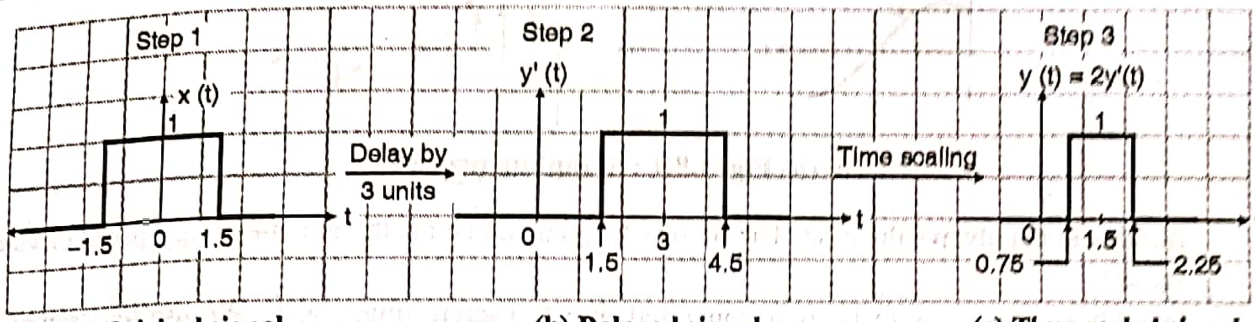
Step 3 : Apply time scaling to $y'(t)$:

Now apply time scaling to $y'(t)$ with $a = 2$ to get $y(t) = x(2t - 3)$ as shown in Fig. P. 1.7.1(c). Note that we have obtained the correct signal.



(a) Original signal (b) Delayed signal (c) Time scaled signal

Fig. P. 1.7.1 : The proper order of operations

Part II : Solution by violating the precedence rule :

Step 4 : Draw the original signal :

Refer Fig. P. 1.7.1(d).

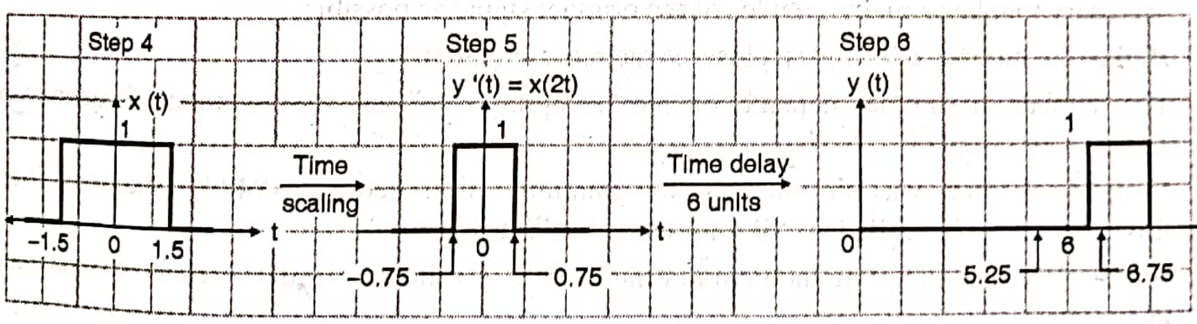
Step 5 : Time-scaling :

Refer Fig. P. 1.7.1(e). The time scaled signal is $y'(t) = x(2t)$.

Step 6 : Time-shifting :

Refer Fig. P. 1.7.1(f). The intermediate signal $y'(t)$ has been delayed by 3 time units, to obtain $y(t)$.

But $y(t) = 2y'(t) = x[2(t-3)] = x[2t-6]$ which is not the desired signal.

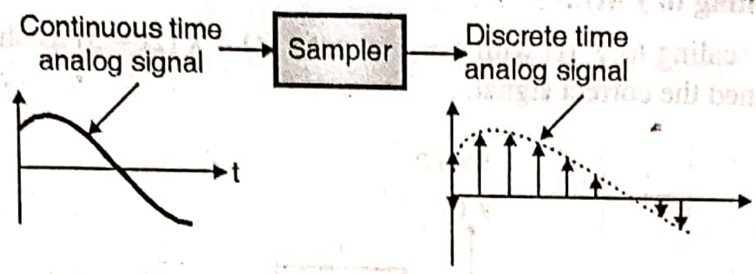


(d) (e) (f)

Fig. P. 1.7.1 : Solution by violation of precedence rule

1.8 Sampling Process :

- In the pulse modulation and digital modulation systems, the signal to be transmitted must be in the discrete time form.
- If the message signal is coming from a digital source (e.g. a digital computer) then it is in the proper for a digital communication system to be processed.



(L-156) **Fig. 1.8.1 : Sampling process**

- But this is not always the case. The message signal can be analog in nature (e.g. speech or video signal).
- In such a case it has to be first converted into a discrete time signal. We use the "sampling process" to do this.
- Thus using the sampling process we convert a continuous time signal into a discrete time signal.
- For the sampling process to be of practical utility it is necessary to choose the sampling rate properly. The sampling process should satisfy the following requirements :
  1. Sampled signal should represent the original signal faithfully.
  2. We should be able to reconstruct the original signal from its sampled version.
- Fig. 1.8.1 summarizes the sampling process.
- Thus sampling is the process of converting a continuous analog signal to a discrete analog signal and the sampled signal is the discrete time representation of the original analog signal.

1.9 Sampling Theorem for Low Pass Signals in Time Domain :

- In order to represent the original message signal "faithfully" (without loss of information), it is necessary to take as many samples of the original signal as possible.
- Higher the number of samples, closer is the representation.
- The number of samples depends on the "sampling rate" and the maximum frequency of the signal to be sampled.
- Sampling theorem was introduced to the communication theory in 1949 by Shannon. Therefore this theorem is also called as "**Shannon's sampling theorem**".
- The statement of sampling theorem in time domain, for the bandlimited signals of finite energy is as follows :

Statement :

1. If a finite energy signal x (t) contains no frequencies higher than "W" Hz (i.e. it is a band limited signal) then it is completely determined by specifying its values at the instants of time which are spaced (1/2W) seconds apart.
2. If a finite energy signal x (t) contains no frequency components higher than "W" Hz then it may be completely recovered from its samples which are spaced (1/2W) seconds apart.

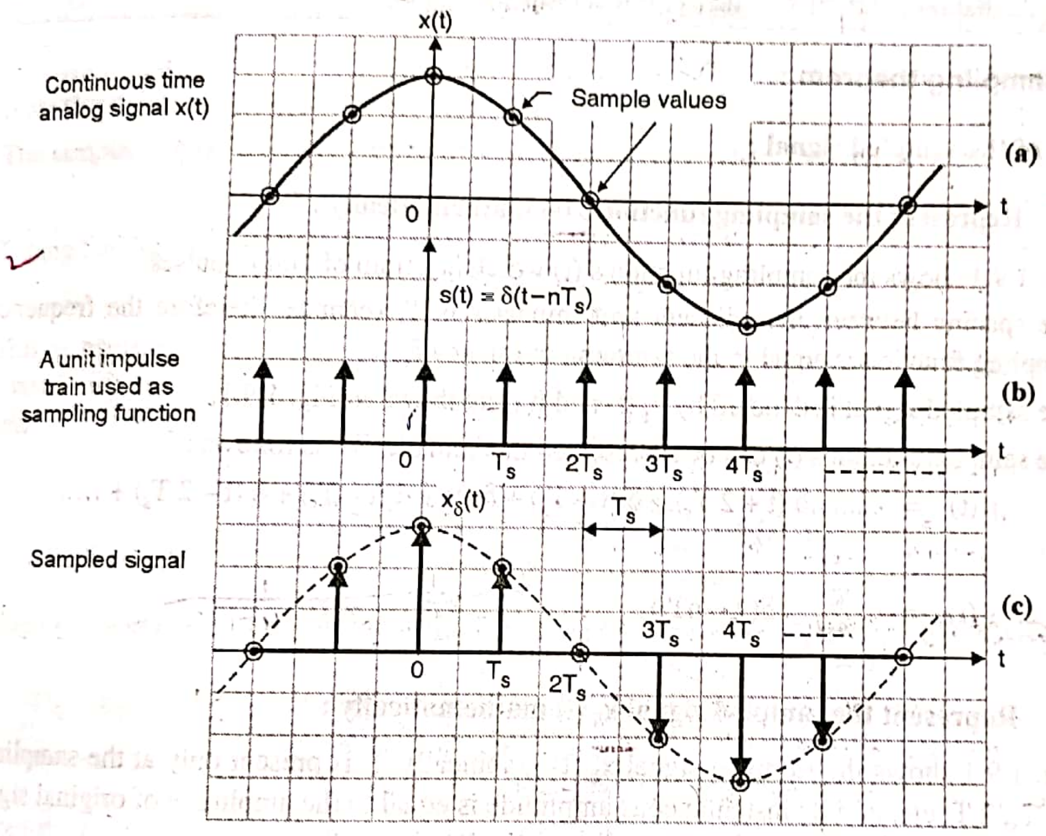
Combined statement of sampling theorem : A continuous time signal x (t) can be completely represented in its sampled form and recovered back from the sampled form if the sampling frequency $f_s \geq 2W$ where "W" is the maximum frequency of the continuous time signal x (t).

1.9.1 Proof of Sampling Theorem :

Let us now prove the sampling theorem in time domain. The assumptions made for this proof are as follows :

Assumptions :

- Let x (t) be a continuous time analog signal as shown in Fig. 1.9.1.



