Last time: parabolic trajectory & escape velocity

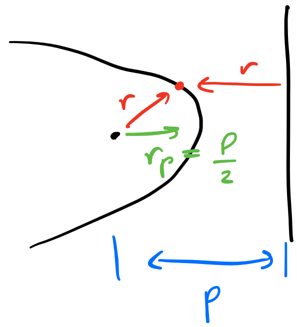
Recap: (1) $v = \sqrt{\frac{2\mu}{r}}$

(2) $2\sqrt{\frac{\mu}{p^3}}\,(t - t_p) = \tan\frac{\theta}{2} + \frac{1}{3}\tan^3\frac{\theta}{2}$

Define: Parabolic Mean Anomaly

let $M_p = \sqrt{\frac{\mu}{p^3}}\, t = \frac{\mu^2}{h^3} t$

side thing



Last time cont'd:

Example 3 { Sat. on parabolic escape
perigee speed 10 km/s

Find dist. from Earth center 6 hrs after perigee.

plan: find r_p → calc. t & LHS of θ Eq. → solve θ → solve r

$$v_{esc} = \sqrt{\frac{2\mu}{r_p}} \rightarrow r_p = \frac{2\mu}{v_{esc}^2}$$

$$r_p = \frac{2(3.986 \times 10^5)}{(10)^2} = 7972 \text{ km}$$

At perigee, $h = r_p v_p = 79720 \text{ km}^2/\text{s}$, use $h^2 = p\mu$

$$\therefore \text{LHS} = 2\sqrt{\frac{\mu}{p^3}}\,(t - \cancel{t_p}^{\,0}) = 2\sqrt{\frac{\mu^2}{h^3}}\, t$$

→ continue w/ posn eqn

Form of #2 → let $x = \tan(\frac{\theta}{2})$

$$\therefore 2M_p = x + \frac{1}{3}x^3$$

$$x^3 + 3x - 6M_p = 0 \quad (3)$$

factor, calculator, matlab, etc.

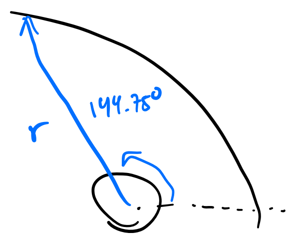
Closed form Solution:

$$x = \left(3M_p + \sqrt{(3M_p)^2+1}\right)^{1/3} - \left(3M_p + \sqrt{(3M_p)^2+1}\right)^{-1/3}$$

example:

$$M_p = \frac{\mu^2}{h^3} t = \frac{\left(3.986 \times 10^5 \ \text{km}^3/\text{s}^2\right)^2}{\left(79720 \ \text{km}^2/\text{s}\right)^3}\left(6 \text{ hr} \cdot 3600 \tfrac{\text{s}}{\text{hr}}\right)$$

$$= 6.7737 \text{ rad}$$



$$x = \tan\frac{\theta}{2} = \left(3(6.7737) + \sqrt{(3 \cdot 6.7737)^2+1}\right)^{1/3} - \left(\qquad '' \qquad \right)^{-1/3}$$

$$= 3.1481 = \tan\frac{\theta}{2}$$

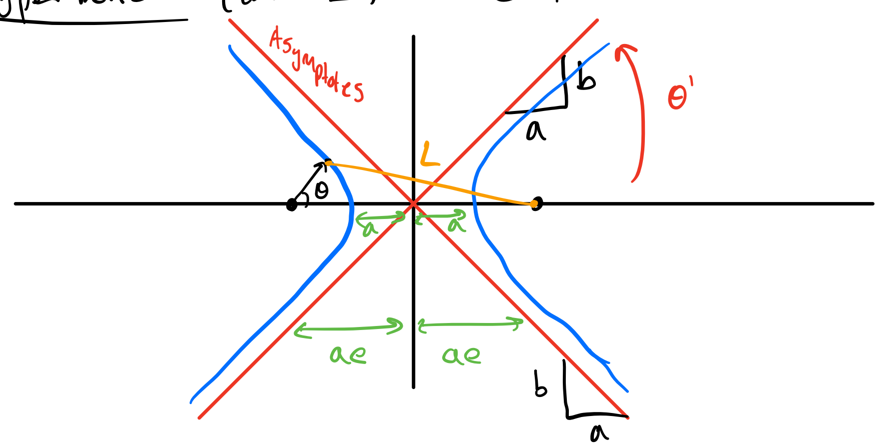
$$\theta = 2\tan^{-1}(3.1481) = 144.75^\circ$$

Now: use trajectory

$$r = \frac{p}{1+\cos\theta} = \underbrace{\frac{h^2}{\mu}\frac{1}{1+\cos\theta}}_{\text{have } h}$$

$$\therefore \ r = \frac{(79720)^2}{(3.986 \times 10^5)} \frac{1}{1+\cos(144.75^\circ)} = \boxed{86{,}476 \text{ km} = r}$$

