PARUL UNIVERSITY
Faculty of Engineering & Technology
Department of Applied Sciences & Humanities
1st year B.Tech Programme (All branches)
Mathematics-I (Subject Code: 303191101)
UNIT-2 First Order Differential Equation

- **Differential Equation:**

  Ordinary differential equation

  An ordinary differential equation is an equation which contains derivatives of a dependent variable, $y(x)$, w.r.t. only one independent variable.

  For example:

1. $\frac{dy}{dx} = \cos x$
2. $\frac{d^2y}{dx^2} + 4y = 0$
3. $x^2\frac{d^3y}{dx^3} + 2e^x\frac{d^2y}{dx^2} = (x^2+2)y^2$

  Partial differential equation

  A partial deferential equation is an equation which contains partial derivatives of a dependent variables $f(x,y)$, w.r.t. two or more independent variables.

  For example:

1. $\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$
2. $\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2}$
3. $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

- **Order**

  The order of a differential equation is the order of the highest derivative occurring in that equation.

- **Degree**

  The degree of a differential equation is the highest index of the highest order derivative.

- **Examples:**

Sr. No	Differential equation	Order	Degree
1.	$\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3 + y = 0$	**2**	**1**
2.	$3\left(\frac{d^3y}{dx^3}\right)^3 + \left(\frac{d^2y}{dx^2}\right) + \frac{dy}{dx} = e^{-x}\sin x$	**3**	**3**
3.	$\frac{\left(1+\left(\frac{dy}{dx}\right)^2\right)^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} = 1$ OR $\left(1+\left(\frac{dy}{dx}\right)^2\right)^3 = \left(\frac{d^2y}{dx^2}\right)^2$	**2**	**2**

Formation of Differential Equation:

The differential equation can be formed by differentiation of ordinary equation and eliminating the arbitrary constants by taking minimum number of differentiation of ordinary equation.

To eliminate n number of arbitrary constants, we require n number of more equations besides the given equation. This leads us n[th] order derivatives. Generally if equation consists n number of arbitrary constants, then its differential equation has order n.

Examples: Form the differential equations of the following equations.

1. $y = (c_1 + c_2 x)e^x$

Solution: $y = c_1 e^x + c_2 x e^x$ ... (1)

Differentiating …(1) with respect to x

$$y' = \frac{dy}{dx} = c_1 e^x + c_2 x e^x + c_2 e^x$$

$$y' = \frac{dy}{dx} = y + c_2 e^x \quad \text{... (2)}$$

$\Rightarrow c_2 e^x = y' - y$

Differentiating …(2) with respect to x

$$y'' = \frac{d^2y}{dx^2} = y' + c_2 e^x \quad \text{... (3)}$$

$\Rightarrow y'' = y' + y' - y$

$\Rightarrow y'' - 2y' + y = 0$ is required ODE.

2. Form the differential equation of all circles touching the x-axis at the origin.

Solution: Let the centre of such circle be (0,a) , hence the equation of circle is

$$(x - 0)^2 + (y - a)^2 = a^2$$

$$\Rightarrow x^2 + y^2 - 2ay = 0 \quad \text{... (1)}$$

$$\Rightarrow a = \frac{x^2 + y^2}{2y}$$

Differentiating …(1) w.r.t. x, we have

$$2x + 2y\frac{dy}{dx} - 2a\frac{dy}{dx} = 0 \quad \text{... (2)}$$

$$\Rightarrow 2x + y'(2y - 2a) = 0$$

$$\Rightarrow y' = \frac{-2x}{2y - 2a} = \frac{-2x}{2y - 2\left(\frac{x^2+y^2}{2y}\right)} = \frac{2xy}{x^2 - y^2}$$

$y' = \frac{2xy}{x^2-y^2}$ is required ODE.

3. Verify that $y = e^{-x}(a\cos x + b\sin x)$ is a solution of $y'' + 2y' + 2y = 0$, where a and b are constants.

$$y = e^{-x}(a\cos x + b\sin x) \quad \text{... (1)}$$

$$y' = -e^{-x}(a\cos x + b\sin x) + e^{-x} = -e^{-x}[a\cos x + b\sin x + a\sin x - b\cos x] \quad \text{... (2)}$$

$$y'' = -e^{-x}[-a\cos x - b\sin x + a\sin x - b\cos x] + e^{-x}[a\sin x - b\cos x + a\cos x + b\sin x]$$

$$y'' = e^{-x}[2a\sin x - 2b\cos x] \quad \text{... (3)}$$

By substituting $y, y' and\ y''$ in $y'' + 2y' + 2y$

$= e^{-x}[2a\sin x - 2b\cos x] - 2e^{-x}[a\cos x + b\sin x + a\sin x - b\cos x] + 2e^{-x}(a\cos x + b\sin x)$

$= e^{-x}[2a\sin x - 2b\cos x - 2a\sin x + 2b\cos x - 2a\cos x - 2b\sin x + 2a\cos x + 2b\sin x]$

$= e^{-x}[0] = 0.$

Hence, $y = e^{-x}(a\cos x + b\sin x)$ is a solution of $y'' + 2y' + 2y = 0$.

Exercise:

1. Form the differential equation from $y = e^x(c_1\cos x + c_2\sin x)$.

2. Form the differential equation of the cardioids $r = b(1 + \cos\theta)$.

3. Verify that $y = \sqrt{cx}$is the solution of $y' = \frac{y}{2x}$

Initial Value Problem: A particular solution can be obtained from a general solution by an initial condition$y(x_0) = y_0$ which determines the value of the arbitrary constant c. An ordinary differential equation with initial condition is known as **Initial Value Problem.**

$$y' = f(x, y)\,; y(x_0) = y_0$$

Exact Differential Equation:

A first order differential equation of the form $M(x, y)dx + N(x, y)dy = 0$ is called an exact differential equation if for some function $z(x, y)$, $dz = M(x, y)dx + N(x, y)dy$.