(D-408) **Fig. 1.9.1 : Sampling of a continuous time signal x (t)**

- Let x (t) be a signal with finite energy and infinite duration.
- Let x (t) be a strictly bandlimited signal. That means it does not contain any frequency components above "W" Hz.
- Let s (t) be the sampling function as shown in Fig. 1.9.1. It is a train of unit impulses, spaced by a period of T_s seconds. This sampling function samples the original signal at a rate of "f_s" samples per second. Therefore "T_s" represents the sampling period such that,

$$T_s = \frac{1}{f_s} = \text{Sampling period} \quad \text{...(1.9.1)}$$

$$\text{and} \quad f_s = \frac{1}{T_s} = \text{Sampling rate.}$$

Procedure to be followed :

- We are going to follow the steps given below to prove the sampling theorem :

Step 1 : Represent the sampling function s (t) mathematically.

Step 2 :	Represent the sampled signal x_δ (t) mathematically.
Step 3 :	Obtain the Fourier transform of the sampled signal.
Step 4 :	Prove that the sampled signal x_δ (t) completely represents x (t).
Step 5 :	Represent x (t) as summation of sinc functions (interpolation).
Step 6 :	Graphical representation of the interpolation process.
Step 7 :	Actual recovery of x (t) using an ideal low pass filter.

Part 1 : Sampling theorem :

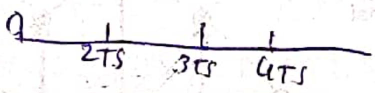
Spectrum of the sampled signal :

Step 1 : Represent the sampling function s (t) mathematically :

- Fig. 1.9.1 shows the sampling function s (t) which is a train of unit impulses.
- The spacing between the adjacent unit impulses is T_s seconds, therefore the frequency of the sampling function is equal to the sampling frequency f_s.
- The sampled signal is denoted by x_δ (t) and it is as shown in Fig. 1.9.1.
- The sample function s (t) can be represented mathematically as follows :

$$s(t) = \ldots\ldots\ldots \delta(t+2T_s) + \delta(t+T_s) + \delta(t) + \delta(t-T_s) + \delta(t-2T_s) + \ldots\ldots$$

$$\therefore\ s(t) = \sum_{n=-\infty}^{\infty} \delta(t-nT_s) \qquad \ldots(1.9.2)$$



Step 2 : Represent the sampled signal x_δ (t) mathematically :

- Fig. 1.9.1 shows the sampled signal x_δ (t) graphically. It is present only at the sampling instants i.e. T_s, 2 T_s etc. and its instantaneous amplitude is equal to the amplitude of original signal x (t) at the sampling instants.
- This is shown by the encircled points in Fig. 1.9.1. Let us represent the instantaneous amplitude of x (t) at the various sampling points t = n T_s as x (n T_s). This is the amplitude of the encircled points of Fig. 1.9.1.
- Looking at the sampled signal x_δ (t) we can say that the sampled signal is obtained by multiplying x (t) and s (t).

$$\therefore\ x_\delta(t) = x(t) \times s(t) = x(nT_s) \times s(t) \qquad \ldots(1.9.3)$$

- Substituting the expression for s (t) from Equation (1.9.2) we get the mathematical expression for the sampled signal x_δ (t) as,

$$x_\delta(t) = \sum_{n=-\infty}^{\infty} x(nT_s) \cdot \delta(t-nT_s) \qquad \ldots(1.9.4)$$

Step 3 : Obtain the Fourier transform of the sampled signal :

- The fourier transform of a train of impulses (dirac delta function) is given by,

$$X(f) = f_0 \sum_{n=-\infty}^{\infty} \delta(f - nf_0)$$

F funcn.

Same

- Here we have the similar pulse train as sampling function s (t). Therefore the Fourier transform of the sampling function is given by,

All signal.

$$S(f) = f_s \sum_{n=-\infty}^{\infty} \delta(f - nf_s) \quad ...(1.9.5)$$

- Note that f_0 has been replaced by f_s in the above equation.
- The sampled signal in the time domain is represented as product of x (t) and s (t).

$$\text{i.e. } x_\delta(t) = x(t) \times s(t) \quad ...(1.9.6)$$

- Taking the Fourier transform of both the sides we get,

$$\text{i.e. } X_\delta(f) = X(f) * S(f) \quad ...(1.9.7)$$

FT

- This is because the Fourier transform of the product of two signals in the time domain is the convolution of their Fourier transforms. Substituting the value of S(f) from Equation (1.9.5) we get,

$$X_\delta(f) = X(f) * \left[f_s \sum_{n=-\infty}^{\infty} \delta(f - nf_s) \right] \quad ...(1.9.8)$$

where * denotes convolution. Interchanging the orders of convolution and summation results in :

$$X_\delta(f) = f_s \sum_{n=-\infty}^{\infty} X(f) * \delta(f - nf_s) \quad ...(1.9.9)$$

- From the properties of delta function, we find that the convolution of X (f) and $\delta(f - nf_s)$ is equal to $X(f - nf_s)$. Hence the above equation can be simplified as follows :

$$\text{F.T. of the sampled signal, } X_\delta(f) = f_s \sum_{n=-\infty}^{\infty} X(f - nf_s) \quad ...(1.9.10)$$

where X (f) = Fourier transform of the original signal x (t).

Conclusion from Equation (1.9.10) :

1. The term $X(f - nf_s)$ in Equation (1.9.10) represents the shifted version of the spectrum X (f) of the original signal x (t). Thus depending on the value of "n" (which extends from $-\infty$ to $+\infty$) we will get infinite number of original spectrums X (f) centered at frequencies $0, \pm f_s, \pm 2f_s, \pm 3f_s, \pm 4f_s$ ...etc. In other words,

$$X(f - nf_s) = X(f) \text{ at } f = 0, \pm f_s, \pm 2f_s, \pm 3f_s \quad ...(1.9.11)$$

2. This concept will be clear if we open Equation (1.9.10) and write the terms separately as shown below.

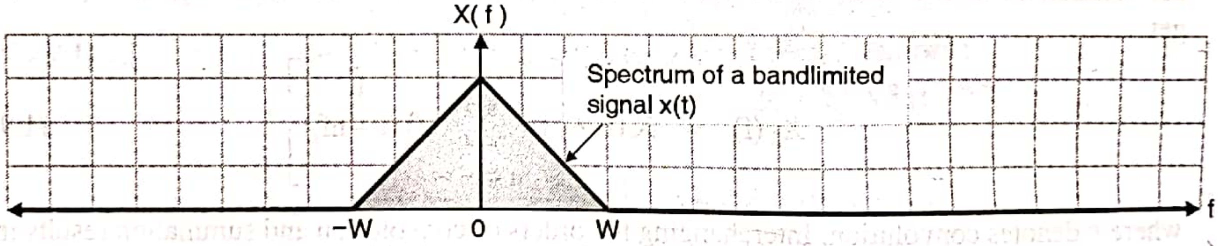
Now open the summation sign in Equation (1.9.10) to get, (D-409)

$$X_\delta(f) = \ldots\ldots + \underbrace{f_s X(f+2f_s)}_{X(f)\text{ shifted left by }2f_s} + \underbrace{f_s X(f+f_s)}_{X(f)\text{ shifted left by }f_s} + \underbrace{f_s X(f)}_{\text{Spectrum }X(f)} + \underbrace{f_s X(f-f_s)}_{X(f)\text{ shifted right by }f_s} + \ldots\ldots$$

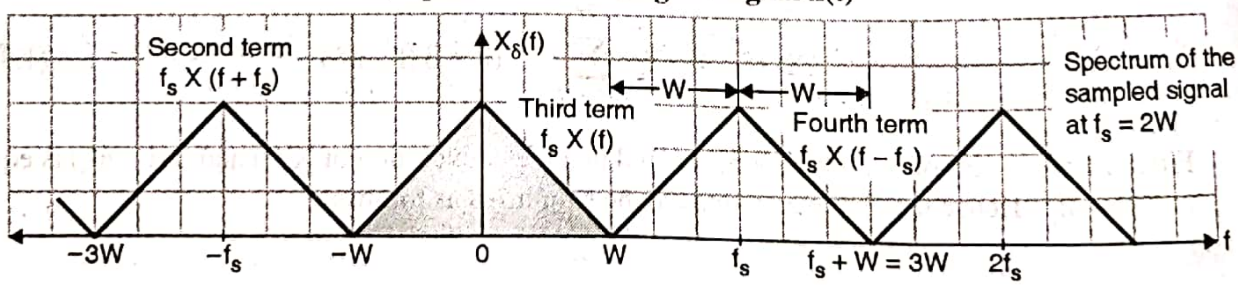
The spectrum $X_\delta(f)$ of the sampled signal is plotted as shown in Fig. 1.9.2.

Equation (1.9.10) can also be written as :

$$X_\delta(f) = f_s X(f) + \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} f_s X(f - nf_s) \quad \ldots (1.9.12)$$



(a) Spectrum of the original signal x(t)



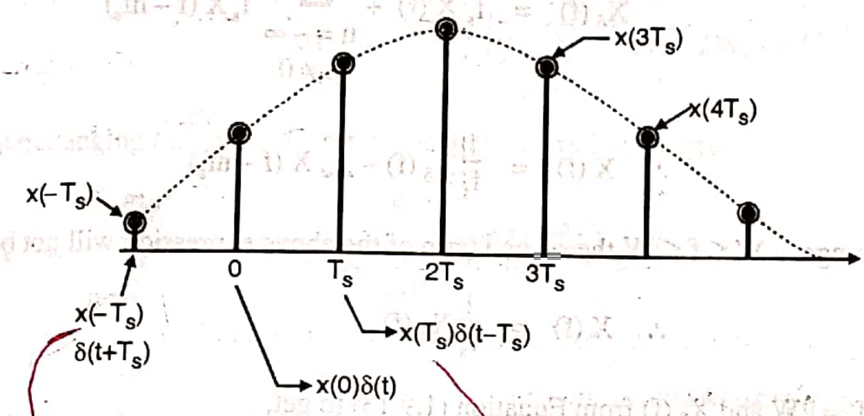
(b) Spectrum of the sampled signal $x_\delta(t)$ with $f_s = 2W$, i.e. Sampling is done exactly at Nyquist rate

(D-410) **Fig. 1.9.2**

Comment :

From Equation (1.9.12) we conclude that the process of uniform sampling of a signal in the time domain results in a periodic spectrum in the frequency domain with a period equal to the sampling rate f_s.

