UNIVERSITY OF BRISTOL

Fall 2024 Week 6 Exam

FACULTY OF ENGINEERING

SEMT30003
Computational Neuroscience

TIME ALLOWED:
1 hour

This paper contains TWO parts.

[A] The first contains 5 short questions. Each question is worth 2 MARKS. All short questions should be attempted. (10 MARKS)

[B] The second contains 3 in-depth questions. Each of these is worth 10 MARKS. The BEST 2 in-depth questions will be used for assessment. (20 MARKS)

The maximum for this paper is 30 MARKS.

Word counts are not mandatory, they are provided as hints to help you avoid spending more time than needed. You may exceed the advised word count if you wish. You may also use diagrams and mathematics as appropriate in short answer questions.

Other Instructions:
Calculators must have the Faculty of Engineering Seal of Approval.

TURN OVER ONLY WHEN TOLD TO START WRITING

Section A: Five short questions—answer all questions

QA.1 [2 marks] Which of these statements best matches the simplifying heuristic known as Dale's principle (pick one):

1. Most dendrites receive either exclusively excitatory or exclusively inhibitory inputs.
2. Single neurons typically use the same set of chemical neurotransmitters at all their axonal terminals.
3. A single neuron releases either excitatory or inhibitory neurotransmitters depending on the identity of the post-synaptic target at each synapse.
4. When neuron A excites neuron B, and persistently contributes to making B spike, the strength of the synapse from A $\rightarrow$ B is decreased.

Answer: 2

QA.2 [2 marks] Which of the following are types of glutamate receptors in the brain (select ALL that apply):

1. GABA
2. AMPA
3. 5HT-2A
4. NMDA
5. nAChR
6. Muscarinic

Answer: 1 point each for (2) and (4); -$\frac{1}{3}$ mark for each selected incorrect answer.

QA.3 [2 marks] Neurons fire an action potential when the membrane voltage rises above approximately -55 mV, which opens voltage-gated ion channels that allow sodium to enter the cell and raise the voltage further.

i [1 mark] What eventually stops the membrane voltage from continuing to rise? (≈5–25 words)

Answer: Above 0 mV the sodium channels inactivate, cutting off the excitatory conductance. Around -20 mV voltage gated potassium channels open, repolarizing the membrane. Full marks for “sodium inactivation” or “potassium channels open”. Half marks for vague “voltage-gated ion channels change their conductances”. Zero marks for incorrect attempts, e.g. reversing the role of potassium and sodium.

ii [1 mark] What causes the refractory period following the action potential? (≈5–25 words)

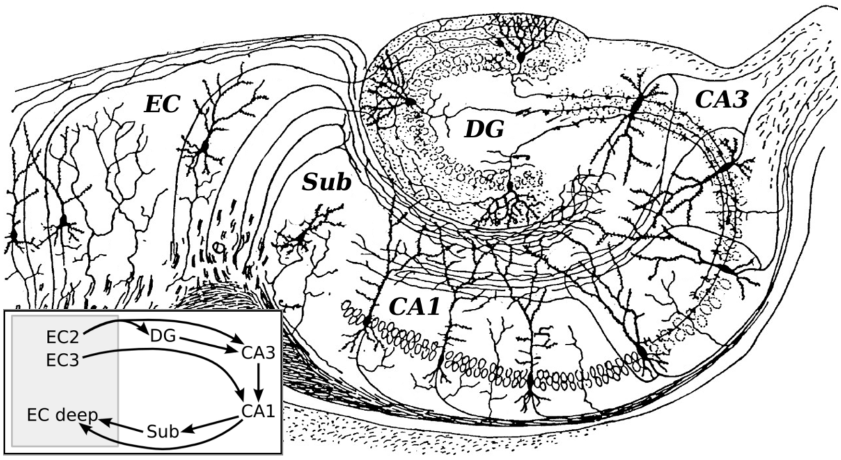
Answer: Potassium channels take time to inactivate, and continue to pass current that hyperpolarises the membrane. Inactivated sodium channels need time to deinactivate so that they can open to trigger a spike. $\frac{3}{4}$ marks for explanation of only one ion, full marks for both.

