Parul University

Faculty of Engineering & Technology

Department of Applied Sciences and Humanities

1st Year B.Tech Programme (All Branches)

Mathematics – 1 (303191101)

Unit – 4 Sequence and Series (Lecture Note)

Sequence:

- Limit of a sequence
- Convergence & Divergence of a sequence
- Oscillatory sequence
- Sandwich/Squeezing theorem for sequences
- Convergence properties of sequence
- Monotonic sequence(Monotonic increasing & Monotonic decreasing)
- Alternating sequence
- Bounded & Unbounded sequence.

Series:

- Convergence, Divergence & Oscillatory series
- Some properties of infinite series
- Telescoping series
- Geometric series
- p-series, Integral test
- Comparison test
  (i)Direct
  (ii)Limit Comparison
- D'Alembert ratio test
- Cauchy's root test
- Alternating series
- Leibnitz test

- **Sequence:**

  A sequence is a function whose domain is the set of positive integers.

  It is generally written as $a_1, a_2, a_3 .... a_n, ...$

- If the number of terms in a sequence is infinite, it is called infinite sequence otherwise it is said to be finite sequence

$$e.g.\ 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots \qquad ; \quad 1, -1, 2, -2, \ldots$$

- **Limit of a sequence:**

  Let $\{a_n\}$ be a sequence.

  A real number l is said to be the limit of the sequence $\{a_n\}$; if for every $\varepsilon > 0$, there exist an integer N such that $n \geq N \Rightarrow |a_n - l| < \varepsilon$

  If such a number exists then we write

$$\lim_{n\to\infty} a_n = l.$$

- **Convergence, Divergence & Oscillations of a Sequence:**
  - A sequence $\{a_n\}$ is said to be convergent if the sequence has finite limit.

$$i.e.\ if \quad \lim_{n\to\infty} a_n = finite.$$

  - A sequence $\{a_n\}$ is said to be divergent if the sequence has infinite limit.

$$i.e. \qquad if \ \lim_{n\to\infty} a_n = \pm\infty.$$

  For example, $\lim_{n\to\infty} \frac{1}{n} = 0$, $\lim_{n\to\infty} \frac{n+1}{n} = 1$, $\lim_{n\to\infty} 2n = \infty$,

  - A sequence $\{a_n\}$ is said to be oscillatory if the sequence is neither convergent nor divergent. For example, let

$$\{u_n\} = \left\{(-1)^n + \frac{1}{2^{\frac{1}{n}}}\right\}$$

$$\lim_{n\to\infty} u_n \begin{matrix} = 2\,, if\ n\ is\ even \\ = 0\,, if\ n\ is\ odd \end{matrix}$$

  Since the limit is not unique, the sequence is oscillatory.

- **Convergence properties of sequences:**
  - Let $\{a_n\}$ and $\{b_n\}$ be two convergent sequences and k be any real number, then the following sequences will also converge.

1) $\{a_n + b_n\}$ With $\lim_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty}(a_n) + \lim_{n\to\infty}(b_n)$
2) $\{ka_n\}$ With $\lim_{n\to\infty}(ka_n) = k \lim_{n\to\infty}(a_n)$
3) $\{a_nb_n\}$ With $\lim_{n\to\infty}(a_nb_n) = \left(\lim_{n\to\infty}(a_n)\right)\left(\lim_{n\to\infty}(b_n)\right)$

4) $\left\{\frac{a_n}{b_n}\right\}$ With $\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \frac{\lim_{n\to\infty}(a_n)}{\lim_{n\to\infty}(b_n)}$; $\left(if \lim_{n\to\infty}(b_n) \neq 0\right)$

❖ **Some Important Formula:**

$\lim_{n\to\infty} \frac{ln(n)}{n} = 0$	$\lim_{n\to\infty} \sqrt[n]{n} = 1$
$\lim_{n\to\infty} x^{\frac{1}{n}} = 1 (x > 0)$	$\lim_{n\to\infty} x^n = 0 (\lvert x\rvert < 1)$
$\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x \quad (any\ x)$	$\lim_{n\to\infty} \left(\frac{x^n}{n!}\right) = 0 \quad (any\ x)$

Que.: Applying the definition, show that $\left\{\frac{1}{n}\right\}$ converges 0 as $n \to \infty$.

To prove: Let $\epsilon > 0$, we must show that there exists an integer N such that for all n,

$$n > N \Longrightarrow \left|\frac{1}{n} - 0\right| < \epsilon$$

Solution: Let $\epsilon > 0$ be given.

Let N be an integer such that $N > \frac{1}{\epsilon}$.

$$\begin{aligned} n \geq N \quad &\Longrightarrow n \geq N > \frac{1}{\epsilon} \\ &\Longrightarrow n > \frac{1}{\epsilon} \\ &\Longrightarrow \left|\frac{1}{n} - 0\right| = \frac{1}{n} < \epsilon \\ &\Longrightarrow \left|\frac{1}{n} - 0\right| < \epsilon \end{aligned}$$

$$\therefore \lim_{n\to\infty} \frac{1}{n} = 0$$

Que.: Test the Convergence of the following sequences.

