Interest Rate Modeling with Retarded Langevin Equations

Markus Hess

Markus-Hess@gmx.net

https://orcid.org/0000-0002-4969-1178

April 6, 2022

Abstract. In this paper, we present an arithmetic short rate model based on generalized Langevin equations. The innovative feature of the model is that it accounts for memory effects in interest rate markets via the involved Langevin processes. In this setup, we provide a representation for the related zero-coupon bond price and infer its risk-neutral time dynamics. We also deduce the associated forward rate dynamics, the latter being of Heath-Jarrow-Morton type. We further establish a measure change to the risk-adjusted forward measure and propose a market-consistent calibration procedure. We finally derive a pricing formula for a European call option written on the zero-coupon bond by Fourier transform methods.

Keywords: short rate; forward rate; zero-coupon bond price; option pricing; forward measure; market-consistent calibration; generalized/retarded Langevin equation

MSC 2010 codes: 60G44, 60G51, 60H10, 91B30, 91B70, 91G20, 91G30

JEL classifications: C02, G12

1 Introduction

In this paper, we propose an innovative short rate model based on the generalized stochastic Langevin equation. For this reason, let us start by giving a brief review of the historical Langevin equation introduced by the French physicist Paul Langevin in 1908. Based on heuristic arguments, Langevin postulated a stochastic model to describe the forces acting on a particle moving in a fluid. More precisely, the proposed equation yields a stochastic description of a moving particle driven by Brownian motion. Langevin assumed that there are two forces acting on a Brownian particle of mass $m > 0$, namely

- a viscous friction force $-m\,\eta\,v(t)$ characterized by the friction coefficient $\eta > 0$ and the velocity $v(t)$ of the particle at time $t \geq 0$, as well as
- a fluctuating force $F(t)$ representing the continual impacts of the fluid molecules on the moving particle.

In this setup, the fluctuating force $F(t)$ is assumed to be independent of the velocity $v(t)$, while $F(t)$ is considered as an external force, also called Langevin force. In the one-dimensional case, the famous Langevin equation reads as

$$m\, a(t) = -m\, \eta\, v(t) + F(t) \tag{1.1}$$

where $a(t) = \dot{v}(t)$ denotes the acceleration of the moving particle at time $t \geq 0$. Hence, by Newton's second axiom (the so-called *lex secunda*), the left hand side of (1.1) describes the kinematic force of the particle at time $t \geq 0$. Obviously, Equation (1.1) is equivalent to the differential equation

$$\frac{dv(t)}{dt} = -\eta\, v(t) + \frac{1}{m} F(t)$$

where $t \geq 0$. Langevin further assumed that the fluctuating force is random and modeled it as a Gaussian white noise, i.e.

$$F(t) = \dot{W}(t) = \frac{dW(t)}{dt}$$

where $W(t)$ constitutes a standard Brownian motion (BM). Consequently, he arrived at the stochastic differential equation (SDE)

$$dv(t) = -\eta\, v(t)\, dt + m^{-1}\, dW(t) \tag{1.2}$$

which constitutes a BM-driven zero-reverting Ornstein-Uhlenbeck (OU) equation with constant mean-reversion speed $\eta > 0$ and constant volatility coefficient m^{-1}. Historically, the Langevin equation (1.1), respectively (1.2), is the first example of a stochastic differential equation, i.e. a differential equation involving a random term with specified statistical properties. It is well-known (and not surprising) that the solution $v(\cdot)$ of the SDE (1.2) itself constitutes a stochastic process.

Let us now move on to the generalized Langevin equation which will be the main modeling tool in the present paper. The generalized Langevin equation is also called retarded Langevin equation and it reads in the one-dimensional case as

(1.3)

$$m\, a(t) = -m \int_0^t \eta(t-s)\, v(s)\, ds + F(t)$$

wherein $m > 0$ is the mass of the particle, $a(\cdot)$ denotes the acceleration of the particle, $\eta(\cdot)$ is a dissipative memory kernel, $v(\cdot)$ indicates the velocity of the particle, and $F(\cdot)$ constitutes a Gaussian driving force (a white noise) modeling the random fluctuations of the particle over time. Equation (1.3) is equivalent to

(1.4)

$$dv(t) = -\left(\int_0^t \eta(t-s)\, v(s)\, ds \right) dt + \frac{1}{m} dW(t)$$

which constitutes a stochastic integro-differential equation. At this step, it is interesting to compare the generalized/retarded Langevin equation (1.4) with the standard Langevin equation provided in (1.2). If we do so, we see that retardation effects are newly taken into account via the parameter integral appearing in the drift of (1.4). This parameter integral is given by the convolution product of the memory kernel $\eta(\cdot)$ and the velocity $v(\cdot)$ of the particle. To read more on the Langevin equation, its history, and related applications, we refer to [3], [4], [10], [11], [14], [16], [22], [25], and [27]. For more information on (deterministic and stochastic) parameter integrals, we refer to [1], [2], [18], and the Appendix of the present paper.

Let us now turn our attention to a concrete application of the generalized Langevin equation in an interest rate market context. Inspired by (1.4), we introduce the stochastic Langevin process $X = (X_t)_{t\geq 0}$ which satisfies a generalized/retarded Langevin SDE of the form

(1.5)

$$dX_t = \left(\int_0^t m(t-s)\, X_s\, ds \right) dt + \xi(t)\, dW_t$$

where $m(\cdot)$ is a time-dependent and deterministic memory kernel, $\xi(\cdot) > 0$ constitutes a time-dependent and deterministic volatility function, while $W = (W_t)_{t\geq 0}$ denotes a standard BM. Having Eq. (1.5) at hand, we propose to model the short rate process $r = (r_t)_{t\geq 0}$ via

$$r_t = \mu(t) + X_t$$

where $\mu(\cdot)$ is a deterministic seasonality function, and X follows the SDE (1.5). In this model, the retarded Langevin SDE (1.5) accounts for memory effects in the (random) interest rate via the involved memory kernel $m(\cdot)$. An economical motivation for including retardation effects in an interest rate model could be the following: Imagine that the banks belonging to the US Federal Reserve System (shortly, the Fed) increase/decrease interest rates significantly. Then it happens very often in real-world financial markets that other central banks (e.g. the European Central Bank) also increase/decrease interest rates in a similar (or sometimes opposite) direction. As the reaction of the other central banks usually takes place with a time delay of several hours or even days, it makes sense to incorporate such retardation effects by proposing an interest rate model depending on retarded Langevin equations.

In the sequel, we give a brief survey of different interest rate models proposed in the literature. Frequently applied short rate models are, for example, the Cox-Ingersoll-Ross (CIR) model [12], the Hull-White model [20], or the Vasicek model [28]. In [20] and [28] the short rate process is modeled by a BM-driven SDE of OU-type. As a consequence, the short rate process is normally distributed in these models and may become arbitrarily negative. In the recent years, there indeed appeared negative interest rates from time to time, but the negative rates usually were small. Further on, in [12] the short rate is modeled by a so-called square-root process of OU-type. This approach leads to a mean-reverting, strictly positive and chi-square distributed short rate process. Also in [6] the authors propose a model which accounts for positive interest rates. More precisely, in [6] it is assumed that the instantaneous short rate process evolves as the exponential of an OU process with time dependent coefficients. As a consequence, the short rate process in [6] follows a log-normal distribution. In [13] the short rate process is modeled by a geometric BM approach and thus, only takes strictly positive values. Moreover, in [24] post-crisis short rate models in an extended multiple-curve framework are presented. A very detailed overview on different short rate models and their properties can be found in the textbooks [8], [15], and [26]. See, in particular, Section 3 in [8], Sections 5.2.1 and 5.4 in [15], and Section 3 in [26].

The most famous forward rate model certainly is the Heath-Jarrow-Morton (HJM) model proposed in [17]. Therein, the instantaneous forward rate process is modeled by an arithmetic BM-driven SDE approach. In [5] the HJM model is extended to a jump-diffusion setup wherein the forward rate process is affected by diffusion and jump noise. HJM-type models are also treated in Section 5 in [8], Section 6 in [15], Section 5 in [18], and Section 3.2 in [26].