Hyperbolic (case III) $\rightarrow e > 1$



use r, a, e, θ, L to show:

$$r = \frac{a(e^2-1)}{1+e\cos\theta}$$

trajectory for hyperbola

$$y = \pm \frac{b}{a} x$$ Asymptotes (cartesian)

$$\tan \theta' = \frac{b}{a}$$

Note as $r \to \infty$, $\cos\theta = -\frac{1}{e}$ polar asymtotes

Relate cartesian → polar of asymtotes

$$\frac{1}{e^2} = \left(1 + \frac{b^2}{a^2}\right)^{-1}$$

$$b^2 = a^2(e^2 - 1)$$ hyperbolic

to get speed: 1) $\bar{v} = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta$

$$v^2 = \dot{r}^2 + (r\dot{\theta})^2$$

2) $r^2\dot{\theta} = h$ w/ $r = \dfrac{a(e^2-1)}{1+e\cos\theta}$

Algebra: $\boxed{v = \sqrt{\mu\left(\frac{2}{r} + \frac{1}{a}\right)}}$ hyperbolic "vis viva"

as $r \to \infty$, $v \to \sqrt{\frac{\mu}{a}}$ Finite speed @ $r \to \infty$

Now, position

Similar process to ellipse

use: $r = \dfrac{a(e^2-1)}{1+e\cos\theta}$, $r^2 \dfrac{d\theta}{dt} = h$, $h^2 = \mu a(e^2-1)$

take derivatives, chain rule, sep. vars, etc.

Define: Hyperbolic eccentric anomaly $\equiv F = H$ in slides

in polar: in cartesian

$$\cosh H = \frac{r+a}{ae} \qquad \sinh H = \frac{y}{b}, \quad \cosh H = \frac{x}{a}$$

(recall $\sinh x = \frac{1}{2}(e^x - e^{-x})$, $\cosh x = \frac{1}{2}(e^x + e^{-x})$)

Define: hyperbolic Mean anomaly $\equiv M_H$

$$M_H = \sqrt{\frac{\mu}{a^3}}\, t = \frac{\mu^2}{h^3}(e^2-1)^{3/2}\, t \rightarrow (t - t_p),\ t_p = 0$$

shape   energy

Kepler's $\rightarrow$ Hyperbola

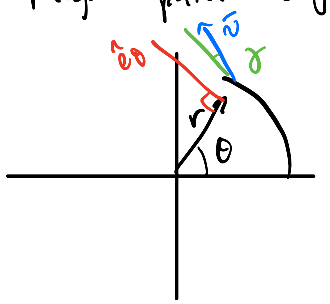
$$\boxed{M_H = e \sinh H - H}$$

H in radians

Relate back to θ

$$\boxed{\tan\frac{\theta}{2} = \sqrt{\frac{e+1}{e-1}}\tanh\left(\frac{H}{2}\right)}$$

Flight path angle



$$\cos\gamma = \frac{r\dot{\theta}}{|v|}$$

$$\text{Algebra} \rightarrow \cos\gamma = \sqrt{\frac{a^2(e^2-1)}{r(2a+r)}}$$

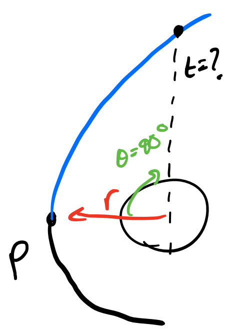
$$\bar{h} = \bar{r} \times \bar{v}$$

<u>Example</u>: Satellite in circ. orbit, $r = 10{,}000$ km

$\rightarrow$ Speed boost to $1.5\times$ initial speed

$\rightarrow$ puts into hyperbolic orbit (Δt is small)

$\rightarrow$ Find time req'd to get to $\theta = 90°$



<u>Path</u>:
1) Find v_{circ} & $1.5\, v_{circ}$
2) Find orbit params
3) use $\theta \rightarrow H$
4) use $H \rightarrow M_H \rightarrow t$

1) Circ: $v_c = \sqrt{\frac{\mu}{r}} = \sqrt{\frac{3.986 \times 10^5}{10{,}000 \text{ km}}} = 6.3135 \text{ km/s}$

Boost: $v_{boost} = 1.5\, v_c = 9.4702 \text{ km/s}$

2) Hyperbolic "vis viva": $v^2 = \frac{2\mu}{r} + \frac{\mu}{a} \rightarrow a = \mu \left(v^2 - \frac{2\mu}{r}\right)^{-1}$

$$a = 3.98 \times 10^5 \left((9.4702)^2 - \frac{2(3.986 \times 10^5)}{10{,}000} \right)^{-1}$$

At closest approach

$a = 40{,}000 \text{ Km}$

$\therefore$ From geometry, $e = \frac{r}{a} + 1$

$$e = \frac{10{,}000}{40{,}000} + 1 = 1.25 = e$$

3) Kepler's $\rightarrow$ get H: $\tanh\left(\frac{H}{2}\right) = \sqrt{\frac{e-1}{e+1}} \tan \frac{\theta}{2}$

$$= \sqrt{\frac{.25}{2.25}} \tan \frac{90°}{2} = 0.333 \text{ rad}$$

$\therefore\ H = 2 \tanh^{-1}(.333) = 0.69315 \text{ rad}$

4) Kepler $\sqrt{\frac{\mu}{a^3}} (t - \cancel{t_p}^{\,0}) = e \sinh H - H$

$$t = \sqrt{\frac{a^3}{\mu}} (e \sinh H - H)$$

$$t = \sqrt{\frac{40{,}000^3}{3.986 \times 10^5}} \left(1.25 \sinh(.69315) - .69315\right)$$

$$\rightarrow \boxed{t = 3097 \text{ s}}$$

$\rightarrow$ can show w/ eqns: circle is faster than hyp. for 90°

$t_c = 2488 \text{ s}$