1.2 MATRICES

1.2.1. Definition :

A collection of numbers arranged in a rectangular array (i.e. in row and column wise) is called a **Matrix.** If the array has m number of rows and n number of columns then the size of the matrix is $m\times n$.

A general form of a $m\times n$ matrix A is

$$A=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}=\left[a_{ij}\right]_{m\times n}$$

or,

$$A=\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & b_{m2} & \cdots & a_{mn} \end{pmatrix}=\left(a_{ij}\right)_{m\times n}$$

Note. (1) The numbers a_{11}, a_{12} (in general a_{ij}) are called elements of the matrix. a_{ij} belongs to the i th row and j th column of the matrix. The element a_{11} is called the **leading element** of the matrix. The diagonal containing the elements a_{11}, a, a_{33} ... is called Principal Diagonal of the Matrix.

Example : $A=\begin{bmatrix} 1 & 2 & 7 & 1 \\ -1 & 3 & 4 & 5 \\ 0 & 2 & -4 & 0 \end{bmatrix}$ is a matrix of size 3×4. 1, 2, 7, ... are elements of the matrix A.

(2) Matrix is an arrangment of numbers. It does not have any real value.

(3) $\left(a_{ij}\right)_{m\times n}$ stands for short form of matrix.

1.2.2. Equality of two Matrices.

Two matrices $\left(a_{ij}\right)_{m\times n}$ and $\left(b_{ij}\right)_{m\times n}$ are said to be equal if they have same size and $a_{ij}=b_{ij}$ for all values of i and j.

Example : If $X=\begin{pmatrix} 2 & 1 & 4 \\ 3 & 1 & 0 \end{pmatrix}, Y=\begin{pmatrix} 2 & 1 & 4 \\ 7 & 1 & 0 \end{pmatrix}$ then $X\neq Y$.

1.2.3. Types of Matrices :

Row Matrix : A matrix with a single row is called a Row Matrix. For example the matrix

$A=\begin{pmatrix} 1 & 3 & 4 & 0 & -1 \end{pmatrix}$ is a row matrix of size 1×5

Column Matrix : A matrix with a single column is called a column matrix. For example the matrix

$A=\begin{pmatrix} 6 \\ 2 \\ 5 \end{pmatrix}$ is a column matrix of size 3×1.

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Diagonal Matrix.

A square matrix whose off-diagonal terms are 0 is called diagonal matrix.

Example : $\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 4 \end{pmatrix}$ is a diagonal matrix.

Square Matrix : A matrix having same number of rows and columns is called a square matrix.

A square matrix of size $n \times n$ is called ***n* th order** square matrix.

Example. $\begin{pmatrix} 1 & 9 & 4 \\ 0 & -1 & 3 \\ -2 & 5 & 7 \end{pmatrix}$ is a 3rd order square matrix. The diagonal formed by the elements 1, –1 and 7 is called **Principal Diagonal** of the square matrix.

Identity Matrix. A square matrix is said to be identity matrix if its diagonal elements are 1 and other elements are 0. It is denoted by I.

Example : $I_{3\times3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is a 3rd-order identity matrix.

Null Matrix. A matrix is said to be a null matrix if all the elements of it are zero. It is denoted by O.

Example : $O_{3\times2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ is a null matrix.

Upper Triangular Matrix. A square matrix is said to be upper triangular if every term below each term of its principal diagonal is zero

Example : $\begin{bmatrix} 3 & 1 & 5 \\ 0 & 2 & 0 \\ 0 & 0 & 6 \end{bmatrix}$ is a 3rd order upper triangular matrix.

Lower Triangular Matrix.

A square matrix is said to be lower triangular if every term above each term of its principal diagonal is zero.

Example : $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 7 & 3 & 0 & 0 \\ 0 & 8 & 4 & 0 \\ 1 & 6 & 7 & -1 \end{bmatrix}$ is a 4th order lower triangular matrix.

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1.2.4. Matrix Algebra.

Addition of two Matrices : Two matrices $A=(a_{ij})_{m\times n}$ and $B=(b_{ij})_{m\times n}$ are said to be conformable for addition if they have same size; their sum $A+B$ is defined as $A+B=(a_{ij}+b_{ij})_{m\times n}$.

Example. $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}+\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix}=\begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \\ a_{31}+b_{31} & a_{32}+b_{32} \end{bmatrix}$

Multiplication of a Matrix by a scalar (number) : If $A=(a_{ij})_{m\times n}$ be a matrix and k be a number then the product kA is defined as $kA=(ka_{ij})_{m\times n}$.

Example : $6\begin{pmatrix} 7 & 1 & -5 \\ 2 & 1 & 0 \end{pmatrix}=\begin{pmatrix} 42 & 6 & -30 \\ 12 & 6 & 0 \end{pmatrix}$

Note. The matrix $-A$ is nothing but $(-1)A$.

Subtraction of two Matrices : Two matrices $A=(a_{ij})_{m\times n}$ and $B=(b_{ij})_{m\times n}$ are said to be conformable for subtraction if they have same size; their difference $A-B$ is defined as $A-B=(a_{ij}-b_{ij})_{m\times n}$

Example : $\begin{pmatrix} 3 & 1 & 4 \\ -1 & 0 & 2 \end{pmatrix}-\begin{pmatrix} 1 & 5 & 6 \\ 2 & 1 & 9 \end{pmatrix}=\begin{pmatrix} 2 & -4 & -2 \\ -3 & -1 & -7 \end{pmatrix}$

Product of two Matrices : Two matrices A and B are conformable for the product AB if the number of columns of A is equal to the number of rows in B. That is if A is of size $m\times n$ then B must be of size $n\times p$. Then product AB would be a matrix of size $m\times p$ defined by $AB=(c_{ij})_{m\times p}$ where $c_{ij}=a_{i1}b_{1j}+a_{i2}b_{2j}+\ldots+a_{in}b_{nj}$

Example : (1) If $A=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{pmatrix}_{4\times 3}$ then A is of size 4×3.

For getting AB, B a matrix must be of 3 rows and any number of columns.

Let $B=\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{pmatrix}_{3\times 2}$ having size 3×2.

Then AB will be of size 4×2 and $AB=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{pmatrix}\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{pmatrix}$

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$$= \begin{pmatrix} a_{11}b_{11}+a_{12}b_{21}+a_{13}b_{31} & a_{11}b_{12}+a_{12}b_{22}+a_{13}b_{32} \\ a_{21}b_{11}+a_{22}b_{21}+a_{23}b_{31} & a_{21}b_{12}+a_{22}b_{22}+a_{23}b_{32} \\ a_{31}b_{11}+a_{32}b_{21}+a_{33}b_{31} & a_{31}b_{12}+a_{32}b_{22}+a_{33}b_{32} \\ a_{41}b_{11}+a_{42}b_{21}+a_{43}b_{31} & a_{41}b_{12}+a_{42}b_{22}+a_{43}b_{32} \end{pmatrix}_{4\times 2} = (c_{ij})_{4\times 2}$$

where $c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j}$ for $i = 1, 2, 3, 4$ and $j = 1, 2, 3, 4$.

e.g. $c_{32} = a_{31}\, b_{12} + a_{32}\, b_{22} + a_{33}\, b_{32}$ etc.

