Ann. Rev. Fluid Mech. 1982. 14:395–442

NUMERICAL METHODS IN FREE-SURFACE FLOWS

Ronald W. Yeung

Department of Applied Mathematics[1], University of Adelaide, Adelaide, South Australia 5001 and Department of Ocean Engineering, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139

1. INTRODUCTION

Free boundaries occur in a wide variety of physical phenomena—jets, cavities, seepage of groundwater, ice melting in water, gravity waves, just to name a few. All share the common feature that the domain of interest has an unknown boundary, on which a double condition is to be imposed. In this category of problems, gravity waves stand out with the characteristic that both restoring and inertial forces are important. This results in a complex, yet often fascinating, interplay between potential and kinetic energies. It is in the context of surface gravity waves, or water waves, that we use the term free surface in this article. A more general title is chosen in hope of stimulating some mutually beneficial exchange with other free-surface workers.

Even with the neglect of surface tension and viscosity, the theory of water waves is not a simple subject. Traditionally, the theory is classified into two areas: one based on systematic expansions, with the amplitude-to-wave-length ratio taken as a small parameter (Stokes 1847), the other based on the assumption of small depth-to-wavelength ratio (Boussinesq 1871, Rayleigh 1876). A comprehensive account of the more recent developments in each may be found in the reviews of Yuen & Lake (1980) and Miles (1980). The complexity of the subject of water waves compounds with the introduction of a body into the wave field, and direct numerical solution is often necessary.

We focus our attention on both linear and nonlinear problems, the former in the context of infinitesimal-wave theory. Existence and uniqueness theorems do not generally exist for these problems, particularly the

[1]Australian-American Educational Foundation exchange visitor, Spring 1981.

0066-4189/82/0115-0395$02.00

nonlinear ones. Unfortunately, these problems also define an area of a great deal of engineering interest. Nevertheless, driven by an increasing demand for the utilization and the resources of the seas, hydrodynamicists have moved ahead to develop the necessary computational tools and have provided some useful engineering solutions. Many problems, physical and numerical, remain unsolved. The days for full-scale numerical simulation are still ahead.

The nontrivial nature of some of the free-surface flow problems to be discussed is perhaps well illustrated by a concluding remark of Southwell & Vaisey (1946). Noted for his pioneering works in the development and application of relaxation methods to a large variety of physical problems, Southwell remarked after attempting several "free-streamline" problems in the cited paper: "Problems concerned with 'free' stream-lines (with a double condition) are among the hardest yet attempted in this series; an essential instability (on this boundary) makes any tentative solution liable to diverge; 'graded nets' have proved indispensable." Southwell & Vaisey were considering the nonlinear steady flow about a gliding plate in an era of "human computers." They were perplexed by the absence of waves downstream. Southwell's views must still be shared in part by many modern workers. Even today, the precise role of a numerical radiation condition in nonlinear flow is not clearly understood.

This survey covers and contrasts the numerical methods that are actively being used to solve free-surface flow problems. The literature cited is slanted towards body-wave problems because of the author's own interests. However, relevant techniques related to "free-wave" calculations are also mentioned to illustrate different facets of a methodology. The companion article of Schwartz & Fenton (1982) in this volume, addressing various aspects of highly nonlinear waves, is also relevant to the general subject area of free-surface flow. In order to make the scope of the work more manageable, and to provide a more distinct contrast of the methodologies, we restrict our attention to problems based on a potential or stream-function formulation. The use of the velocity potential is justifiable only when the inertial forces dominate the viscous forces. Cases of bluff bodies in steady motion and small bodies in a wave field of large amplitude are well-known exceptions. Methods based on the use of the primitive-variables equation (velocity-pressure formulations) are not included here, but merit a separate review.

Three major categories of methods are reviewed here: finite differences, finite elements, and boundary-integral equations. In an era of continuous evolution of numerical methods, it is sometimes difficult (and risky) to classify them. The more outstanding methods, in fact, have a hybrid character that incorporates better features of the others. Besides, finite difference with respect to time is used inevitably in most methods for unsteady problems; discretization techniques in integral-equation methods could be regarded as a finite-element treatment of the boundary. But the confusion appears to disappear when the categorization is taken to mean the manner in which the field equation is tackled in the computational (interior) domain. Spectral methods, because of their limitations in dealing with arbitrary body geometries, are not extensively covered. For completeness, the relevant equations are first recalled in Section 2, followed by sections that describe the individual features of each type of method. The emphasis is on formulation and techniques. Details regarding implementation are generally available from the cited references or text materials.

2. GOVERNING EQUATIONS

We assume at the outset that the fluid is incompressible, inviscid, and lacking surface tension. The flow is assumed to be irrotational except in regions behind the body where a thin vortex sheet may exist because of the generation of lift. In this latter case, an appropriate "cut" in the fluid domain Ω must be chosen to render the velocity potential single-valued. Let $Oxyz$ be a moving coordinate system with absolute velocity $\mathbf{U} = (U,0,0)$. The Oxz plane is taken to coincide with the "still" water surface, $\mathcal{F}_o$, and the y axis points upwards. The fluid disturbance at time t is thus described by a velocity potential $\phi(x,y,z,t)$ with the fluid velocity $\mathbf{u}$ given by $\nabla\phi$. It is well known that ϕ is a solution of Laplace's equation and that the momentum equations yield Bernoulli's integral. Hence

$$\nabla^2\phi(\mathbf{x}, t) = 0 \qquad \text{for } \mathbf{x} = (x, y, z) \text{ in } \Omega\,, \tag{2.1}$$

$$\frac{p(\mathbf{x}, t)}{\rho} + \phi_t - U\phi_x + \tfrac{1}{2}|\nabla\phi|^2 \tag{2.2}$$

where p is the fluid pressure, ρ the density, and g the acceleration of gravity. In (2.2) and henceforth, subscripts will be used to denote partial derivatives, and the arguments shown for the first field variable are understood to apply to all others.

The fluid is bounded on top by a free surface $\mathcal{F}$ described by, say, $y = \eta(x,z,t)$, where η is the free-surface elevation, internally by the surface of a rigid body $\mathcal{H}$, which may or may not intersect $\mathcal{F}$, and below by a bottom surface $\mathcal{B}$, defined by $y = -h(x,z,t)$, with the depth h normally taken as constant. The boundary contours that define Ω will be denoted by $\partial\Omega$. The kinematic boundary conditions on $\mathcal{H}$ or $\mathcal{B}$ are

$$\left.\frac{\partial\phi}{\partial n}\right|_{\mathcal{B},\mathcal{H}} = V_n\,, \tag{2.3}$$

where $\mathbf{n}$ is an exterior unit normal to Ω. V_n in (2.3) is the normal velocity of the boundary surface, which can be considered prescribed on $\mathcal{H}$, but vanishes on $\mathcal{B}$. We note in passing that it is convenient to take $\mathbf{U}$ as the translational velocity of the body, in which case $\mathcal{H}$ is time independent if the body motion is steady. On the free surface $\mathcal{F}$, the kinematic and dynamic boundary conditions can be obtained respectively by noting that $y = \eta$ is a material surface, and by setting $p(x,\eta,t)$ to the applied pressure (in excess of atmosphere) p_0. Hence,

$$\frac{D\eta}{Dt} = \phi_y \qquad \text{on } \mathcal{F}\ , \tag{2.4a}$$

$$\frac{D\phi}{Dt} = \tfrac{1}{2}|\nabla\phi|^2 - g\eta - p_0/\rho \qquad \text{on } \mathcal{F}\ , \tag{2.4b}$$

where $D/Dt = \partial/\partial t + (\nabla\phi - \mathbf{U})\cdot\nabla$ is the material derivative in a moving frame. Note that (2.4a) is essentially a statement of (2.3) with V_n being the normal velocity of the free surface. A more traditional formulation in a stationary frame can be recovered simply by setting $\mathbf{U} = 0$ in the above. For the purpose of later usage, it is worthwhile to express (2.4) in their full form:

$$\eta_t\,(x, z, t) = \phi_y\,(x, \eta, z, t) + (U - \phi_x)\eta_x - \phi_z\,\eta_z\ , \tag{2.5a}$$

$$\phi_t\,(x, \eta, z, t) = -g\eta(x, z, t) + U\phi_x - \tfrac{1}{2}|\nabla\phi|^2\ , \tag{2.5b}$$

where we have taken $p_0 = 0$ for simplicity. Equation (2.1) with boundary conditions (2.3) and (2.4) defines the nonlinear initial-boundary-value problem of the motion of water waves in the presence of a rigid body. Existence proofs for the nonlinear problem are available only for a few special cases. The order of the differential equations, however, suggests that initial conditions are appropriate for the following: $\phi(\mathbf{x},0^-)$, $\eta(x,z,0)$ and η_t. Once ϕ is solved, (2.2) can be used to evaluate $p(\mathbf{x},t)$, in particular on the body surface, whereby physical quantities of interest such as force or moment can be calculated. Various specializations of the foregoing general formulation are next reviewed.

STEADY-MOTION PROBLEMS If the motion has achieved a steady state in the moving frame, the free-surface conditions are simply (2.5), with the left-hand side set equal to zero:

$$\phi_n\,(x, \eta, z) = -U\eta_x\ , \qquad U\phi_x = g\eta + \tfrac{1}{2}|\nabla\phi|^2\ . \tag{2.6a,b}$$

However, since the problem in this situation is kinematically equivalent to that due to a uniform flow of velocity $-U$ about a fixed $\mathcal{H}$, it is common to introduce a total potential Φ defined by

$$\Phi = -Ux + \phi\ .$$

The condition (2.3) and (2.6) are now replaced by

$$\left.\frac{\partial \Phi}{\partial n}\right|_{\mathcal{B},\mathcal{H}} = 0\,, \tag{2.7}$$

$$\left.\frac{\partial \Phi}{\partial n}\right|_{\mathcal{F}} = 0\,, \qquad \tfrac{1}{2}|\nabla\Phi|^2 + g\eta = \tfrac{1}{2}U^2\,. \tag{2.8a,b}$$

In two dimensions, similar formulation can be written in terms of a stream function Ψ, where $\mathbf{u} = (\Psi_y - U, -\Psi_x, 0)$. The equivalent conditions are then:

$$\Psi|_{\mathcal{F}} = 0\,, \qquad \left.\tfrac{1}{2}\left(\frac{\partial \Psi}{\partial n}\right)^2\right|_{\mathcal{F}} + g\eta = \tfrac{1}{2}U^2\,, \tag{2.9a,b}$$

$$\Psi|_{\mathcal{B}} = Uh(x=+\infty)\,. \tag{2.10}$$

However, the body $\mathcal{H}$ takes on a stream-function value that is unknown.

In steady-state problems where the flow is subcritical, physical observations indicate that waves occur only downstream. Thus "asymptotic conditions" of the following form may be stated:

$$\phi \to o(1) \qquad \text{as } x \to \infty \tag{2.11}$$

and

$$\phi \to W(x, y, z) + C \qquad \text{as } x \to -\infty\,, \tag{2.12}$$

where $W(x,y,z)$ is a wave-like solution, C a constant, both of which are not a priori known. Such conditions are not necessary in the initial-value formulation.

LINEARIZED STEADY-MOTION PROBLEMS If U is considered to be of $O(1)$, ϕ and η of $O(\epsilon)$, where the small number ϵ is either the Froude number based on the submergence of the disturber (cases of deep submergences) or the longitudinal surface slope of the disturber (cases of thin or slender bodies), the nonlinear terms involving ϕ and η in (2.6) could be discarded. The linearized free-surface conditions to be satisfied on $y = 0$ are therefore

$$-U\phi_x + g\eta = 0\,, \qquad U\eta_x + \phi_y = 0 \qquad \text{on } \mathcal{F}_0, \tag{2.13a,b}$$

which can be combined to yield

$$\phi_{xx} + \kappa\phi_y = 0 \qquad \text{on } \mathcal{F}_0\,, \tag{2.14}$$

where $\kappa \equiv g/U^2$. If the body is not deeply submerged, a consistent linearization procedure requires that the condition (2.3) on $\mathcal{H}$ be satisfied in some linearized manner also. A detailed account of the formalism of such a perturbation expansion can be found in Wehausen (1973) and more recently in Dern (1977). Many workers, however, prefer to use (2.3) in its exact form together with (2.14). If $\mathcal{H}$ intersects $\mathcal{F}_0$, the boundary-value problem (2.1), (2.3), and (2.12) is known as the Neumann-Kelvin problem, for which neither existence nor uniqueness proofs have been established.

In the last decade, another type of linearized theory has been developed (Ogilvie 1968, Baba & Takekuma 1975, Newman 1976) as an attempt to model low-speed steady flow over full bodies, such as a tanker hull. The fully nonlinear equation (2.8) may be rewritten as follows:

$$\Phi_s \left(\tfrac{1}{2}\Phi_s^2 \right)_s = - g\Phi_y \qquad \text{on } y = \eta\,, \tag{2.15}$$

where s is the arc-length along the streamline under consideration. If Φ is decomposed as a sum of $\Phi^{(r)}$ and $\phi^{(w)}$ where $\Phi^{(r)}$ satisfies (2.1), (2.7) and $\Phi_y{}^{(r)} = 0$ on $y = 0$, the so-called double-body potential, and $\phi^{(w)}$ is a wave-like potential, the resulting linear free-surface condition for $\phi^{(w)}$ reduces to

$$\begin{aligned} &\frac{\partial}{\partial s}(\Phi_s^{(r)2}\phi_s^{(w)}) + g\phi_y^{(w)} \\ &= - (\Phi_s^{(r)2}\Phi_{ss}^{(r)} + g\eta^{(r)}\Phi_{yy}^{(r)}) \qquad \text{on } y = 0\,, \end{aligned} \tag{2.16}$$

where $\eta^{(r)} = (U^2 - \Phi_s{}^{(r)2})/2g$ and s is now the double-body streamline coordinate. The variable coefficients in (2.16) are indicative of a physical process in which the wave-like motion is being convected downstream by the double-body potential; the latter can be determined by a number of well-established means.

LINEARIZED OSCILLATORY-MOTION PROBLEMS If the disturber has no forward velocity and conducts oscillatory motion about some equilibrium position, the small parameter for linearization is the amplitude of the body motion. Equation (2.5) then reduces to

$$\eta_t\,(x, z, t) = \phi_y\,, \qquad \phi_t\,(x, 0, z, t) = -g\eta \tag{2.17a,b}$$

or

$$\phi_{tt}\,(x, 0, z, t) + g\phi_y = 0\,, \tag{2.18}$$

which are the linearized conditions of classical water-wave theory. The body condition (2.3) may be satisfied at its equilibrium position.

If the motion is time-harmonic of the $e^{-i\omega t}$ type, where $i = \sqrt{-1}$, and ω is the angular frequency, the standard decomposition of the form $\phi(\mathbf{x},t$ $= \varphi(\mathbf{x})e^{-i\omega t}$, $V_n(\mathbf{x},t$
from (2.3) and (2.18) that the boundary conditions for the (time-) complex spatial potential $\varphi(\mathbf{x})$ are

$$\varphi_n = v_n\,, \qquad \text{on } \mathcal{B}, \mathcal{H}, \tag{2.19}$$

$$-\frac{\omega^2}{g}\varphi(x, 0, z) + \varphi_y = 0\,, \tag{2.20}$$

the latter being a condition of mixed type on $\mathcal{F}_0$. As before, φ must satisfy (2.1). Furthermore, in order to obtain outgoing waves, a radiation condition must be imposed. This can be stated as

$$\varphi_x \mp ik\varphi \to 0 \qquad \text{as } x \to \pm\infty\ , \tag{2.21}$$

$$(kr)^{1/2}(\varphi_r - ik\varphi) \to 0 \qquad \text{as } r(=\sqrt{x^2+z^2}) \to \infty\ , \tag{2.22}$$

for the two- and three-dimensional cases, respectively. The wave number k in (2.21, 2.22) is the solution of the equation $\omega^2 = gk \tanh kh$. We remark that (2.19)–(2.22) represents the class of radiation and diffraction problems that one customarily encounters in the study of motion of floating bodies, and wave forces on structures. An Existence and Uniqueness theorem for this class of problem has been given by John (1950) under rather restrictive hypotheses of the body geometry. A more general proof has recently been given by Lenoir & Martin (1981).

Not considered specifically in this article is the case of a steadily translating body that undergoes periodic oscillations about some mean position, the so-called ship-motion problem. A recent review of Newman (1978) covered this in some detail.

3. FINITE-DIFFERENCE METHODS

The use of finite-difference techniques in solving partial-differential equations is a well-established one. There is a substantial and rapidly growing amount of literature addressing aspects of stability, convergence, and efficiency of the various types of differencing schemes available. The literature also abounds in fluid-mechanics applications. Indeed, finite-difference methods may very well be regarded as the backbone of many of today's flow-simulation codes (Patterson 1978, D. Chapman 1978). General features of such numerical simulation techniques are well discussed by Emmons (1972), Orszag & Israeli (1974), and MacCormack & Lomax (1979). Details of the methods and relevant analysis are available from texts such as Richtmyer & Morton (1967), Roache (1976), and Ames (1977). The present section covers only those aspects that are particularly relevant to the implementation of finite-difference methods in free-surface flow problems. By inspection, it is evident that the treatment of the field equation in a potential or stream-function formulation is relatively straightforward. The complication is associated with the determination of the free-surface location in conjunction with the satisfaction of the boundary conditions and the field equation, all preferably simultaneously.

