Pricing an Option written on an Option

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Abstract. Compound options are options for which the underlying is another option. In other words, a compound option is an option written on an option. In this paper, we present two new approaches to compound option pricing. The first approach relies on Malliavin calculus methods and the Clark-Ocone formula, while the second is based on Fourier transform techniques. We finally apply our theoretical results to several practical examples.

Keywords: compound option; option pricing; stochastic volatility; stochastic differential equation; Malliavin calculus; Clark-Ocone formula; Fourier transform

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1 Introduction

Let us start by giving a brief introduction to compound option pricing. Suppose that we are given a European option with underlying stock price process $S = (S_t)_{t\in[0,T]}$ and payoff $F = f(S_T)$ at maturity $T > 0$. Here, $f: \mathbb{R}^+ \longrightarrow \mathbb{R}$ constitutes a deterministic payoff function. For the sake of notational simplicity, we assume that the interest rate equals zero. Then, in accordance to the risk-neutral pricing theory, the option price P at any time $t \leq T$ prior to maturity is given by

$$P_t = \mathbb{E}_{\mathbb{Q}}(f(S_T)|\mathcal{F}_t) \tag{1.1}$$

where $\mathbb{Q}$ indicates a risk-neutral pricing measure, and $\mathcal{F} = (\mathcal{F}_t)_{t\in[0,T]}$ is a filtration modeling the information flow in the market. It is important to note that we interpret the option price $P = (P_t)_{t\in[0,T]}$ as a time-continuous stochastic process. Now, suppose that there is another option with price process $\Pi = (\Pi_t)_{t\in[0,\tau]}$ available in the market (a so-called compound option) whose underlying is the process P. More precisely, let us assume that the payoff of the compound option is given by $G = g(P_\tau)$ at maturity $\tau < T$. Here, G constitutes an $\mathcal{F}_\tau$-measurable random variable, $g: \mathbb{R} \longrightarrow \mathbb{R}$ is a deterministic payoff function, and the underlying price process P satisfies (1.1). Hence, compound options are options for which the underlying is another option. In other words, a compound option is an option written on an option. In accordance to the risk-neutral pricing theory, the price Π of the compound option at any time $t \in [0,\tau]$ prior to maturity is given by

$$\Pi_t = \mathbb{E}_{\mathbb{Q}}(g(P_\tau)|\mathcal{F}_t)\,. \tag{1.2}$$

As it holds $\tau < T$ by assumption, the initial option Π (the so-called overlying) matures previously to the second option P (called the underlying or mother option). The compound option Π is also called split-fee option, since its exercise payoff involves the value of another option, namely P_τ.

Compound options are frequently used in corporate finance and also in currency or fixed income markets. Typical examples of compound options are call on call (CoC), call on put (CoP), put on call (PoC), or put on put (PoP) options. It is the purpose of the present paper to obtain representations for the option price processes $P = (P_t)$ and $\Pi = (\Pi_t)$ introduced in (1.1) and (1.2), respectively.

In what follows, we give a brief overview on related literature. Compound options have initially been introduced by Geske [8] in the context of evaluating plain-vanilla options written on a share of common stocks. In [8] it is shown that the deduced compound option price formula contains the Black-Scholes formula as a special case. In [10] the theory of compound options is applied to the pricing of American options in jump-diffusion models. The author derives analytical valuation formulas and applies the results to extendible options. Based on Fourier transform techniques, in [9] the evaluation of European compound options in a stochastic volatility model is investigated. Moreover, in [11] the author infers closed-form formulas for the prices of compound call options written on power options and currency call options. In [13] two alternative approaches to derive the value of a compound option are presented, while in [18] the evaluation of compound options is accomplished by a summation of a series of multi-normal distribution functions. In [19] the pricing of compound and extendible options under a mixed fractional Brownian motion model with jumps is investigated. The author also provides related numerical simulation results. To get more information on option pricing techniques (not necessarily in a compound option context) we refer to [3], [6], [7], Sections 9.1 and 11.1.3 in [4], and Section 2.7 in [12].

The paper is organized as follows. In Section 2 we formulate the financial market model and introduce a measure change from the physical to a risk-neutral pricing measure. We also provide the corresponding conditions ensuring the arbitrage-freeness of the market. In Section 3 we derive the time dynamics of the underlying option price process by two different methods: once by an application of Malliavin calculus and the Clark-Ocone formula in Section 3.1, and once by an application of Fourier transform techniques in Section 3.2. Moreover, Section 4 is devoted to the pricing of the overlying compound option. In Section 5 we apply the theoretical results obtained in the previous sections to several practical examples.

2 Formulation of the financial market model

In this section, we present the mathematical formulation of the underlying financial market model. Let $\left(\Omega, \mathfrak{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P}\right)$ be a filtered probability space satisfying the usual hypotheses, i.e. $\mathfrak{F}$ denotes a sigma-algebra augmented by all $\mathbb{P}$-null sets, and $\mathcal{F} = (\mathcal{F}_t)_{t\in[0,T]}$ with $\mathcal{F}_T \subset \mathfrak{F}$ constitutes a complete and right-continuous filtration. Here, $\mathbb{P}$ is the physical probability measure, and $T > 0$ denotes a finite time horizon. We assume that $\mathcal{F}_0$ is trivial. In this setup, for all $t \in [0,T]$ we introduce a risk-less bank account β which evolves according to the ordinary differential equation

$$d\beta(t) = r(t)\,\beta(t)\,dt \tag{2.1}$$

with initial value $\beta(0) > 0$ and deterministic short rate $r: [0,T] \longrightarrow \mathbb{R}$. The solution of (2.1) reads as

$$\beta(t) = \beta(0)\exp\left\{\int_0^t r(s)\,ds\right\}$$

whereas Itô's formula yields

$$d\beta^{-1}(t) = -r(t)\,\beta^{-1}(t)\,dt \tag{2.2}$$

for all $t \in [0,T]$. Let us further introduce a stock price process $S = (S_t)_{t\in[0,T]}$ (a risky asset) satisfying the stochastic differential equation (SDE)

(2.3) $$dS_t = S_t\,[\mu(t)\,dt + \xi(t)\,dW_t]$$

where the initial value $S_0 > 0$ is constant, $\mu: [0,T] \longrightarrow \mathbb{R}$ constitutes a deterministic and continuous drift function, $\xi: [0,T] \longrightarrow \mathbb{R}^+ :=]0,\infty[$ is a deterministic and continuous volatility function, and the process $W = (W_t)_{t\in[0,T]}$ denotes a one-dimensional $\mathcal{F}$-adapted standard Brownian motion (BM) under $\mathbb{P}$. In this setting, we assume that

$$\int_0^T \{|r(t)| + |\mu(t)| + \xi^2(t)\}\,dt < \infty$$

and define the filtration $\mathcal{F} = (\mathcal{F}_t)$ for all $t \in [0,T]$ via

(2.4) $$\mathcal{F}_t := \mathcal{F}_t^W := \sigma\{W_s : 0 \le s \le t\}\,.$$

In the following, we introduce a probability measure change to an equivalent risk-neutral martingale measure $\mathbb{Q}$. To this end, for all $t \in [0,T]$ we define the (strictly positive) Radon-Nikodym density process

