PARUL UNIVERSITY
FACULTY OF ENGINEERING & TECHNOLOGY
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Unit-3 Propositional Logic
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Introduction:

The rules of logic give precise meaning to mathematical statements. These rules are used to distinguish between valid and invalid mathematical arguments. Which helps to understand and to construct correct mathematical arguments. Besides the importance of logic in understanding mathematical reasoning, logic has numerous applications to computer science. These rules are used in the design of computer circuits, the construction of computer programs, the verification of the correctness of programs, and in many other ways. Furthermore, software systems have been developed for constructing some, but not all, types of proofs automatically

Overview:

- **Syntax, semantics, propositions**
- **Basic connectives and truth tables**
  - **Negation**
  - **Conjunction**
  - **Disjunction**
  - **Exclusive or**
  - **Implication**
  - **Biconditional**
- **Precedence of logical operators**
- **Converse, contrapositive, and inverse of an implication**
- **Logic and bit operations**
- **Logical equivalence: the laws of logic, logical implication**
- **Propositional satisfiability**
- **Predicates**
- **Quantifiers**
  - **Universal quantifier**
  - **The existential quantifier**
  - **The uniqueness quantifier**
  - **Negating quantified expressions**
- **Rules of inference**

Definition- Syntax
The structure of statements in a computer language is said to be a **syntax**.

Definition- Semantics
In computer science, the term **semantic**s refers to the meaning of language constructs, as opposed to their form.

Definition-Proposition
A **proposition** is a declarative sentence (that is, a sentence that declares a fact) that is either true or false, but not both.

For example, all the following declarative sentences are propositions.
1. Washington, D.C., is the capital of the United States of America.
2. Toronto is the capital of Canada.
3. $1 + 1 = 2$.
4. $2 + 2 = 3$.

Propositions 1 and 3 are true, whereas 2 and 4 are false.

The following are not propositions.
1. What time is it?
2. Read this carefully.
3. $x + 1 = 2$.
4. $x + y = z$.

Sentences 1 and 2 are not propositions because they are not declarative sentences. Sentences 3 and 4 are not propositions because they are neither true nor false. Note that each of sentences 3 and 4 can be turned into a proposition if we assign values to the variables.

Definition-Propositional Variables
Variables that represent propositions are known as **propositional variables** (or **statement variables**) The conventional letters used for propositional variables are $p, q, r, s, \ldots$

Definition-Truth value
The **truth value** of a proposition is true, denoted by T, if it is a true proposition, and the truth value of a proposition is false, denoted by F, if it is a false proposition.

Definition-Propositional Logic:
The area of logic that deals with propositions is called the **propositional calculus** or **propositional logic**.

Definition-Compound Propositions:
Many mathematical statements are constructed by combining one or more propositions. New propositions, called **compound propositions**, are formed from existing propositions using logical operators.

BASIC CONNECTIVES AND TRUTH TABLES

Definition- Connectives
The logical operators that are used to form new propositions from two or more existing propositions. These logical operators are also called **connectives**.

Definition- Negation
Let p be a proposition. The $\boldsymbol{negation\ of}$ p, denoted by $\neg p$ (also denoted by p), is the statement "It is not the case that p."
The proposition $\neg p$ is read "$not\ p$." The truth value of the negation of p, $\neg p$, is the opposite of the truth value of p.

For example, consider the proposition "$Michael's\ PC\ runs\ Linux$".
The negation is "$It\ is\ not\ the\ case\ that\ Michael's\ PC\ runs\ Linux$."
This negation can be more simply expressed as "$Michael's\ PC\ does\ not\ run\ Linux$."

The table on right displays the **truth table** for the negation of a proposition p. This table has a row for each of the two possible truth values of a proposition p. Each row shows the truth value of $\neg p$ corresponding to the truth value of p for this row.

p	$\neg p$
T	F
F	T

Remark:
The negation of a proposition can also be considered as the result of the operation of the **negation operator** on a proposition. The negation operator constructs a new proposition from a single existing proposition.

Definition- Conjunction

Let p and q be propositions. The $\boldsymbol{conjunction}$ of p and q, denoted by $p \wedge q$, is the proposition "p and q." The conjunction $p \wedge q$ is true when both p and q are true and is false otherwise.
The table on right displays the **truth table** for the $\boldsymbol{conjunction}$ of p and q.

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Note that in logic the word "but" sometimes is used instead of "and" in a conjunction.

For example, the statement "$The\ sun\ is\ shining, but\ it\ is\ raining$" is another way of saying "$The\ sun\ is\ shining\ and\ it\ is\ raining$."

For example, p is the proposition "$Rebecca's\ PC\ has\ more\ than\ 16\ GB\ free\ hard\ disk\ space$" and q is the proposition "$The\ processor\ in\ Rebecca's\ PC\ runs\ faster\ than\ 1\ GHz$."

The conjunction of these propositions, $p \wedge q$, is the proposition "$Rebecca's\ PC\ has\ more\ than\ 16\ GB\ free\ hard\ disk\ space, and\ the\ processor\ in\ Rebecca's\ PC\ runs\ faster\ than\ 1\ GHz$."
This conjunction can be expressed more simply as "$Rebecca's\ PC\ has\ more\ than\ 16\ GB$

$free\ hard\ disk\ space, and\ its\ processor\ runs\ faster\ than\ 1\ GHz$."

For this conjunction to be true, both conditions given must be true. It is false, when one or both of these conditions are false.

Definition- Disjunction

Let p and q be propositions. The ***disjunction*** of p and q, denoted by $p \vee q$, is the proposition "p or q." The disjunction $p \vee q$ is false when both p and q are false and is true otherwise.
The table on right displays the **truth table** for the ***disjunction*** of p and q.

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Remark:

- The use of the connective *or* in a disjunction corresponds to one of the two ways the word *or* is used in English, namely, as an **inclusive or**. A disjunction is true when at least one of the two propositions is true.

  For instance, the inclusive or is being used in the statement

  "*Students who have taken calculus or computer science can take this class.*"

  Here, the proposition means that students who have taken both calculus and computer science can take the class, as well as the students who have taken only one of the two subjects.

  For example, p is the proposition " $Rebecca's\ PC\ has\ more\ than\ 16\ GB\ free\ hard\ disk\ space$" and q is the proposition "$The\ processor\ in\ Rebecca's\ PC\ runs\ faster\ than\ 1\ GHz$."

  The disjunction of p and q, $p \vee q$, is the proposition

  "$Rebecca's\ PC\ has\ at\ least\ 16\ GB\ free\ hard\ disk\ space, or\ the\ processor\ in\ Rebecca's\ PC\ runs\ faster\ than\ 1\ GHz.$"

  This proposition is true when Rebecca's PC has at least 16 GB free hard disk space, when the PC's processor runs faster than 1 GHz, and when both conditions are true. It is false when both of these conditions are false, that is, when Rebecca's PC has less than 16 GB free hard disk space and the processor in her PC runs at 1 GHz or slower.

