Estimation for high-frequency data under parametric market microstructure noise

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Abstract

We develop a general class of noise-robust estimators based on the existing estimators in the nonnoisy high-frequency data literature. The microstructure noise is a parametric function of the limit order book. The noise-robust estimators are constructed as plug-in versions of their counterparts, where we replace the efficient price, which is non-observable, by an estimator based on the raw price and limit order book data. We show that the technology can be applied to five leading examples where, depending on the problem, price possibly includes infinite jump activity and sampling times encompass asynchronicity and endogeneity.

Keywords: functionals of volatility; high-frequency covariance; high-frequency data; limit order book; parametric market microstructure noise

1 Introduction

It is now widely acknowledged that the availability of high-frequency data has led to a more accurate description of financial markets. Over the past decades, empirical studies have unveiled several aspects of the frictionless efficient price. Accordingly, the assumptions on the latter have been gradually weakened to the extent that it is common nowadays to represent it as a general Itô semi-martingale including jumps with infinite activity. Moreover, the sampling times are also often considered as asynchronous, random, and even sometimes endogenous, i.e. possibly correlated with the efficient price. The accessibility of high-frequency data has also shed light on the frictions, or so-called market microstructure noise (MMN), which get prominent as the sampling frequency increases. As a matter of fact, realized volatility (i.e. summing the square returns), which is efficient in the absence of frictions becomes badly biased when the frequency increases. This was visible on the signature plot in Andersen

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et al (2001a). A typical challenge that faces a theoretical statistician today is to incorporate jumps, asynchronicity, endogeneity and frictions into the model.

A frequently used set-up is

$$\underbrace{Z_{t_i}}_{\text{observed price}} = \underbrace{X_{t_i}}_{\text{efficient price}} + \underbrace{\epsilon_{t_i}}_{\text{MMN}}, \tag{1}$$

where ϵ_{t_i} is i.i.d. and latent. In two nice and independent papers, Li et al (2016) and Chaker (2017), and subsequently Clinet and Potiron (2019b) and Clinet and Potiron (2019a), consider the following parametric form for the noise to estimate volatility:

$$\underbrace{Z_{t_i}}_{\text{observed price}} = \underbrace{X_{t_i}}_{\text{efficient price}} + \underbrace{\phi(Q_{t_i}, \theta_0)}_{\text{parametric noise}} + \underbrace{\epsilon_{t_i}}_{\text{residual noise}}, \tag{2}$$

where Q_{t_i} is the information from the limit order book and ϕ is a function known to the statistician. A simple and familiar example was introduced in Roll (1984), and specified in e.g. Hasbrouck (2002), where

$$\phi(Q_{t_i}, \theta_0) = I_{t_i}\theta_0,\tag{3}$$

with I_{t_i} corresponding to the trade direction, i.e. 1 if the transaction at time t_i is buyer initiated and -1 if seller initiated, and θ_0 standing for half of the effective spread. In Glosten and Harris (1988), the extension includes the trading volume V_{t_i} and takes on the form

$$\phi(Q_{t_i}, \theta_0) = I_{t_i} \theta_0^{(1)} + I_{t_i} V_{t_i} \theta_0^{(2)}. \tag{4}$$

A different model features information about the quoted spread S_{t_i} , where

$$\phi(Q_{t_i}, \theta_0) = I_{t_i} S_{t_i} \theta_0. \tag{5}$$

This model can be seen as an updated time-varying Roll model, as the quoted spread is nowadays available in the structure of current limit order book markets, whereas it was not observed at the time when Roll model was proposed.

There are two regimes related to the parametric model (2), i.e. the null residual noise and non zero residual noise. To estimate volatility, the cited papers rely on a plug-in procedure. In a first step, they provide estimators of the parameter θ_0 and establish fast convergence rate which satisfies

$$N(\widehat{\theta} - \theta_0) = O_{\mathbb{P}}(1), \tag{6}$$

where N stands for the number of observations and pre estimate the efficient price via

$$\widehat{X}_{t_i} = Z_{t_i} - \phi(Q_{t_i}, \widehat{\theta}). \tag{7}$$

In a second step, one can apply a "usual" estimator of volatility, considering the observed price as in fact the pre estimated efficient price. More specifically, in case of absence of residual noise, the cited

papers implement realized volatility and retrieve efficiency of the method. In the presence of residual noise, they also provide residual noise robust estimators.

In this paper, we will assume the null residual noise regime, which we agree is quite a strong assumption (at first glance). Indeed, from a theoretical statistician standpoint, the non zero residual noise regime, of which the common set-up (1) is a particular case, is obviously more challenging. Nonetheless, the original papers Li et al (2016) and Chaker (2017) most likely wanted to select empirically variables from the limit order book that fully explain the MMN. Actually, in their empirical study on four stocks and one day, Li et al (2016) find that the residual noise of models such as (3) and (4) accounts for 20-30% of the total MMN variance, which is quite low and yet not negligible. Chaker (2017) proposes and implements on a full year of one stock from the New York Stock Exchange tests for the absence of residual noise. She finds rejection rate around 15-25\% for (3), and 10-30\% in the case of (4), here again quite nice results but not indicating the absence of residual noise. More recently, implemented on a month with 31 constituents from the CAC 40, Clinet and Potiron (2019b) find that the "best" model among several competitors from the financial economics literature is (5), with related residual noise accounting for as low as 1% of the MMN variance, and results in line with previous findings for the other models. Finally, in an extensive study on 50 stocks randomly selected from the S&P 500 during the period 2009-2017, Clinet and Potiron (2019a) exhibit (5) as the model explaining the most variance of the MMN, with residual noise accounting for (almost) 0% of the total MMN variance. Those two empirical studies back up the null residual noise regime.

When implementing a non noise-robust procedure with high frequency data, it is often the case that the applied statistician faces a dilemma in using tick-by-tick data on the statistical principle that one should not throw away data, or subsampling -say every five minutes- in respect to the limited theoretical assumptions. We argue that the plug-in approach is a cheap method that kills two birds with one stone. On the one hand, it provides the theoretical statistician with a simple and transparent method for adding MMN in his theory. On the other hand, this will be useful for the applied statistician as he/she will be able to use tick-by-tick data when implementing the related estimator. This strategy is actually successfully used in Andersen et al (2019). In particular, our paper enlightens the theoretical aspect of the plug-in approach.

To do so, we describe the general framework as follows. If we define the horizon time as T, one typically seeks to estimate the random integrated parameter

$$\Xi = \int_0^T \xi_t dt,\tag{8}$$

where the spot parameter ξ_t is a stochastic process which can correspond to the volatility, the high-frequency covariance, functionals of volatility and volatility of volatility, employing a given data-based estimator $\widetilde{\Xi}(X_{t_0}, \dots, X_{t_N})$. In the absence of noise, $\widetilde{\Xi}$ usually enjoys a stable central limit theorem of the form

$$N^{\kappa}(\widetilde{\Xi} - \Xi) \to \mathcal{MN}(AB, AVAR),$$
 (9)

where $\kappa > 0$ corresponds to the rate of convergence, and $\mathcal{MN}(AB, AVAR)$ designates a mixed normal distribution of random bias AB and random variance AVAR (due to the fact that the parameter itself is

random). In addition, for the purpose of practical implementation, one typically provides a related studentized central limit theorem, i.e. data-based statistics $\widetilde{AB}(X_{t_0}, \dots, X_{t_N})$ and $\widetilde{AVAR}(X_{t_0}, \dots, X_{t_N})$ such that

$$N^{\kappa} \frac{\widetilde{\Xi} - N^{-\kappa} \widetilde{AB} - \Xi}{\sqrt{\widetilde{AVAR}}} \to \mathcal{N}(0, 1). \tag{10}$$

Accordingly, when observations are contaminated by the parametric noise, we propose to exploit the corresponding class of plug-in estimators to estimate the integrated parameter. They are constructed as $\widehat{\Xi} = \widetilde{\Xi}(\widehat{X}_{t_0}, \cdots, \widehat{X}_{t_N})$, $\widehat{AB} = \widetilde{AB}(\widehat{X}_{t_0}, \cdots, \widehat{X}_{t_N})$ and $\widehat{AVAR} = \widehat{AVAR}(\widehat{X}_{t_0}, \cdots, \widehat{X}_{t_N})$. This plug-in approach seems to be traced back to the framework of the model with uncertainty zones from Robert and Rosenbaum (2010) and Robert and Rosenbaum (2012).

The main contribution of this paper is presented in Section 4, where we state that under parametric noise the central limit theorems (9) and (10) still hold when we substitute the estimators by their related plug-in version in five leading examples of the literature. Depending on the problem at hand, price possibly features jumps with infinite activity and sampling times include asynchronicity and endogeneity. The first example considers the threshold realized volatility inspired by Andersen et al (2001b), Barndorff-Nielsen and Shephard (2002b) and Mancini (2009). Technically, we extend the central limit theory of realized volatility under endogenous sampling in Li et al (2014), which includes no jumps to allow for jumps with infinite activity. The second example deals with the threshold bipower variation, which was originally with no threshold in Barndorff-Nielsen and Shephard (2004), and from Corsi et al (2010) and Vetter (2010). In the third example, we discuss the Hayashi and Yoshida (2005) estimator to estimate high-frequency covariance. The fourth example is devoted to the local estimator from Jacod and Rosenbaum (2013) which estimates functionals of volatility. Finally, we focus on the estimator of volatility of volatility introduced in Vetter (2015) in the last example.

In all those examples, the only required assumption on $\widehat{\theta}$ to obtain (9) and (10) is the fast convergence (6), which is already obtained in a general setting where price process features big jumps in Li et al (2016), so that our contribution in that respect boils down to adding possible small jumps. Moreover, the asymptotic properties in both equations remain unchanged, whereas the rate of convergence is slower in the i.i.d latent noise case. It means that the parametric noise assumption induces faster rates of convergence than the i.i.d condition, but it is fair to say that we play a different game in this paper as plug-in estimators exploit supplementary data available from the limit order book.

The rest of this paper is structured as follows. Section 2 introduces the model. Section 3 is devoted to the estimation. The five examples are developed in Section 4. We conclude in Section 5. Proofs can be found in Section 6.

2 Model

Almost all the quantities defined in what follows are multi-dimensional. Accordingly, the notation $x^{(k)}$ refers to the k-th component of x. We define the horizon time as T > 0, and the (possibly random)

number of observations¹ as N. The observation times, which satisfy $0 \le t_0^{(k)} \le ... \le t_N^{(k)} \le T$, are possibly asynchronous, i.e. they may differ from one price component to the next (see Section 4.3), and endogenous, i.e. correlated with X_t (as in Section 4.1 and Section 4.3). When observations are regular and synchronous, we have $\Delta_i t := t_i - t_{i-1} = T/n := \Delta$ (as in Section 4.2, Section 4.4 and Section 4.5), which implicitly means that N = n and t_i are 1-dimensional, although the price process can be multi-dimensional.

In view of the empirical findings described in the introduction, it is natural to specify (2) as the "pure" parametric noise model via

$$\underbrace{Z_{t_i}}_{\text{observed price}} = \underbrace{X_{t_i}}_{\text{efficient price}} + \underbrace{\phi(Q_{t_i}, \theta_0)}_{\text{parametric noise}},$$
(11)

where the parameter $\theta_0 \in \Theta \subset \mathbb{R}^l$ with Θ a compact set, the impact function ϕ is known of class C^3 in its second argument, and $Q_{t_i} \in \mathbb{R}^q$ includes observable information² at the observation time t_i from the limit order book such as the aforementioned trade type (Roll (1984)), trading volume (Glosten and Harris (1988)) and quoted bid-ask spread, but also possibly the duration time between two trades (Almgren and Chriss (2001)), the quoted depth (Kavajecz (1999)), the order flow imbalance (Cont et al (2014)), etc. In practice, ϕ could be always chosen as (5), although we do not specify this particular model in the paper for generality purposes. Further discussion is available in: Black (1986), Hasbrouck (1993), O'hara (1995), Madhavan et al (1997), Madhavan (2000), Stoll (2000) and Hasbrouck (2007) among other prominent works. One can also look at the review from Diebold and Strasser (2013). Finally, on the grounds that the one-lag autocorrelation in mid price returns is often found positive empirically, Andersen et al (2017) extend the usual martingale-plus-noise setting to allow for positivity in the one-lag serial autocorrelation. Note that the model of (11), without residual noise, is theoretically interesting because it allows to adapt the existing methods by plugging in the estimated price in place of the existing estimator.

Finally, we assume that

$$\max_{i,j,k} |Q_{t_i^{(k)}}^{(k,j)}| = O_{\mathbb{P}}(1), \tag{12}$$

where $Q_{t_i^{(k)}}^{(k)} = (Q_{t_i^{(k)}}^{(k,1)}, \cdots, Q_{t_i^{(k)}}^{(k,j_k)})$ corresponds to the information related to $X^{(k)}$ at time $t_i^{(k)}$. The latent d-dimensional log-price X_t possibly including jumps and its related d^2 -dimensional spot volatility

¹All the defined quantities are implicitly or explicitly indexed by n (except for the integrated parameter which does not depend on n). For example N should be thought and considered as N_n . Consistency and convergence in law refer to the behavior as $n \to \infty$. A full specification of the model also involves the stochastic basis $\mathcal{B} = (\Omega, \mathbb{P}, \mathcal{F}, \mathbf{F})$, where \mathcal{F} is a σ -field and $\mathbf{F} = (\mathcal{F}_t)_{t \in [0,T]}$ is a filtration, which will be example-specific. We assume that all the processes (including the integrated parameter ξ_t) are \mathbf{F} -adapted (either in a continuous or discrete meaning for Q_{t_i}) and that the observation times t_i are \mathbf{F} -stopping times. Also, when referring to Itô-semimartingale and stable convergence in law, we automatically mean that the statement is relative to \mathbf{F} . Finally, we assume in (13) that W is also a Brownian motion under the larger filtration $\mathcal{H}_t = \mathcal{F}_t \vee \sigma\{Q_{t_i}, 0 \le i \le N\}$.

²Note that we do not assume that Q_t exists for any $t \in [0, T] - \{t_0, \dots, t_N\}$ as it is often the case in the i.i.d setting, see, e.g., the framework in Jacod et al (2009).

 $c_t = \sigma_t \sigma_t^T$ are Itô-semimartingales of the form

$$X_{t} = X_{0} + \int_{0}^{t} b_{s} ds + \int_{0}^{t} \sigma_{s} dW_{s} + \int_{0}^{t} \int_{\mathbb{R}} \delta(s, z) \mathbf{1}_{\{||\delta(s, z)|| \le 1\}} (\mu - \nu)(ds, dz)$$

$$+ \int_{0}^{t} \int_{\mathbb{R}} \delta(s, z) \mathbf{1}_{\{||\delta(s, z)|| > 1\}} \mu(ds, dz),$$

$$\int_{0}^{t} \int_{\mathbb{R}} \delta(s, z) \mathbf{1}_{\{||\delta(s, z)|| > 1\}} \mu(ds, dz),$$

$$\int_{0}^{t} \int_{\mathbb{R}} \delta(s, z) \mathbf{1}_{\{||\delta(s, z)|| > 1\}} \mu(ds, dz),$$
(13)

$$c_{t} = c_{0} + \int_{0}^{t} \widetilde{b}_{s} ds + \int_{0}^{t} \widetilde{\sigma}_{s} dW'_{s} + \int_{0}^{t} \int_{\mathbb{R}} \widetilde{\delta}(s, z) \mathbf{1}_{\{||\widetilde{\delta}(s, z)|| \leq 1\}} (\mu - \nu)(ds, dz)$$

$$+ \int_{0}^{t} \int_{\mathbb{R}} \widetilde{\delta}(s, z) \mathbf{1}_{\{||\widetilde{\delta}(s, z)|| > 1\}} \mu(ds, dz),$$

$$(14)$$

where W_t is a d-dimensional Brownian motion and W'_t is a d^2 -dimensional Brownian motion possibly correlated with W_t , the d-dimensional b_t and d^2 -dimensional \widetilde{b}_t drifts are locally bounded, σ_t and the d^2 -dimensional $\widetilde{c}_t = \widetilde{\sigma}_t \widetilde{\sigma}_t^T$ are locally bounded, μ is a Poisson random measure on $\mathbb{R}^+ \times E$ where E is an auxiliary Polish space, with the related intensity measure, i.e. the nonrandom predictable compensator, $\nu(dt, dz) = dt \otimes \lambda(dz)$ for some σ -finite measure λ on \mathbb{R}^+ . Finally, $\delta = \delta(\omega, t, z)$ (respectively $\widetilde{\delta}$) is a predictable \mathbb{R}^d -valued ($\mathbb{R}^{d \times d}$ -valued) function on $\Omega \times \mathbb{R}^+ \times \mathbb{R}$ such that locally $\sup_{\omega,t} || \delta(\omega,t,z) ||^r \le \gamma(z)$ ($\sup_{\omega,t} || \widetilde{\delta}(\omega,t,z) ||^{\widetilde{r}} \le \gamma(z)$) for some nonnegative bounded λ -integrable function γ and some $r \in [0,1)$ ($r \in [0,1)$) ($r \in [0,1)$) ($r \in [0,1)$) Furthermore, we define the "genuine" drift as $b'_t = b_t - \int \delta(t,z) \mathbf{1}_{\{||\delta(t,z)|| \le 1\}} \lambda(dz)$, the continuous part of X_t as

$$X_t' = X_0 + \int_0^t b_s' ds + \int_0^t \sigma_s dW_s,$$

and the jump part as $J_t = \sum_{s < t} \Delta X_s$. Key to our analysis is the decomposition

$$X_t = X_t' + J_t. (15)$$

3 Estimation under parametric noise

3.1 Integrated parameter estimation

The object of interest can be the integrated volatility, etc. In the non-noisy version of the problem, the typical scenario is such that the high-frequency data user has a data-based estimator $\tilde{\Xi}(X_{t_0}, \dots, X_{t_N})$ of (8), such as the standard realized volatility (RV), i.e. $RV = \sum_{i=1}^{N} \Delta_i X^2$ where $\Delta_i A = A_{t_i} - A_{t_{i-1}}$, and possibly a related central limit theorem and a studentized version of it. In all generality, they respectively take the form of

$$N^{\kappa}(\widetilde{\Xi} - \Xi) \to \mathcal{MN}(AB, AVAR),$$
 (16)

³Here the restriction r < 1 follows from Jacod and Rosenbaum (2013). Indeed, even for the realized volatility problem, (16) may not happen in the case r > 1. Indeed, it yields a different optimal rate of convergence as shown in Jacod and Reiss (2014) (of the form $N^{\kappa} \log N$ for some $\kappa > 0$). Moreover, as explained in their Remark 3.4, a CLT is not even achievable in some cases. The case r = 1 is let aside. Such bordercase is examined in Vetter (2010) when considering the bipower variation.

where $\kappa > 0$ corresponds to the rate of convergence, and

$$N^{\kappa} \frac{\widetilde{\Xi} - N^{-\kappa} \widetilde{AB} - \Xi}{\sqrt{\widetilde{AVAR}}} \to \mathcal{N}(0, 1), \tag{17}$$

where $\widetilde{AB}(X_{t_0}, \dots, X_{t_N})$ and $\widetilde{AVAR}(X_{t_0}, \dots, X_{t_N})$ are also data-based statistics which respectively correspond to the asymptotic bias and the asymptotic variance estimator. The aim of this section is to equip the high-frequency data user with noise-robust estimators which are based on $\widetilde{\Xi}$.