(2) If $A = \begin{pmatrix} 3 & 5 & 2 \\ 3 & 1 & 0 \end{pmatrix}_{2\times 3}$ and $B = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 1 & 3 & 2 \end{pmatrix}_{3\times 3}$ then $AB = \begin{pmatrix} 3 & 5 & 2 \\ 3 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 1 & 3 & 2 \end{pmatrix}$

$$= \begin{pmatrix} 3\times 1+5\times 4+2\times 1 & 3\times 2+5\times 5+2\times 3 & 3\times 3+5\times 6+2\times 2 \\ 3\times 1+1\times 4+0\times 1 & 3\times 2+1\times 5+0\times 3 & 3\times 3+1\times 6+0\times 2 \end{pmatrix} = \begin{pmatrix} 25 & 37 & 43 \\ 7 & 11 & 15 \end{pmatrix}_{2\times 3}$$

1.2.5. Laws on Matrix Algebra.

(1) *Commutative law for addition :* If A and B are conformable for addition then $A + B = B + A$

(2) *Associative law for addition :* If A, B and C are conformable for addition then $(A+B)+C = A+(B+C)$

(3) *Distributive law of scalar multiplication on matrix addition :* If A and B are conformable for addition and k is a number then $k(A+B) = kA + kB$.

(4) *Addition law with Null Matrix :* If A is a matrix which is conformable for $A+0$ then $A+0=A$

(5) $A - A = 0$

(6) *Commutative law for multiplication :* If A and B are two matrices then AB and BA may not be equal (even if they are conformable for multiplication)

So in general matrix multiplication is non–commutative. Thus, in general, $AB \neq BA$

Example : $A = \begin{bmatrix} 1 & 3 \\ 5 & 0 \end{bmatrix}$, $B = \begin{bmatrix} -7 & 1 \\ 2 & 3 \end{bmatrix}$.

Then $AB = \begin{bmatrix} 1 & 3 \\ 5 & 0 \end{bmatrix}\begin{bmatrix} -7 & 1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} -1 & 10 \\ -35 & 5 \end{bmatrix}$ and $BA = \begin{bmatrix} -7 & 1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 5 & 0 \end{bmatrix} = \begin{bmatrix} -2 & -21 \\ 17 & 6 \end{bmatrix}$ showing $AB \neq BA$

(7) *Associative law for product :* If A and B are conformable for the product AB ; B and C are conformable for the product BC, then $A(BC) = (AB)C$

(8) *Distributive law for Product :* If A and B are conformable for the product AB ; B and C are conformable for the sum $B + C$, then $A(B+C) = AB + AC$.

(9) *Product with Identity Matrix :* If A is a matrix and I is an identity matrix such that AI is conformable then $AI = A$.

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1.2.6. Transpose of a Matrix : Let A be a matrix of size $m\times n$. Then the matrix A^T obtained by interchanging the rows and columns of A is called transpose of A. Thus if $A=(a_{ij})_{m\times n}$ then its transpose $A^T=(a'_{ij})_{n\times m}$ where $a'_{ij}=a_{ji}$

Example : If $A=\begin{bmatrix}2 & 5\\ -1 & 2\\ 3 & 4\end{bmatrix}$ then its transpose $A^T=\begin{pmatrix}2 & -1 & 3\\ 5 & 2 & 4\end{pmatrix}$.

Properties of Transposed Matrix.

If A and B are two matrices then

(1) $(A^T)^T=A$ (2) $(A+B)^T=A^T+B^T$ (3) $(A-B)^T=A^T-B^T$

(4) $(kA)^T=kA^T$ (5) $(AB)^T=B^TA^T$

Illustration of Property (5) :

Let $A=\begin{pmatrix}1 & -1 & 2\\ 2 & 1 & 0\end{pmatrix}$ and $B=\begin{pmatrix}1 & 3\\ 2 & 0\\ -1 & 1\end{pmatrix}$ Here $AB=\begin{pmatrix}1 & -1 & 2\\ 2 & 1 & 0\end{pmatrix}\begin{pmatrix}1 & 3\\ 2 & 0\\ -1 & 1\end{pmatrix}=\begin{pmatrix}-3 & 5\\ 4 & 6\end{pmatrix}$

$\therefore (AB)^T=\begin{pmatrix}-3 & 4\\ 5 & 6\end{pmatrix}$ ... (:)

Again $B^TA^T=\begin{pmatrix}1 & 2 & -1\\ 3 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 2\\ -1 & 1\\ 2 & 0\end{pmatrix}=\begin{pmatrix}1-2-2 & 2+2+0\\ 3+0+2 & 6+0+0\end{pmatrix}=\begin{pmatrix}-3 & 4\\ 5 & 6\end{pmatrix}$ ... (2)

From (1) and (2) we see $(AB)^T=B^TA^T$

1.2.7. Symmetric and Skew–Symmetric Matrices.

A square matrix $A=(a_{ij})_{n\times n}$ is called symmetric if $A^T=A$ i.e. if $a_{ij}=a_{ji}$ for all values of i and j

Example : $A=\begin{pmatrix}2 & -1 & 4\\ -1 & 7 & 8\\ 4 & 8 & 6\end{pmatrix}$ is a symmetric matrix because here $A^T=A$.

A square matrix $A=(a_{ij})_{n\times n}$ is called skew–symmetric matrix if $A^T=-A$ i.e. if $a_{ij}=-a_{ji}$ for all values of i and j.

Example : $A=\begin{pmatrix}0 & 3 & -1\\ -3 & 0 & -4\\ 1 & 4 & 0\end{pmatrix}$ is skew–symmetric because $A^T=\begin{pmatrix}0 & -3 & 1\\ 3 & 0 & 4\\ -1 & -4 & 0\end{pmatrix}=-\begin{pmatrix}0 & 3 & -1\\ -3 & 0 & -4\\ 1 & 4 & 0\end{pmatrix}=-A$

Note. All the elements of the principal diagonal of a skew–symmetric matrix must be 0.

****Theorem** :Any matrix can be expressed as sum of a symmetric matrix and a skew symmetric matrix.

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$$=\begin{pmatrix} 6-2-3 & 12-6-6 & 18-6-12 \\ -1+1 & -2+3 & -3+3 \\ -1+0+1 & -2+0+2 & -3+4 \end{pmatrix}=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}=I$$

Similarly we can show $\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 3 \\ 1 & 2 & 4 \end{pmatrix}\begin{pmatrix} 6 & -2 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}=I$

So, $\begin{pmatrix} 6 & -2 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$ is inverse of $\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 3 \\ 1 & 2 & 4 \end{pmatrix}$

Invertible Matrix :

We can prove that every square matrix may not have an inverse. A matrix which has inverse is called an invertible matrix.

Theorem 1. A non singular matrix A is invertible and its inverse, $A^{-1}=\dfrac{1}{\det(A)}Adj\,(A)$

Proof. Beyond the scope of the book.

