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ARFIMAX and ARFIMAX-TARCH Realized Volatility Modeling

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Abstract

ARFIMAX models are applied in estimating the intra-day realized volatility of the CAC40 and DAX30 indices. Volatility clustering and asymmetry characterize the logarithmic realized volatility of both indices. ARFIMAX model with time-varying conditional heteroscedasticity is the best performing specification and, at least in the case of DAX30, provides statistically superior next trading day's realized volatility forecasts.

Keywords: ARFIMAX, Realized Volatility, TARCH, Volatility Forecasting.

JEL: C22, C32, C53, G15.

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Abstract

ARFIMAX models are applied in estimating the intra-day realized volatility of the CAC40 and DAX30 indices. Volatility clustering and asymmetry characterize the logarithmic realized volatility of both indices. ARFIMAX model with time-varying conditional heteroscedasticity is the best performing specification and, at least in the case of DAX30, provides statistically superior next trading day's realized volatility forecasts.

1. Introduction

Andersen and Bollerslev (1998) first stated that the volatility estimates based on intra-day returns are more accurate than those based one daily data and introduced the *realized volatility*. The concept of the realized volatility is based on the *integrated volatility*. The integrated volatility, $\sigma_t^{2(IV)}$, aggregated over the time interval $(t-1, t)$:

$$\sigma_t^{2(IV)} = \int_{t-1}^t \sigma^2(x) dx, \quad (1)$$

is a latent variable which is not observable. Integrated volatility's volatility named *integrated quarticity* is $\sigma_t^{2(IQ)} = \int_{t-1}^t 2\sigma^4(x) dx$. The integrated volatility is asymptotically normally distributed:

$$\frac{\sqrt{m} \left(\sigma_t^{2(IV)} - \int_{t-1}^t \sigma^2(x) dx \right)}{\sqrt{\int_{t-1}^t 2\sigma^4(x) dx}} \xrightarrow{d} N(0,1).^1 \quad (2)$$

The volatility at a lower frequency is computed using data available at a higher frequency. Thus, the integrated volatility over the time interval $(t-1, t)$ can be consistently estimated by the

trading day's t realized volatility, $\sigma_t^{2(RV)} = \sum_{j=1}^{m-1} \left(\log(P_{(j+1/m),t}) - \log(P_{(j/m),t}) \right)^2$, which is defined as the sum of squared log-returns observed over m intra-day time intervals, for $\{P_{(m),t}\}_{j=1}^m$ denoting the asset prices over m intra-day time intervals of day t . The realized volatility converges in

¹ For details about $\sigma_t^{2(IV)}$ and $\sigma_t^{2(IQ)}$ see Barndorff-Nielsen and Shephard (2005) and references therein.

probability to the integrated volatility as $m \rightarrow \infty$, or

$$p \lim_{m \rightarrow \infty} \left(\sum_{j=1}^{m-1} \left(\log(P_{(j+1/m),t}) - \log(P_{(j/m),t}) \right)^2 \right) = \sigma_t^{2(IV)}.^2$$

Fractionally integrated autoregressive moving average with exogenous variables (ARFIMAX) models, introduced by Granger (1980), were proposed to model the long memory property of the realized volatility. Ebens (1999) proposed the application of ARFIMAX models in realized volatility modelling and they, subsequently, applied by Bollerslev and Wright (2001), Giot and Laurent (2004), Koopman et al. (2005) and Angelidis and Degiannakis (2009) among others. As concerns the point forecasts of volatility, the findings provide evidence in favour of realized volatility ARFIMAX models rather than autoregressive conditional heteroskedasticity (ARCH) framework of modeling daily log-returns. Corsi et al. (2005) noted that the volatility of S&P500 index futures volatility also exhibits time-variation and proposed the estimation of an ARFIMAX model that accounts for conditional heteroskedasticity.

In the present study, an ARFIMAX model is extended to account for volatility clustering as well as for the asymmetric relation between realized volatility and volatility of realized volatility. The unobservable term of an ARFIMAX specification is modelled as an asymmetric ARCH process. The new model framework, named ARFIMAX-TARCH model, is applied for CAC40 and DAX30 stock indices. In-sample as well as out-of-sample analysis provide statistically significant evidence in favour of the new model specification. Thus, in risk management applications, the volatility's conditional volatility should be taken into consideration.

The manuscript is divided in six sections. In section 2, descriptive information of the CAC40 and DAX30 realized volatility measures is provided. Section 3 lays out the ARFIMAX and ARFIMAX-TARCH specifications. The in-sample and out-of-sample model evaluation is investigated in sections 4 and 5, respectively, and section 6 concludes.

2. CAC40 and DAX30 Realized Volatility Properties

Tick by tick linearly interpolated prices of the CAC40 and DAX30 indices were obtained from Olsen and Associates for the period of July 1995 to December 2003. The sampling frequency should be as high as the market microstructure features do not induce bias to volatility estimator. In order to avoid market microstructure frictions without lessening the accuracy of the continuous record asymptotics, in most of the studies, such as Andersen and Bollerslev (1998),

² See chapter 1.5 in Karatzas and Shreve (1988) and Barndorff-Nielsen and Shephard (2005).

Andersen et al. (1999, 2001b) and Kayahan et al. (2002), a sampling frequency of five minutes is used.

The realized intraday volatility at day t is computed as in Martens (2002) and Koopman et al. (2005):

$$\sigma_t^{2(RV)} = \frac{\hat{\sigma}_{oc}^2 + \hat{\sigma}_{co}^2}{\hat{\sigma}_{oc}^2} \sum_{j=1}^{m-1} \left(\log(P_{(j+1/m),t}) - \log(P_{(j/m),t}) \right)^2, \quad (3)$$

where $\{P_{(m),t}\}_{j=1}^m$ are the five-minute linearly interpolated prices at trading day t with $m = 103$ ($m = 134$) observations per day for the CAC40 (DAX30) index, $\hat{\sigma}_{oc}^2 = T^{-1} \sum_{t=1}^T \left(\log(P_{(1),t}) - \log(P_{(1/m),t}) \right)^2$ is the open to close sample variance and $\hat{\sigma}_{co}^2 = T^{-1} \sum_{t=1}^T \left(\log(P_{(1/m),t}) - \log(P_{(1),t-1}) \right)^2$ is the close to open sample variance. The factor $\sigma_{oc}^{-2}(\sigma_{oc}^2 + \sigma_{co}^2)$ accounts for changes in the asset prices during the hours that the stock market is closed without inserting the noisy effect of daily returns.

