EMA550 SP25 Kyle Adler

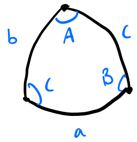
Rel. $\bar{a}$ & $\bar{v}$

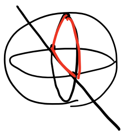
$$\bar{v}_P = \bar{v}_{O'} + (\bar{v}_P)_{xyz} + \bar{\omega} \times \bar{r}_{P/O'}$$
$$\bar{a}_P = \bar{a}_{O'} + (\bar{a}_P)_{xyz} + \bar{\alpha} \times \bar{r}_{P/O'} + \bar{\omega} \times (\bar{\omega} \times \bar{r}_{P/O'}) + 2\bar{\omega} \times (\bar{v}_P)_{xyz}$$

$\sin(90° - \theta) = \cos\theta$ | $\sin(-\theta) = -\sin(\theta)$

$\cos(90° - \theta) = \sin\theta$ | $\cos(-\theta) = \cos(\theta)$

Spherical trig:





Properties

1) Sum of two sides > third side
2) $A+B+C > 180°$
3) $A, B, C < 180°$

A, B, C - angles of orientation between curves

a, b, c - angles of arcs @ center of great circle

Cosine formulas
- Relating a side and the opposite angle

$$\cos a = \cos b \cos c + \sin b \sin c \cos A$$
$$\cos b = \cos a \cos c + \sin a \sin c \cos B$$
$$\cos c = \cos a \cos b + \sin a \sin b \cos C$$

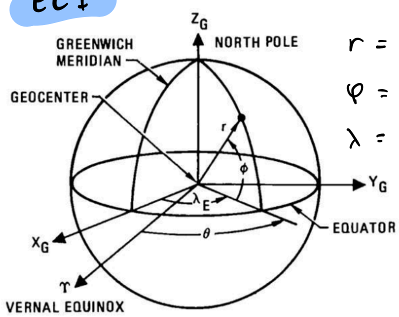
- Relating angles with adjacent angles and the opposite side

$$\cos A = -\cos B \cos C + \sin B \sin C \cos a$$
$$\cos B = -\cos A \cos C + \sin A \sin C \cos b$$
$$\cos C = -\cos A \cos B + \sin A \sin B \cos c$$

Law of Sines

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}$$

ECI

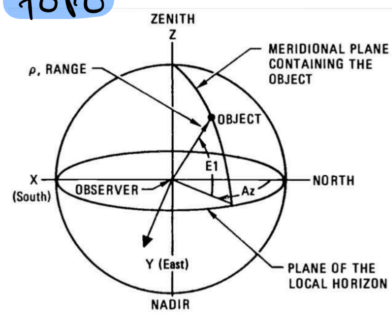


r = range

φ = latitude

λ = longitude

TOPO



x axis - southward

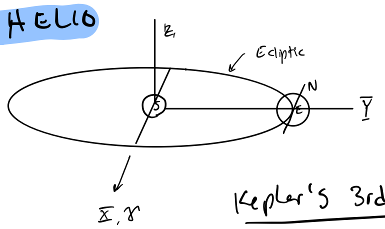
y axis - Eastward

ρ = range

El = angle above horizon

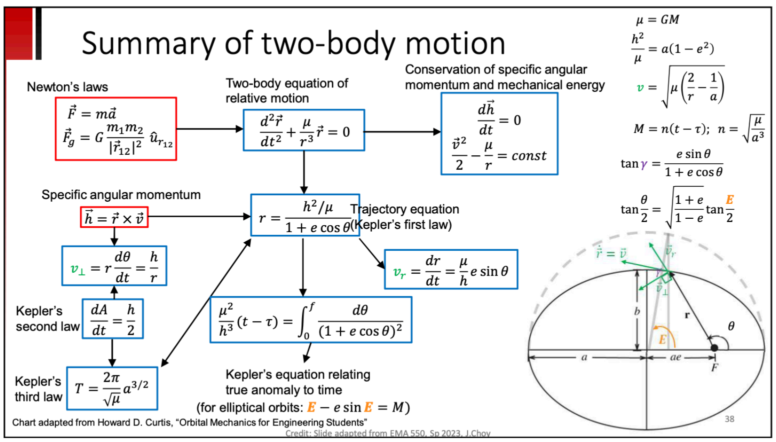
Az = Eastward angle from north

HELIO



Kepler's 3rd Law:

