Math Worksheet, Computational Neuroscience 2024

- This worksheet contains examples mathematics problems for the midterm.
- It is not a practice exam; It does not address *everything*; There will be further questions about the math in courseworks 1, 2.
- The exam format will be 2–5 multiple-choice or short-answer "fact" and "understanding" questions, followed by 3 in-depth multi-part mathematical questions. Marks from the best 2-out-of-3 will be used. Most students will only have time to attempt the 2 they feel most confident in.
- If you get stuck or something seems weird, it's probably a typo (email to report).

Q1: Consider the ODE

$$\dot{v} = g \cdot (D - v) + B, \tag{1}$$

where g, D, and B are constants. Show that

$$v(t) = \alpha(t) \cdot v_0 + [1 - \alpha(t)] \cdot v_\infty \tag{2}$$

is a solution to ODE (1) with initial conditions $v(t = 0) = v_0$, where

$$\begin{aligned} \alpha(t) &= e^{-gt} \\ v_\infty &= D + B/g \end{aligned} \tag{3}$$

(Any approach permitted).

Approach 1: Verify

This is meant to be a faster approach for students less-practices in solving ODEs directly, but this is not the only or best method — any other approach is fine. We are given

$$\begin{aligned} v(t) &= \alpha(t) \cdot v_0 + [1 - \alpha(t)] \cdot v_\infty \\ &= v_\infty + e^{-gt}(v_0 - v_\infty); \end{aligned} \tag{4}$$

Treat this as an ansatz and differentiate to verify:

$$\begin{aligned} \dot{v} &= -ge^{-gt}(v_0 - v_\infty) \\ &= -g(v - v_\infty) \\ &= -g[v - (D + B/g)] \\ &= g[D + B/g - v] \\ &= g(D - v) + B \quad \blacksquare \end{aligned} \tag{5}$$

Approach 2: Using integrating factor. Your response can be brifer than this one.

Arrange (1) to apply integrating factor method:

$$\begin{aligned} \dot{v} &= g \cdot (D - v) + B \\ \dot{v} + gv &= \underbrace{gD + B}_{gv_\infty} \end{aligned} \tag{6}$$

The integrating factor is $\exp(\int_{dt} g) = e^{gt}$; multiply through:

$$\begin{aligned} e^{gt}[\dot{v} + gv] &= e^{gt} g v_\infty \\ e^{gt}\dot{v} + g e^{gt} v &= e^{gt} g v_\infty \\ \frac{\mathrm{d}}{\mathrm{d}t}[e^{gt} v] &= e^{gt} g v_\infty \end{aligned} \tag{7}$$

Integrate both sides and collect terms:

$$\begin{aligned} \int_{dt} \frac{\mathrm{d}}{\mathrm{d}t}[e^{gt} v] &= \int_{dt} e^{gt} g v_\infty \\ e^{gt} v(t) &= v_\infty e^{gt} + c \\ v(t) &= v_\infty + c e^{-gt} \end{aligned} \tag{8}$$

Evaluate at $t = 0$ to get constant c:

$$\begin{aligned} v_0 &= v_\infty + c e^{-g \cdot 0} = v_\infty + c \\ c &= v_0 - v_\infty \\ v &= v_\infty + \underbrace{e^{-gt}}_{\alpha(t)}(v_0 - v_\infty) \\ v &= v_\infty + \alpha(t) v_0 - \alpha(t) v_\infty \\ v &= \alpha(t) v_0 + [1 - \alpha(t)] v_\infty \quad \blacksquare \end{aligned} \tag{9}$$

Q2: Provide the solution to $\tau\dot{v} = c - v$ and evaluate $v(t)$ at time $t = 10$ for $c = 0$, $\tau = 10$, and $v(0) = -70$.

The first part ("provide solution") is similar to Q1 and we are not asked to do redundant work. If you can intuit the solution immediately, simply write it. Otherwise, any approach valid for Q1 will also get you here. If you're not 100% confident, you can differentiate to verify.

$$v(t) = c + e^{-t/\tau}[c - v(0)]$$

Evaluate at given values:

$$\begin{aligned} v(10) &= 0 + e^{-10/10}[0 - (-70)] && \text{(plugging in constants' values)} \\ &= \tfrac{1}{e} \cdot 70 && \text{(evaluating)} \end{aligned} \tag{10}$$

There's no need to use a calculator to go any further (a correct numeric answer with incorrect algebra might raise suspicion). Any variation algebraically equivalent to the above will get full marks.

Note: this reminds us that the time constant τ is the time it takes for the distance between $v(0)$ and c to decay to $1/e$ of its original value.

Q3:

You are asked to write computer code to numerically integrate a linear first-order ODE for a neuron's membrane voltage v, being driven by a known time-varying input $u(t)$

$$C\dot{v} = \frac{1}{R}(E - v) + u(t), \tag{11}$$

where C, R, and E are constants.

Q3a: (facts/trivia) Is forward Euler a first or second-order method?

First

Q3b: (facts/trivia) What is the order of the error in forward Euler in terms of the time step Δt?