Finite-difference methods are most suitable, or at least simplest to implement, for boundary geometry that is rectilinear. For nonlinear problems, the free-surface boundary will not usually intersect the mesh system at grid points that are regularly spaced. This is similar to the difficulty caused by the presence of an arbitrarily shaped body in the fluid, but with the further complication that such intersections are time dependent. The use of "irregular stars" near the fluid boundaries is generally called for, but they are inferior in accuracy compared with the regular ones in the rest of the field. To achieve an accuracy near the boundary that is the same order as the field, one normally would have either to refine the mesh locally or use a difference formula of a higher-order accuracy. Either approach will complicate the numerical procedure substantially. Boundary conditions are known to have a strong influence on the accuracy of the flow solutions; their proper treatments are thus imperative. With the presence of a free surface that is unknown a priori, we thus have the unfortunate situation that the regions that demand the greatest accuracy are precisely those where it is hardest to achieve. The use of boundary-fitted coordinates, which allow the problem to be solved in a mapped domain composed solely of rectangles, will overcome this type of difficulty but at the expense of introducing some other complexities.

In many practical applications, one is interested only in the steady-state solution. Because of the difficulty of implementing a *nonlinear* radiation condition, if one indeed exists, at the outflow boundary, most workers prefer to solve an initial-value problem and obtain the large-time asymptotic behavior. The assumption is that such an asymptotic solution is equivalent to the steady state. For nonlinear problems, there is no reason a priori why the asymptotic solutions are independent of initial condition. Besides, it is conceivable that phenomena such as wave breaking may interrupt the solution process, before steady state is reached.

The initial-value approach has the advantage that the free-surface conditions can be used to advance the solution in time. The choice of the finite-difference form is critical; seemingly minor modifications can sometimes have drastic effects on the stability and convergence characteristics of the system. Stability is not equivalent to accuracy. Overly stable finite-difference forms can introduce such a large amount of artificial viscosity that the final results are completely distorted.

To monitor the accuracy of the computations, it is useful to apply an energy check on the numerical solution as time progresses. Let E be the total energy of the fluid *motion* in the domain Ω. In a fixed reference frame ($U=0$) the following energy theorem has been given by John (1949):

$$\frac{1}{\rho}\frac{dE}{dt} = \frac{d}{dt}\int_{\Omega} [\tfrac{1}{2}|\nabla\phi|^2 + gy]d\Omega \tag{3.1a}$$

$$= \int_{\Omega} \nabla\phi_t \cdot \nabla\phi d\Omega - \int_{\partial\Omega}\left(\frac{p}{\rho} + \phi_t\right)V_n\, d\partial\Omega \tag{3.1b}$$

$$= \int_{\partial\Omega} \phi_t(\phi_n - V_n)\, d\partial\Omega - \int_{\partial\Omega} \frac{p}{\rho} V_n\, d\partial\Omega\,, \tag{3.1c}$$

where the Transport Theorem and (2.2) have been used in arriving at the second equality, and the divergence theorem at the third. V_n here is the normal velocity of the boundary surface. Note that because of the free-surface conditions, the boundary $\mathcal{F}$ does not contribute to the right-hand side of (3.1c), as is to be expected from physical reasoning. Since the second integral of (3.1c) over $\mathcal{H}$ is the rate of work done by the body $\dot{W}_{\mathcal{H}}$, we obtain

$$\begin{aligned}\dot{W}_{\mathcal{H}} &= \dot{E} + \int_{\Sigma} [pV_n - \rho\phi_t\,(\phi_n - V_n)]d\partial\Omega \\ &= \rho\,\frac{d}{dt}\left[\frac{1}{2}\int_{\partial\Omega} \phi\phi_n\, d\partial\Omega + \frac{g}{2}\int_{\mathcal{F}_0} \eta^2\, dxdz\right] \\ &\quad - \int_{\Sigma} [pV_n - \rho\phi_t\,(\phi_n - V_n)]d\partial\Omega\,.\end{aligned} \tag{3.2}$$

The last integral of (3.2) vanishes when Σ is a stationary rigid surface, otherwise it accounts for the rate of work by Σ and the energy flux out of the control volume Ω. Regardless of the type of numerical method, (3.2) is particularly convenient for computation, since it involves boundary quantities that are normally either known or being sought after anyway. Linearized forms of (3.2) have been used by Haussling & Van Eseltine (1974) in two dimensions, and Ohring & Telste (1977) in three dimensions. One useful specialization of (3.2) is the case of steady motion of a body, for which $\dot{E} = 0$ in the moving frame of reference. In this case, if we choose a vertical plane Σ^- perpendicular to the direction of motion downstream and take advantage of the vanishing of disturbances far upstream, (3.2) can be simplified to

$$R = \tfrac{1}{2}\rho\int_{\Sigma^-} (\phi_y^2 + \phi_z^2 - \phi_x^2)d\partial\Omega + \tfrac{1}{2}\rho g\int_{\Sigma^-\cap\mathcal{F}_0} \eta^2 dz\,, \tag{3.3}$$

where R is the steady-state resistance, and the location of Σ^- is arbitrary. This equation can provide a consistent check of the local calculations around the body with the downstream-outflow behavior. An alternate derivation of (3.3) based on momentum considerations was given by Wehausen (1973). Equation (3.3), like (3.2), is exact within the assumptions of potential flow. Neither one has been implemented in checking nonlinear calculations.

Finite-Difference Forms

In the interest of notational simplicity, we discuss here the treatment of two-dimensional problems only; three-dimensional analogues of what follow are often quite self-evident. The notation $\phi(i\Delta x,\, j\Delta y,\, n\Delta t) \equiv \phi_{ij}^n$ will be used; in particular $j = j^*$ will denote the grid points on the free surface (or $y = 0$, in the case of linearized problems). The five-point central-differencing formula of Laplace's equation (at any time index) is

$$\phi_{ij} = \tfrac{1}{4}(\phi_{i+1,\,j} + \phi_{i-1,\,j} + \phi_{i,\,j-1} + \phi_{i,\,j+1}) + O(\Delta x^2)\,, \tag{3.4}$$

where Δx and Δy are assumed to be the same. The coefficients in front of the "neighborhood points" will be more complex (see, for example, Forsythe & Wasow 1960), if the points are not equally spaced around (i,j) and the subsequent five-point formula of the same points is only of $O(\Delta x)$ accuracy. Equation (3.4) is generally solved by successive over-relaxation based on the following rearranged formula:

$$\begin{aligned}\phi_{ij}^{(k+1)} = \phi_{ij}^{(k)} + \frac{\omega}{4}[\phi_{i+1,\,j}^{(k)} + \phi_{i-1,\,j}^{(k+1)} + \phi_{i,\,j+1}^{(k)} + \phi_{i,\,j-1}^{(k+1)}\\ - 4\phi_{ij}^{(k)}]\,, \qquad i\uparrow, j\uparrow\end{aligned} \tag{3.4a}$$

where the superscript in parenthesis indicates the stage of iteration, with i,j being taken in an ascending sequence, and ω is the *relaxation* parameter. Dirichlet conditions on the boundary are thus easily handled by (3.4a); but Neumann or mixed-type conditions require a redefinition of the coefficients on the right-hand side. Because of its ease of implementation, (3.4a) is preferred by most workers (von Kerczek & Salvesen 1974, Chan & Hirt 1974, R. Chapman 1976). Higher-order methods have been used by Ohring (1975), who also utilized a direct method due to Buzbee et al. (1971) for handling the irregular stars on the solid boundary.

Being first-order in time, the free-surface conditions (2.5a,b) are particularly useful for advancing η and ϕ on the free boundary. Various difference forms are conceivable. For illustration, we consider the case of linearized conditions (2.17a,b). A few obvious schemes are

$$\eta^{n+1} = \eta^n + \Delta t\,\phi_y^n\,, \qquad \phi^{n+1} = \phi^n - \frac{g\Delta t}{2}(\eta^{n+1} + \eta^n)\,; \tag{3.5a}$$

$$\eta^{n+1/2} = \eta^{n-1/2} + \Delta t\,\phi_y^n\,, \qquad \phi^{n+1} = \phi^n - g\Delta t\,\eta^{n+1/2}\,; \tag{3.5b}$$

$$\begin{aligned}\eta^{\overline{n+1}} &= \eta^n + \Delta t\,\phi_y^n\,, \qquad \phi^{\overline{n+1}} = \phi^n - g\Delta t\,\eta^n\,,\\ \eta^{n+1} &= \eta^n + \frac{\Delta t}{2}(\phi_y^n + \phi_y^{\overline{n+1}}), \qquad \phi^{n+1} = \phi^n - \frac{g\Delta t}{2}(\eta^{\overline{n+1}} + \eta^n)\,;\end{aligned} \tag{3.5c}$$

$$\eta^{n+1} = \eta^n + \frac{\Delta t}{2}(\phi_y^n + \phi_y^{n+1}), \qquad \phi^{n+1} = \phi^n - \frac{g\Delta t}{2}(\eta^{n+1} + \eta^n)\,; \tag{3.5d}$$

where all values of ϕ are understood to be at j^*. The first two are one-step explicit schemes that use the current information at n or $n-½$ to advance the solution. (3.5b) is a staggered system, which is actually centered at n and is thus $O(\Delta t^2)$ in accuracy. (3.5c) is a two-step predictor-corrector method without iteration. $\phi_y^{\overline{n+1}}$ however has to be determined based on

$\phi^{\overline{n+1}}$ of the predictor step. (3.5d) is the so-called Euler modified method, which is implicit, thus requiring a simultaneous solution of ϕ and η via ϕ_y at $n+1$.

An assessment of the stability characteristics of these schemes may be made by a *simplified* von Neumann analysis. Let $\phi(x, y, n\Delta t)$ be of the form $\phi^n e^{ikx+ky}$ where k is a wave number; thus $\phi_y^n = k\phi^n$. If (ϕ^{n+1}, η^{n+1}) is now written in terms of an amplification matrix G multiplying (ϕ^n, η^n), the necessary condition for stability is that the spectral radius of G be less than 1 (Lax & Richtmyer 1956). The eigenvalues for the various cases of (3

ber $f \equiv kg\Delta t^2/2$ as parameter:

$$|\lambda| \geqslant (1 + f)^{1/2}, \tag{3.6a}$$

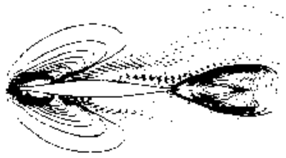
$$|\lambda| = 1, \quad \text{for } f \leqslant 2; \quad |\lambda| \leqslant f - 1 + \sqrt{f(f-2)}, \quad \text{for } f > 2, \tag{3.6b}$$

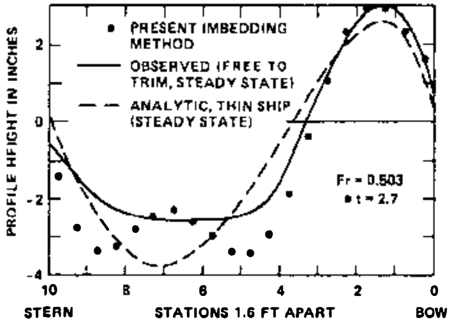
$$|\lambda| = (1 + f^2)^{1/2}, \tag{3.6c}$$

$$|\lambda| = 1, \quad \text{for all } f. \tag{3.6d}$$

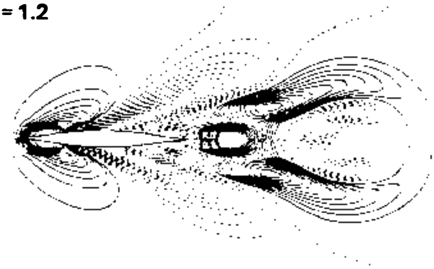
This simple, yet original, analysis is rather informative. It can be seen that the implicit scheme is the only unconditionally stable scheme for all k. In particular, we observe that (3.5a) is always unstable. The staggered system (3.5b), which has been used by Chapman (1976), is conditionally stable if short waves are not excited. Chapman actually made some quasi-three-dimensional calculations for a vertical surface-piercing plate moving steadily at an angle of attack. Within the assumptions of high-Froude-number theory, the solution at each station was treated two-dimensionally in the cross plane, with "time" being taken as the longitudinal position of the station divided by the forward speed. The two-step scheme of (3.5c) has been used by Yen et al. (1977) to advance a two-dimensional finite-element solution for a moving pressure distribution. Though only marginally unstable, $|\lambda| = 1 + O(\Delta t^4)$, this scheme is clearly unsuitable for long-time (steady-state) calculations. Such instability is apparently the cause of the sudden drag increase in Yen's calculation at large value of time.

Implicit schemes similar to (3.5d) are preferred by most workers (Chan & Hirt 1974, Ohring & Telste 1977, Haussling & Coleman 1979) because of the stability property. Further, since $|\lambda| = 1$ there is no artificial damping introduced, at least according to linear analysis. Zero damping, however, does not imply the absence of phase errors, the latter being critical in phenomena involving interference effects such as wave resistance. The fully nonlinear equations (2.5a, 2.5b) are best treated in a manner similar to (3.5d). In a moving coordinate system, strongly convective terms are present. Upwind differencing is usually used to "preserve" the transportative property, but its effect has been seldom analyzed. The judgement is often an indirect one based on a comparison of physical quantities of t = 0.6

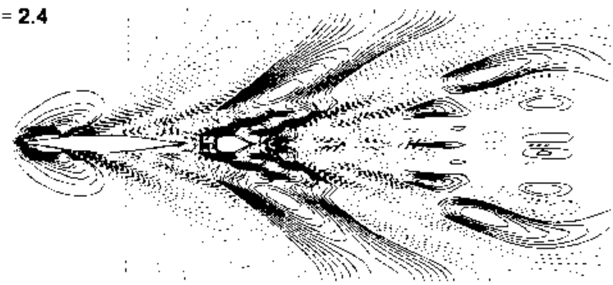




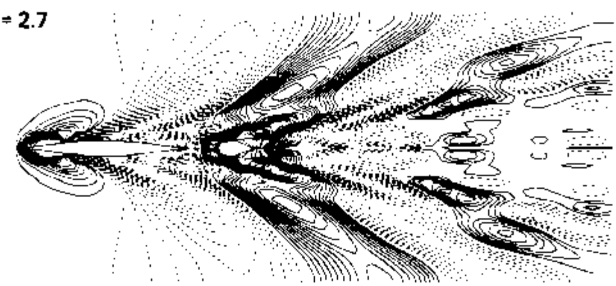
t = 1.2



t = 2.4



t = 2.7



interest, such as force and wave-elevation, with experimental results, which naturally include other factors not modeled in the original equations.

The general nonlinear conditions (2.5a,b) can be solved iteratively using the following algorithm or minor variants of it. The difference equations corresponding to (3.5d) are now written as

$$(\eta^{n+1,\,k+1}-\eta^{n})/\Delta t = \tfrac{1}{2}(\phi_y^n+\phi_y^{n+1,\,k}) + F(\eta^n,\eta^{n+1,\,k},\phi^{n+1,\,k})\,, \tag{3.7a}$$

$$(\phi^{n+1,\,k+1}-\phi^{n})/\Delta t = -\tfrac{1}{2}g(\eta^{n+1,\,k+1}+\eta^n) + G(\eta^n,\eta^{n+1,\,k+1},\phi^{n+1,\,k})\,, \tag{3.7b}$$

where F and G are the remaining terms in (2.5a,b). The second superscript in (3.7) is the stage index of the iterations within a time step. Thus, starting with known values of ϕ in Ω and values of η at step n, η and ϕ (on $\mathcal{F}$) are first updated successively by (3.7). The new values of ϕ are next used as a Dirichlet condition on $\mathcal{F}$, and (3.4a) is solved with the body boundary condition (2.3) and some appropriate conditions upstream and downstream. A new ϕ_y^{n+1} is now calculable and (3.7) can thus be repeated to complete another stage of iteration, and so on, until converged values of η and ϕ are achieved for the time step $n+1$.

Figure 1, taken from Ohring & Telste (1977), gives free-surface contour plots generated by a moving ship that starts from rest. The free-surface conditions used by the authors are linearized; the body condition is, however, exact. A rigid computation box of dimensions $6 \times 1.6 \times 1$ body lengths was used. The development of the Kelvin wave pattern is clearly visible in these plots, but the authors had difficulty obtaining steady-state results. The oscillatory behavior of the wave-resistance coefficient reported in this work could be caused by the sloshing modes of the "tank." Ohring & Telste (also Haussling & Coleman 1979) reported the presence of "wiggles" in their solution and numerical filtering was necessary.

If the ultimate solution desired is that corresponding to a steadily translating body in or near a free surface, it would appear that a direct attack on Equations (2.8a,b) or (2.9a,b) is perhaps more appropriate. This approach has been apparently less successful. In two-dimensions, as far as the free-surface conditions are concerned, it is more convenient to use either a stream-function or potential formulation that includes the free stream. The kinematic condition is then merely a Neumann or Dirichlet type. A unique procedure was given by von Kerczek & Salvesen (1974),

Figure 1 Time-sequence plots of the free-surface contours generated by an impulsively starting ship (from Ohring & Telste 1977). Numerical calculations satisfy an exact body boundary condition whereas "analytic" results are based on Equation (5.8*a*). Fr is the Froude number based on forward speed, and t the number of ship lengths travelled divided by Fr.

who considered the nonlinear problem of steady flow about a submerged point vortex. Proceeding in a downstream direction, these authors successively adjust each "free-surface node" so that the dynamic boundary condition can be satisfied. Since interaction occurred among nodes, they utilized an interpolation polynomial, extending about one half of a wavelength downstream, to represent the free surface and determine the amount of correction based on the pressure residual at the node under consideration. Each node adjustment required a repetitive solution of (3.4a) until the pressure residual was acceptable. The process is conceivably not the most efficient, but was apparently convergent. The authors reported that they were able to generate waves to within 15 percent of the steepest periodic waves of Stokes (1880a). In a subsequent paper on subcritical flow in shallow water (Salvesen & von Kerczek 1978) these authors noted the necessity of accounting for an upstream surge when comparing numerical results with experiments based on the depth Froude number. Also noted was the interesting result that for a given Froude number, downstream waves tend to shorten rather than lengthen, as in the case of deep water, when a disturber increases in strength.

