Parul University

Faculty of Engineering & Technology

Department of Applied Sciences and Humanities

1st Year B. Tech Programme (All Branches)

Mathematics – 1 (303191101) 2023-24

Unit – 6 Multivariable Calculus (Lecture Notes)

Overview:

In this unit, we will study about limit and Continuity, partial differentiation of first order and higher order, chain rule, implicit function, Jacobian. We will study further about Application for Partial Derivatives that includes tangent plane and normal line, extreme values of a function, use of Lagrange's Multiplier and Taylor's and Maclaurin's series for two variables.

Objective:

At the end of this unit, you will be able to understand

- Functions of Several Variables
- Limit and Continuity
- Use of Partial Differentiation
- Application of Chain Rule
- Properties of Jacobian
- How to find out Tangent Plane and Normal Line
- Method to find extreme values of a function
- Method of Lagrange's multipliers
- Expansion using Taylor Series

Functions of Two Variables

Until now, we have only considered functions of a single variable $y = f(x)$.

For example, the area function of a square is its side square which include only one variable, that is only length.

However, many real-world functions consist of two (or more) variables.

Example:

- The area function of a rectangular shape depends on both its width and its height.
- The pressure of a given quantity of gas varies with respect to the temperature of the gas and its volume.
- Volume of cylinder is $v = \pi r^2 h$, where r and h are two independent variables and v is dependent variable.

  Therefore, we require partial derivative for the function, which depends more than one independent variable.

Limit of a Function of Two Variables

Definition: For every number $\epsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all (x, y) in the domain of f,

$$|f(x, y) - L| < \epsilon \ wherever \ 0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta$$

then, the function $f(x, y)$ is said to approach the **limit** L as (x, y) approaches (x_0, y_0), and we write,

$$\lim_{(x,y) \to ((x_0, y_0))} f(x, y) = L$$

To evaluate limit following methods are applicable: -

(i) By direct substitution: - If example is regarding limit $(x, y) \to (x_0, y_0)$ and on direct substitution we obtain finite value.

(ii) By definition of limit: - If in example it is mentioned.

(iii) By path if example is regarding $(x, y) \to (0,0)$ and on direct substitution we obtain an

indeterminate form (mostly $\left(\frac{0}{0}\right)$)

Explanation:

<u>Path 1</u>: Evaluate $\lim_{x \to 0} \left\{ \lim_{y \to 0} f(x, y) \right\}$

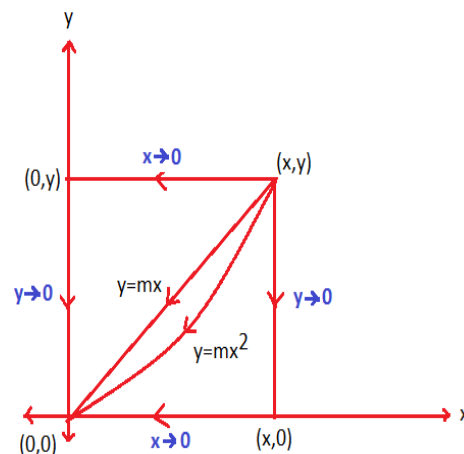<u>Path 2</u>: Evaluate $\lim_{y \to 0} \left\{ \lim_{x \to 0} f(x, y) \right\}$

If limit along path 1 and path 2 are same then

proceed further otherwise limit doesn't exist.

<u>Path 3</u>: Evaluate the limit along $y = mx$ as $x \to 0$.

If limit along path 1, path 2 and path 3 are same then proceed further otherwise limit doesn't exist.

<u>Path 4</u>: Evaluate the limit along $y = mx^2$ as $x \to 0$.

If limit along all the paths is same then limit exist.

Solved Examples:

1. Evaluate $\lim\limits_{(x,y)\to(1,2)} \dfrac{3x^2y}{x^2+y^2+5}$ ------------> **Simple example by direct substitution**

Sol:

$$\lim_{(x,y)\to(1,2)} \frac{3x^2y}{x^2+y^2+5}$$

$$= \frac{3(1)^2(2)}{(1)^2+2^2+5}$$

$$= \frac{6}{10}$$

$$= \frac{3}{5}$$

$$\therefore \lim_{(x,y)\to(1,2)} \frac{3x^2y}{x^2+y^2+5} = \frac{3}{5}$$

2. Applying the definition of limit, show that $\lim\limits_{(x,y)\to(0,0)} \dfrac{4xy^2}{x^2+y^2} = 0.$

Sol.

Let $\varepsilon > 0$ be given.

We want to find a $\delta > 0$ such that

$\left|\dfrac{4xy^2}{x^2+y^2} - 0\right| < \varepsilon$ whenever $0 < \sqrt{(x-0)^2+(y-0)^2} < \delta$

Since, $y^2 \le x^2 + y^2$

$$\frac{y^2}{x^2+y^2} \le 1$$

$$\frac{4|x|y^2}{x^2+y^2} \le 4|x| = 4\sqrt{x^2} \le 4\sqrt{x^2+y^2}$$

Now, $\left|\dfrac{4xy^2}{x^2+y^2} - 0\right| = \dfrac{4|x|y^2}{x^2+y^2} \le 4\sqrt{x^2+y^2} < 4\delta$

By taking, $4\delta = \varepsilon$

$\left|\dfrac{4xy^2}{x^2+y^2} - 0\right| < \varepsilon$ whenever $0 < \sqrt{(x-0)^2+(y-0)^2} < \delta$

$\therefore \lim\limits_{(x,y)\to(0,0)} \dfrac{4xy^2}{x^2+y^2} = 0$ by definition.

3. Evaluate $\lim\limits_{(x,y)\to(0,0)} \dfrac{x-y}{x+y}$ **by Path method**

Sol: By putting $(x,y) \to (0,0)$ we obtain an indeterminate form $\left(\left(\dfrac{0}{0}\right)\right)$

Therefore, we will apply the Path method to evaluate the limit of a function

Path 1:

$$\lim_{x \to 0} \left\{ \lim_{y \to 0} f(x, y) \right\} = \lim_{x \to 0} \left\{ \lim_{y \to 0} \left(\frac{x - y}{x + y} \right) \right\} = \lim_{x \to 0} \left(\frac{x}{x} \right) = 1$$

Path 2:

$$\lim_{y \to 0} \left\{ \lim_{x \to 0} f(x, y) \right\} = \lim_{y \to 0} \left\{ \lim_{x \to 0} \left(\frac{x - y}{x + y} \right) \right\} = \lim_{y \to 0} \left(-\frac{y}{y} \right) = (-1) = -1$$

Since both the limits are different, the limit does not exist.

4. Evaluate $\lim_{(x,y) \to (0,0)} \frac{x^3 + y^3}{x^2 + y^2}$

Sol: By putting $(x, y) \to (0,0)$ we obtain an indeterminate form $\left(\left(\frac{0}{0} \right) \right)$

Therefore, we will apply the Path method to evaluate the limit of a function

Path 1:

$$\lim_{x \to 0} \left\{ \lim_{y \to 0} f(x, y) \right\} = \lim_{x \to 0} \left\{ \lim_{y \to 0} \left(\frac{x^3 + y^3}{x^2 + y^2} \right) \right\} = \lim_{x \to 0} \left(\frac{x^3}{x^2} \right) = \lim_{x \to 0} x = 0$$

Path 2:

$$\lim_{y \to 0} \left\{ \lim_{x \to 0} f(x, y) \right\} = \lim_{y \to 0} \left\{ \lim_{x \to 0} \left(\frac{x^3 + y^3}{x^2 + y^2} \right) \right\} = \lim_{y \to 0} \left(\frac{y^3}{y^2} \right) = \lim_{y \to 0} y = 0$$

Path 3:

Put $y = mx$, as $x \to 0$

$$\lim_{x \to 0} \left(\frac{x^3 + (mx)^3}{x^2 + (mx)^2} \right) = \lim_{x \to 0} \left(\frac{x^3 + m^3 x^3}{x^2 + m^2 x^2} \right) = \lim_{x \to 0} x \left(\frac{1 + m^3}{1 + m^2} \right) = 0$$

Path 4:

Put $y = mx^2$, as $x \to 0$

$$\lim_{x \to 0} \left(\frac{x^3 + (mx^2)^3}{x^2 + (mx^2)^2} \right) = \lim_{x \to 0} \left(\frac{x^3 + m^3 x^6}{x^2 + m^2 x^4} \right) = \lim_{x \to 0} x \left(\frac{1 + m^3 x^3}{1 + m^2 x^2} \right) = 0$$

Since, limit along all the paths are same.

Hence, the limit exists and its value is 0.