$$T^2 \propto a^3$$



Solve E iteratively:

$$f(E) = E - e\sin E - M$$
$$f'(E) = \frac{df}{dE} = 1 - e\cos E$$
$$E_{i+1} = E_i - \frac{E_i - e\sin E_i - M}{1 - e\cos E_i}$$

Find θ from r, a, e

→ trajectory, select $\cos\theta$ root based on quadrant & if sat. moving towards P ($\gamma < 0$) or away from P ($\gamma > 0$)

→ $\theta = 2\pi - \cos^{-1}(\ )$

Time to strike earth: Find E_1, E_2 for current pos & earth radius

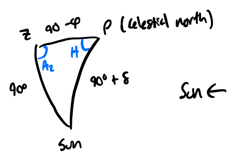
$M \rightarrow \Delta t = t_1 - t_2$

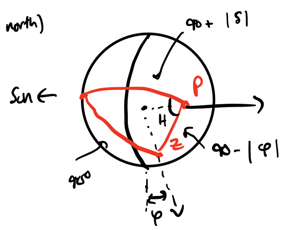
Find Sunrise time

- construct spherical triangle:
- H: Sun south (noon)

  12:00 ± H
- $\Delta T = \left(\frac{Lat}{360°}\right)(24 hrs)$

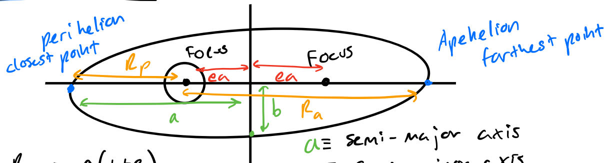
e.g. Noon time = [Noon GMT − (ST shift)] + H long





→ choose δ root based on expectations

Elliptical orbit $(e<1)$



$a \equiv$ semi-major axis
$b \equiv$ semi-minor axis
$e \equiv$ eccentricity

$\therefore\ R_a = a(1+e)$
$R_p = a(1-e)$

→ Speed: $v = \sqrt{\mu\left(\frac{2}{r} - \frac{1}{a}\right)}$ vis viva

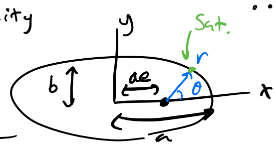
Ellipse details

equation $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$

$r \equiv$ dist. from focus to curve
$\theta \equiv$ angle from x axis to $\vec{r}$ vector

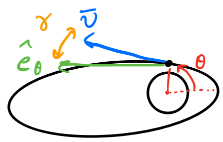
$\therefore\ x = ae + r\cos\theta$, $y = r\sin\theta$

$b = a\sqrt{1-e^2}$



$$\boxed{r(\theta) = \frac{a(1-e^2)}{1+e\cos\theta}}$$

→ Flight path angle γ



$\cos\gamma = \sqrt{\frac{a^2(1-e^2)}{r(2a-r)}}$

$\tan\gamma = \frac{e\sin\theta}{1+e\cos\theta}$

$\gamma > 0$: moving away

→ Mean anomaly: $n = \frac{2\pi}{T} = \sqrt{\frac{\mu}{a^3}}$, $M = n(\overbrace{t-\tau}^{\text{time since perigee}})$

→ Eccentric anomaly: $\tan\frac{\theta}{2} = \sqrt{\frac{1+e}{1-e}}\tan\frac{E}{2}$, $M = E - e\sin E$

Parabolic orbit $(e=1)$

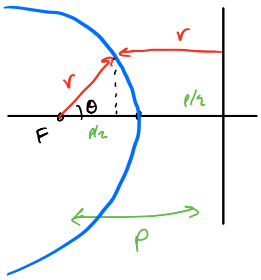
$r = \frac{p}{1+\cos\theta}$ $\quad p = \frac{h^2}{\mu}$

