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Kinematic Analysis of 7-DOF Manipulators

Abstract

This article presents a kinematic analysis of seven-degree-of-freedom serial link spatial manipulators with revolute joints. To uniquely determine the joint angles for a given end-effector position and orientation, the redundancy is parameterized by a scalar variable that defines the angle between the arm plane and a reference plane. The forward kinematic mappings from joint space to end-effector coordinates and arm angle and the augmented Jacobian matrix that gives end-effector and arm angle rates as functions of joint rates are presented. Conditions under which the augmented Jacobian becomes singular are also given and are shown to correspond to the arm being either at a kinematically singular configuration or at a nonsingular configuration for which the arm angle ceases to parameterize the redundancy.

1. Introduction

Robot manipulators that have more joint degrees of freedom (DOFs) than the minimum number needed to perform some tasks of interest are referred to as "redundant" to indicate the existence of the excess degrees of freedom. Although redundancy is obviously a task-dependent concept, manipulators with greater than six DOFs are usually called redundant, because the classic problem of end-effector position and orientation control for a "spatial manipulator" can be handled by a 6-DOF robot arm (Hollerbach 1984; Burdick 1988). Redundancy can be exploited for a variety of applications—singularity avoidance, collision avoidance, enhancement of mechanical advantage, manipulability enhancement, subtask performance, and so on—which greatly increase the flexibility and use of robot arms.

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The Robotics Research arms form a family of commercially available 7-DOF revolute joint serial link manipulators that offer one extra degree of joint space redundancy over that needed for the basic task of end-effector placement and orientation. These arms can be described as having a spherical-revolute-spherical (i.e., 3R-1R-3R) joint arrangement that can be referred to as *anthropomorphic* (Hollerbach 1984). In this article, a scalar parameterization, ψ, is given of the redundancy. For the Robotics Research arm, ψ is defined as the "arm angle" (a natural redundancy parameter for anthropomorphic arms [Hollerbach 1984]), which is the angle between the plane passing through the arm and a reference plane. The forward kinematic mappings from joint space to end-effector coordinates and ψ are then given. We also present the augmented Jacobian, J^A, which gives end-effector rates and $\dot{\psi}$ as functions of joint rates. Although in this article a particular emphasis is placed on the Robotics Research arms (in particular, the model K-1207 arm) and the use of the arm angle for resolving redundancy, many of the results developed here apply equally to any arbitrary 7-DOF manipulator with the use of *any* appropriate redundancy-resolving scalar parameter, not necessarily the arm angle.

Because of the displacement of joint axes ("nonzero joint offsets"), the Robotics Research arms have no known analytic closed-form inverse kinematic solutions for specified end-effector coordinates and redundancy parameter ψ. Consequently, the importance of differential kinematics (i.e., of the Jacobian) is increased, because at this time most approaches to solving the inverse kinematics and controlling the end-effector motions of nonsolvable arms are based on differential methods. For example, a "resolved-rate" (or "inverse Jacobian") kinematic control approach used for nonredundant arms (Whitney 1969) has been extended for general task space control (Baillieul 1985; Seraji and Colbaugh 1990), as well as to find

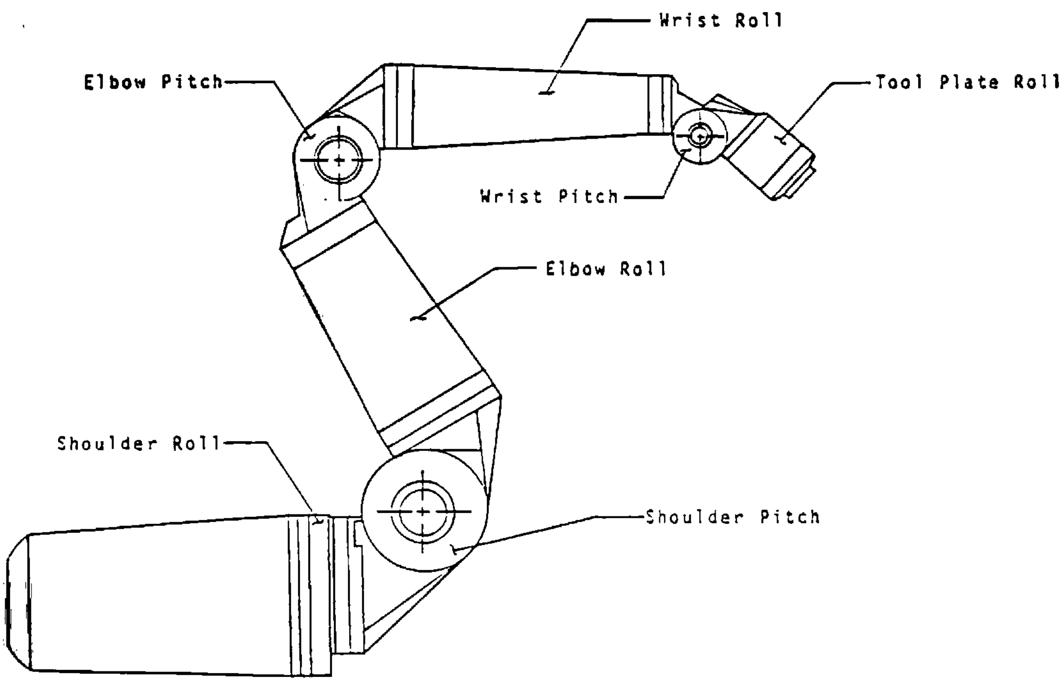

Fig. 1. Robotics Research model K-1207 arm.

joint angles for a 7-DOF arm given end-effector coordinates and ψ (Seraji et al. 1991; Long 1992). Similarly, an augmented "Jacobian transpose" approach has been used for redundant arm control in task space (Seraji 1989). Alternatively, end-effector motions can be controlled using pseudoinverse techniques (Liegeois 1977), which are based on the use of the pseudoinverse of the end-effector Jacobian rather than on using the inverse (or transpose) of the augmented Jacobian.

In order to control the 7-DOF arm motion in Cartesian task space while simultaneously controlling ψ, it is generally required that the augmented Jacobian remains nonsingular. The singularities of the augmented Jacobian are of two types: the "kinematic singularities" of the end effector itself and additional "algorithmic singularities" corresponding to arm configurations for which the augmented Jacobian is singular even if the end effector is in a kinematically nonsingular configuration. Therefore, for the purposes of simultaneously controlling end-effector motions and ψ, it is desirable to find both the algorithmic and kinematic singularities for the augmented Jacobian. We give an analytic expression for an algorithmic singularity measure appropriate for the augmented Jacobian derived in this article and discuss conditions for which the Robotics Research K-1207 arm and the related "zero offset" arm of Hollerbach (1984) are algorithmically and kinematically singular (i.e., the conditions for which the augmented Jacobian becomes singular). Because of space limitations, most proofs are omitted from this article. A more complete version of the article containing all the proofs and some numerical examples can be found in Kreutz-Delgado et al. (1990).

2. Forward Kinematics

2.1. Mapping from Joint Space to End-Effector Coordinates

The Robotics Research model K-1207 arm is a 7-DOF manipulator with nonzero offsets (denoted by the nonzero link lengths $a_i, i = 1, \cdots, 6$) at each of the joints, as shown in Figures 1 through 3. Denavit-Hartenberg (D-H) link frame assignments are given in accordance with the convention described in Craig (1986) and Yoshikawa (1990). This assignment results in the interlink homogeneous transformation matrix

$$
{}^{i-1}T_i = \begin{pmatrix} {}^{i-1}R_i & {}^{i-1}P_i \\ 0^T & 1 \end{pmatrix}
= \begin{pmatrix} \cos\theta_i & -\sin\theta_i & 0 & a_{i-1} \\ \sin\theta_i \cdot \cos\alpha_{i-1} & \cos\theta_i \cdot \cos\alpha_{i-1} & -\sin\alpha_{i-1} & -d_i \cdot \sin\alpha_{i-1} \\ \sin\theta_i \cdot \sin\alpha_{i-1} & \cos\theta_i \cdot \sin\alpha_{i-1} & \cos\alpha_{i-1} & d_i \cdot \cos\alpha_{i-1} \\ \mathbf{0} & \mathbf{0} & 0 & 1 \end{pmatrix}
$$

where θ_i denotes the ith joint angle (Craig 1986). The D-H parameters for the K-1207 arm are given in Table 1.

