3.6

Graph Coloring

Consider the problem of determining the least number of colors that can be used to color a map so that adjacent regions never have the same color.

For example, for the map shown below, four colors are sufficient, but three colors are not enough.

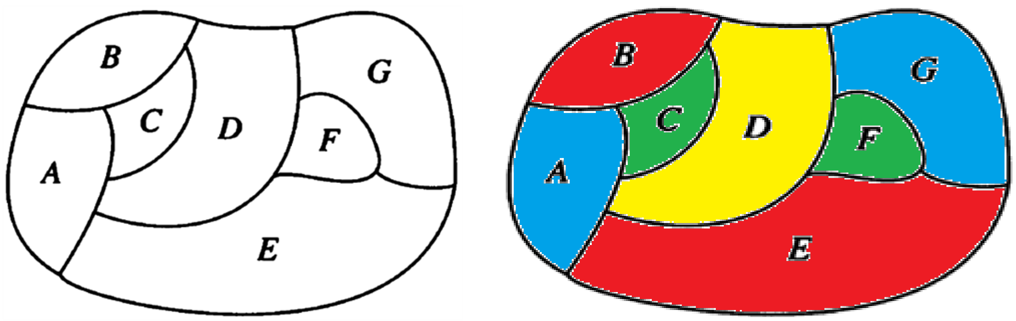


Figure-1

For the map shown below three colors are sufficient but two are not enough.

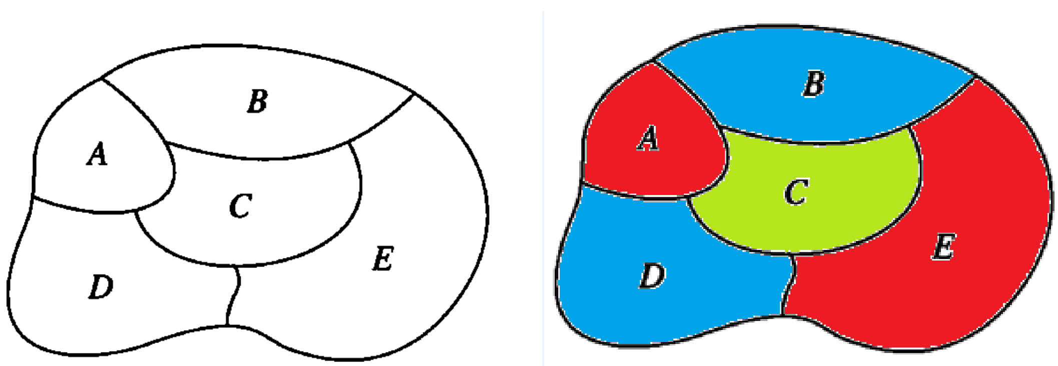
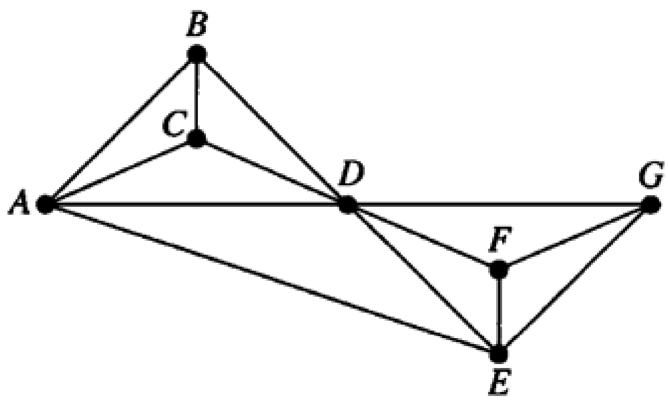
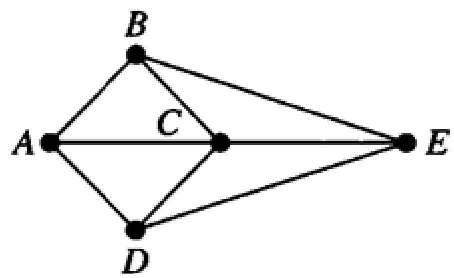


Figure-2

Representation of maps: Each map in the plane can be represented by a graph. Each region of the map is represented by a vertex and edges connect two vertices if the regions represented by these vertices have a common border. Two regions that touch at only one point are not considered adjacent. The graph so obtained corresponding to the map is called the ***dual graph*** of the map. It is clear that the dual graph of maps, so constructed are planar (That is, the dual graph in the plane corresponding to a map is a planar graph).

