$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & -4 & -40 \\ 0 & 1 & 0 & -1 & 5 \\ 0 & 0 & 1 & -2/4 & 75/4 \\ 0 & 0 & 0 & 1 & 22.5 \end{array}\right] \quad \begin{array}{l} R_4 \to R_4 + 4R_3 \\ R_4 \to R_4\left(\frac{1}{12}\right) \end{array}$$

$T_4 = 22.5$

$T_3 - T_4\left(\frac{2}{4}\right) = 75/4$

$T_3 = 30$

$T_2 = 5 + T_4 = 27.5$

$T_1 = -40 - T_3 - 4T_4$

$T_1 = \underline{20}$

So,

$T_1 = 20$

$T_2 = 27.5$

$T_3 = 30$

$T_4 = 22.5$

Unit:- 6

Multivariable Calculus

Tutorial - 2

1. Evaluate the following limits, if exists:

a) $\lim\limits_{(x,y)\to(-1,2)} \dfrac{x^3+y^3}{x^2+y^2}$

Soln Path - I

$$\lim_{\substack{x\to(-1)\\ y\to 2}} \frac{x^3+y^3}{x^2+y^2} = \lim_{x\to -1} \frac{x^3+8}{x^2+4} = \frac{-1+8}{1+4} = \frac{7}{5}$$

$$\lim_{\substack{x\to -1\\ y\to 2}} f(x,y) = \frac{(-1)^3+2^3}{(-1)^2+2^2} = \frac{-1+8}{1+4} = \frac{7}{5} \text{ Ans}$$

(b) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2+y^2+1}{3+x^2+3y^2}$

Soln

$$\lim_{\substack{x\to 0\\ y\to 0}} f(x,y) = \frac{1}{3} \text{ Ans}$$

(c) $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{x^2+y}$

Soln $\lim\limits_{\substack{x\to 0\\ y\to 0}} f(x,y) = \dfrac{0}{0}$ No Defined

Then Path - I

$x \to 0, y \to 0$

$$= \lim_{\substack{x \to 0 \\ y \to 0}} f(x,y) = \lim_{\substack{x \to 0 \\ y \to 0}} \frac{xy}{x^2+y} = \lim_{x \to 0} \frac{0}{x^2} = 0$$

~~lim~~ Path - II

$y \to 0, x \to 0$

$$= \lim_{\substack{y \to 0 \\ x \to 0}} \frac{xy}{x^2+y} = \lim_{y \to 0} \frac{0}{y} = 0$$

Path - III

$$= \lim_{\substack{x \to 0 \\ y \to mx}} \frac{xy}{x^2+y} = \frac{x\,mx}{x^2+mx} \lim_{x \to 0} = \lim_{x \to 0} \frac{mx^2}{x^2+mx}$$

$$= \lim_{x \to 0} \frac{mx}{x+m} = \frac{0}{m} = 0$$

Path - IV

$$= \lim_{\substack{x \to 0 \\ y \to mx^2}} \frac{xy}{x^2+y} = \lim_{x \to 0} \frac{x\,mx^2}{x^2+mx^2}$$

$$= \lim_{x \to 0} \frac{mx}{1+m} = \frac{0}{1+m} = 0$$

Hence, All path have same limit the limit of the $f(x,y)$ = $\lim_{(x,y) \to (0,0)} \frac{\cancel{x^3+y^2}\,xy}{x^2+y} = 0$.

d. $\lim_{(x,y) \to (0,0)} \frac{2x^2y^2}{x^4+y^2}$

Sol^n

$$\lim_{(x,y) \to (0,0)} f(x,y) = \frac{0}{0}$$

Path - I

$x \to 0, y \to 0$

$$\lim_{\substack{x \to 0 \\ y \to 0}} \frac{2x^2y^2}{x^4+y^2} = \lim_{x \to 0} \frac{0}{x^4} = 0$$

~~lim~~ Path - II

$y \to 0, x \to 0$

$$\lim_{\substack{y \to 0 \\ x \to 0}} \frac{2x^2y^2}{x^4+y^2} = \lim_{y \to 0} \frac{0}{y^2} = 0$$

Path III

$$\lim_{\substack{x \to 0 \\ y \to mx}} \frac{2x^2y^2}{x^4+y^2} = \lim_{x \to 0} \frac{2x^2m^2x^2}{x^4+m^2x^2} = \lim_{x \to 0} \frac{2x^2m^2x^2}{x^2(x^2+m^2)}$$

$$= \lim_{x \to 0} \frac{m^2x^2}{x^2+m^2} = \lim_{x \to 0} \frac{0}{m^2} = 0$$

Path IV

$$\lim_{\substack{x \to 0 \\ y \to mx^2}} \frac{2x^2y^2}{x^4+y^2} = \lim_{x \to 0} \frac{2x^2(mx^2)^2}{x^4+(mx^2)^2} = \lim_{x \to 0} \frac{2x^2m^2x^4}{x^4(1+m^2)}$$

$$= \lim_{x \to 0} \frac{x^2}{1+m^2} = \frac{0}{1+m^2} = 0$$

Limits exists at $(0,0)$

e. $\lim_{(x,y)\to(0,0)} \frac{x^2y}{y^2+x^4}$

Sol^n

$= \frac{0}{0}$ Not Defined

Path - I

$x\to 0, y\to 0$

$\lim_{\substack{x\to 0\\ y\to 0}} \frac{x^2y}{y^2+x^4} = \lim \frac{0}{y^2} \lim_{x\to 0} \frac{0}{x^4} = 0$

Path II

$\lim_{\substack{y\to 0\\ x\to 0}} \frac{x^2y}{y^2+x^4} \lim_{y\to 0} \frac{0}{y^2} = 0$

Path - III

$\lim_{\substack{x\to 0\\ y\to mx}} \frac{x^2y}{y^2+x^4} = \lim_{x\to 0} \frac{x^2mx}{mx^2+x^4} = \lim_{x\to 0} \frac{x^2mx}{x^2(m+x^2)}$

$= \lim_{x\to 0} \frac{mx}{m+x^2} = \frac{0}{m} = 0$

Path - IV

$\lim_{\substack{x\to 0\\ y\to mx^2}} \frac{x^2y}{y^2+x^4} = \lim_{x\to 0} \frac{x^2mx^2}{(mx^2)^2+x^4} = \lim_{x\to 0} \frac{mx^4}{m^2x^4+x^4}$

$= \lim_{x\to 0} \frac{m}{m^2+1} = \frac{m}{m^2+1}$

- In Path - IV limit depends on m and which is not same so, the limit does not exist at (0,0).