(2.5)

$$Z_t := \frac{d\mathbb{Q}}{d\mathbb{P}}\bigg|_{\mathcal{F}_t} := \mathcal{E}\left(\int_0^{\cdot} \gamma(s)\,dW_s\right)_t := \exp\left\{\int_0^t \gamma(s)\,dW_s - \frac{1}{2}\int_0^t \gamma^2(s)\,ds\right\}$$

where $Z_0 = 1$ and $\gamma: [0,T] \longrightarrow \mathbb{R}$ is a deterministic and continuous $\mathcal{L}^2$-function which is called market price of risk. Taking Itô's formula into account, we deduce the SDE

$$dZ_t = Z_t\,\gamma(t)\,dW_t$$

such that $Z = (Z_t)_{t\in[0,T]}$ constitutes an $\mathcal{F}$-adapted local $\mathbb{P}$-martingale. Since it holds $\gamma(\cdot) \in \mathcal{L}^2[0,T]$ by assumption, the corresponding Novikov condition is fulfilled (cf. Theorem 2.43 in [16]). Hence, we observe $\mathbb{E}_{\mathbb{P}}[Z_t] \equiv 1\ \forall t \in [0,T]$ such that the strictly positive Doléans-Dade exponential Z constitutes a true $\mathbb{P}$-martingale. Here, $\mathbb{E}_{\mathbb{P}}$ denotes the expectation operator under $\mathbb{P}$. Applying Girsanov's theorem (cf. Theorem 12.21 in [5]), we state that $\mathbb{Q}$ forms a probability measure on $\mathcal{F}$ which is equivalent to $\mathbb{P}$ and that the stochastic process

(2.6)

$$\left(B_t := W_t - \int_0^t \gamma(s)\,ds\right)_{t\in[0,T]}$$

constitutes an $\mathcal{F}$-adapted standard Brownian motion under $\mathbb{Q}$. In the next step, we introduce the discounted stock price process $\tilde{S} = (\tilde{S}_t)$ which is defined via $\tilde{S}_t := \beta^{-1}(t)\,S_t$ for all $t \in [0,T]$. Using integration by parts, (2.2), (2.3), and (2.6), we obtain the SDE

$$d\tilde{S}_t = \tilde{S}_t\,[\{\mu(t) - r(t) + \xi(t)\,\gamma(t)\}\,dt + \xi(t)\,dB_t]$$

under $\mathbb{Q}$. Due to no-arbitrage theory, the discounted stock price $\tilde{S}$ must form a martingale under the risk-neutral probability measure $\mathbb{Q}$ (cf. Proposition 9.2 in [4], Section 2.1.3 in [12], Sections 1.1.2 and 3.1 in [16]). For this reason, we require the drift restriction

(2.7) $$0 = \mu(t) - r(t) + \xi(t)\,\gamma(t)$$

which implies the $\mathbb{Q}$-martingale dynamics

$$d\tilde{S}_t = \tilde{S}_t\,\xi(t)\,dB_t$$

for all $t \in [0, T]$. Moreover, a combination of (2.3), (2.6) and (2.7) yields the SDE

$$dS_t = S_t \, [r(t) \, dt + \xi(t) \, dB_t]$$

under $\mathbb{Q}$. So far, we have pursued a standard approach which is well-known from the Black-Scholes theory and its extensions to additive process setups. For the sake of simplicity, we assume from now on that the short rate equals zero, i.e. $r(t) \equiv 0$ for all $t \in [0, T]$, which – by the way – is not an unrealistic assumption with view on the current low-interest-rate market environment. Consequently, we obtain for all $t \in [0, T]$ the geometric SDE under $\mathbb{Q}$

$$dS_t = S_t \, \xi(t) \, dB_t$$

which possesses the strictly positive solution

$$S_t = S_0 \, e^{X_t} \tag{2.8}$$

with a real-valued and normally distributed *additive process* (sometimes called *Lévy-type process*)

$$X_t := \int_0^t \xi(s) \, dB_s - \frac{1}{2} \int_0^t \xi^2(s) \, ds \tag{2.9}$$

admitting independent but non-stationary increments under $\mathbb{Q}$ due to the time-dependency of the involved volatility function. See [17], Section 14.1 in [4], or Section 9.1 in [5] to read more on additive processes. We further recall that (2.7) is equivalent to

$$\gamma(t) = -\frac{\mu(t)}{\xi(t)} \tag{2.10}$$

such that the equivalent martingale measure $\mathbb{Q}$ is uniquely determined through (2.5) and (2.10). Eventually note that for all $t \in [0, T]$ it holds

$$\mathcal{F}_t^B := \sigma\{B_s : 0 \le s \le t\} = \sigma\{W_s : 0 \le s \le t\} = \mathcal{F}_t^W = \mathcal{F}_t \tag{2.11}$$

since the market price of risk $\gamma(\cdot)$ in (2.6), respectively in (2.10), is deterministic.

3 The time dynamics of the underlying option price process

Let us consider a European option written on the stock price process S with payoff $F = f(S_T)$ at maturity $T > 0$. Here, $F \in \mathcal{L}^2(\mathbb{P}) \cap \mathcal{L}^2(\mathbb{Q})$ constitutes an $\mathcal{F}_T$-measurable random variable, $f: \mathbb{R}^+ \longrightarrow \mathbb{R}$ is a deterministic payoff function, and the underlying stock price process $S = (S_t)_{t \in [0,T]}$ satisfies (2.8)-(2.9). In accordance to the risk-neutral pricing theory (cf. Eq. (9.12) in [4]), the option price P at any time $t \in [0, T]$ prior to maturity is given by

$$P_t = \frac{\beta(t)}{\beta(T)} \, \mathbb{E}_{\mathbb{Q}}(F | \mathcal{F}_t) = \exp\left\{ - \int_t^T r(s) \, ds \right\} \, \mathbb{E}_{\mathbb{Q}}(f(S_T) | \mathcal{F}_t)$$

where β denotes the bank account introduced in (2.1), and $\mathbb{Q}$ indicates the risk-neutral pricing measure defined by (2.5). As we assumed in Section 2 that the short rate equals zero, we find

$$P_t = \mathbb{E}_{\mathbb{Q}}(F | \mathcal{F}_t) = \mathbb{E}_{\mathbb{Q}}(f(S_T) | \mathcal{F}_t) \tag{3.1}$$

for all $t \in [0, T]$. With view on our upcoming investigations, it is important to note that we consider the option price $P = (P_t)_{t \in [0,T]}$ as a time-continuous stochastic process. In this context, we interpret the option as a 'traded asset' into which any market participant can invest. We further recall that the discounted option price process $\tilde{P}_t := \beta^{-1}(t)\, P_t = P_t$ constitutes a $\mathbb{Q}$-martingale, as required by no-arbitrage theory. In the following, we derive the time dynamics of the option price process (P_t) by two different methods: once by an application of Malliavin calculus in Section 3.1, and once by an application of Fourier transform techniques in Section 3.2.

3.1 Option pricing with Malliavin calculus methods

First of all, we recall the following well-known results.

Theorem 3.1 (Clark-Ocone formula; Theorems 4.1 and 13.28 in [5]**).** *Let $F \in \mathcal{L}^2(\mathbb{P})$ be an $\mathcal{F}_T$-measurable random variable. Then (in the setup introduced in Section 2) F can be represented as*

$$F = \mathbb{E}_{\mathbb{P}}[F] + \int_0^T \mathbb{E}_{\mathbb{P}}(\mathcal{D}_s F | \mathcal{F}_s)\, dW_s$$

where $\mathcal{D}_s F$ denotes the Malliavin derivative of F at time $s \in [0, T]$. (See Section 3 in [5] to read more on the Malliavin derivative and its properties.)

