ME 364: Elementary Heat Transfer

Professor Mike Wagner

1343 ERB

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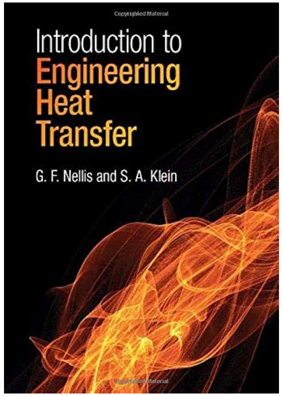
Course Information

- Read the syllabus!
- TA's:
  - Justin Paddock (Section 301) jpaddock@wisc.edu
  - Viraj Tamraparni (Sections 303, 304) tamraparni@wisc.edu
  - Peilai Li (Sections 305, 306) pli267@wisc.edu
- Class structure
  - Traditional lectures (M, W)
  - Discussion section (T, R) Solve problems in class
  - Weekly homework; individual, require more thought
  - 2 midterms in class, 1 final
- Textbook: "Introduction to Engineering Heat Transfer," Nellis & Klein, 2021



Course Information: Lectures

- Lecture skeleton notes provided on Canvas
- Recordings available within a day or two
- Instructor notes won’t be published (see recordings)
- Lecture attendance policy:
  - Students who attend lecture almost always do better than those that don’t
  - Don’t be a distraction to other students
- Please ask questions!

Course Information: Discussion

- Solve relatively straightforward assignments in class
- Complete the assignment before you leave
- Assignments are graded for completion and accuracy
- Attendance is required
- Problems often use EES
    - https://remote.engr.wisc.edu

Course Information: Homework

- Weekly homework assignments
  - Due by 11:59 PM
  - Due dates given on course schedule (Canvas)
- Typically 1-2 problems, multiple parts, involve theory and analysis
- Most problems use EES in some way
- Intended to be interesting and practical
- 20 points, graded for correctness, show your work

Course Information: Exams

- 2 in-class midterms; 60 minutes
- 1 final exam; 120 minutes
- Closed book, but note sheets allowed
  - 2 pages per exam, front and back (2 for 1st exam, then 4, then 6)
- Conceptual questions and worked problems
- Calculator probably needed

Course Information: **How to get help**

- Office hours: Wednesday 10-11, Thursday 10-11 (Zoom)
- Post questions to Piazza
- Contacting me
  - Email: mjwagner2@wisc.edu
  - Try to respond quickly, but not always possible. If no response within ~24 hours, it is okay to send me a reminder.
  - Stopping by my office is welcome, but it’s best to check beforehand
  - Any requests should keep in mind fairness to other students

Course Information: **How to do well**

- No single thing dominates the grading
- Get into a routine: Lecture → Discussion → Homework
- Office hours are useful even if you’re not struggling
- Use EES early and often
- Refresh yourself on solutions to differential equations
- Develop intuition: predict behavior *before* you solve the problem

Lecture 1

- Relation of Heat Transfer to Thermodynamics
- Modes of Heat Transfer

Relation to thermodynamics

THERMO:
- HEAT, WORK, IN/OUT, RELATE TO TEMP
- DOES NOT PREDICT TRANSIENT TEMP

XFER:
- HOW HEAT MOVES

$\dot{i}$: ENTHALPY

1^{ST} LAW OF THERMO:

- IN A CLOSED SYS., ENERGY BALANCES

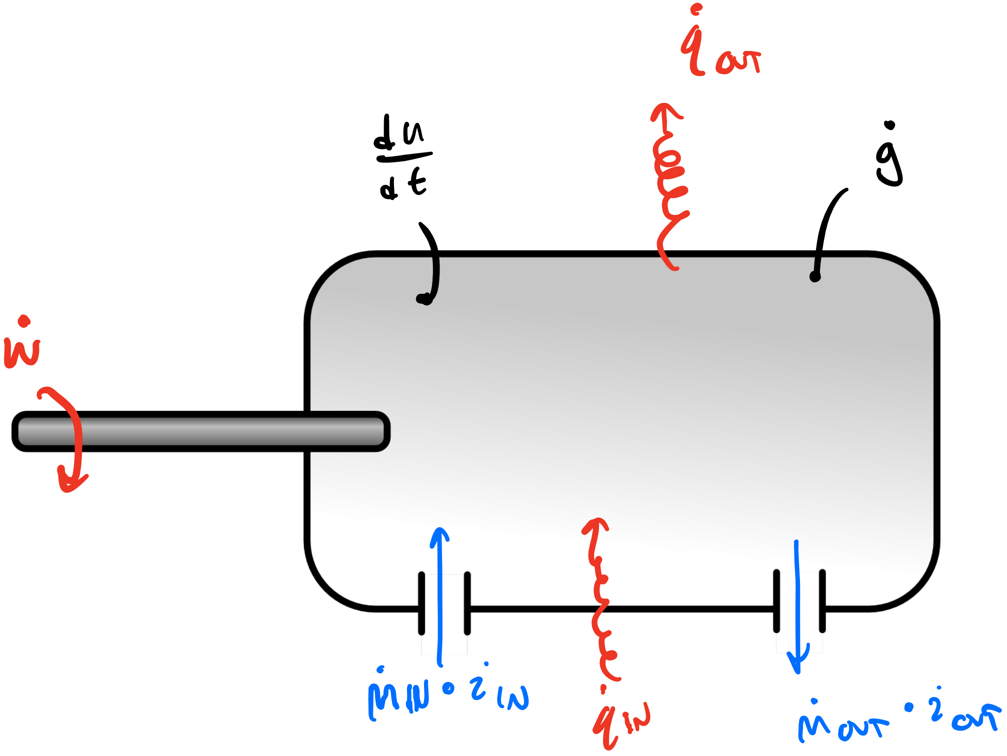
$$E_{IN} - E_{OUT} + E_{GEN} - E_{STORED} = \Delta U$$

(FOR SOME Δt DURATION)

$$\dot{E}_{IN} - \dot{E}_{OUT} + \dot{E}_{GEN} - \dot{E}_{ST} = \frac{dU}{dt}$$

(FOR SOME MOMENT)

$$\dot{E}_{IN} = \dot{m}_{IN}\, i_{IN} + \dot{q}_{IN}$$

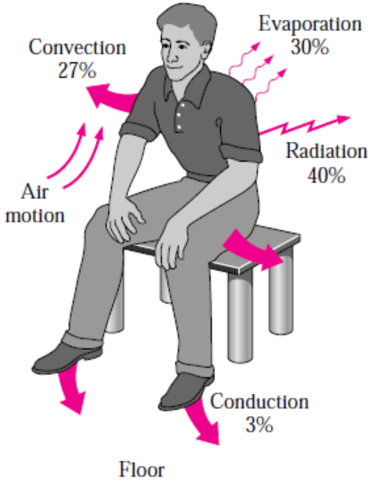


Relation to thermodynamics

2ND LAW OF THERMO: HEAT FLOWS SPONTANEOUSLY FROM HOT TO COLD RESERVOIRS
"CLAUSIUS"
↳ NO WORK REQUIRED

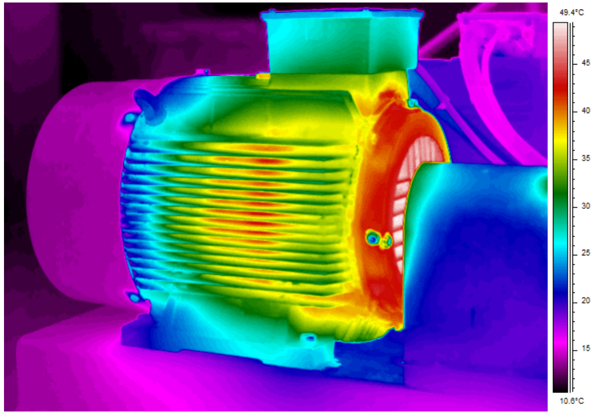
↳ IRREVERSIBILITY

https://mypdh.engineer/lessons/3-heat-transfer-from-the-human-body/



– IMPORTANCE OF THERMAL MANAGEMENT

Heat transfer is part of virtually every physical process



https://iactthermography.org/condition-monitoring/

https://www.energy.gov/energysaver/thermographic-inspections

3 modes of heat transfer

THIS CLASS

HEAT FLOW ⟷ TEMP

Conduction | Convection | Radiation

Definition: Energy transfer due to interaction between microscale energy carriers within a material

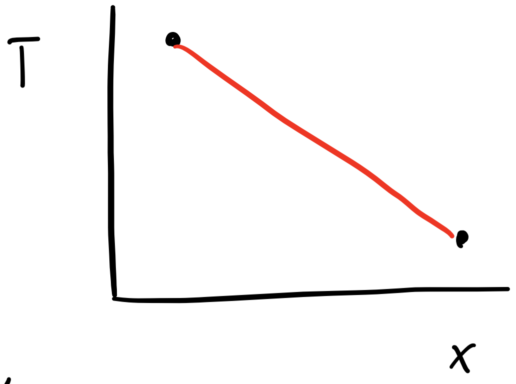
FOURIER'S LAW

$$\dot{q}'' = -k\,\nabla T \xrightarrow{1D} -k \cdot \frac{dT}{dx}$$

DIFFERENTIAL

$\dot{q}$ PER AREA W/m^2

CONDUCTIVITY $W/m \cdot K$



TERMINOLOGY

q	J	HEAT TRANSFER	ENERGY
$\dot{q}$	$J/s \rightarrow W$	HEAT TRANSFER RATE	POWER
$\dot{q}''$	W/m^2	HEAT FLUX	POWER/AREA

3 modes of heat transfer

CONDUCTION + ADVECTION

Conduction | <u>Convection</u> | Radiation

Definition: Heat transfer from a surface to a moving fluid

RATE EQUATION: <u>NEWTON'S LAW OF COOLING</u>

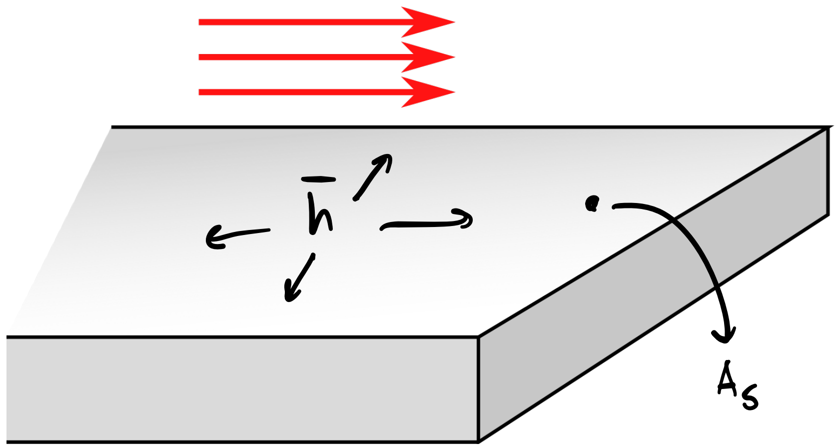
$$\dot{q}_{conv} = \bar{h} \cdot A_s \cdot (T_s - T_\infty)$$

m^2

SURFACE TEMP [K]

FREE STREAM TEMP [K]

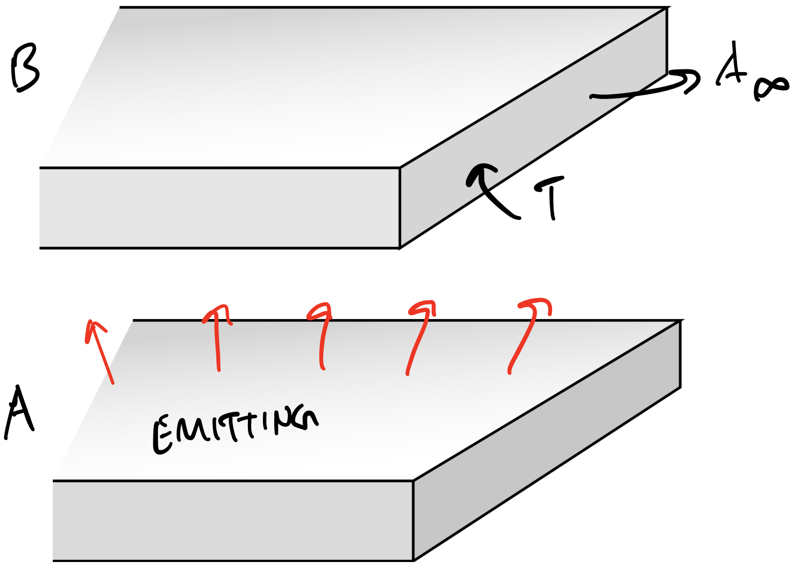
HEAT XFER COEFF (AVG) $\left[\frac{W}{m^2 K} \right]$



3 modes of heat transfer

Conduction Convection <u>Radiation</u>

Definition: Heat transfer via electromagnetic waves (photons) from surface A to B at a distance



RATE EQUATION : STEFAN - BOLTZMANN

$$\dot{q}_{RAD} = \sigma \cdot \varepsilon \cdot A_s \cdot \left(T_s^4 - T_\infty^4\right)$$

EMITTING SURFACE AREA [m^2]

AMBIENT TEMP

S.B. CONST.

5.67×10^{-8} [W/m^2K^4]

SURFACE EMISSIVITY

SURF. TEMP.

PROBLEM SOLVING METHODOLOGY

1) SKETCH PROBLEM & LIST ALL KNOWNS ON SKETCH
(CONVERT UNITS IF NECESSARY)

2) LIST ASSUMPTIONS → ESTABLISH LIMITATIONS
MAY NEED OWN ASSUMPTIONS → JUSTIFY W/ CALCULATIONS

3) ANALYSIS – BALANCES, RATE EQ'S, PROPERTIES, ETC.
↳ ADDITIONAL SKETCHES
→ BUILD SYSTEM OF EQUATIONS

4) SOLUTION – IDENTIFY SOLUTION STRATEGY
↳ SEQUENTIAL/EXPLICIT? ITERATION?

5) DISCUSSION/EXPLORATION – EXAMINE SOLUTION, PRESENT CONCLUSIONS
→ SANITY CHECKS, CHANGE INPUT, CHECK OUTPUT

Lecture 2

1-D Steady State Conduction

Last time…

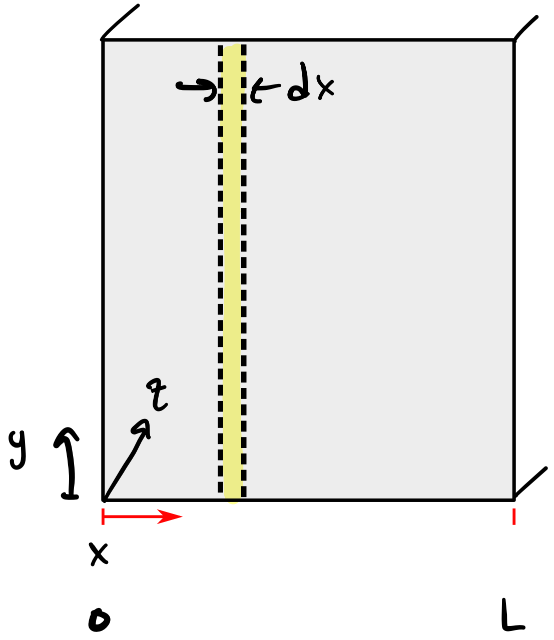
3 modes of heat transfer and rate equations

- Conduction $\dot{q}'' = -k \cdot \nabla \mathrm{T} \rightarrow -k \cdot \frac{dT}{dx}$
- Convection $\dot{q}_{conv} = \overline{h} \cdot A_s \cdot (T_s - T_\infty)$
- Radiation $\dot{q}_{rad} = \sigma \cdot \varepsilon \cdot A_s \cdot (T_S^4 - T_\infty^4)$

Definitions

Domain: The set of input values for which a function is defined

Control volume: The subject of analysis in which laws of conservation apply



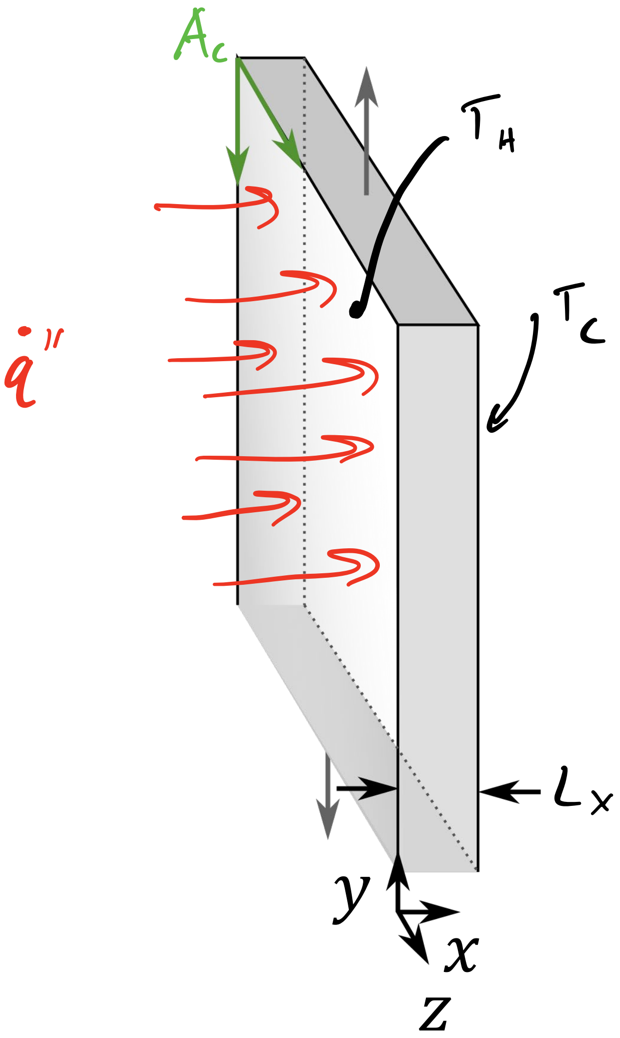
$$\dot{q}'' = -k \nabla T$$

CARTESIAN: $\dot{q}'' = -k\left(\frac{\partial T}{\partial x} + \frac{\partial T}{\partial y} + \frac{\partial T}{\partial z}\right)$ ← PDE

↳ 1D: $\dot{q}'' = -k \frac{dT}{dx}$ ← ODE

Conduction analysis in 1D Plane Wall

- FINITE THICKNESS
- ∞ LENGTH → NEGLECT END FX



$$\dot{q} = \dot{q}'' \cdot A_c$$

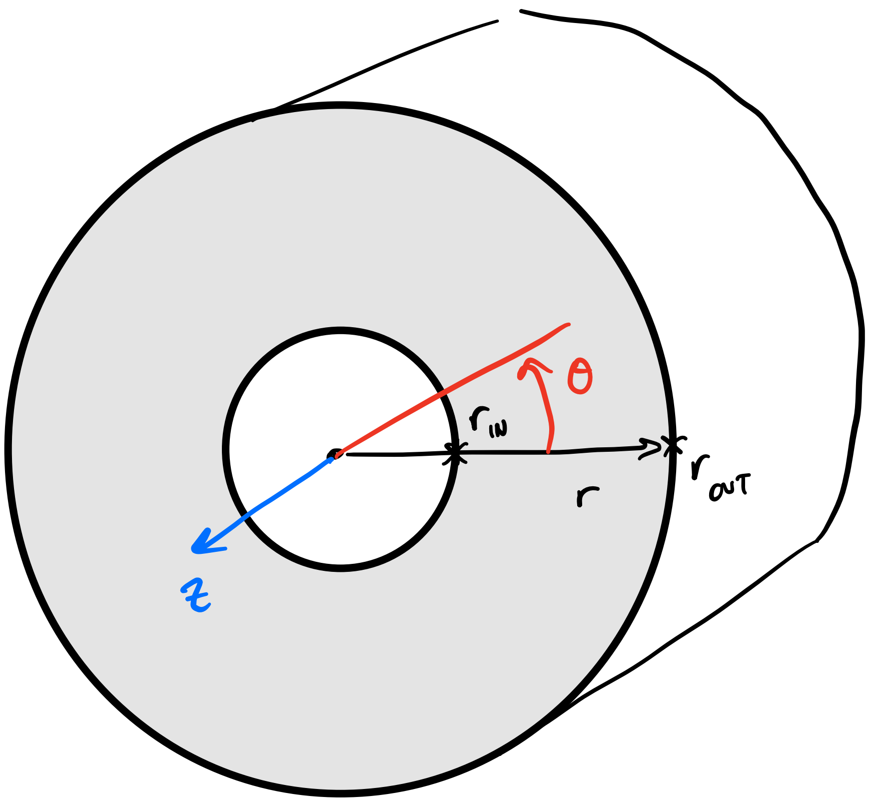
$$\dot{q} = \underline{-k \cdot A_c} \cdot \frac{dT}{dx}$$

CONST, IN THIS CASE
NOT ALWAYS

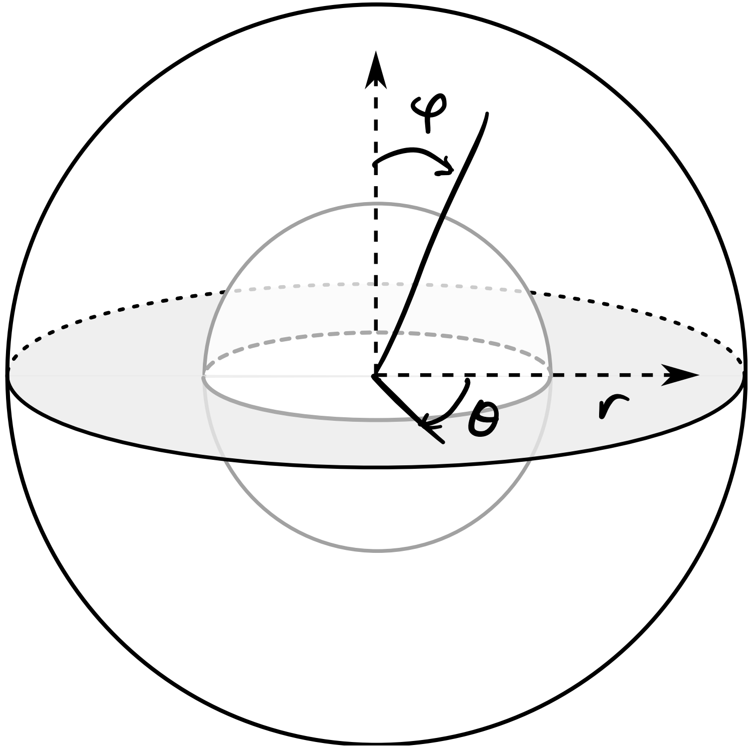
Define **adiabatic**:
Boundary across which heat does not flow

Analysis in other coordinate systems

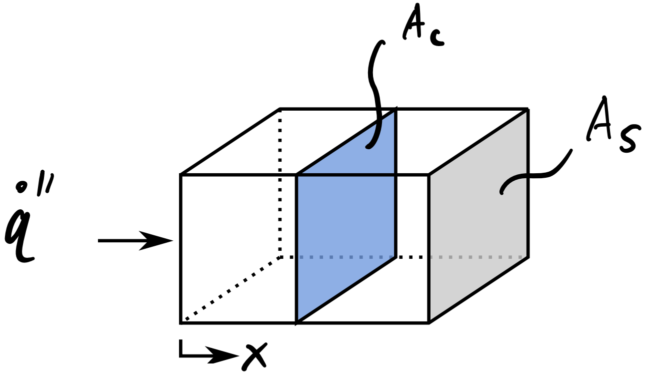
Cylindrical



Spherical



Analysis in other coordinate systems

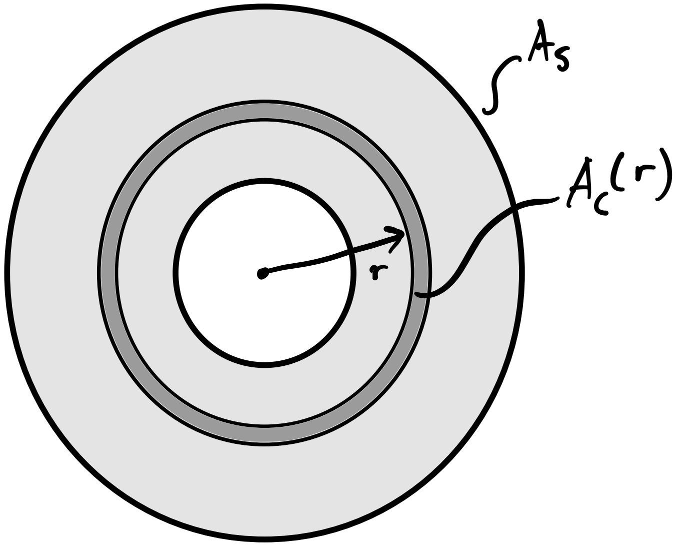


CARTESIAN: $A_c = A_S$ (GENERALLY)

CYLINDRICAL: $A_c \neq A_S$

$$A_{cyl} = 2\pi r L$$

$$A_{sph} = 4\pi r^2$$



Conduction analysis methodology (1)

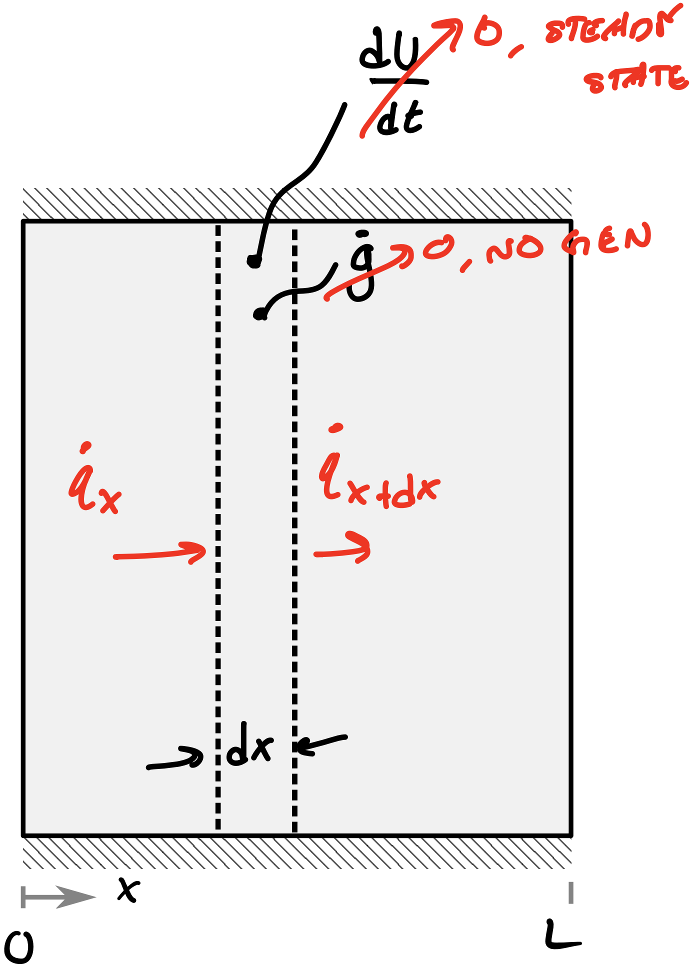
$\dot{g}$: "GENERATION"

~ 7 steps

1) DEFINE DOMAIN
2) DRAW CONTROL VOLUME
3) PERFORM CV E-BAL

IN + GEN = OUT + STORED

$$\dot{q}_x = \dot{q}_{x+dx}$$



Conduction analysis methodology (2)

4) DETERMINE HEAT FLOW AT $x+dx$ AS $dx \to 0$

$$f(x) = \sum_{n=0}^{\infty} \frac{d^n f(x)}{dx^n} \cdot \frac{1}{n!} dx^n = f(x) + \frac{df(x)}{dx} dx + \underbrace{\frac{d^2 f(x)}{dx^2} \frac{1}{2!} dx^2 + \ldots}_{\text{GOES AWAY}}$$

THEN: $$\dot{q}_{x+dx} = \dot{q}_x + \frac{d\dot{q}_x}{dx} dx$$

RECALL: $$\dot{q}_x = \dot{q}_{x+dx}$$

$$\rightarrow \dot{q}_x = \dot{q}_x + \frac{d\dot{q}_x}{dx} dx$$

$$\therefore \frac{d\dot{q}_x}{dx} = 0$$

Conduction analysis methodology (3)

5) SUB IN A RATE EQUATION

$$\dot{q} = -kA_c \frac{dT}{dx}$$

$$\frac{d}{dx}\left(-kA_c \frac{dT}{dx}\right) = 0 \qquad \boxed{\therefore \frac{d^2T}{dx^2} = 0}$$

GOV. DIFF. EQ.

Conduction analysis methodology (4)

6) FIND SOLUTION FOR TEMP.

$$\frac{d^2T}{dx^2} = 0 \rightarrow \int d\,\frac{dT}{dx} = \int 0\,dx \rightarrow \frac{dT}{dx} = C$$

$$\int dT = \int C\,dx \rightarrow \boxed{T = C_1 x + C_2}$$

Conduction analysis methodology (5)

7) APPLY BC'S TO SOLVE FOR C_1, C_2

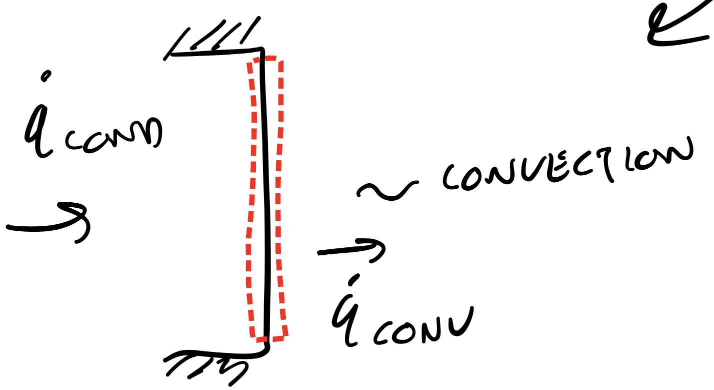
BC TYPE 1: SPECIFIED TEMP (DIRICHLET) $T_{x=0} = T_H$

BC TYPE 2: SPECIFIED FLUX (NEUMANN) $\dot{q}_{x=L} = \text{const}$

BC TYPE 3: MATCHED RATE (ROBIN) $\bar{h} A_s (T_{x=L} - T_\infty) = -k A_c \frac{dT}{dx}\Big|_{x=L}$

$\downarrow$

$T(x)$ FUNCTION



$\bar{h} A_s (C_1 \underset{L}{x} + C_2 - T_\infty) = -k A_c C_1$

Lecture 3

Thermal Resistance Analogy

HW 2G → SYMBOLIC EXPRESSION - CHECK PIAZZA

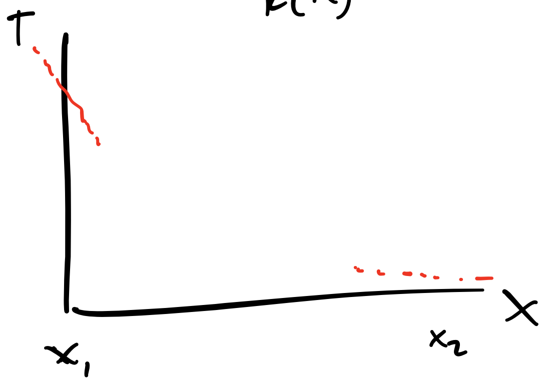
11:59PM

HW 1 DUE FRIDAY - SINGLE PDF

CONST CONST

$$\dot{q} = -k A_c \frac{dT}{dx}$$

$k(x) \rightarrow C_1 x + C_2$

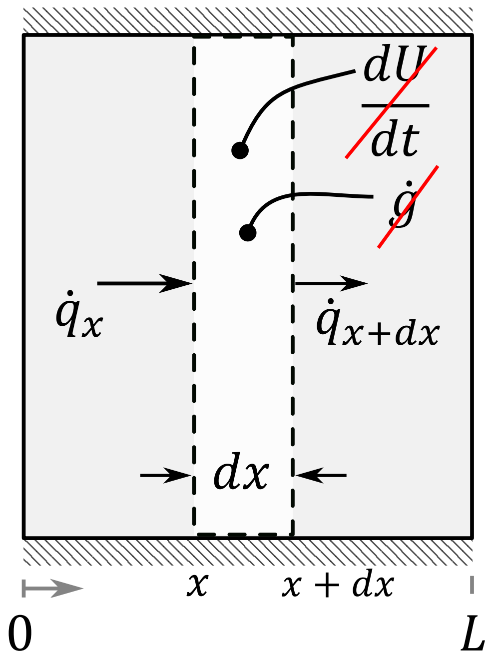


Last time…

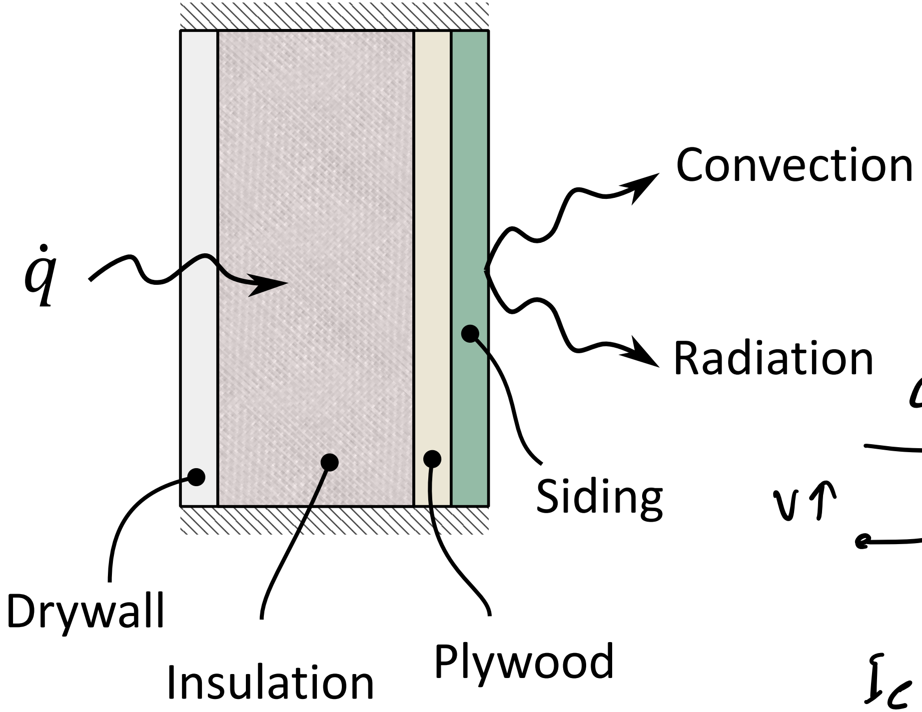
Control volume analysis

Derive temperature equation from heat flow

1) Define domain
2) Draw control volume (CV)
3) Perform CV energy balance
4) Determine heat flow at $x + dx$ as $dx \to 0$
5) Substitute rate equation into the ODE
6) Find solution for $T(x)$
7) Apply boundary conditions and solve for unknown constants
   A. Dirichlet (specify temperature)
   B. Neumann (specify flux)
   C. Robin (matched rate)



Motivation for using thermal resistances



Use the thermal resistance model to simplify analysis

HELPS SIMPLIFY ANALYSIS

→ WOULD NEED TO SOLVE 4 DIFFERENT SETS OF EQNS, BC'S, ETC

CIRCUITS	HEAT XFER
$V\uparrow \xrightarrow{I_e} V\downarrow$, resistor R_e	$T_H \xrightarrow{\dot{q}} T_c$, resistor R
$I_c = \dfrac{V\uparrow - V\downarrow}{R_e}$	$\dot{q} = \dfrac{T_H - T_c}{R}$

Deriving an expression for resistance R

Recall Fourier's law: $\dot{q} = -k \cdot A_c \cdot \frac{dT}{dx}$

$$\frac{d\dot{q}}{dx} = 0 \rightarrow \frac{d}{dx}\left(-\cancel{k}\cancel{A_c}\frac{dT}{dx}\right) = 0$$

CONST

$$\hookrightarrow T = C_1 x + C_2$$

BC1) $T_{x=0} = T_H \rightarrow T_H = C_1(0) + C_2$

$\rightarrow T_H = C_2$

BC2) $T_{x=th} = T_C \rightarrow T_C = C_1\, th + C_2$

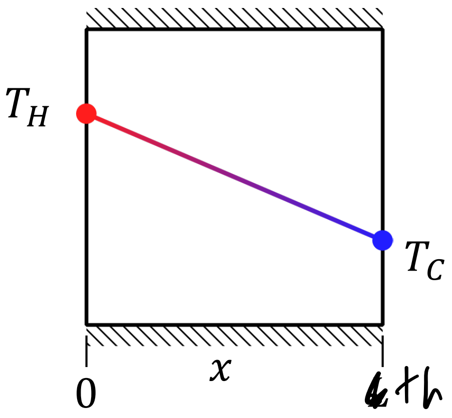
$$C_1 = \left(\frac{T_C - T_H}{th}\right)$$

$$T = \left(\frac{T_C - T_H}{th}\right) \cdot x + T_H$$

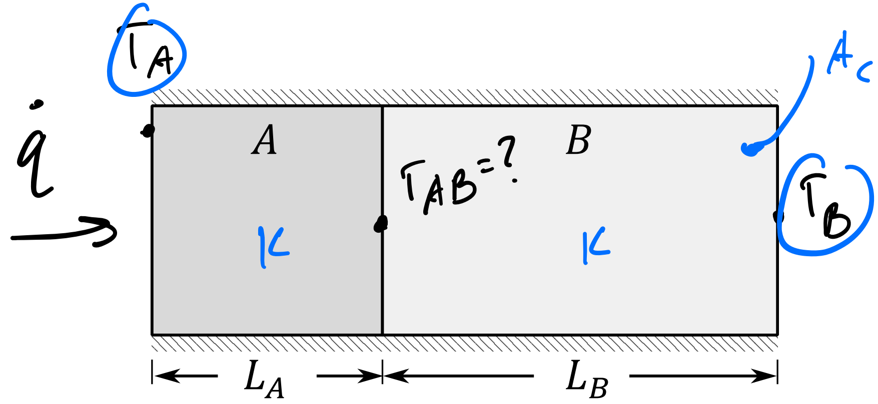
$$\dot{q} = -kA_c \frac{d}{dx}\left(\frac{T_C - T_H}{th} \cdot x + T_H\right)$$

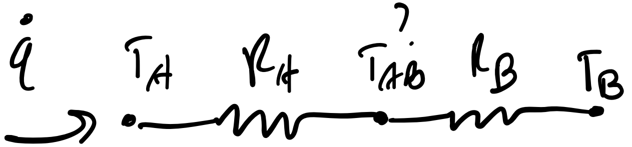
$$= -\frac{kA_c}{th}\left(T_C - T_H\right) \Rightarrow \boxed{R = \frac{th}{kA_c}}$$

(PLANE WALL)



Example: Resistance model of a wall with 2 materials





$$\dot q = \frac{T_A - T_B}{R_{TOT}}$$

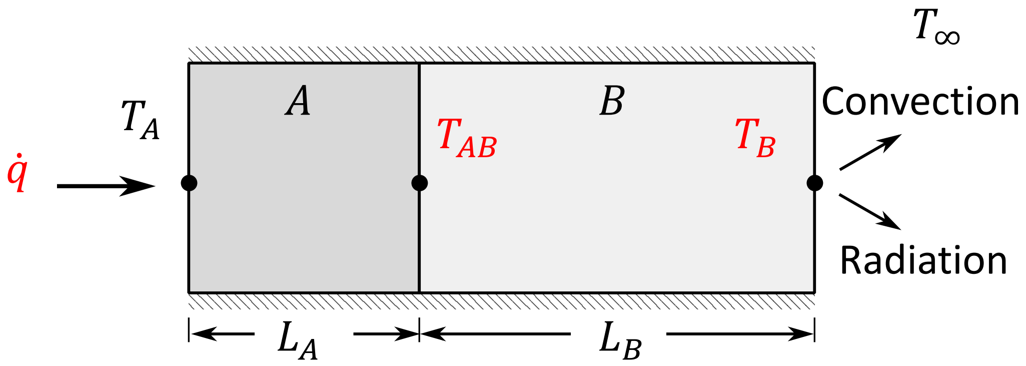
$$R_{TOT} = \underbrace{R_A}_{\frac{L_A}{kA_c}} + \underbrace{R_B}_{\frac{L_B}{kA_c}}$$

$$\dot q = \frac{T_A - T_{AB}}{R_A}$$

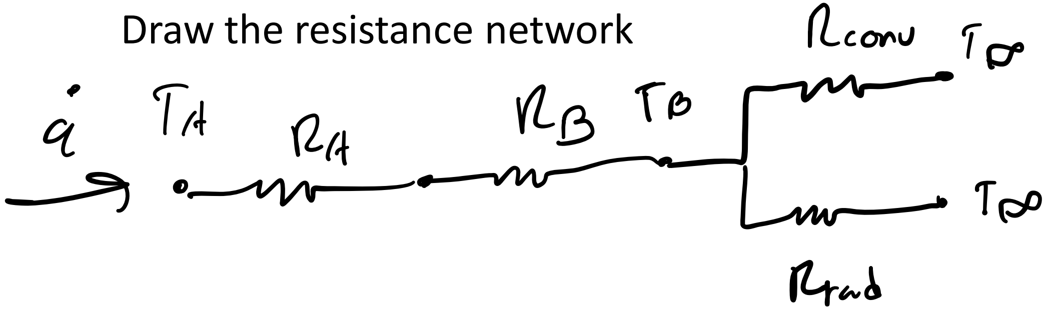
SOLVE FOR THIS

Example: Resistance model of a wall with 2 materials **(modified)**

Heat flow in series and parallel:



Draw the resistance network

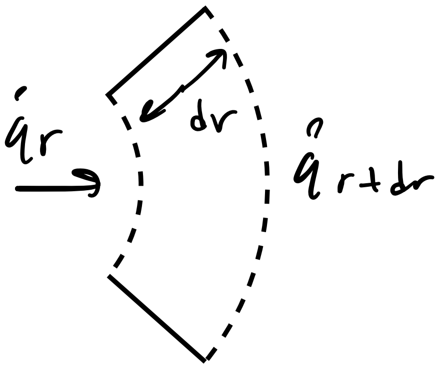


R_{TOTAL} ($T_A \to T_\infty$)

$$= R_A + R_B + \left[\frac{1}{R_{conv}} + \frac{1}{R_{rad}} \right]^{-1}$$

Conduction resistances in other coordinate systems (1)

Cylindrical



$$R_{cyl} = \frac{\ln(R_{in}/R_{out})}{k \cdot 2\pi L}$$

$$\dot{q}_r - \dot{q}_{r+dr} = 0$$

$$\dot{q}_r - \left(\dot{q}_r + \frac{d\dot{q}_r}{dr}dr\right) = 0$$

$$\rightarrow \frac{d\dot{q}_r}{dr} = 0$$

$$\dot{q} = -kA_c(r)\frac{dT}{dr} \rightarrow -k2\pi r L\frac{dT}{dr}$$

$$\frac{d\dot{q}_r}{dr} = \frac{d}{dr}\left[-k2\pi r L\frac{dT}{dr}\right] = 0 \rightarrow \int \frac{d}{\cancel{dr}}\left[r\frac{dT}{dr}\right]\cancel{dr} = \int 0\,dr$$

$$r\frac{dT}{dr} = C_1 \rightarrow \int dT = \int \frac{C_1}{r}dr$$

$$\rightarrow T = C_1\ln(r) + C_2$$

Conduction resistances in other coordinate systems (2)

Cylindrical

BC1) $T_{r=r_{in}} = T_H$

BC2) $T_{r=r_{out}} = T_C$

$$T(r) = \frac{(T_C - T_H)\ln\left(\frac{r}{r_{in}}\right)}{\ln\left(\frac{r_{out}}{r_{in}}\right)} + T_H$$

$$\dot{q} = \underbrace{\frac{k \cdot 2\pi L}{\ln\left(\frac{r_{in}}{r_{out}}\right)}}_{\frac{1}{R_{cyl}}} (T_H - T_C)$$

$$\rightarrow R_{cyl} = \frac{\ln\left(\frac{r_{in}}{r_{out}}\right)}{k\,2\pi L}$$

$$\boxed{R_{sph} = \frac{\frac{1}{r_{in}} - \frac{1}{r_{out}}}{4\pi k}}$$

Convection and radiation resistances (1)

$$\dot{q} = \bar{h} A_s (T_H - T_C) \qquad R_{conv} = \frac{1}{\bar{h} A_s}$$

RADIATION

$$\dot{q}_{rad} = \sigma \varepsilon A_s \underbrace{(T_H^4 - T_C^4)} \rightarrow (T_H^2 + T_C^2)\underbrace{(T_H^2 - T_C^2)}$$

$$\dot{q}_{rad} = \underbrace{\sigma \varepsilon A_s \cdot (T_H^2 + T_C^2)(T_H + T_C)}_{1/R_{rad}}(T_H - T_C)$$

Convection and radiation resistances (2)

UNKNOWN

$$R_{rad} = \frac{1}{\sigma \cdot \varepsilon \cdot A_s \cdot (\mathrm{T_H^2} + \mathrm{T_C^2})(\mathrm{T_H} + \mathrm{T_C})}$$

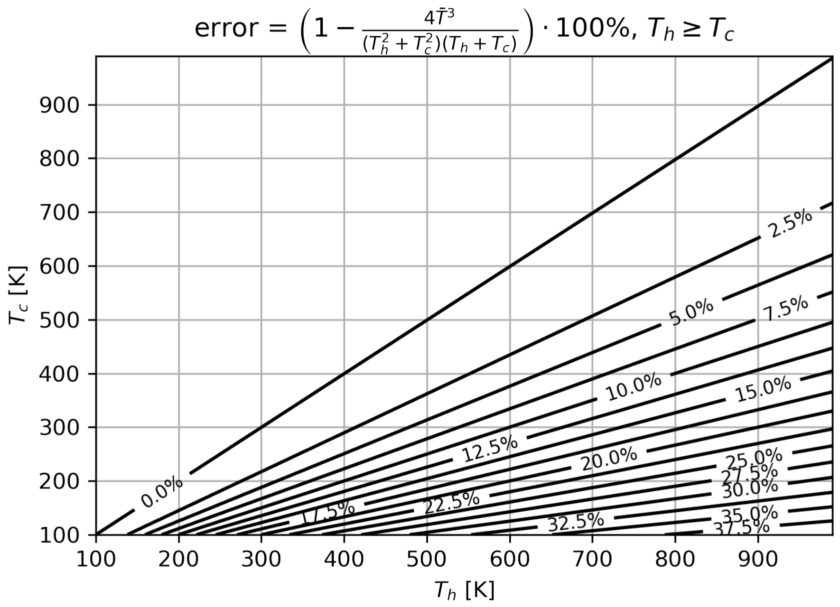
SOLVING: 1) GUESS VALUE FOR UNKNOWN TEMPS (T_H)

2) COMPUTE RESISTANCE

3) SOLVE FOR $\dot{q}$

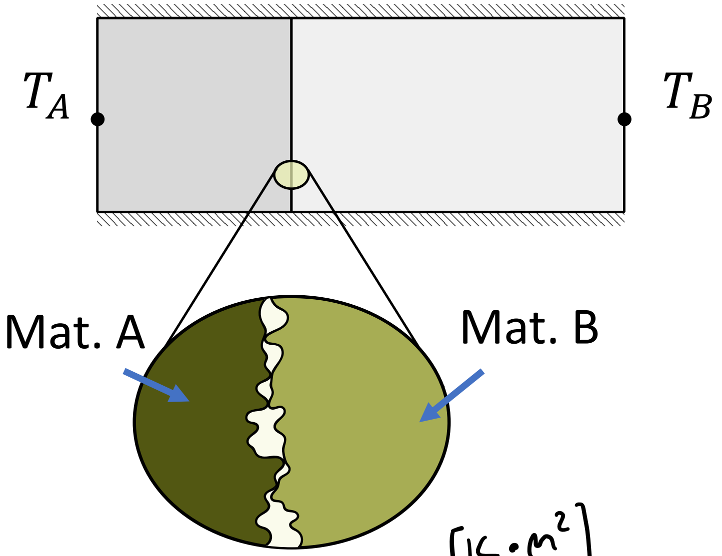
4) GO BACK & UPDATE T_H

$$R_{rad} \approx \frac{1}{\sigma \varepsilon A_s \cdot 4 \bar{T}^3}, \quad \bar{T} = \frac{T_H + T_C}{2}$$

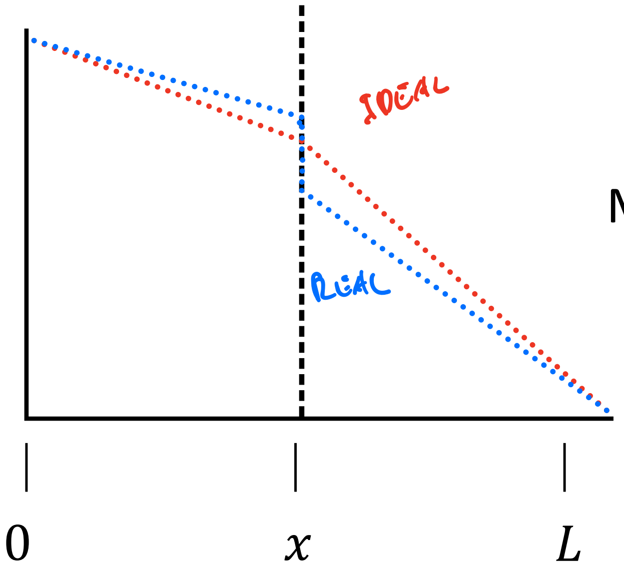


Contact and fouling resistances

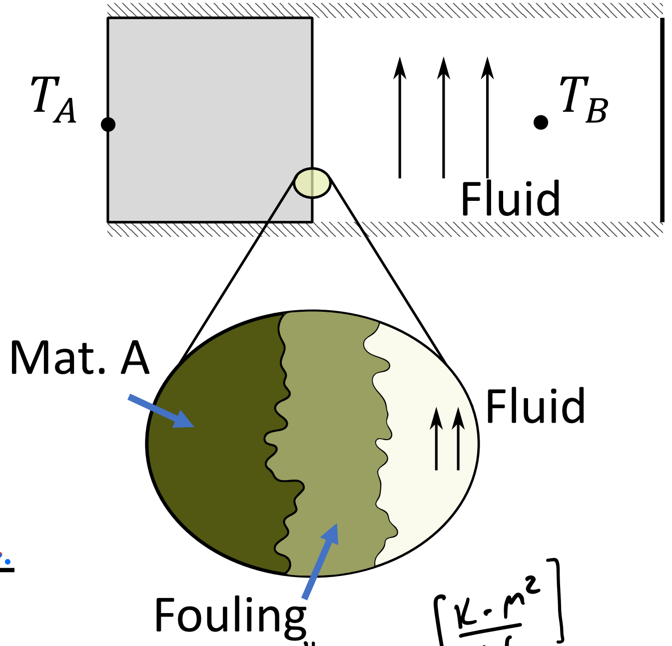
Contact resistance:
Imperfect contact (conduction) between materials



$$R_c = \frac{R''_c}{A_s} \rightarrow \left[\frac{K \cdot m^2}{W}\right]$$



Fouling resistance:
Additional conduction due to film buildup between surface and fluid (convection)



$$R_F = \frac{R''_F}{A_s} \rightarrow \left[\frac{K \cdot m^2}{W}\right]$$

Lecture 4

Thermal Generation

Last time…

Series vs. parallel:

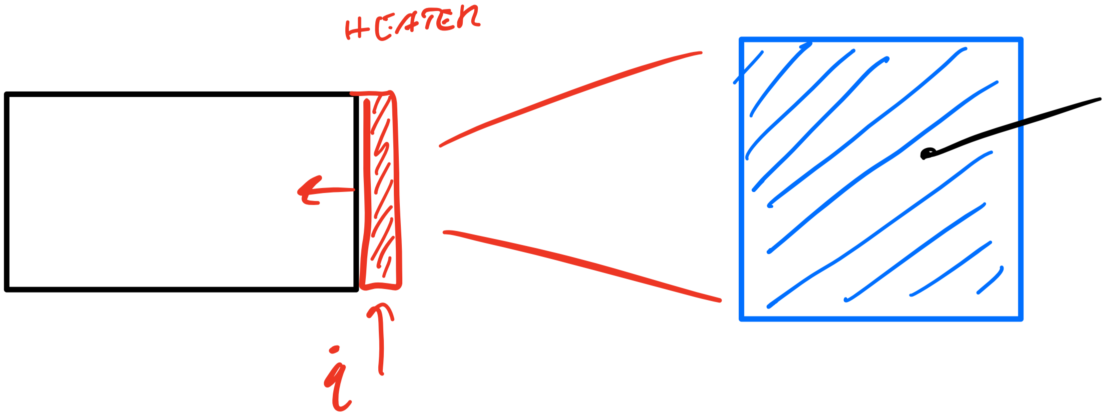
$$R_{series} = R_1 + R_2 + \cdots R_N$$

$$R_{parallel} = \left[\frac{1}{R_1} + \frac{1}{R_2} + \cdots + \frac{1}{R_N}\right]^{-1}$$

The thermal resistance model $\dot{q} = \dfrac{T_H - T_C}{R}$

Conduction	$R_{cart} = \dfrac{th}{k \cdot A_c}$	$R_{cyl} = \dfrac{\ln\left(\frac{r_{in}}{r_{out}}\right)}{k \cdot 2\pi L}$	$R_{sph} = \dfrac{\frac{1}{r_{in}} - \frac{1}{r_{out}}}{4\pi k}$
Convection	$R_{conv} = \dfrac{1}{\overline{h} \cdot A_s}$		
Radiation	$R_{rad} = \dfrac{1}{\sigma \cdot \varepsilon \cdot A_s \cdot \left(\mathrm{T_H^2} + \mathrm{T_C^2}\right)(\mathrm{T_H} + \mathrm{T_C})}$		$R_{rad} \approx \dfrac{1}{\sigma \cdot \varepsilon \cdot A_s \cdot 4\overline{T}^3}$
Contact	$R_c = \dfrac{R_c''}{A_s}$		
Fouling	$R_f = \dfrac{R_f''}{A_s}$		

Definition: Thermal generation is the volumetric addition of heat to a CV



$\dot{g}'''\left[\frac{W}{m^3}\right] > 0$

e.g. $\dot{g}'''(x, y, z)$

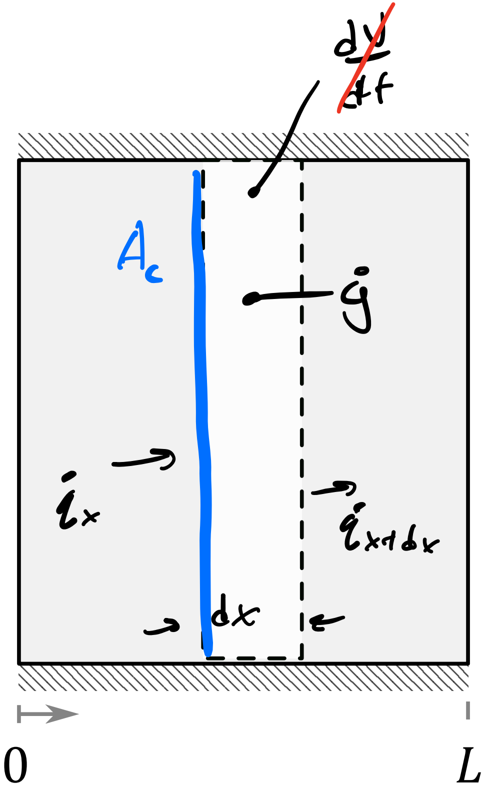
EXAMPLES

- CHEMICALLY REACTING MEDIA
- NUCLEAR REACTION
- ABSORPTION OF RADIATION
- VISCOUS SHEAR
- ELECTRICAL DISSIPATION - $\dot{g} = I^2 R_e$

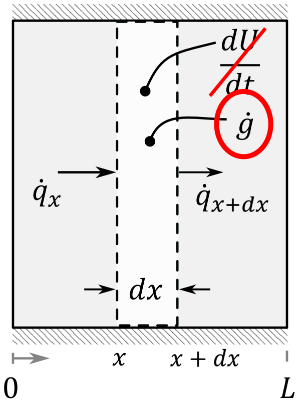
Analysis methodology – with generation (1)

Steps are the same as before!

1) Define domain
2) Draw control volume (CV)
3) Perform CV energy balance
4) Determine heat flow at $x + dx$ as $dx \to 0$
5) Substitute rate equation into the ODE
6) Find solution for $T(x)$
7) Apply boundary conditions and solve for unknown constants



Analysis methodology – with generation (2)



3) Perform CV energy balance

IN + GEN = OUT + ~~STORED~~

$$\dot{q}_x + \dot{g}''' A_c \cdot dx = \dot{q}_{x+dx}$$

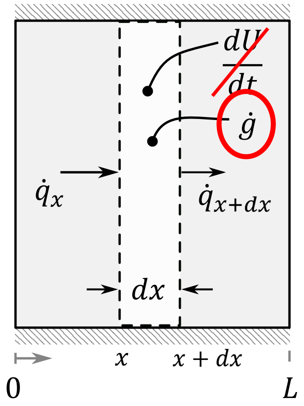
4) Find heat flow as $dx \to 0$

$$\cancel{\dot{q}_x} + \dot{g}''' \cdot A_c \cdot d\cancel{x} = \cancel{\dot{q}_x} + \frac{d\dot{q}_x}{dx} \cancel{dx} \rightarrow \frac{d\dot{q}}{dx} = \dot{g}''' \cdot A_c$$

5) Substitute rate equation

$$\underbrace{\dot{g}'''}_{-k} \cdot \cancel{A_c} = \frac{d}{dx}\left[-\cancel{k} \cdot \cancel{A_c} \cdot \frac{dT}{dx}\right] \rightarrow \boxed{\frac{d^2T}{dx^2} = -\frac{\dot{g}'''}{k}}$$

Analysis methodology – with generation (3)



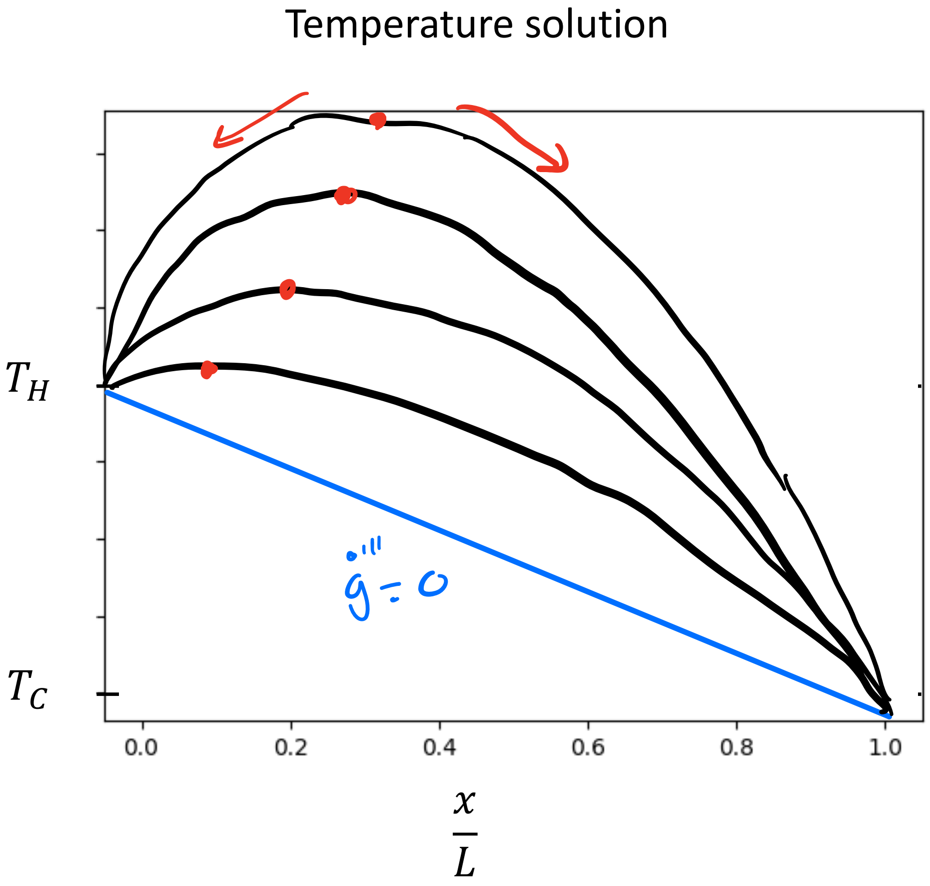
6) Find solution for T

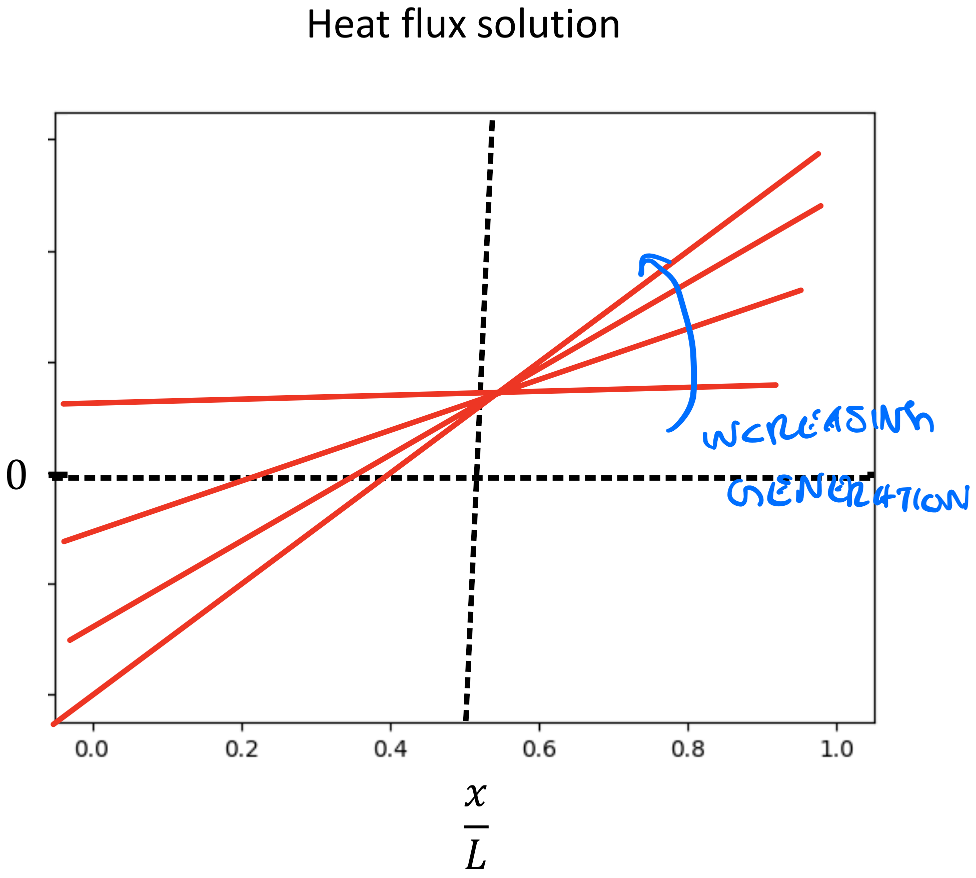
$$\int d\left(\frac{dT}{dx}\right) = \frac{-\dot{g}'''}{k}\int dx$$

$$\frac{dT}{dx} = \frac{-\dot{g}'''}{k}x + C_1$$

$$\int dT = \int\left[\frac{-\dot{g}'''}{k}x + C_1\right]dx$$

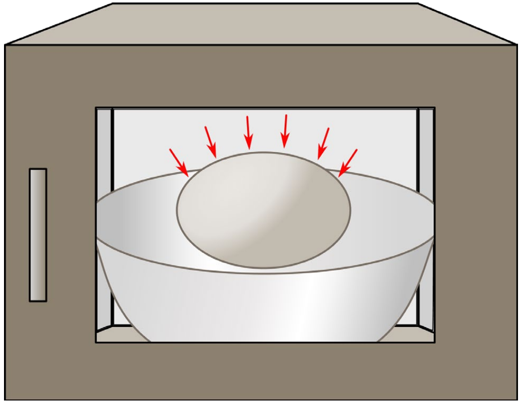
$$T(x) = \underbrace{\frac{-\dot{g}'''}{2k}x^2}_{\text{GEN. TERM}} + \underbrace{C_1 x + C_2}_{\text{LINEAR}}$$


Temperature solution
T_H
T_C
0.0
0.2
0.4
0.6
0.8
1.0
$\frac{x}{L}$
$\dot{g}''' = 0$

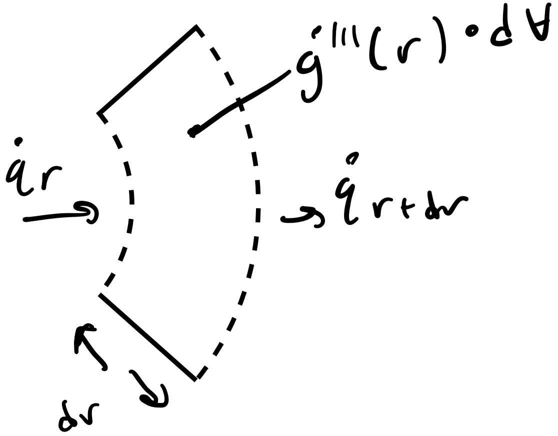

Heat flux solution
0
0.0
0.2
0.4
0.6
0.8
1.0
$\frac{x}{L}$
INCREASING
GENERATION

Nonuniform generation (1)

Example: Egg in a microwave (spherical)



$$\dot{g}'''(r) = r \cdot \alpha$$



$$\dot{q}_r - \dot{q}_{r+dr} + \dot{g}'''(r)\cdot d\forall = 0$$

$$\cancel{\dot{q}_r} - \left(\cancel{\dot{q}_r} + \frac{d\dot{q}_r}{dr}dr\right) + r\cdot\alpha\cdot 4\pi r^2 dr = 0$$

Fourier's

$$\dot{q} = -k \cdot 4\pi r^2 \frac{dT}{dr}$$

$$\frac{d\dot{q}}{dr} = r\cdot\alpha\cdot 4\pi r^2$$

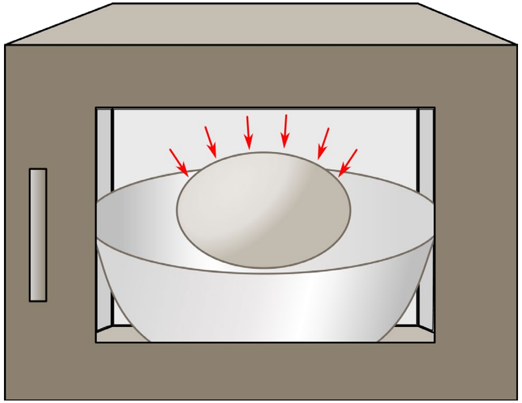
$$\frac{d}{dr}\left[-k\,\cancel{4\pi}\, r^2 \frac{dT}{dr}\right] = r\cdot\alpha\cdot\cancel{4\pi}\, r^2$$

$$\boxed{\frac{d}{dr}\left[r^2\frac{dT}{dr}\right] = -\frac{\alpha}{k} r^3}$$

→ SEPARATE, INTEGRATE

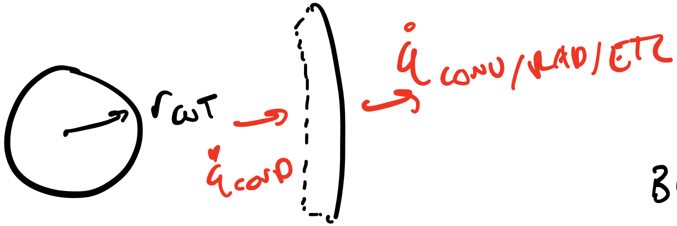
Nonuniform generation (2)

Example: Egg in a microwave (spherical)



$$\dot{g}'''(r) = r \cdot \alpha$$

$$T = -\frac{\alpha}{12 \cdot k} r^3 - \frac{C_1}{r} + C_2$$



$$\boxed{T = \frac{-\alpha}{12k} r^3 - \frac{C_1}{r} + C_2}$$

BC1) @ $r=0$, $\dot{q}=0$

$$\dot{q}_{r=0} = 0, \quad \left.\frac{dT}{dr}\right|_{r=0} = 0$$

BC2) MATCHED RATE

$$\dot{q}_{r=r_{out}} = \bar{h} A_s (T_{r=r_{out}} - T_\infty)$$

BC1: $\frac{d}{dr}\left[-\frac{\alpha}{12k} r^3 - \frac{C_1}{r} + C_2\right]_{r=0} = 0$

$$\left[\frac{dT}{dr} = \frac{-\alpha}{4k} r^2 + \frac{C_1}{r^2}\right]_{r=0} \rightarrow C_1 = 0$$

BC2: $-kA_c \left.\frac{dT}{dr}\right|_{r=r_{out}} = \bar{h} A_s (T_{r=r_{out}} - T_\infty)$

$$k\left(\frac{\alpha}{4k}\right) r_{wt}^2 = \bar{h}\left(-\frac{\alpha}{12k} r_{wt}^3 + c_2 - T_\infty\right)$$

Lecture 5

Extended Surfaces

Last time…

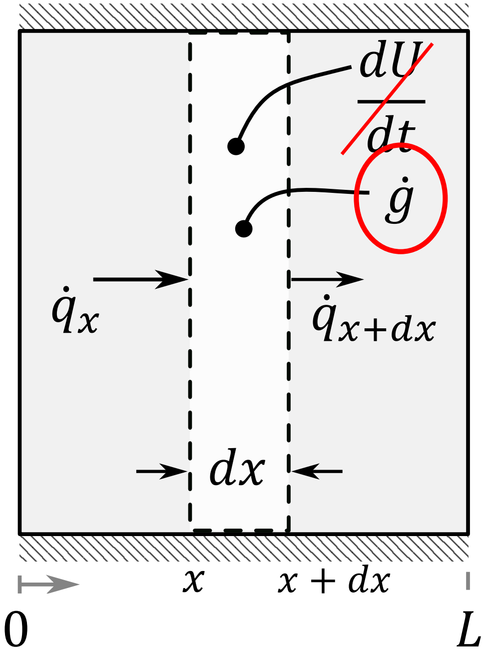
1D steady-state conduction with generation

$$T = -\frac{\dot{g}'''}{2 \cdot k} \cdot x^2 + C_1 \cdot x + C_2$$

Solution approaches linear form as $\dot{g}''' \rightarrow 0$

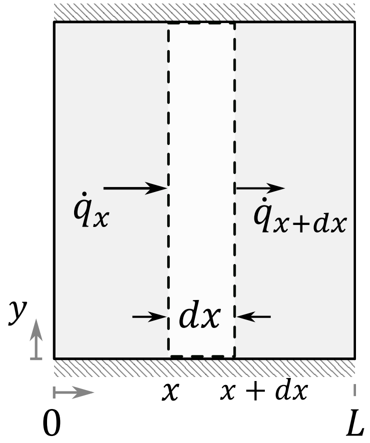
ODE involves the derivative $\frac{d\dot{q}}{dx} = const$

Went through an example involving nonuniform generation

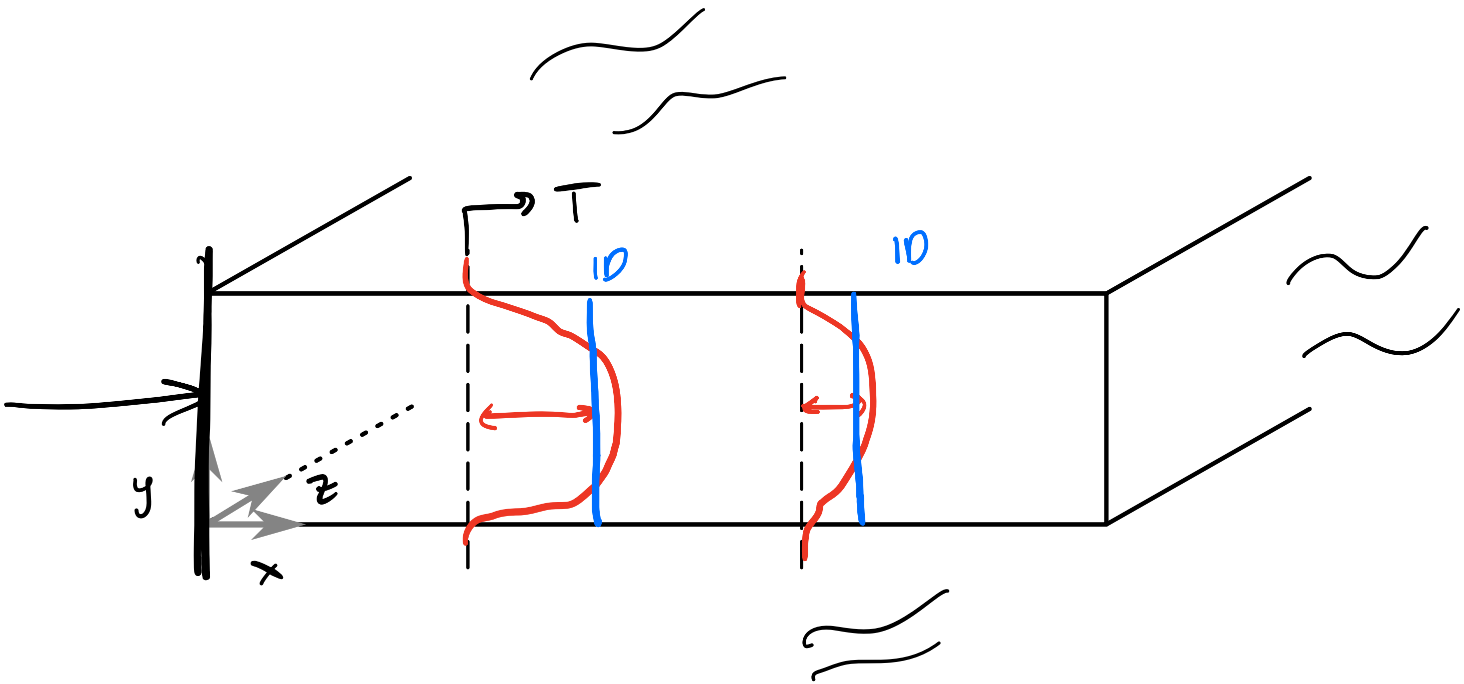


Extended surfaces introduce heat loss along the domain

Heat only flows in the x-direction

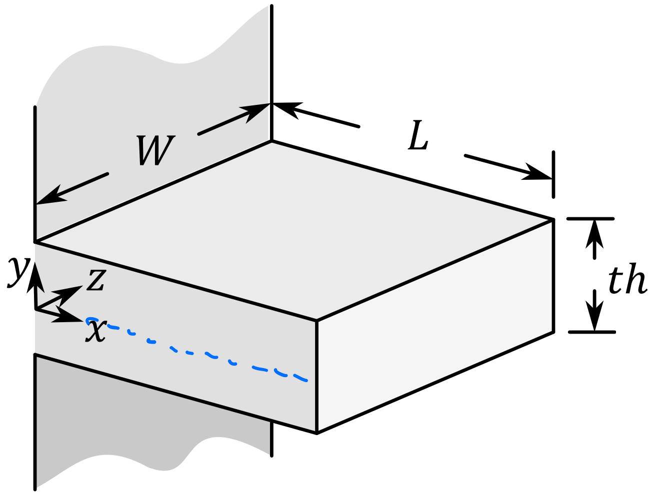


NOW: ALLOWING HEAT TO TRAVEL OUTWARDS



WANT 1D MODEL INSTEAD OF 2D

Extended surface approximation



BIOT NUMBER

$$BIOT = \frac{\Delta T_{COND}}{\Delta T_{SURR}} < 0.1$$

$$Bi = \frac{RES. \ IN \ y}{RES. \ TO \ SURR} << 1 \rightarrow 0.1$$

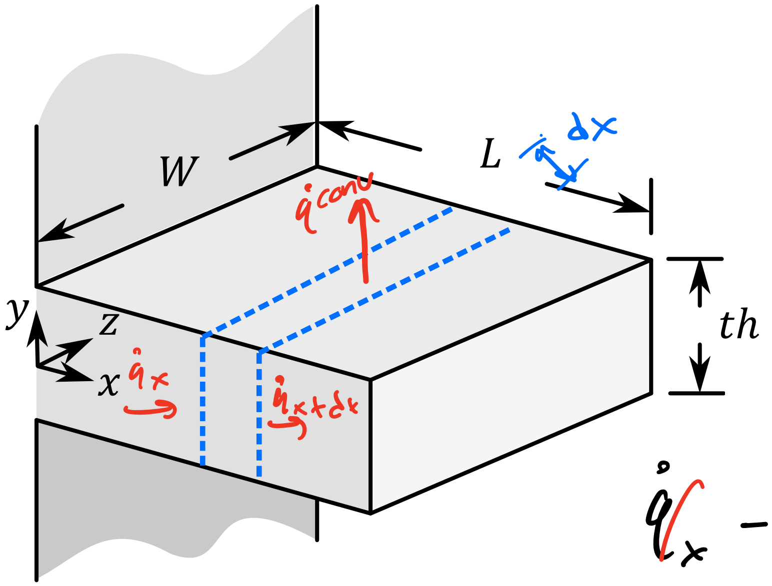
$$R_{COND} = \frac{th}{KA}$$

$$Bi = \frac{R_{COND,y}}{R_{CONV}}$$

$$R_{COND,y} = \frac{th}{2 \cdot K \cdot W \cdot L} \qquad R_{CONV} = \frac{1}{\bar{h} \cdot W \cdot L}$$

$$Bi = \frac{th}{2} \cdot \frac{2WL\bar{h}}{KWL} = \boxed{Bi = \frac{th \cdot \bar{h}}{2K}}$$

Analytical solution for temperature in an extended surface (1)



$$\dot q_x - \dot q_{x+dx} - \dot q_{conv}(x) = 0$$

$$\dot q_{conv}(x) = \bar h\, dA_s\left(T(x) - T_\infty\right)$$

$$= \bar h \cdot perimeter \cdot dx\left(T - T_\infty\right)$$

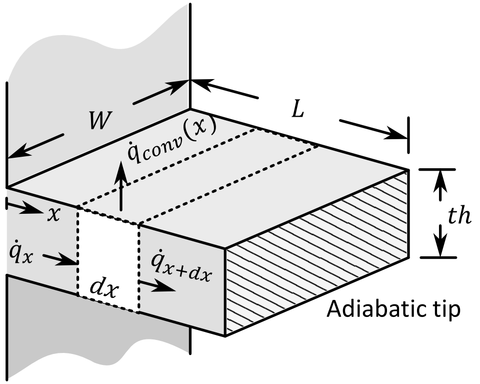
$$\cancel{\dot q_x} - \left(\cancel{\dot q_x} + \frac{d\dot q_x}{dx}\cancel{dx}\right) - \bar h\, per\, \cancel{dx}\left(T - T_\infty\right) = 0$$

$$-\frac{d\dot q}{dx} - per \cdot \bar h\left(T - T_\infty\right) = 0 \rightarrow -\frac{d}{dx}\left(-kA_c\frac{dT}{dx}\right) - per \cdot \bar h\left(T - T_\infty\right) = 0$$

$$\rightarrow \frac{d^2T}{dx^2} - \frac{per \cdot \bar h}{kA_c}T = -\frac{per\,\bar h}{kA_c}T_\infty$$

LINEAR
2ND ORDER
NON-HOMOGENEOUS

Analytical solution for temperature in an extended surface (2)



$$\boxed{\frac{d^2T}{dx^2} - \frac{per\cdot\bar{h}}{k\cdot A_c}T = -\underbrace{\frac{per\cdot\bar{h}}{k\cdot A_c}T_\infty}_{\text{NON-HOMOGENOUS}}}$$

<u>Solving a non-homogeneous, linear ODE</u>

Goal: Find a function $T(x)$ that satisfies the ODE $\longrightarrow$ $T(x) = T_h(x) + T_p(x)$

METHODS:

1) INSPECTION
2) METHOD OF UNDETERMINED COEFFICIENTS (MUC)

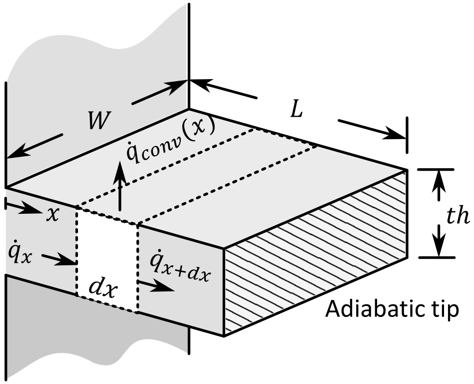
STEPS

1. WRITE ODE IN STANDARD FORM
2. GUESS FORM OF T_p TO BE UNKNOWN CONSTANTS MULTIPLYING THE <u>FORM</u> OF RHS & ITS DERIV.S

GUESS $T_p(x) = C_3$

$$\cancel{\frac{d^2(C_3)}{dx^2}} - \cancel{\frac{per\cdot\bar{h}}{k\cdot A_c}}(C_3) = -\cancel{\frac{per\cdot\bar{h}}{k\cdot A_c}}T_\infty \rightarrow C_3 = T_\infty$$

Analytical solution for temperature in an extended surface (3)



More on Method of Undetermined Coefficients (MUC):

GUESS FORM OF RHS + DERIVS.

<u>ODE</u>

<u>Guess for particular solution</u>

PLUG IN

$$\ast \quad \frac{d^2T}{dx^2} - \beta T = \underbrace{-\beta T_\infty}_{\text{CONST.}} - \frac{x^2}{\beta} \qquad T_p(x) = C_3 + C_4 x^2 + C_5 x$$

$$\frac{d^2T}{dx^2} - \beta T = -\frac{\sin(\omega x)}{\beta} \qquad T_p(x) = C_3 \sin(\omega x) + C_4 \cos(\omega x)$$

$$\frac{d^2T}{dx^2} - \beta T = -\frac{e^{\alpha x}}{\beta} \qquad T_p(x) = C_3 e^{\alpha x}$$

$$\ast: \frac{d^2}{dx^2}\left[C_3 + C_4 x^2 + C_5 x\right] - \beta\left[C_3 + C_4 x^2 + C_5 x\right] = -\beta T_\infty - \frac{x^2}{\beta}$$

$$\underbrace{2C_4 - \beta C_3}_{} + \underbrace{\beta C_4 x^2}_{} + \underbrace{\beta C_5 x}_{} = \underbrace{-\beta T_\infty}_{} - \underbrace{\frac{x^2}{\beta}}_{}$$

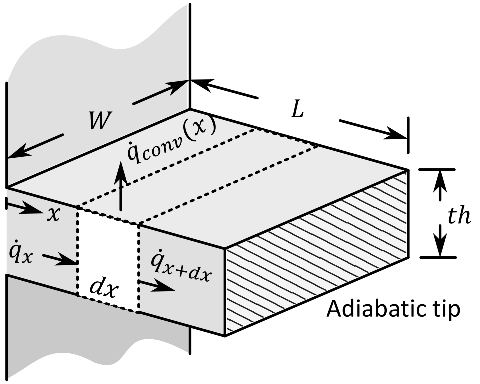
1) $2C_4 - \beta C_3 = -\beta T_\infty$

2) $\beta C_5 x = 0$

3) $\beta C_4 x^2 = -\frac{x^2}{\beta}$

SOLVE

Analytical solution for temperature in an extended surface (4)



$$\frac{d^2T}{dx^2} - \frac{per \cdot \bar{h}}{k \cdot A_c} T = -\frac{per \cdot \bar{h}}{k \cdot A_c} T_\infty$$

<u>Solving a non-homogeneous, linear ODE</u>

Goal: Find a function $T(x)$ that satisfies the ODE $\longrightarrow$ $T(x) = T_h(x) + T_p(x)$

$$\frac{d^2T}{dx^2} - \frac{per \cdot \bar{h}}{k \cdot A_c} T = 0$$

LET $m^2 = \frac{per \cdot \bar{h}}{k \cdot A_c}$

$$T_h(x) = C_1 \exp(mx) + C_2 \exp(-mx)$$

$$T(x) = T_h + T_p = \boxed{C_1 \exp(mx) + C_2 \exp(-mx) + T_\infty}$$

GENERAL SOLUTION IN TEMP.

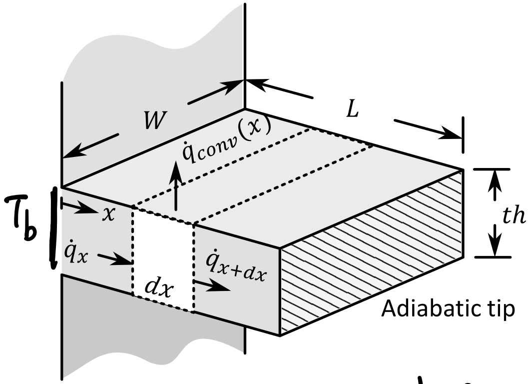
Solutions to common linear homogeneous ODE's

Table 3.1 Solutions to some common linear homogeneous ODEs.

Entry	Linear homogeneous ODE	Solution
1	$\frac{d^2T_h}{dx^2} - m^2 T_h = 0$	$T_h = C_1 \exp(m\,x) + C_2 \exp(-m\,x)$ $T_h = C_1 \cosh(m\,x) + C_2 \sinh(m\,x)$
2	$\frac{d^2T_h}{dx^2} + m^2 T_h = 0$	$T_h = C_1 \cos(m\,x) + C_2 \sin(m\,x)$
3	$\frac{dT_h}{dt} + \frac{T_h}{\tau} = 0$	$T_h = C_1 \exp\left(-\frac{t}{\tau}\right)$

More on pp. 196 of Nellis & Klein…

Analytical solution for temperature in an extended surface (5)



$$\boxed{T(x) = C_1 \exp(mx) + C_2 \exp(-mx) + T_\infty} \leftarrow T_h(x) + T_p(x)$$

BC 1) $T_{x=0} = T_b$, BC 2) $\dot{q}_{x=L} = 0 = \frac{dT}{dx}$

1) $T_b = C_1 \exp(0) + C_2 \exp(0) + T_\infty \rightarrow \boxed{T_b = C_1 + C_2 + T_\infty}$

2) $\frac{d}{dx}\left[C_1 \exp(mx) + C_2 \exp(-mx) + T_\infty\right] = 0$

$$\left[C_1 m \exp(mx) - C_2 m \exp(-mx)\right]_{x=L} = 0$$

$$C_1 = \frac{C_2 m \exp(-mL)}{m \exp(mL)} \rightarrow \boxed{C_1 = \frac{(T_b - T_\infty)\exp(-mL)}{\exp(mL) + \exp(-mL)}}$$

Analytical solution for temperature in an extended surface (6)

$$\underbrace{\frac{T(x) - T_\infty}{T_b - T_\infty}} = \frac{\exp\left[-mL\left(1-\frac{x}{L}\right)\right] + \exp\left[mL\left(1-\frac{x}{L}\right)\right]}{\exp(mL) + \exp(-mL)}$$

NORMALIZED TEMP

$0 \leq \Upsilon \leq 1$

$$\rightarrow \quad m = \sqrt{\frac{per \cdot \bar{h}}{k \cdot A_c}}$$

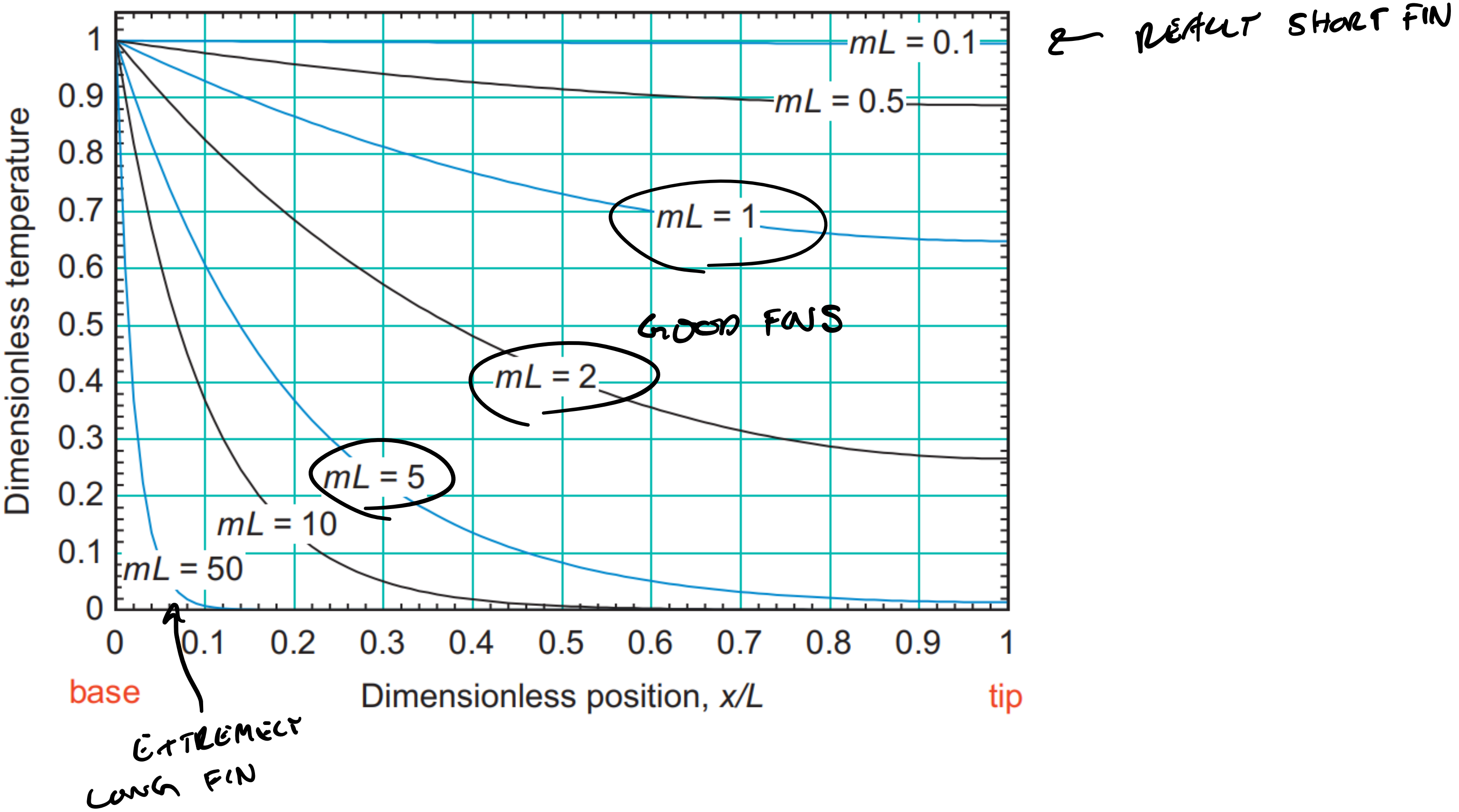
$$\cosh(A) = \frac{\exp(A) + \exp(-A)}{2} \qquad \sinh(A) = \frac{\exp(A) - \exp(-A)}{2}$$

$$\frac{T(x) - T_\infty}{T_b - T_\infty} = \frac{\cosh\left[mL\left(1-\frac{x}{L}\right)\right]}{\cosh(mL)}$$

ADIABATIC
CONST. C.S.
CONVECTION ONLY

Analytical solution for temperature in an extended surface (7)

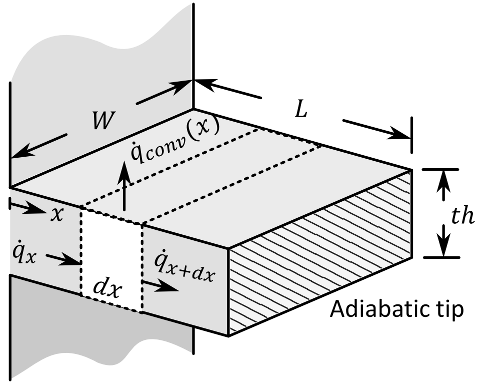
Solution for adiabatic tip



Lecture 6

Fin performance and fin efficiency

Last time…



Extended surface analysis

Used Biot number to justify extended surface approximation:

$$Bi = \frac{R_{cond,y}}{R_{surr}} < 0.1$$

Found solution to non-homogeneous ODE

$$T(x) = T_h(x) + T_p(x)$$

Derived solution for constant-area fin with adiabatic tip, known base temperature:

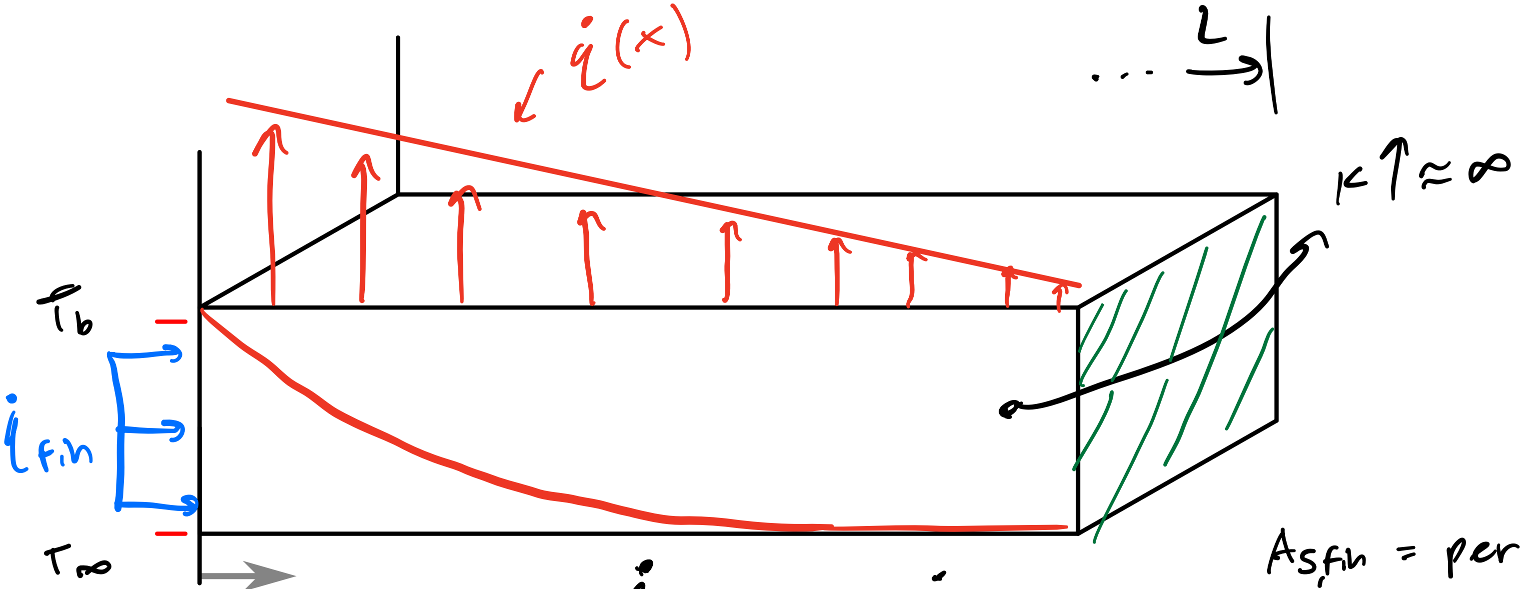
$$\frac{T(x) - T_\infty}{T_b - T_\infty} = \frac{\cosh\left[mL\left(1 - \frac{x}{L}\right)\right]}{\cosh(mL)}$$

Defined "fin constant" mL

where $\mathrm{m} = \sqrt{\frac{per \cdot \bar{h}}{k \cdot A_c}}$

Fin efficiency

DEF: $\eta_{fin} = \dfrac{\text{heat xfer to fin}}{\text{max heat xfer from fin}}$



$A_{s,fin} = per \cdot L$

$$\eta_{fin} = \frac{\dot{q}_{fin}}{\dot{q}_{fin,max}} = \frac{\dot{q}_{fin}}{\bar{h} A_{s,fin} (T_b - T_\infty)}$$

IF ENTIRE SURFACE @ $T = T_b$

Computing fin efficiency for a given geometry (1)

From last time, solving for $T(x)$

$$\frac{T(x)-T_\infty}{T_b-T_\infty}=\frac{\cosh\left[mL\left(1-\frac{x}{L}\right)\right]}{\cosh(mL)}$$

IN = OUT, EASIER THAN INT. OVER SURF

$$\dot{q}_{fin} = -kA_{c,fin}\left.\frac{dT}{dx}\right|_{x=0} = -kA_c\frac{d}{dx}\left[\frac{\cosh\left[mL\left(1-\frac{x}{L}\right)\right]}{\cosh(mL)}\left(T_b-T_\infty\right)+T_\infty\right]_{x=0}$$

$$\dot{q}_{fin} = kA_c m\left(T_b-T_\infty\right)\overbrace{\frac{\sinh(mL)}{\cosh(mL)}}^{\tanh(mL)}$$

Computing fin efficiency for a given geometry (2)

Then subbing into the efficiency equation:

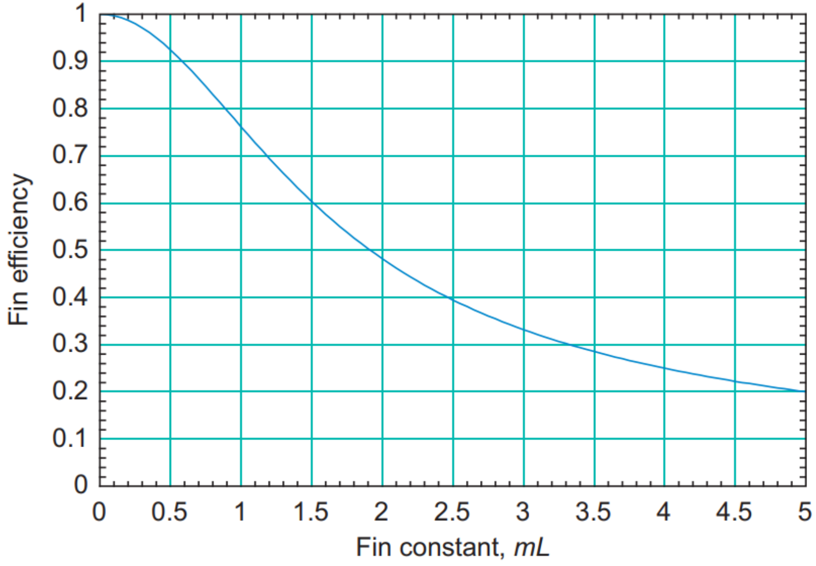
ONLY DEPENDS ON GEOMETRY

$$\eta_{fin} = \frac{kA_c\sqrt{\frac{per\cdot\bar{h}}{k\cdot A_c}}\,(T_b - \cancel{T_\infty})\tanh(mL)}{\bar{h}A_s(T_b - \cancel{T_\infty})} \rightarrow \boxed{\eta_{fin} = \frac{\tanh(mL)}{mL}}$$

ADB. TIP

CONST. C.S.

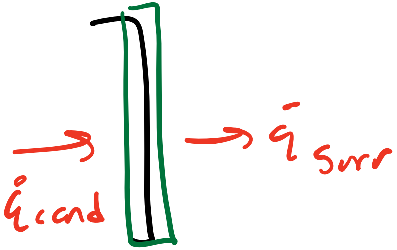
See Table 3.2, pp. 204 for more derived fin efficiencies



Fin resistance

$$\eta_{fin} = \frac{\dot{q}_{fin}}{\bar{h}A_s(T_b-T_\infty)} \rightarrow \dot{q}_{fin} = \underbrace{\eta_{fin}\,\bar{h}\,A_s}_{\frac{1}{R_{fin}}}(T_b-T_\infty)$$

$$R_{fin} = \frac{1}{\eta_{fin}\bar{h}A_s} = \frac{1}{\eta_{fin}\bar{h}\,per\cdot L}$$

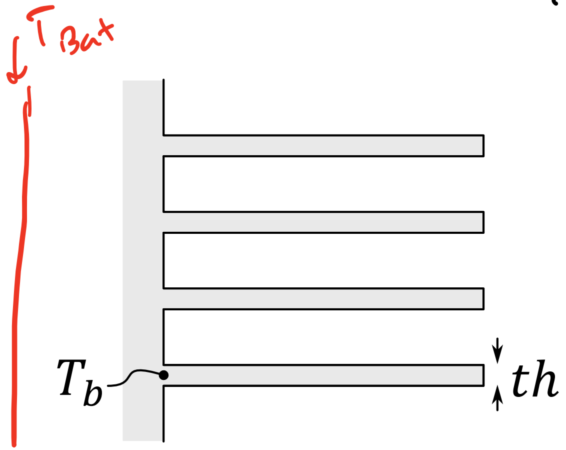


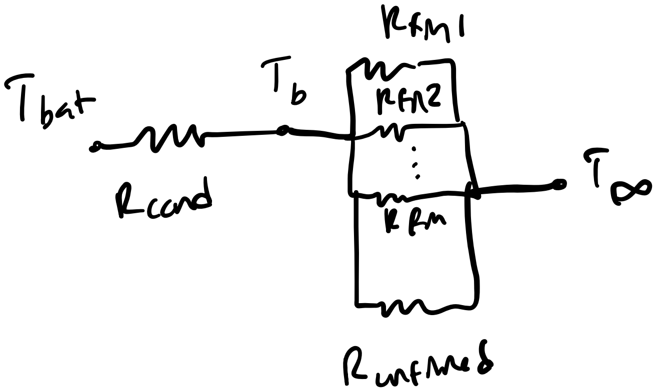
Convection from the fin tip

BC1) $T_{x=0} = T_b$ BC2) $-k\cancel{A_c}\frac{dT}{dx}\Big|_{x=L} = \bar{h}\cancel{A_{tip}}(T_{x=L}-T_\infty)$

$$\eta_{fin} = \frac{\tanh(mL) + mLAR_{tip}}{mL[1+mLAR_{fin}\tanh(mL)](1+AR_{fin})}$$

Fin arrays





$$R_{TOT} = \left[\frac{1}{R_{unfinned}} + \frac{1}{R_{f1}} + \frac{1}{R_{f2}} + \dots + \frac{1}{R_{fm}}\right]^{-1}$$

Lecture 7

Numerical solutions to 1-D conduction

Last time…

Fin efficiency and fin performance

Fin efficiency is independent of temperature

$$\eta_{fin} = \frac{\tanh(mL)}{mL}$$

Fin resistance captures all resistances in the fin

$$R_{fin} = \frac{1}{\eta_{fin} \cdot \bar{h} \cdot A_s}$$

Array resistance includes finned and unfinned portions in parallel

$$R_{tot} = \left[\frac{1}{R_{unfinned}} + \frac{N}{R_f}\right]^{-1}$$

Motivation for numerical solutions

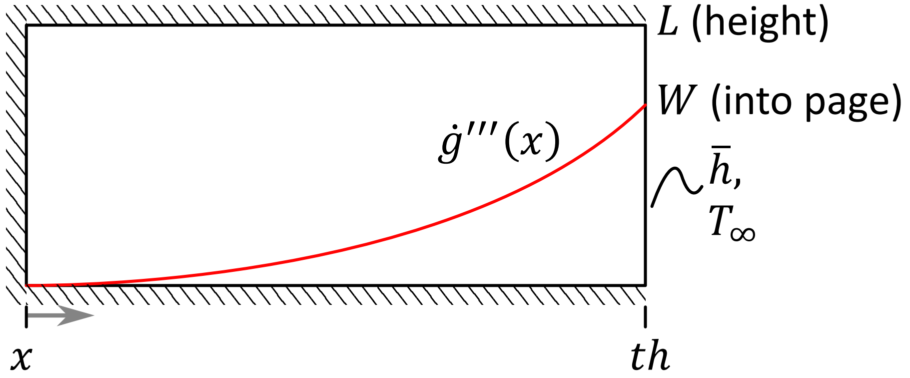
Analytical solution	**Numerical solution**
Exact	Approximate
Limited applicability	Very flexible
Require ODE techniques	Require algebraic techniques
Computationally efficient	Computationally expensive

Numerical solution method

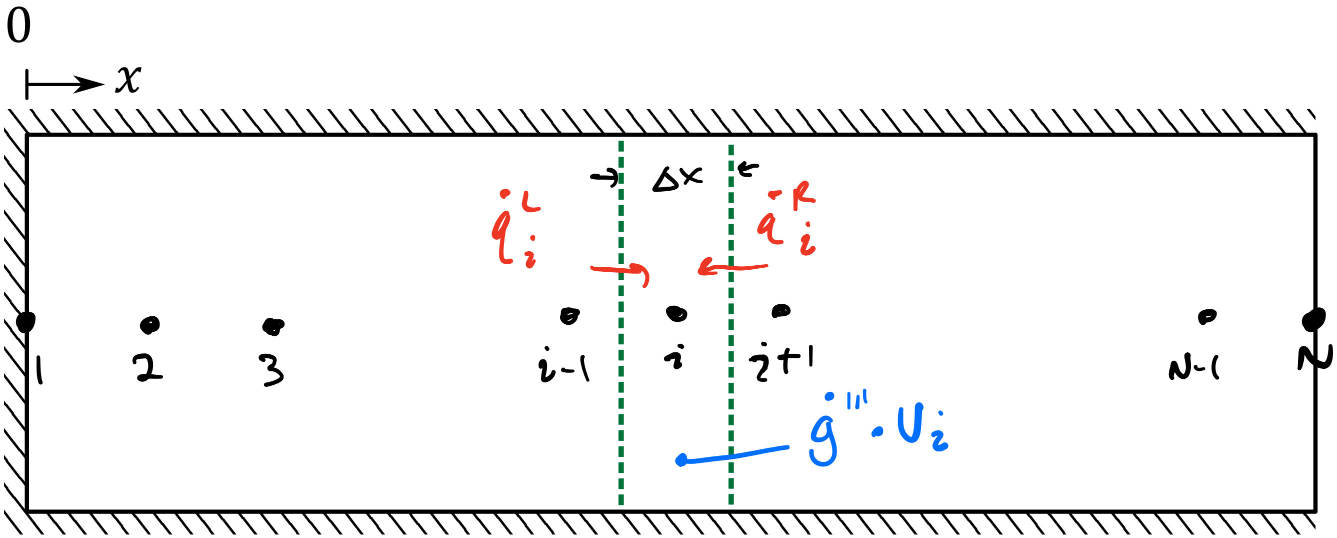
1. Discretize the domain in to N finite CV's
2. Perform energy balance on each CV "node"
3. Represent heat flows using a resistance model
4. Solve system of algebraic equations for temperature at each node
5. Verify solution

1-D numerical solution example

Plane wall with nonuniform generation



1. Discretize the domain in to N finite CV's



$\Delta x = \frac{th}{N-1}$ NODE POSITIONS: $x_i = \Delta x(i-1)$, where $i = 1 \ldots N$

1-D numerical solution example

2. Perform energy balance on each CV "node"

FOR INTERNAL NODES

IN + GEN = ~~OUT~~ + ~~STORED~~

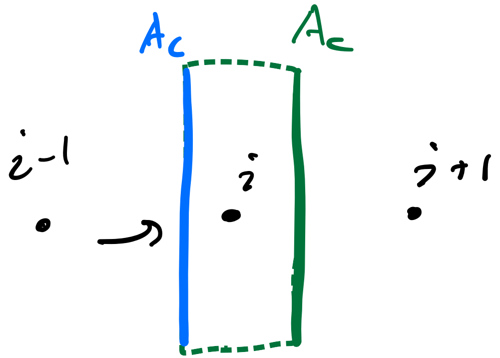
$$\dot{q}_i^R + \dot{q}_i^L + \dot{g}''' \cdot V_i = 0 \quad \forall\, i \in 2:N-1$$

FOR ALL IN SET

1-D numerical solution example

$$\boxed{\dot{q}_i^R + \dot{q}_i^L + \dot{g}'''(x_i) \cdot V_i = 0 \;\; \forall\, i = 1 \dots N}$$

3. Represent heat flows using a resistance model **(repeat for end nodes)**



$$\dot{q}_i^L = \frac{(T_{i-1} - T_i)}{\Delta x} k \underline{A_c}$$

$$\dot{q}_i^R = \frac{(T_{i+1} - T_i)}{\Delta x} k \underline{A_c}$$

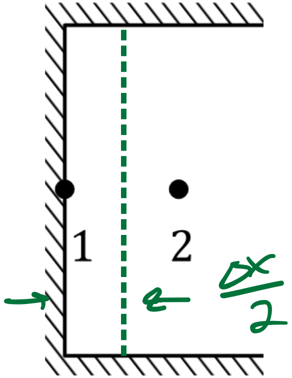
$$\dot{g}_i = \dot{g}'''(x_i) \cdot \Delta x \cdot A_c$$

$$k A_c \frac{T_{i-1} - T_i}{\Delta x} + k A_c \frac{T_{i+1} - T_i}{\Delta x} + \dot{g}'''(x_i) \Delta x A_c = 0 \;\; \forall\, i \in 2 : N-1$$

1-D numerical solution example

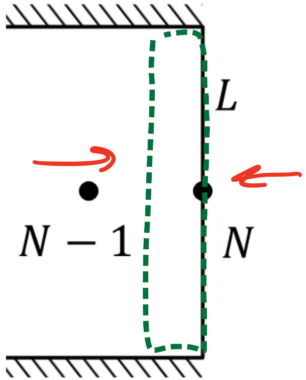
$$\boxed{\dot{q}_i^R + \dot{q}_i^L + g'''(x_i) \cdot V_i = 0 \;\; \forall\, i = 1 \ldots N}$$

3. Represent heat flows using a resistance model



$\dot{q}_1^L = 0$

$$0 + kA_c\left(\frac{T_2 - T_1}{\Delta x}\right) + \dot{g}'''(x_1) \cdot \frac{\Delta x}{2} A_c = 0$$



ORDER MUST MATCH DIRECTION OF HEAT FLOW

$$kA_c \frac{T_{N-1} - T_N}{\Delta x} + \bar{h} A_c (T_\infty - T_N) + \dot{g}'''(x_N) \frac{\Delta x}{2} A_c = 0$$

1-D numerical solution example

4. Solve system of algebraic equations for temperature at each node

N eqns → Energy Bal @ Ea NODE

N unkns → T_i, $i \in 1:N$

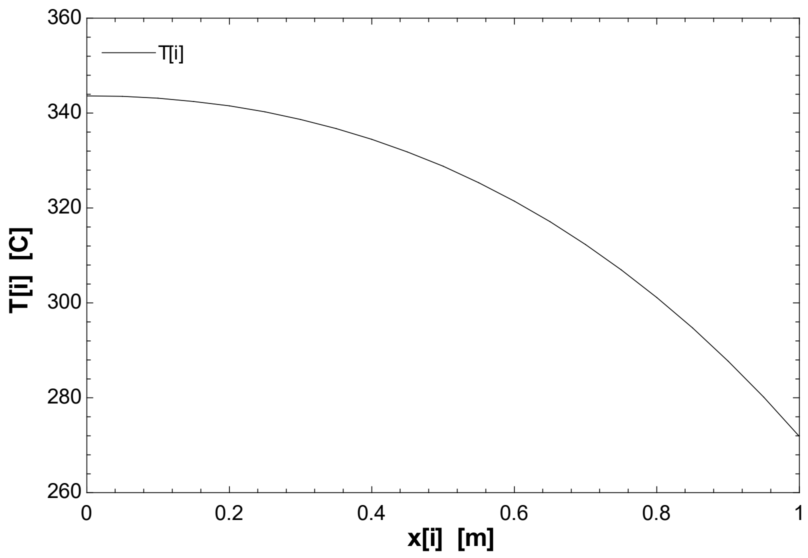
$$\left.\begin{matrix} \\ \\ \end{matrix}\right\} \underline{x} = \underline{\underline{A}}^{-1} \cdot \underline{b}$$

$$\underline{\underline{A}} \cdot \underline{x} = \underline{b}$$

$$\begin{bmatrix} -\frac{kA_s}{\Delta x} & \frac{kA_s}{\Delta x} & 0 & 0 & \cdots & 0 \\ \frac{kA_s}{\Delta x} & -\frac{2kA_s}{\Delta x} & \frac{kA_s}{\Delta x} & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & & \\ 0 & & & & & \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ \vdots \\ T_N \end{bmatrix} = \begin{bmatrix} -\dot{g}'''(x_1)\frac{\Delta x}{2} wL \\ -\dot{g}'''(x_2)\Delta x wL \\ \\ \\ \end{bmatrix}$$

1-D numerical solution example

```
Function g_dot_tp(x)
        g_dot_tp := exp(x / (1[m])) * (100 [W/m^3])
End

k = 1 [W/m-K]
th = 1 [m]
W = 1 [m]
L = 1 [m]
h_bar = 1 [W/m^2-K]
T_infty = 100 [C]

N = 21

DELTAx = th / (N-1)

Duplicate i=1,N
        x[i] = DELTAx * (i-1)
End

"Internal nodes"
Duplicate i=2,N-1
        k*W*L/DELTAx * (T[i-1] + T[i+1] - 2*T[i]) + g_dot_tp(x[i])*DELTAx * W * L = 0
End

"Node 1"
"This is the equation we used in class, assuming adiabatic boundary"
k * W * L / DELTAx * (T[2]-T[1]) + g_dot_tp(x[1]) * W * L * DELTAx/2 = 0
"Node N"
k * W * L / DELTAx * (T[N-1] - T[N]) + h_bar * W * L * (T_infty - T[N]) + g_dot_tp(x[N]) * W * L * DELTAx / 2 = 0
```



Lecture 8

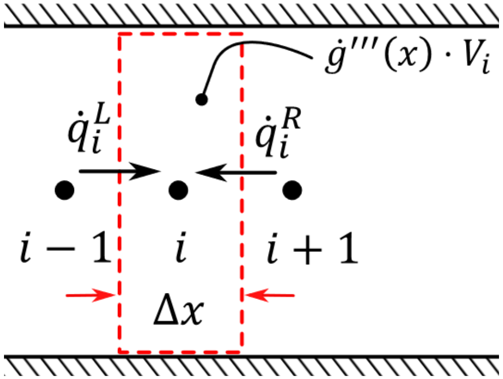
Numerical solutions to 2-D conduction

- know how to derive node equations

Last time…

Numerical solutions to 1-D steady-state conduction

1. Discretize the domain in to N finite CV's
2. Perform energy balance on each CV "node"
3. Represent heat flows using a resistance model
4. Solve system of algebraic equations for temperature at each node
5. Verify solution



2-D numerical solutions

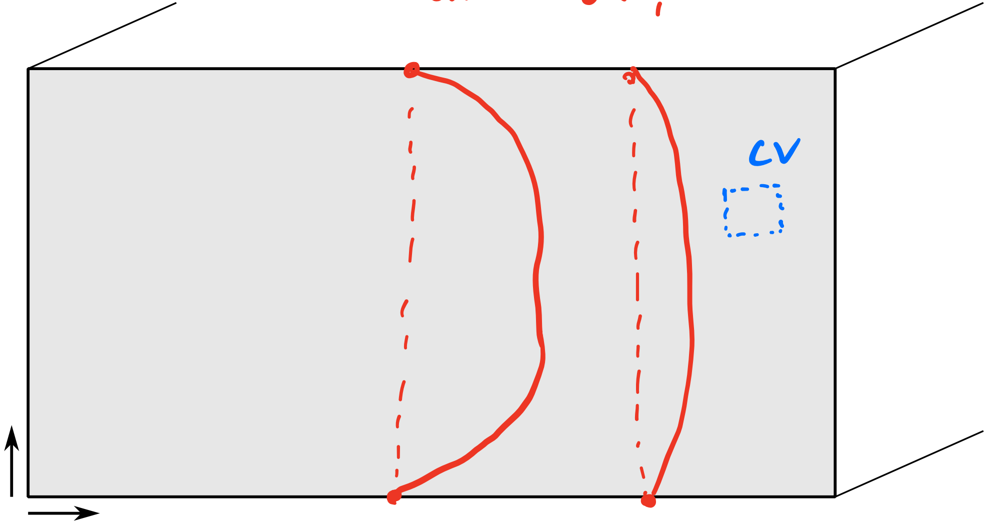
Analytical basis:

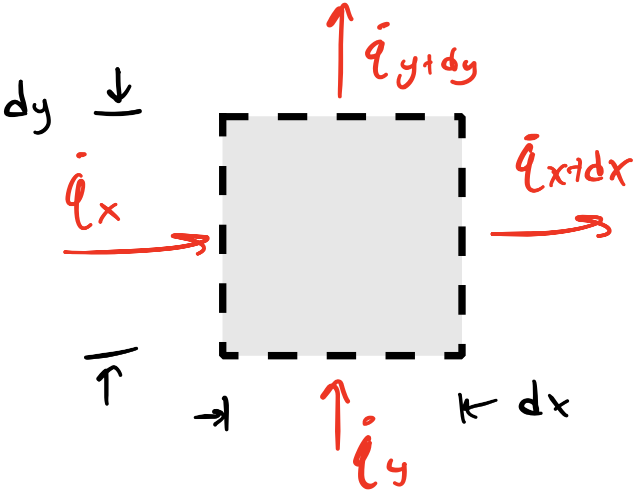
Fourier's Law: $\dot{q}'' = -k\nabla T$

$$= -k\left(\frac{\partial T}{\partial x}\hat{i} + \frac{\partial T}{\partial y}\hat{j} + \cancel{\frac{\partial T}{\partial z}\hat{k}}\right)$$

2D

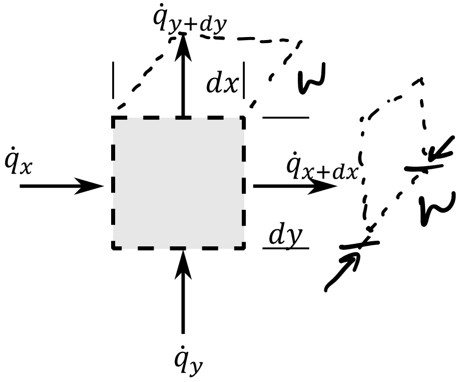
Non-uniform, Bi >> 0.1





2-D numerical solutions

Analytical basis:



IN + GEN = OUT + STORED

$$\cancel{\dot{q}_x} + \cancel{\dot{q}_y} = \dot{q}_{x+dx} + \dot{q}_{y+dy}$$

$$\cancel{\dot{q}_x} + \frac{\partial \dot{q}_x}{\partial x}dx \qquad \cancel{\dot{q}_y} + \frac{\partial \dot{q}_y}{\partial y}dy$$

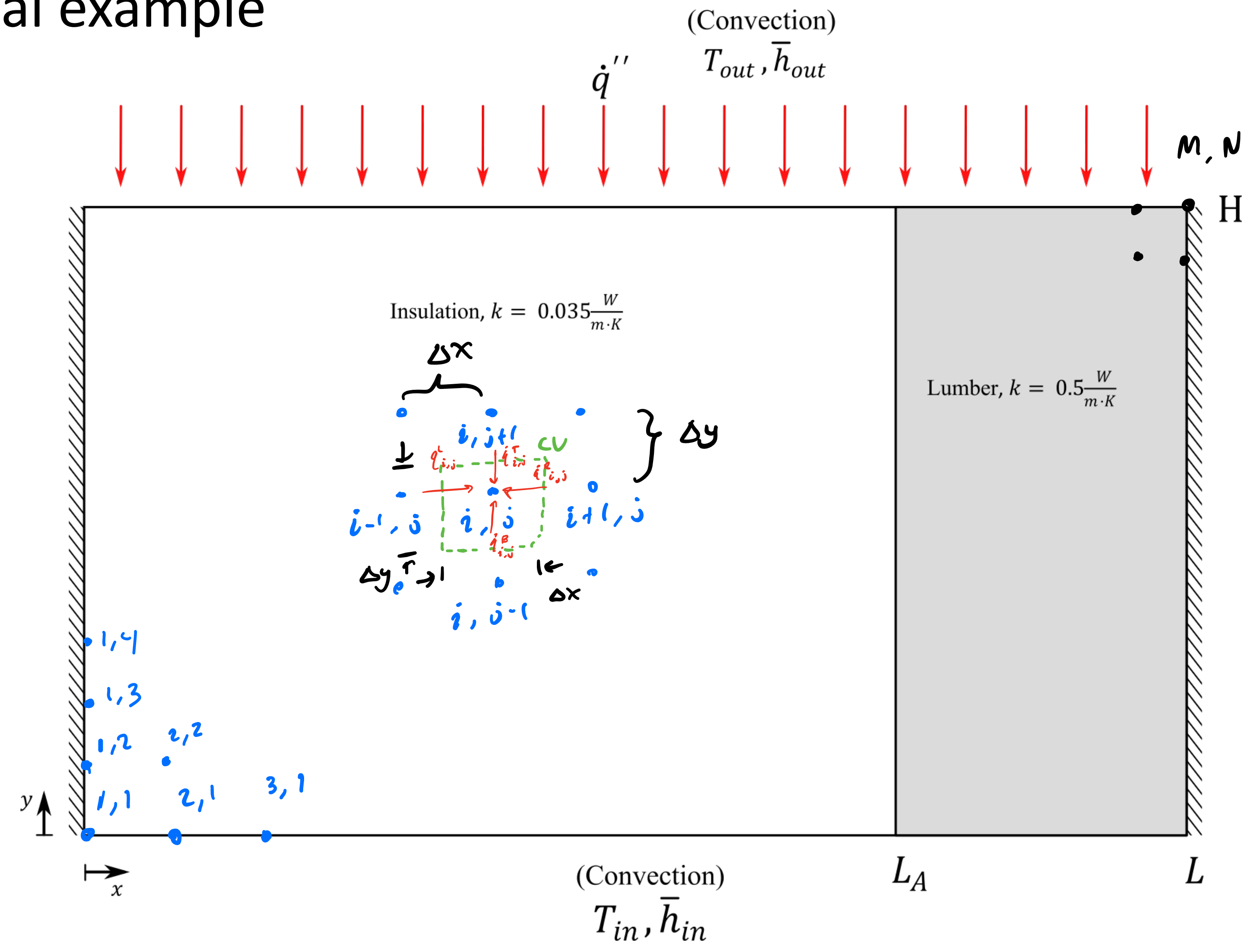
$$\frac{\partial \dot{q}_x}{\partial x}dx + \frac{\partial \dot{q}_y}{\partial y}dy = 0$$

$$\frac{\partial}{\partial x}\left[-\cancel{k}\,\cancel{w}\,\cancel{dy}\,\frac{\partial T}{\partial x}\right]\cancel{dx} + \frac{\partial}{\partial y}\left[-\cancel{k}\,\cancel{w}\,\cancel{dx}\,\frac{\partial T}{\partial y}\right]\cancel{dy} = 0$$

$$\boxed{\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0}$$

2-D numerical example

Composite wall



2-D numerical example

1) Discretize the domain

M nodes in x
N " y

$\left.\begin{matrix} M \text{ nodes in } x \\ N \text{ nodes in } y \end{matrix}\right\}$ Total $= M \cdot N$

$\Delta x = \dfrac{L}{M-1}$

$\Delta y = \dfrac{H}{N-1}$

$x_i = (i-1)\Delta x$

$y_j = (j-1)\Delta y$

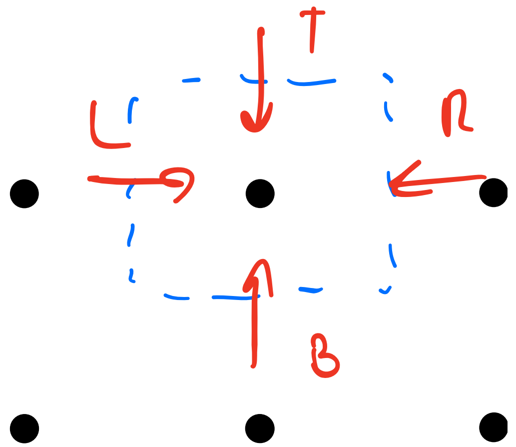
Conductivity

```
function(x)
   if (x ≤ L_A)
        k[i] = 0.035
   else
        k[i] = 0.5
   endif
end function
```

2-D numerical example

const. conductivity

2) Energy balance on internal nodes



$$\dot{q}^{R}_{ij} + \dot{q}^{L}_{ij} + \dot{q}^{T}_{ij} + \dot{q}^{B}_{ij} = 0$$

$$\forall\ i \in (2:M-1),\ j \in (2:N-1)$$

$$\dot{q}^{R}_{ij} = k \frac{w \Delta y}{\Delta x} (T_{i+1,j} - T_{ij}$$

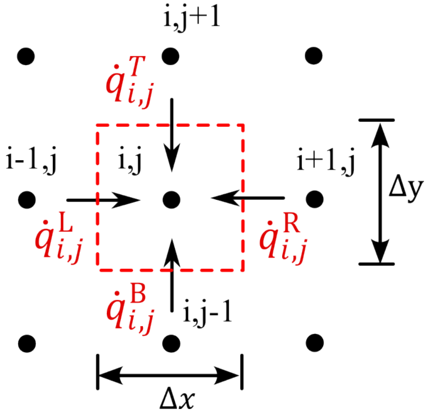
$$\dot{q}^{L}_{ij} = k w \frac{\Delta y}{\Delta x} (T_{i-1,j} - T_{ij})$$

$$\dot{q}^{T}_{ij} = k w \frac{\Delta x}{\Delta y} (T_{i,j+1} - T_{ij})$$

$\dot{q}^{B}_{ij}$ = same pattern

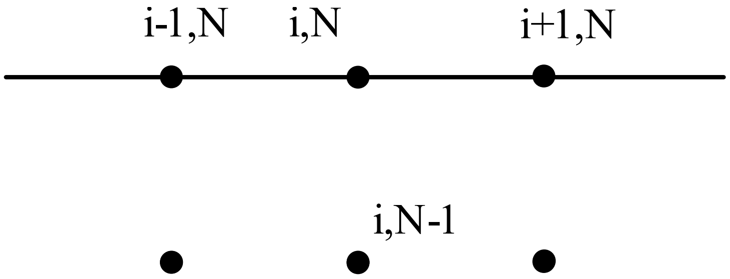
2-D numerical example

2) Revised energy balance for our composite problem



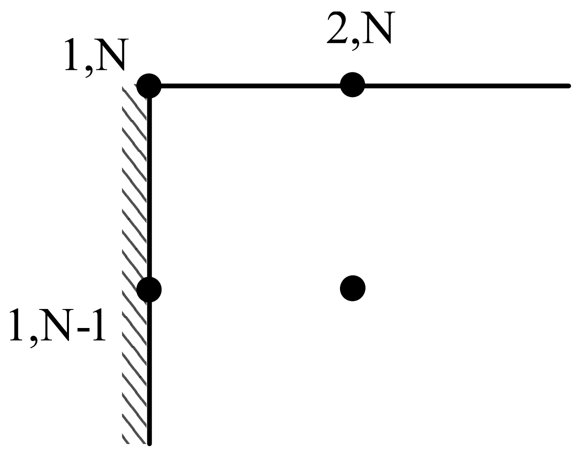
2-D numerical example

3) Energy balance on boundary nodes



2-D numerical example

3) Energy balance on corner nodes

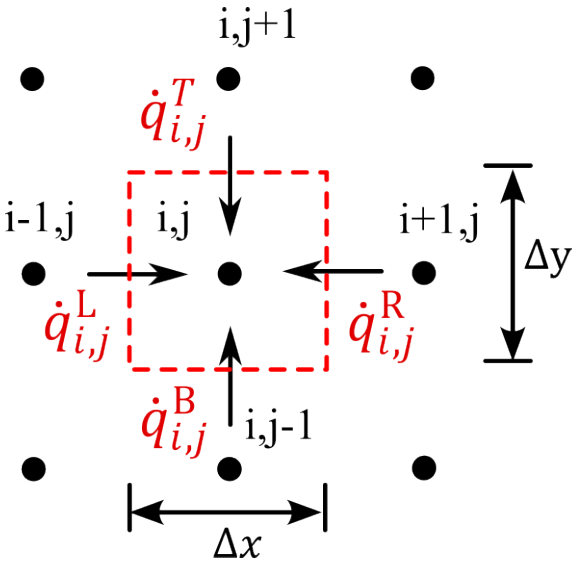


2-D numerical example

Formulation if conductivity is a single value

2) Energy balance on internal nodes

$$\dot{q}^R_{i,j} + \dot{q}^L_{i,j} + \dot{q}^T_{i,j} + \dot{q}^B_{i,j} = 0 \;\; \forall\, i \in 2 \dots M-1;\; j \in 2 \dots N-1$$



$$\dot{q} = k \cdot \frac{area\ for\ conduction}{distance\ to\ conduct}(Node\ temperature\ difference)$$

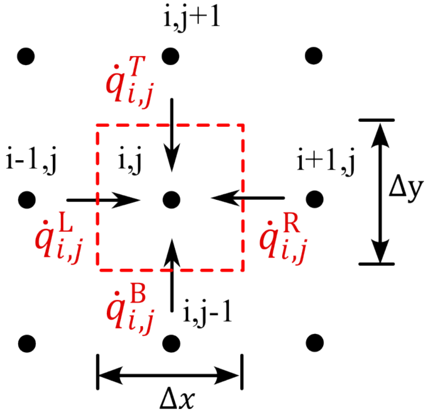
$$\left.\begin{aligned} \dot{q}^R_{i,j} &= k_i\, W \frac{dy}{dx}\left(T_{i+1,j} - T_{i,j}\right) \\ \dot{q}^L_{i,j} &= k_i\, W \frac{dy}{dx}\left(T_{i-1,j} - T_{i,j}\right) \end{aligned}\right\} k_i\, W \frac{dy}{dx}\left(T_{i+1,j} + T_{i-1,j} - 2 \cdot T_{i,j}\right)$$

$$\left.\begin{aligned} \dot{q}^T_{i,j} &= k_i\, W \frac{dx}{dy}\left(T_{i,j+1} - T_{i,j}\right) \\ \dot{q}^B_{i,j} &= k_i\, W \frac{dx}{dy}\left(T_{i,j-1} - T_{i,j}\right) \end{aligned}\right\} k_i\, W \frac{dx}{dy}\left(T_{i,j+1} + T_{i,j-1} - 2 \cdot T_{i,j}\right)$$

$$\cancel{k_i W} \frac{dx}{dy}\left(T_{i,j+1} + T_{i,j-1} - 2 \cdot T_{i,j}\right) + \cancel{k_i W} \frac{dy}{dx}\left(T_{i+1,j} + T_{i-1,j} - 2 \cdot T_{i,j}\right) = 0$$

$$\forall\, i \in 2 \dots M-1;\; j \in 2 \dots N-1$$

2-D numerical example



2) Revised energy balance for our composite problem

Δx

$$\dot{q}^{R}_{i,j} = \frac{(k_i + k_{i+1})}{2} W \frac{dy}{dx}\left(T_{i+1,j} - T_{i,j}\right)$$

Δy

$$\dot{q}^{L}_{i,j} = \frac{(k_i + k_{i-1})}{2} W \frac{dy}{dx}\left(T_{i-1,j} - T_{i,j}\right)$$

Conductivity/Area shared between node eqn's Needs to be evaluated @ boundary

k only depends on i

$$\left\{\begin{aligned} \dot{q}^{T}_{i,j} &= k_i\, W \frac{dx}{dy}\left(T_{i,j+1} - T_{i,j}\right) \\ \dot{q}^{B}_{i,j} &= k_i\, W \frac{dx}{dy}\left(T_{i,j-1} - T_{i,j}\right) \end{aligned}\right\} \; k_i\, W \frac{dx}{dy}\left(T_{i,j+1} + T_{i,j-1} - 2 \cdot T_{i,j}\right)$$

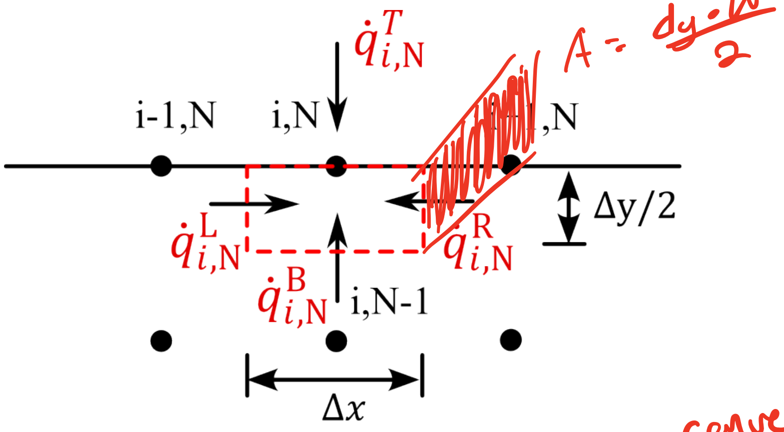
$$k_i \frac{dx}{dy}\left(T_{i,j+1} + T_{i,j-1} - 2 \cdot T_{i,j}\right) + \frac{(k_i + k_{i+1})}{2} \frac{dy}{dx}\left(T_{i+1,j} - T_{i,j}\right) + \frac{(k_i + k_{i-1})}{2} \frac{dy}{dx}\left(T_{i-1,j} - T_{i,j}\right) = 0$$

$$\forall\, i \in 2 \ldots M-1;\; j \in 2 \ldots N-1$$

2-D numerical example

For top-side nodes 'j=N'

3) Energy balance on boundary nodes



convection on top

$$\dot{q}_{i,N}^{R} + \dot{q}_{i,N}^{L} + \dot{q}_{i,N}^{T} + \dot{q}_{i,N}^{B} = 0 \ \ \forall\, i \in 2 \ldots M-1$$

$$\dot{q}_{i,N}^{R} = \frac{(k_i + k_{i+1})}{2} W \frac{dy}{2 \cdot dx} \left(T_{i+1,N} - T_{i,N}\right)$$

$$\dot{q}_{i,N}^{L} = \frac{(k_i + k_{i-1})}{2} W \frac{dy}{2 \cdot dx} \left(T_{i-1,j} - T_{i,j}\right)$$

$$\dot{q}_{i,N}^{B} = k_i\, W \frac{dx}{dy} \left(T_{i,N-1} - T_{i,N}\right)$$

$$\dot{q}_{i,N}^{T} = \bar{\mathrm{h}}_{out}\, W\, dx \left(T_{out} - T_{i,N}\right) + \dot{q}'' W\, dx$$

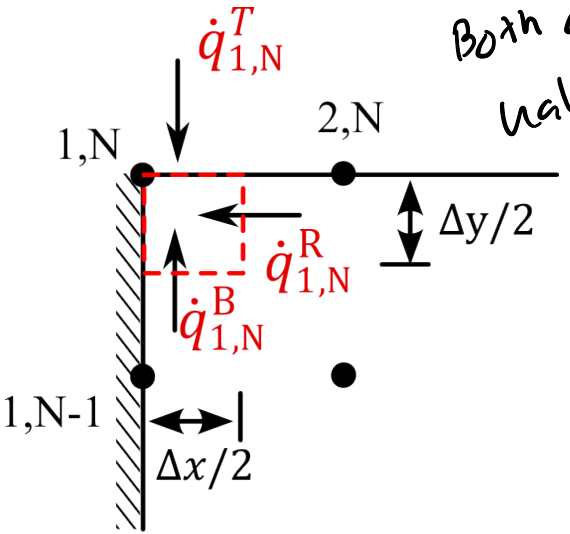
$$\frac{(k_i + k_{i+1})}{2} \frac{dy}{2 \cdot dx} \left(T_{i+1,N} - T_{i,N}\right) + \frac{(k_i + k_{i-1})}{2} \frac{dy}{2 \cdot dx} \left(T_{i-1,j} - T_{i,j}\right) + k_i \frac{dx}{dy} \left(T_{i,N-1} - T_{i,N}\right)$$
$$+ \bar{\mathrm{h}}_{out}\, dx \left(T_{out} - T_{i,N}\right) + \dot{q}''\, dx = 0 \qquad \forall\, i \in 2 \ldots M-1$$

Repeat for Left, Right, Bottom boundaries

2-D numerical example

For top-left corner 1,N

3) Energy balance on corner nodes



Both areas halved

$$\dot{q}^{R}_{1,N} + \dot{q}^{T}_{1,N} + \dot{q}^{B}_{1,N} = 0$$

$$\dot{q}^{R}_{1,N} = k_1 W \frac{dy}{2 \cdot dx} \left(T_{2,N} - T_{1,N}\right)$$

$$\dot{q}^{L}_{1,N} = 0$$

$$\dot{q}^{B}_{1,N} = k_1\, W \frac{dx}{2 \cdot dy} \left(T_{1,N-1} - T_{1,N}\right)$$

$$\dot{q}^{T}_{1,N} = \bar{\mathrm{h}}_{out}\, W \frac{dx}{2} \left(T_{out} - T_{1,N}\right) + \dot{q}'' W \frac{dx}{2}$$

Repeat for top-right, bottom-left, bottom-right corners

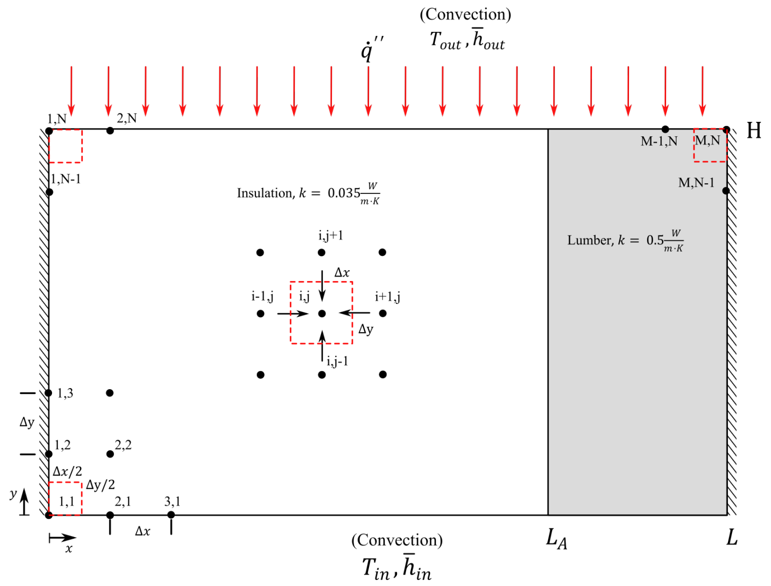
Yields a system of equations $M \times N$

Lecture 9

0-D Transient Analysis (Non-Steady-State)

Last time…

Numerical solutions to 2-D steady-state conduction



1. Discretize the domain into $\mathrm{M} \times N$ finite CV's
2. Perform energy balance on internal nodes $i = 2 \ldots M-1;\;\; j = 2 \ldots N-1$
3. Perform energy balance on top, bottom, left, right edges
4. Perform energy balance on top-left, top-right, bottom-left, bottom-right corners
5. Solve system of algebraic equations for temperature at each node
6. Verify solution

Non-steady state problems

T & $\dot{q}$ can vary w/ time

$T(\underbrace{x, y, z}_{\text{uniform/unimportant}}, t) \rightarrow T(t) \rightarrow$ Lumped capacitance

↳ uniform temp. in space, not in time

= "0-D" problem

$$Bi = \frac{R_{cond,int}}{R_{surr}} < 0.1$$

$$R_{cond,int} = \frac{L_{cond}}{k A_{cond}}$$

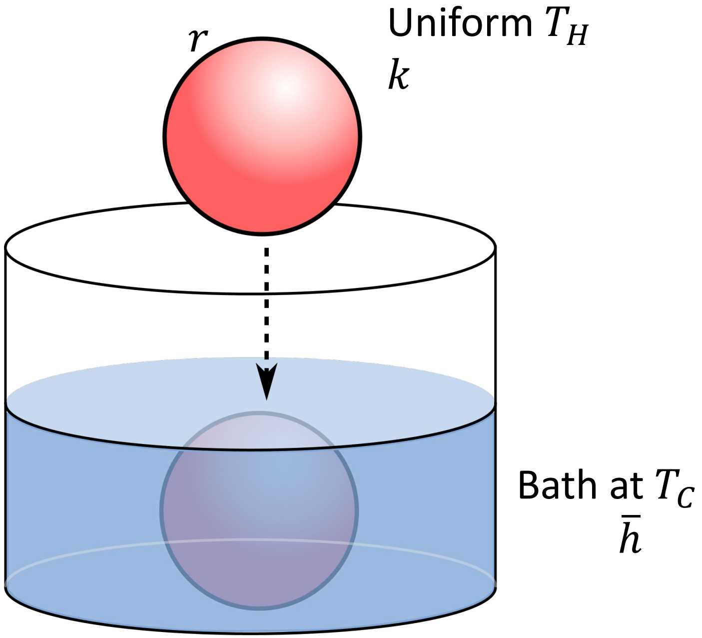
$$L_{cond} = \frac{Vol}{A_s}$$

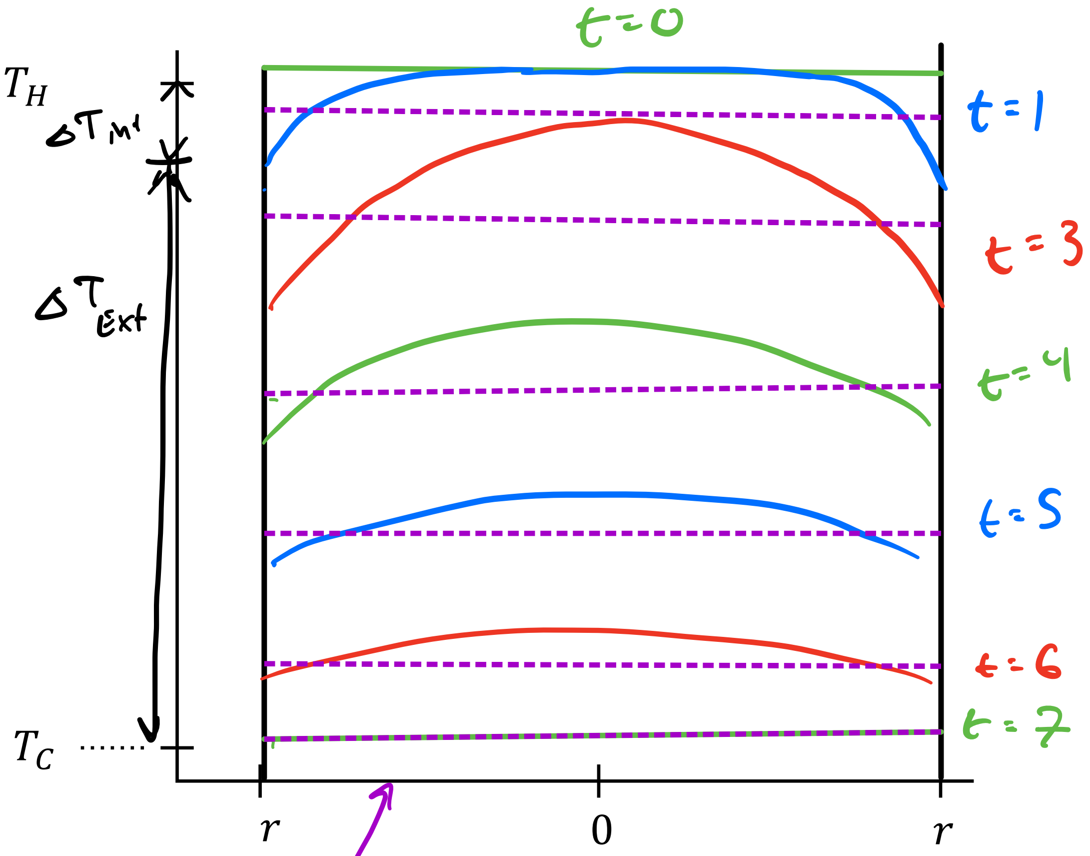
$$A_{cond} = A_s$$

→ sphere:

$$L_{cond} = \frac{4\pi r_0^3}{3} \cdot \frac{1}{4\pi r_0^2} = \frac{r_0}{3}$$

$$Bi = \frac{r_0}{3kA_s} \cdot \frac{\bar{h} A_s}{1} = \boxed{\frac{r_0 \bar{h}}{3k}}$$





avg temp. lines, error characterized by biot

Lumped capacitance time constant

$$\tau_{LC} = R_{surr} \cdot C_{obj}$$

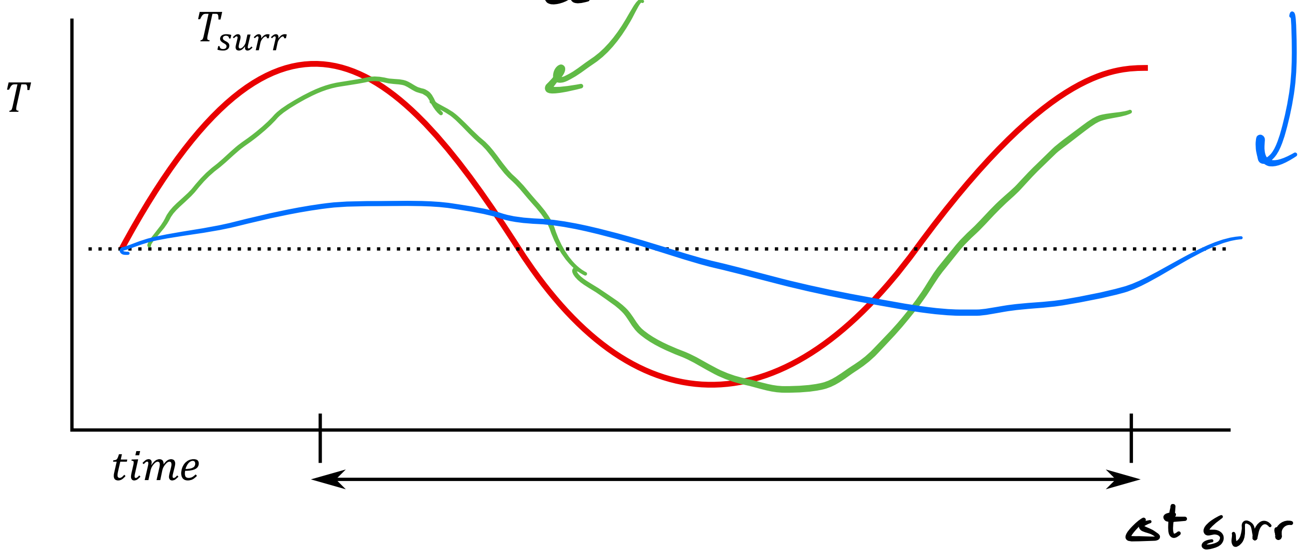
res. to surr.

thermal capacitance of object

$$\tau_{LC} = \frac{1}{\bar{h} A_s} \cdot mc$$

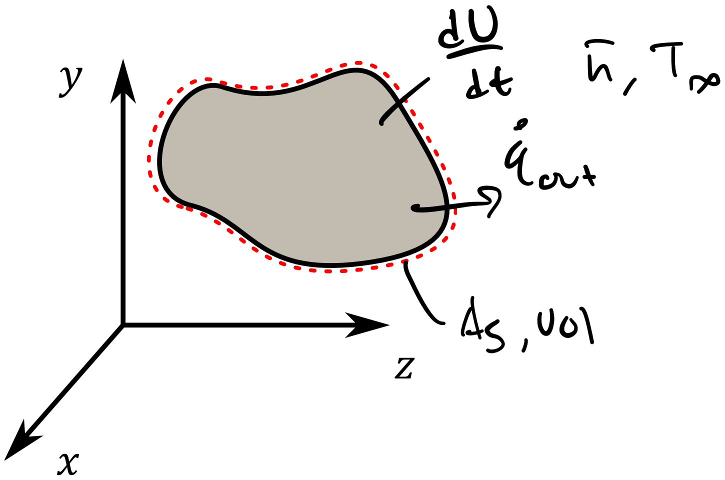
$$\left[\frac{\cancel{m^2}\cancel{K}}{W}\right]\left[\frac{1}{\cancel{m^2}}\right]\left[\cancel{kg}\right]\left[J/\cancel{kg}\text{-}\cancel{K}\right]$$

$\tau_{LC} << \Delta t_{surr}$ $\qquad$ $\tau_{LC} >> \Delta t_{surr}$



0-D Transient Problem Analysis Approach

1) Control volume analysis



~~IN~~ + ~~GEN~~ = OUT + STORED

$$\dot{q}_{out} + \frac{dU}{dt} = 0$$

$$\dot{q}_{out} = \bar{h} A_s (T(t) - T_\infty)$$

$$\frac{dU}{dt} = \frac{d(mu)}{dt} = m\frac{du}{dT}\cdot\frac{dT}{dt} = mc\frac{dT}{dt}$$

$$\bar{h} A_s (T - T_\infty) + mc\frac{dT}{dt} = 0$$

$$\frac{dT}{dt} + \underbrace{\frac{\bar{h}A_s}{mc}}_{=\frac{1}{\tau_L}} T = \underbrace{\frac{\bar{h}A_s}{mc}} T_\infty$$

0-D Transient Problem Analysis Approach (1)

2) Solve the (non-homogeneous) ODE

$$\boxed{\frac{dT}{dt} + \frac{1}{\tau_{LC}} T = \frac{1}{\tau_{LC}} T_\infty}$$

need IC's instead of BC's

$$T(t=0) = T_{ini}$$

$$T = T_h + T_p$$

$$\frac{d}{dt}(T_h + T_p) + \frac{1}{\tau_{LC}}(T_h + T_p) = \frac{1}{\tau_{LC}} T_\infty$$

H) $$\frac{dT_h}{dt} + \frac{1}{\tau_{LC}} T_h = 0$$

$$\int \frac{dT_h}{dt} dt = -\int \frac{T_h}{\tau_{LC}} dt$$

$$\Rightarrow T_h = \exp\left(-\frac{t}{\tau_{LC}} + C_1\right)$$

$$= T_h = C_1 \exp\left(-\frac{t}{\tau_{LC}}\right)$$

Full soln

P) $$\frac{dT_p}{dt} + \frac{1}{\tau_{LC}} T_p = \frac{1}{\tau_{LC}} T_\infty$$

$\rightarrow$ GUESS $T_p = \text{const} = C_1$

$\therefore T_p = T_\infty$

0-D Transient Problem Analysis Approach (2)

2) Solve the (non-homogeneous) ODE ✓

3) Enforce the boundary condition

$$T = c_1 \exp\left(-\frac{t}{\tau_{LC}}\right) + T_\infty$$

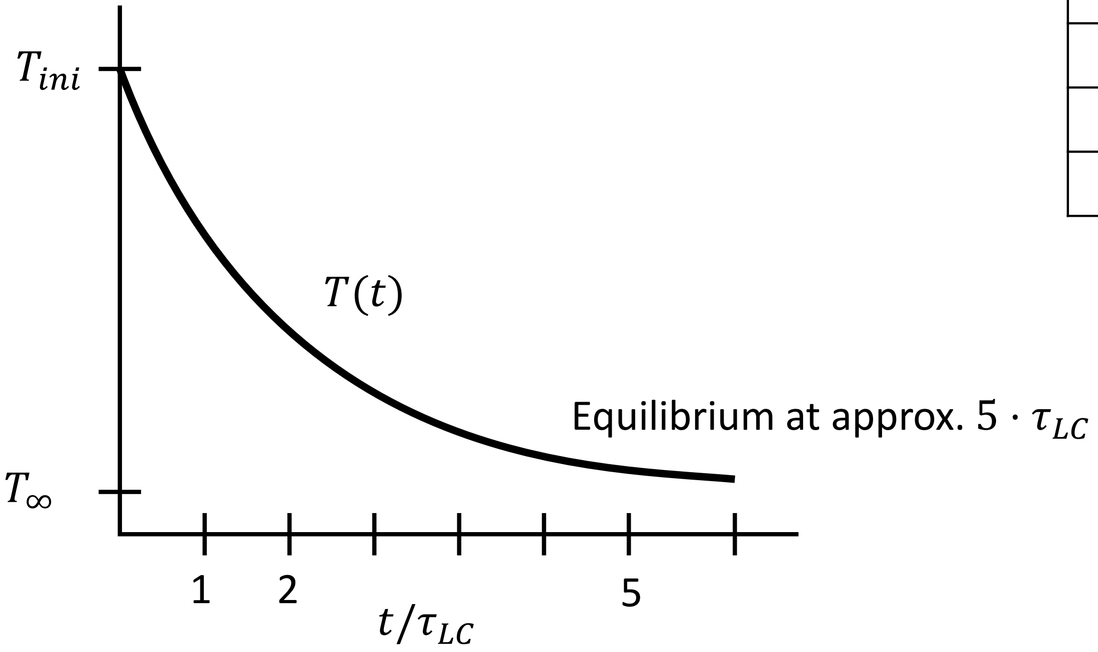
$$T_{ini} = c_1 \exp\left(-\frac{0}{\tau_{LC}}\right) + T_\infty = c_1 + T_\infty$$

$$\boxed{T = (T_{ini} - T_\infty)\exp\left(-\frac{t}{\tau_{LC}}\right) + T_\infty}$$

In terms of non-dimensional temperature:

$$T = (T_{ini} - T_\infty) \exp\left[-\frac{t}{\tau_{LC}}\right] + T_\infty$$

$$\frac{T - T_\infty}{T_{ini} - T_\infty} = \exp\left[-\frac{t}{\tau_{LC}}\right]$$

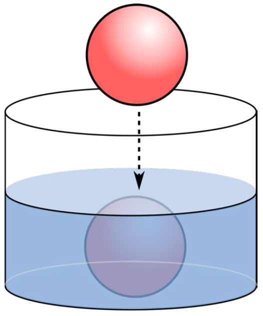


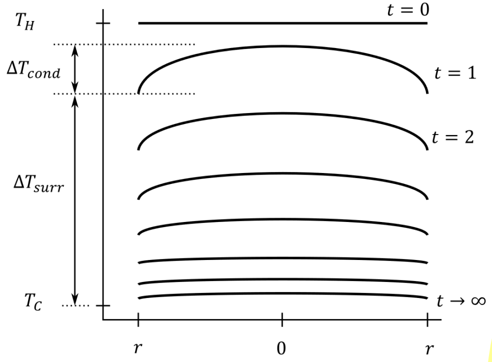
t/τ_{LC}	$\exp\left(-\frac{t}{\tau_{LC}}\right)$
1	0.367
2	0.135
3	0.050
5	0.007
10	0.00005

Lecture 10

1-D Transient Conduction

Last time…





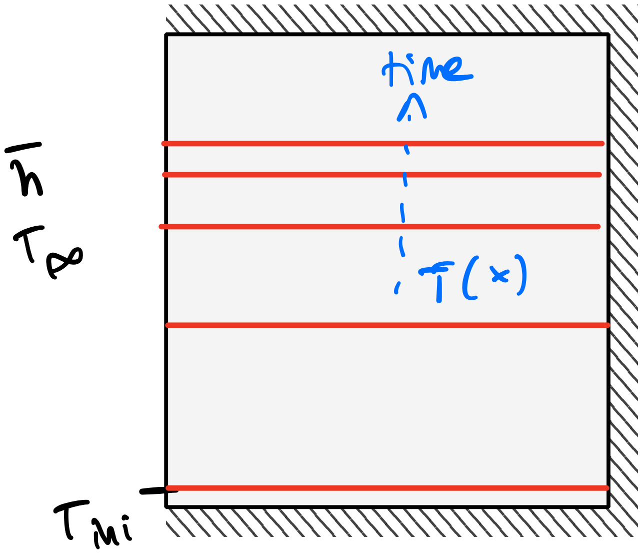
0-D Transient Analysis

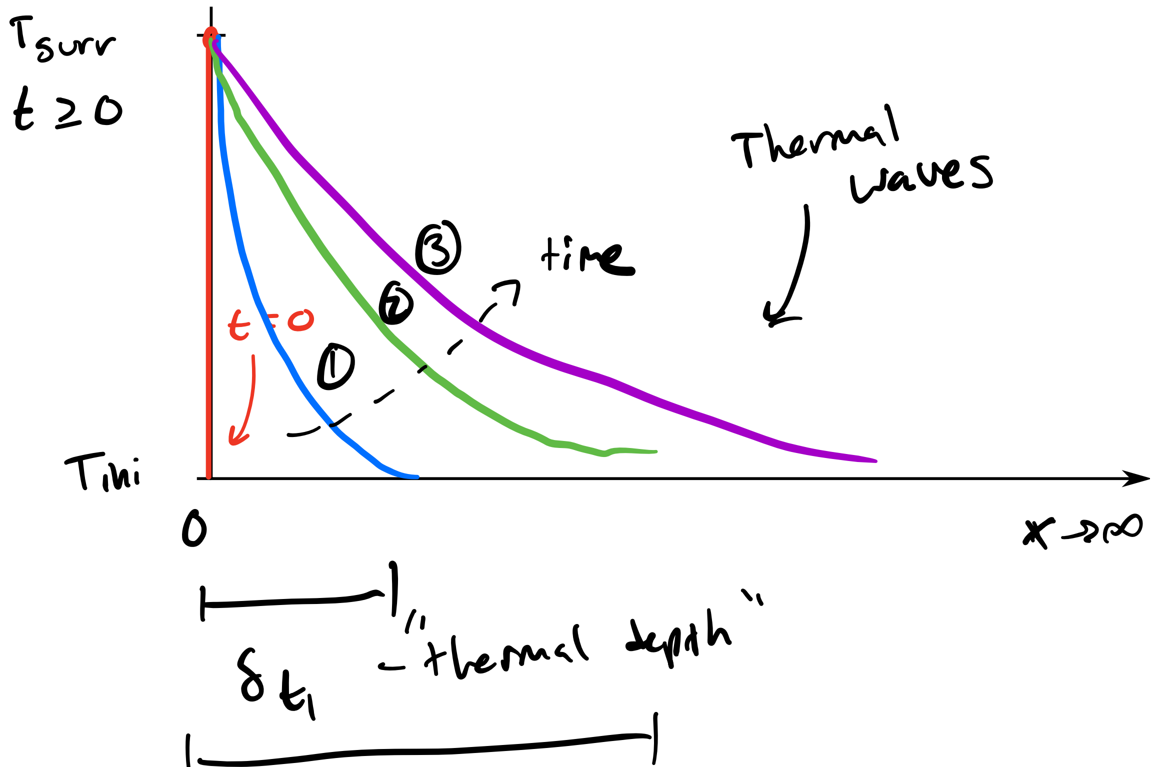
$L_{cond} = \frac{V}{A_s}$

- Biot number for lumped bodies: $Bi = \frac{R_{cond,int}}{R_{surr}}$ ← Justify assumption of uniform internal temp
- Lumped capacitance time constant: $\tau_{LC} = R_{surr} \cdot C_{obj}$ ← time for object to react
  - Equilibrium at $5 \cdot \tau_{LC}$ ← EQ Time
- Developed analytical model for temperature
  - Example: $T = (T_{ini} - T_\infty) \exp\left[-\frac{t}{\tau_{LC}}\right] + T_\infty$

1D Transient conduction

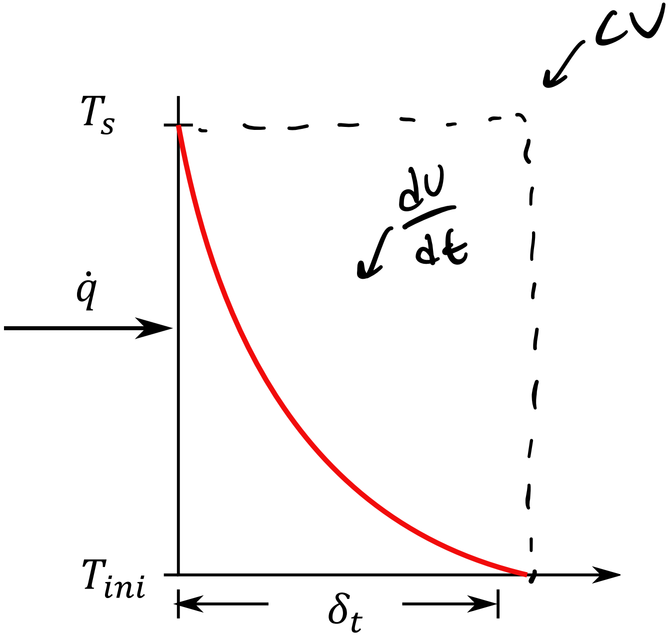
Semi-infinite plane wall





How long does it take thermal wave to be felt at some position x?

δt_2



$\dot{q} - \dfrac{dU}{dt} = 0$

$\dot{q} = \dfrac{T_s - T_{ini}}{R_{pw}}$, $\quad R_{pw} = \dfrac{\delta_t}{k A_c}$

$\dfrac{dU}{dt} = c\rho V \dfrac{dT}{dt}$, $\quad V = A_c \cdot \delta_t$, $\quad \dfrac{T_s - T_{ini}}{2}$

$$k A_c \left(\frac{T_s - T_{ini}}{\delta_t}\right) \approx \frac{d}{dt}\left[\frac{(T_s - T_{ini})}{2} \rho c \delta_t A_c\right]$$

$$\rightarrow \frac{2k}{\rho c} \approx \delta_t \cdot \frac{d\delta_t}{dt}$$

thermal diffusivity

$\alpha = k/\rho c \quad [\frac{m^2}{s}]$

$$2\alpha \approx \delta_t \frac{d\delta_t}{dt} \rightarrow \int 2\alpha\, dt \approx \int \delta_t\, d\delta_t$$

$$\rightarrow 2\alpha t \approx \delta_t^2/2 \Rightarrow \boxed{\delta_t \approx 2\sqrt{\alpha t}}$$

How far can a thermal wave travel in a given amount of time?

thermal depth not $\propto T$

$f(\rho, c, k)$

"L" $L = 2\sqrt{\alpha t} \rightarrow \boxed{t = \frac{L^2}{4\alpha}} \rightarrow \tau_{diff}$ [s]

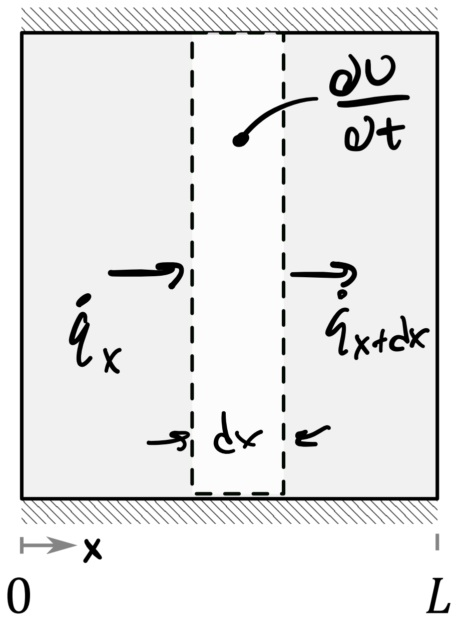
time for:

"internal thermal equilibrium"

Symbol	τ_{LC}	τ_{diff}
Describes	Temperature response to surroundings	Internal temperature equilibrium
Formula	$R_{surr} \cdot C_{obj}$	$\frac{L^2}{4\alpha}$
Response to forcing function	$1 \cdot \tau_{LC}$	$\frac{1}{5} \cdot \tau_{diff}$
Time to equilibrium	$5 \cdot \tau_{LC}$	$1 \cdot \tau_{diff}$

time response occurs

Analytical solutions to 1D transient problems



1) CV analysis

$$IN + \cancel{GEN}^{\,0} = OUT + STORED$$

2) E-bal

$$\dot q_x = \dot q_{x+dx} + \frac{\partial U}{\partial t}$$

$$\cancel{\dot q_x} = \cancel{\dot q_x} + \frac{\partial \dot q_x}{\partial x}dx + \frac{\partial U}{\partial t}$$

$$\frac{\partial \dot q_x}{\partial x}\cancel{dx} = -\rho c A_c \cancel{dx}\frac{\partial T}{\partial t}$$

Fourier's

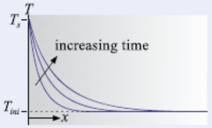
$$\frac{\partial}{\partial x}\left[\underset{-k}{\underline{-k}}\cancel{A_c}\frac{\partial T}{\partial x}\right] = \underbrace{\underset{-k}{\underline{-\rho}}}_{1/\alpha}\cancel{A_c}\frac{\partial T}{\partial t}$$

$$\rightarrow \boxed{\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}} \quad \text{Gov. PDE in } T$$

Table 6.1 Solutions to semi-infinite body problems.

Boundary condition | **Analytical solution and function in EES**

(a)

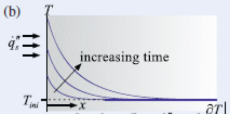


step change in surface temp.: $T_{x=0} = T_s$

$$\frac{T - T_{ini}}{T_s - T_{ini}} = 1 - \mathrm{erf}\left(\frac{x}{2\sqrt{\alpha\, t}}\right) \qquad \dot{q}''_{x=0} = \frac{k}{\sqrt{\pi\, \alpha\, t}}\left(T_s - T_{ini}\right)$$

T = SemiInf1(T_ini, T_s, alpha, x, time)

(b)

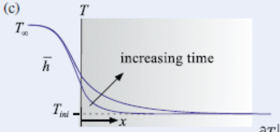


surface heat flux: $\dot{q}''_s = -k \left.\frac{\partial T}{\partial x}\right|_{x=0}$

$$T - T_{ini} = \frac{\dot{q}''_s}{k}\left[2\sqrt{\frac{\alpha\, t}{\pi}}\exp\left(-\frac{x^2}{4\alpha\, t}\right) - x\, \mathrm{erfc}\left(\frac{x}{2\sqrt{\alpha\, t}}\right)\right]$$

T = SemiInf2(T_ini, q``_dot_s, k, alpha, x, time)

(c)

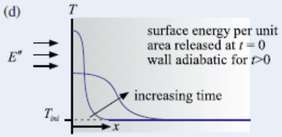


convection to fluid: $\bar{h}\left(T_\infty - T_{x=0}\right) = -k \left.\frac{\partial T}{\partial x}\right|_{x=0}$

$$\frac{T - T_{ini}}{T_\infty - T_{ini}} = \mathrm{erfc}\left(\frac{x}{2\sqrt{\alpha\, t}}\right) - \exp\left(\frac{\bar{h}\, x}{k} + \frac{\bar{h}^2\, \alpha\, t}{k^2}\right)\mathrm{erfc}\left(\frac{x}{2\sqrt{\alpha\, t}} + \frac{\bar{h}}{k}\sqrt{\alpha\, t}\right)$$

T = SemiInf3(T_ini, T_infinity, h_bar, k, alpha, x, time)

(d)

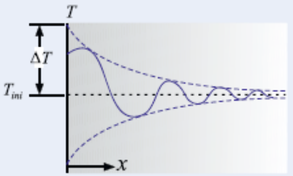


surface energy pulse: $\lim_{t, \Delta t \to 0} \dot{q}_s \Delta t = E''$

$$T - T_{ini} = \frac{E''}{\rho\, c \sqrt{\pi\, \alpha\, t}}\exp\left(-\frac{x^2}{4\alpha\, t}\right)$$

T = SemiInf4(T_ini, E``, rho, c, alpha, x, time)

(e)



sinisoidal surface temp. at quasi-steady state

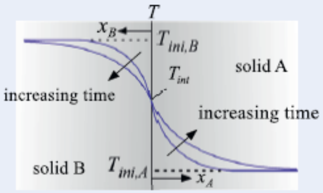
$T_{x=0} = T_{ini} + \Delta T \sin(\omega t)$

$$T - T_{ini} = \Delta T \exp\left(-x\sqrt{\frac{\omega}{2\alpha}}\right)\sin\left(\omega t - x\sqrt{\frac{\omega}{2\alpha}}\right)$$

$$\dot{q}''_{x=0} = k \Delta T \sqrt{\frac{\omega}{\alpha}}\sin\left(\omega t + \frac{\pi}{4}\right)$$

T = SemiInf5(T_ini, DeltaT, omega, alpha, x, time)

(f)



contact between two semi-infinite solids

$$T_{int} = \frac{\sqrt{k_A\, \rho_A\, c_A}\, T_{ini,A} + \sqrt{k_B\, \rho_B\, c_B}\, T_{ini,B}}{\sqrt{k_A\, \rho_A\, c_A} + \sqrt{k_B\, \rho_B\, c_B}}$$

$$\frac{T_A - T_{ini,A}}{T_{int} - T_{ini,A}} = 1 - \mathrm{erf}\left(\frac{x_A}{2\sqrt{\alpha_A t}}\right),$$

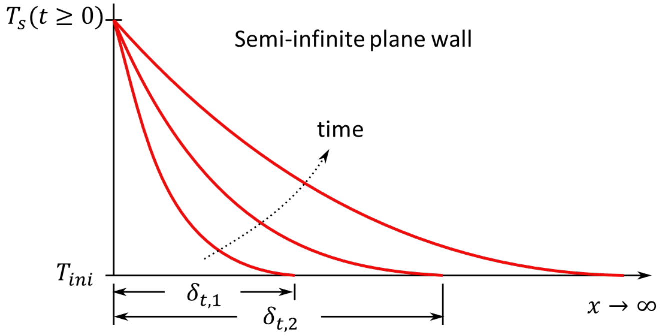
$$\frac{T_B - T_{ini,B}}{T_{int} - T_{ini,B}} = 1 - \mathrm{erf}\left(\frac{x_B}{2\sqrt{\alpha_B t}}\right)$$

T_{ini} = initial temperature	t = time relative to surface disturbance	k = conductivity
x = position from surface	ρ = density	c = specific heat capacity
α = thermal diffusivity	erf() = Gaussian error function	erfc() = complementary error function

Lecture 11

Numerical solutions to transient problems

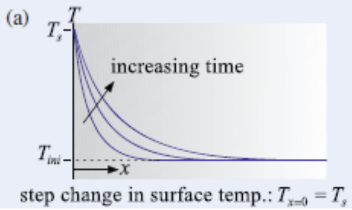
Last time…



1-D Transient Analysis

approx. model

- Derived thermal wave depth in a plane wall: $\delta_t \approx 2\sqrt{\alpha\, t}$
- Diffusive time constant τ_{diff} quantifies time to internal equilibrium ($\tau_{LC} \rightarrow$ external equilibrium)
- Derived governing PDE for 1-D transient conduction (hard to solve analytically!) $\alpha \dfrac{\partial^2 T}{\partial x^2} = \dfrac{\partial T}{\partial t}$
- Introduced 1D transient problem solutions in EES
    - Make sure BC's match your problem! ← can use
    - Numerical solutions often required

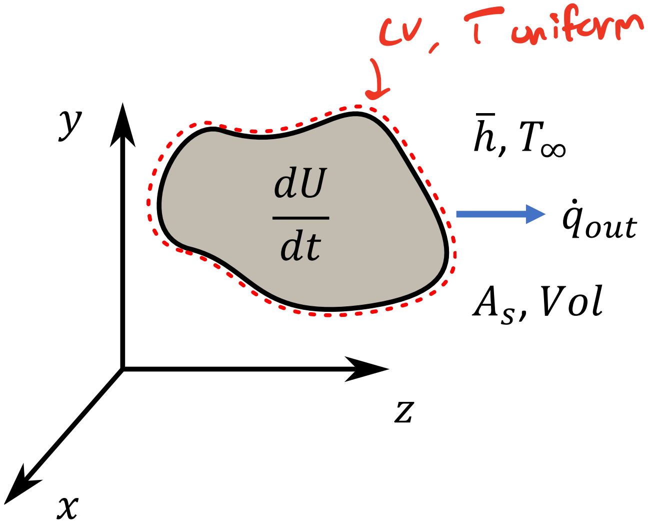


$$\frac{T - T_{ini}}{T_s - T_{ini}} = 1 - \text{erf}\left(\frac{x}{2\sqrt{\alpha\, t}}\right) \qquad \dot{q}''_{x=0} = \frac{k}{\sqrt{\pi\, \alpha\, t}}(T_s - T_{ini})$$

T = SemiInf1(T_ini, T_s, alpha, x, time)

Numerical solutions to transient problems

Recall 0D problem



$$\underbrace{\frac{dT}{dt} + \frac{1}{\tau_{LC}} T = \frac{1}{\tau_{LC}} T_\infty}_{\text{gov. diff. eq.}} \rightarrow \frac{dT}{dt} = f(T, t)$$

State equation: relates time rate of change of a variable (T) to the system state (current t, T)

- Discretize the time domain

$$0 \leq t \leq t_{sim}$$

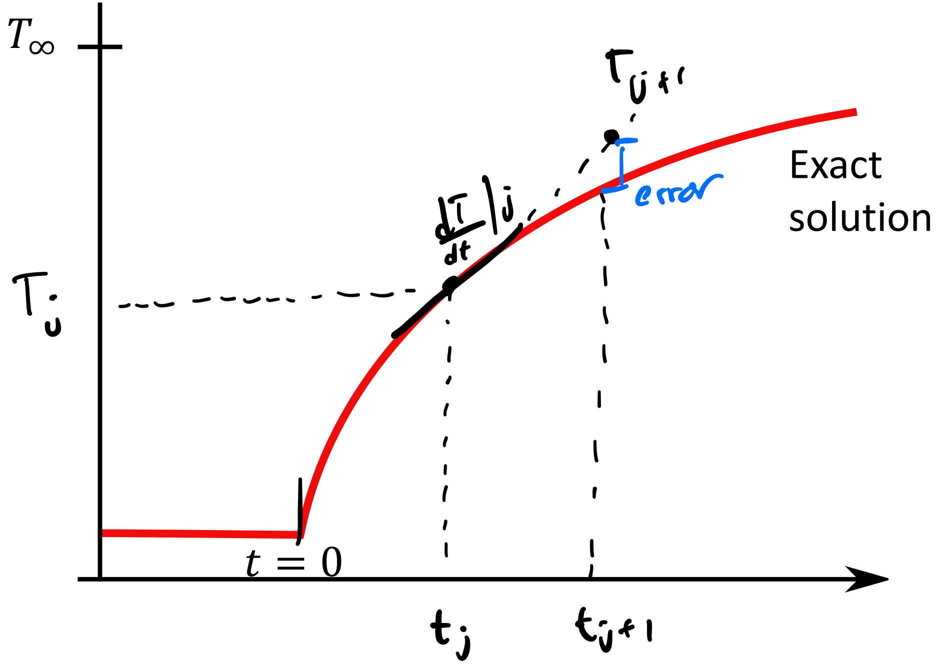
$$N_{steps} \rightarrow \Delta t = \frac{t_{sim}}{N-1}$$

$$t_j = (j-1)\Delta t \quad \forall j \in 1 \ldots N$$

BC: Initial temp $:= T_{ini}$

$$\rightarrow T_1 = T_{ini}$$

Simplest method: Forward finite difference (Euler's method)



$$\left.\frac{dT}{dt}\right|_1 = \frac{(T_2 - T_1)}{\Delta t} \rightarrow \left.\frac{dT}{dt}\right|_j = \frac{(T_{j+1} - T_j)}{\Delta t}$$

$$T_{j+1} = \underbrace{T_j}_{\text{current}} + \overbrace{\underbrace{\left.\frac{dT}{dt}\right|_j}_{\text{deriv}} \cdot \underbrace{\Delta t}_{\text{timestep}}}^{\text{Projection}}$$

$$\frac{dT}{dt} + \frac{T}{\tau_{Lc}} = \frac{T_\infty}{\tau_{Lc}}$$

depends on air

$$\left.\frac{dT}{dt}\right|_j = \frac{(T_\infty - T_j)}{\tau_{Lc}}$$

state eqn

$$T_{j+1} = T_j + \frac{\Delta t}{\tau_{LC}}\left(T_\infty - T_j\right) \quad \forall\, j \in (1:N-1)$$

features of Euler's method:

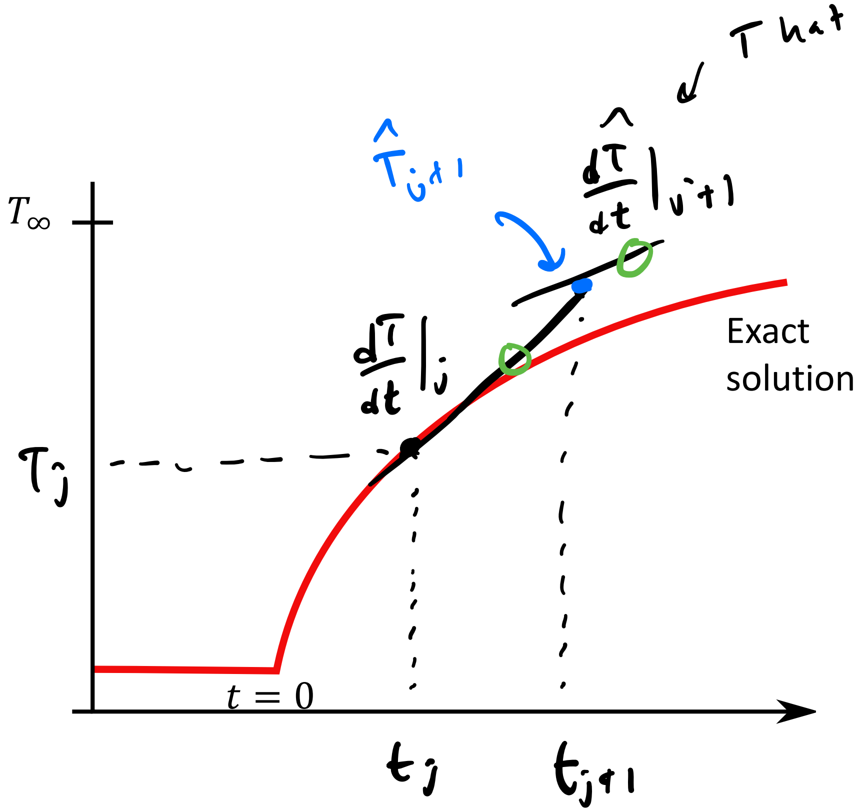
- N equations, N unknowns
- Explicit (know all information)
- Prone to oscillation if $\Delta t \leq \Delta t_{crit}$
- Requires small time steps → usually computationally expensive

$$T_{j+1} = T_j - \frac{\Delta t}{\tau_{LC}} T_j + \frac{\Delta t}{\tau_{LC}} T_\infty \rightarrow T_j \underbrace{\left[1 - \frac{\Delta t}{\tau_{LC}}\right]}_{\text{Must be } > 0 \text{ to be stable}} + \frac{\Delta t}{\tau_{LC}} T_\infty$$

$$\rightarrow \Delta t \leq \underbrace{\tau_{LC}}_{t_{crit}}$$

τ_{LC} for entire domain here (0-D)

Predictor-corrector method (Heun's)



$$\left(\frac{dT}{dt}\Big|_j + \frac{d\hat{T}}{dt}\Big|_{j+1}\right)\frac{1}{2}$$

$$T_{j+1} = T_j + \left[\frac{dT}{dt}\Big|_j + \frac{d\hat{T}}{dt}\right]\frac{1}{2}\cdot\Delta t$$

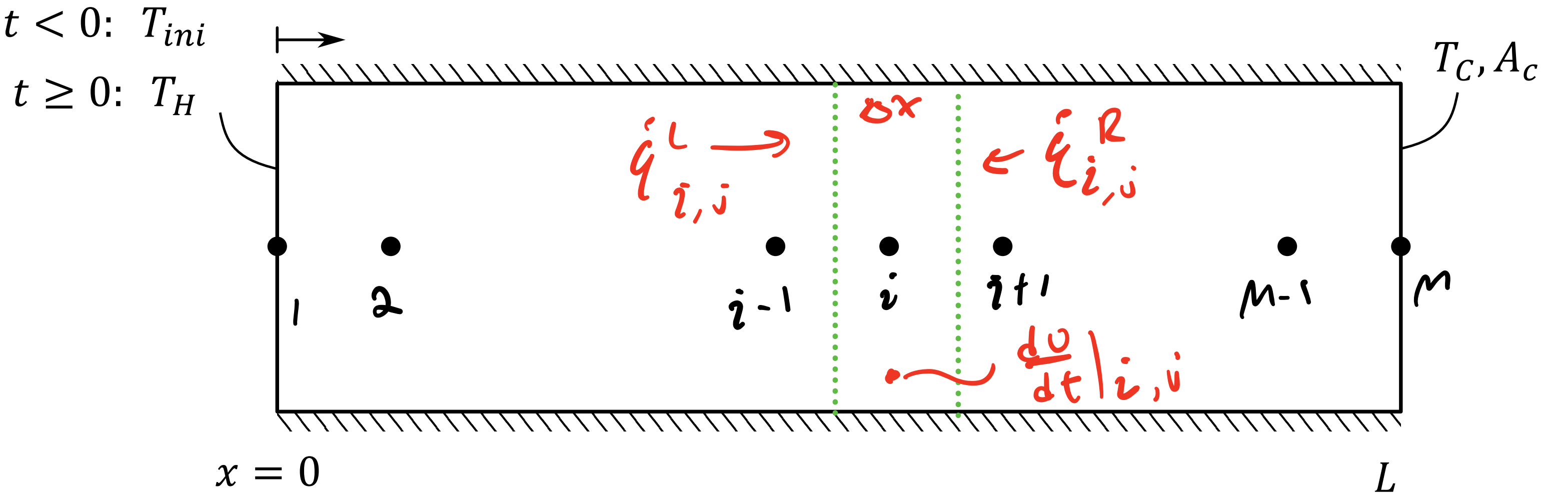
$$\frac{dT}{dt}\Big|_j = \frac{T_\infty - T_j}{\tau_{LC}}$$

$$\hat{T}_{j+1} = T_j + \frac{dT}{dt}\Big|_j \cdot \Delta t$$

$$\frac{d\hat{T}}{dt}\Big|_{j+1} = \frac{T_\infty - \hat{T}_{j+1}}{\tau_{LC}}$$

- Good stability w/ this technique
  → Reducing the step reduces error

Extending to 1D numerical solutions



Discretize domain: $\Delta x = \frac{L}{M-1}$, $x_i = \Delta x(i-1) \;\; \forall\, i \in (1:M)$

$\Delta t = \frac{t_{sim}}{N-1}$, $t_j = \Delta t(j-1) \;\; \forall\, j \in (1:N)$

Energy balance (Internal nodes)

$$\dot{q}^{L}_{i,j} + \dot{q}^{R}_{i,j} - \frac{dU}{dt}\Big|_{i,j} = 0$$

combine & solve for $\frac{dT}{dt}$

Rates:

$$\dot{q}^{L}_{i,j} = \frac{kA_c}{\Delta x}\left(T_{i-1,j} - T_{i,j}\right)$$

$$\dot{q}^{R}_{i,j} = \frac{kA_c}{\Delta x}\left(T_{i+1,j} - T_{i,j}\right)$$

$$\frac{dU}{dt}\Big|_{i,j} = \rho A_c \Delta x\, c \frac{dT}{dt}\Big|_{i,j}$$

$$\frac{dT}{dt}\Big|_{ij} = \frac{kA_c}{\Delta x \underbrace{\rho A_c \Delta x}_{\text{mass}} c}\left(T_{i-1,j} + T_{i+1,j} - 2T_{ij}\right)$$

state eqn

$$\boxed{\frac{dT}{dt}\Big|_{ij} = \frac{\alpha}{\Delta x^2}\left(T_{i-1,j} + T_{i+1,j} - 2T_{ij}\right)}$$

$$T_{i,j+1} = T_{ij} + \frac{dT}{dt}\Big|_{ij} \cdot \Delta t$$

$$\forall\, i \in (2:M-1);\ \forall\, j \in (1:N-1)$$

Re at for end nodes

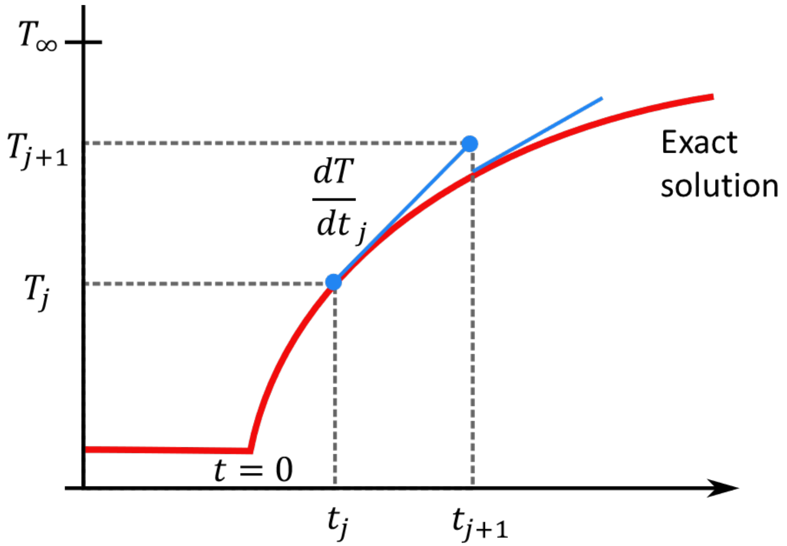
Lecture 12

Introduction to Convection: Laminar Flow

Last time…

Numerical Solutions to Transient Problems

- Estimate change in temperature using state equation
  - Example: $\left.\frac{dT}{dt}\right._j = \frac{T_\infty - T_j}{\tau_{LC}}$



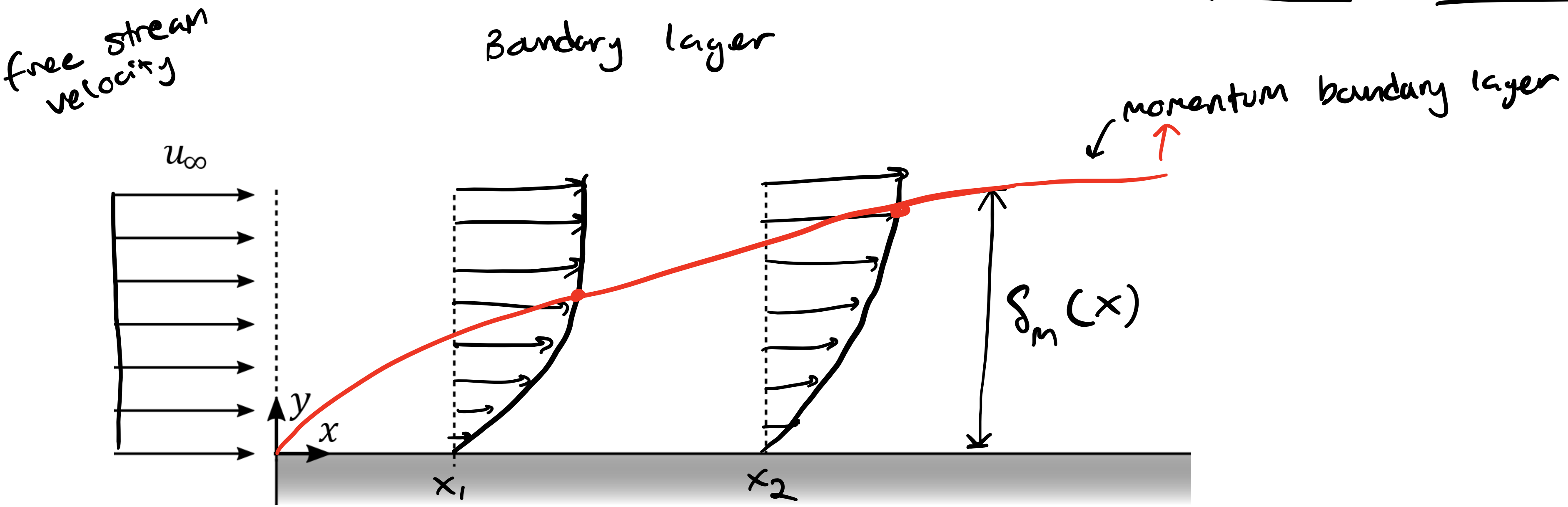
- Discussed methods for stepping forward in time
  - Euler's method (explicit): $$T_{j+1} = T_j + \left.\frac{dT}{dt}\right|_j \Delta t$$
  - Heun's method (predictor-corrector) $$T_{j+1} = T_j + \left[\left.\frac{dT}{dt}\right._j + \left.\frac{d\hat{T}}{dt}\right._{j+1}\right]\frac{\Delta t}{2}$$
- Can extend to 1-D transient

Convection and the rate equation

$$\dot{q}_{conv} = \bar{h} A_s (T_s - T_\infty)$$

Not a fluid property

Define convection: Heat transport between a solid surface and a fluid in motion via conduction and advection



Velocity boundary layer: Region near surface where fluid velocity deviates significantly from free stream velocity

$$u(y) \leq 0.99\, u_\infty$$

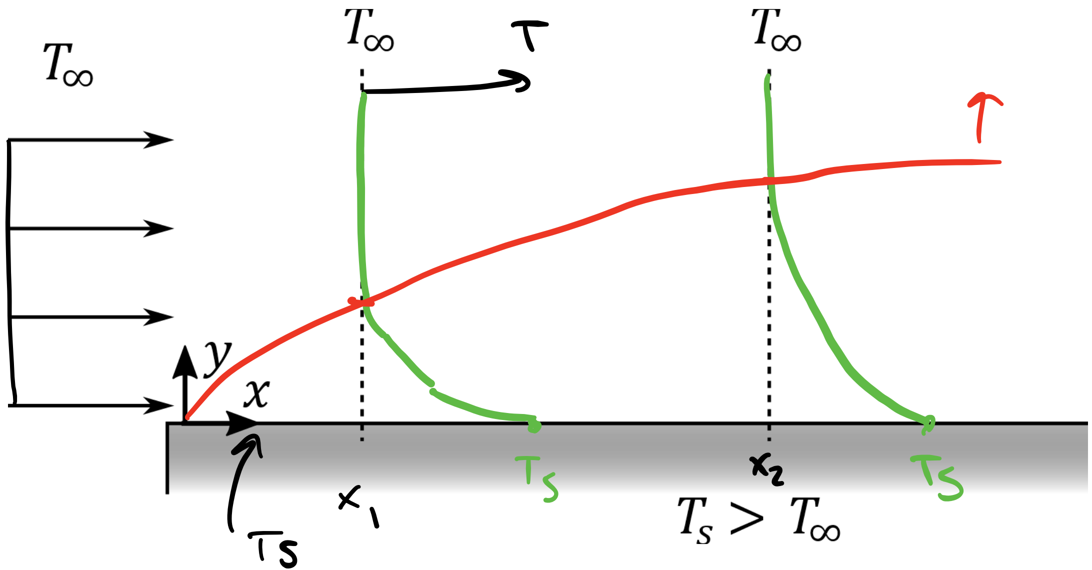
Viscous shear:

$$\tau = \mu \frac{\partial u}{\partial y} \approx \mu \frac{u_\infty}{\delta_M(x)}$$

Thermal boundary layer:

$T_s > T_\infty$

$$\frac{T_s - T(y)}{T_s - T_\infty} \geq 0.01$$



Estimating boundary layer growth

$\delta_t \approx 2\sqrt{\alpha t} \rightarrow t \approx \frac{x}{u_\infty}$

$\delta_t \approx 2\sqrt{\alpha \cdot \frac{x}{u_\infty}}$

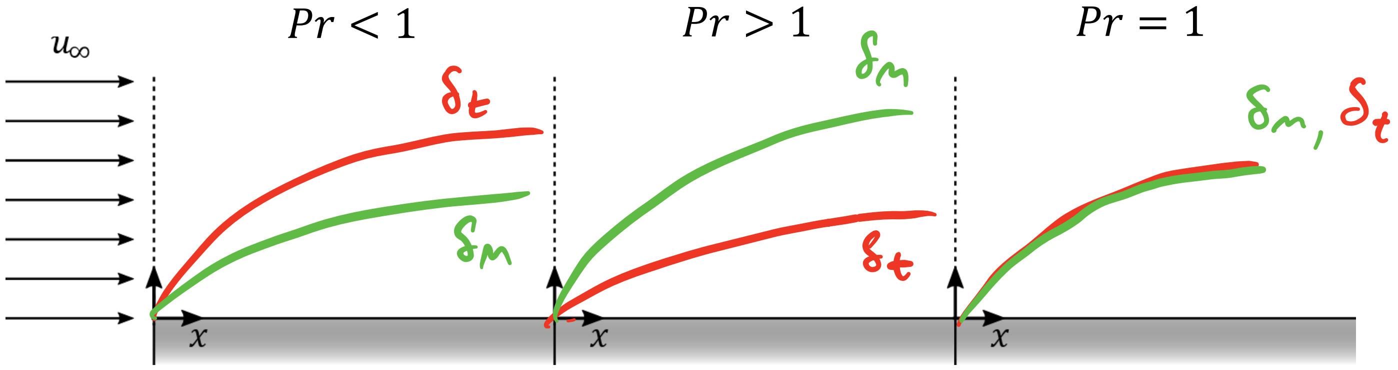
for momentum, $\nu = \frac{\mu}{\rho} \rightarrow$ kinematic viscosity $\left[\frac{m^2}{s}\right]$

$\delta_m \approx 2\sqrt{\nu \frac{x}{u_\infty}}$

Define: Prandtl number

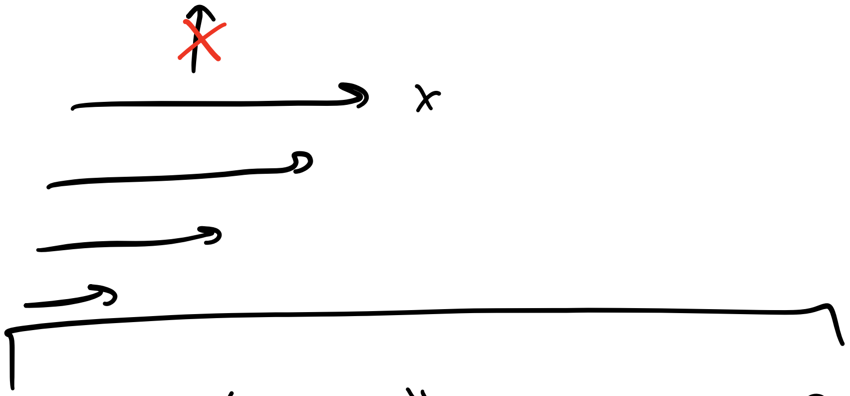
$$Pr = \frac{\text{momentum transport}}{\text{heat transport}} = \left(\frac{\delta_m}{\delta_t}\right)^2 = \frac{\left(2\sqrt{\nu\, x/u_\infty}\right)^2}{\left(2\sqrt{\alpha \frac{x}{u_\infty}}\right)^2} = \frac{\nu}{\alpha} = \frac{\mu c}{k}$$

$$\boxed{Pr = \frac{\mu c}{k}}$$



Estimating heat transfer coefficient $\bar{h}$

For laminar flow, $\dot{q}_y''$ is conduction



"laminar" → sheets of fluid

$$\dot{q}_y'' = -k\frac{dT}{dy} \approx k\frac{\Delta T}{\Delta y}$$

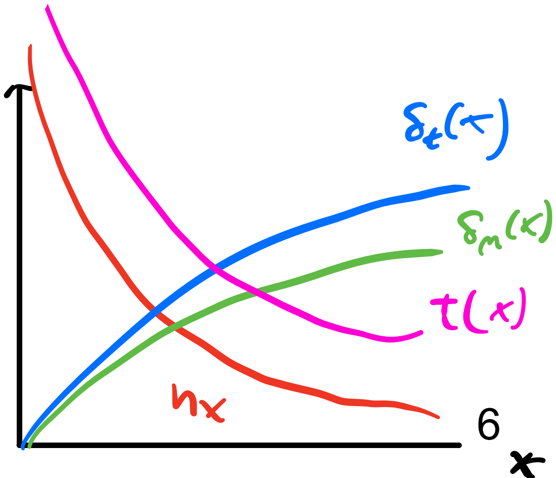
$$R''_{conv} = \frac{1}{\bar{h}\cancel{A}}$$

$$\dot{q}_y'' \approx k\frac{(T_s - T_\infty)}{\delta_t(x)}$$

$$\dot{q}'' \approx \frac{(T_s - T_\infty)}{R_{conv}}$$

$$\bar{h} = \frac{k}{\bar{\delta}_t} \rightarrow \downarrow h_x \approx \frac{k}{\delta_t(x)} \uparrow$$

relationship between fluid properties & BL thickness



Reynolds #

$$Re = \frac{\rho U_\infty L_{char}}{\mu}$$ ← char. length

→ flow over flat plate

$$\underbrace{Re_x}_{} = \frac{\rho U_\infty \cdot x}{\mu}$$

Momentum BL

$$\delta_M \approx 2\sqrt{\nu \cdot \frac{x}{U_\infty}} \rightarrow \frac{\delta_M}{x} \approx \frac{2}{x}\sqrt{\nu\frac{x}{U_\infty}} = 2\sqrt{\frac{\nu}{xU_\infty}}$$

$$\nu = \frac{\mu}{\rho} \rightarrow \frac{\delta_M}{x} \approx 2\sqrt{\underbrace{\frac{\mu}{x \cdot U_\infty \cdot \rho}}_{1/Re_x}} \rightarrow \delta_M \approx \frac{2x}{\sqrt{Re_x}}$$

Self-similar model

$$\boxed{\delta_M = \frac{4.916\ x}{Re_x^{0.5}}}$$

Thermal BL:

$$\delta_t \approx \frac{2x}{\sqrt{Re_x \cdot Pr}} \rightarrow \boxed{\delta_t = \frac{4.916\ x}{Re_x^{0.5}\ Pr^{1/3}}}$$

Non-dimensional expressions for τ and h_x

$$\tau = \underbrace{\tfrac{1}{2}\rho u_\infty^2}_{\text{momentum}} C_f \qquad \text{use: } \tau \approx \mu \frac{u_\infty}{\delta_m}, \quad \delta_m \approx \frac{2x}{\sqrt{Re_x}}$$

$$C_f \approx 2\frac{\mu}{\rho u_\infty \delta_m}$$

$$C_f \approx \frac{2\mu}{\rho u_\infty}\frac{\sqrt{Re_x}}{2x} \rightarrow C_f \approx \frac{\mu}{\rho u_\infty x}\sqrt{Re_x} \rightarrow C_f \approx \frac{1}{\sqrt{Re_x}}$$

approx.

$$\boxed{C_f = \frac{0.664}{\sqrt{Re_x}}}$$

exact

Nusselt # (non-dimensional HTC)

$$Nu_x = \frac{h_x \cdot x}{k} \qquad \text{since: } h \approx \frac{k}{\delta_t} \qquad \delta_t \approx \frac{2x}{\sqrt{Re}\sqrt{Pr}}$$

$$\rightarrow \frac{x}{Nu_x} \approx \frac{2x}{\sqrt{Re}\sqrt{Pr}} \rightarrow Nu_x \approx \frac{1}{2}\sqrt{Re \cdot Pr}, \quad \boxed{Nu_x = 0.332\, Re^{0.5} \cdot Pr^{1/3}}$$

approx

exact

Reynolds analogy

$$Nu_x \approx \frac{x}{\delta_t}, \quad \delta_t \approx \frac{x}{Nu_x}$$

$$C_f \approx \frac{2\mu}{\rho u_\infty \delta_m} \rightarrow \delta_m \approx \frac{2\mu}{\rho u_\infty C_f}$$

$$\delta_t = \delta_m \quad (Pr \approx 1) \rightarrow \frac{x}{Nu_x} \approx \frac{2\mu}{\rho u_\infty C_f}$$

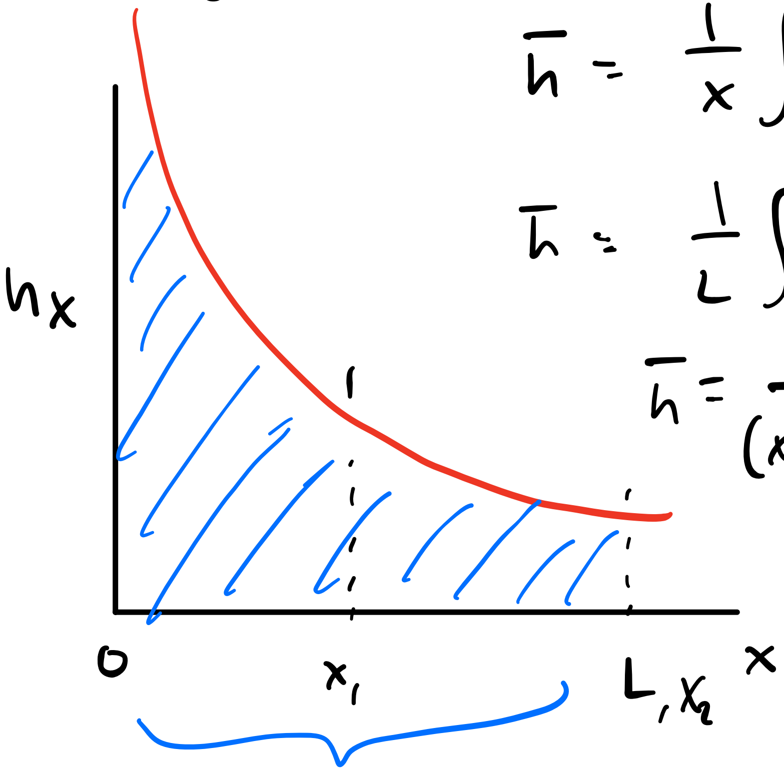
$$Nu_x \approx \underbrace{\frac{\rho u_\infty x}{\mu}}_{Re_x} \cdot C_f \rightarrow Nu_x \approx \frac{C_f Re}{2}$$

better

Chilton-Colburn

$$\boxed{Nu_x \approx \frac{Pr^{1/3} C_f Re_x}{2}}$$

Local versus average values



$$\bar{h} = \frac{1}{x}\int_0^x h_x\,dx$$

avg HTC

$$\bar{h} = \frac{1}{L}\int_0^L h_x\,dx$$

$$\bar{h} = \frac{1}{(x_2 - x_1)}\int_{x_1}^{x_2} h_x\,dx$$

$k \rightarrow f(T)$

evaluate properties @ T_{film}

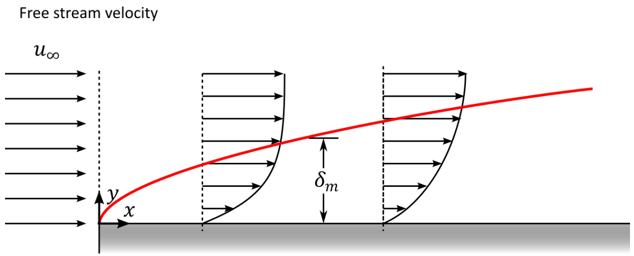
$$T_{film} = \frac{T_s + T_\infty}{2}$$

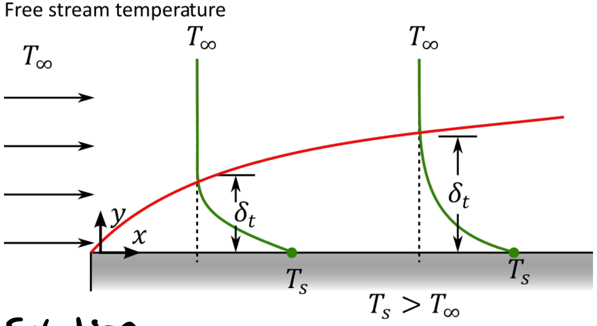
Lecture 13

Non-dimensional Parameters and Turbulent Flow

Last time…

Introduction to Convection: Laminar Flow





- Momentum (velocity) and thermal boundary layers develop near surface
- Prandtl number relates fluid's ability to transport heat and momentum

$$Pr = \frac{\nu}{\alpha} = \frac{\mu\, c}{k}$$

- H.T.C. related to BL thickness $\quad h_x \approx \dfrac{k}{\delta_{t,x}}$
- Exact models for BL thickness
- Non-dimensional HTC and shear stress

"Self-Similar" Solution ✓

$$\delta_m = \frac{4.916\, x}{\sqrt{Re_x}} \qquad \delta_t = \frac{4.916\, x}{\sqrt{Re_x} \cdot Pr^{1/3}}$$

$$Nu_x = 0.332\ \mathrm{Re}_x^{0.5}\ \mathrm{Pr}^{0.333} \qquad C_{f,x} = \frac{0.664}{\sqrt{Re_x}}$$

Local; flow over external flat plate

Dimensionless variables

what we're looking for in correlations

$$C_f = C_f\left(\tilde{x}, Re_L, \frac{\partial \tilde{P}}{\partial \tilde{x}}\right)$$

$$\bar{C}_f = \bar{C}_f\left(Re_L, \frac{\partial \tilde{P}}{\partial \tilde{x}}\right)$$

neglect

$$Nu = Nu\left(\tilde{x}, Re_L, \frac{\partial \tilde{P}}{\partial \tilde{x}}, Pr, E_c\right)$$

$$\bar{Nu} = \bar{Nu}\left(Re_L, \frac{\partial \tilde{P}}{\partial \tilde{x}}, Pr, E_c\right)$$

continuity: $\frac{\partial \tilde{u}}{\partial \tilde{x}} + \frac{\partial \tilde{v}}{\partial \tilde{y}} = 0$ $\quad \left(\tilde{u} = \frac{u}{u_\infty}\right)$

x-mom: $\tilde{u}\frac{\partial \tilde{u}}{\partial \tilde{x}} + \tilde{v}\frac{\partial \tilde{u}}{\partial \tilde{y}} = -\frac{d\tilde{p}_\infty}{d\tilde{x}} + \frac{1}{Re_L}\frac{\partial^2 \tilde{u}}{\partial \tilde{y}^2}$

energy: $\tilde{u}\frac{\partial \tilde{\theta}}{\partial \tilde{x}}$ ...

"Eckert #" viscous dissipation

$$E_c = \frac{u_\infty^2}{c(T_\infty - T_s)} \rightarrow \text{if } \frac{E_c}{Re} \text{ is small}$$

↳ neglect viscous dissipation

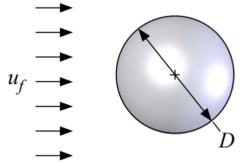
"Peclet #"

$$Pe = Re \cdot Pr$$

if Pe small, cannot neglect conduction in direction of flow (slow flow, cond in x)

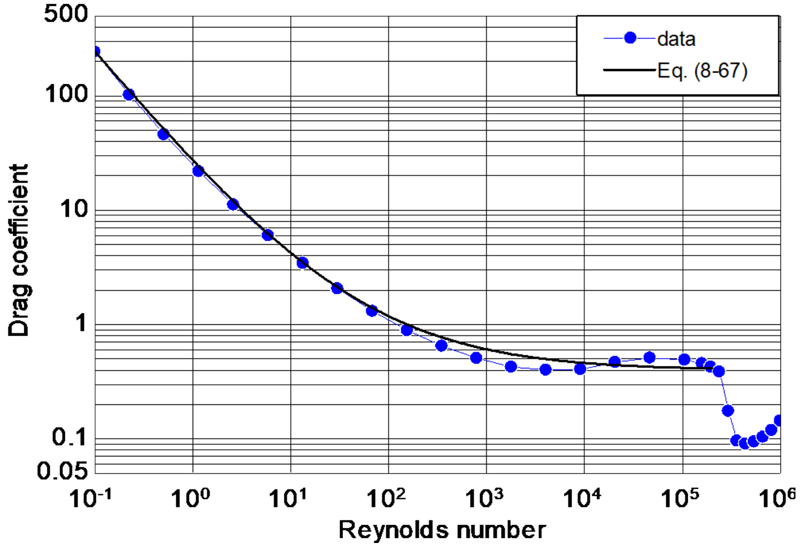
(neglect when Pe "large")

$$C_D = \frac{2F_D}{\rho u_\infty^2 A_p}$$



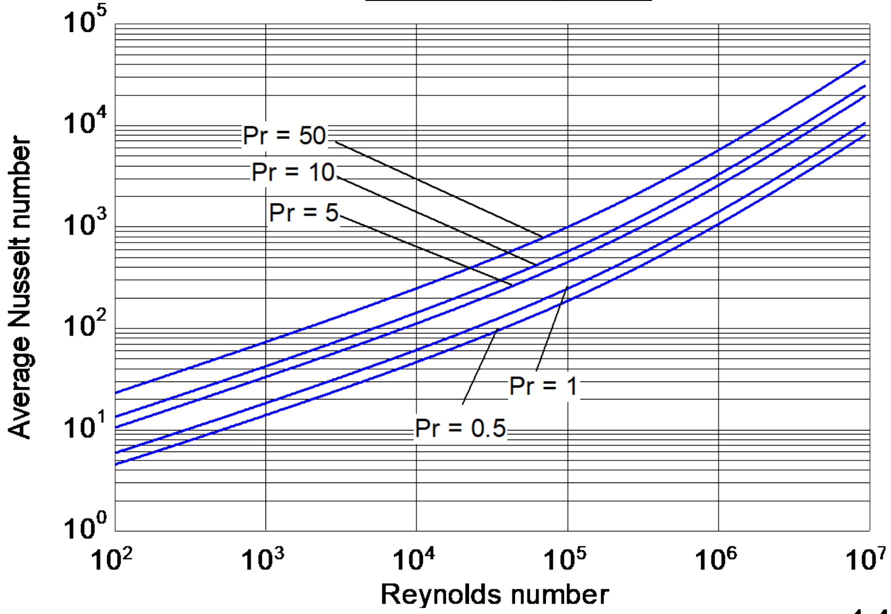
$$\bar{C}_D = \bar{C}_D(Re_L)$$

$$\bar{C}_D = \bar{C}_D(Re_L)$$



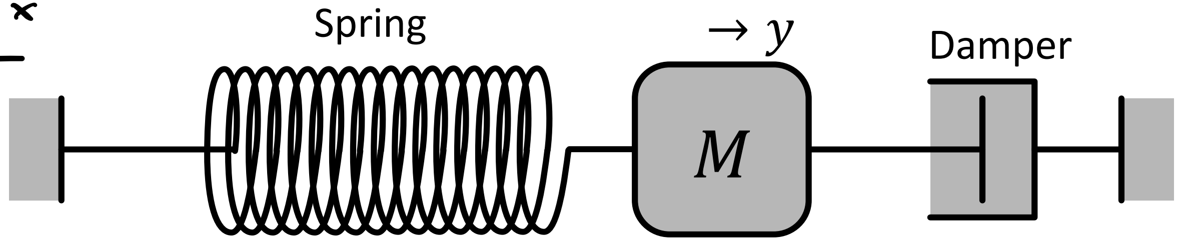
$$\overline{Nu} = \overline{Nu}(Re_L, Pr)$$

Flow over a cylinder

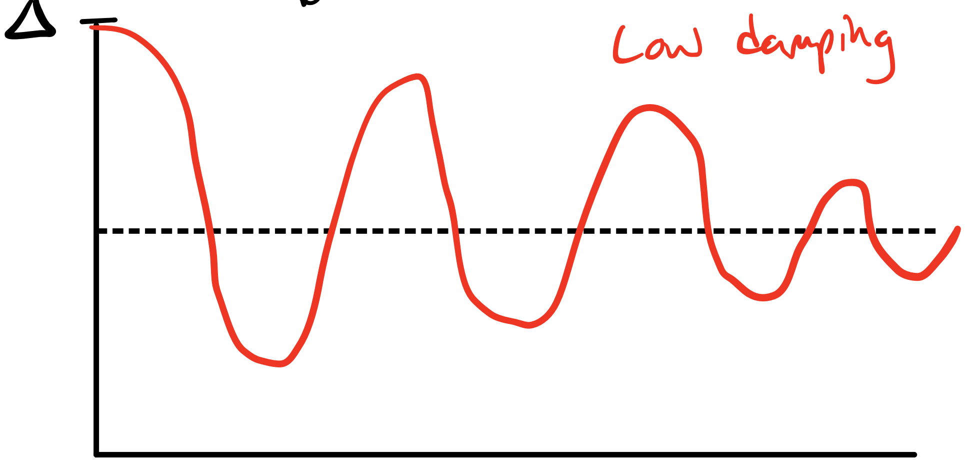


Turbulent Flow

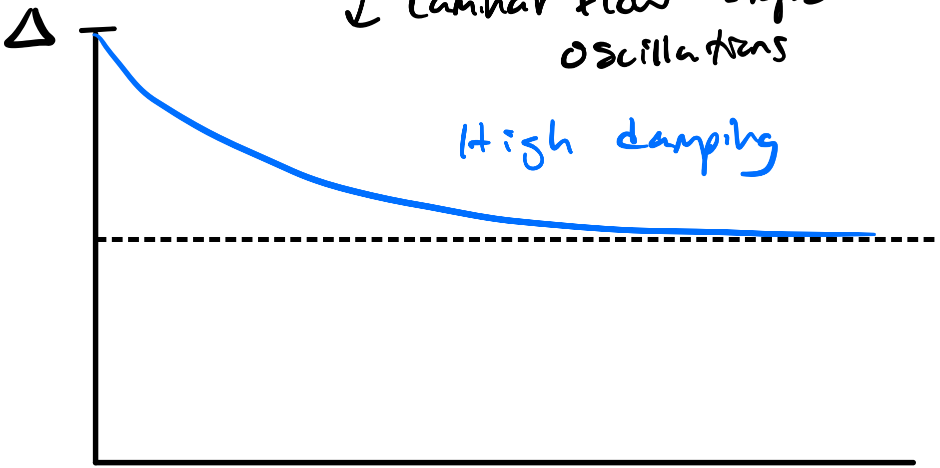
$$Re = \frac{\text{Inertial forces}}{\text{viscous forces}} = \frac{\rho u_\infty x}{\mu}$$



↓ turbulent flow



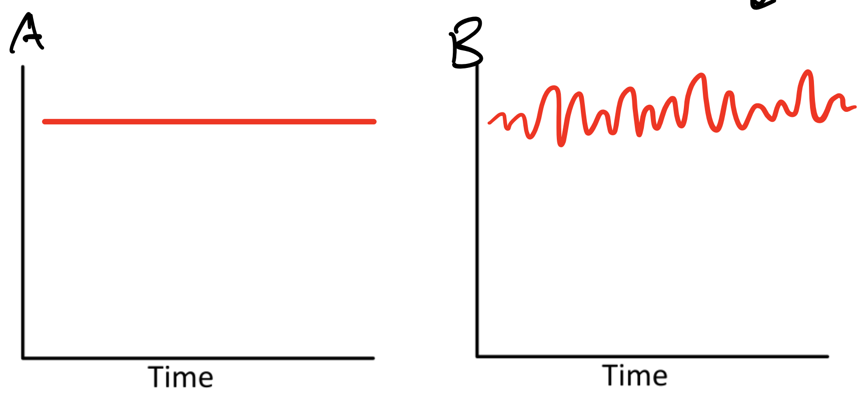
↓ laminar flow stops oscillations

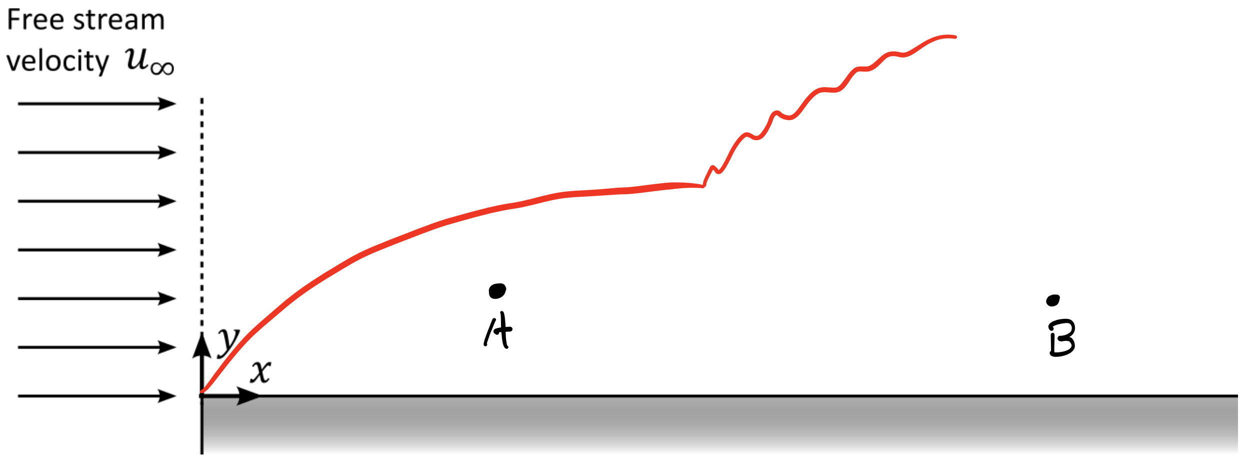


https://www.youtube.com/watch?v=e1TbkLIDWys

The fluid path in turbulent flow

velocity as $f(t)$ oscillating around a value





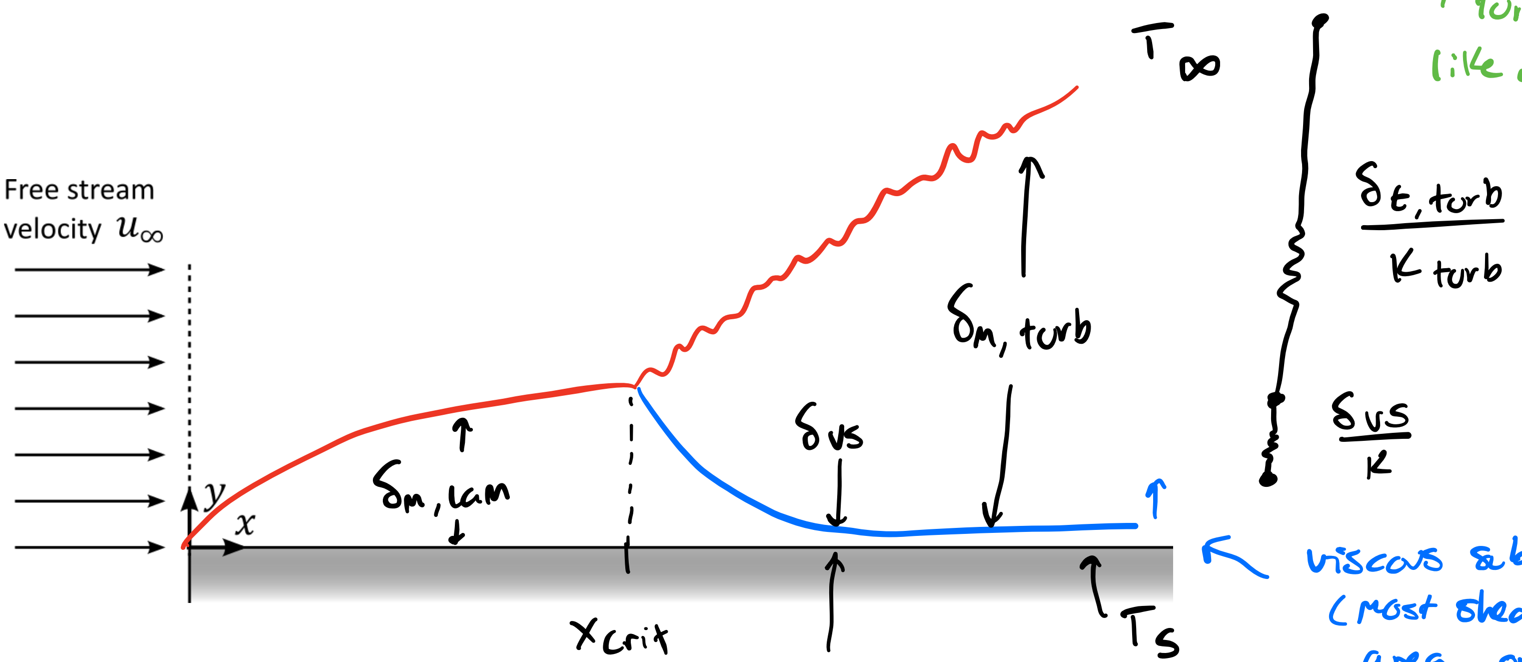
Transistion to turbulence

$$Re \geq Re_{crit} \approx 5e5$$

$$Re_{crit} = \frac{\rho u_\infty \cdot x_{crit}}{\mu}$$

inertial forces overcome viscous damping forces at Re_{crit}

Heat transfer in the turbulent boundary layer



k_{turb} (eff. conductivity) $\gg k$

likewise $\mu_{turb} \gg \mu$

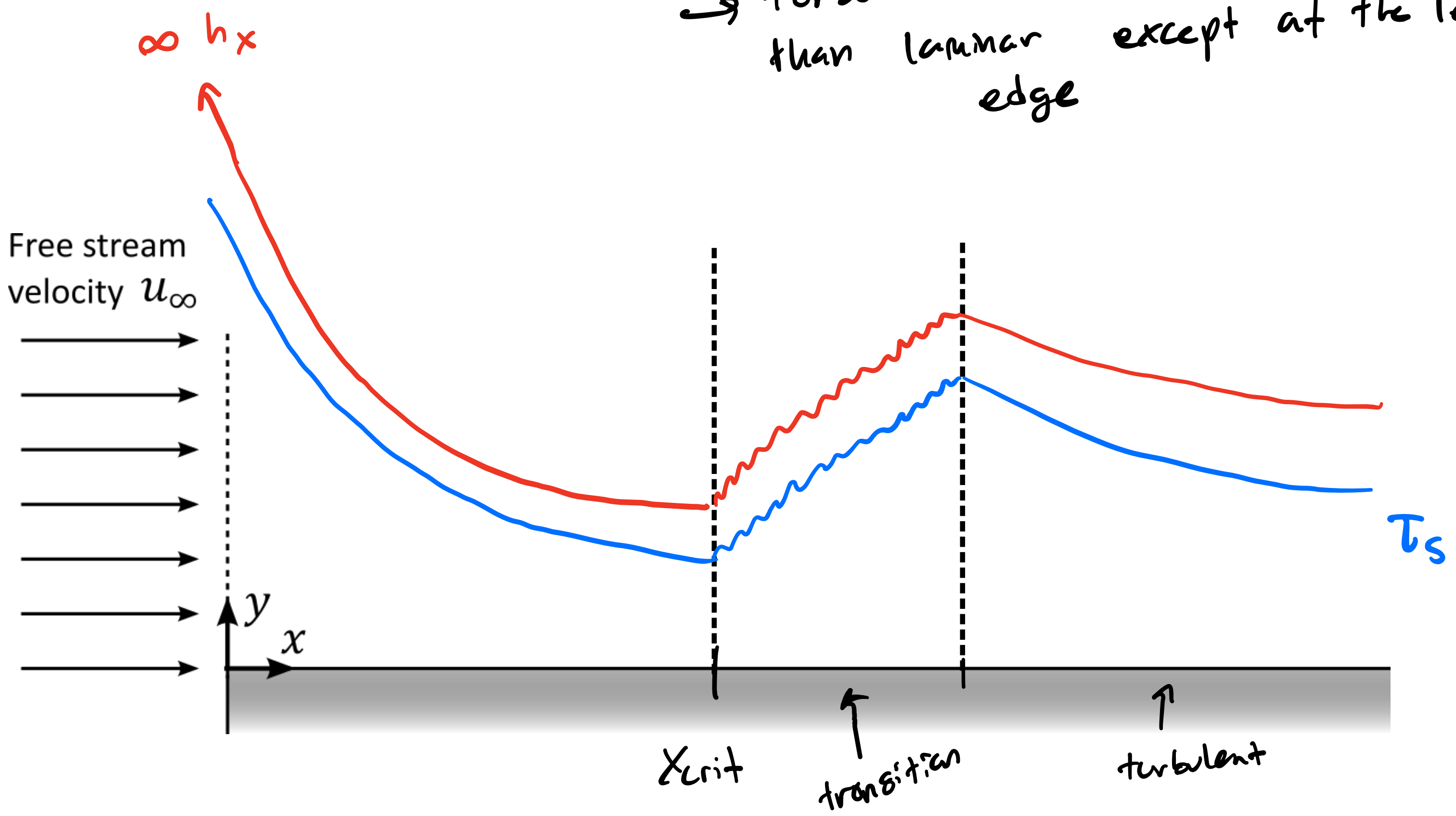
$\delta_{vs} \ll \delta_{lam} \ll \delta_{turb}$

viscous sublayer (most shear occurs, small area of laminar flow)

$$R''_{conv} = R''_{vs} + R''_{turb} = \frac{\delta_{vs}}{k} + \cancel{\frac{\delta_{t,turb}}{k_{turb}}} \text{ eff.} \to 0 \quad \to \quad q''_{turb} \approx \frac{k(T_s - T_\infty)}{\delta_{vs}}$$

Turbulent and laminar flow compared

→ turbulent heat transfer is better than laminar except at the leading edge



const. temp

Table 8.1 Summary of correlations for a smooth, isothermal flat plate.

Flow condition	Parameter	Local value	Average value
laminar, $Re_x < Re_{crit}$ $Re_L < Re_{crit}$	friction coefficient	$C_f = \dfrac{0.664}{\sqrt{Re_x}}$	$\bar{C}_f = \dfrac{1.328}{\sqrt{Re_L}}$
	Nusselt number	$Nu_x = \dfrac{0.3387\, Re_x^{1/2}\, Pr^{1/3}}{\left[1+\left(\frac{0.0468}{Pr}\right)^{2/3}\right]^{1/4}}$	$\overline{Nu}_L = \dfrac{0.6774\, Pr^{1/3}\, Re_L^{1/2}}{\left[1+\left(\frac{0.0468}{Pr}\right)^{2/3}\right]^{1/4}}$
turbulent*, $Re_x > Re_{crit}$ $Re_L > Re_{crit}$	friction coefficient	$C_f = \dfrac{0.027}{Re_x^{1/7}}$	$\bar{C}_f = \dfrac{1}{Re_L}\left[1.328\, Re_{crit}^{0.5} + 0.0315\left(Re_L^{6/7} - Re_{crit}^{6/7}\right)\right]$
	Nusselt number	$Nu_x = 0.0135\, Re_x^{6/7}\, Pr^{1/3}$	$\overline{Nu}_L = \dfrac{0.6774\, Pr^{1/3}\, Re_{crit}^{1/2}}{\left[1+\left(\frac{0.0468}{Pr}\right)^{2/3}\right]^{1/4}} + 0.0158\, Pr^{1/3}\left(Re_L^{6/7} - Re_{crit}^{6/7}\right)$

* Note that the average correlations in the turbulent region include integration of the local correlations for laminar flow through the laminar region.

Example: Calculating average HTC from correlation

$$Nu_x = \frac{h_x\, x}{k}$$

$$Re_x = \frac{\rho\, u_\infty x}{\mu}$$

$$\bar{h} = \frac{1}{L}\int_0^L h_x\, dx \qquad Nu_x = 0.332\, Re_x^{0.5}\, Pr^{1/3}$$

$$\bar{h} = \frac{k}{L}\int_0^L \frac{1}{x}\, 0.332\, Re_x^{0.5}\, Pr^{1/3}\, dx$$

$$\rightarrow \bar{h} = \frac{k}{L}\, 0.332\, Pr^{1/3}\left(\frac{\rho u_\infty}{\mu}\right)^{0.5}\int_0^L x^{-1/2}\, dx$$

$$\rightarrow \bar{h} = 0.664\, k\, Pr^{1/3}\left(\frac{\rho u_\infty}{\mu L}\right)^{1/2}$$

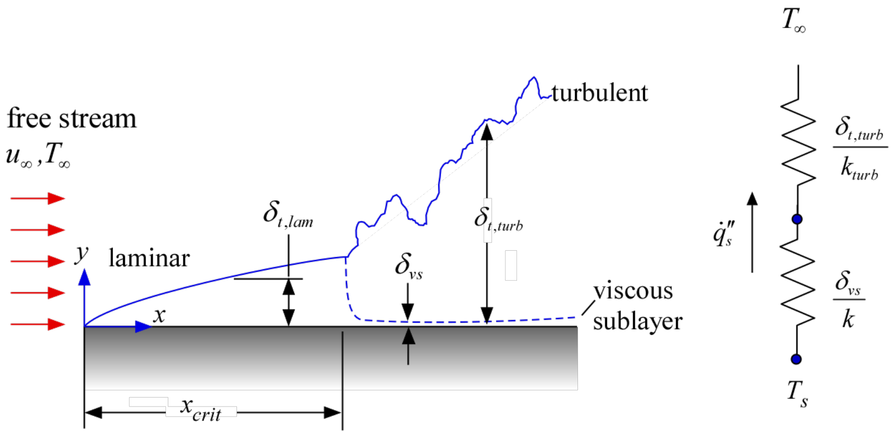
$$\bar{h} = \frac{1}{x_{crit}}\int_0^{x_{crit}} h_{x,lam}\, dx + \frac{1}{L - x_{crit}}\int_{x_{crit}}^{L} h_{x,turb}\, dx$$

Lecture 14

External Forced Convection Applications

Last time…

Non-dimensional Parameters and Turbulent Flow



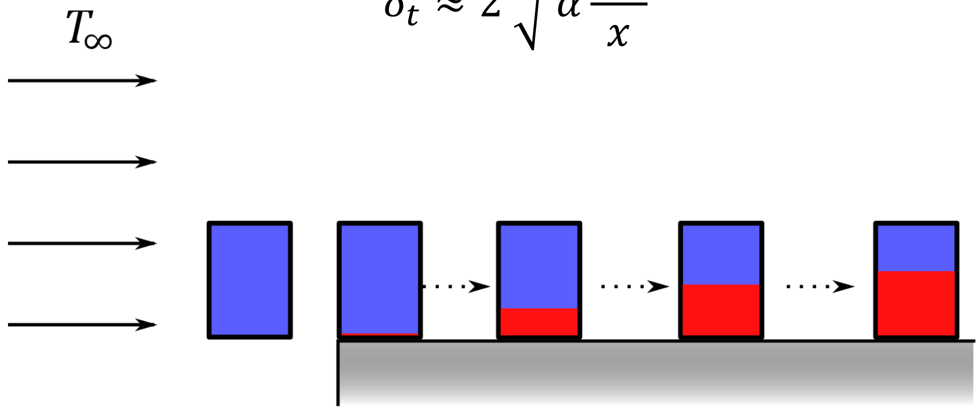
- Non-dimensional parameter groups arise from conservation equations
- Turbulent flow occurs when inertial forces overcome viscous forces
- Viscous sublayer dominates transport characteristics
- Average HTC includes contributions from entire plate length (multiple correlations)

When should we use approximate solution or approximate solution?

What makes solution approximate?

Thermal wave analogy considers “block” of fluid moving in x

$$\delta_t \approx 2\sqrt{\alpha\frac{u_\infty}{x}}$$



Net effect: Approx. solution **underestimates** h_x

Correlation is more accurate if:

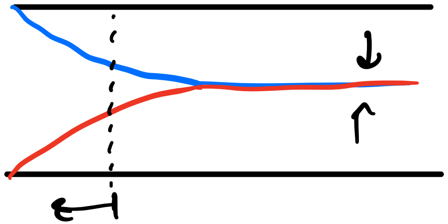
- Available for geometry
- Within valid ND parameter range (Re, Pr, Ec, etc)
- Properly formulated (local, avg., flux, const T., etc.)

Methodology for solving convection problems

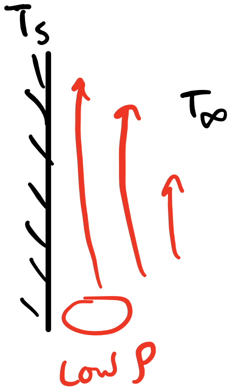
1. Understand flow situation and geometry
2. Evaluate fluid properties
3. Identify the correlation
4. Evaluate the correlation using ND parameters
5. Compute dimensional result

Methodology for solving convection problems

Internal: Boundary layers merge

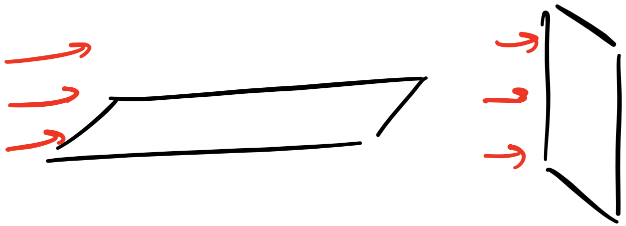


1. Understand flow situation and geometry



- Internal vs External flow
- Forced vs Natural
- local vs average
- Turbulent vs laminar
- Surface boundary condition (const. T vs. const $\dot{q}''$)
- Geometry

may not know at first



Methodology for solving convection problems

2. Evaluate fluid properties

$$T_{film} = \frac{T_s - T_\infty}{2}$$

"≈ avg temp in BL"

3. Identify correlation

- EES, textbook, paper, experiment / approx. model

4. Evaluate correlation using ND parameters

Re_x, Pr, Re_L, Ec, etc. → plug in

↳ local vs. average / use fluid properties

Methodology for solving convection problems

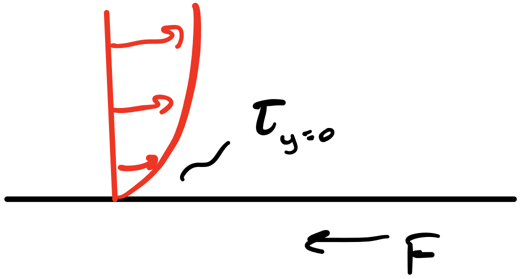
5. Compute dimensional parameters

$$\bar{h} = \overline{Nu}\,\frac{k}{L} \qquad \dot{q} = \bar{h} A_s (T_s - T_\infty)$$

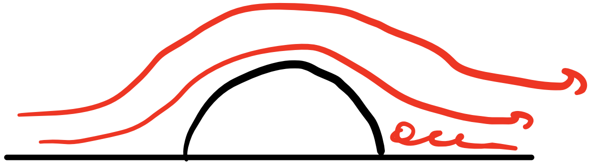
$$\tau = \frac{C_f \rho u_\infty^2}{2} \qquad F = \tau \cdot A_s$$

What's the difference between C_f and C_D?

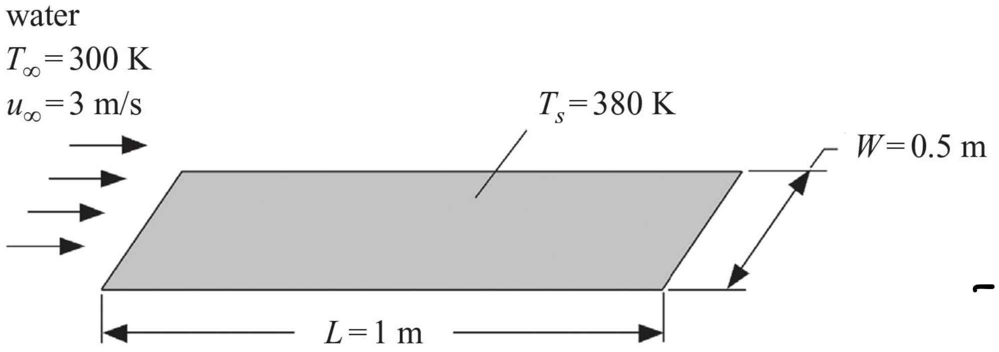
Cf : force exerted onto surface from fluid shear



Cp : force due to pressure difference & friction



Correlation example: Calculate total heat transfer from plate to water



1. Classify

- flat plate
- forced convection
- isothermal (T_s constant/uniform)
- external flow
- average HTC, not local
- flow type ?

2. Evaluate fluid properties

$$T_{film} = \frac{T_s + T_\infty}{2} = \frac{300K + 380K}{2} = 340K$$

$$\rho = 978\frac{kg}{m^3} \qquad k = 0.66\frac{W}{m \cdot K}$$

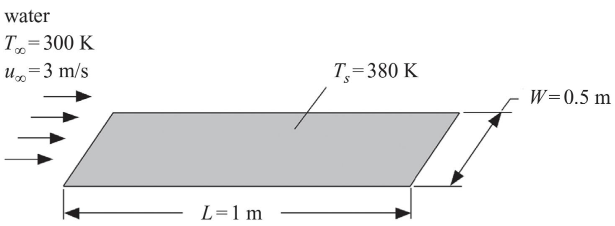
$$\mu = 4.2 \times 10^{-4}\ Pa \cdot s \qquad c = 4190\frac{J}{kg \cdot K}$$

$$Re_{crit} = \frac{\rho u_\infty x_{crit}}{\mu}$$

$$x_{crit} = \frac{5E5 \cdot 4.2E\text{-}4}{3 \cdot 978} = 0.072\ (m)$$

→ laminar & turbulent!

Correlation example: Calculate total heat transfer from plate to water



3. Choose correlation

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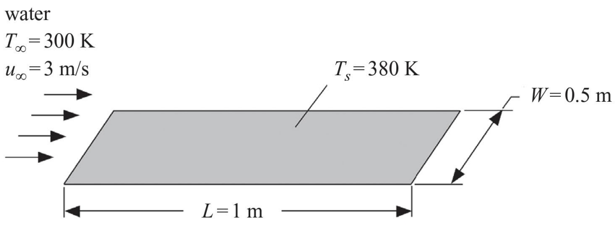
* Note that the average correlations in the turbulent region include integration of the local correlations for laminar flow through the laminar region.

Correlation example: Calculate total heat transfer from plate to water



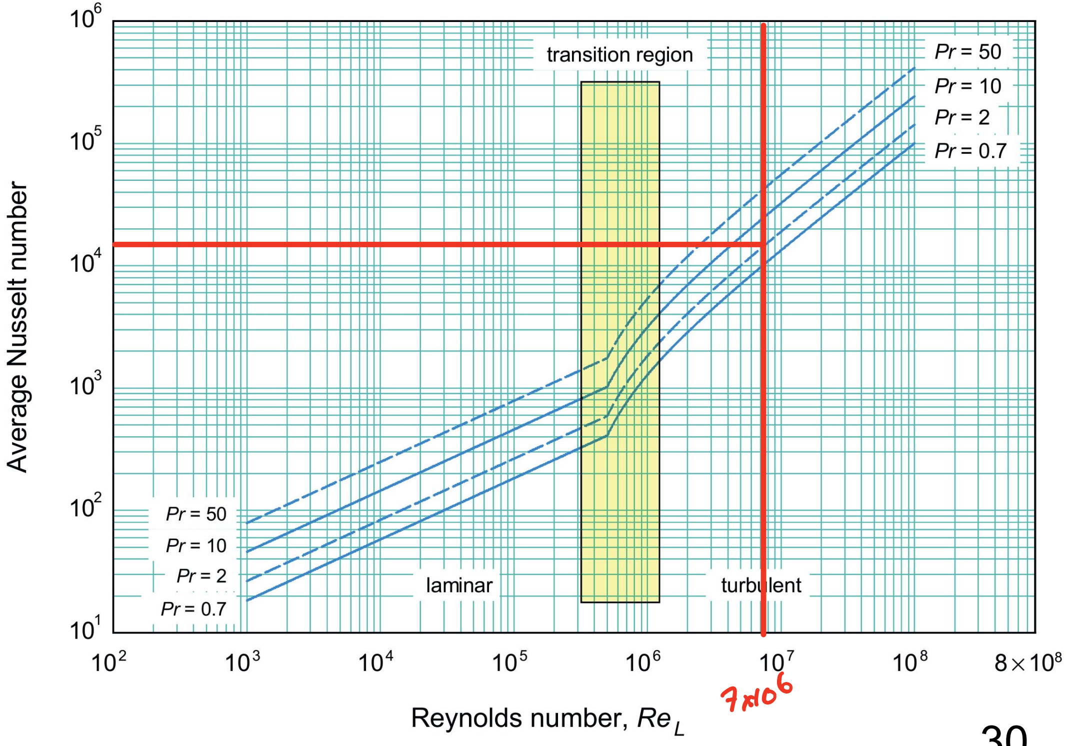
4. Evaluate correlation

$Re_{crit} = 5E5$

$Pr = \frac{\nu}{\alpha} = 2.69$

$Re_L = \frac{\rho u_\infty L}{\mu} = 7\times10^6$

$\overline{Nu} \approx 1\times10^4$



5. Evaluate dimensional values

$\overline{Nu} = \frac{\bar{h}\cdot L_{char}}{k}$ $= L$

$\boxed{\bar{h} = 6600 \frac{W}{m^2K}}$

$q = \bar{h}\cdot A_s(T_s - T_\infty) = \boxed{264{,}000\ [W]}$

Lecture 15

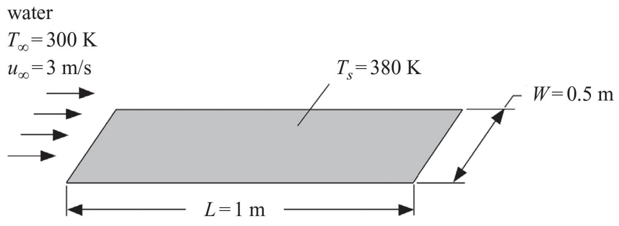
Internal Forced Convection

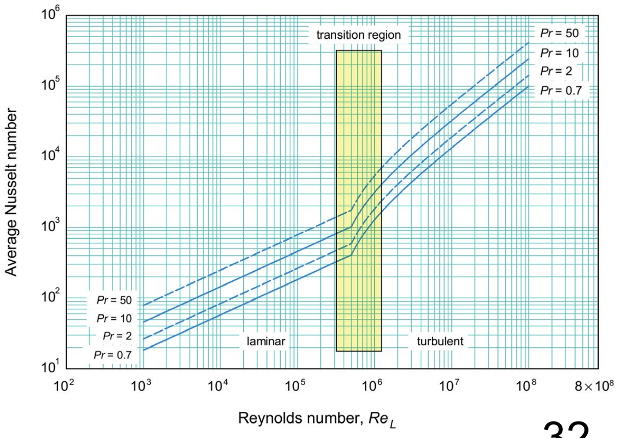
Last time…

External Forced Convection Applications

- Methodology for solving convection problems
  1. Understand flow situation and geometry
  2. Evaluate fluid properties
  3. Identify the correlation
  4. Evaluate the correlation using ND parameters
  5. Compute dimensional result

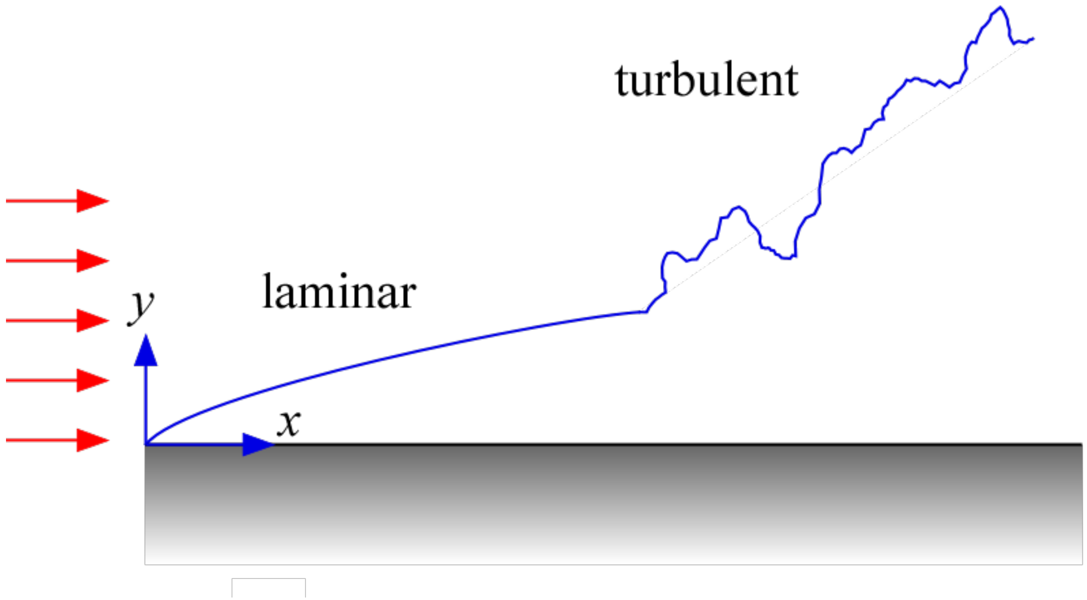
- Considered example: flow over flat plate



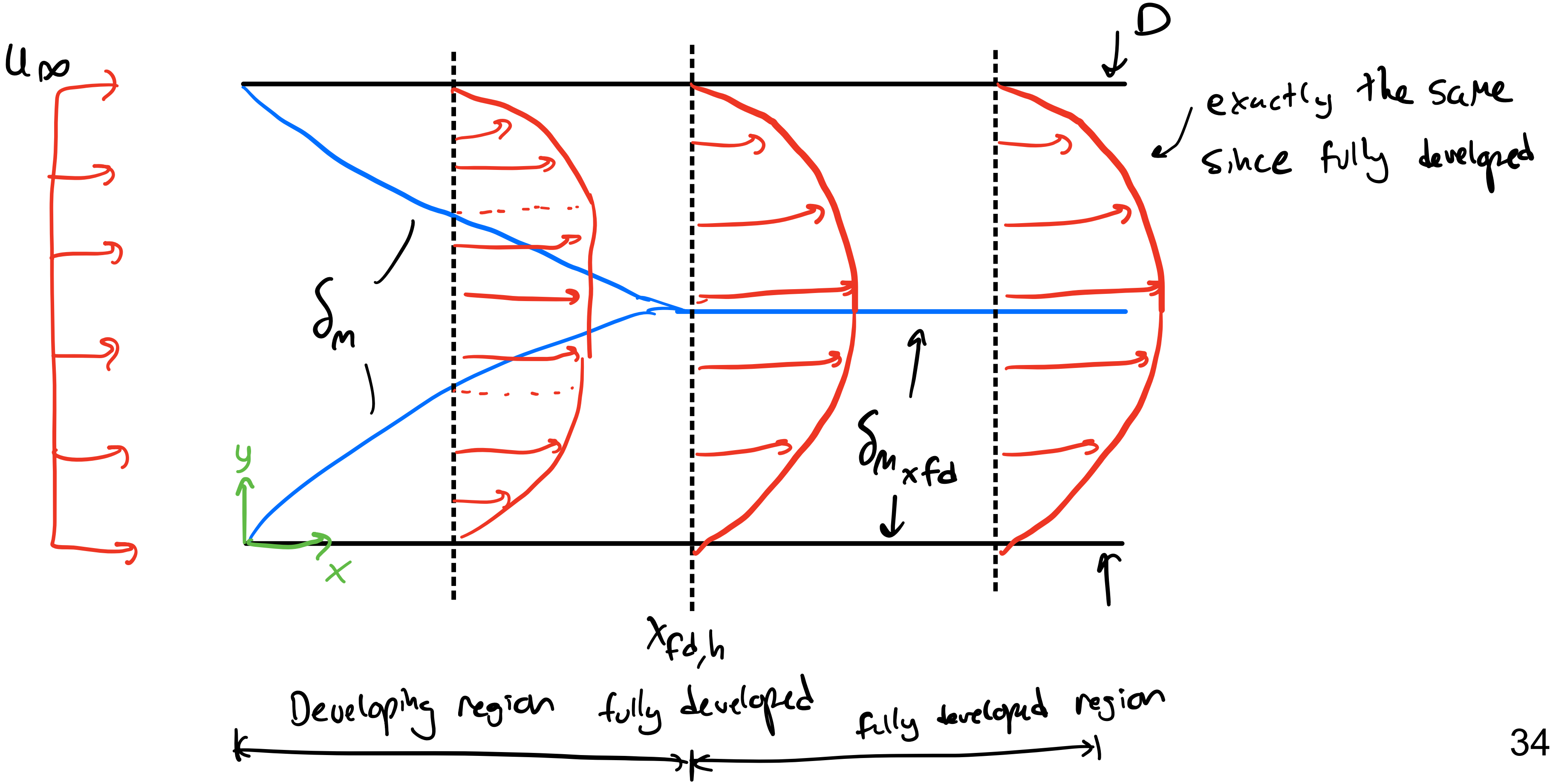


Recall external flow behavior:

- Grows indefinitely with position x
- Transitions to turbulent when $Re_x \geq Re_{crit} \approx 5 \times 10^5$



Internal flow exists when boundary layers are constrained



Beyond the hydrodynamic entry length:

$$x_{fd}, \quad \delta_M = \frac{D}{2}$$

Recall Re # over flat plate

$$Re = \frac{\rho u_\infty x}{\mu} \rightarrow \frac{\text{Inertial}}{\text{viscous}} \quad \Bigg| \quad Re_{D_h} = \frac{\rho \bar{u} D_h}{\mu}$$

$$D_h = \frac{4 \cdot A_c}{Per}$$

↙ cross-sectional area

↑ wetted perimeter

$$\bar{u} = \int_{A_c} u \, dA_c$$

$$\bar{u} = \frac{\dot{V}}{A_c} = \frac{\dot{m}}{\rho A_c}$$

Predicting hydrodynamic entry length

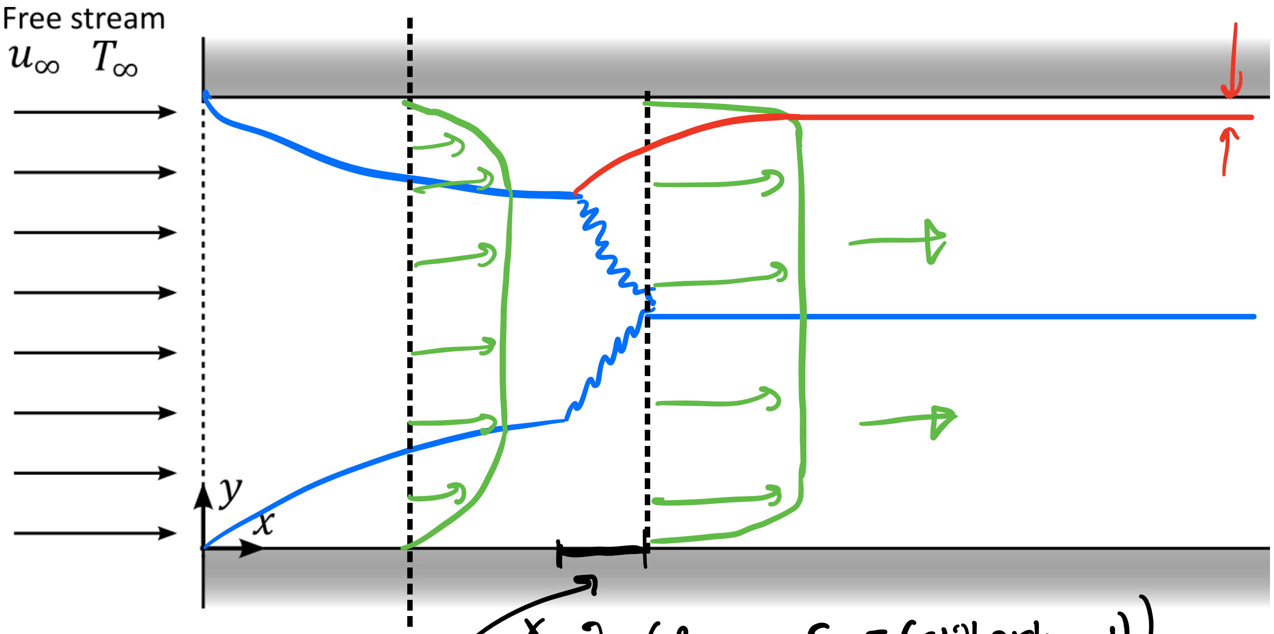
$$\delta_{m,lam} \approx 2\sqrt{\nu t} \rightarrow 2\sqrt{\nu \frac{x}{u_m}} \rightarrow \frac{D}{2} \approx 2\sqrt{\nu \frac{x_{fd,h}}{u_m}}$$

$$x_{fd,h} \approx \frac{D_h^2}{16} \cdot \frac{u_m \rho}{\mu} \rightarrow \boxed{x_{fd,h} \approx \frac{Re_{D_h} \cdot D_h}{16}}$$

$$Re_{crit} \rightarrow 5e5$$

$$Re_{crit,D_h} \rightarrow 2300$$

Turbulent internal flow



x_{crit} ($Re_{crit} = 5e5$ (still external))

transition

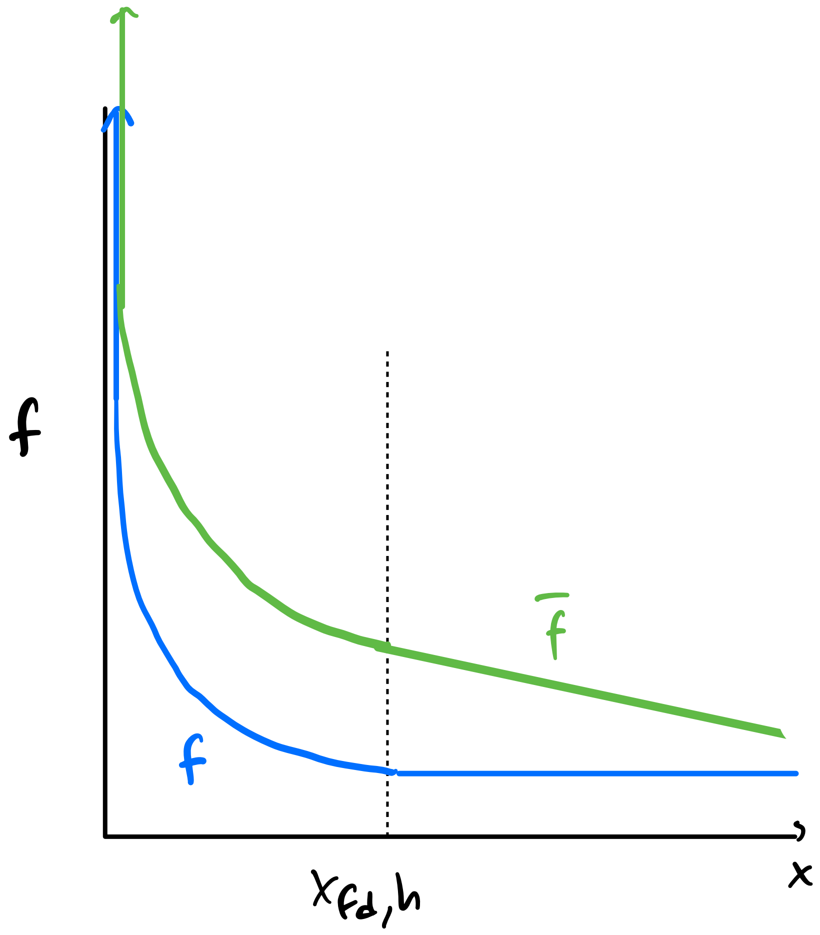
$\mapsto$ Fully developed (turbulent)

← δ_{vs} does not grow when fully developed

τ_s much higher for turbulent than laminar

$\rightarrow \dfrac{dP}{dx} \uparrow\uparrow$

Pressure considerations for internal flow



friction factor

local: "Moody" → Darcy chart

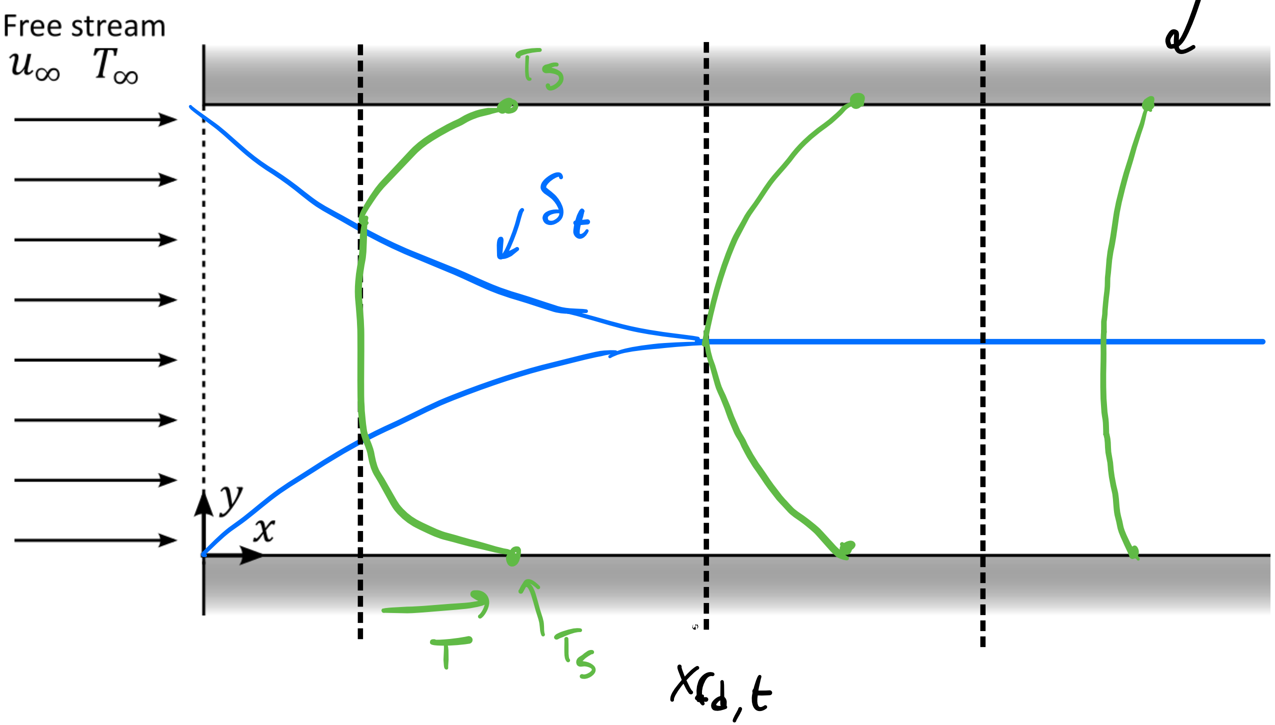
$$f = \left(-\frac{dP}{dx}\right)\frac{2D_h}{\rho u_m^2} \qquad u_m = \text{mean velocity}$$

Average → "Apparent"

$$\bar{f} = \Delta P \cdot \frac{2D_h}{L\rho u_m^2}, \quad \Delta P = P_{x=0} - P_{x=L}$$

Thermal considerations for internal flow

Continued heat addition since $T_s = const$



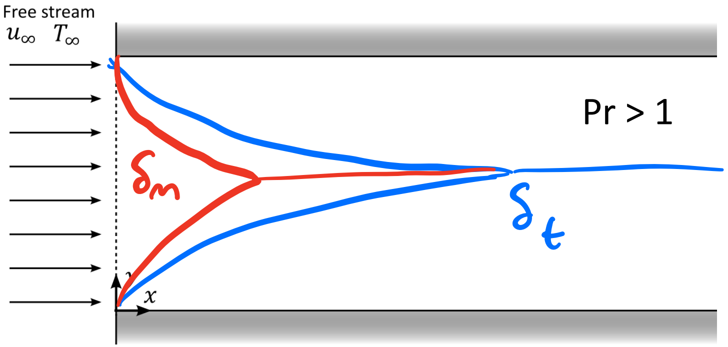
Mean fluid temperature

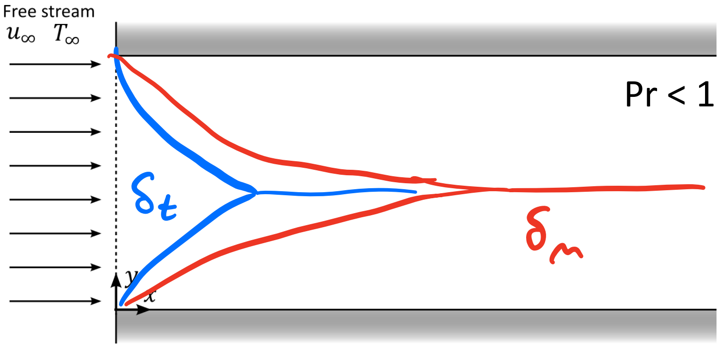
$$T_m(x) = \frac{1}{\dot{V}} \int_{A_c} T(x,y)\, u(x,y)\, dA_c$$

Convection rate equation

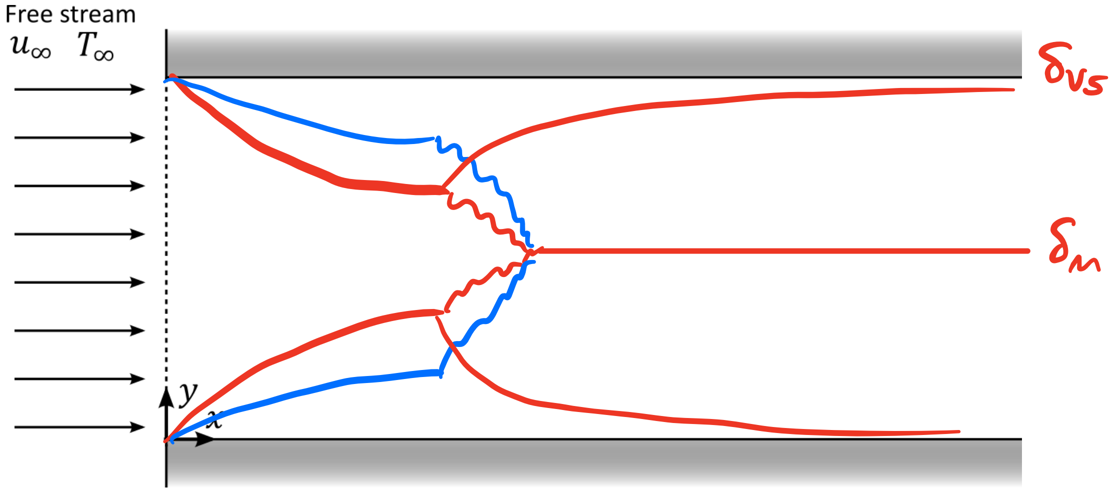
$$\dot{q}_x'' = h_x \left(T_{s,x} - T_{m,x}\right)$$

Thermal entry length



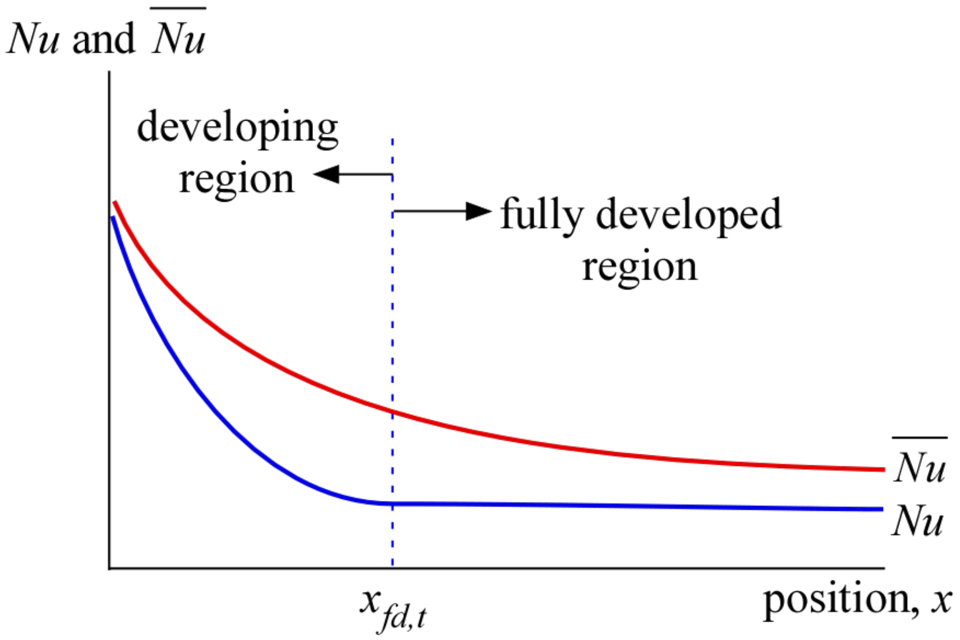


Transition to turbulence



bulk movement (eddies) dominates both fluid & heat xfer outside of VS, so both transition together

Nusselt number for internal flow



Local Nusselt number:

$$Nu_x = \frac{h_x\, D_h}{k}$$

Average Nusselt number:

$$\overline{Nu} = \frac{\bar{h}\, D_h}{k}$$

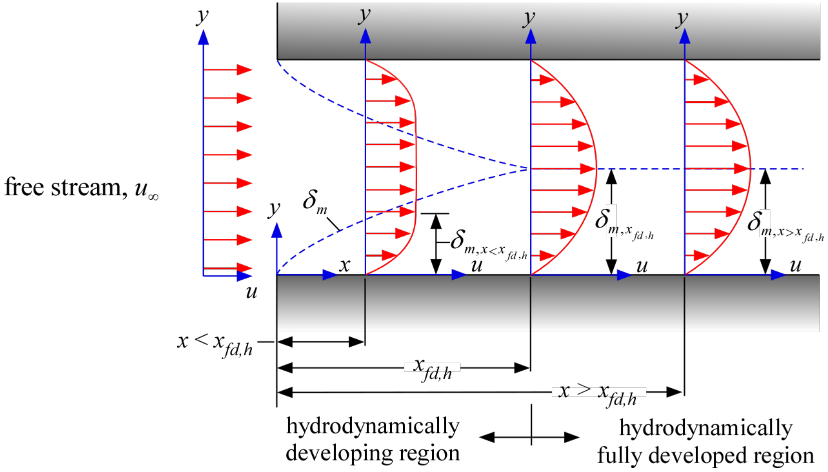
Lecture 16

Internal Convection Correlations

Internal Flow Energy Balance

Last time…

Internal Forced Convection



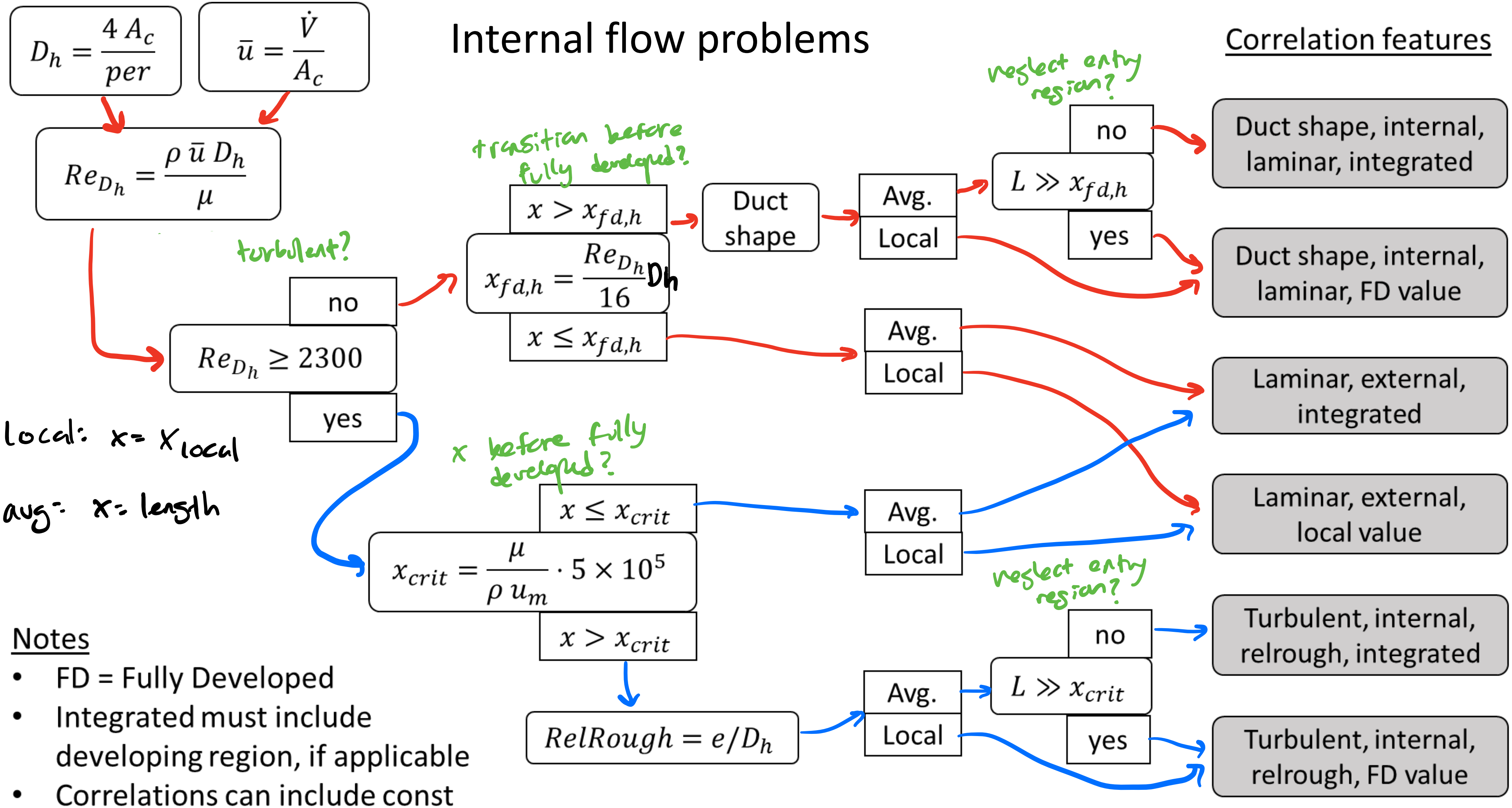
- Occurs when boundary layers growth is constrained
- Fully developed flow $\dfrac{x_{fd,h}}{D_h} \approx \dfrac{Re_{D_h}}{16}$
- Re defined in terms of hydraulic diameter $D_h = 4 \cdot \dfrac{A_c}{per}$
- Transition to turbulence must occur before full development, otherwise always laminar

$$Re_{D_h,crit} \approx 2{,}300$$

- Friction factor: Local="Darcy", chart is "Moody"; Average is "Apparent FF"

$$f = \left(-\frac{dp}{dx}\right)\frac{2\,D_h}{\rho\,u_m^2} \qquad \bar{f} = \Delta P\,\frac{2\,D_h}{L\,\rho\,u_m^2} \qquad \Delta P = P_{x=0} - P_{x=L}$$

Internal flow problems



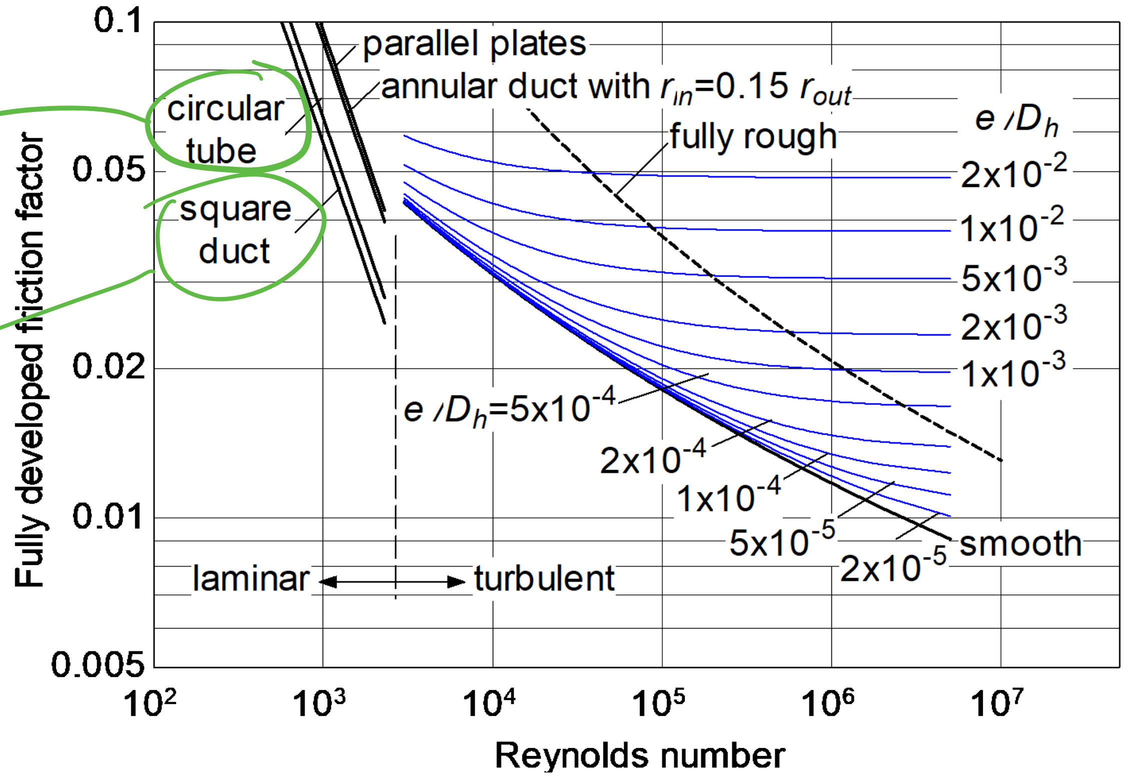
Notes

- FD = Fully Developed
- Integrated must include developing region, if applicable
- Correlations can include const T or $\dot{q}''$ flavors

Correlation usage

friction factor
fully developed

$$f_{fd,circ} = \frac{64}{Re_{Dh}}$$



Rectangular

$$f_{fd} = 96 \cdot \left(1 - 1.355\, A_R + 1.947\, A_R^2 - 1.701\, A_R^3 + 0.956\, A_R^4 - 0.254\, A_R^5\right)$$

A_R = aspect ratio

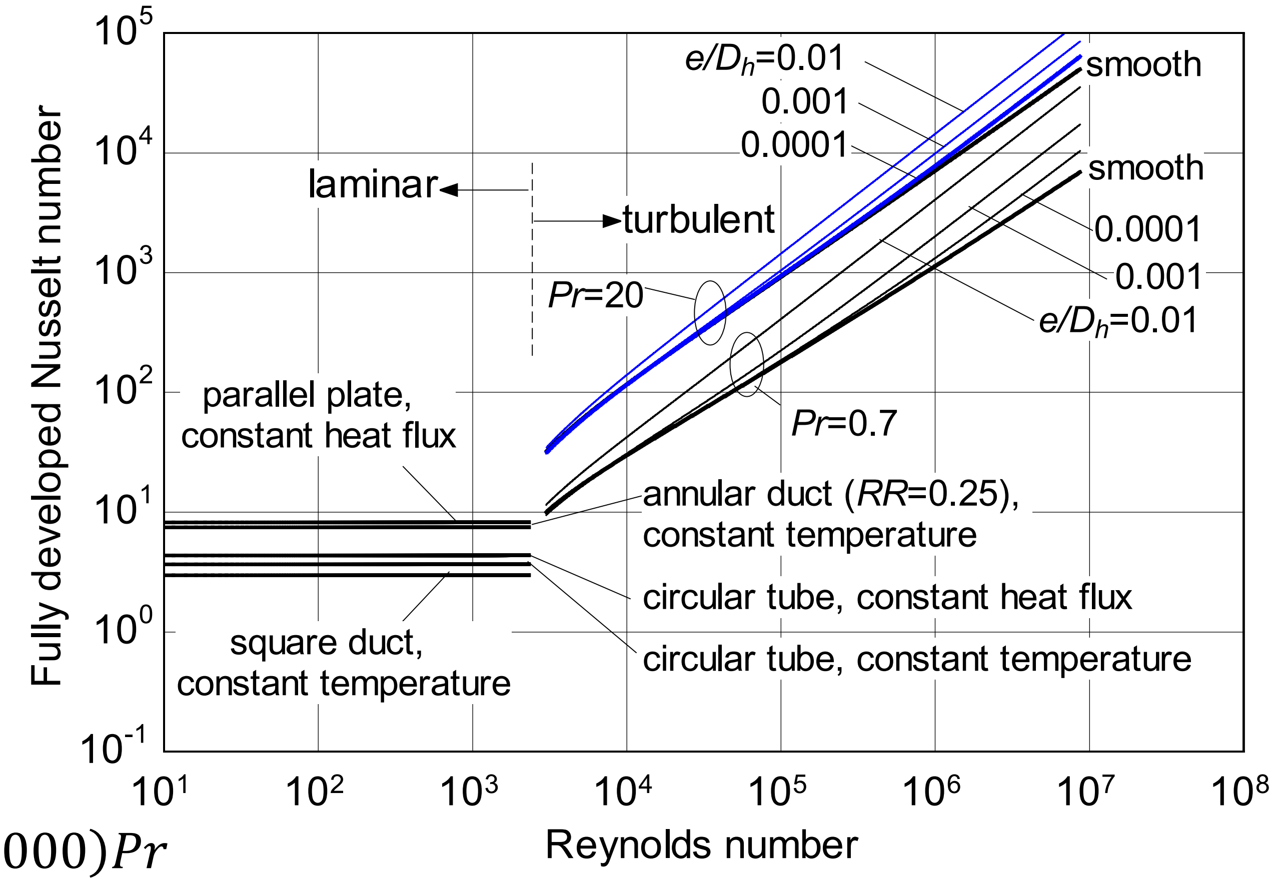
Correction for developing region: $\bar{f} = f_{fd}\left(1 + (D_h/L)^{0.7}\right)$

Correlation usage

Circular duct

$Nu_{fd} = 4.36 \rightarrow$ const flux

$Nu_{fd} = 3.66 \rightarrow$ const temp



Turbulent

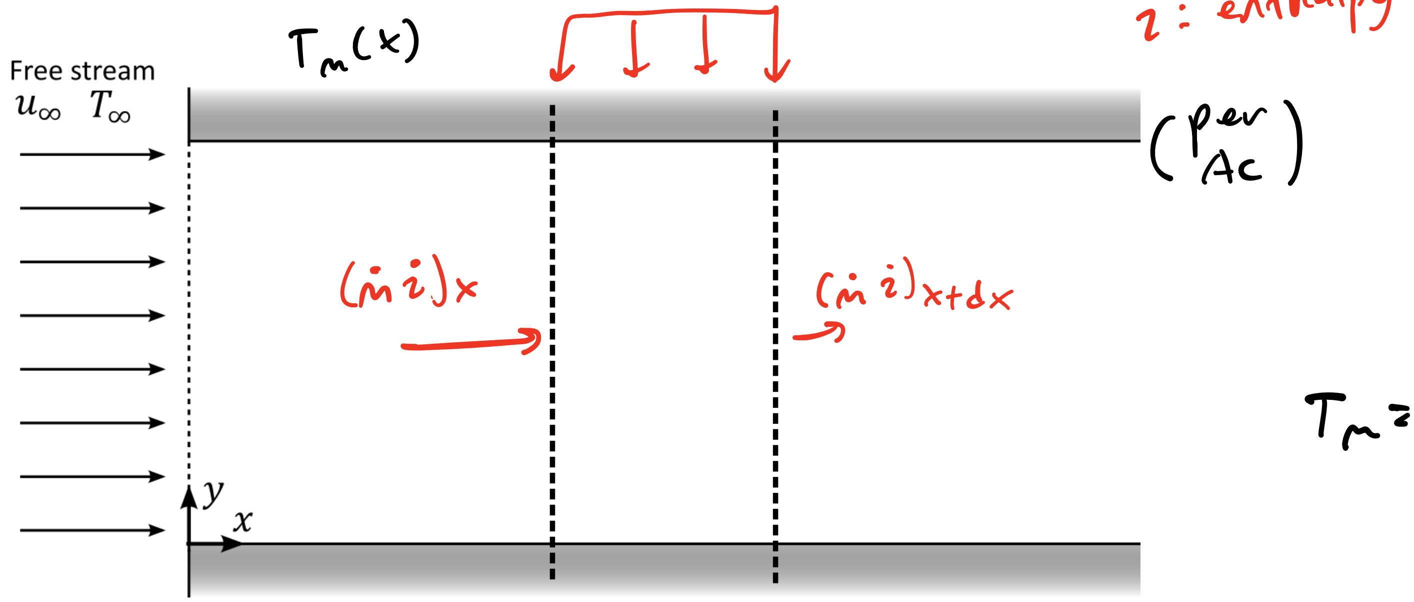
$$Nu_{fd} = \frac{\left(\frac{f_{fd}}{8}\right)(Re - 1000)Pr}{1 + 12.7(Pr^{0.66} - 1)\sqrt{\frac{f_{fd}}{8}}}$$

Gnielinski

$2300 \leq Re \leq 5e6$

$0.5 \leq Pr \leq 1000$

Energy Balance



T_m = mean temp

$$(\dot{m}i)_x + q''\cdot per\cdot dx - (\dot{m}i)_{x+dx} = 0$$

$$\cancel{(\dot{m}i)_x} + q''\cdot per\cdot \cancel{dx} - \left[\cancel{(\dot{m}i)_x} + \frac{d(\dot{m}i)_x}{dx}\,\cancel{dx}\right] = 0$$

$$\frac{d(\dot{m}i)_x}{dx} = q''\cdot per \rightarrow \boxed{\dot{m}c_p\frac{dT_m}{dx} = q''\cdot per}$$

Energy Balance – fluid temperature solution

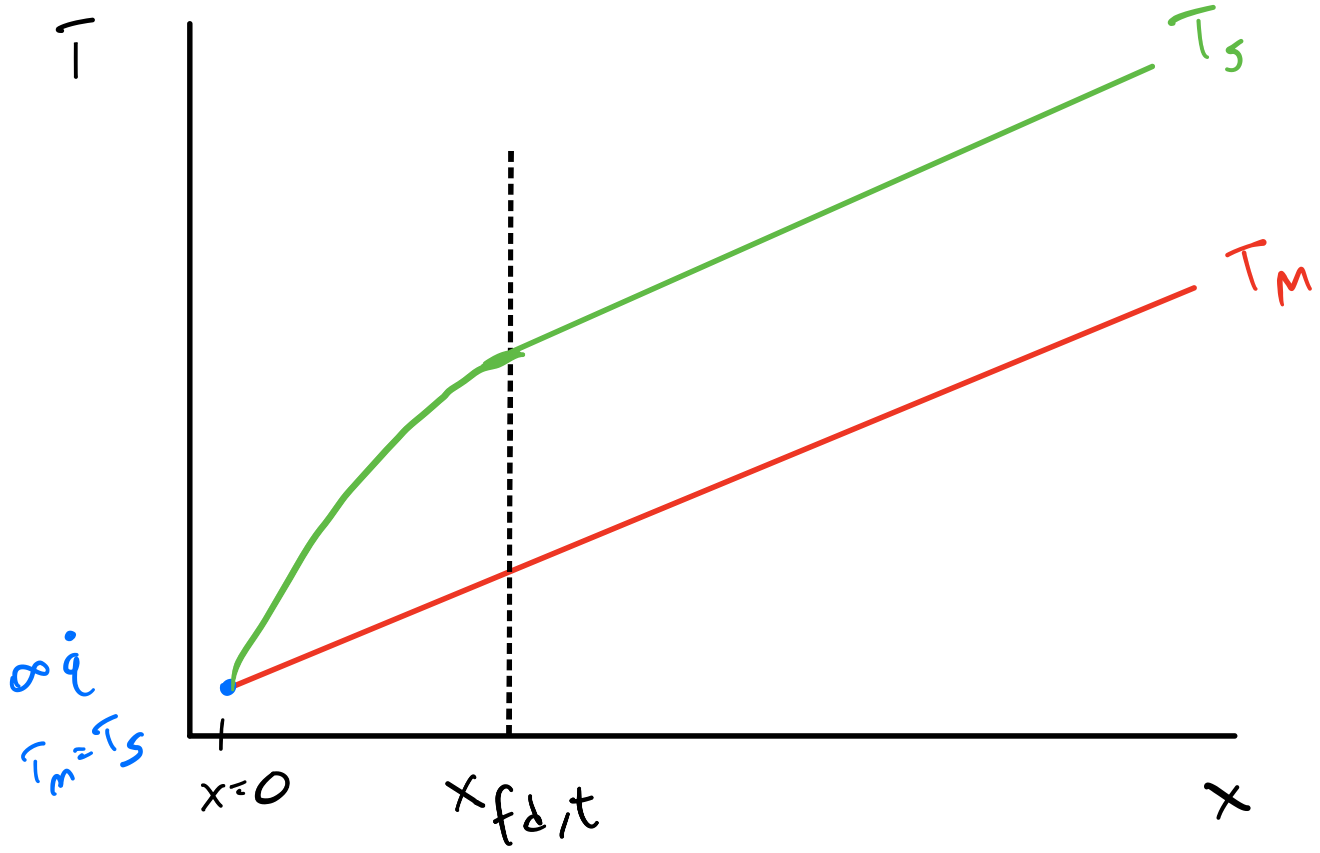
$$\dot{m}\, c_p\, \frac{dT_m}{dx} = \dot{q}'' \cdot per \quad \rightarrow \quad \frac{dT_m}{dx} = \frac{\dot{q}'' \cdot per}{\dot{m} c_p}$$

$$T_m = \frac{\dot{q}'' \cdot per}{\dot{m} c_p} x + C_1 \quad , \quad BC1: T_{in} = C_1 \quad (@\, x=0)$$

$$\rightarrow \boxed{T_m = \frac{\dot{q}'' per}{\dot{m} c_p} x + T_{in}}$$

Energy Balance – Surface temperature solution

$$T_s = T_m + \frac{q''}{hx}$$



Energy Balance – Specified surface temperature

$$\dot{m}\, c \frac{dT_m}{dx} = h_x\, per\, (T_s - T_m) \rightarrow \frac{dT_m}{dx} + \frac{h_x\, per}{\dot{m}\, c} T_m = \frac{h_x\, per}{\dot{m}\, c} T_s$$

$$T_m = T_h + T_p$$

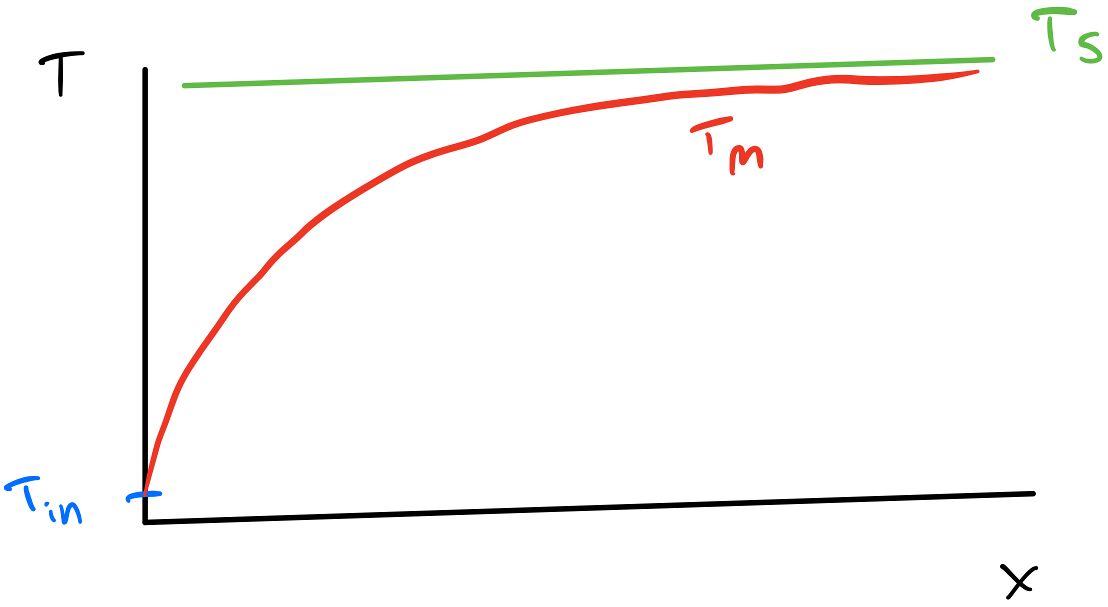
H) $\frac{dT_h}{dx} + \frac{h_x\, per}{\dot{m} c} T_h = 0 \rightarrow T_h = \exp(-h_x\, per/(\dot{m} c) \cdot x)\, C_1$

P) $T_p = T_s \rightarrow T = \exp\left(-\frac{h_x\, per}{\dot{m} C} x\right) C_1 + T_s$

BC) $T_m(x=0) = T_{in} \rightarrow C_1 = T_{in} - T_s$

Energy Balance – Specified surface temperature

$$T_m = (T_{in} - T_s) \exp\left(-\frac{h_x\, per}{\dot{m}\, c_p} \cdot x\right) + T_s$$



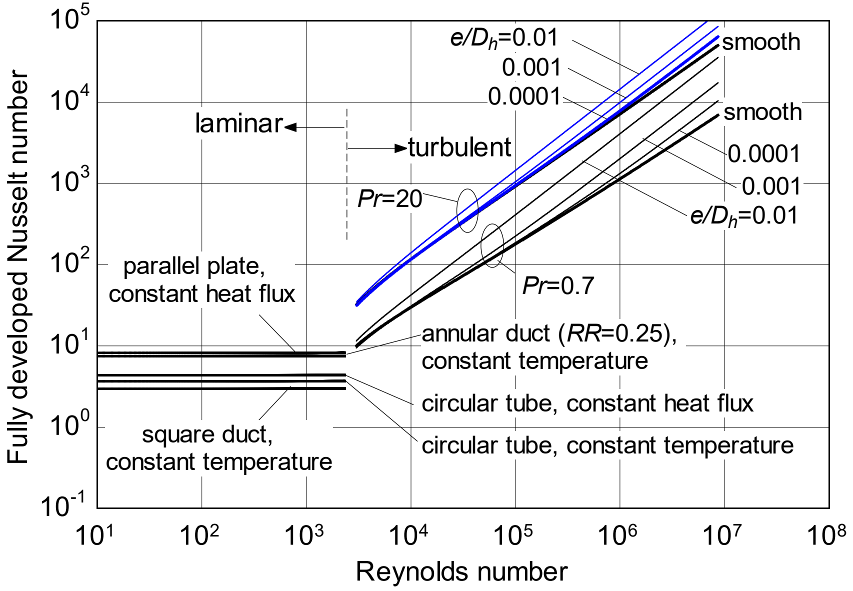
Lecture 17

Free Convection

(Natural)

Last time…

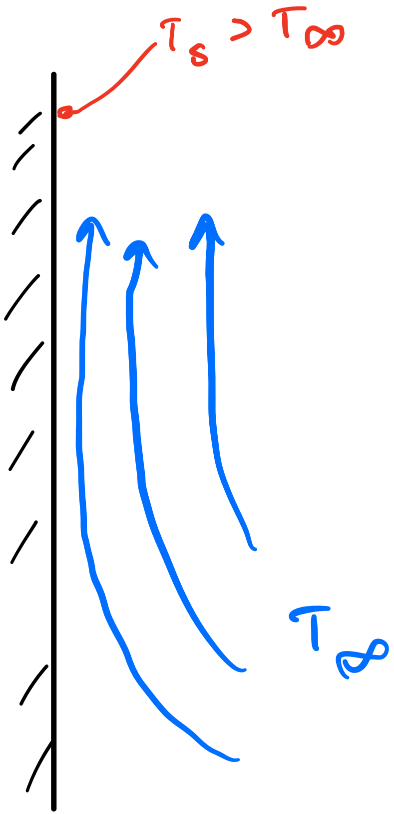
Internal Convection Correlations
Internal Flow Energy Balance

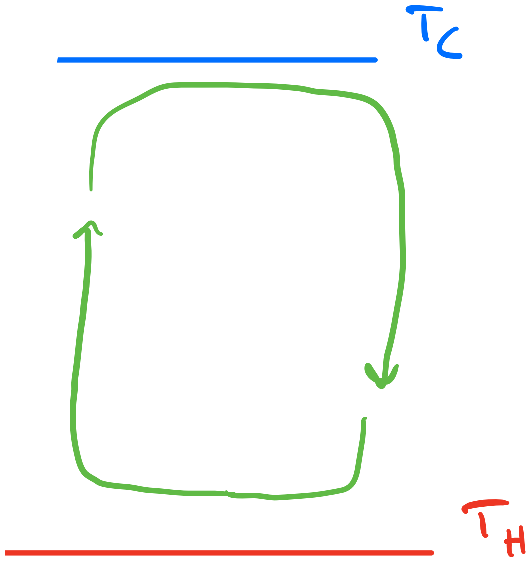


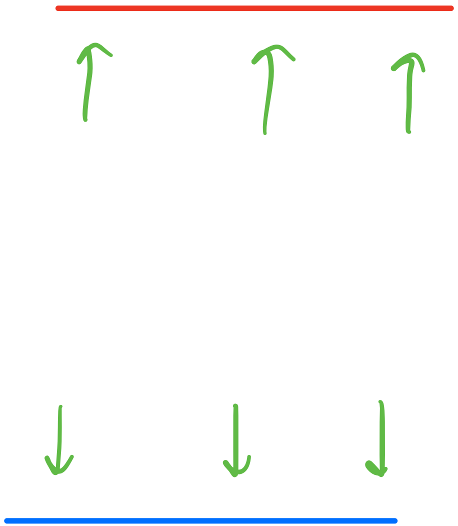
- Internal flow classification diagram
- Correlations for friction factor, Nusselt number
- Internal energy balance → Control volume analysis on differential fluid element
  - Solutions for specified flux $$T_m = \frac{\dot{q}'' \cdot per}{\dot{m} \cdot c_p} x + T_{in}$$
  - Specified surface temperature $$T_m = (T_{in} - T_s) \exp\left(-\frac{h_x \, per}{\dot{m}\, c_p} \cdot x\right) + T_s$$

Free ("natural") convection

Buoyancy driven flow







Characteristic velocity

- u_m / $\bar{u}$ for internal, u_∞ for external
- Natural convection: ?

$$u_{char} = \sqrt{gL\beta(T_s - T_\infty)}$$

volumetric thermal expansion coefficient

$$\beta = -\frac{1}{\rho}\left(\frac{\partial \rho}{\partial T}\right)_{p=const}$$

$$p = \rho RT$$

$$\rightarrow \rho = \frac{p}{RT}$$

$$\beta = -\frac{RT}{p} \cdot \frac{\partial}{\partial T}\left[\frac{p}{RT}\right]$$

$$-\frac{RT}{p} \cdot \left(-\frac{p}{RT^2}\right) \rightarrow \boxed{\beta = \frac{1}{T}}$$

Non-dimensional parameters for buoyancy-driven flow

$$u_{char} = \sqrt{g\,L\,\beta\,(T_s - T_\infty)}$$

$$Re = \frac{\rho\, L_{char}\, u_{char}}{\mu} \rightarrow Re = \frac{\rho L_{char}}{\mu}\sqrt{gL\beta(T_s - T_\infty)}$$

$$Re^2 = \frac{g L_{char}^3 \beta |T_s - T_\infty|}{\nu^2} = Gr \quad \text{"Grashuf"}$$

"Rayleigh"

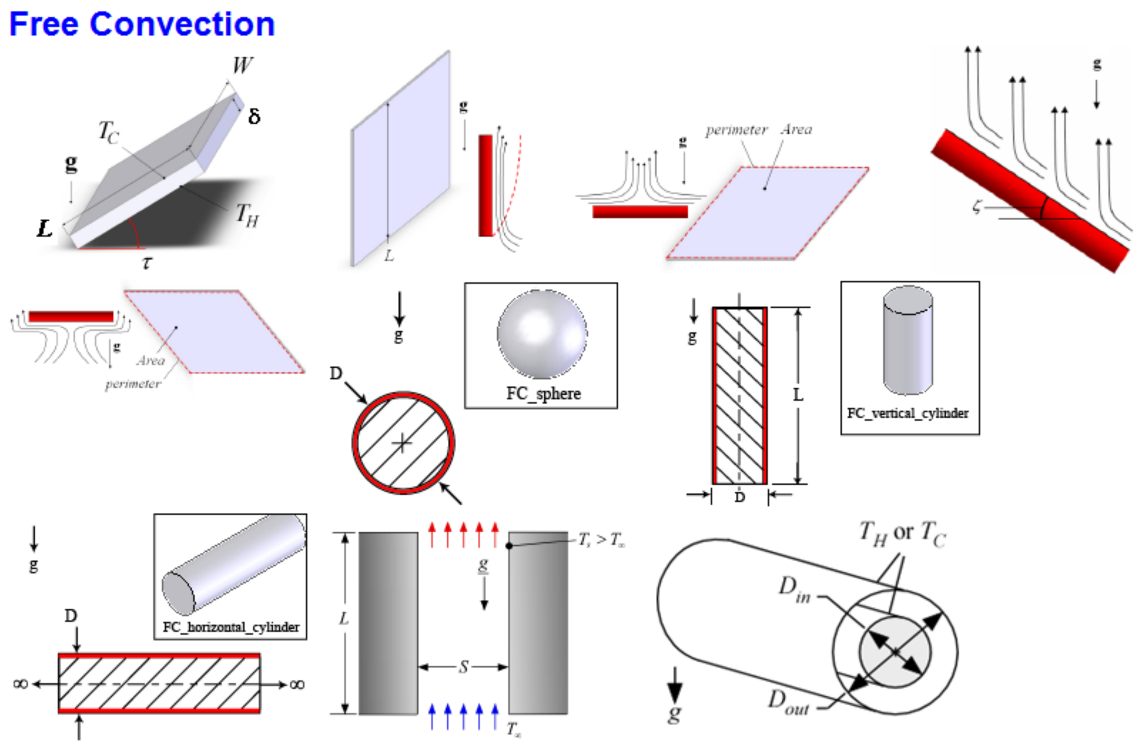
$$Ra = Gr \cdot Pr$$

$$\overline{Nu} = \frac{\bar{h} \cdot L_{char}}{k}$$

Free convection correlations

Considerations

- Laminar vs. turbulent
- Geometry
- Direction of gravity
- $T_s > T_\infty$ | $T_s < T_\infty$
- Internal vs. External
- const. flux vs. const. temp.



Free convection correlation: Vertical plate

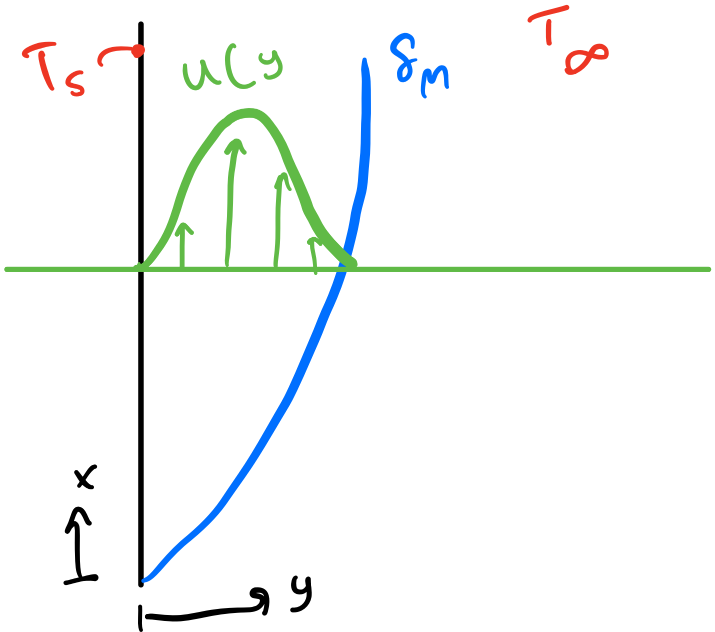
$$\overline{Nu} = \left[0.825 + \frac{0.387\, Ra^{\frac{1}{6}}}{\left(1 + \left(\frac{0.492}{Pr}\right)^{\frac{9}{16}}\right)^{\frac{8}{27}}} \right]^2$$

Where:

Vertical plate height

$$Ra = \frac{g\, L^3\, \beta\, (T_s - T_\infty)}{\nu\, \alpha}$$

$Ra_{crit} \approx 10^9$ → turbulent



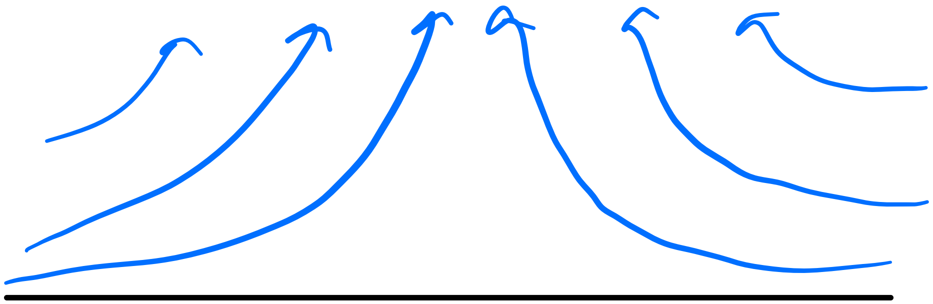
Free convection correlation: Horizontal plate

$$\overline{Nu}_{lam} = \frac{1.4}{\ln\left(1 + \frac{1.4}{0.835\, C_{lam}\, Ra^{\frac{1}{4}}}\right)}$$

$$C_{lam} = \frac{0.671}{\left(1 + \left(\frac{0.492}{Pr}\right)^{\frac{9}{16}}\right)^{\frac{4}{9}}}$$

$$\overline{Nu}_{turb} = C_{turb} \cdot Ra^{\frac{1}{3}}$$

$$C_{turb} = 0.14\left(\frac{1 + 0.0107\, Pr}{1 + 0.01\, Pr}\right)$$



$$\overline{Nu} = \left(\overline{Nu}_{lam}^{\,10} + \overline{Nu}_{turb}^{\,10}\right)^{1/10}$$

Combined convection

Method to combine natural and forced convection.
Larger typically dominates with some transition region

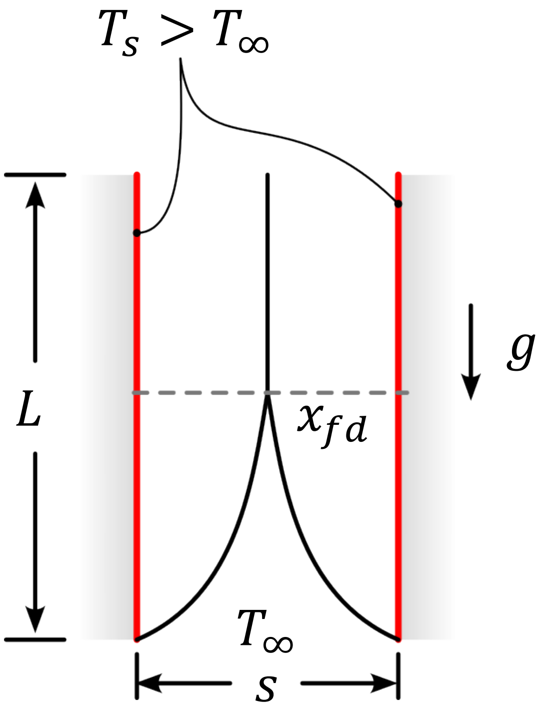
$$\bar{h}_{comb} = \left(\bar{h}_{forced}^{m} + \bar{h}_{free}^{m}\right)^{1/m} \qquad m \approx 3$$

If natural and forced convection act in opposite directions:

$$\bar{h}_{comb} = \left(\max\left[\bar{h}_{forced}, \bar{h}_{free}\right]^{m} - \min\left[\bar{h}_{forced}, \bar{h}_{free}\right]^{m}\right)^{1/m}$$

Internal free convection

Boundary layers merge within internal channel



One side

$$\bar{h} = \frac{\dot{q}}{A_s(T_s - T_\infty)}$$

One side

Surroundings (Not T_m)

$$\overline{Nu} = \frac{\bar{h}\, s}{k} \qquad Ra = \frac{g\, s^3\, \beta\, (T_s - T_\infty)}{\nu\, \alpha}$$

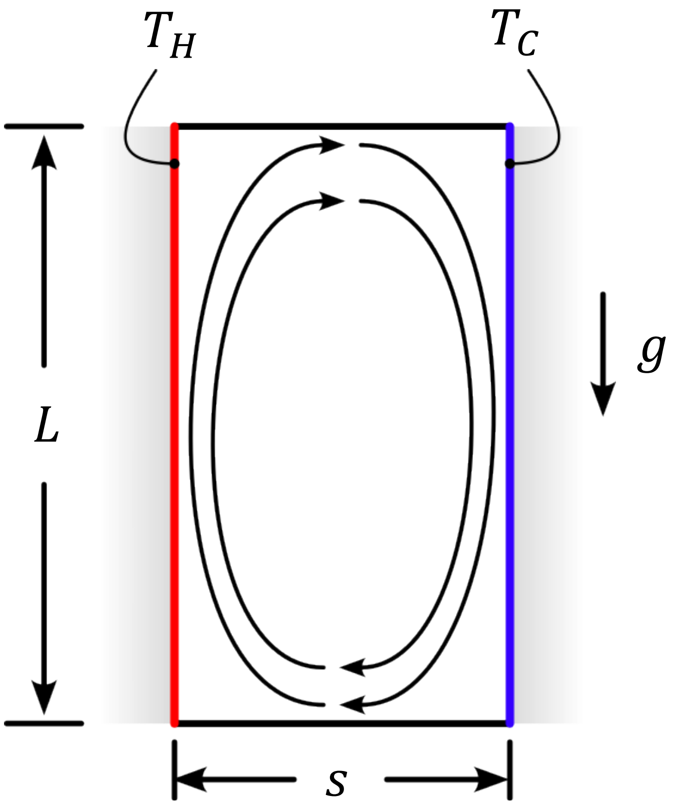
$\overline{Nu}$ given by Elenbaas correlation:
(FC_vertical_channel_ND)

$$\overline{Nu} = \frac{Ra}{24}\frac{s}{L}\left[1 - \exp\left(\frac{35}{Ra}\frac{L}{s}\right)\right]^{0.75}$$

Enclosure free convection

Circulation is affected by top/bottom surfaces

E.g.: Double-pane windows



$$\overline{Nu} = \frac{\bar{h}\, s}{k} \qquad \bar{h} = \frac{\dot{q}}{W\, L\, (T_H - T_C)}$$

HT from hot to cold surface

<u>MacGregor and Emory:</u>

$$\overline{Nu} = 0.42\, Ra^{\frac{1}{4}}\, Pr^{0.012} \left(\frac{L}{s}\right)^{-0.3} \qquad \text{For } 10^4 < Ra < 10^7$$

$$\overline{Nu} = 0.046\, Ra^{\frac{1}{3}} \qquad \text{For } 10^6 < Ra < 10^9$$

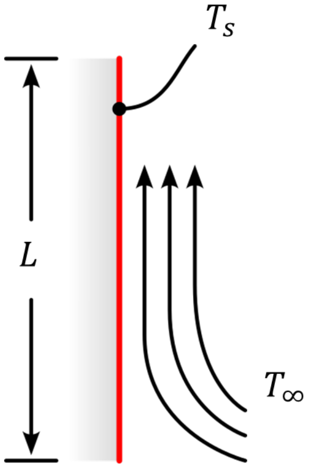
Additional correction for tilt angle $\neq 90°$

Lecture 18

Boiling

Last time…

Free convection



- Flows driven by gradients in local fluid density
- Characteristic velocity related to temperature difference
- Described by non-dimensional parameters

$$u_{char} = \sqrt{g\, L\, \beta\, (T_s - T_\infty)}$$

  - Grashof number $$Gr = Re^2 = \frac{g\, L_{char}^3\, \beta\, (T_s - T_\infty)}{\nu^2} \quad = \frac{\nu}{\alpha}$$
  - Rayleigh number $$Ra = Gr \cdot Pr$$
- Correlations for vertical, horizontal plates given
- Approach for combined convection $$\bar{h}_{comb} = \left(\bar{h}_{forced}^m + \bar{h}_{free}^m\right)^{\frac{1}{m}}$$

Boiling

Pool boiling: Fluid motion due to local density gradients

Flow boiling: Fluid motion driven in part by external force

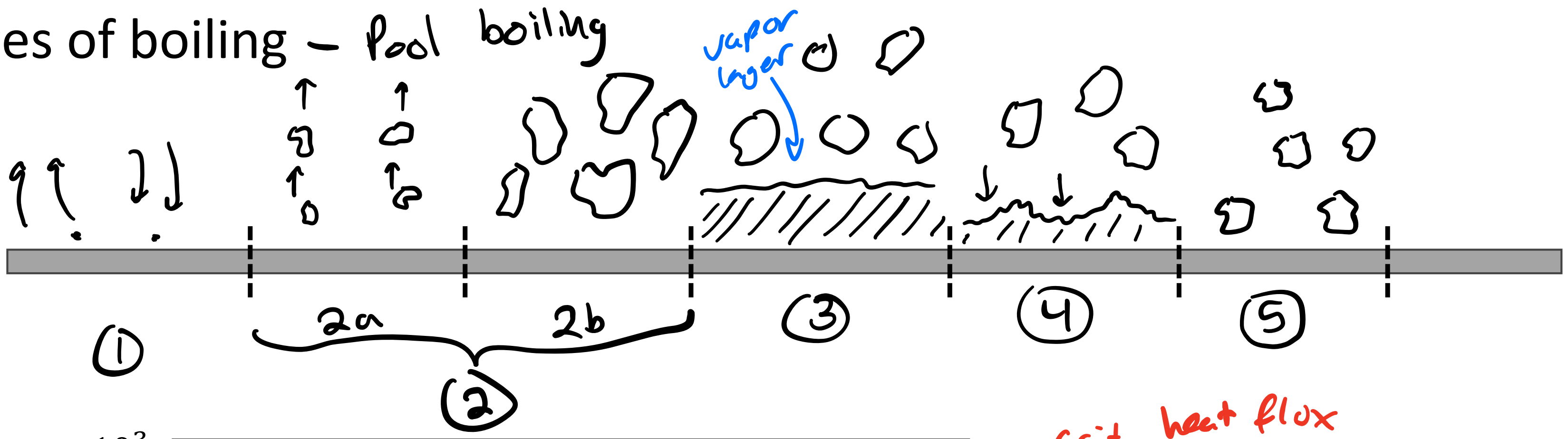
Heat flow into fluid: $\dot{q}''_s = \bar{h}\,\Delta T_E$ ← Excess temp. difference

$$\Delta T_E = T_s - T_{sat}$$

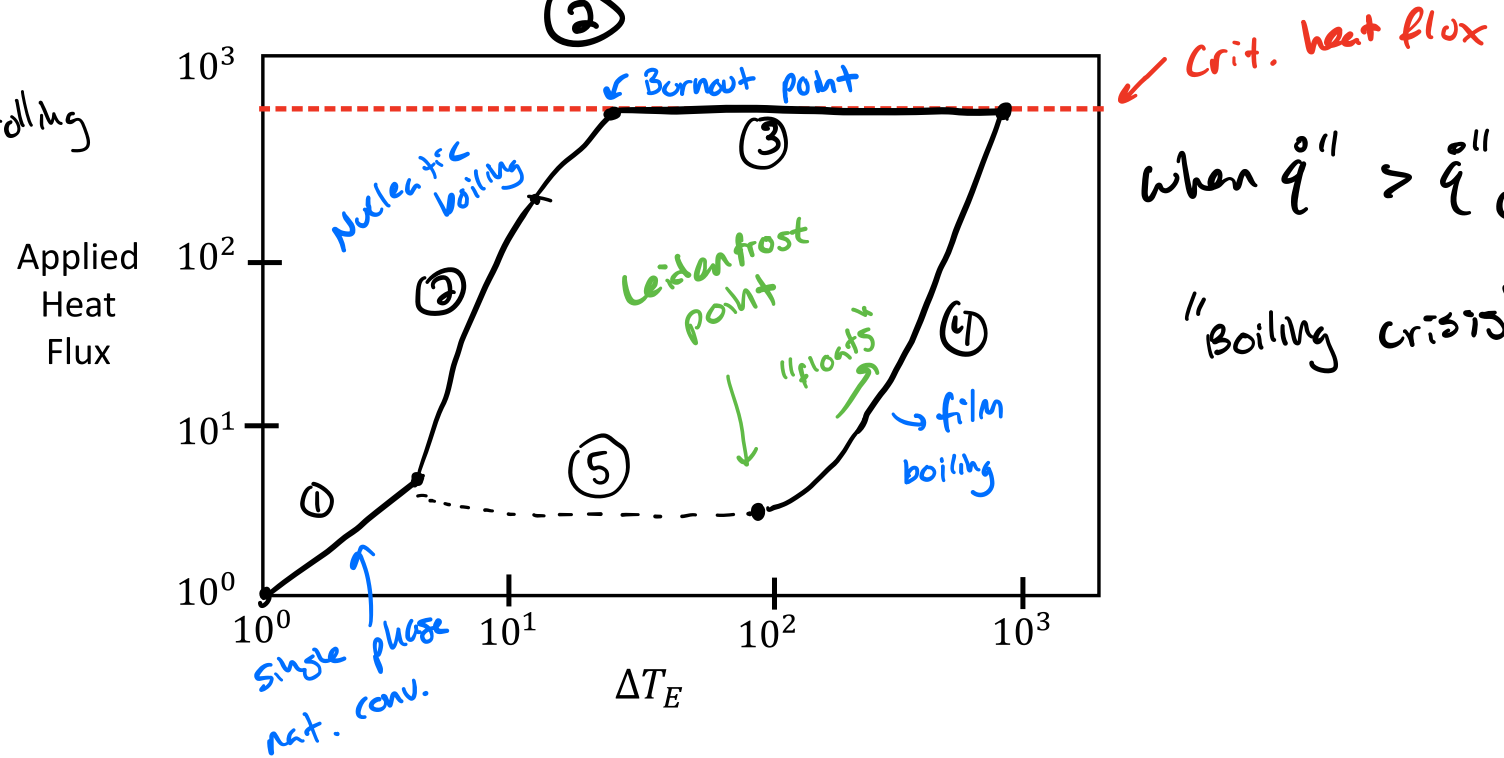
↑ surf. temp ↑ saturation temp

Typically really high $\bar{h}$ values → high heat flux w/ small ΔT

Stages of boiling – Pool boiling



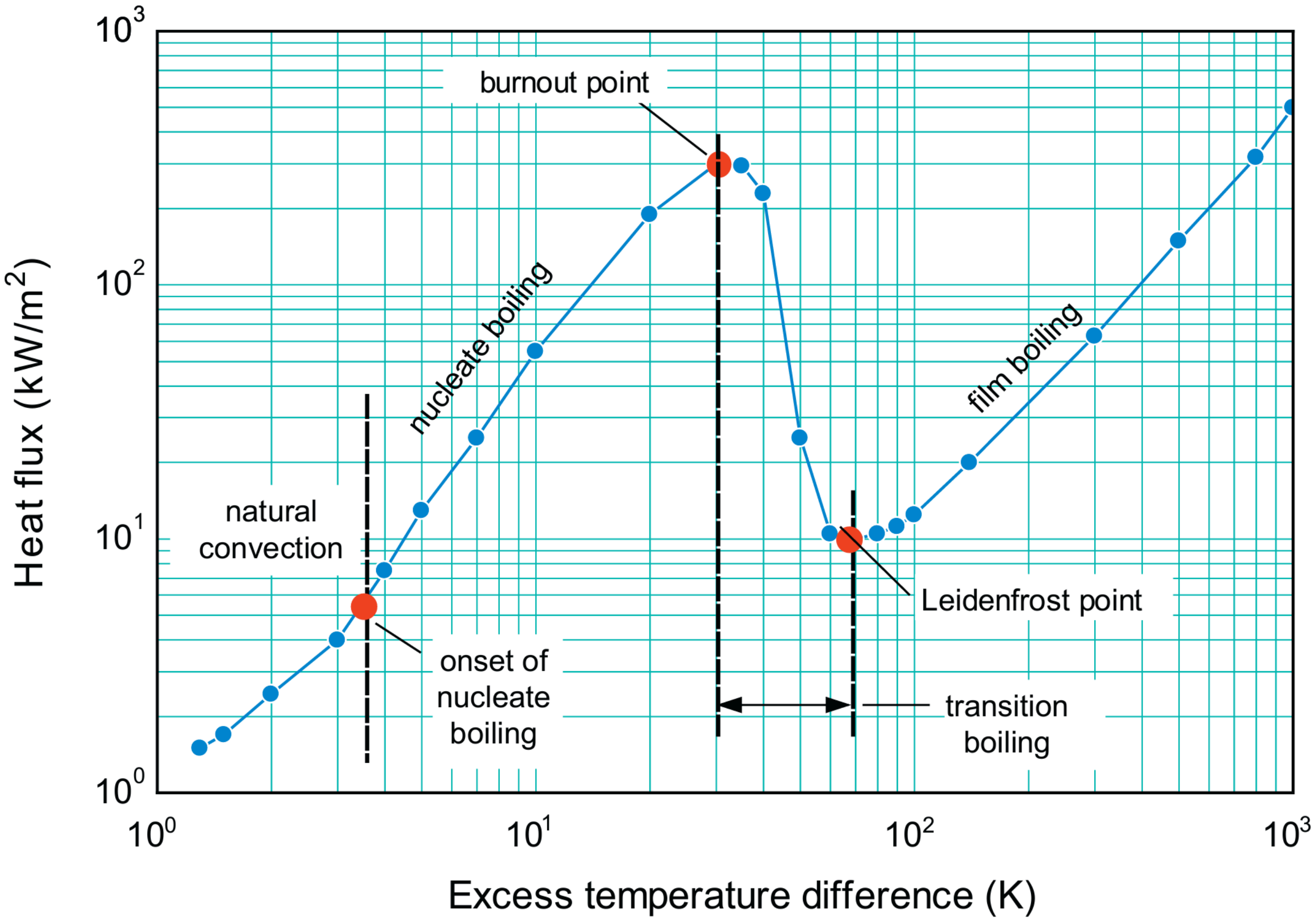
controlling



when $\dot{q}'' > \dot{q}''_{crit}$

"Boiling crisis"

Stages of boiling: Alternate view



Stages of boiling: Alternate view

Nucleate boiling correlation:

$$\dot{q}''_{s,nb} = \mu_{l,sat}\,\Delta i_{vap}\,\sqrt{\frac{g\left(\rho_{l,sat} - \rho_{v,sat}\right)}{\sigma_{ST}}} \cdot \left(\frac{C_{l,sat}\,\Delta T_E}{C_{nb}\,\Delta i_{vap}\,Pr_{l,sat}^{n}}\right)^3$$

$l \rightarrow$ saturated liquid
$v \rightarrow$ saturate vapor
$\sigma_{ST} \rightarrow$ liquid-vapor surface tension
$C_{nb} \rightarrow$ Experimental coefficient. Use $C_{nb} = 0.013$ or Table 11.1
$g \rightarrow$ gravity
$\Delta i_{vap} \rightarrow$ Enthalpy of vaporization

Critical heat flux

"critical_heat_flux" in EES

$$\dot{q}''_{s,crit} = C_{crit}\, \Delta i_{vap}\, \rho_{v,sat} \left[\frac{\sigma_{ST}\, g\left(\rho_{l,sat} - \rho_{v,sat}\right)}{\rho_{v,sat}^2}\right]^{\frac{1}{4}}$$

$C_{crit} = C_{crit}(\tilde{L})$, where $\tilde{L} = \frac{L_{char}}{L_{nb}}$

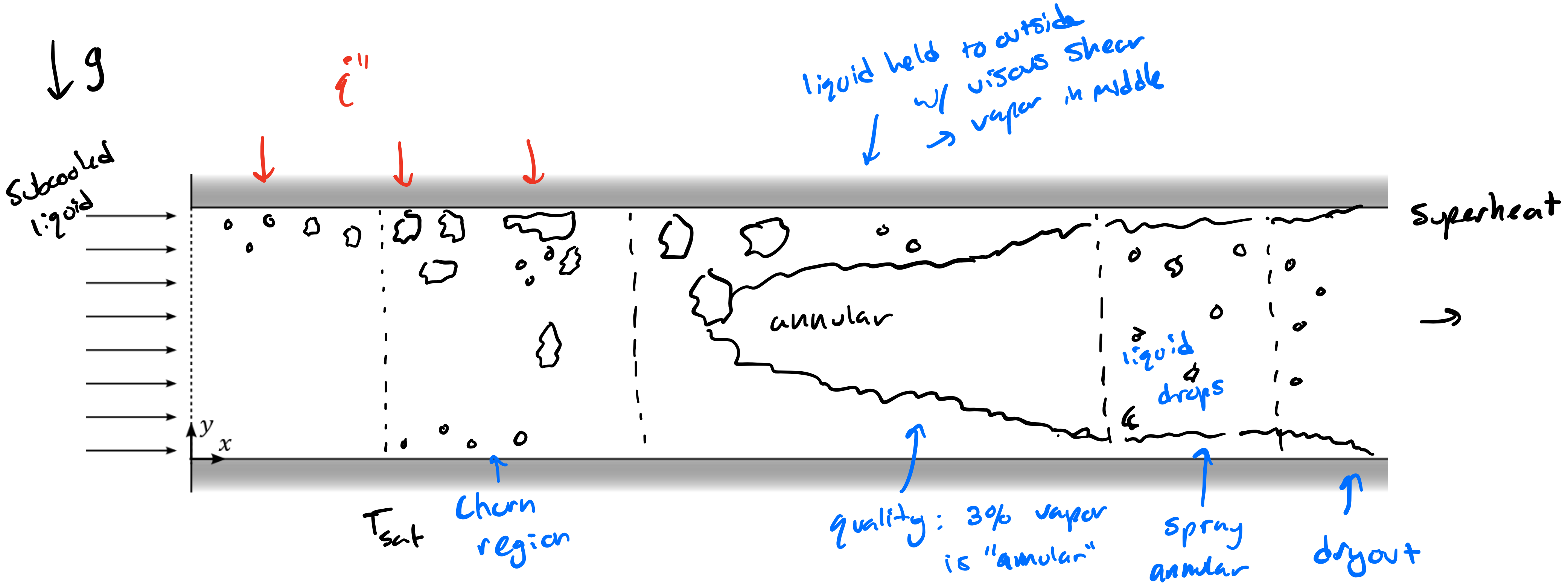
$$L_{nb} = \sqrt{\frac{\sigma_{ST}}{g\left(\rho_{l,sat} - \rho_{v,sat}\right)}}$$

See Table 11.2 for C_{crit}, definition of L_{char}

Table 11.2 Values of C_{crit} for Eq. (11.4) for various heater geometries, from Mills (1992).

Geometry	C_{crit}	Characteristic length	Range of $\tilde{L}$
large flat plate	0.15	width or diameter	$\tilde{L} > 27$
small flat plate	$\frac{9\pi L_{nb}^2}{5A_s}$	width or diameter	$9 < \tilde{L} < 20$
large horizontal cylinder	0.12	cylinder radius	$\tilde{L} > 1.2$
small horizontal cylinder	$\frac{0.12}{\tilde{L}^{0.25}}$	cylinder radius	$0.15 < \tilde{L} < 1.2$
large sphere	0.11	sphere radius	$4.26 < \tilde{L}$
small sphere	$\frac{0.227}{\tilde{L}^{0.5}}$	sphere radius	$0.15 < \tilde{L} < 4.26$
large, finite body	≈ 0.12		

Flow boiling



Flow boiling correlation

Shah flow boiling correlation

$$\bar{h} = \bar{h}(Co, Bo, Fr)$$

Co → convection #

$$Co = \left(\frac{1}{\chi} - 1\right)^{0.8} \sqrt{\frac{\rho_{v,sat}}{\rho_{l,sat}}}$$

x = steam quality (vap. mass fraction)

Bo → Boiling #

$$Bo = \frac{\dot{q}_s''}{G\,\Delta i_{vap}}$$

$G = \dot{m}/A_c$ "mass flux"

Fr → Froude #

$$Fr = \frac{G^2}{\rho_{l,sat}^2\, g\, D_h}$$

Correlation is multiplier on single-phase turbulent flow

$$\bar{h} = \tilde{h}\, h_l$$

Shah

Gnielinski for turbulent flow, liquid phase

Flow boiling correlation

$$N = \begin{cases} Co & \text{for vertical tubes or horizontal tubes with } Fr > 0.04 \\ 0.38\ Co\ Fr^{-0.3} & \text{for horizontal tubes with } Fr \leq 0.04. \end{cases} \tag{11.16}$$

$$\tilde{h}_{cb} = 1.8N^{-0.8} \tag{11.17}$$

Shah flow boiling correlation Implementation

$$\tilde{h}_{nb} = \begin{cases} 230\sqrt{Bo} & \text{if } Bo \geq 0.3 \times 10^{-4} \\ 1 + 46\sqrt{Bo} & \text{if } Bo < 0.3 \times 10^{-4} \end{cases} \tag{11.18}$$

$$\tilde{h}_{bs,\,1} = \begin{cases} 14.70\sqrt{Bo}\exp\left(2.74N^{-0.1}\right) & \text{if } Bo \geq 11 \times 10^{-4} \\ 15.43\sqrt{Bo}\exp\left(2.74N^{-0.1}\right) & \text{if } Bo < 11 \times 10^{-4} \end{cases} \tag{11.19}$$

$$\tilde{h}_{bs,\,2} = \begin{cases} 14.70\sqrt{Bo}\exp\left(2.74N^{-0.15}\right) & \text{if } Bo \geq 11 \times 10^{-4} \\ 15.43\sqrt{Bo}\exp\left(2.74N^{-0.15}\right) & \text{if } Bo < 11 \times 10^{-4} \end{cases} \tag{11.20}$$

$$\tilde{h} = \begin{cases} \mathrm{Max}(\tilde{h}_{cb}, \tilde{h}_{bs,\,2}) & \text{if N} \leq 0.1 \\ \mathrm{Max}(\tilde{h}_{cb}, \tilde{h}_{bs,\,1}) & \text{if } 0.1 < N \leq 1.0\,. \\ \mathrm{Max}(\tilde{h}_{cb}, \tilde{h}_{nb}) & \text{if } N > 1.0 \end{cases} \tag{11.21}$$

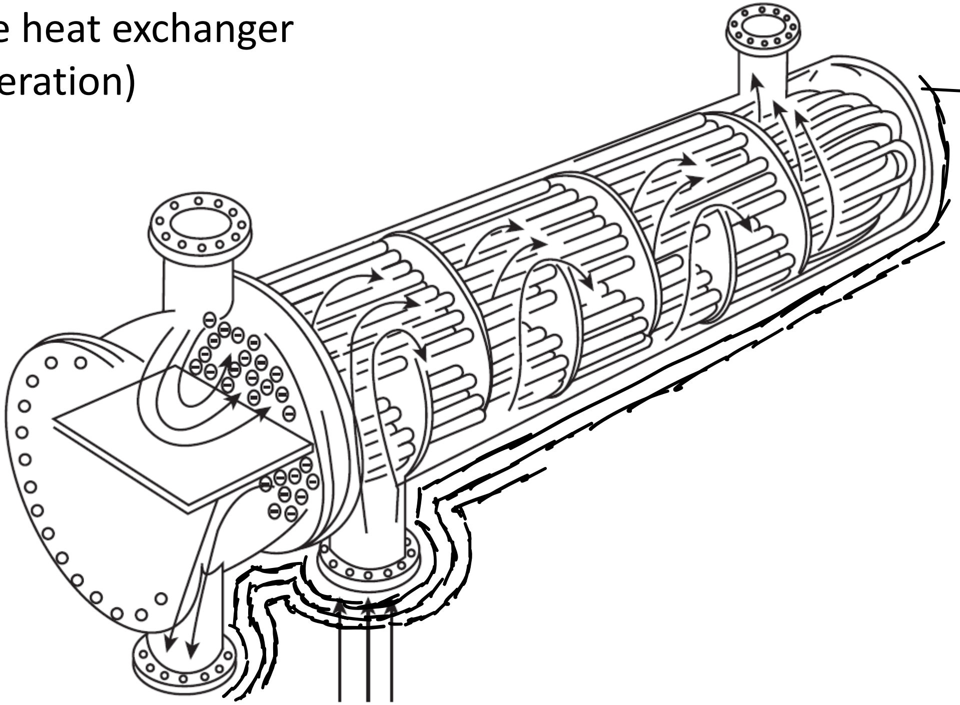
Lecture 19

Introduction to heat exchangers

Heat Exchangers

Define: A heat exchanger is a device used to continuously transfer heat from one fluid stream to another

Shell & tube heat exchanger
(power generation)



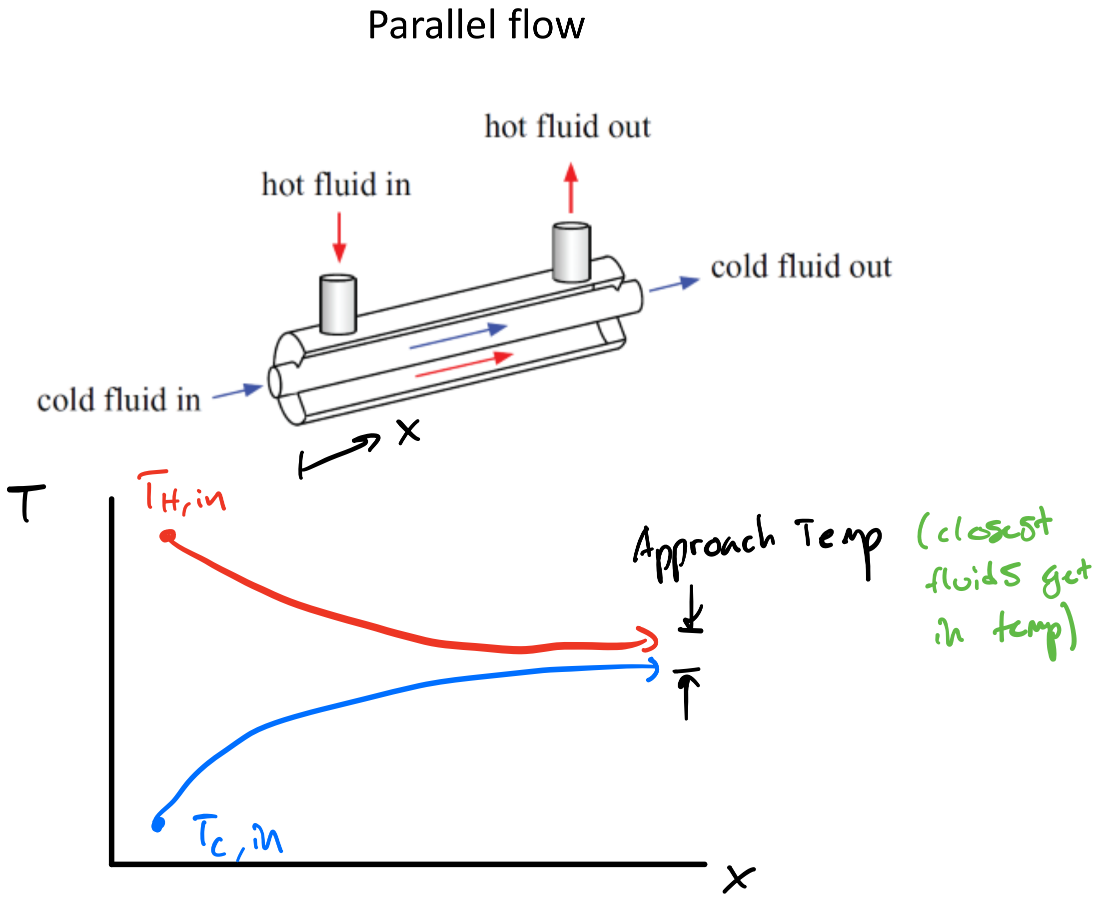
Examples:

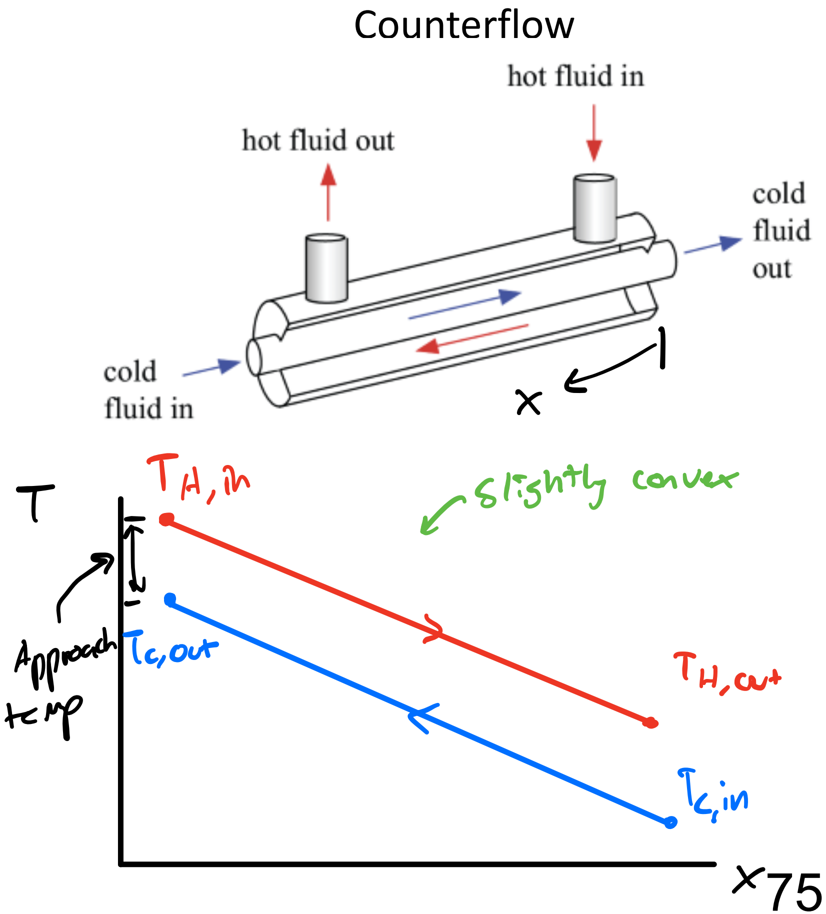
- Refrigerator
- Furnace/AC
- Power cycles
- Radiators
- ...

Heat exchanger flow configuration

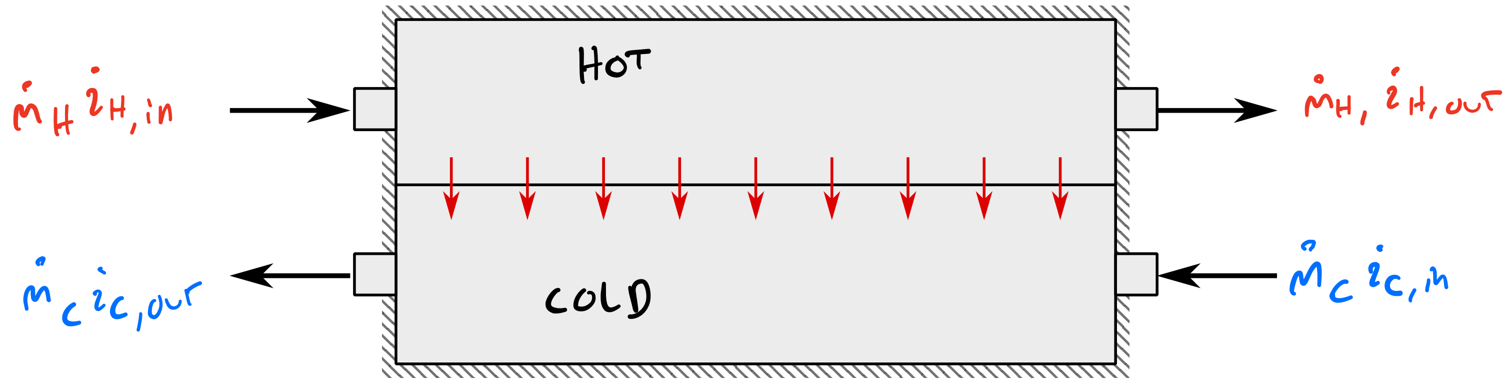
directions of flows

Heat exchanger performance depends strongly on flow configuration





Heat exchanger energy balance



$$\dot{m}_H i_{H,in} + \dot{m}_c i_{c,in} - \dot{m}_H i_{H,out} - \dot{m}_c i_{c,out} = 0$$

Incompressible $c(T - T_{ref})$

$$\dot{m}_H c_H (T_{H,in} - T_{H,out}) = \dot{m}_c c_c (T_{c,out} - T_{c,in})$$

Heat exchanger definitions

$\dot{m} \cdot c \left[\frac{W}{K}\right]$ "capacitance rate" $= \dot{C}$

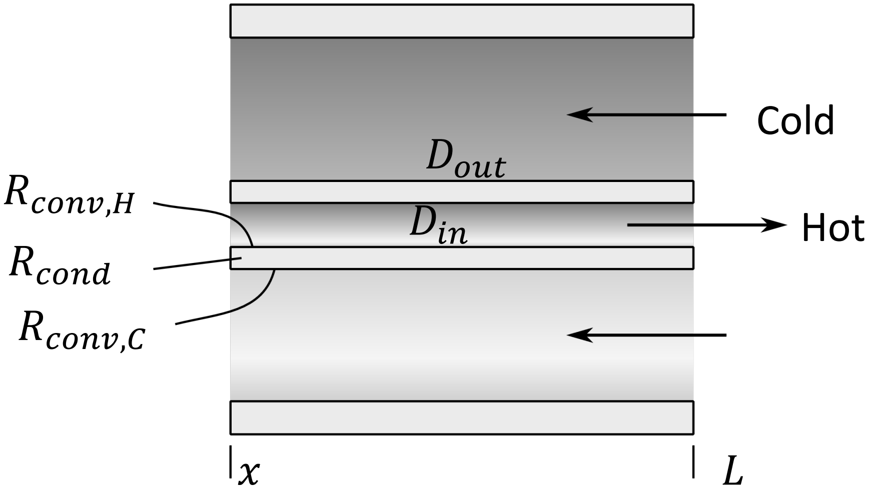
$\dot{C}$

$$\boxed{\dot{q} = \dot{C}_H (T_{H,in} - T_{H,out}) = \dot{C}_C (T_{C,out} - T_{C,in})}$$

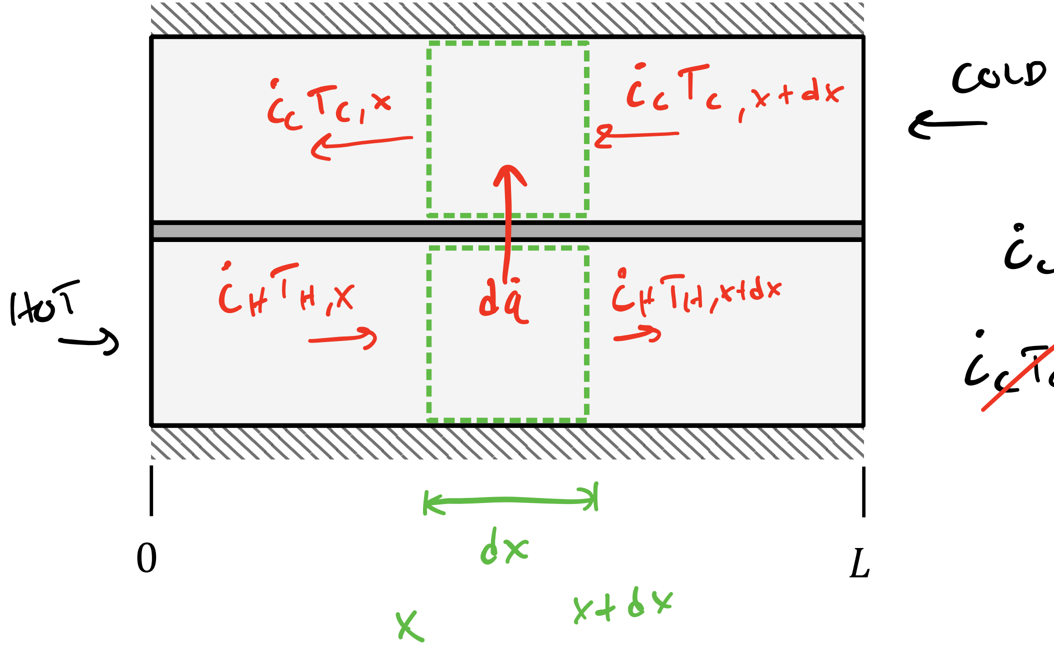
$UA = \frac{1}{R_{tot}}$ "conductance"   High $UA \rightarrow$ Good HX

$$R_{tot} = R_{conv,H} + R_{cond} + R_{conv,C}$$

$$= \frac{1}{\bar{h}_H \pi D_{in} L} + \frac{\ln\left(\frac{D_{out}}{D_{in}}\right)}{2\pi k_t L} + \frac{1}{\bar{h}_C \pi D_{out} L}$$



Heat exchanger energy balance (1)



How do we incorporate UA?

Cold side E-bal: IN − OUT = 0

$$\dot{C}_c T_{c,x+dx} - \dot{C}_c T_{c,x} + d\dot{q} = 0$$

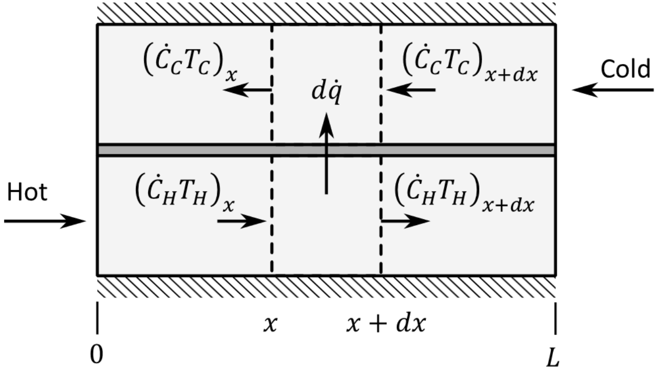
$$\cancel{\dot{C}_c T_{c,x}} + \dot{C}_c \frac{dT_{c,x}}{dx} dx - \cancel{\dot{C}_c T_{c,x}} + d\dot{q} = 0$$

$$d\dot{q} = -\dot{C}_c \frac{dT_{c,x}}{dx} dx$$

Hot side:

$$\cancel{\dot{C}_H T_{H,x}} - \dot{C}_H \cancel{T_{H,x}} - \frac{dT_{H,x}}{dx} dx - d\dot{q} = 0 \rightarrow d\dot{q} = -\dot{C}_H \frac{dT_{H,x}}{dx} dx$$

Heat exchanger energy balance (2)



Use local heat transfer

$$d\dot{q} = \frac{T_H - T_C}{R_{CV}} \qquad R_{CV} = R_{tot} \cdot \frac{L}{dx}$$

$$d\dot{q} = (T_H - T_C)\frac{UA \cdot dx}{L} \rightarrow E\cdot bal: \quad \frac{T_H - T_C}{L} UA\, \cancel{dx} = -\dot{C}_H \frac{dT_{H,x}}{dx} \cancel{dx}$$

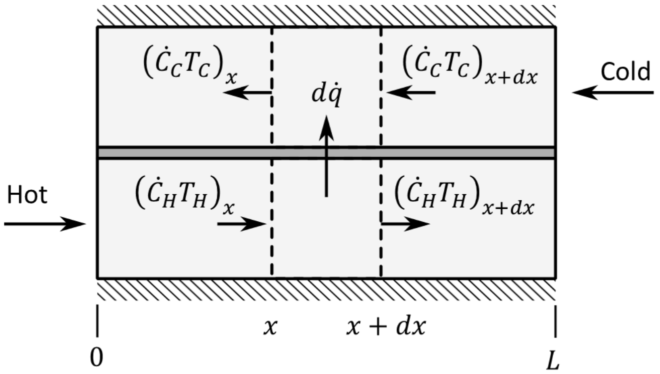
Hot

$$\frac{dT_{H,x}}{dx} = -\frac{UA}{\dot{C}_H L}(T_{H,x} - T_{C,x})$$

COLD

$$\frac{dT_{C,x}}{dx} = -\frac{UA}{\dot{C}_C L}(T_{H,x} - T_{C,x})$$

Heat exchanger energy balance (3)



$$\frac{d}{dx}\underbrace{\left[T_{H,x} - T_{C,x}\right]}_{\theta_x} = -\frac{UA}{L}\left(T_{H,x} - T_{C,x}\right)\left(\frac{1}{\dot{C}_H} - \frac{1}{\dot{C}_C}\right)$$

$$\frac{d\theta}{dx} = -\frac{UA}{L}\,\theta\left(\frac{1}{\dot{C}_H} - \frac{1}{\dot{C}_C}\right) \rightarrow \int_{\theta_{x=0}}^{\theta_{x=L}} \frac{d\theta}{\theta} = -\frac{UA}{L}\left(\frac{1}{\dot{C}_H} - \frac{1}{\dot{C}_C}\right)\int_{x=0}^{x=L} dx$$

$$\ln\left(\frac{\theta_{x=L}}{\theta_{x=0}}\right) = -\frac{UA}{\cancel{L}}\left(\frac{1}{\dot{C}_H} - \frac{1}{\dot{C}_C}\right)\cdot \cancel{L}$$

Counterflow HX

$$\boxed{\ln\left(\frac{T_{H,out} - T_{C,in}}{T_{H,in} - T_{C,out}}\right) = -UA\left(\frac{1}{\dot{C}_H} - \frac{1}{\dot{C}_C}\right)}$$

Parallel flow:

$$\ln\left(\frac{T_{H,out} - T_{C,out}}{T_{H,in} - T_{C,in}}\right) = -UA\left(\frac{1}{\dot{C}_C} + \frac{1}{\dot{C}_H}\right)$$

Lecture 20

Effectiveness-NTU Solutions

Statements of HX performance

Log mean temperature difference

$$\dot{q} = UA \cdot \Delta T_{LM} \leftrightarrow \dot{q} = \bar{h} A_s (T_H - T_C)$$

Counterflow:

$$\dot{q} = \dot{C}_C (T_{C,out} - T_{C,in})$$

$$\dot{q} = \dot{C}_H (T_{H,in} - T_{H,out})$$

$$\dot{C}_C = \frac{\dot{q}}{(T_{C,out} - T_{C,in})}$$

$$\dot{C}_H = \frac{\dot{q}}{(T_{H,in} - T_{H,out})}$$

Sub into UA

Statements of HX performance

$$\ln\left(\frac{T_{H,out} - T_{C,in}}{T_{H,in} - T_{C,out}}\right) = -UA\left(\frac{1}{\dot{C}_H} - \frac{1}{\dot{C}_C}\right) \qquad \dot{C}_C = \frac{\dot{q}}{(T_{C,out} - T_{C,in})} \qquad \dot{C}_H = \frac{\dot{q}}{(T_{H,in} - T_{H,out})}$$

$$\ln\left(\frac{T_{H out} - T_{C in}}{T_{H in} - T_{C out}}\right) = -UA\left(\frac{T_{H in} - T_{H out} - T_{C out} + T_{C in}}{\dot{q}}\right)$$

$$\dot{q} = -UA\underbrace{\left[\frac{T_{H out} - T_{H in} + T_{C out} - T_{C in}}{\ln\left(\frac{T_{H out} - T_{C in}}{T_{H in} - T_{C out}}\right)}\right]}_{\Delta T_{LM}}$$

log mean temp difference

Effectiveness-NTU solutions (ε-NTU)

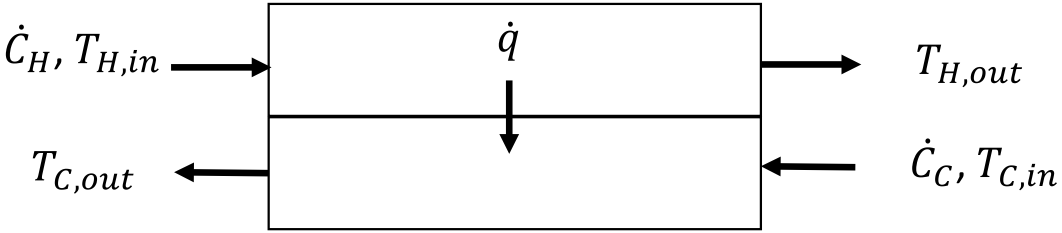
$\varepsilon \rightarrow$ effectiveness $[0 \ldots 1]$

NTU $\rightarrow$ Number of transfer units $[0 \ldots \infty]$

What is the maximum rate of heat transfer?

$$\dot{q}_{max,H} = \dot{C}_H \left(T_{H,in} - T_{C,in}\right)$$

$$\dot{q}_{max,C} = \dot{C}_C \left(T_{H,in} - T_{C,in}\right)$$



$$\dot{q}_{max} = \underbrace{\min\left(\dot{C}_H, \dot{C}_C\right)}_{\dot{C}_{min}} \cdot \left(T_{H,in} - T_{C,in}\right)$$

Effectiveness-NTU solutions (ε-NTU)

$$\varepsilon = \frac{\dot{q}}{\dot{q}_{max}} \quad \begin{matrix}\text{(actual)} \\ \text{(theoretical / limit)}\end{matrix} \qquad \varepsilon = \frac{\dot{q}}{\dot{C}_{min}(T_{H,in} - T_{C,in})}$$

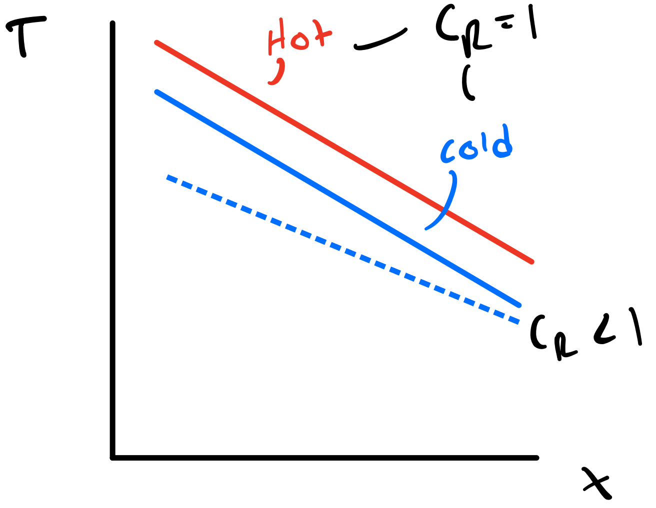
NTU - Non dimensional "size" of the HX

$$NTU = \frac{UA}{\dot{C}_{min}}$$

Capacitance ratio

$$\dot{C}_R = \frac{\dot{C}_{min}}{\dot{C}_{max}} = \frac{\min(\dot{C}_H, \dot{C}_C)}{\max(\dot{C}_H, \dot{C}_C)}$$

$[0 \ldots 1]$



Characterizing HX performance without temperatures

$$\ln\left(\frac{T_{H,out} - T_{C,in}}{T_{H,in} - T_{C,out}}\right) = -UA\left(\frac{1}{\dot{C}_H} - \frac{1}{\dot{C}_C}\right)$$

Counterflow example

$$T_{C,out} = T_{C,in} + \frac{\dot{q}}{\dot{C}_C} \rightarrow T_{C,out} = T_{C,in} + \frac{\varepsilon \cdot \dot{C}_{min}(T_{H,in} - T_{C,in})}{\dot{C}_C}$$

$$T_{H,out} = T_{H,in} - \frac{\dot{q}}{\dot{C}_H} \rightarrow T_{H,out} = T_{H,in} - \frac{\varepsilon \cdot \dot{C}_{min}(T_{H,in} - T_{C,in})}{\dot{C}_H}$$

$$\ln\left[\frac{\cancel{(T_{H,in} - T_{C,in})}\left(1 - \frac{\varepsilon \dot{C}_{min}}{\dot{C}_H}\right)}{\cancel{(T_{H,in} - T_{C,in})}\left(1 - \frac{\varepsilon \dot{C}_{min}}{\dot{C}_C}\right)}\right] = -UA\left[\frac{1}{\dot{C}_H} - \frac{1}{\dot{C}_C}\right]$$

Counterflow HX performance equation

pick one, doesn't end up mattering

$\dot{C}_c = \dot{C}_{min}$

$\dot{C}_h = \dot{C}_{max}$

$$\ln\left(\frac{1-\frac{\varepsilon\,\dot{C}_{min}}{\dot{C}_H}}{1-\frac{\varepsilon\,\dot{C}_{min}}{\dot{C}_C}}\right) = -UA\left(\frac{1}{\dot{C}_H}-\frac{1}{\dot{C}_C}\right)$$

$$\ln\left(\frac{1-\varepsilon\overbrace{\frac{\dot{C}_{min}}{\dot{C}_{max}}}^{C_R}}{1-\varepsilon\underbrace{\frac{\dot{C}_{min}}{\dot{C}_{min}}}_{=1}}\right) = -\underbrace{\frac{UA}{\dot{C}_{min}}}_{NTU}\left(\underbrace{\frac{\dot{C}_{min}}{\dot{C}_{max}}}_{C_R}-\underbrace{\frac{\dot{C}_{min}}{\dot{C}_{min}}}_{1}\right)\cdot\dot{C}_{min}$$

if $C_R = 1$:

$$\varepsilon = \frac{NTU}{1+NTU}$$

$$\boxed{\ln\left(\frac{1-\varepsilon C_R}{1-\varepsilon}\right) = -NTU\,(C_R-1)} \rightarrow \varepsilon = \frac{1-\exp(-NTU(1-C_R))}{1-C_R\exp(-NTU(1-C_R))}$$

Table 12.2 Effectiveness–*NTU* relations for various heat exchanger configurations in the form effectiveness as a function of number of transfer units and capacity ratio.

Flow arrangement		$\varepsilon\,(NTU, C_R)$
One fluid (or any configuration with $C_R = 0$)		$\varepsilon = 1 - \exp(-NTU)$
Counter-flow		$\varepsilon = \begin{cases} \dfrac{1-\exp[-NTU(1-C_R)]}{1-C_R\exp[-NTU(1-C_R)]} & \text{for } C_R < 1 \\ \dfrac{NTU}{1+NTU} & \text{for } C_R = 1 \end{cases}$
Parallel-flow		$\varepsilon = \dfrac{1-\exp[-NTU(1+C_R)]}{1+C_R}$
Cross-flow	both fluids unmixed	$\varepsilon = 1 - \exp\left[\dfrac{NTU^{0.22}}{C_R}\{\exp(-C_R NTU^{0.78}) - 1\}\right]$
	both fluids mixed	$\varepsilon = \left[\dfrac{1}{1-\exp(-NTU)} + \dfrac{C_R}{1-\exp(-C_R NTU)} - \dfrac{1}{NTU}\right]^{-1}$
	$\dot{C}_{max}$ mixed and $\dot{C}_{min}$ unmixed	$\varepsilon = \dfrac{1-\exp[C_R\{\exp(-NTU) - 1\}]}{C_R}$
	$\dot{C}_{min}$ mixed and $\dot{C}_{max}$ unmixed	$\varepsilon = 1 - \exp\left[-\dfrac{1-\exp(-C_R\,NTU)}{C_R}\right]$

See Table pp 792 for more

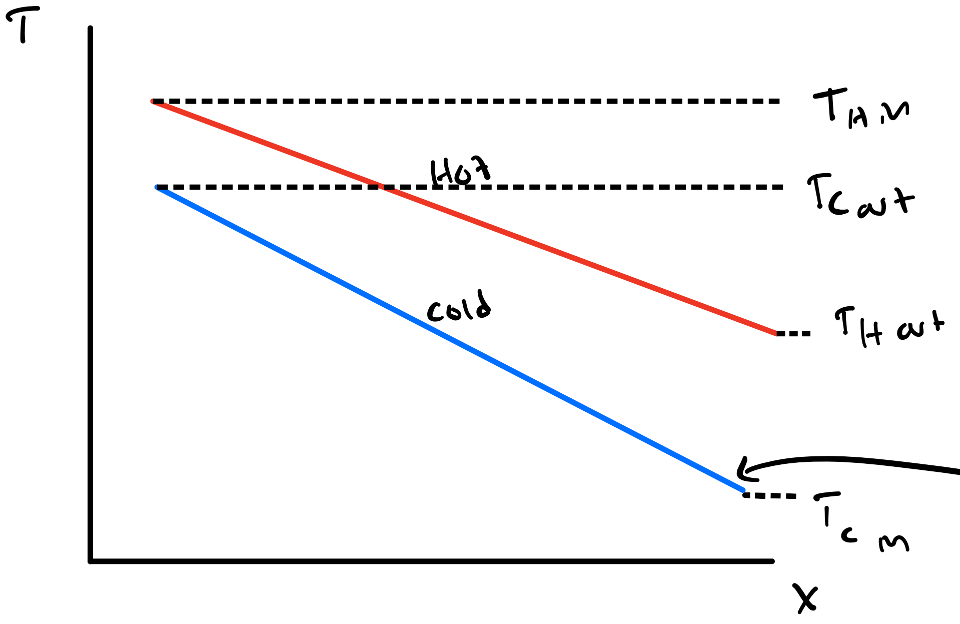
Solutions programmed in EES

Solution given in terms of ε or NTU (Table 12.3)

Understanding ε

lower cap. rate → req. larger ΔT for $\dot{q}$

$$\varepsilon = \frac{\Delta T_{C\,min}}{\Delta T_{max}}$$



$\Delta T_{C\,min}$

ΔT_{max}

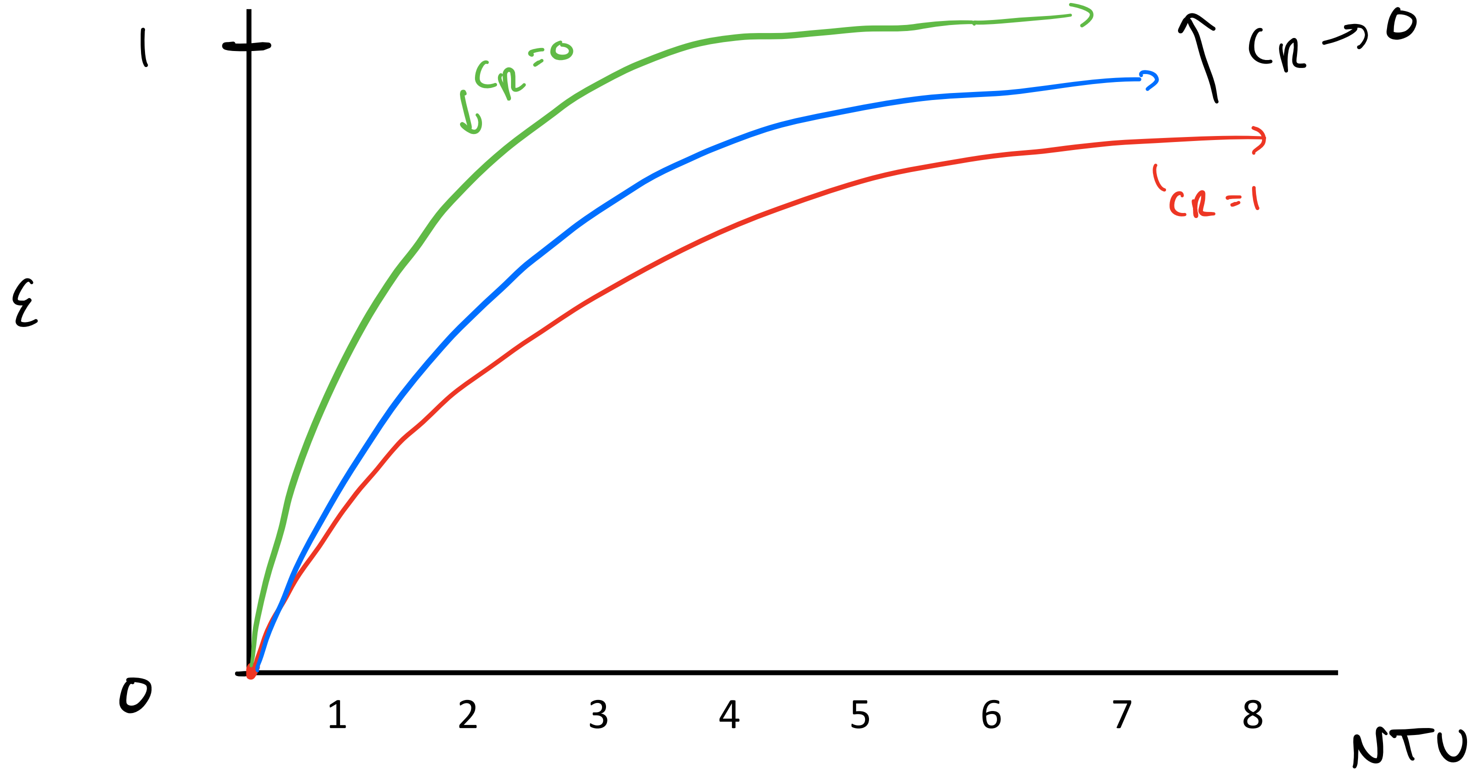
lower cap. rate
(absorbs less heat for given temp)
changes temp most

Limiting cases

Counterflow

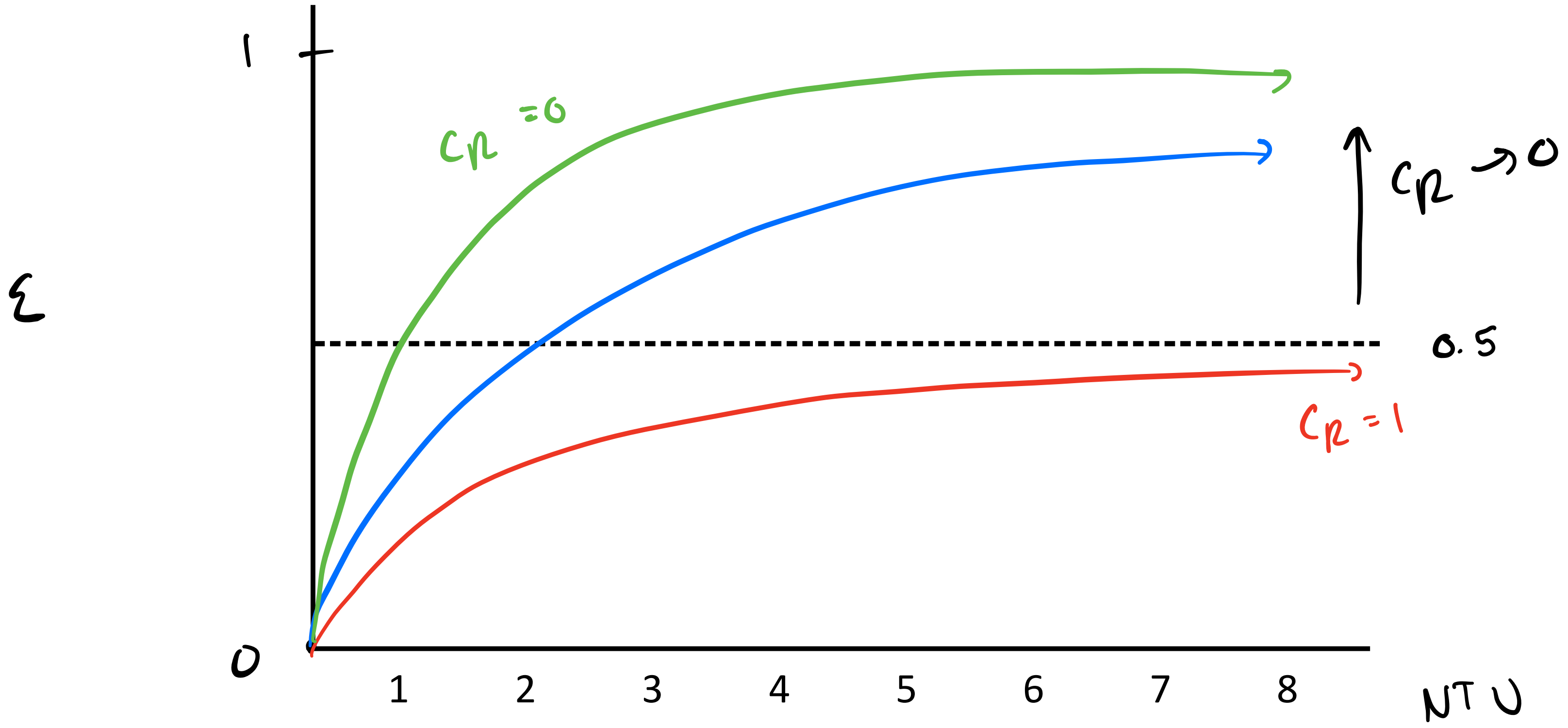
What happens when $NTU \to 0$, or $\to \infty$?

What happens when $C_R <$ [redacted]?



Effectiveness in parallel flow heat exchangers

What happens when $C_R = 1$? Other values?



Lecture 21

Radiation Concepts and Blackbody Radiation

Last time…

Effectiveness-NTU Solutions

- One method for estimating heat transfer: log mean temperature difference $\dot{q} = UA \cdot \Delta T_{lm}$
- Effectiveness is ratio of actual heat transfer to maximum limit $\varepsilon = \dfrac{\dot{q}}{\dot{C}_{min}\left(T_{H,in} - T_{C,in}\right)}$
- NTU is non-dimensional size of heat exchanger $NTU = \dfrac{UA}{\dot{C}_{min}}$
- System of equations needed to solve HX problem:
  - Unknowns: $T_{c,out}, T_{h,out}, \dot{q}$
  - Equations: 2 energy balances; conductance → NTU → effectiveness → Actual heat transfer
- Discussed behavior of effectiveness vs NTU at different capacitance ratios C_R

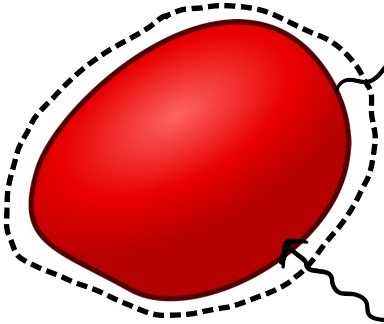
Basis for radiation rate equation

S.B. const. — surface area

$$\dot{q}_{rad} = \sigma \varepsilon A_s \left(T_s^4 - T_\infty^4 \right)$$

emissivity — surf. T — surr. T

$$\dot{q}_{rad} = \underbrace{\sigma \varepsilon A_s \underbrace{\left(T_s^2 + T_\infty^2 \right)\left(T_s + T_\infty \right)}_{\approx 4 \bar{T}^3}}_{\approx \bar{h}_{rad} \cdot A_s} \left(T_s - T_\infty \right)$$



$\dot{q}_{s_{out}} = \sigma \varepsilon A_s T_s^4$

$\dot{q}_{s,in} = \sigma \varepsilon A_s T_\infty^4$

Net: $\dot{q}_s = \sigma \varepsilon A_s \left(T_s^4 - T_\infty^4 \right)$

Blackbody emissive power

Max power emitted from surface:

$$\frac{\dot{q}}{A_s} = \sigma T^4 \qquad \sigma = 5.67 \times 10^{-8} \ \frac{W}{m^2 - K^4}$$

[K]

σT^4 assumes:

- Emitted power over all wavelengths (total spectral)
- " " " " directions (total hemispherical)
- Emissivity is $\varepsilon = 1$ at all wavelengths (black body)

emissivity is equal to absorptivity

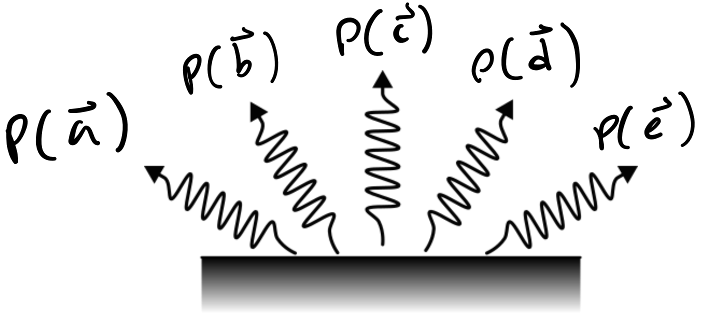
Define Radiation: Energy transfer to or from a surface via electromagnetic waves

- Surface phenomenon
- Non-local
- can travel through a vacuum / transparent media
- Power of a "ray" of radiation is related to EM frequency

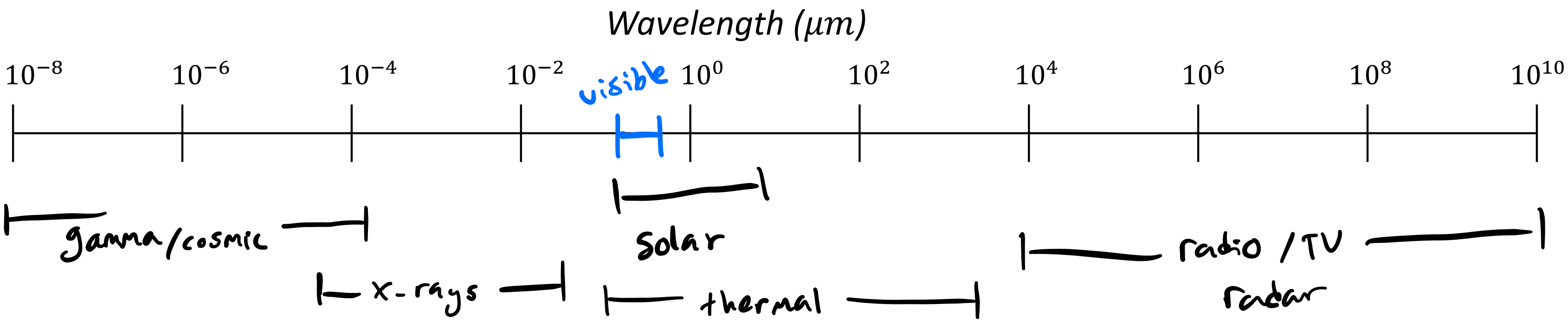
Radiation characteristics

$\rho(\vec{a}) = \rho(\vec{b}) \ldots$ "Diffuse surface"

Radiation is fully independant of direction!



Spectral (wavelength) dependence



Blackbody emissive spectral power

$T_{sun} = 5800\,K$

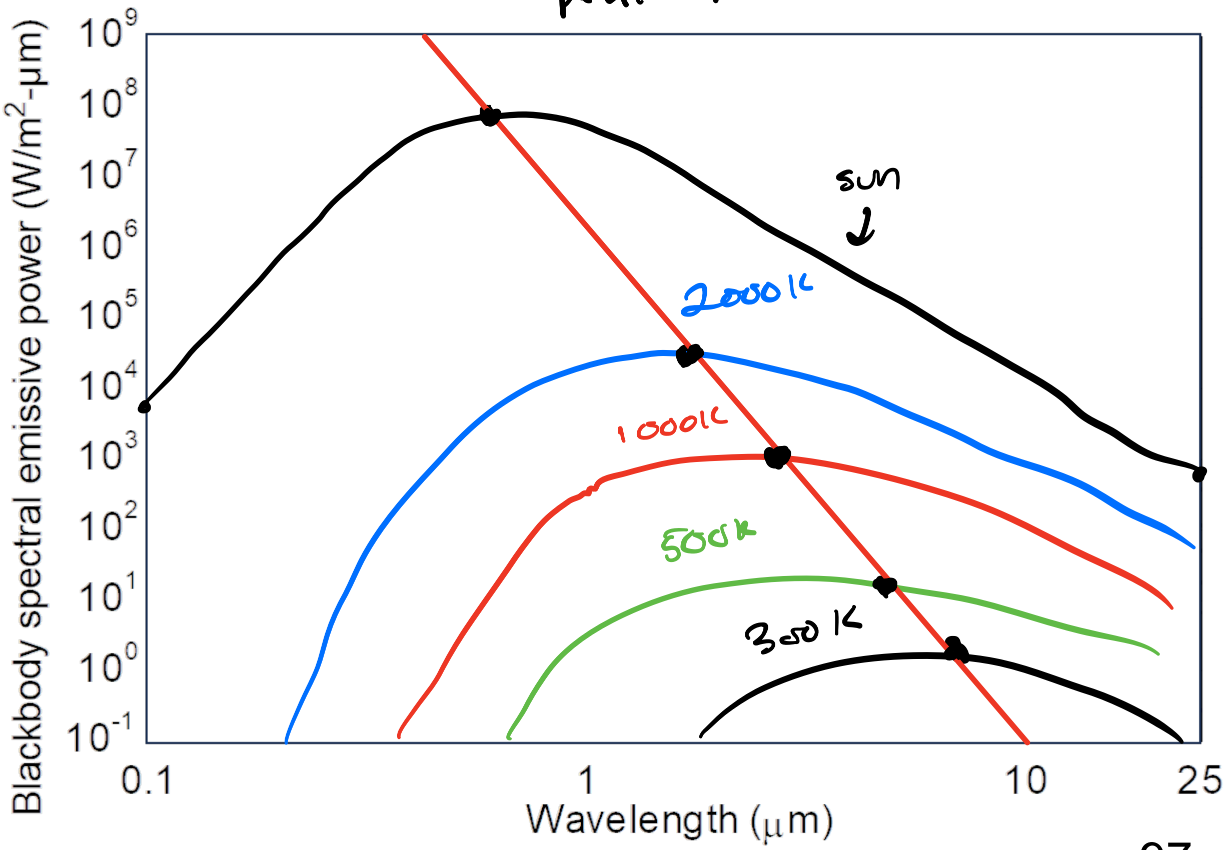
$E_{b,\lambda}$ blackbody spectral emissive power $\left[\frac{W}{m^2\,\mu m}\right]$

Wien's displacement law

$$\lambda_{max} \cdot T = 2898\ \mu m\text{-}K$$

definition of temperature

Difference in wavelength of peak radiation



Planck's Law

$$E_{b,\lambda} = \frac{C_1}{\lambda^5\left(\exp\left(\frac{C_2}{\lambda T}\right)-1\right)}$$

$C_1 = 3.742 \times 10^8 \ \frac{W}{\mu m^4 \cdot m^2}$

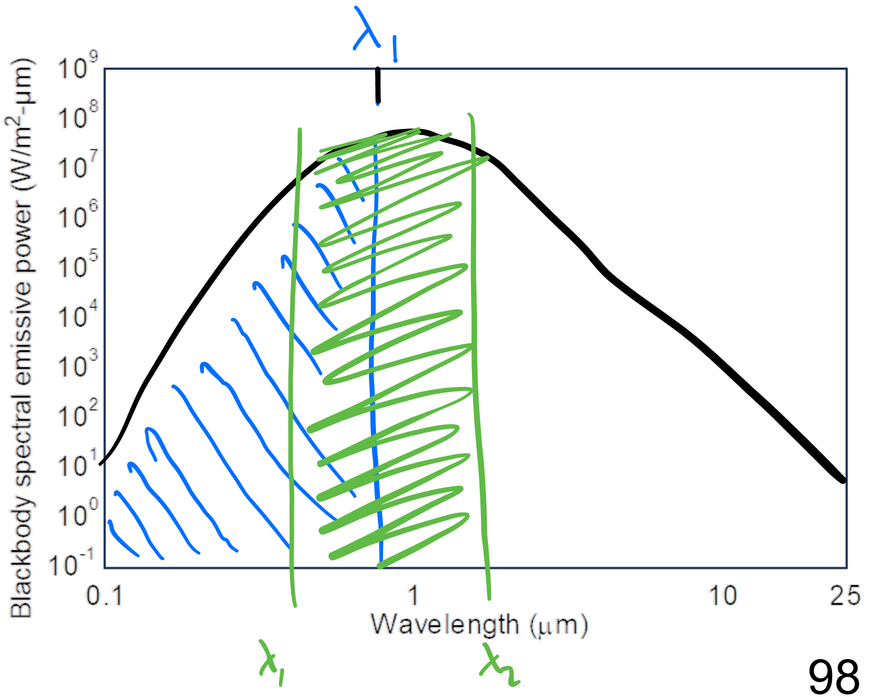
$C_2 = 14388 \ \mu m \cdot K$

$\frac{W}{m^2 \cdot \mu m}$

Total blackbody emissive power

$$E_b = \int_0^\infty E_{b,\lambda}\, d\lambda = \sigma T^4$$

$$E_{b,0-\lambda} = \int_0^{\lambda_1} \frac{C_1}{\lambda^5\left(\exp\left(\frac{C_2}{\lambda T}\right)-1\right)}\, d\lambda$$



External fractional function

$$F_{0-\lambda_1} = \frac{\mathcal{E}_{b,0-\lambda_1}}{\mathcal{E}_b} = \frac{\int_0^{\lambda_1} \frac{c_1}{\lambda^5\left(\exp\left(\frac{c_2}{\lambda T}\right)-1\right)} d\lambda}{\sigma T^4}$$

$$u = \lambda T \rightarrow d\lambda = \frac{du}{T}$$

$$F_{0-\lambda_1} = \int_0^{\lambda_1 T} \frac{c_1}{\sigma (\lambda T)^5 \left(\exp\left(\frac{c_2}{\lambda T}\right)-1\right)} d(\lambda T)$$

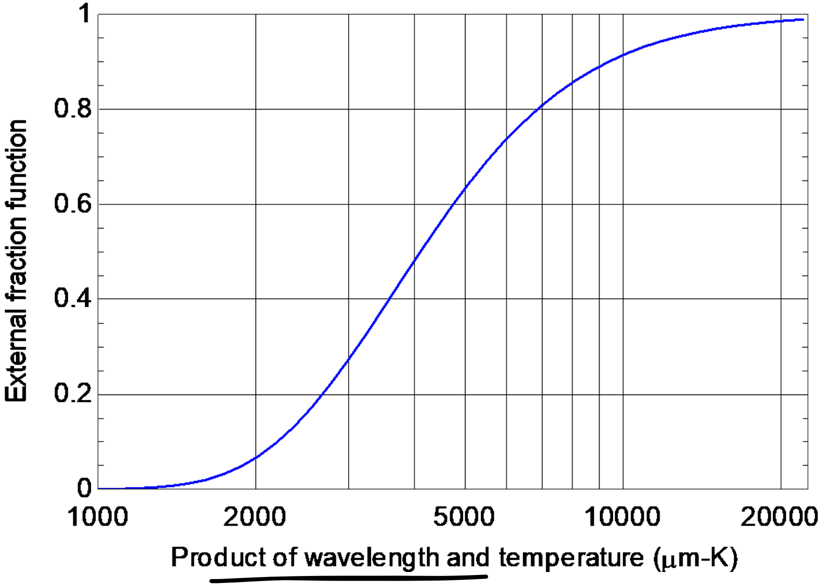
$$F(\lambda_2 T) - F(\lambda_1 T)$$

$0.38\ \mu m \rightarrow 0.78\ \mu m$

$\rightarrow F(0.78 \cdot 5800) - F(0.38 \cdot 5800)$

$= 0.56 - 0.1 = \boxed{0.46}$

46 % visible from sun



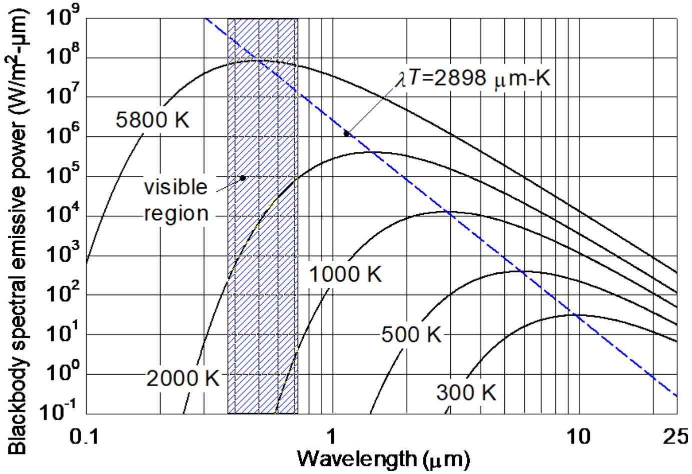
λT

Lecture 22

Radiation View Factors

Last time…

Radiation Concepts and Blackbody Radiation



$$\frac{\dot{q}}{A_s} = \sigma T^4$$

- Blackbody radiation; all surfaces emit EM radiation
- Planck's law predicts radiation from BB at given wavelength, temperature

$$E_{b,\lambda} = \frac{C_1}{\lambda^5 \left(\exp\left(\frac{C_2}{\lambda T}\right) - 1\right)}$$

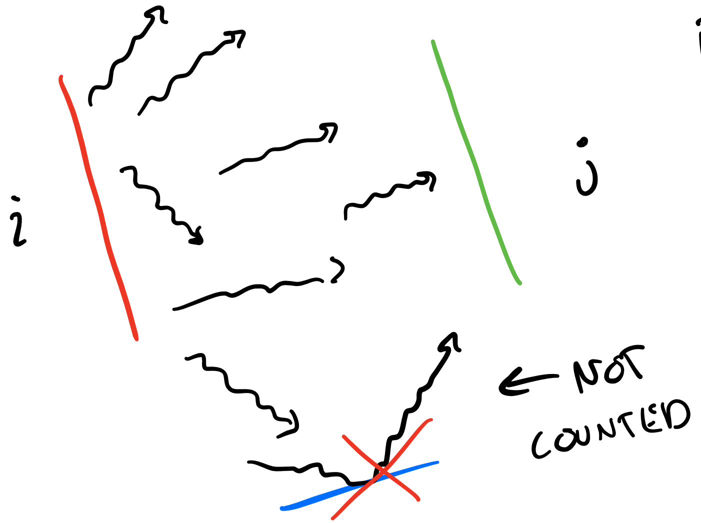
can predict most likely wavelength given T

- Wien's displacement law relates temperature and peak emissive wavelength
- External fractional function used to determine fraction of power emitted in given wavelength band from a surface at some temperature

Radiation view factors

View factor

$$F_{ij} = \frac{\text{radiation leaving surface } i \text{ that goes directly to } j}{\text{total radiation leaving } i}$$

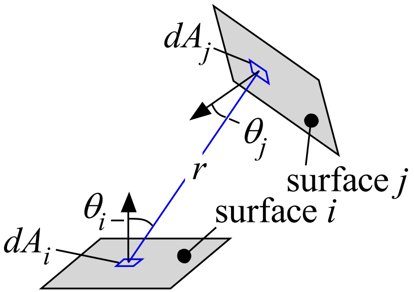


$$\dot{q}_{ij} = F_{ij}\,\sigma A_i \left(T_i^4 - T_j^4\right)$$ ↙ Blackbody ($\varepsilon = 1$)

$$R^g_{ij} = \frac{1}{F_{ij} \cdot A_i} \left[\frac{1}{m^2}\right]$$ "geometric resistance"

→ different than thermal Res.

Evaluating view factors



$$A_i\,F_{i,j} = A_j\,F_{j,i} = \int_{A_j}\int_{A_i} \frac{\cos(\theta_i)\cos(\theta_j)}{\pi\,r^2}\,dA_i\,dA_j$$

Methods for evaluating view factors

A.) Inspection

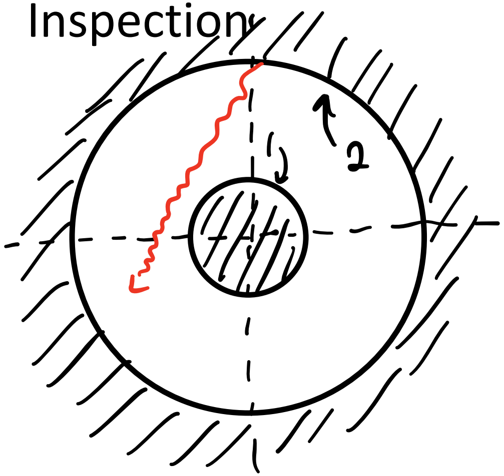
B.) Rules

1. Enclosure
2. Reciprocity
3. Consolidation
4. Symmetry
5. Hottel's crossed strings method

C.) Analysis

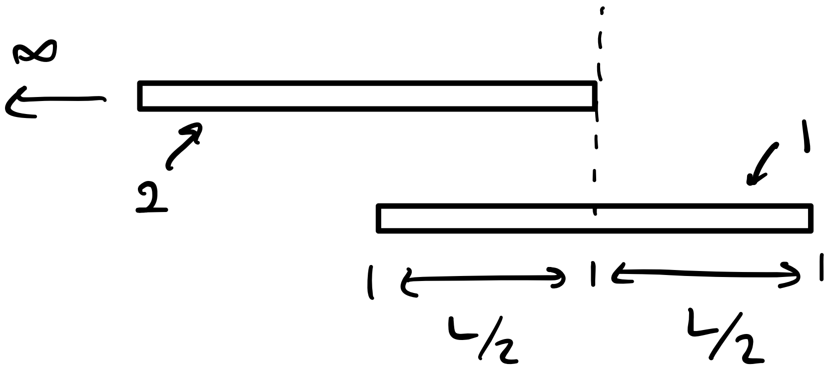
1. View factor libraries
2. Monte-Carlo ray tracing

Inspection



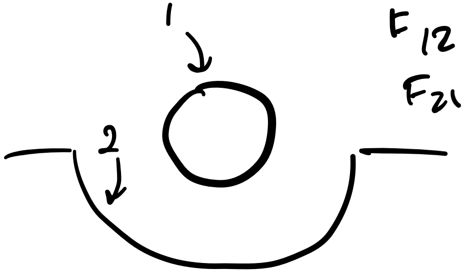
$F_{12} = 1$

$F_{21} = ?$



$F_{12} = 0.5$

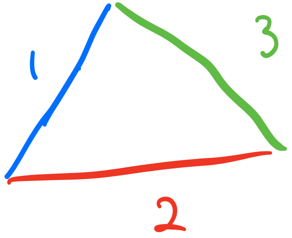
$F_{21} = 0$



$F_{12} = 0.5$

$F_{21} = ?$

View factor rules: Enclosure



(internal surfaces)

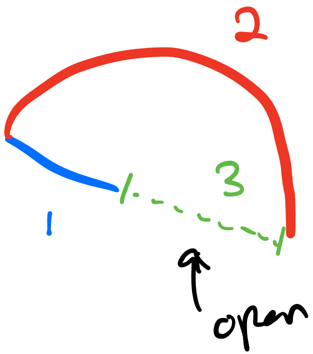
maybe if 1: $\int$

$$F_{12} + F_{13} + F_{11} = 1$$

$$F_{21} + F_{22} + F_{23} = 1$$

$$\sum_{j=1}^{n} F_{ij} = 1 \quad \forall\, i \in (1:n)$$

– can solve if you know all but one



– define "virtual surface"

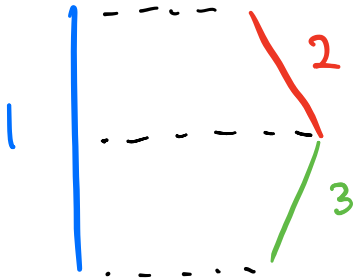
$$F_{22} \neq 0$$

View factor rules: Consolidation

Want to know from 1 to both 2 & 3

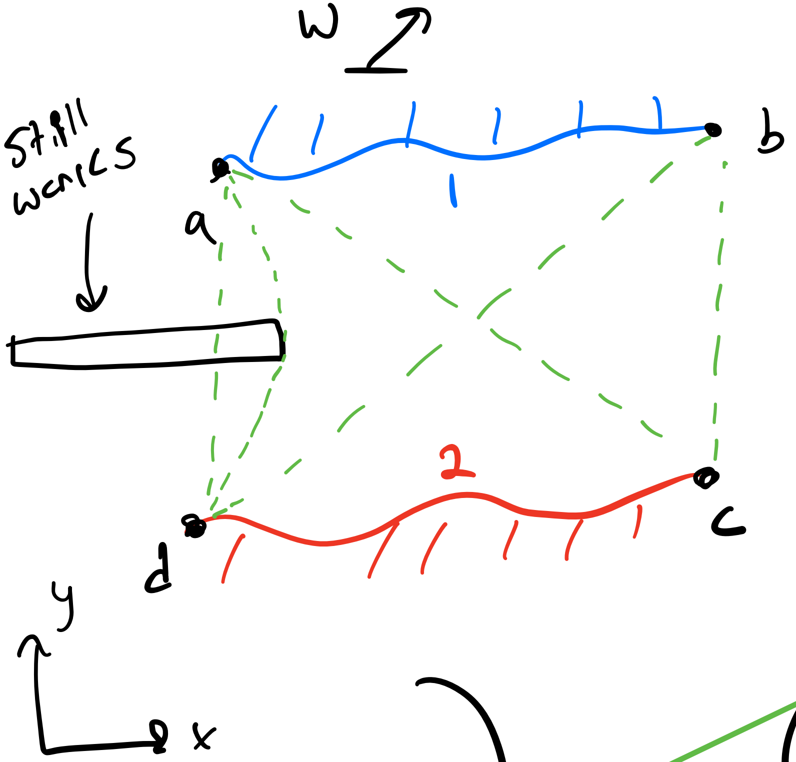
$$F_{1,2+3} = F_{12} + F_{13}$$

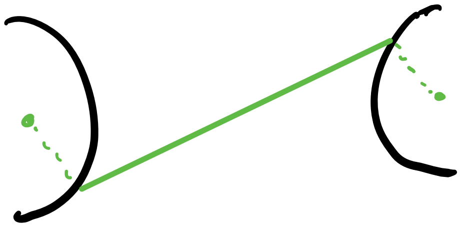
View factor rules: Symmetry



$$F_{12} = F_{13}$$

View factor rules: Hottel's crossed strings method





$$A_1 F_{12} = A_2 F_{21} = w\left[\frac{\Sigma L_{crossed} - \Sigma L_{uncrossed}}{2}\right]$$

$$w\left[\frac{(L_{ac} + L_{bd}) - (L_{ad} + L_{bc})}{2}\right]$$

- works for 2D problems
- works for weird surfaces
- works for inserted plate

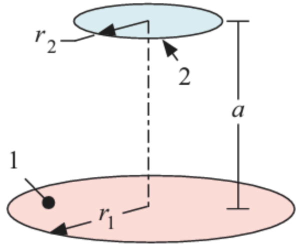
View factor analysis

Libraries

http://www.thermalradiation.net/indexCat.html

built in

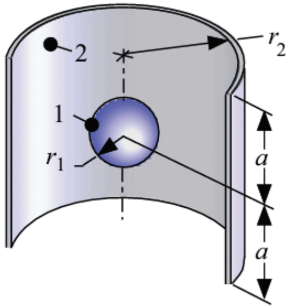
Coaxial disks



$$F_{1,2} = \frac{1}{2}\left[S - \sqrt{S^2 - 4\left(\frac{r_2}{r_1}\right)^2}\right]$$

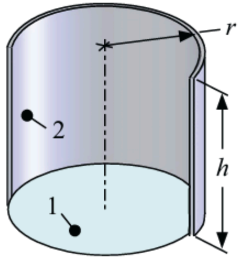
$$S = 1 + \frac{1 + \left(\frac{r_2}{a}\right)^2}{\left(\frac{r_1}{a}\right)^2}$$

Sphere to a cylinder



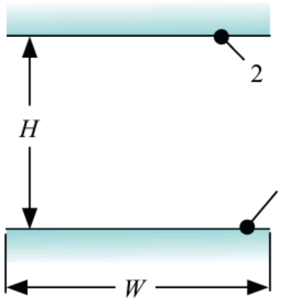
$$F_{1,2} = \frac{1}{\sqrt{1 + \left(\frac{r_2}{a}\right)^2}}$$

Base of cylinder to internal sides



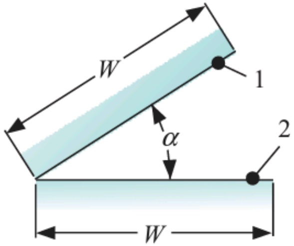
$$F_{1,2} = \frac{h}{r}\left(\sqrt{1 + \left(\frac{h}{2r}\right)^2} - \frac{h}{2r}\right)$$

Parallel plates ($L>>W$)



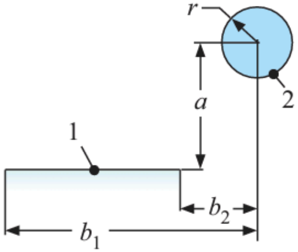
$$F_{1,2} = \sqrt{1 + \left(\frac{H}{W}\right)^2} - \frac{H}{W}$$

Plates joined at an angle ($L>>W$)



$$F_{1,2} = 1 - \sin\left(\frac{\alpha}{2}\right)$$

Plate to cylinder ($L>>r,b_1$)



$$F_{1,2} = \frac{r}{(b_1 - b_2)}\left[\tan^{-1}\left(\frac{b_1}{a}\right) - \tan^{-1}\left(\frac{b_2}{a}\right)\right]$$

Monte Carlo ray tracing

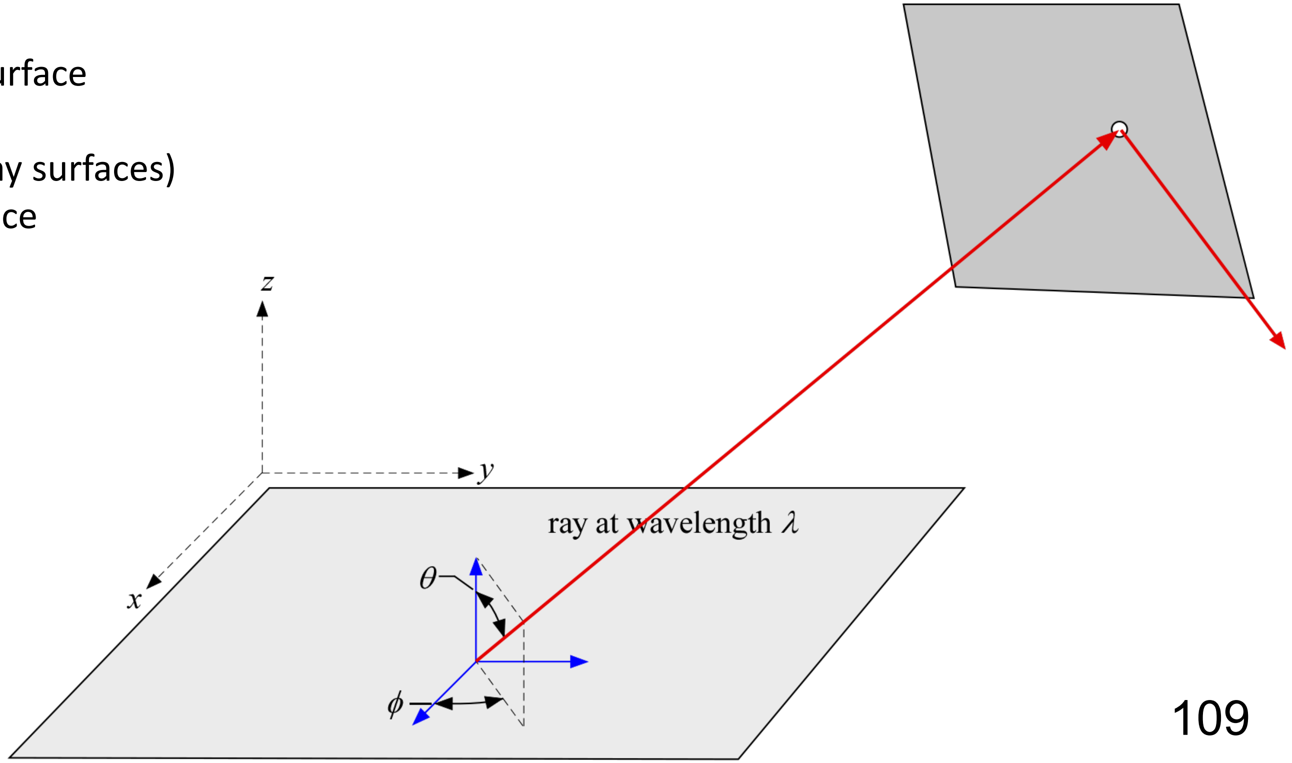
Characteristics:

- Uses random sampling to estimate value of interest
- Stochastic, as opposed to deterministic
- Accommodates real surfaces (gray, semigray, specular, odd geometries)

Method

1. Randomly select location on emitting surface
2. Randomly select direction to emit
3. Select a wavelength for the ray (non-gray surfaces)
4. Determine whether ray hits target surface
5. Repeat for many rays

$$F_{1,2} \approx \frac{\#\ \text{hits}}{\#\ \text{rays}}$$



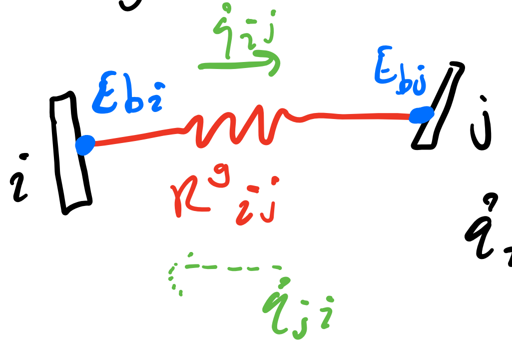
Lecture 23

Blackbody Radiation Exchange

Blackbody radiation exchange

Recall def. emissive power

$$E_b = \int_0^\infty E_{b,\lambda}\, d\lambda = \sigma T^4 \cdot \cancelto{1}{\varepsilon} \rightarrow \text{B.B. say } \varepsilon = 1$$



$$\dot{q}_{ij} = -\dot{q}_{ji}$$

Rate eqn for radiation

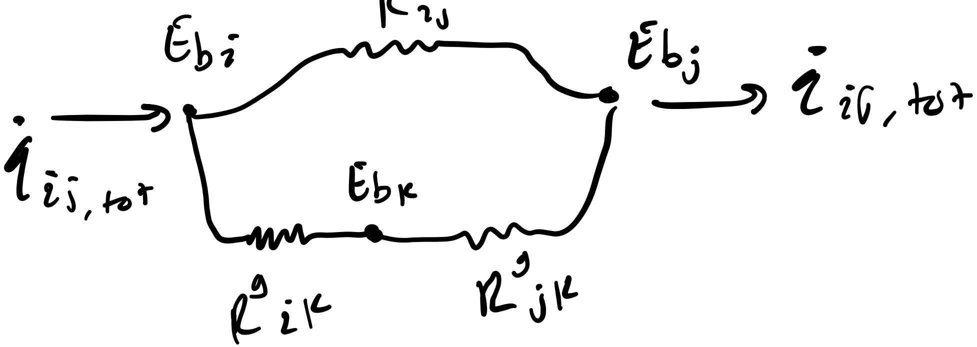
$$\dot{q}_{ij} = F_{ij} A_i (\sigma T_i^4 - \sigma T_j^4)$$

$$= \underbrace{F_{ij} A_i}_{1/R^g_{ij}} (E_{bi} - E_{bj})$$

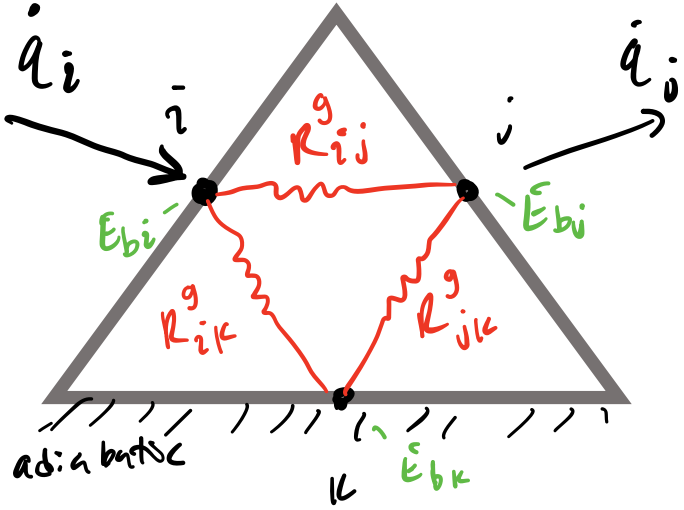
$$\rightarrow \dot{q}_{ij} = \frac{E_{bi} - E_{bj}}{R^g_{ij}}$$

Radiation surface networks

Assume isothermal → use as nodes



3-surface network

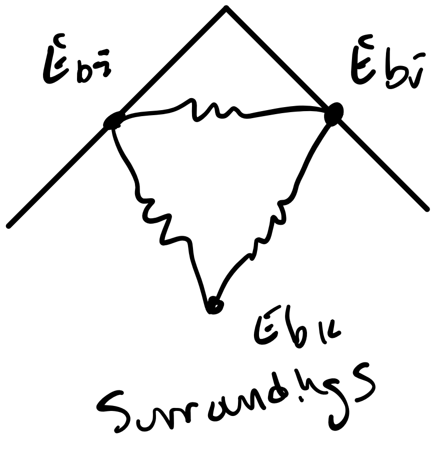


$$\dot{q}_{ij,tot} = \frac{E_{bi} - E_{bj}}{R^g_{ij,tot}}$$

$$R^g_{ij,tot} = \left[\frac{1}{R^g_{ij}} + \frac{1}{R^g_{ik} + R^g_{jk}}\right]^{-1}$$

Radiation surface networks

3-surface network, including “surroundings”



→ use enclosure rule

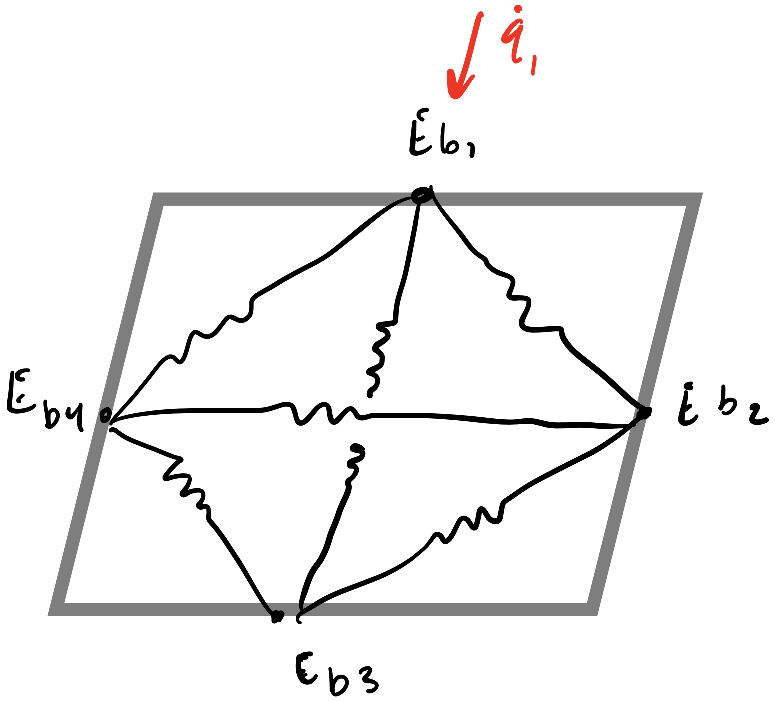
→ surroundings:

- blackbody ($\varepsilon = 1$)
- emit @ T_{surr}
- area → ∞

$$F_{ik} A_i = F_{ki} A_k$$

$$F_{ki} = \frac{F_{ik} A_i}{A_k} \rightarrow A_k \rightarrow \infty \rightarrow F_{ki} \rightarrow 0$$

Extending to N-surfaces



System of equations:

$$\dot{q}_i = \sum_{j=1}^{N} A_i F_{ij} \sigma (T_i^4 - T_j^4) \quad \forall\ i \in 1:N$$

$$\dot{q}_i = \sum_{j=1}^{N} A_i F_{ij} (E_{bi} - E_{bj}) \quad \forall\ i \in 1:N$$

↑ gives us N equations

Don't know:

$$\left.\begin{array}{l} E_{b,i} \rightarrow N \\ \dot{q}_i \rightarrow N \end{array}\right\} 2 \times N \text{ unknowns}$$

Nodal boundary conditions

Boundary condition types:

- Need N addit'l equations

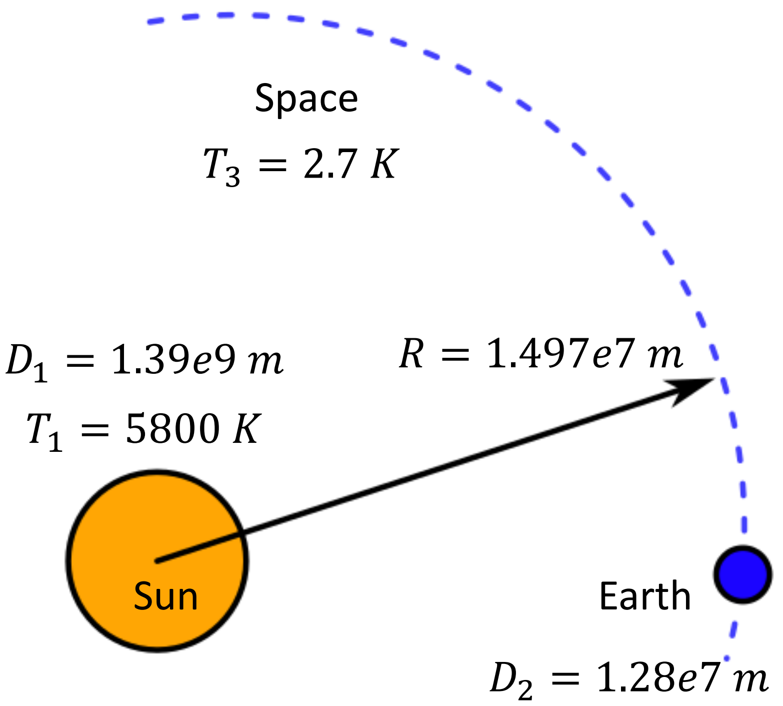
1) Known temperature → $\sigma T^4 = E_b$

2) Known emissive power → E_b

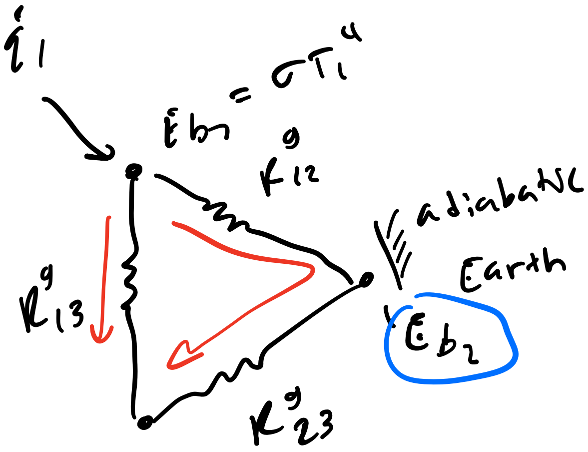
3) Known surface heat transfer rate → $\dot{q}$

Example

Estimating the temperature of the Earth



Radiation exchange example



$E_{b_3} = \sigma T_3^4$

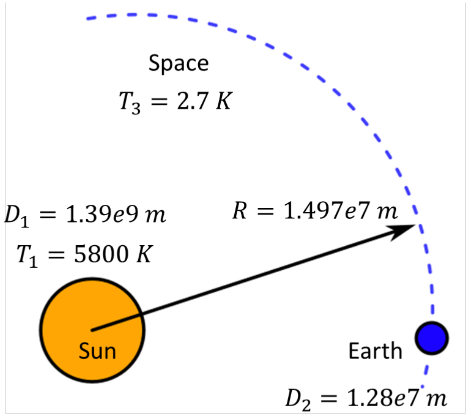
$$E_{b_1} = \sigma T_1^4 = (5.67e^{-8})(5800)^4$$
$$= 6.417e7 \ \frac{W}{m^2}$$

$$E_{b_2} = \sigma T_2^4$$

$$E_{b_3} = \sigma T_3^4 = 3.013\times10^{-6} \ W/m^2$$

$$A_1 = 4\pi\left(\frac{D_1}{2}\right)^2 \ldots$$

$$A_2 = \ldots$$



Radiation exchange example

View factors

sun → earth

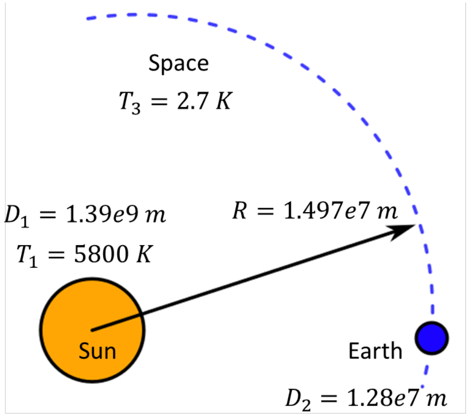
$$F_{12} = \frac{\left(\pi \frac{D_2^2}{4}\right)}{4\pi R^2} = 4.591 \times 10^{-10}$$

$$F_{21} = F_{12}\frac{A_1}{A_2} \approx 0$$

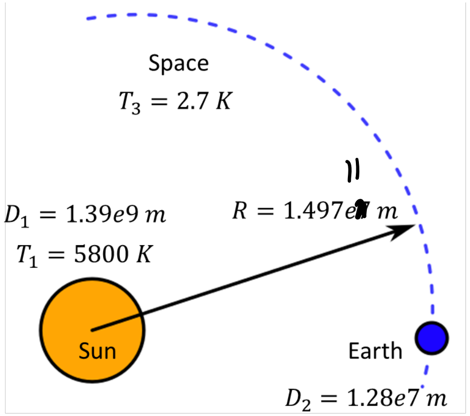
– negligible impact on T_{sun}

$$F_{23} \approx 1$$

$$F_{13} \approx 1 - F_{12} - \cancel{F_{12}}^{\,0}$$



Radiation exchange example



Resistances

$$R^g_{12} = \frac{1}{A_1 F_{12}} = 3.628 \times 10^{-10}\ \frac{1}{m^2}$$

$$R^g_{13} = \frac{1}{A_1 F_{13}} = 1.647 \times 10^{-19}\ \frac{1}{m^2}$$

$$R^g_{23} = \frac{1}{A_2 F_{23}} = 1.455 \times 10^{-15}\ \frac{1}{m^2}$$

Energy balance

surface 2 adiabatic

$$\dot{q}_{12} = \dot{q}_{23}$$

$$\dot{q}_{12} = \frac{E_{b_1} - E_{b_3}}{R^g_{12} + R^g_{23}} = 1.769 \times 10^{-7}$$

$$\dot{q}_{12} = \frac{E_{b_1} - E_{b_2}}{R^g_{12}}$$

$$E_{b_2} = E_{b_1} - \dot{q}_{12} R^g_{12}$$
$$= 346\ W/m^2 = \sigma T_2^4$$

$$T_2 = 279.4\ K \rightarrow 43.3^\circ F$$

Actual: $59^\circ F$

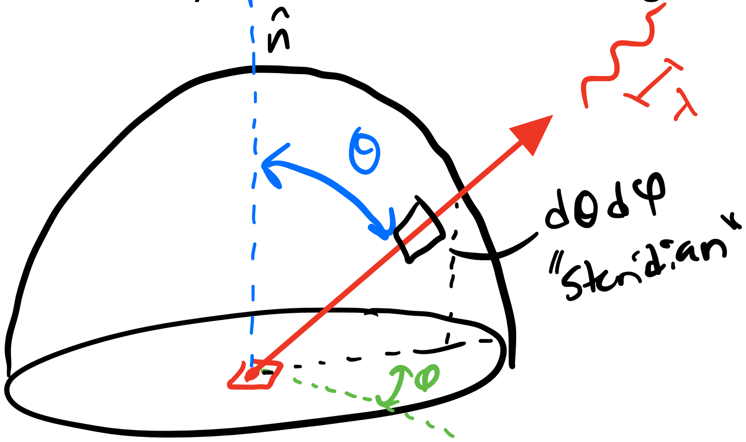
Lecture 24

Emission from Real Surfaces

Kirchoff’s Law

Real surfaces

Emissivity is a function of wavelength and direction



$\varepsilon_{\lambda\theta\varphi} \rightarrow$ spectral, directional emissivity

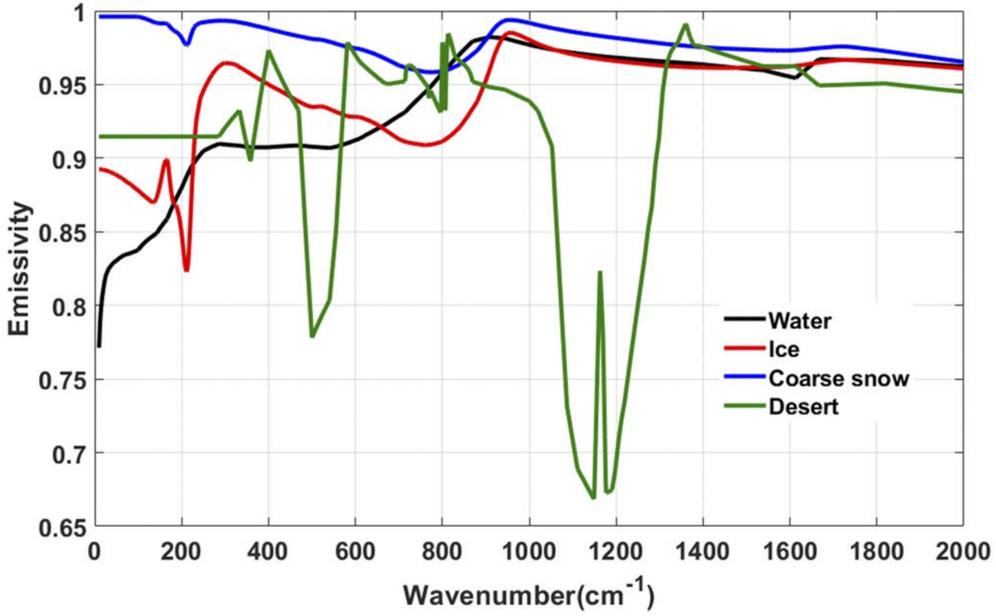
- spherical coordinates
- θ angle of displacement from normal $\hat{n}$
- $\varepsilon_{\lambda\theta\varphi}$ can depend on θ
- $\varepsilon_{\lambda\theta\varphi} = \dfrac{I_{\lambda\theta\varphi}}{I_{\lambda\varphi}}$

__Intensity:__

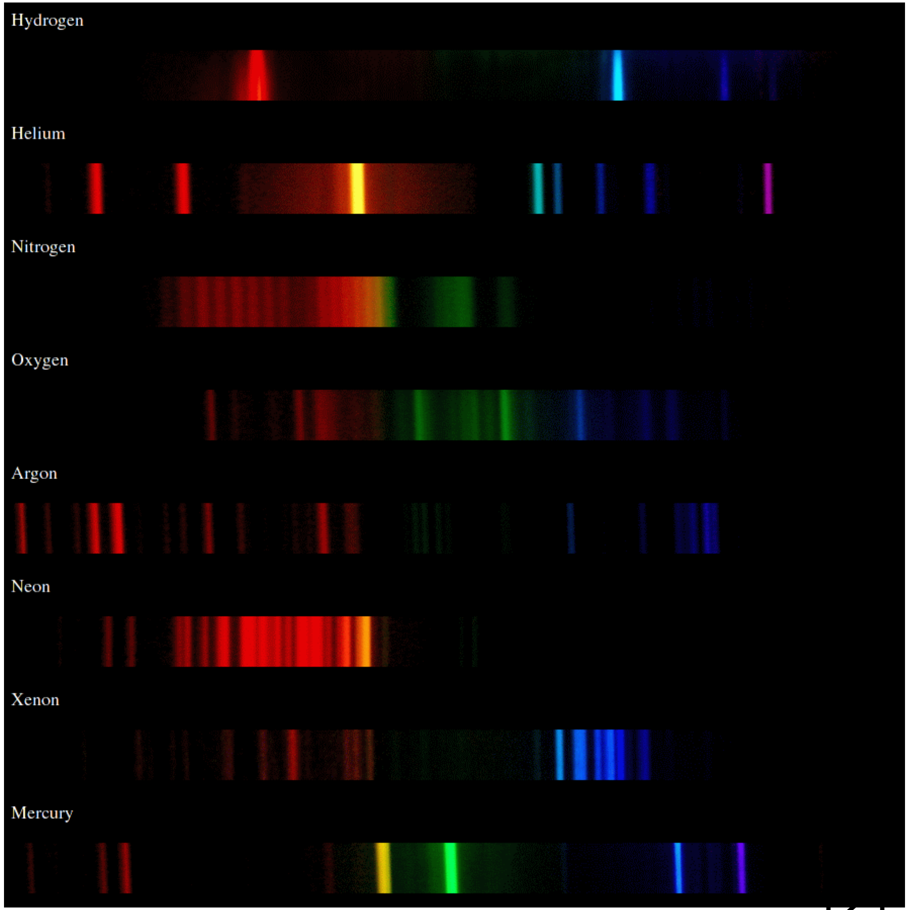
$I_{\lambda\theta\varphi} \rightarrow$ units $\dfrac{W}{m^2 \cdot sr}$

Spectral dependence

Emission and absorption characteristics relate to chemical composition

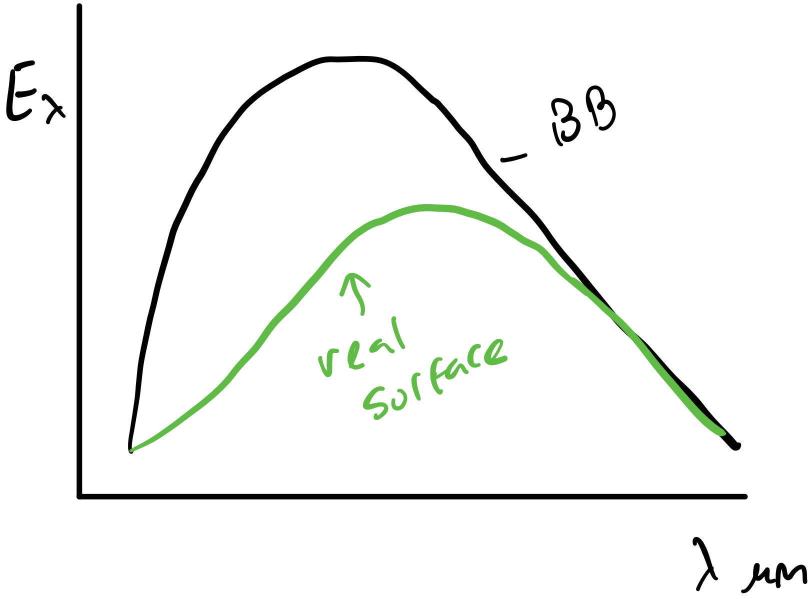


https://doi.org/10.1175/JCLI-D-17-0125.1



http://spiff.rit.edu/classes/phys200/lectures/spectra/spectra.html

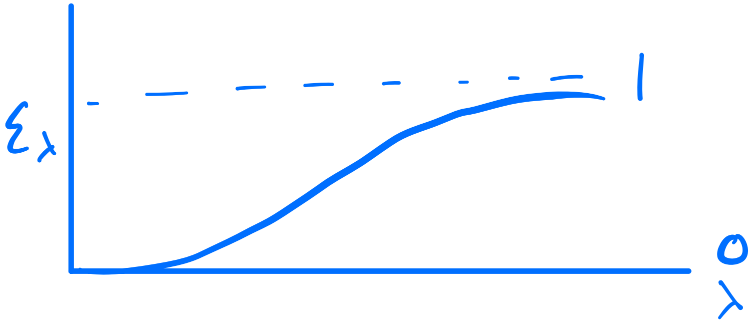
Expressions of emissivity



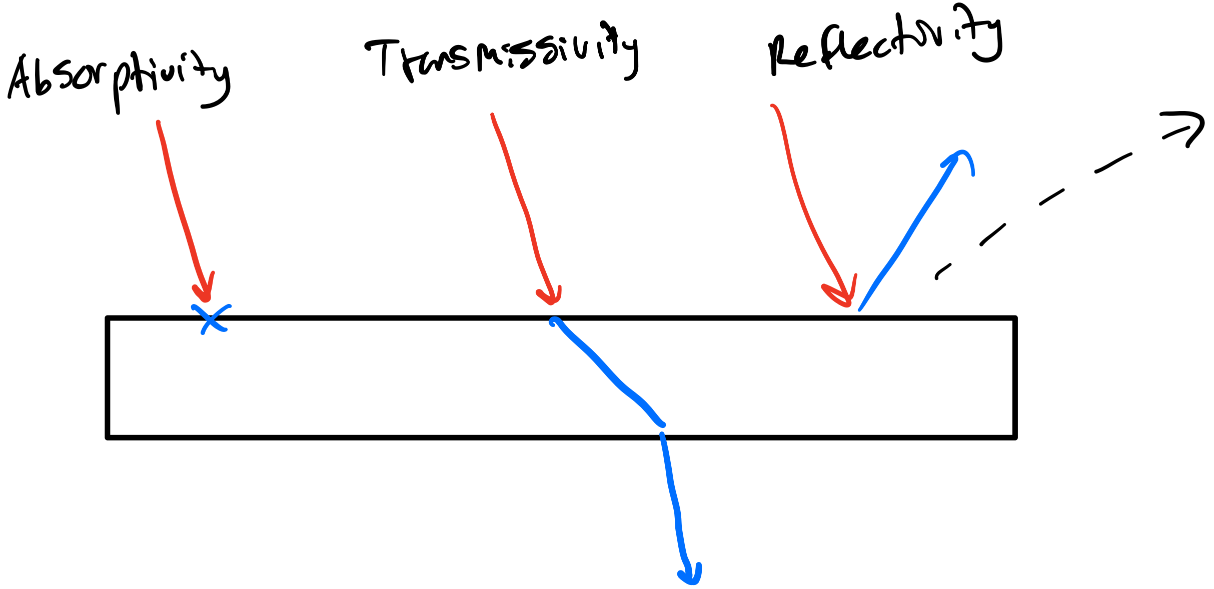
$\varepsilon_{\lambda\theta\varphi}$ → spectral directional

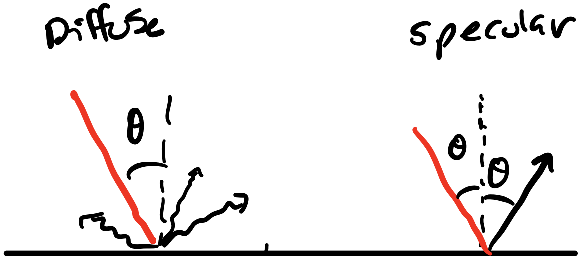
ε_λ → spectral hemispherical emissivity "diffuse surface"

ε → total hemispherical emissivity "diffuse grey surface



Real surface characteristics





$$1 = \alpha_{\lambda\theta\varphi} + \tau_{\lambda\theta\varphi} + \rho_{\lambda\theta\varphi}$$

Diffuse: $\alpha_\lambda + \tau_\lambda + \rho_\lambda = 1$

Diffuse grey: $\alpha + \tau + \rho = 1$

Kirchoff's law

$$\alpha_{\lambda\theta\varphi} = \varepsilon_{\lambda\theta\varphi}$$

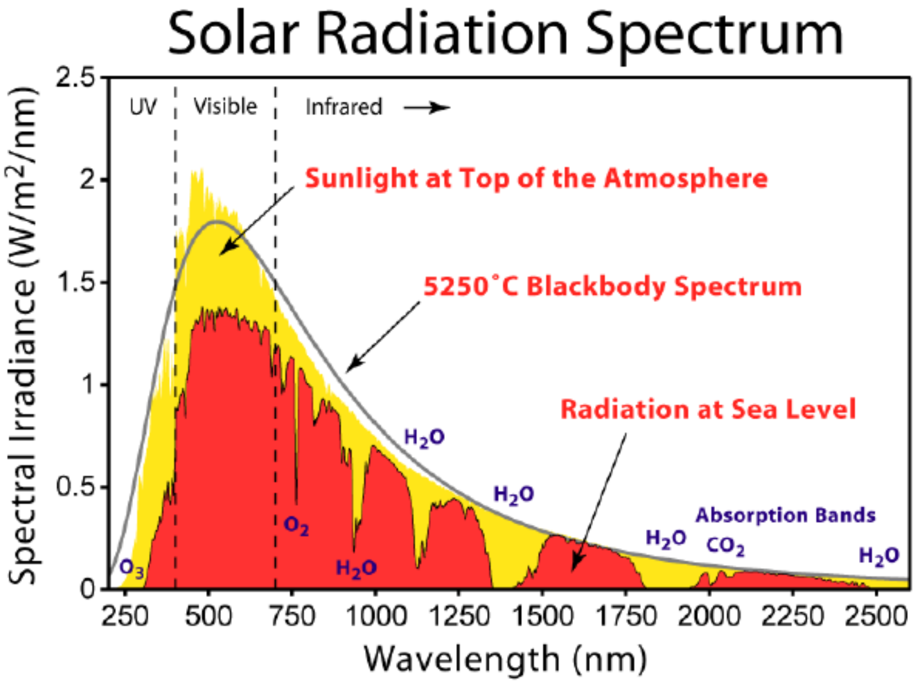
also for real surfaces!

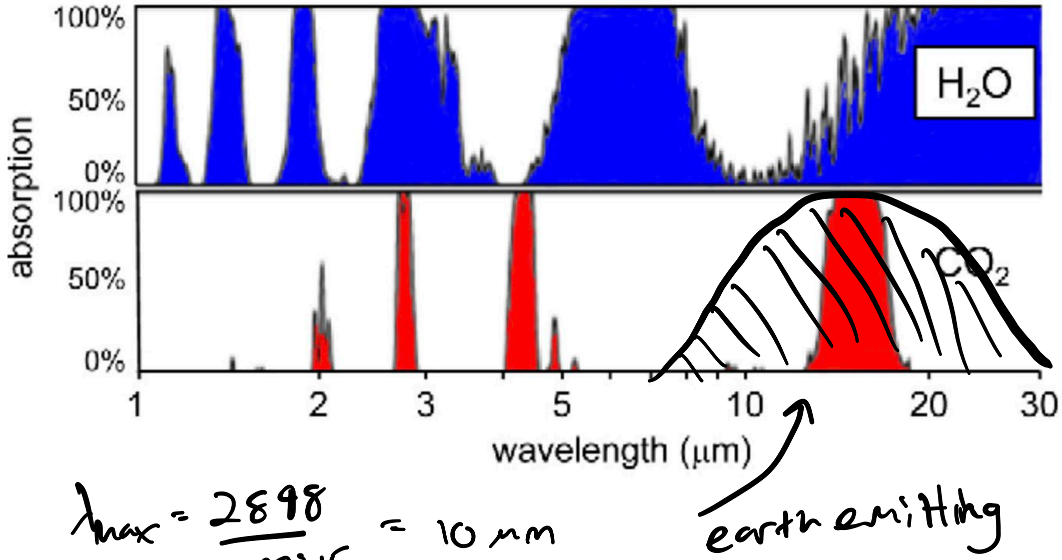
Diffuse

$$\alpha_\lambda = \varepsilon_\lambda$$

Diffuse grey

$$\alpha = \varepsilon$$





$$\lambda_{max} = \frac{2898}{15+273{,}15} = 10\ \mu m$$

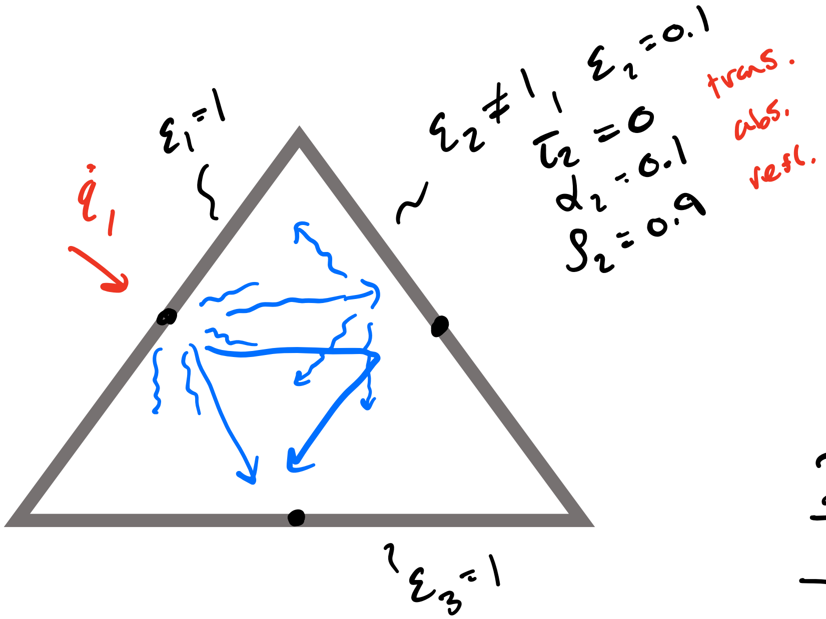
Lecture 25

Radiation Exchange Between Gray Surfaces

ε doesn't depend on λ, but $\varepsilon \neq 1$

like B.B.

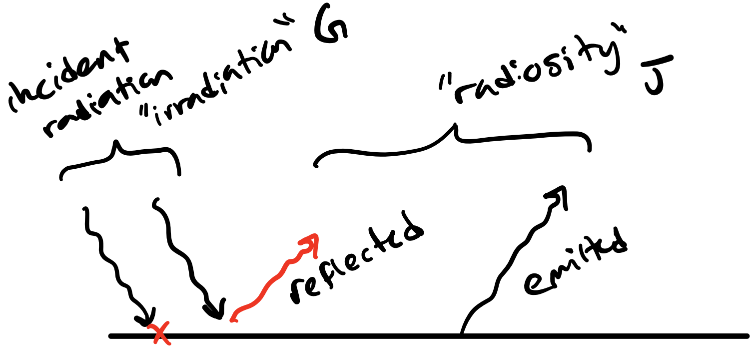
Diffuse Gray Surface Radiation Exchange



- At given temp, a real surface emits less than a black body

power emitted $E = E_b \cdot \varepsilon = \varepsilon \sigma T^4$

$\therefore$ BB emissive model breaks down



Irradiation

- Radiation from all sources $j = 1:N$ hitting surface i
- Rad. from $j \rightarrow i$ can be emitted from j or reflected from j

Radiosity

(leaving i)

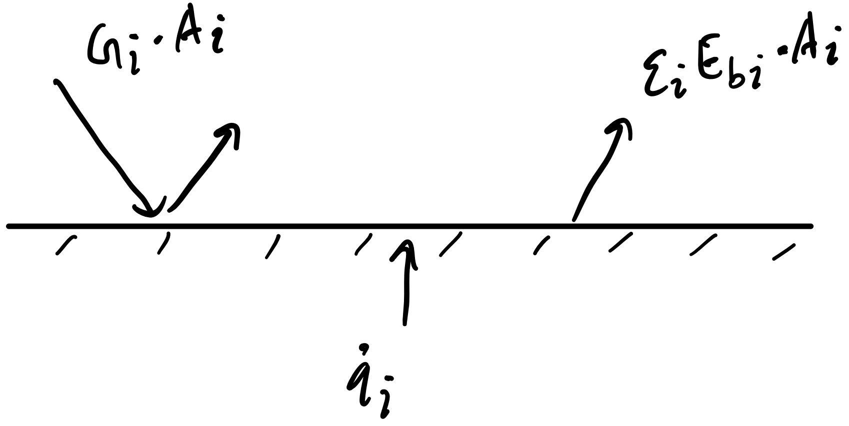
$$J_i \left[\frac{W}{m^2}\right] = \overbrace{(1-\varepsilon_i)\,G_i}^{\text{reflected power}} + \overbrace{\varepsilon_i E_{bi}}^{\text{emitted power}}$$

emissivity — irradiation — BB emissive power

Assumes $\tau_i = 0$

$$G_i A_i + \dot{q}_i - J_i A_i = 0$$

$$\rightarrow \dot{q}_i = J_i A_i - G_i A_i$$



Radiosity – relation to emissivity

Radiosity equation: $J_i = (1-\varepsilon_i)\, G_i + \varepsilon_i\, E_{b,i}$

Rearrange for G: $G_i = \dfrac{J_i - \varepsilon_i E_{bi}}{1-\varepsilon_i}$

$$\dot{q}_i = J_i A_i - \left(\frac{J_i - \varepsilon_i E_{bi}}{1-\varepsilon_i}\right) A_i \rightarrow J_i A_i \cdot \frac{(1-\varepsilon_i)}{(1-\varepsilon_i)} - \left(\frac{J_i - \varepsilon_i E_{bi}}{1-\varepsilon_i}\right) A_i$$

$$\dot{q}_i = \frac{\cancel{J_i A_i} - J_i A_i \varepsilon_i - \cancel{J_i A_i} + \varepsilon_i A_i E_{bi}}{(1-\varepsilon_i)}$$

$$\dot{q}_i = \underbrace{\frac{A_i \varepsilon_i}{(1-\varepsilon_i)}}_{1/\text{Resistance}} \underbrace{(E_{bi} - J_i)}_{\text{potential}}$$

Surface Resistance

$$R_i^s = \frac{1-\varepsilon_i}{A_i\ \varepsilon_i} \left[\frac{1}{m^2}\right]$$

surface resistance

R_i^s

when $\varepsilon \to 1$:

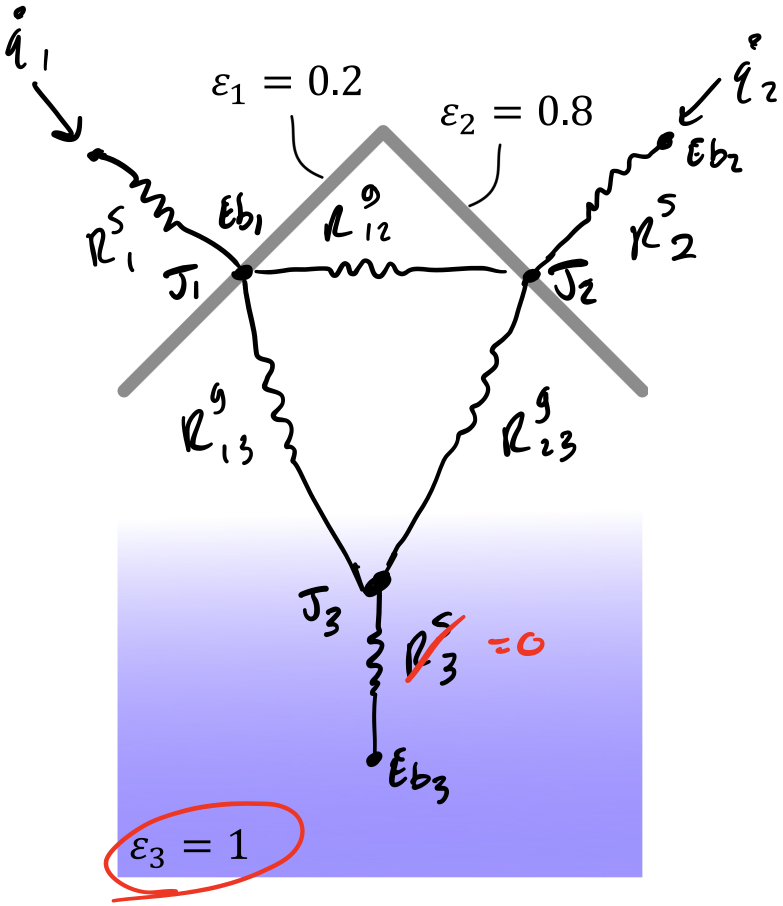
$\frac{1-1}{A_i \cdot 1} \to 0$ resistance

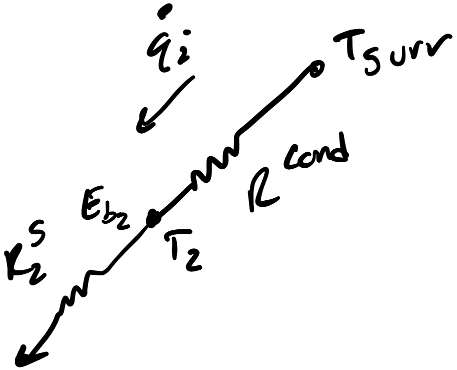
$\dot{q}_i = \frac{E_{bi} - J_i}{R_i^s} \to E_{bi} = J_i$ (for B.B)

Multi-surface example

$$R^g_{12} = \frac{1}{A_1 F_{12}}$$

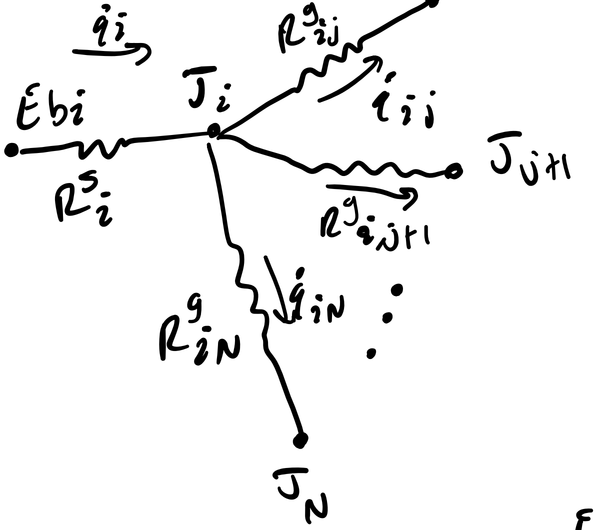
$$R^S_1 = \frac{1-\varepsilon_1}{A_1 \varepsilon_1}$$





Multi-surface example

Energy balance on a single node for N surfaces



Heat flow @ each node:

$$\dot{q}_i = \frac{E_{bi} - J_i}{R^s_i}$$

Between nodes:

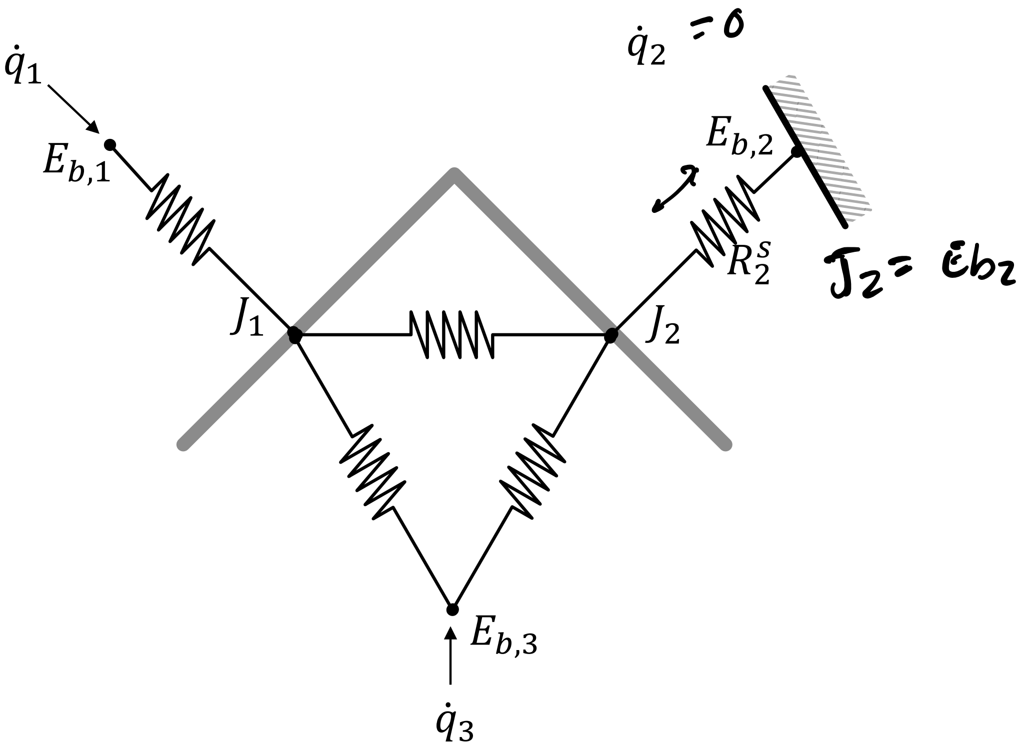
$$\dot{q}_{ij} = \frac{J_i - J_j}{R^g_{ij}}$$

Energy balance

$$\frac{E_{bi} - J_i}{R^s_i} = \sum_{j=1}^{N} \frac{J_i - J_j}{R^g_{ij}} \quad \forall\, i \in 1{:}N$$

Re-radiating surfaces

What is the radiosity of an adiabatic surface?



Heat exchangers