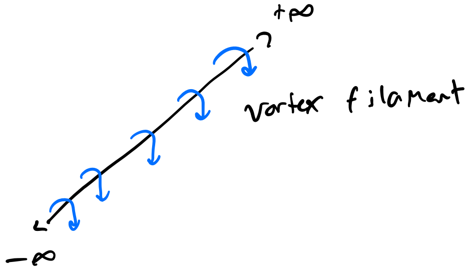
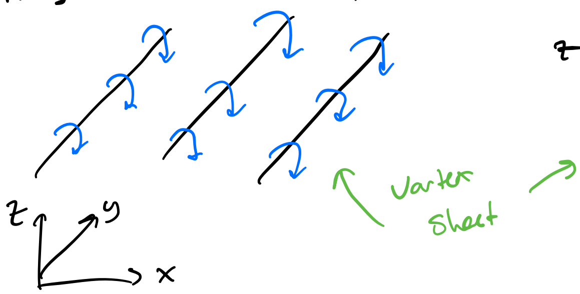
vortex panels & sheets

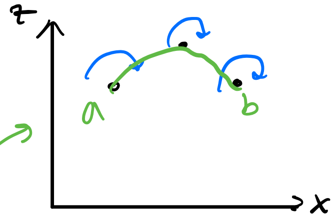
free vortex

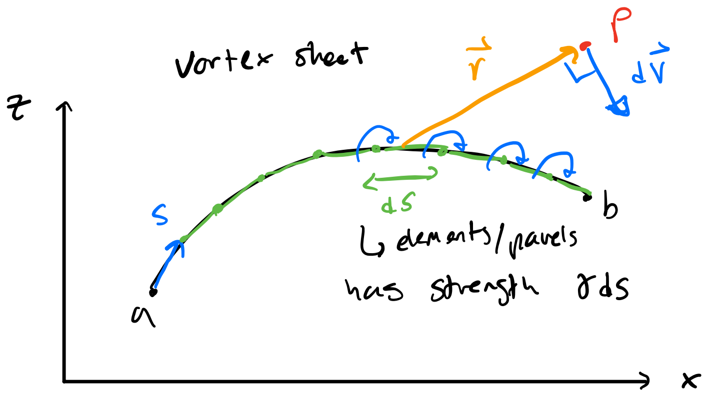
imagine a line in/out paper



Arrange a stack of filaments: vortex sheet







- element ds induces a small velocity at point P
- $d\vec{V} = -\dfrac{\gamma ds}{2\pi r}$ $\perp$ to $\vec{r}$

  (from free vortex analysis, only has V_θ component)
- as we move from $a \to b$, both $\vec{r}$ & $d\vec{V}$ change direction
- velocity potential at P due to ds

$$d\varphi = -\frac{\gamma ds}{2\pi}\theta$$

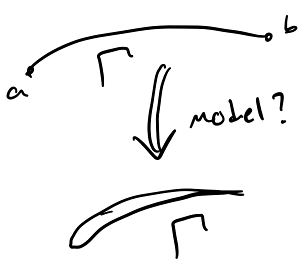
Potential function at point $P(x,z)$ due to vortex sheet

$$\boxed{\varphi(x,z) = -\frac{1}{2\pi}\int_a^b \theta\gamma ds}$$

what is γ?

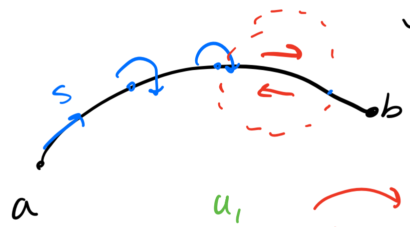
$$\boxed{\Gamma = \int_a^b \gamma ds}$$

Circulation of vortex sheet

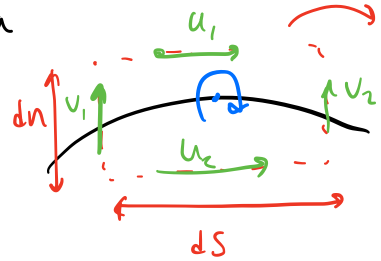


Recall

$$L = \rho V_\infty \Gamma$$



velocity values immediately above/below sheet → discontinuity in U_s above/below



C, closed rectangular contour

$$\Gamma = -\oint_C \vec{V}\cdot d\vec{s} \rightarrow \Gamma = -\left[v_2\,dn - u_1\,ds - v_1\,dn + u_2\,ds\right]$$

