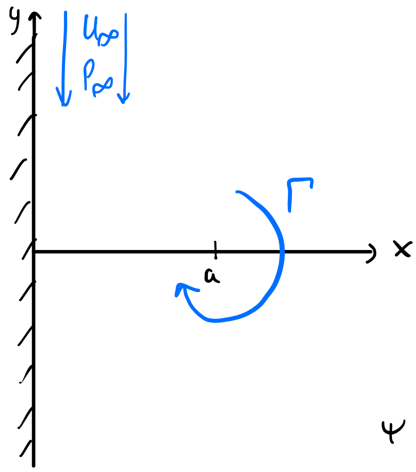
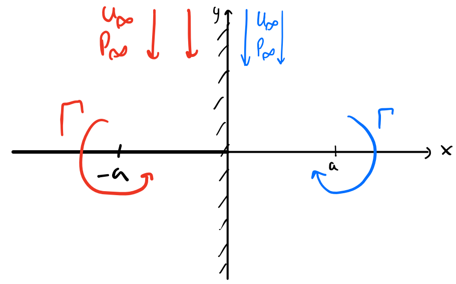
Problem 1 Clockwise vortex of strength Γ distance a from wall in free stream U_∞



1a) Find stream function $\psi(x,y)$

– mirror image vortex



$$\psi = \psi_v + \psi_{v,m} + \psi_u$$

$$\underbrace{\psi_v = \frac{\Gamma}{2\pi}\ln r}_{\substack{r^2 = x^2+y^2 \\ x = x+a}}, \quad \underbrace{\psi_{v,m} = -\frac{\Gamma}{2\pi}\ln r}_{x = x-a}, \quad \psi_u = V_\infty x$$

$$\boxed{\psi = \frac{\Gamma}{2\pi}\left[\ln\sqrt{(x+a)^2+y^2} - \ln\sqrt{(x-a)^2+y^2}\right] + U_\infty x}$$

1b) Find $\varphi(x,y)$

$$\varphi = \varphi_v + \varphi_{v,m} + \varphi_u$$

$$\varphi_v = -\frac{\Gamma}{2\pi}\theta, \quad \varphi_{v,m} = \frac{\Gamma}{2\pi}\theta, \quad \varphi_u = -U_\infty y$$

$x = x+a$ $\qquad$ $x = x-a$

$$\boxed{\varphi = \frac{\Gamma}{2\pi}\left[-\arctan\left(\frac{y}{x+a}\right) + \arctan\left(\frac{y}{x-a}\right)\right] - U_\infty y}$$

1c) Find u_x, u_y @ $(x,y) = (0,a)$

$$u_x = \frac{\partial}{\partial y}\left[\frac{\Gamma}{2\pi}\left[\ln\sqrt{(x+a)^2+y^2} - \ln\sqrt{(x-a)^2+y^2}\right] + U_\infty x\right]$$

$1/2$

$$u_x(x,y) = \frac{\Gamma}{4\pi}\left[\frac{2y}{(x+a)^2+y^2} - \frac{2y}{(x-a)^2+y^2}\right]$$

@ $x=0$, $y=a$:

$$u_x(0,a) = \frac{\Gamma}{4\pi}\left[\frac{2a}{2a^2} - \frac{2a}{2a^2}\right] = \boxed{0 = u_x(0,a)}$$

→ also by symmetry

$$u_y = -\frac{\partial \psi}{\partial x} = -\frac{\partial}{\partial x}\left[\frac{\Gamma}{4\pi}\left[\ln((x+a)^2+y^2) - \ln((x-a)^2+y^2)\right] + u_\infty x\right]$$

$$u_y = -\frac{\Gamma}{4\pi}\left[\frac{2x+2a}{(x+a)^2+y^2} - \frac{2x-2a}{(x-a)^2+y^2}\right] - u_\infty$$

$$u_y(0,a) = -\frac{\Gamma}{4\pi}\left[\frac{2a}{2a^2} + \frac{2a}{2a^2}\right] - u_\infty = \boxed{-\frac{\Gamma}{2\pi a} - u_\infty = u_y(0,a)}$$

(d) Find force per unit span on wall $-a < y < a$

$$|u_w| = u_y(0,y) = -\frac{\Gamma}{\cancel{4}\pi}\left[\frac{\cancel{4}a}{a^2+y^2}\right] - u_\infty$$

Bernoulli: $P_\infty + \frac{1}{2}\rho u_\infty^2 = P_w + \frac{1}{2}\rho |u_w|^2$

$$P_w(y) = P_\infty + \frac{1}{2}\rho\left(u_\infty^2 - |u_w|^2\right)$$

$$P_w(y) = P_\infty + \frac{1}{2}\rho\left(\cancel{u_\infty^2} - \left[\frac{\Gamma^2 a^2}{\pi^2(a^2+y^2)^2} + \frac{2u_\infty \Gamma}{\pi(a^2+y^2)} + \cancel{u_\infty^2}\right]\right)$$

$$P_w(y) = P_\infty + \frac{1}{2}\rho\left[\frac{2u_\infty \Gamma}{\pi(a^2+y^2)} - \frac{\Gamma^2 a^2}{\pi^2(a^2+y^2)^2}\right]$$

$$\frac{\text{Force}}{\text{unit depth}} = \int_{-a}^{a} P_w(y)\,dy = \int_{-a}^{a} P_\infty\,dy + \int_{-a}^{a} \frac{\rho u_\infty \Gamma}{\pi(a^2+y^2)}\,dy - \int_{-a}^{a} \frac{\Gamma^2 a^2}{\pi^2(a^2+y^2)^2}\,dy$$

$$= \left[P_\infty y\right]_{-a}^{a} + \left(\frac{1}{a}\arctan\left(\frac{y}{a}\right)\cdot\frac{\rho u_\infty \Gamma}{\pi}\right]_{-a}^{a} - \left[\frac{1}{2a^2}\left[\frac{y}{a^2+y^2} + \frac{1}{a}\arctan\left(\frac{y}{a}\right)\right]\cdot\frac{\Gamma^2 a^2}{\pi^2}\right]_{-a}^{a}$$

$$= 2aP_\infty + \frac{\rho u_\infty \Gamma}{\pi}\left[\left(\frac{1}{a}\right)\left(\frac{\pi}{4}\right) - \left(\frac{1}{a}\right)\left(-\frac{\pi}{4}\right)\right] - \frac{\Gamma^2 a^2}{2a^2\pi^2}\left[\left(\frac{a}{2a^2} + \frac{1}{a}\left(\frac{\pi}{4}\right)\right) - \left(\frac{-a}{2a^2} + \frac{1}{a}\left(-\frac{\pi}{4}\right)\right)\right]$$

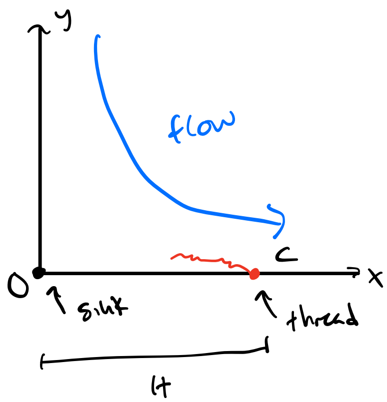
$$= 2aP_\infty + \frac{\rho u_\infty \Gamma}{\cancel{\pi}}\left(\frac{2\cancel{\pi}}{4a}\right) - \frac{\Gamma^2 a^{\cancel{2}}}{\cancel{2}\pi^2}\left[\frac{\cancel{2}\cancel{a}}{\cancel{2}a^{\cancel{2}}} + \frac{\cancel{2}\pi}{4\cancel{a}}\right]$$

$$= 2aP_\infty + \frac{\rho u_\infty \Gamma}{2a} - \frac{\Gamma^2 a}{\pi^2}\left[\frac{1}{2} + \frac{\pi}{4}\right]$$

$$\boxed{F_\omega = 2aP_\infty + \frac{\rho u_\infty \Gamma}{2a} - \frac{\Gamma^2 a}{2\pi^2} - \frac{\Gamma^2 a}{4\pi}}$$

Problem 2

corner flow, $\psi = Axy$



Sink: volumetric rate of q per unit depth

Find velocity potential & velocity components in terms of x, y, A & q

Find condition (A & q) s.t. thread extends toward origin

$$\varphi_{total} = \varphi_{corner} + \varphi_{sink}$$

$$\boxed{\varphi = \frac{A}{2}(x^2 - y^2) - \frac{q}{2\pi}\ln\sqrt{x^2+y^2}}$$

$$\boxed{\begin{aligned} u &= \frac{\partial\varphi}{\partial x} = Ax - \frac{q}{2\pi}\frac{x}{x^2+y^2} \\ v &= \frac{\partial\varphi}{\partial y} = -Ay - \frac{q}{2\pi}\frac{y}{x^2+y^2} \end{aligned}}$$

towards sink: $v = 0$, u negative, at $(H, 0)$

At $y = 0$, $v = 0$

$$u = Ax - \frac{q}{2\pi}\frac{x}{x^2+y^2} < 0 \rightarrow \frac{q}{2\pi}\frac{x}{x^2+y^2} > Ax$$

$$\rightarrow \frac{q}{2\pi}\frac{\cancel{H}}{H^{\cancel{2}}} > AH \quad \rightarrow \boxed{A < \frac{q}{2\pi H^2}}$$

