The first unit

Counting Rules and Probability

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General Objective:

The principle goal of this unit is to give trainee the basic principle of counting as permutations, combinations and arrangement and define the random experiment, Sample space, events, Conditional probability, independence between events and Bayes theorem.

Detailed Objectives:
At the end of this unit, trainee will be able to
- Find the total number of outcomes in a sequence of events, using the fundamental counting rule.
- -Find the number of ways that r objects can be selected from n objects, using the permutation rule.
- Find the number of ways that r objects can be selected from n objects, without regard to order, using the combination rule.
- Find the probability of an event, using the counting rule.
- Determine sample spaces and find the probability of an event
- Find the conditional probability of an event.

Time needed for this unit is 12h.

1-The Basic Principle of Counting, Permutations, Combinations.

The basic principle of counting will be fundamental to all our work. It states that if one experiment can result in any of m possible outcomes and if another experiment can result in any of n possible outcomes, then there are $m\ n$ possible outcomes of the two experiments

The basic principle of counting
Suppose that two experiments are to be performed. Then if experiment 1 can result in any one of m possible outcomes and if, for each outcome of experiment 1, there are n possible outcomes of experiment 2, then together there are $m\ n$ possible outcomes of the two experiments.

EXAMPLE 1

A small community consists of 10 women, each of whom has 3 children. If one woman and one of her children are to be chosen as mother and child of the year, how many different choices are possible?

Solution. By regarding the choice of the woman as the outcome of the first experiment and the subsequent choice of one of her children as the outcome of the second experiment, we see from the basic principle that there are 10 * 3 = 30 possible choices.

When there are more than two experiments to be performed, the basic principle can be generalized.

The generalized basic principle of counting
If r experiments that are to be performed are such that the first one may result in any of n_1 possible outcomes; and if, for each of these n_1 possible outcomes, there are n_2 possible outcomes of the second experiment; and if, for each of the possible outcomes of the first two experiments, there are n_3 possible outcomes of the third experiment; and if . . . , then there is a total of $n_1 \cdot n_2 \cdots n_r$ possible outcomes of the r experiments.

EXAMPLE 2

How many different batting orders are possible for a baseball team consisting of 9 players?

Solution. There are 9! = 362,880 possible batting orders.

EXAMPLE 3

A class in probability theory consists of 6 men and 4 women. An examination is given, and the students are ranked according to their performance. Assume that no two students obtain the same score.
(a) How many different rankings are possible?
(b) If the men are ranked just among themselves and then the women just among themselves, how many different rankings are possible?

Solution. (a) Because each ranking corresponds to a particular ordered arrangement of the 10 people, the answer to this part is
10! = 3,628,800.
(b) Since there are 6! possible rankings of the men among themselves and 4! possible rankings of the women among themselves, it follows from the basic principle that there are $(6!)(4!) = (720)(24) = 17,280$ possible rankings in this case.

EXAMPLE 4

Ms. Jones has 10 books that she is going to put on her bookshelf. Of these, 4 are mathematics books, 3 are chemistry books, 2 are history books, and 1 is a language book. Ms. Jones wants to arrange her books so that all the books dealing with the same subject are together on the shelf. How many different arrangements are possible?

Solution. There are 4! 3! 2! 1! arrangements such that the mathematics books are first in line, then the chemistry books, then the history books, and then the language book. Similarly, for each possible ordering of the subjects, there are 4! 3! 2! 1! Possible arrangements. Hence, as there are 4! possible orderings of the subjects, the desired answer is
4! 4! 3! 2! 1! = 6912.

We shall now determine the number of permutations of a set of n objects when certain of the objects are indistinguishable from each other. To set this situation straight in our minds, consider the following example.

EXAMPLE 5

How many different letter arrangements can be formed from the letters *PEPPER*?

Solution. We first note that there are 6! permutations of the letters $P_1E_1P_2P_3E_2R$ when the 3P's and the 2E's are distinguished from each other. However, consider any one of these permutations—for instance, $P_1P_2E_1P_3E_2R$. If we now permute the P's among themselves and the E's among themselves, then the resultant arrangement would still be of the form *PPEPER*. That is, all 3! 2! Permutations

$$\begin{gathered} P_1P_2E_1P_3E_2R/P_1P_2E_2P_3E_1R \\ P_1P_3E_1P_2E_2R/P_1P_3E_2P_2E_1R \\ P_2P_1E_1P_3E_2R/P_2P_1E_2P_3E_1R \\ P_2P_3E_1P_1E_2R/P_2P_3E_2P_1E_1R \\ P_3P_1E_1P_2E_2R/P_3P_1E_2P_2E_1R \\ P_3P_2E_1P_1E_2R/P_3P_2E_2P_1E_1R \end{gathered}$$

are of the form *PPEPER*. Hence, there are 6!/(3! 2!) = 60 possible letter arrangements of the letters *PEPPER*.

