Recap

Case 1

Given $\begin{cases} u_e \text{ known, const. across stages} \\ \varepsilon \quad \text{"} \quad \text{"} \quad \text{"} \\ \text{Assigned } M_{01},\ M_L,\ n \end{cases}$

Find $\lambda_i = \dfrac{M_{0(i+1)}}{M_{0i} - M_{0(i+1)}}$ which $\begin{cases} \text{maximizes } \dfrac{u_n}{u_e} = \sum_{i=1}^{n} F(\lambda_i) \\ \text{verifies } \ln \dfrac{M_{01}}{M_L} = \sum_{i=1}^{n} G(\lambda_i) \end{cases}$

→ calculus of variations, maximize $L(\lambda_i) = F(\lambda_i) + \alpha G(\lambda_i)$

Set $\dfrac{\partial L}{\partial \lambda_i} = 0$

⋮

$$\frac{\cancel{\varepsilon+\lambda_i}\ 1}{1+\lambda_i}\,\frac{(\varepsilon+\cancel{\lambda_i})-(1+\cancel{\lambda_i})}{(\varepsilon+\lambda_i)^{\cancel{2}}} + \alpha\,\frac{\cancel{1+\lambda_i}\ 1}{\lambda_i}\,\frac{1+\cancel{\lambda_i}-\cancel{\lambda_i}}{(1+\lambda_i)^{\cancel{2}}} = 0$$

$$\frac{\varepsilon-1}{(1+\lambda_i)(\varepsilon+\lambda_i)} + \underbrace{\frac{\alpha}{\lambda_i}\,\frac{1}{1+\lambda_i}}_{\text{split into simple fractions}} = 0$$

$+\lambda_i \ -\lambda_i$

$$\frac{(\varepsilon+\lambda_i)-(1+\lambda_i)}{(1+\lambda_i)(\varepsilon+\lambda_i)} + \underbrace{\frac{A}{\lambda_i} + \frac{B}{1+\lambda_i}}_{A(1+\lambda_i)+B\lambda_i=0} = 0$$

$A = \alpha$

$B = -\alpha$

↓

$$\frac{1}{1+\lambda_i} - \frac{1}{\varepsilon+\lambda_i} + \frac{\alpha}{\lambda_i} - \frac{\alpha}{1+\lambda_i}$$

... → $$\frac{1-\alpha}{1+\lambda_i} + \frac{\alpha}{\lambda_i} = \frac{1}{\varepsilon+\lambda_i}$$

$$\frac{\lambda_i - \cancel{\alpha\lambda_i} + \alpha + \cancel{\alpha\lambda_i}}{\lambda_i(1+\lambda_i)} = \frac{1}{\varepsilon+\lambda_i}$$ Flip

$$\frac{(1+\lambda_i)\lambda_i}{\lambda_i+\alpha} = \varepsilon+\lambda_i$$

$$\lambda_i + \cancel{\lambda_i^2} = \varepsilon\lambda_i + \cancel{\lambda_i^2} + \varepsilon\alpha + \alpha\lambda_i$$

$$\lambda_i = \frac{\alpha\varepsilon}{1-\alpha-\varepsilon} = \lambda = \text{constant} \quad (1)$$

→ similar stages $(\lambda_i = \lambda,\ \varepsilon_i = \varepsilon)$

⇒ optimal stages are similar

α still unknown

But can avoid calculating α by imposing

$$\underbrace{\frac{M_e}{M_{01}}}_{\text{Both known}} = \prod_{i=1}^{n} \frac{\lambda_i}{1+\lambda_i} = \prod_{i=1}^{n}\left(\frac{\lambda}{1+\lambda}\right) = \left(\frac{\lambda}{1+\lambda}\right)^n$$

on 9/30

$$\rightarrow \lambda = \frac{\left(\frac{M_e}{M_{01}}\right)^{1/n}}{1-\left(\frac{M_e}{M_{01}}\right)^{1/n}} = \frac{1}{\left(\frac{M_{01}}{M_e}\right)^{1/n} - 1}$$

} now in terms of quantities we know

If we use this value of λ in $\frac{u_n}{u_e} = \sum_{i=1}^{n} \ln\left(\frac{1+\lambda_i}{\varepsilon+\lambda_i}\right)$

we get back (✱✱) from 9/30

Case 2

G
- u_e known & const. across stages
- ε_i known & variable " "
- M_{01}, M_e, n known