→ trajectory: same as ellipse

$r = \frac{a(1-e^2)}{1+e\cos\theta}$

→ speed:

$v_{escape} = \sqrt{\frac{2\mu}{r}}$

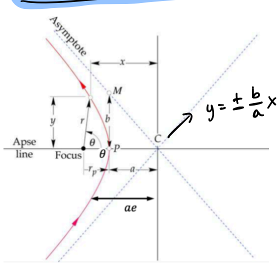


$$\sqrt{\frac{\mu}{p^3}}(t-\tau) = \frac{1}{2}\tan\frac{\theta}{2} + \frac{1}{6}\tan^3\frac{\theta}{2}$$

- We define the mean anomaly, M_p, of the parabola as
$$M_p \equiv \sqrt{\frac{\mu}{p^3}}(t-\tau) = \frac{\mu^2}{h^3}(t-\tau)$$
- Given t, we can find f by solving the cubic equation
$$\frac{1}{2}\tan\frac{\theta}{2} + \frac{1}{6}\tan^3\frac{\theta}{2} - M_p = 0$$
  - Solution:
$$\tan\frac{\theta}{2} = z - \frac{1}{z}$$
$$z = \left(3M_p + \sqrt{1+(3M_p)^2}\right)^{1/3}$$

Hyperbolic orbit $(e>1)$

$b^2 = a^2(e^2-1)$



$y = \pm\frac{b}{a}x$

→ trajectory:

$r = \frac{a(e^2-1)}{1+e\cos\theta}$ $\quad r_p = a(e-1)$

→ speed:

$v = \sqrt{\frac{2\mu}{r} + \frac{\mu}{a}}$ $\quad v \to \sqrt{\frac{\mu}{a}}$ as $r \to \infty$

→ Hyperbolic anomaly H, M_h

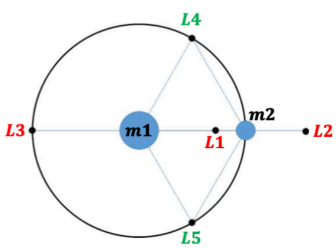
$r = a(e\cosh H - 1)$

$M_h = \sqrt{\frac{\mu}{a^3}}(t-\tau) = e\sinh H - H$

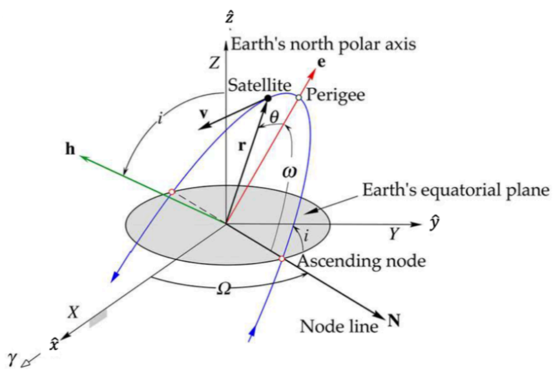
$\tanh\frac{H}{2} = \left(\frac{e-1}{e+1}\right)\tan\frac{\theta}{2}$

Lagrange points

$\bar{v} = \bar{a} = \bar{0}$



MCS



Practical orbits

$$\frac{d\Omega}{dt} = \dot{\Omega} = \frac{-3J_2 n r_e^2 \cos i}{2a^2(1-e^2)^2}$$

$$\frac{d\omega}{dt} = \dot{\omega} = \frac{3J_2 n r_e^2}{4a^2(1-e^2)^2}(4 - 5\sin^2 i)$$

