مقدمة

الحمد لله الذي علَّم بالقلم، علَّم الإنسان ما لم يعلم، والصلاة والسلام على من بُعث مُعلماً للناس وهادياً وبشيراً، وداعياً إلى الله بإذنه وسراجاً منيراً؛ فأخرج الناس من ظلمات الجهل والغواية، إلى نور العلم والهداية، نبينا ومعلمنا وقدوتنا الأول محمد بن عبدالله وعلى آله وصحبه أجمعين، أما بعد:

تسعى المؤسسة العامة للتدريب التقني والمهني لتأهيل الكوادر الوطنية المدربة القادرة على شغل الوظائف التقنية والفنية والمهنية المتوفرة في سوق العمل السعودي، ويأتي هذا الاهتمام نتيجة للتوجهات السديدة من لدن قادة هذا الوطن التي تصب في مجملها نحو إيجاد وطن متكامل يعتمد ذاتياً على الله ثم على موارده وعلى قوة شبابه المسلح بالعلم والإيمان من أجل الاستمرار قدماً في دفع عجلة التقدم التنموي، لتصل بعون الله تعالى لمصاف الدول المتقدمة صناعياً.

وقد خطت الإدارة العامة للمناهج خطوة إيجابية تتفق مع التجارب الدولية المتقدمة في بناء البرامج التدريبية، وفق أساليب علمية حديثة تحاكي متطلبات سوق العمل بكافة تخصصاته لتلبي تلك المتطلبات، وقد تمثلت هذه الخطوة في مشروع إعداد المعايير المهنية الوطنية ومن بعده مشروع المؤهلات المهنية الوطنية، والذي يمثل كل منهما في زمنه، الركيزة الأساسية في بناء البرامج التدريبية، إذ تعتمد المعايير وكذلك المؤهلات لاحقاً في بنائها على تشكيل لجان تخصصية تمثل سوق العمل والمؤسسة العامة للتدريب التقني والمهني بحيث تتوافق الرؤية العلمية مع الواقع العملي الذي تفرضه متطلبات سوق العمل، لتخرج هذه اللجان في النهاية بنظرة متكاملة لبرنامج تدريبي أكثر التصاقاً بسوق العمل، وأكثر واقعية في تحقيق متطلباته الأساسية.

وتتناول هذه الحقيبة التدريبية "**........**" لمتدربي برنامج "......" في المعاهد الصناعية الثانوية ومعاهد العمارة والتشييد، موضوعات حيوية تتناول كيفية اكتساب المهارات اللازمة لهذا البرنامج لتكون مهاراتها رافداً لهم في حياتهم العملية بعد تخرجهم من هذا البرنامج. والإدارة العامة للمناهج وهي تضع بين يديك هذه الحقيبة التدريبية تأمل من الله عز وجلً أن تسهم بشكل مباشر في تأصيل المهارات الضرورية اللازمة، بأسلوب مبسط خالٍ من التعقيد.

والله نسأل أن يوفق القائمين على إعدادها والمستفيدين منها لما يحبه و يرضاه ؛ إنه سميع مجيب الدعاء.

الإدارة العامة للمناهج

Course Name Mathematics 2 **Course Code** MATH 302

Course Description:

This course covers sequences and series, especially focusing on power series and Taylor's formula. Give students an understanding of Fourier series and Fourier transform, and provide students with practice in their application and interpretation in a range of situations. Help understanding how single-variable calculus generalizes to higher dimensions and Learn Green's theorem. Treatment of numerical methods including numerical solution of equations, interpolation method, approximation of functions, numerical integration, and differentiation, ...

General Objective:

The primary objective of the course is to develop the basic understanding of the mathematics that underlies modern science.

Detailed Objectives:

Trainee will be able to:

1- Evaluate limits of sequences, know basic limits and determine the limits of some simple recursively defined sequences.
2- Apply series tests (Divergence Test, Comparison and Limit Comparison Tests, Ratio Test, Alternating Series Test, …) to determine whether a particular series converges or diverges.
3- Determine the radius and interval of convergence for a power series and describe when they can be differentiated and integrated term-by-term.
4- Represent functions as Taylor series and Maclaurin series.
5- Approximate functions using Taylor polynomials and partial sums of infinite series.
6- Compute the coefficients of Fourier series for a periodic function.
7- Find the sum of a Fourier series of a continuous or regular numerical function at a given point.
8- Approximate functions using trigonometric polynomials (in particular the Fourier polynomial) and partial sums of infinite series.
9- Calculate Fourier transforms for a variety of simple functions.
10- Apply Fourier analysis to solve various engineering problems.
11- Evaluate double and triple integrals, and learn their use to compute volume, surface area, etc.
12- Use Green's Theorem in the Plane.
13- Analyze and Solve problems using Numerical Methods.

CHAPTER 1:

Infinite Sequences and Series

1.1 Sequences

Introduction to Sequences

Goals:

Define a sequence.

Determine if a given sequence is arithmetic or geometric.

A sequence is an ordered list of terms or elements. And we say also a sequence can be thought of as a list of numbers written in a definite order $a_1, a_2, a_3, a_4, a_5, \ldots, a_n, \ldots$. Each term in a sequence is identified by its location in the sequence.

a_1 is the first term.

a_2 is the second term.

a_3 is the third term.

a_4 is the fourth term.

and so on...

EXAMPLES 1: Some sequences can be defined by giving a formula for the nth term. In the following examples we give three descriptions of the sequence: one by using the preceding notation, another by using the defining formula, and a third by writing out the terms of the sequence. Notice that doesn't have to start at 1.

(i)	$\{\frac{n}{n+1}\}_{n=1}^{\infty}$	$a_n = \frac{n}{n+1}$	$\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots, \frac{n}{n+1}, \ldots\}$
(ii)	$\left\{\frac{(-1)^n(n+1)}{3^n}\right\}$	$a_n = \frac{(-1)^n(n+1)}{3^n}$	$\{-\frac{2}{3}, \frac{3}{9}, -\frac{4}{27}, \frac{5}{81}, \ldots, \frac{(-1)^n(n+1)}{3^n}, \ldots\}$
(iii)	$\{\sqrt{n-3}\}_{n=3}^{\infty}$	$a_n = \sqrt{n-3}, n \geqslant 3$	$\{0, 1, \sqrt{2}, \sqrt{3}, \ldots, \sqrt{n-3}, \ldots\}$
(iv)	$\{\cos\frac{n\pi}{6}\}_{n=0}^{\infty}$	$a_n = \cos\frac{n\pi}{6}, n \geqslant 0$	$\{1, \frac{\sqrt{3}}{2}, \frac{1}{2}, 0, \ldots, \cos\frac{n\pi}{6}, \ldots\}$
(v)	$\{\frac{n}{n+1}\}_{n=1}^{\infty}$	$a_n = \frac{n}{n+1}$	$\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots, \frac{n}{n+1}, \ldots\}$

A sequence can be finite or infinite.

A finite sequence $\{a_1, a_2, a_3, a_4, a_5, \ldots, a_n\}$ is a function with domain 1, 2, 3, ..., n.

1,3,5,7,9

An infinite sequence $\{a_1, a_2, a_3, a_4, a_5, \ldots, a_n, \ldots\}$ is a function with domain 1, 2, 3, 4, ...

1,3,5,7,9,. . .

A sequence is often expressed as a rule or formula.

$a_n = 3n - 5$

$a_n = 2^n + 1$

EXAMPLE 2: Find the first four terms of sequence $a_n = 3n - 5$.

SOLUTION $a_1 = 3(1) - 5 = -2$, $a_2 = 3(2) - 5 = 1$, $a_3 = 3(3) - 5 = 4$, $a_4 = 3(4) - 5 = 7$

$$-2,1,4,7$$

EXAMPLE 3: Find the first four terms of sequence $a_n = 2^n + 1$

SOLUTION $a_1 = 2^{(1)} + 1 = 3$, $a_2 = 2^{(2)} + 1 = 5$, $a_3 = 2^{(3)} + 1 = 9$, $a_4 = 2^{(4)} + 1 = 17$

$$3,5,9,17$$

A sequence can also be expressed as a recursive formula, this means each term in the sequence is based on previous terms, not just n. For example: $a_n = 4a_{n-1} - 3; a_1 = 3$.

$$a_2 = 4a_1 - 3 = 4(3) - 3 = 9$$

$$a_3 = 4a_2 - 3 = 4(9) - 3 = 33$$

$$3,9,33$$

Another example: $a_n = a_{n-2} + a_{n-1}; a_1 = 1, a_2 = 2$

$$a_3 = a_1 + a_2 = 1 + 2 = 3$$

$$a_4 = a_2 + a_3 = 2 + 3 = 5$$

$$1,2,3,5$$

There are different types of sequences.

