Differential Equations

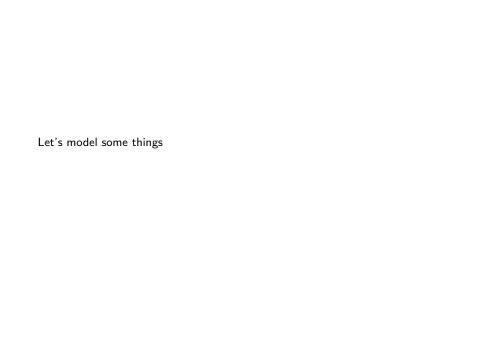
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M Rule

Learning outcomes:

- Review ODEs
- ► Set exam expectations
- ► Numerical methods
- ► Recall how to solve 1st-order linear ODEs
- ► Be ready to approach numerical integration in labs and coursework

A pen-and-paper exam for computational neuroscience...looks quite a lot like a pen-and-paper exam for mathematical neuro



$$\begin{array}{lll} C\,\dot{v} = -g_{\rm AMPA} & v + & g_{\ell}(E_{\ell} - v \) & ({\rm Passive \ neuronal \ membrane}) \\ \dot{s} = -k_{\rm leave} & s + & k_{\rm bind} \left[x\right] \left(1 - s \) & ({\rm Neurotransmitter \ binding \ to \ a \ receptor}) \\ \dot{m} = -\beta_{m}(v)m + & \alpha_{m}(v) \left(1 - m\right) & ({\rm Voltage-gated \ channel}) \\ \dot{u} = -\gamma & u + \gamma f({\rm W}u + s) \left(1 - \rho u\right) & ({\rm Recurrent \ rate \ network}) \end{array}$$

$$x(t) = x_{\infty} + e^{-t/\tau}(x_0 - x_{\infty})$$

given
$$x(t=0) = x_0$$

$$\begin{split} \dot{x} &= \frac{\mathrm{d}}{\mathrm{d}t} \big\{ x_{\infty} + e^{-t/\tau} (x_0 - x_{\infty}) \big\} \\ &= -\frac{1}{\tau} e^{-t/\tau} (x_0 - x_{\infty}) \\ &= -\frac{1}{\tau} [x - x_{\infty}] \end{split}$$
 Verify

$$\Rightarrow \quad \tau \dot{x} = -x + x_{\infty}$$

 $\tau \dot{x} = -x + x_{\infty}$

$$\partial_x \tau \dot{x} = -1 < 0 \implies \text{ is stable}$$

Check stability

See First-order, scalar, linear, ordinary differential equation(s): Exam appendix slide online, as well as past exam problems, for how this might appear on an exam.



Nonlinear, first-order, autonomous, scalar ODEs

$$\dot{x}=f(x,u)$$
 Nonlinear, autonomous ODE solve $f(x)=0$ for $x=x^*$ Find fixed point x^* ? Check stability at x^* ?

Bivariate nonlinear ODE stability analysis

$$\begin{pmatrix} \dot{v} \\ \dot{w} \end{pmatrix} = \begin{pmatrix} f(v, w) \\ g(v, w) \end{pmatrix}$$

$$J(v^*, w^*) = \begin{bmatrix} \partial_v f(v^*, w^*) & \partial_w f(v^*, w^*) \\ \partial_v f(v^*, w^*) & \partial_w g(v^*, w^*) \end{bmatrix}$$
 Check stability

Check stability using eigenvalues of \boldsymbol{J}

One form of the Quadratic Integrate-and-Fire (QIF) model is given by

$$\dot{v} = v^2 - v + I$$

For an (unknown) constant current I, find its fixed points in v

$$\dot{v} = v^2 - v + I = 0$$

$$v = \frac{1}{2}(1 \pm \sqrt{1 - 4I})$$

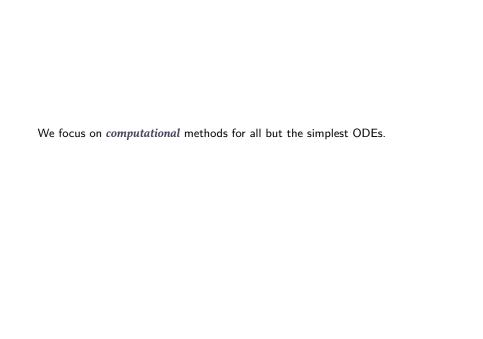
If I = 0 v = 0 is a fixed point. Is this fixed point stable?

