Option Pricing when Investors are (Only) Sufficiently Rational

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Abstract

This research adapts the Black-Scholes option pricing model that is widely used in practice to a world where investors only form sufficiently rational expectations (expectations that deviate from perfection without creating arbitrage opportunities). Within the no-arbitrage interval of market values that may exist in the presence of transaction costs, we utilize recent findings from the brain sciences to show that option prices will reflect the Black-Scholes formula but with the risk-free rate replaced with a higher rate. We measure the improvement that this modification brings by calibrating the adjusted Black-Scholes model with S&P 500 index options. We show that extending the brain-centric approach to models with stochastic volatility and jumps leads to the same modification: replace the risk-free rate with a higher rate.

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The Black and Scholes (1973) option pricing model which is widely used in practice and in research makes numerous assumptions, including the existence of a frictionless market (and a lognormally distributed price for the underlying stock) that lead to a unique option price. However, empirical studies have long indicated that market prices deviate from the Black-Scholes model option values (Rubinstein 1994), and adaptations of the model (such as by allowing for price jumps, stochastic variance, and other deviations from lognormal distributions as well as different priced factors) have not been found to fully explain the prices of options observed in the marketplace (e.g., see Bates (2003), Song and Xiu (2016), and Constantinides, Czerwonko, and Perrakis (2020)). Leland (1985) showed that, in the presence of market frictions (transaction costs), a no-arbitrage range of option prices might exist instead of a unique price, and Constantinides and Perrakis (2002) derived a tighter interval of potential market values that could exist even if returns were characterized by distributional jumps and a stochastic variance.[1] By utilizing findings from neuroscience research, this research seeks to determine the particular price that might be realized in the marketplace within the no-arbitrage bounds of possible values.

According to Muth (1961), prices should reflect rational expectations based on perfect analysis of all available information, which might indicate option prices would precisely equal the Black-Scholes model values if returns on stocks underlying options were lognormally distributed and return variances could be estimated perfectly.[2] However, in a world with market imperfections like transaction costs, as well as with uncertainty about return distributions and about parameter values like variances, investors may only have a strong enough incentive (given limited brain resources) to determine the value of an option

[1] Wallmeier (2021) has found empirical evidence that option prices stay within such rational bounds.
[2] Black and Scholes (1973) showed that, with lognormally distributed returns on the assets underlying options, their option pricing model would hold within the context of common asset pricing theories based on betas or covariances with priced factors, such as the market portfolio return in Sharpe's (1964) Capital Asset Pricing Model (CAPM), because returns on options are a derivative function of those of the underlying asset. By extension, the Black-Scholes formula would also hold for other widely accepted models having pricing kernels linearly related to covariances with other factors such as indicated by Merton's Intertemporal Capital Asset Pricing Model (ICAPM), Ross's (1976) Arbitrage Pricing Theory (APT) that allows for the many different beta factors (like those examined in Fama and French (2016) to be priced, and Breeden's (1979) Consumption Capital Asset Pricing Model (CCAPM).

within the bounds of a no-arbitrage interval.[3] In this article, we use recent findings from brain sciences to show that such sufficiently rational expectations lead to an approximate Black-Scholes model valuation that differs from the classical Black-Scholes formula in a small way.[4] We show that this brain-centric perspective can play a role in resolving the main option pricing puzzles. We measure the improvement by calibrating the adjusted Black-Scholes model with S&P 500 index options. We demonstrate that the same adjustment extends to other option pricing models, such as those with stochastic volatility (SV) and stochastic volatility with jumps (SVJ).

In the past decade and a half, researchers have started to look at how the brain solves its internal resource allocation problem (see Alonso et al (2014) and references therein).[5] One of the most intriguing findings from neuroscience research is that the brain is a predicting organ[6] that actively predicts rather than just passively awaits incoming information.[7] That is, information processing has a critical *top-down* aspect to it. Specifically, the brain creates a knowledge structure or a schema by synthesizing similar past experiences. Incoming information triggers this schema, which then makes an initial prediction.[8] This initial expectation is then adjusted with incoming information by spending brain resources.[9] Such integration of *top-down* prediction from the activated schema and

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[3] As indicated by Garleanu et al (2009) and Kang and Park (2008), investor demand drives market prices. For example, Bollen and Whaley (2004) have shown that institutional demand for portfolio insurance may drive up the market prices of index puts. Bakshi, Madan, and Panayotov (2010) have discovered evidence that far out-of-the-money calls may be bid up in price by short sellers seeking protection from large upward price movements. Figlewski and Malik (2014) hypothesize that the market values of some call (put) options may be higher because of excessive demand from investors seeking leveraged (short) positions on underlying assets.
[4] The risk-free rate is replaced with a higher rate in the European call option formula, which increases with the risk-premium on the underlying stock. For a European put option, apart from replacing the risk-free rate with a higher rate, an extra additive term appears which increases with the risk-premium on the underlying stock
[5] McKenzie, R. (2018) argues that pushing the notion of scarcity inside the human brain provides a foundation for combining the neoclassical and behavioral approaches in economics.
[6] Chapter 4 in Feldman, B. (2020) synthesizes and reviews this literature. The idea that the brain is a prediction engine is very old and is typically traced back to the German physicist and physiologist Hermann von Helmholtz in neuroscience literature. Recent advances in brain sciences have transformed this idea into a dominant paradigm for thinking about the brain (See Clark, A. (2015) and Hohwy, J. (2014) among others).
[7] That's why, drinking water almost immediately quenches your thirst even though it takes at least 20 minutes for water to reach your blood stream.
[8] As any activity in the brain requires resources, choosing a schema to make an initial prediction requires resources as well. However, as this resource requirement is expected to be much smaller than the resources spent in attempting to appropriately adjust the initial predictions, we ignore it for modelling convenience.
[9] A sample of neuroscience literature that explores the role of schemas includes Tse et al (2007), van Kesteren et al (2010), Tse et al (2011), van Kesteren et al (2012), Ghosh and Gilboa (2014), Ghosh et al (2014), Brod et al (2015), Spalding et al (2015), Sweegers et al (2015), Gilboa and Marlatte (2017), and Ohki and Takei (2018).

bottom-up information streaming-in is a critical aspect of forming expectations.[10] Schemas are critical to brain's operations as they play a major role in economizing on resources by setting proper initial predictions (what to expect at a restaurant, how to put petrol in your car etc). In fact, a break-down in schema reliance has been associated with autism spectrum disorder (ASD).[11] What are the implications of the brain-centric view for option pricing? As a call option is a levered position in the underlying stock, it is reasonable to consider that the initial predictions about the reward and risk of a call option are influenced by the reward and risk of the underlying stock. It is these initial predictions that the brain attempts to appropriately adjust with incoming information by attempting to optimally spend its limited resources.

The brain faces a hard-limit on energy supply (about 20% of the body's total energy in-take), so it faces the internal problem of allocating its limited energy supply across various tasks (including basic visual and auditory functions, motor tasks such as picking up a coffee mug, as well as cognitive tasks such as figuring out how much a call option is worth). While neurons perform multiple tasks, there is task specialization in the sense that, once a neuron is assigned to a task, it becomes part of a system of neurons that becomes exclusively responsible for the task. Neurons in a system continue to fire together till the task is completed or resources allocated to the system are exhausted. Each system competes for resources that are allocated by the 'central executive system' (CES) located in areas in the lateral prefrontal cortex (LPFC) with task performance dependent on resources allocated to that system. A resource shortfall implies reduced task performance (Alonso et al 2014). As explained by Bossaerts (2009), neuroscience studies have found that the brain assigns a separate set of neurons to evaluate the reward from any investment, and this system of communicating brain cells is in a different area of the brain than those used to analyze risk.[12] The implication for option pricing is that reward and risk forecasting are

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The notion of schemas has a long history (Barlett 1932, Bransford and Johnson 1972, Anderson and Pearson 1984). See Hampson and Morris (1996), Anderson (2000), and Pankin (2013) for an overview of schema theory.

[10] For example, if you see a dog in your neighbour's front yard, the schema of a pet may be activated leading to an initial prediction that you are about to have a friendly interaction. However, if incoming information does not match the initial prediction, for example, the dog appears to be aggressive, then you may adjust these expectations.

[11] See Loth et al (2011).

[12] In particular, one region of the brain (the striatum which includes the nucleus accumbens) processes the expected value of a payoff on an investment (Tobler et al 2006), while other areas of the brain (the angular cingulate cortex and anterior insula/inferior frontal gyrus) analyze risk (Fukunaga et al 2018). There is some

separate tasks with individual task performance dependent on resource allocation decisions in the brain.

Overall, these findings suggest the following regarding a call option: Initial predictions regarding reward and risk are influenced by the underlying stock. These initial predictions are adjusted with incoming information by spending brain resources. Different investors make different adjustments based on the resource allocation decisions in their brains. We impose the condition that only those investors survive against whom arbitrage opportunities are not created. That is, they have reward and risk expectations that lead to a value within the no-arbitrage interval as defined by market fictions.

The above considerations lead to a call option pricing formula in which the risk-free rate is replaced by a higher rate. The adjusted formula generates the implied volatility skew, and contributes to the puzzles of zero-beta-straddle (Coval and Shumway 2001), covered-call writing (Whaley 2002), and leverage-adjusted returns (Constantinides et al 2013). Calibrating the adjusted formula with recent data on S&P 500 index options quantifies the improvement that the adjustment brings. Moving beyond Black-Scholes, we show that the same adjustment holds for other popular models such as Heston SV and Bates SVJ models.

1. The Predictive Brain and the Black-Scholes Formula

As indicated by expected utility maximization theories utilized in common asset pricing models (Breeden 1979), the value C_t in any period t of a call option on any asset like a stock (to any investor) is equal to the expected payoff on the stock divided by a discount rate, which reflects the investor's subjective expectations and utility of payoffs in different future states of the world:

$$C_t = \frac{E_t'[dC_t]}{\overline{r_c}'dt} \qquad (1.1)$$

evidence that uncertainty is processed by the brain in the same way as known risks through probability assessments/assignments (Nagel et al. 2018). A special sector of the brain (the ventral medial prefrontal cortex) utilizes these informative inputs to create an integrated value, but another area of the brain (the dorsal anterior midcingulate cortex) converts that value into an overall utility to the individual (Kurmianingsih and Mullette-Gillman 2016).

where $E_t'[dC]$ is the instantaneous expected payoff from the call option and $\overline{r_c}'dt$ is the corresponding discount-rate (the superscript $'$ is used to indicate that these are subjective assessments of reward and risk).