An arithmetic sequence is a sequence that has the pattern of adding the same value to determine consecutive terms.

We say arithmetic sequences have a common difference, $d = a_n - a_{n-1}$,example: 37,31,25,19,13,7

$$d = a_2 - a = 31 - 37 = -6$$

A geometric sequence is a sequence that has the pattern of multiplying by a constant to determine consecutive Terms.

We say geometric sequences have a common ratio, $r = \frac{a_n}{a_{n-1}}$, For example: $9,3,1,\frac{1}{3},\frac{1}{9},\ldots$

$r = \frac{a_3}{a_2} = \frac{1}{3}$

Arithmetic Sequences

Goals

Determine the nth term of an arithmetic sequence.
Determine the common difference of an arithmetic sequence.
Determine the formula for an arithmetic sequence.

An arithmetic sequence is a sequence that has the pattern of adding a constant to determine consecutive terms.
We say arithmetic sequences have a common difference.

$$d == a_n - a_{n-1}$$
$$a_n = a_1 + (n-1)\,d$$

$$a_n = a_{n-1} + d$$
$$-5, -1, 3, 7, 11, \ldots$$

A sequence is a function.

EXAMPLE 4: What is the domain and range of the following sequence $\mathbf{9, 6, 3, 0, -3, -6}$.

SOLUTION $\boldsymbol{D}: \{\mathbf{1, 2, 3, 4, 5, 63}\}$ $\boldsymbol{R}: \{\mathbf{-6, -3, 0, 3, 6, 93}\}$

EXAMPLE 4: Given the formula for the arithmetic sequence, determine the first 3 terms and then the 8th term. Also state the common difference. $a_n = -4n + 3$

SOLUTION $a_1 = -4(1) + 3 = -1$

$a_2 = -4(2) + 3 = -5$

$a_3 = -4(3) + 3 = -9$
$a_5 = -4(8) + 3 = -29$
$d = -5 - (-1) = -5 + 1 = -4$

EXAMPLE 4: Given the arithmetic sequence, determine the formula and the 12th term. $\mathbf{-2, 1.5, 5, 8.5, 12, 15.5}, \ldots$

SOLUTION $a_n = a_1 + (n-1)\, d$

$a_1 = -2$ $d = a_n - a_{n-1}$ $d = 12 - 8.5$ $d = 3.5$
$a_n = -2 + (n-1)(3.5)$ $a_n = -2 + 3.50n - 3.5$ $a_n = 3.5n - 5.5$
$a_{12} = 3.5(12) - 5.5$ $a_{12} = 36.5$

Geometric Sequences

Goals

Determine the nth term of a geometric sequence.
Determine the common ratio of a geometric sequence.
Determine the formula for a geometric sequence.

A geometric sequence is a sequence that has the pattern of multiplying by a constant to determine consecutive terms.

We say geometric sequences have a common ratio. $r = \frac{a_n}{a_{n-1}}$

$$a_n = a_1 r^{n-1}$$

$$a_n = a_{n-1} r$$

$$-2, -6, -18, -54, -162$$

EXAMPLE 5: What is the domain and range of the following sequence $-12, 6, -3, \frac{3}{2}, -\frac{3}{4}$ What is r?

SOLUTION $r = -\frac{1}{2}$ $D: \{1,2,3,4,5\}$ $R: \{-12, -3, -\frac{3}{4}, \frac{3}{2}, 6\}$

EXAMPLE 6: Given the formula for the geometric sequence, determine the first 2 terms and then the 5th term. Also state the common ratio.

SOLUTION $r = 2$

$a_n = 3\,(2)^n$
$a_1 = 3\,(2)^1 = 6$
$a_2 = 3\,(2)^2 = 12$
$a_5 = 3\,(2)^5 = 96$

EXAMPLE 6: Given the geometric sequence $\frac{1}{3}, \frac{2}{9}, \frac{4}{27}, \frac{8}{81}, \dots$, determine the formula. Then determine the 6th term.

SOLUTION $r = \frac{2}{3}$

$$a_n = \frac{1}{3}\left(\frac{2}{3}\right)^{n-1}$$

$$a_6 = \frac{1}{3}\left(\frac{2}{3}\right)^5 = \frac{2^5}{3^6} = \frac{32}{729}$$

EXAMPLE 7: Given the information about the geometric sequence $a_0 = 5, a_1 = \frac{40}{9}, a_2 = \frac{320}{81}$, determine the formula for the nth term.

SOLUTION $r = \frac{8}{9}$

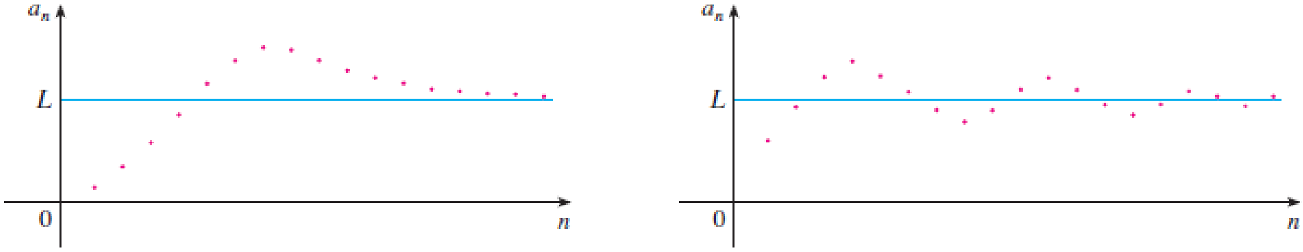
$$a_n = 5\left(\frac{8}{9}\right)^n$$

Introduction to Limits of a Sequence

Goal

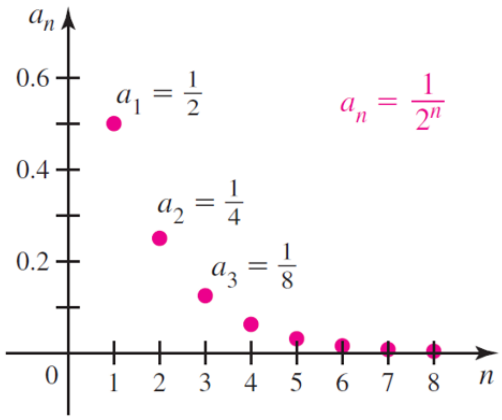
Determine whether a sequence converges or diverges.

Definition: Limit of a Sequence If the terms of a sequence $\{a_n\}$ approaches a number L, as n increases, then $\lim_{n\to\infty} a_n = L$ Sequences that have limits converge. Sequences that do not have limits diverge.

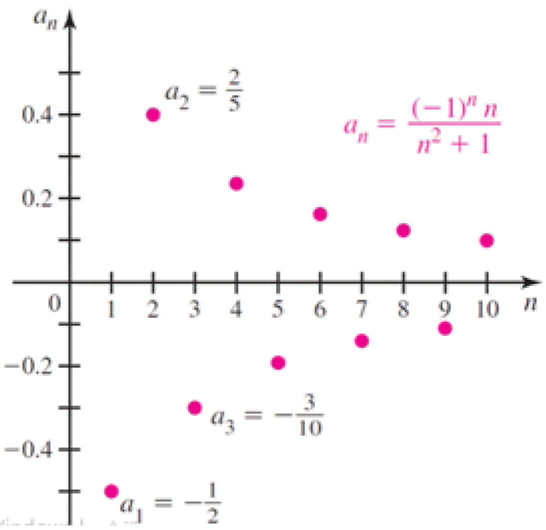


The graphs of two sequences that have the limit L.

EXAMPLE 8: Consider the following graphs of sequences.

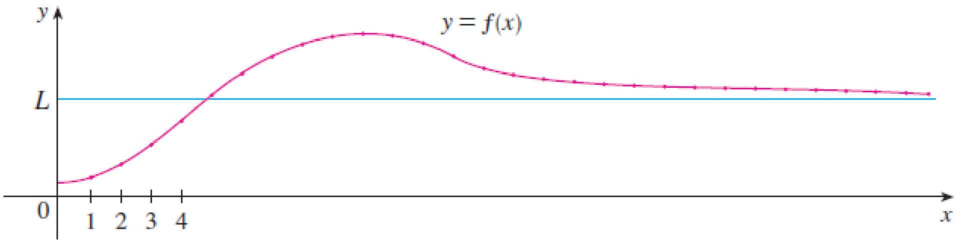


EXAMPLE 9: Consider the following graphs of sequences.



