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Problem 1 Neglect pressure & gravity

1a) Show that $\frac{M_p}{M_L} = \frac{(1-\varepsilon)\left[1-e^{-\Delta u/u_e}\right]}{-\varepsilon + e^{-\Delta u/u_e}}$

$M_0 = M_L + M_p + M_s$, $\varepsilon = \frac{M_s}{M_p+M_s}$, $\lambda = \frac{M_L}{M_p+M_s}$, $R = \frac{1+\lambda}{\varepsilon+\lambda}$

$M_p + M_s = \frac{M_L}{\lambda}$ $\qquad M_s = \varepsilon(M_p+M_s) = \varepsilon\frac{M_L}{\lambda}$

$$M_p = \frac{M_L}{\lambda} - M_s = \frac{M_L}{\lambda} - \varepsilon\frac{M_L}{\lambda} \rightarrow \frac{M_p}{M_L} = \frac{1-\varepsilon}{\lambda}$$

→ Need λ in terms of $e^{\Delta u/u_e}$

$\Delta u = u_e \ln R$

$$\frac{\Delta u}{u_e} = \ln\left(\frac{1+\lambda}{\varepsilon+\lambda}\right) \rightarrow e^{\Delta u/u_e}(\varepsilon+\lambda) = 1+\lambda$$

$$\rightarrow e^{\Delta u/u_e}\varepsilon - 1 = \lambda\left(1-e^{\Delta u/u_e}\right) \rightarrow \lambda = \frac{e^{\Delta u/u_e}\varepsilon - 1}{1-e^{\Delta u/u_e}}$$

→ combine:

$$\frac{M_p}{M_L} = \frac{1-\varepsilon}{\lambda} = \frac{(1-\varepsilon)\left[1-e^{\Delta u/u_e}\right]}{\varepsilon e^{\Delta u/u_e}-1} \cdot \frac{-e^{-\Delta u/u_e}}{-e^{-\Delta u/u_e}}$$

$$\rightarrow \boxed{\frac{M_p}{M_L} = \frac{(1-\varepsilon)\left[1-e^{-\Delta u/u_e}\right]}{-\varepsilon + e^{-\Delta u/u_e}}}$$

→ see matlab plot #1

1b) what can ou conclude about the sign of $\varepsilon - e^{-\Delta u/u_e}$?

If the sign of $\varepsilon - e^{-\Delta u/u_e}$ is positive, then $\frac{M_p}{M_L}$ becomes negative, which is impossible. so the sign of $\varepsilon - e^{-\Delta u/u_e}$ must be negative for $\frac{M_p}{M_L}$ to be physical.

1c) Neglecting drag, find the max limit ε for a single stage rocket with $I_{sp} = 432.5$, $\Delta u = 8765$ m/s

$\rightarrow$ Since $\varepsilon - e^{-\Delta u/u_e} < 0$ to be physical:

$$-e^{-\Delta u/u_e} < -\varepsilon \rightarrow \varepsilon < e^{-\Delta u/u_e}$$

$$u_e = g_e I_{sp} = 4238 \text{ m/s}$$

$$\rightarrow \varepsilon < e^{-8765/4238} \rightarrow \boxed{\varepsilon < 0.126}$$

Problem 2

3-stage rocket: $M_{0_1} = 18000$ kg, $M_L = 3000$ kg
$\varepsilon_1 = 0.062$, $\varepsilon_2 = 0.130$, $\varepsilon_3 = 0.161$, $u_e = 2200$ m/s

Find optimal $M_{p_1}, M_{p_2}, M_{p_3}$, & u_3. Report value of α

$\rightarrow$ mass optimization case 2

From lecture: $\lambda_i = \frac{\alpha \varepsilon_i}{1 - \alpha - \varepsilon_i}$ $\rightarrow$ Find α that verifies ratio

$$\rightarrow \frac{M_L}{M_{0_1}} = \prod_{i=1}^{n} \frac{\alpha \varepsilon_i}{\alpha \varepsilon_i + 1 - \alpha - \varepsilon_i}$$

$$\rightarrow \frac{3000}{18000} = \frac{\alpha(0.062)}{\alpha(0.062) + 1 - \alpha - 0.062} \cdot \frac{\alpha(0.130)}{\alpha(0.130) + 1 - \alpha - 0.130} \cdot \frac{\alpha(0.161)}{\alpha(0.161) + 1 - \alpha - 0.161}$$

