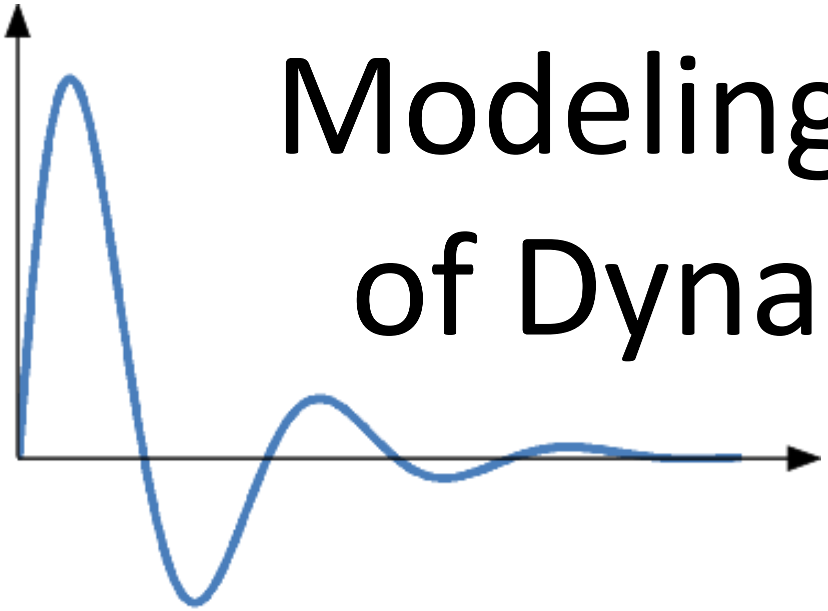
Modeling and Analysis of Dynamic Systems



Modeling & Analysis of Dynamic Systems

- Topic Overview -

- Dynamic System Modeling 2.1 – 2.4
- Dynamic Systems – Differential Equations 3.1 – 3.6
- Laplace Transform and Linear Differential Equations 3.1
- Transfer Functions 3.1.2
- Block Diagram Modeling 3.2.1 – 3.2.2
- Linear System Response Characteristics 3.3
- Time Domain Specifications 3.4
- Effects of System Poles and Zeros 3.5

ME 340 Review

Note: We will cover material very quickly. Important to carefully review material yourself (**FPE** Chapters 2 and 3)

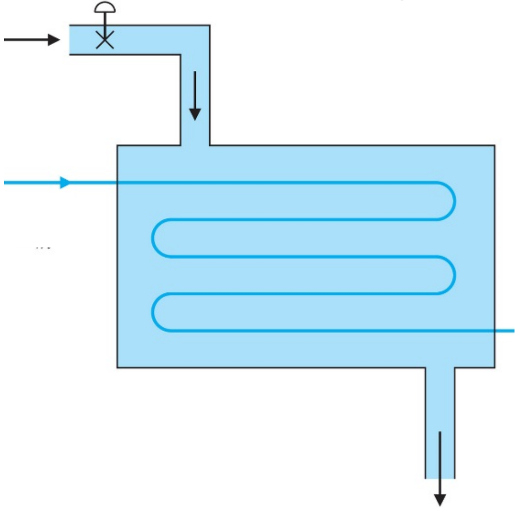
Dynamic System Modeling

2.1-2.4

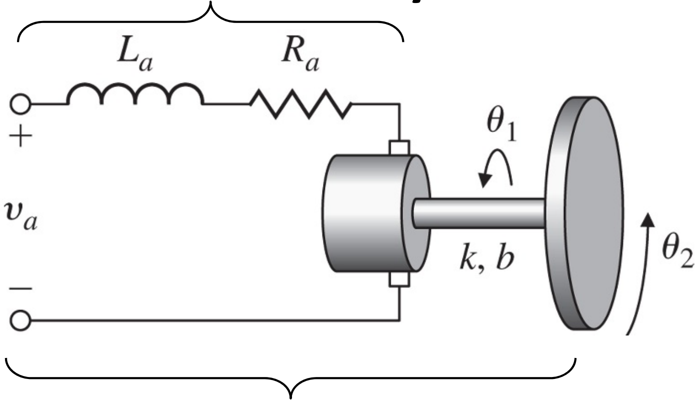
LUMPED PARAMETER MODEL

- Probably the **most important step** in design
- Control **approach** is a strong **function of system** characteristics – reflected in modeling
- In this class, will **spend** little to **no time** on **modeling**
- Please review background when necessary (e.g. ME 340)

Fluid / thermal systems

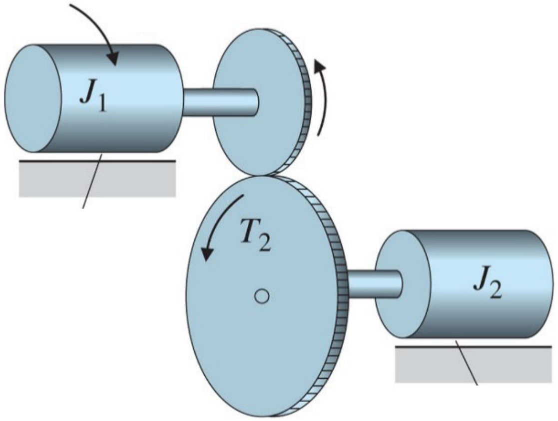


Electrical Systems



Hybrid Systems

Mechanical Systems



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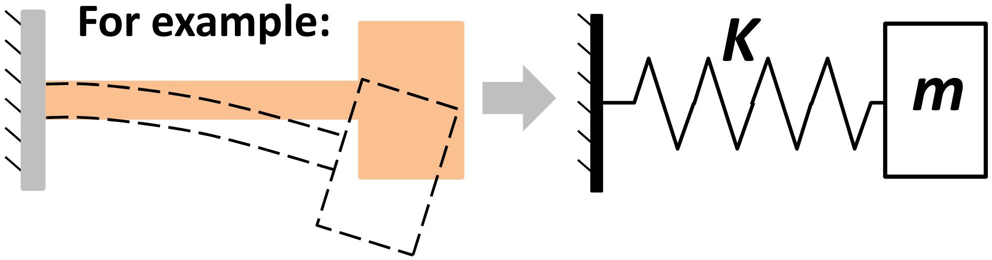
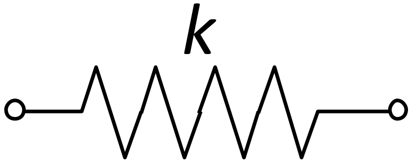
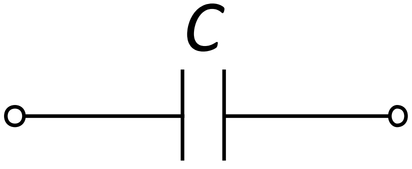
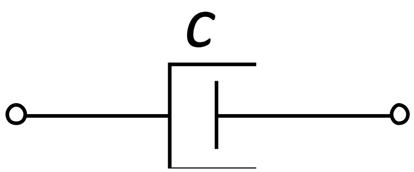
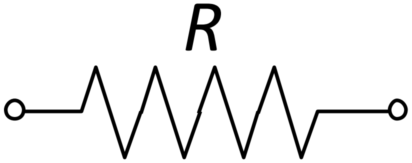
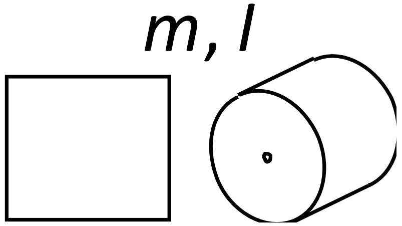
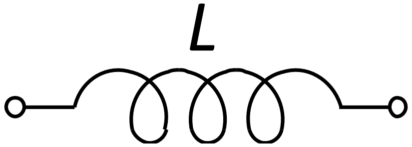
Dynamic Systems – Differential Equations

- We use **lumped parameter modeling**

For example:



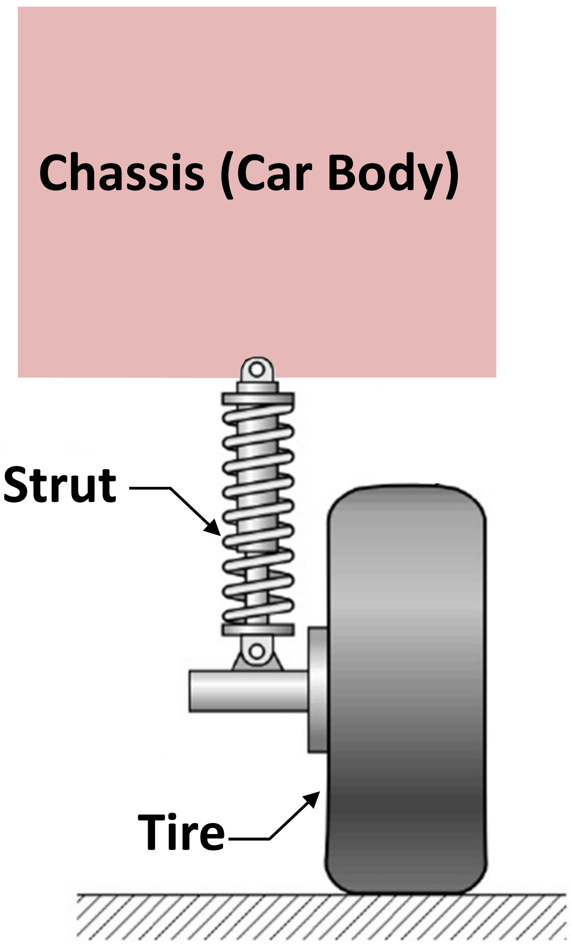
Mechanical	Electrical	Fluid	Thermal
 $F = kx$	 $i = C\frac{dv}{dt}$ $v = \frac{1}{C}\int i$	C	C
 $F = c\frac{dx}{dt}$	 $v = iR$	R	R
 $F = m\frac{d^2x}{dt^2}$ $M = I\frac{d^2\theta}{dt^2}$	 $v = L\frac{di}{dt}$	L	Not applicable

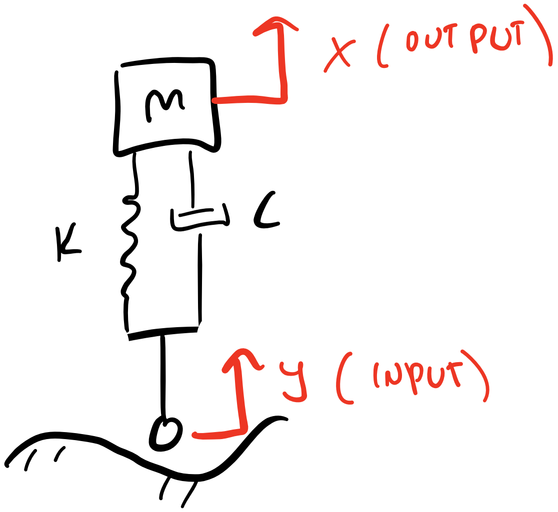
- These models are described by differential equations
- For **linear systems**, their dynamics are governed by **linear (constant coefficient) differential equations**

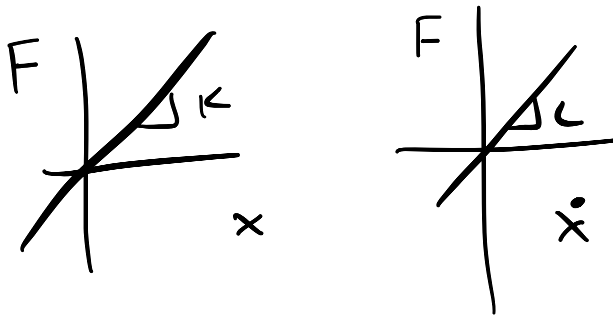
Dynamic Systems – Differential Equations

Example: Develop a model and the associated equations of motion for the mechanical system shown below

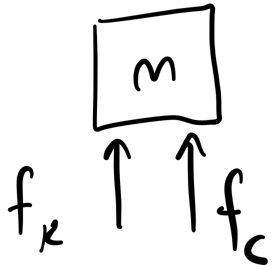
Equation of motion: - 2ND LAW, FBD/KD OR EULER-LAGRANGE







$F = ma$



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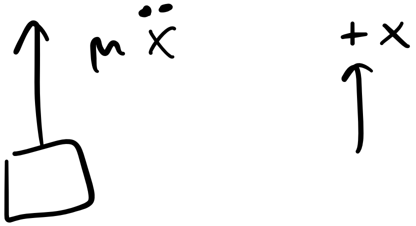


$f_k + f_c = m\ddot{x}$

$f_k = K(y-x)$, $f_c = c(\dot{y}-\dot{x})$

$$f_k + f_c = m\ddot{x}$$

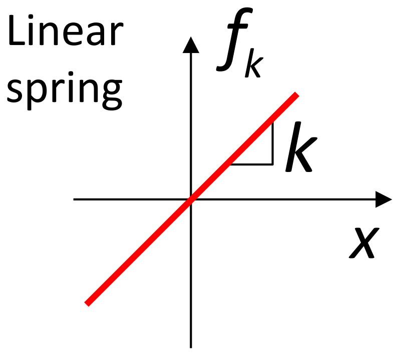
$$\boxed{m\ddot{x} + kx + c\dot{x} = ky + c\dot{y}}$$ EOM

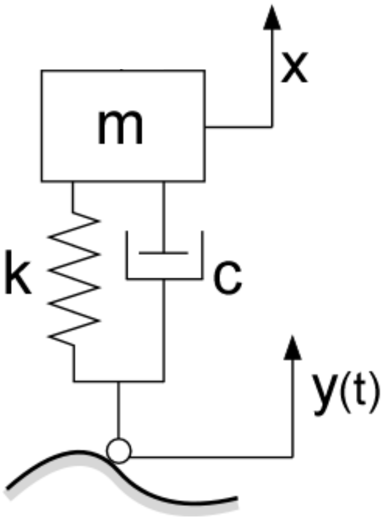
LIN. CONST. COEFF. ORD. DIFF. EQ.

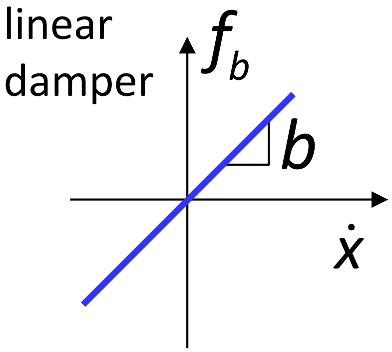
Dynamic Systems – Differential Equations

Example: continued

Mechanical Dynamics Lumped Parameter Model:



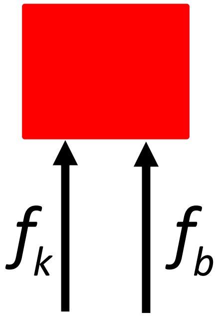




Newton's 2nd Law:

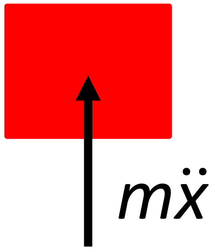
$$\sum F = m\ddot{x}$$

Free Body Diagram:



$=$

Kinetic Diagram:



Spring force on mass:

$$f_k = k(y - x)$$

Damper force on mass:

$$f_b = b(\dot{y} - \dot{x})$$

$$f_k + f_b = m\ddot{x}$$

$$k(y - x) + b(\dot{y} - \dot{x}) = m\ddot{x}$$

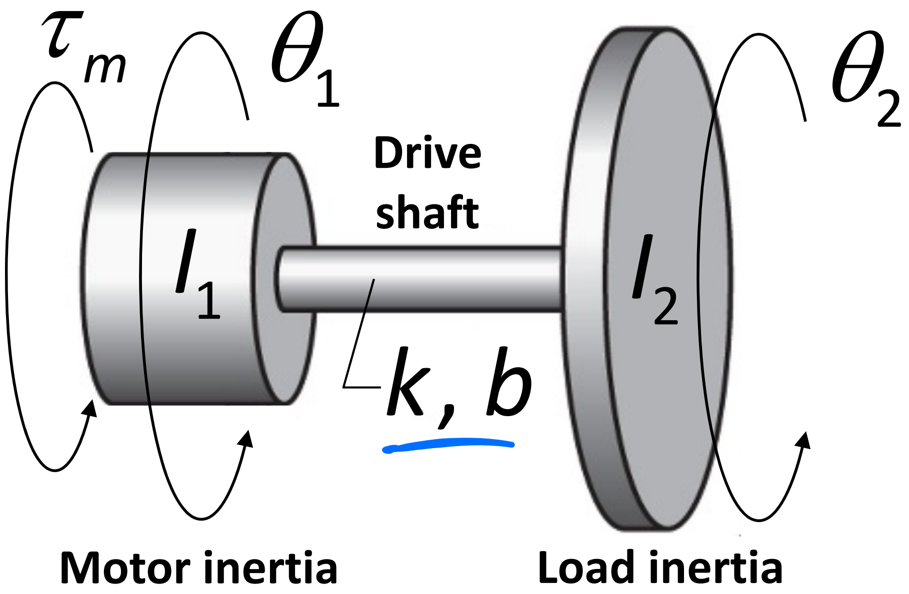
Equation of motion: $\boxed{m\ddot{x} + b\dot{x} + kx = b\dot{y} + ky}$

Linear, constant coefficient differential equation (for a linear, time invariant system)

Dynamic Systems – Differential Equations

Example: Develop a model and the associated equations of motion for the mechanical system shown below

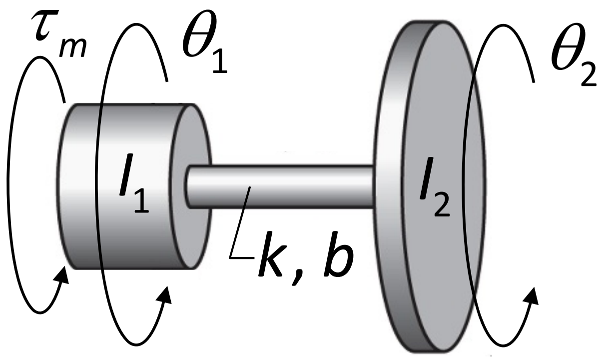
K, b : COMPLIANCE, DAMPING

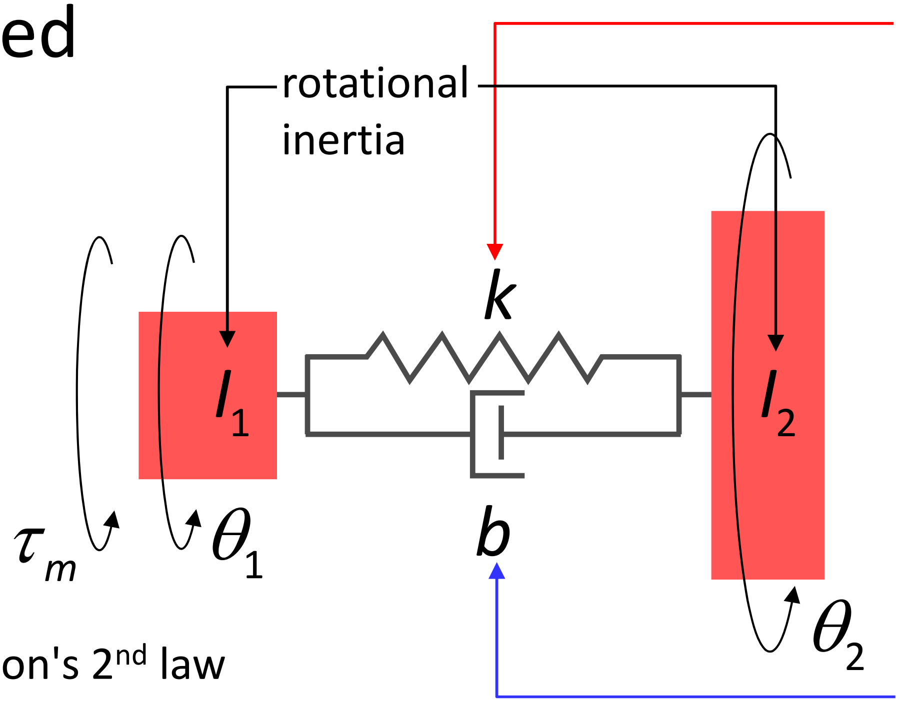


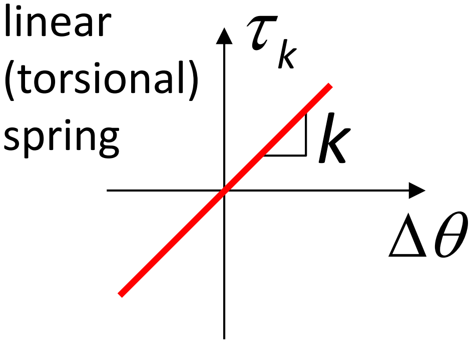
Dynamic Systems – Differential Equations

Example: continued

Mechanical Dynamics Lumped Parameter Model:

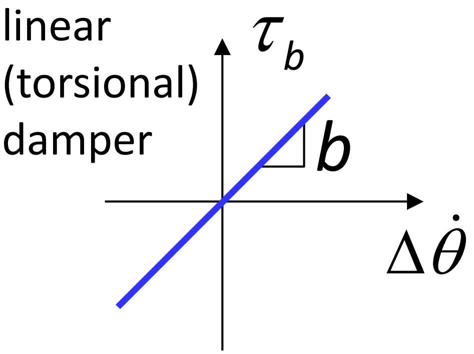




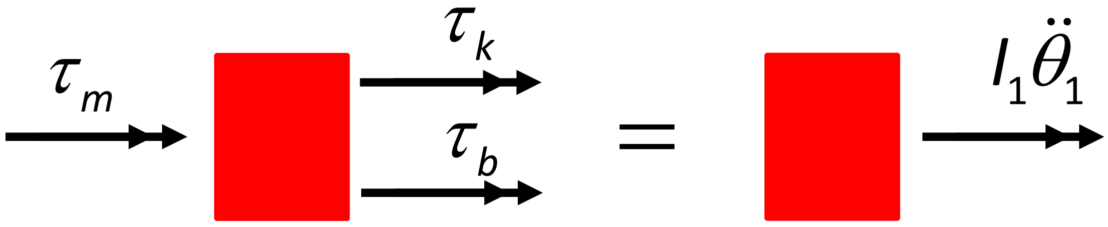


$$\sum \tau = I\alpha$$

Newton's 2nd law (rotation about fixed axis)



Inertia, I_1:



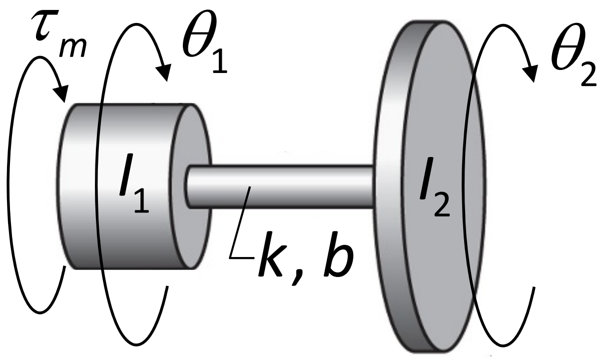
$$\tau_m + \underbrace{k\left(\theta_2 - \theta_1\right)}_{\tau_k} + \underbrace{b\left(\dot{\theta}_2 - \dot{\theta}_1\right)}_{\tau_b} = I_1\ddot{\theta}_1$$

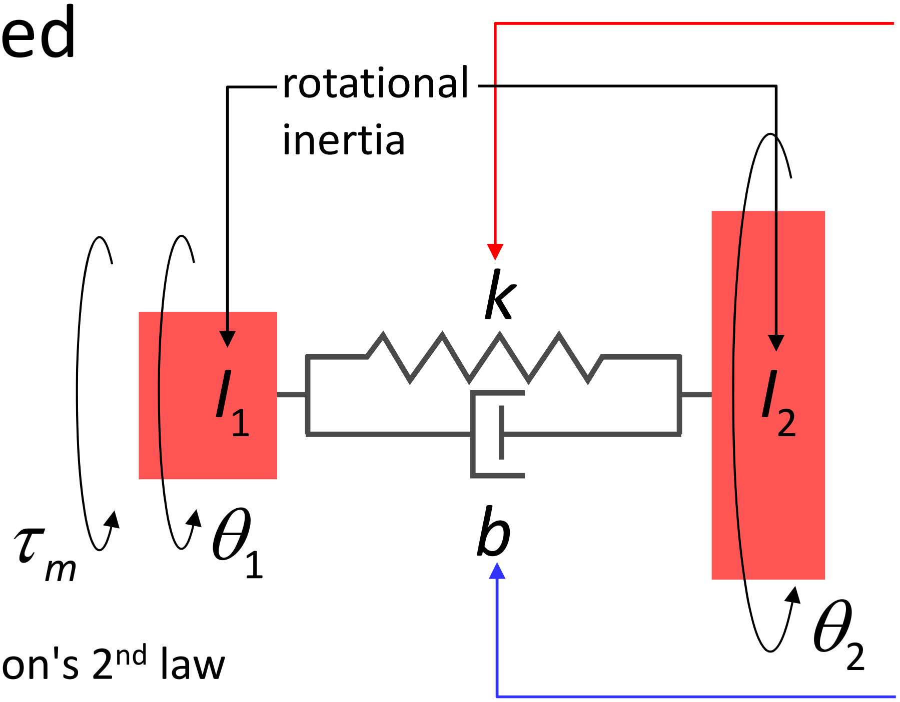
Differential Equations: $\boxed{I_1\ddot{\theta}_1 + b\dot{\theta}_1 - b\dot{\theta}_2 + k\theta_1 - k\theta_2 = \tau_m}$

Dynamic Systems – Differential Equations

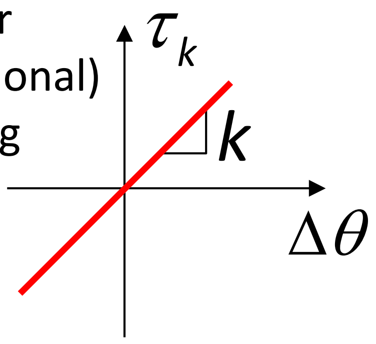
Example: continued

Mechanical Dynamics Lumped Parameter Model:

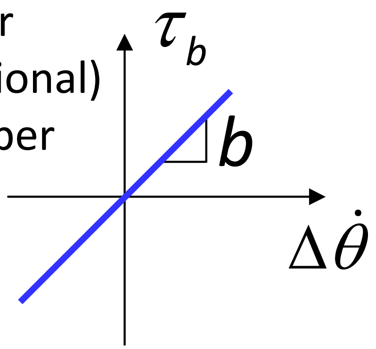




linear (torsional) spring



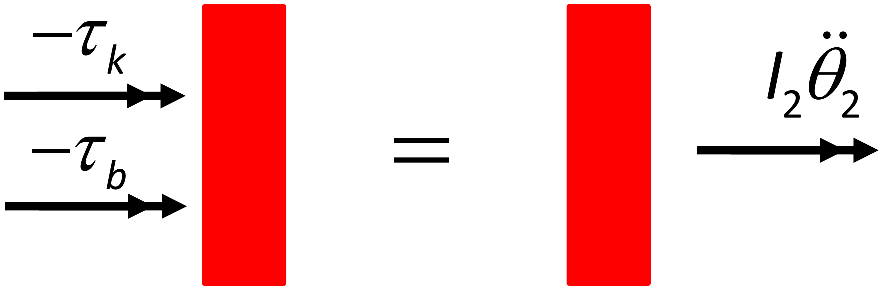
linear (torsional) damper



$$\sum \tau = I\alpha$$

Newton's 2nd law (rotation about fixed axis)

Inertia, I_2:



$$-\underbrace{k\left(\theta_2-\theta_1\right)}_{-\tau_k}-\underbrace{b\left(\dot{\theta}_2-\dot{\theta}_1\right)}_{-\tau_b}=I_2\ddot{\theta}_2$$

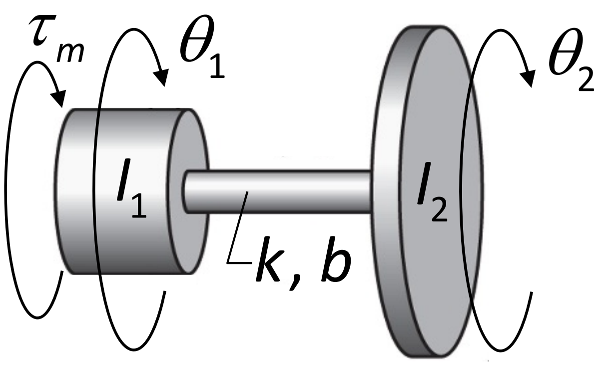
Differential Equations:

$$\boxed{I_2\ddot{\theta}_2-b\dot{\theta}_1+b\dot{\theta}_2-k\theta_1+k\theta_2=0}$$

Dynamic Systems – Differential Equations

Example: continued

Mechanical Dynamics
Lumped Parameter Model:



Equations of motion:

$$I_1\ddot{\theta}_1 + b\dot{\theta}_1 - b\dot{\theta}_2 + k\theta_1 - k\theta_2 = \tau_m$$

$$I_2\ddot{\theta}_2 - b\dot{\theta}_1 + b\dot{\theta}_2 - k\theta_1 + k\theta_2 = 0$$

Note: number of equations = number of degrees of freedom

Equations of motion: System of linear (constant coefficient) differential equations

- Dependent variables on left-side
- Independent (input) variables on right-side

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Laplace Transform

Why Laplace transform? $\mathscr{L}\left[f(t)\right]=\int_{-\infty}^{+\infty}f(t)e^{-st}dt$

- To find the solution of initial value problems of linear differential equations
- Mathematical basis for majority of control system analysis and design techniques
  (e.g. root locus, frequency domain analysis)

Laplace Transform: Definition

Definition: $\mathscr{L}\left[f(t)\right]=\int_{-\infty}^{+\infty} f(t)e^{-st}dt$

One-sided (or *unilateral*) Laplace transform

$$\mathscr{L}_{-}\left[f(t)\right]=F(s)=\int_{0-}^{\infty} f(t)e^{-st}dt$$

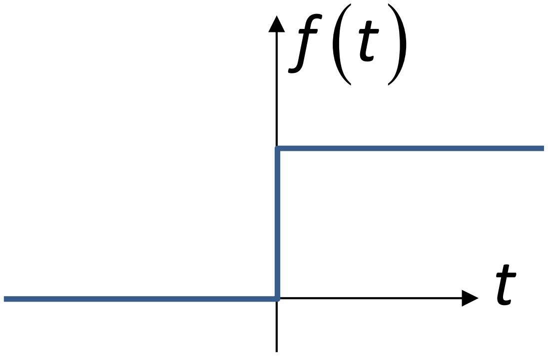
$\mathscr{L}\left[f(t)\right]\Bigg\{$

- Function of *s*
- Not all functions have Laplace transform. Integral must converge (e.g. $f(t)\neq e^{t^2}$)

Laplace Transform

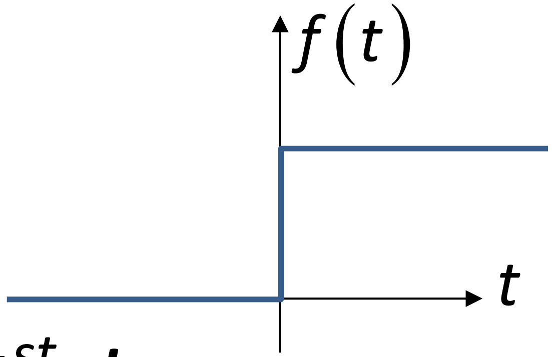
Example: Evaluate the Laplace transform of $f\left(t\right)=1\left(t\right)$ for $t\geq 0$

$$F\left(s\right)=\int_{0-}^{\infty}f\left(t\right)e^{-st}dt$$



Laplace Transform

Example: Evaluate the Laplace transform of $f(t)=1(t)$ for $t\geq 0$



$$F(s)=\int_{0-}^{\infty} f(t)e^{-st}dt$$

$$F(s)=\int_{0-}^{\infty} 1(t)e^{-st}dt=\int_{0-}^{\infty} e^{-st}dt$$

$$F(s)=-\tfrac{1}{s}e^{-st}\Big|_{t=0-}^{t=\infty}$$

$$F(s)=-\tfrac{1}{s}\left[e^{-\infty}-e^{0}\right]=-\tfrac{1}{s}[0-1]$$

$$F(s)=\tfrac{1}{s} \longleftrightarrow f(t)=1(t)$$

Laplace transform pair

Laplace Transform

Example: Evaluate the Laplace transform of $f\left(t\right)=e^{-at}$ for $t\geq 0$

$$F\left(s\right)=\int_{0-}^{\infty} f\left(t\right)e^{-st}dt$$

$$\vdots$$

$$F\left(s\right)=\frac{1}{\left(s+a\right)} \xleftrightarrow{\textbf{Laplace transform pair}} f\left(t\right)=e^{-at}$$

Laplace Transform

example: Evaluate the Laplace transform of $f(t)=e^{-at}$ for $t \geq 0$

$$F(s)=\int_{0-}^{\infty} f(t)e^{-st}dt$$

$$=\int_{0-}^{+\infty} e^{-at}e^{-st}dt$$

$$=\int_{0-}^{\infty} e^{-(s+a)t}dt$$

$$=-\frac{1}{(s+a)}e^{-(s+a)t}\Bigg|_{t=0-}^{t=\infty}$$

$$=-\frac{1}{(s+a)}\left[e^{-(s+a)\infty}-e^{-(s+a)0}\right]\Bigg|$$

$$F(s)=\frac{1}{(s+a)} \underset{\textit{Laplace transform pair}}{\longleftrightarrow} f(t)=e^{-at}$$

Laplace Transform Table (FPE 7 Ed.)

Number	$F(s)$	$f(t), t \geq 0$	
1	1	$\delta(t)$	unit *impulse*
2	$1/s$	$1(t)$	unit *step*
3	$1/s^2$	t	unit *ramp*
4	$2!/s^3$	t^2	unit *acceleration*
5	$3!/s^4$	t^3	
6	$m!/s^{m+1}$	t^m	
7	$\dfrac{1}{s+a}$	e^{-at}	exponential
8	$\dfrac{1}{(s+a)^2}$	te^{-at}	
9	$\dfrac{1}{(s+a)^3}$	$\dfrac{1}{2!}t^2e^{-at}$	
10	$\dfrac{1}{(s+a)^m}$	$\dfrac{1}{(m-1)!}t^{m-1}e^{-at}$	
11	$\dfrac{a}{s(s+a)}$	$1-e^{-at}$	

Laplace Transform Table (FPE 7 Ed.)

