|z| > 1$$

$$\boxed{u[n] \xleftrightarrow{Z.T} \frac{1}{1-z^{-1}} \; ; \; |z| > 1}$$

(Pb) $x[n] = -u[-n-1]$

$$X[z] = \sum_{n=-\infty}^{\infty} -u[-n-1] \cdot z^{-n}$$

$$u[n] = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases} \qquad u[-n-1] = \begin{cases} 1 & (-n-1) \geq 0 \Rightarrow n \leq -1 \\ 0 & (-n-1) < 0 \Rightarrow n > -1 \end{cases}$$

$$X[z] = \sum_{n=-\infty}^{\infty} -u[-n-1]\, z^{-n}$$

$$= -\sum_{n=-\infty}^{-1} z^{-n} = -\sum_{n=1}^{\infty} z^{+n} = -\frac{z}{1-z} = -\frac{1}{\frac{1}{z}-1}$$

$$= \frac{1}{1-z^{-1}}$$

$$\boxed{-u[-n-1] \xleftrightarrow{Z.T} \frac{1}{1-z^{-1}} \; ; \; |z|<1}$$

(Pb) $x[n] = u[-n]$

$$X[z] = \sum_{n=-\infty}^{\infty} u[-n] \cdot z^{-n}$$

$$u[n] = \begin{cases} 1 & n \geq 0 \\ 0 & n<0 \end{cases} \qquad u[-n] = \begin{cases} 1 & n \leq 0 \\ 0 & n>0 \end{cases}$$

$$X[z] = \sum_{n=-\infty}^{0} z^{-n} = \sum_{n=0}^{\infty} z^{n} = \frac{1}{1-z} \; ; \; |z|<1$$

$$X[z] = \frac{1}{z[z^{-1}-1]} = \frac{z^{-1}}{z^{-1}-1} = \frac{-z^{-1}}{1-z^{-1}}$$

$$\boxed{\begin{array}{ll} u[-n] \xleftrightarrow{Z.T} \dfrac{1}{1-z} & |z|<1 \\ u[-n] \xleftrightarrow{Z.T} \dfrac{-z^{-1}}{1-z^{-1}} & |z|<1 \end{array}}$$

(Pb) $x[n] = a^{|n|}$

ⓐ $0<a<1$ ⓑ $1<a<\infty$

Ans:- ⓐ $x[n] = a^{|n|} = \begin{cases} a^{n} & n \geq 0 \\ a^{-n} & n<0 \end{cases}$

$$x[n] = \underbrace{a^{n} u[n]}_{\downarrow} + \underbrace{a^{-n} u[-n-1]}$$

$u[n] \rightarrow$ Right sided

$u[-n-1] \rightarrow$ Left sided

$$\frac{1}{1-az^{-1}} \; ; \; |z|>a \qquad \frac{-1}{1-a^{-1}z^{-1}} \; ; \; |z|<a$$

Scanned by CamScanner

$$a^n u[n] \longleftrightarrow \frac{1}{1-az^{-1}} \quad |z|>a$$

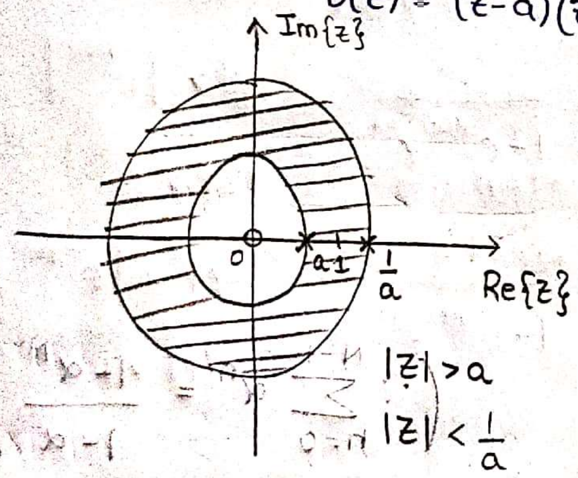
$$-a^n u[-n-1] \longleftrightarrow \frac{1}{1-az^{-1}} \quad |z|<a$$

