MATH: Operations Research **2014-15 First Term**

Handout 1: Introduction to Linear Programming

Instructor: Junfeng Yang (jfyang@nju.edu.cn) *September 2, 2014*

Contents

1.1 简介

线性规划 (Linear Programming, LP) 是运筹学中研究较早、发展较快、应用广泛、方法较成熟的一个重要分支。它研究线性约束条件下线性目标函数的极值问题的数学理论和方法。

求解线性不等式系统至少可以追溯至法国数学家 Fourier (1768-1830). 1939年苏联数学家 Kantorovich 在《生产组织与计划中的数学方法》一书中提出线性规划问题，未引起重视。1947年美国数学家 Dantzing 提出求解线性规划的单纯形法，为这门学科奠定了基础。1947年美国数学家 von Neumann 提出对偶理论，开创了线性规划的许多新的研究领域，扩大了它的应用范围和解题能力。1951年美国经济学家 Koopman 把线性规划应用到经济领域，为此与 Kantorovich 一起获1975年诺贝尔经济学奖。20世纪50年代后对线性规划进行大量的理论研究，并涌现出一大批新的算法，如对偶单纯形法、灵敏度分析、参数规划问题、互补松弛定理、分解算法等。线性规划的研究成果还直接推动了其他数学规划问题包括整数规划、随机规划和非线性规划的算法研究。由于数字电子计算机的发展，出现了许多线性规划软件，如 CPLEX, 可以很方便地求解大规模、超大规模的线性规划问题。1979年苏联数学家 Khachian 提出解线性规划问题的椭球算法，并证明它是多项式时间算法。1984年美国贝尔电话实验室的印度数学家 Karmarkar 提出解线性规划问题的新的多项式时间算法：内点法, 既具有好的理论性质，又有很高的的数值计算效率。

1.2 线性规划的例子

Example 1.2.1 (生产问题) 某工厂在计划期内要安排生产*I*、*II*两种产品，已知生产单位产品所需要的设备台时，*A*、*B*两种原材料的消耗，以及资源的限制情况，如下表所示。该工厂每生产一单位产品*I*可获利*2*元，每生产一单位产品*II*可获利*3*元，问工厂应分别生产多少个*I*产品和*II*产品才能使工厂获利最大？

	产品*I*	产品*II*	资源总量
设备	*1*	2	*8*
材料*A*	*4*	*0*	*16*
材料*B*	*0*	*4*	*12*

设x_1, x_2分别表示计划期产品*I*、*II*的产品，则有下面线性规划模型：

$$
\begin{aligned}
\max z = 2x_1 + 3x_2 & \\
s.t.\ x_1 + 2x_2 &\le 8 \\
4x_1 \qquad &\le 16 \\
4x_2 &\le 12 \\
x_1, x_2 &\ge 0.
\end{aligned}
$$

Example 1.2.2 (饮食问题) 为了合理、健康、均衡饮食，人体每天必须必须摄入m中不同的营养元素，其中第i $(i = 1, 2, \ldots, m)$ 种营养元素的日摄入量等于（不得少于） b_i 个单位. 另外，市场上有n种不同的食物，其中第j 种食物的单价为c_j, $j = 1, 2, \ldots, n$. 假设每单位的第j中食物中营养元素i 的含量为a_{ij} $(i = 1, 2, \ldots, m;$ $j = 1, 2, \ldots, n)$, 如何合理安排每种食物的购买量使得既能达到营养需求又最经济？

设第j种食物的购买量为x_j, 则有下面线性规划模型：

$$
\begin{aligned}
\min z = c_1x_1 + c_2x_2 + \ldots + c_nx_n & & \\
s.t.\ a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= (\geq) & b_1 \\
a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= (\geq) & b_2 \\
&\ldots & \\
a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= (\geq) & b_m \\
x_1, x_2, \ldots, x_n &\geq & 0.
\end{aligned}
$$

Example 1.2.3 (运输问题) 某资源有m个产地，产量分别为$a_1, a_2, \ldots, a_m$; 有n个需求地，需求量分别为$b_1, b_2, \ldots, b_n$. 每单位资源从产地i运往需求地j的运费为c_{ij}. 如何安排运输方案既能满足需求又最经济？

设从产地i运往需求地j的量为x_{ij}, 则有下面线性规划模型：

$$
\begin{aligned}
\min z = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij} & & \\
s.t.\ \sum_{j=1}^{n} x_{ij} &= a_i,\ i = 1, 2, \ldots, m; \\
\sum_{i=1}^{m} x_{ij} &= b_j,\ j = 1, 2, \ldots, n; \\
x_{ij} &\geq 0,\ i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n.
\end{aligned}
$$

(Implication: $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$.)

Example 1.2.4 (世界杯投标问题) 世界杯开赛前组织赌局，规则如下. 任何球迷均可以赌阿根廷、巴西、意大利、德国与法国五支球队中的某一支球队或某几支球队中的一支会最终赢得冠军，并愿意以一定价格购买此种组合的股票若干。例如：球迷*#1* 愿意以每股*0.75*元的价格购买*10*股，赌阿根廷、巴西、意大利中的一支球队赢得冠军。赌局的组织者可以决定卖给球迷*#1* 此种股票若干。世界杯结束后，若冠军确由此三支球队中的某一支球队夺得，赌局的组织者每卖给球迷*#1* 一股此种股票就需付给他*1*元。假设订单信息已知，如何决定卖给每个球迷多少股？

Order	*#1*	*#2*	*#3*	*#4*	*#5*	...
Argentina	*1*	*0*	*1*	*1*	*0*	...
Brazil	*1*	*0*	*0*	*1*	*1*	...
Italy	*1*	*0*	*1*	*1*	*0*	...
Germany	*0*	*1*	*0*	*1*	*1*	...
France	*0*	*0*	*1*	*0*	*0*	...
Bidding Price p	*0.75*	*0.35*	*0.4*	*0.95*	*0.75*	...
Quantity limit q	*10*	*5*	*10*	*10*	*5*	...
Order fill x	x_1	x_2	x_3	x_4	x_5	...

令 A 表示如下矩阵

$$
A = \begin{pmatrix}
1 & 0 & 1 & 1 & 0 & \cdots \\
1 & 0 & 0 & 1 & 1 & \cdots \\
1 & 0 & 1 & 1 & 0 & \cdots \\
0 & 1 & 0 & 1 & 1 & \cdots \\
0 & 0 & 1 & 0 & 0 & \cdots
\end{pmatrix}.
$$

若考虑在最坏情形下盈利，则有如下模型:

$$\max_{x}\{p^T x - \|Ax\|_\infty:\ s.t.\ 0 \le x \le q\}.$$

该模型可转化为如下线性规划问题：

$$\max_{x,z}\left\{p^T x - z:\ s.t.\ Ax \le z\mathbf{1},\ 0 \le x \le q\right\},$$

这里 $\mathbf{1} = (1, 1, \ldots, 1)^T$.

1.3 图解法

对于只有两个变量的线性规划问题（如上述生产问题），可通过画图的方法求出其最优解. 从图解法不难看出，对于一般的线性规划问题可能出现的情况有 (1) 最优解唯一; (2) 最优解无穷多; (3) 无界解; (4) 约束集为空. 另外，若线性规划有有限最优解，则最优解一定可以在约束集的边界上取得。

1.4 线性规划的一般形式与标准形式

线性规划的一般形式为：

$$\begin{aligned}
\min z = c_1x_1 + c_2x_2 + \ldots + c_nx_n & & \\
s.t.\ a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &\quad \le (=, \ge) & b_1 \\
a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &\quad \le (=, \ge) & b_2 \\
\cdots\ \cdots & & \\
a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &\quad \le (=, \ge) & b_m \\
(x_1, x_2, \ldots, x_n &\quad \ge & 0.)
\end{aligned}$$

其中$x_1, x_2, \ldots, x_n$称为决策变量. 线性规划的标准形式为：

$$\begin{aligned}
\min z = c_1x_1 + c_2x_2 + \ldots + c_nx_n & & \\
s.t.\ a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &\quad = & b_1 \\
a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &\quad = & b_2 \\
\cdots\ \cdots & & \\
a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &\quad = & b_m \\
x_1, x_2, \ldots, x_n &\quad \ge & 0.
\end{aligned}$$

非标准形式的线性规划均可转化为标准形式：

1. $\max\{f(x) : x \in S\} = -\min\{-f(x) : x \in S\}$;
2. “$\le$"：引入松弛变量;
3. “$\ge$"：引入剩余变量;
4. 自由变量可通过两种方法处理.

线性规划标准形式的矩阵向量形式：令$c=(c_1,\ldots,c_n)^T$, $x=(x_1,\ldots,x_n)^T$, $b=(b_1,\ldots,b_m)^T$,

$$A=\begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{pmatrix}_{m\times n}$$

则线性规划的标准形式可表示为

$$\begin{aligned} \min z = c^T x & \\ s.t.\ Ax &= b \\ x &\geq 0. \end{aligned}$$

其中x称为决策变量（向量），c称为价值系数（向量），b称为资源向量，A称为约束条件系数矩阵。令

$$a_1=\begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix}, a_2=\begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix}, \ldots, a_n=\begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix}$$

线性规划的标准形式也可表示为：

$$\begin{array}{ll} \min & z=c^T x \\ s.t. & \sum_{j=1}^n x_j a_j = b \\ & x \geq 0. \end{array}$$

Assumption 1.4.1 (基本假设) *(1) $m<n$ 且 rank$(A)=m$; (2) $b\geq 0$.*

1.5 作业

1. 用图解法求解如下线性规划问题:

$$\begin{aligned} \max x_1+3x_2 & \\ s.t.\ 5x_1+10x_2 &\leq 50 \\ x_1+x_2 &\geq 1 \\ x_2 &\leq 4 \\ x_1,x_2 &\geq 0. \end{aligned}$$

(a)

$$\begin{array}{ll} \min & z=x_1+1.5x_2 \\ s.t. & x_1+3x_2\geq 3 \\ & x_1+x_2\geq 2 \\ & x_1\geq 0, x_2\geq 0. \end{array}$$

(b)

$$\begin{array}{ll} \max & z=2x_1+2x_2 \\ s.t. & x_1-x_2\geq -1 \\ & -0.5x_1+x_2\leq 2 \\ & x_1\geq 0, x_2\geq 0. \end{array}$$

(c)

$$\begin{array}{ll} \max & z=x_1+x_2 \\ s.t. & x_1-x_2\geq 0 \\ & 3x_1-x_2\leq -3 \\ & x_1\geq 0, x_2\geq 0. \end{array}$$

(d)

2. Let f be a piecewise linear function given by $f(x)=\max\{c_1^T x+d_1,\ldots,c_p^T x+d_p\}$. Convert the following problem to a linear programming problem:

$$\min_x\{f(x):\ s.t.\ Ax=b, x\geq 0\}.$$

3. 下面数学规划模型在信号处理中被称为Basis Pursuit模型：

$$\min_{x\in R^n}\{\|x\|_1: \ s.t. \ Ax=b\},$$

其中$\|x\|_1 := \sum_{i=1}^n |x_i|$, $A\in R^{m\times n}$(一般$m\ll n$), $b\in R^m$. 在稀疏信号处理中，该模型通常用来逼近下面NP-难的组合优化问题：

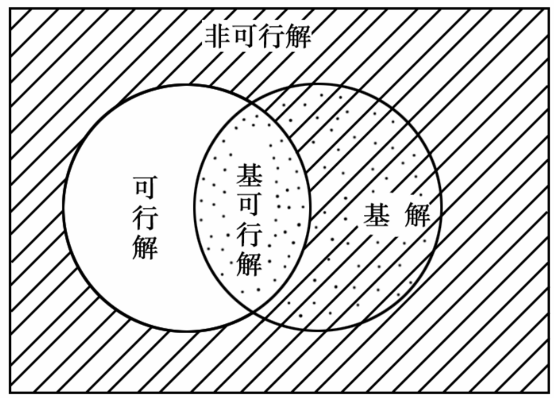
$$\min_{x\in R^n}\{\|x\|_0: \ s.t. \ Ax=b\},$$

其中$\|x\|_0$表示x的非零元素的个数。试将Basis Pursuit模型转化为线性规划模型。

1.6 基本概念

- 可行域、可行解、最优解
  - 可行域（集）：$\mathcal{F}=\{x\in R^n: Ax=b, x\geq 0\}$;
  - 可行解：$x\in\mathcal{F}$ (对应一个可行的决策);
  - 最优（可行）解：可行域中使得目标函数达到最优值的点;
- 基、基向量、基变量
  - 基：设B是A的一个m阶非奇异子矩阵，则B的列向量构成R^m的一组基, 称B是线性规划问题的一个基;
  - 基向量/非基向量：B的列向量称为基向量，A的其他列向量称为非基向量;
  - 基变量/非基变量：与基向量相应的x的分量称为基变量，x的其他分量称为非基变量（为书写方便，经常令x_B表示基变量，x_N表示非基变量，此时，$(x_B, x_N)^T$为x的一个重排）;
- 基解、基可行解、可行基
  - 基解: 令所有非基变量为0，则$Ax=b$变为$Bx_B=b$. 不妨设x的前m个分量为基变量，即$x_1, x_2, \ldots, x_m$. 称$x=(B^{-1}b, 0)^T$为线性方程组$Ax=b$的基解. 由基解的定义可知：
    1. 基解的非零分量个数至多为m
    2. 若某基解的非零分量个数小于m，则称其为退化的基解
    3. 若某基解非退化，则其非零分量对应的x的分量（共m个）为基变量
    4. 基解的非零分量对应的A的列向量线性无关
    5. 基解的数量不超过C_n^m 个
    6. 基解的非零分量可能为正、也可能为负
  - 基可行解: 若基解的非零分量均为正，则称该基解为基可行解.
    1. 基可行解的数量不超过基解的数量，从而不超过C_n^m个
    2. 若某基可行解的非零元素（即大于零的元素）的个数小于m，则称其为退化的基可行解.
  - 可行基: 对应于基可行解的基称为可行基。
- 最优基可行解: 若线性规划问题的某最优解是基解，则称其为最优基可行解.

不同概念之间的关系如下图：



1.7 线性规划的基本定理

Theorem 1.1 (线性规划的基本定理) 线性规划问题

$$\begin{aligned} \min z = c^T x & \\ s.t.\ Ax &= b \\ x &\geq 0. \end{aligned}$$

其中$A \in R^{m\times n}$满足$rank(A) = m$. 则

1. 若线性规划问题存在可行解，则一定存在基可行解;
2. 若线性规划问题存在最优解，则一定存在最优基可行解.

Proof: （1）设$x = (x_1, x_2, \ldots, x_n)^T$为一个可行解, 即$Ax = b$, $x \geq 0$. 记A的列向量为$a_1, a_2, \ldots, a_n$. 则$Ax = b$又可表示为

$$x_1 a_1 + \ldots + x_n a_n = b.$$

设x共有p个分量大于零，且不妨设该p个分量是最前面p个，从而有

$$x_1 a_1 + \ldots + x_p a_p = b.$$

不妨设$\{a_1, \ldots, a_p\}$不全为零（否则, 该向量组无极大无关组，$b = 0$. 此时，显然有$x = 0$为基可行解）. 下面分两种情况讨论

1. $a_1, \ldots, a_p$线性无关：此时显然有$p \leq m$. 若$p = m$, 则x为基解；若$p < m$，则x为退化的基解。证毕。
2. $a_1, \ldots, a_p$线性相关：存在不全为零的系数$y_1, \ldots, y_p$使得

$$y_1 a_1 + \ldots + y_p a_p = 0.$$

   不妨设至少有一系数$y_i > 0$.

$\forall \epsilon \in R$, 都有

$$(x_1 - \epsilon y_1) a_1 + \ldots + (x_p - \epsilon y_p) a_p = b.$$

令$y = (y_1, \ldots, y_p, 0, \ldots, 0)^T$, 则有$A(x - \epsilon y) = b, \forall \epsilon \in R$. 令$\epsilon = \min_i\{x_i/y_i : y_i > 0\}$, 则有$x - \epsilon y \geq 0$, 从而$x - \epsilon y$ 为可行解. 另外，$x - \epsilon y$ 的非零分量（正分量）的个数至多为$p - 1$.

如必要，重复上述过程（每次至少消去一个正的分量，得到的解仍为可行解），直至得到一个可行解，其非零元素对应的列向量组线性无关（此时，非零元素对应的列向量组即为初始向量组$\{a_1, \ldots, a_p\}$ 的极大无关组）。此时可归结为第一种情况。证毕。[1]

（2）设$x = (x_1, x_2, \ldots, x_n)^T$为一个最优可行解($Ax = b$, $x \geq 0$, 且目标函数值达到最优). 记A的列向量为$a_1, a_2, \ldots, a_n$. 则$Ax = b$又可表示为

$$x_1 a_1 + \ldots + x_n a_n = b.$$

设x共有p个分量大于零，且不妨设该p个分量是最前面p个，从而有

$$x_1 a_1 + \ldots + x_p a_p = b.$$

不妨设$\{a_1, \ldots, a_p\}$不全为零（否则, 该向量组无极大无关组，$b = 0$. 此时，可证$x = 0$为最优基可行解. $c^T x = c_B^T x_B + c_N^T x_N$. 必有最优值$c_B^T x_B = 0$. 否则，若$c_B^T x_B > 0$ 可取可行解εx ($\varepsilon \in (0, 1)$), 使目标函数值更小，此与x最优相矛盾；若$c_B^T x_B < 0$, 可取$\varepsilon x, \varepsilon \to +\infty$，此时无界解，与条件矛盾）. 下面分两种情况讨论

1. $a_1, \ldots, a_p$线性无关：此时显然有$p \leq m$. 若$p = m$, 则x为基解；若$p < m$，则x为退化的基解。总之，x为最优基可行解. 证毕。

2. $a_1, \ldots, a_p$线性相关：存在不全为零的系数$y_1, \ldots, y_p$使得

   $$y_1 a_1 + \ldots + y_p a_p = 0.$$

   不妨设至少有一系数$y_i > 0$.

$\forall \epsilon \in R$, 都有

$$(x_1 - \epsilon y_1) a_1 + \ldots + (x_p - \epsilon y_p) a_p = b.$$

令$y = (y_1, \ldots, y_p, 0, \ldots, 0)^T$, 则有$A(x - \epsilon y) = b, \forall \epsilon \in R$. 当$|\epsilon|$充分小时，$x - \epsilon y \geq 0$，从而$x - \epsilon y$可行。由于$x$最优，所以$c^T x \leq c^T (x - \epsilon y), \forall |\epsilon|$充分小。这意味着$c^T y = 0$.

令$\epsilon = \min_i\{x_i/y_i : y_i > 0\}$, 则有$x - \epsilon y \geq 0$, 从而$x - \epsilon y$为可行解, 且最优. 另外，$x - \epsilon y$ 的非零分量（正分量）的个数至多为$p - 1$. 如必要，重复上述过程（每次消去一个正的分量，得到的解仍为最优可行解），直至得到一个最优可行解，其非零元素对应的列向量组线性无关。此时可归结为第一种情况。 ■

Remark 1.7.1 (几点说明) *（1）线性规划可能没有最优解（无界解）; (2)* 线性规划有可能有最优的非基可行解（此时一定有无穷多解）*; (3)* 线性规划的基可行解的总数不超过$C_n^m = \frac{n!}{m!(n-m)!}$ 个. 逐个计算并比较函数值的方法仅在理论上可行!

1.8 几何解释

Definition 1.2 (凸集) 设$C \subset R^n$. 称C为凸集，如果对$\forall x \in C, y \in C$ 以及$\forall \alpha \in (0, 1)$, 都有

$$\alpha x + (1 - \alpha) y \in C.$$

[1]每次消去（至少）一个正分量后，相应的列向量组中也消去（至少）一个向量，并且消去前后，向量组的极大无关组保持不变。这保证了一定可以通过此过程得到一个可行解，其非零元素对应的列向量组线性无关（并且为初始向量组的极大无关组）。

Theorem 1.3 *Consider linear programming in standard form. If nonempty, the feasible set*

$$\mathcal{F} = \{x \in R^n : Ax = b, x \geq 0\}$$

is convex.

Definition 1.4 (凸集的极点) 设C 为R^n 中凸集, $x \in C$. 如果对任意的$y \in C, z \in C$, $y \neq z$, 以及$\alpha \in (0,1)$, 都有$x \neq \alpha y + (1-\alpha)z$, 则称$x$ 为C 的极点或顶点。

Definition 1.5 (凸组合) 设$x_i \subset R^n$, $\alpha_i \in [0,1]$, $i = 1, 2, \ldots, k$, 并且$\sum_{i=1}^k \alpha_i = 1$, 其中$k$为任意正整数。称$x = \sum_{i=1}^k \alpha_i x_i$ 为$x_1, x_2, \ldots, x_k$的凸组合。

Definition 1.6 (凸包, convex hull) 设S 为R^n 的子集。R^n 中包含S 的最小的凸集称为S 的凸包，记为$conv(S)$.

Theorem 1.7 *If $\mathcal{F}$ is bounded, then $\mathcal{F} = \text{conv(extreme points of } \mathcal{F})$.*

Remark 1.8.1 *More general result: A closed bounded convex subset of R^n is equal to the closed convex hull of its extreme points.*

Theorem 1.8 (基可行解与极点的等价性) 设*rank*$(A) = m$, $\mathcal{F} = \{x \in R^n : Ax = b, x \geq 0\}$. $x \in \mathcal{F}$为线性规划的基可行解当且仅当x为$\mathcal{F}$的极点。

Proof: **必要性**：设$x = (x_1, \ldots, x_m, 0, \ldots, 0)^T$为线性规划的基可行解. 则

$$x_1 a_1 + \ldots + x_m a_m = b,$$

并且$a_1, \ldots, a_m$线性无关。假设存在$y, z \in \mathcal{F}$, $y \neq z$, 以及$\alpha \in (0,1)$, 使得$x = \alpha y + (1-\alpha)z$. 易见$y$和$z$的最后面$n-m$个分量全部为0. 因此有

$$y_1 a_1 + \ldots + y_m a_m = b$$

以及

$$z_1 a_1 + \ldots + z_m a_m = b.$$

由$a_1, \ldots, a_m$的线性无关可得$x = y = z$. 由反证法，必要性得证。

充分性：设$x \in \mathcal{F}$ 为极点，并且不妨设x的全部非零分量（即正分量）为最前面k个。则$x_1, \ldots, x_k > 0$,

$$x_1 a_1 + \ldots + x_k a_k = b.$$

为了证明x是基可行解, 只需证明$a_1, \ldots, a_k$线性无关。假设$a_1, \ldots, a_k$线性相关，则存在非平凡组合

$$y_1 a_1 + \ldots + y_k a_k = 0.$$

定义$y = (y_1, \ldots, y_k, 0, \ldots, 0)^T$. 取$\epsilon$充分小，则有$x + \epsilon y \geq 0$, $x - \epsilon y \geq 0$. 由$Ay = 0$得$x + \epsilon y \in \mathcal{F}$和$x - \epsilon y \in \mathcal{F}$. 显然有$x = \frac{1}{2}(x + \epsilon y) + \frac{1}{2}(x - \epsilon y)$. 此与$x \in \mathcal{F}$为极点矛盾。由反证法，充分性得证。 ■

Corollary 1.9 *If $\mathcal{F} = \{x \in R^n : Ax = b, x \geq 0\}$ is nonempty, it has at least one extreme point.*

Corollary 1.10 *If there is a finite optimal solution to a linear programming problem, there is a finite optimal solution which is an extreme point of the constraint set.*

Corollary 1.11 *The constraint set $\mathcal{F}$ possesses at most a finite number of extreme points.*

Summary 1.8.1 *Basic points to know:*

1. 线性规划问题的可行域$\mathcal{F}$是凸集，可能有界，也可能无界。
2. 若线性规划问题有最优解，必定可以在某顶点上达到最优。
3. $\mathcal{F}$有有限个顶点，每个基可行解对应一个顶点。
4. 若$\mathcal{F}$非空有界，则线性规划问题一定有最优解。
5. 若$\mathcal{F}$无界，则线性规划问题可能无最优解，也可能有最优解。若有最优解，也必定在某顶点上达到。
6. 虽然顶点数目是有限的，但当m, n较大时，"枚举法"是行不通的，所以要继续讨论如何有效寻找最优解的方法。

单纯形法：从一个基可行解（顶点）移动到另一个基可行解（顶点）, ... 在移动的过程中，目标函数值逐步改进，直至最优基可行解.

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MATH: Operations Research **2014-15 First Term**

Handout 2: 解线性规划的单纯形法

Instructor: Junfeng Yang *September 14, 2014*

Contents

2.1 Simplex (单纯形)

Definition 2.1 (general position) *A set of $d+1$ points $\{x_1, x_2, \ldots, x_{d+1}\}$ in R^d is said to be in general position if*

$$\begin{vmatrix} 1 & 1 & \ldots & 1 \\ x_1 & x_2 & \ldots & x_{d+1} \end{vmatrix} \neq 0.$$

Definition 2.2 (d-dimensional simplex, d 维单纯形) *In R^d, the convex hull of $d+1$ points in general position is called a d-dimensional simplex.*

Example 2.1.1 *A zero-dimensional simplex is a point; a one-dimensional simplex is a line segment; a two-dimensional simplex is a triangle and its interior; and a three-dimensional simplex is a tetrahedron and its interior.*

2.2 Pivots (旋转运算)

2.2.1 First interpretation

The linear equality constraints of LP (in standard form) are

$$\begin{array}{rcl} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n & = & b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n & = & b_2 \\ \ldots & \ldots & \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n & = & b_m. \end{array}$$

In matrix form, they appear as $Ax = b$. Denote the ith row rector of A by a^i, $i = 1, 2, \ldots, m$. These constraints can be rewritten as

$$\begin{array}{rcl} a^1 x & = & b_1 \\ a^2 x & = & b_2 \\ & \ldots & \\ a^m x & = & b_m. \end{array}$$

Recall that we assume $m < n$ and rank$(A) = m$. Without loss of generality, we assume that the first m columns of A are linearly independent. Through Gaussian reduction, the linear equality constraints become the so-called *canonical form* (典范形式):

$$\begin{array}{cccccccc} x_1 & & & & + & y_{1,m+1}x_{m+1} + y_{1,m+2}x_{m+2} + \cdots + y_{1,n}x_n & = & y_{10} \\ & x_2 & & & + & y_{2,m+1}x_{m+1} + y_{2,m+2}x_{m+2} + \cdots + y_{2,n}x_n & = & y_{20} \\ & & \ddots & & & \vdots & \vdots & \vdots \\ & & & x_m & + & y_{m,m+1}x_{m+1} + y_{m,m+2}x_{m+2} + \cdots + y_{m,n}x_n & = & y_{m0}. \end{array}$$

Corresponding to this canonical form, $x_1, \ldots, x_m$ are called basic variables, the other variables are nonbasic, and the basic solution is

$$x_1 = y_{10}, \ldots, x_m = y_{m0}, x_{m+1} = 0, \ldots, x_n = 0,$$

or in vector form $x = (y_0, 0)^T$, where $y_0 = (y_{10}, \ldots, y_{m0})^T$. The system is called in canonical form if by some reordering of the equations and the variables its coefficients (and right-hand side) take the form

$$\begin{array}{cccccccc}
1 & & & & y_{1,m+1} & y_{1,m+2} & \cdots & y_{1,n} & y_{10} \\
& 1 & & & y_{2,m+1} & y_{2,m+2} & \cdots & y_{2,n} & y_{20} \\
& & \ddots & & \vdots & \vdots & \ddots & \vdots & \vdots \\
& & & 1 & y_{m,m+1} & y_{m,m+2} & \cdots & y_{m,n} & y_{m0}
\end{array}$$

The question solved by pivoting is this: given a system in canonical form, suppose a basic variable is to be made nonbasic and a nonbasic variable is to be made basic, what is the new canonical form corresponding to the new set of basic variables? Suppose in the canonical system we wish to replace the basic variable x_p, $1 \le p \le m$, by the nonbasic variable x_q, $q > m$. This can be done if and only if y_{pq} is nonzero; it is accomplished by dividing row p by y_{pq} to get a unit coefficient for x_q in the pth equation, and then subtracting suitable multiples of row p from each of the other rows in order to get a zero coefficient for x_q in all other equations.