The following are the dual graphs corresponding to maps in fig.1 and fig.2 respectively.





The problem of coloring the regions of a map is equivalent to the problem of coloring the vertices of the dual graph so that no two adjacent vertices in this graph have the same color.

Graph coloring: A ***coloring*** of a simple graph is the assignment of a color to each vertex of the graph so that no two adjacent vertices are assigned the same color.

For most graphs a coloring can be found that uses fewer colors than the number of vertices in the graph. What is the least number of colors necessary?

Chromatic number: The ***chromatic number*** of a graph is the least number of colors needed for a coloring of this graph.

The chromatic number of a graph G is denoted by $\chi(G)$. (where χ is the Greek letter ***chi***).

Note that the chromatic number of a planar graph is same as the minimum number of colors required to color a planar map so that no two adjacent regions are assigned the same color. This question has been studied for more than a 100 years. The answer is provided by one of the most famous theorems in mathematics.

Theorem 1: ***The Four Color Theorem: The chromatic number of a planar graph is no greater than four.***

The Four Color Theorem was originally posed as a conjecture in the 1850's. It was finally proved by the American mathematicians ***Kenneth Appel*** and ***Wolfgang Haken*** in 1976. Prior to 1976, many incorrect proofs were published, often with hard to find errors. In addition, many futile attempts were made to construct counter examples by drawing maps that require more than four colors.

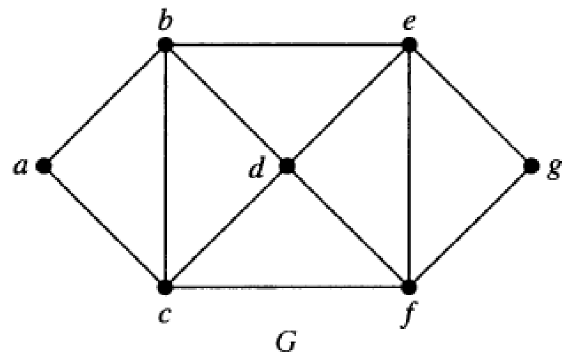
Perhaps the most notorious fallacious proof in all of mathematics is the incorrect proof of Four Color Theorem published in 1879 by a London barrister and a mateur mathematician, Alfred Kempe. Mathematicians accepted his proof as correct until 1890, when Percy Heawood found an error that made Kempe's argument incomplete. However, Kempe's line of reasoning turned out to be the basis of the successful proof given by Appel and Haken. Their proof relies on a careful case-by-case analysis carried out by computer. They showed that if the Four Color Theorem was false, there would have to be a counterexample of one of approximately 2000 different types, and they then showed that none of these types exists. They used over 1000 hours of computer time in their proof. This proof generated a large amount of controversy, because computers played such an important role in it. For example, could there be an error in a computer program that led to incorrect results? Was their argument really a proof it is depended on what could be unreliable computer output?

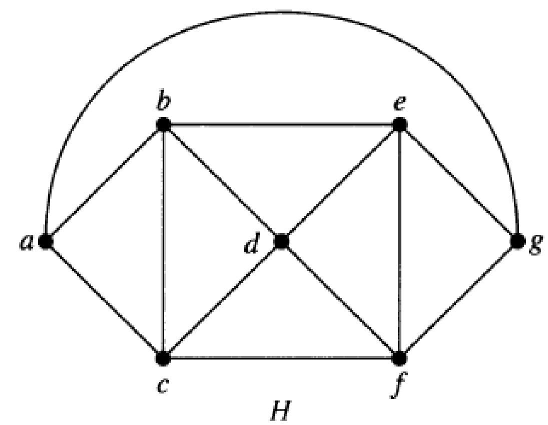
Note:

(i) The Four Color Theorem applies only to planar graphs
(ii) Nonplanar graphs can have arbitrarily large chromatic numbers.

