Parul University

Faculty of Engineering & Technology

Department of Applied Sciences and Humanities

1st Year B. Tech Programme (All Branches)

Mathematics – 1 (303191101)

Unit – 1 Improper Integral (Lecture Note)

OVERVIEW:

In most of applications of Engineering and Science there occurs special function, like gamma function, beta function etc, which are in the form of integrals which are of special types in which the **limits of integration are infinity** or **the integrand becomes unbounded within the limits**. Such types of integrals are known as **improper integrals**. Beta and gamma functions are very fundamental and hold great importance in various branches of Engineering and physics.

❖ **Course Outcome** :

- Identify various types of Improper Integration.
- Add together infinitely many numbers.
- Represent a differentiable function f(x) as an infinite sum of powers of x.
- Decide on convergence or divergence of a wide class of series.
- To answer at least about the convergence or divergence of integral when integral is not easily evaluated using techniques known.

❖ **Improper integrals:** The integral $\int_a^b f(x)dx$, is called improper integral if

i) one or both limits of integration are infinite.

ii) Function f(x) becomes infinite at a point within or at the end points of the interval of integration.

Examples:

1. $\int_1^\infty xe^{-x}dx$,is an improper integral due to infinite limit.

2. $\int_0^3 \frac{e^{-x}}{\sqrt{x}}dx$,is an improper integral as the integrand tends to ∞ as $x \to 0$.

3. $\int_{-1}^4 \frac{1}{x-1}dx$,is an improper integral as the integrand is unbounded as $x \to 1$.

4. $\int_0^5 xe^{3x}dx$,is a proper integral.

- ❖ **Improper integrals are classified into three kinds.**
  - Type –I
  - Type –II
  - Type –III
- ❖ **<u>Improper integrals of the first kind(Type-I):</u>**

If in the definite integral $\int_a^b f(x)dx$,a or b or both a and b are infinite, then the integral is called improper integral of Type-I.

(1) If f(x) is continuous on [a, ∞), then $\int_a^{\infty} f(x)dx = \lim_{b\to\infty} \int_a^b f(x)dx$

(2) If f(x) is continuous on (−∞, b], then $\int_{-\infty}^{b} f(x)dx = \lim_{a\to-\infty} \int_a^b f(x)dx$

(3) If f(x) is continuous on (−∞, ∞), then $\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{0} f(x)dx+\int_0^{\infty} f(x)dx$,

Or

$$\int_{-\infty}^{\infty} f(x)dx = \lim_{a\to-\infty} \int_a^0 f(x)dx + \lim_{b\to\infty} \int_0^b f(x)dx$$

Evaluate 1)

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$$
$$= \int_{-\infty}^{0} \frac{dx}{1+x^2} + \int_0^{\infty} \frac{dx}{1+x^2}$$
$$= \lim_{a\to-\infty} \int_a^0 \frac{1}{1+x^2}dx + \lim_{b\to\infty} \int_0^b \frac{1}{1+x^2}dx$$
$$= \lim_{a\to-\infty} [\tan^{-1} x]_a^0 + \lim_{b\to\infty} [\tan^{-1} x]_0^b$$
$$= \lim_{a\to-\infty} [\tan^{-1} 0 - \tan^{-1} a] + \lim_{b\to\infty} [\tan^{-1} b - \tan^{-1} 0]$$
$$= 0 - \tan^{-1}(-\infty) + \tan^{-1}(\infty) - 0$$
$$= \frac{\pi}{2} + \frac{\pi}{2}$$
$$= \pi$$

Evaluate 2) $\int_0^{\infty} \frac{dx}{1+x^2}$

Evaluate 3) $\int_1^{\infty} \frac{1}{x^2} dx$

Evaluate 4) $\int_{-\infty}^{0} x\sin x dx$

Evaluate 5) $\int_1^\infty \frac{1}{\sqrt{x}} dx$

$$= \lim_{a\to\infty} \int_1^a \frac{1}{\sqrt{x}} dx$$
$$= \lim_{a\to\infty} \left[2\sqrt{x}\right]_1^a$$
$$= \lim_{a\to\infty} \left[2\sqrt{a} - 2\right]$$
$$\to \infty$$

Note: P integral $\int_1^\infty \frac{1}{x^p} dx$, converges when p >1 and diverges when p$\leq$ 1

Evaluate 6) Show that $\int_1^\infty \frac{logx}{x^2}$ Converges and obtain it value.

<u>EXERCISE:</u>

Evaluate:

1. $\int_2^\infty \frac{(x+3)}{(x-1)(x^2+1)} dx$
2. $\int_{-\infty}^\infty e^x dx$
3. $\int_0^\infty x^2 \, e^{-x} dx$
4. $\int_1^\infty \frac{e^{-\sqrt{x}}}{\sqrt{x}} dx$

❖ **<u>Improper Integrals of the second type (Type-II):</u>**

If in the definite integral $\int_a^b f(x)dx$, the integrand f(x) becomes infinite at x=a or x=b or at one or more points within the interval (a, b), then the integral is called improper integral of Type-II.

(1) If f(x) is unbounded at x=a then

$$\int_a^b f(x)dx = \lim_{c\to a} \int_c^b f(x)dx$$

(2) If f(x) is unbounded at x=b then

$$\int_a^b f(x)dx = \lim_{c\to b} \int_a^c f(x)dx$$

(3) If f(x) is unbounded at x=aand x=b then

$$\int_a^b f(x)dx = \lim_{c1\to a} \int_{c1}^0 f(x)dx + \lim_{c2\to b} \int_0^{c2} f(x)dx$$

The improper integral is said to converge (or exist) when limit in R.H. S. of 1), 2) ,3) exist or finite. Otherwise, it is said to diverge.

Evaluate 1)

$$\int_0^3 \frac{1}{\sqrt{3-x}}$$
$$= \lim_{a\to 3} \int_0^a \frac{1}{\sqrt{3-x}} dx$$
$$= \lim_{a\to 3} \left[-2\sqrt{3-x}\right]_0^a$$
$$= \lim_{a\to 3} \left[-2\sqrt{3-a} + 2\sqrt{3}\right]$$
$$= 2\sqrt{3}$$

Evaluate 2) $\int_0^{\frac{\pi}{2}} secx dx$

Evaluate 3) $\int_0^5 \frac{1}{(x-2)^2} dx$

Evaluate 4) $\int_{-1}^1 \frac{1}{x^{\frac{2}{3}}} dx$

Improper Integral of third kind (Type-III): It is a definite integral in which one or both limits of integration are infinite, and the integrand become infinite at one or more points within or at the end points of the interval of integration. Thus it is a combination of the first kind and the second kind.

For example: $\int_0^\infty \frac{1}{x^2} dx$ is an improper integral of the third kind as the upper limit of integration is infinite and integrand $\frac{1}{x^2}$ is infinite at x = 0.

Evaluate: $\int_0^\infty \frac{1}{x^2} dx$

Gamma function

The function of n (n>0) defined by the integral $\int_0^\infty e^{-x} x^{n-1} dx$ is called gamma function and is denoted by Γn i.e $\Gamma n = \int_0^\infty e^{-x} x^{n-1} dx$.

❖ **Properties of gamma function:**

1) $\Gamma n+1 = n\Gamma n$
2) $\Gamma 1 = 1$
3) $\Gamma n+1 = n!$ N is a positive integer.
4) $\Gamma n = 2\int_0^\infty e^{-x^2} x^{2n-1} dx$.

❖ **Examples**

Evaluate 1) $\int_0^\infty e^{-\sqrt{x}} \sqrt[4]{x} dx$

assume $\sqrt{x} = t$

$$\frac{1}{2\sqrt{x}} dx = dt$$
$$dx = 2t\,dt$$

$$\int_0^\infty e^{-t}\left(t^2\right)^{1/4} dt = \int_0^\infty e^{-t}(t)^{1/2} dt$$
$$= \int_0^\infty e^{-t}(t)^{3/2-1} dt$$
$$= \Gamma\frac{3}{2} = \frac{1}{2}\sqrt{\pi}$$

$assume\ \sqrt{x} = t$

$$\frac{1}{2\sqrt{x}}\,dx = dt$$

$$dx = 2t\,dt$$

$$\int_0^\infty e^{-t}\left(t^2\right)^{1/4} dt = \int_0^\infty e^{-t}\,(t)^{1/2}\,dt$$

$$= \int_0^\infty e^{-t}\,(t)^{3/2-1}\,dt$$

$$= \overline{\big)}\frac{3}{2} = \frac{1}{2}\sqrt{\pi}$$

Evaluate 2) $\int_0^\infty \sqrt{x}e^{-\sqrt[3]{x}}\,dx$

Evaluate 3) $\int_0^\infty x^4\, e^{-x^2}dx$

EXERCISE:

1) $\int_0^\infty e^{-x^4}\,dx$

2) $\displaystyle\int_0^\infty e^{-x^2}\,\sqrt{x^3}dx$

3) $\int_0^\infty 5^{-4x^2}\,dx$

Beta function

The function of m and n defined by the integral (m, n>0) $\int_0^1 x^{m-1}\,(1-x)^{n-1}dx$ is called the Beta function and is denoted by $\beta(m,n)$.

i.e$\beta(m,n) = \int_0^1 x^{m-1}\,(1-x)^{n-1}dx$

Properties of Beta Function:

1) $\boldsymbol{\beta(m,n) = \beta(n,m)}$
2) $\boldsymbol{\beta(m,n) = 2\int_0^{\pi/2} sin^{2m-1}\theta cos^{2n-1}\theta d\theta}$

Examples

Evaluate 1)$\int_0^3 (3-\mathrm{x})^{(\frac{1}{2})}\,\mathrm{x}^{\frac{5}{2}}\mathrm{dx}$

$assume\ x = 3t$

$dx = 3dt$

$x = 0 \Rightarrow t = 0$

$x = 3 \Rightarrow t = 1$

$$\int_0^1 (3-3t)^{1/2}(3t)^{5/2}\,3\,dt$$

$$= 3^4 \int_0^1 (1-t)^{1/2}(t)^{5/2}\,dt$$

$$= 3^4 \int_0^1 (1-t)^{3/2-1}(t)^{7/2-1}\,dt$$

$$= 3^4 \beta\left(3/2, 7/2\right)$$

Evaluate 2) $\int_0^1 \frac{xdx}{\sqrt{1-x^5}}$

Evaluate 3) $\int_0^{2a} x^2\sqrt{2ax - x^2}\,dx$

❖ **EXERCISE:**

1) $\int_0^{\frac{\pi}{4}} \cos^3 2x \sin^4 4x\, dx$

2) $\int_0^{\infty} \frac{x^{m-1}}{(a+bx)^{m+n}}\, dx$

3) Prove that $\int_0^{\infty} \frac{x^8(1-x^6)}{(1+x)^{24}}\, dx$

Properties of Beta and Gamma Function:

1) $\beta(m, n) = \frac{\Gamma m \Gamma n}{\Gamma m+n}$

2) $\frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right) = \int_0^{\pi/2} sin^p\theta cos^q\ \theta d\theta$

3) $\int_0^{\pi/2} sin^p\theta cos^q\theta d\theta = \frac{1}{2}\frac{\Gamma\left(\frac{p+1}{2}\right)\ \Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{p+q+2}{2}\right)}$

4) $\Gamma\left(\frac{1}{2}\right) = \sqrt{\Pi}$

Examples

Evaluate 1) $\int_3^7 \sqrt[4]{(x-3)(7-x)}\ \text{dx}$

Evaluate 2) $\int_0^{\pi/2} \frac{d\theta}{\sqrt{sin\theta}} \int_0^{\frac{\pi}{2}} \sqrt{sin\theta}\, d\theta$

Evaluate 3) $\int_0^{\pi/2} \frac{d\theta}{\sqrt{1-\frac{1}{2}\sin^2\theta}}$

PARUL UNIVERSITY
Faculty of Engineering & Technology
Department of Applied Sciences & Humanities
1st year B.Tech Programme (All branches)
Mathematics-II (Subject Code: 203191102)
UNIT-2 First Order Differential Equation

- **Differential Equation:**
Ordinary differential equation

An ordinary differential equation is an equation which contains derivatives of a dependent variable, $y(x)$, w.r.t. only one independent variable.

For example:

1. $\frac{dy}{dx} = \cos x$
2. $\frac{d^2y}{dx^2} + 4y = 0$
3. $x^2 \frac{d^3y}{dx^3} + 2e^x \frac{d^2y}{dx^2} = (x^2+2)y^2$

Partial differential equation

A partial deferential equation is an equation which contains partial derivatives of a dependent variables $f(x, y)$, w.r.t. two or more independent variables.

For example:

1. $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$
2. $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$
3. $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

- **Order**
The order of a differential equation is the order of the highest derivative occurring in that equation.

- **Degree**

The degree of a differential equation is the highest index of the highest order derivative.

- **Examples:**

Sr. No	Differential equation	Order	Degree
1.	$\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^3+y=0$	**2**	**1**
2.	$3\left(\frac{d^3y}{dx^3}\right)^3+\left(\frac{d^2y}{dx^2}\right)+\frac{dy}{dx}=e^{-x}\sin x$	**3**	**3**
3.	$\frac{\left(1+\left(\frac{dy}{dx}\right)^2\right)^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}=1$ **OR** $\left(1+\left(\frac{dy}{dx}\right)^2\right)^3=\left(\frac{d^2y}{dx^2}\right)^2$	**2**	**2**

Initial Value Problem: A particular solution can be obtained from a general solution by an initial condition $y(x_0)=y_0$ which determines the value of the arbitrary constant c. An ordinary differential equation with initial condition is known as **Initial Value Problem.**

$$y'=f(x,y) \qquad y(x_0)=y_0$$

Exact Differential Equation: A first order differential equation of the form

$$M(x,y)dx+N(x,y)dy=0$$

is called an exact differential equation if

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Therefore, the solution of the exact differential equation is

$$\int_{y\,is\,const} M(x,y)dx+\int_{terms\,containing\,only\,y} N(x,y)dy=c$$

For example:

1. Solve $(x^3+3xy^2)dx+(3x^2y+y^3)dy=0$ Solution: $(x^3+3xy^2)dx+(3x^2y+y^3)dy=0$ $M=(x^3+3xy^2)$ and $N=(3x^2y+y^3)$ $\frac{\partial M}{\partial y}=6xy$ and $\frac{\partial N}{\partial x}=6xy$ Since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, the equation is exact. Therefore the general solution is	2. Solve $\frac{dy}{dx}+\frac{y\cos x+\sin y+y}{\sin x+x\cos y+x}=0$ Solution: $\frac{dy}{dx}+\frac{y\cos x+\sin y+y}{\sin x+x\cos y+x}=0$ $(y\cos x+\sin y+y)dx+(\sin x+x\cos y+x)dy=0$ $M=(y\cos x+\sin y+y)$ and $N=(\sin x+x\cos y+x)$

$$\int_{y\,is\,const} M(x,y)dx + \int_{terms\,containing\,only\,y} N(x,y)dy = c$$ $$i.e. \int (x^3 + 3xy^2)dx + \int y^3 dy = c$$ $$\therefore \frac{x^4}{4} + 3y^2 \frac{x^2}{2} + \frac{y^4}{4} = c$$	$$\frac{\partial M}{\partial y} = \cos x + \cos y + 1$$ and $$\frac{\partial N}{\partial y} = \cos x + \cos y + 1$$ Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the equation is exact. Therefore the general solution is $$\int_{y\,is\,const} M(x,y)dx + \int_{terms\,containing\,only\,y} N(x,y)dy = c$$ $$i.e. \int (y\cos x + \sin y + y)dx + \int 0dy = c$$ $$\therefore y\sin x + x\sin y + xy = c$$
3.Solve $((x+1)e^x - e^y)dx - xe^y dy = 0, \qquad y(1) = 0$ Solution: $((x+1)e^x - e^y)dx - xe^y dy = 0$ $$M = ((x+1)e^x - e^y) \quad \text{and} \quad N = -xe^y$$ $$\frac{\partial M}{\partial y} = -e^y \quad \text{and} \quad \frac{\partial N}{\partial y} = -e^y$$ Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the equation is exact. Therefore the general solution is $$\int_{y\,is\,const} M(x,y)dx + \int_{terms\,containing\,only\,y} N(x,y)dy = c$$ $$i.e. \int ((x+1)e^x - e^y)dx + \int 0dy = c$$ $$\therefore (x+1)e^x - e^x - xe^y = c$$ Now given that y(1)=0 Therefore substituting x=1, y=0 in the solution $$e - 1 = c$$ Therefore the solution is $$\therefore (x+1)e^x - e^x - xe^y = e - 1$$	**Examples:** 1.Solve $(2xy + e^y)dx + (x^2 + xe^y)dy = 0, \qquad y(1) = 1$ 2.Solve $ye^x dx + (2y + e^x)dy = 0, \qquad y(0) = -1$ 3.Solve $\frac{dy}{dx} = \frac{y+1}{e^y(y+2) - x}$

Non exact Differential Equation OR Reducible to exact diff. Equation:

If $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ then the given equation is not exact.

Therefore, by multiplying the given equation with integrating factor reduces it to exact. There are four cases for finding the integrating factor.

Case-:1

If the given differential equation is homogeneous with $Mx + Ny \neq 0$ then

I.F =

Case-2:

If the given differential equation is of the form $f(x,y)ydx + g(x,y)xdy = 0$ with $Mx - Ny \neq 0$, then

I.F =

Case-3:

If $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)$ is a function of x alone, say f(x), then

I.F =

Case-4:

If $\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$ is a function of y alone, say g(y), then

I.F =

For examples:

1. Solve $(xy - 2y^2)dx - (x^2 - 3xy)dy = 0$

Solution: $(xy - 2y^2)dx - (x^2 - 3xy)dy = 0$

$M = (xy - 2y^2)$ and

$N = (x^2 - 3xy)$

$\frac{\partial M}{\partial y} = x - 4y$ and

$\frac{\partial N}{\partial x} = 2x - 3y$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, the equation is not exact.

Therefore

$Mx + Ny = x^2y - 2xy^2 - x^2y + 3xy^2 = xy^2 \neq 0$

$$\therefore I.F = \frac{1}{Mx + Ny} = \frac{1}{xy^2}$$

Multiplying throughout by I.F, the equation becomes

2. Solve $\left(x^4 + y^4\right) dx - xy^3 \ dy = 0$

$M = x^4 + y^4$ $\quad N = -xy^3$

$\frac{\partial M}{\partial y} = 4y^3$ $\quad \frac{\partial N}{\partial x} = -y^3$

$$\therefore I.F. = \frac{1}{Mx + Ny} = \frac{1}{x^5 + xy^4 - xy^4} = \frac{1}{x^5}$$

$$\left(\frac{1}{x} + \frac{y^4}{x^5}\right)dx - \frac{y^3}{x^4}dy = 0$$

$$\int\left(\frac{1}{x} + \frac{y^4}{x^5}\right)dx = c$$

$$\ln x - \frac{1}{4}\left(\frac{y}{x}\right)^4 = c$$

3. Solve $(x^2y^2 + 2)ydx + (2 - x^2y^2)xdy = 0$

Solution: $(x^2y^2 + 2)ydx + (2 - x^2y^2)xdy = 0$

$M = (x^2y^2 + 2)y$ and $N = (2 - x^2y^2)x$

$$\frac{(xy-2y^2)}{xy^2}dx-\frac{(x^2-3xy)}{xy^2}dy=0$$

$$\therefore\left(\frac{1}{y}-\frac{2}{x}\right)dx+\left(-\frac{x}{y^2}+\frac{3}{y}\right)dy=0$$

$$\therefore M'=\left(\frac{1}{y}-\frac{2}{x}\right) \qquad N'=\left(-\frac{x}{y^2}+\frac{3}{y}\right)$$

$$\therefore\frac{\partial M'}{\partial y}=-\frac{1}{y^2} \qquad \frac{\partial N'}{\partial x}=-\frac{1}{y^2}$$

$$\therefore\frac{\partial M'}{\partial y}=\frac{\partial N'}{\partial x}$$

which is exact.
Therefore the solution is

$$\int_{y\,is\,const} M'(x,y)dx+\int_{terms\,containing\,only\,y} N'(x,y)dy=c$$

$$i.e.\int\left(\frac{1}{y}-\frac{2}{x}\right)\;dx+\int\frac{3}{y}dy=c$$

$$\therefore\frac{x}{y}-2\log x+3\log y=c$$

$$\frac{\partial M}{\partial y}=3x^2y^2+2$$

and

$$\frac{\partial N}{\partial x}=2-3x^2y^2$$

Since $\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$, the equation is not exact.
Therefore

$$Mx-Ny=x^3y^3+2xy-2xy+x^3y^3=2x^3y^3\neq 0$$

$$\therefore I.F=\frac{1}{Mx-Ny}=\frac{1}{2x^3y^3}$$

Multiplying throughout by I.F, the equation becomes

$$\frac{(x^2y^2+2)y}{2x^3y^3}dx+\frac{(2-x^2y^2)x}{2x^3y^3}dy=0$$

$$\therefore\frac{1}{2}\left(\frac{1}{x}+\frac{2}{x^3y^2}\right)dx+\frac{1}{2}\left(\frac{2}{x^2y^3}-\frac{1}{y}\right)dy=0$$

$$\therefore M'=\frac{1}{2}\left(\frac{1}{x}+\frac{2}{x^3y^2}\right) \qquad N'=\frac{1}{2}\left(\frac{2}{x^2y^3}-\frac{1}{y}\right)$$

$$\therefore\frac{\partial M'}{\partial y}=-\frac{2}{x^3y^3} \qquad \frac{\partial N'}{\partial x}=-\frac{2}{x^3y^3}$$

$$\therefore\frac{\partial M'}{\partial y}=\frac{\partial N'}{\partial x}$$

which is exact.
Therefore the solution is

$$\int_{y\,is\,const} M'(x,y)dx+\int_{terms\,containing\,only\,y} N'(x,y)dy=c$$

$$i.e.\int\frac{1}{2}\left(\frac{1}{x}+\frac{2}{x^3y^2}\right)\;dx+\int-\frac{1}{y}dy=c$$

$$\therefore\frac{1}{2}\left(\log x-\frac{1}{x^2y^2}\right)-\frac{1}{2}\log y=c$$

$$\therefore\log x-\log y-\frac{1}{x^2y^2}=c$$

$$\therefore\log\left(\frac{x}{y}\right)-\frac{1}{x^2y^2}=c$$

3.Solve $(2x\log x-xy)dy+2ydx=0$

Solution: $2ydx+(2x\log x-xy)dy=0$

$$M=2y$$

and

$$N=(2x\log x-xy)$$

$$\frac{\partial M}{\partial y}=2$$

and

$$\frac{\partial N}{\partial x}=2\log x+2-y$$

Since $\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$, the equation is not exact.

4.Solve $\left(\frac{y}{x}\sec y-\tan y\right)dx+(\sec y\log x-x)dy=0$

Solution: $\left(\frac{y}{x}\sec y-\tan y\right)dx+(\sec y\log x-x)dy=0$

$$M=\left(\frac{y}{x}\sec y-\tan y\right)$$

and

$$N=(\sec y\log x-x)$$

Therefore

$$\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)=\frac{2-2\log x-2+y}{2x\log x-xy}$$

$$=-\frac{1}{x}=f(x)$$

$$\therefore I.F=e^{\int -\frac{1}{x}dx}=e^{-\log x}=\frac{1}{x}$$

Multiplying throughout by I.F, the equation becomes

$$\frac{1}{x}(2y)dx+\frac{1}{x}(2x\log x-xy)dy=0$$

$$\therefore \frac{2y}{x}dx+(2\log x-y)dy=0$$

$$\therefore M'=\frac{2y}{x} \qquad N'=2\log x-y$$

$$\therefore \frac{\partial M'}{\partial y}=\frac{2}{x} \qquad \frac{\partial N'}{\partial x}=\frac{2}{x}$$

$$\therefore \frac{\partial M'}{\partial y}=\frac{\partial N'}{\partial x}$$

which is exact.
Therefore the solution is

$$\int\limits_{y\, is\, const} M'(x,y)dx+\int\limits_{terms\, containing\, only\, y} N'(x,y)dy=c$$

$$i.e. \int\frac{2y}{x}dx+\int -ydy=c$$

$$\therefore 2y\log x-\frac{y^2}{2}=c$$

$$\frac{\partial M}{\partial y}=\frac{1}{x}\sec y+\frac{y}{x}\sec y\tan y-\sec^2 y$$

and $\dfrac{\partial N}{\partial x}=\dfrac{\sec y}{x}-1$

Since $\dfrac{\partial M}{\partial y}\neq\dfrac{\partial N}{\partial x}$, the equation is not exact.
Therefore

$$\frac{1}{M}\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)=\frac{\frac{\sec y}{x}-1-\frac{\sec y}{x}-\frac{y}{x}\sec y\tan y+\sec^2}{\frac{\sec y}{x}-\tan y}$$

$$=-\tan y=f(y)$$

$$\therefore I.F=e^{\int -\tan y dy}=e^{-\log\sec x}=\sec^{-1} y=\cos y$$

Multiplying throughout by I.F, the equation becomes

$$\cos y\left(\frac{y}{x}\sec y-\tan y\right)dx+\cos y(\sec y\log x-x)$$

$$\therefore \left(\frac{y}{x}-\sin y\right)dx+(\log x-x\cos y)dy=0$$

$$\therefore M'=\left(\frac{y}{x}-\sin y\right) \qquad N'=(\log x-x\cos y)$$

$$\therefore \frac{\partial M'}{\partial y}=\frac{1}{x}-\cos y \qquad \frac{\partial N'}{\partial x}=\frac{1}{x}-\cos y$$

$$\therefore \frac{\partial M'}{\partial y}=\frac{\partial N'}{\partial x}$$

which is exact.
Therefore the solution is

$$\int\limits_{y\, is\, const} M'(x,y)dx+\int\limits_{terms\, containing\, only\, y} N'(x,y)dy=c$$

$$i.e. \int\left(\frac{y}{x}-\sin y\right)dx+\int 0dy=c$$

$$\therefore y\log x-x\sin y=c$$

Example:

1. $(xy^3+y)dx+2(x^2y^2+x+y^4)dy=0$
2. $x^2ydx-(x^3+y^3)dy=0$
3. $(x^2+y^2+1)dx-2xydy=0$
4. $xe^x(dx-dy)+e^xdx+ye^ydy=0$

❖ Linear Differential Equation

A first order differential Equation of the form $\dfrac{dy}{dx}+P(x)y=Q(x)$ or $\dfrac{dx}{dy}+P(y)x=Q(y)$

is called Linear Differential Equation.

Differential equation	Integrating factor	General solution
$\frac{dy}{dx}+P(x)y=Q(x)$	$I.F=e^{\int P(x)dx}$	$y(I.F)=\int Q(x)(I.F)dx+c$
$\frac{dx}{dy}+P(y)x=Q(y)$	$I.F=e^{\int P(y)dy}$	$x(I.F)=\int Q(y)(I.F)dy+c$

For Example:

1.Solve $\frac{dy}{dx}+2y\tan x=\sin x$

Solution: $\frac{dy}{dx}+2y\tan x=\sin x$

The equation is linear equation

Therefore, comparing with $\frac{dy}{dx}+P(x)y=Q(x)$

$$P(x)=2\tan x, \qquad Q(x)=\sin x$$

$$IF=e^{\int P(x)dx}=e^{\int 2\tan x dx}=e^{\log\sec^2 x}=\sec^2 x$$

Therefore the general solution is

$$y(I.F)=\int Q(x)(I.F)dx+c$$

$$\therefore y(\sec^2 x)=\int \sin x\sec^2 x dx+c$$

$$\therefore y(\sec^2 x)=\int \tan x\sec x dx+c$$

$$\therefore y(\sec^2 x)=\sec x+c$$

2.Solve $(x+1)\frac{dy}{dx}-y=e^{3x}(x+1)^2$

Solution:

$$(x+1)\frac{dy}{dx}-y=e^{3x}(x+1)^2$$

$$\frac{dy}{dx}-\frac{y}{(x+1)}=e^{3x}(x+1)$$

The equation is linear equation

Therefore, comparing with

$$\frac{dy}{dx}+P(x)y=Q(x)$$

$$P(x)=-\frac{1}{x+1}, \qquad Q(x)=e^{3x}(x+1)$$

$$IF=e^{\int P(x)dx}=e^{\int -\frac{1}{x+1}dx}=e^{\log(x+1)^{-1}}=\frac{1}{x+1}$$

Therefore the general solution is

$$y(I.F)=\int Q(x)(I.F)dx+c$$

$$\therefore y\frac{1}{x+1}=\int e^{3x}(x+1)\frac{1}{x+1}dx+c$$

$$\therefore y\frac{1}{x+1}=\int e^{3x}dx+c$$

$$\therefore y\frac{1}{x+1}=\frac{e^{3x}}{3}+c$$

3.Solve $\frac{dy}{dx}+\frac{4x}{1+x^2}y=\frac{1}{(x^2+1)^3}$

Solution:

$$\frac{dy}{dx}+\frac{4x}{1+x^2}y=\frac{1}{(x^2+1)^3}$$

The equation is linear equation

4.Solve $y'+y\tan x=\sin 2x, \qquad y(0)=1$

Solution: $y'+y\tan x=\sin 2x$

The equation is linear equation

Therefore comparing with $\frac{dy}{dx}+P(x)y=Q(x)$

$$P(x)=\tan x, \qquad Q(x)=\sin 2x$$

<table>
<tr>
<td>

Therefore comparing with $\frac{dy}{dx}+P(x)y=Q(x)$

$$P(x)=\frac{4x}{1+x^2},\qquad Q(x)=\frac{1}{(1+x^2)^3}$$

$$IF=e^{\int P(x)dx}=e^{\int\frac{4x}{1+x^2}dx}=e^{\log(1+x^2)^2}=(1+x^2)^2$$

Therefore the general solution is

$$y(I.F)=\int Q(x)(I.F)dx+c$$

$$\therefore y(1+x^2)^2=\int\frac{1}{(1+x^2)^3}(1+x^2)^2dx+c$$

$$\therefore y(1+x^2)^2=\int\frac{1}{(1+x^2)}dx+c$$

$$\therefore y(1+x^2)^2=\tan^{-1}x+c$$

</td>
<td>

$$IF=e^{\int P(x)dx}=e^{\int\tan x dx}=e^{\log\sec x}=\sec x$$

Therefore the general solution is

$$y(I.F)=\int Q(x)(I.F)dx+c$$

$$\therefore y(\sec x)=\int\sin 2x\sec x dx+c$$

$$\therefore y(\sec x)=\int\sin x dx+c$$

$$\therefore y(\sec x)=-\cos x+c$$

Given $y(0)=1$

$$\therefore 1=-1+c$$

$$\therefore c=2$$

$$\therefore y(\sec x)=-\cos x+2$$

</td>
</tr>
</table>

For example:

1. Solve $\frac{dy}{dx}+\frac{1}{x^2}y=6e^{\frac{1}{x}}$
2. Solve $(1+y^2)dx=(\tan^{-1}y-x)dy$
3. Solve $\frac{dy}{dx}+\frac{3y}{x}=\frac{\sin x}{x^3}$

❖ Non Linear Differential Equation or Bernoulli Equation

The equation of the form

$$\frac{dy}{dx}+Py=Qy^n \qquad (1)$$

where P and Q are functions of x or constants is known as Bernoulli's Equation.

Dividing (1) by y^n

$$y^{-n}\frac{dy}{dx}+\frac{P}{y^{n-1}}=Q \qquad (2)$$

Put $\frac{1}{y^{n-1}}=v$

$$\frac{1-n}{y^n}\frac{dy}{dx}=\frac{dv}{dx}$$

$$\frac{1}{y^n}\frac{dy}{dx}=\frac{1}{1-n}\frac{dv}{dx}$$

Substituting in (2)

$$\frac{dv}{dx}+(1-n)Pv=Q$$

which is linear form.

For Example

1.Solve $x\frac{dy}{dx}+y=x^3y^6$	2.Solve $x\frac{dy}{dx}+y=y^2\log x$
Solution: $x\frac{dy}{dx}+y=x^3y^6$ Dividing both the sides by y^6 $y^{-6}x\frac{dy}{dx}+y^{-5}=x^3 \qquad (1)$ Taking $y^{-5}=v$ $\therefore -5y^{-6}\frac{dy}{dx}=\frac{dv}{dx}$ Therefore from (1) $\therefore \frac{dv}{dx}-\frac{5}{x}v=-5x^2$ $\therefore IF=e^{\int -\frac{5}{x}dx}=x^{-5}$ Therefore the general solution is $v(I.F)=\int Q(x)(I.F)dx+c$ $\therefore vx^{-5}=\int -5x^2x^{-5}dx+c$ $\therefore vx^{-5}=\int -5x^{-3}dx+c$ $\therefore vx^{-5}=\frac{-5x^{-2}}{-2}+c$ $\therefore y^{-5}x^{-5}=\frac{5x^{-2}}{2}+c$	Solution: $x\frac{dy}{dx}+y=y^2\log x$ Dividing both the sides by xy^2 $y^{-2}\frac{dy}{dx}+\frac{1}{xy}=\frac{\log x}{x} \qquad (1)$ Taking $y^{-1}=v$ $\therefore -y^{-2}\frac{dy}{dx}=\frac{dv}{dx}$ Therefore from (1) $\therefore \frac{dv}{dx}-\frac{1}{x}v=-\frac{\log x}{x}$ $\therefore IF=e^{\int -\frac{1}{x}dx}=x^{-1}=\frac{1}{x}$ Therefore the general solution is $v(I.F)=\int Q(x)(I.F)dx+c$ $\therefore v\frac{1}{x}=\int\left(-\frac{\log x}{x}\right)\frac{1}{x}dx+c$ $\therefore v\frac{1}{x}=-\int \log x\frac{1}{x^2}dx+c$ $\therefore y^{-1}x^{-1}=\frac{1}{x}\log x+\frac{1}{x}+c$

2. Solve $x\dfrac{dy}{dx}=4x^3y^2+y \qquad y(0)=2$ Solution: $x\dfrac{dy}{dx}=4x^3y^2+y$ $\therefore \dfrac{dy}{dx}-\dfrac{1}{x}y=4x^2y^2$ Dividing both the sides by y^2 $y^{-2}\dfrac{dy}{dx}-\dfrac{1}{x}y^{-1}=4x^2 \qquad (1)$ Taking $y^{-1}=v$ $\therefore y^{-1}\dfrac{dy}{dx}=-\dfrac{dv}{dx}$ Therefore from (1) $\therefore \dfrac{dv}{dx}+\dfrac{v}{x}=-4x^2$ $\therefore IF=e^{\int \frac{1}{x}dx}=x$ Therefore the general solution is $v(I.F)=\int Q(x)(I.F)dx+c$ $\therefore vx=\int -4x^2xdx+c$ $\therefore vx^{-5}=\int -4x^3dx+c$ $\therefore vx=\dfrac{-4x^4}{4}+c$ $\therefore vx=-x^4+c$ $\therefore \dfrac{x}{y}=-x^4+c$ $\because y(0)=2$ $\therefore 0=0+c \Rightarrow c=0$ $\therefore$ General Solution is $\dfrac{x}{y}=-4x^2$	3. Solve $\dfrac{dy}{dx}+\dfrac{1}{x}=\dfrac{e^y}{x^2}$ Solution: $\dfrac{dy}{dx}+\dfrac{1}{x}=\dfrac{e^y}{x^2}$ Dividing both the sides by e^y $e^{-y}\dfrac{dy}{dx}+\dfrac{e^{-y}}{x}=\dfrac{1}{x^2} \qquad (1)$ Taking $e^{-y}=v$ $\therefore e^{-y}\dfrac{dy}{dx}=-\dfrac{dv}{dx}$ Therefore from (1) $\therefore \dfrac{dv}{dx}-\dfrac{v}{x}=-\dfrac{1}{x^2}$ $\therefore IF=e^{\int -\frac{1}{x}dx}=\dfrac{1}{x}$ Therefore the general solution is $v(I.F)=\int Q(x)(I.F)dx+c$ $\therefore v\dfrac{1}{x}=\int -\dfrac{1}{x^2}\dfrac{1}{x}dx+c$ $\therefore v\dfrac{1}{x}=\int -x^{-3}dx+c$ $\therefore v\dfrac{1}{x}=\dfrac{x^{-2}}{2}+c$ $\therefore v\dfrac{1}{x}=x^{-2}+c$ $\therefore \dfrac{e^{-y}}{x}=x^{-2}+c$

Examples:

1. Solve $\dfrac{dy}{dx}+y\tan x=y^3\sec x$
2. Solve $y^4dx=\left(x^{-\frac{3}{4}}-y^3x\right)dy$

<u>Differential equations of first order but not of first degree:</u>

We shall study first order differential equation of higher degree. We shall denote the derivative $\frac{dy}{dx} = p$. For a given differential equation of first order but of higher degree, three cases may arises.

Case1- First-Order Equations of Higher Degree Solvable for p

Let F(x, y, p) = 0 can be solved for p and can be written as

$(p-q_1(x,y))\ (p-q_2(x,y))\ \ldots\ldots\ldots\ (p-q_n(x,y)) = 0$

Equating each factor to zero we get equations of the first order and first degree.

One can find solutions of these equations by the methods discussed in the previous chapter. Let their solution be given as:

$\square_i(x,y,c_i)=0,\ i=1,2,3\ \ldots\ldots\ldots n$ (1)

Therefore the general solution can be expressed in the form

$\square_1(x,y,c_1)\ \square_2(x,y,c_2)\ldots\ldots\ldots\square_n(x,y,c) = 0$ (2)

where c in any arbitrary constant.

