CHAPTER 6

Fourier Series

Syllabus :

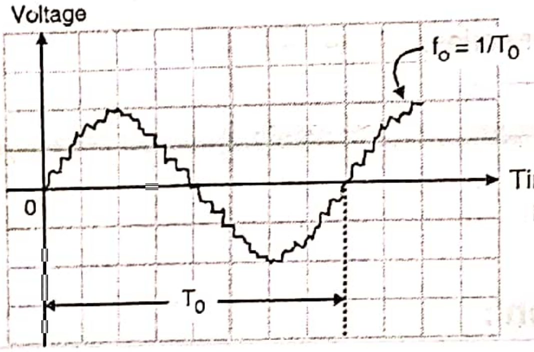
Representation of periodic functions, Fourier series.

6.1 Frequency Domain Representation :

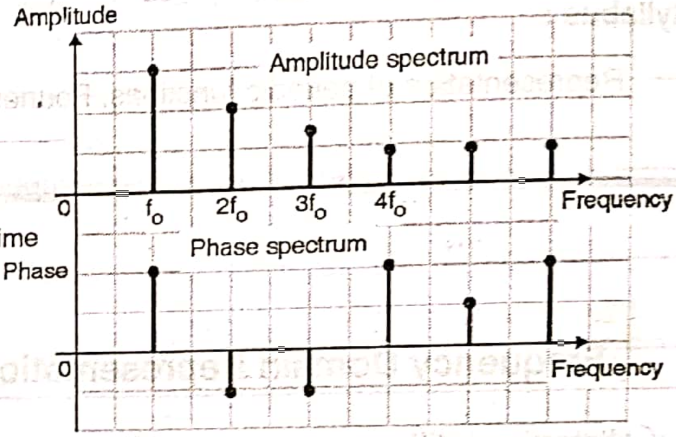
- All the signals till now were drawn with respect to time. That means time "t" was considered as a variable. The representation of signal with respect to time is called as its **time domain representation.**
- The time domain representation of the signal is not sufficient for its analysis. Hence we have to use the frequency domain representation of the signal for the sake of analysis.
- In the frequency domain representation, the variable plotted on the X-axis is frequency "f" rather than "t".
- The signal represented in the frequency domain is called as the **line spectrum**. The line spectrum consists of two graphs namely :
  1. **Amplitude spectrum :** A graph of amplitude on Y-axis versus frequency on X-axis.
  2. **Phase spectrum :** A graph of phase on Y-axis versus frequency on X-axis.

  The signal x(t) and its line spectrum are shown in Fig. 6.1.1.
- The graph of instantaneous signal voltage plotted on Y-axis versus time plotted on X-axis is called as the time domain representation.
- It is as shown in Fig. 6.1.1(a). The time domain representation gives us the following information :
  1. Shape of the signal
  2. Its frequency
  3. Type of the signal (periodic or nonperiodic).
  4. One cycle period.

- But we do not know anything about what frequency components are present and in what proportion they have been mixed in order to obtain the particular shape of the signal.
- All this information can be obtained from the line spectrum of a signal.
- Line spectrum [Fig. 6.1.1(b)] is the representation of the same signal x(t), now in the frequency domain.
- It can be obtained by using either Fourier series or Fourier transform. It consists of the amplitude and phase spectrums of the signal.
- The line spectrum indicates the amplitude and phase of various frequency components present in the given signal. The line spectrum enables us to analyze and synthesize a signal.



(a) Time domain representation



(b) Line spectrum

Fig. 6.1.1

6.1.1 How to Plot Line Spectrum ?

- The line spectrum is useful in knowing about the existence and amplitudes/phases of various frequency components present in a given signal x(t).
- The important conventions about the line spectra are as follows :

1. In all the spectral drawings, the independent variable plotted on the x-axis is frequency "f" in Hz and not ω.
2. Phase angle is always measured with respect to the cosine waves. That means it is measured with respect to the positive real axis of the phasor diagram. Hence it is necessary to convert sinewaves to cosines using the following standard identity :

$$\sin \omega t = \cos(\omega t - 90^\circ) \quad ...(6.1.1)$$

3. The amplitude is always considered as a "positive" quantity. So if negative signs appear for a phase angle, they should be absorbed in the phase change to keep amplitude positive. This is explained in the following expression,

$$-A \cos \omega t = A \cos(\omega t \pm 180^\circ) \quad ...(6.1.2)$$

- The additional phase change of $\pm$ 180° converts the negative amplitude "– A" to positive amplitude "+ A". We can choose a phase shift of either + 180° or – 180°, as the effect is going to be the same.

Ex. 6.1.1 : Sketch the line spectrum of the following signal :

$$m(t) = 3 - 5\cos(40\pi t - 30^\circ) + 4\sin 120\pi t$$

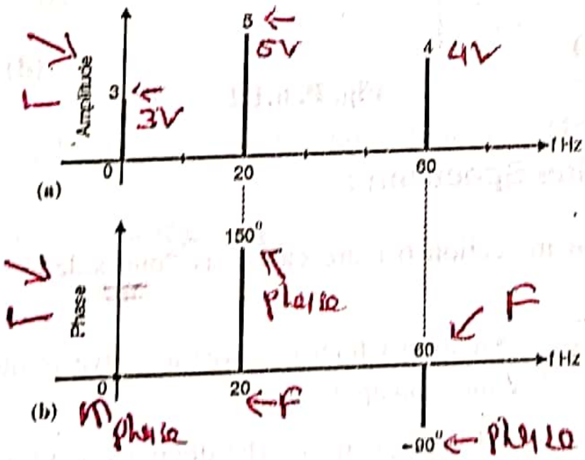
Soln. : In the given signal, the first term represents a dc term which has a zero frequency. The other two terms can be written as follows :

1. First term : $3 = 3\cos 2\pi 0 t$ as $f = 0$
2. Second term : $-5\cos(40\pi t - 30^\circ) = 5\cos(2\pi 20 t - 30^\circ + 180^\circ) = 5\cos(2\pi 20 t + 150^\circ)$
   Thus the negative amplitude has been made positive by adding a phase angle of 180°.
3. Third term : $4\sin 120\pi t = 4\sin 2\pi 60 t = 4\cos(2\pi 60 t - 90^\circ)$.
   Thus the sine term has been converted to the cosine term by adding a phase shift of – 90°.
   The amplitudes, frequencies and phase angles of the three terms are listed in the Table P. 6.1.1.

Table P. 6.1.1

Sr. No.	Term	Amplitude	Frequency	Phase
1.	$3\cos 2\pi 0 t$	3 V	0 Hz	0°
2.	$5\cos(2\pi 20 t + 150^\circ)$	5 V	20 Hz	150°
3.	$4\cos(2\pi 60 t - 90^\circ)$	4 V	60 Hz	– 90°

With the help of Table P. 6.1.1 we can plot the line spectra as shown in Fig. P. 6.1.1.



(a) Amplitude spectrum

(b) Phase spectrum

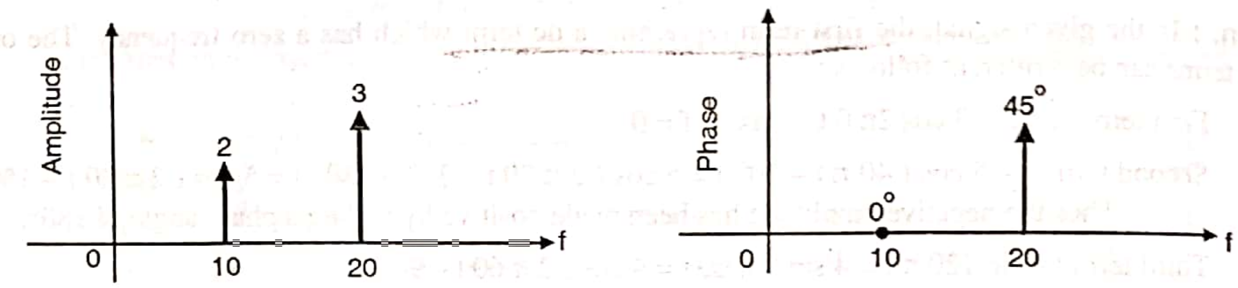
Fig. P. 6.1.1

Ex. 6.1.2 : Show single sided frequency representations of :

1. $x(t) = 3\cos\left(40\pi t + \frac{\pi}{4}\right) + 2\cos(20\pi t)$
2. $x(t) = 3\sin(100\pi t) + \cos(50\pi t)$

Soln. :

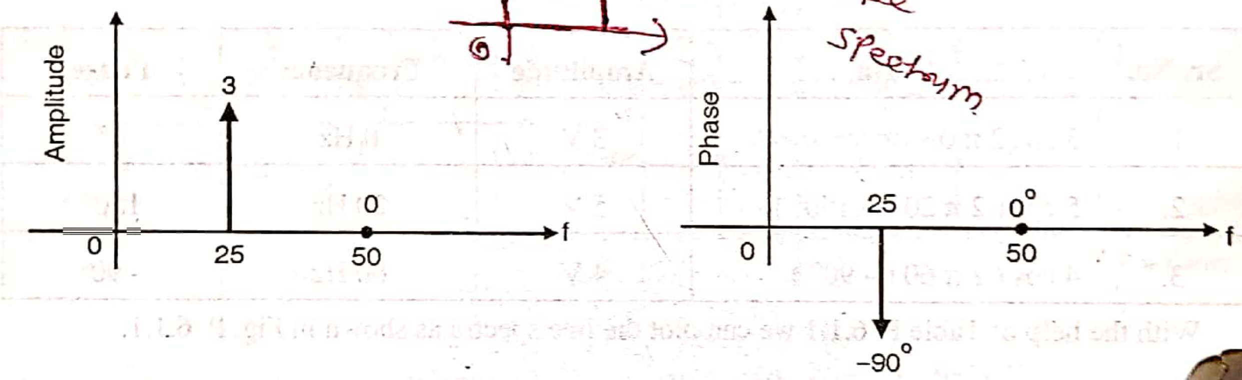
1.



(a) (b)

Amplitude
A single side spectrum
0

2.



(c) (d)

Fig. P. 6.1.2

6.1.2 Double Sided Line Spectrum :

- The line spectra drawn in section 6.1 are called as "one sided" or "positive frequency" line spectra.
- But the other spectral representation which involves negative frequencies as well proves to be more valuable in may communication applications.
- It is called as the double sided line spectrum. The double sided line spectra can be obtained by modifying the single sided spectrum as shown in Fig. 6.1.2. Both the line spectra shown in this figure correspond to a signal x (t) = A cos ($2\pi f_o$ t).

Conclusions :

The conclusions drawn from Fig. 6.1.2 are as follows :

1. Looking at Fig. 6.1.2(a) i.e. the amplitude spectrum, we conclude that in the single sided spectrum there is only one frequency component present at $f = f_o$ with an amplitude A.

   Whereas in double sided line spectra, two frequency components f_o and $-f_o$ are present with amplitude (A/2) but no change in polarity.

2. The single sided phase spectrum contains only one component at f_0 with phase ϕ. But double sided spectrum contains two components at f_0 and $-f_0$ with phases equal to ϕ and $-\phi$ respectively. Thus phase shift remains unchanged but they have opposite signs.

3. From Fig. 6.1.2(a) it is clear that the double sided amplitude spectrum has an even symmetry and Fig. 6.1.2(b) shows that the double sided phase spectrum has an odd symmetry.

- The double sided line spectrum representation is very useful in mathematical analysis of a signal. The negative frequency components shown in the double sided spectrums are not practically present. They are fictitious frequency components.

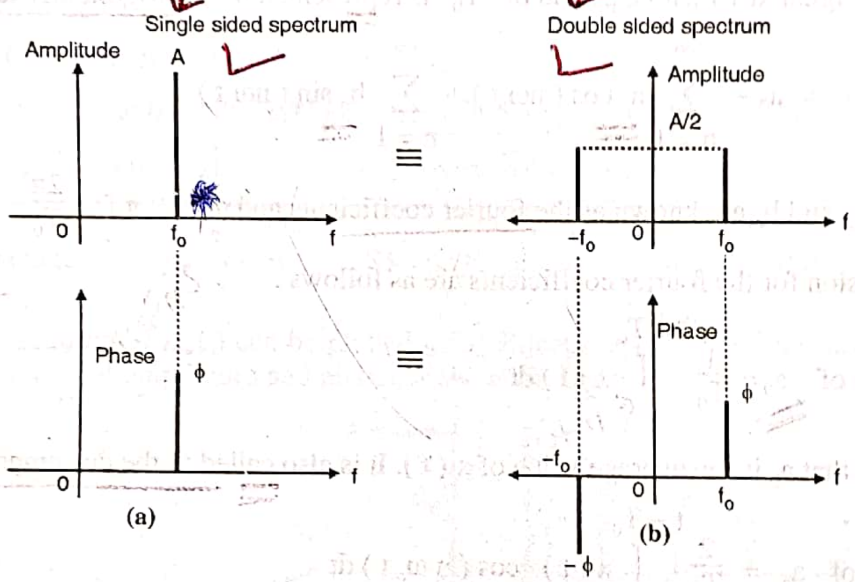


Fig. 6.1.2 : Equivalence between single and double sided spectrums

6.2 Review of CT Fourier Series :

Need of CT fourier series :

- Sinewaves and cosinewaves are the basic building functions for any periodic signal.
- That means any periodic signal basically consists of sinewaves having different amplitudes of different frequencies and having different relative phase shifts.
- Fourier series represents a periodic waveform in the form of sum of infinite number of sine and cosine terms. It is a representation of the signal in a time domain series form.
- Fourier series is a "tool" used to analyze any periodic signal. After the "analysis" we obtain the following information about the signal :
  1. What all frequency components are present in the signal ?
  2. Their amplitudes and
  3. The relative phase difference between these frequency components.

  All the "frequency components" are nothing else but sinewaves at those frequencies.

6.2.1 Types of Fourier Series :

- There are three types of fourier series used for the analysis of periodic signals. They are :
  1. Trigonometric or quadrature fourier series
  2. Polar fourier series
  3. Exponential fourier series.

  Let us understand them one-by-one.

6.2.2 Trigonometric or Quadrature Fourier Series :

- A periodic signal x(t) with a period of "T_0" is represented by the trigonometric fourier series as,

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t) + \sum_{n=1}^{\infty} b_n \sin(n\omega_0 t) \qquad \text{...(6.2.1)}$$

where a_0, a_n and b_n are known as the fourier coefficients and $\omega_0 = 2\pi f_0 = \frac{2\pi}{T_0}$

The expression for the fourier coefficients are as follows :

1. The value of $a_0 = \frac{1}{T_0} \int_{t}^{t+T_0} x(t)\, dt$ ...(6.2.2)

This shows that a_0 is the average value of x (t). It is also called as the dc component of x(t).

2. The value of $a_n = \frac{2}{T_0} \int_{t}^{t+T_0} x(t) \cdot \cos(n\omega_0 t)\, dt$ ...(6.2.3)

3. The value of $b_n = \frac{2}{T_0} \int_{t}^{t+T_0} x(t) \cdot \sin(n\omega_0 t)\, dt$

Equation (6.2.1) can be expanded as follows :

$$x(t) = \underbrace{a_0}_{\text{DC value of } x(t)} + \underbrace{a_1 \cos \omega_0 t}_{\text{Fundamental}} + \underbrace{a_2 \cos 2\omega_0 t}_{\text{Second harmonic}} + \cdots + \underbrace{b_1 \sin \omega_0 t}_{\text{Fundamental component}} + \underbrace{b_2 \sin 2\omega_0 t}_{\text{Second harmonic}} + \cdots$$

Conclusions :

1. The above expression is suitable to plot the line spectrum.
2. The first term "a_0" has a zero frequency. Hence it is called as the **dc component of x(t)**. It is also called as the average value or dc value of x (t).
3. The sine and cosine terms at frequency f_0 are called fundamental components. The terms at frequency $2 f_0$ are called as the second harmonic components and so on.

6.2.3 Polar Fourier Series :

The polar fourier series is $\boxed{\text{derived}}$ from the trigonometric fourier series by combining the sine and cosine terms of same frequency. The polar Fourier series representation of x(t) is as follows :

$$x(t) = C_0 + \sum_{n=1}^{\infty} C_n \cos(n\omega_0 t + \phi_n) \quad ...(6.2.5)$$

where, $$C_n = \left[a_n^2 + b_n^2\right]^{1/2} \quad ...(6.2.6)$$

and $$\phi_n = -\tan^{-1}\left[\frac{b_n}{a_n}\right] \quad ...(6.2.7)$$

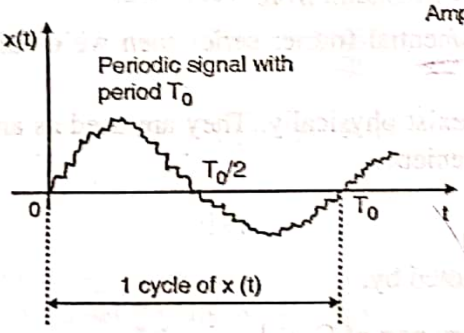
and C_0 = Average value of x (t) = a_0

Equation (6.2.5) can be expanded by opening the summation sign as follows :

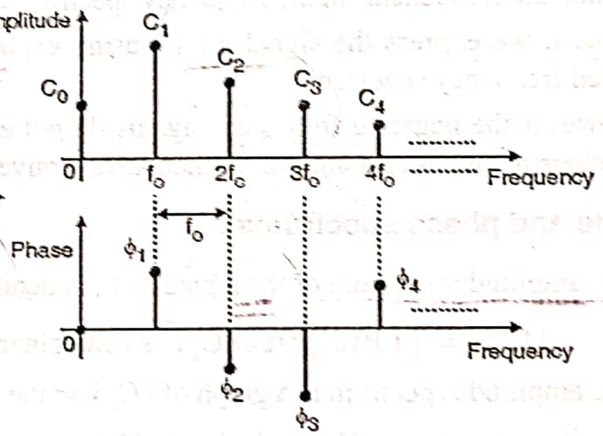
$$x(t) = \underbrace{C_0}_{\text{Average}} + \underbrace{C_1 \cos(\omega_0 t + \phi_1)}_{\text{Fundamental}} + \underbrace{C_2 \cos(2\omega_0 t + \phi_2)}_{\text{Second harmonic}} + \underbrace{C_3 \cos(3\omega_0 t + \phi_3)}_{\text{Third harmonic}} + \text{.........} \quad ...(6.2.8)$$

Line spectrum :

- The line spectrum of x(t) can be plotted using Equation (6.2.8). A line spectrum of x(t) with arbitrary values of amplitudes and phases is shown in Fig. 6.2.1.



(a) Time domain representation of x(t)



(b) Line spectrum of x(t)

Fig. 6.2.1 : Line spectrum using polar fourier series

- As seen from Fig. 6.2.1(b) the frequency spectrum of a continuous signal is discrete in nature. The frequency components f_0, 2 f_0, 3 f_0.... etc. are called as the "spectral components".
- The adjacent spectral components are spaced by "f_0" from each other. As the spectrum is consisting of vertical lines, (C_1, C_2, ...) this spectrum is called as the line spectrum.

6.2.4 Exponential Fourier Series [or Complex Exponential Fourier Series] :

- The sine and cosine terms can be expressed in terms of the exponential terms using Euler's equations as follows :

$$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2} \quad ...(6.2.9)$$

$$\text{and } \sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2} \quad ...(6.2.10)$$

- Substituting the sine and cosine functions in terms of exponential function in the expression for the quadrature Fourier series i.e. in Equation (6.2.1), we can obtain another type of Fourier series called the exponential Fourier series.

A periodic signal x (t) is expressed in the exponential fourier series form as follows :

Exponential Fourier series :

$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{j2\pi n t/T_0} \quad ...(6.2.11)$$

$$\text{where, } C_n = \frac{1}{T_0} \int_{t}^{t+T_0} x(t) \cdot e^{-j2\pi n t/T_0} dt \quad ...(6.2.12)$$

As seen from the Equation (6.2.11), this fourier series consists of exponential terms. Therefore it is called as the exponential Fourier series.

Concept of negative frequency :

- In Equation (6.2.11), observe that "n" is extending from $-\infty$ to $+\infty$ instead of from 0 to ∞. Due to this the frequencies in the frequency spectrum will extend from $-\infty$ to $+\infty$.
- Thus if we express the signal x (t) using exponential fourier series then we obtain a double sided frequency spectrum.
- However the negative frequency signals do not exist physically. They are used as an important mathematical concept and for mathematical convenience.

Amplitude and phase spectrums :

- The amplitude spectrum of the signal x(t) is denoted by,

$$|C_n| = [(\text{Real part of } C_n)^2 + (\text{Imaginary part of } C_n)^2]^{1/2} \quad ...(6.2.13)$$

- The amplitude spectrum is a graph of $|C_n|$ on the y-axis versus frequency $f = n f_o$ on the x-axis.

The phase spectrum of x (t) is denoted by,

$$\phi_n = \arg(C_n) = \tan^{-1}\left[\frac{\text{Imaginary part of } C_n}{\text{Real part of } C_n}\right] \quad ...(6.2.14)$$

- The phase spectrum is a graph of ϕ_n on the y-axis versus frequency $f = n f_o$ on the x-axis.
- The amplitude spectrum is a symmetric or even function. That means $|C_n| = |C_{-n}|$. But the phase spectrum is an asymmetric or odd function. That means $\arg(C_n) = -\arg(C_{-n})$.

Advantages of exponential Fourier series :

1. Instead of finding a_0, a_n and b_n, we have to find only the value of one Fourier coefficient i.e. C_n.
2. It is easier to find and plot the amplitude and phase responses.
3. The exponential F.S. allows us to plot the double sided spectrum, because it is possible to plot the negative frequencies.

6.2.5 Dirichlet Conditions for the Existence of Fourier Series :

The Fourier series stated in the Equations (6.2.1) and (6.2.5) will exist if and only if the periodic signal x(t) satisfies the following conditions. These are known as the "Dirichlet" conditions. They are as follows :

1. The periodic signal x(t) and its integrals are finite and single valued in the interval (t to t + T_0), i.e. over a period of one cycle T_0.
2. x(t) must have only finite number of discontinuities in the given interval of time.
3. x(t) should have only finite number of maxima and minima in the given interval of time.
4. The function x(t) is absolutely integrable, that is

$$\int_{-T_0/2}^{T_0/2} |x(t)|\,dt < \infty$$

6.2.5.1 Knowledge of Waveform Symmetry used to Calculate Fourier Coefficient :

If the given waveform has some kind of a symmetry, then it becomes easy to calculate the Fourier coefficients. It is as explained below :

1. Signal with zero mean :

If the mean value of the given signal is zero.

$$\because \text{ i.e. } \quad \text{if} \int_0^T x(t)\,dt = 0$$

$$\text{then} \quad a_0 = 0$$

But there is no change in the expressions of a_n and b_n. A signal having zero mean value is shown in Fig. 6.2.2.