Denoting the coefficients of the new system in canonical form by y'_{ij}, we have explicitly

$$y'_{ij} = \begin{cases} \frac{y_{ij}}{y_{pq}}, & i = p; \\ y_{ij} - \frac{y_{pj}}{y_{pq}} y_{iq}, & i \neq p, \end{cases} \quad j = 1, 2, \ldots, n.$$

The above equations are the pivot equations that frequently arise in LP. The element y_{pq} in the original system is said to be the pivot element (旋转元).

Example 2.2.1 *Consider the system in canonical form:*

$$\begin{array}{cccccc}
1 & & & 1 & 1 & -1 & 5 \\
& 1 & & 2 & -3 & 1 & 3 \\
& & 1 & -1 & 2 & -1 & -1
\end{array}$$

Find the basic solution having basic variables x_4, x_5, x_6.

2.2.2 Second interpretation

Denote the columns of A by $a_1, a_2, \ldots, a_n$. Suppose that the system is already in canonical form:

a_1	a_2	$\ldots$	a_m	a_{m+1}	a_{m+2}	$\ldots$	a_n	b
1	0	$\ldots$	0	$y_{1,m+1}$	$y_{1,m+2}$	$\ldots$	$y_{1,n}$	y_{10}
0	1	$\ldots$	0	$y_{2,m+1}$	$y_{2,m+2}$	$\ldots$	$y_{2,n}$	y_{20}
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
0	0	$\ldots$	1	$y_{m,m+1}$	$y_{m,m+2}$	$\ldots$	$y_{m,n}$	y_{m0}

Clearly, $a_j = y_{1j}a_1 + y_{2j}a_2 + \ldots + y_{mj}a_m$ for all j from 1 to n. Suppose we wish to replace the basic variable x_p, $1 \le p \le m$, by the nonbasic variable x_q, $q > m$ (note that this can be done if and only if $y_{pq} \neq 0$). It follows from $a_q = \sum_{i=1}^m y_{iq} a_i$ that

$$a_p = \frac{1}{y_{pq}} \left(a_q - \sum_{i=1, i \neq p}^{m} y_{iq} a_i \right).$$

Thus, for all j, it holds that

$$a_j = \sum_{i=1}^{m} y_{ij} a_i = \sum_{i=1, i \neq p}^{m} \left(y_{ij} - \frac{y_{iq}}{y_{pq}} y_{pj} \right) a_i + \frac{y_{pj}}{y_{pq}} a_q.$$

This again implies the same pivoting equations

$$y'_{ij} = \begin{cases} \frac{y_{pj}}{y_{pq}}, & i = p; \\ y_{ij} - \frac{y_{pj}}{y_{pq}} y_{iq}, & i \neq p, \end{cases} \quad j = 1, 2, \ldots, n.$$

2.3 Determination of vector to leave basis

Recall that the feasible region of LP is $\mathcal{F} = \{x \in R^n : Ax = b, x \geq 0\}$. Starting from a basic feasible solution, how can we generate a new basic solution that maintains feasibility?

In general, it is impossible to arbitrarily specify a pair of variables whose roles are to be interchanged and expect to maintain feasibility. However, it is possible to first arbitrarily specify which nonbasic variable is to become basic and then determine (according to certain rule) which basic variable should become nonbasic.

Assumption 2.3.1 *Every basic feasible solution is nondegenerate.*[1]

Suppose $x = (x_1, \ldots, x_m, 0, \ldots, 0)^T$ is a basic feasible solution (thus, $x_i > 0$, $i = 1, 2, \ldots, m$). It holds

$$x_1 a_1 + \ldots + x_m a_m = b.$$

Suppose we have decided to bring a_q ($q > m$) into basis, and in terms of the current basis, a_q can be represented as

$$a_q = y_{1q} a_1 + \ldots + y_{mq} a_m.$$

Let $\varepsilon \geq 0$. It follows that

$$(x_1 - \varepsilon y_{1q}) a_1 + \ldots + (x_m - \varepsilon y_{mq}) a_m + \varepsilon a_q = b.$$

- If $y_{iq} \leq 0$ for $i = 1, 2, \ldots, m$, it follows that $\mathcal{F}$ is unbounded.
- If otherwise, setting $\varepsilon = \min_i \{x_i / y_{iq} : y_{iq} > 0\}$ will give us a new basis with a_q replacing a_p, where p corresponds to the minimizing index defining ε. We note that p is uniquely determined under the nondegenerate assumption. The basic solution corresponding to the new basis maintains feasibility.

Suppose we start with a basic feasible solution determined by

a_1	a_2	$\ldots$	a_m	a_{m+1}	a_{m+2}	$\ldots$	a_n	b
1	0	$\ldots$	0	$y_{1,m+1}$	$y_{1,m+2}$	$\cdots$	$y_{1,n}$	y_{10}
0	1	$\ldots$	0	$y_{2,m+1}$	$y_{2,m+2}$	$\cdots$	$y_{2,n}$	y_{20}
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
0	0	$\ldots$	1	$y_{m,m+1}$	$y_{m,m+2}$	$\cdots$	$y_{m,n}$	y_{m0}

[1]This assumption is only for convenience of discussion. If it is violated, the simplex method can be amended properly.

Feasibility (and nondegeneracy) implies that $y_{i0} > 0$ for $i = 1, 2, \ldots, m$. We have determined to bring a_q into basis. Note that $x_i/y_{iq} = y_{i0}/y_{iq}$, $i = 1, 2, \ldots, m$, and

$$p = \arg\min_i \{x_i/y_{iq} = y_{i0}/y_{iq} : y_{iq} > 0\}.$$

a_1	a_2	$\ldots$	a_m	a_{m+1}	$\ldots$	a_q	$\ldots$	a_n	b
1	0	$\ldots$	0	$y_{1,m+1}$	$\ldots$	$y_{1,q}$	$\ldots$	$y_{1,n}$	y_{10}
0	1	$\ldots$	0	$y_{2,m+1}$	$\ldots$	$y_{2,q}$	$\ldots$	$y_{2,n}$	y_{20}
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$
0	0	$\ldots$	1	$y_{m,m+1}$	$\ldots$	$y_{m,q}$	$\ldots$	$y_{m,n}$	y_{m0}

Example 2.3.1 *If we bring a_4 into basis, which one should leave?*

a_1	a_2	a_3	a_4	a_5	a_6	b
1	0	0	2	4	6	4
0	1	0	1	2	3	3
0	0	1	-1	2	1	1

2.4 Determination of vector to enter basis

Till far, we have not taken into account the objective function $c^T x$. The objective function determines which nonbasic vector is to become basic at the current basic feasible solution.

Suppose the current basic feasible solution is $x = (x_B, 0)^T = (y_{10}, \ldots, y_{m0}, 0, \ldots, 0)^T$, and correspondingly we have a canonical form tableau:

a_1	a_2	$\ldots$	a_m	a_{m+1}	a_{m+2}	$\ldots$	a_n	b
1	0	$\ldots$	0	$y_{1,m+1}$	$y_{1,m+2}$	$\ldots$	$y_{1,n}$	y_{10}
0	1	$\ldots$	0	$y_{2,m+1}$	$y_{2,m+2}$	$\ldots$	$y_{2,n}$	y_{20}
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
0	0	$\ldots$	1	$y_{m,m+1}$	$y_{m,m+2}$	$\ldots$	$y_{m,n}$	y_{m0}

The basic and nonbasic variables are linked by

$$x_i = y_{i0} - \sum_{j=m+1}^{n} y_{ij} x_j, \; i = 1, 2, \ldots, m.$$

Therefore, the objective function reads

$$\begin{aligned} z = c^T x &= \sum_{i=1}^{m} c_i \left(y_{i0} - \sum_{j=m+1}^{n} y_{ij} x_j \right) + \sum_{j=m+1}^{n} c_j x_j \\ &= \sum_{i=1}^{m} c_i y_{i0} + \sum_{j=m+1}^{n} \left(c_j - \sum_{i=1}^{m} c_i y_{ij} \right) x_j \\ &\triangleq z_0 + \sum_{j=m+1}^{n} (c_j - z_j) x_j, \end{aligned}$$

where $z_0 := \sum_{i=1}^m c_i y_{i0}$ and $z_j := \sum_{i=1}^m c_i y_{ij}$. This is the fundamental relation required to determine the pivot column, i.e., the nonbasic vector to become basic. The most important relations in deriving the simplex method:

$$\begin{cases} x_i = y_{i0} - \sum_{j=m+1}^n y_{ij} x_j, & \text{for } i = 1, 2, \ldots, m; \\ z = z_0 + \sum_{j=m+1}^n (c_j - z_j)\, x_j, & \text{where } z_j = \sum_{i=1}^m c_i y_{ij}. \end{cases}$$

Q: Starting at a basic feasible solution, which variable should enter basis?

A: Since our objective is minimization, we choose nonbasic variable x_j to enter basis if and only if $c_j - z_j < 0$, in which case increase x_j from 0 to a positive value will decrease the value of the objective function.

Q: What if more than one nonbasic variables are such that $c_j - z_j < 0$?

A: Various strategies can be utilized. The simplest one is to choose the one with the most negative $c_j - z_j$.

Q: What if $c_j - z_j \geq 0$ for all nonbasic variables?

A: In this case, the current basic feasible solution is already optimal.

Q: If x_q is to enter basis ($c_q - z_q < 0$), which one should leave?

A: x_p should leave, where $p = \arg\min_i \{y_{i0}/y_{iq} : y_{iq} > 0\}$. Under the nondegeneracy assumption, p is uniquely determined. The resulting new basic feasible solution will have objective function value smaller than z_0.

Q: What if $\arg\min_i \{y_{i0}/y_{iq} : y_{iq} > 0\}$ is not a singleton?

A: Degeneracy happens. If the simplex method is not amended properly, cycle may happen. In many applications, anticycling procedures are unnecessary (since cycle may not happen). However, many codes incorporate anticycling strategies for safety.

Q: What happens if $c_q - z_q < 0$ but $y_{iq} \leq 0$ for all $i \in \{1, 2, \ldots, m\}$?

A: $\mathcal{F}$ is unbounded, and the objective function can be made arbitrarily small (towards minus infinity).

Theorem 2.3 (Improvement of basic feasible solution) *(1) Given a nondegenerate basic feasible solution with corresponding objective value z_0, suppose that for some j there holds $c_j < z_j$. Then there is a feasible solution with objective value $z < z_0$. (2) If a_j can be substituted for some vector in the original basis to yield a new basic feasible solution, this new solution will have $z < z_0$. (3) If a_j cannot be substituted to yield a basic feasible solution, then $\mathcal{F}$ is unbounded and $c^T x$ can be made arbitrarily small (towards minus infinity) in $\mathcal{F}$.*

Theorem 2.4 (Optimality condition) *Any basic feasible solution satisfying $c_j \geq z_j$ for all j is optimal.*

The quantities $r_j \triangleq c_j - z_j = c_j - \sum_{i=1}^m c_i y_{ij}$, $j = m+1, \ldots, m$, are referred to as *relative cost coefficients*, which measure the cost of a variable relative to the current basis.

Example 2.4.1 *Take the diet problem as an example in which the nutritional requirements must be met exactly.*

a_1	a_2	$\ldots$	a_m	a_{m+1}	$\ldots$	a_q	$\ldots$	a_n	b
1	*0*	$\ldots$	*0*	$y_{1,m+1}$	$\ldots$	$y_{1,q}$	$\ldots$	$y_{1,n}$	y_{10}
0	*1*	$\ldots$	*0*	$y_{2,m+1}$	$\ldots$	$y_{2,q}$	$\ldots$	$y_{2,n}$	y_{20}
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$
0	*0*	$\ldots$	*1*	$y_{m,m+1}$	$\ldots$	$y_{m,q}$	$\ldots$	$y_{m,n}$	y_{m0}

In this example, q represents a certain food. The food q can be synthetically replaced by the foods in the basis, i.e.,

$$a_q = y_{1,q}a_1 + y_{2,q}a_2 + \ldots + y_{m,q}a_m.$$

$r_q = c_q - z_q = c_q - \sum_{i=1}^m c_i y_{iq} < 0$ *implies that the price c_q of food q is cheaper than the price $\sum_{i=1}^m c_i y_{iq}$ of the* synthetic food q. *In this case, bring food q into basis will reduce the total cost. In this sense, for each j, $j = m+1, \ldots, n$, $z_j = \sum_{i=1}^m c_i y_{ij}$ is also called* synthetic price.

2.5 Simplex tableau

The standard form LP is equivalent to an augmented problem of the form

$$\min_{x,z} \left\{ z : s.t. \begin{pmatrix} A & 0 \\ c^T & -1 \end{pmatrix} \begin{pmatrix} x \\ z \end{pmatrix} = \begin{pmatrix} b \\ 0 \end{pmatrix}, \; x \geq 0 \right\}$$

An initial simplex tableau to this problem is given by

a_1	a_2	$\ldots$	a_m	a_{m+1}	$\ldots$	a_q	$\ldots$	a_n	a_{n+1}	b
1	0	$\ldots$	0	$y_{1,m+1}$	$\ldots$	$y_{1,q}$	$\ldots$	$y_{1,n}$	0	y_{10}
0	1	$\ldots$	0	$y_{2,m+1}$	$\ldots$	$y_{2,q}$	$\ldots$	$y_{2,n}$	0	y_{20}
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$	$\vdots$
0	0	$\ldots$	1	$y_{m,m+1}$	$\ldots$	$y_{m,q}$	$\ldots$	$y_{m,n}$	0	y_{m0}
c_1	c_2	$\ldots$	c_m	c_{m+1}	$\ldots$	c_q	$\ldots$	c_n	-1	0

Through Gaussian reduction, the last row can be transformed to

$$(0, \ldots, 0, r_{m+1}, \ldots, r_q, \ldots, r_n, -1, -z_0).$$

The second last column will not change if the exchange of basis vectors happen among the first n columns and can be deleted. The initial simplex tableau takes the form

a_1	a_2	$\ldots$	a_m	a_{m+1}	$\ldots$	a_q	$\ldots$	a_n	b
1	0	$\ldots$	0	$y_{1,m+1}$	$\ldots$	$y_{1,q}$	$\ldots$	$y_{1,n}$	y_{10}
0	1	$\ldots$	0	$y_{2,m+1}$	$\ldots$	$y_{2,q}$	$\ldots$	$y_{2,n}$	y_{20}
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$
0	0	$\ldots$	1	$y_{m,m+1}$	$\ldots$	$y_{m,q}$	$\ldots$	$y_{m,n}$	y_{m0}
0	0	$\ldots$	0	r_{m+1}	$\ldots$	r_q	$\ldots$	r_n	$-z_0$

The basic feasible solution corresponding to this tableau is

$$x_i = \begin{cases} y_{i0}, & 1 \leq i \leq m; \\ 0, & m+1 \leq i \leq n. \end{cases}$$

By nondegeneracy assumption, we have $y_{i0} > 0$, $i = 1, 2, \ldots, m$. The corresponding objective value is z_0. Suppose we have selected $y_{p,q}$ to be the next pivot element (a_q to leave and a_p to enter basis), i.e.,

$$r_q < 0 \;\text{ and }\; y_{p,0}/y_{p,q} = \min_{i=1,2,\ldots,m} \{y_{i,0}/y_{i,q} : y_{i,q} > 0\}.$$

Recall that the current simplex tableau is

$$
\begin{array}{ccccc|ccccc}
a_1 & \dots & a_p & \dots & a_m & \dots & a_j & \dots & a_q & \dots \\
\hline
1 & & & & & \dots & y_{1,j} & \dots & y_{1,q} & \dots \\
& \ddots & & & & & \vdots & & \vdots & \\
& & 1 & & & \dots & y_{p,j} & \dots & (y_{p,q}) & \dots \\
& & & \ddots & & & \vdots & & \vdots & \\
& & & & 1 & \dots & y_{m,j} & \dots & y_{m,q} & \dots \\
\hline
0 & \dots & 0 & \dots & 0 & \dots & r_j & \dots & r_q & \dots
\end{array}
$$

Via Gaussian reduction, it is transformed to

$$
\begin{array}{ccccc|ccccc}
a_1 & \dots & a_p & \dots & a_m & \dots & a_j & \dots & a_q & \dots \\
\hline
1 & \dots & -\frac{1}{y_{p,q}} y_{1,q} & \dots & 0 & \dots & y_{1,j} - \frac{y_{p,j}}{y_{p,q}} y_{1,q} & \dots & 0 & \dots \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots & \\
0 & \dots & \frac{1}{y_{p,q}} & \dots & 0 & \dots & \frac{y_{p,j}}{y_{p,q}} & \dots & 1 & \dots \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots & \\
0 & \dots & -\frac{1}{y_{p,q}} y_{m,q} & \dots & 1 & \dots & y_{m,j} - \frac{y_{p,j}}{y_{p,q}} y_{m,q} & \dots & 0 & \dots \\
\hline
0 & \dots & -\frac{1}{y_{p,q}} r_q & \dots & 0 & \dots & r_j - \frac{y_{p,j}}{y_{p,q}} r_q & \dots & 0 & \dots
\end{array}
$$

The new basis is $\{a_1, \dots, a_{p-1}, a_{p+1}, \dots, a_m, a_q\}$. It can be shown that elements in the last row corresponding to nonbasic vectors are equal to the new relative cost coefficients. In fact, for $j \in \{m+1, \dots, n\}$, $j \neq q$, it holds that

$$
\begin{aligned}
r_j - \frac{y_{p,j}}{y_{p,q}} r_q &= c_j - \sum_{i=1}^{m} c_i y_{ij} - \frac{y_{p,j}}{y_{p,q}} \left(c_q - \sum_{i=1}^{m} c_i y_{iq} \right) \\
&= c_j - \sum_{i=1, i\neq p}^{m} c_i \left(y_{ij} - \frac{y_{p,j}}{y_{p,q}} y_{iq} \right) - c_q \frac{y_{p,j}}{y_{p,q}}.
\end{aligned}
$$

While for the new nonbasic vector a_p, it holds that

$$
-\frac{1}{y_{p,q}} r_q = -\frac{1}{y_{p,q}} \left(c_q - \sum_{i=1}^{m} c_i y_{iq} \right) = c_p - \sum_{i=1, i\neq p}^{m} c_i \left(-\frac{y_{iq}}{y_{p,q}} \right) - c_q \frac{1}{y_{p,q}}.
$$

2.6 The simplex method

The computational procedure of the simplex method is as follows.

1. Form an initial simplex tableau corresponding to a basic feasible solution. The relative cost coefficients can be found by row reduction and should also be appended as a last row.
2. If each $r_j \geq 0$, stop; the current basic feasible solution is optimal.
3. Select q such that $r_q < 0$ to determine which nonbasic variable is to become basic.
4. Calculate the ratios y_{i0}/y_{iq} for $y_{iq} > 0$, $i = 1, \dots, m$. If no $y_{iq} > 0$, stop; the problem is unbounded. Otherwise, select p as the index i corresponding to the minimum ratio.
5. Pivot on the pqth element, updating all rows including the last by Gaussian reduction. Return to step 2.

Remark 2.6.1 *(1) The process terminates only if optimality is achieved or unboundedness is discovered. (2) If neither condition is discovered at a given basic feasible solution, then the objective can be strictly decreased. (3) Since there are only a finite number of possible basic feasible solutions, and no basis repeats because of the strictly decreasing objective, the algorithm must reach a basis satisfying one of the two terminating conditions.*

Example 2.6.1 *Solve the following LP by simplex method.*

$$\begin{aligned} \max 3x_1 + x_2 + 3x_3 & \\ s.t.\ 2x_1 + x_2 + x_3 &\leq 2 \\ x_1 + 2x_2 + 3x_3 &\leq 5 \\ 2x_1 + 2x_2 + x_3 &\leq 6 \\ x_1, x_2, x_3 &\geq 0. \end{aligned}$$

First, we transform it to standard form

$$\min -3x_1 - x_2 - 3x_3 + 0x_4 + 0x_5 + 0x_6$$
$$s.t. \begin{pmatrix} 2 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 1 & 0 \\ 2 & 2 & 1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_6 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \\ 6 \end{pmatrix}$$
$$x_i \geq 0,\ i = 1, \ldots, 6.$$

The initial simplex tableau is

	a_1	a_2	a_3	a_4	a_5	a_6	b
	2	1	1	1	0	0	2
	1	2	3	0	1	0	5
	2	2	1	0	0	1	6
r^T	-3	-1	-3	0	0	0	0

Elements can be selected to pivot:

	a_1	a_2	a_3	a_4	a_5	a_6	b
	(2)	(1)	1	1	0	0	2
	1	2	(3)	0	1	0	5
	2	2	1	0	0	1	6
r^T	-3	-1	-3	0	0	0	0

For hand calculation, we select (1). Thus a_2 will enter basis and a_4 will leave. Through Gaussian reduction, we get the second simplex tableau

	a_1	a_2	a_3	a_4	a_5	a_6	b
	2	1	1	1	0	0	2
	-3	0	1	-2	1	0	1
	-2	0	-1	-2	0	1	2
r^T	-1	0	-2	1	0	0	2

Elements can be selected to pivot:

	a_1	a_2	a_3	a_4	a_5	a_6	b
	(2)	*1*	*1*	*1*	*0*	*0*	*2*
	-3	*0*	*(1)*	*-2*	*1*	*0*	*1*
	-2	*0*	*-1*	*-2*	*0*	*1*	*2*
r^T	*-1*	*0*	*-2*	*1*	*0*	*0*	*2*

We select (1). Thus a_3 will enter basis and a_5 will leave. The third simplex tableau

	a_1	a_2	a_3	a_4	a_5	a_6	b
	5	*1*	*0*	*3*	*-1*	*0*	*1*
	-3	*0*	*1*	*-2*	*1*	*0*	*1*
	-5	*0*	*0*	*-4*	*1*	*1*	*3*
r^T	*-7*	*0*	*0*	*-3*	*2*	*0*	*4*

Elements can be selected to pivot:

	a_1	a_2	a_3	a_4	a_5	a_6	b
	(5)	*1*	*0*	*(3)*	*-1*	*0*	*1*
	-3	*0*	*1*	*-2*	*1*	*0*	*1*
	-5	*0*	*0*	*-4*	*1*	*1*	*3*
r^T	*-7*	*0*	*0*	*-3*	*2*	*0*	*4*

We select (5). Thus a_1 will enter basis and a_2 will leave. The fourth simplex tableau

	a_1	a_2	a_3	a_4	a_5	a_6	b
	1	*1/5*	*0*	*3/5*	*-1/5*	*0*	*1/5*
	0	*3/5*	*1*	*-1/5*	*2/5*	*0*	*8/5*
	0	*1*	*0*	*-1*	*0*	*1*	*4*
r^T	*0*	*7/5*	*0*	*6/5*	*3/5*	*0*	*27/5*

$r \geq 0$*! So, we have reached an optimal solution:*

$$x^* = (1/5, 0, 8/5, 0, 0, 4)^T.$$

The optimal objective function value (of the modified "min" problem) is $-27/5$.

2.7 Homework

用单纯形法求解

$$\begin{aligned} \min \quad & -2x_1 - 4x_2 - x_3 - x_4 \\ s.t. \quad & x_1 + 3x_2 + x_4 \leq 4 \\ & 2x_1 + x_2 \leq 3 \\ & x_2 + 4x_3 + x_4 \leq 3 \\ & x_i \geq 0,\ i = 1, 2, 3, 4. \end{aligned}$$

References

[Luenberger-Ye] David G. Luenberger and Yinyu Ye Linear and nonlinear programming.

MATH: Operations Research **2014-15 First Term**

Handout 3: 单纯形法的进一步讨论

Instructor: Junfeng Yang *September 17, 2014*

Contents

3.1 Slack variables

A basic feasible solution is sometimes immediately available for linear programming (LP for short) problems, e.g., in LP problems with constraints of the form

$$Ax \le b,\ x \ge 0,$$

where $b \ge 0$. A basic feasible solution is immediately available by introducing slack variables and transform the original problem to

$$\begin{aligned} \min \quad & c^T x + 0^T y \\ s.t. \quad & Ax + y = b \\ & x \ge 0,\ y \ge 0. \end{aligned}$$

However, an initial basic feasible solution is not always apparent.

3.2 Artificial variables

To find a basic feasible solution to

$$\begin{aligned} \min \quad & c^T x \qquad\qquad (\mathbf{LP}) \\ s.t. \quad & Ax = b,\ x \ge 0, \end{aligned}$$

where $b \ge 0$, we first solve an auxiliary LP:

$$\begin{aligned} \min \quad & \textstyle\sum_{i=1}^m y_i \qquad\qquad (\mathbf{auxLP}) \\ s.t. \quad & Ax + y = b,\ x \ge 0,\ y \ge 0. \end{aligned}$$

Here $y = (y_1, \ldots, y_m)^T \in R^m$ is a vector of artificial variables. It is easy to see that (1) if **LP** has a feasible solution, then **auxLP** has a minimum value of zero with $y = 0$; and (2) if **LP** has no feasible solution, then the minimum value of **auxLP** is greater than 0.

Note that **auxLP** is already in canonical form with basic feasible solution $(x, y) = (0, b)$. Suppose **auxLP** is solved by the simplex method, a basic feasible solution is obtained at each step. If the minimum value of **auxLP** is zero, then the final basic solution will have $y = 0$. In this case, the x-part will give a basic feasible solution to the system $Ax = b, x \ge 0$.

In practice, one can first introduce slack variables and then introduce less number of artificial variables if necessary.

Example 3.2.1 *Find a basic feasible solution to*

$$\begin{aligned} 2x_1 + x_2 + 2x_3 &= 4 \\ 3x_1 + 3x_2 + x_3 &= 3 \\ (x_1, x_2, x_3)^T &\ge 0. \end{aligned}$$

Introduce artificial variables $x_4 \ge 0$, $x_5 \ge 0$ and an objective function $x_4 + x_5$. The initial tableau is

	x_1	x_2	x_3	x_4	x_5	b
	2	1	2	1	0	4
	3	3	1	0	1	3
c^T	0	0	0	1	1	0

To initiate the simplex procedure we must update the last row (via Gaussian reduction) so that it has zeros under the basic variables. This yields the first tableau:

$$\begin{array}{ccccccc} & x_1 & x_2 & x_3 & x_4 & x_5 & b \\ & 2 & 1 & 2 & 1 & 0 & 4 \\ & (3) & 3 & 1 & 0 & 1 & 3 \\ r^T & -5 & -4 & -3 & 0 & 0 & -7 \end{array}$$

Pivoting in the column having the most negative bottom row component as indicated, we obtain the second tableau:

$$\begin{array}{ccccccc} & x_1 & x_2 & x_3 & x_4 & x_5 & b \\ & 0 & -1 & (4/3) & 1 & -2/3 & 2 \\ & 1 & 1 & 1/3 & 0 & 1/3 & 1 \\ r^T & 0 & 1 & -4/3 & 0 & 5/3 & -2 \end{array}$$

Pivoting one more time, we obtain the next and also the final tableau:

$$\begin{array}{ccccccc} & x_1 & x_2 & x_3 & x_4 & x_5 & b \\ & 0 & -3/4 & 1 & 3/4 & -1/2 & 3/2 \\ & 1 & 5/4 & 0 & -1/4 & 1/2 & 1/2 \\ r^T & 0 & 0 & 0 & 1 & 1 & 0 \end{array}$$

Both of the artificial variables have been driven out of the basis, thus reducing the value of the objective function to zero and leading to the basic feasible solution to the original problem $(x_1, x_2, x_3) = (1/2, 0, 3/2)$.

3.3 Two-Phase Method

Using artificial variables, we solve a general LP problem by a two-phase method:

1. Phase I: Artificial variables are introduced and a basic feasible solution is found (or it is determined that no feasible solutions exist) for an auxiliary LP. Note that artificial variables need be introduced only in those equations that do not contain slack variables.
2. Phase II: Using the basic feasible solution resulting from phase I, the original LP can be solved by the simplex method.