Where, $M = \frac{\partial Z}{\partial x}, N = \frac{\partial z}{\partial y}$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$is a sufficient condition for $M(x, y)dx + N(x, y)dy = 0$ to be exact differential equation.

Therefore, the solution of the exact differential equation is

$$z = \int_{y\ is\ const.} M(x, y)dx + \int_{terms\ containing\ only\ y} N(x, y)dy \qquad \text{or}$$

$$z = \int_{x\ is\ const.} N(x, y)dy + \int_{terms\ containing\ only\ x} M(x, y)dx$$

For example:

1. Solve $\boldsymbol{2xydx + (1 + x^2)dy = 0}$.	2. Solve $\boldsymbol{(x + \sin y)dx + (x\cos y - 2y)dy = 0}$
Solution:$M = 2xy\ and\ N = 1 + x^2$ $$\frac{\partial M}{\partial y} = 2x\ \frac{\partial N}{\partial x} = 2x \quad \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = 2x$$ ∴The given ODE is exact. ∴Solution of given ODE is given by $$\int_{y\ is\ const.} M\,dx + \int_{terms\ containing\ only\ y} N\,dy = c$$ $$\Rightarrow \int 2xy\,dx + \int 1\,dy = c$$ $\Rightarrow x^2y + y = c$ which is required solution.	**Solution:** $M = (x + \sin y)\ , N = (x\cos y - 2y)$ $\frac{\partial M}{\partial y} = \cos y, \frac{\partial N}{\partial x} = \cos y \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = \cos y$ ∴The given ODE is exact. ∴Solution of given ODE is given by $$\int_{x\ is\ const.} N\,dy + \int_{terms\ containing\ only\ x} M\,dx = c$$ $$\Rightarrow \int (x\cos y - 2y)\,dy + \int x\ dx = c$$ $$\Rightarrow -x\ \sin y - y^2 + \frac{x^2}{2} = c$$ $\frac{x^2}{2} - x\ \sin y - y^2 = c$ is required solution.

3. Solve $(x^3 + 3xy^2)dx + (3x^2y + y^3)dy = 0,$

$$y(1) = 1$$

Solution: $M = (x^3 + 3xy^2), N = (3x^2y + y^3)$

$$\frac{\partial M}{\partial y} = 6xy \quad \frac{\partial N}{\partial x} = 6xy \quad \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = 6xy$$

∴The given ODE is exact.

∴Solution of given ODE is given by

$$\int\limits_{y \text{ is const.}} M\,dx + \int\limits_{\text{terms containing only } y} N\,dy = c$$

$$\Rightarrow \int (x^3 + 3xy^2)\,dx + \int (y^3)\,dy = c$$

$$\Rightarrow \frac{x^4}{4} + \frac{3x^2y^2}{2} + \frac{y^4}{4} = c$$

$x^4 + 6x^2y^2 + y^4 = 4c$ is required general solution.

Using Initial condition, we have $1 + 6 + 1 = 4c \Rightarrow c = 2$

The particular solution is $x^4 + 6x^2y^2 + y^4 = 8$.

4. Solve $\frac{dy}{dx} + \frac{y\cos x + \sin y + y}{\sin x + x\cos y + x} = 0$

Solution:

$$(y\cos x + \sin y + y)dx + (\sin x + x\cos y + x)dy = 0$$

$$M = y\cos x + \sin y + y, N = \sin x + x\cos y + x$$

$$\frac{\partial M}{\partial y} = \cos x + \cos y + 1 \,, \frac{\partial N}{\partial x} = \cos x + \cos y + 1$$

$$\Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = \cos x + \cos y + 1$$

∴The given ODE is exact.
Therefore the general solution is

$$\int\limits_{x \text{ is const.}} N\,dy + \int\limits_{\text{terms containing only } x} M\,dx = c$$

$$\int (\sin x + x\cos y + x)dy + \int 0\,dx = c$$

$y\,\sin x + x\sin y + xy = c$ is required solution.

Exercise: Solve the following ODE

1. $(2xy + e^y)dx + (x^2 + xe^y)dy = 0, y(1) = 1.$
2. $(ye^x)dx + (2y + e^x)dy = 0, y(0) = -1$
3. $\frac{dy}{dx} = \frac{y+1}{e^y(y+2)-x}$

Non exact Differential Equation OR Reducible to exact diff. Equation:

If $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ then the given equation is not exact.

Therefore, by multiplying the given equation with integrating factor reduces it to exact. There are four cases for finding the integrating factor.

Case-:1

If the given differential equation is homogeneous with $Mx + Ny \neq 0$ then

$$\boxed{\text{I.F} = \frac{1}{Mx+Ny}}$$

Case-2:

If the given differential equation is of the form $f(x,y)ydx + g(x,y)xdy = 0$ with $Mx - Ny \neq 0$, then

$$\boxed{\text{I.F} = \frac{1}{Mx-Ny}}$$

Case-3:

If $\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)$ is a function of x alone, say f(x), then

$$\boxed{\text{I.F} = e^{\int f(x)dx}}$$

Case-4:

If $\frac{1}{M}\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)$ is a function of y alone, say g(y), then

$$\boxed{\text{I.F} = e^{\int g(y)dy}}$$

For examples:

1. Solve $(xy-2y^2)dx-(x^2-3xy)dy=0$

Solution: $(xy-2y^2)dx-(x^2-3xy)dy=0$

$M=(xy-2y^2)$ and

$N=(x^2-3xy)$

$\frac{\partial M}{\partial y}=x-4y$ and

$\frac{\partial N}{\partial x}=2x-3y$

Since $\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$, the equation is not exact.

