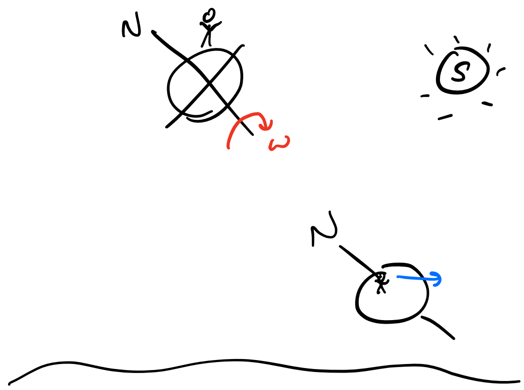
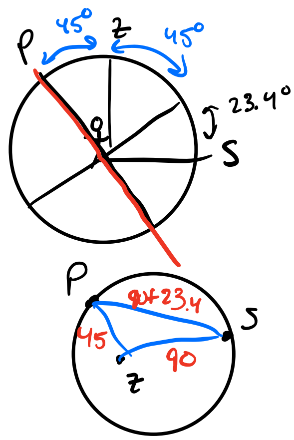
Sunrise

NOON





Last time

n-body problem → 3 conclusions
- → sys. moves in straight line
- → const. angular momentum
- → energy conserved

Let's justify 2-body problem:

For one body → $m_i \ddot{R} = \dfrac{G m_i m_j}{r_{ij}^2} \hat{u}_{ij}$

define reduced mass

$$\mu_r \equiv \frac{m_i m_j}{m_i + m_j}$$

eventually
$i \to 1$
$j \to 2$

Consider masses in solar sys.

Sun $\quad M_s = 1.99 \times 10^{30}\ kg = 333{,}000\ M_E$

Jupiter $\quad (\mu_r)_{Jupiter/Sun} = \dfrac{(333{,}000)(317.9)}{333{,}000 + 317.9} M_E = 317.7\ M_E$

Moon $\quad M_{Moon} = 7.348 \times 10^{22}\ kg = 0.0123\ M_E$

$$(\mu_r)_{Moon/Earth} = \frac{(0.0123)(1)}{1 + 0.0123} M_E = 0.01215$$

$\mu_{r_{sat/Earth}} = m_{satellite}$

These vast differences → two-body problem

Later: Find "sphere of influence"

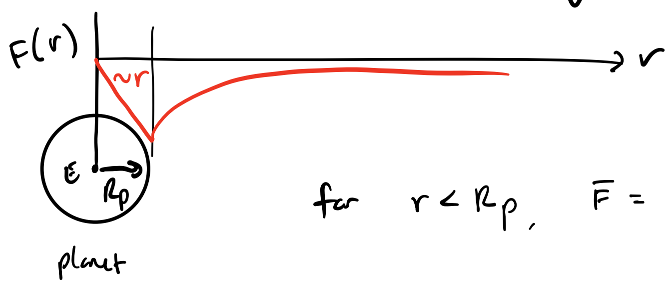
using particle analysis → Earth ≠ particle

(see prussing 3.13)

consider body: R, potential of gravity

$$V = -\frac{G M_1 M_2}{r}$$

since $\bar{F} = -\frac{\partial V}{\partial r}\hat{u}_r$



given:

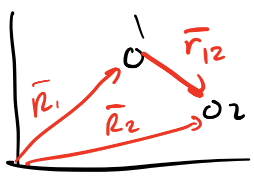
for $r < R_p$, $\bar{F} = \dfrac{-G M_1 \left(\frac{r}{R_p}\right)^3 M_2}{r^2}\hat{u}_r$

$r > R_p$, $\bar{F} = \dfrac{-G M_1 M_2}{r^2}\hat{u}_r$

Remarks:

- satellite's $R > R_p$
- Treat all as particles
- Assume spherical

Let's derive EOM: go from absolute $\bar{R}$ to relative $\bar{r}_{ij} \to \bar{r}_{12}$



$$\ddot{\bar{r}}_{12} = \frac{G M_1 M_2}{r_{12}^2}\hat{u}_{R},$$ use $\mu_r = \frac{M_1 M_2}{M_1 + M_2}$

- drop "12"

$$\mu_r \ddot{\bar{r}} = -\frac{G M_1 M_2}{r^2}\hat{u}_r \left(\frac{r}{r}\right)$$

define $\mu = G(M_1 + M_2)$

$\mu \neq \mu_r$

standard form $$\ddot{\bar{r}} + \frac{\mu}{r^3}\bar{r} = 0 \quad (1)$$

orbital motion → rotation, take cross product

$$\bar{r} \times \ddot{\bar{r}} + \cancel{\bar{r} \times \frac{\mu \bar{r}}{r^3}}^{\,0} = 0 \qquad \bar{r} \parallel \bar{r}$$

ang. mom, per unit mass

$$\therefore \frac{d}{dt}(\bar{r} \times \dot{\bar{r}}) = 0 \implies \bar{r} \times \dot{\bar{r}} = \text{const} = \bar{h}$$

$$(\bar{H} = \bar{r} \times m\bar{v})$$

Angular momentum = const:

→ implies planar orbital motion

- cross (1) w/ $\bar{h}$ (trying to get trajectories)

$$\underbrace{\ddot{\bar{r}} \times \bar{h}}_{\text{Can show } = \frac{d}{dt}(\dot{\bar{r}} \times \bar{h})} = -\frac{\mu}{r^3} \bar{r} \times \bar{h} = \mu \frac{d}{dt}\left(\frac{\bar{r}}{r}\right)$$

use $\bar{A} \times \bar{B} \times \bar{C} = (\bar{A} \cdot \bar{C})\bar{B} - (\bar{A} \cdot \bar{B})\bar{C}$

$$\frac{d}{dt}(\dot{\bar{r}} \times \bar{h}) = \mu \frac{d}{dt}\left(\frac{\bar{r}}{r}\right)$$

both have $\frac{d}{dt}$

take integral, add const.