Proposition 3.2. *Let $F \in \mathcal{L}^2(\mathbb{P}) \cap \mathcal{L}^2(\mathbb{Q})$ be an $\mathcal{F}_T$-measurable random variable and $\mathbb{Q}$ be the risk-neutral probability measure introduced in (2.5). Moreover suppose that $Z_T F \in \mathcal{L}^2(\mathbb{P})$ as well as*

$$\mathbb{E}_{\mathbb{Q}}[|F|] < \infty\,, \qquad \mathbb{E}_{\mathbb{Q}}\left[\int_0^T (\mathcal{D}_s F)^2\, ds\right] < \infty\,.$$

Then, under $\mathbb{Q}$, the random variable F can be represented as

(3.2)

$$F = \mathbb{E}_{\mathbb{Q}}[F] + \int_0^T \mathbb{E}_{\mathbb{Q}}(\mathcal{D}_s F | \mathcal{F}_s^B)\, dB_s = \mathbb{E}_{\mathbb{Q}}[F] + \int_0^T \mathbb{E}_{\mathbb{Q}}(\mathcal{D}_s F | \mathcal{F}_s)\, dB_s\,.$$

Proof. Recall that $\gamma(\cdot)$ in (2.6) is deterministic, which ensures that (2.11) is in force in our setup. Then the result is a direct consequence of Theorems 4.5 and 6.41 in [5]. ■

To read more on the applicability of the Clark-Ocone formula under probability measure changes, we refer to Section 4.2 in [5] and, in particular, to Remark 4.6 in [5]. Also recall Theorem 6.41 in [5] in this context. For more information on Malliavin calculus and its applications, we refer − besides [5] − to [1], [14], and [15]. We proceed with computing the conditional expectation appearing in (3.1).

Proposition 3.3. *Let $F = f(S_T)$ be an $\mathcal{F}_T$-measurable random variable and denote by $\mathcal{D}_s F$ the Malliavin derivative of F at time $s \in [0, T]$. Then the option price process P in (3.1) can for all $t \in [0, T]$ be represented as*

(3.3)

$$P_t = P_0 + \int_0^t \Sigma_s\, dB_s$$

with deterministic initial value $P_0 = \mathbb{E}_{\mathbb{Q}}[F]$ and $\mathcal{F}$-adapted stochastic volatility process

$$\Sigma_s := \mathbb{E}_{\mathbb{Q}}(\mathcal{D}_s F | \mathcal{F}_s)\,. \qquad (3.4)$$

Proof. We substitute (3.2) into (3.1) and hereafter exploit the $(\mathcal{F}, \mathbb{Q})$-martingale property of the appearing Brownian integral. This procedure directly leads us to the assertion. ■

Remark 3.4. From the integral representation (3.3) we immediately deduce the $(\mathcal{F}, \mathbb{Q})$-martingale dynamics

$$dP_t = \Sigma_t \, dB_t \tag{3.5}$$

with terminal value $P_T = F$ and initial value $P_0 = \mathbb{E}_{\mathbb{Q}}[F]$. Thus, (3.5) actually constitutes a forward-backward SDE (FBSDE). Further note that the option price process $P = (P_t)_{t\in[0,T]}$ follows an arithmetic modeling approach with stochastic volatility.

In the next step, we aim at providing a more explicit expression for the volatility process Σ defined in (3.4). To this end, we take (2.8)-(2.9) along with the chain rule for Malliavin derivatives (see Theorem 3.5 in [5]) into account and obtain for all $t \in [0, T]$

$$\mathcal{D}_t(S_T) = S_0 \, \mathcal{D}_t(e^{X_T}) = S_T \, \mathcal{D}_t(X_T) = S_T \, \xi(t) \, . \tag{3.6}$$

In the remainder of Section 3.1, we assume that the payoff function $f(\cdot)$ appearing in (3.1) is continuously differentiable, i.e. $f \in \mathcal{C}^1(\mathbb{R}^+)$. A renewed application of the chain rule then yields

$$\mathcal{D}_t F = \mathcal{D}_t f(S_T) = f'(S_T) \, \mathcal{D}_t(S_T) = f'(S_T) \, S_T \, \xi(t) \tag{3.7}$$

for all $t \in [0, T]$. Hence, the volatility process $(\Sigma_t)_{t\in[0,T]}$ defined in (3.4) can be expressed as

$$\Sigma_t = \xi(t) \, \mathbb{E}_{\mathbb{Q}}(S_T \, f'(S_T) | \mathcal{F}_t) \tag{3.8}$$

such that the integral equation (3.3) can be rewritten as

$$P_t = P_0 + \int_0^t \xi(s) \, \mathbb{E}_{\mathbb{Q}}(S_T \, f'(S_T) | \mathcal{F}_s) \, dB_s \tag{3.9}$$

for all $t \in [0, T]$.

3.2 Option pricing with Fourier transform methods

In this section, we derive the time dynamics of the option price process P by an alternative method involving Fourier transform techniques. To begin with, we recall that the Fourier transform and the inverse Fourier transform of a deterministic function $\theta: \mathbb{R} \longrightarrow \mathbb{R}$ with $\theta \in \mathcal{L}^1(\mathbb{R})$ are given by, respectively,

$$\hat{\theta}(y) = (2\pi)^{-1} \int_{\mathbb{R}} \theta(x) \, e^{-iyx} \, dx \; , \qquad \theta(x) = \int_{\mathbb{R}} \hat{\theta}(y) \, e^{iyx} \, dy \tag{3.10}$$

where $i^2 = -1$. We proceed with computing the conditional expectation appearing in (3.1). Taking (2.8) into account, we get

$$P_t = \mathbb{E}_{\mathbb{Q}}(f(S_T) | \mathcal{F}_t) = \mathbb{E}_{\mathbb{Q}}\big(f\big(\delta(X_T)\big) \big| \mathcal{F}_t\big) = \mathbb{E}_{\mathbb{Q}}(q(X_T) | \mathcal{F}_t) \tag{3.11}$$

where we introduced the deterministic functions

$$\delta(x) \coloneqq S_0 \, e^x \, , \qquad q(x) \coloneqq (f \circ \delta)(x) \, .$$

In the remainder of Section 3.2, we assume that the payoff function $f(\cdot)$ is such that $q(\cdot) \in \mathcal{L}^1(\mathbb{R})$. Then we can apply the inverse Fourier transform on (3.11) which yields

(3.12)

$$P_t = \int_{\mathbb{R}} \hat{q}(y)\, \mathbb{E}_{\mathbb{Q}}(e^{iyX_T} | \mathcal{F}_t)\, dy$$

where the Fourier transform of $q(\cdot)$ is given by

(3.13)

$$\hat{q}(y) = (2\pi)^{-1} \int_{\mathbb{R}} f(S_0\, e^x)\, e^{-iyx}\, dx\,.$$

With respect to the independent increment property of the additive process X defined in (2.9), we obtain

$$\mathbb{E}_{\mathbb{Q}}(e^{iyX_T} | \mathcal{F}_t) = e^{iyX_t}\, \mathbb{E}_{\mathbb{Q}}\left(e^{iy[X_T - X_t]} \middle| \mathcal{F}_t\right) = e^{iyX_t}\, \mathbb{E}_{\mathbb{Q}}\left[e^{iy(X_T - X_t)}\right]$$

where the resulting usual expectation can be computed to

$$\mathbb{E}_{\mathbb{Q}}\left[e^{iy(X_T - X_t)}\right] = \exp\left\{\frac{iy\,(iy-1)}{2} \int_t^T \xi^2(s)\, ds\right\}.$$

Merging the two latter equations into (3.12), we arrive at the representation

(3.14)

$$P_t = \int_{\mathbb{R}} \hat{q}(y)\, e^{iyX_t + \lambda(t,T,y)}\, dy$$

where we introduced the complex-valued deterministic function

(3.15)

$$\lambda(t,T,y) \coloneqq \frac{iy\,(iy-1)}{2} \int_t^T \xi^2(s)\, ds$$

for notational convenience. We recall that option pricing with inverse Fourier transforms has initially been proposed in [3]. Also see Section 11.1.3 in [4], Section 1.4 in [6], [7], or [9] to read more on this topic. (Option pricing and hedging with inverse Laplace transforms is investigated in Section 6.2 in [16].) We summarize our recent findings in the subsequent proposition.