1. $\left\{\frac{n^2+n}{2n^2-n}\right\}$ Solution: Let $a_n = \frac{n^2+n}{2n^2-n}$ $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^2+n}{2n^2-n}$ $= \lim_{n\to\infty} \frac{1+\frac{1}{n}}{2-\frac{1}{n}} = \frac{1}{2}$ As the value of limit is finite the	2. $\{2^n\}$ Solution: Let $a_n = 2^n$ $\lim_{n\to\infty} a_n = \lim_{n\to\infty} 2^n$ $= \infty$ As the value of limit of the sequence is infinite the $\{2^n\}$ is divergent. 3. $\{2-(-1)^n\}$ Solution: Let $a_n = 2-(-1)^n$

$\left\{\frac{n^2+n}{2n^2-n}\right\}$ is convergent. 4. $\{\sqrt{n+1}-\sqrt{n}\}$ Solution: Let $a_n = \sqrt{n+1}-\sqrt{n} \times \frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}}$ $= \frac{n+1-n}{\sqrt{n+1}+\sqrt{n}}$ $= \frac{1}{\sqrt{n+1}+\sqrt{n}}$ $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{\sqrt{n+1}+\sqrt{n}}$ $= 0$ As the value of limit is finite the sequence is convergent.	$\lim_{n\to\infty} a_n = \lim_{n\to\infty} 2-(-1)^n$ $= 2-1 = 1, if\ n\ is\ even.$ $= 2-(-1) = 3, if\ n\ is\ odd.$ Since limit is not unique, the $\{2-(-1)^n\}$ is oscillatory. **Exercise:** Test the convergence of the following sequences. 1. $\left\{\frac{n}{n^2+1}\right\}$ 2. $\{e^n\}$ 3. $\{1+(-1)^n\}$

- **Monotonic sequence:**
  - A sequence $\{a_n\}$ is said to be monotonically increasing if $a_n \le a_{n+1}$ for each value of n.

$$a_n - a_{n+1} \le 0$$

  - A sequence $\{a_n\}$ is said to be monotonically decreasing if $a_n \ge a_{n+1}$ for each value of n.
  - A sequence $\{a_n\}$ is said to be strictly increasing if $a_n < a_{n+1}$ for each value of n.
  - A sequence $\{a_n\}$ is said to be strictly decreasing if $a_n > a_{n+1}$ for each value of n.
  - A sequence $\{a_n\}$ is said to be monotonic if it is either increasing or decreasing.
- **Bounded & unbounded sequence:**
  - A sequence $\{a_n\}$ is said to be bounded above if there is a real number M such that $a_n \le M, for\ all\ n \in \mathbb{N}$. M is said to be an upper bound of the sequence.
  - A sequence $\{a_n\}$ is said to be bounded below if there is a real number m such that
    $a_n \ge m, forall\ n \in \mathbb{N}$. m is said to be a lower bound of the sequence.

- A sequence $\{a_n\}$ is said to be <u>bounded</u> if it is both bounded above and bounded below.
- A sequence $\{a_n\}$ is said to be <u>unbounded</u> if it is not bounded.

1) $a_n = n$

$a_n = 1,2,3,4,\ldots$

Also $a_n \geq 1 \Rightarrow a_n \; is \; bounded \; below.$

2) $a_n = \frac{n}{n+1}$

$a_n = \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots$

$a_n \geq \frac{1}{2}, bounded \; below$

$a_n < 1, bounded \; above$

$\frac{1}{2} \leq a_n < 1$

$a_n \; is \; bounded.$

3) $a_n = \frac{1}{n}$

$\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \ldots \ldots$

$a_n \leq 1, bounded \; above$

$a_n > 0, bounded \; below$

It is bounded.

4) $a_n = (-1)^n$

5) $a_n = (-1)^n . n$

unbounded.

❖ <u>Note that</u>

- If $\{a_n\}$ is bounded above and increasing then it is convergent.
- If $\{a_n\}$ is unbounded above and increasing then it is divergent to ∞.
- If $\{a_n\}$ is bounded below and decreasing then it is convergent.
- If $\{a_n\}$ is unbounded below and decreasing then it is divergent to $-\infty$.

1) The sequence n^2 is **increasing sequence**

1, 4, 9,16,……

Increasing sequence

2) $\frac{1}{2^n}$ is **decreasing sequence**

❖ **Sandwich theorem:**

Let $\{a_n\}, \{b_n\}, \{c_n\}$ be sequences of real numbers such that

$$(i) c_n \leq a_n \leq b_n \ ; \ \forall\, n \geq n_0, for\ some\ n_0\ \ and$$

$$(ii) \lim_{n\to\infty} c_n = l = \lim_{n\to\infty} b_n$$

$$then\ \lim_{n\to\infty} a_n = l$$

Que. Show that the sequence $\left\{\frac{\sin n}{n}\right\}_{n=1}^{\infty}$ converges to 0.

Solution:

We know that $-1 \leq sinn \leq 1 \Rightarrow -\frac{1}{n} \leq \frac{\sin n}{n} \leq \frac{1}{n}$

Further, $\lim_{n\to\infty}\left(-\frac{1}{n}\right) = 0$ and $\lim_{n\to\infty}\frac{1}{n} = 0.$

$\therefore$ By sandwich theorem, $\lim_{n\to\infty}\frac{\sin n}{n} = 0$

Example: Test the convergence of the series $a_n = \frac{n}{n^2+1}$.

Solution:

$$a_n = \frac{n}{n^2+1}$$

$$a_{n+1} = \frac{n+1}{(n+1)^2+1}$$

$$a_n - a_{n+1} = \frac{n}{n^2+1} - \frac{n+1}{(n+1)^2+1} > 0$$

$$a_n - a_{n+1} > 0$$

It is decreasing squence.

$$a_n = \frac{1}{2}, \frac{2}{5}, \frac{3}{10}, \dots..$$

$$a_n \leq \frac{1}{2}, \quad a_n > 0$$

$0 < a_n \leq \frac{1}{2}.$

It is bounded.

Every monotonically bounded sequence is convergent.

❖ **Infinite Series:**

The sum of an infinite sequence of numbers is called **infinite Series**

e.g. $a_1 + a_2 + a_3 + \dots\dots\dots + a_n + \dots\dots = \sum_{n=1}^{\infty} a_n$

- $S_n = a_1 + a_2 + a_3 + \cdots + a_n$ is called n^th partial sum of the series.
- The convergence of infinite series depends on the convergence of the corresponding infinite sequence of partial sums.
- The infinite series is

Convergent	If $\lim_{n\to\infty} S_n = S\ (finite)$
Divergent	If $\lim_{n\to\infty} S_n = \infty\ \ or - \infty$
Oscillatory	If $\lim_{n\to\infty} S_n = niether\ finite\ nor\ \pm\infty$
Oscillating finitely	If value fluctuates within finite range
Oscillating infinitely	If value fluctuates within $\infty\ and - \infty$

- If a series $\sum_{n=1}^{\infty} a_n$converges to S then we say that the sum of the series is S and we write $S = \sum_{n=1}^{\infty} a_n$

❖ Convergence properties of series:

Let $\sum a_n$and $\sum b_n$ be two convergent series and k be any real number, then the following series will also converge.