The Limit of a Sequence

Theorem $If \quad \lim_{x\to\infty} f(x) = L \quad and\ f(n) = a_n$, when n is an integer, then $\lim_{n\to\infty} a_n = L$



In particular, since we know that $\lim_{x\to\infty}(1/x^r) = 0$ when $r > 0$ we have $\lim_{n\to\infty}\frac{1}{n^r} = 0$ if $r > 0$

If $\lim_{n\to\infty} a_n = \infty$,then the sequence $\{a_n\}$ is divergent but in a special way. We say that $\{a_n\}$ diverges to ∞.

Limit Laws for Sequences

If $\{a_n\}$ and $\{b_n\}$ are convergent sequences and c is a constant, then

$$\lim_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty}(a_n - b_n) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty} c a_n = c \lim_{n\to\infty} a_n \lim_{n\to\infty} c = c$$

$$\lim_{n\to\infty}(a_n b_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} \quad if \ \lim_{n\to\infty} b_n \neq 0$$

$$\lim_{n\to\infty} a_n^p = [\lim_{n\to\infty} a_n]^p \quad if \quad p > 0 \ and \ a_n > 0$$

EXAMPLE 10: Determine if the sequence $a_n = \frac{3n-2}{n-1}$ converges of diverges.

SOLUTION Divide numerator and denominator by the highest power of n that occurs in the denominator and then use the Limit Laws.

$$\lim_{n\to\infty}\frac{3n-2}{n-1} = \lim_{n\to\infty}\frac{3-\frac{2}{n}}{1-\frac{1}{n}} = \frac{\lim_{n\to\infty} 3 - \lim_{n\to\infty}\frac{2}{n}}{\lim_{n\to\infty} 1 - \lim_{n\to\infty}\frac{1}{n}} = \frac{3-0}{1-0} = \frac{3}{1} = 3$$

$\lim_{n\to\infty} a_n = 3$ is Converges

EXAMPLE 11: Is the sequence $a_n = \frac{n}{\sqrt{10+n}}$ convergent or divergent?

SOLUTION As in Example 10, we divide numerator and denominator by n:

$$\lim_{n\to\infty}\frac{n}{\sqrt{10+n}} = \lim_{n\to\infty}\frac{1}{\sqrt{\frac{10}{n^2}+\frac{1}{n}}} = \infty$$

because the numerator is constant and the denominator approaches 0. So $\{a_n\}$ is divergent.

EXAMPLE 12: Calculate $\lim_{n\to\infty}\frac{\ln n}{n}$.

SOLUTION Notice that both numerator and denominator approach infinity. We can't apply l'Hospital's Rule directly because it applies not to sequences but to functions of a real variable. However, we can apply l'Hospital's Rule to the related function $f(x) = (\ln x)/x$ and obtain

$$\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{n\to\infty} \frac{1/x}{1} = 0$$

Therefore, we have

$$\lim_{n\to\infty} \frac{\ln n}{n} = 0$$

$$If \ \lim_{n\to\infty} |a_n| = 0 \quad , \qquad then \ \lim_{n\to\infty} a_n = 0$$

EXAMPLE 13: Determine whether the sequence $a_n = (-1)^n$ is convergent or divergent.

SOLUTION If we write out the terms of the sequence, we obtain

$$\{-1,1,-1,1,-1,1,\ldots.\}$$

Since the terms oscillate between 1 and -1 infinitely often, a_n does not approach any number. Thus $\lim\limits_{n\to\infty} (-1)^n$ does not exist; that is, the sequence $\{(-1)^n\}$ is divergent.

EXAMPLE 14: Evaluate $\lim\limits_{n\to\infty} \frac{(-1)^n}{n}$ if it exists.

SOLUTION We first calculate the limit of the absolute value:

$$\lim_{n\to\infty} \left|\frac{(-1)^n}{n}\right| = \lim_{n\to\infty} \frac{1}{n} = 0$$

The following theorem says that if we apply a continuous function to the terms of a convergent sequence, the result is also convergent.

EXAMPLE 15: Determine if the sequence $a_n = -2 + (-1)^n$ converges or diverges.

SOLUTION $-3, -1, -3, -1, -3, -1, \ldots \ldots \ldots$

$\lim\limits_{n\to\infty} a_n$ does not exist Diverges

EXAMPLE 16: Determine if the sequence $a_n = \frac{\ln n}{5n}$ converges or diverges.

SOLUTION $$\lim_{x\to\infty} \frac{\ln x}{5x} = \lim_{n\to\infty} \frac{1/x}{5} = 0$$

Therefore, we have

$\lim\limits_{n\to\infty} \frac{\ln n}{5n} = 0$ Converges.

The sequence $\{r^n\}$ is convergent if $-1 < r \leq 1$ and divergent for all other values of r.

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & if \quad -1 < r \le 1 \\ 1 & if \quad r = 1 \end{cases}$$

EXAMPLE 17: Determine if the sequence $a_n = \frac{1}{(-2)^n}$ converges of diverges.

SOLUTION $|a_n| = \frac{1}{2^n}$

$$\lim_{x\to\infty} \frac{1}{2^x} = 0$$

$\lim_{n\to\infty} |a_n| = 0$ Converges

Squeeze Theorem

$$If\ \{a_n\}, \{b_n\}, and\ \{c_n\}\ are\ sequences\ such\ that$$

$$a_n \le\ b_n \le c_n \qquad for\ all\ n\ and$$

$$\lim_{n\to\infty} a_n\ = L \qquad\qquad \lim_{n\to\infty} c_n\ = L$$

$$then$$

$$\lim_{n\to\infty} b_n\ = L$$

EXAMPLE 18: Determine if the sequence $b_n = \frac{\sin n}{2n}$ converges of diverges.

SOLUTION $b_n = \frac{\sin n}{2n}$

$$a_n = \frac{-1}{2n} \qquad\qquad c_n = \frac{1}{2n}$$

$$\frac{-1}{2n} \le \frac{\sin n}{2n} \le \frac{1}{2n}$$

$$\lim_{n\to\infty} \frac{-1}{2n} = 0 \qquad\qquad \lim_{n\to\infty} \frac{1}{2n} = 0$$

$\therefore \lim_{n\to\infty} \frac{\sin n}{2n} = 0$ converges

EXAMPLE 19: Discuss the convergence of the sequence $a_n = n!/n^n$.

where n! $= n! = 1 \cdot 2 \cdot 3 \cdot \cdots \cdot n$.

SOLUTION Both numerator and denominator approach infinity as $n \to \infty$ ` but here we have no corresponding function for use with l'Hospital's Rule (x! is not defined when x is not an integer). Let's write out a few terms to get a feeling for what happens to a_n as n gets large:

$a_1 = 1$ $\qquad a_2 = \frac{1\cdot 2}{2\cdot 2}$ $\qquad a_3 = \frac{1\cdot 2\cdot 3}{3\cdot 3\cdot 3}$

$$a_n = \frac{1 \cdot 2 \cdot 3 \cdot \cdots \cdot n}{n \cdot n \cdot n \cdot \cdots \cdot n}$$

It appears from these expressions that the terms are decreasing and perhaps approach 0. To confirm this, observe from Equation 8 that

$$a_n = \frac{1}{n}\left(\frac{2 \cdot 3 \cdot \cdots \cdot n}{n \cdot n \cdot \cdots \cdot n}\right)$$

Notice that the expression in parentheses is at most 1 because the numerator is less than (or equal to) the denominator. So

$$0 < a_n \leq \frac{1}{n}$$

We know that $1/n \quad \to 0 \ \ as \ \ n \to \infty$. Therefore $a_n \to 0$ as $n \to \infty$ by the Squeeze Theorem.

EXAMPLE 20: Use the Squeeze Theorem to determine if the sequence $b_n \ = \frac{5n}{n!}$ converges or diverges.

SOLUTION $\quad b_n \ = \frac{5n}{n!}$

$a_n \ = \ 0 \qquad\qquad c_n = \frac{5}{n}\left(\frac{5^4}{4!}\right)$

$\lim\limits_{n\to\infty} 0 = 0 \qquad\qquad \lim\limits_{n\to\infty} \frac{5}{n}\left(\frac{5^4}{4!}\right) = 0$

$\therefore \lim\limits_{n\to\infty} \frac{5n}{n!} = 0$ converges

Definition A sequence $\{a_n\}$ is called **increasing** if $a_n < a_{n+1}$ for all $n \geq 1$,that is, $a_1 < a_2 < a_3 \cdots$. It is called **decreasing** if $a_n > a_{n+1}$ for all $n \geq 1$.A sequence is **monotonic** if it is either increasing or decreasing.