Remark 3.3.1 *Note that in Phase I, we try to find a basic feasible solution via solving a LP. In this sense, finding a basic feasible solution is as difficult as solving the original LP. Indeed, this is true because solving LP is equivalent to solving a system of linear inequalities (will be made accurate later).*

Example 3.3.1 *Consider the problem*

$$\begin{aligned} \min \quad & 4x_1 + x_2 + x_3 \\ s.t. \quad & 2x_1 + x_2 + 2x_3 = 4 \\ & 3x_1 + 3x_2 + x_3 = 3 \\ & (x_1, x_2, x_3) \geq 0 \end{aligned}$$

A basic feasible solution has been found in Example 3.2.1. Deleting columns corresponds to artificial variables and replacing the last row by the new cost coefficients, we obtain the initial tableau

	x_1	x_2	x_3	b
	0	$-3/4$	1	$3/2$
	1	$5/4$	0	$1/2$
c^T	4	1	1	0

Transforming the last row so that zeros appear in the basic columns, we get the first tableau

	x_1	x_2	x_3	b
	0	$-3/4$	1	$3/2$
	1	$(5/4)$	0	$1/2$
r^T	0	$-13/4$	0	$-7/2$

Keep iterating, we arrive at

	x_1	x_2	x_3	b
	$3/5$	0	1	$9/5$
	$4/5$	1	0	$2/5$
r^T	$12/5$	0	0	$-11/5$

Thus the optimal solution is $(x_1, x_2, x_3) = (0, 2/5, 9/5)$.

Example 3.3.2 (A free variable problem)

$$\begin{aligned} \min \quad & -2x_1 + 4x_2 + 7x_3 + x_4 + 5x_5 \\ s.t. \quad & -x_1 + x_2 + 2x_3 + x_4 + 2x_5 = 7 \\ & -x_1 + 2x_2 + 3x_3 + x_4 + x_5 = 6 \\ & -x_1 + x_2 + x_3 + 2x_4 + x_5 = 4 \\ & x_1 \ \textit{free}, (x_2, x_3, x_4, x_5) \geq 0. \end{aligned}$$

Since x_1 *is free, it can be eliminated by solving for* x_1 *in terms of the other variables from the first equation and substituting everywhere else. This can all be done with the simplex tableau as follows:*

	x_1	x_2	x_3	x_4	x_5	b
	(-1)	1	2	1	2	7
	-1	2	3	1	1	6
	-1	1	1	2	1	4
c^T	-2	4	7	1	5	0

Equivalent problem:

	x_1	x_2	x_3	x_4	x_5	b
	1	-1	-2	-1	-2	-7
	0	1	1	0	-1	-1
	0	0	-1	1	-1	-3
c^T	0	2	3	-1	1	-14

Multiplying the two equality constraints by -1 so that the right hand side become positive and introducing x_6 and x_7, we obtain the initial tableau of phase I:

$$\begin{array}{ccccccccc} & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & b \\ & -1 & -1 & 0 & 1 & 1 & 0 & 1 \\ & 0 & 1 & -1 & 1 & 0 & 1 & 3 \\ c^T & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{array}$$

First tableau of phase I:

$$\begin{array}{cccccccc} & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & b \\ & -1 & -1 & 0 & (1) & 1 & 0 & 1 \\ & 0 & 1 & -1 & 1 & 0 & 1 & 3 \\ r^T & 1 & 0 & 1 & -2 & 0 & 0 & -4 \end{array}$$

Second tableau of phase I:

$$\begin{array}{cccccccc} & -1 & -1 & 0 & 1 & 1 & 0 & 1 \\ & (1) & 2 & -1 & 0 & -1 & 1 & 2 \\ r^T & -1 & -2 & 1 & 0 & 2 & 0 & -2 \end{array}$$

Final tableau of phase I:

$$\begin{array}{cccccccc} & 0 & 1 & -1 & 1 & 0 & 1 & 3 \\ & 1 & 2 & -1 & 0 & -1 & 1 & 2 \\ r^T & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{array}$$

Now we go back to the equivalent reduced problem

$$\begin{array}{cccccc} & x_2 & x_3 & x_4 & x_5 & b \\ & 0 & 1 & -1 & 1 & 3 \\ & 1 & 2 & -1 & 0 & 2 \\ c^T & 2 & 3 & -1 & 1 & -14 \end{array}$$

First tableau of phase II:

$$\begin{array}{cccccc} & 0 & 1 & -1 & 1 & 3 \\ & 1 & (2) & -1 & 0 & 2 \\ r^T & 0 & -2 & 2 & 0 & -21 \end{array}$$

Final tableau of phase II:

$$\begin{array}{cccccc} & -1/2 & 0 & -1/2 & 1 & 2 \\ & 1/2 & 1 & -1/2 & 0 & 1 \\ r^T & 1 & 0 & 1 & 0 & -19 \end{array}$$

Optimal solution: $(x_2, x_3, x_4, x_5) = (0, 1, 0, 2)$. The free variable x_1 and the optimal function value can then be computed.

3.4 Big-M Method

It is possible to combine the two phases of the two-phase method into a single procedure by the big-M method. Given the linear program in standard form

$$\begin{aligned} \min \quad & c^T x \\ s.t. \quad & Ax = b \\ & x \geq 0, \end{aligned}$$

one forms the approximating problem

$$\begin{array}{rc} \min & c^T x + M\sum_{i=1}^m y_i \\ s.t. & Ax + y = b \\ & x \geq 0 \\ & y \geq 0. \end{array}$$

In this problem $y = (y_1, y_2, \ldots, y_m)^T$ is a vector of artificial variables and $M > 0$ is a large constant. The term $M\sum_{i=1}^m y_i$ serves as a penalty term for nonzero y_i's. If this auxiliary LP problem is solved by the simplex method, the following conclusions are true:

1. If an optimal solution is found with $y = 0$, then the corresponding x is an optimal basic feasible solution to the original problem.
2. If for every $M > 0$ an optimal solution is found with $y \neq 0$, then the original problem is infeasible. Or equivalently, if the original problem is feasible, then the modified LP must have optimal solution with $y = 0$ for some $M > 0$.
3. If for every $M > 0$ the approximating problem is unbounded, then the original problem is either unbounded or infeasible.
4. Suppose that the original problem has a finite optimal value $V(\infty)$. Let $V(M)$ be the optimal value of the approximating problem. Then $V(M) \leq V(\infty)$.
5. For $M_1 \leq M_2$ we have $V(M_1) \leq V(M_2)$.
6. There exists $M_0 > 0$ such that for $M \geq M_0$, $V(M) = V(\infty)$, and hence the big-M method will produce the right solution for large enough values of M.

3.5 Matrix form of the simplex method

Let $B \in R^{m\times m}$ be a basis matrix, i.e., B is a nonsingular submatrix of A. As usual, we assume that B consists of the first m columns of A. Then by partitioning A, x and c as

$$A = (B, D),\ x^T = (x_B^T, x_D^T),\ c^T = (c_B^T, c_D^T),$$

the standard LP can be rewritten as

$$\begin{array}{rc} \min & c_B^T x_B + c_D^T x_D \\ s.t. & Bx_B + Dx_D = b \\ & x_B \geq 0, x_D \geq 0. \end{array}$$

Letting $x_D = 0$, we obtain a basic solution $x = (B^{-1}b, 0)$ to $Ax = b$. If B is a feasible basis, then the basic solution $x = (B^{-1}b, 0)$ is also feasible. The basic and nonbasic variables are related by

$$x_B = B^{-1}b - B^{-1}Dx_D. \tag{3.1}$$

Deleting basic variables in the objective function yields

$$z = c_B^T(B^{-1}b - B^{-1}Dx_D) + c_D^T x_D = c_B^T B^{-1}b + (c_D^T - c_B^T B^{-1}D)x_D. \tag{3.2}$$

The equations (3.1) and (3.2) are the two most important relations in deriving the simplex method. The vector

$$r_D^T := c_D^T - c_B^T B^{-1} D$$

is the relative cost vector for nonbasic variables (for basic variables, the relative cost is always zero). The components of this vector will be used to determine which vector to bring into basis.

The initial simplex tableau takes the form

$$\begin{bmatrix} A & b \\ c^T & 0 \end{bmatrix} = \begin{bmatrix} B & D & b \\ c_B^T & c_D^T & 0 \end{bmatrix}$$

which is, in general, not in canonical form and does not correspond to a point in the simplex procedure. If the matrix B is used as a basis, then the corresponding tableau becomes

$$T = \begin{bmatrix} I & B^{-1}D & B^{-1}b \\ 0 & c_D^T - c_B^T B^{-1} D & -c_B^T B^{-1} b \end{bmatrix}$$

which is the matrix form of the simplex method.

3.6 The revised simplex method

Experience based on extensive computation: the simplex method converges to an optimal basic feasible solution in around $m \sim 1.5m$ pivot operations. In particular, if $m \ll n$, i.e., if the matrix A has far fewer rows than columns, pivots will occur in only a small fraction of the columns during the course of optimization. Since the other columns are not explicitly used, it appears that the work expended in calculating the elements in these columns after each pivot is, in some sense, wasted effort. The revised simplex method aims to avoid unnecessary calculations.

Given the inverse B^{-1} of the current basis, and the current solution $x_B = y_0 = B^{-1}b$, do the following:

1. Calculate the current relative cost coefficients $r_D^T = c_D^T - c_B^T B^{-1} D$. This can best be done by first calculating $\lambda^T = c_B^T B^{-1}$ and then the relative cost vector $r_D^T = c_D^T - \lambda^T D$. If $r_D \geq 0$, the current solution is optimal.
2. Determine which vector a_q is to enter the basis by selecting, say, the most negative cost coefficient; and calculate $y_q = B^{-1} a_q$ which gives the vector a_q expressed in terms of the current basis.
3. If no y_{iq} (the ith component of y_q) is greater than 0, stop; the problem is unbounded. Otherwise, calculate the ratios y_{i0}/y_{iq} for $y_{iq} > 0$ to determine which vector is going to leave the basis.
4. Update B^{-1} and the current solution $B^{-1}b$. Return to Step 1.

The update of B^{-1} can be done by the usual pivot operations applied to an array consisting of B^{-1} and y_q, where the pivot is the appropriate element in y_q, as explained below. Suppose the current basis is $B = (a_1, a_2, \ldots, a_m)$, a_q is going to enter basis and a_p is going to leave.

$$\begin{aligned} A &= (B, D) \xrightarrow{B^{-1}} (I, B^{-1}D) \\ &= (B^{-1}a_1, \ldots, B^{-1}a_m, B^{-1}a_{m+1}, \ldots, B^{-1}a_n) \\ &= (e_1, \ldots, e_p, \ldots, e_m, y_{m+1}, \ldots, y_q, \ldots, y_n) \\ &\xrightarrow{Q} (Qe_1, \ldots, Qe_p, \ldots, Qe_m, Qy_{m+1}, \ldots, Qy_q, \ldots, Qy_n) \\ &= (e_1, \ldots, e_{p-1}, Qe_p, e_{p+1}, \ldots, e_m, \\ &\quad\; Qy_{m+1}, \ldots, Qy_{q-1}, e_p, Qy_{q+1}, \ldots, Qy_n). \end{aligned}$$

Now the new basis is $\bar{B} = (a_1, \ldots, a_{p-1}, a_{p+1}, \ldots, a_m, a_q)$. Clearly, it holds that

$$(QB^{-1})\bar{B} = (e_1, \ldots, e_{p-1}, e_{p+1}, \ldots, e_m, e_p),$$

which is a permutation of the identity matrix. So, we update B^{-1} by applying the same Gaussian reduction as we do for reducing y_q to e_p.

Example 3.6.1 (demonstrate the revised simplex method)

$$\begin{aligned} &\min -3x_1 - x_2 - 3x_3 + 0x_4 + 0x_5 + 0x_6 \\ &s.t. \begin{pmatrix} 2 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 1 & 0 \\ 2 & 2 & 1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_6 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \\ 6 \end{pmatrix} \\ &\qquad x_i \geq 0,\ i = 1, \ldots, 6. \end{aligned}$$

The initial tableau of the simplex method:

	a_1	a_2	a_3	a_4	a_5	a_6	b
	2	1	1	1	0	0	2
	1	2	3	0	1	0	5
	2	2	1	0	0	1	6
c^T	-3	-1	-3	0	0	0	0

Let Ind_B and Ind_D be the set of indices of basic and nonbasic variables, respectively. The initial tableau of the the revised simplex method:

Ind_B	B^{-1}			x_B
4	1	0	0	2
5	0	1	0	5
6	0	0	1	6

$Ind_B = \{4, 5, 6\}$ and $Ind_D = \{1, 2, 3\}$. Compute relative cost coefficients:

$$\lambda^T = c_B^T B^{-1} = (0, 0, 0)B^{-1} = (0, 0, 0), \quad r_D^T = c_D^T - \lambda^T D = (-3, -1, -3).$$

We decide to bring a_2 into basis (to simplify the hand calculation). The representation of a_2 under the current basis B is given by

$$y_2 = B^{-1}a_2 = (1, 2, 2)^T.$$

Thus, we have

Ind_B	B^{-1}			x_B	y_2
4	1	0	0	2	(1)
5	0	1	0	5	2
6	0	0	1	6	2

Select (1) as pivot element (thus a_2 enters basis and a_4 leaves). Update B^{-1}, x_B, Ind_B and Ind_D:

Ind_B	B^{-1}			x_B
2	1	0	0	2
5	-2	1	0	1
6	-2	0	1	2

$Ind_B = \{2,5,6\}$ and $Ind_D = \{1,3,4\}$. Compute relative cost coefficients:

$$\lambda^T = c_B^T B^{-1} = (-1,0,0)B^{-1} = (-1,0,0), \quad r_D^T = c_D^T - \lambda^T D = (-1,-2,1).$$

Bring a_3 into basis (corresponds to -2 in r_D). Compute the representation of a_3 under current basis B: $y_3 = B^{-1}a_3 = (1,1,-1)^T$. Thus, we have

Ind_B	B^{-1}			x_B	y_3
2	1	0	0	2	1
5	-2	1	0	1	(1)
6	-2	0	1	2	-1

Select (1) as pivot element (thus a_3 enters basis and a_5 leaves). Update B^{-1}, x_B, Ind_B and Ind_D:

Ind_B	B^{-1}			x_B
2	3	-1	0	1
3	-2	1	0	1
6	-4	1	1	3

$Ind_B = \{2,3,6\}$ and $Ind_D = \{1,4,5\}$. Compute relative cost coefficients:

$$\lambda^T = c_B^T B^{-1} = (-1,-3,0)B^{-1} = (3,-2,0), \quad r_D^T = c_D^T - \lambda^T D = (-7,-3,2).$$

Bring a_1 into basis (corresponds to -7 in r_D). Compute the representation of a_1 under current basis B: $y_1 = B^{-1}a_1 = (5,-3,-5)^T$. Thus, we have

Ind_B	B^{-1}			x_B	y_1
2	3	-1	0	1	(5)
3	-2	1	0	1	-3
6	-4	1	1	3	-5

Select (5) as pivot element (thus a_1 enters basis and a_2 leaves). Update B^{-1}, x_B, Ind_B and Ind_D:

Ind_B	B^{-1}			x_B
1	$3/5$	$-1/5$	0	$1/5$
3	$-1/5$	$2/5$	0	$8/5$
6	-1	0	1	4

$Ind_B = \{1,3,6\}$ and $Ind_D = \{2,4,5\}$. Compute relative cost coefficients:

$$\lambda^T = c_B^T B^{-1} = (-3,-3,0)B^{-1} = (-6/5,-3/5,0), \quad r_D^T = c_D^T - \lambda^T D = (7/5,6/5,3/5).$$

Since $r_D \geq 0$, we conclude that the solution $x = (1/5,0,8/5,0,0,4)^T$ is optimal.

3.7 Homework

用两阶段法求解

$$\begin{aligned}
\min \quad & -3x_1 + x_2 + 3x_3 - x_4 \\
s.t. \quad & x_1 + 2x_2 - x_3 + x_4 = 0 \\
& 2x_1 - 2x_2 + 3x_3 + 3x_4 = 9 \\
& x_1 - x_2 + 2x_3 - x_4 = 6 \\
& x_i \geq 0, \ i = 1,2,3,4.
\end{aligned}$$

References

[Luenberger-Ye] David G. Luenberger and Yinyu Ye Linear and nonlinear programming.

MATH: Operations Research **2014-15 First Term**

Handout 4: Linear programming duality theory – Weak duality

Instructor: Junfeng Yang *September 21, 2014*

Contents

4.1 Introduction

Associated with every LP (linear program), and intimately related to it, is a corresponding dual LP. The primal and the dual problems are constructed from the same underlying cost and constraint coefficients. If the primal is to minimize, then the dual is to maximize, and vice versa. The optimum function values of both the primal and the dual, if finite, are equal. The variables of the dual problem can be interpreted as prices associated with the constraints of the primal problem. The variables of the dual problem are also intimately related to the calculation of the relative cost coefficients in the simplex method. In summary, studying the dual LP sharpens our understanding.

4.2 Dual problem of LP and weak duality

It is convenient to assume that the minimum value of any real valued function over an empty set is $+\infty$, while the maximum value of any real valued function over an empty set is $-\infty$, i.e.,

$$+\infty = \min_x \{f(x) :\ s.t.\ x \in \emptyset\} \quad \text{and} \quad -\infty = \max_x \{f(x) :\ s.t.\ x \in \emptyset\}.$$

In this course, we use the above as convention.

4.2.1 Standard form

Notation 4.2.1

- *$A \in R^{m\times n}$, $b \in R^m$ and $c \in R^n$;*
- *$\mathcal{F}_1 := \{x \in R^n : Ax = b, x \geq 0\}$;*
- *$R^n_+ := \{x \in R^n : x \geq 0\}$.*

Consider LP in standard form:

$$p_1^* := \min_{x\in R^n} \{c^T x : s.t.\ Ax = b, x \geq 0\}. \tag{Primal-LP1}$$

The Lagrange function associated with Primal-LP1 is defined by

$$\mathcal{L}(x, \lambda, \mu) = c^T x - \lambda^T (Ax - b) - \mu^T x,$$

which is a function from $R^n \times R^m \times R^n$ to R. For any $x \in \mathcal{F}_1$, $\lambda \in R^m$ and $\mu \in R^n_+$, it is clear that

$$\mathcal{L}(x, \lambda, \mu) \leq c^T x.$$

Therefore, for any $\lambda \in R^m$ and $\mu \in R^n_+$, we have

$$\inf_{x\in R^n} \mathcal{L}(x, \lambda, \mu) \leq \inf_{x\in\mathcal{F}_1} \mathcal{L}(x, \lambda, \mu) \leq \inf_{x\in\mathcal{F}_1} c^T x = p_1^*,$$

i.e., $\inf_{x\in R^n} \mathcal{L}(x, \lambda, \mu)$ is a lower bound of the optimal function value p_1^*. It is natural to raise a question like what is the best low bound of p_1^* obtained in this way? In fact, the best lower bound of p_1^* that can be so obtained is

$$\begin{aligned}
& \sup\nolimits_{\lambda\in R^m,\mu\in R^n_+} \inf_{x\in R^n} \mathcal{L}(x, \lambda, \mu) \\
= \ & \sup\nolimits_{\lambda\in R^m,\mu\in R^n_+} \begin{cases} b^T\lambda, & \text{if } c - A^T\lambda - \mu = 0; \\ -\infty, & \text{o.w.} \end{cases} \\
= \ & \sup\nolimits_{\lambda\in R^m,\mu\in R^n_+} \{b^T\lambda : s.t.\ c - A^T\lambda - \mu = 0\} \\
= \ & \sup\nolimits_{\lambda\in R^m} \{b^T\lambda : s.t.\ A^T\lambda \leq c\}.
\end{aligned}$$

The following problem is called the *dual problem* of Primal-LP1:

$$d_1^* := \max_{\lambda \in R^m} \{b^T\lambda : s.t.\ A^T\lambda \le c\}. \qquad \text{(Dual-LP1)}$$

Theorem 4.1 (Weak Duality) *Let p_1^* and d_1^* be, respectively, defined as above. Then, $d_1^* \le p_1^*$.*

Corollary 4.2

1. *If x and λ are feasible for Primal-LP1 and Dual-LP1, respectively, then $b^T\lambda \le d_1^* \le p_1^* \le c^Tx$.*
2. *If x_0 and λ_0 are feasible for Primal-LP1 and Dual-LP1, respectively, and if $b^T\lambda_0 = c^Tx_0$, then x_0 and λ_0 are optimal for their respective problems. Furthermore, $b^T\lambda_0 = d_1^* = p_1^* = c^Tx_0$.*
3. *If $p_1^* = -\infty$, i.e., Primal-LP1 is feasible and unbounded below, then Dual-LP1 must be infeasible.*
4. *If $d_1^* = +\infty$, i.e., Dual-LP1 is feasible and unbounded above, then Primal-LP1 must be infeasible.*

4.2.2 Another form

Notation 4.2.2 $\mathcal{F}_2 := \{x \in R^n : Ax \ge b, x \ge 0\}$.

Consider LP in the form:

$$p_2^* := \min_{x \in R^n} \{c^Tx : s.t.\ Ax \ge b, x \ge 0\}. \qquad \text{(Primal-LP2)}$$

The Lagrange function associated with Primal-LP2 is defined by

$$\mathcal{L}(x, \lambda, \mu) = c^Tx - \lambda^T(Ax - b) - \mu^Tx,$$

which is a function from $R^n \times R^m \times R^n$ to R. For any $x \in \mathcal{F}_2$, $\lambda \in R_+^m$ and $\mu \in R_+^n$, it is clear that

$$\mathcal{L}(x, \lambda, \mu) \le c^Tx.$$

Therefore, for any $\lambda \in R_+^m$ and $\mu \in R_+^n$, it holds that

$$\inf_{x \in R^n} \mathcal{L}(x, \lambda, \mu) \le \inf_{x \in \mathcal{F}_2} \mathcal{L}(x, \lambda, \mu) \le \inf_{x \in \mathcal{F}_2} c^Tx = p_2^*,$$

i.e., $\inf_{x \in R^n} \mathcal{L}(x, \lambda, \mu)$ is a lower bound of the optimal function value p_2^*. The best lower bound of p_2^* that can be so obtained is

$$\begin{aligned}
& \sup_{\lambda \in R_+^m, \mu \in R_+^n} \inf_{x \in R^n} \mathcal{L}(x, \lambda, \mu) \\
= \ & \sup_{\lambda \in R_+^m, \mu \in R_+^n} \begin{cases} b^T\lambda, & \text{if } c - A^T\lambda - \mu = 0; \\ -\infty, & \text{o.w.} \end{cases} \\
= \ & \sup_{\lambda \in R_+^m, \mu \in R_+^n} \{b^T\lambda : s.t.\ c - A^T\lambda - \mu = 0\} \\
= \ & \sup_{\lambda \in R^m} \{b^T\lambda : s.t.\ A^T\lambda \le c, \lambda \ge 0\}.
\end{aligned}$$

The following problem is called the *dual problem* of Primal-LP2:

$$d_2^* := \max_{\lambda \in R^m} \{b^T\lambda : s.t.\ A^T\lambda \le c, \lambda \ge 0\}. \qquad \text{(Dual-LP2)}$$

Theorem 4.3 (Weak Duality) *Let p_2^* and d_2^* be, respectively, defined as above for Primal-LP2 and Dual-LP2. Then, $d_2^* \leq p_2^*$.*

Corollary 4.4 *1. If x and λ are, respectively, feasible for Primal-LP2 and Dual-LP2, then $b^T\lambda \leq d_2^* \leq p_2^* \leq c^T x$.*

2. If x_0 and λ_0 are, respectively, feasible for Primal-LP2 and Dual-LP2, and if $b^T\lambda_0 = c^T x_0$, then x_0 and λ_0 are optimal for their respective problems. Furthermore, $b^T\lambda_0 = d_2^* = p_2^* = c^T x_0$.

3. If $p_2^* = -\infty$, i.e., Primal-LP2 is feasible and unbounded below, then Dual-LP2 must be infeasible.

4. If $d_2^* = +\infty$, i.e., Dual-LP2 is feasible and unbounded above, then Primal-LP2 must be infeasible.

4.2.3 More general form

Notation 4.2.3
- *$A \in R^{m_1 \times n}$, $b \in R^{m_1}$, $c \in R^n$;*
- *$E \in R^{m_2 \times n}$, $f \in R^{m_2}$, $G \in R^{m_3 \times n}$, $h \in R^{m_3}$;*
- *$I, J, K \subset \{1, 2, \dots, n\}$;*
- *$\mathcal{F}_3 := \{x \in R^n : Ax = b,\ Ex \geq f,\ Gx \leq h,\ x_I \geq 0,\ x_J \leq 0\}$.*

Consider LP in the form:

$$p_3^* := \min_{x \in R^n} \left\{ c^T x \;\middle|\; \begin{array}{l} Ax = b,\ Ex \geq f,\ Gx \leq h \\ x_I \geq 0,\ x_J \leq 0,\ x_K \ \text{free} \end{array} \right\}. \tag{Primal-LP3}$$

The associated Lagrange function is defined by

$$\mathcal{L}(x, u, v, w, \lambda, \mu) = c^T x - u^T(Ax - b) - v^T(Ex - f) - w^T(h - Gx) - \lambda^T x_I - \mu^T(-x_J),$$

which is a function from $R^n \times R^{m_1} \times R^{m_2} \times R^{m_3} \times R^{|I|} \times R^{|J|}$ to R. For any $x \in \mathcal{F}_3$, $u \in R^{m_1}, v \in R_+^{m_2}, w \in R_+^{m_3}, \lambda \in R_+^{|I|}$ and $\mu \in R_+^{|J|}$, it is clear that

$$\mathcal{L}(x, u, v, w, \lambda, \mu) \leq c^T x.$$

Therefore, for any $u \in R^{m_1}, v \in R_+^{m_2}, w \in R_+^{m_3}, \lambda \in R_+^{|I|}$ and $\mu \in R_+^{|J|}$, it holds that

$$\inf_{x \in R^n} \mathcal{L}(x, u, v, w, \lambda, \mu) \leq \inf_{x \in \mathcal{F}_3} \mathcal{L}(x, u, v, w, \lambda, \mu) \leq \inf_{x \in \mathcal{F}_3} c^T x = p_3^*,$$

i.e., $\inf_{x \in R^n} \mathcal{L}(x, u, v, w, \lambda, \mu)$ is a lower bound of p_3^*. The best lower bound of p_3^* that can be so obtained is

$$\begin{array}{rll} & \sup_{u \in R^{m_1}, v \geq 0, w \geq 0, \lambda \geq 0, \mu \geq 0} & \inf_{x \in R^n} \mathcal{L}(x, u, v, w, \lambda, \mu) \\ = & \sup_{u \in R^{m_1}, v \in R_+^{m_2}, w \in R_+^{m_3}} & \left\{ \begin{array}{l} b^T u + f^T v - h^T w, \\ \text{if} \left\{ \begin{array}{l} (c - A^T u - E^T v + G^T w)_I \geq 0, \\ (c - A^T u - E^T v + G^T w)_J \leq 0, \\ (c - A^T u - E^T v + G^T w)_K = 0. \end{array} \right. \\ -\infty, \text{o.w.} \end{array} \right. \end{array}$$

The following problem is called the *dual problem* of Primal-LP3:

$$d_3^* := \max_{u,v,w} \left\{ b^T u + f^T v - h^T w \;\middle|\; \begin{array}{c} (c - A^T u - E^T v + G^T w)_I \geq 0, \\ (c - A^T u - E^T v + G^T w)_J \leq 0, \\ (c - A^T u - E^T v + G^T w)_K = 0, \\ u \in R^{m_1}, v \in R_+^{m_2}, w \in R_+^{m_3}. \end{array} \right\} \tag{Dual-LP3}$$

Theorem 4.5 (Weak Duality) *Let p_3^* and d_3^* be, respectively, defined as above for Primal-LP3 and Dual-LP3. Then, $d_3^* \leq p_3^*$.*

Corollary 4.6 *1. If x and (u, v, w) are feasible for Primal-LP3 and Dual-LP3, respectively, then*

$$b^T u + f^T v - h^T w \leq d_3^* \leq p_3^* \leq c^T x.$$

2. If x_0 and (u_0, v_0, w_0) are, resp., feasible for Primal-LP3 and Dual-LP3, and if $b^T u_0 + f^T v_0 - h^T w_0 = c^T x_0$, then x_0 and (u_0, v_0, w_0) are optimal for their respective problems. Furthermore, $d_3^* = p_3^* = c^T x_0$.

