Recap

n-stages
$g = D = 0$

$u_{e\,i} = u_e = \text{const.}$
$u_e = u_{eq}$ (neglect $p_e - p_a$)

$$u_n = u_e \ln \prod_{i=1}^{n} R_i = u_e \ln \prod_{i=1}^{n} \frac{1+\lambda_i}{\varepsilon_i + \lambda_i}$$

Similar stages:
$\lambda_i = \lambda = \text{const}$
$\varepsilon_i = \varepsilon = \text{const}$

$$\xi = \frac{1+\lambda}{\lambda} = \left(\frac{M_0}{M_L}\right)^{1/n}$$

$$\frac{u_n}{u_e} = n \ln \frac{\xi}{\varepsilon(\xi - 1) + 1} \qquad (\star')$$

$$\frac{M_0}{M_L} = \left(\frac{1-\varepsilon}{\exp\left(-\frac{1}{n}\frac{u_n}{u_e}\right) - \varepsilon}\right)^n$$

plot $\frac{M_{0_1}}{M_L}$ vs. n to study $\begin{cases} \text{feasibility of low } n \\ \quad \text{(simpler/safer \& cheaper)} \\ \text{advantage of high } n \end{cases}$

Criterion: want $\frac{M_{0_1}}{M_L}$ as low as possible

→ pick: $\varepsilon = 0.1$

$\frac{u_n}{u_e}$ based upon $\begin{cases} \text{desired } u_n \\ \text{available } u_e \text{ based on:} \\ \quad \text{- type of propellant} \\ \quad \text{- engine "choice"} \end{cases}$

→ Fig. 10.8

Remarks:

i) By rights, should <u>not</u> plot solid lines b/c n is discrete
– but helps visualize trends

For $\frac{u_n}{u_e} = 3$

ii) @ $n=1$, $\frac{M_{01}}{M_L} \to \infty$

cannot reach $u_n = 3u_e$ w/ single stage

iii) @ $n=2$, $\frac{M_{01}}{M_L} \approx 54$ large gain

@ $n=3$, $\frac{M_{01}}{M_L} \approx 37$

iv) From $n=4$ to $n=10$

$34 \leq \frac{M_{01}}{M_L} \leq 30$ Small gains

v) All the same for $\frac{u_n}{u_e} = 6$

Note: Still based on "similar" stages

Return to ($\star'$)

$$\frac{u_n}{u_e} = n \ln \frac{\xi}{\varepsilon(\xi - 1) + 1}$$

Replace $\xi = \left(\frac{M_{01}}{M_L}\right)^{1/n}$

$$\frac{u_n}{u_e} = n \ln \frac{\left(\frac{M_{01}}{M_L}\right)^{1/n}}{\varepsilon\left(\left(\frac{M_{01}}{M_L}\right)^{1/n} - 1\right) + 1} \qquad (\star\star)$$

Plot to study effect of n upon $\frac{u_n}{u_e}$

Pick $\varepsilon = 0.1$, use $\frac{M_{01}}{M_L}$ as param.

$\to$ Fig 10.9

Plot $\underbrace{\dfrac{u_{n\text{-stages}}}{u_{\text{single stage}}}}_{\text{left axis}}$ vs. n (assuming $u_{ei} = u_e = \text{const}$

$\underbrace{\text{EQ } (\star\star)}_{\text{Right axis}}$

Remarks

For $\dfrac{M_{01}}{M_L} = 4{,}000$

i) @ $n=4$, $\dfrac{u_n}{u_e} \approx 6$ & $\dfrac{u_n}{u_{ss}} = 2.5$

ii) All curves w/ finite $\dfrac{M_{01}}{M_L}$ saturate

past some value of n, $\dfrac{u_n}{u_\infty}$ no longer increases

§ 4.3 optimization of mass distribution

How best to partition structural & propellant masses across stages (Appendix 8 in book)

For all the following cases: $g = D = 0$, $P_e = P_a \to u_e = u_{eq}$

Case 1 Given $\begin{cases} u_e \text{ known, const. across stages} \\ \varepsilon \quad '' \quad '' \quad '' \\ \text{Assigned } M_{01}, M_L, n \end{cases}$

Find: $\lambda_i = \dfrac{M_{0(i+1)}}{M_{0i} - M_{0(i+1)}}$ for $i = 1 \ldots n$

that maximize $\dfrac{u_n}{u_e}$

At each stage, $\Delta u_i = u_e \ln R_i = u_e \ln \dfrac{1+\lambda_i}{\varepsilon + \lambda_i}$

$\to$ over n stages: $\dfrac{u_n}{u_e} = \sum_{i=1}^{n} \dfrac{\Delta u_i}{u_e} = \sum_{i=1}^{n} \underbrace{\ln \dfrac{1+\lambda_i}{\varepsilon+\lambda_i}}_{F(\lambda_i)} = \sum_{i=1}^{n} F(\lambda_i)$

Recall $\frac{\mu_e}{\mu_{01}} = \prod_{i=1}^{n} \frac{\lambda_i}{1+\lambda_i}$

$$\ln \frac{\mu_e}{\mu_{01}} = \ln \prod_{i=1}^{n} \frac{\lambda_i}{1+\lambda_i} = \sum_{i=1}^{n} \ln \underbrace{\frac{\lambda_i}{1+\lambda_i}}_{G(\lambda_i)} = \sum_{i=1}^{n} G(\lambda_i)$$

Must find λ_i's that @ once:

- maximize $\frac{u_n}{u_e} = \sum_{i=1}^{n} F(\lambda_i)$
- verify $\ln \frac{\mu_e}{\mu_{01}} = \sum_{i=1}^{n} G(\lambda_i)$

this type of problem belongs to the field of "calculus of variations"

simplest procedure:

maximize new function $\mathcal{L}(\lambda_i) = F(\lambda_i) + \alpha\, G(\lambda_i)$

α is unknown const. called Lagrange multiplier

Thus, look for $\frac{\partial \mathcal{L}}{\partial \lambda_i} = 0$ trust RB that it is a max

$$\frac{\partial \mathcal{L}}{\partial \lambda_i} = \frac{\partial F}{\partial \lambda_i} + \alpha \frac{\partial G}{\partial \lambda_i}$$

$$= \frac{\partial}{\partial \lambda_i}\left(\ln \frac{1+\lambda_i}{\varepsilon+\lambda_i}\right) + \alpha \frac{\partial}{\partial \lambda_i}\left(\ln \frac{\lambda_i}{1+\lambda_i}\right)$$

$$= \frac{\varepsilon+\lambda_i}{1+\lambda_i} \cdot \frac{1\cdot(\varepsilon+\lambda_i) - (1+\lambda_i)\cdot 1}{(\varepsilon+\lambda_i)^2} + \alpha\, \frac{1+\lambda_i}{\lambda_i} \cdot \frac{1\cdot(1+\lambda_i) - \lambda_i \cdot 1}{(1+\lambda_i)^2} = 0$$

SET