2. Check whether the given function is continuous or not, if yes then find points of continuity.

a. $f(x,y) = \begin{cases} \frac{x^2y^2}{2x^2+y^2}, & \text{if } (x,y) \neq (0,0) \\ 1, & \text{if } (x,y) = (0,0) \end{cases}$

Sol^n

(i) $f(x,y)$ exists

$f(0,0) = 1$

(ii) $\lim_{(x,y)\to(0,b)} f(x,y)$

Path - I

$= \lim_{\substack{x\to 0\\ y\to 0}} \frac{x^2y^2}{2x^2+y^2} = \frac{0}{2x^2} = 0$

Path - II

$\lim_{\substack{y\to 0\\ x\to 0}} \frac{x^2y^2}{2x^2+y^2} = \lim_{y\to 0} \frac{0}{y^2} = 0$

(iii) $\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{2x^2+y^2} \neq 1 = f(0,0)$

Three conditions is not satisfied

So, $f(x,y)$ is not continuous at $(0,0)$.

b. $f(x,y) = \frac{xy}{x^2+xy+y^2}$, if $(x,y) \neq (0,0)$
if $(x,y) = (0,0) = 0$

Sol^n

(i) $f(x,y)$ = exist
$f(0,0) = 0$

(ii) $\lim_{(x,y)\to(a,b)} f(x,y)$

Path-I

$= \lim_{\substack{x\to0\\y\to0}} \frac{xy}{x^2+xy+y^2} = \lim_{x\to0} \frac{0}{x^2} = 0$

Path-II

$= \lim_{\substack{x\to0\\y\to0}} \frac{xy}{x^2+xy+y^2} = \lim_{y\to0} \frac{0}{y^2} = 0$

Path-III

$= \lim_{\substack{x\to0\\y\to mx}} \frac{xy}{x^2+xy+y^2} = \lim_{x\to0} \frac{xmx}{x^2mx^2+m^2x^2}$

$= \lim_{x\to0} \frac{mx^2}{x^2(1+m+m^2)} = \lim_{x\to0} \frac{m}{1+m+m^2}$

$= \frac{m}{1+m+m^2}$

limit depends on m

So, limit does not exist

Two condition are not satisfied

Hence

$f(x,y)$ is not continuous at $(0,0)$.

c. $f(x,y) = \frac{x^2y}{x^2+y^2}$ if $(x,y) \neq (0,0)$
if $(x,y) = (0,0)$

Sol^n

(i) $f(x,y)$ = exist
$f(0,0)$

(ii) $\lim_{(x,y)\to(a,b)} f(x,y)$

Path-I

$= \lim_{\substack{x\to0\\y\to0}} \frac{x^2y}{x^2+y^2} = \frac{0}{x^2} = 0$

Path - II

$$\lim_{\substack{y\to 0\\ x\to 0}} \frac{x^2 y}{x^2+y^2} = \lim_{y\to 0} \frac{0}{y^2} = 0$$

limit ~~does not~~ exit

Two condition does not satisfied

So,

$f(x,y)$ is not continuous at $(0,0)$

d. $f(x,y) = \dfrac{xy}{\sqrt{x^2+y^2}}$ if $(x,y) \neq (0,0)$

0 if $(x,y) = (0,0)$

Soln

(i) $f(x,y)$ exist at $f(0,0) = 0$

(ii) $\lim_{(x,y)\to(a,b)} f(x,y)$

Path - I

$$\lim_{\substack{x\to 0\\ y\to 0}} \frac{xy}{\sqrt{x^2+y^2}} = \lim_{x\to 0} \frac{0}{x} = 0$$

Path - II

$$\lim_{\substack{y\to 0\\ x\to 0}} \frac{xy}{\sqrt{x^2+y^2}} = \lim_{y\to 0} \frac{0}{y} = 0$$

Path - II

$$\lim_{\substack{x\to 0\\ y\to mx}} \frac{xy}{\sqrt{x^2+y^2}} = \lim_{x\to 0} \frac{xmx}{\sqrt{x^2+m^2x^2}} = \lim_{x\to 0} \frac{mx^2}{x\sqrt{1+m^2}}$$

$$= \lim_{x\to 0} \frac{mx}{\sqrt{1+m^2}} = \frac{m\times 0}{\sqrt{1+m^2}} = 0$$

Path - III

$$\lim_{\substack{x\to 0\\ y\to mx^2}} \frac{xy}{\sqrt{x^2+y^2}} = \lim_{x\to 0} \frac{xmx^2}{\sqrt{x^2+m^2x^4}} = \lim_{x\to 0} \frac{mx^3}{x\sqrt{1+m^2x^2}}$$

$$= \lim_{x\to 0} \frac{mx^2}{\sqrt{1+mx^2}} = \frac{m\times 0}{\sqrt{1+m\times 0}} = \frac{0}{1} = 0$$

limit exist,

~~But~~ Also $\lim_{(x,y)\to(0,0)} \dfrac{xy}{\sqrt{x^2+y^2}} = 0 = f(x,y)$

Hence $f(x,y)$ is ~~dis~~ continuous at $(0,0)$.

3. Find the first order partial derivatives of the following functions.

a. $f(x,y) = y\sin(xy)$ at $(0, \frac{\pi}{2})$

Soln I^{st} order.

