CHAPTER 3

C.T. Linear Time Invarient (LTI) Systems

Syllabus :

Impulse response characterization and convolution integral for CT-LTI system, Signal responses to CT-LTI system, Properties of convolution, LTI system response properties from impulse response.

3.1 Impulse Response of LTI System :

Impulse response is the response of a relaxed LTI system to unit impulse δ(t). It is shown in Fig. 3.1.1.

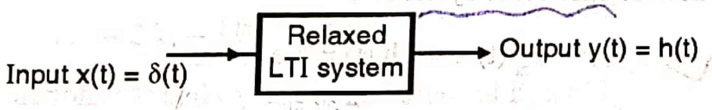

Fig. 3.1.1

LTI system is said to be initially relaxed system if zero input produced zero output. In case of certain system; if we apply zero input then output is not zero. Some value of output is obtained. Such systems are called as non-relaxed systems.

An impulse response is useful to evaluate the zero-state response of LTI system. Consider a first order system described by the difference equation.

$$y'(t) + a\,y(t) = x(t) \qquad ...(3.1.1)$$

Its characteristic equation is,

$$s + a = 0 \qquad ...(3.1.2)$$

We are assuming initially relaxed system. Thus x (t) = 0 and here order of system is 1. Thus power of s is 1.

This characteristic equation produces a natural response.

$$Y_{zi}(t) = k\,e^{-at} \quad \text{for } t \geq 0 \qquad ...(3.1.3)$$

Here k is constant.

To find impulse response directly; we have to solve the difference equation.

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$y'(t) + a\,y(t) = 0$, subject to zero initial conditions.

But an impulse response produces a sudden change in the state. For impulse response of first order system, note that $h(0) = 1$. This is because $\delta(t) = 1$ for $t = 0$. Thus finding impulse response of $y'(t) + a\,y(t) = x(t)$ is then equivalent to solving homogeneous equation $y'(t) + a\,y(t) = 0$, with $y(0) = 1$.

Similarly impulse response of second order system

$y''(t) + a_1 y'(t) + a_2 y(t) = x(t)$ can be calculated by solving the homogeneous equation,

$y''(t) + a_1 y'(t) + a_2 y(t) = 0$ with initial conditions $y(0) = 0$ and $y'(0) = 1$.

Solved problems :

Ex. 3.1.1 : Find impulse response of system described by the equation $2y'(t) + 3y(t) = x(t)$

Soln. : Given equation is the first order difference equation.

Its characteristics equation is,

$$2s + 3 = 0 \Rightarrow s + \frac{3}{2} = 0$$

$$\therefore \quad s = -\frac{3}{2} \text{ and } a = \frac{3}{2}$$

It represents the root of given system.

Now natural response for first order system is,

$h(0) = 1$

$$Y_{zi}(t) = \boxed{h(t) = k\,e^{-at}} \qquad \text{...(1)}$$

$$\therefore \quad h(t) = k\,e^{-3/2t} \qquad \text{...(2)}$$

But we know that $h(0) = 1$

Putting $t = 0$ in Equation (2) we get,

$h(0) = k\,e^0 \qquad \therefore \quad k = 1$

Putting this value in Equation (1)

$$h(t) = e^{-3/2t} \quad \text{for } t \geq 0$$

$$\therefore \quad h(t) = e^{-\frac{3}{2}t}\,u(t)$$

This is the impulse response of system.

Ex. 3.1.2 : Find impulse response of a system described by :

$y''(t) + 5y'(t) + 6y(t) = x(t)$

Soln. :

Given, $\quad y''(t) + 5y'(t) + 6y(t) = x(t)$

This is the second order difference equation.

Its characteristic equation is given by,

$$s^2 + 5s + 6 = 0 \quad \text{...(1)}$$

Thus roots are at,

$$s_1 = -2 \text{ and } s_2 = -3$$

$$\therefore \quad a_1 = 2 \text{ and } a_2 = 3$$

Now natural response is given by,

$$h(t) = k_1 e^{-a_1 t} + k_2 e^{-a_2 t}$$

$$\therefore \quad h(t) = k_1 e^{-2t} + k_2 e^{-3t} \quad \text{...(2)}$$

For second order system we have initial conditions,

$$h(0) = 0 \quad \text{and} \quad h'(0) = 1$$

Putting $t = 0$ in Equation (2) we get,

$$h(0) = k_1 + k_2$$

$$\therefore \quad k_1 + k_2 = 0 \quad \text{...(3)}$$

$h(0) = 0$

Differentiating Equation (2) with respect to t we get,

$$h'(t) = -2k_1 e^{-2t} - 3k_2 e^{-3t} \quad \text{...(4)}$$

$h'(0) = 1$

Putting $t = 0$ we get,

$$h'(t) = -2k_1 - 3k_2$$

$$\therefore \quad -2k_1 - 3k_2 = 1 \quad \text{...(5)}$$

We will solve Equations (4) and (5) to obtain values of k_1 and k_2.

From Equation (3) we get,

$$k_1 = -k_2$$

Putting this value in Equation (5),

$$+2k_2 - 3k_2 = 1 \qquad \therefore \quad k_2 = -1$$

$$\text{and} \quad k_1 = 1$$

Putting these values in Equation (2) we get,

$$h(t) = e^{-2t} - e^{-3t} \quad \text{for } t \geq 0$$

$$\therefore \quad h(t) = e^{-2t}\, u(t) - e^{-3t}\, u(t)$$

This is impulse response of given system.

3.2 Convolution Integral for CT LTI System :

Convolution is a mathematical operation which is used as a tool by communication engineers for system analysis, probability theory and transform calculations. Convolution can be performed in time as well as frequency domain.

The convolution of two functions x(t) and y (t) is defined as,

$$\text{Convolution : } x(t) * y(t) = \int_{-\infty}^{\infty} x(\tau) \cdot y(t-\tau)\, d\tau \qquad ...(3.2.1)$$

Equation (3.2.1) is called as convolution integral. Note that the independent variable is "t" which is same as the independent of the functions x (t) and y (t) which are being convolved. The integration is always performed with respect to a dummy variable such as τ and t is treated as a constant so far as the integration is concerned. The process of convolution involves following operations of y (τ) while the signal x (τ) remains unchanged :

1. Folding or time reversal to obtain y ($-\tau$).
2. Time shifting the folded signal y ($-\tau$) to obtain y ($t-\tau$).
3. Multiplication of x (τ) and y ($t-\tau$).
4. Integration of the product term x (τ) $\cdot$ y ($t-\tau$).

Proof : Consider an LTI continuous time system with

h (t) = Impulse response of the system
x (t) = Input signal
y (t) = Output signal

Writing the impulse response of the system in progressive form, i.e. Impulse response h (t) of an LTI system is defined to be response of system to unit impulse δ (t) [δ (t) $\rightarrow$ unit impulse at t = 0]

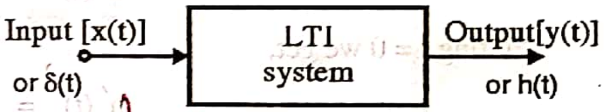


Fig. 3.2.1

By **time invarient property** of LTI system, response to impulse applied at any time "nT" [δ ($t-\tau$) is h ($t-\tau$)].

It indicates that if unit impulse is delayed by 'τ' then impulse response is also delayed by the same amount.

According to the definition of linear convolution,

$$x(t) * y(t) = \int_{-\infty}^{\infty} x(\tau) \cdot y(t-\tau)\, d\tau$$

In this case we can write,

$$x(t) * \delta(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

Here h ($t-\tau$) is response of LTI system to input δ (t).

$$\therefore\ Y(t) = x(t) * \delta(t) = \int_{-\infty}^{\infty} x(\tau) \cdot h(t-\tau)\, d\tau$$

3.2.1 Representation of C.T. Signal in terms of Impulses :

Consider a signal x (t) which will be first approximated as a set of pulse of duration T as shown in Fig. 3.2.2. The function can be then expressed as,

$$x(t) = \sum_{t=-\infty}^{\infty} x(tT)\, P_T(t-tT)$$

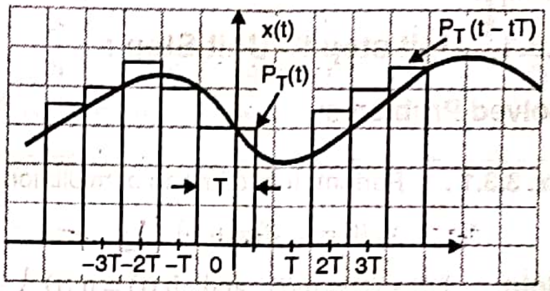


Fig. 3.2.2

Where $x(tT)$ = Is weight of signal, (t is an integer)

At every value of time; weight of signal is multiplied by pulse of duration T.

Note that, this approximation is valid even for small value of T.

The above equation can be written as,

$$\therefore \quad x(t) = \sum_{t=-\infty}^{\infty} x(tT)\, T\, [P_T(t-tT)/T]$$

$$\text{But} \quad \left[\lim_{T \to 0} P_T(t-tT)/T\right] \to \delta(t-tT)$$

∴ The signal x (t) can be expressed as :

$$\therefore \quad x(t) = \sum_{t=-\infty}^{\infty} x(tT)\, \delta(t-tT) \qquad \ldots(3.2.2)$$

Where T = Pulse width

f_s = Sampling frequency = 1/T

T is very small and f_s is high compared to the frequencies present in the signal.

3.2.2 Properties of Convolution Integral :

Some of the important properties of convolution integral are given below.

(a) Commutative property :

If $x_1(t)$ and $x_2(t)$ are continuous time signals then,

$$x_1(t) * x_2(t) = x_2(t) * x_1(t) \qquad \ldots(3.2.3)$$

(b) Distributive property :

$$x_1(t) * [x_2(t) + x_3(t)] = x_1(t) * x_2(t) + x_1(t) * x_3(t) \qquad \ldots(3.2.4)$$

Where $x_3(t)$ is another signal.

(c) Associative property :

This property states that,

$$x_1(t) * [x_2(t) * x_3(t)] = [x_1(t) * x_2(t)] * x_3(t) \qquad \ldots(3.2.5)$$

3.3 Computation of Convolution Integral using Graphical Method :

In this section, we will perform convolution integral of two signals with the help of some examples.

3.3.1 Unit step to Unit Step :

Solved Problems :

Ex. 3.3.1 : Perform the following convolution operation in time domain :

$x_1(t) = x_2(t) = u(t)$

Soln. : Let $x(t) = u(t)$ and $h(t) = u(t)$

According to the definition of convolution,

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)$$

Given $\quad x(t) = h(t) = u(t)$

$\therefore \quad x(\tau) = h(\tau) = u(\tau)$.

It is unit step as shown in Fig. P. 3.3.1(a).

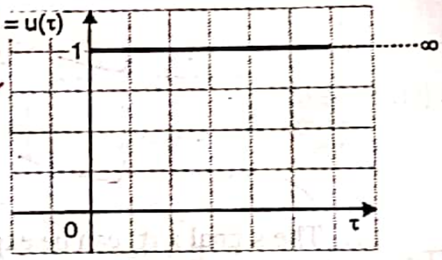


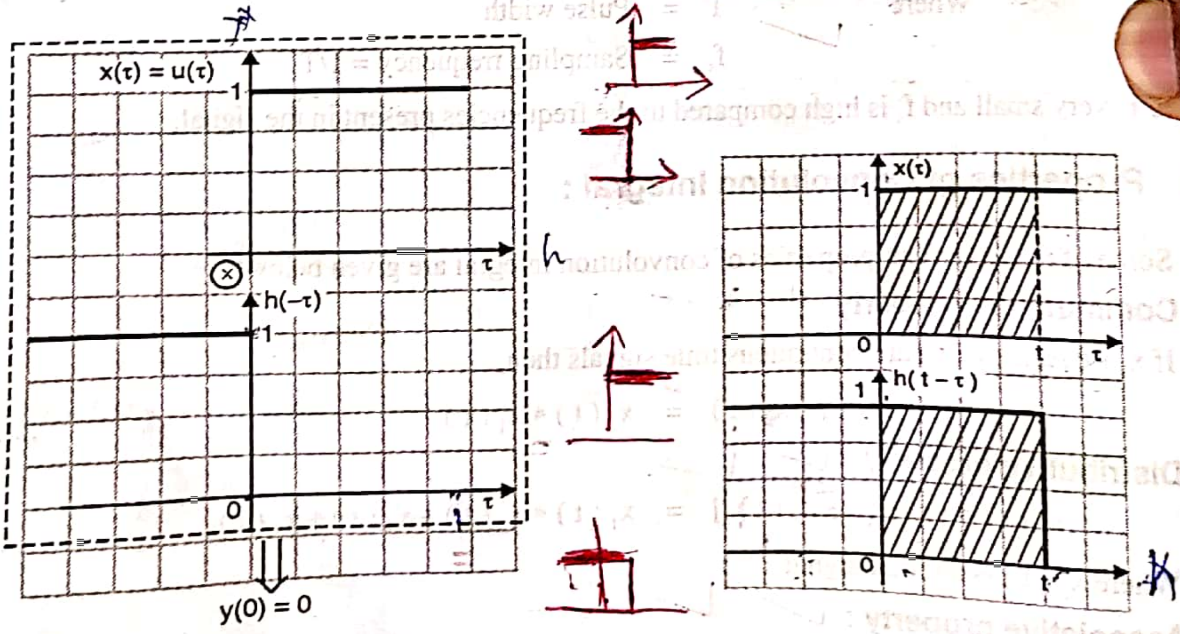
Fig. P. 3.3.1(a)

We will perform the convolution for different ranges.

Consider t = 0 : Thus putting $t = 0$ in Equation (1) we get,

$$y(0) = \int_{-\infty}^{\infty} x(\tau)\, h(-\tau) \qquad \ldots(2)$$

This operation is shown in Fig. P. 3.3.1(b).



(b) (c)

Fig. P. 3.3.1

There is no overlapping. $\therefore$ $y(0) = 0$ for $t = 0$.

Note : The overlapping should be for some range. Here overlapping is only at $t = 0$.

Consider $t > 0$:

As per Equation (1) we have the term $h(t - \tau)$. It is $h(-\tau + t)$. It indicates folding and delay operation. It is obtained by shifting $h(-\tau)$ towards right. This operation is shown in Fig. P. 3.3.1(c).

There is overlapping in the range '0' to 't'. And the value of $x(\tau)$ and $h(t - \tau)$ is 1.

Thus, Equation (1) becomes,

$$y(t) = \int_0^t 1\, d\tau = [\tau]_0^t \qquad \therefore y(t) = t \quad \text{for } t > 0.$$

Consider $t < 0$:

That means value of 't' is negative. Thus second term in Equation (1) becomes $h(-t - \tau)$. It indicates folding and advanced operation. It is obtained by shifting $h(-\tau)$ towards left by 't' as shown in Fig. P. 3.3.1(d).

There is no overlapping. Thus result of integration is zero.

Thus total convolution is, $y(t) = t$. for $t > 0$.

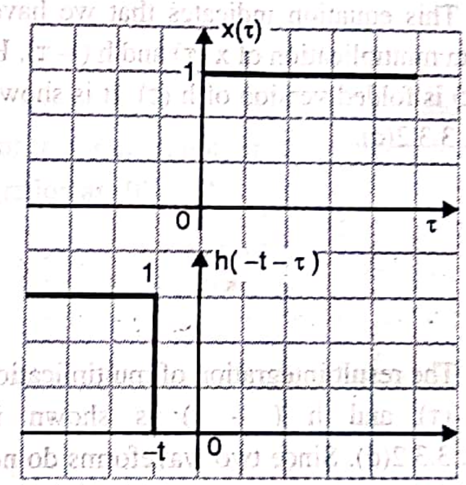


Fig. P. 3.3.1(d)

3.3.2 Unit Step to Exponential :

Ex. 3.3.2 : Perform the convolution operation between two functions in time domain.

$$x_1(t) = u(t) \text{ and } x_2(t) = e^{-t} \quad t \geq 0.$$

Soln. : We will denote the given functions using normal notations.

$$\text{Let } x_1(t) = h(t) = u(t)$$

$$\text{and } x_2(t) = x(t) = e^{-t} u(t).$$

We will obtain the result of convolution using graphical method.

We have the equation of convolution,

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t - \tau)\, d\tau \qquad \text{...(1)}$$

We have $x(t) = e^{-t} u(t)$ $\therefore$ $x(\tau) = e^{-\tau} u(\tau)$ and $h(t) = u(t)$

$\therefore$ $h(\tau) = u(\tau)$

The graph of $x(\tau)$ and $h(\tau)$ is shown in Figs. P. 3.3.2(a) and (b) respectively

First we will calculate y (t) at t = 0. Thus putting t = 0 in Equation (1) we get,

$$y(0) = \int_{-\infty}^{\infty} x(\tau)\, h(-\tau)\, d\tau \quad \dots (2)$$

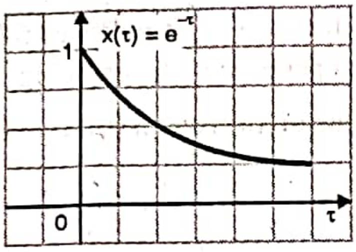


Fig. P. 3.3.2(a)

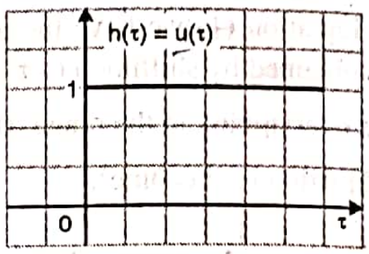


Fig. P. 3.3.2(b)

This equation indicates that we have to perform multiplication of x (τ) and h (– τ). Here h (– τ) is folded version of h (τ). It is shown in Fig. P. 3.3.2(c).

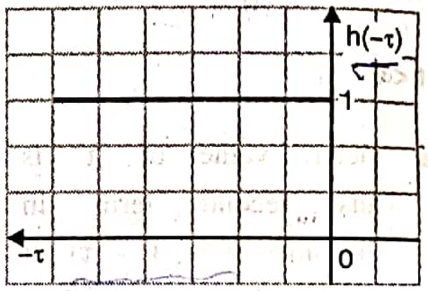


Fig. P. 3.3.2(c)

The result integration of multiplication of x (τ) and h (– τ) is shown in Fig. P. 3.3.2(d). Since two waveforms do not overlap; this result is zero.

$$\therefore \quad y(0) = 0 \quad \dots (3)$$

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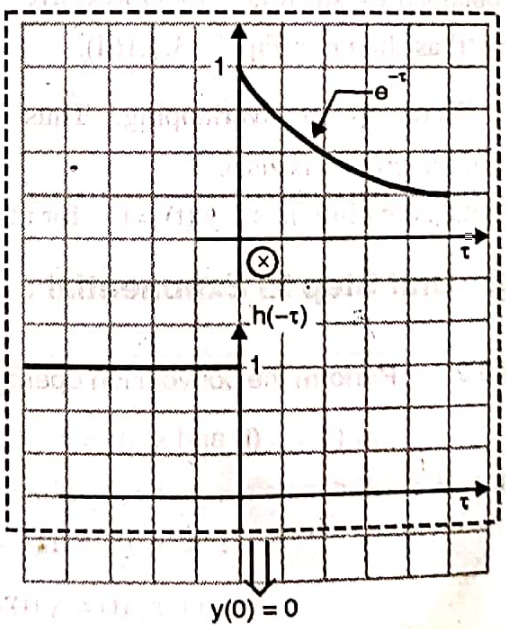


Fig. P. 3.3.2(d)

Now we will consider other possible ranges of 't'. First we will consider $t > 0$. Then Equation (1) becomes.

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau \qquad \text{...(4)}$$

Note that 't' is positive since it is greater than '0'. Consider the second term that is $h(t-\tau)$. It can be written as $h(-\tau+t)$

This indicates folded and delay operation. That mean we have to shift the signal $h(-\tau)$ towards right by 't'. This operation is shown in Fig. P. 3.3.2(e).

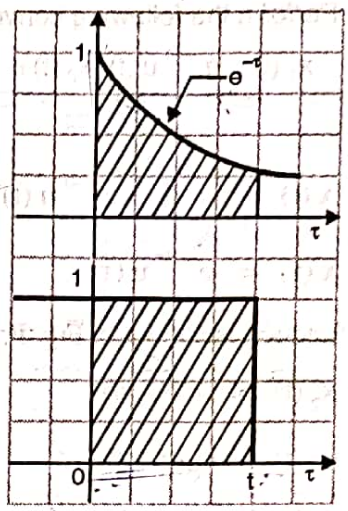


Fig. P. 3.3.2(e)

- Here the value (magnitude) of $h(t-\tau)$ is 1. There is overlapping of two signals in the range '0' to 't'. As per Equation (4) we will get the result of integration, only in the region where two signals overlap. Thus we will make the limits of integration as '0' to 't'.

$$\therefore\ y(t) = \int_0^t x(\tau)\cdot 1\, d\tau$$

$$\therefore\ y(t) = \int_0^t e^{-\tau}\, d\tau = \left[-1\, e^{-\tau}\right]_0^t = -\left[e^{-t} - e^{0}\right]$$

$$\therefore\ y(t) = 1 - e^{-t} \qquad \text{... for } t > 0 \qquad \text{...(5)}$$

Now consider the range $t < 0$. That means 't' is negative. Thus Equation (1) becomes,

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(-t-\tau)\, d\tau.$$

Consider the second term $h(-t-\tau)$. It can be written as $h(-\tau-t)$. It indicates folding and avance operation. It is obtained by shifting $h(-\tau)$ towards left. This operation is shown in Fig. P. 3.3.2(f).

There is no overlapping of signals. Thus result of convolution is zero.

$$\therefore\ y(t) = 0 \qquad \text{for } t < 0.$$

Thus output of convolution is,

$$y(t) = \begin{cases} 1 - e^{-t} & \text{for } t > 0 \\ 0 & \text{for } t \le 0 \end{cases}$$

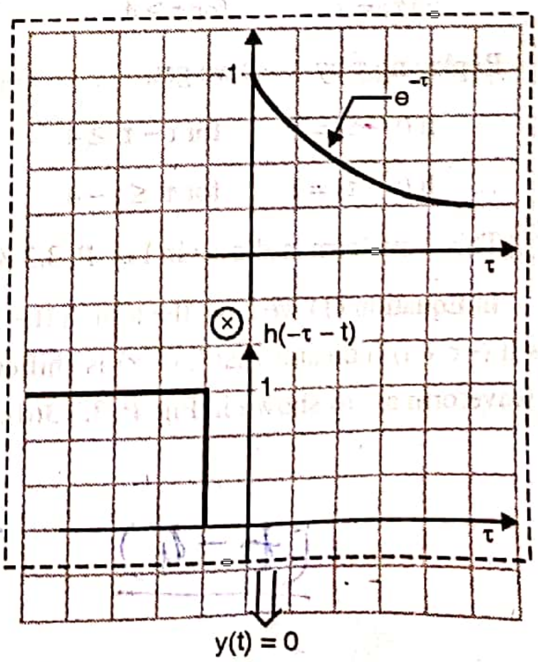


Fig. P. 3.3.2(f)

Ex. 3.3.3 : Perform the following convolution operation of two functions in time domain.

$$x_1(t) = e^{-4t}\, u(t),\; x_2(t) = u(t-4)$$

t-4

Soln. :

Let $\quad x(t) = x_1(t) = e^{-4t}\, u(t),$

$\therefore \quad x(\tau) = e^{-4\tau}\, u(\tau)$

This function is shown in Fig. P. 3.3.3(a).

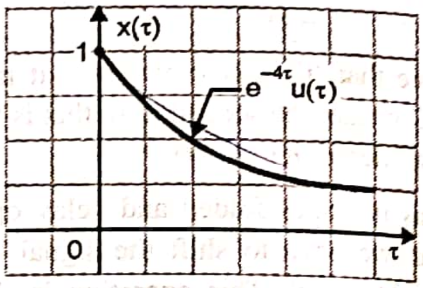


Fig. P. 3.3.3(a)

Here $x_2(t) = u(t-4)$

Let $\quad h(t) = u(t-4)$

$\therefore \quad h(\tau) = u(\tau - 4)$

It indicates unit step delayed by '4'. It is shown in Fig. P. 3.3.3(b).

$(\tau - 4)$

$\tau - (-4)$

We have equation of convolution,

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau \quad \text{...(1)}$$

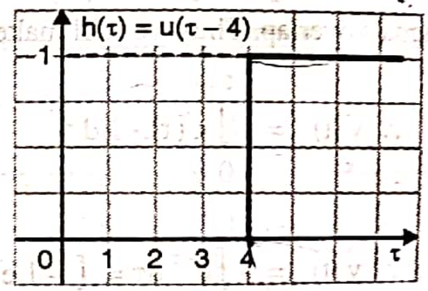


Fig. P. 3.3.3(b)

Given $\quad h(\tau) = u(\tau - 4) = 1 \quad \text{...(2)}$

for $\tau \geq 4$ as shown in Fig. P. 3.3.3(b).

But in Equation (1) we want the term $h(t-\tau)$.

From Equation (2) we have,

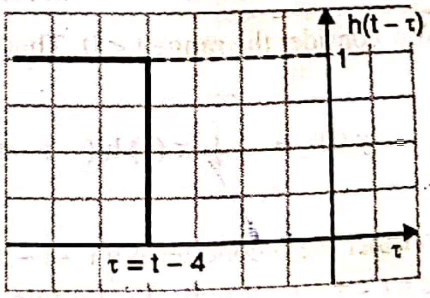
$h(\tau) = 1 \qquad \text{for } \tau \geq 4$

Replacing τ by $t - \tau$ we get,

h(t)

$h(t-\tau) = 1 \qquad \text{for } t - \tau \geq 4$

$\therefore \quad h(t-\tau) = 1 \qquad \text{for } \tau \leq t - 4$



This waveform is shown in Fig. P. 3.3.3(c).

Fig. P. 3.3.3(c)

In Equation (1) we have the term $h(t-\tau)$. It is same as $h(-\tau + t)$. Here $h(-\tau)$ is folded unit step. Thus $h(-\tau + t)$ indicates that $h(-\tau)$ is shifted towards right by 't'. For any arbitrary value of 't' ; the two waveform are as shown in Fig. P. 3.3.3(d).

$(t-4)$ $\quad \tau - 4 = \tau > 0$

$\tau - 4 =$

$\tau =$ $\quad \tau = T - 4$

As we go on increasing the value of 't' then the waveform h (t − τ) will shift towards right. But always there is overlap in the range $\tau = 0$ to $\tau = t - 4$.

Thus we can write,

$$y(t) = \int_0^{t-4} x(\tau)\, h(t-\tau)\, d\tau$$

$$= \int_0^{t-4} e^{-4\tau} \cdot 1\, d\tau = -\left[\frac{e^{-4\tau}}{4}\right]_0^{t-4}$$

$$= -\frac{1}{4}\left[\frac{e^{-4\tau}}{4}\right]_0^{t-4} = -\frac{1}{4}\left[e^{-4(t-4)} - e^0\right]$$

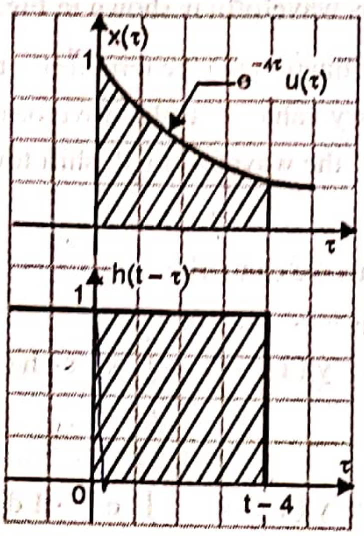


Fig. P. 3.3.3(d)

$$\therefore\ y(t) = \frac{1}{4} - \frac{1}{4}\, e^{-4(t-4)}$$

Ex. 3.3.4 : Evaluate continuous time (CT) convolution integral of $y(t) = e^{-2t}u(t) * u(t+2)$.

Soln. :

$$\text{Let } x(t) = e^{-2t}u(t)$$

$$\therefore\ x(\tau) = e^{-2\tau}u(\tau)$$

This function is shown in Fig. P. 3.3.4(a).

and let $h(t) = u(t+2)$

$$\therefore\ h(\tau) = u(\tau+2)$$

$t = 2 + \tau$

It indicates unit step advanced by 2 units. It is shown in Fig. P. 3.3.4(b). We have equation of convolution.

$$\therefore\ y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau \quad ...(1)$$

Here $h(\tau) = u(\tau+2) = 1$ for $\tau \geq -2$ as shown in Fig. P. 3.3.4(b).

But in Equation (1) we want the term $h(t-\tau)$

We have,

$$h(\tau) = 1 \quad \text{for } \tau \geq -2 \quad ...(2)$$

Replacing τ by $t - \tau$ we get,

$$h(t-\tau) = 1 \quad \text{for } t-\tau \geq -2$$

$$\therefore\ h(t-\tau) = 1 \quad \text{for } \tau \leq t+2 \quad ...(3)$$

This waveform is shown in Fig. P. 3.3.4(c).

In Equation (1) the term $h(t-\tau)$ indicates that $h(-\tau)$ is shifted towards right by 't' positions. For any arbitrary value of 't' the waveform will overlap as shown in Fig. P. 3.3.4(d). As we go on increasing value of 't' the waveform will shift towards right. But always there will be overlap in the range $\tau = 0$ to $t+2$.

Thus we can write,

$$y(t) = \int_0^{t+2} x(\tau)\, h(t-\tau)\, d\tau$$

$$\therefore\ y(t) = \int_0^{t+2} e^{-2\tau}\cdot 1\, d\tau = -\left[\frac{e^{-2\tau}}{2}\right]_0^{t+2} = -\frac{1}{2}\left[e^{-2\tau}\right]_0^{t+2} = -\frac{1}{2}\left[e^{-2t-4} - e^0\right]$$

$$\therefore\ y(t) = -\frac{1}{2}\left[e^{-2t}\cdot e^{-4} - e^0\right]$$

$$\therefore\ y(t) = \frac{1}{2} - \frac{1}{2} e^{-4}\cdot e^{-2t}$$

...Ans.