Therefore

$Mx+Ny=x^2y-2xy^2-x^2y+3xy^2=xy^2\neq 0$

$$\therefore I.F=\frac{1}{Mx+Ny}=\frac{1}{xy^2}$$

Multiplying throughout by I.F, the equation becomes

$$\frac{(xy-2y^2)}{xy^2}dx-\frac{(x^2-3xy)}{xy^2}dy=0$$

$$\therefore\left(\frac{1}{y}-\frac{2}{x}\right)dx+\left(-\frac{x}{y^2}+\frac{3}{y}\right)dy=0$$

$$\therefore M'=\left(\frac{1}{y}-\frac{2}{x}\right)\qquad N'=\left(-\frac{x}{y^2}+\frac{3}{y}\right)$$

$$\therefore\frac{\partial M'}{\partial y}=-\frac{1}{y^2}\qquad\frac{\partial N'}{\partial x}=-\frac{1}{y^2}$$

$$\therefore\frac{\partial M'}{\partial y}=\frac{\partial N'}{\partial x}$$

which is exact.

Therefore the solution is

2. Solve $\left(x^4+y^4\right)dx-xy^3\ dy=0$

$M=x^4+y^4$ $\qquad N=-xy^3$

$\frac{\partial M}{\partial y}=4y^3$ $\qquad \frac{\partial N}{\partial x}=-y^3$

$$\therefore I.F.=\frac{1}{Mx+Ny}=\frac{1}{x^5+xy^4-xy^4}=\frac{1}{x^5}$$

$$\left(\frac{1}{x}+\frac{y^4}{x^5}\right)dx-\frac{y^3}{x^4}dy=0$$

$$\int\left(\frac{1}{x}+\frac{y^4}{x^5}\right)dx=c$$

$$\ln x-\frac{1}{4}\left(\frac{y}{x}\right)^4=c$$

3. Solve $(x^2y^2+2)ydx+(2-x^2y^2)xdy=0$

Solution: $(x^2y^2+2)ydx+(2-x^2y^2)xdy=0$

$M=(x^2y^2+2)y$ and $N=(2-x^2y^2)x$

$\frac{\partial M}{\partial y}=3x^2y^2+2$ and

$\frac{\partial N}{\partial x}=2-3x^2y^2$

Since $\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$, the equation is not exact.

Therefore

$Mx-Ny=x^3y^3+2xy-2xy+x^3y^3=2x^3y^3\neq 0$

$$\int_{y\,is\,const} M'(x,y)dx + \int_{terms\,containing\,only\,y} N'(x,y)dy = c$$

$$i.e.\ \int\left(\frac{1}{y}-\frac{2}{x}\right)\ dx+\int\frac{3}{y}dy=c$$

$$\therefore \frac{x}{y}-2\log x+3\log y=c$$

$$\therefore I.F=\frac{1}{Mx-Ny}=\frac{1}{2x^3y^3}$$

Multiplying throughout by I.F, the equation becomes

$$\frac{(x^2y^2+2)y}{2x^3y^3}dx+\frac{(2-x^2y^2)x}{2x^3y^3}dy=0$$

$$\therefore \frac{1}{2}\left(\frac{1}{x}+\frac{2}{x^3y^2}\right)dx+\frac{1}{2}\left(\frac{2}{x^2y^3}-\frac{1}{y}\right)dy=0$$

$$\therefore M'=\frac{1}{2}\left(\frac{1}{x}+\frac{2}{x^3y^2}\right) \qquad N'=\frac{1}{2}\left(\frac{2}{x^2y^3}-\frac{1}{y}\right)$$

$$\therefore \frac{\partial M'}{\partial y}=-\frac{2}{x^3y^3} \qquad \frac{\partial N'}{\partial x}=-\frac{2}{x^3y^3}$$

$$\therefore \frac{\partial M'}{\partial y}=\frac{\partial N'}{\partial x}$$

which is exact.

Therefore the solution is

$$\int_{y\,is\,const} M'(x,y)dx + \int_{terms\,containing\,only\,y} N'(x,y)dy = c$$

$$i.e.\ \int\frac{1}{2}\left(\frac{1}{x}+\frac{2}{x^3y^2}\right)\ dx+\int-\frac{1}{y}dy=c$$

$$\therefore \frac{1}{2}\left(\log x-\frac{1}{x^2y^2}\right)-\frac{1}{2}\log y=c$$

$$\therefore \log x-\log y-\frac{1}{x^2y^2}=c$$

$$\therefore \log\left(\frac{x}{y}\right)-\frac{1}{x^2y^2}=c$$

3.Solve $(2x\log x-xy)dy+2ydx=0$

Solution: $2ydx+(2x\log x-xy)dy=0$

$M=2y$ and

$N=(2x\log x-xy)$

$$\frac{\partial M}{\partial y}=2 \quad \text{and} \quad \frac{\partial N}{\partial x}=2\log x+2-y$$

Since $\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$, the equation is not exact.

Therefore

$$\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)=\frac{2-2\log x-2+y}{2x\log x-xy}$$

$$=-\frac{1}{x}=f(x)$$

$$\therefore I.F=e^{\int-\frac{1}{x}dx}=e^{-\log x}=\frac{1}{x}$$

Multiplying throughout by I.F, the equation becomes

4.Solve $\left(\frac{y}{x}\sec y-\tan y\right)dx+(\sec y\log x-x)dy=0$

Solution: $\left(\frac{y}{x}\sec y-\tan y\right)dx+(\sec y\log x-x)dy=0$

$$M=\left(\frac{y}{x}\sec y-\tan y\right) \quad \text{and}$$

$N=(\sec y\log x-x)$

$$\frac{\partial M}{\partial y}=\frac{1}{x}\sec y+\frac{y}{x}\sec y\tan y-\sec^2 y$$

and $\frac{\partial N}{\partial x}=\frac{\sec y}{x}-1$

Since $\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$, the equation is not exact.