Note: The use of the connective *or* in a disjunction corresponds to one of the two ways the word *or* is used in English, namely, in an inclusive way. Thus, a disjunction is true when at least one of the two propositions in it is true. Sometimes, we use *or* in an exclusive sense. When the exclusive or is used to connect the propositions p and q, the proposition "$\boldsymbol{p\ or\ q\ (but\ not\ both)}$" is obtained. This proposition is true when p is true and q is false, and when p is false and q is true. It is false when both p and q are false and when both are true.

Definition- Exclusive OR

Let p and q be propositions. The ***exclusive or*** of p and q, denoted by $p \oplus q$, is the proposition that is true when exactly one of p and q is true and is false otherwise.
The table on right displays the **truth table** for the ***exclusive or*** of p and q

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

Definition- Conditional Statement (Implication)

Let p and q be propositions. The ***conditional statement*** $p \rightarrow q$ is the proposition "$if\ p, then\ q$." The conditional statement $p \rightarrow q$ is false when p is true and q is false, and true otherwise.
The table on right displays the **truth table** for the ***conditional statement*** $p \rightarrow q$.

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

- In the conditional statement $p \rightarrow q$, p is called the *hypothesis* (or *antecedent* or *premise*) and q is called the *conclusion* (or *consequence*).
- The statement $p \rightarrow q$ is called a conditional statement because $p \rightarrow q$ asserts that q is true on the condition that p holds. A conditional statement is also called an **implication**.
- Note that the statement $p \rightarrow q$ is true when both p and q are true and when p is false (no matter what truth value q has).
- Because conditional statements play such an essential role in mathematical reasoning, a variety of terminology is used to express $p \rightarrow q$. You will encounter most if not all of the following ways to express this conditional statement:

"$if\ p, then\ q$"
"$if\ p, q$"
"$p\ is\ sufficient\ for\ q$"
"$q\ if\ p$"
"$q\ when\ p$"
"$a\ necessary\ condition\ for\ p\ is\ q$"
"$q\ unless\ \neg p$"
"$p\ implies\ q$"
"$p\ only\ if\ q$"
"$a\ sufficient\ condition\ for\ q\ is\ p$"
"$q\ whenever\ p$"
"$q\ is\ necessary\ for\ p$"
"$q\ follows\ from\ p$"

Definition- Biconditional Statement

Let p and q be propositions. The ***biconditional statement*** $p \leftrightarrow q$ is the proposition "$p\ if\ and\ only\ if\ q.$" The biconditional statement $p \leftrightarrow q$ is true when p and q have the same truth values, and is false otherwise. Biconditional statements are also called *bi-implications*.

The table on right displays the **truth table** for the ***biconditional statement*** $p \leftrightarrow q$

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

For example, let p be the statement "$You\ can\ take\ the\ flight$," and let q be the statement "$You\ buy\ a\ ticket.$"

Then $p \leftrightarrow q$ is the statement "$You\ can\ take\ the\ flight\ if\ and\ only\ if\ you\ buy\ a\ ticket.$"

This statement is true if p and q are either both true or both false, that is, if you buy a ticket and can take the flight or if you do not buy a ticket and you cannot take the flight. It is false when p and q have opposite truth values, that is, when you do not buy a ticket, but you can take the flight and when you buy a ticket but you cannot take the flight.

Exercise-1

(i) "q whenever p" is Biconditional statement. {T/F} **[Winter 2019 – 20]**

(ii) The negation of the statement "A circle is an ellipse" is________ **[Summer/Winter 2023 – 24]**

(iii) The negation of the statement "New Delhi is a city." is________. **[Winter 2023 – 24]**

(iv) If the truth value of $p \vee q$ is F then the truth value of $(\neg p \wedge \neg q)$ is ________. **[Winter 2021 – 22]**

(v) Complete the truth table **[Winter 2023 – 24] [Summer 2023 – 24]**

p	$\neg(\neg p)$
T	
F	

(vi) Let p and q be the propositions

p : I bought a lottery ticket this week.

q : I won the million dollar jackpot.

Express each of these propositions as an English sentence.

a) $\neg p$ **b)** $p \vee q$ **c)** $p \rightarrow q$

d) $p \wedge q$ **e)** $p \leftrightarrow q$ **f)** $\neg p \rightarrow \neg q$

g) $\neg p \wedge \neg q$ **h)** $\neg p \vee (p \wedge q)$

(vii) Let p and q be the propositions

p: It is below freezing. q: It is snowing.

Write these propositions using p and q and logical connectives. (including negations).

a) It is below freezing and snowing.

b) It is below freezing but not snowing.

c) It is not below freezing and it is not snowing.

d) It is either snowing or below freezing (or both).

e) If it is below freezing, it is also snowing.

f) Either it is below freezing or it is snowing, but it is not snowing if it is below freezing.

g) That it is below freezing is necessary and sufficient for it to be snowing.

PRECEDENCE OF LOGICAL OPERATORS

Generally parentheses are used to specify the order in which logical operators in a compound proposition are to be applied.

For instance, $(p \vee q) \wedge (\neg r)$ is the conjunction of $p \vee q$ and $\neg r$.

However, to reduce the number of parentheses, note that the negation operator is applied before all other logical operators. This means that $\neg p \wedge q$ is the conjunction of $\neg p$ and q, namely, $(\neg p) \wedge q$, not the negation of the conjunction of p and q, namely $\neg (p \wedge q)$.

Another general rule of precedence is that the conjunction operator takes precedence over the disjunction operator, so that $p \wedge q \vee r$ means $(p \wedge q) \vee r$ rather than $p \wedge (q \vee r)$.

The conditional and biconditional operators $\rightarrow$ and $\leftrightarrow$ have lower precedence than the conjunction and disjunction operators, $\wedge$ and $\vee$. Consequently, $p \vee q \rightarrow r$ is the same as $(p \vee q) \rightarrow r$.

The table on right displays the precedence levels of the logical operators, $\neg, \wedge, \vee, \rightarrow, and\ \leftrightarrow$.

Operator	*Precedence*
$\neg$	1
$\wedge$ $\vee$	2 3
$\rightarrow$ $\leftrightarrow$	4 5

Example 1: Construct the truth table of the compound proposition $(p \vee \neg q) \rightarrow (p \wedge q)$.

Solution:

p	q	$\neg q$	$p \vee \neg q$	$p \wedge q$	$(p \vee \neg q) \rightarrow (p \wedge q)$
T	T	F	T	T	T
T	F	T	T	F	F
F	T	F	F	F	T
F	F	T	T	F	F

Exercise-2

1. Construct a truth table for each of these compound propositions.

   a) $(p \rightarrow q) \leftrightarrow (\neg q \rightarrow \neg p)$
   b) $(p \rightarrow q) \rightarrow (q \rightarrow p)$
   c) $p \oplus (p \vee q)$
   d) $(p \rightarrow q) \rightarrow (q \rightarrow p)$ **[Summer 2021-2022]**

CONVERSE, CONTRAPOSITIVE, AND INVERSE

Definition-Converse
Consider a conditional statement $p \rightarrow q$.
The proposition $q \rightarrow p$ is called the **converse** of $p \rightarrow q$.