The class of market models was introduced in [7]. For instance, the popular LIBOR model belongs to this modeling class. Market models will not be treated in the present article and thus, the interested reader is referred to [7], Part III in [8], or Section 11 in [15] to read more on this topic.

In the present paper, we propose an innovative interest rate model which is based on generalized/retarded Langevin equations. More precisely, in our arithmetic approach (2.1)-(2.2), the short rate process is modeled by a deterministic shift function plus a weighted sum of BM-driven retarded Langevin processes. We also provide a representation for the related zero-coupon bond price and derive its risk-neutral time dynamics. In addition, we infer the dynamics of the instantaneous forward rate, the latter being of HJM-type. We further establish a probability measure change to an associated forward measure and deduce a condition under which the forward rate model can be market-consistently calibrated. The analytical tractability of our model is eventually illustrated by the derivation of a pricing formula for a European call option written on the zero-coupon bond.
The paper is organized as follows. In Section 2 we present the mathematical formulation of our arithmetic short rate model based on generalized Langevin equations and compare the model with other approaches in the literature. Section 3 is devoted to the pricing of zero-coupon bonds associated with the proposed short rate model. In Section 4 we investigate forward rate modeling and provide a market-consistent calibration procedure. Section 5 is dedicated to the pricing of a call option written on the zero-coupon bond.

2 Short rate modeling with Langevin equations

In this section, we present the mathematical formulation of our arithmetic short rate model based on generalized Langevin equations. Let $\big(\Omega, \mathfrak{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P}\big)$ be a filtered probability space satisfying the usual hypotheses, i.e. $\mathfrak{F}$ denotes a sigma-algebra augmented by all $\mathbb{P}$-null sets, and $\mathcal{F} = (\mathcal{F}_t)_{t\in[0,T]}$ with $\mathcal{F}_T \subset \mathfrak{F}$ constitutes a right-continuous and complete filtration. Here, $\mathbb{P}$ is the physical/real-world probability measure, and $T > 0$ denotes a finite time horizon. We assume that $\mathcal{F}_0$ is trivial. In this setup, for arbitrary $n \in \mathbb{N}$, we suppose that the short rate process $r = (r_t)_{t\in[0,T]}$ evolves according to

(2.1)

$$r_t = \mu(t) + \sum_{k=1}^{n} w_k \, X_t^k$$

where $\mu\colon [0,T] \longrightarrow \mathbb{R}$ constitutes a bounded and deterministic $\mathcal{C}^1$-function (representing some "deterministic shift", cf. Section 3.8 in [8]), $(w_k)_{k=1}^n$ is a sequence of deterministic weights, and the factor processes $X^1, \dots, X^n$ satisfy generalized/retarded Langevin SDEs of the form

(2.2)

$$dX_t^k = \left(\int_0^t m_k(t-s)\, X_s^k \, ds\right) dt + \xi_k(t)\, dW_t^k$$

for all $t \in [0,T]$ and $k \in \{1, \dots, n\}$. Herein, the continuous and Lebesgue-integrable deterministic functions $m_k\colon [0,T] \longrightarrow \mathbb{R}$ are called memory kernels, $\xi_k\colon [0,T] \longrightarrow]0,\infty[$ denote deterministic and Lebesgue-square-integrable volatility functions in $\mathcal{C}^1[0,T]$, and the processes $W^1, \dots, W^n$ constitute independent and $\mathcal{F}$-adapted (one-dimensional) standard Brownian motions under $\mathbb{P}$.

Remark 2.1. We stress that the existence and uniqueness of the solution to the generalized Langevin SDE (2.2) is well-studied (cf. e.g. Theorem 4.6 in [23]). More precisely, due to Lemma 1 in [4] we know that the SDE (2.2) admits a unique solution for every $k \in \{1, \dots, n\}$, since the involved memory kernels fulfill $m_k(\cdot) \in \mathcal{L}^1[0,T]$ by assumption.

Remark 2.2. We state that our modeling approach in (2.1)-(2.2) was inspired by the arithmetic multi-factor short rate model introduced in [19]. However, note that there are no Langevin equations present in [19], whereas the factor processes $X^1, \dots, X^n$ therein are modeled – instead of (2.2) – by zero-reverting pure-jump OU processes of the form

$$dX_t^k = -\lambda_k \, X_t^k \, dt + \sigma_k \, dL_t^k$$

where $\lambda_k > 0$ denote constant mean-reversion speeds, $\sigma_k > 0$ are constant volatility coefficients, and $L^1, \dots, L^n$ constitute increasing pure-jump Lévy processes, so-called time-homogeneous subordinators. An advantage of the model proposed in [19] over our model (2.1)-(2.2) is that the (mean-reverting) short rate process is bounded from below, i.e. in [19] it holds $r_t \geq \mu(t)$ a.s. for all $t \in [0, T]$. This property is no longer valid in our setup, as the factor processes in (2.2) are driven by Brownian motions. On the other hand, the model proposed in [19] does not account for memory effects, as there are no memory kernels present in the underlying factor dynamics.

Assumption (A). Without loss of generality, we set $n = 1$ and $w_1 = 1$ from now on in order to keep the exposition of the paper short and the involved formulas compact. We emphasize that all our further computations likewise work in the multi-factor setting where $n > 1$.

In accordance to Assumption (A), we will work in the remainder of the paper under the following single-factor model specification:

$$r_t = \mu(t) + X_t \tag{2.3}$$

where X is a BM-driven retarded Langevin process satisfying an SDE of the form

$$dX_t = \left(\int_0^t m(t-s) \, X_s \, ds \right) dt + \xi(t) \, dW_t \tag{2.4}$$

with deterministic initial value $X_0 = x_0 \in \mathbb{R}$. As a consequence, for all $t \in [0, T]$ the short rate process $r = (r_t)$ follows the time dynamics

$$dr_t = \left(\dot{\mu}(t) + \int_0^t m(t-s) \, X_s \, ds \right) dt + \xi(t) \, dW_t$$

with deterministic initial value

$$r_0 = \mu(0) + x_0 \, .$$

From now on, we assume that the filtration $\mathcal{F} = (\mathcal{F}_t)$ is for all $t \in [0, T]$ defined by the natural Brownian filtration, i.e.

$$\mathcal{F}_t \coloneqq \mathcal{F}_t^W \coloneqq \sigma\{W_s : 0 \leq s \leq t\} \tag{2.5}$$

wherein σ denotes the sigma-algebra generator. In the next step, we compute the solution of the Langevin SDE (2.4). For this purpose, we introduce the deterministic C^1-function $h: [0, T] \longrightarrow \mathbb{R}$ which is the unique solution to the ordinary differential equation (ODE)

$$\dot{h}(t) = \int_0^t m(t-s) \, h(s) \, ds \tag{2.6}$$

with normalized initial value $h(0) = 1$. Herein, $m(\cdot)$ denotes the memory kernel appearing in (2.4). The function $h(\cdot)$ is sometimes called the differential resolvent of $m(\cdot)$ in the literature (cf. e.g. [4]). We further introduce the Laplace transform of the memory kernel $m(\cdot) \in \mathcal{L}^1[0,\infty[$ which is defined through

(2.7)

$$\widehat{m}(z) := \int_0^\infty m(t)\, e^{-zt}\, dt$$

for all complex numbers $z \in \mathbb{C}$ such that the integral in (2.7) is well-defined. We are now prepared to prove the subsequent result.

Proposition 2.3. *Assume that $m(\cdot)$, $h(\cdot)$ and $\xi(\cdot)$ are such that*

(2.8)

$$\int_0^T \int_0^T m^2(t-u)\, h^2(u-s)\, \xi^2(s)\, du\, ds < \infty$$

for all $t \in [0,T]$. Then for all $t \in [0,T]$ the stochastic process

(2.9)

$$X_t = x_0\, h(t) + \int_0^t h(t-s)\, \xi(s)\, dW_s$$

is the unique solution to the generalized/retarded Langevin SDE (2.4). The function $h(\cdot)$ in (2.9) is implicitly determined through its Laplace transform

$$\hat{h}(z) = \frac{1}{z - \widehat{m}(z)}$$

where $\widehat{m}(z)$ is such as defined in (2.7).

