Problem 1

1(a) Neglect thickness distribution, determine expression for zero-lift angle of attack ($\alpha_{L=0}$) of a 4-digit NACA camberline in terms of the three parameters m, p, θ_p.

conditions:

$$C_l = 2\pi(\alpha - \alpha|_{L=0}) \qquad \alpha|_{L=0} = -\frac{1}{\pi}\int_0^{\pi} \frac{dz}{dx}(\cos\theta_0 - 1)\, d\theta_0$$

$$\frac{dz/c}{dx/c} = \begin{cases} \frac{2m}{p^2}(p - x/c) & 0 \le x/c \le p \\ \frac{2m}{(1-p)^2}(p - x/c) & p < x/c \le 1 \end{cases}$$

Change of variables for $\frac{dz}{dx}$:

$$x = \frac{c}{2}(1 - \cos\theta_0)$$

$$\rightarrow 1 - \frac{2x}{c} = \cos\theta_0$$

$$\rightarrow \theta_0 = \operatorname{acos}\left(1 - \frac{2x}{c}\right)$$

Bounds:

$$\frac{x}{c} = 0 \rightarrow \theta_0\ 0$$

$$\frac{x}{c} = p \rightarrow \theta_0 = \operatorname{acos}(1-2p) = \theta_p$$

$$\frac{x}{c} = 1 \rightarrow \theta_0 = \operatorname{acos}(-1) = \pi$$

$$\rightarrow \frac{dz/c}{dx/c} = \begin{cases} \frac{2m}{p^2}\left(p - \frac{1}{2}(1 - \cos\theta_0)\right) & 0 \le \theta_0 \le \operatorname{acos}(1-2p) \\ \frac{2m}{(1-p)^2}\left(p - \frac{1}{2}(1 - \cos\theta_0)\right) & \operatorname{acos}(1-2p) < \theta_0 \le \pi \end{cases}$$

$$\alpha|_{L=0} = -\frac{1}{\pi}\int_0^{\pi} \frac{dz}{dx}(\cos\theta_0 - 1)\, d\theta_0$$

$$\rightarrow \alpha|_{L=0} = -\frac{1}{\pi}\int_0^{\theta_p} \frac{2m}{p^2}\left(p - \frac{1}{2}(1-\cos\theta_0)\right)(\cos\theta_0 - 1)\, d\theta_0$$
$$- \frac{1}{\pi}\int_{\operatorname{acos}(1-2p)}^{\pi} \frac{2m}{(1-p)^2}\left(p - \frac{1}{2}(1-\cos\theta_0)\right)(\cos\theta_0 - 1)\, d\theta_0$$

$$\alpha|_{L=0} = -\frac{2m}{p^2\pi}\int_0^{\theta_p}\left(p - \frac{1}{2} + \frac{1}{2}\cos\theta_0\right)(\cos\theta_0 - 1)\, d\theta_0$$
$$- \frac{2m}{(1-p)^2\pi}\int_{\theta_p}^{\pi}\left(p - \frac{1}{2} + \frac{1}{2}\cos\theta_0\right)(\cos\theta_0 - 1)\, d\theta_0$$

$$\alpha|_{L=0} = \frac{-2m}{p^2\pi}\int_0^{\theta p}\left(p\cos\theta_0 - \tfrac{1}{2}\cos\theta_0 + \tfrac{1}{2}\cos^2\theta_0 - p + \tfrac{1}{2} - \tfrac{1}{2}\cos\theta_0\right)d\theta_0$$

$$- \frac{2m}{(1-p)^2\pi}\int_{\theta p}^{\pi}\left(p\cos\theta_0 - \tfrac{1}{2}\cos\theta_0 + \tfrac{1}{2}\cos^2\theta_0 - p + \tfrac{1}{2} - \tfrac{1}{2}\cos\theta_0\right)d\theta_0$$

$$\alpha|_{L=0} = \frac{-2m}{p^2\pi}\left[p\sin\theta_0 - \tfrac{1}{2}\sin\theta_0 + \tfrac{1}{2}\left(\tfrac{1}{2}\theta_0 + \tfrac{1}{4}\sin 2\theta_0\right) - p\theta_0 + \frac{\theta_0}{2} - \tfrac{1}{2}\sin\theta_0\right]_0^{\theta p}$$