Problem 3 Lifting flow over cylinder

$$\psi = (V_\infty r\sin\theta)(1 - R^2/r^2) + \frac{\Gamma}{2\pi}\ln\frac{r}{R}$$

can be decomposed into $\psi = \psi_1 + \psi_2 + \psi_3$

3a) Find ψ_1, ψ_2, ψ_3 in both (x,y) & (r,θ)

$$\psi_1(r,\theta) = V_\infty r\sin\theta \quad , \quad \psi_1(x,y) = V_\infty y$$

$$\psi_2(r,\theta) = -V_\infty r\sin\theta\,\frac{R^2}{r^2} \quad , \quad \psi_2(x,y) = -V_\infty R^2\cdot\frac{y}{x^2+y^2}$$

$$\psi_3(r,\theta) = \frac{\Gamma}{2\pi}\ln\frac{r}{R} \quad , \quad \psi_3(x,y) = \frac{\Gamma}{2\pi}\ln\frac{\sqrt{x^2+y^2}}{R}$$

3b) Find corresponding $\varphi_1, \varphi_2, \varphi_3$ in (r,θ) & (x,y). Construct $\varphi(x,y)$ & $\varphi(r,\theta)$

1) $V_{r_1} = \frac{1}{r}\frac{\partial\psi}{\partial\theta} = \frac{1}{r}(V_\infty r\cos\theta) = V_\infty\cos\theta = \frac{\partial\varphi}{\partial r}$

$V_{\theta_1} = -\frac{\partial\psi}{\partial r} = -(V_\infty\sin\theta) = -V_\infty\sin\theta = \frac{1}{r}\frac{\partial\varphi}{\partial\theta}$

$\rightarrow \varphi_1(r,\theta) = V_\infty r\cos\theta \quad , \quad \varphi_1(x,y) = V_\infty x$

2) $V_{r_2} = \frac{1}{r}\frac{\partial\psi}{\partial\theta} = \frac{1}{r}\left(-V_\infty r\cos\theta\,\frac{R^2}{r^2}\right) = -V_\infty\cos\theta\,\frac{R^2}{r^2} = \frac{\partial\varphi}{\partial r}$

$V_{\theta_2} = -\frac{\partial\psi}{\partial r} = -\left(\cancel{+}V_\infty\sin\theta\left(\cancel{-}\frac{R^2}{r^2}\right)\right) = -V_\infty\sin\theta\,\frac{R^2}{r^2} = \frac{1}{r}\frac{\partial\varphi}{\partial\theta}$

$\rightarrow \varphi_2(r,\theta) = V_\infty\cos\theta\,\frac{R^2}{r} \quad , \quad \varphi_2(x,y) = V_\infty R^2\,\frac{x}{x^2+y^2}$

3) $V_{r_3} = \frac{1}{r}\frac{\partial\psi}{\partial\theta} = 0 = \frac{\partial\varphi}{\partial r}$

$V_{\theta_3} = -\frac{\partial\psi}{\partial r} = -\frac{\Gamma}{2\pi}\frac{1}{r} = \frac{1}{r}\frac{\partial\varphi}{\partial\theta}$

$\rightarrow \varphi_3(r,\theta) = -\frac{\Gamma}{2\pi}\theta \quad , \quad \varphi_3(x,y) = -\frac{\Gamma}{2\pi}\arctan\left(\frac{y}{x}\right)$

$$\rightarrow \boxed{\varphi(r,\theta) = V_\infty r\cos\theta + V_\infty\cos\theta\,\frac{R^2}{r} + -\frac{\Gamma}{2\pi}\theta}$$

$$\rightarrow \boxed{\varphi(x,y) = V_\infty x + V_\infty R^2\,\frac{x}{x^2+y^2} - \frac{\Gamma}{2\pi}\arctan\left(\frac{y}{x}\right)}$$

3c) Modify 3b) to include freestream @ α

$$\varphi_{fs}(x,y) = V_\infty(x\cos\alpha + y\sin\alpha)$$

$$\varphi_{fs}(r,\theta) = V_\infty r(\cos\theta\cos\alpha + \sin\theta\sin\alpha)$$

$$\boxed{\begin{aligned}\varphi(r,\theta) &= V_\infty r\cos\theta + V_\infty\cos\theta\frac{R^2}{r} + -\frac{\Gamma}{2\pi}\theta + V_\infty r(\cos\theta\cos\alpha + \sin\theta\sin\alpha)\\ \varphi(x,y) &= V_\infty x + V_\infty R^2\frac{x}{x^2+y^2} - \frac{\Gamma}{2\pi}\arctan\left(\frac{y}{x}\right) + V_\infty(x\cos\alpha + y\sin\alpha)\end{aligned}}$$

Problem 4 Matlab

Input: V_∞, ρ, α, Γ

Lifting flow over cylinder

$$\psi = (V_\infty r\sin\theta)(1 - R^2/r^2) + \frac{\Gamma}{2\pi}\ln\frac{r}{R}$$

$$\rightarrow \psi(x,y) = V_\infty(y\cos\alpha + x\sin\alpha)\left(1 + \frac{R^2}{x^2+y^2}\right) + \frac{\Gamma}{2\pi}\ln\left(\frac{\sqrt{x^2+y^2}}{R}\right)$$

$$u = \frac{\partial\psi}{\partial y} \qquad v = -\frac{\partial\psi}{\partial x}$$