Compressible Flow: Overview

Thermo properties:

h = enthalpy

p = pressure

ρ = density

T = Temperature

$h = c_p T$

↑ spec. heat @ p=const

Reference conditions

- freestream condition

$V_\infty,\ P_\infty,\ T_\infty,\ \rho_\infty,\ h_\infty,$

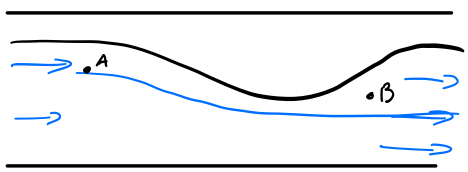
- Total or stagnation condition:

Defined as condition where fluid particles are slowed to zero velocity adiabatically

h_0 = total enth., p_0 = total press., ρ_0, T_0.

$u_0 = 0$

- property of any flow



flow is moving, but still has total h, p, etc.

Mach no. = $M_\infty = \dfrac{V_\infty}{a_\infty}$

speed of sound = a_∞

Assume: Inviscid, steady & irrotational

Goal: examine governing equations
continuity, momentum, energy

Apply assumptions to develop a simple model

Regimes

$M_\infty < 0.3$ => Incompressible

$0.3 < M_\infty < 0.85$ => Compressible, subsonic

(no shocks; similar to incompressible flow, smooth streamlines, no discontinuities; will have changes in ρ, T; will include energy eqn.)

$0.85 < M_\infty < 1.2$ => Transonic
most challenging!

$1.2 < M_\infty$ => supersonic
can have shocks/discontinuities

$5 < M_\infty$ => hypersonic
Chemistry eqns!!

Derivation of velocity-potential eqn

Goal: Develop eqn for compressible flow in terms of φ $(\vec{V} = \nabla\varphi)$ } using conservation laws: continuity & momentum

compressible continuity: $\dfrac{\partial(\rho u)}{\partial x} + \dfrac{\partial(\rho v)}{\partial y} = 0$ (steady)

$$\rho\frac{\partial u}{\partial x} + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + \rho\frac{\partial v}{\partial y} = 0 \; ; \quad u = \frac{\partial\varphi}{\partial x}, \quad v = \frac{\partial \varphi}{\partial y}$$

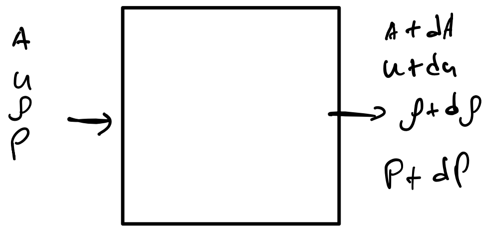
$$\rho\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial\varphi}{\partial x}\frac{\partial\rho}{\partial x} + \frac{\partial\varphi}{\partial y}\frac{\partial\rho}{\partial y} + \rho\frac{\partial^2\varphi}{\partial y^2} = 0$$

$$\boxed{\rho\left(\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2}\right) + \frac{\partial\varphi}{\partial x}\frac{\partial\rho}{\partial x} + \frac{\partial\varphi}{\partial y}\frac{\partial\rho}{\partial y} = 0}$$

→ next: sub. $\dfrac{\partial\rho}{\partial x}$ & $\dfrac{\partial\rho}{\partial y}$

need these from momentum

Quasi-1D flow



⇒ mom-bal on CV

$$PA + \rho u^2 A + p\,dA = (P+dP)(A+dA) + (\rho+d\rho)(u+du)^2(A+dA)$$

Next: expand terms; drop H.O.T.

$\boxed{dP = -\rho u\, du}$ quasi-1D momentum eqn.

$$V^2 = u^2 + v^2$$

$$dP = -\rho V dV = -\frac{\rho}{2}d(V^2) = -\frac{\rho}{2}d(u^2+v^2)$$

sub φ

$$dP = -\frac{\rho}{2}d\left[\left(\frac{\partial\varphi}{\partial x}\right)^2 + \left(\frac{\partial\varphi}{\partial y}\right)^2\right]$$

To find $d\rho$:

speed of sound: $a^2 = \dfrac{dp}{d\rho}$ → $dp = a^2 d\rho$

$$\Rightarrow \boxed{d\rho = \frac{-\rho}{2a^2}d\left[\left(\frac{\partial\varphi}{\partial x}\right)^2 + \left(\frac{\partial\varphi}{\partial y}\right)^2\right]}$$

→ $$\frac{\partial\rho}{\partial x} = \frac{-\rho}{a^2}\left(\frac{\partial\varphi}{\partial x}\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial\varphi}{\partial y}\frac{\partial^2\varphi}{\partial x\partial y}\right)$$