QA.4 [3 marks] The neurotransmitter GABA is inhibitory: It lowers the membrane voltage of the postsynaptic cells. What makes the voltage response inhibitory rather than excitatory? Explain in terms of the role of ionotropic channels and the relative concentration of chloride ions inside versus outside the cell in this process. (≈25-100 words)

Answer: Minimum (gist) required for full marks

1 mark GABA is inhibitory when the chloride ion concentration is higher outside the cell
1 mark Open GABA channels let negatively-charged chloride ions enter, decreasing membrane voltage.
Half mark removed for each false statement (suggests scattershot approach or misunderstanding).
Expect text generally similar to a subset of this explanation: *Ion pumps move chloride ions out of the cell, causing the extracellular concentration of Cl^- to be larger than the intracellular concentration. Entropic forces favour diffusion of Cl^- inward to equalise this concentration difference. Opening the GABA channel increases the chloride conductance, allowing negatively charged chloride ions to enter the cell and decrease the voltage. The excitatory voltage-gated sodium and calcium channels in neurons open above a certain voltage. Making the membrane voltage more negative makes it harder raise the voltage enough to trigger spikes, and is hence inhibitory. If the concentration gradient were reversed, then GABA would have an excitatory effect, as opening the GABA channels would allow negative charges to flow out of the cell and raise the membrane voltage.*

QA.5 [3 marks] This an annotated sketch of the circuity of the hippocampus and surrounding structures by Santiago Ramón y Cajal. Circle the region that is thought to play an important role in pattern completion thanks to recurrent self-excitation in the local circuitry.



CA3 and only CA3 should be circled for full marks. Interpret erased, crossed out answers charitably unless there appears to be a deliberate attempt to present an ambiguous response to game the marking.

In the unlikely event that an over-eager student annotates more than just CA3, for example discussing the entire procedure of encoding, pattern completion, and recall, give full marks only when CA3 is indicated as the site of densest local recurrent excitatory connections.

Section B: In-depth questions—answer TWO questions

QB.1 [10 marks] This question is about attractor networks

(a) [4 marks] What is an attractor network? (≈25–50 words)

In the broadest sense, an attractor network system composed of neuron-like units with recurrent connections, and recurrent dynamics that converge to one of several stable fixed points. Some attractor networks, like the Hopfield network, are derived using an "energy" function. Network dynamics in time (e.g. the Hopfield update rule) are chosen so that they decrease this energy, until reaching a stable local minima.

No students commented that the synchronous Hopfield update rule does not technically minimise energy, as it can be come trapped in cycles. These cycles are, however, attracting (and indeed some attractor dynamics in the brain may attract to cycles or manifolds). An asynchronous update (flipping one neuron at a time), and/or certain choices of how to treat units that are exactly at threshold, can recover the exact energy minimisation view. This level of depth was not expected as it was not expressly highlighted during teaching.

Any answer resembling this, or implicitly describing a system with these properties, gets full marks. Answers that could be reasonably inferred from information already given in the exam like "the thing that CA3 is" or "the thing that stores memories" get partial marks provided nothing factually incorrect is stated.

Consider a Hopfield network model:

- Network states are a column vector $\boldsymbol{x} = \{x_1, x_2, \ldots, x_K\}^\top$ for K units ($K = 2$ mostly here) States are either positive or negative one $x_i \in \{-1, 1\}$.
- Network weights are a symmetric matrix $\mathbf{W} \in \mathbb{R}^{K\times K}$. Assume that the weight matrix is set to zero along its diagonal $w_{ii} = 0$.
- Threshold is defined such that non-negative activation triggers a 1 output

$$\Theta(r) = \begin{cases} 1 & \text{if } r \geq 0 \\ -1 & \text{otherwise.} \end{cases}$$

(b) [3 marks] Write the mathematical expression of the energy of a pattern in the Hopfield network (or the equivalent in words). State whether (remembered) patterns stored in the weight matrix should have higher or lower energy compared to random patterns.