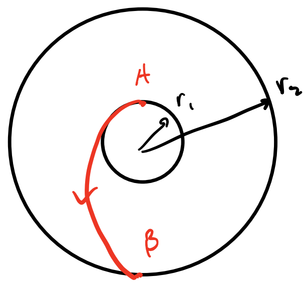
$$n = 2\pi/T$$

$$J_2 = 1.082626683 \times 10^{-3} \text{ (for Earth)}$$

Ground tracks:

$$T = \frac{360° - \Delta N}{15°/hr}$$

between asc. nodes



Hohmann transfer

time: $T_t = \left(\frac{1}{2}\right) 2\pi \sqrt{\frac{a_t^3}{\mu}} = \pi \sqrt{\frac{(r_1+r_2)^3}{8\mu}}$

t = "transfer"

$$\Delta v_A = v_{t_A} - v_{c_1}$$

$$= \sqrt{\mu\left(\frac{2}{r_1} - \frac{1}{a}\right)} - \sqrt{\frac{\mu}{r_1}} = \boxed{\sqrt{\frac{\mu}{r_1}}\left(\sqrt{\frac{r_2}{a}} - 1\right)}$$

algebra

$$\Delta v_B = v_{c_2} - v_{t_B}$$

$$= \sqrt{\frac{\mu}{r_2}} - \sqrt{\mu\left(\frac{2}{r_2} - \frac{1}{a}\right)} = \boxed{\sqrt{\frac{\mu}{r_2}}\left(1 - \sqrt{\frac{r_1}{a}}\right)}$$

$$\Delta v_{total} = \Delta v = |\Delta v_A| + |\Delta v_B|$$

Split plane change:

$$\Delta i = 60° = \alpha_1 + \alpha_2$$

$$\Delta v_1^2 = v_{t_A}^2 + v_{c_1}^2 - 2 v_{t_A} v_{c_1} \cos \alpha_1$$

$$\Delta v_2^2 = v_{t_B}^2 + v_{c_2}^2 - 2 v_{t_B} v_{c_2} \cos \alpha_2$$

$$\rightarrow \alpha_2 = 60° - \alpha_1$$

phasing

$$T_{ph} = \frac{1}{2} T_{target} = \frac{1}{2} \frac{2\pi}{\sqrt{\mu}} a^{3/2}$$

$$a_{ph} = \left(\frac{T\sqrt{\mu}}{2\pi}\right)^{2/3}$$

$$\Delta v = \frac{\Delta\theta}{2\pi} \frac{\mu}{3 v_0 a_0 N}$$

a_0, v_0 = current orbit & speed

N = # of orbits req'd for interception (incl. decimal)

Cost of transfer

$h = \bar{r} \times \bar{v}$

specific orbit energy $\mathcal{E} = \frac{1}{2} v^2$

change in energy for transfer:

$$\Delta\mathcal{E} = -\frac{\mu}{2a_2} + \frac{\mu}{2a_1}$$

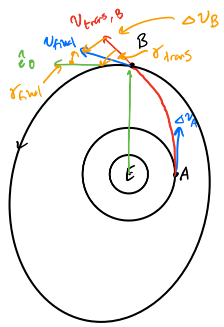
a_1 = initial orbit

a_2 = target orbit

$$\mathcal{E}_{cost} \cong \frac{\Delta v^2}{2}$$

Bi-elliptic

To decrease time of Hohmann transfer:



need v_B, triangle not necessarily right

use Law of cosines for Δv_B

$$\Delta v_B^2 = v_{final}^2 + v_{trans,B}^2 - 2 v_{final} v_{trans,B} \cos(\gamma_t - \gamma_f)$$

γ_t = flight path ∡ of transfer @ B

γ_f = " " " target orbit @ B

$$v = \sqrt{\mu\left(\frac{2}{r} - \frac{1}{a}\right)}$$

Transfer 1 ☆

$$a_1 = \frac{r_A + r_B}{2}$$

$$e_1 = \frac{r_B - r_A}{r_B + r_A}$$

Transfer 2 ☆

$$a_2 = \frac{r_C + r_B}{2}$$

$$e_2 = \frac{r_B - r_C}{r_B + r_C}$$

Calculate Δv's

$$\Delta v_1 = v_{t_1 A} - v_{c_1} = \sqrt{\mu\left(\frac{2}{r_A} - \frac{2}{r_A + r_B}\right)} - \frac{\mu}{r_A}$$