$\rightarrow$ matlab vpa solve: $\boxed{\alpha = 0.8164}$

$$\rightarrow \lambda_1 = \frac{(0.8164)(0.062)}{1 - 0.8164 - 0.062} = \boxed{0.416 = \lambda_1}$$

$$\rightarrow \lambda_2 = \frac{(0.8164)(0.130)}{1 - 0.8164 - 0.130} = \boxed{1.980 = \lambda_2}$$

$$\rightarrow \lambda_3 = \frac{(0.8164)(0.161)}{1 - 0.8164 - 0.161} = \boxed{5.816 = \lambda_3}$$

$$R_1 = \frac{1+\lambda_1}{\varepsilon_1 + \lambda_1} = \frac{1+0.416}{0.062+0.416} = 2.9623 = \frac{M_{o_1}}{M_{b_1}}$$

$$M_{p_1} = M_{o_1} - M_{b_1} \rightarrow M_{p_1} = M_{o_1} - \frac{M_{o_1}}{2.9623} = \boxed{11924 = M_{p_1}} \text{ kg}$$

$$R_2 = \frac{1+\lambda_2}{\varepsilon_2 + \lambda_2} = 1.412 = \frac{M_{o_2}}{M_{b_2}}$$

$\rightarrow$ Find M_{o_2}: $\lambda_1 = \dfrac{M_{o_2}}{M_{o_1} - M_{o_2}}$

$$\rightarrow \lambda_1 M_{o_1} - \lambda_1 M_{o_2} = M_{o_2} \rightarrow M_{o_2} = \frac{\lambda_1 M_{o_1}}{1+\lambda_1} = 5288$$

$$M_{p_2} = M_{o_2} - M_{b_2} \rightarrow M_{p_2} = M_{o_2} - \frac{M_{o_2}}{1.412} = \boxed{1543 = M_{p_2}} \text{ kg}$$

$$R_3 = \frac{1+\lambda_3}{\varepsilon_3 + \lambda_3} = 1.140 = \frac{M_{o_3}}{M_{b_3}}$$

$\rightarrow M_{o_3}$: $\lambda_2 = \dfrac{M_{o_3}}{M_{o_2} - M_{o_3}}$

$$\rightarrow M_{o_3} = \frac{\lambda_2 M_{o_2}}{1+\lambda_2} = 3513.6$$

$$M_{p_3} = M_{o_3} - M_{b_3} \rightarrow M_{p_3} = M_{o_3} - \frac{M_{o_3}}{1.140} = \boxed{431.5 = M_{p_3}} \text{ kg}$$

$$U_n = U_e \ln \prod_{i=1}^{n} R_i$$

$$\rightarrow U_3 = 2200 \ln [2.9623 \cdot 1.412 \cdot 1.140] = \boxed{3436.4 = U_3} \text{ m/s}$$

Problem 3 3-stage rocket:

$u_3 = 11200$ m/s

$m_\ell = 3000$ kg

$u_{e_1} = 2900$ m/s, $\varepsilon_1 = 0.050$

$u_{e_2} = 4200$ m/s, $\varepsilon_2 = 0.071$

$u_{e_3} = 4200$ m/s, $\varepsilon_3 = 0.191$

Find m_{01} & $m_{p_1}, m_{p_2}, m_{p_3}$ to minimize $\frac{m_{01}}{m_\ell}$

→ lecture case 3

↳ $\lambda_i = \frac{1-\varepsilon_i R_i}{R_i - 1}$, $R_i = \frac{1+\alpha u_{ei}}{\alpha u_{ei}\varepsilon_i}$, $u_n = \sum_{i=1}^{n} u_{ei} \ln\left(\frac{1+\alpha u_{ei}}{\alpha u_{ei}\varepsilon_i}\right)$