Illustration : Find the inverse of the matrix $A=\begin{pmatrix} 1 & 2 & 1 \\ 1 & -1 & 1 \\ 1 & 3 & -1 \end{pmatrix}$

$$\text{Here det }(A)=\begin{vmatrix} 1 & 2 & 1 \\ 1 & -1 & 1 \\ 2 & 3 & -1 \end{vmatrix}=1\begin{vmatrix} -1 & 1 \\ 3 & -1 \end{vmatrix}-2\begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix}+1\begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix}=9\neq 0$$

$\therefore$ A is invertible. Now as determined earliar, $adj\ A=\begin{pmatrix} -2 & 5 & 3 \\ 3 & -3 & 0 \\ 5 & 1 & -3 \end{pmatrix}$

$$\therefore\ A^{-1}=\frac{1}{\det(A)}adj\,(A)=\begin{pmatrix} -2/9 & 5/9 & 1/3 \\ 1/3 & -1/3 & 0 \\ 5/9 & 1/9 & -1/3 \end{pmatrix}$$

Note : We can verify the above result by multiplying it with A and getting identity matrix I.

Theorem 2. For invertible matrices A and B;

(1) $\left(A^{-1}\right)^{-1}=A$

(2) AB is invertible and $(AB)^{-1}=B^{-1}A^{-1}$

(3) A^T is invertible and $\left(A^T\right)^{-1}=\left(A^{-1}\right)^T$

Proof. Beyond the scope of the book.

Illustration :

If $A=\begin{bmatrix} 2 & 4 \\ 1 & 3 \end{bmatrix}$, $B=\begin{bmatrix} -1 & 0 \\ 5 & 1 \end{bmatrix}$ verify $(AB)^{-1}=B^{-1}A^{-1}$

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Here $\det(A) = \begin{vmatrix} 2 & 4 \\ 1 & 3 \end{vmatrix} = 2 \neq 0$. Therefore A is invertible matrix

Now $adj\,(A) = \begin{bmatrix} 3 & -4 \\ -1 & 2 \end{bmatrix}$

$\therefore A^{-1} = \frac{1}{\det(A)} adj\,(A) = \frac{1}{2}\begin{bmatrix} 3 & -4 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 3/2 & -2 \\ -1/2 & 1 \end{bmatrix}$

Again $\det(B) = -1$ and $adj\,(B) = \begin{bmatrix} 1 & 0 \\ -5 & -1 \end{bmatrix}$

$\therefore B^{-1} = \begin{bmatrix} -1 & 0 \\ 5 & 1 \end{bmatrix}$

$\therefore B^{-1}A^{-1} = \begin{bmatrix} -1 & 0 \\ 5 & 1 \end{bmatrix}\begin{bmatrix} \frac{3}{2} & -2 \\ -1/2 & 2 \end{bmatrix} = \begin{bmatrix} -3/2 & 2 \\ 7 & -9 \end{bmatrix}$

Also $AB = \begin{bmatrix} 2 & 4 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 5 & 1 \end{bmatrix} = \begin{bmatrix} 18 & 4 \\ 14 & 3 \end{bmatrix}$

$\therefore \det\,(AB) = -2$ and $adj(AB) = \begin{bmatrix} 3 & -4 \\ -14 & 18 \end{bmatrix}$

$\therefore (AB)^{-1} = \frac{Adj\,(AB)}{\det\,(AB)} = \begin{bmatrix} -3/2 & 2 \\ 7 & -9 \end{bmatrix}$

$(AB)^{-1} = B^{-1}A^{-1}$ is verified.

1.2.11. Illustrative Examples.

Example. 1. Find the value of x, y, z and t which satisfy the equation

$$\begin{bmatrix} x+3 & x+2y \\ z-1 & 4t-6 \end{bmatrix} = \begin{bmatrix} 0 & -7 \\ 3 & 2t \end{bmatrix}$$ [W.B.S.C 2003]

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Solution. Since $\begin{bmatrix} x+3 & x+2y \\ z-1 & 4t-6 \end{bmatrix} = \begin{bmatrix} 0 & -7 \\ 3 & 2t \end{bmatrix}$

therefore
$x+3=0 \quad (1)$
$x+2y=-7 \quad (2)$
$z-1=3 \quad (3)$
$4t-6=2t \quad (4)$

From (1) we get $x=-3$. Putting this in (2) we get $-3+2y=-7$ or, $2y=3-7=-4$ $\therefore y=-2$

From (3) we get $z=3+1=4$. From (4) we get $4t-2t=6$ or, $2t=6$ or, $t=3$

Thus $x=-3, y=-2, z=4, t=3$.

Example. 2. Given $A=\begin{pmatrix} 2 & 3 \\ 5 & 6 \\ 7 & -7 \end{pmatrix}, B=\begin{pmatrix} 3 & 8 & 9 \\ 2 & -1 & 1 \\ 1 & 3 & 3 \end{pmatrix}$ examine whether $A+B, AB$ and BA are defined. Find them where they are defined.

Solution. A is of size 3×2, B is of size 3×3. Since they are not of same size, $A+B$ is not defined.

Since the number of columns of $A(=2)$ and the number of rows of $B(=3)$ are unequal, AB is not defined.

Again since number of columns of $B(=3)$ and the number of rows of $A(=3)$ are same, BA is defined and

$$BA=\begin{pmatrix} 3 & 8 & 9 \\ 2 & -1 & 1 \\ 1 & 3 & 3 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ 5 & 6 \\ 7 & -7 \end{pmatrix}=\begin{pmatrix} 6+40+63 & 9+48-63 \\ 4-5+7 & 6-6-7 \\ 2+15+21 & 3+18-21 \end{pmatrix}=\begin{pmatrix} 109 & -6 \\ 6 & -7 \\ 38 & 0 \end{pmatrix}$$

Example. 3. Find the values of x, y, z and t for which $\begin{pmatrix} x+y & y-z \\ 5-t & 7+x \end{pmatrix}=\begin{pmatrix} t-x & z-t \\ z-y & x+z+t \end{pmatrix}$

Solution. Since $\begin{pmatrix} x+y & y-z \\ 5-t & 7+x \end{pmatrix}=\begin{pmatrix} t-x & z-t \\ z-y & x+z+t \end{pmatrix}$

$\therefore x+y=t-x \Rightarrow 2x+y-t=0$ ... (1)

$y-z=z-t \Rightarrow y+t-2z=0$ ... (2)

$5-t=z-y \Rightarrow y-t-z=-5$ ... (3)

$7+x=x+z+t \Rightarrow t+z=7$ ... (4)

Solving x, y, z and t from (1), (2), (3) and (4) we get $x=1, y=2, z=3$ and $t=4$.

Example. 4. If $\begin{pmatrix} y & 1 \\ 3 & x \end{pmatrix}+\begin{pmatrix} x & 1 \\ -1 & -y \end{pmatrix}=\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ then find the values of x and y.

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Solution. Given $\begin{pmatrix} y & 1 \\ 3 & x \end{pmatrix} + \begin{pmatrix} x & 1 \\ -1 & -y \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$

or, $\begin{pmatrix} y+x & 1+1 \\ 3-1 & x-y \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ or, $\begin{pmatrix} y+x & 2 \\ 2 & x-y \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$

This implies $y+x=1$ and $x-y=1$

Adding we get $2x=2$ $\quad \therefore x=1 \quad \therefore y+1=1 \qquad \therefore y=0$.