Number	$F(s)$	$f(t), t \geq 0$
12	$\dfrac{a}{s^2(s+a)}$	$\dfrac{1}{a}(at - 1 + e^{-at})$
13	$\dfrac{b-a}{(s+a)(s+b)}$	$e^{-at} - e^{-bt}$
14	$\dfrac{s}{(s+a)^2}$	$(1 - at)e^{-at}$
15	$\dfrac{a^2}{s(s+a)^2}$	$1 - e^{-at}(1 + at)$
16	$\dfrac{(b-a)s}{(s+a)(s+b)}$	$be^{-bt} - ae^{-at}$
17	$\dfrac{a}{s^2+a^2}$	$\sin at$
18	$\dfrac{s}{s^2+a^2}$	$\cos at$
19	$\dfrac{s+a}{(s+a)^2+b^2}$	$e^{-at}\cos bt$
20	$\dfrac{b}{(s+a)^2+b^2}$	$e^{-at}\sin bt$
21	$\dfrac{a^2+b^2}{s[(s+a)^2+b^2]}$	$1 - e^{-at}\left(\cos bt + \dfrac{a}{b}\sin bt\right)$

sinusoidal terms

exponentially modulated sinusoidal terms

Laplace Transform

Laplace transform definition $\mathscr{L}\left[f(t)\right]=\int_{-\infty}^{+\infty} f(t)e^{-st}dt$

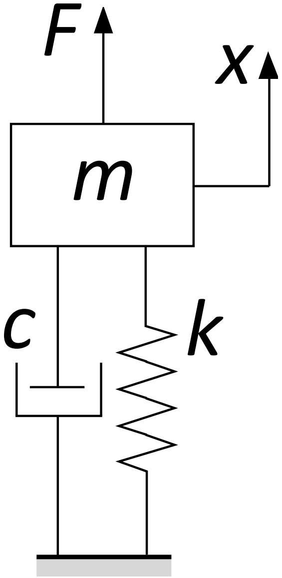
$\mathscr{L}\left[f(t)\right]$ or $F(s)$ $\Big\{$ function of s

- We will use the Laplace transform to find the solution to *linear constant coefficient* differential equations (e.g. dynamic system equations of motion))
- Mathematical basis for majority of control system analysis and design techniques
  (e.g. root locus, frequency domain)

Function of time $f(t)$	**Laplace transform** $F(s)$
$\delta(t)$	1
$1(t)$	$1/s$
t	$1/s^2$
e^{-at}	$\dfrac{1}{s+a}$

Solution of ODE using Laplace Transform

e.g.



$$m\ddot{x}+c\dot{x}+kx=F(t)$$

initial conditions $\begin{cases} x(0)=x_0 \\ \dot{x}(0)=\dot{x}_0 \end{cases}$

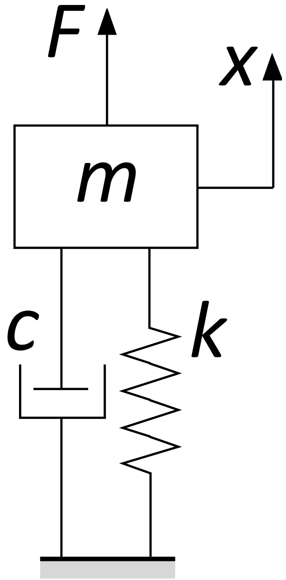
What is the Laplace transform of the ODE?

$$\mathscr{L}\left[m\ddot{x}+c\dot{x}+kx\right]=\mathscr{L}\left[F(t)\right]=F(s)$$

- What is the Laplace transform of a <u>derivative</u> (w.r.t. to time)
- What is the Laplace transform of <u>linear combination of terms</u>

Solution of ODE using Laplace Transform

e.g.



$$m\ddot{x}+c\dot{x}+kx=F(t)$$

initial conditions $\begin{cases} x(0)=x_0 \\ \dot{x}(0)=\dot{x}_0 \end{cases}$

What is the Laplace transform of the ODE?

$$\mathscr{L}[m\ddot{x}+c\dot{x}+kx]=\mathscr{L}[F(t)]=F(s)$$

We'll show that

$$\mathscr{L}[m\ddot{x}+c\dot{x}+kx]=(ms^2+cs+k)\underset{\mathscr{L}[x(t)]}{\underset{\uparrow}{X}}(s)+g(s)$$

Leading to … $(ms^2+cs+k)X(s)=F(s)-g(s)$

$$X(s)=\frac{F(s)-g(s)}{ms^2+cs+k+\cdots}$$

$$\overbrace{\mathscr{L}^{-1}[X(s)]}^{x(t)}=\mathscr{L}^{-1}\left[\frac{F(s)-g(s)}{ms^2+cs+k+\cdots}\right]$$

Laplace Transform Properties

- The Laplace transform has specific properties that make it useful in the solution of linear differential equations

Linearity property:

$$\mathscr{L}\left[a\cdot f(t)+b\cdot g(t)\right]=a\cdot\mathscr{L}\left[f(t)\right]+b\cdot\mathscr{L}\left[g(t)\right]$$

Derivative property:

$$\mathscr{L}\left[\frac{d^n f}{dt^n}\right]=s^n F(s) \text{ + initial condition terms}$$

- Good – can use Laplace transform of simple functions to build Laplace transforms of complex functions

Laplace Transform: Linearity Property

- **Linearity:** If given two functions of time, *f(t)* and *g(t)*

$$\mathcal{L}\left[f(t)\right]=F(s) \quad \text{and} \quad \mathcal{L}\left[g(t)\right]=G(s)$$

- The Laplace transform of $\mathcal{L}\left[a\cdot f(t)+b\cdot g(t)\right]$

$$\begin{aligned}\mathcal{L}\left[a\cdot f(t)+b\cdot g(t)\right]&=\int_{0-}^{\infty}\left(a\cdot f(t)+b\cdot g(t)\right)e^{-st}dt\\&=\underbrace{a\int_{0-}^{\infty}f(t)e^{-st}dt}_{a\cdot F(s)}+\underbrace{b\int_{0-}^{\infty}g(t)e^{-st}dt}_{g\cdot G(s)}\\&=\quad a\cdot F(s)\quad+\quad g\cdot G(s)\end{aligned}$$

- The linearity property

$$\mathcal{L}\left[a\cdot f(t)+b\cdot g(t)\right]=a\cdot\mathcal{L}\left[f(t)\right]+b\cdot\mathcal{L}\left[g(t)\right]$$

Laplace Transform: Derivative Property

- **Derivative:** The Laplace transform of $\dot{f}(t)$

$$F(s)=\int_{0-}^{\infty} f(t)e^{-st}dt$$

$$\mathscr{L}\left[\dot{f}(t)\right]=\int_{0-}^{\infty}\frac{df}{dt}e^{-st}dt$$

Integration by parts $\int udv = uv - \int vdu$ where $\begin{cases} u=e^{-st} \text{ and } du=-se^{-st} \\ dv=\dfrac{df}{dt} \text{ and } v=f(t) \end{cases}$

$$\mathscr{L}[\dot{f}(t)]=f(t)e^{-st}\Big|_{0-}^{\infty}-\int_{0-}^{\infty}f(t)\left(-se^{-st}\right)dt$$

$$=-f(0) \qquad +s\underbrace{\int_{0-}^{\infty}f(t)e^{-st}dt}_{F(s)}$$

$$\mathscr{L}\left[\dot{f}(t)\right]=sF(s)-f(0)$$

Laplace Transform: Derivative Property

First derivative:

$$\mathscr{L}\left[\dot{f}(t)\right] = \underline{sF(s)} - \underbrace{f(0)}_{\text{from initial conditions}}$$

Second derivative:

$$\mathscr{L}\left[\ddot{f}(t)\right] = \underline{s^2F(s)} \underbrace{- sf(0) - \dot{f}(0)}_{\text{from initial conditions}}$$

Third derivative:

$$\mathscr{L}\left[\dddot{f}(t)\right] = \underline{s^3F(s)} \underbrace{- s^2f(0) - s\dot{f}(0) - \ddot{f}(0)}_{\text{from initial conditions}}$$

n

$$\mathscr{L}\left[\frac{d^n f}{dt^n}\right] = \underset{\text{+ initial condition terms}}{\underline{s^nF(s)}}$$

Laplace transform of an equation with derivatives (i.e. differential equation) has *X*(*s*), assorted terms of *s*, and constants

Laplace Transform: Derivative Property

integration:

$$\mathscr{L}\left[\int f(t)dt\right] = \frac{F(s)}{s}$$ + initial condition terms

$$\mathscr{L}\left[\int\int\cdots\int f(t)dt\right] = \frac{F(s)}{s^n}$$ + initial condition terms

Laplace Transform - Summary

Laplace transform: $\mathscr{L}\left[f(t)\right]=\int_{-\infty}^{+\infty} f(t)e^{-st}dt$

$f(t)$	$F(s)$
$\delta(t)$	1
$1(t)$	$1/s$
t	$1/s^2$

Derivative property:

$$\mathscr{L}\left[\dot{f}(t)\right]=sF(s)-f(0)$$

$$\mathscr{L}\left[\ddot{f}(t)\right]=s^2F(s)-sf(0)-\dot{f}(0)$$

$$\mathscr{L}\left[\dddot{f}(t)\right]=s^3F(s)-s^2f(0)-s\dot{f}(0)-\ddot{f}(0)$$

$$\mathscr{L}\left[\frac{d^nf}{dt^n}\right]=s^nF(s) \text{ + initial condition terms}$$

Linearity property:

$$\mathscr{L}\left[a\cdot f(t)+b\cdot g(t)\right]=a\cdot\mathscr{L}\left[f(t)\right]+b\cdot\mathscr{L}\left[g(t)\right]$$

Laplace Transform: Linear Diff EQs

Example: Evaluate the Laplace transform of the output, *x(t)*, of the following initial value problem

$$3\ddot{x}+2\dot{x}+x=4t \quad \text{with initial conditions: } x(0)=3 \text{ and } \dot{x}(0)=2$$

TAKE LAPLACE

$\hookrightarrow \mathcal{L}[4t] = 4\underbrace{(1/s^2)}$ FROM $\mathcal{L}$ TABLE

$\hookrightarrow \mathcal{L}[x(t)] = X(s)$

$x(0)=3$ KNOWN

$\hookrightarrow \mathcal{L}[2\dot{x}(t)] = 2(sX(s) - x(0))$

$\dot{x}(0)=2$

$\hookrightarrow \mathcal{L}[3\ddot{x}(t)] = 3(s^2X(s) - sx(0) - \dot{x}(0))$

$$\Rightarrow 3(s^2X(s)-3s-2)+2(sX(s)-3)+X(s)=4/s^2$$

GROUP & ISOLATE $X(s)$

$$\rightarrow \quad X(s) = \frac{9s^3 + 12s^2 + 4}{s^2(3s^2 + 2s + 1)}$$

THIS IS THE LAPLACE XFORM OF $x(t)$

Laplace Transform: Linear Diff EQs

example: Evaluate the Laplace transform of the output, *x(t)*, of the following initial value problem

$$3\ddot{x}+2\dot{x}+x=4t$$ *with initial conditions:* $x(0)=3$ and $\dot{x}(0)=2$

$$\mathscr{L}[4t]=4\left(1/s^{2}\right)$$

$$\mathscr{L}\left[x(t)\right]=X(s)$$

$$\mathscr{L}\left[2\dot{x}(t)\right]=2\left(sX(s)-x(0)\right)$$

$$\mathscr{L}\left[3\ddot{x}(t)\right]=3\left(s^{2}X(s)-sx(0)-\dot{x}(0)\right)$$

Assemble terms

$$3\left(s^{2}X(s)-3s-2\right)+2\left(sX(s)-3\right)+X(s)=4/s^{2}$$

Laplace Transform: Linear Diff EQs

example: continued

Assemble terms

$$3\left(s^2X(s)-3s-2\right)+2\left(sX(s)-3\right)+X(s)=4/s^2$$

Group terms and isolate *X(s)*

$$\left(3s^2+2s+1\right)X(s)-9s-12=4/s^2$$

$$\left(3s^2+2s+1\right)X(s)=\frac{9s^3+12s^2+4}{s^2}$$

$$X(s)=\frac{9s^3+12s^2+4}{s^2\left(3s^2+2s+1\right)}$$

If I could find $x(t)$ where $\mathscr{L}[x(t)] = X(s)$ then I have found the solution to:

$3\ddot{x}+2\dot{x}+x=4t \quad x(0)=3$ and $\dot{x}(0)=2$

Laplace Transform: Linear Diff EQs

Example: Evaluate the Laplace transform of the output, *x(t)*, of the following initial value problem

$$\ddot{x} + 2\dot{x} - 3x = 6\dot{f} + 2f \qquad x(0) = \dot{x}(0) = 1,\ f(0) = 0, \text{ and } f(t) = 2\delta(t)$$

$$\rightarrow \quad \mathcal{L} = \quad (s^2 + 2s + 3)X(s) = (6s+2)F(s)$$

$$\Rightarrow \quad X(s) = \frac{\overbrace{12s+4}^{\text{FROM IC's}}}{\underbrace{s^2+2s-3}_{\text{CHARACTERISTIC EQN } (\Delta(s))}}$$

PARTIAL FRAC EXP.

$$X(s) = \frac{12s+4}{s^2+2s-3}$$

```
>> roots([1 2 -3])
```

$$+1 \quad \cancel{2} \quad -3$$

$$x(s) = \frac{12s+4}{s^2+2s-3} = \frac{c_1}{s+3} + \frac{c_2}{s-1} = \frac{(c_1+c_2)s + (3c_2 - c_1)}{(s-1)(s+3)}$$

$$\left.\begin{aligned} c_1 + c_2 &= 12 \\ 4 &= 3c_2 - c_1 \end{aligned}\right\} \begin{aligned} c_1 &= 8 \\ c_2 &= 4 \end{aligned}$$

$$\hookrightarrow x(s) = \frac{8}{s+3} + \frac{4}{s-1}$$

NOW: INVERSE LAPLACE

VIA TABLE: $x(t) = 8e^{-3t} + 4e^{t}$

EXPONENTS CORRESPOND TO ROOTS OF CHARACTERISTIC EQN

Laplace Transform: Linear Diff EQs

Example: Evaluate the Laplace transform of the output, *x(t)*, of the following initial value problem

$$\ddot{x} + 2\dot{x} - 3x = 6\dot{f} + 2f \qquad x(0) = \dot{x}(0) = 1,\ f(0) = 0, \text{ and } f(t) = 2\delta(t)$$

$$\mathcal{L}[2f] = 2F(s)$$

$$\mathcal{L}\left[6\dot{f}\right] = 6\left(sF(s) - f(0)\right) = 6sF(s)$$

$$\mathcal{L}[3x] = 3X(s)$$

$$\mathcal{L}[2\dot{x}] = 2\left(sX(s) - x(0)\right) = 2sX(s) - 2$$

$$\mathcal{L}\left[\ddot{x}(t)\right] = s^2X(s) - sx(0) - \dot{x}(0) = s^2X(s) - s - 1$$

Assemble terms

$$\left(s^2X(s) - s - 1\right) + \left(2sX(s) - 2\right) + 3X(s) = 6sF(s) + 2F(s)$$

Laplace Transform: Linear Diff EQs

example: continued

Forcing term: $f\left(t\right)=2\delta\left(t\right) \longrightarrow \mathscr{L}\left[2\delta\left(t\right)\right]=2$

$$\left(s^2X\left(s\right)-s-1\right)+\left(2sX\left(s\right)-2\right)+3X\left(s\right)=6sF\left(s\right)+2F\left(s\right)$$

$$\left(s^2X\left(s\right)-s-1\right)+\left(2sX\left(s\right)-2\right)+3X\left(s\right)=12s+4$$

Group terms and isolate *X(s)*

$$\left(s^2+2s+3\right)X\left(s\right)-s-3=12s+4$$

$$\left(s^2+2s+3\right)X\left(s\right)=13s+7$$

$$X\left(s\right)=\frac{13s+7}{s^2+2s+3}$$

If I could find $x(t)$ where $\mathscr{L}[x(t)] = X(s)$ then I have found the solution to:

$\ddot{x}+2\dot{x}-3x=6\dot{f}+2f \qquad x(0)=\dot{x}(0)=1,\ f(0)=0$, and $f\left(t\right)=2\delta\left(t\right)$

Solving IVP using Laplace Transforms

1. Take Laplace transform of differential equation *including* initial conditions and forcing terms

$$\mathcal{L}\left[\ddot{x}+a\dot{x}+bx=c\right]$$

2. Solve for dependent variable $X\left(s\right)=\dfrac{N\left(s\right)}{D\left(s\right)}$

Laplace transform useful because

1. Transforms ODE to algebraic equation in *s* where we can isolate *X*(*s*)
2. Includes the initial condition in algebraic equation

3. Apply inverse Laplace transform (using table entries to find *x(t)*)

4. If necessary, use partial fraction expansion to split into separate terms for each root

Partial Fraction Expansion

Least Common Denominator Method:

For real, distinct roots only. (For complex and repeating roots, see FPE Appendix A)

$$X(s)=\frac{N(s)}{D(s)}=\frac{N(s)}{(s+r_1)(s+r_2)\cdots(s+r_n)}=\underbrace{\frac{C_1}{(s+r_1)}}+\cdots\underbrace{\frac{C_n}{(s+r_n)}}$$

D(s) in factored form

Laplace transform from table

e.g.

$$X(s)=\frac{As+B}{(s+r_1)(s+r_2)}=\frac{C_1}{(s+r_1)}+\frac{C_2}{(s+r_2)}=\frac{C_1(s+r_2)+C_2(s+r_1)}{(s+r_1)(s+r_2)}$$

Equate terms in *N(s)* to find coefficients C_i

$$As+B=(C_1+C_2)s+(C_1r_2+C_2r_1)$$

$$\begin{aligned} A&=C_1+C_2 \\ B&=C_1r_2+C_2r_1 \end{aligned} \quad \Rightarrow \quad \begin{aligned} C_1&=-(B-Ar_1)/(r_1-r_2) \\ C_2&=(B-Ar_2)/(r_1-r_2) \end{aligned}$$

Solving IVP using Laplace Transforms

Example: Given the initial value problem:

$$\ddot{x}+8\dot{x}+12x=12 \qquad x(0)=0 \text{ and } \dot{x}(0)=4$$

Solving IVP using Laplace Transforms

Example: Given the initial value problem:

$$\ddot{x}+8\dot{x}+12x=12 \qquad x(0)=0 \text{ and } \dot{x}(0)=4$$

the resulting Laplace transform:

$$s^2X(s)-4+8sX(s)+12X(s)=\tfrac{12}{s}$$

group terms

$$\left(\underbrace{s^2+8s+12}_{\Delta(s)\ \longleftarrow\ \textbf{Characteristic equation}}\right)X(s)-4=\tfrac{12}{s}$$

isolate *X*(*s*)

$$X(s)=\frac{4s+12}{s\left(\underbrace{s^2+8s+12}_{\Delta(s)\ \longleftarrow\ \textbf{Characteristic equation}}\right)}$$

Evaluate the output, *x(t)*, by taking the inverse Laplace transform of *X(s)*

Solving IVP using Laplace Transforms

Example: continued

$$X(s)=\frac{4s+12}{s\left(s^2+8s+12\right)}=\frac{4s+12}{s(s+2)(s+6)}$$

$$=\frac{C_1}{s}+\frac{C_2}{(s+2)}+\frac{C_3}{(s+6)}$$

Note: you can find the roots using the Matlab command `roots`. For example: $s^2+8s+12$

```
>> roots([1  8  12])

ans = 2.0000
      6.0000
```

$$X(s)=\frac{C_1(s+2)(s+6)+C_2 s(s+6)+C_3 s(s+2)}{s(s+2)(s+6)}$$

$$X(s)=\frac{\left(C_1+C_2+C_3\right)s^2+\left(8C_1+6C_2+2C_3\right)s+12C_1}{s(s+2)(s+6)}=\frac{4s+12}{s(s+2)(s+6)}$$

Solving for C_1, C_2, and C_3

$$\begin{aligned}12C_1&=12\\8C_1+6C_2+2C_3&=4\\C_1+C_2+C_3&=0\end{aligned}\;\Rightarrow\;\begin{aligned}C_1&=1\\C_2&=-0.5\\C_3&=-0.5\end{aligned}$$

Note: can use Matlab:

$$\begin{bmatrix}12&0&0\\8&6&2\\1&1&1\end{bmatrix}\begin{bmatrix}C_1\\C_2\\C_3\end{bmatrix}=\begin{bmatrix}12\\4\\0\end{bmatrix}$$

```
A = [1   1   1
     12  11   3
     20  10   2];
b = [1; 14; 49];
x = inv(A)*b
```

or

```
x = A\b
```

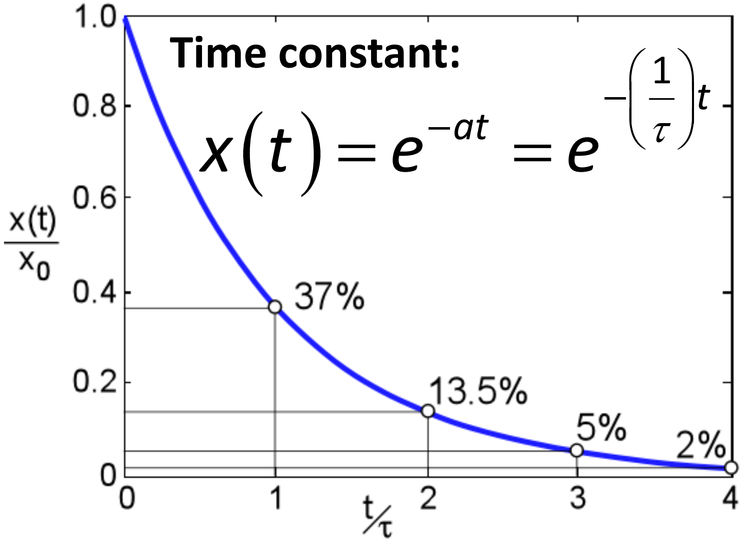
Solving IVP using Laplace Transforms

Example: continued

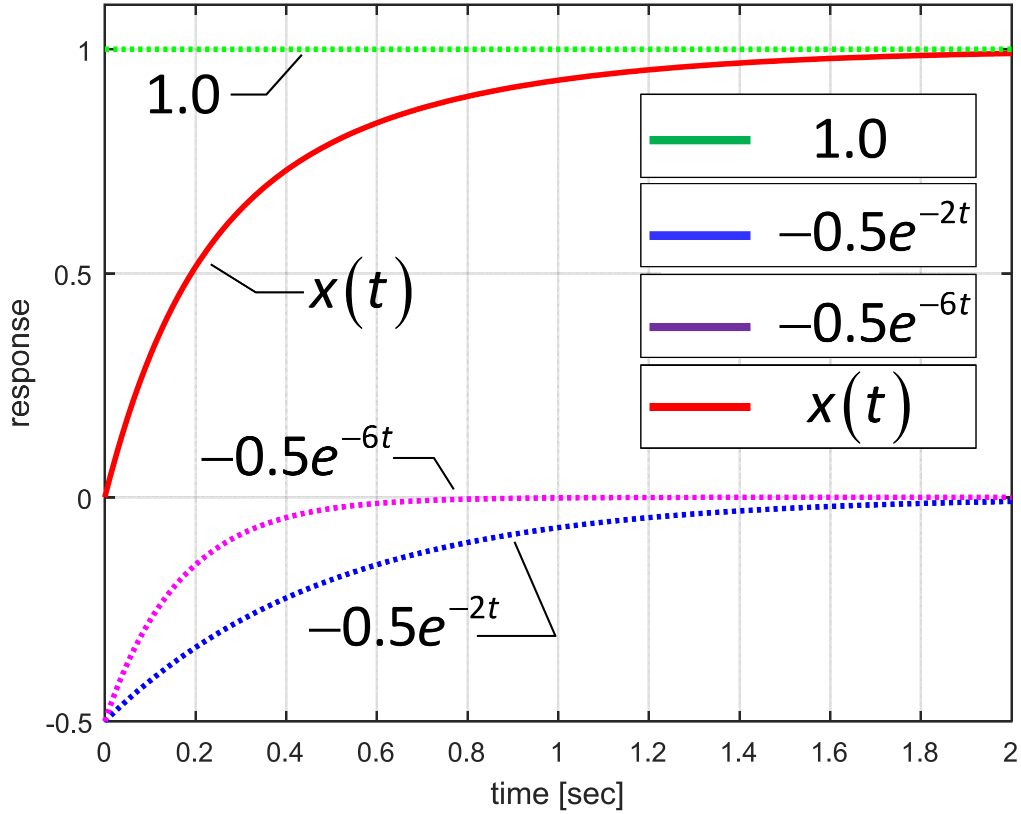
$$X(s)=\frac{4s+12}{s\left(s^2+8s+12\right)}=\frac{C_1}{s}+\frac{C_2}{(s+2)}+\frac{C_3}{(s+6)}=\frac{1}{s}-\frac{0.5}{(s+2)}-\frac{0.5}{(s+6)}$$

$$x(t)=\mathcal{L}^{-1}\left(X(s)\right)=\mathcal{L}^{-1}\left(\frac{1}{s}\right)-0.5\mathcal{L}^{-1}\left(\frac{1}{s+6}\right)-0.5\mathcal{L}^{-1}\left(\frac{1}{s+2}\right)$$

$$x(t)=\underset{\uparrow\ \text{From forcing}}{1}\;\underbrace{-0.5e^{-2t}-0.5e^{-6t}}_{\text{From roots of }\Delta(s)}$$


Time constant:

$$x(t)=e^{-at}=e^{-\left(\frac{1}{\tau}\right)t}$$



Solving IVP using Laplace Transforms

1. Take Laplace transform of differential equation *including* initial conditions and forcing terms

$$\mathscr{L}\left[\ddot{x}+a\dot{x}+bx=c\right]$$

2. Solve for dependent variable $X(s)=\dfrac{N(s)}{D(s)}$

3. Apply inverse Laplace transform (using table entries to find *x(t)*)

4. If necessary, use partial fraction expansion to split into separate terms for each root

Solving IVP using Laplace Transforms

Example: Given the initial value problem:

$\ddot{x}+2\dot{x}-3x=6\dot{f}+2f$ $x(0)=0,\ \dot{x}(0)=0,$ and $y(0)=0$

and $f\left(t\right)=2\delta\left(t\right)$

Solving IVP using Laplace Transforms

Example: continued

$$\ddot{x}+2\dot{x}-3x=6\dot{f}+2f \qquad x(0)=0,\ \dot{x}(0)=0,\ \text{and}\ \ y(0)=0$$

$$\text{and}\ \ f(t)=2\delta(t)$$

the resulting Laplace transform:

$$\left(s^2+2s-3\right)X(s)=\left(6s+2\right)F(s)$$

The Laplace transform of $f(t)$ = $2\delta(t)$

$$\mathscr{L}\left(2\delta(t)\right)=2$$

Substituting in and isolation $X(s)$

$$\underbrace{\left(s^2+2s-3\right)}_{\Delta(s)}X(s)=2\left(6s+2\right)$$

$$X(s)=\frac{12s+4}{s^2+2s-3}$$

Characteristic equation

Solving IVP using Laplace Transforms

Example: continued

Perform a partial fraction expansion. First evaluate the roots of the denominator

$$X(s)=\frac{12s+4}{s^2+2s-3} \longrightarrow s=\frac{-2\pm\sqrt{2^2-4(-3)}}{2}$$
$$=+1 \text{ and } -3$$

Note: you can find the roots using the Matlab command `roots`. For example: s^2+2s-3

```
>> roots([1  2  -3])

ans = -3.0000
       1.0000
```

Partial fraction expansion:

$$X(s)=\frac{12s+4}{s^2+2s-3}=\frac{C_1}{s+3}+\frac{C_2}{s-1} \quad =\frac{(C_1+C_2)s+(3C_2-C_1)}{(s-1)(s+3)}$$

Equate coefficients and solve for C_1 and C_2

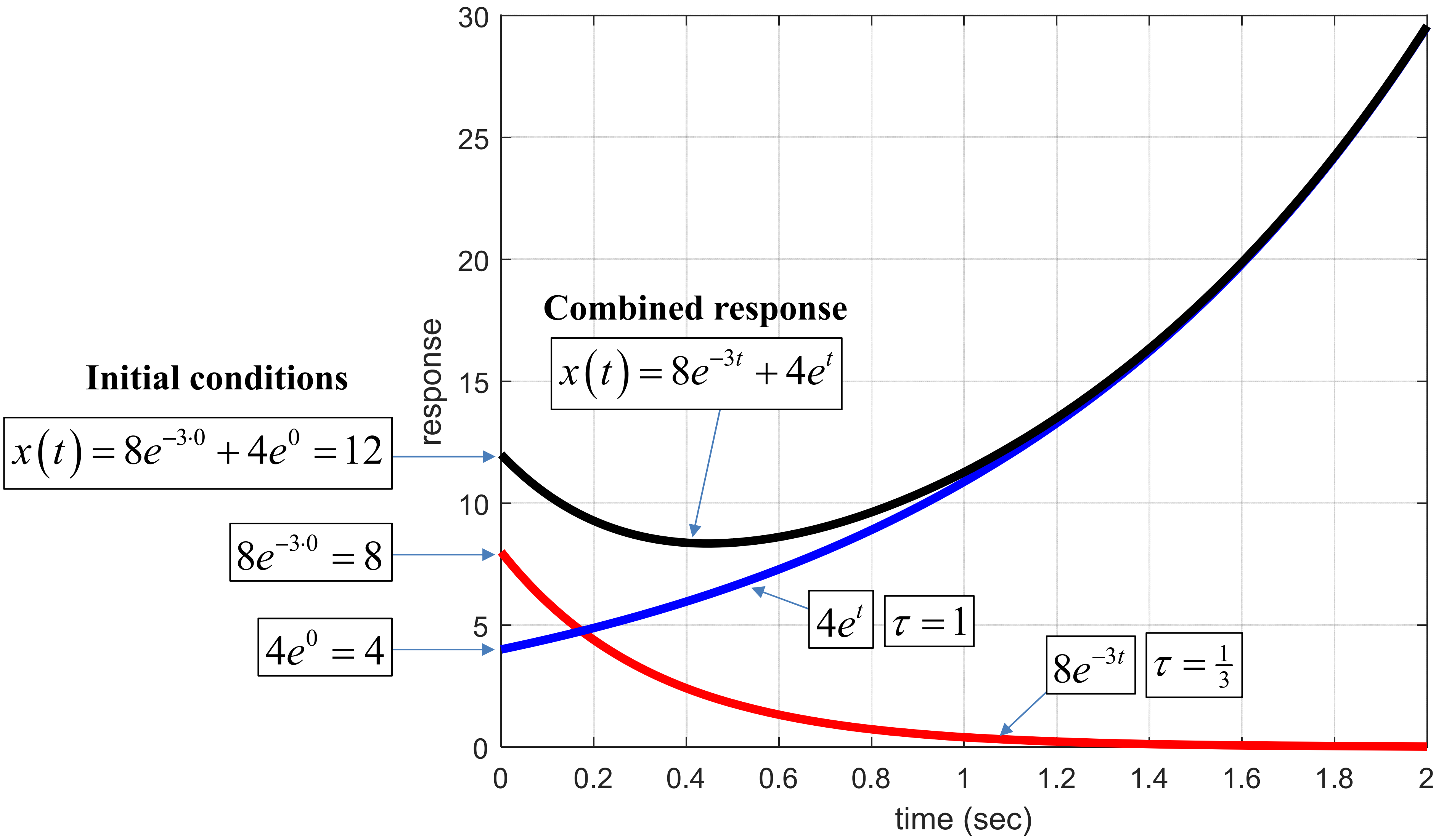
$$\left.\begin{aligned} C_1+C_2&=12 \\ 3C_2-C_1&=4 \end{aligned}\right\} \begin{aligned} C_1&=8 \\ C_2&=4 \end{aligned} \qquad X(s)=\frac{8}{s+3}+\frac{4}{s-1}$$

Inverse Laplace transform (via table) $x(t)=8e^{-3t}+4e^{t}$

Solving IVP using Laplace Transforms

Example: continued

Inverse Laplace transform (via table) $x\left(t\right)=8e^{-3t}+4e^{t}$



Combined response

$x\left(t\right)=8e^{-3t}+4e^{t}$

Initial conditions

$x\left(t\right)=8e^{-3\cdot 0}+4e^{0}=12$

$8e^{-3\cdot 0}=8$

$4e^{0}=4$

$4e^{t}$ $\tau=1$

$8e^{-3t}$ $\tau=\frac{1}{3}$

Summary – IVP Solution with Laplace Transforms

1. Take Laplace transform of differential equation *including* initial conditions and forcing terms

$$\mathscr{L}\left[\ddot{x}+a\dot{x}+bx=c\right]$$

Linearity property:

$$\mathscr{L}\left[af(t)+bg(t)\right]=aF(s)+bG(s)$$

Derivative property:

$$\mathscr{L}\left[\frac{d^n f}{dt^n}\right]=s^n F(s)$$

+ I.C.

2. Solve for $X(s)=\frac{N(s)}{D(s)}$

3. If necessary, use partial fraction expansion

$$X(s)=\frac{N(s)}{D(s)}=\frac{N(s)}{(s+r_1)(s+r_2)\cdots(s+r_n)}=\frac{C_1}{(s+r_1)}+\cdots\frac{C_n}{(s+r_n)}$$

4. Apply inverse Laplace transform (using table entries)

Characteristic Equation

Characteristic equation determine system dynamic characteristics

Examples:

$$\ddot{x}+4\dot{x}+3x=0$$

$x(0)=1$ and $\dot{x}(0)=2$

$$\ddot{x}-4x=1$$

$x(0)=2$ and $\dot{x}(0)=0$

$$2\dot{x}+6x=0$$

$x(0)=1$

Laplace transform

$$\underbrace{\left(s^2+4s+3\right)}_{\Delta(s)}X\left(s\right)-2s-5=0$$

$$\underbrace{\left(s^2-4\right)}_{\Delta(s)}X\left(s\right)-2=1/s$$

$$2\underbrace{\left(s+3\right)}_{\Delta(s)}X\left(s\right)-2=0$$

isolate $X(s)$

$$X\left(s\right)=\frac{2s+5}{s^2+4s+3}$$

$$X\left(s\right)=\frac{2s+1}{s\left(s^2-4\right)}$$

$$X\left(s\right)=\frac{1}{s+3}$$

partial fraction expansion

$$=\underbrace{\frac{C_1}{\left(s+1\right)}+\frac{C_2}{\left(s+3\right)}}_{\text{From roots of }\Delta(s)}$$

$$=\frac{C_1}{\underset{\text{From forcing}}{s}}+\underbrace{\frac{C_2}{\left(s+2\right)}+\frac{C_3}{\left(s-2\right)}}_{\text{From roots of }\Delta(s)}$$

$$=\frac{C_1}{\underbrace{s+3}_{\text{From roots of }\Delta(s)}}$$

Final Value Theorem

Recall derivative theorem $$\mathscr{L}\left[\dot{f}(t)\right]=\int_0^\infty \dot{f}(t)e^{-st}dt=sF(s)-f(0)$$

Take limit of $s\rightarrow 0$ both sides: $$\lim_{s\rightarrow 0}\int_0^\infty \dot{f}(t)e^{-st}dt=\lim_{s\rightarrow 0}\left[sF(s)-f(0)\right]$$

$$\int_0^\infty \dot{f}(t)dt=\lim_{s\rightarrow 0}\left[sF(s)\right]-f(0)$$

$$f(\infty)-f(0)=\lim_{s\rightarrow 0}\left[sF(s)\right]-f(0)$$

- *Final* value theorem:

Value of f(t) as $t\rightarrow\infty$ →

$$\boxed{f(\infty)=\lim_{s\rightarrow 0}\left[sF(s)\right]}$$

Valid if:

- degree of numerator (of *F(s)*) is less than degree of denominator
- Roots of *Δ(s)* have negative real parts (i.e. system is stable)

Final Value Theorem

Example: find the steady-state value of *x(t)* for the IVP shown using the final value theorem

$\ddot{x} + 5\dot{x} + 2x = 6y + \dot{y}$ I.C.s $x(0) = 0$ $\dot{x}(0) = 0$

Input: $y(t) = 2u_s(t)$

Final Value Theorem

Example: find the steady-state value of *x(t)* for the IVP shown using the final value theorem

$$\ddot{x} + 5\dot{x} + 2x = 6y + \dot{y}$$

I.C.s $x(0) = 0 \quad \dot{x}(0) = 0$

Use final value theorem:

$$x(\infty) = \lim_{s \to 0} \left[sX(s) \right]$$

Input: $y(t) = 2u_s(t)$

Evaluate *X(s)*

$$\mathscr{L}[\ddot{x} + 5\dot{x} + 2x] = \mathscr{L}[6y + \dot{y}]$$

$$\left(s^2 + 5s + 2\right) X(s) = (s + 6) Y(s)$$

$$X(s) = \frac{s+6}{\left(s^2 + 5s + 2\right)} Y(s)$$

Final Value Theorem

Example: find the steady-state value of *x(t)* for the IVP shown using the final value theorem

$$\ddot{x}+5\dot{x}+2x=6y+\dot{y} \qquad \text{I.C.s} \quad x(0)=0 \quad \dot{x}(0)=0$$

Use final value theorem:

$$x(\infty)=\lim_{s\to 0}\left[sX(s)\right]$$

Input: $y(t) = 2u_s(t)$

Evaluate $x(\infty)$

$$X(s)=\frac{s+6}{\left(s^2+5s+2\right)}Y(s) \quad \text{where} \quad Y(s)=\frac{2}{s} \quad \text{for} \quad y(t)=2u_s(t)$$

$$X(s)=\frac{s+6}{\left(s^2+5s+2\right)}\frac{2}{s} \quad =\frac{2s+12}{s^3+5s^2+2s} \quad \text{for} \quad y(t)=2u_s(t)$$

$$x(\infty)=\lim_{s\to 0}\left[s\frac{2s+12}{s^3+5s^2+2s}\right] =\lim_{s\to 0}\left[\frac{2s+12}{s^2+5s+2}\right] \quad =6$$

Modeling & Analysis of Dynamic Systems

- Topic Overview -

- Dynamic System Modeling 2.1 – 2.4
- Dynamic Systems – Differential Equations 3.1 – 3.6
- Laplace Transform and Linear Differential Equations 3.1
- ➡ Transfer Functions 3.1.2
- Block Diagram Modeling 3.2.1 – 3.2.2
- Linear System Response Characteristics 3.3
- Time Domain Specifications 3.4
- Effects of System Poles and Zeros 3.5

Transfer Function

- **Transfer Function:** a succinct encapsulation of system dynamics – which is equivalent to the system ODE
- Transfer function is defined as the ratio of output to input e.g. *X(s)/F(s)*

Transfer Function →
$$T(s) = \frac{X(s)}{F(s)}$$

- Derivation of system transfer function – *an example*
  - Given a system: $\ddot{x} + a\dot{x} + bx = c\dot{f} + df$ $\qquad \underbrace{x(0) = 0 \;\; \dot{x}(0) = 0}$

I.C.s are set to zero: only interested in response due to input

  - Take Laplace transform of both sides

$$\underbrace{\left(s^2 + as + b\right)}_{\textbf{characteristic equation}} X(s) = (cs + d)F(s)$$

$$T(s) = \frac{X(s)}{F(s)} = \frac{(cs + d)}{\left(s^2 + as + b\right)}$$

← **Characteristic equation** $\Delta(s)$

Transfer Function

- The transfer function can be used to find the response of the system to an arbitrary forcing function, *f(t)*

- For example - given the system transfer function & forcing function

$$T(s) = \frac{X(s)}{F(s)} \qquad f(t) = 3t \longrightarrow F(s) = \frac{3}{s^2}$$

$$X(s) = T(s)F(s) = T(s)\frac{3}{s^2}$$

$$x(t) = \mathscr{L}^{-1}\left[T(s)\frac{3}{s^2}\right]$$

Transfer Function

- The transfer function representation is equivalent to the system ODE - *i.e. you can go back and forth easily*

e.g. $$T(s)=\frac{10s+5}{s^2+4s+5}=\frac{X(s)}{F(s)}$$

Cross-multiply and perform inverse transform

$$(s^2+4s+5)X(s)=(10s+5)F(s)$$

Inverse Laplace transform
(easy – because I.C.s were zero)

$$\ddot{x}+4\dot{x}+5x=10\dot{f}+5f$$

Remember that:

$$\dot{x} \longrightarrow sX(s)+\text{I.C. terms}$$

$$\ddot{x} \longrightarrow s^2X(s)+\text{I.C. terms}$$

$$\frac{d^nx}{dt^n} \longrightarrow s^nX(s)+\text{I.C. terms}$$

Set equal to zero – forced response only

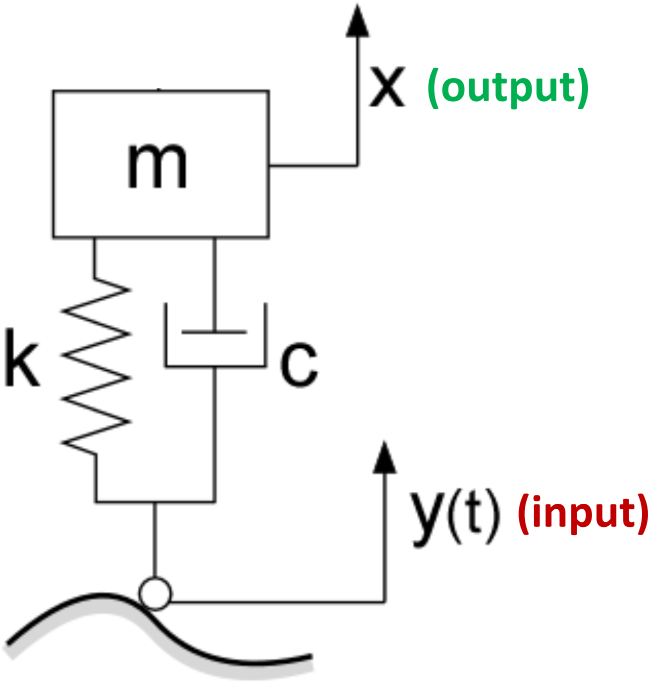
Transfer Function

Example: For the adjacent system, evaluate the transfer function

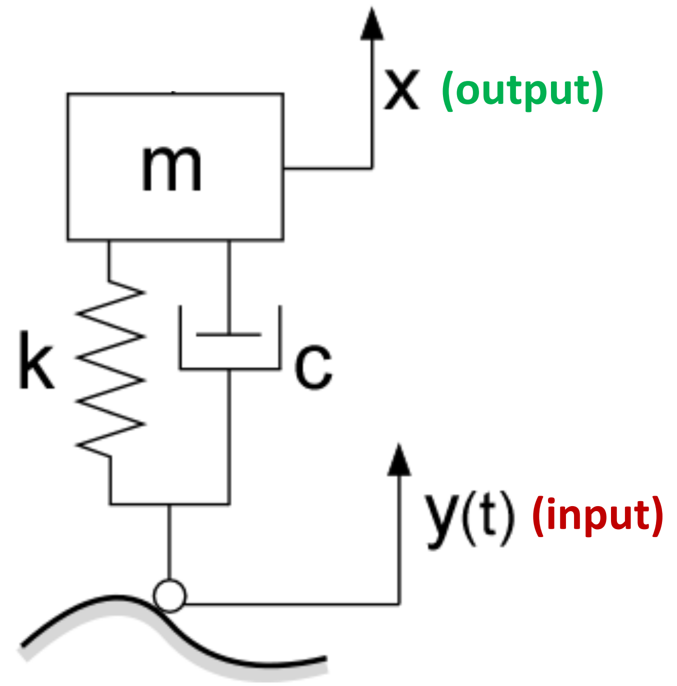
$$T(s) = \frac{X(s)}{Y(s)}$$

Given its equation of motion:

$$m\ddot{x} + c\dot{x} + kx = c\dot{y} + ky$$



Transfer Function



Example: For the adjacent system, evaluate the transfer function

$$T(s)=\frac{X(s)}{Y(s)}$$

Given its equation of motion:

$$m\ddot{x}+c\dot{x}+kx=c\dot{y}+ky$$

Take Laplace transform (with initial conditions set equal to zero)

$$\left(ms^2+cs+k\right)X(s)=\left(cs+k\right)Y(s)$$

Form transfer function

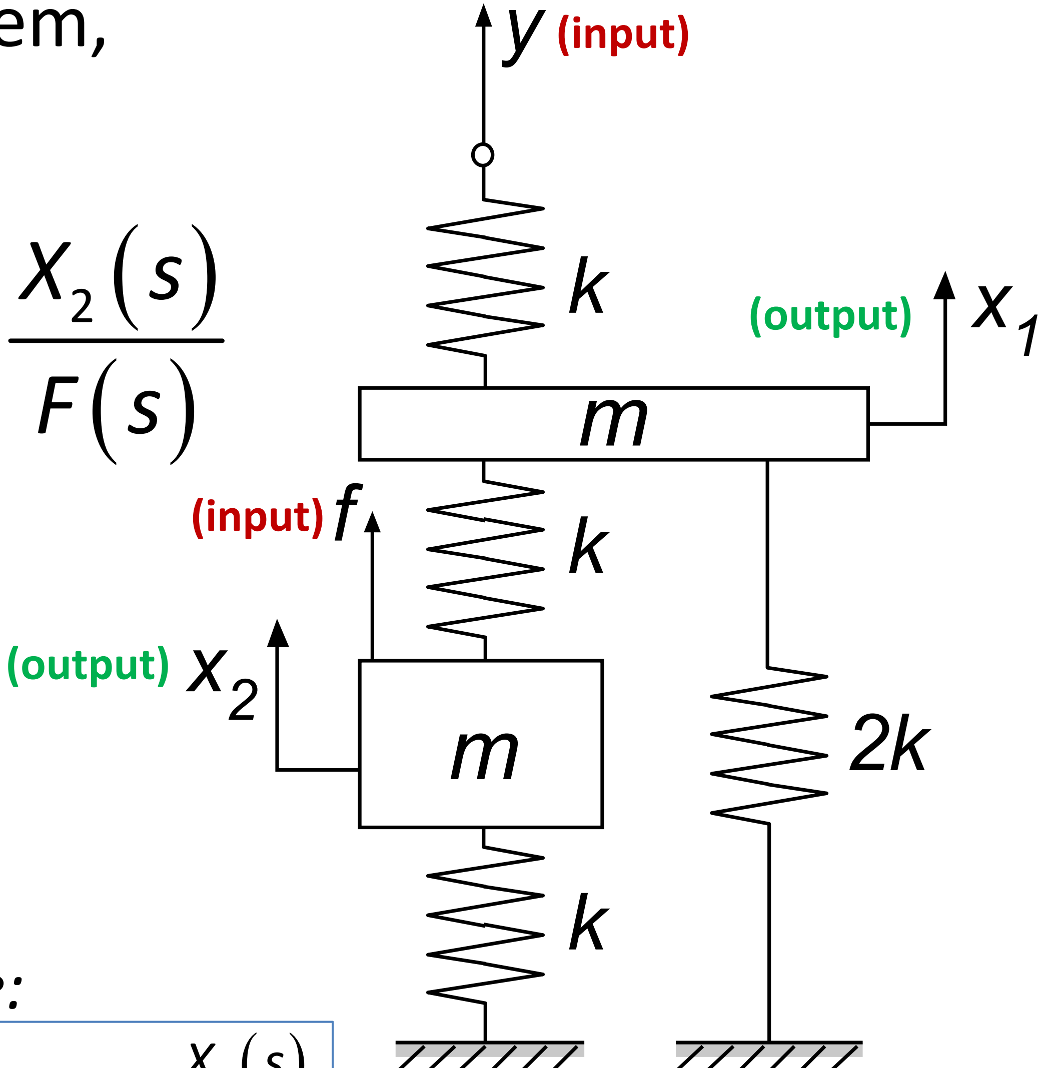
$$\frac{X(s)}{Y(s)}=\frac{cs+k}{ms^2+cs+k}$$