One obvious approach to solving the nonlinear conditions (2.9) is to use Newton's method iteratively. This is best carried out using a numerical formulation that has implicitly satisfied the field equation, e.g. an integral-equation method or a spectral method. Finite-difference or finite-element methods are not particularly suitable in this respect. Newton's method used in conjunction with a spectral representation based on Fourier series has recently been used by Reinecke & Fenton (1981) for calculating profiles of free waves (see also Schwartz 1981 in Section 5). They report that convergence to 12 decimal places within a few iterations was possible! The initial "guess" was taken simply as linear Stokes waves, but other extrapolative means were found necessary for the case of very steep waves.

Since Dean's (1965) work, the stream-function formulation with a Fourier representation has been used extensively for evaluating properties of nonlinear periodic waves (see also Dean 1974, Chaplin 1980). This particular approach becomes ineffective when a physical body is present in the wave field, in which case a simple spectral representation is no longer available. Further, neither the value of the stream function itself nor its normal derivative is known on the body boundary.

Boundary-Fitted Coordinates

The basic idea of boundary-fitted coordinates is to transform the physical boundaries of a problem to coordinate lines in a mapped space, wherein finite-difference computations can be conveniently carried out in a regular mesh system without extensive interpolation along the boundary. In estuary hydrodynamics, the simple transformation in the vertical direction,

$\zeta = (\eta - y)/(\eta + h)$, has been used quite successfully for some time (Freeman et al. 1972, Noye et al. 1981). More elaborate techniques were considered by Winslow (1966) for interior flows. Thompson et al. (1974) have further developed the idea for exterior and multi-body flows. If (ξ,ζ) denote the mapped coordinates in two dimensions, the generating system is typically taken as the solution of the Poisson equation (Thompson et al. 1976):

$$\xi_{xx} + \xi_{yy} = P(\xi,\zeta), \qquad \zeta_{xx} + \zeta_{yy} = Q(\xi,\zeta), \tag{3.8}$$

with Dirichlet conditions (viz. ξ,ζ being constant) on $(x,y) \in \mathcal{B}$, Σ, and $\mathcal{F}$. Here P,Q are preselected functions that control the density of coordinate lines. Since the objective is to work in the mapped domain, (3.8) is actually solved with the dependent and independent variables interchanged:

$$\alpha x_{\xi\xi} - 2\beta x_{\xi\zeta} + \gamma x_{\zeta\zeta} = -J^2[x_\xi P(\xi,\zeta) + x_\zeta Q(\xi,\zeta)],$$
$$\alpha y_{\xi\xi} - 2\beta y_{\xi\zeta} + \gamma y_{\zeta\zeta} = -J^2[y_\xi P(\xi,\zeta) + y_\zeta Q(\xi,\zeta)], \tag{3.9}$$

where α, β, and γ are relatively simple functions of the first derivatives of x and y with respect to ξ and ζ, and J is the Jacobian $\partial(x,y)/\partial(\xi,\zeta)$. The coupled quasi-linear elliptic system (3.9) must be solved at every instant of time if the boundary contours, such as $\mathcal{F}$, are time dependent. Figure 2, taken from Haussling & Coleman (1979), shows an example of the mapping of a doubly connected region of the flow onto a H-shaped (ξ,ζ) plane.

The formulation of the problem is completed by transforming the field equation and boundary conditions into the (ξ,ζ) plane using the usual rules for implicit functions. Equation (2.1), for example, would now read as

$$\alpha\phi_{\xi\xi} - 2\beta\phi_{\xi\zeta} + \gamma\phi_{\zeta\zeta} + J^2(P\phi_\xi + Q\phi_\zeta) = 0, \tag{3.10}$$

which is fundamentally similar to (3.9). At each time step, the *transformed* expressions of the free-surface condition (2.5a,b) can be used to advance η and ϕ as described earlier. However, this must now be followed by a successive over-relaxation solution of (3.9), and then of (3.10), the latter being subject to condition (2.3), and it is necessary to apply this procedure iteratively because of the implicit nature of (3.7). It is evident that the original problem with difficult boundary conditions has been exchanged for one having complicated field equations but straight-line boundaries. It is not clear which method is more efficient for the same degree of accuracy. For free-surface related flows no comparisons have been made on a one-to-one basis at the writing of this article.

The boundary-fitted coordinate techniques have been successfully applied by Shanks & Thompson (1977), Haussling & Coleman (1979), and Chan & Chan (1980). Shanks & Thompson sought a direct solution of the Navier-Stokes equation (primitive-variable form) for a translating hydrofoil at very low Reynolds numbers (<100). Although the practical implications of conducting flow calculations at such low Reynolds numbers is unclear, the results were impressive and physically plausible.

Using the velocity-potential formulation, Haussling & Coleman calculated the fluid motion due to a submerged circular cylinder undergoing either translating or swaying motion. These authors reported difficulties in obtaining steady-state results for the case of translatory motion because incipient breaking waves were encountered in the transient stage. We note in passing that the convective terms in the free-surface conditions of Haussling & Coleman (1979) were of the wrong signs. This could be merely typographical since the numerical results appear reasonable from the physical standpoint.

As an alternative to boundary-fitted coordinates method, it is worthwhile to mention that Multi-Level Adaptive Techniques, which provide very efficient means of solving flexible multi-level grid-structure systems, can be quite effective in dealing with boundary contours that require fine resolution. A more detailed account may be found in Brandt et al. (1980).

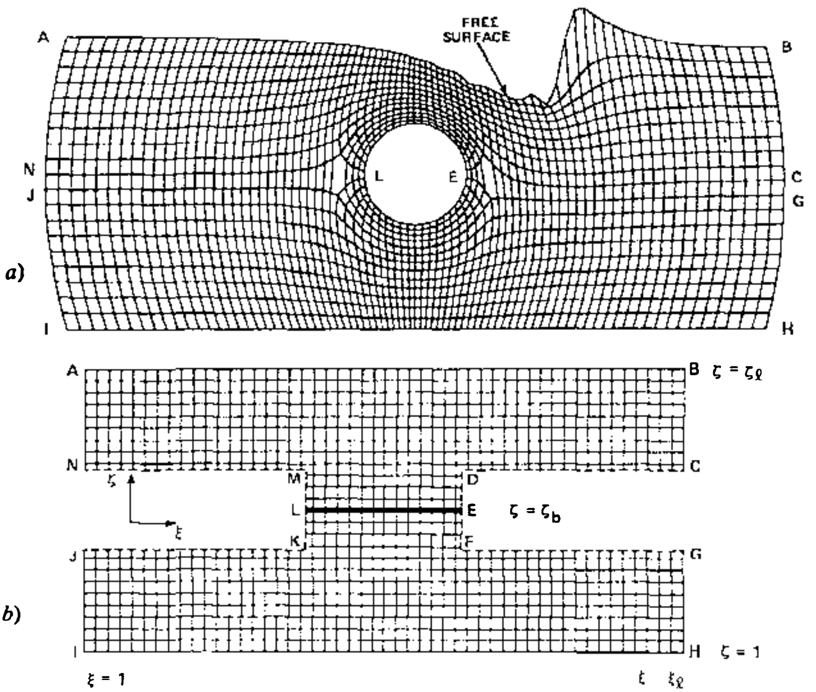


Figure 2 (*a*) Free-surface profile and boundary-fitted coordinates in the physical plane for a cylinder translating in water of finite depth. (*b*) The corresponding transformed computational region. (From Haussling & Coleman 1979)

Radiation Conditions

In the interest of keeping the computation region to a minimum, it is always desirable to truncate Ω by some control boundary Σ. Surely, a simple approach is to take a sufficiently large Σ beyond which disturbances can be considered vanishingly small. But such a boundary must necessarily expand in time as the disturbances propagate outwards. An alternative approach is to map the point at infinity to a finite value. Grosch & Orszag (1977) indicated that such mappings were not particularly useful for solutions that oscillate out to infinity. The ideal Σ would be one that is close to the region of interest but at the same time "transparent" to the flow. In practice, it is necessary to prescribe some sort of boundary conditions. What then is the appropriate boundary condition that will have a minimal effect on the interior solution? This has always been a "sore point" in computational fluid mechanics. There is no absolutely satisfactory answer to this nontrivial difficulty and the search still continues (see, for example, Hedstrom 1979, Rudy & Strikwerda 1980, Carmerlengo & O'Brien 1980).

For steadily propagating free waves that are periodic in space, planes of symmetry serve well as truncation boundaries on which symmetry conditions can be applied. In unsteady flow a helpful artifice is to assume the solution to be of the form $Q(x,y,t) = Q(x-ct,y)$, with Q representing ϕ or η. This leads to the well-known one-dimensional Sommerfeld condition (1949):

$$\frac{\partial Q}{\partial t} + c\,\frac{\partial Q}{\partial x} = 0\,, \tag{3.11}$$

where c is the phase velocity of the right-propagating waves. If the process is time harmonic $Q = \overline{Q}e^{-i\omega t}$, $\overline{Q}$ being complex in time, (3.11) reduces to (2.21) for $\overline{Q}$, which has been applied directly as an open-boundary condition by Chenault (1970), Bai & Yeung (1974), Zienkiewicz & Bettess (1975). It is noteworthy that (3.11) is not indicative of the presence of other solutions that "propagate" at infinite velocity because of the elliptic nature of the field equation. A successful way of accounting for them is by a matching procedure discussed in Sections 4 and 5.

Equation (3.11) is a standard transportive equation, for which forward-time upwind-differencing is known to be stable if the Courant number is less than unity (Thom & Apelt 1961). Its implementation is straightforward if c is known. However, in nonlinear problems the dispersion characteristics is generally unknown, and hence also c. Nevertheless, one may assume that (3.11) is locally correct with c being a slowly varying function of space and time. Orlanski (1976) proposed determining c numerically based on (3.11) at neighborhood points just interior to the boundary Σ and at an earlier time step. The boundary values of Q on Σ can then be advanced by applying (3.11) at the present time step but this time on Σ, using the value of c just determined. Orlanski derived a three-time-level formula based on centered-time and upwind-space differencing as an approximation to (3.11). Following Orlanski, Chan (1977) used a similar approximation but with upwind-centered time differencing, and noted that his scheme was non-dissipative for constant c. By taking advantage of this condition and body-fitted coordinates in three-dimensions Chan & Chan (1980) attempted the unsteady problem of nonlinear flow about a general ship hull. Although it is premature to judge the success of the reported computations at this time, the work is perhaps the most ambitious undertaking of all those reported here.

Two additional points concerning (3.11) are noteworthy. Because of stability considerations, the *numerically* feasible value of c must be less than $\Delta x/\Delta t$ (the Courant number $c\Delta t/\Delta x$ being unity). This imposes an upper limit on the fastest wave that a given spatial and temporal gridsize can represent. Equation (3.11) is physically reflective for two-dimensional plane waves at oblique incidence to Σ. Engquist & Majda (1977) have developed a hierarchy of *absorbing* boundary conditions for hyperbolic equations in two dimensions. Their leading-order result was (3.11), which is exact for one-dimensional waves. The next order, however, was given by the condition

$$c\,\frac{\partial^2 Q}{\partial x \partial t} + c^2\,\frac{\partial^2 Q}{\partial t^2} - \frac{1}{2}\,\frac{\partial^2 Q}{\partial y^2} = 0\,. \tag{3.12}$$

Equation (3.12) was shown to reduce the reflection coefficient from 17% to 3% at 45° incidence. Procedures for implementing this are apparently more complex but seems worth the effort.

It is of interest to note that in a moving coordinate system of velocity U, (3.11) becomes $Q_t + (c\text{-}U)Q_x = 0$, which yields no useful information in the steady-state limit. The difficulty of imposing a "radiation condition" in a steady-state formulation is hence apparent. However, in their calculations of two-dimensional waves, von Kerczek & Salvesen (1974) simply imposed a uniform-flow condition upstream and an extrapolative condition at the downstream (outflow) boundary. Although such an outflow condition did not actually satisfy the field equations, the results appeared to suggest that the irregularities created at the outflow boundary had little upstream effects. It is unclear to what extent this was associated with the "upwind" iteration algorithm used by these authors in the successive overrelaxation procedure.

An alternative technique to absorbing conditions is the use of artificial damping in part of the fluid domain or boundary. This has received some attention in the literature (e.g. Arakawa & Mintz 1974, Israeli & Orszag

1974, Chan 1975). The technique usually gives rise to a nontrivial problem of "designing" a damping pad, whose absorption spectrum must be wide, but whose effects on the phase of the fluid response is to be minimal.

4. THE METHODS OF FINITE ELEMENTS

Of the three major categories of methods discussed in this paper, the finite-element method is generally considered the newest comer. Introduced and developed in the 1960s, it has established itself as the mainstream computational tool for stress analysis of structures of complicated geometry. It was soon adapted to numerous other areas of applied mechanics. Extensive reviews of its general successes in fluid-flow problems have been given by Shen (1977) and Norrie & de Vries (1978). Applications pertaining to water-wave radiation and diffraction were also covered recently by Mei (1978).

The finite-element and finite-difference methods share the common feature that both tackle the field equation directly. In fact, if linear interpolative elements based on a uniform grid system were used, Zienkiewicz (1975) showed that the variational form of the Laplace operator yields an expression identical with the five-point central-differencing formula. However, with the introduction of curvilinear or isoparametric elements, the finite-element approach allows one to cope with arbitrary boundary geometry with little loss of accuracy. Although this particular advantage over finite-differencing appears to be modestly reduced when boundary-fitted coordinates are exploited as described earlier, finite-element methods still retain superiority in flexibility, particularly in cases where "superelements" can be introduced to overcome any local irregularities or difficult behavior at infinity.

In its most general form, the underlying principle of the finite-element method is one based on the method of weighted residuals. The usual procedure consists of first subdividing the domain of interest into a mesh of finite-sized subregions, within each of which the solution is represented by some convenient choice of trial functions, usually polynomials, with unknown parameters to be determined. The trial functions are determined by substituting them into the field equation and requiring the integrated error or residuals based on certain weighting functions to vanish. An integration by parts is normally performed to reduce the inter-element continuity requirement of the trial functions and to incorporate any inhomogeneous boundary conditions that are as yet not satisfied. For the Laplace operator, the integrability requirement after such steps amounts merely to an inter-element continuity of the function being sought. The weighting functions, also known as test functions, can be chosen in a variety of ways; each leads to a different type of method of weighted residuals. One particular choice is to make the space of test functions identical with that of the trial functions. This is known as Galerkin's method. Since each member of the test-function space yields one condition from the residual criterion, there are just enough equations to determine the unknown parameters in the problem. The formulation described in the foregoing is commonly referred to as the "weak formulation." In contrast, a "strong formulation" is one based on the existence of a variational principle where a functional is made stationary. Corresponding to each variational principle, it is possible to find the equivalent Galerkin procedure, but not always vice versa. For linear problems, the matrix resulting from a variational principle is always symmetric, thus reducing storage requirement and solution time. Such an advantage disappears when the problem is nonlinear, although the variational principle itself can be used to provide valuable insight into the dynamics of the physical system.

Variational Methods

The fully nonlinear problem defined by Equations (2.1), (2.3), and (2.5) with $U = 0$ can be considered equivalent to taking the variation of the following Lagrangian function:

$$J(\phi, \eta) = -\int dt \left\{ \int_{\Omega} p \, d\Omega + \int_{\mathcal{B} \cup \Sigma \cup \mathcal{H}} V_n \phi \, d\partial\Omega \right\}, \tag{4.1}$$

where p is the pressure in excess of atmospheric, as given by (2.2). Bateman (1932) has used the pressure as the Lagrangian to obtain the equations of motion of an inviscid incompressible fluid. However, Luke (1967) was the first to point out that the free-surface boundary conditions follow from (4.1). If one considers ϕ and η as independent variations and keeps in mind that η is a function of x, z, and t, then

$$\delta J = \int dt \left\{ -\int_{\Omega} \nabla^2 \phi \delta\phi \, d\Omega + \int_{\mathcal{B} \cup \Sigma \cup \mathcal{H}} (\phi_n - V_n) \, \delta\phi \, d\partial\Omega \right.$$
$$\left. + \int_{\mathcal{F}} \left(\phi_n - \frac{\eta_t}{[1 + \eta_x^2 + \eta_z^2]^{1/2}} \right) \delta\phi \, d\partial\Omega - \int_{\mathcal{F}} p \, \delta\eta \, d\partial\Omega \right\}, \tag{4.2}$$

where $\delta\phi$ is subject to no variations on the time boundaries. The last term in (4.2) arises from a variation in the physical domain Ω. It can be seen that all boundary conditions are satisfied *naturally* in (4.2). Luke's original form does not contain the second term in (4.1) since he assumed $\delta\phi = 0$ on the boundary Σ. In spite of its apparent simplicity, (4.1) has not been used for numerical computations. However, extensive use of this variational principle has been made by Whitham (1967, 1970) to examine the dispersion characteristics of nonlinear water waves. An alternative form that is dynamically equivalent to (4.1) has been given by Miles (1977).

