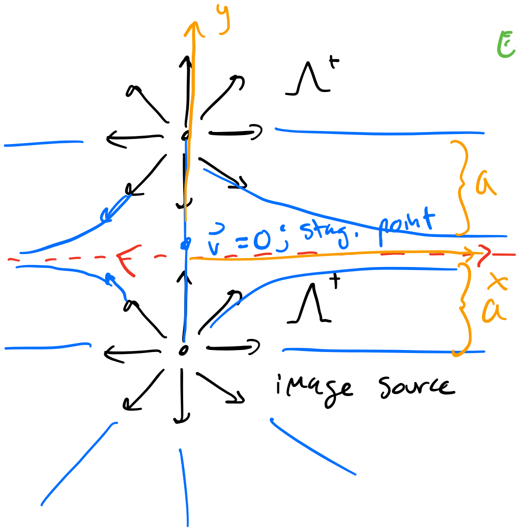
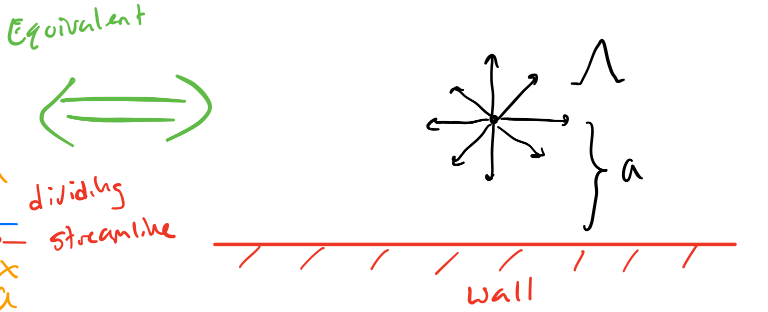
Lecture 17 Method of images





To form velocity potential, super impose $\varphi_1 + \varphi_2$ (source + image source)

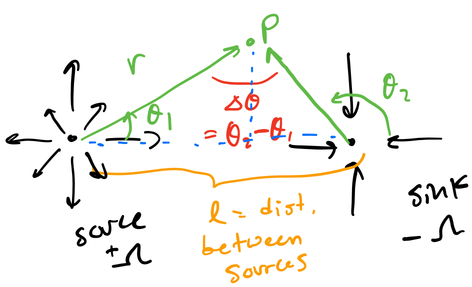
Need to translate source to $(0, a)$
and image to $(0, -a)$

$$\varphi = \frac{\Lambda}{2\pi}\ln(r - r_1) + \frac{\Lambda}{2\pi}\ln(r - r_2)$$

$$r_1^2 = (x - x_1)^2 + (y - y_1)^2$$

$$\varphi = \frac{\Lambda}{2\pi}\ln\sqrt{x^2+(y-a)^2} + \frac{\Lambda}{2\pi}\ln\sqrt{x^2+(y+a)^2}$$

Lecture 18: Doublet



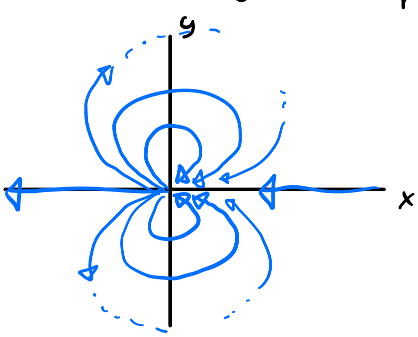
$$\psi = \frac{\Lambda}{2\pi}\theta_1 - \frac{\Lambda}{2\pi}\theta_2$$

$$\rightarrow \psi = -\frac{\Lambda}{2\pi}\Delta\theta$$

Next, a doublet is defined as
$l \to 0$ and $\Lambda \cdot l$ = constant = K

From geometry: $\Delta\theta = \dfrac{l\sin\theta}{r - l\cos\theta}$ $\Rightarrow$ sub into ψ: $\psi = -\dfrac{\overset{K}{\Lambda\, l}\sin\theta}{2\pi\,(r - l\cos\theta)}$