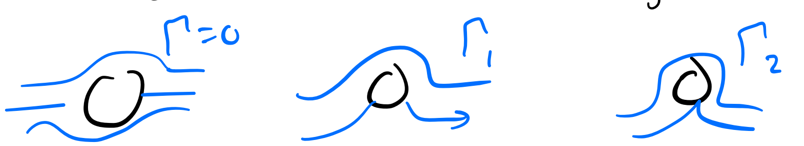
$$\Gamma = (u_1 - u_2)\,ds + \cancel{(v_1 - v_2)}\,dn = \gamma\,ds$$

Let $dn \to 0$ → $\boxed{\gamma = (u_1 - u_2)}$

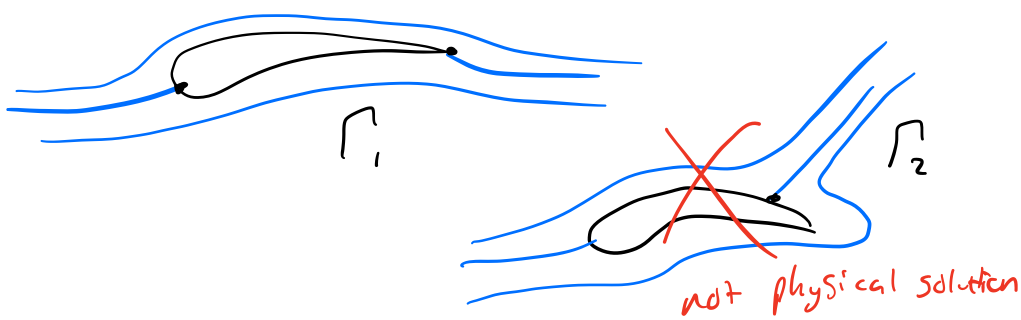
Important!
- local "jump" in velocity
- critical to vortex panel method!

Kutta condition

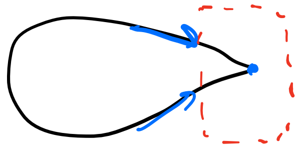
- How much circulation do I want on my airfoil?



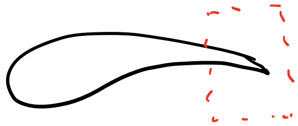
physical solution



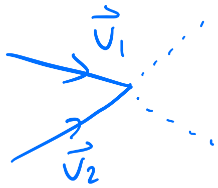
consider 2 geometries:



finite TE

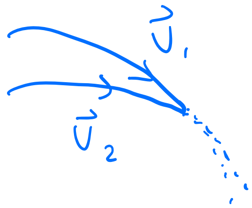


cusp TE



$\boxed{\vec{V}_1 = \vec{V}_2 = 0 \text{ at TE}}$

condition for physical sol'n



$\boxed{\vec{V}_1 = \vec{V}_2 \quad (\text{mag. } \& \text{ dir.})}$

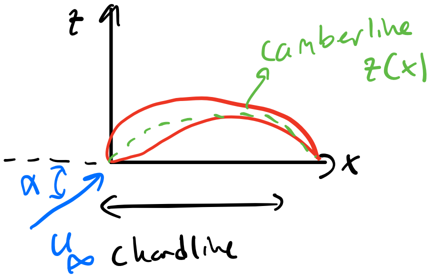
$\boxed{\gamma(TE) = V_1 - V_2 = 0}$

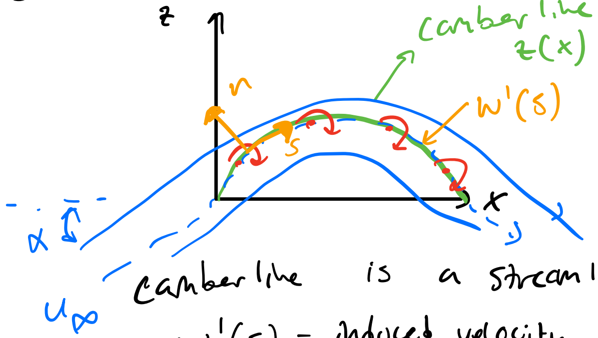
↑ Kutta condition

Notes on K.C.

