KINEMATIC ANALYSIS OF 7 DOF ANTHROPOMORPHIC ARMS

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ABSTRACT

This paper presents a kinematic analysis of anthropomorphic seven degree-of-freedom serial link spatial manipulators with revolute joints. To uniquely determine joint angles for a given end-effector position and orientation, the redundancy is parameterized by a scalar variable which corresponds to the angle between the arm plane and a reference plane. The forward kinematic mappings from joint-space to end-effector coordinates and arm angle, and the augmented Jacobian matrix which gives end-effector and arm angle rates as functions of joint rates, are given. Conditions under which the augmented Jacobian becomes singular are given and are shown to correspond to the arm being either at a kinematically singular configuration or at a nonsingular configuration for which the arm angle ceases to parameterize the redundancy.

1. INTRODUCTION

Robot manipulators which have more joint degrees-of-freedom (DOF) than the minimum number needed to perform some tasks of interest are referred to as "redundant" to indicate the existence of the excess degrees-of-freedom. Although redundancy is obviously a task-dependent concept, manipulators with greater than six DOF are usually called redundant since the classical problem of end-effector position and orientation control for a "spatial manipulator" can be handled by a six DOF robot arm [1,2]. Redundancy can be exploited for a variety of applications — singularity avoidance, collision avoidance, enhancement of mechanical advantage, manipulability enhancement, subtask performance, and so on — which greatly increase the flexibility and use of redundant robot arms.

The Robotics Research arms form a family of commercially available 7 DOF revolute joint serial link manipulators which offer one degree of joint-space redundancy over that needed for the fundamental task of end-effector placement and orientation. These arms can be described as having a spherical-revolute-spherical (i.e., 3R-1R-3R) joint arrangement which can be described as anthropomorphic after [1]. In this paper, a scalar parameterization, ψ, is given of the redundancy. In particular, for the Robotics Research arm, ψ is defined as the "arm angle" (a natural redundancy parameter for anthropomorphic arms [1]), which is the angle between the plane passing through the arm and a reference plane. The forward kinematic mappings from joint-space to end-effector coordinates and ψ are then given. We also present the augmented Jacobian, J^A, which gives end-effector rates and $\dot{\psi}$ as a function of joint rates. Although in this paper a particular emphasis is placed on the Robotics Research arms (in particular, the model K-1207 arm) and the use of the arm angle for resolving redundancy, many of the results developed here apply equally to any arbitrary 7 DOF manipulator with the use of *any* appropriate redundancy resolving scalar parameter, and not necessarily the arm angle.

Due to the displacement of joint axes ("non-zero joint offsets"), the Robotics Research arms have no known analytic closed-form inverse kinematic solutions for specified end-effector coordinates and redundancy parameter ψ. Consequently, the importance of differential kinematics, i.e. of the Jacobian, is increased since at this time most approaches to solving the inverse kinematics, and controlling the end-effector motions, of nonsolvable arms are based on differential methods. For example, a "resolved-rate" (or "inverse Jacobian") kinematic control approach [3], based on the augmented Jacobian, can be used for general task-space control as well as to find joint angles for given end-effector coordinates and ψ. Similarly, an augmented "Jacobian-transpose" approach can be used for control [4]. Alternatively, end-effector motions can be controlled using pseudoinverse techniques [5] which are based on the use of the pseudoinverse of the end-effector Jacobian rather than on using the inverse (or transpose) of the augmented Jacobian.

In order to control the 7 DOF arm motion in Cartesian task-space while simultaneously controlling ψ, it is generally required that the augmented Jacobian remains nonsingular. The singularities of the augmented Jacobian are of two types: the kinematic singularities of the end-effector itself, and additional "algorithmic singularities" corresponding to arm configurations for which the augmented Jacobian is singular even if the end-effector is in a kinematically nonsingular configuration. Therefore, for the purposes of simultaneously controlling end-effector motions and ψ, it is desirable to find both the algorithmic and kinematic singularities for the augmented Jacobian. We give an analytic expression for an algorithmic singularity measure appropriate for the augmented Jacobian given in this paper, and present conditions for which the Robotics Research K-1207 arm and the related "zero offset" arm of [1], are algorithmically and kinematically singular — i.e., the conditions for which the augmented Jacobian becomes singular.

Because of space limitations, the discussions are necessarily brief and proofs are omitted. A complete description containing all the proofs may be found in [6].

2. FORWARD KINEMATICS

2.1. Mapping from Joint-Space to End-Effector Coordinates

The Robotics Research model K-1207 arm is a 7 DOF manipulator with nonzero offsets (denoted by the nonzero link lengths a_i, $i = 1, \cdots, 6$) at each of the joints, as shown in Figures 1-3. Denavit-Hartenberg (D-H) link frame assignments are given in accordance with the convention described in [7]. The D-H parameters for the K-1207 arm are given in Table 1. The associated interlink homogeneous transformation matrices, ${}^{i-1}T_i$, $i = 1, \cdots 7$, are easily found from the D-H parameter values listed in Table 1. If the link length parameters a_i, $i = 1, \cdots, 6$ are set to zero, the 7 DOF all-revolute anthropomorphic arm described in [1] is retrieved and we call this arm the "zero-offset" arm. Rather than construct and implement the symbolic expression, it is more efficient to numerically compute the forward kinematic function ${}^0T_7$ via a link-by-link iteration of the form

$$ {}^0T_i = {}^0T_{i-1} \cdot {}^{i-1}T_i \,, \quad i = 1, \cdots, 7 \qquad (2.1) $$

exploiting special structural properties of the homogeneous transformation matrices during each link update [8,9]. Furthermore, it is useful to explicitly have the interlink homogeneous transformations, ${}^{i-1}T_i$, since important quantities (such as the vectors w, e, and p defined later) can then be computed as by-products. In fact, such quantities are often direct by-products of the intermediate steps of the iteration (2.1).