1) $\sum (a_n \pm b_n)$ with $\sum (a_n \pm b_n) = \sum a_n \pm \sum b_n$
2) $\sum ka_n$ With $\sum ka_n = k \sum a_n$

❖ Telescoping series:

A series is said to be telescoping if while writing the n^th partial sum all terms except first and last vanish.

Que: Check the convergence of the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$.

Solution: Here, $a_n = \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$.

$$\frac{1}{1}\left(\frac{1}{n} - \frac{1}{n+1}\right)$$

Therefore the partial sum is given by,

$$s_n = a_1 + a_2 + \cdots + a_{n-1} + a_n$$

$$= \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{n-1} - \frac{1}{n}\right)$$
$$+ \left(\frac{1}{n} - \frac{1}{n+1}\right)$$
$$= 1 - \frac{1}{n+1}$$
$$\therefore S = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \left(1 - \frac{1}{n+1}\right) = 1$$
$$\therefore \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$$

It is convergent.

For example: $\frac{1}{n(n+3)} = \frac{1}{3}\left(\frac{1}{n} - \frac{1}{n+3}\right)$

Que. Find the Sum of the series $\log 2 + \log\frac{3}{2} + \log\frac{4}{3} + \cdots + \infty$

Solution:

$$S_n = \log 2 + \log\frac{3}{2} + \log\frac{4}{3} + \cdots + \log\frac{n+1}{n}$$
$$= \log\left(2 \cdot \frac{3}{2} \cdot \frac{4}{3} \cdot \ldots \cdot \frac{n+1}{n}\right)$$
$$S_n = \log(n+1)$$
$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \log(n+1)$$
$$= \log\infty$$
$$= \infty$$

As it is infinite therefore the series is divergent.

Que: Find the Sum of the series $\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \infty$

Solution: $a_n = \frac{n}{(n+1)!} = \frac{(n+1)-1}{(n+1)!} = \frac{1}{n!} - \frac{1}{(n+1)!}$

Therefore the partial sum is given by,

$$s_n = a_1 + a_2 + \cdots + a_{n-1} + a_n$$
$$= \left(\frac{1}{1!} - \frac{1}{2!}\right) + \left(\frac{1}{2!} - \frac{1}{3!}\right) + \cdots + \left(\frac{1}{(n-1)!} - \frac{1}{n!}\right)$$
$$+ \left(\frac{1}{n!} - \frac{1}{(n+1)!}\right)$$
$$= 1 - \frac{1}{(n+1)!}$$

$$\therefore S = \lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(1 - \frac{1}{(n+1)!}\right) = 1$$

$$\therefore \frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots = \sum_{n=1}^{\infty} \frac{n}{(n+1)!} = 1$$

❖ **Geometric Series:**

An infinite series in the form $a + ar + ar^2 + \cdots + ar^{n-1} + \cdots$ is said to be a geometric series.

It converges to $\frac{a}{1-r}$ if $|r| < 1$ i.e. $\sum_{n=1}^{\infty} a\, r^{n-1} = \frac{a}{1-r}, |r| < 1$

If $|r| \geq 1$ then the series diverges.

If $r = -1$ then series is oscillatory.

Que. Discuss the convergence of $\sum_{n=0}^{\infty} 2^n$

Solution:

Given series, $\sum_{n=0}^{\infty} 2^n = 2^0 + 2^1 + 2^2 + \cdots$ is a geometric series with $a = 1$ and $r = 2$

$$r = \frac{2}{1} = 2, \;\; r = \frac{4}{2} = 2$$

Since $r = 2 > 1,$ the series is divergent.

Que. Check the convergence of a series $\frac{1}{3^0} - \frac{1}{3} + \frac{1}{3^2} - \frac{1}{3^3} + \frac{1}{3^4} + \cdots$.Also find sum.

Solution:

$$S_n = 1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{27} + \cdots$$

$$r = \frac{a_2}{a_1} = -\frac{\frac{1}{3}}{1} = -\frac{1}{3}$$

$$r = \frac{a_3}{a_2} = \frac{\frac{1}{9}}{-\frac{1}{3}} = -\frac{1}{3}$$

$$r = -\frac{1}{3}$$

Here the series is geometric series with $a = 1$ and and $|r| = \frac{1}{3}$

Since, $|r| = \frac{1}{3} < 1,$ the series is convergent.

$Sum = \frac{a}{1-r} = \frac{1}{1-\left(-\frac{1}{3}\right)} = \frac{1}{\frac{4}{3}} = \frac{3}{4}.$

Que. Discuss the convergence of $\sum_{n=1}^{\infty} \frac{3^{2n}}{4^{2n}}$

Solution: Since,

$$\sum_{n=1}^{\infty} \frac{3^{2n}}{4^{2n}} = \sum_{n=1}^{\infty} \frac{(3^2)^n}{(4^2)^n} = \sum_{n=1}^{\infty} \frac{(9)^n}{(16)^n} = \sum_{n=1}^{\infty} \left(\frac{9}{16}\right)^n$$

is a geometric series with $a = \frac{9}{16}$ and $r = \frac{9}{16}$.