$\tau + 2 = t$

$\tau =$

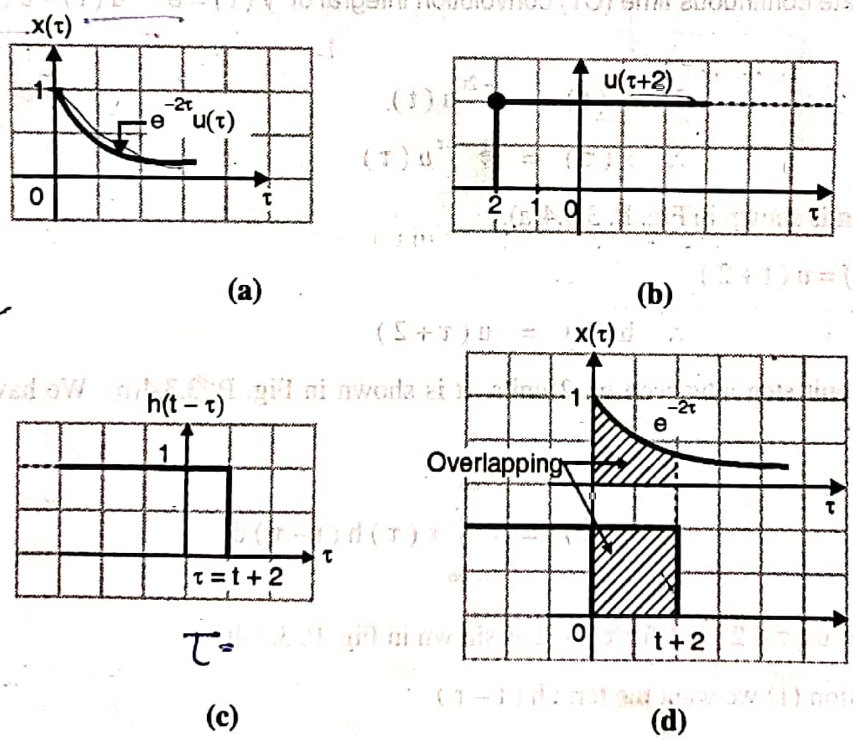


Fig. P. 3.3.4

3.3.3 Exponential to Exponential :

Ex. 3.3.5 : For an LTI system with unit impulse response $h(t) = e^{-2t}u(t)$, determine output to the input $x(t) = e^{-t}u(t)$.

Soln. :

We have, $y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$

Given : $x(t) = e^{-t} u(t) \quad \therefore x(\tau) = e^{-\tau} u(\tau)$

and $h(t) = e^{-2t} u(t) \quad \therefore h(\tau) = e^{-2\tau} u(\tau)$

These signals are as shown in Fig. P. 3.3.5(a) and Fig. P. 3.3.5(b) respectively.

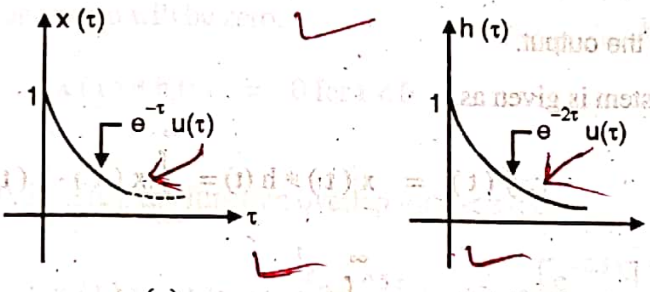


Fig. P. 3.3.5

Now $h(t-\tau) = e^{-2(t-\tau)} u(t-\tau) = e^{-2(-\tau+t)} u(-\tau+t)$

The term $u(-\tau + t)$ represents folded unit step $u(-\tau)$ and it is existing between $\tau = -\infty$ to $\tau = t$

It is shown in Fig. P. 3.3.5(c)

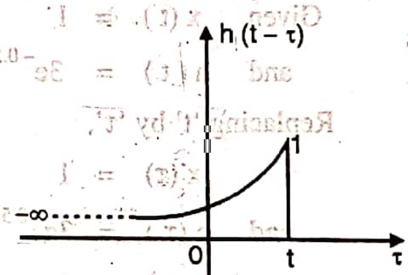


Fig. P. 3.3.5(c)

Case I : When t < 0

For t < 0; Fig. P. 3.3.5(a) and Fig. P. 3.3.5(c) will not overlap

$$\therefore y(t) = 0$$

Case II : When t > 0

This condition is shown in Fig. P. 3.3.5(d)

Here overlapping is in the range 0 to 't'.

$$\therefore y(t) = \int_0^t e^{-\tau} u(\tau) \cdot e^{-2(-\tau+t)} u(t-\tau) \cdot d\tau$$

Amplitude of $u(\tau)$ and $u(t-\tau)$ are 1.

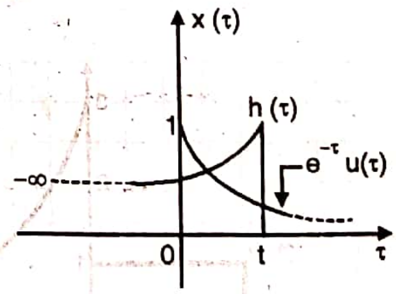


Fig. P. 3.3.5(d)

$$\therefore y(t) = \int_0^t e^{-\tau} e^{-2(-\tau+t)} d\tau$$

$$\therefore y(t) = \int_0^t e^{-\tau} \cdot e^{+2\tau} \cdot e^{-2t} d\tau = e^{-2t} \int_0^t e^{\tau} d\tau = e^{-2t} [e^{\tau}]_0^t$$

$$\therefore y(t) = e^{-2t} [e^{t} - 1]$$

3.3.4 Exponential to Rectangular :

Ex. 3.3.6 : Impulse response of an LTI system is given by :

$$h(t) = \begin{cases} 3e^{-0.5t} & ; \quad t \geq 0 \\ 0 & ; \quad \text{Otherwise} \end{cases}$$

Find the system output due to the input :

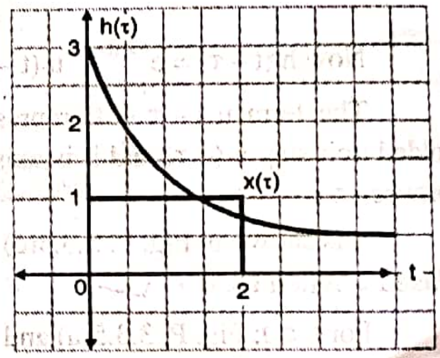
$$x(t) = \begin{cases} 1 & ; \quad 0 \leq t \leq 2 \\ 0 & ; \quad \text{Otherwise} \end{cases}$$

Also sketch the output.

Soln. : Output of LTI system is given as,

$$\therefore y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau) \cdot h(t-\tau)\, d\tau$$

$$\text{OR} \quad y(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$$



(a)

Given, $x(t) = 1$ for $0 \leq t \leq 2$

and $h(t) = 3e^{-0.5t}$ for $t \geq 0$

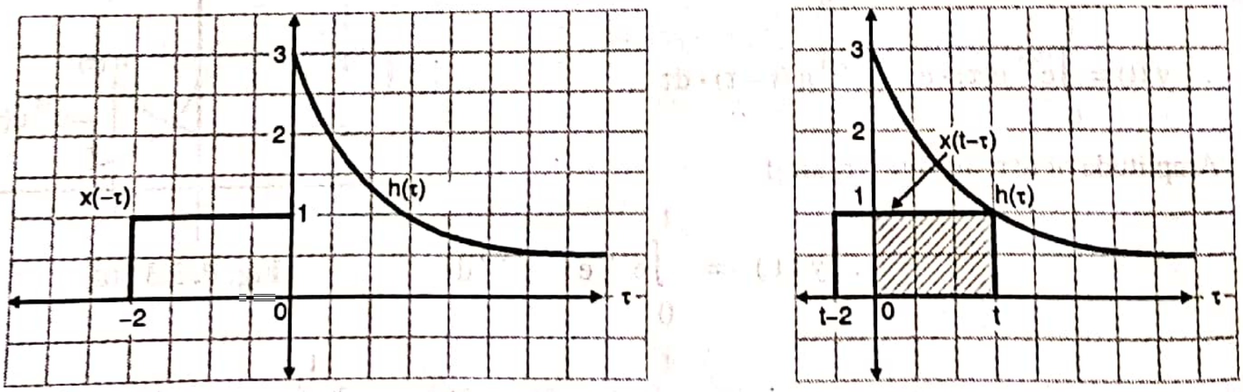
Replacing 't' by 'τ',

$x(\tau) = 1$ for $0 \leq t \leq 2$

and $h(\tau) = 3e^{-0.5\tau}$ for $\tau \geq 0$

$x(t-\tau) = 1$ for $0 \leq t-\tau \leq 2$ i.e. $t-2 \leq \tau$

(a) Signal $x(-\tau)$ and $h(\tau)$:



(b) (c)

Fig. P. 3.3.6

(b) $h(\tau)$ remains unchanged but $x(\tau)$ is folded to obtain $x(-\tau)$

(c) $h(\tau)$ remains unchanged, but $x(-\tau)$ is shifted i.e. $x(t-\tau)$.

(d) $x(-\tau)$ keeps shifting.

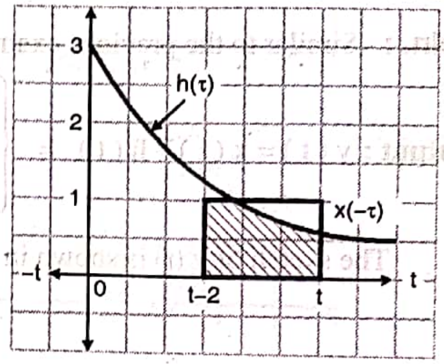


Fig. P. 3.3.6(d)

For $t < 0$:

As seen from Fig. P. 3.3.6(c), the function do not overlap for $t < 0$ hence their product, hence their product will be zero and the integration also will be zero.

$$\therefore \quad x(t) * h(t) = 0 \text{ for } t < 0$$

For $0 < \tau < t$,

As seen from Fig. P. 3.3.6(d), the function overlap for $0 < \tau < t$ so,

$$\therefore x(t) * h(t) = \int_0^t 3e^{-0.5\tau} \cdot 1 \cdot d\tau = 3\left[\frac{e^{-0.5\tau}}{-0.5}\right]_0^t$$

$$= -3 \times 2\,[e^{-0.5t} - e^0]$$

$$= 6\,[1 - e^{-0.5t}]$$

For $t > 2$:

From Fig. P. 3.3.6(b), the function overlap for $(t-2) < \tau < t$,

$$\therefore \quad x(t) * h(t) = \int_{t-2}^{t} 3e^{-0.5\tau} \cdot d\tau$$

$$= 3\left[\frac{e^{-0.5\tau}}{-0.5}\right]_{t-2}^{t} = -3 \times 2\,[e^{-0.5t} - e^{-0.5(t-2)}]$$

$$= -6\,[e^{-0.5t} - e^{-0.5t} \cdot e^1]$$

$$= -6\,e^{-0.5t}[1 - 2.716]$$

$$\therefore \quad y(t) = x(t) * h(t) = 10.296\, e^{-0.5t}$$

Ex. 3.3.7 : Impulse response of an LTI system is given by :

$$h(t) = \begin{cases} e^{-2t} & ; \ t \geq 0 \\ 0 & ; \ \text{otherwise} \end{cases}$$

Find the system output due to the input :

$$x(t) = \begin{cases} A & ; \ 0 \leq t \leq 2 \\ 0 & ; \ \text{otherwise} \end{cases}$$

Also sketch the output.

Soln. : Similar to the previous example.

Output : $y(t) = x(t) * h(t) = \begin{cases} 0 & \text{for } t < 0 \\ \frac{A}{2}[1 - e^{-2t}] & \text{for } 0 < t < 2 \\ 26.799\, e^{-2t} & \text{for } t > 2 \end{cases}$

The sketch of y (t) is shown in Fig. P. 3.3.7

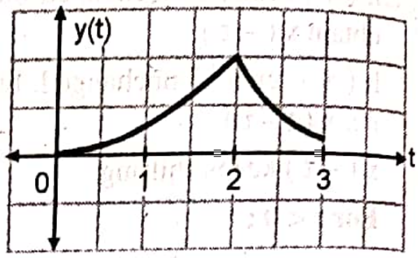


Fig. P. 3.3.7

3.3.5 Unit Step to Rectangular :

Ex. 3.3.8 : Obtain the convolution of :

$x(t) = u(t)$ and $h(t) = 1$ for $-1 \le t \le 1$.

Soln. :

Given : $x(t) = u(t) \quad \therefore x(\tau) = u(\tau)$

and $h(t) = 1$ for $-1 \le t \le 1$

$\therefore h(\tau) = 1$ for $-1 \le \tau \le 1$

According to the definition of convolution.

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$$

Here $x(t-\tau) = u(t-\tau) = u(-\tau + t)$

These two signals are shown in Fig. P. 3.3.8(a) and Fig. P. 3.3.8(b) respectively.

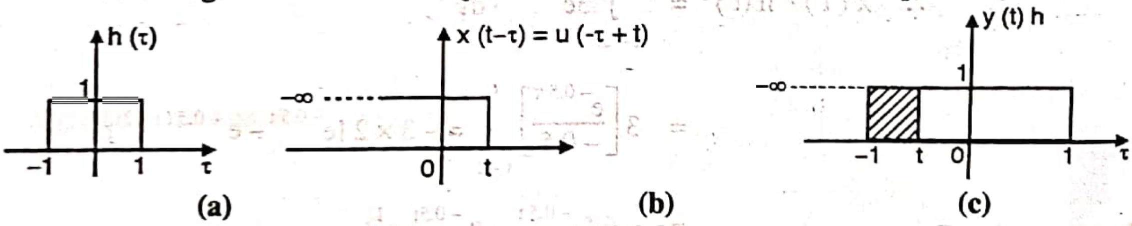


Fig. P. 3.3.8

Case - I : When $-1 < t < 0$:

This condition is shown in Fig. P. 3.3.8(c)

Here overlapping is in the range – 1 to t.

$$\therefore y(t) = \int_{-1}^{t} u(\tau) \cdot 1\, d\tau = \int_{-1}^{t} d\tau = [\tau]_{-1}^{t}$$

$$\therefore y(t) = t + 1$$

Case - II : When $0 < t < 1$:

It is shown in Fig. P. 3.3.8(d).

Here again overlapping is in the same range – 1 to t.

$$\therefore y(t) = t + 1$$

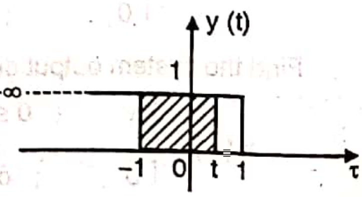


Fig. P. 3.3.8(d)

Case - III : When t > 1 :

It is shown in Fig. P. 3.3.8(e).

For any value of t which is greater than 1 there is always overlapping from – 1 to + 1.

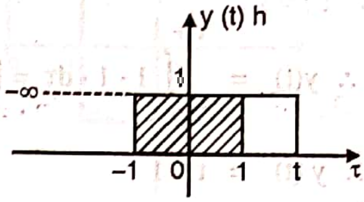


Fig. P. 3.3.8(e)

$$\therefore\ y(t) = \int_{-1}^{1} 1 \cdot d\tau = [\tau]_{-1}^{1}$$

$$\therefore\ y(t) = 2$$

3.3.6 Rectangular to Rectangular :

Ex. 3.3.9 : Obtain convolution integral of :

$x(t) = 1$ for $-1 \le t \le 1$

$h(t) = 1$ for $0 \le t \le 2$

Soln. : We can write,

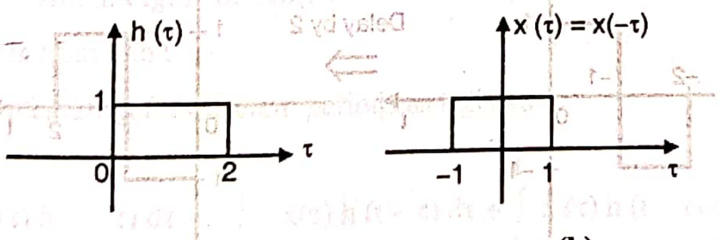
$$x(\tau) = 1 \quad \text{for } -1 \le \tau \le 1$$

$$\text{and } h(\tau) = 1 \quad \text{for } 0 \le \tau \le 2$$

According to equation of convolution,

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$$

The signals $h(\tau)$ and $x(-\tau)$ are shown in Fig. P. 3.3.9(a) and Fig. P. 3.3.9(b) respectively.



(a) (b)

Fig. P. 3.3.9

Case - I : When t < – 1

This condition is shown in Fig. P. 3.3.9(c).

There is no overlapping so convolution is zero.

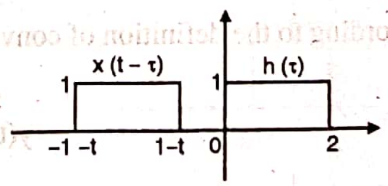


Fig. P. 3.3.9(c)

Case - II : When 0 < t < 2

This condition is shown in Fig. P. 3.3.9(d).

The overlapping is in the range 0 to t + 1

$$y(t) = \int_{0}^{t+1} x(\tau)\, h(t-\tau)\, d\tau$$

$$\therefore\ y(t) = \int_{0}^{t+1} 1 \cdot 1 \cdot d\tau = [\tau]_{0}^{t+1}$$

$$\therefore\ y(t) = t+1$$

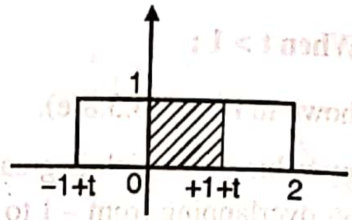


Fig. P. 3.3.9(d)

Case - II : When t > 2

This condition is shown in Fig. P. 3.3.9(e).

Overlapping is in the range $-1+t$ to 2.

$$\therefore\ y(t) = \int_{-1+t}^{2} 1 \cdot 1 \cdot d\tau = [\tau]_{-1+t}^{2} = 2+1-t = 3-t$$

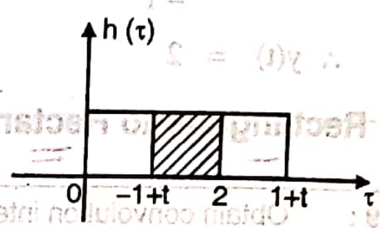


Fig. P. 3.3.9(e)

Ex. 3.3.10 : Evaluate the convolution integral $x(t) * x(2-t)$ where $x(t)$ is shown in Fig. P. 3.3.10(a).

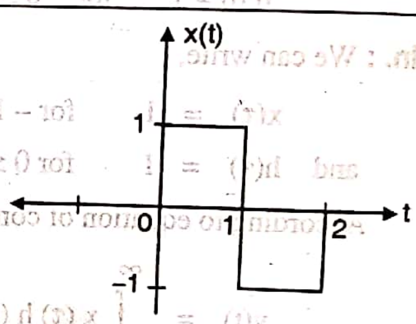


Fig. P. 3.3.10(a)

Soln. :

Let $h(t) = x(2-t) = x(-t+2)$

It is shown in Fig. P. 3.3.10(b).

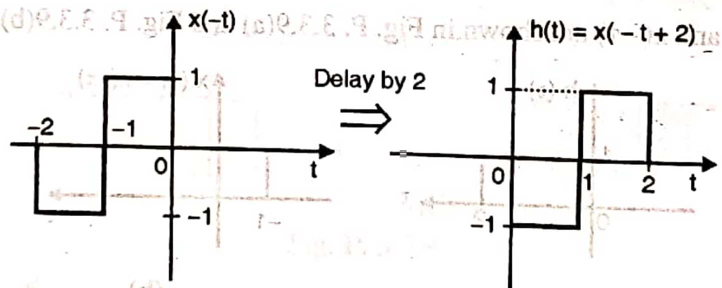


Fig. P. 3.3.10(b)

According to the definition of convolution,

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

The signals $x(\tau)$ and $h(-\tau)$ are shown in Fig. P. 3.3.10(c) and P. 3.3.10(d) respectively.

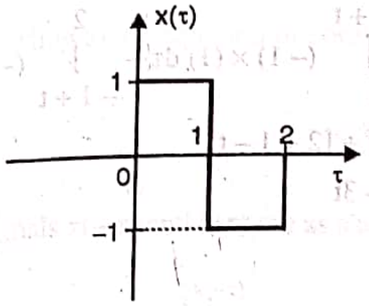


Fig. P. 3.3.10(c)

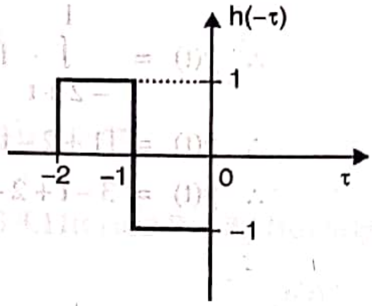


Fig. P. 3.3.10(d)

We will consider following cases for convolution.

Case 1 : When t < 0 :

Here h (– τ) will shift towards left. There will not be any overlapping. Thus y (t) = 0.

Case 2 : When 0 < t < 1 :

This condition is shown in Fig. P. 3.3.10(e).

Here overlapping is from 0 to t

$$\therefore \; y(t) = \int_0^t x(\tau)\, h(t-\tau)\, d\tau$$

$$\therefore \; y(t) = \int_0^t 1 \cdot (-1) \cdot d\tau = -t$$

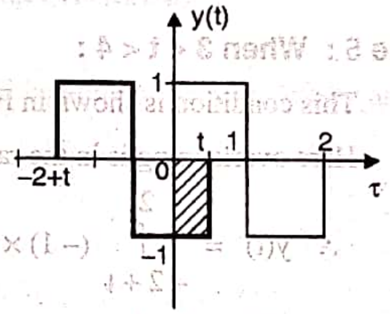


Fig. P. 3.3.10(e)

Case 3 : When 1 < t < 2 :

This condition is shown in Fig. P. 3.3.10(f).

Here overlapping is from 0 to t

We will perform integration for different periods as follows :

$$y(t) = \int_0^{t-1} x(\tau)\, h(t-\tau)\, d\tau + \int_{t-1}^{1} x(\tau)\, h(t-\tau)\, d\tau + \int_1^t x(\tau)\, h(t-\tau)\, d\tau$$

$$\therefore \; y(t) = \int_0^{t-1} 1 \times (1)\, d\tau + \int_{t-1}^{1} 1 \times (-1)\, d\tau + \int_1^t (-1) \times (-1)\, d\tau$$

$$\therefore \; y(t) = -(t-1) - [1-t+1] + [t-1]$$

$$\therefore \; y(t) = -t+1-2+t+t-1$$

$$\therefore \; t(t) = t-2$$

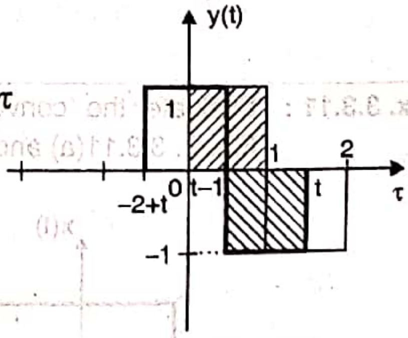


Fig. P. 3.3.10(f)

Case 4 : When 2 < t < 3 :

This condition is shown in Fig. P. 3.3.10(g). Here overlapping is from – 2 + t to 2.

$$\therefore\ y(t) = \int_{-2+t}^{1} 1 \times (1)\, d\tau + \int_{1}^{-1+t} (-1) \times (1)\, d\tau + \int_{-1+t}^{2} (-1) \times (-1)\, d\tau$$

$$\therefore\ y(t) = [1+2-t] - [-1+t-1] + [2+1-t]$$

$$\therefore\ y(t) = 3-t+2-t+3-t = 8-3t$$

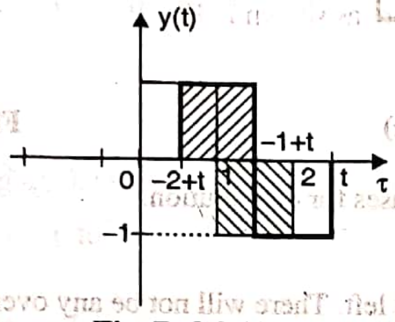


Fig. P. 3.3.10(g)

Case 5 : When 3 < t < 4 :

This condition is shown in Fig. P. 3.3.10(h).

Here overlapping is in the range $-2+t$ to 2.

$$\therefore\ y(t) = \int_{-2+t}^{2} (-1) \times (1)\, d\tau = -[2+2-t]$$

$$\therefore\ y(t) = t-4$$

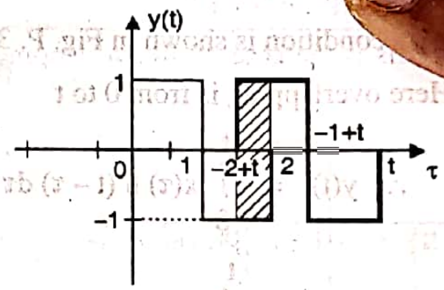


Fig. P. 3.3.10(h)

Case 6 : When t > 4 :

There will not be any overlapping

$$\therefore\ y(t) = 0$$

The result of convolution is as follows.

$$y(t) = \begin{cases} 0 & \text{For } t < 0 \\ -t & \text{For } 0 < t < 1 \\ t-2 & \text{For } 1 < t < 2 \\ 8-3t & \text{For } 2 < t < 3 \\ t-4 & \text{For } 3 < t < 4 \\ 0 & \text{For } t > 4 \end{cases}$$

Ex. 3.3.11 : Evaluate the convolution integral for input x(t) and impulse response h(t) shown in Fig. P. 3.3.11(a) and Fig. P. 3.3.11(b).

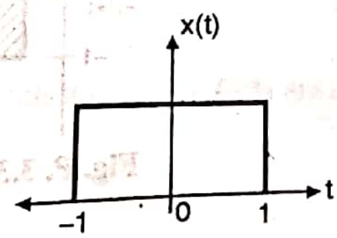


Fig. P. 3.3.11(a)

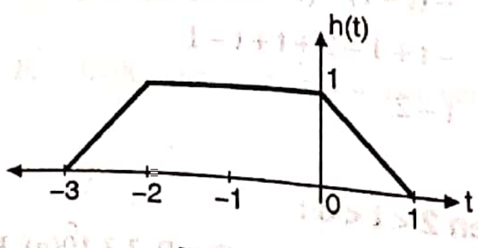


Fig. P. 3.3.11(b)

Soln. : According to the equation of convolution integral,

$$u(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau \qquad ....(1)$$

The signals $x(-\tau)$ and $h(\tau)$ are as shown in Fig. P. 3.3.11(c) and P. 3.3.11(d) respectively.

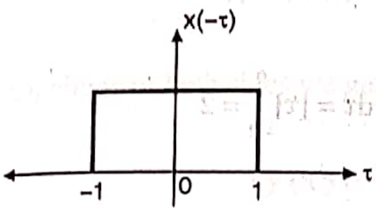


Fig. P. 3.3.11(c)

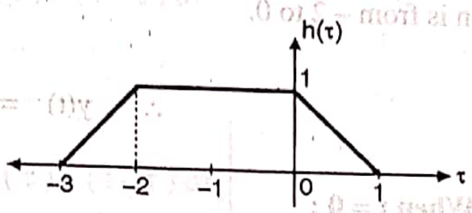


Fig. P. 3.3.11(d)

Case 1 : When $t < -4$:

This is the case of no overlapping. Because signal $x(-\tau)$ will be shifted towards, left by amount $t < -4$.

$$\therefore \quad y(t) = 0$$

Case 2 : $-2 < t < 0$:

This conditions is shown in Fig. P. 3.3.11(e).

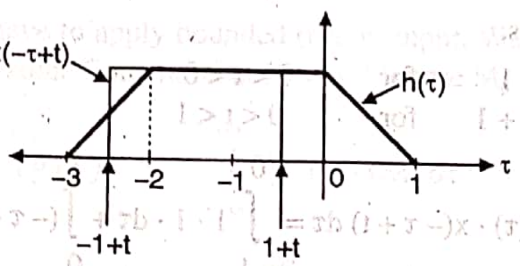


Fig. P. 3.3.11(e)

Here overlapping is from $-1+t$ to $1+t$.

The equation of $h(\tau)$ for this range is,

$$h(\tau) = \begin{cases} \tau + 3 & \text{for} \quad (-1+t) < \tau < -2 \\ 1 & \text{for} \quad -2 < \tau < 1+t \end{cases}$$

$$\therefore y(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$$

$$= \int_{-1+t}^{-2} (\tau+3)\cdot 1\, d\tau + \int_{-2}^{1+t} 1 \cdot 1 \cdot d\tau = \left[\frac{\tau^2}{2} + 3\tau\right]_{-1+t}^{2} + [\tau]_{-2}^{1+t}$$

$$= \left[\left(\frac{4}{2}+6\right) - \left(\frac{(-1+t)^2}{2} + 3(-1+t)\right)\right] + [1+t+2]$$

$$= 8-\left[\frac{(-1+t)^2}{2}-3+3t\right]+3+t$$

$$\therefore\ y(t) = 8-\frac{(-1+t)^2}{2}+3-3t+3+t = 14-2t-\frac{(-1+t)^2}{2}$$

Case 3 : t = – 1 :

When $t=-1$ then $x(-\tau+t) = x(-\tau-1)$ and this is the case of full overlapping. So range of integration is from – 2 to 0.

$$\therefore\ y(t) = \int_{-2}^{0} 1\cdot 1\cdot d\tau = [\tau]_{-2}^{0} = 2$$

Case 4 : When t = 0 :

This condition is shown in Fig. P. 3.3.11(f).

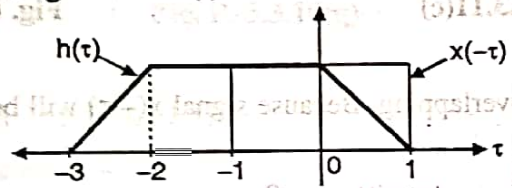


Fig. P. 3.3.11(f)

Here overlapping is from – 1 to + 1.

Now equation of h(τ) is,

$$h(\tau) = \begin{cases} 1 & \text{for } -1<t<0 \\ -\tau+1 & \text{for } 0<t<1 \end{cases}$$

$$\therefore\ y(t) = \int_{-\infty}^{\infty} h(\tau)\cdot x(-\tau+t)\, d\tau = \int_{-1}^{0} 1\cdot 1\cdot d\tau + \int_{0}^{1}(-\tau+1)\cdot 1\, d\tau$$

$$= [\tau]_{-1}^{0} - \left[\frac{\tau^2}{2}-\tau\right]_{0}^{1} = 1-\left[\left(\frac{1}{2}+1\right)-(0)\right] = \frac{3}{2}$$

Case 5 : When 1 < t < 0 :

This condition is shown in Fig. P. 3.3.11(g).