Definition- Contrapositive
The **contrapositive** of $p \rightarrow q$ is the proposition $\neg q \rightarrow \neg p$.

Definition- Inverse
The proposition $\neg p \rightarrow \neg q$ is called the **inverse** of $p \rightarrow q$.

Of these three conditional statements formed from $p \rightarrow q$, only the contrapositive always has the same truth value as $p \rightarrow q$.

Example-2:
What is the contrapositive, the converse, and the inverse of the given conditional statement,

$$\text{“}The\ home\ team\ wins\ whenever\ it\ is\ raining?\text{”}$$

Solution:
Because “q whenever p” is one of the ways to express the conditional statement $p \to q$, the original statement can be rewritten as

$$\text{“}If\ it\ is\ raining, then\ the\ home\ team\ wins.\text{”}$$

Let p be the statement "$it\ is\ raining$" and q be the statement "$the\ home\ team\ wins$". Hence, the following.

Conditional statement:	$p \to q$	“$If\ it\ is\ raining, then\ the\ home\ team\ wins.$”
Contrapositive:	$\neg q \to \neg p$	“$If\ the\ home\ team\ does\ not\ win, then\ it\ is\ not\ raining.$”
Converse:	$q \to p$	“$If\ the\ home\ team\ wins, then\ it\ is\ raining.$”
Inverse:	$\neg p \to \neg q$	“$If\ it\ is\ not\ raining, then\ the\ home\ team\ does\ not\ win.$”

Exercise-3:

1. State the converse, contrapositive and Inverse of the proposition, “If it snow tonight, then I will stay at home.” **[Winter 2019 – 20] [Winter 2021 – 22]**
2. The converse of the statement “If x < y, then x + 5 < y + 5” is____
3. The contrapositive of the statement “If my car is in the repair shop, then I cannot go to the market.” is___________ **[Winter 2023 – 24]**
4. What is the contrapositive, the converse and the inverse of the conditional statement “The home team wins whenever it is raining?” **[Summer 2023 – 24] [Winter 2022 - 23]**
5. The inverse of conditional statement p → q is ______ **[Summer 2023 – 24]**
6. The contrapositive of conditional statement $p \to q$ is ________ **[Winter 2022 - 23]**
7. The converse of the statement “If x > y, then x + a > y + a” is____ **[Winter 2023 – 24]**
8. The contrapositive of the statement “If 7 is greater than 5, then 8 is greater than 6” is___________ **[Winter 2023 – 24]**

LOGIC AND BIT OPERATIONS

Definition-Bit:

Computers represent information using bits. A **bit** is a symbol with two possible values, namely, 0 (zero) and 1 (one). This meaning of the word bit comes from *b*inary dig*it*, because zeros and ones are the digits used in binary representations of numbers.

A bit can be used to represent a truth value, because there are two truth values, namely, ***true*** and ***false***. **1** represents **T (true)**, **0** represents **F (false)**.

Definition-Boolean variable:
A variable is called a **Boolean variable** if its value is either true or false. Consequently, a Boolean variable can be represented using a bit.

Definition-Bit Operations
bit operations correspond to the logical connectives. By replacing true by a one and false by a zero in the truth tables for the operators ∧, ∨, and ⊕, the truth tables for the corresponding bit operations are obtained.

We will also use the notation *OR*, *AND*, and *XOR* for the operators $\vee$, $\wedge$, and $\oplus$, as is done in various programming languages.

Truth table for bit operators is as following.

x	y	$x \vee y$	$x \wedge y$	$x \oplus y$
0	0	0	0	0
0	1	1	0	1
1	0	1	0	1
1	1	1	1	0

Definition- Bit string
A ***bit string*** is a sequence of zero or more bits. The ***length of*** this ***string*** is the number of bits in the string.

For example 101010011 is a bit string of length nine

Definition- Bitwise OR, Bitwise AND, and bitwise XOR
The **bitwise *OR***, **bitwise *AND***, and **bitwise *XOR*** of two strings of the same length can be defined to be the strings that have as their bits the *OR*, *AND*, and *XOR* of the corresponding bits in the two strings, respectively.

The symbols $\vee, \wedge, and \oplus$ are used to represent the bitwise *OR*, bitwise *AND*, and bitwise *XOR* operations, respectively.

Example-3: Find the bitwise *OR*, bitwise *AND*, and bitwise *XOR* of the bit strings 01 1011 0110 and 11 0001 1101.

Solution.

01 1011 0110	
11 0001 1101	
11 1011 1111	bitwise *OR*
01 0001 0100	bitwise *AND*
10 1010 1011	bitwise *XOR*

Exercise-4:

1. Find the bitwise *OR*, bitwise *AND*, and bitwise *XOR* of each of these pairs of bit strings.
   a) 101 1110, 010 0001
   b) 1111 0000, 1010 1010
2. How many different bit strings of length ten are there? **[Winter 2017 – 18]**
3. (101 1110) $\oplus$ (010 0001) =____________ **[Summer 2023 – 24]**
4. The bitwise XOR of each of the following pair of bit strings: 1101 1001 1011; 1011 0010 1010 will be ____________________. **[Winter 2018 – 19]**
5. The bitwise XOR of each of the following pair of bit strings: 1101 1011 1011; 1001 0110 1010 will be ____. **[Summer 2018 – 19]**
6. How many different bit strings of length 4 are there? **[Summer 2018 – 19]**
7. Find the bitwise OR, bitwise And & bitwise XOR of each of the following pairs of bit strings: (a) 1111 0000, 1010 1010 (b) 00 0111 0001, 10 0100 1000 **[Summer 2021-2022]**

LOGICAL EQUIVALENCE: THE LAWS OF LOGIC, LOGICAL IMPLICATION

Definition- Tautology
A compound proposition that is always true, no matter what the truth values of the propositional variables that occur in it, is called a ***tautology***.

Definition- Contradiction
A compound proposition that is always false is called a ***contradiction***.

Definition- Contingency
A compound proposition that is neither a tautology nor a contradiction is called a ***contingency***.
For example, since $p \lor \neg p$ is always true, it is a tautology. And as $p \land \neg p$ is always false, it is a contradiction.

Definition- Logical Equivalent
The compound propositions p and q are called $logically\ equivalent$ if $p \leftrightarrow q$ is a tautology.
The notation $p \equiv q$ denotes that p and q are logically equivalent.
In other words, compound propositions that have the same truth values in all possible cases are called **logically equivalent**.

Remark:
- The symbol $\equiv$ is not a logical connective, and $p \equiv q$ is not a compound proposition but rather is the statement that $p \leftrightarrow q$ is a tautology.
- The symbol $\Leftrightarrow$ is sometimes used instead of $\equiv$ to denote logical equivalence.
- One way to determine whether two compound propositions are equivalent is to use a truth table. In particular, the compound propositions p and q are equivalent if and only if the columns giving their truth values agree.

Some standard logical equivalences are given in the following table.