5. Evaluate $\lim_{(x,y) \to (0,0)} \frac{2xy}{x^2 + y^2}$ **by Path method.**

Sol: By putting $(x, y) \to (0,0)$ we obtain an indeterminate form $\left(\left(\frac{0}{0} \right) \right)$

Therefore, we will apply the Path method to evaluate the limit of a function

Path 1:

$$\lim_{(x,y) \to (0,0)} \left(\frac{2xy}{x^2 + y^2} \right) = 0$$

Path 2:

$$\lim_{(x,y) \to (0,0)} \left(\frac{2xy}{x^2 + y^2} \right) = 0$$

Path 3:

Put $y = mx$, as $x \to 0$

$$\lim_{x \to 0} \left(\frac{2x(mx)}{x^2 + (mx)^2} \right) = \lim_{x \to 0} \left(\frac{2mx^2}{x^2 + m^2 x^2} \right) = \frac{2m}{1 + m^2}$$

As the limit depends on m and m is not fixed, the limit doesn't exist.

Exercise:

1. Evaluate $\lim\limits_{(x,y) \to (0,0)} \frac{x^2 + y}{3x + y^2}$

2. Applying the definition of limit, show that $\lim\limits_{(x,y) \to (0,0)} \frac{3x^2 y}{x^2 + y^2} = 0$

3. Evaluate $\lim\limits_{(x,y) \to (0,0)} \frac{x^2 y}{x^4 + y^2}$

4. Evaluate $\lim\limits_{(x,y) \to (0,0)} \frac{xy}{\sqrt{x^2 + y^2}}$

Continuity of function of two Variables

A function $f(x, y)$ is **continuous at the point** (x_0, y_0) if it satisfies following properties,

1. $f(x, y)$ is defined at (x_0, y_0)

2. $\lim\limits_{(x,y) \to (x_0, y_0)} f(x, y)$ exists

3. $\lim\limits_{(x,y) \to (x_0, y_0)} f(x, y) = f(x_0, y_0)$

Solved Examples:

1. Discuss the continuity of

$$f(x, y) = \frac{x^2 y}{x^3 + y^3}, \qquad (x, y) \neq (0,0)$$

$$0 \qquad (x, y) = (0,0)$$

at point $(0, 0)$. [Summer 2023]

Sol: Here, $f(x, y)$ is defined at $(0,0)$ and $f(0,0) = 0$

Path 1:

$$\lim_{(x,y) \to (0,0)} \left(\frac{x^2 y}{x^3 + y^3} \right) = 0$$

Path 2:

$$\lim_{(x,y) \to (0,0)} \left(\frac{x^2 y}{x^3 + y^3} \right) = 0$$

Path 3: Put $y = mx$, as $x \to 0$

$$\lim_{x \to 0} \left(\frac{x^2(mx)}{x^3 + (mx)^3} \right) = \lim_{x \to 0} \left(\frac{mx^3}{x^3 + m^3 x^3} \right) = \frac{m}{1 + m^3}$$

As the limit depends on m and m is not fixed, the limit doesn't exist. Therefore, limit doesn't exist.

Hence, $f(x, y)$ is discontinuous at (0,0).

2. Discuss the continuity of

$$f(x, y) = \frac{x^2 - y^2}{\sqrt{x^2 + y^2}}, \quad (x, y) \neq (0,0)$$

$$0 \qquad (x, y) = (0,0)$$

at point $(0, 0)$. [Winter 2023]

Sol: Here, $f(x, y)$ is defined at (0,0) and $f(0,0) = 0$.

Path 1:

$$\lim_{x \to 0}\left[\lim_{y \to 0} \frac{x^2 - y^2}{\sqrt{x^2 + y^2}}\right] = \lim_{x \to 0} \frac{x^2}{x} = x = 0$$

Path 2:

$$\lim_{y \to 0}\left[\lim_{x \to 0} \frac{x^2 - y^2}{\sqrt{x^2 + y^2}}\right] = \lim_{y \to 0}\left(-\frac{y^2}{y}\right) = \lim_{y \to 0}(-y) = 0$$

Path 3: Put $y = mx$, as $x \to 0$

$$\lim_{x \to 0}\left(\frac{x^2 - (mx)^2}{\sqrt{x^2 + (mx)^2}}\right) = \lim_{x \to 0}\left(\frac{x^2(1 - m^2)}{x\sqrt{1 + m^2}}\right) = 0$$

Path 4: Put $y = mx^2$, as $x \to 0$

$$\lim_{x \to 0}\left(\frac{x^2 - (mx^2)^2}{\sqrt{x^2 + (mx^2)^2}}\right) = \lim_{x \to 0}\left(\frac{x^2(1 - m^2x^2)}{x\sqrt{1 + m^2x^2}}\right) = 0$$

Since, limit along all the paths are same.

Therefore, limit exist.

Also, $\lim\limits_{(x,y) \to (0,0)} \frac{x^2 - y^2}{\sqrt{x^2 + y^2}} = 0 = f(0,0)$

Hence, $f(x, y)$ is continuous at (0,0).

3. Check whether the given function is continuous at origin or not, if yes then find point of continuity.

$$f(x, y) = \frac{x + y}{\sqrt{x} - \sqrt{y}} \qquad (x, y) \neq (0,0)$$

$$-1 \qquad (x, y) = (0,0)$$

Sol: Here, $f(x, y)$ is defined at (0,0) and $f(0,0) = -1$

Path 1:

$$\lim_{x \to 0}\left[\lim_{y \to 0} \frac{x + y}{\sqrt{x} - \sqrt{y}}\right] = \lim_{x \to 0} \frac{x}{\sqrt{x}} = \sqrt{x} = 0$$

Path 2:

$$\lim_{y\to 0}\left[\lim_{x\to 0}\frac{x+y}{\sqrt{x}-\sqrt{y}}\right] = \lim_{y\to 0}\left(\frac{y}{-\sqrt{y}}\right) = \lim_{y\to 0}(-\sqrt{y}) = 0$$

Path 3: Put $y = mx$, as $x \to 0$

$$\lim_{x\to 0}\left(\frac{x+y}{\sqrt{x}-\sqrt{y}}\right) = \lim_{x\to 0}\left(\frac{x+mx}{\sqrt{x}-\sqrt{mx}}\right) = \lim_{x\to 0}\sqrt{x}\left(\frac{1+m}{1-\sqrt{m}}\right) = 0$$

Path 4: Put $y = mx^2$, as $x \to 0$

$$\lim_{x\to 0}\left(\frac{x+y}{\sqrt{x}-\sqrt{y}}\right) = \lim_{x\to 0}\left(\frac{x+mx^2}{\sqrt{x}-\sqrt{mx^2}}\right) = \lim_{x\to 0}\left(\frac{x+mx^2}{\sqrt{x}-\sqrt{m}x}\right) = \lim_{x\to 0}\frac{x}{\sqrt{x}}\left(\frac{1+mx}{1-\sqrt{m}x}\right) = 0$$

Since, limit along all the paths are same.

Therefore, limit exist.

But, $\lim_{(x,y)\to(0,0)}\frac{x+y}{\sqrt{x}-\sqrt{y}} = 0 \neq -1 = f(0,0)$

Hence, $f(x,y)$ is discontinuous at (0,0).

Exercise:

1. Discuss the continuity of

$$f(x,y) = \frac{x}{3x+5y}, \qquad (x,y) \neq (0,0)$$

$$1, \qquad (x,y) = (0,0)$$

2. Show that

$$f(x,y) = \frac{x^2 y}{y^2 - x^2}, \qquad (x,y) \neq (0,0)$$

$$0, \qquad (x,y) = (0,0)$$

is continuous at origin.

3. Discuss the continuity of $f(x,y) = \begin{cases} \dfrac{x^2 - y^2}{x^2 + y^2}; & (x,y) \neq (0,0) \\ 0; & (x,y) = (0,0) \end{cases}$ at origin.   **[Winter 2023]**

[Summer 2023]

4. Show that the function $f(x,y) = \dfrac{2x^2 y}{x^4 + y^2}$ has no limit as (x, y) approaches to (0,0) .

[Summer 2023].