Here overlapping is from – 1 + t to 1

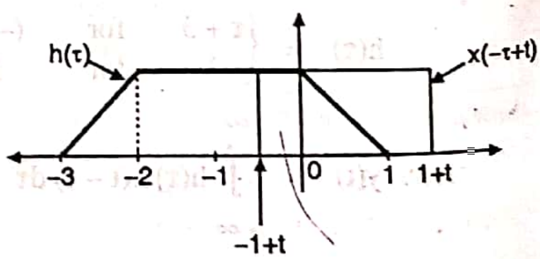


Fig. P. 3.3.11(g)

$$\therefore\ y(t) = \int_{-1+t}^{0} 1\cdot 1\, d\tau + \int_{0}^{1} 1\cdot(-\tau+1)\, d\tau$$

$$= [\tau]_{-1+t}^{0} + \left[\frac{-\tau^2}{2}+\tau\right]_{0}^{1} = t-1+\frac{1}{2}$$

$$\therefore\ y(t) = t-\frac{1}{2}$$

3.4 Stability for CT Systems :

LTI system is BIBO stable if bounded input produces bounded output. We have equation of convolution integral,

$$y(t) = x(t) * h(t)$$

$$\therefore \quad y(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau \qquad \ldots(3.4.1)$$

Taking magnitude of both sides we get,

$$|y(t)| = \left| \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau \right| \qquad \ldots(3.4.2)$$

If we take magnitude sign inside integration then equality sign becomes ≤.

$$\therefore \quad |y(t)| \le \int_{-\infty}^{\infty} |h(\tau)\, x(t-\tau)\, d\tau|$$

$$\therefore \quad |y(t)| \le \int_{-\infty}^{\infty} |h(\tau)|\, |x(t-\tau)\, d\tau| \qquad \ldots(3.4.3)$$

To check the stability we have to apply bounded (finite) input. We will assume that the value of input x (t) is less than or equal to some finite number M_x. In this case $M_x < \infty$. Thus we can write,

$$|y(t)| \le \int_{-\infty}^{\infty} |h(\tau)|\, M_x\, d\tau$$

$$\therefore \quad |y(t)| \le M_x \int_{-\infty}^{\infty} |h(\tau)|\, d\tau \qquad \ldots(3.4.4)$$

We know that M_x is finite number. Now to obtain bounded (finite) output; integration term in Equation (3.4.3) should be finite.

$$\therefore \quad \int_{-\infty}^{\infty} |h(\tau)|\, d\tau < \infty \qquad \ldots(3.4.5)$$

This is the condition for stability of LTI system. So LTI system is BIBO stable if its impulse response is absolutely summable.

3.5 LTI System Properties from Impulse Response (Step Response) :

We have the equation of convolution,

$$y(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau \qquad \ldots(3.5.1)$$

Again input applied is unit step.

$$\text{Thus } x(t) = u(t) = 1 \text{ for } t \geq 0.$$

$$\therefore \quad x(t-\tau) = u(t-\tau) = 1 \text{ for } t \geq \tau \text{ that means } \tau \leq t.$$

Putting this value and modifying the limits of Equation (3.4.6).

$$y(t) = \int_{-\infty}^{t} h(\tau) \cdot 1\, d\tau$$

$$\therefore \quad y(t) = \int_{-\infty}^{t} h(\tau)\, d\tau \qquad \ldots(3.5.2)$$

This is the step response for continuous time system.

Ex. 3.5.1 : Impulse response of the RC circuit is given as $h(t) = \frac{1}{RC} e^{-t/RC} u(t)$

Find the unit step response of the circuit.

Soln. :

Impulse response is the response of circuit when applied input is unit impulse $\delta(t)$, And unit step response means output of system when applied input is $u(t)$.

Now step response is related to impulse response as,

$$s(t) = \int_{\infty}^{t} h(\tau)\, d\tau$$

$$\text{Give, } h(t) = \frac{1}{RC} e^{-t/RC} u(t)$$

$$\therefore \quad h(\tau) = \frac{1}{RC} e^{-\tau/RC} u(\tau)$$

Due to multiplication of $u(\tau)$ the limits of integration will be from 0 to t

$$= \int_{0}^{t} \frac{1}{RC} e^{-\tau/RC}\, d\tau$$

$$= \frac{1}{RC} \left[\frac{e^{-\tau/RC}}{-\frac{1}{RC}} \right]_0^t = -[e^{-t/RC} - e^{o}]$$

$$\therefore \quad s(t) = 1 - e^{-t/RC}$$

Ex. 3.5.2 : Find the step response of the given LTI system :

$h(t) = t\, x(t+1)\, u(t)$

Soln. :

Given : $h(t) = t\,x(t+1)\,u(t)$

Now step response is output of system when applied input is unit step that means $x(t) = u(t)$.

$$\therefore \quad x(t+1) = u(t+1)$$

$$\therefore \quad h(t) = t\,u(t+1)\,u(t)$$

Now step response is given by,

$$s(t) = \int_{-\infty}^{t} h(\tau)\,d\tau$$

$$\therefore \quad s(t) = \int_{-\infty}^{t} \tau\,u(\tau+1)\cdot u(\tau)\,d\tau$$

The multiplication of $u(\tau+1)\cdot u(\tau)$ gives the range, $\tau = -1$ to ∞.

$$\therefore \quad s(t) = \int_{-1}^{t} \tau\,d\tau = \left[\frac{\tau^2}{2}\right]_{-1}^{t}$$

$$\therefore \quad s(t) = \frac{t^2}{2} - \frac{(-1)^2}{2}$$

$$\therefore \quad s(t) = \frac{t^2-1}{2}$$

Ex. 3.5.3 : Test the stability of the LTI systems, whose impulse responses are given below :

1. $h(t) = e^{-5|t|}$ 2. $h(t) = e^{4t}\,u(t)$.

Soln. :

1. According to the condition of stability,

$$\int_{-\infty}^{\infty} |h(t)|\,dt < \infty$$

Given, $h(t) = e^{-5|t|}$

$$\therefore \int_{-\infty}^{\infty} |h(t)|\,dt = \int_{-\infty}^{\infty} e^{-5|t|}\,dt$$

$$= \int_{-\infty}^{0} e^{5t}\,dt + \int_{0}^{\infty} e^{-5t}\,dt$$

$$= \left[\frac{1}{5}\cdot e^{5t}\right]_{-\infty}^{0} + \left[\frac{-1}{5}\cdot e^{-5t}\right]_{0}^{\infty} = \left[\frac{1}{5} - 0\right] + \left[0 + \frac{1}{5}\right]$$

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$$= \frac{2}{5} < \infty$$

Output is bounded; so system is stable.

2. **Given :** $h(t) = e^{4t}\, u(t)$

$$\int_{-\infty}^{\infty} |h(t)|\, dt = \int_{-\infty}^{\infty} |e^{4t}\, u(t)|\, dt$$

$$= \int_{0}^{\infty} e^{4t}\, dt = \left[\frac{e^{4t}}{4}\right]_0^{\infty} = \infty$$

Since output is not bounded; system is unstable.

Ex. 3.5.4 : Determine unit step response of the system whose impulse response is given as $h(t) = 3\,tu(t)$

Soln. :

The step response is related to impulse response as,

$$s(t) = \int_{-\infty}^{t} h(\tau)\, d\tau$$

Here $h(t) = 3\,t\,u(t)$

$\therefore\ h(\tau) = 3\,\tau\, u(\tau)$

Due to unit step, $u(\tau)$; the limits of integration will be from 0 to t.

$$\therefore\ s(t) = \int_{-\infty}^{t} 3\,\tau\, d\tau = 3\left[\frac{\tau^2}{2}\right]_0^{t}$$

$$\therefore\ s(t) = \frac{3}{2}t^2$$

This is the step response of system.

3.5.1 Exponential Response :

- Consider an LTI system having impulse response h (t) driven by an input x (t) such that,

$$x(t) = e^{j2\pi ft} \quad ...(3.5.3)$$

- Using Equation (3.5.3), the system response is given by,

$$y(t) = \int_{-\infty}^{\infty} h(\tau) \cdot e^{j2\pi f(t-\tau)}\, d\tau \quad ...(3.5.4)$$

$$\therefore\ y(t) = e^{j2\pi ft} \int_{-\infty}^{\infty} h(\tau)\, e^{-j2\pi f\tau}\, d\tau \quad ...(3.5.5)$$

1\. Let us define $$H(f) = \int_{-\infty}^{\infty} h(\tau)\, e^{-j2\pi f\tau}\, d\tau$$

$$\therefore \quad y(t) = H(f)\, e^{j2\pi ft} \qquad ...(3.5.6)$$

Thus the response of an LTI system to a complex exponential function of frequency f is therefore the same complex exponential function multiplied by a constant coefficient $H(f)$, which is called as the transfer function of the system.

- The transfer function H (f) and the impulse function h (t) form a fourier transform pair. This is indicated as follows :

$$H(f) = \int_{-\infty}^{\infty} h(t)\, e^{-j2\pi ft}\, dt \qquad ...(3.5.7)$$

$$\text{and} \quad h(t) = \int_{-\infty}^{\infty} H(f) \cdot e^{j2\pi ft}\, df \qquad ...(3.5.8)$$

1100 1100
1300 1300
800 800
3200 3400

Alternative definition of transfer function :

- The transfer function is defined alternatively as follows :

$$H(f) = \left. \frac{y(t)}{x(t)} \right|_{x(t) = e^{j2\pi ft}} \qquad ...(3.5.9)$$

- We know that in time domain,

$$y(t) = h(t) * x(t)$$

- Taking the fourier transform of both sides we get,

$$Y(f) = H(f) \cdot X(f) \qquad ...(3.5.10)$$

Thus it is possible to describe an LTI system simply in the frequency domain.

3.6 Invertibility :

- A system is said to be invertible if the input of the system can be recovered from the system output.
- A set of oparations will be needed to recover the input from output which may be viewed as a second system connected in cascade with the given system, such that the output signal of the second system is equal to the input signal of the given system.

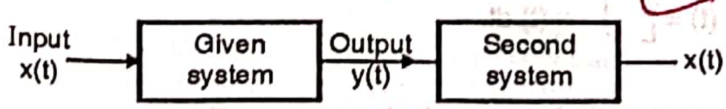

Fig. 3.6.1(a) : Concept of invertibility

- In order to represent the invertibility mathematically assume that operator H represents a C.T. given system with input signal x(t) producing the output signal y(t).

- Let y(t) be applied to a second C.T. system represented by the operator H^{-1} as shown in Fig. 3.6.1(b). Then the output signal of the second system is given by :

$$\text{Output of second system} = H^{-1}\{y(t)\} = H^{-1}\{H\,x(t)\}$$
$$= H^{-1} H\{x(t)\}$$

- If we want the output of second system to equal to the original input signal x(t), then it is necessary that

$$H^{-1} H = I \qquad \text{...(3.6.1)}$$

Where I denotes the identity operator.

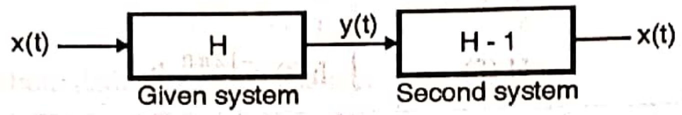

Fig. 3.6.1(b) : Invertibility expressed mathematically

- The operator H^{-1} is called as **the inverse operator** and the second system is called as the **inverse system.**
- However it is important to understand that H^{-1} is not the receprocal of H. It is inverse of H.
- There has to be a one-to-one mapping between the input and output signals for a system to be invertible.
- That means an invertible system produces distinct outputs for distinct inputs.
- All the discussion done for the C.T. invertible system is applicable to the D.T. systems as well.
- The property of invertibility is very important in designing the **communication systems.** When a signal travels over a communication channel, it gets distorted due to physical characteristics of the channel.
- An **equalizer** can be used to compensate for this distortion and get back the original signal. The equalizer is included on the receiver side.
- An equalizer is an inverse system of the communication channel.

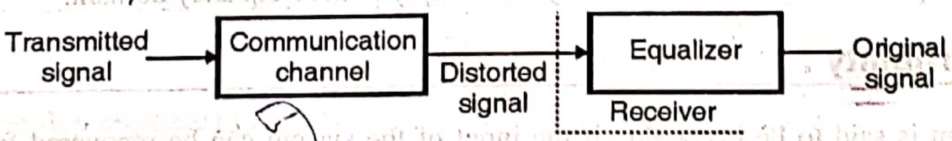

Fig. 3.6.2 : Equalizer is an inverse system of the communication channel

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Ex. 3.6.1 : For an inductor the input output relation is as follows :

$$y(t) = \frac{1}{L}\int_{-\infty}^{t} x(t)\,dt$$

Find the operation that represents its inverse system.

Soln. :

$$H = \frac{1}{L} \int_{-\infty}^{t} dt$$

$$\therefore H^{-1} = L \times \frac{d}{dt}$$ ...Ans.

This is the operator of the inverse system so that

$$H^{-1} H = I$$

3.6.1 Comparisons :

Causal system and non-causal system :

Sr. No.	Causal system	Non-causal system
1.	A system is said to be causal system if output at any instant of time depends only on present and past inputs. But the output does not depend on future input.	A system is said to be non-causal system if its output depends not only on present and past inputs but also on future inputs.
2.	Causal system is practically realizable.	Non-causal system is practically not realizable.
3.	Condition for causality, $h(t) = 0$ for $t < 0$. i.e. $h(t)$ is zero for all negative values of t only.	For non-causal system $h(t) \neq 0$ for $t < 0$. i.e. $h(t)$ exist for negative values of t also.
4.	Example : All real time systems.	Examples : Population growth, Weather broadcasting etc.

Time varient system and time invarient system :

Sr. No.	Time varient System	Time invarient system
1.	A system is time varient if its input-output characteristics changes with time.	A system is time invarient system if its input-output characteristic does not changes with time.
2.	If $x(n) \xrightarrow{T} y(n)$ then for time varient system, $x(n-k) \xrightarrow{T} y(n-k)$	If $x(n) \xrightarrow{T} y(n)$ then for TVI system, $x(n-k) \xrightarrow{T} y(n-k)$
3.	e.g. : $y(n) = x(-n)$.	e.g. : $y(n) = x(2n)$.

Static system and dynamic system :

Sr. No.	Static System	Dynamic System
1.	It is a system in which output at any instant of time depends on input sample at the same time.	It is a system in which output at any instant of time depends on input sample at the same time as well as at other inputs.
2.	Output does not depend on delayed $[x(n-k)]$ or advanced $[x(n+k)]$ inputs.	Output depends on present $[x(n)]$, delayed $[x(n-k)]$ and advanced $[x(n+k)]$ inputs.
3.	These are memoryless systems.	These system has a memory.
4.	e.g. : $y(n) = 5x(n)$	e.g. : $y(n) = x(n) + 5x(n-1)$

Review Questions

Q. 1 Define convolution integral and state its properties.

Q. 2 How any arbitrary signal can be expressed in terms of summation of weighted unit impulses ?

Q. 3 Define the term 'step response'. How it can be expressed in terms of impulse response for CT system ?

Q. 4 State and prove the condition of stability.

Q. 5 What is correlation ?

Q. 6 Define autocorrelation of energy signals.

Q. 7 Define cross correlation of energy signals.

Q. 8 Define auto and cross correlation of power signals.

6300
3000
10000
17
11700
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22,700
$\boxed{10000}$
Now

5 DT LTI Systems

Syllabus :

Impulse response characterization and convolution sum, Causal signal response to DT-LTI systems, Properties of convolution summation, Impulse response of DT-LTI system, DT-LTI system properties from impulse response, System analysis from difference equation model.

5.1 Impulse Response of DT-LTI System :

It is also called as linear convolution and it is applicable for discrete time systems.

5.1.1 Representation of Discrete Time Signal as Weighted Impulses :

In this sub-section we will show that LTI system can be completely characterized by its impulse response. We will be using following notations related to discrete time systems.

$x(n) \Rightarrow$ Input sequence

$h(n) \Rightarrow$ Impulse response of system

$y(n) \Rightarrow$ Output sequence

Consider input sequence,

$$x(n) = \{1, 2, \underset{\uparrow}{1}, 2, 1\} \qquad ...(5.1.1)$$

This sequence is represented as shown in Fig. 5.1.1(a).

The different values of x (n) are as follows :

$x(-2) = 1, x(-1) = 2,$

$x(0) = 1, x(1) = 2, x(2) = 1$

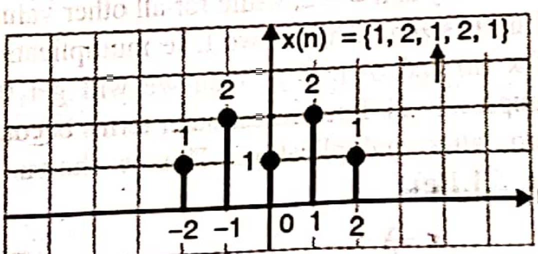


Fig. 5.1.1(a) : Input sequence x (n)

Now consider unit impulse δ (n). It is expressed as,

$$\delta(n) = \begin{cases} 1 & \text{for } n = 0 \\ 0 & \text{elsewhere} \end{cases} \quad ...(5.1.2)$$

That means its value is 1 at n = 0 and its value is zero for all other values of 'n'. Such unit impulse is as shown in Fig. 5.1.1(b).

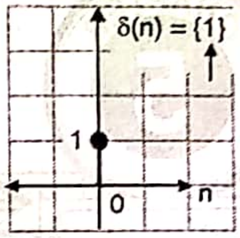


Fig. 5.1.1(b) : Unit impulse, δ (n)

If we want to delay this unit impulse by '2' samples then it is denoted by δ (n – 2)

$$\text{And} \quad \delta(n-2) = \begin{cases} 1 & \text{for } n = 2 \\ 0 & \text{otherwise} \end{cases} \quad ...(5.1.3)$$

This sequence is shown in Fig. 5.1.1(c).

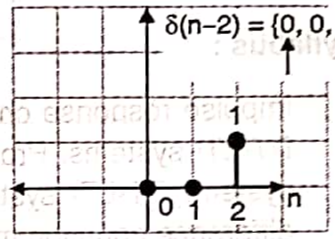


Fig. 5.1.1(c) : Impulse sequence δ (n – 2)

Similarly if we want to advance this unit impulse by '2' samples then it is denoted by δ (n + 2).

$$\therefore \quad \delta(n+2) = \begin{cases} 1 & \text{for } n = 2 \\ 0 & \text{otherwise} \end{cases} \quad ...(5.1.4)$$

This sequence is shown in Fig. 5.1.1(d).

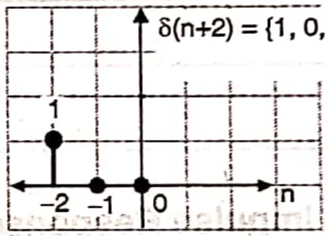


Fig. 5.1.1(d) : Impulse sequence δ (n + 2)

In general, if δ (n) is delayed by 'k' samples then we can write,

$$\therefore \quad \delta(n-k) = \begin{cases} 1 & \text{for } n = k \\ 0 & \text{otherwise} \end{cases} \quad ...(5.1.5)$$

0 → 0
1 → 1 (time)

Decomposition of input sequence x (n) :

The given input sequence is,

$$x(n) = \{1, 2, \underset{\uparrow}{1}, 2, 1\}$$

Its range is from – 2 to +2. Now decomposition means obtaining a particular sample from given sequence x(n). Suppose we want to obtain sample at n = – 2 that means x (– 2). Then take the multiplication of x (n) and δ (n + 2). We know that the value of δ (n + 2) is one only at n = – 2, while for all other values of n, it is zero. Thus if we take multiplication of x (n) and δ (n + 2) then we will get the sample x (– 2). Because all other terms become zero, after multiplication. This is shown in Fig. 5.1.1(e).

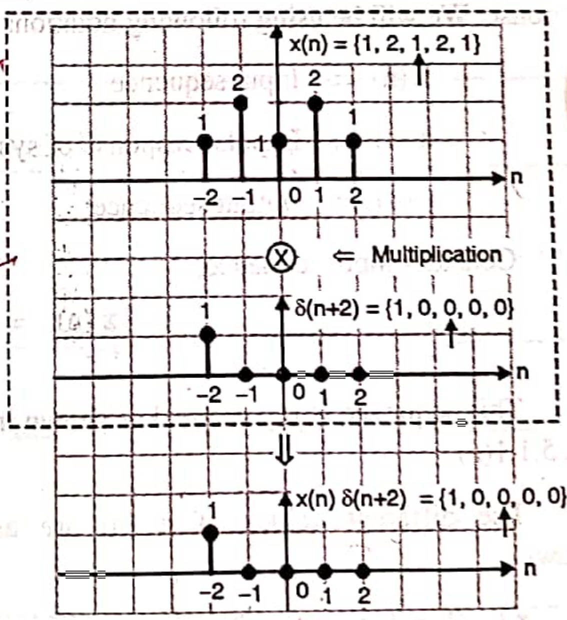


Fig. 5.1.1(e) : Decomposed input signal sample at n = – 2

0 → 0
1 → 1 (time)

Similarly other decomposed input signal samples are obtained.

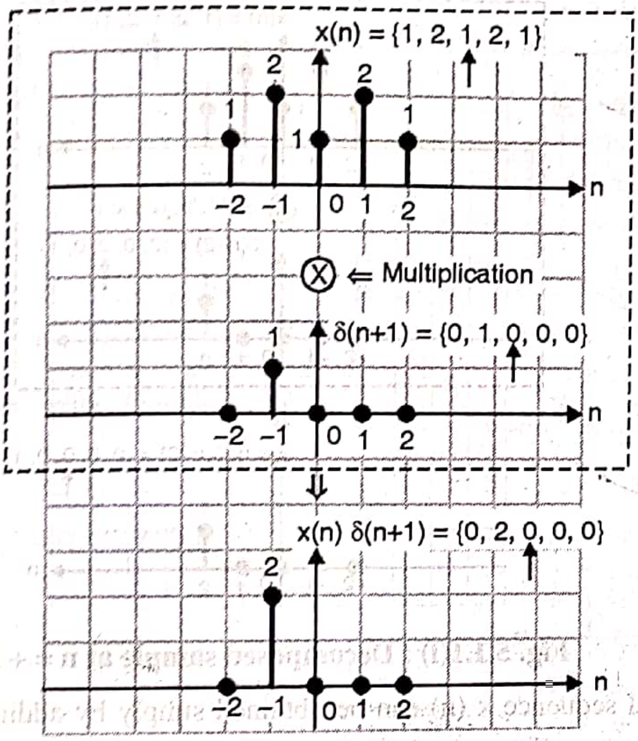


Fig. 5.1.1(f) : Decomposed sample at n = – 1

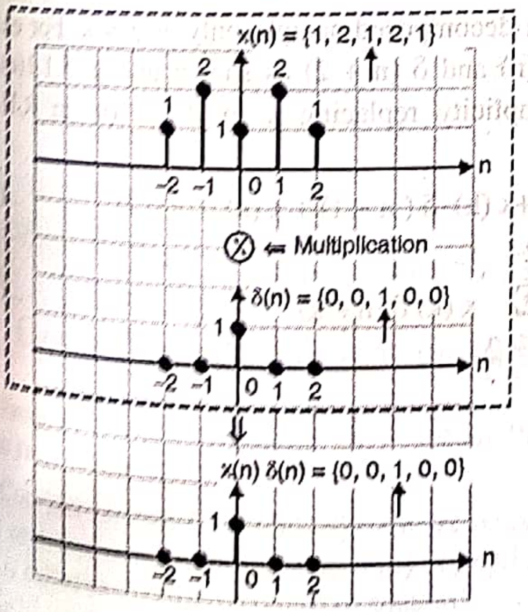


Fig. 5.1.1(g) : Decomposed sample at n = 0

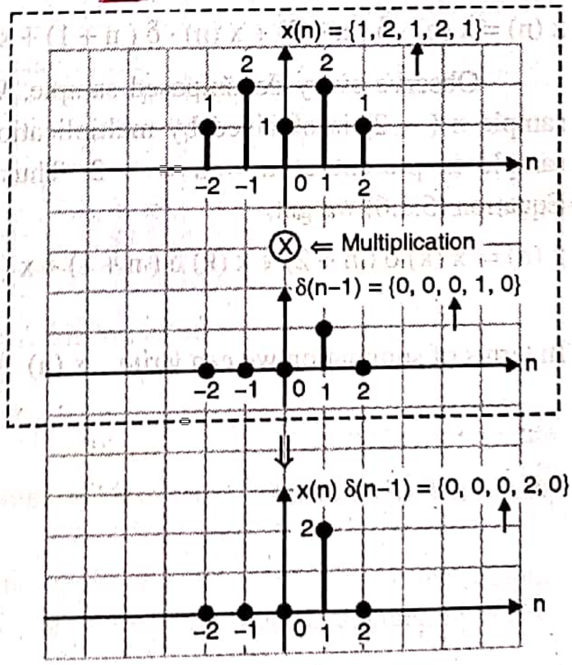


Fig. 5.1.1(h) : Decomposed sample at n = + 1

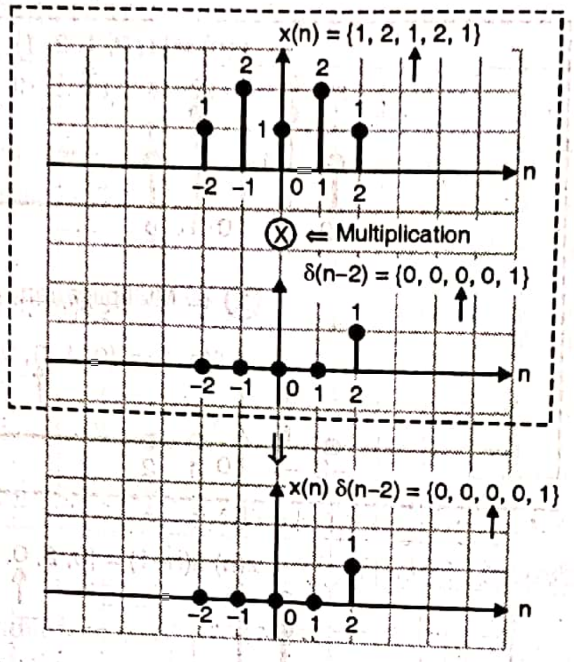


Fig. 5.1.1(i) : Decomposed sample at n = + 2

Now the original sequence x (n) can be obtained simply by adding all decomposed samples as shown in Fig. 5.1.2.

Mathematically we can write,

$$x(n) = x(n) \cdot \delta(n+2) + x(n) \cdot \delta(n+1) + x(n) \cdot \delta(n) + x(n) \cdot \delta(n-1) + x(n) \cdot \delta(n-2) \quad ...(5.1.6)$$

Observe every decomposed sample. We get a decomposed sample only at n = k. For example sample x (– 2) is obtained by multiplication of x (n) and δ (n + 2) as shown in Fig. 5.1.1(e). This sample is present at n = k = – 2. Thus for simplicity replacing x (n) by x (k) in R.H.S. of Equation (5.1.6) we get,

$$x(n) = x(k)\,\delta(n+2) + x(k)\,\delta(n+1) + x(k)\,\delta(n) + x(k)\,\delta(n-1) + x(k)\,\delta(n-2) \quad ...(5.1.7)$$

In terms of summation we can write, $x(n) = \sum_{k=-2}^{2} x(k)\,\delta(n-k)$ ...(5.1.8)

In Equation (5.1.7), the amplitude of every 'δ' term is 1. We are multiplying every unit impulse (advanced or delayed) by corresponding amplitude of input sample. This is called as scaling of unit samples. Thus Equations (5.1.7) and (5.1.8) indicates that **input sequence x (n) is represented as sum of weighted (or scaled) unit impulses.**

In Equation (5.1.8) the limits of summation are from – 2 to + 2 ; this is because input sequence x (n) is present from n = – 2 to n = + 2. In general we can write Equation (5.1.8) as,

$$x(n) = \sum_{k=-\infty}^{\infty} x(k)\,\delta(n-k) \qquad ...(5.1.9)$$

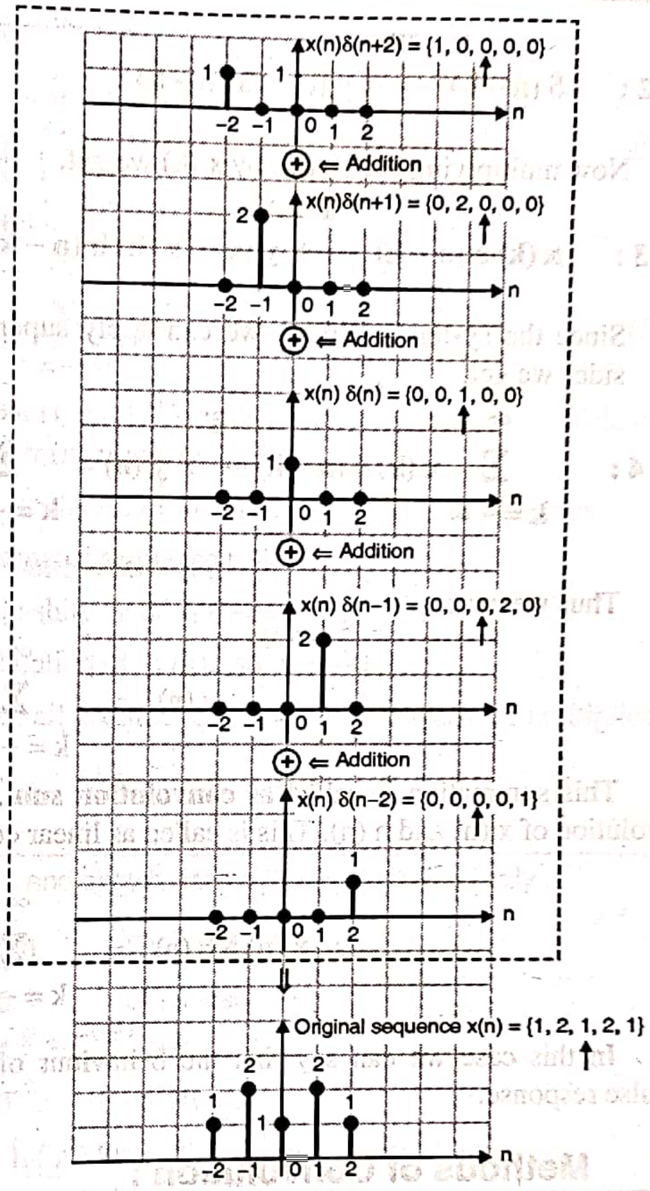


Fig. 5.1.2 : x (n) = Addition of all decomposed samples

5.2 Linear Convolution (Convolution Sum) :

Proof of linear convolution Or Proof of **"LTI system is completely characterized by unit impulse response h (n)":**

Consider relaxed LSI (LTI) system. A relaxed system means if input x (n) is zero then output y (n) is zero. Let us say unit impulse δ (n) is applied to this system then its output is denoted by h (n). "h (n)" is called as impulse response of system.

Step 1 : $\delta(n) \xrightarrow{T} y(n) = h(n)$

Since the system is time invariant, if we delay input by 'k' samples then output should be delayed by same amount.