Characteristic equation $\Delta(s)$

Transfer Functions

Example: For the adjacent system, evaluate the transfer functions

$$T_1(s) = \frac{X_1(s)}{F(s)} \quad \text{and} \quad T_2(s) = \frac{X_2(s)}{F(s)}$$



Note: number of possible transfer functions equals the (# of inputs) x (# of outputs)

In this case:

$$T_1(s) = \frac{X_1(s)}{F(s)} \quad T_2(s) = \frac{X_2(s)}{F(s)}$$

$$T_3(s) = \frac{X_1(s)}{Y(s)} \quad T_4(s) = \frac{X_2(s)}{Y(s)}$$

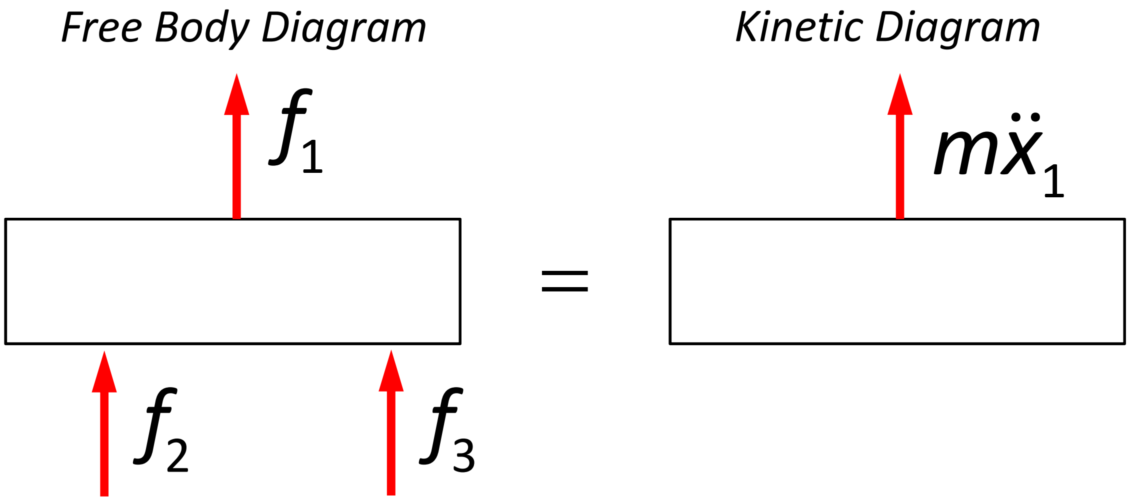
Transfer Functions

Example: continued

Newton's Law
(Planar Dynamics)

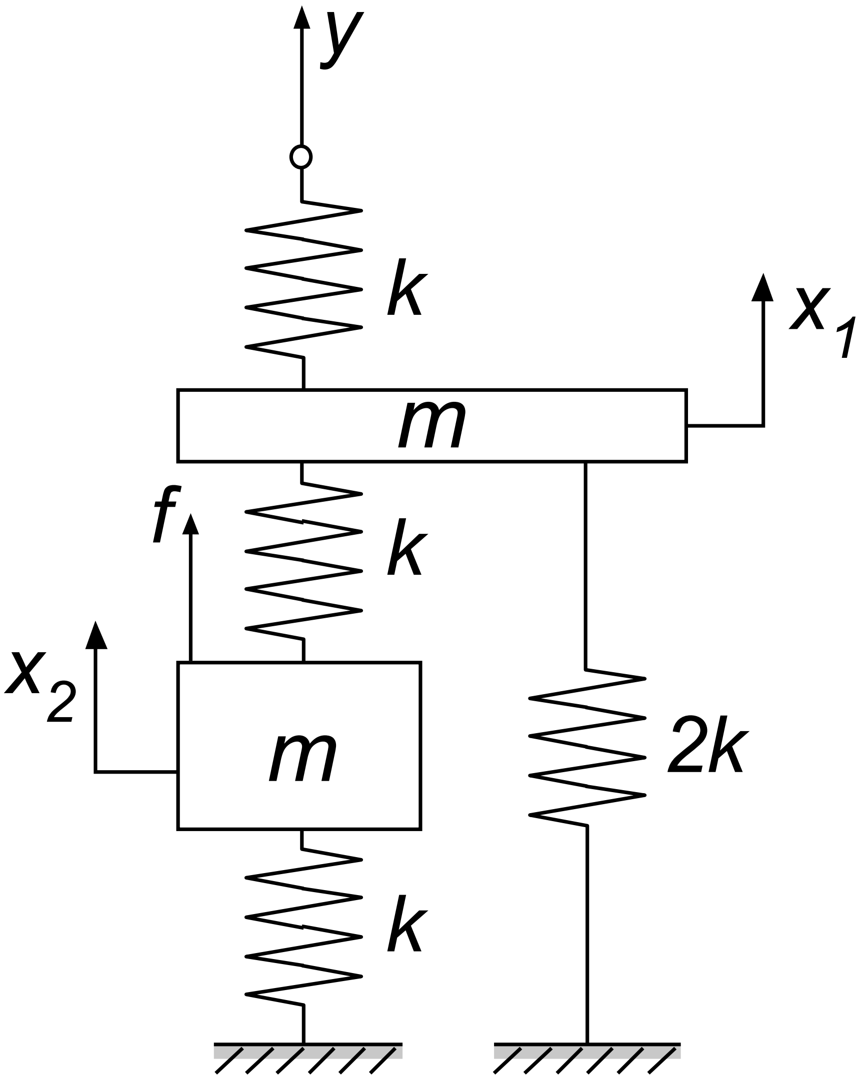
$$\vec{F} = m\vec{a}$$
$$\tau = I\alpha$$

x_1 degree of freedom:



$$\underbrace{k\left(y - x_1\right)}_{f_1} + \underbrace{k\left(x_2 - x_1\right)}_{f_2} + \underbrace{2k\left(-x_1\right)}_{f_3} = m\ddot{x}_1$$

$$\boxed{m\ddot{x}_1 + 4kx_1 - kx_2 = ky}$$



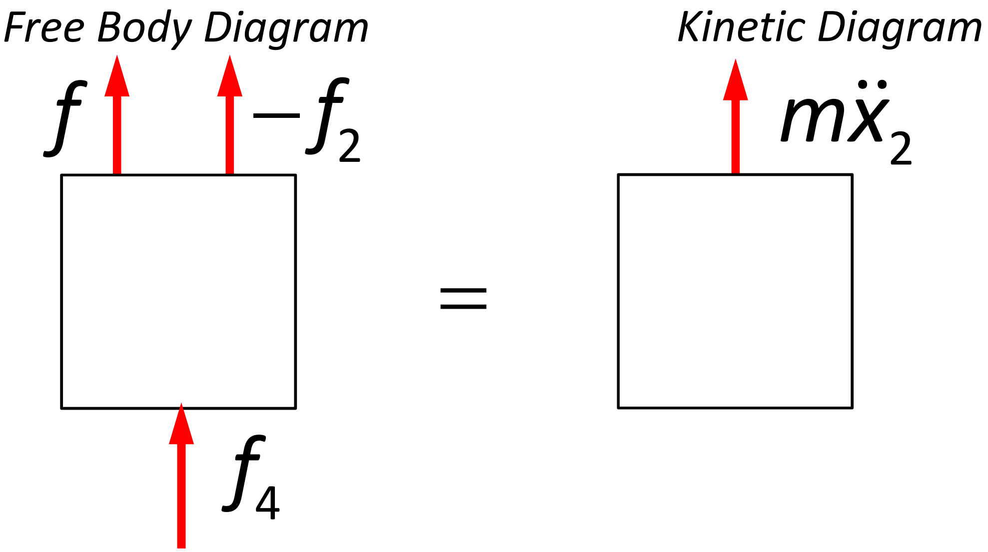
Transfer Functions

Example: continued

Newton's Law
(Planar Dynamics)

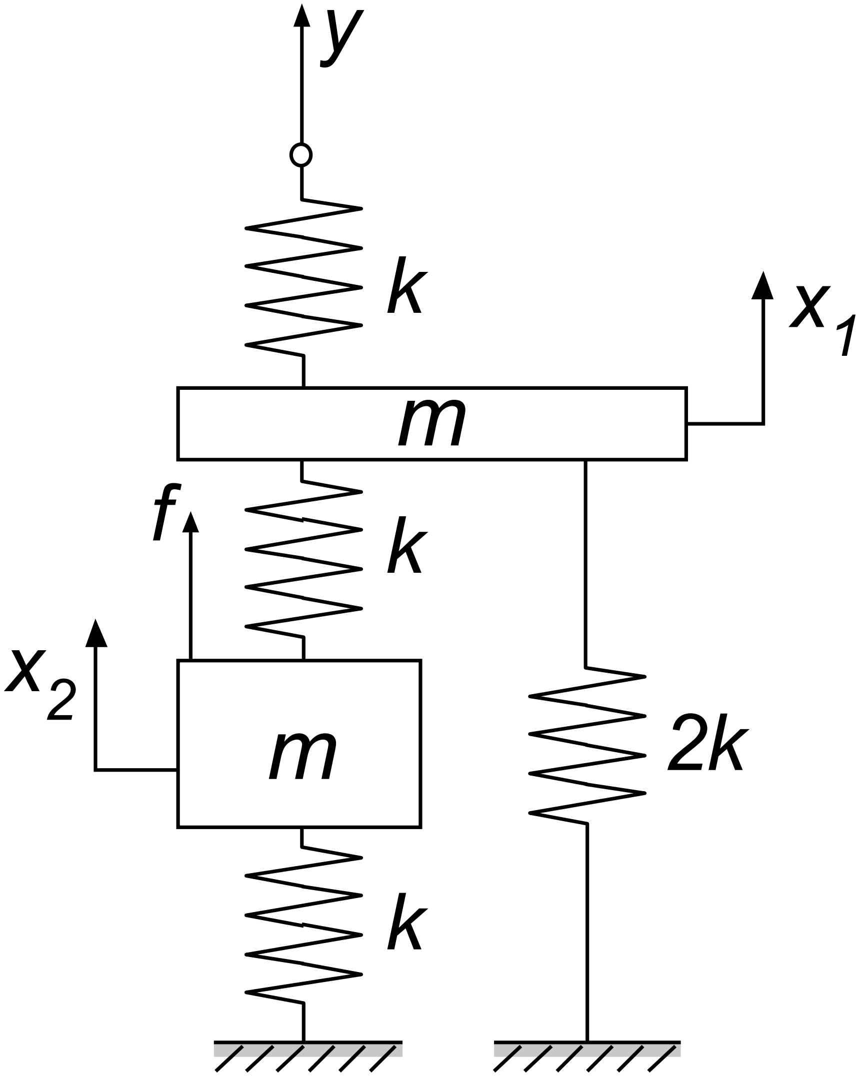
$$\vec{F} = m\vec{a}$$
$$\tau = I\alpha$$

x_2 degree of freedom:



$$f - \underbrace{k\left(x_2 - x_1\right)}_{-f_2} + \underbrace{k\left(-x_2\right)}_{f_4} = m\ddot{x}_2$$

$$\boxed{m\ddot{x}_2 - kx_1 + 2kx_2 = f}$$



Transfer Functions

Example: continued

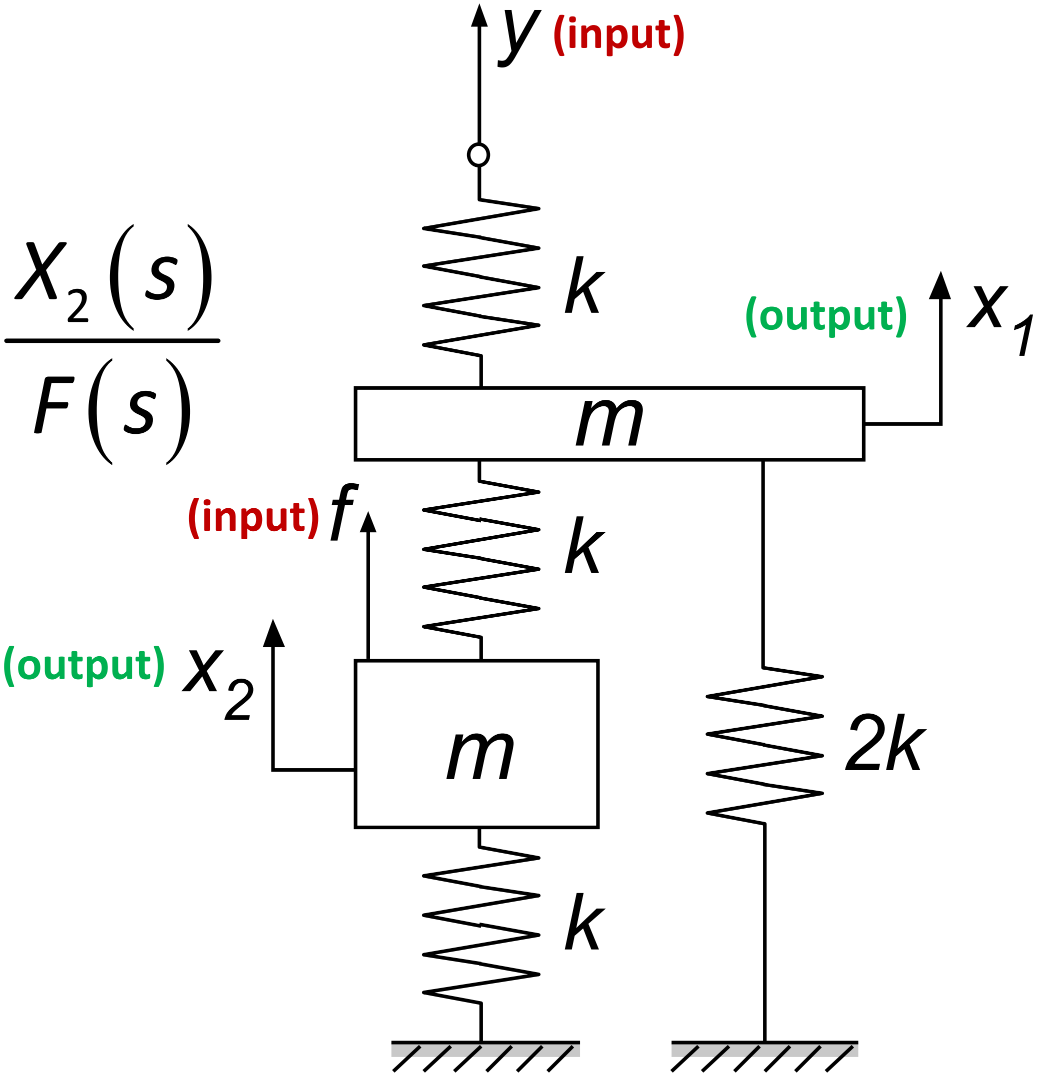
Evaluate the transfer functions:

$$T_1(s) = \frac{X_1(s)}{F(s)} \quad \text{and} \quad T_2(s) = \frac{X_2(s)}{F(s)}$$

Equations of motion:

$$m\ddot{x}_2 - kx_1 + 2kx_2 = f$$

$$m\ddot{x}_1 + 4kx_1 - kx_2 = ky$$



Transfer Functions

Example: continued

Equations of motion: Take the LT (setting I.C.s = 0)

$$m\ddot{x}_1+4kx_1-kx_2=ky \longrightarrow \left(ms^2+4k\right)X_1(s)-kX_2(s)=\overset{\overset{0}{\downarrow}}{kY}(s) \quad (1)$$

$$m\ddot{x}_2-kx_1+2kx_2=f \longrightarrow \left(ms^2+2k\right)X_2(s)-kX_1(s)=F(s) \quad (2)$$

We're evaluating $T_1(s)=\frac{X_1(s)}{F(s)}$ so set other inputs equal to zero

Solve for $X_1(s)$: From (1) $X_2(s)=\left(\frac{ms^2+4k}{k}\right)X_1(s)$

Substitute into (2) $$\left(ms^2+2k\right)\left(\frac{ms^2+4k}{k}\right)X_1(s)-kX_1(s)=F(s)$$

Isolate $X_1(s)$ $$\left(\frac{m^2s^4+6mks^2+7k^2}{k}\right)X_1(s)=F(s)$$

$$\boxed{\frac{X_1(s)}{F(s)}=\frac{k}{m^2s^4+6mks^2+7k^2}} \longleftarrow \Delta(s) \longleftarrow$$

Characteristic equation $\Delta(s)$

Transfer Functions

Example: continued

Equations of motion: Take the LT (setting I.C.s = 0)

$$m\ddot{x}_1+4kx_1-kx_2=ky \longrightarrow \left(ms^2+4k\right)X_1(s)-kX_2(s)=k\overset{\overset{0}{\downarrow}}{Y}(s) \quad (1)$$

$$m\ddot{x}_2-kx_1+2kx_2=f \longrightarrow \left(ms^2+2k\right)X_2(s)-kX_1(s)=F(s) \quad (2)$$

We're evaluating $T_2(s)=\frac{X_2(s)}{F(s)}$ so set $Y(s)$ equal to zero

Solve for $X_2(s)$: From (1) $X_1(s)=\left(\frac{k}{ms^2+4k}\right)X_2(s)$

Substitute into (2) $\left(ms^2+2k\right)X_2(s)-k\left(\frac{k}{ms^2+4k}\right)X_2(s)=F(s)$

Isolate $X_1(s)$ $\left(\frac{m^2s^4+6mks^2+7k^2}{ms^2+4k}\right)X_2(s)=F(s)$

$$\boxed{\frac{X_2(s)}{F(s)}=\frac{ms^2+4k}{m^2s^4+6mks^2+7k^2}} \longleftarrow \Delta(s) \longleftarrow$$

Characteristic equation $\Delta(s)$

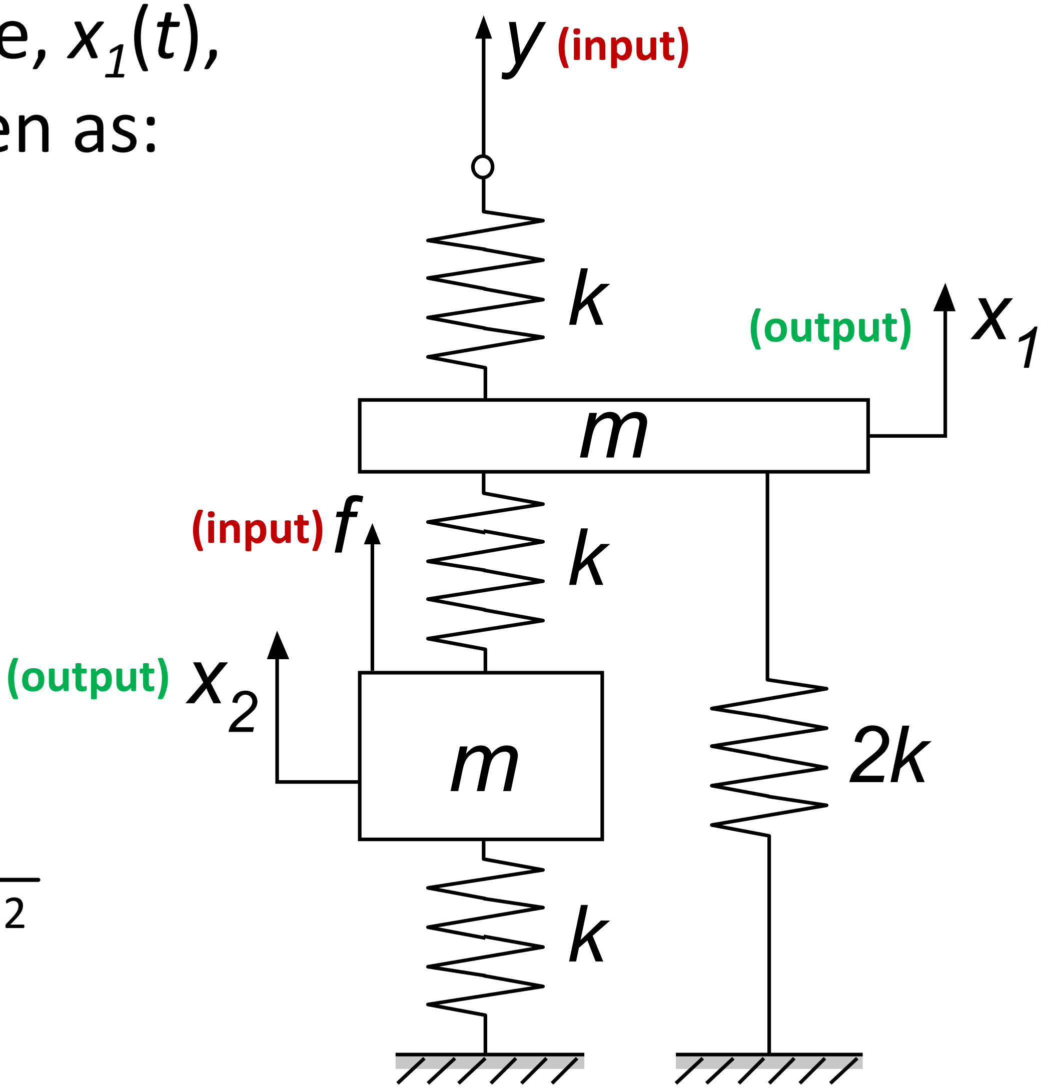
Transfer Functions

Example: Evaluate the response, $x_1(t)$, when the input force, f(t) is given as:

Unit impulse: $f(t) = \delta(t)$

Unit step: $f(t) = 1(t)$

Sinusoidal: $f(t) = \sin(t)$



Transfer functions:

$$\frac{X_2(s)}{F(s)} = \frac{ms^2 + 4k}{m^2s^4 + 6mks^2 + 7k^2}$$

$$\frac{X_1(s)}{F(s)} = \frac{k}{m^2s^4 + 6mks^2 + 7k^2}$$

Transfer Functions

Example: continued

Brief introduction to Matlab

Dynamic system analysis in Matlab

```
% system parameters
m = 1; k = 8;
```

$$T_1(s) = \frac{k}{m^2 s^4 + 6mks^2 + 7k^2}$$

```
% form transfer function T1
s = tf('s');
sysT1 = k/(m^2*s^4 + 6*m*k*s^2 + 7*k^2);
% alternatively
num = [k];
den = [m^2   0   6*m*k   0   7*k^2];
sysT1 = tf(num,den);

% simulate and plot the impulse response
tend = 50;
impulse(sysT1,tend);
title('Response to a unit impulse')
```

Define Laplace variable

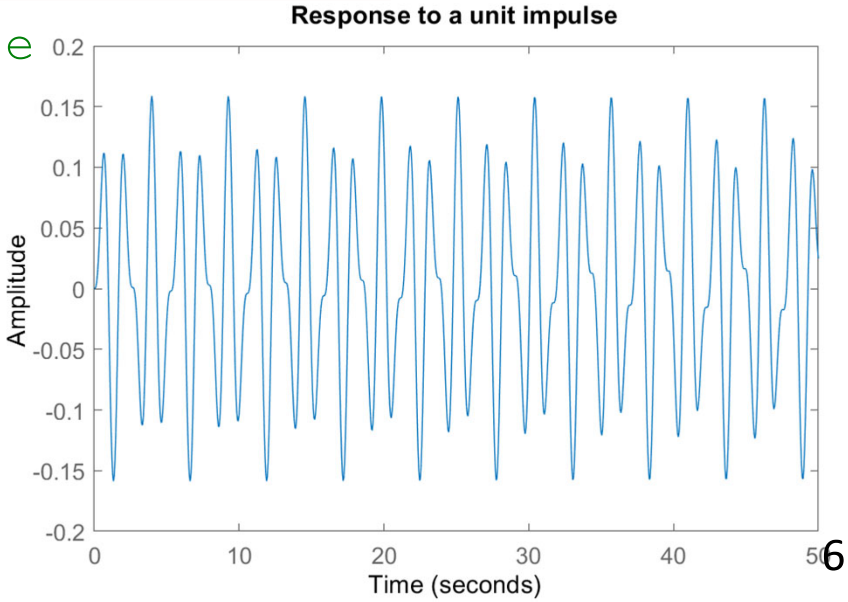
Define transfer function

Denominator coefficients

Numerator coefficients

Define transfer function

Impulse function



Transfer Functions

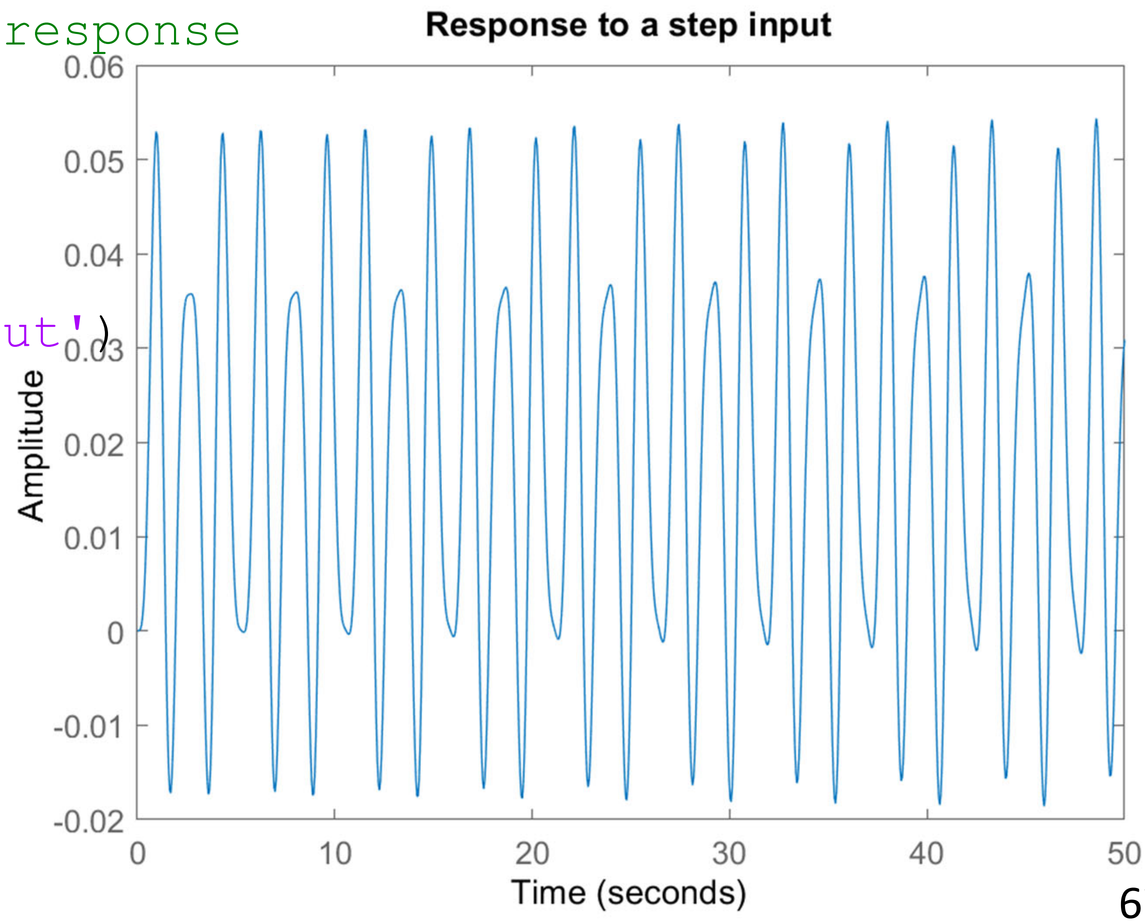
Example: continued

```
% simulate and plot the step response
tend = 50;
figure
step(sysT1,tend);
title('Response to a step input’)
```

step function

```
% alternatively for the step response
t = linspace(0,tend,1000)';
u = ones(size(t));
[y,t] = lsim(sysT1,u,t);
figure;  plot(t,y,'r')
title('Response to a step input')
xlabel('time [sec]');
ylabel('y');
```

Linear simulation

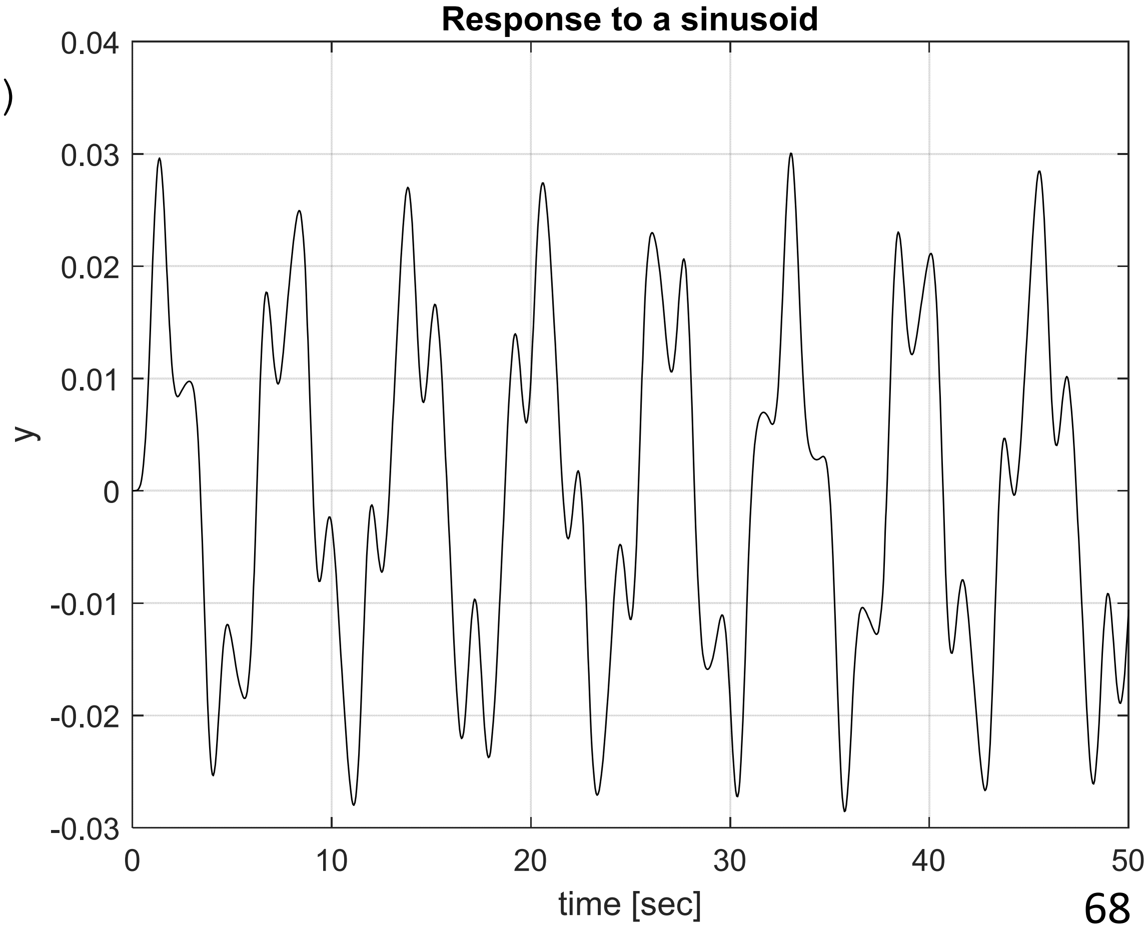


Transfer Functions

Example: continued

```
% simulate and plot the response to a sinusoidal input
t = linspace(0,tend,1000)';
u = sin(t);
[y,t] = lsim(sysT1,u,t);
figure;  plot(t,y,'k')
xlabel('time [sec]');
ylabel('y');
title('Response to a sinusoid')
grid on
```

Linear simulation function



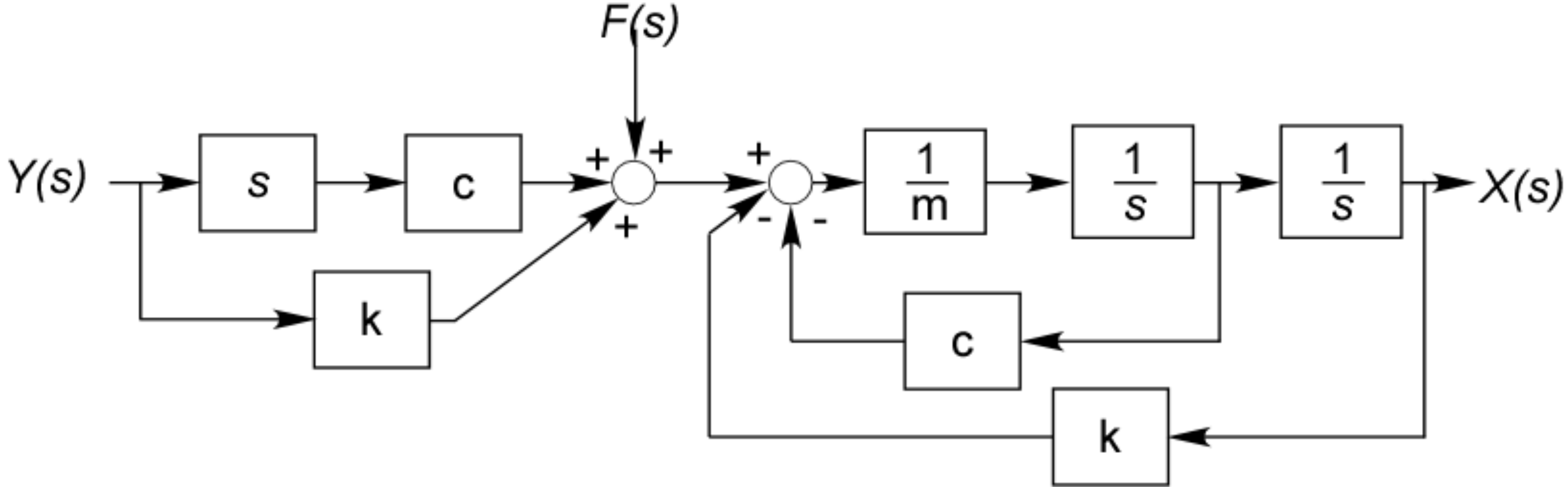
Modeling & Analysis of Dynamic Systems

- Topic Overview -

- Dynamic System Modeling 2.1 – 2.4
- Dynamic Systems – Differential Equations 3.1 – 3.6
- Laplace Transform and Linear Differential Equations 3.1
- Transfer Functions 3.1.2
- ➡ Block Diagram Modeling 3.2.1 – 3.2.2
- Linear System Response Characteristics 3.3
- Time Domain Specifications 3.4
- Effects of System Poles and Zeros 3.5

Block Diagrams

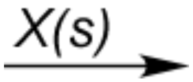
- Block diagram is a graphical representation of a dynamic system (in the Laplace or *s*-domain)



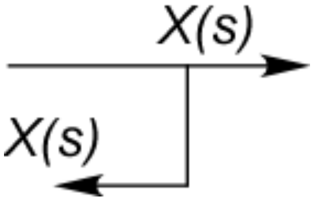
- Used to construct complete model of system from separate parts
- Used extensively in control system analysis and design / simulation

Block Diagram Elements

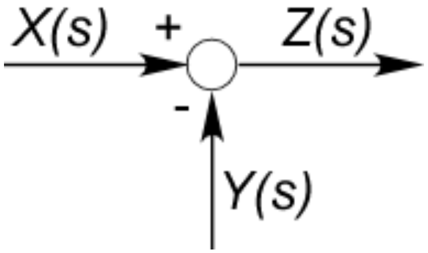
Signal:



Takeoff point:

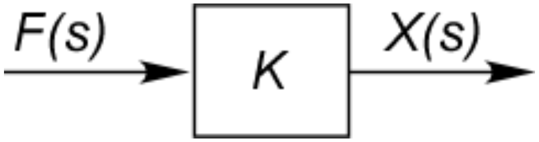


Summation:

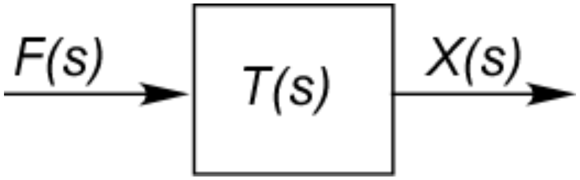


$$Z(s) = X(s) - Y(s)$$

Gain (multiplier):

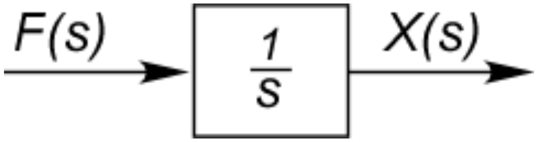


$$X(s) = KF(s)$$



$$X(s) = T(s)F(s)$$

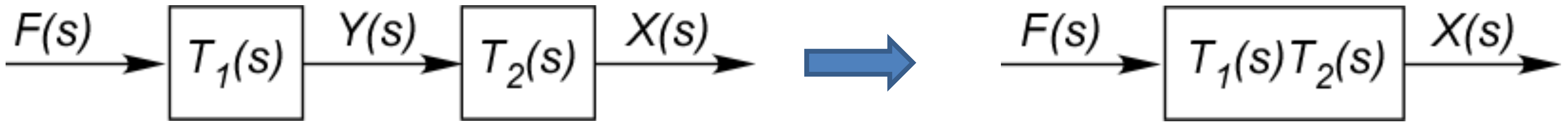
Integrator:



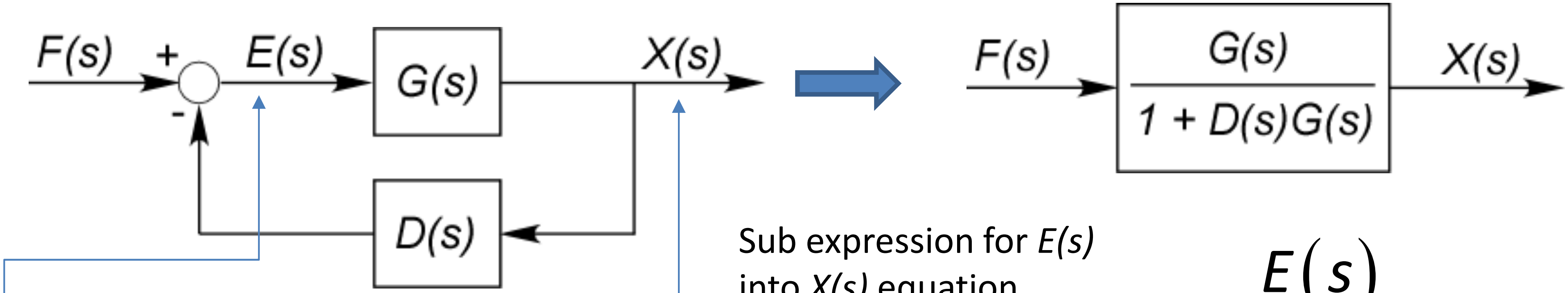
$$x(t) = \int f(t)\,dt$$

Block Diagram Reduction

Blocks in Series:



Blocks in Negative Feedback:



$$X(s)=E(s)G(s)$$

$$E(s)=F(s)-X(s)D(s)$$

Sub expression for *E(s)* into *X(s)* equation

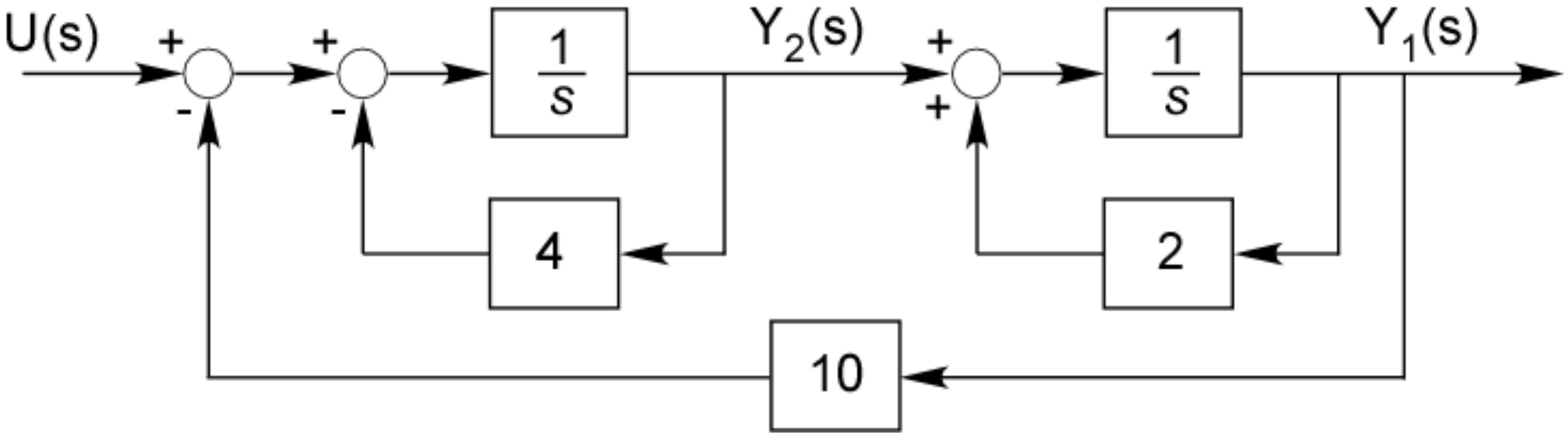
$$X(s)=\left[\overbrace{F(s)-X(s)D(s)}^{E(s)}\right]G(s)$$

Expand, collect terms and form transfer function

$$\frac{X(s)}{F(s)}=\frac{G(s)}{1+D(s)G(s)}$$

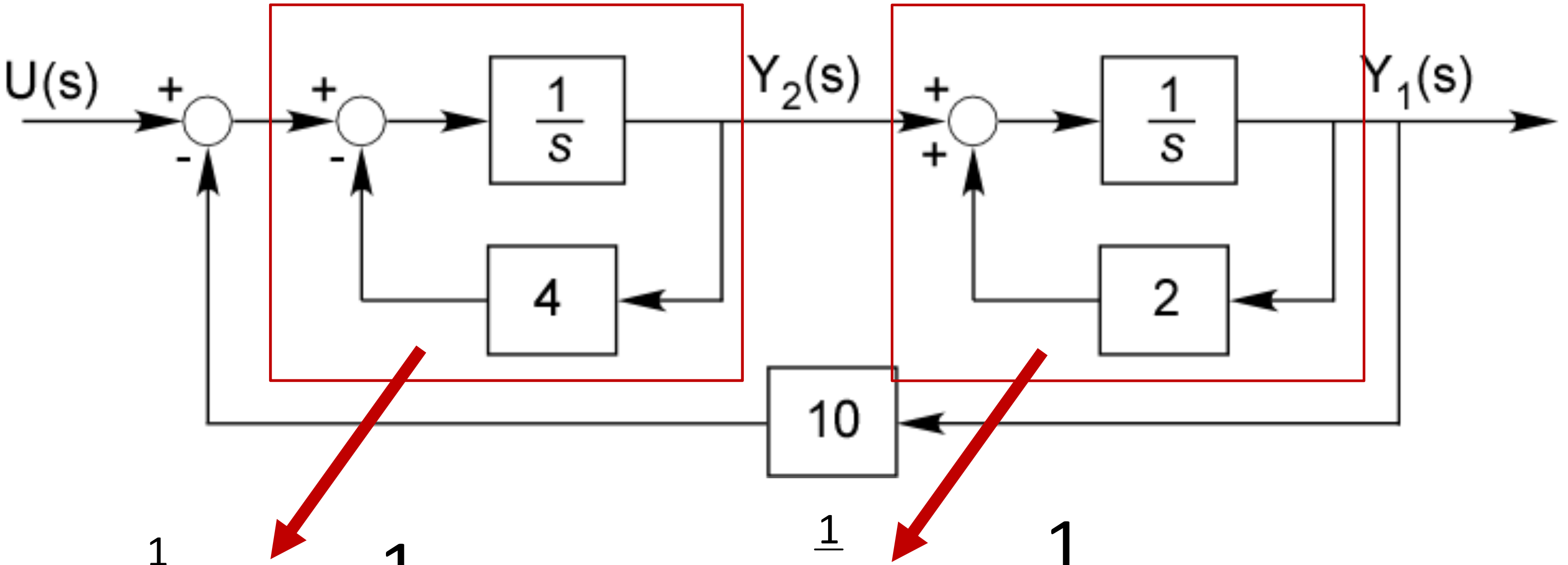
Block Diagram Reduction

Example: Evaluate the transfer function $\dfrac{Y_1(s)}{U(s)}$



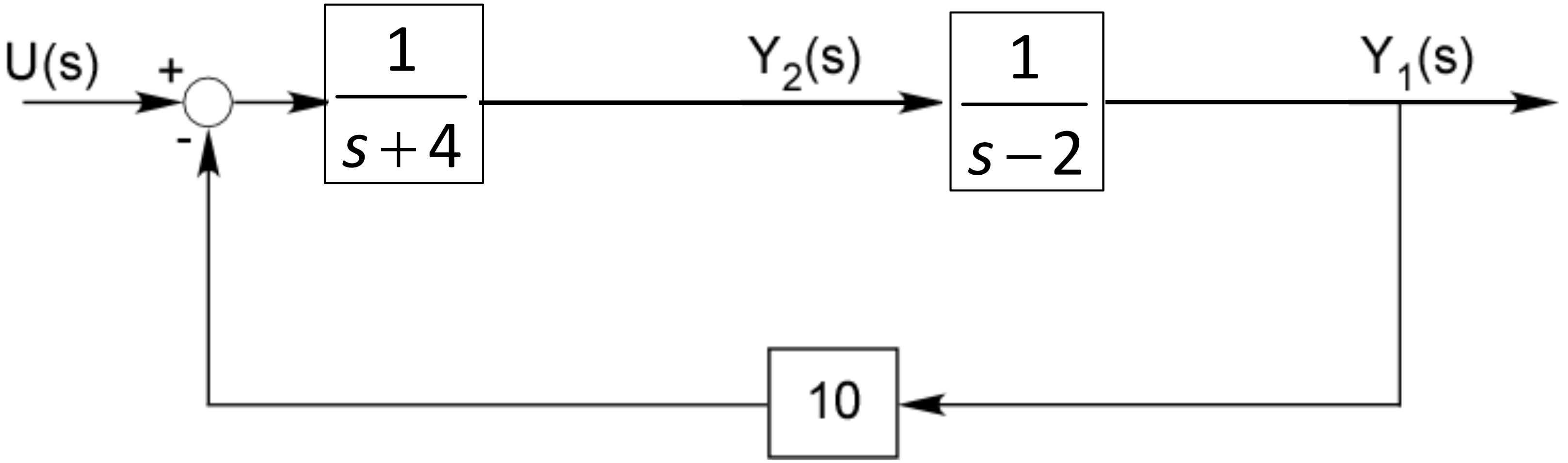
Block Diagram Reduction

Example: Evaluate the transfer function $\dfrac{Y_1(s)}{U(s)}$



$$\frac{\frac{1}{s}}{1+4\frac{1}{s}}=\frac{1}{s+4}$$

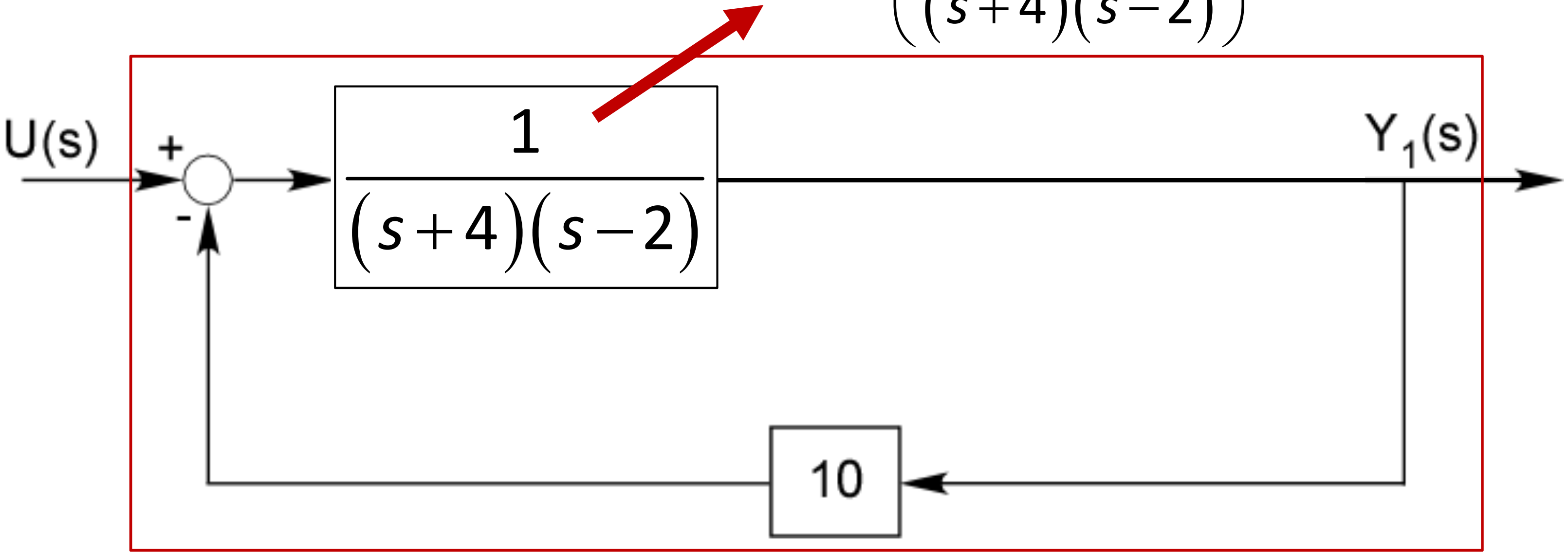
$$\frac{\frac{1}{s}}{1-2\frac{1}{s}}=\frac{1}{s-2}$$

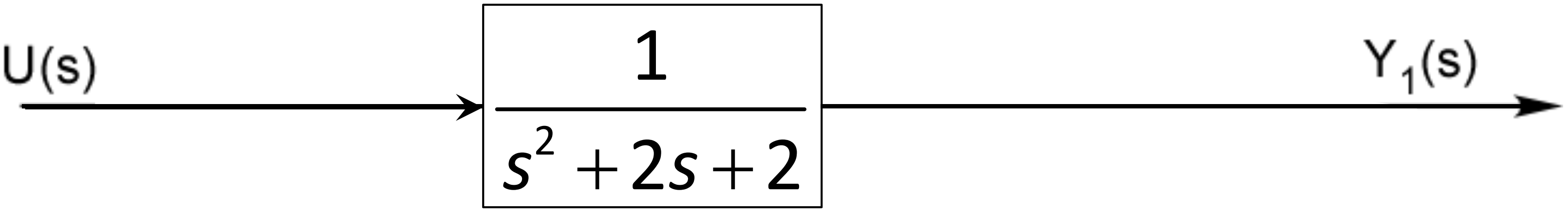


Block Diagram Reduction

Example: continued

$$\frac{\dfrac{1}{(s+4)(s-2)}}{1+10\left(\dfrac{1}{(s+4)(s-2)}\right)} = \frac{1}{s^2+2s+2}$$





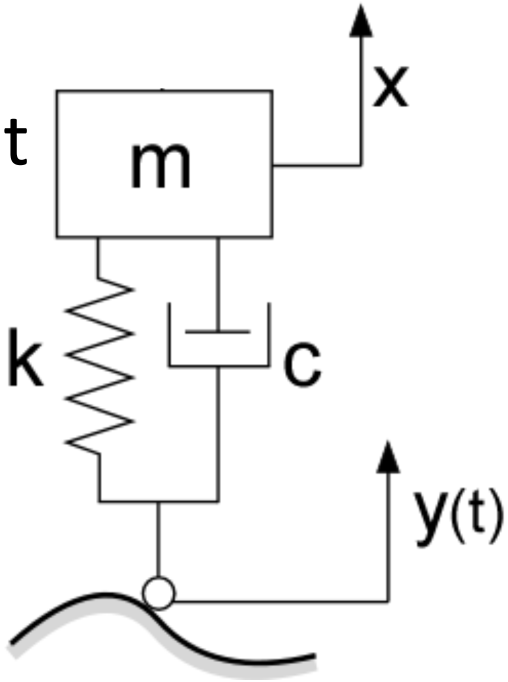
$$\frac{Y(s)}{U(s)} = \frac{1}{s^2+2s+2}$$

Block Diagram Construction

- A system block diagram is not unique
- A useful procedure to construct block diagrams:
  - Find system equations of motion
  - In each equation, solve for highest derivative of dependent variable – and use as input to integrator block
  - Form lower derivatives using integrator blocks

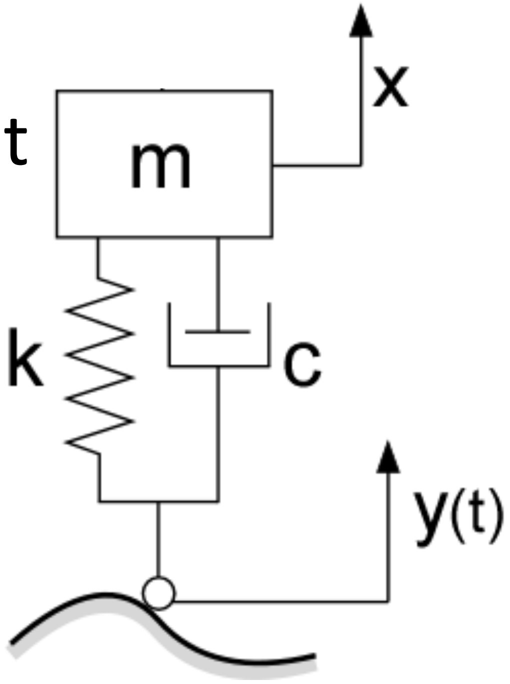
Block Diagram Construction

Example: Construct block diagram of system shown with *f* and *y* as inputs and *x* as the output



Block Diagram Construction

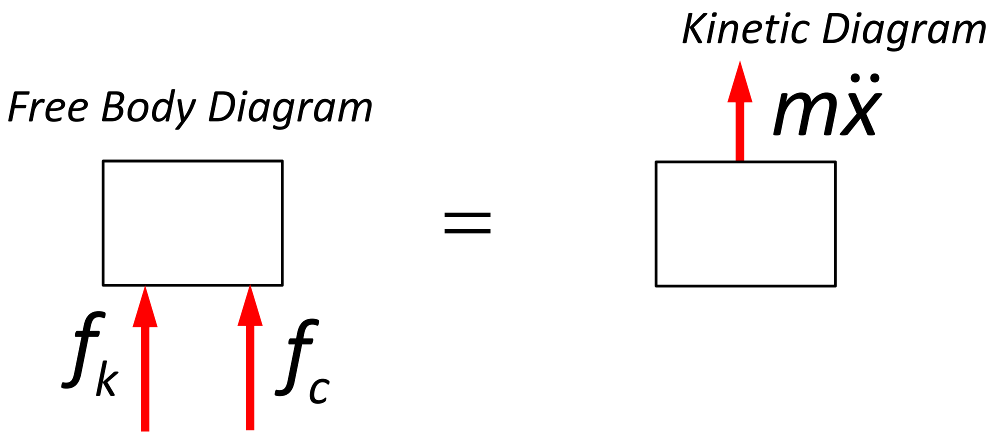
Example: Construct block diagram of system shown with *f* and *y* as inputs and *x* as the output



Newton's Law
(Planar Dynamics)

$$\vec{F} = m\vec{a}$$

$$\tau = I\alpha$$


Free Body Diagram

Kinetic Diagram

$$\underbrace{k\left(y-x\right)}_{f_k}+\underbrace{c\left(\dot{y}-\dot{x}\right)}_{f_c}=m\ddot{x}_2$$

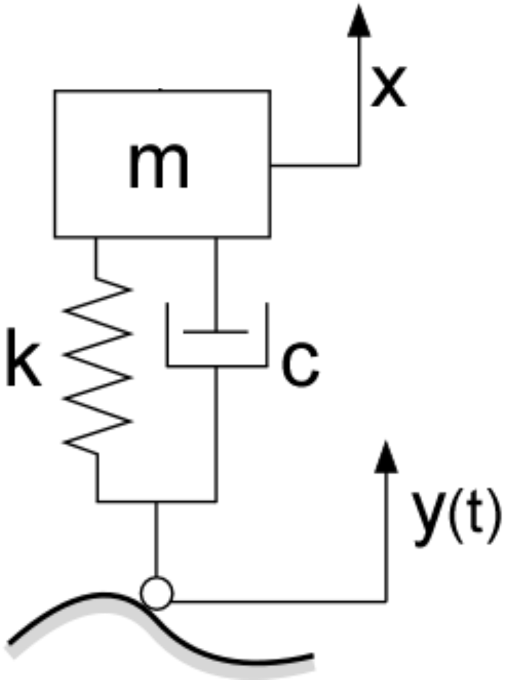
Equation of motion:

$$\boxed{m\ddot{x}+c\dot{x}+kx=c\dot{y}+ky}$$

Block diagram?

Block Diagram Construction

Example: continued

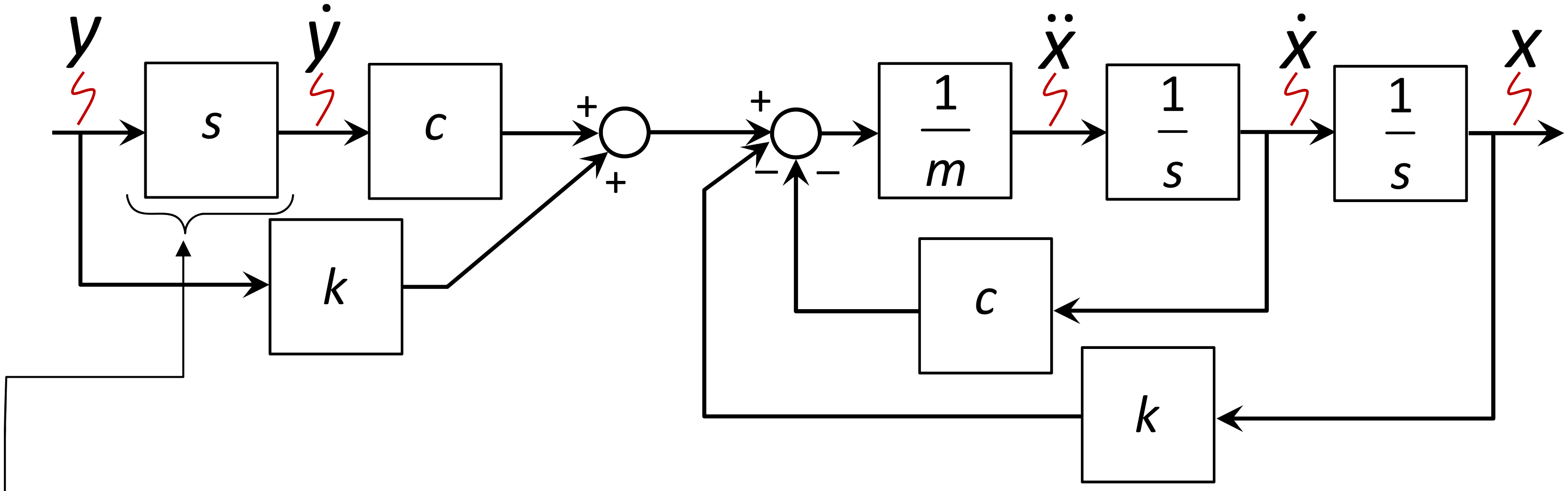


Equation of motion:

$$m\ddot{x} + c\dot{x} + kx = c\dot{y} + ky$$

$$\ddot{x} = \tfrac{1}{m}\left(-c\dot{x} - kx + c\dot{y} + ky\right)$$

Note: the solution is not unique

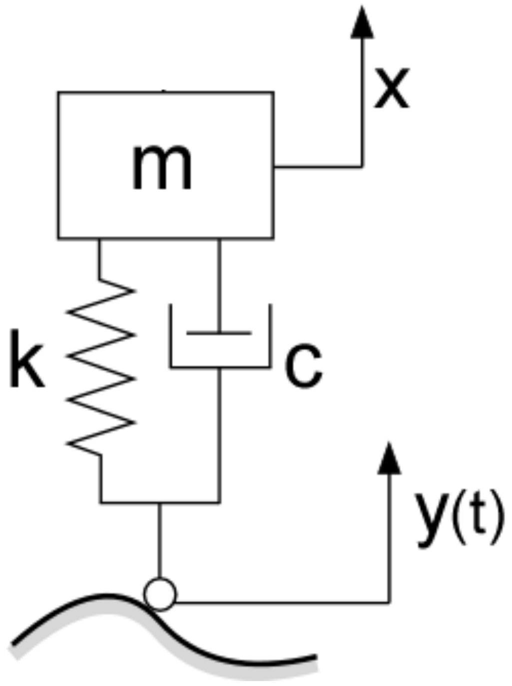


FYI: Simulation software (e.g. Matlab/Simulink), doesn't like **derivative** terms. See following slides in posted lecture notes

Block Diagram Construction

Example: continued (optional)

Equation of motion:



$$m\ddot{x} + c\dot{x} + kx = c\dot{y} + ky$$

Alternatively (to avoid use of derivative):

Evaluate the transfer function *X(s)/Y(s)*

$$\left(ms^2 + cs + k\right)X\left(s\right) = \left(cs + k\right)Y\left(s\right)$$

$$\frac{X\left(s\right)}{Y\left(s\right)} = \underbrace{\left(\frac{1}{ms^2 + cs + k}\right)}_{}\underbrace{\left(cs + k\right)}_{}$$

Define intermediate variable *z*

$$\frac{X\left(s\right)}{Z\left(s\right)} = \left(cs + k\right) \Rightarrow x = c\dot{z} + kz$$

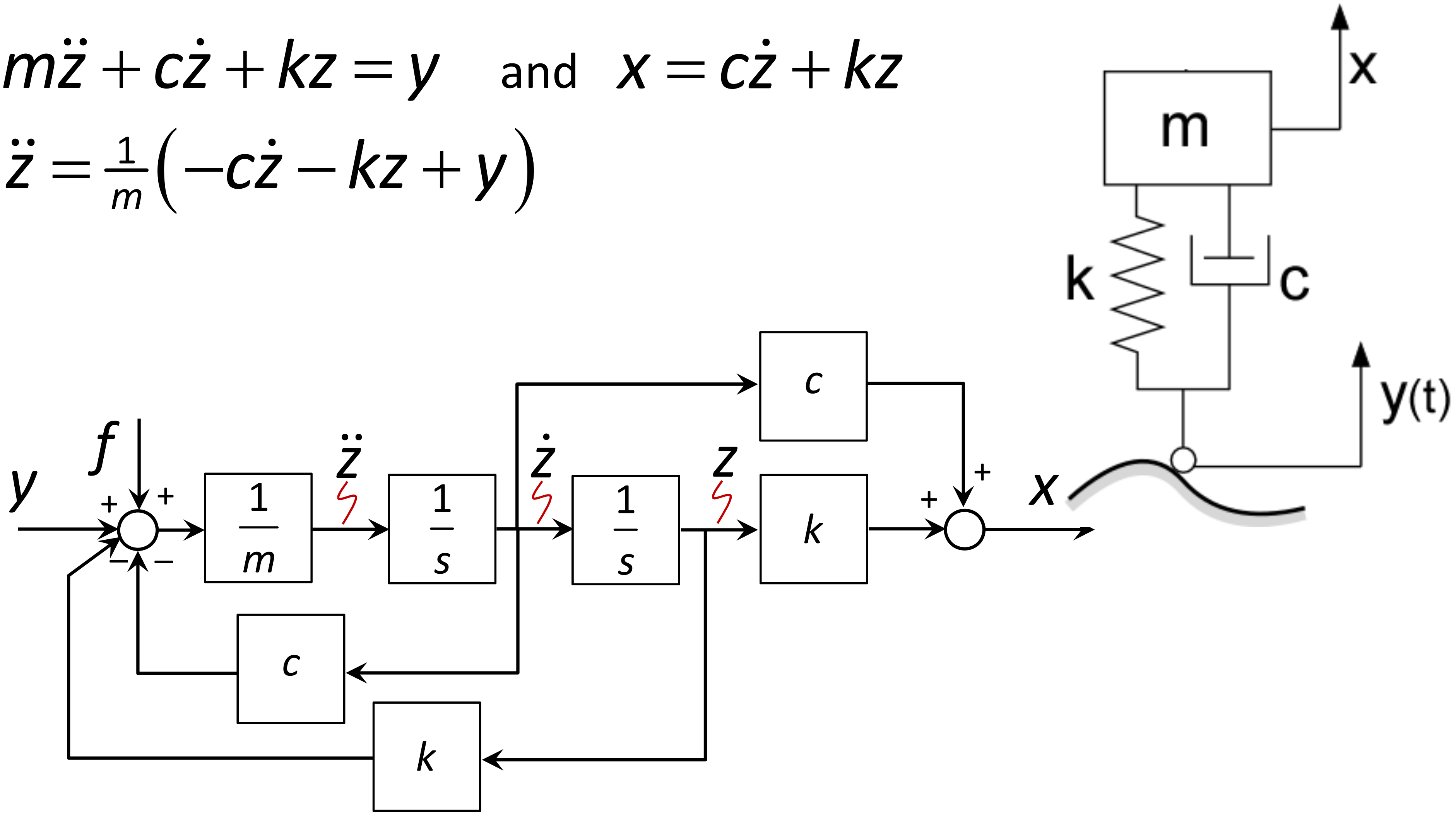
$$\frac{Z\left(s\right)}{Y\left(s\right)} = \frac{1}{ms^2 + cs + k} \Rightarrow m\ddot{z} + c\dot{z} + kz = y$$

Block Diagram Construction

Example: continued (optional)

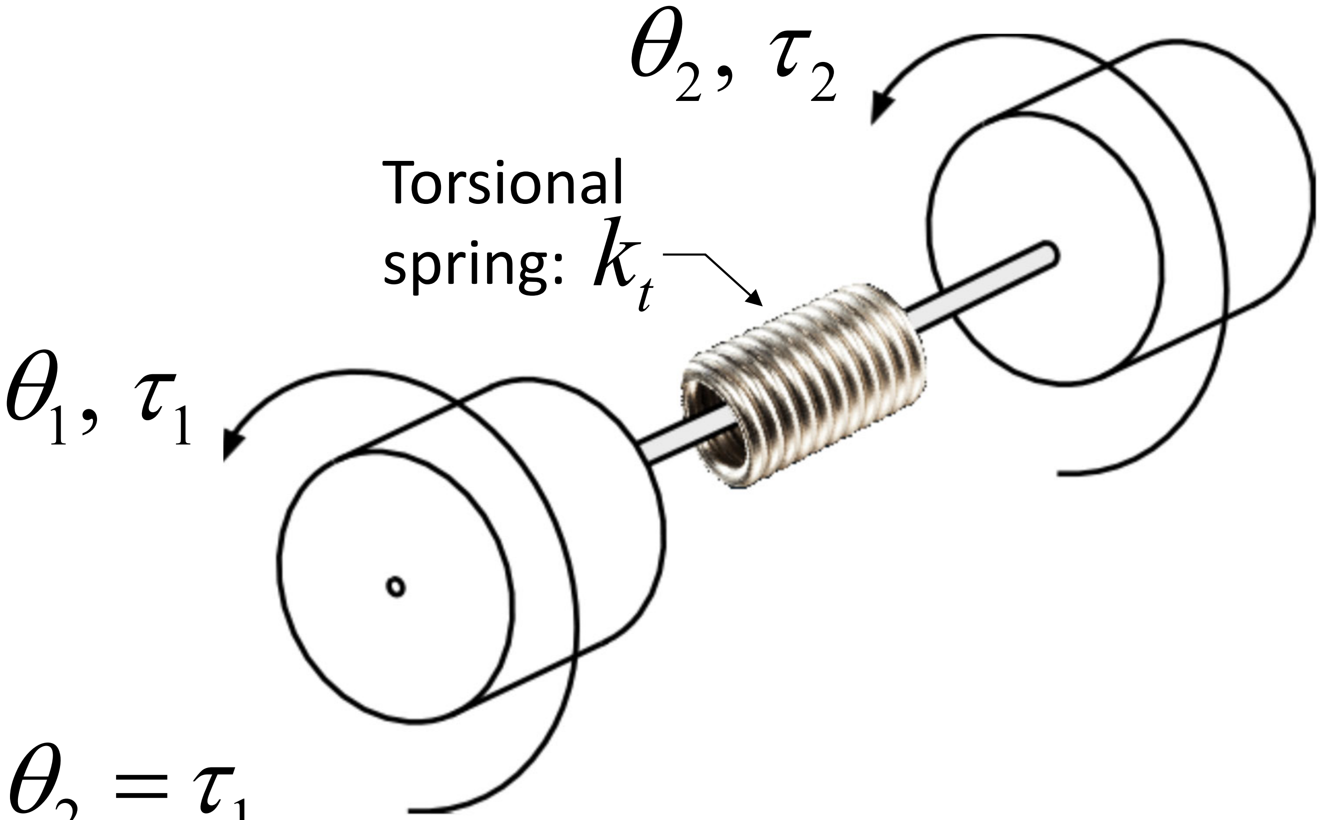
$$m\ddot{z}+c\dot{z}+kz=y \quad \text{and} \quad x=c\dot{z}+kz$$

$$\ddot{z}=\tfrac{1}{m}\left(-c\dot{z}-kz+y\right)$$



Block Diagram Construction

Example: Construct block diagram of system shown with τ_1 and τ_2 as inputs and θ_1 and θ_2 as the output



Equations of motion:

$$I_1\ddot{\theta}_1 + k_t\theta_1 - k_t\theta_2 = \tau_1$$

$$I_2\ddot{\theta}_2 - k_t\theta_1 + k_t\theta_2 = \tau_2$$

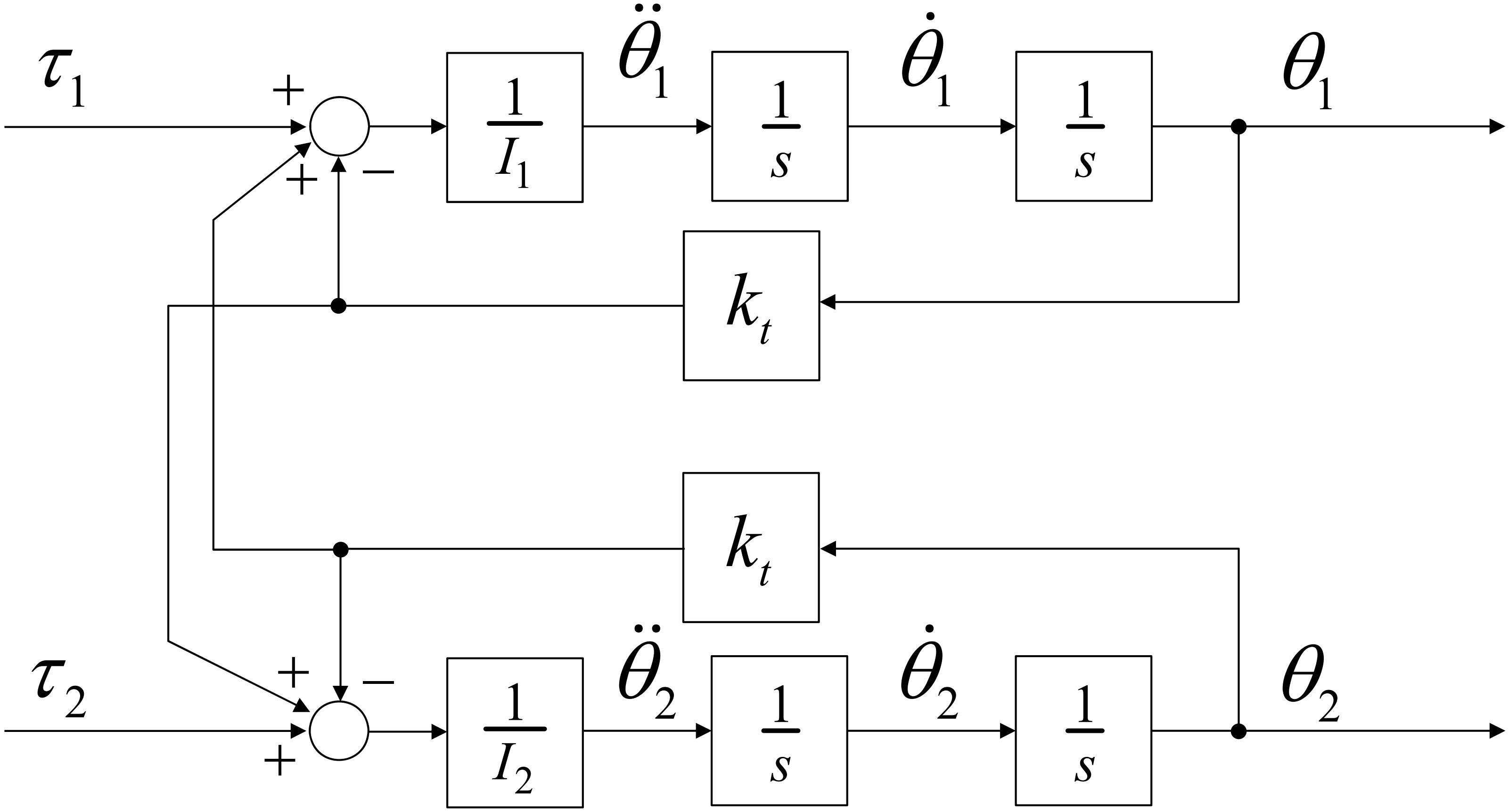
Block Diagram Construction

Example: continued

Equations of motion:

$$I_1\ddot{\theta}_1 + k_t\theta_1 - k_t\theta_2 = \tau_1 \longrightarrow \ddot{\theta}_1 = \tfrac{1}{I_1}\left(\tau_1 - k_t\theta_1 + k_t\theta_2\right)$$

$$I_2\ddot{\theta}_2 - k_t\theta_1 + k_t\theta_2 = \tau_2 \longrightarrow \ddot{\theta}_2 = \tfrac{1}{I_2}\left(\tau_2 + k_t\theta_1 - k_t\theta_2\right)$$



Matlab Block Diagram Construction

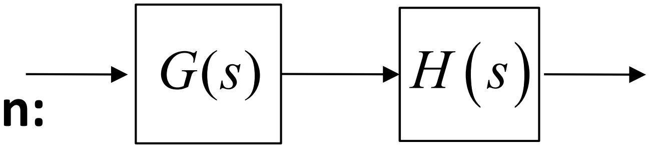
$$G(s)=\frac{1}{s+3} \qquad H(s)=\frac{2s}{s^2+3}$$

```
s = tf('s');
G = 1/(s + 3);
H = 2*s/(s^2 + 3);
```

Alternatively …

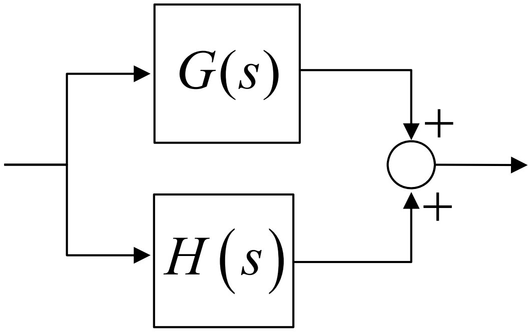
```
num = [1];
den = [1 3];
G = tf(num,den);
```

Series Connection:



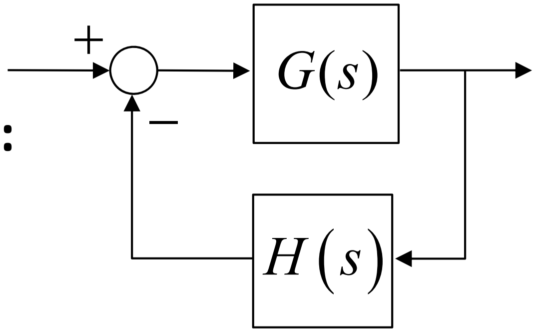
```
sys = G*H

                     2 s
sys = ----------------------
      s^3 + 3 s^2 + 3 s + 90
```

Parallel Connection:



```
sys = G + H
          3 s^2 + 6 s + 3
sys = ---------------------
      s^3 + 3 s^2 + 3 s + 9
```

Feedback:



```
sys = feedback(G,H)
              s^2 + 3
sys = ---------------------
      s^3 + 3 s^2 + 5 s + 9
```

`feedback` assumes negative feedback.
For positive feedback use `feedback(sys1,sys1,+1)`

For complex systems use **Simulink**

Modeling & Analysis of Dynamic Systems

- Topic Overview -

- Dynamic System Modeling 2.1 – 2.4
- Dynamic Systems – Differential Equations 3.1 – 3.6
- Laplace Transform and Linear Differential Equations 3.1
- Transfer Functions 3.1.2
- Block Diagram Modeling 3.2.1 – 3.2.2
- ➡ Linear System Response Characteristics 3.3
- Time Domain Specifications 3.4
- Effects of System Poles and Zeros 3.5

Response Characteristics: $\Delta(s)$

Recall, solution of *x(t)* is expressed as:

$$x(t) = C_1 e^{s_1 t} + C_2 e^{s_2 t} + \cdots + C_n e^{s_n t}$$

where $s_1, s_2, \ldots, s_n$ are the roots of the denominator of $X(s)$

➡ *Question: where do the roots come from?*

Prior example: $\ddot{x} + 8\dot{x} + 12x = 12 \qquad x(0) = 0$ and $\dot{x}(0) = 4$