[Insert Table 1 about here]

Table 1 lists descriptive information of the logarithmic realized volatility, $\log(\sigma_t^{2(RV)})$. Lilliefors' (1967) and Anderson and Darling's (1954) statistics reject the null hypothesis that the logarithmic realized variance is normally distributed. Literature provides empirical evidence that the distribution of the daily returns, $y_t = \log(P_t/P_{t-1})$, standardized by the square root of the realized volatility, $\{y_t/\sigma_t^{(RV)}\}_{t=1}^T$, is very close to the normal one, but, it is statistically distinguishable from the normal distribution³. The histograms of $\{y_t/\sigma_t^{(RV)}\}_{t=1}^T$ are plotted in Figure 1. According to Lilliefors and Anderson-Darling normality tests in Table 3, $\{y_t/\sigma_t^{(RV)}\}_{t=1}^T$ is normally distributed, in the case of DAX30, at 1% level of significance. In the case of CAC40, the null hypothesis, that the empirical distribution of $\{y_t/\sigma_t^{(RV)}\}_{t=1}^T$ is the normal one, is rejected at any reasonable level of significance.

Figure 2 presents the square root of realized volatility. On the right-hand axis, the daily index prices are plotted to present the asymmetric relationship between index log-returns and changes in realized volatility.

[Insert Figure 1 about here]

³ See for example Andersen et al. (2001a).

[Insert Figure 2 about here]

3. ARFIMAX and ARFIMAX-TARCH Intra-Day Volatility Models

An ARFIMAX(k, d, l) model for the realized volatility, $\sigma_t^{2(RV)}$, can be presented as:

$$\begin{aligned} (1 - c(L))(1 - L)^d (\log(\sigma_t^{2(RV)}) - x_t' \beta) &= (1 + \delta(L)) \varepsilon_t, \\ \varepsilon_t &\sim N(0, \sigma_\varepsilon^2), \end{aligned} \quad (4)$$

where $c(L) = \sum_{i=1}^k c_i L^i$, $\delta(L) = \sum_{i=1}^l \delta_i L^i$, L is the backward-shift operator,

$$(1 - L)^d = \sum_{j=0}^{\infty} \frac{\Gamma(j - d) L^j}{\Gamma(j + 1) \Gamma(-d)} = \sum_{j=0}^{\infty} L^j \prod_{k=0}^j \frac{k - 1 - d}{k},$$

$\Gamma(\cdot)$ is the Gamma function, x_t is a vector of explanatory variables and β is a vector of unknown parameters.

As, we want to investigate whether the volatility of realized volatility exhibits time-variation, the ARFIMAX model is extended to account for conditional heteroskedasticity⁴. An ARFIMAX model with time varying conditional variance for the realized volatility, $\sigma_t^{2(RV)}$, can be presented as:

$$\begin{aligned} (1 - c(L))(1 - L)^d (\log(\sigma_t^{2(RV)}) - x_t' \beta) &= (1 + \delta(L)) \varepsilon_t \\ \varepsilon_t &= \sigma_t z_t \\ \sigma_t^2 &= g(\theta | I_{t-1}) \\ z_t &\sim N(0, 1), \end{aligned} \quad (5)$$

where I_{t-1} is the information set available in time $t - 1$, σ_t^2 is a time-varying, positive and measurable function of I_{t-1} , θ is a vector of unknown parameters and $g(\cdot)$ is a functional form of I_{t-1} . In our case, an asymmetric conditional variance specification is considered to account for asymmetric relationship between realized volatility and its volatility. Therefore an ARFIMAX(k, d, l)-TARCH(p, q) model for the realized volatility is proposed:

$$\begin{aligned} (1 - c(L))(1 - L)^d (\log(\sigma_t^{2(RV)}) - x_t' \beta) &= (1 + \delta(L)) \varepsilon_t, \\ \varepsilon_t &= \sigma_t z_t, \\ \sigma_t^2 &= a_0 + \sum_{i=1}^q a_i \varepsilon_{t-i}^2 + \sum_{i=1}^q (\gamma_i I(\varepsilon_{t-i} < 0) \varepsilon_{t-i}^2) + \sum_{i=1}^p b_i \sigma_{t-i}^2, \end{aligned} \quad (6)$$

⁴ Baillie et al. (1996) firstly proposed an ARFIMAX-ARCH model to analyze monthly consumer price index inflation. Hauser and Kunst (1998) also applied ARFIMAX-ARCH model to analyze monthly Swiss one-month Euromarket interest rates.

where $z_t \sim N(0,1)$. $I(\cdot)$ denotes the indicator function, i.e. $I(\varepsilon_{t-i} < 0) = 1$ if $\varepsilon_{t-i} < 0$, and $I(\varepsilon_{t-i} < 0) = 0$, otherwise. The Threshold-ARCH(p, q), or TARCH(p, q), specification, introduced by Glosten et al. (1993), allows positive, ($\varepsilon_{t-i} > 0$), and negative, ($\varepsilon_{t-i} < 0$), innovations to have differential effects on the volatility of the realized volatility.