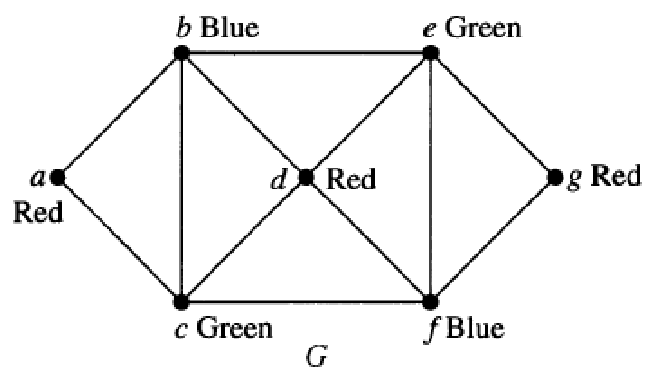
To show that the chromatic number of a graph is k, first, we must show that the graph can be colored with k colors (and this can be done by constructing such a coloring) and then show that the graph cannot be colored using fewer than k colors.

Example 1: What are the chromatic numbers of the graphs G and H shown below.



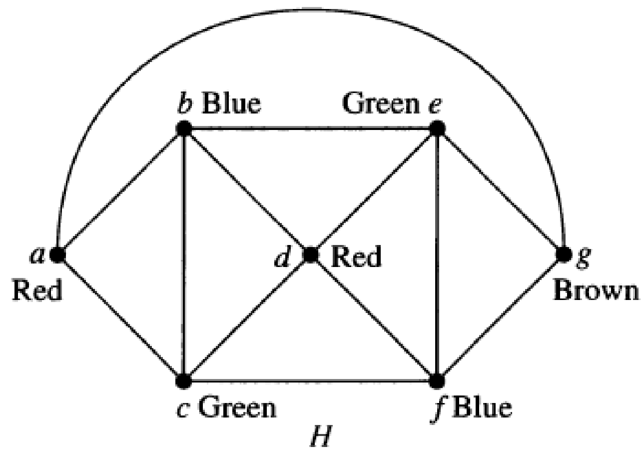


Solution: Notice that the vertices a, b and c in G are pairwise adjacent. Therefore, we need at least three colors. Thus, the chromatic number of G is at least three.



Now, assign colors red, blue and green to the vertices a, b and c respectively. Since d is adjacent to b(blue) and c(green), d must be colored red. Further, e must be colored green because it is adjacent to b(blue) and e(red) and f must be colored blue because it is adjacent to d(red) and e(green). Finally, g must be colored red because it is adjacent to f(blue) and e(green). Thus, G can be colored using exactly three colors. Therefore $\chi(G) = 3$.

Notice that the graph H is made up of the graph G with an edge $\{a, g\}$. Any attempt to color H using three colors must follow the same steps as that used to color G, except at the last stage, i.e., coloring the vertex g.

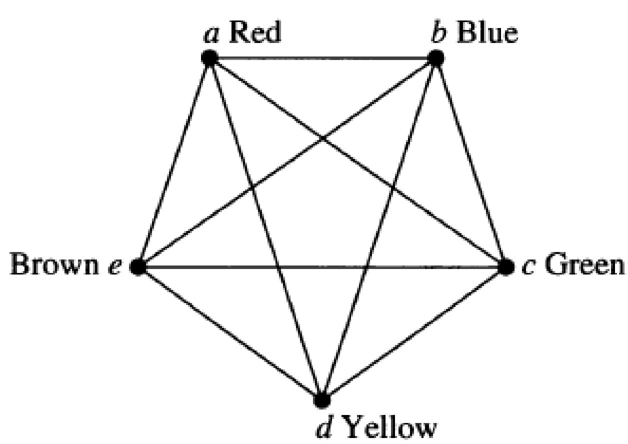


Now observe that g is adjacent to vertices a(red), e(green) and f(blue). This shows that a fourth color, say brown, needs to be used. Thus, H can be colored using exactly 4 colors. Therefore, $\chi(H) = 4$.