As in the original Black and Scholes (1973) option pricing model, we initially assume that the underlying stock follows geometric Brownian motion:

$$dS_t = \mu S_t dt + \sigma S_t dW_t \tag{1.1a}$$

where μ and σ are instantaneous expected underlying stock return and standard deviation respectively, and W_t is a Wiener process.

Ito's lemma implies:

$$dC_t = \left(\frac{\partial C_t}{\partial t} + \mu S_t \frac{\partial C_t}{\partial S} + \frac{\sigma^2 S_t^2}{2}\frac{\partial^2 C_t}{\partial S^2}\right) dt + \ \sigma S_t \frac{\partial C_t}{\partial S} dW_t \tag{1.1b}$$

We closely follow Alonso et al (2014), and Siddiqi and Murphy (2020a) (2020b) in setting-up the resource allocation problem inside the human brain as follows. Valuing a call option requires two tasks. Task 1 is estimating the expected payoff or the numerator in (1.1), whereas Task 2 requires forecasting risk which leads to the denominator in (1.1). Apart from these two cognitive tasks, we combine all other non-cognitive or motor tasks (such as lifting an object or looking in a certain direction) that the brain may be engaged in at the time of analysis and refer to this aggregate as Task 0. Each task is performed by a separate brain system, which alone is responsible for that task. That is, system 1 performs Task 1, system 2 performs Task 2, and system 0 performs Task 0. Systems are made-up of neurons, which demand resources from the 'central executive system' (CES). The CES allocates resources based on relative task important and complexity. Resource deficit implies underperformance in the task.

In what follows, we suppress the time subscripts for simplicity whenever doing so does not create any confusion. We assume that the initial prediction for expected payoff from a call option, $E'[dC]$, is influenced by the expected payoff from the underlying stock,

$E[dS]$.[13] This initial prediction is then adjusted with incoming information by spending brain resources. This is, Task 1 assigned to the relevant system of neurons is:

$$E'[dC] = E[dS] - m_1 D_1 \tag{1.2}$$

where $D_1 = E[dS] - E[dC]$ denotes the correct adjustment needed, and m_1 is the fraction of correct adjustment achieved.[14] When resource need is fully met, $m_1 = 1$, thus leading to successful completion of the task resulting in rational expectations. For simplicity and without any loss in generality, we equate the fraction of task completion with the fraction of resource needs being met. So, when resource needs are not fully met, that is, when the resources allocated, x_1, are less than the resource need, θ_1, $m_1 = \frac{x_1}{\theta_1}$. [15]

From (1.1a):

$$E[dS] = \mu S dt \tag{1.2a}$$

From (1.1b):

$$E[dC] = \left(\frac{\partial C}{\partial t} + \mu S \frac{\partial C}{\partial S} + \frac{\sigma^2 S^2}{2} \frac{\partial^2 C}{\partial S^2}\right) dt \tag{1.2b}$$

Substituting (1.2a) and (1.2b) in (1.2) and simplifying:

$$E'[dC] = m_1 \left(\frac{\partial C}{\partial t} + \mu S \frac{\partial C}{\partial S} + \frac{\sigma^2 S^2}{2} \frac{\partial^2 C}{\partial S^2}\right) dt + (1 - m_1)\mu S dt \tag{1.2c}$$

So, the (subjective) instantaneous expected payoff from a call option is a weighted average of the correct underlying stock expected payoff and the correct call expected payoff. Note, that we do not assume that the initial prediction coincides with the expected stock payoff or $E[dS] = \mu S dt$. We only assume that the initial prediction is such that the expected stock payoff influences the expectations about call payoffs. So, m_1 could be positive at the stage

[13] For simplicity and tractability, throughout this article, we assume that rational expectations are formed about the reward and risk of the underlying stock.
[14] The adjustment entails applying the following transformation to state-wise stock payoffs: $max(S - K, 0)$ where K is the strike price. This is a relatively simple adjustment which most beginning students of option pricing learn to apply quite quickly.
[15] As shown later in this section, more than required resources are never allocated in the optimization problem of brain's internal resources.

of initial prediction as well. Adjusting the initial prediction then means taking a positive m_1 towards $m_1 = 1$.

The standard asset pricing equation in continuous time (see Cochrane (2005), chapter 1) is:

$$E\left[\frac{dP_t}{P_t}\right] = rdt - E\left[\frac{d\Lambda_t}{\Lambda_t}\frac{dP_t}{P_t}\right] \qquad (1.3)$$

where P_t is the price of an asset, rdt is the risk-free rate over dt, and $\Lambda_t = e^{-\beta t}u'(c_t)$ is the 'discount-factor' in continuous time ($u'(c_t)$ is the marginal utility of consumption and β is the time-discount factor).

Applying (1.3) to the underlying stock results in:

$$E\left[\frac{dS_t}{S_t}\right] = rdt - E\left[\frac{d\Lambda_t}{\Lambda_t}\frac{dS_t}{S_t}\right] \qquad (1.3b)$$

As mentioned earlier, we have assumed that rational expectations are formed about the underlying stock.

The application of (1.3) to the call option leads to:

$$E'\left[\frac{dC_t}{C_t}\right] = rdt - E'\left[\frac{d\Lambda_t}{\Lambda_t}\frac{dC_t}{C_t}\right] \qquad (1.3c)$$

where the superscript $'$ indicates that the expectations are subjective (can be different from rational expectations). Note, that $E'\left[\frac{dC_t}{C_t}\right]$ is equal to $\overline{r_c}'dt$ in the denominator of (1.1).

To estimate $\overline{r_c}'dt$, the brain needs to estimate the risk of the call option, $E'\left[\frac{d\Lambda_t}{\Lambda_t}\frac{dC_t}{C_t}\right]$. The task of risk estimation (Task 2) is assigned to a separate system of neurons dedicated to the task, which makes an initial prediction influenced by underlying risk, $E\left[\frac{d\Lambda_t}{\Lambda_t}\frac{dS_t}{S_t}\right]$, and then spends brain resources in attempting to appropriately adjust it:

$$E'\left[\frac{d\Lambda_t}{\Lambda_t}\frac{dC_t}{C_t}\right] = E\left[\frac{d\Lambda_t}{\Lambda_t}\frac{dS_t}{S_t}\right] - m_2 D_2 \qquad (1.4)$$

where $D_2 = E\left[\frac{d\Lambda_t}{\Lambda_t}\frac{dS_t}{S_t}\right] - E\left[\frac{d\Lambda_t}{\Lambda_t}\frac{dC_t}{C_t}\right]$, and m_2 is the fraction of correct adjustment achieved.[16] When the resource need is completely met, $m_2 = 1$, thus leading to successful completion of the task culminating in fully rational expectations. When there are insufficient brain resources to fully evaluate risk[17], that is, when the resources allocated, x_2, are less than the resource need, θ_2, $m_2 = \frac{x_2}{\theta_2}$.

(1.4) can be re-arranged as:

$$E'\left[\frac{d\Lambda_t}{\Lambda_t}\frac{dC_t}{C_t}\right] = (1 - m_2)E\left[\frac{d\Lambda_t}{\Lambda_t}\frac{dS_t}{S_t}\right] + m_2 E\left[\frac{d\Lambda_t}{\Lambda_t}\frac{dC_t}{C_t}\right] \qquad (1.4b)$$

(1.4b) shows that the (subjective) risk of a call option is the weighted average of the actual underlying stock risk and the call option risk under rational expectations. As before, m_2 could be positive at the stage of initial prediction, which means that brain resources are spent in taking an already positive m_2 toward 1. If Task 2 reaches completion, that is when $m_2 = 1$, then fully rational expectations about the risk of a call option are formed.

From (1.1a) and (1.3b):

$$E\left[\frac{d\Lambda_t}{\Lambda_t}\frac{dS_t}{S_t}\right] = \sigma E\left[\frac{d\Lambda_t}{\Lambda_t}dW_t\right] = (r - \mu)dt = -\delta dt \qquad (1.4c)$$

where δ is the risk-premium on the underlying stock.

From (1.1b) and (1.4c):

$$E\left[\frac{d\Lambda_t}{\Lambda_t}\frac{dC_t}{C_t}\right] = \sigma\frac{S}{C}\frac{\partial C}{\partial S}E\left[\frac{d\Lambda_t}{\Lambda_t}dW_t\right] = \frac{S}{C}\frac{\partial C}{\partial S}E\left[\frac{d\Lambda_t}{\Lambda_t}\frac{dS_t}{S_t}\right] = -\frac{S}{C}\frac{\partial C}{\partial S}\delta dt \qquad (1.4d)$$

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[16] Lieder et al. (2018, 2020) have shown that, when engaged in estimating the value of any variable(s), people tend to rationally start with a benchmark and then make a guesstimated adjustment in the direction of the true value, typically settling on an estimate somewhere between the anchor and the actual correct number. Such a partial adjustment results when the brain has insufficient information/resources to fully analyse a problem, i.e., here when $m_2 < 1$.

[17] Although the brain seems well-suited to analyze diffusion processes (Montague and Berns 2002), errors in evaluation are often made due to having insufficient resources to perfectly carry out multiple tasks when faced with an overwhelming amount of information that must be analyzed (Fehr and Rangel 2011). Given the possibility of underlying asset price jumps (Todorov 2010) and distributional uncertainty in general (Drechsler 2013), it is especially difficult to have sufficient brain resources to evaluate option risks and their impact on investor utility. The difficulty of evaluating the utility of stochastic risky outcomes has long perplexed financial experts/researchers, leading to a pricing kernel puzzle that remains unresolved (Cuesdenu and Jackwerth 2018), with existing models being unable to fully explain the full spectrum of observed option prices and returns (Liu 2021).

