524 HW6 Kyle Adler

Problem 1 Instantaneous & "mean" prop. efficiencies η, η_m

1a) Plot expressions as $f\left(\frac{u}{u_e}\right)$ and $f\left(\frac{\Delta u}{u_e}\right)$, $0 \leq \frac{u}{u_e} = \frac{\Delta u}{u_e} \leq 10$

$$\eta = \frac{2\frac{u}{u_e}}{1+\left(\frac{u}{u_e}\right)^2} \qquad \eta_m = \frac{1}{\Delta u/u_e} \ln\left(1+\left(\frac{\Delta u}{u_e}\right)^2\right)$$

→ Matlab Figure 1

1b) Find $\frac{u}{u_e}$ & $\frac{\Delta u}{u_e}$ that maximize η & η_m

$$\frac{d}{d\frac{u}{u_e}}(\eta) = \frac{\left[1+\left(\frac{u}{u_e}\right)^2\right]\cdot 2 - 2\frac{u}{u_e}\left(2\frac{u}{u_e}\right)}{\left[1+\left(\frac{u}{u_e}\right)^2\right]}$$

$$= \frac{2+2\left(\frac{u}{u_e}\right)^2 - 4\left(\frac{u}{u_e}\right)^2}{1+2\left(\frac{u}{u_e}\right)^2+\left(\frac{u}{u_e}\right)^4} = \frac{2-2\left(\frac{u}{u_e}\right)^2}{\left[1+\left(\frac{u}{u_e}\right)^2\right]} = 0$$

$$0 = \cancel{2}\left[1-\left(\frac{u}{u_e}\right)^2\right] \rightarrow \max(\eta) \text{ @ } \frac{u}{u_e} = 1$$

$$\rightarrow \eta_{max} = \eta\left(\frac{u}{u_e}=1\right) = \frac{2}{2} = 1$$

η_m: Matlab vpa solve: $\max(\eta_m)$ @ $\frac{\Delta u}{u_e}$

1c) η is maximized when rocket velocity is equal to the exhaust velocity.

→ Question - who cares about power in exhaust gasses w.r.t. ground? It provides the same power to rocket.

Problem 2 A gas with γ flows from reservoir at stag. cond. P_0, T_0, through insulated, frictionless duct. varying cross section, no work.

Insulated, frictionless: $\dot{Q} = 0$, isentropic

2a) Energy, show that $u_{max} = a_0 \sqrt{\frac{2}{\gamma - 1}}$

$$\cancel{\frac{d}{dt}\int_{CV}\left(e + \frac{u^2}{2} + gz\right)\rho\, d\forall} + \int_{CS}\left(h + \frac{u^2}{2} + \cancel{gz}\right)\rho\, \underline{u}_{rel}\cdot d\underline{A} = \cancel{\dot{Q}} - \cancel{\int_{CS} p\, \underline{u}_b \cdot d\underline{A}}^{\,0} - \cancel{\dot{W}}$$

$$\rightarrow \quad -\left[h_0 + \frac{u_0^2}{2}\right]\cancel{\dot{m}} + \left[h + \frac{u^2}{2}\right]\cancel{\dot{m}} = 0$$

$$u = \sqrt{2(h_0 - h)}$$

$\boxed{u_{max} \text{ @ } h = 0}$

$$u_{max} = \sqrt{2 c_p T_0}$$

$$c_p = c_v + R \qquad \gamma = \frac{c_p}{c_v}$$

$$c_p = \frac{c_p}{\gamma} + R$$

$$c_p\left(1 - \frac{1}{\gamma}\right) = R$$

$$c_p = \frac{R}{1 - \frac{1}{\gamma}} = \frac{R\gamma}{\gamma - 1}$$

$$a_0 = \sqrt{\gamma R T_0}$$

$$\rightarrow \quad u_{max} = \sqrt{\frac{2 R \gamma T_0}{\gamma - 1}} = \boxed{a_0 \sqrt{\frac{2}{\gamma - 1}} = u_{max}}$$

2b) M @ $u = u_{max}$ $\qquad M_{max} = \frac{u_{max}}{a}$

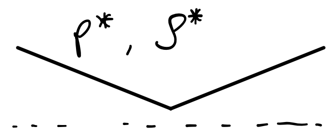
$a = \sqrt{\gamma R T}$ $\qquad T \rightarrow 0$ as all energy converted to velocity

$$\boxed{\begin{array}{l} \therefore\ a \rightarrow 0, \text{ while } u_{max} > 0 \\ \therefore\ M_{max} \rightarrow \infty \end{array}}$$

Problem 3

3a) Develop expressions for p/p^*, $\frac{\rho}{\rho^*}$, T/T^* in terms of M. & Find M in terms of $\frac{u}{u^*}$

At $*$: Sonic throat



$$\frac{p}{p^*} = \frac{\frac{p}{p_0}}{\frac{p^*}{p_0}} = \frac{\left(1+\frac{\gamma-1}{2}M^2\right)^{\frac{-\gamma}{\gamma-1}}}{\left(1+\frac{\gamma-1}{2}\right)^{\frac{-\gamma}{\gamma-1}}}$$

$$\boxed{\frac{p}{p^*} = \left(\frac{1+\frac{\gamma-1}{2}M^2}{1+\frac{\gamma-1}{2}}\right)^{\frac{\gamma}{1-\gamma}}}$$

$$\rightarrow \frac{\rho}{\rho^*} = \frac{\frac{\rho}{\rho_0}}{\frac{\rho^*}{\rho_0}} = \frac{\left(1+\frac{\gamma-1}{2}M^2\right)^{\frac{1}{1-\gamma}}}{\left(1+\frac{\gamma-1}{2}\right)^{\frac{1}{1-\gamma}}} \rightarrow \boxed{\frac{\rho}{\rho^*} = \left(\frac{1+\frac{\gamma-1}{2}M^2}{1+\frac{\gamma-1}{2}}\right)^{\frac{1}{1-\gamma}}}$$

$$\rightarrow \frac{T}{T^*} = \ldots = \left(\frac{1+\frac{\gamma-1}{2}M^2}{1+\frac{\gamma-1}{2}}\right)^{-1} \rightarrow \boxed{\frac{T}{T^*} = \frac{1+\frac{\gamma-1}{2}}{1+\frac{\gamma-1}{2}M^2}}$$

M in terms of $\frac{u}{u^*}$

$$u = Ma \qquad u^* = a_*$$

$$\rightarrow \frac{u}{u^*} = \frac{Ma}{a_*} = \frac{M\sqrt{\cancel{\gamma R}T}}{\sqrt{\cancel{\gamma R}T_*}} = M\sqrt{\frac{T}{T_*}}$$

$$\rightarrow \frac{u}{u_*} = M\sqrt{\frac{1+\frac{\gamma-1}{2}}{1+\frac{\gamma-1}{2}M^2}} \rightarrow \frac{M^2+M^2\frac{\gamma-1}{2}}{1+\frac{\gamma-1}{2}M^2} = \left(\frac{u}{u_*}\right)^2$$

$$\rightarrow M^2\left(1+\frac{\gamma-1}{2}\right) = \left(\frac{u}{u_*}\right)^2 + \left(\frac{u}{u_*}\right)^2\frac{\gamma-1}{2}M^2$$

$\rightarrow M^2\left(1+\frac{\gamma-1}{2}-\frac{\gamma-1}{2}\left(\frac{u}{u_*}\right)^2\right)=\left(\frac{u}{u_*}\right)^2$

$$\rightarrow \boxed{M=\frac{u/u_*}{\sqrt{1+\frac{\gamma-1}{2}\left(1-(u/u_*)^2\right)}}} \rightarrow \frac{u/u_*}{-\frac{\gamma-1}{2}\frac{u}{u_*}}$$

3b) Limits that M imposes on u/u_*

As $M\rightarrow\infty$, occurs when denominator $\rightarrow 0$

$\therefore\ 1+\frac{\gamma-1}{2}\left(1-(u/u_*)^2\right)=0$

$\rightarrow 1-(u/u_*)^2=-\frac{2}{\gamma-1}$

$\rightarrow u/u_*=\sqrt{1+\frac{2}{\gamma-1}} \rightarrow \boxed{u/u_*=\sqrt{\frac{\gamma+1}{\gamma-1}}}$

2a): $a_0\sqrt{\frac{2}{\gamma-1}}=u_{max}$ $\rightarrow \frac{u_{max}}{u_{star}}=\frac{a_0\sqrt{\frac{2}{\gamma-1}}}{a_*}$

$\rightarrow \frac{u_{max}}{u_*}=\sqrt{\frac{2}{\gamma-1}}\frac{a_0}{a_*}=\sqrt{\frac{2}{\gamma-1}}\frac{\sqrt{\cancel{\gamma R}T_0}}{\sqrt{\cancel{\gamma R}T_*}}=\sqrt{\frac{2}{\gamma-1}}\sqrt{\frac{T_0}{T_*}}$

$\frac{T_*}{T_0}=\frac{1}{1+\frac{\gamma-1}{2}}=\frac{2}{\gamma+1}$ $\rightarrow \frac{u_{max}}{u_*}=\sqrt{\frac{\cancel{2}}{\gamma-1}\cdot\frac{\gamma+1}{\cancel{2}}}$

$\rightarrow \boxed{\frac{u_{max}}{u_*}=\sqrt{\frac{\gamma+1}{\gamma-1}}}$ ✓ consistent between 2a) and 3b)

3c) Plot

→ Matlab figure 2

Problem 4 Air, $P = 1\,atm$, $T = 298.15\,K$

$u = 0$

$\rightarrow M = 0$

$$\frac{p^*}{p} = \left(\frac{1+\frac{\gamma-1}{2}(0)^2}{1+\frac{\gamma-1}{2}}\right)^{\frac{-\gamma}{1-\gamma}} = \left(\frac{1}{1+\frac{\gamma-1}{2}}\right)^{\frac{\gamma}{\gamma-1}} = 0.156$$

$$\rightarrow \boxed{p^* = 0.156\ atm}$$

$$\frac{T^*}{T} = \frac{1+\frac{\gamma-1}{2}(0)^2}{1+\frac{\gamma-1}{2}} = \frac{1}{1+\frac{\gamma-1}{2}} = 0.588 \rightarrow \boxed{T^* = 175\ K}$$

$$\rho = \frac{P}{RT}, \quad R = 287 \rightarrow \rho = \frac{101300}{287 \cdot 298.15} = 1.184\ \frac{kg}{m^3}$$

$$\frac{\rho^*}{\rho} = \left(\frac{1}{1+\frac{\gamma-1}{2}}\right)^{\frac{1}{\gamma-1}} = 0.265 \rightarrow \boxed{\rho^* = 0.314\ \frac{kg}{m^3}}$$

$$u^* = a^*, \quad a^* = \sqrt{\gamma R T^*} = \sqrt{(1.4)(287)(175)} = \boxed{u^* = 265.17\ \frac{m}{s}}$$