Equivalence	*Name*
$p \land \mathbf{T} \equiv p$ $p \lor \mathbf{F} \equiv p$	Identity laws
$p \lor \mathbf{T} \equiv \mathbf{T}$ $p \land \mathbf{F} \equiv \mathbf{F}$	Domination laws
$p \lor p \equiv p$ $p \land p \equiv p$	Idempotent laws
$\neg(\neg p) \equiv p$	Double negation law
$p \lor q \equiv q \lor p$ $p \land q \equiv q \land p$	Commutative laws
$(p \lor q) \lor r \equiv p \lor (q \lor r)$ $(p \land q) \land r \equiv p \land (q \land r)$	Associative laws
$p \lor (q \land r) \equiv (p \lor q) \land (p \lor r)$ $p \land (q \lor r) \equiv (p \land q) \lor (p \land r)$	Distributive laws
$\neg(p \land q) \equiv \neg p \lor \neg q$ $\neg(p \lor q) \equiv \neg p \land \neg q$	De Morgan's laws
$p \lor (p \land q) \equiv p$ $p \land (p \lor q) \equiv p$	Absorption laws
$p \lor \neg p \equiv \mathbf{T}$ $p \land \neg p \equiv \mathbf{F}$	Negation laws

Example-4:
Show that $\neg(p \lor q) \equiv \neg p \land \neg q$ are logically equivalent. **[Winter 2021 – 22]**

Solution.

p	q	$(p \lor q)$	$\neg(p \lor q)$	$\neg p$	$\neg q$	$\neg p \land \neg q$
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

The truth tables for these compound propositions are displayed in the table. Because the truth values of the compound propositions $\neg(p \lor q)$ and $\neg p \land \neg q$ agree for all possible combinations of the truth values of p and q, it follows that $\neg(p \lor q) \leftrightarrow (\neg p \land \neg q)$ is a tautology.
Hence, these compound propositions are logically equivalent.

Example-5:
Show that $p \rightarrow q$ and $\neg p \lor q$ are logically equivalent.

Solution.

p	q	$p \rightarrow q$	$\neg p$	$\neg p \lor q$
T	T	T	F	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Since the truth values of $\neg p \lor q$ and $p \rightarrow q$ agree, they are logically equivalent.

Example-6: Show that $\neg(p \lor (\neg p \land q))$ and $\neg p \land \neg q$ are logically equivalent by developing a series of logical equivalences. **[Summer 2021-2022]**
Solution.

$$
\begin{aligned}
\neg(p \lor (\neg p \land q)) &\equiv \neg p \land \neg(\neg p \land q) && \text{by the second De Morgan law}\\
&\equiv \neg p \land [\neg(\neg p) \lor \neg q] && \text{by the first De Morgan law}\\
&\equiv \neg p \land (p \lor \neg q) && \text{by the double negation law}\\
&\equiv (\neg p \land p) \lor (\neg p \land \neg q) && \text{by the second distributive law}\\
&\equiv \boldsymbol{F} \lor (\neg p \land \neg q) && \text{because } \neg p \land p \equiv \boldsymbol{F}\\
&\equiv (\neg p \land \neg q) \lor \boldsymbol{F} && \text{by the commutative law for disjunction}\\
&\equiv \neg p \land \neg q && \text{by the identity law for } \mathbf{F}
\end{aligned}
$$

Consequently $\neg(p \lor (\neg p \land q))$ and $\neg p \land \neg q$ are logically equivalent.

Example-7: Show that $(p \land q) \rightarrow (p \lor q)$ is a tautology. **[Summer 2017 – 18]**

Solution.

$$
\begin{aligned}
(p \land q) \rightarrow (p \lor q) &\equiv \neg(p \land q) \lor (p \lor q) && \text{Because } p \rightarrow q \textit{ and } \neg p \lor q \text{ are equivalent}\\
&\equiv (\neg p \lor \neg q) \lor (p \lor q) && \text{by the first De Morgan law}
\end{aligned}
$$

$\equiv T \lor T$	by the associative and commutative laws for disjunction
$\equiv T$	$\neg p \lor p \quad \equiv$ T and the commutative law for disjunction
	by the domination law

Consequently $(p \land q) \rightarrow (p \lor q)$ is a tautology.

Exercise-5

1. Which of the following is knowns as Idempotent Law? **[Winter 2022 - 23]**
   (a) $p \lor p \equiv p$ (b) $p \lor T \equiv p$
   (c) $p \lor q \equiv p \land q$ (d) $p \land F \equiv F$
2. Which of the following is known as Commutative Laws?
   (a) $p \lor q \equiv q \lor p$. (b) $p \lor T \equiv p$. (c) $p \lor p \equiv p$. (d) $p \land F \equiv F$ **[Summer 2023 – 24]**
3. The truth value of $p \land (\sim p)$ is tautology. (T/F) **[Summer 2017 – 18]**
4. The truth value of $p \lor \sim p$ is not a Tautology? (True/False) **[Winter 2022 - 23]**
5. The truth value of $p \land \neg q$ is not a tautology. T/F **[Winter 2021 – 22]**
6. $p \land q\ (p \lor q)$ is known as **[Winter 2021 – 22]**

   a) Distributive Law b) Idempotent Law c) Absorption Law d) Domination Law

7. Prove by constructing truth table, **[Winter 2019 – 20]**
   (i) $(p \land q) \rightarrow (p \lor q)$ is a tautology.
   (ii) Show that $\neg (p \lor q) \equiv \neg p \land \neg q$ are logically equivalent.
8. Prove the *distributive law* of disjunction over conjunction using truth table.
9. Show that $p \leftrightarrow q$ and $(p \land q) \lor (\neg p \land \neg q)$ are logically equivalent.
10. Show that $p \rightarrow q$ and $\neg q \rightarrow \neg p$ is logically equivalent**.** **[Winter 2022 - 23]**

PROPOSITIONAL SATISFIABILITY

Definition- Propositional satisfiable
A compound proposition is **satisfiable** if there is an assignment of truth values to its variables that makes it true.

Definition- Propositional unsatisfiable
When no such assignments exists, that is, when the compound proposition is false for all assignments of truth values to its variables, the compound proposition is **unsatisfiable.**

Definition- Solution of Propositional Satisfiable
The particular assignment of truth values that makes a compound proposition true is called a s**olution** of this particular satisfiability problem.

Remark:

- A compound proposition is **unsatisfiable** if and only if its **negation is a tautology**.

Example-8: Determine whether each of the compound propositions is satisfiable.

$$(p \leftrightarrow q) \land (\neg p \leftrightarrow q)$$

Solution.

p	q	$(p \leftrightarrow q)$	$\neg p$	$(\neg p \leftrightarrow q)$	$(p \leftrightarrow q) \wedge (\neg p \leftrightarrow q)$
T	T	T	F	F	F
T	F	F	F	T	F
F	T	F	T	T	T
F	F	T	T	F	F

Since, $(p \leftrightarrow q) \wedge (\neg p \leftrightarrow q)$ is true when p is false and q is true, it is satisfiable.

PREDICATES

Definition- Predicates

Consider a statement that cannot be verified to be true or false until the values of the variables are not specified. Such statements can be divided in two parts, one which contains the variables, known as subject and the other which refers to a property that the subject of the statement can have, is known as the predicate.