$$- \frac{2m}{(1-p)^2\pi}\left[p\sin\theta_0 - \tfrac{1}{2}\sin\theta_0 + \tfrac{1}{2}\left(\tfrac{1}{2}\theta_0 + \tfrac{1}{4}\sin 2\theta_0\right) - p\theta_0 + \frac{\theta_0}{2} - \tfrac{1}{2}\sin\theta_0\right]_{\theta p}^{\pi}$$

$$\boxed{\begin{aligned}\alpha|_{L=0} = &-\frac{2m}{p^2\pi}\left[p\sin\theta_p - \sin\theta_p + \tfrac{1}{2}\left(\tfrac{1}{2}\theta_p + \tfrac{1}{4}\sin(2\theta_p)\right) - p\theta_p + \frac{\theta_p}{2}\right] \\ &-\frac{2m}{(1-p)^2\pi}\left[\left(\tfrac{1}{2}\left(\frac{\pi}{2}\right) - p\pi + \frac{\pi}{2}\right) - \left(p\sin\theta_p - \sin\theta_p + \tfrac{1}{2}\left(\tfrac{1}{2}\theta_p + \tfrac{1}{4}\sin(2\theta_p)\right) - p\theta_p + \frac{\theta_p}{2}\right)\right]\end{aligned}}$$

1b) Find C_l of NACA 2412 $(\alpha = 0°)$

$\rightarrow$ $\alpha|_{L=0}$ equation, plug in $m = \frac{2}{100}$ $p = \frac{4}{10}$, $\theta_p = a\cos(1-2p)$

$\rightarrow$ $\alpha|_{L=0} = -0.0363$ rad $= -2.08°$

$$C_l = 2\pi(\cancel{\alpha}^{\,0} - \alpha|_{L=0}) = 2\pi(0.0363) = \boxed{0.2278 = C_l}$$

1c) $C_l = 2\pi(8°\text{ to rad} - \alpha|_{L=0}) = \boxed{1.1051 = C_l'}$

1d) Compared to the experimental data, this result is extremely close, since $\alpha|_{L=0} \approx -2°$ and $C_l(\alpha=8°) \approx 1.1$, alebeit slightly smaller, which implies some neglected real-world effects, as expected.

Problem 2

2a) Write two different expressions for infinitesimal lift per unit span dL' generated by an element dx of the vortex sheet that lies along the chord of thin, asymmetric profile @ $\alpha = 0$ (one w/ local pressure difference & one w/ local circulation $\gamma(x)$)

→ Local pressure difference:

$$L' = c\,(P_\ell(x) - P_u(x))$$

$$\rightarrow \boxed{dL' = (P_\ell(x) - P_u(x))\,dx}$$

→ local circulation $\gamma(x)$:

Kutta - Joukowski: $L = \rho V \Gamma$

$$\rightarrow \boxed{dL' = \rho V_\infty \gamma(x)\,dx}$$

2b) Relate pressure coefficient to $\gamma(x)$

$$C_p(x) = \frac{P_\ell(x) - P_u(x)}{\frac{1}{2}\rho V_\infty^2}$$

$$\rightarrow \frac{dL'}{dx} = P_\ell(x) - P_u(x) = \rho V_\infty \gamma(x)$$

$$\rightarrow C_p(x) = \frac{\cancel{\rho} \cancel{V_\infty} \gamma(x)}{1/2\,\cancel{\rho} V_\infty^{\cancel{2}}} = \boxed{\frac{2\gamma(x)}{V_\infty} = C_p(x)}$$

2c) General 4-digit NACA @ α, Find A_0, A_1, A_n using only α, geometry (m, p, t), & n.

