Feynman on Fermat's Last Theorem

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Richard Feynman was probably one of the most talented physicists of the 20th century. He was known for having a tremendous mathematical and physical intuition that allowed him to deconstruct complex concepts and approach problems from first principles. There are countless anecdotes that show Feynman's genius, from his ability as an undergrad at MIT to use his own methods to solve seemingly untreatable integrals to coming up with his own derivation of the Schrödinger equation as a grad student in Princeton. While reading more about Feynman's derivation of the Schrödinger equation in Schweber's book QED and the Men who made it I ended up finding a mention to an undated two-page manuscript written by Feynman about Fermat's Last Theorem. The manuscript doesn't appear in the book but Schweber casts some light on Feynman's approach which I will try to explain here in more detail.

Fermat claimed in the 17th century that if n is a positive integer greater than 2, the equation $x^n + y^n = z^n$ does not admit integer non-trivial solutions, i.e. a solution where all three x, y and z are non-zero. This statement is universally known as "Fermat's Last Theorem" (or FLT), and the equation therein is called "the Fermat equation".

During more than three and half centuries this difficult problem received the attention of many mathematicians of great fame, such as, among others, L. Euler, Legendre, P.G.L. Dirichlet, E.E. Kummer, and more recently D.R. Heath-Brown, G. Frey, and A. Wiles, who finally solved the problem.

Schweber doesn't mention the date of the manuscript but since Feynman died in 1988 and Andrew Wiles published the proof of the Theorem in 1995 we know that when Feynman wrote it FLT was still one of the most famous open problems in mathematics. What's interesting about the manuscript is that Feynman's approach to the problem is purely probabilistic. He starts by

calculating the probability that a number N is a perfect n^{th} power. To do this we need to calculate the distance between $\sqrt[n]{N}$ and $\sqrt[n]{N+1}$, where N is a large integer (I will explain later why we are doing this)
$d = \sqrt[n]{N+1} - \sqrt[n]{N} = \sqrt[n]{N}\sqrt[n]{1+\frac{1}{N}} - \sqrt[n]{N} = \sqrt[n]{N}\left(\sqrt[n]{1+\frac{1}{N}} - 1\right)$ If we now use the power expansion $(1+x)^k = 1 + kx + \frac{k(k-1)}{2}x^2 + \ldots$ for $-1 < x < 1$ $d = \sqrt[n]{N}\left(\left(1 + \frac{1}{n}\frac{1}{N} + \frac{\frac{1}{n}(\frac{1}{n}-1)}{2}\frac{1}{N^2} + \ldots\right) - 1\right)$ where $k = \frac{1}{n}$ and $x = \frac{1}{N}$. Note that we can use the power expansion since $\frac{1}{N} < 1$. Taking the limit $N \to \infty$ and preserving only the larger terms of the expansion we end up with $d \approx \frac{\sqrt[n]{N}}{nN}$

Note that $d \approx \frac{\sqrt[n]{N}}{nN} = \frac{1}{n\underbrace{\sqrt[n]{N}\ldots\sqrt[n]{N}}_{n-1 \text{ times}}} < 1$ since $n > 1$, $\sqrt[n]{N} > 1$ and so $n\sqrt[n]{N}\ldots\sqrt[n]{N} > 1$.

Feynman then writes "the probability that N is a perfect n^{th} power is $\frac{\sqrt[n]{N}}{nN}$". He didn't explain how he got to this conclusion so here is what I think his thought process was. If N is a perfect power $N = z^n$, there exists at least one integer ($\sqrt[n]{N} = z$) in the interval $[\sqrt[n]{N}, \sqrt[n]{N+1}]$. Since the distance between all consecutive integers is 1 the probability that $[\sqrt[n]{N}, \sqrt[n]{N+1}]$ contains an integer is the ratio of the length of the intervals between two integers and the distance between $\sqrt[n]{N}$ and $\sqrt[n]{N+1}$: $\frac{d}{1}$. A good way to visualize this is imagining a line where the distance between all consecutive integers is 1 meter. If someone drops a ruler of length d meter on top of the line the probability the ruler "hits" an integer is $\frac{d \text{ meter}}{1 \text{meter}} = d \approx \frac{\sqrt[n]{N}}{nN}$.