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(y\sin(xy))$$

$$f_x = y\cos xy \cdot y$$

$$f_x = y^2\cos xy$$

$$f_y = y\cos xy \cdot x + \sin xy$$

$$= xy\cos xy + \sin xy$$

Then

$$f_x(0, \tfrac{\pi}{2}) = \frac{\pi^2}{4}\cos 0 = \frac{\pi^2}{4}$$

$$f_y(0, \tfrac{\pi}{2}) = \sin 0 = 0$$

b. $f(x,y) = x^2 + 3xy + y - 1$ at $(4, -5)$

Soln

$$f(x,y) = x^2 + 3xy + y - 1$$

$$f_x = 2x + 3y$$

$$f_y = 3x + 1$$

Then

$$f_x(4,-5) = 2\times 4 + 3(-5) = -7$$

$$f_y(4,-5) = 3\times 4 + 1 = 13$$

c. $f(x,y,z) = \frac{x}{y+z}$ at $(3,2,1)$

Soln

$$f(x,y,z) = \frac{x}{y+z}$$

$$f_x = \frac{1}{y+z}$$

$$f_y = \frac{-x}{(y+z)^2}$$

$$f_z = \frac{-x}{(y+z)^2}$$

Then

$$f_x(3,2,1) = \frac{1}{2+1} = \frac{1}{3}$$

$$f_y(3,2,1) = \frac{-3}{(2+1)^2} = -\frac{1}{3}$$

$$f_z(3,2,1) = \frac{-3}{(2+1)^2} = -\frac{1}{3}$$

d. $f(x,y,z) = e^{(x^2+y^2+1)}$ at $(1,1,1)$

Soln

$$f(x,y,z) = e^{(x^2+y^2+1)}$$

$$f_x = e^{(x^2+y^2+1)} \cdot 2x$$

$$f_y = e^{(x^2+y^2+1)} \cdot 2y$$

$$f_z = e^{(x^2+y^2+1)} \cdot 0 = 0$$

Then

$$f_x(1,1,1) = e^{(1+1+1)} \cdot 2$$
$$= 2e^3$$

$$f_y(1,1,1) = e^{(1+1+1)} \cdot 2 = 2e^3$$

$$f_z(1,1,1) = 0$$ Ans

4. Find all second order partial derivatives of the following function

Soln

a. $x^2 y \sin x$

Soln

$$f = x^2 y \sin x$$

$$f_x = 2xy\sin x + x^2 y \cos x$$

$$f_{xx} = 2y\sin x + 2xy\cos x + 2xy\cos x - x^2 y \sin x$$
$$= 2y\sin x + 4xy\cos x - x^2 y \sin x$$

$f_y = x^2 \sin x$ | $f_{xy} = 2x\cos x$

$f_{yy} = 0$ | $f_{yx} = 2x\sin x + x^2\cos x$

b. $x^3 y \sin y$

Soln let

$$u = x^3 y \sin y$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = 6xy\sin y$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial y}\left(x^3 y\cos y + x^3 \sin y\right)$$
$$= x^3\cos y - x^3 y\sin y + x^3\cos y$$
$$= 2x^3\cos y - x^3 y\sin y$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}\left(x^3 y\cos y + x^3\sin y\right)$$
$$= 3x^2 y\cos y + 3x^2 \sin y$$

c. $x^3 + y^3$

Soln let $f(x^3+y^3)$

$$f_{xx} = 6x$$
$$f_{yy} = 6y$$
$$f_{xy} = \cancel{3x^2+3y^2} = 0$$
$$f_{yx} = 0$$

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial y}(6xy\sin y)$$
$$= 6xy\cos y + 6x\sin y$$

d. $\sin(xy)$

Sol

$u = \sin(xy)$

$\frac{\partial^2 u}{\partial x^2} = -\sin(xy)y^2 \quad = -y^2(\sin(xy))$

$\frac{\partial^2 u}{\partial y^2} = -\sin(xy)x^2 = -x^2\sin(xy)$

$\frac{\partial^2 u}{\partial x \partial y} = -\sin(xy)\,xy \quad = -xy\sin(xy)$

$\frac{\partial^2 u}{\partial y \partial x} = -xy\sin(xy)$

5. Find $\frac{\partial^3 u}{\partial x \partial y \partial z}$ for $u = e^{5xyz}$

Sol

$u = e^{5xyz}$

$\frac{\partial u}{\partial z} = e^{5xyz} \cdot 5xy$

$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial z}\right) = e^{5xyz} \cdot 5x + 5xy \cdot e^{5xyz} \cdot 5xz$

$\frac{\partial^2 u}{\partial y \partial z} = 5x e^{5xyz} + 25x^2yz\, e^{5xyz}$

$= e^{5xyz}(5x + 25x^2yz)$

$\frac{\partial}{\partial x}\left(\frac{\partial^2 u}{\partial y \partial z}\right) = \frac{\partial}{\partial x} e^{5xyz}(5x + 25x^2yz)$

$= e^{5xyz}(5x + 25x^2yz)\,5yz + e^{5xyz}(5 + 50xyz)$

$= e^{5xyz}(5xyz + 25x^2y^2z^2) + e^{5xyz}(5 + 50xyz)$

$= e^{5xyz}(55xyz + 25x^2y^2z^2 + 5)$

$= e^{5xyz}(25x^2y^2z^2 + 55xyz + 5)$ Ans

6. If $u(x,y) = \log(\sqrt{x^2+y^2})$ then show that

$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

Sol

$u = \log(\sqrt{x^2+y^2})$

L.H.S.

$\frac{\partial u}{\partial x} = \frac{1}{x^2+y^2} \cdot 2x$

$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{2x}{x^2+y^2}\right)$

$= \frac{(x^2+y^2)2 - 2x(2x)}{(x^2+y^2)^2}$

$$= \frac{2x^2+2y^2-4x^2}{(x^2+y^2)^2}$$

$$= \frac{2y^2-2x^2}{(x^2+y^2)^2}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{2y}{x^2+y^2}\right)$$

$$= \frac{(x^2+y^2)2 - 2y(2y)}{(x^2+y^2)^2}$$

$$= \frac{2(x^2+y^2)-4y^2}{(x^2+y^2)^2}$$

$$= \frac{2x^2-2y^2}{(x^2+y^2)^2}$$

~~Also~~ L.H.S

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{2y^2-2x^2}{(x^2+y^2)^2} + \frac{2x^2-2y^2}{(x^2+y^2)^2}$$

$$= \frac{0}{(x^2+y^2)^2}$$

$$= 0$$

L.H.S = R.H.S H.P

7. a. If $u = \frac{e^{(x+y+z)}}{e^x+e^y+e^z}$ then show that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 2u$.

Soln

$$u = \frac{e^{(x+y+z)}}{e^x+e^y+e^z}$$

L.H.S

$$\frac{\partial u}{\partial x} = \frac{(e^x+e^y+e^z)(e^{x+y+z}) - e^{(x+y+z)} \cdot e^x}{(e^x+e^y+e^z)^2}$$

$$= \frac{(e^y+e^z)\, e^{(x+y+z)}}{(e^x+e^y+e^z)^2}$$

$$\frac{\partial u}{\partial y} = \frac{(e^x+e^y+e^z)\, e^{x+y+z} - e^{x+y+z} \cdot e^y}{(e^x+e^y+e^z)^2}$$

$$= \frac{(e^x+e^z)\, e^{x+y+z}}{(e^x+e^y+e^z)^2}$$

$$\frac{\partial u}{\partial z} = \frac{(e^x+e^y+e^z)\, e^{x+y+z} - e^{x+y+z} \cdot e^z}{(e^x+e^y+e^z)^2}$$

$$= \frac{(e^x+e^y)\, e^{x+y+z}}{(e^x+e^y+e^z)^2}$$

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=\frac{e^{x+y+z}\left(2e^{x}+2e^{y}+2e^{z}\right)}{\left(e^{x}+e^{y}+e^{z}\right)^{2}}$$

$$=\frac{e^{x+y+z}\,2\left(e^{x}+e^{y}+e^{z}\right)}{\left(e^{x}+e^{y}+e^{z}\right)^{2}}$$

