Stability analysis of linear systems

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1 Stability analysis

- The state-space model of LTI systems is given by

$$\dot{x}(t) = Ax(t) + Bu(t). \tag{1}$$

- In control theory, a system is referred to as an **autonomous** system when the control input is identically zero, i.e., $u(t) \equiv 0$ for all $t \geq 0$.
- In such cases, the system's evolution depends only on its internal dynamics and initial state.
- The autonomous system corresponding to the state-space model is represented as follows:

$$\dot{x}(t) = Ax(t). \tag{2}$$

☒ The following definitions introduce the concepts of stability and asymptotic stability:

Definition 1. Asymptotic Stability is the ability of an autonomous system to drive its state **towards** a desired equilibrium point (often set as the origin) over time, even if the system starts from non-zero initial conditions.

- Based on the above definition, system (2) can be said to be **asymptotically stable at the origin if** the following condition holds:

$$\lim_{t\to\infty} x(t) = 0, \text{ for any } x(0) \neq 0. \tag{3}$$

Definition 2. **Stability** means that the system state remains **near** the equilibrium point for all time, but it does **not necessarily** converge to the equilibrium.

- Based on the above definition, system (2) can be said to be **stable at the origin if** the following condition holds:

$$\|x(t)\| \leq \epsilon, \text{ for all } t \geq 0. \tag{4}$$

Example 1.1. Determine the stability of the following two systems:

1) $\dot{x}(t) = 0, \;\; x(0) = \delta < \epsilon$

2) $\dot{x}(t) = -x(t), \;\; x(0) = \delta < \epsilon$

1.1 Hurwitz stability criterion

Example 1.2. Find the eigenvalue of A given by

$$A = \begin{bmatrix} -1 & -2 \\ 1 & -4 \end{bmatrix}.$$

Example 1.3. Find the eigenvalue of A given by

$$A = \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix}.$$

Example 1.4. Find the eigenvalue of A given by

$$A = \begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix}.$$

☒ The following presents the Hurwitz stability criterion:

The autonomous system (2) is asymptotically stable **if and only if** the system matrix $A \in \mathbb{R}^{n\times n}$ has all its eigenvalues in the open left-half complex plane $\mathbb{C}^{-}$.

- This criterion can be rewritten as follows: system (2) is asymptotically stable if and only if

$$\mathbf{Re}\{\lambda_i\} < 0,\ \forall i = 1, 2, \cdots, n.$$

Example 1.5. **Assess** the stability of $\dot{x}(t) = Ax(t)$ with A in Examples 1.2, 1.3, and 1.4.

Example 1.6. **Find** the range of k such that $\dot{x}(t) = Ax(t)$ is asymptotically stable, where

$$A = \begin{bmatrix} -1+k & -2 \\ 1 & -4 \end{bmatrix}.$$

1.2 Lyapunov stability

- The Lyapunov stability is a well-known method for evaluating the stability of

$$\dot{x}(t) = Ax(t).$$

- The first step in applying the Lyapunov stability method is to **choose** a suitable Lyapunov function.
- In linear systems, the standard choice for a Lyapunov function is as follows:

$$V(t) = x^T(t)Px(t) \tag{5}$$

where $P \in \mathbb{R}^{n \times n}$ must be **symmetric** and **positive definite**.

- Since $P = P^T > 0$, the Lyapunov function becomes a positive real-valued function:

$$V(t) = x^T(t)Px(t) > 0, \ \forall x(t) \neq 0.$$

☒ The following lemma presents the Lyapunov stability criterion:

Lemma 1 (Lyapunov stability criterion). System (2) can be said to be asymptotically stable at the origin **if** it holds that

$$\dot{V}(t) < 0. \tag{6}$$

Proof of Lemma: Assume that condition (6) holds. Then, $V(t)$ is strictly monotonically decreasing. In other words, $V(t)$ converges to zero as time goes to infinity:

Since $P = P^T > 0$, this equality is satisfied when the state $x(t)$ converges to the origin, written as follows:

As a result, according to Definition 2, we can conclude that system (2) is asymptotically stable at the origin. ∎

- Next, we will attempt to transform the Lyapunov stability criterion in (6) into a set of linear matrix inequalities (LMIs).
- This is because LMIs can be solved through numerical tools such as MATLAB and other optimization toolboxes.

☒ The following definition provides a simple explanation of the LMI.

Definition 3. An LMI denotes a matrix inequality that is linear in the decision variables. For instance, it can be said that for the decision variable X,

- $AX + X^T A^T + B < 0$: matrix inequality
- $X^T AX + B < 0$: matrix inequality.

☒ The following lemma provides a set of LMIs for the Lyapunov stability criterion of (2).

Lemma 2. System (2) is asymptotically stable at the origin **if** there exists a matrix variable $P = P^T > 0$ that satisfies

$$A^T P + PA < 0. \tag{7}$$

Proof of Lemma: From (5), it follows that

$$
\begin{aligned}
\dot{V}(t) &= \dot{x}^T(t)Px(t) + x^T(t)P\dot{x}(t) \\
&= x^T(t)A^TPx(t) + x^T(t)PAx(t) \\
&= x^T(t)\Big(\qquad \Big)x(t).
\end{aligned}
$$

Thus, if (7) holds, it is satisfied that $\dot{V}(t) < 0$. Furthermore, based on Lemma 1, we can conclude that system (2) is asymptotically stable at the origin. ∎

Example 1.7. To simplify the stability analysis, let us set $P = P^T > 0$ as follows:

$$P = \begin{bmatrix} p_1 & 0 \\ 0 & p_2 \end{bmatrix}$$

where p_1 and p_2 are positive scalars. **Provide** a possible pair of p_1 and p_2 to show that the the following system is asymptotically stable:

$$\dot{x}(t) = Ax(t), \; A = \begin{bmatrix} -1 & -2 \\ 1 & -4 \end{bmatrix}.$$

Solution to Example 1.4:

$PA + A^T P$

$$= \begin{bmatrix} -p_1 & -2p_1 \\ p_2 & -4p_2 \end{bmatrix} + \begin{bmatrix} -p_1 & p_2 \\ -2p_1 & -4p_2 \end{bmatrix} = \begin{bmatrix} -2p_1 & p_2 - 2p_1 \\ p_2 - 2p_1 & -8p_2 \end{bmatrix}.$$

To simplify the process of finding a solution, let us set $p_2 = 2p_1$. Then, it is obtained that

$$PA + A^T P = \begin{bmatrix} \quad \end{bmatrix}.$$

As a valid candidate that ensures $P > 0$, we may simply choose $p_1 = 1$ (i.e., $p_2 = 2$), which satisfies the following condition:

Therefore, the given autonomous system is asymptotically stable because there exists $P = P^T > 0$ satisfying the Lyapunov stability criterion. $\blacksquare$

Example 1.8. The following inequality provides the exponential stability criterion:

$$\dot{V}(t) < -\alpha V(t), \ \alpha > 0.$$

Please formulate a set of linear matrix inequalities to assess this stability criterion.