Therefore

$$\frac{1}{M}\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)=\frac{\frac{\sec y}{x}-1-\frac{\sec y}{x}-\frac{y}{x}\sec y\tan y+\sec^2}{\frac{\sec y}{x}-\tan y}$$

$$=-\tan y=f(y)$$

$$\frac{1}{x}(2y)dx+\frac{1}{x}(2x\log x-xy)dy=0$$

$$\therefore \frac{2y}{x}dx+(2\log x-y)dy=0$$

$$\therefore M'=\frac{2y}{x} \qquad N'=2\log x-y$$

$$\therefore \frac{\partial M'}{\partial y}=\frac{2}{x} \qquad \frac{\partial N'}{\partial x}=\frac{2}{x}$$

$$\therefore \frac{\partial M'}{\partial y}=\frac{\partial N'}{\partial x}$$

which is exact.
Therefore the solution is

$$\int\limits_{y\,is\,const} M'(x,y)dx+\int\limits_{terms\,containing\,only\,y} N'(x,y)dy=c$$

$$i.e. \int\frac{2y}{x}dx+\int -y\,dy=c$$

$$\therefore 2y\log x-\frac{y^2}{2}=c$$

$$\therefore I.F=e^{\int -\tan y\,dy}=e^{-\log\sec x}=\sec^{-1}y=\cos y$$

Multiplying throughout by I.F, the equation becomes

$$\cos y\left(\frac{y}{x}\sec y-\tan y\right)dx+\cos y(\sec y\log x-x)$$

$$\therefore \left(\frac{y}{x}-\sin y\right)dx+(\log x-x\cos y)dy=0$$

$$\therefore M'=\left(\frac{y}{x}-\sin y\right) \qquad N'=(\log x-x\cos y)$$

$$\therefore \frac{\partial M'}{\partial y}=\frac{1}{x}-\cos y \qquad \frac{\partial N'}{\partial x}=\frac{1}{x}-\cos y$$

$$\therefore \frac{\partial M'}{\partial y}=\frac{\partial N'}{\partial x}$$

which is exact.
Therefore the solution is

$$\int\limits_{y\,is\,const} M'(x,y)dx+\int\limits_{terms\,containing\,only\,y} N'(x,y)dy=c$$

$$i.e. \int\left(\frac{y}{x}-\sin y\right)dx+\int 0\,dy=c$$

$$\therefore y\log x-x\sin y=c$$

Example:

1. $(xy^3+y)dx+2(x^2y^2+x+y^4)dy=0$

2. $x^2ydx-(x^3+y^3)dy=0$ **[Summer 2023]**

3. $(x^2+y^2+1)dx-2xydy=0$

4. $xe^x(dx-dy)+e^xdx+ye^ydy=0$

❖ Linear Differential Equation

A first order differential Equation of the form $\frac{dy}{dx}+P(x)y=Q(x)$ or $\frac{dx}{dy}+P(y)x=Q(y)$

is called Linear Differential Equation.

Differential equation	Integrating factor	General solution
$\frac{dy}{dx}+P(x)y=Q(x)$	$I.F=e^{\int P(x)dx}$	$y(I.F)=\int Q(x)(I.F)dx+c$
$\frac{dx}{dy}+P(y)x=Q(y)$	$I.F=e^{\int P(y)dy}$	$x(I.F)=\int Q(y)(I.F)dy+c$

For Example:

1.Solve $\frac{dy}{dx}+2y\tan x=\sin x$	2.Solve $(x+1)\frac{dy}{dx}-y=e^{3x}(x+1)^2$

Solution: $\dfrac{dy}{dx} + 2y\tan x = \sin x$

The equation is linear equation

Therefore, comparing with $\dfrac{dy}{dx} + P(x)y = Q(x)$

$$P(x) = 2\tan x, \qquad Q(x) = \sin x$$

$$IF = e^{\int P(x)dx} = e^{\int 2\tan x dx} = e^{\log \sec^2 x} = \sec^2 x$$

Therefore the general solution is

$$y(I.F) = \int Q(x)(I.F)dx + c$$

$$\therefore y(\sec^2 x) = \int \sin x \sec^2 x dx + c$$

$$\therefore y(\sec^2 x) = \int \tan x \sec x dx + c$$

$$\therefore y(\sec^2 x) = \sec x + c$$

Solution:

$$(x+1)\frac{dy}{dx} - y = e^{3x}(x+1)^2$$

$$\frac{dy}{dx} - \frac{y}{(x+1)} = e^{3x}(x+1)$$

The equation is linear equation

Therefore, comparing with

$$\frac{dy}{dx} + P(x)y = Q(x)$$

$$P(x) = -\frac{1}{x+1}, \qquad Q(x) = e^{3x}(x+1)$$

$$IF = e^{\int P(x)dx} = e^{\int -\frac{1}{x+1}dx} = e^{\log(x+1)^{-1}} = \frac{1}{x+1}$$

Therefore the general solution is

$$y(I.F) = \int Q(x)(I.F)dx + c$$

$$\therefore y\frac{1}{x+1} = \int e^{3x}(x+1)\frac{1}{x+1}dx + c$$

$$\therefore y\frac{1}{x+1} = \int e^{3x}dx + c$$

$$\therefore y\frac{1}{x+1} = \frac{e^{3x}}{3} + c$$

3.Solve $\dfrac{dy}{dx} + \dfrac{4x}{1+x^2}y = \dfrac{1}{(x^2+1)^3}$

Solution:

$$\frac{dy}{dx} + \frac{4x}{1+x^2}y = \frac{1}{(x^2+1)^3}$$

The equation is linear equation

Therefore comparing with $\dfrac{dy}{dx} + P(x)y = Q(x)$

$$P(x) = \frac{4x}{1+x^2}, \qquad Q(x) = \frac{1}{(1+x^2)^3}$$

$$IF = e^{\int P(x)dx} = e^{\int \frac{4x}{1+x^2}dx} = e^{\log(1+x^2)^2} = (1+x^2)^2$$