EXAMPLE 21: The sequence $\left\{\frac{3}{n+1}\right\}$ is decreasing because

The right side is smaller because it has a larger denominator.$\frac{3}{n+5} > \frac{3}{(n+1)+5} = \frac{3}{n+6}$

and so $a_n > a_{n+1}$ for all $n \geq 1$

EXAMPLE 22: Show that the sequence $a_n = \frac{n}{n^2+1}$ is decreasing.

SOLUTION We must show that $a_{n+1} < a_n$ that is,

$$\frac{n+1}{(n+1)^2+1} < \frac{n}{n^2+1}$$

This inequality is equivalent to the one we get by cross-multiplication:

$$\frac{n+1}{(n+1)^2+1} < \frac{n}{n^2+1} \quad \Leftrightarrow (n+1)(n^2+1) < n[(n+1)^2+1]$$
$$\Leftrightarrow n^3+n^2+n+1 < n^3+2n^2+2n$$
$$\Leftrightarrow 1 < n^2+n$$

Since $n \geq 1$ we know that the inequality $n^2+n > 1$ is true. Therefore $a_{n+1} < a_n$ and so $\{a_n\}$ is decreasing.

Definition A sequence $\{a_n\}$ is **bounded above** if there is a number M such that
$a_n \leq M$ for all $n \geq 1$
It is **bounded below** if there is a number m such that
$m \leq a_n$ for all $n \geq 1$
If it is bounded above and below, then $\{a_n\}$ is a **bounded sequence**.

Monotonic Sequence Theorem Every bounded, monotonic sequence is convergent.

EXAMPLE 23: determine whether or not the sequence $a_n = \frac{2n-3}{3n+4}$ is bounded

SOLUTION $-\frac{1}{7}, \frac{1}{10}, \frac{3}{13}, \frac{5}{16}, \frac{7}{19}, \frac{9}{22}, \ldots$

the sequence is increasing, it is monotonic and bounded

bounded below $a_n \geq -\frac{1}{7}$ and bounded above $a_n < \frac{2}{3}$

EXAMPLE 23: determine whether or not the sequence $a_n = n(-1)^n$ is bounded

SOLUTION $-1, 2, -3, 4, -5, 6, \ldots$

the sequence is not monotonic and bounded

1.1 EXERCISES

EXERCISES:

1) List the first five terms of the sequence.

(i) $a_n = \frac{2^n}{2n+1}$

(ii) $a_n = \frac{n^2-1}{n^2+1}$

(iii) $a_n = \frac{(-1)^{n-1}}{5^n}$

(iv) $a_n = \cos\frac{n\pi}{2}$

(v) $a_n = \frac{1}{(n+1)!}$

(vi) $a_n = \frac{(-1)^n n}{n!+1}$

(vii) $a_1 = 1$, $a_{n+1} = 5a_n - 3$

(viii) $a_1 = 6$, $a_{n+1} = \frac{a_n}{n}$

2) Find a formula for the general term an of the sequence, assuming that the pattern of the first few terms continues.

(i) $\{\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \frac{1}{10}, \ldots\}$

(ii) $\{4, -1, \frac{1}{4}, -\frac{1}{16}, \frac{1}{64}, \ldots\}$

(iii) $\{-3, 2, -\frac{4}{3}, \frac{8}{9}, -\frac{16}{27}, \ldots\}$

(iv) $\{5, 8, 11, 14, 17, \ldots\}$

(v) $\{\frac{1}{2}, -\frac{4}{3}, \frac{9}{4}, -\frac{16}{5}, \frac{25}{6}, \ldots\}$

(vi) $\{1, 0, -1, 0, 1, 0, -1, 0, \ldots\}$

3) Determine whether the sequence converges or diverges. If it converges, find the limit.

(1) $a_n = \frac{3+5n^2}{n+n^2}$

(2) $a_n = \frac{3+5n^2}{1+n}$

(3) $a_n = \frac{n^4}{n^3-2n}$

(4) $a_n = 2 + (0.86)^n$

(5) $a_n = 3^n 7^{-n}$

(6) $a_n = \frac{3\sqrt{n}}{\sqrt{n}+2}$

(7) $a_n = e^{-1/\sqrt{n}}$

(8) $a_n = \frac{4^n}{1+9^n}$

(9) $a_n = \sqrt{\frac{1+4n^2}{1+n^2}}$

(10) $a_n = \cos\left(\frac{n\pi}{n+1}\right)$

(11) $a_n = \frac{n^2}{\sqrt{n^3+4n}}$

(12) $a_n = e^{2n/(n+2)}$

(13) $a_n = \frac{(-1)^n}{2\sqrt{n}}$

(14) $a_n = \frac{(-1)^{n+1} n}{n+\sqrt{n}}$

(15) $\{\frac{(2n-1)!}{(2n+1)!}\}$

(16) $\{\frac{\ln n}{\ln 2n}\}$

(17) $\{\sin n\}$

(18) $a_n = \frac{\tan^{-1} n}{n}$

(19) $\{n^2 e^{-n}\}$

(20) $a_n = \ln(n+1) - \ln n$

(21) $a_n = \frac{\cos^2 n}{2^n}$

(22) $a_n = \sqrt[n]{2^{1+3n}}$

(23) $a_n = n\sin(1/n)$

(24) $a_n = 2^{-n}\cos n\pi$

(25) $a_n = (1 + \frac{2}{n})^n$

(26) $a_n = \sqrt[n]{n}$

(27) $a_n = \ln(2n^2+1) - \ln(n^2+1)$

(28) $a_n = \frac{(\ln n)^2}{n}$

(29) $a_n = \arctan(\ln n)$

(30) $a_n = n - \sqrt{n+1}\sqrt{n+3}$

(31) $\{0,1,0,0,1,0,0,0,1,\ldots\}$

(32) $\{\frac{1}{1},\frac{1}{3},\frac{1}{2},\frac{1}{4},\frac{1}{3},\frac{1}{5},\frac{1}{4},\frac{1}{6},\ldots\}$

(33) $a_n = \frac{n!}{2^n}$

(34) $a_n = \frac{(-3)^n}{n!}$

4)Determine whether the sequence is increasing, decreasing, or not monotonic. Is the sequence bounded?

(i) $a_n = \cos n$

(ii) $a_n = \frac{1}{2n+3}$

(iii) $a_n = \frac{1-n}{2+n}$

(iv) $a_n = n(-1)^n$

(v) $a_n = 2 + \frac{(-1)^n}{n}$

(vi) $a_n = 3 - 2ne^{-n}$

1.2 Series

Arithmetic Series

Goals

Define a series.

Determine the partial sum of an arithmetic series.

Summing or Adding the terms of an arithmetic sequence creates what is called a series.

Sequence: 2, 4, 6, 8, 10, 12

Series: 2 + 4 + 6 + 8 + 10 + 12

Formulas to find the nth Partial Sum of an Arithmetic Sequence $S_n = \frac{n}{2}(a_1 + a_n)$ or $S_n = \frac{n[2a_1+(n-1)d]}{2}$

EXAMPLE 1: Let's see if we can figure out where these formulas come from.

1 + 2 + 3 + 4 + 5 +...... + 96 + 97 +98 +99 + 100

Sum of 101 $\rightarrow$ $S_n = \frac{n}{2}(a_1 + a_n) \rightarrow S_{100} = \frac{100}{2}(1 + 100) \rightarrow 50(101) = 5050$

EXAMPLE 2: Determine the sum of the arithmetic series 3 + 8 + 11 + ... + 73 .

SOLUTION $d = a_{n-}a_{n-1}$, $a_n = a_1 + (n - 1)d$, $d = 5$, $n = 15$, $a_1 = 3$, $a_n = 73$

$S_n = \frac{n}{2}(a_1 + a_n) \rightarrow S_{15} = \frac{15}{2}(3 + 73) \rightarrow S_{15} = 570$.

EXAMPLE 3: Determine the sum of the arithmetic series $a_n = -4n + 3$; $n = 20$.

SOLUTION $S_n = \frac{n}{2}(a_1 + a_n) \rightarrow S_{20} = \frac{20}{2}\big(-1 + (-77)\big) \rightarrow S_{20} = -780$

Geometric Series

Goal

Determine the partial sum of a geometric series.

Summing or Adding the terms of a geometric sequence creates what is called a series.

Sequence: 2, 4, 8, 16, 32, 64

Series: 2 + 4 + 8 + 16 + 32 + 64

Partial Sum of a first nth terms of a geometric series.

$$S_n = \frac{a_1(1 - r^n)}{1 - r}$$

EXAMPLE 4: Determine the sum of the geometric series 3 +6 + 12 + ... + 1536 .