Expect something similar $-\frac{1}{2}\boldsymbol{x}^\top\mathbf{W}\boldsymbol{x}$ or $-\frac{1}{2}\sum_{ij} x_i x_j W_{ij}$. Over-thought answers that are mathematically equivalent in context are also valid. The $\frac{1}{2}$ is optional and no points are lost if it is missing. [+2] points for correct expression and sign or [+1] point for correct expression but incorrect sign.

Any statement indicating lower energy is more stable, or (successfully) stored patterns have lower energy, or equivalent, gets [1] mark even if no mathematical expression is given.

(c) [3 marks] In a Hopfield network with three neurons there are two patterns to store

$$\boldsymbol{x}_1 = (1, -1, -1) \text{ and } \boldsymbol{x}_2 = (1, 1, 1);$$

Calculate the weight matrix storing *both* patterns $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$. Use the approach form coursework 1, which calculates the weight matrix for all patterns that are stored in a single step.

$$\begin{aligned} \boldsymbol{x}_1\boldsymbol{x}_1^\top = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix}, \qquad \boldsymbol{x}_2\boldsymbol{x}_2^\top = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \\ \mathbf{W} = \tfrac{1}{2}(\boldsymbol{x}_1\boldsymbol{x}_1^\top + \boldsymbol{x}_2\boldsymbol{x}_2^\top - 2\mathbf{I}) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \end{aligned} \tag{1}$$

Full marks for any answer proportional to this (positive scalar factors do not change the dynamics in the Hopfield model we study). Answers correctly solving for weights w_{12} w_{23} w_{13} individually were also valid. Partial marks off for not setting the diagonal to zero (this is stated in the definition of the network being considered, above).

A common mistake involved taking the outer product of the two patterns, e.g. $\boldsymbol{x}_1\boldsymbol{x}_2^\top$ in some form. Some answers included this incorrect approach, but with cancellation of errors leading to partly correct solutions. These could not receive credit.

QB.2 [10 marks] This question is about linear ordinary differential equations.

(a) [3 marks] You are given a model of a neuron's membrane voltage, whose state v evolves in time according the ordinary differential equation

$$C\dot{v} = \tfrac{1}{R}(E - v) + I. \tag{2}$$

The variables C, R, E, and I are all constants. Re-arrange Equation (2) in the format

$$\tau\dot{v} = v_\infty - v \tag{3}$$

by finding the values of the constants τ and v_∞ in terms of C, R, E, and I.

$$\begin{aligned} C\dot{v} &= \tfrac{1}{R}(E - v) + I \\ (RC)\dot{v} &= (E + RI) - v \\ v_\infty &= E + RI \\ \tau &= RC \\ \tau\dot{v} &= v_\infty - v \end{aligned} \tag{4}$$

Throughout this question, no marks will be deducted for writing v_∞ as $\bar{v}$ or other sensible substitutions with notation in common in computational neuroscience.

(b) [3 marks] If v starts at value v_0 at time $t = 0$, show that Equation (3) has a solution of the form $v(t) = e^{-at}(v_0 - v_\infty) + v_\infty$ and state the relationship between a and τ.

To show this, students may integrate the ODE using any valid technique, or simply confirm by differentiation or other proof that this form is a correct solution. Here are some examples:

Show equivalent by differentiating to verify:

$$\begin{aligned} v(t) &= e^{-at}(v_0 - v_\infty) + v_\infty \\ \dot{v} &= -a\{e^{-at}v_0 - e^{-at}v_\infty\} \\ \tfrac{1}{a}\dot{v} &= \underbrace{e^{-at}(v_\infty - v_0)}_{=-v(t)+v_\infty} \\ &\Rightarrow \\ \tfrac{1}{a}\dot{v} &= v_\infty - v(t), && \text{setting parameters } \alpha = 1/\tau \text{ gives} \\ \tau\dot{v} &= v_\infty - v(t) && \text{recovering the original ODE. } \blacksquare \end{aligned} \tag{5}$$

Relationship: $\alpha = 1/\tau$.

