Hohmann xfer recap

- Two-impulse maneuver w/ minimum ΔV

- Categorize maneuvers based on cost of ΔV & time
(fuel)

Energy of one orbit vs another

relative EOM (two body)

distance between bodies

$$\frac{d\bar{r}}{dt} \cdot \left(\frac{d^2 r}{dt^2} + \frac{\mu}{r^3} \bar{r} = 0 \right)$$

① ②

dot w/ velocity to get E

math magic

① use: $\frac{d}{dt}\underbrace{\left(\frac{d\bar{r}}{dt} \cdot \frac{d\bar{r}}{dt}\right)}_{|\bar{v}|^2} = \underline{2 \frac{d\bar{r}}{dt} \cdot \frac{d^2\bar{r}}{dt^2}}$

$$\therefore \frac{1}{2}|\bar{v}|^2 = \frac{d\bar{r}}{dt} \cdot \frac{d^2\bar{r}}{dt^2}$$

time rate of change of mechanical energy (kinetic)

assumes same dir.

② $\frac{\mu}{r^3} \bar{r} \cdot \frac{d\bar{r}}{dt} = \frac{\mu}{r^3} r \frac{dr}{dt} = \frac{\mu}{r^2} \frac{dr}{dt} = -\mu \frac{d}{dt}\left(\frac{1}{r}\right)$

time rate of change of potential energy

$$\therefore \frac{d}{dt}\left(\frac{1}{2}|\bar{v}|^2\right) - \mu \frac{d}{dt}\left(\frac{1}{r}\right) = 0$$

$$\frac{d}{dt}\left(\frac{1}{2}|\bar{v}|^2 - \frac{\mu}{r}\right) = 0 \rightarrow \frac{1}{2}|\bar{v}|^2 - \frac{\mu}{r} = \text{const.} = \mathcal{E}$$

specific mechanical energy

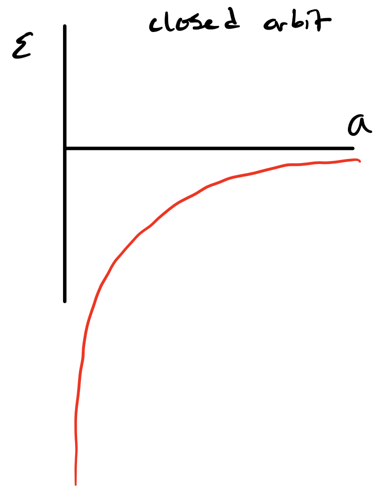
For a closed orbit

$\mathcal{E} = \frac{1}{2}|\bar{v}|^2 - \frac{\mu}{r}$, use $v^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right)$, $a = r$ for circle

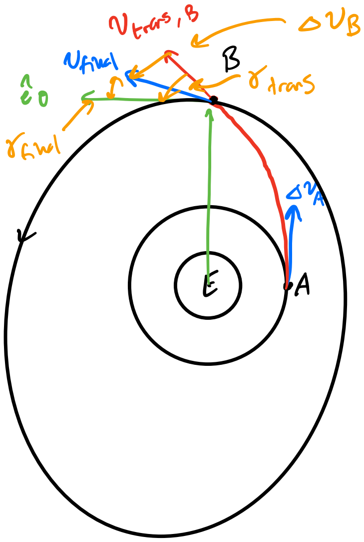
(or use $h = rv$, $r = h^2/\mu$)

$$\mathcal{E} = \frac{1}{2}\mu\left(\frac{2}{r} - \frac{1}{\cancel{a}\, r}\right) - \frac{\mu}{r} = \frac{\mu}{2r} - \frac{\mu}{r} = -\frac{\mu}{2r} \quad \boxtimes$$

$\therefore$ For orbits, $\underset{\text{orbit}}{\mathcal{E}} = -\dfrac{\mu}{2a}$, $\underset{\Delta v}{\mathcal{E}_{cost}} = \frac{1}{2}|\bar{v}|^2$



To decrease time of Hohmann transfer:



need v_B, triangle not necessarily right

use Law of cosines for Δv_B

$$\Delta v_B^2 = v_{final}^2 + v_{trans,B}^2 - 2 v_{final} v_{trans,B} \cos(\gamma_t - \gamma_f)$$

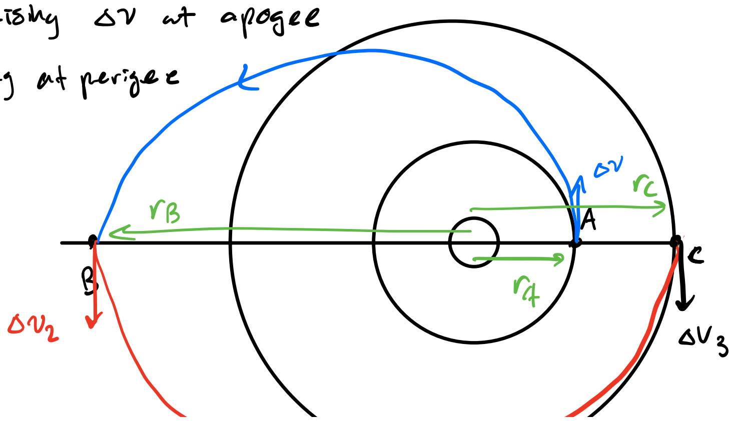
γ_t = flight path $\measuredangle$ of transfer @ B

