Chap. 1: Mathematical description of linear systems and their properties

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Sections

4.2 For nonlinear systems with control input

- Let us consider the following nonlinear state-space model with control input $u(t)$:

$$\dot{x}(t) = f(x(t), u(t)) = \begin{bmatrix} \quad \end{bmatrix} \in \mathbb{R}^n. \tag{10}$$

- As shown here, the nonlinear function vector depends on the state $x(t)$ and the input $u(t)$.
- In this case as well, it should be assumed that the nonlinear function $f_i(x, u)$ has $\mathcal{C}^1$ continuity.

- Then, nonlinear system (10) can be linearized at (ϕ, ψ) using the following steps:
  - Choose an equilibrium point (ϕ, ψ) such that $\lim_{t\to\infty} x(t) = \phi$ and $\lim_{t\to\infty} u(t) = \psi$.
  - Obtain the following system matrices:

$$A = \boxed{}$$

$$B = \boxed{}.$$

- Then, the linearized state-space model is given by

$$\dot{z}(t) = Az(t) + Bv(t)$$

where

$$z(t) = x(t) - \phi,\ v(t) = u(t) - \psi.$$

Example 4.2. Consider the following nonlinear system:

$$\ddot{y}(t) = -y(t) + y^3(t) + u(t).$$

In this system, $y(t)$ and $u(t)$ converge to 1 and 0, respectively, as time goes to infinity.

a) Obtain the nonlinear state-space representation

b) Find the linearized state-space model.

Sol.: a) Setting $x_1 = y$ and $x_2 = \dot{y}$, the state-space representation is given by

$$\dot{x} = \begin{bmatrix} \boxed{\qquad\qquad\qquad} \end{bmatrix} =: f\left(x, u\right)$$

b) Let us choose ϕ and ψ as

$$\phi_1 = \lim_{t\to\infty} x_1(t) = \boxed{\quad} \implies \quad \phi_2 = \dot{\phi}_1 = \boxed{\quad}$$

$$\psi = \lim_{t\to\infty} u(t) = \boxed{\quad}$$

Then, the equilibrium point is given as

$$\begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = \begin{bmatrix} \boxed{\quad} \end{bmatrix}, \ \psi = \boxed{\quad}$$

Since

$$\frac{\partial f}{\partial x} = \begin{bmatrix} \quad \end{bmatrix}, \quad \frac{\partial f}{\partial u} = \begin{bmatrix} \quad \end{bmatrix},$$

the linearlized system at $(\phi, \psi) = \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, 0 \right)$ is given by

$$A = \begin{bmatrix} \quad \end{bmatrix}, \quad B = \begin{bmatrix} \quad \end{bmatrix}.$$

$\blacksquare$

Example 4.3. Let us consider the following nonlinear system:

$$\ddot{r}(t) = r(t)\dot{\theta}^2(t) - \frac{k}{r^2(t)} + u_1(t)$$

$$\ddot{\theta}(t) = -2\frac{\dot{\theta}(t)\dot{r}(t)}{r(t)} + \frac{1}{r(t)}u_2(t).$$

In this system, $r(t)$ and $\dot{\theta}(t)$ converge to constant r_0 and constant $w_0 = \sqrt{\frac{k}{r_0^3}}$, respectively, as time goes to infinity. Construct A and B.

Sol.: Then defining that $x_1(t) = r(t)$, $x_2(t) = \dot{r}(t)$, $x_3(t) = \theta(t)$, and $x_4(t) = \dot{\theta}(t)$, we can obtain

$$\begin{cases} \dot{x}_1(t) = \\ \dot{x}_2(t) = \\ \dot{x}_3(t) = \\ \dot{x}_4(t) = \end{cases} \tag{11}$$