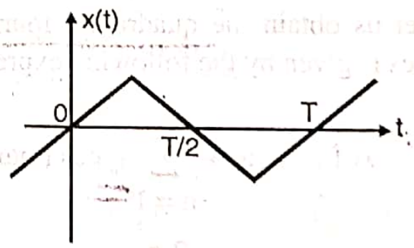


Fig. 6.2.2

2. An even symmetry :

If the given signal x(t) is an even signal,

$$\text{i.e.} \quad x(t) = x(-t)$$

$$\text{Then} \quad b_n = 0 \quad \dots \text{for all } n$$

3. An odd symmetry :

If the given signal x (t) is an odd signal.

$$\text{i.e.} \quad x(t) = -x(-t)$$

$$\text{Then} \quad a_0 = 0,\ a_n = 0 \quad \dots \text{for all } n$$

4. Half wave symmetry :

If the given signal, x (t) has half wave symmetry

$$\text{i.e.} \quad \text{if } x(t) = -x\left(t + \frac{T_0}{2}\right)$$

$$\text{Then} \quad a_0 = 0,\ a_n = 0 \quad \ldots \text{ for n even}$$

$$b_n = 0 \quad \ldots \text{ for n even}$$

Thus if we know the symmetry of the given waveform, then the work involved in finding out the Fourier coefficients is reduced considerably.

6.2.6 Examples on Fourier Series :

Ex. 6.2.1 : Obtain the quadrature fourier series for the rectangular pulse train shown in Fig. P. 6.2.1.

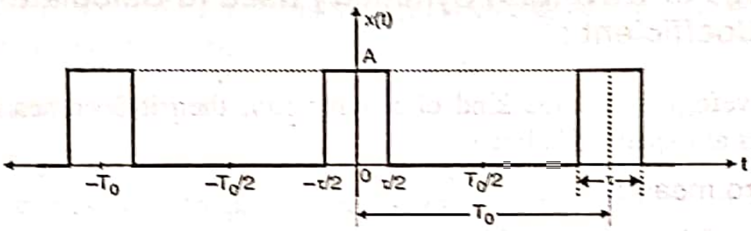


Fig. P. 6.2.1

Soln. : Let us obtain the quadrature fourier series for the given rectangular pulse. The quadrature fourier series is given by the following expression :

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega_o t) + \sum_{n=1}^{\infty} b_n \sin(n\omega_o t) \quad \ldots(1)$$

Substituting $\omega_o = 2\pi f_o = \frac{2\pi}{T_0}$ into Equation (1) we get,

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left[\frac{2\pi n t}{T_0}\right] + \sum_{n=1}^{\infty} b_n \sin\left[\frac{2\pi n t}{T_0}\right] \quad \ldots(2)$$

To find the fourier coefficients, we must consider one complete cycle of x (t) for integration.

Here we will consider one cycle from $t = -\frac{T_0}{2}$ to $t = \frac{T_0}{2}$. Let us obtain the fourier coefficients now.

1. To obtain the value of a_0 :

$$a_0 = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t)\, dt$$

$$= \frac{1}{T_0} \int_{-\tau/2}^{\tau/2} x(t)\, dt \qquad \text{.....As } x(t) \text{ exists from } -\tau/2 \text{ to } \tau/2 \text{ only}$$

$$= \frac{1}{T_0} \int_{-\tau/2}^{\tau/2} A\, dt \qquad \text{.....As } x(t) = A \text{ from } -\tau/2 \text{ to } \tau/2$$

$$\therefore \quad a_0 = \frac{A\tau}{T_0} \qquad ...(3)$$

2. **To obtain the value of a_n :** From the Equation (6.2.3), a_n is given as,

$$a_n = \frac{2}{T_0} \int_{-T_0/2}^{T_0/2} x(t) \cdot \cos\left[\frac{2\pi n t}{T_0}\right] dt = \frac{2}{T_0} \int_{-\tau/2}^{\tau/2} A \cos\left[\frac{2\pi nt}{T_0}\right] dt$$

$$= \frac{2A}{T_0} \cdot \frac{1}{\frac{2\pi n}{T_0}} \left[\sin\left(\frac{2\pi nt}{T_0}\right)\right]_{-\tau/2}^{\tau/2} = \frac{A}{\pi n}\left[\sin\left(\frac{2\pi n\tau}{2T_0}\right) - \sin\left(\frac{-2\pi n\tau}{2T_0}\right)\right]$$

$$= \frac{A}{\pi n}\left[\sin\left(\frac{\pi n\tau}{T_0}\right) + \sin\left(\frac{\pi n\tau}{T_0}\right)\right]$$

$$\therefore \quad a_n = \frac{2A}{\pi n}\sin\left[\frac{\pi n\tau}{T_0}\right] \qquad ...(4)$$

3. **To obtain the value of b_n :** From the Equation (6.2.4), b_n is given as,

$$b_n = \frac{2}{T_0} \int_{-T_0/2}^{T_0/2} x(t) \cdot \sin\left[\frac{2\pi n t}{T_0}\right] dt = \frac{2}{T_0} \int_{-\tau/2}^{\tau/2} A \sin\left[\frac{2\pi nt}{T_0}\right] dt$$

$$= \frac{-2A}{T_0} \cdot \frac{1}{\frac{2\pi n}{T_0}} \left[\cos\left(\frac{2\pi nt}{T_0}\right)\right]_{-\tau/2}^{\tau/2} = \frac{-A}{\pi n}\left[\cos\left(\frac{2\pi n\tau}{2T_0}\right) - \cos\left(\frac{-2\pi n\tau}{2T_0}\right)\right]$$

$$\therefore \quad b_n = \frac{-A}{\pi n}\left[\cos\left(\frac{\pi n\tau}{T_0}\right) - \cos\left(\frac{\pi n\tau}{T_0}\right)\right]$$

$$\therefore \quad b_n = 0 \qquad ...(5)$$

4. Substitute the values of a_0, a_n and b_n in the Equation (2) we get the quadrature fourier series for $x(t)$ as,

$$x(t) = \frac{A\tau}{T_0} + \sum_{n=1}^{\infty} \frac{2A}{n\pi} \sin\left[\frac{n\pi t}{T_0}\right] \cos\left[\frac{2\pi nt}{T_0}\right] \qquad \text{...Ans.}$$

Ex. 6.2.2 : Obtain the exponential fourier series for the rectangular pulse train shown in Fig. P. 6.2.2(a) and sketch the spectrum.

Soln. :

Step 1 : Find a_0 :

$$a_0 = \frac{1}{T}\int_0^{T/2} A\,dt = \frac{A}{T}[t]_0^{T/2} = \frac{A}{T}\times\frac{T}{2} = \frac{A}{2} \quad ...(1)$$

Step 2 : Find a_n and b_n :

$$a_n = \frac{2}{T}\int_0^{T/2} A\cos\left[\frac{2\pi nt}{T}\right]dt = \frac{2}{T}A\times\frac{T}{2\pi n}\left[\sin\frac{2\pi nt}{T}\right]_0^{T/2}$$

$$\therefore\ a_n = \frac{A}{n\pi}\left[\sin\frac{2\pi nt}{2T} - \sin 0\right] = \frac{A}{n\pi}[\sin \pi n]$$

$$\therefore\ a_n = 0 \quad ...[\because \sin n\pi = 0 \text{ for all } n] \quad ...(2)$$

$$\therefore\ b_n = \frac{2}{T}\int_0^{T/2} A\sin\left[\frac{2\pi nt}{T}\right]dt$$

$$= \frac{-2A}{T}\times\frac{T}{2\pi n}\left[\cos\frac{2\pi nt}{T}\right]_0^{T/2}$$

$$= \frac{-A}{n\pi}\left[\cos\frac{2\pi nT}{2T} - \cos 0\right] = \frac{-A}{n\pi}[\cos n\pi - 1]$$

For n even $\cos n\pi = 1$ $\therefore\ b_n = 0$

For n odd $\cos n\pi = -1$ $\therefore\ b_n = \frac{2A}{n\pi}$ ...(3)

Step 3 : Write the Fourier series :

The trigonometric Fourier series for x (t) is

$$x(t) = a_0 + \sum_{n=\text{odd}} b_n \sin(n\omega_o t)$$

$$\therefore\ x(t) = \frac{A}{2} + \sum_{n\ \text{odd}} \frac{2A}{n\pi}(n\omega_o t) \quad ...\ \text{Ans.}$$

Ex. 6.2.12 : Find the exponential Fourier series and plot the magnitude and phase spectrum for the periodic signal x (t).

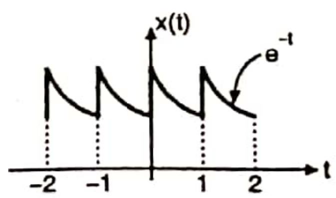


Fig. P. 6.2.12

Soln. :

Exponential Fourier series is given by,

$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{j2\pi nt/T_0} \quad \text{Where} \quad C_n = \frac{1}{T_0}\int_{t}^{t+T_0} x(t)\, e^{-j2\pi nt/T_0}\, dt$$

The given signal is expressed as, $x(t) = e^{-t},\ 0 < t < 1$

To obtain value of C_n :

$$C_n = \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t)\, e^{-j2\pi nt/T_0}\, dt$$

Here $x(t) = e^{-t}$ for $0 < t < 1$ and $T_0 = 1$

$$\therefore C_n = \frac{1}{1}\int_0^1 e^{-t}\cdot e^{-j2\pi nt}\, dt = \int_0^1 e^{-(1+j2\pi n)t}\, dt$$

$$= \left[\frac{e^{-(1+j2\pi n)t}}{-(1+j2\pi n)}\right]_0^1 = \frac{1}{-(1+j2\pi n)}\left[e^{-(1+j2\pi n)} - e^0\right]$$

$$\therefore C_n = \frac{1}{(1+j2\pi n)}\left[1 - e^{-1}\cdot e^{j2\pi n}\right]$$

Here $e^{-1} = 0.632$ and $e^{j2\pi n} = \cos 2\pi n + j\sin 2\pi n = 1$

$$\therefore C_n = \frac{0.632}{1+j2\pi n}$$

To obtain the Fourier series :

Putting the value of C_n in the equation of x (t) we get,

$$x(t) = \sum_{n=-\infty}^{\infty} \frac{0.632}{1+j2\pi n} e^{j2\pi nt/T_0}$$

Spectrum of signal :

$$\text{Here} \quad |C_n| = \frac{0.632}{\sqrt{1+(2\pi n)^2}}$$

The phase spectrum is given by,

$$\phi_n = \tan^{-1}\left\{\frac{Im}{Re}\right\} = \tan^{-1}\left\{\frac{Im}{Re}\right\} \text{for numerator} - \tan^{-1}\left\{\frac{Im}{Re}\right\} \text{for denominator}$$

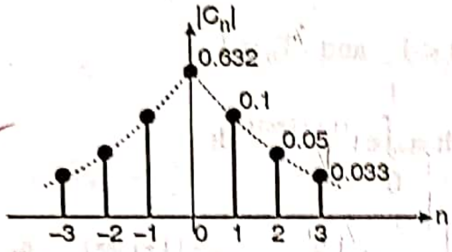
$$\therefore \phi_n = \tan^{-1}\{0\} - \tan^{-1}\{2\pi n\}$$

$$\therefore \phi_n = -\tan^{-1}\{2\pi n\}$$

The following table shows calculation of magnitude and phase

Value of n	$\lvert C_n \rvert$	ϕ_n (Radian)
– 3	0.033	1.51
– 2	0.05	1.49
– 1	0.1	1.14
0	0.632	0
1	0.1	– 1.14
2	0.05	– 1.49
3	0.033	– 1.51

The magnitude and phase spectrum are shown in Figs. P. 6.2.12(a) and P. 6.2.12(b) respectively.



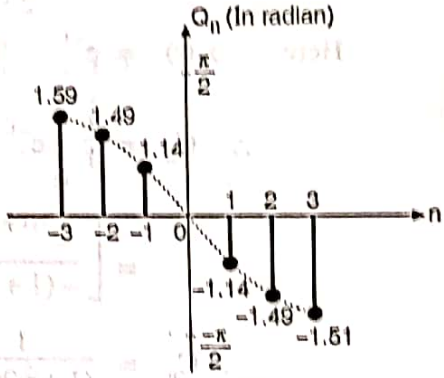


Fig. P. 6.2.12

Ex. 6.2.13 : Find trignometric fourier series coefficient of the following function. Also find its exp... series coefficient.

$x(t) = 4 + 2\cos(20\pi t) + \cos(40\pi) + 3\sin(20\pi t)$

Soln. :

Given $\quad x(t) = 4 + 2\cos(20\pi t) + \cos(40\pi) + 3\sin(20\pi t)$

We have,

$$\therefore X(\omega) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos \omega_0 t + \sum_{n=1}^{\infty} b_n \sin n\,\omega_0 t$$

$$\text{Here } a_0 = 2$$

$$a_n = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t) \cos n\,\omega_0\, t\, dt$$

$$\text{and } b_n = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t) \sin n\,\omega_0 t$$

Scanned with CamScanner

Ex. 6.2.17 : Find exponential Fourier series coefficient of the signal shown in Fig. P. 6.2.17.

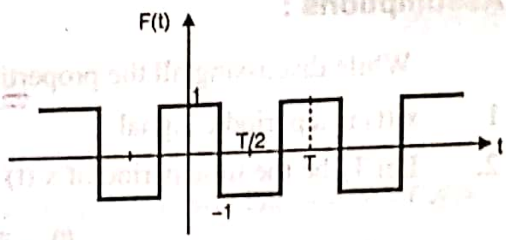


Fig. P. 6.2.17

Soln. : The coefficient of exponential series is given by,

$$C_n = \frac{1}{T}\int_{t}^{t+T} x(t)\, e^{-j2\pi n/T}\, dt$$

$$\therefore\ C_n = \frac{1}{T}\int_{-T/2}^{T/2} 1\cdot e^{-j2\pi n t/T}\, dt$$

$$= \frac{1}{T}\cdot\frac{1}{\frac{-j2\pi n}{T}}\left[e^{-j2\pi n t/T}\right]_{-T/2}^{T/2}$$

$$= \frac{1}{n\pi}\frac{1}{-j2}\left[e^{-\frac{j2\pi n T}{2T}} - e^{j2\pi nT/2T}\right]$$

$$= \frac{1}{n\pi}\left[\frac{e^{+j\pi n} - e^{-j\pi n}}{2j}\right]$$

$$= \frac{1}{n\pi}\sin \pi n \qquad \text{...since } \sin x = \frac{e^{jx}-e^{-jx}}{2j}$$

$$= \frac{\sin \pi n}{\pi n}$$

$$= \sin c\, n\pi \qquad \text{...since } \sin cx = \frac{\sin \pi\lambda}{\pi\lambda}$$

The exponential Fourier series is given by,

$$x(t) = \sum_{n=-\infty}^{\infty} \sin c(n\pi)\, e^{j2\pi nt/T}$$

6.3 Properties of C.T. Fourier Series :

In this section we are going to discuss the following important properties of C.T. Fourier series :

1. Linearity
2. Time shifting
3. Time reversal
4. Time scaling
5. Multiplication
6. Conjugation and conjugate symmetry.
7. Parseval's relation.

Assumptions :

While discussing all the properties, we will have the following common assumptions.

1. x (t) is a periodic signal.
2. Let T_0 be the time period of x (t) and let the fundamental frequency be given by,

$$\omega_0 = 2\pi / T_0$$

3. Let the Fourier series coefficient of x(t) be denoted by a_n. Then the relation between x (t) and a_n is given by,

$$x(t) \xleftrightarrow{FS} a_n$$

6.3.1 Linearity :

- Let x (t) and y (t) denote two periodic signals both having a period T_0. Let their Fourier coefficients be denoted by a_n and b_n respectively.

That means,

Fourier series

$$x(t) \xleftrightarrow{FS} a_n \qquad x(t) \xleftrightarrow{FS} a_n$$

$$y(t) \xleftrightarrow{FS} b_n \qquad y(t) \xleftrightarrow{FS} b_n$$

- Then the linear combination z (t) or x(t) and y(t) will have Fourier coefficients C_n such that

$$z(t) \xleftrightarrow{FS} C_n$$

where $z(t) = Ax(t) + By(t)$

and $C_n = Aa_n + Bb_n$

That means,

$$z(t) = [Ax(t) + By(t)] \xleftrightarrow{FS} C_n = [Aa_n + Bb_n] \quad ...(6.3.1)$$

Where A and B are arbitrary contents.

Proof :

Let x(t) and y(t) be expressed in terms of exponential Fourier series as

$$x(t) = \sum_{n=-\infty}^{\infty} a_n e^{j2\pi nt/T_0}$$

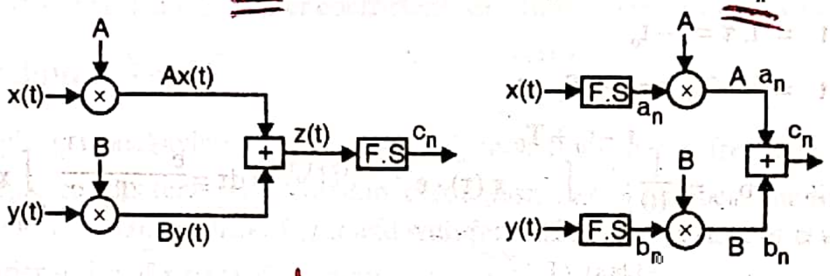
$$y(t) = \sum_{n=-\infty}^{\infty} b_n e^{j2\pi nt/T_0}$$

$$\therefore \quad Z(t) = A\,x(t) + B\,y(t) = A \sum_{n=-\infty}^{\infty} a_n e^{j2\pi nt/T_0} + B \sum_{n=-\infty}^{\infty} a_n e^{j2\pi nt/T_0}$$

$$\therefore \quad z(t) \xleftrightarrow{FS} C_n \;=\; Aa_n + B\,b_n$$ ...Proved.

Graphical representation :

- The linearity property can be illustrated graphically as shown in Fig. 6.3.1.
- Fig. 6.3.1(a) shows the LHS. Here z(t) is first obtained as A x(t) + B y(t) and a F.S. of z(t) is obtained at the end.
- Fig. 6.3.1(b) shows the RHS of linearity property in which Fourier series of x(t) and y(t) is obtained first then the multiplication and addition takes place to obtain C_n.



(a) LHS of linearity property **(b) RHS of linearity property**

Fig. 6.3.1 : Graphical illustration of linearity property

- This shows that the sequence in which operations carried out on a linear combination is not important.
- The linearity property can be extended easily to a combination of an arbitrary number of signals of period T_0.

6.3.2 Time Shifting :

- Let x(t) be a periodic signal with a period T_0 and fourier coefficients a_n.

  i.e. $$x(t) \xleftrightarrow{F.S.} a_n$$

- Then the time shifted signal $x(t - t_0)$ will have the Fourier coefficients of $e^{-jk\omega_o t_o a_n}$. That means,

$$x(t - t_0) \xleftrightarrow{FS} .\, e^{-jk\omega_o t_o a_n} \qquad ...(6.3.2)$$

- Note that when a time shift is applied to a periodic signal, the period T_0 of the signal should remain unchanged.

Proof :

- As per the definition of exponential Fourier series,

$$x(t) = \sum_{n=-\infty}^{\infty} a_n\, e^{j 2\pi n t / T_0},$$

$$\text{Where} \quad a_n = \frac{1}{T_0} \int_{t}^{t+T_0} x(t)\, e^{-j 2\pi n t / T_0}\, dt \qquad ...(6.3.3)$$

- Similarly the Fourier series coefficients b_n of the time shifted signal $x(t - t_o)$ is given by,

$$b_n = \frac{1}{T_0} \int_{t}^{t+T_0} x(t - t_o)\, e^{-j 2\pi n t / T_0}\, dt$$

- Let $(t - t_o) = \tau$

$$\therefore \quad t = \tau + t_o$$

$$\therefore \quad dt = d\tau$$

For $t = t, \tau = t - t_o$

For $t = t + T_0, \tau = t + T_0 - t_o$.

$$\therefore \quad b_n = \frac{1}{T_0} \int_{t-t_o}^{t-t_o+T_0} x(\tau) \cdot e^{-j 2\pi n (\tau + t_o)/T_0}\, d\tau = \frac{e^{-j 2\pi n t_o / T_0}}{T_0} \int_{T_0} x(\tau) \cdot e^{-j 2\pi n t}\, d\tau$$

$$b_n = e^{-j 2\pi n t_o / T_0}\, a_n$$ ...Proved.

Conclusion :

When a periodic signal is time shifted, then the magnitude of its Fourier series coefficients remain unchanged

i.e. $|a_n| = |b_n|$

6.3.3 Time Reversal :

This property states that if

$$x(t) \xleftrightarrow{FS} a_n$$

Then the time reversed version of x(t) i.e. $x(-t)$ forms a Fourier series pair with a_{-n}

$$x(-t) \longleftrightarrow a_{-n}$$

Meaning :

- The meaning of time reversal is that, the time reversal applied to C.T. signal results in the time reversal of the corresponding sequence of Fourier series coefficients.
- If x(t) is even, then the Fourier series coefficients will also be even. That means, if $x(t) = x(-t)$, then $a_n = a_{-n}$.
- Similarly if x(t) is odd, then a_n will also be odd. That means if $x(t) = -x(-t)$ then $a_n = -a_{-n}$.

Proof :

$$x(t) = \sum_{n=-\infty}^{\infty} a_n e^{j 2\pi n t / T_0}$$ ...Definition of Fourier series

Replace t by – t to get

$$x(-t) = \sum_{n=-\infty}^{\infty} a_n e^{-j2\pi n t/T_0}$$ Substitute $n = -m$ to get

$$x(-t) = \sum_{m=-\infty}^{\infty} a_{-m} e^{j2\pi m t/T_0} \quad ...(6.3.4)$$

This expression shows that the Fourier coefficients of a time reversed signal x (– t) are a_{-m}.

6.3.4 Time Scaling :

- x(t) is the original signal having a time period of T_0 and a fundamental frequency of $\omega_o = 2\pi/T_0$.
- Then $x(\alpha t)$ represents its time scaled version. α is a positive real number. The time scaled signal $x(\alpha t)$ is periodic with a time period T_0/α and with fundamental frequency of $\alpha\,\omega_o$.
- Then the Fourier series of $x(\alpha t)$ is given by,

$$x(\alpha t) = \sum_{n=-\infty}^{\infty} a_n e^{j2\pi n \alpha t/T_0}$$

Proof :

$$\text{Let } x(t) = \sum_{n=-\infty}^{\infty} a_n e^{j2\pi n t/T_0} \quad ...\text{Definition}$$

Substitute $t = \alpha t$ to get, $x(\alpha t) = \sum_{n=-\infty}^{\infty} a_n e^{j2\pi n \alpha t/T_0}$ **...Proved.**

Conclusion :

- Expand the expression for $x(\alpha t)$ as follows.

$$x(\alpha t) = ...a_{-1}e^{-j2\pi\alpha t/T_0} + \underbrace{a_0}_{\text{DC}} + \underbrace{a_1 e^{j2\pi\alpha t/T_0}}_{\text{Fundamental}} + \underbrace{a_2 e^{j4\pi\alpha t/T_0}}_{\text{Second harmonic}} + \quad ...(6.3.5)$$

- This shows that the Fourier coefficients remain unchanged, but time scaling of each harmonic component has taken place.

