Frequency response

- Bode plot: Frequency response
- Magnitude ratio over range of forcing frequencies

$$G(s) = \frac{(s-z_1)(s-z_2)(s-z_m)}{(s-p_1)(s-p_2)(s-p_n)}$$

$$G(j\omega) = \left[\frac{(s-z_1)(s-z_2)(s-z_m)}{(s-p_1)(s-p_2)(s-p_n)}\right]_{s=j\omega}$$

Polar: $$(s-r_i)_{s=j\omega} = M(\underbrace{\cos\varphi + j\sin\varphi}_{e^{j\varphi} = \cos\varphi + j\sin\varphi})$$

$$= Me^{j\varphi}$$

$$\rightarrow G(j\omega) = \frac{B_1 e^{j\psi_1} \; B_2 e^{j\psi_2} \ldots}{A_1 e^{j\varphi_1} \; A_2 e^{j\varphi_2} \ldots}$$

$$\rightarrow G(j\omega) = \left[\frac{B_1 B_2 \ldots}{A_1 A_2 \ldots}\right] e^{j[(\psi_1 + \psi_2 \ldots) - (\varphi_1 + \varphi_2 \ldots)]}$$

Pole-zero factorization

$$G(s) = K \frac{(s-z_1)(s-z_2)\ldots}{(s-p_1)(s-p_2)\ldots}$$

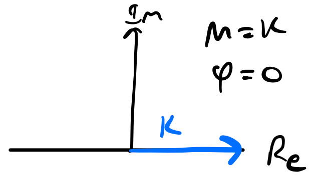
p_1, p_2, p_3 poles

z_1, z_2, z_3 zeros

$$G(s) = K \frac{(s-z_1)(s-z_2)\ldots}{(s-p_1)(s-p_2)\ldots} = K_0 \underbrace{\frac{(\tau_{z_1}s+1)(\tau_{z_2}s+1)\ldots}{(\tau_{p_1}s+1)(\tau_{p_2}s+1)\ldots}}_{\text{Bode form}}$$

Simple gain: K

Frequency response of K: $\Bigg\} \rightarrow \begin{array}{l} G(s) = K \\ G(j\omega) = K \end{array}$



$M = K$
$\varphi = 0$

Magnitude ratio (gain)

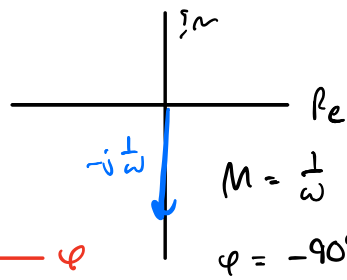
$$M = |G(j\omega)| = K$$

phase:

$$\varphi = \angle G(j\omega) = 0^\circ$$

Integrator: $\frac{1}{s}$: Frequency response: $G(s) = \frac{1}{s}$

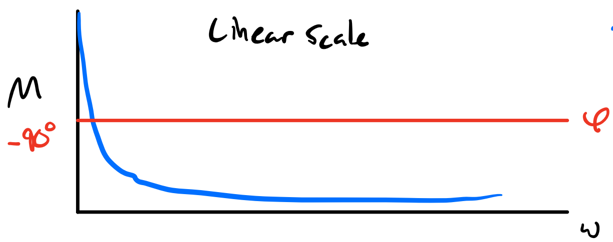
$$G(j\omega) = \frac{1}{j\omega} \cdot \frac{j}{j} = -\frac{1}{\omega} j$$



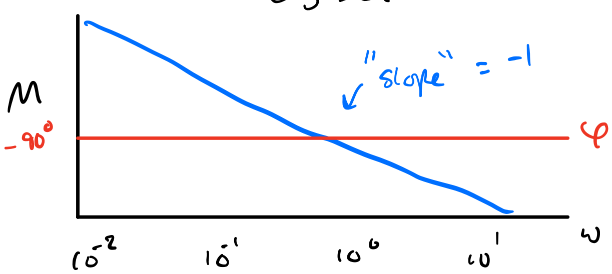
$M = \frac{1}{\omega}$

$\varphi = -90^\circ$

Linear Scale



Log scale



Decibel - alternative magnitude unit

the ratio $\frac{B}{A} \rightarrow 20 \log_{10} \frac{B}{A}$

$M(\omega)$	$m(\omega)$
0.1	-20 db
1	0 db
10	+20 db
100	+40 db

± 1 slope on log-log scale

$\updownarrow$

± 20 db / decade

Frequency response: $\frac{1}{s^n}$ $\rightarrow G(s) = \frac{1}{s^n} = \prod_{1}^{n} \frac{1}{s}$

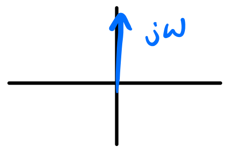
For $\frac{1}{s}$: $M = \frac{1}{\omega}$ $M = |G(j\omega)| = \prod_{1}^{n} \frac{1}{\omega} = \frac{1}{\omega^n}$

$\log M = -\log \omega$ $\log M = -n \log \omega$

phase:

$\varphi = \angle G(j\omega) = -90^\circ$ $\varphi = -n(90)^\circ$

Frequency response: s $G(s) = s$ $G(j\omega) = j\omega$



$M = \omega$

$\varphi = 90^\circ$

Multiple differentiation s^n

$M = \omega$ $M = \omega^n$

For s: For s^n:

$\log M = \log \omega$ $\log M = n \log \omega$

$\varphi = 90^\circ$ $\varphi = n(90)^\circ$

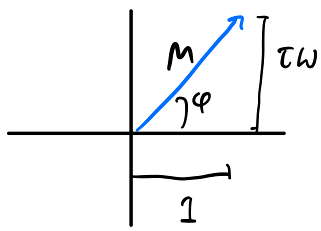
Frequency response: 1st order: $G(s) = \tau s + 1$ (Bode form)

$G(j\omega) = 1 + \tau \omega j$

$M = \sqrt{1 + \tau^2 \omega^2}$

$\log M = \frac{1}{2} \log(1 + \tau^2 \omega^2)$

$\varphi = \tan^{-1}(\omega \tau)$



Low frequency response: $\frac{1}{2}\log(1) = 0 \rightarrow M = 1$

$\varphi = 0$

High freq.: $M \approx \underbrace{\log(\tau)}_{\text{const.}} + \underbrace{\log(\omega)}_{\text{+1 slope}}$

$\varphi = \tan^{-1}(\omega\tau) \approx \tan^{-1}(\infty) = 90^\circ$