Scientific Computing
Derivation of the Crank-Nicolson method
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The Crank-Nicolson method enables solutions to the diffusion equation to be computed with an accuracy of $O((\Delta t)^2)$. These notes will show how the Crank-Nicolson method can be derived by introducing a second grid for time that is offset from the original grid by half a time step. This idea of using two grids that are offset from one another is called grid staggering, and the combination of the two grids is called a *staggered grid*. Staggered grids are incredibly useful, especially when using the finite-difference method to solve systems of differential equations, such as those describing fluid flow or the deformation of solids.

To derive the Crank-Nicolson method, we let the points $t_n = n\Delta t$ where $n = 0, 1, 2, \dots$ denote the original grid for the time variable t. Now introduce a second grid for time by letting $t_{n+1/2} = n(\Delta t/2)$. We now evaluate the diffusion equation on the second grid for time to get

$$\left.\frac{\partial u}{\partial t}\right|_{t=n(\Delta t)/2} = D\left.\frac{\partial^2 u}{\partial x^2}\right|_{t=n(\Delta t/2)} \tag{1}$$

As before, we now let $u_i^n \simeq u(x_i, t_n)$ denote the numerical approximation to the true solution. The second derivative on the right-hand side of (1) is discretised as usual using central differences as

$$\left.\frac{\partial^2 u}{\partial x^2}\right|_{t=n(\Delta t/2)} = \frac{u_{i+1}^{n+1/2} - 2u_i^{n+1/2} + u_{i-1}^{n+1/2}}{(\Delta x)^2} + O((\Delta x)^2). \tag{2}$$

However, the time derivative on the left-hand side of (1) is now discretised using the*central* difference formula, which is second-order accurate in Δt. Therefore, we let

$$\left.\frac{\partial u}{\partial t}\right|_{t=n(\Delta t)/2} = \frac{u_i^{n+1} - u_i^n}{\Delta t} + O((\Delta t)^2). \tag{3}$$

Combining everything so far gives

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \frac{D}{(\Delta x)^2}\left(u_{i+1}^{n+1/2} - 2u_i^{n+1/2} + u_{i-1}^{n+1/2}\right) + O((\Delta t)^2, (\Delta x)^2). \tag{4}$$

This equation is not very helpful right now because it involves the solution at three different points in time (t_n, $t_{n+1/2}$, and t_{n+1}). The reason this is problematic is because (4) does not tell us how to compute the solution at the half time steps $t_{n+1/2}$; it only provides a formula for computing the solution at integer time steps t_1, t_2, t_3, etc. To get around this issue, we use linear interpolation and write $u_i^{n+1/2} = (u_i^{n+1} + u_i^n)/2$. Making use of this in (4) leads to the Crank-Nicolson formula

$$\begin{aligned}\frac{u_i^{n+1} - u_i^n}{\Delta t} =& \frac{D}{2(\Delta x)^2}\left(u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}\right) \\ &+ \frac{D}{2(\Delta x)^2}\left(u_{i+1}^{n} - 2u_i^{n} + u_{i-1}^{n}\right) + O((\Delta t)^2, (\Delta x)^2).\end{aligned} \tag{5}$$

At this point you might wonder what error has been introduced by using linear interpolation to compute the solution at half time steps. Using Taylor expansions and following the approach laid out in the supplementary notes from Week 19, one can show that $u_i^{n+1/2} = (u_i^{n+1} + u_i^n)/2 + O((\Delta t)^2)$. Hence, the error introduced by linear interpolation is the same order as that introduced by the discretisation of the time derivative; therefore, (5) holds.