2.2. Mapping from Joint-Space to Arm Angle

When the arm is in a kinematically nonsingular configuration, there will generally exist one excess joint degree-of-freedom for the task of end-effector control since there are seven joint angles available to position and orient the end-effector — a task which requires only six degrees-of-freedom. As a result, for a fixed end-effector frame, there is generally a one-dimensional subset of joint space (a "self-motion") which maps to this configuration. The extra degree-of-freedom represented by a self-motion can be used to attain some additional task requirement, provided that this task can be performed independently of end–effector placement [4,10]. Furthermore, the imposition of an auxillary task constraint can provide sufficient additional information to uniquely determine the joint angles (within the multiplicity of solutions represented by the pose [3,4]). This scalar additional task variable is denoted by ψ in this paper and is assumed to be a parameterization of the self-motions which map to a given end-effector frame. We say that the "basic task" of end-effector placement has been *augmented* by the "additional task" represented by ψ. In essence, the concept of the forward kinematic map is generalized to be the mapping from $\theta \in R^7$ to $({}^0T_7, \psi)$.

Although ψ can be any additional scalar parameter which is independent of end-effector frame, we define and use the "arm angle" to resolve the manipulator redundancy. Refer to Figures 1 and 2 where $S = O_1$, $E = O_4$, and $W = O_7$ denote the origins of link frames 1, 4, and 7 attached to the shoulder, elbow and wrist respectively. The arm angle ψ is defined by the angle from the reference plane containing the unit vector $\hat{V}$ and the shoulder-wrist line SW to the shoulder-elbow-wrist plane SEW in the right-hand sense about the vector $w = W - S$. With the arm angle ψ as a parameterization of manipulator redundancy, a self-motion is described by a rotation of the plane SEW about the line SW. Note that the arm angle ψ is undefined when the wrist point W is anywhere on the line through the shoulder point S and containing $\hat{V}$ — even though this is generally not a singular configuration — since in this case the reference plane is not uniquely defined. The angle ψ is also undefined when e and w are colinear since then the plane SEW is not uniquely defined. In the latter case, the arm is either nearly fully outstretched, or folded, and is therefore near or at an "elbow singular" configuration [2,11]. In such cases, ψ ceases to parameterize the redundancy.

To derive the forward kinematic function which gives ψ as a function of joint angles, again consider Figure 2. Let $w = W - S$, $e = E - S$, and let $\hat{V}$ denote an arbitrary fixed unit vector, e.g. the unit vector in the vertical direction of the base frame. Let the projection of e onto w be given by $d = \widehat{w}(\widehat{w}^T e)$, $\widehat{w} = w/\|w\|$. The minimum distance from the line SW to the point E is along the vector $p = e - d = (I - \widehat{w}\widehat{w}^T)e$. The reference plane is the plane which contains both w and the unit vector $\widehat{V}$. The unit vector in the reference plane which is orthogonal to w is given by $\widehat{\ell} = \ell/\|\ell\|$, with $\ell = (w \times \widehat{V}) \times w$. We also define the unit vector $\widehat{p} = p/\|p\|$. Note that e, w, $\widehat{w}$, d, p, $\widehat{p}$, ℓ, and $\widehat{\ell}$ can be computed during the forward kinematics iteration (2.1), as will be discussed subsequently. The vector ℓ, or equivalently $\widehat{\ell}$, is treated as a free vector which can slide along the line SW. In particular, ℓ is moved along the line SW until its base is in contact with the base of vector p at the point d (see Figure 2), so that ψ is the angle from ℓ to p. The forward kinematic function which maps the joint angles θ to the arm angle ψ is [6]:

$$ \psi = \operatorname{atan2}(\widehat{w}^T(\widehat{V} \times p), \widehat{V}^T p) \qquad (2.2) $$

For the special case of the zero-offset arm discussed in [1], corresponding to $a_i = 0, i = 1, \ldots, 6$, (2.2) reduces to

$$ \psi = \operatorname{atan2}\left(S_2 S_3 S_4 ,\ \frac{d_5 S_4}{\|w\|} \left[C_2 S_4 + S_2 C_3 (1 + C_4) \right] \right) \qquad (2.3) $$

where $\|w\| = (d_3^2 + d_5^2 + 2 d_3 d_5 C_4)^{1/2}$. Note that (2.2) is undefined when both arguments are simultaneously zero. This occurs when the arm is in a configuration for which e and w are colinear, or for which the wrist point W is directly on the line through $\hat{V}$ and containing the shoulder point S. These indeterminacies are discussed above, and are due to the inability to uniquely define the arm plane SEW or the reference plane, respectively. The augmented forward kinematics mapping, $\theta \rightarrow ({}^0T_7, \psi)$, is given by (2.1) and (2.2). The quantities $\widehat{w}$ and $p = e - d = (I - \widehat{w}\widehat{w}^T)e$ are first computed during the iteration (2.1), after which ψ is computed by (2.2).