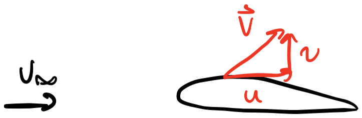
$$\frac{\partial\rho}{\partial y} = \frac{-\rho}{a^2}\left(\frac{\partial\varphi}{\partial x}\frac{\partial^2\varphi}{\partial x\partial y} + \frac{\partial\varphi}{\partial y}\frac{\partial^2\varphi}{\partial y^2}\right)$$

$$\boxed{\left[1-\frac{1}{a^2}\left(\frac{\partial\varphi}{\partial x}\right)^2\right]\frac{\partial^2\varphi}{\partial x^2}+\left[1-\frac{1}{a^2}\left(\frac{\partial\varphi}{\partial y}\right)^2\right]\frac{\partial^2\varphi}{\partial y^2}-\frac{2}{a^2}\left(\frac{\partial\varphi}{\partial x}\right)\left(\frac{\partial\varphi}{\partial y}\right)\left(\frac{\partial^2\varphi}{\partial x\partial y}\right)=0}$$

Exact equation: velocity-potential Eqn.

1. Solve numerically for φ
2. compute u, v
3. compute a, M
4. use isentropic relations to get P, T, ρ, etc.

Linearized velocity-potential Eqn



perturbation theory: $u = u_\infty + \hat{u}$

$v = \hat{v}$

$\varphi = U_\infty x + \hat{\varphi}$

$\hat{u} = \frac{\partial\hat{\varphi}}{\partial x}$; $\hat{v} = \frac{\partial\hat{\varphi}}{\partial y}$

$\frac{\partial\varphi}{\partial x} = u_\infty + \frac{\partial\hat{\varphi}}{\partial x}$

$\frac{\partial\varphi}{\partial y} = \frac{\partial\hat{\varphi}}{\partial y}$

Rewrite exact equation in these terms

$$\boxed{\left[a^2-(u_\infty-\hat{u})\right]\frac{\partial\hat{u}}{\partial x}+(a^2+\hat{v}^2)\frac{\partial\hat{v}}{\partial y}-2(u_\infty+\hat{u})\hat{v}\frac{\partial\hat{u}}{\partial x}=0}$$ Eq. (1)

thermo: h_0 (total enth.)

energy eqn: $h_{o_1} = h_{o_2}$, rewrite as $c_p T_\infty + \frac{u_\infty^2}{2} = c_p T + \frac{V^2}{2}$

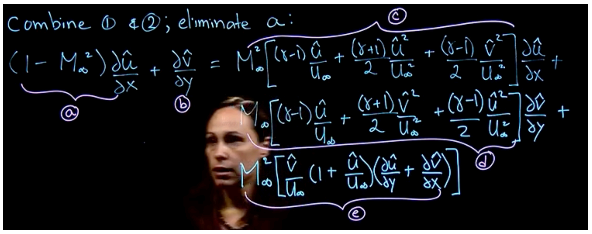
Speed of sound for ideal gas: $a = \sqrt{\gamma R T}$

$$V^2 = (u_\infty + \hat{u})^2 + \hat{v}^2$$

combine thermo eqns:

$$\boxed{\frac{a_\infty^2}{\gamma-1}+\frac{u_\infty^2}{2}=\frac{a^2}{\gamma-1}+\frac{(u_\infty+\hat{u})^2+\hat{v}^2}{2}}$$ eq. (2)

Combine & eliminate a:



assume small perturbations:

$\hat{u} << u_\infty$; $\frac{\hat{u}}{u_\infty} << 1$; $\frac{\hat{v}}{u_\infty} << 1$

$\frac{\hat{u}^2}{u_\infty^2} <<< 1$; $\frac{\hat{v}}{u_\infty^2} <<< 1$

* valid for thin airfoils & small α

Reduce to:

$$\boxed{(1-M_\infty^2)\frac{\partial \hat{u}}{\partial x} + \frac{\partial \hat{v}}{\partial y} = 0}$$

$$\boxed{(1-M_\infty^2)\frac{\partial^2 \hat{\varphi}}{\partial x^2} + \frac{\partial^2 \hat{\varphi}}{\partial y} = 0}$$

same eqn, Linearized or Approx. Eqn
(remember assumptions)

$M_\infty < 0.8$ OR $M_\infty > 1.2$
(avoid transonic)

Apply small perturb. assump.:

ⓒ << ⓐ ;

if $M_\infty < 0.8$ or $M_\infty > 1.2$,

ⓓ << ⓑ , e << ⓐ,ⓑ

Let $\beta = 1 - M_\infty^2$

$$\boxed{\beta^2 \frac{\partial^2 \hat{\varphi}}{\partial x^2} + \frac{\partial^2 \hat{\varphi}}{\partial y^2} = 0}$$

Solution method depends on β^2 / M_∞

- if $M_\infty \to 0$, $\beta^2 \to 1$
  Recover incomp. eqn.
- if $M_\infty < 1$, $\beta^2 +$
  elliptic PDE
- if $M_\infty > 1$, $\beta^2 -$
  hyperbolic PDE

Pressure coeff. of comp. flow

$$c_p = \frac{p - p_\infty}{\frac{1}{2}\rho_\infty u_\infty^2} \qquad q_\infty = \frac{1}{2}\rho_\infty u_\infty^2 = \frac{\gamma}{2} p_\infty M_\infty^2$$

$$\boxed{C_p = \frac{2}{\gamma M_\infty^2}\left(\frac{p}{p_\infty} - 1\right)}$$

exact eqn

next, write (p/p_∞) in terms of $\hat{u}, \hat{v}$ $(\hat{\varphi})$

energy eqn: $$T + \frac{V^2}{2c_p} = T_\infty + \frac{u_\infty^2}{2c_p}$$

$$c_p = \frac{\gamma R}{\gamma - 1} \quad \& \quad a = \sqrt{\gamma R T}$$

subs: $$\frac{T}{T_\infty} - 1 = \left(\frac{\gamma - 1}{2}\right)\left(\frac{u_\infty^2 - V^2}{a_\infty^2}\right)$$

$$V^2 = (u_\infty + \hat{u})^2 + \hat{v}^2$$

Isentropic: $$\frac{p}{p_\infty} = \left(\frac{T}{T_\infty}\right)^{\frac{\gamma}{\gamma-1}}$$

$$\frac{p}{p_\infty} = \underbrace{\left\{1 - \frac{\gamma - 1}{2} M_\infty^2 \left[\frac{2\hat{u}}{u_\infty} + \frac{\hat{u}^2 + \hat{v}^2}{u_\infty^2}\right]\right\}^{\frac{\gamma}{\gamma-1}}}_{= (1 - \varepsilon)^{\gamma/\gamma - 1}}$$

ε = a small value

binomial expansion

$$(1 - \varepsilon)^{\frac{\gamma}{\gamma-1}} \approx 1 - \frac{\gamma}{\gamma - 1}\varepsilon + \dots$$

sub $\frac{p}{p_\infty}$ into c_p

$$\rightarrow \boxed{C_p = \frac{-2\hat{u}}{u_\infty}}$$

Linearized/approx. C_p

works for subsonic & supersonic

e.g. Subsonic flow $M_\infty < 0.8$

1. solve elliptic PDE for $\hat{\varphi}$
2. $\hat{u}$ & $\hat{v}$
3. $C_p = \frac{-2\hat{u}}{U_\infty} = \frac{-2}{U_\infty}\frac{\partial \hat{\varphi}}{\partial x}$

can show that $C_p = -\frac{1}{\beta}\frac{2}{U_\infty}\frac{\partial \bar{\varphi}}{\partial x}$

$\bar{\varphi}$ = incompressible solution

$$\boxed{C_p = \frac{C_{p,0}}{\beta}}$$

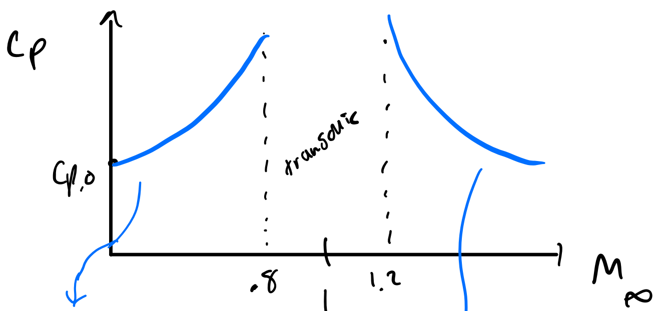
$C_{p,0}$ = incomp. C_p

True for subsonic, $M_\infty < 0.8$

$$\boxed{C_l = \frac{C_{l,0}}{\beta}}$$

Prandtl - Glauert compressibility correction

works pretty well up to $M_\infty < 0.7$



$C_p \sim \frac{1}{\sqrt{1-M_\infty^2}}$

$C_p \sim \frac{1}{\sqrt{M_\infty^2 - 1}}$ (haven't done this)