5. Discuss the continuity of $f(x,y) = \begin{cases} \dfrac{2xy}{x^2 + y^2}; & (x,y) \neq (0,0) \\ 0; & (x,y) \neq (0,0) \end{cases}$ at Origin.   **[Summer 2023]**

Partial Derivatives:

The partial derivative of $f(x, y)$ with respect to x at the point (x_0, y_0) is

$$\left(\frac{\partial f}{\partial x}\right)_{(x_0, y_0)} = \lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h}, \quad provided\ the\ limit\ exists.$$

The partial derivative of $f(x, y)$ with respect to y at the point (x_0, y_0) is

$$\left(\frac{\partial f}{\partial y}\right)_{(x_0, y_0)} = \lim_{h \to 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h}, \quad provided\ the\ limit\ exists.$$

Second order partial derivative:

Two successive partial differentiations of $f(x, y)$ with respect to x (holding y constant) is denoted by

$\frac{\partial^2 f}{\partial x^2}$ or $f_{xx}(x, y)$. That is, we define

$$f_{xx}(x, y) = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$$

Similarly, two successive partial differentiations of $f(x, y)$ with respect to y (holding x constant) is

denoted by $\frac{\partial^2 f}{\partial y^2}$ or $f_{yy}(x, y)$. That is, we define

$$f_{yy}(x, y) = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$$

We use $\frac{\partial^2 f}{\partial x \partial y}$ to mean differentiate first with respect to y then with respect to x, and we use $\frac{\partial^2 f}{\partial y \partial x}$ to mean

differentiate first with respect to x then with respect to y.

Note:

$f_{xy}(x, y) = \frac{\partial^2 f}{\partial y \partial x}$ and $f_{yx}(x, y) = \frac{\partial^2 f}{\partial x \partial y}$ are known as mixed order partial derivatives.

The Mixed Derivative / Clairaut's Theorem

If $f(x, y)$ and its partial derivative f_x, f_y, f_{xy}, f_{yx} are

  (i)   defined throughout an open region containing a point (a, b) and

  (ii)   they are all continuous at (a, b),

then $f_{xy}(a, b) = f_{yx}(a, b)$.

THEOREM: If a function $f(x, y)$ is differentiable at (x_0, y_0) then $f(x, y)$ is continuous at (x_0, y_0).

Solved Examples:

1. **Find $\frac{\partial f}{\partial z}$ at (1,2,3) for $f(x, y, z) = x^2 y z^2$ using the definition.**

Sol: Here, $\frac{\partial f}{\partial z} = \lim_{h \to 0} \frac{f(x_0, y_0, z_0 + h) - f(x_0, y_0, z_0)}{h}$

$$\left(\frac{\partial f}{\partial z}\right)_{(1,2,3)} = \lim_{h\to 0} \frac{f(1,2,3+h)-f(1,2,3)}{h}$$

$$= \lim_{h\to 0} \frac{2(9+6h+h^2)-18}{h}$$

$$= \lim_{h\to 0}(12+2h)$$

$$= 12$$

2. **If $f(x,y) = x^3 + y^3 - 2xy^2$. Find all second order partial derivatives of $f(x,y)$ at $(1,-1)$**

Sol: Here, $f_x(x,y) = 3x^2 - 2y^2 f_y(x,y) = 3y^2 - 4xy$

$$f_{xx}(x,y) = 6x$$
$$f_{yy}(x,y) = 6y - 4x$$
$$f_{xy}(x,y) = -4y,$$
$$f_{yx}(x,y) = -4y$$

Then,

$$f_{xx}(1,-1) = 6$$
$$f_{yy}(1,-1) = -10$$
$$f_{xy}(1,-1) = 4$$
$$f_{yx}(1,-1) = 4$$

3. **If $u = log(tanx + tany + tanz)$, then show that**

$$sin2x\frac{\partial u}{\partial x} + sin2y\frac{\partial u}{\partial y} + sin2z\frac{\partial u}{\partial z} = 2$$

Sol: u = log (tan x + tan y + tan z)

Differentiating u partially w.r.t. x, y and z,

$$\frac{\partial u}{\partial x} = \frac{1}{tan\ x + tan\ y + tan\ z}sec^2x$$

$$\frac{\partial u}{\partial y} = \frac{1}{tan\ x + tan\ y + tan\ z}.sec^2y$$

$$\frac{\partial u}{\partial z} = \frac{1}{tan\ x + tan\ y + tan\ z}.sec^2z$$

Hence,

$$sin\ 2x\ \frac{\partial u}{\partial x} + sin\ 2y\ \frac{\partial u}{\partial y} + sin\ 2z\ \frac{\partial u}{\partial z}$$

$$= \frac{2sinxcosxsec^2x + 2sinycosysec^2y + 2sinzcoszsec^2z}{tan\ x + tan\ y + tan\ z}$$

$$= \frac{2(\tan x + \tan y + \tan z\;)}{\tan x + \tan y + \tan z}$$

$$= 2$$

4. If $u(x, y, z) = e^{3xyz}$ **show that** $\frac{\partial^3 u}{\partial x \partial y \partial z} = (3 + 27xyz + 27x^2y^2z^2)e^{3xyz}$

Sol: $u(x, y, z) = e^{3xyz}$

Differentiating u w.r.t z,

$$\frac{\partial u}{\partial z} = 3xye^{3xyz}$$

Differentiating $\frac{\partial u}{\partial z}$ w.r.t y,

$$\frac{\partial^2 u}{\partial y \partial z} = 3x \frac{\partial}{\partial y}(ye^{3xyz})$$
$$= 3x(e^{3xyz}.1 + ye^{3xyz}.3xz)$$
$$= e^{3xyz}(3x + 9x^2yz)$$

Differentiating $\frac{\partial^2 u}{\partial y \partial z}$ w.r.t x,

$$\frac{\partial^3 u}{\partial x \partial y \partial z} = \frac{\partial}{\partial x}[[\, e^{3xyz}(3x + 9x^2yz)]$$
$$= e^{3xyz}(3 + 18xyz) + (3x + 9x^2yz).e^{3xyz}.3yz$$
$$= e^{3xyz}(3 + 18xyz + 9xyz + 27x^2y^2z^2)$$
$$= e^{3xyz}(3 + 27xyz + 27x^2y^2z^2)$$

Exercise:

1. Find the values of $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ at point (4,-5) if $f(x, y) = x^2 + 3xy + y - 1$.

2. If $z = x + y^x$, prove that $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$

3. Find f_{yxyz} if $f(x,y,z) = 1 - 2xy^2z + x^2y$.

4. Find the $f_x(1,2)$ for the function $f(x,y) = x^2y + xy^2$ **[Winter 2022]**

5. If $f(x,y) = x^3 + y^3 - 2xy^2$, find all second order partial derivatives of $f(x,y)$ at $(1,-1)$.
   [Summer 2023]

6. . Find the value of $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ at point $(1,2)$ if $f(x,y) = x^2 + 2xy + 3y^2 - 1$. **[Winter 2023]**

Homogeneous function:

A function $u = f(x, y)$ is said to be homogeneous function of degree 'n' in x and y if degree of each term of $u = f(x, y)$ is n.

Thus, for a homogeneous function f of degree n; $f(tx, ty) = t^n f(x, y)$

<u>**Note:**</u>

Also, if the function $u = f(x, y)$ is a homogeneous function of degree 'n' in x and y

then it can be written as $u = x^n \emptyset \left(\frac{y}{x}\right) or \ u = y^n \emptyset \left(\frac{x}{y}\right)$

<u>**Euler's theorem for the function of two independent variables:**</u>

Statement: If u is a homogeneous function of degree n in x and y, then

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$$

Proof: Let $u = f(x, y)$ be a homogeneous function of degree 'n' in x and y, then it can be

written as $\quad\quad u = x^n \emptyset \left(\frac{y}{x}\right) \quad$ _____(1)

Differentiate (1) partially w.r.t x, we get

$$\frac{\partial u}{\partial x} = x^n \frac{\partial}{\partial x}\left[\emptyset\left(\frac{y}{x}\right)\right] + \emptyset\left(\frac{y}{x}\right)\frac{\partial}{\partial x}(x^n)$$

$$= x^n \emptyset'\left(\frac{y}{x}\right)\frac{\partial}{\partial x}\left(\frac{y}{x}\right) + \emptyset\left(\frac{y}{x}\right)(nx^{n-1})$$

$$= x^n \emptyset'\left(\frac{y}{x}\right)\left(-\frac{y}{x^2}\right) + \emptyset\left(\frac{y}{x}\right)(nx^{n-1})$$

$$= -yx^{n-2}\emptyset'\left(\frac{y}{x}\right) + nx^{n-1}\emptyset\left(\frac{y}{x}\right)$$

$$\Rightarrow x\frac{\partial u}{\partial x} = -yx^{n-1}\emptyset'\left(\frac{y}{x}\right) + nx^n\emptyset\left(\frac{y}{x}\right) \quad \text{_____(2)}$$

Differentiate (1) partially w.r.t y, we get

$$\frac{\partial u}{\partial y} = x^n \frac{\partial}{\partial y}\left[\emptyset\left(\frac{y}{x}\right)\right]$$

$$= x^n \emptyset'\left(\frac{y}{x}\right)\frac{\partial}{\partial y}\left(\frac{y}{x}\right)$$

$$= x^n \emptyset'\left(\frac{y}{x}\right)\left(\frac{1}{x}\right)$$

$$= x^{n-1}\emptyset'\left(\frac{y}{x}\right)$$

$$\Rightarrow y\frac{\partial u}{\partial y} = yx^n\emptyset'\left(\frac{y}{x}\right) \quad \text{_____(3)}$$

Adding (2) and (3) we get,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = -yx^{n-1}\emptyset'\left(\frac{y}{x}\right) + nx^n\emptyset\left(\frac{y}{x}\right) + yx^n\emptyset'\left(\frac{y}{x}\right) = nx^n\emptyset\left(\frac{y}{x}\right)$$

$$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$$

<u>**Euler's theorem for the function of three independent variables:**</u>

Statement: If u is a homogeneous function of degree n in x, y and z, then

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = nu$$

[Similar to **Euler's theorem for the function of two independent variables**]

Cor.1 If u is a homogeneous function of degree n in x and y, then

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = n(n-1)u.$$

Proof: Let $u = f(x, y)$ be a homogeneous function of degree 'n' in x and y, then by Euler's theorem for homogeneous functions

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = nu \underline{\hspace{3cm}} (1)$$

Differentiate (1) partially w.r.t x, we get

$$\Rightarrow x \left[\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} \right) \right] + \frac{\partial u}{\partial x}\left[\frac{\partial}{\partial x}(x) \right] + y \left[\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y} \right) \right] = n \frac{\partial}{\partial x}(u)$$

$$\Rightarrow x \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} + y \left[\frac{\partial^2 u}{\partial x \partial y} \right] = n \frac{\partial u}{\partial x}$$

On multiplying both sides by x, we get

$$\Rightarrow x^2 \frac{\partial^2 u}{\partial x^2} + x \frac{\partial u}{\partial x} + xy \left[\frac{\partial^2 u}{\partial x \partial y} \right] = nx \frac{\partial u}{\partial x} \underline{\hspace{3cm}} (2)$$

Differentiate (1) partially *w.r.t. y*, we get

$$\Rightarrow x \left[\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x} \right) \right] + \frac{\partial u}{\partial y}\left[\frac{\partial}{\partial y}(y) \right] + y \left[\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y} \right) \right] = n \frac{\partial}{\partial y}(u)$$

$$\Rightarrow x \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial u}{\partial y} + y \frac{\partial^2 u}{\partial y^2} = n \frac{\partial u}{\partial y}\left[since, \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x} \right]$$

On multiplying both sides by y, we get

$$\Rightarrow xy \left[\frac{\partial^2 u}{\partial x \partial y} \right] + y \frac{\partial u}{\partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = ny \frac{\partial u}{\partial y} \underline{\hspace{3cm}} (3)$$

Adding (2) and (3) we get,

$$x^2 \frac{\partial^2 u}{\partial x^2} + x \frac{\partial u}{\partial x} + xy \left[\frac{\partial^2 u}{\partial x \partial y} \right] + xy \left[\frac{\partial^2 u}{\partial x \partial y} \right] + y \frac{\partial u}{\partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = nx \frac{\partial u}{\partial x} + ny \frac{\partial u}{\partial y}$$

$$\Rightarrow x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} + x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = n \left(x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} \right)$$

$$\Rightarrow x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} + nu = n^2 u$$

$$\Rightarrow x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = n^2 u - nu$$

$$\Rightarrow x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = n(n-1)u.$$

Cor.2 If $\emptyset(u) = f(x, y)$ is a homogeneous function of degree n in x and y, then

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = n \frac{f(u)}{f'(u)} \text{[Without proof]}$$

Cor.3 If $\emptyset(u) = f(x,y)$ is a homogeneous function of degree n in x and y, then

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = g(u)[g'(u) - 1]$$

where, $g(u) = n \frac{f(u)}{f'(u)}.$ [Without proof]

Note: Corollary 2 and 3 are also known as Modified Euler's theorem of first and second order respectively.

Solved Examples:

1. If $u = \frac{1}{x^2} + \frac{1}{xy} + \frac{logx - logy}{x^2+y^2}$, prove that

$$(i)\ x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = -2u \quad (ii) x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = 6u$$

Sol: $u = f(x,y) = \frac{1}{x^2} + \frac{1}{xy} + \frac{log\frac{x}{y}}{x^2+y^2}$

Replacing x by tx and y by ty,

$$f(tx, ty) = \frac{1}{t^2x^2} + \frac{1}{txty} + \frac{log\frac{tx}{ty}}{t^2x^2+t^2y^2}$$

$$= \frac{1}{t^2}\left[\frac{1}{x^2} + \frac{1}{xy} + \frac{log\frac{x}{y}}{x^2+y^2}\right]$$

$$= t^{-2}f(x,y)$$

Hence, u is a homogeneous function of degree -2

By Euler's Theorem,

$$(i)\ x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$$

$$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = -2u$$

$$(ii)\ \frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = n(n-1)u$$

$$i.e\ x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = -2(-2-1)u = 6u$$

2. If $u = tan^{-1}(\frac{x^2+y^2}{x+y})$, **prove that** , [Winter 2023][Summer 2023]

$$(i)x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{sin2u}{2}$$

$$(ii)x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = -2sin^3ucosu.$$

Sol: $u = tan^{-1}(\frac{x^2+y^2}{x+y})$

Replacing x by tx and y by ty,

$$u = tan^{-1}[t(\frac{x^2+y^2}{x+y})]$$

u is a non-homogeneous function. But $tan\ u = (\frac{x^2+y^2}{x+y})$ is a homogeneous function of degree 1.

By Modified Euler's Theorem,

Let $f(u) = tanu$

$$(i)x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\frac{f(u)}{f'(u)}$$

$$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 1.\frac{tanu}{sec^2u} = sinu\ cosu = \frac{sin2u}{2}$$

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = g(u)[g'(u) - 1]$$

where, $g(u) = n\frac{f(u)}{f'(u)} = 1.\frac{tanu}{sec^2u} = \frac{sin2u}{2}$

$$\Rightarrow g'(u) = cos2u$$

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \frac{sin2u}{2}(cos2u - 1)$$

$$= sinucosu(cos2u - 1)$$

$$= sinucosu(-2sin^2u)$$

$$= -2sin^3ucosu$$

Exercise:

1. If $u = y^2e^{\frac{y}{x}} + x^2\left(\frac{x}{y}\right)$, show that

$$(i)x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2u$$

$$(ii)x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 2u$$

2. If $u = sin^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$, then prove that

$$(i)x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}tanu$$

$$(ii)x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \frac{1}{4}(tan^3u - tanu)$$

3. If $u = \tan^{-1}\left(\frac{x^2+y^2}{x-y}\right)$, Show that: $x\frac{\partial u}{\partial x} + x\frac{\partial u}{\partial y} = \frac{1}{2}\sin 2u$ **[Summer 2022]**

4. If $u = \tan^{-1}\left(x^2 + 2y^2\right)$, Show that: **[Winter 2022]**

$(i)\ x\frac{\partial u}{\partial x} + x\frac{\partial u}{\partial y} = \sin 2u$

$$(ii) \quad x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = 2\sin u \cos 3u$$

5. If $u = y^2 e^{\frac{y}{x}} + x^2 \left(\frac{x}{y} \right)$, show that,   **[Summer 2023]**

$$(i) \quad x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 2u$$

$$(ii) \quad x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = 2u$$

6. If $u = \tan^{-1} \left(\frac{x^2 + y^2}{x + y} \right)$ show that,   **[Winter 2021][Winter 2023][Summer 2023]**
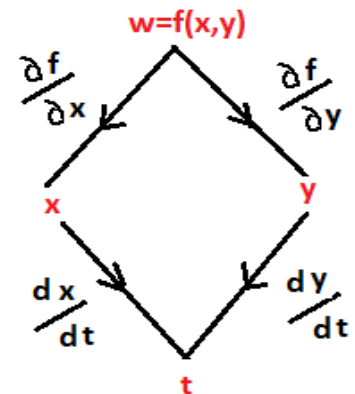
$$(i) \quad x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \sin u \cos u$$

$$(ii) \quad x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = -2\sin^3 u \cos u$$

Chain Rule for Function of Two Independent Variable

If $w = f(x, y)$ has continuous partial derivative f_x, f_y and if

$x = x(t), \quad y = y(t)$ are differentiable function of t, then the

composite $w = f(x(t), y(t))$ is a differentiable function of

t then,

$$\frac{df}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt}$$

Chain Rule for Function of Three Independent Variables

If $w = f(x, y, z)$ is differentiable and $x, y \ and \ z$ are

differentiable function of t then w is a differentiable

function of t then,

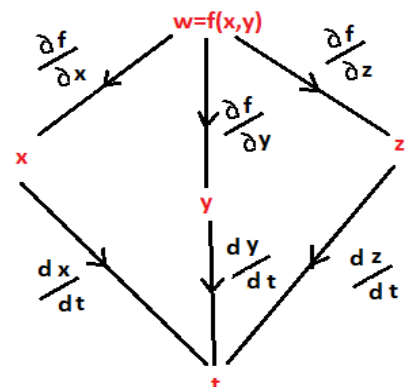$$\frac{df}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial f}{\partial z} \cdot \frac{dz}{dt}$$

Chain Rule for Function of Two Independent Variable and Three Intermediate Variable

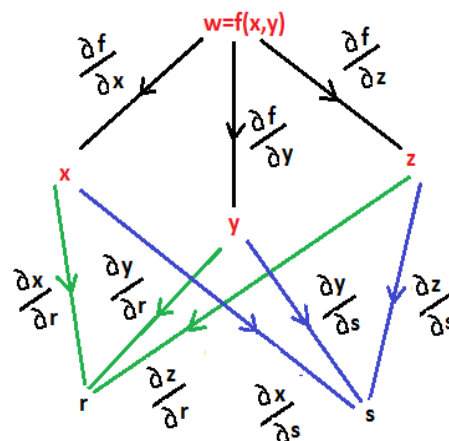Suppose that $w = f(x, y, z), x = g(r, s), y = h(r, s)$ and

$z = k(r, s)$. If all four functions are differentiable then w

has partial derivative with respect to r and s given by

the formulas

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial r} + \frac{\partial w}{\partial z} \cdot \frac{\partial z}{\partial r}$$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial s} + \frac{\partial w}{\partial z} \cdot \frac{\partial z}{\partial s}$$

Solved Examples:

1. Let $z = x^2 y^3$, where $x = t^2$ and $y = t$, then verify chain rule by expressing z in terms of t.

Sol: Here, the chain rule is $\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$

$$= (2xy^3)(2t) + (3x^2 y^2)(1)$$

$$= (2t^2 t^3)(2t) + (3t^4 t^2)$$

$$= 4t^6 + 3t^6$$

$$= 7t^6 - - - - - - -(1)$$

Also

$$z = x^2 y^3$$

$$z = (t^4)(t^3) = t^7$$

$$\frac{dz}{dt} = 7t^6 - - - - - - -(2)$$

Hence, from (1) and (2), the chain rule is verified.

2. If $u = xy^2 + yz^3, x = \log t, y = e^t, z = t^2$ find $\frac{du}{dt}$ at $t = 1$.

Sol: $u = xy^2 + yz^3, x = \log t, y = e^t, z = t^2$

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt}$$

$$= (y^2)\frac{1}{t} + (2xy + z^3)e^t + (3yz^2)2t$$

Substituting x ,y and z,

$$\frac{du}{dt} = 2(e^{2t})\frac{1}{t} + \left(2(logt) \ \ e^t + t^6\right)e^t + 3e^t t^4.2t$$

Substituting $t = 1$,

$$\frac{du}{dt} = 2e^2 + e^1 + 6e^1$$

$$= 2e^2 + 7e^1$$

3.If $u = f(x - y, y - z, z - x)$, show that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$ **[Winter 2021] [Summer 2023]**

Sol: Let $x - y = l, y - z = m, z - x = n$

$$\frac{\partial l}{\partial x} = 1, \qquad \frac{\partial m}{\partial x} = 0, \qquad \frac{\partial n}{\partial x} = -1,$$

$$\frac{\partial l}{\partial y} = -1, \qquad \frac{\partial m}{\partial y} = 1, \qquad \frac{\partial n}{\partial y} = 0,$$

$$\frac{\partial l}{\partial z} = 0, \qquad \frac{\partial m}{\partial z} = -1, \qquad \frac{\partial n}{\partial z} = 1$$

$$u = f(x - y, y - z, z - x) = f(l, m, n)$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial l}\frac{\partial l}{\partial x} + \frac{\partial u}{\partial m}\frac{\partial m}{\partial x} + \frac{\partial u}{\partial n}\frac{\partial n}{\partial x}$$

$$= \frac{\partial u}{\partial l}.1 + \frac{\partial u}{\partial m}.0 + \frac{\partial u}{\partial n}.-1$$

$$= \frac{\partial u}{\partial l} - \frac{\partial u}{\partial n} \ \ - - - - - - -(1)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial l}\frac{\partial l}{\partial y} + \frac{\partial u}{\partial m}\frac{\partial m}{\partial y} + \frac{\partial u}{\partial n}\frac{\partial n}{\partial y}$$

$$= \frac{\partial u}{\partial l}.-1 + \frac{\partial u}{\partial m}.1 + \frac{\partial u}{\partial n}.0$$

$$= -\frac{\partial u}{\partial l} + \frac{\partial u}{\partial m} \ \ - - - - - - -(2)$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial l}\frac{\partial l}{\partial z} + \frac{\partial u}{\partial m}\frac{\partial m}{\partial z} + \frac{\partial u}{\partial n}\frac{\partial n}{\partial z}$$

$$= \frac{\partial u}{\partial l}.0 + \frac{\partial u}{\partial m}.-1 + \frac{\partial u}{\partial n}.1$$

$$= -\frac{\partial u}{\partial m} + \frac{\partial u}{\partial n} - - - - - - (3)$$

Adding Eqs. (1),(2) and (3),

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$$

4. If $w = x^2 + y^2$, Where $x = r - s$ and $y = r + s$ then find $\frac{\partial w}{\partial r}$ and $\frac{\partial w}{\partial s}$ in terms of r and s.

[Summer 2022]

Solution:

Here, the chain rule is $\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x} * \frac{\partial x}{\partial r} + \frac{\partial w}{\partial y} * \frac{\partial y}{\partial r}$

$$= (2x)(1) + (2y)(1)$$

$$= (2(r - s)) + 2(r + s) = 4r$$

Now for $\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x} * \frac{\partial x}{\partial s} + \frac{\partial w}{\partial y} * \frac{\partial y}{\partial s}$

$$= (2x)(-1) + (2y)(1)$$

$$= (-2(r - s)) + 2(r + s) = 4s$$

Exercise:

1. If $u = f(x^2 + 2yz, y^2 + 2xz)$, then find value of $(y^2 - xz)\frac{\partial u}{\partial x} + (x^2 - yz)\frac{\partial u}{\partial y} + (z^2 - xy)\frac{\partial u}{\partial z}$.

2. If $u = f\left(\frac{x}{y}, \frac{y}{z}, \frac{z}{x}\right)$, prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$

3. If $u = f(r)$ where $r^2 = x^2 + y^2$, prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r) + \frac{1}{r}f'(r)$.

4. If $u = x^2 y + y^2 z + z^2 x$ then find out $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}$.   **[Summer 2022]**

Implicit Differentiation:

Suppose that $f(x, y)$ is differentiable and that the equation $f(x, y) = 0$ defines y as a

differentiable function of x. Then at any point where $f_y \neq 0$,

$$\frac{dy}{dx} = -\frac{f_x}{f_y}$$

Solved Examples:

1. If $y\sin x = x\cos y$, **find** $\frac{dy}{dx}$.

Sol: Let $f(x,y) = y\sin x - x\cos y$

$$\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$$

$$= -\frac{y\cos x - \cos y}{\sin x + x\sin y}$$

$$= \frac{\cos y - y\cos x}{\sin x + x\sin y}$$

2. If $(\cos x)^y = (\sin y)^x$, **find** $\frac{dy}{dx}$.

Sol: $(\cos x)^y = (\sin y)^x$

Taking log on both the sides,

$$y\log\cos x = x\log\sin y$$

Let $f(x,y) = y\log\cos x - x\log\sin y$

$$\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$$

$$= -\frac{\frac{y}{\cos x}(-\sin x) - \log\sin y}{\log\cos x - \frac{x}{\sin y}(\cos y)}$$

$$= \frac{y\tan x + \log\sin y}{\log\cos x - x\cot y}$$

Exercise:

1. If $x^3 + y^3 + 3xy = 1$ then, find $\frac{dy}{dx}$.

2. If $x^y + y^x = c$ then, find $\frac{dy}{dx}$.

3. If $f(x,y) = c$ then, find $\frac{dy}{dx}$ **[Summer 2023]**

Jacobian:

If u and v are continuous and differentiable functions of two independent variables x and y,

then the Jacobian of u,v with respect to x,y and is denoted by

$$J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix}$$

Similarly, if u,v and w are continuous and differentiable functions of three independent

variables x,yand z, then the Jacobian of u,v,w with respect to x,y,z and is denoted by

$$J = \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} & \dfrac{\partial u}{\partial z} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} & \dfrac{\partial v}{\partial z} \\ \dfrac{\partial w}{\partial x} & \dfrac{\partial w}{\partial y} & \dfrac{\partial w}{\partial z} \end{vmatrix}$$

Properties of Jacobian:

1. If u and v are functions of x and y, then $J.J' = 1$, where $J = \frac{\partial(u,v)}{\partial(x,y)}$ and $J' = \frac{\partial(x,y)}{\partial(u,v)}$ **[Summer 2023]**

2. If u,v are functions of r,s and r,s are in turn functions of x,y then $\frac{\partial(u,v)}{\partial(x,y)} = \frac{\partial(u,v)}{\partial(r,s)} \cdot \frac{\partial(r,s)}{\partial(x,y)}$

Solved Examples:

1. Find the Jacobian$\frac{\partial(u,v)}{\partial(x,y)}$ for $u = x^2 - y^2, v = 2xy.$

Sol:$u = x^2 - y^2, \ v = 2xy$

$$\frac{\partial u}{\partial x} = 2x, \qquad \frac{\partial v}{\partial x} = 2y$$

$$\frac{\partial u}{\partial y} = -2y, \quad \frac{\partial v}{\partial y} = 2x$$

$$J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} 2x & -2y \\ 2y & 2x \end{vmatrix}$$

$$= 4(x^2 + y^2)$$

2. Find the Jacobian $\frac{\partial(u,v)}{\partial(x,y)}$ **for** $u = x - y, v = x + y$**. Also, verify that** $J.J' = 1$**.**

Sol: $u = x - y, \quad v = x + y$

$$\frac{\partial u}{\partial x} = 1, \qquad \frac{\partial v}{\partial x} = 1$$

$$\frac{\partial u}{\partial y} = -1, \qquad \frac{\partial v}{\partial y} = 1$$

$$J = \frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix}$$

$$= 2$$

Now, $x = \frac{u+v}{2}, y = \frac{v-u}{2}$

$$\frac{\partial x}{\partial u} = \frac{1}{2}, \qquad \frac{\partial y}{\partial u} = -\frac{1}{2}$$

$$\frac{\partial x}{\partial v} = \frac{1}{2}, \qquad \frac{\partial y}{\partial v} = \frac{1}{2}$$

$$J' = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} \dfrac{1}{2} & \dfrac{1}{2} \\ -\dfrac{1}{2} & \dfrac{1}{2} \end{vmatrix}$$

$$= \frac{1}{2}$$

$$JJ' = 1.$$

Exercise:

1. Find the Jacobian $\frac{\partial(u,v)}{\partial(x,y)}$ for $u = x\sin y, v = y\sin x$.

2. If $u = 2xy, v = x^2 - y^2$ and $x = r\cos\theta, y = r\sin\theta$ then, evaluate $\frac{\partial(u,v)}{\partial(r,\theta)}$ **[Winter 2021]**

3. If $x = u + 3v, y = v - u$ then evaluate the value of $\dfrac{\partial(u,v)}{\partial(x,y)}$.   **[Winter 2022]**

APPLICATIONS OF PARTIAL DERIVATIVES

<u>Tangent Plane and Normal Line</u>

Intuitively, it seems clear that, in a plane, only one line can be tangent to a curve at a point. However, in three-dimensional space, many lines can be tangent to a given point. If these lines lie in the same plane, they determine the tangent plane at that point. A more intuitive way to think of a tangent plane is to assume the surface is smooth at that point (no corners). Then, a tangent line to the surface at that point in any direction does not have any abrupt changes in slope because the direction changes smoothly. Therefore, in a small-enough neighborhood around the point, a tangent plane touches the surface at that point only.
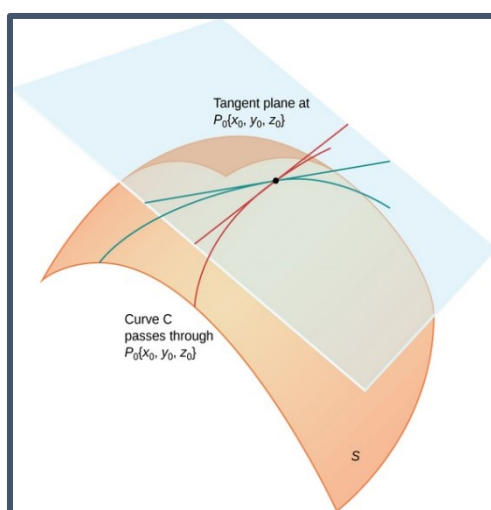
For a tangent plane to a surface to exist at a point on that surface, it is sufficient for the function that defines the surface to be differentiable at that point. We define the term tangent plane here and then explore the idea intuitively.

Definition: The **tangent plane** at the point $P_0(x_0, y_0, z_0)$ on surface $f(x, y, z) = 0$ of the differentiable function f is the plane through P_0 normal to $\nabla f|_{P_0}$. It is given by

$$f_x(P_0)(x - x_0) + f_y(P_0)(y - y_0) + f_z(P_0)(z - z_0) = 0$$

The **normal line** of the surface at P_0 is the line through P_0 parallel to $\nabla f|_{P_0}$. It is given by

$$\frac{x - x_0}{f_x(P_0)} = \frac{y - y_0}{f_y(P_0)} = \frac{z - z_0}{f_z(P_0)}$$

<u>**Solved Examples:**</u>

1. Find the equation of the tangent plane and normal line to the surface $x^2 + y^2 + z^2 = 3$

at the point (1,1,1).   **[Winter 2021][Summer 2023]**

Sol: Here, $f(x, y, z) = x^2 + y^2 + z^2 - 3$

$f_x(x, y, z) = 2x$, $f_x(1,1,1) = 2$

$f_y(x, y, z) = 2y$, $f_y(1,1,1) = 2$

$f_z(x, y, z) = 2z$, $f_z(1,1,1) = 2$

Hence, the equation of the tangent plane at $(1,1,1)$ is

$$(x - 1)2 + (y - 1)2 + (z - 1)2 = 0$$

$$x + y + z = 3$$

The equation of normal line is $\frac{x-1}{2} = \frac{y-1}{2} = \frac{z-1}{2}$.

2. Find the equation of the tangent plane and normal line to the surface $z + 8 = xe^y \cos z$

at the point (8,0,0).

Sol: Let $f(x, y, z) = xe^y \cos z - z - 8$

$f_x(x, y, z) = e^y \cos z$, $f_x(8,0,0) = 1$

$f_y(x, y, z) = xe^y \cos z$, $f_y(8,0,0) = 8$

$f_z(x, y, z) = \sin z - 1$, $f_z(8,0,0) = -1$

Hence, the equation of the tangent plane at $(8,0,0)$ is

$$1(x - 8) + 8(y - 0) - 1(z - 0) = 0$$

$$x - 8 + 8y - z = 0$$

$$x + 8y - z - 8 = 0$$

The equation of normal line is $\frac{x-8}{1} = \frac{y-0}{8} = \frac{z-0}{-1}$

<u>**Exercise:**</u>

1. Find the equations of tangent plane and normal line to the surface $z = x^2 + 3y^2 - 4$

    at (1,1,0).

2. Find the equations of tangent plane and normal line to surface $2x^2 + y^2 + 2z = 3$ at $(2,1,-3)$.

3. Find the equation of tangent plane and normal line to the paraboloid $z = x^2 + y^2$ at (1,1,2).

[Winter 2022]

<u>**Local Maximum and Local Minimum**</u>

Let $f(x, y)$ be defined on a region R containing the point (a, b). Then

1. $f(a, b)$ is a local maximum value of f if $f(a, b) \geq f(x, y)$ for all domain points (x, y) in an open disk centered at (a, b).
2. $f(a, b)$ is a local minimum value of f if $f(a, b) \leq f(x, y)$ for all domain points (x, y) in an open disk centered at (a, b).

<u>**Second Derivative Test for Local Extreme Values**</u>

Suppose that $f(x, y)$ and its first and second partial derivative are continuous throughout

a disk centered at (a, b) and that $f_x(a, b) = f_y(a, b) = 0$. Then, for

$$r = \frac{\partial^2 f}{\partial x^2}, \ s = \frac{\partial^2 f}{\partial x \partial y}, \ t = \frac{\partial^2 f}{\partial y^2}$$

(i) f has a local maximum at (a, b) if $r < 0 \ and \ rt - s^2 > 0$ at (a, b)
(ii) f has a local minimum at (a, b) if $r > 0 \ and \ rt - s^2 > 0$ at (a, b)
(iii) f has a saddle point at (a, b) if $rt - s^2 < 0$ at (a, b)
(iv) The test has no conclusion at $(a, b) if \ rt - s^2 = 0$ at (a, b).