6.3.5 Frequency Shifting :

Statement :

If $x(t) \overset{Fs}{\leftrightarrow} C_{xn}$

Then $x(t) \cdot e^{j2\pi n_o t/T_o} = y(t) \overset{Fs}{\leftrightarrow} C_{yn} = C_{x\,(n-n_o)}$

Meaning : Shifting of frequency component by 'n_o' positions is equivalent to multiplying signal x(t) by $e^{j2\pi n_o t/T_o}$

Proof : According to the definition of C_n,

$$C_{yn} = \frac{1}{T_o} \int_{t}^{t+T_o} y(t)\, e^{-j2\pi nt/T_o}\, dt$$

$$\text{But } y(t) = x(t) \cdot e^{j2\pi n_o t/T_o}$$

$$\therefore C_{yn} = \frac{1}{T_o} \int_{t}^{t+T_o} x(t)\, e^{j2\pi n_o t/T_o}\, e^{-j2\pi nt/T_o}\, dt$$

$$= \frac{1}{T_o} \int_{t}^{t+T_o} x(t)\, e^{-j2\pi_n (n-n_o) t/T_o}\, dt$$

$$\therefore C_{yn} = C_{x\,(n-n_o)}$$

6.3.6 Differentiation :

Statement : If $x(t) \overset{Fs}{\leftrightarrow} C_n$

then $\frac{d}{dt} x(t) \overset{Fs}{\leftrightarrow} j\frac{2\pi n}{T_o} \cdot C_n$

Meaning : Differentiation in time domain is equivalent to multiplying its fourier coefficient by $\frac{j2\pi n}{T_o}$

Proof : According to the definition of fourier series.

$$x(t) = \sum_{n=1}^{\infty} C_n \cdot e^{j2\pi nt/T_o}$$

Taking derivative with respect to time,

$$\frac{d}{dt} x(t) = \sum_{n=1}^{\infty} C_n \cdot \frac{d}{dt} [e^{j2\pi nt/T_o}]$$

$$= \sum_{n=1}^{\infty} C_n \cdot e^{j2\pi nt/T_o} \frac{j2\pi n}{T_o}$$

$$\frac{d}{dt} x(t) = \sum_{n=1}^{\infty} \left[\frac{j2\pi n}{T_o} C_n\right] \cdot e^{j2\pi nt/T_o}$$

Comparing it with the definition of fourier series,

$$\frac{d}{dt} x(t) \overset{Fs}{\leftrightarrow} \frac{j2\pi n}{T_o} \cdot C_n$$

6.4 Fourier Series for Discrete Time Periodic Signals (DTFS) :

Consider x (n) is a discrete time periodic signal i.e.

$$x(n) = x(n+N) \quad \text{... for all n.}$$

The FS representation for x (n) consists of N harmonically related exponential function.

$$e^{j2\pi kn/N} \quad \text{...}k = 0, 1, \text{...} N-1$$

$$\therefore \quad x(n) = \sum_{k=0}^{N-1} C_k e^{j2\pi kn/N}$$

where C_k = Coefficients in series expansion.

The equation for fourier coefficient is,

$$C_k = \frac{1}{N} \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N} \quad \text{...}k = 0, \text{...} N-1$$

This equation is called analysis equation

∴ Synthesis equation,

$$x(n) = \sum_{k=0}^{N-1} C_k e^{j2\pi kn/N} \quad \text{...(6.4.1)}$$

Analysis equation

$$C_k = \frac{1}{N} \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N} \quad \text{...}k = 0, 1 \text{...} N-1 \quad \text{...(6.4.2)}$$

Equation (6.4.1) provide discrete time fourier series. C_k provides amplitude and phase associated with frequency component of x (n) in the frequency domain.

Review Questions

Q. 1 What do you mean by frequency analysis of a signal ?

Q. 2 What is the use of analysis of a signal ?

Q. 3 What is a phase spectrum ?

Q. 4 What is the line spectrum ?

Q. 5 State and prove following properties of CTFS :

1. Time shifting 2. Scaling 3. Differentiation 4. Parseval's theorem

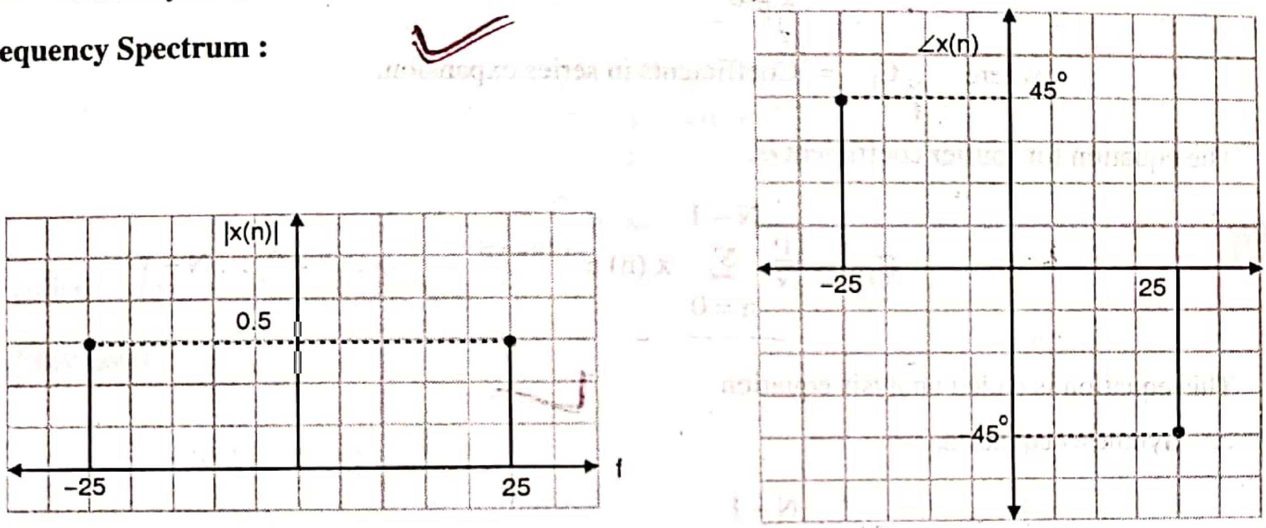
Q. 6 Explain analogy between CTFS and DTFS.

6.5 Examples for Practice :

Ex. 6.5.1 : Find and plot two sided frequency spectrum of $x(t) = \cos\left(50\pi t - \frac{\pi}{4}\right)$

Ans. : Sovle it yourself

Frequency Spectrum :



(a) Double sided amplitude spectrum (b) Double sided phase spectrum

Fig. P. 6.5.1

Ex. 6.5.2 : Expand the periodic gate function shown in Fig. P. 6.5.2 by the exponential Fourier series.

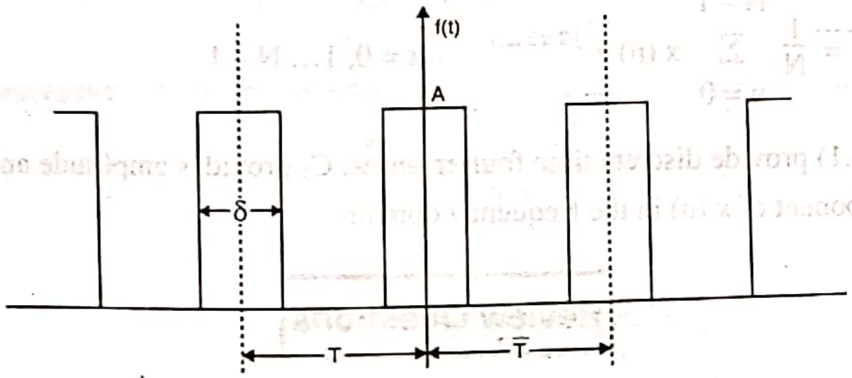


Fig. P. 6.5.2

Ans. : Solve it yourself

$$x(t) = \sum_{n=-\infty}^{\infty} \frac{\delta A}{T_0} \sin c\left[\frac{n\delta}{T_0}\right] e^{j2\pi nt/T_0} \quad \text{...Ans.}$$

Ex. 6.5.3 : Consider spectrum of x (t) given below :

From given spectrum, write

1. Function x (t)
2. Fourier series coefficient of x (t) as $C_0, C_1 \ldots\ldots C_n$
3. Total power of x (t)

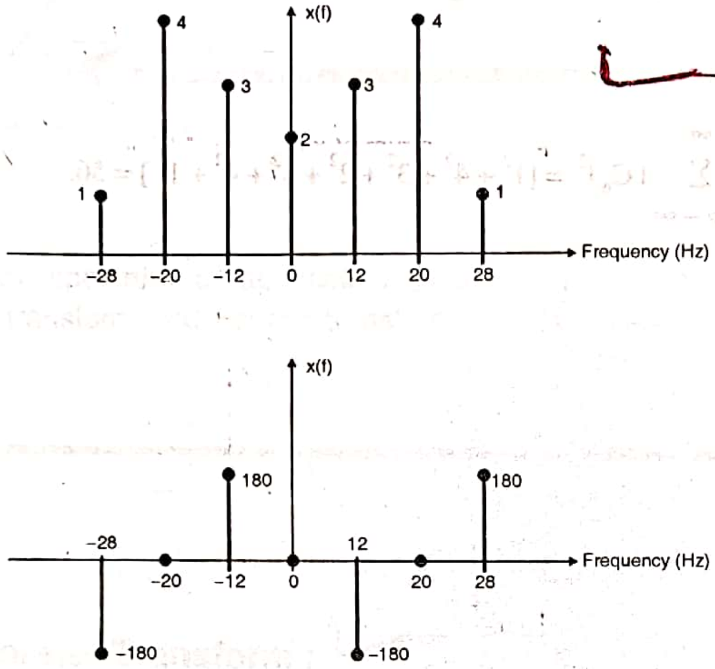


Fig. P. 6.5.3

Soln. : For the spectrum shown in Fig. P. 6.5.3.

(a) Function x (t) can be derived as,

1. First Term :

Amplitude at 0Hz is 2.

2. Second Term :

Amplitude = 3 and phase is – 180°at 12 Hz.

Therefore the second term in cosine form is, $3 \cos (2\pi\, 12\, t - 180^\circ)$

3. Third Term :

Amplitude = 4, and phase - 0° at 20 Hz.

Therefore the third term is, $4 \cos (2\pi\, 20t + 0^\circ)$

(iv) Fourth Term :

Amplitude = 1 and phase = 180° at 28 Hz.

$\therefore$ 4th term is, $\cos (2\pi\, 28t + 180^\circ)$

Hence, $x(t) = 2 + 3 \cos (2\pi\, 12\, t - 180^\circ) + 4 \cos (2\pi\, 20t + 0^\circ) + \cos (2\pi\, 28t + 180^\circ)$

(b) Fourier series coefficients of x (t)

$$C_0 = 2$$
$$C_1 = 3$$
$$C_2 = 4$$
$$C_3 = 1$$

(c) $$P = \sum_{n=-\infty}^{\infty} |C_n|^2 = [1^2 + 4^2 + 3^2 + 2^2 + 3^2 + 4^2 + 1^2] = 56.$$

□□□

CHAPTER 7

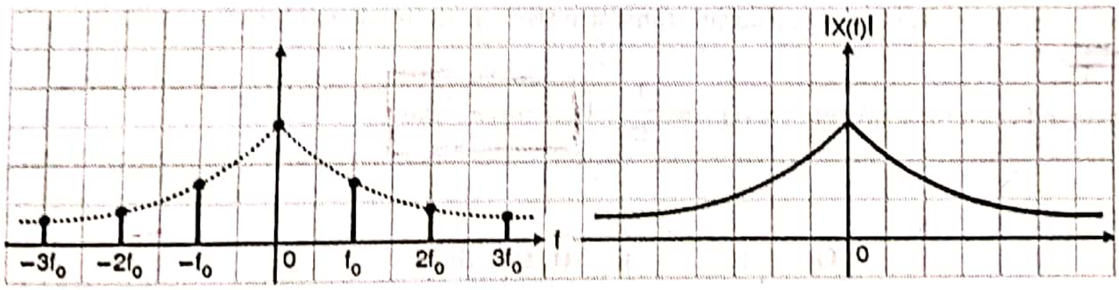
Continuous Time Fourier Transform (CTFT)

Syllabus :

Frequency spectrum of aperiodic signals, Fourier transform, Relation between Laplace transform and Fourier transform and its properties, Introduction to DTFT and DFT.

7.1 C.T. Fourier Transform :

- Till now we have seen how to represent the periodic signals extended over the interval $(-\infty, \infty)$, using the Fourier series. Non-periodic time limited signal can also be represented by the Fourier series.



(a) Line spectrum showing vertical spectral lines at f_0 $2f_0$, ...

(b) Continuous spectrum as $f_0 \to 0$

Fig. 7.1.1

- However the non-periodic signals which extend from $-\infty$ to ∞ can be represented more conveniently using the "Fourier Transform" in the frequency domain.
- It is possible to find the Fourier transform of periodic signal as well. For the non-periodic signals $T_0 \to \infty$. Hence the frequency $f_0 = \frac{1}{T_0} \to 0$. Therefore the difference between the spectral components which is f_0 (as seen in the line spectrum) becomes extremely small and they come very close to each other. Due to this the frequency spectrum of non-periodic signals appears to be continuous as shown in Figs. 7.1.1(a) and (b).

Scanned with CamScanner

7.1.1 Necessity of Fourier Transform :

- Any signal is made up by addition of elementary signals (sine and cosine signals) which are at different frequencies, have different amplitudes and relative phases.
- Using the Fourier transform we can plot the amplitude and phase spectrums of the given signal which provide us all the information about amplitudes and relative phases of such elementary signals.
- Thus Fourier transform can be used for the "analysis" of a signal. It is used for transformation from the time domain to frequency domain.
- The F.T. can also be used for analysis of LTI systems as discussed later on.

7.1.2 Definition of Fourier Transform :

The Fourier transform of a signal x (t) is defined as follows :

$$\text{Fourier transform :} \quad X(\omega) = \int_{-\infty}^{\infty} \boxed{x(t)\, e^{-j\omega t}\, dt}$$

$$\text{OR} \quad X(f) = \int_{-\infty}^{\infty} \boxed{x(t)\, e^{-j2\pi f t}\, dt} \qquad ...(7.1.1)$$

These equations are known as "analysis" equations. (Ift)

7.1.3 Definition of Inverse Fourier Transform : (IFT)

The signal x (t) can be obtained back from its Fourier transform X (f) by using the inverse Fourier transform. The inverse Fourier transform (IFT) is defined as follows :

$$\text{Inverse Fourier transform :} \quad x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \boxed{X(\omega)\, e^{j\omega t}\, d\omega}$$

$$\text{Or} \quad x(t) = \int_{-\infty}^{\infty} X(f)\, e^{j2\pi f t}\, df \qquad ...(7.1.2)$$

Representation :

The signal x(t) and its Fourier transform X(f) form a Fourier transform pair which can be represented as,

$$x(t) \overset{F}{\longleftrightarrow} X(f) \qquad ...(7.1.3)$$

The other way to represent it is as follows :

$$X(t) = F[x(t)] \qquad ...(7.1.4)$$

or it can be represented as,

$$X(t) = F^{-1} [X(f)] \qquad ...(7.1.5)$$

The Fourier transform is a complex function of frequency f. Therefore it is possible to express it in the complex exponential form as follows :

$$X(f) = |X(f)| \cdot e^{j\theta(f)} \quad ...(7.1.6)$$

In this expression :

$|X(f)|$ = The amplitude spectrum of x (t) and

$\theta(f)$ = The phase spectrum.

The amplitude spectrum is a graph of amplitude versus frequency, whereas the phase spectrum is a graph of phase angle versus frequency.

7.1.4 Conditions for the Existence of Fourier Transform :

- We have studied the **Dirichlet conditions** for Fourier series. Similarly these conditions should be satisfied by a signal x (t), then only it is possible to obtain the Fourier transform of x (t).
- For the periodic signals the integration is obtained over one period however for the periodic signals, it will be obtained over a range $-\infty$ to ∞.
- The signal x(t) will have to satisfy the following conditions so that it's Fourier transform can be obtained :

1. The function x (t) should be single valued in any finite time interval T.
2. It should have a finite number of discontinuities in any finite interval T.
3. The function x (t) should have a finite number of maxima and minima in any finite interval of time T.
4. The function x (t) should be an absolutely integrable function.

That means $$\int_{-\infty}^{\infty} |x(t)|\, dt < \infty \quad ...(7.1.7)$$

The conditions stated above are sufficient conditions, but they are not the necessary conditions.

7.1.5 Amplitude and Phase Spectrums :

- The amplitude and phase spectrums are continuous and not discrete in nature. Out of them, the amplitude spectrum of a real valued function x (t) exhibits an even symmetry.

$$\therefore \quad X(f) = X(-f) \quad ...(7.1.8)$$

- And the phase spectrum has an odd symmetry. That means,

$$\theta(f) = -\theta(-f) \quad ...(7.1.9)$$

7.2 Properties of Fourier Transform :

Some of the important properties of the Fourier transform are listed as follows :

- Linearity or superposition
- Time scaling
- Duality or symmetry
- Time shifting
- Frequency shifting
- Area under x (t)
- Area under X (f)
- Differentiation in time domain
- Integration in time domain
- Conjugate function
- Multiplication in time domain (Multiplication theorem)
- Convolution theorem.

Let us understand these properties one-by-one.

7.2.1 Property 1 : Linearity or Superposition :

- If $x_1(t) \overset{F}{\leftrightarrow} X_1(f)$ and $x_2(t) \overset{F}{\leftrightarrow} X_2(f)$ represent the Fourier transform pairs and if a_1 and a_2 are constants then we can write,

$$[a_1 x_1(t) + a_2 x_2(t)] \overset{F}{\leftrightarrow} [a_1 X_1(f) + a_2 X_2(f)] \quad ...(7.2.1)$$

- That means the linear combination of inputs gets transformed into linear combination of their Fourier transforms.
- This property can be used to obtain the Fourier transform of a complicated function say $x(t)$ by decomposing it in the form of sum of simpler functions, say $x_1(t)$ and $x_2(t)$.

Proof of the Property :

$$F[a_1 x_1(t) + a_2 x_2(t)] = \int_{-\infty}^{\infty} [a_1 x_1(t) + a_2 x_2(t)] \boxed{e^{-j2\pi ft} dt}$$

$$= \int_{-\infty}^{\infty} a_1 x_1(t)\, e^{-j2\pi ft} dt + \int_{-\infty}^{\infty} a_2 x_2(t) \cdot e^{-j2\pi ft} dt$$

$$= a_1 X_1(f) + a_2 X_2(f)$$

...Proved.

7.2.2 Property 2 : Time Scaling :

Let $x(t)$ and $X(f)$ form a Fourier transform pair and let "α" be a constant. Then the time scaling property states that,

$$x(\alpha t) \overset{F}{\leftrightarrow} \frac{1}{|\alpha|} X(f/\alpha)$$

Meaning :

1. $x(\alpha t)$ represents a time scaled signal and $X(f/\alpha)$ represents the frequency scaled signal or scaled frequency spectrum.
2. For $\alpha < 1$, $x(\alpha t)$ represents a compressed signal but $X(f/\alpha)$ represents an expanded version of $X(f)$.
3. And for $\alpha > 1$, $x(\alpha t)$ will be an expanded signal in the time domain. But its Fourier transform $X(f/\alpha)$ represents a compressed version of $X(f)$.

Thus compression in the time domain results in expansion of frequency spectrum whereas expansion in the time domain results in compression of the frequency spectrum.

Proof of scaling property :

"α" being a constant, can be positive or negative. i.e. $\alpha > 0$ or $\alpha < 0$. Let us find the F.T. considering both the possibilities.

1. "α" positive ($\alpha > 0$) :

$$F[x(\alpha t)] = \int_{-\infty}^{\infty} x(\alpha t)\, e^{-j2\pi ft}\, dt$$

$$\text{Substitute } \tau = \alpha t, \text{ then } d\tau = \alpha\, dt$$

$$\therefore \quad dt = \frac{d\tau}{\alpha}$$

$$\therefore \quad F[x(\alpha t)] = \int_{-\infty}^{\infty} x(\tau) e^{-j2\pi f \frac{\tau}{\alpha}} \frac{d\tau}{\alpha} = \frac{1}{\alpha} \int_{-\infty}^{\infty} x(\tau) e^{-j2\pi (f/\alpha)\tau}\, d\tau$$

$$= \frac{1}{\alpha} X(f/\alpha) \qquad \text{...(7.2.3)}$$

2. $\alpha < 0$ i.e. "α" negative :

$$F[x(\alpha t)] = \int_{-\infty}^{\infty} x(\alpha t)\, e^{-j2\pi ft}\, d\tau$$

$$\text{Substitute } \tau = \alpha t \quad \therefore \; d\tau = \alpha\, dt$$

$$\therefore \quad dt = d\tau/\alpha$$

$$\therefore \quad F[x(\alpha t)] = \frac{1}{\alpha} \int_{-\infty}^{\infty} x(\tau)\, e^{-j2\pi (f/\alpha)\tau}\, d\tau$$

But as $\alpha < 0$ i.e. negative,

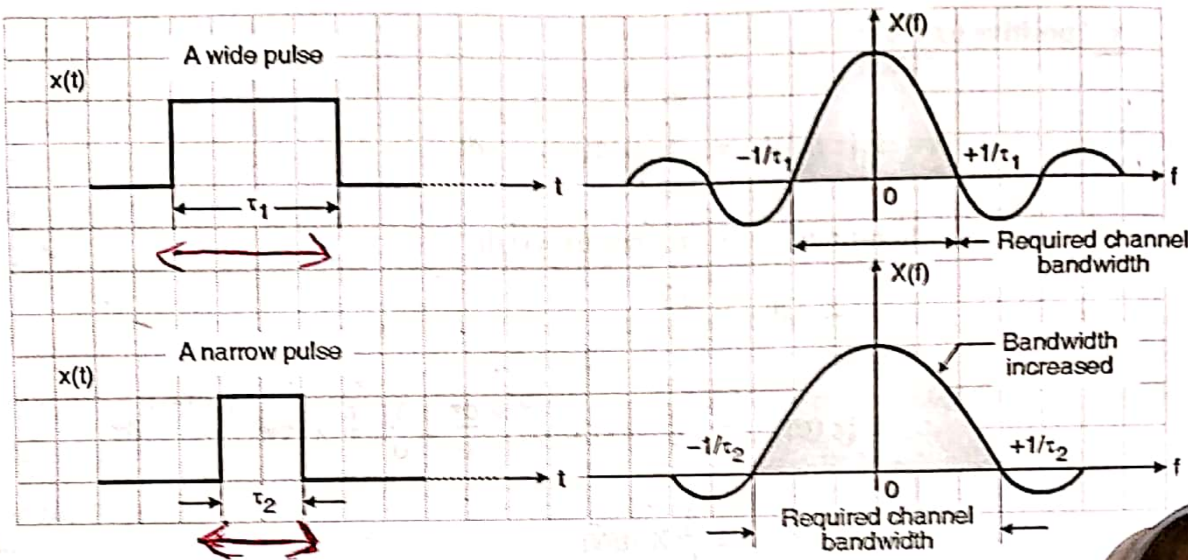
$$F[x(\alpha t)] = -\frac{1}{\alpha} X(f/\alpha) \qquad \text{...(7.2.4)}$$

Combining the results obtained in Equations (7.2.3) and (7.2.4) we can write,

$$F[x(\alpha t)] = \frac{1}{|\alpha|} X(f/\alpha) \qquad \text{...(7.2.5)}$$

Significance of scaling property in communication system :

- In digital communication systems a train of digital pulses is transmitted. The number of such pulses transmitted per second is called as the signalling rate. The signalling rate should be as high as possible in order to transmit as much information as possible.
- But with increase in signalling rate, the width of each pulse will reduce i.e. compression in time domain will take place.
- To transmit narrow pulses without introducing signal distortion, the channel bandwidth needs to be increased. i.e. expansion in the frequency domain will take place.



