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Assignment - 1

1 a) $2x-3y-z=3$

$x-2y-z=4$

$5x-4y-3z=-2$

Sol Augmented matrix $(Ax=B)$

$$\left[\begin{array}{ccc|c} 2 & -3 & -1 & 3 \\ 1 & -2 & -1 & 4 \\ 5 & -4 & -3 & -2 \end{array}\right]$$

$R_1 \to R_1 - R_2$

$$\left[\begin{array}{ccc|c} 1 & -1 & 0 & -1 \\ 1 & -1 & -1 & 4 \\ 3 & -4 & -3 & -2 \end{array}\right]$$

$R_2 \to R_2 - R_1$ $\quad$ $R_3 \to R_3 - 5R_1$

$$\left[\begin{array}{ccc|c} 1 & -1 & 0 & -1 \\ 0 & -1 & -1 & 3 \\ 0 & 1 & -3 & 3 \end{array}\right]$$

$(-R_2)$ $\quad$ $R_3 \to R_3 + R_2$

$$\left[\begin{array}{ccc|c} 1 & -1 & 0 & -1 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & -4 & 6 \end{array}\right] \quad \left[\begin{array}{ccc|c} 1 & -1 & 0 & -1 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 1 & -3/2 \end{array}\right]$$

$z = \frac{-3}{2}$ $\quad$ $y+z=-3$

$y = -3 + \frac{3}{2} = \frac{-3}{2}$

$x - y = -1$

$x = -1 + \frac{3}{2} = \frac{1}{2}$

Unique solution

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Q2 $x+2y-z=1$
$x+y+2z=9$
$2x+y-z=2$

Sol
$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 1 & 1 & 2 & 9 \\ 2 & 1 & -1 & 2 \end{array}\right]$$

$R_2 \to R_2 - R_1$ $R_3 \to R_3 - 2R_1$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & -1 & 3 & 8 \\ 0 & -1 & 1 & 0 \end{array}\right]$$

$R_3 \to R_3 + 3R_2$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & -3 & -8 \\ 0 & 0 & -8 & -24 \end{array}\right]$$

$R_3 \to \left(\frac{-1}{8}\right)$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & -3 & -8 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$z=3$ $y-3z=-8$
$y=-8-3(-3)$
$=1$

$x+2y-z=1$
$x=1-2y+z$
$=1-2+3$
$=1+1$
$=2$

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C $2x-y-z=2$
$x+2y+z=2$
$4x-7y-4z=1$

Sol
$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 2 \\ 1 & 2 & 1 & 2 \\ 4 & -7 & 4 & 1 \end{array}\right]$$

$R_1 \to R_1 - R_2$

$$\left[\begin{array}{ccc|c} 1 & -3 & 0 & 0 \\ 1 & 2 & 1 & 2 \\ 4 & -7 & -4 & 1 \end{array}\right]$$

$R_2 \to R_2 - R_1$ $R_3 \to R_3 - 4R_1$

$$\left[\begin{array}{ccc|c} 1 & -3 & 0 & 0 \\ 0 & 5 & 1 & 2 \\ 0 & 5 & 4 & 2 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & -3 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 5 & -4 & 2 \end{array}\right]$$

$R_3 \to R_2$ $R_2 \, 1/5$ $R_3(1/5)$

$$\left[\begin{array}{ccc|c} 1 & -3 & 0 & 0 \\ 0 & 1 & -4/5 & 2/5 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$z=0$ $y=2/5$
$x-3y=0$
$x=3\left(\frac{2}{5}\right)$

$=\frac{6}{5}$

2 By solve by Gauss elimination

a) $-2y+3z=1$

$3x+6y-3z=-2$

$6x+6y+3z=5$

Sol

$$\left[\begin{array}{ccc|c} 0 & -2 & 3 & 1 \\ 3 & 6 & -3 & -2 \\ 6 & 6 & 3 & 5 \end{array}\right]$$

$R_1 \to R_3$

$$\left[\begin{array}{ccc|c} 0 & -6 & 3 & 1 \\ 3 & 6 & -3 & -2 \\ 0 & -2 & 3 & 5 \end{array}\right]$$

$R_1 \to R_1 - R_2$

$$\left[\begin{array}{ccc|c} 3 & 0 & 2 & 3 \\ 3 & 6 & -3 & -2 \\ 0 & -2 & 3 & 5 \end{array}\right]$$

$R_2 \to R_2 - R_1$

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & 6 & -9 & -5 \\ 0 & -2 & 3 & 5 \end{array}\right]$$

$R_2 \to R_2/6$

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & 1 & -3/2 & -5/6 \\ 0 & -2 & 3 & 5 \end{array}\right]$$

$R_3 \to R_3 + 2R_2$

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & 1 & -3/2 & -5/6 \\ 0 & 0 & 0 & 10/3 \end{array}\right]$$

$0 \neq \frac{10}{3}$

No solution

b) $x+y+2z=8$

$-x-2y+3z=1$

$3x-7y+4z=10$

sol

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ -1 & -2 & 3 & 1 \\ 3 & -7 & 4 & 10 \end{array}\right]$$

$R_2 \to R_2 + R_1$ $\quad R_3 \to R_3 - 3R_1$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & -1 & 5 & 9 \\ 0 & -10 & -2 & -14 \end{array}\right]$$

$-R_2$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & -10 & -2 & -14 \end{array}\right]$$

$R_3 \to R_3 + 10R_2$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & 52 & -104 \end{array}\right]$$

$R_3 \to \frac{1}{52}$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$z=2$ $\quad y-5z=-9$ $\quad x+y+2z=8$