Step 2 : $\delta(n-k) \xrightarrow{T} y(n) = h(n-k)$

Now multiplying both sides by x (k) we get,

Step 3 : $x(k)\,\delta(n-k) \xrightarrow{T} y(n) = x(k)\,h(n-k)$

Since the system is linear; we can apply superposition theorem. So taking summation at both sides we get,

Step 4 : $\sum_{k=-\infty}^{\infty} x(k)\,\delta(n-k) \xrightarrow{T} y(n) = \sum_{k=-\infty}^{\infty} x(k)\,h(n-k)$

Thus we have,

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\,h(n-k) \qquad \text{...(5.2.1)}$$

This summation is called as **convolution sum**. In this case output y (n) is obtained by taking convolution of x (n) and h (n). This is called as linear convolution and it is denoted by symbol '*'.

$$\therefore\ x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k)\,h(n-k)$$

In this case we can say that the behaviour of LTI system is completely characterized by its impulse response.

5.3 Methods of Convolution :

The different methods used for the computation of linear convolution are as follows :

1. Graphical method
2. Using mathematical equation of convolution
3. Tabulation method
4. Multiplication method

5.3.1 Graphical Method :

The linear convolution of two sequence is given by,

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\,h(n-k) \qquad \text{...(5.3.1)}$$

We will have to calculate output for different values of 'n'

For n = 0 we get,

$$y(0) = \sum_{k=-\infty}^{\infty} x(k)\,h(-k) \qquad \text{...(5.3.2)}$$

Note that $h(-k)$ indicates folding of $h(k)$.

For $n = 1$ we get,

$$y(1) = \sum_{k=-\infty}^{\infty} x(k)\, h(1-k) \qquad ...(5.3.3)$$

Here the term $h(1-k)$ can be written as $h(-k+1)$. Thus Equation (5.3.3) becomes,

$$y(1) = \sum_{k=-\infty}^{\infty} x(k)\, h(-k+1) \qquad ...(5.3.4)$$

Here $h(-k+1)$ indicates shifting of folded signal $h(-k)$. It indicates that $h(-k)$ is delayed by '1' sample. Similarly for other values of 'n' output $y(n)$ is calculated.

Thus different operations involved in the calculation of linear convolution are as follows :

1. **Folding operation :** It indicates folding of sequence $h(k)$.
2. **Shifting operation :** It indicates time shifting of $h(-k)$ eg. $h(-k+1)$.
3. **Multiplication :** It indicates multiplication of $x(k)$ and $h(n-k)$.
4. **Summation :** It indicates addition of all product terms obtained because of multiplication of $x(k)$ and $h(n-k)$.

Solved Problems :

Ex. 5.3.1 : For the following signals, determine and sketch convolution $y(n)$ graphically :

$$x(n) = \frac{1}{3}n \qquad 0 \le n \le 6$$
$$= 0 \qquad \text{otherwise}$$
$$\text{and } h(n) = 1 \qquad -2 \le n \le 2$$
$$= 0 \qquad \text{otherwise}$$

Soln. : First we will write sequences $x(n)$ and $h(n)$ by putting different values of n.

Sequence $x(n)$:

Here range of $x(n)$ is from $n = 0$ to $n = 6$ and $x(n) = \frac{1}{3}n$

For $n = 0 \Rightarrow x(0) = \frac{1}{3} \times 0 = 0$ For $n = 1 \Rightarrow x(1) = \frac{1}{3} \times 1 = \frac{1}{3}$

For $n = 2 \Rightarrow x(2) = \frac{1}{3} \times 2 = \frac{2}{3}$ For $n = 3 \Rightarrow x(3) = \frac{1}{3} \times 3 = \frac{3}{3}$

For $n = 4 \Rightarrow x(4) = \frac{1}{3} \times 4 = \frac{4}{3}$ For $n = 5 \Rightarrow x(5) = \frac{1}{3} \times 5 = \frac{5}{3}$

For $n = 6 \Rightarrow x(6) = \frac{1}{3} \times 6 = \frac{6}{3}$

Thus x (n) can be written as,

$$x(n) = \{ x(0), x(1), x(2), x(3), x(4), x(5), x(6)\}$$

$$\therefore \quad x(n) = \left\{ \underset{\uparrow}{0}, \frac{1}{3}, \frac{2}{3}, \frac{3}{3}, \frac{4}{3}, \frac{5}{3}, \frac{6}{3} \right\} \quad \text{...(1)}$$

Now h (n) is given as,

$$h(n) = 1 \quad \text{for } -2 \leq n \leq 2$$
$$= 0 \quad \text{otherwise}$$

Thus range of h (n) is from – 2 to +2

$$\therefore \quad h(n) = \{h(-2), h(-1), h(0), h(1), h(2)\}$$

$$\therefore \quad h(n) = \{1, 1, \underset{\uparrow}{1}, 1, 1\} \quad \text{...(2)}$$

Now the sequences x (k) and h (k) are obtained by replacing 'n' by 'k'.

$$\therefore x(k) = \left\{ \underset{\uparrow}{0}, \frac{1}{3}, \frac{2}{3}, \frac{3}{3}, \frac{4}{3}, \frac{5}{3}, \frac{6}{3} \right\} \quad \text{...(3)}$$

$$\text{and } h(k) = \{1, 1, \underset{\uparrow}{1}, 1, 1\} \quad \text{...(4)}$$

The linear convolution of x (k) and h (k) is given by,

$$y(n) = x(k) * h(k) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k) \quad \text{...(5)}$$

Range of 'n' :

Lower limit of y (n) i.e. $y_l = x_l + h_l = 0 + (-2) = -2$ (Addition of lower range of x (n) and h (n))

Higher limit of y (n) i.e. $y_h = x_h + h_h = 6 + 2 = 8$ (Addition of upper range of x (n) and h (n))

Thus range of n is from – 2 to + 8.

Range of 'k' :

Range of 'k' is similar to x (k). Thus range of 'k' is from 0 to 6.

Now using Equation (5) we will obtain n for different values of n from – 2 to 8.

$$y(n) = \sum_{k=0}^{6} x(k)\, h(n-k) \quad \text{...(6)}$$

Calculation of y (– 2) :

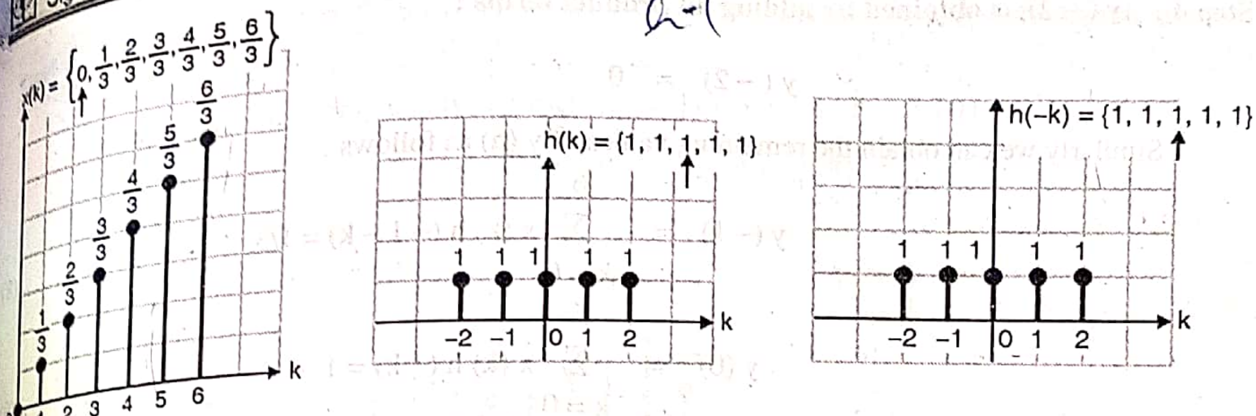
Putting n = – 2 in Equation (6) we get,

$$y(-2) = \sum_{k=0}^{6} x(k)\, h(-2-k) \quad \text{...(7)}$$

Step 1 : First plot x (k) as shown in Fig. P. 5.3.1(a) :

Plot of h (k) is shown in Fig. P. 5.3.1(b). From this obtain h (– k) which is folded version of as h (k) as shown in Fig. P. 5.3.1(c). Observe that h (k) and h (– k) are same in this case.

h-(



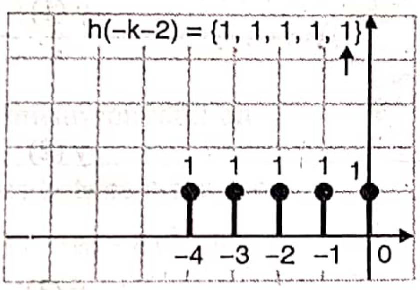
(a) Sequence x (k)

(b) Sequence h (k)

(c) Sequence h (– k)

Fig. P. 5.3.1

Step 2 : h (– 2 – k) is same as h (– k – 2). It indicates advance version of h (– k). It is obtained by shifting h (– k) towards left by '2' samples. This sequence is shown in Fig. P. 5.3.1(d).



– (k+2)

Fig. P. 5.3.1(d) : Sequence h (– k – 2)

Step 3 : The product of x (k) and h (– k – 2) is shown in Fig. P. 5.3.1(e) :

– (k+2)

2< 3324

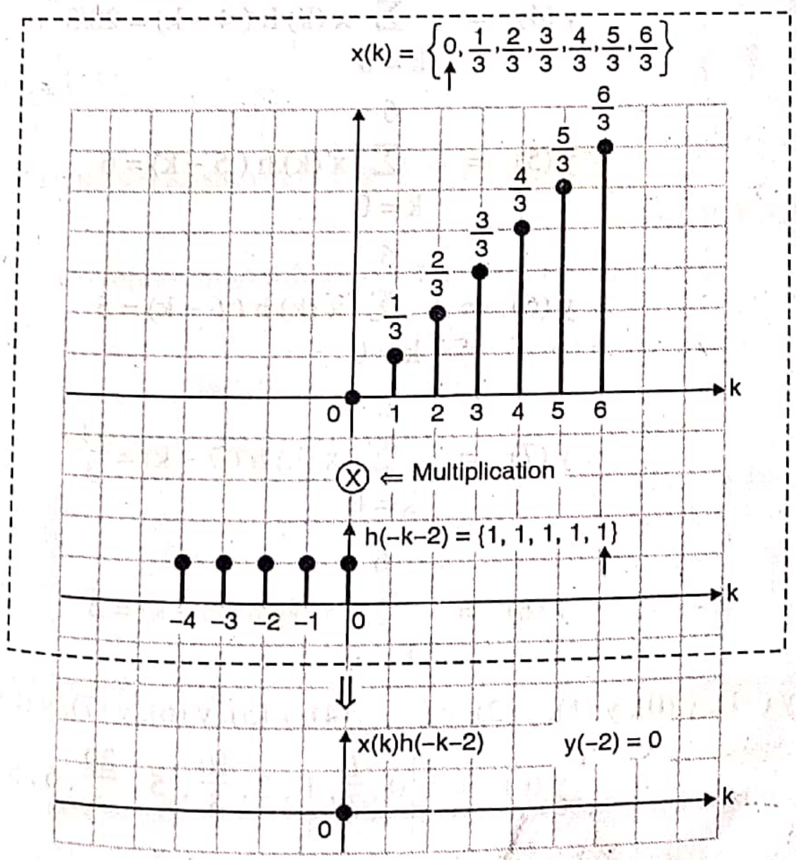


Fig. P. 5.3.1(e) : Product of x (k) and h (– k – 2)

Step 4 : **$y(-2)$ is obtained by adding all product terms :**

$$\therefore \quad y(-2) = 0$$

Similarly we can obtain the remaining values of y (n) as follows :

$$y(-1) = \sum_{k=0}^{6} x(k)\, h(-1-k) = 1/3 \quad \text{...(8)}$$

$$y(0) = \sum_{k=0}^{6} x(k)\, h(-k) = 1 \quad \text{...(9)}$$

$$y(1) = \sum_{k=0}^{6} x(k)\, h(1-k) = 2 \quad \text{...(10)}$$

$$y(2) = \sum_{k=0}^{6} x(k)\, h(2-k) = 10/3 \quad \text{...(11)}$$

$$y(3) = \sum_{k=0}^{6} x(k)\, h(3-k) = 5 \quad \text{...(12)}$$

$$y(4) = \sum_{k=0}^{6} x(k)\, h(4-k) = 20/3 \quad \text{...(13)}$$

$$y(5) = \sum_{k=0}^{6} x(k)\, h(5-k) = 6$$

$$y(6) = \sum_{k=0}^{6} x(k)\, h(6-k) = 5$$

$$y(7) = \sum_{k=0}^{6} x(k)\, h(7-k) = \frac{11}{3} \quad \text{...(16)}$$

$$y(8) = \sum_{k=0}^{6} x(k)\, h(8-k) = 2 \quad \text{...(17)}$$

$\therefore$ y (n) = { y (–2), y (–1), y (0), y (1), y (2), y (3), y (4), y (5), y (6), y (7), y (8) }

$$\therefore \quad y(n) = \left\{0, \frac{1}{3}, \underset{\uparrow}{1}, 2, \frac{10}{3}, 5, \frac{20}{3}, 6, 5, \frac{11}{3}, 2\right\}$$

Note : Arrow always indicates the sample at n = 0. Thus arrow should be marked at sample y (0).

The sequence y (n) is graphically represented as shown in Fig. P. 5.3.1(f).

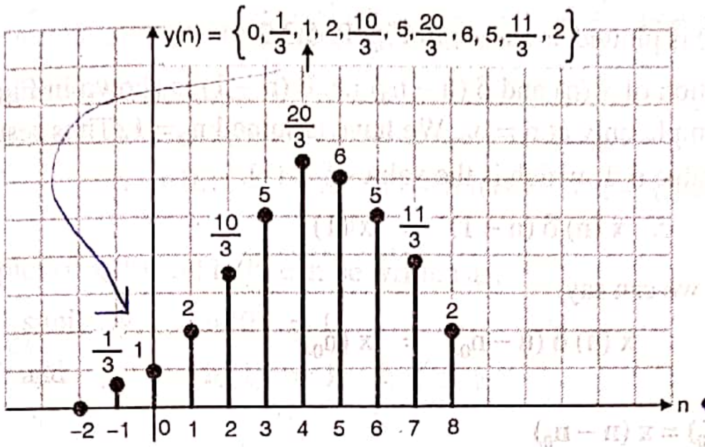


Fig. P. 5.3.1(f) : Result of linear convolution

Ex. 5.3.2 : Prove and explain graphically the difference between relations :

1. $x(n)\,\delta(n-n_0) = x(n_0)$ 2. $x(n) * \delta(n-n_0) = x(n-n_0)$

Soln. :

1. $\mathbf{x(n)\,\delta(n-n_0) = x(n_0)}$

Here x (n) is input sequence. Let the input sequence be given by,

$$x(n) = \{1, 1, \underset{\uparrow}{2}, 1, 1\}$$

This sequence is plotted as shown in Fig. P. 5.3.2(a).

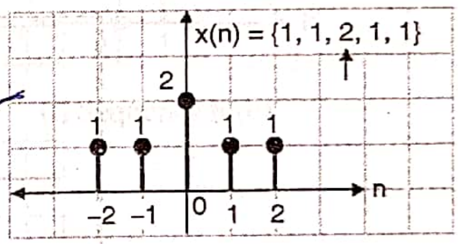
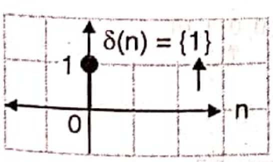


Fig. P. 5.3.2(a) : Sequence x (n)

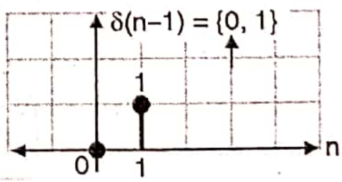
We know that δ (n) represent unit impulse. It is defined as,

$$\delta(n) = 1 \quad \text{for } n = 0$$
$$= 0 \quad \text{elsewhere}$$

Sequence δ (n) is shown in Fig. P. 5.3.2(b).



(b) Unit impulse, δ (n)



(c) Sequence δ (n – 1)

Fig. P. 5.3.2

Now $\delta(n-n_0)$ indicates delay of δ (n) by no units. So sequence $\delta(n-n_0)$ can be represented as,

$$\delta(n-n_0) = 1 \quad \text{for } n = n_0$$
$$= 0 \quad \text{for } n \neq n_0$$

For example, say $n_0 = 1$. Thus we get,

$$\delta(n-1) = 1 \quad \text{for } n = 1$$
$$= 0 \quad \text{for } n \neq n_0$$

The sequence $\delta(n-1)$ is plotted as shown in Fig. P. 5.3.2(c).

Now multiplication of $x(n)$ and $\delta(n-n_0)$ i.e. $\delta(n-1)$ is shown in Fig. P. 5.3.2(d). Here we are getting the resultant sample only at $n = n_0$. We have assumed $n_0 = 1$. Thus result of multiplication is the sample at $n = 1$. This value is 1; which is the value of $x(1)$.

$$\therefore \quad x(n)\,\delta(n-1) = x(1)$$

Thus in general we can say,

$$x(n)\,\delta(n-n_0) = x(n_0)$$

Hence proved.

2. $x(n) * \delta(n-n_0) = x(n-n_0)$

Here '*' indicates convolution. So we have to calculate convolution of $x(n)$ and $\delta(n-n_0)$. According to the definition of convolution,

$$y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k)\,h(n-k) \quad \text{...(1)}$$

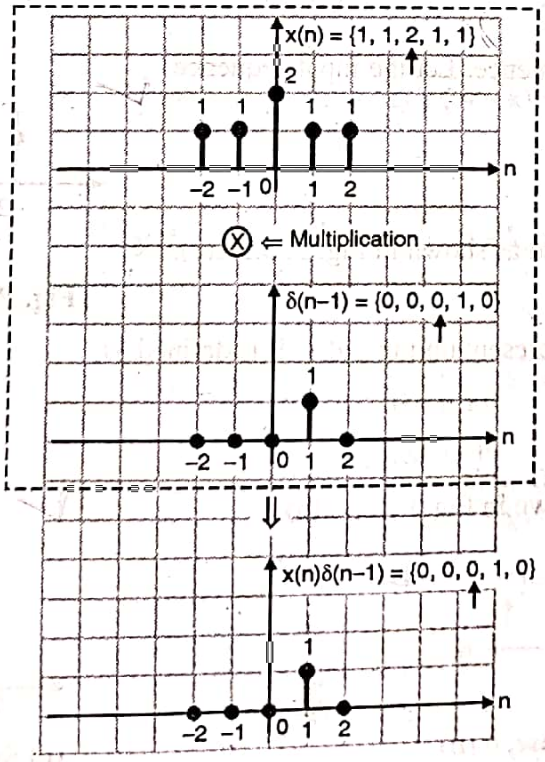


Fig. P. 5.3.2(d) : Product of $x(n)\,\delta(n-n_0)$

Let $\quad h(n) = \delta(n-n_0) \quad$ ...(2)

Now assume two sequence as follows,

$$x(n) = \{1, \underset{\uparrow}{1}, 1\} \text{ and } h(n) = \delta(n - n_0) = \{0, \underset{\uparrow}{1}\}$$

also assume that value of $n_0 = 1$

Note : We cannot assume any arbitrary sequence for h (n). Since $h(n) = \delta(n - n_0)$; it should be in terms of delayed unit impulse.

Step 1 : Sequences x (k) and h (k) can be written by directly replacing 'n' by 'k' :

$$\therefore\ x(k) = \{1, \underset{\uparrow}{1}, 1\} \text{ and } h(k) = \delta(k - n_0) = \{\underset{\uparrow}{0}, 1\}$$

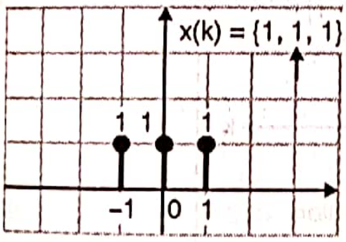
Thus different samples of x (k) and h (k) can be written as,

$x(-1) = 1$ similarly, $h(0) = 0$

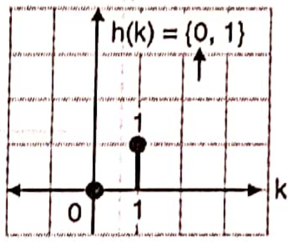
$x(0) = 1$ and $h(1) = 1$

and $x(1) = 1$

Plot of x (k) and h (k) is shown in Fig. P. 5.3.2(e).



(e) Sequence x (k)



(f) Sequence h (k)

Fig. P. 5.3.2

Step 2 : Now range of 'n' is calculated as follows :

$$\text{lower value of } n \Rightarrow y_l = x_l + h_l = -1 + 0 = -1$$

$$\text{and higher value of } n \Rightarrow y_h = x_h + h_h = 1 + 1 = 2$$

Thus range of 'n' is from – 1 to 2. That means we have to calculate sequence y (n) from y (– 1) to y (2).

Range of 'k' is same as x (k). Thus 'k' varies from – 1 to + 1.

Step 3 : Putting the value of 'k' Equation (1) can be written as,

$$y(n) = \sum_{k=-1}^{1} x(k)\, h(n-k) \qquad ...(3)$$

To calculate y (n); put values of 'n' from – 1 to + 2.

Calculation of y (– 1) :

Putting n = – 1 in Equation (3) we get,

$$y(-1) = \sum_{k=-1}^{1} x(k)\, h(-1-k) \qquad ...(4)$$

Here $h(-1-k)$ is same as $h(-k-1)$ · $h(-k)$ is folded version of $h(k)$ and $h(-k-1)$ indicates advance of $h(-k)$ by 1 sample. This sequence is shown in Fig. P. 5.3.2(g).

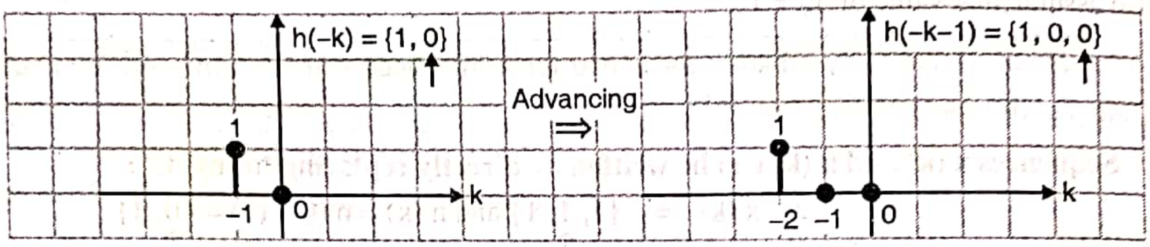


Fig. P. 5.3.2(g) : Sequence $h(-k-1)$

The multiplication of $x(k)$ and $h(-k-1)$ is shown in Fig. P. 5.3.2(h).

$y(-1)$ is obtained by adding all product terms.

$$\therefore \quad y(-1) = 0$$

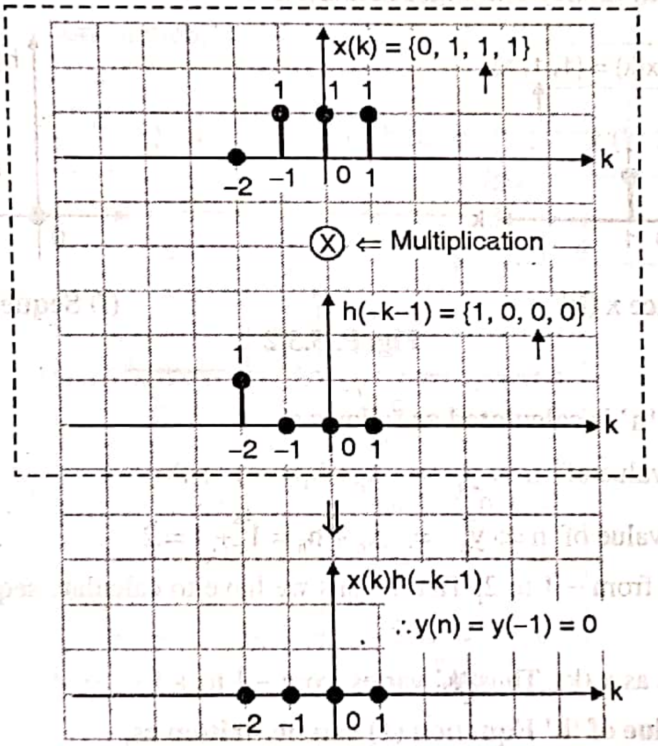


Fig. P. 5.3.2(h) : Product of $x(k)$ and $h(-k-1)$

Similarly we can calculate the remaining values of $y(n)$,

$$y(0) = \sum_{k=-1}^{1} x(k)\,h(-k) = 1 \qquad \text{...(5)}$$

$$y(1) = \sum_{k=-1}^{1} x(k)\,h(1-k) = 1 \qquad \text{...(6)}$$

$$y(2) = \sum_{k=-1}^{1} x(k)\, h(2-k) = 1 \qquad ...(7)$$

The output sequence y (n) is,

$$y(n) = \{y(-1), y(0), y(1), y(2)\}$$

$$\therefore \; y(n) = \{0, \underset{\uparrow}{1}, 1, 1\}$$

This sequence is plotted in Fig. P. 5.3.2(i).

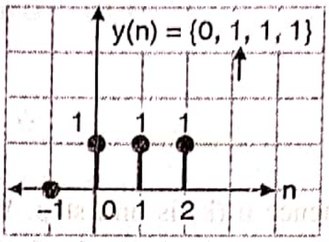


Fig. P. 5.3.2(i) : Output y (n)

Now the output sequence y (n) can also be written as,

$$\therefore \; y(n) = \{\underset{\uparrow}{1}, 1, 1\}$$

Observe graph of x (k) shown in Fig. P. 5.3.2(e) and graph of y (i) shown in Fig. P. 5.3.2(i). It shows that output y (n) is obtained by delaying x (k) by '1' sample.

$$\therefore \; y(n) = x(k-1)$$

But x (k) is same as x (n).

$$\therefore \; y(n) = x(n-1)$$

Initially we have assumed $n_0 = 1$

$$\therefore \; y(n) = x(n-n_0)$$

$$\therefore \; y(n) = x(n) * \delta(n-n_0) = x(n-n_0)$$

Hence proved.

Note : This indicates that the convolution of input sequence x (n) with delayed unit impulse is equivalent to delay of input sequence by corresponding samples (n_0).

Ex. 5.3.3 : Use discrete convolution to find the response to the input $x(n) = a^n u(n)$ of the LTI system with impulse response $h(n) = b^n u(n)$.

Soln. :

Given : $x(n) = a^n u(n)$ and $h(n) = b^n u(n)$.

Sequences x (k) and h (k) can be directly written as,

$x(k) = a^k u(k)$ and $h(k) = b^k u(k)$

According to definition of linear convolution we have,

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k) \qquad ...(1)$$

We have $h(k) = b^k u(k)$ ...(2)

Sequence h (n – k) is obtained by replacing k by n – k in Equation (2)

$$\therefore \quad h(n-k) = b^{n-k} u(n-k) \quad \text{...(3)}$$

Thus Equation (1) becomes, $$y(n) = \sum_{k=-\infty}^{\infty} a^k u(k) \cdot b^{n-k} u(n-k)$$

$$\therefore \quad y(n) = \sum_{k=-\infty}^{\infty} a^k \cdot b^{n-k} u(k)\, u(n-k) \quad \text{...(4)}$$

Sequence u (k) is unit step. While u (n – k) = u (– k + n) indicates delay of u (– k) by 'n' samples. This sequence is obtained as shown in Fig. P. 5.3.3.

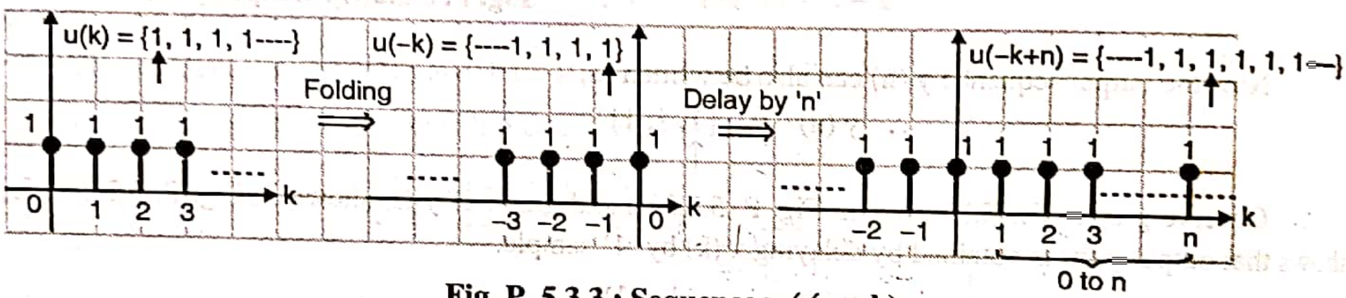


Fig. P. 5.3.3 : Sequence u ((n – k)

Now the product of u (k) and u (n – k) is shown in Fig. P. 5.3.3(a).

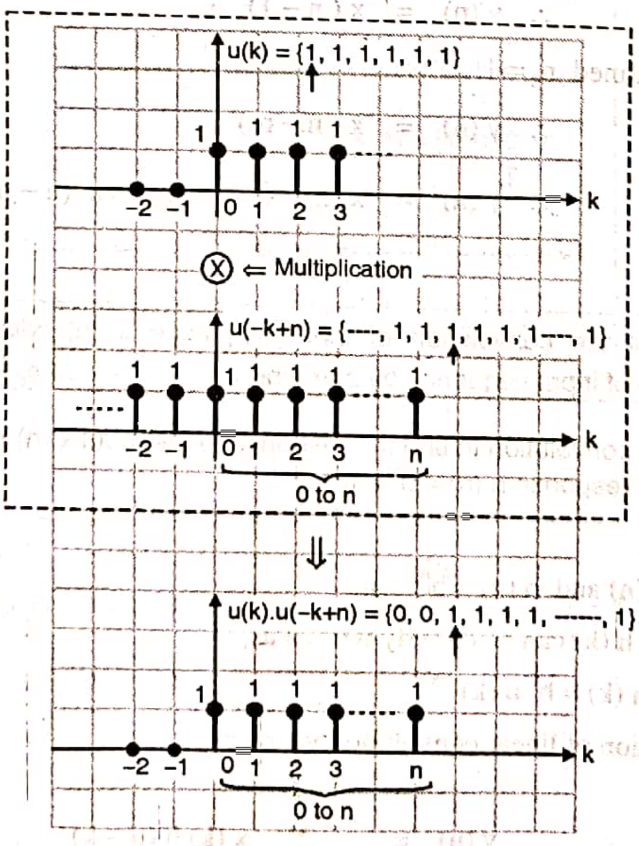


Fig. P. 5.3.3(a) : Product of u (k) and u (n – k)

This multiplication i.e. u (k) · u ((n – k) indicates that the sequence is present only for the range 0 to n. So we will change the limits of summation in Equation (4). New limits of k will be from 0 to n.

$$\therefore \; y(n) = \sum_{k=0}^{n} a^k \cdot b^{n-k} u(k) \cdot u(n-k) \qquad ...(5)$$

But the term u (k) · u (n – k) = 1.

$$\therefore \; y(n) = \sum_{k=0}^{n} a^k \cdot b^{n-k} \qquad ...(6)$$

$$\text{Now } b^{n-k} = b^n \cdot b^{-k} = \frac{b^n}{b^k}$$

$$\therefore \; y(n) = \sum_{k=0}^{n} a^k \frac{b^n}{b^k} \qquad ...(7)$$

Since the summation is in terms of 'k' we can take the term b^n outside the summation.

$$\therefore \; y(n) = b^n \sum_{k=0}^{n} \frac{a^k}{b^k}$$

$$\therefore \; y(n) = b^n \sum_{k=0}^{n} \left(\frac{a}{b}\right)^k \qquad ...(8)$$

Use standard equation of summation,

$$\sum_{n=N_1}^{N_2} a^n = \begin{cases} \dfrac{a^{N_1} - a^{N_2+1}}{1-a} & \text{For } a \neq 1 \\ N_2 - N_1 + 1 & \text{For } a = 1 \end{cases} \qquad ...(9)$$

Compare Equations (8) and (9).

