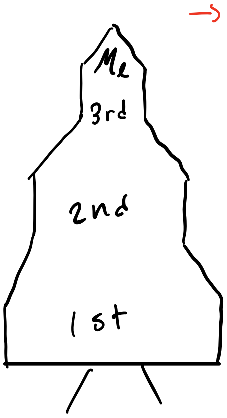
Recap

- wasteful to expend fuel to accelerate semi-empty tanks
- for fixed J, as $M(t)\downarrow$, $a(t)\uparrow$, may exceed limits

→ Discard stages & optimally match engine of each stage to that stage's limits

→ not always the best option, adds complexity



- stages numbered in order of firing
- for each stage, payload ≡ mass of all subsequent stages including M_L

we will:

i) assess advantage / limits of # of stages

ii) Develop optimization approaches for 3 distinct cases

Example: compare final velocities attainable with single & two-stage rockets

Single-stage: Assume $D = g = 0$

$$\begin{cases} M_0 = 15000 \text{ kg} \\ M_L = 1000 \text{ kg} \\ M_s = 2000 \text{ kg} \\ u_e = 3048 \text{ m/s} \end{cases}$$

$$\lambda = \frac{M_L}{M_0 - M_L} = \frac{1}{14} = 0.0714 \qquad R = \frac{1+\lambda}{\varepsilon + \lambda} = 5$$

$$\varepsilon = \frac{M_s}{M_0 - M_L} = \frac{2}{14} = 0.143 \qquad u = u_e \ln R = 4906 \text{ m/s}$$

Two-stage: Assume for comparison

$$\begin{cases} M_{0_1} = 15000 \text{ kg} \\ M_L = 1000 \text{ kg} \\ M_{s_1} + M_{s_2} = 2000 \text{ kg} \\ u_e = 3048 \text{ m/s} \end{cases}$$

$$\left.\begin{array}{l} \lambda_1 = \lambda_2 \\ \varepsilon_1 = \varepsilon_2 \end{array}\right\}$$ Stages w/ equal λ's & ε's called "similar"

First find mass partition

$$\lambda_1 = \lambda_2 \rightarrow \frac{M_{02}}{M_{01} - M_{L_1}} = \frac{M_{L_2}}{M_{02} - M_{L_2}}$$

$$M_{02}^2 - \cancel{M_{02} M_L} = M_{01} M_L - \cancel{M_{02} M_L}$$

$$M_{02} = \sqrt{M_{01} M_L} = \sqrt{15 E6} = 3873$$

$$\lambda_1 = \lambda_2 = \frac{1000}{3873} = 0.348$$

$$\begin{cases} \varepsilon_1 = \varepsilon_2 \\ M_{s_1} + M_{s_2} = 2000 \text{ kg} \end{cases} \qquad \frac{2000 - M_{s_2}}{15000 - 3873}^{\,\varepsilon_1} = \frac{M_{s2}}{3873 - 1000}^{\,\varepsilon_2}$$

$$\rightarrow M_{s_2} = 411 \text{ kg}, \quad M_{s_1} = 1589 \text{ kg}$$

then $R_1 = R_2 = \dfrac{1+\lambda}{\varepsilon+\lambda} = \dfrac{1+0.348}{0.143+0.348} = 2.745$

Each stage provides Δu of

$\Delta u = u_e \ln R$ For this case, u_e & R are the same for both stages

$$\rightarrow \Delta u = 2 u_e \ln R = 2(3048)\ln(2.745) = 6160 \text{ m/s}$$

For <u>this</u> case, $u_{2stage} = 1.26\, u_{1stage}$

" " $u_{e\,2stage} = u_{e\,1stage}$

$R_{2stage} < R_{1stage}$

$u_{2stage} > u_{1stage}$

General n-stage case, $g = D = 0$

- Study effect of n on $\dfrac{M_{01}}{M_L}$, $\dfrac{u_n}{u_e}$

Each stage delivers the velocity increment

$$\Delta u_i = u_{ei} \ln R_i$$

Final velocity

$$u_n = \sum_{i=1}^{n} \Delta u_i = \sum_{i=1}^{n} u_{ei} \ln R_i$$

For now, assume $u_{ei} = u_e = \text{const}$

$$\rightarrow u_n = u_e \sum_{i=1}^{n} \ln R_i = u_e \ln \prod_{i=1}^{n} R_i$$

$$= u_e \ln \prod_{i=1}^{n} \frac{1+\lambda_i}{\varepsilon_i + \lambda_i}$$

Assume similar stages: $\lambda_i = \lambda = \text{const}$, $\varepsilon_i = \varepsilon = \text{const}$

$$\rightarrow R_i = R = \frac{1+\lambda}{\varepsilon+\lambda}$$

$$\Rightarrow \quad u_n = u_e \ln \prod_{i=1}^{n} R_i = u_e \ln R^n = u_e\, n \ln R$$

$$u_n = n\, u_e \ln \frac{1+\lambda}{\varepsilon+\lambda} \quad \star$$

Express $\frac{M_\ell}{M_{01}}$ to the stage payload ratios

$$\frac{M_{01}}{M_\ell} = \frac{M_{01}}{M_{02}} \cdot \frac{M_{02}}{M_{03}} \cdots \frac{M_{0i}}{M_{0(i+1)}} \cdots \frac{M_{0(n-1)}}{M_{0n}} \cdot \frac{M_{0n}}{M_{0\ell}}$$

Remember $\lambda_i = \frac{M_{0(i+1)}}{M_{0i} - M_{0(i+1)}}$

$$\rightarrow \frac{M_{0i}}{M_{0(i+1)}} = \frac{1+\lambda_i}{\lambda_i} \qquad \rightarrow \frac{M_{01}}{M_\ell} = \prod_{i=1}^{n} \frac{1+\lambda_i}{\lambda_i}$$

For similar stages $\lambda_i = \lambda = \text{const}$

$$\frac{M_{01}}{M_\ell} = \left(\frac{1+\lambda}{\lambda}\right)^n \quad \text{OR} \quad \left(\frac{M_{01}}{M_\ell}\right)^{1/n} = \frac{1-\lambda}{\lambda} \equiv \xi$$

$\rightarrow \lambda = \frac{1}{\xi - 1}$ then use $\star$, rewrite:

$$\frac{u_n}{u_e} = n \ln \frac{1+\lambda}{\varepsilon+\lambda} = n \ln \frac{1 + \frac{1}{\xi-1}}{\varepsilon + \frac{1}{\xi-1}} = n \ln \frac{\xi}{\varepsilon(\xi-1)+1} \quad (\star+)$$

then:

$$\frac{1}{n}\frac{u_n}{u_e} = \ln \frac{\xi}{\varepsilon(\xi-1)+1} \longrightarrow \exp\left(\frac{1}{n}\frac{u_n}{u_e}\right) = \frac{\xi}{\varepsilon(\xi-1)+1}$$

$$\varepsilon(\xi-1)+1 = \xi \exp\left(-\frac{1}{n}\frac{u_n}{u_e}\right)$$

$$1-\varepsilon = \xi \exp\left(-\frac{1}{n}\frac{u_n}{u_e}\right) - \varepsilon$$

$$\rightarrow \xi = \frac{1-\varepsilon}{\exp\left(-\frac{1}{n}\frac{u_n}{u_e}\right) - \varepsilon} = \left(\frac{M_{01}}{M_\ell}\right)^{1/n}$$

$$\rightarrow \quad \frac{\mu_{01}}{\mu_l} = \left[\frac{1-\varepsilon}{\exp\left(-\frac{1}{n}\frac{u_n}{u_c}\right) - \varepsilon} \right]^n$$