Example 2: What is the chromatic number of K_n?

Solution: A coloring of K_n can be constructed using n colors by assigning a different color to each vertex. Since every two vertices are adjacent in K_n , no two vertices can be assigned the same color. This shows that K_n cannot be colored using fewer than n colors.Thus, K_n colored using exactly n colors. Therefore, $\chi(K_n) = n$.

A coloring of K_5 using five colors is shown below:

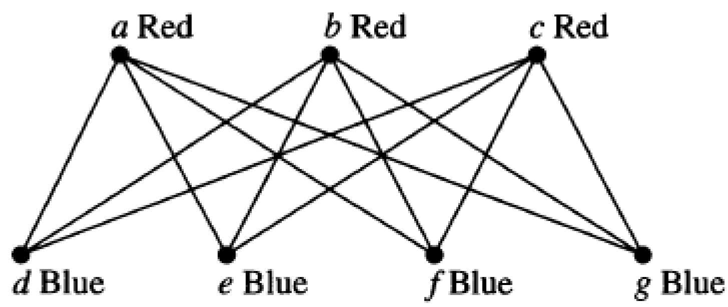


A coloring of K_5

Note: Recall that K_n is nonplanar when $n \geq 5$ and this result $\chi(K_n) = n$ does not contradict the Four Color Theorem.

Example 3: Show that $\chi(K_{m,n}) = 2$.

Solution: Since $K_{m,n}$ is a bipartite graph, its $m + n$ vertices are partitioned into V_1 and V_2 with $|V_1| = m$ and $|V_2| = n$. We color the vertices of V_1 with one color and the vertices of V_2 with a second color. Because edges connect only a vertex of V_1 and a vertex of V_2 , no two adjacent vertices have the same color. This shows that $K_{m,n}$ can be colored using exactly two colors. Therefore, $\chi(K_{m,n}) = 2$.



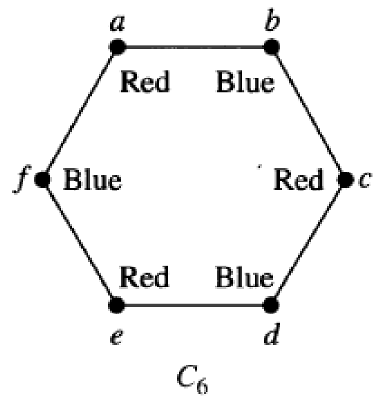
A coloring of $K_{3,4}$

Note: The chromatic number of a bipartite graph is **two.**

Example 4: Determine the chromatic number of C_n(the cycle graph with n vertices) $n \geq 3$.

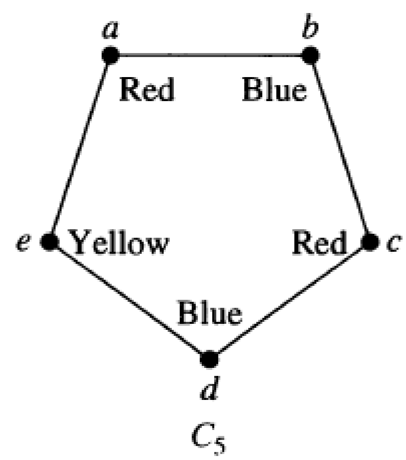
Solution: We consider two cases.

(i) Let n be even. Choose a vertex and color it red. Proceed around the graph in a clockwise direction (because C_n is planar) coloring the second vertex blue, the third vertex red and so on. Finally, the n^{th} vertex can be colored blue, because the two vertices adjacent to it, namely, $(n-1)^{\text{th}}$ and the first vertices, are both colored red. Thus, $\chi(C_n) = 2$, when n is an even positive integer with $n \geq 4$.