Example. 5. If $A = \begin{pmatrix} 2 & -3 & -5 \\ -1 & 4 & 5 \\ 1 & -3 & -4 \end{pmatrix}, B = \begin{pmatrix} -1 & 3 & 5 \\ 1 & -3 & -5 \\ -1 & 3 & 5 \end{pmatrix}$ and $C = \begin{pmatrix} 2 & -2 & 4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{pmatrix}$ verify

(i) $A+(B-C)=(A+B)-C$ $\qquad$ (ii) $A(B+C)=AB+AC$

Solution. (i) Now, $$B-C = \begin{pmatrix} -1 & 3 & 5 \\ 1 & -3 & -5 \\ -1 & 3 & 5 \end{pmatrix} - \begin{pmatrix} 2 & -2 & 4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{pmatrix} = \begin{pmatrix} -1-2 & 3+2 & 5-4 \\ 1+1 & -3-3 & -5-4 \\ -1-1 & 3+2 & 5+3 \end{pmatrix} = \begin{pmatrix} -3 & 5 & 1 \\ 2 & -6 & -9 \\ -2 & 5 & 8 \end{pmatrix}$$

$$\therefore A+(B-C) = \begin{pmatrix} 2 & -3 & -5 \\ -1 & 4 & 5 \\ 1 & -3 & -4 \end{pmatrix} + \begin{pmatrix} -3 & 5 & 1 \\ 2 & -6 & -9 \\ -2 & 5 & 8 \end{pmatrix} = \begin{pmatrix} 2-3 & -3+5 & -5+1 \\ -1+2 & 4-6 & 5-9 \\ 1-2 & -3+5 & -4+8 \end{pmatrix} = \begin{pmatrix} -1 & 2 & -4 \\ 1 & -2 & -4 \\ -1 & 2 & 4 \end{pmatrix} \quad \dots (1)$$

Again $$A+B = \begin{pmatrix} 2 & -3 & -5 \\ -1 & 4 & 5 \\ 1 & -3 & -4 \end{pmatrix} + \begin{pmatrix} -1 & 3 & 5 \\ 1 & -3 & -5 \\ -1 & 3 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\therefore (A+B)-C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 2 & -2 & 4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{pmatrix} = \begin{pmatrix} -1 & 2 & -4 \\ 1 & -2 & -4 \\ -1 & 2 & 4 \end{pmatrix} \quad \dots (2)$$

From (1) and (2) we have $A+(B-C)=(A+B)-C$

(ii) Now, $$B+C = \begin{pmatrix} -1 & 3 & 5 \\ 1 & -3 & -5 \\ -1 & 3 & 5 \end{pmatrix} + \begin{pmatrix} 2 & -2 & 4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 9 \\ 0 & 0 & -1 \\ 0 & 1 & 2 \end{pmatrix}$$

$$\therefore A(B+C) = \begin{pmatrix} 2 & -3 & -5 \\ -1 & 4 & 5 \\ 1 & -3 & -4 \end{pmatrix} \begin{pmatrix} 1 & 1 & 9 \\ 0 & 0 & -1 \\ 0 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 2\times1+(-3)\times0+(-5)\times0 & 2\times1+(-3)\times0+(-5)\times1 & 2\times9+(-3)\times(-1)+(-5)2 \\ (-1)1+4\times0+5\times0 & -1\times1+4\times0+5\times1 & -1\times9+4\times(-1)+5\times2 \\ 1\times1+(-3)\times0+(-4)\times0 & 1\times1+(-3)\times0+(-4)\times1 & 1\times9+(-3)(-1)+(-4)2 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & -3 & 11 \\ -1 & 4 & -3 \\ 1 & -3 & 4 \end{pmatrix} \quad \dots (1)$$

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Now, $AB=\begin{pmatrix}2&-3&-5\\-1&4&5\\1&-3&-4\end{pmatrix}\begin{pmatrix}-1&3&5\\1&-3&-5\\-1&3&5\end{pmatrix}=\begin{pmatrix}-2-3+5&6+9-15&10+15-25\\1+4-5&-3-12+15&-5-20+25\\-1-3+4&3+9-12&5+15-20\end{pmatrix}=\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}$

and $AC=\begin{pmatrix}2&-3&-5\\-1&4&5\\1&-3&-4\end{pmatrix}\begin{pmatrix}2&-2&4\\-1&3&4\\1&-2&-3\end{pmatrix}=\begin{pmatrix}4+3-5&-4-9+10&8-12+15\\-2-4+5&2+12-10&-4+16-15\\2+3-4&-2-9+8&4-12+12\end{pmatrix}=\begin{pmatrix}2&-3&11\\-1&4&-3\\1&-3&4\end{pmatrix}$

$\therefore AB+AC=\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}+\begin{pmatrix}2&-3&11\\-1&4&-3\\1&-3&4\end{pmatrix}=\begin{pmatrix}2&-3&11\\-1&4&-3\\1&-3&4\end{pmatrix}$ ... (2)

From (1) and (2) we get $A(B+C)=AB+AC$

Example. 6. Determine the matrices A and B where $A+2B=\begin{pmatrix}1&2&0\\6&-3&3\\-5&3&1\end{pmatrix}$ and $2A-B=\begin{pmatrix}2&-1&5\\2&-1&6\\0&1&2\end{pmatrix}$.

Solution. We multiply the second relation by 2 and get $4A-2B=2\begin{pmatrix}2&-1&5\\2&-1&6\\0&1&2\end{pmatrix}$

or, $4A-2B=\begin{pmatrix}4&-2&10\\4&-2&12\\0&2&4\end{pmatrix}$ ... (1)

The first relation $A+2B=\begin{pmatrix}1&2&0\\6&-3&3\\-5&3&1\end{pmatrix}$ ... (2)

Adding (1) and (2) we get $5A=\begin{pmatrix}4&-2&10\\4&-2&12\\0&2&4\end{pmatrix}+\begin{pmatrix}1&2&0\\6&-3&3\\-5&3&1\end{pmatrix}$

or, $5A=\begin{pmatrix}5&0&10\\10&-5&15\\-5&5&5\end{pmatrix}$ or, $A=\frac{1}{5}\begin{pmatrix}5&0&10\\10&-5&15\\-5&5&5\end{pmatrix}$ or, $A=\begin{pmatrix}1&0&2\\2&-1&3\\-1&1&1\end{pmatrix}$

From (2), $2B=\begin{pmatrix}1&2&0\\6&-3&3\\-5&3&1\end{pmatrix}-A=\begin{pmatrix}1&2&0\\6&-3&3\\-5&3&1\end{pmatrix}-\begin{pmatrix}1&0&2\\2&-1&3\\-1&1&1\end{pmatrix}=\begin{pmatrix}0&2&-2\\4&-2&0\\-4&2&0\end{pmatrix}$

or, $B=\frac{1}{2}\begin{pmatrix}0&2&-2\\4&-2&0\\-4&2&0\end{pmatrix}=\begin{pmatrix}0&1&-1\\2&-1&0\\-2&1&0\end{pmatrix}$

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Example. 7. If $A=\begin{pmatrix}0&1&2&3\\3&2&1&0\end{pmatrix}$ and $B=\begin{pmatrix}0&3\\1&2\\2&1\\3&0\end{pmatrix}$ then verify whether the commutative law $AB=BA$ is true.

Solution. We see that A is of size 2×4 and B is of size 4×2. So AB has size 2×2 and BA has size 4×4.