Proposition 3.5. *Let $f(\cdot)$ be a deterministic payoff function, $(P_t)_{t\in[0,T]}$ be the option price process introduced in (3.1), and $(S_t)_{t\in[0,T]}$ be the stock price process given by (2.8). Suppose that $q(x) = f(S_0\, e^x)$ is a deterministic $\mathcal{L}^1$-function with Fourier transform $\hat{q}(\cdot)$ given in (3.13). Then the option price P_t at any time $t \in [0,T]$ prior to maturity can be represented as*

(3.16)

$$P_t = \int_{\mathbb{R}} \hat{q}(y)\, e^{A_t(y)}\, dy$$

with a complex-valued $\mathcal{F}$-adapted stochastic process $A = \left(A_t(\cdot)\right)_{t\in[0,T]}$ defined for all $y \in \mathbb{R}$ via

(3.17) $$A_t(y) \coloneqq iy\, X_t + \lambda(t,T,y)$$

where X and λ are such as claimed in (2.9) and (3.15), respectively.

In the following, we derive the time dynamics of the option price process P given in (3.16). To this end, we take (3.17), (2.9) and (3.15) into account and infer the (complex-valued) $\mathbb{Q}$-dynamics

(3.18)

$$dA_t(y) = \frac{y^2}{2}\xi^2(t)\,dt + iy\,\xi(t)\,dB_t = iy\,\xi(t)\,dB_t - \frac{1}{2}i^2y^2\,\xi^2(t)\,dt$$

which implies the geometric SDE

(3.19)
$$d\left(e^{A_t(y)}\right) = e^{A_t(y)}\,iy\,\xi(t)\,dB_t$$

due to Itô's formula. Note that (3.18) possesses the structure of an exponent of a Doléans exponential, such that the process $\left(e^{A_t(y)}\right)_{t\in[0,T]}$ in (3.19) constitutes an $(\mathcal{F},\mathbb{Q})$-martingale for every $y \in \mathbb{R}$. (As it holds $\xi \in \mathcal{L}^2[0,T]$ by assumption, the corresponding Novikov condition is fulfilled.) Hence, also the option price process $(P_t)_{t\in[0,T]}$ in (3.16) constitutes an $(\mathcal{F},\mathbb{Q})$-martingale which stands in accordance to its definition in (3.1). Furthermore, a combination of (3.16) and (3.19) leads us to the $(\mathcal{F},\mathbb{Q})$-martingale dynamics

(3.20)
$$dP_t = \Sigma_t\,dB_t$$

with a complex-valued $\mathcal{F}$-adapted stochastic volatility process

(3.21)

$$\Sigma_t = \xi(t)\int_{\mathbb{R}} \hat{q}(y)\,e^{A_t(y)}\,iy\,dy\,.$$

Note that the deterministic initial value of the FBSDE (3.20) reads as

$$P_0 = \mathbb{E}_{\mathbb{Q}}[F] = \mathbb{E}_{\mathbb{Q}}[f(S_T)] = \int_{\mathbb{R}} \hat{q}(y)\,e^{\lambda(0,T,y)}\,dy$$

whereas the terminal value is given by $P_T = F = f(S_T)$. Not surprisingly, the option price dynamics found in (3.5) and (3.20) coincide. Nevertheless, we obtained different representations for the involved stochastic volatility process Σ in (3.8) and (3.21), respectively. By the way, equaling the representations given in (3.8) and (3.21) yields for all $t \in [0,T]$ the equality

$$\mathbb{E}_{\mathbb{Q}}(S_T\,f'(S_T)|\mathcal{F}_t) = \int_{\mathbb{R}} \hat{q}(y)\,e^{A_t(y)}\,iy\,dy$$

due to the uniqueness of the coefficients in Lévy-Itô decompositions (cf. e.g. [4], [12], [17]). Summing up, we conclude that also in the current section dedicated to Fourier pricing techniques, the option price process (P_t) again satisfies (3.3), while in (3.21) we found an alternative representation for the involved stochastic volatility coefficient (Σ_t). We finally take (3.17) and (2.8) into account and deduce the equality

(3.22)
$$e^{A_t(y)} = e^{\lambda(t,T,y)}\,(S_t/S_0)^{iy}$$

whereas (3.18) implies the integral representation

(3.23)

$$A_t(y) = \lambda(0,T,y) + \frac{y^2}{2}\int_0^t \xi^2(s)\,ds + iy\int_0^t \xi(s)\,dB_s$$

being valid for all $t \in [0,T]$ and $y \in \mathbb{R}$.

4 Pricing compound options

In this section, we investigate the evaluation of another option (a so-called compound option) whose underlying is the option price P introduced in Section 3. More precisely, we recently consider another European option written on the option price process P with payoff $G = g(P_\tau)$ at maturity $\tau > 0$. Here, $G \in \mathcal{L}^2(\mathbb{P}) \cap \mathcal{L}^2(\mathbb{Q})$ constitutes an $\mathcal{F}_\tau$-measurable random variable, $g: \mathbb{R} \longrightarrow \mathbb{R}$ is a deterministic payoff function, and the underlying price process $P = (P_t)_{t\in[0,T]}$ satisfies (3.9), respectively (3.20)-(3.21). In accordance to the risk-neutral pricing theory, the price Π_t of the compound option at any time $t \in [0,\tau]$ prior to its maturity is given by

$$\Pi_t = \mathbb{E}_{\mathbb{Q}}(G|\mathcal{F}_t) = \mathbb{E}_{\mathbb{Q}}(g(P_\tau)|\mathcal{F}_t) \tag{4.1}$$

where $\mathbb{Q}$ indicates the risk-neutral pricing measure introduced in (2.5). From now on, we assume that $\tau < T$ which appears reasonable from an economical perspective. Hence, the initial option Π (the so-called overlying) matures previously to the second option P (called the underlying). The compound option Π is also called split-fee option, as its exercise payoff involves the value of another option, namely P_τ. Numerous concrete examples of compound options will be presented in Section 5. In what follows, we aim at deriving more explicit representations for the option price process $\Pi = (\Pi_t)_{t\in[0,\tau]}$ introduced in (4.1) by two different methods: once by an application of Malliavin calculus in Section 4.1, and once by an application of Fourier transform techniques in Section 4.2.

4.1 Compound option pricing with Malliavin calculus methods

In accordance to Proposition 3.2, we presently assume that $Z_\tau G \in \mathcal{L}^2(\mathbb{P})$ as well as

$$\mathbb{E}_{\mathbb{Q}}[|G|] < \infty\,, \qquad \mathbb{E}_{\mathbb{Q}}\left[\int_0^\tau (\mathcal{D}_s G)^2\, ds\right] < \infty\,,$$

where $Z = (Z_t)$ indicates the Radon-Nikodym density process defined in (2.5). Then, we can represent the payoff G by the Clark-Ocone formula leading us to

$$G = \mathbb{E}_{\mathbb{Q}}[G] + \int_0^\tau \mathbb{E}_{\mathbb{Q}}(\mathcal{D}_s G|\mathcal{F}_s)\, dB_s\,.$$

Merging the latter equation into (4.1), we obtain

$$\Pi_t = \mathbb{E}_{\mathbb{Q}}[G] + \int_0^t \mathbb{E}_{\mathbb{Q}}(\mathcal{D}_s G|\mathcal{F}_s)\, dB_s \tag{4.2}$$

for all $0 \le t \le \tau < T$. Hence, the option price process Π satisfies the $(\mathcal{F},\mathbb{Q})$-martingale dynamics

$$d\Pi_t = \mathbb{E}_{\mathbb{Q}}(\mathcal{D}_t G|\mathcal{F}_t)\, dB_t \tag{4.3}$$

with deterministic initial value $\Pi_0 = \mathbb{E}_{\mathbb{Q}}[G]$ and $\mathcal{F}_\tau$-measurable terminal value $\Pi_\tau = G$. In the remainder of Section 4.1, we assume that the payoff function g appearing in (4.1) is continuously differentiable, i.e. $g \in \mathcal{C}^1(\mathbb{R})$. Then, we can apply Theorem 3.5 in [5] which yields

$$\mathcal{D}_t G = \mathcal{D}_t g(P_\tau) = g'(P_\tau)\, \mathcal{D}_t(P_\tau) \tag{4.4}$$

for all $t \in [0,\tau]$. The remaining challenge now consists in finding an adequate way to compute the appearing Malliavin derivative $\mathcal{D}_t(P_\tau)$ where the process P satisfies (3.9), respectively (3.20)-(3.21).