SOLUTION $r = \frac{a_n}{a_{n-1}} \rightarrow r = 2$, $a_n = a_1 r^{n-1} \rightarrow n = 10$

$S_n = \frac{a_1(1-r^n)}{1-r} \rightarrow S_{10} = \frac{3(1-2^{10})}{1-2} \rightarrow S_{10} = 3069$

EXAMPLE 5: Determine the sum of the geometric series $a_n = 2(-3)^{n-1}$; $n = 5$.

SOLUTION $S_n = \frac{a_1(1-r^n)}{1-r} \rightarrow S_5 = \frac{2(1-(-3)^5)}{1-(-3)} \rightarrow S_5 = 122$

Introduction to Infinite Series

Goal

Define a convergent and divergent infinite series.

Determine if a series is convergent or divergent.

Infinite Series

If we add an infinite sequence, we have an infinite series.

Infinite Sequence

$a_1, a_2, a_3, a_4 \cdots a_n \cdots$

Infinite Series

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + a_4 + \cdots + a_n + \cdots$$

which is called an infinite series (or just a series) and is denoted, for short, by the symbol

$$\sum_{n=1}^{\infty} a_n \quad or \quad \sum a_n$$

Definition Given a series

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + a_4 + \cdots + a_n + \cdots, let\ S_n\ denote\ its\ nth\ partil\ sum:$$

$$S_n = \sum_{i=1}^{\infty} a_i = a_1 + a_2 + a_3 + a_4 + \cdots + a_n + \cdots$$

If the sequence $\{S_n\}$ is convergent and $\lim_{n\to\infty} S_n = S$ exists as a real number, then the series $\sum a_n$ is called **convergent** and we write

$$a_1 + a_2 + a_3 + a_4 + \cdots + a_n + \cdots = S \qquad or \qquad \sum a_n = S$$

The number S is called the **sum** of the series. If the sequence $\{S_n\}$ is divergent, then the series is called **divergent**. Its sequence of partial sums {S} has the terms

$S_1 = a_1$
$S_2 = a_1 + a_2$
$S_3 = a_1 + a_2 + a_3$
$\vdots$

$$S_n = a_1 + a_2 + a_3 + a_4 + \cdots + a_n = \sum_{i=1}^{\infty} a_i\,, \quad for\ n = 1,2,3, \ldots$$

EXAMPLE 4: Suppose we know that the sum of the first n terms of the series $\sum_{n=1}^{\infty} a_n$ an is

$$S_n = a_1 + a_2 + a_3 + a_4 + \cdots + a_n = \frac{2n}{3n+5}$$

Then the sum of the series is the limit of the sequence $\{S_n\}$:

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{2n}{3n+5} = \lim_{n\to\infty} \frac{2}{3+\frac{5}{n}} = \frac{2}{3}$$

In Example 4 we were given an expression for the sum of the first n terms, but it's usually not easy to find such an expression.

EXAMPLE 5: To Determine if an Infinite Series Converges, we could consider the sequence of partial sums.

$$\sum_{n=1}^{\infty} n^2 = 1 + 4 + 9 + 16 + 25 + \cdots + n^2 + \cdots$$

$S_1 = 1$

$S_2 = 1+4 = 5$

$S_3 = 1+4+9=14$

$S_4 = 1+4+9+16 = 30$

The partial sums create a sequence

1,5, 14, 30, ...

Converge or Diverge?

EXAMPLE 6: To Determine if an Infinite Series Converges, we could consider the sequence of partial sums.

$$\sum_{n=1}^{\infty} \frac{3}{10^n} = \frac{3}{10^1} + \frac{3}{10^2} + \frac{3}{10^3} + \frac{3}{10^4} + \frac{3}{10^5} + \cdots + \frac{3}{10^n} + \cdots$$

$S_1 = 0.3$

$S_2 = 0.3+0.03 = 0.33$

$S_3 = 0.3+0.03+0.003 = 0.333$

$S_4 = 0.3+0.03+0.003+0.0003 = 0.3333$

The partial sums create a sequence

0.3, 0.33, 0.333, 0.3333, ...

Converge or Diverge?

The geometric series

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + \cdots\cdots$$

is convergent if $|r| < 1$ and its sum is

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \qquad\qquad |r| < 1$$

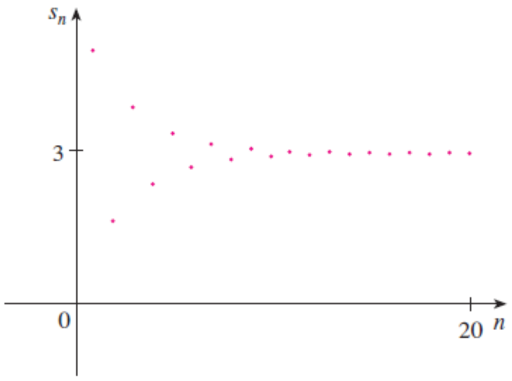
If $|r| \geq 1$, the geometric series is divergent.

EXAMPLE 7: Find the sum of the geometric series.

$$5+\frac{10}{3}+\frac{20}{9}+\frac{40}{27}+\cdots$$

SOLUTION The first term is $a=5$ and the common ratio is $r=-\frac{2}{3}$. Since $|r|=\frac{2}{3}<1$, the series is convergent and its sum is $5+\frac{10}{3}+\frac{20}{9}+\frac{40}{27}+\cdots=\frac{5}{1-\left(-\frac{2}{3}\right)}=3$

n	s_n
1	5.000000
2	1.666667
3	3.888889
4	2.407407
5	3.395062
6	2.736626
7	3.175583
8	2.882945
9	3.078037
10	2.947975



EXAMPLE 8: Is the series $\sum_{n=1}^{\infty} 2^{2n}3^{1-n}$ convergent or divergent?

SOLUTION Let's rewrite the nth term of the series in the form ar^{n-1}

$$\sum_{n=1}^{\infty} 2^{2n}3^{1-n}=\sum_{n=1}^{\infty}(2^2)^n 3^{-(n-1)}=\sum_{n=1}^{\infty}\frac{4^n}{3^{n-1}}=\sum_{n=1}^{\infty} 4(\frac{4}{3})^{n-1}$$

We recognize this series as a geometric series with $a=4$ and $r=\frac{4}{3}$. Since $r>1$, the series diverges

EXAMPLE 9: Show that the **harmonic series**

$$\sum_{n=1}^{\infty}\frac{1}{n}=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots$$

is divergent.

SOLUTION For this particular series it's convenient to consider the partial sums S_2, S_4, S_8, S_{16}, S_{32}, . . . and show that they become large.

$$s_2=1+\frac{1}{2}$$

$$s_4=1+\frac{1}{2}+(\frac{1}{3}+\frac{1}{4})>1+\frac{1}{2}+(\frac{1}{4}+\frac{1}{4})=1+\frac{2}{2}$$

$$s_8=1+\frac{1}{2}+(\frac{1}{3}+\frac{1}{4})+(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8})$$

$$>1+\frac{1}{2}+(\frac{1}{4}+\frac{1}{4})+(\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8})$$

$$=1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}=1+\frac{3}{2}$$

$$s_{16}=1+\frac{1}{2}+(\frac{1}{3}+\frac{1}{4})+(\frac{1}{5}+\cdots+\frac{1}{8})+(\frac{1}{9}+\cdots+\frac{1}{16})$$

$$>1+\frac{1}{2}+(\frac{1}{4}+\frac{1}{4})+(\frac{1}{8}+\cdots+\frac{1}{8})+(\frac{1}{16}+\cdots+\frac{1}{16})$$

$$=1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}=1+\frac{4}{2}$$

Similarly, $s_{32} > 1 + \frac{5}{2}$, $s_{64} > 1 + \frac{6}{2}$, and in general

$$s_{2^n} > 1 + \frac{n}{2}$$

This shows that $s_{2^n} \to \infty$ as $n \to \infty$ and so $\{s_n\}$ is divergent. Therefore, the harmonic series diverges.

__Theorem__ If the series $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$.

Divergent Test for a Series

__Theorem__ If $\lim_{n\to\infty} a_n$ does not exist or if $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

If the series is not divergent, then it is convergent, and so $\lim_{n\to\infty} a_n = 0$

EXAMPLE 10: Show that the series $\sum_{n=1}^{\infty} \frac{n^2}{5n^2+4}$ diverges.

SOLUTION $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^2}{5n^2+4} = \lim_{n\to\infty} \frac{1}{5+4/n^2} = \frac{1}{5} \neq 0$

So the series diverges by the Test for Divergence.