Example 1 **Solve** $xy\left(\frac{dy}{dx}\right)^2 + (x^2+y^2)\frac{dy}{dx} + xy = 0$ **(1)**

Solution: This is first-order differential equation of degree 2. Let $p = \frac{dy}{dx}$

Equation (1) can be written as

$xy\ p^2+(x^2+y^2)\ p+xy=0$

$xyp^2 + x^2p + y^2p + xy = 0$

$xp(yp + x) + y(yp + x) = 0$

$(xp + y)(yp + x) = 0$

This implies that

$xp+y=0,\ yp+x=0$

$p = \frac{-y}{x}$ $\quad\quad$ $p = \frac{-x}{y}$

$\frac{dy}{dx} = \frac{-y}{x}$ $\quad\quad\quad\quad$ $\frac{dy}{dx} = \frac{-x}{y}$

By solving equations, we get

$xy=c_1$ and

$x^2+y^2=c_2$ respectively

[$x\frac{dy}{dx}+y=0$ or $\frac{dy}{dx}+\frac{1}{x}y=0,$ Integrating factor

$I(x)=e^{\int\frac{1}{x}dx}=e^{\log x}.$ This gives

$y.x = +o.x\,dx + c_1$ or $xy=c_1$]

[$y\frac{dy}{dx}+x=0,$ or $ydy+xdx=0$

By integration we get $\frac{1}{2}y^2+\frac{1}{2}x^2=c$

or $x^2+y^2 = c_2$, $c_2 >0$, $-\sqrt{c2} \leq x \leq \sqrt{c2}$]

The general solution can be written in the form

$(x^2+y^2-c_2)\,(xy-c_1)=0$

It can be seen that none of the nontrivial solutions belonging to $xy=c_1$ or $x^2+y^2=c_2$ is valid on the whole real line.

Example 2: Solve $\left(\frac{dy}{dx}\right)^2 - \frac{dy}{dx}x + y = 0$

Example 3: Solve $\left(\frac{dy}{dx}\right)^2 - 5y + 6 = 0$

Example 4: Solve $x^2\left(\frac{dy}{dx}\right)^2 + xy\frac{dy}{dx} - 6y^2 = 0$

Case 2: Differential equations Solvable for y

Let the differential equation given by (1) be solvable for y. Then y can be expressed as a function x and p, that is,

$$y = f(x, p)$$

Differentiating with respect to x we get

$$\frac{dy}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial p}.\frac{dp}{dx}$$

is a first order differential equation of first degree in x and p.

Let solution be expressed in the form

$$\varphi(x, p, c) = 0$$

The solution of equation is obtained by eliminating p from equation. If elimination of p is not possible then together may be considered parametric equations of the solutions with p as a parameter.

Example 1: Solve y^2-1-p^2=o

Solution: It is clear that the equation is solvable for y, that is

$$y = \sqrt{1+p^2}$$

By differentiating with respect to x we get

$$\frac{dy}{dx} = \frac{1}{2}\frac{1}{\sqrt{1+p^2}}.2p\frac{dp}{dx}$$

or $$p = \frac{p}{\sqrt{1+p^2}}\frac{dp}{dx}$$

or $$p\left[1 - \frac{1}{\sqrt{1+p^2}}\frac{dp}{dx}\right] = 0$$

gives p=o or $1-\frac{p}{\sqrt{1+p^2}}\frac{dp}{dx}=0$

By solving p=0 we get

y=1

By $1-\frac{1}{\sqrt{1+p^2}}\frac{dp}{dx}=0$

we get a separable equation in variables p and x.

$$\frac{dp}{dx}=\sqrt{1+p^2}$$

By solving this we get

p=sinh (x+c)

By eliminating p we obtain

y=cos h (x+c) is a general solution.

Solution y=1 of the given equation is a singular solution as it cannot be obtained by giving a particular value to c in solution.

Example 2: $y = 2px + p^4x^4$

Example 3: $y = x\left(\frac{dy}{dx}\right)^2 + \frac{dy}{dx}$

Example 4: $\left(\frac{dy}{dx}\right)^2 + \frac{dy}{dx} = e^y$

<u>Case 3: Differential Equations Solvable for x</u>

Let equation F(x, y, p) = 0 be solvable for x, that is x=f(y,p) ..

Then as argued in the previous section for y we get a function ¬ such that

¬(y, p, c) = 0

By eliminating p from F(x, y, p) = 0 we get a general solution.

<u>Example 1:</u>

Solve $x\left(\frac{dy}{dx}\right)^3 - 12\frac{dy}{dx} - 8 = 0$

Solution: Let $p = \frac{dy}{dx},\quad then$

$xp^3-12p-8=0$

It is solvable for x, that is,

$$x = \frac{12p+8}{p^3} = \frac{12}{p^2} + \frac{8}{p^3} \qquad \ldots \qquad (1)$$

Differentiating (3.18) with respect to y, we get

$$\frac{dx}{dy} = -2\frac{12}{p^3}\frac{dp}{dy} - 3\frac{8}{p^4}\frac{dp}{dy}$$

$$or\ \frac{1}{p} = -\frac{24}{p^3}\frac{dp}{dy} - \frac{24}{p^4}\frac{dp}{dy}$$

$$or \quad dy = \left(-\frac{24}{p^2} - \frac{24}{p^3}\right)dp$$

$$or \quad y = +\frac{24}{p} + \frac{12}{p^2} + c \qquad (2)$$

(1) and (2) constitute parametric equations of solution of the given differential equation.

Example 2: $y = 2px + y^2 p^3$

Example 3: $y^2 \log y = xyp + p^2$

<u>Equations of the First Degree in x and y – Lagrange's and Clairaut's Equation</u>.

Let Equation F(x, y, p) = 0 be of the first degree in x and y, then

$y = x\prod_1(p) + \prod_2(p)$ … (1)

Equation (1) is known as Lagrange's equation.

If $\prod_1(p) = p$ then the equation

$y = xp + \prod_2(p)$ .. (2)

is known as Clairaut's equation

By differentiating (1) with respect to x, we get

$$\frac{dy}{dx} = \varphi_1(p) + x\varphi_1'(p)\frac{dp}{dx} + \varphi_2'(p)\frac{dp}{dx}$$

$$\text{or} \quad p - \varphi_1(p) = (x\varphi_1'(p) + \varphi_2'(p))\frac{dp}{dx} \qquad \ldots \qquad (3)$$

From (3) we get

$$(x+\varphi_2'(p))\frac{dp}{dx}=0 \quad \text{for } \Pi_1(p)=p$$

This gives

$$\frac{dp}{dx}=0 \quad \text{or} \quad x+\Pi_2'(p)=0$$

$$\frac{dp}{dx}=0 \quad \text{gives } p=c \text{ and}$$

by putting this value in (2) we get

$y=cx+\Pi_2(c)$

This is a general solution of Clairaut's equation.

The elimination of p between

$x+\Pi_2'(p)=0$ and (2) gives a singular solution.

If $\Pi_1(p) \neq p$ for any p, then we observe from (3) that

$\frac{dp}{dx}\neq 0$ everywhere. Division by

$[p-\varphi_1(p)]\frac{dp}{dx}$ in (3) gives

$$\frac{dx}{dp}-\frac{\varphi_1'}{p-\varphi_1(p)}x=\frac{\varphi_2'(p)}{p-\varphi_1(p)}$$

which is a linear equation of first order in x and thus can be solved for x as a function of p, which together with (2) will form a parametric representation of the general solution of (2)

Example 1: Solve $\left(\frac{dy}{dx}-1\right)\left(y-x\frac{dy}{dx}\right)=\frac{dy}{dx}$

Solution: Let $p=\frac{dy}{dx}$ then,

$(p-1)(y-xp)=p$

This equation can be written as

$$y = xp + \frac{p}{p-1}$$

Differentiating both sides with respect to x we get

$$\frac{dp}{dx}\left[x - \frac{1}{(p-1)^2}\right] = 0$$

Thus either $\frac{dp}{dx} = 0$ or

$$x - \frac{1}{(p-1)^2} = 0$$

$\frac{dp}{dx} = 0$ gives p=c

Putting p=c in the equation we get

$$y = cx + \frac{c}{c-1}$$

(y-cx)(c-1)=c

which is the required solution.

Exercises

Solve the following differential equations

1. $\left(\frac{dy}{dx}\right)^3 = \frac{dy}{dx}e^{2x}$

2. $y(y-2)p^2 - (y-2x+xy)p+x=0$

3. $-\left(\frac{dy}{dx}\right)^2 + 4y - x^2 = 0$

4. $\left(\frac{dy}{dx} + y + x\right)\left(x\frac{dy}{dx} + y + x\right)\left(\frac{dy}{dx} + 2x\right) = 0$

5. $y + x\frac{dy}{dx} - x^4\left(\frac{dy}{dx}\right)^2 = 0$

6. $\left(x\frac{dy}{dx}-y\right)\left(y\frac{dy}{dx}+x\right)=h^2\frac{dy}{dx}$

7. $y\left(\frac{dy}{dx}\right)^2+(x-y)\frac{dy}{dx}=x$

8. $x\left(\frac{dy}{dx}\right)^2-2y\frac{dy}{dx}+ax=0$

9. $\left(\frac{dy}{dx}\right)^2=y-x$

10. $xy\left(y-x\frac{dy}{dx}\right)=x+y\frac{dy}{dx}$

MATHEMATICAL MODELING

POPULATION GROWTH

One of the simplest models of population growth is based on the observation that when populations (people, plants, bacteria, and fruit flies, for example) are not constrained by environmental limitations, they tend to grow at a rate that is proportional to the size of the population—the larger the population, the more rapidly it grows.

To translate this principle into a mathematical model, suppose that $y = y(t)$ denotes the population at time t. At each point in time, the rate of increase of the population with respect to time is dy/dt, so the assumption that the rate of growth is proportional to the population is described by the differential equation

$$\frac{dy}{dt}=ky \tag{1}$$

where k is a positive constant of proportionality that can usually be determined experimentally. Thus, if the population is known at some point in time, say $y = y_0$ at time $t = 0$, then a general formula for the population $y(t)$ can be obtained by solving the initial-value problem

$$\frac{dy}{dt}=ky, \quad y(0)=y_0$$

PHARMACOLOGY

When a drug (say, penicillin or aspirin) is administered to an individual, it enters the bloodstream and then is absorbed by the body over time. Medical research has shown that the amount of a drug that is present in the bloodstream tends to decrease at a rate that is proportional to the amount of the drug present—the more of the drug that is present in the bloodstream, the more rapidly it is absorbed by the body.

To translate this principle into a mathematical model, suppose that $y = y(t)$ is the amount of the drug present in the bloodstream at time t. At each point in time, the rate of change in y with respect to t is dy/dt, so the assumption that the rate of decrease is proportional to the amount y in the bloodstream translates into the differential equation

$$\frac{dy}{dt} = -ky \tag{2}$$

where k is a positive constant of proportionality that depends on the drug and can be determined experimentally. The negative sign is required because y decreases with time. Thus, if the initial dosage of the drug is known, say $y = y_0$ at time $t = 0$, then a general formula for $y(t)$ can be obtained by solving the initial-value problem

$$\frac{dy}{dt} = -ky, \quad y(0) = y_0$$

EXAMPLE 5 In an oil refinery, a storage tank contains 2000 gal of gasoline that initially has 100 lb of an additive dissolved in it. In preparation for winter weather, gasoline containing 2 lb of additive per gallon is pumped into the tank at a rate of 40 gal/min. The well-mixed solution is pumped out at a rate of 45 gal/min. How much of the additive is in the tank 20 min after the pumping process begins (Figure 16.7)?

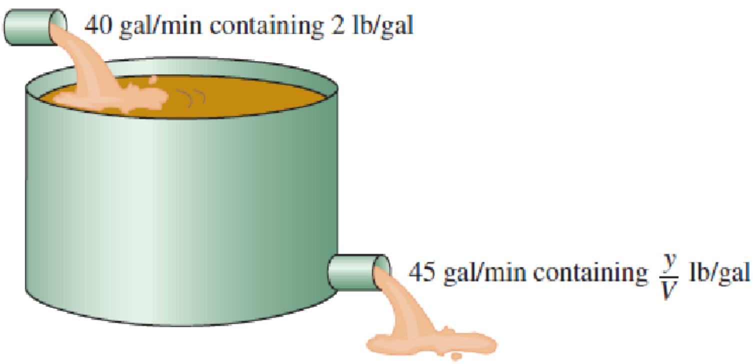


FIGURE 16.7 The storage tank in Example 5 mixes input liquid with stored liquid to produce an output liquid.

Solution Let y be the amount (in pounds) of additive in the tank at time t. We know that $y = 100$ when $t = 0$. The number of gallons of gasoline and additive in solution in the tank at any time t is

$$V(t) = 2000 \text{ gal} + \left(40 \frac{\text{gal}}{\text{min}} - 45 \frac{\text{gal}}{\text{min}}\right)(t \text{ min})$$
$$= (2000 - 5t) \text{ gal}.$$

Therefore,

$$\text{Rate out} = \frac{y(t)}{V(t)} \cdot \text{outflow rate} \qquad \text{Eq. (9)}$$
$$= \left(\frac{y}{2000 - 5t}\right) 45 \qquad \text{Outflow rate is 45 gal/min and } v = 2000 - 5t.$$
$$= \frac{45y}{2000 - 5t} \frac{\text{lb}}{\text{min}}.$$

Also,

$$\text{Rate in} = \left(2 \frac{\text{lb}}{\text{gal}}\right)\left(40 \frac{\text{gal}}{\text{min}}\right)$$
$$= 80 \frac{\text{lb}}{\text{min}}. \qquad \text{Eq. (10)}$$

The differential equation modeling the mixture process is

$$\frac{dy}{dt} = 80 - \frac{45y}{2000 - 5t}$$

in pounds per minute.

To solve this differential equation, we first write it in standard form:

$$\frac{dy}{dt} + \frac{45}{2000 - 5t}\,y = 80.$$

Thus, $P(t) = 45/(2000 - 5t)$ and $Q(t) = 80$. The integrating factor is

$$\begin{aligned} v(t) &= e^{\int P\,dt} = e^{\int \frac{45}{2000-5t}\,dt} \\ &= e^{-9\ln(2000-5t)} \qquad 2000 - 5t > 0 \\ &= (2000 - 5t)^{-9}. \end{aligned}$$

Multiplying both sides of the standard equation by $v(t)$ and integrating both sides gives

$$\begin{aligned} (2000 - 5t)^{-9} \cdot \left(\frac{dy}{dt} + \frac{45}{2000 - 5t}\,y\right) &= 80(2000 - 5t)^{-9} \\ (2000 - 5t)^{-9}\frac{dy}{dt} + 45(2000 - 5t)^{-10}\,y &= 80(2000 - 5t)^{-9} \\ \frac{d}{dt}\left[(2000 - 5t)^{-9}y\right] &= 80(2000 - 5t)^{-9} \\ (2000 - 5t)^{-9}y &= \int 80(2000 - 5t)^{-9}\,dt \\ (2000 - 5t)^{-9}y &= 80 \cdot \frac{(2000 - 5t)^{-8}}{(-8)(-5)} + C. \end{aligned}$$

The general solution is

$$y = 2(2000 - 5t) + C(2000 - 5t)^{9}.$$

Because $y = 100$ when $t = 0$, we can determine the value of C:

$$\begin{aligned} 100 &= 2(2000 - 0) + C(2000 - 0)^{9} \\ C &= -\frac{3900}{(2000)^{9}}. \end{aligned}$$

The particular solution of the initial value problem is

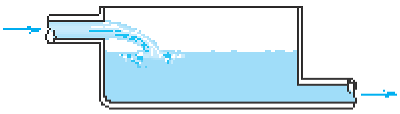
$$y = 2(2000 - 5t) - \frac{3900}{(2000)^9}(2000 - 5t)^9.$$

The amount of additive 20 min after the pumping begins is

$$y(20) = 2[2000 - 5(20)] - \frac{3900}{(2000)^9}[2000 - 5(20)]^9 \approx 1342 \text{ lb}.$$

Mixing problem

Example: The tank in the figure contains 200 litres of water in which 40 lb of salt are dissolved. Five litres of brine, each containing 2lb of dissolved salt, run into the tank per minute, and the mixture, kept uniform by stirring, runs out at the same rate. Find the amount of salt $y(t)$ in the tank at any time t.



Tank

Solution: The time rate of change $y' = \frac{dy}{dt}$ of $y(t)$ equals the inflow of salt minus the outflow.

The inflow is 10lb/min. $y(t)$ is the total amount of salt in the tank. The tank always contains 200 litres because 5 litres flow in and 5 litres flow out per minute.

Thus 1 litre contains $\frac{y(t)}{200}$ lb of salt. Hence the 5 out flowing litres contains $\frac{5y(t)}{200} = \frac{y(t)}{40} = 0.025y(t)$ lb of salt. This is the outflow. The time rate of change y' is the balance:

$$y' = \text{salt inflow rate - salt outflow rate}$$

Since the inflow of salt is 10lb/min and the outflow is $0.025y(t)\ lb/\min$, this equation becomes

$$y' = 10 - 0.025y$$

By separation of variables, the equation will become

$$y' = 10 - 0.025y = -0.025(y - 400)$$

$$\therefore \frac{dy}{y - 400} - 0.025dt$$

$$\ln(y - 400) = 0.025t + a$$

$$y - 400 = ce^{-0.025t}$$

Initially $y(0)=40$

$$40-400=c$$
$$c=-360$$

$$y-400=-360e^{-0.025t}$$

<u>Newton's Law of Cooling:</u>

According to this law, the temperature of a body changes at a rate which is proportional to the difference in temperature between that of the surroundings and heat of the body itself.

If θ_0 is the temperature of the surroundings and θthat of the body at any time t, then

$\frac{d\theta}{dt}=-k(\theta-\theta_0)$, where k is a constant .

<u>Example:</u> Suppose that you turn off the heat in your home at night 2 hours before you go to bed; call this time t=0. If the temperature T at t=0 is $66^{\circ}F$ and at the time you go to bed (t=2) has dropped to $63^{\circ}F$, what temperature can you expect in the morning, say, 8 hours later (t=10)? Of course, this process of cooling off will depend on the outside temperature T_A , which we assume to be constant at $32^{\circ}F$.

Solution: The equation $\frac{dT}{dt}=k(T-T_A)=k(T-32)$ is the equation of Newton's Law of cooling.

$$\frac{dT}{(T-32)}=kdt$$
$$\therefore \ln|T-32|=kt+c$$
$$\therefore T(t)=32+ce^{kt}$$

Given initial condition T(0)=66

$$\therefore T(0)=32+c=66$$
$$\therefore c=34$$

Also T(2)=63

$$\therefore T(2)=32+34e^{2k}=63$$
$$\therefore e^{2k}=\frac{63-32}{34}=0.9117$$
$$\therefore k=-0.0461$$

Now after 10 hours of shutdown t=10, the temperature will be

$$\therefore T(10)=32+34e^{-0.0461(10)}=53.4^{0}F$$

Self study

Modelling with Exponential Functions

Many processes that occur in nature, such as population growth, radioactive decay, heat diffusion, and numerous others, can be modelled using exponential functions. Logarithmic functions are used in models for the loudness of sounds, the intensity of earthquakes, and many other phenomena.

REMARK: Notice that the formula for population growth is the same as that for continuously compounded interest. In fact, the same principle is at work in both cases: The growth of a population (or an investment) per time period is proportional to the size of the population (or the amount of the investment). A population of 1,000,000 will increase more in one year than a population of 1000; in exactly the same way, an investment of $1,000,000 will increase more in one year than an investment of $1000.

EXPONENTIAL GROWTH MODEL: A population that experiences exponential growth increases according to the model

$n(t) = n_0e^{rt}$ where t is time,

$n(t)$ is the population size at time t, n_0 is the initial size of the population, and r is the relative rate of growth (expressed as a proportion of the population).

Example: Exponential growth or decay: The differential equation of exponential growth or decay is governed by

$$y' = ky$$

Solution:

$y' = ky$ is in variable separable form

$$\therefore \frac{1}{y}\frac{dy}{dx} = k$$

$$\therefore \frac{1}{y}dy = kdx$$

Integrating both the sides

$$\therefore \int \frac{1}{y}dy = k\int dx + \log(c)$$

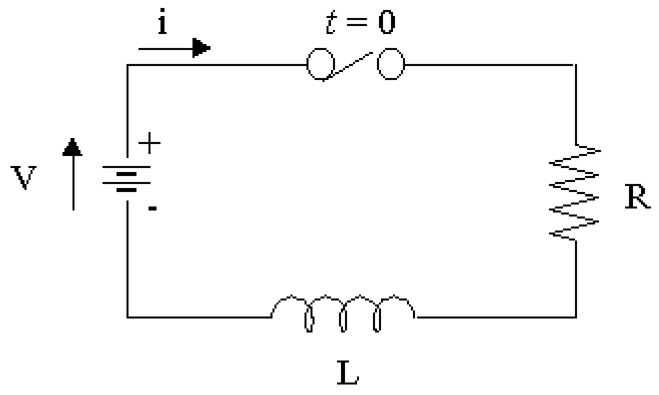
$$\therefore \log(y) = kx + \log(c)$$

$$\therefore y = ce^{kx}$$

Electric Circuit

1. **RL series circuit :**

Consider a circuit containing resistance R and inductance L in series with a voltage source E.



RL circuit diagram

Let i be the current flowing in the circuit at any time t. Then by Kirchhoff's first law, we have sum of voltage drops across R and L=E

i.e., $Ri + L\frac{di}{dt}$=E or $\frac{di}{dt} + \frac{R}{L}i = \frac{E}{L}$ .... (1)

This is Leibnitz's linear equation.

I.F.=$e^{\int \frac{R}{L}dt} = e^{\frac{Rt}{L}}$

therefore, its solution is

$$i(I.F.) = \int \frac{E}{L}(I.F.)dt + c$$

Or $ie^{\frac{Rt}{L}} = \int \frac{E}{L}E^{\frac{Rt}{L}}dt + c = \frac{E}{L}\frac{L}{R}e^{\frac{Rt}{L}} + c$ whence $i = \frac{E}{R} + ce^{-\frac{Rt}{L}}$ .... (2)

If initially there is no current in the circuit, i.e. i=0, when t=0, we have $c = -\frac{E}{R}$

Thus (2) becomes $i = \frac{E}{R}(1 - e^{-\frac{Rt}{L}})$ which shows that i increases with t and attains the maximum value E/R.

Example: The current *i(t)* flowing in an R-L circuit is governed by the equation $\left[L\frac{di}{dt} + Ri = E_0 \sin \omega t\right]$ **where R is the constant resistance, L is the constant inductance and** $E_0 \sin \omega t$ **is the voltage at time *t*,** E_0 **and** ω **being constants. Find the current at any time assuming that initially it is zero.**

Solution: The given equation can be written as $\frac{di}{dt} + \frac{R}{L}i = \frac{E_0}{L}\sin \omega t$

$$\therefore I.F = e^{\int \frac{R}{L}dt} = e^{\frac{Rt}{L}}$$

Therefore general solution is

$$i(t)e^{\frac{Rt}{L}} = \int \frac{E_0}{L}\sin \omega t e^{\frac{Rt}{L}}dt + c$$

$$\therefore i(t) = \frac{E_0}{R^2 + w^2L^2}\left[R\sin \omega t - \omega L\cos \omega t\right] + ce^{-\frac{Rt}{L}}$$

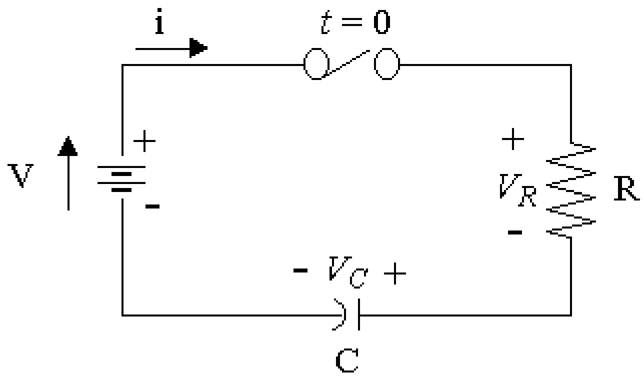
Given the initial condition is $i(0)=0$, which gives

$$c = \frac{\omega E_0 L}{R^2 + w^2 L^2}$$

Therefore the current i at any time t is given by

$$\therefore i(t) = \frac{E_0}{R^2 + w^2 L^2}\left[R\sin\omega t - \omega L\cos\omega t + \omega L e^{-\frac{Rt}{L}} \right]$$

2. RC series circuit :



In an RC circuit, the **capacitor** stores energy between a pair of plates. When voltage is applied to the capacitor, the charge builds up in the capacitor and the current drops off to zero.

Case: 1 Constant Voltage

The voltage across the resistor and capacitor are as follows

$V_R = Ri$ and $V_C = \frac{1}{C}\int idt$

Kirchhoff's voltage law says the total voltages must be zero. So applying this law to a series RC circuit results in the equation:

$$Ri + \frac{1}{C}\int idt = V$$

One way to solve this equation is to turn it into a **differential equation**, by differentiating throughout with respect to t:

$$R\frac{di}{dt} + \frac{i}{C} = 0$$

Solving the equation gives us:

$$i = \frac{V}{R}e^{-\frac{t}{RC}}$$

Case:2 Variable Voltage

We need to solve variable voltage cases in q, rather than in i, since we have an integral to deal with if we use i.

We have $i = \frac{dq}{dt}$ and $q = \int idt$

So the equation $Ri + \frac{1}{C}\int \quad idt = V$ becomes

$$R\frac{dq}{dt} + \frac{q}{C} = V$$

Example: The current *i(t)* flowing in an R-C circuit is governed by the equation $\left[R\frac{dI}{dt} + \frac{1}{C}I = \frac{dE}{dt}\right]$ **where R is the constant resistance, C is the capacitance and E(t) is the periodic electromotive force at time *t*. Find the current at any time assuming that E is constant and** $E(t) = E_0 \sin \omega t$.

Solution: $\left[R\frac{dI}{dt} + \frac{1}{C}I = \frac{dE}{dt}\right]$

$\left[\frac{dI}{dt} + \frac{1}{RC}I = \frac{1}{R}\frac{dE}{dt}\right]$ which is linear equation.

$$\therefore I.F = e^{\int \frac{1}{RC}dt} = e^{\frac{t}{RC}}$$

Therefore general solution is

$$I(t)e^{\frac{t}{RC}} = \int \frac{1}{R}e^{\frac{t}{RC}}\frac{dE}{dt}dt + c$$

Given E=0, then $\frac{dE}{dt} = 0$, which gives

$$I(t) = ce^{-\frac{t}{RC}}$$

Given $E(t) = E_0 \sin \omega t$, then $\frac{dE}{dt} = \omega E_0 \cos \omega t$, which gives

$$\therefore I(t) = ce^{-\frac{t}{RC}} + \frac{\omega E_0 C}{1 + (wRC)^2}\left[\cos \omega t + \omega RC \sin \omega t\right]$$

Parul University

Faculty of Engineering & Technology

Department of Applied Sciences and Humanities

1st Year B. Tech Programme (All Branches)

Mathematics– 1(303191101)

Unit – 1 MATRICES (Lecture Note)

Matrix:

A Matrix is a rectangular array of numbers (or functions) enclosed in brackets. These number or functions are called entries or elements of the matrix.

For example:

$$\begin{bmatrix} 4 & -2 & 1 \\ 0 & 3 & 5 \end{bmatrix}, \begin{bmatrix} \sin x & \cos x \\ -\cos x & \sin x \end{bmatrix}$$

Trace of a matrix:

If A is a square matrix, the trace of A, denoted by $tr(A)$and is defined to be the sum of entries on the main diagonal of A. The trace of A is undefined if$A$ is not a square matrix.

For example:

If $A = \begin{bmatrix} 4 & 5 \\ 10 & 6 \end{bmatrix}$, then $tr(A) = 4 + 6 = 10$.

Symmetric matrix: - For any **square** matrix A, if $A = A^T$,then it is known as symmetric matrix.

For example:

If $A = \begin{bmatrix} 1 & 3 & 4 \\ 3 & 2 & 7 \\ 4 & 7 & 4 \end{bmatrix}$ $then \ \ A^T = \begin{bmatrix} 1 & 3 & 4 \\ 3 & 2 & 7 \\ 4 & 7 & 4 \end{bmatrix}$

Here, we can see that so$A = A^T$; hence $A$is symmetric matrix.

Skew-symmetric matrix: - For any **square** matrix A, if $A = -A^T$ then it is known as Skew symmetric matrix.

For example:

$$A = \begin{bmatrix} 0 & -3 & 4 \\ 3 & 0 & 7 \\ -4 & -7 & 0 \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} 0 & 3 & -4 \\ -3 & 0 & -7 \\ 4 & 7 & 0 \end{bmatrix} = -\begin{bmatrix} 0 & -3 & 4 \\ 3 & 0 & 7 \\ -4 & -7 & 0 \end{bmatrix} = -A^T$$

Here, we can see that $A = -A^T$; so $A$is skew-symmetric matrix.

Singular and non-singular matrix: -

For any square matrix A, if$|A| \neq 0$, then it is known as non-singular matrix and if $|A| = 0$ then it is known as singular matrix.

Example 1: - If $A = \begin{bmatrix} 2 & 1 \\ 8 & 4 \end{bmatrix} \Rightarrow |A| = 8 - 8 = 0 \Rightarrow \text{Singular Matrix}$

Example 2: - If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \Rightarrow |A| = 4 - 6 = -2 \neq 0 \Rightarrow \text{Non} - \text{Singular Matrix}$

Orthogonal Matrix: The matrix is said to be an orthogonal matrix if the product of a matrix and its transpose gives an identity value. i.e.$AA^T = I$

Example: Given$A$ is an orthogonal matrix because

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \text{ Then } A^T = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } AA^T = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

System of linear equation

Linear Equations: Any straight line in the xy-plane can be represented algebraically by equation of the form $ax + by = c$, where a&b are real numbers.

A **system of linear equation** is a collection of one or more linear equations involving the same variables.

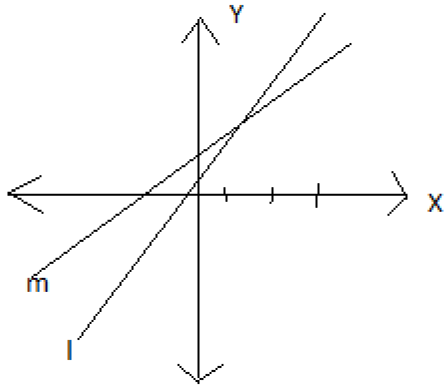
A linear system of m linear equations in n variables: An arbitrary system of m linear equations in n variables $x_1, x_2, x_3, \ldots\ldots x_n$ is a set of equations of the form

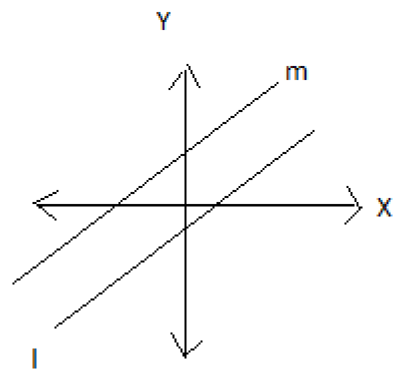
$$\sum_{j=1}^{n} a_{ij}x_j = b_i \ (i = 1,2,3, \ldots, m, j = 1,2,3, \ldots, n)$$

A system of linear equations has either

1. No solutions, or
2. Exactly one solution, or
3. Infinitely many solutions

<u>Geometrical representation</u>

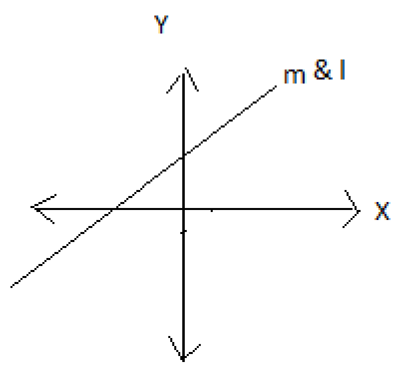




$x - 2y = -1$ .... l $x - 2y = -1$ l

$-x + 3y = 3$ .... m $-x + 2y = 3$ .... m

Exactly one solution **No solution**



$x - 2y = -1$ l

$-x + 2y = 1$ .... m

Infinitely many solutions

<u>Notes</u>

(i) The system is said to be consistent if we get infinitely many solutions or unique solution.

(ii) The system is said to be inconsistent if we get No solution.

Augmented matrix

A system of m equations in n unknowns can be abbreviated by writing only the rectangular array of numbers.

$$\begin{bmatrix} a_{11} a_{12} \cdots a_{1n} | b_1 \\ a_{21} a_{22} \cdots a_{2n} | b_2 \\ \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \\ a_{m1} a_{m2} \ldots a_{mn} | b_m \end{bmatrix}$$

This is known as augmented matrix.

For example: Find the augmented matrix for each of the following system of linear equations:

$$\begin{aligned} 2x_1 + \quad +2x_3 &= 1 \\ 3x_1 - x_2 + 4x_3 &= 7 \\ 6x_1 + x_2 - x_3 &= 0 \end{aligned}$$

Then, augmented matrix is given by $\begin{bmatrix} 2 & 0 & 2 & |1 \\ 3 & -1 & 4 & |7 \\ 6 & 1 & -1 & |0 \end{bmatrix}$.

Condition of Consistency for non-homogeneous system:

(1) If there is a zero row to left of the augmentation bar but the last entry of this row is non-zero then the system has **no solution**.

For example: $\begin{bmatrix} 1 & 0 & 2|1 \\ 0 & 1 & 4|7 \\ \mathbf{0} & \mathbf{0} & \mathbf{0}|\mathbf{4} \end{bmatrix}$

(2) If at least one of the columns on the left of the augmentation bar has zero element pivot entry, then the system has **infinitely many solutions**.

For example: $\begin{bmatrix} 1 & 0 & 2|1 \\ 0 & 1 & 4|7 \\ \mathbf{0} & \mathbf{0} & \mathbf{0}|\mathbf{0} \end{bmatrix}$

(3) If all the rows having the leading entry 1 then the system has **unique solution.**

For example: $\begin{bmatrix} 1 & 0 & 2|1 \\ 0 & 1 & 4|7 \\ \mathbf{0} & \mathbf{0} & \mathbf{1}|\mathbf{4} \end{bmatrix}$

Row-Echelon (RE)form and Row-Reduced Echelon (RRE) formof a matrix

Definition: A rectangular matrix is in row-echelon form (or echelon form) if it has the following three properties:

1. The **first** element in each row must be **non-zero** and equals to **1**, that is called **leading entry 1**.
2. All the **leading 1's** must be on the **right-hand** side of the matrix.
3. If any **zero row** is available, then it must be **below** to the **all-leading 1**.

If the matrix satisfies the **4th property** (i.e., In each column except leading 1 if all entries are zero) then **row-echelon form (RE form)** becomes **row-reducedechelon form (RRE form)**.

Example 1: Which of the following matrices are in row-echelon or echelon form?

$$(a)\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\quad(b)\begin{bmatrix}1&2&0\\0&1&0\\0&0&0\end{bmatrix}\quad(c)\begin{bmatrix}1&0&0\\0&1&0\\0&2&0\end{bmatrix}\quad(d)\begin{bmatrix}1&1&0&1&1\\0&2&0&2&2\\0&0&0&3&3\\0&0&0&0&4\end{bmatrix}\quad(e)\begin{bmatrix}1&0&0&0\\0&1&1&0\\0&0&0&0\\0&0&0&1\end{bmatrix}$$

Example 2: Which of the following matrices are in reduced row-echelon or reduced echelon form?

$$(a)\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}(b)\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}(c)\begin{bmatrix}0&1&0\\0&0&1\\0&0&0\end{bmatrix}(d)\begin{bmatrix}1&0&0\\0&0&1\\0&0&0\end{bmatrix}(e)\begin{bmatrix}1&0&0\\0&0&0\\0&0&1\end{bmatrix}$$

$$(f)\begin{bmatrix}0&1&0\\1&0&0\\0&0&0\end{bmatrix}\quad(g)\begin{bmatrix}0&0\\0&0\end{bmatrix}$$

Methods of solving system of linear equations

Gauss-Elimination Method

Gauss-Jordan Method

Row-Echelon (RE) Form

Row-Reduced Echelon (REE) Form

Examples: Solve the following system using Gauss-Elimination Method

Case-1: Unique solution

Example 1: Solve the following system by gauss- Elimination method

$$x + y + z = 6$$
$$x + 2y + 3z = 14$$
$$2x + 4y + 7z = 30$$

Solution:

The matrix form of the given system is

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 2 & 4 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 14 \\ 30 \end{bmatrix}$$

The augmented matrix is

$$[A|B] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 14 \\ 2 & 4 & 7 & 30 \end{array}\right]$$

Now, to convert the given augmented matrix in row-echelon form we apply elementary operations as following.

$$R_2 \to R_2 - R_1, R_3 \to R_3 - 2R_1$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 8 \\ 0 & 2 & 5 & 18 \end{array}\right]$$

$$R_3 \to R_3 - 2R_2$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

The corresponding system of equation is

$$x + y + z = 6$$
$$y + 2z = 8$$
$$z = 2$$

By using back substitution of $z = 2$ in $y + 2z = 8$, we get $y = 4$ and $z = 2$ & $y = 4$ in $x + y + z = 6$ we get $x = 0$.