3. If $p_3^* = -\infty$, i.e., Primal-LP3 is feasible and unbounded below, then Dual-LP3 must be infeasible.

4. If $d_3^* = +\infty$, i.e., Dual-LP3 is feasible and unbounded above, then Primal-LP3 must be infeasible.

4.2.4 Summary

$$\begin{array}{rl} \min & c^T x \\ \text{s.t.} & Ax = b \\ & x \geq 0. \end{array}$$

(a) Primal-LP1

$$\begin{array}{rl} \max & b^T \lambda \\ \text{s.t.} & \lambda \text{ free} \\ & A^T \lambda \leq c. \end{array}$$

(b) Dual-LP1

$$\begin{array}{rl} \min & c^T x \\ \text{s.t.} & Ax \geq b \\ & x \geq 0. \end{array}$$

(c) Primal-LP2

$$\begin{array}{rl} \max & b^T \lambda \\ \text{s.t.} & \lambda \geq 0 \\ & A^T \lambda \leq c. \end{array}$$

(d) Dual-LP2

$$\begin{array}{rl} \min & c^T x \\ \text{s.t.} & Ax = b \\ & Ex \geq f \\ & Gx \leq h \\ & x_I \geq 0 \\ & x_J \leq 0 \\ & x_K \text{ free.} \end{array}$$

(e) Primal-LP3

$$\begin{array}{rl} \max & b^T u + f^T v - h^T w \\ \text{s.t.} & u \in R^{m_1} \\ & v \in R_+^{m_2} \\ & w \in R_+^{m_3} \\ & (c - A^T u - E^T v + G^T w)_I \geq 0 \\ & (c - A^T u - E^T v + G^T w)_J \leq 0 \\ & (c - A^T u - E^T v + G^T w)_K = 0. \end{array}$$

(f) Dual-LP3

(g) Relation of primal and dual values.

Figure 4.1: Summary on Primal and Dual LPs.

Variables in the dual programs are called dual variables of the primal programs. Now we focus on the most general form Primal-LP3 and its dual problem. It is easy to see that (1) if an inequality constraint in the primal problem is changed to equality, then the corresponding dual variable will be freed; (2) If some of the components of x_I or x_J in the primal problem are freed, then the corresponding inequalities in the dual problem will become equality; Similar remarks apply to other constraints in the primal and the counterparts in the dual.

Theorem 4.7 *For any LP, the dual of the dual is itself.*

Proof: Take the standard form LP for example. Its dual problem is

$$d_1^* := \max\{b^T\lambda : s.t.\ A^T\lambda \le c\}.$$

Let $\mathcal{F}_D := \{\lambda \in R^m : A^T\lambda \le c\}$ and $\mathcal{L} : R^m \times R^n \to R$ be defined by

$$\mathcal{L}(\lambda, x) = b^T\lambda - x^T(A^T\lambda - c).$$

For any $\lambda \in \mathcal{F}_D$ and $x \in R^n_+$, it holds that $b^T\lambda \le \mathcal{L}(\lambda, x)$. Therefore, for any $x \in R^n_+$, it holds that

$$d_1^* = \sup_{\lambda\in\mathcal{F}_D} b^T\lambda \le \sup_{\lambda\in\mathcal{F}_D} \mathcal{L}(\lambda, x) \le \sup_{\lambda\in R^m} \mathcal{L}(\lambda, x),$$

i.e., $\sup_{\lambda\in R^m} \mathcal{L}(\lambda, x)$ is an upper bound of the dual optimal function value d_1^*. It can be shown that the best upper bound of d_1^* that can be so obtained is

$$\inf_{x\in R^n_+} \sup_{\lambda\in R^m} \mathcal{L}(\lambda, x) = \inf_{x\in R^n_+} \{c^T x : s.t.\ Ax = b\},$$

which is exactly the primal LP in standard form. Thus, the dual problem of $\max\{b^T\lambda : s.t.\ A^T\lambda \le c\}$ is

$$\min_x \{c^T x : s.t.\ Ax = b, x \ge 0\}.$$

For other forms of LP, the derivations are similar. ■

Remark 4.2.1 *If the objective is to minimize, then construct lower bound of the optimal value, while if the objective is to maximize, then construct upper bound. This is the key in deriving the dual problems. Lagrange function which incorporates the constraints into the objective function plays the central role.*

4.3 Explanations of dual LP

4.3.1 Diet problem

The diet problem takes the form

$$\begin{array}{rc}
\min & c_1x_1 + c_2x_2 + \ldots + c_nx_n \\
s.t. & a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \ge b_1 \\
 & a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \ge b_2 \\
 & \ldots \\
 & a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \ge b_m \\
 & x_1, x_2, \ldots, x_n \ge 0.
\end{array}$$

Here c_j represents the unit price of food j, $j = 1, 2, \ldots, n$; each b_i represents the minimum requirement of nutrition i, $i = 1, 2, \ldots, m$; a_{ij} represents the quantity of nutrition i that can be provided by a unit of food j, $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$; Each row of A corresponds to a certain nutrition, while each column of A corresponds to a certain food.

Suppose each nutrition is made into pill by a drug company. The unit price of the ith nutrition is λ_i, $i = 1, \ldots, m$. How to determine λ so that the m types of pills are competitive with the n types of real foods and meanwhile the revenue is maximized?

The dual of the diet problem:

$$\begin{array}{rc} \max & b_1\lambda_1 + b_2\lambda_2 + \ldots + b_m\lambda_m \\ s.t. & a_{11}\lambda_1 + a_{21}\lambda_2 + \ldots + a_{m1}\lambda_m \leq c_1 \\ & a_{12}\lambda_1 + a_{22}\lambda_2 + \ldots + a_{m2}\lambda_m \leq c_1 \\ & \ldots \\ & a_{1n}\lambda_1 + a_{2n}\lambda_2 + \ldots + a_{mn}\lambda_m \leq c_n \\ & \lambda_1, \lambda_2, \ldots, \lambda_m \geq 0. \end{array}$$

Note that each row of A^T corresponds to a certain food, while each column of A^T corresponds to a certain nutrition.

4.3.2 Transportation problem

The transportation problem takes the form

$$\begin{array}{rc} \min & \sum_{i=1}^m \sum_{j=1}^n c_{ij}x_{ij} \\ s.t. & \sum_{j=1}^n x_{ij} = a_i,\ i = 1, 2, \ldots, m; \\ & \sum_{i=1}^m x_{ij} = b_j,\ j = 1, 2, \ldots, n; \\ & x_{ij} \geq 0,\ i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n. \end{array}$$

Here each c_{ij} represents the unit transportation cost from ith origin to jth destination, $i = 1, 2, \ldots, m$, $j = 1, \ldots, n$; each a_i represents the total amount of product needs to be shipped out at the ith origin, $i = 1, 2, \ldots, m$; each b_j represents the total amount of product needs to be shipped in at the jth destination, $j = 1, 2, \ldots, n$.

Suppose that an entrepreneur believes he can do better and plans to buy the product at all origins and sell it to all destinations. The price he is willing to buy the product at the ith origin is u_i, $i = 1, 2, \ldots, m$; The price he is willing to sell the product at the jth origin is v_j, $j = 1, 2, \ldots, n$; How to determine the prices u and v so that his offer is competitive and meanwhile the revenue is maximized?

The dual of the transportation problem:

$$\begin{array}{rc} \max & \sum_{j=1}^n b_jv_j - \sum_{i=1}^m a_iu_i \\ s.t. & v_j - u_i \leq c_{ij} \\ & i = 1, 2, \ldots, m; \\ & j = 1, 2, \ldots, n. \end{array}$$

4.3.3 World Cup auction problem

Consider the World Cup auction problem. The data is give as below.

Order	#1	#2	#3	#4	#5	...
Argentina	1	0	1	1	0	...
Brazil	1	0	0	1	1	...
Italy	1	0	1	1	0	...
Germany	0	1	0	1	1	...
France	0	0	1	0	0	...
Bidding Price p	0.75	0.35	0.4	0.95	0.75	...
Quantity limit q	10	5	10	10	5	...
Order fill x	x_1	x_2	x_3	x_4	x_5	...

The LP model:

$$\max_{x,z}\{p^T x - z:\ s.t.\ Ax \le z\mathbf{1},\ 0 \le x \le q\}.$$

The dual problem is

$$\min_{y,\lambda}\{q^T y:\ s.t.\ A^T\lambda + y \ge p,\ \mathbf{1}^T\lambda = 1, \lambda \ge 0, y \ge 0\}.$$

The dual variable λ can be interpreted as the price of the teams, i.e., λ_i is the price of the ith team. The following relations should be understandable:

1. $x_j > 0$ implies that $a_j^T\lambda \le p_j$
2. $0 < x_j < q_j$ implies that $a_j^T\lambda = p_j$
3. $a_j^T\lambda < p_j$ implies that $x_j = q_j$
4. $a_j^T\lambda > p_j$ implies that $x_j = 0$.

An optimal solution to the World Cup auction problem with the five players is given in the following table:

Order	#1	#2	#3	#4	#5	price λ
Argentina	1	0	1	1	0	0.2
Brazil	1	0	0	1	1	0.35
Italy	1	0	1	1	0	0.2
Germany	0	1	0	1	1	0.25
France	0	0	1	0	0	0
Bidding Price p	0.75	0.35	0.4	0.95	0.75	–
Quantity limit q	10	5	10	10	5	–
Order fill x	5	5	5	0	5	–

References

[Luenberger-Ye] David G. Luenberger and Yinyu Ye Linear and nonlinear programming.

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Handout 5: Linear programming duality theory – Strong duality

Instructor: Junfeng Yang *September 28, 2014*

Contents

5.1 Preliminaries

Definition 5.1 (线性类) *A set L in R^n is said to be a* linear variety*, if given any $x, y \in L$, we have $\lambda x + (1-\lambda)y \in L$ for all $\lambda \in R$.*

Let L be a linear variety in R^n. Clearly, for any $x_0 \in L$, the set $L - x_0 := \{x - x_0 : x \in L\}$ is a subspace of R^n. The dimension of L is then defined as that of $L - x_0$.

Definition 5.2 (超平面) *A hyperplane in R^n is an $(n-1)$-dimensional linear variety.*

A hyperplane is a largest linear variety in a space, other than the entire space itself.

Theorem 5.3 *Let $0 \neq a \in R^n$ and $c \in R$. The set $H := \{x \in R^n : a^T x = c\}$ is a hyperplane in R^n. On the other hand, any hyperplane in R^n can be represented in this form.*

Proof: Let H be a hyperplane. Take $x_1 \in H$ and define $M := H - x_1$. Then, M is a subspace. Since H is a hyperplane, the dimension of M is $n-1$. Thus, $M^\perp$ is one-dimensional subspace. Take $0 \neq a \in M^\perp$ and let $c = a^T x_1$. It can be verified that $H = \{x \in R^n : a^T x = c\}$. ■

Definition 5.4 *Let $0 \neq a \in R^n$ and $c \in R$. Corresponding to the hyperplane $H := \{x \in R^n : a^T x = c\}$ are the positive and negative closed half spaces*

$$H_+ = \{x \in R^n : a^T x \geq c\}, \quad H_- = \{x \in R^n : a^T x \leq c\}$$

and the positive and negative open half spaces

$$H_+^\circ = \{x \in R^n : a^T x > c\}, \quad H_-^\circ = \{x \in R^n : a^T x < c\}.$$

Definition 5.5 (凸多胞形) *A set which can be expressed as the intersection of a finite number of closed half spaces is called a convex polytope.*

Corollary 5.6 *The feasible set $\mathcal{F} = \{x \in R^n : Ax = b, x \geq 0\}$ of standard form LP is a convex polytope since $\mathcal{F}$ can be reduced to*

$$\{x = (x_B, x_D) \in R^m \times R^{n-m} : x_B = B^{-1}b - B^{-1}Dx_D \geq 0, x_D \geq 0\},$$

where $A = [B, D]$ and B is a basis matrix.

Definition 5.7 (凸多面体) *A nonempty bounded polytope is called a polyhedron.*

Theorem 5.8 (分离超平面定理) *Let C be a convex set in R^n and let y be a point exterior to the closure of C. Then there is a nonzero vector $a \in R^n$ such that*

$$a^T y < \inf_{x \in C} a^T x.$$

Proof: Define $\delta = \inf_{x \in C} \|x - y\| > 0$. There is an x_0 on the boundary of C such that $\|x_0 - y\| = \delta$. This follows because the continuous function $f(x) = \|x - y\|$ achieves its minimum over any closed and bounded set and it is clearly only necessary to consider x in the intersection of the closure of C and the sphere of radius 2δ centered at y. We shall show that $a = x_0 - y \neq 0$ satisfies the conclusion of the theorem.

Let $x \in C$. For any $\alpha \in [0, 1]$, the point $x_0 + \alpha(x - x_0) \in \text{cl}(C)$ (the closure of C) and thus

$$\|x_0 + \alpha(x - x_0) - y\|^2 \geq \|x_0 - y\|^2.$$

By expanding the left hand side and letting $\alpha \to 0+$, we obtain $(x_0 - y)^T(x - x_0) \geq 0$, which implies

$$(x_0 - y)^T x \geq (x_0 - y)^T x_0 = (x_0 - y)^T y + (x_0 - y)^T (x_0 - y) = (x_0 - y)^T y + \delta^2.$$

Thus, $\inf_{x \in C}(x_0 - y)^T x \geq (x_0 - y)^T y + \delta^2 > (x_0 - y)^T y$. Setting $a = x_0 - y$ completes the proof.

Corollary 5.9 *The hyperplane $H_1 := \{x \in R^n : a^T x = a^T y\}$ contains y and contains C in its open half space $(H_1)^\circ_+$.*

Corollary 5.10 *The hyperplane*

$$H_2 := \left\{ x \in R^n : a^T x = a^T \frac{x_0 + y}{2} \right\}$$

separates y and C because

$$a^T y < a^T \frac{x_0 + y}{2} < a^T x$$

for any $x \in C$. Clearly, separating hyperplanes are not unique.

Theorem 5.11 (Projection onto closed convex set) *Let C be a closed convex set in R^n. For any $y \in R^n$, there is a* **unique** *point $x_0 \in C$ such that $\|y - x_0\| = \inf_{x \in C} \|y - x\|$. The point x_0, denoted by $Proj_C(y)$, is called the projection of y onto C and satisfies*

$$(Proj_C(y) - y)^T (x - Proj_C(y)) \geq 0, \ \ \forall\, x \in C.$$

Proof: From the proof of the separating hyperplane theorem, there is a $x_0 \in C$ such that $\|y - x_0\| = \inf_{x \in C} \|y - x\|$ and, for any $x \in C$, it hods that $(x_0 - y)^T(x - x_0) \geq 0$. If there is $x_0' \in C$ different from x_0 and servers the same role as x_0, then $(x_0 - y)^T(x_0' - x_0) \geq 0$ and $(x_0' - y)^T(x_0 - x_0') \geq 0$ lead to a contradiction. ■

Theorem 5.12 (支撑超平面定理) *Let C be a convex set in R^n and let y be a boundary point of C. Then there is a nonzero vector $a \in R^n$ such that*

$$a^T y \leq \inf_{x \in C} a^T x,$$

i.e., the hyperplane $H = \{x \in R^n : a^T x = a^T y\}$ contains y and contains C in its closed half space H_+. H is thus called the supporting hyperplane of C at y.

Proof: Let $\{y_k\}$ be a sequence of vectors, exterior to the closure of C, converging to y. Let $\{a_k\}$ be the sequence of corresponding vectors constructed in the separating hyperplane theorem, normalized so that $\|a_k\| = 1$, such that

$$a_k^T y_k < \inf_{x \in C} a_k^T x.$$

Taking subsequence if necessary, we assume that $\{a_k\}$ converges to a. Thus, for any $x \in C$, it holds that

$$a^T y = \lim_{k \to \infty} a_k^T y_k \leq \lim_{k \to \infty} a_k^T x = a^T x.$$

The theorem follows by taking infimum on the right hand side with respect to x in C. ■

Definition 5.13 (Affine hull, 仿射包) *Let S be a subset of R^n. The affine hull of S, denoted by* $\mathrm{aff}(S)$*, contains all the affine combinations of points in S, i.e.,*

$$\mathrm{aff}(S) := \left\{ x = \sum_{i=1}^{k} \alpha_i x_i \in R^n \;\middle|\; \begin{array}{c} x_i \in S, \alpha_i \in R, i = 1, 2, \ldots, k; \\ \sum_{i=1}^{k} \alpha_i = 1; \\ k \text{ is any positive integer.} \end{array} \right\}$$

Equivalently, $\mathrm{aff}(S)$ *is the smallest affine set containing S.*

Definition 5.14 (Relative interior, 相对内部) *Let $B := \{x \in R^n : \|x\| \leq 1\}$ be the unit ball in R^n. The relative interior of a convex set C in R^n is defined as*

$$ri(C) := \{x \in \mathrm{aff}(C) : \exists \varepsilon > 0, (x + \varepsilon B) \cap \mathrm{aff}(C) \subset C\}.$$

Definition 5.15 (Cone, 锥) *A set C is called a cone if $x \in C$ implies $\alpha x \in C$ for all $\alpha \geq 0$. A cone that is also convex is called a convex cone.*

Definition 5.16 (Generated cone, 生成锥) *Let $\{x_1, x_2, \ldots, x_m\}$ be a set of m points in R^n. Then*

$$C := \left\{ x = \sum_{i=1}^{m} \alpha_i x_i \in R^n : \alpha_i \geq 0, i = 1, 2, \ldots, m \right\}$$

is a **closed convex cone***, which is called the cone generated by the set of points $\{x_1, x_2, \ldots, x_m\}$.*

Theorem 5.17 (分离超平面定理) *Let C_1 and C_2 be convex sets in R^n with no common relative interior points, i.e., the only common points of C_1 and C_2, if any, are boundary points. Then there is a hyperplane separating C_1 and C_2. In particular, there exist a nonzero vector $a \in R^n$ such that*

$$\sup_{x_1 \in C_1} a^T x_1 \leq \inf_{x_2 \in C_2} a^T x_2.$$

Proof: Consider the set $G = C_2 - C_1$. It is easy to see that G is convex and that 0 is not a relative interior point of G. Hence, the separating hyperplane theorem or the supporting hyperplane theorem applies and gives the appropriate hyperplane. ■

5.2 Farkas lemma

Lemma 5.18 (Farkas Lemma - 1st form) *Let $A \in R^{m \times n}$ and $b \in R^m$. Define*

$$\mathcal{X} := \{x \in R^n : Ax = b, x \geq 0\} \quad \textit{and} \quad \mathcal{Y} := \{y \in R^m : A^T y \leq 0, b^T y > 0\}.$$

Then, one and only one of the sets $\mathcal{X}$ and $\mathcal{Y}$ is nonempty (empty).

Proof: **"$\mathcal{X} \neq \emptyset \Rightarrow \mathcal{Y} = \emptyset$"**: Assume $\mathcal{Y} \neq \emptyset$. Take $x \in \mathcal{X}$ and $y \in \mathcal{Y}$. Then, $0 \geq (A^T y)^T x = y^T A x = y^T b > 0$ gives a contradiction.

"$\mathcal{X} = \emptyset \Rightarrow \mathcal{Y} \neq \emptyset$": Define $S := \{Ax \in R^m : x \geq 0\}$, which is a nonempty polyhedral set and hence it is **closed** and convex. $\mathcal{X} = \emptyset$ implies that $b \notin S$. According to separating hyperplane theorem, there is a $y \in R^m$ such that

$$b^T y < \inf_{z \in S} y^T z = \inf_{x \geq 0} y^T A x.$$

It follows from $\inf_{x\geq 0} y^T Ax \leq 0$ that $b^T y < 0$. Furthermore, $A^T y \geq 0$ must hold because otherwise

$$b^T y < \inf_{x\geq 0} y^T Ax = -\infty,$$

which is impossible because $b^T y$ is a fixed constant. Thus, $\mathcal{Y} \neq \emptyset$ since $-y \in \mathcal{Y}$. ■

Remark 5.2.1 *$\mathcal{Y} \neq \emptyset \implies \mathcal{X} = \emptyset$. Thus, a point $y \in \mathcal{Y}$ is called an infeasibility certificate for Primal-LP1. Furthermore, $\mathcal{Y} \neq \emptyset$ implies that Dual-LP1 is either infeasible ($d_1^* = -\infty$) or feasible but unbounded ($d_1^* = +\infty$).*

Lemma 5.19 (Farkas Lemma - 2nd form) *Let $A \in R^{m\times n}$ and $c \in R^n$. Define*

$$\mathcal{X} := \{x \in R^n : Ax = 0, x \geq 0, c^T x < 0\} \quad \textit{and} \quad \mathcal{Y} := \{y \in R^m : A^T y \leq c\}.$$

Then, one and only one of the sets $\mathcal{X}$ and $\mathcal{Y}$ is nonempty (empty).

Proof: Proof the 2nd form of Farkas Lemma is left to yourself. ■

Remark 5.2.2 *$\mathcal{X} \neq \emptyset \implies \mathcal{Y} = \emptyset$. Thus, a point $x \in \mathcal{X}$ is called an infeasibility certificate for Dual-LP1. Furthermore, $\mathcal{X} \neq \emptyset$ implies that Primal-LP1 is either infeasible ($p_1^* = +\infty$) or feasible but unbounded ($p_1^* = -\infty$).*

The two forms of Farkas Lemma discussed above correspond to Primal-LP1 and Dual-LP1. Farkas Lemma has several other forms corresponds to nonstandard form LPs and their dual problems. The implications of these variants and their corresponding proofs are similar to what we have presented here.

5.3 Strong duality

Theorem 5.20 (Strong Duality for the pair (Primal-LP1, Dual-LP1)) *If either of the problems Primal-LP1 or Dual-LP1 has a finite optimal solution, so does the other, and the corresponding values of the objective functions are equal.*

Proof: Because either problem can be converted to standard form and the roles of primal and dual are reversible (dual to each other), it is sufficient to assume that Primal-LP1 has a finite optimal solution and show that Dual-LP1 has a solution with the same value.

Assume Primal-LP1 has a finite optimal value p_1^*. Define

$$C = \{(r, w) \in R^{m+1} : r = tp_1^* - c^T x, w = tb - Ax, x \geq 0, t \geq 0\}.$$

It can be shown that C is a closed convex cone. First, we claim that $(1, 0) \notin C$ (note that here $(1, 0) = (1, \mathbf{0})$). Otherwise, there exist $x_0 \geq 0$, $t_0 \geq 0$ such that $1 = t_0 p_1^* - c^T x_0$ and $0 = t_0 b - Ax_0$. If $t_0 > 0$, then x_0/t_0 is feasible for Primal-LP1. Then, $1/t_0 = p_1^* - c^T(x_0/t_0) \leq 0$ is a contradiction. If $t_0 = 0$, then $c^T x_0 = -1$, $Ax_0 = 0$ and $x_0 \geq 0$. Let x be any feasible solution of Primal-LP1. Then, for any $\alpha \geq 0$, $x + \alpha x_0$ is also a feasible solution of Primal-LP1. The corresponding function value is

$$c^T(x + \alpha x_0) = c^T x + \alpha c^T x_0,$$

which goes to $-\infty$ as $\alpha \to +\infty$. This contradicts to the fact that p_1^* is finite. In summary, $(1, 0) \notin C$.

Since C is a closed convex set and $(1, 0) \notin C$, by the separating hyperplane theorem, there is a nonzero vector $(s, \lambda) \in R \times R^m$ and a constant **const** such that

$$s = (s, \lambda)^T(1, 0) < \textbf{const} = \inf\{sr + \lambda^T w : (r, w) \in C\}.$$

Since C is a cone, $sr + \lambda^T w \geq 0$ must hold for any $(r, w) \in C$. Otherwise, if there were $(r, w) \in C$ such that $sr + \lambda^T w < 0$, then $\alpha(r, w) \in C$ for any $\alpha \geq 0$. It thus follows that

$$s < \inf\{\alpha(sr + \lambda^T w) : (r, w) \in C\} = -\infty,$$

which is clearly a contradiction since s is a fixed constant. Therefore, $\mathbf{const} \geq 0$. On the other hand, $\mathbf{const} \leq 0$ since $(0, 0) \in C$. As a result, $\mathbf{const} = 0$ and $s < 0$. Without loss of generality, we assume that $s = -1$.

From the above, there exists $\lambda \in R^m$ such that $-r + \lambda^T w \geq 0$ for all $(r, w) \in C$. Equivalently, using the definition of C,

$$(c^T - \lambda^T A)x - tp_1^* + t\lambda^T b \geq 0$$

for all $x \geq 0$ and $t \geq 0$. Setting $t = 0$ yields $A^T\lambda \leq c$, which implies that λ is feasible for Dual-LP1. Setting $x = 0$ and $t = 1$ yields $p_1^* \leq b^T\lambda$, which in view of the weak duality and its corollary shows that λ is optimal for Dual-LP1. Therefore, $p_1^* = d_1^*$ and no duality gap. ■

Remark 5.3.1 *Alternatively, after proving $(1, \mathbf{0}) \notin C$, we can use* **Farkas Lemma** *(2nd form). In fact, $(1, \mathbf{0}) \notin C$ implies that the system*

$$(A, -b)\begin{bmatrix} x \\ t \end{bmatrix} = 0, \quad \begin{bmatrix} x \\ t \end{bmatrix} \geq 0, \quad \begin{bmatrix} c \\ -p_1^* \end{bmatrix}^T \begin{bmatrix} x \\ t \end{bmatrix} = -1$$

has no feasible solution. According to **Farkas Lemma** *(2nd form), there exists $\lambda \in R^m$ such that*

$$(A, -b)^T\lambda = \begin{bmatrix} A^T \\ -b^T \end{bmatrix}\lambda \leq \begin{bmatrix} c \\ -p_1^* \end{bmatrix},$$

or equivalently, $A^T\lambda \leq c$ and $b^T\lambda \geq p_1^*$. $A^T\lambda \leq c$ implies that λ is feasible for Dual-LP1, while $b^T\lambda \geq p_1^*$, together with weak duality, implies that $d_1^* = p_1^*$.

5.4 The optimality system

The following result is a corollary of strong duality.

Theorem 5.21 *Consider the pair of linear programs Primal-LP1 and Dual-LP1. Assume that both are feasible. Then, $x \in R^n$ and $(\lambda, s) \in R^m \times R^n$ are, respectively optimal for Primal-LP1 and Dual-LP1 if and only if the following conditions are all satisfied:*

$$\begin{aligned} \textbf{Primal feasibility}: &\quad Ax = b, &\quad x \geq 0 \\ \textbf{Dual feasibility}: &\quad A^T\lambda + s = c, &\quad s \geq 0 \\ \textbf{Complementarity}: &\quad c^T x - b^T\lambda = 0 &\quad (or\ \ s^T x = 0). \end{aligned}$$

5.5 Summary

The primal and dual LPs can be infeasible simultaneously. A trivial example: $\min_x\{x : \ s.t.\ 0 \cdot x \geq 1\}$, whose dual problem is $\max_\mu\{\mu : \ s.t.\ 0 \cdot \mu = 1, \mu \geq 0\}$.

$$\begin{array}{rl} \min & -4x_1 + 2x_2 \\ \text{s.t.} & -x_1 + x_2 \geq 2 \\ & x_1 - x_2 \geq 1 \\ & x_1, x_2 \geq 0 \end{array}$$

(a) Primal LP

$$\begin{array}{rl} \max & 2\lambda_1 + \lambda_2 \\ \text{s.t.} & -\lambda_1 + \lambda_2 \leq -4 \\ & \lambda_1 - \lambda_2 \leq 2 \\ & \lambda_1, \lambda_2 \geq 0 \end{array}$$

(b) Dual LP

Figure 5.1: It can be verified that the primal and the dual LPs are infeasible simultaneously.

Table 5.1: All possible cases for Primal-LP1 and Dual-LP1. "unbounded" means feasible and unbounded below for minimization and feasible and unbounded above for maximization.