So AB and BA are of unequal size. $\therefore AB \neq BA$

Example. 8. Give an example of two matrices $A\neq 0, B\neq 0$ though $AB=0$

Solution. We set an example $A=\begin{pmatrix}1&2&0\\1&1&0\\-1&4&0\end{pmatrix}$ and $B=\begin{pmatrix}0&0&0\\0&0&0\\2&3&9\end{pmatrix}$.

Here we see $A\neq 0, B\neq 0$ but $AB=\begin{pmatrix}1&2&0\\1&1&0\\-1&4&0\end{pmatrix}\begin{pmatrix}0&0&0\\0&0&0\\2&3&9\end{pmatrix}=\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}=O$

*** Example. 9.** Find, where possible, $A-B$, AB and BA, with reasons where impossible, when

$A=\begin{bmatrix}4&2&-1\\3&-7&1\end{bmatrix}$ and $B=\begin{bmatrix}2&3\\-3&0\\-1&5\end{bmatrix}$. Hence verify whether $AB=BA$ is valid.

Solution. Size of A is 2×3 and that of B is 3×2. Since they are of unequal sizes so $A-B$ is not possible.

AB is possible since number of column of A and number of rows of B are same.

$$\therefore AB=\begin{pmatrix}4&2&-1\\3&-7&1\end{pmatrix}\begin{pmatrix}2&3\\-3&0\\-1&5\end{pmatrix}=\begin{pmatrix}8-6+1&12+0-5\\6+21-1&9+0+5\end{pmatrix}=\begin{pmatrix}3&7\\26&14\end{pmatrix} \quad \cdots \quad (1)$$

BA is possible, since number of columns of B and number of rows of A are same.

$$\therefore BA=\begin{pmatrix}2&3\\-3&0\\-1&5\end{pmatrix}\begin{pmatrix}4&2&-1\\3&-7&1\end{pmatrix}=\begin{pmatrix}8-9&4-21&-2+3\\-12+0&-6+0&3+0\\-4+15&-2-35&1+5\end{pmatrix}=\begin{pmatrix}17&-17&1\\-12&-6&3\\11&-37&6\end{pmatrix} \quad \cdots \quad (2)$$

From (1) and (2) it is obvious that $AB\neq BA$

Example. 10. If $A=\begin{pmatrix}1&3\\2&4\end{pmatrix}, B=\begin{pmatrix}3&-1\\4&2\end{pmatrix}$, find a matrix X such that $A+X=B$

Solution. $A+X=B$ or, $X=B-A$

or, $X=\begin{pmatrix}3&-1\\4&2\end{pmatrix}-\begin{pmatrix}1&3\\2&4\end{pmatrix}=\begin{pmatrix}2&-4\\2&-2\end{pmatrix}$ $\therefore X=\begin{pmatrix}2&-4\\2&-2\end{pmatrix}$

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Example. 11. If $A=\begin{pmatrix}0&1&0\\-1&0&2\\0&0&1\end{pmatrix}, U=\begin{pmatrix}1\\1\\0\end{pmatrix}$, find A^2U

Solution. $A^2=AA=\begin{pmatrix}0&1&0\\-1&0&2\\0&0&1\end{pmatrix}\begin{pmatrix}0&1&0\\-1&0&2\\0&0&1\end{pmatrix}=\begin{pmatrix}0-1+0&0+0+0&0+2+0\\0+0+0&-1+0+0&0+0+2\\0+0+0&0+0+0&0+0+1\end{pmatrix}=\begin{pmatrix}-1&0&2\\0&-1&2\\0&0&1\end{pmatrix}$

$$\therefore A^2U=\begin{pmatrix}-1&0&2\\0&-1&2\\0&0&1\end{pmatrix}\begin{pmatrix}1\\1\\0\end{pmatrix}=\begin{pmatrix}-1+0+0\\0-1+0\\0+0+0\end{pmatrix}=\begin{pmatrix}-1\\-1\\0\end{pmatrix}$$

Example. 12. Let $A=\begin{bmatrix}a&b&c\\d&c&-4\\5&-6&7\end{bmatrix}$ and $B=\begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}$. If $AB=BA$ find the values of a, b, c and d. [W.B.S.C. 2005]

Solution. $AB=\begin{bmatrix}a&b&c\\d&c&-4\\5&-6&7\end{bmatrix}\begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}=\begin{bmatrix}c&b&a\\-4&c&d\\7&-6&5\end{bmatrix}$ and $BA=\begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}\begin{bmatrix}a&b&c\\d&c&-4\\5&-6&7\end{bmatrix}=\begin{bmatrix}5&-6&7\\d&c&-4\\a&b&c\end{bmatrix}$

Given $AB=BA$ or, $\begin{bmatrix}c&b&a\\-4&c&d\\7&-6&5\end{bmatrix}=\begin{bmatrix}5&-6&7\\d&c&-4\\a&b&c\end{bmatrix}$

This gives $c=5$, $b=-6$, $a=7$

$-4=d$ $c=c$, $d=-4$

$7=a$ $-6=b$, $5=c$

$\therefore a=7, b=-6, c=5, d=-4$

Example. 13. Show that $(A+B)^2\neq A^2+2AB+B^2$ where $A=\begin{pmatrix}-1&2\\0&1\end{pmatrix}$ and $B=\begin{pmatrix}1&0\\-1&2\end{pmatrix}$

Solution. $A+B=\begin{pmatrix}-1&2\\0&1\end{pmatrix}+\begin{pmatrix}1&0\\-1&2\end{pmatrix}=\begin{pmatrix}0&2\\-1&3\end{pmatrix}$

$$\therefore (A+B)^2=(A+B)(A+B)=\begin{pmatrix}0&2\\-1&3\end{pmatrix}\begin{pmatrix}0&2\\-1&3\end{pmatrix}=\begin{pmatrix}0-2&0+6\\0-3&-2+9\end{pmatrix}=\begin{pmatrix}-2&6\\-3&7\end{pmatrix} \quad \ldots (1)$$

Again $A^2=\begin{pmatrix}-1&2\\0&1\end{pmatrix}\begin{pmatrix}-1&2\\0&1\end{pmatrix}=\begin{pmatrix}1+0&-2+2\\0+0&0+1\end{pmatrix}=\begin{pmatrix}1&0\\0&1\end{pmatrix}$

$$B^2=\begin{pmatrix}1&0\\-1&2\end{pmatrix}\begin{pmatrix}1&0\\-1&2\end{pmatrix}=\begin{pmatrix}1+0&0+0\\-1-2&0+4\end{pmatrix}=\begin{pmatrix}1&0\\-3&4\end{pmatrix}$$

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$$AB=\begin{pmatrix}-1 & 2\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ -1 & 2\end{pmatrix}=\begin{pmatrix}-1-2 & 0+4\\ 0-1 & 0+2\end{pmatrix}=\begin{pmatrix}-3 & 4\\ -1 & 2\end{pmatrix}$$

$$\therefore A^2+2AB+B^2=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}+2\begin{pmatrix}-3 & 4\\ -1 & 2\end{pmatrix}+\begin{pmatrix}1 & 0\\ -3 & 4\end{pmatrix}=\begin{pmatrix}-4 & 8\\ -5 & 9\end{pmatrix} \qquad \ldots (2)$$

From (1) and (2) we see $(A+B)^2 \neq A^2+2AB+B^2$

Example. 14. If A and B be two matrices such that $A+B$ and AB are both defined prove that A and B are both square matrices of same order.