3. DIFFERENTIAL KINEMATICS

3.1. Manipulator End-Effector Jacobian, J^{ee}

To present actual values for the end-effector Jacobian, J^{ee}, it is first necessary to choose a velocity reference point, as well as a frame in which to represent the vectorial quantities which define the columns of the Jacobian. In this section, to simplify notation, we will suppress the trailing superscript and write the end-effector Jacobian simply as $J = J^{ee}$. When a velocity reference point, a, and a representation frame, $\mathcal{F}_r$, have been chosen, we write ${}^rJ_a = {}^rJ_a^{ee}$.

Let ω_a and v_a be the angular and linear velocities of a coordinate frame, $\mathcal{F}_a$, located at a point a and fixed with respect to the manipulator end-effector. The point a is known as a "velocity reference point" of the end-effector. The Jacobian, $J_a(\theta) \in R^{6\times7}$, relates joint rates to the rate of change of frame $\mathcal{F}_a$ via the linear relationship $(\omega_a^T, v_a^T)^T = J_a(\theta)\dot{\theta}$ and is given by [2]

$$J_a = \begin{pmatrix} \widehat{z}_1 & \cdots & \widehat{z}_7 \\ \widehat{z}_1 \times P_{a,1} & \cdots & \widehat{z}_7 \times P_{a,7} \end{pmatrix} \tag{3.1}$$

In (3.1), $\widehat{z}_i$ denotes the unit vector corresponding to the z-axis of link frame i (i.e. of $\mathcal{F}_i$) while $P_{a,i} \equiv P_{a,O_i} \equiv a - O_i$ is the vector from the origin, O_i, of link frame i to the point a. Note that $P_{i,i} = 0$.

Let $\mathcal{F}_b$ denote an alternative frame fixed with respect to the end-effector and located at the velocity reference point b. The relationship between joint rates and the rate of change of $\mathcal{F}_b$ is given by $(\omega_b^T, v_b^T)^T = J_b\dot{\theta}$. Let $\mathcal{F}_r$ and $\mathcal{F}_s$ be frames which are *not* necessarily fixed with respect to the end-effector. Note that we have defined a and b to be end-effector reference points, i.e. to be fixed with respect to the end–effector, while we have placed no constraints on r and s.

The Jacobian, ${}^rJ_a$, giving the rate of change of $\mathcal{F}_a$ represented in $\mathcal{F}_r$, is related to ${}^sJ_b$, the Jacobian giving the rate of change of frame $\mathcal{F}_b$ represented in frame $\mathcal{F}_s$, by [8]

$${}^rJ_a = \begin{pmatrix} {}^rR_s & 0 \\ 0 & {}^rR_s \end{pmatrix} \cdot \begin{pmatrix} I & 0 \\ {}^s\tilde{P}_{a,b} & I \end{pmatrix} \cdot {}^sJ_b \tag{3.2}$$

where, for a 3-vector x, $\tilde{x}$ denotes the 3×3 skew symmetric matrix defined by $\tilde{x}y = x \times y$ for every $y \in R^3$ and $P_{a,b} = a - b$. When $s = b$, we write ${}^bP_a \equiv {}^bP_{a,b}$ to bring the notation of this section in line with the notation of Section 2. It is straightforward to show from (3.2) that

$$\det[{}^rJ_a(\theta)\,{}^rJ_a(\theta)^T] = \det[{}^sJ_b(\theta)\,{}^rJ_b(\theta)^T] \tag{3.3}$$

for every a, b, r, and s. Since an $m \times n$ matrix M, $m < n$, is full rank if and only if $\det[MM^T] \neq 0$, (3.3) shows that the singularity of a manipulator Jacobian is independent of the choice of velocity reference point and representation frame, and is a function purely of the manipulator configuration variables θ. This is true for spatial manipulators and is not necessarily, and usually not, true in other cases.

An important aspect of the decomposition (3.2) is that s and b can often be chosen to make the Jacobian matrix have a particularly simple structure for the purposes of singularity analysis, efficient evaluation, and efficient inversion. For example, in [9] an algorithm for the efficient computation of ${}^0J_0$ is given. Note that J_0 does *not* give the velocity of the base frame, $\mathcal{F}_0$, as a function of joint rates — indeed, in most cases the base is assumed fixed and the base frame origin, O_0, cannot be a velocity reference point for the moving end-effector. Instead, J_0 is viewed as giving the velocity of a reference frame fixed with respect to the end-effector and instantaneously coincident with the base frame origin, O_0. The computation of

$${}^0J_0 = \begin{pmatrix} {}^0\widehat{z}_1 & {}^0\widehat{z}_2 & \cdots & {}^0\widehat{z}_7 \\ {}^0\widehat{z}_1 \times {}^0P_{0,1} & {}^0\widehat{z}_2 \times {}^0P_{0,2} & \cdots & {}^0\widehat{z}_7 \times {}^0P_{0,2} \end{pmatrix} \tag{3.4}$$

where ${}^kP_{i,j} = {}^{O_k}P_{i,j}$ and $P_{i,j} = P_{O_i,O_j} = O_j - O_i$, naturally fits in with the forward kinematics iteration (2.1), since from

$${}^0T_i = \begin{pmatrix} {}^0R_i & {}^0P_i \\ 0^T & 1 \end{pmatrix}$$

${}^0P_i \equiv {}^0P_{i,0}$, we can obtain ${}^0P_{0,i} = -{}^0P_i$ and ${}^0\widehat{z}_i \times {}^0P_{0,i}$ where ${}^0\widehat{z}_i = {}^0R_ie_3$, $e_3 = (0,0,1)^T$. Having ${}^0J_0$, ${}^0J_7$ can then be found from

$${}^0J_7 = \begin{pmatrix} I & 0 \\ {}^0\tilde{P}_7 & I \end{pmatrix} \cdot {}^0J_0 \tag{3.5}$$

Note that (3.5) is just a special case of (3.2). The symbolic forms of ${}^0J_0$ and ${}^0J_7$ have been found from this procedure, but these expressions are complex and provide little insight.