If the flow is *steady* in a moving frame of reference and ϕ is the disturbance potential, the functional J in (4.1) can be written as

$$J(\phi\,,\eta) = \int_{\Omega} (\tfrac{1}{2}|\nabla\phi|^2 - U\,\phi_x)d\Omega - \int_{\Sigma} (Un_x - V_n)\phi\; d\partial\Omega + \int_{\mathcal{F}_0} \tfrac{1}{2}g\eta^2\, dxdz\,, \tag{4.3}$$

where V_n is the efflux associated with Σ, which is generally unknown. By taking the variation of (4.3), it is easy to show that conditions (2.1), (2.3), and (2.6) are satisfied naturally. If a boundary Σ could be found such that $\delta\phi = 0$, i.e. if ϕ were satisfied as an essential boundary condition on Σ, then (4.3) implies

$$J = \tfrac{1}{2}\boldsymbol{\phi}^T\mathbf{A}(\boldsymbol{\eta})\boldsymbol{\phi} + \mathbf{B}(\boldsymbol{\eta})\boldsymbol{\phi} + \mathbf{C}(\boldsymbol{\eta})\,, \tag{4.4}$$

where $\boldsymbol{\phi}$ and $\boldsymbol{\eta}$ represent the unknown vectors of nodal values in Ω and $\mathcal{F}$ and the superscript T represents the transpose. In (4.4) $\mathbf{A}$ is the global matrix after assembly, $\mathbf{B}$ and $\mathbf{C}$ are vectors. Taking advantage of the quadratic nature of (4.4) in $\boldsymbol{\phi}$, we can extremize J with respect to $\boldsymbol{\phi}$, holding $\boldsymbol{\eta}$ constant, and obtain

$$\mathbf{A}(\boldsymbol{\eta})\boldsymbol{\phi} = -\,\mathbf{B}(\boldsymbol{\eta})\,, \tag{4.5}$$

which is linear in $\boldsymbol{\phi}$ and can be solved to obtain $\boldsymbol{\phi} = -\mathbf{A}^{-1}\mathbf{B}$. If this is substituted back in (4.4) and the resulting expression is extremized with respect to $\boldsymbol{\eta}$, the following nonlinear equation in $\boldsymbol{\eta}$ results:

$$\tfrac{1}{2}\mathbf{B}'(\boldsymbol{\eta})\boldsymbol{\phi} = \mathbf{C}'(\boldsymbol{\eta})\,, \tag{4.6}$$

which can be solved iteratively or by linearization. It is useful to note that the quantities $\mathbf{B}'(\boldsymbol{\eta})$ and $\mathbf{C}'(\boldsymbol{\eta})$ will involve only the layer of elements on the free surface, but their dependence on $\boldsymbol{\eta}$ is algebraically complicated. If the values of $\boldsymbol{\eta}$ obtained from (4.6) differ from those assumed in (4.5), it is necessary to adjust the finite-element mesh in the neighborhood of the free surface and repeat the procedure again. This is basically the so-called moving-net of computations (see, for example, Sarpkaya & Hiriart 1975).

The difficulty of finding a proper truncation boundary is obvious. To circumvent this, Yim (1975) extended (4.3) to a hybrid formulation that accounts for the flow exterior to Σ. The exterior flow is based on linear theory, but the compatibility of the solutions at the interfacing surface Σ was not addressed. No calculations were presented by Yim.

It is sometimes more convenient to consider a total potential Φ that includes the incident stream. If $\tfrac{1}{2}U^2$ is the Bernoulli constant, the functional corresponding to (4.3) for Φ is given by

$$J(\Phi\,,\eta) = \int_{\Omega} \tfrac{1}{2}|\nabla\Phi|^2\, d\Omega - \int_{\Sigma} V_n\Phi\; d\partial\Omega + \tfrac{1}{2}\int_{\mathcal{F}_0} (g\eta^2 - U^2\eta)dxdz\,, \tag{4.7}$$

where we note that homogeneous boundary conditions on $\mathcal{F}$, $\mathcal{H}$, and $\mathcal{B}$ are all satisfied naturally. Apparently unaware of (4.7), Chan & Larock (1973) and Larock & Taylor (1976) calculated the exit flow from an orifice under gravity by using (4.7) without the last term. Instead, an essential condition for the potential on the free surface was imposed. This potential was estimated by integrating the velocity, whose variation was determined by using the dynamic boundary condition. The position of the free surface was successively adjusted to satisfy the no-flux condition. In their examples, Chan & Larock reported that convergence to four decimal places of accuracy occurs within ten iterations.

Complementary to (4.7), a variational principle exists in terms of the stream function in two-dimensional flow. This can be obtained by considering the Lagrangian $p + \frac{1}{2}|\nabla\Psi|^2$, where Ψ is the stream function. The functional now is given by

$$I(\Psi , \eta) = \int_{\Omega} \tfrac{1}{2}|\nabla\Psi|^2 \, d\Omega + \tfrac{1}{2} \int_{\mathcal{F}_0} (U^2\eta - g\eta^2) dx , \tag{4.8}$$

where Ψ is either prescribed or its normal derivatives assumed to vanish on $\partial\Omega$. If Ψ and η are treated as independent variations then

$$\begin{aligned} \delta I = & - \int_{\Omega} \nabla^2\Psi \, \delta\Psi \, d\Omega + \int_{\partial\Omega} \frac{\partial\Psi}{\partial n} \, \delta\Psi \, d\partial\Omega \\ & + \int_{\mathcal{F}_0} [\tfrac{1}{2}(U^2 + |\nabla\Psi|^2) - g\eta] \delta\eta \, dx , \end{aligned} \tag{4.9}$$

which does not appear to satisfy the dynamic boundary condition on $\mathcal{F}$. However, to preserve mass conservation on the free surface, any variation in Ψ on $\mathcal{F}$ must be made in relation to $\delta\eta$. Hence, by noting that $\delta\Psi ds = - \Psi_n \, \delta\eta dx$ on $\mathcal{F}$, we obtain

$$\int_{\mathcal{F}} \frac{\partial\Psi}{\partial n} \, \delta\Psi \, d\partial\Omega = - \int_{\mathcal{F}_0} |\nabla\Psi|^2 \delta\eta \, dx , \tag{4.10}$$

which yields the correct dynamic condition when combined with the third term of (4.9).

Related forms of (4.8) were first given by O'Carroll (1976) and Betts (1979). Computations based on this formulation have been made by Betts (1979) and Aitchison (1980) for nonlinear flow over weirs. The normal procedure is first to determine the critical flux value for a given U and weir geometry. The flux is, in fact, the value of the stream function on $\mathcal{B}$, if Ψ is taken to be zero on the free surface $\mathcal{F}$; both can be specified as essential conditions. Aitchison argued that the critical flux could be determined by observing that the upstream waves reversed in phase as the critical value was approached. For any flux less than critical, she next repeatedly relocated the downstream truncation boundary until the upstream waves disappeared. In the present formulation, (4.8) requires the tangential velocity Ψ_n on the truncation boundary to vanish completely. These are essentially surfaces of constant potential alluded to earlier. In the presence of disturbances generated by an obstruction, such surfaces need not be planar, as Aitchison had assumed. Very reasonable results were obtained by Betts & Assaat (1980) when they applied the procedure to the calculation of nonlinear *free* waves that are periodic in space, and vertical planes of symmetry do exist.

Hybrid Formulations

The term hybrid method is used here to designate methods that employ different solution techniques in two or more subregions of the problem. A common approach is to take advantage of the availability of analytical solutions in flow regions where the geometry is relatively simple. Regions where the geometry is arbitrary or complicated will still be handled by finite elements in the usual context. Such an approach generally allows the user to reduce the number of mesh points and unknowns, with a resulting decrease in storage requirement and computational time. A further advantage is that the analytical solutions can be chosen to permit a rational treatment of the effects of radiation. However, the solutions of the different subdomains must be properly matched together. The idea is quite similar to the analytical theory of matched asymptotics, except that the matching process is often a direct one across the common boundaries of the "inner" and "outer" solution. The hybrid concept as described is not only restricted to finite-element methods. Similar applications can also be found in integral-equation (Section 5) and finite-difference methods (Shaw 1975).

The more successful applications of hybrid methods have so far been restricted to linearized problems where analytical solution in the exterior regions could be obtained without too much difficulty. In particular, treatments for steady flow in a uniform stream or time-harmonic flows with the linearized conditions (2.14) or (2.20) have been quite well established. A convenient way of classifying them is based on their formulation: a "weak" one or a "strong" one in the context of finite-element terminology. We shall discuss them in just that order.

The majority of the methods are based on Galerkin's formulation. Consider the fluid domain to be defined by an interior region Ω^i and an exterior region Ω^e with Σ as the separating boundary. One particular approach to the steady-flow problem, as given by Bai (1977), is to require the *interior* potential $\phi^{(i)}$ to satisfy the following Galerkin statement:

$$\int_{\Omega^i} \nabla^2\phi^{(i)}\,\psi^{(i)}\,d\Omega + \int_{\partial\Omega} \phi_n^{(i)}\,\psi^{(i)}\,d\partial\Omega - \int_{\Omega^i} \nabla\phi^{(i)}\cdot\nabla\psi^{(i)}\,d\Omega$$

$$= \int_{\mathcal{H}} V_n\phi^{(i)}\,\psi^{(i)}\,d\partial\Omega + \int_{\Sigma} \phi_n^{(e)}\,\psi^{(i)}\,d\partial\Omega + \kappa^{-1}\int_{\mathcal{F}_0} \phi_x^{(i)}\,\psi_x^{(i)}\,dxdz$$

$$- \kappa^{-1}\left[\oint_{\mathcal{F}_0\cap\mathcal{H}} \phi_x^{(i)}\,\psi^{(i)}\,n_x ds + \oint_{\mathcal{F}_0\cap\Sigma} \phi_x^{(e)}\,\psi^{(i)}\,n_x ds\right]$$

$$= 0\,, \qquad \text{for all } \psi^{(i)} \text{ in } \Omega^i\,, \tag{4.11}$$

where $\psi^{(i)}$ is a member of the test-function space. The terms with the factor κ^{-1} are consequences of integration by parts of the free-surface condition (2.14). We observe that the derivatives of $\phi^{(i)}$ on Σ are matched with those of $\phi^{(e)}$ in the exterior region as if the latter were given. We note that in two-dimensional problems, the notation $\oint[\;]n_x ds$ in (4.11) should be interpreted as an evaluation of the quantity in [ ] at the designated intersection points. Bai went through a development similar to (4.11) in Ω^e for the *exterior* potential $\phi^{(e)}$ and test function $\psi^{(e)}$, both assumed to take the form of eigenfunctions, but his results can be simplified and stated as

$$\int_{\Sigma} (\phi_n^{(e)}\,\psi^{(e)} - \psi_n^{(e)}\,\phi^{(i)})d\partial\Omega - \kappa^{-1}\int_{\mathcal{F}_0\cap\Sigma} [\phi_x^{(e)}\,\psi^{(e)} - \psi_x^{(e)}\,\phi^{(i)}]n_x ds$$

$$= 0\,, \qquad \text{for all } \psi^{(e)}\,. \tag{4.12}$$

To obtain a better understanding of this condition, we restrict our attention to the case of two-dimensional flow, but what follows can be generalized to three dimensions without much difficulty. First, we observe that in the exterior region(s), a complete and orthonormal set of eigenfunctions $\psi_p^{(e)}$, $p=1,2,\ldots$ exists with the following property:

$$< \psi_p^{(e)}\,, \psi_q^{(e)} > \equiv \int_{-h}^{0} \psi_p^{(e)}\psi_q^{(e)}\,dy - \kappa^{-1}\psi_p^{(e)}\psi_q^{(e)}\Big|_{y=0}$$

$$= \delta_{pq}\,, \qquad p\,, q \geqslant 1\,, \tag{4.13}$$

where δ_{pq} is the Kronecker delta. If $\phi^{(e)}$ is expressed as an eigenfunction expansion of the form:

$$\phi^{(e)} = \sum_{j=1}^{\infty} \alpha_j\psi_j^{(e)}\,, \tag{4.14}$$

where the α_j 's are unknown coefficients, it is rather straightforward to show, using (4.13), that (4.12) is completely equivalent to the condition

$$\alpha_j = <\psi_j^{(e)}\,, \phi^{(i)}\Big|_{\Sigma}>\,, \qquad j = 1\,, 2\,, \ldots. \tag{4.15}$$

In simple terms, (4.15) or (4.12) states that the exterior potential has been expanded in a series of orthogonal eigenfunctions, using the interior potential at Σ as a Dirichlet condition. The determination of $\phi^{(i)}$ on Σ can be completed by noting that the space of $\psi^{(i)}$ in (4.11) is identical with that of $\phi^{(i)}$, and $\phi_n^{(e)}$ (as well as $\phi_x^{(e)}$) is expressible in terms of the α_j's. A more explicit illustration of this methodology of matching can be found in Yeung (1981). The radiation condition in this problem can be imposed as follows: the absence of waves in the upstream exterior region is enforced by requiring the particular coefficient associated with the wave mode to vanish; this yields an extra condition in the form of (4.15) that compensates for the *apparent* indeterminancy of the phase angle of the downstream waves at Σ. An analogous situation arose in the hybrid integral-equation method of Yeung & Bouger (1979). The Galerkin formulation of (4.11) and (4.12) has been used by Bai (1977, 1978) to calculate flow about two-dimensional bodies under a free surface and a steadily moving ship in a canal. With this method, Bai (1979) also derived an approximate formula that corrects for blockage effects of ship models in a towing-tank experiment.

A rather different sort of approach for this very same problem was used by Yamamoto & Kagemoto (1976). Instead of starting with the linearized boundary-value problem, they made use of the functional (4.3) for the interior region Ω^i with the assumptions that V_n and η were known on Σ, the former being satisfied as a natural condition, the latter as an essential condition. If the contributions from the free-surface integral in (4.3) were expanded about $\mathcal{F}_0$ (i.e. $y = 0$), and only the leading-order terms (ϕ, η assumed small) were kept, the following functional was obtained:

$$J^{(i)}(\phi^{(i)}, \eta^{(i)}) = \int_{\Omega^i} (\tfrac{1}{2}|\nabla\phi^{(i)}|^2 - U\phi_x^{(i)})d\Omega + \int_{\Sigma} (Un_x - \phi_n^{(e)})\phi^{(i)}\, d\partial\Omega + \int_{\mathcal{F}_0} \left(\frac{g}{2}\eta^{(i)2} - U\phi_x^{(i)}\eta^{(i)}\right)d\partial\Omega\,. \tag{4.16}$$

The first variation of (4.16) with respect to $\phi^{(i)}$ and $\eta^{(i)}$ can now be taken and manipulated to yield:

$$\begin{aligned}\delta J^{(i)} = &-\int_{\Omega^i} \nabla^2\phi^{(i)}\delta\phi^{(i)}\, d\Omega + \int_{\mathcal{B}\cup\mathcal{H}} (\phi_n^{(i)} - Un_x)\, \delta\phi^{(i)}\, d\partial\Omega \\ &+ \int_{\Sigma} (\phi_n^{(i)} - \phi_n^{(e)})\delta\phi^{(i)}\, d\partial\Omega \\ &+ \int_{\mathcal{F}_0} [(\phi_y^{(i)} + U\eta_x^{(i)})\delta\phi^{(i)} + (g\eta^{(i)} - U\phi_x^{(i)})\delta\eta^{(i)}]d\partial\Omega \\ &+ \oint_{\mathcal{F}_0\cap(\Sigma\cup\mathcal{H})} (Un_x\eta^{(i)})\delta\phi^{(i)}\, ds\,. \end{aligned} \tag{4.17}$$

Thus, it is clear that all boundary conditions of the problem are satisfied naturally, except for the last term. Yamamoto & Kagemoto pointed out that this "inconsistent" term was caused by the linearization process, which makes mass conservation impossible along the intersection contours $\mathcal{F}_0 \cap (\Sigma \cup \mathcal{H})$. By examining the nonlinear terms discarded, they argued the necessity of introducing an auxilliary condition along these contours to supplement (4.17). The final form used by Yamamoto & Kagemoto is a Galerkin-type equation given by

$$\delta J^{(i)} - \oint_{\mathcal{F}_0 \cap \mathcal{H}} \phi_n^{(i)} \, \eta^{(i)} \, \delta\phi^{(i)} \, ds - \oint_{\mathcal{F}_0 \cap \Sigma} U n_x \, \eta^{(e)} \, \delta\phi^{(i)} \, ds = 0 \,, \tag{4.18}$$

where the additional integrals of (4.18) can be seen to combine with the last term of (4.17) to yield mass conservation. No details on the treatment of the second term of (4.18), which is nonlinear, were given. In the exterior region, a solution representation based on (4.14) was used by these authors. The matching of the interior and exterior potentials across Σ was accomplished by a collocation method on Σ. The authors reported some results for the two-dimensional flow about a vertical plate piercing the free surface. Their formulation appears to generate a wave profile of equal heights on both surfaces of the plate. No results were available to allow a direct comparison with those based on Bai's procedure.

The linearized steady flow about a submerged two-dimensional body was solved also by Mei & Chen (1976) using hybrid elements. The formulation was one that utilized the variational principles for the time-harmonic problems described below. Fictitious radiation and scattering problems based on the free-surface condition (2.14) were introduced. The unknown "incident-wave" amplitude of the scattering problem was determined so that the upstream waves vanished completely when combined with the radiation problem. Mei & Chen have also outlined how the procedure can be generalized to three-dimensional problems.

For the class of *time-harmonic* problems defined by (2.3), (2.19), and (2.20), a variety of hybrid techniques based on the Galerkin procedure have been successfully implemented. The usual approach is to construct a functional of the weak form for the interior potential. For this purpose, it is customary to consider $\varphi_n^{(i)}$ as prescribed (and given by $\varphi_n^{(e)}$) on Σ. This will match the normal derivatives and couple the interior problem with the unknown parameters of the exterior representations, which could be in the form of a source distribution (Berkhoff 1972), or an eigenfunction expansion (Chenot 1975). The additional conditions for the determination of the exterior parameters can then be obtained by equating the exterior and interior potential at Σ, with a criterion based on either collocation, least square, or Galerkin. For the last case, if an orthonormal expansion exists in Ω^e, (4.15) results.

An alternative approach based on Galerkin's procedure has been adopted by Lenoir & Jami (1978), who considered φ on Σ as an essential condition, and related this potential to that on $\mathcal{H}$ by using the Green function of the problem (see Section 5), which, by definition, satisfies the radiation condition as required. The idea is originally due to McDonald & Wexler (1972) and has the advantage that if Σ is sufficiently far from $\mathcal{H}$, the singular terms of the Green functions are more amenable to numerical treatment. This coupling condition between Σ and $\mathcal{H}$, however, brought back the unpleasant feature of irregular frequencies associated with the use of Green functions (see section 5 for details). Further, the choice of such a coupling condition is not unique. One can presumably relate the normal derivative on Σ to that on $\mathcal{H}$, instead of the potential, or in fact any linear combination of them.

