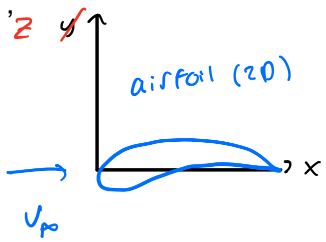
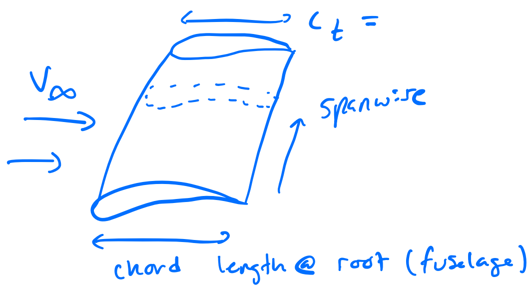
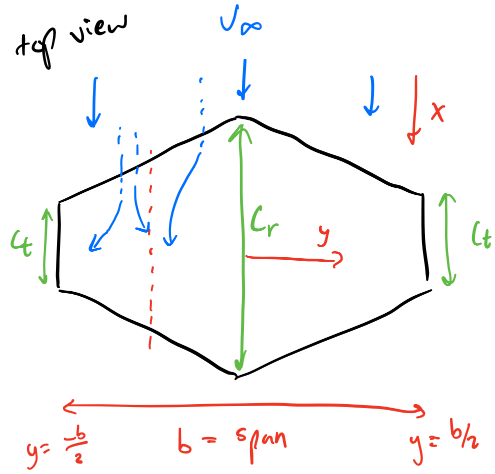
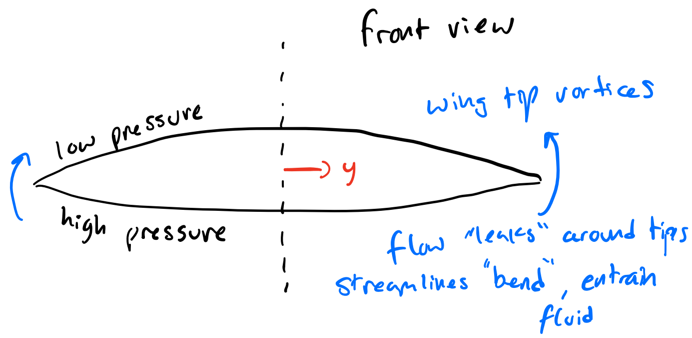






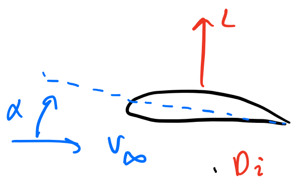
S = planform area

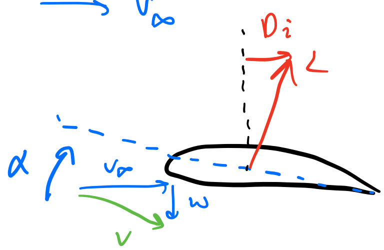
- c(y) will usually vary
- airfoil profile may also vary across span
- variations generate flow in <u>spanwise</u> direction: $\underline{u} = u\hat{i} + v\hat{j} + w\hat{k}$



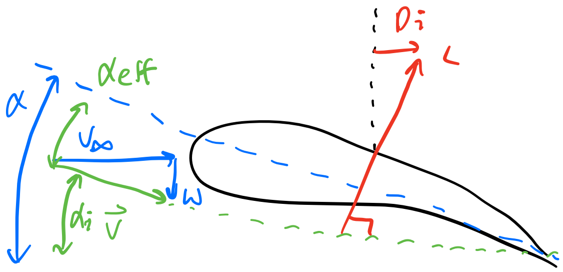
vortices induce a downward velocity called "down wash"

W = down wash





D_i = induced drag

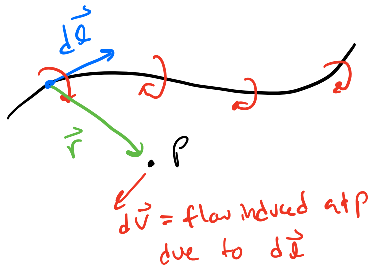


- new velocity vector $\vec{V}$, as seen by airfoil
- new angle of attack, α_{eff}, " " " "
- α_i = induced angle of attack : $\alpha_i + \alpha_{eff} = \alpha$
- Lift vector $\perp$ to $\vec{V}$ creates induced drag

