Stability analysis of linear systems

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- Hurwitz stability criterion (for linear systems)
- Lyapunov stability criterion (for linear systems)
- Robust stability criterion (for uncertain linear systems)
- $\mathcal{H}_\infty$ stability criterion (for linear systems with disturbances)

1.4 $\mathcal{H}_\infty$ stability criterion

- The $\mathcal{H}_\infty$ stability criterion is applied to the following system:

$$\begin{cases} \dot{x}(t) = Ax(t) + Nw(t) \\ z(t) = Gx(t) \end{cases} \tag{18}$$

where $w(t) \in \mathbb{R}^d$ denotes the external disturbances, and $z(t) \in \mathbb{R}^q$ denotes the performance output.

- The $\mathcal{H}_\infty$ stability criterion is used to verify whether
  - system (18) is asymptotically stable, and
  - system (18) has an $\mathcal{H}_\infty$ performance level.
- To handle the $\mathcal{H}_\infty$ performance problem, it is essential to assume $w(t) \in \mathcal{L}_2$ as follows:

$$\boxed{} < \infty.$$

- Let us recall the asymptotic stability criterion and the $\mathcal{H}_\infty$ performance condition.

  - Asymptotical stability criterion:

    $$\dot{V}(t) < 0$$

    which is valid for the case where $x(0) \neq 0$ and $w(t) = 0$.

  - $\mathcal{H}_\infty$ performance condition:

    $$\int_0^\infty z^T(\tau)z(\tau)d\tau \leq \gamma^2 \int_0^\infty w^T(\tau)w(\tau)d\tau$$

    which is valid for the case where $x(0) = 0$ and $w(t) \neq 0$.

The following lemma provides the $\mathcal{H}_\infty$ stability criterion for system (18).

Lemma 5. System (18) is asymptotically stable and has an $\mathcal{H}_\infty$ performance level, if the following condition holds:

$$\dot{V}(t) + z^T(t)z(t) - \gamma^2 w^T(t)w(t) < 0 \tag{19}$$

where γ denotes the $\mathcal{H}_\infty$ performance level (or disturbance attenuation level).

Proof:

(i) Asymptotic stability:

- In the context of asymptotic stability analysis, the influence of disturbances is typically disregarded.
- Thus, letting $w(t) = 0$, it follows from (19) that

$$\dot{V}(t) < \boxed{} < 0 \tag{20}$$

which guarantees the asymptotic stability of system (18).

(ii) $\mathcal{H}_\infty$ performance:

- By integrating both sides of (19), we obtain the following expression:

$$\int_0^\infty z^T(\tau)z(\tau)d\tau - \gamma^2 \int_0^\infty w^T(\tau)w(\tau)d\tau < \boxed{}.$$

- In the context of $\mathcal{H}_\infty$ performance analysis, the influence of the initial condition is disregarded. That is, letting $x(0) = 0$, it follows that

$$V(0) = x^T(0)Px(0) = \boxed{}$$

- Thus, it holds that

$$\int_0^{\infty} z^T(\tau)z(\tau)d\tau - \gamma^2 \int_0^{\infty} w^T(\tau)w(\tau)d\tau < \boxed{\qquad\qquad} < 0 \qquad (21)$$

which ensures the $\mathcal{H}_\infty$ performance condition.

- Therefore, following up on **(i)** and **(ii)**, it can be concluded that inequality (19) serves as the $\mathcal{H}_\infty$ stability criterion.

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The following theorem provides a set of linear matrix inequalities (LMIs) corresponding to the $\mathcal{H}_\infty$ stability criterion in (19).

Theorem 2. System (18) is asymptotically stable and has an $\mathcal{H}_\infty$ performance level, if there exist a symmetric matrix $P \in \mathbb{R}^{n\times n}$ and a positive scalar γ satisfying

$$P > 0 \tag{22}$$

$$\begin{bmatrix} \mathbf{He}\{PA\} + G^T G & PN \\ N^T P & -\gamma^2 I \end{bmatrix} < 0 \tag{23}$$

where the value of γ quantifies the $\mathcal{H}_\infty$ performance level, which corresponds to the system's disturbance attenuation capability.

Proof:

- By incorporating (18), the $\mathcal{H}_\infty$ stability criterion in (19) is expressed as follows:

$$x^T(t)P(\boxed{\qquad\qquad}) + (\boxed{\qquad\qquad})^T Px(t) + x^T \boxed{\quad} x(t) - \gamma^2 w^T(t)w(t) < 0. \quad (24)$$

- Furthermore, condition (24) can be rearranged as follows:

$$\begin{bmatrix} x(t) \\ w(t) \end{bmatrix}^T \begin{bmatrix} \boxed{\qquad\qquad} & PN \\ N^T P & -\gamma^2 I \end{bmatrix} \begin{bmatrix} x(t) \\ w(t) \end{bmatrix} < 0.$$

- Finally, condition (23) follows as a result.

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Example 1.13. Using the robust control toolbox in MATLAB, **analyze** the $\mathcal{H}_\infty$ stability of system (8) with

$$A = \begin{bmatrix} -1 & -2 \\ 1 & -4 \end{bmatrix}, \; N = \begin{bmatrix} 1 \\ 0.1 \end{bmatrix}, \; G = \begin{bmatrix} 1 & 0 \end{bmatrix}, \; \gamma = 0.64.$$

What happens when γ becomes less than 0.64?

Example 1.14. Let us consider the following uncertain linear system with disturbances:

$$\begin{cases} \dot{x}(t) = (A + \Delta A)x(t) + Nw(t) \\ z(t) = Gx(t). \end{cases}$$

Please **obtain** a set of linear matrix inequalities (LMIs) that ensures the robust $\mathcal{H}_\infty$ stability for this system.