Proof. The proof essentially follows the same lines as the proof of Proposition 1 in [4]. Applying the Leibniz formula for Brownian Volterra integrals (see e.g. Section 1 in [18], or Proposition A.3 in the Appendix), we can compute the time differential of the process X given in (2.9) as follows

(2.10)

$$dX_t = \left(x_0\, \dot{h}(t) + \int_0^t \dot{h}(t-s)\, \xi(s)\, dW_s \right) dt + \xi(t)\, dW_t$$

where we used the equality $h(0) = 1$. Now let $X = (X_t)$ and $\tilde{X} = (\tilde{X}_t)$ be two solutions of the SDE (2.10) with identical initial values $X_0 = \tilde{X}_0 = x_0$. Then, parallel to [4], we conclude from (2.10) that it holds $X_t = \tilde{X}_t$ $\mathbb{P}$-a.s. for all $t \in [0,T]$. Hence, X and $\tilde{X}$ are indistinguishable, such that the process X given in (2.9) is the *unique* solution of the SDE (2.10). Moreover, from (2.6) we immediately deduce the equalities

(2.11)

$$\dot{h}(t) = \int_0^t m(t-a)\, h(a)\, da\ , \qquad \dot{h}(t-s) = \int_0^{t-s} m(t-s-u)\, h(u)\, du$$

where $0 \leq s \leq t \leq T$. Plugging (2.11) into (2.10), we obtain

(2.12)

$$dX_t = \left(x_0 \int_0^t m(t-a)\, h(a)\, da + \int_0^t \int_0^{t-s} m(t-s-u)\, h(u)\, du\, \xi(s)\, dW_s \right) dt + \xi(t)\, dW_t$$

for all $t \in [0, T]$. We next perform a change of variables inside the du-integral appearing in the drift of (2.12), and substitute

$$u \coloneqq u(a) \coloneqq a - s\,, \qquad du/da = 1\,, \qquad a_1 = s\,, \qquad a_2 = t\,.$$

Then (2.12) translates into

(2.13)

$$dX_t = \left(x_0 \int_0^t m(t-a)\, h(a)\, da + \int_0^t \int_s^t m(t-a)\, h(a-s)\, da\, \xi(s)\, dW_s \right) dt + \xi(t)\, dW_t$$

where $t \in [0, T]$. Taking Fubini's theorem into account, we infer the equality

$$\int_0^t \int_s^t m(t-a)\, h(a-s)\, da\, \xi(s)\, dW_s = \int_0^t \int_0^a m(t-a)\, h(a-s)\, \xi(s)\, dW_s\, da$$

where $0 \leq s \leq a \leq t \leq T$. Hence, Eq. (2.13) can be rewritten as

$$dX_t = \left(\int_0^t m(t-a) \left\{ x_0\, h(a) + \int_0^a h(a-s)\, \xi(s)\, dW_s \right\} da \right) dt + \xi(t)\, dW_t$$

where the expression inside the curly brackets just equals X_a due to (2.9). Thus, we arrive at

$$dX_t = \left(\int_0^t m(t-a)\, X_a\, da \right) dt + \xi(t)\, dW_t$$

which coincides with (2.4). Hence, we conclude that (2.10) and (2.4) are indeed equivalent. Since (2.9) is the unique solution to (2.10), while the differential resolvent $h(\cdot)$ satisfying the ODE (2.6) is unique, too, the Langevin process X given in (2.9) is indeed the unique solution to the generalized/retarded Langevin SDE (2.4). Furthermore, from (2.8) we deduce that

$$\int_0^t h^2(t-s)\, \xi^2(s)\, ds < \infty$$

for all $t \in [0, T]$ such that the Brownian Volterra integral

$$\int_0^t h(t-s)\, \xi(s)\, dW_s$$

appearing in (2.9) is well-defined. On the other hand, for all $0 \leq s \leq t \leq T$ it holds

$$\begin{aligned} \int_0^t \dot{h}^2(t-s)\, \xi^2(s)\, ds &= \int_0^t \left(\int_0^{t-s} m(t-s-u)\, h(u)\, du \right)^2 \xi^2(s)\, ds \\ &\leq \int_0^t \int_0^{t-s} m^2(t-s-u)\, h^2(u)\, du\, \xi^2(s)\, ds \end{aligned}$$

wherein we used (2.11). We next perform a change of variables inside the appearing du-integral and substitute

$$u \coloneqq u(a) \coloneqq a - s\,, \qquad du/da = 1\,, \qquad a_1 = s\,, \qquad a_2 = t\,,$$

which yields

$$\begin{aligned}\int_0^t \dot{h}^2(t-s)\,\xi^2(s)\,ds &\le \int_0^t \int_s^t m^2(t-a)\,h^2(a-s)\,da\,\xi^2(s)\,ds \\ &\le \int_0^T \int_0^T m^2(t-a)\,h^2(a-s)\,da\,\xi^2(s)\,ds < \infty\end{aligned}$$

where the last expression is finite due to (2.8). Thus, also the Brownian Volterra integral

$$\int_0^t \dot{h}(t-s)\,\xi(s)\,dW_s$$

appearing in the drift of the SDE (2.10) is well-defined, which concludes the proof. ■

Remark 2.4. Merging (2.9) into (2.3), we get the following short rate representation under $\mathbb{P}$

(2.14)

$$r_t = \mu(t) + x_0\,h(t) + \int_0^t h(t-s)\,\xi(s)\,dW_s$$

where $t \in [0,T]$. Hence, for all $t \in [0,T]$ the short rate process is normally distributed under $\mathbb{P}$ with mean

$$\varepsilon(t) \coloneqq \mathbb{E}_{\mathbb{P}}[r_t] = \mu(t) + x_0\,h(t)$$

and variance

$$v^2(t) \coloneqq \mathbb{Var}_{\mathbb{P}}[r_t] = \int_0^t h^2(t-s)\,\xi^2(s)\,ds\,.$$

We write

$$r_t \sim \mathcal{N}\big(\varepsilon(t), v^2(t)\big)$$

under $\mathbb{P}$. As a consequence, the short rate process r may take *arbitrary* negative values, which is an undesired property with view on empirical data. However, the probability for negative rates can for all $t \in [0,T]$ be computed as follows

$$\mathbb{P}(r_t < 0) = \mathbb{P}\{\omega \in \Omega : r_t(\omega) < 0\} = \Phi\left(\frac{0 - \varepsilon(t)}{v(t)}\right) = 1 - \Phi\left(\frac{\varepsilon(t)}{v(t)}\right)$$

where $\Phi(\cdot)$ denotes the cumulative distribution function (cdf) of the standard normal distribution. We conclude that the probability for negative rates is small, whenever the ratio $\varepsilon(t)/v(t)$ is large. Thus, an applicant using the model in practice should adjust the model ingredients $\mu(\cdot)$, $h(\cdot)$, $m(\cdot)$, $\xi(\cdot)$ and x_0 in such a way that the ratio $\varepsilon(\cdot)/v(\cdot)$ takes large values. In this context, we stress that negative interest rates can be observed from time to time in real-world financial markets nowadays. For this reason, it is not necessary to ensure a priori that a proposed short rate model only takes strictly positive values. Also recall that the famous short rate models proposed in [20] and [28] admit negative interest rates, too.

In the next step, we compute the characteristic function of $r = (r_t)$ which is defined via

$$\Psi_{r_t}(u) \coloneqq \mathbb{E}_{\mathbb{P}}[e^{iur_t}]$$

where $i^2 = -1$, $u \in \mathbb{R}$ and $t \in [0,T]$. Using (2.14), a straightforward calculation yields

$$\Psi_{r_t}(u) = \exp\left\{iu[\mu(t) + x_0\, h(t)] - \frac{u^2}{2}\int_0^t h^2(t-s)\, \xi^2(s)\, ds\right\}.$$

We conclude the current section by giving some concrete examples for the memory kernel $m(\cdot)$ introduced in (2.4). The following examples are taken from Section 4.2 in [27].

Example 2.5. (a) A typical choice for the memory kernel is to take $m(t) = t^{-\alpha}$ where $\alpha \in]0,1[$ is constant. This case is also studied in Example 1 in [4], wherein the authors provide a power series representation for the related differential resolvent $h(\cdot)$. In this context, we recall that $m(\cdot)$ and $h(\cdot)$ are linked via the differential equation (2.6).