The equilibrium point is chosen as follows:

$$\phi_1 = \lim_{t\to\infty} x_1(t) = \qquad \Longrightarrow \qquad \phi_2 = \dot{\phi}_1 = $$

$$\phi_4 = \lim_{t\to\infty} x_4(t) = \qquad \Longrightarrow \qquad \phi_3 = \int \phi_4 dt = $$

$$\Longrightarrow$$

That is, ϕ is given by

$$\phi = \begin{bmatrix} \quad \\ \quad \\ \quad \\ \quad \end{bmatrix}, \ \psi = \begin{bmatrix} \quad \\ \quad \end{bmatrix}, \ w_0 := \sqrt{\frac{k}{r_0^3}}.$$

Then the linearized system matrices are given by

$$A = \frac{\partial f}{\partial x}(t,x)\Big|_{x=\phi,u=\psi} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 3w_0^2 & 0 & 0 & \\ 0 & 0 & 0 & 1 \\ 0 & & 0 & 0 \end{bmatrix}$$

$$B = \frac{\partial f}{\partial u}(t,x)\Big|_{x=\phi,u=\psi} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & \end{bmatrix}.$$

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5 Transition between Laplace transform (LT) and state-space (SS) representations

5.1 From SS to LT

- The Laplace transform of the following LTI system:

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

is given by

$$sX(s) - x(0) = AX(s) + BU(s)$$
$$Y(s) = CX(s) + DU(s).$$

- Accordingly, it is obtained that

$$Y(s) = C(sI - A)^{-1}x(0) + \underbrace{\left(\boxed{\qquad\qquad\qquad}\right)}_{=:\,G(s)} U(s).$$

- When the initial condition is zero (i.e., $x(0) = 0$), it is given that

$$Y(s) = \boxed{\quad} U(s).$$

- For this reason, $G(s)$ is referred to as the transfer function (matrix) and is given in the following form:

$$G(s) = \begin{bmatrix} g_{11}(s) & \cdots & g_{1m}(s) \\ \vdots & & \vdots \\ g_{p1}(s) & \cdots & g_{pm}(s) \end{bmatrix} \in \mathbb{C}^{\boxed{\quad}}.$$

- Specifically, the (i, j)th element of $G(s)$, $g_{ij}(s)$, has the following rational function representation:

$$g_{ij}(s) = \frac{n_{ij}(s)}{d_{ij}(s)}.$$

Definition 3. In control theory and systems engineering, rational functions of the form $g(s) = n(s)/d(s)$ can be classified as follows:

1. $g(s)$ is if $\mathbf{deg}(n(s)) \leq \mathbf{deg}(d(s))$.
2. $g(s)$ is if $\mathbf{deg}(n(s)) < \mathbf{deg}(d(s))$.
3. $g(s)$ is if $\mathbf{deg}(n(s)) > \mathbf{deg}(d(s))$.

- In $G(s)$, it basically holds that $\mathbf{deg}(n_{ij}(s)) \leq \mathbf{deg}(d_{ij}(s))$ for all i and j. Thus, $g_{ij}(s)$ is said to be .

Example 5.1. Find the transfer function matrix $G(s)$ for

$$A = \begin{bmatrix} -1 & 0 \\ 2 & 1 \end{bmatrix}, \; B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \; C = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \; D = 0.$$

Sol.: It is clear that

$$sI - A = \begin{bmatrix} \boxed{} \end{bmatrix} - \begin{bmatrix} -1 & 0 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} \boxed{} \end{bmatrix}.$$

Since $\mathbf{det}(sI - A) = (s+1)(s-1) = s^2 - 1$, we have

$$(sI - A)^{-1} = \frac{1}{\boxed{}} \begin{bmatrix} \boxed{} \end{bmatrix} = \begin{bmatrix} \boxed{} \end{bmatrix}.$$

Therefore, the transfer function is given by

$$G(s) = C(sI - A)^{-1}B = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} \boxed{} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

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5.2 From LT to SS

- Our goal is to construct $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, and $D \in \mathbb{R}^{p\times m}$ from a given transfer function matrix $G(s) \in \mathbb{R}^{p\times m}$.
- Since the sizes p and m are given, but the size n cannot be determined from $G(s)$, obtaining the quadruple (A, B, C, D) from $G(s)$ is not unique.
- The following example shows a possible way to find the system matrices from the given transfer function.