Laplace transform:

$$X(s) = \frac{4s+12}{s\underbrace{\left(s^2+8s+12\right)}_{\textit{Characteristic equation, } \Delta(s)}} \qquad = \frac{1}{s} + \frac{-\frac{1}{2}}{s+2} + \frac{-\frac{1}{2}}{s+6}$$

← **Residues**: from I.C.s and input

input

Roots of $\Delta(s)$

$\Delta(s)$: Laplace transform with I.C.s set equal to 0

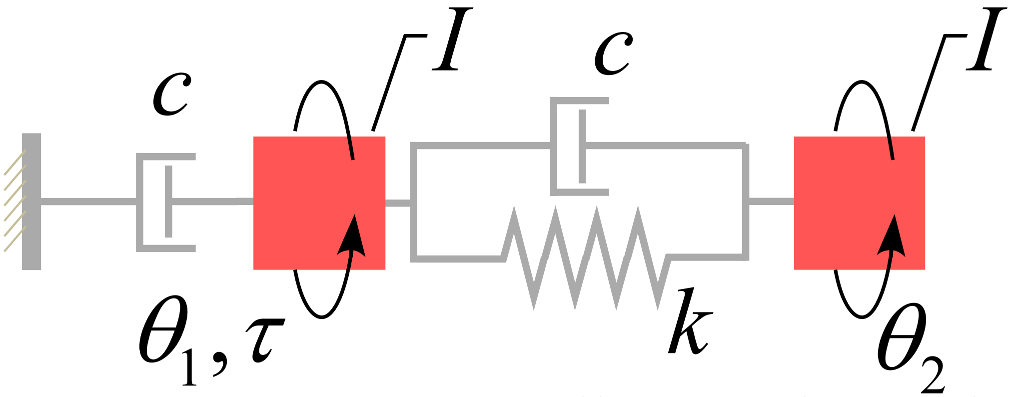
Inverse Laplace transform (solution):

$$x(t) = 1 \underbrace{- \tfrac{1}{2} e^{-2t} - \tfrac{1}{2} e^{-6t}}_{\text{Homogeneous solution}}$$

Response controlled by roots of $\Delta(s)$
(and ICs, inputs)

Response Characteristics: $\Delta(s)$

Example #2: Response of θ_2 due to $\tau = \delta(t)$:



$I = 2\ [\text{kg-m}^2]$

$c = 30\ [\text{N-m/r/s}]$

$k = 50\ [\text{N-m/r}]$

Equations of motion:
$$\begin{cases} I\ddot{\theta}_1 + 2c\dot{\theta}_1 - c\dot{\theta}_2 + k\theta_1 - k\theta_2 = \tau \\ I\ddot{\theta}_2 + c\dot{\theta}_2 - c\dot{\theta}_1 + k\theta_2 - k\theta_1 = 0 \end{cases}$$

Form transfer functions $\theta_1(s)/\tau(s)$ and $\theta_2(s)/\tau(s)$

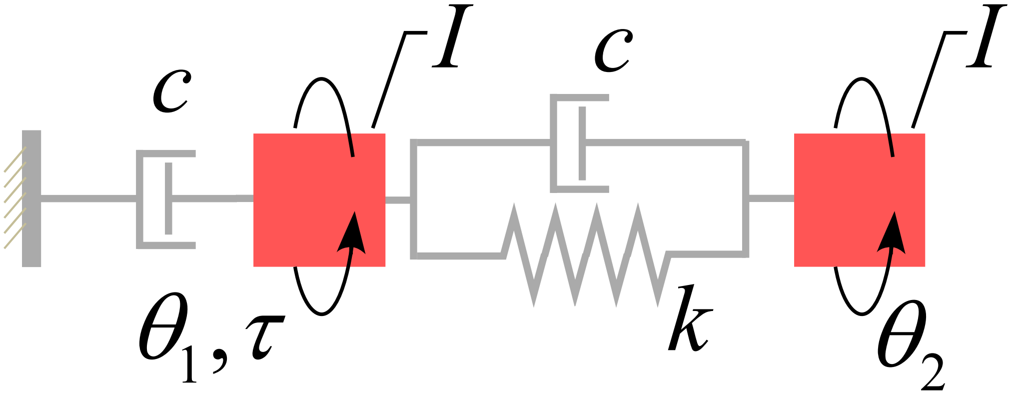
Take the Laplace transform of the equations of motion:

$$\left(Is^2 + 2cs + k\right)\theta_1(s) - \left(cs + k\right)\theta_2(s) = \tau(s)$$

$$\left(Is^2 + cs + k\right)\theta_2(s) - \left(cs + k\right)\theta_1(s) = 0$$

Response Characteristics: $\Delta(s)$

Example #2: Response of θ_2 due to $\tau = \delta(t)$:



$I = 2\ [\text{kg-m}^2]$

$c = 30\ [\text{N-m/r/s}]$

$k = 50\ [\text{N-m/r}]$

Solve for $\theta_1(s)/\tau(s)$ and $\theta_2(s)/\tau(s)$

$$\frac{\theta_1(s)}{\tau(s)} = \frac{Is^2 + cs + k}{(I^2)s^4 + (3Ic)s^3 + (c^2 + 2Ik)s^2 + (ck)s}$$

$$\frac{\theta_2(s)}{\tau(s)} = \frac{cs + k}{(I^2)s^4 + (3Ic)s^3 + (c^2 + 2Ik)s^2 + (ck)s}$$

$\Delta(s)$

Response Characteristics: $\Delta(s)$

Let’s examine response of θ_2:

$$\frac{\theta_2(s)}{\tau(s)} = \frac{30s+50}{4s^4+180s^3+1100s^2+1500s} \longleftarrow \Delta(s)$$

Evaluate response of θ_2 due to $\tau = \delta(t)$

$$\theta_2(s) = \frac{30s+50}{4s^4+180s^3+1100s^2+1500s}\,\overset{\downarrow\,\longleftarrow\,\tau(s)=1}{\tau(s)}$$

Roots of $\Delta(s)$: `r = roots([4 180 1100 1500])` ⟶

```
r = -38.0278
     -5.0000
     -1.9722
```

Partial fraction expansion:

$$\frac{\theta_2(s)}{\tau(s)} = \frac{C_1}{s+1.97} + \frac{C_2}{s+5} + \frac{C_3}{s+38.03}$$

After some algebra …

$$\frac{\theta_2(s)}{\tau(s)} = -\frac{0.021}{s+1.97} + \frac{0.25}{s+5} - \frac{0.229}{s+38.03}$$

Inverse Laplace transform:

$$\theta_2(t) = -0.021e^{-1.97t} + 0.25e^{-5t} - 0.229e^{-38.03t}$$

Response controlled by roots of $\Delta(s)$

Response Characteristics: Roots of $\Delta(s)$

Recall, solution of *x(t)* is expressed as:

$$x(t) = C_1 e^{s_1 t} + C_2 e^{s_2 t} + \cdots + C_n e^{s_n t}$$

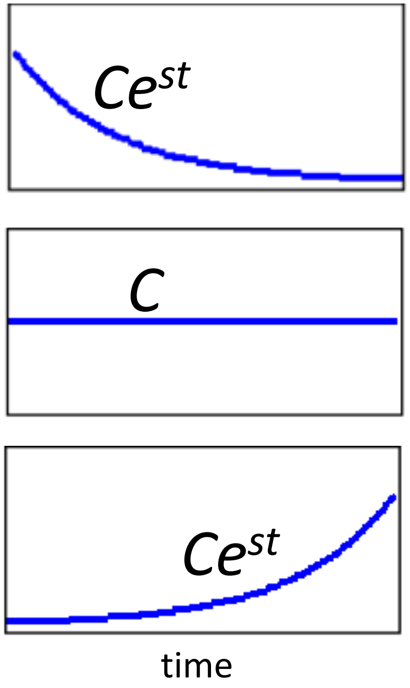
Roots can be real or complex conjugate pairs*

$$\Delta(s) = s^n + a_{n-1}s^{n-1} + \cdots + a_0 = 0$$

Response when root is real:

$$x(t) = \underbrace{C}_{\text{Due to I.C.s}} e^{s_i t}$$

$$\text{if} \begin{cases} s_i < 0: & x(t) \\ s_i = 0: & x(t) \\ s_i > 0: & x(t) \end{cases}$$



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* **complex conjugate root theorem**: if *P* is a polynomial in one variable with real coefficients, and *a* + *bi* is a root of *P*, then its complex conjugate *a* − *bi* is also a root of *P*.

Response Characteristics: Roots of $\Delta(s)$

Recall, solution of *x(t)* is expressed as:

$$x(t) = C_1 e^{s_1 t} + C_2 e^{s_2 t} + \cdots + C_n e^{s_n t}$$

Roots can be real or complex conjugate pairs

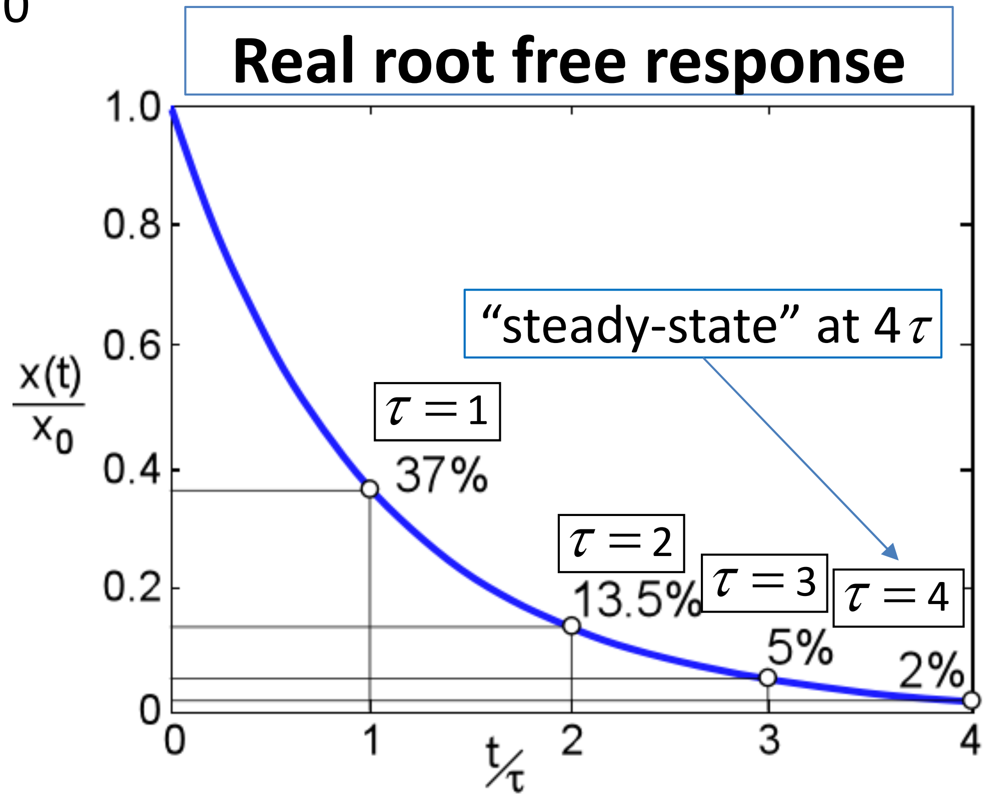
$$\Delta(s) = s^n + a_{n-1} s^{n-1} + \cdots + a_0 = 0$$

Response when root is real:

$$x(t) = \underbrace{C}_{\text{Due to I.C.s}} e^{s_i t}$$

$$x(t) = C e^{-t/\tau_i}$$

time constant: $\tau_i = -\frac{1}{s_i}$



Response Characteristics: Roots of $\Delta(s)$

Recall, solution of *x(t)* is expressed as:

$$x(t) = C_1 e^{s_1 t} + C_2 e^{s_2 t} + \cdots + C_n e^{s_n t}$$

Roots can be real or complex conjugate pairs

$$\Delta(s) = s^n + a_{n-1} s^{n-1} + \cdots + a_0 = 0$$

Response when roots are complex conjugates pairs:

$$x(t) = C_1 e^{st} + C_2 e^{s^* t} \quad \text{where} \quad \begin{aligned} s &= \sigma + j\omega \\ s^* &= \sigma - j\omega \end{aligned} \quad \text{where} \quad j = \sqrt{-1}$$

the inverse Laplace transform has complex exponents and coefficients

Can be expressed in terms of real coefficients as

$$x(t) = Ce^{\sigma t} \sin(\omega t + \phi) \quad \text{where} \quad \phi = \tan^{-1}\left(-\frac{\mathrm{Re}\{C_1\}}{\mathrm{Im}\{C_1\}}\right)$$

Note: ω and ω_d are used interchangeably

Note: C_1 and C_2 are complex conjugates

$$C = 2\sqrt{\mathrm{Re}\{C_1\}^2 + \mathrm{Im}\{C_1\}^2}$$

Complex Conjugate Roots: $x(t) = D_1 e^{st} + D_2 e^{s^* t}$

With complex conjugate roots, the inverse Laplace transform has complex exponents and coefficients

➡ While mathematically correct complex terms are difficult to interpret

$$x(t) = D_1 e^{st} + D_2 e^{s^* t} \quad \textit{where} \quad \begin{aligned} s &= \sigma + j\omega \\ s^* &= \sigma - j\omega \end{aligned} \qquad \text{where } j = \sqrt{-1}$$

$$= D_1 e^{(\sigma + j\omega)t} + D_2 e^{(\sigma - j\omega)t}$$

$$= D_1 e^{\sigma t} e^{j\omega t} + D_2 e^{\sigma t} e^{-j\omega t}$$

Using Euler's theorem: $e^{j\theta} = \cos\theta + j\sin\theta$

and applying to the terms $e^{j\omega t}$ and $e^{-j\omega t}$

$$x(t) = D_1 e^{\sigma t}\left(\cos\omega t + j\sin\omega t\right) + D_2 e^{\sigma t}\left(\cos(-\omega t) + j\sin(-\omega t)\right)$$

$$= e^{\sigma t}\left[\underbrace{(D_1 + D_2)}\cos\omega t + \underbrace{j(D_1 - D_2)}\sin\omega t\right]$$

must be real

Optional derivation

Complex Conjugate Roots: $x(t)=D_1e^{st}+D_2e^{s^*t}$

$$x(t)=e^{\sigma t}\left(\underbrace{(D_1+D_2)}\cos\omega t+\underbrace{j(D_1-D_2)}\sin\omega t\right)$$

must be real

$$\left.\begin{aligned}\operatorname{Im}\{D_1+D_2\}=0 &\longrightarrow \operatorname{Im}\{D_1\}=-\operatorname{Im}\{D_2\}\\ \operatorname{Im}\{j(D_1-D_2)\}=0 &\longrightarrow \operatorname{Re}\{D_1\}=\operatorname{Re}\{D_2\}\end{aligned}\right\}$$

D_1 and D_2 are complex conjugates

letting $D_1=D$ and $D_2=D^*$

$$x(t)=e^{\sigma t}\left((D+D^*)\cos\omega t+j(D-D^*)\sin\omega t\right)$$

$$=e^{\sigma t}\left(2\operatorname{Re}\{D\}\cos\omega t-2\operatorname{Im}\{D\}\sin\omega t\right)$$

Coefficients are now real

Optional derivation

Complex Conjugate Roots: $x(t)=D_1e^{st}+D_2e^{s^*t}$

$$x(t)=e^{\sigma t}\left(2\mathrm{Re}\{D\}\cos\omega t-2\mathrm{Im}\{D\}\sin\omega t\right)$$

Coefficients are now real

We want to get this into the form

$$\begin{aligned}x(t)&=Ce^{\sigma t}\sin(\omega t+\phi)\\&=e^{\sigma t}\Big(\underbrace{C\cdot\sin\phi}_{2\mathrm{Re}\{D\}}\cdot\cos\omega t+\underbrace{C\cdot\cos\phi}_{-2\mathrm{Im}\{D\}}\cdot\sin\omega t\Big)\end{aligned}$$

$$\left.\begin{aligned}\sin\phi&=2\mathrm{Re}\{D\}/C\\\cos\phi&=-2\mathrm{Im}\{D\}/C\end{aligned}\right\}\begin{aligned}\tan\phi=\tfrac{\sin\phi}{\cos\phi}&=-\mathrm{Re}\{D\}/\mathrm{Im}\{D\}\\\underline{\phi=\tan^{-1}\left(-\mathrm{Re}\{D\}/\mathrm{Im}\{D\}\right)}\end{aligned}$$

$$\sin^2\phi+\cos^2\phi=4\mathrm{Re}\{D\}^2/C^2+4\mathrm{Im}\{D\}^2/C^2=1$$

$$\underline{C=2\sqrt{\mathrm{Re}\{D\}^2+\mathrm{Im}\{D\}^2}}$$

Optional derivation

Complex Conjugate Roots: $x(t) = D_1 e^{st} + D_2 e^{s^* t}$

With complex conjugate roots, the inverse Laplace transform has complex exponents and coefficients

$$x(t) = De^{st} + D^* e^{s^* t} \quad \text{where} \quad \begin{aligned} s &= \sigma + j\omega \\ s* &= \sigma - j\omega \end{aligned} \qquad \text{where } j = \sqrt{-1}$$

Can be expressed in terms of real coefficients as

$$x(t) = Ce^{\sigma t} \sin(\omega t + \phi)$$

$$\text{where} \qquad \phi = \tan^{-1}\left(-\frac{\operatorname{Re}\{D\}}{\operatorname{Im}\{D\}}\right)$$

$$C = 2\sqrt{\operatorname{Re}\{D\}^2 + \operatorname{Im}\{D\}^2}$$

Optional derivation

Response Characteristics: Roots of $\Delta(s)$

Recall, solution of *x(t)* is expressed as:

$$x(t) = C_1 e^{s_1 t} + C_2 e^{s_2 t} + \cdots + C_n e^{s_n t}$$

Roots can be real or complex conjugate pairs

$$\Delta(s) = s^n + a_{n-1} s^{n-1} + \cdots + a_0 = 0$$

Response when roots are complex:

$$x(t) = C_1 e^{st} + C_2 e^{s^* t}$$

From I.C.s

$\vdots$

$$x(t) = Ce^{\sigma t} \sin(\omega_d t + \phi)$$

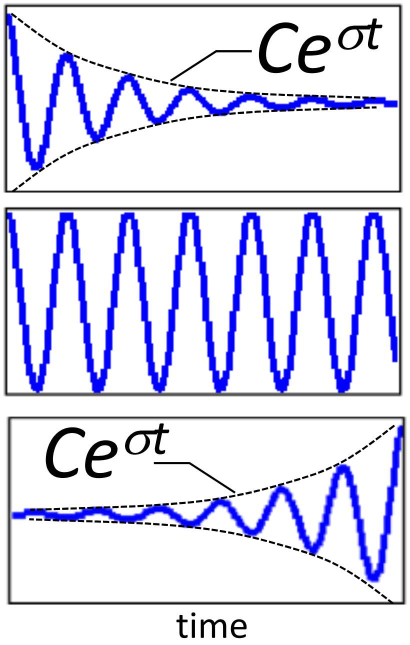
where $s = \sigma \pm j\omega_d$

if

Re{s} < 0: *x(t)*

Re{s} = 0: *x(t)*

Re{s} > 0: *x(t)*

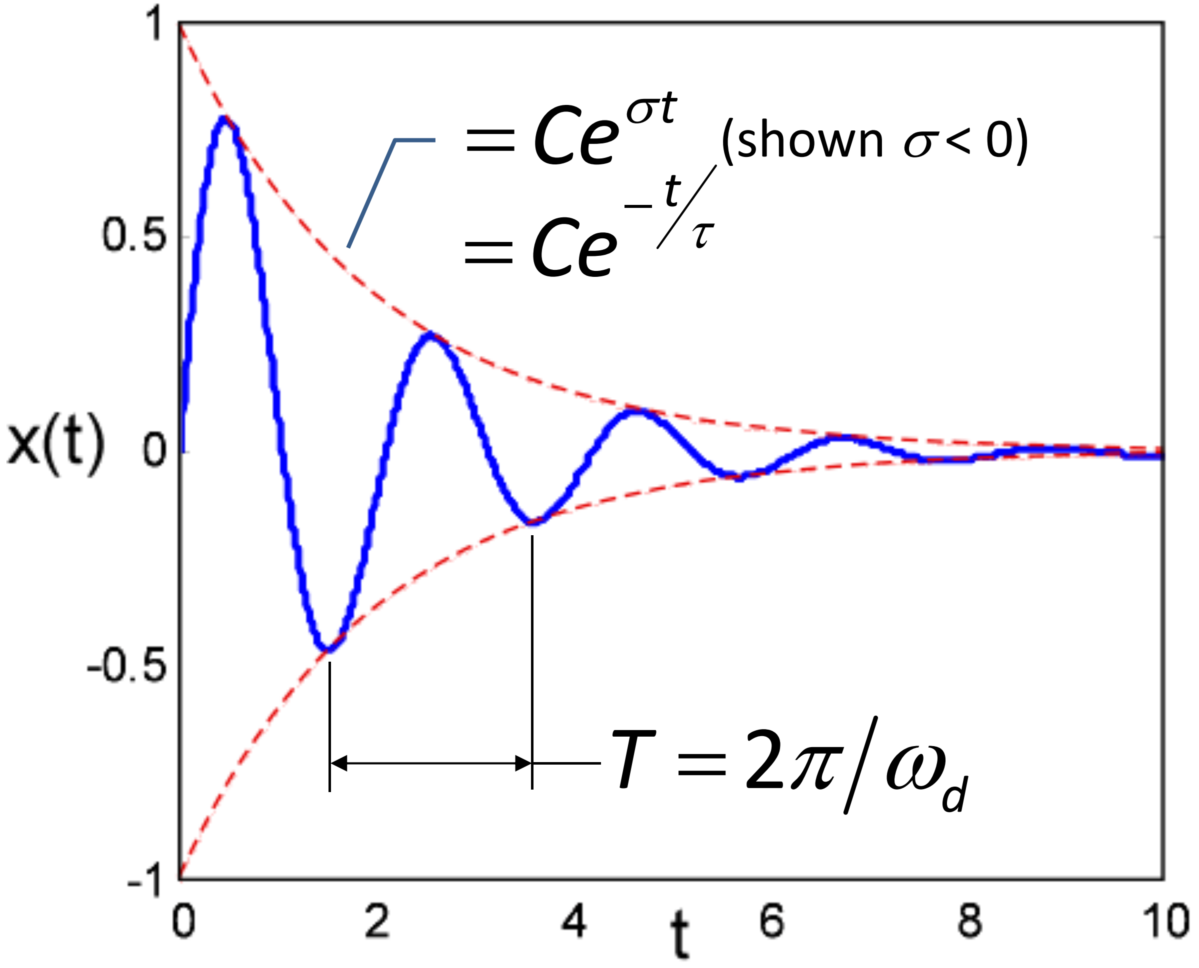


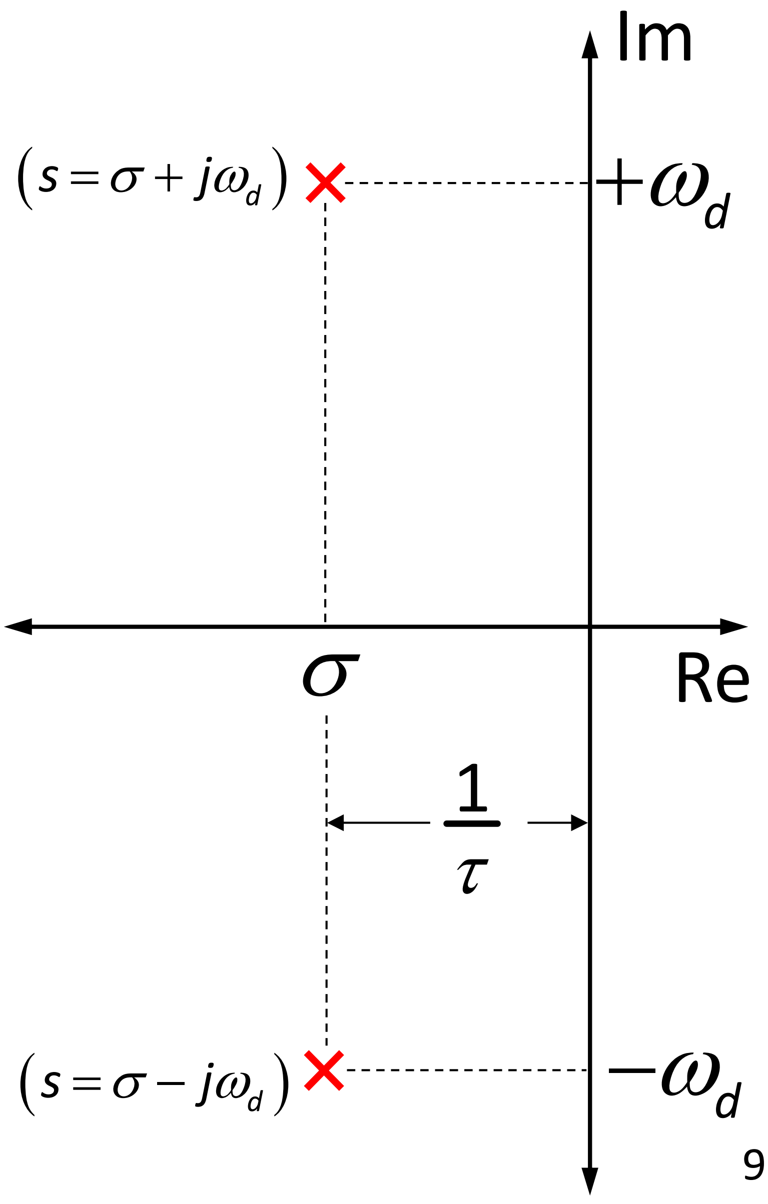
Response Characteristics: Roots of $\Delta(s)$

$$x(t) = Ce^{\sigma t}\sin\left(\omega_d t + \phi\right)$$

where $s = \sigma \pm j\omega_d$

Very useful to visualize roots on the complex plane (or *s*-plane)

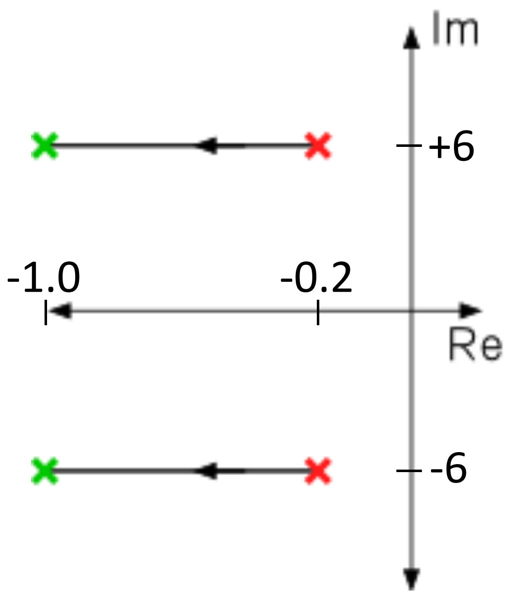


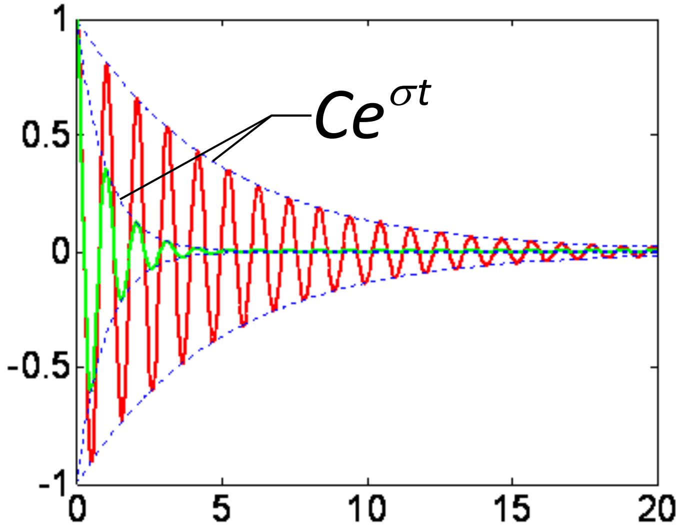


Response Characteristics: Roots of $\Delta(s)$

Change real part, σ (decrease τ):

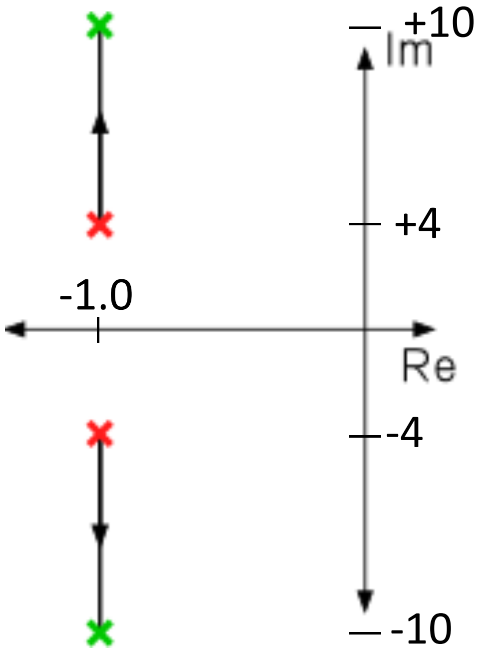
$$\tau = \begin{cases} 5 \text{ sec} \\ 1 \text{ sec} \end{cases} \quad \omega_d = 6 \text{ r/s}$$

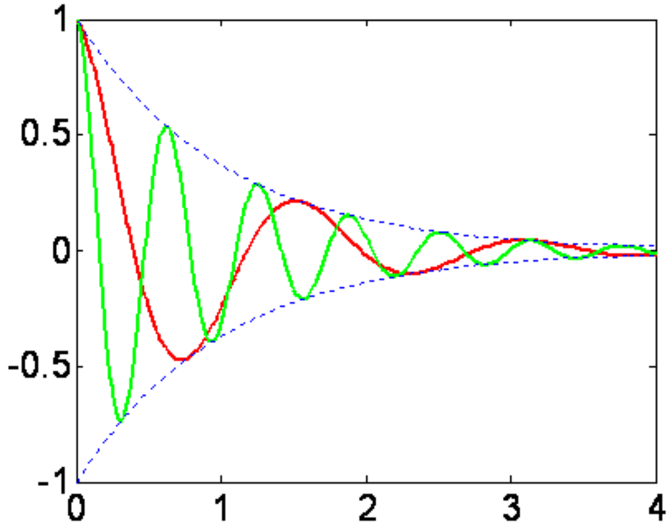




Change imaginary part, ω_d:

$$\omega_d = \begin{cases} 4 \text{ r/s} \\ 10 \text{ r/s} \end{cases} \quad \tau = 1 \text{ sec}$$





Response Characteristics: Roots of $\Delta(s)$

$$x(t)=Ce^{\sigma t}\sin\left(\omega_d t+\phi\right) \quad \textit{where} \quad s=\sigma\pm j\omega_d$$

- While describing response with σ and ω_d is useful, developing a normalized description is also desired

Start by evaluating (portion of) characteristic equation attributable to $s=\sigma\pm j\omega_d$

$$\left(s-\sigma-j\omega_d\right)\left(s-\sigma+j\omega_d\right)=0$$

$$s^2-2\sigma s+\underbrace{\left(\sigma^2+\omega_d^2\right)}=0$$

Define *damping ratio*:

$$\zeta \triangleq -\frac{\sigma}{\omega_n} \rightarrow \sigma=-\zeta\omega_n$$

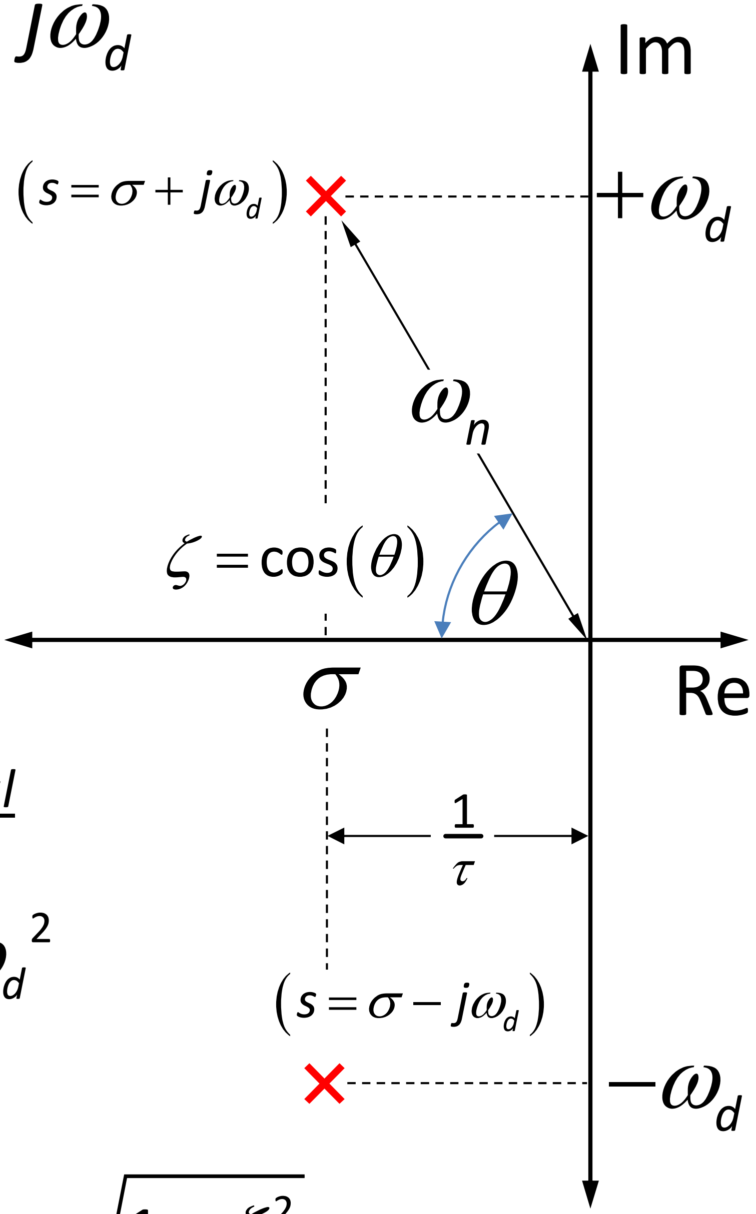
Define *undamped natural frequency*:

$$\omega_n^2 \triangleq \sigma^2+\omega_d^2$$

$$\boxed{s^2+2\zeta\omega_n s+\omega_n^2=0}$$

Evaluate roots as function of ω_n and ζ to equate to complex roots $s=\sigma\pm j\omega_d$

$$\begin{cases}\omega_d=\omega_n\sqrt{1-\zeta^2}\\ \sigma=-\zeta\omega_n\end{cases}$$



Response Characteristics: Roots of $\Delta(s)$

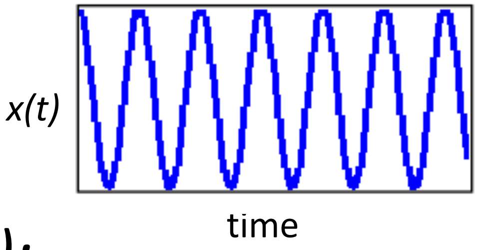
$$x(t) = Ce^{\sigma t}\sin\left(\omega_d t + \phi\right) \quad \textit{where } s = \sigma \pm j\omega_d$$

normalized form: $s^2 + 2\zeta\omega_n s + \omega_n^{\ 2} \ = 0$

when $\zeta = 0$ (undamped):

$$s^2 + \omega_n^{\ 2} = 0 \Rightarrow s = \pm j\omega_n$$

$$x(t) = C\sin\left(\omega_n t + \phi\right)$$

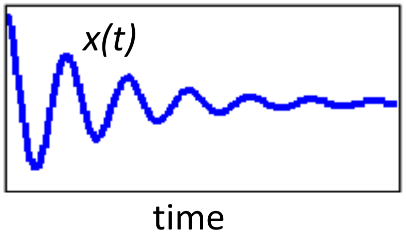


when $0 < \zeta < 1$ (underdamped):

$$s^2 + 2\zeta\omega_n s + \omega_n^{\ 2} = 0$$

$$\Rightarrow s = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$$

$$x(t) = Ce^{\sigma t}\sin\left(\omega_d t + \phi\right)$$

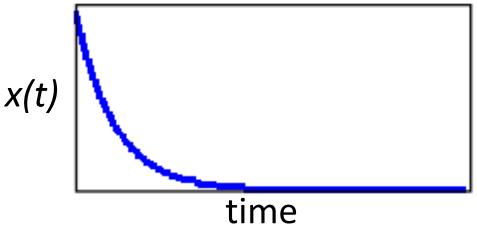


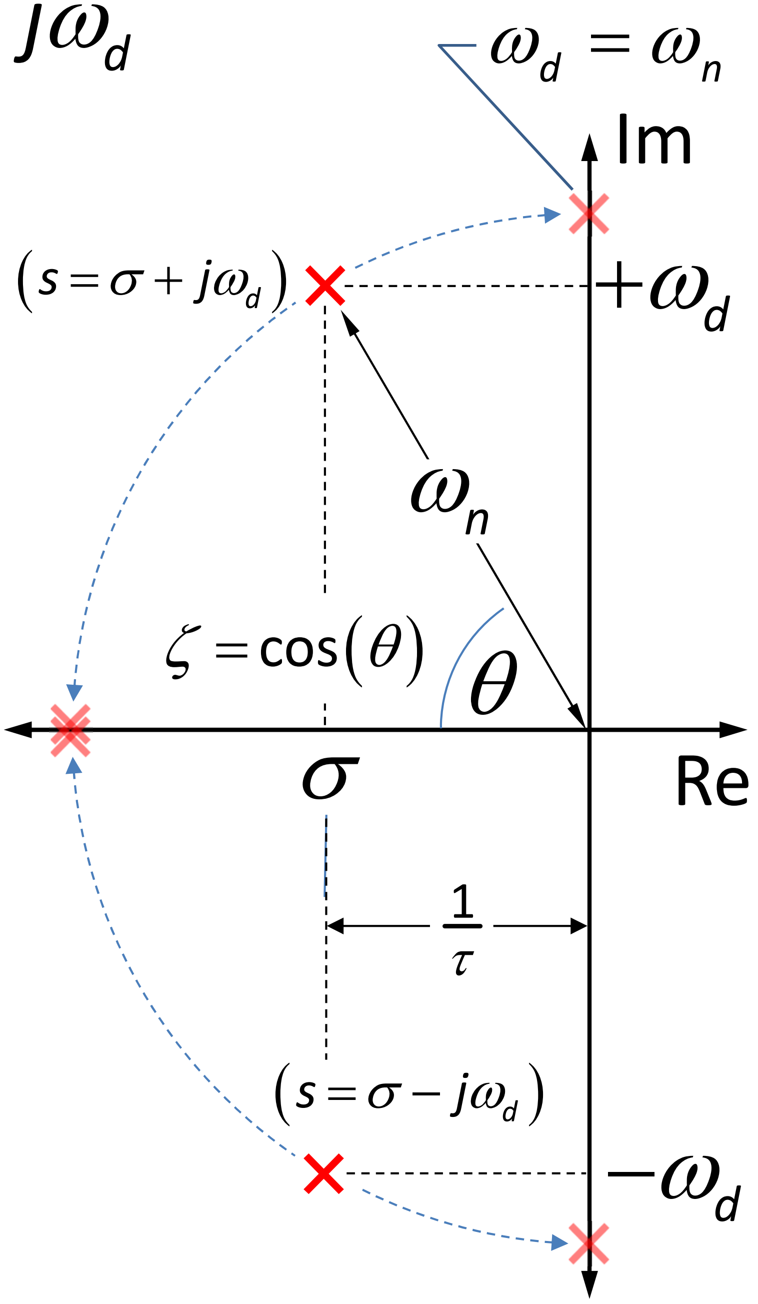
when $\zeta = 1$ (critically damped):

$$s^2 + 2\omega_n s + \omega_n^{\ 2} = 0$$

$$\Rightarrow s = -\omega_n \ \textit{(repeated)}$$

$$x(t) = (C_1 + C_2 t)e^{st}$$





Response Characteristics: Roots of $\Delta(s)$

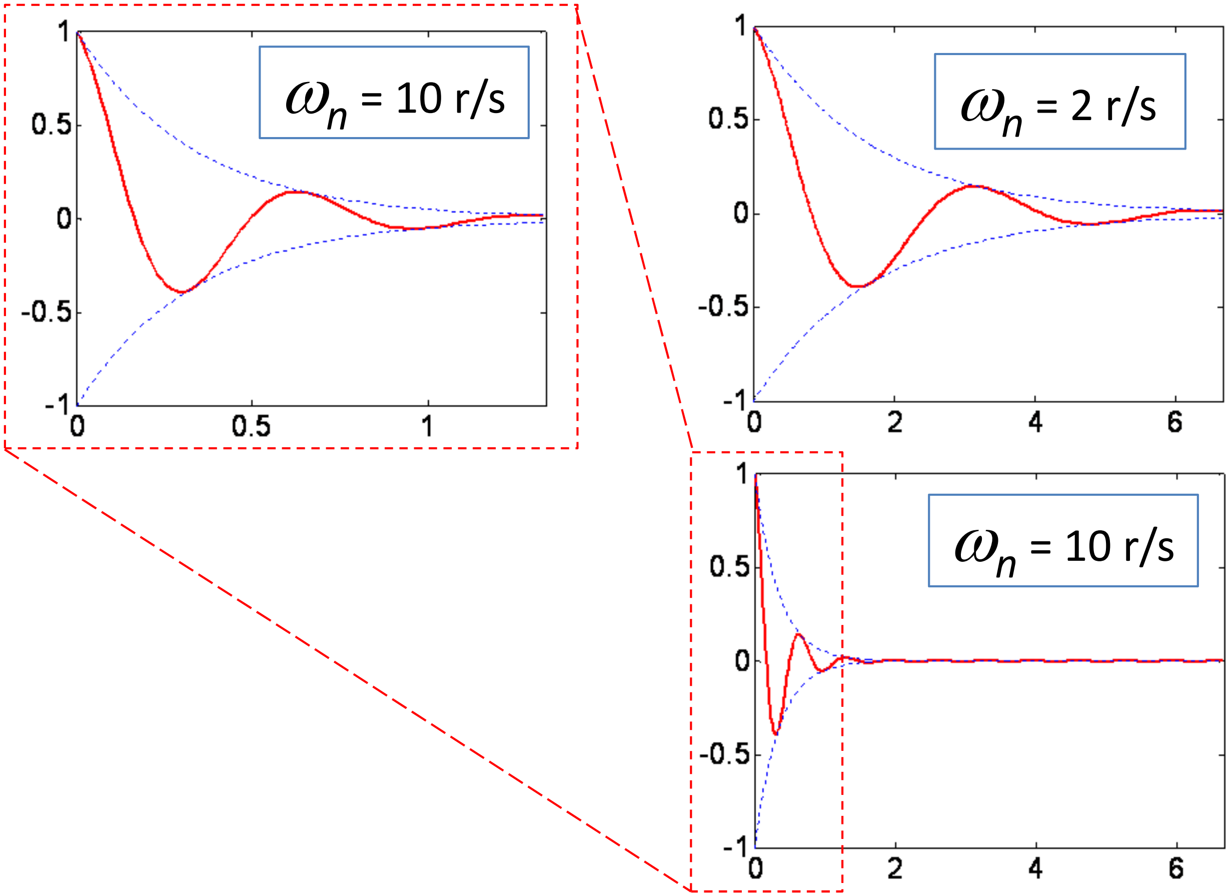
- Damping ratio is a measure of system damping *normalized* to the natural frequency