$x = 0, y = 4, z = 2$ is **unique solution** of given

Case-2: No solution

Example 2: Solve the following system of equationby Gauss elimination.

$$-2b + 3c = 1$$
$$3a + 6b - 3c = -2$$
$$6a + 6b + 3c = 5$$

Solution:

The matrix form of the given system is

$$\begin{bmatrix} 0 & -2 & 3 \\ 3 & 6 & -3 \\ 6 & 6 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 5 \end{bmatrix}$$

The augmented matrix is

$$\left[\begin{array}{ccc|c} 0 & -2 & 3 & 1 \\ 3 & 6 & -3 & -2 \\ 6 & 6 & 3 & 5 \end{array}\right]$$

$$R_1 \leftrightarrow R_2$$

$$\left[\begin{array}{ccc|c} 3 & 6 & -3 & -2 \\ 0 & -2 & 3 & 1 \\ 6 & 6 & 3 & 5 \end{array}\right]$$

$$R_1 \to \left({}^{1}/_{3}\right)R_1$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & -2/3 \\ 0 & -2 & 3 & 1 \\ 6 & 6 & 3 & 5 \end{array}\right]$$

$$R_3 \to R_3 - 6R_1$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & -2/3 \\ 0 & -2 & 3 & 1 \\ 0 & -6 & 9 & 9 \end{array}\right]$$

$$R_3 \to R_3 - 3R_2$$

system.

Case-3: Infinitely many solutions

Example 3: Solve the following system by Gauss elimination method.

$$4x - 2y + 6z = 8$$
$$x + y - 3z = -1$$
$$15x - 3y + 9z = 21$$

Solution:

The matrix form of the given system is

$$\begin{bmatrix} 4 & -2 & 6 \\ 1 & 1 & -3 \\ 15 & -3 & 9 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ -1 \\ 21 \end{bmatrix}$$

The augmented matrix is

$$[A|B] = \left[\begin{array}{ccc|c} 4 & -2 & 6 & 8 \\ 1 & 1 & -3 & -1 \\ 15 & -3 & 9 & 21 \end{array}\right]$$

$$R_1 \leftrightarrow R_2$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -3 & -1 \\ 4 & -2 & 6 & 8 \\ 15 & -3 & 9 & 21 \end{array}\right]$$

$$R_2 \to R_2 - 4R_1, R_3 \to R_3 - 15R_1$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -3 & -1 \\ 0 & -6 & 18 & 12 \\ 0 & -18 & 54 & 36 \end{array}\right]$$

$$R_2 \to (-1/6)R_1, R_3 \to (-1/6)R_3$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -3 & -1 \\ 0 & 1 & -3 & -2 \\ 0 & 1 & -3 & -2 \end{array}\right]$$

$$R_3 \to R_3 - R_2$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -3 & -1 \\ 0 & 1 & -3 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & -2/3 \\ 0 & -2 & 3 & 1 \\ 0 & 0 & 0 & 6 \end{array}\right]$$

The system of linear equation is

$a + 2b - c = -2/3$

$-2b + 3c = 1$

$0a + 0b + 0c = 6$ is not possible.

This shows that the system has **no solution**.

Example 4:

Solve the following system by gauss elimination method.

$$\frac{-1}{x} + \frac{3}{y} + \frac{4}{z} = 30$$

$$\frac{3}{x} + \frac{2}{y} - \frac{1}{z} = 9$$

$$\frac{2}{x} - \frac{1}{y} + \frac{2}{z} = 10$$

Solution:

Let $u = \frac{1}{x}, v = \frac{1}{y}, w = \frac{1}{z}$

Then the system of equations

$$-u + 3v + 4w = 30$$

$$3u + 2v - w = 9$$

$$2u - v + 2w = 10$$

The matrix form of the system is

$$\begin{bmatrix} -1 & 3 & 4 \\ 3 & 2 & -1 \\ 2 & -1 & 2 \end{bmatrix}\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 30 \\ 9 \\ 10 \end{bmatrix}$$

The augmented matrix is

The corresponding system of equations is

$$x + y - 3z = -1$$

$$y - 3z = -2$$

Assigning the free variable z an arbitrary value t,

$y = 3t - 2,$

$$x = -1 - 3t + 2 + 3t = 1$$

Hence, $x = 1, y = 3t - 2, z = t$ is solution of the given system of equations.

Since t is arbitrary real number, The system has **infinitely many solutions.**

Example 5: Consider the following system

$$x + y + z = 6$$

$$x + 2y + 3z = 10$$

$$x + 2y + \lambda z = \mu$$

For what values of λ and μ the system has (i) infinitely many solutions (ii) unique solution and (iii) no solution.

<u>Solution:</u> The Augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & \lambda & \mu \end{array}\right]$$

$$R_2 \to R_2 - R_1, R_3 \to R_3 - R_1$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & \lambda - 1 & \mu - 6 \end{array}\right]$$

$$R_3 \to R_3 - R_2$$

$$\left[\begin{array}{ccc|c} -1 & 3 & 4 & 30 \\ 3 & 2 & -1 & 9 \\ 2 & -1 & 2 & 10 \end{array}\right]$$

$$R_1 \to (-1)R_1$$

$$\left[\begin{array}{ccc|c} 1 & -3 & -4 & -30 \\ 3 & 2 & -1 & 9 \\ 2 & -1 & 2 & 10 \end{array}\right]$$

$$R_2 \to R_2 - 3R_1, R_3 \to R_3 - 2R_1$$

$$\left[\begin{array}{ccc|c} 1 & -3 & -4 & -30 \\ 0 & 11 & 11 & 99 \\ 0 & 5 & 10 & 70 \end{array}\right]$$

$$R_2 \to \left(\frac{1}{11}\right) R_2, R_3 \to \left(\frac{1}{5}\right) R_3$$

$$\left[\begin{array}{ccc|c} 1 & -3 & -4 & -30 \\ 0 & 1 & 1 & 9 \\ 0 & 1 & 2 & 14 \end{array}\right]$$

$$R_3 \to R_3 - 2R_2$$

$$\left[\begin{array}{ccc|c} 1 & -3 & -4 & -30 \\ 0 & 1 & 1 & 9 \\ 0 & 0 & 1 & 5 \end{array}\right]$$

$$\begin{bmatrix} 1 & 1 & 1 & | & 6 \\ 0 & 1 & 2 & | & 4 \\ 0 & 0 & \lambda-3 & | & \mu-10 \end{bmatrix}$$

(i) If $\lambda - 3 = 0$ and $\mu - 10 = 0$, that is if $\lambda = 3$ and $\mu = 10$ then the system has infinitely many solutions.

(ii)If$\lambda - 3 = 0$ then the system has a unique solution. That is $\lambda \neq 3$ and μ can possess any real value.

(iii)If $\lambda - 3 = 0$ and$\mu - 10 \neq 0$, that is if $\lambda = 3$ and $\mu \neq 10$ then the system does not have any solution.

The corresponding system of equations is

$$u - 3v - 4w = -30$$

$$v + w = 9$$

$$w = 5$$

By doing back substitution we get
$$v + 5 = 9 \Rightarrow v = 4 \Rightarrow y = \frac{1}{4}$$

$$u - 12 - 20 = -30 \Rightarrow u = 2 \Rightarrow x = \frac{1}{2}$$

Hence, $x = \frac{1}{2}, y = \frac{1}{4}, z = \frac{1}{5}$ is required **unique solution** of the system.

Exercise: Solve the following system of equations by using Gauss elimination method.

(1) $x + y + 2z = 9$

$2x + 4y - 3z = 1$

$3x + 6y - 5z = 0$

Ans:

$x = 1, y = 2, z = 3$

(2)$3x + y - 3z = 13$

$2x - 3y + 7z = 5$

$2x + 19y - 47z = 32$

Ans: No solution as the augmented matrix in row-echelon form is

$$\begin{bmatrix} 1 & 1/3 & -1 & | & 13/3 \\ 0 & 1 & -27/11 & | & 1 \\ 0 & 0 & 0 & | & 5 \end{bmatrix}$$

(3) $2x + 2y + 2z = 0$

$-2x + 5y + 2z = 1$

$8x + y + 4z = -1$

Ans: Infinitely many solutions.The solution set is $\{(\frac{-3k-1}{7}, \frac{1-4k}{7}, k)/$ k $\in$R}.

Examples: Solve the following system using Gauss-Jordan Method

Case-1: Unique Solution

(1) $x + y + 2z = 8$

$-x - 2y + 3z = 1$

$3x - 7y + 4z = 10$

Solution: The matrix form of the system is

$$\begin{bmatrix} 1 & 1 & 2 \\ -1 & -2 & 3 \\ 3 & -7 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 1 \\ 10 \end{bmatrix}$$

The augmented matrix is

$$[A|B] = \left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ -1 & -2 & 3 & 1 \\ 3 & -7 & 4 & 10 \end{array}\right]$$

$$R_2 \to R_2 + R_1, R_3 \to R_3 - 3R_1$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & -1 & 5 & 9 \\ 0 & -10 & -2 & -14 \end{array}\right]$$

$$R_2 \to (-1)R_2$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & -10 & -2 & -14 \end{array}\right]$$

Case-2: Infinitely many Solutions

(2) $x + 2y - 3z = -2$

$3x - y - 2z = 1$

$2x + 3y - 5z = -3$

Solution:The matrix form of the system is

$$\begin{bmatrix} 1 & 2 & -3 \\ 3 & -1 & -2 \\ 2 & 3 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix}$$

The augmented matrix is

$$[A|B] = \left[\begin{array}{ccc|c} 1 & 2 & -3 & -2 \\ 3 & -1 & -2 & 1 \\ 2 & 3 & -5 & -3 \end{array}\right]$$

$$R_2 \to R_2 - 3R_1, R_3 \to R_3 - 2R_1$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & -2 \\ 0 & -7 & 7 & 7 \\ 0 & -1 & 1 & 1 \end{array}\right]$$

$$R_2 \to (-1/7)R_2$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & -2 \\ 0 & 1 & -1 & -1 \\ 2 & -1 & 1 & 1 \end{array}\right]$$

$$R_3 \to R_3 + 10R_2$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & -52 & -104 \end{array}\right]$$

$$R_3 \to (-1/52)R_3$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$$R_2 \to R_2 + 5R_3, R_1 \to R_1 - 2R_3$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 4 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$$R_1 \to R_1 - R_2$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

The corresponding system of equation is

$x = 3, y = 1, z = 2$ which is a unique solution of the given system of equations.

Case-3: No Solution

(3)$x + y + z = 1$

$$3x - y - z = 4$$

$$x + 5y + 5z = -1$$

Solution:The matrix form of the system is

$$\begin{bmatrix} 1 & 1 & 1 \\ 3 & -1 & -1 \\ 1 & 5 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ -1 \end{bmatrix}$$

$$R_3 \to R_3 + R_2$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & -2 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$R_1 \to R_1 - 2R_2$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The corresponding system of equations gives

$$x - z = 0$$

$$y - z = -1$$

Assigning the free variable z an arbitrary value t,

$$z = t$$

$$x = z = t$$

$$y = z - 1 = t - 1$$

Hence, $x = t, y = t - 1, z = t$ is solution of the given system of equations and since t is arbitrary real number, The system has **infinitely many solutions.**

Exercise: Solve the following system of equations by using Gauss- Jordan method.

(1) $2y + 3z = 7$ Ans: Unique solution

$3x + 6y - 12z = -3$ $x = -1, y = 2, z = 1$

$5x - 2y + 2z = -7$

<table>
<tr>
<td>

The augmented matrix is

$$[A|B] = \begin{bmatrix} 1 & 1 & 1 & | & 1 \\ 3 & -1 & -1 & | & 4 \\ 1 & 5 & 5 & | & -1 \end{bmatrix}$$

$$R_2 \to R_2 - 3R_1, R_3 \to R_3 - R_1$$

$$\begin{bmatrix} 1 & 1 & 1 & | & 1 \\ 0 & -4 & -4 & | & 1 \\ 0 & 4 & 4 & | & -2 \end{bmatrix}$$

$$R_3 \to R_3 + R_2$$

$$\begin{bmatrix} 1 & 1 & 1 & | & 1 \\ 0 & -4 & -4 & | & 1 \\ 0 & 0 & 0 & | & -1 \end{bmatrix}$$

Observe the 3rd row in last matrix gives

$0x + 0y + 0z = -1$ which is not possible.

This shows that the system has **no solution**.

</td>
<td>

(2) $-2y + 3z = 1$

$3x + 6y - 3z = -2$

$6x + 6y + 3z = 5$

Ans: No Solution

as the augmented matrix in row-echelon form is

$$\begin{bmatrix} 1 & 2 & -1 & | & -2/3 \\ 0 & 1 & -3/2 & | & -1/2 \\ 0 & 0 & 0 & | & 6 \end{bmatrix}$$

(3) $x - y + z = 1$

$2x + y - z = 2$

$5x - 2y + 2z = 5$

Ans: Infinitely many solutions.The solution set is $\{(1, k, k)/\ k \in R\}$.

</td>
</tr>
</table>

<u>Practice Problem:</u>

1. Solve the system of linear equations using Gauss-Jordan method (Winter 2017)

$$x + 2y + z = 5;\ -x - y + z = 2; y + 3z = 1$$

Solution: Matrix form of the system is

$$\begin{bmatrix} 1 & 2 & 1 \\ -1 & -1 & 1 \\ 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix}$$

Let $A = \begin{bmatrix} 1 & 2 & 1 \\ -1 & -1 & 1 \\ 0 & 1 & 3 \end{bmatrix}$ $\quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ $\quad B = \begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix}$

$$[A|B] = \begin{bmatrix} 1 & 2 & 15 \\ -1 & -1 & 12 \\ 0 & 1 & 31 \end{bmatrix}$$

$$R_2 \to R_2 + R_1$$

$$\sim\begin{bmatrix}1 & 2 & 15\\0 & 1 & 27\\0 & 1 & 31\end{bmatrix}$$

$$R_1 \to R_1 - 2R_2$$

$$R_3 \to R_3 - R_2$$

$$\sim\begin{bmatrix}1 & 0 & -3 -9\\0 & 1 & 2 \;\; 7\\0 & 0 & 1 \; -6\end{bmatrix}$$

$$R_1 \to R_1 + 3R_3$$

$$R_2 \to R_2 - 2R_3$$

$$\sim\begin{bmatrix}1 & 0 & 0 -27\\0 & 1 & 0 \;\; 19\\0 & 0 & 1 \; -6\end{bmatrix}$$

$\therefore$ The solution of the system is $x = -27, y = 19, z = -6$

2) Solve the system of linear equations $x - 2y + z = 1$; $-x + y - z = 0$; $2x - y + z = -1$ using Gauss Elimination method. (Winter -2108)

The Matrix form of the system of equation

$$\begin{bmatrix}1 & -2 & 1\\-1 & 1 & -1\\2 & -1 & 1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}1\\0\\-1\end{bmatrix}$$

Let A= $\begin{bmatrix}1 & -2 & 1\\-1 & 1 & -1\\2 & -1 & 1\end{bmatrix} X + \begin{bmatrix}x\\y\\z\end{bmatrix}\; B = \begin{bmatrix}1\\0\\-1\end{bmatrix}$

$$[A|B] = \begin{bmatrix}1 & -2 & 1 \;\; 1\\-1 & 1 & -1 \; 0\\2 & -1 & 1 \; -1\end{bmatrix}$$

$$R_2 \to R_1 + R_2 R_3 \to R_3 - 2R_1$$

$$\sim\begin{bmatrix}1 & -2 & 1 \;\; 1\\0 & -1 & 0 \;\; 1\\0 & 3 & -1 -3\end{bmatrix}$$

$$R_2 \to -R_2$$

$$\sim\begin{bmatrix}1 & -2 & 1 \;\; 1\\0 & 1 & 0 \; -1\\0 & 3 & -1 -3\end{bmatrix}$$

$$R_3 \to R_3 - 3R_2$$

$$\sim\begin{bmatrix}1 & -2 & 1 & 1\\ 0 & 1 & 0 & -1\\ 0 & 0 & -1 & 0\end{bmatrix}$$

$$-z = 0 => z = 0$$

$$y = -1$$

$$x - 2y + z = 1$$

$$x = -1$$

Hence, the required solution is $x = -1, y = -1 \ and \ z = 0$

3) Solve the system of linear equation using Gauss Elimination method

$$x + y + z = 6; x + 2y + 3z = 14; 2x + 4y + 7z = 30 (\text{Winter- } 2019)$$

$$[A|B] = \begin{bmatrix}1 & 1 & 1\,6\\ 1 & 2 & 314\\ 2 & 4 & 730\end{bmatrix}$$

$$R_2 \to R_2 - R_1 R_3 \to R_3 - 2R_1$$

$$\sim\begin{bmatrix}1 & 1 & 1\,6\\ 0 & 1 & 2\,8\\ 0 & 2 & 518\end{bmatrix}$$

$$R_3 \to R_3 - 2R_2$$

$$\sim\begin{bmatrix}1 & 1 & 16\\ 0 & 1 & 28\\ 0 & 0 & 12\end{bmatrix}$$

$$z = 2$$

$$y + 2z = 8$$

$$\therefore y = 4$$

$$x = 0$$

Therefore, required solution is $x = 0; y = 4; z = 8$

Exercise: (i) Solve the system of linear equations using Gauss Elimination method

$$x + y + 2z = 9;\ \ 2x + 4y - 3z = 1;\ \ 3x + 6y - 5z = 0 \qquad (\text{Summer } 2022)$$

Answer : $x = 1; y = 2;\ \ z = 3$

(ii) Solve the system of linear equation using Gauss Jordan method:

$$-x + 3y + 4z = 30; \;\; 3x + 2y - z = 9; \;\; 2x - y + 2z = 10 \qquad \text{(Summer 2019)}$$

Answer: $x = 2; y = 4; z = 5$

HOMOGENEOUS EQUATIONS

A system of linear equations in terms of $x_{1,}x_{2,}x_{3,} \dots x_n$ having the matrix form AX=O, where A is $m \times n$ coefficient matrix, X is $n \times 1$ column matrix, O is a $m \times 1$ zero column matrix is called a system of homogeneous equations.

For example:(i) $x + y + z = 0$

$$x + 2y - z = 0$$
$$x + 3y + 2z = 0$$

(ii)$x + y = 0$

$$x + 2y = 0$$

Homogeneous equations are never inconsistent. They always have the solution “all variables = 0”. The solution (0, 0, …, 0) is often called the **trivial solution**. Any other solution is called **nontrivial solution.**

Example-1: Solve the following system:

$$4x + 3y - z = 0$$

$$3x + 4y + z = 0$$

$$5x + y - 4z = 0$$

Solution:The matrix form of the system is

$$\begin{bmatrix} 4 & 3 & -1 \\ 3 & 4 & 1 \\ 5 & 1 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The augmented matrix is

$$[A|B]$$

$$\left[\begin{array}{ccc|c} 4 & 3 & -1 & 0 \\ 3 & 4 & 1 & 0 \\ 5 & 1 & -4 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 4 & 3 & -1 & 0 \\ 3 & 4 & 1 & 0 \\ 5 & 1 & -4 & 0 \end{array}\right] R_1 \to \frac{1}{4} R_1$$

=

$$\left[\begin{array}{ccc|c} 1 & 3/4 & -1/4 & 0 \\ 3 & 4 & 1 & 0 \\ 5 & 1 & -4 & 0 \end{array}\right] \begin{array}{l} R_2 \to R_2 - 3R_1, \\ R_3 \to R_3 - 5R_1 \end{array}$$

=

$$\left[\begin{array}{ccc|c} 1 & 3/4 & -1/4 & 0 \\ 0 & 7/4 & 7/4 & 0 \\ 0 & -11/4 & -11/4 & 0 \end{array}\right] R_2 \to \frac{4}{7} R_2$$

=

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Example-2: Solve the following system

$$\begin{aligned} -2x + 2y - 3z &= 0 \\ 2x + y - 6z &= 0 \\ -x - 2y + 2z &= 0 \\ 3x + y + 4z &= 0 \end{aligned}$$

Solution:

$$\left[\begin{array}{ccc|c} -2 & 2 & -3 & 0 \\ 2 & 1 & -6 & 0 \\ -1 & -2 & 2 & 0 \\ 3 & 1 & 4 & 0 \end{array}\right] R_1 \to -\frac{1}{2} R_1$$

=

$$\left[\begin{array}{ccc|c} 1 & -1 & 3/2 & 0 \\ 2 & 1 & -6 & 0 \\ -1 & -2 & 2 & 0 \\ 3 & 1 & 4 & 0 \end{array}\right] \begin{array}{l} R_2 \to R_2 - 2R_1, \\ R_3 \to R_3 + R_1 \\ R_4 \to R_4 - 3R_1 \end{array}$$

=

$$\left[\begin{array}{ccc|c} 1 & -1 & 3/2 & 0 \\ 0 & 3 & -9 & 0 \\ 0 & -3 & 7/2 & 0 \\ 0 & 4 & -1/2 & 0 \end{array}\right] R_2 \to \frac{1}{3} R_2$$

=

$$\left[\begin{array}{ccc|c} 1 & -1 & 3/2 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & -3 & 7/2 & 0 \\ 0 & 4 & -1/2 & 0 \end{array}\right] \begin{array}{l} R_1 \to R_1 + R_2, \\ R_3 \to R_3 + 3R_2 \\ R_4 \to R_4 - 4R_2 \end{array}$$

=

$x - z = 0,\ y + z = 0,\ 0 = 0.$

The last equation does not give any information about the equations**.**

Let

$z = k \Rightarrow y = -k \ and \ x = k$.

$\therefore the \ solution \ set \ is \{(k, -k, k) / k \in R\}$

Exercise: Solve the following system of equations.

(1)

$$x + y - z + w = 0$$
$$x - y + 2z - w = 0$$
$$3x + y \quad + w = 0$$

Ans: Infinitely many solutions.

The solution set is $\{(t/4, -7t/4, t) / \ t \in R\}$.

(2) $2x + y + 3z = 0$

$$x + 2y \quad = 0$$
$$y + z = 0$$

Ans: Trivial solution

$x = 0, y = 0, z = 0$

$$\left[\begin{array}{ccc|c} 1 & 0 & -3/2 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & -11/2 & 0 \\ 0 & 0 & 23/2 & 0 \end{array}\right] R_3 \to -\frac{2}{11} R_3$$

=

$$\left[\begin{array}{ccc|c} 1 & 0 & -3/2 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 23/2 & 0 \end{array}\right] \begin{array}{l} R_1 \to R_1 + 3/2R_3, \\ R_2 \to R_2 + 3R_3 \\ R_4 \to R_4 - 23/2R_3 \end{array}$$

=

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The required solution is $x = 0, y = 0, z = 0$ which is trivial solution.

Rank of a Matrix

The positive integer r is said to be a rank of a matrix A if it possesses the following properties:

(1) There is at least one minor of order r which is non-zero.

(2) Every minor of order greater than r is zero.

- **Notes:**

1. Rank of matrix A is denoted by $\rho(A)$
2. The rank of matrix remains unchanged by elementary transformation
3. $\rho(A^T) = \rho(A)$
4. The rank of the product of two matrices always less than or equal to the rank of either matrix(i.e., $\rho(AB) \le \rho(A)$ or $\rho(AB) \le \rho(B)$).

Methods for finding Rank of a Matrix

❖ **Method-1: Rank of a Matrix by Determinant Matrix**

Consider a square matrix A of order r.

- **Step-1:** Find the determinant of A. If $det(A) \neq 0$ then $\rho(A) = r$. Otherwise $\rho(A) < r$.
- **Step-2:** Find the all-possible minors of order $r-1$. If any one of them is non-zero then order is $r-1$, otherwise $\rho(A) < r-1$.
- **Step-3:** By continuing this process upto the non-zero determinant.

Example 1: Find the rank the following matrices by determinant method:

(1) $\boldsymbol{A = \begin{bmatrix} 2 & 3 & 4 \\ 4 & 3 & 1 \\ 1 & 2 & 4 \end{bmatrix}}$

Solution: Given, $A = \begin{bmatrix} 2 & 3 & 4 \\ 4 & 3 & 1 \\ 1 & 2 & 4 \end{bmatrix}$ then $det(A) \neq 0$. Hence, the $\rho(A) = 3$

(2) $\boldsymbol{A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \end{bmatrix}}$

Solution: Given, $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \end{bmatrix}$ then $det(A) = 0$. Hence, the rank of A is less than 3.

Now, minor of 1 $= \begin{vmatrix} 3 & 4 \\ 5 & 7 \end{vmatrix} = 21 - 20 = 1 \neq 0$. Hence, $\rho(A) = 2$.

(3) $\boldsymbol{A = \begin{bmatrix} 4 & 2 & 3 \\ 8 & 4 & 6 \\ -2 & -1 & -\frac{3}{2} \end{bmatrix}}$

Solution: Given, $A = \begin{bmatrix} 4 & 2 & 3 \\ 8 & 4 & 6 \\ -2 & -1 & -\frac{3}{2} \end{bmatrix}$ then $det(A) = 0$. Hence $\rho(A) < 3$.

Consider all the minors of order 2, i.e.,

$$\begin{vmatrix}4 & 2\\8 & 4\end{vmatrix} = 0, \begin{vmatrix}2 & 3\\4 & 6\end{vmatrix} = 0, \begin{vmatrix}4 & 3\\8 & 6\end{vmatrix} = 0, \begin{vmatrix}4 & 2\\-2 & -1\end{vmatrix} = 0, \begin{vmatrix}2 & 3\\-1 & -\frac{3}{2}\end{vmatrix} = 0, \begin{vmatrix}4 & 3\\-2 & -\frac{3}{2}\end{vmatrix} = 0$$

Here, all the minors of order 2 are zero. There rank is less than 2. Hence, $\rho(A) = 1$.

(4) $\boldsymbol{A} = \begin{bmatrix} \mathbf{1} & \mathbf{2} & \mathbf{-1} & \mathbf{-4} \\ \mathbf{2} & \mathbf{4} & \mathbf{3} & \mathbf{5} \\ \mathbf{-1} & \mathbf{-2} & \mathbf{6} & \mathbf{-7} \end{bmatrix}$

Solution: Here, the order of matrix A is 3×4. Hence the rank of A is maximum 3 as we can find the square matrix of order 3. Therefore, consider all the minors of order 3, i.e.,

$$\begin{vmatrix}1 & 2 & -1\\2 & 4 & 3\\-1 & -2 & 6\end{vmatrix} = 0, \begin{vmatrix}2 & -1 & -4\\4 & 3 & 5\\-2 & 6 & -7\end{vmatrix} = 0, \begin{vmatrix}1 & 2 & -4\\2 & 4 & 5\\-1 & -2 & -7\end{vmatrix} = 0, \begin{vmatrix}1 & -1 & -4\\2 & 3 & 5\\-1 & 6 & -7\end{vmatrix} = -120$$

Here, one minor of rank 3 is not equal to zero. Hence, $\rho(A) = 3$.

❖ **<u>Method-2: Rank of a Matrix by Row Echelon Form</u>**

The Rank of a Matrix in Row Echelon Form is equal to the number of non-zero rows of the matrix.

For example: $= \begin{vmatrix}1 & 3 & -1\\0 & 1 & 4\\0 & 0 & 0\end{vmatrix}$; the matrix A is in Row Echelon form with two non-zero rows. Hence, rank of matrix A is 2.

Example 1: Find the rank the following matrices by reducing to Row Echelon Form:

(1) $\boldsymbol{A} = \begin{bmatrix} \mathbf{5} & \mathbf{3} & \mathbf{14} & \mathbf{4} \\ \mathbf{0} & \mathbf{1} & \mathbf{2} & \mathbf{1} \\ \mathbf{1} & \mathbf{-1} & \mathbf{2} & \mathbf{0} \end{bmatrix}$ Solution: Given $A = \begin{bmatrix} 5 & 3 & 14 & 4 \\ 0 & 1 & 2 & 1 \\ 1 & -1 & 2 & 0 \end{bmatrix}$	**(2)** $\boldsymbol{A} = \begin{bmatrix} \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{-1} \\ \mathbf{-2} & \mathbf{-1} & \mathbf{-3} & \mathbf{-1} \\ \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{1} \\ \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{-1} \end{bmatrix}$ Solution: Given

By applying row-operations $R_{13}: \begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & 1 & 2 & 1 \\ 5 & 3 & 14 & 4 \end{bmatrix}$ $R_3 - 5R_1: \begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 8 & 4 & 4 \end{bmatrix}$ $R_3 - 8R_2: \begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & -12 & -4 \end{bmatrix}$ $\left(-\frac{1}{12}\right)R_3: \begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & \frac{1}{3} \end{bmatrix}$ The equivalent matrix is in Row-Echelon Form. Number of non-zero rows $= 3$ .Hence, $\rho(A) = 3$	$A = \begin{bmatrix} 1 & 2 & 3 & -1 \\ -2 & -1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix}$ By applying row-operations $R_2 + 2R_1, R_3 - R_1: \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 3 & 3 & -3 \\ 0 & -2 & -2 & 2 \\ 0 & 1 & 1 & -1 \end{bmatrix}$ $R_{24}: \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 1 & 1 & -1 \\ 0 & -2 & -2 & 2 \\ 0 & 3 & 3 & -3 \end{bmatrix}$ $R_3 + 2R_2, R_4 - 3R_2: \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ The equivalent matrix is in Row-Echelon Form. Number of non-zero rows $= 2$.Hence, $\rho(A) = 2$

Exercise: (1) Find the ranks of A, B, AB and verify $\rho(AB) \leq \rho(A)$ or $\rho(AB) \leq \rho(B)$) where

$A = \begin{vmatrix} 2 & 4 & 1 \\ 3 & 6 & 2 \\ 4 & 8 & 3 \end{vmatrix}, B = \begin{vmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 4 & 3 & 5 \end{vmatrix}$

(2) Find the rank the following matrices by reducing to Row Echelon Form:

$$\text{(I) } A=\begin{bmatrix} 1 & 1 & -1 & 1 \\ 1 & -1 & 2 & -1 \\ 3 & 1 & 0 & 1 \end{bmatrix}, \text{ (II) } A=\begin{bmatrix} 3 & -2 & 0 & -1 & -7 \\ 0 & 2 & 2 & 1 & -5 \\ 1 & -2 & -3 & -2 & 1 \\ 0 & 1 & 2 & 1 & 6 \end{bmatrix}$$

Important Results

(1) If $\rho(A) \neq \rho(A|B)$ then the system is inconsistent.

(2) If $\rho(A) = \rho(A|B)$ then the system is consistent.

(3) If $\rho(A) < n$ then there are infinitely many solutions (n is the number of unknowns)

(4) If $\rho(A) = n$ then there is a unique solution.

Example: Find the number of parameters in the general solution of $AX = O$ if A is a 5×7 matrix of rank 3.

Solution: Here, $\rho(A) = 3$ and $n = 7$. Hence, number of parameters = $n - \rho(A) = 7 - 3 = 4$.

Eigen values and Eigen vectors

Let A be $n \times n$ matrix, then there exists a real number λ and a nonzero vector X such that

$$AX = \lambda X$$

then, λ is called as the eigen value or characteristic value or proper roots of the matrix A, and X is called as eigen vector or characteristic vector or real vector corresponding to eigen value λ of the matrix A.

Notes

1. An eigen vector is never the zero vector.
2. The matrix [A $-\lambda I_n$] is known as the **characteristic matrix** of A.
3. The determinant of (A $-\lambda I_n$) after expansion gives the polynomial in λ, it is known as the **characteristic polynomial** of the matrix A of order $n \times n$ and is of degree n.

4. $|A - \lambda I_n|$=0 is called the **characteristic equation** of matrix A.
5. The root of the characteristic equation is known as **characteristic value** or **eigenvalue** of the matrix.
6. The set of all characteristic roots (eigen values) of the matrix A is called the **spectrum of A.**
7. Let A be $n \times n$ matrix and λ be an eigen value for A. Then the set $E_\lambda = \{X/AX = \lambda X\}$ is called the **eigen space of λ.**

Results

1. The eigen values of a diagonal matrix are its diagonal elements.
2. The sum of eigen values of an $n \times n$ matrix is its trace and their product is $|A|$.
3. For the upper triangular (lower triangular) $n \times n$matrix A, the eigen values are its diagonal elements.

Example 1: If $A = \begin{vmatrix} 1 & 4 \\ 3 & 2 \end{vmatrix}$, find the eigen values for the given matrices:

(i) A, (ii) A^T, (iii) A^{-1}, (iv) $4A^{-1}$, (v) A^2, (vi) $A^2 - 2A + I$,

(vii) $A^3 + 2I$

Solution: Given, $A = \begin{vmatrix} 1 & 4 \\ 3 & 2 \end{vmatrix}$

The characteristic equation of matrix A is

$$|A - \lambda I_2| = 0$$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 4 \\ 3 & 2-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)(2-\lambda) - 12 = 0 \Rightarrow \lambda^2 - 3\lambda - 10 = 0 \Rightarrow (\lambda - 5)(\lambda + 2) = 0$$

$$\therefore \lambda = 5 \text{ or } \lambda = -2$$

Eigenvalues of $A = \lambda$	$5, -2$
Eigenvalues of $A^T = \lambda^T$	$5, -2$
Eigenvalues of $A^{-1} = \lambda^{-1}$	$\frac{1}{5}, -\frac{1}{2}$

Eigenvalues of $4A^{-1} = 4\lambda^{-1}$	$\frac{4}{5}, -2$
Eigenvalues of $A^2 = \lambda^2$	$25, 4$
Eigenvalues of $A^2 - 2A + I = \lambda^2 - 2\lambda + 1$	$16, 9$
Eigenvalues of $A^3 + 2I = \lambda^3 + 2$	$127, -6$

Example 2: Find the eigen values of $A = \begin{vmatrix} 3 & 2 \\ 3 & 8 \end{vmatrix}$

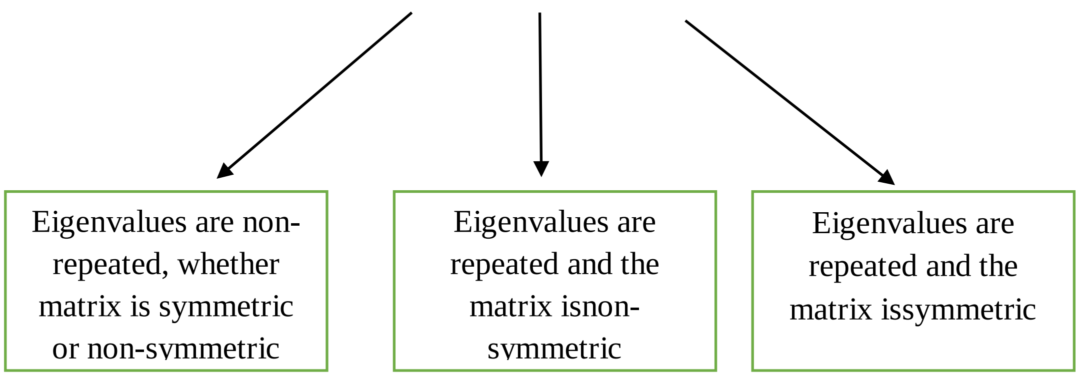
Solution: Given $A = \begin{vmatrix} 3 & 2 \\ 3 & 8 \end{vmatrix}$, then the characteristic equation of matrix A is

$$|A - \lambda I_2| = 0$$

$$\Rightarrow \begin{vmatrix} 3-\lambda & 2 \\ 3 & 8-\lambda \end{vmatrix} = 0 \Rightarrow (3-\lambda)(8-\lambda) - 6 = 0 \Rightarrow \lambda^2 - 11\lambda + 18 = 0 \Rightarrow (\lambda - 9)(\lambda - 2) = 0$$

$$\therefore \lambda_1 = 9 \text{ or } \lambda_2 = 2$$

Types of Eigen Values



Type-1Type-2 Type-3

Example 3: Find the eigen values and eigen vector of the matrix A= $\begin{bmatrix} -2 & -8 & -12 \\ 1 & 4 & 4 \\ 0 & 0 & 1 \end{bmatrix}$	
Solution: The characteristic equation is $\|A - \lambda I_n\|$=0	$= \left[\begin{array}{ccc\|c} 1 & 8/3 & 4 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] R_1 \to R_1 - 8/3R_2$

$$\begin{vmatrix} -2-\lambda & -8 & -13 \\ 1 & 4-\lambda & 4 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0$$

$$\therefore \lambda^3 - S_1\lambda^2 + S_2\lambda - |A| = 0$$

$S_1 = tr(A) = -2+4+1 = 3$

$S_2 =$ Sum of minors of diagonal entries

$$= \begin{vmatrix} 4 & 4 \\ 0 & 1 \end{vmatrix} + \begin{vmatrix} -2 & -12 \\ 0 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 4 \\ 0 & 0 \end{vmatrix} = 4-2+0 = 2$$

$$|A| = -2(4) + 8(1) - 12(0) = -8+8 = 0$$

$\therefore$ characteristic equation is

$$\lambda^3 - 3\lambda^2 + 2\lambda = 0$$

$$\Rightarrow \lambda\left(\lambda^2 - 3\lambda + 2\right) = 0$$

$$\Rightarrow \lambda = 0 \text{ or } \lambda^2 - 3\lambda + 2 = 0$$

$$\Rightarrow \lambda = 0 \text{ or} (\lambda-2)(\lambda-1) = 0$$

$$\Rightarrow \lambda = 0 \text{ or } \lambda = 1 \text{or } \lambda = 2$$

Here, one can observe that all eigenvalues are non-repeated and matrix is non-symmetric.

