Recap

$$T = \dot{m} u_e + (P_e - P_a) A_e$$

Rocket eqn. general:

$$T = Mg\cos\theta + D + M(t)\frac{du(t)}{dt}$$

If $g = D = 0$ $\rightarrow$ $T = M(t)\frac{du(t)}{dt}$ $\rightarrow$ $\Delta u(t) = u_{eq}\ln\frac{M_0}{M(t)}$

For $t = t_b$, $\Delta u(t_b) = u_{eq}\ln\frac{M_0}{M_b} = u_{eq}\ln\frac{M_0}{M_0 - M_p}$ ... etc

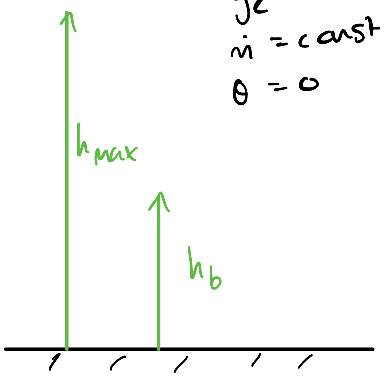
For $\left|\begin{array}{l} D = 0 \\ g \neq 0, \text{const} \\ \theta = 0 \text{ (vertical traj.)} \end{array}\right.$ Following the same algebra as general case,

$$du = -\frac{dM}{M}u_{eq} - g\,dt$$

$$\rightarrow \Delta u(t) = u_{eq}\ln\frac{M_0}{M(t)} - gt$$

Example: $\underbrace{\text{sounding rocket}}_{\text{making measurements}}$

Given $\left\{\begin{array}{l} D = 0 \\ \left.\begin{array}{l} u_e = \text{const} \\ P_e = P_a = \text{const} \end{array}\right\} u_{eq} = u_e \\ g_e = \text{const} \\ \dot{m} = \text{const} \\ \theta = 0 \end{array}\right.$ Find $\left\{\begin{array}{l} \text{Height @ } t_b \\ \text{max height} \end{array}\right.$



$$h_b = \int_0^{t_b} u(t)\,dt$$

$$u(t) = -u_e\ln\frac{M(t)}{M_0} - g_e t \qquad (1)$$

$$\dot{m} = \frac{M_p}{t_b} = \frac{M_0 - M_b}{t_b}$$

$$\rightarrow M(t) = M_0 - \dot{m}t = M_0 - (M_0 - M_b)\frac{t}{t_b}$$

$\swarrow$ $\frac{1}{n} = \underline{M_b}$

$$u(t) = -u_e \ln \left[1 - \underbrace{\left(1 - \frac{1}{R}\right)}_{\frac{R-1}{R}} \frac{t}{t_b}\right]^{''} - \overset{m_0}{g_e} t$$

Recall $\int \ln x \, dx = x \ln x - x$

Call $\frac{R-1}{R} \cdot \frac{1}{t_b} = b$ s.t. $\ln\left[1 - \frac{R-1}{R} \frac{t}{t_b}\right] = \ln(1 - bt)$

$$\rightarrow \quad x = 1 - bt$$

$$dx = -b\,dt \rightarrow dt = -\frac{dx}{b}$$

$$\rightarrow h_b = \int_0^{t_b} \left[-u_e \ln(1 - bt) - g_e t\right] dt$$

$$= \int_1^{1 - b t_b} -u_e \ln x \left(-\frac{dx}{b}\right) - \int_0^{t_b} g_e t \, dt$$

$$= \frac{u_e}{b} \int_1^{1 - b t_b} \ln x \, dx - \int_0^{t_b} g_e t \, dt$$

$$= \underbrace{u_e \frac{R t_b}{R-1}}_{\frac{u_e}{b}} \left[\underbrace{\left(1 - \frac{R-1}{R t_b} t\right)}_{x = 1 - bt} \ln \underbrace{\left(1 - \frac{R-1}{R t_b} t\right)}_{x = 1 - bt} - \underbrace{\left(1 - \frac{R-1}{R t_b} t\right)}_{x = 1 - bt} \right]_0^{t_b} - g_e \frac{t_b^2}{2}$$

$$= u_e \frac{R t_b}{R-1} \left[\left(1 - 1 + \frac{1}{R}\right) \ln\left(1 - 1 + \frac{1}{R}\right) - \left(1 - 1 + \frac{1}{R}\right) - (-1) \right] - g_e \frac{t_b^2}{2}$$

$$\dots$$

$$= \boxed{-\frac{u_e t_b}{R-1} \ln R + u_e t_b - g_e \frac{t_b^2}{2} = h_b} \quad (2)$$

To go from h_b to h_{max}, use energy

kinetic → potential

For $t \geq t_b$, $M(t) = M_b$ - const

@ h_b, $u = u_b$, @ h_{max} $u = 0$

$$\frac{M_b u_b^2}{2} = M_b g_e (h_{max} - h_b)$$

$$\rightarrow h_{max} = \frac{u_b^2}{2g_e} + h_b$$

From (1) $u_b = u(t_b) = u_e \ln R - g_e t_b$ (3)

$$\rightarrow h_{max} = \frac{1}{2g_e}(u_e \ln R - g_e t_b^2) - \frac{u_e t_b}{R-1} \ln R + u_e t_b - g_e \frac{t_b^2}{2}$$

...

$$\boxed{h_{max} = \frac{u_e^2 \ln^2 R}{2g_e} - u_e t_b \left(\frac{R}{R-1} \ln R - 1 \right)} \quad (4)$$

Remarks

- h_{max} decreases as t_b increases

  Best to minimize burn (acceleration phase)
  when this occurs w/in gravitational field

  physically: Reduce energy consumed (wasted) on simply
  lifting propellant

For fixed desired final velocity, reducing t_b requires higher acceleration -> higher thrust

Problems:

i) increased stresses on structure

ii) Higher thrust requires higher $\dot{m}$ -> larger & heavier pumps, valves, pipes, etc.

iii) As u ↑ while rocket is still in atmosphere, $D(\propto u^2)$ is no longer negligible

Note: if g & $D = 0$, t_b has no effect on δu

(Recall in that case, $\Delta u = u_e \ln R$)