$$=\frac{2e^{x+y+z}}{e^{x}+e^{y}+e^{z}}$$

$$=2u$$

L.H.S = R.H.S H.P

b. If $u=\log(x^3+y^3+z^3-3xyz)$ then prove that

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u=\frac{-9}{(x+y+z)^2}$$

Sol

$u=\log(x^3+y^3+z^3-3xyz)$

Diff w.r.t. x, y, z respectively

$$\frac{\partial u}{\partial x}=\frac{3x^2-3yz}{x^3+y^3+z^3-3xyz}$$

$$\frac{\partial u}{\partial y}=\frac{3y^2-3xz}{x^3+y^3+z^3-3xyz}$$

$$\frac{\partial u}{\partial z}=\frac{3z^2-3xy}{x^3+y^3+z^3-3xyz}$$

Now

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=\frac{3x^2-3yz+3y^2-3xz+3z^2-3xy}{x^3+y^3+z^3-3xyz}$$

$$=\frac{3x^2+3y^2+3z^2-3xy-3yz-3xz}{x^3+y^3+z^3-3xyz}$$

$$=\frac{3(x^2+y^2+z^2-xy-yz-xz)}{x^3+y^3+z^3-3xyz}$$

$$=\frac{3(x^2+y^2+z^2-xy-yz-xz)}{(x+y+z)(x^2+y^2+z^2-xy-yz-xz)}$$

$$=\frac{3}{x+y+z}$$

L.H.S

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u=\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)$$

$$=\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{3}{x+y+z}\right)$$

$$=\frac{-3}{(x+y+z)^2}+\frac{-3}{(x+y+z)^2}+\frac{-3}{x+y+z}$$

$$=\frac{-9}{(x+y+z)^2}$$

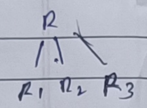
L.H.S = R.H.S H.P

8. If resistors of R_1, R_2, R_3 ohms are connected in parallel to make an R-ohm resistor, the value of R can be found from the eqn $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$

Find the value of $\frac{\partial R}{\partial R_2}$ when $R_1 = 30\Omega$, $R_2 = 45\Omega$ and $R_3 = 90\Omega$

Sol.

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$$



D.w.r.t. R_2 on both side

$$\frac{\partial}{\partial R_2}\left(\frac{1}{R}\right) = \frac{\partial}{\partial R_2}\left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right)$$

Using the chain rule and considering that R is a function of R_2,

$$-\frac{1}{R^2} \cdot \frac{\partial R}{\partial R_2} = -\frac{1}{R_2^2}$$

$$\frac{\partial R}{\partial R_2} = \frac{R^2}{R_2^2}$$

Given.

$R_1 = 30\Omega$, $R_2 = 45\Omega$ & $R_3 = 90\Omega$

$$\frac{1}{R} = \frac{1}{30} + \frac{1}{45} + \frac{1}{90}$$

$$\frac{1}{R} = \frac{3+2+1}{90} = \frac{6}{90} = \frac{1}{15}$$

$$R = 15\Omega$$

Now

$$\frac{\partial R}{\partial R_2} = \frac{R^2}{R_2^2} = \frac{15^2}{45^2} = \frac{225}{2025} = \frac{1}{9} = 0.11\Omega$$

The value of $\frac{\partial R}{\partial R_2}$ is $\frac{1}{9} = 0.11\Omega$

9. If $u(x,y) = \log\left(\frac{x^2+y^2}{x+y}\right)$, then prove that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 1$$

Sol.

$$u = \log\left(\frac{x^2+y^2}{x+y}\right)$$

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\log\left(\frac{x^2+y^2}{x+y}\right)$$

$$= \frac{x+y}{x^2+y^2} \cdot \frac{(x+y)2x - (x^2+y^2)}{(x+y)^2}$$

$$= \frac{(x+y)(2x^2+2xy-x^2-y^2)}{(x^2+y^2)(x+y)^2}$$

$$= \frac{x^2-y^2+2xy}{(x^2+y^2)(x+y)}$$

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial x}\log\left(\frac{x^2+y^2}{x+y}\right)$$

$$\frac{\partial u}{\partial x} = \frac{x+y}{x^2+y^2} \cdot \frac{(x+y)\,2y - (x^2+y^2)}{(x+y)^2}$$

$$= \frac{2xy + 2y^2 - x^2 - y^2}{(x^2+y^2)(x+y)}$$

$$= \frac{y^2 - x^2 + 2xy}{(x^2+y^2)(x+y)}$$

L.H.S.

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{x(x^2-y^2+2xy)}{(x^2+y^2)(x+y)} + \frac{y(y^2-x^2+2xy)}{(x^2+y^2)(x+y)}$$

$$= \frac{x^3 - xy^2 + 2x^2y + y^3 - x^2y + 2xy^2}{(x^2+y^2)(x+y)}$$

$$= \frac{x^3+y^3+x^2y+xy^2}{x^3+y^3+x^2y+xy^2}$$

$$= 1$$

L.H.S = R.H.S H.P

10. If $u = x^3y^2\sin^{-1}\left(\frac{y}{x}\right)$, show that a. $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 5u$

b. $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 20u$

Sol

$$u = x^3y^2\sin^{-1}\left(\frac{y}{x}\right)$$

$$\frac{\partial u}{\partial x} = 3x^2y^2\sin^{-1}\left(\frac{y}{x}\right) + x^3y^2\frac{x}{\sqrt{x^2-y^2}} \cdot \frac{-y}{x^2}$$

$$= 3x^2y^2\sin^{-1}\left(\frac{y}{x}\right) - \frac{x^3y^3}{\sqrt{x^2-y^2}} \cdot \frac{x}{x^2}$$

$$= 3x^2y^2\sin^{-1}\left(\frac{y}{x}\right) - \frac{x^2y^3}{\sqrt{x^2-y^2}}$$

$$\frac{\partial u}{\partial y} = 2x^3y\sin^{-1}\left(\frac{y}{x}\right) + x^3y^2\frac{x}{\sqrt{x^2-y^2}} \cdot \frac{1}{x}$$

$$= 2x^3y\sin^{-1}\left(\frac{y}{x}\right) + \frac{x^3y^2}{\sqrt{x^2-y^2}}$$

L.H.S

$$= x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$$

$$= 3x^3y^2\sin^{-1}\left(\frac{y}{x}\right) - \frac{x^3y^3}{\sqrt{x^2-y^2}} + 2x^3y^2\sin^{-1}\left(\frac{y}{x}\right) + \frac{x^3y^3}{\sqrt{x^2-y^2}}$$

$$= 3x^3y^2\sin^{-1}\left(\frac{y}{x}\right) + 2x^3y^2\sin^{-1}\left(\frac{y}{x}\right)$$