When $\lambda_1 = 0$

$$[A-\lambda I \mid O] = \left[\begin{array}{ccc|c} -2 & -8 & -12 & 0 \\ 1 & 4 & 4 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] R_1 \to -1/2R_1$$

$$= \left[\begin{array}{ccc|c} 1 & 4 & 6 & 0 \\ 1 & 4 & 4 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] R_2 \to R_2 - R_1$$

$$= \left[\begin{array}{ccc|c} 1 & 4 & 6 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$= \left[\begin{array}{ccc|c} 1 & 0 & 4 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

We suppose $z = k, y = 0, x + 4z = 0$

$$\therefore z = k, y = 0, x = -4z = -4k$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -4k \\ 0 \\ k \end{bmatrix} = k\begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix}$$

Therefore, eigen vector space for $\lambda_1 = 0$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix}$$

When $\lambda_3 = 2$

$$[A-\lambda I \mid O] = \left[\begin{array}{ccc|c} -2-2 & -8 & -12 & 0 \\ 1 & 4-2 & 4 & 0 \\ 0 & 0 & 1-2 & 0 \end{array}\right]$$

$$= \left[\begin{array}{ccc|c} -4 & -8 & -12 & 0 \\ 1 & 2 & 4 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right] R_1 \to -1/4R_1$$

$$= \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 1 & 2 & 4 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right] R_2 \to R_2 - R_1$$

$$= \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right] R_3 \to -R_3$$

Therefore, we suppose

$x+4y+6z=0,-2z=0,y=k$

$\therefore z=0,y=k,x=-4k$

Therefore, eigen vector space is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -4k \\ k \\ 0 \end{bmatrix} = k\begin{bmatrix} -4 \\ 1 \\ 0 \end{bmatrix}$$

Therefore, eigen vector space for $\lambda_1 = 0$ is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -4 \\ 1 \\ 0 \end{bmatrix}$$

When $\lambda_2 = 1$

$$[A-\lambda I \mid O] = \left[\begin{array}{ccc|c} -2-1 & -8 & -12 & 0 \\ 1 & 4-1 & 4 & 0 \\ 0 & 0 & 1-1 & 0 \end{array}\right]$$

$$= \left[\begin{array}{ccc|c} -3 & -8 & -12 & 0 \\ 1 & 3 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] R_1 \to -1/3R_1$$

$$= \left[\begin{array}{ccc|c} 1 & 8/3 & 4 & 0 \\ 1 & 3 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] R_2 \to R_2 - R_1$$

$$= \left[\begin{array}{ccc|c} 1 & 8/3 & 4 & 0 \\ 0 & 1/3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] R_2 \to 3R_2$$

$$= \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \begin{array}{l} R_1 \to R_1 - 3R_3 \\ R_2 \to R_2 - R_3 \end{array}$$

$$= \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

We suppose $z=0, y=k, x+2y=0$

$\therefore z=0, y=k, x=-2z=-2k$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2k \\ k \\ 0 \end{bmatrix} = k\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

Therefore, eigen vector space for $\lambda_3 = 2$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

<u>Algebraic multiplicity and Geometric multiplicity</u>

Let A be $n \times n$ matrix and λ be an eigen value for A. If λ occurs $(k \geq 1)$ times then k is called the **Algebraic multiplicity** of λ, and the number of basis vectors is called **Geometric multiplicity**.

Example: Find eigen values and eigen vectors of the matrix. $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$. Also determine algebraic and geometric multiplicity.

Solution: The characteristic equation is $|A - \lambda I_n| = 0$.

$$\begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & 0-\lambda \end{vmatrix}$$

$$= (-2-\lambda)[(1-\lambda)(-\lambda)-(-2)(-6)] - 2[2(-\lambda)-(-1)(-6)] - 3[2(-2)-(-1)(1-\lambda)]$$

$$= (-2-\lambda)[-\lambda+\lambda^2-2] - 2[-2\lambda-6] - 3[-4+1-\lambda]$$

$$= -\lambda^3 - \lambda^2 + 21\lambda + 45$$

$$= -(\lambda^3 + \lambda^2 - 21\lambda - 45)$$

$$\therefore -(\lambda^3 + \lambda^2 - 21\lambda - 45) = 0$$

$$\therefore \lambda^3 + \lambda^2 - 21\lambda - 45 = 0$$

$$\therefore (\lambda+3)(\lambda^2 - 2\lambda - 15) = 0$$

$$\Rightarrow (\lambda+3)(\lambda+3)(\lambda-5) = 0$$

$$\Rightarrow \lambda_1 = 5, \lambda_2 = -3, \lambda_3 = -3$$

Algebraic Multiplicity of $\lambda = -3$ is 2 and of $\lambda = 5$ is 1.

We solve the following homogeneous system:

$$\therefore [A - \lambda I]X = \begin{bmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & 0-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Case I: When$\lambda_1 = 5$	Case II :When$\lambda_2 = -3, \lambda_3 = -3$

$\therefore [A-\lambda I \mid O] = \left[\begin{array}{ccc} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & 0-\lambda \end{array}\right] \left[\begin{array}{c} 0 \\ 0 \\ 0 \end{array}\right]$ $=$ $\left[\begin{array}{ccc\|c} -7 & 2 & -3 & 0 \\ 2 & -4 & -6 & 0 \\ -1 & 2 & -5 & 0 \end{array}\right] R_1 \leftrightarrow R_3$ $=$ $\left[\begin{array}{ccc\|c} -1 & -2 & -5 & 0 \\ 2 & -4 & -6 & 0 \\ -7 & 2 & -3 & 0 \end{array}\right] R_1 \to -R_1$ $=$ $\left[\begin{array}{ccc\|c} 1 & 2 & 5 & 0 \\ 2 & -4 & -6 & 0 \\ -7 & 2 & -3 & 0 \end{array}\right] \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 + 7R_1 \end{array}$ $=$ $\left[\begin{array}{ccc\|c} 1 & 2 & 5 & 0 \\ 0 & -8 & -16 & 0 \\ 0 & 16 & 32 & 0 \end{array}\right] R_2 \to -1/8 R_2$ $=$ $\left[\begin{array}{ccc\|c} 1 & 2 & 5 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 16 & 32 & 0 \end{array}\right] \begin{array}{l} R_1 \to R_1 - 2R_2 \\ R_3 \to R_3 - 16R_2 \end{array}$ $=$ $\left[\begin{array}{ccc\|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$	$\therefore [A-\lambda I \mid O] = \left[\begin{array}{ccc} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & 0-\lambda \end{array}\right] \left[\begin{array}{c} 0 \\ 0 \\ 0 \end{array}\right]$ $=$ $\left[\begin{array}{ccc\|c} 1 & 2 & -3 & 0 \\ 2 & 4 & -6 & 0 \\ -1 & -2 & 3 & 0 \end{array}\right] \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 + R_1 \end{array}$ $=$ $\left[\begin{array}{ccc\|c} 1 & 2 & -3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ which is in Row-Echelon from. We suppose $x_2 = k_1, x_3 = k_2, x_1 + 2x_2 - 3x_3 = 0 \Rightarrow x_1 = -2k_1 + 3k_2$ Therefore, eigen space is for $\lambda_2 = -3, \lambda_3 = -3$ is $\left\{ k_1(-2,1,0) + k_2(3,0,1) / k_1, k_2 \in R \right\}$ Hence, Geometric multiplicity of $\lambda_2 = -3$ is 2 and of $\lambda = 5$ is 1.

which is in Row-Echelon from. We suppose $\begin{matrix} x_3 = k, x_2 + 2x_3 = 0 \Rightarrow x_2 = -2k, \\ x_1 + x_3 = 0 \Rightarrow x_1 = -k \end{matrix}$ Therefore, eigen space is for $\lambda_1 = 5$ is $\left\{ k\left(-1,-2,1\right) / k \in R \right\}$	

Example: Find eigen values and eigen vectors of the matrix. $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$. Also determine algebraic and geometric multiplicity.

Solution: The characteristic equation is $|A - \lambda I_n| = 0$.

$$\begin{vmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix}$$

$$= -\lambda(\lambda^2 - 1) - 1(-\lambda - 1) + 1(1 + \lambda)$$

$$= -\lambda^3 + 3\lambda + 2 = -(\lambda^3 - 3\lambda - 2)$$

$$\therefore -(\lambda^3 - 3\lambda - 2) = 0$$

$$\therefore \lambda^3 - 3\lambda - 2 = 0$$

$$\Rightarrow (\lambda - 2)(\lambda + 1)^2 = 0$$

$$\Rightarrow \lambda_1 = 2, \lambda_2 = -1, \lambda_3 = -1$$

Algebraic Multiplicity of $\lambda = -1$ is 2 and of $\lambda = 2$ is 1.

Case-1: $\lambda_1 = 2$ $\therefore [A - \lambda I \| O] = \begin{bmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$	Case-2: $\lambda_2 = -1, \lambda_3 = -1$ $\therefore [A - \lambda I \| O] = \begin{bmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$=\left[\begin{array}{ccc\|c} -2 & 1 & 1 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & 1 & -2 & 0 \end{array}\right]$ $R_2 \leftrightarrow R_1$ $=\left[\begin{array}{ccc\|c} 1 & -2 & 1 & 0 \\ -2 & 1 & 1 & 0 \\ 1 & 1 & -2 & 0 \end{array}\right]$ $R_2 \to R_2 + 2R_1$, $R_3 \to R_3 - R_1$ $=\left[\begin{array}{ccc\|c} 1 & -2 & 1 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & 3 & -3 & 0 \end{array}\right]$ $R_2 \to -1/3R_2$ $=\left[\begin{array}{ccc\|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 3 & -3 & 0 \end{array}\right]$ $R_1 \to R_1 + 2R_2$, $R_3 \to R_3 - 3R_2$ $=\left[\begin{array}{ccc\|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ Let $x_3 = k, x_2 - x_3 = 0, \Rightarrow x_2 = k, x_1 - x_3 = 0, x_1 = k$. Therefore, eigen space is for $\lambda_1 = 2$ is $\{k(1,1,1)/k \in R\}$	$=\left[\begin{array}{ccc\|c} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right]$ $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$ $=\left[\begin{array}{ccc\|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ Let $x_3 = k_1, x_2 = k_2,$ $\Rightarrow x_1 + x_2 + x_3 = 0 \Rightarrow x_1 = -k_1 - k_2$. Therefore, eigen space is for $\lambda_2 = -1, \lambda_3 = -1$ is $\{k_1(-1,0,1) + k_2(-1,1,0)/k_1, k_2 \in R\}$ Hence, Geometric Multiplicity of $\lambda_2 = -1$ is 2 and $\lambda_1 = 2$ of is 1.

Example: Determine algebraic and geometric multiplicity ofmatrix $A = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 1 \\ -1 & 2 & 2 \end{bmatrix}$.

<u>Answer:</u> $\lambda = 1,2,2$ therefore algebraic multiplicity of $\lambda = 2$ is 2 and geometric multiplicity is 1. For $\lambda = 1$ A.M. is 1 and G.M. is 1.

<u>Note</u>

Theorem: Every square matrix can be decomposed as a sum of symmetric and skew-symmetric matrices.

Proof: Let A be $m \times n$ matrix.

Let $B = \frac{1}{2}(A + A^T)$ and $C = = \frac{1}{2}(A - A^T)$ be two matrices.

Obviously, $A = B + C$

Now,$B^T = \left[\frac{1}{2}(A + A^T)\right]^T = \frac{1}{2}[(A + A^T)]^T = \frac{1}{2}[A^T + (A^T)^T] = \frac{1}{2}(A^T + A) = B$

As $B^T = B$, B is symmetric.

$$C^T = \left[\frac{1}{2}(A - A^T)\right]^T = \frac{1}{2}[(A - A^T)]^T = \frac{1}{2}[A^T - (A^T)^T] = \frac{1}{2}(A^T - A) = -C$$

Therefore,$C^T = -C$, C is skew- symmetric.

Therefore, A is a sum of symmetric and skew-symmetric matrices.

Caley –Hamilton Theorem

Every square matrix satisfies its own characteristic equation i.e. The theorem states that, for a square matrix A of order n, if $|A - \lambda I_n| = 0$.

Example (i): Verify Caley-Hamilton theorem and hence find the inverse of $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$ and A^4.

Solution: The characteristic equation for given matrix is

$$|A - \lambda I_2| = 0.$$

$\Rightarrow \begin{vmatrix} 1-\lambda & 4 \\ 2 & 3-\lambda \end{vmatrix} = 0$

$\Rightarrow (1-\lambda)(3-\lambda) - 8 = 0$

$\Rightarrow \lambda^2 - 4\lambda - 5 = 0$

Now, by putting $\lambda = A$, we have

$$A^2 - 4A - 5I = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} - 4\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 9 & 16 \\ 8 & 17 \end{bmatrix} - \begin{bmatrix} 4 & 16 \\ 8 & 12 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O$$

Hence, Cayley-Hamilton theorem verified.

Now, by using Cayley-Hamilton theorem, we have

$A^2-4A-5I=0$, by applying A^{-1} on both the sides

$$A^{-1}\left(A^2-4A-5I\right)=A^{-1}(0)$$
$$\Rightarrow A-4I-5A^{-1}=0$$
$$\Rightarrow 5A^{-1}=A-4I$$
$$\Rightarrow A^{-1}=\frac{1}{5}(A-4I)=\frac{1}{5}\left(\begin{bmatrix}1 & 4\\ 2 & 3\end{bmatrix}-4\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right)=\frac{1}{5}\begin{bmatrix}-3 & 4\\ 2 & -1\end{bmatrix}$$

And for A^4, applying A^2 both the sides

$$A^2\left(A^2-4A-5I\right)=A^2(0)$$
$$\Rightarrow A^4-4A^3-5A^2=0$$
$$\Rightarrow A^4=4A^3+5A^2$$
$$\Rightarrow A^4=4\begin{bmatrix}41 & 84\\ 42 & 83\end{bmatrix}+5\begin{bmatrix}9 & 16\\ 8 & 17\end{bmatrix}\Rightarrow A^4=\begin{bmatrix}209 & 416\\ 208 & 417\end{bmatrix}$$

Example (ii): Find the characteristics equationof the matrix $A=\begin{bmatrix}2 & 1 & 1\\ 0 & 1 & 0\\ 1 & 1 & 2\end{bmatrix}$ and hence prove that

$$A^8-5A^7+7A^6-3A^5+A^4-5A^3+8A^2-2A+I=\begin{bmatrix}8 & 5 & 5\\ 0 & 3 & 0\\ 5 & 5 & 8\end{bmatrix}.$$

<u>Solution</u>: The characteristics equation is

$$|A-\lambda I|=\begin{vmatrix}2-\lambda & 1 & 1\\ 0 & 1-\lambda & 0\\ 1 & 1 & 2-\lambda\end{vmatrix}=0$$
$$\therefore \lambda^3-5\lambda^2+7\lambda-3=0$$

By Caley-Hamilton Theorem

$\therefore A^3 - 5A^2 + 7A - 3I = 0$ ....(1)

Now,

$A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$

$= A^5(A^3 - 5A^2 + 7A - 3I) + A(A^3 - 5A^2 + 7A - 3I) + A^2 + A + I$

$= A^2 + A + I$ $\quad$ using (1)

$$\therefore A^2 + A + I = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} + \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 5 & 5 \\ 0 & 3 & 0 \\ 5 & 5 & 8 \end{bmatrix}$$

Exercise: (1)Verify Caley-Hamilton theorem and hence find the inverse of $A = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$ and A^4.

(2) Compute $A^9 - 6A^8 + 10A^7 - 3A^6 + A + I$, where $A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 3 & 1 \\ 1 & 0 & 2 \end{bmatrix}$ (Answer: $\begin{bmatrix} 2 & 2 & 3 \\ -1 & 4 & 1 \\ 1 & 0 & 3 \end{bmatrix}$)

Diagonalization of a matrix:

An $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors.

OR

If $n \times n$ matrix A has a basis of eigenvectors, then $D = P^{-1}AP$ is diagonal, with the eigenvalues of A as the entries on the main diagonal. Here, P is the matrix with these eigenvectors as column vectors.

Also, $D^n = P^{-1}A^nP$ and $A^n = PD^nP^{-1}$

Example (i): Find a matrix P that diagonalizes matrix A and determine $P^{-1}AP$

$$A = \begin{bmatrix} -1 & 4 & -2 \\ -3 & 4 & 0 \\ -3 & 1 & 3 \end{bmatrix}$$

$$A = \begin{bmatrix} -1 & 4 & -2 \\ -3 & 4 & 0 \\ -3 & 1 & 3 \end{bmatrix}$$

Solution (i):

The characteristic equation is $|A - \lambda I_n| = 0$

$$\begin{vmatrix} -1-\lambda & 4 & -2 \\ -3 & 4-\lambda & 0 \\ -3 & 1 & 3-\lambda \end{vmatrix} = 0$$

$$\therefore \lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$$

$$\therefore (\lambda - 1)(\lambda - 2)(\lambda - 3) = 0$$

$$\therefore \lambda = 1,2,3$$

For $\lambda = 1$

$$\therefore [A - \lambda I | O] = \begin{bmatrix} -1-\lambda & 4 & -2 \\ -3 & 4-\lambda & 0 \\ -3 & 1 & 3-\lambda \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

=

$$\left[\begin{array}{ccc|c} -2 & 4 & -2 & 0 \\ -3 & 3 & 0 & 0 \\ -3 & 1 & 2 & 0 \end{array}\right] \; R_1 \to -1/2R_1$$

=

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ -3 & 3 & 0 & 0 \\ -3 & 1 & 2 & 0 \end{array}\right] \; \begin{array}{l} R_2 \to R_2 + 3R_1 \\ R_3 \to R_3 + 3R_1 \end{array}$$

$$= \left[\begin{array}{ccc|c} 1 & -4/3 & 2/3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -3 & 3 & 0 \end{array}\right] \; \begin{array}{l} R_1 \to R_1 + 4/3R_2 \\ R_3 \to R_3 + 3R_2 \end{array}$$

$$= \left[\begin{array}{ccc|c} 1 & 0 & -2/3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\therefore z = k, y - z = 0 \;\&\; x - 2/3z = 0$$

$$\Rightarrow z = k, y = k, x = 2/3k$$

$$\therefore (x,y,z) = k(\frac{2}{3},1,1); k \in R$$

$$\therefore (x,y,z) = 3k(2,3,3); k \in R \quad (\square \; 3k = k')$$

$$E_1 = \{k'(2,3,3) / k' \in R\}$$

For $\lambda = 3$

$$\therefore [A - \lambda I | O] = \begin{bmatrix} -1-\lambda & 4 & -2 \\ -3 & 4-\lambda & 0 \\ -3 & 1 & 3-\lambda \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

=

$$\left[\begin{array}{ccc|c} -4 & 4 & -2 & 0 \\ -3 & 1 & 0 & 0 \\ -3 & 1 & 0 & 0 \end{array}\right] \; R_1 \to -1/4R_1$$

=

$=$

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & -5 & 5 & 0 \end{array}\right] R_2 \to -1/3R_2$$

$=$

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -5 & 5 & 0 \end{array}\right] \begin{array}{l} R_1 \to R_1 + 2R_2 \\ R_3 \to R_3 + 5R_2 \end{array}$$

$=$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\therefore z = k, y - z = 0 \,\&\, x - z = 0$

$\Rightarrow z = k, y = k, x = k$

$\therefore (x, y, z) = k(1,1,1); k \in R$

$E_1 = \{k(1,1,1) / k \in R\}$

For $\lambda = 2$

$$\therefore [A - \lambda I | O] = \begin{bmatrix} -1-\lambda & 4 & -2 \\ -3 & 4-\lambda & 0 \\ -3 & 1 & 3-\lambda \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$=$

$$\left[\begin{array}{ccc|c} -3 & 4 & -2 & 0 \\ -3 & 2 & 0 & 0 \\ -3 & 1 & 1 & 0 \end{array}\right] R_1 \to -1/3R_1$$

$=$

$$\left[\begin{array}{ccc|c} 1 & -4/3 & 2/3 & 0 \\ -3 & 2 & 0 & 0 \\ -3 & 1 & 1 & 0 \end{array}\right] \begin{array}{l} R_2 \to R_2 + 3R_1 \\ R_3 \to R_3 + 3R_1 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1/2 & 0 \\ -3 & 1 & 0 & 0 \\ -3 & 1 & 0 & 0 \end{array}\right] \begin{array}{l} R_2 \to R_2 + 3R_1 \\ R_3 \to R_3 + 3R_1 \end{array}$$

$=$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1/2 & 0 \\ 0 & -2 & 3/2 & 0 \\ 0 & -2 & 3/2 & 0 \end{array}\right] R_2 \to -1/2R_2$$

$=$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1/2 & 0 \\ 0 & 1 & -3/4 & 0 \\ 0 & -2 & 3/2 & 0 \end{array}\right] \begin{array}{l} R_1 \to R_1 + R_2 \\ R_3 \to R_3 + 2R_2 \end{array}$$

$=$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1/4 & 0 \\ 0 & 1 & -3/4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\therefore z = k, y - 3/4z = 0 \,\&\, x - 1/4z = 0$

$\Rightarrow z = k, y = 3/4k, x = 1/4k$

$\therefore (x, y, z) = k(\frac{1}{4}, \frac{3}{4}, 1); k \in R$

$\therefore (x, y, z) = 4k(1,3,4); k \in R \quad (\square\ 4k = k')$

$E_1 = \{k'(1,3,4) / k' \in R\}$

$$\therefore P = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 3 \\ 1 & 3 & 4 \end{bmatrix}$$

$$\therefore P^{-1}AP = D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

= $\left[\begin{array}{ccc\|c} 1 & -4/3 & 2/3 & 0 \\ 0 & -2 & 2 & 0 \\ 0 & -3 & 3 & 0 \end{array}\right] R_2 \to -1/2R_2$	

Example (ii): Find a matrix P that diagonalizes A and determine $P^{-1}AP$where $A = \begin{bmatrix} 1 & 0 \\ 6 & -1 \end{bmatrix}$ Also find A^{10} and find eigenvalues of A^2.	
Solution (ii): $A = \begin{bmatrix} 1 & 0 \\ 6 & -1 \end{bmatrix}$ The characteristic equation is $\|A - \lambda I_n\| = 0$ $\begin{vmatrix} 1-\lambda & 0 \\ 6 & -1-\lambda \end{vmatrix} = 0$ $\therefore (1-\lambda)(-1-\lambda) - 0 = 0$ $\therefore \lambda = 1,-1$ For $\lambda = 1$ $\therefore [A - \lambda I \| O] = \left[\begin{array}{cc\|c} 1-\lambda & 0 & 0 \\ 6 & -1-\lambda & 0 \end{array}\right]$ = $\left[\begin{array}{cc\|c} 0 & 0 & 0 \\ 6 & -2 & 0 \end{array}\right]$	For $\lambda = -1$ $\therefore [A - \lambda I \| O] = \left[\begin{array}{cc\|c} 1-\lambda & 0 & 0 \\ 6 & -1-\lambda & 0 \end{array}\right]$ = $\left[\begin{array}{cc\|c} 2 & 0 & 0 \\ 6 & 0 & 0 \end{array}\right]$ $y = k, 6x = 0 => x = 0$ $\therefore (x, y) = \{k(0,1)/k \in R\}$ Now, $P = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \Rightarrow P^{-1} = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$ $P^{-1}AP = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = D$

Suppose $x = k, 6x - 2y = 0$ $x = k, y = 3k$ $\therefore (x, y) = \{k(1,3)/k \in R\}$	$P^{-1}AP = D = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \Rightarrow A = PDP^{-1}$ $A^{10} = PD^{10}P^{-1} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 1^{10} & 0 \\ 0 & (-1)^{10} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$ $= \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$ $A^{10} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ Eigenvalues of A^2 are: $1^2 = 1$ and $(-1)^2 = 1$.

Quadratic Forms

A homogeneous polynomial of second degree in real variables $x_1, x_2, x_3, \dots, x_n$ is called Quadratic form.

For example:

(i) $ax^2 + 2hxy + by^2$ is a quadratic form in the variables x and y

(ii) $2x_1x_2 + 2x_2x_3 + 2x_3x_1 + {x_3}^2$ is a quadratic form in the variables x_1, x_2, x_3.

A quadratic on R^n is a function Q define on R^n whose value at a vector x in R^n can be computed in n variables $x_1, x_2, x_3, \dots, x_n$ by an expression of the form.

$$\boxed{Q(x) = x^T Ax = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}\ x_i\ x_j}$$

Here, A is known as the coefficient matrix. Where A is an $n \times n$ symmetric matrix and is called matrix of the quadratic form.

Matrix Representation of Quadratic Forms

A quadratic form can be represented as a matrix product.

For example:

(I)

(II)

$$ax^2 + 2hxy + by^2 = \begin{bmatrix} x & y \end{bmatrix}\begin{bmatrix} a & h \\ h & b \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$2x_1x_2 + 2x_2x_3 + 2x_3x_1 + x_3^2 = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Example:

(i) Find a real symmetric matrix C of the quadratic form

$$Q = x_1^{\ 2} + 3x_2^{\ 2} + 2x_3^2 + 2x_1x_2 + 6x_2x_3.$$

Solution: The coefficient matrix of Q is

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 6 \\ 0 & 0 & 2 \end{bmatrix}$$

So, C = symmetric matrix =

$$\left[\frac{1}{2}(A + A^T)\right] = \frac{1}{2}1\left\{\begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 6 \\ 0 & 0 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 0 & 6 & 2 \end{bmatrix}\right\} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 3 & 3 \\ 0 & 3 & 2 \end{bmatrix}$$

(ii) Express the following quadratic forms in matrix notation

$$Q = x^2 - 4xy + y^2$$

Solution:

$$x^2 - 4xy + y^2 = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Transformation (Reduction) of Quadratic form to canonical form OR Diagonalizing Quadratic Forms:

Procedure to Reduce Quadratic form to canonical form:

1. Identity the real symmetric matrix associated with the quadratic form.
2. Determine the eigenvalues of A.
3. The required canonical form is given by

   $$Q(x) = y^T Dy = \lambda_1 y_1^{\ 2} + \lambda_2 y_2^{\ 2} + \lambda_3 y_3^{\ 2} + ... + \lambda_n y_n^{\ 2} \quad \text{…………..} (1)$$

   where $\lambda_1, \lambda_2, \ldots\ldots \lambda_n$ are the eigenvalues of A and $D = P^T AP$. The matrix P is said to orthogonally diagonalize the quadratic form.

   and equation (1) is known as canonical form.

4. Form modal matrix P (where $x = Py$) containing the n eigenvectors of A as n column vectors.

Example: Reduce the quadratic form into canonical form

$Q = 3x^2 + 3z^2 + 4xy + 8xz + 8yz$

<u>Solution:</u>

$$A = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 4 \\ 4 & 4 & 3 \end{bmatrix}$$

Eigenvalues for$A$ are$3, -\frac{4}{3}, -1.$

The canonical form of the given quadratic form is

$$y^T By = \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 0 & -4/3 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = 3y_1^2 - 4/3y_2^2 - y_3^2$$

Nature of quadratic form *Q*

a. Positive definite if Q(x) > 0 for all x$\neq$0,
b. Negative definite if Q(x) < 0 for all x$\neq$0,
c. Indefinite if Q(x) assumes both positive and negative values.
d. Positive semidefinite if Q(x)$\geq$0 for all x.
e. Negative semidefinite if Q(x)$\leq$0 for all x.

OR

a. Positive definite if and only if the eigenvalues of A are positive,
b. negative definite if and only if the eigenvalues of A are positive,
c. Indefinite if and only if A has both positive and negative eigenvalues.
d. Positive semi-definite if and only if A has only non-negative eigenvalues.
e. Indefinite if and only if A has only non-positive eigenvalues.

Example: Describe the nature of quadratic forms.

1. $Q = 3x_1^2 + x_2^2 + x_3^2 + 4x_1x_2 + 4x_2x_3$
2. $Q = 2x_1x_2 + 2x_2x_3 + 2x_2x_1$

****

Parul University

Faculty of Engineering & Technology

Department of Applied Sciences and Humanities

1^{st} Year B.Tech Programme (All Branches)

Mathematics – 1 (303191101)

Unit – 4 Sequence and Series (Lecture Note)

Sequence:

- Limit of a sequence
- Convergence & Divergence of a sequence
- Oscillatory sequence
- Sandwich/Squeezing theorem for sequences
- Convergence properties of sequence
- Monotonic sequence(Monotonic increasing & Monotonic decreasing)
- Alternating sequence
- Bounded & Unbounded sequence.

Series:

- Convergence, Divergence & Oscillatory series
- Some properties of infinite series
- Telescoping series
- Geometric series
- p-series, Integral test
- Comparison test
  (i)Direct
  (ii)Limit Comparison
- D'Alembert ratio test
- Cauchy's root test
- Alternating series
- Leibnitz test

❖ **Sequence:**

A sequence is a function whose domain is the set of positive integers.

It is generally written as $a_1, a_2, a_3 \ldots\ldots\ldots a_n, \ldots\ldots\ldots$

➢ If the number of terms in a sequence is infinite, it is called infinite sequence otherwise it is said to be finite sequence

$$e.g. \; 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots \qquad ; \quad 1, -1, 2, -2, \ldots$$

❖ **Limit of a sequence:**

Let $\{a_n\}$ be a sequence.

A real number l is said to be the limit of the sequence $\{a_n\}$; if for every $\varepsilon > 0$, there exist an integer N such that $n \geq N \Rightarrow |a_n - l| < \varepsilon$

If such a number exists then we write

$$\lim_{n\to\infty} a_n = l.$$

❖ **Convergence, Divergence & Oscillations of a Sequence:**

➢ A sequence $\{a_n\}$ is said to be convergent if the sequence has finite limit.

$$i.e. \; if \quad \lim_{n\to\infty} a_n = \; finite.$$

➢ A sequence $\{a_n\}$ is said to be divergent if the sequence has infinite limit.

$$i.e. \qquad if \; \lim_{n\to\infty} a_n = \pm\infty.$$

For example, $\lim_{n\to\infty} \frac{1}{n} = 0$, $\lim_{n\to\infty} \frac{n+1}{n} = 1$, $\lim_{n\to\infty} 2n = \infty$,

➢ A sequence $\{a_n\}$ is said to be oscillatory if the sequence is neither convergent nor divergent. For example, let

$$\{u_n\} = \left\{(-1)^n + \frac{1}{2^{\frac{1}{n}}}\right\}$$

$$\lim_{n \to \infty} u_n \begin{matrix} = 2 \, , if \; n \; is \; even \\ = \; 0 \, , if \; n \; is \; odd \end{matrix}$$

Since the limit is not unique, the sequence is oscillatory.

❖ **Convergence properties of sequences:**

➢ Let $\{a_n\}$ and $\{b_n\}$ be two convergent sequences and k be any real number, then the following sequences will also converge.

1) $\{a_n + b_n\}$ With $\lim_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty}(a_n) + \lim_{n\to\infty}(b_n)$

2) $\{ka_n\}$ With $\lim_{n\to\infty}(ka_n) = k \lim_{n\to\infty}(a_n)$

3) $\{a_n b_n\}$ With $\lim_{n\to\infty}(a_n b_n) = \left(\lim_{n\to\infty}(a_n)\right)\left(\lim_{n\to\infty}(b_n)\right)$

4) $\left\{\frac{a_n}{b_n}\right\}$ With $\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right)=\frac{\lim_{n\to\infty}(a_n)}{\lim_{n\to\infty}(b_n)}$; $\left(if \lim_{n\to\infty}(b_n)\neq 0\right)$

❖ **Some Important Formula:**

$\lim_{n\to\infty}\frac{ln(n)}{n}=0$	$\lim_{n\to\infty}\sqrt[n]{n}=1$
$\lim_{n\to\infty}x^{\frac{1}{n}}=1(x>0)$	$\lim_{n\to\infty}x^n=0(\lvert x\rvert<1)$
$\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n=e^x \quad (any\ x)$	$\lim_{n\to\infty}\left(\frac{x^n}{n!}\right)=0 \qquad (any\ x)$

Que.: Applying the definition, show that $\left\{\frac{1}{n}\right\}$ converges 0 as $n\to\infty$.

To prove: Let $\epsilon>0$, we must show that there exists an integer N such that for all n,

$$n>N\Longrightarrow\left|\frac{1}{n}-0\right|<\epsilon$$

Solution: Let $\epsilon>0$ be given.

Let N be an integer such that $N>\frac{1}{\epsilon}$.

$$\begin{aligned}
n\geq N &\Longrightarrow n\geq N>\frac{1}{\epsilon}\\
&\Longrightarrow n>\frac{1}{\epsilon}\\
&\Longrightarrow\left|\frac{1}{n}-0\right|=\frac{1}{n}<\epsilon\\
&\Longrightarrow\left|\frac{1}{n}-0\right|<\epsilon
\end{aligned}$$

$$\therefore \lim_{n\to\infty}\frac{1}{n}=0$$

Que.: Test the Convergence of the following sequences.

1. $\left\{\frac{n^2+n}{2n^2-n}\right\}$ Solution: Let $a_n=\frac{n^2+n}{2n^2-n}$ $\lim_{n\to\infty}a_n=\lim_{n\to\infty}\frac{n^2+n}{2n^2-n}$ $=\lim_{n\to\infty}\frac{1+\frac{1}{n}}{2-\frac{1}{n}}=\frac{1}{2}$ As the value of limit is finite the	2. $\{2^n\}$ Solution: Let $a_n=2^n$ $\lim_{n\to\infty}a_n=\lim_{n\to\infty}2^n$ $=\infty$ As the value of limit of the sequence is infinite the $\{2^n\}$ is divergent. 3. $\{2-(-1)^n\}$ Solution: Let $a_n=2-(-1)^n$

$\left\{\frac{n^2+n}{2n^2-n}\right\}$ is convergent. 4.$\{\sqrt{n+1}-\sqrt{n}\}$ Solution: Let $a_n = \sqrt{n+1}-\sqrt{n} \times \frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}}$ $= \frac{n+1-n}{\sqrt{n+1}+\sqrt{n}}$ $= \frac{1}{\sqrt{n+1}+\sqrt{n}}$ $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{\sqrt{n+1}+\sqrt{n}}$ $= 0$ As the value of limit is finite the sequence is convergent.	$\lim_{n\to\infty} a_n = \lim_{n\to\infty} 2-(-1)^n$ $= 2-1 = 1\,, if\ n\ is\ even.$ $= 2-(-1) = 3, if\ n\ is\ odd.$ Since limit is not unique, the $\{2-(-1)^n\}$ is oscillatory. **Exercise:** Test the convergence of the following sequences. 1. $\left\{\frac{n}{n^2+1}\right\}$ 2. $\{e^n\}$ 3.$\{1+(-1)^n\}$

❖ **<u>Monotonic sequence</u>:**

- A sequence $\{a_n\}$ is said to be <u>monotonically increasing</u> if $a_n \le a_{n+1}$ for each value of n.

$$a_n - a_{n+1} \le 0$$

- A sequence $\{a_n\}$ is said to be <u>monotonically decreasing</u> if $a_n \ge a_{n+1}$ for each value of n.
- A sequence $\{a_n\}$ is said to be <u>strictly increasing</u> if $a_n < a_{n+1}$ for each value of n.
- A sequence $\{a_n\}$ is said to be <u>strictly decreasing</u> if $a_n > a_{n+1}$ for each value of n.
- A sequence $\{a_n\}$ is said to be <u>monotonic</u> if it is either increasing or decreasing.

❖ **<u>Bounded & unbounded sequence:</u>**

- A sequence $\{a_n\}$ is said to be <u>bounded above</u> if there is a real number *M* such that $a_n \le M, for\ all\ n \in \mathbb{N}.$ M is said to be an <u>upper bound</u> of the sequence.
- A sequence $\{a_n\}$ is said to be <u>bounded below</u> if there is a real number *m* such that
$a_n \ge m, forall\ n \in \mathbb{N}.m$ is said to be a <u>lower bound</u> of the sequence.

- A sequence $\{a_n\}$ is said to be <u>bounded</u> if it is both bounded above and bounded below.
- A sequence $\{a_n\}$ is said to be <u>unbounded</u> if it is not bounded.

1) $a_n = n$

$a_n = 1,2,3,4, \ldots$

Also $a_n \geq 1 \Rightarrow a_n \; is \; bounded \; below.$

2) $a_n = \frac{n}{n+1}$

$a_n = \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots$

$a_n \geq \frac{1}{2}, bounded \; below$

$a_n < 1, bounded \; above$

$\frac{1}{2} \leq a_n < 1$

$a_n \; is \; bounded.$

3) $a_n = \frac{1}{n}$

$\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \ldots \ldots$

$a_n \leq 1, bounded \; above$

$a_n > 0, bounded \; below$

It is bounded.

4) $a_n = (-1)^n$

5) $a_n = (-1)^n . n$

unbounded.

- **<u>Note that</u>**
  - If $\{a_n\}$ is bounded above and increasing then it is convergent.
  - If $\{a_n\}$ is unbounded above and increasing then it is divergent to ∞.
  - If $\{a_n\}$ is bounded below and decreasing then it is convergent.
  - If $\{a_n\}$ is unbounded below and decreasing then it is divergent to $-\infty$.