4. In-sample Evaluation

The ARFIMAX(k, d, l) model is considered in the form:

$$(1 - c(L))(1 - L)^d (\log(\sigma_t^{2(RV)}) - w_0 - w_1 y_{t-1} - w_2 I(y_{t-1} > 0) y_{t-1}) = (1 + \delta(L)) \varepsilon_t, \quad (7)$$

$$\varepsilon_t \sim N(0, \sigma_\varepsilon^2),$$

where $I(y_t > 0) = 1$ when $y_t > 0$ and $I(y_t > 0) = 0$ otherwise. Parameter w_2 models the asymmetric relationship between realized volatility and previous trading day's log-return⁵. Since, $\varepsilon_t \sim N(0, \sigma_\varepsilon^2)$, the $\exp(\varepsilon_t)$ is log-normally distributed, therefore the unbiased in-sample realized volatility is estimated as:

$$\hat{\sigma}_t^{2(RV^*)} = \exp(\log \hat{\sigma}_t^{2(RV)} + 0.5 \sigma_\varepsilon^2). \quad (8)$$

In order to determine the optimal lag order, model (7) is estimated for $k = 0, 1, 2$ and $l = 0, 1, 2$. The optimal lag order is chosen due to the minimization of Schwarz's (1978) Bayesian criterion (SBC). The SBC , for model i , is computed as:

$$SBC^{(i)} = -2T^{-1} L_T(\{y_t\}; \hat{\psi}^{(T)}) + \tilde{\psi} T^{-1} \log(T), \quad (9)$$

where $\psi = (c_1, \dots, c_k, d, w_0, w_1, w_2, \delta_1, \dots, \delta_l)'$, $L_T(\cdot)$ is the maximized value of the log-likelihood function, $\hat{\psi}^{(T)}$ is the maximum likelihood estimator of ψ based on a sample of size T and $\tilde{\psi}$ denotes the dimension of ψ . The optimal lag orders are ARFIMAX($2, d, 2$) and ARFIMAX($0, d, 1$) for CAC40 and DAX30, respectively. The models are estimated in Doornik and Ooms' (2006) ARFIMA 1.04 package of Ox Metrics.

The ARFIMAX(k, d, l)-TARCH(p, q) model is also considered in the form:

$$(1 - c(L))(1 - L)^d (\log(\sigma_t^{2(RV)}) - w_0 - w_1 y_{t-1} - w_2 I(y_{t-1} > 0) y_{t-1}) = (1 + \delta(L)) \varepsilon_t,$$

$$\varepsilon_t = \sigma_t z_t, \quad (10)$$

$$\sigma_t^2 = a_0 + \sum_{i=1}^q a_i \varepsilon_{t-i}^2 + \sum_{i=1}^q (\gamma_i I(\varepsilon_{t-i} < 0) \varepsilon_{t-i}^2) + \sum_{i=1}^p b_i \sigma_{t-i}^2,$$

⁵ As presented in Figure 2.

$$z_t \sim N(0,1).$$

Model (10) is estimated for $k = 0,1,2$, $l = 0,1,2$, $p = 0,1,2$ and $q = 1,2$. The *SBC* is computed according to equation (9) for $\psi = (c_1, \dots, c_k, d, w_0, w_1, w_2, \delta_1, \dots, \delta_l, a_0, a_1, \dots, a_q, \gamma_1, \dots, \gamma_q, b_1, \dots, b_p)'$. The ARFIMAX(1, d ,1)-TARCH(1,1) model achieves the minimum value of the *SBC* criterion for both indices. The unbiased in-sample realized volatility is also estimated according to (8). The models are estimated in Laurent and Peters' (2006) G@RCH 4.04 package for Ox Metrics.

The estimated parameters of the models that minimize the *SBC* criterion are reported in Table 2. The values of the parameters inform us that i) the asymmetric relation between past return and realized volatility is statistically significant in all cases (coefficient w_2), ii) the fractional integration parameter is statistically insignificant only for Paris stock market in the ARFIMAX(2, d ,2) model (coefficient d), iii) all the conditional variance parameters are statistically significant (coefficients a_1 and b_1), and iv) the asymmetric relationship between realized volatility and its volatility is also statistically significant (coefficient γ_1). The significance of the conditional variance parameters justifies the modeling of realized volatility's conditional variance⁶.

[Insert Table 2 about here]

Figure 3 plots the histograms of logarithmic realized volatility, $\{\log(\sigma_t^{2(RV)})\}_{t=1}^T$ and logarithmic realized volatility standardized by its standard deviation estimated by the ARFIMAX(1, d ,1)-TARCH(1,1) model, $\{\log(\sigma_t^{2(RV)})/\hat{\sigma}_t\}_{t=1}^T$. The $\{\log(\sigma_t^{2(RV)})/\hat{\sigma}_t\}_{t=1}^T$ is much closer to the normal distribution than the $\{\log(\sigma_t^{2(RV)})\}_{t=1}^T$ is, but it is statistically distinguishable from it.

Figure 4 depicts the density graph of daily returns standardized by the in-sample estimated realized standard deviation, $\{y_t/\hat{\sigma}_t^{(RV^*)}\}_{t=1}^T$, whereas normality tests are listed in Table 3. Based on Lilliefors statistic, the hypothesis that $\{y_t/\hat{\sigma}_t^{(RV^*)}\}_{t=1}^T$ is normally distributed can not be rejected only in the case of DAX30 at 1% level of significance. Anderson-Darling statistic rejects the normality in any case.

[Insert Figure 3 about here]

⁶ The standardized residuals obey the assumption of autocorrelation absence.

[Insert Figure 4 about here]

[Insert Table 3 about here]

The average squared distance between realized volatility and its in-sample estimation is measured to evaluate the accuracy of the models in estimating the realized volatility:

$$MSE = T^{-1} \sum_{t=1}^T \left(\sigma_t^{2(RV)} - \hat{\sigma}_t^{2(RV^*)} \right)^2. \quad (11)$$

Hansen and Lunde (2006) have stated that the MSE loss function ensures the equivalence of the ranking of volatility models that is induced by the true volatility and its proxy. The values of the MSE loss functions are reported in Table 4. For both indices, the ARFIMAX(1, d , 1)-TARCH(1, 1) model has the lowest value.