Find α:

$$u_3 = 11200 = u_{e_1}\ln\left(\frac{1+\alpha u_{e_1}}{\alpha u_{e_1}\varepsilon_1}\right) + u_{e_2}\ln\left(\frac{1+\alpha u_{e_2}}{\alpha u_{e_2}\varepsilon_2}\right) + u_{e_3}\ln\left(\frac{1+\alpha u_{e_3}}{\alpha u_{e_3}\varepsilon_3}\right)$$

→ matlab vpasolve: $\alpha = -0.000377$

Find R_i with α known:

$R_1 = \frac{1+\alpha u_{e_1}}{\alpha u_{e_1}\varepsilon_1} = \boxed{1.696 = R_1}$ → $\boxed{R_2 = 5.184}$ → $\boxed{R_3 = 1.927}$

Now find λ_i with R_i:

$\lambda_1 = \frac{1-\varepsilon_1 R_1}{R_1 - 1} = \boxed{1.314 = \lambda_1}$ → $\boxed{\lambda_2 = 0.151}$ → $\boxed{\lambda_3 = 0.681}$

Now find m_{0_3}: $m_{0_4} = m_\ell$

$\lambda_3 = \frac{m_{04}}{m_{03} - m_{04}}$ → $m_{03}\lambda_3 - m_{04}\lambda_3 = m_{04}$ → $m_{03} = \frac{m_{04}(1+\lambda_3)}{\lambda_3}$

→ $m_{03} = 7405$ kg

$R_3 = \frac{m_{03}}{m_{b3}}$, $m_{p_3} = m_{0_3} - m_{b_3} = m_{0_3} - \frac{m_{03}}{R_3} = \boxed{3562 \text{ kg} = m_{p3}}$

$\lambda_2 = \frac{m_{03}}{m_{02} - m_{03}}$ → $m_{02} = \frac{m_{03}(1+\lambda_2)}{\lambda_2} = 56447$ kg

$R_2 = \frac{M_{o2}}{M_{b2}}$, $\quad M_{p2} = M_{o2} - \frac{M_{o2}}{R_2} = \boxed{45558\ \text{Kg} = M_{p2}}$

$\lambda_1 = \frac{M_{o2}}{M_{o1} - M_{o2}} \to M_{o1} = \frac{M_{o2}(1+\lambda_1)}{\lambda_1} = \boxed{99405\ \text{Kg} = M_{o1}}$

$R_1 = \frac{M_{o1}}{M_{b1}}$, $\quad M_{p1} = M_{o1} - \frac{M_{o1}}{R_1} = \boxed{40794\ \text{Kg} = M_{p1}}$

Problem 4

u_{ei}, $\varepsilon = \text{const}$

4a) For assigned $\frac{M_p}{M_{o1}}$, u_{ei}, ε, develop a procedure to find values of λ_i that maximize final velocity

$$G \left\{ \begin{array}{l} \frac{M_p}{M_{o1}} \\ M_{o1} \\ u_{ei} \\ \varepsilon = \text{const} \end{array} \right. \qquad \text{Find } \lambda_i \text{ which} \left\{ \begin{array}{l} \text{maximizes } u_n = \sum_{i=1}^{n} \underbrace{u_{ei} \ln \frac{1+\lambda_i}{\varepsilon+\lambda_i}}_{F(\lambda_i)} \\ \text{verifies } \frac{M_p}{M_{o1}} = \prod_{i=1}^{n} \underbrace{\frac{\lambda_i}{1+\lambda_i}}_{G(\lambda_i)} \end{array} \right.$$