Solution. Since $A+B$ is defined A and B have same size. So let both have size $m\times n$. Again since AB is defined so number of columns of A and the number of rows of B are same. Therefore $n=m$. So A has size $m\times m$ and the size of B is also $m\times m$. Thus A and B are both square matrices of order m.

Example. 15. If $A=\begin{pmatrix}a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\\ a_3 & b_3 & c_3\end{pmatrix}$ show that $(3A)^T=3A^T$

Solution. Since $A=\begin{pmatrix}a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\\ a_3 & b_3 & c_3\end{pmatrix}, 3A=\begin{pmatrix}3a_1 & 3b_1 & 3c_1\\ 3a_2 & 3b_2 & 3c_2\\ 3a_3 & 3b_3 & 3c_3\end{pmatrix}$

$$\therefore (3A)^T=\begin{pmatrix}3a_1 & 3a_2 & 3a_3\\ 3b_1 & 3b_2 & 3b_3\\ 3c_1 & 3c_2 & 3c_3\end{pmatrix}=3\begin{pmatrix}a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{pmatrix}=3A^T$$

Example. 16. If $A=\begin{pmatrix}1 & -1 & 2\\ 2 & 1 & 0\end{pmatrix}$ and $B=\begin{pmatrix}1 & 3\\ 2 & 0\\ -1 & 1\end{pmatrix}$ then examine whether $(AB)^T=B^TA^T$ is valid.

Solution. $AB=\begin{pmatrix}1 & -1 & 2\\ 2 & 1 & 0\end{pmatrix}\begin{pmatrix}1 & 3\\ 2 & 0\\ -1 & 1\end{pmatrix}=\begin{pmatrix}-3 & 5\\ 4 & 6\end{pmatrix}\therefore (AB)^T=\begin{pmatrix}-3 & 4\\ 5 & 6\end{pmatrix} \qquad \ldots (1)$

Again $B^TA^T=\begin{pmatrix}1 & 2 & -1\\ 3 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 2\\ -1 & 1\\ 2 & 0\end{pmatrix}=\begin{pmatrix}1-2-2 & 2+2+0\\ 3+0+2 & 6+0+0\end{pmatrix}=\begin{pmatrix}-3 & 4\\ 5 & 6\end{pmatrix} \qquad \ldots (2)$

From (1) and (2) we see $(AB)^T=B^TA^T$.

Example. 17. If $A=\begin{bmatrix}1 & 2 & 3\\ 6 & 7 & 8\\ 6 & -3 & 4\end{bmatrix}$ and $B=\begin{bmatrix}1 & 2 & 3\\ 3 & 4 & 2\\ 5 & 6 & 1\end{bmatrix}$ show that $(AB)^T=B^TA^T$ and $(A+B)^T=A^T+B^T$

[W.B.S.C 2003]

Solution. Now $AB=\begin{bmatrix}1 & 2 & 3\\ 6 & 7 & 8\\ 6 & -3 & 4\end{bmatrix}\begin{bmatrix}1 & 2 & 3\\ 3 & 4 & 2\\ 5 & 6 & 1\end{bmatrix}=\begin{bmatrix}1+6+15 & 2+8+18 & 3+4+3\\ 6+21+40 & 12+28+48 & 18+14+8\\ 6-9+20 & 12-12+24 & 18-6+4\end{bmatrix}=\begin{bmatrix}22 & 28 & 10\\ 67 & 88 & 40\\ 17 & 24 & 16\end{bmatrix}$

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$$\therefore (AB)^T = \begin{bmatrix} 22 & 67 & 17 \\ 28 & 88 & 24 \\ 10 & 40 & 16 \end{bmatrix} \quad \ldots (1)$$

Now, $A^T = \begin{bmatrix} 1 & 6 & 6 \\ 2 & 7 & -3 \\ 3 & 8 & 4 \end{bmatrix}$ and $B^T = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \\ 3 & 2 & 1 \end{bmatrix}$

$$\therefore B^T A^T = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 6 & 6 \\ 2 & 7 & -3 \\ 3 & -8 & 4 \end{bmatrix} = \begin{bmatrix} 1+6+15 & 6+21+40 & 6-9+20 \\ 2+8+18 & 12+28+48 & 12-12+24 \\ 3+4+3 & 18+14+8 & 18-6+4 \end{bmatrix}$$

$$= \begin{bmatrix} 22 & 67 & 17 \\ 28 & 88 & 24 \\ 10 & 40 & 16 \end{bmatrix} \quad \ldots (2)$$

From (1) and (2) we have $(AB)^T = B^T A^T$

Now, $A + B = \begin{bmatrix} 1 & 2 & 3 \\ 6 & 7 & 8 \\ 6 & -3 & 4 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 2 \\ 5 & 6 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 4 & 6 \\ 9 & 11 & 10 \\ 11 & 3 & 5 \end{bmatrix}$.

$$\therefore (A+B)^T = \begin{bmatrix} 2 & 9 & 11 \\ 4 & 11 & 3 \\ 6 & 10 & 5 \end{bmatrix} \quad \ldots (4)$$

and $A^T + B^T = \begin{bmatrix} 1 & 6 & 6 \\ 2 & 7 & -3 \\ 3 & 8 & 4 \end{bmatrix} + \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \\ 3 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 9 & 11 \\ 4 & 11 & 3 \\ 6 & 10 & 5 \end{bmatrix} \quad \ldots (5)$

From (4) and (5) we have $(A+B)^T = A^T + B^T$

Example. 18. If $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 5 & 6 & 7 \end{bmatrix}$ then show that $A + A^T$ is a symmetric matrix and $A - A^T$ is skew symmetric [*W.B.S.C 2004, 2006, 2013*]

Solution. Given $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 5 & 6 & 7 \end{bmatrix}$ $\quad \therefore A^T = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \\ 3 & 5 & 7 \end{bmatrix}$

$\therefore A + A^T = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 5 & 6 & 7 \end{bmatrix} + \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \\ 3 & 5 & 7 \end{bmatrix} = \begin{bmatrix} 2 & 5 & 8 \\ 5 & 8 & 11 \\ 8 & 11 & 14 \end{bmatrix}$ which is symmetric

Again $A - A^T = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 5 & 6 & 7 \end{bmatrix} - \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \\ 3 & 5 & 7 \end{bmatrix} = \begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & -1 \\ 2 & 1 & 0 \end{bmatrix}$ which is skew symmetric.

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Example. 19. If A be a skew symmetric matrix then prove that A^2 is symmetric.

Solution. Now, $(A^2)^T = (AA)^T = A^T A^T$ [applying the law $(AB)^T = B^T A^T$]

$$= (-A)(-A) \quad [\because A \text{ is skew symmetric}]$$

$$= AA = A^2 \qquad \therefore A^2 \text{ is symmetric.}$$

*

Example. 20. Express the matrix $\begin{pmatrix} 4 & 2 & -3 \\ 1 & 3 & -6 \\ -5 & 0 & -7 \end{pmatrix}$ as the sum of a symmetric and a skew symmetric matrix.

