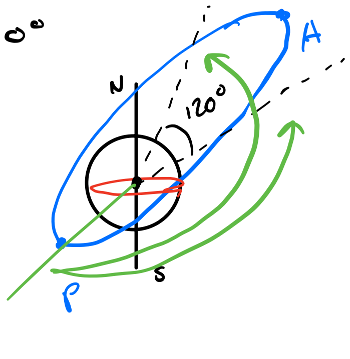
Slide 16: equations for EQ sheet

Example 1: Molniya orbit given $\begin{cases} a = 25200 \text{ km} \\ e = 0.72 \\ T \cong 11 \text{ hr} \end{cases}$

1) Determine fraction of orbit period w/ favorable LOS

Let's use 120°



θ = true anomaly from perigee (θ=0)

$\therefore\ 120° \le \theta \le 240°$

$\frac{2\pi}{3} \le \theta \le \frac{4\pi}{3}$

careful w/ quadrants

Game plan:

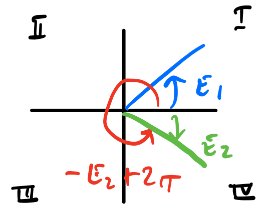
$\theta \rightarrow E \rightarrow M \rightarrow \Delta t,\ t \text{ vs. } T$

use: $\tan\left(\frac{E}{2}\right) = \left(\frac{1-e}{1+e}\right)^{1/2} \tan\left(\frac{\theta}{2}\right)$

$E_1 = 2\,\text{atan}\left(\frac{1-.72}{1+.72}\right)^{1/2} \tan\left(\frac{120°}{2}\right) = \boxed{1.2199 \text{ rad}}$ E_1

$E_2 = 2\,\text{atan}\left(\frac{1-.72}{1+.72}\right)^{1/2} \tan\left(\frac{240°}{2}\right) = -1.2199 \text{ rad}\Big]$ negative!

tangent:



Modify $E_2 = -1.2199 + 2\pi$

$\boxed{E_2 = 5.0633 \text{ rad}}$ E_2

$= 290.1°$

Apply Kepler: $M = E - e\sin E$ use radians!

$M_1 = E_1 - e\sin E_1 = 0.5438$

$M_2 = E_2 - e\sin E_2 = 5.7394$

Mean anomaly $= M = \frac{2\pi}{T}(t - \cancel{t_p}^{\,0}) = \frac{2\pi}{T} t$

$\therefore \quad t_1 = \frac{M_1 T}{2\pi}, \quad t_2 = \frac{M_2 T}{2\pi}$

$\therefore \quad \frac{\Delta t}{T} = \frac{t_2 - t_1}{T} = \frac{M_2 - M_1}{2\pi} = \frac{5.7394 - 0.5438}{2\pi} = 0.827$

$\boxed{82.7\% \text{ of orbital period between } 120^\circ \leq \theta \leq 240^\circ}$

Example 2: given t_1, find θ

→ use root finding (MATLAB)

→ Newton-Raphson method

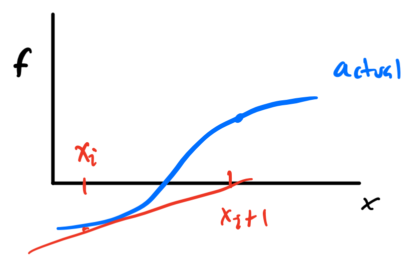
- Kepler → set equal to zero

$$f(E) = 0 = E - e\sin E - M$$

$$\frac{\partial f}{\partial E} = 1 - e\cos E$$

- use 1st order estimates
  - iterate w/

$$E_{i+1} = E_i - \frac{f(E)}{\partial f / \partial E}$$



$$E_{i+1} = E_i - \left(\frac{E_i - e\sin E_i - M}{1 - e\cos E_i} \right) \quad (1)$$

$\therefore$ Example: find θ for 4 hrs of molniya orbit

Take from $\theta_1 = 0$, $\theta_2 =$ 4 hrs later

$$M = nt = \frac{2\pi}{T}(t_2 - \cancel{t_1}^{\,0}) = \frac{2\pi \,(4 \text{ hrs})}{11.04 \text{ hrs}} = 2.28$$

previous → 11.04 hrs

table:

i	E_{i+1}
0	2.6250
1	2.63116
2	2.6315

$(E_0 = M = 2.28)$
assumed initial guess

"converged"

$E = 2.6315 \text{ rad} = 150.8^\circ$

Matlab

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M=2.28 ;
E_0 = M ;
E = fsolve ( @ E ( E -.72*sin(E) -M), E_0 );
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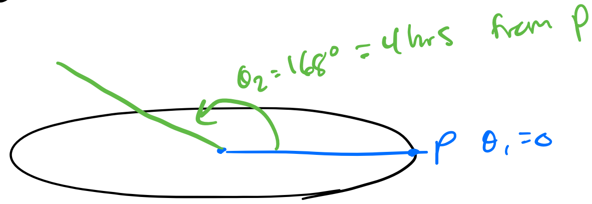
then, $\theta = 2\tan^{-1}\left(\left(\frac{1+.72}{1-.72}\right)^{1/2} \tan\left(\frac{150.8^\circ}{2}\right)\right)$