To estimate the integrated parameter, we first need an estimator of the noise parameter θ_0 defined as $\hat{\theta}$. We assume that $\hat{\theta}$ satisfies

$$N(\widehat{\theta} - \theta_0) = O_{\mathbb{P}}(1). \tag{18}$$

The techniques of this paper are estimator independent and only require (18). In Section 3.2, we provide the form of the estimators from the literature which satisfy (18) (see Proposition 1 below). Based on $\widehat{\theta}$, the efficient price is naturally estimated as

$$\widehat{X}_{t_i} = Z_{t_i} - \phi(Q_{t_i}, \widehat{\theta}). \tag{19}$$

This estimator was already used in Li et al (2016), Chaker (2017) and Clinet and Potiron (2019b). The related plug-in estimator is constructed as

$$\widehat{\Xi} = \widetilde{\Xi}(\widehat{X}_{t_0}, \cdots, \widehat{X}_{t_N}). \tag{20}$$

For instance, in the case of RV, we obtain that $\widehat{RV} = \sum_{i=1}^{N} \Delta_i \widehat{X}^2$. Similarly, we introduce $\widehat{AB} = \widetilde{AB}(\widehat{X}_{t_0}, \dots, \widehat{X}_{t_N})$ and $\widehat{AVAR} = \widetilde{AVAR}(\widehat{X}_{t_0}, \dots, \widehat{X}_{t_N})$.

We end this section with a succinct remark on the theoretical implications of (18). At this point, the reader may notice that the fast rate N^{-1} in (18), which implies the approximation $\widehat{X}_{t_i} = X_{t_i} + \psi_i(\widehat{\theta})$ with $\psi_i(\widehat{\theta}) = \phi(Q_{t_i}, \theta_0) - \phi(Q_{t_i}, \widehat{\theta}) = O_{\mathbb{P}}(N^{-1})$ by (3.1), suggests that the perturbation $\psi_i(\widehat{\theta})$ acts as an additional drift component and therefore could be systematically treated as such in all derivations. There is, however, a fundamental difference between the two quantities, in that drift returns $\Delta_i B = \int_{t_{i-1}}^{t_i} b_s ds$ are typically adapted, hence \mathcal{F}_{t_i} measurable, whereas $\psi_i(\widehat{\theta})$, through $\widehat{\theta}$, depends not only on the additional observations $(Q_{t_j})_{j=0,\dots,N}$ but also on the whole trajectory of the price process X, that is \mathcal{F}_T . This may pose a problem when considering, for instance, terms of the form $A_{i-1}\Delta_i M$ where $\Delta_i M$ is a martingale increment (even for the augmented filtration $\mathcal{H}_t = \mathcal{F}_t \vee \sigma\{Q_{t_i}, 0 \leq i \leq N\}$). Indeed, when $A_{i-1} = \Delta_{i-1}B$, it naturally preserves the martingale structure of $A_{i-1}\Delta_i M$. On the other hand, if $A_{i-1} = \psi_{i-1}(\widehat{\theta})$, such a structure is broken, and additional arguments are necessary in order to retrieve the desired order of the increment $A_{i-1}\Delta_i M$. In this simple example, the problem can be circumvented with a Taylor expansion $\psi_{i-1}(\widehat{\theta}) \approx (\widehat{\theta} - \theta_0)^T \partial_{\theta} \psi_{i-1}(\theta_0) + r_{i-1}(\widehat{\theta})$, using that now $\partial_{\theta} \psi_{i-1}(\theta_0)$ is $\mathcal{H}_{t_{i-1}}$ measurable, and that $r_{i-1}(\widehat{\theta})$ is of order N^{-2} .

3.2 Noise parameter estimation

Several estimators have been proposed by Li et al (2016), Chaker (2017), Clinet and Potiron (2019b) in different settings when we assume a null residual noise $\epsilon_t = 0$. The estimator from Chaker (2017) coincides with the minimum mean square error (MSE) estimator from Li et al (2016) when ϕ is linear (which is the related assumption of the former paper). Moreover, the quasi maximum likelihood estimation (QMLE) from Clinet and Potiron (2019b) reduces to the MSE, due to the Gaussian form of the quasi likelihood function. Accordingly, we review the MSE procedure below and give the related limit theory for the noise parameter estimator.

We assume that $\theta := (\theta_0^{(1)}, \dots, \theta_0^{(d)})$, where for each component $k = 1, \dots, d$ we have $\theta_0^{(k)} := (\theta_0^{(k,1)}, \dots, \theta_0^{(k,l_k)})$, which corresponds to the parameter related to the kth component of the observed price. More specifically, we assume the componentwise form

$$Z_{t_i}^{(k)} = X_{t_i}^{(k)} + \phi(Q_{t_i}^{(k)}, \theta_0^{(k)}). \tag{21}$$

Accordingly, we consider the estimation of $\theta_0^{(k)}$ separately and thus we can assume that d=1 in what follows without loss of generality. The estimator $\hat{\theta}^{(MSE)}$ is given by

$$\widehat{\theta}^{(MSE)} = \underset{\theta \in \Theta}{\operatorname{argmin}} \ Q_N(Z, \theta), \text{ where}$$

$$Q_N(Z, \theta) = \frac{1}{2} \sum_{i=1}^{N} (\Delta_i Z - \mu_i(\theta))^2,$$

where $\mu_i(\theta) = \phi(Q_{t_i}, \theta) - \phi(Q_{t_{i-1}}, \theta)$.

When ϕ is linear, the problem boils down to a linear regression. As a result the estimator admits the explicit form

$$\widehat{\theta}^{(MSE)} = (\mathbb{M}^T \mathbb{M})^{-1} \mathbb{M}^T \Delta Z, \tag{22}$$

where $\Delta Z := (\Delta_1 Z, \dots, \Delta_N Z)$, and as soon as the matrix

$$\mathbb{M} := \left(\Delta_i Q^{(j)}\right)_{1 \le i \le N, 1 \le j \le l}$$

is such that $\mathbb{M}^T \mathbb{M}$ is invertible.

We now recall the limit theory associated to $\widehat{\theta}^{(MSE)}$ under the framework of Li et al (2016) which in particular includes jumps with infinite activity. In the next proposition, Condition **A** assumes the local boundedness of b and σ , the summability of the jump process, and several standard identifiability assumptions of most functions which depend on the parameter θ and the sequence $(Q_{t_i})_{i \in \mathbb{N}}$. Details can be found in Li et al (2016), p. 35.

Proposition 1. (Theorem 1 from Li et al (2016)). Assume Condition A from Li et al (2016). Then

$$N(\widehat{\theta}^{(MSE)} - \theta_0) = O_{\mathbb{P}}(1).$$

4 Applications of the method

In what follows, we state that the plug-in estimators are noise-robust for five leading examples taken from the literature, and that the central limit theorems (9) and (10) hold under parametric noise. In Example 4.1, we study the threshold realized volatility in the case of infinite activity jumps in price and endogeneity in arrival times. We go one step further the central limit theory of realized volatility with in Li et al (2014), which includes no jumps when there is endogeneity in observation times, to allow for jumps with infinite activity. We first state the central limit theorems related to threshold realized volatility, and then the theory associated to the plug-in estimators. In Example 4.2, we consider the threshold bipower variation under infinite activity jumps and regular observations. In Example 4.3, we develop the Hayashi-Yoshida estimator of high-frequency covariance in a no-jump setup, and asynchronous and endogenous observation times. In Example 4.4, we consider the estimation of functionals of volatility when the price can exhibit jumps with infinite activity and observations are regular. Finally, we address the case of volatility of volatility for continuous price and volatility processes and regular observation times in Example 4.5.

4.1 Threshold realized volatility

The parameter is $\xi_t = \sigma_t^2$, and the rate of convergence $\kappa = 1/2$ if observations are not contaminated by the noise. When the price is continuous and observations are regular, a popular estimator of $\Xi = \int_0^T \sigma_s^2 ds$ is RV considered in Andersen et al (2001a), Andersen et al (2001b), but also in Barndorff-Nielsen and Shephard (2002a), Barndorff-Nielsen and Shephard (2002b), Meddahi (2002), etc. Jacod and Protter (1998) showed that

$$n^{1/2} \Big(RV - \int_0^T \sigma_s^2 ds \Big) \to \mathcal{MN} \Big(0, 2T \int_0^T \sigma_s^4 ds \Big).$$

When observations are not regular, the AVAR is equal to $2T \int_0^T \sigma_t^4 dH_t$, where $H_t = \lim T^{-1}N \sum_{t_i \leq t} (t_i - t_{i-1})^2$ is the so-called "quadratic variation of time" (see Zhang (2001) and Mykland and Zhang (2006)), provided that such a quantity exists. When observations are endogenous, Li et al (2014) show that the limit distribution of $n^{1/2}(RV - \Xi)$ includes an asymptotic bias and that the related AVAR is altered. In addition, they prove that the informational content of arrival times can be useful to estimate the asymptotic bias and the AVAR.

Our aim is to allow for parametric noise in this endogenous setting, while also including jumps in the price process. As far as the authors know, no general theory⁴ includes general endogeneity and jumps, even when observations are not noisy. Accordingly, we first extend the results of Li et al (2014) when adding jumps. Then, we show that the technology of this paper applies in such a general setting, and this part essentially boils down to applying the arguments of Li et al (2016).

Although no theory exists under endogeneity, Theorem 13.2.4 (p. 383) in Jacod and Protter (2011) can be used when observations are regular. We consider a similar threshold RV, originally in the

⁴Remark 6 (p. 36) in Li et al (2016) suggests that the threshold RV estimator can be used under endogeneity, but there is no formal proof and this is limited to the case of jumps with finite activity.

spirit of Mancini (2009) and Mancini (2011), and defined as $\widetilde{\Xi} = \sum_{i=1}^{N} (\Delta_i X)^2 \mathbf{1}_{\{|\Delta_i X| \leq w_i\}}$, where $w_i = \alpha \Delta_i t^{\bar{\omega}}$, $\bar{\omega} \in (1/(2(2-r)), 1/2)$ and $\alpha > 0$ is a tuning parameter. In the next theorem, we provide the related central limit theorem and show that the condition of our paper holds.

Theorem 2. We assume that $\inf_{t \in (0,T]} \sigma_t > 0$. We further suppose that there exists non random \widetilde{u}_t and \widetilde{v}_t such that

$$n \sum_{0 < t, i < t} (\Delta_i X')^4 \to^{\mathbb{P}} \int_0^t \widetilde{u}_s \sigma_s^4 ds, \tag{23}$$

$$n^{1/2} \sum_{0 < t_i \le t} (\Delta_i X')^3 \to^{\mathbb{P}} \int_0^t \widetilde{v}_s \sigma_s^3 ds, \tag{24}$$

where $\tilde{u}_t \sigma_t^4$, $\tilde{v}_t \sigma_t^3$ and $\tilde{v}_t^2 \sigma_t^4$ are integrable, and \tilde{v}_t locally bounded and bounded away from 0. Furthermore, we assume that t_i , b_t , σ_t and δ are generated by finitely many Brownian motions⁵. Finally we assume that $N/n \to^{\mathbb{P}} F$ for some random variable F, and that $n\Delta_i t$ are locally bounded and locally bounded away from 0. Then, stably in law as $n \to \infty$, we have

$$N^{1/2}(\widetilde{\Xi} - \Xi) \to \frac{2}{3} \int_0^T v_s \sigma_s dX_s' + \int_0^T \sqrt{\frac{2}{3}u_s - \frac{4}{9}v_s^2} \sigma_s^2 dB_s,$$
 (25)

where $v_s = \sqrt{F}\widetilde{v}_s$, $u_s = F\widetilde{u}_s$ and B_t is a standard Brownian motion independent of the other quantities⁶. Moreover, we have

$$N^{1/2}(\widehat{\Xi} - \Xi) \to \frac{2}{3} \int_0^T v_s \sigma_s dX_s' + \int_0^T \sqrt{\frac{2}{3}u_s - \frac{4}{9}v_s^2} \sigma_s^2 dB_s.$$
 (26)

Remark 3. If observations are regular, then F = 1, $u_s = 3T$ and $v_s = 0$ for all $s \in [0, T]$. Therefore, (25) and (26) can be specified as

$$n^{1/2}(\widetilde{\Xi} - \Xi) \to \mathcal{M}\mathcal{N}\left(0, 2T \int_0^T \sigma_s^4 ds\right),$$
 (27)

$$n^{1/2}(\widehat{\Xi} - \Xi) \to \mathcal{M}\mathcal{N}\left(0, 2T \int_0^T \sigma_s^4 ds\right).$$
 (28)

We provide now jump-robust estimators of $AB = (2/3) \int_0^T v_s \sigma_s dX_s'$ and $AVAR = \int_0^T (\frac{2}{3}u_s - \frac{4}{9}v_s^2)\sigma_s^4 ds$ based on the non jump-robust estimators provided in Li et al (2014). Accordingly, we chop the data into B blocks of h observations (except for the last block which might include less observations). We set $h = \lfloor n^\beta \rfloor$, where $1/2 < \beta < 1$. We can estimate $v_{t_{hi}}\sigma_{t_{hi}}$ as

$$\widetilde{v\sigma}_{i} = \frac{N^{1/2} \sum_{j=h(i-1)+1}^{hi} (\Delta_{j}X)^{3} \mathbf{1}_{\{|\Delta_{j}X| \leq w_{j}\}}}{\sum_{j=h(i-1)+1}^{hi} (\Delta_{j}X)^{2} \mathbf{1}_{\{|\Delta_{j}X| \leq w_{j}\}}},$$

⁵i.e. we assume that t_i are **G**-stopping times, where $\mathbf{G} = (\mathcal{G}_t)_{t \in [0,T]}$ is a sub-filtration of **F** generated by finitely many Brownian motions, and that b_t , σ_t and δ are adapted to **G**.

⁶Here and in the other theorems, we mean that B_t is independent of the underlying σ -field \mathbf{F} .

and AB and AVAR as

$$\begin{split} \widetilde{AB} &= \sum_{i=1}^{B} \underbrace{\frac{2}{3} \widetilde{v} \widetilde{\sigma}_{i} \Bigg\{ \sum_{j=h(i-1)+1}^{hi} \Delta_{j} X \mathbf{1}_{\{|\Delta_{j}X| \leq w_{j}\}} \Bigg\}}_{\widetilde{AB}_{i}}, \\ \widetilde{AVAR} &= \underbrace{\frac{2N}{3} \sum_{j=h}^{N} (\Delta_{i}X)^{4} \mathbf{1}_{\{|\Delta_{i}X| \leq w_{i}\}} - \sum_{j=h}^{B} \widetilde{AB}_{i}^{2}. \end{split}}$$

Recalling that \widehat{AB} and \widehat{AVAR} are constructed respectively as \widetilde{AB} and \widetilde{AVAR} when replacing X by \widehat{X} , we provide now the studentized version of the previous central limit theorems.

Corollary 4. We have

$$N^{1/2} \frac{\widetilde{\Xi} - N^{-1/2} \widetilde{AB} - \Xi}{\sqrt{\widetilde{AVAR}}} \to \mathcal{N}(0, 1), \tag{29}$$

$$N^{1/2} \frac{\widehat{\Xi} - N^{-1/2} \widehat{AB} - \Xi}{\sqrt{\widehat{AVAR}}} \to \mathcal{N}(0, 1).$$
(30)

Remark 5. If observations are regular, there is no asymptotic bias and AVAR can be estimated using the plug-in estimator of quarticity obtained in Section 4.4. In view of Theorem 11 which implies the consistency of the plug-in estimator, we obtain directly by the stable convergence obtained in Theorem 2 that (30) holds.