$$\dot{v} = v^2 - v$$

$$\partial_v \dot{v} = 2v - 1$$

$$\{\partial_v \dot{v}\}(0) = -1 < 0 \quad \Rightarrow \text{ stable}$$





Too complicated to solve

$$\begin{split} C\dot{v} &= g_{\ell}(E_{\ell} - v) + g_{\mathrm{K}}(t)(E_{\mathrm{K}} - v) + g_{\mathrm{Na}}(t)(E_{\mathrm{Na}} - v) + I\\ g_{\mathrm{K}}(t) &= \bar{g}_{\mathrm{K}}n(t)^4\\ g_{\mathrm{Na}}(t) &= \bar{g}_{\mathrm{Na}}m(t)^3h(t)\\ \dot{n} &= \alpha_n\left(v\right)\cdot\left(1 - n\right) - \beta_n\left(v\right)\cdot n\\ \dot{m} &= \alpha_m(v)\cdot\left(1 - m\right) - \beta_m(v)\cdot m\\ \dot{h} &= \alpha_h\left(v\right)\cdot\left(1 - h\right) - \beta_h\left(v\right)\cdot h \end{split}$$

Use numerical integration

Fourth-order Runge-Kutta (RK4)

Use built-in implementation (Matlab, Scipy)

4th order, truncation error

- ▶ Local $O(\Delta^5)$
- ▶ Global $\mathcal{O}(\Delta^4)$

Forward Euler

$$x_{t+\Delta} \leftarrow x_t + \Delta \cdot \dot{x}(t)$$

 1^{st} order in Δ , truncation error

- ▶ Local ~ $\mathcal{O}(\Delta t^2)$
- ▶ Global $\sim \mathcal{O}(\Delta t)$

Exponential Euler

 1^{st} order in Δ , truncation error

Handles stiff equations, \approx exponential behaviour better

$$\dot{x} = f(x)$$

$$\approx f(x_t) + (x - x_t) \cdot \partial_x f(x)$$

$$\dot{x} = f(x)$$

 $\approx f(x_t) + (x - x_t) \cdot \partial_x f(x)$

 $= [f(x_t) - x_t \partial_x f(x)] + x \partial_x f(x)$

 $\tau \dot{x} \approx \frac{1}{\tau} [x_t \partial_x f(x) - f(x_t)] - x$

 $x_{t+\Lambda} = x_{\infty} + e^{-\Delta/\tau} (x_t - x_{\infty})$

Stochastic?

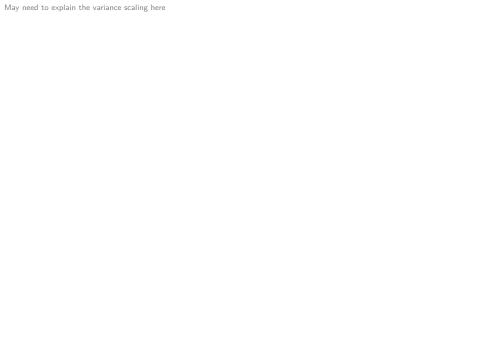
Euler-Maruyama

When you want to add noise

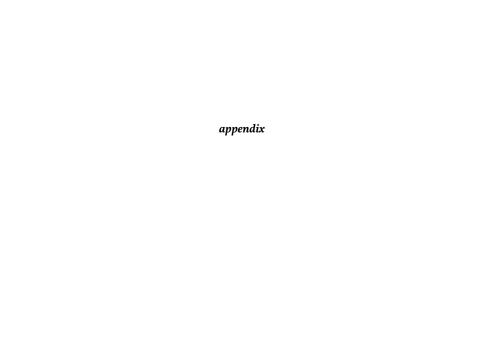
$$\begin{split} \mathrm{d} x &= \mu(x,t)\,\mathrm{d} t + \sigma(x,t)\,\mathrm{d} W \\ x_{n+1} &= x_n + \Delta\,\mu(x,t) + \sigma(x,t)\,\sqrt{\Delta}\,\xi, \qquad \xi_n \sim \mathcal{N}(0,1) \end{split}$$

Why $\sqrt{\Delta}$?