Substituting (1.4c) and (1.4d) in (1.4b):

$$E'\left[\frac{d\Lambda_t}{\Lambda_t}\frac{dC_t}{C_t}\right] = -(1-m_2)\delta dt - m_2\frac{S}{C}\frac{\partial C}{\partial S}\delta dt \tag{1.4e}$$

Substituting (1.4e) in (1.3c):

$$\overline{r_c}' dt = rdt + \delta(1-m_2)dt + \delta m_2\frac{\partial C}{\partial S}\frac{S}{C}dt$$

Writing $m_2 = (1-(1-m_2))$ in the last term of the above equation and substituting $r^* = r + \delta(1-m_2)$ leads to:

$$\overline{r_c}' dt = r^* dt + \frac{\partial C}{\partial S}\frac{S}{C}(\mu - r^*)\, dt \tag{1.4f}$$

where $r^* = r + \delta(1-m_2)$

With rational expectations, that is, when $m_2 = 1$:

$$\overline{r_c}' dt = rdt + \frac{\partial C}{\partial S}\frac{S}{C}(\mu - r)\, dt \tag{1.4g}$$

A comparison of (1.4f) and (1.4g) shows that, in the brain-centric approach, the effect of forming imperfect risk expectations is to replace the risk-free rate with a higher rate. This replacement lowers the discount-rate on the call option, which pushes up the price.

Substituting (1.4f) and (1.2c) in (1.1) and re-arranging leads to:

$$m_1\frac{\partial C}{\partial t} + S\frac{\partial C}{\partial S}\left(r^* - \mu(1-m_1)\right) + m_1\frac{\sigma^2 S^2}{2}\frac{\partial^2 C}{\partial S^2} = r^* C - (1-m_1)\mu S \tag{1.5}$$

where $r^* = r + \delta(1-m_2)$

(1.5) is the general PDE consistent with the brain-centric perspective of the predictive brain.

1.1 When Rational Expectations are Formed

If rational expectations are assumed about both reward and risk, that is, when $m_1 = 1$ and $m_2 = 1$, then the standard Black-Scholes PDE is recovered from (1.5):

$$\frac{\partial C}{\partial t} + rS\frac{\partial C}{\partial S} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 C}{\partial S^2} = rC \tag{1.6}$$

The well-known Black-Scholes formula follows from solving the PDE in (1.6).

1.2 When Investors are Only Sufficiently Rational (Arbitrage Opportunities are Eliminated but Expectations may not be Perfectly Rational)

In this section, we first apply the solution of the brain-resource allocation problem presented in Alonso et al (2014) to option pricing. Then we impose the condition that arbitrage opportunities are eliminated. We show that this amounts to perfectly rational expectations about reward and imperfect expectations about risk.

As in Alonso et al (2014), we consider the realistic case where the CES knows that the resource need of Task 0 (motor task) which is θ_0; however, it does not know θ_1 (the resource need of Task 1) and θ_2 (the resource need of Task 2). The CES is aware of the general distributions from which the resource needs for expected reward (Task 1) and risk forecasting (Task 2) are drawn. We denote these distribution functions by $F^1(\theta_1)$ and $F^2(\theta_2)$ respectively. Assuming that the resource constraint always binds, it follows that the resources allocated to Task 0 equal: $x_0 = k - x_1 - x_2$.

The CES solves the optimization problem of allocating internal brain resources to various tasks by computing a benefit function, u_j for $j \in \{0,1,2\}$ associated with each task. The benefit function takes its maximum value when $x_j = \theta_j$ for $j \in \{0,1,2\}$. That is, the benefit function for a task takes it maximum value when its resource needs are fully met. When $x_j < \theta_j$, there is a loss. When there are too many resources, $x_j > \theta_j$, there is no benefit. It could even be damaging as too much attention could be counterproductive. In any case, we assume that the benefit function is non-increasing when $x_j \geq \theta_j$. Task importance is captured by the parameter α_j for $j \in \{0,1,2\}$.

The CES solves the following optimization problem:

$$max \iint [\alpha_1 u_1(x_1 - \theta_1) + \alpha_2 u_2(x_2 - \theta_2) + \alpha_0 u_0(k - x_1 - x_2 - \theta_0)]\, dF^1 dF^2 \qquad (1.7)$$

s.t

$x_1 \geq 0, x_2 \geq 0, x_1 + x_2 \leq k.$

As shown in Alonso et al (2014), the optimal resource allocation mechanism when the CES is imperfectly informed about the resource needs of cognitive tasks has the following feature:

A resource cap on each task is imposed depending on relative task importance and task complexity. If the cap constrains both tasks then each task hits it cap and that's the amount of resources they get. However, if the cap constrains only one task, then the leftover resources from the unconstrained task are optimally allocated between task 0 and the other constrained task.

If both tasks are constrained, then the expected marginal benefit of allocating resources to each task must be equal to the marginal benefit of allocating resources to task 0. Defining k_1 and k_2 as the minimum caps on Task 1 and Task 2 respectively, in equilibrium:

$$\alpha_1 E[u'(k_1-\theta_1)|\theta_1 \geq k_1] = \alpha_2 E[u'(k_2-\theta_2)|\theta_2 \geq k_2] = \alpha_0 u'(k-k_1-k_2-\theta_0) \quad (1.8)$$

If only one task is constrained, then the leftover resources from the unconstrained task are optimally allocated between task 0 and the constrained task. For example, if Task 1 is unconstrained then the following must be true:

$$\alpha_2 E[u'(y_2(\theta_1)-\theta_2)|\theta_2 \geq y_2(\theta_1)] = \alpha_0 u'(k-\theta_1-y_2(\theta_1)-\theta_0) \quad (1.9)$$

That is, the expected marginal benefit of allocating resources to Task 2 is equal to the marginal benefit of allocating resources to Task 0.

Alonso et al (2014) show that the optimal resource allocation mechanism based on (1.7), (1.8), and (1.9) implies the following resource caps $\overline{x_1^*}$ and $\overline{x_2^*}$:

$$\overline{x_1^*} = \begin{cases} y_1(\theta_2) & if\ \theta_2 < k_2 \\ k_1 & if\ \theta_2 \geq k_2 \end{cases}$$

$$\overline{x_2^*} = \begin{cases} y_2(\theta_1) & if\ \theta_1 < k_1 \\ k_2 & if\ \theta_1 \geq k_1 \end{cases}$$

To illustrate the above result, it is useful to take a simple example with a quadratic benefit function. Setting the benefit function, $u_j = -\left(x_j - \theta_j\right)^2$ for $j \in \{0,1,2\}$, and assuming that resource needs in both Task 1 and Task 2 come from the same uniform distribution over $[0, \bar{\theta}]$, and setting $\theta_0 = k$ for simplicity, we get:

$$\overline{x_1^*} = \begin{cases} y_1(\theta_2) = \dfrac{\alpha_1 \bar{\theta}}{\alpha_1 + 2\alpha_0} - \dfrac{2\alpha_0 \theta_2}{\alpha_1 + 2\alpha_0} & if\ \theta_2 < k_2 \\ k_1 = \left(\dfrac{\alpha_1\alpha_2 + 2\alpha_0\alpha_1 - 2\alpha_0\alpha_2}{\alpha_1\alpha_2 + 2\alpha_0\alpha_1 + 2\alpha_0\alpha_2}\right)\bar{\theta} & if\ \theta_2 \geq k_2 \end{cases} \quad (1.10)$$

$$\overline{x_2^*} = \begin{cases} y_2(\theta_1) = \dfrac{\alpha_2 \bar{\theta}}{\alpha_2 + 2\alpha_0} - \dfrac{2\alpha_0 \theta_1}{\alpha_2 + 2\alpha_0} & if\ \theta_1 < k_1 \\ k_2 = \left(\dfrac{\alpha_1\alpha_2 + 2\alpha_0\alpha_2 - 2\alpha_0\alpha_1}{\alpha_1\alpha_2 + 2\alpha_0\alpha_2 + 2\alpha_0\alpha_1}\right)\bar{\theta} & if\ \theta_1 \geq k_1 \end{cases} \quad (1.11)$$

Two important points to note from (1.10) and (1.11) are as follows:

1) The initial resource caps, k_1 and k_2, depend on relative task importance. If the CES considers a task to be more important, its initial resource cap is set at a higher level.

2) Simple enough tasks are successfully completed as corresponding systems are unconstrained.

Taking a numerical example by setting $\alpha_1 = 2$, $\alpha_2 = 1$, $\alpha_0 = 0.5$, and $\bar{\theta} = 100$, (1.10) and (1.11) lead to:

$$\overline{x_1^*} = \begin{cases} y_1(\theta_2) = 66.66 - \dfrac{\theta_2}{3} & if\ \theta_2 < 20 \\ k_1 = 60 & if\ \theta_2 \geq 20 \end{cases}$$

$$\overline{x_2^*} = \begin{cases} y_2(\theta_2) = 50 - \dfrac{\theta_1}{2} & if\ \theta_1 < 60 \\ k_1 = 20 & if\ \theta_1 \geq 60 \end{cases}$$

As expected, because $\alpha_1 > \alpha_2$, the initial cap is set at a higher value in Task 1 (60) when compared with the initial cap of 20 in Task 2. If the actual resource need in both Task 1 and Task 2 is 50, then Task 1 is successfully completed with 10 units of resources to spare. 5 of these units go towards Task 2. Overall, this makes $m_1 = 1$ and $m_2 = 0.5$.

Next, we impose the condition that market interactions eliminate arbitrage opportunities. We allow for market imperfections in the form of proportional transaction costs. The idea is that different traders would adjust the initial predictions differently with incoming information based on the optimal resource allocation decisions in their respective brains. However, only those traders survive against whom arbitrage opportunities do not exist. Leland (1985) derives option pricing bounds when there are proportional transaction costs. Constantinides and Perrakis (2002) further tighten these bounds by using stochastic dominance arguments.[18] We impose the condition that only those traders survive who value the call option to be within the bounds derived in Constantinides and Perrakis (2002).