Therefore the general solution is

$$y(I.F) = \int Q(x)(I.F)dx + c$$

$$\therefore y(1+x^2)^2 = \int \frac{1}{(1+x^2)^3}(1+x^2)^2 dx + c$$

$$\therefore y(1+x^2)^2 = \int \frac{1}{(1+x^2)}dx + c$$

$$\therefore y(1+x^2)^2 = \tan^{-1} x + c$$

4.Solve $y' + y\tan x = \sin 2x, \qquad y(0) = 1$

Solution: $y' + y\tan x = \sin 2x$

The equation is linear equation

Therefore comparing with $\dfrac{dy}{dx} + P(x)y = Q(x)$

$$P(x) = \tan x, \qquad Q(x) = \sin 2x$$

$$IF = e^{\int P(x)dx} = e^{\int \tan x dx} = e^{\log \sec x} = \sec x$$

Therefore the general solution is

$$y(I.F) = \int Q(x)(I.F)dx + c$$

$$\therefore y(\sec x) = \int \sin 2x \sec x dx + c$$

$$\therefore y(\sec x) = \int \sin x dx + c$$

$$\therefore y(\sec x) = -\cos x + c$$

Given $y(0) = 1$

$$\therefore 1 = -1 + c$$

$$\therefore c = 2$$

$$\therefore y(\sec x) = -\cos x + 2$$

For example:

1. Solve $\frac{dy}{dx}+\frac{1}{x^2}y=6e^{\frac{1}{x}}$
2. Solve $(1+y^2)dx=(\tan^{-1}y-x)dy$
3. Solve $\frac{dy}{dx}+\frac{3y}{x}=\frac{\sin x}{x^3}$

❖ Non Linear Differential Equation or Bernoulli Equation

The equation of the form

$$\frac{dy}{dx}+Py=Qy^n \qquad (1)$$

where P and Q are functions of x or constants is known as Bernoulli's Equation.

Dividing (1) by y^n

$$y^{-n}\frac{dy}{dx}+\frac{P}{y^{n-1}}=Q \qquad (2)$$

Put $\frac{1}{y^{n-1}}=v$

$$\frac{1-n}{y^n}\frac{dy}{dx}=\frac{dv}{dx}$$

$$\frac{1}{y^n}\frac{dy}{dx}=\frac{1}{1-n}\frac{dv}{dx}$$

Substituting in (2)

$$\frac{dv}{dx}+(1-n)Pv=Q$$

which is linear form.

For Example

1.Solve $x\frac{dy}{dx}+y=x^3y^6$ Solution: $x\frac{dy}{dx}+y=x^3y^6$ Dividing both the sides by y^6 $y^{-6}x\frac{dy}{dx}+y^{-5}=x^3 \qquad (1)$ Taking $y^{-5}=v$ $\therefore -5y^{-6}\frac{dy}{dx}=\frac{dv}{dx}$ Therefore from (1)	2.Solve $x\frac{dy}{dx}+y=y^2\log x$ Solution: $x\frac{dy}{dx}+y=y^2\log x$ Dividing both the sides by xy^2 $y^{-2}\frac{dy}{dx}+\frac{1}{xy}=\frac{\log x}{x} \qquad (1)$ Taking $y^{-1}=v$ $\therefore -y^{-2}\frac{dy}{dx}=\frac{dv}{dx}$

$$\therefore \frac{dv}{dx}-\frac{5}{x}v=-5x^2$$

$$\therefore IF=e^{\int -\frac{5}{x}dx}=x^{-5}$$

Therefore the general solution is

$$v(I.F)=\int Q(x)(I.F)dx+c$$

$$\therefore vx^{-5}=\int -5x^2x^{-5}dx+c$$

$$\therefore vx^{-5}=\int -5x^{-3}dx+c$$

$$\therefore vx^{-5}=\frac{-5x^{-2}}{-2}+c$$

$$\therefore y^{-5}x^{-5}=\frac{5x^{-2}}{2}+c$$

Therefore from (1)

$$\therefore \frac{dv}{dx}-\frac{1}{x}v=-\frac{\log x}{x}$$

$$\therefore IF=e^{\int -\frac{1}{x}dx}=x^{-1}=\frac{1}{x}$$

Therefore the general solution is

$$v(I.F)=\int Q(x)(I.F)dx+c$$

$$\therefore v\frac{1}{x}=\int\left(-\frac{\log x}{x}\right)\frac{1}{x}dx+c$$

$$\therefore v\frac{1}{x}=-\int \log x\frac{1}{x^2}dx+c$$

$$\therefore y^{-1}x^{-1}=\frac{1}{x}\log x+\frac{1}{x}+c$$

2. Solve

$$x\frac{dy}{dx}=4x^3y^2+y \qquad y(0)=2$$

Solution: $x\frac{dy}{dx}=4x^3y^2+y$

$$\therefore \frac{dy}{dx}-\frac{1}{x}y=4x^2y^2$$

Dividing both the sides by y^2

$$y^{-2}\frac{dy}{dx}-\frac{1}{x}y^{-1}=4x^2 \qquad (1)$$

Taking $y^{-1}=v$

$$\therefore y^{-1}\frac{dy}{dx}=-\frac{dv}{dx}$$

Therefore from (1)

$$\therefore \frac{dv}{dx}+\frac{v}{x}=-4x^2$$

$$\therefore IF=e^{\int \frac{1}{x}dx}=x$$

Therefore the general solution is

$$v(I.F)=\int Q(x)(I.F)dx+c$$

$$\therefore vx=\int -4x^2xdx+c$$

$$\therefore vx^{-5}=\int -4x^3dx+c$$

$$\therefore vx=\frac{-4x^4}{4}+c$$

$$\therefore vx=-x^4+c$$

$$\therefore \frac{x}{y}=-x^4+c$$

$$\because y(0)=2$$

$$\therefore 0=0+c\Rightarrow c=0$$

3. Solve $\frac{dy}{dx}+\frac{1}{x}=\frac{e^y}{x^2}$

Solution: $\frac{dy}{dx}+\frac{1}{x}=\frac{e^y}{x^2}$

Dividing both the sides by e^y

$$e^{-y}\frac{dy}{dx}+\frac{e^{-y}}{x}=\frac{1}{x^2} \qquad (1)$$

Taking $e^{-y}=v$

$$\therefore e^{-y}\frac{dy}{dx}=-\frac{dv}{dx}$$

Therefore from (1)