1.) Let us first elaborate how far we can get, if we use the representation (3.9). In this case, we apply Corollary 3.19 in [5] and therewith obtain

(4.5)

$$\mathcal{D}_t(P_\tau) = \mathcal{D}_t\left(\int_0^\tau \Sigma_s \, dB_s\right) = \Sigma_t + \int_t^\tau \mathcal{D}_t(\Sigma_s)\, dB_s$$

where the volatility process is given by

$$\Sigma_s = \xi(s)\, \mathbb{E}_{\mathbb{Q}}(S_T\, f'(S_T)|\mathcal{F}_s)$$

due to (3.8). Unfortunately, it seems to be impossible to compute the involved Malliavin derivative

$$\mathcal{D}_t(\Sigma_s) = \xi(s)\, \mathcal{D}_t\left(\mathbb{E}_{\mathbb{Q}}(S_T\, f'(S_T)|\mathcal{F}_s)\right)$$

analytically. Numerical computation methods are needed in order to cope with the appearing expression $\mathcal{D}_t\left(\mathbb{E}_{\mathbb{Q}}(S_T\, f'(S_T)|\mathcal{F}_s)\right)$ for all $0 \le t \le s \le \tau < T$.

2.) Let us now investigate what happens, if we use the representation (3.20)-(3.21). Also in this case, we apply Corollary 3.19 in [5] which leads us to (4.5) again, but now with a volatility process

$$\Sigma_s = i\, \xi(s) \int_{\mathbb{R}} \hat{q}(y)\, e^{A_s(y)}\, y\, dy$$

due to (3.21). Hence, with respect to (4.5), we have to compute the Malliavin derivative

$$\mathcal{D}_t(\Sigma_s) = i\, \xi(s) \int_{\mathbb{R}} \hat{q}(y)\, \mathcal{D}_t\big(e^{A_s(y)}\big)\, y\, dy$$

where for all $0 \le t \le s \le \tau < T$ it holds

$$\mathcal{D}_t\big(e^{A_s(y)}\big) = e^{A_s(y)}\, \mathcal{D}_t\big(A_s(y)\big) = iy\, e^{A_s(y)}\, \mathcal{D}_t(X_s) = iy\, e^{A_s(y)}\, \xi(t)$$

due to the chain rule, (3.17) and (2.9). As a consequence, we obtain

$$\mathcal{D}_t(\Sigma_s) = -\xi(t)\, \xi(s) \int_{\mathbb{R}} \hat{q}(y)\, e^{A_s(y)}\, y^2\, dy$$

which implies

$$\mathcal{D}_t(P_\tau) = \Sigma_t - \xi(t) \int_t^\tau \xi(s) \int_{\mathbb{R}} \hat{q}(y)\, e^{A_s(y)}\, y^2\, dy\, dB_s$$

in accordance to (4.5). Combining the latter equation with (4.3) and (4.4), we arrive at the sophisticated option price dynamics

$$d\Pi_t = \mathbb{E}_{\mathbb{Q}}\left(g'(P_\tau)\left\{\Sigma_t - \xi(t) \int_t^\tau \xi(s) \int_{\mathbb{R}} \hat{q}(y)\, e^{A_s(y)}\, y^2\, dy\, dB_s\right\}\middle|\mathcal{F}_t\right) dB_t$$

where $0 \le t \le \tau < T$. Unfortunately, it appears impossible to compute the involved conditional expectation analytically. Thus, numerical methods are needed in order to compute/approximate the emerging conditional expectation and to simulate related trajectories of the option price process Π.

With view on the recent equations, we conclude that Malliavin calculus methods do not seem to be useful in order to deduce an analytical formula for the option price Π_t introduced in (4.1).

4.2 Compound option pricing with Fourier transform techniques

We now apply Fourier transform techniques to the pricing formula (4.1). To this end, we assume in the remainder of Section 4.2 that the involved payoff function $g(\cdot)$ satisfies $g \in \mathcal{L}^1(\mathbb{R})$. Then we can apply the inverse Fourier transform on (4.1) which yields for all $0 \le t \le \tau < T$

(4.6)

$$\Pi_t = \int_{\mathbb{R}} \hat{g}(y)\, e^{iyP_t}\, \mathbb{E}_{\mathbb{Q}}\big(e^{iy[P_\tau - P_t]} \big| \mathcal{F}_t\big)\, dy$$

where the Fourier transform of g reads as

$$\hat{g}(y) = (2\pi)^{-1} \int_{\mathbb{R}} g(x)\, e^{-iyx}\, dx$$

due to (3.10). In the derivation of (4.6) we used that the process P is $\mathcal{F}$-adapted. The remaining challenge now consists in the computation of the conditional expectation appearing in (4.6). First of all, we use (3.5), respectively (3.20), and obtain

(4.7)

$$\mathbb{E}_{\mathbb{Q}}\big(e^{iy[P_\tau - P_t]} \big| \mathcal{F}_t\big) = \mathbb{E}_{\mathbb{Q}}\left(\exp\left\{iy \int_t^\tau \Sigma_s\, dB_s\right\} \middle| \mathcal{F}_t\right)$$

where $0 \le t \le \tau < T$. At this step, we recall that the involved volatility coefficient $\Sigma = (\Sigma_t)$ is stochastic such that we are facing a stochastic volatility option pricing problem presently. In the sequel, we make use of a similar method as applied in the proof of Proposition 1 provided in Appendix A of [20]. That is, we condition on the stochastic volatility process and get

(4.8)

$$\mathbb{E}_{\mathbb{Q}}\big(e^{iy[P_\tau - P_t]} \big| \mathcal{F}_t\big) =$$

$$\mathbb{E}_{\mathbb{Q}}\left(\exp\left\{iy \int_t^\tau \varrho\, dB_s\right\} \middle| \mathcal{F}_t\right)\Bigg|_{\varrho=\Sigma_s} = \mathbb{E}_{\mathbb{Q}}\left(\mathbb{E}_{\mathbb{Q}}\left[\exp\left\{iy \int_t^\tau \varrho\, dB_s\right\}\right]_{\varrho=\Sigma_s} \middle| \mathcal{F}_t\right) = \mathbb{E}_{\mathbb{Q}}\left(\exp\left\{\frac{i^2 y^2}{2} \int_t^\tau \Sigma_s^2\, ds\right\} \middle| \mathcal{F}_t\right)$$

due to the independent increment property of the appearing Brownian integral with respect to the conditioning filtration $\mathcal{F}$. Merging (4.7) and (4.8) into (4.6), we arrive at the option price formula

(4.9)

$$\Pi_t = \int_{\mathbb{R}} \hat{g}(y)\, e^{iyP_t}\, \mathbb{E}_{\mathbb{Q}}\left(\exp\left\{\frac{i^2 y^2}{2} \int_t^\tau \Sigma_s^2\, ds\right\} \middle| \mathcal{F}_t\right) dy$$

where $0 \le t \le \tau < T$. Herein, the function $\hat{g}$ is such as claimed in the sequel of (4.6), the underlying option price process P satisfies (3.5), respectively (3.20), and the stochastic volatility process Σ is like given in (3.8), respectively in (3.21). As a consequence of the sophisticated structure of Σ_s, it appears impossible to compute the conditional expectation in (4.9) analytically. For this reason, we propose to treat the conditional expectation individually in every practical application, after the payoff functions f and g have been determined concretely.