algebra:

$$v_{t_1 A} = \sqrt{2\mu\left(\frac{r_B}{r_A(r_A + r_B)}\right)}$$

where you're going

where you are

⋮

$$v_{t_1 B} = \sqrt{2\mu\left(\frac{r_A}{r_B(r_A + r_B)}\right)}$$

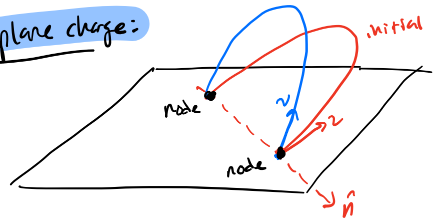
$$v_{t_2 B} = \sqrt{2\mu\left(\frac{r_C}{r_B(r_B + r_C)}\right)}$$

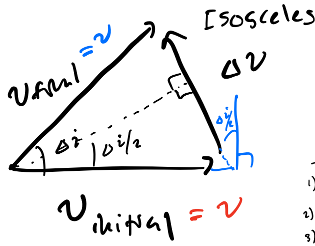
$$v_{t_2 C} = \sqrt{2\mu\left(\frac{r_B}{r_C(r_B + r_C)}\right)}$$

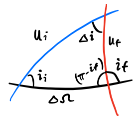
$$\therefore \Delta v_{tot} = \underbrace{|\Delta v_1|}_{v_{t_1 A} - v_{circ,1}} + \underbrace{|\Delta v_2|}_{v_{t_2 B} - v_{t_1 B}} + \underbrace{|\Delta v_3|}_{v_{c_2} - v_{t_2 C}}$$

$v_{circ,1} = v_{A_1}$, $v_{c_2} = v_{circ_2}$

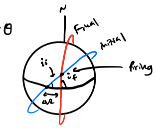
Simple plane change:







let $u_i = \omega + \theta$



Steps for this

1) Find Δi, spherical triangle $(i_i, i_f, \Delta\Omega)$
2) Find θ at firing w/ $u_i = \omega + \theta$ & triangle (i_i, i_f, u_i)
3) Find Δv knowing $\Delta i, a, e$

e.g. $\cos A = -\cos B \cos C + \sin B \sin C \cos a$

$\therefore \frac{\Delta v}{2} = v \sin\left(\frac{\Delta i}{2}\right)$

$$\boxed{\Delta v = 2v\sin\left(\frac{\Delta i}{2}\right)}$$

For a tangential burn $(\gamma = 0)$

For non-tangential burn $(\gamma \neq 0)$ need proj. of v onto θ dir.

$$\therefore \boxed{\Delta v = 2v\sin\left(\frac{\Delta i}{2}\right)\cos\gamma}$$

simple plane change for non-tangential

Bi-elliptic w/ plane change

steps

1) assume circular to start
2) Δv_1 -> large ellipse
3) Δv_2 -> rotates large ellipse
4) Δv_3 -> back to circular orbit

1) initial orbit $v_{c_1} = \sqrt{\frac{\mu}{r_1}}$ pick large $a = \frac{r_1+r_2}{2}$
2) choose r_2 $\Delta v_1 = v_{1t_A} - v_{c_1}$

$v_{1t_A} = \sqrt{\frac{2\mu r_2}{r_1(r_1+r_2)}}$

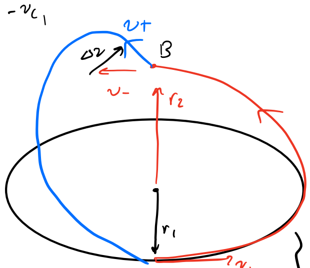
3) plane change

$\Delta v = 2v_{1t_B}\sin\left(\frac{\Delta i}{2}\right)$

$v_{1t_B} = \sqrt{\frac{2\mu r_1}{r_2(r_1+r_2)}}$

4) Final orbit $v_{c_2} = v_{c_1}$

$\Delta v_3 = v_{c_2} - v_{1t_A}$



Lambert

$$\sqrt{\mu}\,\Delta t = a^{3/2}[\alpha - \beta - (\sin\alpha - \sin\beta)]$$

where $\sin\frac{\alpha}{2} = \sqrt{\frac{s}{2a}}$ $\quad \sin\frac{\beta}{2} = \sqrt{\frac{s-c}{2a}}$