Remark 6. (estimating volatility under i.i.d noise) Alternative approaches to estimate integrated volatility under latent i.i.d noise include and are not limited to: the Quasi-Maximum Likelihood Estimator (QMLE) from Aït-Sahalia et al (2005) which was later shown to be robust to time-varying volatility in Xiu (2010), the Two-Scale Realized Volatility in Zhang et al (2005), the multi-Scale realized volatility in Zhang (2006), the pre-averaging approach in Jacod et al (2009), realized kernels in Barndorff-Nielsen et al (2008) and the spectral approach considered in Altmeyer and Bibinger (2015) based on Reiss (2011). Clinet and Potiron (2018) discussed AVAR reduction when considering local estimators. In addition, Li et al (2013) consider endogenous arrival times.

4.2 Threshold bipower variation

Here again $\xi_t = \sigma_t^2$. The bipower variation $BV = \frac{\pi}{2} \sum_{i=2}^N |\Delta_i X| |\Delta_{i-1} X|$ (more generally multipower variation from Barndorff-Nielsen and Shephard (2004) and Barndorff-Nielsen and Shephard (2006)) was originally introduced as an alternative measure robust to finite-activity jumps. In case of regular observations and no jump, Barndorff-Nielsen et al (2006a) and Barndorff-Nielsen et al (2006b) established the central limit theory. See also Kinnebrock and Podolskij (2008) for related development. In case of finite-activity jumps, see also Barndorff-Nielsen et al (2006c).

If jumps exhibit infinite activity, Vetter (2010) shows that BV is no longer consistent, but the jump-robust threshold estimator

$$\widetilde{\Xi} = \frac{\pi}{2} \sum_{i=2}^{N} |\Delta_i X| \mathbf{1}_{\{|\Delta_i X| \le w\}} |\Delta_{i-1} X| \mathbf{1}_{\{|\Delta_{i-1} X| \le w\}}$$

is consistent, where $w = \alpha \Delta^{\bar{\omega}}$, $\bar{\omega} \in (0, 1/2)$. Moreover, he also shows the related central limit theory. See also Corsi et al (2010) for related work. Finally, the general theory (Theorem 13.2.1 (p. 380)) from Jacod and Protter (2011) can be applied too. All those papers have in common that they assume regular observations, and we follow the same setting to show that the techniques of this paper can be used in this example too. We provide the formal result in what follows.

Theorem 7. We have that

$$n^{1/2}(\widehat{\Xi} - \widetilde{\Xi}) \to^{\mathbb{P}} 0.$$
 (31)

In particular, stably in law as $n \to \infty$,

$$n^{1/2}(\widehat{\Xi} - \Xi) \to \frac{\pi}{2} \sqrt{\left(1 + \frac{4}{\pi} - \frac{12}{\pi^2}\right) T} \int_0^T \sigma_s^2 dB_s,$$
 (32)

where B_t is a Brownian motion independent of the other quantities.

In this example, we have that $AVAR = \frac{\pi^2}{4}(1 + \frac{4}{\pi} - \frac{12}{\pi^2})T\int_0^T \sigma_s^4 ds$, which can be estimated by $\widehat{AVAR} = \frac{\pi^2}{4}(1 + \frac{4}{\pi} - \frac{12}{\pi^2})T\widehat{\int_0^T \sigma_s^4 ds}$, where the plug-in estimator of quarticity $\widehat{\int_0^T \sigma_s^4 ds}$ is defined as a particular case of Section 4.4 (i.e $\widehat{\int_0^T \sigma_s^4 ds}$ corresponds to the estimator given in (39) below with $g(x) = x^2$). We also provide the related studentized central limit theorem.

Corollary 8. We have

$$n^{1/2} \frac{\widehat{\Xi} - \Xi}{\sqrt{\widehat{AVAR}}} \to \mathcal{N}(0, 1). \tag{33}$$

4.3 Hayashi-Yoshida estimator of high-frequency covariance

We assume here that X_t is 2-dimensional and that $\xi_t = \rho_t \sigma_t^{(1)} \sigma_t^{(2)}$, where the high-frequency correlation ρ_t satisfies $d\langle W^{(1)}, W^{(2)} \rangle_t = \rho_t dt$. The rate of convergence is $\kappa = 1/2$ in this problem too. We consider that observations are non-synchronous. In this framework and assuming that the price is continuous, Hayashi and Yoshida (2005) bring forward the so-called Hayashi-Yoshida estimator and establish the consistency in case sampling times are independent from the price process. This is extended in an endogenous setting in Hayashi and Kusuoka (2008). The related central limit theory can be found in Hayashi and Yoshida (2008), Hayashi and Yoshida (2011) and Potiron and Mykland (2017), where the latter work considers general endogenous arrival times. See also the remarkable work from Bibinger and Vetter (2015) and Martin and Vetter (2019) in a jumpy setting, and Koike (2014a), Koike (2014b) and Koike (2016) which incorporates jumps, noise and some kind of endogeneity into the model.

As we want to allow for quite exotic endogenous models, we follow Potiron and Mykland (2017). In particular, we assume no jumps in the setup. We describe the hitting boundary with time process (HBT) model introduced in the subsequent paper in what follows. In that model, eight stochastic processes (four of which are actually families of stochastic processes) are of interest, four for each asset. For the index k=1,2, we have the price process $-X_t^{(k)}$ - and three other stochastic processes (two of which are actually families of processes) $-Y_t^{(k)}$, $d_t^{(k)}(s)$ and $u_t^{(k)}(s)$ - related to the observation times of that process. Those four stochastic processes can be correlated, and we further assume that (X_t, Y_t) is a 4-dimensional Itô-process. For the process k=1,2, $Y_t^{(k)}$ stands for the continuous observation time process which drives the observation times related to $X_t^{(k)}$. The others four processes are the down processes $d_t^{(k)}(s)$ and the up processes $u_t^{(k)}(s)$. We assume that the down process takes only negative values and that the up process takes only positive values. A new observation time will be generated whenever one of those two processes is hit by the increment of the observation time process. Then, the increment of the observation time process is hit again. Formally, if we let $\alpha > 0$ stand for the tick size, we define the first observation time as $t_0^{(k)} := 0$ and recursively $t_i^{(k)}$ as

$$t_{i}^{(k)} := \inf \left\{ t > t_{i-1}^{(k)} : \Delta Y_{[t_{i-1}^{(k)}, t]}^{(k)} \notin \left[\alpha d_{t}^{(k)} \left(t - t_{i-1}^{(k)} \right), \alpha u_{t}^{(k)} \left(t - t_{i-1}^{(k)} \right) \right] \right\}, \tag{34}$$

where $\Delta Y_{[a,b]}^{(k)} := Y_b^{(k)} - Y_a^{(k)}.$ We define the Hayashi-Yoshida estimator as

$$\widetilde{\Xi} := \sum_{0 < t_i^{(1)}, \ t_j^{(2)} < T} \Delta_i X^{(1)} \Delta_j X^{(2)} \mathbf{1}_{\left\{ [t_{i-1}^{(1)}, t_i^{(1)}) \cap [t_{j-1}^{(2)}, t_j^{(2)}) \neq \emptyset \right\}}.$$
(35)

In the asymptotic theory, we let $\alpha \to 0$. For the sake of Remark 5 (p. 25) in Potiron and Mykland (2017), α^{-1} is of the same order as $n^{1/2}$. We can now show that the techniques of this paper hold in this case too.

Theorem 9. As the tick size $\alpha \to 0$, we have that

$$\alpha^{-1}(\widehat{\Xi} - \widetilde{\Xi}) \to^{\mathbb{P}} 0. \tag{36}$$

In particular, under the assumptions of Potiron and Mykland (2017), there exist AB and a process AV_t such that stably in law as the tick size $\alpha \to 0$,

$$\alpha^{-1}(\widehat{\Xi} - \Xi) \to AB + \int_0^T (AV_s)^{1/2} dB_s,$$
 (37)

where B_t is a Brownian motion independent of the other quantities, AB and AV_t are defined in Section 4.3 of Potiron and Mykland (2017).

We define \widetilde{AB} and AVAR following respectively (46) and (47) in Potiron and Mykland (2017) (Section 5, p. 28). Note that \widetilde{AB} and \widetilde{AVAR} are already of the right asymptotic order in the sense that $\alpha^{-1}\widetilde{AB} \to^{\mathbb{P}} AB$ and $\alpha^{-2}\widetilde{AVAR} \to^{\mathbb{P}} \int_0^T AV_s ds$ (see (48) and (49) in Corollary 4 of the cited paper). We provide now the studentized version of (37).

Corollary 10. We have

$$\frac{\widehat{\Xi} - \widehat{AB} - \Xi}{\sqrt{\widehat{AVAR}}} \to \mathcal{N}(0, 1). \tag{38}$$

4.4 Functionals of volatility local estimator

The spot parameter is $\xi_t = g(c_t)$ for a given smooth function g on \mathcal{M}_d^+ , the set of all non-negative symmetric $d \times d$ matrices. The problem was initiated by Barndorff-Nielsen and Shephard (2002a). See also Barndorff-Nielsen et al (2006a), Mykland and Zhang (2012) (Proposition 2.17, p. 138) and Renault et al (2017) for related developments. Here, the rate of convergence is $\kappa = 1/2$ again.

Local estimation (Mykland and Zhang (2009), Section 4.1, p. 1421-1426) can make the mentioned estimators efficient. Jacod and Rosenbaum (2013) extended the method in several ways. To do that, they first propose an estimator of the spot volatility \tilde{c}_i , and then take a Riemann sum of $g(\tilde{c}_i)$.

For any matrix $a \in \mathcal{M}_d^+$, the related a^{ij} stands for the (i,j)-component of a. Moreover, for $b \in \mathbb{R}$, [b] stands for the floor of b. Several results are of interest in Jacod and Rosenbaum (2013). In its most useful form (from our point of view), the estimator takes on the form

$$\widetilde{\Xi} = \Delta \sum_{i=1}^{[T/\Delta]-k+1} \left\{ g(\widetilde{c}_i) - \frac{1}{2k} \sum_{j,q,l,m=1}^{d} \partial_{jq,lm}^2 g(\widetilde{c}_i) \left(\widetilde{c}_i^{jl} \widetilde{c}_i^{qm} + \widetilde{c}_i^{jm} \widetilde{c}_i^{ql} \right) \right\}, \tag{39}$$

with

$$\widetilde{c}_i^{lm} = \frac{1}{k\Delta} \sum_{j=0}^{k-1} \Delta_{i+j} X^l \Delta_{i+j} X^m \mathbf{1}_{\{\|\Delta_{i+j} X\| \le w\}},$$

for two sequences of integers k and $w = \alpha \Delta^{\bar{\omega}}$ for some $\alpha > 0$, and

$$\frac{2p-1}{2(2p-r)} \le \bar{\omega} < \frac{1}{2},$$

where we suppose that

$$\|\partial^j g(x)\| \le K(1 + \|x\|^{p-j}), \ j = 0, 1, 2, 3$$
 (40)

for some constants $p \geq 3$, K > 0. In Equation (39), \tilde{c}_i corresponds to an estimator of the spot volatility matrix, the first term is part of the Riemann sum, while the second term is required to remove the asymptotic bias of the first term in $\tilde{\Xi}$, which explodes asymptotically. We show that the associated plug-in estimator $\hat{\Xi}$ enjoys the same limit theory as $\tilde{\Xi}$. More precisely, we have the following result.

Theorem 11. Assume that $k^2\Delta \to 0$, $k^3\Delta \to \infty$. Let $\widetilde{\Xi}'$ be the estimator defined as in (39) where X_t is replaced by its continuous part X_t' . Then, we have the convergence

$$n^{1/2} \left(\widehat{\Xi} - \widetilde{\Xi}' \right) \to^{\mathbb{P}} 0.$$
 (41)

Moreover, stably in law, we have the convergence

$$n^{1/2}\left(\widehat{\Xi} - \Xi\right) \to \int_0^T \sqrt{T\overline{h}(c_s)} dB_s,$$
 (42)

where for $x \in \mathcal{M}_d^+$,

$$\overline{h}(x) = \sum_{j,q,l,m=1}^{d} \partial_{jq} g(x) \partial_{lm} g(x) (x^{jl} x^{qm} + x^{jm} x^{ql}),$$

and where B is a standard Brownian motion independent of the other quantities.

In particular, note that the asymptotic variance in the stable convergence can be expressed as

$$AVAR = T \int_0^T \overline{h}(c_s) ds,$$

so that we naturally define the asymptotic variance estimator as

$$\widehat{AVAR} = T\Delta \sum_{i=1}^{[t/\Delta]-k+1} \overline{h}(\widehat{c}_i).$$

We easily deduce from Corollary 3.7 p. 1471 in Jacod and Rosenbaum (2013) the following studentized version of the above central limit theorem.

Corollary 12. Under the assumptions of the previous theorem, we have the stable convergence in law

$$\frac{n^{1/2}(\widehat{\Xi} - \Xi)}{\sqrt{\widehat{AVAB}}} \to \mathcal{N}(0, 1).$$

Remark 13. (estimation of functionals of volatility under i.i.d noise) Under i.i.d noise, no result with a general function $g(c_t)$ is available. Alternative approaches include: Jacod et al (2010) for even power, Mancino and Sanfelici (2012) and also Andersen et al (2014) in the special case of quarticity, and also Altmeyer and Bibinger (2015) when considering the tricity. See also the work from Potiron and Mykland (2016) (Section 4.2) for a local maximum-likelihood estimation with noise variance vanishing asymptotically.

4.5 Volatility of volatility

In this section we assume that X_t is 1-dimensional and we are interested in the spot parameter $\xi_t = \tilde{\sigma}_t^2$ which corresponds to the so-called volatility of volatility process defined in (14). As far as we know, there is no result in the literature including noise into the model, but in the non-noisy scenario one can consult Vetter (2015) (Theorem 2.5 and Theorem 2.6) and Mykland et al (2012) (Theorem 7 and Corollary 2). We follow here the former author, and aim to show the robustness of Theorem 2.6 when using plug-in estimators. Accordingly, we hereafter assume that both X_t and c_t are continuous processes, i.e. $\delta = \tilde{\delta} = 0$ in (13)-(14). To our knowledge, the case with jumps in X_t and/or c_t remains an open question. The rate of convergence is $\kappa = 1/4$. Introducing the spot volatility estimator⁷ for $i \in \{0, \dots, n-k\}$,

$$\widetilde{c}_i := \frac{n}{k} \sum_{j=1}^k (\Delta_{i+j} X)^2,$$

and the spot quarticity estimator

$$\widetilde{q}_i := \frac{n^2}{3k} \sum_{i=1}^k (\Delta_{i+j} X)^4,$$

⁷Note that the definition of \tilde{c}_i slightly diverges from the previous section.

the author defines the volatility of volatility estimator (see (2.5) on p. 2399 in the cited work) as

$$\widetilde{\Xi} := \sum_{i=0}^{[t/\Delta]-2k} \left\{ \frac{3}{2k} \left(\widetilde{c}_{i+k} - \widetilde{c}_i \right)^2 - \frac{6}{k^2} \widetilde{q}_i \right\}.$$

Letting \hat{c}_i , \hat{q}_i , and $\hat{\Xi}$ be the corresponding plug-in estimators, we obtain the following results.

Theorem 14. Assume that $k = cn^{1/2} + o(n^{1/4})$ for some c > 0. Then stably in law,

$$\sqrt{\frac{n}{k}} \left(\widehat{\Xi} - \Xi \right) \to \sqrt{T} \int_0^T \alpha_s dB_s,$$

where B_t is a Brownian motion independent from the other quantities and

$$\alpha_s^2 = \frac{48}{c^4} \sigma_s^8 + \frac{12}{c^2} \sigma_s^4 \widetilde{\sigma}_s^2 + \frac{151}{70} \widetilde{\sigma}_s^4.$$

Moreover, if we define

$$G^{(1)} = \frac{T}{n} \sum_{i=0}^{[t/\Delta]-k} \widehat{q}_i^2,$$

$$G^{(2)} = T \sum_{i=0}^{[t/\Delta]-2k} \left\{ \frac{3}{2k} (\widehat{c}_{i+k} - \widehat{c}_i)^2 - \frac{6}{k^2} \widehat{q}_i \right\} \widehat{q}_i,$$

$$G^{(3)} = \frac{Tn}{k^2} \sum_{i=0}^{[t/\Delta]-2k} (\widehat{c}_{i+k} - \widehat{c}_i)^4,$$

and finally

$$\widehat{AVAR} = \frac{453}{280}G^{(3)} - \frac{n}{k^2} \frac{486}{35}G^{(2)} - \frac{n^2}{k^4} \frac{1038}{35}G^{(1)},$$

we can derive the following studentized version of the previous central limit theorem.

Corollary 15. Under the assumptions of the previous theorem, we have the stable convergence in law, when k has the optimal rate $c\sqrt{n}$ for c>0

$$n^{1/4} \frac{\widehat{\Xi} - \Xi}{\sqrt{c\widehat{AVAR}}} \to \mathcal{N}(0,1).$$

5 Conclusion

This paper develops plug-in estimators to estimate high-frequency quantities under parametric noise on five different examples. We do not find any particular difficulty when working out the theory of those examples. Another example of application can be found in Andersen et al (2019).

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6 Proofs

6.1 Preliminaries

Due to our assumptions of local boundedness on b_t , b_t , c_t and c_t , (12) and (18), it is sufficient (see, e.g., Lemma 4.4.9 along with Proposition 2.2.1 in Jacod and Protter (2011)) to assume throughout the proofs the following stronger assumption.

