lec20 cont'd: complex conj.

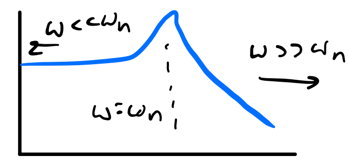
Freq. response w/ complex conjugate pair

$$\boxed{\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}} \quad \text{and} \quad \frac{s^2+2\zeta\omega_n s+\omega_n^2}{\omega_n^2}$$

$$G(j\omega) = \frac{\omega_n^2}{(\omega_n^2-\omega^2)+j(2\zeta\omega_n\omega)} = \frac{1}{\left(1-\frac{\omega^2}{\omega}\right)+j\left(2\zeta\frac{\omega}{\omega_n}\right)}$$

magnitude:

$$M = |G(j\omega)| = \frac{1}{\sqrt{\left(1-\frac{\omega^2}{\omega_n^2}\right)+4\zeta^2\left(\frac{\omega}{\omega_n}\right)^2}} \rightarrow$$

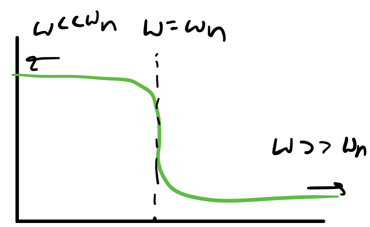


Low freq: $\omega \ll \omega_n$: $M \approx 1 \rightarrow \log M = 0$

High freq: $\omega \gg \omega_n$: $M \approx \frac{1}{\left(\frac{\omega}{\omega_n}\right)^2} \rightarrow \log M \approx -2\log\left(\frac{\omega}{\omega_n}\right)$

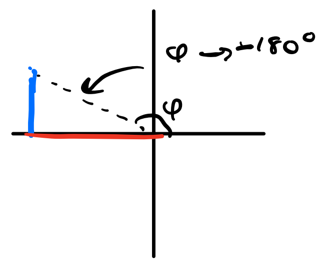
phase: $\angle G(j\omega) = -\tan^{-1}\left(\frac{2\zeta\frac{\omega}{\omega_n}}{1-\frac{\omega^2}{\omega_n^2}}\right) \rightarrow$

denominator



low freq: $\varphi \approx 0°$

High freq: $\varphi \approx -\tan\left(\frac{\overbrace{2\zeta\omega_n}^{Im}}{\underbrace{-\omega}_{Re}}\right) \approx -180°$



$M = |G(j\omega)|$

Find max/min:

$$\frac{dM}{d\omega} = 0 \quad \begin{cases} \omega = \infty \\ \omega = \omega_n\sqrt{1-2\zeta^2} \end{cases} \leftarrow \text{peak response (resonance)}$$

Resonance frequency:

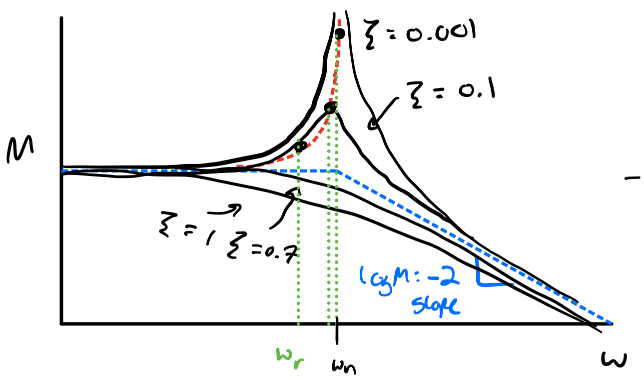
$$\omega_r = \omega_n\sqrt{1-2\zeta^2}$$

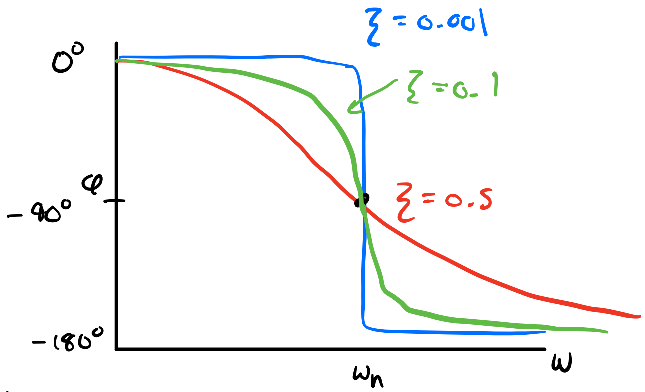
Resonance occurs only if

$$1-2\zeta^2 > 0 \rightarrow \zeta < \frac{\sqrt{2}}{2} \approx 0.707$$

Peak Mag: $M_p = \dfrac{1}{2\zeta\sqrt{1-\zeta^2}}$

note: if $\zeta = 0$, $M_p \rightarrow \infty$





$\omega_r < \omega_n$, very close

Response of numerator: $\dfrac{s^2 + 2\zeta\omega_n s + \omega_n^2}{\omega_n^2}$

Magnitude: $\dfrac{1}{\text{denom. response}}$: low: $\log M = 0$

high: $2\log(\frac{\omega}{\omega_n}$

phase: flip sign

$$m\ddot{x} + c_L\dot{x} + k_L x = f \qquad m=10 \quad c_L=10 \quad k_L=40$$

$$\frac{X(s)}{F(s)} = \frac{1}{ms^2 + c_L s + k_L}$$

$$\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} : \frac{\frac{1}{m}}{s^2 + \frac{c_L}{m}s + \frac{k_L}{m}} = \frac{1}{k_L}\left[\frac{\frac{k_L}{m}}{s^2 + \frac{c_L}{m}s + \frac{k_L}{m}}\right]$$

$$\frac{1}{k_L} = \frac{1}{40} \qquad \frac{X(s)}{F(s)} = \frac{1}{40}\left(\frac{4}{s^2+s+4}\right)$$

$\frac{1}{40}$ term: $\quad M = \frac{1}{40}, \quad \varphi = 0$

$$\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{4}{s^2+s+4} \quad \to \omega_n = \underbrace{2\,\tfrac{r}{s}}$$

$$2\zeta\omega_n = 1 \to \zeta = 0.25 \checkmark$$

$$\omega_r = \omega_n\sqrt{1-2\zeta} \cong 1.87\,\tfrac{r}{s}$$

$$M_p = \frac{1}{2\zeta\sqrt{1-\zeta^2}} \cong 2.1 \; (6.3\,\text{dB})$$

$$20\log_{10}(2.1)$$

$\omega \ll \omega_n$: $\quad M \cong 1, \log M \approx 0$

$\varphi \approx 0^\circ$

$\omega \gg \omega_n$: $\quad M$: -2 slope (-40 dB/decade)

$\varphi \to -180^\circ$

$\omega = \omega_n$: $\quad \varphi = -90^\circ$