3. **Prove that sampled signal x_δ (t) completely represents x (t) :**



(D-411) **Fig. 1.9.3 : Sampled signal x_δ (t)**

- x_δ (t) can be represented in the summation form as follows (Refer Fig. 1.9.3).

$$x_\delta(t) = \ldots x(-T_s)\,\delta(t+T_s) + x(0)\,\delta(t) + x(T_s)\,\delta(t-T_s) + \ldots \qquad \ldots(1.9.12(a))$$

$$x_\delta(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\cdot\delta(t-nT_s)$$

- We can obtain another useful expression for the fourier transform X_δ (f) by taking the fourier transform of both the sides of the equation stated above as,

$$X_\delta(f) = \sum_{n=-\infty}^{\infty} x(nT_s)\, e^{-j2\pi n f T_s} \qquad \ldots(1.9.13)$$

- This equation is the fourier transform of a discrete time signal x_δ (t). Therefore it is called as the discrete fourier transform (DFT). Compare it with the definition of fourier transform of a continuous time signal. i.e.

$$X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi f t}\, dt$$

- As the signal is discrete, the integration sign has been replaced by the summation sign and "t" has been replaced by "nT_s".
- Now consider Equation (1.9.12)

$$X_\delta(f) = f_s X(f) + \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} f_s X(f - nf_s)$$

$$\therefore \quad X(f) = \frac{1}{f_s} X_\delta(f) - \sum X(f - nf_s)$$

- But in the range $-W \leq f \leq W$ the second term of the above expression will not be present

$$\therefore \quad X(f) = \frac{1}{f_s} X_\delta(f) \qquad \text{...(1.9.14)}$$

- Substitute $f_s = 2W$ and $X_\delta(f)$ from Equation (1.9.13) to get,

$$X(f) = \frac{1}{2W} \sum_{n=-\infty}^{\infty} x(nT_s) \cdot e^{-j2\pi n f T_s} \qquad \text{...(1.9.15)}$$

- This is the frequency spectrum of x (t) in terms of x (nT_s) i.e. the sampled signal. Substitute $T_s = 1/2W$ to get,

$$X(f) = \frac{1}{2W} \sum_{n=-\infty}^{\infty} x(n/2W) \cdot e^{-j2\pi n f/2W} \quad .... -W \leq f \leq W \qquad \text{...(1.9.16)}$$

- This equation shows that the spectrum of x (t) is same as the spectrum of $x_\delta(t)$ in the frequency range $-W$ to $+W$. Hence the sampled signal represents the original signal x (t) successfully.

Thus if the sample values x (n/2W) of the signal x (t) are specified for all time, then the Fourier transform X (f) of the original signal is uniquely determined by using the Equation (1.9.16). Because x (t) is related to X (f) by the inverse Fourier transform, it follows that the signal x (t) is itself uniquely determined by the sample values x (n/2W) for $-\infty \leq n \leq \infty$. Or in other words the sequence of samples $\{x(n/2W)\}$ contains all the information of x (t).

Thus we have proved first part of the sampling theorem.

Part 2 of the sampling theorem :

4. Reconstruction of signal from samples :

- This is the second part of the sampling theorem. From Equation (1.9.16) we can obtain x (t) by taking the inverse fourier transform (IFT).

$$x(t) = \text{IFT}\{X(f)\}$$

$$= \text{IFT}\left\{\frac{1}{2W} \sum_{n=-\infty}^{\infty} x(n/2W) .\, e^{-j\pi f n/W}\right\}$$

- Using the definition of inverse Fourier transform,

$$x(t) = \int_{-W}^{W} \frac{1}{2W} \sum_{n=-\infty}^{\infty} x(n/2W) \cdot e^{-j\pi f n/W} e^{j2\pi ft}\, df$$

- Interchanging the order of summation and integration we get,

$$x(t) = \sum_{n=-\infty}^{\infty} x(n/2W) \frac{1}{2W} \int_{-W}^{W} e^{j2\pi f\left(t-\frac{n}{2W}\right)}\, df$$

$$x(t) = \sum_{n=-\infty}^{\infty} x(n/2W) \cdot \frac{1}{2W} \times \frac{1}{j2\pi\left[t-\frac{n}{2W}\right]} \cdot \left[e^{j2\pi f(t-n/2W)}\right]_{-W}^{W}$$

$$x(t) = \sum_{n=-\infty}^{\infty} x(n/2W) \frac{1}{j4\pi W\left[t-\frac{n}{2W}\right]} \cdot \left[e^{j2\pi W(t-n/2W)} - e^{-j2\pi W(t-n/2W)}\right]$$

$$= \sum_{n=-\infty}^{\infty} x(n/2W) \cdot \left[\frac{e^{j2\pi W(t-n/2W)} - e^{-j2\pi W(t-n/2W)}}{j4\pi W\left[t-\frac{n}{2W}\right]}\right]$$

- The term inside the square bracket is a "sine" function.

$$\therefore\ x(t) = \sum_{n=-\infty}^{\infty} x(n/2W) \frac{\sin(2\pi Wt - n\pi)}{(2\pi Wt - n\pi)} \qquad ...(1.9.17)$$

- We can simplify the equation above by using the definition of the "sinc function". The sinc function is defined as :

$$\text{sinc } x = \frac{\sin(\pi x)}{\pi x} \qquad ...(1.9.18)$$

- Therefore Equation (1.9.17) can be written as :

$$x(t) = \sum_{n=-\infty}^{\infty} x(n/2W) \text{ sinc } (2Wt - n) \qquad ...(1.9.19)$$

- Equation (1.9.19) provides an interpolation formula for reconstructing the original signal x (t) from the sequence of sample values {x (n /2W)}. The "sinc" function plays the role of an interpolation function. Each sample x (n / 2W) is multiplied by a delayed version of the interpolation function i.e. sinc function. Then all these resulting waveforms are added to obtain x (t).

5. Graphical representation of the interpolation process :

Let us re-arrange Equation (1.9.19) as follows :

$$x(t) = \sum_{n=-\infty}^{\infty} x(nT_s) \operatorname{sinc} 2W \left(t - \frac{n}{2W}\right)$$

This is because $\frac{1}{2W} = T_s$

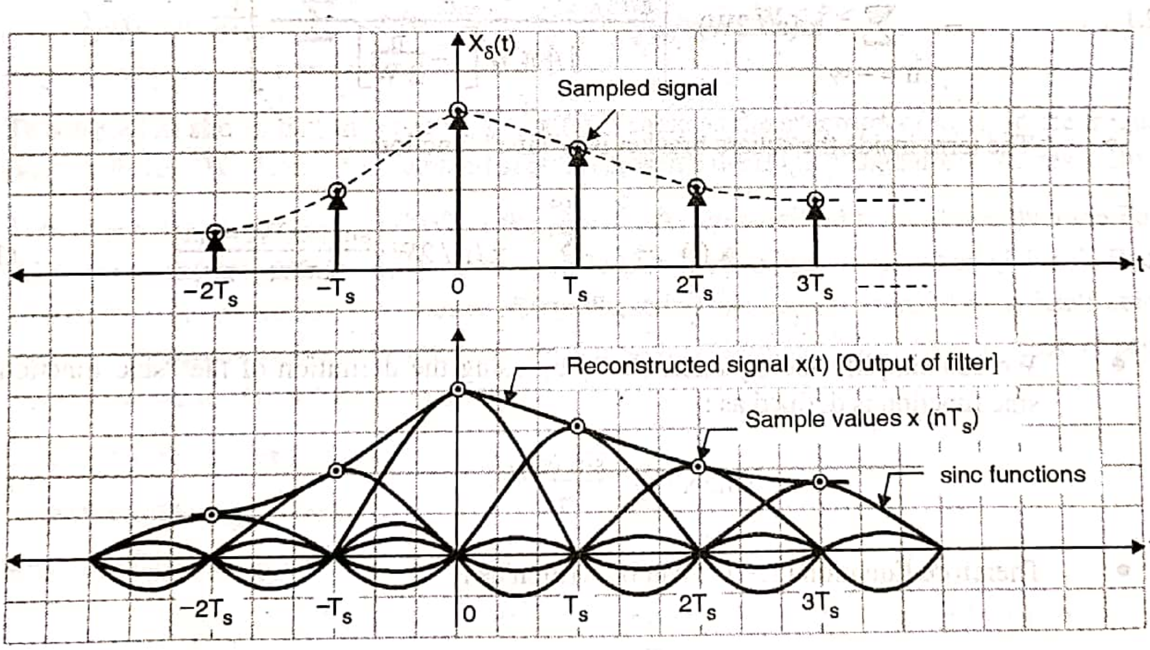
$$\therefore \quad x(t) = \sum_{n=-\infty}^{\infty} x(nT_s) \operatorname{sinc} 2W (t - nT_s) \qquad ...(1.9.20)$$

Let us expand this equation to write,

$$x(t) = x(0) \operatorname{sinc} 2Wt + x(\pm T_s) \operatorname{sinc} 2W (t \pm T_s) + x(\pm 2T_s) \operatorname{sinc} 2W (t \pm 2T_s) + \ldots \qquad .....(1.9.21)$$

(a) First term : x (0) sinc 2Wt :

- This will have a maximum amplitude at t = 0. The maximum amplitude is equal to the sample value x (0) at t = 0. This sinc function will pass through zeros at t = ± 1/2 W, ± 1/4 W ...etc. This is as shown in Fig. 1.9.4.