Compression in time domain ≡ Expansion in frequency domain

Fig. 7.2.1 : Significance of time scaling property

Ex. 7.2.1 : Find the Fourier transform of the decaying exponential pulse shown in Fig. P. 7.2.1.

Soln. : The exponential pulse shown in the Fig. P. 7.2.1 can be represented mathematically as follows :

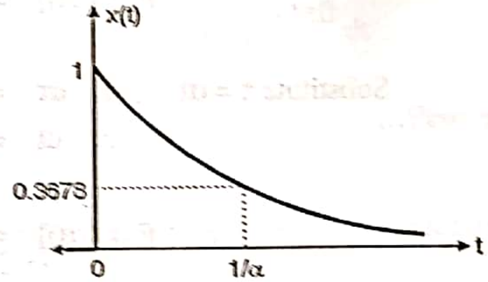


Fig. P. 7.2.1 : Decaying exponential pulse

$$\boxed{x(t) = e^{-\alpha t}} \quad \text{for } t \geq 0 \quad = \quad 0 \quad \text{for } t < 0$$

It can be represented in an alternate way as,

$$x(t) = e^{-\alpha t}\, u(t)$$

The meaning of both the Equations (1) and (2) is the same. This is because u (t) = 1 for t ≥ 0. So multiplying by u (t) does not affect the original function.

To find the Fourier transform :

$$F[x(t)] = \int_{-\infty}^{\infty} e^{-\alpha t}\, u(t)\, e^{-j2\pi f t}\, dt \quad \text{...as per definition of FT.}$$

$$= \int_{0}^{\infty} e^{-\alpha t}\, e^{-j2\pi f t}\, dt = \int_{0}^{\infty} e^{-(\alpha + j2\pi f)t}\, dt = \frac{-1}{(\alpha + j2\pi f)}\left[e^{-(\alpha + j2\pi f)t} \right]_0^{\infty}$$

$$= \frac{-1}{(\alpha + j2\pi f)}\left[e^{-\infty} - e^{0}\right] = \frac{-1}{(\alpha + j2\pi f)}[0 - 1] = \frac{1}{(\alpha + j2\pi f)}$$

This is the required result.

$$\therefore \quad e^{-\alpha t} u(t) \overset{F}{\longleftrightarrow} \frac{1}{(\alpha + j2\pi f)} \quad ...(3)$$

Ex. 7.2.2 : Find the Fourier transform of the exponential pulse shown in Fig. P. 7.2.2(a). Also find the amplitude and phase spectrums for the same.

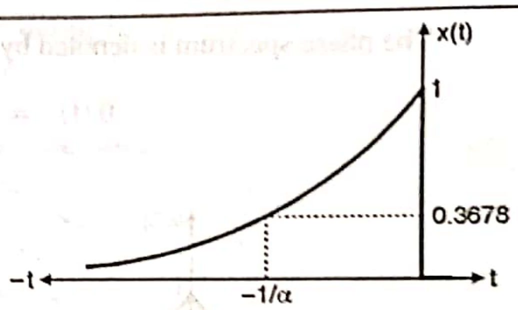


Fig. P. 7.2.2(a)

Soln. : The pulse shown in Fig. P. 7.2.2(a) can be represented as,

$$x(t) = e^{\alpha t} \quad \text{for } t \le 0$$
$$= 0 \quad \text{for } t > 0 \quad ...(1)$$

It can also be represented as,

$$x(t) = e^{\alpha t} u(-t) \quad ...(2)$$

It's Fourier transform is given as,

$$X(f) = \int_{-\infty}^{\infty} e^{\alpha t} u(-t) \cdot e^{-j2\pi f t} dt = \int_{-\infty}^{0} e^{\alpha t} e^{-j2\pi f t} dt = \int_{-\infty}^{0} e^{(\alpha - j2\pi f) t} dt$$

$$= \frac{1}{(\alpha - j2\pi f)} \left[e^{(\alpha - j2\pi f) t} \right]_{-\infty}^{0}$$

$$\therefore \quad X(f) = \frac{1}{(\alpha - j2\pi f)} [e^0 - e^{-\infty}] = \frac{1}{(\alpha - j2\pi f)} [1 - 0] = \frac{1}{(\alpha - j2\pi f)} \quad ...(3)$$

This is the desired result. Thus,

$$e^{\alpha t} u(-t) \overset{F}{\longleftrightarrow} \frac{1}{(\alpha - j2\pi f)} \quad ...(4)$$

To find the amplitude spectrum :

The amplitude spectrum is denoted by $| X(f) |$ and is given by,

$$| X(f) | = [(\text{Real part of } X(f))^2 + (\text{Imaginary part of } X(f))]^{1/2}$$

We have found out that $X(f) = \frac{1}{(\alpha - j2\pi f)} = \frac{1}{(\alpha - j2\pi f)} \times \frac{(\alpha + j2\pi f)}{(\alpha + j2\pi f)} = \frac{\alpha + j2\pi f}{\alpha^2 + (2\pi f)^2}$

$$\therefore \quad \text{Real part of } x(f) = \frac{\alpha}{\alpha^2 + 4\pi^2 f^2} \quad ...(5)$$

$$\text{and imaginary part of } X(f) = \frac{2\pi f}{\alpha^2 + 4\pi^2 f^2} \quad ...(6)$$

Therefore the amplitude spectrum is given as,

$$| X(f) | = \left[\frac{\alpha^2}{(\alpha^2 + 4\pi^2 f^2)^2} + \frac{4\pi^2 f^2}{(\alpha^2 + 4\pi^2 f^2)^2} \right]^{1/2}$$

$$\therefore \quad | X(f) | = \frac{1}{(\alpha^2 + 4\pi^2 f^2)^{1/2}} \quad ...(7)$$

1. The amplitude spectrum $|X(f)| = \frac{2\alpha}{\alpha^2 + (2\pi f)^2}$

2. The phase spectrum $\theta(f) = 0$.

The amplitude and phase spectrums are plotted as shown in Fig. P. 7.2.3(b).

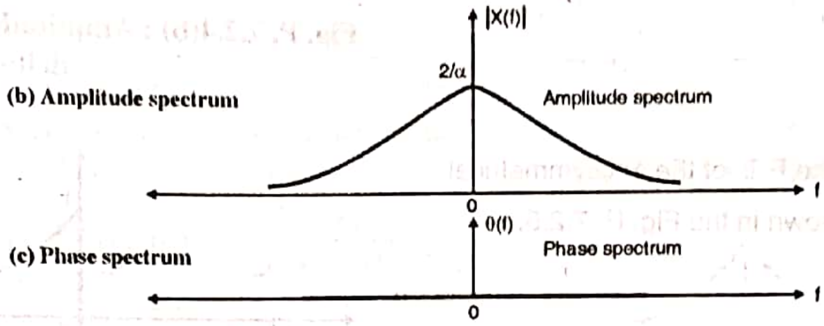


Fig. P. 7.2.3(b)

Ex. 7.2.4 : Obtain the Fourier transform of the delta function shown in Fig. P. 7.2.4(a).

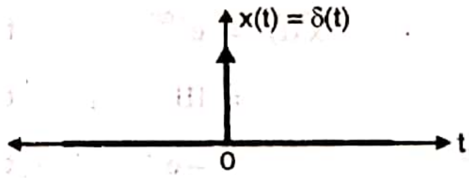


Fig. P. 7.2.4(a) : Delta function

Soln. :

1. By the definition of Fourier transform,

$$X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi ft}\, dt = \int_{-\infty}^{\infty} \delta(t)\, e^{-j2\pi ft}\, dt \qquad ...(1)$$

We cannot substitute the value of $\delta(t)$ directly in the Equation (1) because it is infinitely large at $t = 0$. Therefore let us use the sifting property of the delta function.

2. **Shifting property of delta function :**

The sifting property states that

$$\int_{-\infty}^{\infty} f(t)\, \delta(t - t_d)\, dt = f(t_d) \qquad ...(2)$$

Let us use this property in Equation (1) as follows ;

3. In Equation (2) assume that $t_d = 0$ and $f(t) = e^{-j2\pi ft}$

$$\therefore \quad X(f) = \int_{-\infty}^{\infty} e^{-j2\pi ft} \cdot \delta(t - 0) \quad \therefore \text{ by using Equation (2).}$$

$$X(f) = e^{-j2\pi f t_d}, \quad \text{but } t_d = 0$$

$$\therefore \quad X(f) = e^{-j2\pi \cdot 0} = 1 \qquad ...(3)$$

Thus $\delta(t) \overset{F}{\longleftrightarrow} 1$

The amplitude spectrum of the delta function is as shown in the Fig. P. 7.2.4(b). This shows that the delta function contains all the frequencies from $-\infty$ to ∞ with equal amplitudes. The Fourier transform of a delta function is a dc signal.

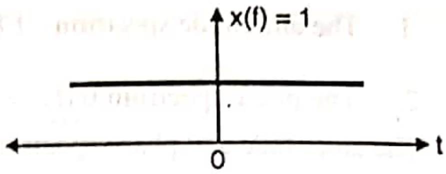


Fig. P. 7.2.4(b) : Amplitude spectrum of delta function

Ex. 7.2.5 : Obtain the F.T. of the antisymmetrical pulse shown in the Fig. P. 7.2.5.

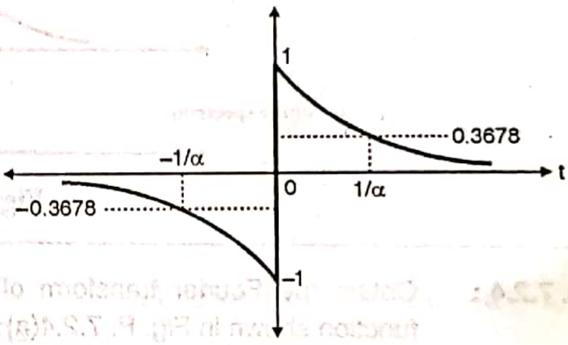


Fig. P. 7.2.5

Soln. :

The antisymmetric pulse can be represented as,

$$x(t) = e^{-\alpha t} \qquad t > 0$$

$$= |1| \qquad t = 0$$

$$= -e^{\alpha t} \qquad t < 0$$

Therefore the Fourier transform is given by,

$$F[x(t)] = \int_{-\infty}^{0^-} -e^{\alpha t} e^{-j2\pi ft} dt + \int_{0^-}^{0^+} |1| e^{-j2\pi ft} dt + \int_{0^+}^{\infty} e^{-\alpha t} e^{-j2\pi ft} dt$$

$$= \int_{-\infty}^{0^-} -e^{(\alpha - j2\pi f)} dt + \int_{0^+}^{\infty} e^{-(\alpha + j2\pi f)t} dt$$

$$= \frac{-1}{\alpha - j2\pi f}\left[e^{(\alpha - j2\pi f)t}\right]_{-\infty}^{0^-} + \frac{1}{-(\alpha + j2\pi f)}\left[e^{-(\alpha + j2\pi f)t}\right]_{0+}^{\infty}$$

$$= \frac{-1}{(\alpha - j2\pi f)}[e^{0} - e^{-\infty}] - \frac{1}{(\alpha + j2\pi f)}[e^{-\infty} - e^{0}]$$

$$= \frac{-1}{(\alpha - j2\pi f)} + \frac{1}{(\alpha + j2\pi f)} = \frac{-\alpha - j2\pi f + \alpha - j2\pi f}{\alpha^2 + (2\pi f)^2}$$

$$\therefore \quad X(f) = \frac{-j4\pi f}{\alpha^2 + (2\pi f)^2} \qquad ...(1)$$

Amplitude spectrum :

The amplitude spectrum of the rectangular function is as shown in Fig. P. 7.2.8(b).

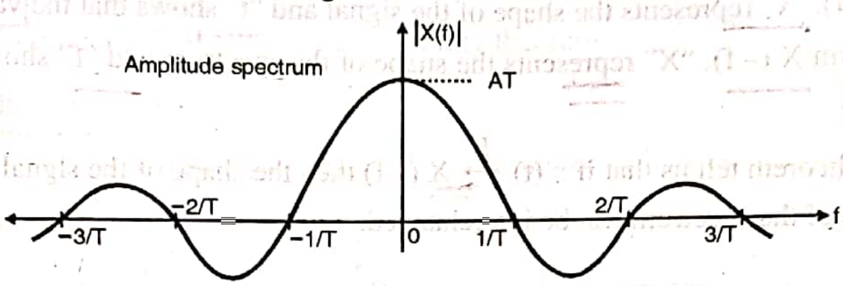


Fig. P. 7.2.8(b) : Amplitude spectrum of a rectangular pulse

As we already know, sinc (0) = 1 $\qquad\therefore\quad$ AT sinc (0) = AT

The sinc function will have zero value for the following values of "fT" :

$$\text{sinc (fT)} = 0 \qquad \text{for fT} = \pm 1, \pm 2, \pm 3, \ldots\ldots$$

$$\text{i.e. for } f = \pm\frac{1}{T}, \pm\frac{2}{T}, \pm\frac{3}{T} \ldots\ldots$$

The phase spectrum has not been shown as it has zero value for all the values of f.

To absorb negative values of | X(f) | in the phase shift :

...he negative amplitude of ...ude spectrum | X (f) | ... made positive by ...cing a phase shift of ± 180° in the phase spectrum. This is as shown in Fig. P. 7.2.8(c). A negative phase shift for positive frequency and positive phase shift for the negative frequency is introduced in order to maintain symmetry of the phase spectrum.

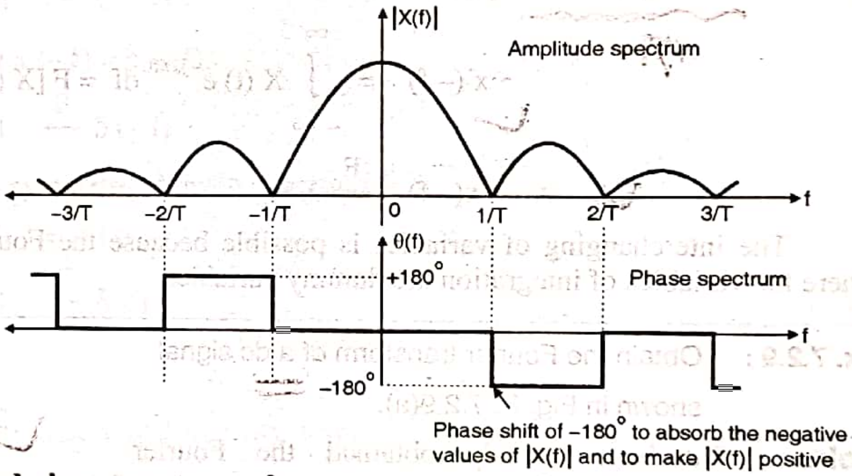


Fig. P. 7.2.8(c) : Amplitude and phase spectrums for a rectangular pulse. Negative values of | X(f) | have been absorbed in the additional phase shift of ± 180° in the phase spectrum

> **Note :** The choice of the phase shift of ± 180° is completely arbitrary.

7.2.3 Property 3 : Duality or Symmetry Property :

This property states that, $\quad$ if $\; x(t) \overset{F}{\longleftrightarrow} X(-f)$

$$\text{Then} \quad X(t) \overset{F}{\longleftrightarrow} x(-f) \qquad \ldots(7.2.6)$$

i.e. t and f can be interchanged.

Meaning :

1. In the term x(t), "x" represents the shape of the signal and "t" shows that the $\boxed{\text{variable is time.}}$
2. And in the term X (– f), "X" represents the shape of the spectrum and "f" shows that the variable $\boxed{\text{is frequency.}}$
3. The Duality theorem tell us that if $x(t) \overset{F}{\longleftrightarrow} X(-f)$ then the shape of the signal in the time domain and the shape of the spectrum can be interchanged.

Proof :

- By definition of the $\boxed{\text{inverse Fourier transform,}}$

$$x(t) = \int_{-\infty}^{\infty} x(t)\, e^{j2\pi ft}\, df$$

- Substitute t = – t in the above equation to get,

$$x(-t) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi ft}\, df$$

- Interchange t and f to get,

$$x(-f) = \int_{-\infty}^{\infty} X(t)\, e^{-j2\pi ft}\, df = F[X(t)]$$

$$\therefore \quad x(-f) \overset{F}{\longleftrightarrow} X(t) \qquad \text{...Proved.}$$

The interchanging of variables is possible because the Fourier transform is a definite integral where the variables of integration are dummy variables.

Ex. 7.2.9 : Obtain the Fourier transform of a dc signal shown in Fig. P. 7.2.9(a).

Soln. : We have already obtained the Fourier transform of the unit impulse or delta function δ (t).

$$\delta(t) \overset{F}{\longleftrightarrow} 1 \qquad \text{...(1)}$$

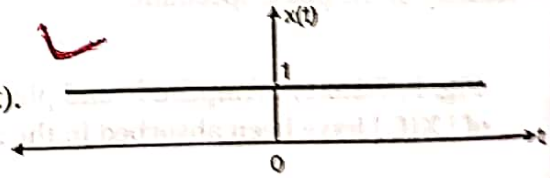


Fig. P. 7.2.9(a) : A dc signal

To solve this example we are going to use the duality property.

Duality property : It states that

$$\text{If} \quad x(t) \overset{F}{\longleftrightarrow} X(-f) \qquad \text{then} \quad X(t) \overset{F}{\longleftrightarrow} x(-f).$$

Step 1 : To verify the condition $x(t) \overset{F}{\longleftrightarrow} X(-f)$:

$$\delta(t) \overset{F}{\longleftrightarrow} 1$$

Hence X (f) = 1. As 1 does not depend on f, X (f) = X (– f) = 1

$$\therefore \quad \delta(t) \overset{F}{\leftrightarrow} X(-f) \qquad \text{...Proved.}$$

As this condition is verified, we can apply duality theorem.

Step 2 : Apply duality theorem :

$$\begin{array}{ccc} \text{If } x(t) \overset{F}{\longleftrightarrow} X(-f) & & \delta(t) \overset{F}{\longleftrightarrow} 1 \\ & \equiv & \\ \text{Then } X(t) \overset{F}{\longleftrightarrow} x(-f) & & \therefore\ 1 \overset{F}{\longleftrightarrow} \delta(-f) \end{array}$$

Consider the statement $1 \overset{F}{\leftrightarrow} \delta(-f)$. As the "δ" function is a symmetrical function,

$$\delta(-f) = \delta(f)$$

$$\therefore \quad 1 \overset{F}{\leftrightarrow} \delta(f) \qquad \text{...Ans.}$$

Thus the dc signal of unity amplitude is transformed into a delta function in the frequency domain as shown in Fig. P. 7.2.9(b).

Hence we can use the duality property as,

$$X(t) \overset{F}{\leftrightarrow} x(-f) \qquad \text{...(2)}$$

In Equation (2), X (t) = 1 and x (– f) = δ (– f)

$$\therefore \quad 1 \overset{F}{\leftrightarrow} \delta(-f) \qquad \text{...(3)}$$

But as the delta function is a symmetrical (even) function

$$\delta(-f) = \delta(f)$$

$$\text{Therefore } 1 \overset{F}{\leftrightarrow} \delta(f) \qquad \text{...Ans.}$$

Meaning :

The above equation states that a dc signal of unity amplitude is transformed into a delta function in the frequency domain.

Important deduction :

The dc signal shown in Fig. P. 7.2.9(a) is represented mathematically as,

$$x(t) = 1 \text{ for } -\infty < t < \infty \qquad \text{...(4)}$$

Therefore its Fourier transform is given as,

$$X(f) = \int_{-\infty}^{\infty} 1 \cdot e^{-j2\pi ft}\, dt \qquad \text{...(5)}$$

But we have already obtained the Fourier transform of a dc signal

$$\therefore \quad X(f) = \delta(f) \qquad \text{...(6)}$$

Substituting this in Equation (5) we get,

$$\int_{-\infty}^{\infty} 1 \cdot e^{-j2\pi ft}\, dt = \delta(f) \quad ...(7)$$

The spectrum of the dc signal is as shown in Fig. P. 7.2.9(b).

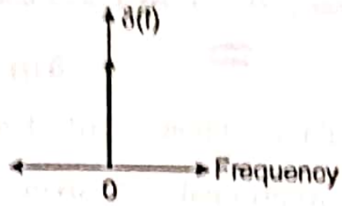


Fig. P. 7.2.9(b) : Spectrum of a dc signal

Ex. 7.2.10 : For the sinc function shown in Fig. P. 7.2.10(a), obtain the Fourier transform and plot the spectrum.