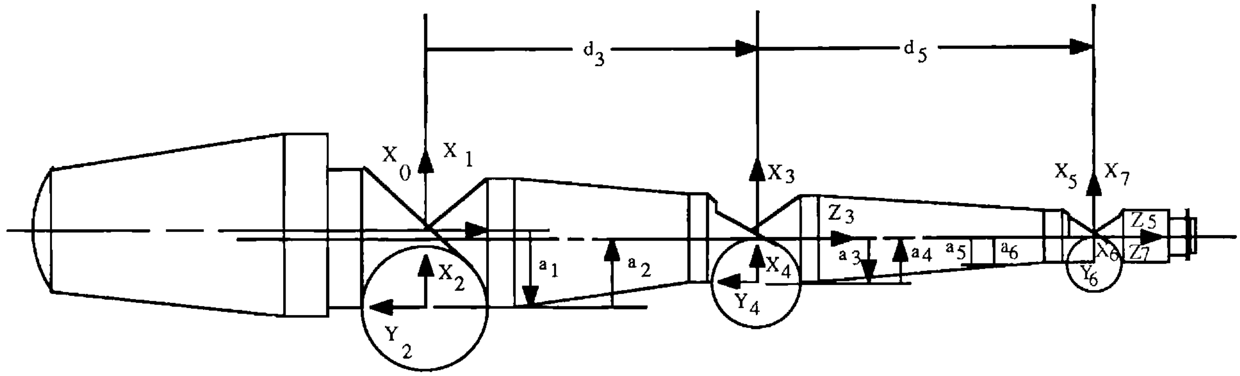

Fig. 2. Robotics Research model K-1207 link frame assignment.

The link frame assignments for the K-1207 are given in Figure 2, where the arm is shown in its zero configuration. The link i coordinate frame is denoted by F_i, with coordinate axes $(\hat{x}_i, \hat{y}_i, \hat{z}_i)$ and origin O_i. The associated interlink homogeneous transformation matrices, ${}^{i-1}T_i, i = 1, \cdots 7$, are easily found from the above expression evaluated for the D-H parameter values listed in Table 1. If the link length parameters $a_i, i = 1, \cdots, 6$ are set to zero, the 7-DOF all-revolute anthropomorphic arm described in Hollerbach (1984) is retrieved; we call this arm the *zero-offset arm*. The forward kinematic function, ${}^0T_7$, which gives the position and orientation of the end effector as a function of the joint angles $\theta = (\theta_1, \cdots, \theta_7)^T$, is ${}^0T_7 = {}^0T_1 \cdots {}^6T_7$. When these multiplications are performed to obtain a symbolic form for ${}^0T_7$, the resulting expression is complex because of the multitude of nonzero joint offsets and the fact that no two consecutive joints axes are parallel. Rather than construct and implement the symbolic expression, it is more efficient to numerically compute the forward kinematic function ${}^0T_7$ via a link-by-link iteration of the form

$$ {}^0T_i = {}^0T_{i-1} \cdot {}^{i-1}T_i, \qquad i = 1, \cdots, 7 \tag{1} $$

exploiting special structural properties of the homogeneous transformation matrices during each link update (Orin and Schrader 1984; Fijany and Bejczy 1988; Long 1992). Furthermore, it is useful to explicitly have the interlink homogeneous transformations, ${}^{i-1}T_i$, as important quantities (such as the vectors w, e, and p defined later) can then be computed. In fact, such quantities are often direct by-products of the intermediate steps of the iteration (1). The iteration (1) is by no means the only possibility for iteratively computing the forward kinematics, and (1) can be modified to yield different intermediate results as needed (for example, the iteration can be done for a reverse iteration with $i = 7, \cdots, 1$) (Orin and Schrader 1984; Fijany and Bejczy 1988). However, it is sufficient for the purposes of this article to focus solely on (1).

Table 1. Denavit-Hartenberg Parameters for the K-1207 Arm

i	α_{i-1}	a_{i-1}	d_i	θ_i	
1	0°	0	0	θ_1	$a_1 = -4.850\,\text{in} = -12.319\,\text{cm}$
2	−90°	a_1	0	θ_2	$a_2 = +4.250\,\text{in} = +10.795\,\text{cm}$
3	+90°	a_2	d_3	θ_3	$a_3 = -3.125\,\text{in} = -7.938\,\text{cm}$
4	−90°	a_3	0	θ_4	$a_4 = -3.125\,\text{in} = -7.938\,\text{cm}$
5	+90°	a_4	d_5	θ_5	$a_5 = -1.937\,\text{in} = -4.920\,\text{cm}$
6	−90°	a_5	0	θ_6	$a_6 = +1.937\,\text{in} = +4.920\,\text{cm}$
7	+90°	a_6	0	θ_7	$d_3 = d_5 = 21.5\,\text{in} = 54.61\,\text{cm}$

2.2. *Mapping from Joint Space to Arm Angle*

When the arm is in a kinematically nonsingular configuration, there will generally exist one excess joint degree of freedom for the task of end-effector control, as there are seven joint angles available to position and orient the end effector—a task that requires only six degrees of freedom. As a result, for a fixed end-effector frame, there is generally a one-dimensional subset of joint space (a "self-motion manifold") that maps to this configuration. Actually, there are finitely many, up to 16 in the most general case, such self-motion manifolds or "poses" (Burdick 1988; Burdick and Seraji 1989). The extra degree of freedom represented by a self-motion manifold can be used to attain some additional task requirement, provided that this task can be performed independently of end-effector placement (Egeland 1987; Seraji 1989). Furthermore, the imposition of an auxiliary task constraint can provide sufficient additional information to uniquely determine the joint angles (Oh et al. 1984; Seraji 1989; Seraji and Colbaugh 1990) (within the multiplicity of solutions represented by the pose). This scalar additional task variable is denoted by ψ in this article and is assumed to be a parameterization of the self-motion manifolds that map to a given end-effector frame. We say that the *basic task* of end-effector placement has been *augmented* by the *additional task* represented by ψ. In essence, the concept of the forward kinematic map is generalized to be the mapping from $\theta \in R^7$ to $({}^0T_7, \psi)$.

Although ψ can be any additional scalar parameter of interest that is independent of end-effector frame, we define and use the "arm angle" to resolve the manipulator redundancy. Refer to Figures 3 and 4, where $S = O_1, E = O_4$, and $W = O_7$ denote the origins of link frames 1, 4, and 7 attached to the shoulder, elbow, and wrist, respectively. The arm angle ψ is defined by the angle from the reference plane containing the unit vector $\hat{V}$ and the shoulder-wrist line SW to the shoulder-elbow-wrist plane SEW in the right-hand sense about the vector $w = W - S$. With the arm angle ψ as a parameterization of manipulator redundancy, a self-motion is described by a rotation of the plane SEW about the line SW. Note that the arm angle ψ is undefined when the wrist point W is anywhere on the line through the shoulder point S containing $\hat{V}$—even though this is generally not a singular configuration—because in this case the reference plane is not uniquely defined. The angle ψ is also undefined when e and w are colinear, as then the plane SEW is not uniquely defined. In the latter case, the arm is either nearly fully outstretched or folded and is therefore near or at an "elbow singular" configuration (Burdick 1988; Wampler 1988b). In such cases, ψ ceases to parameterize the redundancy.

To derive the forward kinematic function that gives ψ as a function of joint angles, again consider Figure 4. Let $w = W - S, e = E - S$, and let $\hat{V}$ denote an arbitrary fixed unit vector (e.g., the unit vector in the vertical direction of the base frame). Let the projection of e onto w be given by $d = \hat{w}(\hat{w}^T e), \hat{w} = w/\|w\|$. The minimum distance from the line SW to the point E is along the vector $p = e - d = (I - \hat{w}\hat{w}^T)e$. The reference plane is the plane that contains both w and the unit vector $\hat{V}$. The unit vector in the reference plane that is orthogonal to w is given by $\hat{\ell} = \ell/\|\ell\|$, with $\ell = (w \times \hat{V}) \times w$. We also define the unit vector $\hat{p} = p/\|p\|$. Note that $e, w, \hat{w}, d, p, \hat{p}, \ell$, and $\hat{\ell}$ can be computed during the forward kinematics iteration (1), as will be discussed subsequently. The vector ℓ, or equivalently $\hat{\ell}$, is treated as a free vector that can slide along the line SW. In particular, ℓ is moved along the line SW until its base is in contact with the base of vector p at the point d (see Figure 4), so that ψ is the angle from ℓ to p. This construction results in

$$C_\psi = \hat{\ell}^T \hat{p}, \quad S_\psi \hat{w} = \hat{\ell} \times \hat{p}, \quad S_\psi = \hat{w}^T(\hat{\ell} \times \hat{p}) \quad (2)$$

where $C_\psi = \cos\psi$ and $S_\psi = \sin\psi$. This gives

$$\tan\psi = \frac{\hat{w}^T(\hat{\ell} \times \hat{p})}{\hat{\ell}^T \hat{p}} = \frac{\hat{w}^T(\ell \times p)}{\ell^T p}. \quad (3)$$

