Chap. 1: Mathematical description of linear systems and their properties

University of Ulsan

Prof. KIM

Sections

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3 Properties of linear systems

- This section introduces the two main properties of linear systems: Superposition and Causality.
- Before discussing these properties, it is necessary to note that the system output $y(t)$ depends on two factors:
  - the initial condition $x(0)$
  - the input signal $u(t)$.
- This can be expressed as the following equation:

$$\begin{aligned} y(t) &= y(x(0), u(t)) \\ &= y(x(0), 0) + y(0, u(t)), \ t \geq 0. \end{aligned} \tag{7}$$

- Furthermore, the output $y(x(0), u(t))$ can be decomposed into the zero-input response and the zero-state response.

- As a simple example, let us consider the following first-order linear system with the initial condition $x(0)$:

$$\begin{aligned}\dot{x}(t) &= ax(t) + bu(t)\\ y(t) &= cx(t).\end{aligned}$$

Then, applying the Laplace transform and its inverse, we can obtain the following output response:

$$y(t) = \underbrace{ce^{at}x(0)}_{:\ \text{zero-input response}} + \underbrace{cb\left(e^{at} * u(t)\right)}_{:\ \text{zero-state response}}.$$

- As shown here, the output $y(t)$ is affected by the initial condition $x(0)$ and the input $u(t)$.
- In addition, as mentioned above, the output $y(t)$ can be decomposed into two types of responses.

Returning to the main point, linear systems satisfy the principle of superposition.

Property 1 (Superposition). Let us suppose that the initial condition and the input are given as follows:

$$
\begin{aligned}
x(0) &= \alpha x^{(1)}(0) + \beta x^{(2)}(0) \\
u(t) &= \gamma u^{(1)}(t) + \delta u^{(2)}(t).
\end{aligned}
$$

Then the output is described as follows:

$$
\begin{aligned}
y(x(0), u(t)) &= y\left(\alpha x^{(1)}(0) + \beta x^{(2)}(0), \gamma u^{(1)}(t) + \delta u^{(2)}(t)\right) \\
&= \alpha y\left(x^{(1)}(0), 0\right) + \beta y\left(x^{(2)}(0), 0\right) \\
&\quad + \gamma y\left(0, u^{(1)}(t)\right) + \delta y\left(0, u^{(2)}(t)\right), \ t \geq 0.
\end{aligned}
$$

Example 3.1. Let us consider a linear system with the following output responses:

input	initial condition	output
$\sin(t)$	$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$	$y_1(t)$
$\cos(t)$	$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$	$y_2(t)$
0	$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$	$y_3(t)$
0	$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$	$y_4(t)$

Find $y(t)$ in terms of $y_1(t)$, $y_2(t)$, $y_3(t)$, and $y_4(t)$ when

$$x(0) = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}, \ u(t) = A\cos(t + \phi).$$

Next, linear systems satisfy the following causality.

Property 2 (Causality). Let us suppose that two identical linear systems are influenced by two inputs $u(t)$ and $\hat{u}(t)$, respectively, and assume that

$$u(t) = \hat{u}(t), \ \forall\ 0 \leq t < T.$$

Then, the output responses satisfy

$$y(x(0), u(t)) = y(x(0), \hat{u}(t)), \ \forall\ 0 \leq t < T.$$

In other words, the causality means that the output $y(t)$ depends only on the post values of the input and not on its future behavior.

4 Linearization

- Linearization is a method of converting a nonlinear system into a linear system based on an equilibrium point or a limited operating region.
- This section presents the linearization process for building linear state-space models for two cases.
  - Sec. 4.1.: for nonlinear systems without control input
  - Sec. 4.2.: for nonlinear systems with control input

Definition 1. Given

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \; y(x) = \begin{bmatrix} y_1(x) \\ \vdots \\ y_m(x) \end{bmatrix}$$

the derivative of y with respect to x is given by

$$\frac{dy}{dx} := \begin{bmatrix} \frac{dy_1}{dx_1} & \cdots & \frac{dy_1}{dx_n} \\ \vdots & \frac{dy_i}{dx_j} & \vdots \\ \frac{dy_m}{dx_1} & \cdots & \frac{dy_m}{dx_n} \end{bmatrix}.$$

As shown here, the (i, j)th element is given by $\frac{dy_i}{dx_j}$.

4.1 For nonlinear systems without control input

- Our goal is to obtain a linearized state-space model from a nonlinear system without input.
- The first thing to do is to convert the given nonlinear system into the nonlinear state-space form:

$$\dot{x}(t) = f(x(t)). \tag{8}$$

- In (8), $x(t) \in \mathbb{R}^n$ denotes the state and $f(x(t))$ denotes the nonlinear function vector:

$$f(x(t)) = \begin{bmatrix} f_1(x(t)) \\ \vdots \\ f_n(x(t)) \end{bmatrix} \in \mathbb{R}^n.$$

- As shown here, the nonlinear function $f_i(x(t))$ does not depend on the input $u(t)$.
- Furthermore, to linearize (8), the following assumption must be satisfied: the nonlinear function $f_i(x)$ has $\mathcal{C}^1$ continuity and it is possible to set the equilibrium point of (8) as $x = \phi$.