For example, if the statement "x is greater than 3" is denoted by $P(x)$, then P denotes the predicate "is greater than 3" and x is the variable.

Definition-Propositional function

The statement $P(x)$ is also said to be the value of the **propositional function** P at x. Once a value has been assigned to the variable x, the statement $P(x)$ becomes a proposition and has a truth value.

Example-9: Let $P(x)$ denote the statement "$x > 3$." What are the truth values of $P(4)$ and $P(2)$?

Solution. P(4): 4>3 which is true.
P(2): 2>3 which is false

Exercise-6

1. Find if the following is a tautology, contradiction or contingency.
   $(p \rightarrow (q \rightarrow r)) \rightarrow ((p \rightarrow q) \rightarrow (p \rightarrow r))$ **[Winter 2023 – 24] [Summer 2023 – 24]**
2. Determine whether each of the compound propositions is satisfiable.
   (a) $(p \vee \neg q) \wedge (q \vee \neg r) \wedge (r \vee \neg p)$
   (b) $(p \vee \neg q) \wedge (q \vee \neg r) \wedge (r \vee \neg p)$ **[Summer 2021-2022] [Winter 2022-23]**
3. Let $Q(x, y)$ denote the statement "$x = y + 3$." What are the truth values of the propositions $Q(1, 2)$ and $Q(3, 0)$?

QUANTIFIERS

When the variables in a propositional function are assigned values, the resulting statement becomes a proposition with a certain truth value. Quantification expresses the extent to which a predicate is true over a range of elements.

In English, the words *all*, *some*, *many*, *none*, and *few* are used in quantifications.

Definition-Predicate calculus

Two types of quantification are discussed here: universal quantification, which tells us that a predicate is true for every element under consideration, and existential quantification, which tells us that there is one or more element under consideration for which the predicate is true. The area of logic that deals with predicates and quantifiers is called the **predicate calculus**.

Definition- Domain

Many mathematical statements assert that a property is true for all values of a variable in a particular domain, called the **domain of discourse** (or the **universe of discourse**), and often just referred to as the **domain**.

Definition- Universal Quantification

Such a statement is expressed using universal quantification. The universal quantification of *P(x)* for a particular domain is the proposition that asserts that *P(x)* is true for all values of *x* in this domain.

Note that the domain specifies the possible values of the variable *x*. The meaning of the universal quantification of *P(x)* changes when we change the domain. The domain must always be specified when a universal quantifier is used; without it, the universal quantification of a statement is not defined.

The *universal quantification* of *P(x)* is the statement

$$\text{“}P(x)\ for\ all\ values\ of\ x\ in\ the\ domain.\text{”}$$

The notation $\forall x\ P(x)$ denotes the universal quantification of *P(x)*.
Here ∀ is called the **universal quantifier.**
We read $\forall x P(x)$ as “$for\ all\ x\ P(x)$” or “$for\ every\ xP(x)$.”
An element for which *P(x)* is false is called a **counter example** of $\forall x\ P(x)$.

Remark: Generally, an implicit assumption is made that all domains of discourse for quantifiers are nonempty. If the domain is empty, then $\forall x\ P(x)$ is true for any propositional function $P(x)$ because there are no elements *x* in the domain for which $P(x)$ is false.

Example-10: Let $P(x)$ be the statement “$x + 1 > x$.” What is the truth value of the quantification $\forall x P(x)$, where the domain consists of all real numbers?

Solution:
Because $P(x)$ is true for all real numbers *x*, the quantification $\forall x P(x)$ is true.

Example-11: Let $Q(x)$ be the statement “$x < 2$.” What is the truth value of the quantification $\forall x\ Q(x)$, where the domain consists of all real numbers?

Solution:
$Q(x)$ is not true for every real number *x*, because, for instance, $Q(3)$ is false. That is, $x = 3$ is a counterexample for the statement $\forall x\ Q(x)$. Thus $\forall x\ Q(x)$ is false.

Exercise-7

1. What is the truth value of $\forall x P(x)$, where $P(x)$ is the statement “$x^2 < 10$” and the domain consists of the positive integers not exceeding 4?
2. What is the truth value of ∀*x(x*2 ≥ *x)* if the domain consists of all real numbers? What is the truth value of this statement if the domain consists of all integers?

Definition-The Existential Quantifier

Many mathematical statements assert that there is an element with a certain property. Such statements are expressed using existential quantification. With existential quantification, we form a proposition that is true if and only if *P(x)* is true for at least one value of *x* in the domain.
The *existential quantification* of *P(x)* is the proposition

$$\text{“}There\ exists\ an\ element\ x\ in\ the\ domain\ such\ that\ P(x).\text{”}$$

We use the notation $\exists x\ P(x)$ for the existential quantification of $P(x)$.
Here ∃ is called the *existential quantifier*.

Remark:

- A domain must always be specified when a statement $\exists x\ P(x)$ is used.
- Furthermore, the meaning of $\exists x\ P(x)$ changes when the domain changes. Without specifying the domain, the statement $\exists x\ P(x)$ has no meaning.
- Besides the phrase "there exists," we can also express existential quantification in many other ways, such as by using the words "for some," "for at least one," or "there is." The existential quantification $\exists x\ P(x)$ is read as
  "$There\ is\ an\ x\ such\ that\ P(x)$,"
  "$There\ is\ at\ least\ one\ x\ such\ that\ P(x)$,"
  or
  "$For\ some\ xP(x)$."
- $\exists x\ P(x)$ is false if and only if $P(x)$ is false for every element of the domain.
- Generally, an implicit assumption is made that all domains of discourse for quantifiers are nonempty. If the domain is empty, then $\exists x\ Q(x)$ is false whenever $Q(x)$ is a propositional function because when the domain is empty, there can be no element *x* in the domain for which $Q(x)$ is true.

Precedence of Quantifiers

The quantifiers $\forall$ and $\exists$ have higher precedence than all logical operators from propositional calculus.

For example, $\forall x\ P(x)\ \vee\ Q(x)$ is the disjunction of $\forall xP(x)$ and $Q(x)$. In other words, it means $\left(\forall xP(x)\right)\ \vee\ Q(x)$ rather than $\forall x\left(P(x) \vee\ Q(x)\right)$.

Example-12: Let *P(x)* denote the statement "$x\ >\ 3$." What is the truth value of the quantification $\exists x\ P(x)$, where the domain consists of all real numbers?

Solution:

Because "$x\ >\ 3$" is sometimes true—for instance, when $x\ =\ 4$—the existential quantification of *P(x)*, which is $\exists xP(x)$, is true.

Example-13: Let *Q(x)* denote the statement "$x = x + 1$.". What is the truth value of the quantification $\exists x\ Q(x)$, where the domain consists of all real numbers?

Solution:

Since $Q(x)$ is false for every real number *x*, the existential quantification of $Q(x)$, which is $\exists x\ Q(x)$, is false.