Dual-LP1	Primal-LP1	$d_1^* \leq p_1^*$
feasible	feasible	$d_1^* = p_1^*$
infeasible	infeasible	$-\infty = d_1^* < p_1^* = +\infty$
unbounded	infeasible	$d_1^* = p_1^* = +\infty$
infeasible	unbounded	$-\infty = d_1^* = p_1^*$

Table 5.2: Primal-LP1 and Dual-LP1: the rest 5 cases cannot happen!

	p_1^* finite	$p_1^* = -\infty$	$p_1^* = +\infty$
d_1^* finite	$\surd$	$\times$	$\times$
$d_1^* = +\infty$	$\times$	$\times$	$\surd$
$d_1^* = -\infty$	$\times$	$\surd$	$\surd$

"$\surd$" means possible, "$\times$" means impossible.

5.6 Relations to the simplex procedure

Theorem 5.22 (Strong duality) *Suppose Primal-LP1 has an optimal basic feasible solution corresponding to the basis B. Then the vector λ satisfying $\lambda^T = c_B^T B^{-1}$ is an optimal solution to Dual-LP1. The optimal values of both problems are equal.*

Proof: Let $A = [B, D]$. Since x is optimal, we have

$$r_D^T = c_D^T - c_B^T B^{-1} D \geq 0.$$

It is easy to verify that λ is dual feasible and $\lambda^T b = c^T x$. Therefore, λ is dual optimal. ■

Where to find the dual optimal solution λ in the final simplex tableau? Suppose we have found an initial basic feasible solution (by phase I of two-phase method, or by introducing slack variables; In all, there is an identity matrix in the reduced form of $[A, b]$). The first and the last simplex tableaus are

$$\begin{bmatrix} (\cdots) & I & (\vdots) \\ (\cdots) & c_I^T & 0 \end{bmatrix} \text{ and } \begin{bmatrix} B^{-1}(\cdots) & B^{-1} & B^{-1}(\vdots) \\ (\cdots) - c_B^T B^{-1}(\cdots) & c_I^T - c_B^T B^{-1} & -c_B^T B^{-1}(\vdots) \end{bmatrix},$$

respectively, where B is the optimal basis. $c_B^T B^{-1}$ is in the final tableau!

$$\begin{array}{rl} \min & -x_1 - 4x_2 - 3x_3 \\ \text{s.t.} & 2x_1 + 2x_2 + x_3 + x_4 = 4 \\ & x_1 + 2x_2 + 2x_3 + x_5 = 6 \\ & x_1, x_2, x_3, x_4, x_5 \geq 0. \end{array}$$

(a) Primal-LP1

$$\begin{array}{rl} \max & 4\lambda_1 + 6\lambda_2 \\ \text{s.t.} & 2\lambda_1 + \lambda_2 \leq -1 \\ & 2\lambda_1 + 2\lambda_2 \leq -4 \\ & \lambda_1 + 2\lambda_2 \leq -3 \\ & \lambda_1, \lambda_2 \leq 0. \end{array}$$

(b) Dual-LP1

Figure 5.2: A pair of primal and dual LPs.

Example 5.6.1 *Consider the pair of primal and dual LPs in Figure 5.2. The first and last simplex tableaus are, respectively,*

$$\begin{array}{cccccc} 2 & 2 & 1 & 1 & 0 & 4 \\ 1 & 2 & 2 & 0 & 1 & 6 \\ -1 & -4 & -3 & 0 & 0 & 0 \end{array} \quad \textit{and} \quad \begin{array}{cccccc} 3/2 & 1 & 0 & 1 & -1/2 & 1 \\ -1 & 0 & 1 & -1 & 1 & 2 \\ 2 & 0 & 0 & 1 & 1 & 10 \end{array}$$

The dual optimal solution is $\lambda^T = (-1, -1)$!

5.7 Explanations of dual variables

5.7.1 Simplex multipliers

In the iteration of the simplex method, the vector $\lambda^T = c_B^T B^{-1}$ changes with B from iteration to iteration, and it is not optimal for the dual problem until an optimal basis B is found for the primal problem. Since λ is used to compute the relative cost vector by

$$r_D^T = c_D^T - \lambda^T D,$$

λ is often called the vector of *simplex multipliers*. Take the diet problem (with equality constraints) for example. Note that each column vector of A corresponds to a certain food. Suppose B is the current basis (consisting certain m foods). All other foods can be synthetically constructed from these m foods. In particular, one unit of the ith nutrient (corresponds to the ith column vector of the identity matrix of order m, denote it by e_i) can be synthetically constructed as

$$e_i = By,$$

where $y = B^{-1}e_i$. The synthetical price of one unit of the ith nutrient is

$$c_B^T y = c_B^T B^{-1} e_i = \lambda^T e_i = \lambda_i.$$

Thus, at each iteration, λ_i represents the *synthetic price* of the ith nutrient under the current basis, and $\lambda^T a_j$ is the *synthetic price* of the jth food. When optimality for primal problem is attained, synthetic price of any food a_j must be no more expensive than the market price c_j, i.e., $\lambda^T A \leq c^T$ (dual feasibility). Simplex method (for primal problem) maintains primal feasibility, 0-duality gap, because

$$c^T x - \lambda^T b = c_B^T x_B - c_B^T B^{-1} b = c_B^T x_B - c_B^T x_B = 0,$$

and keeps improving the primal objective function value until it cannot be improved further, that is, when dual feasibility is met by λ.

5.7.2 Marginal price

Consider $p^*(b) := \min\{c^T x : s.t.\ Ax = b, x \geq 0\}$, where the optimal value is viewed as a function of b. Suppose the optimal basis is B. Primal and dual optimal solutions are $x = (x_B, 0) = (B^{-1}b, 0)$ and $\lambda^T = c_B^T B^{-1}$, respectively. Assuming nondegeneracy (which implies $B^{-1}b > 0$), then small changes in b will not cause the optimal basis to change. This is because

$$B^{-1}b > 0 \quad \text{and} \quad r_D = c_D^T - c_B^T B^{-1} D \geq 0$$

remain unaffected if the change in b is small. Suppose Δb is small enough such that $B(b + \Delta b) > 0$. Then,

$$(x_B + \Delta x_B, 0) = (B^{-1}b + B^{-1}\Delta b, 0)$$

is an optimal solution corresponding to $b + \Delta b$. The increment in the cost function is

$$\Delta p^* = p^*(b + \Delta b) - p^*(b) = c_B^T \Delta x_B = \lambda^T \Delta b,$$

which implies that

$$\lambda = \frac{\partial p^*}{\partial b} = \nabla p^*(b).$$

Therefore, λ gives the sensitivity of the optimal cost with respect to small changes in b. Thus, λ_i is also known as the *marginal price* of the component b_i, since if b_i is changed to $b_i + \Delta b_i$ the optimal value changes by $\lambda_i \Delta b_i$.

5.7.3 Shadow price

Consider the manufacturing problemand its dual given in Figure 5.3. Optimal solutions for primal and dual are, respectively,

$$x = (4, 2)^T \ \text{ and } \ \lambda = (3/2, 1/8, 0)^T.$$

Note that $p^* = b^T \lambda$ and $\partial p^* / \partial b = \lambda$. Therefore, λ represents the sensitivity of p^* with respect to b, given all other conditions (A and c).

From Figure 5.4, it can be seen that

$$\begin{aligned} p^* := \max \quad & 2x_1 + 3x_2 \\ \text{s.t.} \quad & x_1 + 2x_2 \leq 8 \\ & 4x_1 \leq 16 \\ & 4x_2 \leq 12 \\ & x_1, x_2 \geq 0. \end{aligned}$$

(a) Primal-LP1

$$\begin{aligned} d^* := \min \quad & 8\lambda_1 + 16\lambda_2 + 12\lambda_3 \\ \text{s.t.} \quad & \lambda_1 + 4\lambda_2 \geq 2 \\ & 2\lambda_1 + 4\lambda_3 \geq 3 \\ & \lambda_1, \lambda_2, \lambda_3 \geq 0. \end{aligned}$$

(b) Dual-LP1

Figure 5.3: Primal and dual of the manufacturing problem.

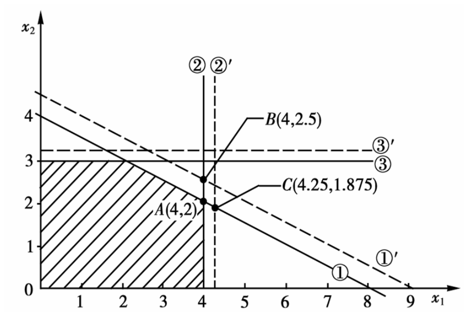

Figure 5.4: Illustration of the manufacturing problem.

1. If available machine hours are increased by 1 (replace "$x_1 + 2x_2 \leq 8$" by "$x_1 + 2x_2 \leq 9$"), then the neat profit will increase by $\lambda_1 = 3/2$.
2. If resource I is increased by 1 (replace "$4x_1 \leq 16$" by "$4x_1 \leq 17$"), then the neat profit will increase by $\lambda_2 = 1/8$.
3. If resource II is increased by 1 (replace "$4x_2 \leq 12$" by "$4x_2 \leq 13$"), then the neat profit remains unchanged ($\lambda_3 = 0$).

If sell out all resource, the selling price should be prime cost plus shadow price. If the market price of b_i is less than its shadow price, then purchase b_i and expand production; If the otherwise, then sell b_i and shrink production; λ is called the *shadow price* of resource b (closely related to current production plan, varies from situation to situation).

Theorem 5.23 *Consider the following pair of LPs:*

$$\min\{c^T x : s.t.\ Ax \geq b, x \geq 0\} \ \ \textit{and} \ \ \max\{b^T \lambda : s.t.\ A^T \lambda \leq c, \lambda \geq 0\}.$$

Let x and λ be, respectively, primal and dual feasible. Denote the jth row of A by a^j. A necessary and sufficient condition that they both be optimal is that for all i and j

- $x_i > 0 \Rightarrow \lambda^T a_i = c_i$
- $x_i = 0 \Leftarrow \lambda^T a_i < c_i$
- $\lambda_j > 0 \Rightarrow a^j x = b_j$
- $\lambda_j = 0 \Leftarrow a^j x > b_j$.

Proof: Left to yourself. ■

Think about the primal LP as the diet problem and give explanations of these four relations.

5.8 Dual simplex method

Dual simplex method maintains dual feasibility, 0-duality gap and keeps improving the dual objective function value until it cannot be improved further (i.e., when primal feasibility is satisfied):

1. starting from an initial dual feasible basic solution, i.e., $x = (B^{-1}b, 0)$, where $B^{-1}b \geq 0$ may not be true but $\lambda^T A = c_B^T B^{-1} A \leq c^T$.
2. at each iteration first determine one variable to leave basis (so that dual objective can be increased)
3. then choose one variable appropriately to enter basis (so that dual feasibility can be maintained)
4. do pivoting and repeat until optimal (i.e., primal feasibility is satisfied).

Given the current dual feasible basis B, i.e., the corresponding dual variable $\lambda^T = c_B^T B^{-1}$ is feasible for the dual problem (i.e., $\lambda^T A \leq c^T$). If $x_B = B^{-1}b \geq 0$, then B is also primal feasible and thus optimal. By assuming nondegeneracy, we see that if B is not primal feasible, then there must be i such that $(x_B)_i < 0$. This i determines which variable is going to leave basis. For simplicity, we assume the current basis contains the first m columns of A. It is easy to verify that

$$\lambda^T a_j \begin{cases} = c_j, & j = 1, 2, \ldots, m; \\ < c_j, & j = m+1, \ldots, n. \end{cases}$$

To develop one cycle of the dual simplex method, we find a new vector $\bar{\lambda}$ such that one of the equalities becomes an inequality and one of the inequality becomes equality, while at the same time increasing the value of the dual objective function. The m equalities in the new solution then determine a new basis. Denote the ith row of B^{-1} by u^i. Then for

$$\bar{\lambda}^T = \lambda^T - \varepsilon u^i,$$

we have

$$\bar{\lambda}^T a_j = \lambda^T a_j - \varepsilon u^i a_j = \lambda^T a_j - \varepsilon y_{ij}.$$

Here y_{ij} is the ijth element of the tableau. Therefore,

$$\begin{cases} \bar{\lambda}^T a_j = c_j, & j \in \{1, \ldots, m\} \setminus \{i\}; \\ \bar{\lambda}^T a_j = c_i - \varepsilon, & j = i; \\ \bar{\lambda}^T a_j = \lambda^T a_j - \varepsilon y_{ij}, & j \in \{m+1, \ldots, n\}. \end{cases}$$

The dual objective value at $\bar{\lambda}$:

$$\bar{\lambda}^T b = \lambda^T b - \varepsilon u^i b = \lambda^T b - \varepsilon (x_B)_i.$$

The above arguments lead to the dual simplex method:

1. Given a dual basic feasible solution x_B. If $x_B \geq 0$, then already optimal. Otherwise, find i such that $(x_B)_i < 0$.
2. If $y_{ij} \geq 0$ for all $j = 1, 2, \ldots, n$, then the dual problem is unbounded above.
3. If $y_{ij} < 0$ for some j, then let

$$\varepsilon_0 = \frac{(A^T\lambda)_k - c_k}{y_{ik}} = \min_j \Big\{ \frac{(A^T\lambda)_j - c_j}{y_{ij}} \ : \ y_{ij} < 0 \Big\}.$$

4. Compute $\bar{\lambda}^T = \lambda^T - \varepsilon_0 u^i$. Form a new basis B by replacing a_i by a_k. Use this basis to determine the new x_B and repeat.

Example 5.8.1 (Dual simplex method)

$$\begin{aligned} \min \quad & 3x_1 + 4x_2 + 5x_3 \\ s.t. \quad & x_1 + 2x_2 + 3x_3 \geq 5 \\ & 2x_1 + 2x_2 + x_3 \geq 6 \\ & x_1, x_2, x_3 \geq 0. \end{aligned}$$

The initial tableau of dual simplex method:

x_1	x_2	x_3	x_4	x_5	b
-1	-2	-3	1	0	-5
(-2)	-2	-1	0	1	-6
3	4	5	0	0	0

($x = (0, 0, 0, -5, -6)^T, c_B^T B^{-1} A = 0 \leq c$.) First choose x_5 to leave basis because -6 is the most negative one among $\{-5, -6\}$; then choose (-2) as pivot element because $3/(-2)$ is the maximum negative ratio among all negative ratios

$$\{3/(-2), 4/(-2), 5/(-1)\}.$$

The second tableau of dual simplex method:

x_1	x_2	x_3	x_4	x_5	b
0	(-1)	-5/2	1	-1/2	-2
1	1	1/2	0	-1/2	3
0	1	7/2	0	3/2	9

Choose (-1) as pivot element because $-2 < 0$ and

$$1/(-1) = \max\{1/(-1), (7/2)/(-5/2), (3/2)/(-1/2)\}.$$

The final tableau of dual simplex method:

x_1	x_2	x_3	x_4	x_5	b
0	1	5/2	-1	1/2	2
1	0	-2	1	-1	1
0	0	1	1	1	11

Optimal solutions for primal and dual LPs are, respectively, $x = (1, 2, 0)^T$ and $\lambda = (-1, -1)^T$.

The primal simplex method maintains primal feasibility, 0-duality gap and keeps decreasing primal objective function value until dual feasibility. On the contrary, the dual simplex method maintains dual feasibility, 0-duality gap and keeps increasing dual objective function value until primal feasibility.

References

[Luenberger-Ye] David G. Luenberger and Yinyu Ye Linear and nonlinear programming.

MATH: Operations Research **2014-15 First Term**

Handout 6: A Brief Introduction to Conic Linear Programming

Instructor: Junfeng Yang *October 12, 2014*

Contents

6.1 Conic linear programming

Let $\mathcal{X}$ and $\mathcal{Y}$ be, respectively, n and m dimensional real Euclidean spaces. The following problem is called conic linear programming:

$$\begin{aligned} \textbf{CLP}: \quad p^* := \min_{x \in \mathcal{X}} \ & \langle c, x \rangle \\ s.t.\ & \mathcal{A}x = b \\ & x \in K, \end{aligned}$$

where $\mathcal{A} : \mathcal{X} \to \mathcal{Y}$ is a linear mapping, $K \subset \mathcal{X}$ is a cone, $b \in \mathcal{Y}$ and $c \in \mathcal{X}$. Clearly, the objective function is linear, the constraint set is the intersection of an affine set with a cone. All difficulties are hidden in the cone.

Linear programming is a special case of CLP with $\mathcal{X} = R^n$, $\mathcal{Y} = R^m$, $K = R^n_+$ and $\langle x, y \rangle = x^T y$.

6.2 Semidefinite programming

Semidefinite programming is another special case of CLP. Let S^n and S^n_+ be, respectively, the sets of all symmetric and symmetric positive semidefinite matrices of order n, i.e.,

$$\begin{aligned} S^n &:= \{X \in R^{n\times n} : X^T = X\} \\ S^n_+ &:= \{X \in R^{n\times n} : X \succeq 0\}. \end{aligned}$$

The notation "$X \succeq 0$" means that $X \in S^n$ and X is positive semidefinite. Clearly, S^n_+ is a cone. Let S^n be endowed with an inner product

$$\langle X, Y \rangle = \sum_{i,j=1}^{n} X_{ij} Y_{ij} = \text{tr}(X^T Y), \ \ \forall\, X, Y \in S^n.$$

Here "tr" means trace. Given a set of symmetric matrices $\{A_1, \ldots, A_m\} \subset S^n$, the following defines a linear mapping $\mathcal{A}$ from S^n to R^m:

$$\mathcal{A}X := \begin{pmatrix} \langle A_1, X \rangle \\ \vdots \\ \langle A_m, X \rangle \end{pmatrix}, \ \ \forall\, X \in S^n.$$

Let $C \in S^n$ and $b \in R^m$. The following problem is a generalization of LP and is called semidefinite programming:

$$\begin{aligned} \textbf{SDP}: \quad \min_{X \in S^n} \ & \langle C, X \rangle \\ s.t.\ & \mathcal{A}X = b \\ & X \succeq 0. \end{aligned}$$

SDP is a special case of **CLP** with $\mathcal{X} = S^n$, $\mathcal{Y} = R^m$, $K = S^n_+$ and $\langle X, Y \rangle = \text{tr}(X^T Y)$.

Example 6.2.1 (The Max-Cut Problem) *Let $G = (V, E)$ be a graph with $n = |V|$ vertices, $w : E \to R$ be an edge weight function on G, i.e.,*

$$w_{ij} = \begin{cases} w((i,j)), & \textit{if } (i,j) \in E; \\ 0, & \textit{o.w.} \end{cases}$$

Let $W = (w_{ij})_{i,j=1,2,\ldots,n}$ be the matrix whose ij-th entry is w_{ij}. A cut is a partition of V into two sets $S \subset V$ and $V \setminus S$. The size of the cut $(S, V \setminus S)$ is

$$\texttt{size}(S) := \sum_{i \in S,\, j \in V \setminus S} w_{ij}.$$

The max-cut problem aims to find a cut $(S, V \setminus S)$ of a weighted graph (V, E, w) such that $\texttt{size}(S)$ is maximized among all cuts. The max-cut problem is well known to be NP-hard!

Now we give a reformulation of the max-cut problem. For any $S \subset V$, define

$$x = (x_1, \ldots, x_n)^T, \text{ where } x_i = \begin{cases} 1, & \text{if } i \in S; \\ -1, & \text{if } i \notin S. \end{cases}$$

Then the max-cut problem can be formulated as

$$\begin{aligned} \max t &:= \frac{1}{2} \sum_{i<j} w_{ij}(1 - x_i x_j) \\ s.t. &\quad x_i \in \{-1, 1\}, \ \forall\, i \in V. \end{aligned}$$

Let $\mathbf{1} := (1, 1, \ldots, 1)^T \in R^n$ and define $C := \frac{1}{4}(Diag(W\mathbf{1}) - W)$. It can be shown that

$$t = x^T C x.$$

In fact,

$$\begin{aligned} t &= \frac{1}{2} \sum_{i<j} w_{ij}(1 - x_i x_j) = \frac{1}{4}\left(2 \sum_{i<j} w_{ij} x_i^2 - 2 \sum_{i<j} w_{ij} x_i x_j\right) \\ &= \frac{1}{4}\left(\sum_i \sum_j w_{ij} x_i^2 - \sum_i \sum_j w_{ij} x_i x_j\right) \\ &= \frac{1}{4}\left(\sum_i \sum_j w_{ij} x_i^2 - x^T W x\right) \\ &= \frac{1}{4}\left(\sum_i (W\mathbf{1})_i x_i^2 - x^T W x\right) = \frac{1}{4}\left(x^T Diag(W\mathbf{1}) x - x^T W x\right) \\ &= \frac{1}{4} x^T \left(Diag(W\mathbf{1}) - W\right) x = x^T C x \left(= constant - \frac{1}{4} x^T W x\right). \end{aligned}$$

If we define $X = xx^T \in S_+^n$, then $t = x^T C x = tr(x^T C x) = tr(C x x^T) = \langle C, X \rangle$. Also, the constraint $x_i \in \{1, -1\}$ is equivalent to $Diag(X) = \mathbf{1}$.

Lemma 6.1 *$X \in S_+^n$ and $rank(X) = 1 \Longleftrightarrow X = xx^T$ for some nonzero $x \in R^n$.*

According to the above lemma, the max-cup problem can be reexpressed as

$$\begin{aligned} \max &\quad \langle C, X \rangle \\ s.t. &\ Diag(X) = \mathbf{1} \\ &\ rank(X) = 1 \\ &\ X \succeq 0. \end{aligned}$$

By considering the following SDP relaxation problem, we can get an upper bound on the optimal value of the max-cut problem:

$$\begin{aligned} \max \quad & \langle C, X \rangle \\ s.t.\ & Diag(X) = \mathbf{1} \\ & X \succeq 0. \end{aligned}$$

Remark 6.2.1 *SDP problems have a particular structure that makes their solution computationally tractable by interior-point methods. Some selected applications among many:*

- *linear matrix inequalities (LMI) arising from systems and control;*
- *engineering applications such as structural optimization and signal processing;*
- *SDP is now widely used in the relaxations of NP-hard combinatorial optimization problems (max cut, max clique, 0-1 integer LP, etc);*
- *polynomial optimization where, under certain mild assumptions, a sequence of increasing order of SDP relaxations can converge to a globally optimal solution;*
- *robust optimization with uncertain data;*

The widespread applications of SDP have stimulated great demands on robust SDP solvers. Now SDP problems of moderate size (roughly, $n \leq 5000$) can be solved relatively efficiently on PC. SDP solvers: SDPT3, Sedumi, SDPA, ...

6.3 Second order cone programming

Definition 6.2 (Second order cone) *The second-order cone or Lorentz cone of order n is defined by*

$$Q^n := \{(t, x) \in R \times R^n : \|x\|_2 \leq t\}.$$

The following problem often arises in robust optimization:

$$\begin{aligned} \mathbf{SOCP}: \quad \min_{x \in R^n} \quad & c^T x && (6.1) \\ s.t.\ & \|A_i x - b_i\|_2 \leq c_i^T x + d_i, && (6.2) \\ & i = 1, 2, \ldots, m, && (6.3) \end{aligned}$$

where $c \in R^n$, $A_i \in R^{n_i \times n}$, $b_i \in R^{n_i}$, $c_i \in R^n$, $d_i \in R$, $i = 1, 2, \ldots, m$. By introducing auxiliary variables, this problem can be transformed to one with linear objective function, linear equality constraints and second order cone constraints.

In robust linear programming, the following problem is considered:

$$\begin{aligned} \min \quad & c^T x \\ s.t. \quad & a_i^T x \leq b_i, \quad \forall\, a_i \in \mathcal{E}_i := \{\bar{a}_i + P_i u : \|u\|_2 \leq 1\}, \\ & i = 1, \ldots, m, \end{aligned}$$

where $P_i \succeq 0$. The robust linear constraint can be expressed as

$$\max\{a_i^T x : a_i \in \mathcal{E}_i\} = \bar{a}_i^T x + \|P_i x\|_2 \leq b_i,$$

which is evidently a second-order cone constraint. Hence, the robust linear programming can be expressed as the following SOCP

$$\begin{aligned} \min \quad & c^T x \\ s.t.\ & \bar{a}_i^T x + \|P_i x\|_2 \le b_i \\ & i = 1, 2, \ldots, m. \end{aligned}$$

6.4 Connections among LP, SOCP and SDP

Clearly, SOCP reduces to LP by letting $A_i \equiv 0$ and $b_i \equiv 0$, $i = 1, 2, \ldots, m$.
Note that

$$\|x\|_2 \le t \Longleftrightarrow \begin{pmatrix} t & x^T \\ x & tI \end{pmatrix} \succeq 0.$$

Therefore, the SOCP problem (6.1) is equivalent to an SDP:

$$\begin{aligned} \min_{x \in R^n} \quad & c^T x \\ s.t. \quad & \begin{pmatrix} c_i^T x + d_i & (A_i x - b_i)^T \\ A_i x - b_i & (c_i^T x + d_i) I \end{pmatrix} \succeq 0, \\ & i = 1, 2, \ldots, m. \end{aligned}$$

It is also obvious that SDP reduces to LP by restricting X to be a diagonal matrix.

6.5 Convex optimization is a CLP

General convex optimization (CO) minimizes a convex function over a convex set. An important property of CO is that any local optimal solution is also global optimal. We consider CO in the following form

$$\begin{aligned} \mathbf{CO}: \quad \min \quad & f(x) \\ s.t. \quad & c_i(x) \le 0 \\ & i = 1, 2, \ldots, m, \end{aligned}$$

where f and c_i, $i = 1, 2, \ldots, m$, are all convex functions on R^n. Clearly, **CO** is equivalent to

$$\begin{aligned} \min_{x,\alpha} \quad & \alpha \\ s.t. \quad & f(x) - \alpha \le 0 \\ & c_i(x) \le 0 \\ & i = 1, 2, \ldots, m. \end{aligned}$$

The new constraint functions remain convex in (x, α). Thus, it is sufficient to consider CO with linear objective function:

$$\begin{aligned} \mathbf{CO}: \quad \min \quad & \langle c, x \rangle \\ s.t. \quad & c_i(x) \le 0 \\ & i = 1, 2, \ldots, m. \end{aligned}$$

For $i = 1, 2, \ldots, m$, we define $C_i := \{(t, x) \in R \times R^n : t > 0, c_i(x/t) \leq 0\}$. It can be shown that each C_i is a convex cone (not closed in general). By noting that $c_i(x) \leq 0$ is equivalent to $(1, x) \in C_i$, the above CO can be rewritten as a CLP:

$$\begin{array}{cc} \min & \langle (0, c), (t, x) \rangle \\ s.t. & \langle (1, 0), (t, x) \rangle = 1 \\ & (t, x) \in C_1 \cap C_2 \cap \ldots \cap C_m. \end{array}$$

6.6 General constrained optimization is a CLP

General constrained optimization takes the form

$$\min\{f(x) : s.t.\, x \in Q \subset R^n\},$$

where Q is generally described by a set of (nonlinear) equality or inequality constraints. Clearly, the above problem is equivalent to

$$\min\{\alpha : s.t.\, x \in Q, f(x) \leq \alpha\},$$

Thus, it is sufficient to consider general constrained optimization with linear objective function

$$\min\{\langle c, x \rangle : s.t.\, x \in Q\}.$$

It is clear that

$$\begin{aligned} x \in Q &\Longleftrightarrow (x, 1) \in \{(y, 1) : y \in Q\} \subset R^{n+1} \\ &\Longleftrightarrow (x, 1) \in \{(ty, t) : y \in Q, t = 1\} \\ &\Longleftrightarrow (x, t) \in K \cap \{(y, t) : y \in R^n, t = 1\}, \end{aligned}$$

where $K := \{(ty, t) : y \in Q, t \geq 0\}$ is a cone. Therefore, the general constrained problem

$$\min\{\langle c, x \rangle : s.t.\, x \in Q\}$$

can be rewritten as

$$\begin{array}{cc} \min & \langle (c, 0), (x, t) \rangle \\ s.t. & \langle (x, t), (0, 1) \rangle = 1 \\ & (x, t) \in K. \end{array}$$

6.7 Short summary

The standard form of conic linear programming is

$$\textbf{CLP}: \quad p^* := \min_{x \in \mathcal{X}} \ \langle c, x \rangle \quad s.t.\ \mathcal{A}x = b, \quad x \in K.$$

CLP is a very general model for mathematical programming. It consists linear programming, semidefinite programming, second order cone programming as special cases. General convex optimization, or even nonlinear optimization problems, can be reformulated as CLP. As a result, the study on algorithms for CLP is usually case by case. The cone K could be cross product of several types of cones. Many others cones exist, e.g., matrix norm cone. All difficulties of CLP are hidden in the cone. Different properties of K can be exploited, e.g., symmetry, homogeneity, convexity, etc.

