Mathematical description of linear systems and their properties

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Contents

1 Preliminaries

1.1 Notations

- $\mathbb{R}$: the set of real numbers.
- $\mathbb{R}^+ = \{x \in \mathbb{R} : x \geq 0\}$: the set of nonnegative real numbers.
- $\mathbb{Z}$: the set of integers.
- $\mathbb{Z}^+ = \{x \in \mathbb{Z} : x \geq 0\}$: the set of nonnegative integers.
- $\mathbb{R}^n$: the set of the n-dimensional real vectors. For instance,

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$$

which means $x_i \in \mathbb{R}$ for all $i \in \{1, 2, ..., n\}$.

- $\mathbb{R}^{m\times n}$: the set of $m \times n$ real matrices. For instance,

$$A = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

which means that each element a_{ij} satisfies that $a_{ij} \in \mathbb{R}$ for all $i \in \{1, 2, ..., m\}$ and $j \in \{1, 2, ..., n\}$.

2 State-space Model

The linear time-invariant ordinary differential equations can be described as the following state-space model:

$$\left\{ \begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{aligned} \right. \tag{1}$$

where

- $x(t) : [0, \infty) \to \mathbb{R}^n$ is the state vector,
- $u(t) : [0, \infty) \to \mathbb{R}^m$ is the control input vector, and
- $y(t) : [0, \infty) \to \mathbb{R}^p$ is the output vector.

Accordingly, the system matrices satisfy

$$A \in \mathbb{R}^{n\times n},\ B \in \mathbb{R}^{n\times m},\ C \in \mathbb{R}^{p\times n},\ \text{and}\ D \in \mathbb{R}^{p\times m}.$$

Example 2.1. Let us consider a simple RLC electrical circuit given in Figure 1.

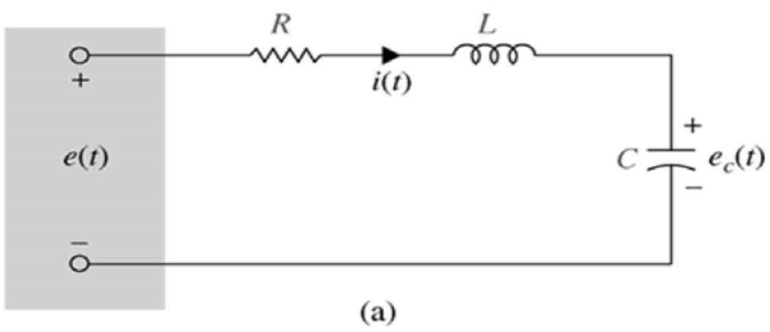
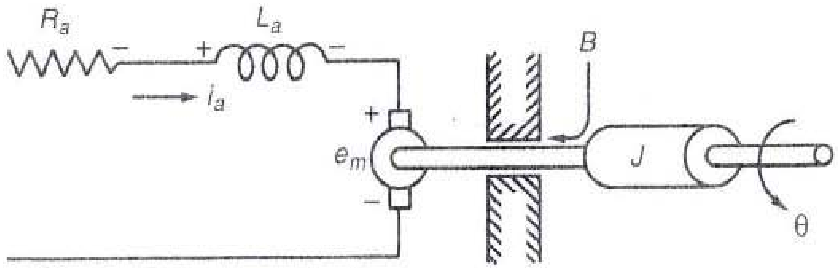


Figure 1: RLC electrical circuit.

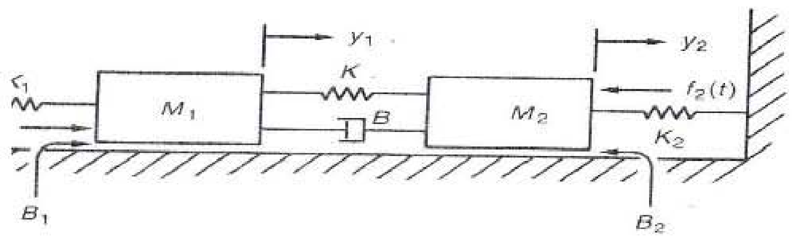
Example 2.2. Let us consider a simplified model of an armature voltage-controlled dc servomotor consisting of a stationary field and a rotating armature and load.



Assignment 2.1. Newton's second law yields the following coupled second-order ordinary differential equations:

$$\begin{aligned} M_1\ddot{y}_1 + (B+B_1)\dot{y}_1 + (K+K_1)y_1 - B\dot{y}_2 - Ky_2 &= f_1(t) \\ M_2\ddot{y}_2 + (B+B_2)\dot{y}_2 + (K+K_2)y_2 - B\dot{y}_1 - Ky_1 &= -f_2(t) \end{aligned} \tag{6}$$

with initial data $y_1(0)$, $y_2(0)$, $\dot{y}_1(0)$, and $\dot{y}_2(0)$.



Please express (6) as a state-space model.

Example 2.3. Simulate the following system using simulink tool:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 4 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & 3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}.$$