For our purposes, only the call option upper bound derived in Constantinides and Perrakis (2002) is relevant.[19] The call upper-bound[20] is obtained as follows: If the proportional transaction costs associated with stock purchase and stock sale are z_1 and z_2 respectively, then the expected call payoff is divided by the discount-rate on the underlying stock with the answer then multiplied by $\left(\frac{1+z_1}{1-z_2}\right)$.

That is, the European call option upper bound, $\bar{C}$, is:

$$\bar{C}_t = \left(\frac{1+z_1}{1-z_2}\right)\frac{E[C_{t+1}|S_t]}{R_s} \tag{1.12}$$

where $R_s = 1 + r_s$ is the gross expected return (discount-rate) on the underlying stock.

[18] These bounds are related to option pricing bounds derived in Perrakis and Ryan (1984) and further extended by Ritchken (1985), Levy (1985), Perrakis (1986) (1988), Ritchken and Kuo (1988), and Constantinides and Zariphopoulou (1999) (2001). A key advantage of these bounds is that even though they work for lognormal distribution, they do not assume it; hence, stochastic volatility and jumps can be accommodated as well. In contrast, Leland's (1985) model assumed constant volatility, and a similarly loose upward bound has been found when his model was updated/adapted to allow for stochastic volatility (Nguyen and Pergamenschhikov 2017).

[19] The lower bound in Constantinides and Perrakis (2002) is below the Black-Scholes formula. As the adjusted Black-Scholes price is always above the Black-Scholes price, the lower bound is always met.

[20] See proposition 1 in Constantinides and Perrakis (2002).

If the underlying stock follows geometric Brownian motion, then the above bound translates into the following:[21]

$$\bar{C} = \left\{SN(d_1^*) - Ke^{-\mu(T-t)}N(d_2^*)\right\}\left(\frac{1+z_1}{1-z_2}\right) \tag{1.13}$$

where $d_1^* = \frac{ln\left(\frac{S}{K}\right)+\left(\mu+\frac{\sigma^2}{2}\right)(T-t)}{\sqrt{\sigma(T-t)}}$, *and* $d_2^* = \frac{ln\left(\frac{S}{K}\right)+\left(\mu-\frac{\sigma^2}{2}\right)(T-t)}{\sqrt{\sigma(T-t)}}$

In other words, the European call option upper bound is obtained by replacing the risk-free rate, r, with μ, in the Black-Scholes price, and multiplying the resulting price with the round-trip transaction cost of trading in the stock, $\left(\frac{1+z_1}{1-z_2}\right)$.

In the brain-centric framework:

$$C = \frac{\{m_1^* E[dC] + (1-m_1^*)E[dS]\}}{\left\{r^* + \frac{\partial C}{\partial S}\frac{S}{C}(\mu - r^*)\right\}dt} \tag{1.14}$$

where $r^* = r + \delta(1-m_2^*)$, m_1^* and m_2^* are the fractions of required resources allocated to reward and risk expectations respectively by the CES (obtained by solving the optimization problem of the brain's internal resources as discussed earlier).

Proposition 1 (Sufficiently Rational Expectations: Eliminating Arbitrage Opportunities) ***The European call option price in the brain-centric approach, always lies within the no-arbitrage bounds, as long as the required resources to form rational expectations are allocated to the reward estimation task. That is, forming rational expectations in the risk estimation task is not necessary if risk expectations are in accord with the predictive brain.***

Proof

If $m_1^* = 1$ and $m_2^* < 1$, then it immediately follows from (1.14) that the call option price is greater than the Black-Scholes price. This is because the numerator in (1.14) is equal to the numerator that is obtained with the Black-Scholes approach (rational expectations);

[21]Although not explicitly indicated by Constantinides and Perrakis (2002), those authors did apply equation (1.13) in Table 1 of their paper to illustrate the relative tightness of the upper bound. That equation (1.13) follows from equation (1.12) for lognormally distributed stocks can easily be shown.

however, the denominator is smaller. Hence, the lower bound (which is below the Black-Scholes price) is always satisfied. If $m_1^* = 1$ and $m_2^* < 1$, the PDE corresponding to the call price in (1.14) is:

$$\frac{\partial C}{\partial t} + r^* S \frac{\partial C}{\partial S} + \frac{\sigma^2 S^2}{2} \frac{\partial^2 C}{\partial S^2} = r^* C \tag{1.15}$$

The solution of (1.15) is an adjusted Black-Scholes formula in which the risk-free rate, r, is replaced with a higher rate, r^*. The upper-bound in (1.13) is the Black-Scholes price with the risk-free rate, r, replaced with a higher rate, μ, and the resulting price multiplied by a factor bigger than 1, $\left(\frac{1+z_1}{1-z_2}\right)$. As the call price rises with the risk-free rate, and $r^* \leq \mu$, it follows that the call option price resulting from (1.15) is always less than the call upper-bound.

▪

Proposition 1 shows that, if a trader forms rational expectations about call payoffs, and her initial risk expectations are influenced by the risk of the underlying stock with an attempt made to adjust it in the right direction without reaching full adjustment, then arbitrage opportunities are not created against such a trader. That is, a trader may survive just by forming rational expectations about the expected call payoff while deviating from perfectly rational expectations about risk in a specific way consistent with the brain-centric notion of the predictive brain.

It is plausible to expect that sufficiently rational expectations are formed that keep the price within the no-arbitrage interval while deviating from the idea of perfection in expectations. This is all the more likely, as picking the Black-Scholes price from the interval (which coincides with rational expectations) does not lead to any specific advantage as any price within the interval prevents arbitrage opportunities. So, an investor may only choose to spend brain resources as far as sufficiently rational expectations are formed without seeking perfection.

The Brain's Internal Resource Allocation Problem

(Graphical Representation of the Optimal Solution)

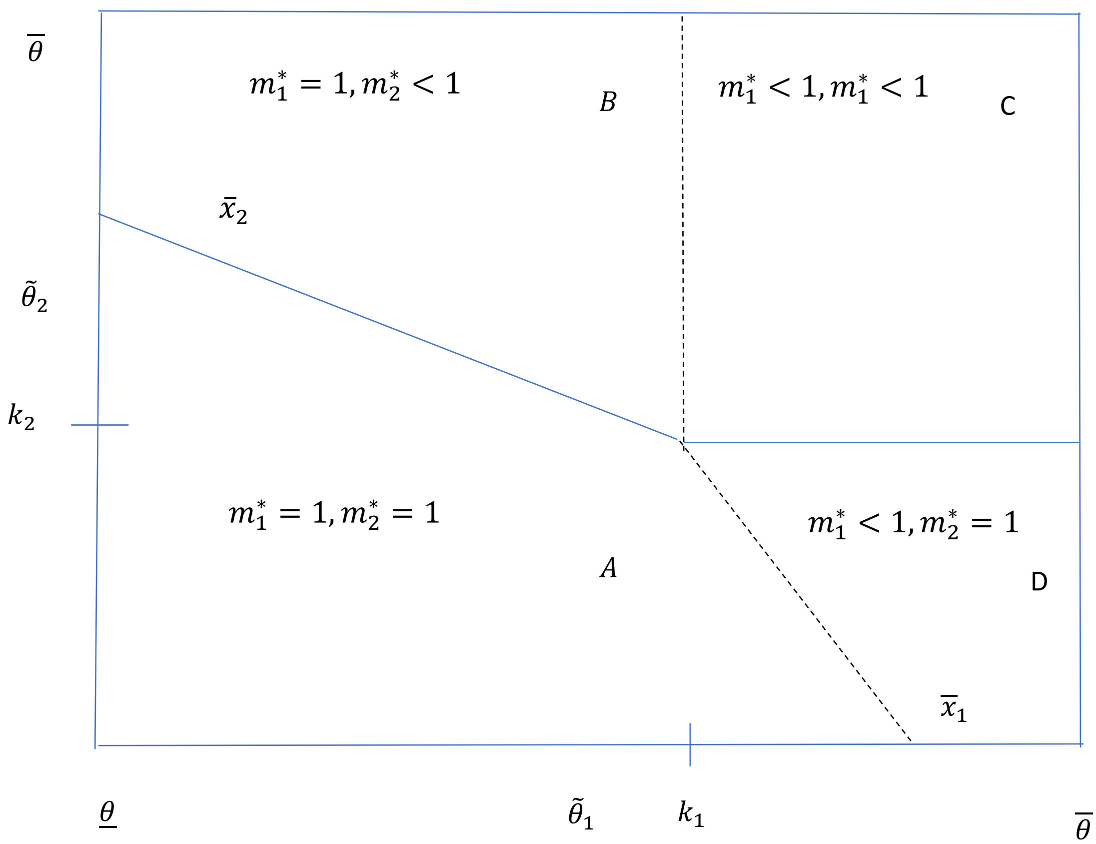

Figure 1 For reward estimation, the actual resource need for forming rational expectations, $\tilde{\theta}_1$, is randomly drawn from a distribution, F, over $[\underline{\theta}, \overline{\theta}]$. Similar holds for the resource need of risk estimation, $\tilde{\theta}_2$. The optimal resource allocation has the property that resource caps k_1 and k_2 are respectively set for the two tasks. If the resource needs exceed the caps in both tasks, then we are in region C where $m_1^* < 1$ and $m_2^* < 1$. If the 'reward estimation' task is completed with the cap, k_1, not reached, then some of the leftover resources are optimally allocated to the 'risk estimation' task. That's why, the resource allocation to 'risk estimation' task, $\overline{x}_2$, has a non-zero slope when $\tilde{\theta}_1 < k_1$. Similarly, resource allocation to 'reward estimation' task, $\overline{x}_1$, exceeds k_1 when $\tilde{\theta}_2 < k_2$. If the market is constrained to always conform to perfectly rational expectations in both tasks, then we are restricted to stay in region A. However, if sufficiently rational expectations are also allowed (Proposition 1), then the region of interest expands to A+B.