Definition: The Uniqueness Quantifier

The **uniqueness quantifier**, denoted by $\exists!$ or $\exists_1$.
The notation $\exists!\ x\ P(x)$ [or $\exists_1 x\ P(x)$] states "There exists a unique x such that $P(x)$ is true."

For instance, $\exists!\ x(x\ -\ 1\ =\ 0)$, where the domain is the set of real numbers, states that there is a unique real number *x* such that $x - 1 = 0$. This is a true statement, as $x = 1$ is the unique real number such that $x\ -\ 1\ =\ 0$.

Definition-Negating Quantified Expressions

P(x) is the statement "*x* has taken a course in calculus" and the domain consists of the students in your class $\forall xP(x)$ denotes the statement

$$"Every\ student\ in\ your\ class\ has\ taken\ a\ course\ in\ calculus."$$

The negation of this statement is

$$\text{"It is not the case that every student in your class has taken a course in calculus."}$$

This is equivalent to

$$\text{"There is a student in your class who has not taken a course in calculus."}$$

And this is simply the existential quantification of the negation of the original propositional function, namely,

$$\exists x \neg P(x).$$

Which gives the following logical equivalence:

$$\neg \forall x P(x) \equiv \exists x \neg P(x).$$

Similarly,

$$\neg \exists x Q(x) \equiv \forall x \neg Q(x).$$

These are known as De Morgan's laws of Quantifiers.

Negation	*Equivalent Statement*	*When Is Negation True?*	*When False?*
$\neg\exists x P(x)$	$\forall x \neg P(x)$	For every x, $P(x)$ is false.	There is an x for which $P(x)$ is true.
$\neg\forall x P(x)$	$\exists x \neg P(x)$	There is an x for which $P(x)$ is false.	$P(x)$ is true for every x.

Example-14: What are the negations of the statements $\forall x(x^2 > x)$ and $\exists x(x^2 = 2)$?

Solution:
The negation of $\forall x(x^2 > x)$ is the statement $\neg\forall x(x^2 > x)$, which is equivalent to $\exists x \neg(x^2 > x)$. This can be rewritten as $\exists x(x^2 \leq x)$.
The negation of $\exists x(x^2 = 2)$ is the statement $\neg\exists x(x^2 = 2)$, which is equivalent to $\forall x \neg(x^2 = 2)$. This can be rewritten as $\forall x(x^2 \neq 2)$.
The truth values of these statements depend on the domain.

Example-15: Express the statement "Every student in this class has studied calculus" using predicates and quantifiers.

Solution:
Let $S(x)$ represent the statement that "person x is in this class";
$C(x)$ represent the statement that "x has studied calculus."
Then the given statement can be expressed as "For every person x, if person x is a student in this class then x has studied calculus." Which can be written as :

$$\forall x\big(S(x) \rightarrow C(x)\big)$$

Exercise-8:

1. What is the truth value of $\exists x\, P(x)$, where $P(x)$ is the statement "$x^2 > 10$" and the universe of discourse consists of the positive integers not exceeding 4?
2. Express the statements "Some student in this class has visited Mexico" and "Every student in this class has visited either Canada or Mexico" using predicates and quantifiers.
3. $P(x)$ is the statement: "$P(x)$ for all values of x in the domain" is known as _________ quantifier. **[Winter 2022 - 23]**

RULES OF INFERENCE

Proofs in mathematics are valid arguments that establish the truth of mathematical statements. By an **argument**, we mean a sequence of statements that end with a conclusion.

By **valid**, we mean that the conclusion, or final statement of the argument, must follow from the truth of the preceding statements, or **premises**, of the argument.

That is, an argument is valid if and only if it is impossible for all the premises to be true and the conclusion to be false.

To deduce new statements from statements we already have, we use rules of inference which are templates for constructing valid arguments.

Rules of inference are our basic tools for establishing the truth of statements.

Definition- Argument in Propositional Logic
An $\boldsymbol{argument}$ in propositional logic is a sequence of propositions.

Definition- Premises
All other than the final proposition in the argument are called $\boldsymbol{premises}$

Definition- Conclusion
The final proposition is called the $\boldsymbol{conclusion.}$

Definition-Valid Argument
An argument is $\boldsymbol{valid}$ if the truth of all its premises implies that the conclusion is true.
An argument form is $\boldsymbol{valid}$ no matter which particular propositions are substituted for the propositional variables in its premises, the conclusion is true if the premises are all true.

Definition- Argument Form
An $\boldsymbol{argument\ form}$ in propositional logic is a sequence of compound propositions involving propositional variables.
The following table gives the rules of inference.

Rule of Inference	*Tautology*	*Name*
$\begin{array}{rl} & p \\ & p \to q \\ \hline \therefore & q \end{array}$	$(p \wedge (p \to q)) \to q$	Modus ponens
$\begin{array}{rl} & \neg q \\ & p \to q \\ \hline \therefore & \neg p \end{array}$	$(\neg q \wedge (p \to q)) \to \neg p$	Modus tollens
$\begin{array}{rl} & p \to q \\ & q \to r \\ \hline \therefore & p \to r \end{array}$	$((p \to q) \wedge (q \to r)) \to (p \to r)$	Hypothetical syllogism
$\begin{array}{rl} & p \vee q \\ & \neg p \\ \hline \therefore & q \end{array}$	$((p \vee q) \wedge \neg p) \to q$	Disjunctive syllogism
$\begin{array}{rl} & p \\ \hline \therefore & p \vee q \end{array}$	$p \to (p \vee q)$	Addition
$\begin{array}{rl} & p \wedge q \\ \hline \therefore & p \end{array}$	$(p \wedge q) \to p$	Simplification
$\begin{array}{rl} & p \\ & q \\ \hline \therefore & p \wedge q \end{array}$	$((p) \wedge (q)) \to (p \wedge q)$	Conjunction
$\begin{array}{rl} & p \vee q \\ & \neg p \vee r \\ \hline \therefore & q \vee r \end{array}$	$((p \vee q) \wedge (\neg p \vee r)) \to (q \vee r)$	Resolution

Example-16: State which rule of inference is the basis of the following argument:
"It is below freezing now. Therefore, it is either below freezing or raining now."

Solution:
Let p be the proposition "It is below freezing now" and q the proposition "It is raining now." Then this argument is of the form

$$\begin{array}{rl} & p \\ \hline \therefore & p \vee q \end{array}$$

This is an argument that uses the addition rule.

Example-17:
State which rule of inference is the basis of the following argument: "It is below freezing and raining now. Therefore, it is below freezing now."

Solution:
Let p be the proposition "It is below freezing now," and let q be the proposition "It is raining now." This argument is of the form

$$\begin{array}{rl} & p \wedge q \\ \hline \therefore & p \end{array}$$

This argument uses the simplification rule.

Example-18:
State which rule of inference is used in the argument:
If it rains today, then we will not have a barbecue today. If we do not have a barbecue today, then we will have a barbecue tomorrow. Therefore, if it rains today, then we will have a barbecue tomorrow.

Solution:
Let p be the proposition "It is raining today," let q be the proposition "We will not have a barbecue today," and let r be the proposition "We will have a barbecue tomorrow." Then this argument is of the form

$$\begin{array}{rl} & p \rightarrow q \\ & q \rightarrow r \\ \hline \therefore & p \rightarrow r \end{array}$$

Hence, this argument is a hypothetical syllogism.