$$\therefore \frac{dv}{dx}-\frac{v}{x}=-\frac{1}{x^2}$$

$$\therefore IF=e^{\int -\frac{1}{x}dx}=\frac{1}{x}$$

Therefore the general solution is

$$v(I.F)=\int Q(x)(I.F)dx+c$$

$$\therefore v\frac{1}{x}=\int -\frac{1}{x^2}\frac{1}{x}dx+c$$

$$\therefore v\frac{1}{x}=\int -x^{-3}dx+c$$

$$\therefore v\frac{1}{x}=\frac{x^{-2}}{2}+c$$

$$\therefore v\frac{1}{x}=x^{-2}+c$$

$$\therefore \frac{e^{-y}}{x}=x^{-2}+c$$

$\therefore$ General Solution is $\frac{x}{y} = -4x^2$	

Examples:

1. Solve $\frac{dy}{dx} + y\tan x = y^3 \sec x$
2. Solve $y^4 dx = \left(x^{-\frac{3}{4}} - y^3 x \right) dy$

<u>Differential equations of first order but not of first degree:</u>

We shall study first order differential equation of higher degree. We shall denote the derivative $\frac{dy}{dx} = p$. For a given differential equation of first order but of higher degree, three cases may arises.

Case1- First-Order Equations of Higher Degree Solvable for p

Let $F(x, y, p) = 0$ can be solved for p and can be written as

$$\left(p - q_1(x,y)\right)\left(p - q_2(x,y)\right) \ldots \ldots \ldots \ldots .. \left(p - q_n(x,y)\right) = 0$$

Equating each factor to zero we get equations of the first order and first degree.

One can find solutions of these equations by the methods discussed in the previous chapter. Let their solution be given as:

$$F_i(x, y, c_i) = 0; \;\; i = 1,2,3, \ldots .. n$$

Therefore, the general solution can be expressed in the form

$$F_1(x, y, c_1) F_2(x, y, c_2) \ldots \ldots \ldots \ldots . F_n(x, y, c_n) = 0$$

where c_i is any arbitrary constant.

Example 1 **Solve** $xy\left(\frac{dy}{dx}\right)^2 + (x^2 + y^2)\frac{dy}{dx} + xy = 0$ **(1)**

Solution: This is first-order differential equation of degree 2. Let$p = \frac{dy}{dx}$

Equation (1) can be written as

$$xyp^2 + (x^2 + y^2)p + xy = 0$$

$$xyp^2 + x^2p + y^2p + xy = 0$$

$$xp(yp + x) + y(yp + x) = 0$$

$$(yp + x)(xp + y) = 0$$

$$(yp + x) = 0 \quad or \quad (xp + y) = 0$$

$$y\frac{dy}{dx} + x = 0 \qquad or \qquad x\frac{dy}{dx} + y = 0$$

$$\frac{dy}{dx} = -\frac{x}{y} \qquad or \qquad \frac{dy}{dx} = -\frac{y}{x}$$

By solving equations, we get

$$x^2 + y^2 = c_1 \qquad or \qquad yx = c_2$$

Therefore, the general solution can be written in the form

$$(x^2 + y^2 - c_1)(yx - c_2) = 0$$

It can be seen that none of the nontrivial solutions belonging to $xy=c_1$ or $x^2+y^2=c_2$ is valid on the whole real line.

Example 2: Solve $\left(\frac{dy}{dx}\right)^2 - \frac{dy}{dx}x + y = 0$

Example 3: Solve $\left(\frac{dy}{dx}\right)^2 - 5y + 6 = 0$

Example 4: Solve $x^2\left(\frac{dy}{dx}\right)^2 + xy\frac{dy}{dx} - 6y^2 = 0$

<u>Case 2: Differential equations Solvable for y</u>

Let the differential equation given by (1) be solvable for y. Then y can be expressed as a function x and p, that is, $y = f(x, p)$

Differentiating with respect to x we get

$$\frac{dy}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial p}.\frac{dp}{dx}$$

is a first order differential equation of first degree in x and p.

Let solution be expressed in the form

$$\varphi(x, p, c) = 0$$

The solution of equation is obtained by eliminating p from equation. If elimination of p is not possible then together may be considered parametric equations of the solutions with p as a parameter.

Example 1: Solve $y^2 - 1 - p^2 = 0$

Solution: It is clear that the equation is solvable for y, that is

$$y = \sqrt{1 + p^2}$$

By differentiating with respect to x we get

$$\frac{dy}{dx}=\frac{1}{2}\frac{1}{\sqrt{1+p^2}}.2p\frac{dp}{dx}$$

or $$p=\frac{p}{\sqrt{1+p^2}}\frac{dp}{dx}$$

or $$p\left[1-\frac{1}{\sqrt{1+p^2}}\frac{dp}{dx}\right]=0$$

gives $p=0$or $$1-\frac{p}{\sqrt{1+p^2}}\frac{dp}{dx}=0$$

By solving $p=0$we gety=c

By $$1-\frac{1}{\sqrt{1+p^2}}\frac{dp}{dx}=0$$

we get a separable equation in variables p and x.

$$\frac{dp}{dx}=\sqrt{1+p^2}$$

By solving this we get

$p=sinh(x+c)$

By eliminating p we obtain

$y=cosh(x+c)$ is a general solution.

Solution $y=c$ of the given equation is a singular solution as it cannot be obtained by giving a particular value to c in solution.

Example 2: $y=2px+p^4x^4$

Example 3: $y=x\left(\frac{dy}{dx}\right)^2+\frac{dy}{dx}$

Example 4: $\left(\frac{dy}{dx}\right)^2+\frac{dy}{dx}=e^y$

Case 3: Differential Equations Solvable for x

Let equation$F(x,y,p)=0$ be solvable for x, that is$x=f(y,p)$
Then as argued in the previous section for y we get a function such that $F(y,p,c)=0$

By eliminating p from $F(x,y,p)=0$ we get a general solution.