Goal: modify 2D thin airfoil theory to account for w

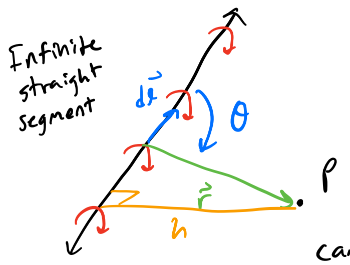
estimate w (value) ; compute new c_l, c_D, c_M

Curved vortex filament



$$\rightarrow \boxed{d\vec{V} = \frac{\Gamma}{4\pi} \frac{d\vec{l} \times \vec{r}}{|\vec{r}|^3}}$$

Biot-Savart Law



side note:

$$d\vec{B} = \frac{\mu I}{4\pi} \frac{d\vec{l} \times \vec{r}}{|\vec{r}|^3}$$

E & M analog

$\vec{B}$ = mag. field

I = current, μ = permeability

geometric relations

$$r = \frac{h}{\sin\theta}$$

$$dl = \frac{-h}{\sin^2\theta} d\theta$$

can show for straight:

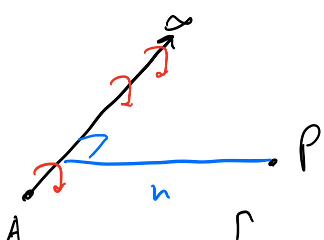
$$\vec{V}(P) = \int_{-\infty}^{\infty} \frac{\Gamma}{4\pi} \frac{d\vec{l} \times \vec{r}}{|\vec{r}|^3} \rightarrow \text{simplify w/ geometry}$$

$$|\vec{V}| = V = \frac{\Gamma}{4\pi} \int_{-\infty}^{\infty} \frac{\sin\theta}{r^2} d\theta$$

$$\boxed{V = \frac{\Gamma}{2\pi h}}$$

Mag. of velocity $\perp$ distance h from infinite straight segment

Semi-infinite filament



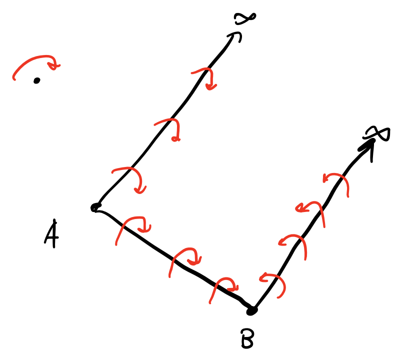
$$V = \frac{\Gamma}{4\pi h}$$

Helmholtz vortex theorems:

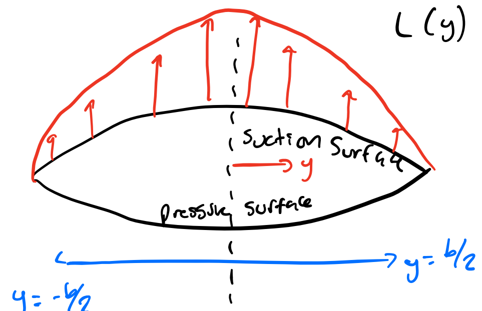
1. Strength of vortex filament is const. along length
2. A vortex filament cannot end in a fluid

   It must:
   - extend to ∞
   - extend to solid boundary (like A)

     OR
   - combine & form a closed path



Lift distribution



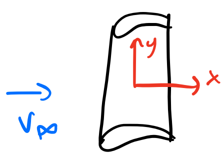
front view

$L(y)$ = lift distribution

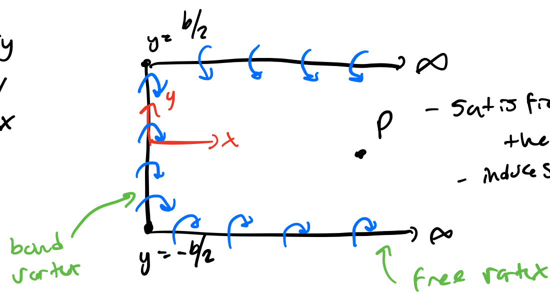
$$L = \rho V_\infty \Gamma$$

$$L(y) = \rho V_\infty \Gamma(y)$$

Prandtl's lifting line theory



replace w/ bound vortex $\Longrightarrow$

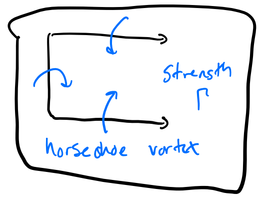


- Satisfies Helmholtz therems
- induces downwash

$w(y)$ = induced velocity at $x = 0$,

or as experienced by wing bound vortex

- bound vortex does not induce velocity on itself, but it is influenced by free vortices