6.8 Dual problem of CLP and duality

Definition 6.3 (Dual cone) *Let K be a cone in $\mathcal{X}$. The set $K^* := \{y \in \mathcal{X} : \langle y, x\rangle \geq 0, \ \ \forall x \in K\}$ is a cone and is called the dual cone of K.*

Theorem 6.4 *Let K be any cone. The dual cone K^* is convex.*

Definition 6.5 (Self-dual cone) *A cone K is called self-dual if $K^* = K$.*

Homework. Prove that the nonnegative orthant R^n_+, the SDP cone S^n_+ and the second order cone $Q^n = \{(t, x) : \|x\|_2 \leq t\}$ are all self-dual.

Define $\mathcal{F} := \{x \in \mathcal{X} : \mathcal{A}x = b, x \in K\}$, and $\mathcal{L} : \mathcal{X} \times R^m \times \mathcal{X}$ by

$$\mathcal{L}(x, y, \mu) := \langle c, x\rangle - y^T(\mathcal{A}x - b) - \langle \mu, x\rangle.$$

For any $x \in \mathcal{F}$, $y \in R^m$ and $\mu \in K^*$, it holds that

$$\mathcal{L}(x, y, \mu) \leq \langle c, x\rangle.$$

Therefore, for any $y \in R^m$ and $\mu \in K^*$, it holds that

$$\inf_{x\in\mathcal{X}} \mathcal{L}(x, y, \mu) \leq \inf_{x\in\mathcal{F}} \mathcal{L}(x, y, \mu) \leq \inf_{x\in\mathcal{F}} \langle c, x\rangle = p^*,$$

i.e., $\inf_{x\in\mathcal{X}} \mathcal{L}(x, y, \mu)$ is a lower bound of p^*. The best lower bound of p^* that can be so obtained is

$$\begin{aligned}
&\sup_{y\in R^m,\mu\in K^*} \inf_{x\in\mathcal{X}} \mathcal{L}(x, y, \mu) \\
=&\sup_{y\in R^m,\mu\in K^*} \begin{cases} b^T y, & \text{if } c - \mathcal{A}^*y - \mu = 0; \\ -\infty, & \text{o.w.} \end{cases} \\
=&\sup_{y\in R^m} \{b^T y : c - \mathcal{A}^*y \in K^*\}.
\end{aligned}$$

As a result, we call the following conic linear programming the dual problem of **CLP**:

$$\text{Dual-CLP}: \quad d^* := \max_{y\in R^m} \{b^T y : \ s.t.\ c - \mathcal{A}^*y \in K^*\}.$$

Theorem 6.6 (Weak duality) *Let p^* and d^* be, respectively, defined for* **CLP** *and* **Dual-CLP**. *Then, $d^* \leq p^*$.*

Weak duality implies that a feasible solution to either problem yields a bound on the optimal value of the other problem. We call $\langle c, x\rangle - b^T y$ the duality gap.

Corollary 6.7 *Let x and y be, respectively, feasible for* **CLP** *and* **Dual-CLP**. *If $\langle c, x\rangle = b^T y$, then x and y are optimal for respective problems.*

A big question is that whether the reverse is also true. That is the strong duality: given x optimal for **CLP**, is there a y that is feasible for **Dual-CLP** and also satisfies $\langle c, x\rangle = b^T y$?

Let $(K, K^*) = (R^n_+, R^n_+)$. According to Farkas lemma, the following systems are alternative:

$$\{x \in R^n : Ax = b, x \in K\} \quad v.s. \quad \{y \in R^m : -A^T y \in K^*, b^T y > 0\}.$$

$$\{x \in R^n : Ax = 0, x \in K, \langle c, x\rangle < 0\} \quad v.s. \quad \{y \in R^m : c - A^T y \in K^*\}.$$

For general closed convex cone K and its dual cone K^*, are the following systems still alternative

$$\{x \in \mathcal{X} : \mathcal{A}x = b, x \in K\} \quad v.s. \quad \{y \in R^m : -\mathcal{A}^* y \in K^*, b^T y > 0\}?$$

$$\{x \in \mathcal{X} : \mathcal{A}x = 0, x \in K, \langle c, x\rangle < 0\} \quad v.s. \quad \{y \in R^m : c - \mathcal{A}^* y \in K^*\}?$$

Here $\mathcal{A}^*$ represents the adjoint operator of $\mathcal{A}$, i.e., $\mathcal{A}^*$ satisfies

$$\langle \mathcal{A}x, y\rangle = \langle x, \mathcal{A}^* y\rangle, \ \forall x \in \mathcal{X}, y \in \mathcal{Y}.$$

Let K be a **general closed convex cone** and K^* be its dual cone. Define

$$\begin{aligned}
S_1 &:= \{x \in \mathcal{X} : \mathcal{A}x = b, x \in K\} \\
S_2 &:= \{y \in R^m : -\mathcal{A}^* y \in K^*, b^T y > 0\}; \\
T_1 &:= \{x \in \mathcal{X} : \mathcal{A}x = 0, x \in K, \langle c, x\rangle < 0\} \\
T_2 &:= \{y \in R^m : c - \mathcal{A}^* y \in K^*\}.
\end{aligned}$$

Then we have

$$\begin{aligned}
S_1 \neq \emptyset &\Longrightarrow S_2 = \emptyset \quad \text{but} \quad S_1 \neq \emptyset \not\Leftarrow S_2 = \emptyset; \\
T_1 \neq \emptyset &\Longrightarrow T_2 = \emptyset \quad \text{but} \quad T_1 \neq \emptyset \not\Leftarrow T_2 = \emptyset,
\end{aligned}$$

which implies that S_1 and S_2 cannot be simultaneously nonempty but can be simultaneously empty, and similarly for T_1 and T_2. This is because $\{\mathcal{A}x : x \in K\}$ and $\{\mathcal{A}^* y + s : y \in R^m, s \in K^*\}$ are, though convex, not necessarily closed.

Example 6.8.1 *Let $\varepsilon > 0$ and*

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \ A_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ b = \begin{pmatrix} 0 \\ 2\varepsilon \end{pmatrix},$$

A linear mapping $\mathcal{A} : S^2 \to R^2$ is defined by

$$\mathcal{A}X = \begin{pmatrix} \langle A_1, X\rangle \\ \langle A_2, X\rangle \end{pmatrix} = \begin{pmatrix} x_{11} \\ 2x_{12} \end{pmatrix}.$$

Note that $\mathcal{A}^* : R^2 \to S^2$ is given by

$$\mathcal{A}^* y = y_1 A_1 + y_2 A_2.$$

It can be shown that (1) for any $X \succeq 0$, $\mathcal{A}X \neq b$, i.e., $S_1 := \{X \in S^2 : \mathcal{A}X = b, X \succeq 0\} = \emptyset$; and (2) $S_2 := \{y \in R^2 : -\mathcal{A}^* y \succeq 0, b^T y > 0\} = \emptyset$. In this example, the set $\{\mathcal{A}X : X \succeq 0\}$ is not closed, and b is on the boundary of this set. Therefore, b cannot be strictly separated from this set. This is why strong duality can fail for general conic linear programming.

Example 6.8.2 (SDP example with duality gap)

$$A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, A_2 = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}, C = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, b = \begin{pmatrix} 0 \\ 2 \end{pmatrix}.$$

It can be shown that $-2 = d^* < p^* = 0$.

Theorem 6.8 (Farkas Lemma for general closed convex cone) *Let K be a* **general closed convex cone** *and K^* be its dual cone. Define*

$$
\begin{aligned}
S_1 &:= \{x \in \mathcal{X} : \mathcal{A}x = b, x \in K\} \\
S_2 &:= \{y \in R^m : -\mathcal{A}^* y \in K^*, b^T y > 0\}; \\
T_1 &:= \{x \in \mathcal{X} : \mathcal{A}x = 0, x \in K, \langle c, x\rangle < 0\} \\
T_2 &:= \{y \in R^m : c - \mathcal{A}^* y \in K^*\}.
\end{aligned}
$$

Then, we have

1. *$S_1 \neq \emptyset \implies S_2 = \emptyset$, $T_1 \neq \emptyset \implies T_2 = \emptyset$;*
2. *If there exists $y \in R^m$ such that $-\mathcal{A}^* y \in int(K^*)$ (the interior of K^*), then $S_1 \neq \emptyset \Leftarrow S_2 = \emptyset$ (or equivalently, $S_1 = \emptyset \implies S_2 \neq \emptyset$);*
3. *If there exists $x \in int(K)$ such that $\mathcal{A}x = 0$, then $T_1 \neq \emptyset \Leftarrow T_2 = \emptyset$ (or equivalently, $T_1 = \emptyset \implies T_2 \neq \emptyset$).*

Theorem 6.9 (Strong duality) *Let $\mathcal{F}_P$ and $\mathcal{F}_D$ be the feasible sets of* **CLP** *and* **Dual-CLP***, respectively. Assume that both $\mathcal{F}_P$ and $\mathcal{F}_D$ are nonempty and at least one of them has an interior, i.e.,*

$$\exists x \in int(K) \text{ such that } \mathcal{A}x = b$$

or

$$\exists y \in R^m \text{ such that } c - \mathcal{A}^* y \in int(K^*).$$

Then, $x \in \mathcal{F}_P$ is optimal for **CLP** *and $y \in \mathcal{F}_D$ is optimal for* **Dual-CLP** *if any only if $\langle c, x\rangle = b^T y$.*

Theorem 6.10 1. *If one of* **CLP** *or* **Dual-CLP** *is unbounded, then the other has no feasible solution.*

2. *If* **CLP** *and* **Dual-CLP** *are both feasible, then both have bounded optimal objective values and the optimal objective values may have a duality gap.*
3. *If one of* **CLP** *and* **Dual-CLP** *has a strictly or interior feasible solution and it has an optimal solution, then the other is feasible and has an optimal solution with the same optimal value.*

Theorem 6.11 (Optimality system for CLP) *Assume strong duality holds. Then, x and y are optimal for respective problems if and only if there exists $s \in K^*$ such that the following conditions are satisfied:*

$$
\begin{aligned}
\textbf{Primal feasibility}: &\quad \mathcal{A}x = b, \quad x \in K \\
\textbf{Dual feasibility}: &\quad \mathcal{A}^* y + s = c, \quad s \in K^* \\
\textbf{Complementarity}: &\quad \langle c, x\rangle - b^T y = 0 \ (\textit{or}, \langle s, x\rangle = 0).
\end{aligned}
$$

MATH: Operations Research **2014-15 First Term**

Handout 8: 博弈论简介

Instructor: Junfeng Yang *October 28, 2014*

Contents

8.1 博弈论简介

博弈论，又称对策论，是自古以来的政治家和军事家都很注意研究的问题。20世纪40年代形成并发展起来的。1944年von Neumann 与Morgenstern 的《博弈论与经济行为》一书出版，标志着现代系统博弈理论的初步形成。20世纪50年代，Nash 建立了非合作博弈的“纳什均衡" 理论，标志着博弈的新时代开始，是纳什在经济博弈论领域划时代的贡献。1994年纳什获得了诺贝尔经济学奖。他提出的纳什均衡概念在非合作博弈理论中起着核心作用。由于纳什均衡的提出和不断完善，为博弈论广泛应用于经济学、管理学、社会学、政治学、军事科学等领域奠定了坚实的理论基础。

对策论亦称竞赛论或博弈论，是研究具有斗争或竞争性质现象的数学理论和方法。一般认为，它是现代数学的一个新分支，是运筹学的一个重要学科。对策论发展的历史并不长，但由于它研究的问题与政治、经济、军事活动乃至一般的日常生活等有着密切联系，并且处理问题的方法具有明显特色，所以日益引起广泛注意。在日常生活中，经常会看到一些相互之间具有斗争或竞争性质的行为，如下棋、打牌、体育比赛等。还比如战争活动中的双方，都力图选取对自己最有利的策略，千方百计去战胜对手。在政治方面，国际间的谈判，各种政治力量之间的斗争，各国际集团之间的斗争等无一不具有斗争的性质。在经济活动中，各国之间、各公司企业之间的经济谈判，企业之间为争夺市场而进行的竞争等，举不胜举。

战国时期，有一天齐王提出要与田忌赛马，双方约定从各自的上、中、下三个等级的马中各选一匹参赛，每匹马均只能参赛一次，每一次比赛双方各出一匹马，负者要付给胜者千金。已经知道，在同等级的马中，田忌的马不如齐王的马，而如果田忌的马比齐王的马高一等级，则田忌的马可取胜。当时，田忌手下的一个谋士孙膑给他出了个主意：每次比赛时先让齐王牵出他要参赛的马，然后来用下马对齐王的上马，用中马对齐王的下马，用上马对齐王的中马。比赛结果，田忌二胜一负，夺得千金。由此看来，两个人各采取什么样的出马次序对胜负是至关重要的。

对策行为的三个基本要素

1. 局中人：在一个对策行为(或一局对策)中，有权决定自己行动方案的对策参加者，称为局中人。一般要求一个对策中至少要有两个局中人。并且，局中人都是贪婪且理智的。如在“齐王赛马”的例子中，局中人是齐王和田忌。

2. 策略集：一局对策中，可供局中人选择的一个实际可行的完整的行动方案称为一个策略。参加对策的每一局中人，都有自己的策略集。一般，每一局中人的策略集中至少应包括两个策略。

   在“齐王赛马”的例子中，如果用(上，中，下)表示以上马、中马、下马依次参赛这样一个次序，这就是一个完整的行动方案，即为一个策略。可见，局中人齐王和田忌各自都有6个策略：(上，中，下)、(上，下，中)、(中，上，下)、(中，下，上)、(下，中，上)、(下，上，中)。

3. 赢得函数(支付函数)：在一局对策中，各局中人选定的策略形成的策略组称为一个局势，即若s_i 是第i个局中人的一个策略，则n个局中人的策略组

   $$s = (s_1, s_2, \dots, s_n)$$

   就是一个局势。全体局势的集合S 可用各局中人策略集的笛卡儿积表示，即

   $$S = S_1 \times S_2 \times \dots \times S_n.$$

   当一个局势出现后，对策的结果也就确定了。也就是说，对任一局势$s \in S$, 局中人i 可以得到一个赢得值$H_i(s)$. 显然，$H_i(s)$ 是局势s 的函数，称为第i 个局中人的赢得函数。

在齐王与田忌赛马的例子中，局中人集合为$I = \{1, 2\}$, 齐王和田忌的策略集可分别用

$$S_1 = \{\alpha_1, \alpha_2, \dots, \alpha_6\} \text{ 和 } S_2 = \{\beta_1, \beta_2, \dots, \beta_6\}$$

表示。这样，齐王的任一策略α_i 和田忌的任一策略β_j 就形成了一个局势s_{ij}. 如果$\alpha_1 =$ (上，中，下)，$\beta_1 =$ (上，中，下)，则在局势s_{11} 下齐王的赢得值为$H_1(s_{11}) = 3$, 田忌的赢得值为 $H_2(s_{11}) = -3$.

对策问题的分类: 根据局中人的个数，分二人对策和多人对策；根据各局中人的赢得函数的代数和是否为零，分零和对策与非零和对策；根据各局中人之间是否允许合作，分为合作对策和非合作对策；根据局中人的策略集中的策略个数，分有限对策和无限对策；根据策略的选择是否与时间有关，分静态对策与动态对策；根据对策模型的数学特征，分矩阵对策、连续对策、微分对策、阵地对策、凸对策、随机对策等；其他分类方式。

二人有限零和对策 在众多的对策模型中，占有重要地位是二人有限零和对策，又称矩阵对策。这类对策是到目前为止在理论研究和求解方法方面都比较完善的一个对策分支。矩阵对策可以说是一类最简单的对策模型，其研究思想和方法十分具有代表性，体现了对策论的一般思想和方法，且矩阵对策的基本结果也是研究其他对策模型的基础。

8.2 矩阵对策的数学模型

在矩阵对策中，一般用I, II 分别表示两个局中人，并设局中人I有m 个纯策略$\{\alpha_1, \ldots, \alpha_m\}$，局中人II 有$n$ 个纯策略$\{\beta_1, \ldots, \beta_n\}$. 则局中人I, II 的纯策略集分别为

$$S_1 = \{\alpha_1, \ldots, \alpha_m\},\ S_2 = \{\beta_1, \ldots, \beta_n\}.$$

当局中人I 选定纯策略α_i, 局中人II 选定纯策略β_j 后，就形成了一个局势(α_i, β_j). 对任一局势(α_i, β_j), 记局中人I的赢得值为a_{ij}. 称

$$A = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{pmatrix}$$

为局中人I 的赢得矩阵(或局中人II 的支付矩阵)。由于假定对策为零和的，故局中人II 的赢得矩阵就是$-A$.

在齐王赛马的例子中:

田忌的策略

齐王的赢得矩阵	上中下	上下中	中上下	中下上	下中上	下上中
齐王的策略 上中下	3	1	1	1	1	−1
上下中	1	3	1	1	−1	1
中上下	1	−1	3	1	1	1
中下上	−1	1	1	3	1	1
下中上	1	1	−1	1	3	1
下上中	1	1	1	−1	1	3

当局中人I, II 的纯策略集，以及局中人I的赢得矩阵A 确定后，一个矩阵对策也就给定了。通常，将一个矩阵对策记成

$$G = \{I, II; S_1, S_2; A\} \text{ 或 } G = \{S_1, S_2; A\}.$$

Example 8.2.1 设有一矩阵对策$G = \{S_1, S_2; A\}$, 其中

$$S_1 = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\},\ S_2 = \{\beta_1, \beta_3, \beta_3\}.$$

$$A = \begin{pmatrix} -6 & 1 & -8 \\ 3 & 2 & 4 \\ 9 & -1 & -10 \\ -3 & 0 & 6 \end{pmatrix}.$$

如何求解两个局中人的最优纯策略？理智的局中人应采取“理智的行为”。

Definition 8.1 设有矩阵对策$G = \{S_1, S_2; A\}$, 其中

$$S_1 = \{\alpha_1, \ldots, \alpha_m\},\ S_2 = \{\beta_1, \ldots, \beta_n\},\ A = (a_{ij})_{m\times n}.$$

若存在$i^* \in \{1, 2, \ldots, m\}$, $j^* \in \{1, 2, \ldots, n\}$ 使得

$$a_{i^*j^*} = \max_i \min_j a_{ij} = \min_j \max_i a_{ij}$$

成立，则称$(\alpha_{i^*}, \beta_{j^*})$为矩阵对策$G$在纯策略下的解（或平衡局势)，$\alpha_{i^*}$ 与β_{j^*} 分别称为局中人*I, II* 的最优纯策略。记$V_G = a_{i^*j^*}$, 并称其为矩阵对策的值。

Remark 8.2.1 在矩阵对策中两个局中人都采取最优纯策略*(*如果最优纯策略存在*)*才是理智的行动。

Example 8.2.2 矩阵对策$G = \{S_1, S_2; A\}$, 其中

$$S_1 = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\},\ S_2 = \{\beta_1, \beta_3, \beta_3\}.$$

$$A = \begin{pmatrix} -6 & 1 & -8 \\ 3 & 2 & 4 \\ 9 & -1 & -10 \\ -3 & 0 & 6 \end{pmatrix}.$$

局中人的最优纯策略为(α_2, β_2), 因为

$$a_{22} = \max_j \min_i a_{ij} = \min_i \max_j a_{ij}.$$

元素a_{22} 为所在行的最小元素、所在列的最大元素。

Theorem 8.2 (纯策略意义下有解的条件) 矩阵对策$G = \{S_1, S_2; A\}$ 在纯策略意义下有解的充分必要条件为：存在纯局势$(\alpha_{i^*}, \beta_{j^*})$ 使得

$$a_{ij^*} \le a_{i^*j^*} \le a_{i^*j}$$

对一切$i \in \{1, \ldots, m\}$ 以及$j \in \{1, \ldots, n\}$成立。

Proof: 充分性：因为对任意的i, j 都有$a_{ij^*} \le a_{i^*j^*} \le a_{i^*j}$, 因此

$$\max_i a_{ij^*} \le a_{i^*j^*} \le \min_j a_{i^*j}.$$

由于$\min_j \max_i a_{ij} \le \max_i a_{ij^*}$ 和 $\min_j a_{i^*j} \le \max_i \min_j a_{ij}$ 因此有

$$\min_j \max_i a_{ij} \le a_{i^*j^*} \le \max_i \min_j a_{ij}.$$

另外，下式永远成立

$$\max_i \min_j a_{ij} \le \min_j \max_i a_{ij}.$$

综上有

$$a_{i^*j^*} = \min_j \max_i a_{ij} = \max_i \min_j a_{ij}.$$

必要性：矩阵对策在纯策略意义下有解的定义：存在$i^* \in \{1, 2, \ldots, m\}, j^* \in \{1, 2, \ldots, n\}$ 使得

$$a_{i^*j^*} = \max_i \min_j a_{ij} = \min_j \max_i a_{ij}$$

成立. 因此有

$$a_{i^*j^*} = \min_j a_{i^*j} = \max_i a_{ij^*}.$$

对一切$i \in \{1, \ldots, m\}$ 以及$j \in \{1, \ldots, n\}$, 都有

$$a_{ij^*} \leq \max_i a_{ij^*} = a_{i^*j^*} = \min_j a_{i^*j} \leq a_{i^*j}.$$

■

Definition 8.3 (鞍点) 设$f(x, y)$为一个定义在$x \in A$ 及 $y \in B$ 上的实值函数，如果存在$x^* \in A, y^* \in B$ 使得对一切$x \in A$ 及 $y \in B$ 都有

$$f(x, y^*) \leq f(x^*, y^*) \leq f(x^*, y),$$

则称(x^*, y^*) 为函数f的鞍点。

由上述定理可知，矩阵对策在纯策略意义下有解的充分必要条件是：$a_{i^*j^*}$ 为矩阵A的鞍点（也称为对策的鞍点）。

Remark 8.2.2 纯策略意义下达到最优解时的直观解释：设$a_{i^*j^*}$ 为矩阵A的鞍点。当局中人I 选择策略α_{i^*}时，局中人II 为减少损失，只会选择策略β_{j^*}; 当局中人II 选择策略β_{j^*} 时，局中人I 为增加所得，也只会选择策略α_{i^*}.

Example 8.2.3 设有一矩阵对策$G = \{S_1, S_2; A\}$, 其中

$$S_1 = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\},\ S_2 = \{\beta_1, \beta_3, \beta_3, \beta_4\}.$$

$$A = \begin{pmatrix} 6 & 5 & 6 & 5 \\ 1 & 4 & 2 & -1 \\ 8 & 5 & 7 & 5 \\ 0 & 2 & 6 & 2 \end{pmatrix}.$$

求对策的解。

矩阵对策的解不一定唯一。对策的解具有如下性质

Theorem 8.4

- 无差别性：若$(\alpha_{i_1}, \beta_{j_1})$ 和$(\alpha_{i_2}, \beta_{j_2})$ 都是矩阵对策G的解，则$a_{i_1j_1} = a_{i_2j_2}$.
- 可交换性：若$(\alpha_{i_1}, \beta_{j_1})$ 和$(\alpha_{i_2}, \beta_{j_2})$ 都是矩阵对策G的解，则$(\alpha_{i_1}, \beta_{j_2})$ 与$(\alpha_{i_2}, \beta_{j_1})$ 也是解。

Remark 8.2.3 矩阵对策的值是唯一的，即当局中人I 采用构成解的最优纯策略时，能保证他的赢得V_G不依赖于对方的纯策略。

8.3 矩阵对策的混合策略

局中人存在纯策略意义下的最优解的充要条件是

$$\max_j \min_i a_{ij} = \min_i \max_j a_{ij}.$$

当$\max_j \min_i a_{ij} < \min_i \max_j a_{ij}$, 不存在纯策略意义下的最优解，此时局中人如何进行博弈？

Example 8.3.1 设有一矩阵对策G, 局中人I 的赢得矩阵为

$$A = \begin{pmatrix} 3 & 6 \\ 5 & 4 \end{pmatrix}.$$

局中人应采取何种决策进行博弈？

设有矩阵对策$G = \{S_1, S_2; A\}$, 其中$S_1 = \{\alpha_1, \ldots, \alpha_m\}$, $S_2 = \{\beta_1, \ldots, \beta_n\}$, $A = (a_{ij})_{m\times n}$. 记

$$S_1^* = \left\{x \in R^m : x \geq 0, \sum\nolimits_{i=1}^m x_i = 1\right\}$$
$$S_2^* = \left\{y \in R^n : y \geq 0, \sum\nolimits_{j=1}^n y_j = 1\right\}.$$

则S_1^* 与S_2^* 分别称为局中人I 和II 的混合策略集(或策略集)；$x \in S_1^*$ 与$y \in S_2^*$ 分别称为局中人I 和II 的混合策略(或策略)，称(x, y) 为一个混合局势(或局势)。局中人I 的赢得函数为

$$E(x, y) = \sum_{i=1}^m \sum_{j=1}^n a_{ij} x_i y_j = x^T A y.$$

得到的新对策记成$G^* = \{S_1^*, S_2^*, E\}$, 称其为对策$G$的混合扩充。

Remark 8.3.1 *1.* 纯策略是混合策略的特例。例如局中人I 的纯策略α_k 可理解为混合策略$x = e_k$.

2. 混合策略x 可理解为当两个局中人多次重复进行对策G 时，局中人I 分别采取纯策略$\alpha_1, \ldots, \alpha_m$ 的频率.

3. 若只进行一次对策，混合策略x 可理解为局中人I 对各纯策略的偏爱程度.

当局中人I 采取某混合策略x 时，在最不利的情况下他只能赢得

$$\min_{y\in S_2^*} E(x, y).$$

因此，局中人I 会采取混合策略使得在最不利情况下的赢得最大化，即

$$\max_{x\in S_1^*} \min_{y\in S_2^*} E(x, y).$$

同理，局中人II 需考虑如下问题

$$\min_{y\in S_2^*} \max_{x\in S_1^*} E(x, y).$$

上面极大极小、极小极大问题均有意义，因为E 关于(x, y) 连续，且S_1^*, S_2^* 为有界闭集。下式恒成立：

$$\max_{x\in S_1^*} \min_{y\in S_2^*} E(x, y) \leq \min_{y\in S_2^*} \max_{x\in S_1^*} E(x, y).$$

Definition 8.5 设$G^* = \{S_1^*, S_2^*, E\}$ 是矩阵对策$G = \{S_1, S_2, A\}$ 的混合扩充。如果

$$\max_{x \in S_1^*} \min_{y \in S_2^*} E(x,y) = \min_{y \in S_2^*} \max_{x \in S_1^*} E(x,y),$$

则称其值，记为V_G, 为对策G 的值。称使上面等式成立的混合局势(x^*, y^*) 为矩阵对策G 在混合策略意义下的解*(*或简称解*)*，x^* 与y^* 分别称为局中人*I* 和*II* 的最优混合策略*(*或简称最优策略*)*.

注： 一般，对$G = \{S_1, S_2, A\}$ 及其混合扩充$G^* = \{S_1^*, S_2^*, E\}$ 不加区别，通常都用$G = \{S_1, S_2, A\}$ 表示。当G 在纯策略意义下无解时，自动认为讨论的是在混合策略意义下的解，相应的局中人I 的赢得函数为$E(x,y)$.

当G 在纯策略意义下有解时，上述V_G 定义与前面定义一致。设G 在混合策略意义下的解为(x^*, y^*), 则$V_G = E(x^*, y^*)$.