Since $r = \frac{9}{16} < 1$, it is convergent. Further it converges to $\frac{a}{1-r} = \frac{\left(\frac{9}{16}\right)}{\left(1-\left(\frac{9}{16}\right)\right)} = \frac{9}{7}$

Que. Check the convergence of $\sum_{n=1}^{\infty} \frac{4^n + 5^n}{6^n}$

Solution:

$$\sum C_n = \sum_{n=1}^{\infty}\left[\left(\frac{4}{6}\right)^n + \left(\frac{5}{6}\right)^n\right] = \sum_{n=1}^{\infty}\left(\frac{4}{6}\right)^n + \sum_{n=1}^{\infty}\left(\frac{5}{6}\right)^n = \sum a_n + \sum b_n$$

where $a_n = \left(\frac{4}{6}\right)^n$ *and* $b_n = \left(\frac{5}{6}\right)^n$

For $\sum a_n$, $r = \left(\frac{4}{6}\right) < 1$, hence $\sum a_n$ is convergent. And $\sum a_n = \frac{\left(\frac{4}{6}\right)}{\left(1-\frac{4}{6}\right)} = \frac{4}{2} = 2.$

Similarly, for $\sum b_n$, $r = \left(\frac{5}{6}\right) < 1$ so $\sum b_n$ is also convergent.

And $\sum b_n = \frac{\left(\frac{5}{6}\right)}{\left(1-\frac{5}{6}\right)} = 5$

Thus, the sum of $\sum a_n + \sum b_n$ is also convergent. $i.e. \sum c_n$ is convergent.
Further, $\sum c_n = \sum a_n + \sum b_n = 2 + 5 = 7$

Exercise:

1) Find the sum of $\sum_{n\to 1}^{\infty} \frac{3^{n-1}-1}{6^{n-1}}$
2) Find the sum of $\sum_{n\to 1}^{\infty} \frac{4^n+1}{6^n}$
3) Prove that $1 + \frac{2}{3} + \frac{4}{9} + \frac{8}{27} + \frac{16}{81} + \cdots$ converges and find its sum.
4) Prove that $5 - \frac{10}{3} + \frac{20}{9} - \frac{40}{27} + \cdots$ converges and find its sum.

❖ P-Series Test

The Series $\sum_{n=0}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \le 1$.

Exercise: Check if the following series is convergent or divergent.

1. $\sum \frac{1}{x^3}$ 2. $\sum \frac{1}{x^{-3}}$ 3. $\sum \frac{1}{x}$ 4. $\sum \frac{1}{x^{\frac{3}{4}}}$

❖ **<u>Zero test of Divergence (Divergence test):</u>**

If $\lim_{n\to\infty} a_n \neq 0$ then $\sum_{n=1}^{\infty} a_n$ must be divergent

Note: If $\lim_{n\to\infty} a_n = 0$ then nothing can be said about convergence of the series $\sum_{n=1}^{\infty} a_n$ .We have to apply another test for convergence

Que. Test the convergence of following series.

1) $\sum_{n=1}^{\infty} n \sin\frac{1}{n}$

Solution:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} n \sin\frac{1}{n} = \lim_{n\to\infty} \frac{\sin\left(\frac{1}{n}\right)}{\left(\frac{1}{n}\right)} = 1 \neq 0$$

Hence, by zero test, the series is divergent.

2) $\sqrt{\frac{1}{2}} + \sqrt{\frac{2}{3}} + \sqrt{\frac{3}{4}} + \cdots \infty$

Solution:

Here, $a_n = \sqrt{\frac{n}{n+1}}$

$$\therefore \lim_{n\to\infty} a_n = \lim_{n\to\infty} \sqrt{\frac{n}{n+1}} = \lim_{n\to\infty} \sqrt{\frac{n}{n\left(1+\frac{1}{n}\right)}} = \lim_{n\to\infty} \sqrt{\frac{1}{\left(1+\frac{1}{n}\right)}} = \sqrt{\frac{1}{(1+0)}}$$

$$= 1 \neq 0$$

Hence, by zero test, the series is divergent.

Que. Prove that $\sum_{n=1}^{\infty} \frac{n^2-1}{n^2+1}$ is divergent.

Solution:

$$\therefore \lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^2-1}{n^2+1} = \lim_{n\to\infty} \frac{n^2\left(1-\frac{1}{n^2}\right)}{n^2\left(1+\frac{1}{n^2}\right)} = \lim_{n\to\infty} \frac{\left(1-\frac{1}{n^2}\right)}{\left(1+\frac{1}{n^2}\right)} = \frac{(1-0)}{(1+0)} = 1$$

$$\neq 0$$

Hence, by zero test, the series is divergent.

❖ **<u>Direct Comparison Test</u>**

Let $\sum a_n$ be a series with no negative terms.

(a) $\sum a_n$ converges if there is a convergent series $\sum c_n$ with $a_n \le c_n$ for all $n > N$, for some integer N.

(b) $\sum a_n$ diverges if there is a divergent series of nonnegative terms $\sum d_n$ with $a_n \ge d_n$ for all $n > N$, for some integer N.

❖ **<u>Limit Comparison Test</u>**

Suppose that $a_n > 0$ and $b_n > 0$ for all $forall\ n \ge N$ (N an integer).

(a) If $\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0$, then $\sum a_n$ and $\sum b_n$ both converge or both diverge.

(b) If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\sum b_n$ converges, then $\sum a_n$ converges.

(c) If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.

Note: $b_n = \dfrac{Highest\ power\ term\ in\ numerator}{Highest\ power\ term\ in\ denomerator}$

Que. for what value of p does the series $\frac{2}{1^p} + \frac{3}{2^p} + \frac{4}{3^p} + \cdots$ is convergent?

Solution:

Here, $a_n = \frac{n+1}{n^p} = \frac{n\left(1+\frac{1}{n}\right)}{n^p} = \frac{\left(1+\frac{1}{n}\right)}{n^{(p-1)}} = \frac{1}{n^{(p-1)}}\left(1+\frac{1}{n}\right)$.

Let $b_n = \frac{1}{n^{(p-1)}}$. Then $\frac{a_n}{b_n} = \left(1+\frac{1}{n}\right)$

$$\therefore \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right) = 1 + 0 = 1 \ne 0$$

$\therefore \sum a_n$ and $\sum b_n$ both converges or diverges together.

$\sum b_n = \sum \frac{1}{n^{(p-1)}}$ converges for $p - 1 > 1$, $i.e.$ for $p > 2$ and it diverge otherwise.

