Last time: Two-body problem

$\bar{h} = \vec{r} \times \bar{v}$

$\frac{dA}{dt} = \frac{h}{2}$

Trajectory: ① $r(\theta) = \frac{h^2/\mu}{1+e\cos\theta}$

② $r(\theta) = \frac{a(1-e^2)}{1+e\cos\theta}$

$\frac{h^2}{\mu} = a(1-e^2)$

$\mu = GM$

$h = r^2 \frac{d\theta}{dt}$

Period: $T = 2\pi \left(\frac{a^3}{\mu}\right)^{1/2}$

Speed: $v^2 = \mu \left(\frac{2}{r} - \frac{1}{a}\right)$

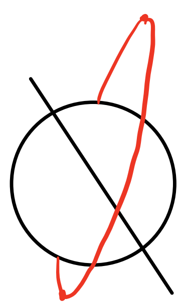
Dir.: $\cos\gamma = \frac{a\sqrt{1-e^2}}{\sqrt{r(2a-r)}}$

$\tan\gamma = \frac{e\sin\theta}{1+e\cos\theta}$

Example 3: Molniya orbit

→ def: large e, large $i \equiv$ ang. of inclination

→ stays near apogee



given: $e = 0.72$, $a = 25{,}200$ km

find: T, altitudes of apogee, perigee, flight path angle γ for $\theta = 90°$ and $r = 30{,}000$ km

1) $T = 2\pi\left(\frac{a^3}{\mu}\right)^{1/2}$, $\mu_E = \mu = 3.986 \times 10^5$ km^3/s^2

$= 2\pi\sqrt{\frac{25200}{3.986E5}} = 39811$ s $= \boxed{11.06 \text{ hr}}$

2) Altitudes

$$alt_p = R_p - R_E = \overbrace{a(1-e)}^{7056} - R_E$$

$$= 25200(1-0.72) - 6378 = \boxed{678 \text{ km}}$$

$$alt_a = R_a - R_E = \overbrace{a(1+e)}^{43,344} - R_E$$

$$= 25200(1+.72) - 6378 = \boxed{36,966 \text{ km}}$$

3) Speeds $\quad v^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right)$

$$\rightarrow v_p = \sqrt{\mu\left(\frac{2}{r_p} - \frac{1}{a}\right)} = 3.986 \times 10^5 \left(\frac{2}{7056} - \frac{1}{25200}\right)$$

$$= \boxed{9.86 \text{ km/s}}$$

sanity check: LEO ~ 7 km/s

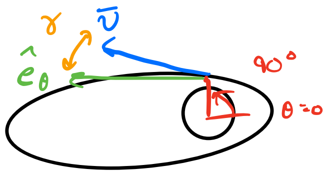
$$v_a = \sqrt{\mu\left(\frac{2}{r_a} - \frac{1}{a}\right)} = \boxed{1.61 \text{ km/s}}$$

4) a) γ when $\theta = 90^\circ$

$$\tan\gamma = \frac{e\sin\theta}{1+e\cos\theta} = e$$

1

0



$$\gamma = \operatorname{atan}(.72) = .624 \text{ rad}$$

$$= 35.8^\circ \ @\ \theta = 90^\circ$$

b) At $r = 30,000$ km, $\gamma = ?$

$$\cos\gamma = \frac{a\sqrt{1-e^2}}{\sqrt{r(2a-r)}} = \frac{(25200)\sqrt{1-.72^2}}{\sqrt{30000(2(25200)-30000)}}$$

$$\cos\gamma = .707$$

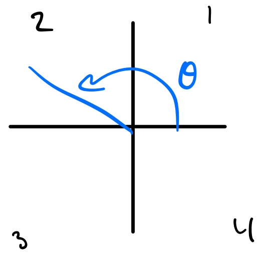
$$\rightarrow \gamma = 45^\circ$$

Find θ w/ $r = 45^\circ$:

use $r(\theta) = \dfrac{a(1-e^2)}{1+e\cos\theta}$ $\rightarrow \cos\theta = \dfrac{a(1-e^2)-r}{re}$

$\cos\theta = -.7221$

$\rightarrow \theta = 136.23^\circ$



Derive Kepler $\rightarrow$ time   Lecture 6

Recall ang. momentum $r^2\dot{\theta} = h = \text{const}$

θ = "true anomaly"

$\therefore \ \dot{\theta} = \dfrac{h}{r^2} = \dfrac{h}{p^2}(1+e\cos\theta)^2$, $p = a(1-e^2)$

$$\frac{d\theta}{dt} = \frac{h}{p^2}(1+e\cos\theta)^2$$

$$\rightarrow \int_0^\theta \frac{d\theta}{(1+e\cos\theta)^2} = \int_{t_p}^{t} \frac{h}{p^2}\,dt$$