Here $N_1 = 0$, $N_2 = n$ and $a = \left(\frac{a}{b}\right)$

$$\therefore \; y(n) = b^n \cdot \left[\frac{\left(\frac{a}{b}\right)^0 - \left(\frac{a}{b}\right)^{n+1}}{1 + \frac{a}{b}} \right] \quad \text{... for } \frac{a}{b} \neq 1$$

$$\therefore \; y(n) = b^n \left[\frac{1 - \left(\frac{a}{b}\right)^{n+1}}{1 - \frac{a}{b}} \right] \qquad ...(10)$$

Equation (10) gives convolution of a^n u (n) and b^n u (n).

Ex. 5.3.4 : Obtain the expression for convolution of unit step sequence with finite duration sequence.

Soln. : We will assume some sequence x (n) as,

$$x(n) = \{\underset{\uparrow}{2}, 2, 1\} \quad ...(1)$$

Note that this is finite duration sequence.

We have to obtain convolution of x (n) with unit step u (n).

$$\text{So let, } h(n) = u(n) \quad ...(2)$$

The equations of h (k) and x (k) can be directly written as,

$$\therefore \; x(k) = \{\underset{\uparrow}{2}, 2, 1\} \text{ and } h(k) = u(k) = \{\underset{\uparrow}{1}, 1, 1, \ldots \infty\}$$

According to the definition of convolution we have,

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k) \quad ...(3)$$

To obtain range of y (n) :

Lowest index of y (n) $\Rightarrow y_l = x_l + h_l = 0 + 0 = 0$

Highest index of y (n) $\Rightarrow y_h = x_h + h_h = 2 + \infty = \infty$

Thus range of y (n) is from '0' to '∞'. That means we have to obtain y (n) for different values of n from 0 to ∞.

Now the range of k is same as x (k). So range of k is from 0 to 2.

Calculation of y (0) :

y (0) is calculated by putting n = 0 in Equation (3).

$$\therefore \; y(0) = \sum_{k=0}^{2} x(k)\, h(-k) \quad ...(4)$$

The sequence x (k) is shown in Fig. P. 5.3.4(a). Sequence h (k) = u (n) is shown in Fig. P. 5.3.4(b). Now h (– k) is folded version of h (k). It is obtained by folding unit step u (n). Sequence h (– k) is shown in Fig. P. 5.3.4(c).

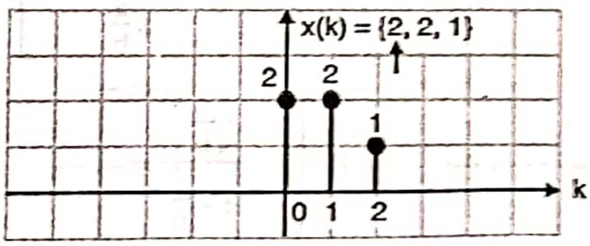


Fig. P. 5.3.4(a) : Sequence $x(k) = \{\underset{\uparrow}{2}, 2, 1\}$

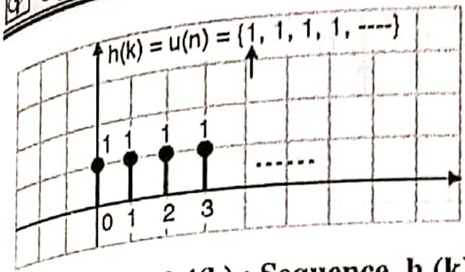
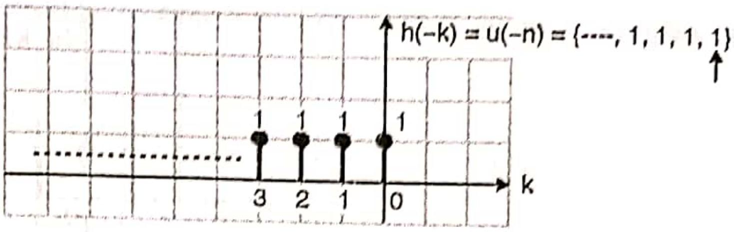


Fig. P. 5.3.4(b) : Sequence h (k) = u (n)



(c) : Sequence h (− k) = u (− n)

Fig. P. 5.3.4

According to Equation (4) y (0) is obtained by multiplying x (k) by h (− k) and then adding all product terms. This is shown in Fig. P. 5.3.4(d).

$$\therefore \quad y(0) = 2$$

Calculation of y (1) :

Putting n = 1 in Equation (3) we get,

$$y(n) = \sum_{k=0}^{2} x(k)\, h(1-k) \qquad \text{...(5)}$$

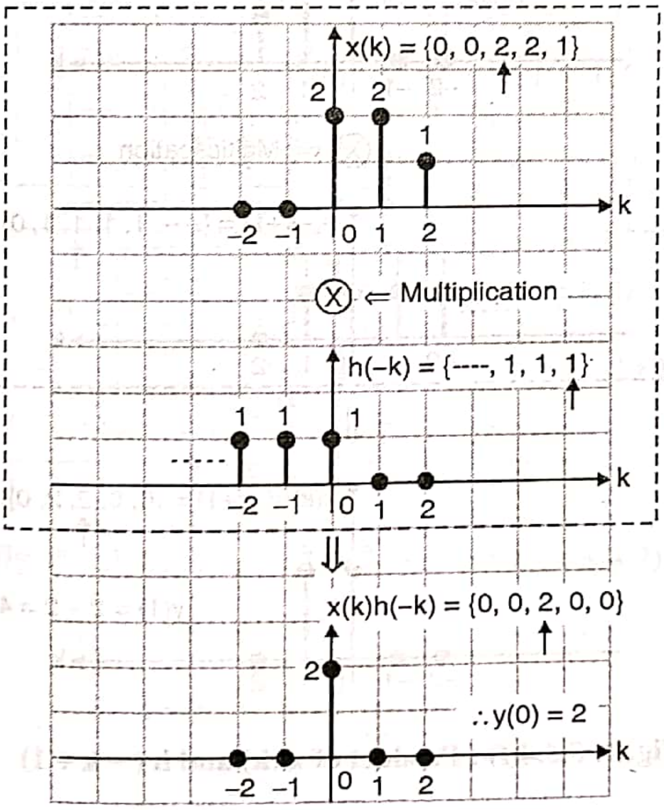


Fig. P. 5.3.4(d) : Product of x (k) and h (− k)

Here h (1 − k) is same as h (− k + 1). It indicates delay of h (− k) by '1' sample. This sequence is shown in Fig. P. 5.3.4(e).

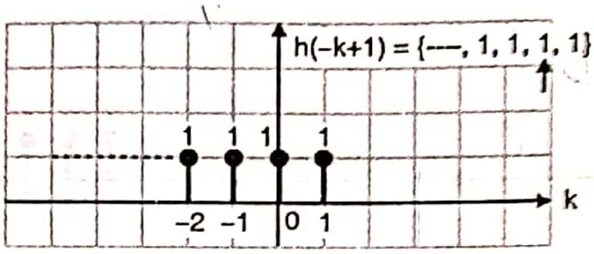


Fig. P. 5.3.4(e) : Sequence h (– k + 1)

The multiplication of x (k) and h (– k + 1) is shown in Fig. P. 5.3.3(f).

Now y (1) is obtained by adding all product terms.

$$\therefore \; y(1) = (2\times 1)+(2\times 1) = 2+2 = 4$$

$$\therefore \; y(1) = 4$$

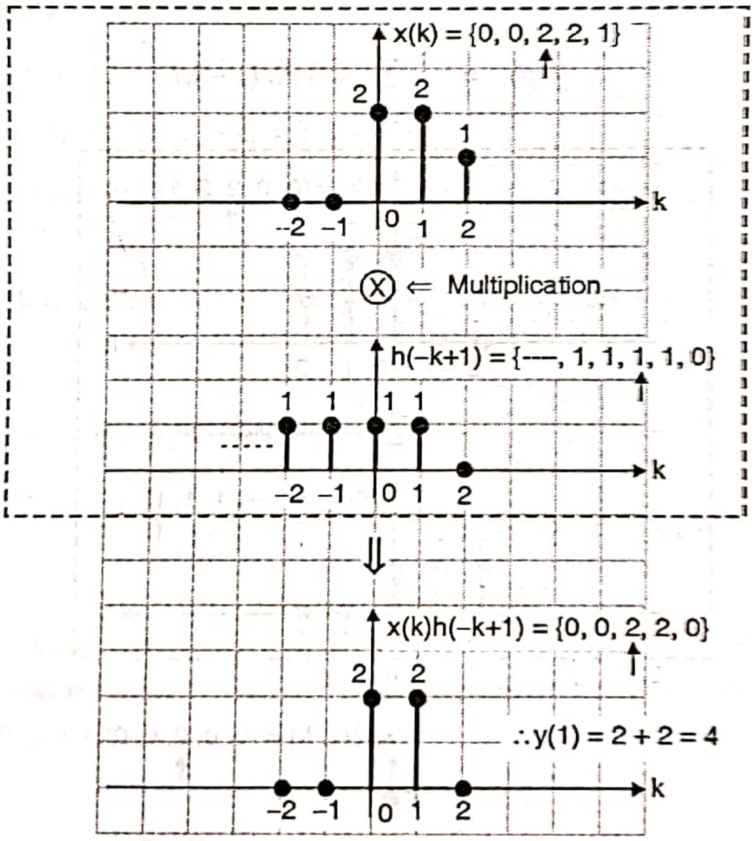


Fig. P. 5.3.4(f) : Product of x (k) and h (– k + 1)

Calculation of y (2) : Putting n = 2 in Equation (3) we get,

$$y(2) = \sum_{k=0}^{2} x(k)\,h(2-k) \quad ...(6)$$

$h(2-k)$ is same as $h(-k+2)$. It is obtained by shifting $h(-k)$ towards right by '2' samples as shown in Fig. P. 5.3.4(g).

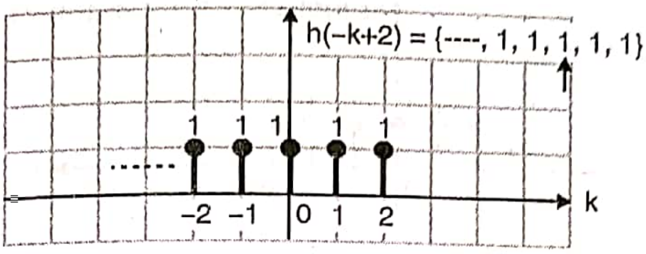


Fig. P. 5.3.4(g) : Sequence $h(-k+2)$

The product of $x(k)$ and $h(-k+2)$ is shown in Fig. P. 5.3.4(h).

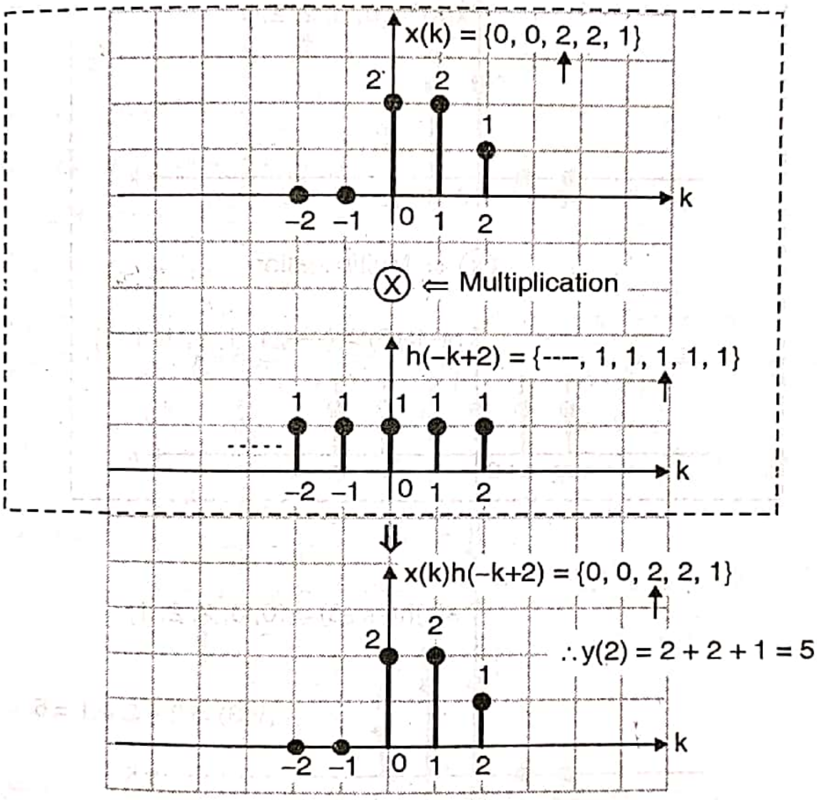


Fig. P. 5.3.4(h) : Product of $x(k)$ and $h(-k+2)$

$$\therefore \quad y(2) = (2\times 1) + (2\times 1) + (1\times 1) = 2+2+1 = 5$$

$$\therefore \quad y(2) = 5$$

Calculation of y (3) :

Putting $n = 3$ in Equation (3) we get,

$$y(3) = \sum_{k=0}^{2} x(k)\, h(3-k) \qquad \text{...(7)}$$

Sequence h (3 – k) is same as h (– k + 3). It is obtained by shifting h (– k) towards right by 3 samples. This sequence is shown in Fig. P. 5.3.4(i).

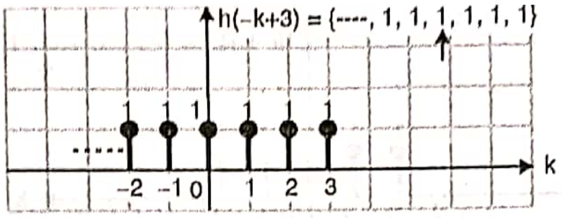


Fig. P. 5.3.4(i) : Sequence h (– k + 3)

The product of x (k) and h (– k + 3) is shown in Fig. P. 5.3.4(j).

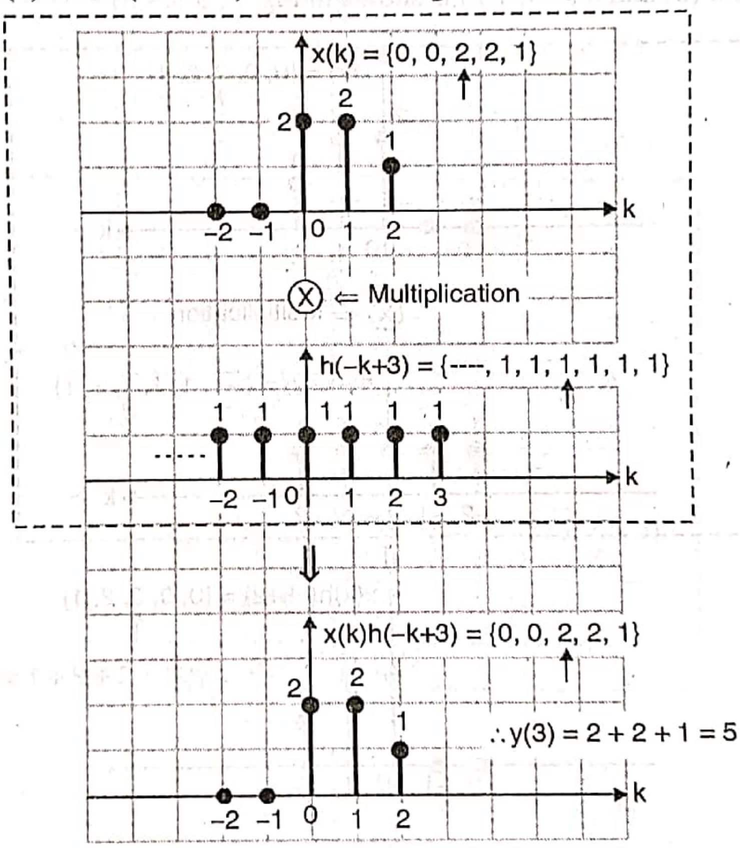


Fig. P. 5.3.4(j) : Product of x (k) and h (– k + 3)

$$\therefore \quad y(3) = (2 \times 1) + (2 \times 1) + (1 \times 1) = 2 + 2 + 1 = 5$$

$$\therefore \quad y(3) = 5$$

Since unit step is infinite sequence; it is clear that all the next samples of y (n) will have magnitude equal to 5. This is because every time we will shift h (– k) towards right by 1 sample. So for every value of n the product of x (k) and delayed h (– k) will remain same.

$\therefore \quad y(4) = 5, \quad y(5) = 5, \quad y(6) = 5 \ldots$

Now the resultant output y (n) can be written as,

$$y(n) = \{ y(0), y(1), y(2), y(3), \ldots\ldots\ldots \infty \}$$

$$\therefore \quad y(n) = \{\underset{\uparrow}{2}, 4, 5, 5, 5, \ldots\} \quad \ldots(8)$$

General expression : The given sequence x (n) is ,

$$\therefore \quad x(n) = \{\underset{\uparrow}{2}, 2, 1\} \text{ and } h(n) = u(n) = \{\underset{\uparrow}{1}, 1, 1, 1, \ldots\}$$

$$\therefore \quad x(0) = 2 \qquad \therefore \ h(0) = u(0) = 1$$

$$x(1) = 2 \qquad h(1) = u(1) = 1$$

and $\quad x(2) = 1 \qquad h(2) = u(2) = 1$

$$h(3) = u(3) = 1 \ldots$$

Now observe Equations (6) and (7) carefully. We can write the generalized expression to obtain Equation (8) as follows,

$$y(n) = x(0)\,u(n) + x(1)\,u(n-1) + x(2)\,u(n-2) + \ldots \quad \ldots(9)$$

Let us verify this equation.

To calculate y (0) put n = 0 in Equation (9).

$$\therefore \ y(0) = x(0)\,u(0) + x(1)\,u(-1) + x(2)\,u(-2)$$

$$\therefore \ y(0) = (2\times 1) + (2\times 0) + (1\times 0) = 2$$

This is because $u(0) = 1$, $u(-1) = 0$ and $u(-2) = 0$. This can be observed from unit step shown in Fig. P. 5.3.4(k).

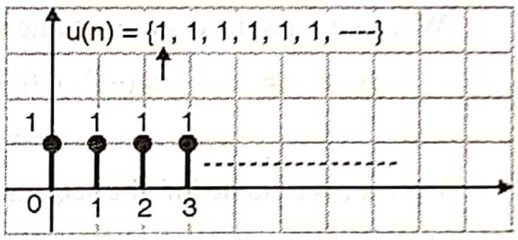


Fig. P. 5.3.4(k) : Unit step u (n)

Similarly putting n = 1 in Equation (9) we get,

$$\therefore \quad y(1) = x(0)\,u(1) + x(1)\,u(1) + x(2)\,u(-1)$$

$$\therefore \quad y(1) = (2\times 1) + (2\times 1) + (1\times 0) = 2 + 2 + 0$$

$$\therefore \quad y(1) = 4$$

Similarly other values can be verified.

Ex. 5.3.5 : Obtain linear convolution of two discrete time signals given as :

$x(n) = u(n)$

$h(n) = a^n u(n),\ a < 1$

Soln. :

Given x (n) = u (n) i.e. unit step which is shown in Fig. P. 5.3.4(k). $h(n) = a^n u(n)$ is an exponential signal. Here a^n is exponential signal which is multiplied by u (n). We know that u (n) is present only for positive values of 'n'. That means from n = 0 to n = ∞. So this multiplication indicates that a^n is present for positive values of 'n'. It will not affect the magnitude of a^n, because magnitude of u (n) is 1. This sequence a^n is shown in Fig. P. 5.3.5(a).

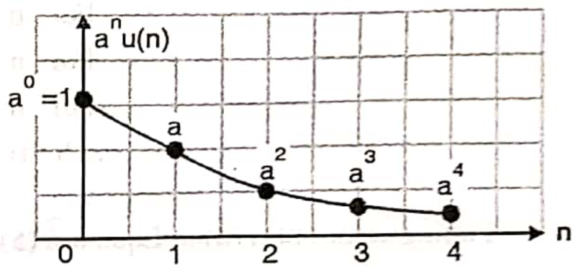


Fig. P. 5.3.5(a) : Exponential sequence a^n

Recall the equation of convolution,

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k) \quad ...(1)$$

In this equation h (n – k) is delayed version of h (k). Instead of that we can delay x (k) and keep h (k) as it is.

$$\therefore \quad y(n) = \sum_{k=-\infty}^{\infty} h(k)\, x(n-k) \quad ...(2)$$

This is commutative property of convolution which will be discussed later in this chapter. We will use Equation (2) in this example because it is easy to delay u (n) than a^n u (n).

Now we have derived generalized expression of convolution for finite sequence with an infinite sequence. This equation is,

$$y(n) = x(0)\,u(n) + x(1)\,u(n-1) + x(2)\,u(n-2) + \ldots \quad ...(3)$$

When both sequences are infinite then Equation (3) can be easily modified as,

$$y(n) = \ldots\ldots x(-2)\,u(n+2) + x(-1)\,u(n+1) + x(0)\,u(n) + x(1)\,u(n-1) + x(2)\,u(n-2) + \ldots \quad ...(4)$$

Here x (n) is some infinite sequence and h (n) = u (n).

Equation (4) is based on Equation (1). In this case input x (k) is kept as it is and h (k) is delayed. But if we want to use Equation (2) to obtain linear convolution then,

$$y(n) = \ldots\ldots h(-2)\,u(n+2) + h(-1)\,u(n+1) + h(0)\,u(n) + h(1)\,u(n-1) + h(2)\,u(n-2) + \ldots$$

In Equation (5) we are taking x (n) = u (n).

The given equation of h (n) is,

$$h(n) = a^n u(n) \quad ...(6)$$

$$\therefore h(n) = \begin{cases} a^n & \text{for } n \geq 0 \\ 0 & \text{for } n < 0 \end{cases} \quad ...(7)$$

Sequence h (n) can be written by putting values of n in Equation (7). That means,

$$\text{For } n = 0 \Rightarrow h(n) = h(0) = a^0 = 1$$

$$\text{For } n = 1 \Rightarrow h(n) = h(1) = a^1 = a$$

$$\text{For } n = 2 \Rightarrow h(n) = h(2) = a^2$$

$$\text{For } n = 3 \Rightarrow h(n) = h(3) = a^3$$

$$\therefore h(n) = \{\underset{\uparrow}{1}, a, a^2, a^3 \ldots\} \quad ...(8)$$

Putting values of h (n) in Equation (5) we get,

$$\therefore \quad y(n) = 1\,u(n) + a\,u(n-1) + a^2\,u(n-2) + a^3\,u(n-3) + \ldots \quad ...(9)$$

Equation (9) gives the convolution of given sequences. To calculate sequence $y(n)$, put different values of n in Equation (9).

$$\therefore \; y(0) = u(0) + a\,u(-1) + a^2 u(-2) + a^3 u(-3)$$

$$\text{Here } u(0) = 1,\; u(-1) = u(-2) = u(-3) = 0$$

$$\therefore \; y(0) = 1$$

Putting $n = 1$ in Equation (9) we get,

$$y(1) = 1\,u(1) + a\,u(0) + a^2 u(-1) + a^3 u(-2)$$

$$\text{Here } u(1) = u(0) = 1 \text{ and } u(-1) = u(-2) = 0$$

$$\therefore \; y(1) = 1 + a$$

Now putting $n = 2$ in Equation (9) we get,

$$y(2) = u(2) + a\,u(1) + a^2 u(0) + a^3 u(-1)$$

$$\text{Here } u(2) = u(1) = u(0) = 1 \text{ and } u(-1) = 0$$

$$\therefore \; y(2) = 1 + a + a^2$$

Similarly for the n^{th} sample, $y(n)$ can be written as,

$$\therefore \; y(n) = 1 + a + a^2 + a^3 + \ldots\ldots + a^n \quad \ldots(10)$$

In terms of summation Equation (10) can be written as,

$$y(n) = \sum_{k=0}^{n} a^k \quad \ldots(11)$$

Now use standard equation of summation,

$$\sum_{n=N_1}^{N_2} a^n = \begin{cases} \dfrac{a^{N_1} - a^{N_2+1}}{1-a} & \text{For } a \neq 1 \\ N_2 - N_1 + 1 & \text{For } a = 1 \end{cases} \quad \ldots(12)$$

In the present example $a < 1$ thus $a \neq 1$.

Now from Equation (11),

$$N_1 = \text{lower limit of summation} = 0$$

$$N_2 = \text{upper limit of summation} = n$$

Putting these values in the standard equation of summation (Equation (12)) we get,

$$y(n) = \frac{a^0 - a^{n+1}}{1-a}$$

$$\therefore \; y(n) = \frac{1 - a^{n+1}}{1-a}$$

5.3.2 Calculation using Basic Equation of Convolution :

In the graphical method, we studied how to obtain convolution of two signals, using graphs. In that method, every time it is required to draw the plot of signal. So that is tedious method. One more simple method is to calculate the convolution of two signals using basic equation of convolution.

How to solve ?

According to the definition of convolution we have,

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k) \quad \text{...(5.3.5)}$$

Step 1 : **Write down the equations of x (k) and h (k)**

Step 2 : **Decide range of 'n'**

We have studied, how to decide range of 'n'. Let us recall those equations.

Lowest index of 'n' for y (n) $\Rightarrow$ Lowest index of x (n) + Lowest index of h (n)

$$\therefore \quad y_l = x_l + h_l \quad \text{...(5.3.6)}$$

and Highest index of 'n' for y (n) $\Rightarrow$ Highest index of x (n) + Highest index of h (n)

$$\therefore \quad y_h = x_h + h_h \quad \text{...(5.3.7)}$$

Decide range of 'k' :

Range of k is same as range of x (k).

Step 3 : **Once the range of 'n' and 'k' is decided; then put different values of 'n' in the equation of y (n).**

Ex. 5.3.6 : Compute linear convolution of the following :

$x(n) = \{ \underset{\uparrow}{1}, 1, 1, 1 \}$ and $h(n) = \{ \underset{\uparrow}{1}, 1, 1, 1 \}$

Soln. : We have,

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k) \quad \text{...(1)}$$

Step 1 : **The sequences x (k) and h (k) can be written as,**

$$x(k) = \{ \underset{\uparrow}{1}, 1, 1, 1 \} \quad \text{and} \quad h(k) = \{ \underset{\uparrow}{1}, 1, 1, 1 \}$$

$\therefore x(0) = 1$	and	$h(0) = 1$
$x(1) = 1$		$h(1) = 1$
$x(2) = 1$		$h(2) = 1$
$x(3) = 1$		$h(3) = 1$

-2 -1 0 1 2

Step 2 : **Range of 'n' for y (n) is,**

Lowest index : $y_l = x_l + h_l = 0 + 0 = 0$

Highest index : $y_h = x_h + h_h = 3 + 3 = 6$

Thus range of y (n) is from y (0) to y (6).

Now range of 'k' is same as x (k).

Thus range of 'k' is from k = 0 to k = 3.

Step 3 : **Putting the value of 'k' in Equation (1) we get,**

$$y(n) = \sum_{k=0}^{3} x(k)\, h(n-k) \qquad \ldots(2)$$

Calculation of y (0) :

Putting n = 0 in Equation (2) we get,

$$y(0) = \sum_{k=0}^{3} x(k)\, h(0-k) = \sum_{k=0}^{3} x(k)\, h(-k)$$

Expanding the summation by putting values of 'k' we get,

$$y(0) = x(0)\,h(-0) + x(1)\,h(-1) + x(2)\,h(-2) + x(3)\,h(-3)$$

$$\therefore\ y(0) = (1\times 1) + (1\times 0) + (1\times 0) + (1\times 0) \ldots \text{as } h(-1) = h(-2) = h(-3) = 0$$

$$\therefore\ y(0) = 1$$

Calculation of y (1) :

Putting n = 1 in Equation (2) we get,

$$y(1) = \sum_{k=0}^{3} x(k)\, h(1-k)$$

Expanding the summation we get,

$$\therefore\ y(1) = x(0)\,h(1) + x(1)\,h(0) + x(2)\,h(-1) + x(3)\,h(-2)$$

$$\therefore\ y(1) = (1\times 1) + (1\times 1) + (1\times 0) + (1\times 0)\,.$$

$$y(1) = 1 + 1 \quad \ldots\ \text{as } h(-1) = h(-2) = 0$$

$$\therefore\ y(1) = 2$$

Similarly we can calculate the remaining values as follows :

$$y(2) = \sum_{k=0}^{3} x(k)\, h(2-k) = 3$$

$$y(3) = \sum_{k=0}^{3} x(k)\, h(3-k) = 4$$

$$y(4) = \sum_{k=0}^{3} x(k)\, h(4-k) = 3$$

$$y(5) = \sum_{k=0}^{3} x(k)\, h(5-k) = 2$$

$$y(6) = \sum_{k=0}^{3} x(k)\, h(6-k) = 1$$

Step 4 : The result of convolution y (n) is given by,

$$y(n) = \{y(0), y(1), y(2), y(3), y(4), y(5), y(6)\}$$

$$\therefore\ y(n) = \{\underset{\uparrow}{1}, 2, 3, 4, 3, 2, 1\}$$

Note that arrow is marked at the sample y (0).