$\therefore \sum a_n = \sum \frac{n+1}{n^p}$ converges for $p \ge 2$ and it diverge otherwise.

Que. Test the convergence of

$$\sum_{n=1}^{\infty} \frac{2n^2 + 2n}{5 + n^5}$$

Solution:

Here, $a_n = \frac{2n^2+2n}{5+n^5} = \frac{n^2\left(2+\frac{2}{n}\right)}{n^5\left(\frac{5}{n^5}+1\right)} = \frac{1}{n^3}\frac{\left(2+\frac{2}{n}\right)}{\left(\frac{5}{n^5}+1\right)}$.

Let $b_n = \frac{n^2}{n^5} = \frac{1}{n^3}$. Then $\frac{a_n}{b_n} = \frac{\left(2+\frac{2}{n}\right)}{\left(\frac{5}{n^5}+1\right)}$

$$\therefore \lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}\frac{\left(2+\frac{2}{n}\right)}{\left(\frac{5}{n^5}+1\right)}=\frac{(2+0)}{(0+1)}=2\neq 0$$

$\therefore \sum a_n$ and $\sum b_n$ both converges or diverges together.

Now, $\sum b_n=\sum\frac{1}{n^3}$ is a $p-series$ with $p=3>1.$ Hence, it is convergent.

$\therefore \sum a_n=\sum\frac{2n^2+2n}{5+n^5}$ converges. [by comparison test]

Que: Test the convergence of $\sum_{n=1}^{\infty}\frac{\sqrt{n}}{n^2+1}$

Sol: $a_n=\frac{\sqrt{n}}{n^2+1}$

$$b_n=\frac{\sqrt{n}}{n^2}=\frac{1}{n^{2-\frac{1}{2}}}=\frac{1}{n^{\frac{3}{2}}}$$

$$\lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}\frac{\frac{\sqrt{n}}{n^2+1}}{\frac{1}{n^{\frac{3}{2}}}}=\lim_{n\to\infty}n^{\frac{3}{2}}\frac{\sqrt{n}}{n^2+1}=\lim_{n\to\infty}\frac{n^2}{n^2+1}=\lim_{n\to\infty}\frac{n^2}{n^2\left(1+\frac{1}{n^2}\right)}=1\neq 0$$

$\therefore \sum a_n$ and $\sum b_n$ both converges or diverges together.

$$b_n=\frac{1}{n^{\frac{3}{2}}}\text{ , }By\ p-series\ ,p=\frac{3}{2}>1, it\ is\ convergent.$$

By Limit comparision Test, $\sum a_n$ *is convergent*.

Que. Test the convergence of the series

$$\sum_{n=1}^{\infty}\frac{n^p}{\sqrt{n+1}+\sqrt{n}}$$

Solution:

Here, $a_n=\frac{n^p}{\sqrt{n+1}+\sqrt{n}}=\frac{n^p}{n^{\frac{1}{2}}\left(\sqrt{1+\frac{1}{n}}+1\right)}=\frac{1}{n^{\frac{1}{2}-p}}\frac{1}{\left(1+\sqrt{1+\frac{1}{n}}\right)}$.

Let $b_n=\frac{n^p}{n^{\frac{1}{2}}}=\frac{1}{n^{\frac{1}{2}-p}}$. Then $\frac{a_n}{b_n}=\frac{1}{\left(1+\sqrt{1+\frac{1}{n}}\right)}$

$$\therefore \lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}\frac{1}{\left(1+\sqrt{1+\frac{1}{n}}\right)}=\frac{1}{(1+\sqrt{1+0})}=\frac{1}{2}\neq 0$$

$\therefore \sum a_n$ and $\sum b_n$ both converges or diverges together.

Now, $\sum b_n=\sum\frac{1}{n^{\frac{1}{2}-p}}$ is a $p-series$ which converges for $\frac{1}{2}-p>1$, $i.e.$ for $p<-\frac{1}{2}$ and diverges otherwise.

$\therefore \sum a_n = \sum \frac{n^p}{\sqrt{n+1}+\sqrt{n}}$ also converges for $p < -\frac{1}{2}$ and diverges otherwise.[by comparison test]

Que. Test the convergence of the series

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots$$

Solution:

Here, $a_n = \frac{1}{n.(n+1)} = \frac{1}{n^2} \frac{1}{\left(1+\frac{1}{n}\right)}$.

Let $b_n = \frac{1}{n^2}$. Then $\frac{a_n}{b_n} = \frac{1}{\left(1+\frac{1}{n}\right)}$

$$\therefore \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)} = \frac{1}{(1+0)} = 1 \neq 0$$

$\therefore \sum a_n$ and $\sum b_n$ both converges or diverges together.

Now, $\sum b_n = \sum \frac{1}{n^2}$ is a $p - series$ with $p = 2 > 1$. Hence, it is convergent.

$\therefore \sum a_n = \sum \frac{1}{n(n+1)}$ converges. [by comparison test]

Que. Test the convergence of

$$\sum_{n=1}^{\infty} \frac{1}{1^2 + 2^2 + 3^2 + \cdots + n^2}$$

Solution:

$$a_n = \frac{1}{1^2 + 2^2 + 3^2 + \cdots + n^2} = \frac{1}{\sum n^2} = \frac{1}{\left(\frac{n(n+1)(2n+1)}{6}\right)}$$

$$= \frac{1}{n^3} \frac{6}{1\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)}$$

$$= \frac{6}{n.n.n\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)}$$

$$(n^2+n)(2n+1) = (2n^3+n^2+2n^2+n) = 2n^3+3n^2+n$$

$$= n^3(2+\frac{3}{n}+\frac{1}{n^2})$$

Let $b_n = \frac{1}{n^3}$. Then $\frac{a_n}{b_n} = \frac{6}{\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)}$

$$\therefore \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{6}{\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)} = \frac{6}{(1+0)(2+0)} = 3 \neq 0$$

$\therefore \sum a_n$ and $\sum b_n$ both converges or diverges together.

