Recap

$$\begin{cases} \rho u A = \dot{m} \quad \text{OR} \quad \dfrac{d\rho}{\rho} + \dfrac{du}{u} + \dfrac{dA}{A} = 0 \\ \rho u \dfrac{du}{dx} = -\dfrac{dP}{dx} \\ dh + u\,du = 0 \quad \text{OR} \quad h + \dfrac{u^2}{2} = h_0 \\ P = \rho R T \end{cases}$$

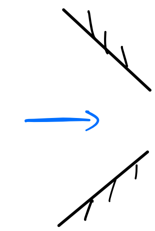
$h_0 \equiv$ stag. enthalpy

$$a = \underbrace{\sqrt{\left(\frac{\partial P}{\partial \rho}\right)_s}}_{\text{general}} = \underbrace{\sqrt{\gamma P/\rho} = \sqrt{\gamma R T}}_{\text{perfect gas}}$$

$$M = \frac{u}{a}$$

$$\frac{dP}{\rho u^2}\left(1 - M^2\right) = \frac{dA}{A} \Rightarrow \text{sign of } dP \text{ depends on } \begin{cases} \text{sign of } dA \\ \text{" of } (1-M^2) \end{cases}$$

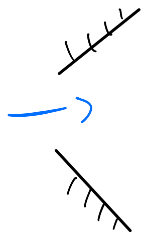
M < 1 case



$dA < 0 \Rightarrow \underbrace{dP < 0}_{\text{"nozzle"}} \Rightarrow du > 0$

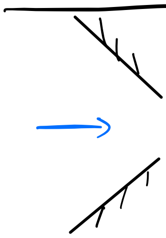
entirely analogous to incomp. (ρ = const) case

Not quantitatively though only sign

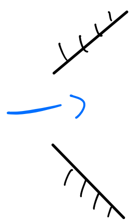


$dA > 0 \Rightarrow \underbrace{dP > 0}_{\text{"diffuser"}} \Rightarrow du < 0$

M > 1 case



$dA < 0 \Rightarrow \underbrace{dP > 0}_{\text{diffuser}} \Rightarrow du < 0$



$dA > 0 \Rightarrow \underbrace{dP < 0}_{\text{nozzle}} \Rightarrow du > 0$

Area – Mach # Relations

Start from Energy:

$h + \frac{u^2}{2} = h_0$ (adiabatic flow)

$h = c_p T$ ($c_p = \text{const}$; $h(T=0) = 0$)

Recall $\begin{cases} a^2 = \gamma R T \rightarrow T = \frac{a^2}{\gamma R} \\ c_p = \frac{\gamma}{\gamma - 1} R \\ M = \frac{u}{a} \end{cases}$

$c_p T + \frac{u^2}{2} = c_p T_0$ ÷ by c_pT

$\frac{T_0}{T} = 1 + \frac{u^2}{2 \frac{\cancel{\gamma}}{\gamma-1} \cancel{R} \frac{a^2}{\cancel{\gamma R}}} \rightarrow \boxed{\frac{T_0}{T} = \left(1 + \underbrace{\frac{\gamma-1}{2} M^2}_{>0}\right)} \geq 1 \Rightarrow \boxed{T < T_0}$

(I) stag. temp is highest

For isentropic (adiabatic + frictionless)

$$\frac{T_1}{T_2} = \left(\frac{P_1}{P_2}\right)^{\frac{\gamma}{\gamma-1}} = \left(\frac{\rho_1}{\rho_2}\right)^{\frac{1}{\gamma-1}}$$

then

$$\boxed{\begin{aligned} \frac{P_0}{P} &= \left(1 + \frac{\gamma-1}{2} M^2\right)^{\frac{\gamma}{\gamma-1}} \quad (\mathrm{II}) \\ \frac{\rho_0}{\rho} &= \left(1 + \frac{\gamma-1}{2} M^2\right)^{\frac{1}{\gamma-1}} \quad (\mathrm{III}) \end{aligned}}$$