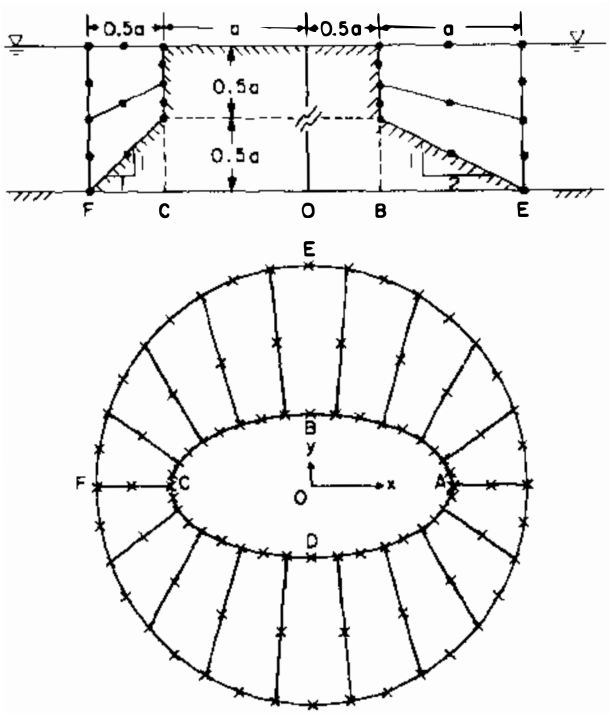
Variational principles specifically applicable to a *hybrid formulation* have been obtained for time-harmonic problems by Bai & Yeung (1974) and Chen & Mei (1974). These have been applied successfully to a variety of two- and three-dimensional problems of water-wave radiation and diffraction (see Mei 1978 for more references). The aforementioned principles can be shown to be special cases of the following:

$$J(\varphi^{(i)}, \varphi^{(e)}) = \tfrac{1}{2}\int_{\Omega} (\nabla\varphi^{(i)})^2 d\Omega - \frac{\omega^2}{2g}\int_{\mathcal{F}_0} \varphi^{(i)2}\, d\partial\Omega - \int_{\mathcal{H}} v_n \varphi^{(i)}\, d\partial\Omega$$
$$- \int_{\Sigma} [\varphi_n^{(e)}(\varphi^{(i)} - \tfrac{1}{2}\varphi^{(e)}) + k(\varphi^{(i)} - \varphi^{(e)})(\varphi_n^{(i)} - \varphi_n^{(e)})] d\partial\Omega\,, \tag{4.19}$$

where k could be any arbitrary value in $[0,1]$ and $\varphi^{(e)}$ is assumed to satisfy all conditions in Ω^e. In spite of the apparently non-unique form of J, the extremum is always given by the third integral. The variation of (4.19) can be easily shown to yield all boundary conditions, including the matching of φ and φ_n, naturally. In particular, the first and last term can be combined to yield the following Galerkin form:

$$\int_{\Sigma} \{(\varphi_n^{(i)} - \varphi_n^{(e)})\,[\delta\varphi^{(i)}(1-k) + \delta\varphi^{(e)}k]$$
$$- (\varphi^{(i)} - \varphi^{(e)})\,[\delta\varphi_n^{(i)}k + \delta\varphi_n^{(e)}(1-k)]\}\, d\partial\Omega\,, \tag{4.20}$$

the interpretation of which in the context of weighted residuals is rather obvious. The computational capabilities of the type of hybrid methods discussed here is illustrated by Figure 3. Taken from Yue et al. (1978), the figure shows the (time-) complex free-surface amplitude around an elliptic island on a circular base due to unit-amplitude waves at two different angles of incidence. The formulation used is one related to (4.19) with $k = 0$.



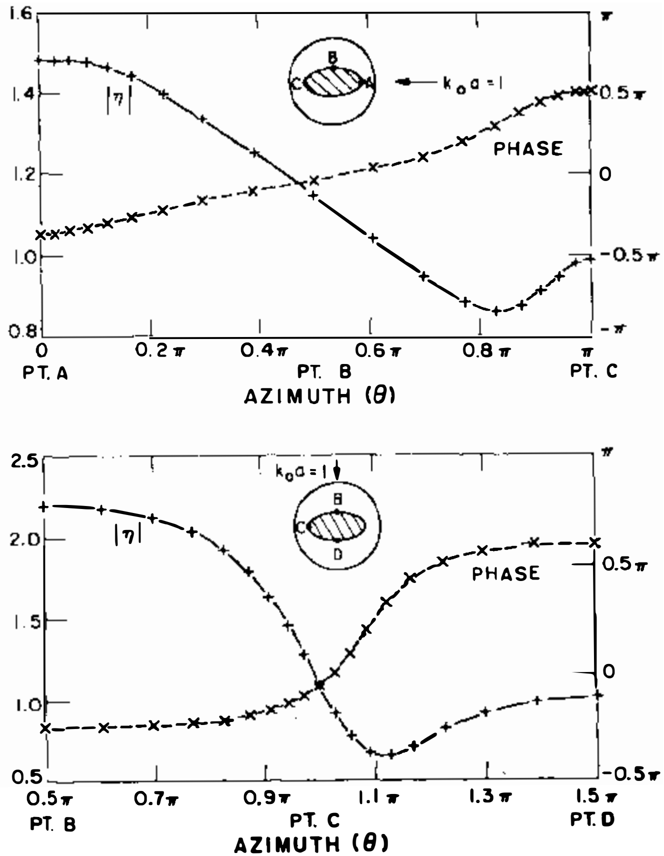


Figure 3 Wave diffraction around an elliptical island with a circular base using a hybrid-element method (from Yue et al 1978). (*a*) Geometric specifics of the island and finite-element grid; (*b*) Run-up, η, on the island due to (unit-amplitude) waves at two different angles of incidence, k_0 being the wave number.

The use of special boundary elements for tackling time-dependent domains has received considerable attention lately in finite-element methods (Banerjee & Butterfield 1979). For unsteady water-wave problems, Wellford & Ganaba (1980) have implemented a pseudo-variational principle due to Pian & Tong (1968).

$$I(\phi\ , V_n) = \tfrac{1}{2}\int_{\Omega} |\nabla\phi|^2 d\Omega - \int_{\mathcal{F}} V_n\,(\phi-\tilde{\phi})d\partial\Omega + \int_{\mathcal{H}\cup\mathcal{B}} \phi V_n\, d\partial\Omega\ , \tag{4.23}$$

where $\tilde{\phi}$ and V_n on $\mathcal{F}$ is the potential and its normal derivative respectively. Equation (4.23) can be used to determine V_n on $\mathcal{F}$ at successive time steps if $\mathcal{F}$ and $\tilde{\phi}$ are known. Starting with known initial values, Wellford & Ganaba advanced $\mathcal{F}$ and $\tilde{\phi}$ in time by an explicit scheme similar to (3.5a), which gave rise to numerical instabilities. Artificial damping was introduced as an attempt to eliminate unwarranted oscillations. The approach used here appears promising but the method needs further refinements. Licht (1980) considered the simpler unsteady problem with the linearized free-surface condition. A fully implicit scheme is used to advance ϕ and η, with (3.11) being used as the open-boundary condition. He experimented with a number of constant values of c to compare his numerical results with existing ones, but offered no insight on the choice of c for a general problem.

5. BOUNDARY-INTEGRAL EQUATION METHODS

The treatment of problems in potential theory by integral-equation methods is classical. The use of "single-layer" and "double-layer" distributions to solve problems of Neumann and Dirichlet type, respectively, are well known from texts such as Kellogg (1929). These formulations lead to Fredholm integral equations of the second kind, for which a rather complete mathematical theory of existence and uniqueness has been developed (see Mikhlin 1957). Less seems to be known about integral equations of the first kind, but it is generally accepted that a well-posed problem will lead to physically plausible numerical solutions. A distinct advantage of an integral-equation formulation over the space-discretization formulations of Sections 3 and 4 is that the space dimension of the problem is reduced by one. Of course, some "fundamental" solution must first be obtained. Free-surface flows seem particularly suitable for integral-equation treatments since physical quantities of primary interest, such as wave height and fluid pressure, are required only on the boundaries. Space-discretization techniques would appear inefficient and wasteful, since they yield massive amounts of interior data that are normally of minor use.

Integral equations for linearized free-surface flows in the presence of bodies were available in the 1950s from the works of John (1950) and Stoker (1957). Very little progress was made towards solving them, except for the few cases of geometry that were described by special coordinates. The works of Hess & Smith (1964, 1967) and their collaborators marked the beginning of a new era of computation. With the advent of the high-speed computer and of the discretization techniques developed by these authors, calculations of flows about arbitrarily shaped bodies in an infinite fluid became possible. The successes in aerodynamics were not matched at an equal pace by those in free-surface hydrodynamics. The delay was associated with the need to develop accurate and efficient means of evaluating and integrating (in space) the complicated Green functions associated with the free-surface conditions. Earlier works were often marred by uncertainties in numerical accuracy, particularly those in three dimensions.

A small departure from tradition was made by Yeung in 1973 (see Bai & Yeung 1974), who pointed out that one could obtain the "usual" wave-like solution by a direct application of Green's theorem, using just the infinite-fluid source. This has, in fact, opened up a new avenue of formulation in body-wave problems. To distinguish the use of such "simple-source" functions from traditional Green functions that satisfy the linearized free-surface conditions and the radiation condition, we shall use the term *Green function* to mean specifically the latter. Literature in both areas is growing. The power of integral-equation methods in treating nonlinear problems has been well demonstrated by Longuet-Higgins & Cokelet (1976). These authors developed a time-stepping algorithm, based on a Lagrangian description of the free-surface fluid particles, for studying the evolution of the profiles of breaking waves. Inverse methods, based on reversing the roles of dependent and independent variables, have recently been found to be quite effective in treating a special class of nonlinear two-dimensional problems, where the body is part of the fluid boundary.

Are boundary-integral equation methods more efficient than space-discretization methods? We shall provide a guideline for answering this nontrivial, but recurrent, question. Let us assume that the major phases of the computational efforts consist of 1. the generagion of a matrix associated with the discretization and 2. the direct "inversion" of this matrix. Suppose that K and N are the total number of unknowns corresponding to integral-equation and space discretization respectively, and that m is the bandwidth associated with the latter method (m being K for the former, which gives rise to a full matrix). Suppose further that β (and γ) is the ratio of the time required to generate a typical matrix coefficient to that required in an inversion step for the integral-equation (and space-discretization) method. Then, clearly, we could argue that the former is superior if

$$K^2(K+3\beta) \leqslant 3Nm(m+\gamma)\ ,$$

where the first term on each side represents the effort involved in the inversion process, and the second in the generation process. Noting that m is $O(N^{1/2})$ and $O(N^{2/3})$ for two- and three-dimensional problems respectively, we obtain

$$(K+3\beta) \leqslant 3e^2 \begin{cases} 1+\gamma N^{-1/2}\ , & \text{for two dimensions}\ , \quad (5.2a) \\ N^{1/3}+\gamma N^{-1/3}\ , & \text{for three dimensions}\ , \quad (5.2b) \end{cases}$$

where $e \equiv N/K$. Equation (5.2) says that for small K, corresponding to the situation of using Green functions, integral-equation methods are superior if the coefficient generation index β is less than $O(e^2)$ in two dimensions and $O(e^2N^{1/3})$ in three dimensions. This is certainly true in the two-dimensional case. In three dimensions, an inevitable increase in β, together with a moderate decrease in e, actually makes the various methods more competitive. In the other limit where K is $O(N^{1/2})$ and $O(N^{2/3})$, in two and three dimensions, integral-equation formulation is always a viable alternative when β is $O(K^2)$ and $O(K^{3/2})$. This simplified analysis does not, of course, account for a host of other factors, such as the use of relaxation iterative techniques, the non-constant value of β within the matrix, the effective use of a hybrid formulation to minimize the computational domain, and the manner in which the double condition on the free surface can be satisfied. Nevertheless, (5.2) should provide a rational basis for making comparisons among various methods. For example, one can then speak of the typical β value in an integral-equation formulation, or the different γ values for various space-discretization methods. We emphasize that boundary-integral methods are always superior in data-storage requirement. Space-discretization techniques saturate the rapid-access memory very quickly; peripheral data storage and transfer are costly in time and effort.

Methods Based on Green's Functions

The method of Green's functions is quite powerful in linearized problems whenever a Green function satisfying the appropriate linearized free-surface conditions and other auxilliary conditions can be derived. A rather complete collection of them is available from Wehausen & Laitone (1960). Integral equations for the velocity potential can generally be derived from Green's third identity:

$$\frac{1}{2}\phi(P) = \int_{\partial\Omega} \phi(Q)G(P,Q)\, d\partial\Omega_Q - \int_{\partial\Omega} \phi G_\nu\, d\partial\Omega_Q\ , \quad P \in \partial\Omega\ , \qquad (5.3)$$

where Q is the "source" point, and P the field point, assumed to approach $\partial\Omega$ from the interior of Ω. Here, the subscript ν denotes the operator $\mathbf{n}(Q) \cdot \nabla$. The Green function is assumed to be of the form $[r^{-1}+H(P,Q)]/4\pi$ in three dimensions, $[-\log r + H]/2\pi$ in two dimensions, with r being $|PQ|$ and H a harmonic function chosen to satisfy the appropriate boundary conditions of the problem. As an example, for time-harmonic problems, if H is constructed so that G satisfies (2.20), (2.19) on $\mathcal{B}$, and the radiation condition (2.21), the boundary $\partial\Omega$ in (5.3) can be shown to reduce to $\mathcal{H}$ only.

A variety of integral equations exist in the literature of body-wave hydrodynamics. These variants are all related to the assumptions made about the behavior of the potential $\overline{\phi}$ inside the body. We shall denote this interior domain by $\overline{\Omega}$; it "intersects" with $\mathcal{F}$ if $\mathcal{H}$ is surface-piercing. For illustration, consider the *time-harmonic* problems just mentioned. Applications of (5.3) in Ω and $\overline{\Omega}$, with P in Ω, can be combined to yield

$$\frac{1}{2}\begin{pmatrix} \varphi\,(P) \\ \overline{\varphi}\,(P) \end{pmatrix} = \int_{\mathcal{H}} (\varphi_\nu - \overline{\varphi})G\,d\partial\Omega - \int_{\mathcal{H}} (\varphi - \overline{\varphi})G_\nu d\partial\Omega\,, \quad \begin{cases} P \in \partial\Omega \\ P \in \partial\overline{\Omega}\,. \end{cases} \tag{5.4}$$

Since the interior potential has no physical meaning, one could choose $\overline{\varphi}$ identically zero in $\overline{\Omega}$, hence also $\overline{\varphi}_\nu$. This choice yields the so-called Green's mixed distribution, or, equivalently,

$$\frac{1}{2}\varphi(P) + \int_{\mathcal{H}} \varphi G_\nu d\partial\Omega = \int_{\mathcal{H}} v_n G\,d\partial\Omega\,, \qquad P \in \partial\Omega\,, \tag{5.5}$$

which is a Fredholm integral equation of the second kind. Alternatively, with $\overline{\varphi} = \varphi$ a source distribution of strength $\sigma = \varphi_\nu - \overline{\varphi}_\nu$ results, whereas with $\overline{\varphi}_\nu = \varphi_\nu$, a normal-dipole distribution of strength $\mu = \overline{\phi} - \phi$ follows. If the Neumann condition on $\mathcal{H}$ is imposed, the integral equation for σ is

$$\frac{1}{2}\sigma(P) + \int_{\mathcal{H}} \sigma G_n\,d\partial\Omega = v_n(P)\,, \qquad P \in \partial\Omega\,, \tag{5.6}$$

whose kernel is related to that of the mixed distribution by a similarity transformation. Equation (5.6) is the version "preferred" by most workers (Frank 1967, Lebreton & Margnac 1968, Garrison & Rao 1971, Faltinsen & Michelsen 1974, and others), probably because of the influence from earlier aerodynamic literature. Fundamentally, (5.5) and (5.6) involve the same amount of computational effort (G and G_n or G_ν), since φ, being proportional to the pressure, is the desired quantity. Actually, (5.5) has a slight advantage over (5.6), and we shall address that momentarily. The mixed distribution has been used by Potash (1971) and Macaskill (1977); the latter treated the two-dimensional problem of unequal depths by matching at a common boundary. A representative work using dipole distribution is that of Chang & Pien (1975), who noted that (5.5) yields the following integral equation of the first kind for μ, in terms of the interior potential $\bar{\varphi}$:

$$\int_{\mathcal{H}} \mu G_\nu \, d\partial\Omega = \frac{1}{2}\,\bar{\varphi}\,(P)\,, \tag{5.7}$$

where $\bar{\varphi}(P)=Uy$ for $\varphi_\nu=\bar{\varphi}_\nu=n_y$, and so on for other rigid-body motions. For an arbitrary φ_ν this procedure breaks down since $\bar{\varphi}$ is unknown. Note, however, in the case of rigid-body motion, $\varphi(P)$ $(=\bar{\varphi}-\mu)$ is known immediately once μ is obtained by solving (5.7). Thus, in this sense, the dipole-distribution method can be 50% more efficient than the other two methods. Being dominant *off* the diagonal, the matrix equations of (5.7) are much less amenable to iterative solutions than those of (5.5) and (5.6). An integral equation for μ can also be obtained by taking the normal derivative of (5.4), but this results in second derivatives of the Green function, which compounds the already difficult task of evaluating a highly oscillatory Green function.