$$-a^{-n} u[-n-1] \longleftrightarrow \frac{1}{1-a^{-1}z^{-1}} \quad |z|<\frac{1}{a}$$

$$\boxed{\begin{aligned} X[z] &= \frac{1}{1-az^{-1}} - \frac{1}{1-a^{-1}z^{-1}} \quad ; \quad a<|z|<\frac{1}{a} \\ X[z] &= \left(\frac{a^2-1}{a}\right) \frac{z}{(z-a)\left(z-\frac{1}{a}\right)} \quad ; \quad a<|z|<\frac{1}{a} \end{aligned}}$$

$K = \dfrac{a^2-1}{a}$ $\quad N(z) = z \Rightarrow$ Zeroes at $z=0$

$D(z) = (z-a)\left(z-\frac{1}{a}\right) \Rightarrow$ Poles at $z = a, \frac{1}{a}$




$a = 0.5$

$\dfrac{1}{a} = \dfrac{K}{1/2} = 2$

ROC $\longrightarrow a<|z|<\frac{1}{a}$

(b) $x[n] = a^{|n|} \quad ; \quad 1<a<\infty$

$$= \begin{cases} a^n & n\geq 0 \\ a^{-n} & n<0 \end{cases}$$

$$x[n] = a^n u[n] - \left[-a^{-n} u[-n-1]\right]$$

$$a^n u[n] \longleftrightarrow \frac{1}{1-az^{-1}} \quad |z|>a$$

$$-a^{-n} u[-n-1] \longleftrightarrow 1/1-a^{-1}z^{-1} \quad |z|<\frac{1}{a}$$

$a > 1 \Rightarrow a = 2$

$$\frac{1}{a} = \frac{1}{2} = 0.5$$

Common Roc doesnot exists,

$\therefore$ z-Transform won't exist for $x[n] = a^{|n|}$ ; $1 < a <$

$$\boxed{\begin{array}{l} a^{|n|} \text{ for } 0 < a < 1 \Rightarrow z.T \text{ exists} \\ a^{|n|} \text{ for } 1 < a < \infty \Rightarrow z.T \text{ doesn't exists.} \end{array}}$$

(Pb) $x[n] = a^n \quad 0 \le n \le N-1$ ; $X[z] = ?$

$$X[z] = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

$$= \sum_{n=0}^{N-1} a^n z^{-n}$$

$$= \sum_{n=0}^{N-1} (az^{-1})^n$$

$$= \frac{1-(az^{-1})^N}{1-az^{-1}} \qquad \left(\because \sum_{n=0}^{N-1} \alpha^n = \frac{1-\alpha^N}{1-\alpha}\right)$$

$$X[z] = \frac{1}{z^{N-1}} \cdot \frac{z^N - a^N}{z-a}$$

(Pb) $x[n] = \begin{cases} 1 & 0 \le n \le N-1 \\ 0 & \text{otherwise} \end{cases}$

(Pb) $x[n] = \sin(\omega_0 n)\, u(n)$

Ans:- $x[n] = \left[\dfrac{e^{j\omega_0 n} - e^{-j\omega_0 n}}{2j}\right] u[n]$

$$Z\{x[n]\} = \frac{1}{2j} Z\left\{e^{j\omega_0 n} u[n]\right\} - \frac{1}{2j} Z\left\{e^{-j\omega_0 n} u[n]\right\}$$

We know that, $a^n u(n) \longleftrightarrow \dfrac{1}{1-az^{-1}} = |z| > |a|$

① $(e^{j\omega_0})^n u(n) \longleftrightarrow \dfrac{1}{1-e^{j\omega_0} z^{-1}}$ $\quad |z| > |e^{j\omega_0}|$