$$c = \sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos\Delta\theta}$$

$$s = \frac{r_1 + r_2 + c}{2}$$

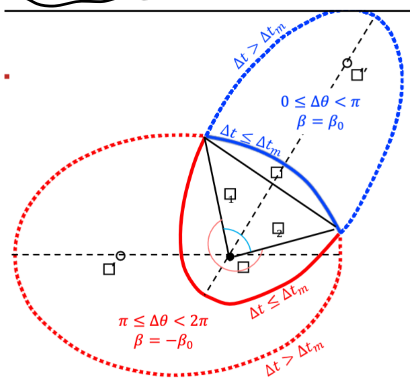
$$\Delta t_p = \frac{1}{3}\sqrt{\frac{2}{\mu}}\left[s^{\frac{3}{2}} - \operatorname{sgn}(\sin\Delta\theta)(s-c)^{\frac{3}{2}}\right]$$

$$\Delta t_m = \sqrt{\frac{s^3}{8\mu}}\left(\pi - \sqrt{\frac{s-c}{s}} + \sin\sqrt{\frac{s-c}{s}}\right)$$

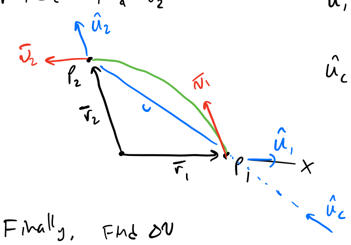
1. Calculate parabolic transfer time Δt_p.
   - If $\Delta t > \Delta t_p \to$ elliptical transfer. Otherwise, orbit is parabolic or hyperbolic
2. Calculate Δt_m and determine quadrant of α.
   $\Delta t \le \Delta t_m \to \alpha = \alpha_0$; else $\alpha = 2\pi - \alpha_0$
   ↑ principal solution
3. Determine quadrant of β.
   $0 \le \Delta\theta < \pi \to \beta = \beta_0$; else $\beta = -\beta_0$
4. Numerically solve Lambert's equation for a unique value of a.

Eccentricity: $p = a(1-e^2) = \frac{4a(s-r_1)(s-r_2)}{c^2}\sin^2\left(\frac{\alpha+\beta}{2}\right)$

Solve for e



If Need $\bar{v}_1$ & $\bar{v}_2$



$\hat{u}_1 = \frac{\bar{r}_1}{|\bar{r}_1|}$ $\quad \hat{u}_2 = \frac{\bar{r}_2}{|\bar{r}_2|}$

$\hat{u}_c = \frac{\bar{r}_2 - \bar{r}_1}{c}$

$A = \sqrt{\frac{\mu}{4a}}\cot\left(\frac{\alpha}{2}\right)$

$B = \sqrt{\frac{\mu}{4a}}\cot\left(\frac{\beta}{2}\right)$

$\bar{v}_1 = (B+A)\hat{u}_c + (B-A)\hat{u}_1$

$\bar{v}_2 = (B+A)\hat{u}_c - (B-A)\hat{u}_2$

Finally, Find Δv

(local x axis to P_1)

ex. if P_1 is circular orbit, $\bar{v}_{init} = 0\hat{i} + v_{c_1}\hat{j}$

$\therefore \Delta\bar{v}_1 = \bar{v}_1 - \bar{v}_{init}$, $|\Delta v_1| = \sqrt{\Delta\bar{v}_1 \cdot \Delta\bar{v}_1}$

$\Delta v_1 \to$ gets you on transfer

Rocketry:

$m\dot{\bar{v}} = -b\bar{c} + \bar{F}_{ext}$

$\frac{\Delta m}{m_0} = 1 - e^{(-\Delta v/c)}$ "mass consumption"