State-feedback control systems

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Contents

- Useful lemmas
- State-feedback control design
- LQR state-feedback control design
- Robust state-feedback control design
- $\mathcal{H}_\infty$ state-feedback control design

5 $\mathcal{H}_\infty$ state-feedback control

- The state-feedback controller is developed with the objective of attenuating the impact of disturbances on the closed-loop system, based on full state measurements.
- To begin with, let us consider the following linear system subject to external disturbances belonging to the $\mathcal{L}_2$ space:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) + Nw(t) \\ z(t) = Cx(t) + Du(t). \end{cases} \tag{31}$$

- For (31), the following state-feedback control law is adopted:

$$u(t) = Kx(t)$$

where K is the control gain to be designed later.

- As a result, the closed-loop control system is described as follows:

$$\begin{cases} \dot{x}(t) = \bar{A}x(t) + Nw(t) \\ z(t) = \bar{C}x(t) \end{cases} \tag{32}$$

where

$$\bar{A} = A + BK, \ \bar{C} = C + DK.$$

Theorem 4. System (31) is asymptotically stable and has an $\mathcal{H}_\infty$ performance level, if there exist matrices $\bar{P} = \bar{P}^T > 0$ and $\bar{K}$, and a scalar $\gamma > 0$ such that the following condition is satisfied:

$$\left[\begin{array}{cc:c} \mathbf{He}\{A\bar{P} + B\bar{K}\} & (\star) & (\star) \\ N^T & -\gamma^2 I & 0 \\ \hdashline C\bar{P} + D\bar{K} & 0 & -I \end{array}\right] < 0. \tag{33}$$

Furthermore, the control gain can be recovered as follows:

$$K = \bar{K}\bar{P}^{-1}.$$

Proof:

- Let us recall the following the $\mathcal{H}_\infty$ stability condition:

$$\dot{V}(t) + z^T(t)z(t) - \gamma^2 w^T(t)w(t) < 0. \tag{34}$$

- Then, condition (34) is described as

$$\begin{aligned} x^T(t)P\Big(\bar{A}x(t) + Nw(t)\Big) + \Big(\bar{A}x(t) + Nw(t)\Big)^T Px(t) \\ + x^T\bar{C}^T\bar{C}x(t) - \gamma^2 w^T(t)w(t) < 0. \end{aligned} \tag{35}$$

- Furthermore, (35) can be rewritten as

$$\begin{bmatrix} x(t) \\ w(t) \end{bmatrix}^T \begin{bmatrix} \boxed{} \end{bmatrix} \begin{bmatrix} x(t) \\ w(t) \end{bmatrix} < 0.$$

- Thus, condition (34) holds if

$$\begin{bmatrix} \mathbf{He}\{P\bar{A}\} + \bar{C}^T\bar{C} & PN \\ N^TP & -\gamma^2 I \end{bmatrix} < 0. \tag{36}$$

- According to Lemma 3, (36) is equivalent to

$$\boxed{\qquad\qquad} \begin{bmatrix} \mathbf{He}\{P\bar{A}\} + \bar{C}^T\bar{C} & PN \\ N^TP & -\gamma^2 I \end{bmatrix} \boxed{\qquad\qquad} < 0 \tag{37}$$

which leads to

$$\begin{bmatrix} \mathbf{He}\{\bar{A}P^{-1}\} + P^{-1}\bar{C}^T\bar{C}P^{-1} & N \\ N^T & -\gamma^2 I \end{bmatrix} < 0. \tag{38}$$

- By Lemma 2 (called the Schur complement), condition (38) can be transformed into

$$\begin{bmatrix} \quad \end{bmatrix} < 0. \tag{39}$$

- By substituting $\bar{A} = A + BK$ and $\bar{C} = C + DK$, and letting $\bar{P} = P^{-1}$ and $\bar{K} = KP^{-1}$, it is derived that

$$\left[\begin{array}{cc:c} \mathbf{He}\{A\bar{P} + B\bar{K}\} & (\star) & (\star) \\ N^T & -\gamma^2 I & 0 \\ \hdashline C\bar{P} + D\bar{K} & 0 & -I \end{array}\right] < 0.$$

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