(b) Another classical choice for the memory kernel is to take $m(t) = e^{-\alpha t}$ where $\alpha \in]0,\infty[$ is constant. In this case, the Laplace transform of the memory kernel [cf. (2.7)] can be computed to $\hat{m}(z) = (z+\alpha)^{-1}$ where $z \in \mathbb{C} \setminus \{-\alpha\}$. Consequently, the Laplace transform of the differential resolvent $h(\cdot)$ takes the form $\hat{h}(z) = (z+\alpha)/(z^2 + \alpha z - 1)$ due to Proposition 2.3.

3 Pricing of zero-coupon bonds

Our aim in this section is to evaluate zero-coupon bonds associated with the short rate model introduced in the previous section. As bonds are priced under a risk-neutral measure, $\mathbb{Q}$ say, we need to introduce a probability measure change from the physical measure $\mathbb{P}$ to an equivalent martingale measure $\mathbb{Q}$. For this purpose, we define for all $t \in [0,T]$ the Radon-Nikodym density process

(3.1)

$$Z_t \coloneqq \frac{d\mathbb{Q}}{d\mathbb{P}}\bigg|_{\mathcal{F}_t} \coloneqq \mathcal{E}\left(\int_0^{\cdot} \gamma(s)\, dW_s\right)_t \coloneqq \exp\left\{\int_0^t \gamma(s)\, dW_s - \frac{1}{2}\int_0^t \gamma^2(s)\, ds\right\}$$

where $\mathcal{E}(\cdot)$ denotes the Doléans exponential (cf. Section 4.1.5 in [15], Section 1.5.7 in [21]), and $\gamma: [0,T] \longrightarrow \mathbb{R}$ is a deterministic $\mathcal{L}^2$-function which is called market price of risk (cf. Section 4.3.2 in [15]). Taking Itô's formula (see Theorem 4.3 in [15], Theorem 1.5.3.1 in [21], Section 2.1.4 in [26]) into account, we find

$$dZ_t = Z_t\, \gamma(t)\, dW_t$$

for all $t \in [0,T]$ such that the process $Z = (Z_t)_{t\in[0,T]}$ constitutes a local martingale under $\mathbb{P}$. Since it holds $\gamma(\cdot) \in \mathcal{L}^2[0,T]$ by assumption, the corresponding Novikov condition is fulfilled. Hence, the strictly positive local $\mathbb{P}$-martingale Z satisfies $\mathbb{E}_{\mathbb{P}}[Z_t] \equiv Z_0 = 1\ \forall\, t \in [0,T]$ such that Z constitutes a true $\mathbb{P}$-martingale. Applying Girsanov's theorem (see e.g. Section 1.7.3 in [21], Section 2.1.4 in [26], Theorem 4.6 in [15]), we state that $\mathbb{Q}$ constitutes a probability measure on $(\mathcal{F}_t)_{t\in[0,T]}$ being equivalent to $\mathbb{P}$ and that the process

(3.2)

$$\left(B_t \coloneqq W_t - \int_0^t \gamma(s)\, ds\right)_{t\in[0,T]}$$

is an $\mathcal{F}$-adapted standard BM under $\mathbb{Q}$. As $\gamma(\cdot)$ is deterministic, it holds

$$\mathcal{F}_t = \mathcal{F}_t^W = \mathcal{F}_t^B \coloneqq \sigma\{B_s : 0 \le s \le t\}$$

due to (2.5) and (3.2). We further substitute (3.2) into (2.14) and obtain the following short rate representation under $\mathbb{Q}$

(3.3)

$$r_t = \mu(t) + x_0\, h(t) + \int_0^t h(t-s)\, \xi(s)\, \gamma(s)\, ds + \int_0^t h(t-s)\, \xi(s)\, dB_s$$

being valid for all $t \in [0,T]$. Hence, we observe that the memory kernel $m(\cdot)$ enters the short rate representation (3.3) via the involved function $h(\cdot)$. This statement is also true for the short rate representation under $\mathbb{P}$ provided in (2.14).

Remark 3.1. We mention that our BM-driven short rate model is complete, as soon as the market price of risk $\gamma(\cdot)$ has been determined. However, the current (single-factor) model can easily be made incomplete, if we return to the original multi-factor specification proposed in (2.1)-(2.2) containing one underlying, but n driving noises. Alternatively, the single-factor model in (2.3) also becomes incomplete, if we assume that ξ and W in (2.4) are vector-valued.

Further on, we introduce a bank account β with stochastic interest rate r satisfying

(3.4) $$d\beta_t = r_t\, \beta_t\, dt$$

with normalized initial capital $\beta_0 = 1$. The solution of (3.4) is for all $t \in [0,T]$ given by

(3.5)

$$\beta_t = \exp\left\{\int_0^t r_s\, ds\right\}$$

(cf. Definition 1.1.1 in [8]) while it holds

(3.6) $$d\beta_t^{-1} = -r_t\, \beta_t^{-1}\, dt\,.$$

In this setup, the zero-coupon bond price at time $t \in [0,T]$ with maturity T is defined as

(3.7) $$P_t(T) \coloneqq \beta_t\, \mathbb{E}_{\mathbb{Q}}(\beta_T^{-1} | \mathcal{F}_t)\,.$$

Merging (3.5) into (3.7), we get for all $t \in [0,T]$ the well-known bond price representation

(3.8)

$$P_t(T) = \mathbb{E}_{\mathbb{Q}}\left(\exp\left\{-\int_t^T r_s\, ds\right\}\middle|\mathcal{F}_t\right)$$

(cf. Eq. (3.2) in [8], Eq. (5.1) in [15]). Note that $P_t(T) > 0$ $\mathbb{Q}$-a.s. $\forall\, t \in [0,T]$ by construction and that $P_t(t) = 1\ \forall\, t \in [0,T]$.

Lemma 3.2. *Define for all* $0 \le u \le t \le T$ *the deterministic functions*

(3.9)

$$\theta(u,t) \coloneqq -\int_u^t h(s-u)\, ds\,, \qquad \eta(t) \coloneqq \int_0^t [\mu(u) + x_0\, h(u) - \xi(u)\, \gamma(u)\, \theta(u,t)]\, du\,.$$

Then, for all $t \in [0,T]$ *the integrated short rate process can be expressed under* $\mathbb{Q}$ *as*

(3.10)

$$I_t := \int_0^t r_s \, ds = \eta(t) - \int_0^t \xi(u)\, \theta(u,t)\, dB_u \,.$$

Proof. We substitute (3.3) into the definition of I_t and obtain

$$I_t = \int_0^t [\mu(s) + x_0\, h(s)]\, ds + \int_0^t \int_0^s h(s-u)\, \xi(u)\, \gamma(u)\, du\, ds + \int_0^t \int_0^s h(s-u)\, \xi(u)\, dB_u\, ds \,.$$

Applying Fubini's theorem twice yields

$$I_t = \int_0^t [\mu(s) + x_0\, h(s)]\, ds - \int_0^t \xi(u)\, \gamma(u)\, \theta(u,t)\, du - \int_0^t \xi(u)\, \theta(u,t)\, dB_u$$

where we identified the function $\theta(u,t)$ introduced in (3.9). We eventually notice that the drift part of the latter equation just equals $\eta(t)$ defined in (3.9), which completes the proof. ■

Obviously, it holds $I_t = \log \beta_t$ with $I_0 = 0$ due to (3.5), as well as $dI_t = r_t\, dt = d\beta_t/\beta_t$ due to (3.4). We also have $\beta_t = e^{I_t}$ which implies

(3.11) $$P_t(T) = e^{I_t}\, \mathbb{E}_{\mathbb{Q}}(e^{-I_T} | \mathcal{F}_t)$$

due to (3.7). Having these equations at hand, we obtain the subsequent result.

Proposition 3.3. *For all $t \in [0,T]$ the bond price process P can be represented under $\mathbb{Q}$ as*

(3.12)

$$P_t(T) = \exp\left\{\eta(t) - \eta(T) + \frac{1}{2}\int_t^T \xi^2(u)\, \theta^2(u,T)\, du + \int_0^t \xi(u)\, [\theta(u,T) - \theta(u,t)]\, dB_u\right\}$$

where the functions η and θ are such as claimed in (3.9).

