* Unit : 03 * * "Matrices" *

* NOTES *

(1) <•> **Matrix :** A set of mn elements (real or Complex arranged in a rectangular array of m rows and n Columns is called a matrix of Order $[m \times n]$.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1m} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2m} \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$$

$$A = [a_{ij}]$$

$$i \le j \le m$$

$$i \le j \le n$$

•> $\Rightarrow$ If $a_{ij} \in R$ $\therefore$ A is called Real number Matrix.

•> $\Rightarrow$ If $a_{ij} \in C$ $\therefore$ A is called Complex Matrix.

(2) <•> **Transpose of Matrix :-**

$$A = [a_{ij}]_{m \times n}$$

$$A^T = [a_{ji}]_{n \times m}$$

$$(A^T)^T = A.$$

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Ques. (Example): $A = \begin{bmatrix} 1 & 3 & 4 \\ 0 & 5 & 7 \end{bmatrix}_{2\times3}$ $\to R_1$, $\to R_2$; columns C_1, C_2, C_3.

Solⁿ:- $A^T = \begin{bmatrix} 1 & 0 \\ 3 & 5 \\ 4 & 7 \end{bmatrix}_{3\times2}$

(3) <•> Symmetric Matrix:

$A = A^T$? No! Incomplete

Let A be a square Matrix with order n and $A = A^T$ the A is called Symmetric Matrix.

$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}_{2\times2}$ [Null Matrix]

$B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2\times2}$ [Identity Matrix]

$A = [a_{ij}]_{n\times m}$

$A^T = [a_{ji}]_{m\times n}$

$A = A^T$ $\quad 1 \le i \le n$

$1 \le j \le n$

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$\Rightarrow a_{ij} = a_{ji} \quad \forall\, i,j$ [for All].

(4) <•> Diagonal Matrix:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{1n} \\ a_{21} & a_{22} & a_{23} & a_{2n} \\ a_{31} & a_{32} & a_{33} & a_{3n} \\ a_{1m} & a_{2m} & \cdots & a_{mn} \end{bmatrix}_{n\times m}$$

(•) Diagonal Elements:

In Matrix A entries of the form a is called as diagonal Elements.

(•) Diagonal elements of Square-Symmetric Matrix is zero.

$$C = \begin{bmatrix} 0 & -4 & 5 \\ 4 & 0 & 0 \\ -5 & 0 & 0 \end{bmatrix}_{3\times3}$$

(5) <•> Trace of Matrix:

Let $A_{n\times n}$ be a matrix the trace of A is Sum of diagonal elements of Matrix $A_{2\times2}$.

$Tr(A) = a_{11} + a_{22} + a_{33} \cdots + a_{nn}$

$A = \begin{bmatrix} 1 & 5 \\ 0 & 3 \end{bmatrix}$ $\underset{(A)}{Tr} = 4$ [3+1]

$A_{n\times n}$ $B_{n\times n}$

$(A\times B)_{n\times n}$; Trace $(A+B)$

$A = [a_{ij}]_{n\times n}$

$B = [b_{ij}]_{n\times n}$

$A+B = [a_{ij} + b_{ij}]_{n\times n}$

Trace $(A+B) = a_{11} + b_{11} + a_{22} + b_{22} + \dots a_{nn} + b_{nn}$.

$= Tr(A) + Tr(B)$.

Questions

Ques: Check wheather the following matrices are Singular or Non-Singular:

1.) $A = \begin{bmatrix} 4 & 0 & 0 \\ 5 & 7 & 0 \\ -6 & 0 & -3 \end{bmatrix}$

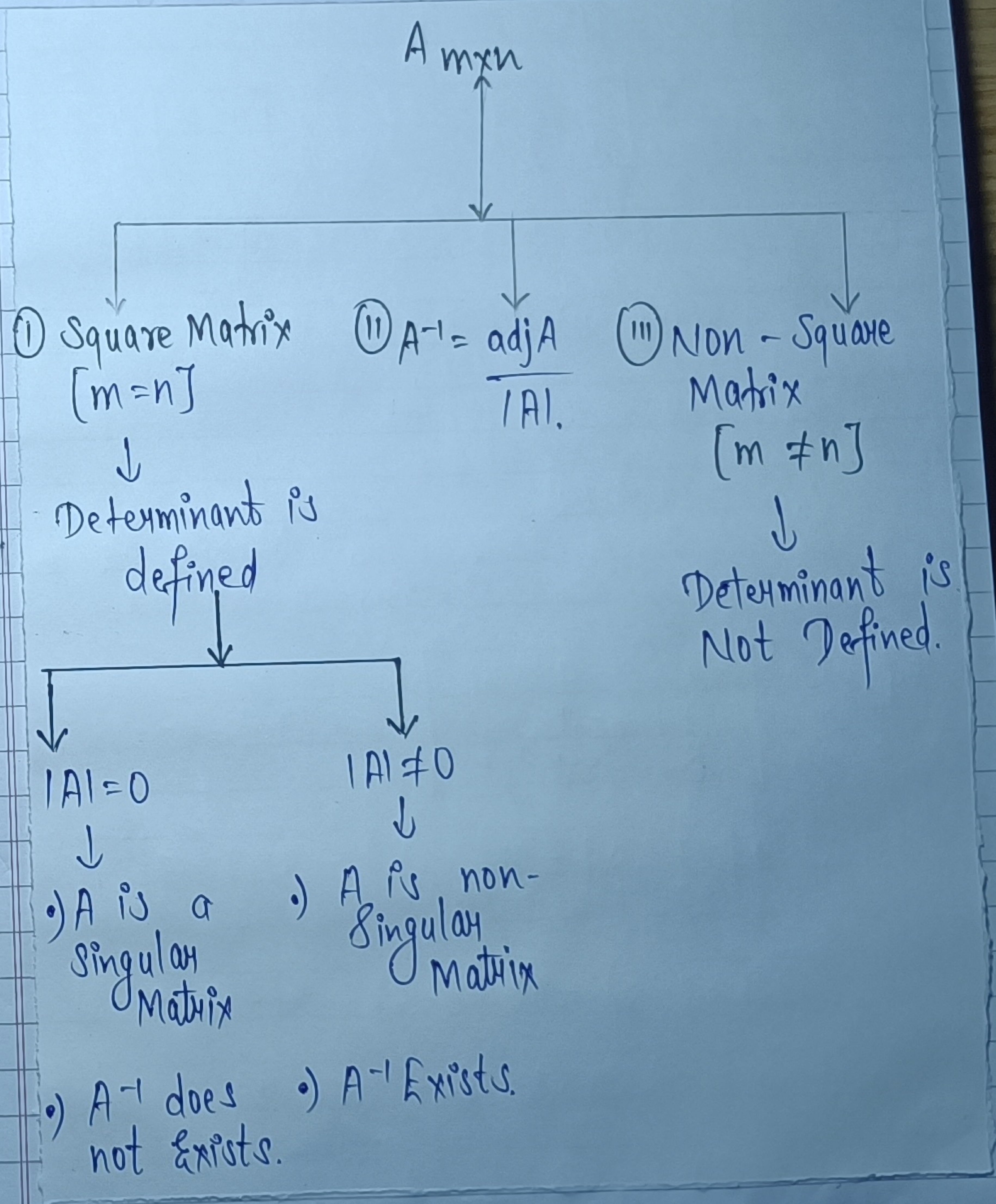
$\therefore 4 \begin{bmatrix} 7 & 0 \\ 0 & -3 \end{bmatrix} + 0 \begin{bmatrix} 5 & 0 \\ -6 & -3 \end{bmatrix} - 0 \begin{bmatrix} 5 & 7 \\ -6 & 0 \end{bmatrix}$

$\therefore 4(-21-0)$

$\therefore 4(-21)$

$\therefore -84$

$\therefore$ This Matrix is Non-Singular Matrix.



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ii.) $B = \begin{bmatrix} -4 & -3 & -7 \\ 4 & 5 & 8 \\ 0 & 7 & 9 \end{bmatrix}$

$\therefore -4\begin{vmatrix} 5 & 8 \\ 7 & 9 \end{vmatrix} - 3\begin{vmatrix} 4 & 8 \\ 0 & 9 \end{vmatrix} + 7\begin{vmatrix} 4 & 5 \\ 0 & 7 \end{vmatrix}$

$\therefore -4(45-56) - 3(36-0) + 7(28-0)$

$\therefore -4(-11) - 3(36) + 7(28)$

$\therefore 44 - 108 + 196$

$\therefore 44 + 88$

$\therefore 132$ $\quad\therefore$ This Matrix is a Non-Singular Matrix.

iii.) $C = \begin{bmatrix} -4 & -7 & 4 \\ 9 & -7 & 0 \\ 8 & -7 & 1 \end{bmatrix}$ $\quad + \; ; \; -$

$\therefore 4\begin{vmatrix} -7 & 0 \\ -7 & 1 \end{vmatrix} - 7\begin{vmatrix} 9 & 0 \\ 8 & 1 \end{vmatrix} - 4\begin{vmatrix} 9 & -7 \\ 8 & -7 \end{vmatrix}$

$\therefore 4(-7+0) - 7(9-0) - 4(-63+56)$

$\therefore 4(-7) - 7(9) - 4(-7)$

$\therefore -28 - 63 + 28$

$\therefore -63$ $\quad\therefore$ This Matrix is Non-Singular Matrix.

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iv.) $D = \begin{bmatrix} 6 & 8 & 8 \\ 8 & -6 & -7 \\ -7 & -7 & 6 \end{bmatrix}$

$\therefore 6\begin{vmatrix} -6 & -7 \\ -7 & 6 \end{vmatrix} + 8\begin{vmatrix} 8 & -7 \\ -7 & 6 \end{vmatrix} - 8\begin{vmatrix} 8 & -6 \\ -7 & -7 \end{vmatrix}$

$\therefore 6(-36-49) + 8(48-49) - 8(-56-42)$

$\therefore 6(-85) + 8(-1) - 8(-98)$

$\therefore -510 - 8 + 784$

$\therefore -518 + 784$

$\therefore 266$ $\quad\therefore$ This Matrix is Non-Singular Matrix.