Example 5.2. Find a quadruple (A, B, C, D) from the following transfer function:

$$G(s) = \begin{bmatrix} \dfrac{4s - 10}{2s + 1} & \dfrac{3}{s + 2} \\ \dfrac{1}{(2s + 1)(s + 2)} & \dfrac{s + 1}{(s + 2)^2} \end{bmatrix}$$

with $p = 2$ and $m = 2$.

Sol.:

(Step 1) Decompose the proper rational function matrix $G(s)$ into a strictly proper rational function matrix $\bar{G}(s)$ and a constant matrix M in the following manner:

$$G(s) = \underbrace{\left[\begin{array}{c:c} \boxed{\qquad\qquad} & \dfrac{3}{s+2} \\ \hdashline \dfrac{1}{(2s+1)(s+2)} & \dfrac{s+1}{(s+2)^2} \end{array}\right]}_{=:\,\bar{G}(s)} + \underbrace{\left[\boxed{\qquad}\right]}_{=:\,M}.$$

(Step 2) To unify the denominators of all elements of $\bar{G}(s)$, determine a common denominator polynomial $d(s)$:

$$d(s) = \boxed{\qquad\qquad\qquad} = s^3 + \frac{9}{2}s^2 + 6s + 2.$$

(Step 3) Then, based on $d(s)$, $\bar{G}(s)$ can be rewritten as

$$\begin{aligned}\bar{G}(s) &= \frac{1}{d(s)}\left[\qquad\qquad\qquad\qquad \right] \\ &= \left[\qquad \right]\frac{s^2}{d(s)} + \left[\qquad \right]\frac{s}{d(s)} + \left[\qquad \right]\frac{1}{d(s)}.\end{aligned}$$

(Step 4) Define the size of $x(t)$ in the following manner:

$$\begin{aligned} n &= (\text{degree of } d(s)) \times (\text{number of inputs}) \\ &= \boxed{} \end{aligned}$$

Thus, $x(t) \in \mathbb{R}^6$ and

$$\begin{bmatrix} X_1(s) \\ X_2(s) \\ X_3(s) \\ X_4(s) \\ X_5(s) \\ X_6(s) \end{bmatrix} = \mathcal{L}\left\{ \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \\ x_5(t) \\ x_6(t) \end{bmatrix} \right\}, \; U(s) = \mathcal{L}\{u(t)\}.$$

(Step 5) Define

$$\begin{bmatrix} X_5(s) \\ X_6(s) \end{bmatrix} := \frac{1}{d(s)} U(s)$$

$$\begin{bmatrix} X_3(s) \\ X_4(s) \end{bmatrix} := s \begin{bmatrix} X_5(s) \\ X_6(s) \end{bmatrix} = \boxed{\quad} U(s)$$

$$\begin{bmatrix} X_1(s) \\ X_2(s) \end{bmatrix} := s^2 \begin{bmatrix} X_5(s) \\ X_6(s) \end{bmatrix} = s \begin{bmatrix} X_3(s) \\ X_4(s) \end{bmatrix} = \boxed{\quad} U(s).$$

(Step 6) From $d(s)\begin{bmatrix} X_5(s) \\ X_6(s) \end{bmatrix} = U(s)$ and $d(s) = s^3 + \frac{9}{2}s^2 + 6s + 2$,

it follows that

$$s^3\begin{bmatrix} \quad \end{bmatrix} + \frac{9}{2}s^2\begin{bmatrix} \quad \end{bmatrix} + 6s\begin{bmatrix} \quad \end{bmatrix} + 2\begin{bmatrix} \quad \end{bmatrix}$$

$$= s\begin{bmatrix} \quad \end{bmatrix} + \frac{9}{2}\begin{bmatrix} \quad \end{bmatrix} + 6\begin{bmatrix} \quad \end{bmatrix} + 2\begin{bmatrix} \quad \end{bmatrix} = U(s)$$