Ex. 5.3.7 : Compute the linear convolution of following :

$x(n) = 1$ and $h(n) = \{2, 1, 2, 1\}$

Soln. :

Step 1 : The sequences x (k) and h (k) can be written as,

$$x(k) = \{\underset{\uparrow}{1}\} \text{ and } h(k) = \{\underset{\uparrow}{2}, 1, 2, 1\}$$

$$\therefore\ x(0) = 1 \qquad \therefore\ h(0) = 2$$

$$h(1) = 1$$

$$h(2) = 2$$

$$h(3) = 1$$

Note that arrow is not present in the given sequences of x (n) and h (n). So by default it is at first (starting) position.

Step 2 : Range of 'n' for y (n) is,

Lowest index : $y_l = x_l + h_l = 0 + 0 = 0$

Highest index : $y_h = x_h + h_h = 0 + 3 = 3$

Thus range of y (n) is from y (0) to y (3).

Step 3 : According to definition of convolution we have,

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k) \qquad ...(1)$$

In this case h (n – k) indicates folding and delaying h (– k). Since x (k) is smaller sequence than h (k); it will be convenient to use following equation,

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)\, x(n-k) \qquad ...(2)$$

Here range of 'k' will be as per the sequence h(k). Thus range of 'k' is from 0 to 3.

Calculation of y (0) :

Putting n = 0 in Equation (2) we get,

$$y(0) = \sum_{k=0}^{3} h(k)\, x(-k)$$

Expanding the summation we get,

$$y(0) = h(0)\,x(0) + h(1)\,x(-1) + h(2)\,x(-2) + h(3)\,x(-3)$$

$$\therefore\ y(0) = (2\times 1) + (1\times 0) + (2\times 0) + (1\times 0)$$

$$\therefore\ y(0) = 2 + 0 + 0 + 0 \qquad \text{... as } x(-1) = x(-2) = x(-3) = 0$$

$$\therefore\ y(0) = 2$$

Similarly we can calculate the remaining values of y (n)

$$\therefore\ y(1) = 1$$

$$\therefore\ y(2) = 2$$

$$\therefore\ y(3) = 1$$

Step 4 : **Now the result of convolution can be written as,**

$$y(n) = \{y(0), y(1), y(2), y(3)\}$$

$$\therefore\ y(n) = \{\underset{\uparrow}{2}, 1, 2, 1\}$$

Note : Here given sequence $x(n) = \{\underset{\uparrow}{1}\}$. We know that δ (n) is the unit impulse and is expressed as, $\delta(n) = \{\underset{\uparrow}{1}\}$. That means we have calculated convolution of $\{\underset{\uparrow}{2}, 1, 2, 1\}$ with unit impulse. Observe the result of convolution. It is the same as $\{\underset{\uparrow}{2}, 1, 2, 1\}$. Thus we can conclude that the convolution of any sequence with unit impulse produces the same sequence.

5.3.3 Tabulation Method of Linear Convolution :

This is the simplest method of performing linear convolution.

Let $x(n) = \{\underset{\uparrow}{x(0)}, x(1), x(2)\}$ and $h(n) = \{\underset{\uparrow}{h(0)}, h(1), h(2)\}$

Step 1 : **Form the matrix as shown in Fig. 5.3.1 :**

	$x(0)$	$x(1)$	$x(2)$
$h(0)$			
$h(1)$			
$h(2)$			

(Columns: $x(n)$; rows: $h(n)$)

Fig. 5.3.1 : Matrix of x (n) and h (n)

Note that we can interchange the positions of x (n) and h (n).

Step 2 : **Multiply the corresponding elements of x (n) and h (n) as shown in Fig. 5.3.2 :**

	$x(0)$	$x(1)$	$x(2)$
$h(0)$	$h(0)x(0)$	$h(0)x(1)$	$h(0)x(2)$
$h(1)$	$h(1)x(0)$	$h(1)x(1)$	$h(1)x(2)$
$h(2)$	$h(2)x(0)$	$h(2)x(1)$	$h(2)x(2)$

Fig. 5.3.2 : Multiplication of elements x (n) and h (n)

Step 3 : **Separate out the elements diagonally as shown in Fig. 5.3.3 :**

	$x(0)$	$x(1)$	$x(2)$
$h(0)$	$h(0)x(0)$	$h(0)x(1)$	$h(0)x(2)$
$h(1)$	$h(1)x(0)$	$h(1)x(1)$	$h(1)x(2)$
$h(2)$	$h(2)x(0)$	$h(2)x(1)$	$h(2)x(2)$

Fig. 5.3.3 : Diagonally separation of elements

Step 4 : **Simply add the elements in that particular block. This addition gives corresponding values of y (n).**

Here range of 'n' is : $y_l = x_l + h_l = 0 + 0 = 0$

and $y_h = x_h + h_h = 2 + 2 = 4$

Thus we get, $y(0) = h(0)\,x(0)$

$$y(1) = h(1)x(1) + h(0)x(1)$$

$$y(2) = h(2)x(0) + h(1)x(1) + h(0)x(2)$$

$$y(3) = h(2)x(1) + h(1)x(2)$$

and $$y(4) = h(2)x(2)$$

After calculating all these values; write down the result of convolution as,

$$y(n) = \{\underset{\uparrow}{y(0)}, y(1), y(2), y(3), y(4)\}$$

Ex. 5.3.8 : Compute the convolution $y(n) = x(n) * h(n)$

where $x(n) = \{1, \underset{\uparrow}{1}, 0, 1, 1\}$ and $h(n) = \{1, -2, -3, \underset{\uparrow}{4}\}$

$-2 + (-3) = -5$

$2 + 1 = 3$

Soln. : Here x (n) contains '5' samples and h (n) has '4' samples. To make the length of x (n) and h (n) same; rewrite the sequence h (n) as follows :

$$h(n) = \{1, -2, -3, \underset{\uparrow}{4}, 0\}$$

We can add zeros at the end or at the beginning of sequence. This will not affect the given sequence. This method of adding zeros, to adjust the length of sequence is called as **zero-padding.**

Range of 'n' : The range of 'n' for y (n) is calculated as follows :

$x(n) = \{1, 1, \underset{\uparrow}{0}, 1, 1\}$

$h(n) = \{1, -2, -3, \underset{\uparrow}{4}, 0\}$

Lowest index of y (n) ⇒ $y_l = x_l + h_l = -2 + (-3) = -5$

and Highest index of y (n) ⇒ $y_h = x_h + h_h = 2 + 1 = 3$

Thus y (n) varies from y (– 5) to y (3). Now using tabulation method the convolution is obtained as shown in Fig. P. 5.3.8.

h(n) \ x(n)	1	1	0	1	1
1					
–2					
–3					
4					
0					

(1 1 0 1 1)(1 2 3 4)

-5 -4 -3 -2 -1 0 1 2 3

Fig. P. 5.3.8(a) : Matrix of x (n) and h (n)

	1	1	0	1	1
1	1 × 1	1 × 1	1 × 0	1 × 1	1 × 1
– 2	– 2 × 1	– 2 × 1	– 2 × 0	– 2 × 1	– 2 × 1
– 3	– 3 × 1	– 3 × 1	– 3 × 0	– 3 × 1	– 3 × 1
4	4 × 1	4 × 1	4 × 0	4 × 1	4 × 1
0	0 × 1	0 × 1	0 × 0	0 × 1	0 × 1

Fig. P. 5.3.8(b) : Convolution of x (n) and h (n)

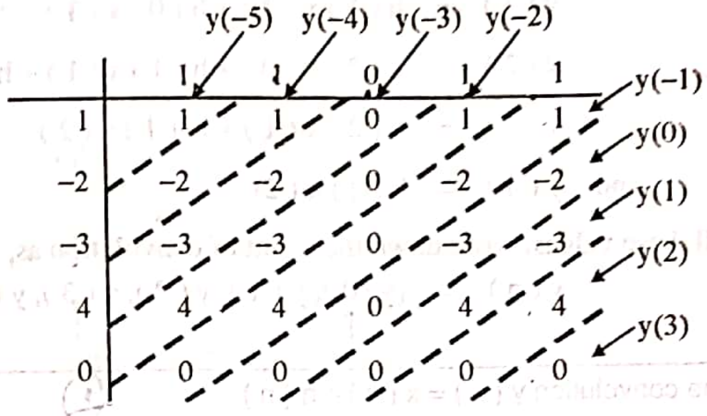


Fig. P. 5.3.8(c) : Diagonally separation of elements

Different values of y (n) are calculated by adding corresponding elements as follows :

$$y(-5) = 1$$
$$y(-4) = -2+1=-1$$
$$y(-3) = -3-2+0=-5$$

$y(-2) = 4-3+0+1=2$ $\qquad$ $y(-1) = 0+4+0-2+1=3$

$y(0) = 0+0-3-2=-5$ $\qquad$ $y(1) = 0+4-3=1$

$y(2) = 0+4=4$ $\qquad$ $y(3) = 0$

Thus result of convolution y (n) is ,

$$y(n) = \{y(-5), y(-4), y(-3), y(-2), y(-1), \underset{\uparrow}{y(0)}, y(1), y(2), y(3)\}$$

lowest =

$$\therefore \; y(n) = \{1, -1, -5, 2, 3, \underset{\uparrow}{-5}, 1, 4, 0\}$$

Neglecting the last '0' term we have,

$$y(n) = \{1, -1, -5, 2, 3, \underset{\uparrow}{-5}, 1, 4\}$$

Ex. 5.3.9 : Compute $y(n) = x(n) * h(n)$

If $x(n) = h(n) = \{\underset{\uparrow}{1}, 2, -1, 3\}$

Soln. :

0+0 =0

3+3 =0

Range of n :

Lowest range of y (n) ⇒ $y_l = x_l + h_l = 0 + 0 = 0$

Highest range of y (n) ⇒ $y_h = x_h + h_h = 3 + 3 = 6$

Using tabular method; the convolution is obtained as shown in Fig. P. 5.3.9.

	1 ⇐ y(0)	2 ⇐ y(1)	–1 ⇐ y(2)	3	
1	1	2	–1	3	⇐ y(3)
2	2	4	–2	6	⇐ y(4)
–1	–1	–2	1	–3	⇐ y(5)
3	3	6	–3	9	⇐ y(6)

Fig. P. 5.3.9 : Linear convolution of x (n) and h (n)

$$y(n) = \{\underset{\uparrow}{1}, 4, 2, 2, 13, -6, 9\}$$

Ex. 5.3.10 : The impulse response of linear time invarient is :

$h(n) = \{1, \underset{\uparrow}{2}, 1, -1\}$

Determine the response of system to the input

$x(n) = \{1, \underset{\uparrow}{2}, 3, 1\}$

-1+1 = 0

2+2 =

Soln. : The response of the system is,

$y(n) = x(n) * h(n)$

Range of n :

Lowest range of y (n) $\Rightarrow y_l = x_l + h_l = 0 + 0 = 0$

Highest range of y (n) $\Rightarrow y_h = x_h + h_h = 3 + 3 = 6$

Using tabulation method; the convolution is obtained as shown in Fig. P. 5.3.10.

	1 ⇐ y(0)	2 ⇐ y(1)	1 ⇐ y(2)	–1	
1	1	2	1	–1	⇐ y(3)
2	2	4	2	–2	⇐ y(4)
3	3	6	3	–3	⇐ y(5)
1	1	2	1	–1	⇐ y(6)

Fig. P. 5.3.10 : Linear convolution of x (n) and h (n)

$$\therefore \quad y(n) = \{\underset{\uparrow}{1}, 4, 8, 8, 3, -2, -1\}$$

Ex. 5.3.11 : Use discrete convolution to find the response to the input x (n) of the LTI system with impulse response h (n) for the corresponding x (n) shown in Fig. P. 5.3.11(a) and (b).

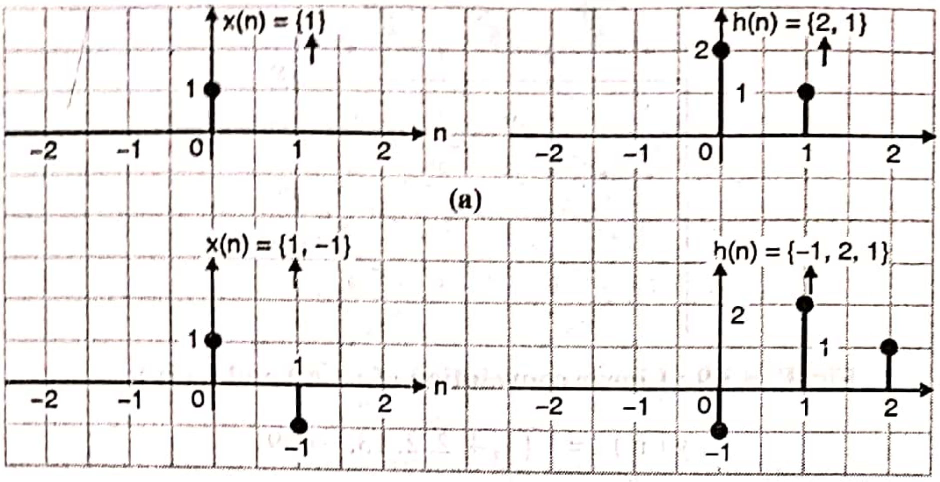


Fig. P. 5.3.11 : Given sequence x (n) and h (n)

Ans. :

$$\therefore \quad y(n) = \{-\underset{\uparrow}{1}, 3, -1, -1\}$$

Ex. 5.3.12 : Find out output of system if input x (n) and impulse response h (n) are given by, $x(n) = n, 0, \leq n \leq 3$, $h(n) = \{2, 3, 1\}$

Soln. : Given $x(n) = n, 0 \leq n \leq 3$.

We will put values of n to obtain sequence x (n).

For $n = 0 \Rightarrow x(0) = 0$

For $n = 1 \Rightarrow x(1) = 1$

For $n = 2 \Rightarrow x(2) = 2$

For $n = 3 \Rightarrow x(3) = 3$

$$\therefore \quad x(n) = \{\underset{\uparrow}{0}, 1, 2, 3\} \text{ and } h(n) = \{\underset{\uparrow}{2}, 3, 1\} = \{\underset{\uparrow}{2}, 3, 1, 0\}$$

Range of 'n' in y (n) is n = 0 to 6.

We will use tabular method to obtain output.

$$y(n) = x(n) * h(n)$$

$$\therefore \quad y(n) = \{\underset{\uparrow}{0}, 2, 7, 13, 11, 3, 0\}$$

	0	1	2	3
2	0	2	4	6
3	0	3	6	9
1	0	1	2	3
0	0	0	0	0

Fig. P. 5.3.12

This is the output of system.

Ex. 5.3.13 : Find the following convolution :

if $x(n) = \{2, 3, \underset{\uparrow}{1}, 4\}$

$h(n) = \{-1, 2, \underset{\uparrow}{3}\}$

and $y(n) = x(n) * h(n)$.

Soln. :

Given $x(n) = \{2, \underset{\uparrow}{3}, 1, 4\}$

and $h(n) = \{-1, 2, \underset{\uparrow}{3}\} = \{-1, 2, \underset{\uparrow}{3}, 0\}$

Here range of 'n' in y (n) is n = – 3 to 2.

We will use tabular method to obtain

$y(n) = x(n) * h(n)$

	2	6	2	8
–1	–2	–3	–1	–4
2	4	6	2	8
3	6	9	3	12
0	0	0	0	0

Fig. P. 5.3.13

$$\therefore\ y(n) = \{-2, 1, 11, \underset{\uparrow}{7}, 11, 12, 0\}$$

5.3.4 Multiplication Method of Linear Convolution :

This is another easy method to obtain convolution of two sequences. This method is similar to multiplication of multidigit numbers. Write down the two sequences, x (n) and h (n) and obtain the multiplication by usual method. The result of multiplication will be equal to the convolution of two sequences. We will solve one example by this method.

Ex. 5.3.14 : Obtain linear convolution of following sequences :

$x(n) = \{1, \underset{\uparrow}{2}, 1, 2\}$ and $h(n) = \{2, 2, -1, \underset{\uparrow}{1}\}$

Soln. :

First we will decide the range of 'n' for y (n).

Lowest index $\Rightarrow y_l = x_l + h_l = -1 + (-2) = -3$

Highest index $\Rightarrow y_h = x_h + h_h = 2 + 1 = 3$

Thus y (n) varies from y (– 3) to y (3)

We will solve this problem using multiplication method.

				x(n) :- 1	2	1	2	
Multiplication				h(n) :- ⊗ 2	2	−1	1	
				1	2	1	2	⇐ x(n) is multiplied by (1)
Addition → ⊕			−1	−2	−1	−2	0	⇐ x(n) is multiplied by (−1) Zero digit similar to normal multiplication
Addition → ⊕		2	4	2	4	0	0	⇐ x(n) is multiplied by (2) Two zero digits similar to normal multiplication
Addition → ⊕	2	4	2	4	0	0	0	⇐ x(n) is multiplied by (2) Three zero digits similar to normal multiplication
	2	6	5	5	5	−1	2	⇐ These values are obtained by adding all digits in corresponding columns.

This result of multiplication is same as linear convolution. We know that range of $y(n)$ is from $y(-3)$ to $y(3)$. So the first term from L.H.S. represents the value of $y(-3)$. Now mark the arrow at the position of $y(0)$. Thus output $y(n)$ can be written as,

$$y(n) = \{2, 6, 5, \underset{\uparrow}{5}, 5, -1, 2\}$$

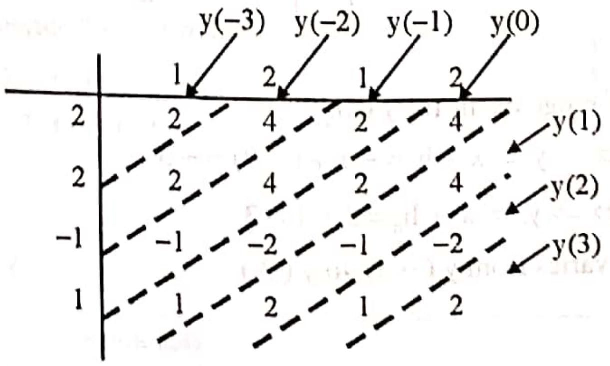
Let us verify this result using tabulation method shown in Fig. P. 5.3.14.

h(n) \ x(n)	1	2	1	2
2				
2				
−1				
1				

(a)

	1	2	1	2
2	(2 × 1)	(2 × 2)	(2 × 1)	(2 × 2)
2	(2 × 1)	(2 × 2)	(2 × 1)	(2 × 2)
−1	(− 1 × 1)	(− 1 × 2)	(− 1 × 1)	(− 1 × 2)
1	(1 × 1)	(1 × 2)	(1 × 1)	(1 × 2)

(b)



(c)

Fig. P. 5.3.14 : Convolution of $x(n) = \{1, 2, 1, \underset{\uparrow}{2}\}$ and $h(n) = \{2, 2, -\underset{\uparrow}{1}, 1\}$

Thus using tabulation method we get,

$y(-3) = 2$ $\quad$ $y(-2) = 2+4=6$

$y(-1) = -1+4+2=5$ $\quad$ $y(0) = 1-2+2+4=5$

$y(3) = 2$ $\quad$ $y(2) = 1-2=-1$

$$\therefore \; y(n) = \{2, 6, 5, \underset{\uparrow}{5}, 5, -1, 2\}$$

5.4 Properties of Linear Convolution :

We have discussed how to obtain the convolution of two sequences. In this sub-section we will study some important properties of linear convolution. These properties are :

1. Commutative property
2. Associative property
3. Distributive property.

1. Commutative property :

Statement : It states that linear convolution is commutative operation. That means

$$x(n) * h(n) = h(n) * x(n)$$

Proof : The linear convolution of $x(n)$ and $h(n)$ is given by,

$$y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k) \qquad \text{...(5.4.1)}$$

In this equation $h(n-k)$ is folded delayed sequence; while $x(k)$ is unchanged.

Put $m = n-k$ as new index of summation in Equation (5.4.1).

$$\therefore \; k = n-m$$

Limits will change as follows :

When $k = -\infty \Rightarrow n-m = -\infty$ $\quad \therefore \; -m = -\infty - n$ $\quad \therefore \; m = \infty$

When $k = +\infty \Rightarrow n-m = \infty$ $\quad \therefore \; -m = \infty - n$ $\quad \therefore \; m = -\infty$

Putting these values in Equation (5.4.1) we get,

$$y(n) = \sum_{m=+\infty}^{-\infty} x(n-m)\, h(m) \qquad \text{...(5.4.2)}$$

In case of summation operation limits from $+\infty$ to $-\infty$ are same as $-\infty$ to $+\infty$. Thus changing limits of summation we get,

$$y(n) = \sum_{m=-\infty}^{\infty} x(n-m)\, h(m) \qquad \text{...(5.4.3)}$$

In Equation (5.4.3), m is dummy variable. It can be replaced by any other variable. So replacing 'm' by 'k' in Equation (5.4.3) we get,

$$y(n) = \sum_{k=-\infty}^{\infty} x(n-k)\, h(k) \qquad \text{...(5.4.4)}$$

In this equation $x(n-k)$ is folded and delayed sequence; while sequence $h(k)$ is unchanged. As per the definition of convolution Equation (5.4.4) can be written as,

$$y(n) = \sum_{k=-\infty}^{\infty} h(k) \cdot x(n-k) = h(n) * x(n) \qquad \text{...(5.4.5)}$$

Thus from Equations (5.4.1) and (5.4.5) we get,

$$y(n) = x(n) * h(n) = h(n) * x(n) \qquad \text{...(5.4.6)}$$

Hence proved that linear convolution is commutative.

Graphical presentation :

Graphically it is presented as shown in Fig. 5.4.1.

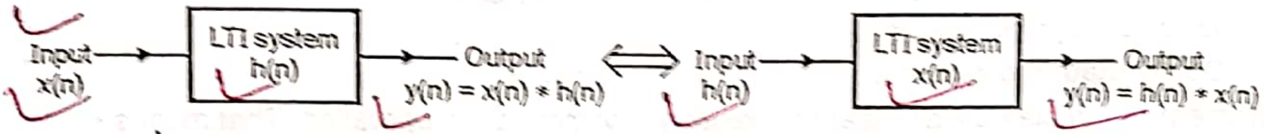


Fig. 5.4.1 : $x(n) * h(n) = h(n) * x(n)$

2. Associative property :

Statement : It states that $[x(n) * h_1(n)] * h_2(n) = x(n) * [h_1(n) * h_2(n)]$

Explanation : We know that '*' indicates convolution operation. Every convolution operation indicates that certain input is applied to the LTI system and output is the convolution of input signal and the impulse response of system.

Now consider L.H.S. term. That is the term, $[x(n) * h_1(n)] * h_2(n)$

Let this output is y (n)

$$\therefore \quad y(n) = [x(n) * h_1(n)] * h_2(n) \qquad \text{...(5.4.7)}$$

There are two convolution operations. The bracket term indicates one convolution of $x(n)$ and $h_1(n)$ Let this output be denoted by $y_1(n)$. Thus Equation (5.4.7) becomes,

$$y(n) = y_1(n) * h_2(n) \qquad \text{...(5.4.8)}$$

The convolution $y_1(n) = x(n) * h_1(n)$ indicates that, we are applying input to LTI system whose impulse response is $h_1(n)$. The output of system is $y_1(n)$. Such system is represented in Fig. 5.4.2(a).

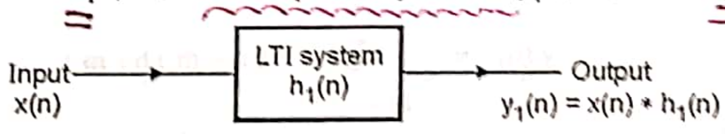


Fig. 5.4.2(a) : $y_1(n) = x(n) * h_1(n)$

Now according to Equation (5.4.8); $y(n)$ is obtained by computing convolution of $y_1(n)$ and $h_2(n)$. Here $y_1(n)$ acts as input signal to the LTI system whose impulse response is $h_2(n)$. The output of this system is $y(n)$. Diagrammatically it is represented as shown in Fig. 5.4.2(b).

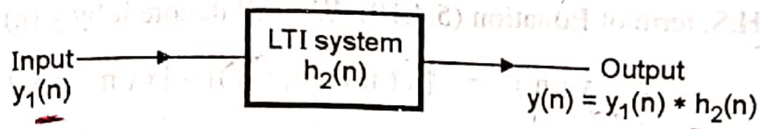


Fig. 5.4.2(b) : $y(n) = y_1(n) * h_2(n)$

Thus to implement Equation (5.4.1) we have to connect (cascade) two systems shown in Figs. 5.4.1 and 5.4.2(a). This cascade connection is shown in Fig. 5.4.2(c).

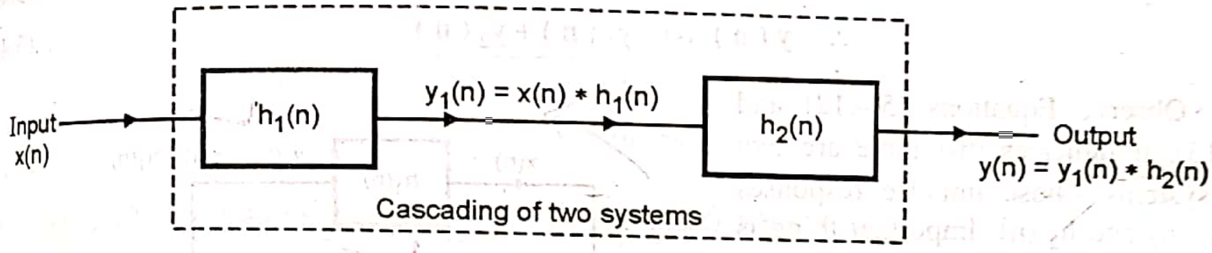


Fig. 5.4.2(c) : $y(n) = [x(n) * h_1(n)] * h_2(n)$

Now consider the R.H.S. term $x(n) * [h_1(n) * h_2(n)]$. We can design one LTI system whose impulse response $h(n)$ is the convolution of $h_1(n)$ and $h_2(n)$. That means convolution of two impulse responses. Thus $h(n) = h_1(n) * h_2(n)$.

$$\therefore \quad x(n) * [h_1(n) * h_2(n)] = x(n) * h(n) \qquad ...(5.4.9)$$

The diagrammatic representation of Equation (5.4.9) is shown in Fig. 5.4.2(d).

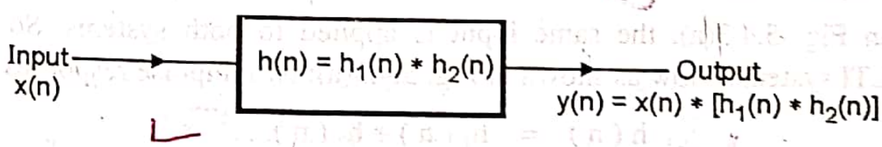


Fig. 5.4.2(d)

Meaning of associative property :

This property indicates that we can replace cascade combination of LTI systems by a single system whose impulse response is convolution of individual impulse responses. It is represented in Fig. 5.4.2(e).

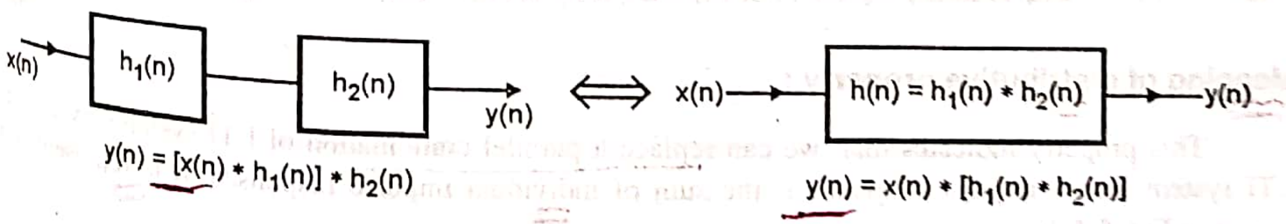


Fig. 5.4.2(e) : Meaning of associative property

3. **Distributive property :**

Statement : It states that linear convolution is distributive. That means

$$x(n) * [h_1(n) + h_2(n)] = [x(n) * h_1(n)] + [x(n) * h_2(n)] \qquad ...(5.4.10)$$

Proof : Consider the R.H.S. term of Equation (5.4.10). We will denote it by y (n).

$$\therefore \quad y(n) = [x(n) * h_1(n)] + [x(n) * h_2(n)] \quad ...(5.4.11)$$

Every bracket term in Equation (5.4.11) indicates the convolution operation. Thus we will denote output of each convolution as follows :

$$\therefore \quad y_1(n) = x(n) * h_1(n) \quad ...(5.4.12)$$

$$\text{and} \quad y_2(n) = x(n) * h_2(n) \quad ...(5.4.13)$$

$$\therefore \quad y(n) = y_1(n) + y_2(n) \quad ...(5.4.14)$$

Observe Equations (5.4.12) and (5.4.13). It indicates that there are two LTI systems whose impulse responses are h_1 (n) and h_2 (n). Important thing is that, the same input signal x (n) is applied to both systems. While Equation (5.4.14) shows that we have to add the outputs of these two LTI system. This process is represented diagrammatically as shown in Fig. 5.4.3(a).

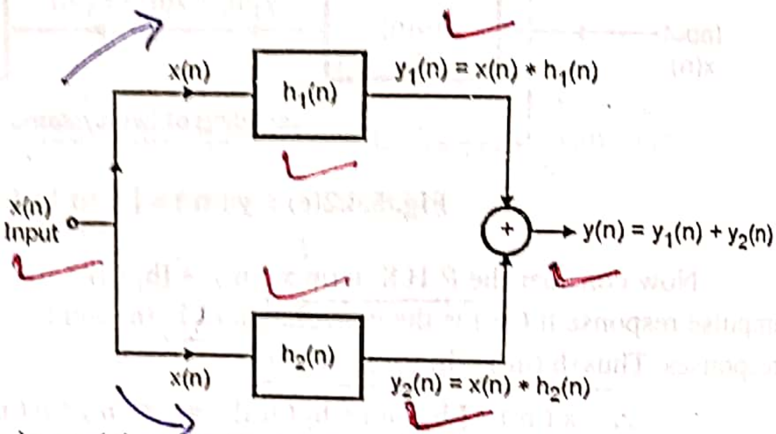


Fig. 5.4.3(a) : $y(n) = y_1(n) + y_2(n) = [x(n) * h_1(n)] + [x(n) * h_2(n)]$

As shown in Fig. 5.4.3(a), the same input is applied to both systems. So this is a parallel connection of two LTI systems. Now as shown in Fig. 5.4.3(a); two impulse responses are added.

$$\therefore \quad h(n) = h_1(n) + h_2(n) \quad ...(5.4.15)$$

Thus we can design a single LTI system having impulse response $h(n) = h_1(n) + h_2(n)$.