The integral equations (5.5–5.7) are all plagued by the presence of irregular frequencies when $\mathcal{H}$ is surface-piercing. This was first pointed out by John (1950) in the context of source distributions for floating-body problems. More recent references related to similar difficulties in accoustic radiation may be found in Jones (1974) and Mei (1978). Here, it suffices to mention that the difficulty is associated with the vanishing of the Fredholm determinant of (5.5–5.7). We recall that a representation based on either a mixed or source distribution imposes a Dirichlet condition on $\partial\bar{\Omega}$, and that the dipole distribution imposes a Neumann condition on $\partial\bar{\Omega}$. If the homogeneous equations associated with these interior problems have either unbounded or nontrivial solution, which happens at a discrete set of "resonant frequencies" in a closed "basin," these representations break down. Note that (5.5) and (5.6) have the same irregular frequencies since the kernel of one is the "transpose" of the other. Further, a subtle distinction between the mixed and source representation is that the inhomogeneous term of the former is orthogonal to the interior eigensolutions at these frequencies, whereas that of the latter is generally not. By the Fredholm alternatives (see, for example, Delves & Walsh 1974), Equation (5.5) actually has a solution, though not unique, whereas (5.6) has none. Neither situation is entirely desirable, but it has been observed that in actual numerical computations (Adachi & Ohmatsu 1979), the mixed distribution yields less "perturbed" results than the source one. After all, it is difficult to "hit" the irregular frequencies precisely, since they are unknown a priori.

Various means of circumventing this difficulty exist. Overspecifying the interior problem by putting a "lid" on $\mathcal{F}$, is one effective alternative (Kobus

1976). Modifying the kernel function G by adding other concentrated singularities in $\overline{\Omega}$ (Ogilvie & Shin 1978, Sayer & Ursell 1977) is another. Recent work of Ursell (1980a) shows that the choice of the coefficient multiplying the concentrated singularities in the modified G is not entirely arbitrary.

For linearized *steady-motion problems,* the integral equation analogous to (5.4) is

$$\frac{1}{2}\begin{pmatrix}\phi\,(P)\\ \overline{\phi}\,(P)\end{pmatrix} = \int_{\mathcal{H}}[(\phi_\nu-\overline{\phi}_\nu)G-(\phi-\overline{\phi})G_\nu]d\partial\Omega + \kappa^{-1}\oint_{\mathcal{H}\cap\mathcal{F}_0}[(\phi-\overline{\phi})G_x-(\phi_x-\overline{\phi}_x)G]n_x\,ds\,, \tag{5.8}$$

where G is assumed to satisfy (2.3) on $\mathcal{B}$, (2.14), and the radiation condition (2.12). The additional term, known commonly as the *line integral* in the Neumann-Kelvin problem of ship hydrodynamics, is associated with an integration by parts of (2.14). Note now that the requirement $\overline{\phi}_\nu = \phi_\nu$ does not result in purely dipole terms, since $\overline{\phi}_x = \phi_x$ is not subsequently implied. On the other hand, a representation consisting purely of sources (surface and "line") is possible. Numerous attempts have been made to solve (5.8), but none made to investigate the possible existence of irregular frequencies. A common misconception has been that the line integral is highly singular. Actually, when the "source" point of G is on $\mathcal{F}_0$, the Green function is merely a weakly singular pressure point. Furthermore, such a singularity may not even exist because of possible cancellation with the edge behavior of the surface distribution. Practical implications or logical inconsistency put aside, it seems that the Neumann-Kelvin problem deserves more thorough attention on the analytical side, especially in view of the amount of effort invested in solving it. Ursell (1980b) examined the two-dimensional problem in some detail and concluded that a "least singular solution" can exist if the source density at the edge of the "surface" is the same as that which arises from the intersection point $\mathcal{H}\cap\mathcal{F}_0$. The implication of this in three dimensions is not immediately obvious, because the normal derivatives and the longitudinal derivatives are not identical, as in the case of two dimensions. However, it is precisely this kind of analysis that would elucidate the exact mathematical difficulty of such a formulation. The works of Guével et al. (1977) and Tsutsumi (1979) are, perhaps, representative of the state of the art of such numerical endeavors. Paradoxically, no two independent works exist, regardless of the numerical methods, that are in agreement with each other.

The "difficulty" associated with the line integral disappears if $\mathcal{H}$ does not intersect with $\mathcal{F}_0$ or if $\mathcal{H}$ is taken as a vertical plate on the centerplane where a linearized body condition is applied. In the latter case, (5.8) can be easily shown to yield

$$\phi(P) = 2\int_{\mathcal{H}} V_n(Q)G(P, Q)\, dxdz(Q)\,, \tag{5.8a}$$

which formed the basis of the well-known work of Michell (1898) on the wave-resistance of thin bodies.

An integral-equation formulation for *time-dependent* problems involving the condition (2.18) was developed only relatively recently (Finkelstein 1957). While (5.3) is still the basis of the formulation, the extra parameter (time) leads to a memory integral, the computational effort of which is quite extensive. Aside from (2.18), the unsteady Green function satisfies two homogeneous initial conditions on the free surface. With the assumptions that the fluid disturbances were initially zero, it can be shown (Yeung 1982) that the integral equation for $\phi(P, t)$ is given by

$$\begin{aligned}\frac{1}{2}\phi(P, t) - \int_{\mathcal{H}} \phi(Q, t)G^{\circ}_{\nu}(P, Q)\, d\partial\Omega_Q = \\ - \int_{\mathcal{H}} V_n(Q, t)G^{\circ}(P, Q)\, d\partial\Omega_Q \\ + \int_0^t d\tau \int_{\mathcal{H}} [\phi(Q, \tau)H_{\tau\nu}(P, Q, t-\tau) - V_n(Q, \tau)H_{\tau}]d\partial\Omega\,, \\ \text{for } P \in \partial\Omega\,,\end{aligned} \tag{5.9}$$

where $G^{\circ} = 4\pi(1/r - 1/r')$, for three dimensions, with r' being the distances from P to the image point of Q about $y=0$, and a similar modification of G applies also to two dimensions. The unsteady-wave effects are all imbedded in the harmonic function H_{τ}, which has the property $H_{\tau}(P,Q,0) = 0$. Thus, the "memory effects" occur only on the right-hand side of the integral equation, but (5.9) needs to be solved at each t. Yeung discussed also the physical meaning of some of the initial conditions that were not quite clearly stated in Finkelstein's original work. Figure 4 shows the transient heave motion of a freely floating cylinder with unit initial displacement. Since $V_n(t)$ is unknown, it must be determined simultaneously from the rigid-body dynamics equation. The fine agreement with experiment seen here is consistent with the fact that hydrodynamic forces in the frequency domain have been well predicted, in the past, by time-harmonic linear theory. An existence and uniqueness theorem for such transient-motion problems has been given by Beale (1977).

Earlier work of Daoud (1975) considered a related formulation for an expanding wedge, as well as the difficulties involved in evaluating the wave kernel. R. B. Chapman (1979) considered the same problem as Yeung, but represented the fluid motion by a discrete set of wave harmonics. The method used to advance the wave motion was quite efficient, but the wave number and frequency distribution must be carefully chosen to represent a non-reflecting exterior medium.

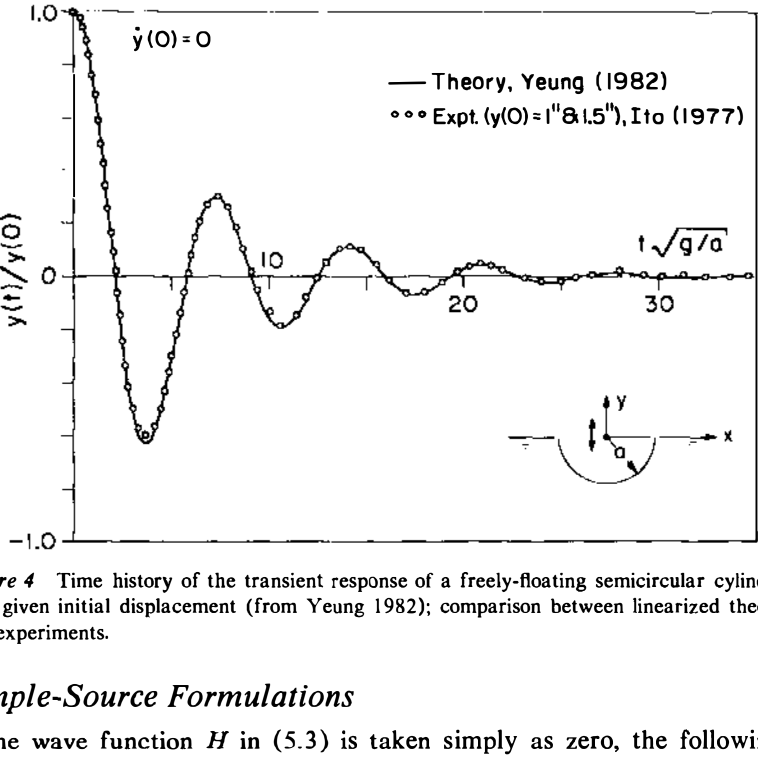


Figure 4 Time history of the transient response of a freely-floating semicircular cylinder with given initial displacement (from Yeung 1982); comparison between linearized theory and experiments.

Simple-Source Formulations

If the wave function H in (5.3) is taken simply as zero, the following relation results in two dimensions:

$$\pi\phi(P) = \int_{\partial\Omega} [\phi_\nu(Q)\log\frac{1}{r}(P, Q) - \phi\frac{\partial}{\partial\nu}\log\frac{1}{r}]d\partial\Omega_Q\,, \qquad P \in \partial\Omega\,, \tag{5.10}$$

which represents a distribution of simple (free-space) sources and dipoles on the entire boundary. Alternatively, if we introduce the complex potential $W(Z) = \phi + i\psi$, with $Z = x + iy$, Cauchy's integral formula yields

$$\pi i W(Z) = ⨍_{\partial\Omega} \frac{W(Z')}{Z'-Z}\,dZ'\,, \qquad Z'\,, Z \in \partial\Omega\,, \tag{5.11}$$

where Z is the field point and the integral ⨍ is to be interpreted in the sense of Cauchy principal values. Equation (5.11) is completely equivalent to stating that ϕ and ψ each satisfies (5.10). This can be shown easily by letting $Z'-Z = re^{i\theta}$, θ being a polar angle, and by making use of the Cauchy-Riemann relations. Equation (5.11) is, of course, not applicable in three dimensions whereas (5.10) holds with $\log r^{-1}$ replaced by $\frac{1}{2} r^{-1}$.

The simple relation (5.10) is the basis of the hybrid methods developed by Yeung (Bai & Yeung 1974, Yeung 1975) for linearized free-surface flows. Actually, Yeung treated the exterior domain as if it were interior, but with an appropriate matching of the interior and exterior solutions across the open boundary Σ. Consider the time-harmonic problem; then $\varphi_\nu^{(i)}$ is either known because of (2.19) or expressible in terms of $\varphi^{(i)}$ because of (2.20) on $\mathcal{B} \cap \mathcal{H} \cap \mathcal{F}_0$. Thus (5.10) is a Fredholm integral equation of the second kind on those boundaries. If the exterior potential $\varphi^{(e)}$ is given in terms of an eigenfunction series of the form (4.14), a direct matching of $\varphi^{(e)}$ with $\varphi^{(i)}$ as well as their normal derivatives on a vertical boundary Σ yields

$$\int_\Sigma [\varphi^{(i)}\frac{\partial}{\partial \nu}\log r - \varphi_\nu^{(i)} \log r] d\partial\Omega$$
$$= \sum_{j=1}^{\infty} \alpha_j \int_\Sigma [\psi_j^{(e)} \frac{\partial}{\partial \nu}\log r - \psi_{jx}^{(e)} \log r] d\partial\Omega\,, \tag{5.12}$$

which occurs on the right-hand side of (5.10). The resulting integral over the vertical boundary is relatively easy to evaluate. A similar substitution of $\varphi^{(i)}$ on the left-hand side of (5.10) leads to an identity that can be used to determine the coefficients α_j by collocation on Σ. The first term in the eigenfunction series corresponds to a propagating-wave solution. Note that if only one term is taken in (5.12), it is effectively the same as applying the radiation condition (2.21). In its most general form, (5.12) accounts for all nonpropagative disturbances.

This *hybrid* integral-equation formulation was applied to both two- and three-dimensional problems by Yeung. It is free from irregular frequencies. It can handle bottoms of unequal asymptotic depths in two dimensions. It is also obvious for the case of constant depth that the integral over $\mathcal{B}$ is completely eliminated (see Bai & Yeung 1974) if $\log r$ in (5.10) is simply replaced by $\log rr''$, where r'' is the distance from P to the image point of Q about $y = -h$. The disadvantage of this method, when compared with the traditional Green-function formulation, is that K in (5.2) is now considerably larger, but this is offset by a much smaller β. Yeung & Bouger (1979) also demonstrated that this method is equally applicable to the linearized steady-motion problems defined by (2.3), (2.14), and (2.12). The boundary condition (2.14), however, requires a higher-order interpolation function than before. In this work, the steady-state radiation condition was rationally treated by simply omitting the wave-like eigenfunction upstream and keeping it downstream.

Berhault (1980) considered obtaining the solution of the integral equation of Yeung by a variational formulation. His procedure yields a symmetric matrix but requires a double integration of the source function. Harten & Efrony (1978) proposed a further subdivision of $\Omega^{(i)}$, with similar matchings as described, to generate a block structure in the coefficient matrix. Efficiency could be improved, they reported, by an order of magnitude. Domain subdividing formulated with (5.10) in mind appears to combine the best features of boundary-integral methods (simplicity) and space-discretization techniques (banded matrix). Earlier, Harten (1975) extended this simple-source formulation to tackle time-dependent linearized flows in an *enclosed* domain. Such interior problems, of course, do not have the difficulty associated with open boundaries. Soh (1976, 1980) considered a vortex-sheet representation of the linearized unsteady problem by using (5.11) with $W(Z)$ replaced by the complex velocity $(u-iv)$. In terms of these velocities, (2.17) implies

$$\eta_t(x, t) = v(x, 0), \qquad u_t(x, t) = -g\eta_x, \tag{5.13a,b}$$

which permits the advancement of η and u at each value of t, with v related to u by (5.11). Soh used an explicit scheme for (5.13), which appeared to cause a steady increase in phase error. Quantities outside of the truncation boundary were treated by extrapolation. A related approach had been proposed earlier by Zaroodny & Greenberg (1973), but the computations presented were too crude for drawing any conclusions. The simple-source formulation was used in a similar manner by Mattioli (1978) for solving the two-dimensional shallow-water equations.

The most successful calculations of unsteady two-dimensional waves were carried out by Longuet-Higgins & Cokelet (1976), who used the integral-equation formulation (5.10) for ϕ, the dynamic free-surface condition (2.4b) (with $U = 0$), and a Lagrangian description of the free-surface fluid particles:

$$\frac{D\mathbf{x}}{Dt} = \nabla\phi. \tag{5.14}$$

We note in passing that the vertical component of (5.14) is basically (2.4a). The periodic behavior of the wave system in x was accounted for by using the following transformation (Nekrasov 1921): $\zeta = e^{ikZ}$, where ζ is the complex variable in the mapped plane, and k the wave number associated with the periodicity. While the use of (5.10) is not new, their time-stepping procedure is quite interesting. With the wave profile and the potential on $\mathcal{F}$ given initially by a procedure such as Schwartz's (1974) for steady progressive waves, ϕ_n for a number of "marker particles" on $\mathcal{F}$ can be obtained by solving (5.10). This is an integral equation of the first kind. The Lagrangian values of ϕ and $\mathbf{x}$ of these marker points are next advanced by using (2.4b) and (5.14), since their right-hand sides, which involve velocities, are now known. Actually, these authors used a fourth-order predictor-corrector method, which required another solution of (5.11) at the corrector stage, using the predictor value. With the new values of ϕ and new location of $\mathcal{F}$, the cycle can be repeated. The procedure is remarkably simple, for a complicated nonlinear problem as such. The authors exercised care in attaining a consistent order of accuracy in all of their intermediate calculations. Figure 5 shows the development of a plunging breaker when the authors apply a surface pressure [p_0 in (2.4b)] for half a period. Since a Lagrangian description of the surface is used, the calculations can proceed without any difficulty even after the wave front becomes vertical. Note that the straining motion of the primary waves near the wave crests causes the marker particles to "bunch together," thus providing the very desirable (but unexpected) effect of improving resolution near a sharp-curvature region. The authors pointed out that grid-scale oscillations occurred in their calculations, but conjectured that they could be physical in origin. Numerical smoothing was used to eliminate these unwarranted oscillations.

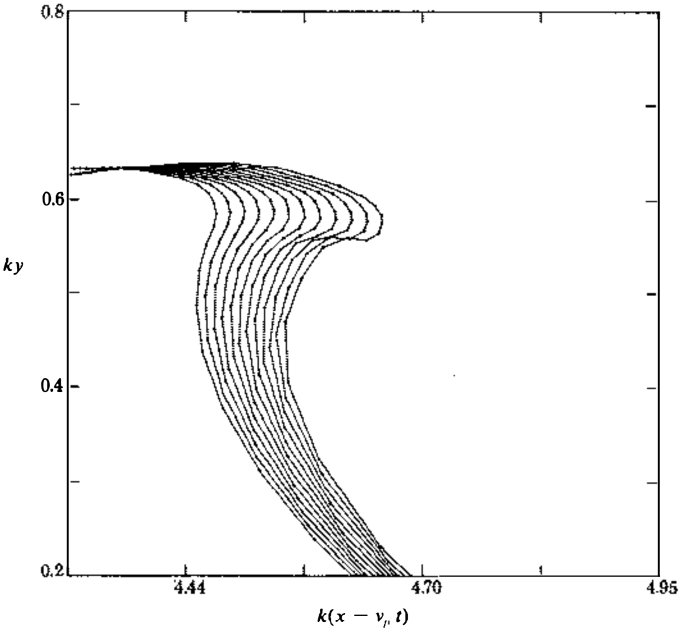


Figure 5 Theoretical wave profiles of a plunging wave breaker generated by applying a pressure distribution on a progressive wave train for one half of a wave period (from Longuet-Higgins & Cokelet 1976). The non-dimensional surface pressure used was $kp/\rho g = 0.146$. Profiles shown correspond to $4.712 < (kg)^{1/2}t < 4.928$, where k is the wave number. v_p is the phase velocity given by $(g/k)^{1/2}$. Marker particles on the free surface were initially distributed evenly along the wave length.