Example-19:
Show that the premises "If you send me an e-mail message, then I will finish writing the program," "If you do not send me an e-mail message, then I will go to sleep early," and "If I go to sleep early, then I will wake up feeling refreshed" lead to the conclusion "If I do not finish writing the program, then I will wake up feeling refreshed."

Solution:
Let p be the proposition "You send me an e-mail message," q the proposition "I will finish writing the program," r the proposition "I will go to sleep early," and s the proposition "I will wake up feeling refreshed." Then the premises are $p \rightarrow q, \neg p \rightarrow r, and\ r \rightarrow s$. The desired conclusion is $\neg q \rightarrow s$.We need to give a valid argument with premises $p \rightarrow q$, $\neg p \rightarrow r$, and $r \rightarrow s$ and conclusion $\neg q \rightarrow s$.
This argument form shows that the premises lead to the desired conclusion.

PROOFS AND TECHNIQUES

Definition-Theorem

A **theorem** is a statement that can be shown to be true.

- In mathematical writing, the term theorem is usually reserved for a statement that is considered at least somewhat important.
- Less important theorems sometimes are called **propositions**.
- A theorem may be the universal quantification of a conditional statement with one or more premises and a conclusion.

Definition-Proof
We demonstrate that a theorem is true with a **proof**.
A proof is a valid argument that establishes the truth of a theorem.

Definition- Axioms:
The statements used in a proof can include **axioms** (or **postulates**), which are statements we assume to be true the premises, if any, of the theorem, and previously proven theorems.

Definition-Lemma

A less important theorem that is helpful in the proof of other results is called a **lemma** (plural lemmas or lemmata).

Definition-Corollary
A **corollary** is a theorem that can be established directly from a theorem that has been proved.

Definition-Conjecture
A conjecture is a statement that is being proposed to be a true statement, usually based on some partial evidence, a heuristic argument, or the intuition of an expert.

Definition-Direct Proof
A direct proof shows that a conditional statement $p \rightarrow q$ is true by showing that if p is true, then q must also be true, so that the combination p true and q false never occurs.

For example: The integer n is even if there exists an integer k such that $n = 2k$, and n is odd if there exists an integer k such that $n = 2k + 1$.

Example-20 Give a direct proof of the theorem **"If n is an odd integer, then n^2 is odd."**

Solution:

- → Note that this theorem states $\forall nP((n) \rightarrow Q(n))$, where P(n) is "n is an odd integer" and Q(n) is "n^2 is odd." As we have said, we will follow the usual convention in mathematical proofs by showing that P(n) implies Q(n), and not explicitly using universal instantiation.
- → To begin a direct proof of this theorem, we assume that the hypothesis of this conditional statement is true, namely, we assume that n is odd.
- → By the definition of an odd integer, it follows that $n = 2k + 1$, where k is some integer. We want to show that n^2 is also odd.
- → We can square both sides of the equation $n = 2k + 1$ to obtain a new equation that expresses n^2.
- → When we do this, we find that $n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$. By the definition of an odd integer, we can conclude that n^2 is an odd integer (it is one more than twice an integer).
- → Consequently, we have proved that if n is an odd integer, then n^2 is an odd integer.

Definition- Indirect Proofs:
Proofs of theorems of this type that are not direct proofs, that is, that do not start with the premises and end with the conclusion, are called **Indirect proofs**.

Definition-Proof by Contraposition:
An extremely useful type of indirect proof is known as **Proof by contraposition**.
Proofs by contraposition make use of the fact that the conditional statement $p \rightarrow q$ is equivalent to its contrapositive, $\neg q \rightarrow \neg p$.

Example-21: Prove that if n is an integer and $3n + 2$ is odd, then n is odd. **[Winter 2022 - 23]**

Solution:

- → We first attempt a direct proof. To construct a direct proof, we first assume that $3n + 2$ is an odd integer.

- → This means that 3n + 2 = 2k + 1 for some integer k. Can we use this fact to show that n is odd? We see that 3n + 1 = 2k, but there does not seem to be any direct way to conclude that n is odd. Because our attempt at a direct proof failed, we next try a proof by contraposition.
- → The first step in a proof by contraposition is to assume that the conclusion of the conditional statement "If 3n + 2 is odd, then n is odd" is false; namely, assume that n is even.
- → Then, by the definition of an even integer, n = 2k for some integer k. Substituting 2k for n, we find that 3n + 2 = 3(2k) + 2 = 6k + 2 = 2(3k + 1). This tells us that 3n + 2 is even (because it is a multiple of 2), and therefore not odd.
- → This is the negation of the premise of the theorem Because the negation of the conclusion of the conditional statement implies that the hypothesis is false, the original conditional statement is true.
- → Our proof by contraposition succeeded; we have proved the theorem "If 3n + 2 is odd, then n is odd."

Definition-Rational Number
The real number r is **rational** if there exist integers p and q with q ≠ 0 such that r = p/q.

Definition-Irrational Number:
A real number that is not rational is called **irrational.**

Example-22: Prove that the sum of two rational numbers is rational. (Note that if we include the implicit quantifiers here, the theorem we want to prove is "For every real number r and every real number s, if r and s are rational numbers, then r + s is rational.

Solution:

- → We first attempt a direct proof. To begin, suppose that r and s are rational numbers. From the definition of a rational number, it follows that there are integers p and q, with q ≠ 0, such that r = p/q, and integers t and u, with u≠ 0, such that s = t/u.
- → Can we use this information to show that r + s is rational? The obvious next step is to add r = p/q and s = t/u, to obtain $r + s = \frac{p}{q} + \frac{t}{u} = \frac{pu+qt}{qu}$ Because q ≠ 0 and u ≠0, it follows that $qu \neq 0$. Consequently, we have expressed r + s as the ratio of two integers, pu + qt and qu, where $qu \neq 0$.
- → This means that r + s is rational. We have proved that the sum of two rational numbers is rational; our attempt to find a direct proof succeeded.

Definition-Proofs by Contradiction
The statement r ∧¬r is a contradiction whenever r is a proposition, we can prove that p is true if we can show that ¬p → (r ∧¬r) is true for some proposition r. Proofs of this type are called **proofs by contradiction**.

Example-23: Show that at least four of any 22 days must fall on the same day of the week.

Solution:

- → Let p be the proposition "At least four of 22 chosen days fall on the same day of the week." Suppose that ¬p is true.
- → This means that at most three of the 22 days fall on the same day of the week. Because there are seven days of the week, this implies that at most 21 days could have been chosen, as for each of the days of the week, at most three of the chosen days could fall on that day.
- → This contradicts the premise that we have 22 days under consideration.
- → That is, if r is the statement that 22 days are chosen, then we have shown that

$\neg p \rightarrow (r \wedge \neg r)$.

→ Consequently, we know that p is true. We have proved that at least four of 22 chosen days fall on the same day of the week.