$|z| > 1$

② $(e^{-j\omega_0})^n u(n) \longleftrightarrow \dfrac{1}{1-e^{-j\omega_0} z^{-1}}$ $\quad |z| > |e^{-j\omega_0}|$

$|z| > 1$

$$X[z] = \frac{1}{2j}\left[\frac{1}{1-e^{j\omega_0} z^{-1}} - \frac{1}{1-e^{-j\omega_0} z^{-1}}\right] \quad |z| > 1$$

$$X[z] = \frac{1}{2j}\left[\frac{1-e^{-j\omega_0} z^{-1} - 1 + e^{j\omega_0} z^{-1}}{(1-e^{j\omega_0} z^{-1})(1-e^{-j\omega_0} z^{-1})}\right] \quad |z| > 1$$

$$X[z] = \frac{z^{-1}\left[\dfrac{e^{j\omega_0} - e^{-j\omega_0}}{2j}\right]}{1-2z^{-1}\left[\dfrac{e^{j\omega_0} + e^{-j\omega_0}}{2}\right] + z^{-2}} \quad |z| > 1$$

$$X[z] = \frac{z^{-1}\sin(\omega_0)}{1-2z^{-1}\cos(\omega_0)+z^{-2}} \qquad |z|>1$$

$$\boxed{\sin(\omega_0 n)\,u[n] \xleftrightarrow{Z.T} \frac{z^{-1}\sin(\omega_0)}{1-2z^{-1}\cos(\omega_0)+z^{-2}} \qquad |z|>1}$$

(Pb) $x[n] = \cos(\omega_0 n)\,u[n]$

$$x(n) = \left(\frac{e^{j\omega_0 n}+e^{-j\omega_0 n}}{2}\right) u[n]$$

$$Z\{x[n]\} = \frac{1}{2} Z\{e^{j\omega_0 n}u[n]\} + \frac{1}{2} Z\{e^{-j\omega_0 n}u[n]\}$$

We know that, $a^n u[n] \longleftrightarrow \frac{1}{1-az^{-1}} \quad |z|>|a|$

$$(e^{j\omega_0})^n u[n] \longleftrightarrow \frac{1}{1-e^{j\omega_0}z^{-1}} \qquad |z|>|e^{j\omega_0}|$$

$$|z|>1$$

$$(e^{-j\omega_0})^n u[n] \longleftrightarrow \frac{1}{1-e^{-j\omega_0}z^{-1}} \qquad |z|>|e^{-j\omega_0}|$$

$$|z|>1$$

$$X[z] = \frac{1}{2}\left[\frac{1}{1-e^{j\omega_0}z^{-1}} + \frac{1}{1-e^{-j\omega_0}z^{-1}}\right] \qquad |z|>1$$

$$X[z] = \frac{1}{2}\left[\frac{1-e^{-j\omega_0}z^{-1}+1-e^{j\omega_0}z^{-1}}{(1-e^{j\omega_0}z^{-1})(1-e^{-j\omega_0}z^{-1})}\right]$$

$$= \frac{1}{2}\left[\frac{2-(e^{j\omega_0}+e^{-j\omega_0})z^{-1}}{(1-e^{j\omega_0}z^{-1})(1-e^{-j\omega_0}z^{-1})}\right]$$

$$= \frac{1-z^{-1}\cos(\omega_0)}{1-2z^{-1}\cos\omega_0+z^{-2}} \quad ; \ |z|>1$$

$$\boxed{\cos(\omega_0 n)\,u[n] \xleftrightarrow{Z.T} \frac{1-z^{-1}\cos(\omega_0)}{1-2z^{-1}\cos(\omega_0)+z^{-2}} \; ; \; |z|>1}$$

(Pb) Determine the Z-Transform & Roc of the following finite duration signals.