Note If we find that $\lim_{n\to\infty} a_n \neq 0$, we know that $\sum a_n$ is divergent. If we find that $\lim_{n\to\infty} a_n = 0$, we know nothing about the convergence or divergence of $\sum a_n$.

Remember if $\lim_{n\to\infty} a_n = 0$, the series o an might converge or it might diverge.

__Theorem__ If Σa_n and Σb_n are convergent series, then so are the series $\Sigma c a_n$ (where c is a constant), $\Sigma(a_n + b_n)$, and $\Sigma(a_n - b_n)$, and

(i) $\sum_{n=1}^{\infty} c a_n = c \sum_{n=1}^{\infty} a_n$

(ii) $\sum_{n=1}^{\infty} (a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$

(iii) $\sum_{n=1}^{\infty} (a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$

EXAMPLE 11: Find the sum of the series $\sum_{n=1}^{\infty} \left(\frac{3}{n(n+1)} + \frac{1}{2^n}\right)$

SOLUTION The series $\Sigma 1/2^n$ is a geometric series with $a = \frac{1}{2}$ and $r = \frac{1}{2}$, so

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{\frac{1}{2}}{1 - \frac{1}{2}} = 1$$

$$\sum_{n=1}^{\infty} \left(\frac{3}{n(n+1)} + \frac{1}{2^n}\right) = 3 \sum_{n=1}^{\infty} \frac{1}{n(n+1)} + \sum_{n=1}^{\infty} \frac{1}{2^n}$$

$$= 3 \cdot 1 + 1 = 4$$

Note A finite number of terms doesn't affect the convergence or divergence of a series. For instance, suppose that we were able to show that the series

$$\sum_{n=4}^{\infty} \frac{n}{n^3+1}$$

is convergent. Since

$$\sum_{n=4}^{\infty} \frac{n}{n^3+1} = \frac{1}{2} + \frac{2}{9} + \frac{3}{28} + \sum_{n=4}^{\infty} \frac{n}{n^3+1}$$

1.2 EXERCISES

1)Determine whether the geometric series is convergent or divergent. If it is convergent, find its sum.

$a)\ \ 3-4+\frac{16}{3}-\frac{64}{9}+\cdots$

$b)\ 4+3+\frac{9}{4}+\frac{27}{16}+\cdots$

$c)\ 10-2+0.4-0.08+\cdots$

$d)\ 2+0.5+0.125+0.03125+\cdots$

$e)\ \sum_{n=1}^{\infty} 12(0.73)^{n-1}$

$f)\ \sum_{n=1}^{\infty} \frac{5}{\pi^n}$

$g)\ \sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{4^n}$

$h)\ \sum_{n=0}^{\infty} \frac{3^{n+1}}{(-2)^n}$

$i)\ \sum_{n=1}^{\infty} \frac{e^{2n}}{6^{n-1}}$

$j)\ \sum_{n=1}^{\infty} \frac{6\cdot 2^{2n-1}}{3^n}$

2) Determine whether the series is convergent or divergent. If it is convergent, find its sum.

$a)\ \frac{1}{3}+\frac{1}{6}+\frac{1}{9}+\frac{1}{12}+\frac{1}{15}+\cdots$

$b)\ \frac{1}{3}+\frac{2}{9}+\frac{1}{27}+\frac{2}{81}+\frac{1}{243}+\frac{2}{729}+\cdots$

$c)\ \sum_{n=1}^{\infty} \frac{2+n}{1-2n}$

$d)\ \sum_{k=1}^{\infty} \frac{k^2}{k^2-2k+5}$

$e)\ \sum_{n=1}^{\infty} 3^{n+1}4^{-n}$

$f)\ \sum_{n=1}^{\infty} [(-0.2)^n+(0.6)^{n-1}]$

$g)\ \sum_{n=1}^{\infty} [(-0.2)^n+(0.6)^{n-1}]$

$h)\ \sum_{n=1}^{\infty} \frac{1}{4+e^{-n}}$

$i)\ \sum_{n=1}^{\infty} \frac{2^n+4^n}{e^n}$

$j)\ \sum_{k=1}^{\infty} (\sin\ 100)^k$

$k)\ \sum_{n=1}^{\infty} \frac{1}{1+(\frac{2}{3})^n}$

$l)\ \sum_{n=1}^{\infty} \ln\left(\frac{n^2+1}{2n^2+1}\right)$

$m)\ \sum_{k=0}^{\infty} (\sqrt{2})^{-k}$

1.3 Series Tests I

The *p*-series test

The *p*-series

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

is convergent if $p > 1$ and divergent if $p \leq 1$.

EXAMPLE 1:

(a) The series

$$\sum_{n=1}^{\infty} \frac{1}{n^3} = \frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \cdots$$

is convergent because it is a p-series with $p = 3 > 1$

(b) The series

$$\sum_{n=1}^{\infty} \frac{1}{n^{1/3}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n}} = 1 + \frac{1}{\sqrt[3]{2}} + \frac{1}{\sqrt[3]{3}} + \frac{1}{\sqrt[3]{4}} + \cdots$$

is divergent because it is a p-series with $p = \frac{1}{3} < 1$

The Comparison Test

Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.

(i) If $\sum b_n$ is convergent and $a_n \leq b_n$ for all n, then $\sum a_n$ is also convergent.
(ii) If $\sum b_n$ is divergent and $a_n \geq b_n$ for all n, then $\sum a_n$ is also divergent.

EXAMPLE 2: Determine whether the series

$$\sum_{n=1}^{\infty} \frac{5}{2n^2 + 4n + 3}$$

converges or diverges.

SOLUTION For large n the dominant term in the denominator is $2n^2$, so we compare the given series with the series $\Sigma 5/(2n^2)$. Observe that

$$\frac{5}{2n^2 + 4n + 3} < \frac{5}{2n^2}$$

because the left side has a bigger denominator. (In the notation of the Comparison Test, a_n is the left side and b_n is the right side.) We know that

$$\sum_{n=1}^{\infty} \frac{5}{2n^2} = \frac{5}{2} \sum_{n=1}^{\infty} \frac{1}{n^2}$$

is convergent because it's a constant times a p-series with $p = 2 > 1$. Therefore

$$\sum_{n=1}^{\infty} \frac{5}{2n^2 + 4n + 3}$$

is convergent by the Comparison Test.

Note Although the condition $a_n \leqslant b_n$ or $a_n \geqslant b_n$ in the Comparison Test is given for all n, we need verify only that it holds for $n \geqslant N$, where N is some fixed integer, because the convergence of a series is not affected by a finite number of terms. This is illustrated in the next example.

EXAMPLE 3: Test the series

$$\sum_{k=1}^{\infty} \frac{\ln k}{k}$$

for convergence or divergence.

SOLUTION We can test it by comparing it with the harmonic series. Observe that $\ln k > 1$ for $k \geqslant 30$ and so

$$\frac{\ln k}{k} > \frac{1}{k} k \geqslant 3$$

We know that $\Sigma 1/k$ is divergent (p-series with $p = 1$). Thus the given series is divergent by the Comparison Test.

The Limit Comparison Test

Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms. If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c$$

where c is a finite number and $c > 0$, then either both series converge or both diverge.

EXAMPLE 4: Test the series

$$\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$$

for convergence or divergence.

SOLUTION We use the Limit Comparison Test with $a_n = \frac{1}{2^n-1}$ $b_n = \frac{1}{2^n}$ and obtain

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1/(2^n - 1)}{1/2^n} = \lim_{n\to\infty} \frac{2^n}{2^n - 1} = \lim_{n\to\infty} \frac{1}{1 - 1/2^n} = 1 > 0$$

Since this limit exists and $\Sigma 1/2^n$ is a convergent geometric series, the given series converges by the Limit Comparison Test.

EXAMPLE 5: Determine whether the series

$$\sum_{n=1}^{\infty} \frac{2n^2+3n}{\sqrt{5+n^5}}$$

converges or diverges.

SOLUTION The dominant part of the numerator is $2n^2$ and the dominant part of the denominator is $\sqrt{n^5} = n^{5/2}$. This suggests taking

$$a_n = \frac{2n^2+3n}{\sqrt{5+n^5}} \quad b_n = \frac{2n^2}{n^{5/2}} = \frac{2}{n^{1/2}}$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{2n^2+3n}{\sqrt{5+n^5}} \cdot \frac{n^{1/2}}{2} = \lim_{n\to\infty} \frac{2n^{5/2}+3n^{3/2}}{2\sqrt{5+n^5}}$$

$$= \lim_{n\to\infty} \frac{2+\frac{3}{n}}{2\sqrt{\frac{5}{n^5}+1}} = \frac{2+0}{2\sqrt{0+1}} = 1$$

Since $\Sigma b_n = 2\Sigma 1/n^{1/2}$ is divergent ($p -$ series with $p = \frac{1}{2} < 1$), the given series diverges by the Limit Comparison Test.