1) The sequence n^2 is **increasing sequence**

1, 4, 9,16,……

Increasing sequence

2) $\frac{1}{2^n}$ is **decreasing sequence**

- ❖ **Sandwich theorem:**

Let $\{a_n\}, \{b_n\}, \{c_n\}$ be sequences of real numbers such that

$$(i) c_n \leq a_n \leq b_n \ ; \ \forall \ n \geq n_0, for\ some\ n_0 \ \ and$$

$$(ii) \lim_{n\to\infty} c_n = l = \lim_{n\to\infty} b_n$$

$$then \ \lim_{n\to\infty} a_n = l$$

Que. Show that the sequence $\left\{\frac{\sin n}{n}\right\}_{n=1}^{\infty}$ converges to 0.

Solution:

We know that $-1 \leq sinn \leq 1 \Rightarrow -\frac{1}{n} \leq \frac{\sin n}{n} \leq \frac{1}{n}$

Further, $\lim\limits_{n\to\infty}\left(-\frac{1}{n}\right) = 0$ and $\lim\limits_{n\to\infty}\frac{1}{n} = 0.$

$\therefore$ By sandwich theorem, $\lim\limits_{n\to\infty}\frac{\sin n}{n} = 0$

Example: Test the convergence of the series $a_n = \frac{n}{n^2+1}$ **.**

Solution:

$$a_n = \frac{n}{n^2+1}$$

$$a_{n+1} = \frac{n+1}{(n+1)^2+1}$$

$$a_n - a_{n+1} = \frac{n}{n^2+1} - \frac{n+1}{(n+1)^2+1} > 0$$

$$a_n - a_{n+1} > 0$$

It is decreasing squence.

$$a_n = \frac{1}{2}, \frac{2}{5}, \frac{3}{10}, \dots ..$$

$$a_n \leq \frac{1}{2}, \quad a_n > 0$$

$0 < a_n \leq \frac{1}{2}.$

It is bounded.

Every monotonically bounded sequence is convergent.

- ❖ **Infinite Series:**

The sum of an infinite sequence of numbers is called **infinite Series**

e.g. $a_1 + a_2 + a_3 + \dots\dots\dots + a_n + \dots\dots = \sum_{n=1}^{\infty} a_n$

- $S_n = a_1 + a_2 + a_3 + \cdots + a_n$ is called n^th partial sum of the series.
- The convergence of infinite series depends on the convergence of the corresponding infinite sequence of partial sums.
- The infinite series is

Convergent	If $\lim_{n\to\infty} S_n = S\ (finite)$
Divergent	If $\lim_{n\to\infty} S_n = \infty\ \ or - \infty$
Oscillatory	If $\lim_{n\to\infty} S_n = niether\ finite\ nor\ \pm \infty$
Oscillating finitely	If value fluctuates within finite range
Oscillating infinitely	If value fluctuates within $\infty\ and - \infty$

- If a series $\sum_{n=1}^{\infty} a_n$converges to S then we say that the sum of the series is S and we write $S = \sum_{n=1}^{\infty} a_n$

❖ Convergence properties of series:

Let $\sum a_n$and $\sum b_n$ be two convergent series and k be any real number, then the following series will also converge.

1) $\sum (a_n \pm b_n)$ with $\sum (a_n \pm b_n) = \sum a_n \pm \sum b_n$
2) $\sum ka_n$ With $\sum ka_n = k \sum a_n$

❖ Telescoping series:

A series is said to be telescoping if while writing the n^th partial sum all terms except first and last vanish.

Que: Check the convergence of the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$.

Solution: Here, $a_n = \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$.

$$\frac{1}{1}\left(\frac{1}{n} - \frac{1}{n+1}\right)$$

Therefore the partial sum is given by,

$$s_n = a_1 + a_2 + \cdots + a_{n-1} + a_n$$

$$= \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{n-1} - \frac{1}{n}\right)$$
$$+ \left(\frac{1}{n} - \frac{1}{n+1}\right)$$
$$= 1 - \frac{1}{n+1}$$
$$\therefore S = \lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(1 - \frac{1}{n+1}\right) = 1$$
$$\therefore \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$$

It is convergent.

For example: $\frac{1}{n(n+3)} = \frac{1}{3}\left(\frac{1}{n} - \frac{1}{n+3}\right)$

Que. Find the Sum of the series $\log 2 + \log\frac{3}{2} + \log\frac{4}{3} + \cdots + \infty$

Solution:

$$S_n = \log 2 + \log\frac{3}{2} + \log\frac{4}{3} + \cdots + \log\frac{n+1}{n}$$
$$= \log\left(2 \cdot \frac{3}{2} \cdot \frac{4}{3} \cdot \ldots \cdot \frac{n+1}{n}\right)$$
$$S_n = \log(n+1)$$
$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \log(n+1)$$
$$= \log\infty$$
$$= \infty$$

As it is infinite therefore the series is divergent.

Que: Find the Sum of the series $\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \infty$

Solution: $a_n = \frac{n}{(n+1)!} = \frac{(n+1)-1}{(n+1)!} = \frac{1}{n!} - \frac{1}{(n+1)!}$

Therefore the partial sum is given by,

$$s_n = a_1 + a_2 + \cdots + a_{n-1} + a_n$$
$$= \left(\frac{1}{1!} - \frac{1}{2!}\right) + \left(\frac{1}{2!} - \frac{1}{3!}\right) + \cdots + \left(\frac{1}{(n-1)!} - \frac{1}{n!}\right)$$
$$+ \left(\frac{1}{n!} - \frac{1}{(n+1)!}\right)$$
$$= 1 - \frac{1}{(n+1)!}$$

$$\therefore S = \lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(1 - \frac{1}{(n+1)!}\right) = 1$$

$$\therefore \frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots = \sum_{n=1}^{\infty} \frac{n}{(n+1)!} = 1$$

❖ **Geometric Series:**

An infinite series in the form $a + ar + ar^2 + \cdots + ar^{n-1} + \cdots$ is said to be a geometric series.

It converges to $\frac{a}{1-r}$ if $|r| < 1$ i.e. $\sum_{n=1}^{\infty} a\, r^{n-1} = \frac{a}{1-r}, |r| < 1$

If $|r| \geq 1$ then the series diverges.

If $r = -1$ then series is oscillatory.

Que. Discuss the convergence of $\sum_{n=0}^{\infty} 2^n$

Solution:

Given series, $\sum_{n=0}^{\infty} 2^n = 2^0 + 2^1 + 2^2 + \cdots$ is a geometric series with $a = 1$ and $r = 2$

$$r = \frac{2}{1} = 2,\;\; r = \frac{4}{2} = 2$$

Since $r = 2 > 1,$ the series is divergent.

Que. Check the convergence of a series $\frac{1}{3^0} - \frac{1}{3} + \frac{1}{3^2} - \frac{1}{3^3} + \frac{1}{3^4} + \cdots$.Also find sum.

Solution:

$$S_n = 1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{27} + \cdots$$

$$r = \frac{a_2}{a_1} = -\frac{\frac{1}{3}}{1} = -\frac{1}{3}$$

$$r = \frac{a_3}{a_2} = \frac{\frac{1}{9}}{-\frac{1}{3}} = -\frac{1}{3}$$

$$r = -\frac{1}{3}$$

Here the series is geometric series with $a = 1$ and and $|r| = \frac{1}{3}$

Since, $|r| = \frac{1}{3} < 1,$ the series is convergent.

$Sum = \frac{a}{1-r} = \frac{1}{1-\left(-\frac{1}{3}\right)} = \frac{1}{\frac{4}{3}} = \frac{3}{4}.$

Que. Discuss the convergence of $\sum_{n=1}^{\infty} \frac{3^{2n}}{4^{2n}}$

Solution: Since,

$$\sum_{n=1}^{\infty} \frac{3^{2n}}{4^{2n}} = \sum_{n=1}^{\infty} \frac{(3^2)^n}{(4^2)^n} = \sum_{n=1}^{\infty} \frac{(9)^n}{(16)^n} = \sum_{n=1}^{\infty} \left(\frac{9}{16}\right)^n$$

is a geometric series with $a = \frac{9}{16}$ and $r = \frac{9}{16}$.

Since $r = \frac{9}{16} < 1$, it is convergent. Further it converges to $\frac{a}{1-r} = \frac{\left(\frac{9}{16}\right)}{\left(1-\left(\frac{9}{16}\right)\right)} = \frac{9}{7}$

Que. Check the convergence of $\sum_{n=1}^{\infty} \frac{4^n + 5^n}{6^n}$

Solution:

$$\sum C_n = \sum_{n=1}^{\infty} \left[\left(\frac{4}{6}\right)^n + \left(\frac{5}{6}\right)^n\right] = \sum_{n=1}^{\infty} \left(\frac{4}{6}\right)^n + \sum_{n=1}^{\infty} \left(\frac{5}{6}\right)^n = \sum a_n + \sum b_n$$

where $a_n = \left(\frac{4}{6}\right)^n$ *and* $b_n = \left(\frac{5}{6}\right)^n$

For $\sum a_n$, $r = \left(\frac{4}{6}\right) < 1$, hence $\sum a_n$ is convergent. And $\sum a_n = \frac{\left(\frac{4}{6}\right)}{\left(1-\frac{4}{6}\right)} = \frac{4}{2} = 2.$

Similarly, for $\sum b_n$, $r = \left(\frac{5}{6}\right) < 1$ so $\sum b_n$ is also convergent.

And $\sum b_n = \frac{\left(\frac{5}{6}\right)}{\left(1-\frac{5}{6}\right)} = 5$

Thus, the sum of $\sum a_n + \sum b_n$ is also convergent. $i.e.\sum c_n$ is convergent.

Further, $\sum c_n = \sum a_n + \sum b_n = 2 + 5 = 7$

Exercise:

1) Find the sum of $\sum_{n\to 1}^{\infty} \frac{3^{n-1}-1}{6^{n-1}}$
2) Find the sum of $\sum_{n\to 1}^{\infty} \frac{4^n+1}{6^n}$
3) Prove that $1 + \frac{2}{3} + \frac{4}{9} + \frac{8}{27} + \frac{16}{81} + \cdots$ converges and find its sum.
4) Prove that $5 - \frac{10}{3} + \frac{20}{9} - \frac{40}{27} + \cdots$ converges and find its sum.

❖ <u>P-Series Test</u>

The Series $\sum_{n=0}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$.

Exercise: Check if the following series is convergent or divergent.

1. $\sum \frac{1}{x^3}$ 2. $\sum \frac{1}{x^{-3}}$ 3. $\sum \frac{1}{x}$ 4. $\sum \frac{1}{x^{\frac{3}{4}}}$

❖ Zero test of Divergence (Divergence test):

If $\lim_{n\to\infty} a_n \neq 0$ then $\sum_{n=1}^{\infty} a_n$ must be divergent

Note: If $\lim_{n\to\infty} a_n = 0$ then nothing can be said about convergence of the series $\sum_{n=1}^{\infty} a_n$ .We have to apply another test for convergence

Que. Test the convergence of following series.

1) $\sum_{n=1}^{\infty} n \sin\frac{1}{n}$

Solution:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} n \sin\frac{1}{n} = \lim_{n\to\infty} \frac{\sin\left(\frac{1}{n}\right)}{\left(\frac{1}{n}\right)} = 1 \neq 0$$

Hence, by zero test, the series is divergent.

2) $\sqrt{\frac{1}{2}} + \sqrt{\frac{2}{3}} + \sqrt{\frac{3}{4}} + \cdots\ \infty$

Solution:

Here, $a_n = \sqrt{\frac{n}{n+1}}$

$$\therefore \lim_{n\to\infty} a_n = \lim_{n\to\infty} \sqrt{\frac{n}{n+1}} = \lim_{n\to\infty} \sqrt{\frac{n}{n\left(1+\frac{1}{n}\right)}} = \lim_{n\to\infty} \sqrt{\frac{1}{\left(1+\frac{1}{n}\right)}} = \sqrt{\frac{1}{(1+0)}}$$

$$= 1 \neq 0$$

Hence, by zero test, the series is divergent.

Que. Prove that $\sum_{n=1}^{\infty} \frac{n^2-1}{n^2+1}$ is divergent.

Solution:

$$\therefore \lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^2-1}{n^2+1} = \lim_{n\to\infty} \frac{n^2\left(1-\frac{1}{n^2}\right)}{n^2\left(1+\frac{1}{n^2}\right)} = \lim_{n\to\infty} \frac{\left(1-\frac{1}{n^2}\right)}{\left(1+\frac{1}{n^2}\right)} = \frac{(1-0)}{(1+0)} = 1$$

$$\neq 0$$

Hence, by zero test, the series is divergent.

❖ Direct Comparison Test

Let $\sum a_n$ be a series with no negative terms.

(a) $\sum a_n$ converges if there is a convergent series $\sum c_n$ with $a_n \le c_n$ for all $n > N$, for some integer N.

(b) $\sum a_n$ diverges if there is a divergent series of nonnegative terms $\sum d_n$ with $a_n \ge d_n$ for all $n > N$, for some integer N.

❖ **Limit Comparison Test**

Suppose that $a_n > 0$ and $b_n > 0$ for all $forall\ n \ge N$ (N an integer).

(a) If $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = c > 0$, then $\sum a_n$ and $\sum b_n$ both converge or both diverge.

(b) If $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\sum b_n$ converges, then $\sum a_n$ converges.

(c) If $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = \infty$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.

Note: $b_n = \dfrac{Highest\ power\ term\ in\ numerator}{Highest\ power\ term\ in\ denomerator}$

Que. for what value of p does the series $\frac{2}{1^p} + \frac{3}{2^p} + \frac{4}{3^p} + \cdots$ is convergent?

Solution:

Here, $a_n = \frac{n+1}{n^p} = \frac{n\left(1+\frac{1}{n}\right)}{n^p} = \frac{\left(1+\frac{1}{n}\right)}{n^{(p-1)}} = \frac{1}{n^{(p-1)}}\left(1 + \frac{1}{n}\right)$.

Let $b_n = \frac{1}{n^{(p-1)}}$. Then $\frac{a_n}{b_n} = \left(1 + \frac{1}{n}\right)$

$$\therefore \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right) = 1 + 0 = 1 \ \neq 0$$

$\therefore \sum a_n$ and $\sum b_n$ both converges or diverges together.

$\sum b_n = \sum \frac{1}{n^{(p-1)}}$ converges for $p - 1 > 1$, $i.e.$ for $p > 2$ and it diverge otherwise.

$\therefore \sum a_n = \sum \frac{n+1}{n^p}$ converges for $p \ge 2$ and it diverge otherwise.

Que. Test the convergence of

$$\sum_{n=1}^{\infty} \frac{2n^2 + 2n}{5 + n^5}$$

Solution:

Here, $a_n = \frac{2n^2+2n}{5+n^5} = \frac{n^2\left(2+\frac{2}{n}\right)}{n^5\left(\frac{5}{n^5}+1\right)} = \frac{1}{n^3}\frac{\left(2+\frac{2}{n}\right)}{\left(\frac{5}{n^5}+1\right)}$.

Let $b_n = \frac{n^2}{n^5} = \frac{1}{n^3}$. Then $\frac{a_n}{b_n} = \frac{\left(2+\frac{2}{n}\right)}{\left(\frac{5}{n^5}+1\right)}$

$$\therefore \lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{\left(2+\frac{2}{n}\right)}{\left(\frac{5}{n^5}+1\right)} = \frac{(2+0)}{(0+1)} = 2 \neq 0$$

$\therefore \sum a_n$ and $\sum b_n$ both converges or diverges together.

Now, $\sum b_n = \sum \frac{1}{n^3}$ is a $p - series$ with $p = 3 > 1.$ Hence, it is convergent.

$\therefore \sum a_n = \sum \frac{2n^2+2n}{5+n^5}$ converges. [by comparison test]

Que: Test the convergence of $\sum_{n=1}^{\infty}\frac{\sqrt{n}}{n^2+1}$

Sol: $a_n = \frac{\sqrt{n}}{n^2+1}$

$$b_n = \frac{\sqrt{n}}{n^2} = \frac{1}{n^{2-\frac{1}{2}}} = \frac{1}{n^{\frac{3}{2}}}$$

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{\frac{\sqrt{n}}{n^2+1}}{\frac{1}{n^{\frac{3}{2}}}} = \lim_{n\to\infty} n^{\frac{3}{2}}\frac{\sqrt{n}}{n^2+1} = \lim_{n\to\infty}\frac{n^2}{n^2+1} = \lim_{n\to\infty}\frac{n^2}{n^2\left(1+\frac{1}{n^2}\right)} = 1 \neq 0$$

$\therefore \sum a_n$ and $\sum b_n$ both converges or diverges together.

$$b_n = \frac{1}{n^{\frac{3}{2}}}\text{ , }By\ p - series\ ,p = \frac{3}{2} > 1, it\ is\ convergent.$$

By Limit comparision Test, $\sum a_n\ is\ convergent.$

Que. Test the convergence of the series

$$\sum_{n=1}^{\infty}\frac{n^p}{\sqrt{n+1}+\sqrt{n}}$$

Solution:

Here, $a_n = \frac{n^p}{\sqrt{n+1}+\sqrt{n}} = \frac{n^p}{n^{\frac{1}{2}}\left(\sqrt{1+\frac{1}{n}}+1\right)} = \frac{1}{n^{\frac{1}{2}-p}}\frac{1}{\left(1+\sqrt{1+\frac{1}{n}}\right)}$.

Let $b_n = \frac{n^p}{n^{\frac{1}{2}}} = \frac{1}{n^{\frac{1}{2}-p}}$. Then $\frac{a_n}{b_n} = \frac{1}{\left(1+\sqrt{1+\frac{1}{n}}\right)}$

$$\therefore \lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{1}{\left(1+\sqrt{1+\frac{1}{n}}\right)} = \frac{1}{(1+\sqrt{1+0})} = \frac{1}{2} \neq 0$$

$\therefore \sum a_n$ and $\sum b_n$ both converges or diverges together.

Now, $\sum b_n = \sum \frac{1}{n^{\frac{1}{2}-p}}$ is a $p - series$ which converges for $\frac{1}{2} - p > 1$, $i.e.$ for $p < -\frac{1}{2}$ and diverges otherwise.

$\therefore \sum a_n = \sum \frac{n^p}{\sqrt{n+1}+\sqrt{n}}$ also converges for $p < -\frac{1}{2}$ and diverges otherwise.[by comparison test]

Que. Test the convergence of the series

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots$$

Solution:

Here, $a_n = \frac{1}{n.(n+1)} = \frac{1}{n^2}\frac{1}{\left(1+\frac{1}{n}\right)}$.

Let $b_n = \frac{1}{n^2}$. Then $\frac{a_n}{b_n} = \frac{1}{\left(1+\frac{1}{n}\right)}$

$$\therefore \lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{n}\right)} = \frac{1}{(1+0)} = 1 \neq 0$$

$\therefore \sum a_n$and $\sum b_n$ both converges or diverges together.

Now, $\sum b_n = \sum \frac{1}{n^2}$ is a $p - series$ with $p = 2 > 1$. Hence, it is convergent.

$\therefore \sum a_n = \sum \frac{1}{n(n+1)}$ converges. [by comparison test]

Que. Test the convergence of

$$\sum_{n=1}^{\infty}\frac{1}{1^2+2^2+3^2+\cdots+n^2}$$

Solution:

$$a_n = \frac{1}{1^2+2^2+3^2+\cdots+n^2} = \frac{1}{\sum n^2} = \frac{1}{\left(\frac{n(n+1)(2n+1)}{6}\right)}$$

$$= \frac{1}{n^3}\frac{6}{1\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)}$$

$$= \frac{6}{n.n.n\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)}$$

$$(n^2+n)(2n+1) = (2n^3+n^2+2n^2+n) = 2n^3+3n^2+n$$

$$= n^3(2+\frac{3}{n}+\frac{1}{n^2})$$

Let $b_n = \frac{1}{n^3}$. Then $\frac{a_n}{b_n} = \frac{6}{\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)}$

$$\therefore \lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{6}{\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)} = \frac{6}{(1+0)(2+0)} = 3 \neq 0$$

$\therefore \sum a_n$and $\sum b_n$ both converges or diverges together.

Now, $\sum b_n = \sum \frac{1}{n^3}$ is a $p - series$ with $p = 3 > 1$. Hence, it is convergent.

$\therefore \sum a_n = \sum \frac{1}{1^2+2^2+3^2+\cdots+n^2}$ converges. [by comparison test]

❖ Ratio Test(D' Alembert Ratio Test)

Let $\sum a_n$ be a series with positive terms and suppose that $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L$

Then (a) the series converges if $L < 1$

(b)the series diverges if $L > 1$,

(c) the test is fail if $L = 1$

Que. Test the convergence of a series $\sum \frac{1}{n!}$

Solution:

Here $a_n = \frac{1}{n!} \Rightarrow a_{n+1} = \frac{1}{(n+1)!}$ and

$\frac{a_{n+1}}{a_n} = \frac{n!}{(n+1)!} = \frac{1}{n+1}$

$$\therefore L = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{1}{n+1} = 0 < 1$$

Hence, by ratio test, given series is convergent.

Que. Test the convergence of the series $\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots$

Solution:

Here $a_n = \frac{n}{(n+1)!}$

$$\Rightarrow a_{n+1} = \frac{n+1}{(n+2)!} and \ \frac{a_{n+1}}{a_n} = \frac{n+1}{(n+2)!}\frac{(n+1)!}{n} = \frac{(n+1)!}{(n+2)(n+1)!}\frac{n+1}{n}$$

$$= \frac{1}{n+2}\left(1 + \frac{1}{n}\right)$$

$$\therefore L = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{1}{n+2}\left(1 + \frac{1}{n}\right) = 0(1+0) = 0 < 1$$

Hence, by ratio test, given series is convergent.

Que. Test the convergence of the series $\sum_{n=0}^{\infty} \frac{4^n - 1}{3^n}$

Solution:

Here $a_n = \frac{4^n-1}{3^n}$

$$\Rightarrow a_{n+1} = \frac{4^{n+1} - 1}{3^{n+1}} and$$

$$\frac{a_{n+1}}{a_n} = \frac{4^{n+1}-1}{3^{n+1}} \frac{3^n}{4^n - 1} = \frac{3^n}{3^{n+1}} \frac{4^n\left(4 - \frac{1}{4^n}\right)}{4^n\left(1 - \frac{1}{4^n}\right)} = \frac{1}{3}\left(\frac{4 - \frac{1}{4^n}}{1 - \frac{1}{4^n}}\right)$$

$$\therefore L = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{1}{3}\left(\frac{4 - \frac{1}{4^n}}{1 - \frac{1}{4^n}}\right) = \frac{1}{3}\left(\frac{4-0}{1-0}\right) = \frac{4}{3} > 1$$

Hence, by ratio test, given series is divergent.

Que. Example: Test the convergence of the series $\sum_{n=0}^{\infty} \frac{n3^n(n+1)!}{2^n n!}$

Solution:

$$a_n = \frac{n3^n(n+1)!}{2^n n!}$$

$$= n(n+1)\left(\frac{3}{2}\right)^n$$

$$\Rightarrow a_{n+1} = (n+1)(n+2)\left(\frac{3}{2}\right)^{n+1} \quad and$$

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)(n+2)\left(\frac{3}{2}\right)^{n+1}}{n(n+1)\left(\frac{3}{2}\right)^n} = \frac{(n+2)}{n}\left(\frac{3}{2}\right)$$

$$= \left(1 + \frac{2}{n}\right)\left(\frac{3}{2}\right)$$

$$\therefore L = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \left(1 + \frac{2}{n}\right)\left(\frac{3}{2}\right) = (1+0)\left(\frac{3}{2}\right) = \frac{3}{2} > 1$$

Hence, by ratio test, given series is divergent.

Que. Test the convergence of

$$\sum_{n=1}^{\infty} \frac{1}{1^2 + 2^2 + 3^2 + \cdots + n^2}$$

Solution:

$$a_n = \frac{1}{1^2 + 2^2 + 3^2 + \cdots + n^2} = \frac{1}{\sum n^2} = \frac{1}{\left(\frac{n(n+1)(2n+1)}{6}\right)}$$

$$= \frac{6}{n(n+1)(2n+1)}$$

$$\Rightarrow a_{n+1} = \frac{6}{(n+1)(n+2)(2(n+1)+1)} = \frac{6}{(n+1)(n+2)(2n+3)} \, and$$

$$\frac{a_{n+1}}{a_n} = \frac{6}{(n+1)(n+2)(2n+3)} \frac{n(n+1)(2n+1)}{6} = \frac{n(2n+1)}{(n+2)(2n+3)}$$

$$= \frac{\left(2+\frac{1}{n}\right)}{\left(1+\frac{2}{n}\right)\left(2+\frac{3}{n}\right)}$$

$$\therefore L = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\left(2+\frac{1}{n}\right)}{\left(1+\frac{2}{n}\right)\left(2+\frac{3}{n}\right)}$$

$$= \frac{(2+0)}{(1+0)(2+0)} = 1$$

Hence, by ratio test fails.

We need to use some other test to check the convergence of the series.

Using comparison test as follows:

$$a_n = \frac{1}{1^2+2^2+3^2+\cdots+n^2} = \frac{1}{\sum n^2} = \frac{1}{\left(\frac{n(n+1)(2n+1)}{6}\right)} = \frac{1}{n^3} \frac{6}{1\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)}$$

Let $b_n = \frac{1}{n^3}$. Then $\frac{a_n}{b_n} = \frac{6}{\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)}$

$$\therefore \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{6}{\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)} = \frac{6}{(1+0)(2+0)} = 3 \neq 0$$

$\therefore \sum a_n$ and $\sum b_n$ both converges or diverges together.

Now, $\sum b_n = \sum \frac{1}{n^3}$ is a $p - series$ with $p = 3 > 1$. Hence, it is convergent.

$\therefore \sum a_n = \sum \frac{1}{1^2+2^2+3^2+\cdots+n^2}$ converges. [by comparison test]

Que. Test the convergence of the series $2 + \frac{3}{2}x + \frac{4}{3}x^2 + \frac{5}{4}x^3 + \cdots$

Solution: Here $a_n = \frac{n+1}{n} x^{n-1}$

$$\Rightarrow a_{n+1} = \frac{n+2}{n+1} x^n \; and \; \frac{a_{n+1}}{a_n} = \frac{(n+2)x^n}{n+1} \frac{n}{(n+1)x^{n-1}} = \frac{n^2\left(1+\frac{2}{n}\right)}{n^2\left(1+\frac{1}{n}\right)^2} x$$

$$= \frac{\left(1+\frac{2}{n}\right)}{\left(1+\frac{1}{n}\right)^2} x$$

$$\therefore L = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\left(1+\frac{2}{n}\right)}{\left(1+\frac{1}{n}\right)^2} x$$

$$= \frac{(1+0)}{(1+0)^2} x = x$$

Hence, by ratio test, given series is (i) convergent if $x < 1$

(ii) divergent if $x > 1$

For $x = 1$.

$$a_n = \frac{n+1}{n} = 1 + \frac{1}{n}$$

$$\Rightarrow \lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right) = (1+0)$$

$$= 1 \neq 0$$

$\therefore$ By zero test, given series diverges for $x = 1$.

Hence, by ratio test, given series is (i) convergent if $x < 1$

(ii) divergent if $x \geq 1$

❖ <u>Root Test (Cauchy Root Test)</u>

Let $\sum a_n$ be a series with $a_n \geq 0$ for $n \geq N$ for some N and suppose that

$$\lim_{n\to\infty} |a_n|^{1/n} = L$$

Then (a) the series converges if $L < 1$

(b) the series diverges if $L > 1$,

(c) the test fails if $L = 1$

Que. Test the convergence of series $\sum_{n=1}^{\infty} \frac{3^n}{2^{n+3}}$

Solution:

$$a_n = \frac{3^n}{2^{n+3}} = \frac{1}{8}\left(\frac{3}{2}\right)^n$$

$$\Rightarrow |a_n|^{\frac{1}{n}} = \left|\frac{1}{8}\left(\frac{3}{2}\right)^n\right|^{\frac{1}{n}} = \frac{1}{8^{\frac{1}{n}}}\left(\frac{3}{2}\right)$$

$$\Rightarrow L = \lim_{n\to\infty} |a_n|^{\frac{1}{n}} = \lim_{n\to\infty} \frac{1}{8^{\frac{1}{n}}}\left(\frac{3}{2}\right)$$

$$= \frac{1}{8^0}\left(\frac{3}{2}\right) = \frac{3}{2} > 1$$

Hence, by root test, given series is divergent.

Que. Test the convergence of series $\sum_{n=1}^{\infty}\left(\frac{n}{2n+5}\right)^n$

Solution:

$$a_n = \left(\frac{n}{2n+5}\right)^n = \left(\frac{1}{2+\frac{5}{n}}\right)^n$$

$$\Rightarrow |a_n|^{\frac{1}{n}} = \left|\left(\frac{1}{2+\frac{5}{n}}\right)^n\right|^{\frac{1}{n}} = \left(\frac{1}{2+\frac{5}{n}}\right)$$

$$\Rightarrow L = \lim_{n\to\infty}|a_n|^{\frac{1}{n}} = \lim_{n\to\infty}\left(\frac{1}{2+\frac{5}{n}}\right) = \left(\frac{1}{2+0}\right)$$

$$= \frac{1}{2} < 1$$

Hence, by root test, given series is convergent.

Que: $\left(\frac{2^2}{1^2}-\frac{2}{1}\right)^{-1}+\left(\frac{3^3}{2^3}-\frac{3}{2}\right)^{-2}+\left(\frac{4^4}{3^4}-\frac{4}{3}\right)^{-3}+\cdots..$

Sol: $\boldsymbol{a_n = \left[\left(\frac{n+1}{n}\right)^{n+1}-\frac{n+1}{n}\right]^{-n}}$

$$L = \lim_{n\to\infty}|a_n|^{\frac{1}{n}} = \left[\left(\frac{n+1}{n}\right)^n\left(\frac{n+1}{n}\right)-\frac{n+1}{n}\right]^{-1}$$

$$= \left[\left(\frac{1+\frac{1}{n}}{1}\right)^n\left(\frac{1+\frac{1}{n}}{1}\right)-\frac{1+\frac{1}{n}}{1}\right]^{-1}$$

$$= [e.1-1]^{-1}$$

$= \frac{1}{e-1}$<1

Hence, by root test, given series is convergent.

Que. Test the convergence of series $\sum_{n=1}^{\infty}\left(1+\frac{1}{\sqrt{n}}\right)^{-n^{\frac{3}{2}}}$

Solution:

$$a_n = \left(1+\frac{1}{\sqrt{n}}\right)^{-n^{\frac{3}{2}}}$$

$$\Rightarrow |a_n|^{\frac{1}{n}} = \left|\left(1+\frac{1}{\sqrt{n}}\right)^{-n^{\frac{3}{2}}}\right|^{\frac{1}{n}} = \left(1+\frac{1}{\sqrt{n}}\right)^{-\left(n^{\frac{3}{2}}\right)\left(n^{-1}\right)} = \left(1+\frac{1}{\sqrt{n}}\right)^{-\sqrt{n}} = \left(\left(1+\frac{1}{\sqrt{n}}\right)^{\sqrt{n}}\right)^{-1}$$

$$\left(\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x \quad (any x)\right)$$

$$\Rightarrow L = \lim_{n\to\infty}|a_n|^{\frac{1}{n}} = \lim_{n\to\infty}\left(1+\frac{1}{\sqrt{n}}\right)^{-\sqrt{n}} = \lim_{n\to\infty}\left(\left(1+\frac{1}{\sqrt{n}}\right)^{\sqrt{n}}\right)^{-1} = (e^1)^{-1} = \frac{1}{e} < 1$$

Hence, by root test, given series is convergent.

Que. Test the convergence of the series $\sum_{n=1}^{\infty}\left(\frac{n+2}{n+3}\right)^n x^n$

Solution:

$$a_n = \left(\frac{n+2}{n+3}\right)^n x^n$$

$$\Rightarrow |a_n|^{\frac{1}{n}} = \left|\left(\frac{n+2}{n+3}\right)^n x^n\right|^{\frac{1}{n}} = \left(\frac{n+2}{n+3}\right)x$$

$$\Rightarrow L = \lim_{n\to\infty}|a_n|^{\frac{1}{n}} = \lim_{n\to\infty}\left(\frac{n+2}{n+3}\right)x = \lim_{n\to\infty}\left(\frac{1+\frac{2}{n}}{1+\frac{3}{n}}\right)x$$

$$= \left(\frac{1+0}{1+0}\right)x = x$$

Hence, by root test, given series is (i) convergent if $x < 1$

(ii) divergent if $x > 1$.

For $x = 1$.

$$a_n = \left(\frac{n+2}{n+3}\right)^n = \left(\frac{1+\frac{2}{n}}{1+\frac{3}{n}}\right)^n = \frac{\left(1+\frac{2}{n}\right)^n}{\left(1+\frac{3}{n}\right)^n}$$

$$\Rightarrow \lim_{n\to\infty} a_n$$

$$= \lim_{n\to\infty}\frac{\left(1+\frac{2}{n}\right)^n}{\left(1+\frac{3}{n}\right)^n} = \frac{\lim_{n\to\infty}\left(1+\frac{2}{n}\right)^n}{\lim_{n\to\infty}\left(1+\frac{3}{n}\right)^n} = \frac{(e)^2}{(e)^3} = \frac{1}{e}$$

$$\neq 0$$

$\therefore$ By zero test, given series diverges for $x = 1$.

Hence, by root test, given series is (i) convergent if $x < 1$

(ii) divergent if $x \geq 1$.

<u>Alternative series</u>

A series in which the terms are alternatively positive and negative is called an alternating Series.$e.g.\ 1-4+9-16+\cdots$

❖ **<u>Leibnitz Test</u>**

The infinite Series $a_1 - a_2 + a_3 - \ldots$ in which the terms are alternatively positive and negative is convergent if (i) $a_n \geq a_{n+1}$ i.e. series is decreasing (ii) $\lim_{n\to\infty} a_n = 0$

Note: If $\lim_{n\to\infty} a_n \neq 0$ then $\sum_{n=1}^{\infty} a_n$ is oscillatory.

Que. Test the convergence of the series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$

Solution:

Here $u_n = \frac{(-1)^{n+1}}{n} \quad u_{n+1} = \frac{(-1)^{n+2}}{n+1}$

$|u_n| = \frac{1}{n} \quad |u_{n+1}| = \frac{1}{n+1}$

1)

$$|u_n| - |u_{n+1}| = \frac{1}{n} - \frac{1}{n+1}$$
$$= \frac{n+1-n}{n(n+1)}$$
$$= \frac{1}{n(n+1)} \succ 0$$

$|u_n| - |u_{n+1}| \succ 0 \Rightarrow |u_n| \succ |u_{n+1}|$

Thus each term is less than its preceding term.

Now

2)

$$\lim_{n\to\infty} |u_n| = \lim_{n\to\infty} \frac{1}{n} = 0$$

Thus by Leibnitz's test the alternating series is convergent.

Que. Test the convergence of the series $2 - \frac{3}{2} + \frac{4}{3} - \frac{5}{4} + \cdots$

Solution:

Here $u_n = \frac{(-1)^{n+1}(n+1)}{n} \quad u_{n+1} = \frac{(-1)^{n+2}(n+2)}{n+1}$

$|u_n| = \frac{(n+1)}{n} \quad |u_{n+1}| = \frac{n+2}{n+1}$

1)

$$|u_n| - |u_{n+1}| = \frac{n+1}{n} - \frac{n+2}{n+1}$$

$$= \frac{(n+1)^2 - n(n+2)}{n(n+1)}$$

$$= \frac{1}{n(n+1)} \succ 0$$

$$|u_n| - |u_{n+1}| \succ 0 \Rightarrow |u_n| \succ |u_{n+1}|$$

Thus each term is less than its preceding term.

Now

2)

$$n \xrightarrow{\lim} \infty |u_n| = n \xrightarrow{\lim} \infty \frac{n+1}{n}$$

$$= n \xrightarrow{\lim} \infty \frac{n\left(1+\frac{1}{n}\right)}{n}$$

$$= 1 \neq 0$$

Thus by Leibnitz's test the alternating series is oscillating.

Que. Test the convergence of the series $\sum_{n=1}^{\infty} \frac{(-1)^n x^{n+1}}{2n-1}$

Solution:

$$u_n = \frac{(-1)^{n+1} x^{n+1}}{2n-1} \quad u_{n+1} = \frac{(-1)^{n+2} x^{n+2}}{2n+1}$$

$$|u_n| = \frac{x^{n+1}}{2n-1} \quad |u_{n+1}| = \frac{x^{n+2}}{2n+1}$$

1)

$$|u_n| - |u_{n+1}| = \frac{x^{n+1}}{2n-1} - \frac{x^{n+2}}{2n+1}$$

$$= \frac{(2n+1)x^{n+1} - x^{n+2}(2n-1)}{(2n-1)(2n+1)}$$

$$= \frac{x^{n+1}[(2n+1)-(2n-1)x]}{(4n^2-1)} \succ 0$$

$$|u_n| - |u_{n+1}| \succ 0 \Rightarrow |u_n| \succ |u_{n+1}|$$

Now

2)

$$n \xrightarrow{\lim} \infty |u_n| = n \xrightarrow{\lim} \infty \frac{x^{n+1}}{2n-1}$$

$$= 0 \qquad if \quad x < 1$$

Thus by Leibnitz's test the alternating series is convergent.

