Recap

Rocket eqn

$$\Delta u = u_{eq} \ln R$$

If $P_e = P_a$ $\quad u_{eq} = u_e$

§4 Math Analysis

→ First study single-stage, then multi-stage

Objective: maximize $\begin{cases} M_\ell \\ \Delta u \end{cases}$

$$\epsilon \equiv \frac{M_s}{M_p + M_s} = \text{structural coeff.}$$

$\epsilon \uparrow$ if $\begin{cases} M_p \downarrow \\ \text{or} \\ M_s \uparrow \end{cases}$ ... $\left[\text{if } \epsilon \uparrow \rightarrow R \downarrow \rightarrow \Delta u \downarrow \right]$

to obtain a large Δu, need to minimize ϵ → little structure, lots of propellant

Develop 2 relationships:

$$\underbrace{\frac{M_\ell}{M_0}}_{\text{"overall" mass ratio}} = f_1\left(\frac{\Delta u}{u_e}\right), \qquad \epsilon = f_2\left(\frac{\Delta u}{u_e}\right)$$

Throughout, assume $u_{eq} = u_e$ → $\Delta u = u_e \ln R$

Start from $M_\ell = M_0 - \underbrace{(M_{eng} + M_{tank})}_{M_s} - M_p$

then
$$\frac{M_\ell}{M_0} = 1 - \frac{M_{eng}}{M_0} - \frac{M_{tank}}{M_p}\frac{M_p}{M_0} - \frac{M_p}{M_0}$$

$$= 1 - \frac{M_{eng}}{M_0} - \left(\frac{M_{tank}}{M_p} + 1\right)\frac{M_p}{M_0} \qquad (1)$$

From $\Delta u = u_e \ln R$, $\quad R = e^{\Delta u / u_e}$

Recall $R = \frac{M_0}{M_b} = \frac{M_0}{M_0 - M_p} \rightarrow \frac{M_0 - M_p}{M_0} = 1 - \frac{M_p}{M_0} = \frac{1}{R}$

only for single-stage

$$\rightarrow 1 - \frac{M_p}{M_0} = e^{-\Delta u / u_e} \rightarrow \frac{M_p}{M_0} = 1 - e^{-\Delta u / u_e} \qquad (2)$$

$$\to \frac{1}{M_0} = \frac{1}{M_p}\left(1 - e^{-\Delta u/u_e}\right) \qquad (2')$$

plug (2) into (1)

$$\to \frac{M_\ell}{M_0} = \underbrace{1 - \frac{M_{ens}}{M_0} - \left(\frac{M_{tank}}{M_p} + 1\right)\left(1 - e^{-\Delta u/u_e}\right)}_{f_1\left(\frac{\Delta u}{u_e}\right)} \qquad (3)$$

Recall $\varepsilon \equiv \frac{M_s}{M_p + M_s} = \frac{M_{tank} + M_{ens}}{M_p + M_{tank} + M_{ens}}$ Divide top & bottom by M_0

$$\to \varepsilon \equiv \frac{\frac{M_{tank}}{M_0} + \frac{M_{ens}}{M_0}}{\frac{M_p}{M_0} + \frac{M_{tank}}{M_0} + \frac{M_{ens}}{M_0}}$$

use (2') in numerator & (2) in denominator

$$\to \varepsilon \equiv \frac{\frac{M_{tank}}{M_p}\left(1 - e^{-\Delta u/u_e}\right) + \frac{M_{ens}}{M_0}}{\underbrace{\left(1 + \frac{M_{tank}}{M_p}\right)\left(1 - e^{-\Delta u/u_e}\right) + \frac{M_{ens}}{M_0}}_{f_2\left(\frac{\Delta u}{u_e}\right)}}$$

Plots for assigned values of u_e, $\frac{M_{tank}}{M_p}$, $\frac{M_{ens}}{M_0}$

i) the lower Δu, the larger $\frac{M_\ell}{M_0}$, i.e. most of the mass is payload. As $\Delta u \uparrow$, $\frac{M_\ell}{M_0} \downarrow$: need lots of propellant & payload becomes smaller portion of M_0

ii) the lower Δu, the higher ε, i.e. of the non-payload mass, only a small fraction is propellant
As $\Delta u \uparrow$, $\varepsilon \downarrow$: fraction of non-payload mass occupied by the propellant increases

iii) For large Δu, $\varepsilon \neq \varepsilon(\Delta u)$: proportions of tank, engine & propellant masses become fixed

iv) For both $\frac{M_p}{M_0}$ & ε, values are higher for H_2 engine than hydrocarbon engine

on p. 478: $\lambda = \frac{M_e}{M_0 - M_e}$ – "payload ratio" correct

on p. 479-480, $\frac{M_e}{M_0}$ also called "payload ratio" wrong

4.2 Multi-stage

In many cases, $M_e < M_p$ and can't do much about it

Sometime, $M_e < M_p$ this need not be the case:

i) wasteful to expend propellant to accelerate semi-empty tanks when fraction of prop. already burned.

ii) When tanks are full, structure ok w/ engine @ full thrust b/c high mass ↔ low accel.

when tanks half-empty, low mass ↔ high accel, possibly exceeding limits of structure/payload

- one solution: lower thrust (throttle engine)
- better: discard entire portions of vehicle & optimally match upper stage to acceleration limits of remaining portion