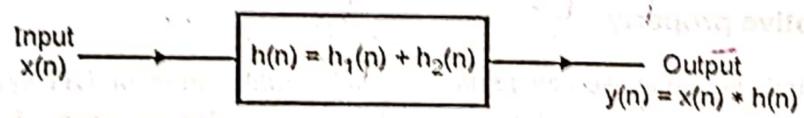


Fig. 5.4.3(b) : $y(n) = x(n) * h(n) = x(n) * [h_1(n) + h_2(n)]$

Meaning of distributive property :

This property indicates that, we can replace a parallel combination of LTI systems by a single LTI system whose impulse response is the sum of individual impulse responses. It is represented as shown in Fig. 5.4.3(c).

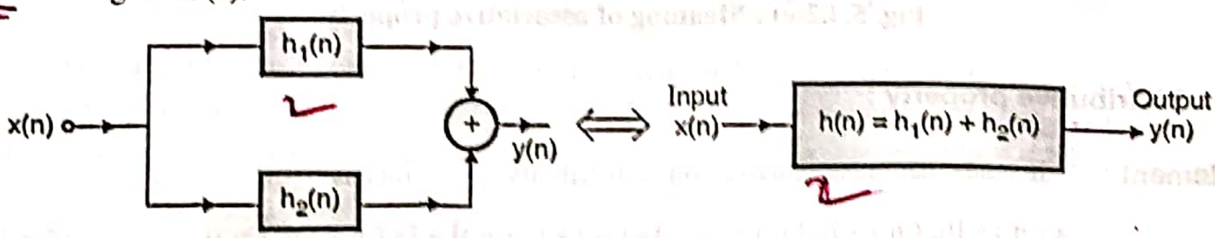


Fig. 5.4.3(c) : Meaning of distributive property

Summary of properties :

Table 5.4.1 shows the summary of properties of linear convolution.

Table 5.4.1

Sr. No.	Name of property	Mathematical equation
1.	Commutative	$x(n) * h(n) = h(n) * x(n)$
2.	Associative	$[x(n) * h_1(n)] * h_2(n) = x(n) * [h_1(n) * h_2(n)]$
3.	Distributive	$x(n) * [h_1(n) + h_2(n)] = [x(n) * h_1(n)] + [x(n) * h_2(n)]$

5.5 Impulse Response of Interconnected Systems :

In case of complicated operations; it is necessary to interconnect different subsystems. There are three types of interconnections as follows :

1. Cascade or series interconnection 2. Parallel interconnection 3. Feedback interconnection

1. Cascade or series interconnection :

In this case two systems are connected in series. That means output of one system is connected to the input of other system. This is shown in Fig. 5.5.1.

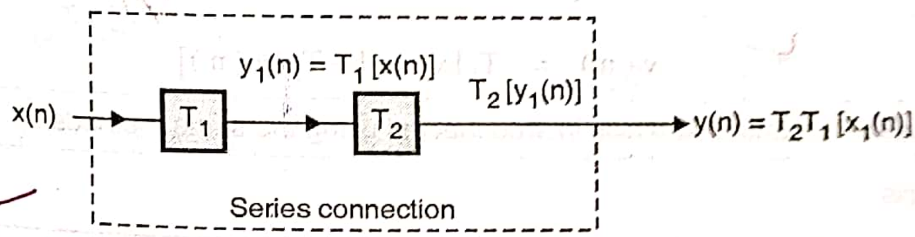


Fig. 5.5.1 : Cascade or series interconnection

Here T_1 and T_2 represents impulse responses of systems 1 and 2 respectively. Thus output of system-1 is,

$$y_1(n) = T_1[x(n)]$$

This is applied as input to system 2. Thus total output is,

$$y(n) = T_2[y_1(n)]$$

$$\therefore \quad y(n) = T_2T_1[x(n)]$$

2. Parallel connection :

Two systems can be connected in parallel as shown in Fig. 5.5.2. Here same input, $x(n)$ is applied to two systems.

Thus output of system-1 is,

$$y_1(n) = T_1[x(n)]$$

and output of system-2 is,

$$y_2(n) = T_2[x(n)]$$

Finally, these two outputs are added by using adder block. Thus the total output is,

$$y(n) = y_1(n) + y_2(n)$$

$$\therefore \quad y(n) = T_1[x(n)] + T_2[x(n)]$$

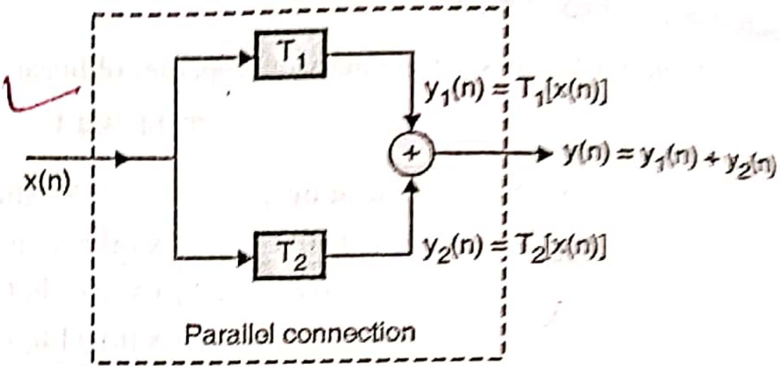


Fig. 5.5.2 : Parallel interconnection

3. Feedback Interconnection :

As the name indicates, a feedback connection is done from output to the input. This interconnection is as shown in Fig. 5.5.3.

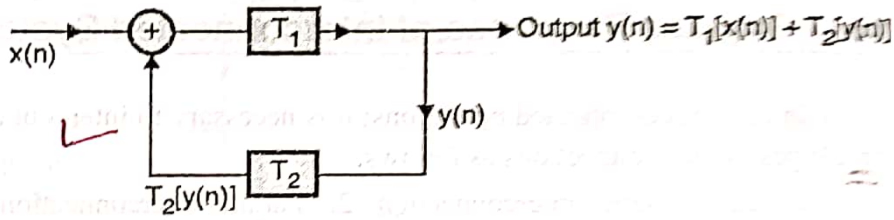


Fig. 5.5.3 : Feedback interconnection

Here output y (n) is applied to system-2 (T_2). Its output is T_2 [y (n)]. This signal is added with input x (n). So the output of adder is x (n) + T_2 [y (n)]. This signal is applied to system-1 (T_1). Thus the total output is,

$$y(n) = T_1[x(n)] + T_2[y(n)]$$

> **Note :** Continuous time systems are also interconnected using the same methods.

Solved Problems :

Ex. 5.5.1 : Find overall impulse response of the system shown in Fig. P. 5.5.1. Also plot the result.

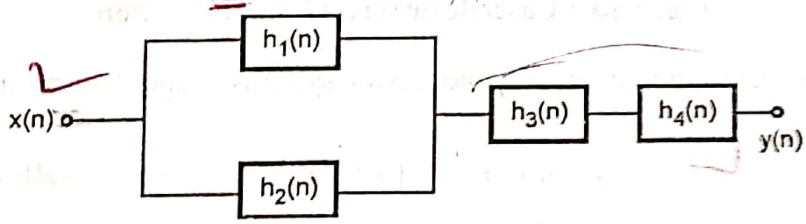


Fig. P. 5.5.1

$$h_1(n) = \{3, 4, \underset{\uparrow}{2}, 1\} \qquad h_3(n) = \{-2, \underset{\uparrow}{4}, 1\}$$

-2 -1 0

$$h_2(n) = \{2, \underset{\uparrow}{1}, 3, 1\} \qquad h_4(n) = \delta(n)$$

Soln. :

Here $h_1(n)$ and $h_2(n)$ are in parallel and $h_3(n)$ and $h_4(n)$ are in series; so applying the reduction rules the system can be drawn as shown in Fig. P. 5.5.1(a).

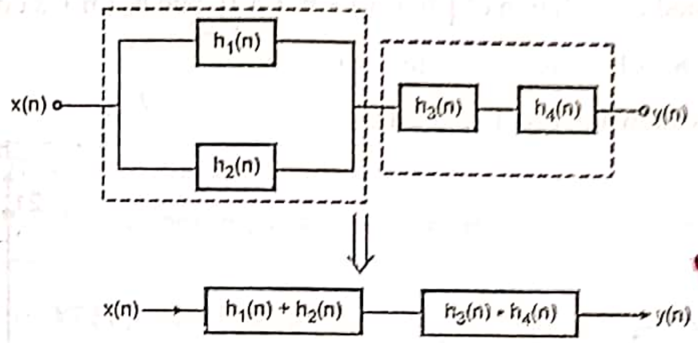


Fig. P. 5.5.1(a)

But $h_4(n) = \delta(n)$ which is unit impulse. The convolution of any function with respect to unit impulse results the same function.

$$\therefore \quad h_3(n) * h_4(n) = h_3(n)$$

Thus the block schematic can be drawn as shown in Fig. P. 5.5.1(b).

Now two blocks are in series so we can draw the block diagram as shown in Fig. P. 5.5.1(c).

$$\text{Thus} \quad h(n) = [h_1(n) + h_2(n)] * h_3(n)$$

Now we will calculate h (n).

The sequence $h_1(n) + h_2(n)$ is shown in Fig. P. 5.5.1(d).

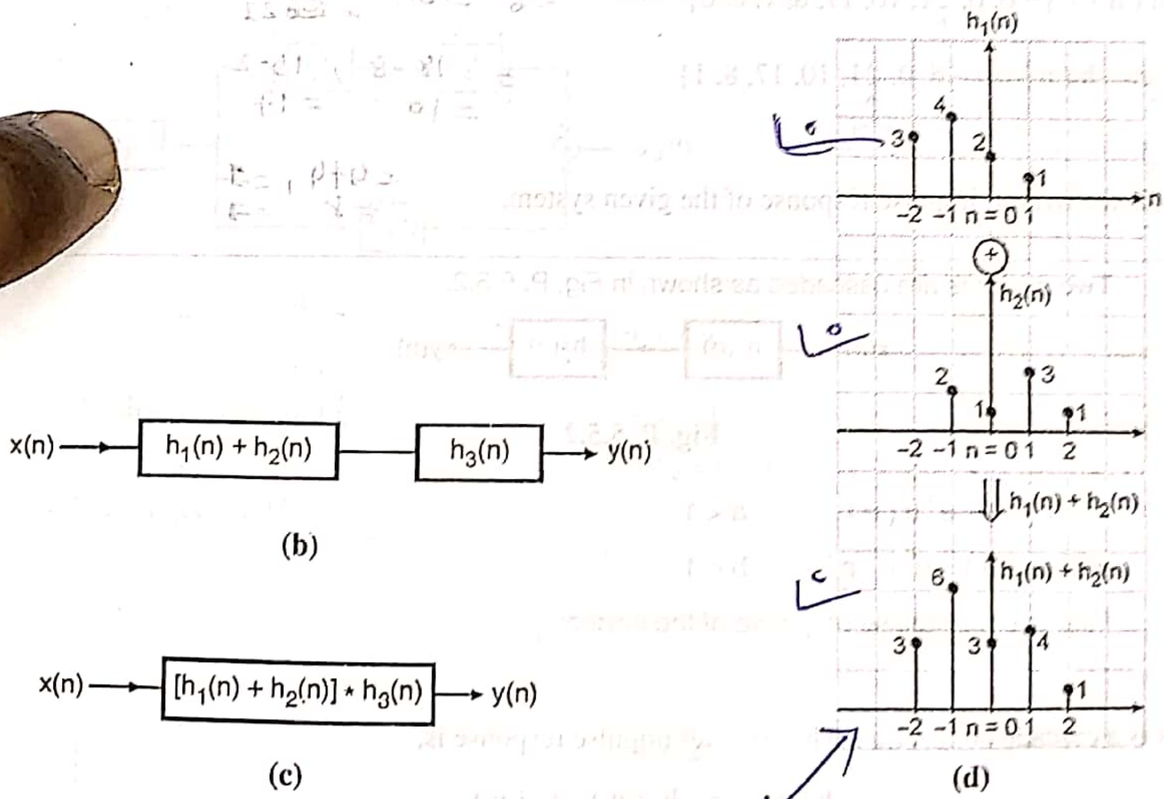


Fig. P. 5.5.1

$$\therefore \quad h_1(n) + h_2(n) = \{3, 6, \underset{\uparrow}{3}, 4, 1\}$$

and we have $h_3(n) = \{-\underset{\uparrow}{2}, 4, 1\} = \{-\underset{\uparrow}{2}, 4, 1, 0, 0\}$

Using tabular method convolution of $[h_1(n) + h_2(n)]$ and $h_3(n)$ is obtained as follows.

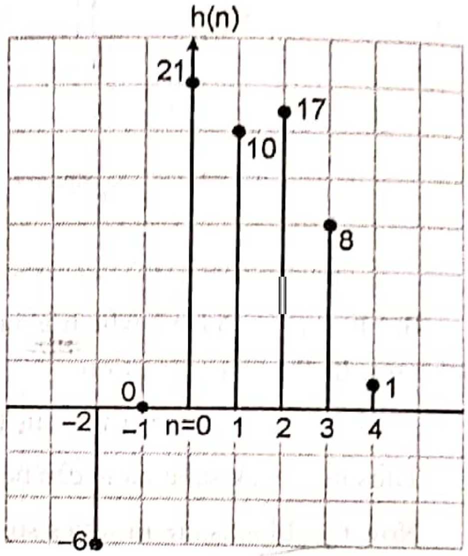
Here the range of 'n' is h (n) is n = − 2 to n = 6.

The plot of h (n) is shown in Fig. P. 5.5.1(f).

$\boxed{20 \text{ to } 26}$

	3	6	3	4	1
−2	−6	−12	−6	−8	−2
4	12	24	12	16	4
1	3	6	3	4	1
0	0	0	0	0	0
0	0	0	0	0	0

(e)



(f)

Fig. P. 5.5.1

$\therefore \quad h(n) = \{-6, 0, \underset{\uparrow}{21}, 10, 17, 8, 1, 0, 0\}$

$\therefore \quad h(n) = \{-6, 0, \underset{\uparrow}{21}, 10, 17, 8, 1\}$

−6, 12−12, 27−6
= −6 = 0 = 21
= 18−8, 19−2
= 10 = 17
= 4+4, = 1
= 8 = 1

This is the overall impulse response of the given system.

Ex. 5.5.2 : Two systems are cascaded as shown in Fig. P. 5.5.2.

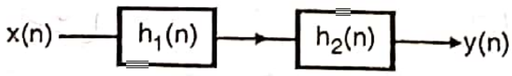


Fig. P. 5.5.2

If $h_1(n) = a^n u(n)$, $a < 1$

and $h_2(n) = b^n u(n)$, $b < 1$

Find overall impulse response of the system.

Soln. :

This is a cascade connection. Thus overall impulse response is,

$$h(n) = h_1(n) * h_2(n)$$

Here $h_1(n) = a^n u(n)$ and $h_2(n) = b^n u(n)$

Thus h (n) is obtained by performing convolution of $a^n u(n)$ and $b^n u(n)$. We have already obtained this convolution in Ex. 5.3.3.

$$\therefore \quad h(n) = b^n \left[\frac{1-\left(\frac{a}{b}\right)^{n+1}}{1-\frac{a}{b}} \right]$$

Ex. 5.5.3 : For the LTI system in the Fig. P. 5.5.3.

(a) Express the system impulse response as a function of impulse responses of the subsystems.

(b) $h_1(t) = h_2(t) = 5\,\delta(t)$; $h_3(t) = h_4(t) = u(t)$. Find the impulse response of the system.

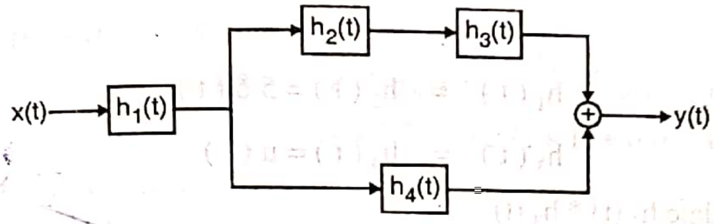


Fig. P. 5.5.3

Soln. :

(a) Here, $h_2(t)$ and $h_3(t)$ are in series. Let the combination of $h_2(t)$ and $h_3(t)$ is $h'(t)$.

$$\therefore \quad h'(t) = h_2(t) * h_3(t) \quad \text{...(1)}$$

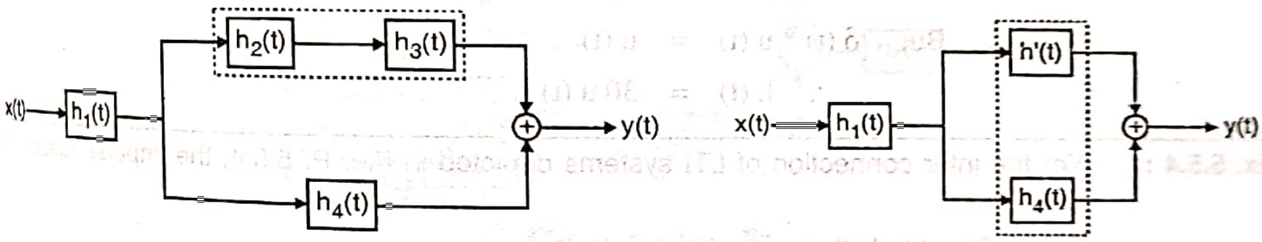


Fig. P. 5.5.3

As shown in above Fig. P. 5.5.3(a), $h'(t)$ and $h_4(t)$ are in parallel. Let, combination is $h''(t)$

$$\therefore \quad h''(t) = h_4(t) + h'(t) \quad \text{...(2)}$$

From Equation (1),

$$\therefore \quad h''(t) = h_4(t) + [h_2(t) * h_3(t)] \quad \text{...(3)}$$

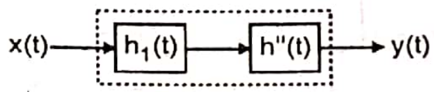


Fig. P. 5.5.3(c)

As shown in above Fig. P. 5.5.3(c), $h_1(t)$ and $h''(t)$ is in series. Let, combination is $h'''(t)$.

$$\therefore \quad h'''(t) = h_1(t) * h''(t)$$

From Equation (3).

$$h'''(t) = h_1(t) * \{h_4(t) + [h_2(t) * h_3(t)]\} \qquad ...(4)$$

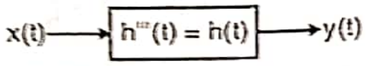


Fig. P. 5.5.3(d)

Thus, expression for system impulse response as a function of impulse responses of the system is,

$$h(t) = h_1(t) * \{h_4(t) + [h_2(t) * h_3(t)]\}$$

(b) Given :

$$h_1(t) = h_2(t) = 5\,\delta(t)$$

$$h_3(t) = h_4(t) = u(t)$$

First we will calculate $h_2(t) * h_3(t)$

$$\therefore \quad h_2(t) * h_3(t) = 5\,\delta(t) * u(t)$$

But convolution of any sequence with $\delta(t)$ produces the same sequence.

$$\therefore \quad h_2(t) * h_3(t) = 5\,[\delta(t) * u(t)] = 5\,u(t)$$

$$\text{Now} \quad h_4(t) + [h_2(t) * h_3(t)] = u(t) + 5\,u(t) = 6\,u(t)$$

$$h(t) = h_1(t) * \{h_4(t) + [h_2(t) * h_3(t)]\} = 5\,\delta(t) * 6\,u(t)$$

$$\text{But} \quad \delta(t) * u(t) = u(t)$$

$$\therefore \quad h(t) = 30\,u(t)$$

Ex. 5.5.4 : For the inter connection of LTI systems depicted in Fig. P. 5.5.4, the impulse responses are :

$h_1(t) = \delta(t-1)$; $h_2(t) = e^{-2t}\,u(t)$

$h_3(t) = \delta(t-1)$; $h_4(t) = e^{-3(t+2)}\,u(t+2)$.

Obtain equation of overall impulse response of the system.

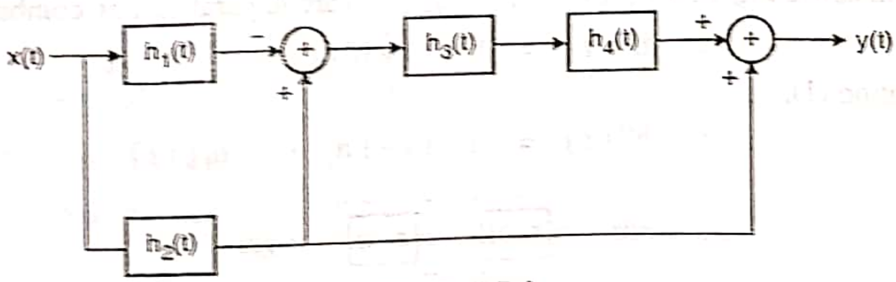


Fig. P. 5.5.4

Soln. :

The path in which block is absent ; is considered as $\delta(t)$. Now if the systems are in parallel, they are added and if the systems are in cascade then convolution is performed. The total impulse response can be expressed as,

$$h_T(t) = \{[h_2(t) - h_1(t)] * [(h_3(t) * h_4(t)] + \delta(t)\} \quad ...(1)$$

Consider the term $h_3(t) * h_4(t)$

$$\therefore \quad h_3(t) * h_4(t) = \delta(t-1) * e^{-3(t+2)} u(t+2) \quad ...(2)$$

We have the standard identity

$$x(t) * \delta(t - t_0) = x(t - t_0)$$

$$\therefore \quad h_3(t) * h_4(t) = e^{-3(t+2)} \cdot u(t+2) * \delta(t-1) = e^{-3(t+1)} \cdot u(t+1)$$

Putting this value in Equation (1) we get,

$$h_T(t) = \{[e^{-2t} u(t) - \delta(t-1)] * [e^{-3(t+1)} \cdot u(t+1) + \delta(t)]\}$$

Using distributive property we get,

$$h_T(t) = e^{-2t} u(t) * e^{-3(t+1)} \cdot u(t+1) + e^{-2t} u(t) * \delta(t) - e^{-3(t+1)} \cdot u(t+1) * \delta(t-1) - \delta(t) * \delta(t-1)$$

Now we have, $x(t) * \delta(t) = x(t)$.

$$\therefore \quad h_T(t) = [e^{-2t} u(t) * e^{-3(t+1)} \cdot u(t+1)] + e^{-2t} u(t) - e^{-3t} u(t) - \delta(t)$$

This is overall impulse response of system.

Ex. 5.5.5 : Find the expression for impulse response relating the input x(t) to the output y(t) in terms of impulse for each subsystem for the LTI system shown in Fig. P. 5.5.5.

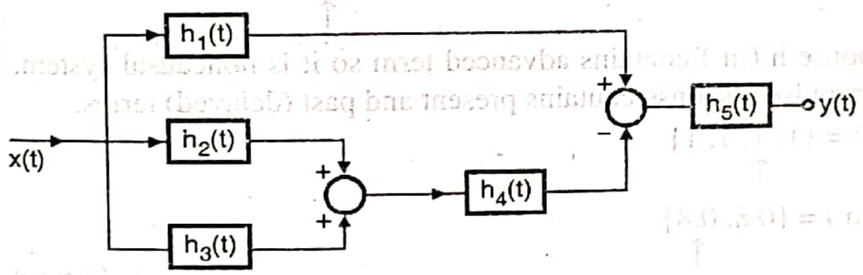


Fig. P. 5.5.5

Soln. : Here $h_2(t)$ and $h_3(t)$ are in parallel. So these blocks can be combined by performing addition, that means $h_2(t) + h_3(t)$.

This block is in series with $h_4(t)$. So it can be combined by performing convolution. That means, $[h_2(t) + h_3(t)] * h_4(t)$.

This block is in parallel with $h_1(t)$. Thus system response is,

$h_1(t) - \{[h_2(t) + h_3(t)] * h_4(t)\}$

Now this block is in series with $h_5(t)$. Thus overall impulse response is,

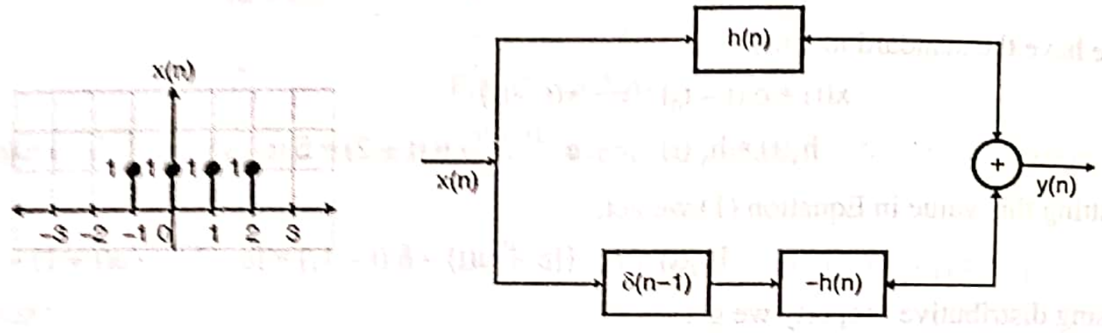
$$h_T(t) = [h_1(t) - \{[h_2(t) + h_3(t)] * h_4(t)\}] * h_5(t)$$

Ex. 5.5.6 : Consider an LTI system with input and output related by $y(n) = 0.8[x(n+1) + x(n)]$

1. Find impulse response h (n)
2. Is this system causal ? Why ?
3. Determine the system response y (n) for input shown in Fig. P. 5.5.6(a).

4. Consider interconnections of LTI system shown in Fig. P. 5.5.6(b). Find overall impulse response of the total system.

5. Solve for overall response of the total system for the same input x (n).



(a) (b)

Fig. P. 5.5.6

Soln. :

1. Given $y(n) = 0.8\,x(n+1) + 0.8\,x(n)$

Impulse response of a system means the response (output) of a system when unit impulse, $\delta(n)$ is applied at input.

$$\therefore \quad h(n) = 0.8\,\delta(n+1) + 0.8\,\delta(n)$$

$$\therefore \quad h(n) = (0.8,\ \underset{\uparrow}{0.8})$$

2. Impulse response h (n) contains advanced term so it is noncausal system. Because a system is causal if its impulse response contains present and past (delayed) terms.

3. Given $x(n) = \{1, \underset{\uparrow}{1}, 1, 1\}$

We have $h(n) = \{0.8, \underset{\uparrow}{0.8}\}$

Now response of system is,

$$y(n) = x(n) * h(n)$$

Here lower limit for y (n) is $-1 + (-1) = -2$

	1	1	1	1
0.8	0.8	0.8	0.8	0.8
0.8	0.8	0.8	0.8	0.8
0	0	0	0	0
0	0	0	0	0

$$\therefore y(n) = \{0.8, \underset{\uparrow}{1.6}, 1.6, 1.6, 0.8\}$$

4. As shown in Fig. P. 5.5.6(b). $\delta(n-1)$ and $-h(n)$ are in series and this series combination is in parallel with h (n). We will denote the total impulse response by $h_T(n)$.

$$\therefore \quad h_T(n) = \{\delta(n-1) * (-h(n))\} + h(n)$$

$$\text{Let } h_1(n) = \delta(n-1) * [-h(n)]$$

$$\therefore\ h_1(n) = \{\underset{\uparrow}{0}, 1\} * \{-0.8, \underset{\uparrow}{-0.8}\}$$

The lower limit of $h_1(n)$ is $0 + (-1) = -1$

	-0.8	0.8
0	0	0
1	-0.8	-0.8

$$\therefore\ h_1(n) = \{0, \underset{\uparrow}{-0.8}, -0.8\}$$

$$\text{Now } h_T(n) = h_1(n) + h(n)$$

This addition is shown in Fig. P. 5.5.6(c).

$$\therefore\ h_T(n) = \{0.8, \underset{\uparrow}{0}, -0.8\}$$

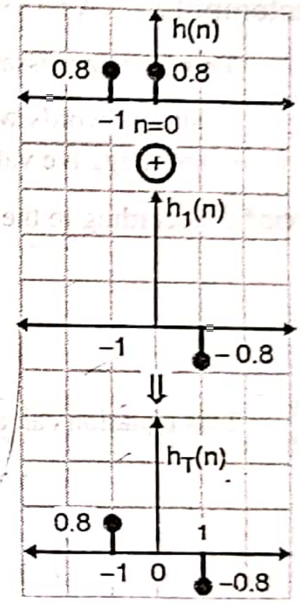


Fig. P. 5.5.6(c)

5. We have $x(n) = \{1, \underset{\uparrow}{1}, 1, 1\}$

Now $y(n) = x(n) * h_T(n)$

The lower limit of $y(n)$ is $-1 + (-1) = -2$

	1	1	1	1
0.8	0.8	0.8	0.8	0.8
0	0	0	0	0
-0.8	-0.8	-0.8	-0.8	-0.8
0	0	0	0	0

$$\therefore\ y(n) = \{0.8, 0.8, \underset{\uparrow}{0}, 0, -0.8, -0.8, 0\}$$

5.6 System Properties in Terms of Impulse Response :

In this section we will study impulse response and different properties of LTI systems.

5.6.1 Causal or Non-causal :

We have studied that a discrete time system is causal if output of system depends only on present and past inputs and not on the present input. In this section we will prove that LTI system is causal if its impulse response $h(n) = 0$ for $n < 0$.

Statement : A LTI system is causal if and only if $h(n) = 0$ for $n < 0$.

This is the necessary and sufficient condition for causality of LTI system.

In simple words we can write this statement as "A LTI system is causal if its impulse response, $h(n) = 0$ for negative values of n".

Proof : According to the equation of linear convolution we have,

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\,h(n-k) \qquad \text{...(5.6.1)}$$

This equation can also be written as,

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)\,x(n-k) \qquad \text{...(5.6.2)}$$

Here limits of summation are $k = -\infty$ to $+\infty$. This summation can be expressed into two parts as follows :

$$y(n) = \sum_{k=0}^{\infty} h(k)\,x(n-k) + \sum_{k=-\infty}^{-1} h(k)\,x(n-k) \qquad \text{...(5.6.3)}$$

Note that total range of k is from $-\infty$ to $+\infty$. In the first term we have taken the range of k from 0 to ∞. Thus for second term the range of k is from -1 to $-\infty$. Because we have covered $k = 0$ in the first term. So for the second summation the range of k should not be from $k = 0$ to $-\infty$ but it should be from $k = -1$ to $-\infty$. Now expanding the terms of summation we get,

$$\begin{aligned} y(n) = {} & [h(0)\,x(n) + h(1)\,x(n-1) + h(2)\,x(n-2) + \dots] \\ & + [h(-1)\,x(n+1) + h(-2)\,x(n+2) + h(-3)\,x(n+3) + \dots] \end{aligned} \qquad \text{...(5.6.4)}$$

Let us consider the 'present' condition is at $n = 0$.