$\theta = 2.93 \text{ rad} = 168^\circ$



Orbit examples:

LEO: $< 2{,}000$ Km (1200 mi)

MEO: $2{,}000$ Km $<$ orbit $< 36{,}000$ Km

HEO: $> 36{,}000$ Km

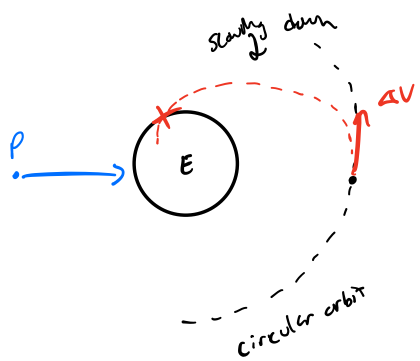
Geostationary is a specific type of geosynchronous
(at equator) (24hr period)

Parabolic & hyperbolic orbits

Thus far: ellipse $r = \frac{a(1-e^2)}{1+e\cos\theta} = \frac{h^2/\mu}{1+e\cos\theta}$ $\quad 0 \le e < 1$

circular

HW2: retrofire burn



apply $\Delta V \to$ limit?

Case I: Rectilinear

case II: Parabolic orbit

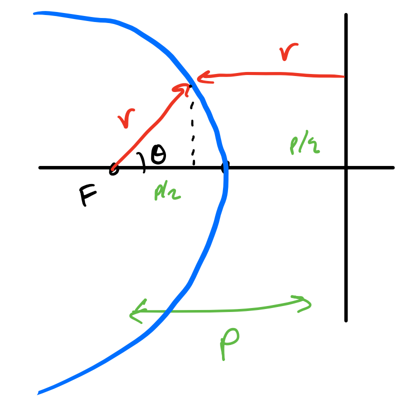
case III: Hyperbolic orbit

Case I: Rectilinear, a finite, $e = 1$, $p = h = 0$

"Active transfer"

Flat ellipse

case II: Parabolic, $a \to \infty$, $e = 1$, p = finite



dist. p

Parabola: locus of pts equidistant between focus pt & line

$\therefore \; r + r\cos\theta = p$

$$\boxed{r = \frac{p}{1+\cos\theta}}$$

Parabola trajectory

to get speed, use polar velocity $\bar{v} = \dot{r}\,\hat{e}_r + r\dot{\theta}\,\hat{e}_\theta$

to show $v^2 = \frac{2\mu}{r}$

$$\boxed{v = \sqrt{\frac{2\mu}{r}}}$$

or, vis viva

$v = \sqrt{\mu\left(\frac{2}{r} - \frac{1}{a}\right)}$

Let $a \to \infty$

Suggest as $r \to \infty$, $v \to 0$

Can use "escape velocity" $\equiv$ vel. req'd for $r \to \infty$ on parabola

e.g. if circular $r = a$

$v_{circ} = \sqrt{\frac{\mu}{r}}$

$\therefore \quad v_{esc} = \sqrt{2}\, v_{circ}$

for r at that moment

<u>to get orbit position</u>, generally use $r^2\dot{\theta} = h = \sqrt{p\mu}$

and use $r = \dfrac{p}{1+\cos\theta} = \dfrac{p}{2\cos^2(\frac{\theta}{2})} = \dfrac{p}{2}\sec^2\left(\frac{\theta}{2}\right)$

plug into h: $\left(\dfrac{p}{2}\right)^2 \sec^4\left(\dfrac{\theta}{2}\right)\dfrac{d\theta}{dt} = \sqrt{p\mu}$ integrate

$$\boxed{2\sqrt{\frac{\mu}{p^3}}\,(t-t_p) = \tan\left(\frac{\theta}{2}\right) + \frac{1}{3}\tan^3\left(\frac{\theta}{2}\right)}$$

p = parab. focal pt.

parabola posn (θ) w/ time

<u>Example</u> 3 { Sat. on parabolic escape
perigee speed 10 km/s

Find dist. from Earth center 6 hrs after perigee.

plan: find r_p → calc. t & LHS of θ Eq. → solve θ → solve r

$v_{esc} = \sqrt{\dfrac{2\mu}{r_p}} \rightarrow r_p = \dfrac{2\mu}{v_{esc}^2}$

$r_p = \dfrac{2(3.986\times10^5)}{(10)^2} = 7972$ km

At perigee, $h = r_p v_p = 79720$ km^2/s, use $h^2 = p\mu$

$\therefore$ LHS $= 2\sqrt{\dfrac{\mu}{p^3}}\,(t - \cancelto{0}{t_p}) = 2\sqrt{\dfrac{\mu^2}{h^3}}\; t$

→ continue w/ posn eqn