<u>**Examples:**</u>

1. Find the extreme value of $x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$ **[Summer 2022][Winter 2023]**

Sol: Let $f(x, y) = x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$

$$\frac{\partial f}{\partial x} = 3x^2 + 3y^2 - 30x + 72$$

$$\frac{\partial f}{\partial y} = 6xy - 30y$$

For extreme values, $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$

$3x^2 + 3y^2 - 30x + 72 = 0$ and $6xy - 30y = 0$

$x^2 + y^2 - 10x + 24 = 0$ and $6y(x - 5) = 0$

$x^2 + y^2 - 10x + 24 = 0$ and $y = 0, x = 5$

When $y = 0$, we have $x^2 - 10x + 24 = 0$

$$x = 4,6$$

and when $x = 5$, we have $25 + y^2 - 50 + 24 = 0$

$$y = \pm 1$$

Therefore, the stationary points are $(4,0), (6,0), (5,1), (5,-1)$

$$r = \frac{\partial^2 f}{\partial x^2} = 6x - 30$$

$$s = \frac{\partial^2 f}{\partial x \partial y} = 6y$$

$$t = \frac{\partial^2 f}{\partial y^2} = 6x - 30$$

(x,y)	r	s	t	$rt - s^2$	Conclusion	$f(x,y)$
$(4,0)$	$-6 < 0$	0	-6	$36 > 0$	Maximum	112
$(6,0)$	$6 > 0$	0	6	$36 > 0$	Minimum	108
$(5,1)$	0	6	0	$-36 < 0$	Saddle Point	--
$(5,-1)$	0	-6	0	$-36 < 0$	Saddle Point	--

2.Find the points on the surface $z^2 = x^2 + y^2$ that are closed to P(1,1,0).

Sol: Let $A(x, y, z)$ be any point on the surface, then by distance formula, the distance d

Between A and P is given by $d = \sqrt{(x - 1)^2 + (y - 1)^2 + z^2}$

$$d^2 = (x - 1)^2 + (y - 1)^2 + z^2$$
$$= 2x^2 + 2y^2 - 2x - 2y + 2 = f \ (say)$$

$$f_x(x,y) = 4x - 2, \qquad f_y(x,y) = 4y - 2$$

For extreme values, $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$

$$4x - 2 = 0, \quad 4y - 2 = 0$$
$$x = \frac{1}{2}, \quad y = \frac{1}{2}$$

So, $(\frac{1}{2}, \frac{1}{2})$ is a stationary point.

$$r = \frac{\partial^2 f}{\partial x^2} = 4$$

$$s = \frac{\partial^2 f}{\partial x \partial y} = 0$$

$$t = \frac{\partial^2 f}{\partial y^2} = 4$$

$$rt - s^2 = (4)(4) - 0 = 16 > 0$$

Hence, function is minimum at $\left(\frac{1}{2}, \frac{1}{2}\right)$

Minimum value of given surface $z^2 = x^2 + y^2$ at $\left(\frac{1}{2}, \frac{1}{2}\right)$

$$= \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

So, $z = \pm \frac{1}{2}$.

Exercise:

1. Discuss the maxima and minima of the function $(x, y) = x^2 + y^2 + 6x + 12$.

Ans. $f_{min} = 3$ **at** $(-3, 0)$

2. Find the extreme values of the function $f(x, y) = x^3 + 3xy^2 - 3x^2 - 3y^2 + 7$.

   [Winter 2022] [Summer 2023]

Ans. $f_{max} = 7$ **at (0,0)** $f_{min} = 3$ **at (2,0)**

MAXIMA AND MINIMA WITH CONSTRAINED VARIABLES

The Method of Lagrange Multipliers:

The method of Lagrange multipliers allows us to maximize or minimize function with a constraint.

Given a function $f(x, y, z)$ subject to the constraint $\emptyset(x, y, z) = 0$_____(1),

Solve by following

   Steps:

1. Construct an equation $f(x, y, z) + \lambda\emptyset(x, y, z) = 0$_____(2)

   where, λ is a variable called Lagrange multiplier.

2. Differentiate Eq. (2) partially w.r.t x, y, z to obtain

$$\frac{\partial f}{\partial x} + \lambda \frac{\partial \emptyset}{\partial x} = 0 \underline{\hspace{2cm}}(3)$$

$$\frac{\partial f}{\partial y} + \lambda \frac{\partial \emptyset}{\partial y} = 0 \underline{\hspace{2cm}}(4)$$

$$\frac{\partial f}{\partial z} + \lambda \frac{\partial \emptyset}{\partial z} = 0 \underline{\hspace{2cm}}(5)$$

3. Solve and eliminate λ from Eqs. (1), (3), (4), (5) to obtain the stationary points (x, y, z).

4. Substitute the stationary points (x, y, z) into f to see where, f attains its maximum and minimum values.

Note: For function of two independent variables, the condition is similar, but without the variable z.

Solved Examples:

1. Find the greatest and smallest values that the function $f(x,y) = xy$ takes on the ellipse $\frac{x^2}{8} + \frac{y^2}{2} = 1$.

Sol: Let $f(x,y,z) = xy$

and $\emptyset(x,y,z) = \frac{x^2}{8} + \frac{y^2}{2} - 1 = 0$ _____(1)

Let the equation be $xy + \lambda\left(\frac{x^2}{8} + \frac{y^2}{2} - 1\right) = 0$ _____(2)

Differentiating Eq. (2) partially w.r.t x,

$$y + \frac{\lambda x}{4} = 0 \text{ _____(3)}$$

Differentiating Eq. (2) partially w.r.t y,

$$x + \lambda y = 0 \text{ _____(4)}$$

From Eqs .(3),(4),

$$\frac{-4y}{x} = \frac{-x}{y}$$

$$\Rightarrow 4y^2 = x^2$$

Substituting x^2 in Eq.(1),

$$\frac{4y^2}{8} + \frac{y^2}{2} = 1$$

$$\Rightarrow y^2 = 1$$

$$\Rightarrow y = \pm 1$$

$$\Rightarrow x = \pm 2$$

Therefore, the function $f(x,y) = xy$ takes extreme values on the ellipse at four points $(2,1), (-2,1), (-2,-1), (2,-1)$

The maximum value is $xy = 2$ and minimum value is $xy = -2$.

2. Find the maximum value of $x^2 y^3 z^4$, subject to the condition $x + y + z = 5$.

Sol: Let $f(x,y,z) = x^2 y^3 z^4$

and $\emptyset(x,y,z) = x + y + z - 5 = 0$ _____(1)

Let the equation be $x^2 y^3 z^4 + \lambda(x + y + z - 5) = 0$ _____(2)

Differentiating Eq.(2) partially w.r.t x,

$$2xy^3z^4 + \lambda = 0$$

$\lambda = -2xy^3z^4$ _____(3)

Differentiating Eq.(2) partially w.r.t y,

$$3x^2y^2z^4 + \lambda = 0$$

$\lambda = -3x^2y^2z^4$ _____(4)

Differentiating Eq.(3) partially w.r.t z,

$$4x^2y^3z^3 + \lambda = 0$$

$\lambda = -4x^2y^3z^3$ _____(5)

From Eqs.(3), (4),(5),

$$-2xy^3z^4 = -3x^2y^2z^4 = -4x^2y^3z^3$$

$$\Rightarrow 2yz = 3xz = 4xy$$

$\Rightarrow y = \dfrac{3}{2}x$ and $z = 2x$

Substituting y and z in Eq.(1),

$$x + \frac{3}{2}x + 2x = 5$$

$$\Rightarrow 9x = 10$$

$$\Rightarrow x = \frac{10}{9}$$

$$y = \frac{3}{2}\left(\frac{10}{9}\right) = \frac{5}{3}$$

$$z = 2\left(\frac{10}{9}\right) = \frac{20}{9}$$

Maximum value of $x^2y^3z^4 = \left(\frac{10}{9}\right)^2 \left(\frac{5}{3}\right)^3 \left(\frac{20}{9}\right)^4 = \dfrac{(2^{10})(5^9)}{3^{15}}$.