In general, the same reasoning as that used in Example 5 shows that there are $\frac{n!}{n_1!\,n_2!\ldots n_r!}$ different permutations of n objects, of which n_1 are alike, n_2 are alike, . . . , n_r are alike.

EXAMPLE 6

A chess tournament has 10 competitors, of which 4 are Russian, 3 are from the United States, 2 are from Great Britain, and 1 is from Brazil. If the tournament result lists just the nationalities of the players in the order in which they placed, how many outcomes are possible?

Solution. There are $\frac{10!}{4!3!2!1!} = 12{,}600$ possible outcomes.

EXAMPLE 7

How many different signals, each consisting of 9 flags hung in a line, can be made from a set of 4 white flags, 3 red flags, and 2 blue flags if all flags of the same color are identical?

Solution. There are $\frac{9!}{4!3!2!} = 1260$ different signals.

2- COMBINATIONS

We are often interested in determining the number of different groups of r objects that could be formed from a total of n objects. For instance, how many different groups of 3 could be selected from the 5 items A, B, C, D, and E? To answer this question, reason as follows: Since there are 5 ways to select the initial item, 4 ways to then select the next item, and 3 ways to select the final item, there are thus $5 \cdot 4 \cdot 3$ ways of selecting the group of 3 when the order in which the items are selected is relevant. However, since every group of 3—say, the group consisting of items A, B, and C—will be counted 6 times (that is, all of the permutations *ABC, ACB, BAC, BCA, CAB*, and *CBA* will be counted when the order of selection is relevant), it follows that the total number of groups that can be formed is

$$\frac{5 \cdot 4 \cdot 3}{3 \cdot 2 \cdot 1} = 10$$

In general, as $n(n-1) \cdots (n-r+1)$ represents the number of different ways that a group of r items could be selected from n items when the order of selection is relevant, and as each group of r items will be counted $r!$ times in this count, it follows that the number of different groups of r items that could be formed from a set of n items is

$$\frac{n\,(n-1)\cdots(n-r+1)}{r!} = \frac{n!}{(n-r)!r!}$$

Notation and terminology
We define $\binom{n}{r}$, for $r \leq n$, by $$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$ and say that $\binom{n}{r}$ represents the number of possible combinations of *n* objects taken *r* at a time. It can be noted as $C_n^r = \binom{n}{r}$

Thus, $\binom{n}{r}$ represents the number of different groups of size *r* that could be selected from a set of *n* objects when the order of selection is not considered relevant.

EXAMPLE 8

A committee of 3 is to be formed from a group of 20 people. How many different committees are possible?

Solution. There are $\binom{20}{3} = \frac{20\cdot19\cdot18}{3\cdot2\cdot1} = 1140$ possible committees.

EXAMPLE 9

From a group of 5 women and 7 men, how many different committees consisting of 2 women and 3 men can be formed?

Solution. As there are $\binom{5}{2}$ possible groups of 2 women, and $\binom{7}{3}$ possible groups of 3 men, it follows from the basic principle that there are $\binom{5}{2}\binom{7}{3} = \left(\frac{5\cdot4}{2\cdot1}\right)\frac{7\cdot6\cdot5}{3\cdot2\cdot1} = 350$ possible committees consisting of 2 women and 3 men.

3- Random experiment, Sample space, Events, Axioms of probability.

Random experiment

A random experiment is an experiment in which the outcome varies in an unpredictable fashion when the experiment is repeated under the same conditions.

Examples

- Experiment E1: Toss a coin three times and note the sequence of heads and tails.
- Experiment E2: Toss a coin three times and note the number of heads
- Experiment E3: A block of information is transmitted repeatedly over a noisy channel until an error-free block arrives at the receiver. Count the number of transmissions required.
- Experiment E4: Measure the lifetime of a given computer memory chip in a specified environment.

Since random experiments do not consistently yield the same result, it is necessary to determine what the set of possible results can be.

Sample space

The sample space S of a random experiment is defined as the set of all possible outcomes.

Note: When we perform a random experiment, one and only one outcome occurs.

The samples spaces corresponding to the experiments in the last Example are given below:

- Experiment E_1: Toss a coin three times and note the sequence of heads and tails.

- Experiment E_2: Toss a coin three times and note the number of heads

- Experiment E_3: A block of information is transmitted repeatedly over a noisy channel until an error-free block arrives at the receiver. Count the number of transmissions required.

- Experiment E_4: Measure the lifetime of a given computer memory chip in a specified environment.