The result (3) can be simplified somewhat. Defining $g = w \times \hat{V}$, we have $\ell = g \times w$, and we note that $\ell^T g = \hat{V}^T g = 0$. This means that ℓ and $\hat{V}$ are coplanar, both lying in the reference plane. Because, in general, the reference plane is spanned by $\hat{V}$ and $\hat{w}$, we have

$$\hat{\ell} = \alpha\hat{V} + \beta\hat{w}, \quad \alpha = 1/(\hat{\ell}^T\hat{V}), \quad \beta = -\alpha\hat{w}^T\hat{V}. \quad (4)$$

Substituting this result into (2) gives

$$C_\psi = \alpha\hat{V}^T\hat{p}, \qquad S_\psi = \alpha\hat{w}^T(\hat{V} \times \hat{p}),$$

which can be used with (3) to obtain

$$\tan\psi = \frac{\hat{w}^T(\hat{V} \times \hat{p})}{\hat{V}^T\hat{p}} = \frac{\hat{w}^T(\hat{V} \times p)}{\hat{V}^T p}. \quad (5)$$

Equation (5) immediately gives the forward kinematic function that maps the joint angles θ to the arm angle ψ as:

$$\psi = \mathrm{atan2}(\hat{w}^T(\hat{w}^T(\hat{V} \times p), \hat{V}^T p) \quad (6)$$

where $\mathrm{atan2}(y, x)$ is defined as in Craig (1986). For the special case of the zero-offset arm discussed in Hollerbach (1984), corresponding to $a_i = 0, i = 1, \ldots, 6$, (6) reduces to

$$\psi = \mathrm{atan2}\left(S_2S_3S_4, \frac{d_5S_4}{\|w\|}[C_2S_4 + S_2C_3(1 + C_4)]\right), \quad (7)$$

where $\|w\| = (d_3^2 + d_5^2 + 2d_3d_5C_4)^{1/2}$. Equation (6) is undefined when both arguments are simultaneously zero. This occurs when the arm is in a configuration for which e and w are colinear or for which the wrist point W is directly on the line through $\hat{V}$ and containing the shoulder point S. These indeterminacies have been discussed previously and are a result of the inability to uniquely define the arm plane SEW or the reference plane, respectively.

The augmented forward kinematics mapping, $\theta \to ({}^0T_7, \psi)$, is given by (1) and (6). The quantities $\hat{w}$ and $p = e - d = (I - \hat{w}\hat{w}^T)e$ are first computed during the iteration (1), after which ψ is computed by (6). Note that, with

$${}^0T_4 = \begin{pmatrix} {}^0R_4 & {}^0P_4 \\ 0^T & 1 \end{pmatrix} \quad \text{and} \quad {}^0T_7 = \begin{pmatrix} {}^0R_7 & {}^0P_7 \\ 0^T & 1 \end{pmatrix},$$

quantities that are directly computed during the iteration (1), the representations of e and w in the base (link 0) frame F_0 are precisely ${}^0e = {}^0P_4$ and ${}^0w = {}^0P_7$ (Long 1992). Also note that $\hat{V}$ is a constant vector that can be expressed in a frame that gives it a particularly simple form such as $(0, 0, 1)^T$ or $(1, 0, 0)^T$.

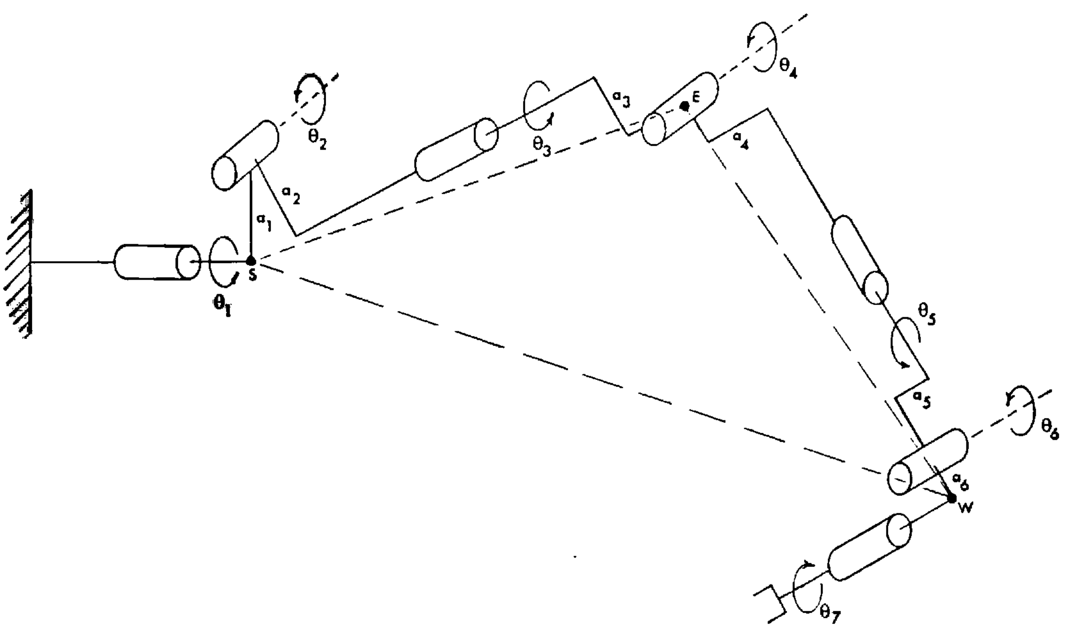

Fig. 3. Definition of the arm plane SEW.

3. Differential Kinematics

3.1. Manipulator End-Effector Jacobian J^{ee}

To present actual values for the end-effector Jacobian, J^{ee}, it is first necessary to choose a velocity reference point, as well as a frame in which to represent the vectorial quantities that define the columns of the Jacobian. In this section, to simplify notation, we will suppress the trailing superscript and write the end-effector Jacobian simply as $J = J^{ee}$. When a velocity reference point, a, and a representation frame, F_r, have been chosen, we write ${}^rJ_a = {}^rJ_a^{ee}$.

Let w_a and v_a be the angular and linear velocities of a coordinate frame, F_a, located at a point a and fixed with respect to the manipulator end effector. The point a will be called a *velocity reference point* of the end effector. The Jacobian, $J_a(\theta) \in R^{6\times 7}$, relates joint rates to the rate of change of frame F_a via the linear relationship $(w_a^T, v_a^T)^T = J_a(\theta)\dot{\theta}$ and is given by Whitney (1969):

$$J_a = \begin{pmatrix} \widehat{z}_1 & \cdots & \widehat{z}_7 \\ \widehat{z}_1 \times P_{a,1} & \cdots & \widehat{z}_7 \times P_{a,7} \end{pmatrix}. \tag{8}$$

In (8), $\widehat{z}_i$ denotes the unit vector corresponding to the z-axis of link frame i (i.e., of F_i), whereas $P_{a,i} \equiv P_{a,O_i} \equiv a - O_i$ is the vector from the origin O_i, of link frame i to the point a. Note that $P_{i,i} = 0$.

Let F_b denote an alternative frame fixed with respect to the end effector and located at the velocity reference point b. The relationship between joint rates and the rate of change of F_b is given by $(w_b^T, v_b^T)^T = J_b\dot{\theta}$. Let F_r and F_s be frames that are *not* necessarily fixed with respect to the end effector. The representations of w_a and v_a in frame F_r are denoted by ${}^rw_a$ and ${}^rv_a$. Similarly, ${}^sw_b$ and ${}^sv_b$ are the representations of w_b and v_b in F_s. Note that we have defined a and b to be end-effector reference points (i.e., to be fixed with respect to the end effector), whereas we have placed no constraints on r and s.

The Jacobian, ${}^rJ_a$, giving the rate of change of F_a represented in F_r, is related to ${}^sJ_b$, the Jacobian giving the rate of change of frame F_b represented in frame F_s, by Fijany and Bejczy (1988):

$${}^rJ_a = \begin{pmatrix} {}^rR_s & 0 \\ 0 & {}^rR_s \end{pmatrix} \cdot \begin{pmatrix} I & 0 \\ {}^s\tilde{P}_{a,b} & I \end{pmatrix} \cdot {}^sJ_b \tag{9}$$

where, for a 3-vector $x, \tilde{x}$ denotes the 3×3 skew symmetric operator defined by $\tilde{x}y = x \times y$ for every $y \in R^3$ and $P_{a,b} = a - b$. When $s = b$, we write ${}^bP_a \equiv {}^bP_{a,b}$. Note that ${}^rR_s \in R^{3\times 3}$ is a rotation matrix, represented in frame F_r, that gives the orientation of frame F_s with respect to frame F_r. Common choices of ${}^rJ_a$ are given by ${}^7J_7$, and ${}^0J_7$. It is straightforward to show from (9) that

$$\det[{}^rJ_a(\theta){}^rJ_a(\theta)^T] = \det[{}^sJ_b(\theta){}^sJ_b(\theta)^T] \tag{10}$$

for any a, b, r, and s. Because an $m \times n$ matrix $M, m < n$, is full rank if and only if $\det[MM^T] \neq 0$, (10) shows that the singularity of a manipulator Jacobian is independent of the choice of velocity reference point and representation frame and is a function purely of the manipulator configuration variables θ. This is true for spatial manipulators and is not necessarily, and usually not, true in other cases.

