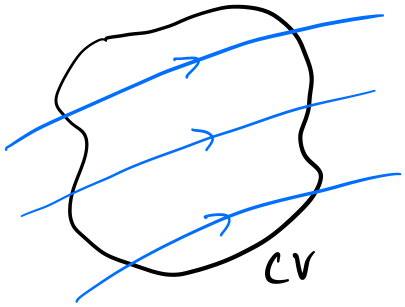
1) Conservation of mass



CV: Arbitrary shape, size, position

For now, assume that CV is <u>fixed</u> in time

Mass inside CV : $M_{CV} = \int_{CV} \rho \, d\forall$

Rate of change $\frac{dM_{CV}}{dt} = \frac{d}{dt}\int_{CV} \rho \, d\forall$

significance : M_{CV} may change in time because:

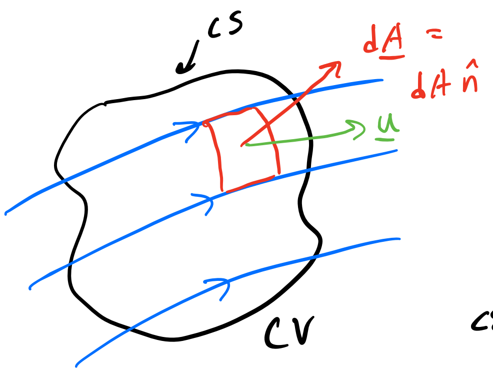
1) Portion of CV where $\rho \neq 0$ varies in time
   e.g. water tank being filled / emptied
2) ρ changes everywhere inside CV
   e.g. gas tank being emtied / filled
3) Shape / size / location of CV vary in time

In turn, changes must be due to mass flow in/out of CV: $\dot{m}$

conservation of mass : $\frac{dM_{CV}}{dt} + \dot{m} = 0$

which implies $\dot{m} < 0$ when entering CV

mass carried in/out of CV by component of velocity locally $\perp$ to surface of CV



$\dot{m} = \int_{cs} \rho \underline{u} \cdot d\underline{A}$

Dot product
1) selects <u>only</u> $\perp$ comp. of $\underline{u}$
2) Accounts for $\pm \leftrightarrow$ out/in

cs = surface enclosing CV

Conservation of mass rewrites:

$$\frac{d}{dt}\int_{CV} \rho \, d\forall + \int_{CS} \rho \underline{u} \cdot d\underline{A} = 0$$

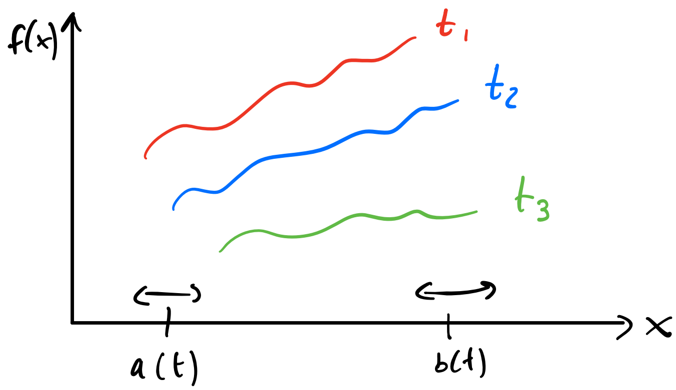
If CV distorts (shrinks, expands, moves in time)

for $\dot{m}$ term, we must only consider velocity of fluid relative to the CS

$$\boxed{\frac{d}{dt}\int_{CV} \rho \, d\forall + \int_{CS} \rho \underline{u}_{rel} \cdot d\underline{A} = 0 \qquad (1)}$$