Hansen's (2005) superior predictive ability (SPA) hypothesis testing is used to investigate whether the model with the lowest MSE value provides statistically superior realized volatility estimates. Let i be the model with the lowest MSE value, $MSE_t^{(i^*)}$ is the value of the MSE function at time t of a competing model i^* , for $i^* = 1, \dots, M$ and $X_t^{(i, i^*)} = MSE_t^{(i)} - MSE_t^{(i^*)}$. The null hypothesis that $E(X_t^{(i, 1)}, \dots, X_t^{(i, M)})' \leq 0$ is tested with the statistic

$$T^{SPA} = \max_{i^*=1, \dots, M} \frac{\sqrt{M} \bar{X}_{i^*}}{\sqrt{Var(\sqrt{M} \bar{X}_{i^*})}}, \text{ where } \bar{X}_{i^*} = T^{-1} \sum_{t=1}^T X_t^{(i, i^*)}.^7$$

According to Table 4, the ARFIMAX-TARCH specification is superior to the ARFIMAX one. The null hypothesis, that the ARFIMAX(1, d , 1)-TARCH(1, 1) model is not outperformed by its competing model, is not rejected.

[Insert Table 4 about here]

5. Out-of-sample Evaluation

In the present section, the ability of the ARFIMAX(k, d, l) and ARFIMAX(k, d, l)-TARCH(p, q) models to predict next trading day's volatility is investigated. In total, for $k = 0, 1, 2$, $l = 0, 1, 2$, $p = 0, 1, 2$ and $q = 1, 2$ lag orders, 56 model specifications are considered. Based on a rolling sample of $\tilde{T} = 1000$ trading days, each model's parameter vector ψ is re-

⁷ The estimation of $Var(\sqrt{M} \bar{X}_{i^*})$ and the p-value of the T^{SPA} are obtained by using the bootstrap method of Politis and Romano (1994). Hansen (2005) provided a program for the computation of the SPA criterion, which is written in Ox Metrics package.

estimated every trading day and \tilde{T} one-day-ahead volatility forecasts are computed⁸, for $\tilde{T} = T - \tilde{T}$. The one-day-ahead conditional standard deviation forecasts are computed as:

$$\sigma_{t+1|t}^{(RV^*)} = \sqrt{\exp(\log(\sigma_{t+1|t}^{2(RV)}) + 0.5\sigma_{\varepsilon}^{2(r)})}. \quad (12)$$

The distance between realized volatility and next day's predicted volatility is measured by the predicted mean squared error loss function:

$$PMSE = \tilde{T}^{-1} \sum_{t=1}^{\tilde{T}} \left(\sigma_{t+1}^{2(RV)} - \sigma_{t+1|t}^{2(RV^*)} \right)^2, \quad (13)$$

and the SPA hypothesis test is applied. In the ARFIMAX(k, d, l) specification, the $PMSE$ is minimized for $k = 1$ and $l = 0$ in the CAC40 case, and for $k = l = 2$ in the case of DAX30. In the ARFIMAX(k, d, l)-TARCH(p, q) framework, the $PMSE$ is minimized for $k = 1$, $l = 2$, $p = 2$, $q = 1$ in the CAC40 case, and for $k = p = q = 1$, $l = 0$ in the case of DAX30. Table 5 presents the p -values of the SPA test for the null hypothesis that the ARFIMAX-TARCH model is not outperformed by the ARFIMAX one. In the case of the DAX30 the ARFIMAX($1, d, 0$)-TARCH($1, 1$) model is superior to the ARFIMAX($2, d, 2$) one. As concerns the CAC40 index, the ARFIMAX($1, d, 2$)-TARCH($2, 1$) model does not achieve the lowest value in the $PMSE$ loss function, but it is not statistically inferior to its competitor, the ARFIMAX($1, d, 0$) model.

[Insert Table 5 about here]

Figure 5 presents the returns scaled by the one-day-ahead realized standard deviation forecasts, $\{y_t / \sigma_{t|t-1}^{(RV^*)}\}_{t=1}^{\tilde{T}}$. According to Table 3, the null hypothesis, that the daily returns standardized by the one-day-ahead realized standard deviation forecasts are normally distributed, is not rejected only for the ARFIMAX($2, d, 2$) model in the DAX30 case. Giot and Laurent (2004) noticed that for the CAC40⁹ and SP500 indices, the $\{y_t / \sigma_t^{(RV)}\}_{t=1}^T$ are normally distributed, but the $\{y_t / \sigma_{t|t-1}^{(RV^*)}\}_{t=1}^T$ are not normally distributed. In the present dataset both series $\{y_t / \sigma_t^{(RV)}\}_{t=1}^T$ and $\{y_t / \sigma_{t|t-1}^{(RV^*)}\}_{t=1}^T$ are not normally distributed but they have an almost normal distribution as in Andersen et al. (2001a).

[Insert Figure 5 about here]

⁸ $\tilde{T} = 1135$ for CAC40 and $\tilde{T} = 1136$ for DAX30.

⁹ CAC40 was analyzed for the period 1995-1999 and fifteen-minute intraday prices were taken into account.

6. Conclusion

ARFIMAX and ARFIMAX-TARCH models were applied in estimating and forecasting the intra-day realized volatility of the CAC40 and DAX30 indices for the period of July 1995 to December 2003. ARFIMAX-TARCH model takes into consideration the dynamics of realized volatility's volatility. The integrated volatility, $\sigma_t^{2(IV)}$, is estimated by the realized volatility, $\sigma_t^{2(RV)}$, in equation (3), whereas the conditional variance of the logarithmic realized volatility, $\hat{\sigma}_t^2$, in equation (10), can be regarded as an estimation of integrated quarticity, $\sigma_t^{2(IQ)}$. The significance of the parameters of the TARCH specification provides evidence in favor of modeling the asymmetric relationship between realized volatility and its volatility.

The in-sample evaluation indicates that the ARFIMAX-TARCH specification clearly outperforms the ARFIMAX one. In the out-of-sample evaluation, the ARFIMAX-TARCH model is superior to the ARFIMAX one for the DAX30. In the case of the CAC40 index, the ARFIMAX-TARCH model is not statistically inferior to its competitor.