ⓐ $x[n] = \{\underset{\uparrow}{1}, 2, 6, -2, 0, 3\}$

n=0 1 2 3 4 5

Ans:- By definition,

$$X[z] = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n} = \sum_{n=0}^{5} x[n]\, z^{-n}$$

$$X[z] = x[0]z^{0} + x[1]z^{-1} + x[2]z^{-2} + x[3]z^{-3} + x[4]z^{-4} + x[5]z^{-5}$$

$$X[z] = 1 + 2z^{-1} + 6z^{-2} - 2z^{-3} + 0 + 3z^{-5}$$

Roc : Entire z-plane except $z=0$

$\because$ $X[z]$ becomes unbounded for $z=0$.

(b) $x[n] = \{1, 2, \underset{\uparrow}{6}, -2, 0, 3\}$

$$X[z] = 1\cdot z^{2} + 2z^{1} + 6z^{0} - 2z^{-1} + 0 + 3z^{-3}$$

Roc: Entire z-plane except $z=0$ & $z=\infty$

$\because$ $X[z]$ becomes unbounded for both $z=0$ & $z=\infty$

(c) $x[n] = \{\underset{\uparrow}{0}, 0, 1, 2, 6, -2, 3\}$

$$X[z] = z^{-2} + 2z^{-3} + 6z^{-4} - 2z^{-5} + 3z^{-6}$$

Roc: Entire z-plane except $z=0$

$\because$ $X[z] = \infty$ for $z=0$.

(d) $x[n] = \{1, 2, 6, -2, 0, \underset{\uparrow}{3}\}$

$$X[z] = z^5 + 2z^4 + 6z^3 - 2z^2 + 3$$

Roc is entire z-plane except $z = \infty$

$\because$ $X[z]$ becomes unbounded for $z = \infty$.

## Properties of z-Transform:

1. **Linearity:**

If $x_1(n) \longleftrightarrow X_1[z]$ with ROC $= R_1$

and $x_2(n) \longleftrightarrow X_2[z]$ with ROC $= R_2$

then $a x_1(n) + b x_2(n) \longleftrightarrow a X_1[z] + b X_2[z]$

with Roc containing $R_1 \cap R_2$

2. **Time shifting:**

If $x[n] \longleftrightarrow X[z]$ with Roc $= R$

then $x[n-n_0] \longleftrightarrow z^{-n_0} X[z]$

with Roc $= R$, except for the possible addition and deletion of $z = 0$ or $z = \infty$

3. **Scaling in z-domain:**

If $x[n] \longleftrightarrow X[z]$ with Roc $= R$

then $z_0^n \, x[n] \longleftrightarrow X\left[\frac{z}{z_0}\right]$ with Roc $= |z_0| R$

'$z_0$' scaling factor

4. **Time Reversal:**

If $x[n] \longleftrightarrow X[z]$ with Roc $= R$

then $x[-n] \longleftrightarrow X\left[\frac{1}{z}\right] = X[z^{-1}]$ with Roc $= \frac{1}{R}$.

5. Differentiation in z-domain:

If $x(n) \longleftrightarrow X[z]$ with $ROC = R$

then $n x(n) \longleftrightarrow -z \frac{dX[z]}{dz}$ with $ROC = R$

6. Time Expansion:

Let 'm' be a positive integer

$$x_{<m>}[n] = \begin{cases} x\left[\frac{n}{m}\right] & \text{if 'n' is a multiple of 'm'.} \\ 0 & \text{if 'n' is not a multiple of 'm'.} \end{cases}$$

$x_{<m>}[N]$ is obtained from $x(n)$ by placing $(m-1)$ zeroes between successive values of the original signal

$x_{<m>}[N] \Rightarrow$ slowed version of $x[n]$

If $x(n) \longleftrightarrow X[z]$ with $ROC = R$

then $x_{<m>}[n] \longleftrightarrow X[z^m]$ with $ROC = R^{1/m}$

7. Convolution:

If $x_1[n] \longleftrightarrow X_1[z]$ with $ROC = R_1$

$x_2[n] \longleftrightarrow X_2[z]$ with $ROC = R_2$

then $x_1[n] * x_2[n] \longleftrightarrow X_1[z] \cdot X_2[z]$

with Roc containing $R_1 \cap R_2$

Prop-6: Example:

$x(n) = \{1, 2, 3, 4\}$

$x_{<2>}[n] = x\left[\frac{n}{2}\right]$

$(m-1) = (2-1)$ zeroes should placed b/w two successive samples

$x_{<2>}[n] = \{1, 0, 2, 0, 3, 0, 4\}$ ($\because$ Padding zeroes)

8. Accumulation:

If $x[n] \longleftrightarrow X[z]$ with Roc = R

then $\sum_{k=-\infty}^{n} x[k] \longleftrightarrow \frac{1}{1-z^{-1}} X[z]$

with Roc atleast $R \cap \{|z|>1\}$

9. First difference:

If $x[n] \longleftrightarrow X[z]$ with Roc = R

then $x[n] - x[n-1] \longleftrightarrow (1-z^{-1}) X[z]$

with Roc atleast $R \cap \{|z|>0\}$

10. Conjugation and Conjugate Symmetry:

If $x[n] \longleftrightarrow X[z]$ with Roc = R

then $x^*[n] \longleftrightarrow X^*[z^*]$ with Roc = R.

Ex:1 Determine the Z.T & Roc of the signal

$$x[n] = a^n u[n] - a^n u[n-1]$$

Ans:- By definition,

$$X[z] = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} a^n u[n] z^{-n} - \sum_{n=-\infty}^{\infty} a^n u[n-1] \cdot z^{-n}$$

$$= \sum_{n=0}^{\infty} (az^{-1})^n - \sum_{n=1}^{\infty} (az^{-1})^n$$

$$= \frac{1}{1-az^{-1}} - \frac{az^{-1}}{1-az^{-1}} \; ; \quad |az^{-1}| < 1$$

$$|z| > |a|$$

$$= \frac{1-az^{-1}}{1-az^{-1}} = 1 \Rightarrow X[z] = 1$$ Roc is entire z-plane

Note:- The sequence $a^n u[n]$ & $a^n u[n-1]$ both have Roc defined by $|z| > |a|$

But the signal $x[n] = a^n u[n] - a^n u[n-1] = \delta[n]$ is finite duration signal. $\{\delta[n]\}$

$\downarrow$

& its Z-T is $X[z] = 1$ which has Roc i.e. the entire z-plane

$$a^n[u[n] - u[n-1]] = a^n \delta[n]$$

$$= a^n \delta[n]\Big|_{n=0}$$

$$= a^0 \delta[n]$$

$$= \delta[n]$$

Ex:2 Determine the Z.T & Roc of the following signals.

(a) $x[n] = \delta[n-K]$

We know that, $\delta[n] \longleftrightarrow 1$ Roc: Entire z-plane

$\delta[n-K] \longleftrightarrow 1 \cdot z^{-K}$ Roc: Entire z-plane Except $z=0$

(b) $x[n] = \delta[n+K]$

$\delta[n] \longleftrightarrow 1$ Roc: Entire z-plane

$\delta[n+K] \longleftrightarrow 1 \cdot z^{K}$ Roc: Entire z-plane except $z=\infty$

(c) $x[n] = 2\delta[n+2] + 3\delta[n] - 5\delta[n-1] + 3\delta[n-2]$

$X[z] = 2z^2 + 3 - 5z^{-1} + 3z^{-2}$

Roc: Entire z-plane except $z=0$ & $z=\infty$

(d) $x[n] = u[n] - u[n-10]$

$$X[z] = \frac{1}{1-z^{-1}} - \frac{z^{-10}}{1-z^{-1}}$$

$$= \frac{1-z^{-10}}{1-z^{-1}} = \frac{z}{z^{10}}\left[\frac{z^{10}-1}{z-1}\right] = \frac{1}{z^9}\left[\frac{z^{10}-1}{z-1}\right]$$

Roc: Entire z-plane except $z=0$