Unit - 3

Matrices

Q.1.A Identify which of the following matrices are symmetric, skew symmetric

(a) $\begin{bmatrix} -1 & 2 & 2 \\ 2 & 3 & 4 \\ 2 & 4 & -2 \end{bmatrix}$

Soln

$$A = \begin{bmatrix} -1 & 2 & 2 \\ 2 & 3 & 4 \\ 2 & 4 & -2 \end{bmatrix} \qquad A^T = \begin{bmatrix} -1 & 2 & 2 \\ 2 & 3 & 4 \\ 2 & 4 & -2 \end{bmatrix}$$

Here,

We can see that $A = A^T$ so A is symmetric matrix

(b) $\begin{bmatrix} 0 & 2 & -3 \\ -2 & 0 & 1 \\ 3 & -1 & 0 \end{bmatrix}$

Soln

Skew symmetric matrix

$A = -A^T$

$$A^T = \begin{bmatrix} 0 & -2 & 3 \\ 2 & 0 & -1 \\ -3 & 1 & 0 \end{bmatrix}$$

$$A' = -\begin{bmatrix} 0 & 2 & -3 \\ -2 & 0 & 1 \\ 3 & -1 & 0 \end{bmatrix}$$

~~$A' = -A^T$~~ $A^T = -A$

So, Above matrix is skew symmetric.

Q.2 Which of the following matrices are in row-echelon form, reduced row echelon form or both? Justify your answer.

(a) $\begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 7 \end{bmatrix}$

SoM

$$A = \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 7 \end{bmatrix}$$

Matrix A is neither row-echelon nor reduced-row echelon form because the condition of row echleon form is

(i) The first element in each row must be non zero and equal to 1 i.e called leading 1.

(ii) All the leading 1 must be on the righ-hand side of the matrix.

Matrix A not satisfied second condition.

Here,

It is not row echleon form and Reduced row echelon form.

b. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = B$

SoM Conditions

(i) The first element in each row must be non zero and equal to 1 i.e called leading 1.

(ii) All the leading 1 must be on the righ-hand side of the matrix

(iii) If any zero row is avaliable then it must be below to the all leading 1.

All three condition are satisfied.

Here

Matrix B is Row echelon form

Condition for Reduce Row-Echelon form

(i) Matrix is echelon form

(ii) Below & Above of leading element 1 must be unity

(iii) there must be zero

Condition satisfied

so, matrix B is also Reduced Row-Echelon form

(c) $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = C$

Soln

Zero matric is not to be row-echelon form and Reduced row-echelon form.

(d) $\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & , & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} = D$

Soln

For row-echelon form zero-row must be below to the all leading 1.

matrix D does not satisfied the condition

So, D matrix is neither row-echelon form and nor Reduced-echelon form.

(e) $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = E$

Soln Conditions: (i) leading element of each row must be non-zero and equal to 1 i.e called leading (1)

(ii) All the leading 1 must be on the right-hand side of the matrix

(iii) If any zero row is present then it must be at last row.

All three condition are satisfied then

Matrix E is row echelon form as well Reduced echelon form

1.B. Check whether the matrix $\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is orthogonal or not.

Soln

Orthogonal matrix

$$AA^T = I$$

$$A = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad A^T = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$AA^T = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

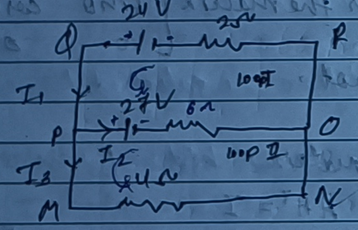
$$= \begin{bmatrix} \cos^2\theta + \sin^2\theta + 0 & \cos\theta\sin\theta - \sin\theta\cos\theta + 0 & 0 \\ \sin\theta\cos\theta - \cos\theta\sin\theta + 0 & \sin^2\theta + \sin^2\theta + 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$AA' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$AA' = I$

So, The given matrix is orthogonal matrix

Q3. In the circuit show below find the currents I_1, I_2 and I_3 in the loops.