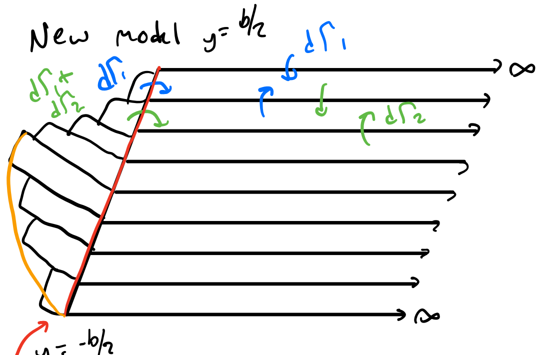


$$w(y) = \underbrace{\frac{-\Gamma}{4\pi(b/2+y)}}_{\text{Left free vortex}} \underbrace{- \frac{\Gamma}{4\pi(b/2-y)}}_{\text{right free vortex}} \rightarrow \underbrace{w(y) = \frac{-\Gamma}{4\pi} \frac{b}{\left(\left(\frac{b}{2}\right)^2 - y^2\right)}}_{\text{Model / theory for downwash}}$$

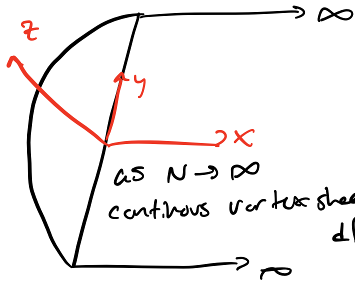
as $y \to b/2$, $w \to \infty \Rightarrow$ not capturing all physics

$\rightarrow$ must refine model

New model $y = b/2$



N horseshoe vortices



as $N \to \infty$
continuous vortex sheet
$d\Gamma = \left(\frac{d\Gamma}{dy}\right) dy$

$\Gamma(y) = ? \rightarrow L(y)$

Lifting line model: lifting line replaces wing
distribution of vortices, not constant as previous model

Recall Biot-Savart: velocity @ y_0 induced by the semi-infinite trail of vorticity at y

Generic: $w = \dfrac{\Gamma}{4\pi h}$

In this case: $dw = \dfrac{-d\Gamma}{4\pi(y_0 - y)}$

$$\boxed{dw = \frac{-\left(\frac{d\Gamma}{dy}\right)dy}{4\pi(y_0 - y)}}$$

$\Rightarrow$ integrate to get total contribution $\rightarrow$

$$\boxed{w(y_0) = \frac{-1}{4\pi}\int_{-b/2}^{b/2} \frac{(d\Gamma/dy)\,dy}{y_0 - y}}$$

velocity induced by entire vortex sheet
$\rightarrow$ is downwash on lifting line (wing)

Recall downwash diagram:

$$\alpha_i(y_0) = \operatorname{atan}\left(-\frac{w(y_0)}{V_\infty}\right)$$

gives induced angle of attack.
Assume small angles: $\boxed{\alpha_i(y_0) \approx \dfrac{-w(y_0)}{V_\infty}}$

$C_\ell = 2\pi(\alpha - \alpha|_{L=0})$
$\downarrow$
old model, 2D airfoils

$$\boxed{\alpha_i(y_0) = \frac{1}{4\pi V_\infty}\int_{-b/2}^{b/2} \frac{(d\Gamma/dy)\,dy}{y_0 - y}}$$

New model: $C_\ell = 2\pi(\alpha_{eff} - \alpha|_{L=0})$

$\alpha_{eff}(y_0) = \alpha - \alpha_i(y_0) \rightarrow C_\ell(y)$

$$L(y_0) = L' = \frac{1}{2}\rho V_\infty^2 \underbrace{c(y_0)}_{\text{chord length}} \underbrace{C_\ell(y_0)}_{\text{lift coeff.}} = \text{lift per unit span}$$

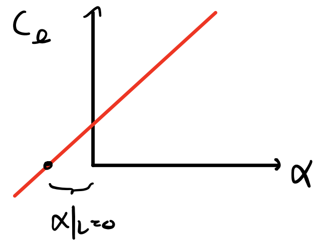
$$L' = \rho V_\infty \Gamma(y_0) \quad \rightarrow \text{combine \& solve for } C_\ell$$

$$\rightarrow \quad C_\ell = \frac{2\Gamma(y_0)}{V_\infty \, c(y_0)}$$

Combine w/ $C_\ell = 2\pi(\alpha_{eff} - \alpha|_{L=0})$ & solve for α_{eff}:

$$\boxed{\alpha(y_0) = \underbrace{\frac{\Gamma(y_0)}{\pi V_\infty c(y_0)} + \alpha|_{L=0}}_{\alpha_{eff}} + \underbrace{\frac{1}{4\pi V_\infty}\int_{-b/2}^{b/2} \frac{(d\Gamma/dy)\,dy}{y_0 - y}}_{\alpha_i}}$$