$y=-9-5z$ $\quad x=8-y+2z$

$=-9+5(2)$ $\quad =8-1-4$

$=-9+10$ $\quad =8-5$

$=1$ $\quad =3$

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c $2x+y+5z=4$
$3x-2y+2z=2$
$5x-8y-4z=1$

sol
$$\left[\begin{array}{ccc|c} 2 & 1 & 5 & 4 \\ 3 & -2 & 2 & 2 \\ 5 & -8 & -4 & 1 \end{array}\right]$$

$R_1 \to R_1 - R_2$
$$\left[\begin{array}{ccc|c} -1 & 3 & 3 & 2 \\ 3 & -2 & 2 & 2 \\ 5 & -8 & -4 & 1 \end{array}\right]$$

$R_1(-1)$
$$\left[\begin{array}{ccc|c} 1 & 3 & -3 & -2 \\ 0 & 1 & 11 & 8 \\ 0 & 1 & 11 & 11 \end{array}\right]$$

$R_3 \to R_3 - R_2$
$$\left[\begin{array}{ccc|c} 1 & 3 & -3 & -2 \\ 0 & 1 & 11 & 8 \\ 0 & 0 & 0 & 3 \end{array}\right]$$

No sol

3 Investigate for what value of λ and μ the system of simultaneous eq

$x+y+z=8$ i> No sol
$x+2y+3z=10$ ii> Unique sol^n
$x+2y+\lambda z=4$ iii> Infinite many sol^n

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sol
$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 8 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & \lambda & \mu \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 8 \\ 0 & 1 & 2 & 2 \\ 0 & 1 & \lambda-1 & \mu-8 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 8 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & \lambda-3 & \mu-10 \end{array}\right] \quad R_3 \to R_3 - R_2$$

i>	ii>	iii>
$\lambda-3=0$	$\lambda-3\neq 0$	$\lambda-3=0$
$\lambda=3$	$\lambda\neq 3$	$\lambda=3$
$\mu-10=0$	$\mu-10\neq 0$	$\mu-10=0$
$\mu\neq 10$	$\mu\neq 10$	$\mu=10$

3 Find the Rank of matrix by determinant method

$$A=\begin{bmatrix} 1 & 2 & -1 & -4 \\ 2 & 4 & 3 & 5 \\ -1 & -2 & 6 & 7 \end{bmatrix}$$

sol
$$\begin{vmatrix} 2 & -1 & -4 \\ 4 & 3 & 5 \\ -2 & 6 & -7 \end{vmatrix}$$

$2(-21\times 30)+1(28+10)-4(24+6)$
$-102+18-120$
$-140\neq 0$
det $|A|$ is not zero
$|A|=3$

4 Find the Rank of matrix by row echelon form

$$A \begin{bmatrix} 3 & -2 & 0 & -1 \\ 0 & 2 & 2 & 1 \\ 1 & -2 & 3 & 2 \\ 0 & 1 & 2 & 1 \end{bmatrix}$$

sol

$$\begin{bmatrix} 1 & 2 & -6 & -5 \\ 0 & 2 & 2 & 1 \\ 1 & -2 & 3 & 2 \\ 0 & 1 & 2 & 1 \end{bmatrix} \quad R_1 \to R_1 - 2R_3$$

$$\begin{bmatrix} 1 & 1 & -6 & -5 \\ 0 & 2 & 2 & 1 \\ 0 & -4 & 9 & 7 \\ 0 & 1 & 2 & 1 \end{bmatrix} \quad R_3 \to R_3 - R_1$$

$$\begin{bmatrix} 1 & 1 & -6 & -5 \\ 0 & 1 & 0 & 0 \\ 0 & -4 & 9 & 1 \\ 0 & 1 & 2 & 1 \end{bmatrix} \quad R_2 \to R_2 - R_1$$

$$\begin{bmatrix} 1 & 0 & -6 & -5 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 9 & 7 \\ 0 & 0 & 2 & 1 \end{bmatrix} \quad \begin{matrix} R_3 \to R_3 - R_2 \\ R_4 \to R_4 - R_2 \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & -6 & -5 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 2 & 1 \end{bmatrix} \quad R_3 \to R_3 - 4R_4$$

$$\begin{bmatrix} 1 & 0 & -6 & -5 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 5 \end{bmatrix} \quad R_4 \to R_4 - 2R_3$$

$$\begin{bmatrix} 1 & 0 & -6 & -5 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad R_4 \to \frac{R_4}{5}$$

$$\begin{bmatrix} 1 & 0 & -6 & -5 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\rho(A) =$ Non zero row

$= 4$

5 Find Eigen Value of the Matrix

$$A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$$

sol

$$\begin{bmatrix} 2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{bmatrix}$$

$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

$S_1 = -1$

$S_2 = -12 - 3 - 6 = 21$

$S_3 = -2(-12) - 2(-6) - 3(-4+1)$
$= -45$

$\lambda^3 + \lambda^2 - 21\lambda - 45 = 0$

$(\lambda+3)(\lambda^2 - 2\lambda - 15) = 0$

$(\lambda+3)(\lambda-5)(\lambda+3)$

$\lambda = -3, -3, 5$

For $\lambda = -3$

$$\begin{bmatrix} -1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad R_1 \to R_1 - 2R_1, \; R_2 \to R_3 + R_1$$

$z = k_1 \qquad y = k_2$

$x = -2k_2 + 3k_1$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2k_2 + 3k_1 \\ k_2 \\ k_1 \end{bmatrix}$$

Eigen ~~value~~ vector of $\lambda = -3$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} k_2 + \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} k_1$$