let $l \to 0$



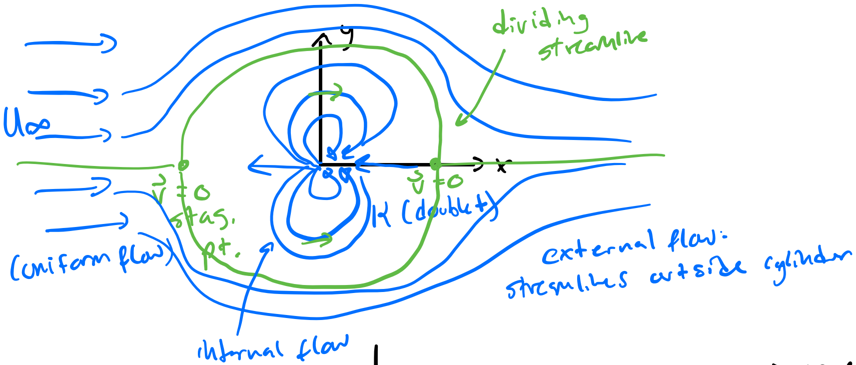
$$\boxed{\begin{aligned}\psi &= -\frac{K}{2\pi}\frac{\sin\theta}{r}\\ \rightarrow \varphi &= \frac{K}{2\pi}\frac{\cos\theta}{r}\end{aligned}}$$

Doublet along x axis

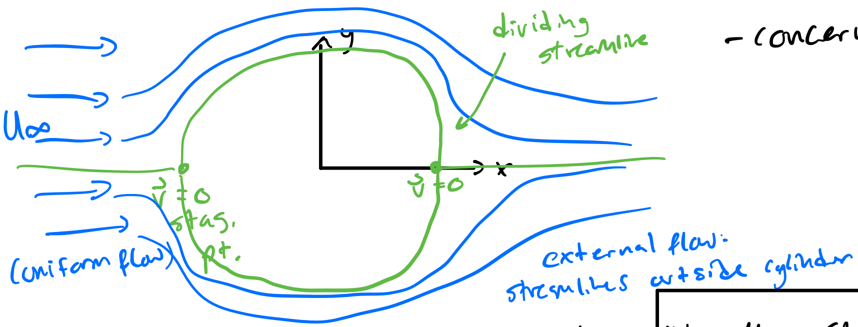
Doublet does have orientation, can be rotated!

r

Lecture 19: Cylinder flow

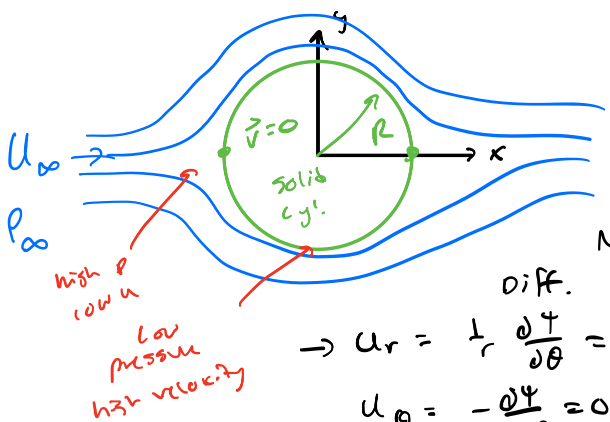


- ignore internal flow, replacing it with a solid body (cylinder)
- concerned w/ external flow only



mathematically:

$$\boxed{\begin{aligned} \psi &= U_\infty r \sin\theta - \frac{K}{2\pi}\frac{\sin\theta}{r} \\ \varphi &= U_\infty r \cos\theta + \frac{K}{2\pi}\frac{\cos\theta}{r} \end{aligned}}$$



Question: What is R?

