Scientific Computing and Optimisation
Implicit-explicit (IMEX) methods
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Consider a nonlinear reaction-diffusion equation given by

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} + q(u). \tag{1}$$

We have seen two ways of numerically solving this equation. The explicit Euler method is the easiest to implement, but the time-step restriction means that a large number of time steps will be needed. The implicit Euler method allows larger time steps to be used, but requires solving a nonlinear algebraic system of equations at each time step. If solving the nonlinear algebraic system using Newton's method, then the Jacobian of the nonlinear system needs to be calculated.

Implicit-explicit methods, or IMEX methods, harness the simplicity of explicit methods and the numerical stability of implicit methods. The idea behind these methods is to treat nonlinear terms explicitly, and treat diffusion terms implicitly. The end result will be a system of *linear* algebraic equations that have to be solved at each time step, removing the need to use a nonlinear algebraic solver.

To see how this works, let's assume the PDE in (1) is being solved with two Dirichlet boundary conditions. The discussion below can be generalised to other combinations of boundary conditions. After discretising in space, the PDE in (1) becomes

$$\frac{\partial \boldsymbol{u}}{\partial t} = \frac{D}{(\Delta x)^2}\left[\mathbf{A}^{DD}\boldsymbol{u} + \boldsymbol{b}^{DD}\right] + \boldsymbol{q}(\boldsymbol{u}), \tag{2}$$

where $\boldsymbol{u} = (u_1, u_2, \ldots, u_{N-1})^T$. In the IMEX method, the diffusion term $\mathbf{A}^{DD}\boldsymbol{u}$ is evaluated using the solution at the next time level, i.e. $\boldsymbol{u}^{n+1}$, whereas the nonlinear source term is evaluated using the solution at the current time level, e.g. $\boldsymbol{u}^n$. Therefore, the IMEX scheme for (2) is

$$\frac{\boldsymbol{u}^{n+1} - \boldsymbol{u}^n}{\Delta t} = \frac{D}{(\Delta x)^2}\left[\mathbf{A}^{DD}\boldsymbol{u}^{n+1} + \boldsymbol{b}^{DD}\right] + \boldsymbol{q}(\boldsymbol{u}^n). \tag{3}$$

After some rearranging and introducing $C = D\Delta t/(\Delta x)^2$, we obtain

$$\left(\mathbf{I} - C\mathbf{A}^{DD}\right)\boldsymbol{u}^{n+1} = \boldsymbol{u}^{n+1} + C\boldsymbol{b}^{DD} + \Delta t\boldsymbol{q}(\boldsymbol{u}^n). \tag{4}$$

This is a linear system of algebraic equations that can be solved for $\boldsymbol{u}^{n+1}$. In fact, this is basically the same linear system that arises when the implicit Euler method is applied to the linear diffusion equation; now, there's just an extra term on the right-hand side coming from the source term.