(**H**) We have that b_t , \widetilde{b}_t , c_t and \widetilde{c}_t are bounded. Moreover, there exists K > 0 such that $\|\widehat{\theta} - \theta_0\| \le K/n$, and $\max_{i,j,k} \left| Q_{t^{(k)}}^{(k,j)} \right| \le K$.

Since the last two properties on $\widehat{\theta}$ and Q are not directly implied by Proposition 2.2.1 from Jacod and Protter (2011), we now detail a general localization procedure in the next proposition, which we apply to the above particular cases in Corollary 17. In the next lemma, if A is a random event, \overline{A} stands for $\Omega - A$.

Proposition 16. (Localization) Let $(A_n^K)_{n\in\mathbb{N},K\in\mathbb{R}_+}$ be a doubly-indexed family of events such that $\lim_{K\to+\infty}\sup_{n\in\mathbb{N}}\mathbb{P}[\overline{A_n^K}]=0$. Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of \mathbb{R}^d -valued random variables for some $d\geq 1$, and X another \mathbb{R}^d -valued random variable, and assume that either of the following properties hold.

- 1. (Local convergence in probability) For any $K \geq 0$, $(X_n X)\mathbb{1}_{A_n^K} \to^{\mathbb{P}} 0$.
- 2. (Local convergence in distribution) For any $K \geq 0$, for any f continuous and bounded, $\mathbb{E}[f(X_n)\mathbb{1}_{A_n^K}] \to \mathbb{E}[f(X)]$.

Then we have respectively

- 1. $X_n \to^{\mathbb{P}} X$.
- 2. $X_n \to^d X$.

Proof. We prove the convergence in probability first. Fix $\epsilon > 0$ and $\eta > 0$, and note that

$$\begin{split} \mathbb{P}[|X_n - X| \geq \eta] \leq \mathbb{P}\left[|X_n - X| \mathbb{1}_{A_n^K} \geq \frac{\eta}{2}\right] + \mathbb{P}\left[|X_n - X| \mathbb{1}_{\overline{A_n^K}} \geq \frac{\eta}{2}\right] \\ \leq \mathbb{P}\left[|X_n - X| \mathbb{1}_{A_n^K} \geq \frac{\eta}{2}\right] + \mathbb{P}\left[\overline{A_n^K}\right]. \end{split}$$

By taking K large enough, we can assume that the second term in the right-hand side is dominated by ϵ . Next, by taking n large enough, we may assume the first term to be smaller than ϵ as well by the local convergence in probability. This proves $X_n \to^{\mathbb{P}} X$. Next we prove the convergence in distribution. We have

$$\begin{split} |\mathbb{E}[f(X_n)] - \mathbb{E}[f(X)]| &= |\mathbb{E}[f(X_n)\mathbb{1}_{A_n^K}] - \mathbb{E}[f(X)] + \mathbb{E}[f(X_n)\mathbb{1}_{\overline{A_n^K}}]| \\ &\leq |\mathbb{E}[f(X_n)\mathbb{1}_{A_n^K}] - \mathbb{E}[f(X)]| + C\mathbb{P}\left[\overline{A_n^K}\right] \end{split}$$

for some constant C using the boundedness of f. Again, taking K large enough makes the third term arbitrary small, and then taking $n \to +\infty$ makes the difference between the first two terms tend to 0, wich proves $X_n \to^d X$.

Corollary 17. When proving the consistency of the estimator $\widehat{\Xi}$ toward Ξ or the asymptotic normality $n^{\kappa}(\widehat{\Xi} - \Xi) \to \mathcal{MN}(AB, AVAR)$, we may assume that there exists K > 0 (which may be arbitrary large) such that $\|\widehat{\theta} - \theta_0\| \le K/n$, and $\max_{i,j,k} \left| Q_{t_i^{(k,j)}}^{(k,j)} \right| \le K$.

Proof. We show the case $\|\widehat{\theta} - \theta_0\| \leq K/n$, the case $\max_{i,j,k} |Q_{t_i^{(k)}}^{(k,j)}| \leq K$ being the same. For the consistency, we apply the previous proposition with $X_n = \widehat{\Xi}$, $X = \Xi$, and $A_n^K = \left\{\|\widehat{\theta} - \theta_0\| \leq K/n\right\}$. By hypothesis (6), $n(\widehat{\theta} - \theta_0)$ is stochastically bounded which exactly means that $\lim_{K \to +\infty} \sup_{n \in \mathbb{N}} \mathbb{P}[n\|\widehat{\theta} - \theta_0\| \geq K] = 0$ (recall that $\widehat{\theta}$ depends on n). For the central limit theory, apply the local convergence distribution with $X_n = n^K(\widehat{\Xi} - \Xi)$, $X \sim \mathcal{MN}(AB, AVAR)$, and again $A_n^K = \left\{\|\widehat{\theta} - \theta_0\| \leq K/n\right\}$. \square

All along the proofs, C is a constant that may vary from one line to the next. We further provide some notation related to the decomposition (13) of the efficient price, i.e. that

$$X_{t} = X_{0} + \int_{0}^{t} b_{s} ds + \int_{0}^{t} \sigma_{s} dW_{s} + \int_{0}^{t} \int_{\mathbb{R}} \delta(s, z) \mathbf{1}_{\{||\delta(s, z)|| \leq 1\}} (\mu - \nu)(ds, dz)$$

$$+ \int_{0}^{t} \int_{\mathbb{R}} \delta(s, z) \mathbf{1}_{\{||\delta(s, z)|| > 1\}} \mu(ds, dz),$$

$$:= X_{0} + B_{t} + M_{t}^{c} + M_{t}^{d} + J_{t}^{b}.$$

$$(43)$$

Note that in this decomposition M_t^c (resp. M_t^d) is a continuous (resp. purely discontinuous) local martingale (see the discussion in Section 2.1.2 in Jacod and Protter (2011)). Finally, we introduce $\Delta_i X(\theta) := \Delta_i X + \psi_i(\theta)$ where $\psi_i(\theta) := \mu_i(\theta_0) - \mu_i(\theta)$. In particular, note that $\Delta_i \hat{X} = \Delta_i X(\hat{\theta})$. Similarly we define $\Delta_i X'(\theta) := \Delta_i X' + \psi_i(\theta)$ and $\Delta_i \hat{X}' = \Delta_i X'(\hat{\theta})$, corresponding to the estimated increments when the jump part J has been removed. Moreover, \mathbb{E}_s is defined as the conditional expectation given \mathcal{F}_s .

6.2 Proof of Theorem 2

For this proof, due to our assumptions in Theorem 2 and using the same argument as for Assumption (H) we further make the following assumption.

(H') We have that $n\Delta_i t$ and \tilde{v}_t are bounded and bounded away from 0.

Note that (25) is a particular case of (26) when $\phi = 0$. In what follows, we directly prove the general case (26). First of all, as $N/n \to^{\mathbb{P}} F$, it is sufficient to show the stable convergence in law

$$n^{1/2}(\widetilde{\Xi} - \Xi) \to \frac{2}{3} \int_0^T \widetilde{v}_s \sigma_s dX_s' + \int_0^T \sqrt{\frac{2}{3}} \widetilde{u}_s - \frac{4}{9} \widetilde{v}_s^2 \sigma_s^2 dB_s. \tag{44}$$

Second, note that if we can prove that

$$n^{1/2} \sum_{i=1}^{N} \left(\Delta_{i} \widehat{X} \right)^{2} \mathbf{1}_{\{|\Delta_{i} \widehat{X}| \le w_{i}\}} = n^{1/2} \sum_{i=1}^{N} \left(\Delta_{i} X' \right)^{2} + o_{\mathbb{P}}(1), \tag{45}$$

then (25) holds in view of Theorem 1 (p. 585) in Li et al (2014) together with the assumptions of Theorem 2. Accordingly, we show (45) in what follows. On the account of the decomposition (15), we have

$$n^{1/2} \sum_{i=1}^{N} \left(\Delta_{i} \widehat{X} \right)^{2} \mathbf{1}_{\{|\Delta_{i} \widehat{X}| \leq w_{i}\}} = n^{1/2} \sum_{i=1}^{N} \left(\Delta_{i} \widehat{X}' \right)^{2} \mathbf{1}_{\{|\Delta_{i} \widehat{X}| \leq w_{i}\}} + 2n^{1/2} \sum_{i=1}^{N} \Delta_{i} \widehat{X}' \Delta_{i} J \mathbf{1}_{\{|\Delta_{i} \widehat{X}| \leq w_{i}\}},$$

$$+ n^{1/2} \sum_{i=1}^{N} \Delta_{i} J^{2} \mathbf{1}_{\{|\Delta_{i} \widehat{X}| \leq w_{i}\}},$$

$$:= I + II + III.$$

We will show in what follows that $I = n^{1/2} \sum_{i=1}^{N} (\Delta_i X')^2 + o_{\mathbb{P}}(1)$, $II = o_{\mathbb{P}}(1)$, and $III = o_{\mathbb{P}}(1)$. We start with I. By definition, we have

$$I = n^{1/2} \sum_{i=1}^{N} \left(\Delta_i \widehat{X}' \right)^2 - n^{1/2} \sum_{i=1}^{N} \left(\Delta_i \widehat{X}' \right)^2 \mathbf{1}_{\{|\Delta_i \widehat{X}| > w_i\}}.$$

We show now that $n^{1/2} \sum_{i=1}^N \left(\Delta_i \widehat{X}'\right)^2 \mathbf{1}_{\{|\Delta_i \widehat{X}| > w_i\}} = o_{\mathbb{P}}(1)$. We have that

$$n^{1/2} \sum_{i=1}^{N} \left(\Delta_{i} \widehat{X}' \right)^{2} \mathbf{1}_{\{|\Delta_{i} \widehat{X}| > w_{i}\}} \leq n^{1/2} \sum_{i=1}^{N} \left(\Delta_{i} \widehat{X}' \right)^{2} \mathbf{1}_{\{|\Delta_{i} \widehat{X}'| > w_{i}/2\}} + n^{1/2} \sum_{i=1}^{N} \left(\Delta_{i} \widehat{X}' \right)^{2} \mathbf{1}_{\{|\Delta_{i} J| > w_{i}/2\}}$$

$$:= A + B.$$

We first deal with A. By the domination $\mathbf{1}_{\{|\Delta_i \widehat{X}'| > w_i/2\}} \leq 2^k |\Delta_i \widehat{X}'|^k w_i^{-k}$, we have for any k > 0:

$$|A| \le Cn^{1/2} \sum_{i=1}^{N} w_i^{-k} |\Delta_i \widehat{X}'|^{2+k}. \tag{46}$$

Now, note that by Assumption (**H**) along with the fact that ψ_i is C^3 in θ and that Θ is a compact set, we easily obtain that for any $k \geq 1$, $|\psi_i(\widehat{\theta})|^k \leq Cn^{-k}$. From here, by Assumption (**H**') we deduce by Burkhölder-Davis-Gundy inequality that

$$\mathbb{E}|\Delta_i \hat{X}'|^k \le C(n^{-k/2} + n^{-k}) \le Cn^{-k/2},\tag{47}$$

and so we can conclude that taking k large enough, $A = o_{\mathbb{P}}(1)$ as a result of the boundedness of $n\Delta_i t$, and $N/n \to F$.

Now, we deal with B. Remark that by (H') and Hölder's inequality we have

$$|B| \leq 2n^{1/2} \sum_{i=1}^{N} \left(\Delta_{i} \widehat{X}'\right)^{2} |\Delta_{i} J| |w_{i}|^{-1}$$

$$\leq Cn^{1/2+\bar{\omega}} \sum_{i=1}^{N} \left(\Delta_{i} \widehat{X}'\right)^{2} |\Delta_{i} J|$$

$$\leq Cn^{1/2+\bar{\omega}} \left(\sum_{i=1}^{N} \left(\Delta_{i} \widehat{X}'\right)^{2p}\right)^{1/p} \left(\sum_{i=1}^{N} |\Delta_{i} J|^{q}\right)^{1/q}$$

where 1/p + 1/q = 1 and p, q > 1. By (46) we get $\left(\sum_{i=1}^N \left(\Delta_i \widehat{X}'\right)^{2p}\right)^{1/p} = O_{\mathbb{P}}(n^{1/p-1})$ and since q > 1, we also have $\sum_{i=1}^N |\Delta_i J|^q = O_{\mathbb{P}}(1)$ because the jumps are summable. Indeed, note first that by application of Theorem 3.3.1, Case A, p.70 from Jacod and Protter (2011) under assumption (A-c), with $f(x) = |x|^q = o(x)$ for $x \to 0$ since q > 1, we have the convergence $\sum_{i=1}^N |\Delta_i J|^q \to^{\mathbb{P}} \sum_{0 < s \le T} |\Delta_s J|^q$. The stochastic boundedness of the left-hand side will therefore be proved if we show that the limit is finite almost surely. We can write

$$\sum_{0 < s \le T} |\Delta_s J|^q = \sum_{0 < s \le T} |\Delta_s J|^q \mathbf{1}_{\{|\Delta_s J| \ge 1\}} + \sum_{0 < s \le T} |\Delta_s J|^q \mathbf{1}_{\{|\Delta_s J| < 1\}}.$$

The first term of the right-hand side is clearly finite since there is only a finite number of jumps larger than 1 on the interval [0,T]. Moreover, for the second term, using that $|x|^q < |x|$ for $x \in [0,1)$ when q > 1, and using that the jumps are summable yields

$$\sum_{0 < s \le T} |\Delta_s J|^q \mathbf{1}_{\{|\Delta_s J| < 1\}} \le \sum_{0 < s \le T} |\Delta_s J| < +\infty \text{ a.s.}$$

Overall this yields $B = O_{\mathbb{P}}(n^{1/p+\bar{\omega}-1/2})$, which tends to 0 as soon as p is taken larger than $(1/2-\bar{\omega})^{-1}$, which is possible since $\bar{\omega} < 1/2$. Now we conclude for I by showing that we have

$$n^{1/2} \sum_{i=1}^{N} \left(\Delta_i \widehat{X}' \right)^2 = n^{1/2} \sum_{i=1}^{N} \left(\Delta_i X' \right)^2 + o_{\mathbb{P}}(1). \tag{48}$$

Note that

$$n^{1/2} \sum_{i=1}^{N} \left(\left(\Delta_i \widehat{X}' \right)^2 - \left(\Delta_i X' \right)^2 \right) = 2n^{1/2} \sum_{i=1}^{N} \Delta_i X' \psi_i(\widehat{\theta}) + n^{1/2} \sum_{i=1}^{N} \psi_i(\widehat{\theta})^2,$$

and the second term in the right-hand side of the equation is negligible as a direct consequence of the domination $|\psi_i(\widehat{\theta})| \leq C/n$. We show now that the first term is also negligible. By the mean value theorem, we also have for some $\overline{\theta} \in [\theta_0, \widehat{\theta}]$ that

$$n^{1/2} \sum_{i=1}^{N} \Delta_{i} X' \psi_{i}(\widehat{\theta}) = n^{1/2} (\widehat{\theta} - \theta_{0})^{T} \sum_{i=1}^{N} \Delta_{i} X' \partial_{\theta} \psi_{i}(\theta_{0}) + \frac{n^{1/2} (\widehat{\theta} - \theta_{0})^{T}}{2} \sum_{i=1}^{N} \Delta_{i} X' \partial_{\theta}^{2} \psi_{i}(\overline{\theta}) (\widehat{\theta} - \theta_{0}) (49)$$

Using that $\hat{\theta} - \theta_0 = O_{\mathbb{P}}(1/n)$, and the fact that $\|\partial_{\theta}^2 \psi(\overline{\theta})\| \leq C$ we deduce that the second term is negligible. Finally, note that $\sum_{i=1}^N \Delta_i X' \partial_{\theta} \psi_i(\theta_0)$ can be decomposed as the sum of $\sum_{i=1}^N \Delta_i \check{B} \partial_{\theta} \psi_i(\theta_0)$, where $\check{B}_t = \int_0^t b_s' ds$, and which is easily proved to be negligible given the local boundedness of b and δ , and $\sum_{i=1}^N \Delta_i M^c \partial_{\theta} \psi_i(\theta_0)$, which is a sum of martingale increments with respect to the filtration $\mathcal{H}_t = \mathcal{F}_t \vee \sigma\{Q_{t_i}, 1 \leq i \leq N\}$. Thus, by (2.2.35) in Jacod and Protter (2011), proving that this term tends to 0 boils down to showing that

$$n^{-1} \sum_{i=1}^{N} \mathbb{E}\left[(\Delta_i M^c)^2 \|\partial_{\theta} \psi_i(\theta_0)\|^2 \right] \to 0,$$

which is immediate since $\|\partial_{\theta}\psi_i(\theta_0)^2\| \leq C$, $N/n \to^{\mathbb{P}} F$ and $\mathbb{E}(\Delta_i M^c)^2 \leq C/n$ by Assumption (H').