Proof. Using (3.10) and (3.11), we obtain

$$P_t(T) = e^{I_t - \eta(T)}\, \mathbb{E}_{\mathbb{Q}}\left(\exp\left\{\int_0^T \xi(u)\, \theta(u,T)\, dB_u\right\} \middle| \mathcal{F}_t\right)$$

where I_t satisfies (3.10). The latter equation is equivalent to

$$P_t(T) = \exp\left\{I_t - \eta(T) + \int_0^t \xi(u)\, \theta(u,T)\, dB_u\right\} \mathbb{E}_{\mathbb{Q}}\left[\exp\left\{\int_t^T \xi(u)\, \theta(u,T)\, dB_u\right\}\right]$$

due to the independent increment property of the involved Brownian integral. The appearing usual expectation can be computed to

$$\mathbb{E}_{\mathbb{Q}}\left[\exp\left\{\int_t^T \xi(u)\, \theta(u,T)\, dB_u\right\}\right] = \exp\left\{\frac{1}{2}\int_t^T \xi^2(u)\, \theta^2(u,T)\, du\right\}$$

which implies

$$P_t(T) = \exp\left\{I_t - \eta(T) + \frac{1}{2}\int_t^T \xi^2(u)\, \theta^2(u,T)\, du + \int_0^t \xi(u)\, \theta(u,T)\, dB_u\right\}.$$

We finally substitute (3.10) into the latter equation and end up with (3.12). ■

Remark 3.4. Note that for all $0 \leq u \leq t \leq T$ the difference $\theta(u,T) - \theta(u,t)$ appearing in (3.12) can be expressed in integral form as

(3.13)

$$\theta(u,T) - \theta(u,t) = -\int_t^T h(s-u)\, ds$$

due to (3.9). Moreover, for all $0 \leq u \leq t \leq T$ we find the equalities

(3.14)

$$\theta(t,t) = 0\,, \qquad \partial_t \theta(u,t) = -h(t-u)\,, \qquad \partial_t \theta^2(u,t) = -2\,\theta(u,t)\, h(t-u)$$

where ∂_t denotes the derivative operator with respect to t.

We are now prepared to derive the risk-neutral time dynamics of the zero-coupon bond price process.

Proposition 3.5. *Under $\mathbb{Q}$ the bond price process $\big(P_t(T)\big)_{t\in[0,T]}$ satisfies the time dynamics*

(3.15)

$$\frac{dP_t(T)}{P_t(T)} = r_t\, dt + \xi(t)\, \theta(t,T)\, dB_t$$

where θ is the deterministic function defined in (3.9).

Proof. To begin with, we introduce the $\mathcal{F}$-adapted, real-valued and normally distributed stochastic process

(3.16)

$$A_t \coloneqq A_t(T) \coloneqq \eta(t) - \eta(T) + \frac{1}{2}\int_t^T \xi^2(u)\, \theta^2(u,T)\, du + \int_0^t \xi(u)\, [\theta(u,T) - \theta(u,t)]\, dB_u$$

where $t \in [0,T]$. Then (3.12) can be rewritten as

(3.17) $$P_t(T) = e^{A_t(T)}$$

for all $t \in [0,T]$. Using the Leibniz formula for Brownian Volterra integrals (see Section 1 in [18], or Proposition A.3 in the Appendix) as well as (3.14), we obtain the SDE

(3.18)

$$dA_t = \left(\dot{\eta}(t) - \frac{1}{2}\,\xi^2(t)\,\theta^2(t,T) + \int_0^t h(t-u)\,\xi(u)\, dB_u\right) dt + \xi(t)\,\theta(t,T)\, dB_t\,.$$

We next apply Itô's formula on (3.17), take (3.18) into account, and deduce the time dynamics

$$\frac{dP_t(T)}{P_t(T)} = \left(\dot{\eta}(t) + \int_0^t h(t-u)\,\xi(u)\, dB_u\right) dt + \xi(t)\,\theta(t,T)\, dB_t$$

being valid for all $t \in [0,T]$. An application of the Leibniz formula for parameter integrals (see Section 1 in [18], or Proposition A.1 in the Appendix) further yields

(3.19)

$$\dot{\eta}(t) = \mu(t) + x_0\, h(t) + \int_0^t h(t-u)\, \xi(u)\, \gamma(u)\, du$$

where we used (3.9) and (3.14). We finally substitute the latter equation into the derived bond price dynamics and infer the SDE

$$\frac{dP_t(T)}{P_t(T)} = \left\{\mu(t) + x_0\, h(t) + \int_0^t h(t-u)\, \xi(u)\, \gamma(u)\, du + \int_0^t h(t-u)\, \xi(u)\, dB_u\right\} dt + \xi(t)\, \theta(t,T)\, dB_t$$

where the expression inside the curly brackets just equals r_t due to (3.3). Hence, the proof is complete. ■

With respect to (2.6), we notice that the memory kernel m implicitly enters the bond price dynamics in (3.15) via the function θ which contains the differential resolvent h.

Corollary 3.6. *The solution of the geometric SDE (3.15) reads for all $t \in [0,T]$ as*

(3.20)

$$P_t(T) = P_0(T)\, \beta_t \exp\left\{\int_0^t \theta(s,T)\, \xi(s)\, dB_s - \frac{1}{2}\int_0^t \theta^2(s,T)\, \xi^2(s)\, ds\right\}$$

where β_t is given by (3.5) and the deterministic initial value satisfies

(3.21)

$$P_0(T) = \exp\left\{-\eta(T) + \frac{1}{2}\int_0^T \theta^2(s,T)\, \xi^2(s)\, ds\right\}$$

due to (3.12).

Remark 3.7 (discounted bond price). For every $t \in [0,T]$ we define the discounted bond price process

(3.22) $$\hat{P}_t(T) \coloneqq P_t(T)/\beta_t$$

where $\hat{P}_0(T) = P_0(T)$. From (3.7) we then deduce

$$\hat{P}_t(T) = \mathbb{E}_{\mathbb{Q}}(\beta_T^{-1} | \mathcal{F}_t)$$

such that the process $\hat{P}_\cdot(T)$ constitutes an $\mathcal{F}$-adapted martingale under $\mathbb{Q}$ for every fixed maturity time T, as required by no-arbitrage theory (cf. [5], [8], [17]). Plugging (3.20) into (3.22), we further obtain

$$\hat{P}_t(T) = P_0(T)\, \mathcal{E}\left(\int_0^\cdot \theta(s,T)\, \xi(s)\, dB_s\right)_t$$

where $\mathcal{E}$ denotes the Doléans exponential generator defined in (3.1). A straightforward calculation finally leads us to the $(\mathcal{F},\mathbb{Q})$-martingale dynamics

$$d\hat{P}_t(T) = \hat{P}_t(T)\, \theta(t,T)\, \xi(t)\, dB_t$$

being valid for all $t \in [0,T]$.

4 Forward rate modeling

In this section, we derive a representation for the instantaneous forward rate associated with our short rate model based on retarded Langevin equations. With reference to Definition 1.4.2 in [8], Eq. (2.1) in [15], or Eq. (3.24) in [26], we define the instantaneous forward rate at time t with maturity T via

(4.1)
$$f_t(T) := -\partial_T \log P_t(T)$$

where $t \in [0,T]$ and ∂_T denotes the derivative operator with respect to T. We recall that Eq. (4.1) is equivalent to the integral representation

(4.2)
$$P_t(T) = \exp\left\{-\int_t^T f_t(u)\, du\right\}$$

and that it holds $f_t(t) = r_t$ for all $t \in [0,T]$ (cf. e.g. [8], [15], [26]). In the present setup, we get the following HJM-type result.

Proposition 4.1. *Under the risk-neutral measure $\mathbb{Q}$, the instantaneous forward rate at time t with fixed maturity T satisfies the HJM-type equation*

(4.3)
$$f_t(T) = f_0(T) - \int_0^t \theta(s,T)\, h(T-s)\, \xi^2(s)\, ds + \int_0^t h(T-s)\, \xi(s)\, dB_s$$

where $t \in [0,T]$ and $f_0(T) = -\partial_T \log P_0(T)$. Herein, θ is the deterministic function defined in (3.9) and h denotes the differential resolvent introduced in (2.6). [See Remark 4.4 for a more detailed representation of the initial value $f_0(T)$.]