Remark 4.1. There is an alternative method to infer (4.8), which involves the enlarged filtration

$$\mathcal{G}_{t,\tau} \coloneqq \mathcal{F}_t \vee \sigma\{\Sigma_s : t \le s \le \tau\}$$

where $0 \le t \le \tau < T$. In accordance to the tower property of conditional expectations, we find

$$\mathbb{E}_{\mathbb{Q}}\left(e^{iy[P_\tau - P_t]}\middle|\mathcal{F}_t\right) = \mathbb{E}_{\mathbb{Q}}\left(\mathbb{E}_{\mathbb{Q}}\left(\exp\left\{iy\int_t^\tau \Sigma_s\, dB_s\right\}\middle|\mathcal{G}_{t,\tau}\right)\middle|\mathcal{F}_t\right) = \mathbb{E}_{\mathbb{Q}}\left(\exp\left\{\frac{i^2y^2}{2}\int_t^\tau \Sigma_s^2\, ds\right\}\middle|\mathcal{F}_t\right),$$

since – as long as we condition on $\mathcal{G}_{t,\tau}$ – we can treat the process (Σ_s) path-wise and thus, handle it like a deterministic function. We refer to [2] where this method is used in a related stochastic volatility context.

5 Examples

In this section, we apply the theoretical results derived in the previous sections to several practical examples of compound options.

Example 5.1 (power options). In the first example, we assume that the payoff function f in (3.1) is defined via $f(x) \coloneqq x^n$ where $n \in \mathbb{N}$ is fixed. Then it holds $F = (S_T)^n$ as well as $f'(x) = n\, x^{n-1}$. With reference to (3.8) and (2.8), we obtain

$$\Sigma_t = n\,\xi(t)\,\mathbb{E}_{\mathbb{Q}}((S_T)^n|\mathcal{F}_t) = n\,\xi(t)\,(S_t)^n\,\mathbb{E}_{\mathbb{Q}}\left(e^{n[X_T - X_t]}\middle|\mathcal{F}_t\right)$$

for all $t \in [0,T]$. Since the additive process X defined in (2.9) possesses independent increments with respect to $\mathcal{F}$, the appearing conditional expectation reduces to a usual expectation which can be computed to

$$\mathbb{E}_{\mathbb{Q}}\left(e^{n[X_T - X_t]}\middle|\mathcal{F}_t\right) = \mathbb{E}_{\mathbb{Q}}\left[e^{n(X_T - X_t)}\right] = \exp\left\{\frac{n(n-1)}{2}\int_t^T \xi^2(s)\, ds\right\}$$

due to (2.9). Hence, we get for all $t \in [0,T]$

$$\Sigma_t = n\,\xi(t)\,(S_t)^n\,\exp\left\{\frac{n(n-1)}{2}\int_t^T \xi^2(s)\, ds\right\}.$$

On the other hand, by similar computation methods as used in the latter equations, we deduce

(5.1)

$$P_t = \mathbb{E}_{\mathbb{Q}}((S_T)^n|\mathcal{F}_t) = (S_t)^n\,\exp\left\{\frac{n(n-1)}{2}\int_t^T \xi^2(s)\, ds\right\}$$

which implies for all $t \in [0,T]$ the equality

(5.2) $$\Sigma_t = n\,\xi(t)\,P_t\,.$$

Substituting (5.2) into (3.5), we infer the geometric SDE

$$dP_t = n\,\xi(t)\,P_t\,dB_t$$

whose solution reads for all $t \in [0,T]$ as

(5.3)

$$P_t = P_0\exp\left\{n\int_0^t \xi(s)\, dB_s - \frac{n^2}{2}\int_0^t \xi^2(s)\, ds\right\}.$$

By the way, if we take $n = 1$ in (5.1), then we are led to the equality chain

$$P_t = \mathbb{E}_{\mathbb{Q}}(S_T|\mathcal{F}_t) = S_t$$

where the second equality stands in accordance to the $(\mathcal{F},\mathbb{Q})$-martingale property of the process S. With view on the first equality, we can interpret P as the forward price of S. Also note that the Fourier transform methods presented in Section 3.2 cannot be applied in the current power option case, since the function $q(x) = f(S_0\, e^x) = (S_0)^n e^{nx}$ is not an element of $\mathcal{L}^1(\mathbb{R})$. Let us now turn our attention to the compound option introduced in (4.1). We presently assume that the payoff function g in (4.1) is defined via $g(x) := x^m$ where $m \in \mathbb{N}$ is fixed. Then it holds $G = (P_\tau)^m$ such that (4.1) takes the form

$$\Pi_t = \mathbb{E}_{\mathbb{Q}}((P_\tau)^m | \mathcal{F}_t) \tag{5.4}$$

for all $0 \le t \le \tau < T$. We now put (5.3) [with t therein replaced by τ] into (5.4) and obtain by straightforward calculations

$$\Pi_t = (P_t)^m \ \exp\left\{n^2 \frac{m(m-1)}{2} \int_t^\tau \xi^2(s)\, ds\right\}$$

$$= (S_t)^{nm} \ \exp\left\{\frac{nm}{2}\left((n-1)\int_t^T \xi^2(s)\, ds + n(m-1)\int_t^\tau \xi^2(s)\, ds\right)\right\}$$

wherein the second equality holds due to (5.1). Finally note that if we take $m = 1$ in (5.4), then we are led to the equality chain

$$\Pi_t = \mathbb{E}_{\mathbb{Q}}(P_\tau | \mathcal{F}_t) = P_t$$

where the second equality stands in accordance to the $(\mathcal{F},\mathbb{Q})$-martingale property of the process P.

Example 5.2 (call on put). Let us assume that the underlying option is a European put option written on the stock price S with constant strike price $K > 0$ and payoff $F = f(S_T) := [K - S_T]^+$ at maturity $T > 0$. That is, the payoff function is presently given by $f(x) = [K - x]^+$. Taking (2.8) into account, we observe

$$f(S_T) = [K - S_0\, e^{X_T}]^+ = [K - \delta(X_T)]^+ = (f \circ \delta)(X_T) = q(X_T)$$

where δ and q are the deterministic functions introduced in the sequel of (3.11), while X is the real-valued additive process defined in (2.9). Note that $q(x) = [K - S_0\, e^x]^+ \notin \mathcal{L}^1(\mathbb{R})$ such that we shall work with the dampened function $\zeta(x) := e^{-ax}\, q(x) \in \mathcal{L}^1(\mathbb{R})$ where $a < 0$ is a real-valued dampening parameter. We then obtain

$$P_t = \mathbb{E}_{\mathbb{Q}}(f(S_T)|\mathcal{F}_t) = \mathbb{E}_{\mathbb{Q}}(q(X_T)|\mathcal{F}_t) = \mathbb{E}_{\mathbb{Q}}(e^{aX_T}\, \zeta(X_T)|\mathcal{F}_t)$$

for all $t \in [0,T]$. Applying the inverse Fourier transform on the latter equation yields