Example 1:

Solve $x\left(\frac{dy}{dx}\right)^3 - 12\frac{dy}{dx} - 8 = 0$

Solution: Let $p = \frac{dy}{dx},\ then$

xp^3-12p-8=0

It is solvable for x, that is,

$$x = \frac{12p+8}{p^3} = \frac{12}{p^2} + \frac{8}{p^3} \qquad \dots \qquad (1)$$

Differentiating (3.18) with respect to y, we get

$$\frac{dx}{dy} = -2\frac{12}{p^3}\frac{dp}{dy} - 3\frac{8}{p^4}\frac{dp}{dy}$$

$$or\frac{1}{p} = -\frac{24}{p^3}\frac{dp}{dy} - \frac{24}{p^4}\frac{dp}{dy}$$

$$or \quad dy = \left(-\frac{24}{p^2} - \frac{24}{p^3}\right)dp$$

$$or \quad y = +\frac{24}{p} + \frac{12}{p^2} + c \qquad (2)$$

(1) and (2) constitute parametric equations of solution of the given differential equation.

Example 2: $y = 2px + y^2 p^3$

Example 3: $y^2 \log y = xyp + p^2$

Equations of the First Degree in x and y – Lagrange's and Clairaut's Equation.

Let Equation F(x, y, p) = 0 be of the first degree in x and y, then

$$y = x\phi_1(p) + \phi_2(p) \qquad \dots \qquad (1)$$

Equation (1) is known as Lagrange's equation.

If $\phi_1(p) = p$ then the equation

$$y = xp + \phi_2(p) \qquad .. \qquad (2)$$

is known as Clairaut's equation

By differentiating (1) with respect to x, we get

$$\frac{dy}{dx} = \varphi_1(p) + x\varphi_1'(p)\frac{dp}{dx} + \varphi_2'(p)\frac{dp}{dx}$$

or $p-\varphi_1(p)=(x\varphi_1'(p)+\varphi_2'(p))\dfrac{dp}{dx}$ ... (3)

From (3) we get

$(x+\varphi_2'(p))\dfrac{dp}{dx}=0$ for $\phi_1(p)=p$

This gives

$\dfrac{dp}{dx}=0$ or $x+\phi_2'(p)=0$

$\dfrac{dp}{dx}=0$ gives p = c and

by putting this value in (2) we get

$y=cx+\phi_2(c)$

This is a general solution of Clairaut's equation.

The elimination of p between

$x+\phi_2'(p)=0$ and (2) gives a singular solution.

If $\phi_1(p) \neq p$ for any p, then we observe from (3) that

$\dfrac{dp}{dx}\neq 0$ everywhere. Division by

$[p-\varphi_1(p)]\dfrac{dp}{dx}$ in (3) gives

$$\frac{dx}{dp}-\frac{\varphi_1'}{p-\varphi_1(p)}x=\frac{\varphi_2'(p)}{p-\varphi_1(p)}$$

which is a linear equation of first order in x and thus can be solved for x as a function of p, which together with (2) will form a parametric representation of the general solution of (2)

<u>Example 1</u>: Solve $\left(\dfrac{dy}{dx}-1\right)\left(y-x\dfrac{dy}{dx}\right)=\dfrac{dy}{dx}$

Solution: Let $p=\dfrac{dy}{dx}$ then,

$(p-1)(y-xp)=p$

This equation can be written as

$$y = xp + \frac{p}{p-1}$$

Differentiating both sides with respect to x we get

$$\frac{dp}{dx}\left[x - \frac{1}{(p-1)^2}\right] = 0$$

Thus either $\frac{dp}{dx} = 0$ or

$$x - \frac{1}{(p-1)^2} = 0$$

$\frac{dp}{dx} = 0$ gives p=c

Putting p=c in the equation we get

$$y = cx + \frac{c}{c-1}$$

(y-cx)(c-1)=c

which is the required solution.

Exercises

Solve the following differential equations

1. $\left(\frac{dy}{dx}\right)^3 = \frac{dy}{dx}e^{2x}$

2. $y(y-2)p^2 - (y-2x+xy)p+x=0$

3. $-\left(\frac{dy}{dx}\right)^2 + 4y - x^2 = 0$

4. $\left(\frac{dy}{dx} + y + x\right)\left(x\frac{dy}{dx} + y + x\right)\left(\frac{dy}{dx} + 2x\right) = 0$

5. $y + x\frac{dy}{dx} - x^4\left(\frac{dy}{dx}\right)^2 = 0$

6. $\left(x\frac{dy}{dx} - y\right)\left(y\frac{dy}{dx} + x\right) = h^2\frac{dy}{dx}$

7. $$y\left(\frac{dy}{dx}\right)^2+(x-y)\frac{dy}{dx}=x$$

8. $$x\left(\frac{dy}{dx}\right)^2-2y\frac{dy}{dx}+ax=0$$

9. $$\left(\frac{dy}{dx}\right)^2=y-x$$

10. $$xy\left(y-x\frac{dy}{dx}\right)=x+y\frac{dy}{dx}$$

MATHEMATICAL MODELING

POPULATION GROWTH

One of the simplest models of population growth is based on the observation that when populations (people, plants, bacteria, and fruit flies, for example) are not constrained by environmental limitations, they tend to grow at a rate that is proportional to the size of the population—the larger the population, the more rapidly it grows.

To translate this principle into a mathematical model, suppose that $y = y(t)$ denotes the population at time t. At each point in time, the rate of increase of the population with respect to time is dy/dt, so the assumption that the rate of growth is proportional to the population is described by the differential equation

$$\frac{dy}{dt}=ky \tag{1}$$

where k is a positive constant of proportionality that can usually be determined experimentally. Thus, if the population is known at some point in time, say $y = y_0$ at time $t = 0$, then a general formula for the population $y(t)$ can be obtained by solving the initial-value problem

$$\frac{dy}{dt}=ky, \quad y(0)=y_0$$

PHARMACOLOGY

When a drug (say, penicillin or aspirin) is administered to an individual, it enters the bloodstream and then is absorbed by the body over time. Medical research has shown that the amount of a drug that is present in the bloodstream tends to decrease at a rate that is proportional to the amount of the drug present—the more of the drug that is present in the bloodstream, the more rapidly it is absorbed by the body.