Proof. We substitute (3.20) into (4.1) and obtain

$$f_t(T) = -\partial_T \log P_0(T) - \int_0^t \partial_T\theta(s,T)\, \xi(s)\, dB_s + \frac{1}{2}\int_0^t \partial_T\theta^2(s,T)\, \xi^2(s)\, ds\,.$$

Taking (4.1) and (3.14) into account, we observe that the latter equation is equivalent to the assertion claimed in (4.3). ■

Remark 4.2. Sticking to the notation introduced in the seminal paper [17], Eq. (4.3) can for all $t \in [0,T]$ be expressed (in differential form) as

$$df_t(T) = \alpha_t(T)\, dt + \sigma_t(T)\, dB_t$$

with deterministic (drift and volatility) functions

$$\alpha_t(T) := -\theta(t,T)\, h(T-t)\, \xi^2(t) = -\theta(t,T)\, \xi(t)\, \sigma_t(T)\,, \qquad \sigma_t(T) := h(T-t)\, \xi(t)\,.$$

Due to this parallelism, we call Proposition 4.1 an "HJM-type result". We refer – besides [17] – to [5], Section 5 in [8], Section 6 in [15], and Section 3.2 in [26] to read more on the Heath-Jarrow-Morton forward rate theory and its applications. Also see Section 5 in [18] in this context, wherein an anticipative HJM modeling approach including stochastic Volterra integrals is presented.

Remark 4.3 (forward measure). Let us define the Radon-Nikodym density process Z^* via

$$Z_t^* := Z_t^*(T) := \frac{d\mathbb{Q}^*}{d\mathbb{Q}}\bigg|_{\mathcal{F}_t} := \mathcal{E}\left(\int_0^{\cdot} \theta(s,T)\,\xi(s)\,dB_s\right)_t$$

where $t \in [0,T]$ and $\mathcal{E}$ denotes the Doléans exponential generator introduced in (3.1). We further assume that the corresponding Novikov condition

$$\int_0^T \theta^2(s,T)\,\xi^2(s)\,ds < \infty$$

is in force. From Girsanov's theorem we then deduce that $\mathbb{Q}^*$ is a probability measure on $\mathcal{F} = (\mathcal{F}_t)_{t\in[0,T]}$ being equivalent to $\mathbb{Q}$ and that the process

(4.4)

$$\left(B_t^* := B_t - \int_0^t \theta(s,T)\,\xi(s)\,ds\right)_{t\in[0,T]}$$

constitutes an $\mathcal{F}$-adapted standard BM under $\mathbb{Q}^*$. Combining (4.3) with (4.4), we infer the $(\mathcal{F},\mathbb{Q}^*)$-martingale representation

$$f_t(T) = f_0(T) + \int_0^t h(T-s)\,\xi(s)\,dB_s^*$$

being valid for all $t \in [0,T]$. Hence, we conclude that the probability measure $\mathbb{Q}^*$ constitutes the so-called T-forward measure (respectively, the forward-risk-adjusted measure) related to the instantaneous forward rate $f_t(T)$ given in (4.3). Since it holds $f_t(t) = r_t$ for all $t \in [0,T]$, we further find under $\mathbb{Q}^*$

$$\mathbb{E}_{\mathbb{Q}^*}(r_T|\mathcal{F}_t) = \mathbb{E}_{\mathbb{Q}^*}(f_T(T)|\mathcal{F}_t) = f_t(T)$$

for all $t \in [0,T]$, which stands in accordance to Proposition 2.5.2 in [8], Lemma 7.2 in [15], and Eq. (4.7) in [26]. Finally note that for all $t \in [0,T]$ it holds

$$Z_t^* = \frac{P_t(T)}{\beta_t\,P_0(T)} = \frac{\hat{P}_t(T)}{P_0(T)} = \frac{\mathbb{E}_{\mathbb{Q}}(\beta_T^{-1}|\mathcal{F}_t)}{\mathbb{E}_{\mathbb{Q}}[\beta_T^{-1}]}$$

(recall Remark 3.7), where $\hat{P}$ denotes the discounted bond price, and β is the bank account process. From the last equality we conclude that the density process $Z^* = (Z_t^*)_{t\in[0,T]}$ constitutes an $\mathcal{F}$-adapted $\mathbb{Q}$-martingale by construction, such that there is actually no need to impose any Novikov condition. See Section 2.5 in [8], Section 7 in [15], or Section 4.1 in [26] to read more on forward measures and their properties.

Remark 4.4 (market-consistent calibration). In the sequel, we illustrate how our forward rate model can be calibrated to a given term structure observed in the real-world interest rate market. This procedure is often called market-consistent calibration in the literature (cf. e.g. pp. 54-56 in [8]). For this purpose, we denote by $f_0^M(T)$ the initial forward rate which can be read off in the market. If $f_0(T) = f_0^M(T)$ and hence, if $P_0(T) = P_0^M(T)$ for all maturity times $T > 0$, then the underlying model is called market-consistent. By putting (3.21) into (4.1), we get

$$f_0(T) = \partial_T\left(\eta(T) - \frac{1}{2}\int_0^T \theta^2(s,T)\,\xi^2(s)\,ds\right).$$

Using (3.19), (3.14), and the Leibniz formula for parameter integrals (see Proposition A.1 in the Appendix), the latter equation translates into

$$f_0(T) = \mu(T) + x_0\, h(T) + \int_0^T h(T-s)\, \xi(s)\, [\gamma(s) + \theta(s,T)\, \xi(s)]\, ds\,.$$

Hence, we conclude that the model ingredients $\mu(\cdot)$, x_0, $h(\cdot)$, $m(\cdot)$, $\xi(\cdot)$, and $\gamma(\cdot)$ should be chosen in such a way that today's forward rate $f_0^M(T)$ (observed in the market) is matched, i.e.

(4.5)

$$f_0^M(T) = \mu(T) + x_0\, h(T) + \int_0^T h(T-s)\, \xi(s)\, [\gamma(s) + \theta(s,T)\, \xi(s)]\, ds$$

for all maturity times $0 < T < \infty$. Since there are five unknowns and just one equation, the model ingredients $\mu(\cdot)$, x_0, $m(\cdot)$, $\xi(\cdot)$, and $\gamma(\cdot)$ are of course not uniquely determined by (4.5). In practice, the calibration procedure could be as follows:

1. Choose a deterministic initial value $x_0 \in \mathbb{R}$ of the Langevin SDE (2.4).
2. Choose a deterministic memory kernel $m(\cdot)$. [The corresponding differential resolvent $h(\cdot)$ is then simultaneously determined via (2.6).]
3. Choose a deterministic and strictly positive volatility function $\xi(\cdot)$.
4. Choose a deterministic market price of risk function $\gamma(\cdot)$.
5. Read off today's forward rate $f_0^M(T)$ in the market for as many different maturity times T as possible.
6. Then, at these discrete maturity times T, the function $\mu(\cdot)$ is determined through the equation

$$\mu(T) = f_0^M(T) - x_0\, h(T) - \int_0^T h(T-s)\, \xi(s)\, [\gamma(s) + \theta(s,T)\, \xi(s)]\, ds\,.$$

   (In the context of the latter equation, we refer to (4.12) in [8], wherein a related equation associated with a two-factor short rate model is derived.)

7. By using interpolation techniques (e.g. spline interpolation), an approximation of the function $t \longmapsto \mu(t)$ can eventually be obtained for all $t \in\,]0,T]$ via

$$\mu(t) = f_0^M(t) - x_0\, h(t) - \int_0^t h(t-s)\, \xi(s) \left[\gamma(s) - \xi(s) \int_s^t h(u-s)\, du\right] ds$$

   with initial value $\mu(0) = r_0 - x_0 = f_0^M(0) - x_0$.

All in all, we conclude that the forward rate model in (4.3) can be market-consistently calibrated to a given term structure $f_0^M(T)$ by choosing the function $t \longmapsto \mu(t)$ as in the equation appearing in Step 7.