$\lambda = 5$

$$\begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 5 \\ -2 & 4 & 6 \\ 7 & -2 & 3 \end{bmatrix} \quad R_1 \to R_3, \; -R_1, R_2(-1), 7R_3$$

$$\begin{bmatrix} 1 & 2 & 5 \\ 0 & 8 & 16 \\ 0 & -16 & -32 \end{bmatrix} \quad R_2 \to R_2 + 2R_1, \; R_3 \to R_3 - 7R_1$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 5 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \quad R_2 \to R_3, \; R_2 \to \frac{1}{8} R_2, \; R_3 \to \frac{1}{16} R_3$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 5 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 5 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad R_3 \to R_3 - R_2$$

$z = k \qquad y + 2k = 0 \qquad x + 2y + 5z = 0$

$y = 2k \qquad x = -3z$

Eigen vector

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix} k$$

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6 Find the value of u which satisfy the equation $A^{100} x = u$, where

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 0 & -2 & -2 \\ 1 & 1 & 0 \end{bmatrix}$$

Sol $A^K = P D^K P^{-1}$

$$\begin{bmatrix} 2-\lambda & 1 & -1 \\ 0 & -2-\lambda & -2 \\ 1 & 1 & -\lambda \end{bmatrix} = 0$$

$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

$S_1 = 0$

$S_2 = 2+1-4 = -1$

$S_3 = 4-2-2 = 0$

$\lambda^3 - \lambda - 0 = 0$

$\lambda(\lambda^2 - 1) = 0$

Eigen value

$\lambda = 0, -1, 1$

We know

$A^n X = \lambda^n X$

$\lambda = 0, \quad u = 0$

$\lambda = -1 \quad u = 1$

$\lambda = 1, \quad u = 1$

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7 Determine the algebraic multiplicity and geometric multiplicity of matrix

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

sol

$$\begin{bmatrix} 2-\lambda & 1 & 1 \\ 1 & 2-\lambda & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$\lambda^3 - S_1\lambda^2 + S_2\lambda + S_3 = 0$

$S_1 = 5$

$S_2 = 2+2+2+1-1$

$= 7$

$S_3 = 4-1 = 3$

$\lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0$

$(\lambda-1)(\lambda^2 - 4\lambda + 3) = 0$

$(\lambda-1)(\lambda-3)(\lambda-1) = 0$

$\lambda = 1, 1, 3$

AM of $\lambda = 1$ is 2 and $\lambda = 3$ is 1

$\lambda = 1$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$ $R_2 \to R_2 - R_1$

$z = K_1 \quad y = K_2$

$x = -K_1 - K_2$

Eigenvector

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -k_1-k_2 \\ k_2 \\ k_1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} k_1 + \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} k_2$$

GM of $\lambda = 1$ is 2

$$\lambda = 3 \quad \left[\begin{array}{ccc|c} -1 & 1 & 1 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 0 & -2 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & -2 & 0 \end{array}\right] \quad \begin{array}{l} R_1(-1) \\ R_2 \to R_2 + R_1 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad R_2 \to \frac{R_2}{2}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$y = k \quad x = k$

GM of $\lambda = 3$ is 1

8 If $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$ find eigen value of following matrix

i) $A^3 + I$

$$A^2 = A.A \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix} \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & 12 & 6 \\ 6 & 13 & 6 \\ 6 & 12 & 7 \end{bmatrix}$$

$$A^3 = A.A^2$$

$$\begin{bmatrix} 7 & 12 & 6 \\ 6 & 13 & 6 \\ 6 & 12 & 7 \end{bmatrix} \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 32 & 62 & 31 \\ 31 & 63 & 31 \\ 31 & 62 & 32 \end{bmatrix}$$

$$\begin{bmatrix} 32-\lambda & 62 & 31 \\ 31 & 63-\lambda & 31 \\ 31 & 62 & 32-\lambda \end{bmatrix} = 0$$

$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

$S_1 = 127$

$S_2 = 156 + 63 + 156 = 375$

$S_3 = 4992 - 6860 - 2883$

$= 249$

$\lambda^3 - 127\lambda^2 + 375\lambda - 249 = 0$

$\lambda - 1\ (\lambda^2 - 126\lambda + 249) = 0$ Eigen value is 1, 1, 124

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ii A^{-1}

By Cayley Hamilton

$$\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix} = 0$$

$S_1 = 7$

$S_2 = 4+3+4 = 11$

$S_3 = 2 \times 4 (-2)(1) - 1 = 5$

$A^3 - 7A^2 + 11A - 5 = 0$

$A^{-1}(A^3 - 7A^2 - 11A - 5) = 0$

$A^2 - 7A - 11I - 5A^{-1} = 0$

$A^{-1} = \frac{1}{5}(A^2 - 7A - 11I)$

$$A^{-1} = \frac{1}{5}\begin{bmatrix} 7 & 12 & 6 \\ 6 & 13 & 6 \\ 6 & 12 & 7 \end{bmatrix} - 7\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix} + \begin{bmatrix} 11 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 11 \end{bmatrix}$$

$$= \frac{1}{5}\begin{bmatrix} 21 & -2 & -1 \\ -1 & 2 & -1 \\ -1 & -2 & 4 \end{bmatrix}$$

iii> $A^2 - 2A + I$

$$A^2 - 2A + I = \begin{bmatrix} 7 & 12 & 6 \\ 7 & 13 & 6 \\ 6 & 12 & 7 \end{bmatrix} - 2\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 8 & 4 \\ 5 & 8 & 4 \\ 4 & 8 & 4 \end{bmatrix}$$

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$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