$$P_t = \int_{\mathbb{R}} \hat{\zeta}(y)\, \mathbb{E}_{\mathbb{Q}}\big(e^{(a+iy)X_T}\big|\mathcal{F}_t\big)\, dy \tag{5.5}$$

where the Fourier transform of $\zeta(x) = e^{-ax}\, [K - S_0\, e^x]^+$ is for every fixed $a < 0$ given by

$$\hat{\zeta}(y) = \frac{K}{2\pi\, (a + iy)\, (a + iy - 1)} \left(\frac{S_0}{K}\right)^{a+iy}$$

due to (3.10). Comparing (5.5) with the corresponding general expression in (3.12), we see that ζ and $a + iy$ in (5.5) play the role of q and iy in (3.12), respectively. Hence, due to Proposition 3.5, the put option price P at any time $t \in [0,T]$ prior to maturity can be represented as

$$P_t = \int_{\mathbb{R}} \hat{\zeta}(y)\, e^{\tilde{A}_t(y)}\, dy$$

with a complex-valued $\mathcal{F}$-adapted stochastic process $\tilde{A}$ defined via

$$\tilde{A}_t(y) := (a + iy)\, X_t + \tilde{\lambda}(t, T, y)$$

where X is given by (2.9) and

$$\tilde{\lambda}(t, T, y) := \frac{(a + iy)(a + iy - 1)}{2} \int_t^T \xi^2(s)\, ds$$

due to (3.15). In accordance to (3.20) and (3.21), the put option price P then satisfies for all $t \in [0, T]$ the $(\mathcal{F}, \mathbb{Q})$-martingale dynamics

$$dP_t = \tilde{\Sigma}_t\, dB_t$$

with complex-valued $\mathcal{F}$-adapted stochastic volatility process

(5.6)

$$\tilde{\Sigma}_t = \xi(t) \int_{\mathbb{R}} \hat{\zeta}(y)\, e^{\tilde{A}_t(y)}\, (a + iy)\, dy$$

and deterministic initial value

$$P_0 = \int_{\mathbb{R}} \hat{\zeta}(y)\, e^{\tilde{\lambda}(0,T,y)}\, dy\,.$$

Further on, we suppose that the overlying compound option is a European call option written on the put option price P with constant strike price $k > 0$ and payoff $G = g(P_\tau) := [P_\tau - k]^+$ at maturity τ where $\tau < T$. That is, the payoff function presently reads as $g(x) = [x - k]^+$ which is not an element of $\mathcal{L}^1(\mathbb{R})$. Thus, we introduce the dampened function $\varphi(x) := e^{-\alpha x} g(x) \in \mathcal{L}^1(\mathbb{R})$ where $\alpha > 0$ is a real-valued dampening parameter. With respect to (4.1), we then obtain

$$\Pi_t = \mathbb{E}_{\mathbb{Q}}(g(P_\tau)|\mathcal{F}_t) = \mathbb{E}_{\mathbb{Q}}(e^{\alpha P_\tau} \varphi(P_\tau)|\mathcal{F}_t)$$

for all $0 \le t \le \tau < T$. Applying the inverse Fourier transform on the latter equation yields

(5.7)

$$\Pi_t = \int_{\mathbb{R}} \hat{\varphi}(y)\, e^{(\alpha+iy)P_t}\, \mathbb{E}_{\mathbb{Q}}\left(e^{(\alpha+iy)(P_\tau - P_t)}\middle|\mathcal{F}_t\right) dy$$

where the Fourier transform of $\varphi(x) = e^{-\alpha x}\, [x - k]^+$ can be computed to

$$\hat{\varphi}(y) = \frac{e^{-(\alpha+iy)k}}{2\pi\, (\alpha + iy)^2}$$

due to (3.10). Comparing (5.7) with the corresponding expression provided in (4.6), we observe that φ and $\alpha + iy$ in (5.7) play the role of g and iy in (4.6), respectively. Hence, taking (4.9) into account, we conclude that the compound option price Π can for all $0 \le t \le \tau < T$ be expressed as

$$\Pi_t = (2\pi)^{-1} \int_{\mathbb{R}} \frac{e^{(\alpha+iy)(P_t - k)}}{(\alpha + iy)^2}\, \mathbb{E}_{\mathbb{Q}}\left(\exp\left\{\frac{(\alpha + iy)^2}{2} \int_t^\tau \tilde{\Sigma}_s^2\, ds\right\}\middle|\mathcal{F}_t\right) dy$$

where $\tilde{\Sigma}$ is the stochastic volatility process claimed in (5.6).

Example 5.3 (digital options). Let us consider a digital option (with notional equal to one) written on the stock price process S with payoff $F = f(S_T) := \mathbb{1}_{[M_1,M_2]}(S_T)$ at maturity $T > 0$. That is, the payoff function is now given by $f(x) = \mathbb{1}_{[M_1,M_2]}(x)$ where $\mathbb{1}$ denotes the indicator function and $0 < M_1 < M_2 < \infty$ are constants. Taking (2.8) into account, we observe

$$f(S_T) = \mathbb{1}_{[M_1,M_2]}(S_0\, e^{X_T}) = \mathbb{1}_{[M_1,M_2]}\big(\delta(X_T)\big) = (f \circ \delta)(X_T) = q(X_T)$$

where $\delta(x) = S_0\, e^x$ and $q(x) = \mathbb{1}_{[M_1,M_2]}(S_0\, e^x)$ are deterministic functions, while X is the real-valued additive process defined in (2.9). Note that $q \in \mathcal{L}^1(\mathbb{R})$ can be expressed as $q(x) = \mathbb{1}_{[m_1,m_2]}(x)$ with finite constants $m_j := \log\big(M_j/S_0\big)$, $j \in \{1,2\}$. Applying the inverse Fourier transform yields

(5.8)

$$P_t = \mathbb{E}_{\mathbb{Q}}(f(S_T)|\mathcal{F}_t) = \mathbb{E}_{\mathbb{Q}}(q(X_T)|\mathcal{F}_t) = \int_{\mathbb{R}} \hat{q}(y)\, \mathbb{E}_{\mathbb{Q}}(e^{iyX_T}|\mathcal{F}_t)\, dy$$

where the Fourier transform of q is given by

$$\hat{q}(y) = \frac{e^{-iym_1} - e^{-iym_2}}{2\pi i y} = \frac{1}{2\pi i y}\left\{\left(\frac{S_0}{M_1}\right)^{iy} - \left(\frac{S_0}{M_2}\right)^{iy}\right\}$$

due to (3.10). Note that $\hat{q}(y)$ is not well-defined at $y = 0$. For this reason, we set $\hat{q}(0) := l$ where

$$l := \lim_{y\to 0} \hat{q}(y) = \frac{m_2 - m_1}{2\pi} = \frac{\log(M_2/M_1)}{2\pi}$$

is a real-valued finite constant. Then the function $\hat{q}: \mathbb{R} \to \mathbb{C}$ becomes continuous on the entire real line. Since (5.8) possesses the same structure as the corresponding general expression in (3.12), we can directly apply Proposition 3.5 in the current case. Thus, we conclude that the option price process P satisfies (3.16), respectively (3.20)-(3.21), while $A_t(y)$ is such as defined in (3.17). Moreover, let us assume that the overlying compound option is also a digital option (but written on the option price process P) with payoff $G = g(P_\tau) := \mathbb{1}_{[C_1,C_2]}(P_\tau)$ at maturity τ where $\tau < T$. That is, the payoff function presently reads as $g(x) = \mathbb{1}_{[C_1,C_2]}(x) \in \mathcal{L}^1(\mathbb{R})$ where $0 < C_1 < C_2 < \infty$ are given constants. We then conclude that the compound option price $\Pi_t = \mathbb{E}_{\mathbb{Q}}(g(P_\tau)|\mathcal{F}_t)$ is for all $0 \le t \le \tau < T$ given by (4.9), but now with Fourier transform

$$\hat{g}(y) = \frac{e^{-iyC_1} - e^{-iyC_2}}{2\pi i y}$$

due to (3.10). Note that $\hat{g}(y)$ is not well-defined at $y = 0$. For this reason, we set $\hat{g}(0) := \chi$ where

$$\chi := \lim_{y\to 0} \hat{g}(y) = \frac{C_2 - C_1}{2\pi}$$

is a real-valued finite constant. Then the function $\hat{q}: \mathbb{R} \to \mathbb{C}$ becomes continuous on the entire real line.