Parul University
Faculty of Engineering & Technology
Department of Applied Sciences and Humanities
1^{st} Year B.Tech Programme (All Branches)
Mathematics – 1(203191102)
Unit 5: Fourier Series

(Lecture Note)

Fourier Series is an infinite series representation of periodic function in terms of the trigonometric sine and cosine functions.

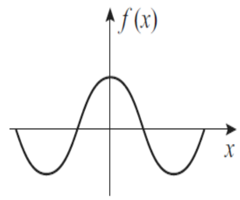
- Most of the single valued functions which occur in applied mathematics can be expressed in the form of Fourier series, which is in terms of sines and cosines.
- Fourier series is a very powerful method to solve ordinary and partial differential equations, particularly with periodic functions appearing as non-homogeneous terms.
- Taylor's series expansion is valid only for functions which are continuous and differentiable. Fourier series is possible not only for continuous functions but also for periodic functions, functions which are discontinuous in their values and derivatives.
- Further, because of the periodic nature, Fourier series constructed for one period is valid for all values.

Prerequisites:

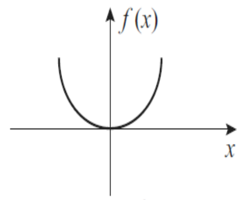
Even Function: A function $f(x)$ is said to be even, if $f(-x) = f(x)$, for all x.

- Even Function is always symmetric about $Y - axis$.

Eg: $cosx, xsinx, \ x^2$, etc



$f(x) = \cos x$

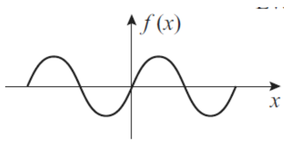


$f(x) = x^2$

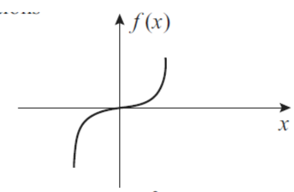
Odd Function: A function $f(x)$ is said to be odd if $f(-x) = \ -f(x)$, for all x.

- Odd function is always symmetric about $X - axis$.

Eg: x, x^3 , $x \ cos \ x, sin \ x$, etc



$f(x) = \sin x$



$f(x) = x^3$

Properties of Definite Integrals:

1. $\int_a^b f(x)dx = F(b) - F(a)$
2. $\int_a^b f(x)dx = \ -\int_b^a f(x)dx$
3. $\int_{-a}^a f(x)dx = \ 2\int_0^a f(x)dx$,if f(x) is even function

   $= \ 0$, if f(x) is odd function
4. $\int_a^b f(x)dx = \ \int_a^b f(t)dt$

Basic Trigonometry

1. $sin \ n\pi \ = 0$
2. $cos \ n\pi \ = (-1)^n$
3. $cos \ (n+1) \ \pi \ = \ cos \ (n-1)\pi \ = \ (-1)^{n+1}$
4. $2sinA \ cosB \ = \ sin \ (A \ + B) \ + \ sin \ (A - B)$
5. $2sinA \ sinB \ = \ cos(A - B) - \ cos \ (A + B)$
6. $2cosA \ cosB \ = \ cos \ (A + B) \ + \ cos(A - B)$
7. $cos(^{n\pi}/_2) \neq 0$

Periodic Functions:

A function $f(x)$ is called periodic if it is defined for all real x and if there is some positive number p such that $f(x+p)=f(x)$ where p is known as **period** of $f(x)$. If a periodic function $f(x)$ has a smallest period p is called **fundamental period** of $f(x)$. eg $sinx, cosx$ has fundamental period of 2π. If l is a fixed number, then $sin(2\pi x/l)$ and $cos(2\pi x/l)$ have period l

Convergence of the Fourier Series (Dirichlet's Conditions)

A function $f(x)$ can be represented by a complete set of orthogonal functions within the interval $(c, c+2l)$. The Fourier series of the function $f(x)$ exists only if the following conditions are satisfied:

(i) $f(x)$ is periodic, $i.e., f(x) = f(x+2l)$, where $2l$ is the period of the function $f(x)$.

(ii) $f(x)$ and its integrals are finite and single-valued.

(iii) $f(x)$ has a finite number of discontinuities, $i.e., f(x)$ is piecewise continuous in the interval $(c, c+2l)$.

(iv) $f(x)$ has a finite number of maxima and minima.

These conditions are known as Dirichlet's conditions.

- **Fourier series:** Fourier series of periodic function $f(x)$ defined in interval $(c, c+2l)$ with fundamental period $2l$ **is given by**

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi}{l}x+b_n\sin\frac{n\pi}{l}x\right)$$

$$a_0=\frac{1}{l}\int_c^{c+2l}f(x)dx$$

$$a_n=\frac{1}{l}\int_c^{c+2l}f(x)\cos\left(\frac{n\pi x}{l}\right)dx,\quad n=1,2,\ldots$$

$$b_n=\frac{1}{l}\int_c^{c+2l}f(x)\sin\left(\frac{n\pi x}{l}\right)dx\quad n=1,2,\ldots$$

Example:1 Find the Fourier Series of $f(x) = x$ in the interval $(0,2\pi)$

Solution: Here $c = 0 \; and \; c + 2l = 2\pi$

$$\therefore l = \pi$$

The Fourier Series of given function is given by

$$\boldsymbol{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi}{l} x + b_n \sin \frac{n\pi}{l} x \right)}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\, x + b_n \sin nx)$$

Now, find

$$\boldsymbol{a_0} = \frac{1}{\pi} \int_c^{c+2l} f(x)\, dx = \frac{1}{\pi} \int_0^{2\pi} x\, dx = \frac{1}{\pi} \left[\frac{x^2}{2} \right]_0^{2\pi} = 2\pi$$

Also,

$$\boldsymbol{a_n} = \frac{1}{\pi} \int_c^{c+2l} f(x) \cos nx\, dx$$

$$= \frac{1}{\pi} \int_0^{2\pi} x \cos nx\, dx,$$

$$= \frac{1}{\pi} \left[x \left(\frac{\sin nx}{n} \right) - (1) \left(-\frac{\cos nx}{n^2} \right) \right]_0^{2\pi}$$

$$= \frac{1}{\pi} \left[0 + \frac{\cos 2n\pi}{n^2} - 0 - \frac{1}{n^2} \right] = \frac{1}{\pi}(0) = 0$$

$$\boldsymbol{b_n} = \frac{1}{\pi} \int_c^{c+2l} f(x) \sin nx\, dx$$

$$= \frac{1}{\pi} \int_0^{2\pi} x \sin nx\, dx$$

$$= \frac{1}{\pi} \left[x \left(-\frac{\cos nx}{n} \right) - (1) \left(-\frac{\sin nx}{n^2} \right) \right]_0^{2\pi}$$

$$= \frac{1}{\pi} \left[-\frac{2\pi}{n} \right] = -\frac{2}{n}$$

Sub. The value of $a_0, a_n and\; b_n$ in $f(x)$

We get

$$f(x) = \pi - 2\sum_{n=1}^{\infty} \frac{1}{n} \sin nx$$

Example:2 Find the Fourier Series of $f(x) = x^2$ in the interval $(0, 2\pi)$.

Solution: Here $c = 0\ and\ c + 2l = 2\pi$

$$\therefore l = \pi$$

The Fourier Series of given function is given by

$$\boldsymbol{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi}{l} x + b_n \sin \frac{n\pi}{l} x\right)}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\, x + b_n \sin nx)$$

Now, find

$$\boldsymbol{a_0} = \frac{1}{\pi} \int_0^{2\pi} x^2\, dx$$

$$= \frac{8\pi^3}{3}$$

Also,

$$\boldsymbol{a_n} = \frac{1}{\pi} \int_c^{c+2l} f(x)\, dx$$

$$= \frac{1}{\pi} \int_0^{2\pi} x^2 \cos nx\, dx$$

$$= \frac{4}{n^2}$$

$$\boldsymbol{b_n} = \frac{1}{\pi} \int_c^{c+2l} f(x) \sin nx\, dx$$

$$= \frac{1}{\pi} \int_0^{2\pi} x^2 \sin nx\, dx$$

$$= -\frac{4\pi}{n}$$

Hence,

$$f(x) = \frac{4\pi^2}{3} + 4\sum_{n=1}^{\infty}\left(\frac{1}{n^2}\cos n\,x - \frac{\pi}{n}\sin nx\right)$$

Example:3 Obtain the Fourier series to represent f(x)= $e^{ax}(a \neq 0)$in the interval $0 < x < 2\pi$

Solution: Given function $f(x)$ may be expanded in Fourier series as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \ldots\ldots\ldots\ldots(i)$$

Where, $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx$

$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$

$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$

Now, $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx$

$= \frac{1}{\pi}\int_0^{2\pi} e^{ax}dx$

$= \frac{1}{\pi}\left[\frac{e^{ax}}{a}\right]_0^{2\pi}$

$= \frac{1}{a\pi}[e^{2a\pi} - 1]$

$\frac{a_0}{2} = \frac{e^{2a\pi}-1}{2a\pi}$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$

$= \frac{1}{\pi}\int_0^{2\pi} e^{ax}\cos nx\,dx$

Using $\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}\{a\cos bx + b\sin bx\} + c$

$$= \frac{1}{\pi}\left[\frac{e^{ax}}{a^2+n^2}\{a\cos nx + n\sin nx\}\right]_0^{2\pi}$$

$$= \frac{1}{\pi(a^2+n^2)}[e^{ax}\{a\cos 2n\pi + n\sin 2n\pi\} - e^0\{a\cos 0 + n\sin 0\}]$$

As $\cos 2\,n\pi = 1$, $\sin 2n\pi = 0$, $\cos 0 = 1$, $\sin 0 = 0$

$$\therefore\ a_n = \frac{a}{\pi(a^2+n^2)} e^{2a\pi} - 1$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\, dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} e^{ax}\sin nx\, dx$$

Using $\int_0^{2\pi} e^{ax}\sin bx\, dx = \frac{e^{ax}}{a^2+b^2}\{a\sin bx - b\cos bx\} + c$

$$= \frac{1}{\pi}\left[\frac{e^{ax}}{a^2+n^2}\{a\sin nx - n\cos nx\}\right]_0^{2\pi}$$

$$= \frac{1}{\pi(a^2+n^2)}[e^{2a\pi}\{a\sin 2n\pi - n\cos 2n\pi\} - \{a\sin 0 - n\cos 0\}]$$

But, $\cos 2\, n\pi = 1$, $\sin 2n\pi = 0$, $\cos 0 = 0$, $\sin 0 = 0$

$$= \frac{a}{\pi(a^2+n^2)}(-ne^{2a\pi} + n)$$

$$\therefore\ b_n = \frac{n}{\pi(a^2+n^2)}(1 - e^{2a\pi})$$

Substituting the values of $a_0 a_n\ and\ b_n\ in\ (i), we\ have$

$$f(x) = \frac{e^{2a\pi} - 1}{\pi}\left[\frac{1}{2a} + a\sum_{n=1}^{\infty}\frac{\cos nx}{a^2+n^2} - \sum_{n=1}^{\infty}\frac{\text{n}\sin\ nx}{a^2+n^2}\right]$$

Example:4 Find the Fourier series representation of $f(x) = x + |x|$ in the interval $-\pi < x < \pi$.

Solution:

Given function $f(x)$ may be expanded in Fourier series as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n\cos nx + b_n\sin nx]$$

Now,

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} x + |x| \, dx$$

$$= \frac{1}{\pi} \left[\int_{-\pi}^{\pi} x \, dx + \int_{-\pi}^{\pi} |x| \, dx \right]$$

$$= \frac{1}{\pi} \left[0 + 2 \int_{0}^{\pi} |x| \, dx \right]$$

$$= \frac{1}{\pi} [\pi^2] = \pi$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} [x + |x|] \cos nx \, dx$$

$$= \frac{1}{\pi} \left[\int_{-\pi}^{\pi} x \cos nx \, dx + \int_{-\pi}^{\pi} |x| \cos nx \, dx \right]$$

$$= \frac{2}{\pi} \int_{0}^{\pi} |x| \cos nx \, dx$$

$$= \frac{2}{\pi} \left[x \frac{\sin nx}{n} - (-1) \left(-\frac{\cos nx}{n} \right) \right]_0^{\pi}$$

$$= \frac{2}{\pi n^2} [(-1)^n - 1]$$

$$b_n = \frac{1}{\pi} \int_{0}^{2\pi} f(x) \sin nx \, dx$$

$$= \frac{1}{\pi} \int_{0}^{2\pi} [x + |x|] \sin nx \, dx = -\frac{2}{n} (-1)^n$$

Substituting the values of a_0, a_n and b_n in $f(x)$, we have

$$f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \left[\frac{2}{\pi n^2}[(-1)^n - 1]\ \cos nx - \frac{2}{n}(-1)^n \sin nx\right]$$

Example:5 Find a Fourier series for $\boldsymbol{f(x) = x + x^2}$, $\quad -\boldsymbol{\pi} < x < \pi$

Also deduce that

$$\boldsymbol{\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots \ldots \ldots \ldots = \frac{\pi^2}{6}}$$

Solution:

Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n \cos nx + b_n \sin nx]$

be the required Fourier Series.

Where, $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx\, dx$

$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx\, dx$

Now, $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$

$= \frac{1}{\pi}\int_{-\pi}^{\pi} x + x^2 dx$

$= \frac{1}{\pi}\int_{-\pi}^{\pi} xdx + \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\, dx$

As x is an odd function, $\int_{-\pi}^{\pi} xdx = 0$ and

$x^2$is an even function, therefore $\frac{1}{\pi}\int_{-\pi}^{\pi} x^2\, dx = 2\ \int_0^{\pi} x^2 dx$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x^2 dx$$

$= \frac{2}{\pi}\left(\frac{x^3}{3}\right)_0^{\pi}$

$$a_0 = \frac{2\pi^2}{3}$$

$$\frac{a_0}{2} = \frac{\pi^2}{3}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} (x + x^2) \cos nx \, dx$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} x \cos nx \, dx + \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \cos nx \, dx$$

As $x \cos nx$ is an odd function $\int_{-\pi}^{\pi} x \cos nx \, dx = 0$ and $x^2 \cos nx$ is an even function, therefore

$$\int_{-\pi}^{\pi} x^2 \cos nx \, dx = 2 \int_{0}^{\pi} x^2 \cos nx \, dx$$

$$= \frac{2}{\pi} \int_{0}^{\pi} x^2 \cos nx \, dx$$

$$= \frac{2}{\pi} \left[(x^2) \left(\frac{\sin nx}{n} \right) - (2x) \left(-\frac{\cos nx}{n^2} \right) + (2) \left(-\frac{\sin nx}{n^3} \right) \right]_0^{\pi}$$

$$= \frac{2}{\pi} \left[\left(\frac{x^2 \sin nx}{n} \right) + \left(\frac{2x \cos nx}{n^2} \right) - \left(\frac{2 \sin nx}{n^3} \right) \right]_0^{\pi}$$

$$= \frac{2}{\pi} \left[\pi^2 \frac{\sin n\pi}{n} + 2\pi \frac{\cos n\pi}{n^2} - 2 \frac{\sin n\pi}{n^3} - 0 \right]_0^{\pi}$$

But $\sin n\pi = 0$, $\cos n\pi = (-1)^n$

$\therefore \ a_n = \frac{4\,(-1)^n}{n^2}$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} (x + x^2) \sin nx \, dx$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin nx \, dx + \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \sin nx \, dx$$

As $x^2 \sin nx \ \ is\ an\ odd\ function, \int_{-\pi}^{\pi} x^2 \sin nx \, dx = 0$ and $x \sin nx$ is an even function , therefore

$$\int_{-\pi}^{\pi} x \sin nx \, dx = 2 \int_{0}^{\pi} x \sin nx \, dx$$

$$= \frac{2}{\pi}\int_0^\pi x \sin nx \, dx$$

$$= \frac{2}{\pi}\left[(x)\left(-\frac{\cos nx}{n}\right) - (1)\left(-\frac{\sin nx}{n^2}\right)\right]_0^\pi$$

$$= \frac{2}{\pi}\left[\left(-\frac{x\cos nx}{n}\right) + \left(\frac{\sin nx}{n^2}\right)\right]_0^\pi$$

$$= \frac{2}{\pi}\left[-\frac{\pi\cos n\pi}{n} + \frac{\sin n\pi}{n^2} - 0\right]$$

But $\cos n\pi = (-1)^n \; and \; \sin n\pi = 0$

$$\therefore \; b_n = -\frac{2\,(-1)^n}{n}$$

Substituting above value in $f(x)$, we get

$$f(x) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n\cos nx}{n^2} - 2\sum_{n=1}^{\infty}\frac{(-1)^n \sin nx}{n} \ldots\ldots\ldots\ldots\ldots\ldots\ldots..(i)$$

As a required Fourier Series,

At, $x = \pi$,

The sum of the series is

$$f(\pi) = \frac{f(\pi+0) + f\,(\pi-0)}{2}$$

$$= \frac{\pi^2 + \pi + \pi^2 - \pi}{2}$$

$$f(\pi) = \pi^2$$

Putting $x = \pi$ in series (ii), we have

$$f(x) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n\cos n\pi}{n^2} - 2\sum_{n=1}^{\infty}\frac{(-1)^n\sin n\pi}{n^2}$$

But $\cos n\pi = (-1)^n \, and \sin n\pi = 0$

$$\pi^2 = \frac{\pi^2}{3} + 4\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2}\ldots\ldots\ldots\ldots\right]$$

$$\therefore \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots\ldots\ldots\ldots = \frac{\pi^2}{6}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.(ii)$$

$$\therefore \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

Example:6 Expand $f(x) = x\cos x, 0 < x < 2\pi$ as Fourier series

Solution: $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n \cos nx + b_n \sin nx]$

Where, $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx$

$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\, dx$

$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\, dx$

Now, $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx$

$\frac{1}{\pi}\int_0^{2\pi} x\cos x\, dx$

Integrating by parts

$= \frac{1}{\pi}[(x)(sinx) - (1)(-\cos x)]_0^{2\pi}$

$= \frac{1}{\pi}[x\sin x + \cos x]_0^{2\pi}$-4

$= \frac{1}{\pi}[2\pi\sin 2\pi + \cos 2\pi - son\ 0 - \cos 0]$

$\cos 2\pi = 1, \sin 2\pi = 0, \cos 0 = 1, \sin 0 = 0$

$= \frac{1}{\pi}[1-1]$

$a_0 = 0$

$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\, dx$

$= \frac{1}{\pi}\int_0^{2\pi} xcos\ x\ \cos nx\, dx$

$= \frac{1}{2\pi}\int_0^{2\pi} x(2cosnx\ \cos x)\, dx$

But 2CC= C+C

$= \frac{1}{2\pi}\int_0^{2\pi} x\{\cos(n+1)x + \cos(n-1)x\}dx$

$= \frac{1}{2\pi}\int_0^{2\pi}\left[\cos(n+1)x\, dx + \int_0^{2\pi} x\{\cos(n-1)x\, dx\}\right]$

Integrating both terms by parts,

$$= \frac{1}{2\pi}\left[(x)\left(\frac{\sin(n+1)x}{n+1}\right) - (1)\left(-\frac{\cos(n+1)x}{(n+1)^2}\right) + (x)\left(\frac{\sin(n-1)x}{n-1}\right) - (1)\left(-\frac{\cos(n-1)x}{(n-1)^2}\right)\right]_0^{2\pi}$$

Where $n \neq 1$

$$= \frac{1}{2\pi}\left[\frac{\mathrm{x}\sin(n+1)x}{n+1} + \frac{\cos(n+1)x}{(n+1)^2} + \frac{\mathrm{x}\sin(n-1)x}{n-1} + \frac{\cos(n-1)x}{(n-1)^2}\right]_0^{2\pi}$$

As $\sin(n \pm 1)\pi = 0$, $\cos(n \pm 1)\pi = 1$

$$= \frac{1}{2\pi}\left[\frac{1}{(n+1)^2} + \frac{1}{(n-1)^2} - \frac{1}{(n+1)^2} - \frac{1}{(n-1)^2}\right]$$

$$\therefore\ a_n = 0$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\, dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} xcos\ x\ \sin nx\, dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x(2sinnx\ \cos x)\, dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x\{\sin(n+1)x + \sin(n-1)x\}dx$$

But , 2SC=S+S,

$$= \frac{1}{2\pi}\left[\int_0^{2\pi} x\sin(n+1)x\, dx\ + \int_0^{2\pi} x\sin(n-1)x\, dx\ \right]$$

Integrating both terms by parts,

$$= \frac{1}{2\pi}\left[(x)\left(-\frac{\cos(n+1)x}{n+1}\right) - (1)\left(-\frac{\sin(n+1)x}{(n+1)^2}\right) + (x)\left(-\frac{\cos(n-1)x}{n-1}\right) - (1)\left(-\frac{\sin(n-1)x}{(n-1)^2}\right)\right]_0^{2\pi}$$

Where $n \neq 1$

$$= \frac{1}{2\pi}\left[-\frac{\mathrm{x}\cos(n+1)x}{n+1} + \frac{\sin(n+1)x}{(n+1)^2} - \frac{\mathrm{x}\cos(n-1)x}{n-1} + \frac{\sin(n-1)x}{(n-1)^2}\right]_0^{2\pi}$$

As $\cos 2(n \pm 1)\pi = 1$, $\sin 2(n \pm 1)\pi = 0$

$$= \frac{1}{2\pi}\left[-\frac{2\pi}{n+1} - \frac{2\pi}{n-1}\right]$$

$$\therefore\ b_n = -\frac{2n}{n^2-1}\ where\ (n \neq 1)$$

Here $a_n\ and\ b_n$ cannot be calculated for $n = 1$.

Again, $a_n = \frac{1}{\pi}\int_0^{2\pi} xcos\, x\ \cos nx\, dx$

Putting $n = 1$

$$a_1 = \frac{1}{\pi}\int_0^{2\pi} xcos^2x\, dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x(1 + cos2x)\, dx$$

$$= \frac{1}{2\pi}\left[\int_0^{2\pi} x\, dx + \int_0^{2\pi} xcos2x)\right]$$

$$= \frac{1}{2\pi}\left[\frac{x^2}{2} + (x)\left(\frac{\sin 2x}{2}\right) - (1)\left(\frac{-cos2x}{4}\right)\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[\frac{x^2}{2} + \frac{x\sin 2x}{2} + \frac{cos2x}{4}\right]_0^{2\pi}$$

As $\sin 4\pi = 0$, $\cos 4\pi = 1$

$$= \frac{1}{2\pi}\left[2\pi^2 + \frac{1}{4} - \frac{1}{4}\right]$$

$$a_1 = \pi$$

And, $b_n = \frac{1}{\pi}\int_0^{2\pi} xcos\, x\ \sin nx\, dx$

Putting $n = 1$

$$= \frac{1}{2\pi}\int_0^{2\pi} x(2\sin x\cos x)\, dx$$

$$\frac{1}{2\pi}\int_0^{2\pi} xsin2x\, dx$$

Integrating by Parts,

$$= \frac{1}{2\pi}\left[(x)\left(-\frac{\cos 2x}{2}\right) - (1)\left(-\frac{sin2x}{4}\right)\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[-\frac{x\cos 2x}{2} + \frac{\sin 2x}{4}\right]_0^{2\pi}$$

As $\sin 4\pi = 0$, $\cos 4\pi = 1$

$$= \frac{1}{2\pi}\left[-\frac{2\pi}{2}\right]$$

$$b_1 = -\frac{1}{2}$$

Now, $f(x) = \frac{a_0}{2} + a_1\cos x + \sum_{n=2}^{\infty} a_n \cos nx + b_1 \sin x + \sum_{n=2}^{\infty} b_n \sin nx$

Substituting the above values, we get

$f(x) = \pi \cos x - \frac{1}{2}\sin x - 2\ \sum_{n=2}^{\infty} \frac{n}{n^2-1}\sin nx$

$\therefore\ x\cos x = \pi\cos x - \frac{\sin x}{2} - 2\left[\frac{2}{3}\sin 2x + \frac{3}{8}\sin 3x + \frac{4}{15}\sin 4x + \cdots .\right]$

Which is desired Fourier series of $f(x) = x\cos x$

Example:7 Find the Fourier Series of the function

$$\boldsymbol{f}(\boldsymbol{x}) = \begin{cases} x & -1 < x < 0 \\ 2 & 0 < x < 1 \end{cases}$$

Solution: Here $l = 1$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n \cos n\pi x + b_n \sin n\pi x]$$

$$a_0 = \frac{1}{l}\int_{c}^{c+2l} f(x)dx$$

$$= \int_{-1}^{1} f(x)dx$$

$$= \int_{-1}^{0} x\,dx + \int_{0}^{1} 2\,dx$$

$$= -\frac{1}{2} + 2 = 3/2$$

$$a_n = \frac{1}{l}\int_{c}^{c+2l} f(x)\cos\frac{n\pi x}{l}dx$$

$$= \int_{-1}^{0} x\cos n\pi x\,dx + \int_{0}^{1} 2\cos n\pi x\,dx$$

$$= \left[x\left(\frac{\sin n\pi x}{n\pi}\right) - (1)\left(\frac{\cos n\pi x}{n^2\pi^2}\right)\right]_{-1}^{0} + \left(\frac{\sin n\pi x}{n\pi}\right)_{0}^{1}$$

$$= \frac{1}{n^2\pi^2}[1 - (1)^n]$$

$$b_n = \frac{1}{l}\int_{c}^{c+2l} f(x)\frac{\sin n\pi x}{n\pi}dx$$

$$= \left[x\left(-\frac{\cos n\pi x}{n\pi}\right) - (1)\left(-\frac{\sin n\pi x}{n^2\pi^2}\right)\right]_{-1}^{0} + \left(\frac{\cos n\pi x}{n\pi}\right)_{0}^{1}$$

$$= \frac{1}{n\pi}[2 - 3(-1)^n]$$

Hence,

$$f(x) = \frac{3}{4} + \sum_{n=1}^{\infty}\left[\frac{1}{n^2\pi^2}[1-(1)^n]\cos n\pi x + \frac{1}{n\pi}[2-3(-1)^n]\sin n\pi x\right]$$

Exercise:

1. **Find the Fourier series of the periodic functions $f(x)$ with period 2π defined as follows: f(x) =**$\begin{cases} 0 & -\pi < x < 0 \\ x & 0 < x < \pi \end{cases}$

Answer: $f(x) = \frac{\pi}{4} + \frac{1}{\pi}\sum_{n=1}^{\infty}[(-1)^n - 1]\frac{\cos nx}{n^2} - \sum_{n=1}^{\infty}(-1)^n\frac{\sin nx}{n}$

2. **Find the Fourier Series of the function $f(x) = 4 - x^2$ in the interval $(0,2)$**

Answer: $a_0 = \frac{4}{3}, a_n = -\frac{4}{n^2\pi^2}, b_n = \frac{4}{n\pi}$

3. **Find the Fourier Series of the function $f(x) = \frac{1}{2}(\pi - x)$ in the interval $(0, 2\pi)$.**
   Hence deduced that $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$
   Answer: $a_0 = a_n = 0, b_n = \frac{1}{n}$
4. Find the Fourier series of periodic functions with period 2, which are given below:

$$f(x) = \begin{cases} \pi; \ for 0 \le x \le 1 \\ \pi(2-x); \ for 1 \le x \le 2 \end{cases}$$

ANS: $\left[a_0 = \frac{3\pi}{4} \ , \ a_n = \frac{1}{\pi n^2}((-1)^n - 1) \ , \ b_n = \frac{1-2(-1)^n}{n}\right]$

Fourier Series for even and odd Function.

$\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$, if $f(x)$ is even function

$= 0$, if $f(x)$ is odd function

Example:7 Find the Fourier Series of the function $f(x) = |x|$ in the interval $(-\pi, \pi)$.

Solution:

Here $f(x) = |x|$ is even function , therefore for the given interval $(-\pi, \pi)$, $b_n = 0$

$$a_0 = \frac{2}{l}\int_0^L f(x)dx$$

$$= \frac{2}{\pi}\int_0^\pi |x|\, dx$$

$$= \frac{2}{\pi}\left(\frac{x^2}{2}\right)_0^\pi = \pi$$

$$a_n = \frac{2}{l}\int_0^L f(x)\cos\frac{n\pi x}{l}\, dx$$

$$a_n = \frac{2}{\pi}\int_0^\pi |x|\cos nx\, dx$$

$$= \frac{2}{\pi}\left[x\left(\frac{\sin nx}{n}\right) + \frac{\cos nx}{n^2}\right]_0^\pi$$

$$= \frac{2}{\pi}\left(\frac{\cos nx}{n^2} - \frac{\cos 0}{n^2}\right)$$

$$= \frac{2}{n^2\pi}[(-1)^n - 1]$$

Hence

$$f(x) = |x| = \frac{\pi}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{[(-1)^n - 1]}{n^2}\cos nx$$

Example:8 Expand $f(x) = |\sin x|, -\pi \le x \le \pi$ as Fourier series.

Solution:

The function $f(x) = |\sin x|$is an even function

$\therefore\ b_n = 0$

For even function the required Fourier Series is given by

$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \ldots\ldots\ldots\ldots\ldots\ldots\ldots..(i)$

Where,

$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx$

$f(x) = |\sin x| = \sin x\, , 0 < x < \pi$

Now, $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$

As $f(x)$ is even function

$a_0 = \frac{2}{\pi}\int_{0}^{\pi} |\sin x| dx$

$= \frac{2}{\pi}\int_{0}^{\pi} \sin x\, dx$

$= \frac{2}{\pi}[-\cos x]_0^{\pi}$

$a_0 = \frac{2}{\pi}(2)$

$\therefore \frac{a_0}{2} = \frac{2}{\pi}$

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx$

Since $f(x)\cos nx$ is even function

$a_n = \frac{2}{\pi}\int_{0}^{\pi} \cos nx \sin x\, dx$

But 2CS= S-S

$= \frac{1}{\pi}\int_{0}^{\pi} [\sin(n+1)x = \sin(n-1)x]dx$

$= \frac{1}{\pi}\left[\frac{\cos(n-1)x}{n-1} - \frac{\cos(n+1)x}{n+1}\right]_0^{\pi}$, (Where, $n \neq 1$)

Since $\cos(n \pm 1)\pi = (-1)^{n+1}$

$= \frac{1}{\pi}\left[\frac{(-1)^{n+1}}{n-1} - \frac{(-1)^{n+1}}{n+1} - \frac{1}{n-1} + \frac{1}{n+1}\right] \ldots \ldots \ldots \ldots \ldots \ldots .. (1)$

Where n is even, $(-1)^{n+1} = -1$

$\therefore\ a_n = \frac{2}{\pi}\left[\frac{1}{n+1} + \frac{1}{n-1}\right]$

$a_n = -\frac{4}{\pi(n^2-1)}$

Where n is odd, $(-1)^{n+1} = 1$, from (1)

$= \frac{1}{\pi}\left[\frac{1}{n-1} - \frac{1}{n+1} - \frac{1}{n-1} + \frac{1}{n+1}\right]$

$\therefore\ a_n = 0$

Thus ,$a_n = 0$,if n is odd

$= -\frac{4}{\pi(n^2-1)}$, if $n$is even

When $n = 1$

$a_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos x\,dx$

$= \frac{2}{\pi}\int_0^{\pi} |\sin x| cosx\,dx$

$= \frac{2}{\pi}\int_0^{\pi} \sin 2x\,dx$

$= \frac{2}{\pi}\left[-\frac{\cos 2x}{2}\right]_0^{\pi}$

$a_1 = 0$

$\therefore\ series\ (i) becomes,$

$f(x) = \frac{2}{\pi} - \frac{4}{\pi}\sum_{\substack{n=2\\ n-even}}^{\infty} \frac{\cos nx}{n^2-1}$

Half Range Cosine – Sine Series

The Fourier series of an even function of period of 2*l* is a "Fourier cosine series" $(-\boldsymbol{l}, \boldsymbol{l})$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi}{l}x$$

$$a_0 = \frac{2}{l}\int_0^L f(x)dx \quad , \quad a_n = \frac{2}{l}\int_0^L f(x)\cos\frac{n\pi x}{l}\,dx$$

The Fourier series of an odd function of period of 2L is a "Fourier sine series" $(-\boldsymbol{l}, \boldsymbol{l})$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{l}x$$

$$b_n = \frac{2}{l}\int_0^L f(x)\sin\frac{n\pi x}{l}\,dx$$

The length of the interval (a, b) is $l = b - a$.

