Differential Equations

Computational Neuroscience
University of Bristol

M Rule

Learning outcomes:

- Review ODEs
- Set exam expectations
- Numerical methods
- Recall how to solve 1^{st}-order linear ODEs
- Be ready to approach numerical integration in labs and coursework

A pen-and-paper exam for computational neuroscience...looks quite a lot like a pen-and-paper exam for mathematical neuro

Let's model some things

$$
\begin{aligned}
C\dot{v} &= -g_{\mathrm{AMPA}}\, v + g_\ell(E_\ell - v) && \text{(Passive neuronal membrane)}\\
\dot{s} &= -k_{\mathrm{leave}}\, s + k_{\mathrm{bind}}[x](1 - s) && \text{(Neurotransmitter binding to a receptor)}\\
\dot{m} &= -\beta_m(v)m + \alpha_m(v)(1 - m) && \text{(Voltage-gated channel)}\\
\dot{\boldsymbol{u}} &= -\gamma\, \boldsymbol{u} + \gamma f(\mathbf{W}\boldsymbol{u} + \boldsymbol{s})(1 - \rho\boldsymbol{u}) && \text{(Recurrent rate network)}
\end{aligned}
$$

First-order, scalar, linear, ordinary differential equation(s)

$$\tau \dot{x} = -x + x_\infty$$

$$x(t) = x_\infty + e^{-t/\tau}(x_0 - x_\infty) \qquad \text{given } x(t=0) = x_0$$

$$\begin{aligned} \dot{x} &= \tfrac{\mathrm{d}}{\mathrm{d}t}\left\{x_\infty + e^{-t/\tau}(x_0 - x_\infty)\right\} \qquad \text{Verify} \\ &= -\tfrac{1}{\tau} e^{-t/\tau}(x_0 - x_\infty) \\ &= -\tfrac{1}{\tau}\left[x - x_\infty\right] \\ \Rightarrow \quad \tau \dot{x} &= -x + x_\infty \qquad \blacksquare \end{aligned}$$

$$\partial_x \tau \dot{x} = -1 < 0 \quad \Rightarrow \quad \text{is stable} \qquad \text{Check stability}$$

See *First-order, scalar, linear, ordinary differential equation(s): Exam* appendix slide online, as well as past exam problems, for how this might appear on an exam.

Can the lecturer solve using an integration factor correctly while on stage?

Perhaps come back to this if there is time

Nonlinear, first-order, autonomous, scalar ODEs

$\dot{x} = f(x, u)$	Nonlinear, autonomous ODE
solve $f(x) = 0$ for $x = x^*$	Find fixed point x^*?
$\partial_x f(x^*) < 0$?	Check stability at x^*?

Bivariate nonlinear ODE stability analysis

$$\begin{pmatrix} \dot{v} \\ \dot{w} \end{pmatrix} = \begin{pmatrix} f(v, w) \\ g(v, w) \end{pmatrix}$$

$$\mathbf{J}(v^*, w^*) = \begin{bmatrix} \partial_v f(v^*, w^*) & \partial_w f(v^*, w^*) \\ \partial_v f(v^*, w^*) & \partial_w g(v^*, w^*) \end{bmatrix}$$

Check stability using eigenvalues of $\mathbf{J}$

Analytic solutions not required

One form of the Quadratic Integrate-and-Fire (QIF) model is given by

$$\dot{v} = v^2 - v + I$$

For an (unknown) constant current I, find its fixed points in v

$$\dot{v} = v^2 - v + I = 0$$

$$v = \frac{1}{2}(1 \pm \sqrt{1 - 4I})$$

If $I = 0$ $v = 0$ is a fixed point. Is this fixed point stable?

$$\dot{v} = v^2 - v$$

$$\partial_v \dot{v} = 2v - 1$$

$$\{\partial_v \dot{v}\}(0) = -1 < 0 \quad \Rightarrow \text{ stable}$$

Lecturer may need to explain that $\partial_v \dot{v}$ bit here?

We focus on ***computational*** methods for all but the simplest ODEs.

Too complicated to solve

$$
\begin{aligned}
C\dot{v} &= g_\ell(E_\ell - v) + g_{\mathrm{K}}(t)(E_{\mathrm{K}} - v) + g_{\mathrm{Na}}(t)(E_{\mathrm{Na}} - v) + I \\
g_{\mathrm{K}}(t) &= \bar{g}_{\mathrm{K}} n(t)^4 \\
g_{\mathrm{Na}}(t) &= \bar{g}_{\mathrm{Na}} m(t)^3 h(t) \\
\dot{n} &= \alpha_n(v) \cdot (1 - n) - \beta_n(v) \cdot n \\
\dot{m} &= \alpha_m(v) \cdot (1 - m) - \beta_m(v) \cdot m \\
\dot{h} &= \alpha_h(v) \cdot (1 - h) - \beta_h(v) \cdot h
\end{aligned}
$$