- must choose a Γ to allow flow to exit smoothly @ TE
- If finite TE, $\vec{V}_{TE} = 0$
- If cusp TE, $\vec{V}_1 = \vec{V}_2$ @ TE
- $\gamma_{TE} = 0$
- Real-life / physical flows, Kutta C. automatically satisfied. (They have a B.L. w/ no-slip condition)

Thin airfoil Theory (P1)

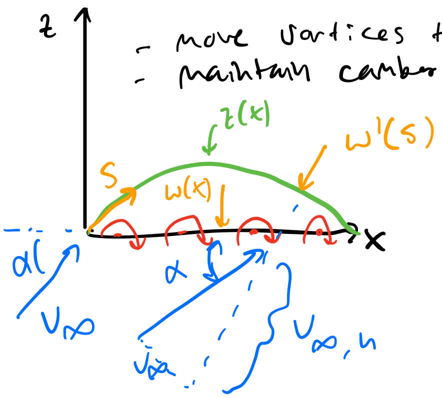




camber line is a streamline

$w'(s)$ = induced velocity $\perp$ to $z(x)$ due to vortices

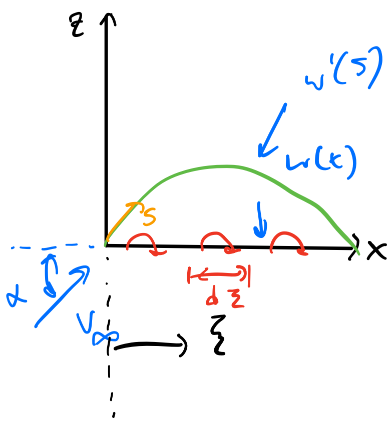
- move vortices to chordline
- maintain camberline as streamline



$w(x)$ = induced velocity $\perp$ to chordline (due to vortices)

$$\boxed{\begin{array}{l}\text{Impose } V_{\infty,n} + w'(s) = 0 \\ \text{for camberline to be a streamline}\end{array}}$$

$$V_{\infty,n} = U_\infty \sin\left[\alpha + \operatorname{atan}\left(-\frac{dz}{dx}\right)\right]$$



$\gamma = \gamma(\xi)$

$$dw = -\frac{\gamma(\xi)\,d\xi}{2\pi(x-\xi)} \qquad w(x) = -\int_0^c \frac{\gamma(\xi)\,d\xi}{2\pi(x-\xi)}$$

Next: small angle approximation

$$w(x) \approx w'(s)$$

$$V_{\infty,n} \approx V_\infty(\alpha - dz/dx) \quad (\sin x \approx x)$$

Using assumptions, sub into B.C.

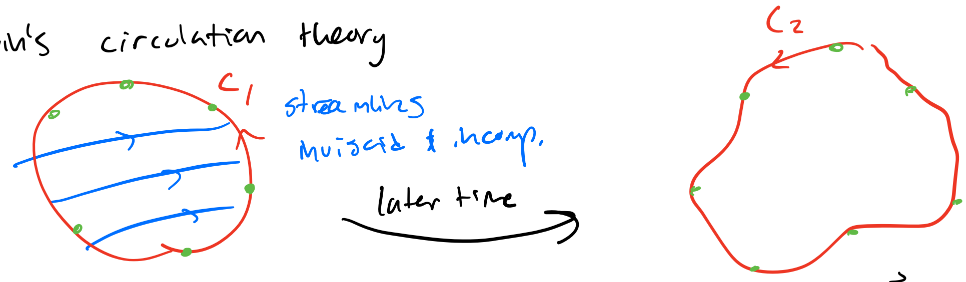
$$\boxed{V_\infty\left(\alpha - \frac{dz}{dx}\right) = \int_0^c \frac{\gamma(\xi)\,d\xi}{2\pi(x-\xi)}}$$

γ unknown!

How to solve for $\gamma \Rightarrow$ <u>next</u>

$\gamma(x=c) = 0$ (Kutta)