γ_f : " " " target orbit @ B

Next transfer option "Bi-elliptic"

<u>Bi-elliptic</u>

- initial $\Delta v \rightarrow$ close to v_{esc} (biggest ellipse we can make)
- perigee - raising Δv at apogee
- apogee lowering at perigee



$$v = \sqrt{\mu\left(\frac{2}{r} - \frac{1}{a}\right)}$$

Transfer 1

$$a_1 = \frac{r_A + r_B}{2}$$

$$e_1 = \frac{r_B - r_A}{r_B + r_A}$$

Transfer 2

$$a_2 = \frac{r_C + r_B}{2}$$

$$e_2 = \frac{r_B - r_C}{r_B + r_C}$$

Calculate Δv's

$$\Delta v_1 = v_{t_1 A} - v_{c_1} = \sqrt{\mu\left(\frac{2}{r_A} - \frac{2}{r_A + r_B}\right)} - \frac{\mu}{r_A}$$

algebra:

where you're going

$$v_{t_1, A} = \sqrt{2\mu\left(\frac{r_B}{r_A(r_A + r_B)}\right)}$$

where you are

⋮

$$v_{t_1 B} = \sqrt{2\mu\left(\frac{r_A}{r_B(r_A + r_B)}\right)}$$

$$v_{t_2 B} = \sqrt{2\mu\left(\frac{r_C}{r_B(r_B + r_C)}\right)}$$

$$v_{t_2 C} = \sqrt{2\mu\left(\frac{r_B}{r_C(r_B + r_C)}\right)}$$

$$\therefore \Delta v_{tot} \quad \underbrace{|\Delta v_1|}_{v_{t_1 A} - v_{circ,1}} + \underbrace{|\Delta v_2|}_{v_{t_2 B} - v_{t_1 B}} + \underbrace{|\Delta v_3|}_{v_{c_2} - v_{t_2 C}}$$

$v_{circ,1} = v_{a_1}$, $\quad v_{c_2} = v_{circ 2}$

Remarks

- lots of wasted energy (last burn, $\Delta v < 0$, slows down)
- practical when $r_C \gg r_A$
- $r_B > r_A \rightarrow$ required, as $r_B \to \infty$ efficiency ↑ but so does time
- $T = \frac{1}{2}\left(2\pi\sqrt{\frac{a_1^3}{\mu}} + 2\pi\sqrt{\frac{a_2^3}{\mu}}\right)$

Example 1 Bi-elliptic

$r_A = 7000$ km

Earth: $\mu = 3.986 \times 10^5$ km^3/s^2

$r_C = 105000$ km

$r_B = 210000$ km (choose)

Note: $\frac{r_C}{r_A} \sim 15$

$r_C \cancel{\gg} r_A$ Bi-elliptic or hohmann?

Bi-elliptic:

Transfer 1: $\left\{ \begin{array}{l} v_{circ,1} = \sqrt{\frac{\mu}{r_A}} = 7.546 \text{ km/s} \\ v_{t1A} = \sqrt{\frac{2\mu r_B}{r_A(r_A+r_B)}} = 10.498 \text{ km/s} \end{array} \right.$ − + Δv_1

Transfer 2: $\left\{ \begin{array}{l} v_{t1B} = \sqrt{\frac{2\mu r_A}{r_B(r_A+r_B)}} = 0.35 \text{ km/s} \\ v_{t2B} = 1.12 \text{ km/s} \\ v_{t2C} = 2.24 \text{ km/s} \end{array} \right.$ − + Δv_2

+ Δv_3 −

Final orbit: $v_{circ_2} = \sqrt{\frac{\mu}{r_C}} = 1.95$ km/s

$\therefore \Delta v_{total} = |\Delta v_1| + |\Delta v_2| + |\Delta v_3|$

$= 4.028$ km/s

time $= \frac{1}{2}(T_{t_1} + T_{t_2}) = 5.6$ days

As an exercise, compute Hohmann Δv, time for r_A to r_C

$\Delta v_{tot,\,hohmann} = 4.0463$ km/s

$\left(\Delta v_{tot} - \Delta v_{BE} \right) \times 100 = .4\%$ less than H

But! $t_H = \frac{1}{2} T_{H_t} = 0.763$ days $\ll T_{Bi\text{-}elliptic}$