Coloring of C_6

(ii) Let n be odd. Choose a vertex and color it red. Proceed as in the above case in the clockwise direction. Finally we note that the n^{th} vertex is adjacent to two vertices of different colours, namely, the first vertex (red) and $(n-1)^{\text{th}}$ vertex(blue). Hence, a third color must be used. Thus $\chi(C_n) = 3$, when n is an odd positive integer with $n \geq 3$.



Coloring of C_5

Summarising, when n is a positive integer, $n \geq 3$.

$$\chi(C_n) = \begin{cases} 2, & \text{if } n \text{ is even} \\ 3, & \text{if } n \text{ is odd} \end{cases}$$

Note: The best algorithm known for finding the chromatic number of a graph have exponential worst-case time complexity (in the number of vertices of the graph).

Applications of Graph Colorings

Graph coloring has a variety of applications to problems involving scheduling and assignments.

Scheduling Final Examinations:

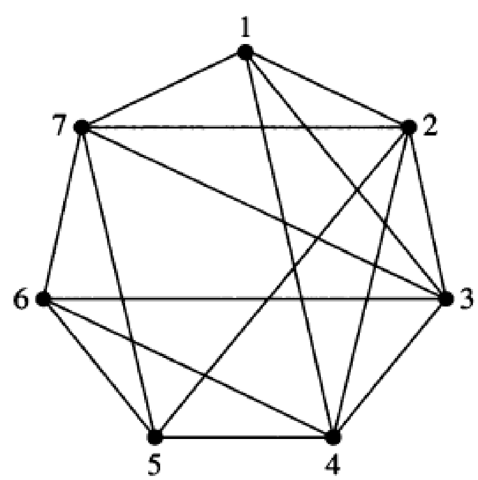
How can the final exams at a university be scheduled so that no student has two exams at the same time.

Solution: This scheduling problem can be solved using a graph model, with vertices representing courses and with an edge between two vertices if there is a common student in the courses they represent. Each time slot for a final exam is represented by a different color. A scheduling of the exams corresponds to a coloring of the associated graph.

For example, suppose that there are seven finals to be scheduled. Suppose that the courses are numbered as 1,2,3,…,7. Suppose that the following pairs of courses have common students:

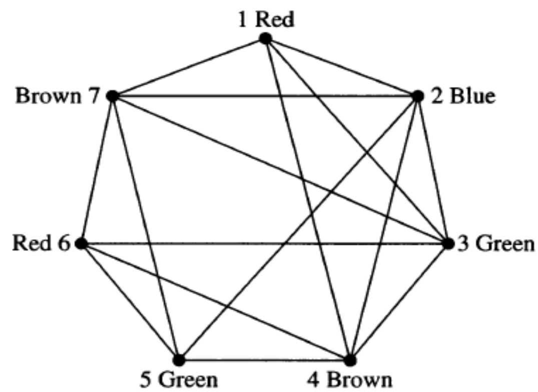
$$1,2\ ;\ 1,3\ ;\ 1,4\ ;\ 1,7\ ;\ 2,3\ ;2,4\ ;2,5\ ;\ 2,7\ ;$$
$$3,4\ ;\ 3,6\ ;\ 3,7\ ;\ 4,5\ ;\ 4,6\ ;5,6\ ;5,7\ ;\ 6,7\ .$$

The graph associated with this set of classes is shown below:



The graph representing the scheduling of final exams

A scheduling consists of a coloring of this graph. A coloring of the graph is shown below:



Notice that the chromatic number of this graph 4. Therefore, four time slots are needed and the associated schedule is given below:

Time period	Courses
I	1,6
II	2
III	3,4
IV	4,7

Algorithm (The Welsh-Powell algorithm)