- Find location of stagnation pts

Need to find expression for velocity

Diff. φ (or ψ)

$$\rightarrow u_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = 0 \rightarrow \frac{1}{r}\left(U_\infty r\cos\theta - \frac{K}{2\pi}\frac{\cos\theta}{r}\right) = 0 \qquad (1)$$

$$u_\theta = -\frac{\partial \psi}{\partial r} = 0 \rightarrow -\left(U_\infty \sin\theta + \frac{K}{2\pi}\frac{\sin\theta}{r^2}\right) = 0 \qquad (2)$$

From (1), $r^2 = \frac{K}{2\pi U_\infty}$, from (2) $u_\theta = -\sin\theta\left(1 + \frac{R^2}{r^2}\right) = 0$

$\theta = 0, \pi$, etc.

$$\boxed{R = \sqrt{\frac{K}{2\pi U_\infty}}}$$

$\rightarrow \psi = U_\infty r \sin\theta - R^2 U_\infty \frac{\sin\theta}{r}$ (replaced $\frac{\kappa}{2\pi}$ with $R^2 U_\infty$)

What is pressure distribution along surface of cylinder?

$P(\theta)$? Apply Bernoulli's eqn $P_\infty + \frac{1}{2}\rho U_\infty^2 = p_s + \frac{1}{2}\rho |u_s|^2$

$|u_s|^2 = u_r^2 + u_\theta^2$

$u_r = u_\infty \cos\theta \left(1 - R^2/r^2\right)$

$u_\theta = -u_\infty \sin\theta \left(1 + R^2/r^2\right)$

if $r = R$ then $u \rightarrow u_{surface}$

$\rightarrow$ only $f(\theta)$

$\rightarrow \boxed{|u_s|^2 = 4 u_\infty \sin^2\theta}$ $\boxed{P_s(\theta) = P_\infty + \frac{1}{2}\rho\left(U_\infty^2 - |u_s|^2\right)}$

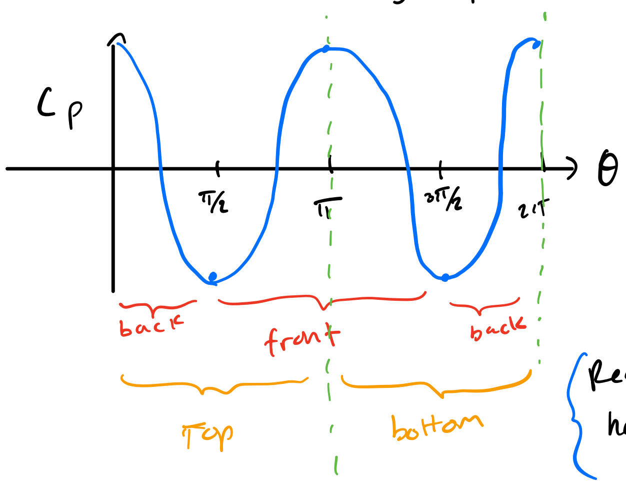
- sub into bernoulli, solve for P_s

pressure coefficient

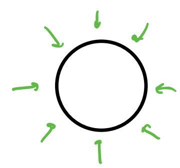
$$\boxed{C_p = \frac{P_s - P_\infty}{\frac{1}{2}\rho U_\infty^2} = 1 - \frac{|u_s|^2}{U_\infty^2}}$$

This expression valid for all external potential flows (airfoils, etc)

The velocity expression $|u_s|$ will change.



Potential flow over stationary cylinder

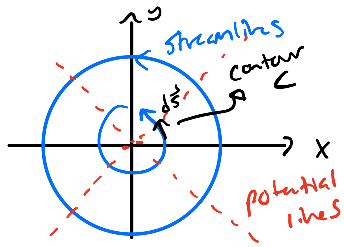


- Lift force? $L = 0$ by symmetry

- Drag force $D = 0$

Realistic flows are NOT potential flows however $L = 0$, $D \neq 0$. Real cylinders have drag!

Lecture 20 Free vortex



$V_r = 0$

$V_\theta \neq 0$

What is V_θ?

(incomp.)