$$= 3u + 2u$$

$$= 5u$$

L.H.S = R.H.S H.P

b. $x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = 2u$

Sol

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left(3x^2y^2 \sin^{-1}\left(\frac{y}{x}\right) + \frac{x^2y^3}{\sqrt{x^2-y^2}}\right)$$

$$= 6xy^2 \sin^{-1}\left(\frac{y}{x}\right) + 3x^2y^2 \frac{x}{\sqrt{x^2-y^2}}\left(\frac{-y}{x^2}\right) \cancel{\sqrt{x^2-y^2}\,2xy^3} -$$

$$\left(\frac{\sqrt{x^2-y^2}\,2xy^3 - \frac{x^2y^3 \cdot 2x}{2\sqrt{x^2-y^2}}}{\left(\sqrt{x^2-y^2}\right)^2}\right)$$

$$= 6xy^2 \sin^{-1}\left(\frac{y}{x}\right) - \frac{3xy^3}{\sqrt{x^2-y^2}} - \frac{\left(2xy^3(x^2-y^2) - x^3y^3\right)}{\left(\sqrt{x^2-y^2}\right)^3}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial y}\left(2x^3y \sin^{-1}\left(\frac{y}{x}\right) + \frac{x^3y^2}{\sqrt{x^2-y^2}}\right)$$

$$= 2x^3 \sin^{-1}\left(\frac{y}{x}\right) + \frac{2x^3y}{\sqrt{x^2-y^2}} \cdot \frac{1}{x} + \frac{\sqrt{x^2-y^2} \cdot 2x^3y - x^3y^2 \cdot \frac{1}{2\sqrt{x^2-y^2}}(-2y)}{\left(\sqrt{x^2-y^2}\right)^2}$$

$$= 2x^3 \sin^{-1}\left(\frac{y}{x}\right) + \frac{2x^3y \cdot 1}{\sqrt{x^2-y^2}} \cdot \frac{1}{x} + \frac{(x^2-y^2)2x^3y + x^3y^3}{\left(\sqrt{x^2-y^2}\right)^3}$$

$$= 2x^3 \sin^{-1}\left(\frac{y}{x}\right) + \frac{2x^2y}{\sqrt{x^2-y^2}} + \frac{(x^2-y^2)2x^3y + x^3y^3}{\left(\sqrt{x^2-y^2}\right)^3}$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}\left(2x^3y \sin^{-1}\left(\frac{y}{x}\right) + \frac{x^3y^2}{\sqrt{x^2-y^2}}\right)$$

$$= 6x^2y \sin^{-1}\left(\frac{y}{x}\right) + 2x^3y \frac{1}{\sqrt{x^2-y^2}}\left(\frac{-y}{x^2}\right) +$$

$$\frac{\sqrt{x^2-y^2} \cdot 3x^2y^2 - x^3y^2 \cdot \frac{1}{2\sqrt{x^2-y^2}} \cdot 2x}{\left(\sqrt{x^2-y^2}\right)^2}$$

$$= \frac{6x^2y \sin^{-1}\left(\frac{y}{x}\right) - \frac{x^4y}{\sqrt{x^2-y^2}}}{\left(\sqrt{x^2-y^2}\right)^2}$$

$$= 6x^2y \sin^{-1}\left(\frac{y}{x}\right) + \frac{2x^2y^2}{\sqrt{x^2-y^2}}(-1) +$$

$$\frac{(x^2-y^2)3x^2y^2 - x^4y^2}{\left(\sqrt{x^2-y^2}\right)^2}$$

$$= 6x^2y \sin^{-1}\left(\frac{y}{x}\right) - \frac{2x^2y^2}{\sqrt{x^2-y^2}} + \frac{(x^2-y^2)3x^2y^2 - x^4y^2}{\left(\sqrt{x^2-y^2}\right)^3}$$

L.H.S

$$= x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2}$$

$$= x^2\left[6xy^2\sin^{-1}\left(\frac{y}{x}\right) - \frac{3xy^3}{\sqrt{x^2-y^2}} - \frac{2x^3y^3 - 2xy^5 - x^3y^3}{(\sqrt{x^2-y^2})^3}\right] +$$

$$2xy\left[6x^2y\sin^{-1}\left(\frac{y}{x}\right) - \frac{2x^2y^2}{\sqrt{x^2-y^2}} + \frac{3x^4y^2 - 3x^2y^4 - x^4y^2}{(\sqrt{x^2-y^2})^3}\right] +$$

$$y^2\left[2x^3\sin^{-1}\left(\frac{y}{x}\right) + \frac{2x^2y}{\sqrt{x^2-y^2}} + \frac{2x^5y + x^3y^3 - 2x^3y^3}{(\sqrt{x^2-y^2})^3}\right]$$

$$= \left[6x^3y^2\sin^{-1}\left(\frac{y}{x}\right) - \frac{3x^3y^3}{\sqrt{x^2-y^2}} - \frac{x^3y^3 - 2x^3y^5}{(\sqrt{x^2-y^2})^3}\right] +$$

$$\left[12x^3y^2\sin^{-1}\left(\frac{y}{x}\right) - \frac{4x^3y^3}{\sqrt{x^2-y^2}} + \frac{4x^5y^3 - 6x^3y^5}{(\sqrt{x^2-y^2})^3}\right]$$

$$+ \left[2x^3y^2\sin^{-1}\left(\frac{y}{x}\right) + \frac{2x^3y^3}{\sqrt{x^2-y^2}} + \frac{2x^5y^3 - x^3y^5}{(\sqrt{x^2-y^2})^3}\right]$$

$$= 6x^3\sin^{-1}\left(\frac{y}{x}\right) + 12x^3y^2\sin^{-1}\left(\frac{y}{x}\right) + 2x^3y^2\sin^{-1}\left(\frac{y}{x}\right)$$

$$= 6u + 12u + 2u$$

$$= 20u$$

L.H.S = R.H.S H.P

11. For $u = \tan^{-1}\left(\frac{y}{x}\right)$ show that a. $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

b. $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$

SOM

a. L.H.S

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)$$

$$\frac{\partial u}{\partial x} = \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \frac{-y}{x^2}$$

$$= \frac{x^2}{x^2+y^2} \cdot \frac{-y}{x^2}$$

$$= \frac{-y}{x^2+y^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{y}{(x^2+y^2)^2} \cdot 2x$$

$$= \frac{2xy}{(x^2+y^2)^2}$$

$$\frac{\partial u}{\partial y} = \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \frac{1}{x} = \frac{x^2}{x^2+y^2} \cdot \frac{1}{x}$$

$$= \frac{x}{x^2+y^2}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{x}{(x^2+y^2)}\right)$$