Theorem 8.6 矩阵对策$G = \{S_1, S_2; A\}$ 在混合策略意义下有解的充分必要条件为：存在$x^* \in S_1^*$, $y^* \in S_2^*$ 使得(x^*, y^*) 为$E(x,y)$ 的一个鞍点，即对一切$x \in S_1^*$ 以及$y \in S_2^*$ 都有

$$E(x, y^*) \le E(x^*, y^*) \le E(x^*, y).$$

Proof: 与纯策略情形完全一致，只需$E(x,y) \leftrightarrow a_{ij}$, $x \leftrightarrow i$, $y \leftrightarrow j$. ■

Example 8.3.2 求矩阵对策$G = \{S_1, S_2; A\}$ 的解，其中

$$A = \begin{pmatrix} 3 & 6 \\ 5 & 4 \end{pmatrix}.$$

在纯策略意义下无解。设$x = (x_1, x_2)^T$, $y = (y_1, y_2)^T$ 为局中人*I, II* 的混合策略，混合策略集为$\{(u,v)^T \in R^2 : u, v \ge 0, u + v = 1\}$. 局中人*I*的赢得函数为

$$E(x,y) = x^T A y = -4(x_1 - 1/4)(y_1 - 1/2) + 9/2.$$

则易见$x^* = (1/4, 3/4)^T$, $y^* = (1/2, 1/2)^T$为$E(x,y)$的鞍点，为两局中人的最优混合策略。$V_G = 9/2$.

8.4 矩阵对策的基本理论

一般矩阵对策在纯策略意义下的解往往是不存在的; 一般矩阵对策在混合策略意义下的解总是存在的; 通过一个构造性的证明，导出求解矩阵对策的基本方法：线性规划方法.

当局中人I 取纯策略α_i 时，其赢得函数记为$E(i,y)$, 即

$$E(i,y) = \sum_{j=1}^{n} a_{ij} y_j.$$

当局中人II 取纯策略β_j 时，其赢得函数记为$E(x,j)$, 即

$$E(x,j) = \sum_{i=1}^{m} a_{ij} x_i.$$

因此有

$$E(x,y)=\sum_{i=1}^{m}\sum_{j=1}^{n}a_{ij}x_iy_j=\sum_{i=1}^{m}\left(\sum_{j=1}^{n}a_{ij}y_j\right)x_i=\sum_{i=1}^{m}E(i,y)x_i,$$
$$=\sum_{j=1}^{n}\left(\sum_{i=1}^{m}a_{ij}x_i\right)y_j=\sum_{j=1}^{n}E(x,j)y_j.$$

Theorem 8.7 矩阵对策$G=\{S_1,S_2;A\}$ 在混合策略意义下有解的充分必要条件为：存在$x^*\in S_1^*$, $y^*\in S_2^*$ 使得对任意$i=1,2,\ldots,m$ 以及$j=1,2,\ldots,n$ 都有

$$\sum_{j=1}^{n}a_{ij}y_j^*=E(i,y^*)\le E(x^*,y^*)\le E(x^*,j)=\sum_{i=1}^{m}a_{ij}x_i^*.$$

Proof: 必要性：首先，存在$x^*\in S_1^*$, $y^*\in S_2^*$ 使得对一切$x\in S_1^*$ 以及$y\in S_2^*$ 都有

$$E(x,y^*)\le E(x^*,y^*)\le E(x^*,y).$$

纯策略是混合策略的特例，取$x=e_i, y=e_j$ ($i=1,2,\ldots,m, j=1,2,\ldots,n$) 即可。

充分性：对一切$x\in S_1^*$ 以及$y\in S_2^*$ 都有

$$E(x,y^*)=\sum_{i=1}^{m}E(i,y^*)x_i\le E(x^*,y^*)\sum_{i=1}^{m}x_i=E(x^*,y^*)$$
$$E(x^*,y)=\sum_{j=1}^{n}E(x^*,j)y_j\ge E(x^*,y^*)\sum_{j=1}^{n}y_j=E(x^*,y^*).$$

因此

$$E(x,y^*)\le E(x^*,y^*)\le E(x^*,y).$$

矩阵对策G 在混合策略意义下有解。 ■

Theorem 8.8 (von Neumann, 1944) 任一矩阵对策$G=\{S_1,S_2;A\}$ 在混合策略意义下一定有解。

Proof: 只需证明，存在x^* 与y^* 使得

$$\sum_{j=1}^{n}a_{ij}y_j^*\le E(x^*,y^*),\ i=1,\ldots,m,$$
$$\sum_{i=1}^{m}a_{ij}x_i^*\ge E(x^*,y^*),\ j=1,\ldots,n,$$
$$\sum_{i=1}^{m}x_i^*=1,\qquad x^*\ge 0,$$
$$\sum_{j=1}^{n}y_j^*=1,\qquad y^*\ge 0.$$

为此考虑如下一对线性规划问题

$$\begin{aligned}\max_{x,u} \quad & u \\ \text{s.t.} \sum_{i=1}^{m} & a_{ij}x_i \geq u,\, j = 1,\ldots,n, \\ & \sum_{i=1}^{m} x_i = 1, \quad x \geq 0.\end{aligned}$$

$$\begin{aligned}\min_{y,v} \quad & v \\ \text{s.t.} \sum_{j=1}^{n} & a_{ij}y_j \leq v,\, i = 1,\ldots,m, \\ & \sum_{j=1}^{n} y_j = 1, \quad y \geq 0.\end{aligned}$$

矩阵形式：

$$\text{P:} \quad \begin{aligned}\max_{x,u} \quad & u \\ \text{s.t. } & A^T x \geq u\mathbf{1}_n \\ & \mathbf{1}_m^T x = 1 \\ & x \geq 0.\end{aligned}$$

$$\text{D:} \quad \begin{aligned}\min_{y,v} \quad & v \\ \text{s.t. } & Ay \leq v\mathbf{1}_m \\ & \mathbf{1}_n^T y = 1 \\ & y \geq 0.\end{aligned}$$

此二问题互为对偶，且显然均可行。由强对偶定理，存在(x^*, u^*) 与(y^*, v^*) 分别为原问题与对偶问题的最优解，且$u^* = v^*$. 记$w^* := u^* = v^*$. 由可行性得$x^* \in S_1^*, y^* \in S_2^*$, 且

$$E(i, y^*) = \sum_{j=1}^{n} a_{ij}y_j^* \leq w^* \leq \sum_{i=1}^{m} a_{ij}x_i^* = E(x^*, j).$$

另外，

$$E(x^*, y^*) = \sum_{i=1}^{m} E(i, y^*)x_i^* \leq w^* \sum_{i=1}^{m} x_i^* = w^*,$$
$$E(x^*, y^*) = \sum_{j=1}^{n} E(x^*, j)y_j^* \geq w^* \sum_{j=1}^{n} y_j^* = w^*.$$

因此，$w^* = E(x^*, y^*)$ 且

$$E(i, y^*) \leq E(x^*, y^*) \leq E(x^*, j).$$

即(x^*, y^*) 是对策G 的解。 ■

记矩阵对策G 的解集为$T(G)$, 对策的值为$V(G)$.

Theorem 8.9 设有两个矩阵对策$G_1=\{S_1,S_2;A_1\}$ 与$G_2=\{S_1,S_2;A_2\}$, 其中$A_1=(a_{ij})$, $A_2=(a_{ij}+L)$, L 为任一常数，则有

- $V_{G_2}=V_{G_1}+L$;
- $T_{G_1}=T_{G_2}$.

Theorem 8.10 设有两个矩阵对策$G_1=\{S_1,S_2;A\}$ 与$G_2=\{S_1,S_2;\alpha A\}$, 其中$\alpha>0$ 为任一常数，则有

- $V_{G_2}=\alpha V_{G_1}$;
- $T_{G_1}=T_{G_2}$.

8.5 解矩阵对策的线性规划法

上述定理表明，求矩阵对策G 的解可通过解如下一对线性规划得到：

$$\begin{aligned}\text{P:}\qquad \max_{x,v}\quad & v\\ \text{s.t. } & A^Tx\geq v\mathbf{1}_n\\ & \mathbf{1}_m^Tx=1\\ & x\geq 0.\end{aligned}$$

$$\begin{aligned}\text{D:}\qquad \min_{y,v}\quad & v\\ \text{s.t. } & Ay\leq v\mathbf{1}_m\\ & \mathbf{1}_n^Ty=1\\ & y\geq 0.\end{aligned}$$

不妨设对策值$v^*=V(G)>0$ (否则, 可考虑$A\longleftarrow A+|v^*|+1$). 作变换$x\leftarrow x/v$, $y\leftarrow y/v$, 将P 与D 等价转化为

$$\begin{aligned}\text{P':}\qquad \min_{x}\quad & \mathbf{1}_m^Tx\\ \text{s.t. } & A^Tx\geq \mathbf{1}_n\\ & x\geq 0.\end{aligned}$$

$$\begin{aligned}\text{D':}\qquad \max_{y}\quad & \mathbf{1}_n^Ty\\ \text{s.t. } & Ay\leq \mathbf{1}_m\\ & y\geq 0.\end{aligned}$$

设已求得P' 与D'的最优解，分别为x^*,y^*. 令$v^*=1/\mathbf{1}_m^Tx^*$. 则对策G的解$T(G)=v^*(x^*,y^*)$, 对策的值$V(G)=v^*$.

Example 8.5.1 求矩阵对策$G=\{S_1,S_2;A\}$ 的解，其中

$$A=\begin{pmatrix}3&6\\5&4\end{pmatrix}.$$

解：该对策问题的解可通过解如下线性规划模型得到：

$$
\begin{aligned}
\text{P'}: \quad & \min_x \quad x_1 + x_2 \\
& \text{s.t. } 3x_1 + 5x_2 \geq 1 \\
& \quad 6x_1 + 4x_2 \geq 1 \\
& \quad x \geq 0.
\end{aligned}
$$

$$
\begin{aligned}
\text{D'}: \quad & \max_y \quad y_1 + y_2 \\
& \text{s.t. } 3y_1 + 6y_2 \leq 1 \\
& \quad 5y_1 + 4y_2 \leq 1 \\
& \quad y \geq 0.
\end{aligned}
$$

用单纯形法解对偶问题, 第一张表

y_1	y_2	y_3	y_4	b
3	6	1	0	1
5	4	0	1	1
-1	-1	0	0	0

最后一张表

y_1	y_2	y_3	y_4	b
0	1	5/18	-1/6	1/9
1	0	-2/9	1/3	1/9
0	0	1/18	1/6	2/9

D' 的最优解与最优函数值分别为$(1/9, 1/9)$, $2/9$. P' 的最优解与最优函数值分别为$(1/18, 1/6)$, $2/9$. 对策G的值为$v^* = V(G) = 9/2$. 对策的解为

$$
\begin{aligned}
x^* &= v^*(1/18, 1/6) = (1/4, 3/4) \\
y^* &= v^*(1/9, 1/9) = (1/2, 1/2).
\end{aligned}
$$

8.6 作业

一位陌生的美女主动过来和你搭讪，并要求和你一起玩个游戏。美女提议："让我们各自亮出硬币的一面，或正或反。如果我们都是正面，那么我给你3元，如果我们都是反面，我给你1元，剩下的情况你给我2元就可以了。"这个游戏公平吗？用线性规划法求双方的最优混合策略。

8.7 其他类型的对策

8.7.1 n 人有限非合作博弈

前面讨论的矩阵博弈属于二人有限零和博弈：

- 二人：只有两个局中人
- 有限：每个局中人的纯策略集为有限集
- 零和：局中人I, II 的赢得函数之和为零
- 非合作：不允许结盟

Nash 将von Neumann 关于二人有限零和博弈推广到n 人有限非零和非合作博弈：

- n 人：n 个局中人, $n \geq 2$
- 有限：每个局中人的纯策略集为有限集
- 非零和：局中人$1, 2, \ldots, n$ 的赢得函数之和不必为零
- 非合作：不允许局中人进行结盟，不允许局中人对支付（赢得）进行再分配

二人有限非零和博弈, 也称为双矩阵博弈。设局中人I, II 的纯策略集分别为

$$\{\alpha_i : i = 1, 2, \ldots, m\}, \quad \{\beta_j : j = 1, 2, \ldots, n\}.$$

赢得矩阵分别为$A, B \in R^{m\times n}$. 非零和意味着不要求对所有的(i,j) 都成立$a_{ij} + b_{ij} = 0$. 若对某(i,j)成立$a_{ij} > 0, b_{ij} > 0$, 此时称为双赢。显然，在纯策略意义下不一定总是有解。如果两局中人分别选择混合策略$x \in R^m$ $(x \geq 0, \mathbf{1}^T x = 1)$ 与$y \in R^n$ $(y \geq 0, \mathbf{1}^T y = 1)$, 则赢得分别为

$$\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}x_i y_j, \quad \sum_{i=1}^{m}\sum_{j=1}^{n} b_{ij}x_i y_j.$$

分别用S_1^* 与S_2^* 表示两局中人的混合策略集。如果存在混合策略$x^* \in S_1^*$ 与$y^* \in S_2^*$ 使得

$$\begin{aligned}\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}x_i^* y_j^* &= \max_{x\in S_1^*}\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}x_i y_j^*,\\ \sum_{i=1}^{m}\sum_{j=1}^{n} b_{ij}x_i^* y_j^* &= \max_{y\in S_2^*}\sum_{i=1}^{m}\sum_{j=1}^{n} b_{ij}x_i^* y_j,\end{aligned}$$

则称(x^*, y^*)为此双矩阵博弈的一个Nash 平衡（均衡）点. 此时，两局中人都不能通过单方改变自己的策略来获得更大的利益。

Theorem 8.11 (Nash, 1950) 在混合策略意义下，任何n $(n \geq 2)$ 人有限非零和非合作博弈必存在平衡点*(*在平衡点处，任何局中人均不能通过单方改变自己的策略来获得更大的利益*)*。

Example 8.7.1 (囚徒困境(Prisoner's Dilemma)) 属双矩阵博弈。局中人I 与II 的赢得矩阵分别为

$$\begin{pmatrix} -1 & -12 \\ 0 & -3 \end{pmatrix} \quad \textit{and} \quad \begin{pmatrix} -1 & 0 \\ -12 & -3 \end{pmatrix}.$$

纳什均衡点存在且唯一：双方均坦白，各获刑三个月。

大猪 \ 小猪	action	wait
action	(5,1)	(4,4)
wait	(9,-1)	(0,0)

Table 8.1: Win > Tie > Lose > Crash.

I \ II	swerve/退让	straight/进攻
swerve/退让	(Tie,Tie)	(Lose,Win)
straight/进攻	(Win,Lose)	(Crash,Crash)

Example 8.7.2 (智猪博弈(Nash, 1950)) 假设猪圈里有一头大猪、一头小猪。猪圈的一头有猪食槽，另一头安装着控制猪食供应的按钮，按一下按钮会有*10*个单位的猪食进槽，但是谁去按按钮就会首先付出*2*个单位的成本。若大猪先到槽边，大小猪吃到食物的收益比是*9:1;* 同时到槽边，收益比是*7:3;* 小猪先到槽边，收益比是*6:4;* 那么，在两头猪都有智慧的前提下，最终结果是小猪选择等待。 纳什均衡点存在且唯一：即大猪行动，小猪等待。

启示：小企业需学会如何“搭便车”，在某些时候如果能够注意等待，让其他大企业首先开发市场，是一种明智的选择。这时候有所不为才能有所为！

Example 8.7.3 (Game of Chicken) 两人狭路相逢，每人有两个行动选择：一是退下来，一是进攻。如果一方退下来，而对方没有退下来，对方获得胜利，这人就很丢面子；如果对方也退下来，双方则打个平手；如果自己没退下来，而对方退下来，自己则胜利，对方则失败；如果两人都前进，那么则两败俱伤。因此，对每个人来说，最好的结果是，对方退下来，而自己不退。

纳什均衡点存在, 但不唯一：一方退让，一放进攻! 如果一方理智，一方强硬，则理智的一方通常会吃亏。

Hawk-Dove (variant of game of Chicken) *Imagine that two players (animals) are contesting an indivisible resource and both can choose between two strategies: Hawk or Dove. Suppose V is the value of the contested resource, and C is the cost of an escalated fight.*

- *Nash equilibria exist but not unique: one plays hawk and the other plays dove.*
- *If $C \leq V$, the resulting game is not a game of Chicken but is instead a Prisoner's Dilemma. Which one is the equilibrium?*
- 纳什均衡点唯一的博弈，博弈结果可预测；否则，难以预测；但可以断言：不会发生鱼死网破、两败俱伤的情况。
- 在外交上的应用：政治博弈、强硬-让步/谈判

Table 8.2: Assume $C > V > 0$.

I \ II	Dove	Hawk
Dove	$(\frac{V}{2},\frac{V}{2})$	$(0,V)$
Hawk	$(V,0)$	$(\frac{V-C}{2},\frac{V-C}{2})$

8.7.2 二人无限零和博弈

至少有一个局中人的纯策略集为无限集（可数集、不可数集），赢得函数之和恒为零。此时，一个混合策略是指取每个纯策略的概率（分布/密度）。局中人I 的赢得函数可以为如下四种形式

$$
\begin{aligned}
&H(x,y)\\
&H(X,y)=\int_{S_1}H(x,y)\mathrm{d}F_X(x)\\
&H(x,Y)=\int_{S_2}H(x,y)\mathrm{d}F_Y(y)\\
&H(X,Y)=\int_{S_1}\int_{S_2}H(x,y)\mathrm{d}F_X(x)\mathrm{d}F_Y(y).
\end{aligned}
$$

Definition 8.12 如果有

$$
\sup_X\inf_Y H(X,Y)=\inf_Y\sup_X H(X,Y)=V_G,
$$

则称V_G为对策G 的值。称使上式成立的(X^*,Y^*) 为对策的解（平衡点），X^* 与Y^* 分别为两局中人的最优策略。

Theorem 8.13 (X^*,Y^*) 为对策$G=\{S_1,S_2;H\}$ 的解的充要条件为：对任意$X\in\bar{X}$, $Y\in\bar{Y}$, 都有

$$
H(X,Y^*)\leq H(X^*,Y^*)\leq H(X^*,Y).
$$

当$S_1=S_2=[0,1]$, $H(x,y)$ 连续时，称这样的对策为连续对策。

Theorem 8.14 对任何连续对策一定有$\sup_X\inf_Y H(X,Y)=\inf_Y\sup_X H(X,Y)$.

8.7.3 合作博弈

- 局中人可以结盟（合作）
- 博弈结束后，局中人需对赢得进行再分配
- 所有局中人可以形成若干联盟，每个局中人仅能参加一个联盟，联盟的所得需在联盟内再分配
- 一般地，合作可以提高联盟的所得，也可以提高每个局中人的所得
- 联盟能否形成，以何种形式存在，一个局中人是否参加联盟，以及参加哪个联盟，不仅取决于对策的规则，更取决于赢得的再分配方案是否合理
- 合作博弈主要研究联盟形成的条件，联盟的稳定性等
- 通俗地讲，如果某局中人参加一个联盟后的赢得不少于不参加联盟时的赢得，该局中人才会参加结盟。

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MATH: Operations Research **2014-15 First Term**

Handout 9: Mathematical preliminaries

Instructor: Junfeng Yang *November 17, 2014*

Contents

9.1 Vector and matrix norms

Definition 9.1 (vector norm) *A mapping $\|\cdot\| : R^n \to R$ is called a vector norm if and only if*

1. $\|x\| \geq 0$ *for all* $x \in R^n$*, and* $\|x\| = 0$ *if and only if* $x = 0$*;*
2. $\|\alpha x\| = |\alpha|\|x\|$ *for all* $\alpha \in R$ *and* $x \in R^n$*.*
3. $\|x + y\| \leq \|x\| + \|y\|$ *for all* $x, y \in R^n$*.*

Definition 9.2 (matrix norm) *A mapping $\|\cdot\| : R^{m\times n} \to R$ is a matrix norm if and only if*

1. $\|A\| \geq 0$ *for all* $A \in R^{m\times n}$*, and* $\|A\| = 0$ *if and only if* $A = 0$*;*
2. $\|\alpha A\| = |\alpha|\|A\|$ *for all* $\alpha \in R$ *and* $A \in R^{m\times n}$*.*
3. $\|A + B\| \leq \|A\| + \|B\|$ *for all* $A, B \in R^{m\times n}$*.*

Let $x = (x_1, x_2, \ldots, x_n)^T \in R^n$. Some well-known examples of vector norms:

$$\|x\|_2 = \sqrt{\sum_{i=1}^n x_i^2}; \qquad (\ell_2\text{-norm})$$

$$\|x\|_1 = \sum_{i=1}^n |x_i|; \qquad (\ell_1\text{-norm})$$

$$\|x\|_\infty = \max_{i=1,2,\ldots,n} |x_i|; \qquad (\ell_\infty\text{-norm})$$

$$\|x\|_p = \left(\sum_{i=1}^n |x_i|^p\right)^{\frac{1}{p}}, \quad \text{where } 1 \leq p \leq +\infty; \qquad (\ell_p\text{-norm})$$

$$\|x\|_A = \sqrt{x^T A x}, \quad \text{where } A \in S^n_{++}; \qquad \text{(ellipsoid-norm)}$$

($A \in S^n_{++}$ means A is positive definite.)

Let $A \in R^{n\times n}$. Some well-known examples of matrix norms:

$$\|A\|_p = \sup_{x\neq 0} \frac{\|Ax\|_p}{\|x\|_p}, \quad \text{where } 1 \leq p \leq +\infty; \qquad (\text{induced } \ell_p\text{-norm})$$

$$\|A\|_1 = \max\left\{\sum_{i=1}^n |a_{ij}| : j = 1, 2, \ldots, n\right\}; \qquad \text{(maximum column norm)}$$

$$\|A\|_\infty = \max\left\{\sum_{j=1}^n |a_{ij}| : i = 1, 2, \ldots, n\right\}; \qquad \text{(maximum row norm)}$$

$$\|A\|_2 = \sqrt{\lambda_{\max}(A^T A)}; \qquad \text{(spectral norm)}$$

($A \in S^n$ implies $\|A\|_2 = \rho(A) := \max\{|\lambda_i(A)| : \ i = 1, \ldots, n\}$.)

Suppose A is nonsingular, then it holds that

$$\|A^{-1}\|_p = \frac{1}{\inf_{x\neq 0} \frac{\|Ax\|_p}{\|x\|_p}};$$

$$\|A\|_F = \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}^2} = \sqrt{\text{tr}(A^T A)}; \qquad \text{(Frobenius norm)}$$

A vector norm $\|\cdot\|$ and a matrix norm $\|\cdot\|'$ are said to be consistent if, for every $A \in R^{n\times n}$ and $x \in R^n$,

$$\|Ax\| \leq \|A\|'\|x\|.$$

Obviously, the ℓ_p-norm has this property, i.e.,

$$\|Ax\|_p \leq \|A\|_p\|x\|_p.$$

Definition 9.3 (norm equivalence) *Let $\|\cdot\|_\alpha$ and $\|\cdot\|_\beta$ be two arbitrary vector norms on R^n. The two norms are said to be equivalent if there exist $\mu_1, \mu_2 > 0$ such that*

$$\mu_1\|x\|_\alpha \leq \|x\|_\beta \leq \mu_2\|x\|_\alpha, \forall\, x \in R^n.$$

From functional analysis, any two vector norms defined on a finite dimensional space are equivalent. In particular, we have

$$\begin{aligned}
\|x\|_2 \leq \|x\|_1 &\leq \sqrt{n}\|x\|_2 \\
\|x\|_\infty \leq \|x\|_2 &\leq \sqrt{n}\|x\|_\infty \\
\|x\|_\infty \leq \|x\|_1 &\leq n\|x\|_\infty \\
\|x\|_\infty \leq \|x\|_2 &\leq \|x\|_1 \\
\sqrt{\lambda_{\min}(A)}\|x\|_2 \leq \|x\|_A &\leq \sqrt{\lambda_{\max}(A)}\|x\|_2.
\end{aligned}$$

Let $A \in R^{m\times n}$. For matrix norms, the following inequalities hold:

$$\begin{aligned}
\|A\|_2 \leq \|A\|_F &\leq \sqrt{n}\|A\|_2 \\
\frac{1}{\sqrt{n}}\|A\|_\infty \leq \|A\|_2 &\leq \sqrt{m}\|A\|_\infty \\
\frac{1}{\sqrt{m}}\|A\|_1 \leq \|A\|_2 &\leq \sqrt{n}\|A\|_1 \\
\max_{i,j}|a_{ij}| \leq \|A\|_2 &\leq \sqrt{mn}\max_{i,j}|a_{ij}|.
\end{aligned}$$

9.2 Some important inequalities

1. Cauchy-Schwarz inequality: Let $x, y \in R^n$. It holds that

$$|x^T y| \leq \|x\|_2\|y\|_2.$$

"$=$" holds if and only if x and y are linearly dependent.

2. Let A be an $n \times n$ symmetric and positive definite matrix, $x, y \in R^n$, then

$$|x^T A y| \leq \|x\|_A \|y\|_A.$$

"=" holds if and only if x and y are linearly dependent.

3. Let A be an $n \times n$ symmetric and positive definite matrix, $x, y \in R^n$, then

$$|x^T y| \leq \|x\|_A \|y\|_{A^{-1}}.$$

"=" holds if and only if x and $A^{-1}y$ are linearly dependent.

4. Young inequality: Assume that $p, q \in R$, $p, q > 1$ satisfy $1/p + 1/q = 1$. For $a, b \in R$, it holds that

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q},$$

and "=" holds if and only if $a^p = b^q$.

5. Holder inequality: Let $x, y \in R^n$. It holds that

$$|x^T y| \leq \|x\|_p \|y\|_q,$$

where $p, q \in R$, $p, q > 1$ satisfy $1/p + 1/q = 1$.

6. Minkowski inequality: Let $x, y \in R^n$. For $p \geq 1$, it holds that

$$\|x + y\|_p \leq \|x\|_p + \|y\|_p.$$

9.3 Multivariate calculus

Continuity, differentiability, derivative and gradient. Let $f : R^n \to R$ be a mapping.

- f is said to be continuous on R^n if ...
- f is said to be continuously differentiable at $x \in R^n$ if $\frac{\partial f}{\partial x_i}(x)$ exists and is continuous, $i = 1, 2, \ldots, n$.
- The gradient of f at x is defined as

$$\nabla f(x) = \begin{pmatrix} \frac{\partial f}{\partial x_1}(x) \\ \vdots \\ \frac{\partial f}{\partial x_n}(x) \end{pmatrix}.$$

- The derivative of f at x is defined as

$$f'(x) = \left(\frac{\partial f}{\partial x_1}(x), \frac{\partial f}{\partial x_2}(x), \ldots, \frac{\partial f}{\partial x_n}(x) \right) = \nabla f(x)^T.$$

- If f is continuously differentiable at each $x \in R^n$, then f is said to be continuously differentiable on R^n and is denoted by $f \in C^1(R^n)$.

Directional derivative.

- Let $x, d \in R^n$. The directional derivative of f at x in the direction d is defined as

$$f'(x;d) := \lim_{\theta \to 0+} \frac{f(x+\theta d) - f(x)}{\theta} = \nabla f(x)^T d.$$

  It represents how fast f changes at x in the direction d.

- For any $x, x+d \in R^n$, if $f \in C^1(R^n)$, then

$$f(x+d) = f(x) + \int_0^1 \nabla f(x+td)^T d\, dt.$$

  ($h(t) = f(x+td)$. Then, $h(1) = h(0) + \int_0^1 h'(t)dt$.)

$$f(x+d) = f(x) + \nabla f(x+\theta d)^T d, \text{for some } \theta \in (0,1).$$

  ($h(1) = h(0) + h'(\theta)$ for some $\theta \in (0,1)$.)

$$f(x+d) = f(x) + \nabla f(x)^T d + o(\|d\|).$$

  ($h(1) = h(0) + h'(0) + o(1)$.)

Hessian matrix.

- A continuously differentiable function $f : R^n \to R$ is called twice continuously differentiable at $x \in R^n$ if $\frac{\partial^2 f}{\partial x_i \partial x_j}(x)$ exists and is continuous, $i, j = 1, 2, \ldots, n$.

- The Hessian matrix of f is defined as the $n \times n$ symmetric matrix with elements

$$\nabla^2 f(x) = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1 \partial x_1}(x) & \frac{\partial^2 f}{\partial x_1 \partial x_2}(x) & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n}(x) \\ \frac{\partial^2 f}{\partial x_2 \partial x_1}(x) & \frac{\partial^2 f}{\partial x_2 \partial x_2}(x) & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n}(x) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1}(x) & \frac{\partial^2 f}{\partial x_n \partial x_2}(x) & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_n}(x) \end{pmatrix}_{n \times n}$$

- If f is twice continuously differentiable at every point in R^n, then f is said to be twice continuously differentiable on R^n and is denoted by $f \in C^2$.