We now turn to II. As by (47) along with Assumption (H'), we have for any k>0 the inequality $\mathbb{P}\left[|\Delta_i\widehat{X}'|>w_i/2\right]\leq Cn^{k(\bar{\omega}-1/2)}$, we can assume without loss of generality, by taking k sufficiently large, that we can add the indicator $\mathbf{1}_{\{|\Delta_i\widehat{X}'|\leq w_i/2\}}$ in II, i.e. that

$$II = 2n^{1/2} \sum_{i=1}^{N} \Delta_{i} \widehat{X}' \Delta_{i} J \mathbf{1}_{\{|\Delta_{i}\widehat{X}| \leq w_{i}\}} \mathbf{1}_{\{|\Delta_{i}\widehat{X}'| \leq w_{i}/2\}},$$

$$\leq 2n^{1/2} \sum_{i=1}^{N} \Delta_{i} \widehat{X}' \Delta_{i} J \mathbf{1}_{\{|\Delta_{i}J| \leq 3w_{i}/2\}} \mathbf{1}_{\{|\Delta_{i}\widehat{X}'| \leq w_{i}/2\}},$$

so that

$$|II| \leq 2n^{1/2} \sum_{i=1}^{N} |\Delta_i \widehat{X}'| |\Delta_i J|^{1-r} |\Delta_i J|^r \mathbf{1}_{\{|\Delta_i J| \leq 3w_i/2\}} \mathbf{1}_{\{|\Delta_i \widehat{X}'| \leq w_i/2\}},$$

$$\leq Cn^{1/2 - \bar{\omega}(2-r)} \underbrace{\sum_{i=1}^{N} |\Delta_i J|^r}_{O_{\mathbb{P}}(1)},$$

where we recall that r > 0 is the jump index of J. Given that $\bar{\omega} \in (1/(2(2-r)), 1/2)$, we immediately deduce that $II = o_{\mathbb{P}}(1)$. Finally, we can show that $III = o_{\mathbb{P}}(1)$ with the same line of reasoning as for II.

6.3 Proof of Corollary 4

We show (30), as (29) is a particular case where $\phi = 0$. This amounts to proving that \widehat{AB} and \widehat{AVAR} are consistent.

We show first that \widehat{AB} is consistent. As in the previous proofs (in this case this is actually quite easier as we only show the consistency), we can remove the truncation and the parametric noise part and replace $\Delta_i \widehat{X}$ by $\Delta_i X'$. We obtain that

$$\widehat{AB} = \sum_{i=1}^{B} \frac{2}{3} \overline{v} \overline{\sigma}_i (X'_{t_{ih}} - X'_{t_{(i-1)h}}) + o_{\mathbb{P}}(1),$$

where

$$\overline{v}\overline{\sigma}_i = \frac{N^{1/2} \sum_{j=h(i-1)+1}^{hi} (\Delta_j X')^3}{\sum_{j=h(i-1)+1}^{hi} (\Delta_j X')^2}.$$

A Taylor expansion on the function f(x,y) = x/y along with a local version of the convergence (24), the fact that $\sum_{i=1}^{N} \left(\Delta_i X'\right)^2 \to^{\mathbb{P}} \Xi$, that σ_t and v_t are bounded and bounded away from 0 and that $N/n \to^{\mathbb{P}} F$ yields

$$\widehat{AB} = \sum_{i=1}^{B} \frac{2}{3} v_{t_{i-1}} \sigma_{t_{i-1}} (X'_{t_{ih}} - X'_{t_{(i-1)h}}) + o_{\mathbb{P}}(1).$$

Applying Theorem I.4.31 (iii) on p. 47 in Jacod and Shiryaev (2003) together with the fact that σ_t and v_t are bounded and bounded away from 0, we conclude that $\widehat{AB} \to^{\mathbb{P}} AB$.

We show now that \widehat{AVAR} is consistent. In this case we can again by similar arguments remove the truncation and substitute $\Delta_i \widehat{X}$ by $\Delta_i X'$, i.e. it holds that

$$\widehat{AVAR} = \frac{2N}{3} \sum_{i=1}^{N} (\Delta_i X')^4 - \frac{4}{9} \sum_{i=1}^{B} (\overline{v}\overline{\sigma}_i)^2 (X'_{t_{ih}} - X'_{t_{(i-1)h}})^2 + o_{\mathbb{P}}(1).$$

By (23) together with the fact that $N/n \to^{\mathbb{P}} F$, we deduce that

$$\frac{2N}{3} \sum_{i=1}^{N} (\Delta_i X')^4 \to^{\mathbb{P}} \frac{2}{3} \int_0^T u_s \sigma_s^4 ds.$$

Furthermore, using similar techniques as for AB, we obtain that

$$\frac{4}{9} \sum_{i=1}^{B} (\overline{v}\sigma_i)^2 (X'_{t_{ih}} - X'_{t_{(i-1)h}})^2 \to^{\mathbb{P}} \frac{4}{9} \int_0^T v_s^2 \sigma_s^4 ds.$$

We have thus shown that $\widehat{AVAR} \to^{\mathbb{P}} AVAR$.

6.4 Proof of Theorem 7

It is immediate to see that (32) holds as a consequence of (31) along with Theorem 3.3 in Vetter (2010). Accordingly, we show that (31) holds in what follows, i.e. that

$$n^{1/2}\widehat{\Xi} = n^{1/2}\widetilde{\Xi} + o_{\mathbb{P}}(1).$$

First, we show that we can assume without loss of generality that the price process X is continuous, i.e. J=0. To do so, we introduce $\widehat{\Xi}'$ as the estimator applied to X' in lieu of X. We show that

$$n^{1/2}\left(\widehat{\Xi} - \widehat{\Xi}'\right) \to^{\mathbb{P}} 0.$$
 (50)

From (15), we can easily obtain the key decomposition

$$\Delta_{i}\widehat{X} = \Delta_{i}X(\widehat{\theta}) = \underbrace{\Delta_{i}\widecheck{B} + \psi_{i}(\widehat{\theta})}_{\Delta_{i}B'} + \Delta_{i}M^{c} + \Delta_{i}J, \tag{51}$$

and by assumption (**H**), also recall that we have $|\psi_i(\widehat{\theta})| \leq |\sup_{\theta \in \Theta} \partial_{\theta} \psi_i(\theta)| |\widehat{\theta} - \theta_0| \leq C/n$. Thus, remark that all usual conditional moment estimates for $\Delta_i \check{B}$ are also true for $\Delta_i B'$. More precisely, replacing $\Delta_i \check{B}$ by $\Delta_i B'$ and \mathcal{F}_i by $\mathcal{G}_i = \mathcal{F}_i \vee \sigma \{Q_{t_i}, 0 \leq i \leq n\}$ in the proof of Lemma 13.2.6 (p. 384) in Jacod and Protter (2011), all the conditional estimates are preserved and thus the lemma holds true in the presence of the error term $\psi_i(\widehat{\theta})$. Indeed, the three key ingredients for the original proof of Lemma 13.2.6 are the following (with our own notations): defining

$$U_i = \frac{|\Delta_i X'|}{\Delta_n^{1/2}}, V_i = \left(\frac{\int_{t_{i-1}}^{t_i} \int_{\mathbb{R}} \gamma(z)^{1/r} \mu(ds, dz)}{\Delta_n^{\bar{\omega}}}\right) \wedge 1 \text{ and } W_i = \left(\frac{\int_{t_{i-1}}^{t_i} \int_{\mathbb{R}} \gamma(z)^{1/r} \mu(ds, dz)}{\Delta_n^{1/2}}\right) \wedge 1,$$

we have (see (13.2.22)-(13.2.23) in Jacod and Protter (2011), pp.384-385), for any m > 0,

$$\mathbb{E}[(U_i)^m | \mathcal{G}_{i-1}] \le C_m, \tag{52}$$

$$\mathbb{E}[(V_i)^m | \mathcal{G}_{i-1}] \le \Delta_n^{(1-r\bar{\omega})(1\wedge(m/r))} \phi_n, \tag{53}$$

$$\mathbb{E}[(W_i)^m | \mathcal{G}_{i-1}] \le \Delta_n^{(1-r/2)(1\wedge(m/r))} \phi_n, \tag{54}$$

where $C_m > 0$ is a constant possibly depending on m, and ϕ_n is a suitable deterministic sequence tending to 0 as $n \to +\infty$. Note that in the presence of the term $\psi_i(\widehat{\theta})$, that is, if V_i and W_i are unchanged but U_i is changed to $\widehat{U}_i = \frac{|\Delta_i \widehat{X}'|}{\Delta_n^{1/2}}$, the conditional deviations (52)-(54) remain unchanged since

$$\left| \mathbb{E}[(U_i)^m | \mathcal{G}_{i-1}] - \mathbb{E}[(\widehat{U}_i)^m | \mathcal{G}_{i-1}] \right| = O_{\mathbb{P}}(n^{-1}\Delta_n^{-1/2}) \to 0$$

using that $\psi_i(\widehat{\theta}) \leq C/n$. Therefore, Lemma 13.2.6 from Jacod and Protter (2011) still holds when X and X' are respectively replaced by \widehat{X} and \widehat{X}' . Applied with $F(x_1, x_2) = |x_1| |x_2|$, k = 2, p' = s' = 2, s = 1 and $\theta = 0$, this directly yields that for all $q \geq 1$ and for some deterministic sequence a_n going to 0,

$$\mathbb{E}\left||\Delta_{i}\widehat{X}||\Delta_{i-1}\widehat{X}|\mathbf{1}_{\left\{|\Delta_{i}\widehat{X}|\leq w\right\}}\mathbf{1}_{\left\{|\Delta_{i-1}\widehat{X}|\leq w\right\}}-|\Delta_{i}\widehat{X}'||\Delta_{i-1}\widehat{X}'|\mathbf{1}_{\left\{|\Delta_{i}\widehat{X}'|\leq w\right\}}\mathbf{1}_{\left\{|\Delta_{i-1}\widehat{X}'|\leq w\right\}}\right|^{q}\leq Ca_{n}\Delta_{n}^{(2q-r)\bar{\omega}+1},$$

where we have used that q/r > 1 and $\bar{\omega} < 1/2$, and where we recall that $\Delta_i \hat{X}' = \Delta_i X'(\hat{\theta})$. Given the definitions of $\hat{\Xi}$ and $\hat{\Xi}'$, applying the above domination with q = 1, we directly deduce the estimate

$$n^{1/2}\mathbb{E}|\widehat{\Xi} - \widehat{\Xi}'| \le a_n n^{1/2 - (2-r)\bar{\omega}} \to 0,$$

since $\bar{\omega} \in (1/(2(2-r)), 1/2)$. From now on, by (50), we are left to show $n^{1/2}(\widehat{\Xi}' - \widetilde{\Xi}) \to^{\mathbb{P}} 0$. By definition, we have that

$$n^{1/2}\widehat{\Xi}' = \frac{\pi n^{1/2}}{2} \sum_{i=2}^{n} |\Delta_{i}\widehat{X}'| \mathbf{1}_{\{|\Delta_{i}\widehat{X}'| \leq w\}} |\Delta_{i-1}\widehat{X}'| \mathbf{1}_{\{|\Delta_{i-1}\widehat{X}'| \leq w\}},$$

$$= \frac{\pi n^{1/2}}{2} \sum_{i=2}^{n} |(\Delta_{i}X' + \psi_{i}(\widehat{\theta}))(\Delta_{i-1}X' + \psi_{i-1}(\widehat{\theta}))| \mathbf{1}_{\{|\Delta_{i}\widehat{X}'| \leq w\}} \mathbf{1}_{\{|\Delta_{i-1}\widehat{X}'| \leq w\}}.$$

If we introduce $\breve{\Xi} = \frac{\pi}{2} \sum_{i=2}^{n} \left| \Delta_i X' \right| \mathbf{1}_{\{|\Delta_i \widehat{X}'| \leq w\}} \left| \Delta_{i-1} X' \right| \mathbf{1}_{\{|\Delta_{i-1} \widehat{X}'| \leq w\}}$, we have

$$n^{1/2} |\widehat{\Xi}' - \widecheck{\Xi}| = \frac{\pi n^{1/2}}{2} \sum_{i=2}^{n} |\Delta_{i} X'| \mathbf{1}_{\{|\Delta_{i} \widehat{X}'| \le w\}} \left(|\Delta_{i-1} \widehat{X}'| - |\Delta_{i-1} X'| \right) \mathbf{1}_{\{|\Delta_{i-1} \widehat{X}'| \le w\}}$$

$$+ \frac{\pi n^{1/2}}{2} \sum_{i=2}^{n} \left(|\Delta_{i} \widehat{X}'| - |\Delta_{i} X'| \right) \mathbf{1}_{\{|\Delta_{i} \widehat{X}'| \le w\}} |\Delta_{i-1} \widehat{X}'| \mathbf{1}_{\{|\Delta_{i-1} \widehat{X}'| \le w\}}$$

$$= I + II.$$

We prove (31) in two steps in what follows. First, we show that $n^{1/2} |\widetilde{\Xi} - \breve{\Xi}| = o_{\mathbb{P}}(1)$. Second, we prove that $I = o_{\mathbb{P}}(1)$ and $II = o_{\mathbb{P}}(1)$. We have

$$n^{1/2} |\widetilde{\Xi} - \breve{\Xi}| \leq \frac{\pi n^{1/2}}{2} \sum_{i=2}^{n} |\Delta_i X'| |\Delta_{i-1} X'| |\mathbf{1}_{\{|\Delta_i \widehat{X}'| \leq w\}} \mathbf{1}_{\{|\Delta_{i-1} \widehat{X}'| \leq w\}} - \mathbf{1}_{\{|\Delta_i X'| \leq w\}} \mathbf{1}_{\{|\Delta_{i-1} X'| \leq w\}}|,$$

so that by standard inequalities we can deduce $n^{1/2} |\widetilde{\Xi} - \breve{\Xi}| \to^{\mathbb{P}} 0$ if

$$\mathbb{E}\left|\mathbf{1}_{\{|\Delta_i \widehat{X}'| \le w\}} - \mathbf{1}_{\{|\Delta_i X'| \le w\}}\right| \le Cn^{-\beta} \tag{55}$$

for some $\beta > 0$ large enough (where C possibly depends on β). Let us thus show now (55). Introducing $\check{\Delta}$ as the symmetric difference operator, we have

$$\begin{split} \left|\mathbf{1}_{\{|\Delta_i\widehat{X}'|\leq w\}} - \mathbf{1}_{\{|\Delta_iX'|\leq w\}}\right| &= \mathbf{1}_{\{|\Delta_i\widehat{X}'|\leq w\}\widecheck{\Delta}\{|\Delta_iX'|\leq w\}} \\ &\leq \mathbf{1}_{\{|\Delta_iX'-w|\leq |\psi_i(\widehat{\theta})|\}} \\ &\leq \mathbf{1}_{\{|\Delta_iX'-w|\leq C/n\}}. \end{split}$$

Now, letting $\gamma \in (\bar{\omega}, 1/2)$ and q > 0, since $\{|\Delta_i X' - w| \leq C/n\} \cap \{|\Delta_i X'| \leq n^{-\gamma}\} = \emptyset$ for n large enough, we automatically have

$$\mathbf{1}_{\{|\Delta_i X' - w| \le C/n\}} \le \mathbf{1}_{\{|\Delta_i X'| > n^{-\gamma}\}} \le n^{\gamma q} |\Delta_i X'|^q,$$

hence

$$\mathbb{E}\left|\mathbf{1}_{\{|\Delta_i \widehat{X}'| \le w\}} - \mathbf{1}_{\{|\Delta_i X'| \le w\}}\right| \le n^{\gamma q} \mathbb{E}|\Delta_i X'|^q$$
$$\le C n^{q(\gamma - 1/2)},$$

and taking q large enough we get (55). Finally, we prove that $I = o_{\mathbb{P}}(1)$. The proof for II is similar. First note that since X' is continuous and $\psi_i(\widehat{\theta}) < K/n$, we can get rid of the indicator functions in I following the same line of reasoning as for (46). Moreover, following arguments similar to that of (55), I is asymptotically unaffected if $\mathbf{1}_{\{|\psi_{i-1}(\widehat{\theta})|<|\Delta_{i-1}X'|\}}$ is present in the sum. Without loss of generality, we can therefore assume that

$$I = \frac{\pi n^{1/2}}{2} \sum_{i=2}^{n} \left| \Delta_i X' \right| \mathbf{1}_{\{|\psi_{i-1}(\widehat{\theta})| < |\Delta_{i-1}X'|\}} \left(\left| \Delta_{i-1} \widehat{X}' \right| - \left| \Delta_{i-1}X' \right| \right) + o_{\mathbb{P}}(1).$$

Next, we decompose I as follows, using the identity for $|y| \le |x|$, $|x+y| - |x| = y \operatorname{sgn}(x)$ with sgn the usual sign function:

$$I = \frac{\pi n^{1/2}}{2} \sum_{i=2}^{n} \psi_{i-1}(\widehat{\theta}) |\Delta_i X'| \operatorname{sgn}(\Delta_{i-1} X') \mathbf{1}_{\{|\psi_{i-1}(\widehat{\theta})| \le |\Delta_i X'|\}} + o_{\mathbb{P}}(1).$$

Again, the indicator function can be removed since its complement event is negligible (it can be majorated by e.g $C|\Delta_i X'|^p/n^p$ for any p where C possibly depends on p), which yields the approximation

$$I = \frac{\pi n^{1/2}}{2} \sum_{i=2}^{n} \psi_{i-1}(\widehat{\theta}) |\Delta_i X'| \operatorname{sgn}(\Delta_{i-1} X') + o_{\mathbb{P}}(1)$$
$$= \frac{\pi n^{1/2} (\widehat{\theta} - \theta_0)^T}{2} \sum_{i=2}^{n} \partial_{\theta} \psi_{i-1}(\theta_0) |\Delta_i X'| \operatorname{sgn}(\Delta_{i-1} X') + o_{\mathbb{P}}(1)$$

where the second step is another application of the mean value theorem (as in the proof of Theorem 2). Now note that standard arguments yield

$$\mathbb{P}[\operatorname{sgn}(\Delta_{i-1}X') \neq \operatorname{sgn}(\Delta_{i-1}W)] = o_{\mathbb{P}}(n^{-p})$$

and

$$\mathbb{E}||\Delta_i X'| - |\sigma_{t_{i-2}} \Delta_i W||^p \le \mathbb{E}|\Delta_i X' - \sigma_{t_{i-2}} \Delta_i W|^p$$

$$\le C n^{-p}$$

for any p > 0 (where the constant C may depend on p) and where we have used (14), so that using $\hat{\theta} - \theta_0 = O_{\mathbb{P}}(n^{-1})$ gives

$$I = \frac{\pi n^{1/2} (\widehat{\theta} - \theta_0)^T}{2} \sum_{i=2}^n \sigma_{t_{i-2}} \partial_{\theta} \psi_{i-1}(\theta_0) |\Delta_i W| \operatorname{sgn}(\Delta_{i-1} W) + o_{\mathbb{P}}(1)$$

which are conditionally centered and uncorrelated increments, with $\operatorname{Var}\left[|\Delta_i W|\operatorname{sgn}(\Delta_{i-1}W)|\mathcal{F}_{i-2}\right] = O(n^{-1})$, so that $\sum_{i=2}^n \sigma_{t_{i-2}} \partial_\theta \psi_{i-1}(\theta_0) |\Delta_i W| \operatorname{sgn}(\Delta_{i-1}W) = O_{\mathbb{P}}(1)$. Therefore, using again that $\widehat{\theta} - \theta_0 = O_{\mathbb{P}}(n^{-1})$, we have $I \to^{\mathbb{P}} 0$.

