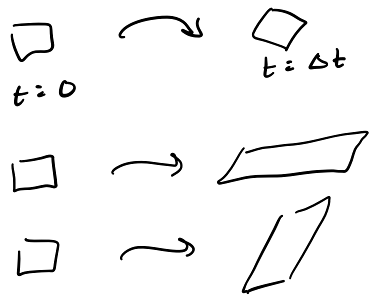
Lecture 9 vorticity & circulation

Kinematics of a fluid particle




Rotation : $\vec{\omega}$ = ang. vel. vector

distortion : $\underline{\varepsilon}$ = $\underbrace{\text{rate of strain tensor}}$

wish to write kinematic changes in terms of the velocity vector

Rotation & distortion

Assuming 2D flow , $\vec{\omega} = \omega \hat{k}$

$$\vec{\omega} = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\hat{k}$$

2D Planar

vorticity $\vec{\xi} = 2\vec{\omega} \rightarrow \boxed{\vec{\xi} = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\hat{k}}$ $\underbrace{\boxed{\vec{\xi} = \vec{\nabla} \times \vec{V}}}$

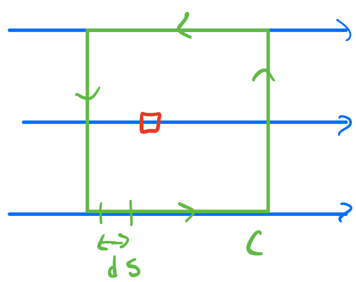
general 3D definition

Rotational flow $\vec{\xi} \neq 0$

Irrotational Flow : fluid particles not rotating

$\vec{\xi} = 0$ , check $\vec{\nabla} \times \vec{V} = 0$

Circulation , $\Gamma$ $\qquad \Gamma = -\oint_C \vec{V} \cdot d\vec{s}$ ← line integral, closed contour C




Split up integral into 4 sides:

right/left side : $\vec{V} \cdot d\vec{s} = 0$

bottom : $\vec{V} \cdot d\vec{s} = +|V|ds$

top : $\vec{V} \cdot d\vec{s} = -|V|ds$

Sum of sides : $\Gamma = 0$

$$\Gamma = -\iint_S (\vec{\nabla} \times \vec{V}) \cdot d\vec{s} \qquad = -\iint_S \vec{\xi} \cdot d\vec{s}$$

← surface bounded by C

Lecture 10 : continuity

Previously: $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$  Assumes: Incomp., 2D

If irrotational, $\vec{\xi} = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0$  (1)

use stream function, sub into (1) (satisfies continuity)

$$\frac{\partial}{\partial x}\left(-\frac{\partial \psi}{\partial x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial \psi}{\partial y}\right) = 0 \rightarrow \boxed{\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0}$$

more general form:

$$\boxed{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \theta^2} = 0}$$

$$\boxed{\nabla^2 \psi = 0}$$

2D Laplace Eqn

Linear PDE

Represents continuity for irrotational, incompressible, and 2D flows

## Lecture 11: Bernoulli

Momentum eqn:

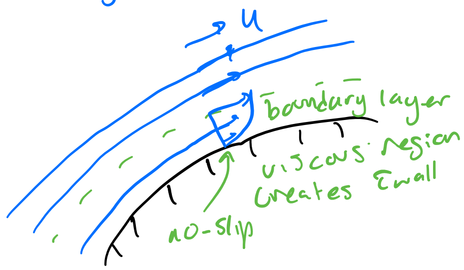
$$(1) \quad \cancel{\frac{\partial(\rho u)}{\partial t}} + \nabla \cdot (\rho u \vec{V}) = -\frac{\partial P}{\partial x} + \mu \cancel{\frac{\partial^2 u}{\partial x^2}}$$

$$(2) \quad \cancel{\frac{\partial(\rho v)}{\partial t}} + \nabla \cdot (\rho v \vec{V}) = -\frac{\partial P}{\partial y} + \mu \cancel{\frac{\partial^2 v}{\partial y^2}}$$

unsteady terms

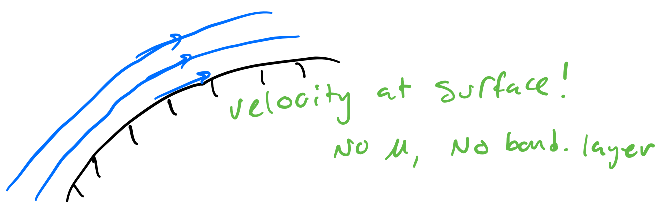
viscous terms

Assumptions:
- 2D
- Incompressible
- steady, $\frac{d}{dt} = 0$
- inviscid: neglect $\tau_{wall}$, $\mu = 0$




Viscous

- Realistic flow




inviscid

- A model of flow

High Re: Small BL

momentum:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial x}$$

$$u\frac{\partial u}{\partial x}dx + v\frac{\partial u}{\partial y}dx = -\frac{1}{\rho}\frac{\partial P}{\partial x}dx$$

Apply to streamline: $v\,dx = u\,dy$

$$u\Big(\underbrace{\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy}_{=\,du}\Big) = -\frac{1}{\rho}\frac{\partial P}{\partial x} \longrightarrow u\,du = -\frac{1}{\rho}\frac{\partial P}{\partial x}dx$$

expand:

$$u\,du = \frac{1}{2}d(u^2) = \frac{1}{2}(u\,du + du \cdot u)$$

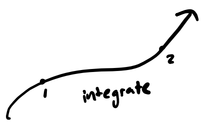
$$(3) \quad \frac{1}{2}d(u^2) = -\frac{1}{\rho}\frac{\partial P}{\partial x}dx \quad \text{(formally x-mom eqn)}$$

same steps for y-mom:

$$(4) \quad \frac{1}{2}d(v^2) = -\frac{1}{\rho}\frac{\partial P}{\partial y}dy$$

Recall $u^2 + v^2 = V^2$ & $dP = \frac{\partial P}{\partial x}dx + \frac{\partial P}{\partial y}dy$