(D-412) **Fig. 1.9.4 : Reconstruction of the original signal x (t) from its samples using the interpolation**

(b) Second term : x (± T_s) sinc 2W (t ± T_s) :

- This sinc function will have maximum amplitude at t = ± T_s. The maximum amplitude is equal to the sample value x (± T_s) at t = ± T_s respectively. Thus sinc 2W (t ± T_s) represents shifted sinc function i.e. "sinc 2Wt" by a period ± T_s. This is as shown in Fig. 1.9.4.

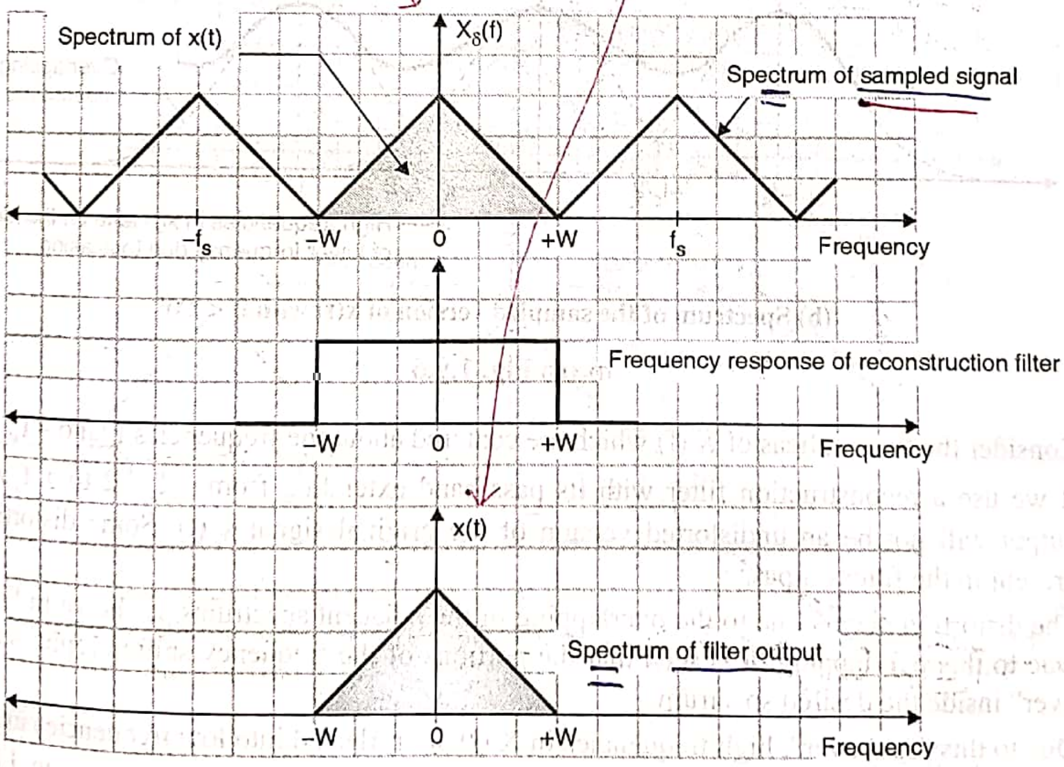
- Similarly the third term, $x(\pm 2T_s)$ sinc $2W(t \pm 2T_s)$ represents shifted sinc function "sinc 2Wt" by a period of $\pm 2T_s$ and so on. We can plot all these sinc functions along with the sampled signal $x_\delta(t)$ as shown in Fig. 1.9.4. Note that the peak amplitude of any sinc function is equal to the corresponding sample value $x(nT_s)$.

6. **Actual reconstruction of the original signal by using a low pass filter :** LS

- Thus the peaks of the sinc pulses represent the amplitudes of the samples.
- The signal x (t) expressed in Equation (1.9.19) is then passed through an ideal low pass filter to recover the original signal x (t). This low pass filter is therefore called as the reconstruction filter. This is shown in the graphical representation of Fig. 1.9.5(a).

(D-413) **Fig. 1.9.5(a) : Reconstruction filter**

- Assume that the cut-off frequency of the ideal low pass filter is adjusted precisely to W Hz. The frequency response of the reconstruction filter is shown in Fig. 1.9.5(b).



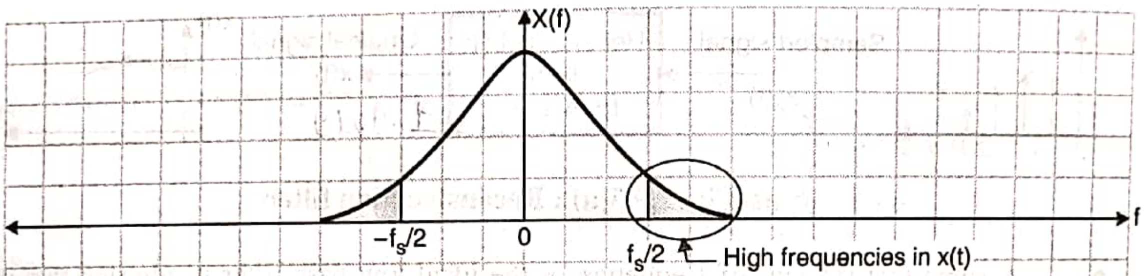
(D-414) **Fig. 1.9.5(b) : Operation of reconstruction**

- (When the sampled signal $x_\delta(t)$ is applied at the input, this filter will allow only the shaded portion in the spectrum of $x_\delta(t)$ to pass through to the output and will block all other frequency components.)

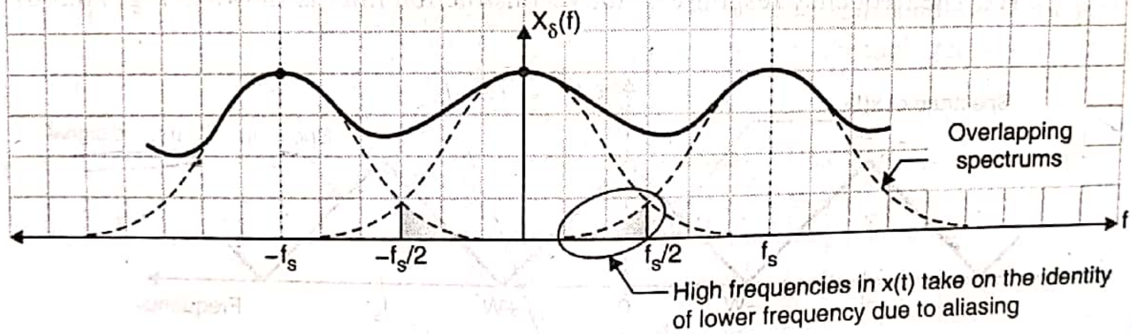
- Thus the frequency components only corresponding to x (t) will be passed through to the output and the original signal x (t) is recovered.

1.9.2 Aliasing or Foldover Error :

- If the signal x (t) is not strictly bandlimited and / or if the sampling frequency f_s is less than 2W, then an error called **aliasing** or **foldover error** is observed. The adjacent spectrums overlap if $f_s < 2W$. This is shown in Fig. 1.9.6(b).
- The signal x (t) is not strictly bandlimited. The spectrum of signal x (t) is shown in Fig. 1.9.6(b).
- The spectrum X_δ (f) of the discrete time signal x_δ (t) is shown in Fig. 1.9.6(b) which is nothing but the sum of X (f) and infinite number of frequency shifted replicas of it as explained earlier.



(a) Spectrum of a continuous time signal x(t)



(b) Spectrum of the sampled version of x(t) with $f_s < 2W$

(D-415) **Fig. 1.9.6**

- Consider the two replicas of X (f) which are centered about the frequencies f_s and $-f_s$.
- If we use a reconstruction filter with its pass-band extending from $-f_s / 2$ to $+f_s / 2$ then its output will not be an undistorted version of the original signal x (t). Some distortion will be present in the filter output.
- The distortion occurs due to the overlapping of the adjacent spectrums as shown in Fig. 1.9.6(b). Due to this overlapping, it is seen that the portions of the frequency shifted replicas are "folded over" inside the desired spectrum.
- Due to this "fold over", high frequencies in X (f) are reflected into low frequencies in X_δ (f). This can be understood by comparing the shaded portions of the spectra shown in Fig. 1.9.6(a) and (b).

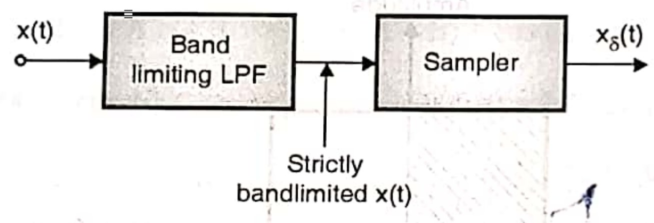
Aliasing : This phenomenon of a high frequency in the spectrum of the original signal x (t), taking on the identity of lower frequency in the spectrum of the sampled signal x_δ (t) is called as aliasing or fold over error.