6.5 Proof of Corollary 8

By the stable convergence of Theorem 7, the proof amounts to showing that \widehat{AVAR} is consistent, which is actually a corollary to Theorem 11 in the special case $g(x) = x^2$.

6.6 Proof of Theorem 9

Following the discussion at the beginning of Appendix A.2 (p. 30) in Potiron and Mykland (2017) and Proposition 1 from Mykland and Zhang (2009), p. 1408, we can assume without loss of generality that the drift b_t is null as the price process X is continuous.

First, note that (37) is a straightforward consequence of (36) together with Theorem 1 (p. 25) in Potiron and Mykland (2017). Consequently, we only need to show (36). We now provide the proof of (36), i.e. that

$$\alpha^{-1}\widehat{\Xi} = \alpha^{-1}\widetilde{\Xi} + o_{\mathbb{P}}(1).$$

First, note that as a result of Remark 5 (p. 25) in Potiron and Mykland (2017), $n^{1/2}$ and α^{-1} are of the same order, and thus it is sufficient to show that

$$n^{1/2}\widehat{\Xi} = n^{1/2}\widetilde{\Xi} + o_{\mathbb{P}}(1).$$

Second, we have to reexpress the Hayashi-Yoshida estimator (35). To do so, we follow the beginning of Section 4.3 in Potiron and Mykland (2017) and introduce some (common) definition in the Hayashi-Yoshida literature. For any positive integer i, we consider the ith sampling time of the first asset $t_i^{(1)}$. We define two related random times, t_i^- and t_i^+ , which correspond respectively to the closest sampling time of the second asset that is strictly smaller than $t_i^{(1)}$, and the closest sampling time of the second asset that is (not necessarily strictly) bigger than $t_i^{(1)}$. Formally, they are defined as

$$t_0^- = 0,$$
 (56)

$$t_i^- = \max\{t_i^{(2)} : t_i^{(2)} < t_i^{(1)}\} \text{ for } i \ge 1,$$
(57)

$$t_i^+ = \min\{t_j^{(2)} : t_j^{(2)} \ge t_i^{(1)}\}. \tag{58}$$

Rearranging the terms in (35) gives us

$$\widetilde{\Xi} = \sum_{t_i^+ < t} \Delta_i X^{(1)} (X_{t_i^+}^{(2)} - X_{t_{i-1}^-}^{(2)}) + o_{\mathbb{P}}(n^{-1/2}).$$
(59)

We deduce that

$$\begin{split} n^{1/2}\widehat{\Xi} &= n^{1/2} \sum_{t_i^+ < t} \Delta_i \widehat{X}^{(1)}(\widehat{X}_{t_i^+}^{(2)} - \widehat{X}_{t_{i-1}^-}^{(2)}) + o_{\mathbb{P}}(1), \\ &= n^{1/2} \widetilde{\Xi} + n^{1/2} \sum_{t_i^+ < t} \psi_i^{(1)}(\widehat{\theta}^{(1)}) \big((\phi(Q_{t_i^+}^{(2)}, \theta_0^{(2)}) - \phi(Q_{t_{i-1}^-}^{(2)}, \theta_0^{(2)})) - (\phi(Q_{t_i^+}^{(2)}, \widehat{\theta}^{(2)}) - \phi(Q_{t_{i-1}^-}^{(2)}, \widehat{\theta}^{(2)})) \big) \\ &+ n^{1/2} \sum_{t_i^+ < t} \Delta_i X^{(1)} \big((\phi(Q_{t_i^+}^{(2)}, \theta_0^{(2)}) - \phi(Q_{t_{i-1}^-}^{(2)}, \theta_0^{(2)})) - (\phi(Q_{t_i^+}^{(2)}, \widehat{\theta}^{(2)}) - \phi(Q_{t_{i-1}^-}^{(2)}, \widehat{\theta}^{(2)})) \big) \\ &+ n^{1/2} \sum_{t_i^+ < t} \psi_i^{(1)}(\widehat{\theta}^{(1)}) (X_{t_i^+}^{(2)} - X_{t_{i-1}^-}^{(2)}) + o_{\mathbb{P}}(1), \\ &:= n^{1/2} \widetilde{\Xi} + I + II + III + o_{\mathbb{P}}(1). \end{split}$$

Our aim is to show that $I = o_{\mathbb{P}}(1)$, $II = o_{\mathbb{P}}(1)$ and $III = o_{\mathbb{P}}(1)$. We start with I. On the account that ϕ is C^3 in θ , and because $\max_i \|Q_{t_i}\|$ is bounded,

$$I \le C n^{1/2} N |\widehat{\theta} - \theta_0|^2,$$

and this is $o_{\mathbb{P}}(1)$ by (18), Remark 5 (p. 25) and Lemma 8 (p. 31) in Potiron and Mykland (2017).

As for II, the proof of Theorem 2 (p. 46) in Li et al (2016) in the volatility case goes through with one change. To prove (69) in the cited paper, since

$$\left((\phi(Q_{t_i^+}^{(2)},\theta_0^{(2)})-\phi(Q_{t_{i-1}^-}^{(2)},\theta_0^{(2)}))-(\phi(Q_{t_i^+}^{(2)},\widehat{\theta}^{(2)})-\phi(Q_{t_{i-1}^-}^{(2)},\widehat{\theta}^{(2)}))\right)$$

is not \mathcal{F}_{t_i} -measurable, we need to use a Taylor expansion around θ_0 . More specifically, let us prove (69) and in line with the notation of the cited paper, we define:

$$F_N(\theta) = \sum_{i=1}^{N^{(1)}} \underbrace{\left(\left(\phi(Q_{t_i^+}^{(2)}, \theta) - \phi(Q_{t_{i-1}^-}^{(2)}, \theta) \right) - \left(\phi(Q_{t_i^+}^{(2)}, \theta_0^{(2)}) - \phi(Q_{t_{i-1}^-}^{(2)}, \theta_0^{(2)}) \right)}_{\chi_i(\theta)} \underbrace{\int_{t_{i-1}^{(1)}}^{t_i^{(1)}} \sigma_t^{(1)} dW_t^{(1)}}_{\Delta M_i^{c,(1)}}.$$

Note now that by the same Taylor expansion as in (49) and the same line of reasoning, we directly get that for $\theta \in \Theta$ such that $|\theta - \theta_0| \le K/N$, for some $\overline{\theta} \in [\theta_0, \theta]$,

$$N^{l}|F_{N}(\theta) - F_{N}(\theta_{0})|^{2l} \leq C_{l}N^{l}|\theta - \theta_{0}|^{2l} \left(\left| \sum_{i=1}^{N^{(1)}} \partial_{\theta} \chi_{i}(\theta_{0}) \Delta M_{i}^{c,(1)} \right|^{2l} + \left| \frac{1}{2} \sum_{i=1}^{N^{(1)}} \partial_{\theta}^{2} \chi_{i}(\overline{\theta}) \Delta M_{i}^{c,(1)} \right|^{2l} |\theta - \theta_{0}|^{2l} \right).$$

Now, using that the first term is a sum of \mathcal{H}_t -martingale increments and Burkholder-Davis-Gundy inequality yields

$$\mathbb{E} \left| \sum_{i=1}^{N^{(1)}} \partial_{\theta} \chi_{i}(\theta_{0}) \Delta M_{i}^{c,(1)} \right|^{2l} \leq C \mathbb{E} \left| \sum_{i=1}^{N^{(1)}} |\partial_{\theta} \chi_{i}(\theta_{0})|^{2} \Delta_{i} t^{(1)} \right|^{l} < C.$$

Similarly, Jensen inequality applied to the measure $(N^{(1)})^{-1} \sum_{i=1}^{N^{(1)}}$, the boundedness of $|\partial_{\theta}^2 \chi_i(\overline{\theta})|$, and direct calculation of moments for $\Delta M_i^{c,(1)}$ yield

$$\mathbb{E} \left| \frac{1}{2} \sum_{i=1}^{N^{(1)}} \partial_{\theta}^{2} \chi_{i}(\overline{\theta}) \Delta M_{i}^{c,(1)} \right|^{2l} \leq C N^{2l-1} \mathbb{E} \sum_{i=1}^{N^{(1)}} \left| \Delta M_{i}^{c,(1)} \right|^{2l} \leq C N^{l}.$$

Combined with $|\theta - \theta_0| \le K/N$, this gives

$$N^{l}\mathbb{E}\sup_{\theta\in\Theta||\theta-\theta_{0}|\leq K/N}|F_{N}(\theta)-F_{N}(\theta_{0})|^{2l}\to 0$$

which is (69) from Li et al (2016). Then, one can proceed as in the proof of Theorem 2 (p. 46) in Li et al (2016).

We turn to III, which is slightly more complicated to deal with. We decompose the increment of the second asset in three parts and rewrite III as

$$\begin{split} III &= n^{1/2} \Big(\sum_{t_i^+ < t} \psi_i^{(1)}(\widehat{\theta}^{(1)}) (X_{t_i^+}^{(2)} - X_{t_i^{(1)}}^{(2)}) + \sum_{t_i^+ < t} \psi_i^{(1)}(\widehat{\theta}^{(1)}) (X_{t_i^{(1)}}^{(2)} - X_{t_{i-1}}^{(2)}) + \sum_{t_i^+ < t} \psi_i^{(1)}(\widehat{\theta}^{(1)}) (X_{t_{i-1}}^{(2)} - X_{t_{i-1}}^{(2)}) \Big) \\ &:= n^{1/2} (III_A + III_B + III_C). \end{split}$$

The problem with III_A is that it is not adapted to a simple filtration. To circumvent this difficulty, we need to rearrange the terms of the sum again. We follow Potiron and Mykland (2017) (Section 4.3) and we define the new sampling times t_i^{1C} as $t_0^{1C} := t_0^{(1)}$, and recursively for i any nonnegative integer

$$t_{i+1}^{1C} := \min \left\{ t_u^{(1)} : \text{ there exists } j \in \mathbb{N} \text{ such that } t_i^{1C} \le t_j^{(2)} < t_u^{(1)} \right\}. \tag{60}$$

In analogy with (56), (57) and (58), we introduce the following times

$$t_0^{1C,-} := 0, (61)$$

$$t_{i-1}^{1C,-} := \max\{t_i^{(2)} : t_i^{(2)} < t_{i-1}^{1C}\} \text{ for } i \ge 2$$
 (62)

$$t_{i-1}^{1C,+} := \min\{t_j^{(2)} : t_j^{(2)} \ge t_{i-1}^{1C}\} \text{ for } i \ge 1.$$
(63)

In light of this definition, we can rewrite III_A as

$$III_{A} = \sum_{t_{i}^{1C,+} < t} \underbrace{\left(\left(\phi(Q_{t_{i}^{1C}}^{(1)}, \widehat{\theta}^{(1)}) - \phi(Q_{t_{i-1}^{1C}}^{(1)}, \widehat{\theta}^{(1)}) \right) - \left(\phi(Q_{t_{i}^{1C}}^{(1)}, \theta_{0}^{(1)}) - \phi(Q_{t_{i-1}^{1C}}^{(1)}, \theta_{0}^{(1)}) \right) \left(X_{t_{i}^{1C,+}}^{(2)} - X_{t_{i}^{1C}}^{(2)} \right)}_{M_{i}(\widehat{\theta}^{(1)})},$$

where $M_i(\theta)$ is $\mathcal{F}_{t_{i+1}^{1C}}$ -measurable. By the mean value theorem, we also have for some $\overline{\theta} \in [\theta_0^{(1)}, \widehat{\theta}^{(1)}]$ that

$$n^{1/2} \sum_{i=1}^{N^{(1)}} M_i(\widehat{\theta}^{(1)}) = n^{1/2} (\widehat{\theta}^{(1)} - \theta_0^{(1)})^T \sum_{i=1}^{N^{(1)}} \partial_{\theta} M_i(\theta_0^{(1)}) + \frac{n^{1/2} (\widehat{\theta}^{(1)} - \theta_0^{(1)})^T}{2} \sum_{i=1}^{N^{(1)}} \partial_{\theta}^2 M_i(\overline{\theta}) (\widehat{\theta}^{(1)} - \theta_0^{(1)}).$$

Following the same line of reasoning as for the proof of (49) in the volatility case, we can show that the two terms go to 0 in probability, so that we have shown that $n^{1/2}III_A = o_{\mathbb{P}}(1)$. The other two terms III_B and III_C do not require rearranging the terms. Specifically, $n^{1/2}III_B$ can be shown $o_{\mathbb{P}}(1)$ following exactly the proof of Theorem 2 (p. 46) in Li et al (2016). Regarding the third term $n^{1/2}III_C$, we can show that it is $o_{\mathbb{P}}(1)$ using a Taylor expansion similarly as for III_A .

6.7 Proof of Corollary 10

Although the quantities introduced are quite involved to formally define \widetilde{AB} and \widetilde{AVAR} , the proof works the same way as for the proof of (30) in Corollary 4, along with techniques and estimates from Potiron and Mykland (2017).

6.8 Proof of Theorem 11

All along this proof, we use the notations k_n , Δ_n , w_n in lieu of respectively k, Δ and w in order to emphasize their dependence on n. We have to show that $n^{1/2}\left(\widehat{\Xi}-\widetilde{\Xi}'\right)=o_{\mathbb{P}}(1)$ where

$$\widehat{\Xi} = \Delta_n \sum_{i=1}^{[T/\Delta_n]-k_n+1} \left\{ g(\widehat{c}_i) - \frac{1}{2k_n} \sum_{j,k,l,m=1}^d \partial_{jk,lm}^2 g(\widehat{c}_i) \left(\widehat{c}_i^{jl} \widehat{c}_i^{km} + \widehat{c}_i^{jm} \widehat{c}_i^{kl} \right) \right\}, \tag{64}$$

with

$$\widehat{c}_{i}^{lm} = \frac{1}{k_{n}\Delta_{n}} \sum_{j=0}^{k_{n}-1} \Delta_{i+j} \widehat{X}^{l} \Delta_{i+j} \widehat{X}^{m} \mathbf{1}_{\{\|\Delta_{i+j}\widehat{X}\| \le w_{n}\}}.$$
(65)

We start by showing that we can assume without loss of generality that X is continuous, i.e replace X by X' in all the expressions. To do so, consider $\widehat{\Xi}'$ and \widehat{c}'_i the estimators applied to the continuous part X' in lieu of X. Without loss of generality, we assume in what follows that X, $\widehat{\theta}$ and θ_0 are 1-dimensional quantities. The multi-dimensional case can be derived by a straightforward adaptation.

Lemma 18. We have

$$n^{1/2}\left(\widehat{\Xi}-\widehat{\Xi}'\right)\to^{\mathbb{P}}0.$$

Proof. Recall that we have the key decomposition

$$\Delta_i \widehat{X} = \Delta_i X(\widehat{\theta}) = \underbrace{\Delta_i \widecheck{B} + \psi_i(\widehat{\theta})}_{\Delta_i B'} + \Delta_i M^c + \Delta_i J, \tag{66}$$

where we recall that $\check{B}_t = \int_0^t b_s' ds$. Now, we apply exactly the same line of reasoning as for the proof of Theorem 7. We replace again $\Delta_i \check{B}$ by $\Delta_i B'$ and \mathcal{F}_i by $\mathcal{G}_i = \mathcal{F}_i \vee \sigma\{Q_{t_i}, 0 \leq i \leq n\}$ in the proof of Lemma 13.2.6 (p. 384) in Jacod and Protter (2011), all the conditional estimates are preserved and thus the lemma remains valid in the presence of the term $\psi_i(\widehat{\theta})$. Applied with $F(x) = x^2$, k = 1, p' = s' = 2, s = 1 and $\theta = 0$, this directly yields that for all $q \geq 1$ and for some deterministic sequence a_n shrinking to 0, we have that

$$\mathbb{E}\left||\Delta_i \widehat{X}|^2 \mathbf{1}_{\left\{|\Delta_i \widehat{X}| \le w_n\right\}} - |\Delta_i \widehat{X}'|^2 \mathbf{1}_{\left\{|\Delta_i \widehat{X}'| \le w_n\right\}}\right|^q \le C a_n \Delta_n^{(2q-r)\bar{\omega}+1}. \tag{67}$$

As a by-product, we also deduce

$$\mathbb{E}\left|\widehat{c}_{i}-\widehat{c}_{i}'\right|^{q} \leq C a_{n} \Delta_{n}^{(2q-r)\bar{\omega}+1-q}.$$
(68)

Moreover, replacing again \mathcal{F}_i by \mathcal{G}_i and $\Delta_i \check{B}$ by $\Delta_i B'$ in the calculation we can also see that the second inequality of (4.10) in Jacod and Rosenbaum (2013) remains true in the presence of $\psi_i(\widehat{\theta})$, that is, introducing $\alpha_i = |\Delta_i \widehat{X}'|^2 - \sigma_{t_i}^2 \Delta_n$, we have

$$|\mathbb{E}[\alpha_i|\mathcal{G}_i]| \le C\Delta_n^{3/2}.\tag{69}$$

Now, remark that by the proof of Lemma 4.4 (p. 1479, case v=1) in Jacod and Rosenbaum (2013), $n^{1/2}(\widehat{\Xi}-\widehat{\Xi}')\to^{\mathbb{P}}0$ is an immediate consequence of our estimates (68) and (69), along with the polynomial condition (40) on g.