Notice that in testing many series we find a suitable comparison series Σb_n by keeping only the highest powers in the numerator and denominator.

1.3 EXERCISES

Determine whether the series converges or diverges.

1. $\sum_{n=1}^{\infty} \frac{1}{n^3+8}$
2. $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}-1}$
3. $\sum_{n=1}^{\infty} \frac{n+1}{n\sqrt{n}}$
4. $\sum_{n=1}^{\infty} \frac{n-1}{n^3+1}$
5. $\sum_{n=1}^{\infty} \frac{9^n}{3+10^n}$
6. $\sum_{n=1}^{\infty} \frac{6^n}{5^n-1}$
7. $\sum_{k=1}^{\infty} \frac{\ln k}{k}$
8. $\sum_{k=1}^{\infty} \frac{k\sin^2 k}{1+k^3}$

9. $\sum_{k=1}^{\infty} \frac{\sqrt[3]{k}}{\sqrt{k^3+4k+3}}$

10. $\sum_{k=1}^{\infty} \frac{(2k-1)(k^2-1)}{(k+1)(k^2+4)^2}$

11. $\sum_{n=1}^{\infty} \frac{1+\cos\ n}{e^n}$

12. $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{3n^4+1}}$

13. $\sum_{n=1}^{\infty} \frac{4^{n+1}}{3^n-2}$

14. $\sum_{n=1}^{\infty} \frac{1}{n^n}$

15. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+1}}$

16. $\sum_{n=1}^{\infty} \frac{2}{\sqrt{n}+2}$

17. $\sum_{n=1}^{\infty} \frac{n+1}{n^3+n}$

18. $\sum_{n=1}^{\infty} \frac{n^2+n+1}{n^4+n^2}$

19. $\sum_{n=1}^{\infty} \frac{\sqrt{1+n}}{2+n}$

20. $\sum_{n=3}^{\infty} \frac{n+2}{(n+1)^3}$

21. $\sum_{n=1}^{\infty} \frac{5+2n}{(1+n^2)^2}$

22. $\sum_{n=1}^{\infty} \frac{n+3^n}{n+2^n}$

23. $\sum_{n=1}^{\infty} \frac{e^n+1}{ne^n+1}$

24. $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{n^2-1}}$

25. $\sum_{n=1}^{\infty} (1+\frac{1}{n})^2 e^{-n}$

26. $\sum_{n=1}^{\infty} \frac{e^{1/n}}{n}$

27. $\sum_{n=1}^{\infty} \frac{1}{n!}$
28. $\sum_{n=1}^{\infty} \frac{n!}{n^n}$
29. $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$
30. $\sum_{n=1}^{\infty} \frac{1}{n^{1+1/n}}$

1.4 Series Tests II

Alternating series

An **alternating series** is a series whose terms are alternately positive and negative. For examples:

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\cdots=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}$$

$$-\frac{1}{2}+\frac{2}{3}-\frac{3}{4}+\frac{4}{5}-\frac{5}{6}+\frac{6}{7}-\cdots=\sum_{n=1}^{\infty}(-1)^{n}\frac{n}{n+1}$$

We see from these examples that the nth term of an alternating series is of the form

$a_n=(-1)^{n-1}b_n$ or $a_n=(-1)^n b_n$

where b_n is a positive number. (In fact, $b_n=|a_n|$) .

The following test says that if the terms of an alternating series decrease toward 0 in absolute value, then the series converges.

Alternating Series Test

If the alternating series

$$\sum_{n=1}^{\infty}(-1)^{n-1}b_n=b_1-b_2+b_3-b_4+b_5-b_6+\cdots \qquad b_n>0$$

satisfies

(i) $b_{n+1}\leqslant b_n$ for all n

(ii) $\lim_{n\to\infty} b_n=0$

then the series is convergent.

EXAMPLE 1: The alternating harmonic series

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots=\sum_{n=1}^{n}\frac{(-1)^{n-1}}{n}$$

Satisfies

(i) $b_{n+1}<b_n$ because $\frac{1}{n+1}<\frac{1}{n}$

(ii) $\lim_{n\to\infty} b_n=\lim_{n\to\infty}\frac{1}{n}=0$

So the series is convergent by the Alternating Series Test.

EXAMPLE 2: The series

$$\sum_{n=1}^{\infty}\frac{(-1)^n 3n}{4n-1}$$

is alternating, but $\lim\limits_{a\to\infty} b_e = \lim\limits_{n\to\infty} \frac{3n}{4n-1} = \lim\limits_{n\to\infty} \frac{3}{4-\frac{1}{n}} = \frac{3}{4}$

so condition (ii) is not satisfied. Instead, we look at the limit of the nth term of the series:

$$\lim_{x\to\infty} a_n = \lim_{n\to\infty} \frac{(-1)^n 3n}{4n-1}$$

This limit does not exist, so the series diverges by the Test for Divergence.

EXAMPLE 3: Test the series

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{n^3+1}$$

for convergence or divergence.

SOLUTION The given series is alternating so we try to verify conditions (i) and (ii) of the Alternating Series Test. Unlike the situation in Example 1, it is not obvious that the sequence given by $b_n = n^2/(n^3+1)$ is decreasing. However, if we consider the related function $f(x) = x^2/(x^3+1)$, we find that $f'(x) = \frac{x(2-x^3)}{(x^3+1)^2}$

Since we are considering only positive x, we see $f'(x) < 0$ if $2 - x^3 < 0$, that is, $x > \sqrt[3]{2}$. Thus f is decreasing on the interval $(\sqrt[3]{2}, \infty)$. This means that $f(n+1) < f(n)$ and therefore $b_{a+1} < b_a$, when $n \geq 2$. (The inequality $b_2 < b_1$ can be verified directly but all that really matters is that the sequence $\{b_e\}$ is eventually decreasing.)

Condition (ii) is readily verified:

$$\lim_{s\to-\infty} b_s = \lim_{s\to-\pi^3+1} = \lim_{k\to-} \frac{\frac{1}{n}}{1+\frac{1}{n^3}} = 0$$

Thus the given series is convergent by the Alternating Series Test.

1.4 EXERCISES

Test the series for convergence or divergence.

1. $\frac{2}{3} - \frac{2}{5} + \frac{2}{7} - \frac{2}{9} + \frac{2}{11} - \cdots$
2. $-\frac{2}{5} + \frac{4}{6} - \frac{6}{7} + \frac{8}{8} - \frac{10}{9} + \cdots$
3. $\frac{1}{\ln 3} - \frac{1}{\ln 4} + \frac{1}{\ln 5} - \frac{1}{\ln 6} + \frac{1}{\ln 7} - \cdots$
4. $\sum\limits_{n=1}^{\infty} \frac{(-1)^{n-1}}{3+5n}$
5. $\sum\limits_{n=0}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n+1}}$
6. $\sum\limits_{n=1}^{\infty} (-1)^n \frac{3n-1}{2n+1}$
7. $\sum\limits_{n=1}^{\infty} (-1)^n \frac{n^2}{n^2+n+1}$
8. $\sum\limits_{n=1}^{\infty} (-1)^n e^{-n}$

9. $\sum_{n=1}^{\infty}(-1)^n \frac{\sqrt{n}}{2n+3}$
10. $\sum_{n=1}^{\infty}(-1)^{n+1} \frac{n^2}{n^3+4}$
11. $\sum_{n=1}^{\infty}(-1)^{n+1}\, ne^{-n}$
12. $\sum_{n=1}^{\infty}(-1)^{n-1}\, e^{2/n}$
13. $\sum_{n=1}^{\infty}(-1)^{n-1} \arctan\, n$
14. $\sum_{n=0}^{\infty} \frac{\sin\,(n+\frac{1}{2})\pi}{1+\sqrt{n}}$
15. $\sum_{n=1}^{\infty} \frac{n\cos\, n\pi}{2^n}$
16. $\sum_{n=1}^{\infty} (-1)^n \sin\,(\frac{\pi}{n})$
17. $\sum_{n=1}^{\infty} (-1)^n \cos\,(\frac{\pi}{n})$
18. $\sum_{n=1}^{\infty} (-1)^n \frac{n^n}{n!}$
19. $\sum_{n=1}^{\infty} (-1)^n(\sqrt{n+1}-\sqrt{n})$

1.5 Series Tests III

Absolute and Conditional Convergence

Definition A series $\sum a_n$ is called **absolutely convergent** if the series of absolute values $\sum|a_n|$ is convergent

EXAMPLE 1: The series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$$

is absolutely convergent because

$$\sum_{n=1}^{\infty} \left|\frac{(-1)^{n-1}}{n^2}\right| = \sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$$

is a convergent p-series $(p = 2)$

EXAMPLE 2: We know that the alternating harmonic series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$
is convergent, but it is not absolutely convergent because the corresponding series of absolute values is
$$\sum_{n=1}^{\infty} \left|\frac{(-1)^{n-1}}{n}\right| = \sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$
which is the harmonic series (p-series with $p = 1$) and is therefore divergent.