$S_1 = 16$

$S_2 = 0 + 0 - 8 = -8$

$S_3 = 0 - 32 + 32 = 0$

$\lambda^3 - 16\lambda^2 - 8\lambda = 0$

$\lambda(\lambda^2 - 16\lambda - 8) = 0$

iv> $A^3 - 3A^2 + A$

$$A^3 - 3A^2 + A = \begin{bmatrix} 32 & 62 & 31 \\ 31 & 63 & 31 \\ 31 & 62 & 32 \end{bmatrix} - 3\begin{bmatrix} 7 & 12 & 6 \\ 6 & 13 & 6 \\ 6 & 12 & 7 \end{bmatrix} + \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 13 & 28 & 14 \\ 14 & 27 & 31 \\ 14 & 28 & 13 \end{bmatrix}$$

$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

$S_1 = 53$

$S_2 = -517 - 27 - 41 = 449$

$S_3 = -6721 + 7056 + 196 = 531$

$\lambda^3 - 53\lambda^2 - 499\lambda - 531 = 0$

$\lambda = 1.4, 60.5, -6.18$

(iv) $A^3 - 3A^2 + A$

$$A^3 - 3A^2 + A = \begin{bmatrix} 32 & 62 & 31 \\ 31 & 63 & 31 \\ 31 & 62 & 32 \end{bmatrix} - 3\begin{bmatrix} 7 & 12 & 6 \\ 6 & 13 & 6 \\ 6 & 12 & 7 \end{bmatrix} + \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 13 & 28 & 14 \\ 14 & 27 & 31 \\ 14 & 28 & 13 \end{bmatrix}$$

The characteristic eq^n

$$\begin{vmatrix} 13-\lambda & 28 & 14 \\ 14 & 27-\lambda & 31 \\ 14 & 28 & 13-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

$S_1 = 53$

$S_2 = -517 - 27 - 41 = -449$

$S_3 = -6721 + 7056 + 196$

$= 531$

$$\lambda^3 - 53\lambda^2 - 449\lambda - 531 = 0$$

Eigen values

$\lambda = -1.43,\ 60.55,\ -6.128$

9. Verify Caley-Hamilton theorem and hence find the inverse $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$ and A^4. Also, express $A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I$ as a linear polynomial

Sol^n

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$$

The characteristic eq^n $|A - \lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & 4 \\ 2 & 3-\lambda \end{vmatrix} = 0$$

$\lambda^2 - \lambda S_1 + S_2 = 0$

$S_1 = 4$

$S_2 = -5$

$\lambda^2 - 4\lambda - 5 = 0$

By Caley Hamilton theorem

$A^2 - 4A - 5 = 0$

$$A^2 = A \cdot A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 9 & 16 \\ 8 & 12 \end{bmatrix}$$

$$A^2 - 4A - 5I = \begin{bmatrix} 9 & 16 \\ 8 & 12 \end{bmatrix} - \begin{bmatrix} 4 & 16 \\ 8 & 12 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 5 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

Hence Calyley Hamilton theorem verified.

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Now

$\because A^2 - 4A - 5I = 0$ by Applying A^{-1} on both sides

$A^{-1}(A^2 - 4A - 5I) = 0$

$5A^{-1} = A - 4I$

$A^{-1} = \frac{1}{5}(A - 4I) = \frac{1}{5}\left[\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} - 4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right]$

$= \frac{1}{5}\begin{bmatrix} -3 & 4 \\ 2 & -1 \end{bmatrix}$

And for A^4 applying A^2 both sides

$A^2(A^2 - 4A - 5I) = A^2(0)$

$A^4 - 4A^3 - 5A^2 = 0$

$A^4 = 4A^3 - 5A^2$

$A^4 = 4\begin{bmatrix} 41 & 84 \\ 42 & 83 \end{bmatrix} + 5\begin{bmatrix} 9 & 16 \\ 8 & 17 \end{bmatrix}$

$A^4 = \begin{bmatrix} 209 & 416 \\ 208 & 417 \end{bmatrix}$

Now

$\because A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I$

$= A^5 - 4A^4 - 5A^3 - 2A^3 + 11A^2 - A - 10I$

$= A^3(A^2 - 4A - 5) - 2A^3 + 11A^2 - A - 10I$

$= -2A^3 + 8A^2 + 3A^2 - 10I$

$= -2A(A^2 - 4A - 5) + 3A^2 - 6A - 10I$

$= 3A^2 - 12A + 6A - 15 - 5$

$= 3(A^2 - 4A - 5)\ 6A - 5 \quad = 6A - 5$

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10. Find the matrix P that diagonalizes matrix A and determine $P^{-1}AP$ where $A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$
Hence, find A^{13}

Sol^n The characteristic eq^n $|A - \lambda I| = 0$

$\begin{vmatrix} 0-\lambda & 0 & -2 \\ 1 & 2-\lambda & 1 \\ 1 & 0 & 3-\lambda \end{vmatrix} = 0$

$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

$S_1 = 5$

$S_2 = \begin{vmatrix} 2 & 1 \\ 0 & 3 \end{vmatrix} + \begin{vmatrix} 0 & -2 \\ 1 & 3 \end{vmatrix} + \begin{vmatrix} 0 & 0 \\ 1 & 2 \end{vmatrix}$

$= 6 + 2 = 8$

$S_3 = -2(-2) = 4$

$\lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0$

$(\lambda - 1)(\lambda^2 - 4\lambda + 4) = 0$

$(\lambda - 1)(\lambda - 2)(\lambda - 2) = 0$

Eigen values of A is

$\lambda = 1, 2, 2$

For $\lambda = 1$

$= \left[\begin{array}{ccc|c} -1 & 0 & -2 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 2 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 2 & 0 \end{array}\right] (-R_1)$