Variant using undetermined coefficients/ansatz

$$\begin{aligned}
v(t) &= A + Be^{\lambda t} && \text{anstaz} \\
\dot{v} &= B\lambda e^{\lambda t} && \text{differentiate} \\
\dot{v} &= \lambda(v - A) && \text{substitute to get ODE} \\
\tau\dot{v} &= v_\infty - v && \text{match coefficients to given ODE} \\
\dot{v} &= -\frac{1}{\tau}(v - v_\infty) && \lambda = -1/\tau \text{ and } A = v_\infty \\
v(t) &= v_\infty + Be^{-t/\tau} && \text{substitute} \\
v(0) &= v_\infty + Be^{-0/\tau} && \text{apply initial condition to get } B \\
B &= v(0) - v_\infty && \text{solve for } B \\
v(t) &= v_\infty + e^{-t/\tau}(v(0) - v_\infty) && \text{substitute } B \ \blacksquare
\end{aligned} \tag{6}$$

Show equivalent by integrating using integration factor

$$\begin{aligned}
\tau\dot{v} &= v_\infty - v && \\
\dot{v} + \tfrac{1}{\tau}v &= \tfrac{1}{\tau}v_\infty && \text{I.F. is } e^{t/\tau} \\
e^{t/\tau}(\dot{v} + \tfrac{1}{\tau}v) &= e^{t/\tau}\tfrac{1}{\tau}v_\infty && \text{Multiply by } e^{t/\tau} \\
e^{t/\tau}v(t) &= \int_{\mathrm{d}t} e^{t/\tau}\tfrac{1}{\tau}v_\infty && \text{Integrate both sides} \\
e^{t/\tau}v(t) &= v_\infty e^{t/\tau} + c && \text{(collect constants to R.H.S.)} \\
v(t) &= v_\infty + ce^{-t/\tau} && \text{Move } e^{t/\tau} \text{ over} \\
v(0) &= v_\infty + ce^{-0/\tau} && \text{evaluate at zero to find } c \\
c &= v_0 - v_\infty && \text{substitute } c \\
v(t) &= v_\infty + e^{-t/\tau}(v_0 - v_\infty) && \text{This matches provided solution. } \blacksquare
\end{aligned} \tag{7}$$

(c) [4 marks] Your colleague wrote computer code to integrate Equation (3) numerically. They used the parameter values $\tau = 10$ ms and $v_\infty = -30$ mV, and initial conditions $v(t = 0) = v_0 = -70$. They used the forward Euler integration method with time step $\Delta t = 5$ ms.

i. [2 marks] What is the order of the error in the Forward Euler method in terms of Δt?

First order

ii. [2 marks] What value for v do you get when calculating $v(t = 5)$ using Forward Euler with $\Delta t = 5$? Does Euler integration over- or under-estimate compared to the true solution of $v(t = 5) \approx -54.26$?

$$\begin{aligned} \dot{v} &= (v_\infty - v)/\tau \\ \dot{v}(0) &= (-30 - -70)/10 = 40/10 = 4 \\ \hat{v}(5) &= v_0 + 5\dot{v}(0) = -70 + 5 \cdot 4 = -70 + 20 = -50 > -54.26 \end{aligned} \tag{8}$$

Forward Euler overestimates (because we have negative curvature, but this detail is not required).

QB.3 This question is about the McCulloch–Pitts neuron and the delta learning rule.

(a) [3 marks] Consider a binary perceptron unit with inputs $\boldsymbol{x} = \{x_1, x_2, \ldots, x_n\}^\top$ and weights $\boldsymbol{w} = \{w_1, w_2, \ldots, w_n\}^\top$, whose output is defined as

$$\hat{y} = \Theta(\textstyle\sum_i w_i x_i) = \Theta(\boldsymbol{w}^\top \boldsymbol{x}), \tag{9}$$

where $\Theta(\cdot)$ is the Heaviside step function (that is, binary threshold)

$$\Theta(r) = \begin{cases} 1 & \text{if } r > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Explain the interpretation of the variables $\boldsymbol{x}$, $\boldsymbol{w}$, and $\boldsymbol{y}$ in terms of both *computation* **and** which aspects of *neuronal signalling and/or anatomy* each corresponds to.