Furthermore, we define the interest rate curve at time $t < T$ with maturity T via

(4.6)

$$R_t(T) := -\frac{\log P_t(T)}{T-t}\,.$$

This object is called "continuously-compounded spot interest rate" in Definition 1.2.3 in [8]. It obviously holds

$$P_t(T) = e^{-(T-t)\, R_t(T)}$$

for all $t \in [0, T[$. We infer the subsequent representation for the interest rate curve.

Proposition 4.5. *For all $t \in [0, T[$ the interest rate curve can be represented as*

$$R_t(T) = \frac{1}{T-t}\left(\int_t^T f_0(u)\, du - \int_0^t \xi^2(s) \int_t^T \theta(s,u)\, h(u-s)\, du\, ds + \int_0^t \xi(s) \int_t^T h(u-s)\, du\, dB_s\right)$$

where $f_0(\cdot)$, θ and h are such as specified in Proposition 4.1.

Proof. We plug (4.2) and (4.3) [with T therein replaced by u] into (4.6) and obtain

$$R_t(T) = \frac{1}{T-t}\int_t^T \left(f_0(u) - \int_0^t \theta(s,u)\, h(u-s)\, \xi^2(s)\, ds + \int_0^t h(u-s)\, \xi(s)\, dB_s\right) du$$

for all $t \in [0, T[$. We ultimately interchange the integration order and arrive at the assertion. ■

5 Option pricing

In this section, we investigate the evaluation of a plain vanilla option written on the zero-coupon bond price. With reference to the risk-neutral pricing theory, the price at time $t \le \tau$ of an option with payoff H_τ at maturity τ reads as

$$C_t = \beta_t\, \mathbb{E}_{\mathbb{Q}}(\beta_\tau^{-1} H_\tau | \mathcal{F}_t) = \mathbb{E}_{\mathbb{Q}}\left(e^{-\int_t^\tau r_s\, ds} H_\tau \middle| \mathcal{F}_t\right) \tag{5.1}$$

where β is the bank account process claimed in (3.5), and $\mathbb{Q}$ denotes the risk-neutral pricing measure (cf. Eq. (3.1) in [8], Section 7.2 in [15]). In the sequel, we consider a European call option written on the bond price process $\left(P_t(T)\right)_{t\in[0,T]}$ with maturity time $T > \tau$. The payoff of the call option written on $P_\tau(T)$ with strike price $K > 0$ and maturity τ is then given by

$$H_\tau = [P_\tau(T) - K]^+ := \max\{0, P_\tau(T) - K\}\,. \tag{5.2}$$

Moreover, we set $i = \sqrt{-1}$ and define the Fourier transform, respectively inverse Fourier transform, of a deterministic function $q\colon \mathbb{R} \longrightarrow \mathbb{R}$ with $q(\cdot) \in \mathcal{L}^1(\mathbb{R})$ via

(5.3)

$$\hat{q}(y) := \frac{1}{2\pi}\int_{\mathbb{R}} q(x)\, e^{-iyx}\, dx\ , \qquad q(x) = \int_{\mathbb{R}} \hat{q}(y)\, e^{iyx}\, dy\,.$$

We then get the following option price formula.

Proposition 5.1 (call option on bond price). *Let β be the bank account process claimed in (3.5). Then the price of a call option with payoff H_τ given in (5.2), strike price $K > 0$, and maturity time τ, can for all $t \le \tau < T$ be expressed as*

(5.4)

$$C_t = \beta_t \int_{\mathbb{R}} \hat{g}(y)\, \psi(y,\tau,T) \exp\left\{\int_0^t \lambda(s,\tau,T,y)\, dB_s + \frac{1}{2}\int_t^\tau \lambda^2(s,\tau,T,y)\, ds\right\} dy$$

with complex-valued deterministic functions

(5.5)

$$\hat{g}(y) = \frac{K^{1-a-iy}}{2\pi\,(a+iy)\,(a-1+iy)},$$

$$\lambda(s,\tau,T,y) := \xi(s)\,[(a+iy)\,\theta(s,T) - (a-1+iy)\,\theta(s,\tau)]\,,$$

$$\psi(y,\tau,T) := \exp\left\{(a-1+iy)\,\eta(\tau) - (a+iy)\,\eta(T) + \frac{a+iy}{2}\int_\tau^T \xi^2(s)\,\theta^2(s,T)\,ds\right\}.$$

Herein, the constant $a > 1$ constitutes an arbitrary real-valued dampening parameter, while the functions θ and η are such as defined in (3.9).

Proof. First of all, we substitute (5.2) and (3.17) into (5.1) and obtain

$$C_t = \beta_t\,\mathbb{E}_\mathbb{Q}\left(\beta_\tau^{-1}\left[e^{A_\tau(T)} - K\right]^+\middle|\mathcal{F}_t\right)$$

for all $0 \le t \le \tau < T$. Since it holds $\beta_t = e^{I_t}$, the latter equation can be rewritten as

$$C_t = \mathbb{E}_\mathbb{Q}\left(e^{I_t - I_\tau}\left[e^{A_\tau(T)} - K\right]^+\middle|\mathcal{F}_t\right)$$

where $(I_t)_t$ denotes the integrated short rate defined in (3.10), and $(A_t)_t$ is the stochastic process introduced in (3.16). Moreover, for all $x \in \mathbb{R}$ we define the deterministic function

$$g(x) := e^{-ax}\,[e^x - K]^+$$

where $a > 1$ is a real-valued and constant dampening parameter ensuring the integrability of the payoff function, i.e. $g(\cdot) \in \mathcal{L}^1(\mathbb{R})$. With this definition at hand, we obtain

$$C_t = \mathbb{E}_\mathbb{Q}\left(e^{I_t - I_\tau + a\,A_\tau(T)}g\big(A_\tau(T)\big)\middle|\mathcal{F}_t\right).$$

With reference to [9], we apply the inverse Fourier transform on the latter equation and get

(5.6)

$$C_t = \beta_t\int_\mathbb{R}\hat{g}(y)\,\mathbb{E}_\mathbb{Q}\left(e^{(a+iy)A_\tau(T) - I_\tau}\middle|\mathcal{F}_t\right)dy$$

for all $0 \le t \le \tau < T$. (Recall that β is $\mathcal{F}$-adapted.) The remaining challenge now consists in the computation of the involved conditional expectation

$$\Xi_t := \mathbb{E}_\mathbb{Q}\left(e^{(a+iy)A_\tau(T) - I_\tau}\middle|\mathcal{F}_t\right).$$

Using (3.10) and (3.16), we find

$$\Xi_t = \psi(y,\tau,T)\,\mathbb{E}_\mathbb{Q}\left(\exp\left\{\int_0^\tau \lambda(s,\tau,T,y)\,dB_s\right\}\middle|\mathcal{F}_t\right)$$

where we identified the functions ψ and λ defined in (5.5). Taking the independent increment property of the appearing Brownian integral under $\mathbb{Q}$ into account, we further deduce

$$\Xi_t = \psi(y,\tau,T)\exp\left\{\int_0^t \lambda(s,\tau,T,y)\,dB_s\right\}\mathbb{E}_\mathbb{Q}\left[\exp\left\{\int_t^\tau \lambda(s,\tau,T,y)\,dB_s\right\}\right]$$

where the usual expectation can for all $0 \leq t \leq \tau < T$ be computed to

$$\mathbb{E}_{\mathbb{Q}}\left[\exp\left\{\int_t^\tau \lambda(s,\tau,T,y)\, dB_s\right\}\right] = \exp\left\{\frac{1}{2}\int_t^\tau \lambda^2(s,\tau,T,y)\, ds\right\}.$$

Putting the latter equations into (5.6), we arrive at (5.4). The expression for the Fourier transform $\hat{g}(y)$ given in (5.5) can be obtained by a straightforward calculation using the definition of the function $g(x)$. ■

With respect to (2.6), we notice that the memory kernel m implicitly enters the call option price formula (5.4) via the involved functions θ and η which both contain the differential resolvent h.

Appendix

For the reader's convenience, we recall some well-known results on deterministic and stochastic Volterra integrals in the sequel.