Prandtl's lifting line theory: unknown is Γ; solve for $\Gamma(y) = \Gamma$

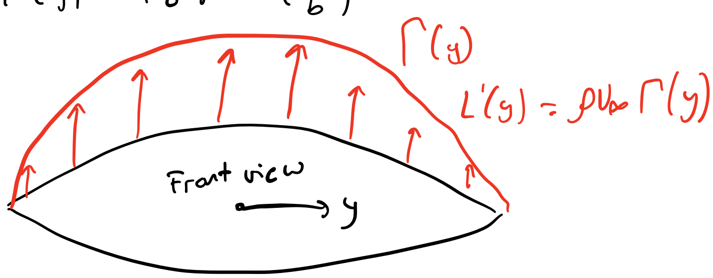


Implementation : solve for $\Gamma(y)$

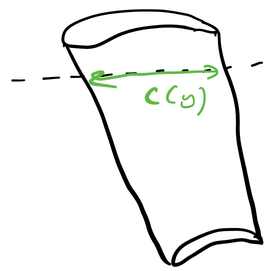
To solve, we will assume a form of $\Gamma(y)$:

A common solution is the <u>Elliptical Lift Distribution</u>

$$\Gamma(y) = \Gamma_0 \sqrt{1 - \left(\frac{2y}{b}\right)^2}$$



$$\Rightarrow L = \int_{-b/2}^{b/2} L'(y)\,dy$$

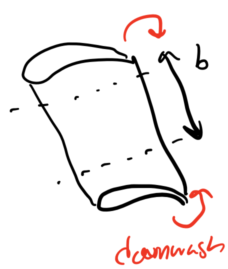


for untwisted wing then $\alpha|_{L=0}$ = const

$$C_L = \frac{L}{\frac{1}{2}\rho V_\infty^2 S} = \frac{L}{q_\infty S}$$

$\hookrightarrow$ For 3D wing

C_ℓ = 2D for an airfoil



* The C_L of a <u>wing</u> will be lower than a 2D model of the same airfoil due to downwash.

$$L'(y) = \rho V_\infty \Gamma(y_0) \; ; \quad L = \int_{-b/2}^{b/2} L'(y)\,dy$$

In addition to <u>decreasing</u> lift, downwash will <u>increase</u> drag.

$$D_i' = L' \sin \alpha_i(y_0) \quad \text{or} \quad D_i' \approx L' \cdot \alpha(y) \text{ for small angles}$$

$$D_i = \int_{-b/2}^{b/2} D_i'(y)\,dy \Rightarrow \text{induced drag on wing, due to downwash}$$

* still have drag from other sources:
  - pressure drag (form drag)
  - shear stresses on surface (skin friction drag)
  - induced drag, due to downwash

<u>Finite wing: General lift distribution</u>

To develop a more general expression, assume the form:

$$\Gamma(\theta) = 2bV_\infty \sum_{n=1}^{N} A_n \sin(n\theta) \qquad \left(\text{change of variables: } y = -b/2 \cos\theta\right)$$

Next, substitute into lifting line: this will determine A_n

$$\alpha(y_0) = \frac{\Gamma(y_0)}{\pi V_\infty c(y_0)} + \alpha|_{L=0} + \frac{1}{4\pi V_\infty}\int_{-b/2}^{b/2} \frac{(d\Gamma/dy)\,dy}{y_0 - y}$$

$$\frac{d\Gamma}{dy} = \frac{d\Gamma}{d\theta}\frac{d\theta}{dy}$$

$$\rightarrow \alpha(\theta_0) = \frac{2b}{\pi c(\theta_0)} \sum_1^N A_n \sin(n\theta_0) + \alpha|_{L=0}(\theta_0) + \underbrace{\frac{1}{\pi}\int_0^\pi \frac{\sum_1^N n A_n \cos n\theta}{\cos\theta - \cos\theta_0}\,d\theta}$$

$$= \int_0^\pi \frac{\cos n\theta\, d\theta}{\cos\theta - \cos\theta_0} = \frac{\pi \sin n\theta_0}{\sin\theta_0}$$

$$\boxed{\rightarrow \alpha(\theta_0) = \frac{2b}{\pi c(\theta_0)} \sum_1^N A_n \sin(n\theta_0) + \alpha|_{L=0}(\theta_0) + \sum_1^N n A_n \frac{\sin n\theta_0}{\sin\theta_0}}$$

A_n unknown ; N unknowns

- θ_0 is a specific location along span of wing
- $b, c,$ are known parameters of geometry
- α is also known