An important aspect of the decomposition (9) is that s and b can often be chosen to make the Jacobian matrix have a particularly simple structure for the purposes of singularity analysis, efficient evaluation, and efficient inversion. For example, in Orin and Schrader (1984), an algorithm for the efficient computation of ${}^0J_0$ is given. Note that J_0 does *not* give the velocity of the base frame, F_0, as a function of joint rates—indeed, in most cases the base in assumed fixed, and the base frame origin, O_0, cannot be a velocity reference point for the moving end effector. Instead, J_0 is viewed as giving the velocity of a reference frame fixed with respect to the end effector and instantaneously coincident with the base frame origin, O_0. The computation of

$$
{}^0J_0 = \begin{pmatrix} {}^0\hat{z}_1 & {}^0\hat{z}_2 & \cdots & {}^0\hat{z}_7 \\ {}^0\hat{z}_1 \times {}^0P_{0,1} & {}^0\hat{z}_2 \times {}^0P_{0,2} & \cdots & {}^0\hat{z}_7 \times {}^0P_{0,2} \end{pmatrix}, \tag{11}
$$

where ${}^kP_{i,j} = {}^{O_k}P_{i,j}$ and $P_{i,j} = P_{O_i,O_j} = O_j - O_i$, naturally fits in with the forward kinematics iteration (1), because from

$$
{}^0T_i = \begin{pmatrix} {}^0R_i & {}^0P_i \\ 0^T & 1 \end{pmatrix},
$$

${}^0P_i \equiv {}^0P_{i,0}$, we can obtain ${}^0P_{0,i} = -{}^0P_i$ and ${}^0\hat{z}_i \times {}^0P_{0,i}$ where ${}^0\hat{z}_i = {}^0R_i e_3, e_3 = (0,0,1)^T$ (Long 1992). Having ${}^0J_0, {}^0J_7$ can then be found from

$$
{}^0J_7 = \begin{pmatrix} I & 0 \\ {}^0\tilde{P}_7 & I \end{pmatrix} \cdot {}^0J_0 \tag{12}
$$

Note that (12) is just a special case of (9). The symbolic forms of ${}^0J_0$ and ${}^0J_7$ can be found from this procedure, but these expressions are complex and provide little insight.

In Fijany and Bejczy (1988), the results in Orin and Schrader (1984) are extended to show that taking $s = O_i$ and $b = O_j$ for an appropriate choice of link frames i and j can result in an expression ${}^iJ_j = {}^{O_i}J_{O_j} = {}^sJ_b$, which is not only efficient to compute, but also simplifies singularity analysis and (for nonredundant manipulators) Jacobian inversion. In particular, to gain insight into the singularity structure of the K-1207 end-effector Jacobian and to obtain an alternative way to construct ${}^0J_7$ we will let $b = 3$ (i.e., let the velocity reference point be the origin of link frame 3) and $s = 3$ (let the reference frame be link frame 3). J_3 should be interpreted as giving the velocity of a fictitious tool frame that is instantaneously coincident with link frame 3. ${}^3J_3$ is found from (8) by taking $P_{a,i} = P_{3,i} = P_{O_3,O_i} = O_3 - O_i$ and representing $\hat{z}_i$ and $P_{3,i}$ in link frame 3 to obtain ${}^3\hat{z}_i$ and ${}^3\hat{z}_i \times {}^3P_{3,i}, {}^3P_{3,i} = {}^{O_3}P_{O_3,O_i}$. The symbolic expression for ${}^3J_3$ found in this manner is given by

$$
{}^3J_3 = \begin{pmatrix}
-S_2C_3 & S_3 & 0 & 0 & S_4 & -C_4S_5 & C_4C_5S_6 + S_4C_6 \\
S_2S_3 & C_3 & 0 & 1 & 0 & C_5 & S_5S_6 \\
C_2 & 0 & 1 & 0 & C_4 & S_4S_5 & C_4C_6 - S_4C_5S_6 \\
d_3S_2S_3 + (a_2C_2 + a_1)S_3 & d_3C_3 & 0 & 0 & 0 & S_4(a_4C_5 + a_5) - d_5C_4C_5 & S_5[C_4(a_5C_6 - d_5s_6 + a_6) + a_4S_4S_6 \\
(d_3S_2 + a_3C_2 + a_1)C_3 & -d_3S_3 & 0 & 0 & -a_3C_4 - a_4 & S_5[a_3S_4d_5] & C_5[S_6(a_3S_4 + d_5) - a_6] - (a_5C_5 + a_4 + a_3C_4)C_6 \\
0 & -a_2 & 0 & a_3 & 0 & C_4(a_4C_5 + a_5) + C_5(d_5S_4 + a_3) & S_5[(a_4C_4S_6 + a_3S_6) + S_4(d_5S_6 - a_5C_6 - a_6)]
\end{pmatrix} \tag{13a}
$$

Note from Table 1 that $a_4 = -a_3, a_6 = -a_5, d_5 = d_3$, and $a_2 = -a_1 + \Delta$, with $\Delta = 0.600$ in. $= 1.524$ cm. Thus ${}^3J_3$ reduces further to

$$
{}^3J_3 = \begin{pmatrix}
-S_2C_3 & S_3 & 0 & 0 & S_4 & -C_4S_5 & C_4C_5S_6 + S_4C_6 \\
S_2S_3 & C_3 & 0 & 1 & 0 & C_5 & S_5S_6 \\
C_2 & 0 & 1 & 0 & C_4 & S_4S_5 & C_4C_6 - S_4C_5S_6 \\
d_3S_2S_3 + S_3[a_1(1 - C_2) + C_2\Delta] & d_3C_3 & 0 & 0 & 0 & S_4(-a_3C_5 + a_5) - d_3C_4C_5 & S_5\{C_4[a_5(C_6 - 1) - d_3S_6] - a_3S_4S_6\} \\
[d_3S_2 + a_1(1 - C_2) + C_2\Delta]C_3 & -d_3S_3 & 0 & 0 & -a_3(C_4 - 1) & -S_5[a_3S_4 + d_3] & C_5[S_6(a_3S_4 + d_3) + a_5] - a_5C_5 + a_3(C_4 - 1)]C_6 \\
0 & a_1 - \Delta & 0 & a_3 & 0 & C_4(-a_3C_5 + a_5) + C_5(d_3S_4 + a_3) & S_5\{a_3S_6(1 - C_4) + S_4[d_3S_6 - a_5(C_6 - 1)]\}
\end{pmatrix} \tag{13b}
$$

Having ${}^3J_3, {}^0J_7$ is found from (9) as

$$
{}^0J_7 = \begin{pmatrix} {}^0R_3 & 0 \\ 0 & {}^0R_3 \end{pmatrix} \cdot \begin{pmatrix} I & 0 \\ {}^3\tilde{P}_{7,3} & I \end{pmatrix} \cdot {}^3J_3 \tag{14}
$$

where ${}^3P_{7,3}$ is given by ${}^3P_{7,3} = {}^3P_4 + {}^3R_4 \cdot {}^4P_5 + {}^3R_4 \cdot {}^4R_5 \cdot {}^5P_6 + {}^3R_4 \cdot {}^4R_5 \cdot {}^5R_6 \cdot {}^6P_7$, or alternatively, ${}^3P_{7,3} =$

${}^0R_3^T({}^0P_{7,0} - {}^0P_{3,0})$, resulting in

$$
{}^3P_{7,3} = \begin{pmatrix} -S_4(a_6S_6 - d_5) + C_4[C_5(a_6C_6 + a_5) + a_4] + a_3 \\ S_5(a_6C_6 + a_5) \\ -C_4(a_6S_6 - d_5) - S_4[C_5(a_6C_6 + a_5) + a_4] \end{pmatrix}, \tag{15}
$$

and ${}^0R_3$ is given by

$$
{}^0R_3 = {}^0R_1 \cdot {}^1R_2 \cdot {}^2R_3 = \begin{pmatrix} C_1C_2C_3 - S_1S_3 & -C_1C_2S_3 - S_1C_3 & C_1S_2 \\ C_1S_3 + S_1C_2C_3 & C_1C_3 - S_1C_2S_3 & S_1S_2 \\ -S_2C_3 & S_2S_3 & C_2 \end{pmatrix}. \tag{16}
$$