Solution. Let $A = \begin{pmatrix} 4 & 2 & -3 \\ 1 & 3 & -6 \\ -5 & 0 & -7 \end{pmatrix} \quad \therefore A^T = \begin{pmatrix} 4 & 1 & -5 \\ 2 & 3 & 0 \\ -3 & -6 & -7 \end{pmatrix}$

Let $B = \frac{1}{2}(A + A^T) = \frac{1}{2}\begin{pmatrix} 8 & 3 & -8 \\ 3 & 6 & -6 \\ -8 & -6 & -14 \end{pmatrix} = \begin{pmatrix} 4 & 3/2 & -4 \\ 3/2 & 3 & -3 \\ -4 & -3 & -7 \end{pmatrix}$ which is a symmetric matrix.

Let $C = \frac{1}{2}(A - A^T) = \frac{1}{2}\begin{pmatrix} 0 & 1 & 2 \\ -1 & 0 & -6 \\ -2 & 6 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{2} & 1 \\ -\frac{1}{2} & 0 & -3 \\ -1 & 3 & 0 \end{pmatrix}$ which is a skew symmetric matrix.

Obviously $A = B + C$ i.e. $\begin{pmatrix} 4 & 2 & -3 \\ 1 & 3 & -6 \\ -5 & 0 & -7 \end{pmatrix} = \begin{pmatrix} 4 & 3/2 & -4 \\ 3/2 & 3 & -3 \\ -4 & -3 & -7 \end{pmatrix} + \begin{pmatrix} 0 & \frac{1}{2} & 1 \\ -\frac{1}{2} & 0 & -3 \\ -1 & 3 & 0 \end{pmatrix}$

Example. 21. Prove that $A^2 - 4A - 5I = 0$ where $A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}$ [*W.B.S.C. 2011*]

Solution. Now, $A^2 = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1+4+4 & 2+2+4 & 2+4+2 \\ 2+2+4 & 4+1+4 & 4+2+2 \\ 2+4+2 & 4+2+2 & 4+4+1 \end{pmatrix} = \begin{pmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{pmatrix}$

$$\text{LHS} = A^2 - 4A - 5I = \begin{pmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{pmatrix} - 4\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix} - 5\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 9-4-5 & 8-8 & 8-8 \\ 8-8 & 9-4-5 & 8-8 \\ 8-8 & 8-8 & 9-4-5 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = O$$

Thus $A^2 - 4A - 5I = O$

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Example. 22. A square matrix A is such that $A^2 = A$ and $(A-A^T)^2 = O$. Prove that $A + A^T = AA^T + A^TA$

Solution. Given $(A-A^T)^2 = 0$ or, $(A-A^T)(A-A^T) = 0$ or, $AA - AA^T - A^TA + A^TA^T = 0$

or, $A^2 - AA^T - A^TA + (AA)^T = 0$ (applying the rule $(AB)^T = B^TA^T$)

or, $A^2 - AA^T - A^TA + (A^2)^T = 0$ or, $A - AA^T - A^TA + A^T = 0 \quad \because A^2 = A$

or, $A + A^T = AA^T + A^TA$

Example. 23. If $A + B = 2B^T$ and $3A + 2B = I$ find A and B where I is the 3rd order identity matrix.

Solution. $A + B = 2B^T$ or, $3A + 3B = 6B^T$ (1) multiplying by 3.

Again we have $3A + 2B = I$ ... (2)

Doing (1) – (2) we get $B = 6B^T - I$ ... (3)

or, $B^T = (6B^T - I)^T = (6B^T)^T - (I)^T = 6(B^T)^T - I = 6B - I$

or, $6B^T = 36B - 6I$ ... (4)

Doing (3) + (4) we get $B + 6B^T = 6B^T - I + 36B - 6I$ or, $35B = 7I$

$$\text{or, } B = \frac{1}{5}I = \frac{1}{5}\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} = \begin{pmatrix}1/5&0&0\\0&1/5&0\\0&0&1/5\end{pmatrix}$$

$$\text{From (1) we get } A = \frac{1}{3}(I - 2B) = \frac{1}{3}\left\{\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} - 2\begin{pmatrix}1/5&0&0\\0&1/5&0\\0&0&1/5\end{pmatrix}\right\} = \begin{pmatrix}1/5&0&0\\0&1/5&0\\0&0&1/5\end{pmatrix}.$$

Example. 24. If A be a mth order symmetric matrix and P be a matrix of size $m\times n$, prove that P^TAP is symmetric.

Solution. Size of A is $m\times m$, size of P is $m\times n$. $\therefore$ Size of AP is $m\times n$.

Again size of P^T is $n\times m$. So P^TAP is defined and its size is $n\times n$, i.e. it is a square matrix.

Now, $(P^TAP)^T = (P^T(AP))^T = (AP)^T(P^T)^T = (AP)^TP = (P^TA^T)P = P^TAP \; (\because A^T = A)$

$\therefore P^TAP$ is symmetric.

Example. 25. If $B^2 = B$ show that $(I-B)^2 = I - B$ and $AB = BA = 0$ where $A = I - B$.

Solution. Now, $(I-B)^2 = (I-B)(I-B) = I - IB - BI + BB = I - B - B + B \; [\because B^2 = B]$
$= I - B$

Again $AB = (I-B)B = IB - B^2 = B - B = 0$. Similarly $BA = 0 \quad \therefore AB = BA = 0$

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Example. 26. For the matrix A satisfying the equation $A^2 - A + I = 0$ find the matrix X such that $AX = I$.

Solution. $A^2 - A + I = 0$ or, $A - A^2 = I$ or, $AI - A^2 = I$

or, $A(I - A) = I$ $\therefore X = I - A$.

Example. 27. Find the value of a in order that $\begin{bmatrix} 2 & 3 & 5 \\ 1 & a & 2 \\ 0 & 1 & -1 \end{bmatrix}$ is a singular matrix. [*W. B. S. C. 2010*]

Solution. Since the given matrix is singular therefore $\begin{bmatrix} 2 & 3 & 5 \\ 1 & a & 2 \\ 0 & 1 & -1 \end{bmatrix} = 0$

or, $2\begin{vmatrix} a & 2 \\ 1 & -1 \end{vmatrix} - 3\begin{vmatrix} 1 & 2 \\ 0 & -1 \end{vmatrix} + 5\begin{vmatrix} 1 & a \\ 0 & 1 \end{vmatrix} = 0$ or, $2(-a-2) - 3(-1-0) + 5(1-0) = 0$

or, $-2a - 4 + 3 + 5 = 0$ or, $-2a + 4 = 0$ $\therefore a = 2$

Example. 28 . Find whether the matrix $\begin{pmatrix} 2 & -3 & 4 \\ 1 & 0 & 1 \\ 0 & -1 & 4 \end{pmatrix}$ is invertible. Find its inverse if possible. Verify the result.

Let $A = \begin{pmatrix} 2 & -3 & 4 \\ 1 & 0 & 1 \\ 0 & -1 & 4 \end{pmatrix}$

Now, det $(A) = \begin{vmatrix} 2 & -3 & 4 \\ 1 & 0 & 1 \\ 0 & -1 & 4 \end{vmatrix} = 2 \times 1 + 3(4 - 0) + 4(-1 - 0)$

$= 10 \neq 0$

$\therefore$ A is invertible, i.e. A^{-1} exists.