O(Δt^2)

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Q3c: You're told that the signal $u(t)$ varies slowly, and is approximately constant $u(t) \approx u_t$ over a short duration between t and $t + \Delta t$, where $\Delta t = 1$ ms. You implement the following (pseudo)code to integrate from $t = 0$ to $t = 1000$ ms based on this assumption:

$$\begin{aligned}
&\gamma = ???\\
&v = v_0\\
&\text{for } t \text{ in } 1 \ldots 1000:\\
&\qquad v_\infty = ???\\
&\qquad \Delta_v = \gamma(v_\infty - v_t)\\
&\qquad v_{t+1} = v_t + \Delta_v
\end{aligned} \tag{12}$$

What expressions belong in the ??? for variables γ and v_∞?

Any route to the correct answer is fine; no need to "prove" or integrate from scratch. You may use any notation (or even words) to express yourself, define intermediate variables, etc. Correct answers without work get full marks (but memorization won't work—this question is not on the exam as such, but does test your readiness).

Solve assuming constant $u(t) \approx u_t$:

$$\begin{aligned}
C\dot{v} &= \tfrac{1}{R}(E - v) + u_t\\
\underbrace{RC}_{\tau}\,\dot{v} &= \underbrace{E + Ru_t}_{v_\infty} - v\\
v(t) &= v_\infty + e^{-t/\tau}(v - v_\infty)
\end{aligned} \tag{13}$$

Every time-step solve forward by Δt using a new u_t:

$$\begin{aligned}
v_{t+1} &= v_\infty + \underbrace{e^{-\Delta_t/\tau}}_{\alpha}(v_t - v_\infty)\\
\Delta_v = v_{t+1} - v_t &= v_\infty + \alpha(v_t - v_\infty) - v_t\\
&= \underbrace{(1-\alpha)}_{\gamma}(v_\infty - v_t)
\end{aligned} \tag{14}$$

Full marks for any technically valid response, even if it is not an exact match to the one above.

Q4: Consider a linear-nonlinear perceptron "neuron" $\hat{y} = \phi(\boldsymbol{w}^\top \boldsymbol{x})$, with inputs $\boldsymbol{x}$, weights $\boldsymbol{w}$ (both column vectors), and output $\hat{y}$.

Q4a: (bookwork / concepts) In coursework 2, we covered the delta learning rule in a context where $\phi(\cdot)$ was a hard-threshold, setting $\hat{y} = 1$ for activation $\boldsymbol{w}^\top \boldsymbol{x} > 0$ and -1 otherwise. *State the weight update equation for the delta rule, and explain how we can view the rule as learning based on the prediction error $y^* - \hat{y}$.*

This answer is longer than needed for full marks. The wording and content does not need to be an exact match to the answer below: Any sincere attempt demonstrating understanding gets full marks.

$$\Delta w_i = \eta \underbrace{(y^* - \hat{y})}_{\text{error}} x_i \tag{15}$$

We can interpret this as the "neuron" comparing a prediction $\hat{y}$ to a supervised target y^*, and calculating the error $y^* - \hat{y}$, which is then correlated with the synaptic inputs. For example, a positive $y^* - \hat{y}$ means our output should have been larger. If an input x_i is positive, we should increase its weight w_i to try to increase $\hat{y}$ and reduce this error for similar inputs $\boldsymbol{x}$ and target output y^* in the future.

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Q4a: Consider a version of linear-nonlinear perceptron that uses $\phi(\cdot) = \exp(\cdot)$ for its nonlinearity. Show that we can recover a learning rule similar to the delta rule by gradient-descent optimization of the following loss function:

$$\begin{aligned} \mathcal{L}(\boldsymbol{w}; \boldsymbol{x}, y^*) &= \hat{y} - y^* \cdot \ln(\hat{y}) \\ \hat{y} &= \phi(r) = e^r \\ r &= \boldsymbol{w}^\top \boldsymbol{x} = \textstyle\sum_i w_i x_i. \end{aligned} \tag{16}$$

$$\begin{aligned} \mathcal{L}(\dots) &= \hat{y} - y^* \cdot \ln(\hat{y}) \\ &= e^r - y^* \cdot r \\ \tfrac{\mathrm{d}}{\mathrm{d}w_k} \mathcal{L}(\dots) &= \tfrac{\mathrm{d}}{\mathrm{d}r} [e^r - y^* \cdot r] \cdot \tfrac{\mathrm{d}r}{\mathrm{d}w_k} \\ &= (e^r - y^*) \cdot x_k \\ &= (\hat{y} - y^*) \cdot x_k \end{aligned} \tag{17}$$

We want to minimize the loss, so a gradient descent update would be the negative of this gradient times some positive learning rate η

$$\Delta w_k = \eta (y^* - \hat{y}) x_k \tag{18}$$

This is the same as the delta rule for the binary perceptron (with the exception that the prediction $\hat{y}$ is calculated differently.)

Full marks for any answer that differentiates $\mathcal{L}$ in terms of the weights and notes similarity to the delta rule. Students may use any notation they wish, e.g. keeping this a vector derivative, etc. A cheeky response might simply be to state $-\frac{\mathrm{d}}{\mathrm{d}w_k}\mathcal{L}(\dots) = (y^* - \hat{y})x_k$. This would not receive full marks unless accompanied by some intermediate steps or notes hinting that the student has verified that it is, at least, true (by inspecting the derivative of $\mathcal{L}$).