The following provides the definition of $\mathcal{C}^1$ continuity.

Definition 2. A function $f_i(x)$ is said to be $\mathcal{C}^1$ continuous at a point $x = a$ if the following conditions are met:

1) (Smoothness) The function $f_i(x)$ is continuous at $x = a$.
2) (Differentiability) The derivative $f_i(x)$ with respect to x exists at $x = a$.

- Based on the above assumption, we can linearize the nonlinear system in (8) using the following steps:
  - Choose an equilibrium point ϕ such that $\lim_{t\to\infty} x(t) = \phi$
  - Define the new state vector as $z(t) = x(t) - \phi$
  - Obtain the following system matrix:

$$A = \frac{\partial f}{\partial x}(x)\bigg|_{x\,=\,\phi} = \left[\begin{array}{ccc} \frac{\partial f_1}{\partial x_1}(x) & \cdots & \frac{\partial f_1}{\partial x_n}(x) \\ \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_1}(x) & \cdots & \frac{\partial f_n}{\partial x_n}(x) \end{array}\right]\Bigg|_{x\,=\,\phi} .$$

- Then, the linearized state-space model is given by

$$\dot{z}(t) = Az(t).$$

Remark 1. In general, $\dfrac{\partial f}{\partial x}(x)$ is called the Jacobian matrix of $f(x)$ with respect to x.

Remark 2. If the equilibrium point $\phi = 0 \in \mathbb{R}^n$, then $z(t) = x(t)$. Accordingly, the linearlized state-space model is given as

$$\dot{x}(t) = Ax(t).$$

Example 4.1. Let us consider a simple pendulum described as

$$\ddot{\theta}(t) + k \sin \theta(t) = 0$$

where $k > 0$ is a constant. For this nonlinear system, construct a linearized state-space model such that $\theta(t)$ converges to 0 as time goes to infinity.

Sol.:

$$\dot{z}(t) = \begin{bmatrix} 0 & 1 \\ -k & 0 \end{bmatrix} z(t).$$

4.2 For nonlinear systems with control input

- Let us consider the following nonlinear state-space model with control input $u(t)$:

$$\dot{x}(t) = f(x(t), u(t)) = \begin{bmatrix} f_1(x(t), u(t)) \\ \vdots \\ f_n(x(t), u(t)) \end{bmatrix} \in \mathbb{R}^n. \tag{10}$$

- As shown here, the nonlinear function depends on the state $x(t)$ and the input $u(t)$.
- In this case as well, it should be assumed that the nonlinear function $f_i(x, u)$ has $\mathcal{C}^1$ continuity and $(x, u) = (\phi, \psi)$ is an equilibrium point of (10).

- Then, nonlinear system (10) can be linearized at (ϕ, ψ) using the following steps:
  - Choose an equilibrium point (ϕ, ψ) such that $\lim_{t\to\infty} x(t) = \phi$ and $\lim_{t\to\infty} u(t) = \psi$.
  - Define the new state and input vectors as

$$z(t) = x(t) - \phi,\ v(t) = u(t) - \psi$$

  - Obtain the following system matrics:

$$A = \frac{\partial f}{\partial x}(x,u)\bigg|_{x\,=\,\phi,\,u\,=\,\psi}$$
$$B = \frac{\partial f}{\partial u}(x,u)\bigg|_{x\,=\,\phi,\,u\,=\,\psi}.$$

- Then, the linearized state-space model is given by

$$\dot{z}(t) = Az(t) + Bv(t).$$

Example 4.2. Consider the following nonlinear system:

$$\ddot{y} = -y + y^3 + u.$$

In this system, y and u converge to 1 and 0, respectively, as time goes to infinity.

a) Obtain the nonlinear state-space representation

b) Find the linearized state-space model.

Sol.:

$$A = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Example 4.3. Let us consider the following nonlinear system:

$$\ddot{r}(t) = r(t)\dot{\theta}^2(t) - \frac{k}{r^2(t)} + u_1(t)$$

$$\ddot{\theta}(t) = -2\frac{\dot{\theta}(t)\dot{r}(t)}{r(t)} + \frac{1}{r(t)}u_2(t).$$

In this system, $r(t)$ and $\dot{\theta}(t)$ converge to constant r_0 and constant $w_0 = \sqrt{\frac{k}{r_0^3}}$, respectively, as time goes to infinity. Consturct A and B.

Sol.:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 3w_0^2 & 0 & 0 & 2r_0w_0 \\ 0 & 0 & 0 & 1 \\ 0 & -\frac{2w_0}{r_0} & 0 & 0 \end{bmatrix}, \; B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & \frac{1}{r_0} \end{bmatrix}.$$