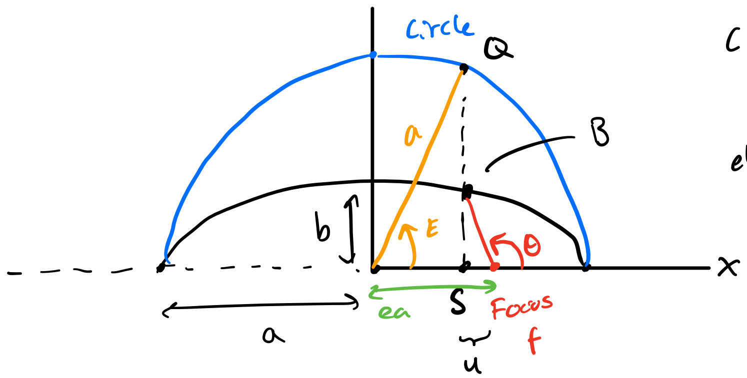
def: $t_p \equiv$ ref. time
usually $t_p = 0$

$$\int_0^\theta \frac{d\theta}{(1+e\cos\theta)^2} = \frac{h}{p^2}(t - t_p)$$

could solve LHS closed form
or numerically

instead let $M \equiv$ "mean anomaly" $= \dfrac{2\pi}{T}t$

idea: circumscribe orbit w/ circle



Circle: $\frac{x_c^2}{a^2} + \frac{y_c^2}{a^2} = 1$

ellipse: $\frac{x_e^2}{a^2} + \frac{y_e^2}{b^2} = 1$

$\therefore \; y_e = \frac{b}{a} \underbrace{y_c}_{a\sin E}$

Define: $E \equiv$ "Eccentric anomaly" for true anomaly θ

then $u = \left| r_{F/o} - r_{S/o} \right| = ea - a\cos(E)$, $y_e = \frac{b}{a}(a\sin E)$

$\rightarrow u = ae - a\cos E$, $y = b\sin E$

1) Take $\frac{d}{dt}$ $\quad \dot{u} = a\sin E\,\dot{E}$, $\quad \dot{y} = b\cos E\,\dot{E}$ $\quad \vec{v} = \dot{u}\,\hat{i} + \dot{y}\,\hat{j}$

2) Relate in h, specifying momentum, $\vec{r} \times \vec{v} = \vec{h}$

magnitude w/ ar $\vec{v}$ is $\vec{h} = y\dot{u} - u\dot{y}$

$\therefore \; h = (b\sin E)(a\sin E\,\dot{E}) - (ae - a\cos E)(b\cos E\,\dot{E})$

$\therefore \; h = ab\dot{E}(1 - e\cos E)$, $\quad \dot{E} = \frac{dE}{dt}$

tie back to trajectory: $\begin{cases} h^2 = \mu a(1-e^2) \\ b = a(1-e^2)^{1/2} \end{cases}$

$$\therefore \; \left(\frac{\mu}{a^3}\right)^{1/2} = \frac{dE}{dt}(1 - e\cos E)$$

Separate variables

$$\int_{t_p}^{t} \left(\frac{\mu}{a^3}\right)^{1/2} dt = \int_0^{E} (1 - \cos E)\, dE$$

$$\left(\mu/a^3\right)^{1/2}(t - t_p) = E - e\sin E$$

Assume: $t_p = 0$ at P

let $n \equiv \frac{2\pi}{T} = \left(\frac{\mu}{a^3}\right)^{1/2}$, $n \equiv$ "mean angular rate"

$\therefore\ M = nt \equiv$ "mean anomaly"

$\therefore\ \boxed{M = E - e\sin E}$ Kepler's Eqn

Functional form: $\left(\frac{\mu}{a^3}\right)^{1/2} t = E - e\sin E$

Remarks:

1) solving: 2 ways

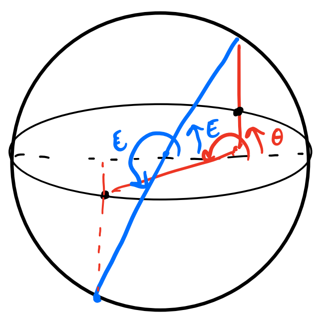
1) given E → calc. t, time since t_p, plug into RHS

2) given t since t_p, calc E

→ plug into LHS, find E

1) numerically iterating

2) E lags θ in quadrant 1 & 2 & leads θ in Q 3 & 4



3) Need θ true anomaly (physical position)

either
$$\begin{cases} \cos E = \dfrac{e + \cos\theta}{1 + e\cos\theta} & \text{careful w/ quadrant!} \\ \tan\left(\dfrac{E}{2}\right) = \left(\dfrac{1-e}{1+e}\right)^{1/2} \tan\dfrac{\theta}{2} \end{cases}$$

4) $r = a(1 - e\cos E)$ another relationship