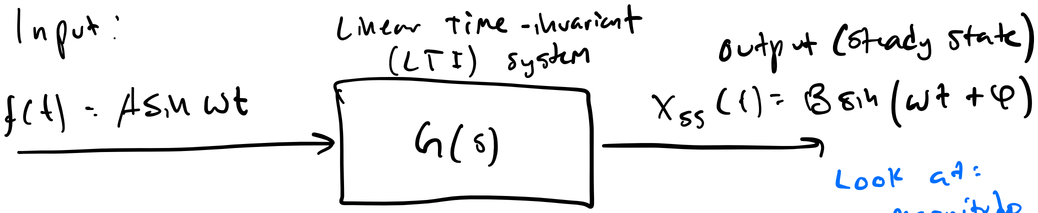
Linear system frequency response

- → shake at certain frequency
- → response will eventually match input w/ different magnitude & phase



Look at:
- magnitude ratio
- phase of response

- → Analysis in frequency domain
- → steady state response
- → very useful in design of control systems

Partial Fractions method

$$G(s) = \frac{2s+4}{(s+1)(s+3)(s+10)} = \frac{C_1}{s+1} + \frac{C_2}{s+3} + \frac{C_3}{s+10}$$

$$G(s)(s+1) = C_1 + \cancel{\frac{(s+1)C_2}{s+3}}^{\,0} + \cancel{\frac{(s+1)C_3}{s+10}}^{\,0}$$

Let $s = -1$

$$G(s)(s+1)\Big|_{s=-1} = C_1$$

← can solve for one term at a time

$$\frac{X(s)}{F(s)} = G(s) = \frac{b(s)}{a(s)} \rightarrow X(s) = \frac{b(s)}{a(s)} F(s) \leftarrow \Delta(s) = (s_o - p_1)\ldots(s_o - p_n)$$

Response due to $f(t) = A\sin(\omega t)$

$$F(s) = \mathcal{L}[f(t)] = \frac{A\omega}{s^2+\omega^2} = \frac{A\omega}{(s+j\omega)(s-j\omega)}$$

$$X(s) = \frac{b(s)}{a(s)} F(s) = \left[\frac{b(s)}{(s_o-p_1)(s_o-p_2)\ldots}\right]\left[\frac{A\omega}{(s+j\omega)(s-j\omega)}\right]$$

Partial fractions:

$$G(s)\underbrace{\frac{A\omega}{s^2+\omega^2}}_{F(s)} = X(s) = \underbrace{\frac{c_1}{(s_o-p_1)} + \ldots + \frac{c_n}{(s_o-p_n)}}_{\text{For stable sys, response will decay to zero}} + \underbrace{\frac{c_A}{(s+j\omega)} + \frac{c_B}{(s-j\omega)}}_{\text{sinusoidal response}}$$

$$X_{ss}(s) = \frac{c_A}{(s+j\omega)} + \frac{c_B}{(s-j\omega)}$$

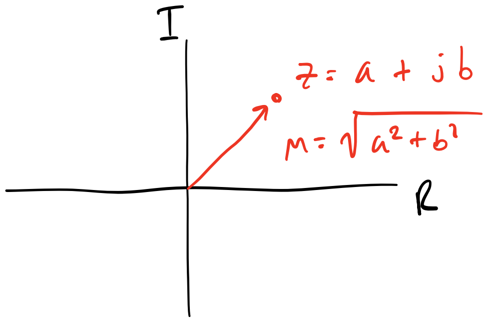
$$\hookrightarrow x_{ss}(t) = c_A e^{-j\omega t} + c_B e^{j\omega t}$$

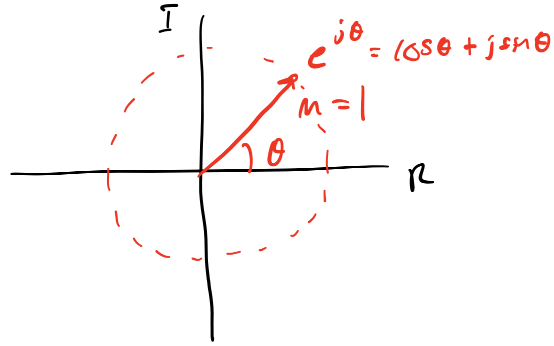
Solve c_A & c_B:

$$G(s)\frac{A\omega}{s^2+\omega^2}(s \pm j\omega) = c_A / c_B$$

$$\Rightarrow \begin{aligned} c_A &= -\frac{A}{2j} G(-j\omega) = \frac{-A}{2j} G(j\omega) e^{-j\varphi} \\ c_B &= \frac{A}{2j} G(j\omega) = \frac{A}{2j} G(j\omega) e^{j\varphi} \end{aligned}$$