$\to$ Maximize $\mathcal{L}(\lambda_i) = F(\lambda_i) + \alpha\, G(\lambda_i)$

$\to$ Set $\frac{\partial \mathcal{L}}{\partial \lambda_i} = 0$

$$\frac{\partial \mathcal{L}}{\partial \lambda_i} = u_{ei} \cdot \frac{\cancel{\varepsilon+\lambda_i}}{1+\lambda_i} \cdot \frac{(\varepsilon+\cancel{\lambda_i}) - (1+\cancel{\lambda_i})}{(\varepsilon+\lambda_i)^{\cancel{2}}} + \alpha \frac{\cancel{1+\lambda_i}}{\lambda_i} \frac{1+\cancel{\lambda_i}-\cancel{\lambda_i}}{(1+\lambda_i)^{\cancel{2}}} = 0$$

$$\to \frac{u_{ei}}{1+\lambda_i} \frac{\varepsilon-1}{\varepsilon+\lambda_i} + \underbrace{\frac{\alpha}{\lambda_i} \cdot \frac{1}{1+\lambda_i}}_{\text{simple fractions}} = 0$$

$+\lambda_i \downarrow -\lambda_i$

$A(1+\lambda_i) + B\lambda_i = 0$

$A = \alpha, \quad B = -\alpha$

$$u_{ei} \frac{(\varepsilon+\lambda_i) - (1+\lambda_i)}{(1+\lambda_i)(\varepsilon+\lambda_i)} + \frac{\alpha}{\lambda_i} - \frac{\alpha}{1+\lambda_i} = 0$$

$$\to \frac{u_{ei}}{1+\lambda_i} - \frac{u_{ei}}{\varepsilon+\lambda_i} + \frac{\alpha}{\lambda_i} - \frac{\alpha}{1+\lambda_i} = 0$$

$$\rightarrow \frac{u_{ei} - \alpha}{1 + \lambda_i} + \frac{\alpha}{\lambda_i} = \frac{u_{ei}}{\varepsilon + \lambda_i}$$

$$\rightarrow \frac{\lambda_i u_{ei} - \cancel{\lambda_i \alpha} + \alpha + \cancel{\alpha \lambda_i}}{\lambda_i (1 + \lambda_i)} = \frac{u_{ei}}{\varepsilon + \lambda_i}$$

$$\rightarrow \frac{\lambda_i (1 + \lambda_i)}{\lambda_i u_{ei} + \alpha} = \frac{\varepsilon + \lambda_i}{u_{ei}} \rightarrow \frac{u_{ei} \lambda_i (1 + \lambda_i)}{\lambda_i u_{ei} + \alpha} = \varepsilon + \lambda_i$$

$$u_{ei} \lambda_i + \cancel{u_{ei} \lambda_i^2} = \varepsilon \lambda_i u_{ei} + \cancel{u_{ei} \lambda_i^2} + \alpha \varepsilon + \alpha \lambda_i$$

$$\lambda_i (u_{ei} - \varepsilon u_{ei} - \alpha) = \alpha \varepsilon$$

$$\rightarrow \boxed{\lambda_i = \frac{\alpha \varepsilon}{u_{ei}(1 - \varepsilon) - \alpha}}$$

plug in & solve in matlab to get α

4b) use verifying equation

$$\frac{M_e}{M_{01}} = \prod_{i=1}^{n} \frac{\lambda_i}{1 + \lambda_i}$$

$\rightarrow$ matlab vpasolve : $\boxed{\alpha = 2062.7}$

$\rightarrow \boxed{\lambda_1 = 1.39}$ $\boxed{\lambda_2 = \lambda_3 = 0.295}$

$\rightarrow M_{0i}, M_{pi}, M_{si}$ values in table

$\rightarrow \boxed{u_3 = 9430 \text{ m/s}}$

4d) Since p4 has a fixed structural coeff. $\varepsilon = 0.191$, whereas p3 has $\varepsilon_1 < \varepsilon_2 < 0.191$, we expect p3 to have lower M_{si} and thus larger M_{pi}. Therefore, p3 should have better performance, resulting in a larger u_{final}. The values in the table agree.