Example-24: Show that the statement "Every positive integer is the sum of the squares of two integers" is false.

Solution:

→ To show that this statement is false, we look for a counterexample, which is a particular integer that is not the sum of the squares of two integers.

→ It does not take long to find a counterexample, because 3 cannot be written as the sum of the squares of two integers.

→ To show this is the case, note that the only perfect squares not exceeding 3 are 02 = 0 and 12 = 1. Furthermore, there is no way to get 3 as the sum of two terms each of which is 0 or 1.

→ Consequently, we have shown that "Every positive integer is the sum of the squares of two integers" is false.

PROOF METHODS AND STRATEGY

Definition-Exhaustive Proof

Some theorems can be proved by examining a relatively small number of examples.
Such proofs are called **exhaustive proofs**, or **proofs by exhaustion** because these proofs proceed by exhausting all possibilities.
An exhaustive proof is a special type of proof by cases where each case involves checking a single example.

Example-25: Prove that $(n + 1)^3 \geq 3n$ if n is a positive integer with $n \leq 4$.

Solution:

→ We use a proof by exhaustion. We only need verify the inequality $(n + 1)^3 \geq 3n$ when n = 1, 2, 3, and 4.

→ For n = 1, we have $(n + 1)^3 = 23 = 8$ and $3n = 31 = 3$; for n = 2, we have $(n + 1)^3 = 33 = 27$ and $3n = 32 = 9$; for n = 3, we have $(n + 1)^3 = 43 = 64$ and $3n = 33 = 27$; and for n = 4, we have $(n + 1)^3 = 53 = 125$ and $3n = 34 = 81$.

→ In each of these four cases, we see that $(n + 1)^3 \geq 3n$. We have used the method of exhaustion to prove that $(n + 1)^3 \geq 3n$ if n is a positive integer with $n \leq 4$.

Definition-Proof by Cases: A proof by cases must cover all possible cases that arise in a theorem.

Example-26: Prove that if n is an integer, then $n^2 \geq n$.

Solution:

→ We can prove that $n^2 \geq n$ for every integer by considering three cases, when $n = 0$, when $n \geq 1$, and when $n \leq -1$. We split the proof into three cases because it is straightforward to prove the result by considering zero, positive integers, and negative integers separately.

Case (i): When $n = 0$, because $0^2 = 0$, we see that $0^2 \geq 0$. It follows that $n^2 \geq n$ is true in this case.
Case (ii): When $n \geq 1$, when we multiply both sides of the inequality $n \geq 1$ by the positive integer n, we obtain $n \cdot n \geq n \cdot 1$. This implies that $n^2 \geq n$ for $n \geq 1$.
Case (iii): In this case $n \leq -1$. However, $n^2 \geq 0$. It follows that $n^2 \geq n$.

- → Because the inequality $n^2 \geq n$ holds in all three cases, we can conclude that if n is an integer, then $n^2 \geq n$.

Definition-Existence Proofs:

- → A theorem of this type is a proposition of the form $\exists x P(x)$, where P is a predicate. A proof of a proposition of the form $\exists x P(x)$ is called an **existence proof**. There are several ways to prove a theorem of this type.
- → Sometimes an existence proof of $\exists x P(x)$ can be given by finding an element a, called a **witness**, such that P(a) is true.
- → This type of existence proof is called **constructive**. It is also possible to give an existence proof that is **nonconstructive**; that is, we do not find an element a such that P(a) is true, but rather prove that $\exists x P(x)$ is true in some other way.

Example-27: Show that there is a positive integer that can be written as the sum of cubes of positive integers in two different ways.

Solution:

- → After considerable computation (such as a computer search) we find that $1729 = 10^3 + 9^3 = 12^3 + 1^3$.
- → Because we have displayed a positive integer that can be written as the sum of cubes in two different ways, we are done.

PROOF STRATEGIES

Definition-Forward proof: Whichever method you choose, you need a starting point for your proof. To begin a direct proof of a conditional statement, you start with the premises. Using these premises, together with axioms and known theorems, you can construct a proof using a sequence of steps that leads to the conclusion. This type of reasoning, called **forward reasoning**.

Definition-Backward proof: Forward reasoning is often difficult to use to prove more complicated results, because the reasoning needed to reach the desired conclusion may be far from obvious. In such cases it may be helpful to use **backward reasoning**.

Example-28: Given two positive real numbers x and y, their **arithmetic mean** is (x + y)/2 and their **geometric mean** is $\sqrt{xy}$. When we compare the arithmetic and geometric means of pairs of distinct positive real numbers, we find that the arithmetic mean is always greater than the geometric mean. [For example, when x = 4 and y = 6, we have$5 = (4 + 6)/2 > \sqrt{4.6} = \sqrt{24}$.] Can we prove that this inequality is always true?

Solution:

- → To prove that $\frac{(x+y)}{2} > \sqrt{xy}$ when x and y are distinct positive real numbers, we can work backward. We construct a sequence of equivalent inequalities. The equivalent inequalities are
- → $\frac{x+y}{2} > \sqrt{xy}$
- → $\frac{(x+y)^2}{4} > xy$
- → $(x+y)^2 > 4xy,$
- → $x^2 + 2xy + y^2 > 4xy,$
- → $x^2 - 2xy + y^2 > 0,$
- → $(x-y)^2 > 0.$

- → Because $(x-y)^2 > 0$ when $x \neq y$, it follows that the final inequality is true. Because all these inequalities are equivalent, it follows that $\frac{(x+y)}{2} > \sqrt{xy}$ when $x \neq y$. Once we have carried out this backward reasoning, we can easily reverse the steps to construct a proof using forward reasoning. We now give this proof. Suppose that x and y are distinct positive real numbers.
- → Then $(x-y)^2 > 0$ because the square of a nonzero real number is positive. Because $(x-y)^2 = x^2 - 2xy + y^2$, this implies that $x^2 - 2xy + y^2 > 0$.
- → Adding 4xy to both sides, we obtain $x^2 + 2xy + y^2 > 4xy$. Because $(x+y)^2 = x^2 + 2xy + y^2$, this means that $(x+y)^2 > 4xy$.
- → Dividing both sides of this equation by 4, we see that $\frac{(x+y)^2}{4} > xy$. Finally, taking square roots of both sides (which preserves the inequality because both sides are positive) yields $\frac{(x+y)}{2} > \sqrt{xy}$.
- → We conclude that if x and y are distinct positive real numbers, then their arithmetic mean (x + y)/2 is greater than their geometric mean $\sqrt{xy}$.

Exercise-9:

1. Use a direct proof to show that, "If x is an even integer then x^2 is an even integer" **[Winter 2021 – 22]**
2. If n is an even integer, then n 2 is odd. (True / False) **[Summer 2023 – 24]**
3. Prove that if n is an integer then $n^2 \geq n$. (Proof by Cases) **[Summer 2023 – 24]**
4. Prove that sum of two rational number is rational. **[Summer 2023 – 24] [Winter 2022-23]**
5. Prove that √6 is an irrational number. Is 6 − √6 a rational number? (Justify your answer). **[Summer/Winter 2023 – 24]**