- Must satisfy both $\nabla \cdot \vec{V} = 0$

and $\nabla \times \vec{V} = 0$ (irrotational)

$$\nabla \cdot \vec{V} = \frac{1}{r}\cancel{\frac{\partial}{\partial r}(rV_r)}^{\,0} + \frac{1}{r}\frac{\partial}{\partial \theta}(V_\theta) = 0$$

$$\frac{\partial V_\theta}{\partial \theta} = 0$$

Thus $V_\theta = fn(r)$

$$\nabla \times \vec{V} = \frac{1}{r}\left(\frac{\partial}{\partial r}(rV_\theta) - \cancel{\frac{\partial V_r}{\partial \theta}}^{\,0}\right)\hat{k} = 0$$

$$\frac{\partial}{\partial r}(rV_\theta) = 0, \quad rV_\theta = \text{const}$$

$$\rightarrow \boxed{V_\theta = \frac{\text{const}}{r}}$$

To find const, look @ circulation Γ

$$\Gamma = -\oint_C \vec{V} \cdot d\vec{s} = \boxed{-V_\theta 2\pi r = \Gamma}$$

use to find const.

$$\text{const} = -\frac{\Gamma}{2\pi}$$

$$\rightarrow \boxed{V_\theta = \frac{-\Gamma}{2\pi}\frac{1}{r}} \quad \rightarrow \Gamma \neq 0$$

A free vortex has circulation!

$$\vec{\zeta} = \vec{\nabla} \times \vec{V} = \left[\frac{1}{r}\underbrace{\frac{\partial}{\partial r}(rV_\theta)}_{=0}\right]\hat{k}$$

Singularity in flow at origin

* Vorticity is zero except at origin (singularity)

$$\Gamma = -\iint_S \vec{\zeta} \cdot d\vec{s} \quad \text{and} \quad \vec{\zeta} = 0 \ \text{everywhere except origin}$$

As radius goes to zero, the value of vorticity at origin must be infinite

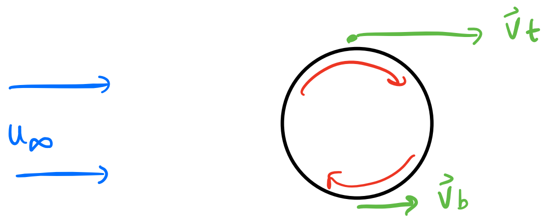
$$\boxed{\vec{\zeta} \rightarrow \infty \ \text{at origin}}$$

Free vortex

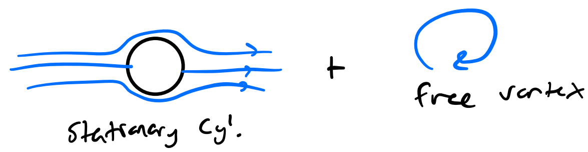
$$\psi = \frac{\Gamma}{2\pi} \ln r$$

$$\varphi = -\frac{\Gamma}{2\pi}\theta$$

Lecture 21 : Rotating cylinder



Potential flow model:

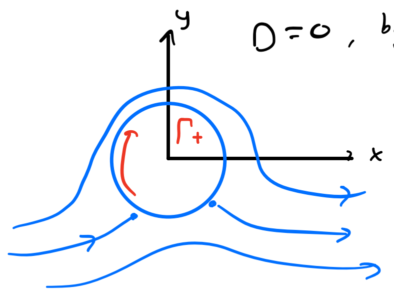


rotating cylinder :

* clockwise rotation generates positive lift: low P on top high P on bottom
* $+\Gamma$ generates + Lift

$$\psi_{rot.cyl} = \psi_{stat.cyl.} + \psi_{vortex}$$

$$\psi = U_\infty r \sin\theta \left(1 - \frac{R^2}{r^2}\right) + \frac{\Gamma}{2\pi}\ln\frac{r}{R}$$



D = 0 , by symmetry front/back

L > 0

What is the lift force?

$\psi \rightarrow$ velocity components $\rightarrow$ Pressure surface $\rightarrow$ Forces L, D

$$U_r\Big|_{r=R} = 0$$

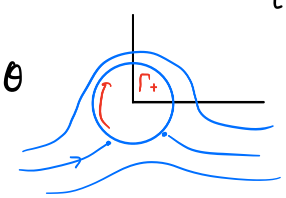
$$U_\theta\Big|_{r=R} = -2U_\infty \sin\theta - \frac{\Gamma}{2\pi R}$$

→ where are stag. pts?