In [8], the results in [9] are extended to show that taking $s = O_i$ and $b = O_j$ for an appropriate choice of link frames i and j can result in an expression ${}^iJ_j = {}^{O_i}J_{O_j} = {}^sJ_b$ which is not only efficient to compute, but which simplifies singularity analysis and (for nonredundant manipulators) Jacobian inversion. In particular, to gain insight into the singularity structure of the K-1207 end-effector Jacobian and to obtain an alternative way to construct ${}^0J_7$ we will let $b = 3$ (i.e., let the velocity reference point be the origin of link frame 3) and $s = 3$ (let the reference frame be link frame 3). J_3 should be interpreted as giving the velocity of a fictitious tool frame which is instantaneously coincident with link frame 3. ${}^3J_3$ is found from (3.1) by taking $P_{a,i} = P_{3,i} = P_{O_3,O_i} = O_3 - O_i$ and representing $\widehat{z}_i$ and $P_{3,i}$ in link frame 3 to obtain ${}^3\widehat{z}_i$ and ${}^3\widehat{z}_i \times {}^3P_{3,i}$, ${}^3P_{3,i} = {}^{O_3}P_{O_3,O_i}$. The symbolic expression for ${}^3J_3$ found in this manner is given by

$$
{}^3J_3 = \begin{pmatrix}
-S_2C_3 & S_3 & 0 & 0 & S_4 & -C_4S_5 & C_4C_5S_6 + S_4C_6 \\
S_2S_3 & C_3 & 0 & 1 & 0 & C_5 & S_5S_6 \\
C_2 & 0 & 1 & 0 & C_4 & S_4S_5 & C_4C_6 - S_4C_5S_6 \\
d_3S_2S_3 + (a_2C_2 + a_1)S_3 & d_3C_3 & 0 & 0 & 0 & S_4(a_4C_5 + a_5) - d_5C_4C_5 & S_5[C_4(a_5C_6 - d_5s_6 + a_6) + a_4S_4S_6] \\
(d_3S_2 + a_2C_2 + a_1)C_3 & -d_3S_3 & 0 & 0 & -a_3C_4 - a_4 & -S_5[a_3S_4 + d_5] & C_5[S_6(a_3S_4 + d_5) - a_6] - (a_5C_5 + a_4 + a_3C_4)C_6 \\
0 & -a_2 & 0 & a_3 & 0 & C_4(a_4C_5 + a_5) + C_5(d_5S_4 + a_3) & S_5[(a_4C_4S_6 + a_3S_6) + S_4(d_5S_6 - a_5C_6 - a_6)]
\end{pmatrix} \tag{3.6}
$$

Having ${}^3J_3$, ${}^0J_7$ is found from (3.2) as

$$
{}^0J_7 = \begin{pmatrix} {}^0R_3 & 0 \\ 0 & {}^0R_3 \end{pmatrix} \cdot \begin{pmatrix} I & 0 \\ {}^3\tilde{P}_{7,3} & I \end{pmatrix} \cdot {}^3J_3 \tag{3.7}
$$

where ${}^3P_{7,3}$ is given by ${}^3P_{7,3} = {}^3P_4 + {}^3R_4 \cdot {}^4P_5 + {}^3R_4 \cdot {}^4R_5 \cdot {}^5P_6 + {}^3R_4 \cdot {}^4R_5 \cdot {}^5R_6 \cdot {}^6P_7$, or alternatively, ${}^3P_{7,3} = {}^0R_3^T({}^0P_{7,0} - {}^0P_{3,0})$. The relative simplicity of (3.6) not only provides an alternative to (3.1) to efficiently compute ${}^0J_7$ via (3.6)—(3.7), but also allows one to gain insight into conditions leading to Jacobian singularity. Note that for the special case of the zero-offset arm discussed in [1], corresponding to $a_1 = \cdots = a_6 = 0$, (3.6) simplifies to

$$
{}^3J_3 = \begin{pmatrix}
-S_2C_3 & S_3 & 0 & 0 & S_4 & -C_4S_5 & C_4C_5S_6 + S_4C_6 \\
S_2S_3 & C_3 & 0 & 1 & 0 & C_5 & S_5S_6 \\
C_2 & 0 & 1 & 0 & C_4 & S_4S_5 & C_4C_6 - S_4C_5S_6 \\
d_3S_2S_3 & d_3C_3 & 0 & 0 & 0 & -d_5C_4C_5 & -d_5C_4S_5S_6 \\
d_3S_2C_3 & -d_3S_3 & 0 & 0 & 0 & -d_5S_5 & d_5C_5S_6 \\
0 & 0 & 0 & 0 & 0 & d_5S_4C_5 & d_5S_4S_5S_6
\end{pmatrix} \tag{3.8}
$$

3.2. Arm Angle Jacobian, J^ψ and Augmented Jacobian, J^A

Let the relationship between the rate of change of *any* scalar additional task variable, ψ, and the joint rates be given by $\dot{\psi} = J^\psi \dot{\theta}$. The "augmented" Jacobian is given by [4]

$$
J^A = \begin{pmatrix} J^{ee} \\ J^\psi \end{pmatrix}
$$

where J^{ee} is the end-effector Jacobian discussed in Section 3.1. For the task of positioning and orienting the end-effector *augmented* by an additional task represented by ψ, the augmented Jacobian relates joint rates to the simultaneous rates of change of the end-effector coordinates and ψ. Having the end-effector Jacobian, J^{ee}, the augmented Jacobian J^A is obtained once J^ψ has been determined for a given task variable ψ. In this section, J^ψ is constructed for the case where ψ describes the arm angle between a reference plane and the elbow plane SEW as defined in Section 2.