$$= \frac{-x}{(x^2+y^2)^2}(+2y)$$

$$= \frac{-2yx}{(x^2+y^2)^2}$$

L.H.S

$$= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

$$= \frac{2xy}{(x^2+y^2)^2} - \frac{2xy}{(x^2+y^2)^2}$$

$$= 0$$

L.H.S = R.H.S H.P

b. ~~Sol~~

Sol L.H.S

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right)$$

$$= \frac{x^2+y^2-x(2x)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

R.H.S

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right)$$

$$= \frac{(x^2+y^2)(-1)+y(2y)}{(x^2+y^2)^2}$$

$$= \frac{y^2-x^2}{(x^2+y^2)^2}$$

L.H.S = R.H.S H.P

12. If $f(x,y,z) = \log(x^2+y^2+z^2)$ then prove $xf_{zx} = zf_{xy}$

Sol

L.H.S

$$\cancel{f_{zx}} = \frac{1}{x^2+y^2+z^2} \cdot \cancel{2x+2z} = \frac{2x+2z}{x^2+y^2+z^2}$$

$$\cancel{xf_{zx}} = \frac{2x^2+2zx}{x^2+y^2+z^2}$$

L.H.S

$$xf_{yz} = \frac{(-2y(2z))x}{(x^2+y^2+z^2)^2} = \frac{-4xyz}{(x^2+y^2+z^2)^2}$$

R.H.S

$$zf_{xy} = \frac{((-2x)\cdot 2y)z}{(x^2+y^2+z^2)^2} = \frac{-4xyz}{(x^2+y^2+z^2)^2}$$

L.H.S = R.H.S

H.P

Unit: 6

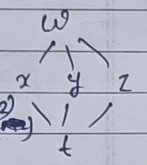
Multivariable Calculus

Tutorial - 3

1. Use the chain Rule to find $\frac{dw}{dt}$ for $w = xe^{y/z}$, $x = t^2$, $y = 1-t$, $z = 1+2t$

Soln

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial t}$$



$$= e^{y/z} \cdot 2t + \frac{xe^{y/z}}{z}(0-1) + xe^{y/z} \cdot \frac{-y}{z^2}(2)$$

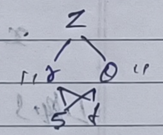
$$= 2te^{y/z} - \frac{txe^{y/z}}{z} - \frac{2xe^{y/z}}{z^2}$$

$$= \left(\frac{2tz^2 - txz - 2x}{z^2}\right)e^{y/z}$$ Ans

2. Use the chain Rule to find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$, $z = e^r\cos\theta$, $r = st$, $\theta = \sqrt{s^2+t^2}$

Soln

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial r}\frac{\partial r}{\partial t} + \frac{\partial z}{\partial \theta}\frac{\partial \theta}{\partial t}$$



$$= e^r\cos\theta\, s + e^r(-\sin\theta)\left(\frac{st}{2\sqrt{s^2+t^2}}\right)$$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial r}\frac{\partial r}{\partial s} + \frac{\partial z}{\partial \theta}\frac{\partial \theta}{\partial s}$$

$$= e^r\cos\theta\, t + e^r(-\sin\theta)\frac{2s}{2\sqrt{s^2+t^2}}$$ Ans

3. Find $\frac{dy}{dx}$ for 1) $x^2 - y^2 + \sin xy = 0$,

Soln

$$f(x,y) = x^2 - y^2 + \sin xy$$

$$f_x = 2x + \cos xy \cdot y$$

$$f_y = -2y + \cos xy \cdot x$$

$$\frac{dy}{dx} = -\frac{f_x}{f_y} = \frac{2x + y\cos xy}{2y - x\cos xy}$$

(2) $x^3 + y^3 = 7xy$

Soln By Implicit Differentiation

$$f(x,y) = x^3 + y^3 - 7xy$$

$$f_x = 3x^2 - 7y$$

$$f_y = 3y^2 - 7x$$

$$\frac{dy}{dx} = -\frac{f_x}{f_y} = -\left(\frac{3x^2 - 7y}{3y^2 - 7x}\right)$$

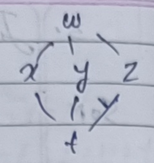
4. Find $\frac{dw}{dt}$ if $w = xy + z$, $x = \cos t$, $y = \sin t$, $z = t$

Soln

$$w = xy + z$$

$$x = \cos t, \quad y = \sin t, \quad z = t$$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial t}$$



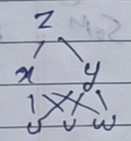
$$= y(-\sin t) + x\cos t + 1$$

$$= -y\sin t + x\cos t + 1$$

5. Use the chain Rule to find the indicated partial derivatives.

a. $z = x^2 + xy^3$, $x = uv^2 + w^3$, $y = u + ve^w$: $\frac{\partial z}{\partial u}, \frac{\partial z}{\partial v}, \frac{\partial z}{\partial w}$ when $u=2, v=1, w=0$

Soln



$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$$

$$= 2xy^3(2uv) + 3xy^2(ve^w)$$

$$\left.\frac{\partial z}{\partial v}\right]_{(2,1,0)} = 4xy^3uv + 3xy^2ve^w = 4xy^3 + 3xy^2$$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$$

$$= 2xy^3(2v) + 3xy^2$$

$$\left.\frac{\partial z}{\partial u}\right]_{(2,1,0)} = 4xy^3v + 3xy^2 = 2xy^3 + 3xy^2$$

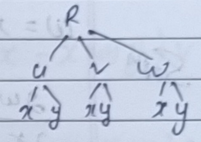
$$\frac{\partial z}{\partial w} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial w} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial w}$$

$$\left.\frac{\partial z}{\partial w}\right]_{(2,1,0)} = 6xy^3w^2 + 3xy^2ve^w = 3xy^2$$

b. $R = \log(u^2+v^2+w^2)$, $u = x+2y$, $v = 2x-y$, $w = 2xy$

$\frac{\partial R}{\partial x}, \frac{\partial R}{\partial y}$ when $x = y = 1$

Soln



$$\frac{\partial R}{\partial x} = \frac{\partial R}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial R}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial R}{\partial w}\frac{\partial w}{\partial x}$$

$$= \frac{2u}{u^2+v^2+w^2} + \frac{2v}{u^2+v^2+w^2}(2) + \frac{2w}{u^2+v^2+w^2}\cdot 2y$$

$$= \frac{2u}{u^2+v^2+w^2} + \frac{4v}{u^2+v^2+w^2} + \frac{4wy}{u^2+v^2+w^2}$$

$$\left.\frac{\partial R}{\partial x}\right]_{(1,1)} = \frac{2u}{u^2+v^2+w^2} + \frac{4v}{u^2+v^2+w^2} + \frac{4w}{u^2+v^2+w^2}$$

$$\frac{\partial R}{\partial y} = \frac{\partial R}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial R}{\partial v}\frac{\partial v}{\partial y} + \frac{\partial R}{\partial w}\frac{\partial w}{\partial y}$$

$$= \frac{2u}{u^2+v^2+w^2}\cdot 2 + \frac{2v}{u^2+v^2+w^2}(-1) + \frac{2w}{u^2+v^2+w^2}\cdot 2x$$

$$= \frac{4u}{u^2+v^2+w^2} - \frac{2v}{u^2+v^2+w^2} + \frac{2wx}{u^2+v^2+w^2}$$

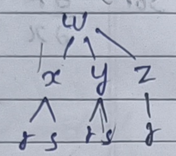
$$\left.\frac{\partial R}{\partial y}\right]_{(1,1)} = \frac{4u}{u^2+v^2+w^2} - \frac{2v}{u^2+v^2+w^2} + \frac{2w}{u^2+v^2+w^2}$$