lecture:

$$A_0 = \alpha - 1/\pi \int_0^{\pi} \frac{dz}{dx}\, d\theta_0$$

$$A_1 = \frac{2}{\pi} \int_0^{\pi} dz/dx \cos\theta_0\, d\theta_0$$

$$A_n = \frac{2}{\pi} \int_0^{\pi} \frac{dz}{dx} \cos n\theta_0\, d\theta_0$$

$$\frac{dz/c}{dx/c} = \frac{dz}{dx} = \begin{cases} \frac{2m}{p^2}\left(p - \frac{1}{2}(1-\cos\theta_0)\right) & 0 \le \theta_0 \le \overbrace{a\cos(1-2p)}^{=\theta_p} \\ \frac{2m}{(1-p)^2}\left(p - \frac{1}{2}(1-\cos\theta_0)\right) & a\cos(1-2p) < \theta_0 \le \pi \end{cases}$$

$$A_0 = \alpha - \frac{1}{\pi}\int_0^{\theta_p} \frac{2m}{p^2}\left(p - \frac{1}{2}(1-\cos\theta_0)\right) d\theta_0 - \frac{1}{\pi}\int_{\theta_p}^{\pi} \frac{2m}{(1-p)^2}\left(p - \frac{1}{2}(1-\cos\theta_0)\right) d\theta_0$$

$$A_0 = \alpha - \frac{2m}{p^2\pi}\int_0^{\theta_p}\left(p - \frac{1}{2} + \frac{1}{2}\cos\theta_0\right) d\theta_0 - \frac{2m}{(1-p)^2\pi}\int_{\theta_p}^{\pi}\left(p - \frac{1}{2} + \frac{\cos\theta_0}{2}\right) d\theta_0$$

$$A_0 = \alpha - \frac{2m}{p^2\pi}\left[\int_0^{\theta_p} p\, d\theta_0 - \int_0^{\theta_p}\frac{1}{2}\, d\theta_0 + \int_0^{\theta_p}\frac{1}{2}\cos\theta_0\, d\theta_0\right] - \frac{2m}{(1-p)^2\pi}\left[\int_{\theta_p}^{\pi} p\, d\theta_0 - \int_{\theta_p}^{\pi} \tfrac{1}{2}\, d\theta_0 + \int_{\theta_p}^{\pi}\frac{1}{2}\cos\theta_0\, d\theta_0\right]$$

$$\boxed{A_0 = \alpha - \frac{2m}{p^2\pi}\left[p\theta_p - \frac{\theta_p}{2} + \frac{1}{2}\sin\theta_p\right] - \frac{2m}{(1-p)^2\pi}\left[p(\pi-\theta_p) - \frac{1}{2}(\pi-\theta_p) + \frac{1}{2}(-\sin\theta_p)\right] \quad \text{where} \quad \theta_p = \arccos(1-2p)}$$

$$A_1 = \frac{2}{\pi}\int_0^{\theta_p}\frac{2m}{p^2}\left(p - \frac{1}{2}(1-\cos\theta_0)\right)\cos\theta_0\, d\theta_0 + \frac{2}{\pi}\int_{\theta_p}^{\pi}\frac{2m}{(1-p)^2}\left(p - \frac{1}{2}(1-\cos\theta_0)\right)\cos\theta_0\, d\theta_0$$

$$A_1 = \frac{4m}{p^2\pi}\int_0^{\theta_p}\left(p - \frac{1}{2} + \frac{1}{2}\cos\theta_0\right)\cos\theta_0\, d\theta_0 + \frac{4m}{(1-p)^2\pi}\int_{\theta_p}^{\pi}\left(p - \frac{1}{2} + \frac{1}{2}\cos\theta_0\right)\cos\theta_0\, d\theta_0$$

$$A_1 = \frac{4m}{p^2\pi}\left[\int_0^{\theta_p} p\cos\theta_0\, d\theta_0 - \int_0^{\theta_p}\frac{1}{2}\cos\theta_0\, d\theta_0 + \int_0^{\theta_p}\frac{1}{2}\cos^2\theta_0\, d\theta_0\right]$$

$$+ \frac{4m}{(1-p)^2\pi}\left[\int_{\theta_p}^{\pi} p\cos\theta_0\, d\theta_0 - \int_{\theta_p}^{\pi}\frac{1}{2}\cos\theta_0\, d\theta + \int_{\theta_p}^{\pi}\frac{1}{2}\cos^2\theta_0\, d\theta_0\right]$$