Now in the case of FLT, $N = x^n + y^n$ and so the probability that $x^n + y^n$ is a perfect perfect n^{th} power is $\frac{\sqrt[n]{x^n+y^n}}{n(x^n+y^n)}$. Of course this probability is for a specific x and y so if we want to calculate the total probability for any $x^n + y^n$ we need to sum over all $x > \mathsf{x}_0$ and $y > \mathsf{y}_0$. Feynman chose to integrate the expression instead of summing it. My assumption is that he chose integrals because they are normally easier to handle than sums and the final result wasn't going to be affected if instead of summing over integers we just integrate over all x and y.

Feynman also chose to do $\mathsf{x}_0 = \mathsf{y}_0$. He ends up with the following expression:

$$\int_{\mathsf{x}_0}^{\infty}\int_{\mathsf{x}_0}^{\infty}\frac{1}{n}(x^n+y^n)^{-1+\frac{1}{n}}dx$$

$$dy = \frac{1}{n\mathsf{x}_0^{n-3}}c_n$$

$$c_n = \int_0^\infty \int_0^\infty (u^n + v^n)^{-1+\frac{1}{n}} du dv$$

To obtain c_n Feynman performs 2 changes of variables. The first one is $\theta = \frac{x - x_0}{x_0}$ $\phi = \frac{y - x_0}{x_0}$

Doing the first change of variables:

$$\int_{\theta(x_0)}^\infty \int_{\phi(x_0)}^\infty f(x(\theta,\phi), y(\theta,\phi)) \left| \frac{\partial(x,y)}{\partial(\theta,\phi)} \right| d\theta d\phi =$$
$$= \int_0^\infty \int_0^\infty \frac{1}{n} x_0^{1-n} ((\theta+1)^n + (\phi+1)^n)^{-1+\frac{1}{n}} x_0^2 d\theta d\phi =$$
$$= \frac{1}{n x_0^{n-3}} \int_0^\infty \int_0^\infty ((\theta+1)^n + (\phi+1)^n)^{-1+\frac{1}{n}} d\theta d\phi$$

where $\left| \frac{\partial(x,y)}{\partial(\theta,\phi)} \right| = \frac{\partial x}{\partial \theta} \frac{\partial y}{\partial \phi} - \frac{\partial x}{\partial \phi} \frac{\partial y}{\partial \theta} = x_0^2$

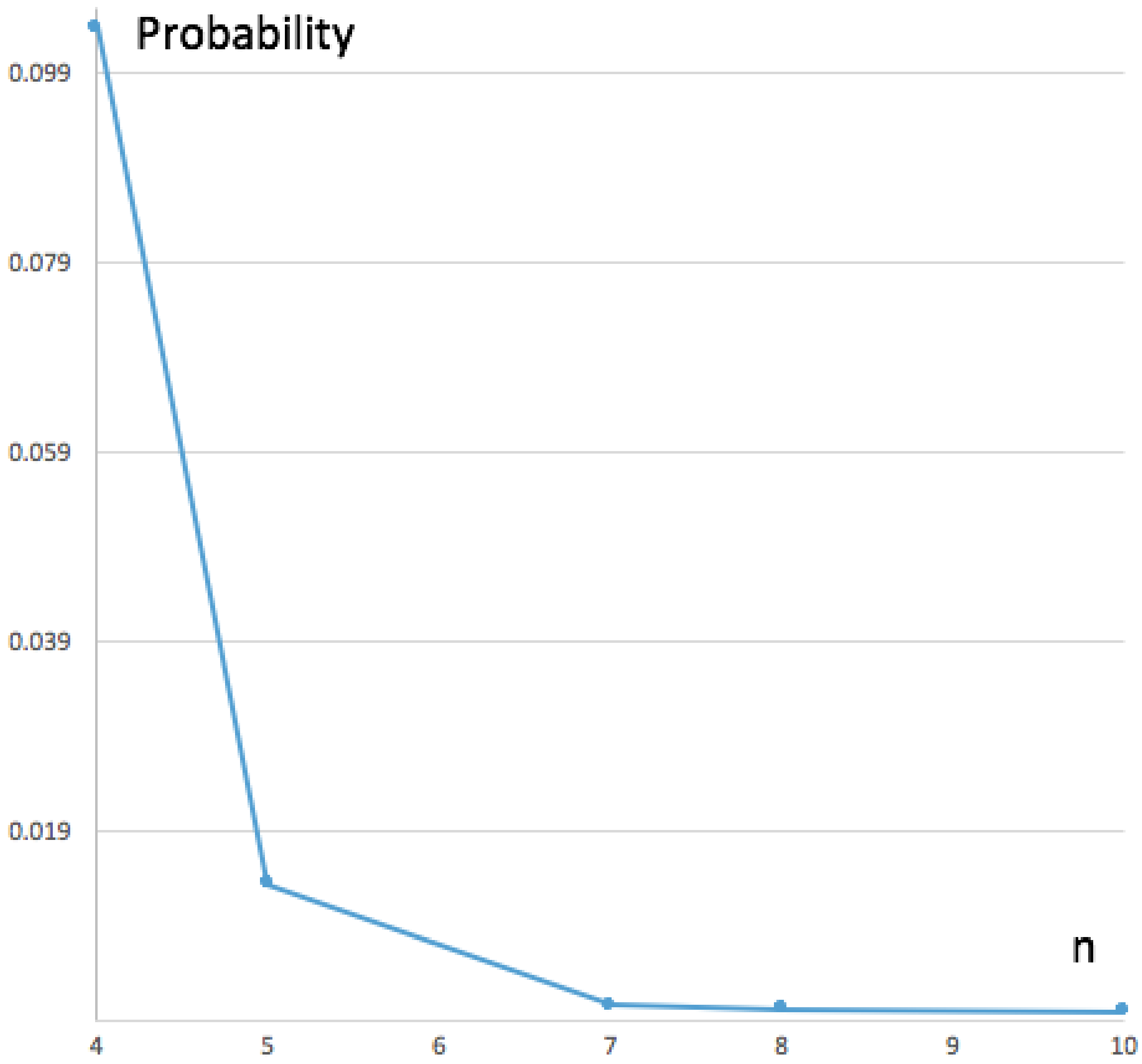
is the Jacobian and $\theta(x_0) = \frac{x_0 - x_0}{x_0} = 0$ $\phi(x_0) = \frac{x_0 - x_0}{x_0} = 0$.