Example:9 Find the Fourier sine series of $f(x) = \pi - x, for\ 0 < x < 3$

Solution: let the required half range sine series be,

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots (i)$$

Where, $b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\, dx$

$$b_n = \frac{2}{\pi}\int_0^{\pi}(\pi - x)\sin nx\, dx$$

Integrating by parts

$$= \frac{2}{\pi}\left[(\pi - x)\left(-\frac{\cos nx}{n}\right) - (-1)\left(-\frac{\sin nx}{n^2}\right)\right]_0^{\pi}$$

$$= -\frac{2}{\pi}\left[(\pi - x)\frac{\cos nx}{n} + \frac{\sin nx}{n^2}\right]_0^{\pi}$$

$$= -\frac{2}{\pi}\left(-\frac{\pi}{n}\right)$$

$$b_n = \frac{2}{n}$$

Substituting above values in (i), we get

$$f(x) = 2\sum_{n=1}^{\infty}\frac{\sin nx}{n}$$

$$f(x) = 2\left[\frac{\sin x}{1} + \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \cdots\ldots\ldots..\right]$$

Example:10. Find the half range of cosine series for $f(x) = \begin{cases} \boldsymbol{x}, & \mathbf{0} < x < \frac{\boldsymbol{\pi}}{\mathbf{2}} \\ \boldsymbol{\pi - x}, & \frac{\boldsymbol{\pi}}{\mathbf{2}} < x < \pi \end{cases}$

Solution: $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots (i)$

Where, $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\, dx$$

Now, $a_0 = \frac{2}{\pi}\left[\int_0^{\frac{\pi}{2}} x\, dx + \int_{\frac{\pi}{2}}^{\pi}(\pi - x)dx\right]$

$$= \frac{2}{\pi}\left[\left(\frac{x^2}{2}\right)_0^{\frac{\pi}{2}} + \left(\pi x - \frac{x^2}{2}\right)_{\frac{x^2}{2}}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[(x^2)_0^{\frac{\pi}{2}} + (2\pi x - x^2)_{\frac{x^2}{2}}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi^2}{4} + \pi^2 - \frac{3\pi^2}{4}\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi^2}{2}\right]$$

$$= \frac{\pi}{2}$$

$$\therefore \frac{a_0}{2} = \frac{\pi}{4}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\, dx$$

$$= \frac{2}{\pi}\left[\int_0^{\frac{\pi}{2}} xcos\ nx\ \ dx + \int_{\frac{\pi}{2}}^{\pi}(\pi - x)\cos nx\, dx\right]$$

$$= \frac{2}{\pi}\left[\left((x)\left(\frac{\sin nx}{n}\right) - (1)\left(-\frac{\cos nx}{n^2}\right)\right)\right]_0^{\frac{\pi}{2}} + \frac{2}{\pi}\left[\left((\pi - x)\left(\frac{\sin nx}{n}\right) - (-1)\left(-\frac{\cos nx}{n^2}\right)\right)\right]_{\frac{\pi}{2}}^{\pi}$$

$$= \frac{2}{\pi}\left[\left(\frac{x\ \sin nx}{n} + \frac{\cos nx}{n^2}\right)\right]_0^{\frac{\pi}{2}} + \frac{2}{\pi}\left[\left((\pi - x)\frac{\sin nx}{n} - \frac{\cos nx}{n^2}\right)\right]_{\frac{\pi}{2}}^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{\pi}{2n}\sin\frac{n\pi}{2} + \frac{1}{n^2}\cos\frac{n\pi}{2} - \frac{1}{n^2} - \frac{(-1)^n}{n^2} - \frac{\pi}{2n}\sin\frac{n\pi}{2} + \frac{1}{n^2}\cos\frac{n\pi}{2}\right]$$

$$\therefore a_n = \frac{2}{n^2\pi}\left[2\cos\frac{n\pi}{2} - 1 - (-1)^n\right]$$

Where n is odd

$$\therefore a_n = \frac{2}{n^2\pi}\cos\frac{n\pi}{2} = 0\left(\cos\frac{n\pi}{2} = 0, where\ n\ is\ odd\right)$$

When n is even from (ii)

$$a_n = \frac{2}{n^2\pi}\left[\cos\frac{n\pi}{2} - 1\right]$$

For $n = 2$, $\quad a_2 = \frac{1}{\pi}[-2] = -\frac{2}{\pi}$

For $n = 4$, $\quad a_4 = \frac{1}{4\pi}[0] = 0$

For $n = 6$, $\quad a_6 = \frac{1}{9\pi}[-2] = -\frac{2}{9\pi}$

Substituting above values in (i), we have

$$f(x) = \frac{\pi}{4} - \frac{2}{\pi}\left[\frac{\cos 2x}{1^2} + \frac{\cos 6x}{3^2} + \cdots \ldots \ldots \ldots .\right]$$

Example:11. If $\boldsymbol{f(x)} = \begin{cases} \boldsymbol{mx} & \boldsymbol{, 0 \le x \le \frac{\pi}{2}} \\ \boldsymbol{m(\pi - x),} & \boldsymbol{\frac{\pi}{2} \le x \le \pi} \end{cases}$, then show that,

$$\boldsymbol{f(x) = \frac{4m}{\pi}\left\{\frac{sinx}{1^2} - \frac{sin3x}{3^2} + \frac{sin5x}{5^2} - \cdots \dots \dots .\right\}}$$

Solution:

Here we have to find half range sine series in $(0, \pi)$

Let the required half range sine series be

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx \dots \dots \dots \dots \dots \dots \dots (i)$$

Where, $b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx \, dx$

$$= \frac{2}{\pi}\int_0^{\frac{\pi}{2}} mx \sin nx \, dx + \frac{2}{\pi}\int_{\frac{\pi}{2}}^{\pi} m(\pi - x) \sin nx \, dx$$

Integrating by parts

$$= \frac{2m}{\pi}\left[(x)\left(-\frac{\cos nx}{n}\right) - (1)\left(-\frac{\sin nx}{n^2}\right)\right]_0^{\frac{\pi}{2}} + \frac{2m}{\pi}\left[(\pi - x)\left(-\frac{\cos nx}{n}\right) - (-1)\left(-\frac{\sin nx}{n^2}\right)\right]_{\frac{\pi}{2}}^{\pi}$$

$$= -\frac{2m}{\pi}\left[x\frac{\cos nx}{n} - \frac{\sin nx}{n^2}\right]_0^{\frac{\pi}{2}} - \frac{2m}{\pi}\left[(\pi - x)\frac{\cos nx}{n} + \frac{\sin nx}{n^2}\right]_{\frac{\pi}{2}}^{\pi}$$

$$= -\frac{2m}{\pi}\left[\frac{\pi}{2n}\cos\left(\frac{n\pi}{2}\right) - \frac{1}{n^2}\sin\left(\frac{n\pi}{2}\right)\right] + \frac{2m}{\pi}\left[\frac{\pi}{2n}\cos\left(\frac{n\pi}{2}\right) + \frac{1}{n^2}\sin\left(\frac{n\pi}{2}\right)\right]$$

$$b_n = \frac{4m}{n^2\pi}\sin\left(\frac{n\pi}{2}\right)$$

Substituting above value in (i), it takes the form

$$f(x) = \frac{4m}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\left(\frac{n\pi}{2}\right)\sin nx$$

$$f(x) = \frac{4m}{\pi}\left\{\frac{sinx}{1^2} - \frac{sin3x}{3^2} + \frac{sin5x}{5^2} - \frac{sin7x}{7^2} + \cdots \dots \dots .\right\}$$

Example:12. Obtain half range sine series to represent $\boldsymbol{f(x) = lx - x^2}$ in the range $\boldsymbol{(0, l)}$

Solution: Let $f(x) = \sum_{n=1}^{\infty} f(x) \sin\frac{n\pi x}{l}$

Where, $b_n = \frac{2}{l}\int_0^{l} f(x) \sin\frac{n\pi x}{l} \, dx$

$$= \frac{2}{l}\int_0^{l} lx - x^2 \sin\frac{n\pi x}{l} \, dx$$

$= \frac{2}{l}\left[(lx - x^2)\left(-\frac{l}{n\pi}\cos\frac{n\pi x}{l}\right) - (l-2x)\left(-\frac{l^2}{n^2\pi^2}\sin\frac{n\pi x}{l}\right) + (-2)\frac{l^3}{n^3\pi^3}\cos\frac{n\pi x}{l}\right]_0^l$

$= \frac{-4l^2}{n^3\pi^3}[\cos n\pi - 1]$

$= \frac{-4l^2}{n^3\pi^3}[(-1)^n - 1]$

$= 0 \ if \ n \ is \ even$

$= \frac{8l^2}{n^3\pi^3} \ if \ n \ is \ odd$

Hence, half range sine series for $f(x)$ is

$$f(x) = \frac{8l^2}{n^3\pi^3}\left[\frac{1}{1^3}\sin\frac{\pi x}{l} + \frac{1}{3^3}\sin\frac{3\pi x}{l} + \frac{1}{5^3}\sin\frac{5\pi x}{l} + \cdots \ldots ..\right]$$

Example:13 Find Fourier series of $f(x) = x^2, 0 < x < c$

Solution:

Let, $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} \ldots \ldots \ldots \ldots \ldots \ldots \ldots .. (i)$

Comparing $0 < x < c \ with \ 0 < x < l, we \ get \ l = c$

Now, $a_0 = \frac{2}{c}\int_0^c f(x)dx$

$= \frac{2}{c}\int_0^c x^2 dx$

$= \frac{2}{c}\left[\frac{x^3}{3}\right]_0^c$

$= \frac{2}{c}\left[\frac{c^3}{3}\right]$

$a_0 = \frac{2c^2}{3}$

$\frac{a_0}{2} = \frac{c^2}{3}$

$a_n = \frac{2}{c}\int_0^c f(x)\cos\left(\frac{n\pi x}{c}\right)dx$

$= \frac{2}{c}\int_0^c x^2\cos\left(\frac{n\pi x}{c}\right)dx$

$= \frac{2}{c}\left[(x^2)\left(\frac{\sin\left(\frac{n\pi x}{c}\right)}{\frac{n\pi}{c}}\right) - (2x)\left(-\frac{\cos\left(\frac{n\pi x}{c}\right)}{\frac{n^2\pi^2}{c^2}}\right) + (2)\left(-\frac{\sin\left(\frac{n\pi x}{c}\right)}{\frac{n^3\pi^3}{c^3}}\right)\right]_0^c$

$= \frac{2}{c}\left[\frac{2c^2}{n^2\pi^2}x\cos\left(\frac{n\pi x}{c}\right)\right]_0^c$

$$= \frac{4c}{n^2\pi^2}\left[x\cos\left(\frac{n\pi x}{c}\right)\right]_0^c$$

$$= \frac{4c}{n^2\pi^2}[c\cos n\pi]$$

$$\therefore\ a_n = \frac{4c^2(-1)^n}{n^2\pi^2}$$

Substituting above values in (i), we get

$$f(x) = \frac{c^2}{3} + \frac{4c^2}{n^2\pi^2}\sum_{n=1}^{\infty}(-1)^n\frac{\cos\left(\frac{n\pi x}{c}\right)}{n^2}$$

- **EXERCISE:**

1. Find a Fourier sine series for $f(x) = k$ in $0 < x < \pi$.
   [ANS: $f(x) = \frac{4k}{\pi}\sum_{n=1}^{\infty}\frac{sinnx}{n}$]

2. Find half range cosine series for

$$f(x) = kx \qquad , 0 \le x \le \frac{l}{2}$$

$$= k\,(l - x) \quad , \frac{l}{2} \le x \le l$$

[ANS: $f(x) = \frac{\pi}{2}\sin x - \frac{8}{\pi}\sum_{n=1}^{\infty}\frac{2n\sin 2nx}{\pi\left(4n^2-1\right)^2}$]

Parul University
Faculty of Engineering & Technology
Department of Applied Sciences and Humanities
1st Year B. Tech Programme (All Branches)
Mathematics – 1 (303191101) 2023-24
Unit – 6 Multivariable Calculus (Lecture Notes)

Overview:

In this unit, we will study about limit and Continuity, partial differentiation of first order and higher order, chain rule, implicit function, Jacobian. We will study further about Application for Partial Derivatives that includes tangent plane and normal line, extreme values of a function, use of Lagrange's Multiplier and Taylor's and Maclaurin's series for two variables.

Objective:

At the end of this unit, you will be able to understand

- Functions of Several Variables
- Limit and Continuity
- Use of Partial Differentiation
- Application of Chain Rule
- Properties of Jacobian
- How to find out Tangent Plane and Normal Line
- Method to find extreme values of a function
- Method of Lagrange's multipliers
- Expansion using Taylor Series

Functions of Two Variables

Until now, we have only considered functions of a single variable y = $f(x)$.

For example, the area function of a square is its side square which include only one variable, that is only length.

However, many real-world functions consist of two (or more) variables.

Example:

- The area function of a rectangular shape depends on both its width and its height.
- The pressure of a given quantity of gas varies with respect to the temperature of the gas and its volume.
- Volume of cylinder is $v = \pi r^2 h$, where r and h are two independent variables and v is dependent variable.

  Therefore, we require partial derivative for the function, which depends more than one independent variable.

Limit of a Function of Two Variables

Definition: For every number $\in > 0$, there exists a corresponding number $\delta > 0$ such that for all (x, y) in the domain of f,

$$|f(x,y) - L| < \epsilon \text{ } wherever \text{ } 0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta$$

then, the function $f(x, y)$ is said to approach the **limit** L as (x, y) approaches (x_0, y_0) , and we write,

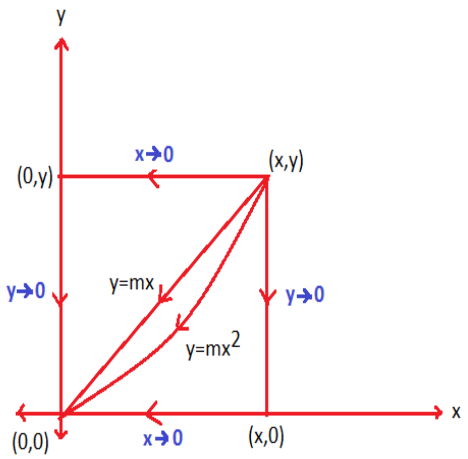
$$\lim_{(x,y)\to((x_0,y_0))} f(x,y) = L$$

To evaluate limit following methods are applicable: -

(i) By direct substitution: - If example is regarding limit $(x, y) \to (x_0, y_0)$ and on direct substitution we obtain finite value.

(ii) By definition of limit: - If in example it is mentioned.

(iii) By path if example is regarding $(x, y) \to (0,0)$ and on direct substitution we obtain an indeterminate form (mostly $\left(\frac{0}{0}\right)$)



Explanation:

Path 1: Evaluate $\lim\limits_{x\to 0}\left\{\lim\limits_{y\to 0} f(x,y)\right\}$

Path 2: Evaluate $\lim\limits_{y\to 0}\left\{\lim\limits_{x\to 0} f(x,y)\right\}$

If limit along path 1 and path 2 are same then proceed further otherwise limit doesn't exist.

Path 3: Evaluate the limit along $y = mx$ as $x \to 0$.

If limit along path 1, path 2 and path 3 are same then proceed further otherwise limit doesn't exist.

Path 4: Evaluate the limit along $y = mx^2$as $x \to 0$.

If limit along all the paths is same then limit exist.

Solved Examples:

1. Evaluate $\lim\limits_{(x,y)\to(1,2)} \frac{3x^2y}{x^2+y^2+5}$ **------------> Simple example by direct substitution**

Sol:

$$\lim_{(x,y)\to(1,2)} \frac{3x^2y}{x^2+y^2+5}$$
$$= \frac{3(1)^2(2)}{(1)^2+2^2+5}$$
$$= \frac{6}{10}$$
$$= \frac{3}{5}$$
$$\therefore \lim_{(x,y)\to(1,2)} \frac{3x^2y}{x^2+y^2+5} = \frac{3}{5}$$

2. Applying the definition of limit, show that $\lim\limits_{(x,y)\to(0,0)} \frac{4xy^2}{x^2+y^2} = 0.$

Sol.

Let $\varepsilon > 0$ be given.

We want to find a $\delta > 0$ such that

$\left|\frac{4xy^2}{x^2+y^2} - 0\right| < \varepsilon$ whenever $0 < \sqrt{(x-0)^2+(y-0)^2} < \delta$

Since, $y^2 \le x^2 + y^2$

$$\frac{y^2}{x^2+y^2} \le 1$$
$$\frac{4|x|y^2}{x^2+y^2} \le 4|x| = 4\sqrt{x^2} \le 4\sqrt{x^2+y^2}$$

Now, $\left|\frac{4xy^2}{x^2+y^2} - 0\right| = \frac{4|x|y^2}{x^2+y^2} \le 4\sqrt{x^2+y^2} < 4\delta$

By taking, $4\delta = \varepsilon$

$\left|\frac{4xy^2}{x^2+y^2} - 0\right| < \varepsilon$ whenever $0 < \sqrt{(x-0)^2+(y-0)^2} < \delta$

$\therefore \lim\limits_{(x,y)\to(0,0)} \frac{4xy^2}{x^2+y^2} = 0$ by definition.

3. Evaluate $\lim\limits_{(x,y)\to(0,0)} \frac{x-y}{x+y}$ **by Path method**

Sol: By putting $(x,y) \to (0,0)$ we obtain an indeterminate form $(\left(\frac{0}{0}\right))$

Therefore, we will apply the Path method to evaluate the limit of a function

Path 1:

$$\lim_{x\to 0}\left\{\lim_{y\to 0} f(x,y)\right\} = \lim_{x\to 0}\left\{\lim_{y\to 0}\left(\frac{x-y}{x+y}\right)\right\} = \lim_{x\to 0}\left(\frac{x}{x}\right) = 1$$

Path 2:

$$\lim_{y\to 0}\left\{\lim_{x\to 0} f(x,y)\right\} = \lim_{y\to 0}\left\{\lim_{x\to 0}\left(\frac{x-y}{x+y}\right)\right\} = \lim_{y\to 0}\left(-\frac{y}{y}\right) = (-1) = -1$$

Since both the limits are different, the limit does not exist.

4. Evaluate $\lim\limits_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^2+y^2}$

Sol: By putting $(x,y) \to (0,0)$ we obtain an indeterminate form $\left(\left(\frac{0}{0}\right)\right)$

Therefore, we will apply the Path method to evaluate the limit of a function

Path 1:

$$\lim_{x\to 0}\left\{\lim_{y\to 0} f(x,y)\right\} = \lim_{x\to 0}\left\{\lim_{y\to 0}\left(\frac{x^3+y^3}{x^2+y^2}\right)\right\} = \lim_{x\to 0}\left(\frac{x^3}{x^2}\right) = \lim_{x\to 0} x = 0$$

Path 2:

$$\lim_{y\to 0}\left\{\lim_{x\to 0} f(x,y)\right\} = \lim_{y\to 0}\left\{\lim_{x\to 0}\left(\frac{x^3+y^3}{x^2+y^2}\right)\right\} = \lim_{y\to 0}\left(\frac{y^3}{y^2}\right) = \lim_{y\to 0} y = 0$$

Path 3:

Put $y = mx$, as $x \to 0$

$$\lim_{x\to 0}\left(\frac{x^3+(mx)^3}{x^2+(mx)^2}\right) = \lim_{x\to 0}\left(\frac{x^3+m^3x^3}{x^2+m^2x^2}\right) = \lim_{x\to 0} x\left(\frac{1+\mathrm{m}^3}{1+\mathrm{m}^2}\right) = 0$$

Path 4:

Put $y = mx^2$, as $x \to 0$

$$\lim_{x\to 0}\left(\frac{x^3+(mx^2)^3}{x^2+(mx^2)^2}\right) = \lim_{x\to 0}\left(\frac{x^3+m^3x^6}{x^2+m^2x^4}\right) = \lim_{x\to 0} x\left(\frac{1+m^3x^3}{1+m^2x^2}\right) = 0$$

Since, limit along all the paths are same.

Hence, the limit exists and its value is 0.

5. Evaluate $\lim\limits_{(x,y)\to(0,0)} \frac{2xy}{x^2+y^2}$ by Path method.

Sol: By putting $(x,y) \to (0,0)$ we obtain an indeterminate form $\left(\left(\frac{0}{0}\right)\right)$

Therefore, we will apply the Path method to evaluate the limit of a function

Path 1:

$$\lim_{(x,y)\to(0,0)}\left(\frac{2xy}{x^2+y^2}\right) = 0$$

Path 2:

$$\lim_{(x,y)\to(0,0)}\left(\frac{2xy}{x^2+y^2}\right) = 0$$

Path 3:

Put $y = mx$, as $x \to 0$

$$\lim_{x\to 0}\left(\frac{2x(mx)}{x^2+(mx)^2}\right) = \lim_{x\to 0}\left(\frac{2mx^2}{x^2+m^2x^2}\right) = \frac{2m}{1+m^2}$$

As the limit depends on m and m is not fixed, the limit doesn't exist.

Exercise:

1. Evaluate $\lim\limits_{(x,y)\to(0,0)} \frac{x^2+y}{3x+y^2}$

2. Applying the definition of limit, show that $\lim\limits_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2} = 0$

3. Evaluate $\lim\limits_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2}$

4. Evaluate $\lim\limits_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}}$

Continuity of function of two Variables

A function $f(x,y)$ is **continuous at the point** $(x_{0,}y_0)$ if it satisfies following properties,

1. $f(x,y)$ is defined at (x_0, y_0)
2. $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y)$ exists
3. $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0)$

Solved Examples:

1. Discuss the continuity of

$$f(x,y) = \begin{matrix} \dfrac{x^2y}{x^3+y^3}, & (x,y)\neq(0,0) \\ 0 & (x,y)=(0,0) \end{matrix}$$

at point $(0,0)$. **[Summer 2023]**

Sol: Here, $f(x,y)$ is defined at $(0,0)$ and $f(0,0) = 0$

Path 1:

$$\lim_{(x,y)\to(0,0)}\left(\frac{x^2y}{x^3+y^3}\right) = 0$$

Path 2:

$$\lim_{(x,y)\to(0,0)}\left(\frac{x^2y}{x^3+y^3}\right) = 0$$

Path 3: Put $y = mx$, as $x \to 0$

$$\lim_{x\to 0}\left(\frac{x^2(mx)}{x^3+(mx)^3}\right) = \lim_{x\to 0}\left(\frac{mx^3}{x^3+m^3x^3}\right) = \frac{m}{1+m^3}$$

As the limit depends on m and m is not fixed, the limit doesn't exist. Therefore, limit doesn't exist. Hence, $f(x,y)$ is discontinuous at (0,0).

2. Discuss the continuity of

$$f(x,y) = \frac{x^2 - y^2}{\sqrt{x^2+y^2}}, \qquad (x,y) \neq (0,0)$$
$$0 \qquad (x,y) = (0,0)$$

at point $(0,0)$. **[Winter 2023]**

Sol: Here, $f(x,y)$ is defined at (0,0) and $f(0,0) = 0$.

Path 1:

$$\lim_{x\to 0}\left[\lim_{y\to 0}\frac{x^2-y^2}{\sqrt{x^2+y^2}}\right] = \lim_{x\to 0}\frac{x^2}{x} = x = 0$$

Path 2:

$$\lim_{y\to 0}\left[\lim_{x\to 0}\frac{x^2-y^2}{\sqrt{x^2+y^2}}\right] = \lim_{y\to 0}\left(-\frac{y^2}{y}\right) = \lim_{y\to 0}(-y) = 0$$

Path 3: Put $y = mx$, as $x \to 0$

$$\lim_{x\to 0}\left(\frac{x^2-(mx)^2}{\sqrt{x^2+(mx)^2}}\right) = \lim_{x\to 0}\left(\frac{x^2(1-m^2)}{x\sqrt{1+m^2}}\right) = 0$$

Path 4: Put $y = mx^2$, as $x \to 0$

$$\lim_{x\to 0}\left(\frac{x^2-(mx^2)^2}{\sqrt{x^2+(mx^2)^2}}\right) = \lim_{x\to 0}\left(\frac{x^2(1-m^2x^2)}{x\sqrt{1+m^2x^2}}\right) = 0$$

Since, limit along all the paths are same.

Therefore, limit exist.

Also, $\lim\limits_{(x,y)\to(0,0)} \frac{x^2-y^2}{\sqrt{x^2+y^2}} = 0 = f(0,0)$

Hence, $f(x,y)$ is continuous at (0,0).

3. Check whether the given function is continuous at origin or not, if yes then find point of continuity.

$$f(x,y) = \frac{x+y}{\sqrt{x}-\sqrt{y}} \qquad (x,y) \neq (0,0)$$
$$-1 \qquad (x,y) = (0,0)$$

Sol: Here,$f(x,y)$ is defined at (0,0) and $f(0,0) = -1$

Path 1:

$$\lim_{x\to 0}\left[\lim_{y\to 0}\frac{x+y}{\sqrt{x}-\sqrt{y}}\right] = \lim_{x\to 0}\frac{x}{\sqrt{x}} = \sqrt{x} = 0$$

Path 2:

$$\lim_{y\to 0}\left[\lim_{x\to 0}\frac{x+y}{\sqrt{x}-\sqrt{y}}\right]=\lim_{y\to 0}\left(\frac{y}{-\sqrt{y}}\right)=\lim_{y\to 0}(-\sqrt{y})\ =0$$

Path 3: Put $y=mx$, as $x\to 0$

$$\lim_{x\to 0}\left(\frac{x+y}{\sqrt{x}-\sqrt{y}}\right)\ =\lim_{x\to 0}\left(\frac{x+mx}{\sqrt{x}-\sqrt{mx}}\right)\ =\lim_{x\to 0}\sqrt{x}\left(\frac{1+m}{1-\sqrt{m}}\right)=0$$

Path 4: Put $y=mx^2$, as $x\to 0$

$$\lim_{x\to 0}\left(\frac{x+y}{\sqrt{x}-\sqrt{y}}\right)\ =\lim_{x\to 0}\left(\frac{x+mx^2}{\sqrt{x}-\sqrt{mx^2}}\right)\ =\lim_{x\to 0}\left(\frac{x+mx^2}{\sqrt{x}-\sqrt{m}x}\right)=\lim_{x\to 0}\frac{x}{\sqrt{x}}\left(\frac{1+mx}{1-\sqrt{mx}}\right)=0$$

Since, limit along all the paths are same.

Therefore, limit exist.

But, $\lim\limits_{(x,y)\to(0,0)}\frac{x+y}{\sqrt{x}-\sqrt{y}}\ =0\neq -1=f(0,0)$

Hence, $f(x,y)$ is discontinuous at (0,0).

Exercise:

1. Discuss the continuity of

$$f(x,y)=\frac{x}{3x+5y},\qquad (x,y)\neq(0,0)$$
$$1,\qquad (x,y)=(0,0)$$

2. Show that

$$f(x,y)=\frac{x^2y}{y^2-x^2},\qquad (x,y)\neq(0,0)$$
$$0,\qquad (x,y)=(0,0)$$

is continuous at origin.

3. Discuss the continuity of $f(x,y)=\begin{cases}\dfrac{x^2-y^2}{x^2+y^2}; & (x,y)\neq(0,0)\\ 0; & (x,y)=(0,0)\end{cases}$ at origin. **[Winter 2023]**

[Summer 2023]

4. Show that the function $f(x,y)=\dfrac{2x^2y}{x^4+y^2}$ has no limit as (x, y) approaches to (0,0) .

[Summer 2023].

5. Discuss the continuity of $f(x,y)=\begin{cases}\dfrac{2xy}{x^2+y^2}; & (x,y)\neq(0,0)\\ 0; & (x,y)\neq(0,0)\end{cases}$ at Origin. **[Summer 2023]**

Partial Derivatives:

The partial derivative of $f(x,y)$ with respect to x at the point (x_0,y_0) is

$$\left(\frac{\partial f}{\partial x}\right)_{(x_0,y_0)} = \lim_{h\to 0}\frac{f(x_0+h,y_0)-f(x_0,y_0)}{h}, \ \textit{provided the limit exists.}$$

The partial derivative of $f(x,y)$ with respect to y at the point (x_0,y_0) is

$$\left(\frac{\partial f}{\partial y}\right)_{(x_0,y_0)} = \lim_{h\to 0}\frac{f(x_0,y_0+h)-f(x_0,y_0)}{h} \ , \ \textit{provided the limit exists.}$$

Second order partial derivative:

Two successive partial differentiations of $f(x,y)$ with respect to *x* (holding *y* constant) is denoted by $\frac{\partial^2 f}{\partial x^2}$ or $f_{xx}(x,y)$.That is, we define

$$f_{xx}(x,y) = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$$

Similarly, two successive partial differentiations of $f(x,y)$ with respect to *y* (holding *x* constant) is denoted by $\frac{\partial^2 f}{\partial y^2}$ or $f_{yy}(x,y)$. That is, we define

$$f_{yy}(x,y) = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$$

We use $\frac{\partial^2 f}{\partial x\partial y}$ to mean differentiate first with respect to y then with respect to x, and we use $\frac{\partial^2 f}{\partial y\partial x}$ to mean differentiate first with respect to x then with respect to y.

Note:

$f_{xy}(x,y) = \frac{\partial^2 f}{\partial y\partial x}$ and $f_{yx}(x,y) = \frac{\partial^2 f}{\partial x\partial y}$ are known as mixed order partial derivatives.

The Mixed Derivative / Clairaut's Theorem

If $f(x,y)$ and its partial derivative f_x, f_y, f_{xy}, f_{yx} are

(i) defined throughout an open region containing a point (a,b) and

(ii) they are all continuous at (a,b),

then $f_{xy}(a,b) = f_{yx}(a,b)$.

THEOREM: If a function $f(x,y)$ is differentiable at (x_0,y_0) then $f(x,y)$ is continuous at (x_0,y_0).

Solved Examples:

1. **Find $\frac{\partial f}{\partial z}$ at (1,2,3) for $f(x,y,z) = x^2yz^2$ using the definition.**

Sol: Here, $\frac{\partial f}{\partial z} = \lim_{h\to 0}\frac{f(x_0,y_0,z_0+h)-f(x_0,y_0,z_0)}{h}$

$$\left(\frac{\partial f}{\partial z}\right)_{(1,2,3)} = \lim_{h \to 0} \frac{f(1,2,3+h) - f(1,2,3)}{h}$$

$$= \lim_{h \to 0} \frac{2(9+6h+h^2) - 18}{h}$$

$$= \lim_{h \to 0} (12 + 2h)$$

$$= 12$$

2. **If $f(x,y) = x^3 + y^3 - 2xy^2$. Find all second order partial derivatives of $f(x,y)$ at $(1,-1)$**

Sol: Here, $f_x(x,y) = 3x^2 - 2y^2 f_y(x,y) = 3y^2 - 4xy$

$$f_{xx}(x,y) = 6x$$
$$f_{yy}(x,y) = 6y - 4x$$
$$f_{xy}(x,y) = -4y,$$
$$f_{yx}(x,y) = -4y$$

Then,

$$f_{xx}(1,-1) = 6$$
$$f_{yy}(1,-1) = -10$$
$$f_{xy}(1,-1) = 4$$
$$f_{yx}(1,-1) = 4$$

3. **If $u = log(tanx + tany + tanz)$, then show that**

$$sin2x \frac{\partial u}{\partial x} + sin2y \frac{\partial u}{\partial y} + sin2z \frac{\partial u}{\partial z} = 2$$

Sol: u = log (tan x + tan y + tan z)

Differentiating u partially w.r.t. x, y and z,

$$\frac{\partial u}{\partial x} = \frac{1}{tan\, x + tan\, y + tan\, z} sec^2 x$$

$$\frac{\partial u}{\partial y} = \frac{1}{tan\, x + tan\, y + tan\, z} .sec^2 y$$

$$\frac{\partial u}{\partial z} = \frac{1}{tan\, x + tan\, y + tan\, z} .sec^2 z$$

Hence,

$$sin\, 2x \frac{\partial u}{\partial x} + sin\, 2y \frac{\partial u}{\partial y} + sin\, 2z \frac{\partial u}{\partial z}$$

$$= \frac{2sinxcosxsec^2 x + 2sinycosysec^2 y + 2sinzcoszsec^2 z}{tan\, x + tan\, y + tan\, z}$$

$$= \frac{2(\tan x + \tan y + \tan z \quad)}{\tan x + \tan y + \tan z}$$

$$= 2$$

4. If $u(x, y, z) = e^{3xyz}$ **show that** $\frac{\partial^3 u}{\partial x \partial y \partial z} = (3 + 27xyz + 27x^2y^2z^2)e^{3xyz}$

Sol: $u(x, y, z) = e^{3xyz}$

Differentiating u w.r.t z,

$$\frac{\partial u}{\partial z} = 3xye^{3xyz}$$

Differentiating $\frac{\partial u}{\partial z}$ w.r.t y,

$$\frac{\partial^2 u}{\partial y \partial z} = 3x \frac{\partial}{\partial y}(ye^{3xyz})$$

$$= 3x(e^{3xyz}.1 + ye^{3xyz}.3xz)$$

$$= e^{3xyz}(3x + 9x^2yz)$$

Differentiating $\frac{\partial^2 u}{\partial y \partial z}$ w.r.t x,

$$\frac{\partial^3 u}{\partial x \partial y \partial z} = \frac{\partial}{\partial x}[[\ e^{3xyz}(3x + 9x^2yz)]]$$

$$= e^{3xyz}(3 + 18xyz) + (3x + 9x^2yz).e^{3xyz}.3yz$$

$$= e^{3xyz}(3 + 18xyz + 9xyz + 27x^2y^2z^2)$$

$$= e^{3xyz}(3 + 27xyz + 27x^2y^2z^2)$$

Exercise:

1. Find the values of $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ at point (4,-5) if $f(x, y) = x^2 + 3xy + y - 1$.
2. If $z = x + y^x, prove\ that\ \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$
3. Find f_{yxyz} if $f(x,y,z) = 1 - 2xy^2z + x^2y$.
4. Find the $f_x(1,2)$ for the function $f(x,y) = x^2y + xy^2$ **[Winter 2022]**
5. If $f(x,y) = x^3 + y^3 - 2xy^2$, find all second order partial derivatives of $f(x,y)$ at $(1,-1)$. **[Summer 2023]**
6. . Find the value of $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ at point $(1,2)$ if $f(x,y) = x^2 + 2xy + 3y^2 - 1$. **[Winter 2023]**

Homogeneous function:

A function $u = f(x, y)$ is said to be homogeneous function of degree '*n*' in *x* and *y* if degree of each term of $u = f(x, y)$ is *n*.

Thus, for a homogeneous function *f* of degree *n;* $f(tx, ty) = t^n f(x, y)$

Note:

Also, if the function $u = f(x, y)$ is a homogeneous function of degree '*n*' in *x* and *y* then it can be written as $u = x^n \emptyset\left(\frac{y}{x}\right)$ $or\ u = y^n \emptyset\left(\frac{x}{y}\right)$

Euler's theorem for the function of two independent variables:

Statement: If *u* is a homogeneous function of degree *n* in *x* and *y*, then

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$$

Proof: Let $u = f(x, y)$ be a homogeneous function of degree '*n*' in *x* and *y,* then it can be written as

$$u = x^n \emptyset\left(\frac{y}{x}\right) \quad \text{\_\_\_\_\_\_\_\_\_\_\_\_(1)}$$

Differentiate (1) partially w.r.t *x*, we get

$$\frac{\partial u}{\partial x} = x^n \frac{\partial}{\partial x}\left[\emptyset\left(\frac{y}{x}\right)\right] + \emptyset\left(\frac{y}{x}\right)\frac{\partial}{\partial x}(x^n)$$

$$= x^n \emptyset'\left(\frac{y}{x}\right)\frac{\partial}{\partial x}\left(\frac{y}{x}\right) + \emptyset\left(\frac{y}{x}\right)(nx^{n-1})$$

$$= x^n \emptyset'\left(\frac{y}{x}\right)\left(-\frac{y}{x^2}\right) + \emptyset\left(\frac{y}{x}\right)(nx^{n-1})$$

$$= -yx^{n-2}\emptyset'\left(\frac{y}{x}\right) + nx^{n-1}\emptyset\left(\frac{y}{x}\right)$$

$$\Rightarrow x\frac{\partial u}{\partial x} = -yx^{n-1}\emptyset'\left(\frac{y}{x}\right) + nx^n\emptyset\left(\frac{y}{x}\right) \quad \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_(2)}$$

Differentiate (1) partially w.r.t *y*, we get

$$\frac{\partial u}{\partial y} = x^n \frac{\partial}{\partial y}\left[\emptyset\left(\frac{y}{x}\right)\right]$$

$$= x^n \emptyset'\left(\frac{y}{x}\right)\frac{\partial}{\partial y}\left(\frac{y}{x}\right)$$

$$= x^n \emptyset'\left(\frac{y}{x}\right)\left(\frac{1}{x}\right)$$

$$= x^{n-1}\emptyset'\left(\frac{y}{x}\right)$$

$$\Rightarrow y\frac{\partial u}{\partial y} = yx^n\emptyset'\left(\frac{y}{x}\right) \quad \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_(3)}$$

Adding (2) and (3) we get,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = -yx^{n-1}\emptyset'\left(\frac{y}{x}\right) + nx^n\emptyset\left(\frac{y}{x}\right) + yx^n\emptyset'\left(\frac{y}{x}\right) = nx^n\emptyset\left(\frac{y}{x}\right)$$

$$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$$

Euler's theorem for the function of three independent variables:

Statement: If *u* is a homogeneous function of degree *n* in *x, y* and *z*, then

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = nu$$

[Similar to **Euler's theorem for the function of two independent variables]**

Cor.1 If u is a homogeneous function of degree n in x and y, then

$$x^2\frac{\partial^2 u}{\partial x^2}+\ 2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=n(n-1)u.$$

Proof: Let $u=f(x,y)$ be a homogeneous function of degree 'n' in x and y, then by Euler's theorem for homogeneous functions

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=nu \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_(1)}$$

Differentiate (1) partially w.r.t x, we get

$$\Rightarrow x\left[\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)\right]+\frac{\partial u}{\partial x}\left[\frac{\partial}{\partial x}(x)\right]+y\left[\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)\right]=n\frac{\partial}{\partial x}(u)$$

$$\Rightarrow x\frac{\partial^2 u}{\partial x^2}+\frac{\partial u}{\partial x}+y\left[\frac{\partial^2 u}{\partial x\partial y}\right]=n\frac{\partial u}{\partial x}$$

On multiplying both sides by x, we get

$$\Rightarrow x^2\frac{\partial^2 u}{\partial x^2}+x\frac{\partial u}{\partial x}+xy\left[\frac{\partial^2 u}{\partial x\partial y}\right]=nx\frac{\partial u}{\partial x}\text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_(2)}$$

Differentiate (1) partially *w.r.t. y*, we get

$$\Rightarrow x\left[\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)\right]+\frac{\partial u}{\partial y}\left[\frac{\partial}{\partial y}(y)\right]+y\left[\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)\right]=n\frac{\partial}{\partial y}(u)$$

$$\Rightarrow x\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial u}{\partial y}+y\frac{\partial^2 u}{\partial y^2}=n\frac{\partial u}{\partial y}\left[since,\frac{\partial^2 u}{\partial x\partial y}=\frac{\partial^2 u}{\partial y\partial x}\right]$$

On multiplying both sides by y, we get

$$\Rightarrow xy\left[\frac{\partial^2 u}{\partial x\partial y}\right]+y\frac{\partial u}{\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=ny\frac{\partial u}{\partial y}\text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_(3)}$$

Adding (2) and (3) we get,

$$x^2\frac{\partial^2 u}{\partial x^2}+x\frac{\partial u}{\partial x}+xy\left[\frac{\partial^2 u}{\partial x\partial y}\right]+xy\left[\frac{\partial^2 u}{\partial x\partial y}\right]+y\frac{\partial u}{\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=nx\frac{\partial u}{\partial x}+ny\frac{\partial u}{\partial y}$$

$$\Rightarrow x^2\frac{\partial^2 u}{\partial x^2}+\ 2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}+x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=n\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)$$

$$\Rightarrow x^2\frac{\partial^2 u}{\partial x^2}+\ 2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}+nu=n^2u$$

$$\Rightarrow x^2\frac{\partial^2 u}{\partial x^2}+\ 2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=n^2u-nu$$

$$\Rightarrow x^2\frac{\partial^2 u}{\partial x^2}+\ 2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=n(n-1)u.$$

Cor.2 If $\emptyset(u)=f(x,y)$ is a homogeneous function of degree n in x and y, then

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=n\frac{f(u)}{f'(u)}\text{[Without proof]}$$

Cor.3 If $\emptyset(u) = f(x, y)$ is a homogeneous function of degree n in x and $y, then$

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = g(u)[g'(u) - 1]$$

where, $g(u) = n\frac{f(u)}{f'(u)}$**.** [Without proof]

Note: Corollary 2 and 3 are also known as Modified Euler's theorem of first and second order respectively.

Solved Examples:

1. If $u = \frac{1}{x^2} + \frac{1}{xy} + \frac{logx - logy}{x^2+y^2}$, prove that

$$(i)\ x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = -2u (ii) x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 6u$$

Sol: $u = f(x, y) = \frac{1}{x^2} + \frac{1}{xy} + \frac{log\frac{x}{y}}{x^2+y^2}$

Replacing x by tx and y by ty,

$$f(tx, ty) = \frac{1}{t^2x^2} + \frac{1}{txty} + \frac{log\frac{tx}{ty}}{t^2x^2+t^2y^2}$$

$$= \frac{1}{t^2}\left[\frac{1}{x^2} + \frac{1}{xy} + \frac{log\frac{x}{y}}{x^2+y^2}\right]$$

$$= t^{-2}f(x, y)$$

Hence, u is a homogeneous function of degree -2

By Euler's Theorem,

$$(i)\ x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$$

$$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = -2u$$

$$(ii)\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = n(n-1)u$$

$$i.e\ x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = -2(-2-1)u = 6u$$

2. If $u = tan^{-1}(\frac{x^2+y^2}{x+y})$**, prove that ,** **[Winter 2023][Summer 2023]**

$$(i) x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{sin2u}{2}$$

$$(ii) x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = -2sin^3u cosu.$$

Sol: $u = tan^{-1}(\frac{x^2+y^2}{x+y})$

Replacing x by tx and y by ty,

$u = tan^{-1}[t(\frac{x^2+y^2}{x+y})]$

u is a non-homogeneous function. But $tan\, u = (\frac{x^2+y^2}{x+y})$is a homogeneous function of degree 1.