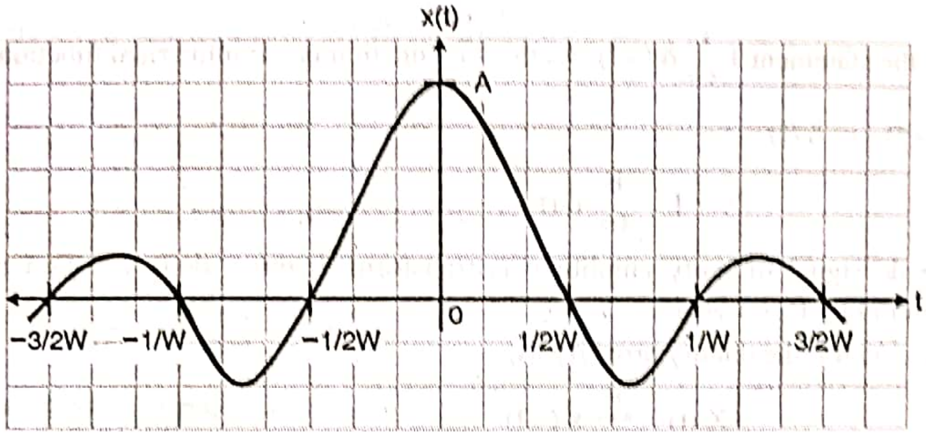


Fig. P. 7.2.10(a) : A sinc pulse

Soln. :

- The sinc signal shown in Fig. P. 7.2.10(a) can be expressed mathematically as :

$$x(t) = A \text{ sinc } (2\,Wt) \quad ...(1)$$

- To evaluate the Fourier transform of this function, we are going to apply the duality and time scaling properties of the Fourier transform. We have obtained the Fourier transform of a rectangular pulse of amplitude A and duration T, as :

$$A \text{ rect } [t/T] \overset{F}{\longleftrightarrow} AT \text{ sinc } (fT) \quad ...(2)$$

- Using the duality property we can write that,

$$AT \text{ sinc } (tT) \overset{F}{\longleftrightarrow} A \text{ rect } [f/T] \quad ...(3)$$

Compare the LHS of Equation (3) with the RHS of Equation (1) which states the expression for x (t), it is observed that T = 2W. Substituting this into Equation (3) we get

$$2\,AW \text{ sinc } (2\,Wt) \overset{F}{\longleftrightarrow} A \text{ rect } [f/2W]$$

$$\therefore \quad A \text{ sinc } (2\,Wt) \overset{F}{\longleftrightarrow} \frac{A}{2W} \text{ rect } [f/2W] \quad ...\text{Ans.}$$

Thus a sinc pulse in the time domain is transformed into a rectangular pulse in the frequency domain. The spectrum of sinc pulse is shown in Fig. P. 7.2.10(b).

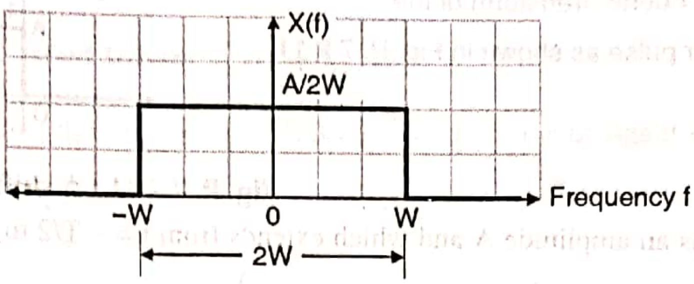


Fig. P. 7.2.10(b) : Spectrum of a sinc pulse

7.2.4 Property 4 : Time Shifting :

The time shifting property states that if x (t) and X (f) form a Fourier transform pair then,

$$x(t - t_d) \overset{F}{\longleftrightarrow} e^{-j2\pi f t_d} X(f) \quad ...(7.2.7)$$

Here the signal $x(t - t_d)$ is a time shifted signal. It is the same signal x (t) only shifted in time.

Proof :

$$F[x(t - t_d)] = \int_{-\infty}^{\infty} x(t - t_d) \cdot e^{-j2\pi f t} dt \quad ...(7.2.8)$$

$$\text{Let } (t - t_d) = \tau,$$

$$\therefore \quad t = t_d + \tau$$

$$\therefore \quad dt = d\tau.$$

Substituting these values in Equation (7.2.8) we get,

$$F[x(t - t_d)] = \int_{-\infty}^{\infty} x(\tau) \cdot e^{-j2\pi f(t_d + \tau)} d\tau = e^{-j2\pi f t_d} \int_{-\infty}^{\infty} x(\tau)\, e^{-j2\pi f \tau} d\tau$$

$$\therefore \quad F[x(t - t_d)] = e^{-j2\pi f t_d} X(f)$$

...Proved.

This shows that the time shifting does not have any effect on the amplitude spectrum, but there is an additional phase shift of $-2\pi f t_d$, which is denoted by the term $e^{-j 2\pi f t_d}$

Significance of time shifting in communication systems :

- If signal x(t) is transmitted by a transmitter, then due to the distance travelled, this signal becomes a time delayed signal $x(t - t_d)$ when it reaches the receiver.
- The time delay "t_d" is dependent on the distance between the transmitter and the receiver.
- The time shifting property explains the effect of such time shifting on the spectrum of the signal. It tells us that there is no effect of time shifting on the amplitude spectrum but there is an additional phase shift of $-2\pi f t_d$.

Ex. 7.2.11 : Obtain the Fourier transform of the rectangular pulse as shown in Fig. P. 7.2.11.

Soln. :

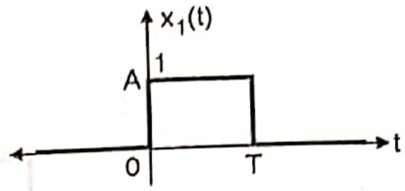


Fig. P. 7.2.11 : A shifted rectangular pulse

1. A pulse which has an amplitude A and which extends from $t = -T/2$ to $t = +T/2$ is expressed as follows :

$$x(t) = A \text{ rect}(t/T) \quad \text{...(1)}$$

2. The rectangular pulse of Fig. P. 7.2.11 is the delayed version of the standard rectangular pulse expressed in Equation (2), with a time delay of T/2. Hence it can be mathematically represented as,

$$x_1(t) = A \text{ rect}\left[\frac{t - T/2}{T}\right] \quad \text{...(2)}$$

3. Using the property of "time shifting" we get its Fourier transform as,

$$F[x_1(t)] = X_1(f) = e^{-j2\pi fT/2} X(f) = e^{-j\pi ft} \cdot AT \text{ sinc}(fT)$$

$$\therefore \quad X_1(f) = AT e^{-j\pi ft} \text{ sinc}(fT) \quad \text{...Ans.}$$

Ex. 7.2.12 : Obtain the Fourier transform of the rectangular pulse as shown in Fig. P. 7.2.12.

Soln. :

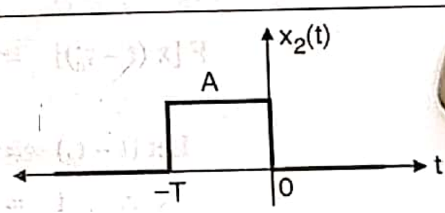


Fig. P. 7.2.12 : A shifted rectangular pulse

1. The rectangular pulse in Fig. P. 7.2.12 is the advanced version of the standard rectangular pulse, with an advance of T/2. Hence the signal is represented as,

$$x_2(t) = A \text{ rect}\left[\frac{t + T/2}{T}\right] \quad \text{...(1)}$$

2. Using the property of time shifting, the Fourier transform of this pulse is given as,

$$X_2(f) = e^{-j2\pi f(-T/2)} \cdot AT \text{ sinc}(fT)$$

$$\therefore \quad X_2(f) = e^{+j\pi fT} AT \text{ sinc}(fT) \quad \text{...Ans.}$$

7.2.5 Property 5 : Area under x(t) :

This property states that the area under the curve x(t) equals the value of its Fourier transform at $f = 0$.

i.e. if $x(t) \overset{F}{\longleftrightarrow} X(f)$ then,

$$\text{Area under } x(t) = \int_{-\infty}^{\infty} x(t)\, dt = X(0) \quad \text{...(7.2.9)}$$

$$\text{Area under } x(t) = \int_{-\infty}^{\infty} x(t)\, dt = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi ft}\, dt \text{ at } f = 0.$$

$$= X(f) \text{ at } f = 0$$

$$\therefore \quad \text{Area under } x(t) = X(0). \qquad \textbf{...Proved.}$$

7.2.6 Property 6 : Area under X (f) :

If $x(t) \overset{F}{\longleftrightarrow} X(f)$ then the area under X (f) is equal to the value of signal x(t) at t = 0.

That means if $x(t) \overset{F}{\longleftrightarrow} X(f)$ then,

$$\text{Area under } X(f) = \int_{-\infty}^{\infty} X(f) \cdot df = x(0) \qquad \text{...(7.2.10)}$$

Proof : By definition of inverse Fourier transform,

$$x(t) = \int_{-\infty}^{\infty} X(f) \cdot e^{j2\pi ft} \, df$$

Substitute t = 0 in this equation to get,

$$x(0) = \int_{-\infty}^{\infty} X(f) \cdot e^{0} \, df = \int_{-\infty}^{\infty} X(f) \, dt$$

The RHS of this equation is the area under X (f). Hence the property is proved.

Ex. 7.2.13 : Show that the total area under the curve of sinc function is equal to 1. That means

$$\int_{-\infty}^{\infty} \text{sinc}(t) \, dt = 1 \qquad \text{...(1)}$$

...n. :

1. The Fourier transform of a sinc pulse is given by,

$$F[\text{sinc}(t)] = \int_{-\infty}^{\infty} \text{sinc}(t) \cdot e^{-j2\pi ft} \, dt \qquad \text{...(2)}$$

2. If we substitute f = 0 in Equation (2) then we will get the area under the sinc function. Therefore let us write the expression for the Fourier transform of sinc (t) and then substitute f = 0 into it to get the area.

$$\therefore \quad F[\text{sinc}(t)] = \text{rect}(f) \qquad \text{...(3)}$$

Thus Fourier transform is a rectangular pulse as shown in Fig. P. 7.2.13.

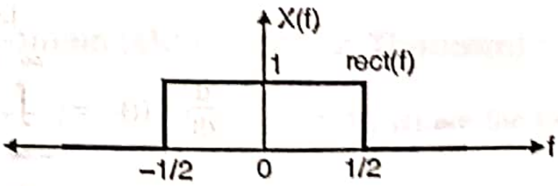


Fig. P. 7.2.13

3. $\therefore$ Area under sinc (t) = rect (0) ...(4)

From Fig. P. 7.2.13, rect (0) = 1

$$\therefore \quad \text{Area under sinc}(t) = \int_{-\infty}^{\infty} \text{sinc}(t)\, dt = 1 \qquad \text{...Proved.}$$

7.2.7 Property 7 : Frequency Shifting :

The frequency shifting characteristics states that if x(t) and X(f) form a Fourier transform pair then,

$$e^{j2\pi f_c t}\, x(t) \xleftrightarrow{F} X(f - f_c) \qquad \text{...(7.2.11)}$$

Here f_c is a real constant.

Proof :

$$F\left[e^{j2\pi f_c t}\, x(t)\right] = \int_{-\infty}^{\infty} e^{j2\pi f_c t}\, x(t)\, e^{-j2\pi f t}\, dt = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi (f - f_c) t}\, dt$$

$$= X(f - f_c) \qquad \text{...Proved.}$$

The term $X(f - f_c)$ represents a shifted frequency spectrum. The whole spectrum is thus shifted right by "f_c" in the frequency domain, when the signal x (t) is multiplied by $e^{j2\pi f_c t}$ in the time domain.

7.2.8 Property 8 : Differentiation in Time Domain :

Some processing techniques involve differentiation and integration of the signal x(t). **This property is applicable if and only if the derivative of x (t) is Fourier transformable.**

Statement : Let $x(t) \xleftrightarrow{F} X(f)$ and let the derivative of x(t) be Fourier transformable. Then,

$$\frac{d}{dt}\, x(t) \xleftrightarrow{F} j2\pi f\, X(f) \qquad \text{...(7.2.12)}$$

Proof : By the definition of inverse Fourier transform,

$$x(t) = \int_{-\infty}^{\infty} X(f)\, e^{+j2\pi f t}\, df$$

Therefore

$$\frac{d}{dt}\, x(t) = \frac{d}{dt}\left[\int_{-\infty}^{\infty} X(f)\, e^{+j2\pi f t}\, df\right] = \int_{-\infty}^{\infty} X(f)\left(\frac{d}{dt}\, e^{+j2\pi f t}\right) df$$

$$\frac{d}{dt}\, x(t) = \int_{-\infty}^{\infty} \left[X(f) \cdot j2\pi f\right] e^{+j2\pi f t}\, df$$

As per the definition of the inverse Fourier transform the term inside the square bracket must be the Fourier transform of $\frac{d}{dt}\, x(t)$.

$$\therefore \quad F\left[\frac{d}{dt}\, x(t)\right] = j\, 2\pi f\, X(f)$$

$$\text{OR} \quad \frac{d}{dt} x(t) \overset{F}{\longleftrightarrow} j\,2\pi f\,X(f).$$ ...Proved.

Meaning :

Differentiating the signal in time domain is equivalent to multiplying its Fourier transform by $(j2\pi f)$ in the frequency domain. Thus differentiation will enhance the high frequency components since $|j2\pi f\,X(f)| > |X(f)|$.

7.2.9 Property 9 : Integration in Time Domain :

Integration in time domain is equivalent to dividing the Fourier transform by $(j2\pi f)$.

i.e. if $x(t) \overset{F}{\longleftrightarrow} X(f)$ and provided that $X(0) = 0$ then,

$$\int_{-\infty}^{t} x(\lambda)\,d\lambda \overset{F}{\longleftrightarrow} \frac{1}{j2\pi f} X(f) \qquad \text{...(7.2.13)}$$

Proof :

In order to avoid confusion let us use a different variable than "t". Therefore let us express x(t) as,

$$x(t) = \frac{d}{dt}\left[\int_{-\infty}^{t} x(\lambda)\,d\lambda\right] \qquad \text{...(7.2.14)}$$

Taking the Fourier transform of both the sides,

$$X(f) = F\left[\frac{d}{dt}\left\{\int_{-\infty}^{t} x(\lambda)\,d\lambda\right\}\right] \qquad \text{...(7.2.15)}$$

Using the property of differentiation we get,

$$X(f) = j\,2\pi f\,F\left[\int_{-\infty}^{t} x(\lambda)\,d\lambda\right]$$

$$\therefore \quad F\left[\int_{-\infty}^{t} x(\lambda)\,d\lambda\right] = \frac{1}{j\,2\pi f} X(f) \qquad \text{...(7.2.16)}$$

This is the desired result. Equation (7.2.16) shows that the integration suppresses the high frequency components because X (f) is being divided by $2\pi f$.

7.2.10 Property 10 : Multiplication in Time Domain (Multiplication Theorem) :

The multiplication theorem states that : If $x_1(t) \overset{F}{\longleftrightarrow} X_1(f)$ and $x_2(t) \overset{F}{\longleftrightarrow} X_2(f)$ are the two Fourier transform pairs then,

$$x_1(t) \cdot x_2(t) \overset{F}{\longleftrightarrow} \int_{-\infty}^{\infty} X_1(\tau) \cdot X_2(f-\tau)\,d\tau \qquad \text{...(7.2.17)}$$

This means that the multiplication of two signals $x_1(t)$ and $x_2(t)$ in the time domain gets transformed into convolution of their Fourier transforms in the frequency domain.

$$\therefore \quad x_1(t) \cdot x_2(t) \overset{F}{\longleftrightarrow} X_1(f) * X_2(f) \qquad ...(7.2.18)$$

Proof : The Fourier transform of the product signal is given by,

$$F[x_1(t) \cdot x_2(t)] = \int_{-\infty}^{\infty} [x_1(t) \cdot x_2(t)]\, e^{-j2\pi f t}\, dt \qquad ...(7.2.19)$$

Let us express $x_2(t)$ as follows :

$$\text{By definition of IFT,} \quad x_2(t) = \int_{-\infty}^{\infty} X_2(f') \cdot e^{+j2\pi f' t}\, df'$$

Here f and f' are not same. This is just to avoid confusion.

Now substitute this value of $x_2(t)$ in the Equation (7.2.19) to get,

$$F[x_1(t) \cdot x_2(t)] = \int_{-\infty}^{\infty} x_1(t) \cdot \int_{-\infty}^{\infty} X_2(f')\, e^{+j2\pi f' t}\, df' \cdot e^{-j2\pi f t}\, dt \qquad ...(7.2.20)$$

$$= \int_{-\infty}^{\infty} x_1(t) \int_{-\infty}^{\infty} X_2(f')\, e^{-j2\pi (f-f') t}\, df'\, dt \qquad ...(7.2.21)$$

Now substitute $\tau = f - f'$

$$= \int_{-\infty}^{\infty} x_1(t) \int_{-\infty}^{\infty} X_2(f-\tau)\, e^{-j2\pi \tau t}\, d\tau\, dt \qquad ...(7.2.22)$$

$$= \int_{-\infty}^{\infty} X_2(f-\tau)\, d\tau \cdot \int_{-\infty}^{\infty} x_1(t)\, e^{-j2\pi\tau t}\, dt \qquad ...(7.2.23)$$

The second term in the above equation represents the Fourier transform of $x_1(t)$. Therefore Equation (7.2.23) can be written as,

$$F[x_1(t)\, x_2(t)] = \int_{-\infty}^{\infty} X_2(f-\tau)\, d\tau\, X_1(\tau)$$

$$\text{OR} \quad = \int_{-\infty}^{\infty} X_1(\tau) \cdot X_2(f-\tau) \cdot d\tau$$

$$\text{Thus} \quad [x_1(t) \cdot x_2(t)] \overset{F}{\longleftrightarrow} X_1(f) * X_2(f) \qquad \text{...Proved.}$$

This is the desired result.

7.2.11 Property 11 : Convolution in the Time Domain (Convolution Theorem) :

This property states that the convolution of signals in the time domain will be transformed into the multiplication of their Fourier transforms in the frequency domain.

$$\text{i.e. } [x_1(t) * x_2(t)] \overset{F}{\longleftrightarrow} X_1(f)\, X_2(f) \qquad \text{...(7.2.24)}$$

Proof :

The convolution of the two signals in the time domain is defined as,

$$X_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\lambda) \cdot x_2(t-\lambda)\, d\lambda \qquad \text{...(7.2.25)}$$

Taking the Fourier transform of the convolution.

$$F[x_1(t) * x_2(t)] = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x_1(\lambda) \cdot x_2(t-\lambda)\, d\lambda \right] e^{-j2\pi f t}\, dt \qquad \text{...(7.2.26)}$$

Multiply and divide the RHS of the Equation (7.2.26) by $e^{-j2\pi f\lambda}$ to get,

$$F[x_1(t) * x_2(t)] = \int_{-\infty}^{\infty} x_1(\lambda) \cdot e^{-j2\pi f\lambda}\, d\lambda \cdot \int_{-\infty}^{\infty} x_2(t-\lambda) \cdot e^{-j2\pi f t} \cdot e^{j2\pi f\lambda}\, dt$$

$$= \int_{-\infty}^{\infty} x_1(\lambda)\, e^{-j2\pi f\lambda}\, d\lambda \int_{-\infty}^{\infty} x_2(t-\lambda)\, e^{-j2\pi f(t-\lambda)}\, dt \qquad \text{...(7.2.27)}$$

Let $(t-\lambda) = m$ in Equation (7.2.26)

$$\therefore \quad F[x_1(t) * x_2(t)] = \int_{-\infty}^{\infty} x_1(\lambda) \cdot e^{-j2\pi f\lambda}\, d\lambda \int_{-\infty}^{\infty} x_2(m) \cdot e^{-j2\pi f m}\, dm$$

Using the definition of the Fourier transform to the RHS we get,

$$F[x_1(t) * x_2(t)] = X_1(f)\, X_2(f) \qquad \text{...(7.2.28)}$$

This is the required result.

Significance of convolution in a communication system :

Consider a communication system with input x(t), output y(t) and impulse response h(t), as shown in Fig. 7.2.2.

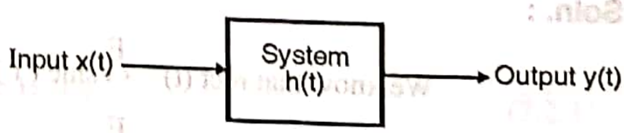


Fig. 7.2.2

The relation between x(t), h(t) and y(t) is as follows,

$$y(t) = x(t) * h(t) \qquad \text{...(7.2.29)}$$

That means y(t) is obtained by taking convolution of x(t) and h (t).

Method I :

Output of a system y(t) can be obtained by taking the convolution of input x(t) and impulse response of the system h (t).

$$y(t) = x(t) * h(t)$$

Method II :

The output y(t) can be obtained using the Fourier transform.

Let $y(t) \overset{F}{\longleftrightarrow} Y(f)$, $x(t) \overset{F}{\longleftrightarrow} X(f)$ and $h(t) \overset{F}{\longleftrightarrow} H(f)$. Then taking the F.T. of Equation (7.2.29) we get

$$Y(f) = X(f) \cdot H(f)$$

Now take IFT to get y (t) = IFT [Y (f)]

Multiplication and taking IFT is simpler than obtaining the convolution. Hence in practice we can use this method to obtain y (t) i.e. output of a system.

A signal cannot be band limited and time limited simultaneously :

- To justify this statement let us first consider a time limited signal and its Fourier transform. A rectangular pulse of duration T and amplitude A is an excellent example of a time limited signal.
- We know that its Fourier transform is a sinc pulse which extends from $f = -\infty$ to $f = +\infty$ eventhough the amplitude of the sinc function goes on decreasing with increase in frequency.
- Thus a time limited signal results into an unlimited spectrum in frequency domain.
- Now consider a rectangular pulse of width "f" in the frequency domain. This is a bandlimited signal. Due to the principle of duality, the inverse Fourier transform of this rectangular pulse is a sinc function in time domain which extends from $t = -\infty$ to $t = +\infty$.
- Thus band limiting in the frequency domain results in a "time unlimited" signal in the time domain. Thus a signal cannot be band limited or time limited simultaneously.

Ex. 7.2.14 : State and prove the convolution property of Fourier transform and find the Fourier transform of the following signal using convolution property :

$x(t) = \text{rect}(t) * \text{rect}(t)$

where

$$\text{rect}(t) = \begin{cases} 1 & \frac{-1}{2} \le t \le \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$

Soln. :

We know that $\text{rect}(t) \overset{F}{\longleftrightarrow} \text{sinc}(f)$.

$\therefore [\text{rect}(t) * \text{rect}(t)] \overset{F}{\longleftrightarrow} \text{sinc}(f)\,\text{sinc}(f) = \text{sinc}^2(f)$ ...**Ans.**

7.2.12 Conjugate Functions :

This property states that, if $x(t) \overset{F}{\longleftrightarrow} X(f)$ and if x (t) is a complex valued function then,

$$x^*(t) \xleftrightarrow{F} X^*(f)$$

Proof :

As per the definition of IFT, we have

$$x(t) = \int_{-\infty}^{\infty} X(f)\, e^{+j2\pi f t}\, df$$

Now take the complex conjugate of both the sides to get,

$$x^*(t) = \left[\int_{-\infty}^{\infty} X(f) \cdot e^{+j2\pi f t}\, df\right]^*$$

$$\therefore \quad x^*(t) = \int_{-\infty}^{\infty} X^*(f)\, e^{-j2\pi f t}\, df \qquad ...(7.2.30)$$

Now replace f by – f to get,

$$x^*(t) = \int_{-\infty}^{\infty} X^*(-f)\, e^{+j2\pi f t}\, df \qquad ...(7.2.31)$$

The RHS of above equation is nothing but the inverse Fourier transform of $X^*(f)$

$$\therefore \quad x^*(t) = F^{-1}[X^*(-f)] \qquad ...(7.2.32)$$

$$\therefore \quad x^*(t) \xleftrightarrow{F} X^*(-f)$$ **...Proved.**

Real and imaginary parts of a time function :

Let x(t) be a complex-valued function which can be represented in terms of its real and imaginary parts as follows :

$$x(t) = \text{Re}[x(t)] + j\,\text{Im}[x(t)] \qquad ...(7.2.33)$$

where Re = Real part of x(t) and

Im = Imaginary part of x(t).