Figure 1 illustrates. It shows the region, A, within which perfectly rational expectations are formed about both reward and risk, $m_1^* = 1$ and $m_2^* = 1$. It also shows the regions where rational expectations are only formed about either reward or risk. Note the region, B, where rational expectations are only formed about reward but not risk: $m_1^* = 1$ and $m_2^* < 1$. With the market behaviour strictly restricted to rational expectations, only region A is relevant.

However, when expectations are allowed to be sufficiently rational (may deviate from rational expectations without creating arbitrage opportunities) then the relevant region for market behaviour is larger: A + B.

We end up in region B if the following holds: Rational expectations about payoffs are formed if the actual resource need for the task, $\tilde{\theta}_1$, is less than the cap, k_1. In that case, $m_1^* = 1$. The leftover resources are then optimally allocated between the 'risk estimation' task and the motor task (Task 0), which leads to an increase in the cap for 'risk estimation' task from k_2 to $y_2(\theta_1) = \frac{\alpha_2 \bar{\theta}}{\alpha_2 + 2\alpha_0} - \frac{2\alpha_0 \theta_1}{\alpha_2 + 2\alpha_0}$ as per (1.11). If the resource need of 'risk estimation' task, $\tilde{\theta}_2$, is still greater than $y_2(\theta_1)$, then $m_2^* < 1$.

2. The Black-Scholes Formula with Sufficiently Rational Expectations

With the condition that expectations can just be sufficiently rational (deviate from perfection without creating arbitrage opportunities), the corresponding PDE can be obtained from (1.5) by setting $m_1^* = 1$:

$$\frac{\partial C}{\partial t} + r^* S \frac{\partial C}{\partial S} + \frac{\sigma^2 S^2}{2} \frac{\partial^2 C}{\partial S^2} = r^* C \tag{2.1}$$

where $r^* = r + \delta(1 - m_2^*)$

Note, that the above PDE is identical to the Black-Scholes PDE with the risk-free rate, r, replaced with a higher rate, r^*. Hence, following the same steps as in the derivation of the Black-Scholes formula with r replaced with r^* leads to the Black-Scholes formula with sufficiently rational expectations.

Proposition 2 presents the solution to (2.1).

Proposition 2 (Call Option: Sufficiently Rational Expectations) ***The Black-Scholes formula for a European call option with sufficiently rational expectations is given by:***

$$\boldsymbol{C = SN(d_1^*) - Ke^{-r^*(T-t)}N(d_2^*)} \qquad (2.1a)$$

where $\boldsymbol{r^* = r + \delta(1 - m_2^*)}$***,*** $\boldsymbol{\delta}$ ***is the risk-premium on the underlying,*** $\boldsymbol{0 < m_2^* \leq 1}$ ***is the fraction of the resource need that has been met for the brain system forecasting risk,*** $\boldsymbol{K}$ ***is the strike price,*** $\boldsymbol{d_1^* = \frac{ln\left(\frac{S}{K}\right)+\left(r^*+\frac{\sigma^2}{2}\right)(T-t)}{\sqrt{\sigma(T-t)}}}$***, and*** $\boldsymbol{d_2^* = \frac{ln\left(\frac{S}{K}\right)+\left(r^*-\frac{\sigma^2}{2}\right)(T-t)}{\sqrt{\sigma(T-t)}}}$

The only difference between the call formula with sufficiently rational expectations and with the standard Black-Scholes formula (which assumes perfectly rational expectations) is replacement of r with a higher rate r^*. When resources needed to form perfectly rational expectations about risk are made available, that is, when $m_2^* = 1$, the formula in proposition 2 converges to the standard Black-Scholes formula. On the other hand, when $m_2 < 1$, investors essentially underestimate the risk of the call option, as shown in equation (1.4b), and hence require a lower return on the option, as shown in equation (1.4f), thus resulting in a higher call option value.[22]

The formula for a European put option, P, is similarly derived below (alternatively, it can be obtained directly from the call formula by applying put-call parity). The price of a put option is given by:

$$P = \frac{E[dP]}{\overline{r_p}'dt} \qquad (2.2)$$

Applying put-call parity and noting that $\frac{\partial P}{\partial S}\frac{S}{P} = \frac{C}{P}\left(\frac{\partial C}{\partial S}\frac{S}{C} - \frac{S}{P}\right)$:

$$\overline{r_p}'dt = rdt + \alpha_C \frac{C}{P}dt + \frac{\partial P}{\partial S}\frac{S}{P}\delta dt \qquad (2.3)$$

[22] In particular, with investors being unable to exactly analyze all the information needed to estimate uncertain payoffs to options (and the relative utility/benefit of option positions in different states of the world that could include underlying asset price jumps, stochastic volatility, and higher order return moments which would be very difficult to estimate/forecast in a dynamically changing environment), they adjust the required return on an option only part of the way toward the correct expected return on the option that would be estimated if returns were lognormally distributed and volatility known with certainty.

where $\alpha_C = \delta(1 - m_2^*)\left(1 - \frac{\partial C}{\partial S}\frac{S}{C}\right) < 0$

From Ito's Lemma:

$$E[dP] = \left(\frac{\partial P}{\partial t} + \mu S\frac{\partial P}{\partial S} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 P}{\partial S^2}\right) dt \quad (2.4)$$

Substituting (2.4) and (2.3) in (2.2):

$$\frac{\partial P}{\partial t} + r^* S\frac{\partial P}{\partial S} + \frac{\sigma^2 S^2}{2}\frac{\partial^2}{\partial S^2} = r^* P - Ke^{-r(T-t)} \quad (2.5)$$

Solving (2.5) leads to the European put option pricing formula with sufficiently rational expectations.

Proposition 3 (Put Option: Sufficiently Rational Expectations) ***The Black-Scholes formula for a European put option with sufficiently rational expectations is given by:***

$$\boldsymbol{Ke^{-r^*(T-t)}N(-d_2^*) - SN(-d_1^*) + Ke^{-r(T-t)}\left(1 - e^{-\delta(1-m_2^*)(T-t)}\right)} \quad (\mathbf{2.5a})$$

where $\boldsymbol{r^* = r + \delta(1 - m_2^*)}$***,*** $\boldsymbol{\delta}$ ***is the risk-premium on the underlying,*** $\boldsymbol{m_2^*}$ ***is the fraction of the resource need that has been met for the brain system forecasting risk of the corresponding call option,*** $\boldsymbol{K}$ ***is the strike price,*** $\boldsymbol{d_1^* = \frac{ln\left(\frac{S}{K}\right) + \left(r^* + \frac{\sigma^2}{2}\right)(T-t)}{\sqrt{\sigma(T-t)}}}$***, and*** $\boldsymbol{d_2^* = \frac{ln\left(\frac{S}{K}\right) + \left(r^* - \frac{\sigma^2}{2}\right)(T-t)}{\sqrt{\sigma(T-t)}}}$

3. Option Pricing Puzzles when Traders are (Only) Sufficiently Rational

Black-Scholes formula when adjusted for sufficiently rational expectations (proposition 2 and 3) generates the implied volatility skew, and potentially contributes to inferior zero-beta-straddle returns, superior covered-call writing returns, as well as leverage-adjusted return puzzles. This is discussed below.

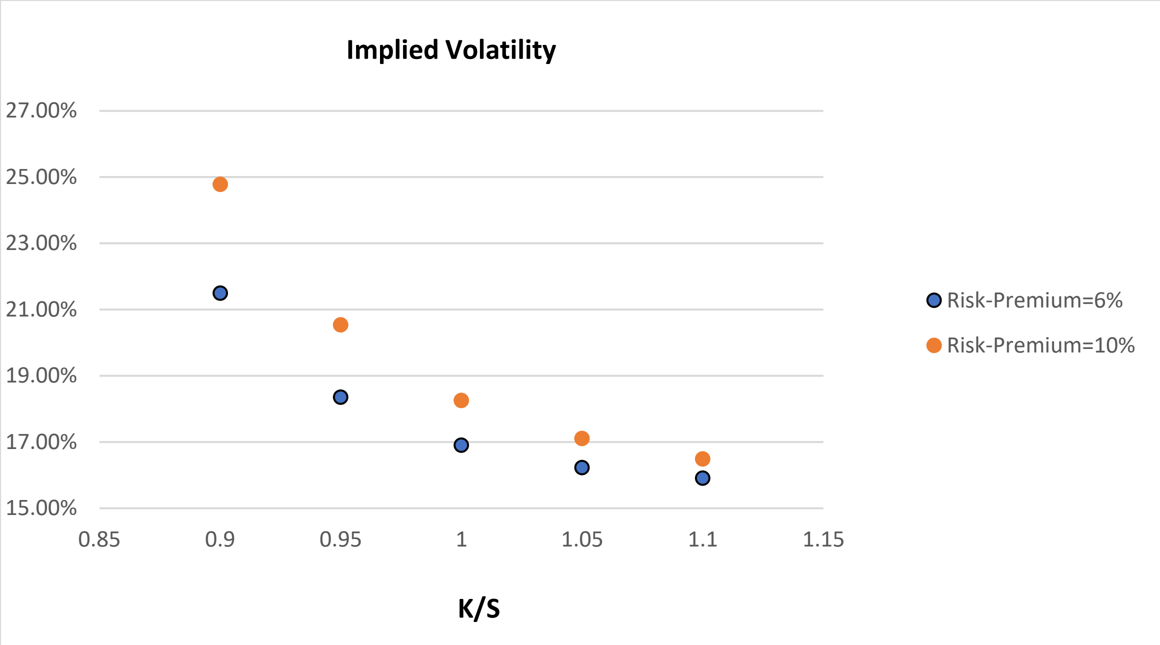

Figure 2 The adjusted formula generates implied volatility skew which steepens as the risk-premium on the underlying increases. The parameter values are: $S = 100, r = 0, T - t = 0.25\ year, \sigma = 15\%$ with the following two values of, δ, the risk-premium on the underlying: 6% and 10%

3.1 Implied Volatility Skew

The option pricing formula adjusted for sufficiently rational expectations (formulas in proposition 2 and 3) generates the implied volatility skew. If market prices are determined by the adjusted formula and the Black-Scholes formula (not adjusted for sufficiently rational expectations) is used to back-out implied volatility, then a skew is observed which steepens as the risk-premium on the underlying increases.[23] Figure 2 provides an illustration.