3.Using Lagrange's multiplier method find shortest and longest distance from the point (1,2,-1) on the sphere $x^2 + y^2 + z^2 = 24$

Sol: Let $f(x, y, z) = (x-1)^2 + (y-2)^2 + (z+1)^2$

and $\emptyset(x, y, z) = x^2 + y^2 + z^2 - 24 = 0$ _____(1)

Let the equation be $(x-1)^2 + (y-2)^2 + (z+1)^2 + \lambda(x^2 + y^2 + z^2 - 24) = 0$ ___(2)

Differentiating Eq.(2) partially w.r.t x,

$$2(x-1) + 2\lambda x = 0$$

$$\lambda = -1 + \frac{1}{x} \underline{\hspace{2cm}}(3)$$

Differentiating Eq.(2) partially w.r.t y,

$2(y-2) + 2\lambda y = 0$

$$\lambda = -1 + \frac{2}{y} \underline{\hspace{2cm}}(4)$$

Differentiating Eq.(3) partially w.r.t z,

$2(z+1) + 2\lambda z = 0$

$$\lambda = -1 + \frac{2}{z} \underline{\hspace{2cm}}(5)$$

From Eqs.(3), (4),(5),

$$-1 + \frac{1}{x} = -1 + \frac{2}{y} = -1 + \frac{2}{z}$$

$$\Rightarrow \frac{1}{x} = \frac{2}{y} = \frac{2}{z}$$

$\Rightarrow y = 2x$ and $z = 2x$

Substituting y and z in Eq.(1),

$$x^2 + (2x)^2 + (2x)^2 = 24$$

$$\Rightarrow 9x^2 = 24$$

$$\Rightarrow x^2 = \frac{24}{9}$$

$$\Rightarrow x = \pm\frac{2\sqrt{6}}{3}$$

$$y = 2\left(\frac{2\sqrt{6}}{3}\right) = \pm\frac{4\sqrt{6}}{3}$$

$$z = 2\left(\frac{2\sqrt{6}}{3}\right) = \pm\frac{4\sqrt{6}}{3}$$

Minimum value of $\sqrt{\left(\frac{2\sqrt{6}}{3} - 1\right)^2 + \left(\frac{4\sqrt{6}}{3} - 2\right)^2 + \left(\frac{4\sqrt{6}}{3} + 1\right)^2} = 20.20$

Maximum value of $\sqrt{\left(-\frac{2\sqrt{6}}{3} - 1\right)^2 + \left(-\frac{4\sqrt{6}}{3} - 2\right)^2 + \left(-\frac{4\sqrt{6}}{3} + 1\right)^2} = 39.79$

Exercise:

1. Find the maximum and minimum values of the function $f(x,y) = 3x + 4y$ on the circle $x^2 + y^2 = 1$ using the method of Lagrange's multipliers.

2. A soldier placed at a point (3,4) wants to shoot the fighter plane of an enemy which is flying along

the curve $y = x^2 + 4$ when it is nearest to him. Find such distance.

3. Find the minimum value of x^2yz^3subject to the condition $2x + y + 3z = 1$ Using Lagrange's

multiplier method.   **[Summer 2022]**

<u>Taylor's Formula for $f(x,y)$ at the point (a,b)</u>

Suppose $f(x,y)$ and its partial derivative are continuous throughout an open rectangular

region R centered at a point (a,b).Then, throughout R ,

$$f(x,y) = f(a,b) + xf_x(a,b) + yf_y(a,b) + \frac{1}{2!}\left(x^2f_{xx}(a,b) + 2xyf_{xy}(a,b) + y^2f_{yy}(a,b)\right)$$

$$+ \frac{1}{3!}\left(x^3f_{xxx}(a,b) + 3x^2yf_{xxy}(a,b) + 3xy^2f_{xyy}(a,b) + y^3f_{yyy}(a,b)\right) + \cdots$$

<u>Taylor's Formula for $f(x,y)$ at origin (Also known as Maclaurin's series)</u>

$$f(x,y) = f(0,0) + xf_x(0,0) + yf_y(0,0) + \frac{1}{2!}\left(x^2f_{xx}(0,0) + 2xyf_{xy}(0,0) + y^2f_{yy}(0,0)\right)$$

$$+ \frac{1}{3!}\left(x^3f_{xxx}(0,0) + 3x^2yf_{xxy}(0,0) + 3xy^2f_{xyy}(0,0) + y^3f_{yyy}(0,0)\right) + \cdots$$

<u>Note:</u> For quadratic expansion, find the taylor's series upto second degree terms and for cubic

expansion , find the taylor's series upto third degree terms.

<u>Solved Examples:</u>

1. Expand $x^2y + 3y - 2$ in powers of $(x - 1)$and $(y + 2)$ upto second degree terms.

   Sol: Let $f(x,y) = x^2y + 3y - 2$

By Taylor's Expansion,

$$f(x,y) = f(a,b) + \left[(x - a)f_x(a,b) + (y - b)f_y(a,b)\right]$$
$$+ \frac{1}{2!}\left[(x - a)^2f_{xx}(a,b) + \quad 2(x - a)(y - b)f_{xy}(a,b) + (y - b)^2f_{yy}(a,b)\right] + \cdots$$

Here, $a = 1, b = -2$

$f(x,y) = x^2y + 3y - 2$, $f(1,-2) = (1)^2(-2) + 3(-2) - 2 = -10$

$\qquad f_x(x,y) = 2xy$, $f_x(1,-2) = 2(1)(-2) = -4$

$\qquad f_y(x,y) = x^2 + 3$, $f_y(1,-2) = (1)^2 + 3 = 4$

$\qquad f_{xx}(x,y) = 2y$, $f_{xx}(1,-2) = 2(-2) = -4$

$\qquad f_{xy}(x,y) = 2x$, $f_{xy}(1,-2) = 2(1) = 2$

$\qquad f_{yy}(x,y) = 0$, $f_{yy}(1,-2) = 0$

Substituting these values in Taylor's Expansion,

$$f(x,y) = -10 + [(x - 1)(-4) + (y + 2)4]$$

$$+\frac{1}{2!}[(x-1)^2(-4) + 2(x-1)(y+2)(2) + (y+2)^2(0)]+\cdots$$

$$x^2y + 3y - 2 = -10 - 4(x-1) + 4(y+2) - 2(x-1)^2 + 2(x-1)(y+2) + \cdots$$

2. Expand $e^x \log (1+y)$ in powers of x and y upto third degree.

Sol: Let $f(x,y) = e^x \log (1+y)$

By Maclaurin's series,

$$f(x,y) = f(0,0) + [xf_x(0,0) + yf_y(0,0)] + \frac{1}{2!}[(x)^2 f_{xx}(0,0) + 2xyf_{xy}(0,0) + (y)^2 f_{yy}(0,0)]$$

$$+\frac{1}{3!}[(x)^3 f_{xxx}(0,0) + 3x^2 y f_{xxy}(0,0) 3xy^2 f_{xyy}(0,0) + (y)^3 f_{yyy}(0,0)]+\cdots$$

$f(x,y) = e^x \log \log (1+y)$,	$f(0,0) = 0$
$f_x(x,y) = e^x \log \log (1+y)$,	$f_x(0,0) = 0$
$f_y(x,y) = \dfrac{e^x}{1+y}$,	$f_y(0,0) = 1$
$f_{xx}(x,y) = e^x \log \log (1+y)$,	$f_{xx}(0,0) = 0$
$f_{xy}(x,y) = \dfrac{e^x}{1+y}$,	$f_{xy}(0,0) = 1$
$f_{yy}(x,y) = -\dfrac{e^x}{(1+y)^2}$,	$f_{yy}(0,0) = -1$
$f_{xxx}(x,y) = e^x \log \log (1+y)$,	$f_{xxx}(0,0) = 0$
$f_{xxy}(x,y) = \dfrac{e^x}{1+y}$,	$f_{xxy}(0,0) = 1$
$f_{xyy}(x,y) = -\dfrac{e^x}{(1+y)^2}$,	$f_{xxy}(0,0) = -1$
$f_{yyy}(x,y) = \dfrac{2e^x}{(1+y)^3}$,	$f_{yyy}(0,0) = 2$

Substituting these values in Taylor's series,

$$f(x,y) = 0 + [x(0) + y(1)] + \frac{1}{2!}[x^2(0) + 2xy(1) + y^2(-1)]$$

$$+\frac{1}{3!}[x^3(0) + 3x^2y(1) + 3xy^2(-1) + y^3(2)] + \cdots$$

$$e^x \log \log (1+y) = y + \frac{1}{2!}(2xy - y^2) + \frac{1}{3!}(3x^2y - 3xy^2 + 2y^3) + \cdots$$

$$= y + xy - \frac{y^2}{2} + \frac{x^2y}{2} - \frac{xy^2}{2} + \frac{y^3}{3} - \cdots$$

Exercise:

1. Expand $x^2 + xy + y^2$ in powers of $(x-1)$ and $(y-2)$ upto second-degree terms.

2. Expand $e^x \cos y$ in powers of x and y upto third degree.