To translate this principle into a mathematical model, suppose that $y = y(t)$ is the amount of the drug present in the bloodstream at time t. At each point in time, the rate of change in y with respect to t is dy/dt, so the assumption that the rate of decrease is proportional to the amount y in the bloodstream translates into the differential equation

$$\frac{dy}{dt} = -ky \tag{2}$$

where k is a positive constant of proportionality that depends on the drug and can be determined experimentally. The negative sign is required because y decreases with time. Thus, if the initial dosage of the drug is known, say $y = y_0$ at time $t = 0$, then a general formula for $y(t)$ can be obtained by solving the initial-value problem

$$\frac{dy}{dt} = -ky, \quad y(0) = y_0$$

EXAMPLE 5 In an oil refinery, a storage tank contains 2000 gal of gasoline that initially has 100 lb of an additive dissolved in it. In preparation for winter weather, gasoline containing 2 lb of additive per gallon is pumped into the tank at a rate of 40 gal/min. The well-mixed solution is pumped out at a rate of 45 gal/min. How much of the additive is in the tank 20 min after the pumping process begins (Figure 16.7)?

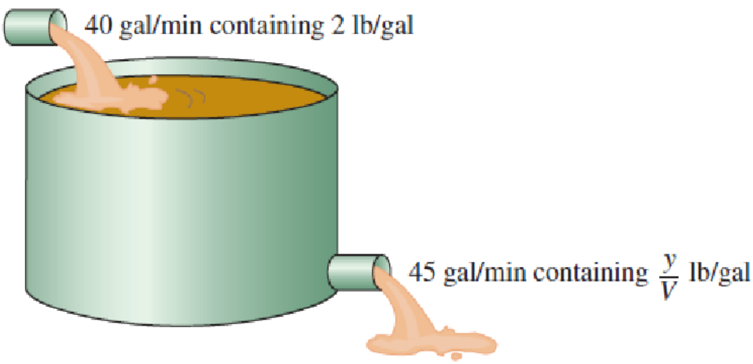

FIGURE 16.7 The storage tank in Example 5 mixes input liquid with stored liquid to produce an output liquid.

Solution Let y be the amount (in pounds) of additive in the tank at time t. We know that $y = 100$ when $t = 0$. The number of gallons of gasoline and additive in solution in the tank at any time t is

$$\begin{aligned} V(t) &= 2000 \text{ gal} + \left(40 \frac{\text{gal}}{\text{min}} - 45 \frac{\text{gal}}{\text{min}}\right)(t \text{ min}) \\ &= (2000 - 5t) \text{ gal}. \end{aligned}$$

Therefore,

$$\begin{aligned} \text{Rate out} &= \frac{y(t)}{V(t)} \cdot \text{outflow rate} && \text{Eq. (9)} \\ &= \left(\frac{y}{2000 - 5t}\right) 45 && \text{Outflow rate is 45 gal/min and } v = 2000 - 5t. \\ &= \frac{45y}{2000 - 5t} \frac{\text{lb}}{\text{min}}. \end{aligned}$$

Also,

$$\begin{aligned} \text{Rate in} &= \left(2 \frac{\text{lb}}{\text{gal}}\right)\left(40 \frac{\text{gal}}{\text{min}}\right) \\ &= 80 \frac{\text{lb}}{\text{min}}. && \text{Eq. (10)} \end{aligned}$$

The differential equation modeling the mixture process is

$$\frac{dy}{dt} = 80 - \frac{45y}{2000 - 5t}$$

in pounds per minute.

To solve this differential equation, we first write it in standard form:

$$\frac{dy}{dt} + \frac{45}{2000 - 5t}\, y = 80.$$

Thus, $P(t) = 45/(2000 - 5t)$ and $Q(t) = 80$. The integrating factor is

$$\begin{aligned} v(t) &= e^{\int P\,dt} = e^{\int \frac{45}{2000-5t}\,dt} \\ &= e^{-9\ln(2000-5t)} \qquad 2000 - 5t > 0 \\ &= (2000 - 5t)^{-9}. \end{aligned}$$

Multiplying both sides of the standard equation by $v(t)$ and integrating both sides gives

$$\begin{aligned} (2000 - 5t)^{-9} \cdot \left(\frac{dy}{dt} + \frac{45}{2000 - 5t}\, y\right) &= 80(2000 - 5t)^{-9} \\ (2000 - 5t)^{-9}\frac{dy}{dt} + 45(2000 - 5t)^{-10}\, y &= 80(2000 - 5t)^{-9} \\ \frac{d}{dt}\left[(2000 - 5t)^{-9} y\right] &= 80(2000 - 5t)^{-9} \\ (2000 - 5t)^{-9} y &= \int 80(2000 - 5t)^{-9}\,dt \\ (2000 - 5t)^{-9} y &= 80 \cdot \frac{(2000 - 5t)^{-8}}{(-8)(-5)} + C. \end{aligned}$$

The general solution is

$$y = 2(2000 - 5t) + C(2000 - 5t)^9.$$

Because $y = 100$ when $t = 0$, we can determine the value of C:

$$100 = 2(2000 - 0) + C(2000 - 0)^9$$

$$C = -\frac{3900}{(2000)^9}.$$

The particular solution of the initial value problem is

$$y = 2(2000 - 5t) - \frac{3900}{(2000)^9}(2000 - 5t)^9.$$

The amount of additive 20 min after the pumping begins is

$$y(20) = 2[2000 - 5(20)] - \frac{3900}{(2000)^9}[2000 - 5(20)]^9 \approx 1342 \text{ lb}.$$