Sum (3) & (4) $\rightarrow$ $\boxed{\frac{1}{2}dV^2 = -\frac{1}{\rho}dP}$




$$\frac{1}{2}\rho\int_1^2 V \cdot dV = \int_1^2 dP \Rightarrow \boxed{P_1 + \frac{1}{2}\rho V_1^2 = P_2 + \frac{1}{2}\rho V_2^2}$$

Bernoulli's Equation

$\rightarrow$ used both x & y momentum

$\rightarrow$ states cons. momentum w/ assumptions

Assumptions:

- 2D
- Incompressible
- steady, $\frac{d}{dt} = 0$
- inviscid: neglect $\tau_{wall}$, $\mu = 0$
- flow along streamline ☆

Note!

Revisit equation:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial x}$$

Rather than assume streamline, assume <u>Irrotational</u>

$$\vec{\nabla} \times \vec{V} = 0 \Rightarrow \frac{\partial v}{\partial x} = \frac{\partial u}{\partial y}$$

use subs to derive Bernoulli!

$\rightarrow$ Assume flow along streamline <u>OR</u> Irrotational

If flow irrotational, then Bernoulli applies to any 2 points!

Lecture 12: velocity potential

$\varphi$ = velocity potential, "phi"

$\vec{v} = \nabla \varphi$

valid for 2D & 3D flows

valid for irrotational flows ☆

irrotational flow $\Longleftrightarrow$ potential flow

unsteady OR steady

compressible OR incomp.

our interests

Recall grad:

$$\nabla \varphi = \frac{\partial \varphi}{\partial x} \hat{\imath} + \frac{\partial \varphi}{\partial y} \hat{\jmath} + \frac{\partial \varphi}{\partial z} \hat{k}$$

$$\nabla \varphi = \frac{\partial \varphi}{\partial r} \hat{r} + \frac{1}{r} \frac{\partial \varphi}{\partial \theta} \hat{\theta} + \frac{\partial \varphi}{\partial z} \hat{z}$$

Assuming incompressible:

$\nabla \cdot \vec{V} = 0$ (from continuity)

combine w/ $\vec{V} = \nabla \varphi$

$$\nabla \cdot (\nabla \varphi) = \boxed{\nabla^2 \varphi = 0}$$

Laplace's Eqn (incomp. + irrot.)

Recall: $\boxed{\nabla^2 \psi = 0}$ (2D, incomp, irrot.)

If flow assumptions satisfied, then velocity potential and stream function satisfy Laplace.

## Lecture 13: Laplace

How do I solve $\nabla^2 \varphi = 0$? ($\nabla^2 \psi = 0$?)

Linear PDE

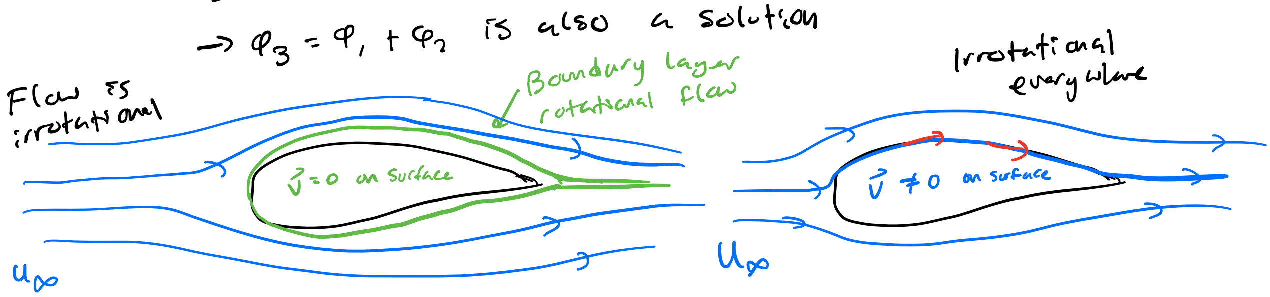
1) Separation of variables
2) Method of characteristics
3) Numerical methods
4) Linear superpositioning

Best method determined by boundary conditions

$\varphi_1$ is a solution to $\nabla^2 \varphi = 0$

$\varphi_2$ is also " "

$\rightarrow \varphi_3 = \varphi_1 + \varphi_2$ is also a solution




Realistic flow

Potential flow model

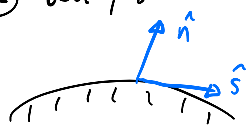
common boundary conditions for Potential flows (Laplace's Eqn)

(1) $u = U_\infty$ as $x, y \to -\infty$ $\Longleftrightarrow$ $u = \frac{\partial \phi}{\partial x}$, $v = \frac{\partial \phi}{\partial y}$

$$\boxed{\frac{\partial \phi}{\partial x} = U_\infty, \quad \frac{\partial \phi}{\partial y} = 0}$$

inlet boundary conditions

(2) Wall/surface boundary condition




$u_n = 0$: no flow thru surface

$u_s$ only: surface is a streamline

$$\boxed{\frac{\partial \psi}{\partial s} = 0}$$

$\vec{v} \cdot \hat{n} = 0$

$\nabla \phi \cdot \hat{n} = 0$ $\rightarrow$ $\boxed{\frac{\partial \phi}{\partial n} = 0}$ enforced along surface

Note: include $+C$ in $\psi / \phi$!