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$$= \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$z = k \quad , y = z = k \quad x + 2z = 0$

$x = -2k$

Eigen Vector of $\lambda = 1$ is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} k$$

$\lambda = 2$

$$\left[\begin{array}{ccc|c} -2 & 0 & -2 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right] \frac{-R_1}{2}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{array}$$

$z = k_1 \quad , y = k_2 \quad x = -k_1$

Eigen Vector of $\lambda = 2$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} k + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} k_2$$

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Diagonal Matrix (D) Modal Matrix (P)

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \qquad P = \begin{bmatrix} -2 & -1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$P^{-1} A^n P = D^n = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$A^n = P D^n P^{-1}$$

$$P = \begin{bmatrix} -2 & -1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} -2 & 0 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

The characteristic eq^n $|A - \lambda I| = 0$

$$\begin{vmatrix} -2-\lambda & 0 & -1 \\ 1 & 1-\lambda & 0 \\ 1 & 0 & 1-\lambda \end{vmatrix} = 0$$

$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

$S_1 = 0$

$S_2 = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + \begin{vmatrix} -2 & -1 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} -2 & 0 \\ 1 & 1 \end{vmatrix}$

$= 1 - 1 - 2 = -2$

$S_3 = \cancel{-2 + 1 = 0} \; -2 + 1 = -1$

$\lambda^3 - 2\lambda - 1 = 0$

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By Caley Hamilton theorem

$A^3 - 2A - I = 0$ Apply A^{-1} both side

$A^{-1}(A^3 - 2A - II) = 0$

$A^2 - 2I - A^{-1} = 0$

$A^{-1} = A^2 - 2I$

$$P^{-1} = \begin{bmatrix} -1 & 0 & -1 \\ 1 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix}$$

$$A^{13} = PD^{13}P^{-1} = \begin{bmatrix} -2 & 0 & -1 \\ -1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}^{13} \begin{bmatrix} -1 & 0 & -1 \\ 1 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 0 & -(2)^{13} \\ -1 & (2)^{13} & 0 \\ 1 & (2)^{13} & (2)^{13} \end{bmatrix} \begin{bmatrix} -1 & 0 & -1 \\ 1 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} -2+0+(2)^{13} & 0 & 2-2(2)^{13} \\ 1+(2)^{13}+0 & (2)^{13} & 1+(2)^{13}+0 \\ -1+(2)^{13}+(2)^{13} & (2)^{13} & -1+(2)^{13}+2(2^{13}) \end{bmatrix}$$

$$= \begin{bmatrix} -(2+2^{13}) & 0 & 2-(2)^{14} \\ 1+2^{13} & 2^{13} & 1+2^{13} \\ -1+2(2^{13}) & 2^{13} & -1+3(2)^{13} \end{bmatrix}$$

$$A^{13} = \begin{bmatrix} 8194 & 0 & -\cancel{8190}\ 16382 \\ 8193 & 8192 & 8193 \\ 16383 & 8192 & 24575 \end{bmatrix}$$

Multivariable Calculus

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11. Describe the nature of quadratic forms

$Q = -3x_1^2 - 3x_2^2 - 3x_3^2 - 2x_1x_2 - 2x_1x_3 + 2x_2x_3$

Sol^n The coefficient matrix of Q is

$$A = \begin{bmatrix} -3 & -1 & -1 \\ -1 & -3 & 1 \\ -1 & 1 & -3 \end{bmatrix}$$

Characteristic eq^n $|A - \lambda I| = 0$

$$\begin{vmatrix} -3-\lambda & -1 & -1 \\ -1 & -3-\lambda & 1 \\ -1 & 1 & -3 \end{vmatrix} = 0$$

$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

$S_1 = -9$

$S_2 = \begin{vmatrix} -3 & 1 \\ 1 & -3 \end{vmatrix} + \begin{vmatrix} -3 & -1 \\ -1 & -3 \end{vmatrix} + \begin{vmatrix} -3 & -1 \\ -1 & -3 \end{vmatrix}$

$= 8 + 8 + 8 = 24$

$S_3 = -24 + 4 + 4 = -16$

$\lambda^3 + 9\lambda^2 + 24\lambda + \cancel{2}16 = 0$

$(\lambda+1)(\lambda^2 + 8\lambda + 16) = 0$

$(\lambda+1)(\lambda+4)(\lambda+4) = 0$

Eigen Values is $\lambda = -1, -4, -4$

Nature of Qudratic form is Definite Negative cuz all eigen values are -ve.

Multivariable Calculus

12. Discuss the continuity of $f(x,y) = \frac{x^3-y^3}{x^3+y^3}, (x,y) \neq (0,0)$ is $(x,y)=(0,0)$

Sol

(i) $f(x,y)$ = exist

$f(0,0) = 15$

(ii) $\lim_{(x,y)\to(a,b)} f(x,y)$

Path-1

$= \lim_{\substack{x=0\\y=0}} \frac{x^3-y^3}{x^3+y^3} = \lim_{x\to 0} \frac{x^3}{x^3} = \lim_{x=0} 1 = 1$

Path II

$= \lim_{\substack{y=0\\x=0}} \frac{x^3-y^3}{x^3+y^3} = \lim_{y=0} \frac{-y^3}{y^3} = \lim_{y=0} -1 = -1$