The complex conjugate of x (t) is given by,

$$x^*(t) = \text{Re}[x(t)] - j\,\text{Im}[x(t)] \qquad ...(7.2.34)$$

Add Equations (7.2.33) and (7.2.34) to get,

$$2\,\text{Re}[x(t)] = x(t) + x^*(t)$$

$$\therefore \quad \text{Re}[x(t)] = \frac{1}{2}\left[x(t) + X^*(t)\right] \qquad ...(7.2.35)$$

Subtract Equations (7.2.34) from (7.2.33) to get,

$$2j\,\text{Im}[x(t)] = x(t) - x^*(t)$$

$$\therefore \quad \text{Im}[x(t)] = \frac{1}{2j}[x(t) - x^*(t)] \qquad ...(7.2.36)$$

Therefore applying the conjugate property we can write that,

$$F\{\text{Re}[x(t)]\} = \frac{1}{2}\{F[x(t)] + F[x^*(t)]\}$$

$$\therefore \quad F\{\text{Re}[x(t)]\} = \frac{1}{2}[X(f) + X^*(-f)] \qquad ...(7.2.37)$$

Similarly,

$$F\{\text{Im}[x(t)]\} = \frac{1}{2j}\{F[x(t)] - F[x^*(t)]\}$$

$$\therefore \quad F\{\text{Im}[x(t)]\} = \frac{1}{2j}[X(f) - X^*(-f)] \qquad ...(7.2.38)$$

Conclusion : From Equation (7.2.38), it is clear that imaginary part of x(t) for a real valued signal x(t) will be 0. Therefore $X(f) = X^*(-f)$, that means X(f) exhibits a "conjugate symmetry".

Ex. 7.2.15 : Determine Fourier Transform of an impulse function. What is the effect on its Fourier Transform it impulse is delayed in time by 3 sec ?

Soln. : For the first part of Example refer Ex. 7.2.4

If the impulse is delayed in time by 3 sec.

Shifting property of delta function :

$$\text{Assume} \quad t_d = 3 \text{ and } f(t) = e^{-j2\pi ft}$$

$$x(t - t_d) \overset{F}{\longleftrightarrow} e^{-j2\pi f t_d} X(f)$$

$$\delta(t) 1 \overset{F}{\longleftrightarrow} 1$$

$$\therefore \quad X(f) = 1$$

$$\therefore \quad \delta(t-3) \overset{F}{\longleftrightarrow} e^{-j2\pi f(3)} X(f)$$

$$\therefore \quad \delta(t-3) \overset{F}{\longleftrightarrow} e^{-j6\pi f} X(f)$$

$$\therefore \quad \delta(t-3) \overset{F}{\longleftrightarrow} e^{-j6\pi f} \qquad \text{...Ans.}$$

Ex. 7.2.16 : Prove that if Fourier transform of a function f (t) is F (ω) then Fourier transform of $-jt\,f(t)$ is $\frac{d}{d\omega}F(\omega)$

Soln. :

Given $\quad f(t) \longleftrightarrow F(\omega)$

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt$$

$$= \frac{\alpha}{\alpha^2 + (2\pi f)^2} + \frac{-j2\pi f}{\alpha^2 + (2\pi f)^2} \quad ...(2)$$

$\therefore$ Amplitude spectrum $|X(f)| = \{[\text{Re}\,X(f)]^2 + [\text{Im}\,X(f)]^2\}^{1/2}$

$$= \left[\frac{\alpha^2}{(\alpha^2 + 4\pi^2 f^2)^2} + \frac{4\pi^2 f^2}{(\alpha^2 + 4\pi^2 f^2)^2}\right]^{1/2} = \left[\frac{\alpha^2 + 4\pi^2 f^2}{(\alpha^2 + 4\pi^2 f^2)^2}\right]^{1/2}$$

$$|X(f)| = \frac{1}{[\alpha^2 + 4\pi^2 f^2]^{1/2}} \quad ...(3)$$

$\therefore$ Square of the amplitude spectrum is given as,

$$|X(f)|^2 = \frac{1}{\alpha^2 + 4\pi^2 f^2} \quad ...(4)$$

To obtain the energy of the signal :

The energy in the frequency range of $-W$ to W is given by,

$$\text{Energy } E = \int_{-W}^{W} \frac{1}{\alpha^2 + 4\pi^2 f^2}\, df \quad ...(5)$$

Let us use the standard identity as,

$$\int \frac{dx}{\alpha^2 + b^2 x^2} = \frac{1}{\alpha b} \tan^{-1}\left(\frac{bx}{\alpha}\right)$$

Hence applying this to Equation (5), $x = f$, $b = 2\pi$

$$E = \frac{1}{2\pi\alpha}\left[\tan^{-1}\left(\frac{2\pi f}{\alpha}\right)\right]_{-W}^{W} = \frac{1}{2\pi\alpha}\left[\tan^{-1}\left(\frac{2\pi W}{\alpha}\right) - \tan^{-1}\left(\frac{-2\pi W}{\alpha}\right)\right]$$

$$= \frac{1}{2\pi\alpha} \times 2\tan^{-1}\left(\frac{2\pi W}{\alpha}\right)$$

Substitute, $W = \dfrac{\alpha}{2\pi}$

$$\therefore E = \frac{1}{\pi\alpha}\tan^{-1}(1) = \frac{1}{\pi\alpha} \times \frac{\pi}{4} = \frac{1}{4\alpha} \quad ...\textbf{Ans.}$$

Ex. 7.3.3 : Calculate the energy of the sinc pulse shown in Fig. P. 7.3.3.

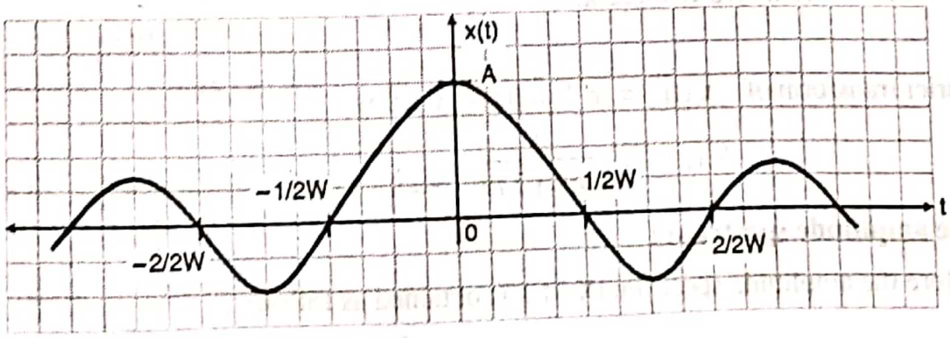


Fig. P. 7.3.3 : Sinc pulse

Soln. : The Fourier transform of the sinc pulse shown in Fig. P. 7.3.3 is given by,

$$X(f) = F[A \text{ sinc } (2Wt)] = \frac{A}{2W} \text{rect } (f/2W) \quad ...(1)$$

The energy of the signal x (t) is given by,

$$E = \int_{-\infty}^{\infty} |X(f)|^2 df \quad ...(2)$$

Substituting the expression for X (f) we get,

$$E = \int_{-\infty}^{\infty} [(A/2W) \text{ rect } (f/2W)]^2 df \quad ...(3)$$

But $\frac{A}{2W}$ rect (f/2W) is a rectangular pulse of duration 2W, extending from f = – W to f = + W with a constant amplitude of (A/2 W).

Therefore Equation (3) gets modified to,

$$E = \int_{-W}^{W} [A/2W]^2 df = \frac{A^2}{4W^2}(W+W)$$

$$\therefore E = \frac{A^2}{2W} \quad ...\text{Ans.}$$

7.4 Solved Examples on C.T. Fourier Transform :

Ex. 7.4.1 : Obtain the Fourier transform of a cosine wave having a frequency f_0 and peak amplitude of unity and plot its spectrum. Refer Fig. P. 7.4.1(a).

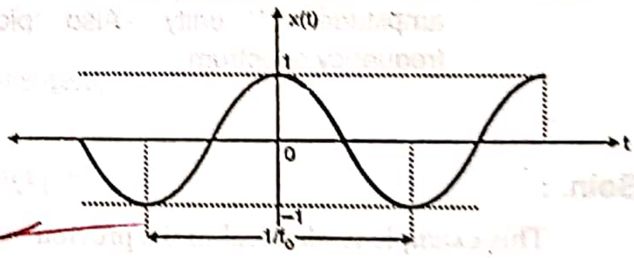


Fig. P. 7.4.1(a) : Given wave

Soln. : A cosine wave can be mathematically represented as,

$$x(t) = A\cos(2\pi f_0 t)$$

But A = 1.

$$\therefore \quad x(t) = \cos(2\pi f_0 t) \quad ...(1)$$

By Euler's identity we can write,

$$\therefore \quad x(t) = \cos(2\pi f_0 t) = \frac{e^{+j2\pi f_0 t} + e^{-j2\pi f_0 t}}{2} \quad ...(2)$$

The Fourier transform of x (t) is given by,

$$X(f) = \int_{-\infty}^{\infty} x(t) \cdot e^{-j2\pi f t} dt$$

Substituting the value of x (t) we get,

$$X(f) = \int_{-\infty}^{\infty} \left[\frac{e^{+j2\pi f_0 t} + e^{-j2\pi f_0 t}}{2}\right] e^{-j2\pi f t}\, dt = \frac{1}{2} \int_{-\infty}^{\infty} \left[e^{-j2\pi (f-f_0)t} + e^{-j2\pi (f+f_0)t}\right] dt$$

$$\therefore \quad X(f) = \frac{1}{2}\int_{-\infty}^{\infty} e^{-j2\pi (f-f_0)t}\, dt + \frac{1}{2}\int_{-\infty}^{\infty} e^{-j2\pi (f+f_0)t}\, dt \qquad ...(3)$$

Refer to Fig. 7.4.1(a) in which we have found the Fourier transform of a dc signal. In that example we have proved that,

$$\int_{-\infty}^{\infty} e^{-j2\pi f t} = \delta(f) \qquad ...(4)$$

Using this property for the RHS of Equation (3) we get,

$$X(f) = \frac{1}{2}\,\delta(f - f_0) + \frac{1}{2}\,\delta(f + f_0) \qquad ...Ans.$$

The frequency spectrum is as shown in the Fig. P. 7.4.1(b) which shows that two impulses are present one at f_0 and the other at $-f_0$.

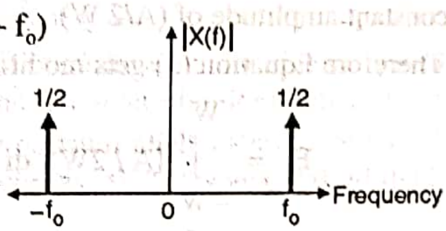


Fig. P. 7.4.1(b) : Spectrum of a cosine wave

Ex. 7.4.2 : Find the Fourier transform of a sinewave having frequency of f_0 and peak amplitude of unity. Also plot its frequency spectrum.

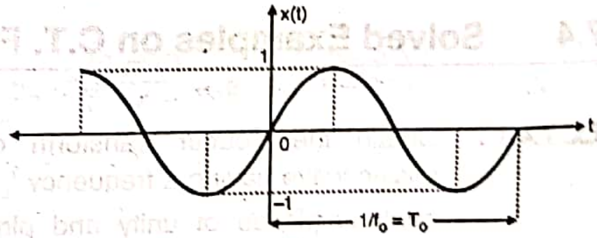


Fig. P. 7.4.2(a) : Given waveform

Soln. :

This example is identical to the previous example. Express the signal x (t) as,

$$x(t) = \sin(2\pi f_0 t) \qquad ...(1)$$

Using Euler's identity express x (t) as follows :

$$x(t) = \sin(2\pi f_0 t) = \frac{e^{+j2\pi f_0 t} - e^{-j2\pi f_0 t}}{2j}$$

The Fourier transform of x (t) is given by,

$$X(f) = \int_{-\infty}^{\infty} \sin(2\pi f_0 t)\, e^{-j2\pi f t}\, dt$$

Substituting the value of sin $(2\pi f_0 t)$ we get,

$$X(f) = \int_{-\infty}^{\infty} \frac{1}{2j}\left[e^{j2\pi f_0 t} - e^{-j2\pi f_0 t}\right] e^{-j2\pi f t}\, dt$$

$\therefore \quad X_1(f) = 200 \text{ sinc } (200 f) + 100 \text{ sinc } (100 f)$ ...Ans.

The amplitude spectrum is shown in Fig. P. 7.4.14(b).

2.

$$x_2(t) = -0.5 \quad \ldots \quad -100 \le t \le -50$$
$$= 2 \quad \ldots \quad -50 \le t \le 50$$
$$= -0.5 \quad \ldots \quad 50 \le t \le 100$$

$$\therefore \quad x_2(f) = \int_{-\infty}^{\infty} x_2(t)\, e^{-j2\pi f}\, dt$$

$$= \int_{-100}^{-50} -0.5\, e^{-j2\pi ft}\, dt + \int_{-50}^{50} 2\, e^{-j2\pi ft}\, dt - \int_{50}^{100} 0.5\, e^{-j2\pi ft}\, dt$$

$$= \frac{+1}{j2\pi f}\left[\frac{(e^{-j100\pi f} - e^{j200\pi f})}{2} - 2(e^{-j100\pi f} - e^{j100\pi f}) + \frac{(e^{-j200\pi f} - e^{j100\pi f})}{2}\right]$$

$$= \frac{(e^{j100\pi f} - e^{-j100\pi f})}{(2\pi f)(2j)} + \frac{2(e^{+j100\pi f} - e^{-j100\pi f})}{(\pi f)(2j)} - \frac{(e^{j200\pi f} - e^{-j200\pi f})}{(2\pi f)(2j)}$$

$$= \frac{\sin(100\pi f)}{2\pi f} + \frac{2\sin(100\pi f)}{\pi f} - \frac{\sin(200\pi f)}{2\pi f}$$

$$= \frac{2.5\sin(100\pi f)}{\pi f} - \frac{\sin(200\pi f)}{2\pi f} = \frac{250\sin(100\pi f)}{100\pi f} - \frac{100\sin(200\pi f)}{200\pi f}$$

$\therefore \quad X_2(f) = 250 \text{ sinc } (100 f) - 100 \text{ sinc } (200 f)$ ...Ans.

7.5 Fourier Transform of a Periodic Signal :

- The Fourier transform of the periodic signals extending from $-\infty$ to ∞ on the time scale cannot be obtained using the equations defined for the non-periodic signals.
- The reason for this is that such functions are not absolutely integrable. Function x(t) should be absolutely integrable is one of the conditions which must be satisfied as discussed earlier in this chapter.

$$\text{i.e.} \quad \int_{-\infty}^{\infty} |x(t)|\, dt < \infty$$

But for a periodic signal : $\int_{-\infty}^{\infty} |x(t)| = \infty$

- Therefore x(t) is not absolutely integrable and hence the Fourier transform for the periodic signal does not exist.

How to get it from an exponential Fourier series ?

However the Fourier transform for a periodic function can be obtained over one cycle period i.e. $-T_0/2$ to $T_0/2$. The procedure to obtain the Fourier transform for periodic signals is as follows :

1. Express the periodic signal $x_p(t)$ in the form of an exponential Fourier series.
2. Take the Fourier transform of both the sides.
3. The Fourier transform of the periodic signal $x_p(t)$ will then be the summation of the Fourier transforms of the individual terms in the Fourier series expansion.

After following this procedure we get the Fourier transform of a periodic signal as follows :

Step 1 : Let $x_p(t)$ be a periodic signal with period of T_0 seconds. It is expressed in the exponential Fourier series form as follows :

$$x_p(t) = \sum_{-\infty}^{\infty} C_n e^{j2\pi n f_0 t} \text{ where } C_n = \frac{1}{T_0}\int_{t}^{t+T_0} x_p(t)\, e^{-j2\pi n f_0 T} \quad \text{...(7.5.1)}$$

Step 2 : Take the Fourier transform of both the sides of Equation (7.5.1) to get,

$$F[x_p(t)] = F\left[\sum_{-\infty}^{\infty} C_n e^{j2\pi n f_0 t}\right] = \int_{-\infty}^{\infty}\left[\sum_{-\infty}^{\infty} C_n e^{j2\pi n f_0 t}\right]\cdot e^{-j2\pi ft}\, dt$$

$$= \sum_{-\infty}^{\infty} C_n \int_{-\infty}^{\infty} e^{j2\pi n f_0 t}\cdot e^{-2\pi f t}\, dt$$

$$= \sum_{-\infty}^{\infty} C_n \int_{-\infty}^{\infty} e^{-j2\pi(f-nf_0)t}\, dt \quad \text{...(7.5.2)}$$

But $\int e^{-j2\pi ft}\, dt = \delta(f)$

$$\therefore \quad F[x_p(t)] = \sum_{-\infty}^{\infty} C_n\cdot\delta(f-nf_0) \quad \text{...(7.5.3)}$$

$$\text{where} \quad C_n = \frac{1}{T_0}\int_{t}^{t+T_0} x_p(t)\cdot e^{-j2\pi n f_0 t}\, dt \quad \text{...(7.5.4)}$$

Conclusions from Equation (7.5.3) :

1. Equation (7.5.3) tells us that the Fourier transform of a periodic signal $x_p(t)$ is in the form of impulses weighted by the factor C_n.

   i.e. $X(f) = \ldots\ldots\ C_{-1}\,\delta(f+f_0) + C_0\,\delta(f) + C_1\,\delta(f-f_0) + C_2\,\delta(f-2f_0) + \ldots\ldots$

4. **Amplitude spectrum :**

The amplitude spectrum is given by,

$$|X(f)| = -4 \operatorname{sinc}(2f)\sin(2\pi f)$$

Ex. 7.5.11 : Find out the Fourier transform of damped sinusoid shown in Fig. P. 7.5.11.

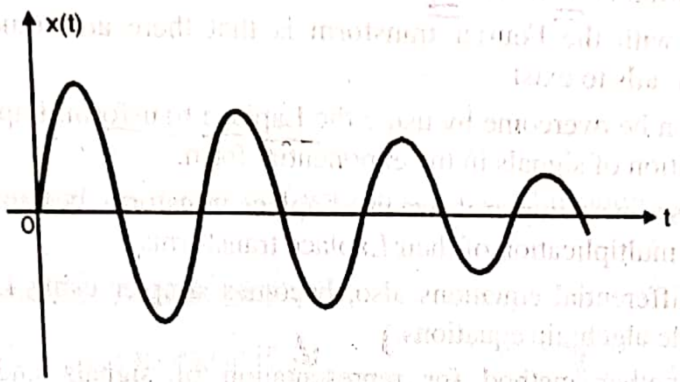


Fig. P. 7.5.11

Soln. : The damped sinusoid is given as,

$$X(t) = e^{-at} \cdot \sin \omega t \cdot u(t)$$

Fourier transform is,

$$X(\omega) = \int_{-\infty}^{\infty} x(t) \cdot e^{-j\omega t} \cdot dt$$

Putting value of X (t) in above equation,

$$X(\omega) = \int_{-\infty}^{\infty} e^{-at} \cdot \sin \omega t \cdot u(t)\, e^{-j\omega t} \cdot dt$$

$$\text{As, } u(t) = 1 \text{ for } t \geq 0$$
$$= 0 \text{ for } t < 0$$

$$X(\omega) = \int_{0}^{\infty} e^{-at} \cdot \sin \omega t \cdot e^{-j\omega t} \cdot dt$$

$$\therefore X(\omega) = \int_{0}^{\infty} e^{-(a+j\omega)t} \cdot \sin \omega t \cdot dt$$

7.6 Merits of Fourier Transform :

1. It is possible to uniquely recover the original time function x (t).
2. We can evaluate the convolutional integrals using F.T.
3. F.T. is very useful in communication systems.

7.7 Limitations of Fourier Transform and Need of LT and ZT :

- The most important limitation of F.T. is that there are many time functions for which the F.T. does not exist, because such functions are not absolutely integrable.
- For the LTI system analysis, if we use the Fourier transform then it becomes a little difficult to obtain the convolution of x (t) and h (t) to obtain the system response.
- Another problem with the Fourier transform is that there are some functions for which the Fourier Transform fails to exist.
- These problems can be overcome by using the Laplace transform. Laplace transform is a general form of representation of signals in the exponential form.
- System analysis becomes simpler if we use Laplace transform, because then convolution of x (t) and h (t) becomes multiplication of their Laplace transforms.
- The solution of differential equations also, becomes simpler using Laplace transform because they become simple algebraic equations.
- Z-transform is another method for representation of signals and systems. Z-transform representation is more general and most important is that it is applicable to stable, unstable, causal, non-causal systems as well.
- It is easier to obtain the system response of an LTI system using the Z-transform because we have to compute the multiplication X (z) and H (z) of the Z-transforms of x (n) and h (n) instead of computing the convolution sum of x (n) and h (n) in the time domain.

7.8 Introduction to DTFT :

- The DSP processors are used to perform frequency analysis of signals.
- To do the frequency analysis, we should convert the time domain signal into the frequency domain.
- For this, the different frequency transformation techniques are used.

What does a Fourier Transform do ?

- Any signal is build up by addition of elementary signals which are at different frequencies, different amplitudes and relative phases.
- Using the Fourier transform we can plot the amplitude and phase spectrums, which provide us all the information about amplitudes and relative phases of such elementary signals.
- Thus Fourier transform can be used for the "analysis" of a signal.
- It is used for transformation from the time domain to frequency domain.

7.9 Representation of DT Signals by Fourier Transform (DTFT) :

Basically the Fourier transform of a-periodic, finite energy discrete time signal is called as DTFT.

Mathematical Equation : Mathematically it is defined as,

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n} \quad ...(7.9.1)$$

Here $X(\omega)$ = Fourier transform of $x(n)$ and $x(n)$ = Discrete time a periodic signal

We can obtain discrete time signal $x(n)$ from $X(\omega)$ by taking inverse Fourier transform.

It is given by,

$$x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega) \cdot e^{j\omega n} \, d\omega \qquad \text{...(7.9.2)}$$

Notation : The Fourier transform of $x(n)$ is denoted by,

$$x(n) \xleftrightarrow{F} X(\omega)$$

Equation (7.9.1) is called as **Analysis Formula (FT)** and Equation (7.9.2) is called as **Synthesis Formula (IFT).**

7.9.1 Condition for Existence of FT :

The Fourier transform is convergent if,

$$\sum_{n=-\infty}^{\infty} |x(n)| < \infty$$

- That means Fourier transform exists only if discrete time signal is absolutely summable. This is necessary and sufficient condition for the existence of Fourier transform.