By Modified Euler's Theorem,

Let $f(u) = tanu$

$$(i) x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\frac{f(u)}{f'(u)}$$

$$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 1.\frac{tanu}{sec^2 u} = sinu\, cosu = \frac{sin2u}{2}$$

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = g(u)[g'(u) - 1]$$

where, $g(u) = n\frac{f(u)}{f'(u)} = 1.\frac{tanu}{sec^2 u} = \frac{sin2u}{2}$

$$\Rightarrow g'(u) = cos2u$$

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \frac{sin2u}{2}(cos2u - 1)$$

$$= sinucosu(cos2u - 1)$$

$$= sinucosu(-2sin^2 u)$$

$$= -2sin^3 ucosu$$

<u>Exercise:</u>

1. If $u = y^2 e^{\frac{y}{x}} + x^2\left(\frac{x}{y}\right)$, show that

$$(i) x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2u$$

$$(ii) x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 2u$$

2. If $u = sin^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$, then prove that

$$(i) x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}tanu$$

$$(ii) x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \frac{1}{4}(tan^3 u - tanu)$$

3. If $u = \tan^{-1}\left(\frac{x^2+y^2}{x-y}\right)$, Show that: $x\frac{\partial u}{\partial x} + x\frac{\partial u}{\partial y} = \frac{1}{2}\sin 2u$ **[Summer 2022]**

4. If $u = \tan^{-1}\left(x^2+2y^2\right)$, Show that**:** **[Winter 2022]**

   $(i)\ x\frac{\partial u}{\partial x} + x\frac{\partial u}{\partial y} = \sin 2u$

$$(ii)\quad x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=2\sin u\cos 3u$$

5. If $u=y^2e^{\frac{y}{x}}+x^2\left(\frac{x}{y}\right)$, show that, **[Summer 2023]**

$$(i)\quad x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=2u$$

$$(ii)\quad x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=2u$$

6. If $u=\tan^{-1}\left(\frac{x^2+y^2}{x+y}\right)$ show that, **[Winter 2021][Winter 2023][Summer 2023]**

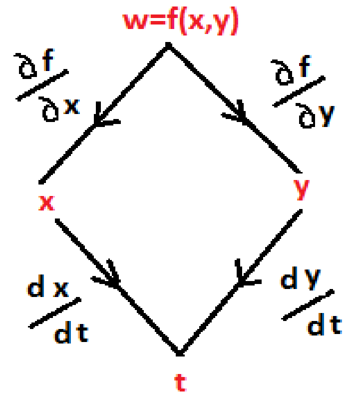
$$(i)\quad x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\sin u\cos u$$

$$(ii)\quad x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=-2\sin^3 u\cos u$$

Chain Rule for Function of Two Independent Variable

If $w=f(x,y)$ has continuous partial derivative f_x, f_y and if $x=x(t),\ \ y=y(t)$ are differentiable function of t, then the composite $w=f(x(t),y(t))$ is a differentiable function of t then,

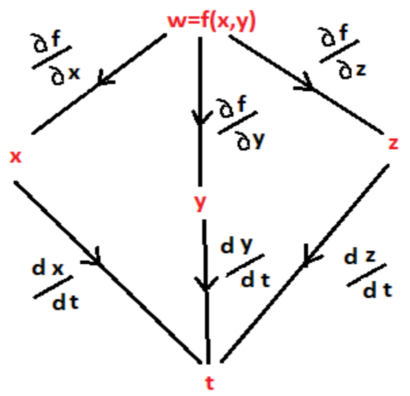
$$\frac{df}{dt}=\frac{\partial f}{\partial x}\cdot\frac{dx}{dt}+\frac{\partial f}{\partial y}\cdot\frac{dy}{dt}$$



Chain Rule for Function of Three Independent Variables

If $w=f(x,y,z)$ is differentiable and $x, y\ and\ z$ are differentiable function of t then w is a differentiable function of t then,

$$\frac{df}{dt}=\frac{\partial f}{\partial x}\cdot\frac{dx}{dt}+\frac{\partial f}{\partial y}\cdot\frac{dy}{dt}+\frac{\partial f}{\partial z}\cdot\frac{dz}{dt}$$



Chain Rule for Function of Two Independent Variable and Three Intermediate Variable

Suppose that $w = f(x,y,z), x = g(r,s), y = h(r,s)$ *and*

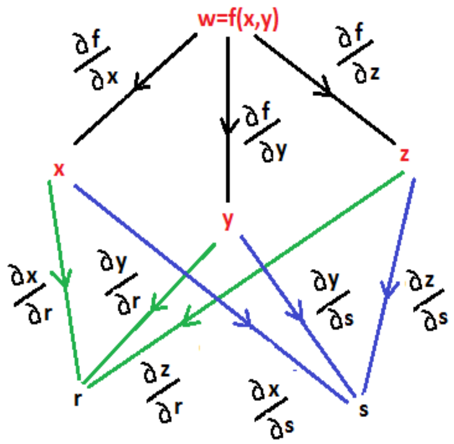
$z = k(r,s)$.If all four functions are differentiable then w

has partial derivative with respect to $r and s$ givenby

the formulas

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial r}$$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial s}$$



Solved Examples:

1. Let $z = x^2y^3$, where $x = t^2$and $y = t$, then verify chain rule by expressing z in terms of t.

Sol: Here, the chain rule is $\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$

$$= (2xy^3)(2t) + (3x^2y^2)(1)$$

$$= (2t^2t^3)(2t) + (3t^4t^2)$$

$$= 4t^6 + 3t^6$$

$$= 7t^6 - - - - - - - -(1)$$

Also

$$z = x^2y^3$$

$$z = (t^4)(t^3) = t^7$$

$$\frac{dz}{dt} = 7t^6 - - - - - - - -(2)$$

Hence, from (1) and (2), the chain rule is verified.

2. If $u = xy^2 + yz^3, x = logt, y = e^t, z = t^2$find $\frac{du}{dt}$ at $t = 1$.

Sol: $u = xy^2 + yz^3, x = logt, y = e^t, z = t^2$

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt}$$

$$= (y^2)\frac{1}{t} + (2xy + z^3)e^t + (3yz^2)2t$$

Substituting x ,y and z,

$$\frac{du}{dt} = 2(e^{2t})\frac{1}{t} + \left(2(logt) \quad e^t + t^6\right)e^t + 3e^t t^4 . 2t$$

Substituting $t = 1$,

$$\frac{du}{dt} = 2e^2 + e^1 + 6e^1$$

$$= 2e^2 + 7e^1$$

3.If $u = f(x - y, y - z, z - x)$, **show that** $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$ **[Winter 2021] [Summer 2023]**

Sol: Let $x - y = l, y - z = m, z - x = n$

$$\frac{\partial l}{\partial x} = 1, \quad \frac{\partial m}{\partial x} = 0, \quad \frac{\partial n}{\partial x} = -1,$$

$$\frac{\partial l}{\partial y} = -1, \quad \frac{\partial m}{\partial y} = 1, \quad \frac{\partial n}{\partial y} = 0,$$

$$\frac{\partial l}{\partial z} = 0, \quad \frac{\partial m}{\partial z} = -1, \quad \frac{\partial n}{\partial z} = 1$$

$$u = f(x - y, y - z, z - x) = f(l, m, n)$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial l}\frac{\partial l}{\partial x} + \frac{\partial u}{\partial m}\frac{\partial m}{\partial x} + \frac{\partial u}{\partial n}\frac{\partial n}{\partial x}$$

$$= \frac{\partial u}{\partial l}.1 + \frac{\partial u}{\partial m}.0 + \frac{\partial u}{\partial n}.-1$$

$$= \frac{\partial u}{\partial l} - \frac{\partial u}{\partial n} \quad --------(1)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial l}\frac{\partial l}{\partial y} + \frac{\partial u}{\partial m}\frac{\partial m}{\partial y} + \frac{\partial u}{\partial n}\frac{\partial n}{\partial y}$$

$$= \frac{\partial u}{\partial l}.-1 + \frac{\partial u}{\partial m}.1 + \frac{\partial u}{\partial n}.0$$

$$= -\frac{\partial u}{\partial l} + \frac{\partial u}{\partial m} --------(2)$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial l}\frac{\partial l}{\partial z} + \frac{\partial u}{\partial m}\frac{\partial m}{\partial z} + \frac{\partial u}{\partial n}\frac{\partial n}{\partial z}$$

$$= \frac{\partial u}{\partial l}.0 + \frac{\partial u}{\partial m}.-1 + \frac{\partial u}{\partial n}.1$$

$$= -\frac{\partial u}{\partial m} + \frac{\partial u}{\partial n} - - - - - - - - (3)$$

Adding Eqs. (1),(2) and (3),

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$$

4. If $w = x^2 + y^2$, Where $x = r - s$ and $y = r + s$ then find $\frac{\partial w}{\partial r}$ $and \frac{\partial w}{\partial s}$ in terms of r and s.

[Summer 2022]

Solution:

Here, the chain rule is $\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x} * \frac{\partial x}{\partial r} + \frac{\partial w}{\partial y} * \frac{\partial y}{\partial r}$

$$= (2x)(1) + (2y)(1)$$

$$= (2(r - s)) + 2(r + s) = 4r$$

Now for $\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x} * \frac{\partial x}{\partial s} + \frac{\partial w}{\partial y} * \frac{\partial y}{\partial s}$

$$= (2x)(-1) + (2y)(1)$$

$$= (-2(r - s)) + 2(r + s) = 4s$$

<u>Exercise:</u>

1. If $u = f(x^2 + 2yz, y^2 + 2xz)$, then find value of $(y^2 - xz)\frac{\partial u}{\partial x} + (x^2 - yz)\frac{\partial u}{\partial y} + (z^2 - xy)\frac{\partial u}{\partial z}$.

2. If $u = f\left(\frac{x}{y}, \frac{y}{z}, \frac{z}{x}\right)$, prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$

3. If $u = f(r)$ where $r^2 = x^2 + y^2$, prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r) + \frac{1}{r} f'(r)$.

4. If $u = x^2 y + y^2 z + z^2 x$ then find out $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}$. **[Summer 2022]**

<u>Implicit Differentiation:</u>

Suppose that $f(x, y)$ is differentiable and that the equation $f(x, y) = 0$ defines y as a differentiable function of x. Then at any point where $f_y \neq 0$,

$$\frac{dy}{dx} = -\frac{f_x}{f_y}$$

Solved Examples:

1. **If** $ysinx = xcosy$, **find** $\frac{dy}{dx}$.

Sol: Let $f(x,y) = ysinx - xcosy$

$$\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$$

$$= -\frac{ycosx - cosy}{sinx + xsiny}$$

$$= \frac{cosy - ycosx}{sinx + xsiny}$$

2. **If** $(cosx)^y = (siny)^x$, **find** $\frac{dy}{dx}$.

Sol: $(cosx)^y = (siny)^x$

Taking log on both the sides,

$$ylogcosx = xlogsiny$$

Let $f(x,y) = ylogcosx - xlogsiny$

$$\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$$

$$= -\frac{\frac{y}{cosx}(-sinx) - logsiny}{logcosx - \frac{x}{siny}(cosy)}$$

$$= \frac{ytanx + logsiny}{logcosx - xcoty}$$

Exercise:

1. If $x^3 + y^3 + 3xy = 1$ then, find $\frac{dy}{dx}$.

2. If $x^y + y^x = c\ then,$ find $\frac{dy}{dx}$.

3.If $f(x,y) = c$ then, find $\frac{dy}{dx}$ **[Summer 2023]**

Jacobian:

If u and v are continuous and differentiable functions of two independent variables x and y, then the Jacobian of u,v with respect to x,y and is denoted by

$$J=\frac{\partial(u,v)}{\partial(x,y)}=\begin{vmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{vmatrix}$$

Similarly, if u,v and w are continuous and differentiable functions of three independent variables x,yand z, then the Jacobian of u,v,w with respect to x,y,z and is denoted by

$$J=\frac{\partial(u,v,w)}{\partial(x,y,z)}=\begin{vmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z}\\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z}\end{vmatrix}$$

Properties of Jacobian:

1. If u and v are functions of x and y, then $J.J' = 1$, where $J = \frac{\partial(u,v)}{\partial(x,y)}$ and $J' = \frac{\partial(x,y)}{\partial(u,v)}$ **[Summer 2023]**

2. If u,v are functions of r,s and r,s are in turn functions of x,y then $\frac{\partial(u,v)}{\partial(x,y)} = \frac{\partial(u,v)}{\partial(r,s)} \cdot \frac{\partial(r,s)}{\partial(x,y)}$

Solved Examples:

1. Find the Jacobian $\frac{\partial(u,v)}{\partial(x,y)}$ for $u = x^2 - y^2, v = 2xy$.

Sol: $u = x^2 - y^2,\ v = 2xy$

$$\frac{\partial u}{\partial x} = 2x, \quad \frac{\partial v}{\partial x} = 2y$$

$$\frac{\partial u}{\partial y} = -2y, \quad \frac{\partial v}{\partial y} = 2x$$

$$J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{vmatrix}$$

$$= \begin{vmatrix}2x & -2y\\ 2y & 2x\end{vmatrix}$$

$$= 4(x^2 + y^2)$$

2. Find the Jacobian $\frac{\partial(u,v)}{\partial(x,y)}$ **for** $u = x - y, v = x + y$**. Also, verify that** $J.J' = 1$.

Sol: $u = x - y,\ v = x + y$

$$\frac{\partial u}{\partial x} = 1, \qquad \frac{\partial v}{\partial x} = 1$$

$$\frac{\partial u}{\partial y} = -1, \qquad \frac{\partial v}{\partial y} = 1$$

$$J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix}$$

$$= 2$$

Now, $x = \frac{u+v}{2}, y = \frac{v-u}{2}$

$$\frac{\partial x}{\partial u} = \frac{1}{2}, \qquad \frac{\partial y}{\partial u} = -\frac{1}{2}$$

$$\frac{\partial x}{\partial v} = \frac{1}{2}, \qquad \frac{\partial y}{\partial v} = \frac{1}{2}$$

$$J' = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{vmatrix}$$

$$= \frac{1}{2}$$

$$JJ' = 1.$$

Exercise:

1. Find the Jacobian $\frac{\partial(u,v)}{\partial(x,y)}$ for $u = x\sin y, v = y\sin x$.

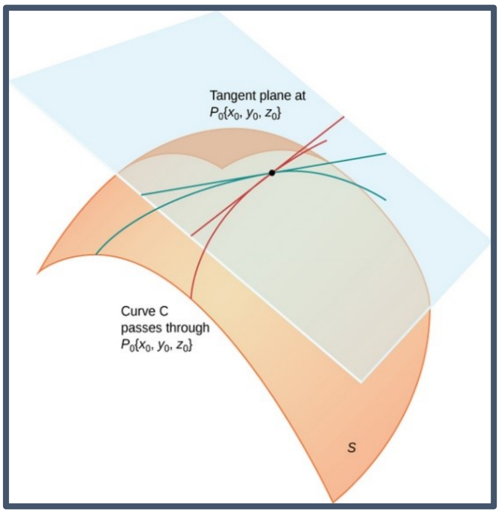
2. If $u = 2xy,\ v = x^2 - y^2$ and $x = r\cos\theta,\ y = r\sin\theta$ then, evaluate $\frac{\partial(u,v)}{\partial(r,\theta)}$ **[Winter 2021]**

3. If $x = u + 3v, y = v - u$ then evaluate the value of $\frac{\partial(u,v)}{\partial(x,y)}$. **[Winter 2022]**

APPLICATIONS OF PARTIAL DERIVATIVES

<u>Tangent Plane and Normal Line</u>

Intuitively, it seems clear that, in a plane, only one line can be tangent to a curve at a point. However, in three-dimensional space, many lines can be tangent to a given point. If these lines lie in the same plane, they determine the tangent plane at that point. A more intuitive way to think of a tangent plane is to assume the surface is smooth at that point (no corners). Then, a tangent line to the surface at that point in any direction does not have any abrupt changes in slope because the direction changes smoothly. Therefore, in a small-enough neighborhood around the point, a tangent plane touches the surface at that point only.



For a tangent plane to a surface to exist at a point on that surface, it is sufficient for the function that defines the surface to be differentiable at that point. We define the term tangent plane here and then explore the idea intuitively.

Definition: The **tangent plane** at the point $P_0(x_0, y_0, z_0)$ on surface $f(x, y, z) = 0$ of the differentiable function f is the plane through P_0 normal to $\nabla f|_{P_0}$. It is given by

$$f_x(P_0)(x - x_0) + f_y(P_0)(y - y_0) + f_z(P_0)(z - z_0) = 0$$

The **normal line** of the surface at P_0 is the line through P_0 parallel to $\nabla f|_{P_0}$. It is given by

$$\frac{x - x_0}{f_x(P_0)} = \frac{y - y_0}{f_y(P_0)} = \frac{z - z_0}{f_z(P_0)}$$

Solved Examples:

1. Find the equation of the tangent plane and normal line to the surface $x^2 + y^2 + z^2 = 3$ **at the point (1,1,1).** **[Winter 2021][Summer 2023]**

Sol: Here, $f(x, y, z) = x^2 + y^2 + z^2 - 3$

$f_x(x, y, z) = 2x$, $f_x(1,1,1) = 2$

$f_y(x, y, z) = 2y$, $f_y(1,1,1) = 2$

$f_z(x, y, z) = 2z$, $f_z(1,1,1) = 2$

Hence, the equation of the tangent plane at (1,1,1) is

$$(x - 1)2 + (y - 1)2 + (z - 1)2 = 0$$

$$x + y + z = 3$$

The equation of normal line is $\frac{x-1}{2} = \frac{y-1}{2} = \frac{z-1}{2}$.

2. Find the equation of the tangent plane and normal line to the surface $z + 8 = xe^y cosz$ **at the point (8,0,0).**

Sol: Let $f(x, y, z) = xe^y cosz - z - 8$

$f_x(x, y, z) = e^y cosz$, $f_x(8,0,0) = 1$

$f_y(x, y, z) = xe^y cosz$, $f_y(8,0,0) = 8$

$f_z(x, y, z) = sinz - 1$, $f_z(8,0,0) = -1$

Hence, the equation of the tangent plane at (8,0,0) is

$$1(x - 8) + 8(y - 0) - 1(z - 0) = 0$$

$$x - 8 + 8y - z = 0$$

$$x + 8y - z - 8 = 0$$

The equation of normal line is $\frac{x-8}{1} = \frac{y-0}{8} = \frac{z-0}{-1}$

Exercise:

1. Find the equations of tangent plane and normal line to the surface $z = x^2 + 3y^2 - 4$

at (1,1,0).

2. Find the equations of tangent plane and normal line to surface $2x^2 + y^2 + 2z = 3$ at $(2,1,-3)$.

3. Find the equation of tangent plane and normal line to the paraboloid $z = x^2 + y^2$ at (1,1,2).

[Winter 2022]

Local Maximum and Local Minimum

Let $f(x, y)$ be defined on a region R containing the point (a, b).Then

1. $f(a, b)$ is a local maximum value of f if $f(a, b) \geq f(x, y)$ for all domain points (x, y) in an open disk centered at (a, b).
2. $f(a, b)$ is a local minimum value of f if $f(a, b) \leq f(x, y)$ for all domain points (x, y) in an open disk centered at (a, b).

Second Derivative Test for Local Extreme Values

Suppose that $f(x, y)$ and its first and second partial derivative are continuous throughout

a disk centered at (a, b) and that $f_x(a, b) = f_y(a, b) = 0$. Then, for

$$r = \frac{\partial^2 f}{\partial x^2},\ s = \frac{\partial^2 f}{\partial x \partial y},\ t = \frac{\partial^2 f}{\partial y^2}$$

(i) f has a local maximum at (a, b) if $r < 0$ and $rt - s^2 > 0$ at (a, b)
(ii) f has a local minimum at (a, b) if $r > 0$ and $rt - s^2 > 0$ at (a, b)
(iii) f has a saddle point at (a, b) if $rt - s^2 < 0$ at (a, b)
(iv) The test has no conclusion at $(a, b) if$ $rt - s^2 = 0$ at (a, b).

Examples:

1. Find the extreme value of $x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$ **[Summer 2022][Winter 2023]**

Sol: Let $f(x, y) = x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$

$$\frac{\partial f}{\partial x} = 3x^2 + 3y^2 - 30x + 72$$

$$\frac{\partial f}{\partial y} = 6xy - 30y$$

For extreme values, $\frac{\partial f}{\partial x} = 0$and $\frac{\partial f}{\partial y} = 0$

$3x^2 + 3y^2 - 30x + 72 = 0$ and $6xy - 30y = 0$

$x^2 + y^2 - 10x + 24 = 0$and$6y(x - 5) = 0$

$x^2 + y^2 - 10x + 24 = 0$and$y = 0, x = 5$

When $y = 0$, we have $x^2 - 10x + 24 = 0$

$$x = 4,6$$

and when $x = 5$, we have $25 + y^2 - 50 + 24 = 0$

$$y = \pm 1$$

Therefore, the stationary points are $(4,0), (6,0), (5,1), (5,-1)$

$$r = \frac{\partial^2 f}{\partial x^2} = 6x - 30$$

$$s = \frac{\partial^2 f}{\partial x \partial y} = 6y$$

$$t = \frac{\partial^2 f}{\partial y^2} = 6x - 30$$

(x, y)	r	s	t	$rt - s^2$	**Conclusion**	$f(x, y)$
$(4,0)$	$-6 < 0$	0	-6	$36 > 0$	Maximum	112
$(6,0)$	$6 > 0$	0	6	$36 > 0$	Minimum	108
$(5,1)$	0	6	0	$-36 < 0$	Saddle Point	--
$(5,-1)$	0	-6	0	$-36 < 0$	Saddle Point	--

2.Find the points on the surface $z^2 = x^2 + y^2$ that are closed to P(1,1,0).

Sol: Let $A(x, y, z)$ be any point on the surface, then by distance formula, the distance d Between A and P is given by $d = \sqrt{(x-1)^2 + (y-1)^2 + z^2}$

$$d^2 = (x-1)^2 + (y-1)^2 + z^2$$
$$= 2x^2 + 2y^2 - 2x - 2y + 2 = f\ (say)$$

$f_x(x, y) = 4x - 2,$ $\quad$ $f_y(x, y) = 4y - 2$

For extreme values, $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$

$$4x - 2 = 0, \quad 4y - 2 = 0$$
$$x = \frac{1}{2}, \quad y = \frac{1}{2}$$

So, $(\frac{1}{2}, \frac{1}{2})$ is a stationary point.

$$r = \frac{\partial^2 f}{\partial x^2} = 4$$

$$s = \frac{\partial^2 f}{\partial x \partial y} = 0$$

$$t = \frac{\partial^2 f}{\partial y^2} = 4$$

$$rt - s^2 = (4)(4) - 0 = 16 > 0$$

Hence, function is minimum at $(\frac{1}{2}, \frac{1}{2})$

Minimum value of given surface $z^2 = x^2 + y^2$at $(\frac{1}{2}, \frac{1}{2})$

$$= \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

So, $z = \pm\frac{1}{2}$.

Exercise:

1. Discuss the maxima and minima of the function$(x, y) = x^2 + y^2 + 6x + 12$.

Ans. $\boldsymbol{f_{min} = 3\ at\ (-3, 0)}$

2. Find the extreme values of the function $f(x, y) = x^3 + 3xy^2 - 3x^2 - 3y^2 + 7$.

   [Winter 2022] [Summer 2023]

Ans. $\boldsymbol{f_{max} = 7}$ **at (0,0)** $\boldsymbol{f_{min} = 3}$ **at (2,0)**

MAXIMA AND MINIMA WITH CONSTRAINED VARIABLES

The Method of Lagrange Multipliers:

The method of Lagrange multipliers allows us to maximize or minimize function with a constraint.

Given a function $f(x, y, z)$ subject to the constraint $\emptyset(x, y, z) = 0$___________(1),

Solve by following

Steps:

1. Construct an equation $f(x, y, z) + \lambda\emptyset(x, y, z) = 0$_____________(2)

   where, λ is a variable called Lagrange multiplier.

2. Differentiate Eq. (2) partially w.r.t x, y, z to obtain

$$\frac{\partial f}{\partial x} + \lambda\frac{\partial \emptyset}{\partial x} = 0 \_\_\_\_\_\_\_\_\_\_\_\_\_ (3)$$

$$\frac{\partial f}{\partial y} + \lambda\frac{\partial \emptyset}{\partial y} = 0 \_\_\_\_\_\_\_\_\_\_\_\_\_ (4)$$

$$\frac{\partial f}{\partial z} + \lambda\frac{\partial \emptyset}{\partial z} = 0 \_\_\_\_\_\_\_\_\_\_\_\_\_ (5)$$

3. Solve and eliminateλ from Eqs. $(1), (3), (4), (5)$ to obtain the stationary points (x, y, z).
4. Substitute the stationary points (x, y, z) into f to see where, f attains its maximum and minimum values.

Note: For function of two independent variables, the condition is similar, but without the variable z.

Solved Examples:

1. Find the greatest and smallest values that the function $f(x, y) = xy$ takes on the ellipse $\frac{x^2}{8} + \frac{y^2}{2} = 1$.

Sol: Let $f(x, y, z) = xy$

and $\emptyset(x, y, z) = \frac{x^2}{8} + \frac{y^2}{2} - 1 = 0$__________(1)

Let the equation be $xy + \lambda\left(\frac{x^2}{8} + \frac{y^2}{2} - 1\right) = 0$__________(2)

Differentiating Eq. (2) partially w.r.t x,

$$y + \frac{\lambda x}{4} = 0 \text{ \_\_\_\_\_\_\_\_\_\_(3)}$$

Differentiating Eq. (2) partially w.r.t y,

$$x + \lambda y = 0 \text{ \_\_\_\_\_\_\_\_\_\_(4)}$$

From Eqs .(3),(4),

$$\frac{-4y}{x} = \frac{-x}{y}$$

$$\Rightarrow 4y^2 = x^2$$

Substituting x^2 in Eq.(1),

$$\frac{4y^2}{8} + \frac{y^2}{2} = 1$$

$$\Rightarrow y^2 = 1$$

$$\Rightarrow y = \pm 1$$

$$\Rightarrow x = \pm 2$$

Therefore, the function $f(x, y) = xy$ takes extreme values on the ellipse at four points $(2,1), (-2,1), (-2,-1), (2,-1)$

The maximum value is $xy = 2$ and minimum value is $xy = -2$.

2. Find the maximum value of $x^2y^3z^4$, subject to the condition $x + y + z = 5$.

Sol: Let $f(x, y, z) = x^2y^3z^4$

and $\emptyset(x, y, z) = x + y + z - 5 = 0$__________(1)

Let the equation be $x^2y^3z^4 + \lambda(x + y + z - 5) = 0$__________(2)

Differentiating Eq.(2) partially w.r.t x,

$$2xy^3z^4 + \lambda = 0$$

$\lambda = -2xy^3z^4$__________(3)

Differentiating Eq.(2) partially w.r.t y,

$$3x^2y^2z^4 + \lambda = 0$$

$\lambda = -3x^2y^2z^4$__________(4)

Differentiating Eq.(3) partially w.r.t z,

$$4x^2y^3z^3 + \lambda = 0$$

$\lambda = -4x^2y^3z^3$__________(5)

From Eqs.(3), (4),(5),

$$-2xy^3z^4 = -3x^2y^2z^4 = -4x^2y^3z^3$$

$$\Rightarrow 2yz = 3xz = 4xy$$

$\Rightarrow y = \frac{3}{2}x$ and $z = 2x$

Substituting y and z in Eq.(1),

$$x + \frac{3}{2}x + 2x = 5$$

$$\Rightarrow 9x = 10$$

$$\Rightarrow x = \frac{10}{9}$$

$$y = \frac{3}{2}\left(\frac{10}{9}\right) = \frac{5}{3}$$

$$z = 2\left(\frac{10}{9}\right) = \frac{20}{9}$$

Maximum value of $x^2y^3z^4 = \left(\frac{10}{9}\right)^2\left(\frac{5}{3}\right)^3\left(\frac{20}{9}\right)^4 = \frac{(2^{10})(5^9)}{3^{15}}$.

3.Using Lagrange's multiplier method find shortest and longest distance from the point (1,2,-1) on the sphere $x^2 + y^2 + z^2 = 24$

Sol: Let $f(x, y, z) = (x-1)^2 + (y-2)^2 + (z+1)^2$

and $\emptyset(x, y, z) = x^2 + y^2 + z^2 - 24 = 0$__________(1)

Let the equation be $(x-1)^2 + (y-2)^2 + (z+1)^2 + \lambda(x^2 + y^2 + z^2 - 24) = 0$ ___(2)

Differentiating Eq.(2) partially w.r.t x,

$$2(x-1) + 2\lambda x = 0$$

$$\lambda = -1 + \frac{1}{x} \text{\_\_\_\_\_\_\_\_\_\_}(3)$$

Differentiating Eq.(2) partially w.r.t y,

$$2(y-2) + 2\lambda y = 0$$

$$\lambda = -1 + \frac{2}{y} \text{\_\_\_\_\_\_\_\_\_\_}(4)$$

Differentiating Eq.(3) partially w.r.t z,

$$2(z+1) + 2\lambda z = 0$$

$$\lambda = -1 + \frac{2}{z} \text{\_\_\_\_\_\_\_\_\_\_}(5)$$

From Eqs.(3), (4),(5),

$$-1 + \frac{1}{x} = -1 + \frac{2}{y} = -1 + \frac{2}{z}$$

$$\Rightarrow \frac{1}{x} = \frac{2}{y} = \frac{2}{z}$$

$\Rightarrow y = 2x$ and $z = 2x$

Substituting y and z in Eq.(1),

$$x^2 + (2x)^2 + (2x)^2 = 24$$

$$\Rightarrow 9x^2 = 24$$

$$\Rightarrow x^2 = \frac{24}{9}$$

$$\Rightarrow x = \pm\frac{2\sqrt{6}}{3}$$

$$y = 2\left(\frac{2\sqrt{6}}{3}\right) = \pm\frac{4\sqrt{6}}{3}$$

$$z = 2\left(\frac{2\sqrt{6}}{3}\right) = \pm\frac{4\sqrt{6}}{3}$$

Minimum value of $\sqrt{\left(\frac{2\sqrt{6}}{3} - 1\right)^2 + \left(\frac{4\sqrt{6}}{3} - 2\right)^2 + \left(\frac{4\sqrt{6}}{3} + 1\right)^2} = 20.20$

Maximum value of $\sqrt{\left(-\frac{2\sqrt{6}}{3} - 1\right)^2 + \left(-\frac{4\sqrt{6}}{3} - 2\right)^2 + \left(-\frac{4\sqrt{6}}{3} + 1\right)^2} = 39.79$

Exercise:

1. Find the maximum and minimum values of the function $f(x, y) = 3x + 4y$ on the circle $x^2 + y^2 = 1$ using the method of Lagrange's multipliers.
2. A soldier placed at a point (3,4) wants to shoot the fighter plane of an enemy which is flying along

the curve $y = x^2 + 4$ when it is nearest to him. Find such distance.

3. Find the minimum value of x^2yz^3 subject to the condition $2x + y + 3z = 1$ Using Lagrange's multiplier method. **[Summer 2022]**

<u>Taylor's Formula for $f(x, y)$ at the point (a,b)</u>

Suppose $f(x, y)$ and its partial derivative are continuous throughout an open rectangular region R centered at a point (a, b).Then, throughout R ,

$$f(x,y) = f(a,b) + xf_x(a,b) + yf_y(a,b) + \frac{1}{2!}\left(x^2 f_{xx}(a,b) + 2xyf_{xy}(a,b) + y^2 f_{yy}(a,b)\right)$$

$$+\frac{1}{3!}\left(x^3 f_{xxx}(a,b) + 3x^2 y f_{xxy}(a,b) + 3xy^2 f_{xyy}(a,b) + y^3 f_{yyy}(a,b)\right) + \cdots$$

<u>Taylor's Formula for $f(x, y)$ at origin (Also known as Maclaurin's series)</u>

$$f(x,y) = f(0,0) + xf_x(0,0) + yf_y(0,0) + \frac{1}{2!}\left(x^2 f_{xx}(0,0) + 2xyf_{xy}(0,0) + y^2 f_{yy}(0,0)\right)$$

$$+\frac{1}{3!}\left(x^3 f_{xxx}(0,0) + 3x^2 y f_{xxy}(0,0) + 3xy^2 f_{xyy}(0,0) + y^3 f_{yyy}(0,0)\right) + \cdots$$

<u>Note:</u> For quadratic expansion, find the taylor's series upto second degree terms and for cubic expansion , find the taylor's series upto third degree terms.

<u>Solved Examples:</u>

1. Expand $x^2y + 3y - 2$ in powers of $(x - 1)$and $(y + 2)$ upto second degree terms.

Sol: Let $f(x, y) = x^2y + 3y - 2$

By Taylor's Expansion,

$$f(x,y) = f(a,b) + \left[(x-a)f_x(a,b) + (y-b)f_y(a,b)\right]$$
$$+\frac{1}{2!}\left[(x-a)^2 f_{xx}(a,b) + \quad 2(x-a)(y-b)f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)\right] + \cdots$$

Here, $a = 1, b = -2$

$f(x,y) = x^2y + 3y - 2,$ $\quad$ $f(1,-2) = (1)^2(-2) + 3(-2) - 2 = -10$

$f_x(x,y) = 2xy,$ $\quad$ $f_x(1,-2) = 2(1)(-2) = -4$

$f_y(x,y) = x^2 + 3,$ $\quad$ $f_y(1,-2) = (1)^2 + 3 = 4$

$f_{xx}(x,y) = 2y,$ $\quad$ $f_{xx}(1,-2) = 2(-2) = -4$

$f_{xy}(x,y) = 2x,$ $\quad$ $f_{xy}(1,-2) = 2(1) = 2$

$f_{yy}(x,y) = 0,$ $\quad$ $f_{yy}(1,-2) = 0$

Substituting these values in Taylor's Expansion,

$$f(x,y) = -10 + [(x-1)(-4) + (y+2)4]$$

$$+\frac{1}{2!}[(x-1)^2(-4)+2(x-1)(y+2)(2)+(y+2)^2(0)]+\ldots$$

$$x^2y+3y-2=-10-4(x-1)+4(y+2)-2(x-1)^2+2(x-1)(y+2)+\cdots$$

2. Expand $e^x log\ (1+y)$in powers of x and y upto third degree.

Sol: Let $f(x,y)=e^x log\ (1+y)$

By Maclaurin's series,

$$f(x,y)=f(0,0)+\left[xf_x(0,0)+yf_y(0,0)\right]+\frac{1}{2!}\left[(x)^2f_{xx}(0,0)+2xyf_{xy}(0,0)+(y)^2f_{yy}(0,0)\right]$$
$$+\frac{1}{3!}\left[(x)^3f_{xxx}(0,0)+3x^2yf_{xxy}(0,0)3xy^2f_{xyy}(0,0)+(y)^3f_{yyy}(0,0)\right]+\ldots$$

$$f(x,y)=e^x\ log\ log\ (1+y)\,, \qquad f(0,0)=0$$
$$f_x(x,y)=e^x\ log\ log\ (1+y)\,, \qquad f_x(0,0)=0$$
$$f_y(x,y)=\frac{e^x}{1+y}, \qquad f_y(0,0)=1$$
$$f_{xx}(x,y)=e^x\ log\ log\ (1+y)\,, \qquad f_{xx}(0,0)=0$$
$$f_{xy}(x,y)=\frac{e^x}{1+y}, \qquad f_{xy}(0,0)=1$$
$$f_{yy}(x,y)=-\frac{e^x}{(1+y)^2}, \qquad f_{yy}(0,0)=-1$$
$$f_{xxx}(x,y)=e^x\ log\ log\ (1+y)\,, \qquad f_{xxx}(0,0)=0$$
$$f_{xxy}(x,y)=\frac{e^x}{1+y}, \qquad f_{xxy}(0,0)=1$$
$$f_{xyy}(x,y)=-\frac{e^x}{(1+y)^2}, \qquad f_{xxy}(0,0)=-1$$
$$f_{yyy}(x,y)=\frac{2e^x}{(1+y)^3}, \qquad f_{yyy}(0,0)=2$$

Substituting these values in Taylor's series,

$$f(x,y)=0+[x(0)+y(1)]+\frac{1}{2!}[x^2(0)+2xy(1)+y^2(-1)]$$
$$+\frac{1}{3!}[x^3(0)+3x^2y(1)+3xy^2(-1)+y^3(2)]+\cdots$$
$$e^x\ log\ log\ (1+y)\ =y+\frac{1}{2!}(2xy-y^2)+\frac{1}{3!}(3x^2y-3xy^2+2y^3)+\cdots$$
$$=y+xy-\frac{y^2}{2}+\frac{x^2y}{2}-\frac{xy^2}{2}+\frac{y^3}{3}-\cdots$$

Exercise:

1. Expand x^2+xy+y^2in powers of $(x-1)$and $(y-2)$ upto second- degree terms.

2. Expand $e^x cosy$ in powers of x and y upto third degree.