Effect of aliasing :

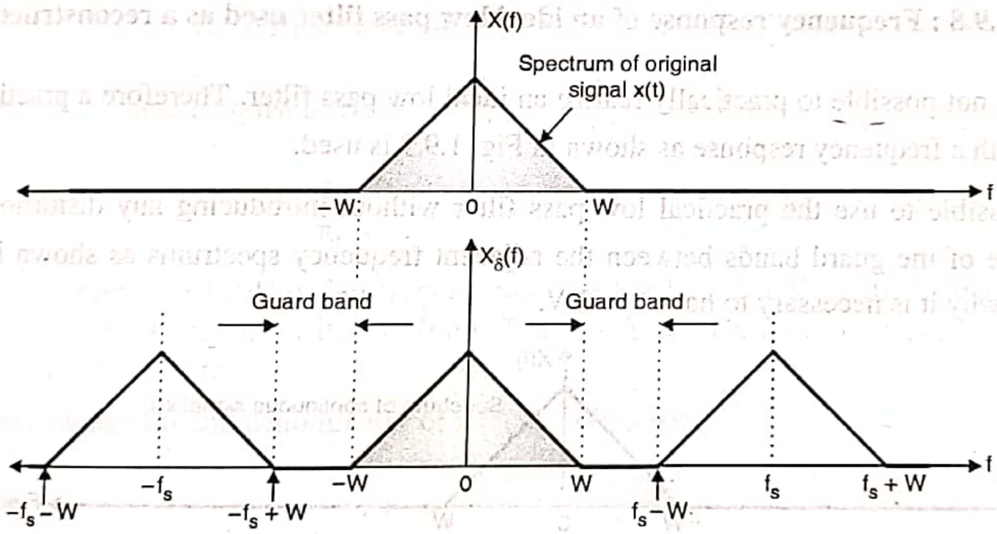
Due to aliasing some of the information contained in the original signal x (t) is lost in the process of sampling.

How to eliminate aliasing ?

- Aliasing can be completely eliminated if we take the following action :
- Use a bandlimiting low pass filter and pass the signal x (t) through it before sampling as shown in Fig. 1.9.7(a).
- This filter has a cutoff frequency at $f_c = W$, therefore it will strictly bandlimit the signal x (t) before sampling takes place. This filter is also called as antialiasing filter or prealias filter.



(D-416) **Fig. 1.9.7(a) : Use of a bandlimiting filter to eliminate aliasing**



(D-417) **Fig 1.9.7(b) : Spectrum of a sampled signal for $f_s > 2W$**

- Increase the sampling frequency f_s to a great extent i.e. $f_s >>> 2W$. Due to this, even though x (t) is not strictly bandlimited, the spectrums will not overlap. A guard band is created between the adjacent spectrums as shown in Fig. 1.9.7(b).

Thus aliasing can be prevented by :

1. Using an antialiasing or prealiasing filter and
2. Using the sampling frequency $f_s > 2W$.

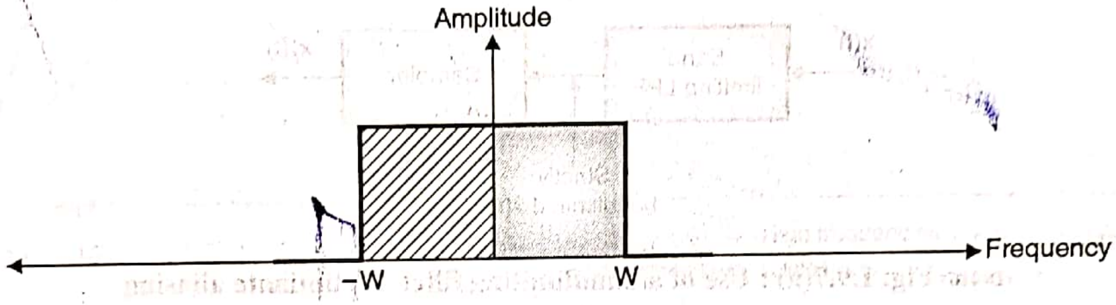
1.9.3 Nyquist Rate and Nyquist Interval :

- The minimum sampling rate of "2W" samples per second for a signal x (t) having maximum frequency of "W" Hz is called as "Nyquist rate". The reciprocal of Nyquist rate i.e. 1/2W is called as the Nyquist interval.

$$\text{Nyquist rate} = 2W \text{ Hz}$$
$$\text{Nyquist interval} = 1/2W \text{ seconds}$$

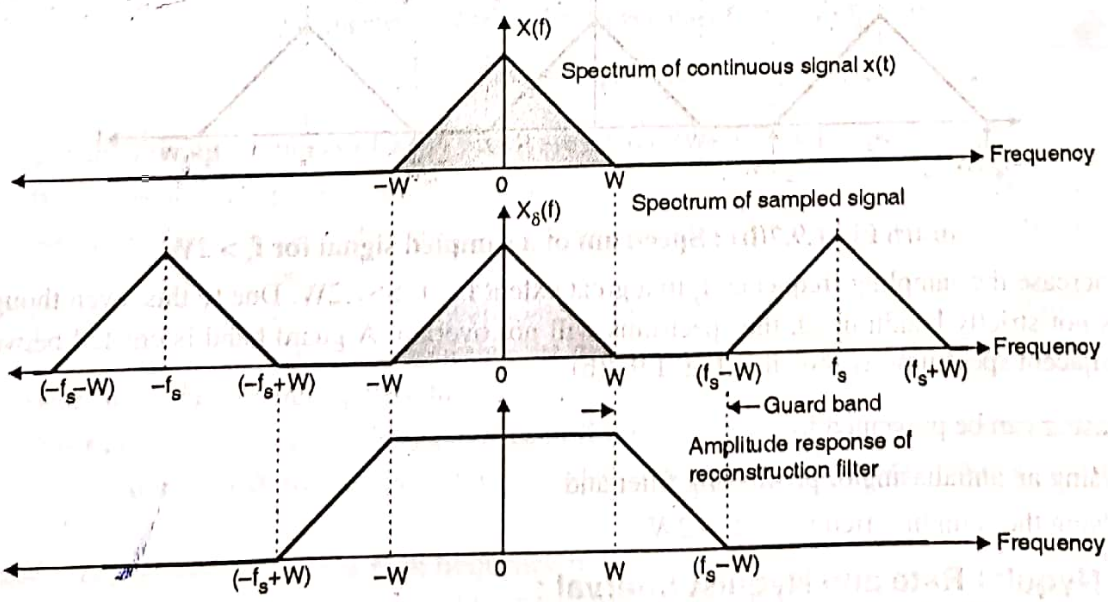
1.9.4 Effect of Nonideal Filter :

- As mentioned earlier the reconstruction filter is a low pass filter. It is expected to pass all the frequencies in the range of (– W to + W) Hz.
- This is because the original signal x (t) is bandlimited to "W" Hz.
- Therefore the frequency response of a reconstruction filter should be as shown in Fig. 1.9.8. This is the frequency response of an ideal low pass filter.



(D-418) **Fig. 1.9.8 : Frequency response of an ideal low pass filter used as a reconstruction filter**

- But it is not possible to practically realize an ideal low pass filter. Therefore a practical low pass filter with a frequency response as shown in Fig. 1.9.9 is used.
- It is possible to use the practical low pass filter without introducing any distortion due to the presence of the guard bands between the adjacent frequency spectrums as shown in Fig. 1.9.8. That is why it is necessary to have $f_s > 2W$.



(D-419) **Fig. 1.9.9 : Amplitude response of a practical reconstruction filter**

1.9.5 Examples on Sampling Theorem for Low Pass Signals :

Ex. 1.9.1 : Find the Nyquist rate and Nyquist interval for each of the following signals :

(a) $x(t) = 5 \cos 1000\pi t \cos 4000\pi t$.

(b) $x(t) = \dfrac{\sin 200\pi t}{\pi t}$.

Soln. :

(a) The given signal x (t) is in the form of product of cosine term. So let us use the following standard expression :

$$2 \cos A \cos B = \cos(A+B) + \cos(A-B)$$

$$5 \cos 1000\pi t \cos 4000\pi t = 2.5 \cos 5000\pi t + 2.5 \cos 3000\pi t$$

$$\therefore \quad x(t) = 2.5 \cos 5000\pi t + 2.5 \cos 3000\pi t \qquad ...(1)$$

From Equation (1) it is clear that the maximum frequency component present in the signal x (t) is of 2500 Hz. In other words x (t) is bandlimited to 2.5 kHz (W = 2.5 kHz).

$$\therefore \quad \text{Nyquist rate} = 2W = 2 \times 2.5 \text{ kHz} = 5 \text{ kHz} \qquad \textbf{...Ans.}$$

$$\text{and Nyquist interval} = \frac{1}{2W} = \frac{1}{5 \times 10^3} = 0.2 \text{ msec} \qquad \textbf{...Ans.}$$

(b) $$x(t) = \frac{\sin 200\pi t}{\pi t}$$

- In order to calculate the Nyquist rate we need to calculate the maximum frequency component present in its spectrum. The spectrum of x (t) can be obtained by taking its fourier transform.

Multiply numerator and denominator of x (t) by 200 to get,

$$x(t) = \frac{200 \sin(200\pi t)}{(200\pi t)}$$

$$\text{But} \quad \frac{\sin \pi t}{\pi t} = \text{sinc } t \quad \therefore \quad \frac{\sin(200\pi t)}{(200\pi t)} = \text{sinc}(200 t)$$

$$\therefore \quad x(t) = 200 \text{ sinc}(200 t)$$

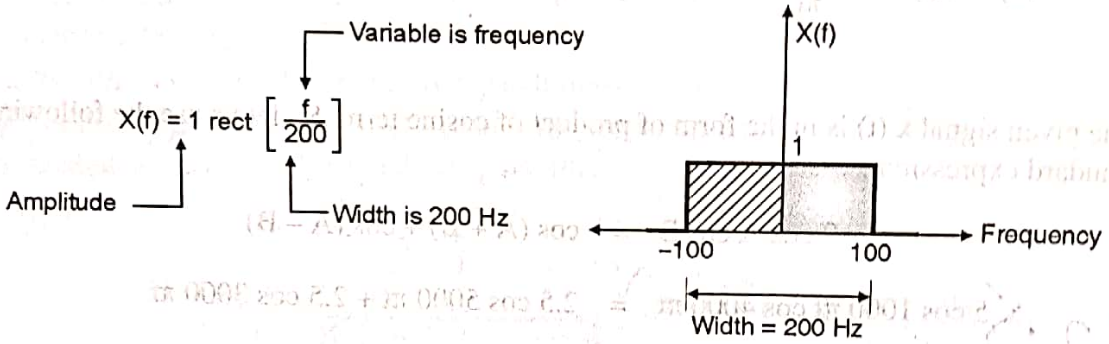
We know that $A \text{ sinc } 2Wt \overset{F}{\leftrightarrow} \dfrac{A}{2W} \text{ rect}\left[\dfrac{f}{2W}\right]$

$$\therefore \quad 200 \text{ sinc}(200 t) \overset{F}{\leftrightarrow} \frac{200}{200} \text{ rect}\left[\frac{f}{200}\right]$$

$$\therefore \quad 200 \text{ sinc } (200\, t) \overset{F}{\leftrightarrow} \text{rect}\left[\frac{f}{200}\right]$$

$$\therefore X(f) = \text{rect}\,[f/200]$$

The spectrum X (f) have been shown in Fig. P. 1.9.1.