Before proceeding further, it is necessary to define the Jacobians $\mathbf{E}$ and $\mathbf{W}$ which relate joint rates to $\dot{e}$ and $\dot{w}$ respectively via $\dot{e} = \mathbf{E}\dot{\theta}$ and $\dot{w} = \mathbf{W}\dot{\theta}$, where e and w are defined in Section 2.2. Note that $\dot{e}$ is the linear velocity of the manipulator elbow point $E = O_4$ and $\dot{w}$ is the linear velocity of the wrist point $W = O_7$. We have

$$
\mathbf{E} = (\,\widehat{z}_1 \times P_{4,1}\,,\ \ \widehat{z}_2 \times P_{4,2}\,,\ \ \widehat{z}_3 \times P_{4,3}\,,\ \ 0,\ \ \cdots,\ \ 0\,) \tag{3.9}
$$

$$
\mathbf{W} = (\,\widehat{z}_1 \times P_{7,1}\,,\ \ \cdots,\ \ \widehat{z}_6 \times P_{7,6}\,,\ \ 0\,) \tag{3.10}
$$

where $P_{i,j} = O_j - O_i$. Note that (3.9) and (3.10) are given in coordinate-free form and that to provide values for $\mathbf{E}$, or $\mathbf{W}$, a choice of reference frame for representing $\widehat{z}_j$ and $P_{i,j}$ must be made.

Proposition 3.1: The relationship between $\dot{\theta}$ and $\dot{\psi}$, where ψ is the arm angle as defined in Section 2.2, is given by

$$
\dot{\psi} = \frac{1}{\|p\|}(\hat{w} \times \hat{p})^T\dot{p} - \frac{1}{\|\ell\|}(\hat{w} \times \hat{\ell})^T\dot{\ell} \tag{3.11a}
$$

$$
= \frac{(\hat{w} \times \hat{p})^T}{\|p\|}\left\{\mathbf{E} - \frac{\hat{w}^Te}{\|w\|}\mathbf{W}\right\}\dot{\theta} + \frac{\hat{V}^Tw}{\|\ell\|}(\hat{w} \times \hat{\ell})^T\mathbf{W}\dot{\theta} \tag{3.11b}
$$

which results in

$$
J^\psi = \frac{(\widehat{w} \times \widehat{p})^T}{\|p\|}\mathbf{E} + \left\{\frac{\widehat{V}^Tw}{\|\ell\|}(\widehat{w} \times \widehat{\ell})^T - \frac{\widehat{w}^Te}{\|w\|\,\|p\|}(\widehat{w} \times \widehat{p})^T\right\}\mathbf{W} \tag{3.12}
$$

Since the arm angle ψ is given by the angle from ℓ to p, it is natural that $\dot{\psi}$ should depend only on $\dot{\ell}$ and $\dot{p}$ as in (3.11a). Equation (3.11a) says that only the components of $\dot{\ell}$ and $\dot{p}$ which result in an instantaneous motion of ℓ and p directly towards or away from each other can produce a change in the arm angle ψ. Note that J^ψ is independent of the reference frame chosen to represent the quantities in the right-hand side of (3.12).

4. SINGULARITIES OF THE AUGMENTED JACOBIAN, J^A

In Section 4.1, we consider the general case of an *arbitrary* seven degree-of-freedom arm with an *arbitrary* scalar redundancy variable ψ. In Section 4.2, the case where ψ parameterizes the arm angle in the sense of Section 2.2 is discussed. Section 4.3 discusses the singular configurations of the 7 DOF K-1207 and zero-offset arms.

4.1 Singularity Measure for the Augmented Jacobian

The main results of this section are Propositions (4.1) and (4.2). Proposition (4.1) gives a condition for singularity of the augmented Jacobian which depends on the values of a scalar measure of kinematic singularity and of a scalar measure of algorithmic singularity. Proposition (4.2) gives an explicit expression for the inverse of the augmented Jacobian. The results in this subsection hold for a general scalar ψ, not just for the case where ψ is the arm angle, and for any 7 DOF arm, not just all-revolute arms.

In [13], a measure of "nearness" of a redundant manipulator to kinematic singularity is defined by

$$
m(\theta) = \left[\det\left(J^{ee}(\theta)J^{ee\,T}(\theta)\right)\right]^{\frac{1}{2}} \tag{4.1}
$$

where $m(\theta)$ is known as "manipulability measure." For $J^{ee} \in R^{6\times 7}$, $m(\theta) \neq 0$ if and only if $\text{rank}(J^{ee}) = 6$, which is true if and only if the arm is not at a kinematic singularity. When the arm is at a kinematic singularity, there is at least one direction in task-space along which the end-effector cannot move instantaneously. The number of independent directions in task-space along which the arm cannot instantaneously move is $6 - \text{rank}(J^{ee})$.