(6) ⟨·⟩ Orthogonal Matrix :- A Square Matrix A is Called Orthogonal if

$$[AA^T = I = A^TA]$$

We know that $AA^{-1} = I = A^{-1}A$.

$[A^{-1} = A^T]$ only for this Matrix.

: Questions :

Ques: Verify if the following matrices are orthogonal and Hence find the Inverse.

- Minor: $a_{11}\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}$
- Cofactor: $a_{ij} = (-1)^{i+j} \times$ minor of a_{ij}
- Adjoint: Transpose of Cofactor.

(I) $A = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{bmatrix}$

$\therefore A^T = \frac{1}{3}\begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ 2 & -2 & -1 \end{bmatrix}$

$A \cdot A^T = I$

$\therefore \frac{1}{9}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ 2 & -2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Prove It $\rightarrow$

$\therefore \frac{1}{9}\begin{bmatrix} 1+4+4 & 2+2-4 & -2+4-2 \\ 2+2-4 & 4+1+4 & -4+2+2 \\ -2+4-2 & -4+2+2 & 4+4+1 \end{bmatrix}$

$\therefore \frac{1}{9}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Inverse of Matrix :-

I. Find Minors
II. Find Cofactors
III. Adjoint.
IV. $A^{-1} = \frac{1}{|A|}$ adj A.

(II) $A = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$A^T = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$I = A^T \cdot A.$

$\therefore \begin{bmatrix} \cos^2\theta + \sin^2\theta & 0 & 0 \\ 0 & \cos^2\theta + \sin^2\theta & 0 \\ 0 & 0 & \end{bmatrix}$

$\therefore \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$

Hence $AA^T = I = A^TA$ i.e A is orthogonal Matrix.

Inverse of Matrix :-

I. Find Minors
II. Find Cofactors
III. Adjoint
IV. $A^{-1} = \frac{1}{|A|}$ adj A.

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(7) <•> Elementary Transportation :~

1.] $R_i \longleftrightarrow R_j$
[Interchange of ith and jth row]

11.] kR_i
[Multiplication of ith row by non-zero k]

111.] $R_i + kR_j$
[Addition of k times the jth row to the ith row].

(8) <•> Equivalence of Matrix :- If B be an m×n matrix Obtained from m×n matrix by elementary transformation of A the A is Called Equivalent to B.

Symbolically we Can write $A \sim B$

Eg: $A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 4 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & 4 \\ b & 1 & 3 \end{bmatrix}$

$\therefore A \sim B$

$D = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 4 & 12 \\ 0 & 4 & 4 \end{bmatrix}$ $\therefore A \sim D$

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(9) <•> Row Echelon form of a Matrix :-

A matrix A is said to be in row echelon form if it satisfies the following Properties

1.] Every zero row of the Matrix A Occurs below a non-zero rows.

11.] The first non-zero number from the left of a non-zero row should be 1. This is Called a lading 1.

111.] for each non-zero row, the lading 1 appears to the right and below any lading 1 in the preceding rows.

Eg: 1.] $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 3 & 0 & 1 \end{bmatrix}$

$\therefore$ No because the 1st non-zero element of 2nd row of matrix A is not 1.

2.] $B = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

$\therefore$ No because 0 row of matrix B occurs above non-zero row of Matrix B.

3.] $C = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$

∴ No because leading 1 of 2nd row appears to the left of leading 1 of 1st row.

4.] $D = \begin{bmatrix} 0 & 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 4 & -7 \end{bmatrix}$

∴ No because leading 1 of 2nd row appears to the left of leading 1 of 1st row.

5.] $E = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

∴ Yes, it is a Echelon form. It satisfies every condition. No non-zero row matrix.

6.] $F = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$

∴ No; Condition 2 not satisfied.

7.] $G = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

∴ Yes; Condition is Satisfied.

Ques: Find a row echelon form of following Matrices:-

1.] $\begin{bmatrix} 0 & -1 & 2 & 3 \\ 2 & 3 & 4 & 5 \\ 1 & 3 & -1 & 2 \\ 3 & 2 & 4 & 1 \end{bmatrix}$

$R_1 \longleftrightarrow R_3$

$\begin{bmatrix} 1 & 3 & -1 & 2 \\ 2 & 3 & 4 & 5 \\ 0 & -1 & 2 & 3 \\ 3 & 2 & 4 & 1 \end{bmatrix}$

$-2R_1 + R_2 \longrightarrow R_2$

$-3R_1 + R_4 \longrightarrow R_4$

$\begin{bmatrix} 1 & 3 & -1 & 2 \\ 0 & -3 & 6 & 1 \\ 0 & -1 & 2 & 3 \\ 0 & -7 & 7 & -5 \end{bmatrix}$

$R_2 \longleftrightarrow R_3$

$\begin{bmatrix} 1 & 3 & -1 & 2 \\ 0 & -1 & 2 & 3 \\ 0 & -3 & 6 & 1 \\ 0 & -7 & 7 & -5 \end{bmatrix}$

$(-1) R_2$

$\begin{bmatrix} 1 & 3 & -1 & 2 \\ 0 & 1 & -2 & -3 \\ 0 & -3 & 6 & 7 \\ 0 & -7 & 7 & -5 \end{bmatrix}$

$3R_2 + R_3 \longrightarrow R_3$

$\begin{bmatrix} 1 & 3 & -1 & 2 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 0 & -8 \\ 0 & 0 & -7 & -26 \end{bmatrix}$

$R_3 \longleftrightarrow R_4$

$\begin{bmatrix} 1 & 3 & -1 & 2 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & -7 & -26 \\ 0 & 0 & 0 & -8 \end{bmatrix}$

$\frac{R_3}{(-7)} \rightarrow R_3 \; ; \; \frac{R_4}{(-8)} \longrightarrow R_4$

$\begin{bmatrix} 1 & 3 & -1 & 2 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 1 & 26/7 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Ques: Find row echelon form of matrix $B=\begin{bmatrix}1 & 2 & -3 & 1\\ -1 & 0 & 3 & 4\\ 0 & 1 & 2 & -1\\ 2 & 3 & 0 & -3\end{bmatrix}$

$$\Rightarrow B=\begin{bmatrix}1 & 2 & -3 & 1\\ -1 & 0 & 3 & 4\\ 0 & 1 & 2 & -1\\ 2 & 3 & 1 & -3\end{bmatrix}$$

$R_1+R_2 \to R_2$
$R_1+R_4 \to R_4$

$$\begin{bmatrix}1 & 2 & -3 & 1\\ 0 & 2 & 0 & 5\\ 0 & 1 & 2 & -1\\ 0 & -1 & 6 & -3\end{bmatrix}$$

$R_2 \leftrightarrow R_3$

$$\begin{bmatrix}1 & 2 & -3 & 1\\ 0 & 1 & 2 & -1\\ 0 & 2 & 0 & 5\\ 0 & -1 & 6 & -5\end{bmatrix}$$

$-2R_2+R_3 \to R_3$
$R_2+R_4 \to R_4$

$$\begin{bmatrix}1 & 2 & -3 & 1\\ 0 & 1 & 2 & -1\\ 0 & 0 & -4 & 7\\ 0 & 0 & 8 & -6\end{bmatrix}$$

$2R_3+R_4 \to R_4$

$$\begin{bmatrix}1 & 2 & -3 & 1\\ 0 & 1 & 2 & -1\\ 0 & 0 & -4 & 7\\ 0 & 0 & 0 & 8\end{bmatrix}$$

$\frac{R_3}{(-4)} \to R_3$, $\frac{R_4}{(8)} \to R_4$

$$\begin{bmatrix}1 & 2 & -3 & 1\\ 0 & 1 & 2 & -1\\ 0 & 0 & 1 & -7/4\\ 0 & 0 & 0 & 1\end{bmatrix}$$

Ques: Find row echelon form of matrix $C=\begin{bmatrix}1 & 1 & 2\\ 2 & 4 & -3\\ 3 & 6 & -5\end{bmatrix}$

$$\Rightarrow C=\begin{bmatrix}1 & 1 & 2\\ 2 & 4 & -3\\ 3 & 6 & -5\end{bmatrix}$$

$-2R_1+R_2 \to R_2$
$-3R_1+R_3 \to R_3$

$$\begin{bmatrix}1 & 1 & 2\\ 0 & 2 & -7\\ 0 & 3 & -11\end{bmatrix}$$

$\frac{R_2}{2} \to R_2$

$-1.5R_2+R_3 \to R_3$

$$\begin{bmatrix}1 & 1 & 2\\ 0 & 1 & -7/2\\ 0 & 0 & -1/2\end{bmatrix}$$

$-2R_3 \to R_3$

$$\begin{bmatrix}1 & 1 & 2\\ 0 & 1 & -7/2\\ 0 & 0 & 1\end{bmatrix}$$

(10) * Matrix A is said to be in reduced row echelon form if each Column that Contains a leading element 1 in row echelon form of the Matrix A has zero energy everywhere else in that Column.

Ques. Find a reduced row echelon form of the following matrices.

1.] $\begin{bmatrix} 3 & 4 & 5 \\ 4 & 5 & 6 \\ 5 & 6 & 7 \end{bmatrix}$

$-R_1 + R_2 \longrightarrow R_2$

$\begin{bmatrix} 3 & 4 & 5 \\ 1 & 1 & 1 \\ 5 & 6 & 7 \end{bmatrix}$

$R_1 \longleftrightarrow R_2$

$\begin{bmatrix} 3 & 4 & 5 \\ 1 & 1 & 1 \\ 5 & 6 & 7 \end{bmatrix}$

$-3R_1 + R_2 \longrightarrow R_2$
$-5R_1 + R_3 \longrightarrow R_3$

$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

$-R_2 + R_1 \longrightarrow R_1$

$\begin{bmatrix} 1 & 0 & 7 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

2.] $\begin{bmatrix} 2 & 2 & 2 \\ -2 & 5 & 2 \\ 8 & 1 & 4 \end{bmatrix}$

$\frac{R_1}{2} \longrightarrow R_1$

$\begin{bmatrix} 1 & 1 & 1 \\ -2 & 5 & 2 \\ 8 & 1 & 4 \end{bmatrix}$

$2R_1 + R_2 \longrightarrow R_2$
$-8R_1 + R_3 \longrightarrow R_3$.

$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 7 & 4 \\ 0 & -7 & -4 \end{bmatrix}$

$R_2 + R_3 \longrightarrow R_3$

$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 7 & 4 \\ 0 & 0 & 0 \end{bmatrix}$

$R_2/7 \longrightarrow R_2$

$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 4/7 \\ 0 & 0 & 0 \end{bmatrix}$

$-R_2 + R_1 \longrightarrow R_1$ $\begin{bmatrix} 1 & 0 & 3/7 \\ 0 & 1 & 4/7 \\ 0 & 0 & 0 \end{bmatrix}$

ii] (·) Rank of a matrix A.

By Definition

By row echelon form of Matrix

[*] Rank of a matrix A is Number of non-zero rows in Row echelon form of Matrix A.

Ques. Find Rank of following matrices :

i] $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 3 & 4 \\ 0 & 5 & -7 & 3 \end{bmatrix}$

$R_1 \longrightarrow R_3$

$\begin{bmatrix} 0 & 5 & -7 & 3 \\ 1 & 0 & 3 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$R_2 \longleftrightarrow R_1$

$\begin{bmatrix} 1 & 0 & 3 & 4 \\ 0 & 5 & -7 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\frac{1}{5} R_2 \longleftrightarrow R_2$

$\begin{bmatrix} 1 & 0 & 3 & 4 \\ 0 & 1 & -7/5 & 3/5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\therefore$ Rank of a Matrix is 2.

11.] $\begin{bmatrix}1&4&3&2\\0&1&0&0\\0&0&1&3\\1&0&4&5\end{bmatrix}$

$-R_1+R_4 \to R_4$

$\begin{bmatrix}1&4&3&2\\0&1&0&0\\0&0&1&3\\0&-4&1&3\end{bmatrix}$

$4R_2+R_4 \to R_4$

$\begin{bmatrix}1&4&3&2\\0&1&0&0\\0&0&1&3\\0&0&1&3\end{bmatrix}$

$-R_3+R_4 \to R_4$

$\begin{bmatrix}1&4&3&2\\0&1&0&0\\0&0&1&3\\0&0&0&0\end{bmatrix}$

∴ is R.E of matrix B.

And it has 3 non-zero row, Therefore rank of R=3.

Ques: Find RE; RRE and rank of following Matrices:-

(1) $\begin{bmatrix}2&3&4\\4&3&1\\1&2&4\end{bmatrix}$

(2) $\begin{bmatrix}1&-1&0&2\\0&1&1&-1\\2&1&2&1\\3&-2&1&6\end{bmatrix}$

(3) $\begin{bmatrix}8&4&3\\2&1&1\\1&2&1\end{bmatrix}$

(4) $\begin{bmatrix}5&-1&5\\0&2&0\\-5&3&-15\end{bmatrix}$

(5) $\begin{bmatrix}3&-3&4\\2&-3&4\\0&-1&1\end{bmatrix}$

(12) <·> System of Linear Equations:

Consider a system with m linear equations and n variables $x_1; x_2; x_3 \ldots\ldots x_n$ as follows

$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_m x_n = b_1$

$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = b_2$

$\vdots$

$a_{m1}x_1 + a_{m2}x_3 + a_{m3}x_3 + \cdots + a_{mn}x_n = b_m$

Matrix representation of System of linear equation can be given as follows:

$A_{n\times n}\, X_{n\times 1} = B_{m\times 1}$

$A = \begin{bmatrix}a_{11}&a_{12}&a_{13}&\cdots&a_{1n}\\a_{21}&a_{22}&a_{23}&\cdots&a_{2n}\\\vdots&&&&\\a_{m1}&a_{m2}&a_{m3}&\cdots&a_{mxn}\end{bmatrix}_{m\times n}$

$X = \begin{bmatrix}x_1\\x_2\\x_n\end{bmatrix}_{n\times 1}$ $B = \begin{bmatrix}b_1\\b_2\\b_m\end{bmatrix}_{m\times 1}$

(13) <·> In System of Linear Equations:

$A_{m\times n}\, X_{n\times 1} = B_{m\times 1} \leftarrow (*)$

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⇒ If $B_{m\times1} = O_{m\times1}$ then (*) is called as system of Homogeneous Linear Equation.

non-homogeneous → $a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = 0$

$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = 0$

$\vdots$

$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n = 0$

$Ax = 0.$

*) Solutions of a system of homogeneous linear equation where there are m equation with n variables.

1.] The System has exactly 1 Solution (Unique Soln)

2.] This Solution is called as trivial Solution.

3.] The System has infinite Solution.

Note: In the System of linear equation $A_{m\times n} X_{n\times1} = O_{m\times1}$ if $m=n$ and determinant of A is not Equal to 0. Then System has Unique Solution. i.e → A^{-1} Exists.

$$A^{-1}\cdot(AX) = A^{-1}\cdot 0$$

$$A^{-1}_{n\times n}(AX) = 0_{n\times1}$$

$$(A^{-1}A)\cdot X = 0_{n\times1}$$

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$$I_{n\times n} \cdot X_{n\times1} = 0_{n\times1}$$

$$X_{n\times1} = 0_{n\times1}.$$

Example :- Solve Each of the following System

i.) $x_1 + 4x_2 + x_3 = 0$
$-x_1 + 5x_2 + 2x_3 = 0$
$x_1 + 3x_2 - 4x_3 = 0.$

The matrix form of Given System is $AX = B$.

$$\begin{bmatrix} 1 & 4 & 1 \\ -1 & 5 & 2 \\ 1 & 3 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$d(A) = 1(-20-6) - 4(4-2) + 1(-3-5)$$
$$= -26 - 8 - 8$$
$$= -42$$

Here $d(A) \neq 0$ it means the Given System has Unique Solution which is $x_1 = 0, x_2 = 0, x_3 = 0$.

Solve each of the following System by Gauss Elimination method.

1) $x + y + 2z = 9$ (Gauss Elimination RE Form).
$2x + 4y - 3z = 1$
$3x + 6y - 5z = 0.$

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The matrix form of given System is $AX=B$

$$A=\begin{bmatrix}1&1&2\\2&4&-3\\3&6&-5\end{bmatrix}_{3\times3} \quad X=\begin{bmatrix}x\\y\\z\end{bmatrix}_{3\times1} \quad B=\begin{bmatrix}9\\1\\0\end{bmatrix}_{3\times1}$$

The Argumented matrix of matrix A is
$M=[A|B]$

$$\sim\left[\begin{array}{ccc|c}1&1&2&9\\2&4&-3&1\\3&6&-5&0\end{array}\right] \quad \begin{array}{l}-2R_1+R_2\to R_2\\-3R+R_3\to R_3\end{array}$$

$$\sim\left[\begin{array}{ccc|c}1&1&2&9\\0&2&-7&-17\\0&3&-11&-27\end{array}\right] \quad \begin{array}{l}-2(R_3)\to R_3\\3(R_2)\to R_2\end{array}$$

$$\sim\left[\begin{array}{ccc|c}1&1&2&9\\0&-6&21&51\\0&6&-22&-54\end{array}\right] \quad R_2+R_3\to R_3$$

$$\sim\left[\begin{array}{ccc|c}1&1&2&9\\0&-6&21&51\\0&0&-1&-3\end{array}\right] \quad \begin{array}{l}-1/6(R_2)\to R_2\\-1(R_3)\to R_3\end{array}$$

$$\sim\left[\begin{array}{ccc|c}1&1&2&9\\0&1&-7/2&-17/2\\0&0&1&3\end{array}\right]$$ is row echlon form of Argumented matrix

The Corresponding System of equation is

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$x+y+2z=9$ --- (1)
$y-\frac{7}{2}z=-17/2$ --- (II)

$[z=3]$ --- (3)

by putting equation (3) in equation 2.
$y-\frac{7}{2}\times3=-\frac{17}{2}$

$[y=2]$ ----- (II)

by putting equation (4) and equation (3) in equ (1)

$x+2+6=9$

$[x=1]$

Here we have $x=1$, $y=2$ and $z=3$ is Soln of given System unique solution.

II) $4x-2y+6z=8$
$x+y-3z=-1$
$15x-3y+9z=21$

[Gauss Jordan (means RRE form) → Reduced Row echelon form?

•> The matrix form of given System is $AX=B$.

$$A=\begin{bmatrix}4&-2&6\\1&1&-3\\15&-3&9\end{bmatrix}_{3\times3} \quad X=\begin{bmatrix}x\\y\\z\end{bmatrix}_{3\times1} \quad B=\begin{bmatrix}8\\-1\\21\end{bmatrix}_{3\times1}$$

•> The argumented matrix of matrix A is
$M=[A|B]$

$$\sim\left[\begin{array}{ccc|c}4&-2&6&8\\1&1&-3&-1\\15&-3&9&21\end{array}\right] \quad R_2\longleftrightarrow R_1$$

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$$\sim \left[\begin{array}{ccc|c} 1 & 1 & -3 & -1 \\ 4 & -2 & 6 & 8 \\ 15 & -3 & 9 & 21 \end{array}\right] \quad \begin{array}{l} -4(R_1) + R_2 \rightarrow R_2 \\ -15(R_1) + R_3 \rightarrow R_3. \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & -3 & -1 \\ 0 & -6 & 18 & 12 \\ 0 & -18 & 54 & 36 \end{array}\right] \quad -3(R_2) \rightarrow R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & -3 & -1 \\ 0 & 18 & -54 & -36 \\ 0 & -18 & 54 & +36 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & -3 & -1 \\ 0 & 18 & -54 & -36 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \tfrac{1}{18}(R_2) \rightarrow R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & -3 & -1 \\ 0 & 1 & -3 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad -1(R_2) + R_1 \rightarrow R_1$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & +1 \\ 0 & 1 & -3 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$ ∴ RRE form of Argumented matrix A.

- The leading element 1 is in column 1 and Column 2 hence the variable x and y are called leading Variable where as the Variable z is called a free Variable. (no leading element in Column 2)

i.e $= z = t$ Where $t \in R$

$y = 3t - 2$ $\quad y = 3z - 2.$

$x = 1$

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∴ Hence $x = 1$ and $y = 3t - 2$ and $z = t$ where $t \in R$ is infinite many Solution of given System.

iii.)
$$\left.\begin{array}{l} x - y + 2z - w = -1 \\ 2x + y - 2z - 2w = -2 \\ -x + 2y - 4z + w = 1 \\ 3x - 3w = -3 \end{array}\right\}$$ Solve Gauss Jordan ....

- The matrix form of Given System is $AX = B$.

$$A = \begin{bmatrix} 1 & -1 & 2 & -1 \\ 2 & 1 & -2 & -2 \\ -1 & 2 & -4 & 1 \\ 3 & 0 & 0 & -3 \end{bmatrix}_{4\times4} \quad B = \begin{bmatrix} -1 \\ -2 \\ 1 \\ 3 \end{bmatrix}_{4\times1} \quad X = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

- The Argumented matrix of matrix A is $M = [A|B]$

$$\sim \left[\begin{array}{cccc|c} 1 & -1 & 2 & -1 & -1 \\ 2 & 1 & -2 & -2 & -2 \\ -1 & 2 & -4 & 1 & 1 \\ 3 & 0 & 0 & -3 & -3 \end{array}\right] \quad \begin{array}{l} \bullet\ -2R_1 + R_2 \rightarrow R_2 \\ \bullet\ R_1 + R_3 \rightarrow R_3 \\ \bullet\ -3R_1 + R_4 \rightarrow R_4. \end{array}$$

$$\sim \left[\begin{array}{cccc|c} 1 & -1 & 2 & -1 & -1 \\ 0 & 3 & -6 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 3 & -6 & 0 & 0 \end{array}\right] \quad \begin{array}{l} \tfrac{1}{3}(R_2) \rightarrow R_2 \\ -1(R_2) + R_4 \rightarrow R_4 \\ R_2 + (-3)R_3 \rightarrow R_3 \end{array}$$

$$\sim \left[\begin{array}{cccc|c} 1 & 1 & 2 & -1 & -1 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad R_1 + R_2 \rightarrow R_1$$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & -1 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

- RRE Form of Argumented matrix A.

- The leading element 1 is in Column 1 and Column 2 hence the Variable x and y is Called leading Variable where variable z and w is Called Free Variable.

i.e $z, w = t_1, t_2$ where $t_1, t_2 \in R$.

$x - w = -1 \Rightarrow x = -1 + t_2$.

$y - 2z = 0 ; y = 2t_1$.

$\therefore$ Hence $x = -1 + t_2 ; y = 2t_1 ; z = t_1$ and $w = t_2$ Where t_1 and $t_2 \in R$ is infinite many Solution of Given System.

12.) $AX = B$.

- Non Homogeneous System
  - i) No Solution
  - ii.) Unique Solⁿ
  - iii.) Infinite Many Solⁿ.

- Homogeneous System
  - i.) Unique Solⁿ
  - ii.) Infinite many Solⁿ.

$AX = B$.

- Consistent System
  - System has Solution
- Inconsistent System
  - No Solution

Important Result :—

i.) $\rho(A) = \rho(A|B)$ then System is Consistent.

- Unique Solution: $\rho(A) = \rho(A|B) = x$
- Infinite many Solution: $\rho(A) = \rho(A|B) < n$.

$\therefore$ Here (n) is number of Column or Variable.

ii.) $\rho(A) \neq \rho(A|B)$ then System is inconsistent i.e, System has no Solution.

Example:

$$x + 2y - 3z = 0$$
$$-x + 3y + 4z = 0$$
$$x - 4y - z = 1.$$

- The matrix form of Given System is $AX = B$.

$$A = \begin{bmatrix} 1 & 2 & -3 \\ -1 & 3 & +4 \\ 1 & -4 & -1 \end{bmatrix}_{3\times3} \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}_{3\times1} \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}_{3\times1}$$

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• The Argumented matrix of matrix A is $M[A/B]$.

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 0 \\ -1 & 3 & 4 & 0 \\ 1 & -4 & -1 & 1 \end{array}\right] \quad \begin{array}{l} R_1 + R_2 \rightarrow R_2 \\ R_2 + R_3 \rightarrow R_3. \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 0 \\ 0 & 5 & 1 & 0 \\ 0 & -1 & 3 & 1 \end{array}\right] \quad R_2 \leftrightarrow R_3$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 0 \\ 0 & -1 & 3 & 1 \\ 0 & 5 & 1 & 0 \end{array}\right] \quad -1(R_2) \rightarrow R_2$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 0 \\ 0 & +1 & -3 & 1 \\ 0 & 5 & 1 & 0 \end{array}\right] \quad -5(R_2) + R_3 \rightarrow R_3$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 0 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 16 & 5 \end{array}\right] \quad \tfrac{1}{16}(R_3) \rightarrow R_3$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 0 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 1 & 5/16 \end{array}\right] \quad \text{R.E form.}$$

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15 <·> Linear Dependence and Linear Independence :-

Let $a_1, a_2, \ldots, a_n$ be the elements and $\alpha_1, \alpha_2, \ldots, \alpha_n$ be Scalars.

If the Equation
$$\alpha_1 a_1 + \alpha_2 a_2 + \cdots + \alpha_n a_n = 0.$$
$\Rightarrow \alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$;
then the elements $a_1, a_2, \ldots, a_n$ are linearly Independent.

* If the Equation; $\alpha_1 a_1 + \alpha_2 a_2 + \cdots + \alpha_n a_n = 0$.
$\Rightarrow$ at least one of $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$ is non-zero
then the elements $a_1, a_2, \ldots, a_n$ are linear dependent.

Example: 1) Take $a_1 = (1,2)$; $a_2 = (7,8)$ Let α_1, α_2 be Scalars Such that :-

$$\Rightarrow \alpha_1 a_1 + \alpha_2 a_2 = 0$$
$$\alpha_1 (1,2) + \alpha_2 (7,8) = (0,0)$$
$$(1\alpha_1 + 2\alpha_1) + (7\alpha_2, 8\alpha_2) = (0,0)$$
$$\{(\alpha_1 + 7\alpha_2), (2\alpha_1 + 7\alpha_2)\} = (0,0)$$

$$\alpha_1 + 7\alpha_2 = 0 \quad \text{------ (1)}$$
$$\alpha_2 + 4\alpha_2 = 0 \quad \text{------ (2)}$$

Taking (1) and (2)

$$3\alpha_2 = 0 \quad \text{------ (3)}$$
$\alpha_2 = 0$ put in equation (1)
$\alpha_1 = 0$.

The Elements a_1; a_2 are Linear Independent.

Example: 2.) Take $a_1 = (-1, 3)$; $a_2 = (1, -3)$

$\Rightarrow$ Let α_1, α_2 be Scalars Such that

$$\alpha_1 a_1 + \alpha_2 a_2 = 0$$
$$\alpha_1(-1, 3) + \alpha_2(1, -3) = (0, 0)$$
$$(-\alpha_1, 3\alpha_1) + (\alpha_2, -3\alpha_2) = (0, 0)$$

$$(-\alpha_1 + \alpha_2)\ (3\alpha_1 - 3\alpha_2) = (0, 0)$$

$$-\alpha_1 + \alpha_2 = 0$$
$$\alpha_1 - \alpha_2 = 0$$
$$\alpha_1 = \alpha_2$$

Take $\alpha_2 = k$, k is any number $(k \neq 0)$
$\therefore \alpha_1 = k$

$\therefore$ Hence a_1, a_2 are linearly Dependent.

Example: 3.) Take $a_1 = (1, 2, 0)$; $a_2 = (2, 3, 0)$, $a_3 = (8, 13, 0)$.

$\Rightarrow$ Let α_1 and α_2 and α_3 be Scalars Such that

$$\alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3 = 0$$
$$\alpha_1(1, 2, 0) + \alpha_2(2, 3, 0) + \alpha_3(8, 13, 0) = (0, 0, 0)$$

$$(\alpha_1 + 2\alpha_2 + 8\alpha_3), (2\alpha_1 + 3\alpha_2 + 13\alpha_3) = (0, 0, 0)$$

$$\alpha_1 + 2\alpha_2 + 8\alpha_3 = 0 \text{ ---- (1)}$$
$$2\alpha_1 + 3\alpha_2 + 13\alpha_3 = 0 \text{ ---- (11)}$$

$$\alpha_1 = -2\alpha_2 - 8\alpha_3 \text{ ---- (3)} \quad \text{but in (2)}$$
$$-4\alpha_2 - 16\alpha_3 + 3\alpha_2 + 13\alpha_3 = 0$$

$$-\alpha_2 - 3\alpha_3 = 0 \text{ ---- (4)}$$
$$\alpha_2 = -3\alpha_3 \text{ ---- (4)}$$

From (3) $[\alpha_1 = -2\alpha_3]$

Take $\alpha_3 = k$; $\alpha_2 = -2k$; $k \neq 0$

The Element a_1; a_2; a_3 are Linear Dependent.

Examine Linear Dependence and Linear Independence for the following :

i.) $(1, 0, 0)$; $(0, 1, 0)$; $(0, 0, 1)$.

ii.) $(1, 1, 0)$; $(1, 1, 1)$; $(1, 0, 1)$

iii.) $(1, 0, 1)$; $(1, 1, 0)$; $(1, 1, -1)$

iv.) $(1, 2, 1)$; $(-1, 1, 0)$; $(5, -1, 2)$

v.) $(1, 5, 2)$; $(0, 0, 1)$; $(1, 1, 0)$.

Sol^n i.) Let $a_1 = (1, 0, 0)$; $a_2 = (0, 1, 0)$ and $a_3(0, 0, 1)$ and α_1, α_2 and α_3 be Scalar Such that

$$\alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3 = 0$$
$$\alpha_1 = 0 \ ; \ \alpha_2 = 0 \ ; \ \alpha_3 = 0$$

The element are linear Independence.

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Sol^n ii.> Let $a_1 = (1,1,0)$, $a_2 = (1,1,1)$, $a_3 = (1,0,1)$
and $\alpha_1, \alpha_2, \alpha_3$ be Scalar Such that
$\alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3 = 0.$

$\alpha_1 + \alpha_2 + \alpha_3 = 0$ ----- (i)
$\alpha_1 + \alpha_2 = 0$ ----- (ii)
$\alpha_2 + \alpha_3 = 0$ ----- (iii)

$[\alpha_2 = -\alpha_3]$ $[\alpha_1 = \alpha_3]$ $[\alpha_3 = 0]$

Hence, the Element are Linear Independence.

Sol^n iii.> Let $\alpha_1, \alpha_2, \alpha_3$ are Scalar Such that:
$\alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3 = 0.$
$\alpha_1 + \alpha_2 + \alpha_3 = 0$ ----- (i)
$\alpha_2 + \alpha_3 = 0$ ----- (ii)
$\alpha_1 - \alpha_3 = 0$ ----- (iii)

$[\alpha_1 = \alpha_3]$; $[\alpha_2 = \alpha_1]$; $[\alpha_3 = 0]$

Hence, the Element are Linear Independence.

Sol^n iv.> Let $\alpha_1, \alpha_2, \alpha_3$ are Scalar Such that
$\alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3 = 0$

$\alpha_1 - \alpha_2 + 5\alpha_3 = 0$ ----- (i)
$2\alpha_1 + \alpha_2 - \alpha_3 = 0$ ----- (ii)
$\alpha_1 + \alpha_3 = 0$ ----- (iii)

$[\alpha_1 = -\alpha_3]$; $[-\alpha_1 + \alpha_2 + \alpha_1 = 0]$; $[\alpha_3 = 0]$
$[\alpha_2 = \alpha_1]$

Hence, the Element is Linear Independence.

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16.> Eigen Values and Eigen Vectors :-

(•) Let A be a $n \times n$ matrix. Then there exist a real number λ and a non-zero Vector X Such that ;

$$[AX = \lambda X]$$

=) Then λ is Called Eigen Values of the A matrix or Characteristic Value.

=) X is Called Eigen Vectors or Characteristic Vector of Eigen Value λ of the matrix A.

Notes: •> An Eigen Vector is never the zero Vector

•> The matrix $[A - \lambda I_n]$ is known as Characteristic matrix of A.

•> The determinant of $[A - \lambda I_n]$ after expansion gives the polynomial in λ it is known as the characteristic polynomial of the matrix A of order n and is of degree n.

•> $|A - \lambda I_n|$ is known as characteristic equation of A.

•> The roots of characteristic equation is known as Characteristic Value of Eigen Value of Matrix.

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- The set of all characteristic roots or eigen value of the matrix A is called the spectrum of A.

- Let A be a $n \times n$ matrix λ be an Eigen Value for a set A then the $E_\lambda = \{X \mid AX = \lambda X\}$ is called the Eigen space of λ.

Some Importance Result :-

1.) The Eigen Values of a diagonal matrix are its diagonal element.

2.) The sum of Eigen value of an $n \times n$ matrix is it trace and their product is $|A|$.

3.) For the upper triangular (lower triangular) matrix A, the Eigen values are its diagonal Element.

Example: If $A = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$, find the given value for the given Matrix.

i.) A ii.) A^T iii.) A^{-1} iv.) $4A^{-1}$ v.) A^2

vi.) $A^2 - 2A + I$ vii.) $A^3 + 2I$.

$\Rightarrow$ The Characteristic Equation is $|A - \lambda I| = 0$

$$\left| \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = 0$$

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$$\left| \begin{bmatrix} 1-\lambda & 4 \\ 3 & 2-\lambda \end{bmatrix} \right| = 0$$

$\Rightarrow (1-\lambda)(2-\lambda) - 12 = 0$
$\Rightarrow \lambda^2 - 3\lambda + 2 - 12 = 0$
$\Rightarrow \lambda^2 - 3\lambda - 10 = 0$
$\Rightarrow \lambda^2 - \lambda(5-2) - 10 = 0$
$\Rightarrow \lambda^2 - 5\lambda + 2\lambda - 10 = 0$
$\Rightarrow \lambda(\lambda - 5) + 2(\lambda - 5) = 0$
$\Rightarrow \lambda = 5 ; \lambda = -2.$

Eigen Values or char value or characteristic eqn. roots are 5 and -2.

Eigen Value of $A \Rightarrow 5 ; -2.$

Eigen Value of $A^T \Rightarrow 5 ; -2.$

Eigen Value of $A^{-1} \Rightarrow 1/5 ; -1/2$

Eigen Value of $4A^{-1} \Rightarrow 4/5 ; -2.$

Eigen Value of $A^2 \Rightarrow 25 ; 4$

Eigen Value of $A^2 - 2A + I \Rightarrow 25 - 10 + 1 \Rightarrow 16 ; 9$

Eigen Value of $A^3 + 2I \Rightarrow 125 + 2 \Rightarrow 127 ; -6.$

Example: Find the Given Eigen Value and Eigen Vector of Matrix :- $A = \begin{bmatrix} -2 & -8 & -12 \\ 1 & 4 & 4 \\ 0 & 0 & 1 \end{bmatrix}$

$\Rightarrow$ The Characteristic equation on Given matrix A=

$|A - \lambda I = 0|$

$$\begin{vmatrix} -2-\lambda & -8 & -12 \\ 1 & 4-\lambda & 4 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0$$

$(-2-\lambda)[(4-\lambda)(1-\lambda)] - 8(-8)(1-\lambda) - 12[0] = 0.$

$\Rightarrow (-2-\lambda)\cdot[\lambda^2 - 5\lambda + 4] + 8 - 8\lambda = 0.$

$\Rightarrow -2\lambda^2 + 10\lambda - 8 - \lambda^3 + 5\lambda^2 - 4\lambda + 8 - 8\lambda = 0$

$\Rightarrow -\lambda^3 + 3\lambda^2 - 2\lambda = 0$

$\Rightarrow \lambda^3 - 3\lambda^2 + 2\lambda = 0$ [Characteristic equation of A]

$\Rightarrow \lambda(\lambda^2 - 3\lambda + 2) = 0$

$\Rightarrow \lambda(\lambda-1)(\lambda-2) = 0$

$\Rightarrow \lambda = 0 ; \lambda = 1 ; \lambda = 2$

The Characteristic Equation is :-

$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$, where

$S_1 =$ trace (A) $= -2+4+1 = 3$

$S_2 =$ Sum of minors of diagonal elements of $A = 4 + (-2) + 0 = 2.$

$S_3 = |A| = 0.$

By this : $\lambda^3 - 3\lambda^2 + 2\lambda = 0.$

$\therefore$ Here one can observe that all the Eigen Value are non-repeated and matrix is non-Symmetric

For $\lambda = 0$

$$A - \lambda I = \begin{bmatrix} -2 & -8 & -12 \\ 1 & 4 & 4 \\ 0 & 0 & 1 \end{bmatrix} \underset{R_1 \to -\frac{1}{2}R_1}{\sim} \begin{bmatrix} 1 & 4 & 6 \\ 1 & 4 & 4 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\underset{R_2 \to R_2 - R_1}{\sim} \begin{bmatrix} 1 & 4 & 6 \\ 0 & 0 & -2 \\ 0 & 0 & 1 \end{bmatrix} \underset{R_2 \to -\frac{1}{2}R_2}{\sim} \begin{bmatrix} 1 & 4 & 6 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\underset{\substack{R_3 \to R_3 - R_2 \\ R_1 \to R_1 - 6R_2}}{\sim} \begin{bmatrix} 1 & 4 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$ Which is in RRE form.

Now $(A - \lambda I)x = 0$

$$\begin{bmatrix} 1 & 4 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x + 4y = 0 ; z = 0$

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -4y \\ y \\ 0 \end{bmatrix} = y \begin{bmatrix} -4 \\ 1 \\ 0 \end{bmatrix}, y \in R.$$

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→ The eigen Vectors Corresponding to eigen Value $\lambda = 0$ are non zero Vectors of the form.

$$X = \begin{bmatrix} -4k \\ k \\ 0 \end{bmatrix} = k \begin{bmatrix} -4 \\ 1 \\ 0 \end{bmatrix} ; k \in R - \{0\}.$$

∴ The Eigen Space form $\lambda = 0$ is given by

$$E_0 = \{k(-4, 1, 0) \mid k \in R\}.$$

16(•) Types of Eigen Values :-

Types 1 ⇒ Eigen Values are non-repeated, whether matrix is Symmetric or non-Symmetric

Types 2 ⇒ Eigen Values are repeated and the matrix is non-Symmetric.

Types 3 ⇒ Eigen Value are repeated and the matrix is symmetric.

Ques: Find the Eigen Values and Corresponding Eigen Vectors of the Given Matrix :-

$$A = \begin{bmatrix} 4 & 3 & -1 \\ 3 & 4 & 1 \\ 5 & 1 & -4 \end{bmatrix}$$

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The Characterisation Equation of Given Matrix

$A = |A - \lambda I|$.

$$\begin{vmatrix} 4-\lambda & 3 & -1 \\ 3 & 4-\lambda & 1 \\ 5 & 1 & -4-\lambda \end{vmatrix} = 0$$

$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

$S_1 =$ Sum of diagonal element of A

$S_2 =$ Sum of minors of diagonal element

$S_3 = |A|$.

$S_1 = 4 + 4 - 4 \Rightarrow 4.$

$S_2 = -21$

$S_3 = 4(-16-1) - 3(-12-5) - 1(3-20)$

$= -68 - 3(-17) - 1(-17)$

$= -68 + 51 + 17$

$\Rightarrow \lambda^3 - 4\lambda^2 - 21\lambda = 0$

$\lambda(\lambda^2 - 4 - 21) = 0$

$\lambda(\lambda^2 - 7\lambda + 3\lambda - 21) = 0$

$\lambda(\lambda - 7)(\lambda + 3) = 0$

$\lambda = 0 ; +7 ; -3$

For $\lambda = 0$

$$A - \lambda I = \begin{bmatrix} 4 & 3 & -1 \\ 3 & 4 & 1 \\ 5 & 1 & -4 \end{bmatrix} \quad R_1 \to \frac{1}{4}(R_1).$$

$$\begin{bmatrix} 1 & 3/4 & -1/4 \\ 3 & 4 & 1 \\ 5 & 1 & -4 \end{bmatrix} \quad R_2 \Rightarrow -3R_1 + R_2$$

$$\begin{bmatrix} 1 & 3/4 & -1/4 \\ 0 & 7/4 & 7/4 \\ 5 & 1 & -4 \end{bmatrix} \quad R_3 \to -5R_1 + R_2$$

$$\begin{bmatrix} 1 & 3/4 & -1/4 \\ 0 & 7/4 & 7/4 \\ 0 & -11/4 & -11/4 \end{bmatrix} \quad \begin{matrix} R_2 \to 7R_2 \\ R_3 \to -4/_{11} R_3 \end{matrix}$$

$$\begin{bmatrix} 1 & 3/4 & -1/4 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{matrix} R_3 \to R_3 - R_2 \\ R_1 \to R_1 - 3/_4 R_2 \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & +1 \\ 0 & 0 & 0 \end{bmatrix}$$

Now; $AX = \lambda X$

$AX - \lambda X = 0$

$(A - \lambda I)X = 0 \iff (A - \lambda I)X = 0$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Leftrightarrow x - z = 0$

$y + z = 0$

$\Leftrightarrow x = z \; ; \; y = -z$

$$\Leftrightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} z \\ -z \\ z \end{bmatrix} = z \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \text{ (Eigen Base Vector)}$$

Eigen Space for $\lambda = 0$ is

$E_0 = \{k(1, -1, 1) \mid k \in \mathbb{R}\}$

17.) Algebric Multiplicity and Geometric Multiplicity :-

$\Rightarrow$ Algebric Multiplicity :- The number of times eigen values are repeated is known as Algebric multiplicity for the respectively Eigen Value and is known as algebraic multiplicity and no. of Eigen Vectors for the corresponding Eigen Value is known as Geometric Multiplicity.

18.) Cayley - Hamilton Theorem :- The Square matrix A Satisfy its own Characteristic Equation.

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0 \quad (3 \times 3)$$

$$\lambda^2 - S_1\lambda + S_2 = 0 \quad (2 \times 2)$$

Example:

i) Cayley - Hamilton Theorem Verify for the Matrix.

$$A = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$ Hence, find the Value of A^{-1} and A^4?

Solution :- $S_1 = \text{trace}(A) = 1 + 3 + 1 = 5$

$S_2 =$ Sum of Minor of diagonal

$$= \begin{vmatrix} 3 & 0 \\ -2 & 1 \end{vmatrix} + \begin{vmatrix} 1 & -2 \\ 0 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ -1 & 3 \end{vmatrix} = 3 + 1 + 5 = 9$$

$$S_3 = \det(A) = \begin{vmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{vmatrix}$$

$$= 1(3) - 2(-1) - 2(2)$$
$$= 3 + 2 - 4$$
$$= 5 - 4$$
$$= 1$$

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$
$$\therefore \lambda^3 - 5\lambda^2 + 9\lambda - 1 = 0$$

We have to prove

$$A^3 - 5A^2 + 9A - I = 0$$

$$A^2 = A \cdot A = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1-2 & 2+6+4 & -2-2 \\ -1-3 & -2+9 & 2 \\ 2 & -6-2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 12 & -4 \\ -4 & 7 & 2 \\ 2 & -8 & 1 \end{bmatrix}$$

$$A^3 = A^2 \cdot A = \begin{bmatrix} -1 & 12 & -4 \\ -4 & 7 & 2 \\ 2 & -8 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -13 & 42 & -2 \\ -11 & 9 & 10 \\ 10 & -22 & -3 \end{bmatrix}$$

$$A^3 - 5A + 9A - I = \quad (1)$$

$$\begin{bmatrix} -13 & 42 & -2 \\ -11 & 9 & 10 \\ 10 & -22 & -3 \end{bmatrix} + \begin{bmatrix} 5 & -60 & 20 \\ 20 & -35 & -10 \\ -10 & 40 & -5 \end{bmatrix} + \begin{bmatrix} 9 & 18 & -18 \\ -9 & 27 & 0 \\ 0 & -18 & 9 \end{bmatrix}$$

$$+ \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence, it is Verified

$$A^3 - 5A^2 + 9A - I = 0.$$

Multiply A^{-1} in (1)

$$A^2 - 5A + 9I - A^{-1} = 0$$
$$A^{-1} = A^2 - 5A + 9I$$

$$= \begin{bmatrix} -1 & 12 & -4 \\ -4 & 7 & 2 \\ 2 & -8 & 1 \end{bmatrix} + \begin{bmatrix} -5 & -10 & 10 \\ 5 & -15 & 0 \\ 0 & 10 & -5 \end{bmatrix} + \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}$$

Multiply A in equation (1)

$$A^4 - 5A^3 + 9A^2 - A = 0$$
$$A^4 = 5A^3 - 9A^2 + A.$$

$$= \begin{bmatrix} -65 & 210 & -10 \\ -55 & 45 & 50 \\ 50 & -110 & -15 \end{bmatrix} + \begin{bmatrix} 9 & -108 & 36 \\ 36 & -63 & -18 \\ -18 & 72 & -9 \end{bmatrix} + \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -55 & 104 & 24 \\ -25 & -15 & 32 \\ 32 & -40 & -23 \end{bmatrix}$$

Ques:- Find Characteristic equation of $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$ and hence Prove that $A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I = \begin{bmatrix} 8 & 5 & 5 \\ 0 & 3 & 0 \\ 5 & 5 & 8 \end{bmatrix}$?

Solution:- $S_1 = \text{trace}(A) = 2+1+2 = 5$

S_2 = Sum of minors of diagonal

$$\begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} = 2+3+2 = 7.$$

$$S_3 = \begin{vmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{vmatrix} = 2(2) - 1(0) + 1(-1) = 4-1 = 3$$

$\lambda^3 - 5\lambda^2 + 7\lambda - 3$

By Cayley Hamilton Theorem we have;

$$A^3 - 5A + 7A - 3I = 0$$

$A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$

$A^5(A^3 - 5A^2 + 7A - 3I) + A(A^3 - 5A^2 + 7A - 3I)$

$+ A^2 + A + I = A^2 + A + I.$

$$A^2 = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} + \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 8 & 5 & 5 \\ 0 & 3 & 0 \\ 5 & 5 & 8 \end{bmatrix}$$

Hence Proved.

Ques:- Compute $A^9 - 6A^8 + 10A^7 - 3A^6 + A + I$, $A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 3 & 1 \\ 1 & 0 & 2 \end{bmatrix}$

Solution:- $S_1 = 1+3+2 = 6$

$$S_2 = \begin{vmatrix} 3 & 1 \\ 0 & 2 \end{vmatrix} + \begin{vmatrix} 1 & 3 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ -1 & 3 \end{vmatrix} = 6-1+5 = 5+5 = 10.$$

$$S_3 = \begin{vmatrix} 1 & 2 & 3 \\ -1 & 3 & 1 \\ 1 & 0 & 2 \end{vmatrix} = 1(6) - 2(-3) + 3(-3) = 6+6-9 = 3.$$

$\lambda^3 - 6\lambda^2 + 10\lambda - 3 = 0.$

By Cayley Hamilton Theorem; $A^3 - 6A^2 + 10A - 3I = 0.$

$A^9 - 6A^8 + 10A^7 - 3A^6 + A + I$

$A^6(A^3 - 6A^2 + 10A - 3I) + A + I.$

$A + I.$

$$\begin{bmatrix} 1 & 2 & 3 \\ -1 & 3 & 1 \\ 1 & 0 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 3 \\ -1 & 4 & 1 \\ 1 & 0 & 3 \end{bmatrix}$$

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19.2.) **Diagonalization** :- Method to convert any square matrix into diagonal matrix is known as diagonalization. We have following steps:-

1.) Find Eigen Value and eigen Vector for the corresponding Eigen Value.

2.) Check A.M = G.M (Necessary Condition)

3.) Generate Matrix P from Eigen Vector

4.) Find P^{-1}

5.) Compute $P^{-1}AP = D$, where D is a diagonal Matrix.

Example: 1.) Find the matrix P that diagonalise Matrix A where $A = \begin{bmatrix} -1 & 4 & -2 \\ -3 & 4 & 0 \\ -3 & 1 & 3 \end{bmatrix}$

Solution: $S_1 = -1+4+3 = 6$

$S_2 = 12-3-6-4+12 = 11.$

$$S_3 = \begin{vmatrix} -1 & 4 & -2 \\ -3 & 4 & 0 \\ -3 & 1 & 3 \end{vmatrix} = -1(12) - 4(-9) - 2(9) = -12+36-18 = 6.$$

$\therefore \lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$

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$$\begin{array}{c|cccc} 1 & 1 & -6 & 11 & -6 \\ & 0 & 1 & -5 & 6 \\ \hline & 1 & -5 & 6 & 0 \end{array}$$

$\lambda = 1 ; \lambda^2 - 5\lambda + 6 = 0$

$\lambda = 1 ; (\lambda-2)(\lambda-3) = 0$

$\lambda = 1 ; 2 ; 3$

$$[A-\lambda I] = \begin{bmatrix} -1-\lambda & 4 & -2 \\ -3 & 4-\lambda & 0 \\ -3 & 1 & 3-\lambda \end{bmatrix}$$

$[\lambda = 1.]$

$$\left[\begin{array}{ccc|c} -2 & 4 & -2 & 0 \\ -3 & 3 & 0 & 0 \\ -3 & 1 & 2 & 0 \end{array}\right]$$

$R_1 \to R_1/-2$

$$\sim \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ -3 & 3 & 0 & 0 \\ -3 & 1 & 2 & 0 \end{array}\right]$$

$R_2 \to R_2 + 3R_1$

$R_3 \to R_3 + 3R_1.$

$$\sim \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -3 & 5 & 0 \end{array}\right]$$

$R_3 \to R_3 + 5R_2.$

$$\sim \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$z = t ; y = t ; x = t.$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$[\lambda = 2]$

$$\left[\begin{array}{ccc|c} -3 & 4 & -2 & 0 \\ -3 & 2 & 0 & 0 \\ -3 & 1 & 1 & 0 \end{array}\right]$$

$R_1 \to R_1/-3$

$$\sim \left[\begin{array}{ccc|c} 1 & -4/3 & 2/3 & 0 \\ -3 & 2 & 0 & 0 \\ -3 & 1 & 1 & 0 \end{array}\right]$$

$R_2 \to R_2 + 3R_1$

$R_3 \to R_3 + 3R_1$

$$\sim \left[\begin{array}{ccc|c} 1 & -4/3 & 2/3 & 0 \\ 0 & -2 & 2 & 0 \\ 0 & -3 & 3 & 0 \end{array}\right]$$

$R_2 \to R_2/-2.$

$$\sim \left[\begin{array}{ccc|c} 1 & -4/3 & 2/3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -3 & 3 & 0 \end{array}\right]$$

$R_3 \to R_3 + 3R_2.$

$$\sim \left[\begin{array}{ccc|c} 1 & -4/3 & 2/3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$z = t;\ y = t;\ x = 2/3\, t$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = t \begin{bmatrix} 2/3 \\ 1 \\ 1 \end{bmatrix}$$

$[\lambda = 3]$

$$\left[\begin{array}{ccc|c} -4 & 4 & -2 & 0 \\ -3 & 1 & 0 & 0 \\ -3 & 1 & 0 & 0 \end{array}\right]$$

$R_1 \to R_1 - 4$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & 1/2 & 0 \\ -3 & 1 & 0 & 0 \\ -3 & 1 & 0 & 0 \end{array}\right]$$

$R_2 \to R_2 + 3R_1$
$R_3 \to R_3 + 3R_1$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & 1/2 & 0 \\ 0 & -2 & 3/2 & 0 \\ 0 & -2 & 3/2 & 0 \end{array}\right]$$

$R_2 \to -1/2\, R_2$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & 1/2 & 0 \\ 0 & 1 & -3/4 & 0 \\ 0 & -2 & 3/2 & 0 \end{array}\right]$$

$R_3 \to R_3 + 2R_2$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & 1/2 & 0 \\ 0 & 1 & -3/4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$z = t;\ y = 3/4\, t;\ x = -1/4\, t$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = t \begin{bmatrix} 1/4 \\ 3/4 \\ 1 \end{bmatrix}$$

λ	A.M	G.M
1	1	1
2	1	1
3	1	1

$$P = \left[\begin{array}{ccc|ccc} 1 & 2/3 & 1/4 & 1 & 0 & 0 \\ 1 & 1 & 3/4 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$R_2 \to R_2 - R_1$
$R_3 \to R_3 - R_1$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 2/3 & 1/4 & 1 & 0 & 0 \\ 0 & 1/3 & 1/2 & -1 & 1 & 0 \\ 0 & 1/3 & 3/4 & -1 & 0 & 1 \end{array}\right]$$

$R_2 \to R_2(3)$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 2/3 & 1/4 & 1 & 0 & 0 \\ 0 & 1 & 3/2 & -3 & 3 & 0 \\ 0 & 1/3 & 3/4 & -1 & 0 & 1 \end{array}\right]$$

$R_1 \to R_1 - 2/3\, R_2$
$R_3 \to R_3 - 1/3\, R_2$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & -3/4 & 3 & -2 & 0 \\ 0 & 1 & 3/2 & -3 & 3 & 0 \\ 0 & 0 & 1/4 & 0 & -1 & 1 \end{array}\right] \quad R_3 \to R_3(4)$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -3/4 & 3 & -2 & 0 \\ 0 & 1 & 3/2 & -3 & 3 & 0 \\ 0 & 0 & 1 & 0 & -4 & 4 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 - 3/2\, R_3 \\ R_1 \to R_1 + 3/4\, R_3 \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -5 & 3 \\ 0 & 1 & 0 & -3 & 9 & -6 \\ 0 & 0 & 1 & 0 & -4 & 4 \end{array}\right]$$

$$P^{-1} = \begin{bmatrix} 3 & -5 & 3 \\ -3 & 9 & -6 \\ 0 & -4 & 4 \end{bmatrix}$$

$$P^{-1}AP = \begin{bmatrix} 3 & -5 & 3 \\ -3 & 9 & -6 \\ 0 & -4 & 4 \end{bmatrix} \begin{bmatrix} -1 & 4 & -2 \\ -3 & 4 & 0 \\ -3 & 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2/3 & 1/4 \\ 1 & 1 & 3/4 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} = D$$

Ques:- Find the Matrix P that diagonalize A where $A = \begin{bmatrix} 1 & 0 \\ 6 & -1 \end{bmatrix}$ also find the Value of A^{10}?

Solution: $S_1 = 1-1 = 0$

$S_2 = \begin{vmatrix} 1 & 0 \\ 6 & -1 \end{vmatrix} = -1$

$\lambda^2 - S_1\lambda + S_2 = 0$

$\lambda^2 - 1 = 0$

$\lambda = 1 ; -1.$

$[A - \lambda I] = \begin{bmatrix} 1-\lambda & 0 \\ 6 & -1-\lambda \end{bmatrix}$

$[\lambda = 1]$

$\left[\begin{array}{cc|c} 0 & 0 & 0 \\ 6 & -2 & 0 \end{array}\right] R_1 \leftrightarrow R_2$

$\sim \left[\begin{array}{cc|c} 6 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right] R_1 \to R_1/6$

$\sim \left[\begin{array}{cc|c} 1 & -1/3 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$y = t \;;\; x = t/3$

$\begin{bmatrix} x \\ y \end{bmatrix} = t\begin{bmatrix} 1/3 \\ 1 \end{bmatrix}$

$[\lambda = -1]$

$\left[\begin{array}{cc|c} 2 & 0 & 0 \\ 6 & 0 & 0 \end{array}\right] R_1 \to R_1/2$

$\sim \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 6 & 0 & 0 \end{array}\right]$

$R_2 \to R_2 - 6R_1$

$\sim \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$y = t \;;\; x = 0.$

$\begin{bmatrix} x \\ y \end{bmatrix} = t\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$P = \begin{bmatrix} 1/3 & 0 \\ 1 & 1 \end{bmatrix}$

$\left[\begin{array}{cc|cc} 1/3 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right]$

$R_1 \to 3R_1$

$\sim \left[\begin{array}{cc|cc} 1 & 0 & 3 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right]$

$R_2 \to R_2 - R_1$

$\sim \left[\begin{array}{cc|cc} 1 & 0 & 3 & 0 \\ 0 & 1 & -3 & 1 \end{array}\right]$

$P^{-1} = \begin{bmatrix} 3 & 0 \\ -3 & 1 \end{bmatrix}$

$P^{-1}AP = \begin{bmatrix} 3 & 0 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 6 & -1 \end{bmatrix}\begin{bmatrix} 1/3 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} 1/3 & 0 \\ 1 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = D$

$A^k = PD^kP^{-1}$

$A^{10} = PD^{10}P^{-1} = \begin{bmatrix} 3 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ -3 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1/3 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = A$

Q.2.) Quadratic Form :- A Homogenous Polynomial of second degree in real variables is called Quadratic forms.

$ax^2 + 2hxy + by^2$ is a Quadratic in the variable x and y.

$2x_1x_2 + 2x_2x_3 + 2x_3x_1 + x_3^2$ is Quadratic form in the variable x_1, x_2 and x_3.

(•) The Matrix representation of Quadratic form $Q(x) = x^TAx$ where x is a column matrix and A is a coefficient matrix.

A is a symmetric matrix and it is called Matrix of the Quadratic form.

$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad [x_1 \; x_2 \; x_3]$

(•) $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

Example: $x^2 - 4xy + y^2$; $Q(x) = [x \;\; y] \begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

$x^2 + 2y^2 + z^2 - 6xy + 4xy^2 - 8zx$; $Q(x) = [x \;\; y \;\; z]$

$\begin{bmatrix} 1 & -3 & 4 \\ -3 & 2 & 2 \\ -4 & 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

(•) $x_1^2 + x_2^2 + x_3^2$; $Q(x) = [x_1 \;\; x_2 \;\; x_3] \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$4x_3^2 + 2x_1x_3 + 2x_2x_1$; $Q(x) = [x_1 \;\; x_2 \;\; x_3] \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 4 \end{bmatrix}$

Q1. <•> **Nature of the Quadratic form :-**

i) If all the Eigen Values of Quadratic form is positive then it is positive definite.

ii) If the Eigen Values of Quadratic form are zero and positive then it is Semipositive definite.

iii) If all the Eigen Values of Quadratic form are negative then it is negative definite.

iv) If all the Eigen Values of Quadratic form are zero then it is Semi negative Definite.

v) If all the Eigen Values of Quadratic form are Negative or Positive both at Same time than it is Indefinite.