Ex. 7.9.1 : Obtain DTFT of unit impulse $\delta(n)$.

Soln. :

Here discrete time sequence is,

$$x(n) = \delta(n) \qquad \text{...(1)}$$

According to the definition of DTFT we have,

$$\text{DTFT}\{x(n)\} = X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

$$\therefore X(\omega) = \sum_{n=-\infty}^{\infty} \delta(n) \times e^{-j\omega n} \qquad \text{...(2)}$$

But we know that $\delta(n)$ exists only at $n = 0$

$$\therefore \delta(n) = 1 \quad \text{for } n = 0$$
$$= 0 \quad \text{otherwise}$$

Thus Equation (2) becomes,

$$X(\omega) = \sum_{\text{at } n=0} 1 \times e^{-j\omega n} = 1 \times e^{0} = 1$$

$$\therefore \delta(n) \xleftrightarrow{FT} 1$$

This is the standard Fourier transform pair.

Ex. 7.9.2 : Obtain DTFT of unit step u (n).

Soln. :

$$\text{Here} \quad x(n) = u(n)$$

According to definition of DTFT,

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

$$\text{DTFT}\{u(n)\} = X(\omega) = \sum_{n=-\infty}^{\infty} u(n) e^{-j\omega n} \quad ...(1)$$

But we know that u (n) exists only for positive values of n.

$$\therefore u(n) = 1 \quad \text{for } n \geq 0$$
$$= 0 \quad \text{for } n < 0$$

Thus we can write the limits of summation, from $n = 0$ to $n = \infty$.

$$\therefore X(\omega) = \sum_{n=0}^{\infty} 1 \times e^{-j\omega n}$$

$$\therefore X(\omega) = \sum_{n=0}^{\infty} [e^{-j\omega}]^n$$

Now, we have the geometric series formula,

$$\sum_{n=0}^{\infty} A^n = A^0 + A + A^2 + A^3 + \ldots = \frac{1}{1-A} \quad \text{if } |A| < 1$$

Let $A = e^{-j\omega}$. Thus Equation (2) can be written as,

$$X(\omega) = \frac{1}{1-e^{-j\omega}} \quad ...(3)$$

Now, if we put $\omega = 0$ in Equation (3) then we get, $X(\omega) = \frac{1}{0} = \infty$. That means at $\omega = 0$; the DTFT obtained by Equation (3) is not valid. So we should obtain $X(\omega)$ for different values of 'ω' other than zero. Rearranging Equation (3) we get,

$$X(\omega) = \frac{1}{e^{-j\omega/2} \times e^{j\omega/2} - e^{-j\omega/2} \times e^{-j\omega/2}} \quad ...(4)$$

Note : $e^{-j\omega/2} \cdot e^{j\omega/2} = e^0 = 1$ and $= e^{-j\omega/2} \cdot e^{-j\omega/2} = e^{-j\omega}$

Taking $e^{-j\omega/2}$ common from the denominator we get,

$$X(\omega) = \frac{1}{e^{-j\omega/2}[e^{j\omega/2} - e^{-j\omega/2}]} \quad ...(5)$$

According to Euler's identity we have,

7.10 Summary of Properties of DTFT :

Table 7.10.1 shows summary of properties of DTFT.

Table 7.10.1

Name of property	Time domain represe7ntation	Frequency domain representation
1. Linearity	$a_1 x_1(n) + a_2 x_2(n)$	$a_1 X_1(\omega) + a_2 X_2(\omega)$
2. Time shifting	$x(n-k)$	$e^{-j\omega k} X(\omega)$
3. Time reversal	$x(-n)$	$X(-\omega)$
4. Convolution theorem	$x_1(n) * x_2(n)$	$X_1(\omega) \cdot X_2(\omega)$
5. Correlation theorem	$r_{x_1 x_2}(n)$	$X_1(\omega) \cdot X_2(-\omega)$
6. Frequency shifting	$e^{j\omega_0 n} \cdot x(n)$	$X(\omega - \omega_0)$
7. Multiplication of two sequences	$x_1(n) \cdot x_2(n)$	$X_3(\omega) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X_1(\lambda) X_2(\Omega - \lambda)\, d\lambda$
8. Differentiation in Frequency domain	$nx(n)$	$j \frac{d}{d\omega} X(\omega)$

7.11 Introduction to DFT :

Definition of DFT :

It is a finite duration discrete frequency sequence which is obtained by sampling one period of fourier transform. Sampling is done at 'N' equally spaced points, over the period extending from $\omega = 0$ to $\omega = 2\pi$.

7.11.1 Mathematical Equations of DFT :

The DFT of discrete sequence $x(n)$ is denoted by $X(k)$. It is given by,

$$X(k) = \sum_{n=0}^{N-1} x(n) \cdot e^{-j2\pi kn/N} \quad ...(7.11.1)$$

Here $k = 0, 1, 2 \ldots N-1$

Since this summation is taken for 'N' points; it is called as **'N' point DFT.**

We can obtain discrete sequence $x(n)$ from its DFT. It is called as inverse discrete fourier transform (IDFT). It is given by,

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k)\, e^{j\,2\pi kn/N} \quad ...(7.11.2)$$

Here $n = 0, 1, 2, \ldots N-1$

This is called as **'N' point IDFT.**

Now we will define the new term 'W' as,

$$W_N = e^{-j2\pi/N} \quad ...(7.11.3)$$

This is called as **twiddle factor**. Twiddle factor makes the computation of DFT a bit easy and fast.

Using twiddle factor we can write equations of DFT and IDFT as follows :

$$X(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{kn} \quad ...(7.11.4)$$

Here $n = 0, 1, 2 \ldots N-1$

$$\text{and } x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\, W_N^{kn} \quad ...(7.11.5)$$

Here $n = 0, 1, 2, \ldots N-1$

7.11.2 Why it is Necessary to have N ≥ L ?

Here N : N-point DFT.
L : Length of the signal x (n).

DFT is a finite duration discrete frequency sequence which is obtained by sampling one period of fourier transform. Sampling is done at 'N' equally spaced points over the period extending from $\omega = 0$ to $\omega = 2\pi$.

N is very important parameter in the definition of DFT and IDFT. 'N' decides the limits of 'k' and 'n'. 'N' also decides the amount of processing time required to calculate N point DFT. The values of 'N' is decided according to the number of elements present in the given sequence x(n). That means the length of sequence

$$x(n) = \{ x(0), x(1), x(2), x(3), x(4), x(5), x(6), x(7) \}$$

Here the last value x (7) is denoted by $x_h(n)$. Normally value of N is taken as $N = x_h(n) + 1$ or $N \geq L$. If it is less than 'L' then it is not possible to compute DFT or IDFT.

Now 'N' is required to be a power of 2. Accordingly the length of sequence x (n) is adjusted by padding zeros at the end.

7.11.3 Relationship between DTFT and DFT :

The DTFT is discrete time Fourier transform and is given by,

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n} \quad ...(7.11.6)$$

The range of ω is from $-\pi$ to π or 0 to 2π.

Now we know that discrete Fourier transform (DFT) is obtained by sampling one cycle of Fourier transform. And DFT of x (n) is given by,

$$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j2\pi kn/N} \qquad \ldots(7.11.7)$$

Comparing Equations (7.11.6) and (7.11.7), we can say that DFT is obtained from DTFT by putting $\omega = \frac{2\pi k}{N}$

$$\therefore\ X(k) = X(\omega)\Big|_{\omega = \frac{2\pi k}{N}}$$

By comparing DFT with DTFT we can write,

1. The continuous frequency spectrum $X(\omega)$ is replaced by discrete fourier spectrum $X(k)$.
2. Infinite summation in DTFT is replaced by finite summation in DFT.
3. The continuous frequency variable is replaced by finite number of frequencies located at $\frac{2\pi k}{NT_s}$; where T_s is sampling time.

Ex. 7.11.1 : Obtain DFT of unit impulse $\delta(n)$.

Soln. :

Here $x(n) = \delta(n)$ ...(1)

According to the definition of DFT we have,

$$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j2\pi kn/N} \qquad \ldots(2)$$

But $\delta(n) = 1$ only at $n = 0$. Thus Equation (2) becomes,

$$X(k) = \delta(0)\, e^{0} = 1$$

$$\therefore\ \delta(n) \xleftrightarrow{\text{DFT}} 1$$

This is the standard DFT pair.

Ex. 7.11.2 : Obtain DFT of delayed unit impulse $\delta(n - n_0)$.

Soln. : We know that $\delta(n - n_0)$ indicates unit impulse delayed by 'n_0' samples.

Here $x(n) = \delta(n - n_0)$ ...(1)

Now we have, $$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j2\pi kn/N} \qquad \ldots(2)$$

But $\delta(n - n_0) = 1$ only at $n = n_0$. Thus Equation (2) becomes,

$$X(k) = 1 \cdot e^{-j2\pi k n_0/N}$$

$$\therefore \quad \delta(n-n_0) = e^{-j2\pi k n_0/N}$$

Similarly we can write,

$$\delta(n+n_0) = e^{j2\pi k n_0/N}$$

Ex. 7.11.3 : Obtain N–point DFT of exponential sequence :

$$x(n) = a^n u(n) \text{ for } 0 \le n \le N-1$$

Soln. : According to the definition of DFT,

$$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j2\pi kn/N} \quad \text{...(1)}$$

Here $x(n) = a^n u(n)$

The multiplication of a^n with $u(n)$ indicates sequence is positive. Putting $x(n) = a^n$ in Equation (1) we get,

$$X(k) = \sum_{n=0}^{N-1} a^n e^{-j2\pi kn/N}$$

$$\therefore \quad X(k) = \sum_{n=0}^{N-1} (a e^{-j2\pi k/N})^n \quad \text{...(2)}$$

Now use the standard summation formula,

$$\sum_{k=N_1}^{N_2} A^k = \frac{A^{(N_1)} - A^{(N_2+1)}}{1-A}$$

Here $N_1 = 0$, $N_2 = N-1$ and $A = ae^{-j2\pi k/N}$

$$\therefore \quad X(k) = \frac{(ae^{-j2\pi k/N})^0 - (ae^{-j2\pi k/N})^{N-1+1}}{1 - ae^{-j2\pi k/N}}$$

$$\therefore \quad X(k) = \frac{1 - a^N e^{-j2\pi k}}{1 - ae^{-j2\pi k/N}} \quad \text{...(3)}$$

Using Euler's identity to the numerator term, we get,

$$e^{-j2\pi k} = \cos 2\pi k - j \sin 2\pi k$$

But k is an integer

$$\therefore \quad \cos 2\pi k = 1 \text{ and } \sin 2\pi k = 0$$

$$\therefore \quad e^{-j2\pi k} = 1 - j0 = 1$$

$$\therefore \quad X(k) = \frac{1-a^N}{1-a\,e^{-j2\pi k/N}}$$

$$\therefore \; a^n u(n) \xleftrightarrow{\text{DFT}} \frac{1-a^N}{1-a\,e^{-j2\pi k/N}}$$

Review Questions

Q. 1 What does a Fourier transform do?

Q. 2 Define Fourier transform.

Q. 3 Define inverse Fourier transform.

Q. 4 What are the different properties of Fourier transforms ?

Q. 5 State and prove the following properties of Fourier transform :

1. Time shifting 2. Frequency shifting 3. Convolution in time domain

Q. 6 State and prove the following properties of Fourier transform :

1. Time scaling 2. Duality 3. Differentiation in time domain

Q. 7 Define unit impulse function and find its Fourier transform.

Q. 8 List out various properties of the Fourier transform and prove modulation theorem stating its significance in analog communication system.

Q. 9 List the merits and limitations of Fourier transform.

Q. 10 Show that compression in the time domain is equivalent to expansion in the frequency domain and vice versa.

Q. 11 State and prove the frequency translation theorem.

Q. 12 Fourier transform is a limiting case of Fourier series by letting the period of periodic function infinite. Justify the statement with an example of periodic gate function.

Q. 13 Explain "a signal cannot be band limited and time limited simultaneously".

7.12 Examples for Practice :

Ex. 7.12.1 : Obtain frequency spectrum of x (t) if $x(t) = 2\delta(t+0.2) - \delta(t-0.2)$

Ans. : $= 2 \cdot 2j \sin(2\pi f_c t)$ where $t = 0.2$ ...Ans.

Ex. 7.12.2 : Find Fourier transform of X (t) where the function X (t) is given by x (t) = U (t) and using properties of Fourier transform find Fourier transform of y (t) where

$y(t) = U(2t) + U(t-1)$

Ans. :

$$y(t) \xleftrightarrow{F} \frac{1}{j\,2\pi f} + \frac{e^{-j\,2\pi f}}{j\,2\pi f}$$ ...Ans.

Ex. 7.12.3 : For a non periodic signal x (t) with F.T. X (ω) i.e.

$$x(t) \xleftrightarrow{\text{F.T.}} X(\omega)$$

determine the F.T. of : 1. $x(t-t_d)$ 2. $\frac{d^2}{dt^2}[x(t)]$

Ans. :

$$F[x(t-t_d)] = e^{-j2\pi f t_d} X(f) \quad \text{...Ans.}$$

Ex. 7.12.4 : Sketch the signal and then find it's Fourier transform :

$$f(t) = 5[U(t+3) + U(t+2) - U(t-2) - U(t-3)]$$

Ans. :

$$f(s) = \frac{5}{\pi f}[\sin(6\pi f) + \sin(4\pi f)] \quad \text{...Ans.}$$

Ex. 7.12.5 : Determine the function f(t) where Fourier transform is shown in Fig. P. 7.12.5.

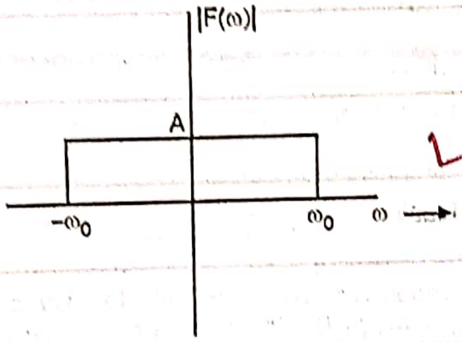


Fig. P. 7.12.5

Ans. :

$$f(t) = \frac{A \sin \omega_0 t}{\pi t} \quad \text{...Ans.}$$

□□□

Unit 3: Fourier, Laplace and z- Transforms:

Fourier series representation of periodic signals, Waveform Symmetries, Calculation of Fourier Coefficients. Fourier Transform, convolution/multiplication and their effect in the frequency domain, magnitude and phase response, Fourier domain duality. The DiscreteTime Fourier Transform (DTFT) and the Discrete Fourier Transform (DFT). Parseval's Theorem. Review of the Laplace Transform for continuous time signals and systems, system functions, poles and zeros of system functions and signals, Laplace domain analysis, solution to differential equations and system behavior. The z-Transform for discrete time signals and systems, system functions, poles and zeros of systems and sequences, z-domain analysis.

Fourier transform provides a valuable technique for frequency domain analysis and design of continuous time signals and LTI systems. While the Z-transform provides a valuable technique for analysis and design of discrete time signals and discrete time LTI systems.

The Z-transform has real and imaginary parts like fourier transform. A plot of imaginary part versus real part is called as Z-plane or complex Z-plane. The poles and zeros of discrete time system are plotted in the complex Z-plane. The pole-zero plot is main characteristic of discrete time LTI systems. We can also check the stability of system using pole-zero plot.

The importance of Z-transform is as follows :

1. Discrete time signals and LTI systems can be completely characterized using Z-transform.
2. The stability of LTI system can be determined using Z-transform.
3. Mathematical calculations are reduced using Z-transform. For example convolution operation is transformed into simple multiplication operation.
4. By calculating Z-transform of given signal, DFT and FT can be determined.
5. Entire family of digital filters can be obtained from one proto-type design using Z-transform.
6. The solution of differential equations can be simplified using Z-transform.

1.2 Z-Transform :

There are two types of Z-transform :

1. Single sided Z-transform
2. Double sided Z-transform.

1. Single sided Z-transform :

Definition : A single sided Z-transform of discrete time signal x (n) is defined as,

$$\boxed{X(Z) = \sum_{n=0}^{\infty} x(n) Z^{-n}} \quad ...(1)$$

2. Double sided Z-transform :

Definition : A double sided Z-transform of discrete time signal x (n) is defined as,

$$\boxed{X(Z) = \sum_{n=-\infty}^{\infty} x(n) Z^{-n}} \quad ...(2)$$

1.2.1 Region of Convergence (ROC) :

In case of Z-transform; the limits of summation are from $n = -\infty$ to $n = \infty$. So if we will expand this summation then we will get an infinite power series. This infinite power series (that means Z-transform) will exist only for those values of Z for which the series attains a finite value. (That means the series converges).

Definition of ROC :

The region of convergence (ROC) of $X(Z)$ is set for all the values of Z for which $X(Z)$ attains a finite value. Everytime when we find the Z-transform, we must indicate its ROC.

Significance of ROC :

1. ROC will decide whether a system (filter) is stable or unstable.
2. ROC also determines the type of sequence that means.
   (i) Causal or non-causal
   (ii) Finite or infinite.

Prob. 1 : Obtain the Z-transform of following finite duration sequences.

1. $x(n) = \{1, 2, 4, 5, 0, 7\}$
2. $x(n) = \{1, 2, 4, 5, 0, \underset{\uparrow}{7}\}$
3. $x(n) = \{1, 2, \underset{\uparrow}{4}, 5, 0, 7\}$

1. Here arrow is not mentioned. So by default it is at first position.

$$\therefore \quad x(n) = \{\underset{\uparrow}{1}, 2, 4, 5, 0, 7\} \qquad \text{...(1)}$$

According to definition of Z-transform we have,

$$X(Z) = \sum_{n=-\infty}^{\infty} x(n) Z^{-n} \qquad \text{...(2)}$$

But x (n) is present from n = 0 to n = 5. The different values are as follows :

$$x(0) = 1 \qquad x(3) = 5$$

$$x(1) = 2 \qquad x(4) = 0$$

$$x(2) = 4 \qquad x(5) = 7$$

Thus we will change the limits of summation from n = 0 to n = 5.

$$\therefore \quad X(Z) = \sum_{n=0}^{5} x(n) Z^{-n} \qquad \text{...(3)}$$

Expanding the summation we get,

$$\therefore \quad X(Z) = x(0)Z^0 + x(1)Z^{-1} + x(2)Z^{-2} + x(3)Z^{-3} + x(4)Z^{-4} + x(5)Z^{-5}$$

$$\therefore \quad X(Z) = 1.Z^0 + 2Z^{-1} + 4Z^{-2} + 5Z^{-3} + 0.Z^{-4} + 7Z^{-5}$$

But $Z^0 = 1$

$$\boxed{\therefore \ X(Z) = 1 + \frac{2}{Z} + \frac{4}{Z^2} + \frac{5}{Z^3} + \frac{7}{Z^5}} \qquad \text{...(4)}$$

(i) Putting Z = 0 in Equation (4) we get,

$$X(Z) = 1 + \frac{2}{0} + \frac{4}{0} + \frac{5}{0} + \frac{7}{0} = 1 + \infty = \infty$$

We know that $\frac{2}{0}, \frac{4}{0}$ etc. are equal to ∞. And $1 + \infty$ is again ∞. Thus Z = 0 is not valid as it results $X(Z) = \infty$.

(ii) Putting $Z = \infty$ in Equation (4) we get,

$$X(Z) = 1 + \frac{2}{\infty} + \frac{4}{\infty} + \frac{5}{\infty} + \frac{7}{\infty} = 1 + 0 = 1$$

Here $\frac{2}{\infty}, \frac{4}{\infty} = 0$. Thus putting $Z = \infty$ we get finite value. So $Z = \infty$ is allowed.

Now ROC of given sequence is written as follows :

ROC : Entire Z-plane except | Z | = 0

This ROC is shown in Fig. U-2.

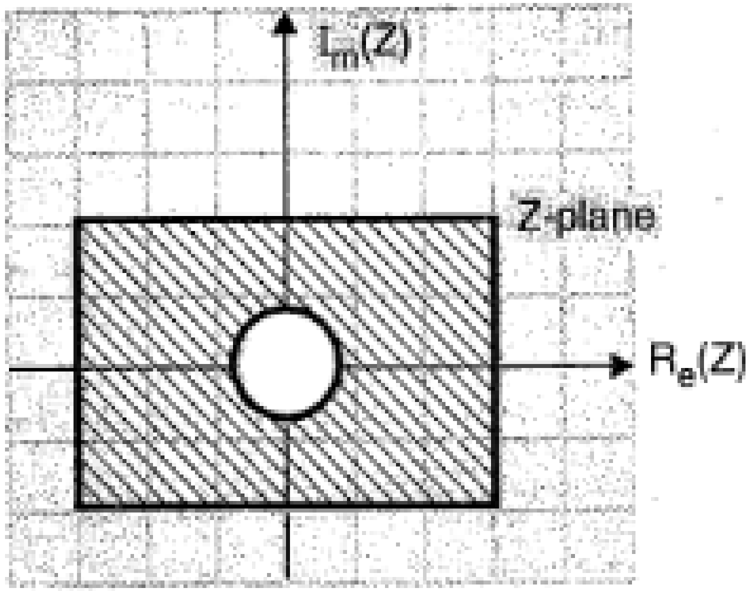


Fig. U-2 : ROC

Note : This sequence is causal, since x (n) is present only for positive values of n. Thus for a causal finite duration sequence ROC is entire Z-plane except | Z | = 0.

2. Given sequence is,

$$x(n) = \{1, 2, 4, 5, 0, \underset{\uparrow}{7}\} \qquad ...(1)$$

According to definition of Z-transform we have,

$$X(Z) = \sum_{n=-\infty}^{\infty} x(n) Z^{-n} \qquad ...(2)$$

Note that the range of given sequence is from $n = -5$ to $n = 0$. The different values of $x(n)$ are :

$$x(-5) = 1 \qquad x(-2) = 5$$
$$x(-4) = 2 \qquad x(-1) = 0$$
$$x(-3) = 4 \qquad x(0) = 7$$

We will change the limits of summation from $n = -5$ to $n = 0$.

$$\therefore \quad X(Z) = \sum_{n=-5}^{0} x(n) Z^{-n} \qquad ...(3)$$

Expanding the summation we get,

$$X(Z) = x(-5) Z^{+5} + x(-4) Z^{+4} + x(-3) Z^{+3} + x(-2) Z^{+2} + x(-1) Z^{+1} + x(0) Z^{0}$$

$$\therefore \quad X(Z) = 1 Z^{5} + 2 Z^{4} + 4 Z^{3} + 5 Z^{2} + 0 Z^{1} + 7 Z^{0}$$

ROC : We will determine ROC by putting $Z = 0$ and $Z = \infty$.