3.2 Zero-Beta-Straddle

The returns from the zero-beta-straddle have been below the risk-free rate (in violation of the Black-Scholes model) (Coval and Shumway 2001). However, when the Black-Scholes

[23] Derman (2002) previously indicated that using a higher interest rate in the Black-Scholes (1973) model could create an option-implied volatility skew, although the theory he developed was based on a hypothesis of undiversified investors (like day traders of the late 1990s) potentially expecting excessively high returns on stocks with above-average trading frequencies (and created deviations from put-call parity).

formula is adjusted for sufficiently rational expectations, returns are below the risk-free rate. Hence, sufficiently rational expectations potentially contribute to this puzzle.

With sufficiently rational expectations:

$$\overline{r_c}' = r + \alpha_C + \frac{\partial C}{\partial S}\frac{S}{C}(\mu - r) \tag{3.1}$$

where $\alpha_C = \delta(1 - m_2^*)\left(1 - \frac{\partial C}{\partial S}\frac{S}{C}\right) < 0$. Note, that with perfectly rational expectations (as in the standard Black-Scholes model), $\alpha_C = 0$.

From (2.3)

$$\overline{r_p}' = r + \alpha_C \frac{C}{P} + \frac{\partial P}{\partial S}\frac{S}{P}(\mu - r) \tag{3.2}$$

Zero-beta-straddle portfolio satisfies the following condition (Coval and Shumway 2001):

$$\theta_C \beta_C + (1 - \theta_C)\beta_P = 0 \tag{3.3}$$

where $\theta_C = \frac{-C\beta_C + S}{P\beta_C - C\beta_{C+S}}$ and $(1 - \theta_C) = \frac{P\beta_C}{P\beta_C - C\beta_C + S}$.

The expected return from the zero-beta-straddle strategy is:

$$\theta_C \overline{r_c}' + (1 - \theta_C)\overline{r_p}' \tag{3.4}$$

Substituting (3.1) and (3.2) in (3.4) and subtracting the portfolio return expected under the Black-Scholes model (with perfectly rational expectations) leads to the following:

$$Excess\ Return = \alpha_C\left[\theta_C + (1 - \theta_C)\frac{C}{P}\right] \tag{3.5}$$

(3.5) is negative as $\alpha_C < 0$.

Proposition 4 (Zero-Beta-Straddle) ***The return from a zero-beta-straddle strategy with sufficiently rational expectations is lower than the risk-free rate. The amount by which the return is lower than the risk-free rate is given by:***

$$\boldsymbol{\alpha_C\left[\theta_C + (1 - \theta_C)\frac{C}{P}\right] = \alpha_C \cdot \frac{S}{P\beta_C - C\beta_C + S}}$$

where $\boldsymbol{\alpha_C = \delta(1 - m_2^*)\left(1 - \frac{\partial C}{\partial S}\frac{S}{C}\right) < 0}$

3.3 Covered Call Writing

The returns from the covered-call writing strategy have been larger than what is expected under the Black-Scholes model (Whaley 2002). Adjusting the Black-Scholes model for sufficiently rational expectations leads to higher returns from the covered-call writing strategy.

Covered-call portfolio has the following return:

$$\frac{S}{S-C}\mu - \frac{C}{S-C}\overline{r_c}' \tag{3.6}$$

Substituting (3.1) in (3.6) and subtracting the corresponding portfolio return under the Black-Scholes model:

$$Excess\ Return = -\frac{S}{S-C}\alpha_C > 0$$

Proposition 5 (Covered-Call Writing) ***Covered-call writing strategy with sufficiently rational expectations generates excess returns over what is expected under the Black-Scholes model. The excess return is given by:***

$$\boldsymbol{-\frac{S}{S-C}\alpha_C > 0}$$

3.4 Leverage Adjusted Returns

Constantinides et al (2013) present the following empirical findings which are inconsistent with the Black-Scholes model:

1) Leverage-adjusted call returns are lower than the return on the underlying.

2) Leverage-adjusted call returns fall as the ratio of strike to spot or K/S rises.

3) Leverage-adjusted put return are higher than the return on the underlying.

4) Leverage-adjusted put returns fall as K/S rises.

The Black-Scholes formula with sufficiently rational expectations generates all of the above phenomena.

Leverage-adjusted call option return is given by:

$$R_{LC} = \left(\frac{1}{\frac{\partial C}{\partial S}\frac{S}{C}}\right) r + \frac{\alpha_C}{\frac{\partial C}{\partial S}\frac{S}{C}} + \delta + \left(1 - \frac{1}{\frac{\partial C}{\partial S}\frac{S}{C}}\right) r$$

$$\Rightarrow R_{LC} = \mu + \frac{\alpha_C}{\frac{\partial C}{\partial S}\frac{S}{C}}$$

$$\Rightarrow R_{LC} = \mu + \frac{\delta(1-\bar{m}_2)\left(1 - \frac{\partial C}{\partial S}\frac{S}{C}\right)}{\frac{\partial C}{\partial S}\frac{S}{C}}$$

$$\Rightarrow R_{LC} = \mu + \frac{\delta(1-\bar{m}_2)}{\frac{\partial C}{\partial S}\frac{S}{C}} - \delta(1-\bar{m}_2) \tag{3.7}$$

It is clear from (3.7) that not only the leverage-adjusted call option return, R_{LC}, is lower than the return on the underlying, μ, but that it also falls further as K/S rises.

Leverage-adjusted put option return is given by:

$$R_{LP} = \left(\frac{1}{\frac{\partial P}{\partial S}\frac{S}{P}}\right) r + \frac{\alpha_C}{\frac{\partial P}{\partial S}\frac{S}{P}} \cdot \frac{C}{P} + \delta + \left(1 - \frac{1}{\frac{\partial P}{\partial S}\frac{S}{P}}\right) r$$

$$\Rightarrow R_{LP} = \mu + \frac{\alpha_C}{\frac{\partial P}{\partial S}\frac{S}{C}} \cdot \frac{C}{P}$$

$$\Rightarrow R_{LP} = \mu + \frac{\delta(1-\bar{m}_2)\left(1 - \frac{\partial C}{\partial S}\frac{S}{C}\right)}{\frac{\partial P}{\partial S}\frac{S}{C}} \cdot \frac{C}{P}$$

$$\Rightarrow R_{LP} = \mu + \left(\frac{K}{S}\right)\frac{\delta(1-\bar{m}_2)e^{-r^*(T-t)}N(d_2)}{1-N(d_1)} \tag{3.8}$$

It follows from (3.8) that the leverage-adjusted put option return is larger than the return on the underlying; however, it falls as K/S rises.

Proposition 6 With sufficiently rational expectations, ***leverage-adjusted call option return is smaller than the return on the underlying and falls further as K/S rises, whereas leverage-adjusted put option return is larger than the return on the underlying; however, it falls as K/S rises.***

5. Model Calibration and Performance

The Black-Scholes formula only has one unobservable, which is the standard deviation, σ, of the underlying stock. Allowing for sufficiently rational expectations also makes the interest rate unobservable, as the risk-free rate, r, in the Black-Scholes formula is replaced with a higher rate, r^*. In this section, we calibrate the Black-Scholes formula with sufficiently rational expectations to quantify the improved capacity of the resources-constrained brain model to explain market prices compared to the perfect forecast Black-Scholes model.

To calibrate the model, we use call option prices on S&P 500 index at close on July 9, 2021 with an expiry of 1 week (on July 16), 2 weeks (on July 23), 3 weeks (July 30), and 4 weeks (August 6). The data is obtained from Barchart.com. The midpoint between bid and ask is used for the option price. The S&P 500 index value on July 9 was 4369.55.[24] We use 'near-the-money' options as defined by Barchart.com. They are the most heavily traded options with high volumes leading to prices that are expected to be as close to efficiency as possible. All strikes from 4320 to 4415 (with a gap of 5) are in 'near-the-money' category. That is, there are 20 call options for each expiry. US 1 month treasury bill rate on July 9 is used for the risk-free rate, which is 0.05% (source: Yahoo Finance).

[24] The index value is adjusted for dividends by subtracting the present value of dividends over the life of the option. That is, for an option contract with a life of 14 days, the present value of dividends is calculated as: $PV(D) = \sum_i^{14} e^{-r*i/365} D_i$ where D_i is the daily dividend and r is the risk-free rate. The present value is then subtracted from the index level to obtain the dividend-exclusive spot index level. The average daily dividend yield estimate in July 2021 is 1.33% (source: https://www.multpl.com/s-p-500-dividend-yield/table/by-month). For simplicity, we use the constant dividend yield of 1.33% to estimate daily dividends over the life of the options as this only introduces a negligible error.

Table 1

A comparison of performance with call options on S&P 500 index on July 9, 2021

	Original Black-Scholes Rational Expectations			**Adjusted Black-Scholes Sufficiently Rational Expectations**			
	σ	SSE	RMSE	σ	r^*	SSE	RMSE
1 week	9.66%	71.8	1.89	7.7%	9.56%	0.424	0.146
2 weeks	10.08%	186.29	3.05	7.19%	10.25%	0.794	0.2
3 weeks	10.91%	290.5	3.81	7.27%	10.64%	0.82	0.2
4 weeks	11.32%	345.61	4.16	7.32%	10.19%	1.0092	0.225
1 month US T-Bill rate is used as the risk-free rate, which was 0.05% on July 9.							

Data Source: Barchart.com

To calibrate the model, we use the simplest approach of minimizing the mean sum-of-squared-differences:

$$MSE(M) = \sum_{i}^{N} \frac{\left(C_i^M(K_i, T_i) - C_i^{Mkt}(K_i, T_i)\right)^2}{N} \tag{5.1}$$

where $C_i^M(K_i, T_i)$ is model price and $C_i^{Mkt}(K_i, T_i)$ is the corresponding market price and N is the number of data points in the sample.