To sum up, from econometric point of view, the daily returns standardized by i) the realized standard deviation, ii) the in-sample estimated realized standard deviation, and iii) the one-day-ahead realized standard deviation forecasts are almost normally distributed but they are statistically distinguishable from the normal distribution. The logarithmic realized volatility standardized by its standard deviation is much closer to the normal distribution than the logarithmic realized volatility but also statistically distinguishable from it.

From economic point of view, in order to obtain more accurate stock index volatility estimations, it is necessary to treat ARFIMAX models for realized volatility, and ARCH modeling for volatility of realized volatility, simultaneously. Thus, when the volatility estimation is required in financial applications, such as risk management, option pricing and portfolio analysis, the time varying conditional heteroscedasticity of volatility should be taken into consideration.

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Figures & Tables

Figure 1. Histograms of daily log-returns standardized by the realized standard deviation, or

$$\{y_t / \sigma_t^{(RV)}\}_{t=1}^T, \text{ where } y_t = \log(P_t / P_{t-1}) \text{ and } \sigma_t^{(RV)} = \sqrt{\frac{\hat{\sigma}_{oc}^2 + \hat{\sigma}_{co}^2}{\hat{\sigma}_{oc}^2} \sum_{j=1}^{m-1} (\log(P_{(j+1/m),t}) - \log(P_{(j/m),t}))^2}.$$

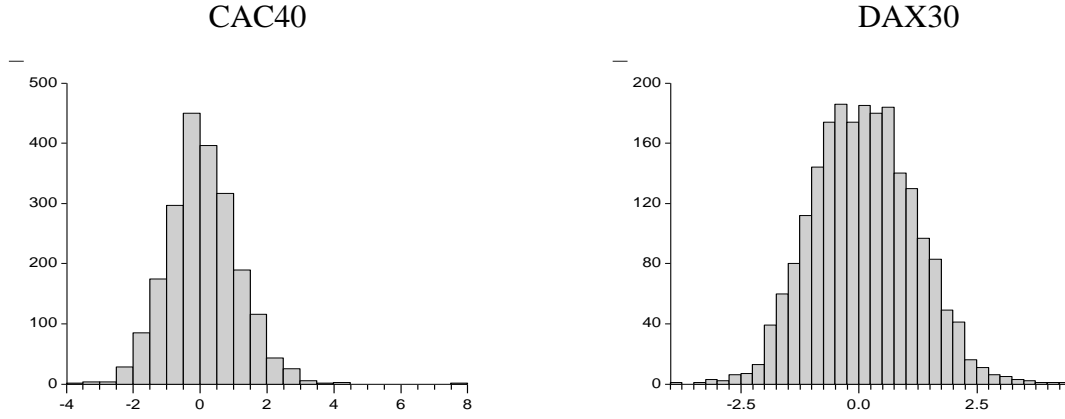


Figure 2. Realized standard deviation, $\{\sigma_t^{(RV)}\}_{t=1}^T$, and index daily prices, $\{P_t\}_{t=1}^T$.

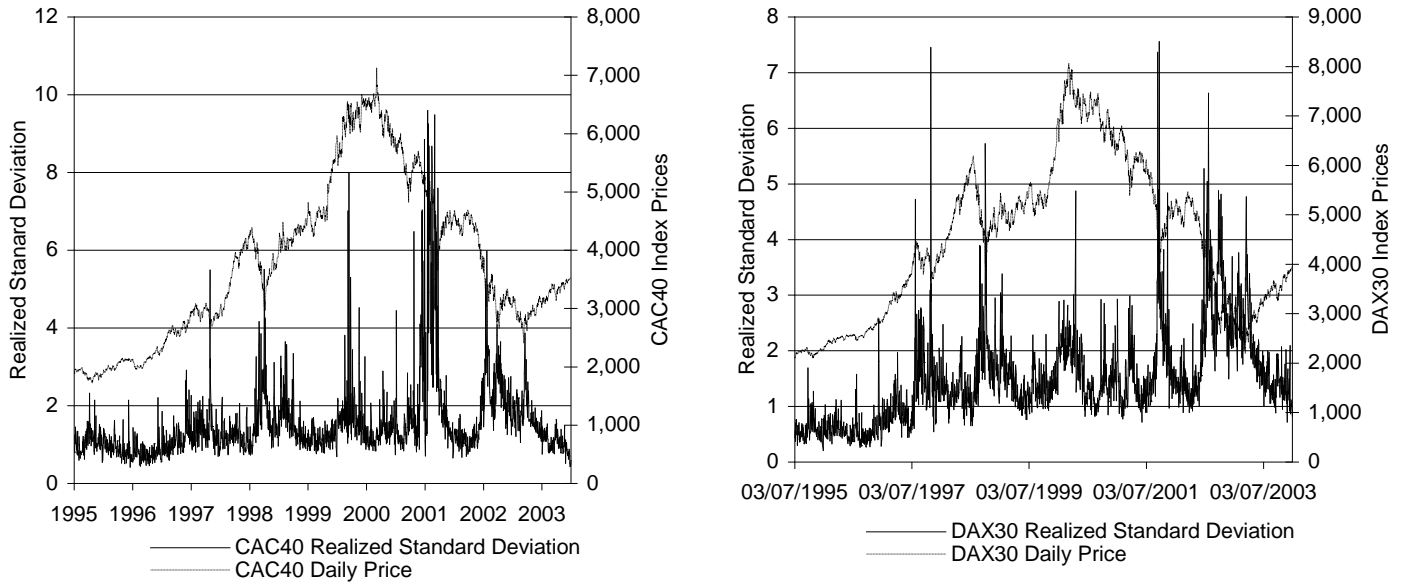


Figure 3. Histogram of logarithmic realized volatility, $\{\log(\sigma_t^{2(RV)})\}_{t=1}^T$, and of logarithmic realized volatility standardized by its standard deviation, $\{\log(\sigma_t^{2(RV)})/\hat{\sigma}_t\}_{t=1}^T$.