$\boldsymbol{x}$: Each x_i is an input arriving at the neuron, and models (vaguely) the concentration of neurotransmitters released by the presynaptic input at a single incoming synapse.

$\boldsymbol{w}$: Each w_i controls the effect of input x_i on the postsynaptic cell. Larger weights have more effect. Positive weights are excitatory inputs. Negative, inhibitory. Larger weights correspond to larger synapses, or synapses with denser post-synaptic ion channels, or any other biochemical changes at the synapses that increases the impact of input x_i on the dendritic membrane voltage.

y: y is the neurons output. Here, 1 models ”spike” and 0 models ”no spike”. This is a very crude way to capture the all-or-nothing binary nature of the action potential in a computational model.

(b) [3 marks] The delta learning rule for updating weight w_k in (9) is

$$\Delta w_k = \eta \cdot \text{error} \cdot x_k. \tag{10}$$

Re-write Equation (10), replacing error with its expression in terms of $\boldsymbol{w}$ and a single pair of supervised training data $(\boldsymbol{x}, y^*)$, where y^* is the target output.

$$\begin{aligned} \Delta w_k &= \eta \cdot (y^* - \hat{y}) \cdot x_k. \\ &= \eta \cdot [y^* - \Theta(\boldsymbol{w}^\top \boldsymbol{x})] \cdot x_k. \end{aligned} \tag{11}$$

Students who did not substitute to obtain an expression depending on $\boldsymbol{x}$ received partial marks. Some students lost partial marks for omitting the threshold $\Theta(\cdot)$, which is necessary as it has been stated in the definition of the perceptron being considered here (Equation 9).

Turn Over for QB.3(c)!

(c) [4 marks] Consider the following function,

$$f(\boldsymbol{w}; \boldsymbol{x}, y^*) = \mathrm{relu}(r) - y^* \cdot r \qquad \text{where } r = \boldsymbol{w}^\top \boldsymbol{x} = \textstyle\sum_i w_i x_i, \tag{12}$$

where $\mathrm{relu}(r) = \mathrm{maximum}(0, r)$ is the “rectified linear unit”, which clips negative values of r to zero.

Suppose that we want to minimise $f(\boldsymbol{w}; \boldsymbol{x}, y^*)$ by adjusting the synaptic weights $\boldsymbol{w}$ (for a given fixed $(\boldsymbol{x}, y^*)$. Show that using gradient descent to change $\boldsymbol{w}$ to minimise Equation (12) leads to the “delta rule” weight update in Equation (10).[1]

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}w_i} f(\boldsymbol{w}; \boldsymbol{x}, y^*) &= \frac{\mathrm{d}}{\mathrm{d}w_i} \mathrm{relu}(r) - y^* \cdot \frac{\mathrm{d}}{\mathrm{d}w_i} r \\ &= [\Theta(r) - y^*] \cdot \frac{\mathrm{d}}{\mathrm{d}w_i} r \\ &= [\hat{y} - y^*] \cdot x_i \end{aligned} \tag{13}$$

With gradient descent, we follow the negative of the gradient, so

$$\Delta w = \eta \cdot (-\frac{\mathrm{d}}{\mathrm{d}w_i}) = \eta \cdot [y^* - \hat{y}] \cdot x_i; \tag{14}$$

This is the delta rule.

Very few students attempted this question, but many who did received full marks. Some students attempted to work from both directions, but lost partial marks for not noting that the derivative of $\mathrm{relu}(\cdot)$ is $\Theta(\cdot)$ to complete the proof. Others lost partial marks for completing the proof by asserting a different (incorrect) derivative for $\mathrm{relu}(\cdot)$.

[1] You may assume that the derivative of $\mathrm{relu}(r)$ at $r = 0$ is zero, that is: $\frac{\mathrm{d}}{\mathrm{d}r} \mathrm{relu}(r)\big|_0 = 0$.