We calibrate the Black-Scholes and the adjusted Black-Scholes models at each maturity. So, 4 sets of calibrations are performed (one for each maturity). The Black-Scholes model only requires that σ be calibrated. The adjusted Black-Scholes model requires calibration of σ and r^*. The calibrated parameter values, sum-of-squared-errors (SSE), and root-mean-squared-errors (RMSE) are reported in Table 1 for each case. An eyeball comparison of SSE and RMSE across the two models appears to show a substantially better fit with sufficiently rational expectations.

To assess, whether the superior performance is statistically significant, we use the standard F statistic for nested models:

$$F = \frac{\left(\frac{SSE1 - SSE2}{p_2 - p_1}\right)}{\frac{SSE_2}{N - p_2}}, \tag{5.2}$$

where $SSE1$ is the SSE in the Black-Scholes model, $SSE2$ is the SSE in the adjusted Black-Scholes model, p_1 is the number of parameters estimated in the Black-Scholes model, p_2 is the number of parameters estimated in the adjusted model, and N is the number of observations in the sample. The test statistic values are: 9.36 (1 week maturity), 12.98 (2 weeks maturity), 19.65 (3 weeks maturity), and 18.97 (4 weeks maturity). The threshold value obtained from an F distribution table with degrees-of-freedom $(p_2 - p_1, N - p_2) = (1, 18)$ is 8.285 at 1% significance level. As the test statistic is greater than the threshold value at every maturity, it follows that the adjusted model is a better fit even at 1% significance level for all maturities.

5.1 The value of m_2

As the higher rate, $r^* = r + \delta(1 - m_2) = m_2 r + (1 - m_2)\mu$ (where μ is the expected return from the underlying), inferring m_2 from the calibrated value of r^* requires estimating μ. The average annual total return (price return plus dividend return) from S&P 500 index since inception (1926-2020) is 12.16%. The average total return over the past 5 years (2016-2020) is 15.86%. The total return from S&P 500 index in 2020 was 18.4%.[25] We infer m_2 based on all three cases. That is, 95-year-average index return, 5-year-average index return, and 2020 index return. The inferred values of m_2, corresponding to each case, are reported in Table 2. The value of m_2 ranges from 0.125 to 0.482. In other words, the initial risk prediction is adjusted from 12.5% to over 48% of the correct distance depending on one's estimate of the underlying return.

[25] Source: https://www.slickcharts.com/sp500/returns

Table 2

Inferred value of m_2 based on historical S&P 500 total returns and the calibrated r^*

	95-year-average	5-year-average	2020
1 week	0.215	0.398	0.482
2 weeks	0.158	0.355	0.444
3 weeks	0.125	0.33	0.423
4 weeks	0.163	0.358	0.447

For investors assuming underlying asset price jumps and stochastic volatility, the resource-constrained brain theory has somewhat different implications, although the modification to the respective valuation model is similar, as shown in the following Section 6.

6. The Brain-Centric Approach with Stochastic Volatility and Jumps

In this section, we consider the brain-centric approach when there is stochastic volatility and/or jumps in asset price underlying an option. By introducing a factor, $\bar{A}_t$, one can relate the risk of a call option with the risk of the underlying as follows:

$$E\left[\frac{d\Lambda_t}{\Lambda_t}\frac{dC_t}{C_t}\right] = \bar{A}_t E\left[\frac{d\Lambda_t}{\Lambda_t}\frac{dS_t}{S_t}\right] \tag{6.1}$$

where, as mentioned earlier, $E\left[\frac{d\Lambda_t}{\Lambda_t}\frac{dC_t}{C_t}\right]$ is the risk of the call option, and $E\left[\frac{d\Lambda_t}{\Lambda_t}\frac{dS_t}{S_t}\right]$ is the risk of the underlying stock.

Suppressing the time-subscript for simplicity of notation, $\bar{A} = \frac{\partial C}{\partial S}\frac{S}{C}$ if the underlying stock follows geometric Brownian motion as in the Black-Scholes (1973) model, but $\bar{A} = \frac{\partial C}{\partial S}\frac{S}{C} + \frac{1}{C}\frac{\partial C}{\partial V}\frac{\lambda(S,V,t)}{\delta}$ if stochastic volatility is introduced as in the Heston (1993) model (see appendix A) where $\lambda(S, V, t)$ is the price of volatility risk. And, $\bar{A} = \frac{\partial C}{\partial S}\frac{S}{C} + \frac{1}{C}\frac{\partial C}{\partial V}\frac{\lambda(S,V,t)}{\delta} + \frac{ER^J}{\delta}$ if jumps are also added as in the Bates (2000) model (see appendix B) where ER^J is the price of jump risk.

In other words, the factor, $\bar{A}_t$, which relates the risk of the underlying with the risk of the call option is determined by the distributional properties of the underlying stock. As we show below, a general result about call option risk can be derived without needing to know the exact form of $\bar{A}_t$.

From (1.3b):

$$E\left[\frac{d\Lambda_t}{\Lambda_t}\frac{dS_t}{S_t}\right] = rdt - E\left[\frac{dS_t}{S_t}\right] = rdt - \mu dt = -\delta dt \tag{6.2}$$

where, as per earlier notation, δ is the risk-premium on the underlying stock and r is the risk-free rate.

It follows from (6.1) that:

$$E\left[\frac{d\Lambda_t}{\Lambda_t}\frac{dC_t}{C_t}\right] = -\bar{A}_t\delta dt \tag{6.3}$$

In the brain-centric approach, the initial risk prediction of the call option in the brain is influenced by the risk of the underlying stock. In accordance with (1.4b), this leads to the subjective risk of the call option, $E'\left[\frac{d\Lambda_t}{\Lambda_t}\frac{dC_t}{C_t}\right]$, being equal to:

$$E'\left[\frac{d\Lambda_t}{\Lambda_t}\frac{dC_t}{C_t}\right] = -(1-m_2)\delta dt - m_2\bar{A}_t\delta dt \tag{6.4}$$

Note, if $\bar{A}_t = \frac{\partial C}{\partial S}\frac{S}{C}$ then (6.4) is the same as (1.4e). In other words, (6.4) can be considered a generalization of (1.4e) when stochastic volatility and/or jumps are introduced.

The subjective expected return on the call option is then:

$$E'\left[\frac{dC_t}{C_t}\right] = \bar{r}_c'dt = rdt - E'\left[\frac{d\Lambda_t}{\Lambda_t}\frac{dC_t}{C_t}\right] = rdt + (1-m_2)\delta dt + m_2\bar{A}_t\delta dt \tag{6.5}$$

Writing, $m_2 = \big(1-(1-m_2)\big)$ in the last term of (6.5) and re-arranging leads to:

$$\bar{r}_c'dt = (r + (1-m_2)\delta)dt + \big(\mu - (r + (1-m_2)\delta)\big)\bar{A}_t dt \tag{6.6}$$

Writing $r^* = r + (1-m_2)\delta$ in (6.6) leads to:

$$\bar{r}_c' dt = r^* dt + (\mu - r^*)\bar{A}_t dt \tag{6.7}$$

If rational expectations are formed about risk, that is, the risk assessment task in the brain gets sufficient resources needed for completion, then, $m_2 = 1$ and:

$$\bar{r}_c dt = r dt + (\mu - r)\bar{A}_t dt \tag{6.8}$$

A comparison of (6.7) and (6.8) indicates that, when the initial risk prediction does not get fully adjusted to reach rational expectations then the effect is to replace the risk-free rate, r, with a higher rate, $r^* = r + (1 - m_2)\delta$. Note, that nothing has been assumed about the distributional properties of the underlying in (6.7) and (6.8) so the result is general and works for any $\bar{A}_t$. The specific value that $\bar{A}_t$ takes depends on whether the distributional properties of the underlying asset prices are those of the Black-Scholes (1973), Heston (1993), or Bates (2000) model (or any another model consistent with asset pricing theory).

If rational expectations are formed about call option payoff, $E_t[dC_t]$, but the initial risk prediction is not fully adjusted to reach rational expectations (that is, 6.7 holds instead of 6.8), then the call price is given by:

$$C_t = \frac{E_t[dC_t]}{r^* dt + (\mu - r^*)\bar{A}_t dt} \tag{6.9}$$

Note, that (6.9) is a completely general formulation of call option price and Black-Scholes (1973), Heston (1993), and Bates (2000) models can be considered specific parametrizations of (6.9). To get to a specific model from (6.9), one needs to specify the stochastic process of the underlying, then apply Ito's lemma to get $E_t[dC_t]$ and $\bar{A}_t$. For example, as shown in section 1, to get to the Black-Scholes PDE with rational expectations, insert $m_2 = 1$, $\bar{A} = \frac{\partial C}{\partial S}\frac{S}{C}$, and $E[dC] = \left(\frac{\partial C}{\partial t} + \mu S \frac{\partial C}{\partial S} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 C}{\partial S^2}\right) dt$ in (6.9) and re-arrange. To get to the Black-Scholes PDE with sufficiently rational expectations, keep $m_2 < 1$, and insert $\bar{A} = \frac{\partial C}{\partial S}\frac{S}{C}$, and $E[dC] = \left(\frac{\partial C}{\partial t} + \mu S \frac{\partial C}{\partial S} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 C}{\partial S^2}\right) dt$ in (6.9).

As further illustration, we show how to arrive at Heston PDE (equation 6 in Heston (1993)) from (6.9). In the Heston (1993) model, the stochastic process of the underlying is:

$$dS_t = \mu S_t dt + S_t \sqrt{V_t} dZ_{1t}$$

$$dV_t = k(\theta - V_t)dt + \sigma\sqrt{V_t}dZ_{2t}$$

$$E[dZ_{1t}dZ_{2t}] = \rho dt$$

So, in the Heston (1993) model, the underlying follows a Black-Scholes type process but with stochastic volatility, V_t, that has a long-run mean of θ. The rate of mean-reversion in volatility is captured by k and σ is volatility-of-volatility. Diffusion in stock price and diffusion in volatility are allowed to have a correlation, ρ.