Now, $\sum b_n = \sum \frac{1}{n^3}$ is a $p - series$ with $p = 3 > 1$. Hence, it is convergent.

$\therefore \sum a_n = \sum \frac{1}{1^2+2^2+3^2+\cdots+n^2}$ converges. [by comparison test]

❖ <u>Ratio Test(D' Alembert Ratio Test)</u>

Let $\sum a_n$ be a series with positive terms and suppose that $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L$

Then (a) the series converges if $L < 1$

(b)the series diverges if $L > 1$,

(c) the test is fail if $L = 1$

Que. Test the convergence of a series $\sum \frac{1}{n!}$

Solution:

Here $a_n = \frac{1}{n!} \Longrightarrow a_{n+1} = \frac{1}{(n+1)!}$ and

$\frac{a_{n+1}}{a_n} = \frac{n!}{(n+1)!} = \frac{1}{n+1}$

$$\therefore L = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{1}{n+1} = 0 < 1$$

Hence, by ratio test, given series is convergent.

Que. Test the convergence of the series $\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots$

Solution:

Here $a_n = \frac{n}{(n+1)!}$

$$\Longrightarrow a_{n+1} = \frac{n+1}{(n+2)!} and \ \frac{a_{n+1}}{a_n} = \frac{n+1}{(n+2)!}\frac{(n+1)!}{n} = \frac{(n+1)!}{(n+2)(n+1)!}\frac{n+1}{n}$$

$$= \frac{1}{n+2}\left(1 + \frac{1}{n}\right)$$

$$\therefore L = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{1}{n+2}\left(1 + \frac{1}{n}\right) = 0(1+0) = 0 < 1$$

Hence, by ratio test, given series is convergent.

Que. Test the convergence of the series $\sum_{n=0}^{\infty} \frac{4^n - 1}{3^n}$

Solution:

Here $a_n = \frac{4^n-1}{3^n}$

$$\Longrightarrow a_{n+1} = \frac{4^{n+1} - 1}{3^{n+1}} and$$

$$\frac{a_{n+1}}{a_n} = \frac{4^{n+1}-1}{3^{n+1}}\frac{3^n}{4^n-1} = \frac{3^n}{3^{n+1}}\frac{4^n\left(4-\frac{1}{4^n}\right)}{4^n\left(1-\frac{1}{4^n}\right)} = \frac{1}{3}\left(\frac{4-\frac{1}{4^n}}{1-\frac{1}{4^n}}\right)$$

$$\therefore L = \lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{1}{3}\left(\frac{4-\frac{1}{4^n}}{1-\frac{1}{4^n}}\right) = \frac{1}{3}\left(\frac{4-0}{1-0}\right) = \frac{4}{3} > 1$$

Hence, by ratio test, given series is divergent.

Que. Example: Test the convergence of the series $\sum_{n=0}^{\infty}\frac{n3^n(n+1)!}{2^n n!}$

Solution:

$$a_n = \frac{n3^n(n+1)!}{2^n n!}$$

$$= n(n+1)\left(\frac{3}{2}\right)^n$$

$$\Rightarrow a_{n+1} = (n+1)(n+2)\left(\frac{3}{2}\right)^{n+1} \quad and$$

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)(n+2)\left(\frac{3}{2}\right)^{n+1}}{n(n+1)\left(\frac{3}{2}\right)^n} = \frac{(n+2)}{n}\left(\frac{3}{2}\right)$$

$$= \left(1+\frac{2}{n}\right)\left(\frac{3}{2}\right)$$

$$\therefore L = \lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\left(1+\frac{2}{n}\right)\left(\frac{3}{2}\right) = (1+0)\left(\frac{3}{2}\right) = \frac{3}{2} > 1$$

Hence, by ratio test, given series is divergent.

Que. Test the convergence of

$$\sum_{n=1}^{\infty}\frac{1}{1^2+2^2+3^2+\cdots+n^2}$$

Solution:

$$a_n = \frac{1}{1^2+2^2+3^2+\cdots+n^2} = \frac{1}{\sum n^2} = \frac{1}{\left(\frac{n(n+1)(2n+1)}{6}\right)}$$

$$= \frac{6}{n(n+1)(2n+1)}$$

$$\Rightarrow a_{n+1} = \frac{6}{(n+1)(n+2)(2(n+1)+1)} = \frac{6}{(n+1)(n+2)(2n+3)} \, and$$

$$\frac{a_{n+1}}{a_n}=\frac{6}{(n+1)(n+2)(2n+3)}\frac{n(n+1)(2n+1)}{6}=\frac{n(2n+1)}{(n+2)(2n+3)}$$

$$=\frac{\left(2+\frac{1}{n}\right)}{\left(1+\frac{2}{n}\right)\left(2+\frac{3}{n}\right)}$$

$$\therefore L=\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{\left(2+\frac{1}{n}\right)}{\left(1+\frac{2}{n}\right)\left(2+\frac{3}{n}\right)}$$

$$=\frac{(2+0)}{(1+0)(2+0)}=1$$

Hence, by ratio test fails.

We need to use some other test to check the convergence of the series.

Using comparison test as follows:

$$a_n=\frac{1}{1^2+2^2+3^2+\cdots+n^2}=\frac{1}{\sum n^2}=\frac{1}{\left(\frac{n(n+1)(2n+1)}{6}\right)}=\frac{1}{n^3}\frac{6}{1\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)}$$

Let $b_n=\frac{1}{n^3}$. Then $\frac{a_n}{b_n}=\frac{6}{\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)}$

$$\therefore \lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}\frac{6}{\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)}=\frac{6}{(1+0)(2+0)}=3\ \neq 0$$

$\therefore \sum a_n$and $\sum b_n$ both converges or diverges together.

Now, $\sum b_n=\sum\frac{1}{n^3}$ is a $p-series$ with $p=3>1$. Hence, it is convergent.