Proposition A.1 (Leibniz formula for parameter integrals). *Denote by $\partial_t := d/dt$ the derivative operator with respect to t and let $a: \mathbb{R} \longrightarrow \mathbb{R}$ and $b: \mathbb{R} \longrightarrow \mathbb{R}$ be continuously differentiable deterministic functions. Assume that the deterministic function $f: \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ with $(s,t) \longmapsto f(s,t)$ is continuous in the first variable and continuously differentiable in the second, in symbols $f \in C^{0,1}(\mathbb{R} \times \mathbb{R})$. Also suppose that $f(s,t)$ and $\partial_t f(s,t)$ are integrable on $\mathbb{R}$ with respect to s for every t. Then it holds*

(A.1)

$$\partial_t\left(\int_{a(t)}^{b(t)} f(s,t)\, ds\right) = \int_{a(t)}^{b(t)} \partial_t f(s,t)\, ds + f(b(t),t)\, b'(t) - f(a(t),t)\, a'(t)$$

for all $t \in \mathbb{R}$.

The subsequent result is an immediate consequence of Proposition A.1.

Corollary A.2. *In the special case where $a(t) \equiv 0$ and $b(t) := t$, we obtain for all $t \in \mathbb{R}$*

(A.2)

$$d\left(\int_0^t f(s,t)\, ds\right) = \left(\int_0^t \partial_t f(s,t)\, ds\right) dt + f(t,t)\, dt$$

where d denotes the t-differential.

We remark that Proposition A.1 and Corollary A.2 still hold true, if the function $f(s,t)$ therein is replaced by a stochastic process $F(s,t) := F(s,t,\omega)$ which fulfills similar continuity and differentiability conditions as the deterministic function $f(s,t)$. On a filtered probability space $(\Omega, \mathfrak{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ the deterministic Leibniz formulas (A.1) and (A.2) can be extended to the following Brownian motion case involving stochastic integrals.

Proposition A.3 (stochastic Leibniz formula; Brownian motion case). *Let $\mathbb{R}^+ := [0,\infty[$ and $W = (W_t)_{t\geq 0}$ be an $\mathcal{F}$-adapted standard BM under $\mathbb{P}$. Assume that the deterministic function $\sigma: \mathbb{R}^+ \times \mathbb{R}^+ \longrightarrow \mathbb{R}$ with $(s,t) \longmapsto \sigma(s,t)$ is continuous in the first variable and continuously differentiable in the second, i.e. $\sigma \in C^{0,1}(\mathbb{R}^+ \times \mathbb{R}^+)$. Also suppose that for all $t \in \mathbb{R}^+$ it holds*

$$\int_0^t \left[\sigma^2(s,t) + \left(\partial_t \sigma(s,t)\right)^2\right] ds < \infty$$

such that the $\mathcal{F}_t$*-adapted Brownian Volterra integrals*

$$\int_0^t \sigma(s,t)\, dW_s\ , \qquad \int_0^t \partial_t \sigma(s,t)\, dW_s$$

are well-defined. Then for all $t \in \mathbb{R}^+$ *we have the equality*

(A.3)

$$d\left(\int_0^t \sigma(s,t)\, dW_s\right) = \left(\int_0^t \partial_t \sigma(s,t)\, dW_s\right) dt + \sigma(t,t)\, dW_t$$

where d *denotes the* t*-differential.*

Equality (A.3) immediately follows from Equation (3.6) in [1].

References

[1] Agram, N., Øksendal, B. (2015) *Malliavin Calculus and Optimal Control of Stochastic Volterra Equations*, Journal of Optimization Theory and Applications, Vol. 167 (3), 1070-1094

[2] Barndorff-Nielsen, O., Benth, F., Veraart, A. (2013) *Modelling Energy Spot Prices by Volatility Modulated Lévy Driven Volterra Processes*, Bernoulli, Vol. 19, 803-845

[3] Barrat, J., Hansen, J. (2003) *Brownian Motion, Diffusion and the Langevin Equation*, in: *Basic Concepts for Simple and Complex Liquids*, 228-256, Cambridge University Press, Cambridge

[4] Benth, F., Khedher, A., Vanmaele, M. (2020) *Pricing of Commodity Derivatives on Processes with Memory*, Risks, Vol. 8 (1), 8010008

[5] Björk, T., Kabanov, Y., Runggaldier, W. (1997) *Bond market structure in the presence of marked point processes*, Mathematical Finance, Vol. 7, 211-239

[6] Black, F., Karasinski, P. (1991) *Bond and Option Pricing when Short Rates are Lognormal*, Financial Analysts Journal, Vol. 47, 52-59

[7] Brace, A., Gatarek, D., Musiela, M. (1997) *The market model of interest rate dynamics*, Mathematical Finance, Vol. 7, 127-155

[8] Brigo, D., Mercurio, F. (2001) *Interest Rate Models – Theory and Practice*, First Edition, Springer, Berlin, Heidelberg

[9] Carr, P., Madan, D. (1999) *Option valuation using the fast Fourier transform*, Journal of Computational Finance, Vol. 2 (4), 61-73

[10] Coffey, W., Kalmykov, Y. (2012) *The Langevin Equation – with Applications to Stochastic Problems in Physics, Chemistry and Electrical Engineering*, Third Edition, Vol. 27, World Scientific, Singapore

[11] Comte, F., Renault, E. (1996) *Long Memory Continuous Time Models*, Journal of Econometrics, Vol. 73 (1), 101-149

[12] Cox, J., Ingersoll, J., Ross, S. (1985) *A theory of the term structure of interest rates*, Econometrica, Vol. 53, 385-407

[13] Dothan, L. (1978) *On the Term Structure of Interest Rates*, Journal of Financial Economics, Vol. 6, 59-69

[14] Farias, R., Ramos, R., Da Silva, L. (2009) *Stochastic Langevin Equations: Markovian and non-Markovian Dynamics*, Physical Review E, Vol. 80 (3), 031143

[15] Filipovic, D. (2009) *Term-Structure Models – A Graduate Course*, Springer Finance, Springer, Berlin, Heidelberg

[16] Fox, R. (1977) *The Generalized Langevin Equation with Gaussian Fluctuations*, Journal of Mathematical Physics, Vol. 18 (12), 2331-2335

[17] Heath, D., Jarrow, R., Morton, A. (1992) *Bond pricing and the term structure of interest rates: a new methodology for contingent claims valuation*, Econometrica, Vol. 60, 77-105

[18] Hess, M. (2018) *The Stochastic Leibniz Formula for Volterra Integrals under Enlarged Filtrations*, SSRN working paper: https://ssrn.com/abstract=3308400

[19] Hess, M. (2020) *A Pure-Jump Mean-Reverting Short Rate Model*, Modern Stochastics: Theory and Applications, Vol. 7 (2), 113-134

[20] Hull, J., White, A. (1990) *Pricing interest-rate-derivative securities*, Review of Financial Studies, Vol. 3 (4), 573-592

[21] Jeanblanc, M., Yor, M., Chesney, M. (2009) *Mathematical Methods for Financial Markets*, Springer, London

[22] Küchler, U., Mensch, B. (1992) *Langevin's stochastic differential equation extended by a time-delayed term*, Stochastics: An International Journal of Probability and Stochastic Processes, Vol. 40 (1-2), 23-42

[23] Lipster, R., Shiryaev, A. (1984) *Statistics of Random Processes I, General Theory*, Second Edition, Springer, Berlin

[24] Morino, L., Runggaldier, W. (2014) *On multicurve models for the term structure*, in *Nonlinear Economic Dynamics and Financial Modeling*, 275-290, Springer

[25] Pavliotis, G. (2014) *The Langevin Equation*, in: *Stochastic Processes and Applications*, Texts in Applied Mathematics, Vol. 60, Springer, New York

[26] Pelsser, A. (2005) *Efficient Methods for Valuing Interest Rate Derivatives*, Third Edition, Springer, London

[27] Takahashi, M. (1996) *Non-ideal Brownian motion, generalized Langevin Equation and its application to the security market*, Financial Engineering and the Japanese Markets, Vol. 3 (2), 87-119

[28] Vasicek, O. (1977) *An equilibrium characterization of the term structure*, Journal of Financial Economics, Vol. 5 (2), 177-188