which leads to

$$s\begin{bmatrix} X_1(s) \\ X_2(s) \end{bmatrix} = -\frac{9}{2}\begin{bmatrix} \quad \end{bmatrix} - 6\begin{bmatrix} \quad \end{bmatrix} - 2\begin{bmatrix} \quad \end{bmatrix} + U(s).$$

$$sX(s) = \underbrace{\left[\begin{array}{cc|cc|cc} & & & & & \\ & & & & & \\ \hline & & 0 & 0 & 0 & 0 \\ & & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & & & 0 & 0 \\ 0 & 0 & & & 0 & 0 \end{array}\right]}_{=A} \left[\begin{array}{c} X_1(s) \\ X_2(s) \\ \hline X_3(s) \\ X_4(s) \\ \hline X_5(s) \\ X_6(s) \end{array}\right] + \underbrace{\left[\begin{array}{cc} & \\ & \\ \hline 0 & 0 \\ 0 & 0 \\ \hline 0 & 0 \\ 0 & 0 \end{array}\right]}_{=B} U(s).$$

(Step 7) From

$$Y(s) = (\bar{G}(s) + M))U(s)$$

it follows that

$$\begin{aligned} Y(s) &= \left[\boxed{\qquad} \right] \frac{s^2}{d(s)} U(s) + \left[\boxed{\qquad\qquad} \right] \frac{s}{d(s)} U(s) \\ &+ \left[\boxed{\qquad\quad} \right] \frac{1}{d(s)} U(s) + \left[\boxed{\quad} \right] U(s) \\ &= \underbrace{\left[\boxed{\quad} \,\middle|\, \boxed{\qquad\quad} \,\middle|\, \boxed{\qquad\quad} \right]}_{=C} X(s) + \underbrace{\left[\boxed{\quad} \right]}_{=D} U(s). \end{aligned}$$

6 Algebraic equivalence

- Let us consider the following LTI system:

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases} \tag{12}$$

- In addition, assume that there exit a nonsingular matrix $T \in \mathbb{R}^{n \times n}$ and a new state vector z, such that it holds that

$$z = Tx. \tag{13}$$

- Then, using (13), the original linear system (12) can be transformed into the following linear state-space system:

$$\begin{cases} \dot{z} = \tilde{A}z + \tilde{B}u \\ y = \tilde{C}z + \tilde{D}u \end{cases} \tag{14}$$

where

$$\tilde{A} = \boxed{}, \quad \tilde{B} = \boxed{}, \quad \tilde{C} = \boxed{}, \quad \tilde{D} = \boxed{}$$

- If the initial condition is chosen as $z(0) = Tx(0)$ and the same input u is applied, then system (14) has an **identical** response to system (12).

Definition 4. Two LTI systems (12) and (14) are said to be **algebraically equivalent** if there exists a nonsingular matrix T such that it holds that

$$\tilde{A} = TAT^{-1},\ \tilde{B} = TB,\ \tilde{C} = CT^{-1}.$$

Definition 5. The map $z = Tx$ is called a **similarity transformation**.

Example 6.1. Given

$$A = \begin{bmatrix} -1 & 2 \\ 1 & -3 \end{bmatrix}, \ B = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \ C = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

find the equivalent system matrices $\tilde{A}$, $\tilde{B}$, and $\tilde{C}$ that satisfy

$$z_1 = 2x_1 + x_2$$
$$z_2 = x_1 + x_2$$

Sol.: From the relationship between x and y, we can obtain the following nonsingular matrix:

$$T = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix} \rightarrow T^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}.$$

Thus,

$$\tilde{A} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix} = \begin{bmatrix} -2 & 3 \\ 1 & -2 \end{bmatrix}$$

$$\tilde{B} = \begin{bmatrix} \quad \\ \quad \end{bmatrix}, \tilde{C} = \begin{bmatrix} \quad & \quad \end{bmatrix}.$$

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