From Ito's lemma:

$$dC_t = \left[\frac{\partial C}{\partial t} + \mu S_t \frac{\partial C}{\partial S} + k(\theta - V_t)\frac{\partial C}{\partial V} + \frac{S_t^2 V_t}{2}\frac{\partial^2 C}{\partial S^2} + \frac{\sigma^2 V_t}{2}\frac{\partial^2 C}{\partial V^2} + \rho\sigma S_t V_t \frac{\partial^2 C}{\partial S \partial V}\right]dt$$
$$+ S_t\sqrt{V_t}\frac{\partial C}{\partial S}dZ_{1t} + \sigma\sqrt{V_t}\frac{\partial C}{\partial V}dZ_{2t} \qquad (6.10)$$

It follows that:

$$E_t[dC_t] = \left[\frac{\partial C}{\partial t} + \mu S_t \frac{\partial C}{\partial S} + k(\theta - V_t)\frac{\partial C}{\partial V} + \frac{S_t^2 V_t}{2}\frac{\partial^2 C}{\partial S^2} + \frac{\sigma^2 V_t}{2}\frac{\partial^2 C}{\partial V^2}\right.$$
$$\left. + \rho\sigma S_t V_t \frac{\partial^2 C}{\partial S \partial V}\right]dt \qquad (6.11)$$

Substituting (6.11), $m_2 = 1$, and $\bar{A}_t = \frac{\partial C}{\partial S}\frac{S}{C} + \frac{1}{C}\frac{\partial C}{\partial V}\frac{\lambda(S,V,t)}{\delta}$ (see appendix A for the derivation) in (6.9) and re-arranging leads to the Heston PDE (equation 6 in Heston (1993)) (time subscripts are suppressed):

$$rC = \frac{\partial C}{\partial t} + rS\frac{\partial C}{\partial S} + [k(\theta - V_t) - \lambda(S, V, t)]\frac{\partial C}{\partial V} + \frac{S^2 V}{2}\frac{\partial^2 C}{\partial S^2} + \frac{\sigma^2 V}{2}\frac{\partial^2 C}{\partial V^2}$$
$$+ \rho\sigma SV\frac{\partial^2 C}{\partial S \partial V} \qquad (6.12)$$

If the initial risk prediction is not fully adjusted to reach rational expectations regarding risk, then keep $m_2 < 1$ in (6.9) and insert (6.11) and $\bar{A}_t = \frac{\partial C}{\partial S}\frac{S}{C} + \frac{1}{C}\frac{\partial C}{\partial V}\frac{\lambda(S,V,t)}{\delta}$ in (6.9) to get a PDE which is equivalent to (6.12) but with r replaced with $r^* = r + (1 - m_2)\delta$.

Proposition 7 ***Extending the Brain-Centric approach to Heston SV model requires replacing the risk-free rate, r, with a higher rate, $r^* = r + (1 - m_2)\delta$ in that model's European call option formula.***

Corollary 7.1 ***Any option pricing model, which is consistent with asset pricing theory (such as Bates SVJ model), can be adjusted for the brain-centric approach by replacing the risk-free rate, r, with a higher rate, $r^* = r + (1 - m_2)\delta$ in that model's European call option formula.***

7. Conclusions

Forming expectations is a cognitive task, and like any task performed in the brain, requires brain resources. Given market imperfections, and the fact that forming perfectly rational expectations about all variables of interest could be a bottomless time sinkhole, it is likely that traders may only be forming sufficiently rational expectations, which are expectations that represent estimates of fully rational expectations without creating arbitrage opportunities. In this article, by using recent findings from brain sciences, we adjust the popular Black-Scholes formula for such sufficiently rational expectations. We show that all that needs to be done to adjust the European call option formula for sufficiently rational expectations, is to replace the risk-free rate with a higher rate. We show that the same adjustment is needed even if stochastic volatility and/or jumps are introduced.

We show that the adjusted formula generates the implied volatility skew and potentially contributes to several well-known option pricing puzzles. A key implication for industry practice is that moving from perfectly rational expectations to sufficiently rational expectations only requires that one additional parameter, a higher rate that replaces the risk-free rate, is also calibrated. As an illustration, we calibrated the adjusted model to demonstrate the improvement that the adjustment brings.

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Appendix A

In the Heston (1993) model, the stochastic process of the underlying is:

$$dS_t = \mu S_t dt + S_t\sqrt{V_t}dZ_{1t}$$

$$dV_t = k(\theta - V_t)dt + \sigma\sqrt{V_t}dZ_{2t}$$

$$E[dZ_{1t}dZ_{2t}] = \rho dt$$

From Ito's lemma:

$$dC_t = \left[\frac{\partial C}{\partial t} + \mu S_t\frac{\partial C}{\partial S} + k(\theta - V_t)\frac{\partial C}{\partial V} + \frac{S_t^2 V_t}{2}\frac{\partial^2 C}{\partial S^2} + \frac{\sigma^2 V_t}{2}\frac{\partial^2 C}{\partial V^2} + \rho\sigma S_t V_t\frac{\partial^2 C}{\partial S\partial V}\right]dt$$
$$+ S_t\sqrt{V_t}\frac{\partial C}{\partial S}dZ_{1t} + \sigma\sqrt{V_t}\frac{\partial C}{\partial V}dZ_{2t} \qquad (A.1)$$

It follows that the risk of a call option is (time-subscripts are suppressed where feasible):

$$E\left[\frac{d\Lambda_t}{\Lambda_t}\frac{dC_t}{C_t}\right] = \frac{S}{C}\frac{\partial C}{\partial S}E\left[\frac{d\Lambda_t}{\Lambda_t}\sqrt{V_t}dZ_{1t}\right] + \frac{1}{C}\frac{\partial C}{\partial V}E\left[\frac{d\Lambda_t}{\Lambda_t}\sigma\sqrt{V_t}dZ_{2t}\right] \qquad (A.2)$$

From the stochastic process of the underlying stock:

$$E\left[\frac{d\Lambda_t}{\Lambda_t}\frac{dS_t}{S_t}\right] = E\left[\frac{d\Lambda_t}{\Lambda_t}\sqrt{V_t}dZ_{1t}\right] \qquad (A.3)$$

From (1.3b):

$$E\left[\frac{d\Lambda_t}{\Lambda_t}\frac{dS_t}{S_t}\right] = E\left[\frac{d\Lambda_t}{\Lambda_t}\sqrt{V_t}dZ_{1t}\right] = rdt - \mu dt = -\delta dt \qquad (A.4)$$

where δ is the equity risk premium.

Similarly, following Heston (1993), the price of volatility risk, $\lambda(S_t, V_t, t)$ can be defined as:

$$\lambda(S_t, V_t, t)dt = -E\left[\frac{d\Lambda_t}{\Lambda_t}\sigma\sqrt{V_t}dZ_{2t}\right] \qquad (A.5)$$

Substituting (A.4) and (A.5) in (A.2):

$$E\left[\frac{d\Lambda_t}{\Lambda_t}\frac{dC_t}{C_t}\right] = -\frac{S}{C}\frac{\partial C}{\partial S}\delta dt - \frac{1}{C}\frac{\partial C}{\partial V}\lambda(S_t, V_t, t)dt \qquad (A.6)$$

Substituting the above in (1.3) leads to:

$$E\left[\frac{dC_t}{C_t}\right] = rdt - E\left[\frac{d\Lambda_t}{\Lambda_t}\frac{dC_t}{C_t}\right] = rdt + \frac{S}{C}\frac{\partial C}{\partial S}\delta dt + \frac{1}{C}\frac{\partial C}{\partial V}\lambda(S_t, V_t, t)dt \qquad (A.7)$$

(A.7) is the expected return from a call option if the underlying stock price process has stochastic volatility as in Heston (1993).

From (6.8):

$$E\left[\frac{dC_t}{C_t}\right] = rdt + \bar{A}_t\delta dt \qquad (A.8)$$

A comparison of (A.7) and (A.8) gives the value of $\bar{A}_t$ as:

$$\bar{A}_t = \frac{S}{C}\frac{\partial C}{\partial S} + \frac{1}{C}\frac{\partial C}{\partial V}\frac{\lambda(S_t, V_t, t)}{\delta} \qquad (A.9)$$

Appendix B

Bates (2000) SVJ model adds jump risk to the diffusive stock price risk and diffusive volatility risk already present in the Heston (1993) model. By Ito's lemma, the dynamics of call option price under Bates (2000) SVJ model are given by:

$$dC_t = \left[\frac{\partial C}{\partial t} + \mu S_t\frac{\partial C}{\partial S} + k(\theta - V_t)\frac{\partial C}{\partial V} + \frac{S_t^2 V_t}{2}\frac{\partial^2 C}{\partial S^2} + \frac{\sigma^2 V_t}{2}\frac{\partial^2 C}{\partial V^2} + \rho\sigma S_t V_t\frac{\partial^2 C}{\partial S\partial V}\right]dt$$
$$+ S_t\sqrt{V_t}\frac{\partial C}{\partial S}dZ_{1t} + \sigma\sqrt{V_t}\frac{\partial C}{\partial V}dZ_{2t} + d\left(\sum_{j=1}^{N_t}\Delta C_{T_j}\right) \qquad (B.1)$$

The additional term $d\left(\sum_{j=1}^{N_t}\Delta C_{T_j}\right)$ in (B.1) (when compared with (A.1)) captures the impact of jumps. Defining the price of jump risk as $E\left[\frac{d\Lambda_t}{\Lambda_t}\frac{d\left(\sum_{j=1}^{N_t}\Delta C_{T_j}\right)}{C}\right] = -ER^J$ and following the same steps as in appendix A leads to:

$$\bar{A}_t = \frac{S}{C}\frac{\partial C}{\partial S} + \frac{1}{C}\frac{\partial C}{\partial V}\frac{\lambda(S_t, V_t, t)}{\delta} + \frac{ER^J}{\delta} \qquad (B.2)$$