Definition A series o an is called conditionally convergent if it is convergent but not absolutely convergent.

Example 2 shows that the alternating harmonic series is conditionally convergent.

Thus it is possible for a series to be convergent but not absolutely convergent. However, the next theorem shows that absolute convergence implies convergence.

Theorem If a series $\sum a_n$ is absolutely convergent, then it is convergent.

EXAMPLE 3: Determine whether the series
$$\sum_{n=1}^{\infty} \frac{\cos n}{n^2} = \frac{\cos 1}{1^2} + \frac{\cos 2}{2^2} + \frac{\cos 3}{3^2} + \cdots$$
is convergent or divergent.

SOLUTION This series has both positive and negative terms, but it is not alternating. (The first term is positive, the next three are negative, and the following three are positive: the signs change irregularly.) We can apply the Comparison Test to the series of absolute values
$$\sum_{n=1}^{\infty} \left|\frac{\cos n}{n^2}\right| = \sum_{n=1}^{\infty} \frac{|\cos\ n|}{n^2}$$
Since $|\cos\ n| \leqslant 1$ for all n, we have
$$\frac{|\cos\ n|}{n^2} \leqslant \frac{1}{n^2}$$
We know that $\Sigma 1/n^2$ is convergent (p-series with $p = 2$) and therefore $\Sigma|\cos\ n|/n^2$ is convergent by the Comparison Test. Thus the given series $\Sigma(\cos\ n)/n^2$ is absolutely convergent and therefore convergent

The following test is very useful in determining whether a given series is absolutely convergent.

The Ratio Test

(i) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then the series $\sum_{n=1}^{\infty}\ a_n$ is absolutely convergent (and therefore convergent).

(ii) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L > 1$ or $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \infty$, then the series $\sum_{n=1}^{\infty}\ a_n$ is divergent.

(iii) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$, the Ratio Test is inconclusive; that is, no conclusion can be drawn about the convergence or divergence of Σa_n

EXAMPLE 4: Test the series $\sum_{n=1}^{\infty}\ (-1)^n \frac{n^3}{3^n}$ for absolute convergence.

SOLUTION We use the Ratio Test with $a_n = (-1)^n n^3/3^n$:
$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{(-1)^{n+1}(n+1)^3}{3^{n+1}}}{\frac{(-1)^n n^3}{3^n}}\right| = \frac{(n+1)^3}{3^{n+1}} \cdot \frac{3^n}{n^3}$$
$$= \frac{1}{3}\left(\frac{n+1}{n}\right)^3 = \frac{1}{3}\left(1 + \frac{1}{n}\right)^3 \to \frac{1}{3} < 1$$
Thus, by the Ratio Test, the given series is absolutely convergent.

EXAMPLE 5: Test the convergence of the series $\sum_{n=1}^{\infty} \frac{n^n}{n!}$.

SOLUTION Since the terms $a_n = n^n/n!$ are positive, we don't need the absolute value signs.

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^{n+1}}{(n+1)!} \cdot \frac{n!}{n^n} = \frac{(n+1)(n+1)^n}{(n+1)n!} \cdot \frac{n!}{n^n} = (\frac{n+1}{n})^n = (1+\frac{1}{n})^n \to e \text{ as } n \to \infty$$

Since $e > 1$, , the given series is divergent by the Ratio Test.

Note Although the Ratio Test works in Example 5, an easier method is to use the test for Divergence. Since $a_n = \frac{n^n}{n!} = \frac{n \cdot n \cdot n \cdot \dots \cdot n}{1 \cdot 2 \cdot 3 \cdot \dots \cdot n} \geqslant n$ it follows that a_n does not approach 0 as $n \to \infty$. Therefore the given series is divergent by the Test for Divergence.

The Root Test

(i) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (and therefore convergent).

(ii) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L > 1$ or $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

(iii) If If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$, the Root Test is inconclusive.

EXAMPLE 6: Test the convergence of the series

$$\sum_{n=1}^{\infty} (\frac{2n+3}{3n+2})^n$$

SOLUTION

$$a_n = (\frac{2n+3}{3n+2})^n$$

$$\sqrt[n]{|a_n|} = \frac{2n+3}{3n+2} = \frac{2+\frac{3}{n}}{3+\frac{2}{n}} \to \frac{2}{3} < 1$$

Thus the given series is absolutely convergent (and therefore convergent) by the Root Test.

1.5 EXERCISES

1) Use the Ratio Test to determine whether the series is convergent or divergent.

1. $\sum_{n=1}^{\infty} \frac{n}{5^n}$
2. $\sum_{n=1}^{\infty} \frac{(-2)^n}{n^2}$
3. $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{3^n}{2^n n^3}$
4. $\sum_{n=0}^{\infty} \frac{(-3)^n}{(2n+1)!}$
5. $\sum_{k=1}^{\infty} \frac{1}{k!}$
6. $\sum_{k=1}^{\infty} ke^{-k}$
7. $\sum_{n=1}^{\infty} \frac{10^n}{(n+1)4^{2n+1}}$

8. $\sum_{n=1}^{\infty} \frac{n!}{100^n}$
9. $\sum_{n=1}^{\infty} \frac{n\pi^n}{(-3)^{n-1}}$
10. $\sum_{n=1}^{\infty} \frac{n^{10}}{(-10)^{n+1}}$
11. $\sum_{n=1}^{\infty} \frac{\cos(n\pi/3)}{n!}$
12. $\sum_{n=1}^{\infty} \frac{n!}{n^n}$
13. $\sum_{n=1}^{\infty} \frac{n^{100}100^n}{n!}$
14. $\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2}$
15. $1 - \frac{2!}{1\cdot 3} + \frac{3!}{1\cdot 3\cdot 5} - \frac{4!}{1\cdot 3\cdot 5\cdot 7} + \cdots$
16. $+ (-1)^{n-1}\frac{n!}{1\cdot 3\cdot 5\cdot \cdots \cdot (2n-1)} + \cdots$
17. $\frac{2}{3} + \frac{2\cdot 5}{3\cdot 5} + \frac{2\cdot 5\cdot 8}{3\cdot 5\cdot 7} + \frac{2\cdot 5\cdot 8\cdot 11}{3\cdot 5\cdot 7\cdot 9} + \cdots$
18. $\sum_{n=1}^{\infty} \frac{2\cdot 4\cdot 6\cdot \cdots \cdot (2n)}{n!}$
19. $\sum_{n=1}^{\infty} (-1)^n \frac{2^n n!}{5\cdot 8\cdot 11\cdot \cdots \cdot (3n+2)}$

2)Use the Root Test to determine whether the series is convergent or divergent.

1. $\sum_{n=1}^{\infty} (\frac{n^2+1}{2n^2+1})^n$
2. $\sum_{n=1}^{\infty} \frac{(-2)^n}{n^n}$
3. $\sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{(\ln\ n)^n}$
4. $\sum_{n=1}^{\infty} (\frac{-2n}{n+1})^{5n}$
5. $\sum_{n=1}^{\infty} (1+\frac{1}{n})^{n^2}$
6. $\sum_{n=0}^{\infty} (\arctan\ n)^{\mathrm{n}}$

3)Use any test to determine whether the series is absolutely convergent, conditionally convergent, or divergent.

Course Name Mathematics 2 **Course Code** MATH 302

1. $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$
2. $\sum_{n=1}^{\infty} (\frac{1-n}{2+3n})^n$
3. $\sum_{n=1}^{\infty} \frac{(-9)^n}{n10^{n+1}}$
4. $\sum_{n=1}^{\infty} \frac{n5^{2n}}{10^{n+1}}$
5. $\sum_{n=2}^{\infty} (\frac{n}{\ln n})^n$
6. $\sum_{n=1}^{\infty} \frac{\sin(n\pi/6)}{1+n\sqrt{n}}$
7. $\sum_{n=1}^{\infty} \frac{(-1)^n \arctan n}{n^2}$
8. $\sum_{n=2}^{\infty} \frac{(-1)^n}{n\ln n}$