Use numerical integration

Fourth-order Runge-Kutta (RK4)

Use built-in implementation (Matlab, Scipy)

4^{th} order, truncation error

- Local $\mathcal{O}(\Delta^5)$
- Global $\mathcal{O}(\Delta^4)$

Forward Euler

$x_{t+\Delta} \leftarrow x_t + \Delta \cdot \dot{x}(t)$

1^{st} order in Δ, truncation error

- Local $\sim \mathcal{O}(\Delta t^2)$
- Global $\sim \mathcal{O}(\Delta t)$

Exponential Euler

Handles stiff equations, ≈ exponential behaviour better

$$
\begin{aligned}
\dot{x} &= f(x) \\
&\approx f(x_t) + (x - x_t) \cdot \partial_x f(x) \\
&= [f(x_t) - x_t \partial_x f(x)] + x \underbrace{\partial_x f(x)}_{-1/\tau}
\end{aligned}
$$

$$
\tau \dot{x} \approx \underbrace{\tfrac{1}{\tau}[x_t \partial_x f(x) - f(x_t)]}_{x_\infty} - x
$$

$$
x_{t+\Delta} = x_\infty + e^{-\Delta/\tau}(x_t - x_\infty)
$$

1st order in Δ, truncation error

Stochastic?

Euler-Maruyama

When you want to add noise

$$dx = \mu(x,t)\,dt + \sigma(x,t)\,dW$$

$$x_{n+1} = x_n + \Delta\,\mu(x,t) + \sigma(x,t)\,\sqrt{\Delta}\,\xi, \qquad \xi_n \sim \mathcal{N}(0,1)$$

Why $\sqrt{\Delta}$?

- Take a and $b \sim \mathcal{N}(0,1)$. Variances add when you add random variables (linearity of expectation). So, if a and b have variance 1, $c = a + b$ has variance 2.
- To sample a zero-mean Gaussian random number with variance $v = \sigma^2$, sample $\xi \sim \mathcal{N}(0,1)$ and multiply by $\sigma = \sqrt{v}$. So $c \leftarrow \sqrt{2} \cdot \xi$, in this example.
- The Wiener process W is defined such that $W_{t+\Delta} - W \sim \mathcal{N}(0, \sigma^2{=}\Delta)$.

May need to explain the variance scaling here *end*

appendix

Why are we reviewing (exponential) Euler if RK4 is better?

You ***do not*** want to manually calculate RK4 on the exam

Intuition for how systems evolve

Sometimes you need to write your own solver

- The built-in solver broke
- Your'e doing somethign unusual

Some learning rules change weights according to an $\approx$ ODE

- Naïve (batch) gradient descent with fixed step size $\eta \sim$ Forward Euler

Exponential Euler

- Exact for certain (linear) models, exact for LIF!
- Good for models $\approx$ linear systems with time-varying parameters (most of neuroscience)

Euler Maruyama

- Often used as there are few built-in alternatives in Matlab/Python (try DifferentialEquations.jl)

First-order, scalar, linear, ordinary differential equation(s): Exam

Exam?

- Prove the above ODE and solution match
- Solve and evaluate this ODE forward in time (initial value problem)
- Does $x(t)$ reach y? If so, when? (first passage time)
- What input will make $x(t_1 > 0) = x^*$? (control)
- Is $x(t)$ stable? What is its fixed point (stability)
- Evaluate using a calculator
- Explain how a specific ODE relates to biology
- Write a 1^{st}-order linear ODE for a biological process
- Set variables in a difficult ODE to constants to get 1^{st}-order linear ODE

Direct solution of first-order ODEs via integrating factor will also be a valid proof on the exam.

$$
\begin{aligned}
\tau\dot{x} &= x_\infty - x & \\
\dot{x} + \tfrac{1}{\tau}x &= \tfrac{1}{\tau}x_\infty & \text{I.F. is } e^{t/\tau} \\
e^{t/\tau}(\dot{x} + \tfrac{1}{\tau}x) &= e^{t/\tau}\tfrac{1}{\tau}x_\infty & \text{Multiply by } e^{t/\tau} \\
e^{t/\tau}x &= \int_{\mathrm{d}t} e^{t/\tau}\tfrac{1}{\tau}x_\infty & \text{Integrate both sides} \\
e^{t/\tau}x &= x_\infty e^{t/\tau} + c & \text{(collect constants to R.H.S.)} \\
x &= x_\infty + ce^{-t/\tau} & \text{Move } e^{t/\tau} \text{ over} \\
x(0) &= x_\infty + ce^{-0/\tau} & \text{evaluate at zero to find } c \\
c &= x_0 - x_\infty & \text{substitute } c \\
x &= x_\infty + e^{-t/\tau}(x_0 - x_\infty) & \text{This matches provided solution.} \blacksquare
\end{aligned}
$$

end