Solution method: • choose N locations (θ_0 values) along wing $\rightarrow$ N equations + N unknowns (A_n's)

• system of equations to solve for A_n

• once we know A_n, sub back into $\Gamma(\theta)$ expression to obtain lift distribution

once $\Gamma(\theta)$ is known:

$$C_L = \frac{2}{V_\infty S}\int_{-b/2}^{b/2} \Gamma(y)\,dy = \frac{2b^2}{S}\sum_1^N A_n \int_0^\pi \sin(n\theta)\sin\theta\, d\theta$$

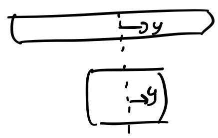
Recall: $\int_0^\pi \sin(n\theta)\sin\theta\, d\theta = \begin{cases} n=1: & \pi/2 \\ n\neq 1: & 0 \end{cases}$

$$\rightarrow \boxed{C_L = A_1 \pi\, b^2/S}$$

b = span

S = planform area

b^2/S = aspect ratio = AR



$$\boxed{C_L = A_1 \cdot \pi \cdot AR}$$

* C_L only depends on A_1 but must have N large enough for accuracy!

same analysis for induced drag, compute $C_{D,i}$

$$C_{D,i} = \frac{2}{V_\infty S}\int_{-b/2}^{b/2} \Gamma(y)\,\alpha_i(y)\,dy = \frac{2b^2}{S}\int_0^\pi \left(\sum_1^N A_n \sin(n\theta)\right)\alpha_i(\theta)\sin\theta\, d\theta$$

Recall α_i given by:

$$\alpha_i(y_0) = \frac{1}{4\pi V_\infty}\int_{-b/2}^{b/2} \frac{(d\Gamma/dy)}{y_0 - y}\,dy \quad \text{OR} \quad \alpha_i(\theta_0) = \frac{1}{\pi}\sum_1^N nA_n \int_0^\pi \frac{\cos n\theta}{\cos\theta - \cos\theta_0}\,d\theta$$

simplifies to: $\boxed{\alpha_i(\theta) = \sum_1^N nA_n \frac{\sin n\theta}{\sin\theta}}$

$$C_{D,i} = \frac{2b^2}{S}\int_0^\pi \left[\sum_1^N A_n \sin n\theta\right]\left[\sum_1^N nA_n \frac{\sin n\theta}{\sin\theta}\right] d\theta$$

$A, A_1,\ A, A_2,\ A_2 A_2,\ A_2 A_3$, etc

only keep w/ same subscript, others go to zero.

use integral: $\int_0^\pi \sin m\theta \sin k\theta\, d\theta = \begin{cases} 0 & m\neq k \\ 1 & m=k \end{cases}$

$$\rightarrow C_{D_i} = \frac{2b^2}{S}\left(\sum_1^N n A_n^2\right)\cdot\frac{\pi}{2} \Rightarrow \boxed{C_{D,i} = \pi \cdot AR \sum_1^N n A_n^2}$$

$$C_{D,i} = \pi\, AR\, A_1^2 \Big[1 + \underbrace{\sum_2^N n\left(\frac{A_n}{A_1}\right)^2}_{\delta\ =\ \text{induced drag factor}}\Big]$$

$$\boxed{C_{D,i} = \frac{C_L^2}{\pi \cdot AR}(1+\delta)}$$

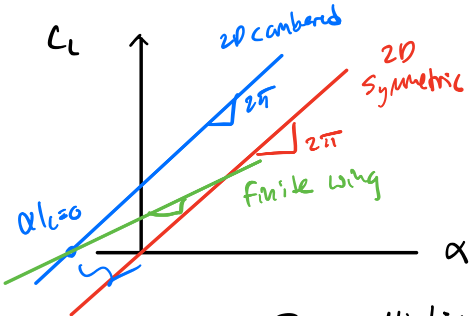
$e = \frac{1}{1+\delta}$ = span efficiency factor

$$\boxed{C_{D,i} = \frac{C_L^2}{\pi \cdot AR \cdot e}}$$

when $\delta = 0$ then $e = 1$, considered the best possible case

minimum amount of induced drag

- this best case is equivalent to the elliptical distribution of Γ



a = lift slope

can compute lift slope for various lift dist. $(\Gamma(y) = ?)$

For elliptical: $a = \frac{a_0}{1 + a_0/(\pi \cdot AR)}$; $a_0 = 2\pi$

General distribution:

$$a = \frac{a_0}{1 + (a_0/\pi \cdot AR)(1+\tau)}$$

τ = induced factor for lift slope

$\tau = 0.05 \rightarrow 0.25$ common range