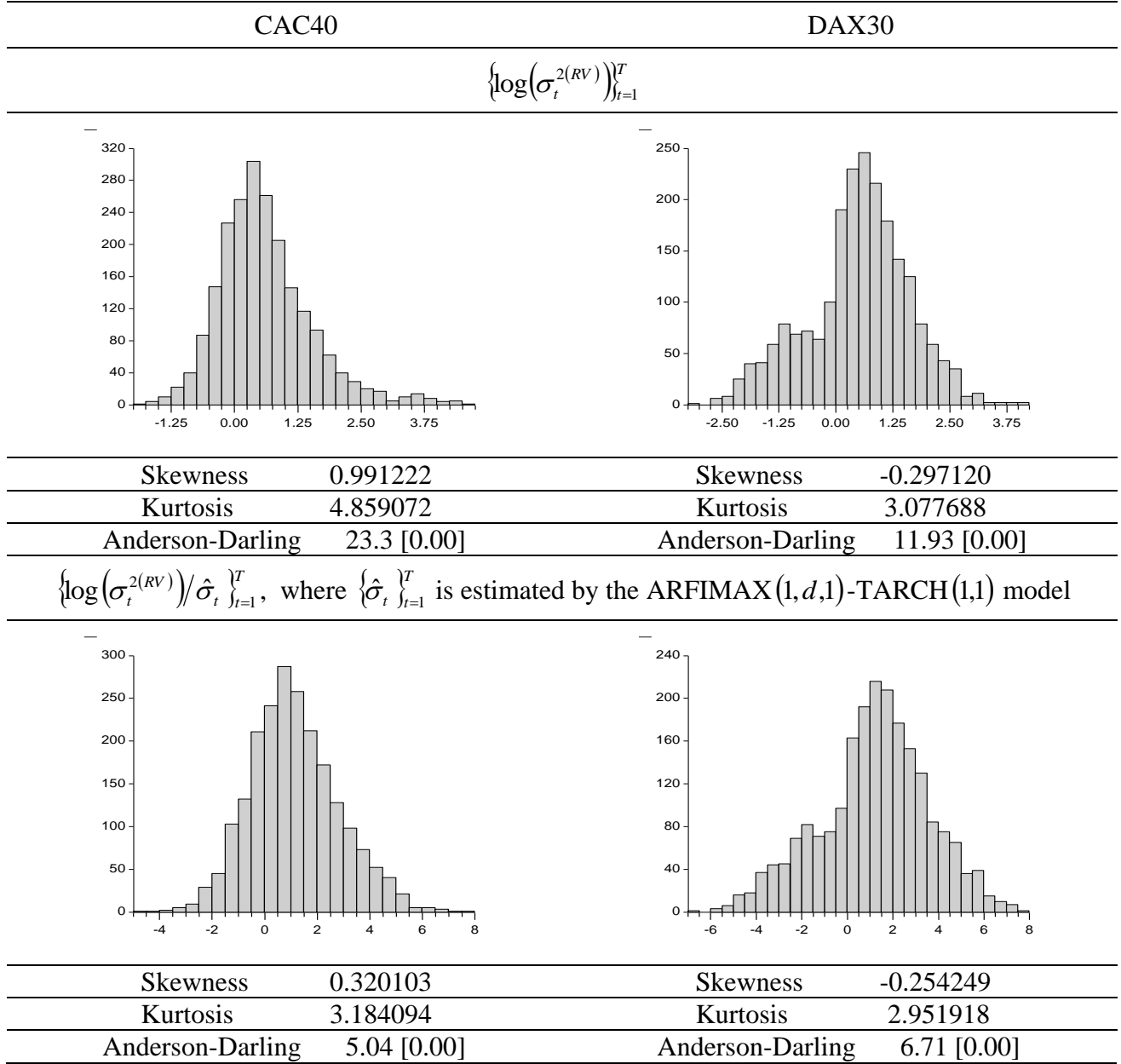


Figure 4. Histogram of daily log-returns standardized by the in-sample estimated realized standard deviation, $\{y_t / \hat{\sigma}_t^{(RV^*)}\}_{t=1}^T$.