The relative simplicity of (13) not only enables one to efficiently compute ${}^0J_7$ via (13)–(16), but also allows one to gain insight into conditions leading to Jacobian singularity if the computation of ${}^0J_7$ follows the computation of ${}^0T_7$ using (8) (Long 1992). Note that for the special case of the zero-offset arm discussed in Hollerbach (1984), corresponding to $a_1 = \cdots = a_6 = 0$, (13a) simplifies to

$$
{}^3J_3 = \begin{pmatrix} -S_2C_3 & S_3 & 0 & 0 & S_4 & -C_4S_5 & C_4C_5S_6 + S_4C_6 \\ S_2S_3 & C_3 & 0 & 1 & 0 & C_5 & S_5S_6 \\ C_2 & 0 & 1 & 0 & C_4 & S_4S_5 & C_4C_6 - S_4C_5S_6 \\ d_3S_2S_3 & d_3C_3 & 0 & 0 & 0 & -d_5C_4C_5 & -d_5C_4S_5S_6 \\ d_3S_2C_3 & -d_3S_3 & 0 & 0 & 0 & -d_5S_5 & d_5C_5S_6 \\ 0 & 0 & 0 & 0 & 0 & d_5S_4C_5 & d_5S_4S_5S_6 \end{pmatrix}, \tag{17}
$$

which can be simplified even further by using the fact that $d_5 = d_3$.

3.2. Arm Angle Jacobian J^ψ and Augmented Jacobian J^A

Let the relationship between the rate of change of any scalar additional task variable, ψ, and the joint rates be given by $\dot{\psi} = J^\psi \dot{\theta}$. The "augmented" Jacobian is given by (Oh et al. 1984; Seraji 1989; Seraji and Colbaugh 1990)

$$
J^A = \begin{pmatrix} J^{ee} \\ J^\psi \end{pmatrix},
$$

where J^{ee} is the end-effector Jacobian discussed in Section 3.1. For the task of positioning and orienting the end effector augmented by an additional task represented by ψ, the augmented Jacobian relates joint rates to the simultaneous rates of change of the end-effector coordinates and ψ. Having the end-effector Jacobian, J^{ee}, the augmented Jacobian J^A is obtained once J^ψ has been determined for a given task variable ψ. In this section, J^ψ is constructed for the case where ψ describes the arm angle between a reference plane and the elbow plane SEW as defined in Section 2.

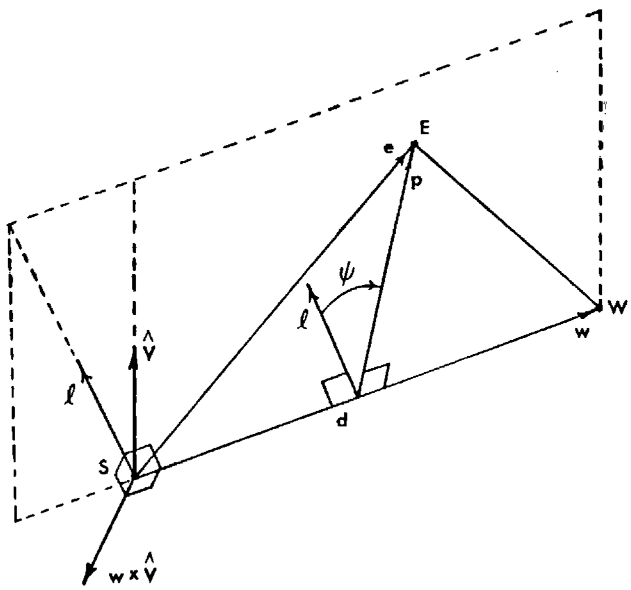

Fig. 4. Definition of the arm angle ψ.

Before proceeding further, it is necessary to define the Jacobians **E** and **W** that relate joint rates to $\dot{e}$ and $\dot{w}$, respectively, via $\dot{e} = \mathbf{E}\dot{\theta}$ and $\dot{w} = \mathbf{W}\dot{\theta}$, where e and w are defined in Section 2.2. Note that $\dot{e}$ is the linear velocity of the manipulator elbow point $E = O_4$, and $\dot{w}$ is the linear velocity of the wrist point $W = O_7$. We have

$$
\mathbf{E} = (\hat{z}_1 \times P_{4,1}, \quad \hat{z}_2 \times P_{4,2}, \quad \hat{z}_3 \times P_{4,3}, \quad 0, \quad \cdots, \quad 0), \tag{18}
$$

$$
\mathbf{W} = (\hat{z}_1 \times P_{7,1}, \quad \cdots, \quad \hat{z}_6 \times P_{7,6}, \quad 0), \tag{19}
$$

where $P_{i,j} = O_j - O_i$. Note that (18) and (19) are given in coordinate-free form and that to provide values for **E**, or **W**, a choice of reference frame for representing $\hat{z}_j$ and $P_{i,j}$ must be made. Also note that, on comparing (8) and (19), $J_7^{ee} = \begin{pmatrix} \cdots \\ \mathbf{W} \end{pmatrix}$, so that any procedure for producing a value for $J_7 = J_7^{ee}$ (such as the one discussed following (11)) automatically results in a value for **W**. Furthermore, just as one can construct **W** from knowledge of ${}^{i-1}T_i, i = 1, \cdots, 7$, one can readily compute values for **E** given the interlink homogeneous transformations ${}^{i-1}T_i$.

Recall the definitions of $\ell, \hat{\ell}, \hat{V}, p, \hat{p}, w, \hat{w}$ and e given in Section 2.2. Also recall, as discussed in Section 2.2, that these quantities can all be computed from knowledge of the interlink homogeneous transformations ${}^{i-1}T_i$, (Long 1992).

PROPOSITION 3.1. The relationship between $\dot{\theta}$ and $\dot{\psi}$, where ψ is the arm angle as defined in Section 2.2, is given by

$$\dot{\psi} = \frac{1}{\|p\|}(\hat{w} \times \hat{p})^T \dot{p} - \frac{1}{\|\ell\|}(\hat{w} \times \hat{\ell})^T \dot{\ell} \tag{20a}$$

$$= \frac{(\hat{w} \times \hat{p})^T}{\|p\|}\left\{\mathbf{E} - \frac{\hat{w}^T e}{\|w\|}\mathbf{W}\right\}\dot{\theta} + \frac{\hat{V}^T w}{\|\ell\|}(\hat{w} \times \hat{\ell})^T \mathbf{W}\dot{\theta}, \tag{20b}$$

which results in

$$J^{\psi} = \frac{(\widehat{w} \times \widehat{p})^T}{\|p\|}\mathbf{E} + \left\{\frac{\widehat{V}^T w}{\|\ell\|}(\widehat{w} \times \widehat{\ell})^T - \frac{\widehat{w}^T e}{\|w\|\,\|p\|}(\widehat{w} \times \widehat{p})^T\right\}\mathbf{W} \tag{21}$$

A proof of Propositon 3.1 can be found in Kreutz-Delgado et al. (1990).

Because the arm angle ψ is given by the angle from ℓ to p, it is natural that $\dot{\psi}$ should depend only on $\dot{\ell}$ and $\dot{p}$ as in (20a). Equation (20a) says that only the components of $\dot{\ell}$ and $\dot{p}$ that result in an instantaneous motion of ℓ and p directly toward or away from each other can produce a change in the arm angle ψ. Based on earlier discussions, it should be obvious that J^{ψ} can be constructed from knowledge of the interlink homogenous transformations ${}^{i-1}T_i$. Also note that J^{ψ} is independent of the reference frame chosen to represent the quantities in the right side of (21).

4. Singularities of Augmented Jacobian J^A

In Section 4.1, we consider the general case of an arbitrary seven-degree-of-freedom arm with an arbitrary scalar redundancy variable ψ. In Section 4.2, the case where ψ parameterizes the arm angle in the sense of Section 2.2 is discussed. Section 4.3 discusses the singular configurations of the 7-DOF K-1207 and zero-offset arms.

4.1. Singularity Measure for the Augmented Jacobian

The main result of this section is Proposition 4.1, which gives a condition for singularity of the augmented Jacobian based on the values of a scalar measure of kinematic singularity and of a scalar measure of algorithmic singularity. As background preparation to the proof of Proposition 4.1, Lemmas 4.1 and 4.2 are first presented. The results in this subsection hold for a general scalar ψ, not just for the case where ψ is the arm angle, and for any 7-DOF arm, not just all-revolute arms.