Random experiments involving the same experimental procedure may have different sample spaces as shown by Experiments E_1 and E_2.

EXERCISE: The three balls numbered 1 to 3 in an urn are drawn at random one at a time until the urn is empty. The sequence of the ball numbers is noted. Find the sample space.

Events

We are usually not interested in the occurrence of specific outcomes, but rather on the occurrence of some event (i.e. whether the outcome satisfies certain conditions).

***EXAMPLE* 10**

Experiment: Determine the value of a voltage waveform at time t_1

S = $(-\infty, +\infty)$.

We might be interested in the event "voltage is negative" which corresponds to $(-\infty, 0)$.

The event occurs if and only if the outcome of the experiment is in this subset.

For this reason we define an event as a subset of S

EXERCISE: Experiment E: A block of information is transmitted repeatedly over a noisy channel until an error-free block arrives at the receiver. Count the number of transmissions required.

S =

A = Fewer than 10 transmissions are required =

EXERCISE: An engineering firm is hired to determine if certain waterways in Virginia are safe for shing. Samples are taken from three rivers.

(a) List the elements of a sample space S, using the letters F for "safe to fish" and N for "not safe to fish".

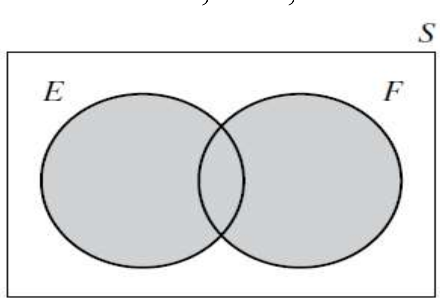
(b) List the elements of S corresponding to event E that at least two of the rivers are safe for fishing.

(c) Define an event that has as its elements the points $\{\text{FFF}, \text{NFF}, \text{FFN}, \text{NFN}\}$.

Set Operations

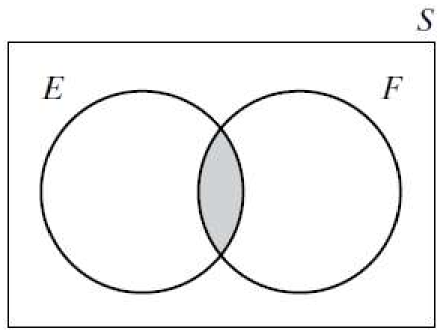
- The union of two events E and F, denoted by E ∪ F, is defined as the set of outcomes that are either in A or in B, or both.

The event E ∪ F occurs if either E, or F, or both E and F occur.



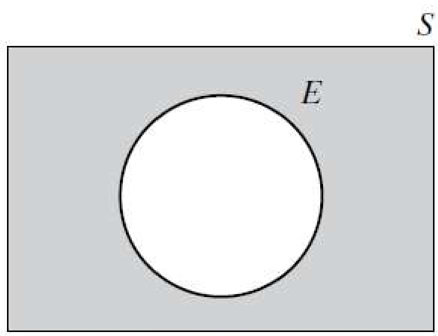
(a) Shaded region: $E \cup F$.

- The intersection of two events E and F, denoted by E ∩ F, is defined as the set of outcomes that are common to E and F.



(b) Shaded region: EF.

- The complement of an event E, denoted by E' or E^C, is defined as the set of all outcomes not in E. The event E' occurs when the event E does not occur and vice versa.



(c) Shaded region: E^c.

EXERCISE:

A deck of playing contains 52 cards. We perform the experiment of randomly selecting one card from the deck.

S = The collection of all 52 cards

Let us consider the following 4 events

A = The card selected is the king of hearts
B = The card selected is a king
C = The card selected is a heart
D = The card selected is a face card

How many outcomes comprising each of these four events?

Solution:

A:

B:

C:
D:

Determine D', B ∩ C, B ∪ C, C ∩ D

The operations of forming unions, intersections, and complements of events obey certain rules similar to the rules of algebra. We list a few of these rules:

Commutative laws E ∪ F = F ∪ E   E F = F E (OR E ∩ F = F ∩ E)
Associative laws (E ∪ F) ∪ G = E ∪ (F ∪ G)   (E F) G = E (F G)
Distributive laws (E ∪ F) ∩ G = (E ∩ G) ∪ (F ∩ G)
(E ∩ F) ∪ G = (E ∪ G) ∩ (F ∪ G)

The following useful relationships between the three basic operations of forming unions, intersections, and complements are known as *DeMorgan's laws*:

$$\left(\bigcup_{i=1}^{n} E_i\right)^C = \bigcap_{i=1}^{n} {E_i}^c$$
$$\left(\bigcap_{i=1}^{n} E_i\right)^C = \bigcup_{i=1}^{n} {E_i}^c$$