Time constant: $\tau = \frac{1}{\zeta\omega_n}$

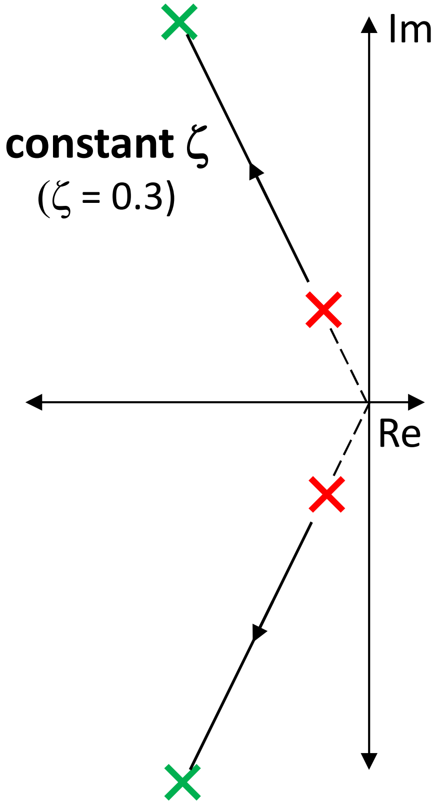
$(\sigma = -\zeta\omega_n)$

Undamped system period: $T = \frac{2\pi}{\omega_n}$

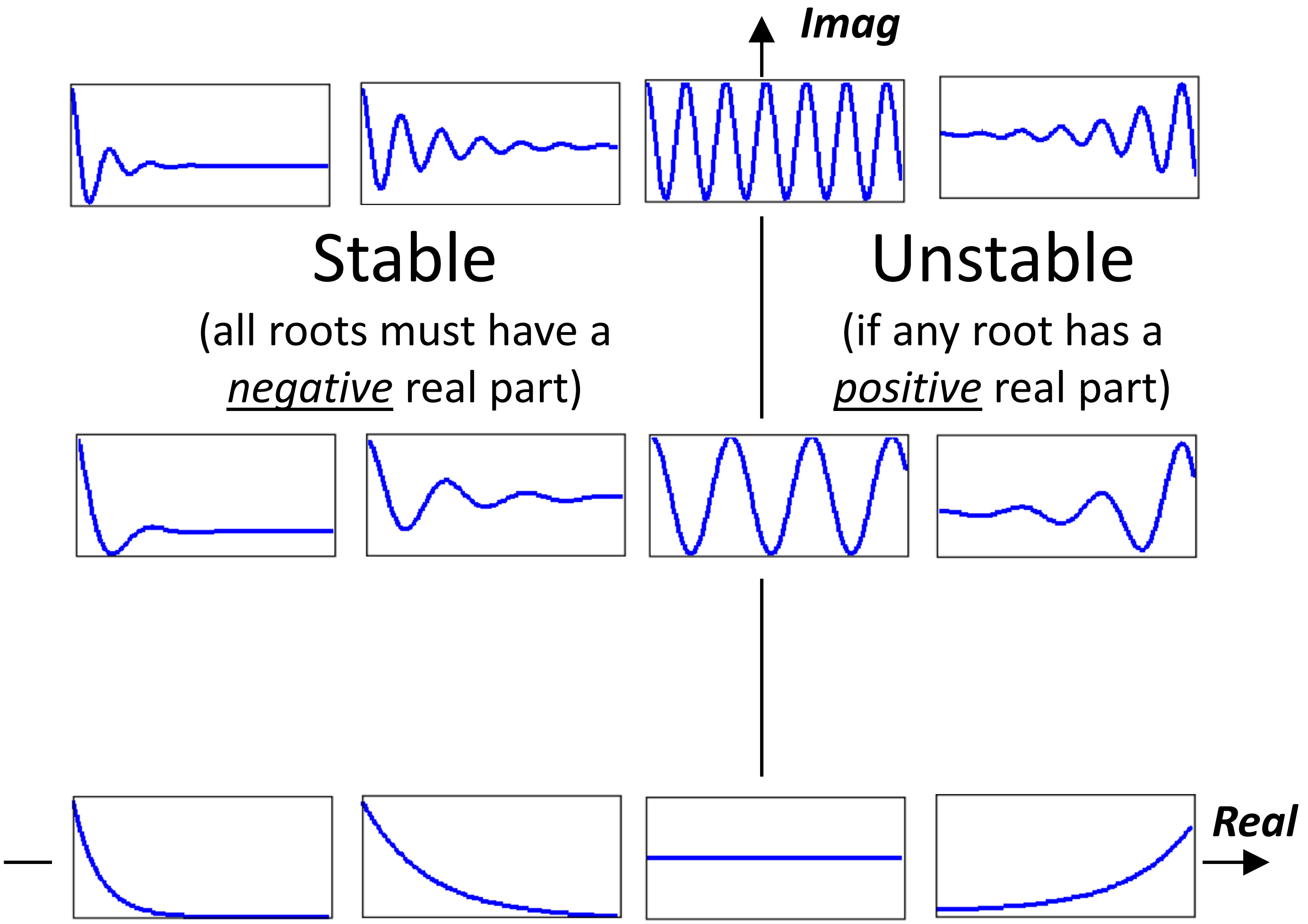
$$\frac{\tau}{T} = \frac{1}{2\pi\zeta} \longrightarrow \tau_{normalized} \propto \frac{1}{\zeta}$$



Example:



Stability and Response Characteristics



Stable

(all roots must have a *negative* real part)

Unstable

(if any root has a *positive* real part)

Negative imaginary parts not shown for clarity (but they're still there!)

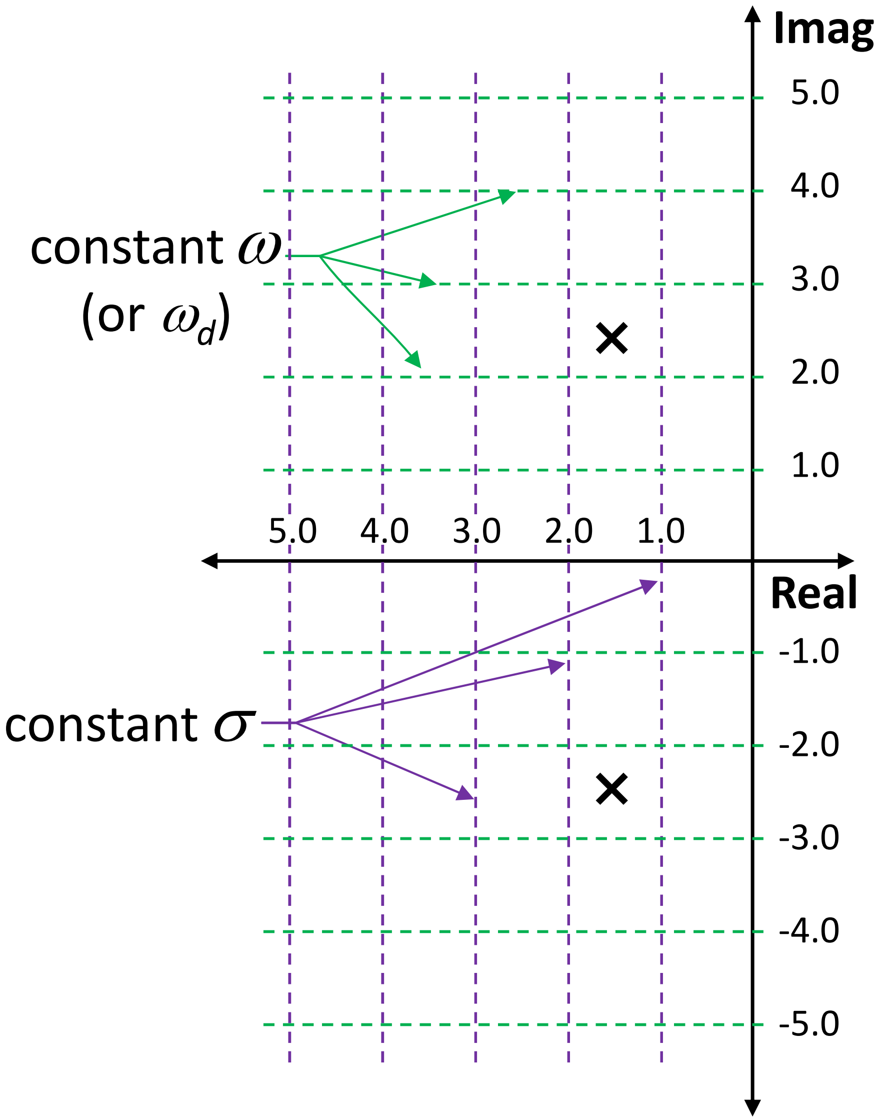
Stability:

- System is stable if and only if $e^{s_i t} \to 0$ for all s_i
- Necessary condition:
  for all s_i $\mathrm{Re}\{s_i\} < 0$

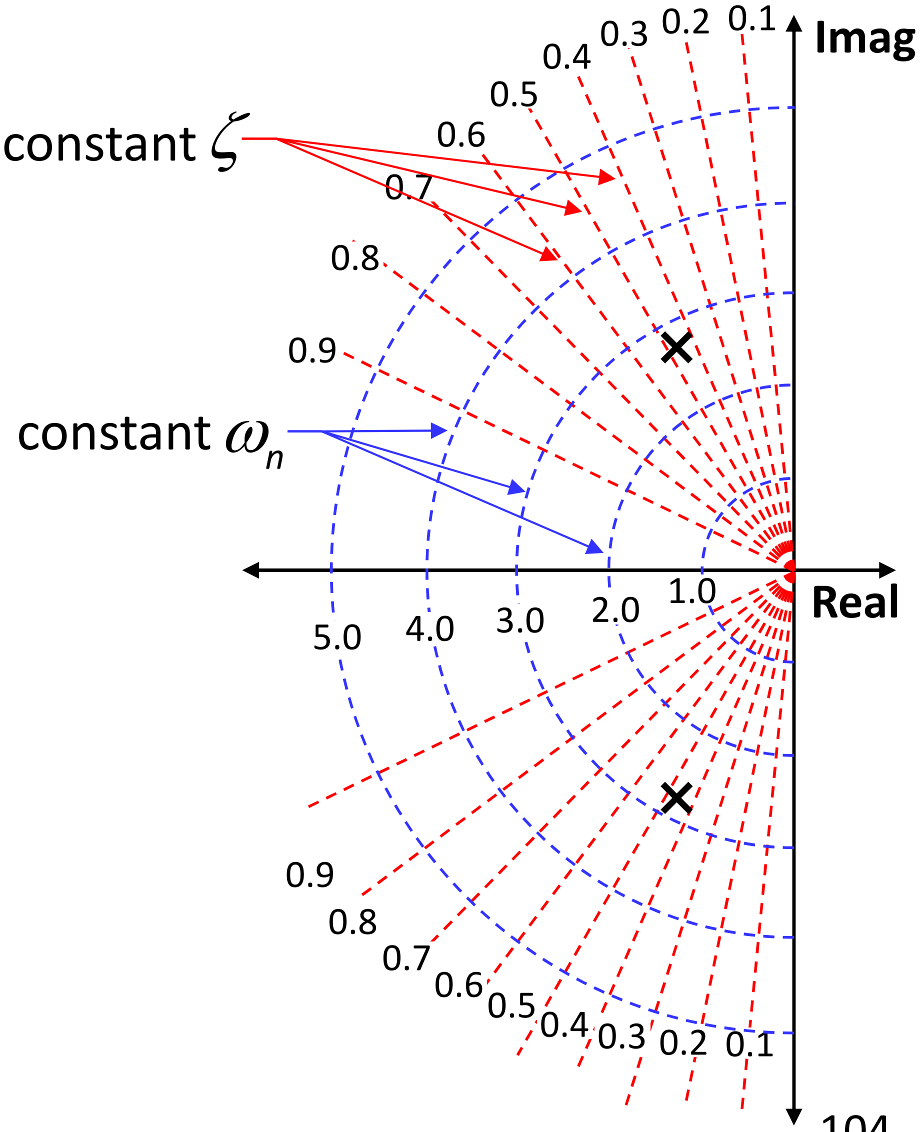
- Roots along real axis
  - exponential behavior – no oscillations
  - magnitude or root is proportional to speed of response
- Roots along imaginary axis
  - pure oscillation, no decay
  - Frequency is proportional to magnitude
- Complex conjugate pairs – oscillatory
  - $\omega = |Im\{s\}|$
  - $\tau = 1/|Re\{s\}|$

Stability and Response Characteristics

Lines of constant ω and σ



Lines of constant ω_n and ζ



Dominant Root Approximation

- Systems of order >2: combination of roots that are:
  - **Distinct real** numbers (i.e. 1st order)
  - **Distinct complex conjugate** pairs (i.e. 2nd order w/ oscillations)
  - **Repeated**, either real or complex
- **Dominant root approximation**
  - Use the roots having the **largest time constant** (smallest real part) to estimate the response
  - Approximation is very sound as long as dominant root[s] is indeed very dominant ➔ far from all other roots of the system

Dominant Root Approximation

System Transfer Function

$$\frac{X(s)}{F(s)} = T(s) = k\frac{(s+z_1)(s+z_2)\cdots(s+z_m)}{(s+p_1)(s+p_2)\cdots(s+p_n)}$$ (factored into individual roots)

Response to step input, $u_s(t)$ $\longrightarrow$ $F(s) = {}^{1}\!/_{s}$

$$X(s) = k\frac{(s+z_1)\cdots(s+z_m)}{(s+p_1)\cdots(s+p_n)}F(s) = k\frac{(s+z_1)\cdots(s+z_m)}{(s+p_1)\cdots(s+p_n)}\left(\frac{1}{s}\right)$$

partial fraction expansion

$$X(s) = \frac{C_0}{s} + \frac{C_1}{(s+p_1)} + \cdots + \frac{C_n}{(s+p_n)}$$

take inverse Laplace transform of each term to evaluate the time domain response

Dominant Root Approximation

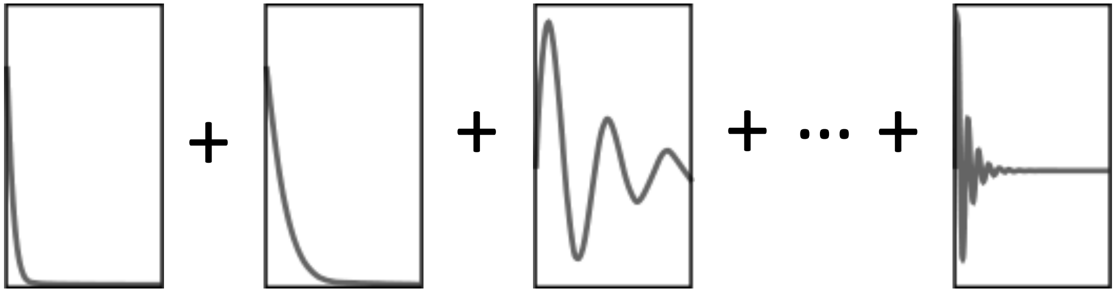
partial fraction expansion

$$X(s)=\frac{C_0}{s}+\frac{C_1}{(s+p_1)}+\cdots+\frac{C_n}{(s+p_n)}$$

constant (x_{ss})

Time domain response

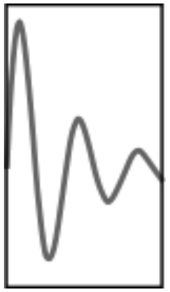
$$x(t)=x_{ss}+$$



slowest root

Approximate system model:

$$x(t)\cong x_{ss}+$$



$$\frac{X(s)}{F(s)}\cong\frac{K}{(s+p)(s+p^*)}\quad\text{or}\quad\frac{K}{s+p}$$

Dominant Root Approximation

Example: $$\frac{d^4y}{dt^4}+15\frac{d^3y}{dt^3}+75\frac{d^2y}{dt^2}+145\frac{dy}{dt}+84y=F$$

Find the dominant root approximation

Dominant Root Approximation

Example: $$\frac{d^4y}{dt^4}+15\frac{d^3y}{dt^3}+75\frac{d^2y}{dt^2}+145\frac{dy}{dt}+84y=F$$

Find the dominant root approximation

Form transfer function $$\frac{Y(s)}{F(s)}=\frac{1}{s^4+15s^3+75s^2+145s+84}$$

Factored characteristic equation $$\Delta(s)=(s+1)(s+3)(s+4)(s+7)$$

```
D_s = [1 15 75 145 84];
r = roots(D_s)
```

`roots`: solves for roots of polynomial

Identify slowest (dominant root(s)) $$\frac{Y(s)}{F(s)}\approx\frac{C}{s+1}$$

Dominant Root Approximation

Example: $$\frac{d^4y}{dt^4}+15\frac{d^3y}{dt^3}+75\frac{d^2y}{dt^2}+145\frac{dy}{dt}+84y=F$$

Find the dominant root approximation

Steady-state response of approximation should be the same a actual system

Full system:

$$y(\infty)=\lim_{s\to 0}\left[s\frac{Y(s)}{F(s)}F(s)\right]=s\left(\frac{1}{s^4+15s^3+75s^2+145s+84}\right)\underbrace{\frac{1}{s}}_{\substack{unit\\step}}$$

$$y(\infty)=\frac{1}{84}$$

set equal

Approx. system: $$y(\infty)=\lim_{s\to 0}\left[s\frac{Y(s)}{F(s)}F(s)\right]=s\left(\frac{C}{s+1}\right)\underbrace{\frac{1}{s}}_{unit\ step}=C\xrightarrow{}C=\frac{1}{84}$$

$$\frac{Y(s)}{F(s)}\approx\frac{C}{s+1}=\frac{\frac{1}{84}}{s+1}\quad\text{or}\quad \dot{y}+y=\left(\tfrac{1}{84}\right)F$$

Dominant Root Approximation

```
%characteristic equation
% and its roots
D_s = [1 15 75 145 84];
r = roots(D_s);

%original system
```

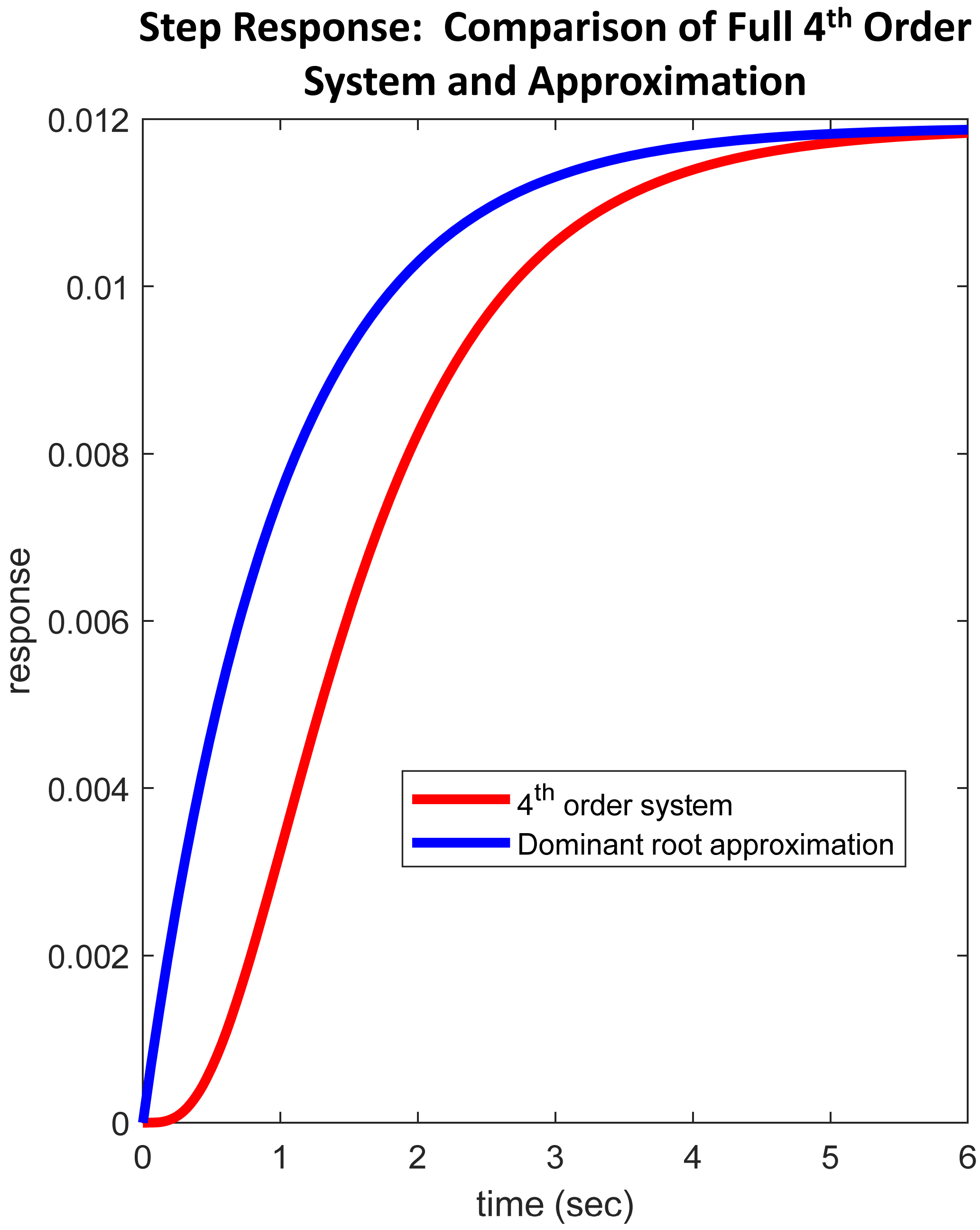
$$\frac{Y(s)}{F(s)} = \frac{1}{s^4 + 15s^3 + 75s^2 + 145s + 84}$$

```
s = tf('s');
sys = 1/(s^4 + 15*s^3 ...
     + 75*s^2 + 145*s + 84);

%approximate system
sysA = (1/84)/(s + 1);
```

$$\frac{Y(s)}{F(s)} \approx \frac{\frac{1}{84}}{s+1}$$

```
%compare step response
step(sys,sysA)
```



Step Response: Comparison of Full 4th Order System and Approximation

Modeling & Analysis of Dynamic Systems

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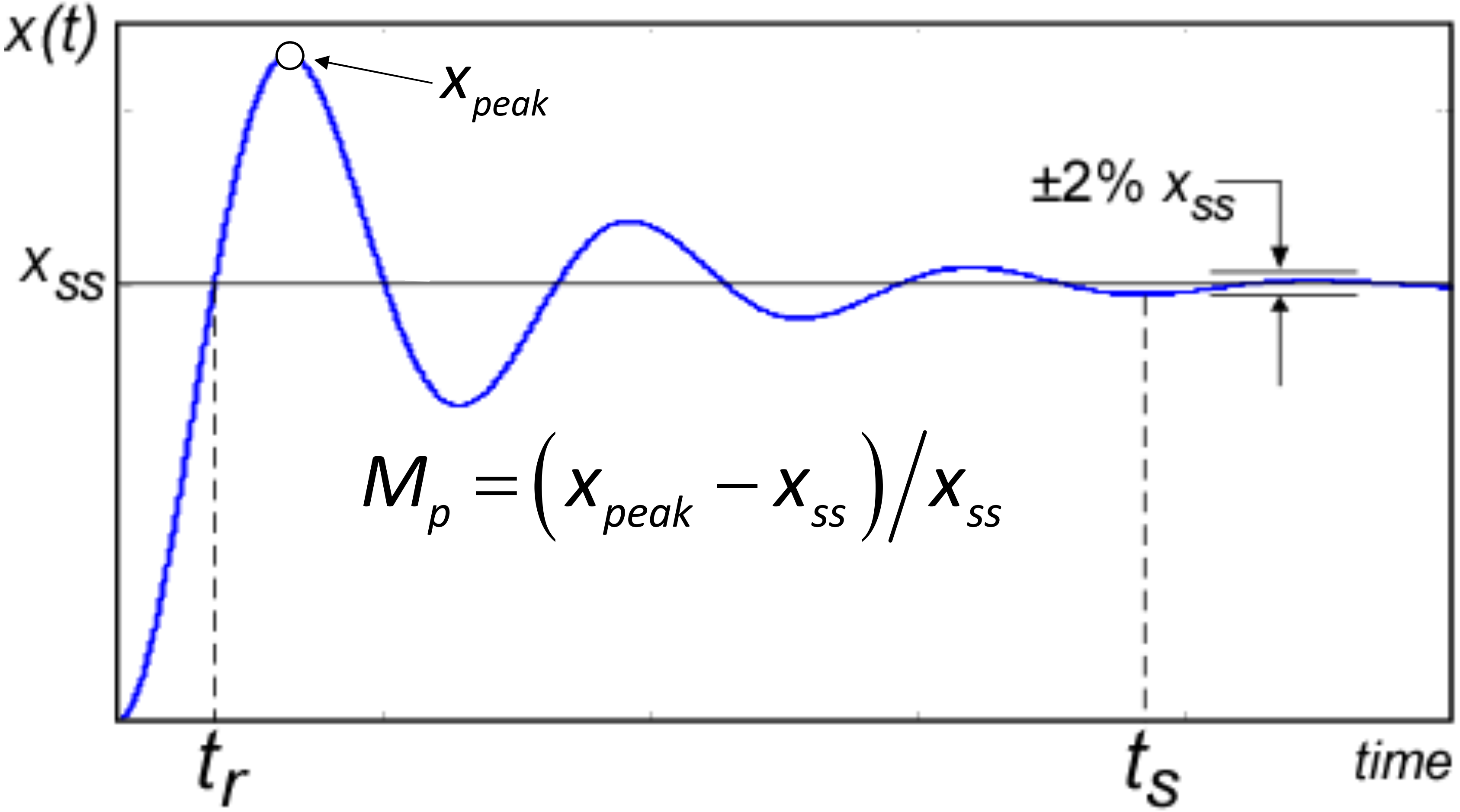
- Linear System Response Characteristics 3.3
- ➡ Time Domain Specifications 3.4
- Effects of System Poles and Zeros 3.5

Time Domain Performance Specifications

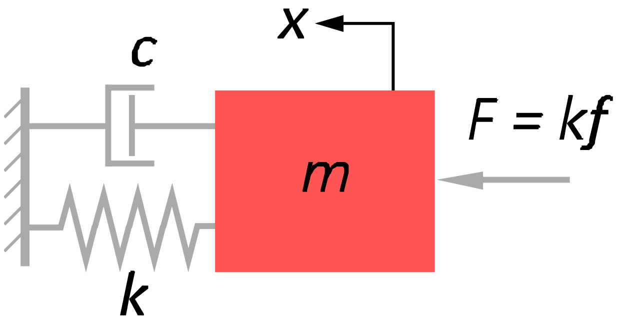
- **Rise time – t_r :** Time for response to reach steady-state, x_{ss}
- **Settling time – t_s :** Time at which response remains within ±1% (or other %) of the steady-state, x_{ss}
- **Overshoot - M_p :** Maximum deviation beyond steady-state, x_{ss}



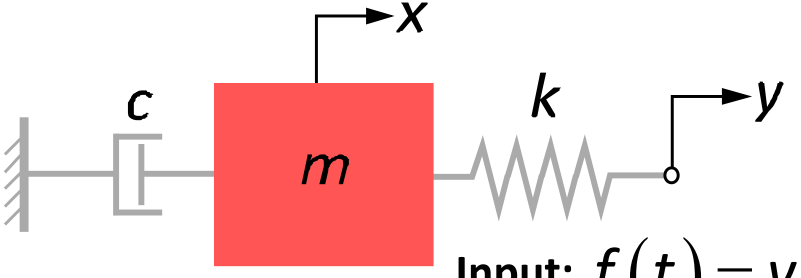
$$M_p = \left(x_{peak} - x_{ss}\right) / x_{ss}$$

Time Domain Performance Specifications

2nd order system:



or



Input: $f(t)=y(t)=1(t)$

(so that unit step steady-state response = 1.0)

$$m\ddot{x}+c\dot{x}+kx=kf\longrightarrow\ddot{x}+\tfrac{c}{m}\dot{x}+\tfrac{k}{m}x=\tfrac{k}{m}f$$

taking the Laplace transform (assume zero I.C.s)

$$\left(s^2+\tfrac{c}{m}s+\tfrac{k}{m}\right)X(s)=\tfrac{k}{m}F(s)$$

Unit step:

$$f(t)=1(t)\rightarrow F(s)=\tfrac{1}{s}$$

put into normalized form: $\tfrac{c}{m}=2\zeta\omega_n s,\ \tfrac{k}{m}=\omega_n{}^2$

$$\left(s^2+2\zeta\omega_n s+\omega_n{}^2\right)X(s)=\omega_n{}^2F(s)\rightarrow X(s)=\frac{\omega_n{}^2}{s^2+2\zeta\omega_n s+\omega_n{}^2}F(s)$$

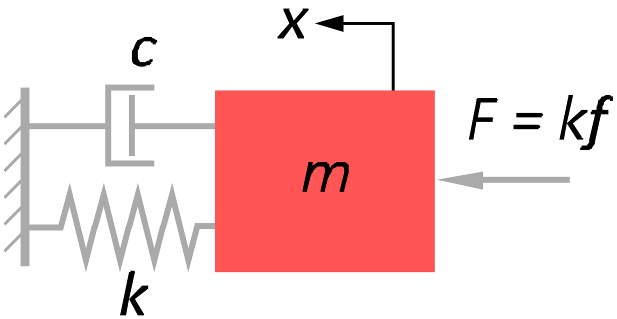
isolate *X(s)* and evaluate the inverse Laplace transform

$$x(t)=\mathscr{L}^{-1}\left[\frac{\omega_n{}^2}{s\left(s^2+2\zeta\omega_n s+\omega_n{}^2\right)}\right]$$

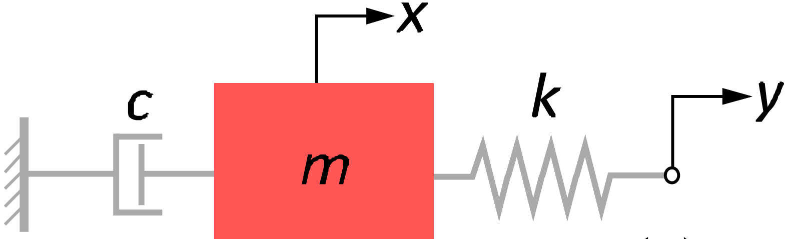
we assume underdamped system $0\le\zeta\le1$

Time Domain Performance Specifications

2nd order system:



or

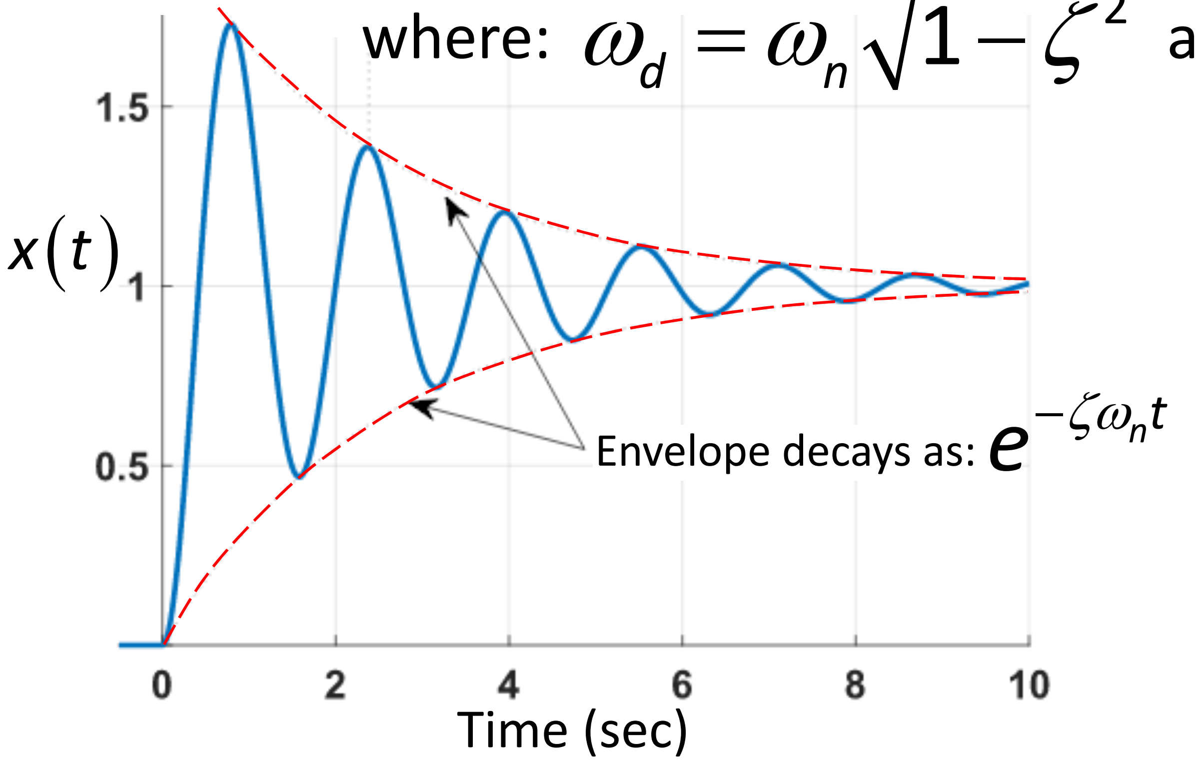


Input: $f(t)=y(t)=1(t)$

After some algebra, partial fraction expansion, and Laplace table lookups …

$$x(t)=1-\frac{1}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin(\omega_d t+\phi)$$ **assume** $0\le\zeta\le1$

where: $\omega_d=\omega_n\sqrt{1-\zeta^2}$ and $\phi=\cos^{-1}(\zeta)$



From *x*(*t*) we can evaluate time domain specifications

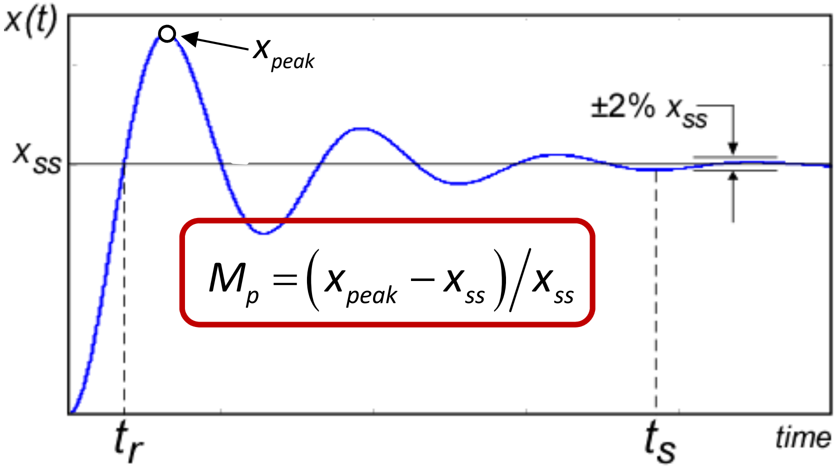
BUT

these are heuristics derived from simple second order system

– **JUST AN APPROXIMATION** –

Overshoot

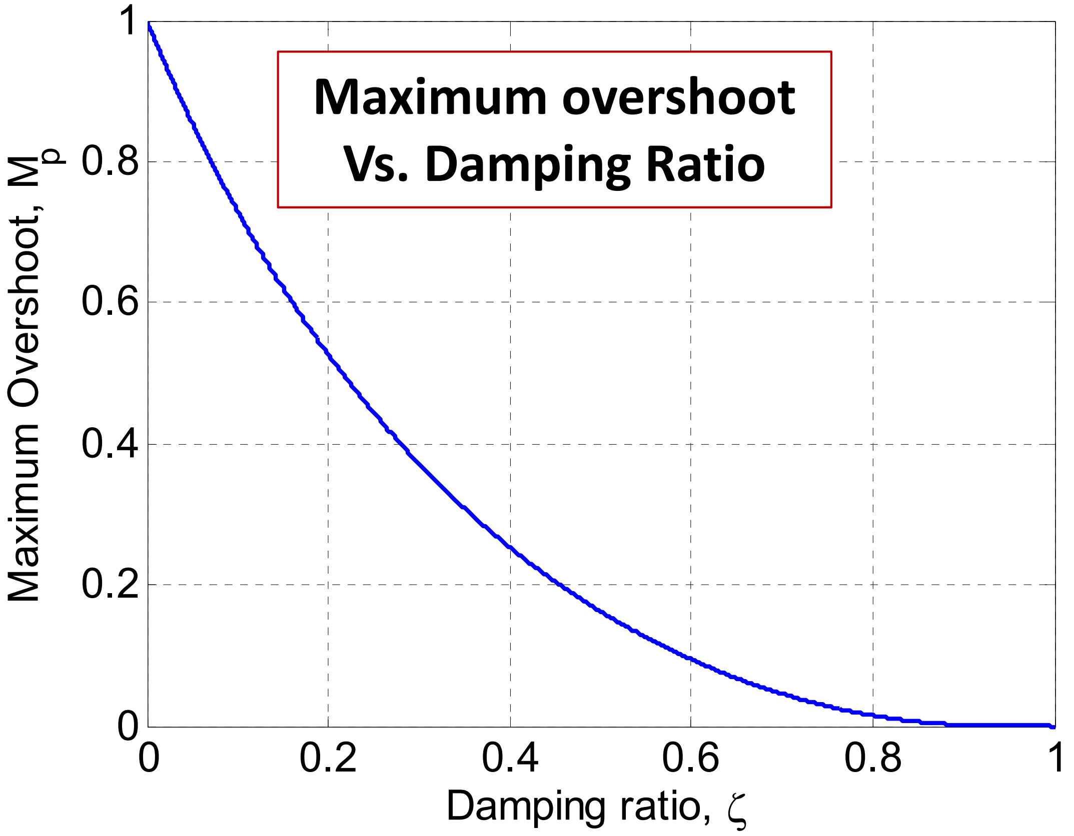
- **Maximum overshoot - M_p :** Maximum deviation beyond steady-state, x_{ss}, expressed as a ratio or percent of x_{ss}



$$M_p = \frac{x_{peak} - x_{ss}}{x_{ss}}$$

$$\vdots$$

$$M_p = e^{-\pi\zeta / \sqrt{1-\zeta^2}}$$



- Maximum overshoot, M_p, is a function of damping ratio, ζ, only
- For $\zeta > 1$, $x(t)$ does not overshoot x_{ss}