In Yoshikawa (1985), a measure of "nearness" of a redundant manipulator to kinematic singularity is defined by the non-negative quantity

$$m(\theta) = \left[\det\left(J^{ee}(\theta) J^{ee^T}(\theta)\right)\right]^{\frac{1}{2}}, \tag{22}$$

where $m(\theta)$ is known as the *manipulability measure*. For $J^{ee} \in R^{6\times 7}, m(\theta) \neq 0$ if and only if $\operatorname{rank}(J^{ee}) = 6$, which is true if and only if the arm is not at a kinematic singularity. When the arm is at a kinematic singularity, there is at least one direction in task space along which the end effector cannot move instantaneously. The number of independent directions in task space along which the arm cannot instantaneously move is $6 - \operatorname{rank}(J^{ee})$.

For $J^{ee} \in R^{6\times 7}$ full rank (i.e., $\operatorname{rank}(J^{ee}) = 6$), it is true that $\dim N(J^{ee}) = 1$, where $N(J^{ee})$ denotes the null space of J^{ee}. J^{ee} full rank also means that its six rows are linearly independent and that some combination of six of the columns of J^{ee} are linearly independent. The 6×6 matrix formed by the six linearly independent columns must be nondegenerate and have a nonvanishing determinant.

With $J^{ee} \in R^{6\times 7}$ full rank, there exists a unit vector $\hat{n} \in R^7$ such that $J^{ee}\hat{n} = 0$. The vector $\hat{n}$ spans the one-dimensional null space $N(J^{ee})$ and gives the instantaneous direction along which one can move in joint space without causing any end-effector motion. That is, $\hat{n}$ gives the locally allowable self-motion of the arm. Moving the kinematically nonsingular arm with a joint space velocity $\dot{\theta} = \alpha \cdot \hat{n}$, for scalar α, will always result in arm motions for which the end effector is fixed. Generally, the most efficient way to obtain $\hat{n}$ is to solve the equation $J^{ee}\hat{n} = 0$ via an iterative numerical technique such as Gaussian elimination. The following lemma shows that, in principle, it is possible to get an analytical expression for the null space spanning vector of a full rank $J^{ee} \in R^{6\times 7}$; however, in general, the resulting expression is generally too complex to be computed easily.

LEMMA 4.1. Let C_i be the cofactor of J^{ee} found from striking out the ith column of J^{ee} (Baillieul 1985; Luh and Gu 1985). Then, for $n \in R^7$ given by $n_i = C_i$, we have that $J^{ee}n = 0$ and

$$\det\begin{pmatrix} J^{ee} \\ n^T \end{pmatrix} = n^T n = m^2, \tag{23}$$

where m is the manipulability measure (22). If, furthermore, J^{ee} is full rank, n is nonzero.

A proof of Lemma 4.1 can be found in Kreutz-Delgado et al. (1990).

Note from (23) that $\|n\| = m$. It is straightforward to show that n of Lemma 4.1 is independent of the choice of velocity reference point and/or representation frame. By exploiting the special structure of ${}^3J_3$ in (17), the analytical expression for n for the zero-offset arm can readily be derived and is given by (31) below.

Having a unit null space spanning vector, $\hat{n} = \pm(n/\|n\|)$, the projector onto the Jacobian null space is given by

$$P_0 = \hat{n}\hat{n}^T, \tag{24a}$$

and the projector onto the orthogonal complement of $N(J^{ee})$ (i.e., onto $N(J^{ee})^\perp = R(J^{ee^T})$) is given by

$$P = I - P_0 = I - \hat{n}\hat{n}^T. \tag{24b}$$

Alternatively, from least squares theory, the projectors are given by

$$P_0 = I - J^{ee+}J^{ee}, \tag{25a}$$

$$P = J^{ee+}J^{ee}, \tag{25b}$$

where

$$J^{ee+} = J^{ee^T}(J^{ee}J^{ee^T})^{-1} \tag{26}$$

is the pseudoinverse of J^{ee}, assuming that J^{ee} is full rank.

LEMMA 4.2. Let $J^{ee} \in R^{6\times 7}$ be full rank; then $J^\psi = c_0\hat{n}^T + c_1^T J^{ee}$ with

$$c_0 = J^\psi \hat{n} \tag{27a}$$

$$c_1^T = J^\psi J^{ee+} \tag{27b}$$

where $\hat{n} \in R^7$ is a unit vector that spans the null space $N(J^{ee})$, and J^{ee+} is the pseudoinverse (26).

Proof. For P_0 given by (24a) and P given by (25b) we have

$$\begin{aligned} J^\psi &= J^\psi(P_0 + P) = J^\psi(\hat{n}\hat{n}^T + J^{ee+}J^{ee}) \\ &= J^\psi\hat{n}\hat{n}^T + J^\psi J^{ee+}J^{ee} = c_0\hat{n}^T + c_1^T J^{ee} \end{aligned}$$

□

PROPOSITION 4.1. For $J^A = \begin{pmatrix} J^{ee} \\ J^\psi \end{pmatrix}$, we have

$$\det J^A = \pm c_0 \cdot m, \tag{28}$$

where m is the manipulability measure (22).

Proof.

$$\begin{aligned} \det J^A &= \det\begin{pmatrix} J^{ee} \\ c_0\hat{n} + c_1^T J^{ee} \end{pmatrix} = \det\begin{pmatrix} J^{ee} \\ c_0\hat{n} \end{pmatrix} = c_0 \det\begin{pmatrix} J^{ee} \\ \hat{n} \end{pmatrix} \\ &= \frac{\pm c_0}{\|n\|}\det\begin{pmatrix} J^{ee} \\ n \end{pmatrix} = \pm c_0 m \end{aligned}$$

The scalar $|c_0|$ is a measure of the nearness to "algorithmic singularity" of the arm. From (27) we observe that c_0 is just the "dot product" of the vectors J^ψ and $\hat{n}$ and is therefore the component of J^ψ along the self-motion direction. Equation (28) says that any arm with one degree of redundancy is singular if and only if $c_0 = 0$ or $m = 0$. The scalar m is a measure of the kinematic singularity and vanishes if and only if the arm is at a kinematic singularity. The scalar $|c_0| = |J^\psi\hat{n}|$ (see (27a)) is zero when the arm is at the "algorithmic singularities" of J^ψ, a result originally derived in Baillieul (1985). Note that when $c_0 = 0$, we have

$$\dot{\psi} = J^\psi\dot{\theta} = c_0\hat{n}^T\dot{\theta} + c_1^T J^{ee}\dot{\theta} = c_1^T J^{ee}\dot{\theta},$$

which shows that $\dot{\psi}$ does not have any dependency on the self-motion vector $\hat{n}$ and is coupled only to the motion of the basic task given by $J^{ee}\dot{\theta}$. Obviously, then, ψ cannot be controlled independently of the basic task when $c_0 = 0$. The requirement that both c_0 and m be nonzero for J^A to be nonsingular can also be seen in the analytical expression for $(J^A)^{-1}$ given below.

PROPOSITION 4.2. For $J^A = \begin{pmatrix} J^{ee} \\ J^\psi \end{pmatrix}$, we have

$$(J^A)^{-1} = \left[\left(I - \frac{\hat{n}J^\psi}{c_0}\right)J^{ee+} \quad \frac{\hat{n}}{c_0}\right] = \left[J^{ee+} - \frac{\hat{n}c_1^T}{c_0} \quad \frac{\hat{n}}{c_0}\right]. \tag{29}$$

Proposition 4.2 is a generalization of a result in Baillieul (1985) and is proved in Kreutz-Delgado et al. (1990).

Obviously for $J^{A^{-1}}$ to exist we must have $c_0 \neq 0$ and $(J^{ee}J^{ee^T})$ invertible, the latter condition being true if and only if $m \neq 0$.

Note that, using (29), $\dot{\theta} = J^{A^{-1}}\begin{pmatrix} \dot{x} \\ \dot{\psi} \end{pmatrix}$ is equivalent to

$$\dot{\theta} = J^{ee+}\dot{x} + \hat{n}\xi,$$

$$\xi = \frac{1}{c_0}(\dot{\psi} - c_1^T\dot{x}),$$

showing that the inverse augmented Jacobian kinematic control can be formulated as a form of pseudoinverse control. The inverse augmented Jacobian control has the special property that it results in cyclic behavior over simply connected task space regions for which J^A is full rank (Wampler 1988a; Seraji 1989). In general, pseudoinverse control does not yield cyclic behavior over simply connected regions (Klein and Huang 1983; Wampler 1988a; Baker and Wampler 1988; Shamir and Yomdin 1988).