$\varphi = \angle G(j\omega)$



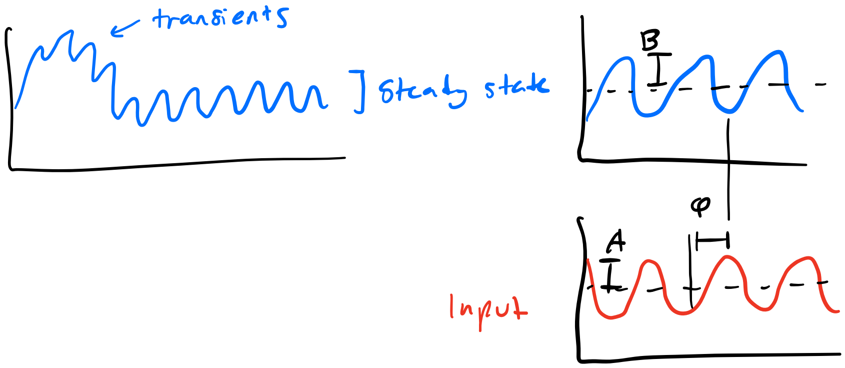


Input: $A\sin(\omega t)$

$$\Rightarrow \quad X_{ss}(t) = \underset{\text{magnitude ratio}}{M}\left[\overset{\text{input magnitude}}{A}\sin(\omega t + \underset{\text{phase angle}}{\varphi})\right] \text{ where } \begin{cases} M = |G(j\omega)| \\ \varphi = \angle G(j\omega) \end{cases}$$

E.g. $\dddot{x} + 5\ddot{x} + 11\dot{x} + 15x = 100\sin(20t)$

Response $x(t)$



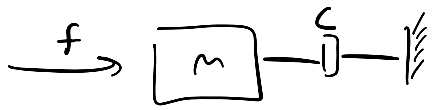
For input frequency ω_0:

$$M = |G(j\omega_0)| = |G(s)|\big|_{s=j\omega_0}$$

$$= \sqrt{Re(G(j\omega_0))^2 + Im(G(j\omega_0))^2}$$

$$\varphi = \text{atan2}\left[\frac{Im(G(j\omega_0))}{Re(G(j\omega_0))}\right]$$

E.g.



$f = \underbrace{10}_{A}\sin(\underbrace{2}_{\omega} t)$

$m\ddot{x} + c\dot{x} = f$

$G(s) = \frac{X(s)}{F(s)} = \frac{1}{ms^2+cs} = \frac{1}{s(ms+c)}$

$G(j\omega) = M_G e^{j\varphi_G}$

$\frac{1}{s} \cdot \frac{1}{ms+c}$

$M_1 e^{j\varphi_1} \cdot M_2 e^{j\varphi_2}$

$= M_1 M_2 e^{j(\varphi_1+\varphi_2)}$

$M_G = M_1 \cdot M_2$

$\varphi_G = \varphi_1 + \varphi_2$

$G(j\omega_0) = \frac{1}{m(j\omega_0)^2 + c(j\omega_0)} = \frac{1}{-m\omega_0^2 + jc\omega_0} \leftarrow M_D e^{j\varphi_D}$

$\to M_D = \sqrt{m^2\omega_0^4 + c^2\omega_0^2}$

$M = \frac{1}{M_D} \cong 0.177$ $\quad \varphi_D = \tan^{-1}\left(\frac{c\omega_0}{-m\omega_0^2}\right)$

$\varphi = -\varphi_D = -135^\circ$ $\quad \varphi_D = 135^\circ$

$x_{ss}(t) = M \cdot A\sin(\omega t + \varphi)$

$\to x_{ss}(t) = 1.77\sin(2t - 2.36)$