Ques: $Q = 3x_1^2 + x_2^2 + x_3^2 + 4x_1x_2 + 4x_2x_3$.

Solution: $Q(x) = [x_1 \;\; x_2 \;\; x_3] \begin{bmatrix} 3 & 2 & 0 \\ 2 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$A = \begin{bmatrix} 3 & 2 & 0 \\ 2 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix}$

$S_1 = 3+1+1 = 5$

$S_2 = -3+3-1 = -1$

$S_3 = \begin{vmatrix} 3 & 2 & 0 \\ 2 & 1 & 2 \\ 0 & 2 & 1 \end{vmatrix} = 3(-3) - 2(2) = -9 - 4 = -13$

$\lambda^3 - 5\lambda^2 - \lambda - 13 = 0$.

So here real Eigen Values are not possible.

Ques: $2x_1x_2 + 2x_2x_3 + 2x_3x_1$.

Solution: $[x_1 \;\; x_2 \;\; x_3] \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ $S_1 = 0$, $S_2 = -1-1-1 = -3$

$S_3 = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} = -1(-1)+1(1) = 2.$

$\lambda^3 - 3\lambda - 2 = 0.$

$$\begin{array}{c|cccc} 2 & 1 & 0 & 3 & -2 \\ & 0 & 2 & 2 & 2 \\ \hline & 1 & 2 & 1 & 0 \end{array}$$

$\lambda = 2$; $\lambda^2 + 2\lambda + 1 = 0$

$(\lambda+1)^2 = 0$

$\lambda = -1, -1$

$\lambda = 2, -1, -1$ ∴ Indefinite Nature.

Ques: $x_1^2 x_2^2 x_3^2 + 2x_1$

Solution: $Q(x) = [x_1 \quad x_2 \quad x_3] \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ $S_1 = 3$; $S_2 = 0$

$S_3 = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 0.$

$\lambda^3 - 3\lambda^2 = 0$

$\lambda^2(\lambda - 3) = 0$

$\lambda = 0, 0, 3$ ∴ Semi Positive Definate.

22.(•) Canonical Form :- To convert the Quadratic form into the diagonal form. This process is known as Canonical Form. In this we have $Q(x) = y^T D y$ where $D = P^T A P$ and P is the Orthogonal Matrix.

Example: Ques ⇒ $Q = 3x^2 + 3z^2 + 4xy + 8xz + 8yz.$

$A = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 4 \\ 4 & 4 & 3 \end{bmatrix}$ $S_1 = 6$

$S_2 = -16 - 7 - 4 = -27.$

$S_3 = \begin{vmatrix} 3 & 2 & 4 \\ 2 & 0 & 4 \\ 4 & 4 & 3 \end{vmatrix} = 3(-16) - 2(-10) + 4(8) = -48 + 20 + 32 = 4$

$\lambda^3 - 6\lambda^2 - 27\lambda + 4 = 0$

$$\begin{array}{c|cccc} -2 & 1 & -6 & -27 & 4 \\ & 0 & 2 & -4 & -8 \\ \hline & 1 & -4 & 8 & -16 \end{array}$$

Ques: Reduce the Quadratic form $x_1^2 + 2x_2^2 + x_3 - 2x_1x_2 + 2x_2x_3$ into the Canonical form ?

Solution: $A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

$S_1 = 1+2+1 = 4$

$S_2 = \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + \begin{vmatrix} 1 & -1 \\ -1 & 2 \end{vmatrix}$

$= 1+1+1$

$= 3$

$S_3 = \begin{vmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 0 & 1 & 1 \end{vmatrix} = 1(1)+1(-1) = 0.$

$\lambda^3(\lambda^2 - 4\lambda + 3) = 0$

$\lambda(\lambda-3)(\lambda-1) = 0$

$\lambda = 0; 1; 3$

$[A - \lambda I] = \begin{vmatrix} 1-\lambda & -1 & 0 \\ -1 & 2-\lambda & 1 \\ 0 & 1 & 1-\lambda \end{vmatrix}$

$\lambda = 0$

$\left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ -1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right]$

$R_2 \to R_2 + R_1.$

$\sim \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right]$

$R_3 \to R_3 - R_2.$

$\sim \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$z = t;\ y = -t;\ x = -t.$

$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$

$[\lambda = 1.]$

$\left[\begin{array}{ccc|c} 0 & -1 & 0 & 0 \\ -1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right]$

$R_2 \longleftrightarrow R_1$

$\sim \left[\begin{array}{ccc|c} -1 & 1 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right]$

$R_2 \to -R_2.$

$\sim \left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right]$

$R_3 \to R_3 - R_2.$

$\sim \left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$z = t;\ y = 0;\ x = t.$

$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

$(\lambda = 3)$

$\left[\begin{array}{ccc|c} -2 & -1 & 0 & 0 \\ -1 & -1 & 1 & 0 \\ 0 & 1 & -2 & 0 \end{array}\right]$

$R_2 \longleftrightarrow R_1$

$\sim \left[\begin{array}{ccc|c} -1 & -1 & 1 & 0 \\ -2 & -1 & 0 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right]$

$R_1 \to R_1$

$\sim \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ -2 & -1 & 0 & 0 \\ 0 & -1 & 2 & 0 \end{array}\right]$

$R_2 \to R_2 + 2R_1.$

$\sim \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 1 & -2 & 0 \end{array}\right]$

$R_3 \to R_3 - R_2.$

$\sim \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$z = t;\ y = 2t;\ x = t.$

$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = t \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$

$P = \begin{bmatrix} -1 & 1 & -1 \\ -1 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix}$

$-1+0+1 = 0$

$-1+0+1 = 0$

$1-2+1 = 0$

Normalized matrix is :-

$P = \begin{bmatrix} -1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \\ -1/\sqrt{3} & 0 & 2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \end{bmatrix}$

$P^T = \begin{bmatrix} -1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ -1/\sqrt{6} & 2/\sqrt{6} & 1/\sqrt{6} \end{bmatrix}$

$P^TAP = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} = D.$

$Q = y^T D Y.$

$= \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

$= y_2^2 + 3y_3^2$ //.

Ques: $x_1^2 + x_2^2 + x_3^2 - 2x_1x_2$

Solution:~ $A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$S_1 = 3$

$S_2 = 1+1+0 = 2$

$S_3 = \begin{vmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1(1)+1(-1) = 0$

$\lambda^3 - 3\lambda^2 + 2\lambda = 0$

$\lambda(\lambda^2 - 3\lambda + 2) = 0$

$\lambda(\lambda-2)(\lambda-1) = 0$

$\lambda = 0; 1; 2.$

$[A - \lambda I] = \begin{bmatrix} 1-\lambda & -1 & 0 \\ -1 & 1-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{bmatrix}$

$(\lambda = 0)$

$\left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$

$R_2 \rightarrow R_2 + R_1$

$\sim \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$

$R_2 \leftrightarrow R_3$

$\sim \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$z = 0;\ y = t;\ x = t.$

$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

$[\lambda = 1]$

$\left[\begin{array}{ccc|c} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$R_1 \leftrightarrow R_2.$

$\sim \left[\begin{array}{ccc|c} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$R_2 \rightarrow R_2.$

$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ $x = 0$, $y = 0$, $z = t$

$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = t \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$[\lambda = 2]$

$\left[\begin{array}{ccc|c} -1 & -1 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right]$

$R_1 \rightarrow -R_1$

$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right]$

$R_2 \rightarrow R_2 + R_1.$

$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right]$

$R_3 \rightarrow -R_2.$

$\sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}$ $R_2 \rightarrow -R_2.$

$y = t;\ z = 0;\ x = -t.$

$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

$P = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

Normalized Matrix is

$P = \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \end{bmatrix}$

$P^T = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \end{bmatrix}$

$P^TAP = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} = D.$

$Q = y^T D Y = [y_1\ y_2\ y_3] \begin{bmatrix} 0 & 0 & 6 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

$= y_2^2 + 2y_3^2.$

⇒ Example of Row Echelon Form :-

$$\begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 7 \end{bmatrix}$$ It is not R.E form

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} ; \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} ; \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ It is zero Matrix.

Ques: $2x_1 - x_2 + x_3 = 3$

$3x_1 - x_2 + 2x_3 = 6$

$-5x_1 + 8x_2 - 4x_3 = 2$

Solution: $$\left[\begin{array}{ccc|c} 2 & -1 & 1 & 3 \\ 3 & -1 & 2 & 6 \\ -5 & 8 & -4 & 2 \end{array}\right]$$ $R_1 \to R_1/2$

$$\sim \left[\begin{array}{ccc|c} 1 & -1/2 & 1/2 & 3/2 \\ 3 & -1 & 2 & 6 \\ -5 & 8 & -4 & 2 \end{array}\right]$$ $R_2 \to R_2 - 3R_1$
$R_3 \to R_3 + 5R_1$.

$$\sim \left[\begin{array}{ccc|c} 1 & -1/2 & 1/2 & 3/2 \\ 0 & 1/2 & -1/2 & 3/2 \\ 0 & 11/2 & -3/2 & 19/2 \end{array}\right]$$ $R_2 \to 2R_2$.

$$\sim \left[\begin{array}{ccc|c} 1 & -1/2 & 1/2 & 3/2 \\ 0 & 1 & -1 & 3 \\ 0 & 11/2 & -3/2 & 19/2 \end{array}\right]$$

$R_3 \to R_3 - \frac{11}{2} R_2$.

$$\sim \left[\begin{array}{ccc|c} 1 & -1/2 & 1/2 & 3/2 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 4 & -7 \end{array}\right]$$

$R_3 \to R_3/4$

$-3/2 + 11/2$

$19/2 - 33/2$.

$$\sim \left[\begin{array}{ccc|c} 1 & -1/2 & 1/2 & 3/2 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 7/4 \end{array}\right]$$

$z = -7/4$; $y = 5/4$; $x = 3$ //.