6. Find $\frac{\partial w}{\partial r}$ and $\frac{\partial w}{\partial s}$ in terms of r and s if

$w = x + 2y + z^2$, $x = r/s$, $y = r^2 + \log s$, $z = 2r$

Soln

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r}$$



$$= \frac{1}{s} + 2 \cdot 2r + 2z \cdot 2$$

$$= 1/s + 4r + 2(2r) \cdot 2$$

$$= \frac{1}{s} + 4r + 8r$$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s}$$

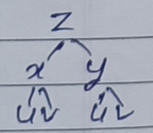
$$= -\frac{r}{s^2} + \frac{2}{s}$$

$$= \frac{-r + 2s}{s^2}$$

7. Find $\frac{\partial z}{\partial u}$ when $u = 0, v = 1$ if $z = \sin xy + x \sin y$

$x = u^2 + v^2$ ~~$- uv$~~ $y = uv$

Soln



$z = \sin xy + x \sin y$

$x = u^2 + v^2$, $y = uv$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$$

$$= (\cos xy \cdot y + \sin y)\, 2u + (\cos xy \cdot x - x \cos y)\, v$$

$u = 0$, $v = 1$

$$\frac{\partial z}{\partial u} = (\cos xy \cdot y + \sin y)\, 2(0) + (\cos xy \cdot x - x \cos y)(1)$$

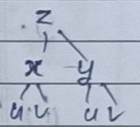
$$= x(\cos xy - \cos y) \text{ Ans}$$

8. If $z = f(x, y)$ where $x = e^u + e^{-v}$, $y = e^{-u} - e^v$,

Prove that $\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial v}$

Soln

L.H.S



$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$$

$$= \frac{\partial z}{\partial x}(e^u) + \frac{\partial z}{\partial y}(e^{-u})$$

$$= \frac{\partial z}{\partial x}e^u - \frac{\partial z}{\partial y}e^{-u}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$$

$$= \frac{\partial z}{\partial x}(-e^{-v}) + \frac{\partial z}{\partial y}(-e^v)$$

$$= -\frac{\partial z}{\partial x}e^{-v} - \frac{\partial z}{\partial y}e^v$$

$$\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}(-e^{-u}) + \frac{\partial z}{\partial y}(-e^{-u}) + \frac{\partial z}{\partial x} e^{-v} + \frac{\partial z}{\partial y} e^{v}$$

$$= (e^{u} + e^{-v}) \frac{\partial z}{\partial x} + (-e^{-u} + e^{v}) \frac{\partial z}{\partial y}$$

$$= (e^{v} + e^{-v}) \frac{\partial z}{\partial x} - (e^{-u} - e^{v}) \frac{\partial z}{\partial y}$$

$$= x \frac{\partial z}{\partial x} - y \frac{\partial z}{\partial y}$$

L.H.S = R.H.S H.P

9. a. Find the Jacobian $\frac{\partial(u,v)}{\partial(x,y)}$ for the following function

(i) $u = x^2 - y^2$; $v = 2xy$

Soln

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 2x & -2y \\ 2y & 2x \end{vmatrix}$$

$$= 4x^2 + 4y^2$$

$$= 4(x^2 + y^2)$$

(ii) $u = \frac{x^2 + y^2}{1 - xy}$; $v = \tan^{-1} x + \tan^{-1} y$

Soln

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{(1-xy)2x + (x^2+y^2)(-1)}{(1-xy)^2} & \frac{(1-xy)2y - (x^2+y^2)(-1)}{(1-xy)^2} \\ \frac{1}{1+x^2} & \frac{1}{1+y^2} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{2x - 2x^2y + x^2 + y^2}{(1-xy)^2} & \frac{2y - 2xy^2 + x^2 + y^2}{(1-xy)^2} \\ \frac{1}{1+x^2} & \frac{1}{1+y^2} \end{vmatrix}$$

$$= \frac{x^2 + y^2 - 2x^2y + 2x}{(1+y^2)(1-xy)^2} - \frac{x^2 + y^2 - 2xy^2 + 2y}{(1-xy)^2(1+x^2)}$$

b. For the transformation $x = e^v \sec u$, $y = e^v \tan u$
Prove that $\frac{\partial(x,y)}{\partial(u,v)} \cdot \frac{\partial(u,v)}{\partial(x,y)} = 1$

Soln L.H.S

$$J = \frac{\partial(x,y)}{\partial(u,v)} \cdot \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} e^v \sec u \tan u & e^v \sec u \\ e^v \sec^2 u & e^v \tan u \end{vmatrix}$$

$$= e^{2v} \sec u \tan^2 u - e^{2v} \sec^3 u$$

$$= e^{2v} \sec u (\sec^2 u - 1) - e^{2v} \sec^3 u$$

$$= e^{2v} \sec^3 u - e^{2v} \sec u - e^{2v} \sec^3 u$$

$$= -e^{2v} \sec u$$

Now.

$x = e^v \sec u$, $y = e^v \tan u$

$x^2 - y^2 = e^{2v} \sec^2 u - e^{2v} \tan^2 u$

$x^2 - y^2 = e^{2v} (\sec^2 u - \tan^2 u)$

$x^2 - y^2 = e^{2v}$

$2v = \log(x^2 - y^2)$ Taking log on both side

$v = \frac{1}{2} \log(x^2 - y^2)$

$$\frac{y}{x} = \frac{e^v \tan u}{e^v \sec u} = \frac{\sin u}{\cos u} \times \frac{\cos u}{1}$$

$\sin u = \frac{y}{x}$

$u = \sin^{-1}\left(\frac{y}{x}\right)$

u & v be the function of x & y

$$J' = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{x}{\sqrt{x^2-y^2}} \cdot \frac{-y}{x^2} & \frac{x}{\sqrt{x^2-y^2}} \cdot \frac{1}{x} \\ \frac{1}{2} \frac{2x}{x^2-y^2} & \frac{1}{2} \frac{(-2y)}{x^2-y^2} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{-y}{x\sqrt{x^2-y^2}} & \frac{1}{\sqrt{x^2-y^2}} \\ \frac{x}{x^2-y^2} & \frac{-y}{x^2-y^2} \end{vmatrix}$$

$$= \frac{y^2}{x(x^2-y^2)^{3/2}} - \frac{x}{(x^2-y^2)^{3/2}}$$

$$= \frac{y^2 - x^2}{x(x^2-y^2)^{3/2}}$$

b. Find the eq^n of the N.L of the sphere $x^2+y^2+z^2=6$ at the point (a,b,c). Show that the N.L passes through the origin.