AXIOMS OF PROBABILITY

If an experiment can result in any one of the N different equally likely outcome, and if exactly n of these outcomes correspond to event A, then the probability of event A is

$$P(A) = \frac{n}{N}$$

Consider an experiment whose sample space is *S*. For each event *E* of the sample space *S*, we assume that a number *P(E)* is defined and satisfies the following three axioms:

Axiom 1

$$0 \leq P(E) \leq 1$$

Axiom 2

$$P(S) = 1$$

Axiom 3

For any sequence of mutually exclusive events $E_1, E_2, \ldots$ (that is, events for which $E_i\, E_j = \emptyset$ *when* $i \neq j$),

$$P\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} P(E_i)$$

We refer to $P(E)$ as the probability of the event E.
Thus, Axiom 1 states that the probability that the outcome of the experiment is an outcome in E is some number between 0 and 1. Axiom 2 states that, with probability 1, the outcome will be a point in the sample space S. Axiom 3 states that, for any sequence of mutually exclusive events, the probability of at least one of these events occurring is just the sum of their respective probabilities.

If we consider a sequence of events $E_1, E_2, \ldots$, where $E_1 = S$ and $E_i = \emptyset$ for $i > 1$, then, because the events are mutually exclusive and because

$$S = \bigcup_{i=1}^{\infty} E_i$$

we have, from Axiom 3,

$$P(S) = \sum_{i=1}^{\infty} P(E_i) = P(S) + \sum_{i=2}^{\infty} P(\emptyset)$$

implying that

$$P(\emptyset) = 0$$

That is, the null event has probability 0 of occurring.

Note that it follows that, for any finite sequence of mutually exclusive events $E_1, E_2, \ldots, E_n$

$$P\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} P(E_i)$$

***EXAMPLE* 11**

If our experiment consists of tossing a coin and if we assume that a head is as likely to appear as a tail, then we would have

$$P(\{H\}) = P(\{T\}) = \frac{1}{2}$$

On the other hand, if the coin were biased and we felt that a head were twice as likely to appear as a tail, then we would have

$$P(\{H\}) = \frac{2}{3} \text{ and } P(\{T\}) = \frac{1}{3}$$

***EXAMPLE* 12**

Suppose that a coin is tossed three times. If we observe the sequence of heads and tails, then there are eight possible outcomes

$$S = \{HHH, HHT, HTH, THH, TTH, THT, HTT, TTT\}:$$

If we assume that the outcomes of S are equiprobable, then the probability of each of the eight elementary events is $\frac{1}{8}$. Let A be the event of obtaining two heads in three tosses.

$$P(A) = P[\{HHT, HTH, THH\}] = \frac{3}{8}$$

***EXAMPLE* 13**

If a die is rolled and we suppose that all six sides are equally likely to appear, then we would have

$$P(\{1\}) = P(\{2\}) = P(\{3\}) = P(\{4\}) = P(\{5\}) = P(\{6\}) = \frac{1}{6}$$

From Axiom 3, it would thus follow that the probability of rolling an even number would equal

$$P(\{2, 4, 6\}) = P(\{2\}) + P(\{4\}) + P(\{6\}) = \frac{1}{2}$$

EXERCISE:

A batch of 6 items contains 4 defective items. Suppose 3 items are selected at random and tested.

What is the probability that exactly 2 of the items tested are defective?

EXERCISE:

A group of three undergraduate and five graduate students are available to _fill certain student government posts. If four students are to be randomly selected from this group, find the probability that exactly two undergraduates will be among the four chosen.

EXERCISE:

The probabilities that a secretary will make 0, 1, 2, 3, 4, or 5 or more mistake in typing a recent report are, respectively, 0.12, 0.25, 0.36, 0.14, 0.09, 0.04

Let A = the secretary is making at most 2 mistakes.

Let B = the secretary is making at least 4 mistakes.

Find $P(A \cup B)$

Note that, since E and E^c are always mutually exclusive and since $E \cup E^c = S$, we have, by Axioms 2 and 3,

$$1 = P(S) = P(E \cup E^c) = P(E) + P(E^c)$$

Proposition 1

- $P(E^c) = 1 - P(E)$
- If $E \subset F$ then $P(E) \leq P(F)$
- $P(E \cup F) = P(E) + P(F) - P(E \cap F)$

EXERCISE:

For married couples living in a certain suburb, the probability that the husband will vote in a school board elections is 0.21, the probability that the wife will vote in the election is 0.28, and the probability that they will both vote is 0.15. What is the probability that at least one will vote.