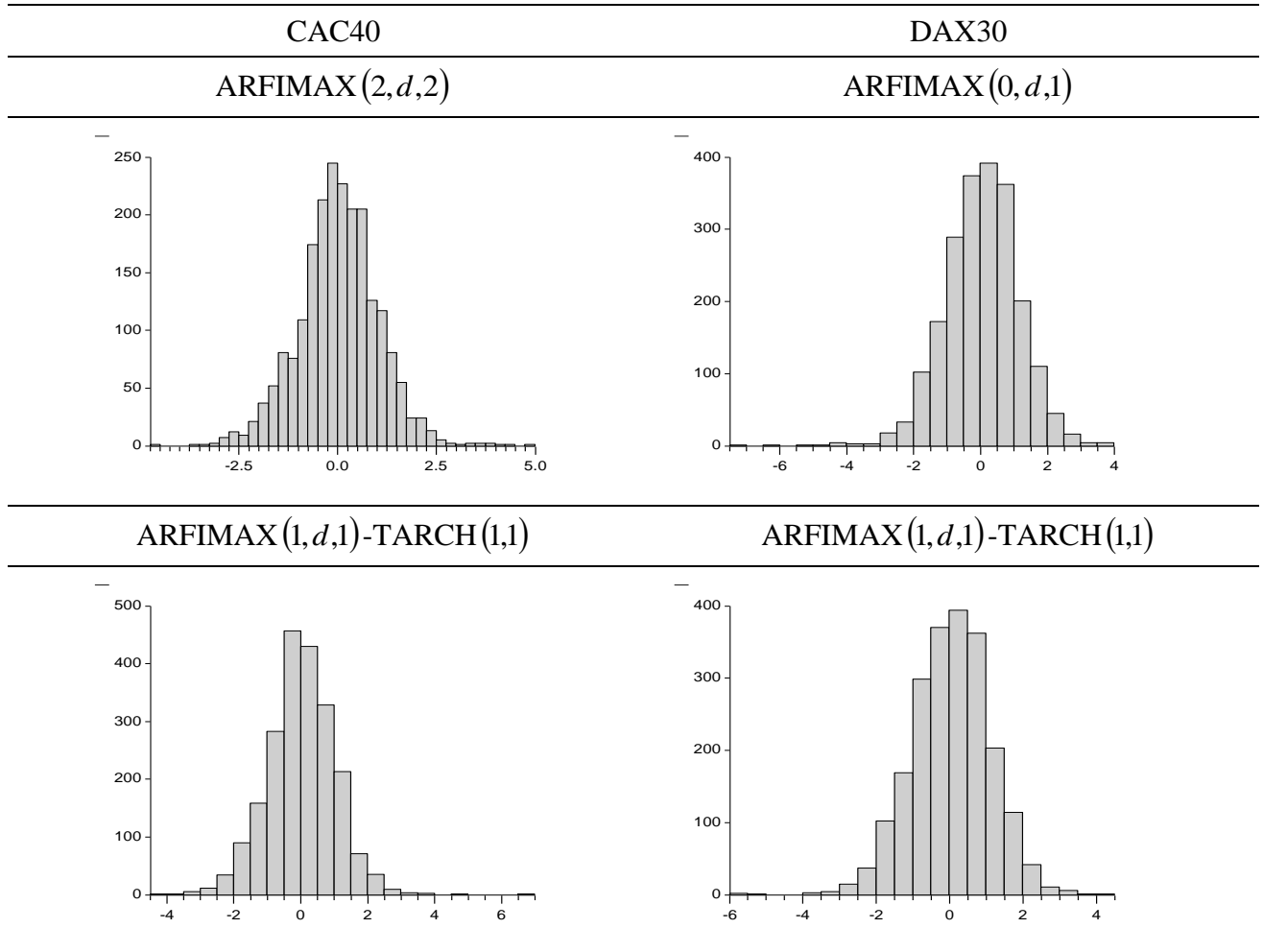


Figure 5. Histogram of daily log-returns standardized by the one-day-ahead realized standard deviation, $\{y_t / \sigma_{t|t-1}^{(RV^*)}\}_{t=1}^{\tilde{T}}$.

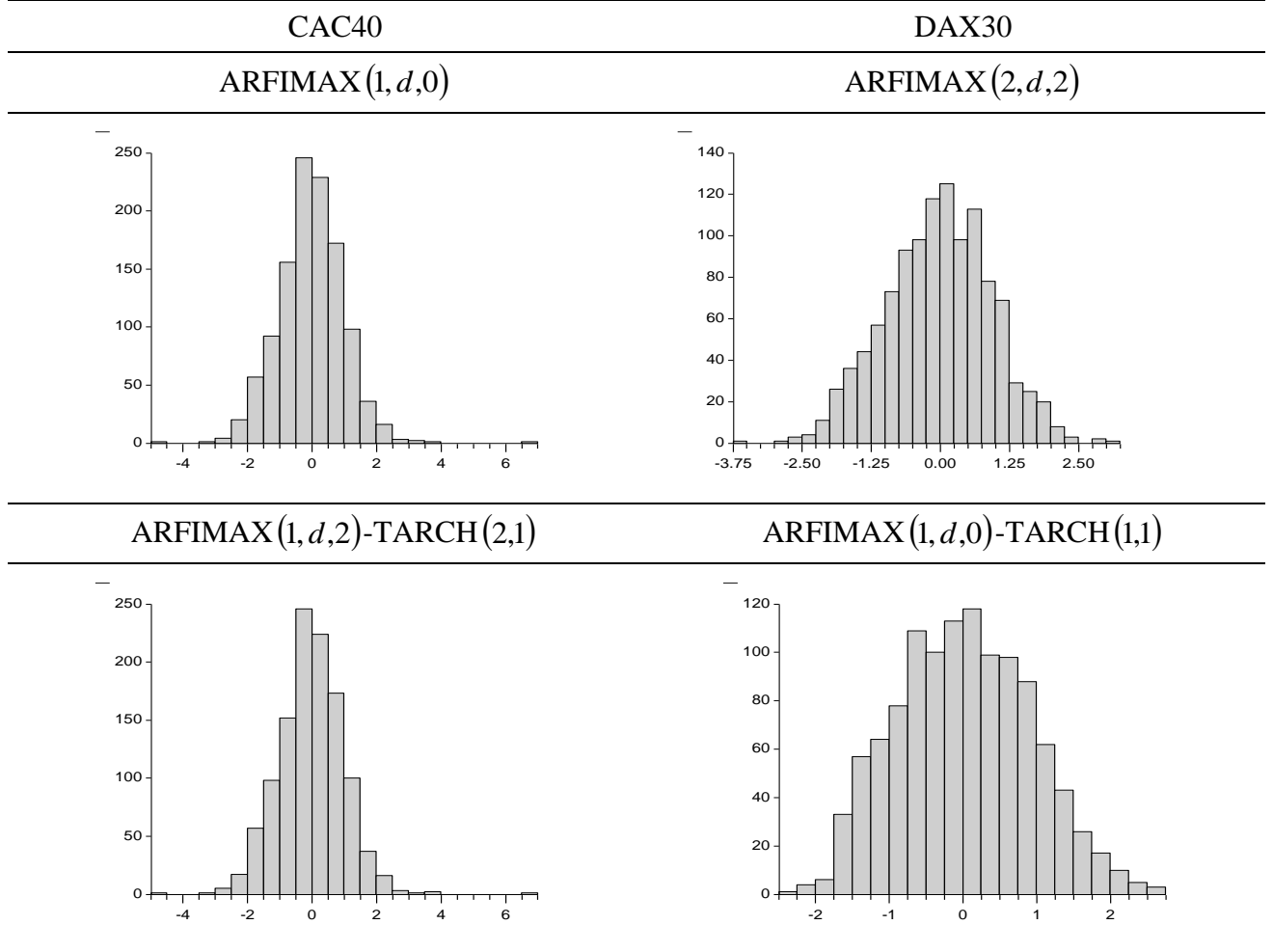


Table 1. Descriptive statistics of the logarithmic realized volatility, $\log(\sigma_t^{2(RV)})$.

