Problem 1 Airfoil, known $\alpha|_{L=0}$, chord length c_a produces circulation Γ_a in freestream V_∞, ρ

1.1) Find expression for α_a

$$C_\ell = 2\pi(\alpha - \alpha|_{L=0}) \qquad L = \rho V_\infty \Gamma$$

$$C_\ell = \frac{L}{\frac{1}{2}\rho V_\infty^2 c} = \frac{\cancel{\rho}\, \cancel{V_\infty} \Gamma}{\frac{1}{2}\cancel{\rho} V_\infty^{\cancel{2}} c_a} = \frac{2\Gamma_a}{V_\infty c_a}$$

$$\alpha = \frac{C_\ell}{2\pi} + \alpha|_{L=0} \rightarrow \boxed{\alpha = \frac{\Gamma_a}{\pi V_\infty c_a} + \alpha|_{L=0}}$$

1.2) Elliptic wing, midspan chord c_0, span b, $\Gamma_{ell}(y=0) = \Gamma_a$

Find $\alpha = \alpha_1$

$$\alpha(y_0) = \frac{\Gamma(y_0)}{\pi V_\infty c(y_0)} + \alpha|_{L=0}(y_0) + \frac{1}{4\pi V_\infty}\int_{-b/2}^{b/2} \frac{(d\Gamma/dy)\,dy}{y_0 - y}$$

$$\rightarrow \Gamma(y) = \Gamma_0\sqrt{1-\left(\frac{2y}{b}\right)^2}, \qquad \frac{d\Gamma}{dy} = \frac{-4\Gamma_0 y}{b^2\sqrt{1-4y^2/b^2}}$$

$$\rightarrow y_0\,(\text{midspan}) = 0 \rightarrow \alpha(y_0 = 0) = \frac{\Gamma_a}{\pi V_\infty c_0} + \alpha|_{L=0} + \frac{1}{4\pi V_\infty}\int_{-b/2}^{b/2}$$

$$\rightarrow \alpha = \frac{\Gamma_a}{\pi V_\infty c_0} + \alpha|_{L=0} - \frac{\cancel{4}\Gamma_a}{\cancel{4}\pi V_\infty}\underbrace{\int_{-b/2}^{b/2} \frac{y}{b^2\sqrt{1-4y^2/b^2}} \cdot \frac{1}{\cancel{y_0} - y}\,dy}_{=\frac{\cancel{\pi}}{2b}}$$

$$\rightarrow \boxed{\alpha_1 = \frac{\Gamma_a}{\pi V_\infty c_0} + \alpha|_{L=0} - \frac{\Gamma_a}{2bV_\infty}}$$

1.3) Non-elliptic wing, c_0, b, $\Gamma(y=0) = \Gamma_a$. Find α_2 if

$$\alpha_i(y) = \frac{\Gamma_a}{2bV_\infty}\left[\frac{7}{8} + \frac{1}{2}\left(\frac{2y}{b}\right)^2\right]$$

$$\boxed{\alpha_2 = \frac{\Gamma_a}{\pi V_\infty c_0} + \alpha|_{L=0} + \frac{7\Gamma_a}{16 b V_\infty}}$$

Problem 2 NACA 0012 (symmetric 2D wing), $c_a = 65\text{ cm}$

$\rho_\infty = 1.225\ \text{kg/m}^3$, $\mu_\infty = 1.789 \times 10^{-5}\ \text{kg/ms}$, $\alpha = \alpha$, $V_\infty = 50\ \text{m/s}$

$L' = 500\ \text{N/m}$

2a) Find α & Γ_a

$$L' = \rho V_\infty \Gamma'(y), \quad \Gamma'(y) = \Gamma_a'$$

$$\rightarrow \Gamma_a' = \frac{L'}{\rho V_\infty} = \boxed{8.16 = \Gamma_a'}$$

$$\Gamma = \pi \alpha c V_\infty \rightarrow \alpha = \frac{\Gamma}{\pi c V_\infty} = \boxed{\alpha = 0.08 = 4.6^\circ}$$

2b) Finite wing, same freestream, $b = 90\text{ cm}$. Find L if

$$\Gamma(y) = \Gamma_0\left[1 - \left(\frac{2y}{b}\right)^2\right], \quad \Gamma_0 = \Gamma_a = \Gamma_a'$$

$$L'(y) = \rho V_\infty \Gamma(y) \quad ; \quad L = \int_{-b/2}^{b/2} L'(y)\,dy$$

$$\rightarrow L = \int_{-b/2}^{b/2} \rho V_\infty \Gamma(y)\,dy = \int_{-b/2}^{b/2} \rho V_\infty \Gamma_a\left[1 - \left(\frac{2y}{b}\right)^2\right] dy$$

$$L = \rho V_\infty \Gamma_a \int_{-b/2}^{b/2} \left[1 - \left(\frac{2y}{b}\right)^2\right] dy = \rho V_\infty \Gamma_a \left[y - \frac{1}{3}\left(\frac{2y}{b}\right)^3 \cdot \frac{1}{2/b}\right]_{-b/2}^{b/2}$$

$$L = \rho V_\infty \Gamma_a \left[\frac{b}{2} - \frac{1}{3}(1)^3 \cdot \frac{b}{2} - \left(-\frac{b}{2} - \frac{1}{3}(-1)^3 \cdot \frac{b}{2}\right)\right]$$

$$L = \rho U_\infty \Gamma_a \left[b - \frac{1}{3}\frac{b}{2} - \left(\frac{1}{3}\frac{b}{2}\right)\right]$$

$$\rightarrow L = \rho U_\infty \Gamma_a \left[b - \frac{b}{3}\right] = \rho U_\infty \Gamma_a \cdot \frac{2b}{3}$$

$$\rightarrow \boxed{L = 300 \ N/m^2}$$

2c) Untwisted elliptic, same c/s as a), $\Gamma_0 = \Gamma_a$, $C_0 = 65cm$, $b = 90\,cm$

Find α to get same lift as b)

Elliptic: $\Gamma(y) = \Gamma_0 \sqrt{1 - \left(\frac{2y}{b}\right)^2}$

$$L = \int_{-b/2}^{b/2} \rho V_\infty \Gamma(y)\, dy = \rho U_\infty \Gamma_0 \underbrace{\int_{-b/2}^{b/2} \sqrt{1 - \left(\frac{2y}{b}\right)^2}\, dy}_{\text{wolfram: } \frac{\pi b}{4}}$$

$$\rightarrow L = \rho U_\infty \Gamma_0 b \cdot \frac{\pi}{4} \rightarrow \Gamma_0 = \frac{4L}{\pi \rho U_\infty b} = \boxed{6.93 = \Gamma_0}$$

$$\alpha(y_0) = \frac{\Gamma(y_0)}{\pi V_\infty C(y_0)} + \underbrace{\cancel{\alpha_{L=0}}}_{0,\ \text{symmetric}} + \frac{1}{4\pi V_\infty}\int_{-b/2}^{b/2} \frac{(d\Gamma/dy)}{y_0 - y}\, dy$$

$$\alpha(y_0 = 0) = \frac{\Gamma_0}{\pi V_\infty C_0} + \frac{\Gamma_0}{4\pi V_\infty} \underbrace{\int_{-b/2}^{b/2} \frac{4y}{b^2\sqrt{1 - \frac{4y^2}{b^2}}}\, dy}_{2\pi/b}$$

$$\rightarrow \alpha = \frac{\Gamma_0}{\pi V_\infty C_0} + \frac{\Gamma_0}{2bV_\infty}$$

$$\rightarrow \boxed{\alpha = 0.145 = 8.3^\circ}$$