- Let $f \in C^2$. For any $x, d \in R^n$, the second directional derivative of f at x in the direction d is defined as

$$f''(x;d) = \lim_{\theta \to 0+} \frac{f'(x+\theta d; d) - f'(x;d)}{\theta},$$

  which is equal to $d^T \nabla^2 f(x) d$.

- For any $x, d \in R^n$, there exists $\theta \in (0,1)$ such that

$$f(x+d) = f(x) + \nabla f(x)^T d + \frac{1}{2} d^T \nabla^2 f(x+\theta d) d,$$

  or

$$f(x+d) = f(x) + \nabla f(x)^T d + \frac{1}{2} d^T \nabla^2 f(x) d + o(\|d\|^2).$$

Vector-valued functions.

- Let $F : R^n \to R^m$ be a vector-valued function, i.e.,

$$F(x) = \begin{pmatrix} f_1(x) \\ \vdots \\ f_m(x) \end{pmatrix}.$$

  F is continuously differentiable at $x \in R^n$ if each f_i is so.

- The derivative $F'(x) \in R^{m\times n}$ of F at x is called the Jacobian matrix of F at x and is defined by

$$\begin{aligned} F'(x) = J(x) &= \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(x) & \frac{\partial f_1}{\partial x_2}(x) & \dots & \frac{\partial f_1}{\partial x_n}(x) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(x) & \frac{\partial f_m}{\partial x_2}(x) & \dots & \frac{\partial f_m}{\partial x_n}(x) \end{pmatrix}_{m\times n} \\ &= \begin{pmatrix} f_1'(x) \\ \vdots \\ f_m'(x) \end{pmatrix} = \begin{bmatrix} \nabla f_1(x), & \nabla f_2(x), & \dots, & \nabla f_m(x) \end{bmatrix}^T. \end{aligned}$$

- If $F : R^n \to R^m$ is continuously differentiable in R^n, then for any $x, x+d \in R^n$, it holds that

$$F(x+d) - F(x) = \int_0^1 J(x+td)d\,dt.$$

Let $f : R^n \to R^m$ and $g : R^m \to R$ be continuously differentiable. Let $h = g \circ f : R^n \to R$. Then the chain rule is

$$h'(x) = g'(f(x))f'(x) \quad \text{or} \quad \nabla h(x) = \nabla f(x) \nabla g(f(x)),$$

where $\nabla f = (\nabla f_1, \nabla f_2, \dots, \nabla f_m) \in R^{n\times m}$.

For general mapping.

- Let E_1, E_2 be two finite dimensional Euclidean spaces;
- Let $\mathcal{M}(E_1, E_2)$ be the set of linear operators from E_1 to E_2.

Let $f : E_1 \longrightarrow E_2$ be a mapping from E_1 to E_2. Then, for any $x \in E_1$, $f'(x)$ is a linear operator from E_1 to E_2, that is

$$f' : E_1 \longrightarrow \mathcal{M}(E_1, E_2).$$

Therefore,

$$\begin{aligned} f'' : \qquad & E_1 \longrightarrow \mathcal{M}\Big(E_1, \mathcal{M}(E_1, E_2)\Big) \\ f''' : E_1 & \longrightarrow \mathcal{M}\Big(E_1, \mathcal{M}\Big(E_1, \mathcal{M}(E_1, E_2)\Big)\Big), \qquad \dots \end{aligned}$$

Assume differentiability. For any $x, d \in E_1$, it holds that

$$f(x+d) = f(x) + \int_0^1 f'(x+td)d\,dt.$$

$$f(x+d) = f(x) + f'(x)d + \frac{1}{2!}f''(x)[dd] + \dots + \frac{1}{n!}f^{(n)}(x)[d^n] + \dots$$

9.4 Convex sets

Definition 9.4 (凸集) 设$C \subset R^n$. 称C为凸集，如果对$\forall x \in C, y \in C$ 以及$\forall \alpha \in (0,1)$, 都有

$$\alpha x + (1-\alpha)y \in C.$$

R^n中凸集的性质：

1. If C is a convex set and $\beta \in R$, the set $\beta C := \{\beta x : x \in C\}$ is convex.
2. If C and D are convex sets, then the set

$$C + D = \{x + y : x \in C, y \in D\}$$

   is convex.
3. The intersection of any collection of convex sets is convex.

Definition 9.5 (凸组合) 设$x_i \subset R^n$, $\alpha_i \in [0,1]$, $i = 1, 2, \ldots, k$, 并且$\sum_{i=1}^{k} \alpha_i = 1$, 其中$k$为任意正整数。称$x = \sum_{i=1}^{k} \alpha_i x_i$ 为$x_1, x_2, \ldots, x_k$的凸组合。

Definition 9.6 (凸包) 设S 为R^n 的子集。R^n 中包含S 的最小的凸集称为S 的凸包，记为$conv(S)$.

Theorem 9.7 (凸包的等价刻画) $conv(S)$由S中点的所有凸组合组成，即

$$conv(S) = \left\{ x = \sum_{i=1}^{k} \alpha_i x_i \in R^n \;\middle|\; \begin{array}{l} x_i \in S; \\ \alpha_i \in [0,1], i = 1, 2, \ldots, k, \\ \sum_{i=1}^{k} \alpha_i = 1; \\ k \text{ is any positive integer.} \end{array} \right\}$$

9.5 Convex functions

Definition 9.8 *Let $C \subset R^n$ be a nonempty convex set and $f : C \to R$.*

- *f is said to be convex on C if*

$$f(\alpha x + (1-\alpha)y) \le \alpha f(x) + (1-\alpha) f(y)$$

  holds for all $x, y \in C$ and $\alpha \in (0,1)$.

- *f is said to be strictly convex on C if*

$$f(\alpha x + (1-\alpha)y) < \alpha f(x) + (1-\alpha) f(y)$$

  holds for all $x, y \in C$, $x \neq y$, and $\alpha \in (0,1)$.

- *f is said to be strongly (or uniformly) convex on C if there exists $\mu > 0$ such that*

$$f(\alpha x + (1-\alpha)y) \le \alpha f(x) + (1-\alpha) f(y) - \frac{1}{2}\mu\alpha(1-\alpha)\|x-y\|^2$$

  holds for all $x, y \in C$ and $\alpha \in (0,1)$.

Theorem 9.9 *Let $C \subset R^n$ be a nonempty open convex set and let $f : C \to R$ be a differentiable function. Then*

- *f is convex if and only if*
$$f(y) \geq f(x) + \nabla f(x)^T (y - x), \ \forall\, x, y \in C.$$

- *f is strictly convex if and only if*
$$f(y) > f(x) + \nabla f(x)^T (y - x), \ \forall\, x, y \in C, x \neq y.$$

- *f is strongly (or uniformly) convex if and only if*
$$f(y) \geq f(x) + \nabla f(x)^T (y - x) + \frac{1}{2}\mu \|x - y\|^2, \ \forall\, x, y \in C,$$
where $\mu > 0$ is a constant.

Proof:

- Convex case. Necessity. For $x, y \in C$, $\alpha \in (0, 1)$, it holds that
$$f(\alpha y + (1 - \alpha)x) \leq \alpha f(y) + (1 - \alpha) f(x),$$
or, equivalently,
$$f(x + \alpha(y - x)) - f(x) \leq \alpha(f(y) - f(x)).$$
Dividing α and letting $\alpha \to 0+$ yield the result.
Sufficiency. Take $x, y \in C$, $\alpha \in (0, 1)$ and let $z = \alpha x + (1 - \alpha)y$.
$$f(x) \geq f(z) + \nabla f(z)^T (x - z)$$
$$f(y) \geq f(z) + \nabla f(z)^T (y - z).$$
α times first plus $1 - \alpha$ times second implies that f is convex.

- Strictly convex case. Necessity. Let $x, y \in C$ ($x \neq y$) and $\alpha \in (0, 1)$. Strict convexity of f implies that
$$f(x + \alpha(y - x)) - f(x) < \alpha(f(y) - f(x)).$$
The results follows from
$$f(x + \alpha(y - x)) - f(x) \geq \alpha \nabla f(x)^T (y - x).$$
Sufficiency. Take $x, y \in C$ ($x \neq y$), $\alpha \in (0, 1)$ and let $z = \alpha x + (1 - \alpha)y$. Clearly $z \neq x$ and $z \neq y$.
$$f(x) > f(z) + \nabla f(z)^T (x - z)$$
$$f(y) > f(z) + \nabla f(z)^T (y - z).$$
α times first plus $1 - \alpha$ times second implies that f is strictly convex.

- Strongly (uniformly) convex case. First, it is easy to show that f is strongly convex with constant $\mu > 0$ if and only if $g = f - \frac{\mu}{2}\| \cdot \|^2$ is convex, i.e.,
$$g(y) \geq g(x) + \nabla g(x)^T (y - x), \ \forall\, x, y \in C,$$
or, equivalently,
$$f(y) \geq f(x) + \nabla f(x)^T (y - x) + \frac{1}{2}\mu \|x - y\|^2, \ \forall\, x, y \in C.$$

■

Theorem 9.10 (Convexity meets differentiability) *Let $C \subset R^n$ be a nonempty open convex set and let $f : C \to R$ be a differentiable function. Then*

- *f is convex if and only if*
$$\langle \nabla f(x) - \nabla f(y), x - y\rangle \geq 0, \ \forall\, x, y \in C.$$
- *f is strictly convex if and only if*
$$\langle \nabla f(x) - \nabla f(y), x - y\rangle > 0, \ \forall\, x, y \in C, x \neq y.$$
- *f is strongly (or uniformly) convex if and only if*
$$\langle \nabla f(x) - \nabla f(y), x - y\rangle \geq \mu\|x - y\|^2, \ \forall\, x, y \in C,$$
where $\mu > 0$ is a constant.

Proof:

- Necessity for all three cases. Suppose f is strongly convex with constant $\mu > 0$. Then, for any $x, y \in C$, it holds that
$$f(y) \geq f(x) + \nabla f(x)^T(y - x) + \frac{1}{2}\mu\|x - y\|^2,$$
$$f(x) \geq f(y) + \nabla f(y)^T(x - y) + \frac{1}{2}\mu\|x - y\|^2.$$
Addition of the two yields
$$\langle \nabla f(x) - \nabla f(y), x - y\rangle \geq \mu\|x - y\|^2.$$
If f is only convex, let $\mu = 0$ in the above. If f is strictly convex, let $\mu = 0$ and $x \neq y$, then "$\geq$" can be replaced by "$>$" in the above three.

- Sufficiency for convex case. For any fixed $x, y \in C$, it holds that
$$f(y) - f(x) = \nabla f(\xi)^T(y - x),$$
where $\xi = x + t(y - x)$ for some $t \in (0, 1)$. Thus
$$\langle \nabla f(\xi) - \nabla f(x), y - x\rangle = \frac{1}{t}\langle \nabla f(\xi) - \nabla f(x), \xi - x\rangle \geq 0,$$
and
$$f(y) - f(x) = \nabla f(\xi)^T(y - x) \geq \nabla f(x)^T(y - x),$$
i.e., f is convex.

- Sufficiency for strictly convex case. Let $x \neq y$ and replace "$\geq$" by "$>$" in the above.

- Sufficiency for strongly convex case. Let
$$\phi(t) = f(x + t(y - x)) = f(u),$$
where $u = x + t(y - x)$, $t \in (0, 1)$. Note that
$$\phi'(t) = \langle \nabla f(u), y - x\rangle, \quad \phi'(0) = \langle \nabla f(x), y - x\rangle,$$
$$\phi'(t) - \phi'(0) = \frac{1}{t}\langle \nabla f(u) - \nabla f(x), u - x\rangle \geq \frac{\mu}{t}\|u - x\|^2 = t\mu\|y - x\|^2.$$
$$\phi(1) - \phi(0) - \phi'(0) = \int_0^1 (\phi'(t) - \phi'(0))dt \geq \frac{\mu}{2}\|y - x\|^2,$$
which, by the definition of ϕ, implies that f is strongly convex.

■

Notation: $A \succeq B$ or $A - B \succeq 0$ means that $A - B$ is positive semi-definite; $A \succ B$ or $A - B \succ 0$ means that $A - B$ is positive definite.

Theorem 9.11 (Convexity meets second order differentiability) *Let $C \subset R^n$ be a nonempty open convex set, and let $f : C \to R$ be twice continuously differentiable. Then*

- *f is convex if and only if $\nabla^2 f(x) \succeq 0$ for any $x \in C$.*
- *f is strictly convex if $\nabla^2 f(x) \succ 0$ for any $x \in C$.*
- *f is uniformly convex with constant $\mu > 0$ if and only if*

$$d^T \nabla^2 f(x) d \geq \mu \|d\|^2, \ \forall \ x \in C, d \in R^n,$$

i.e., $\nabla^2 f$ is uniformly positive definite in C (the minimum eigenvalue of $\nabla^2 f(x)$ is greater or equal to μ for all $x \in C$).

Proof: Hints: Consider either

$$f(y) = f(x) + \nabla f(x)^T (y - x) + \frac{1}{2}(y - x)^T \nabla^2 f(\xi)(y - x),$$

where $\xi \in (x, y)$, or

$$f(y) = f(x) + \nabla f(x)^T (y - x) + \frac{1}{2}(y - x)^T \nabla^2 f(x)(y - x) + o(\|y - x\|^2).$$

■

9.6 Convex optimization

Let $f : R^n \to R$ and $X \subset R^n$. An optimization problem in general form is as follows

$$\begin{aligned} \min_x \quad & f(x) \\ s.t. \ & x \in X. \end{aligned}$$

Definition 9.12 (convex optimization) *The optimization problem $\min\{f(x) : s.t. \ x \in X\}$ is called a convex optimization problem if $X \subset R^n$ is a convex set and $f : X \to R$ is a convex function.*

Definition 9.13 (local minimizer) *1. A point $x^* \in X$ is called a local minimizer if there exists $\delta > 0$ such that*

$$f(x^*) \leq f(x), \forall x \in X \cap \{x \in R^n : \|x - x^*\| < \delta\}.$$

2. A point $x^* \in X$ is called a strict local minimizer if there exists $\delta > 0$ such that

$$f(x^*) < f(x), \forall x \in X \cap \{x \in R^n : \|x - x^*\| < \delta\}, \ x \neq x^*.$$

Definition 9.14 (global minimizer) *1. $x^* \in X$ is a global minimizer if $f(x^*) \leq f(x)$ for all $x \in X$.*

2. *$x^* \in X$ is a strict global minimizer if $f(x^*) < f(x)$ for all $x \in X$, $x \neq x^*$.*

For general optimization problem, it is usually very difficult to find a global optimal solution since any iterative algorithms tend to be trapped around local optimal solutions. The most important feature of convex optimization is that any local optimal solution is also globally optimal.

Theorem 9.15 *Any local optimal solution of a convex optimization problem is also globally optimal.*

9.7 Homework

1. Let $f : R^n \to R$. The epigraph of f is defined by

$$epi(f) := \{(z, x) :\ f(x) \leq z, z \in R, x \in R^n\}.$$

   Prove that f is convex if and only if $epi(f)$ is a convex subset of $R \times R^n$.

2. Suppose $f : R^n \to R$ is a convex function.

   (a) For $z \in R$, the level set of f is defined by

$$L(z) := \{x \in R^n :\ f(x) \leq z\}.$$

   Prove that, if nonempty, $L(z)$ must be convex.

   (b) Define $g : R_{++} \times R^n \to R$ by

$$g(t, x) := t\, f(x/t)$$

   Prove that $g(t, x)$ is convex in (t, x).

3. Let $A \in R^{m \times n}$ and $g : R^m \to R$ be continuously differentiable. Let $h(x) = g(Ax)$ for $x \in R^n$. Prove that

$$h'(x) = g'(Ax)A \quad \text{or} \quad \nabla h(x) = A^T \nabla g(Ax).$$

MATH: Operations Research **2014-15 First Term**

Handout 10: Nonlinear Programming: an Introduction

Instructor: Junfeng Yang *November 17, 2014*

Contents

10.1 Nonlinear Programming Models

Optimization in general form ($f : R^n \to R$, $X \subset R^n$)

$$\begin{aligned} \min \quad & f(x) \\ s.t. \; & x \in X. \end{aligned}$$

Linear programming

$$\begin{aligned} \min \quad & c^T x \\ s.t. \; & Ax = b \\ & x \geq 0. \end{aligned}$$

Nonlinear programming ($f, c_i : R^n \to R$, $i = 1, 2, \ldots, m$)

$$\begin{aligned} \min \quad & f(x) \\ s.t. \; & c_i(x) \leq 0, \, i = 1, 2, \ldots, m, \end{aligned}$$

or

$$\begin{aligned} \min \quad & f(x) \\ s.t. \; & c_i(x) = 0, \, i = 1, 2, \ldots, m_e, \\ & c_i(x) \leq 0, \, i = m_e + 1, 2, \ldots, m, \end{aligned}$$

where at least one of the functions is nonlinear.

Unconstrained optimization $\min_{x \in R^n} f(x)$.

Convex optimization The optimization problem $\min\{f(x) : s.t. \; x \in X\}$ is called a convex optimization problem if $X \subset R^n$ is a convex set and $f : X \to R$ is a convex function.

A commonly studied form of convex optimization

$$\begin{aligned} \min \quad & f(x) \\ s.t. \quad & Ax = b, \\ & c_i(x) \leq 0, \, i = 1, 2, \ldots, m, \end{aligned}$$

where all $c_i : R^n \to R$, $i = 1, 2, \ldots, m$, are convex functions.

Objectives for different problems

- Convex optimization: seek a globally optimal solution;
- General nonlinear programming (objective or constraint functions are not known to be convex): global optimization is too ambitious and local optimization is a compromise that has to be taken. Sometimes, only a KKT/stationary point can be guaranteed.

Things to learn

- Theory: optimality conditions, duality theory
- Study of various numerical algorithms (construction of algorithms, convergence and numerical performance, etc.)
- Applications

10.2 Optimality conditions for unconstrained optimization

Let $f : R^n \to R$. We consider unconstrained optimization

$$\min_{x \in R^n} f(x).$$

We focus on the class of continuously differentiable functions, i.e., $f \in C^1(R^n)$. Most algorithms are constructed based on derivatives (gradient, Hession). Direct algorithms, which do not use derivative information, are also very useful in practical applications.

For any $x, d \in R^n$, the Taylor's expansion tells that

$$f(x + td) = f(x) + t\nabla f(x)^T d + o(t),$$

from which we can see that

$$\exists \delta > 0 \text{ such that } f(x + td) < f(x), \ \forall t \in (0, \delta),$$

if and only if $\nabla f(x)^T d < 0$.

Definition 10.1 (descent direction) *Let $f \in C^1(R^n)$ and $x \in R^n$. A vector $d \in R^n$ is called a descent direction of f at x if*

$$\nabla f(x)^T d < 0.$$

Theorem 10.2 (First order necessary condition) *Let $f \in C^1(R^n)$. If $x^* \in R^n$ is a local minimizer of f, then $\nabla f(x^*) = 0$.*

Proof: Since $x^* \in R^n$ is a local minimizer of f, there is no descent direction at x^*, i.e., for any $d \in R^n$, it holds that

$$\nabla f(x^*)^T d \geq 0.$$

Setting $d = -\nabla f(x^*)$ completes the proof. ■

Definition 10.3 (stationary point) *A point $x^* \in R^n$ is said to be a stationary (or critical) point of a differentiable function f if $\nabla f(x^*) = 0$.*

Theorem 10.4 (Second order necessary condition) *Let $f \in C^2(R^n)$, i.e., $f : R^n \to R$ is twice continuously differentiable. If $x^* \in R^n$ is a local minimizer of f, then $\nabla f(x^*) = 0$ and $\nabla^2 f(x^*) \succeq 0$.*

Proof: From Taylor's expansion, for any $\alpha > 0$ and $d \in R^n$, it holds that

$$\frac{f(x^* + \alpha d) - f(x^*)}{\alpha^2} = \frac{1}{2} d^T \nabla^2 f(x^*) d + o(1).$$

If there exist $d \in R^n$ such that $d^T \nabla^2 f(x^*) d < 0$, then

$$\frac{1}{2} d^T \nabla^2 f(x^*) d + o(1) < 0$$

for $\alpha > 0$ sufficiently small, in which case $f(x^* + \alpha d) < f(x^*)$. This contradicts to the fact that x^* is a local minimizer of f. Thus, $\nabla^2 f(x^*) \succeq 0$. ■

Theorem 10.5 (Second order sufficient condition) *Let $f \in C^2(R^n)$. If $\nabla f(x^*) = 0$ and $\nabla^2 f(x^*) \succ 0$, then x^* is a strict local minimizer of f.*

Proof: Since $f \in C^2(R^n)$ and $\nabla^2 f(x^*) \succ 0$, there exist $\delta > 0$ such that $\nabla^2 f(x) \succ 0$ in $B(x^*, \delta)$ (open neighborhood of x^* with radius δ). For any $d \in R^n$, $0 < \|d\| < \delta$, it holds that

$$f(x^* + d) = f(x^*) + \frac{1}{2} d^T \nabla^2 f(x^* + \theta d) d,$$

for some $\theta \in (0, 1)$. Since $\nabla^2 f(x^* + \theta d) \succ 0$, it is clear that

$$f(x^* + d) > f(x^*),$$

which implies that x^* is a strict local minimizer of f. ∎

For convex function, we have stronger results.

Theorem 10.6 *Let $C \subset R^n$ be a nonempty convex set and $f : C \to R$. Suppose $x^* \in C$ is a local minimizer for $\min_{x \in C} f(x)$. Then*

- *If f is convex, then x^* is also a global minimizer.*
- *If f is strictly convex, then x^* is a unique global minimizer.*

Theorem 10.7 *Let $f : R^n \to R$ be a differentiable convex function. Then x^* is a global minimizer if and only if $\nabla f(x^*) = 0$.*

Proof: Since f is differentiable and convex, for any $x \in R^n$, it holds that

$$f(x) \geq f(x^*) + \nabla f(x^*)^T (x - x^*) = f(x^*).$$

This completes the proof of sufficiency. ∎

10.3 Structure of optimization algorithms

Most optimization algorithms are iterative in nature; Starting at an initial point x_0, an algorithm generates a sequence of points $\{x_k : k = 1, 2, \ldots\}$; The sequence is either finite or infinite. If finite, the last point is the solution/stationary point of the problem; If infinite, generally any limit point of the sequence is a solution of the problem; A desirable optimization algorithm should be able to (1) approach a solution point stably when the current point is far away; and (2) converge quickly to a solution when already close to one. These two points corresponds to global convergence and local convergence of an algorithm.

For unconstrained optimization, the structure of an algorithm is generally as follows.

Algorithm 1 (structure of unconstrained optimization algorithm of line search type) *Initialization: provide initial point x_0, algorithmic parameters, etc.*

1. *Find d_k satisfies $\nabla f(x_k)^T d_k < 0$;*
2. *Find $\alpha_k > 0$ such that $f(x_k + \alpha_k d_k) < f(x_k)$.*
3. *Check stopping criterion. If satisfied, stop; otherwise, repeat.*

This is the structure of line search type methods. Trust region type methods are different.

10.4 Step size rules / Line search

Notation. Use subscript k to count iteration number, i.e., a sequence of points generated by an algorithm will be denoted by $\{x_k : k = 1, 2, \ldots\}$. For simplicity the gradient of f at x is sometimes denoted by $g(x)$, i.e., $g(x) = \nabla f(x)$. Thus, sometimes $\nabla f(x_k)$ is denoted by g_k, i.e., $g_k := \nabla f(x_k)$. Also, occasionally $f(x_k)$ is shortened as f_k, i.e., $f_k := f(x_k)$.

Suppose d_k is a descent direction at the current point x_k, i.e., $g_k^T d_k < 0$ and the iteration formula is

$$x_{k+1} = x_k + h_k d_k.$$

There exist a few step size rules to determine h_k.

1. Predetermined: for examples

$$h_k = h > 0 \text{ (constant step)}, \quad h_k = \frac{h}{\sqrt{k+1}}.$$

(simple, mainly used in gradient method applied to convex and Lipschitz problems.)

2. Exact line search: Find $h_k > 0$ such that

$$h_k = \arg\min_{h \geq 0} f(x_k + h d_k).$$

Recall that $g_{k+1} = \nabla f(x_{k+1}) = \nabla f(x_k + h_k d_k)$. Consequence: $g_{k+1}^T d_k = 0$, i.e., g_{k+1} is perpendicular to d_k. For gradient method, it holds that $d_k = -g_k$ and thus

$$g_{k+1}^T g_k = 0.$$

Exact line search is mainly studied theoretically and rarely used in practice.

3. Goldstein-Armijo line search rule. Let α and β be given parameters which satisfy $0 < \alpha < \beta < 1$. The Goldstein-Armijo rule determines a step size h_k such that

$$\begin{aligned} f(x_{k+1}) &\leq f(x_k) + \alpha h_k g_k^T d_k, \\ f(x_{k+1}) &\geq f(x_k) + \beta h_k g_k^T d_k. \end{aligned}$$

Let $\phi(h) = f(x_k + h d_k)$. The two conditions are equivalent to

$$\begin{aligned} \phi(h_k) &\leq \phi(0) + \alpha \phi'(0) h_k, \quad \text{(suffiicent decrease)} \\ \phi(h_k) &\geq \phi(0) + \beta \phi'(0) h_k, \quad (h_k \text{ not too small}). \end{aligned}$$

Such h_k exists unless $\phi(h)$ ($h \geq 0$) is unbounded below.

4. Wolfe-Powell line search rule. Let $\gamma \in (\alpha, 1)$. The Wolfe-Powell rule determines a step size h_k such that

$$\begin{aligned} f(x_{k+1}) &\leq f(x_k) + \alpha h_k g_k^T d_k, \\ g_{k+1}^T d_k &\geq \gamma g_k^T d_k. \end{aligned}$$

The second condition is equivalent to $\phi'(h_k) \geq \gamma \phi'(0)$. Suppose $\hat{h}_k > 0$ satisfies $f(x_k + \hat{h}_k d_k) = f(x_k) + \alpha \hat{h}_k g_k^T d_k$. Then,

$$\hat{h}_k \nabla f(x_k + \theta_k \hat{h}_k d_k)^T d_k = f(x_k + \hat{h}_k d_k) - f(x_k) = \alpha \hat{h}_k g_k^T d_k.$$

Since $\gamma > \alpha$, the above implies that $h_k := \theta_k \hat{h}_k$ satisfies the second condition.

5. Strong Wolfe-Powell line search rule. The strong Wolfe-Powell rule determines a step size h_k such that

$$f(x_{k+1}) \le f(x_k) + \alpha h_k g_k^T d_k,$$
$$|g_{k+1}^T d_k| \le \gamma |g_k^T d_k|.$$

Theoretically, $\gamma \to 0$ implies exact line search.

6. Backtracking line search. Let $0 < \delta < 1$. Initialize $h_k = \hat{h} > 0$ (e.g., $\hat{h} = 1$), repeat $h_k = \delta \hat{h}$ until

$$f(x_{k+1}) \le f(x_k) + \frac{h_k}{2} g_k^T d_k.$$

(easy to be realized and frequently used in practice.)

7. Curvilinear search. Define a curve $\{x_k(h) : h \ge 0\}$ at x_k which satisfies

$$\frac{\mathrm{d} f(x_k(h))}{\mathrm{d} h}|_{h=0} < 0.$$

At iteration k, search along the curve $\{x_k(h) : h \ge 0\}$ and determine $h_k > 0$ such that certain decrease conditions are satisfied.

8. Nonmonotone line search. Let $0 < \delta < 1$. Initialize $h_k = \hat{h} > 0$ (e.g., $\hat{h} = 1$), repeat $h_k = \delta \hat{h}$ until

$$f(x_{k+1}) \le C_k + \frac{h_k}{2} g_k^T d_k,$$

where $C_k := \max\{f_k, f_{k-1}, \ldots, f_{k-m+1}\}$ and m is a predetermined positive integer.

References

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