Path II ≠ Path I So, limit does not exist

(iii) $f(x,y) = \lim_{(x,y)\to(a,0)} \frac{x^3-y^3}{x^3+y^3} \neq 15 = f(0,0)$

Three conditions is not satisfied

So,

$f(x,y)$ is not continuous at $(0,0)$

13. If $f(x,y,z) = \log(x^2+y^2+z^2)$, Prove that $xf_{yz} = yf_{zx} = zf_{xy}$

Sol

$f_x = \frac{2x}{x^2+y^2+z^2}$

$f_y = \frac{2y}{x^2+y^2+z^2}$

$f_z = \frac{2z}{x^2+y^2+z^2}$

$f_{yz} = \frac{0-2z(2y)}{(x^2+y^2+z^2)^2} = \frac{-4yz}{(x^2+y^2+z^2)^2}$

$xf_{yz} = \frac{-4xyz}{(x^2+y^2+z^2)^2}$

$f_{zx} = \frac{(x^2+y^2+z^2)(0) - 2x(2z)}{(x^2+y^2+z^2)^2} = \frac{-4xz}{(x^2+y^2+z^2)^2}$

$yf_{zx} = \frac{-4xyz}{(x^2+y^2+z^2)^2}$

$f_{xy} = \frac{(x^2+y^2+z^2)(0) - 2y(2x)}{(x^2+y^2+z^2)^2} = \frac{-4xy}{(x^2+y^2+z^2)^2}$

$zf_{xy} = \frac{-4xyz}{(x^2+y^2+z^2)^2}$

So,

$xf_{yz} = yf_{zx} = zf_{xy}$ H.P.

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14 If $f(x,y) = 3x^2 + 2y^3 - 2x^2y^2$. find all second order partial derivatives of $f(x,y)$ at $(2,-1)$.

Soln I^{st} order

$f(x) = 6x - 4xy^2$

$f(y) = 6y^2 - 4x^2y$

~~$f(xy) = -8xy$~~

Second order partial derivatives

$fxx = 6 - 4y^2 \Rightarrow fxx(2,-1) = 6 - 4(-1)^2 = 2$

$fyy = 12y - 4x^2 \Rightarrow fyy(2,-1) = -12 - 16 = -28$

$f(xy) = -8xy \Rightarrow fxy(2,-1) = 16$

$f(yx) = -8xy \Rightarrow f(yx)(2,-1) = 16$

15. If $u = \tan^{-1}(x^2 + 2y^2)$, the show that

(i) $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{20}$ ~~tan u~~ $\sin 2u$

Soln

$x = tx$, $y = ty$

$u = \tan^{-1}(t^2x^2 + 2t^2y^2)$

$\tan u = t^2(x^2 + 2y^2)$

$u = \tan u$ is hom. function with degree $n = 2$

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Now,

By Modified Euler formula

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\frac{f(u)}{f'(u)}$$

$$= 2\frac{\tan u}{\sec^2 u}$$

$$= 2\frac{\sin u}{\cos u}\cos^2 u$$

$$= 2\sin u \cos u$$

$$= \sin 2u \quad \text{H.P}$$

(ii) $x^2\frac{\partial^2 u}{\partial x^2} + xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 2\sin u \cos 3u$

Soln By modified Euler formula

$$x^2\frac{\partial^2 u}{\partial x^2} + xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = g(u)(g'(u) - 1)$$

$$g(u) = n\frac{f(u)}{f'(u)} = \sin 2u$$

Now

$$= \sin 2u(2\cos 2u - 1)$$

$$= 2\sin 2u \cos 2u - \sin 2u$$

$$= \sin 4u - \sin(0) - \sin 2u$$

$$= 2\cos 3u \sin u$$

$$= 2\sin u \cos 3u \quad \text{H.P}$$

16. If $u=\sin^{-1}\left(\frac{x^{1/4}+y^{1/4}}{x^{1/5}+y^{1/5}}\right)$ then prove that

(i) $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{1}{20}\tan u$

Sol^n

$x=tx$ & $y=ty$

$$u=\sin^{-1}\left(\frac{(tx)^{1/4}+(ty)^{1/4}}{(tx)^{1/5}+(ty)^{1/5}}\right)$$

$$\sin u=\frac{t^{1/4}}{t^{1/5}}\left(\frac{x^{1/4}+y^{1/4}}{x^{1/5}+y^{1/5}}\right)$$

$$\sin u=t^{1/20}\left(\frac{x^{1/4}+y^{1/4}}{x^{1/5}+y^{1/5}}\right)$$

$u=\sin u$ is a Hom. fun^n with degree $n=\frac{1}{20}$

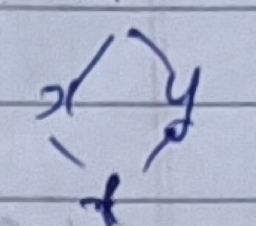
By Modified Euler formula

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=n\frac{f(u)}{f'(u)}$$

$$=\frac{1}{20}\frac{\sin u}{\cos u}$$

$$=\frac{1}{20}\tan u \quad \underline{H.P.}$$

17. If $u=\sin\left(\frac{x}{y}\right)$, $y=t^2$, $x=e^t$ find $\frac{du}{dt}$



Sol^n

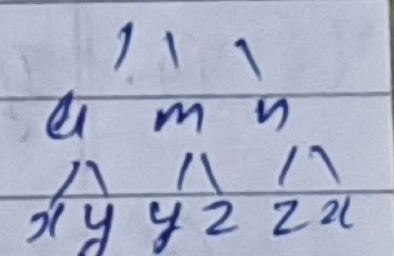
$$\frac{du}{dt}=\frac{\partial u}{\partial x}\frac{dx}{dt}+\frac{\partial u}{\partial y}\frac{dy}{dt}$$

$$=\cos\left(\frac{x}{y}\right)e^t+\cos\left(\frac{x}{y}\right)\left(-\frac{x}{y^2}\right)$$

$$=e^t\cos\left(\frac{x}{y}\right)-\frac{x}{y^2}\cos\left(\frac{x}{y}\right).$$

18. If $u=f(x^2-y^2,\ y^2-z^2,\ z^2-x^2)$ show that

$$\frac{1}{x}\frac{\partial u}{\partial x}+\frac{1}{y}\frac{\partial u}{\partial y}+\frac{1}{z}\frac{\partial u}{\partial z}=0$$