For $J^{ee} \in R^{6\times 7}$ full rank, i.e. $\text{rank}(J^{ee}) = 6$, it is true that $\dim\mathcal{N}(J^{ee}) = 1$, where $\mathcal{N}(J^{ee})$ denotes the null space of J^{ee}. J^{ee} full rank also means that its six rows are linearly independent and that some combination of six of the columns of J^{ee} are linearly independent. The 6×6 matrix formed by the six linearly independent columns must be nondegenerate and have a nonvanishing determinant.

With $J^{ee} \in R^{6\times 7}$ full rank, there exists a unit vector $\hat{n} \in R^7$ such that $J^{ee}\hat{n} = 0$. The vector $\hat{n}$ spans the one-dimensional nullspace $\mathcal{N}(J^{ee})$ and gives the instantaneous direction along which one can move in joint-space without causing any end-effector motion. That is, $\hat{n}$ gives the locally allowable self-motion of the arm. Moving the kinematically nonsingular arm with a joint-space velocity $\dot{\theta} = \alpha\cdot\hat{n}$, for scalar α, will always result in arm motions for which the end-effector is fixed. Generally, the most efficient way to obtain $\hat{n}$ is to solve the equation $J^{ee}\hat{n} = 0$ via an iterative numerical technique such as Gaussian-Elimination. It is possible to get an analytic expression for the null space spanning vector of a nonsingular $J^{ee} \in R^{6\times 7}$, however, in general, the resulting expression is generally too complex to be computed easily [6]. For the zero offset arm a non-unit null space vector is shown to be given by

$$n = d_3d_5S_4\{-d_5C_3S_4S_6, d_5S_2S_3S_4S_6, S_6[S_2(d_3+d_5C_4)+d_5C_2C_3S_4],\\ 0, -S_2[S_6(d_5+d_3C_4)+d_3S_4C_5C_6], -d_3S_2S_4S_5S_6, d_3C_5S_2S_4\}^T$$

Let us now turn to the Jacobian J^{ψ} for the additional task. The vector J^{ψ} can be written in the form

$$J^{\psi} = c_0\hat{n}^T + c_1^T J^{ee} \tag{4.2}$$

for some scalar c_0 and $c_1 \in R^6$.

Lemma: Let $J^{ee} \in R^{6\times 7}$ be full rank, then $J^{\psi} = c_0\hat{n}^T + c_1^T J^{ee}$ with

$$c_0 = J^{\psi}\hat{n} \tag{4.3a}$$
$$c_1^T = J^{\psi}J^{ee+} \tag{4.3b}$$

where $\hat{n} \in R^7$ is a unit vector which spans the null-space $\mathcal{N}(J^{ee})$ and J^{ee+} is the pseudo-inverse $J^{ee+} = J^{eeT}(J^{ee}\cdot J^{eeT})^{-1}$.

Proposition 4.1: For $J^A = \binom{J^{ee}}{J^{\psi}}$, we have

$$\det J^A = \pm c_0 \cdot m \tag{4.4}$$

where m is the manipulability measure (4.1).

The scalar $|c_0|$ is a measure of the nearness to "algorithmic singularity" of the arm. Equation (4.4) says that any arm with one degree-of-redundancy is singular if and only if $c_0 = 0$ or $m = 0$. The scalar m is a measure of the kinematic singularity and vanishes if and only if the arm is at a kinematic singularity. The scalar $|c_0| = |J_{\psi}\hat{n}|$ (see (4.3)) is zero when the arm is at the "algorithmic singularities" of J^{ψ}, a result originally derived in [14]. Note that when $c_0 = 0$

$$\dot{\psi} = J^{\psi}\dot{\theta} = c_0\hat{n}^T\dot{\theta} + c_1^T J^{ee}\dot{\theta} = c_1^T J^{ee}\dot{\theta}$$

which shows that $\dot{\psi}$ does not have any dependency upon the self-motion vector $\hat{n}$ and is coupled only to the motion of the basic task given by $J^{ee}\dot{\theta}$. Obviously, then, ψ cannot be controlled independently of the basic task when $c_0 = 0$. The requirement that both c_0 and m be nonzero for J^A to be nonsingular can be implicitly seen in the analytic expression for $(J^A)^{-1}$ given in the following proposition which is a generalization of a result in [14].

Proposition 4.2: For $J^A = \binom{J^{ee}}{J^{\psi}}$, we have

$$\begin{aligned}(J^A)^{-1} &= \left[\left(I - \frac{\hat{n}J^{\psi}}{c_0}\right)J^{ee+} \ \vdots\ \frac{\hat{n}}{c_0}\right]\\ &= \left[J^{ee+} - \frac{\hat{n}c_1^T}{c_0} \ \vdots\ \frac{\hat{n}}{c_0}\right]\end{aligned} \tag{4.5}$$

Obviously for $J^{A^{-1}}$ to exist we must have $c_0 \neq 0$ and $(J^{ee}J^{ee^T})$ invertible, the latter condition being true if and only if $m \neq 0$.

Note that, using (4.5), $\dot{\theta} = J^{A^{-1}}\binom{\dot{x}}{\dot{\psi}}$ is equivalent to

$$\dot{\theta} = J^{ee+}\dot{x} + \hat{n}\xi$$
$$\xi = \frac{1}{c_0}(\dot{\psi} - c_1^T\dot{x})$$

showing that the inverse augmented Jacobian kinematic control can be formulated as a form of pseudoinverse control. The inverse augmented Jacobian control has the special property that it results in cyclic behavior over simply-connected task-space regions for which J^A is full rank [4,15]. In general, pseudoinverse control does not yield cyclic behavior over simply-connected regions [15-18].