Finally we do the second change of variables $u = \theta + 1$ and $v = \phi + 1$

$$\frac{1}{n x_0^{n-3}} \int_0^\infty \int_0^\infty ((\theta+1)^n + (\phi+1)^n)^{-1+\frac{1}{n}} d\theta d\phi =$$
$$= \frac{1}{n x_0^{n-3}} \int_1^\infty \int_1^\infty (u^n + v^n)^{-1+\frac{1}{n}} du dv$$

I think there's actually a typo in the lower limits of the integral (c_n) that Feynman derived as they should be 1's and not 0's. Note that $u(0) = 0 + 1 = 1$ and $v(0) = 0 + 1 = 1$.

Finally we got an expression for the probability that $z^n = x^n + y^n$ is an integer and we can calculate it for several n's. Setting $x_0 = 2$ we can see that the probability of there being integer solutions to $z^n = x^n + y^n$ ($\frac{1}{n x_0^{n-3}} \int_1^\infty \int_1^\infty (u^n + v^n)^{-1+\frac{1}{n}} du\, dv$) does decrease with increasing n.



Feynman also knew about Sophie Germain's result, who proved in the early 19th century that Fermat's equation has no solution for $n \leq 100$. Since it gets more and more difficult to find a solution as n increases, Feynman tried to calculate the probability of finding a solution to Fermat's equation using the knowledge that there's none for $n \leq 100$.

For sufficiently large n (I invite readers to derive this limit)

$c_n \approx \frac{1}{n}$

Therefore the probability of finding a solution for a particular n is $\frac{1}{n^2 x_0^{n-3}}$ and consequently the probability of finding a solution for any $n > n_0 = 100$ is $\int_{100}^{\infty} \frac{1}{n^2 x_0^{n-3}} dn$. If we calculate the integral for $x_0 = 2$

$$\int_{100}^{\infty} \frac{1}{n^2 2^{n-3}} dn \approx 8.85 \times 10^{-34}$$

which means that the probability is less than 10^{-31}%. Feynman concluded: "for my money Fermat's theorem is true". This is of course not very formal from a mathematical standpoint and is far from the real 110 pages long proof of FLT that took A.Wiles years to put together, notwithstanding it's a really good example of Feynman's scientific approach and genius. As Feynman used to say:

> *the main job of theoretical physics is to prove yourself wrong as soon as possible.*