Velocity

$$\begin{cases} u = Ma \\ a = \sqrt{\gamma R T} \\ T = T_0 \left(1 + \frac{\gamma-1}{2} M^2\right)^{-1} \end{cases}$$

$$\Rightarrow u = M \sqrt{\frac{\gamma R T_0}{1 + \frac{\gamma-1}{2} M^2}} \quad (\mathrm{IV})$$

From (III):

$\rho = \rho_0 \left(1 - \frac{\gamma-1}{2} M^2\right)^{\frac{1}{1-\gamma}}$ ← flip sign

and $\rho_0 = \frac{P_0}{R T_0}$ ← set by inj. pumps ← " " comb. process

Mass flux $\equiv \dfrac{\text{mass flow rate}}{\text{unit Area}}$

$$= \frac{\dot{m}}{A} = \rho u = \underbrace{\frac{p_0}{RT_0}\left(1+\frac{\gamma-1}{2}M^2\right)^{\frac{1}{1-\gamma}}}_{\rho} \cdot \underbrace{M\sqrt{\frac{\gamma R T_0}{1+\frac{\gamma-1}{2}M^2}}}_{u}$$

$$\Rightarrow \frac{p_0\sqrt{\gamma}}{\sqrt{RT_0}}\, M \left[\frac{1}{1+\frac{\gamma-1}{2}M^2}\right]^{\overbrace{\frac{1}{\gamma-1}-\frac{1}{2}}^{=\frac{\gamma+1}{2(\gamma-1)}}} = \frac{\dot{m}}{A}$$

For fixed $\dot{m}$, look for value of M that maximizes $\frac{\dot{m}}{A}$

→ Set $\dfrac{\partial\left(\frac{\dot{m}}{A}\right)}{\partial M} = 0$

→ It happens at $M=1$

Indicate values of A, T, p, ρ, u attained @ $M=1$ w/ * superscript

$A(M=1) = A^*$, $p(M=1) = p^*$ etc.

⇒ $\frac{\dot{m}}{A}$ is max @ $M=1$

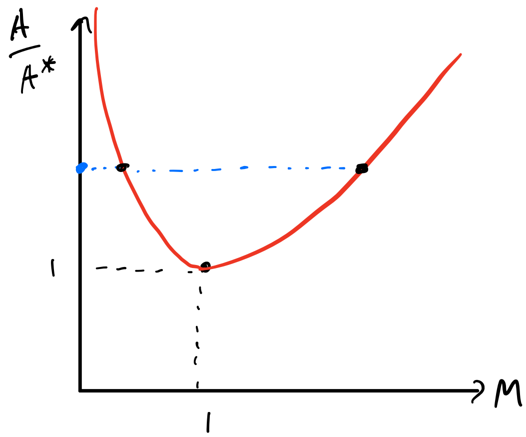
there we have $\dfrac{\dot{m}}{A^*} = \sqrt{\dfrac{\gamma}{RT_0}}\left(\dfrac{2}{\gamma+1}\right)^{\frac{\gamma+1}{2(\gamma-1)}}$ (V)

÷ max value by generic value of mass flux

$$\frac{\frac{\dot{m}}{A^*}}{\frac{\dot{m}}{A}} = \boxed{\frac{A}{A^*} = \frac{1}{M}\left[\frac{2}{\gamma-1}\left(1+\frac{\gamma-1}{2}M^2\right)\right]^{\frac{\gamma+1}{2(\gamma-1)}}}$$

Since $\dot{m}$ = const., $\frac{\dot{m}}{A}$ maximum where A is minimum

→ A^* coincides w/ minimum area "throat"



Remarks

0) Sonic conditions ($M=1$) can only be reached at a throat (area minimum)

i) curve "looks" like a nozzle but x-axis is M <u>not</u> x

ii) Curve has both subsonic & supersonic branches

→ same value of A/A^* may correspond to $M<1$ or $M>1$