An algorithm can be used to color a simple graph. First, list the vertices in $v_1, v_2, \dots, v_n$ in order of non increasing degree so that

$$\deg(v_1) \geq \deg(v_2) \geq \cdots \geq \deg(v_n)$$

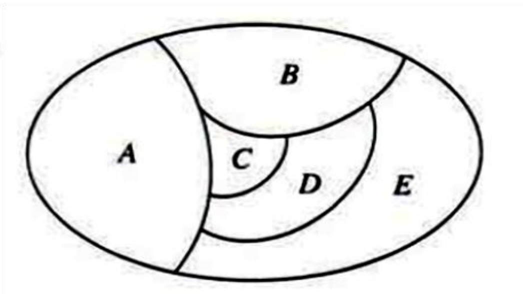
Assign color 1 to v_1 and to the vertex in the list not adjacent to v_1 (if one exists) and successively to each vertex in the list not adjacent to a vertex already assigned color 1. Then assign color 2 to the first vertex in the list not already colored. Successively assign color 2 to vertices in the list that have not already been colored and are not adjacent to vertices assigned color 2. Continue this process until all vertices are colored.

Note: It may not always give the best solution, but it will usually perform better than just coloring the vertices without a plan will.

For problems see $\mathbf{P5}$, $\mathbf{P6}$

P1:

Construct the dual graph for the map given below:

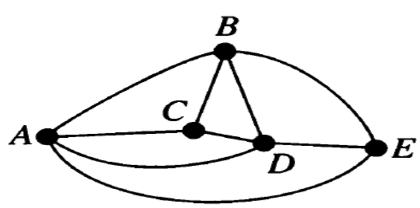


Find the number of colors needed to color the map so that no two adjacent regions have the same color.

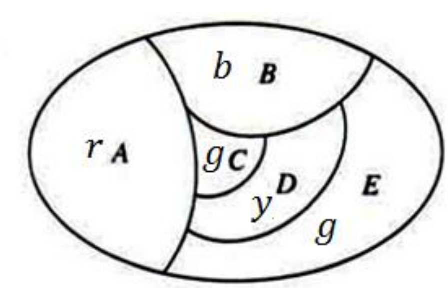
Solution:

Each map in the plane can be represented by a graph. Each region of the map is represented by a vertex. Edges connect two vertices if the regions represented by these vertices have a common border. The resulting graph is called the ***dual graph*** of the map.

The dual graph of the given map is



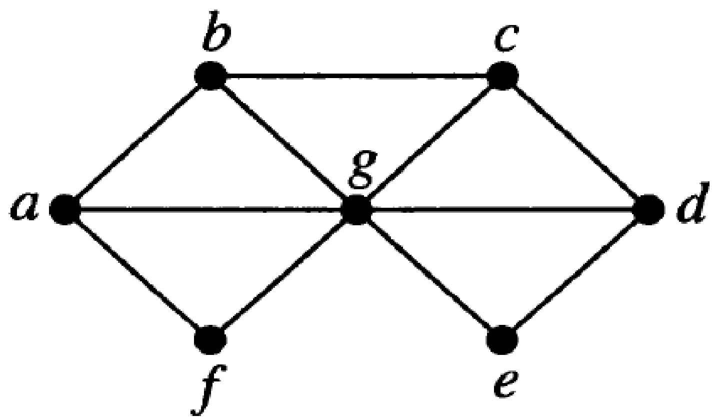
The number of colors required to color the graph is 4



r-red, b-blue , g-green, y-yellow

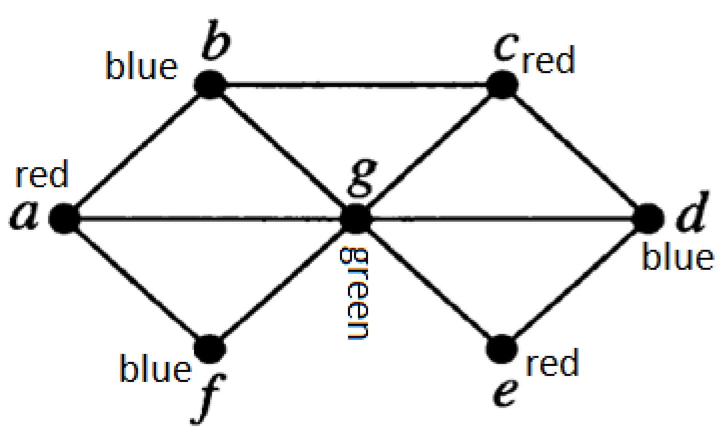
P2:

Find the chromatic number of the graph



Solution:

Assign the color green to g notice that g is adjacent to every vertex. Therefore, no vertex receives the colour green. Assign color red to the vertex a. Observe that the vertices c and e are not adjacent to the vertex a and so we assign color red to the vertices c and e. Now assign color blue to the vertex b and observe that the vertices d and f are not adjacent to b. So, assign the color blue to the vertices d and f.