Soln loop ①

$-27 - 6I_2 - 2I_1 + 24 = 0$

$2I_1 - 6I_2 + 3 = 0$ — ①

loop ②

$-4I_3 + 6I_2 + 27 = 0$

$6I_2 + 4I_3 + 27 = 0$ — ②

At P : $I_1 - I_2 - I_3 = 0$

$$\begin{array}{ccc|c} 2 & -6 & 0 & -3 \\ 0 & 6 & -4 & -27 \\ 1 & -1 & -1 & 0 \end{array}$$

Soln KCL

at P: $I_1 - I_2 - I_3 = 0$. — ①

loop 1

$-27 - 6I_2 - 2I_1 + 240$

$2I_1 + 6I_2 = -3$ — ②

loop 2

$-27 - 6I_2 + 4I_3 = 0$

$6I_2 + 4I_3 = -27$ — ③

$$\left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ -2 & -6 & 4 & -27 \\ 2 & 6 & 0 & -3 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & -8 & 2 & -27 \\ 2 & 6 & 0 & 3 \end{array}\right] \quad R_2 \to R_2 + 2R_1$$

$$\left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 1 & -\frac{1}{4} & -27/8 \\ 0 & 8 & 2 & -3 \end{array}\right] \quad \begin{array}{l} R_2\left(-\frac{1}{8}\right) \\ R_3 \to R_3 - 2R_1 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 1 & -\frac{1}{4} & -\frac{27}{8} \\ 0 & 0 & 1 & \frac{1}{2} \end{array}\right] \quad R_3 \to -\left(\frac{1}{6}\right)$$

$$I_3 = \frac{1}{2} = 0.5 \text{ Amp}$$

$$I_2 + I_3\left(-\frac{1}{4}\right) = -\frac{27}{8}$$

$$I_2 = -\frac{27}{8} + \frac{1}{4}\,\frac{1}{2}$$

$$I_2 = \frac{-27 \times 6 + 1}{48} = \frac{-162 + 8}{48}$$

$$= 3.66 \text{ Amp}$$

$$I_1 - I_2 - I_3 = 0$$

$$I_1 = I_2 + I_3$$

$$I_1 = 0.5 + 3.66$$

$$I_1 = 4.16 \text{ Amp}$$

Q.5 Solve the following systems of equations using Gauss elimination method.

(a) $x - 2y + 2z = 3$

$2x + y + 2z = 4$

$6x + 2y - 2z = 4$

Sol Augmented matrix

$AX = B$

$$\left[\begin{array}{ccc|c} 1 & -2 & 2 & 3 \\ 2 & 1 & 2 & 4 \\ 6 & 2 & -2 & 4 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 2 & 3 \\ 0 & 5 & -2 & -2 \\ 0 & 14 & -14 & -14 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 6R_1 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 2 & 3 \\ 0 & 1 & -2/5 & -2/5 \\ 0 & 14 & -14 & -14 \end{array}\right] \quad R_2 \to \left(\frac{1}{5}\right) R_2$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 2 & 3 \\ 0 & 1 & -2/5 & -2/5 \\ 0 & 0 & -42/5 & -42/5 \end{array}\right] \quad R_3 \to R_3 - 14R_2$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 2 & 3 \\ 0 & 1 & -2/5 & -2/5 \\ 0 & 0 & 1 & 1 \end{array}\right] \left(\frac{-5}{42}\right) R_3$$

$z = 1$

$y - \frac{2}{5}z = -\frac{2}{5}$

$y = -\frac{2}{5} + \frac{2}{5}$

$y = 0$

$x - 2y + 2z = 3$

$x - 2y + 2z + 3$

$x = -2 + 3$

$x = 1$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$ This is unique soln

(b) $2x_1 - x_2 + x_3 = 3$

$3x_1 - x_2 + 2x_3 = 6$

$-5x_1 + 8x_2 - 4x_3 = 2$

Soln

Augmented matrix

$[A|B]$

$$\left[\begin{array}{ccc|c} 2 & -1 & 1 & 3 \\ 3 & -1 & 2 & 6 \\ -5 & 8 & -4 & 2 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & -1/2 & 1/2 & 3/2 \\ 3 & -1 & 2 & 6 \\ -5 & 8 & -4 & 2 \end{array}\right] R_1\left(\frac{1}{2}\right)$$

$$\left[\begin{array}{ccc|c} 1 & -1/2 & 1/2 & 3/2 \\ 0 & 1/2 & \frac{1}{2} & 3/2 \\ -5 & 8 & -4 & 2 \end{array}\right] R_2 \to R_2 - 3R_1$$

$$\left[\begin{array}{ccc|c} 1 & -1/2 & 1/2 & 3/2 \\ 0 & 1 & 1 & 3 \\ 0 & 11/2 & -3/2 & 19/2 \end{array}\right] R_2 \to (2R_2) \quad R_3 \to R_3 + 5R_1$$

$$\left[\begin{array}{ccc|c} 1 & -1/2 & 1/2 & 3/2 \\ 0 & 1 & -1 & -6 \\ 0 & 0 & 8/2 & 22 \end{array}\right] R_3 \to R_3 + 11R_1$$

$$\left[\begin{array}{ccc|c} 1 & -1/2 & 1/2 & 3/2 \\ 0 & 1 & -1 & -6 \\ 0 & 0 & 1 & \frac{11}{2} \end{array}\right] R_3 \to \left(\frac{1}{4} R_3\right)$$