$\therefore \sum a_n=\sum\frac{1}{1^2+2^2+3^2+\cdots+n^2}$ converges. [by comparison test]

Que. Test the convergence of the series $2+\frac{3}{2}x+\frac{4}{3}x^2+\frac{5}{4}x^3+\cdots$

Solution: Here $a_n=\frac{n+1}{n}x^{n-1}$

$$\Rightarrow a_{n+1}=\frac{n+2}{n+1}x^n\ and\ \frac{a_{n+1}}{a_n}=\frac{(n+2)x^n}{n+1}\frac{n}{(n+1)x^{n-1}}=\frac{n^2\left(1+\frac{2}{n}\right)}{n^2\left(1+\frac{1}{n}\right)^2}x$$

$$=\frac{\left(1+\frac{2}{n}\right)}{\left(1+\frac{1}{n}\right)^2}x$$

$$\therefore L = \lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{\left(1+\frac{2}{n}\right)}{\left(1+\frac{1}{n}\right)^2}x$$

$$= \frac{(1+0)}{(1+0)^2}x = x$$

Hence, by ratio test, given series is (i) convergent if $x < 1$

(ii) divergent if $x > 1$

For $x = 1$.

$$a_n = \frac{n+1}{n} = 1 + \frac{1}{n}$$

$$\Rightarrow \lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(1+\frac{1}{n}\right) = (1+0)$$

$$= 1 \neq 0$$

$\therefore$ By zero test, given series diverges for $x = 1$.

Hence, by ratio test, given series is (i) convergent if $x < 1$

(ii) divergent if $x \geq 1$

- **<u>Root Test (Cauchy Root Test)</u>**

Let $\sum a_n$ be a series with $a_n \geq 0$ for $n \geq N$ for some N and suppose that

$$\lim_{n\to\infty}\left|a_n\right|^{1/n} = L$$

Then (a) the series converges if $L < 1$

(b) the series diverges if $L > 1$,

(c) the test fails if $L = 1$

Que. Test the convergence of series $\sum_{n=1}^{\infty}\frac{3^n}{2^{n+3}}$

Solution:

$$a_n = \frac{3^n}{2^{n+3}} = \frac{1}{8}\left(\frac{3}{2}\right)^n$$

$$\Rightarrow |a_n|^{\frac{1}{n}} = \left|\frac{1}{8}\left(\frac{3}{2}\right)^n\right|^{\frac{1}{n}} = \frac{1}{8^{\frac{1}{n}}}\left(\frac{3}{2}\right)$$

$$\Rightarrow L = \lim_{n\to\infty}|a_n|^{\frac{1}{n}} = \lim_{n\to\infty}\frac{1}{8^{\frac{1}{n}}}\left(\frac{3}{2}\right)$$

$$= \frac{1}{8^0}\left(\frac{3}{2}\right) = \frac{3}{2} > 1$$

Hence, by root test, given series is divergent.

Que. Test the convergence of series $\sum_{n=1}^{\infty}\left(\frac{n}{2n+5}\right)^{n}$

Solution:

$$a_n = \left(\frac{n}{2n+5}\right)^n = \left(\frac{1}{2+\frac{5}{n}}\right)^n$$

$$\Rightarrow |a_n|^{\frac{1}{n}} = \left|\left(\frac{1}{2+\frac{5}{n}}\right)^n\right|^{\frac{1}{n}} = \left(\frac{1}{2+\frac{5}{n}}\right)$$

$$\Rightarrow L = \lim_{n\to\infty}|a_n|^{\frac{1}{n}} = \lim_{n\to\infty}\left(\frac{1}{2+\frac{5}{n}}\right) = \left(\frac{1}{2+0}\right)$$

$$= \frac{1}{2} < 1$$

Hence, by root test, given series is convergent.

Que: $\left(\frac{2^2}{1^2}-\frac{2}{1}\right)^{-1} + \left(\frac{3^3}{2^3}-\frac{3}{2}\right)^{-2} + \left(\frac{4^4}{3^4}-\frac{4}{3}\right)^{-3} + \cdots ..$

Sol: $\boldsymbol{a_n} = \left[\left(\frac{n+1}{n}\right)^{n+1} - \frac{n+1}{n}\right]^{-n}$

$$L = \lim_{n\to\infty}|a_n|^{\frac{1}{n}} = \left[\left(\frac{n+1}{n}\right)^n\left(\frac{n+1}{n}\right) - \frac{n+1}{n}\right]^{-1}$$

$$= \left[\left(\frac{1+\frac{1}{n}}{1}\right)^n\left(\frac{1+\frac{1}{n}}{1}\right) - \frac{1+\frac{1}{n}}{1}\right]^{-1}$$

$$= [e.1-1]^{-1}$$

$= \frac{1}{e-1}<1$

Hence, by root test, given series is convergent.

Que. Test the convergence of series $\sum_{n=1}^{\infty}\left(1+\frac{1}{\sqrt{n}}\right)^{-n^{\frac{3}{2}}}$

Solution:

$$a_n = \left(1+\frac{1}{\sqrt{n}}\right)^{-n^{\frac{3}{2}}}$$

$$\Rightarrow |a_n|^{\frac{1}{n}} = \left|\left(1+\frac{1}{\sqrt{n}}\right)^{-n^{\frac{3}{2}}}\right|^{\frac{1}{n}} = \left(1+\frac{1}{\sqrt{n}}\right)^{-\left(n^{\frac{3}{2}}\right)(n^{-1})} = \left(1+\frac{1}{\sqrt{n}}\right)^{-\sqrt{n}} = \left(\left(1+\frac{1}{\sqrt{n}}\right)^{\sqrt{n}}\right)^{-1}$$

$$(\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x \quad (any x))$$

$$\Rightarrow L = \lim_{n\to\infty}|a_n|^{\frac{1}{n}} = \lim_{n\to\infty}\left(1+\frac{1}{\sqrt{n}}\right)^{-\sqrt{n}} = \lim_{n\to\infty}\left(\left(1+\frac{1}{\sqrt{n}}\right)^{\sqrt{n}}\right)^{-1} = (e^1)^{-1} = \frac{1}{e} < 1$$

Hence, by root test, given series is convergent.