F { λ_i's that
- maximize $\frac{u_n}{u_e}$
- verify $\frac{M_e}{M_{01}}$

Using the same algebra as case 1,

$$\lambda_i = \frac{\alpha\varepsilon_i}{1-\alpha-\varepsilon_i} \quad (2)$$

this time λ_i does vary with the stages

→ must find α that verifies ratio use $\frac{\mu_e}{\mu_{01}} = \prod_{i=1}^{n} \frac{\lambda_i}{1+\lambda_i} = \prod_{i=1}^{n} \frac{\frac{\alpha \varepsilon_i}{1-\alpha-\varepsilon_i}}{1+\frac{\alpha \varepsilon_i}{1-\alpha-\varepsilon_i}}$

$$= \prod_{i=1}^{n} \frac{\alpha \varepsilon_i}{\alpha \varepsilon_i + 1 - \alpha - \varepsilon_i} \quad (3)$$

From (3), find α ; From (2), find all λ_i's

Case 3

G $\begin{cases} u_{ci} \text{ known \& variable w/ } i \\ \varepsilon_i \quad " \quad " \quad " \\ u_n, \ n \ \text{known} \end{cases}$

F { λ_i which $\begin{cases} \text{minimize } \frac{\mu_{01}}{\mu_e} \\ \text{verify } u_n = \sum_{i=1}^{n} u_{ei} \ln(R_i) \end{cases}$

Recall $R_i = \frac{1+\lambda_i}{\varepsilon_i + \lambda_i}$

$$\to \lambda_i = \frac{1 - \varepsilon_i R_i}{R_i - 1} \quad (4)$$

Then $\frac{\mu_{01}}{\mu_e} = \prod_{i=1}^{n} \frac{1+\lambda_i}{\lambda_i} = \prod_{i=1}^{n} \left(1 + \frac{1}{\lambda_i}\right)$

$$= \prod_{i=1}^{n} \left(1 + \frac{R_i - 1}{1 - \varepsilon_i R_i}\right)$$

take ln of both sides

$$\ln\left(\frac{\mu_{01}}{\mu_e}\right) = \ln \prod_{i=1}^{n} \left(1 + \frac{R_i - 1}{1 - \varepsilon_i R_i}\right) = \sum_{i=1}^{n} \underbrace{\ln\left(1 + \frac{R_i - 1}{1 - \varepsilon_i R_i}\right)}_{F(R_i)}$$

$$= \sum_{i=1}^{n} F(R_i)$$

(verifying fn)

$$U_n = \sum_{i=1}^{n} U_{e_i} \ln(R_i) = \sum_{i=1}^{n} G(R_i)$$

Must find R_i's that: $\begin{cases} \text{minimize } \sum_{i=1}^{n} F(R_i) \\ \text{verify } U_n = \sum_{i=1}^{n} G(R_i) \end{cases}$

Again, set $\mathcal{L}(R_i) = F(R_i) + \alpha G(R_i)$

minimize $L(R_i)$ by setting $\frac{\partial L}{\partial R_i} = 0$

$$\frac{\partial L}{\partial R_i} = \frac{\partial}{\partial R_i}\left[\ln\left(1 + \frac{R_i - 1}{1 - \varepsilon_i R_i}\right)\right] + \alpha \frac{\partial}{\partial R_i}\left(U_{e_i} \ln R_i\right)$$

$$= \frac{1}{1 + \frac{R_i - 1}{1 - \varepsilon_i R_i}} \cdot \frac{1(1 - \varepsilon_i R_i) - (R_i - 1)(-\varepsilon_i)}{(1 - \varepsilon_i R_i)^2} + \frac{\alpha U_{e_i}}{R_i}$$

$$= \frac{1 - \varepsilon_i R_i + \varepsilon_i R_i - \varepsilon_i}{(1 - \varepsilon_i R_i + R_i - 1)(1 - \varepsilon_i R_i)} + \frac{\alpha U_{e_i}}{R_i}$$

$\vdots$

$$\frac{1}{R_i}\left[\frac{1}{1 - \varepsilon_i R_i} + \alpha U_{e_i}\right] = 0 \rightarrow R_i = \frac{1 + \alpha U_{e_i}}{\alpha U_{e_i} \varepsilon_i} \quad (5)$$

To find α, impose

$$U_n = \sum_{i=1}^{n} U_{e_i} \ln R_i = \sum_{i=1}^{n} U_{e_i} \ln\left(\frac{1 + \alpha U_{e_i}}{\alpha U_{e_i} \varepsilon_i}\right) \quad (6)$$

$\rightarrow$ Find α from (6), R_i from (5), λ_i from (4)

(4) $\left[\lambda_i = \frac{1 - \varepsilon_i R_i}{R_i - 1} \right.$

Careful! for case 3, must use (4) to determine λ_i, not expressions from cases 1 & 2