Problem 1

TWR β_0 $\quad J = \text{const.}$ $\quad D = 0, \; g = g_e$

$u_e = u_{eq}$. $\quad \alpha = \frac{M_p}{M_o}$

(a) Show that for $\dot{m} = \text{const}$, $t_b = \frac{I_{sp}}{\beta_0}\frac{M_p}{M_o}$

$$I_{sp} = \frac{I}{M_p g_e} = \frac{u_{eq}}{g_e}, \qquad \beta = \frac{J}{M_o g_e} = \frac{\dot{m} u_{eq}}{M_o g_e}$$

$$\Delta u(t_b) = u_{eq} \ln R$$

Start from t_b expression:

$$t_b = \frac{M_p}{\dot{m}} \qquad \dot{m} = \frac{\beta_0 M_o g_e}{u_{eq}}$$

$$\rightarrow t_b = \frac{M_p u_{eq}}{\beta_0 M_o g_e} = \boxed{\frac{I_{sp}}{\beta_0} \cdot \frac{M_p}{M_o} = t_b}$$

$\frac{u_{eq}}{g_e} = I_{sp}$

(b) use a) to show that $\frac{\Delta u}{u_e} = \frac{\Delta u}{g_e I_{sp}} = -\left[\ln\left(1 - \frac{M_p}{M_o}\right) + \frac{1}{\beta_0}\frac{M_p}{M_o}\right]$

$\frac{I_{sp}}{\beta_0}\frac{M_p}{M_o} = t_b$ $\quad$ Δu eqn: $\Delta u(t) = u_{eq} \ln \frac{M_o}{M(t)} - g t$

$$\Delta u(t) = u_e \ln \frac{M_o}{M_o - M_p} - g t \rightarrow \text{sub } t_b \text{ eqn}$$

$$\Delta u(t_b) = u_e \ln \frac{M_o}{M_o - M_p} - g_e \frac{I_{sp}}{\beta_0}\frac{M_p}{M_o} \rightarrow \text{sub } u_e = g_e I_{sp}$$

$$\Delta u = g_e \cdot I_{sp}\left[\ln\left(1 - \frac{M_p}{M_o}\right)^{-1} - \frac{1}{\beta_0}\frac{M_p}{M_o}\right]$$

$$\boxed{\frac{\Delta u}{g_e I_{sp}} = -\left[\ln\left(1 - \frac{M_p}{M_o}\right) + \frac{1}{\beta_0}\frac{M_p}{M_o}\right]}$$

(c) Plot 1b) using $\frac{M_p}{M_o}$ as variable & β_0 parameter.

→ See figure 1

(d) Discuss plot, $\beta_0 < 1$ region, effect of $\frac{M_p}{M_o}$ & β_0

For $\beta_0 < 1$, the Δu curves do not have a positive $\frac{\Delta u}{g_e I_{sp}}$ until higher $\frac{M_p}{M_o}$. This makes sense as for $\beta_0 < 0$, the rocket will not begin accelerating until enough propellant mass has reduced M_o such that $\beta(t) > 1$

For all β_0, $\frac{\Delta u}{g_e I_{sp}}$ increases as M_p/M_0 increases, and faster for larger β_0. Logically, the higher initial TWR results in greater acceleration and a "head-start" in Δu.

Problem 2

$a_{max} \to \beta_{max} \equiv \frac{a_{max}}{g_e}$. $D = 0$, $g = g_e$, $u_{eq} = u_e$

2a) Write two a_{max} in terms of thrust and β_{max}, then relate β_0 & β_{max} in terms of $\alpha = \frac{M_p}{M_0}$

$$\Sigma F = ma = J - mg$$

$$\to a = \frac{J}{m} - g_e$$

$$a_{max} = \frac{J \text{ (const)}}{M_{min}} - g_e = \frac{J}{M_b} - g_e = \boxed{\frac{J}{M_0 - M_p} - g_e = a_{max}}$$

$$\boxed{a_{max} = \beta_{max} \cdot g_e}$$

$$\to \beta_{max} g_e = \frac{J}{M_0 - M_p} - g_e$$

$\beta_0 = \frac{J}{M_0 g_e}$

$\to J = \beta_0 M_0 g_e$

$$\cancel{g_e}(\beta_{max} + 1) = \frac{\beta_0 M_0 \cancel{g_e}}{M_0 - M_p}$$

$$(\beta_{max} + 1) = \frac{M_0 \beta_0}{M_0 - M_p}$$

$$\boxed{(\beta_{max} + 1)\left(1 - \frac{M_p}{M_0}\right) = \beta_0}$$

$\to$ plug into (b)

2b)

$$\to \boxed{\frac{\Delta u}{g_e I_{sp}} = -\left[\ln\left(1 - \frac{M_p}{M_0}\right) + \frac{\frac{M_p}{M_0}}{(\beta_{max}+1)\left(1 - \frac{M_p}{M_0}\right)}\right]}$$

2c) See figure 2

2d) Find $\left(\frac{M_p}{M_0}\right)_{max}$ to maximize $\frac{\Delta u}{g_e I_{sp}}$ in terms of β_{max}

maximize

$$\frac{\partial}{\partial \alpha}\frac{\Delta u}{g_e I_{sp}} = \frac{\partial}{\partial \alpha}\left[-\ln(1-\alpha) - \frac{\alpha}{(\beta_{max}+1)(1-\alpha)}\right] = 0$$

$$= \cancel{-}\frac{1}{(1-\alpha)}\cdot(\cancel{-}1) - \frac{\alpha\cdot(-\beta_{max}-1)-(\beta_{max}+1)(1-\alpha)\cdot 1}{[(\beta_{max}+1)(1-\alpha)]^2} = 0$$

→ matlab syms diff solve :

$$\boxed{\alpha_{max} = \frac{\beta_{max}}{\beta_{max}+1} = \left(\frac{M_p}{M_0}\right)_{max}}$$

2e) $\frac{\Delta u}{g_e I_{sp}}$ goes up as M_p/M_0 increases, until it begins to be limited by β_{max}. A high M_p/M_0 results in high Δu, but also high β at end of burn, so it needs a high β_{max} to reach larger Δu values.