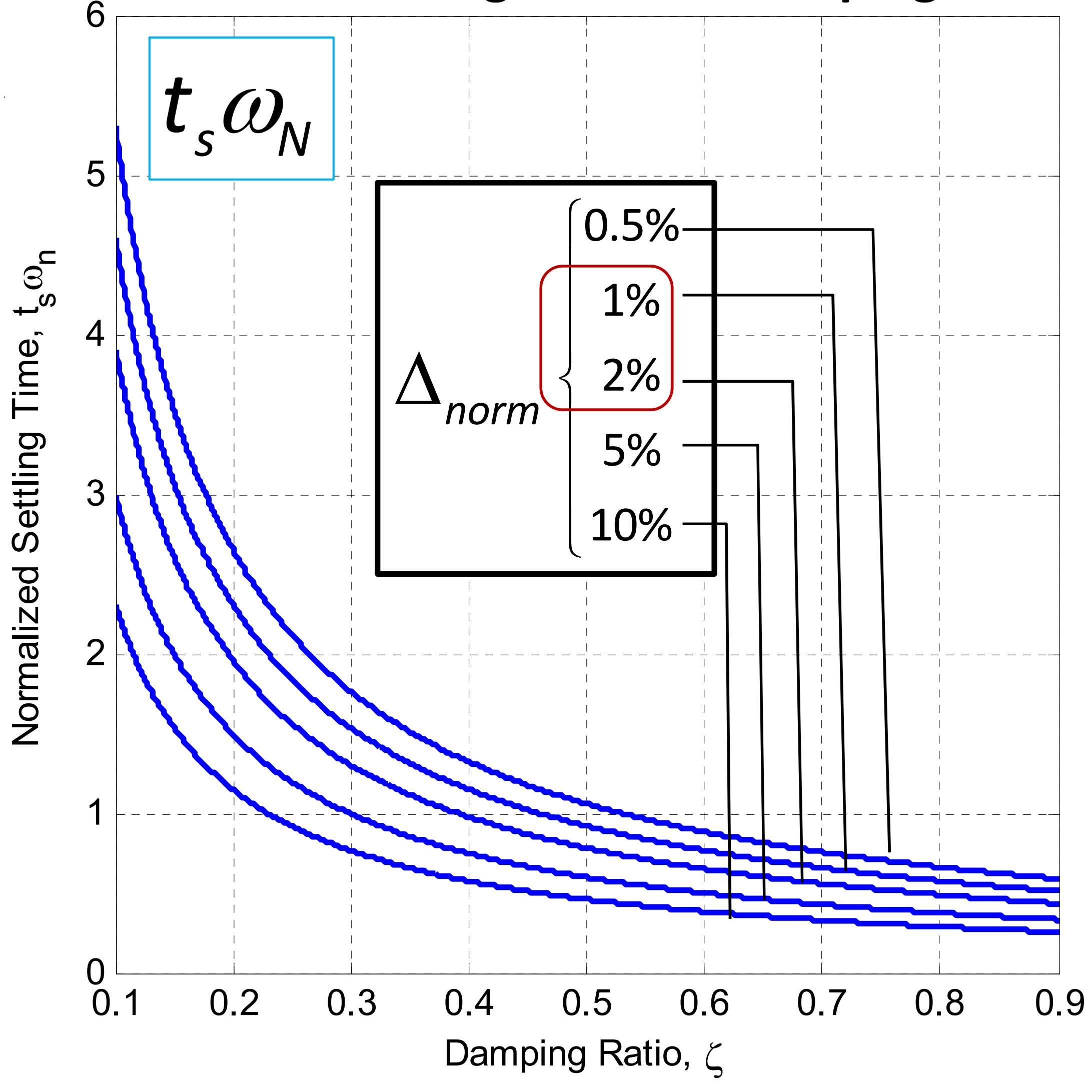
$$\boxed{\zeta = \sqrt{\frac{\left(\ln M_p\right)^2}{\pi^2 + \left(\ln M_p\right)^2}}}$$

- An aside: We can estimate ζ from the maximum overshoot

Settling Time

- **Settling time – t_s :** Time at which response remains within ±1% (or Δ%) of the steady-state, x_{ss}

Normalized Settling Time Vs. Damping Ratio



Exponential envelope:

$$\longrightarrow e^{-\zeta\omega_n t_s} = \Delta$$

$$\ln\left(e^{-\zeta\omega_n t_s}\right) = \ln(\Delta)$$

$$-\zeta\omega_n t_s = \ln(\Delta)$$

$$\Rightarrow t_s = -\frac{\ln\Delta}{\zeta\omega_n} = \frac{\ln\Delta}{\sigma}$$

settling time

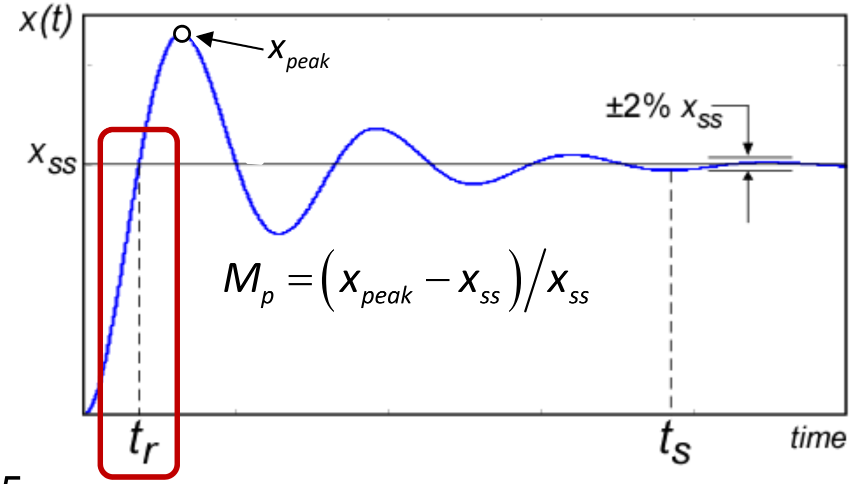
Commonly used specifications

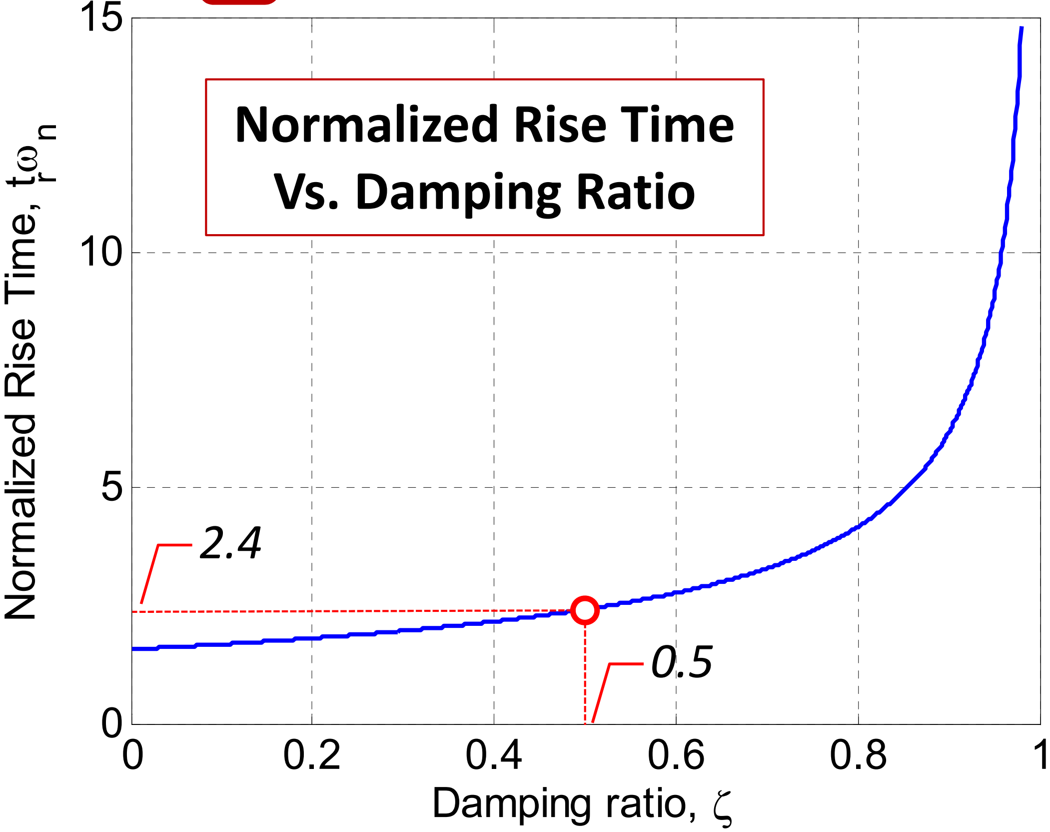
Δ	1%	2%
t_s	$\frac{4.6}{\zeta\omega_n} = \boxed{\frac{4.6}{\sigma}}$	$\frac{3.9}{\zeta\omega_n} = \boxed{\frac{3.9}{\sigma}}$

- Settling time increases without bound as ζ approaches zero

Rise Time

- **Rise time – t_r:** Time for response to reach steady-state, x_{ss}





$$x(t)=1-\underbrace{\frac{1}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin\left(\omega_d t+\phi\right)}$$

$$\frac{1}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\underbrace{\sin\left(\omega_d t_r+\phi\right)}=0$$

$$\sin\left(\underbrace{\omega_d t_r+\phi}\right)=0$$

$$\omega_d t_r+\phi=\pi$$

$$t_r=\frac{\pi-\phi}{\omega_d}=\frac{\pi-\phi}{\omega_n\sqrt{1-\zeta^2}}$$

$$\text{where}\quad \phi=\cos^{-1}\left(\zeta\right)$$

rise time

$$t_r \cong 2.4/\omega_n$$

(ζ = 0.5 and $x(t)$ from 0 to 1)

Time Domain Performance Specifications

Example: Given the time domain specifications below, determine the region of acceptable root locations (in the *s*-plane)

Time domain specifications:

$$\begin{cases} t_s \leq 1.0 & \text{(settling time)} \\ \quad \text{where } \Delta = 1\% & \text{(definition for steady-state)} \\ t_r \leq 0.3 & \text{(rise time)} \\ M_p \leq 25\% & \text{(maximum overshoot)} \end{cases}$$

Time Domain Performance Specifications

Example: continued

Overshoot:

$$M_p \cong e^{-\pi\zeta/\sqrt{1-\zeta^2}} \Longrightarrow \zeta \cong \sqrt{\frac{(\ln M_p)^2}{\pi^2+(\ln M_p)^2}}$$

set $\zeta \geq$ to the result

Settling time:

$$t_s \cong \frac{\ln\Delta}{\zeta\omega_n} = \frac{\ln\Delta}{\sigma} \Longrightarrow \sigma \cong \frac{\ln\Delta}{t_s}$$

set $\sigma \leq$ to the result

Rise time:

$$t_r \cong 2.4/\omega_n \Longrightarrow \omega_n \cong 2.4/t_r$$

set $\omega_n \geq$ to the result

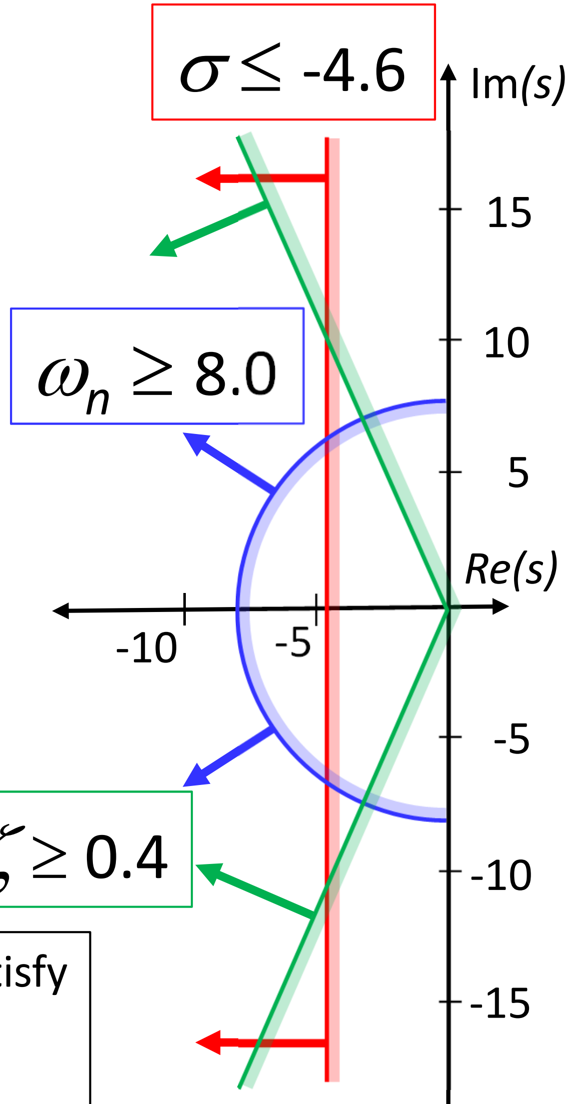
Example:

$t_s \leq 1.0$
$\Delta = 1\%$ $\Longrightarrow$ $\sigma \leq -4.6$

$t_r \leq 0.3$ $\Longrightarrow$ $\omega_n \geq 8.0$

$M_p \leq 25\%$ $\Longrightarrow$ $\zeta \geq 0.4$

- Choose root locations to satisfy time domain specifications
- Keep σ, ω_n, and ζ as small as possible



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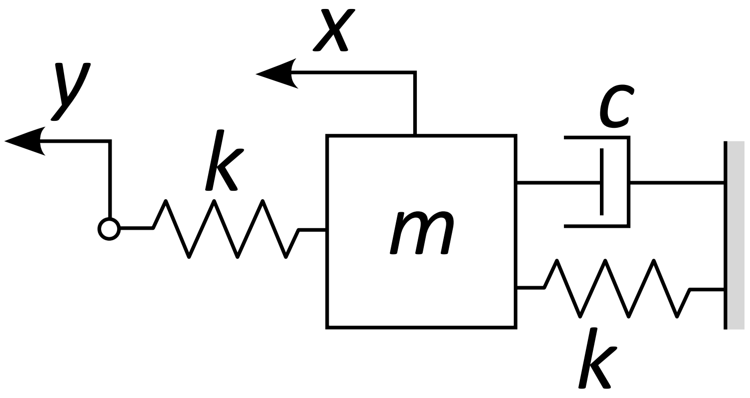
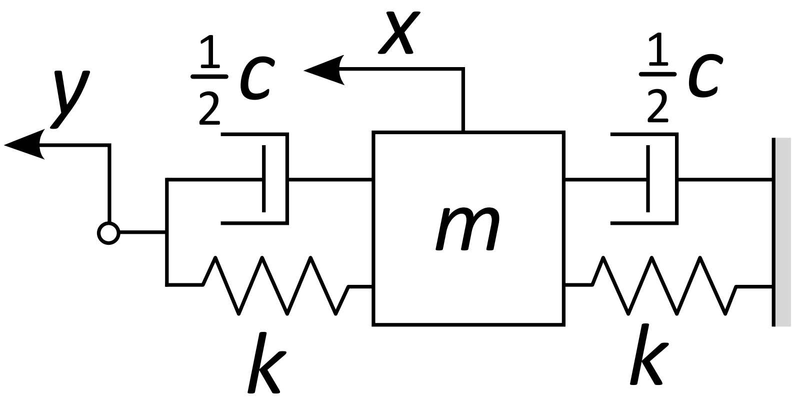
Effects of Zeros and Additional Poles

Step response of a 2nd order system with two distinct (real) poles and effect of adding a real zero





$$m\ddot{x}+c\dot{x}+2kx=ky$$

$$\frac{X(s)}{Y(s)}=\frac{k}{ms^2+cs+2k \longleftarrow \Delta(s)}$$

$$m\ddot{x}+c\dot{x}+2kx=ky+\tfrac{1}{2}c\dot{y}$$

$$\frac{X(s)}{Y(s)}=\frac{\tfrac{1}{2}cs+k \longleftarrow \textit{Real zero}}{ms^2+cs+2k \longleftarrow \Delta(s)}$$

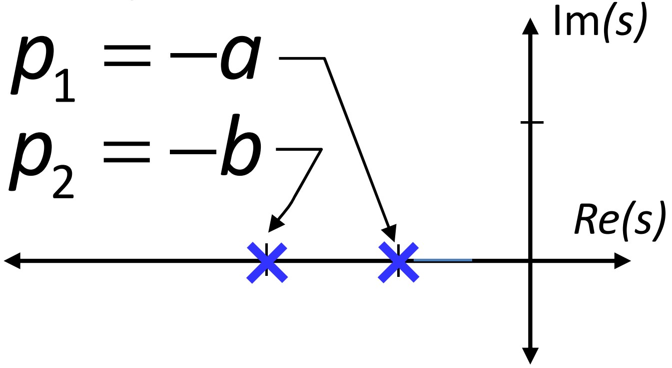
- **Poles** are roots of $\Delta(s)$, denominator: qualitatively determine response (i.e. correspond to terms in partial fraction expansion)

- **Zeros** are roots of numerator: can affect the transient response by affecting contribution of partial fraction terms in a particular response

Effects of Zeros and Additional Poles

Step response of a 2nd order system with two distinct (real) poles and effect of adding a real zero

Two real poles:



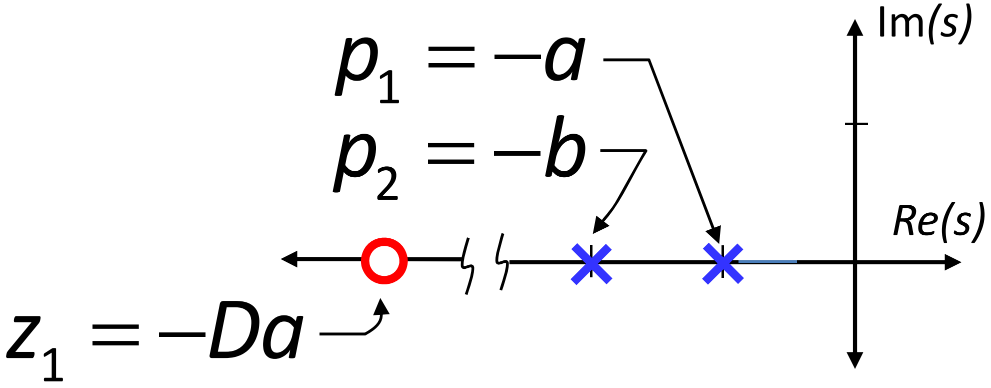
Transfer function (DC gain = 1):

$$\frac{X(s)}{F(s)} = \frac{ab}{(s+a)(s+b)} \leftarrow \Delta(s)$$

System differential equation:

$$\ddot{x} + (a+b)\dot{x} + (ab)x = (ab)f$$

Two real poles + one real zero:



Transfer function (DC gain = 1):

$$\frac{X(s)}{F(s)} = \frac{\left(\frac{b}{D}\right)(s+Da)}{(s+a)(s+b)} \leftarrow \Delta(s)$$

System differential equation:

$$\ddot{x} + (a+b)\dot{x} + (ab)x = (ab)f + \tfrac{b}{D}\dot{f}$$

Effects of Zeros and Additional Poles

2nd order system:	***2nd order system with a zero:***
$\ddot{x}+\left(a+b\right)\dot{x}+\left(ab\right)x=\left(ab\right)f$	$\ddot{x}+\left(a+b\right)\dot{x}+\left(ab\right)x=\left(ab\right)f+\frac{b}{D}\dot{f}$

Take Laplace transform ($f\left(t\right)=1\left(t\right)$ and $x\left(0\right)=\dot{x}\left(0\right)=0$)

$X\left(s\right)=\dfrac{ab}{s\left(s+a\right)\left(s+b\right)}$	$X\left(s\right)=\dfrac{\frac{b}{D}\left(s+Da\right)}{s\left(s+a\right)\left(s+b\right)}$ ← *additional zero*

Partial fraction expansion

$$=\frac{C_1}{s}+\frac{C_2}{s+a}+\frac{C_3}{s+b}$$

$$=\frac{C_1\left(s+a\right)\left(s+b\right)+C_2\left(s\right)\left(s+b\right)+C_3\left(s\right)\left(s+a\right)}{s\left(s+a\right)\left(s+b\right)}$$

$$=\frac{\left(C_1+C_2+C_3\right)s^2+\left(C_1\left(a+b\right)+C_2b+C_3a\right)s+\left(C_1ab\right)}{s\left(s+a\right)\left(s+b\right)}$$

Effects of Zeros and Additional Poles

2^{nd} order system:

$$\ddot{x}+(a+b)\dot{x}+(ab)x=(ab)f$$

2^{nd} order system with a zero:

$$\ddot{x}+(a+b)\dot{x}+(ab)x=(ab)f+\tfrac{b}{D}\dot{f}$$

$\vdots$ *Partial fraction expansion* $\vdots$

$$\frac{ab}{s(s+a)(s+b)}=\frac{(C_1+C_2+C_3)s^2+\left(C_1(a+b)+C_2b+C_3a\right)s+(C_1ab)}{s(s+a)(s+b)}=\frac{\tfrac{b}{D}(s+Da)}{s(s+a)(s+b)}$$

equate numerators and solve for C_i

$$\begin{aligned} C_1+C_2+C_3&=0\\ C_1(a+b)+C_2b+C_3a&=0\\ C_1ab&=ab\end{aligned} \quad\Rightarrow\quad \begin{aligned} C_1&=1\\ C_2&=-\tfrac{b}{b-a}\\ C_3&=\tfrac{b}{b-a}\end{aligned}$$

equate numerators and solve for C_i

$$\begin{aligned} C_1+C_2+C_3&=0\\ C_1(a+b)+C_2b+C_3a&=\tfrac{b}{D}\\ C_1ab&=ab\end{aligned} \quad\Rightarrow\quad \begin{aligned} C_1&=1\\ C_2&=-\frac{b-{}^{b}\!/_{D}}{b-a}\\ C_3&=\frac{a-{}^{b}\!/_{D}}{b-a}\end{aligned}$$

Inverse Laplace transform

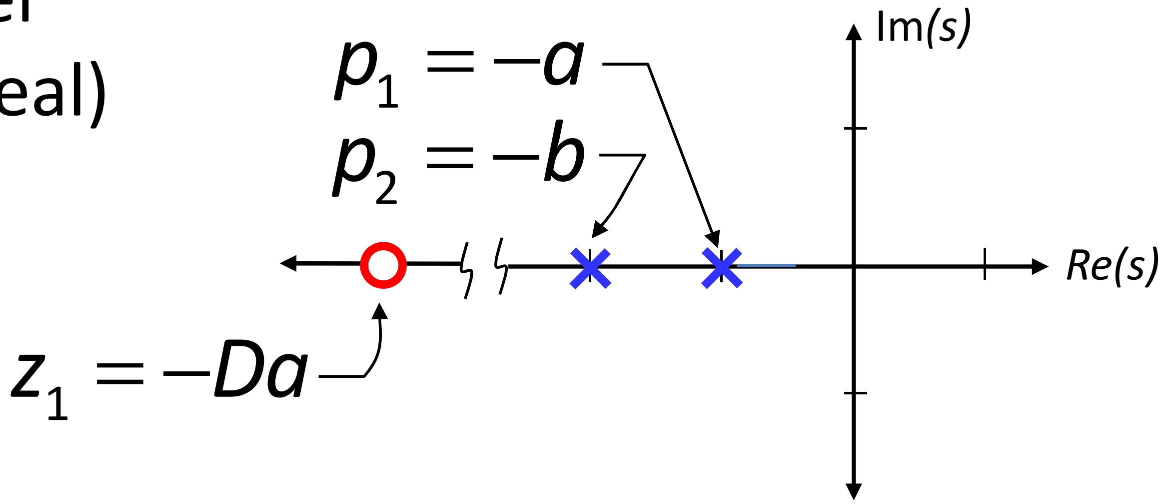
$$x(t)=1-\left(\frac{b}{b-a}\right)e^{-at}+\left(\frac{a}{b-a}\right)e^{-bt}$$

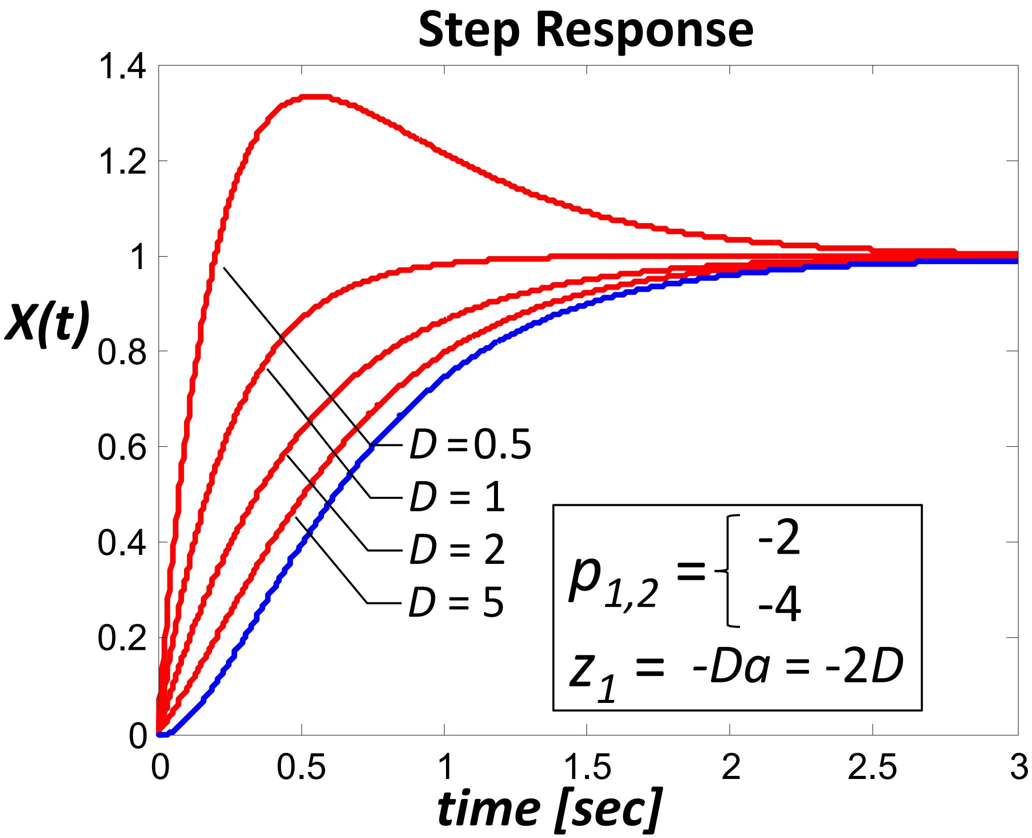
$$x(t)=1-\left(\frac{b\boxed{-{}^{b}\!/_{D}}}{b-a}\right)e^{-at}+\left(\frac{a\boxed{-{}^{b}\!/_{D}}}{b-a}\right)e^{-bt}$$

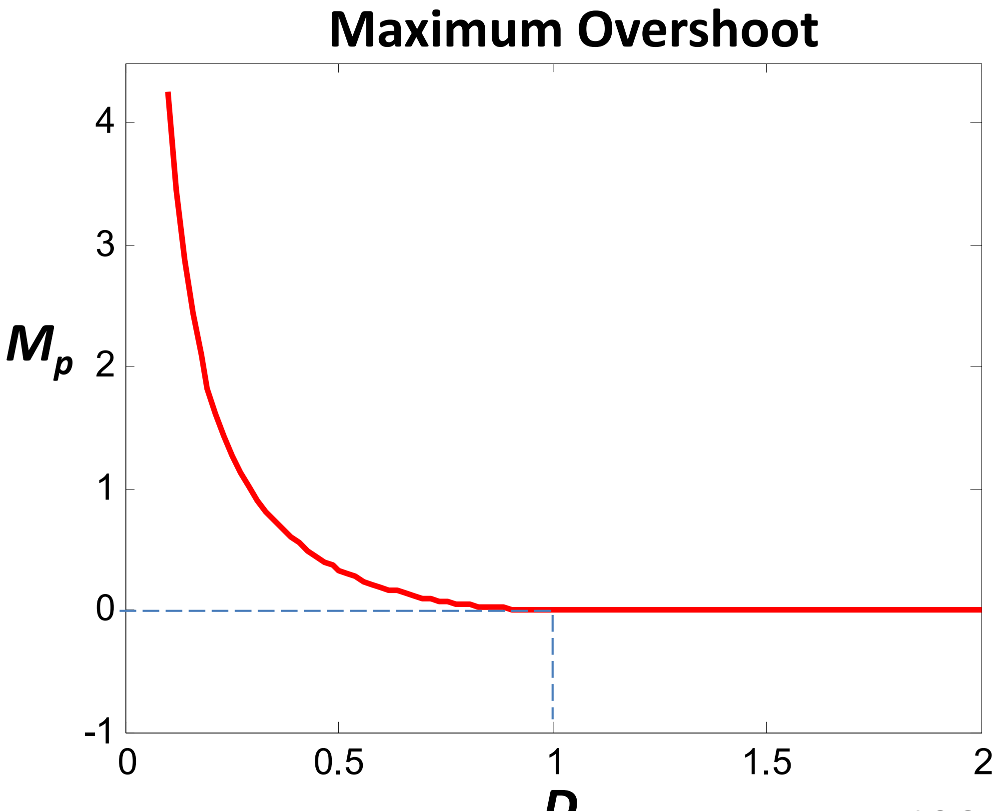
Relative contribution of system roots to transient response is affected by zero at *z = -Da* ⇒ *"Numerator dynamics"*

Effects of Zeros and Additional Poles

Step response of a 2nd order system with two distinct (real) poles and one real zero

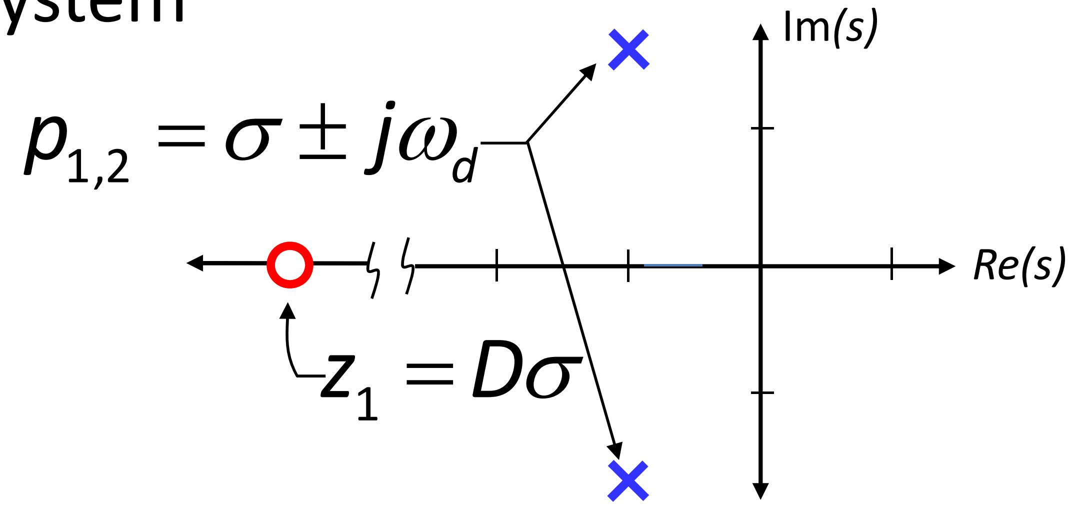






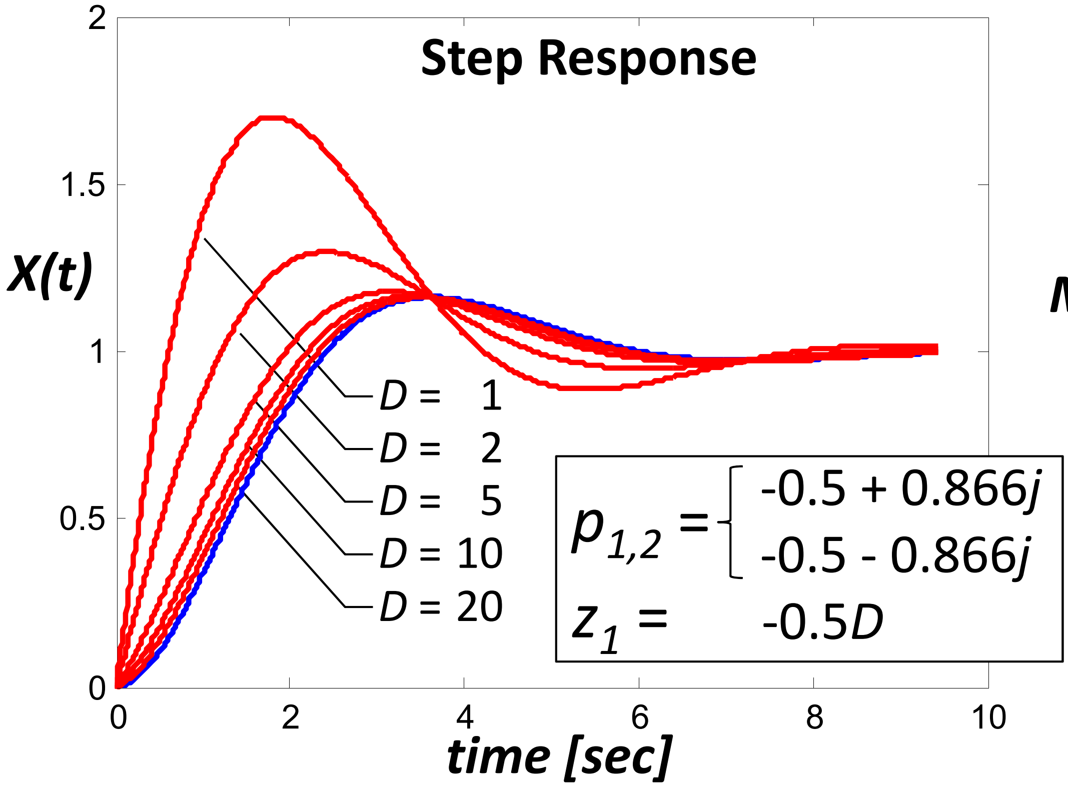
Effects of Zeros and Additional Poles

Step response of a 2^{nd} order system with complex conjugate poles and one real zero



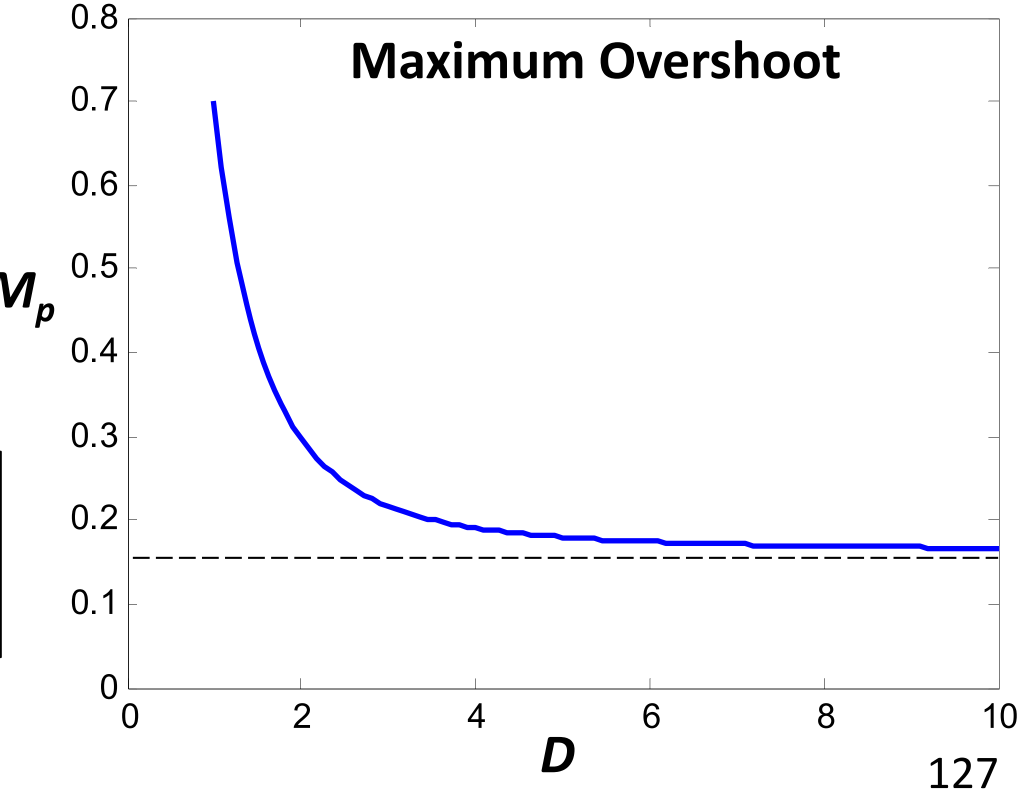
$$p_{1,2} = \sigma \pm j\omega_d$$

$$z_1 = D\sigma$$



$$p_{1,2} = \begin{cases} -0.5 + 0.866j \\ -0.5 - 0.866j \end{cases}$$

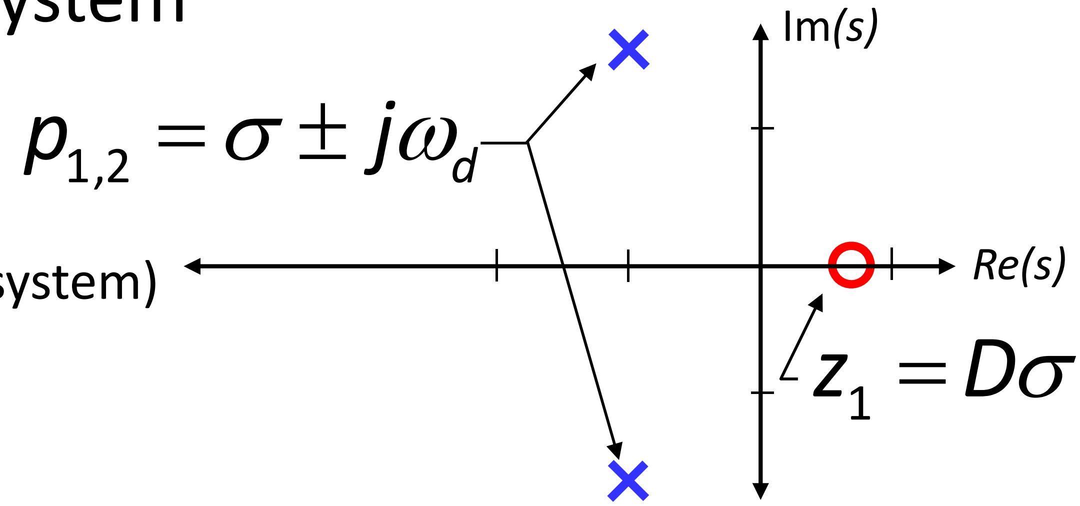
$$z_1 = -0.5D$$



Effects of Zeros and Additional Poles

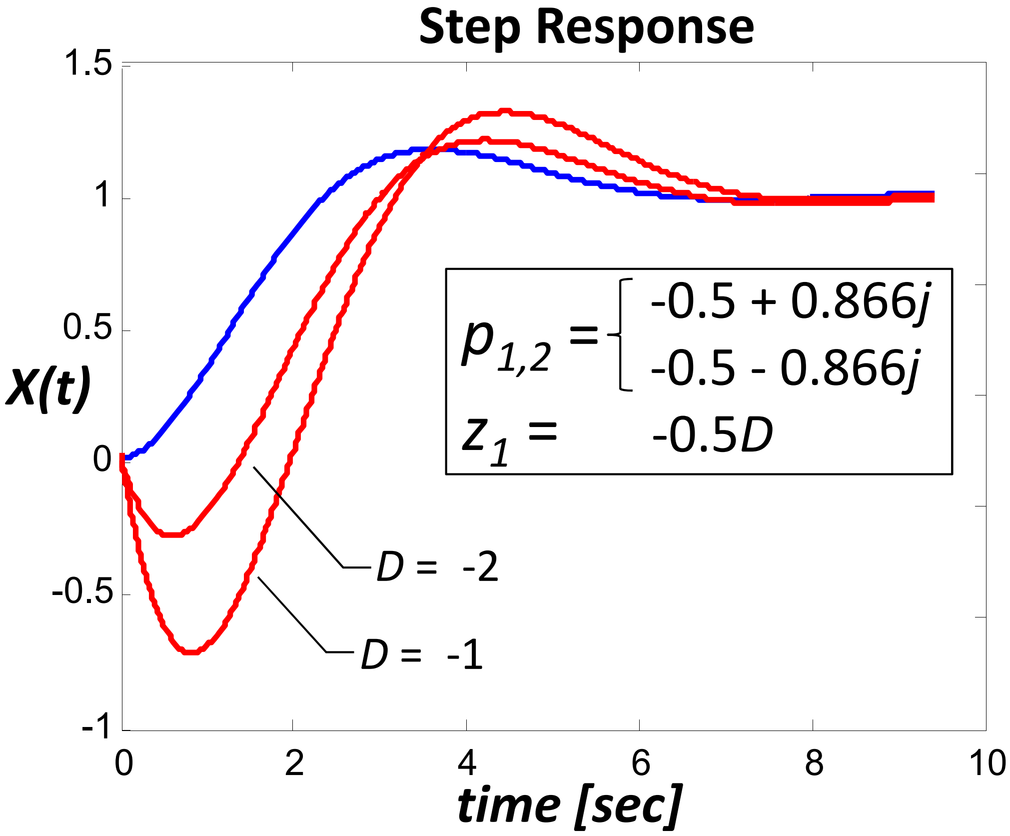
Step response of a 2nd order system with complex conjugate poles and one real RHP zero