From now on, by virtue of Lemma 18, we only have to prove $n^{1/2}(\widehat{\Xi}' - \widetilde{\Xi}') \to^{\mathbb{P}} 0$. We now want to show that in the definition of $\widehat{\Xi}'$, we can substitute \widehat{c}'_i by \overline{c}'_i , where

$$\overline{c}_{i}^{\prime lm} = \frac{1}{k_{n} \Delta_{n}} \sum_{j=0}^{k_{n}-1} \Delta_{i+j} \widehat{X}^{\prime l} \Delta_{i+j} \widehat{X}^{\prime m} \mathbf{1}_{\{|\Delta_{i+j} X^{\prime}| \le w_{n}\}}, \tag{70}$$

that is when the indicator function is applied to X' itself instead of \widehat{X}' . We first state a technical lemma.

Lemma 19. We have, for any $i \in \{1, \dots, n\}$, any $j \in \{1, \dots, 3\}$, and any $q \ge 1$,

$$\mathbb{E}|\partial^j g(\widehat{c}_i')|^q \leq C \text{ and } \mathbb{E}|\partial^j g(\overline{c}_i')|^q \leq C.$$

Proof. In view of (40), it is sufficient to prove that for any $q \geq 1$,

$$\mathbb{E}|\hat{c}_i'|^q \leq C$$
 and $\mathbb{E}|\bar{c}_i'|^q \leq C$.

Moreover, since $|\tilde{c}_i'|^q \leq C(|\tilde{c}_i' - \overline{c}_i'|^q + |\overline{c}_i' - \tilde{c}_i|^q + |\tilde{c}_i'|^q)$, and as $\mathbb{E}|\tilde{c}_i|^q \leq C$ as an easy consequence of (4.11) in Jacod and Rosenbaum (2013) (p. 1476) and the boundedness of c in Assumption (H), it suffices to show the \mathbb{L}_q boundedness of

$$\widehat{c}_{i}' - \overline{c}_{i}' = \frac{1}{k_{n} \Delta_{n}} \sum_{j=0}^{k_{n}-1} |\Delta_{i+j} \widehat{X}'|^{2} \left(\mathbf{1}_{\{|\Delta_{i+j} \widehat{X}'| \le w_{n}\}} - \mathbf{1}_{\{|\Delta_{i+j} X'| \le w_{n}\}} \right)$$
(71)

and

$$\overline{c}_{i}' - \widetilde{c}_{i} \leq \frac{2}{k_{n}\Delta_{n}} \sum_{j=0}^{k_{n}-1} \Delta_{i+j} X' \psi_{i+j}(\widehat{\theta}) \mathbf{1}_{\{|\Delta_{i+j}X'| \leq w_{n}\}} + \frac{1}{k_{n}\Delta_{n}} \sum_{j=0}^{k_{n}-1} \psi_{i+j}(\widehat{\theta})^{2}, \tag{72}$$

$$:= I + II.$$

We first show the \mathbb{L}_q boundedness of (71). First recall that in (55) we proved that

$$\mathbb{E}\left|\mathbf{1}_{\{|\Delta_i\widehat{X}'|\leq w_n\}} - \mathbf{1}_{\{|\Delta_iX'|\leq w_n\}}\right| \leq n^{-\beta}$$

for any $\beta > 0$. Thus, by Cauchy-Schwarz inequality and Jensen's inequality we easily get that $\mathbb{E}|\vec{c}_i - \vec{c}_i|^q \leq C$ considering β large enough.

We prove now the \mathbb{L}_q boundedness of (72). By Jensen's inequality applied to

$$|k_n^{-1}\sum_{j=0}^{k_n-1} \Delta_{i+j} X' \psi_{i+j}(\widehat{\theta})|^q,$$

we have

$$\mathbb{E}|I|^q \le \frac{Cn^q}{k_n} \sum_{j=0}^{k_n-1} \mathbb{E}|\Delta_{i+j}X'|^q \underbrace{|\psi_{i+j}(\widehat{\theta})|^q}_{C/n^q}$$

$$\le Cn^{-q/2}.$$

For II we have

$$\mathbb{E}|II|^q \le \frac{Cn^q}{k_n} \mathbb{E} \sum_{j=0}^{k_n-1} |\psi_{i+j}(\widehat{\theta})|^{2q}$$

$$\le Cn^{-q},$$

and thus this yields the \mathbb{L}_q boundedness of $\overline{c}'_i - \widetilde{c}_i$, which concludes the proof.

Lemma 20. Let $\overline{\Xi}'$ be defined as $\widehat{\Xi}'$ where \widehat{c}'_i is replaced by \overline{c}'_i . Then

$$n^{1/2}\left(\widehat{\Xi}' - \overline{\Xi}'\right) \to^{\mathbb{P}} 0.$$

Proof. We have

$$n^{1/2}\left(\widehat{\Xi}' - \overline{\Xi}'\right) = n^{1/2} \Delta_n \sum_{i=1}^{[T/\Delta_n] - k_n + 1} \left\{ g(\widehat{c}'_i) - g(\overline{c}'_i) \right\}$$

$$+ \frac{n^{1/2} \Delta_n}{2k_n} \sum_{i=1}^{[T/\Delta_n] - k_n + 1} \left\{ h(\overline{c}'_i) - h(\widehat{c}'_i) \right\},$$

$$(73)$$

with $h(x) = 2\partial^2 g(x)x^2$, so that proving our claim boils down to showing that both terms in the right-hand side of (73) are negligible. For the first one, we have

$$\sum_{i=1}^{[T/\Delta_n]-k_n+1} \left| g(\widehat{c}_i') - g(\overline{c}_i') \right| \leq \frac{1}{k_n \Delta_n} \sum_{i=1}^{[T/\Delta_n]-k_n+1} \sum_{j=0}^{k_n-1} \left| \partial g(a_{i,j}) \right| \left| \Delta_{i+j} \widehat{X}' \right|^2 \left| \mathbf{1}_{\left\{ \left| \Delta_{i+j} \widehat{X}' \right| \leq w_n \right\}} - \mathbf{1}_{\left\{ \left| \Delta_{i+j} X' \right| \leq w_n \right\}} \right|$$

for some (random) $a_{i,j}$ such that $|a_{i,j}| \leq |\hat{c}'_i| + |\bar{c}'_i|$ by the mean value theorem. Now, by Lemma 19 and the fact that g is of polynomial growth we get $\mathbb{E}|\partial g(a_{i,j})|^q \leq C$ for any $q \geq 1$, and thus by Cauchy-Schwarz inequality we will have

$$n^{1/2} \Delta_n \sum_{i=1}^{[T/\Delta_n]-k_n+1} \left\{ g(\widetilde{c}_i') - g(\overline{c}_i') \right\} \to^{\mathbb{P}} 0$$

if we can prove that

$$\sum_{i=1}^{[T/\Delta_n]-k_n+1} \sum_{j=0}^{k_n-1} \left(\mathbb{E}\left[|\Delta_{i+j} \widehat{X}'|^4 \left| \mathbf{1}_{\{|\Delta_{i+j} \widehat{X}| \le w_n\}} - \mathbf{1}_{\{|\Delta_{i+j} X'| \le w_n\}} \right| \right] \right)^{1/2} = o(k_n n^{-1/2}),$$

i.e. that

$$\sum_{i=1}^{[T/\Delta_n]-k_n+1} \left(\mathbb{E}\left[|\Delta_i \widehat{X}'|^4 \left| \mathbf{1}_{\{|\Delta_i \widehat{X}'| \le w_n\}} - \mathbf{1}_{\{|\Delta_i X'| \le w_n\}} \right| \right] \right)^{1/2} = o(n^{-1/2}).$$

Recalling $|\Delta_i \widehat{X}'|^4 \leq C(|\Delta_i X'|^4 + |\psi_i(\widehat{\theta})|^4)$, we have that

$$\sum_{i=1}^{[T/\Delta_n]-k_n+1} \left(\mathbb{E}\left[|\Delta_i X'|^4 \left| \mathbf{1}_{\{|\Delta_i \widehat{X}'| \le w_n\}} - \mathbf{1}_{\{|\Delta_i X'| \le w_n\}} \right| \right] \right)^{1/2} = O(n^{-\beta/4}) = o(n^{-1/2})$$

since β can be taken arbitrary big, using again Cauchy-Schwarz inequality along with the fact that $\mathbb{E}|\Delta_i X'|^q \leq C n^{-q/2}$, and (55). Finally, it is immediate to prove

$$\sum_{i=1}^{[T/\Delta_n]-k_n+1} \left(\mathbb{E}\left[|\psi_i(\widehat{\theta})|^4 \left| \mathbf{1}_{\{|\Delta_i \widehat{X}'| \le w_n\}} - \mathbf{1}_{\{|\Delta_i X'| \le w_n\}} \right| \right] \right)^{1/2} = o\left(n^{-1/2} \right),$$

given that $|\psi_i(\widehat{\theta})|^4 \leq K/n^4$. The second term on the right-hand side of (73) is proved in the same way.

In the 1-dimensional setting, we now introduce the following notation for $\theta \in \Theta$:

$$c_i'(\theta) = \frac{1}{k_n \Delta_n} \sum_{j=0}^{k_n - 1} |\Delta_{i+j} X'(\theta)|^2 \mathbb{1}_{\{|\Delta_{i+j} X'| \le w_n\}},$$

where we recall that for any $i \in \{1, \dots, n\}$, $\Delta_i X'(\theta) = \Delta_i X' + \psi_i(\theta)$. Note that $\overline{c}_i' = c_i'(\widehat{\theta})$, and $\widetilde{c}_i = c_i'(\theta_0)$. We define

$$E_n := n^{1/2} \Delta_n \sum_{i=1}^{[T/\Delta_n] - k_n + 1} \left\{ g(\overline{c}_i') - g(\widetilde{c}_i) \right\}.$$

By the mean value theorem along with the chain rule we have for some $\overline{\theta} \in [\theta_0, \widehat{\theta}]$,

$$\begin{split} E_{n} &= \frac{2n^{1/2}}{k_{n}} (\widehat{\theta} - \theta_{0}) \sum_{i=1}^{[T/\Delta_{n}] - k_{n} + 1} \partial g(\widetilde{c}_{i}) \sum_{j=0}^{k_{n} - 1} \Delta_{i+j} X' \partial_{\theta} \psi_{i+j}(\theta_{0}) 1_{\{|\Delta_{i+j} X'| \leq w_{n}\}} \\ &+ \frac{n^{1/2}}{k_{n}} (\widehat{\theta} - \theta_{0})^{2} \sum_{i=1}^{[T/\Delta_{n}] - k_{n} + 1} \partial g(c'_{i}(\overline{\theta})) \sum_{j=0}^{k_{n} - 1} \Delta_{i+j} X'(\overline{\theta}) \partial_{\theta}^{2} \psi_{i+j}(\overline{\theta}) 1_{\{|\Delta_{i+j} X'| \leq w_{n}\}} \\ &+ \frac{n^{1/2}}{k_{n}} (\widehat{\theta} - \theta_{0})^{2} \sum_{i=1}^{[T/\Delta_{n}] - k_{n} + 1} \partial g(c'_{i}(\overline{\theta})) \sum_{j=0}^{k_{n} - 1} \partial_{\theta} \psi_{i+j}(\overline{\theta})^{2} 1_{\{|\Delta_{i+j} X'| \leq w_{n}\}} \\ &+ \frac{2n^{1/2}}{k_{n}^{2} \Delta_{n}} (\widehat{\theta} - \theta_{0})^{2} \sum_{i=1}^{[T/\Delta_{n}] - k_{n} + 1} \partial^{2} g(c'_{i}(\overline{\theta})) \left\{ \sum_{j=0}^{k_{n} - 1} \Delta_{i+j} X'(\overline{\theta}) \partial_{\theta} \psi_{i+j}(\overline{\theta}) 1_{\{|\Delta_{i+j} X'| \leq w_{n}\}} \right\}^{2}, \\ &:= I + II + III + IV. \end{split}$$

We now show that each term is $o_{\mathbb{P}}(1)$.

Lemma 21. We have

$$I = \frac{2n^{1/2}}{k_n} (\widehat{\theta} - \theta_0) \sum_{i=1}^{[T/\Delta_n] - k_n + 1} \partial g(\widetilde{c}_i) \sum_{j=0}^{k_n - 1} \Delta_{i+j} X' \partial_{\theta} \psi_{i+j}(\theta_0) 1_{\{|\Delta_{i+j} X'| \le w_n\}} \to^{\mathbb{P}} 0.$$

Proof. Since Assumption (H) yields $\frac{2n^{1/2}}{k_n}(\widehat{\theta}-\theta_0)=O_{\mathbb{P}}(k_n^{-1}n^{-1/2})$, it suffices to prove that

$$\sum_{i=1}^{[T/\Delta_n]-k_n+1} \partial g(\widetilde{c}_i) \sum_{j=0}^{k_n-1} \Delta_{i+j} X' \partial_{\theta} \psi_{i+j}(\theta_0) 1_{\{|\Delta_{i+j}X| \le w_n\}} = o_{\mathbb{P}}(k_n n^{1/2}).$$
 (74)

Recalling the decomposition $\Delta_{i+j}X' = \Delta_{i+j}\check{B} + \Delta_{i+j}M^c$, we first show that the above term is negligible when $\Delta_{i+j}X'$ is replaced by $\Delta_{i+j}M^c$. In that case, by virtue of the domination $1_{\{|\Delta_{i+j}M^c|\geq w_n\}} \leq w_n^{-1}|\Delta_{i+j}M^c|$, Burkhölder-Davis-Gundy inequality, Hölder's inequality, along with the fact that $|\partial g(\widetilde{c}_i)|$ is \mathbb{L}_q bounded by Lemma 19, the indicator function can be removed without loss of generality. Thus, introducing

$$A_n = \sum_{i=1}^{[T/\Delta_n]-k_n+1} \partial g(\widetilde{c}_i) \sum_{j=0}^{k_n-1} \Delta_{i+j} M^c \partial_{\theta} \psi_{i+j}(\theta_0),$$

and

$$B_n = \sum_{i=1}^{[T/\Delta_n]-k_n+1} \partial g(c_{t_i}) \sum_{j=0}^{k_n-1} \Delta_{i+j} M^c \partial_{\theta} \psi_{i+j}(\theta_0),$$

we show that $A_n - B_n = o_{\mathbb{P}}(k_n n^{1/2})$ and $B_n = o_{\mathbb{P}}(k_n n^{1/2})$ separately. We have for some $\xi_i \in [\widetilde{c}_i, c_{t_i}]$,

$$|A_n - B_n| \le \sum_{i=1}^{[T/\Delta_n] - k_n + 1} |\partial^2 g(\xi_i)| |\widetilde{c}_i - c_{t_i}| \sum_{j=0}^{k_n - 1} |\Delta_{i+j} M^c| |\partial_\theta \psi_{i+j}(\theta_0)|.$$

Moreover, by (4.11) in Jacod and Rosenbaum (2013) (p. 1476), we have the estimate

$$\mathbb{E}\left[\left|\widetilde{c}_{i}-c_{t_{i}}\right|^{2}\right] \leq C\left(k_{n}^{-1}+k_{n}\Delta_{n}\right). \tag{75}$$

Thus, by application of Hölder's inequality, the fact that $\partial^2 g(\xi_i)$ is \mathbb{L}_q bounded by Lemma 19, and that for any $q \geq 1$:

$$\mathbb{E}\left[|\Delta_{i+j}M^c|^q|\partial_\theta\psi_{i+j}(\theta_0)|^q\right] \le C\mathbb{E}\left[|\Delta_{i+j}M^c|^q\right] < Cn^{-q/2},$$

we deduce that

$$\mathbb{E}|A_n - B_n| \le Ck_n n^{1/2} \left(k_n^{-1} + k_n \Delta_n\right)^{1/2} = o_{\mathbb{P}}(k_n n^{1/2}).$$

As for B_n , we note that it can be expressed as a sum of martingale increments with respect to the filtration $\mathcal{H}_t = \mathcal{F}_t \vee \sigma\{Q_{t_i}, i = 0, \dots, n\}$, and we have $B_n = \sum_{i=1}^{[T/\Delta_n]} \chi_i$ with

$$\chi_i = \sum_{l=(i-k_n+1)\wedge 1}^i \partial g(\sigma_{t_l}^2) \partial_{\theta} \psi_i(\theta_0) \Delta_i M^c.$$

Thus, by property (2.2.35) p. 56 in Jacod and Protter (2011), proving that $B_n = o_{\mathbb{P}}(k_n n^{1/2})$ boils down to showing that

$$\widetilde{B}_n := n^{-1} k_n^{-2} \sum_{i=1}^{[T/\Delta_n]} \mathbb{E}\chi_i^2 \to 0.$$
 (76)

Now, using the boundedness of c, we have

$$\mathbb{E}\chi_i^2 \le Ck_n^2 \mathbb{E}\partial_\theta \psi_i(\theta_0)^2 \left(\Delta_i M^c\right)^2$$

$$\le Ck_n^2 n^{-1}.$$

Therefore $\widetilde{B}_n = O_{\mathbb{P}}(n^{-1})$ which proves (76) and thus (74) when replacing $\Delta_{i+j}X'$ by $\Delta_{i+j}M^c$. Finally, the case where we consider the drift term $\Delta_{i+j}\check{B}$ in lieu of $\Delta_{i+j}X'$ follows immediately from the fact that $\mathbb{E}|\Delta_{i+j}\check{B}|^k \leq Cn^{-k}$ for any $k \geq 1$.