EXERCISE:

Disease I and II are prevalent among people in a certain population. It is assumed that 10% of the population will contract disease I sometime during their lifetime, 15% will contract disease II eventually, and 3%

will contract both diseases. Find the probability that a randomly chosen person from this population will contract at least one disease.

EXERCISE:

From past experience a stockbroker believes that under present economic conditions a customer will invest in tax-free bonds with a probability of 0.6, will invest in mutual funds with a probability of 0.3, and will invest in both tax-free bonds and mutual funds, with a probability of 0.15. At this time, _find the probability that a customer will invest

(a) in either tax-free bonds or mutual funds;

(b) in neither tax-free bonds nor mutual funds.

Solution

Consider the events

B: customer invests in tax free bonds

M: customer invests in mutual funds

4- CONDITIONAL PROBABILITIES

Suppose that we toss 2 dice, and suppose that each of the 36 possible outcomes is equally likely to occur and hence has probability $\frac{1}{36}$. Suppose further that we observe that the first die is a 3. Then, given this information, what is the probability that the sum of the 2 dice equals 8? To calculate this probability, we reason as follows: Given that the initial die is a 3, there can be at most 6 possible outcomes of our experiment, namely, (3, 1), (3, 2), (3, 3), (3, 4), (3, 5), and (3, 6). Since each of these outcomes originally had the same probability of occurring, the outcomes should still have equal probabilities. That is, given that the first die is a 3, the (conditional) probability of each of the outcomes (3, 1), (3, 2), (3, 3), (3, 4), (3, 5), and (3, 6) is $\frac{1}{6}$, whereas the (conditional) probability of the other 30 points in the sample space is 0. Hence, the desired probability will be $\frac{1}{6}$.

.

If we let E and F denote, respectively, the event that the sum of the dice is 8 and the event that the first die is a 3, then the probability just obtained is called the *conditional probability that E occurs given that F has occurred* and is denoted by

$$P(E|F)$$

Definition
If $P(F) > 0$, then $$P(E|F)=\frac{P(E\cap F)}{P(F)}$$

***EXAMPLE* 14**

A coin is flipped twice. Assuming that all four points in the sample space $S = \{(h, h), (h, t), (t, h), (t, t)\}$ are equally likely, what is the conditional probability that both flips land on heads, given that (a) the first flip lands on heads? (b) at least one flip lands on heads?

Solution. Let $B = \{(h, h)\}$ be the event that both flips land on heads; let $F = \{(h, h), (h, t)\}$ be the event that the first flip lands on heads; and let $A = \{(h, h), (h, t), (t, h)\}$ be the event that at least one flip lands on heads. The probability for (a) can be obtained from

$$\begin{aligned} P(B|F) &= \frac{P(B\cap F)}{P(F)} \\ &= \frac{P(\{(h,h)\})}{P(\{(h,h),(h,t)\})} \\ &= \frac{\frac{1}{4}}{\frac{2}{4}} = \frac{1}{2} \end{aligned}$$

For (b), we have

$$\begin{aligned} P(B|A) &= \frac{P(B\cap A)}{P(A)} \\ &= \frac{P(\{(h,h)\})}{P(\{(h,h),(h,t),(t,h)\})} \end{aligned}$$

$$= \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{3}$$

Thus, the conditional probability that both flips land on heads given that the first one does is $\frac{1}{2}$, whereas the conditional probability that both flips land on heads given that at least one does is only $\frac{1}{3}$. Many students initially find this latter result surprising. They reason that, given that at least one flip lands on heads, there are two possible results: Either they both land on heads or only one does. Their mistake, however, is in assuming that these two possibilities are equally likely. For, initially, there are 4 equally likely outcomes. Because the information that at least one flip lands on heads is equivalent to the information that the outcome is not *(t, t)*, we are left with the 3 equally likely outcomes *(h, h)*, *(h, t)*, *(t, h)*, only one of which results in both flips landing on heads.

***EXAMPLE* 15**

In the card game bridge, the 52 cards are dealt out equally to 4 players—called East, West, North, and South. If North and South have a total of 8 spades among them, what is the probability that East has 3 of the remaining 5 spades?