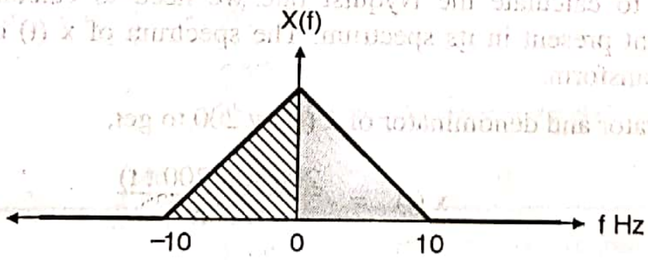
(D-420) **Fig. P. 1.9.1 : Spectrum of the given signal**

From Fig. P. 1.9.1 the maximum frequency in the frequency spectrum is 100 Hz.

$$\text{Nyquist rate} = 2 \times 100 = 200 \text{ Hz} \quad \text{...Ans.}$$

$$\text{And Nyquist interval} = 1/200 = 5 \text{ mS} \quad \text{...Ans.}$$

Ex. 1.9.2 : The spectrum of a continuous time analog signal x (t) is as shown in Fig. P. 1.9.2(a). This signal is sampled at a frequency f_c which is 3/2 times the maximum frequency f_M in the spectrum of x (t). Draw the spectrum of the sampled signal clearly showing the effect of undersampling. If the sampled signal is then passed through an ideal low pass filter having a cutoff frequency $f_c = f_M$ then draw the spectrum of signal recovered at the filter output.



(D-421) **Fig. P. 1.9.2(a) : Spectrum of the continuous analog signal x (t)**

Soln. :

From Fig. P. 1.9.2(a) it is clear that the signal x (t) is bandlimited to 10 Hz i.e.

$$f_M = 10 \text{ Hz} \quad \text{...(1)}$$

Hence the sampling frequency f_s is given by,

$$f_s = \frac{3}{2} \times f_M = 15 \text{ Hz.} \quad \text{...(2)}$$

To draw the spectrum of sampled signal X_δ (f) :

We know that the ideal sampling results in the spectrum X_δ (f) which is expressed mathematically as,

$$X_\delta(f) = f_s \sum_{n=-\infty}^{\infty} X(f - nf_s)$$

$$\therefore \quad X_\delta(f) = f_s X(f) + \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} f_s X(f - nf_s)$$

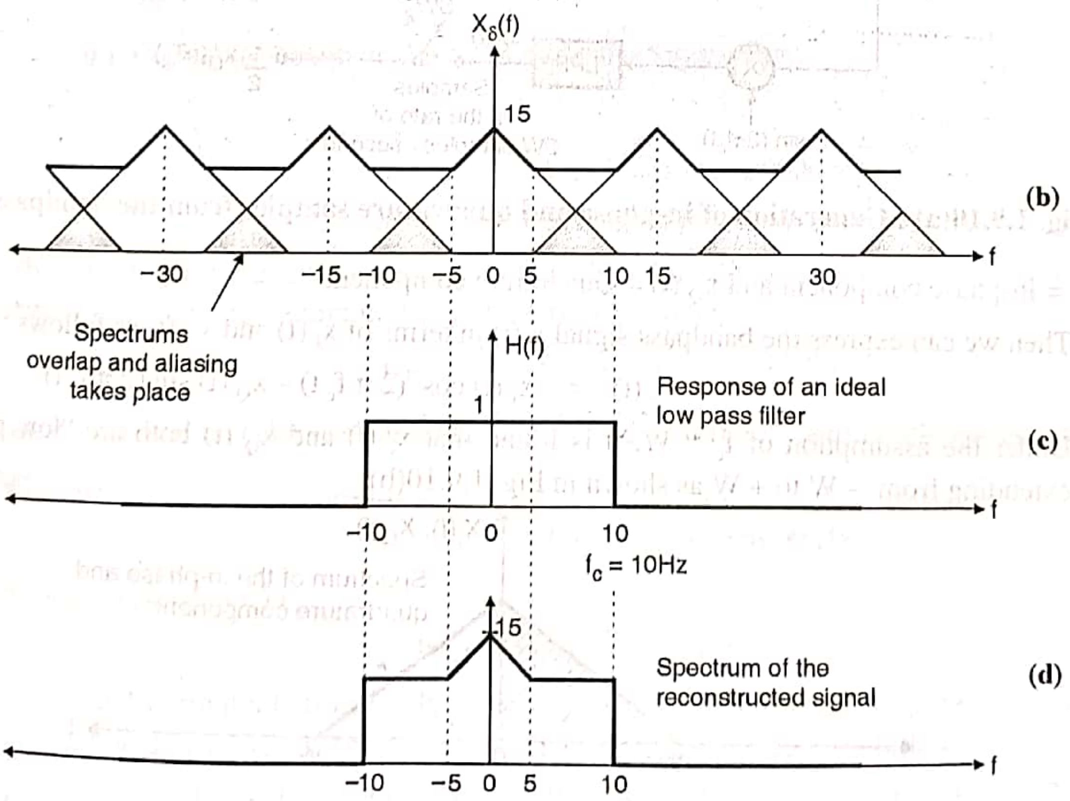
Substitute $f_s = 15$ to get, $X_\delta(f) = 15\,X(f) + 15 \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} X(f - 15n)$ ...(3)

Fig. P. 1.9.2(b) shows the spectrum $X_\delta(f)$.

Observe the overlapping of spectrums as $f_s < 2 f_M$.

Reconstruction of the signal :

- An ideal low pass filter with a cutoff frequency $f_c = f_M = 10$ Hz is being used for reconstruction. The response of this filter is as shown in Fig. P. 1.9.2(c).
- It is going to allow only that portion of $X_\delta(f)$ to pass through, which lies in the frequency range of – 10 Hz to 10 Hz as shown in Fig. P. 1.9.2(d).



(b) Spectrum of sampled signal **(c) Response of an ideal low pass filter** **(d) Output of the filter pass filter**

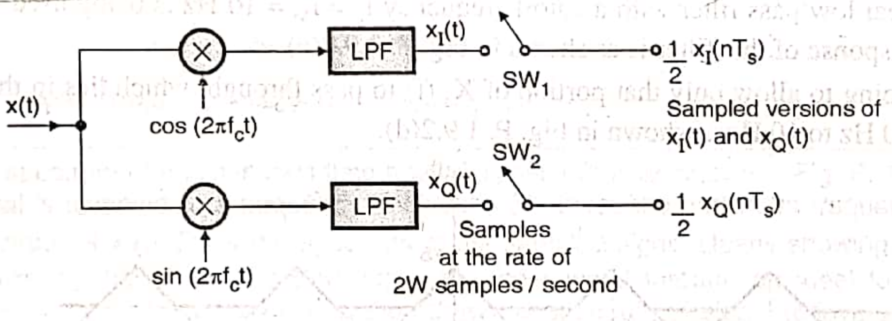
(D-422) **Fig. P. 1.9.2**

1.9.6 Sampling Theorem for Bandpass Signals :

- The sampling theorem for the bandpass signals can be stated as follows :
- A bandpass signal x (t), having a maximum bandwidth of 2W Hz can be completely represented in its sampled form and recovered back from the sampled form if it is sampled at a rate which is at least twice the maximum bandwidth. (i.e. $f_s \geq 4W$.)

1.9.6.1 Quadrature Sampling of Bandpass Signals :

- In this section, we consider a scheme called "quadrature sampling" for the uniform sampling of bandpass signals. This scheme is actually a natural extension of the sampling of low pass signals. The scheme is as follows :
- In this scheme, we do not sample the bandpass signal directly. Instead, before sampling we represent the bandpass signal x (t) in terms of its "in-phase" and "quadrature" components, $x_I(t)$ and $x_Q(t)$ respectively.
- The in-phase and quadrature components can be obtained by multiplying the bandpass signal x (t) by cos $(2\pi f_c t)$ and sin $(2\pi f_c t)$ respectively and then by suppressing the sum frequency components by means of low pass filters as shown in Fig. 1.9.10(a).



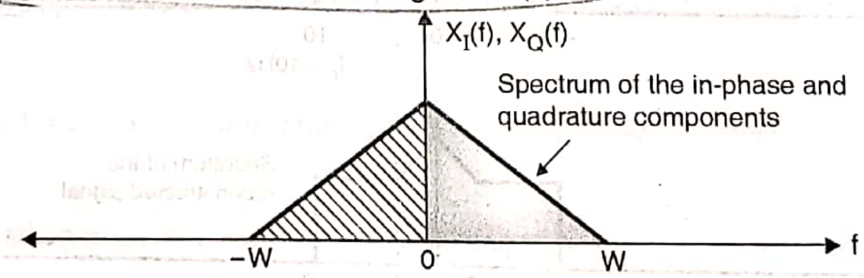
(D-423) **Fig. 1.9.10(a) : Generation of in-phase and quadrature samples from the bandpass signal x (t)**

If $x_I(t)$ = In-phase component and $x_Q(t)$ = Quadrature component.