Problem 3 Single stage, $D = g = 0$, $u_e = u_{eq}$

3a) Find $\frac{M_p}{M_L}$ in terms of mass ratio $R = \frac{M_0}{M_b}$ and payload ratio $\lambda = \frac{M_L}{M_0 - M_L} = \frac{M_L}{M_p + M_s}$

$$\to \boxed{M_L = \lambda(M_0 - M_L)}$$

$$R = \frac{M_0}{M_0 - M_p} = \frac{M_0}{M_b} \to M_0 R - M_p R = M_0$$

$$M_0(R-1) = M_p R$$

$$\boxed{\to M_p = \frac{M_0(R-1)}{R}}$$

$$\rightarrow \frac{M_p}{M_\ell} = \frac{M_0 (R-1)/R}{\lambda (M_0 - M_\ell)}$$

$$\boxed{M_0 = M_\ell + M_p + M_s}$$

$$\boxed{M_0 - M_\ell = M_p + M_s}$$

$$M_b = M_\ell + M_s$$

$$= \frac{M_0 (R-1)}{R\lambda (M_p + M_s)}$$

$$= \frac{(R-1)(M_\ell + M_p + M_s)}{R\lambda (M_p + M_s)} = \frac{(R-1) M_\ell}{R\lambda (M_p + M_s)} + \frac{(R-1)\cancel{(M_p+M_s)}}{R\lambda \cancel{(M_p+M_s)}}$$

λ

$$= \frac{(R-1)\lambda}{R\lambda} + \frac{(R-1)}{R\lambda} = \boxed{\frac{R-1}{R\lambda}(\lambda + 1)} = \frac{M_p}{M_\ell}$$

$$= \frac{R\lambda}{R\lambda} - \frac{\lambda}{R\lambda} + \frac{R-1}{R\lambda}$$

$$= 1 - \frac{1}{R} + \frac{1}{\lambda} - \frac{1}{R\lambda}$$

$$\boxed{\frac{M_p}{M_\ell} = 1 - \frac{1}{R\lambda} - \frac{1}{R} + \frac{1}{\lambda}}$$

3b) Find ε in terms of R & λ

$$R = \frac{1+\lambda}{\varepsilon + \lambda} \qquad (\varepsilon + \lambda) R = 1 + \lambda$$

$$\rightarrow \boxed{\varepsilon = \frac{1+\lambda}{R} - \lambda}$$

3c)

$$\Delta u = u_e \ln R \rightarrow \frac{\Delta u}{u_e} = \ln R$$

$$\rightarrow R = e^{\Delta u / u_e}$$

$$\rightarrow \boxed{\frac{M_p}{M_e} = 1 - \frac{1}{\lambda} e^{-\Delta u/u_e} - e^{-\Delta u/u_e} + \frac{1}{\lambda}}$$

$$\rightarrow \boxed{\varepsilon = (1+\lambda) e^{-\Delta u/u_e} - \lambda}$$

3d) See figure 3.

It appears unphysical because as $\varepsilon \to 0$, the mass of the structure goes to zero, which is extremely unrealistic since as $M_p \uparrow$, realistically M_s will increase as well. A near-weightless rocket (besides payload & prop.) would be able to reach insane ΔV, but it's totally unphysical.

3e) Analytically determine the (limit) max value of $\frac{\Delta u}{u_e}$ for any λ & corresponding $\frac{M_p}{M_e}$

$$\Delta u = u_e \ln(R)$$

$$\rightarrow \frac{\Delta u}{u_e} = \ln(R) \qquad \text{Max } \frac{\Delta u}{u_e} \text{ @ max } R$$

$$R = \frac{1+\lambda}{\varepsilon + \lambda} \qquad \text{max } R \text{ @ } \varepsilon = 0$$

From: $$\frac{M_p}{M_e} = 1 - \frac{1}{\lambda} e^{-\Delta u/u_e} - e^{-\Delta u/u_e} + \frac{1}{\lambda}$$

$$\frac{M_p}{M_e} \lambda = \lambda - e^{-\Delta u/u_e} - \lambda e^{-\Delta u/u_e} + 1$$

$$= \underbrace{\lambda - (1+\lambda)\, e^{-\Delta u / u_e}}_{= -\varepsilon} + 1$$

$$\frac{M_p}{M_e}\lambda = 1 - \varepsilon \rightarrow \frac{M_p}{M_e} = \frac{1 - \cancel{\varepsilon}^{\,0}}{\lambda}$$

$$\rightarrow \boxed{\frac{M_p}{M_e} = \frac{1}{\lambda}}$$

$$\frac{\Delta u}{u_e} = \ln R = \ln\left(\frac{1+\lambda}{\cancel{\varepsilon}_{\,0} + \lambda}\right)$$

$$\rightarrow \boxed{\frac{\Delta u}{u_e} = \ln\left(\frac{1+\lambda}{\lambda}\right)}$$