Lemma 22. We have that $II = o_{\mathbb{P}}(1)$, $III = o_{\mathbb{P}}(1)$, $IV = o_{\mathbb{P}}(1)$.

Proof. Proving the first claim is equivalent to showing that

$$\widetilde{II} := \sum_{i=1}^{[T/\Delta_n]-k_n+1} \partial g(c_i'(\overline{\theta})) \sum_{j=0}^{k_n-1} \Delta_{i+j} X'(\overline{\theta}) \partial_{\theta}^2 \psi_{i+j}(\overline{\theta}) 1_{\{|\Delta_{i+j}X'| \leq w_n\}} = o_{\mathbb{P}}(k_n n^{3/2}).$$

Note, again, that by Assumption (**H**) and the fact that $\overline{\theta}$ belongs to a compact set, we have $|\partial_{\theta}^2 \psi_{i+j}(\overline{\theta})| \leq C$. Thus

$$\mathbb{E}\left|\widetilde{II}\right| \leq C \sum_{i=1}^{[T/\Delta_n]-k_n+1} \mathbb{E}\left[\left|\partial g(c_i'(\overline{\theta}))\right| \sum_{j=0}^{k_n-1} \left|\Delta_{i+j}X'(\overline{\theta})\right|\right] \\
\leq C \sum_{i=1}^{[T/\Delta_n]-k_n+1} \sum_{j=0}^{k_n-1} \left(\mathbb{E}\partial g(c_i'(\overline{\theta}))^2\right)^{1/2} \left(\mathbb{E}|\Delta_{i+j}X'(\overline{\theta})|^2\right)^{1/2} \\
\leq Ck_n n^{1/2} = o_{\mathbb{P}}(k_n n^{3/2}),$$

where we have used Lemma 19, and the fact that for any $q \geq 1$,

$$\mathbb{E}|\Delta_{i+j}X'(\overline{\theta})|^q \le C \left(\mathbb{E}|\Delta_{i+j}X'|^q + \mathbb{E}\left[\underbrace{(\overline{\theta} - \theta_0)^q}_{\le K/n^q} \underbrace{\sup_{\theta \in \Theta} |\partial_{\theta}\psi_i(\theta)|^q}_{\le K}\right] \right) \le C \left(n^{-q/2} + n^{-q}\right). \tag{77}$$

For the second claim, we have (bounding the indicator function from above by 1) the estimate

$$\widetilde{III} \leq \sum_{i=1}^{[T/\Delta_n]-k_n+1} \partial g(c_i'(\overline{\theta})) \sum_{j=0}^{k_n-1} \partial_{\theta} \psi_{i+j}(\overline{\theta})^2$$

$$\leq Ck_n \underbrace{\sum_{i=1}^{[T/\Delta_n]-k_n+1} |\partial g(c_i'(\overline{\theta}))|}_{O_{\mathbb{P}}(n)}$$

$$= O_{\mathbb{P}}(k_n n) = o_{\mathbb{P}}(k_n n^{3/2}),$$

so that $III = o_{\mathbb{P}}(1)$. Finally we show that $IV = o_{\mathbb{P}}(1)$, that is

$$\widetilde{IV} := \sum_{i=1}^{[T/\Delta_n]-k_n+1} \partial^2 g(c_i'(\overline{\theta})) \left\{ \sum_{j=0}^{k_n-1} \Delta_{i+j} X'(\overline{\theta}) \partial_{\theta} \psi_{i+j}(\overline{\theta}) 1_{\{|\Delta_{i+j} X'| \le w_n\}} \right\}^2 = o_{\mathbb{P}}(k_n^2 n^{1/2}). \tag{78}$$

By Cauchy-Schwarz inequality and the fact that $|\partial_{\theta}\psi_{i+j}(\overline{\theta})|^2 \leq C$, we get the domination

$$\mathbb{E}|\widetilde{IV}| \leq Ck_n \mathbb{E}\left[\sum_{i=1}^{[T/\Delta_n]-k_n+1} |\partial^2 g(c_i'(\overline{\theta}))| \sum_{j=0}^{k_n-1} |\Delta_{i+j}X'(\overline{\theta})|^2\right]$$

$$\leq Ck_n \sum_{i=1}^{[T/\Delta_n]-k_n+1} \sum_{j=0}^{k_n-1} \underbrace{\left(\mathbb{E}\partial^2 g(c_i'(\overline{\theta}))^2\right)^{1/2}}_{\leq C} \underbrace{\left(\mathbb{E}|\Delta_{i+j}X'(\overline{\theta})|^4\right)^{1/2}}_{O(n^{-1})}$$

$$\leq Ck_n^2 = o(k_n^2 n^{1/2}),$$

where we have used (77) with q = 4, and we are done.

Similarly we have by the mean value theorem that

$$\frac{n^{1/2}\Delta_n}{k_n} \sum_{i=1}^{[T/\Delta_n]-k_n+1} \left\{ h(\overline{c}_i') - h(\widetilde{c}_i) \right\}$$

is equal to

$$\frac{2n^{1/2}}{k_n^2}(\widehat{\theta} - \theta_0) \sum_{i=1}^{[T/\Delta_n]-k_n+1} \partial h(c_i'(\overline{\theta})) \sum_{j=0}^{k_n-1} \Delta_{i+j} X'(\overline{\theta}) \partial_{\theta} \psi_{i+j}(\overline{\theta}) 1_{\{|\Delta_{i+j}X'| \leq w_n\}}.$$

Lemma 23. We have

$$\frac{n^{1/2}\Delta_n}{k_n} \sum_{i=1}^{[T/\Delta_n]-k_n+1} \left\{ h(\overline{c}_i') - h(\widetilde{c}_i) \right\} \to^{\mathbb{P}} 0.$$

Proof. By Assumption (H) we have

$$\mathbb{E}\left|\frac{n^{1/2}\Delta_n}{k_n}\sum_{i=1}^{[T/\Delta_n]-k_n+1}\left\{h(\overline{c}_i')-h(\widetilde{c}_i)\right\}\right| \leq \frac{C}{n^{1/2}k_n^2}\sum_{i=1}^{[T/\Delta_n]-k_n+1}\sum_{j=0}^{k_n-1}\mathbb{E}\left[|\partial h(c_i'(\overline{\theta}))||\Delta_{i+j}X'(\overline{\theta})|\right].$$

Since ∂h is also of polynomial growth, we deduce as for Lemma 19 that for any $q \geq 1$, $\mathbb{E}|\partial h(c'_i(\overline{\theta}))|^q \leq C$, and so an application of Cauchy-Schwarz inequality yields

$$\mathbb{E}\left|\frac{n^{1/2}\Delta_n}{k_n}\sum_{i=1}^{[T/\Delta_n]-k_n+1}\left\{h(\overline{c}_i)-h(\widetilde{c}_i)\right\}\right| \le C/k_n \to 0.$$

We prove now the theorem.

Proof of Theorem 11. Recall that by Lemma 18 we only need to prove that $n^{1/2}(\widehat{\Xi}' - \widetilde{\Xi}') \to^{\mathbb{P}} 0$. We have

$$n^{1/2}\left(\widehat{\Xi}' - \widetilde{\Xi}'\right) = n^{1/2}\left(\widehat{\Xi}' - \overline{\Xi}'\right) + n^{1/2}\left(\overline{\Xi}' - \widetilde{\Xi}'\right).$$

The first term above is negligible by virtue of Lemma 20. Moreover, since

$$n^{1/2} \left(\overline{\Xi}' - \widetilde{\Xi}' \right) = n^{1/2} \Delta_n \sum_{i=1}^{[T/\Delta_n] - k_n + 1} \left\{ g(\overline{c}_i') - g(\widetilde{c}_i) \right\}$$
$$+ \frac{n^{1/2} \Delta_n}{2k_n} \sum_{i=1}^{[T/\Delta_n] - k_n + 1} \left\{ h(\widetilde{c}_i) - h(\overline{c}_i') \right\},$$

the assertion $n^{1/2}\left(\overline{\Xi}'-\widetilde{\Xi}'\right)\to^{\mathbb{P}}0$ is an immediate consequence of Lemma 21, Lemma 22 and Lemma 23. Combined with Theorem 3.2 (p. 1469, applied to X') in Jacod and Rosenbaum (2013), this yields the central limit theorem.

6.9 Proof of Corollary 12

By Slutsky's Lemma, all we need to prove is that $\widehat{AVAR} \to^{\mathbb{P}} AVAR$. Given the form of \widehat{AVAR} , this can be shown using exactly the same line of reasoning as for the general theorem replacing g by \overline{h} in all our estimates and combining the results with Corollary 3.7 in Jacod and Rosenbaum (2013) in lieu of Theorem 3.2, except that there is no scaling by $n^{1/2}$ in front of the estimates and no bias term. Since the C^3 property of g is only used once when handling the bias term in Lemma 23, the fact that \overline{h} is only of class C^2 is not problematic.

6.10 Proof of Theorem 14 and Corollary 15

In Vetter (2015), the author introduces

$$A_{i} = \frac{2n}{k_{n}} \sum_{j=1}^{k_{n}} \int_{(i+j-1)T/n}^{(i+j)T/n} (X_{s} - X_{(i+j-1)T/n}) dX_{s}$$

and

$$B_i := \frac{n}{k_n} \int_{iT/n}^{(i+k_n)T/n} \sigma_s^2 ds.$$

Accordingly, we define

$$\widehat{A}_{i} := \frac{2n}{k_{n}} \sum_{j=1}^{k_{n}} \left\{ \int_{(i+j-1)T/n}^{(i+j)T/n} (X_{s} - X_{(i+j-1)T/n}) dX_{s} + \psi_{i+j}(\widehat{\theta}) \Delta_{i+j} X \right\},$$

$$\widehat{B}_{i} := \frac{n}{k_{n}} \left\{ \int_{iT/n}^{(i+k_{n})T/n} \sigma_{s}^{2} ds + \sum_{j=1}^{k_{n}} \psi_{i+j}(\widehat{\theta})^{2} \right\},$$

along with the approximated increments for some arbitrary $p \ge 1$ and $1 \le l \le J(p) := [[nt/T - 2k_n]/((p+2)k_n)]$, where [x] is defined as the floor function of x,

$$\widetilde{A}_{i+k_n} - \widetilde{A}_i := \frac{n}{k_n} \sigma_{a_l(p)T/n} \sum_{j=1}^{k_n} \left(\Delta_{i+k_n+j} W^2 - \Delta_{i+j} W^2 \right),$$

and

$$\widetilde{B}_{i+k_n} - \widetilde{B}_i := \frac{n}{k_n} \int_{iT/n}^{(i+k_n)T/n} \widetilde{\sigma}_{a_l(p)T/n} (W'_{(s+k_nT/n)} - W'_s) ds,$$

where $a_l(p) := (l-1)(p+2)k_n$. Note that $\widehat{c}_i = \widehat{A}_i + \widehat{B}_i$, and therefore $\widehat{\Xi}$ can be linked to the above quantities as follows:

$$\widehat{\Xi} = \sum_{i=0}^{[T/\Delta_n]-2k_n} \left\{ \frac{3}{2k_n} (\widehat{A}_{i+k_n} - \widehat{A}_i + \widehat{B}_{i+k_n} - \widehat{B}_i)^2 - \frac{6}{k_n^2} \widehat{q}_i \right\}.$$
 (79)

Remark also that the approximated increments are independent of the information process and of $\hat{\theta}$. Now note that the general proof in Vetter (2015) is conducted in the following two steps.

- Compute an estimate for the deviations $A_{i+k_n} A_i (\widetilde{A}_{i+k_n} \widetilde{A}_i)$, $B_{i+k_n} B_i (\widetilde{B}_{i+k_n} \widetilde{B}_i)$, and $\widetilde{q}_i \int_{t_i}^{t_{i+1}} \sigma_s^4 ds$.
- Systematically use the previous estimate to replace A_i (resp. B_i , \widetilde{q}_i) by its counterpart \widetilde{A}_i (resp. \widetilde{B}_i , $\int_{t_i}^{t_{i+1}} \sigma_s^4 ds$) in all the encountered expressions.

Since \widetilde{A}_i , \widetilde{B}_i and $\int_{t_i}^{t_{i+1}} \sigma_s^4 ds$ are independent of the information process and $\widehat{\theta}$, the second step holds in our setting as well with no modification in the proofs of Vetter (2015). Thus, all we have to do in order to prove the theorem is to adapt the first step replacing A_i , B_i and \widetilde{q}_i by \widehat{A}_i , \widehat{B}_i and \widehat{q}_i . More precisely, we adapt Lemma A.1 and the second equation in the proof of (A.8) p. 2411 (corresponding to the approximation of \widetilde{q}_i by $\int_{t_i}^{t_{i+1}} \sigma_s^4 ds$) in Vetter (2015) as follows (in the next lemma, recall that \widetilde{A}_i and \widetilde{B}_i depend on some parameter $p \geq 1$).

Lemma 24. We have for any $r \ge 1$, $p \ge 1$, and any $i \in \{a_l(p), \dots, a_l(p) + pk_n\}$

$$\mathbb{E}\left[\left|\widehat{A}_{i+k_n} - \widehat{A}_i - (\widetilde{A}_{i+k_n} - \widetilde{A}_i)\right|^r\right] \le C(pn^{-1})^{r/2},$$

$$\mathbb{E}\left[\left|\widehat{B}_{i+k_n} - \widehat{B}_i - (\widetilde{B}_{i+k_n} - \widetilde{B}_i)\right|^r\right] \le C(pn^{-1})^{r/2},$$

 $\mathbb{E}\left[\left|\widehat{A}_{i+k_n} - \widehat{A}_i\right|^r\right] \le Cn^{-r/2},$

and

$$\mathbb{E}\left[\left|\widehat{B}_{i+k_n} - \widehat{B}_i\right|^r\right] \le Cn^{-r/2}.$$

Moreover we have uniformly in $t \in [0, T]$

$$\sqrt{\frac{n}{k_n}} \mathbb{E} \left| \sum_{i=1}^{[t/\Delta_n] - 2k_n} \frac{6}{k_n^2} \widehat{q}_i - \frac{6}{k_n^2} \int_0^t \sigma_s^4 ds \right| = o(1).$$

Proof. By Lemma A.1 in Vetter (2015), it suffices to prove that we have

$$\mathbb{E}\left[\left|\widehat{A}_{i+k_n} - \widehat{A}_i - (A_{i+k_n} - A_i)\right|^r\right] \le C(pn^{-1})^{r/2},$$

and a similar statement for \widehat{B}_i . Since $|\psi_k(\widehat{\theta})| \leq K/n$ for all $1 \leq k \leq n$, we obtain

$$\mathbb{E}\left[\left|\widehat{A}_{i+k_n} - \widehat{A}_i - (A_{i+k_n} - A_i)\right|^r\right] \leq \frac{2^r n^r}{k_n^r} \mathbb{E}\left[\left|\sum_{j=1}^{k_n} \left\{\psi_{i+k_n+j}(\widehat{\theta})\Delta_{i+k_n+j}X - \psi_{i+j}(\widehat{\theta})\Delta_{i+j}X\right\}\right|^r\right],$$

$$\leq \frac{Cn^r}{k_n} \sum_{j=1}^{k_n} \mathbb{E}\left[\left|\psi_{i+k_n+j}(\widehat{\theta})\Delta_{i+k_n+j}X\right|^r + \left|\psi_{i+j}(\widehat{\theta})\Delta_{i+j}X\right|^r\right],$$

$$\leq \frac{C}{k_n} \sum_{j=1}^{k_n} \underbrace{\mathbb{E}\left[\left|\Delta_{i+k_n+j}X\right|^r + \left|\Delta_{i+j}X\right|^r\right],}_{\leq Cn^{-r/2}},$$

$$\leq Cp^{r/2}n^{-r/2},$$

since $p \ge 1$, where we used Jensen's inequality at the second step and the domination $|\psi_i(\widehat{\theta})|^r \le C/n^r$ at the third step. Proving the other three inequalities and the approximation for \widehat{q}_i can be done by similar calculation.

Now, to prove Theorem 14, it is sufficient to follow closely the proof of Theorem 2.6 in Vetter (2015) replacing all occurrences of A_i , B_i and $\sum_{i=1}^{[t/\Delta_n]-2k_n} \frac{6}{k_n^2} \widehat{q}_i$ by \widehat{A}_i , \widehat{B}_i and $\frac{6}{k_n^2} \int_0^t \sigma_s^4 ds$, and accordingly all applications of Lemma A.1 and the approximation for \widehat{q}_i by Lemma 24 above.

A similar line of reasoning yields Corollary 15.

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