- Then we can express the bandpass signal x (t) in terms of $x_I(t)$ and $x_Q(t)$ as follows :

$$x(t) = x_I(t) \cos(2\pi f_c t) - x_Q(t) \sin(2\pi f_c t) \qquad ...(1.9.22)$$

- Under the assumption of $f_c > W$, it is found that $x_I(t)$ and $x_Q(t)$ both are "low pass signals" extending from – W to + W as shown in Fig. 1.9.10(b).



(D-424) **Fig. 1.9.10(b) : Spectrum of the in-phase and quadrature components of x (t)**

- Then both the in-phase and quadrature components are separately sampled at a rate of 2W samples per second by the switches SW_1 and SW_2 as shown in Fig. 1.9.10(a) to obtain the sampled versions of $x_I(t)$ and $x_Q(t)$.

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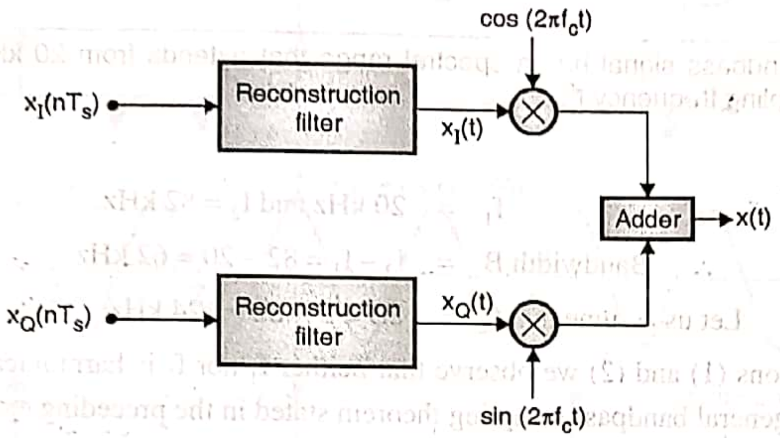
- In order to reconstruct the original bandpass signal from its quadrature sampled version, we first reconstruct the in-phase component $x_I(t)$ and quadrature component $x_Q(t)$ from their respective sampled versions $x_I(nT_s)$ and $x_Q(nT_s)$ by means of reconstruction filters. Then multiply $x_I(t)$ and $x_Q(t)$ by $\cos(2\pi f_c t)$ and $\sin(2\pi f_c t)$ respectively and add the result. The reconstruction process of x (t) is shown in Fig. 1.9.10(c).



(D-425) **Fig. 1.9.10(c) : Reconstruction of the bandpass signal x(t)**

Ex. 1.9.3 : The given signal is $m(t) = 10 \cos 2000 \pi t \cos 8000 \pi t$

(a) What is the minimum sampling rate based on the low pass uniform sampling theorem ?

(b) Repeat (a) based on the bandpass sampling theorem.

Soln. :

(a)

$$m(t) = 10 \cos 2000 \pi t \cos 8000 \pi t$$

$$\therefore \quad m(t) = 5 \cos 10000 \pi t + 5 \cos 6000 \pi t \qquad ...(1)$$

- Thus the highest frequency present in m (t) is W = 5000 Hz. Therefore as per the low pass sampling theorem, the minimum sampling rate is given by,

$$f_{s\,min} = 2W = 2 \times 5000 = 10 \text{ kHz} \qquad ...\textbf{Ans.}$$

(b) Looking at Equation (1) it is clear that the given signal m (t) contains two frequency components which are,

$$f_1 = 3000 \text{ Hz and } f_2 = 5000 \text{ Hz}$$

The bandwidth of m (t) is,

$$B = f_2 - f_1 = 2000 \text{ Hz}$$

- When f_1 and f_2 are not harmonically related to the sampling frequency f_s, the bandpass sampling theorem stated in section 1.9.6 is stated in a more generalized form as follows :
- If a bandpass signal x (t) has a bandwidth "B" and the highest frequency "f_M" then x (t) can be recovered from its sampled version if $f_s = \frac{2f_M}{k}$ where k is the largest integer not exceeding $\frac{f_M}{B}$. All higher sampling rates are not necessarily usable unless they exceed $2f_M$.

$$\text{Thus } f_M = f_2 = 5 \text{ kHz}$$

$$\text{and} \quad B = 2 \text{ kHz}$$

$$\therefore \quad k = \frac{5}{2} = 2.5. \text{ Therefore the value of k is 2}$$

$$\therefore \quad f_s = \frac{2 \times 5 \text{ kHz}}{2} = 5 \text{ kHz} \qquad \text{...Ans.}$$

Ex. 1.9.4 : A bandpass signal has a spectral range that extends from 20 kHz to 82 kHz. Find the sampling frequency f_s.

Soln. :

$$f_1 = 20 \text{ kHz and } f_2 = 82 \text{ kHz.}$$

$$\therefore \quad \text{Bandwidth B} = f_2 - f_1 = 82 - 20 = 62 \text{ kHz} \qquad \text{...(1)}$$

$$\text{Let us assume that } f_s = 2B = 2 \times 62 = 124 \text{ kHz} \qquad \text{...(2)}$$

From Equations (1) and (2) we observe that neither f_1 nor f_2 is harmonically related to f_s. Hence we have to use the general bandpass sampling theorem stated in the preceding example.

$$\therefore \quad k = \frac{f_M}{B} = \frac{82}{62} = 1.32 \rightarrow 1$$

$$\therefore \quad f_s = \frac{2 f_M}{k} = \frac{2 \times 82}{1} = 164 \text{ kHz} \qquad \text{...Ans.}$$

Ex. 1.9.5 : The signals $x_1(t) = 10 \cos(100 \pi t)$ and $x_2(t) = 10 \cos(50 \pi t)$ are both sampled at times $t_n = n / f_s$ where $n = 0, \pm 1, \pm 2, ...$ and the sampling frequency is 75 samples/sec. Show that the two sequences of samples thus obtained are identical. What is this phenomenon called ?

Soln. :

Let us prove that we get the identical sequences of samples by using the graphical method.

The signals $x_1(t)$ and $x_2(t)$ are cosine waves with equal peak amplitudes. Their frequencies are 50 Hz and 25 Hz respectively.

$$\therefore \quad x_1(t) : \text{Has peak voltage} = 10 \text{ V and } f_1 = 50 \text{ Hz}$$

$$\therefore \quad x_2(t) : \text{Has peak voltage} = 10 \text{ V and } f_2 = 25 \text{ Hz}$$

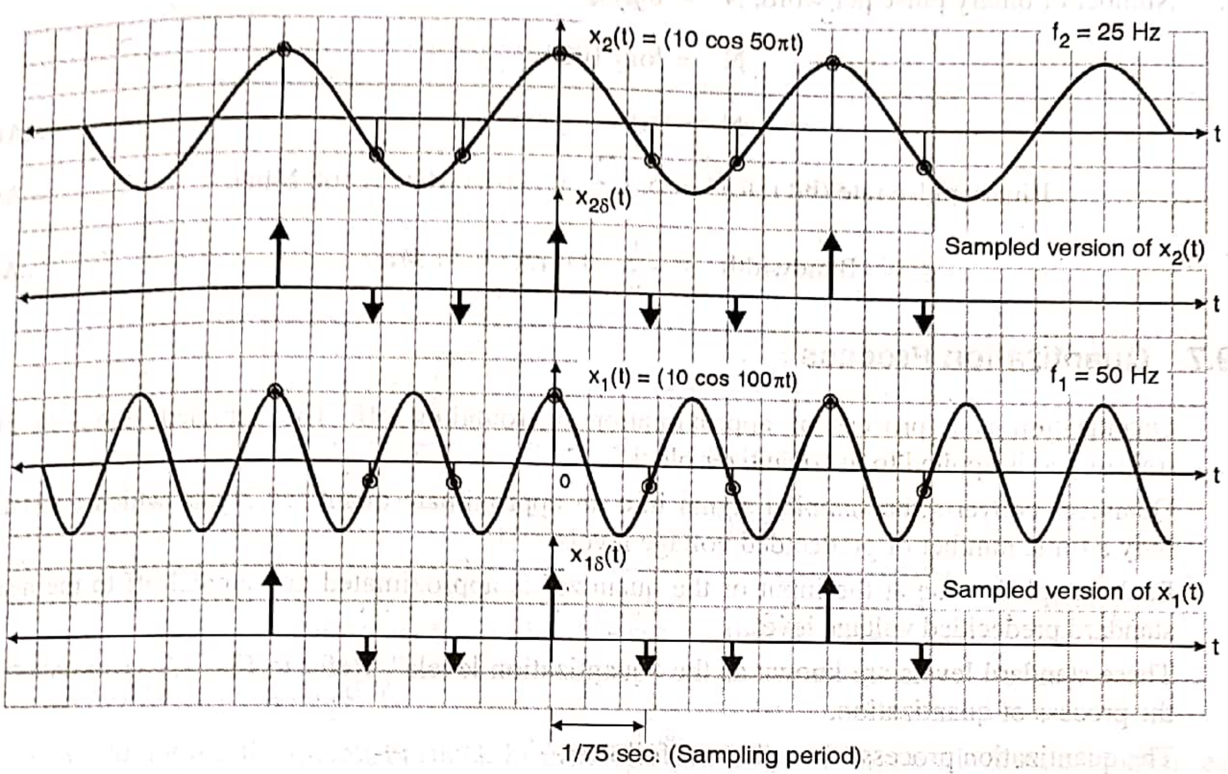
The sampling frequency $f_s = 75$ Hz and sampling period $T_s = 1/75 = 13.33$ mS. Steps to be followed to plot the sampled versions are as follows :

Step 1 : Draw the signals $x_1(t)$, $x_2(t)$ and the sampling function.

Step 2 : Encircle the sample values.