This Lagrangian-Eulerian integral-equation formulation was applied quite successfully by Fenton & Mills (1976) in shallow-water calculations. Faltinsen (1977) considered an extension of this method to the problem of an oscillating body in water of an infinite extent, but encountered much difficulty in eliminating the effects of the truncation boundary. Vinje & Brevig (1980) conducted a similar calculation by assuming that the body was periodic in space, but using (5.11) instead of (5.10). Their preliminary results appeared to be quite different from Faltinsen's, particularly for large-amplitude body motions.

Methods based on simple-source distributions have been used by Gadd (1976) and Dawson (1977) for three-dimensional *steady-motion* of ships. Of particular interest is Dawson's work which utilized the double-body approximation. Dawson assumed that $\phi^{(w)}$ in (2.16) can be represented by a simple source-sheet on $\mathcal{H} \cap \mathcal{F}_0$. This source strength on $\mathcal{H}$ was determined, as usual, by (2.3), but on $\mathcal{F}_0$ by (2.16), using a four-point upstream differencing scheme for the streamwise velocity. Actually, the last term of (2.16) was "neglected" by Dawson in his formulation. However, the computed wave resistance and wave profiles show definite improvement over traditional linear theory. Dawson claimed that upstream differencing was all that needed to satisfy the "radiation condition."

Inverse Formulations

The inverse formulation takes advantage of the fact that the free surface is a stream surface [see (2.9a)]. If the usual roles of the dependent (Φ, Ψ) and independent (x,y) variables are now interchanged, the dynamic free-surface condition (2.9b), though still nonlinear, can be satisfied on a *known* boundary. Stokes (1880b) noted this simplification in his work on steady periodic waves. In terms of the complex potential $W(=\Phi+i\Psi)$, this inverse representation for waves of phase velocity c and wavelength k is simply given by

$$Z(W) = -\frac{W}{c} + i\sum_{j=1}^{\infty} \alpha_j \left(e^{ikW/c} - e^{-2kh}\, e^{-ikW/c}\right), \quad \mathrm{Im}(\alpha_j) = 0, \tag{5.15}$$

which yields the free-surface elevation when evaluated at $\Psi = 0$. In fact, considerable effort by many workers in the ensuing years has been devoted to the determination of the coefficient α_j's so that the dynamic condition can be satisfied. More notable recent work is by Thomas (1968), Schwartz (1974), and Cokelet (1977).

The inverse formulation has also been much used in jets and sluice flows (Larock 1970, Moayeri 1973). Conformal mapping is normally used in problems with physical boundaries to transform both the physical and free boundaries to constant Ψ lines. The resulting problem can be solved by

Hilbert-Riemann techniques or by direct numerical methods. The need for such a mapping is evident from the fact that stagnation points are now singular points in the inverse formulations. Thus, if ζ is the mapped complex variable, the dynamic condition (2.9b) now becomes

$$\left|\frac{d\zeta}{dW}\right|^{-2}\left|\frac{dZ}{d\zeta}\right|^{-2} - U^2 + 2g\,\mathrm{Im}(Z) = 0\,, \tag{5.16}$$

which is more amenable to numerical treatment if $Z'(\zeta)$ is chosen to absorb the singular behavior. The type of geometry that can be handled is therefore rather restrictive. Inverse formulations for three-dimensional flows in a general context were considered by Jeppson (1972), but the resulting field equation was so complex that it was no longer amenable to boundary-integral equation treatment.

Using an inverse formulation, Vanden Broeck & Tuck (1977) considered the problem of steady flow about a semi-infinite body of finite draft. The authors employed the following transformation for a rectangular stern or bow profile

$$\frac{d\zeta}{dW} = -1 + \left(\frac{W}{W+1}\right)^{1/2}\frac{dZ}{dW}\,, \tag{5.17}$$

where all variables were so normalized that $Z = 0$ ($W = 0$) was the contact point between $\mathcal{F}$ and $\mathcal{H}$ and $Z = -id$ ($W = -1$) was the "keel" corner, d to be determined as part of the solution. The radical factor in (5.17) was chosen based on the behavior of the zero Froude-number solution at these points. If Cauchy's integral formula is applied to $\zeta'(W)$ in the complex W plane, it follows that

$$\zeta'(W) = \frac{-1}{2\pi i}\int_{-\infty}^{\infty}\frac{\zeta'(\tilde{\Phi})}{(\tilde{\Phi}-W)}\,d\tilde{\Phi}\,, \tag{5.18}$$

where $\zeta'(\tilde{\Phi})$ is the inverse velocity along the streamline $\Psi = 0$. Or equivalently, after accounting for the body condition and making use of (5.17),

$$x_{\Phi}(\Phi, 0) = \left(\frac{\Phi+1}{\Phi}\right)^{1/2}\left[1 - \frac{1}{\pi}\int_0^{\infty}\left(\frac{\tilde{\Phi}}{\tilde{\Phi}+1}\right)^{1/2}\frac{y_{\Phi}(\tilde{\Phi}, 0)}{\Phi-\tilde{\Phi}}d\tilde{\Phi}\right]. \tag{5.19}$$

Equation (5.19) provides a relation between the horizontal and vertical slopes of the inverse velocity on the free surface because $(x_{\Phi}, y_{\Phi}) = (u,v)/|\mathbf{u}|^2$. The determination of these two unknown functions requires the use of the dynamic condition (5.16), which reduces to

$$y(x_{\Phi}^2+y_{\Phi}^2) - (d-H) = 0\,, \tag{5.20}$$

where H is the nondimensional draft of the body. The nonlinear integro-differential equation (5.19), (5.20) was solved by Vanden Broeck & Tuck for stern flows. A low-Froude-number series was developed and summed near the body, this was used to "seed" and match with a numerical solution of the integral equation based on Newtonian iteration. The stern-flow results presented marked the first published nonlinear body-wave computations, in which both free-surface and body conditions were satisfied exactly. It is also significant to note that these authors were unable to obtain a convergent solution for the bow-flow problem. It was conjectured (also by Dagan & Tulin 1972) that a jet structure developed near the stagnation point. Some ideas on how to handle such a jet situation were also discussed in this paper.

Inverse formulations following this line of approach were recently used by Vanden Broeck & Schwartz (1979) for finite-depth water waves, by Schwartz (1981) for a moving pressure distribution, and by Forbes (1981) for bottom-mounted obstacles. These last two works noted the possibility that wave-free subcritical flows exist at certain geometries, as in the case of linear theories. Both applied zero wave-elevation and uniform-velocity conditions at the upstream end of the integral equation, which was also truncated downstream abruptly. These authors noted that the truncation, as in the original work of Vanden Broeck & Tuck (1977), had only indiscernible effects on the interior solution, except for the occasional presence of grid-scale upstream waves. Is this rather successful procedure restricted to the Cauchy-type integral-equation that is being solved? Or is it a methodology that can be applied to steady problems in general? The question remains unanswered.

6. SUMMARY REMARKS

A variety of numerical methods and techniques for treating linear and nonlinear free-surface flow problems have been reviewed in this article. With the exception of the method of Green functions, all are faced with the difficulty of imposing an effective open boundary condition of some kind. Hybrid methods based on matching an interior numerical solution with an exterior analytical representation appear to be the most rational. Unfortunately, these have been successfully developed only for linearized problems. A one-dimensional Sommerfeld-type condition has been employed quite successfully in a number of instances, but questions related to effects of oblique incidence and of the presence of non-propagative type solutions still remain. In steady-translation problems "asymmetrical" techniques, such as upstream differencing, appear to give the flow enough of a "preferred" direction that waves normally occur only downstream. Work that uses a large computation domain, with zero-disturbance condition at the outer boundary, generally pushes the computation hardware to such a limit that attention to accuracy and convergence seems always lacking.

We have made no attempts to compare the computer time of the various methods, since such numbers will depend on a variety of "behind-the-scenes" factors. But we have proposed in Section 5 a simple formula based on which such comparisons can be made on a more rational basis, at least for linear problems. Although the nonlinear free-surface conditions can be handled by a variety of iterative techniques that are strongly method-dependent, it seems evident that an iterative solution on just the free-boundary surface or contour should be considerably more convenient and efficient than one that couples such iterations with the solution of the field equation. Boundary-integral-equation methods have precisely this advantage, apart from the usual superiority in storage requirement and ease of implementation. Space discretization techniques are generally regarded more suitable for problems where the field is inhomogeneous. Nevertheless, all methods have contributed to the understanding of free-surface flows about bodies.

The complexities associated with free-surface flows are amazing. Recent discoveries related to the modulational instability of weakly nonlinear wavetrains, the absence of a definite end state during the evolution of such unstable wavetrains, and the bifurcation of large-amplitude waves serve well as examples. A thorough understanding of physical phenomena involving nonlinear body-wave problems requires more than direct numerical solutions. Parallel analytical investigation and experimental confirmation of numerical results are highly desirable. The "best" numerical methods to come may well be those that exploit analytical simplifications that are appropriate for the physical phenomenon being examined.

Acknowledgment

Preparation of this article has been supported primarily by the Office of Naval Research, under task NR062-611. Partial support by the AAEF and the NSF is also gratefully acknowledged. It is a pleasure to thank Professor E. O. Tuck for his comments on the manuscript. I would like to dedicate this work to the late Mr. K. C. Chow and his family. This review was completed April, 1981.

Literature Cited

Adachi, H., Ohmatsu, S. 1979. On the influence of irregular frequencies in the integral equation solutions of the time-dependent free-surface problems. *J. Soc. Naval Architects Jpn.* 145:127–36

Aitchison, J. M. 1980. A finite element solution for critical flow over a weir. *Proc. Int. Conf. Finite Elements in Flow Problems, Banff, Alberta, Canada* 2:52–59

Ames, W. F. 1977. *Numerical Methods for Partial Differential Equations.* 2nd ed. New York: XIV+365 pp.

Arakawa, A., Mintz, Y. 1974. The UCLA atmospheric general circulation model. *Univ. Calif. Los Angeles Workshop Notes*

Baba, E., Takekuma, K. 1975. A study on free-surface flow around the bow of slowly moving full forms. *J. Soc. Naval Architects Jpn.* 137:1–10

Bai, K. J. 1977. A localized finite-element method for steady, three-dimensional free-surface flow problems. *Proc. 2nd Int. Conf. Numer. Ship Hydrodyn., Berkeley, Calif.* pp. 78–87

Bai, K. J. 1978. A localized finite-element method for two-dimensional steady potential flows with a free surface. *J. Ship Res.* 22:216–30

Bai, K. J. 1979. Blockage correction with a free surface. *J. Fluid Mech.* 94:433–52

Bai, K. J., Yeung, R. W. 1974. Numerical solutions to free-surface flow problems. *10th Symp. Nav. Hydrodyn., Cambridge, Mass,* pp. 609–33; disc. pp. 634–47

Banerjee, P. K. B., Butterfield, R., eds. 1979. *Developments in Boundary-Element Methods,* Vol. 1. London: Applied Science Publisher. xii+292 pp.

Bateman, H. 1932. *Partial Differential Equations.* Cambridge: Univ. Press. New York: Dover, 1944. xxii+522 pp.

Beale, J. T., 1977. Eigen function expansion for objects floating in an open sea. *Comm. Pure Appl. Math.* 30:283–313

Berhault, C. 1980. An integro-variational method for interior and exterior free surface flow problems. *Appl. Ocean Res.* 2:33–38

Berkhoff, J. C. W. 1972. Computation of combined refraction-diffraction. *Proc. 13th Coastal Engrg. Conf., Vancouver.* pp 471–90

Betts, P. L. 1979. A variational principle in terms of stream function for free-surface flows and its application to the finite element method. *Comput. Fluids.* 7:145–53

Betts, P. L., Assaat, M. I. 1980. Finite element solutions of large amplitude water waves. *Proc. Int. Conf. Finite Elements in Flow Problems, Banff, Alberta, Canada* 2:24–32

Boussinesq, J. 1871. Théorie de l'intumescence liquide appelée onde solitaire ou de translation se propageant dans un canal rectangulaire. *C. R. Acad. Sci.* Paris 72:755–59

Brandt, A., Dendy, J. E., Ruppel, H. 1980. The multigrid method for semi-implicit hydrodynamics codes. *J. Comput. Phys.* 34:348–70

Buzbee, B. L., Dorr, F. W., George, J. A., Golub, G. H. 1971. The direct solution of the discrete Poisson equation on irregular regions. *SIAM J. Numer. Anal.* 8:722–36

Carmerlengo, A. L., O'Brien, J. J. 1980. Open boundary conditions in rotating fluids. *J. Comput. Phys.*35:12–35

Chan, R. K. C. 1975. Two-dimensional time-dependent calculations of large amplitude surface gravity waves due to a surface disturbance. *Proc. 1st Int. Conf. Numer. Ship Hydrodyn. Gaithersburg, Md.,* pp. 315–32

Chan, R. K. C. 1977. Finite difference simulation of the planar motion of a ship. *Proc. 2nd Int. Conf. Numer. Ship Hydrodyn., Berkeley, Calif.,* pp. 39–52

Chan, R. K. C., Chan, F. W-K. 1980. Numerical solutions of transient and steady free-surface flows about a ship of general hull shape. *13th Symp. Naval Hydrodyn., Tokyo, Jpn.,* pp. 257–80

Chan, R. K. C., Hirt, C. W. 1974. Two-dimensional calculations of the motion of floating bodies. *10th Symp. Naval Hydrodyn., Cambridge, Mass.,* pp. 667–82

Chan, S. T. K., Larock, B. E. 1973. Free-surface ideal fluid flows by finite elements. *ASCE J. Hydraul. Div.* 99:(HY6):959–74

Chang, M. S., Pien, P. C. 1975. Hydrodynamic forces on a body moving beneath a free surface. *Proc. 1st Int. Conf. Numer. Ship Hydrodyn.* pp. 530–60

Chaplin, J. R. 1980. Developments of stream-function wave theory. *Coastal Engrg.* 3:179–205

Chapman, D. R. 1978. Computational aerodynamics development and outlook. *AIAA J.* 17:1293–1313

Chapman, R. B. 1976. Free-surface effects for yawed surface-piercing plates. *J. Ship Res.* 20:125–36

Chapman, R. B. 1979. Large amplitude transient motion of two-dimensional floating bodies. *J. Ship Res.* 23:20–31

Chen, H. S., Mei, C. C. 1974. Oscillation and wave forces in a man-made harbor in the open sea. *10th Symp. Naval Hydrodyn., Cambridge, Mass.* pp.573–96

Chenault, W. 1970. *Motion of a ship at the free surface.* MS thesis. Naval Postgraduate School, Monterey, Calif.

Chenot, J. L. 1975. Méthode numérique de calcul du mouvement d'un corps flottant soumis à l'influence d'une houle périodique en théorie linéaire.*Rev. Inst. Fr. Pet.* 30:779–802

Cokelet, E. D. 1977. Steep gravity waves in water of arbitrary uniform depth. *Philos. Trans. R. Soc. London, Ser. A* 286:183–230

Dagan, G., Tulin, M. P. 1972. Two-dimensional free-surface gravity flow past blunt bodies. *J. Fluid Mech.* 51:529–43

Daoud, N. 1975. Potential flow near to a fine ship's bow. *Univ. Mich., Dept. Naval Arch. Mar. Engrg. Rep. No. 177* ii+58pp.

Dawson, C. W. 1977. A practical computer method for solving ship-wave problems. *Proc. 2nd Int. Conf. Numer. Ship Hydrodyn., Berkeley, Calif.,* pp. 32–38

Dean, R. G. 1965. Stream function representation of nonlinear ocean waves. *J. Geophys. Res.* 70:4561–72

Dean, R. G. 1974. Evaluation and development of water wave theories for engineering applications. *U.S. Army Corps Engrs. Coastal Engrg. Res. Rep. Sp. Rep. No. 1.* 534 pp.

Delves, L. M., Walsh, J. 1974. *Numerical Solution of Integral Equations.* Oxford: Clarendon 339 pp.

Dern, J. C. 1977. Existence, uniqueness and regularity of the solution of Neumann-Kelvin problem for two or three dimensional submerged bodies. *Proc. 2nd Int. Conf. Numer. Ship. Hydrodyn., Berkeley, Calif.*, pp. 57–77.

Emmons, H. W. 1972. Critique of numerical modeling of fluid-mechanics phenomena. *Ann. Rev. Fluid Mech.* 2:15–36

Engquist, B., Majda, A. 1977. Absorbing boundary conditions for the numerical simulation of waves. *Math. Comput.* 31(139):629–51

Faltinsen, O. M. 1977. Numerical solution of transient nonlinear free-surface motion outside or inside moving bodies. *Proc. 2nd Int. Conf. Numer. Ship Hydrodyn., Berkeley, Calif.* pp. 347–57

Faltinsen, O. M., Michelsen, F. C. 1974. Motions of large structures in waves at zero Froude number. *Int. Symp. Dynamics of Marine Vehicles and Structures in Waves, Univ. College, London*, pp. 99–114

Fenton, J. D., Mills, D. A. 1976. Shoaling waves: numerical solution of exact equations. *Proc. IUTAM. Symp. Waves on Water of Variable Depth, Canberra. Lecture Notes in Physics* 64:94–101

Finkelstein, A. B. 1957. The initial-value problem for transient water-waves. *Comm. Pure Appl. Math.* 10:511–22

Forbes, L. K. 1981. On the wave resistance of a submerged elliptical body. *J. Engrg. Math.* In press

Forsythe, G. E., Wasow, W. R. 1960. *Finite-Difference Methods for Partial Differential Equations. New York: Wiley.* 444 pp.