Solution. Probably the easiest way to compute the desired probability is to work with the reduced sample space. That is, given that North–South have a total of 8 spades among their 26 cards, there remains a total of 26 cards, exactly 5 of them being spades, to be distributed among the East–West hands. Since each distribution is equally likely, it follows that the conditional probability that East will have exactly 3 spades among his or her 13 cards is

$$\frac{\binom{5}{3}\binom{21}{10}}{\binom{26}{13}}$$

EXERCISE:

An experiment is conducted to examine the relationship between cigarette smoking and cancer. The individuals are randomly selected from the male population of a certain section of the United States. The results are summarized as follows:

	Developing Cancer	
Smoker	Yes	No
Yes	0.05	0.20
No	0.03	0.72

Find P(developing cancer|smoker)

EXERCISE:

Call a household prosperous if its income exceeds $100,000. Call the household educated if the household completed college. Select an American household at random, and let A be the event that the selected household is prosperous and let B be the event that it is educated. According to the Current Population Survey, P(A) = 0.134, P(B) = 0.254, P(A|B) = 0.080.

(a) Find the conditional probability that a household is educated, given that it is prosperous.

(b) Find the conditional probability that a household is prosperous, given that it is educated.

The multiplicative rule:

$P(E|F)=\frac{P(E\cap F)}{P(F)}$ for $P(F) > 0$. Multiplying both sides of Equation by P(F), we obtain

$$P(E \cap F) = P(F)P(E|F).$$

In words, this Equation states that the probability that both *E* and *F* occur is equal to the probability that *F* occurs multiplied by the conditional probability of E given that F occurred. This Equation is often quite useful in computing the probability of the intersection of events.

***EXAMPLE* 16**

Celine is undecided as to whether to take a French course or a chemistry course. She estimates that her probability of receiving an A grade would be $\frac{1}{2}$ in a French course and $\frac{2}{3}$ in a chemistry course. If Celine decides to base her decision on the flip of a fair coin, what is the probability that she gets an A in chemistry?

Solution. Let the event that Celine takes chemistry C and A denote the event that she receives an A in whatever course she takes, then the desired probability is $P(C \cap A)$, which is calculated as follows:

$$P(C \cap A) = P(C)P(A|C) = \left(\frac{1}{2}\right)\left(\frac{2}{3}\right) = \frac{1}{3}$$

EXERCISE:

A focus group of 10 consumers has been selected to view a new TV commercial. After the viewing, 2 members of the focus group will be randomly selected and asked to answer detailed questions about the commercial. The group contain 4 men and 6 women. What is the probability that the 2 chosen to answer questions will both be women?

P(first person is female) =

P(second person is female | first person is female) =

P(both people are female) =

EXERCISE:

A box has three tickets, colored red, white and blue. Two tickets will be drawn at random without replacement. What is the chance of drawing the red and then the white?

EXERCISE:

If we randomly pick two television tubes in succession from a shipment of 240 television tubes of which 15 are defective, what is the probability that they will both be defective?

A generalization of Equation $P(E \cap F) = P(F)\mathrm{P(E|F)}$, which provides an expression for the probability of the intersection of an arbitrary number of events, is sometimes referred to as the *multiplication rule*.

The multiplication rule
$P(E_1 \cap E_2 \cap E_3 \ldots \cap E_n)$ $= P(E_1)P(E_2\vert E_1)P(E_3\vert E_1 \cap E_2) \ldots P(E_n\vert E_1 \cap \ldots \cap E_{n-1})$

To prove the multiplication rule, just apply the definition of conditional probability to its right-hand side, giving

$$P(E_1)\frac{P(E_1 \cap E_2)}{P(E_1)}\frac{P(E_1 \cap E_2 \cap E_3)}{P(E_1 \cap E_2)} \cdots \frac{P(E_1 \cap E_2 \cdots \cap E_n)}{P(E_1 \cap E_2 \cdots \cap E_{n-1})}$$
$$= P(E_1 \cap E_2 \cap E_3 \ldots \cap E_n)$$

5- INDEPENDENT EVENTS

The conditional probability of E given F, is not generally equal to P(E). In other words, knowing that F has occurred generally changes the chances of E's occurrence. In the special cases where P(E|F) does in fact equal P(E), we say that E is independent of F. That is, E is independent of F if knowledge that F has occurred does not change the probability that E occurs.

Since $P(E|F) = P(E \cap F)/P(F)$, it follows that E is independent of F if

$$P(E \cap F)=P(E)\,P(F)$$

By symmetric, whenever *E* is independent of *F*, *F* is also independent of *E*. We thus have the following definition.

Definition
Two events E and F are said to be *independent* if $$P(E \cap F) = P(E)\,P(F)$$ So Two events E and F are said to be *independent* if $$P(E\|F) = P(E) \text{ and } P(F\|E) = P(F)$$ Two events E and F that are not independent are said to be *dependent*.

***EXAMPLE* 17**

A card is selected at random from an ordinary deck of 52 playing cards. If E is the event that the selected card is an ace and F is the event that it is a spade, then E and F are independent.

This follows because $P(E \cap F) = \frac{1}{52}$, whereas $P(E) = \frac{4}{52}$ and $P(F) = \frac{13}{52}$.