Can write alternate form for $\frac{d}{dt}\int_{CV} \rho \, d\forall$

To account for changes w.r.t. time of both ρ & CV



Recall Leibniz rule:

$$\frac{d}{dt}\int_{a(t)}^{b(t)} f(x,t)\, dx = \int_{a(t)}^{b(t)} \frac{\partial f(x,t)}{\partial t}\, dx + f(b,t)\dot{b} - f(a,t)\dot{a}$$

By extension to 3-D

$$\frac{d}{dt}\int_{CV} \rho(\underline{x},t)\, d\forall = \int_{CV} \frac{\partial \rho(x,t)}{\partial t}\, d\forall + \int_{CS} \rho \underline{u}_b \cdot dA$$

$\underline{u}$ of CV boundary

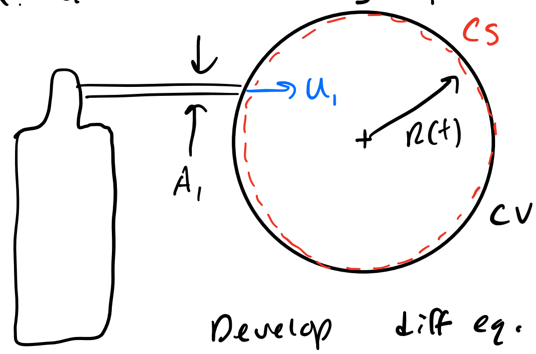
→ Reassembling

$$\int_{CV} \frac{\partial \rho}{\partial t}\, d\forall + \int_{CS} \rho \underline{u}_b \cdot d\underline{A} + \int_{CS} \rho \underline{u}_{rel} \cdot dA = 0 \qquad (2)$$

velocity of   velocity of fluid rel. to boundary

CV boundary

Ex. Gas bottle filling spherical balloon



Assume

A_1 known

$A_1 << $ Area of balloon (CS)

ρ, u both const in time & space

Develop diff eq. for $R(t)$.

Use (1): $\frac{d}{dt}\int_{CV} \rho \, d\forall + \int_{CS} \rho \underline{u}_{rel} \cdot d\underline{A} = 0$

$CV \equiv$ balloon

negative bc dot product

$$\frac{d}{dt}\left(\rho \cdot \frac{4}{3}\pi R^3(t)\right) - \rho u_1 A_1 = 0$$

$$\cancel{\rho} \cdot \frac{4}{\cancel{3}}\pi \cdot \cancel{3} R^2(t) \cdot \dot{R} + \cancel{\rho} u_1 A_1 = 0$$

$$\dot{R} R^2 - \frac{u_1 A_1}{4\pi} = 0$$

Use (2):

$$\int_{CV} \cancel{\frac{\partial \rho}{\partial t}} d\forall + \int_{CS} \rho \, \underline{u}_b \cdot d\underline{A} + \int_{CS} \rho \underline{u}_{rel} \cdot d\underline{A} = 0$$

$\rho =$ const

$\underline{u}_b = \dot{R} \, \hat{e}_R$

$d\underline{A} = dA \, \hat{e}_R$

$\hat{e}_R \cdot \hat{e}_R = 1$

$$0 + \cancel{\rho} \dot{R} \cdot \underbrace{4\pi R^2}_{A} - \cancel{\rho} u_1 A_1 = 0$$

nonlinear 1st order diff eq.

$$\dot{R} R^2 - \frac{u_1 A_1}{4\pi} = 0$$ same result as from (1)

$$\frac{dR}{dt} R^2 = \frac{u_1 A_1}{4\pi}$$

$$R^2 \, dR = \frac{u_1 A_1}{4\pi} dt$$

$$\frac{R^3}{3} = \frac{u_1 A_1}{4\pi} t + C$$ ← from initial condition

2) momentum equation

2.1) Fixed CV

Newton's law:

$$\Sigma \underline{F} = ma$$

$$\Sigma \underline{F} = \frac{d}{dt}\underbrace{(m\underline{u})}_{\text{momentum } \underline{P}} = \frac{dP}{dt}$$

For a moving fluid: