Thermo Fundamentals

Variables of state ≡ Process-independent

Internal energy, $e = e(v, T)$

Ideal gas: $de = C_v\,dT$

$P = \rho R T$, $R = \frac{\bar{R}}{\bar{m}} = 8314\ \frac{J}{kmol\text{-}K}$

Isentropic: $ds = 0$:

$$\frac{T_2}{T_1} = \left(\frac{P_2}{P_1}\right)^{\frac{\gamma-1}{\gamma}}$$

$$\frac{P_2}{P_1} = \left(\frac{\rho_2}{\rho_1}\right)^{\gamma} \leftarrow \text{Not eqn. of state}$$

$$\frac{T_2}{T_1} = \left(\frac{\rho_2}{\rho_1}\right)^{\gamma-1}$$

Sign convention

work by fluid: negative

entering CV: negative

Work / Energy

$de = dq - dw = dq - P\,dv$

$\uparrow$ by fluid

$h \equiv e + Pv = e + \frac{P}{\rho}$

$dh = C_p\,dT$, $C_p = C_v + R$, $\gamma = \frac{C_p}{C_v}$

Reversible: $ds = \frac{dq}{T} \geq 0$

$\rightarrow dq = T\,ds \rightarrow s_2 - s_1 = C_p \ln\frac{T_2}{T_1} - R\ln\frac{P_2}{P_1}$

Conservation of mass (continuity)

$$\frac{d}{dt}\int_{CV} \rho\, d\forall + \int_{CS} \rho \underline{u}_{rel} \cdot d\underline{A} = 0$$

$$\rightarrow \int_{CV} \frac{\partial \rho}{\partial t} d\forall + \int_{CS} \rho \underline{u}_b \cdot d\underline{A} + \int_{CS} \rho \underline{u}_{rel} \cdot d\underline{A} = 0$$

Conservation of momentum

$$\frac{d}{dt}\int_{CV} \rho \underline{u}\, d\forall + \int_{CS} \rho\, \underline{u}\, \underline{u} \cdot d\underline{A} = \int_{CV} \rho \underline{g}\, d\forall - \int_{CS} P\, d\underline{A} + \underline{F_E}$$

Conservation of energy

$$\frac{d}{dt}\int_{CV} \left(e + \frac{u^2}{2} + gz\right) \rho\, d\forall + \int_{CS} \left(h + \frac{u^2}{2} + gz\right) \rho \underline{u}_{rel} \cdot d\underline{A} = \dot{Q} - \int_{CS} P\, \underline{u}_b \cdot dA - \dot{W}_{shaft}$$

Rocket engines

$\mathcal{T} = \dot{m} u_e + (P_e - P_a) A_e = M(t)\, g\cos\theta + D + M(t)\frac{du}{dt}$ - Full form 1-D steady state

$\mathcal{T} = \dot{m} u_e + (P_e - P_a) A_e = \dot{m} u_{eq}$ - 1-D, steady-state, $D = g = 0$

$I = \int F\,dt = u_{eq} M_p \rightarrow I/M_p = u_{eq} = \frac{\mathcal{T}}{\dot{m}}$

$I_{sp} = \frac{I}{M_p g_e} = \frac{u_{eq}}{g_e}$ $\qquad \dot{m} = \rho u A$

Δu & stuff

$\Delta u = u_{eq} \ln\frac{M_0}{M(t)}$

$\Delta u(t_b) = u_{eq} \ln R \rightarrow R = \frac{M_0}{M_b} = \frac{M_0}{M_0 - M_p} = \frac{M_0}{M_s + M_e}$

$$\frac{p}{p_0} = \left(1 + \frac{\gamma-1}{2}M^2\right)^{\frac{\gamma}{1-\gamma}}$$

$$\frac{T}{T_0} = \left(1 + \frac{\gamma-1}{2}M^2\right)^{-1}$$

$$\frac{\rho}{\rho_0} = \left(1 + \frac{\gamma-1}{2}M^2\right)^{\frac{1}{1-\gamma}}$$

$$\frac{p}{p^*} = \left(\frac{1+\frac{\gamma-1}{2}M^2}{1+\frac{\gamma-1}{2}}\right)^{\frac{\gamma}{1-\gamma}}$$

$$\frac{\rho}{\rho^*} = \left(\frac{1+\frac{\gamma-1}{2}M^2}{1+\frac{\gamma-1}{2}}\right)^{\frac{1}{1-\gamma}}$$

$$\frac{T}{T^*} = \frac{1+\frac{\gamma-1}{2}}{1+\frac{\gamma-1}{2}M^2}$$

$$\frac{A}{A^*} = \frac{1}{M}\left[\frac{2}{\gamma+1}\left(1 + \frac{\gamma-1}{2}M^2\right)\right]^{\frac{\gamma+1}{2(\gamma-1)}}$$

$$u_e = \sqrt{\frac{2\gamma\bar{R}}{(\gamma-1)\bar{M}} T_0\left[1-\left(\frac{p_e}{p_0}\right)^{\frac{\gamma-1}{\gamma}}\right]}$$

$$\dot{m} = p_0 A^* \sqrt{\frac{\bar{M}}{\bar{R}T_0}}\left[\gamma\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{\gamma-1}}\right]^{1/2} \quad \text{(isentropic)}$$

$$p_0 = p_e\left(1 - u_e^2\frac{(\gamma-1)\bar{M}}{2\gamma\bar{R}T_0}\right)^{\frac{\gamma}{1-\gamma}}$$

Char. velocity

$$c^* = \frac{p_0 A^*}{\dot{m}} \quad \text{(actual)}$$

$$c^*_{ideal} = \sqrt{\frac{1}{\gamma}\left(\frac{\gamma+1}{2}\right)^{\frac{\gamma+1}{\gamma-1}}\frac{\bar{R}T_0}{\bar{M}}}$$

$$C_J = \frac{J}{p_0 A^*}$$

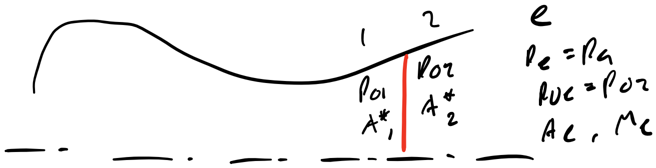
$$\frac{J}{p_0 A^*} = \sqrt{\frac{2\gamma^2}{(\gamma-1)}\left(\frac{2}{(\gamma+1)}\right)^{\frac{\gamma+1}{\gamma-1}}\left[1-\left(\frac{p_e}{p_0}\right)^{\frac{\gamma-1}{\gamma}}\right]} + \left(\frac{p_e}{p_0} - \frac{p_a}{p_0}\right)\frac{A_e}{A^*}$$

Locating shock in nozzle

if $\frac{P_{sler}}{P_0} < \frac{P_a}{P_0} < \frac{P_{sub}}{P_0}$ there is a shock in nozzle

If $P_a = P_{sler}$, shock is on exit plane

If $P_{sler} < P_a$, " " between throat & exit



G $\begin{cases} P_{01} \\ A_t \\ A_e \\ P_e = P_a \end{cases}$ F $\begin{cases} M_e \\ P_{0e} = P_{02} \\ M_1 \text{ just before shock} \\ A_1 \text{ @ shock} \end{cases}$

could iterate by guessing Area!

i) isentropic right of shock

@ Exit: $\frac{P_e}{P_{0e}} = \frac{P_e}{P_{02}} = \left(1 + \frac{\gamma-1}{2} M_e^2\right)^{\gamma/1-\gamma}$

@ Exit: $\frac{A_e}{A^*_2} = \frac{1}{M_e}\left[\frac{2}{\gamma+1}\left(1 + \frac{\gamma-1}{2} M_e^2\right)\right]^{\frac{\gamma+1}{2(\gamma-1)}}$

unknown P_{0e}, M_e, A^*_2

Multiply: $\frac{P_e}{P_{02}} \cdot \frac{A_e}{A^*_2} = \left(1 + \frac{\gamma-1}{2} M_e^2\right)^{\gamma/1-\gamma} \cdot \frac{1}{M_e}\left[\frac{2}{\gamma+1}\left(1 + \frac{\gamma-1}{2} M_e^2\right)\right]^{\frac{\gamma+1}{2(\gamma-1)}}$

$$= \frac{1}{M_e}\left[\frac{2}{\gamma+1}\right]^{\frac{\gamma+1}{2(\gamma-1)}} \left[1 + \frac{\gamma-1}{2} M_e^2\right]^{\underbrace{\frac{\gamma}{1-\gamma} + \frac{\gamma+1}{2(\gamma-1)}}_{= -1/2}}$$

Raise both sides to -2 → get EQ:

$$M_e^4 + b M_e^2 + c = 0$$

$$M_e^2 = -\frac{1}{\gamma-1} + \sqrt{\frac{1}{(\gamma-1)^2} + \frac{2}{(\gamma-1)}\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{\gamma-1}}\left(\frac{P_{02}A_2^*}{P_e A_e}\right)^2}$$

$P_{02}A_2^*$ unknown, but $P_{02}A_2^* = P_{01}A_1^*$

$$\rightarrow \boxed{M_e^2 = -\frac{1}{\gamma-1} + \sqrt{\frac{1}{(\gamma-1)^2} + \frac{2}{(\gamma-1)}\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{\gamma-1}}\left(\frac{P_{01}A_1^*}{P_e A_e}\right)^2}}$$

