Problem 1   Thin airfoil theory

1a) verify that $\gamma(\theta) = 2\alpha V_\infty \frac{1+\cos\theta}{\sin\theta}$ satisfies the fundamental equation of thin airfoil theory for a symmetric airfoil AND satisfies Kutta condition.

Recall fundamental equation:

$$V_\infty\left(\alpha - \cancel{\frac{dz}{dx}}\right) = \frac{1}{2\pi}\int_0^c \frac{\gamma(\xi)\,d\xi}{x-\xi}$$

0, symmetric

Map $x \to \theta_0$: $\quad x = \frac{c}{2}(1-\cos\theta_0)$

$\xi \to \theta$: $\quad \xi = \frac{c}{2}(1-\cos\theta)$

limits: $\xi = c = \frac{c}{2}(1-\cos\theta)$ $\qquad d\xi = \frac{c}{2}\sin\theta\,d\theta$

$\cos\theta = -1$ @ $\theta = \pi$

$\xi = 0 \to \theta = 0$

$$\to V_\infty\alpha = \frac{1}{2\pi}\int_0^\pi \frac{2\alpha V_\infty \frac{1+\cos\theta}{\cancel{\sin\theta}}\cdot\frac{\cancel{c}}{2}\cancel{\sin}\theta\,d\theta}{\frac{\cancel{c}}{2}(1-\cos\theta_0) - \frac{\cancel{c}}{2}(1-\cos\theta)} \cdot \frac{2}{2}$$

$$V_\infty\alpha = \frac{1}{\cancel{2}\pi}\int_0^\pi \frac{\cancel{2}\alpha V_\infty(1+\cos\theta)\,d\theta}{\cancel{1}-\cos\theta_0 - \cancel{1} + \cos\theta} = \frac{1}{\pi}\int_0^\pi \frac{\alpha V_\infty(1+\cos\theta)d\theta}{\cos\theta-\cos\theta_0}$$

$$\to V_\infty\alpha = \frac{V_\infty\alpha}{\pi}\left[\underbrace{\int_0^\pi \frac{d\theta}{\cos\theta-\cos\theta_0}}_{0} + \underbrace{\int_0^\pi \frac{\cos\theta}{\cos\theta-\cos\theta_0}d\theta}_{\text{integral table: } = \frac{\pi\sin\theta_0}{\sin\theta_0} = \pi}\right]$$

$$\to \boxed{V_\infty\alpha = \frac{V_\infty\alpha}{\pi}(0+\pi) = V_\infty\alpha}$$

verify Kutta: $\gamma(TE) = 0 = \gamma(\pi)$

$\gamma(\pi) = 2\alpha V_\infty \frac{1+\cos\pi}{\sin\pi} = 2\alpha V_\infty \frac{0}{0} \to$ L'hospitals

$$\to \boxed{2\alpha V_\infty \frac{-\sin\pi}{\cos\pi} = \frac{-0}{1} = 0 \ \checkmark \text{ verified}}$$

(b) In our model, $\gamma(\theta)$ represents a strength distribution of vortex elements over the airfoil. The elements make up a vortex sheet along the camberline of an airfoil that represents flow around the airfoil. This strength distribution can then be integrated to find the Γ circulation over the entire vortex sheet. Then through Kelvin's circulation theory this can be applied to the airfoil itself