Sol^n let,

$l=x^2-y^2,\ m=y^2-z^2,\ n=z^2-x^2$

$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial l}\frac{\partial l}{\partial x}+\frac{\partial u}{\partial n}\frac{\partial n}{\partial x}$$

$$=\frac{\partial u}{\partial l}(2x)+\frac{\partial u}{\partial n}(-2x)$$

$$\frac{\partial u}{\partial y}=\frac{\partial u}{\partial l}\frac{\partial l}{\partial y}+\frac{\partial u}{\partial m}\frac{\partial m}{\partial y}$$

$$=\frac{\partial u}{\partial l}(-2y)+\frac{\partial u}{\partial m}(2y)$$

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$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial m}\frac{\partial m}{\partial z} + \frac{\partial u}{\partial n}\frac{\partial n}{\partial z}$$

$$= \frac{\partial u}{\partial m}(-2z) + \frac{\partial u}{\partial m}(2z)$$

L.H.S

$$= \frac{1}{x}\frac{\partial u}{\partial x} + \frac{y}{}\frac{\partial u}{\partial y} + \frac{1}{z}\frac{\partial u}{\partial z}.$$

$$= \frac{\partial u}{\partial l}(2) - \frac{\partial u}{\partial n}(2) - \frac{\partial u}{\partial l}(2) + \frac{\partial u}{\partial m}(2) - \frac{\partial u}{\partial m}(2) + \frac{\partial u}{\partial m}(2)$$

$= 0$ H.P

19. If $y^{x^y} = \sin x$ then fin $\frac{dy}{dx}$

Sol^n

~~$f(x,y) = y^{x^y} - \sin x$~~

~~$fx = y^{x^y} \log$~~

Taking log on both side

$x^y \log y = \log \sin x$

Partial derivative with resp. to. x and y

$$f(x) = yx^{y-1}\log y - \frac{\cos x}{\sin x}$$

$$f(y) = x^y \log x \cdot \log y + \frac{x^y}{y}$$

$$\frac{dy}{dx} = \frac{-fx}{fy} = \frac{\cot x - yx^{y-1}\log y}{\frac{x^y}{y} + x^y \log x \cdot \log y}$$

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20. If $u = xyz$, $v = xy + yz + zx$ and $w = x + y + z$ then evaluate $\frac{\partial(u,v,w)}{\partial(x,y,z)}$

Sol^n

$$J = \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

$R_1 \leftrightarrow R_3$

$$= \begin{vmatrix} yz & xz & xy \\ y+z & x+z & x+y \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ y+z & x+z & x+y \\ yz & xz & yx \end{vmatrix}$$

$= xy(x+z) - xz(x+y) - (y+z)yx + yz(x+y) + xz(y+z) - yz(x+z)$

$= x^2y + \cancel{xyz} - x^2z - \cancel{xyz} - y^2x - xyz + xyz + y^2z + xyz + xz^2 - xyz - yz^2$

$= x^2y - x^2z - y^2x + y^2z + xz^2 - yz^2$

$= x^2(y-z) + y^2(z-x) + z^2(x-y)$

21. Find the eq^n of tangents plane and normal line for the surface $xyz = 4$ at $(1,2,2)$

Sol^n

$f(x,y,z) = xyz - 4$

$x_0 = 1$, $y_0 = 2$ & $z_0 = 2$

$fx = yz \Rightarrow f_x(1,2,2) = 4$

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$f_y = xz \Rightarrow f_y(1,2,2) = 2$

$f_z = xy \Rightarrow f_z(1,2,2) = 2$

The Tangent Plane

$f_x(P_0)(x-x_0) + f_y(P_0)(y-y_0) + f_z(P_0)(z-z_0) = 0$

$4(x-1) + 2(y-2) + 2(z-2) = 0$

$4x-4 + 2y-4 + 2z-4 = 0$

$x+y+z-6 = 0$

Normal line

$$\frac{x-x_0}{f_x(P_0)} = \frac{y-y_0}{f_y(P_0)} = \frac{z-z_0}{f_z(P_0)}$$

$$\frac{x-1}{4} = \frac{y-2}{2} = \frac{z-2}{2}$$

22. Show that tangent plane to the surface $x^2 = y(x+z)$ at any point passes through the origin

Sol^n

let Points (a,b,c) $\quad x_0 = a,\ y_0 = b$ & $z_0 = c$

$f(x,y) = x^2 - yx - yz$

$f(x) = 2x - y \Rightarrow f_x(a,b,c) = 2a-b$

$f_y = -x-z \Rightarrow f_y(a,b,c) = -a-c$

$f_z = -y \Rightarrow f_z(a,b,c) = -b$

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$f_x(P_0)(x-x_0) + f_y(P_0)(y-y_0) + f_z(P_0)(z-z_0) = 0$