	CAC40		DAX30	
Mean	0.602		0.498	
Median	0.479		0.567	
Maximum	4.523		4.046	
Minimum	-1.751		-3.171	
Std. Dev.	0.923		1.117	
Skewness	0.9912		-0.2971	
Kurtosis	4.8590		3.0776	
Lilliefors	0.077	[0.00]	0.067	[0.00]
Anderson-Darling	23.3	[0.00]	11.93	[0.00]

P-values are displayed in squared brackets.

Table 2. Estimated parameters of the ARFIMAX(k, d, l)-TARCH(p, q) model:

$$(1 - c(L))(1 - L)^d (\log(\sigma_t^{2(RV)}) - w_0 - w_1 y_{t-1} - w_2 I(y_{t-1} > 0) y_{t-1}) = (1 + \delta(L)) \varepsilon_t$$

$$\sigma_t^2 = a_0 + \sum_{i=1}^q a_i \varepsilon_{t-i}^2 + \sum_{i=1}^q (\gamma_i I(\varepsilon_{t-i} < 0) \varepsilon_{t-i}^2) + \sum_{i=1}^p b_i \sigma_{t-i}^2.$$

	CAC40		DAX30	
	ARFIMAX(2, d , 2)	ARFIMAX(1, d , 1)- TARCH(1,1)	ARFIMAX(0, d , 1)	ARFIMAX(1, d , 1)- TARCH(1,1)
c_1	1.39680 ^a (0.3218)	0.199197 ^a (0.050252)	-	0.194364 ^a (0.045510)
c_2	-0.412986 (0.3041)	-	-	-
d	0.258789 (0.3203)	0.755740 ^a (0.068972)	0.497103 ^a (0.004017)	0.782399 ^a (0.061348)
w_0	0.416191 (0.3620)	0.040664 (0.25369)	0.198627 (2.975)	-1.331032 ^a (0.48103)
w_1	0.0229716 (0.01194)	0.021021 ^b (0.010327)	0.0350528 ^a (0.01099)	0.021232 ^b (0.0096266)
w_2	-0.118749 ^a (0.02080)	-0.104875 ^a (0.020196)	-0.128690 ^a (0.01885)	-0.098282 ^a (0.018332)
δ_1	-1.35808 ^a (0.1985)	-0.640628 ^a (0.071377)	-0.138449 ^a (0.02359)	-0.635783 ^a (0.059226)
δ_2	0.429437 ^b (0.1737)	-	-	-
a_0	-	0.005054 ^b (0.0020817)	-	0.010667 ^a (0.0039589)
a_1	-	0.058264 ^a (0.014591)	-	0.116056 ^a (0.034983)
γ_1	-	-0.054705 ^a (0.018628)	-	-0.077160 ^b (0.032481)
b_1	-	0.945498 ^a (0.017546)	-	0.872181 ^a (0.036273)
SBC	1.526597	1.411473	1.374893	1.291441

Standard errors are reported in parentheses.

^a Indicates that the coefficient is statistically significant at 1% level of significance.

^b Indicates that the coefficient is statistically significant at 5% level of significance.

Table 3. Lilliefors and Anderson-Darling statistics for the hypothesis that the daily returns standardized by i) the realized standard deviation, $\{y_t/\sigma_t^{(RV)}\}_{t=1}^T$, ii) the in-sample estimated realized standard deviation, $\{y_t/\hat{\sigma}_t^{(RV^*)}\}_{t=1}^T$, and iii) the one-day-ahead realized standard deviation forecast, $\{y_t/\sigma_{t|t-1}^{(RV^*)}\}_{t=1}^T$, are normally distributed.

CAC40		DAX30		
$\left\{y_t / \sigma_t^{(RV)}\right\}_{t=1}^T$				
Lilliefors	0.028 [0.00]	0.020 [0.042]		
Anderson-Darling	2.115 [0.00]	0.944 [0.014]		
$\left\{y_t / \hat{\sigma}_t^{(RV^*)}\right\}_{t=1}^T$				
	ARFIMAX(2,d,2)	ARFIMAX(1,d,1)- TARCH(1,1)	ARFIMAX(0,d,1)	ARFIMAX(1,d,1)- TARCH(1,1)
Lilliefors	0.032 [0.00]	0.030 [0.00]	0.020 [0.045]	0.020 [0.041]
Anderson-Darling	3.166 [0.00]	2.695 [0.00]	2.018 [0.00]	1.381 [0.001]
$\left\{y_t / \sigma_{t t-1}^{(RV^*)}\right\}_{t=1}^{\tilde{T}}$				
	ARFIMAX(1,d,0)	ARFIMAX(1,d,2)- TARCH(2,1)	ARFIMAX(2,d,2)	ARFIMAX(1,d,0)- TARCH(1,1)
Lilliefors	0.031 [0.016]	0.032 [0.011]	0.022 [>0.10]	0.029 [0.027]
Anderson-Darling	1.410 [0.001]	1.297[0.002]	0.595 [0.12]	1.413 [0.001]
P-values are displayed in squared brackets.				

Table 4. *MSE* loss functions and the p-value of the SPA test for the null hypothesis that the ARFIMAX(1,d,1)-TARCH(1,1) model provides the best in-sample realized volatility estimation.

Index	ARFIMAX(2,d,2)	ARFIMAX(1,d,1)- TARCH(1,1)	SPA p-value
CAC40	32.569	28.032	[0.53]
	ARFIMAX(0,d,1)	ARFIMAX(1,d,1)- TARCH(1,1)	SPA p-value
DAX30	9.054	7.329	[0.53]

Table 5. *PMSE* loss functions and the p-value of the SPA test for the null hypothesis that the ARFIMAX-TARCH model is not outperformed by the ARFIMAX one.

Index	ARFIMAX(1, d , 0)	ARFIMAX(1, d , 2)- TARCH(2, 1)	SPA p-value
CAC40	45.438	50.838	[0.054]
	ARFIMAX(2, d , 2)	ARFIMAX(1, d , 0)- TARCH(1, 1)	SPA p-value
DAX30	9.769	6.412	[0.55]