To independently control end-effector coordinates simultaneously with ψ, it is clear that it is necessary and sufficient that c_0 and m both be nonzero, in which case the arm can also be controlled in a cyclic manner. It is therefore important to determine the configurations which correspond to $m = 0$ and $c_0 = 0$.

4.2 Arm Angle Algorithmic Singularity Measure

In Section 4.1, results are derived which are applicable to any 7 DOF spatial manipulator with any associated scalar additional task variable ψ. Here, we will obtain the form of c_0 and c_1 for the special case where the additional task is to control the arm angle defined in Section 2.2. Aside from assuming one degree-of-redundancy (i.e. 7 DOF), the spatial manipulator is otherwise unspecified. As noted in Section 4.1, $|c_0|$ provides a measure of nearness to algorithmic singularity.

Proposition 4.3: Let ψ denote the arm angle defined in Section 2. Then, $J_\psi = c_0\hat{n} + c_1^T J^{ee}$ where

$$c_0 = \frac{(\hat{w} \times \hat{p})^T \mathbf{E}\hat{n}}{||p||} \qquad -4.6a)$$

$$c_1 = \left\{ \frac{(\hat{w} \times \hat{p})^T}{||p||}\mathbf{E} + \left(\frac{\hat{V}^T w}{||p||}(\hat{w} \times \hat{\ell})^T - \frac{\hat{w}^T e}{||w||\,||p||}(\hat{w} \times \hat{p}^T) \right) \mathbf{W} \right\} J^{ee+} \qquad -4.6b)$$

Note that there are two possible ways that c_0 as given by (4.6a) can vanish. First, it may be that $\mathbf{E}\hat{n} = 0$, so that a self-motion causes no motion of the elbow point at all. Secondly, it may be that $\mathbf{E}\hat{n} \neq 0$, but the resulting elbow motion is entirely in the plane SEW (see Fig. 1) so that $(\hat{w} \times \hat{p})^T \mathbf{E}\hat{n} = 0$. In either case, it is apparent that a self-motion has nothing to do with a change of the arm angle ψ. Assuming that the arm is not at a kinematic singularity (i.e., $m \neq 0$) the algorithmic singularities are precisely those configurations for which a self-motion causes no change in the arm angle ψ.

4.3 Singular Configurations of the 7 DOF Manipulator

With the above tools in hand, one can proceed to systematically find and categorize the kinematic and algorithmic singularities of the K-1207 and zero-offset arms. In reference [6], a detailed discussion is given of all singularities of the zero-offset arm and a partial listing is given of the K-1207 singularities. Kinematic singularities are found by determining row and column dependencies of (3.6) and (3.8). Algorithmic singularities occur when the conditions for $c_0 = 0$ are satisfied. A complete listing of singular configurations of the zero-offset arm can also be found in [19].

Kinematic Singularities of the Zero-Offset Arm:

- **Elbow Singularity:** $S_4 = 0$. In this case, row 6 of (3.8) identically vanishes.
- **Shoulder Singularity:** $S_2 = 0$ and $C_3 = 0$. In this case, rows 4 and 6 of (3.8) become linearly dependent:

$$\mathbf{j}_6 = -\tan\theta_4 \cdot \mathbf{j}_4$$

- **Wrist/wrist Singularity:** $S_6 = 0$ and $C_5 = 0$. Row 6 of (3.8) identically vanishes.
- **Wrist/shoulder Singularity:** $S_6 = 0$ and $S_2 = 0$. Rows 4, 5, and 6 of (3.8) become linearly dependent.

$$S_3S_4C_5\mathbf{j}_4 + C_3S_4C_5\mathbf{j}_5 + (S_3C_4C_5 + C_3S_5)\mathbf{j}_6 = 0$$

Algorithmic Singularities of the Zero-Offset Arm:

- **Wrist Algorithmic Singularity:** $S_6 = 0$. In this case, joint axes $\hat{z}_5$ and $\hat{z}_7$ are colinear and $\theta_5$and θ_7 are specified by a given end-effector frame only to within a constant: $\theta_5 + \theta_7 =$ constant.
- **Shoulder Algorithmic Singularity:** $S_2 = 0$. Joint axes $\hat{z}_1$ and $\hat{z}_3$ are colinear. Joint variables θ_1 and θ_3 are specified only to within a constant: $\theta_1 + \theta_3 =$ constant.

These appear to be the only algorithmic singularities for the kinematically nonsingular zero-offset arm.

We now list some known kinematic singularities of the K-1207 arm and some near kinematic singularities which correspond to $a_1 + a_2 = 0$ in (3.6).

Kinematic Singularities of the K-1207 Arm

- **Wrist/elbow singularity:** $S_6 = 0$, $S_4 = 0$, and $C_4 = +1$. Columns 3, 5, 7 of expression (3.6) are colinear.
- **Wrist/shoulder near singularity:** $a_1 + a_2 = 0$, $S_6 = 0$, $S_2 = 0$, and $C_2 = +1$. Columns 1, 3 are colinear and 5, 7 are also colinear.
- **Elbow/shoulder near singularity:** $a_1 + a_2 = 0$, $S_4 = 0$, $S_2 = 0$, and $C_2 = +1$. Columns 1, 3, 5 are colinear.