***EXAMPLE* 18**

Two coins are flipped, and all 4 outcomes are assumed to be equally likely. If E is the event that the first coin lands on heads and F the event that the second lands on tails, then E and F are independent, since $P(E \cap F) = P(\{(H, T)\}) = \frac{1}{4}$, whereas $P(E) = P(\{(H,H), (H, T)\}) = \frac{1}{2}$ and $P(F) = P(\{(H, T), (T, T)\}) = \frac{1}{2}$.

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EXERCISE:

Call a household prosperous if its income exceeds \$100; 000. Call the household educated if the household completed college. Select an American household at random, and let A be the event that the selected household is prosperous and let B be the event that it is educated. According to the Current Population Survey, P(A) = 0.134, P(B) = 0.254, P(A|B) = 0.080. Are events A and B independent?

6- BAYES'S FORMULA

Let E and F be events so $P(E|F) = \frac{P(F|E)\,P(E)}{P(F)}$. We may express E as

$$E = (E \cap F) \cup (E \cap F^c)$$

As $E \cap F$ and $E \cap F^C$ are clearly mutually exclusive, we have, by Axiom 3,

$$\begin{aligned} P(E) &= P(E \cap F) + P(E \cap F^c) \\ &= P(E|\mathrm{F})P(F) + P(E|F^c)P(F^c) \\ &= P(E|\mathrm{F})P(F) + P(E|F^c)[1 - P(F)] \end{aligned}$$

This equation states that the probability of the event E is a weighted average of the conditional probability of E given that F has occurred and the conditional probability of E given that F has not occurred—each conditional probability being given as much weight as the event on which it is conditioned has of occurring. This is an extremely useful formula, because its use often enables us to determine the probability of an event by first "conditioning" upon whether or not some second event has occurred. That is, there are many instances in which it is difficult to compute the probability of an event directly, but it is straightforward to compute it once we know whether or not some second event has occurred.

***EXAMPLE* 19**

An insurance company believes that people can be divided into two classes: those who are accident prone and those who are not. The company's statistics show that an accident-prone person will have an accident at sometime within a fixed 1-year period with probability .4, whereas this probability decreases to .2 for a person who is not accident prone. If we assume that 30 percent of the population is accident prone, what is the probability that a new policyholder will have an accident within a year of purchasing a policy?

Solution. We shall obtain the desired probability by first conditioning upon whether or not the policyholder is accident prone. Let A_1 denote the event that the policyholder will have an accident within a year of purchasing the policy, and let A denote the event that the policyholder is accident prone. Hence, the desired probability is given by

$$\begin{aligned} P(A_1) &= P(A_1|\mathrm{A})P(A) + P(A_1|A^c)P(A^c) \\ &= (0.4)(0.3) + (0.2)(0.7) = 0.26 \end{aligned}$$

***EXAMPLE* 20**

Suppose that a new policyholder has an accident within a year of purchasing a policy. What is the probability that he or she is accident prone?

Solution. The desired probability is

$$P(\mathrm{A}|A_1) = \frac{P(\mathrm{A} \cap A_1)}{P(A_1)}$$
$$= \frac{P(\mathrm{A})P(A_1|\mathrm{A})}{P(A_1)}$$
$$= \frac{(0.3)(0.4)}{(0.26)} = \frac{6}{13}$$

***EXAMPLE* 21**

Suppose we are interested in diagnosing cancer in patients who visit a chest clinic.

Let C represent the event "Person has cancer"

Let S represent the event "Person is a smoker"

- We know the probability of the prior event P(C) = 0.1 on the basis of past data (10% of the patients entering the clinic turn out to have cancer).
- We know $\mathrm{P}(\mathrm{S}|\mathrm{C})$ by checking from our record the proportion of smokers among those diagnosed with cancer. Suppose $\mathrm{P}(\mathrm{S}|\mathrm{C})$ = 0.8
- We know $\mathrm{P}(S|C^c)$by checking from our record the proportion of smokers among those not diagnosed with cancer. Suppose $\mathrm{P}(S|C^c)$ = 0.466

P(S) = (0.8)(0.1) + (0.466)(0.9) = 0.5

$\mathrm{P}(\mathrm{C}|\mathrm{S}) = \frac{(0.8)(0.1)}{0.5}$ =0.16

Thus, in light of the evidence that the person is a smoker we revise our prior probability from 0.1 to a posterior probability of 0.16. This is a significance increase, but it is still unlikely that the person has cancer.

EXERCISE:

You ask your neighbor to water a sickly plane while you are on vacation. Without water it will die with probability 0.8; with water it will die with probability 0.15. You are 90 percent certain that your neighbor will remember to water the plant.