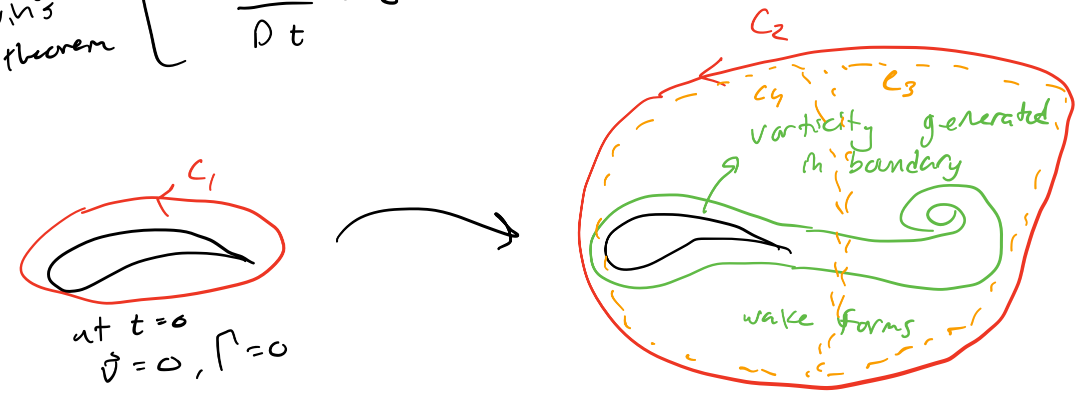
Kelvin's circulation theory



$$\Gamma_1 = -\int_{C_1} \vec{V} \cdot d\vec{s} \qquad \Gamma_2 = -\int_{C_2} \vec{V} \cdot d\vec{s}$$

equivalent for Kelvin's theorem:

- Can show $\Gamma_1 = \Gamma_2$
- vorticity is not created or destroyed
- $$\frac{D\Gamma}{Dt} = 0$$



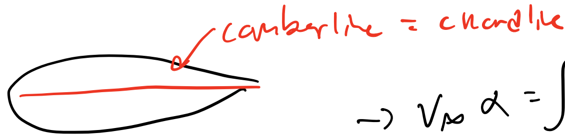
at $t=0$
$\vec{V}=0$, $\Gamma=0$

$\Gamma_2 = 0$

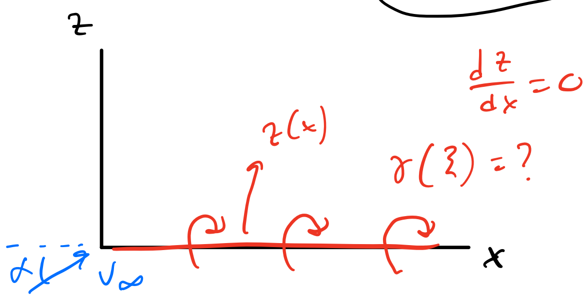
$\Gamma_3 = \Gamma_4$

Thin airfoil theory:

How to solve: $V_\infty \left(\alpha - \frac{dz}{dx}\right) = \int_0^c \frac{\gamma(\xi)\, d\xi}{2\pi(x-\xi)}$ and $\gamma(c)=0$



$$\rightarrow V_\infty \alpha = \int_0^c \frac{\gamma(\xi)\, d\xi}{2\pi(x-\xi)}$$



in our approx., model sym. airfoil as single line, this yields exact sol'n for pot. flow over thin plate

Sol'n method: change of variables

$$\left.\begin{aligned} \xi &= \frac{c}{2}(1-\cos\theta) \\ x &= \frac{c}{2}(1-\cos\theta_0) \end{aligned}\right\} \quad \text{mapping of} \quad \begin{aligned} x &\to \theta_0 \\ \xi &\to \theta \end{aligned}$$

Note change of bounds!!

$\hookrightarrow d\xi = \frac{c}{2}\sin\theta \, d\theta$

$$\rightarrow V_\infty \alpha = \frac{1}{2\pi}\int_0^\pi \frac{\gamma(\theta)\frac{c}{2}\sin\theta\, d\theta}{\frac{c}{2}(1-\cos\theta_0) - \frac{c}{2}(1-\cos\theta)}$$

$$\rightarrow \boxed{V_\infty \alpha = \frac{1}{2\pi}\int_0^\pi \frac{\gamma(\theta)\sin\theta\, d\theta}{(\cos\theta - \cos\theta_0)}}$$

$\hookrightarrow$ integral solver

$$\rightarrow \boxed{\gamma(\theta) = 2\alpha V_\infty \frac{(1+\cos\theta)}{\sin\theta}}$$

At $x=c$, $\theta=\pi \Rightarrow$ div. by zero!

$$\lim_{\theta\to\pi} \frac{1+\cos\theta}{\sin\theta} \approx \frac{\frac{d}{d\theta}(1+\cos\theta)}{\frac{d}{d\theta}(\sin\theta)} = -\frac{\sin\pi}{\cos\pi} = 0 \quad \therefore \text{ Kutta satisfied}$$