To independently control end-effector coordinates simultaneously with ψ, it is clear that it is necessary and sufficient that c_0 and m both be nonzero. It is therefore important to determine the configurations that correspond to $m = 0$ and $c_0 = 0$.

4.2. Arm Angle Algorithmic Singularity Measure

In Section 4.1, results are derived that are applicable to any 7-DOF spatial manipulator with any associated scalar additional task variable ψ. Here, we will obtain the form of c_0 and c_1 for the special case where the additional task is to control the arm angle defined in Section 2.2. Aside from assuming one degree of redundancy (i.e., 7 DOFs), the spatial manipulator is otherwise unspecified. As noted in Section 4.1, $|c_0|$ provides a measure of nearness to algorithmic singularity.

PROPOSITION 4.3. Let ψ denote the arm angle defined in Section 2. Then, $J^{\psi} = c_0\hat{n}^T + c_1^T J^{ee}$ where

$$c_0 = \frac{(\hat{w} \times \hat{p})^T \mathbf{E}\hat{n}}{\|p\|} \tag{30}$$

Proof. Note that $\mathbf{W}\hat{n} = 0$, because a pure self-motion results in a zero wrist motion. Equations (21) and (27) taken together then yield (30). □

There are two possible ways that c_0 as given by (30) can vanish. First, it may be that $\mathbf{E}\hat{n} = 0$, so that a self-motion causes no motion of the elbow point at all. Secondly, it may be that $\mathbf{E}\hat{n} \neq 0$, but the resulting elbow motion is entirely in the plane SEW (see Fig. 3), so that $(\hat{w} \times \hat{p})^T \mathbf{E}\hat{n} = 0$. In either case, it is apparent that self-motion has nothing to do with a change of the arm angle ψ. Assuming that the arm is not at a kinematic singularity (i.e., $m \neq 0$), the algorithmic singularities are precisely those configurations for which a self-motion causes no change in the arm angle ψ.

4.3. Singular Configurations of the 7-DOF Manipulator

In this section, we give conditions that correspond to the zero-offset 7-DOF arm and the Robotics Research K-1207 arm in a kinematic or algorithmic singular configuration. The singular configurations of the zero-offset arm are well understood as a result of the work reported in Wampler (1988b) and Podhorodeski et al. (1989) and the work presented here. Consequently, the results for the zero-offset arm presented here are very complete. In contrast, only a few preliminary results are available for the K-1207 arm at the present time. The singularity analysis for the K-1207 arm presented here is therefore far from definitive and is presented primarily to demonstrate that the insights obtained from the previous sections are applicable to this arm and to focus attention on the issues that remain to be solved.

Singularities of the Zero-Offset Arm

In this section, we discuss the kinematic and algorithmic singularities of the zero-offset arm.

Kinematic Singularities The zero-offset 7-DOF arm of Hollerbach (1984) is kinematically singular when the Jacobian J^{ee} loses row rank or, equivalently, when the rows of (17) become linearly dependent. Note that the Jacobian J^{ee} is also singular when three or more columns become colinear, as there cannot be a combination of six columns that are linearly independent. Singularities of J^{ee} can therefore be found by testing for row or column dependencies.

We have also shown that the null space spanning vector, n, given by Lemma 4.1, is identically zero if and only if the arm is at a kinematic singularity. An alternative way to determine kinematically singular configurations is therefore to compute n and determine the conditions for which every component of n is simultaneously zero. For the K-1207 arm, the resulting expression for n is very complex and gives a deeper insight to determine singular configurations by finding the row dependencies of (13). However, for the zero-offset arm, n is readily computed to be

$$\begin{aligned} n = d_3 d_5 S_4 \Big\{ & -d_5 C_3 S_4 S_6, d_5 S_2 S_3 S_4 S_6, \\ & S_6[S_2(d_3 + d_5 C_4) + d_5 C_2 C_3 S_4], 0, \\ & -S_2[S_6(d_5 + d_3 C_4) + d_3 S_4 C_5 C_6], \\ & -d_3 S_2 S_4 S_5 S_6, d_3 C_5 S_2 S_4 \Big\}^T \end{aligned} \tag{31}$$

It is seen that either by demanding that (31) vanishes identically, or by checking for row or column dependencies of (17), one can find the kinematically singular configurations of the zero-offset arm. From such considerations, we can systematically determine that the following configurations correspond to the zero-offset arm being singularly degenerate. By "singularly degenerate" we mean that only *one* degree of end-effector motion freedom has been lost. Each row of J^{ee} is denoted by $\mathbf{j}_i, i = 1, \ldots, 6$. For the configurations listed it is obvious that $n = 0$, and we comment only on the nature of the Jacobian row dependencies. The singularly degenerate kinematic singularities of the zero-offset arm are as follows:

- *Elbow Singularity*: $S_4 = 0$. In this case, row 6 of (17) identically vanishes.
- *Shoulder Singularity*: $S_2 = 0$ and $C_3 = 0$. In this case, rows 4 and 6 of (17) become linearly dependent:
$$\mathbf{j}_6 = -\tan\theta_4 \cdot \mathbf{j}_4$$
- *Wrist/Wrist Singularity*: $S_6 = 0$ and $C_5 = 0$. Row 6 of (17) identically vanishes.
- *Wrist/Shoulder Singularity*: $S_6 = 0$ and $S_2 = 0$. Rows 4, 5, and 6 of (17) become linearly dependent.
$$S_3 S_4 C_5 \mathbf{j}_4 + C_3 S_4 C_5 \mathbf{j}_5 + (S_3 C_4 C_5 + C_3 S_5)\mathbf{j}_6 = 0$$

The elbow, shoulder, and wrist/wrist singularities have been discussed in Wampler (1988b) and Podhorodeski et al. (1989). The wrist/shoulder singularity is discussed in Pohorodeski et al. (1989).

Note that a necessary condition for the arm to be at a wrist singularity is that $S_6 = 0$. Unlike the six-degree-of-freedom PUMA arm, the wrist singularity of the zero-offset arm is an avoidable kinematic singularity (Burdick 1988; Wampler 1988b)—just ensure that S_2 and C_5 are both non-zero. The shoulder singularity is also avoidable, whereas the elbow singularity is unavoidable (Wampler 1988b). Although it will not be discussed here, the fact that the shoulder and wrist singularities are avoidable means that the zero-offset arm can perform a shoulder or wrist pose flip without encountering a singular configuration. The fact that the elbow singularity is unavoidable means that an elbow pose flip cannot be performed without encountering the elbow singularity (Burdick 1988).

Note that columns 2, 3, and 4 of (17) are always linearly independent, so that rank $(J^{ee}) \geq 3$ for any configuration. This means that at most a singular configuration of the zero-offset arm can be triply degenerate, so that no more than three end-effector degrees of freedom can be lost at a singularity. This result is also given in Podhorodeski et al. (1989). Note that when the arm is simultaneously at the elbow singular and shoulder singular configurations, the arm is only in one singularly degenerate configuration, as row 6 of (17) vanishes, while rows 1 through 5 remain linearly independent. On the other hand, when the arm is simultaneously in the elbow and wrist/shoulder singular configurations, the arm is doubly degenerate, as row 6 vanishes and row 1 becomes linearly dependent on rows 4 and 5. Using (17) one can determine the degeneracy associated with combinations of singular configurations.

A systematic listing can be found in Podhorodeski et al. (1989), where degeneracy is analyzed using screw theory.

Algorithmic Singularities Let us now turn to the consideration of the algorithmic singularities of the zero-offset 7-DOF arm when the additional task is taken to be the arm angle defined in Section 2.2. From (30), it is evident that the arm is at an algorithmic singularity if and only if

$$(\hat{w} \times \hat{p})^T \mathbf{E}\dot{n} = (\hat{w} \times \hat{p})^T \dot{e}_0 = 0,$$

where $\dot{e}_0$ corresponds to an elbow motion for which $\dot{w} = 0$. Note that it is *not* possible that $\dot{e}_0 \neq 0$ while $(\hat{w} \times \hat{p})^T \dot{e}_0 = 0$, as a motion of the point E in the plane SEW *must* correspond to a motion of the point W (i.e., to $\dot{w} \neq 0$) in the zero-offset case. Therefore, for the zero-offset arm, an algorithmic singularity must correspond to a self-motion $\dot{n} \neq 0$ for which $\dot{e}_0 = E\dot{n} = 0$. Because $\dot{n}$ has been only defined in this article for the case when the arm is kinematically nonsingular, we will ignore configurations for which the arm is kinematically singular. Algorithmic singularities of the zero-offset arm are as follows:

- *Wrist Algorithmic Singularity*: $S_6 = 0$. In this case, joint axes $\hat{z}_5$ and $\hat{z}_7$ are colinear, and θ_5 and θ_7 are specified by a given end-effector frame only to within a constant: $\theta_5 + \theta_7 =$ constant.
- *Shoulder Algorithmic Singularity*: $S_2 = 0$. Joint axes $\hat{z}_1$ and $\hat{z}_3$ are colinear. Joint variables θ_1 and θ_3 are specified only to within a constant: $\theta_1 + \theta_3 =$ constant.