(i) Putting $Z = 0$ in Equation (4) we get,

$$X(Z) = 0+0+0+0+7 = 7$$

This is a finite value. So $Z = 0$ is allowed.

(ii) Putting $Z = \infty$ in Equation (4) we get,

$$X(Z) = \infty+\infty+\infty+\infty+7 = \infty+7 = \infty$$

This is because $\infty + 7$ etc. $= \infty$. So $Z = \infty$ is not allowed.

Thus ROC is entire Z-plane except $|Z| = \infty$. This ROC is shown in Fig. U-3.

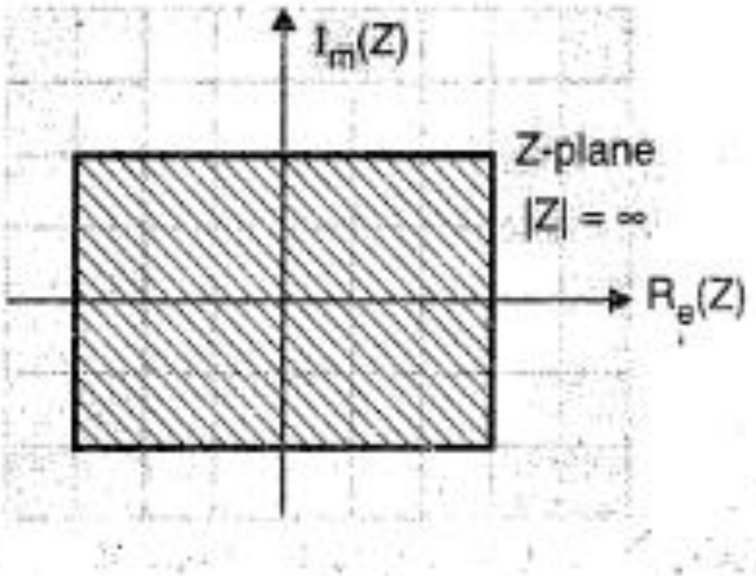


Fig. U-3 : ROC

Note : The given sequence $x(n)$ is anticausal. This is because $x(n)$ is present only for negative values of 'n'. Thus for anticausal finite duration sequence, ROC is entire Z-plane except $|Z| = \infty$

2. Given sequence is

3. Given sequence is,

$$x(n) = \{1, 2, \underset{\uparrow}{4}, 5, 0, 7\} \quad ...(1)$$

According to definition of Z-transform we have,

$$X(Z) = \sum_{n=-\infty}^{\infty} x(n) Z^{-n} \quad ...(2)$$

But x (n) is present from n = – 2 to n = 3. The different values of x (n) are as follows :

$$x(-2) = 1 \qquad x(1) = 5$$

$$x(-1) = 2 \qquad x(2) = 0$$

$$x(0) = 4 \qquad x(3) = 7$$

We will change the limits of summation from n = – 2 to n = 3.

$$\therefore \quad X(Z) = \sum_{n=-2}^{n=3} x(n) Z^{-n} \quad ...(3)$$

Expanding the summation we get,

$$X(Z) = x(-2)Z^{+2} + x(-1)Z^{+1} + x(0)Z^{0} + x(1)Z^{-1} + x(2)Z^{-2} + x(3)Z^{-3}$$

ROC : We will determine ROC by putting Z = 0 and Z = ∞ in Equation (4).

(i) Putting Z = 0 in Equation (4) we get,

$$X(Z) = 0+0+4+\frac{5}{0}+\frac{2}{0}+\frac{7}{0} = 4+\infty = \infty$$

This is because $\frac{5}{0}, \frac{2}{0}$ and $\frac{7}{0} = \infty$

Thus $Z = 0$ is not allowed.

(ii) Putting Z = ∞ in Equation (4) we get,

$$X(Z) = \infty+\infty+4+\frac{5}{\infty}+\frac{2}{\infty}+\frac{7}{\infty} = \infty$$

Thus $Z = \infty$ is not allowed.

ROC : ROC is entire Z-plane except $|Z| = 0$ and $|Z| = \infty$.

This ROC is shown in Fig. U-4.

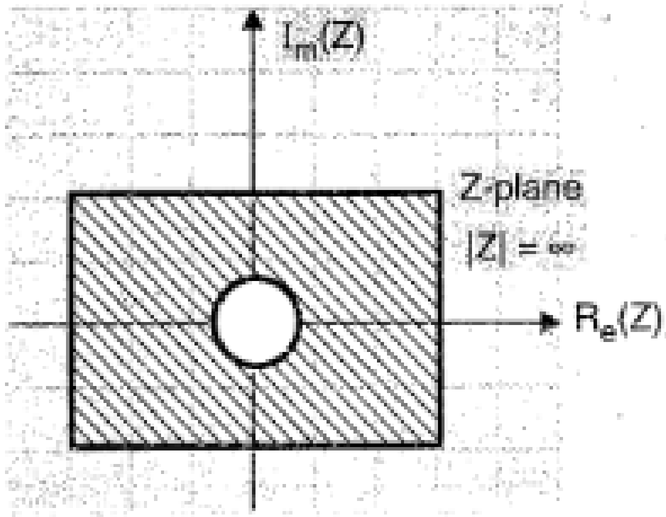


Fig. U-4 : ROC

Note : The given sequence is a bothsided sequence. This is because x(n) is present for both positive and negative values of 'n'. Thus for bothsided finite duration sequence the ROC is entire Z-plane except $|Z| = 0$ and $|Z| = \infty$.

Z-transform of Standard Sequences :

1. Z-transform of unit impulse $\delta(n)$:

ROC : In Equation (3), there is no 'Z' term. So *ROC is entire Z-plane.* That means Z can have any value.

Thus the Z-transform pair is

$$\boxed{\delta(n) \overset{Z}{\longleftrightarrow} 1}$$

2. Z-transform of delayed unit impulse, $\delta(n-k)$:

Here $\delta(n-k)$ is a delayed unit impulse. It indicates that $\delta(n)$ is delayed by 'k' samples. It is shown in Fig. U-6.

It is given by,

$$\delta(n-k) = 1 \quad \text{only at } n = k \text{ and } k > 0$$
$$= 0 \quad \text{otherwise}$$

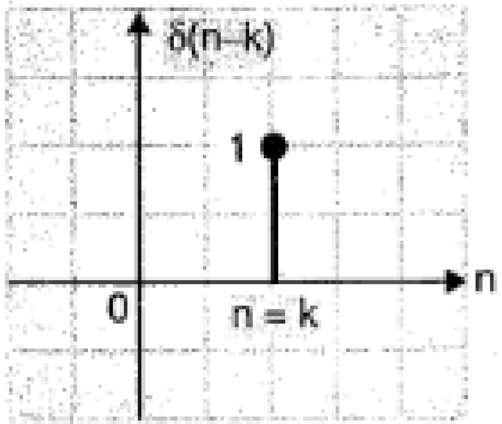


Fig. U-6 : Delayed unit impulse

ROC :

Here $X(Z) = Z^{-k} = \frac{1}{Z^k}$. Since k is positive, $(k > 0)$ for any value of 'Z' (except Z = 0) we will get finite value of $X(Z)$. Thus *ROC is entire Z-plane except Z = 0*. This is because if we put $Z = 0$ then $X(Z) = \infty$. Thus the Z-transform pair is,

$$\boxed{\delta(n-k) \overset{Z}{\longleftrightarrow} Z^{-k}}$$

3. Z-transform of advanced unit impulse , $\delta(n+k)$, $k > 0$:

Here $\delta(n+k)$, $k > 0$ is an advanced unit impulse. It indicates that $\delta(n)$ is advanced by '$+k$' samples. It is shown in Fig. U-7.

It is given by,

$$\delta(n+k) = 1 \quad \text{only at } n = -k,\ k > 0$$
$$= 0 \quad \text{otherwise}$$

ROC :

Here $k > 0$. So we will get finite values of $X(Z)$ for all 'Z' except $Z = \infty$. Because for $Z = \infty$ we get, $X(Z) = (\infty)^k = \infty$. Thus *ROC is entire Z-plane except $Z = \infty$.*

Thus Z-transform pair is.

$$\boxed{\delta(n+k) \overset{Z}{\longleftrightarrow} Z^{+k}}$$

4. Z-transform of unit step, u (n) :

We know that u (n) is unit step as shown in Fig. U-8.

It is given by,

$$u(n) = 1 \quad \text{for } n \geq 0$$
$$= 0 \quad \text{otherwise}$$

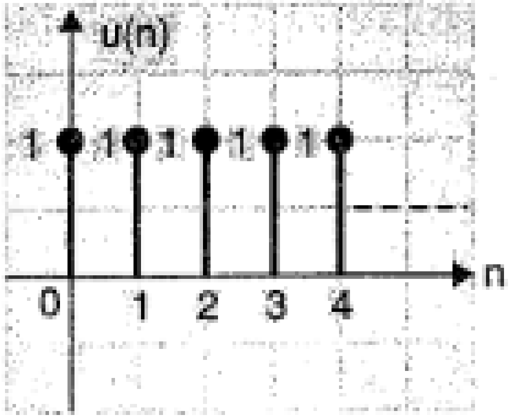


Fig. U-8 : Unit step

According to the definition of Z-transform we have,

$$X(Z) = \sum_{n=-\infty}^{\infty} x(n) Z^{-n} \qquad \ldots(1)$$

Here x (n) = u (n). Since u (n) is present from n = 0 to n = ∞ we will change the limits of summation.

$$\therefore \quad X(Z) = \sum_{n=0}^{\infty} u(n) Z^{-n} \qquad \ldots(2)$$

But the magnitude of u (n) is always 1.

$$\therefore \quad X(Z) = \sum_{n=0}^{\infty} 1 \cdot Z^{-n}$$

$$\therefore \quad X(Z) = \sum_{n=0}^{\infty} (Z^{-1})^n \qquad \text{...(3)}$$

The standard equation of geometric series is,

$$\boxed{\sum_{n=0}^{\infty} A^n = A^0 + A + A^2 + A^3 + \dots = 1 + A + A^2 + A^3 + \dots = \frac{1}{1-A} \text{ if } |A| < 1} \qquad \text{...(4)}$$

Let $A = Z^{-1}$. Thus Equation (3) becomes,

$$X(Z) = \frac{1}{1-A} \quad \text{if} \quad |A| < 1 = \frac{1}{1-Z^{-1}} \text{ if } |Z^{-1}| < 1$$

$\therefore$ Multiplying numerator and denominator by Z we get,

$$X(Z) = \frac{Z}{Z-Z^{-1}Z} \quad \text{if} \quad |Z^{-1}| < 1$$

$$\boxed{\therefore \quad X(Z) = \frac{Z}{Z-1}} \quad \text{if } |Z^{-1}| < 1$$

In Equation (4), the condition "if $|A| < 1$" indicates that if this condition is satisfied then only the geometric series is convergent otherwise not. In the given example we get this condition as, $|Z^{-1}| < 1$. That means this is the condition of ROC.

Thus ROC : $|Z^{-1}| < 1$

that means $\left|\frac{1}{Z}\right| < 1$

$\therefore\ |1| < |Z| \cdot 1$

$\therefore$ ROC is $|Z| > 1$

This ROC is plotted as shown in Fig. U-9.

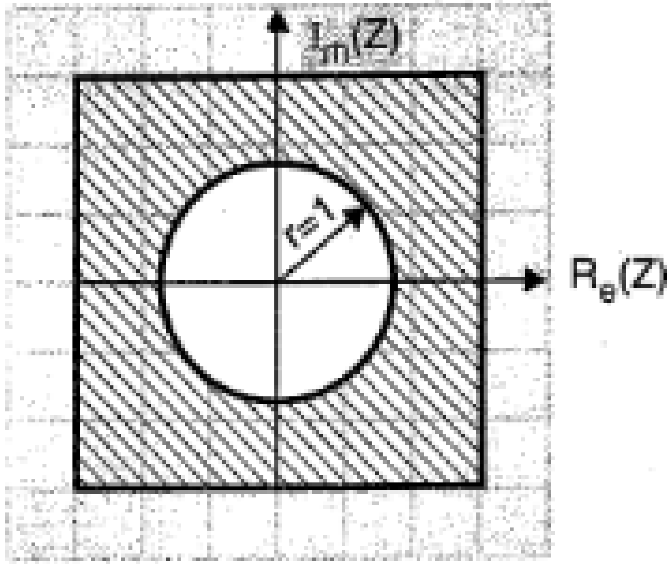


Fig. U-9 : ROC of unit step

Thus ROC is exterior part of circle having radius 1.

Thus the Z-transform pair is,

$$\boxed{u(n) \overset{Z}{\longleftrightarrow} \frac{Z}{Z-1} \text{ if } |Z| > 1}$$

5. Z-transform of unit ramp :

We know that unit ramp sequence is denoted by $r(n)$. It is as shown in Fig. U-10.

Its magnitude is as follows :

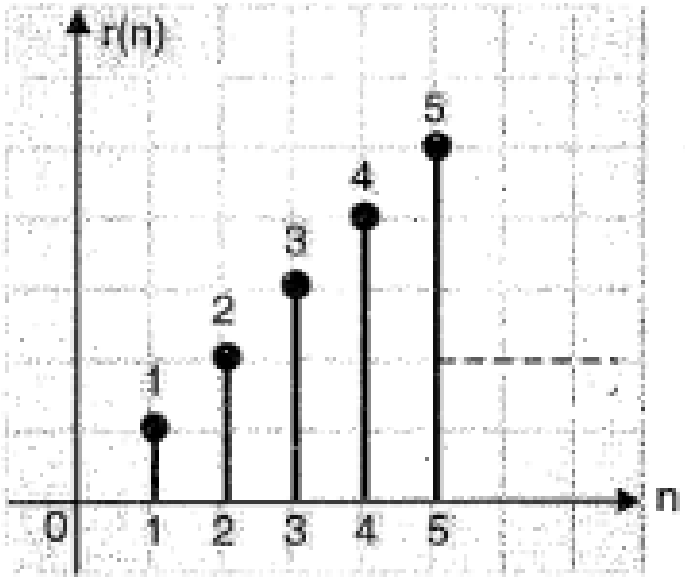


Fig. U-10 : Unit ramp

at $n = 0, \quad r(n) = 0$

at $n = 1, \quad r(n) = 1$

at $n = 2, \quad r(n) = 2$

at $n = 3, \quad r(n) = 3 \ldots$

Thus it is expressed as,

$$r(n) = n \quad \text{for } n \geq 0$$

$$= 0 \quad \text{otherwise} \qquad \ldots(1)$$

Since this sequence is again causal sequence we can write

$$r(n) = n\,u(n) \qquad \ldots(2)$$

ROC : Here the condition "$|Z^{-1}| < 1$" indicates the ROC.

$$\therefore \quad \text{ROC is } \left|\frac{1}{Z}\right| < 1 \Rightarrow 1 < |Z|$$

$$\boxed{\therefore \text{ ROC is } |Z| > 1}$$

Thus ROC is exterior part of circle having radius 1. This ROC is same as shown in Fig. U-9.

Thus the Z-transform pair is

$$\boxed{nu(n) \overset{Z}{\longleftrightarrow} \frac{Z}{(Z-1)^2} \quad \text{if } |Z| > 1}$$

6. Z-transform of right hand exponential sequence :

As the name indicates, it is an exponential sequence present at the right hand that means only for positive values of 'n'. So it is a causal exponential sequence. Thus it is expressed as,

$$x(n) = \alpha^n u(n) = \begin{cases} \alpha^n & \text{for } n \geq 0 \\ 0 & \text{for } n < 0 \end{cases} \qquad \ldots(1)$$

ROC :

This condition $|\alpha Z^{-1}| < 1$ indicates the ROC.

$\therefore$ ROC is $|\alpha Z^{-1}| < 1 \Rightarrow \left|\frac{\alpha}{Z}\right| < 1 \Rightarrow |\alpha| < |Z|$

Thus ROC is $|Z| > |\alpha|$.

It indicates that ROC is exterior part of circle having radius α.

This ROC is shown in Fig. U-12.

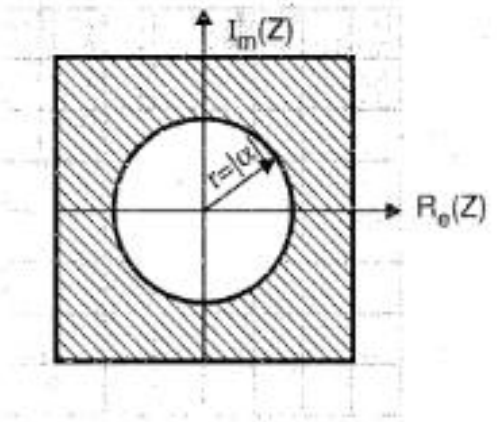


Fig. U-12 : ROC of $\alpha^n u(n)$

Thus we can write Z-transform pair as,

$$\alpha^n u(n) \overset{Z}{\longleftrightarrow} \frac{Z}{Z-\alpha} \quad \text{if } |Z| > |\alpha|$$

7. Z-transform of left handed exponential sequence :

As the name indicates; it is an exponential sequence present on left hand side; that means only at negative values of 'n'. So it is anticausal or non-causal exponential sequence. It is expressed as,

$$x(n) = -\alpha^n u(-n-1) \quad \text{...(1)}$$

$$\boxed{X(Z) = \frac{Z}{Z-\alpha} \quad \text{if} \quad |Z| < |\alpha|}$$

ROC :

Here ROC is $|Z| < |\alpha|$. That means ROC is interior part of circle having radius α. This ROC is shown in Fig. U-15.

Thus Z-transform pair is

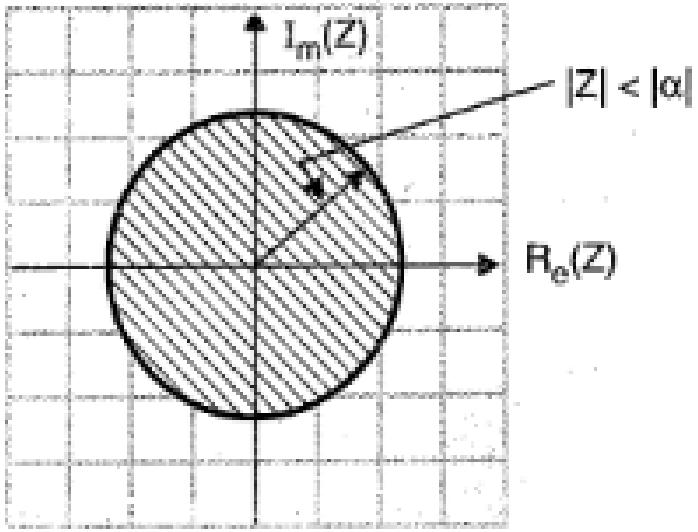
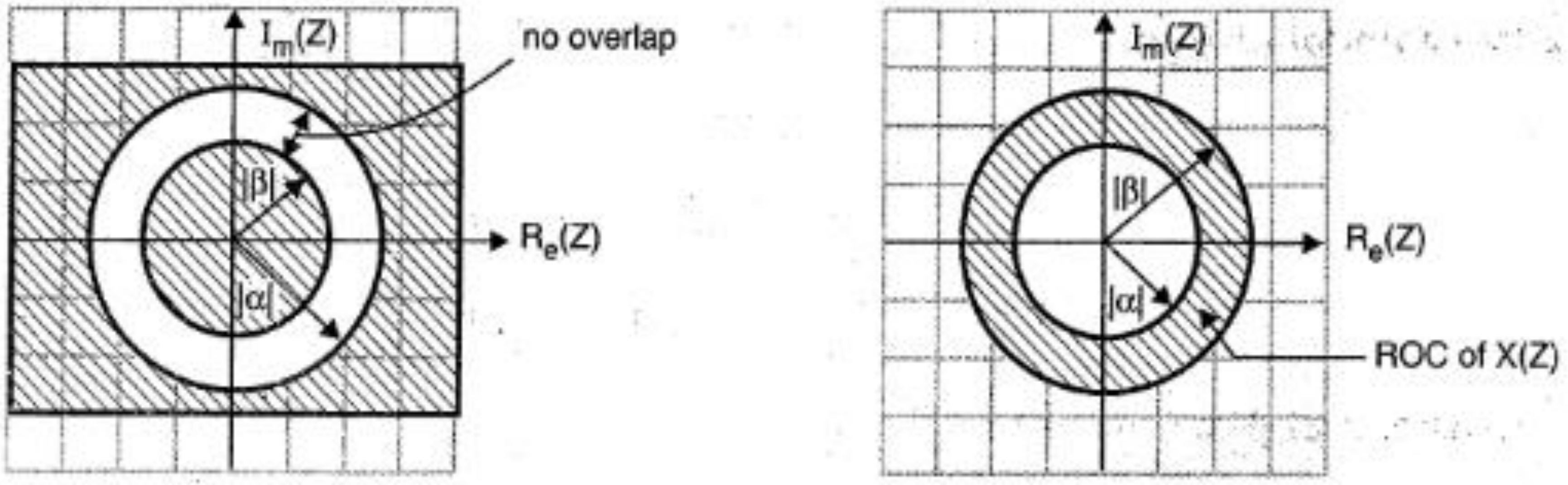


Fig. U-15 : ROC of $-\alpha^n u(-n-1)$

$$\boxed{-\alpha^n u(-n-1) \overset{Z}{\longleftrightarrow} \frac{Z}{Z-\alpha} \quad \text{if } |Z| < |\alpha|}$$

(8) Z-transform of two sided exponential :



(b) ROC for $|\beta| < |\alpha|$ **(c) ROC for $|\beta| > |\alpha|$**

Fig. U-16

In this case there is a ring in the Z-plane. In this case both power series converge simultaneously. That means $X(Z)$ exist.

$$\therefore \; X(Z) = X_1(Z) + X_2(Z) = \frac{Z}{Z-\alpha} - \frac{Z}{Z-\beta}$$

$$\boxed{\therefore \; X(Z) = \frac{Z}{Z-\alpha} + \frac{Z}{\beta - Z} \quad \text{for } |\alpha| < |Z| < |\beta|}$$

1.2.3 Properties of ROC :

We have solved some examples using Z transform and we have discussed ROC in each case. Based on this; the properties of ROC are summarized as follows :

(1) The ROC is a ring, whose center is at origin.

(2) ROC cannot contain any pole.

(3) If ROC of X(Z) includes unit circle then and then only the fourier transform of D.T. sequence x(n) converges.

(4) The ROC must be a connected region.

(5) For a finite duration sequence, x(n); the ROC is entire Z plane except $Z = 0$ and $Z = \infty$.

(6) If x(n) is causal then ROC is exterior part of circle of radius say 'α'.

(7) If x(n) is anticausal then ROC is interior part of circle of radius say 'α'.

(8) If x(n) is two sided sequence then ROC is intersection of two circles of radii 'α' and 'β'.