Que. Test the convergence of the series $\sum_{n=1}^{\infty}\left(\frac{n+2}{n+3}\right)^n x^n$

Solution:

$$a_n = \left(\frac{n+2}{n+3}\right)^n x^n$$

$$\Rightarrow |a_n|^{\frac{1}{n}} = \left|\left(\frac{n+2}{n+3}\right)^n x^n\right|^{\frac{1}{n}} = \left(\frac{n+2}{n+3}\right)x$$

$$\Rightarrow L = \lim_{n\to\infty}|a_n|^{\frac{1}{n}} = \lim_{n\to\infty}\left(\frac{n+2}{n+3}\right)x = \lim_{n\to\infty}\left(\frac{1+\frac{2}{n}}{1+\frac{3}{n}}\right)x$$

$$= \left(\frac{1+0}{1+0}\right)x = x$$

Hence, by root test, given series is (i) convergent if $x < 1$
(ii) divergent if $x > 1$.

For $x = 1$.

$$a_n = \left(\frac{n+2}{n+3}\right)^n = \left(\frac{1+\frac{2}{n}}{1+\frac{3}{n}}\right)^n = \frac{\left(1+\frac{2}{n}\right)^n}{\left(1+\frac{3}{n}\right)^n}$$

$$\Rightarrow \lim_{n\to\infty} a_n$$

$$= \lim_{n\to\infty}\frac{\left(1+\frac{2}{n}\right)^n}{\left(1+\frac{3}{n}\right)^n} = \frac{\lim_{n\to\infty}\left(1+\frac{2}{n}\right)^n}{\lim_{n\to\infty}\left(1+\frac{3}{n}\right)^n} = \frac{(e)^2}{(e)^3} = \frac{1}{e}$$

$$\neq 0$$

$\therefore$ By zero test, given series diverges for $x = 1$.

Hence, by root test, given series is (i) convergent if $x < 1$
(ii) divergent if $x \geq 1$.

<u>Alternative series</u>

A series in which the terms are alternatively positive and negative is called an alternating Series. $e.g.\ 1 - 4 + 9 - 16 + \cdots$

❖ **<u>Leibnitz Test</u>**

The infinite Series $a_1 - a_2 + a_3 - ...$ in which the terms are alternatively positive and negative is convergent if (i) $a_n \geq a_{n+1}$ i.e. series is decreasing (ii) $\lim\limits_{n\to\infty} a_n = 0$

Note: If $\lim\limits_{n\to\infty} a_n \neq 0$ **then** $\sum\limits_{n=1}^{\infty} a_n$ **is oscillatory.**

Que. Test the convergence of the series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$

Solution:

Here $u_n = \frac{(-1)^{n+1}}{n} \quad u_{n+1} = \frac{(-1)^{n+2}}{n+1}$

$|u_n| = \frac{1}{n} \quad |u_{n+1}| = \frac{1}{n+1}$

1)

$$|u_n| - |u_{n+1}| = \frac{1}{n} - \frac{1}{n+1}$$

$$= \frac{n+1-n}{n(n+1)}$$

$$= \frac{1}{n(n+1)} \succ 0$$

$|u_n| - |u_{n+1}| \succ 0 \Rightarrow |u_n| \succ |u_{n+1}|$

Thus each term is less than its preceding term.

Now

2)

$$n \xrightarrow{\lim} \infty |u_n| = n \xrightarrow{\lim} \infty \frac{1}{n} = 0$$

Thus by Leibnitz's test the alternating series is convergent.

Que. Test the convergence of the series $2 - \frac{3}{2} + \frac{4}{3} - \frac{5}{4} + \cdots$

Solution:

Here $u_n = \frac{(-1)^{n+1}(n+1)}{n} \quad u_{n+1} = \frac{(-1)^{n+2}(n+2)}{n+1}$

$|u_n| = \frac{(n+1)}{n} \quad |u_{n+1}| = \frac{n+2}{n+1}$

1)

$$|u_n|-|u_{n+1}| = \frac{n+1}{n} - \frac{n+2}{n+1}$$

$$= \frac{(n+1)^2 - n(n+2)}{n(n+1)}$$

$$= \frac{1}{n(n+1)} \succ 0$$

$$|u_n|-|u_{n+1}| \succ 0 \Rightarrow |u_n| \succ |u_{n+1}|$$

Thus each term is less than its preceding term.

Now

2)

$$\lim_{n\to\infty} |u_n| = \lim_{n\to\infty} \frac{n+1}{n}$$

$$= \lim_{n\to\infty} \frac{n\left(1+\frac{1}{n}\right)}{n}$$

$$= 1 \neq 0$$

Thus by Leibnitz's test the alternating series is oscillating.

Que. Test the convergence of the series $\sum_{n=1}^{\infty} \frac{(-1)^n x^{n+1}}{2n-1}$

Solution:

$$u_n = \frac{(-1)^{n+1} x^{n+1}}{2n-1} \quad u_{n+1} = \frac{(-1)^{n+2} x^{n+2}}{2n+1}$$

$$|u_n| = \frac{x^{n+1}}{2n-1} \quad |u_{n+1}| = \frac{x^{n+2}}{2n+1}$$

1)

$$|u_n|-|u_{n+1}| = \frac{x^{n+1}}{2n-1} - \frac{x^{n+2}}{2n+1}$$

$$= \frac{(2n+1)x^{n+1} - x^{n+2}(2n-1)}{(2n-1)(2n+1)}$$

$$= \frac{x^{n+1}[(2n+1)-(2n-1)x]}{(4n^2-1)} \succ 0$$

$$|u_n|-|u_{n+1}| \succ 0 \Rightarrow |u_n| \succ |u_{n+1}|$$

Now

2)

$$\lim_{n\to\infty} |u_n| = \lim_{n\to\infty} \frac{x^{n+1}}{2n-1}$$

$$= 0 \qquad if \quad x < 1$$

Thus by Leibnitz's test the alternating series is convergent.