(FYI: this is called a non-minimum phase system)



$$p_{1,2} = \sigma \pm j\omega_d$$

$$z_1 = D\sigma$$



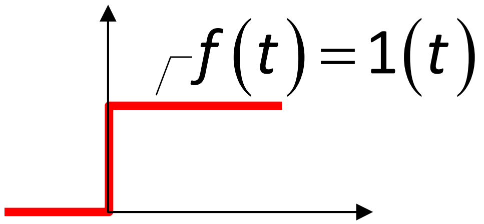
When the zero is in the RHP:

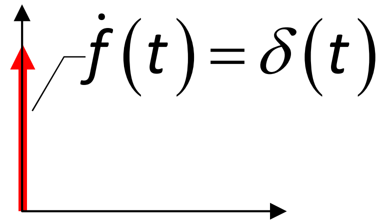
$$\ddot{x} + (a+b)\dot{x} + (ab)x = (ab)f + \tfrac{b}{D}\dot{f}$$

D is negative

$$\ldots = (ab)f - \tfrac{b}{D}\dot{f}$$

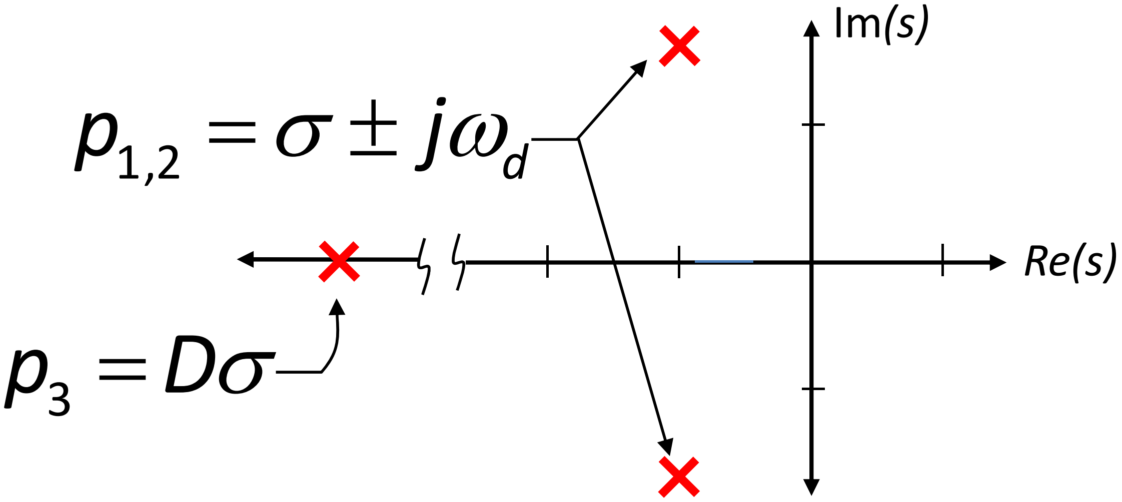
Derivative of a unit step is an impulse at *t* = 0





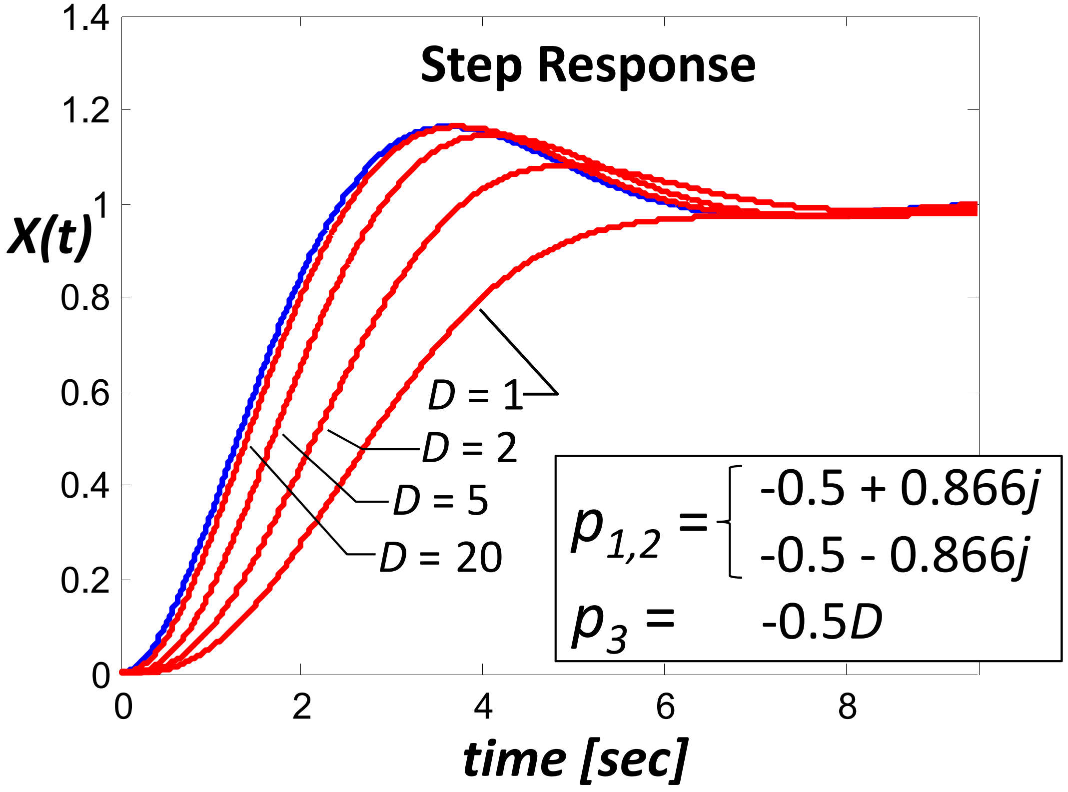
Effects of Zeros and Additional Poles

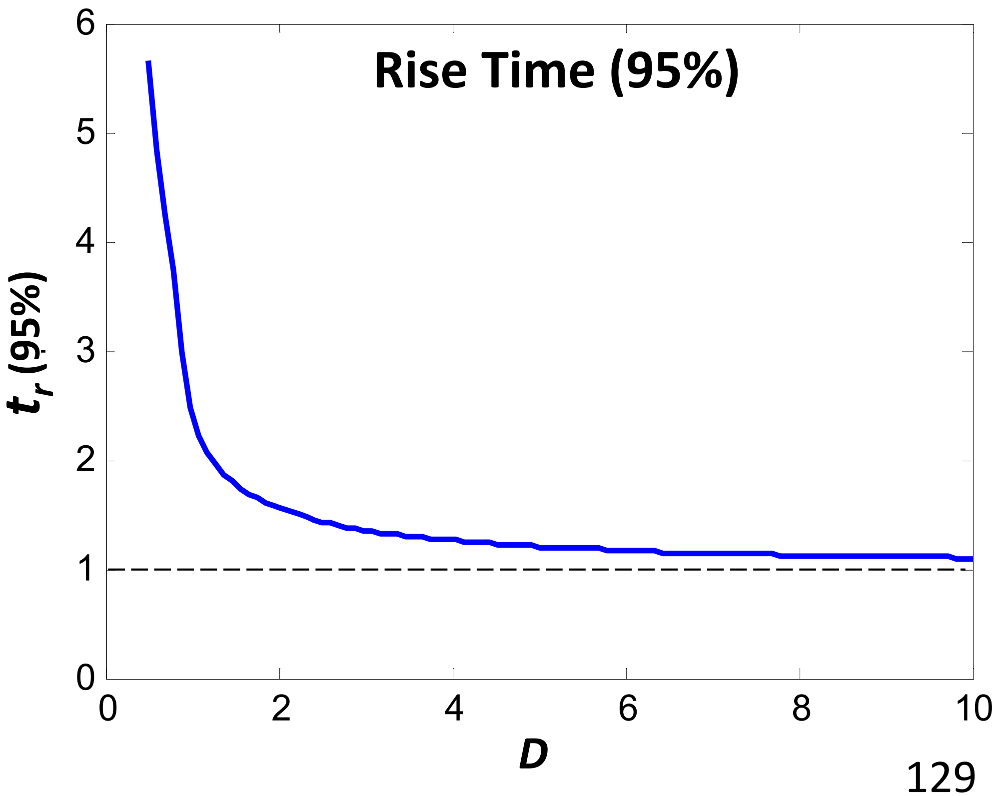
Step response of a 2nd order system with C.C. poles and one additional real pole



$$p_{1,2} = \sigma \pm j\omega_d$$

$$p_3 = D\sigma$$





Effects of Zeros and Additional Poles

Response of system is generally described by its characteristic equation but other factors can effect the specific response

Transfer function: $$T(s)=\frac{N(s)}{D(s)}=\frac{(s+z_1)(s+z_2)\cdots(s+z_m)}{(s+p_1)(s+p_2)\cdots(s+p_n)}\longleftarrow\Delta(s)$$

Response due to forcing $$T(s)=\frac{X(s)}{F(s)}\longrightarrow X(s)=T(s)F(s)$$

$$X(s)=\underbrace{\frac{C_1}{(s+p_1)}+\frac{C_2}{(s+p_2)}}_{\text{e.g. dominant 2nd order roots (poles)}}+\underbrace{\cdots+\frac{C_n}{(s+p_n)}}_{\text{e.g. higher order roots (poles)}}+\text{ forcing terms}$$

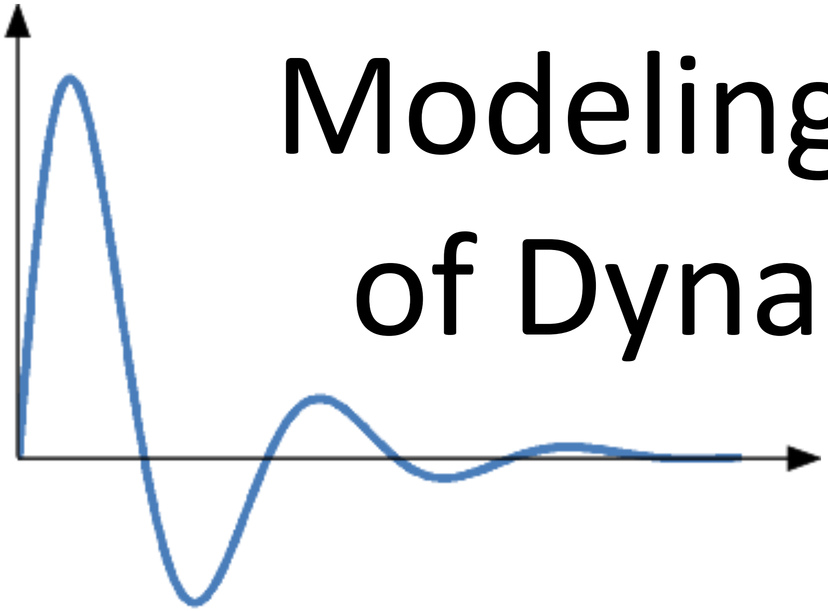
Residues (i.e. C_1, C_2, etc.) determined by:

- Initial conditions
- Forcing terms
- Numerator roots (zeros)

$$x(t)=\overbrace{C_1e^{-p_1t}+C_2e^{-p_1t}}+\overbrace{\cdots+C_ne^{-p_1t}}+\text{ forcing terms}$$

Transient Response is affected by:

- Dominant roots of the characteristic equation (i.e. dominant poles)
- Higher order roots (i.e. poles)
- Numerator roots (i.e. zeros)



Modeling and Analysis of Dynamic Systems

Brief Mathematics Review

Introductory Mathematics

- Mathematical functions commonly found in solution of *Linear Differential Equations*:
  - exponential function:
    ⇨ e^{at} where $e = 2.7182$ ... (irrational)
  - sine & cosine functions:
    ⇨ $\sin(bt)$ or $\cos(bt)$
  - exponentially modulated sine function:
    ⇨ $e^{at}\sin(bt)$

Exponential Function

- e is defined such that:

$$\frac{d}{dx}e^x = e^x \quad \text{for all values of x}$$

- Natural logarithm

$$y = e^x \Longrightarrow \log_e x = y = \ln x$$

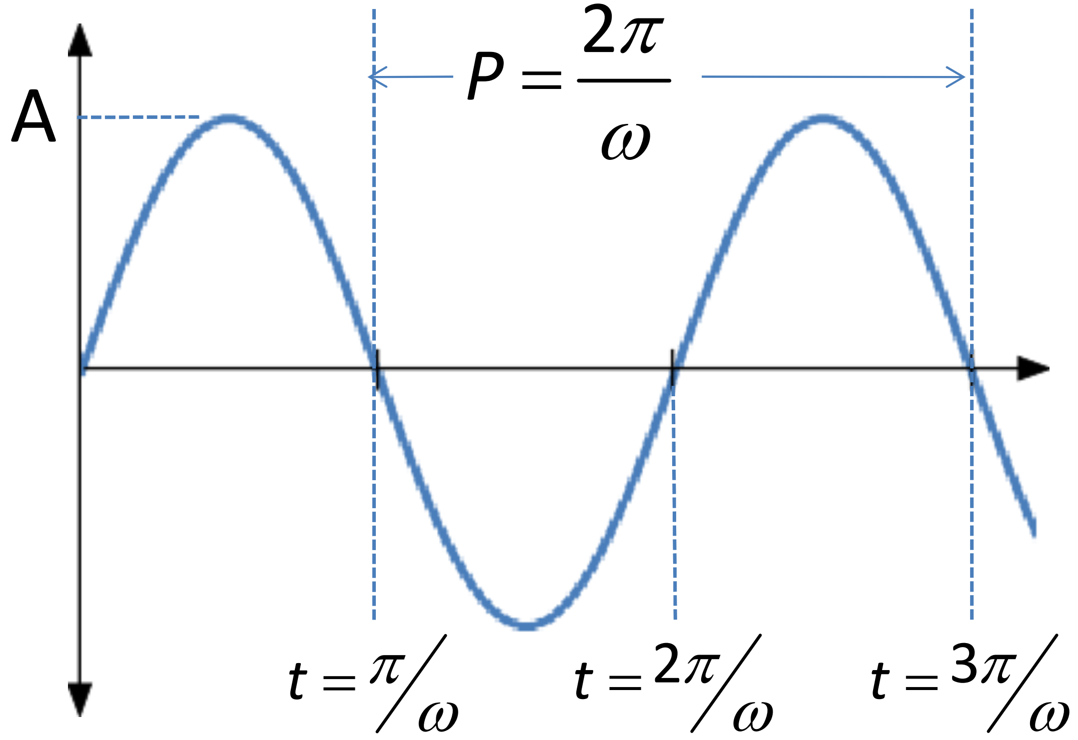
- Logarithm math

$$\ln xy = \ln x + \ln y \qquad \ln \frac{x}{y} = \ln x - \ln y$$

$$\ln x^n = n \ln x$$

Sine & Cosine Functions

- Sine function: $y(x) = A\sin x$
- When function of time: $y(t) = A\sin\omega t$



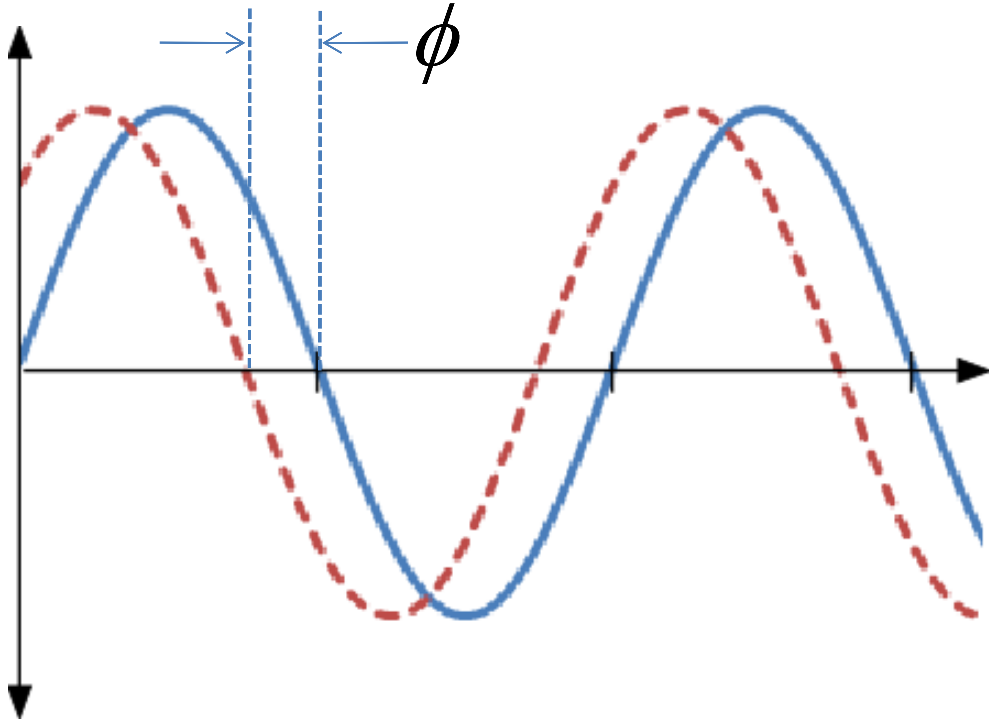
Frequency: ω
- Units: radians/sec

Period: P (or T)
- P = $2\pi/\omega$
- Units: seconds

Sine & Cosine Functions

- Sine function with phase (shifted on time axis):

$$y(t) = A\sin(\omega t + \phi)$$



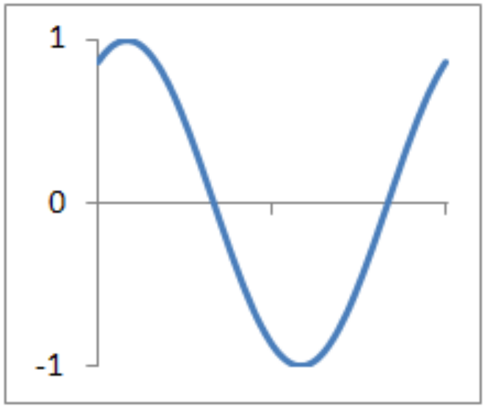
Phase: ϕ

- Units: radians
- *Positive* phase shifts the wave-form to the left

Sine & Cosine Functions

- Sine with phase can be expressed as sum of sine and cosine with no phase

$$A\sin(\omega t+\phi) \;=\; B\sin(\omega t)+C\cos(\omega t)$$



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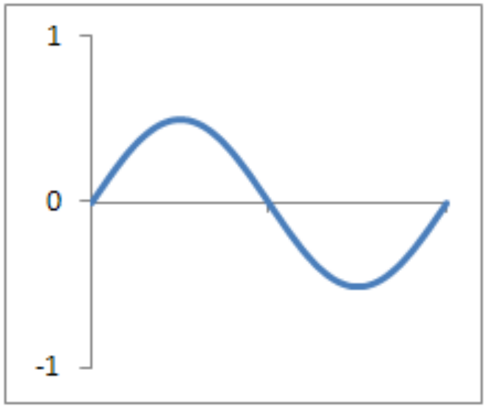

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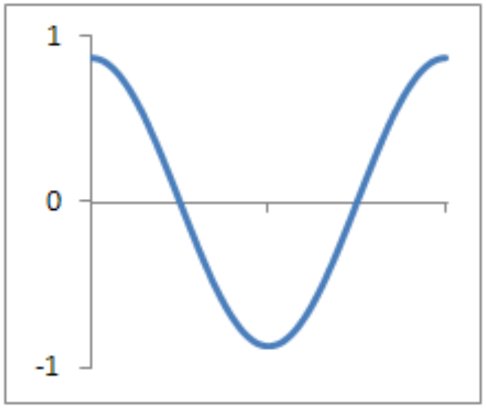


where

$$B=A\cos\phi$$

$$C=A\sin\phi$$

to convert back …

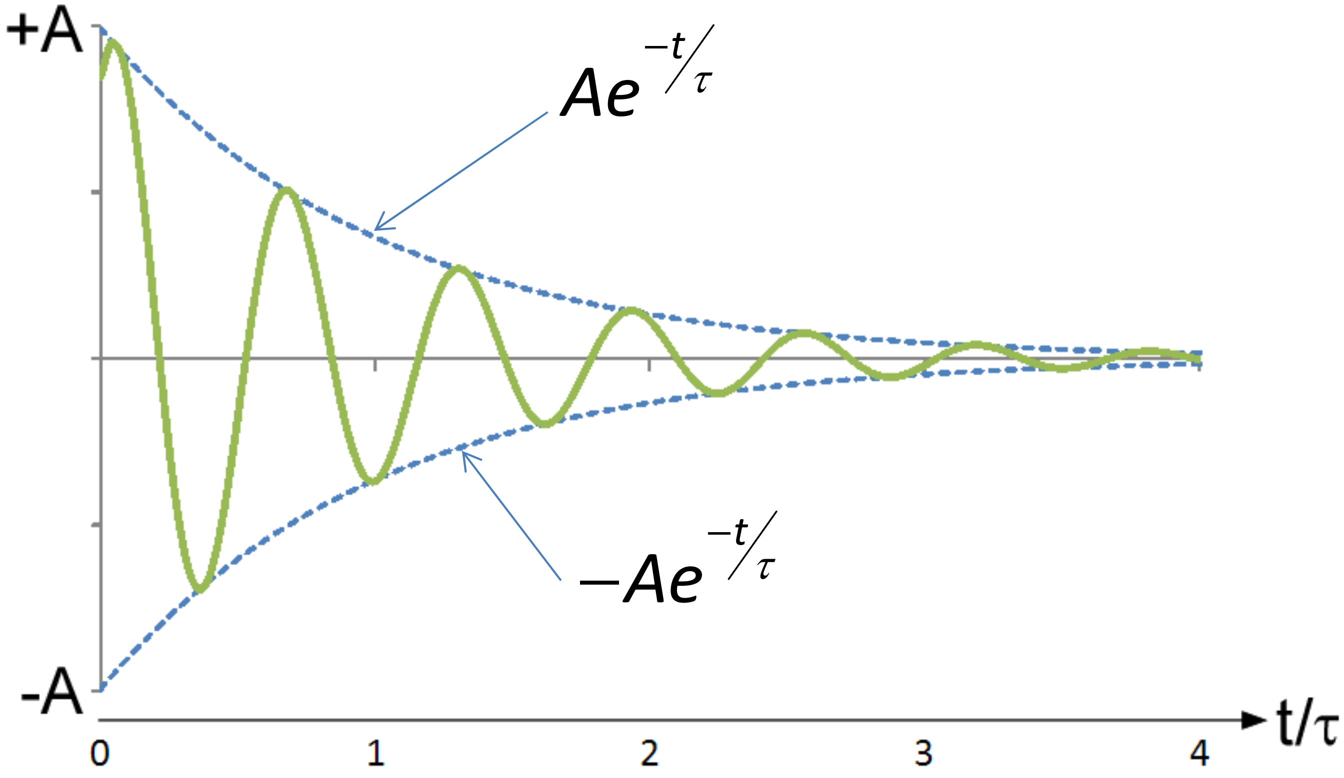
$$A=\pm\sqrt{B^2+C^2}$$

$$\cos\phi={}^{\mathrm{B}}\!/_{A} \quad \text{and} \quad \sin\phi={}^{C}\!/_{A}$$

Exponentially-Modulated Sine

- Product of exponential and sine functions

$$Ae^{-t/\tau}\sin(\omega t+\phi)$$



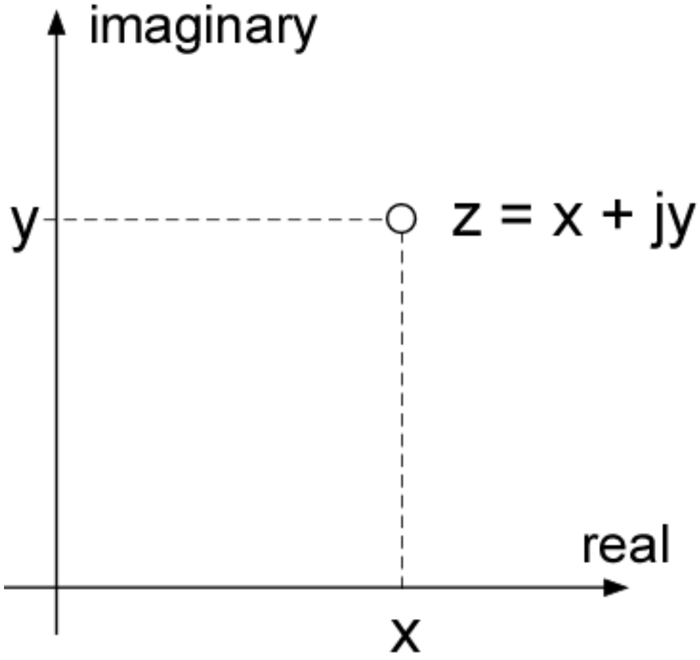
Review – Complex Numbers

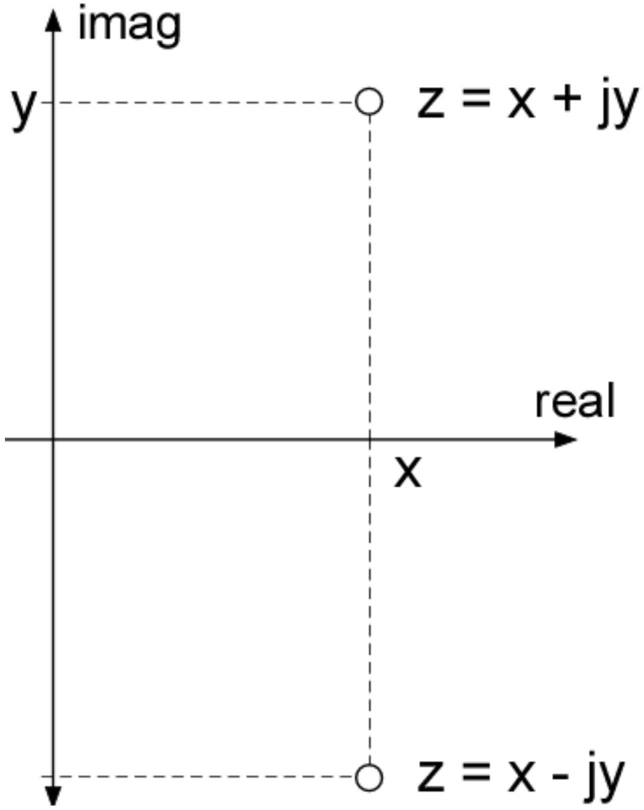
- complex number:

$$z = x + jy \qquad \text{where } j^2 = -1$$

imaginary part

real part





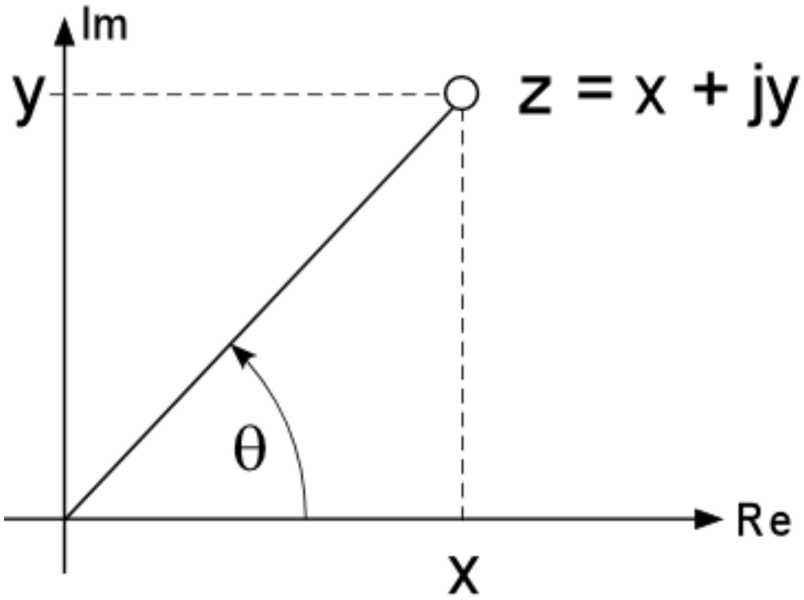
- complex conjugate:

$$z = x + jy \longrightarrow z = x - jy$$

Review – Complex Numbers

- complex number:

$$z = x + jy$$



magnitude:

$$|z| = \sqrt{x^2 + y^2}$$

angle (or phase):

$$\theta = \tan^{-1}\left[\frac{y}{x}\right]$$

- equivalent rectangular form:

$$z = x + jy \longleftrightarrow z = |z|(\cos\theta + j\sin\theta)$$

Review – Complex Numbers

Euler Theorem: $\cos\theta + j\sin\theta = e^{j\theta}$

- Euler's theorem derivation

Power series expansions:

$$e^{x} = 1 + x + \frac{x^{2}}{2!} + \frac{x^{3}}{3!} + \frac{x^{4}}{4!} + \cdots$$

$$\cos\theta = 1 - \frac{\theta^{2}}{2!} + \frac{\theta^{4}}{4!} - \frac{\theta^{6}}{6!} + \cdots$$

$$\sin\theta = \theta - \frac{\theta^{3}}{3!} + \frac{\theta^{5}}{5!} - \frac{\theta^{7}}{7!} + \cdots$$

Review – Complex Numbers

- Euler's theorem derivation (continued)

Use series expansions to form sum:

$$\cos\theta + j\sin\theta = 1 + (j\theta) + \frac{(j\theta)^2}{2!} + \frac{(j\theta)^3}{3!} + \cdots$$

Equivalent to e^x *where* $x = j\theta$

$$e^{(j\theta)} = 1 + (j\theta) + \frac{(j\theta)^2}{2!} + \frac{(j\theta)^3}{3!} + \frac{(j\theta)^4}{4!} + \cdots$$

It follows that …

$$\cos\theta + j\sin\theta = e^{j\theta}$$ **Euler's theorem**

Review – Complex Numbers

- equivalent rectangular form:

$$e^{j\theta} = \cos\theta + j\sin\theta$$

$$z = x + jy \longleftrightarrow z = |z|\overbrace{(\cos\theta + j\sin\theta)}^{\downarrow}$$

$$z = |z|e^{j\theta}$$

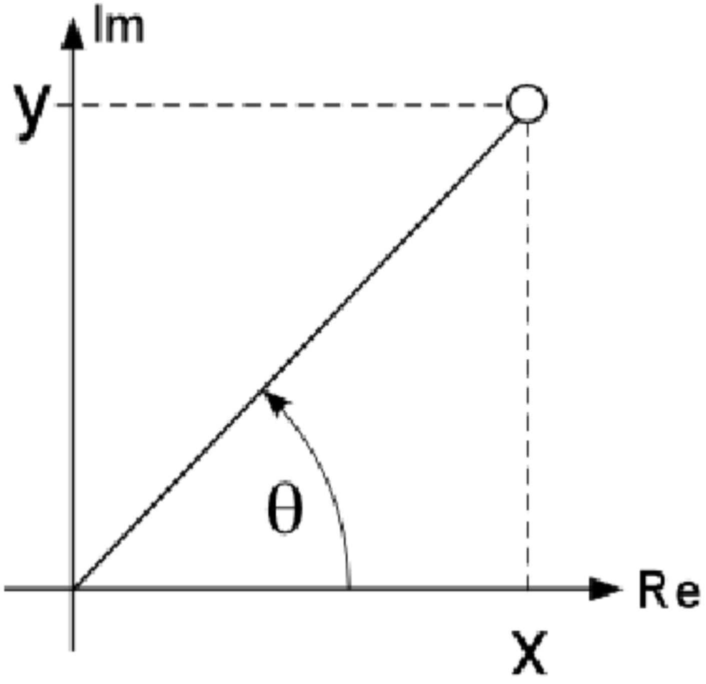
rectangular form:

$$z = x + jy$$

$$z = |z|(\cos\theta + j\sin\theta)$$

polar form:

$$z = |z|e^{j\theta}$$



Review – Complex Numbers

- Addition / subtraction *(easier in rectangular form)*

$$\left.\begin{aligned} z_1 &= x_1 + jy_1 \\ z_2 &= x_2 + jy_2 \end{aligned}\right\} z_1 + z_2 = (x_1 + x_2) + j(y_1 + y_2)$$

- Multiplication/ division *(easier in polar form)*

$$\left.\begin{aligned} z_1 &= |z_1| e^{j\theta_1} \\ z_2 &= |z_2| e^{j\theta_2} \end{aligned}\right\} z_1 z_2 = |z_1||z_2| e^{j(\theta_1 + \theta_2)}$$

Quick derivation of $x(t) = D_1 e^{st} + D_2 e^{s^* t} = Ce^{\sigma t}\sin(\omega t + \phi)$

With complex conjugate roots, the inverse Laplace transform has complex exponents and coefficients

$$x(t) = D_1 e^{st} + D_2 e^{s^* t} \quad \textit{where} \quad \begin{aligned} s &= \sigma + j\omega \\ s^* &= \sigma - j\omega \end{aligned}$$

$$= D_1 e^{(\sigma + j\omega)t} + D_2 e^{(\sigma - j\omega)t}$$

$$= D_1 e^{\sigma t} e^{j\omega t} + D_2 e^{\sigma t} e^{-j\omega t}$$

noting $\boxed{e^{j\theta} = \cos\theta + j\sin\theta}$ (Euler's theorem)

$$x(t) = D_1 e^{\sigma t}\left(\cos\omega t + j\sin\omega t\right) + D_2 e^{\sigma t}\left(\cos(-\omega t) + j\sin(-\omega t)\right)$$

$$= e^{\sigma t}\left[\underbrace{\left(D_1 + D_2\right)}\cos\omega t + j\underbrace{\left(D_1 - D_2\right)}\sin\omega t\right]$$

must be real

Continuing …

Quick derivation of $x(t) = D_1 e^{st} + D_2 e^{s^* t} = Ce^{\sigma t}\sin(\omega t + \phi)$

$$x(t) = e^{\sigma t}\left(\underbrace{\left(D_1 + D_2\right)}\cos\omega t + j\underbrace{\left(D_1 - D_2\right)}\sin\omega t\right)$$

must be real

$$\left.\begin{aligned} \operatorname{Im}\left\{D_1 + D_2\right\} = 0 &\Longrightarrow \operatorname{Im}\left\{D_1\right\} = -\operatorname{Im}\left\{D_2\right\} \\ \operatorname{Im}\left\{j\left(D_1 - D_2\right)\right\} = 0 &\Longrightarrow \operatorname{Re}\left\{D_1\right\} = \operatorname{Re}\left\{D_2\right\} \end{aligned}\right\}$$

D_1 and *D_2* are complex conjugates

letting $D = D_1$

$$x(t) = e^{\sigma t}\left(\left(D + D^*\right)\cos\omega t + j\left(D - D^*\right)\sin\omega t\right)$$

$$= e^{\sigma t}\left(2\operatorname{Re}\{D\}\cos\omega t - 2\operatorname{Im}\{D\}\sin\omega t\right)$$

Coefficients are now real

Continuing …

Quick derivation of $x(t)=D_1e^{st}+D_2e^{s^*t}=Ce^{\sigma t}\sin(\omega t+\phi)$

$$x(t)=e^{\sigma t}\left(2\mathrm{Re}\{D\}\cos\omega t-2\mathrm{Im}\{D\}\sin\omega t\right)$$

Coefficients
are now real

We want to get this into the form

$$\begin{aligned} x(t)&=Ce^{\sigma t}\sin(\omega t+\phi)\\ &=e^{\sigma t}\Big(\underbrace{C\cdot\sin\phi}_{2\mathrm{Re}\{D\}}\cdot\cos\omega t+\underbrace{C\cdot\cos\phi}_{-2\mathrm{Im}\{D\}}\cdot\sin\omega t\Big)\end{aligned}$$

$$\left.\begin{aligned}\sin\phi&=2\mathrm{Re}\{D\}/C\\ \cos\phi&=-2\mathrm{Im}\{D\}/C\end{aligned}\right\}\begin{aligned}\tan\phi=\tfrac{\sin\phi}{\cos\phi}&=-\mathrm{Re}\{D\}/\mathrm{Im}\{D\}\\ \underline{\phi=\tan^{-1}\left(-\mathrm{Re}\{D\}/\mathrm{Im}\{D\}\right)}&\end{aligned}$$

$$\sin^2\phi+\cos^2\phi=4\mathrm{Re}\{D\}^2/C^2+4\mathrm{Im}\{D\}^2/C^2=1$$

$$\underline{C=2\sqrt{\mathrm{Re}\{D\}^2+\mathrm{Im}\{D\}^2}}$$