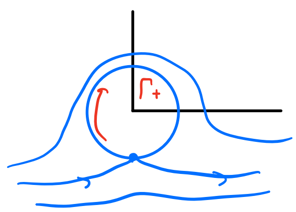
$U_\theta \big|_{r=R} = 0$, solve for θ

$$-\frac{\Gamma}{2\pi R} = 2U_\infty \sin\theta \rightarrow \theta = \arcsin\left[\frac{-\Gamma}{4\pi U_\infty R}\right]$$

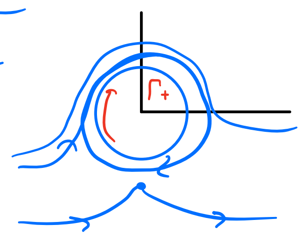
1) if $\frac{\Gamma}{4\pi U_\infty R} < 1$, can obtain 2 values of θ



2) if $\frac{\Gamma}{4\pi U_\infty R} = 1$,



3) if $\frac{\Gamma}{4\pi U_\infty R} > 1$, stag. pt. off of body



$$C_p = 1 - \frac{|u_s|^2}{u_\infty^2} = 1 - \frac{1}{u_\infty^2}\left(-2u_\infty \sin\theta - \frac{\Gamma}{2\pi R}\right)^2$$

$$C_p(\theta) = 1 - 4\sin^2\theta - \frac{2\Gamma \sin\theta}{\pi R u_\infty} - \frac{\Gamma^2}{4\pi^2 R^2 u_\infty^2}$$

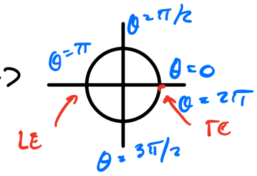
$$C_L = \frac{1}{c}\int_0^c C_{p,l}\,dx - \frac{1}{c}\int_0^c C_{p,u}\,dx \Rightarrow$$

chord $c = 2R$

$x = R\cos\theta$

$dx = -R\sin\theta\, d\theta$

$$C_L = \frac{1}{2R}\int_\pi^{2\pi} C_{p,l}(-R\sin\theta)\,d\theta - \frac{1}{2R}\int_{\pi,\ LE}^{0,\ TE} C_{p,u}(-R\sin\theta)\,d\theta \rightarrow$$



$$C_L = \frac{1}{2\cancel{R}}\left[\int_\pi^{2\pi} C_{p,l}(-\cancel{R}\sin\theta)\,d\theta + \int_0^\pi C_{p,u}(-\cancel{R}\sin\theta)\,d\theta\right]$$

$C_p = C_{p,l} = C_{p,u}$

$$\boxed{C_L = -\frac{1}{2}\int_0^{2\pi} C_p \sin\theta\, d\theta}$$

Trig properties:

$\int_0^{2\pi}\sin\theta\, d\theta = 0$, $\int_0^{2\pi}\sin^2\theta\, d\theta = \pi$, $\int_0^{2\pi}\sin^3\theta\, d\theta = 0$

$$\rightarrow C_L = \cancel{+}\frac{1}{\cancel{2}}\left(\cancel{+}\frac{2\Gamma}{\pi R u_\infty}\right)\cancel{\pi} = \boxed{\frac{\Gamma}{R u_\infty} = C_L}$$

lift coeff. rotating cylinder

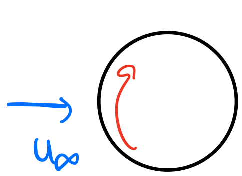
Recall: $C_L = \frac{L}{\frac{1}{2}\rho u_\infty^2 \cdot 2R}$

$$\rightarrow \boxed{L = \rho u_\infty \Gamma}$$

Lift force rotating cylinder

Kutta - Joukowski Theorem

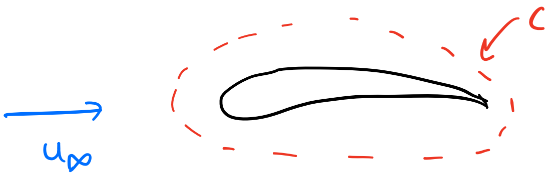
application to other bodies



Γ^+

produces

$L = \rho u_\infty \Gamma$



$\Gamma = \oint \vec{V}\cdot d\vec{s} \rightarrow L = \rho u_\infty \Gamma$