Example 5.4 (call option). Consider the claim

$$F = f(S_T) := [S_T - K]^+$$

where $K > 0$ denotes the constant strike price and $f(x) = [x - K]^+$ is a deterministic payoff function. Note that $f(x)$ is not differentiable at $x = K$ such that we cannot apply the chain rule (provided in Theorem 3.5 in [5]) directly to compute the Malliavin derivative $\mathcal{D}_t F$ in the present example. For this reason, we now apply similar approximation techniques as proposed in Example

4.11 in [5]. That is, following the argumentation on p. 52 in [5], for arbitrary $n \in \mathbb{N}$ we approximate $f(\cdot)$ by $\mathcal{C}^1$-functions $f_n(\cdot)$ which fulfill the properties

- $f_n(x) = f(x)$ for all $x \in]0,\infty[$ with $|x-K| \geq 1/n$,
- $0 \leq f_n'(x) \leq 1$ for all $x \in]0,\infty[$.

At this step, it is worth noting that for all $x \in]0,\infty[$ it holds

$$\lim_{n\to\infty} f_n'(x) = \mathbb{1}_{[K,\infty[}(x)$$

where $\mathbb{1}$ denotes the indicator function. We further put $F_n := f_n(S_T)$ and observe $\mathcal{D}_t F_n \to \mathcal{D}_t F$ as $n \to \infty$ (in the $\mathcal{L}^2(\mathbb{Q})$-sense) due to Theorem 3.3 in [5]. Hence, for all $t \in [0,T]$ we obtain

$$\mathcal{D}_t F = \lim_{n\to\infty} \mathcal{D}_t F_n = \lim_{n\to\infty} \mathcal{D}_t\big(f_n(S_T)\big) = \mathcal{D}_t(S_T) \lim_{n\to\infty} f_n'(S_T)$$

where we used the chain rule (see Theorem 3.5 in [5]) for the last equality. In (3.6) we found $\mathcal{D}_t(S_T) = S_T\, \xi(t)$ which currently implies

$$\mathcal{D}_t F = S_T\, \xi(t) \lim_{n\to\infty} f_n'(S_T) = S_T\, \xi(t)\, \mathbb{1}_{[K,\infty[}(S_T)$$

for all $t \in [0,T]$. As a consequence of the latter equation and (3.4), we deduce

$$\Sigma_t = \mathbb{E}_{\mathbb{Q}}(\mathcal{D}_t F | \mathcal{F}_t) = \xi(t)\, \mathbb{E}_{\mathbb{Q}}\big(S_T\, \mathbb{1}_{[K,\infty[}(S_T) \big| \mathcal{F}_t\big) = \xi(t)\, \mathbb{E}_{\mathbb{Q}}\big(S_T\, \mathbb{1}_{[K,\infty[}(S_T) \big| S_t\big)$$

where the last equality is valid due to the time-inhomogeneous $(\mathcal{F},\mathbb{Q})$-Markov property of the stock price process S given by (2.8)-(2.9). Moreover, from (2.8) we infer $S_T = S_t\, e^{X_T - X_t}$ which yields

$$\Sigma_t = \xi(t)\, \mathbb{E}_{\mathbb{Q}}\big[c\, e^{X_T - X_t}\, \mathbb{1}_{[K,\infty[}(c\, e^{X_T - X_t})\big]\Big|_{c=S_t} = \xi(t)\, \mathbb{E}_{\mathbb{Q}}\big[c\, e^{X_T - X_t}\, \mathbb{1}_{[\log(K/c),\infty[}(X_T - X_t)\big]\Big|_{c=S_t}$$

where we conditioned on the fixed state $c = S_t > 0$. The latter equation is equivalent to

$$\Sigma_t = \xi(t)\, S_t \int_{\log(K/S_t)}^{\infty} e^x\, d\mathbb{Q}^{X_T - X_t}(x)$$

where S_t satisfies (2.8) and the distribution of the increment $X_T - X_t$ under $\mathbb{Q}$ possesses the Lebesgue density

$$d\mathbb{Q}^{X_T - X_t}(x) = \Psi_{\eta,v^2}(x)\, dx$$

for all $x \in \mathbb{R}$. Herein, $\Psi_{\eta,v^2}(\cdot)$ denotes the probability density function (pdf) of the normal distribution with parameters

$$\eta = \eta(t,T) = \mathbb{E}_{\mathbb{Q}}[X_T - X_t] = -\frac{1}{2}\int_t^T \xi^2(s)\, ds\ , \qquad v^2 = v^2(t,T) = \mathbb{V}\mathrm{ar}_{\mathbb{Q}}[X_T - X_t] = -2\,\eta(t,T)\,.$$

Thus, a straightforward calculation (using the properties of the normal distribution) yields

(5.9)

$$\Sigma_t = \xi(t)\, S_t\, \Phi\left(\frac{\log(S_t/K) - \eta(t,T)}{v(t,T)}\right)$$

where $\Phi(\cdot)$ denotes the cumulative distribution function (cdf) of the standard normal distribution. Consequently, from (3.3) we deduce that the underlying of the compound option Π satisfies in the current example

$$P_\tau = P_0 + \int_0^\tau \xi(s)\, S_s\, \Phi\left(\frac{\log(S_s/K) - \eta(s,T)}{v(s,T)}\right) dB_s\,.$$

The price of the corresponding compound option is then given by (4.9) wherein the process Σ satisfies (5.9). Due to the sophisticated structure of (5.9), the conditional expectation appearing in (4.9) cannot be calculated analytically, unfortunately. Hence, numerical approximation methods have to be used in the current example to compute/simulate the compound option price Π_t in (4.9).

Example 5.5 (logarithmic payoff). Consider the claim

$$F = f(S_T) \coloneqq \log(S_T) = \log(S_0) + X_T = \log(S_0) + \int_0^T \xi(s)\, dB_s - \frac{1}{2}\int_0^T \xi^2(s)\, ds$$

with payoff function $f(x) = \log(x)$ which is an element of $C^1(\mathbb{R}^+)$. In the present case, we find

$$P_0 = \mathbb{E}_{\mathbb{Q}}[F] = \log(S_0) - \frac{1}{2}\int_0^T \xi^2(s)\, ds\,, \qquad \mathcal{D}_t F = \mathcal{D}_t(X_T) = \xi(t)\,, \qquad \Sigma_t = \xi(t)$$

for all $t \in [0,T]$. As a consequence, the price dynamics in (3.5) simplifies to

$$dP_t = \xi(t)\, dB_t$$

such that the underlying of the option Π in (4.1) is normally distributed under $\mathbb{Q}$ with

$$P_\tau = P_0 + \int_0^\tau \xi(s)\, dB_s \sim \mathcal{N}\left(P_0, \int_0^\tau \xi^2(s)\, ds\right).$$

Since the underlying P of the option Π in the current example has a very simple structure (normally distributed arithmetic model), it is possible to compute the option price Π_t explicitly for many different payoff functions $g(\cdot)$ in (4.1).

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