It is a coloring of the graph and $\chi(G) = 3$.

P3:

Which graphs have a chromatic number 1.

Solution:

Graphs with no edges will have a chromatic number 1.

P4:

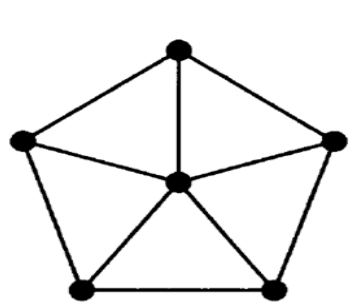
What is the chromatic number of W_n.

Solution:

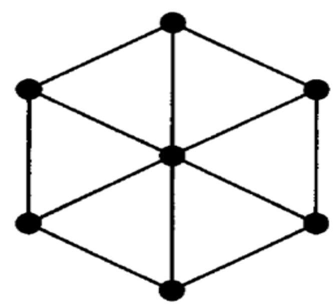
Note that we obtain the wheel graph W_n by adding an additional vertex to the cycle graph C_n , $n \geq 3$ and connecting this additional vertex to each of the n vertices of C_n. Assign a colour 1 to this additional vertex.

Now, no other vertex receives colour 1 because the additional vertex is adjacent to all other vertices. We have to colour C_n. It is known that $\chi(C_n)$ is 2 if n is even and is 3 if n is odd. Thus

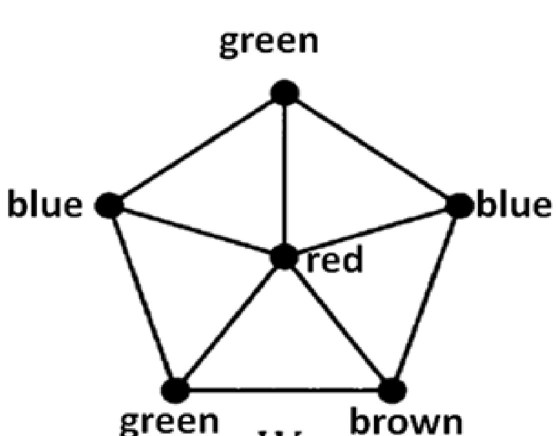
$$\chi(W_n) = \begin{cases} 3 & if\ n\ is\ even\ , n \geq 4 \\ 4 & if\ n\ is\ odd\ , n \geq 3 \end{cases}$$