$z = \frac{11}{2}$

$y - z = -6$

$y = \frac{11}{2} - 6 = -\frac{1}{2}$

$$x - \frac{1}{2}y + \frac{1}{2}z = \frac{3}{2}$$

$$x = \frac{3}{2} + \frac{1}{2}y - \frac{1}{2}z$$

$$x = \frac{3 + y - z}{2}$$

$$x = \frac{3 + \frac{11}{2} - \frac{1}{2}}{2}$$

$$x = \frac{3+5}{2}$$

$$x = \frac{8}{2}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & -1/2 & 1/2 & 3/2 \\ 0 & 1 & -1 & -6 \\ 0 & 11/2 & -3/2 & 19/2 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & -1/2 & \frac{1}{2} & 3/2 \\ 0 & 1 & -1 & -6 \\ 0 & 0 & -8 & 85 \end{array}\right] \quad R_3 = \frac{1}{2}R_3 - 11R_2$$

$$\left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & \frac{1}{2} & 3/2 \\ 0 & 1 & 1 & 6 \\ 0 & 0 & 1 & 85/8 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & \frac{1}{2} & 3/2 \\ 0 & 1 & 1 & 3 \\ 0 & 11 & -3 & 19 \end{array}\right] \quad (2R_3)$$

$$\left[\begin{array}{ccc|c} 1 & -1/2 & 1/2 & 3/2 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & -14 & -14 \end{array}\right] \quad R_3 \to R_3 - 11R_2$$

$$\left[\begin{array}{ccc|c} 1 & -1/2 & 1/2 & 3/2 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & -1 \end{array}\right] \quad \left(-\frac{1}{14}\right)R_3$$

$$z = -1$$

$$y + z = 3$$

$$y = 3 - z$$

$$y = 3 + 1$$

$$y = 4$$

$$x - \frac{1}{2}y + \frac{1}{2}z = \frac{3}{2}$$

$$x = \frac{3}{2} + \frac{1}{2}y + \frac{1}{2}z$$

$$\therefore x = \frac{3 + y + z}{2}$$

$$x = \frac{3 + 4 - 1}{2}$$

$$x = 3$$

$(x, y, z) \to (3, 4, -1)$

unique soln

Q.0 Solve the following systems of linear equations by Gauss Jordan Method.

(a) $x + 2y + z = 0$
$-x - y + z = 2$
$y + 3z = 3$

Soln

The Augmented Matrix

$[A|B]$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ -1 & -1 & 2 & 2 \\ 0 & 1 & 3 & 3 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 3 & 2 \\ 0 & 1 & 3 & 3 \end{array}\right] R_2 \to R_2 + R_1$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 1 \end{array}\right] R_3 \to R_3 - R_2$$

$z = t$
$y + z = 2$
$y = 2 - z$
$y = 2 - t$

$x + 2y + z = 0$
$x = -2y - z$
$x = -2(2-t) - t$
$x = -4 + t - t$
$x = -4$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -4 \\ 2-t \\ t \end{bmatrix}$$

No Soln

b. $x + 2y + z = 5$
$-2x - 2y + z = 2$
$-x + 2z = 1$

Soln

The Augmented Matrix

$[A|B]$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 5 \\ -2 & -2 & 1 & 2 \\ -1 & 0 & 2 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 5 \\ 0 & 2 & 3 & 7 \\ 0 & 2 & 3 & 6 \end{array}\right] \begin{array}{l} R_2 \to R_2 + 2R_1 \\ R_3 \to R_3 + R_1 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 5 \\ 0 & 1 & 1 & 7/2 \\ 0 & 0 & 2 & 1 \end{array}\right] R_3 \to R_3 - R_1$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 5 \\ 0 & 1 & 1 & 7/2 \\ 0 & 0 & 1 & 1/2 \end{array}\right] \frac{1}{2}R_3$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 9/2 \\ 0 & 1 & 0 & 6/2 \\ 0 & 0 & 1 & 1/2 \end{array}\right] \begin{array}{l} R_1 \to R_1 - R_3 \\ R_2 \to R_2 - R_2 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3/2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 1/2 \end{array}\right] R_1 \to R_1 - R_2$$

$$z = \frac{1}{2}, \quad y = 3, \quad x = \frac{3}{2}$$

The Unique Soln

Q7. Solve the following using any method and find the value of d so that the equations have a (a) a non trivial (b) a trivial solution