(a) What is the probability that the plant will be alive when you return?

(b) If it is dead, what is the probability your neighbor forgot to water it?

EXERCISE:

According to the Arizona Chapter of the American Lung Association, 7.0% of the population has lung disease. Of those having lung disease, 90% are smokers; of those not having lung disease, 25.3% are smokers. Determine the probability that a randomly selected smoker has lung disease.

Solution:

EXERCISE:

A certain federal agency employs three consulting firms (A, B, and C) with probabilities 0.40, 0.35, and 0.25, respectively. From past experience it is known that the probability of cost overruns for the firms are 0.05, 0.03, and 0.15 respectively. Suppose a cost overrun is experienced by the agency.

(a) What is the probability that the consulting firm involved is company C?

(b) What is the probability that it is company A?

PROBLEMS

EXERCISE 1:

(a) How many different 7-place license plates are possible if the first 2 places are for letters and the other 5 for number?

(b) Repeat part (a) under the assumption that no letter or number can be repeated in a single license plate.

EXERCISE 2:

Twenty workers are to be assigned to 20 different jobs, one to each job. How many different assignments are possible?

EXERCISE 3:

If 4 Americans, 3 French people, and 3 British people are to be seated in a row, how many seating arrangements are possible when people of the same nationality must sit next to each other?

EXERCISE 4:

A history class contain 7 male students and 5 female students. Find the number of ways that the class can elect:

(a) a class representative

(b) two class representatives, one male and one female

(c) a president and then a vice-president.

EXERCISE 5:

A class contains 10 students with 6 men and 4 women. Find the number of ways:

(a) a 4-member committee can be selected from the students,

(b) a 4-member committee with 2 men and 2 women can be selected.

(c) the class can select a president then vice-president then treasurer, and then secretary.

EXERCISE 6:

For each of the following, list the sample space and tell whether you think the events are equally likely.

(a) Roll two dice; record the sum of the numbers.

(b) A family has 3 children; record the genders in order of birth.
(c) Toss 4 coins; record the number of tails.
(d) Toss a coin 10 times.; record the longest run of heads.

EXERCISE 7:
In a large city, 70% of the households receive a daily newspaper, and 50% of those who receive a daily newspaper have a television set. What is the probability that a randomly selected household will be one that receives a daily newspaper and has a television set?

EXERCISE 8:
A bin contains 100 balls, of which 25 are red, 40 are white, and 35 are black. If two balls are selected from the bin without replacement, what is the probability that one will be red and one will be white?

EXERCISE 9:
Suppose the probability that a U.S. resident has traveled to Canada is 0.18, to Mexico is 0.09, and to both countries s 0.04. What is the probability that an American chosen at random has
(a) traveled to Canada but not to Mexico?
(b) traveled to either Canada or Mexico?
(c) not traveled to either country?

EXERCISE 10:
Employment data at a large company reveal that 72% of the workers are married, that 44% are college graduates, and that half of the grads are married. What is the probability that a randomly chosen worker
(a) is neither married nor a college graduate?
(b) is married but not a college graduate?
(c) is married or a college graduate?

EXERCISE 11:
Seventy percent of kids who visit a doctor have fever, and 30% of kids with fever have sore throats. What is the probability that a kid who goes to the doctor has a fever and a sore throat?

EXERCISE 12:

You pick three cards at random from a deck. Find the probability of each event described below.

(a) You get no aces.

(b) You get all hearts.

(c) The third card is your first black card.

(d) You have at least one diamond.

EXERCISE 13:

There are 90 applications for a job with the news department of a television station. Some of them are college graduates and some are not, some of them have at least three years' experience and some have not, with the exact breakdown being:

	College graduates	Not College graduates
At least three years' experience	18	19
Lee than three years' experience	36	27

In the order in which the applicants are interviewed by the station manager is random, G is the event that the first applicant interviewed is a college graduate, and T is the event that the first applicant interviewed has at least three years' experience, determine

$$P(G), \quad P(T^c), \quad P(G \cap T), \quad P(\mathrm{T}|\mathrm{G}), \quad P(G^c|T^c)$$

EXERCISE 14:

Given the table below, are high blood pressure and high cholesterol independent? Explain.

		Blood Pressure	
		High	OK
Cholesterol	High	0.11	0.21
	OK	0.16	0.52

EXERCISE 15:

Suppose that 23% of adults smoke cigarettes. It is known that 57% of smokers and 13% of nonsmokers develop a certain lung condition by age 60.

(a) Explain how these statistics indicate that lung condition and smoking are not independent.

(b) What is the probability that a randomly selected 60-year-old has this lung condition?