$$A_1 = \frac{4m}{p^2\pi}\left[\left(p\sin\theta_p\right) - \left(\frac{1}{2}\sin\theta_p\right) + \frac{1}{2}\left(\frac{1}{2}\theta_p + \frac{1}{4}\sin 2\theta_p\right)\right]$$

$$+ \frac{4m}{(1-p)^2\pi}\left[\left(-p\sin\theta_p\right) - \frac{1}{2}\left(-\sin\theta_p\right) + \frac{1}{2}\left[\left(\frac{1}{2}\pi\right) - \left(\frac{1}{2}\theta_p + \frac{1}{4}\sin 2\theta_p\right)\right]\right]$$

where $\theta_p = \arccos(1-2p)$

$$A_n = \frac{2}{\pi}\int_0^{\theta_p} \frac{2m}{p^2}\left(p - \frac{1}{2}(1-\cos\theta_0)\right)\cos n\theta_0 \, d\theta_0 + \frac{2}{\pi}\int_{\theta_p}^{\pi} \frac{2m}{(1-p)^2}\left(p - \frac{1}{2}(1-\cos\theta_0)\right)\cos n\theta_0 \, d\theta_0$$

$$A_n = \frac{4m}{p^2\pi}\int_0^{\theta_p}\left(p - \frac{1}{2} + \frac{1}{2}\cos\theta_0\right)\cos n\theta_0 \, d\theta_0 + \frac{4m}{(1-p)^2\pi}\int_{\theta_p}^{\pi}\left(p - \frac{1}{2} + \frac{1}{2}\cos\theta_0\right)\cos n\theta_0 \, d\theta_0$$

$$A_n = \frac{4m}{p^2\pi}\left[\int_0^{\theta_p} p\cos n\theta_0 \, d\theta_0 - \frac{1}{2}\int_0^{\theta_p}\cos n\theta_0 \, d\theta_0 + \frac{1}{2}\int_0^{\theta_p}\cos\theta_0\cos n\theta_0 \, d\theta_0\right]$$

$$+ \frac{4m}{(1-p)^2\pi}\left[\int_{\theta_p}^{\pi} p\cos n\theta_0 \, d\theta_0 - \frac{1}{2}\int_{\theta_p}^{\pi}\cos n\theta_0 \, d\theta_0 + \frac{1}{2}\int_{\theta_p}^{\pi}\cos\theta_0\cos n\theta_0 \, d\theta_0\right]$$

$$\int \cos\theta_0 \cos n\theta_0 = \frac{\sin((1-n)\theta_0)}{2(1-n)} + \frac{\sin((1+n)\theta_0)}{2(1+n)}$$

$a = 1 \quad b = 0 \quad c = n \quad d = 0$

$$A_n = \frac{4m}{p^2\pi}\left[\frac{p}{n}\sin(n\theta_p) - \frac{1}{2}\left(\frac{1}{n}\sin(n\theta_p)\right) + \frac{1}{2}\left(\frac{\sin((1-n)\theta_p)}{2(1-n)} + \frac{\sin((1+n)\theta_p)}{2(1+n)}\right)\right]$$

$$+ \frac{4m}{(1-p)^2\pi}\left[\frac{p}{n}\left(-\sin(n\theta_p)\right) - \frac{1}{2}\left(-\frac{1}{n}\sin(n\theta_p)\right) + \frac{1}{2}\left(-\frac{\sin((1-n)\theta_p)}{2(1-n)} - \frac{\sin((1+n)\theta_p)}{2(1+n)}\right)\right]$$

where $\theta_p = \arccos(1-2p)$

2 d) plot A_n vs. n

2 e) plot C_p vs x w/ n

$$\gamma(\theta) = 2V_\infty\left[A_0 \frac{1+\cos\theta}{\sin\theta} + \sum_{n=1}^{\infty} A_n \sin(n\theta)\right], \qquad \theta = \operatorname{acos}\left(1 - \frac{2x}{c}\right)$$

$\to$ matlab