W_5

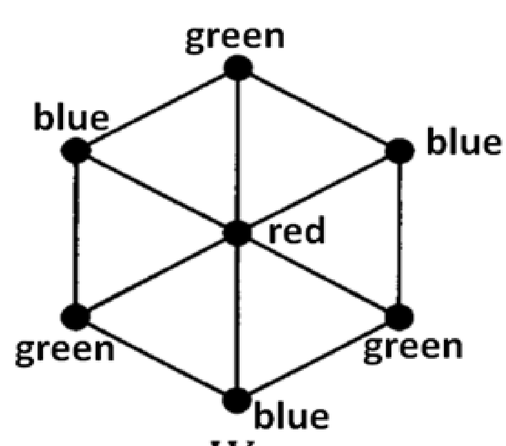


W_6



W_5

$\chi(W_5) = 4$



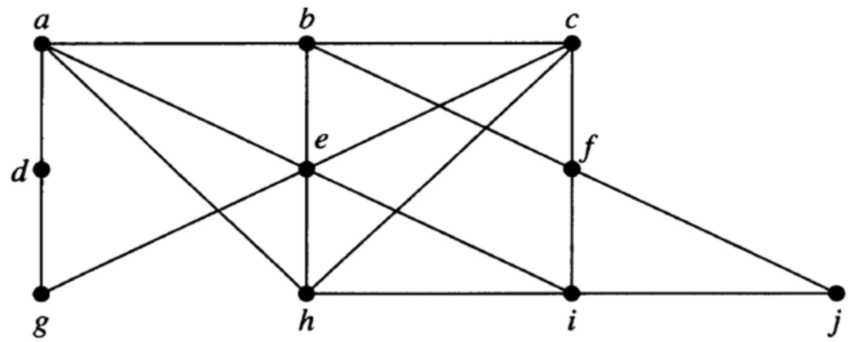
W_6

$\chi(W_6) = 3$

P5:

a) Construct a coloring of the graph G using the algorithm

b) Find the chromatic number of G



Solution:

The given graph is a simple graph. We find a coloring by Welsh-Powell algorithm

List the vertices of in order of non increasing degrees as shown below.

Vertex :	e	a	b	c	f	h	i	d	g	j
Degree :	6	4	4	4	4	4	4	2	2	2

Assign color 1 to the vertex e and to the vertex in the list not adjacent to e, namely the vertex f. The vertex next to f in the list not adjacent to a, e (the vertices colored color 1) is d. Assign color 1 to d. Thus

Vertex :	e	a	b	c	f	h	i	d	g	j
Degree :	6	4	4	4	4	4	4	2	2	2
Color :	1				1			1		

Now, assign color 2 to the vertex a (the first vertex in the list not already colored). Successively assign color 2 to vertices in the list,that have not already been colored, and are not adjacent to vertices assigned color 2.

The vertex not adjacent to a in the list is c, assign color 2 to c. The vertex not adjacent to a and c in the list is i, assign color 2 to i. The vertex not adjacent to a, c, i is g, assign color 2 to g. Thus

Vertex :	e	a	b	c	f	h	i	d	g	j
Degree :	6	4	4	4	4	4	4	2	2	2
Color :	1	2		2	1		2	1	2	

Now assign color 3 to b and proceeding as above we get

Vertex :	e	a	b	c	f	h	i	d	g	j
Degree :	6	4	4	4	4	4	4	2	2	2
Color :	1	2	3	2	1	3	2	1	2	3

The following is a coloring of the graph

Color 1 : e, f, d

Color 2 : a, c, i, g

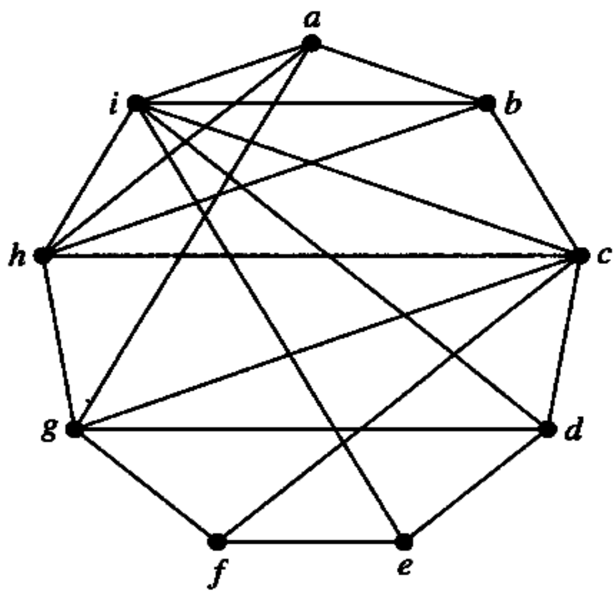
Color 3 : b, h, j

The vertices e, a and b are connected to each other, so at least three colors are need to color G.

Thus, $\chi(G) = 3$.

P6:

Find the chromatic number of the graph



Solution:

Following the steps of the Welsh-Powell algorithm yields the following data.

Vertex :	c	i	g	h	a	b	d	e	f
Degree :	6	6	5	5	4	4	4	3	3
Color :	1	2	2	3	1	4	3	1	3

The vertices c, i, h and b are connected to each other, so at least four colors are needed to color the given graph G. Thus

$$\chi(G) = 4$$