Find $M_e < 1$

ii) $\frac{P_e}{P_{02}} = \frac{P_e}{P_{0e}} = \left(1 + \frac{\gamma-1}{2}M_e^2\right)^{\frac{\gamma}{1-\gamma}}$ $\rightarrow$ Find $P_{0e} = P_{02}$

iii) Across shock: $\frac{P_{02}}{P_{01}} = \left[\frac{(\gamma+1)M_1^2}{2+(\gamma-1)M_1^2}\right]^{\frac{\gamma}{\gamma-1}}\left[\frac{\gamma+1}{2\gamma M_1^2-(\gamma-1)}\right]^{\frac{1}{\gamma-1}}$

$\rightarrow$ Find $M_1 > 1$ (root solver)

iv) $\frac{A_1}{A_1^*} = \frac{1}{M_1}\left[\frac{2}{\gamma+1}\left(1+\frac{\gamma-1}{2}M_1^2\right)\right]^{\frac{\gamma+1}{2(\gamma-1)}}$

$\rightarrow$ Find A_1

Across shock:

Remarks:

i) No shock possible if $M_1 < 1 \rightarrow$ shock can only form in supersonic flow

ii) $M_2 \leq 1$ flow after shock is always subsonic

iii) if $M_1 = 1 \rightarrow M_2 = 1$ infinitely weak shock $\equiv$ acoustic wave

Plug (4) into (3) and get

$$\frac{p_2}{p_1} = 1 + \frac{2\gamma}{\gamma+1}\left(M_1^2 - 1\right) \geq 1$$

plug (4) into (6):

$$\frac{T_2}{T_1} = \frac{\left[2\gamma M_1^2 - (\gamma-1)\right] \cdot \left[(\gamma-1)M_1^2 + 2\right]}{(\gamma+1)^2 M_1^2} \geq 1$$

From (1): $\frac{u_1}{u_2} = \frac{\rho_2}{\rho_1} = \frac{p_2}{\cancel{R}T_2} \cdot \frac{\cancel{R}T_1}{p_1} = \frac{p_2}{p_1} \cdot \frac{T_1}{T_2}$

$$\frac{u_1}{u_2} = \frac{\rho_2}{\rho_1} = \frac{(\gamma+1)M_1^2}{2+(\gamma-1)M_1^2} \geq 1$$

$p, T, \rho \uparrow$ across shock

$u \downarrow$ across shock

About p_0

- flow is isentropic before & after shock

$$\frac{p_1}{p_{01}} = \left(1 + \frac{\gamma-1}{2}M_1^2\right)^{\gamma/1-\gamma}$$

after shock: $\frac{p_2}{p_{02}} = \left(1 + \frac{\gamma-1}{2}M_2^2\right)^{\frac{\gamma}{1-\gamma}}$

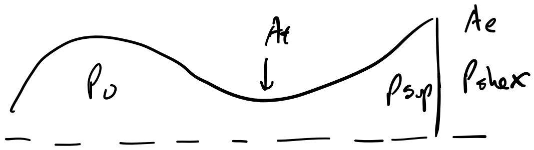
across shock: $M_2 = \left(\frac{1 + \frac{\gamma-1}{2}M_1^2}{\gamma M_1^2 - \frac{\gamma-1}{2}}\right)^{1/2}$

$$\frac{p_2}{p_1} = 1 + \frac{2\gamma}{\gamma+1}\left(M_2^2 - 1\right)$$

then $\frac{p_{02}}{p_{01}} = \frac{p_{02}}{p_2}\frac{p_2}{p_1}\frac{p_1}{p_{01}} = \ldots$

$$\rightarrow \frac{p_{02}}{p_{01}} = \left[\frac{(\gamma+1)M_1^2}{2+(\gamma-1)M_1^2}\right]^{\frac{\gamma}{\gamma-1}} \cdot \left[\frac{\gamma+1}{2\gamma M_1^2 - (\gamma-1)}\right]^{\frac{1}{\gamma-1}} \leq 1$$

To determine P_{shex} from known nozzle geometry



Isentropic supersonic $\quad \dfrac{A_e}{A^*} = \dfrac{A_e}{A_t} = \dfrac{1}{M_{sup}}\left[\dfrac{2}{\gamma+1}\left(1+\dfrac{\gamma-1}{2}M_{sup}^2\right)\right]^{\frac{\gamma+1}{2(\gamma-1)}}$

→ Determine M_{sup}

$$\frac{P_{sup}}{P_0} = \left(1 - \frac{\gamma-1}{2}M_{sup}^2\right)^{\frac{\gamma}{1-\gamma}} \rightarrow \text{Determine } P_{sup}$$

Across shock, $\dfrac{P_{shex}}{P_{sup}} = 1 + \dfrac{2\gamma}{\gamma+1}\left(M_{sup}^2 - 1\right) \rightarrow$ determine P_{shex}

<u>Prop. efficiency</u>

$$\eta = \frac{2\frac{u}{u_e}}{1+\left(\frac{u}{u_e}\right)^2},$$

"Mean propulsion efficiency"

$$\eta_M = \frac{1}{\Delta u/u_e}\ln\left(1+\left(\frac{\Delta u}{u_e}\right)^2\right)$$