Co-factors of $2 = (-1)^{1+1}\begin{vmatrix} 0 & 1 \\ -1 & 4 \end{vmatrix} = 1$

Co-factors of $-3 = (-1)^{1+2}\begin{vmatrix} 1 & 1 \\ 0 & 4 \end{vmatrix} = -4$

Co-factors of $4 = (-1)^{1+3}\begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} = -1$

Similarly Co-factor of $1 = 8$, Co-factor of $0 = 8$

Co-factor of $1 = 2$, Co-factor of $0 = -3$

Co-factor of $-1 = 2$, Co-factor of $4 = 3$

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$$\therefore adj(A) = \begin{pmatrix} 1 & -4 & -1 \\ 8 & 8 & 2 \\ -3 & 2 & 3 \end{pmatrix}^T = \begin{pmatrix} 1 & 8 & -3 \\ -4 & 8 & 2 \\ -1 & 2 & 3 \end{pmatrix}$$

Now $A^{-1} = \dfrac{1}{\det(A)} adj(A) = \dfrac{1}{10}\begin{pmatrix} 1 & 8 & -3 \\ -4 & 8 & 2 \\ -1 & 2 & 3 \end{pmatrix}$ *ans.*

Verification :

Now $= AA^{-1} = \begin{pmatrix} 2 & -3 & 4 \\ 1 & 0 & 1 \\ 0 & -1 & 4 \end{pmatrix} \dfrac{1}{10}\begin{pmatrix} 1 & 8 & -3 \\ -4 & 8 & 2 \\ -1 & 2 & 3 \end{pmatrix}$

$$= \frac{1}{10}\begin{pmatrix} 2 & -3 & 4 \\ 1 & 0 & 1 \\ 0 & -1 & 4 \end{pmatrix}\begin{pmatrix} 1 & 8 & -3 \\ -4 & 8 & 2 \\ -1 & 2 & 3 \end{pmatrix}$$

$$= \frac{1}{10}\begin{pmatrix} 10 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 10 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$$

Example. 29. Find A^{-1} *if* $A = \begin{pmatrix} 1 & 3 \\ 2 & -1 \end{pmatrix}$

Solution. det $(A) = \begin{vmatrix} 1 & 3 \\ 2 & -1 \end{vmatrix} = -1 - 6 = -7 \neq 0$

$\therefore$ A is invertible.

We shall find the Co-factor of every element in the above determinant.

Co-factor of $1 = (-1)^{1+1}|-1| = -1$

Co-factor of $3 = (-1)^{1+2}|2| = -2$

Co-factor of $2 = (-1)^{2+1}|3| = -3$

Co-factor of $-1 = (-1)^{2+2}|1| = 1$

$$\therefore adj(A) = \begin{pmatrix} -1 & -2 \\ -3 & 1 \end{pmatrix}^T = \begin{pmatrix} -1 & -3 \\ -2 & 1 \end{pmatrix}$$

$$\therefore A^{-1} = \frac{1}{\det(A)} adj(A) = \frac{1}{-7}\begin{pmatrix} -1 & -3 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{7} & \frac{3}{7} \\ \frac{2}{7} & -\frac{1}{7} \end{pmatrix}$$

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Example. 30 . If $=\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$ prove that $A^2-4A-5I=O$. Hence find A^{-1}

Solution. Now $\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$

$$=\begin{bmatrix} 1+4+4 & 2+2+4 & 2+4+2 \\ 2+2+4 & 4+1+4 & 4+2+2 \\ 2+4+2 & 4+2+2 & 4+4+1 \end{bmatrix}=\begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix}$$

$$LHS = A^2-4A-5I$$

$$=\begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix}-4\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}-5\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$=\begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix}-\begin{bmatrix} 4 & 8 & 8 \\ 8 & 4 & 8 \\ 8 & 8 & 4 \end{bmatrix}-\begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$=\begin{bmatrix} 9-4-5 & 8--8 & 8-8 \\ 8-8 & 9-4-5 & 8-8 \\ 8-8 & 8-8 & 9-4-5 \end{bmatrix}=\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}=O$$

So, $A^2-4A-5I=0$

From the above relation we get $A^2-4A=5I$

or, $A^2-4AI=5I$ $(\because AI=A)$

or, $A(A-4I)=5I$

or, $\frac{1}{5}A(A-4I)=I$

or, $A\left\{\frac{1}{5}(A-4I)\right\}=I$

So by definition of inverse of A we have

$$A^{-1}=\frac{1}{5}(A-4I)=\frac{1}{5}\left\{\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}-\begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}\right\}=\frac{1}{5}\left\{\begin{pmatrix} -3 & 2 & 2 \\ 2 & -3 & 2 \\ 2 & 2 & -3 \end{pmatrix}\right\}$$

$$A^{-1}=\begin{pmatrix} -3/5 & 2/5 & 2/5 \\ 2/5 & -3/5 & 2/5 \\ 2/5 & 2/5 & -3/5 \end{pmatrix}$$

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Example. 31. Prove that the matrix $\begin{pmatrix} 6 & -2 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$ is an inverse of the matrix $\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 3 \\ 1 & 2 & 4 \end{pmatrix}$

Solution. We see $\begin{pmatrix} 6 & -2 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 3 \\ 1 & 2 & 4 \end{pmatrix}$

$$=\begin{pmatrix} 6-2-3 & 12-6-6 & 18-6-12 \\ -1+1 & -2+3 & -3+3 \\ -1+0+1 & -2-0-2 & -3+4 \end{pmatrix}=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}=I$$

Similarly we can show that

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 3 \\ 1 & 2 & 4 \end{pmatrix}\begin{pmatrix} 6 & -2 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}=I$$

$$\therefore \begin{pmatrix} 6 & -2 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \text{ is inverse of } \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 3 \\ 1 & 2 & 4 \end{pmatrix}$$

Exercise

1. If $A=\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B=\begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}$ evaluate $2A-B$ [*W.B.S.C. 2004*]

2. If $A=\begin{bmatrix} -2 & 3 & 5 \\ -1 & -4 & 5 \\ 1 & -3 & 4 \end{bmatrix}, B=\begin{bmatrix} 1 & 3 & 5 \\ 1 & 3 & -5 \\ -1 & 3 & 5 \end{bmatrix}, C=\begin{bmatrix} 2 & -2 & -4 \\ -1 & -3 & 4 \\ 1 & 2 & 3 \end{bmatrix}$ verify the following laws

(i) $A(B+C)=AB+AC$ (ii) $A+C=C+A$ (iii) $A+(B-C)=(A+B)-C$

3. If $A=\begin{bmatrix} 5 & 1 \\ 7 & 0 \end{bmatrix}$ and $B=\begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}, C=\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ verify the relation $A(BC)=(AB)C$

4. If $A=\begin{bmatrix} 2 & -3 & -5 \\ -1 & 4 & 5 \\ 1 & -3 & -4 \end{bmatrix}, B=\begin{bmatrix} -1 & 3 & 5 \\ 1 & -3 & -5 \\ -1 & 3 & 5 \end{bmatrix}, C=\begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$ show that $AB=0=BA$

5. If $3A-B=\begin{bmatrix} -2 & 6 & 1 \\ -3 & -4 & 7 \\ 3 & -17 & 5 \end{bmatrix}$ and $A+2B=\begin{bmatrix} 4 & -5 & -2 \\ 6 & 8 & -7 \\ 1 & 34 & -10 \end{bmatrix}$ find A and B.

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