These appear to be the only algorithmic singularities for the kinematically nonsingular zero-offset arm.

Recall that the wrist/wrist singularity is an avoidable *kinematic* singularity—i.e., $S_6 = 0$, but $C_5 \neq 0$ will ensure that the wrist/wrist kinematic singularity is avoided (unlike the PUMA arm). The redundant degree of freedom of the 7-DOF arm enables the arm to move out of the wrist/wrist kinematic singularity. At the wrist/wrist kinematic singularity, $S_6 = C_5 = 0$, one can move out of $C_5 = 0$ by the self-motion for which $\theta_5 + \theta_7 =$ constant. During this self-motion $\psi =$ constant (i.e., when $S_6 = 0$, the arm angle ψ has nothing to do with the self-motion). When $S_6 = 0$, it is impossible to control ψ independently of the end-effector frame. The wrist kinematic singularity is avoidable when $S_6 = 0$ because of the available redundancy. However, if the redundancy is needed to control the additional task variable ψ, the wrist/wrist singularity returns as the wrist *algorithmic* singularity. Similarly, the shoulder kinematic singularity $S_2 = C_3 = 0$ is avoidable via a self-motion for which $\theta_1 + \theta_3 =$ constant, but $S_2 = 0$ corresponds to an algorithmic singularity for which the arm angle ψ cannot be controlled independently of end-effector coordinates. Note that when $S_4 = 0$, axes $\hat{z}_3$ and $\hat{z}_5$ are colinear so that $\theta_3 + \theta_5 =$ constant represents allowable self-motions of the arm. However, because $S_4 = 0$ corresponds to a kinematic singularity, this configuration is not listed as an algorithmic singularity. Recall also that for $S_4 = 0, \psi$ is not defined.

Singularities of the Robotics Research K-1207 Arm

We now discuss the kinematic and algorithmic singularities of the K-1207 arm.

Kinematic Singularities A complete systematic determination of the kinematically singular configurations of the K-1207 arm has yet to be performed, although it is believed that an analysis of the row- or column-dependencies of (13) will eventually result in a complete understanding of this arm. For example, it is straightforward to show that columns 2 through 4 are always linearly independent and nonzero so that rank $(J^{ee}) \geq 3$. This means that no singular configuration of the K-1207 arm can ever be more than triply degenerate. In general, because the link offsets a_i are relatively small in size, one expects to find that kinematically singular configurations of the zero-offset arm correspond to near-singular configurations of the K-1207 arm. We now list some known kinematic singularities of the K-1207 arm and some near-kinematic singularities that correspond to $\Delta = 0$ in (13b).

- *Wrist/Elbow Singularity*: $S_6 = 0$, $S_4 = 0$, and $C_4 = -1$. Columns 3, 5, 7 of expression (13b) are colinear.
- *Wrist/Shoulder Near Singularity*: $\Delta = 0$, $S_6 = 0$, $S_2 = 0$, and $C_2 = -1$. Columns 1, 3 are colinear and 5, 7 are also colinear.
- *Elbow/Shoulder Near Singularity*: $\Delta = 0$, $S_4 = 0$, $S_2 = 0$, and $C_2 = +1$. Columns 1, 3, 5 are colinear.

Algorithmic Singularities As for kinematic singularities, the determination of the algorithmic singularities of the K-1207 arm is more complicated than for the zero-offset arm. For instance, it is not clear whether simultaneously it can be true that $\dot{e}_0 = \mathbf{E}\hat{n} \neq 0$ and $(\hat{w} \times \hat{p})^T \dot{e}_0 = 0$. Certainly, though, $\dot{e}_0 = \mathbf{E}\hat{n} = 0$ is a sufficient condition for an algorithmic singularity, which easily results in the following three conditions for an algorithmic singularity.

- *Wrist Algorithmic Singularity*: $S_6 = 0$. Joint axes $\hat{z}_5$ and $\hat{z}_7$ are colinear, and $\theta_5 + \theta_7$ is specified to within a constant, $\theta_5 + \theta_7 =$ constant.
- *Elbow Algorithmic Singularity*: $S_4 = 0$. Joint axes $\hat{z}_3$ and $\hat{z}_5$ are colinear. The sum, $\theta_3 + \theta_5$, is specified only to within a constant, $\theta_3 + \theta_5 =$ constant.
- *Shoulder Near-Algorithmic Singularity*: $\Delta = 0$ and $S_2 = 0$. Joint axes $\hat{z}_1$ and $\hat{z}_3$ are colinear. The sum $\theta_1 + \theta_3$ is specified to within a constant: $\theta_1 + \theta_3 =$ constant.

The wrist algorithmic singularity occurs when $S_6 = 0$ because $a_5 = -a_6$. The elbow algorithmic singularity occurs when $S_4 = 0$ because $a_5 = -a_6$. Unlike the zero-offset arm, the K-1207 arm is kinematically nonsingular when $S_4 = 0$, although the K-1207 arm has an algorithmic singularity for $S_4 = 0$. However, when $S_4 = 0$ the arm is at a near kinematically singular configuration and may, at times, be viewed as being effectively kinematically singular. Strictly speaking, $S_2 = 0$ is an algorithmic singularity only if $\Delta = 0$ and $a_1 = -a_2$. Because $\Delta = a_1 + a_2 = 0.6$ in., which should be compared to a maximal reach in excess of 43 in., the arm is effectively algorithmically singular at $S_2 = 0$.

5. Conclusions

In this article, for the 7-DOF Robotics Research arms and the related zero-offset manipulator of Hollerbach (1984), we have derived the forward kinematic functions that map from joint space to end-effector coordinates and arm angle. We have also constructed the corresponding Jacobians and discussed the nature of the Jacobian null space spanning vector when the Jacobian is full rank. Singularity of the augmented Jacobian is discussed, and a scalar measure of algorithmic singularity is given to complement the kinematic singularity measure of Yoshikawa (1985). Algorithmic and kinematic singular configurations of the zero-offset and K-1207 arms are then obtained. The algorithmic and kinematic singularities of the zero-offset arm are characterized. Future research will focus on categorizing all singular configurations of the K-1207 arm. Future effort will also be concerned with the relationship between singularities and pose (in the sense of Burdick [1988] and Burdick and Seraji [1989]) and with the categorization of singularities into avoidable and nonavoidable singularities. Global solvability of the K-1207 arm is also an important research issue. The zero-offset arm is analytically solvable for all poses (Hollerbach 1984; Burdick 1988), whereas the K-1207 arm appears to have no analytical solution to the inverse kinematics problem.

It must be pointed out that the results of this article apply to robot arms used as "spatial manipulators" (Burdick 1988)—i.e., for the case where the basic task is the spatial placement (position and orientation) of the end effector in Cartesian space. The statements given here about the transformability and singularity of the end-effector Jacobian do not, in general, apply to common special cases, such as a planar manipulator used for positioning in the plane.

The recent work in Klein and Huang (1983), Baillieul (1985), Wampler (1988a), Baker and Wampler (1988), Shamir and Yomdin (1988), Seraji (1989), Seraji and Colbaugh (1990), and Long (1992) has extended our understanding of the capabilities of manipulator configuration control via differential means in fundamental ways. The use of pseudoinverse control results, in general, in the loss of *cyclicity* of manipulator configuration as the end effector cycles through a periodic motion. This means that as the end effector is constrained to move along a well-specified path, the arm may experience unpredictable "self-motion" wanderings. Recently, conditions for a pseudoinverse control law to result in cyclic behavior have been given (Shamir and Yomdin 1988). Furthermore, it is now believed that over a simply connected region of task space every cyclic differential control law can be realized by the so-called "inverse function" controller (Wampler 1988a; Baker and Wampler 1988) or by the use of an (inverse) augmented Jacobian control law

(Seraji 1989; Seraji and Colbaugh 1990). In particular, then, the use of the augmented Jacobian given in this article to find joint rates as a function of end-effector rates and ψ can result in cyclic motions of the K-1207 manipulator. However, cyclicity of motions derived from the augmented Jacobian approach is only guaranteed over a simply connected region of task space if the augmented Jacobian remains nonsingular over this region (Wampler 1988a; Baker and Wampler 1988). It is evident, then, that the categorization of the algorithmic and kinematic singular configurations of a manipulator is useful for planning arm motions that preserve the property of cyclicity.

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