$(2a-b)(x-a) + (-a-c)(y-b_0) + (-b)(z-c) = 0$

Passing through the origin $(x,y,z) = (0,0,0)$

$f_x(P_0)(0-x_0) + f_y(P_0)(0-y_0) + f_z(P_0)(z-z_0) = 0$

$(2a-b)(0-a) + (-a-c)(0-b) + (-b)(0-c) = 0$

$-2a^2 + 2ab + 2bc = 0$

$a^2 - ab - bc = 0$

23. Find the extreme values of the function $f(x,y) = x^2+2y^2$ on the circle $x^2+y^2=1$.

Sol^n

$f(x,y) = x^2+2y^2$

$\phi(x,y) = x^2+y^2-1$

let the eq^n

$f(x,y) + \lambda\phi(x,y) = 0$

$x^2+2y^2 + \lambda(x^2+y^2-1)$

Differ Partially w.r.t. x and y

$2x + 2\lambda x = 0$

$\lambda = -\frac{2x}{2x} = -1$

$4y + 2\lambda y = 0$

$\lambda = -\frac{4y}{2y}$

$\lambda = -2$

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$x^2+y^2=1$

Put $x=0$ $\quad$ $y=0$

$y=\pm 1$ $\quad$ $x=\pm 1$

at point $(0,4)$, $(1,0)$

$Max^m = 2$

$Min = 1$

24. Find the shortest and longest distance from the point $(1,2,-1)$ to the sphere $x^2+y^2+z^2=24$

Sol^n

let $f(x,y,z) = (x-1)^2+(y-2)^2+(z+1)^2 = d^2$

and $\phi(x,y,z) = x^2+y^2+z^2-24=0$ $\quad$ — (1)

let the eq^n

$(x-1)^2+(y-2)^2+(z+1)^2+\lambda(x^2+y^2+z^2-24)=0$

Partially Diff w.r.t. $x, y+z$

$2(x-1)+2\lambda x = 0$

$\lambda = -1+\frac{1}{x}$

$2(y-2)+2\lambda y = 0$

$\lambda = -1+\frac{2}{y}$

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$2(z+1)+2\lambda z = 0$

$\lambda = -1-\frac{1}{z}$

$\frac{1}{x} = \frac{2}{y} = -\frac{1}{z}$

$y=2x$ and $z=-x$

Substituting y and z in Eq^n(1)

$x^2+(2x)^2+(-x)^2 = 24$

$x^2+4x^2+x^2 = 24$

$6x^2 = 24$

$x^2 = \frac{24}{6}$

$x = \pm 4$

$y = \pm 8$

$z = 4$

Shortest distance $\sqrt{(4-1)^2+(8-2)^2+(4+1)^2} = 8.36$ m

longest distance $\sqrt{(-4-1)^2+(-8-2)^2+(4+1)^2} = 12.24$

25. Find maxima and minima of the function $f(x,y) = x^3+y^2-3x-12y+20$

Sol^n

$f_x = 3x^2-3$

$f_y = 2y-12$

$f_x = 0$

$3(x^2-1) = 0$

$x = \pm 1$

$f_y = 0$

$2y - 12 = 0$

$y = 6$

Second derivative

$r = f_{xx} = 6x$

$s = f_{xy} = 0$

$t = f_{yy} = 2$

The stationary Pts are $(1, 6)$ $(-1, 6)$

(x, y)	r	s	t	$rt - s^2 > 0$	conclusion	$f(x, y)$
$(1, 6)$	6	0	2	$12 > 0$	min	~~126~~ -18
$(-1, 6)$	-6	0	2	$-12 < 0$	~~saddle pt~~ max	-14

26. Expand $\tan^{-1}\left(\frac{y}{x}\right)$ in powers of $(x-1)$ and $(y-1)$ using Taylor's expansion.

Soln

$f(x, y) = \tan^{-1}\left(\frac{y}{x}\right)$ $\quad a = 1$ & $b = 1$

Taylor's formula

$$f(x, y) = f(a, b) + (x-a) f_x(a, b) + (y-b) f_y(a, b) + \frac{1}{2!}\left[(x-a)^2 f_{xx}(a, b) + 2(x-a)(y-b) f_{xy}(a, b) + (y-b)^2 f_{yy}(a, b)\right]$$

$$f_x = \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(\frac{-y}{x^2}\right) \Rightarrow f_x(1,1) = \frac{-1}{2}$$

$$f_y = \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(\frac{1}{x}\right) \Rightarrow f_y(1,1) = \frac{1}{2}$$

$$f_{xx} = \frac{2x^2 \cancel{+y^2}(-y) + xy\,2x}{(x^2+y^2)^2} = \frac{-2xy + 2x^2 y}{(x^2+y^2)^2}$$

$f_{xx}(1,1) \Rightarrow 0$

$$f_{yy} = \frac{-x\,2y}{(x^2+y^2)^2} = \frac{-2}{4} = -\frac{1}{2}$$

$$f_{xy} = \frac{x(2x) - (x^2+y^2)}{(x^2+y^2)^2} = \frac{2x^2 - x^2 + y^2}{(x^2+y^2)^2} = \frac{x^2+y^2}{(x^2+y^2)^2}$$

$$f_{xy}(1,1) \Rightarrow \frac{2}{4} = \frac{1}{2} \qquad f(1,1) = 45$$

$$f(x, y) = 45 + (x-1)\left(-\frac{1}{2}\right) + (y-1)\frac{1}{2} + \frac{1}{2}\left[(x-1)^2(0) + 2(x-1)(y-1)\frac{1}{2} + (y-1)^2\left(-\frac{1}{2}\right)\right]$$

$$= 45 - \frac{x-1}{2} + \frac{y-1}{2} + \frac{1}{2}\left[\frac{2xy - 2x + 2 - 2y}{2} - \frac{y^2 - 2y + 1}{2}\right]$$

$$= 45 - \frac{x-1}{2} + \frac{y-1}{2} + \frac{2xy - y^2 - 2x + 1}{2}$$