$$\boxed{\dot{\bar{r}} \times \bar{h} = \mu\left(\frac{\bar{r}}{r} + \bar{e}\right)} \quad (2)$$

using vector identities → 30+ them to show

LHS → h^2 RHS → $\mu r + \mu \bar{r} \cdot \bar{e}$

$$h^2 = \mu r + \mu \bar{r} \cdot \bar{e}$$

$\bar{r} \cdot \bar{e} = |\bar{r}||\bar{e}| \cos\theta$

let $r = r(\theta)$

using dot prod:

$$\therefore \boxed{r(\theta) = \frac{h^2/\mu}{1 + e\cos\theta}} \quad (3)$$

radial orbit position as fn (θ)

θ = "True anomaly"

Recall: Also

$$\boxed{\begin{array}{l} r(\theta) = \dfrac{a(1-e^2)}{1 + e\cos\theta} \\ \therefore h^2/\mu = a(1-e^2) \end{array}}$$

from geometry

Values of e

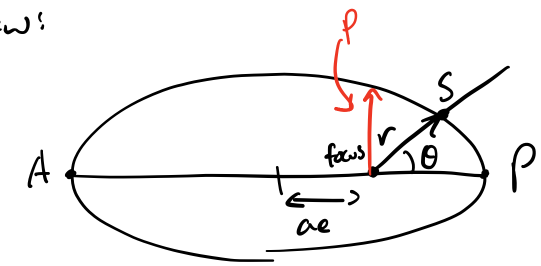
$e = 0$: circular, bound

$0 < e < 1$: ellipse, bound

$e = 1$: Parabolic, unbound, $r \to \infty$, $\theta \to \pi$

$e > 1$: hyperbolic, unbound, $r \to \infty$ @ $\theta < \pi$ ← asymptote

Review:



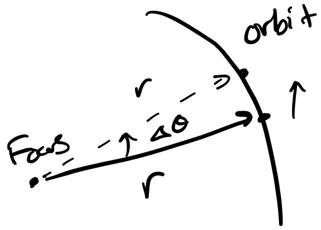
@ $\theta = 0$, $r = r_P = a(1-e)$

$\theta = \pi$, $r = r_A = a(1+e)$

$\theta = \frac{\pi}{2}$, $r = p = a(1-e^2)$

$p \equiv$ "semi latus rectum"

Need time!



↑ time pass, Δt, $\Delta\theta$

Area of triangle $\Delta A = \frac{1}{2}(r)(r\Delta\theta)$

$$\frac{\Delta A}{\Delta t} = \frac{1}{r} r r \frac{\Delta\theta}{\Delta t}$$

$$\frac{dA}{dt} = \frac{1}{2} r^2 \dot{\theta}$$

notice angular momentum

since $\bar{r} \times \dot{\bar{r}} = \bar{h}$

$$\bar{r} \times (\dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta) = \bar{h}$$

$$r^2\dot{\theta}(\hat{e}_r \times \hat{e}_\theta) = \bar{h}$$

$$\therefore \quad r^2\dot{\theta} = |\bar{h}| = h$$

$$\therefore \quad \frac{dA}{dt} = \frac{1}{2} r^2 \dot{\theta} = \frac{1}{2} h = \text{constant!}$$

Also $\frac{dA}{dt} = \frac{I}{A} \Rightarrow T = \frac{A}{dA/dt} = \frac{\pi ab}{h/2}$ ← area of ellipse

$$= \frac{\pi a\left(a\sqrt{1-e^2}\right)}{\frac{1}{2}\left(\sqrt{\mu a}\sqrt{1-e^2}\right)}$$

$$\therefore \boxed{T = 2\pi\left(\frac{a^3}{\mu}\right)^{1/2}}$$

bound circular & elliptical

Example: Find orbit period of ISS, altitude = 200 km

$$\mu = \mu_E = GM_E = G(M_E + \cancel{m_{ISS}}^{\,0}) = 3.986\times10^5 \text{ km}^3/\text{s}^2$$

$$a = R_E + \text{altitude} = 6378 \text{ km} + 200 \text{ km} = 6578 \text{ km}$$

$$T = 2\pi\left(\frac{6578^3}{3.986\times10^5}\right)^{1/2} = 5310 \text{ s} = 88.5 \text{ min}$$

orbit speed? → energy: $(T + V = \text{const})$

total energy = $\boxed{\varepsilon = -\frac{\mu}{2a}}$ ← only depends on a

Use to show $\boxed{v^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right)}$ "vis viva"

for orbit speed
→ depends on radius

Find speed of ISS in circular orbit:

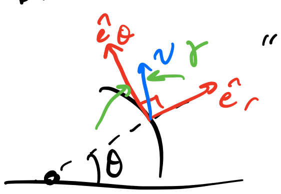
→ $e = 0$ → $r = a = \text{const} = 6578$ km

∴ vis viva: $v = \sqrt{\frac{\mu}{a}}$ for circular!

→ $= 7.78$ km/s

Direction at "r" in orbit given by "Flight path angle"

γ = ang. between $\bar{v}$ and circumferential dir.



$$\boxed{\therefore \cos\gamma = \frac{a\sqrt{1-e^2}}{\sqrt{r(2a-r)}}}$$

$$\boxed{\therefore \tan\gamma = \frac{e\sin\theta}{1+e\cos\theta}}$$