Sol^n

$f(x,y,z) = x^2+y^2+z^2-6=0 \quad x_1=a,\ y_1=b,\ z_1=c$

$f_x = 2x \quad f_y = 2y \quad f_z = 2z$

$f_x(a,b,c) = 2a \quad f_y(a,b,c) = 2b \quad f_z(a,b,c) = 2c$

Normal line

$$\frac{x-a}{2a} = \frac{y-b}{2b} = \frac{z-c}{2c}$$

for $(x,y,z) = (0,0,0)$ we have

$$\frac{0-a}{2a} = \frac{0-b}{2b} = \frac{0-c}{2c}$$

$$\frac{-1}{2} = \frac{-1}{2} = \frac{-1}{2}$$

Hence the normal line pass es through the origin $(0,0,0)$

11. (a). Find the stationary value of x^3+y^3-3axy, a>0

Sol^n

$f(x,y) = x^3+y^3-3axy$

$f_x = 3x^2-3ay$

$f_y = 3y^2-3ax$

$f_x=0$	$f_y=0$
$3x^2-3ay=0$	$3y^2-3ax=0$
$x^2-ay=0$	$y^2-ax=0$
$x=\pm\sqrt{ay}$	$y=\pm\sqrt{ax}$
$x^2=ay$ — (i)	$y^2=ax$ — (ii)

On Sq^2 eq^n (i)

$x^4=a^2y^2 = a^2ax = a^3x$

$x^3=a^3x = x(x^3-a^3)=0$

$x=a \qquad x=0$

$y=a \qquad y=0$

Stationary values (a,a) $(0,0)$

b. Examine the function $x^3y^3(12-3x-4y)$ for extreme values.

Sol^n

$f(x,y) = x^3y^3(12-3x-4y)$

~~$f_x = 3x^2y^3(-3) = -9x^2y^3$~~

~~$f_y = x^3 3y^2(-4) = 12x^3y^2$~~

$f_x = 36x^2y^3 - 12x^3y^3 - 12x^2y^4$

~~$f_y = 36x^3y^2 - 9x^4y^2 - 16x^3y^3$~~

$f_y = 24x^3y - 6x^4y - 8x^3y^2$

$f_x = 0$

$12x^2y^2(3-x-y) = 0$

$x+y = 3$

$x = 3-y$ — (i)

$f_y = 0$

$2x^3y(12-3x-4y) = 0$

$3x+4y = 12$ — (ii)

From eqn (i) and (ii)

$x = 0, y = 3$

Second derivative

$r = \frac{\partial^2 f}{\partial x^2} = 72xy^2 - 36x^2y^2 - 24xy^3$

$s = \frac{\partial^2 f}{\partial x \partial y} = 72x^2y - 24x^3y - 24x^2y^2$

$t = \frac{\partial^2 f}{\partial y^2} = 24x^3 - 6x^4 - 16x^3y$

The stationary Pt are $(0,3)$

x,y	r	s	t	$rt-s>0$	Conclusion	$f(x,y)$
$(0,3)$	0	0	0	$0=0$	Saddle Pt	0

Q.12. a) Find the min^m value of xyz, subject to the condition $2x+2y+2z = 36$

Soln

$f(x,y,z) = xyz$

$\phi(x,y,z) = 2x+2y+2z = 36$ — (1)

$f(x,y,z) + \lambda\phi(x,y,z) = 0$

$xyz + \lambda(2x+2y+2z-36) = 0$ — (2)

Diff partially w.r.t. x, y & z

$\frac{\partial f}{\partial x} + \lambda\frac{\partial \phi}{\partial x} = yz + 2\lambda = 0$

$\lambda = \frac{-yz}{2}$ — (3)

$\frac{\partial f}{\partial y} + \lambda\frac{\partial \phi}{\partial y} = xz + 2\lambda = 0$

$\lambda = \frac{-xz}{2}$ — (4)

$\frac{\partial f}{\partial y} + \lambda\frac{\partial \phi}{\partial y} = xy + 2\lambda = 0.$

$\lambda = \frac{-xy}{2}$ — (5)

From eqn (3), (4), (5).

$\frac{-yz}{2} = \frac{-xz}{2} = \frac{-xy}{2}$

$z = x$; $y = x$

Substituting y and z in eqn (1)

$2x+2x+2x = 36$

$x = \frac{36}{6}$

$x = 6, y = 6, z = 6$

min^m values of xyz is

$= 6\times6\times6 = 216$

b. Find the minm value of x^2+y^2 subject to the condition $ax+by=c$

Sol

$f(x,y) = x^2+y^2$

$\phi(x,y) = ax+by-c = 0$

$f(x,y) + \lambda\phi(x,y) = 0$

$x^2+y^2+\lambda(ax+by-c) = 0$

diff. partially w.r.t. x and y

$\frac{\partial f}{\partial x} + \lambda\frac{\partial \phi}{\partial x} = 2x+\lambda a = 0$

$\lambda = \frac{-2x}{a}$ — ①

$\frac{\partial f}{\partial y} + \lambda\frac{\partial \phi}{\partial y} = 2y+\lambda b = 0$

$\lambda = \frac{-2y}{b}$ — ②

From eqn ① and ②

$\frac{-2x}{a} = \frac{-2y}{b}$

$y = \frac{xb}{a}$

Now

$ax + b\left(\frac{xb}{a}\right) = c$

$a^2x+b^2x = ac$

$x(a^2+b^2) = ac$

$x = \frac{ac}{a^2+b^2}$

$y = \frac{bc}{a^2+b^2}$

Minm value of x^2+y^2

$= \left(\frac{ac}{a^2+b^2}\right)^2 + \left(\frac{bc}{a^2+b^2}\right)^2$

$= \frac{a^2c^2}{(a^2+b^2)^2} + \frac{b^2c^2}{(a^2+b^2)^2}$

$= \frac{a^2c^2+b^2c^2}{(a^2+b^2)^2}$ Ans