Algorithmic Singularities of the K-1207:

- **Wrist Algorithmic Singularity:** $S_6 = 0$. Joint axes $\hat{z}_5$ and $\hat{z}_7$ are colinear and $\theta_5 + \theta_7$ is specified to within a constant, $\theta_5 + \theta_7 =$ constant.
- **Elbow Algorithmic Singularity:** $S_4 = 0$. Joint axes $\hat{z}_3$ and $\hat{z}_5$ are colinear. The sum, $\theta_3 + \theta_5$, is specified only to within a constant, $\theta_3 + \theta_5 =$ constant.
- **Shoulder Near-Algorithmic Singularity:** $a_1 + a_2 = 0$ and $S_2 = 0$. Joint axes $\hat{z}_2$ and $\hat{z}_3$ are colinear. The sum $\theta_3 + \theta_5$ is specified to within a constant: $\theta_3 + \theta_5 =$ constant.

5. CONCLUSIONS

In this paper, for the 7 DOF Robotics Research arms and the related zero-offset manipulator of [1], we have derived the forward kinematic functions which map from joint-space to end-effector coordinates and arm angle. We have also constructed the corresponding Jacobians and discussed the nature of the Jacobian null-space spanning vector when the Jacobian is full rank. Singularity of the augmented Jacobian is discussed and a scalar measure of algorithmic singularity is given to complement the kinematic singularity measure of [13]. Algorithmic and kinematic singular configurations of the zero-offset and K-1207 arms can then be obtained. For instance, the algorithmic and kinematic singularities of the zero-offset arm are given in [21]. Future research will focus on categorizing all singular configurations of the K-1207 arm. Future effort will also be concerned with the relationship between singularities and pose (in the sense of [2]) and with the categorization of singularities into avoidable and nonavoidable singularities. Global solvability of the K-1207 arm is also an important research issue. The zero-offset arm is analytically solvable for all poses [1,2], while the K-1207 arm appears to have no analytic solution to the inverse kinematics problem.

It must be pointed out that the results of this paper apply to robot arms used as "spatial manipulators" [2], i.e., for the case where the basic task is the spatial placement (position and orintation) of the end-effector in Cartesian space. The statements given here about the transformability and singularity of the end-effector Jacobian do not, in general, apply to common special cases such as a planar manipulator used for positioning in the plane.

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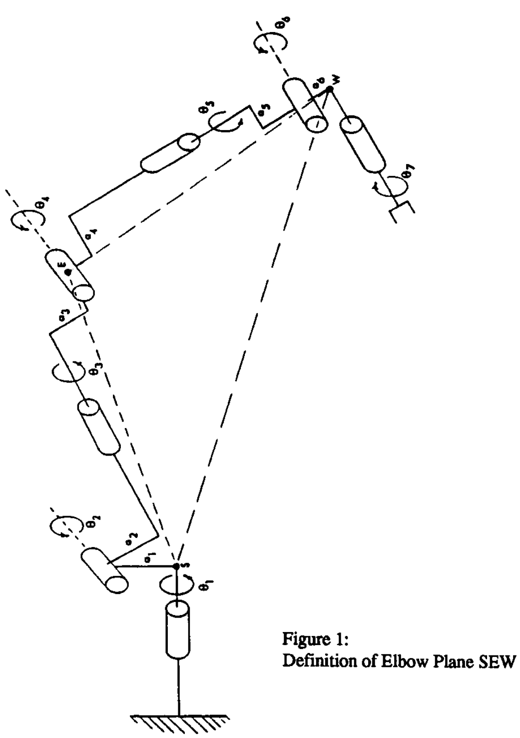

Figure 1:
Definition of Elbow Plane SEW

i	α_{i-1}	a_{i-1}	d_i	θ_i	
1	0°	0	0	θ_1	$a_1 = +4.850$ in $= +12.319$ cm
2	-90°	a_1	0	θ_2	$a_2 = -4.250$ in $= -10.795$ cm
3	+90°	a_2	d_3	θ_3	$a_3 = -3.125$ in $= -\ 7.938$ cm
4	-90°	a_3	0	θ_4	$a_4 = +3.125$ in $= +\ 7.938$ cm
5	+90°	a_4	d_5	θ_5	$a_5 = -1.937$ in $= -\ 4.920$ cm
6	-90°	a_5	0	θ_6	$a_6 = +1.937$ in $= +\ 4.920$ cm
7	+90°	a_6	0	θ_7	$d_3 = d_5 = 21.5$ in $= 54.61$ cm

TABLE 1: Denavit-Hartenberg Parameters of the K-1207 Arm

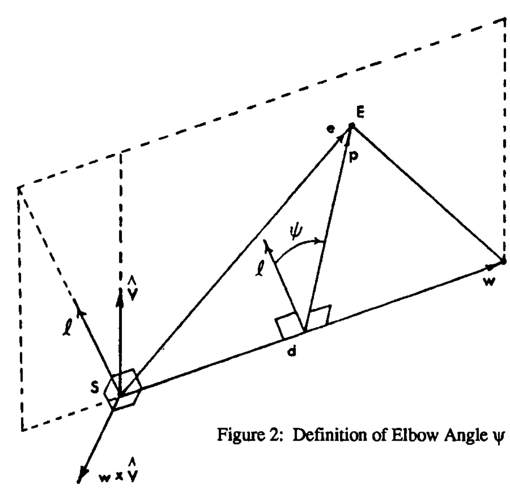

Figure 2: Definition of Elbow Angle ψ