Step 3 : Draw the sampled signals $x_{1\delta}(t)$ and $x_{2\delta}(t)$.

All these waveforms are as shown in Fig. P. 1.9.5.



(D-427) **Fig. P. 1.9.5**

Conclusion :

The sampling frequency of 75 Hz is lower than the Nyquist rate. Because to sample a 50 Hz signal the sampling rate should atleast be 100 Hz. Therefore aliasing takes place. The identical sequence in Fig. P. 1.9.5 are being obtained due to aliasing.

Ex. 1.9.6 : A television signal (video and audio) has a bandwidth of 4.5 MHz. This signal is sampled, quantized and binary coded to obtain a PCM signal :

1. Determine the sampling rate if the signal is to be sampled at a 20% rate above the Nyquist rate.
2. If the samples are quantized into 1024 levels, determine the number of binary pulses required to encode each sample.
3. Determine the binary pulse rate (bits per second) of the binary-coded signal, and the maximum bandwidth required to transmit this signal.

Soln. :

Given :

$$W = 4.5 \text{ MHz}$$

1.

$$\text{Sampling rate} = 1.2 \times \text{Nyquist rate} = 1.2 \times 2\,W$$

$$= 1.2 \times 2 \times 4.5 \times 10^6 = 10.8 \text{ MHz} \quad \text{...Ans.}$$

2. Given that number of quantization levels, Q = 1024

$$\text{But } Q = 2^N$$

$\therefore$ Number of binary pulse per word, $N = \log_2 Q$

$$\therefore \quad N = \log_2 1024$$

$$\therefore \quad N = 10 \quad \text{...Ans.}$$

3. Binary pulse rate (bit rate) $= N f_s = 10 \times 10.8 \text{ MHz} = 108 \text{ Mbps}$ ...Ans.

$$\text{Bandwidth} = \frac{1}{2} \text{ bit rate} = 54 \text{ MHz} \quad \text{...Ans.}$$

1.9.7 Quantization Process :

- Quantization is a process of approximation or rounding off. The sampled signal in PCM transmitted is applied to the quantizer block.
- Quantizer converts the sampled signal into an approximate quantized signal which consists of only a finite number of predecided voltage levels.
- Each sampled value at the input of the quantizer is approximated or rounded off to the nearest standard predecided voltage level.
- These standard levels are known as the "quantization levels". Refer to Fig. 1.9.11 to understand the process of quantization.
- The quantization process takes place as follows :
- The input signal x (t) is assumed to have a peak to peak swing of V_L to V_H volts. This entire voltage range has been divided into "Q" equal intervals each of size "s".

7
- "s" is called as the step size and its value is given as,

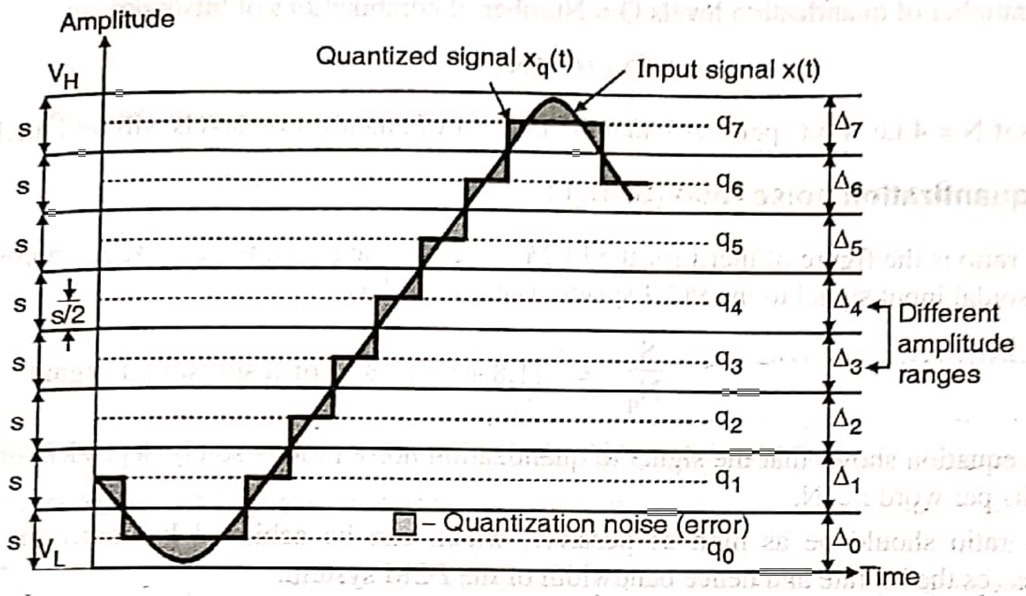
2.

$$s = \frac{V_H - V_L}{Q} \qquad \text{...(1.9.22)}$$

15,

In Fig. 1.9.11, the value of Q = 8

- At the center of these ranges, the quantization levels $q_0, q_1, \ldots q_7$ are placed. Thus the number of quantization levels is Q = 8. The quantization levels are also called as decision thresholds.
- x_q (t) represents the quantized version of x (t). We obtain x_q (t) at the output of the quantizer.
- When x (t) is in the range Δ_0, then corresponding to any value of x (t), the quantizer output will be equal to "q_0".
- Similarly for all the values of x (t) in the range Δ_1, the quantizer output is constant equal to "q_1".
- Thus in each range from Δ_0 to Δ_7, the signal x (t) is rounded off to the nearest quantization level and the quantized signal is produced.
- The quantized signal x_q (t) is thus an approximation of x (t). The difference between them is called as **quantization error or quantization noise.**
- This error should be as small as possible.
- To minimize the quantization error we need to reduce the step size "s" by increasing the number of quantization levels Q.



(L-225) **Fig. 1.9.11 : Process of quantization**

Why is quantization required ?

- If we do not use the quantizer block in the PCM transmitter, then we will have to convert each and every sampled value into a unique digital word.
- This will need a large number of bits per word (N). This will increase the bit rate and hence the bandwidth requirement of the channel.
- To avoid this, if we use a quantizer with only 256 quantization levels then all the sampled values will be finally approximated into only 256 distinct voltage levels.
- So we need only 8 bits per word to represent each quantized sampled value.
- Thus the number of bits per word can be reduced. This will eventually reduce the bit rate and bandwidth requirement.

Quantization error or quantization noise $\in$:

- The difference between the instantaneous values of the quantized signal and input signal is called as quantization error or quantization noise.

$$\in \;=\; x_q(t) - x(t) \qquad \text{...(1.9.23)}$$

- The quantization error is shown by shaded portions of the waveform in Fig. 1.9.11.
- The maximum value of quantization error is $\pm s/2$ where s is step size.
- Therefore to reduce the quantization error we have to reduce the step size by increasing the number of quantization levels i.e. Q.
- The mean square value of the quantization is given by,

$$\text{Mean square value of quantization error} \;=\; \frac{s^2}{12} \qquad \text{...(1.9.24)}$$

- The relation between the number of quantization levels Q and the number of bits per word (N) in the transmitted signal can be found as follows :
- Because each quantized level is to be converted into a unique N bit digital word, assuming a binary coded output signal,

- The number of quantization levels Q = Number of combinations of bits/word.

$$\therefore \quad Q = 2N \qquad ...(1.9.25)$$

- Thus if N = 4 i.e. 4 bits per word then the number of quantization levels will be 2^4 i.e.16.

Signal to quantization noise ratio (SNR_q) :

- This ratio is the figure of merit for the PCM systems. The signal to quantization noise ratio with a sinusoidal input signal to the PCM system is expressed as,

$$\frac{S_i}{N_q} = [1.8 + 6N] \text{ dB} \quad \textbf{For a sinusoidal signal} \qquad ...(1.9.26)$$

- This equation shows that the signal to quantization noise ratio is solely dependent on the number of bits per word i.e. N.
- This ratio should be as high as possible, which can be achieved by increasing N. But this increases the bit rate and hence bandwidth of the PCM system.
- Therefore the number of bits per word is a compromise between high SNR_q and bandwidth requirements.

Review Questions

Q. 1 Define signal.

Q. 2 Give the classification of signals.

Q. 3 Define continuous time (CT) and discrete time (DT) signals.

Q. 4 Differentiate between CT and DT signals.

Q. 5 How is a CT signal represented mathematically ?

Q. 6 How do you obtain a DT signal from a CT signal ?

Q. 7 Define a digital signal.

Q. 8 Define periodic and nonperiodic signals.

Q. 9 State the condition of periodicity.

Q. 10 Compare periodic and nonperiodic signals.

Q. 11 Prove that $x(t) = A \cos \omega_0 t$ is a periodic signal.

Q. 12 Define deterministic and random signals.

Q. 13 Compare deterministic and random signals.

Q. 14 Define even and odd signals.

Q. 15 Compare energy and power signals.

Q. 16 Define signal energy.

Q. 17 Define signal power.

Q. 18 Define multichannel and multidimensional signals.

Q. 19 Define the following signals and write their mathematical expressions :

1. DC signal
2. Exponential signal
3. Rectangular pulse
4. Signum function.

Q. 20 State mathematical expression for unit step and rectangular signals.

Q. 21 State the important properties of delta function.

Q. 22 State the relation between DT unit impulse and DT unit step signals.

Q. 23 Define a unit ramp signal and draw it graphically.

Q. 24 State different types of complex exponential signal.

Q. 25 What is the relation between the complex exponential and sinusoidal signals.

Q. 26 Draw the x(t) = sinc (t) signal graphically.

Q. 27 Define the following :

1. Time shifting.
2. Time scaling

Q. 28 Define folding or time reversal.

Q. 29 What is the difference between amplitude scaling and time scaling ?

Q. 30 State the relationship between unit step and unit impulse signals.

Q. 31 State the precedence rule for time shifting and time scaling.

□□□