Frank, W. 1967. Oscillations of cylinders in or below the free surface of deep fluids. *Naval Ship Res. Dev. Cent. Rep. 2375.* vit 40 pp.

Freeman, N. G., Hale, A. M., Danard, M. B. 1972. A modified sigma equation approach to the numerical modeling of Great Lakes' hydrodynamics. *J. Geophys. Res.* 77:1050–60

Gadd, G. E. 1976. A method of computing the flow and surface wave pattern around full forms. *R. Inst. Naval Arch.* 118: 207–19

Garrison, C. J., Seetharama Rao, V. 1971. Interaction of waves with submerged objects. *Proc. ASCE. J. Waterways, Harbors, Coastal Engrg. Div.* 97:259–77

Grosch, C. E., Orszag, S. A. 1977. Numerical solutions of problems in unbounded regions: coordinate transforms. *J. Comput. Phys.* 25:273–96

Guèvel, M., Delhommeau, G., Cordonnier, J. P. 1977. Numerical solution of the Neumann-Kelvin problem by. the method of singularities. *Proc. 2nd Int. Conf. Numer. Ship Hydrodyn., Berkeley, Calif.*, pp. 107–23

Harten, A. 1975. An efficient differentio-integral equation technique for time-dependent free-surface flows. *Proc 1st Int. Conf. Numer. Ship Hydrodyn. Gaithersburg, Md.*, pp. 717–28

Harten, A. Efrony, S. 1978. A partition technique for the solution of potential flow problems by integral equations methods. *J. Comput. Phys.* 27:71–87

Haussling, H. J., Coleman, R. M. 1979. Nonlinear water waves generated by an accelerated circular cylinder. *J. Fluid Mech.* 92:767–81

Haussling, H. J., Van Eseltine, R. T. 1974. A combined spectral finite-difference method for linear and nonlinear wave problems. *Naval Ship Res. Dev. Cent., Carderock, Md., Rep. 4580.* vii+35 pp.

Hedstrom, G. W. 1979. Nonreflecting boundary conditions for nonlinear hyperbolic systems. *J. Comput. Phys.* 30:222–37

Hess, J. L., Smith A. M. O. 1964. Calculation of non-lifting potential flow about arbitrary three-dimensional bodies. *J. Ship Res.* 8:22–44

Hess, J. L., Smith, A. M. O. 1967. Calculation of potential flow about arbitrary bodies. *Prog. Aeronaut. Sci.* 8:1–137

Israeli, M., Orszag, S. A. 1974. Numerical simulation of radiation boundary conditions by damping. *Flow Res. Kent. Wash. Rep. #30.*

Jeppson, R. W. 1972. Inverse solution to three-dimensional potential flow. *Proc. ASCE, J. Engrg. Mech. Div.* 98:789–812

John, F. 1949. On the motion of floating bodies -I. *Comm. Pure Appl. Math.* 2:13–57

John, F. 1950. On the motion of floating bodies -II. *Comm. Pure Appl. Math.* 3:45–10

Jones, D. A. 1974. Integral equations for the exterior acoustic problem. *Q. J. Mech. Appl. Math.* 27:129–42

Kellogg, O. D. 1929. *Foundations of Potential Theory.* Berlin: Springer, ix+384 pp.

Kobus, J. M. 1976. *Application de la méthode des singlularités au problème des flotteurs cylindriques soumis à des oscillations harmoniques forcées de faible amplitude.* These (Docteur-Ingénieru). Univ. Nantes ix+156 pp.

Larock, B. E. 1970. A theory for free outflow beneath radial gates. *J. Fluid Mech.* 41:851–64

Larock, B. E., Taylor, C. 1976. Computing three-dimensional free-surface flows. *Int. J. Numer. Meth. Engrg.* 10:1143–1152

Lax, P. D., Richtmyer, R. D. 1956. Survey of the stability of linear finite difference equations. *Comm. Pure Appl. Math.* 9:267–93

Lebreton J.-C. Margnac, A. 1968. Calcul des mouvements d'un navire ou d'une plat-forme amarrés dans la houle. *La Houille Blanche* 23:379–89

Lenoir, M., Jami, A. 1978. A variational formulation for exterior problems in linear hydrodynamics. *Comput. Method Appl. Mech. Engrg.* 16:341–59

Lenoir, M., Martin, D. 1981. An application of the principle of limiting absorption to the motions of floating bodies. *J. Math. Anal. Appl.* 79:370–83

Licht, C. 1980. *Etude théorique et numérique de l'évolution d'un système fluide-flotteur.*, Thèse (Docteur-ingénier). Univ. Nantes

Longuet-Higgins, M. S., Cokelet, E. D. 1976. The deformation of steep surface waves on water. I. A numerical method of computation. *Proc. R. Soc. London Ser. A* 350:1–26

Luke, J. C. 1967. A variational principle for a fluid with a frec surface *J. Fluid Mech.* 27:395–97

Macaskill, C. C. 1977. *Numerical solutions of some fluid flow problems by boundary integral-equation techniques.* PhD thesis. Dept. Appl. Math., University of Adelaide, S. A.

MacCormack, R. W., Lomax, H. 1979. Numerical solution of compressible viscous flows. *Ann. Rev. Fluid Mech.* 11:289–316

Mattioli, F. 1978. Wave induced oscillations in harbours of variable depth. *Comput. Fluids,* 6:161–72

McDonald, B. H., Wexler, A. 1972. Finite element solution of unbounded field problems. IEEE Trans. Microwave Theory and Technique, MTT-20:841–47

Mei, C. C. 1978. Numerical methods in water-waves diffraction and radiation. *Ann. Rev. Fluid Mech.* 10:393–416

Mei, C. C., Chen, H. S. 1976. A hybrid element method for steady linearized free-surface flows. *Int. J. Numer. Meth. Engrg.* 10:1153–75

Michell, J. H. 1898. The wave resistance of a ship. *Philos. Mag.* 45(5):106–23

Mikhlin, S. G. 1957. *Integral Equations and their Applications to Certain Problems in Mechanics, Mathematical Physics and Technology.* New York: Macmillan xiv+341 pp.

Miles, J. W. 1977. On Hamilton's principle for surface waves. *J. Fluid Mech.* 83:153–58

Miles, J. W. 1980. Solitary waves. *Ann. Rev. Fluid Mech.* 12:11–43

Moayeri, M. S. 1973. Flow in open channels with smooth curved boundaries. *J. Hydraul. Div., Proc. ASCE* pp. 2217–32

Nekrasov, A. I. 1921. On waves of permanent type. *Izv. Ivanovo-Voznesensk. Politekhn. Inst. I.* 3:52–65

Newman, J. N. 1976. Linearized wave resistance theory. *Proc. Int. Seminar on Wave Resistance, Tokyo, Jpn.* pp 31–44

Newman, J. N. 1978. The theory of ship motions. *Adv. Appl Mech.* 18:221–83

Norrie, D. H., de Vries, G. 1978. A survey of the finite element application in fluid mechanics. *Finite Elements in Fluids vol. 3,* pp. 363–96

Noye, J., May, R. L., Teubner, M. D. 1981. Three-dimensional numerical model of tides in Spencer Gulf. *Ocean Management* 6:137–47

O'Carroll, M. J. 1976. Variational technique for free-streamline problems. *Proc. 2nd Symp. Finite Element Methods in Flow Problems, Ligure, Italy,* pp. 489–95

Ogilvie, T. F. 1968. Wave resistance—The low-speed limit. *Univ. Michigan, Dept. Naval Arch. & Marine Engrg. Rep. No. 2*

Ogilvie, T. F., Shin, Y. S. 1978. Integral-equation solutions for time-dependent free-surface problems. *J. Soc. Naval Architects Jpn.* 143:41–51

Ohring, S. 1975. A fast fourth-order Laplace solver for application to numerical three-dimensional water wave problems. *Proc. 1st Int. Conf. Numer. Ship Hydrodyn., Gaithersburg, Md.,* pp. 641–63

Ohring, S., Telste, J. 1977. Numerical solution of transient three-dimensional ship-wave problems. *Proc. 2nd Int. Conf. Numer. Ship Hydrodyn, Berkeley, Calif.* pp. 88–103

Orlanski, I. 1976. A simple boundary condition for unbounded hyperbolic flows. *J. Comput. Phys.* 21:251–69

Orszag, S. A., Israeli, M. 1974. Numerical simulation of viscous incompressible flows. *Ann. Rev. Fluid Mech.* 6:281–318

Patterson, G. S. 1978. Prospects for computational fluid mechanics. *Ann. Rev. Fluid Mech.* 10:289–300

Pian, T. H., Tong, P. 1968. Basis of finite element analysis for solid continua. *Int. J. Numer. Math. Engrg.* 1:3–28

Potash, R. L. 1971. Second-order theory of oscillating cylinders. *J. Ship. Res.* 14:295–324

Rayleigh, Lord. 1876. On waves. *Philos. Mag.* 1(5):257–79

Reinecke, M., Fenton, J. D. 1981. A Fourier approximation method for steady water waves. *J. Fluid Mech.* 104:119–37

Richtmyer, R. D., Morton, K. W. 1967. *Difference Methods for Initial-Value Problems.* 2nd ed. New York: Wiley Interscience. 405 pp.

Roache, P. J. 1976. *Computational Fluid Dynamics.* Albuquerque, NM: Hermosa. vii+446 pp.

Rudy, D. H., Strikwerda, J. C. 1980. A nonreflecting outflow boundary for subsonic Navier-Stokes Calculations. *J. Comput. Phys.* 36:55–70

Salvesen, N., von Kerczek, C. 1978. Nonlinear aspects of subcritical shallow-water flow past two-dimensional obstructions. *J. Ship Res.* 22:203–11

Sarpkaya, T., Hiriart, G. 1975. Finite element analysis of jet impingement on axisymmetric curved deflectors. In *Finite Elements in Fluids, Vol. 1,* New York: Wiley pp. 265–79

Sayer, P., Ursell, F. 1977. Integral-equation methods for calculating virtual mass in water of finite depth. *Proc. 2nd Int. Conf. Numer. Ship Hydrodyn., Berkeley, Calif.,* pp. 176–84

Schwartz, L. W. 1974. Computer extension and analytic continuation of Stokes' Expansion for gravity waves. *J. Fluid Mech.* 62:553–78

Schwartz, L. W. 1981. Nonlinear solution for an applied overpressure on a moving stream. *J. Engrg. Math.* 15:147–156

Schwartz, L. W., Fenton, J. D. 1982. Strongly nonlinear waves. *Ann. Rev. Fluid Mech.* 14:39–60

Shanks, S. P., Thompson, J. F. 1977. Numerical solution of the Navier-Stokes equations for 2-D hydrofoils in or below a free surface. *Proc. 2nd Int. Conf. Ship Hydrodyn., Berkeley, Calif.,* pp 202–20

Shaw, R. P. 1975. An outer boundary integral equation applied to transient wave scattering in an inhomogeneous medium. *J. Appl. Mech.* 42:147–52

Shen, S.-F. 1977. Finite-element methods in fluid mechanics. *Ann. Rev. Fluid Mech.* 9:421–45

Soh, W. K. 1976. Vortex sheet method for calculation of linearized free surface waves. *Rep. Nav/Arch 76/5.* Sch. Mech. & Indust. Engrg., Univ. New South Wales, 17 pp.

Soh, W. K. 1980. Computer simulation of water waves and the frequencies of a submerged cylinder. *Proc. 7th Australas. Hydraul. Fluid Mech. Conf., Brisbane,* pp. 60–63

Sommerfeld, A. J. W. 1949. *Vorlesungen über theoretische Physik, II. Mechanik der deformierbaren Medien.* Leipzig: Akademie. xi+335 pp.

Southwell, R. V., Vaisey, G. 1946. Relaxation methods applied to engineering problems. XII. Fluid Motion Characterized by free stream-lines. *Philos. Trans. R. Soc. London Ser. A* 240:117–61

Stoker, J. J. 1957. *Water Waves.* New York: Wiley Interscience. xxviii+567 pp.

Stokes, G. G. 1847. On the theory of oscillatory waves. *Trans. Cambridge Philos. Soc.* 8:441–55

Stokes, G. G. 1880a. Consideration relative to the greatest height of oscillatory waves which can be propagated without change of form. *Math. Phys. Pap.* 1:225–28. Cambridge: Univ. Press

Stokes, G. G. 1880b. Supplement to a paper on the theory of oscillatory waves. *Math. Phys. Pap.* 1:314–26. Cambridge: Univ. Press

Thom, A., Apelt, C. J. 1961. *Field Computations in Engineering and Physics.* New York: Van Nostrand 65 pp.

Thomas, J. W. 1968. Irrotational gravity waves of finite height: A numeric study. *Mathematika* 15:139–48

Thompson, J. F., Thames, F. C., Mastin, C. W. 1974. Automatic numerical generation of body-fitted curvilinear coordinate system for field containing any number of arbitrary two-dimensional bodies. *J. Comput. Phys.* 15:299–319

Thompson, J. F., Thames, F. C., Hodge, J. K., Shanks, S. P., Reddy, R. N. Mastin, C. W. 1976. Solutions of Navier-Stokes equations in various flow regimes on fields containing any number of arbitrary bodies using boundary-fitted coordinate systems. *Lecture Notes in Physics* 59:421–27

Tsutsumi, T. 1979. Calculation of the wave resistance of ships by the numerical solution of Neumann-Kelvin problem. *Proc. Workshop on Ship Wave-Resistance Computations, Naval Ship Res. Dev. Cent., Carderock, Md.,* pp. 162–201

Ursell, F. 1980a. Irregular frequencies and the motion of floating bodies. Manuscript, Dept. Mathematics, Manchester Univ.

Ursell, F. 1980b. Mathematical notes on the two-dimensional Kelvin-Neumann problem of ship hydrodynamics. *13th Symp. Naval Hydrodyn., Tokyo,* pp. 245–51

Vanden Broeck, J.-M. Schwartz, L. W. 1979. Numerical computation of steep gravity waves in shallow water. *Phys. Fluids* 22:1868–71

Vanden Broeck, J.-M., Tuck, E. O. 1977. Computation of near-bow or stern flows, using series expansion in the Froude number. *Proc. 2nd Int. Conf. Numer. Ship Hydrodyn., Berkeley, Calif.,* pp. 371–81

Vinje, T. Brevig, P. 1980. Nonlinear, two-dimensional ship motions. *Norwegian Inst. Tech., Rep. on Ships in Rough Seas.* 92 pp.

von Kerczek, C. H., Salvesen, N. 1974. Numerical solutions of two-dimensional nonlinear wave problems. *10th Symp. Naval Hydrodyn., Cambridge, Mass.,* pp. 649–65

Wehausen, J. V. 1971. The motion of floating bodies. *Ann. Rev. Fluid Mech.* 3:237–68

Wehausen, J. V. 1973. Wave resistance of ships. *Adv. Appl. Mech.* 13:93–244

Wehausen, J. V., Laitone, E. V. 1960. Surface waves. *Handbuch der Physik* 9:446–778 Berlin: Springer.
Wellford, C. L., Ganaba, T. 1980. Finite element procedures for fluid mechanics problems involving large free surface motion. *Proc. 3rd Int. Conf. Finite Elements in Flow Problems, Banff, Alberta, Canada,* 2:13–23
Whitham, G. B. 1967. Nonlinear dispersion of water waves. *J. Fluid Mech.* 27:399–412
Whitham, G. B. 1970. Two-timing, variational principles and waves. *J. Fluid Mech.* 44:373–95
Winslow, A. M. 1966. Numerical solution of the quasilinear Poisson equation in nonuniform triangular mesh. *J. Comput. Phys.* 1:149–72
Yamamoto, Y., Kagemoto, H. 1976. Finite-element treatments for free surface waves caused by a body in uniform flow. *Proc. 26th Jpn. Natl. Cong. Appl Mech.*, pp. 549–561
Yen, S. M., Lee, K. D., Akai, T. J. 1977. Finite element and finite difference solutions of nonlinear free surface wave problems. *Proc. 2nd Int. Conf. Numer. Ship Hydrodyn., Berkeley, Calif.*, 305–18
Yeung, R. W. 1975. A hybrid integral-equation method for time-harmonic free-surface flows. *Proc. 1st Int. Conf. Numer. Ship Hydrodyn., Gaithersburg, Md.*, pp. 581–608
Yeung, R. W. 1981. Added mass and damping of a vertical cylinder in finite-depth waters. *Appl. Ocean Res.* 3:119–33
Yeung, R. W. 1982. The transient heaving motions of floating cylinders. *J. Engrg. Math.* In press
Yeung, R. W., Bouger, Y. C. 1979. A hybrid integral-equation method for steady two-dimensional ship waves. *Int. J. Numer. Meth. Engrg.* 14:317–36
Yim, B. 1975. A variational principle associated with a localized finite element technique for steady ship-wave and cavity problems. *Proc. 1st Int. Conf. Numer. Ship Hydrodyn., Gaithersburg, Md.*, pp. 137–53
Yue, D. K. P., Chen, H. S., Mei, C. C. 1978. A hybrid element method for diffraction of water waves by three-dimensional bodies. *Int. J. Numer. Methods in Engrg.* 12:245–66
Yuen, H. C., Lake, B. M. 1980. Instabilities of waves on deep water. *Ann. Rev. Fluid Mech.* 12:303–34
Zaroodny, S. J., Greenberg, M. D. 1973. On a vortex sheet approach to the numerical calculation of water waves. *J. Comput. Phys.* 11:440–46
Zienkiewicz, O. C. 1975. Why finite elements? In *Finite Elements in Fluids Vol. 1*, pp. 1–23. New York: Wiley
Zienkiewicz, O. C., Bettess, P. 1975. Infinite elements in the study of fluid-structure instruction problems. *Proc. 2nd Int. Symp. Computing Meth. Appl. Sci. Engrg, Versailles, France,* pp.