- ▶ Take a and $b \sim \mathcal{N}(0,1)$. Variances add when you add random variables (linearity of expectation). So, if a and b have variance 1, c = a + b has variance 2.
- ▶ To sample a zero-mean Gaussian random number with variance $v = \sigma^2$, sample $\xi \sim \mathcal{N}(0,1)$ and multiply by $\sigma = \sqrt{v}$. So $c \leftarrow \sqrt{2} \cdot \xi$, in this example.
- ▶ The Wiener process W is defined such that $W_{t+\Delta} W \sim \mathcal{N}(0, \sigma^2 = \Delta)$.







Why are we reviewing (exponential) Euler if RK4 is better?

You do not want to manually calculate RK4 on the exam

Intuition for how systems evolve

Sometimes you need to write your own solver

- ► The built-in solver broke
- Your'e doing somethign unusual

Some learning rules change weights according to an \approx ODE

▶ Naïve (batch) gradient descent with fixed step size $\eta \sim$ Forward Euler

Exponential Euler

- Exact for certain (linear) models, exact for LIF!
- ► Good for models ≈ linear systems with time-varying parameters (most of neuroscience)

Euler Maruyama

 Often used as there are few built-in alternatives in Matlab/Python (try DifferentialEquations.jl)

First-order, scalar, linear, ordinary differential equation(s): Exam

Exam?

- Prove the above ODE and solution match
- ► Solve and evaluate this ODE forward in time
- ▶ Does x(t) reach y? If so, when?
- ▶ What input will make $x(t_1 > 0) = x^*$?
- ▶ Is x(t) stable? What is its fixed point
- Evaluate using a calculator
- Explain how a specific ODE relates to biology
- ▶ Write a 1st-order linear ODE for a biological process
- ▶ Set variables in a difficult ODE to constants to get 1st-order linear ODE

(initial value problem)

(first passage time)

(control)

(stability)

Direct solution of first-order ODEs via integrating factor will also be a valid proof on the exam.

proof on the exam.
$$\tau \dot{x} = x_{\infty} - x \\ \dot{x} + \frac{1}{\tau} x = \frac{1}{\tau} x_{\infty} \\ e^{t/\tau} (\dot{x} + \frac{1}{\tau} x) = e^{t/\tau} \frac{1}{\tau} x_{\infty}$$
 I.F. is $e^{t/\tau}$ Multiply by $e^{t/\tau}$

$$\begin{aligned} &+ \frac{1}{\tau} x) = e^{t/t} \frac{1}{\tau} x_{\infty} & \text{Multiply by } e^{t/t} \\ &e^{t/\tau} x = \int_{\mathbb{R}^{+}} e^{t/\tau} \frac{1}{\tau} x_{\infty} & \text{Integrate both sides} \end{aligned}$$

$$e^{t/\tau}x = x_{\infty}e^{t/\tau} + c$$
 (collect constants to R.H.S.)

$$x = x_{\infty} + ce^{-t/\tau}$$
 Move $e^{t/\tau}$ over

$$x = x_{\infty} + cc$$
 Move $c = cc$

$$x(0) = x_{\infty} + ce^{-0/\tau}$$
 evaluate at zero to find c

$$(c) = x_{\infty} + ce^{-v/t}$$
 evaluate at zero to find c
 $(c) = x_{0} - x_{\infty}$ substitute c

$$c=x_0-x_\infty$$
 substitute c $x=x_\infty+e^{-t/\tau}(x_0-x_\infty)$ This matches provided solution.