Thus,

$x(0) \Rightarrow$ Present input

$x(n-1), x(n-2), \dots \Rightarrow$ Delayed inputs that means past inputs.

and $x(n+1), x(n+2), \dots \Rightarrow$ Advanced inputs that means future inputs.

We discussed that for the system to be causal we want only present and past inputs. We do not want future inputs. Thus second bracket in Equation (5.6.4) should be zero to make the system causal.

Here $x(n+1)$, $x(n+2)$ ... are future inputs. We do not want to make input to the system zero. So the second bracket is zero if we adjust $h(-1)$, $h(-2)$, $h(-3)$ ... to be equal to zero. Then in this case we will have only first bracket terms. That means we will have only present and past input terms. And then the system will be causal.

Thus for causality,

$$h(-1) = h(-2) = h(-3) = \ldots = 0 \quad \ldots(5.6.5)$$

That means,

$$h(n) = 0 \quad \text{for } n < 0 \quad \ldots(5.6.6)$$

Similarly for C.T. system, the condition for causality is $h(t) = 0$ for $t < 0$.

Modification of convolution equation :

We have the equation of linear convolution,

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k)$$

$$\text{OR } y(n) = \sum_{k=-\infty}^{\infty} h(k)x(n-k)$$

We have derived that, the LTI system is causal then $h(n) = h(k) = 0$ for n (or k) < 0. Thus if LTI system is causal then the limits of summation will be from $k = 0$ to $+\infty$; because for negative values of 'k', $h(k)$ is zero. Thus for causal LTI system the equation of linear convolution is,

$$y(n) = \sum_{k=0}^{\infty} x(k)h(n-k) \quad \text{OR} \quad y(n) = \sum_{k=0}^{\infty} h(k)x(n-k)$$

And for C.T. system we get,

$$y(t) = \int_0^{\infty} x(\tau)h(t-\tau)\,d\tau \quad \text{or} \quad y(t) = \int_0^{\infty} h(\tau)x(t-\tau)\,d\tau$$

5.6.2 Memory Less (Static) and With Memory (Dynamic) :

A system is memoryless if its outputs at any time depends only on the value of the input at that same time.

Discrete-time LTI system is memory less or static if

$$h(n) = 0 \quad \text{For } n \neq 0.$$

In this case the impulse response has the form,

$$h(n) = k\,\delta(n)$$

Where $k = h(0)$ is a constants, and convolution sum reduces to,

$$y(n) = k\,x(n)$$

Dynamic or with memory : A system is with memory or dynamic if systems present outputs depends on present input as well as input at other time.

Discrete-time LTI system is with memory or dynamic if

$h(n) \neq 0$ For $n \neq 0$.

If a discrete time LTI system has an impulse response h (n) that is not zero for n ≠ 0 then the system has memory.

For C.T. system to be static its impulse response must be in the form,

$$h(t) = c\,\delta(t)$$

5.6.3 Stable Systems :

Earlier we have discussed the stability condition for discrete time system. It states that a discrete time system is stable if bounded input produces bounded output. Here the meaning of word "bounded" is; input and output signals should have some finite magnitude. The magnitude of these signals should not be infinity. Now in this section we will derive the stability condition applicable to LTI systems; in terms of unit sample response.

5.6.3.1 Proof of Stability Criteria For LTI (or LSI) Systems in Terms of Unit Impulse Response :

Statement : LTI system is stable if its impulse response is absolutely summable.

Proof : Consider a linear time invarient system having impulse response h (n). Let x (n) be the input applied to this system. Now according to the definition of convolution; the output of such system is expressed as,

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\,h(n-k) \quad \text{...(5.6.7)}$$

Equation (5.6.7) can also be written as,

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)\,x(n-k) \quad \text{...(5.6.8)}$$

Now first we will define bounded sequence. A discrete time sequence is said to be bounded if the absolute value of every element is less than some finite number. Thus input x (n) will be bounded if $|x(n)|$ is less than some finite number. Let us denote this finite number by M_x. Thus for input signal to be bounded.

$$|x(n)| \leq M_x \quad \text{...(5.6.9)}$$

Here M_x is a finite number; so definitely its value should be less than infinity. Thus Equation (5.6.9) can be written as,

$$|x(n)| \leq M_x < \infty \quad \text{...(5.6.10)}$$

Now taking absolute value of both sides of Equation (5.6.8) we get,

$$|y(n)| = \left| \sum_{k=-\infty}^{\infty} h(k)\,x(n-k) \right| \quad \text{...(5.6.11)}$$

We will read R.H.S. of Equation (5.6.11) as "absolute value of summation of terms". If we take 'Σ' sign outside then the term becomes $\sum_{k=-\infty}^{\infty} |h(k)|\,|x(n-k)|$. This is summation of absolute values of terms. Always absolute value of sum of terms is less than or equal to the sum of absolute values of terms.

$$\text{Thus} \quad \left| \sum_{k=-\infty}^{\infty} h(k)\,x(n-k) \right| \le \sum_{k=-\infty}^{\infty} |h(k)|\,|x(n-k)| \qquad ...(5.6.12)$$

But from Equation (5.6.11); the L.H.S. term is $|y(n)|$.

$$\therefore \quad |y(n)| \le \sum_{k=-\infty}^{\infty} |h(k)|\,|x(n-k)| \qquad ...(5.6.13)$$

Here $x(n-k)$ is delayed input signal. If input is bounded then its delayed version is also bounded. This is because delay or folding is related to time shifting operations. By performing these operations; the magnitude is not changed. Now for bounded input we have,

$$|x(n)| \le M_x$$

$$\therefore \quad |x(n-k)| \le M_x \qquad ...(5.6.14)$$

Putting this value in Equation (5.6.13) we get,

$$|y(n)| \le \sum_{k=-\infty}^{\infty} |h(k)| \cdot M_x$$

$$\therefore \quad |y(n)| \le M_x \sum_{k=-\infty}^{\infty} |h(k)| \qquad ...(5.6.15)$$

We know that M_x is a finite number. We want output $y(n)$ to be bounded. That means $|y(n)|$ should be finite. So from Equation (5.6.15), to obtain finite output we have,

$$\sum_{k=-\infty}^{\infty} |h(k)| < \infty \qquad ...(5.6.16)$$

Here $h(k) = h(n)$ is the impulse response of LTI system. Thus Equation (5.6.16) gives the condition of stability in terms of impulse response of the system.

Now the stability factor is denoted by 's'.

$$\therefore \quad s = \sum_{k=-\infty}^{\infty} |h(k)| < \infty \qquad ...(5.6.17)$$

Thus LTI system is stable if its impulse response is absolutely summable.

5.6.3.2 Solved Problems on Stability :

Ex. 5.6.1 : Determine range of values of parameter 'a' for which the linear time invariant system with impulse response.

$$h(n) = a^n \quad n \ge 0 \text{ and } n \text{ even}$$
$$= 0 \quad \text{otherwise}$$

is stable

Soln. : We know that LTI system is BIBO stable when,

$$\sum_{k=-\infty}^{\infty} |h(k)| < \infty \qquad ...(1)$$

Given equation is $h(n) = a^n$ for $n \geq 0$ and n even and $h(n) = 0$ otherwise. That means sequence $h(n)$ is present from $n = 0$ to $n = \infty$ and for even values of n only. So limits of summation will change from $n = 0$ to $n = \infty$. We will also replace $h(k)$ by $h(n)$. Thus Equation (1) becomes,

$$\sum_{k=-\infty}^{\infty} |h(k)| = \sum_{\substack{n=0 \\ n \text{ even}}}^{\infty} |a^n| \qquad ...(2)$$

Since 'n' is even, to satisfy this condition we will put $n = 2p$. Here value of p is 0, 1, 2 .... ∞. So 'n' is always even for any value of 'p'. Thus Equation (2) becomes,

$$\sum_{k=-\infty}^{\infty} |h(k)| = \sum_{p=0}^{\infty} |a^p|$$

$$= \sum_{p=0}^{\infty} |a|^p$$

Expanding the summation we get,

$$\sum_{k=-\infty}^{\infty} |h(k)| = |a|^0 + |a|^1 + |a|^2 + |a|^3 + ...$$

$$= 1 + |a| + |a|^2 + |a|^3 + ... \qquad ...(3)$$

This is geometric series. For geometric series we have,

$$\sum_{n=0}^{\infty} a^n = 1 + a + a^2 + a^3 + ... = \frac{1}{1-a} \quad \text{if } |a| < 1$$

Thus Equation (3) becomes,

$$\sum_{k=-\infty}^{\infty} |h(k)| = \frac{1}{1-|a|} \quad \text{if } |a| < 1$$

This condition (if $|a| < 1$) indicates that the given series converges to $\frac{1}{1-|a|}$ if and only if $|a| < 1$. Thus for the system to be stable $|a| < 1$ is the condition. Otherwise the system will be unstable.

5.6.4 Invertible Systems and Deconvolution :

A linear time invariant system generates output $y(n)$ by taking convolution of input $x(n)$ and its impulse response $h(n)$. That means $y(n) = x(n) * h(n)$. But in many applications it is required to generate the input signal from its output $y(n)$. This system is called as inverse system.

Invertibility of LTI system :

A system is said to be invertible if input of a system can be recovered from the output of system. To achieve the same input consider two system connected in series; where second system is used to recover the input. The block schematic is shown in Fig. 5.6.1.

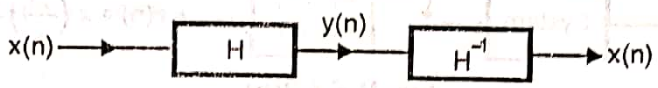


Fig. 5.6.1

Here H represents discrete time system and produces y (n) when x (n) is the input. Then y (n) is applied to the second system. This is also discrete time system denoted by H^{-1}. Here H^{-1} denotes inverse operation.

$$\therefore \quad y(n) = H[x(n)]$$

$$\text{and} \quad H^{-1}[y(n)] = H^{-1}\{H[x(n)\} = H H^{-1}[x(n)]$$

$$\text{But } H H^{-1} = 1$$

$$\therefore \quad H^{-1} y(n) = x(n)$$

Here operator H^{-1} is the inverse operator and the associated system is called as inverse system.

5.6.4.1 Solved Problems on Invertible Systems :

Ex. 5.6.2 : Which of the following systems are invertible ? If invertible find the inverse.

1. $y(t) = x(t-n)$
2. $y(t) = x(2t)$
3. $y(n) = n\,x(n)$

Soln. :

1. Given equation is, $y(t) = x(t-n)$

It indicates that signal x (t) is delayed by 'n' samples. So **it is invertible system.** We can design another system in which x (t) can be made advanced by 'n' samples. Thus inverse system is,

$$Z(t) = y(t+n)$$

This is shown in Fig. P. 5.6.2(a).

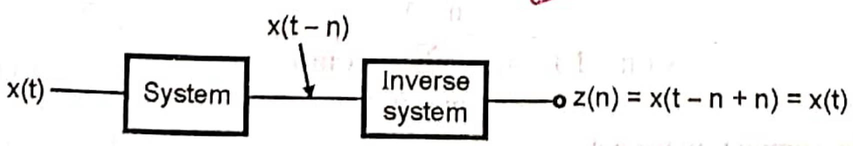


Fig. P. 5.6.2(a)

2. Given, $y(t) = x(2t)$

Input is x (t); so this system indicates that output is obtained by compressing input by the factor 2. There can be another system which will expand input by 2. Thus **it is invertible system** and inverse system is,

$$Z(t) = y\left(\frac{t}{2}\right)$$

This is shown in Fig. P. 5.6.2(b).

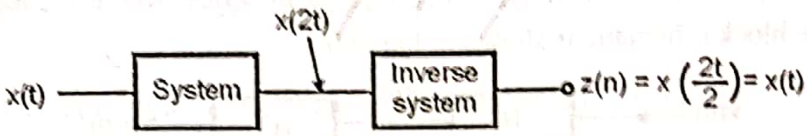


Fig. P. 5.6.2(b)

3. Given, $y(n) = n\,x(n)$

It indicates that input x (n) is multiplied by 'n'. We can design a system which will divide input by 'n'. Thus it is invertible system and inverse system is,

$$Z(n) = \frac{1}{n} y(n)$$

This is shown in Fig. P. 5.6.2(c).

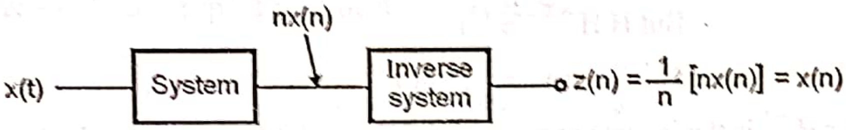


Fig. P. 5.6.2(c)

Ex. 5.6.3 : Consider an accumulator which computes the running sum of all inputs upto present time and expressed as :

$$y(n) = \sum_{m=0}^{n} x(m)$$

Is it an invertible system ? If yes, find inverse of this system. If no give reasons.

Soln. : The given equation is,

$$y(n) = \sum_{m=0}^{n} x(m) \qquad ...(1)$$

Expanding the summation we get,

$$y(n) = x(0) + x(1) + x(2) + ... + x(n) \qquad ...(2)$$

Replacing 'n' by n – 1 in Equation (1) we get,

$$y(n-1) = \sum_{m=0}^{n-1} x(m) \qquad ...(3)$$

Expanding the summation we get,

$$y(n-1) = x(0) + x(1) + x(2) + ... + x(n-1) \qquad ...(4)$$

Invertible system is designed to generate the original input.

If we subtract Equation (4) from Equation (3) then we can generate original input.

$$\therefore \quad x(n) = y(n) - y(n-1) \qquad ...(5)$$

Thus it is invertible system. Equation (5) indicates equation of inverse system.

$$\therefore \quad y(n) = \frac{3}{2}\left[1-\left(\frac{1}{3}\right)^n \cdot \left(\frac{1}{3}\right)\right]$$

$$\therefore \quad y(n) = \frac{3}{2}-\frac{1}{2}\cdot\left(\frac{1}{3}\right)^n$$

This is the step response of system.

5.7 System Analysis from Difference Equation Model :

The output of discrete time system can be calculated using equation of linear convolution.

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k) \qquad \text{...(5.7.1)}$$

Here h (n) is the impulse response of system. By the use of h (n); we can calculate output y (n) for any input x (n) using Equation (5.7.1).

Now in this section we will discuss the difference equation which are useful to implement (realize) the discrete time system. These difference equations are especially used for designing the digital filter.

We discussed that impulse response h (n) is used to calculate output of discrete time system. Now if the system is causal then impulse response h (n) is zero for negative values of n. So for causal LTI system equation of convolution becomes,

$$y(n) = \sum_{k=0}^{\infty} x(k)\, h(n-k) \qquad \text{...(5.7.2)}$$

Equation (5.7.2) can also be written as,

$$y(n) = \sum_{k=0}^{\infty} h(k)\, x(n-k) \qquad \text{...(5.7.3)}$$

Since output depends on impulse response h (n); the number of samples, present in h (n) plays an important role to determine output y (n). The number of samples present in h (n) is called as length of impulse response. Depending upon the length of impulse response; there are two types of discrete time systems :

1. Finite impulse response (FIR) systems. 2. Infinite impulse response (IIR) systems.

1. Finite impulse response systems (FIR) :

As the name indicates; impulse response is having a finite length. That means h (n) contains some finite number of samples. Let the range of h (n) is from – M to + M. Thus equation of convolution for FIR system becomes,

$$y(n) = \sum_{k=-M}^{M} h(k)\, x(n-k) \qquad \text{(Non-causal)} \qquad \text{...(5.7.4)}$$

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Note that in Equation (5.7.4), negative terms are present, so this equation is for the non-causal system. Now for the causal FIR system we will consider only positive samples of h (n). Thus range of h (n) becomes '0' to M instead of – M to M. Equation (5.7.4) becomes,

$$y(n) = \sum_{k=0}^{M} h(k)\, x(n-k) \qquad \text{...(5.7.5)}$$

Note that the summation is taken from 0 to M. So there will be total M + 1 terms in h (n) or h (k). These terms are :

h (0), h (1), h (2) ....h (M).... Number of terms in h (n).

Now expanding Equation (5.7.5) we get,

$$y(n) = h(0)\, x(n) + h(1)\, x(n-1) + h(2)\, x(n-2) + \ldots + h(M)\, x(n-M) \qquad \text{...(5.7.6)}$$

In Equation (5.7.6) x (n) is present input and x (n – 1), x (n – 2) ... etc. are past inputs. Every input is multiplied by corresponding value of h (n). Thus Equation (5.7.6) indicates that output y (n) is weighted sum of inputs.

Note that in this case the system has to view input samples only from x (n) to x (n – M) and not all input samples. These input samples are most recent input samples. Thus this system has to store x (n – 1), x (n – 2) ... x (n – M), past samples in the memory. Since these are finite number of samples; **FIR has limited or finite memory requirement.**

2. Infinite impulse response (IIR) systems :

As the name indicates, impulse response h (n) contain '∞' samples. That means length of h (n) is infinite. Now the equation of convolution for non-causal IIR system becomes.

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)\, x(n-k) \quad \text{...(Non-causal)} \qquad \text{...(5.7.7)}$$

For the causal system, limits of summation will be from 0 to ∞.

$$\therefore \quad y(n) = \sum_{k=0}^{\infty} h(k)\, x(n-k) \quad \text{... (Causal)} \qquad \text{...(5.7.8)}$$

Here the range of h (n) = h (k) is as follows :

h (0), h (1), h (2) ...., h(∞). So this is infinite impulse response system.

Now expanding Equation (5.7.8) we get,

$$y(n) = h(0)\, x(n) + h(1)\, x(n-1) + h(2)\, x(n-2) + \ldots + h(\infty)\, x(n-\infty) \qquad \text{...(5.7.9)}$$

In this case IIR system has to store all past samples from x (n – 1) to x (n – ∞). Thus **IIR system needs infinite memory.** This is practically not possible.

Note : We discussed that FIR system needs limited memory. So it is possible to design FIR systems using convolution equation. But IIR systems cannot be designed using convolution equation. This is because of the infinite memory requirement. So to implement such IIR systems; difference equations are used.

Now depending upon length of h (n) we have classified the systems as FIR and IIR. Similarly depending upon output of system; the discrete time systems are classified as follows :

1. Non-recursive systems 2. Recursive systems.

Why this type of classification is required ?

Equation of convolution indicates that for causal systems; the output depends on present and past input as well as it depends on the impulse response, h (n). But this is not the case always. There are some systems whose present output depends on present and past inputs as well as past (previous) output. For such systems we should take into account, the previous output to calculate present output. This type of classification gives the idea about the requirement of previous output to calculate present output.

1. Non-recursive systems :

Non-recursive systems do not require any past output sample to calculate the present output. Consider the equation of FIR causal system (Equation (5.7.5)). This equation is,

$$y(n) = \sum_{k=0}^{M} h(k)\, x(n-k)$$

To calculate present output we need present and past inputs only as shown by Equation (5.7.6). We will rewrite Equation (5.7.6) as follows :

$$y(n) = h(0)\, x(n) + h(1)\, x(n-1) + \ldots + h(m)\, x(n-M)$$

This equation does not contain any previous output term like y (n – 1), y (n – 2) ... etc. So this is the equation of non-recursive system. Thus we can conclude that **causal FIR system is non-recursive system.** In other words we can say that non-recursive implementation of FIR causal system is possible.

Now for non-causal FIR system; the limits of summation will be from – M to M. Thus this system contain present, past and future inputs. But it does not contain future output. So this is also non-recursive system.

In case of non-recursive systems; past output terms are not required. Past (previous) output terms are obtained by taking feedback from output to the input. Thus **non-recursive system do not require any feedback.**

Is it possible to implement non-recursive IIR system ?

We know that,

Non-recursive $\Leftrightarrow$ No feedback or no past output term present

and IIR $\Leftrightarrow$ Infinite impulse response.

The equation of convolution for causal IIR system is,

$$y(n) = \sum_{k=0}^{\infty} h(k)\, x(n-k)$$

Expanding the summation we get,

$$y(n) = h(0)\, x(n) + h(1)\, x(n-1) + h(2)\, x(n-2) + \ldots + h(\infty)\, x(n-\infty)$$

We have discussed earlier that such systems require to store all past inputs. That means it is required to store infinite input terms in the memory. So infinite memory is required. This is not possible practically. Thus non-recursive implementation of causal IIR system is not possible.

The representation of non-recursive system is shown in Fig. 5.7.1(a).

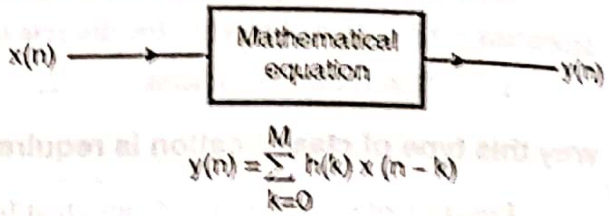

Fig. 5.7.1(a) : Non-recursive system

2. Recursive systems :

A discrete time system in which output y (n) depends on present input, past inputs as well as previous outputs; is called as recursive system.

Let us consider one example. Consider a system represented by block schematic as shown in Fig. 5.7.1(b).

First we will write the equation for system.

1. As shown in Fig. 5.7.1(b); a connection is drawn from output y (n) to the input side. This is a feedback.
2. This feedback signal, y (n) is passed through an unit delay (z^{-1}) block. So the output of delay block is y (n – 1). This is past output compared to y (n).
3. The signal y (n – 1) is multiplied by constant multiplier 'a' to obtain ay (n – 1).
4. An adder is used to add input x (n) and signal ay (n – 1).
5. Now the addition of x (n) and a y (n – 1) produces the output.

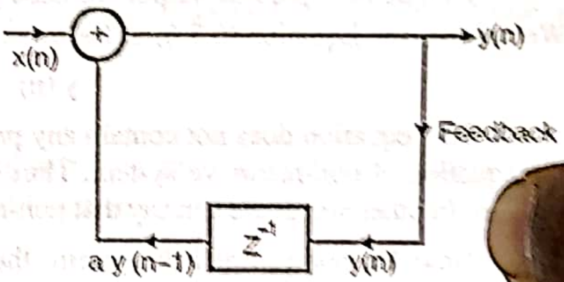

Fig. 5.7.1(b) : Example of recursive system

Thus the output of system is expressed as,

$$y(n) = x(n) + ay(n-1) \quad \text{...(5.7.10)}$$

At instant n = 0 we get,

$$y(0) = x(0) + ay(-1)$$

Here $y(0) \Rightarrow$ present output

$x(0) \Rightarrow$ present input

and $y(-1) \Rightarrow$ past output

Similarly at n = 1 we get,

$$y(1) = x(1) + ay(0)$$

Thus y (0) is past output.

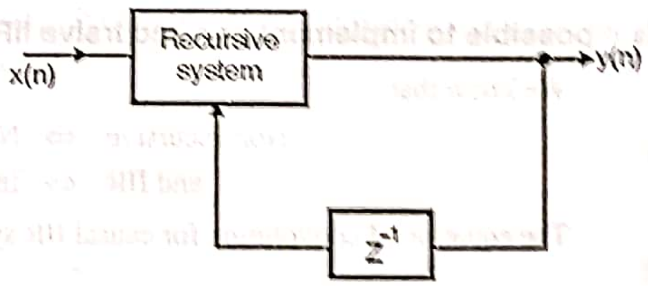

Fig. 5.7.1(c) : Representation of recursive system

So at any instant present output depends on past output. Hence it is recursive system. The representation of recursive system is shown in Fig. 5.7.1(c).

5.7.1 LTI Systems Characterized by Constant Coefficient Difference Equations :

In the previous section, we discussed that, the behaviour of system depends on its impulse response. In this subsection we will discuss how LTI systems are characterized by input-output relations; called as difference equation with constant coefficients.

Consider a recursive system shown in Fig. 5.7.1(b). The equation of output is,

$$y(n) = x(n) + ay(n-1) \qquad ...(5.7.11)$$

This is first order difference equation. Here 'a' is a constant coefficient. Now output y (n) is calculated by putting different values of n in Equation (5.7.11).

$$\text{For } n = 0 \Rightarrow y(0) = x(0) + ay(-1) \qquad ...(1)$$

$$\text{For } n = 1 \Rightarrow y(1) = x(1) + ay(0) \qquad ...(2)$$

Putting value of y (0) from Equation (1) we get,

$$y(1) = x(1) + a[x(0) + ay(-1)]$$

$$\therefore \quad y(1) = x(1) + ax(0) + a^2 y(-1)$$

Rearranging the terms,

$$y(1) = a^2 y(-1) + ax(0) + x(1) \qquad ...(3)$$

$$\text{For } n = 2 \Rightarrow y(2) = x(2) + ay(1) \qquad ...(4)$$

Putting value of y (1) from Equation (3) we get,

$$y(2) = x(2) + a[a^2 y(-1) + ax(0) + x(1)]$$

$$y(2) = x(2) + a^3 y(-1) + a^2 x(0) + ax(1)$$

Rearranging the terms,

$$y(2) = a^3 y(-1) + a^2 x(0) + ax(1) + x(2) \qquad ...(5)$$

Observe Equations (3) and (5) carefully. Now we can directly write the equation for value 'n'.

$$\therefore \quad y(n) = a^{n+1} y(-1) + a^n x(0) + a^{n-1} x(1) + \dots + ax(n-1) + x(n) \qquad ...(5.7.12)$$

In terms of summation Equation (5.7.12) can be written as,

$$y(n) = a^{n+1} y(-1) + \sum_{k=0}^{n} a^k x(n-k) \qquad ...(5.7.13)$$

Note that we have obtained output y (n) by putting only positive values of n. That means for $n \geq 0$. So these are the equations of causal system. Every equation contains the term y (– 1). It is assumed that, we are starting the operation of system at n = 0. Then the term y (– 1) is called as **initial condition of a system.**

Important terms related to Equation (5.7.13) :

We know that Equation (5.7.13) is representing the behaviour of causal system. It consists of two parts as follows :

1. The first term, y (– 1) is called initial condition of the system.

2. The second term, is the response of system to the input x (n).

Now depending upon initial conditions, two types of responses are obtained.

1. Zero state response or forced response :

If initial condition, y (– 1) of the system is zero then the response of the system to the input signal is called as **zero state response.** It is denoted by y_{zs} (n). In this case, whatever output we are getting, that is because of input x (n) only. And since the system is forced to the input; it is also called as **forced response.** Since y (– 1) = 0; the system is said to be relaxed system.

Thus putting initial condition, y (– 1) = 0 in Equation (5.7.13) we get,

$$y(n) = y_{zs}(n) = \sum_{k=0}^{n} a^k x(n-k) \quad \text{... for } n \geq 0 \qquad \text{...(5.7.14)}$$

2. Zero input response or natural response :

As the name indicates; input is not applied to the system and initial condition y (– n) ≠ 0. Thus, this kind of output without applying input x (n) is called as **zero input response** or **natural response.** Since y (– n) ≠ 0, this is called as non-relaxed initial condition. This output is denoted by y_{zi}.

Putting x (n) = 0 in Equation (3.7.13) we get,

$$y(n) = y_{zi}(n) = a^{n+1} y(-1) \qquad \text{...(5.7.15)}$$

This equation indicates that output is not dependent on input x (n) but it depends on initial condition, y (– 1).

3. Total response :

The total response of system is obtained by adding zero state response and zero input response.

$$\therefore \quad \text{Total response, } y(n) = y_{zi}(n) + y_{zs}(n) \qquad \text{...(5.7.16)}$$

Equation (5.7.11) is basically a linear constant coefficient difference equation. The general form of this equation can be expressed as,

$$y(n) = -\sum_{k=1}^{N} a_k\, y(n-k) + \sum_{k=0}^{M} b_k\, x(n-k) \qquad \text{...(5.7.17)}$$

Here a_k and b_k are constant coefficients. And 'N' is the order of difference equation. That means 'N' represents the order of system.

Stability, linearity and time invariance property of recursive systems :

We will discuss these properties of recursive systems described by constant coefficient difference equations.

1. **Stability :** A recursive system is stable when we get a bounded output by applying bounded input.

2. **Linearity :** A recursive system described by Equation (5.7.17) is said to be linear if it satisfies following conditions :

   1. The total response of system is addition of zero input response and zero state response.

   $$\therefore \quad y(n) = y_{zs}(n) + y_{zi}(n)$$

   2. Zero state response should be linear that means principle of superposition should be applicable.
   3. Zero input response should be linear that means principle of superposition should be applicable.

3. **Time invariance :** As shown by Equation (5.7.17), the output y (n) contains constant coefficients a_k and b_k. These coefficients are not affected by the change in time (shifting). So the recursive system is time invariant.

5.7.2 Solution of Linear Constant Coefficient Difference Equation :

In this subsection we will discuss how to obtain exploit solution of a given linear constant coefficient difference equation. Basically there are two methods to obtain this solution.

These methods are as follows :

1. Direct method 2. Indirect method

Indirect method is based on z transformation. Presently we will discuss direct method for obtaining the solution.

In the direct method; it is assumed that the solution consists of two parts :

1. Homogeneous or complementary solution. It is denoted by $y_h(n)$.
2. Particular solution. It is denoted by $y_p(n)$.

Thus the total solution is expressed as,

$$y(n) = y_h(n) + y_p(n) \qquad \text{...(5.7.18)}$$

Homogeneous solution of difference equation :

Recall general form of difference equation (Equation (5.7.17)),

$$\text{It is,} \quad y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k) \qquad \text{...(5.7.19)}$$

In the first term the limits of summation are from k = 1 to N. We can change these limits from k = 0 to N as follows. Take the limits from k = 0 to N then,

$$\sum_{k=0}^{N} a_k y(n-k) = a_0 y(n) + \sum_{k=1}^{N} a_k y(n-k)$$

$$\therefore \quad -\sum_{k=1}^{N} a_k y(n-k) = a_0 y(n) - \sum_{k=0}^{N} a_k y(n-k) \qquad \text{...(5.7.20)}$$

Putting this value in Equation (5.7.19) we get,

$$\therefore \quad 4k = 6k - 4k + 4$$

$$\therefore \quad 2k = 4 \Rightarrow k = 2$$

Putting this value in Equation (3) we get,

$$y_p(n) = 2.2^n u(n)$$

Note : If total solution is asked to calculate, then first obtain $y_h(n)$ and $y_p(n)$ separately. The total solution is $y(n) = y_h(n) + y_p(n)$.

Review Questions

Q. 1 Obtain the equations of impulse response for the first and second order systems.

Q. 2 Derive the equation of linear convolution.

Q. 3 Explain the properties of convolution sum.

Q. 4 Explain how we test BIBO stability of DTLTI system using impulse response ?

Q. 5 Prove that DTLTI system is causal if h (n) = 0 for n < 0.

Q. 6 Explain invertibility of DTLTI system.