$$2x + y + 2z = 0$$
$$x + y + 3z = 0$$
$$4x + 3y + dz = 0$$

Soln The Augmented matrix

$A|B$

$$\left[\begin{array}{ccc|c} 2 & 1 & 2 & 0 \\ 1 & 1 & 3 & 0 \\ 4 & 3 & d & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 2 & 1 & 2 & 0 \\ 1 & 1 & 3 & 0 \\ 4 & 3 & d & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 2 & 1 & 2 & 0 \\ 4 & 3 & d & 0 \end{array}\right] R_{12}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & -1 & -4 & 0 \\ 4 & 3 & d & 0 \end{array}\right] R_2 \to R_2 - 2R_1$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 1 & d-12 & 0 \end{array}\right] R_3 \to R_3 - 4R_1$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & d-16 & 0 \end{array}\right] R_3 \to R_3 - R_2$$

(i) a non trivial

$\lambda - 8 = 0$

$\lambda = 0$

(ii) a trivial solution

$\lambda - 8 \neq 0$

$\lambda \neq 8$

Q.8. (a) Find the value of x so that the rank of matrix A is (i) equal to 3 and (ii) less than 3

$$A = \begin{bmatrix} 3-x & 2 & 2 \\ 1 & 4-x & 0 \\ -2 & -4 & 1-x \end{bmatrix}$$

Sol^n

$$|A| = (3-x)\begin{vmatrix} 4-x & 0 \\ -4 & 1-x \end{vmatrix} - 2\begin{vmatrix} 1 & 0 \\ -2 & 1-x \end{vmatrix} + 2\begin{vmatrix} 1 & 4-x \\ -2 & -4 \end{vmatrix}$$

$$= (3-x)\left[(4-x)(1-x)\right] - 2(1-x) + 2\left[-4 + 2(4-x)\right]$$

$$= (3-x)(4-4x-x+x^2) - 2 + 2x + 2\left[-4 + 8 - 2x\right]$$

$$= 12 - 12x - 3x + 3x^2 - 4x + 4x^2 + x^2 - x^3 - 2(4-2x)$$

$$= x^3 - 8x^2 + 21x - 18 = 0$$

$$(x-2)(x-3)(x+3) = 0$$

(i) equal to 3

$x \neq 2, 3$

(ii) less the three

$x = 2, 3$

b. Find a and b such that the rank of the matrix :-

$$\begin{bmatrix} 1 & -2 & 3 & 1 \\ 2 & 1 & -1 & 2 \\ 6 & -2 & a & b \end{bmatrix} \text{ is } 2$$

Sol^n let

$$A = \begin{bmatrix} 1 & -2 & 3 & 1 \\ 2 & 1 & -1 & 2 \\ 6 & -2 & a & b \end{bmatrix}$$

Order of A is 3×4

$\rho(A) \leq 3$

$$\begin{bmatrix} 1 & -2 & 3 \\ 2 & 1 & -1 \\ 6 & -2 & a \end{bmatrix} = 0$$

$$1(a-2) + 2(2a+6) + 3(-4-6) = 0$$

$$a - 2 + 4a + 12 - 30 = 0$$

$$5a - 20 = 0$$

$$5a = 20$$

$$a = \frac{20}{5}$$

$$a = 4$$

$$\begin{bmatrix} -2 & 3 & 1 \\ 1 & -1 & 2 \\ -2 & a & b \end{bmatrix} = 0$$

$$-2(-b-2a) - 3(b+4) + 1(a-2) = 0$$

$$2b + 4a - 3b - 12 + a - 2 = 0$$

$-b + 5a = 14$

$b = 5a - 14$

If $a = 4$

$b = 5 \times 4 - 14$

$b = 20 - 14$

$b = 6$

If the Rank of Matrix is 2 then ~~the~~ value of (a, b) is $(4, 6)$

b. Find the ranks of the below stated matrix

(i) $\begin{bmatrix} 1 & 2 & 2 & -1 \\ 1 & 4 & 0 & 2 \\ -1 & 0 & -4 & 4 \end{bmatrix}$

Sol^n let

$$A = \begin{bmatrix} 1 & 2 & 2 & -1 \\ 1 & 4 & 0 & 2 \\ -1 & 0 & -4 & 4 \end{bmatrix}$$

Order of A is 3×4

$\rho(A) \leq 3$

Consider the third order minors

$$\begin{bmatrix} 1 & 2 & 2 \\ 1 & 4 & 0 \\ -1 & 0 & -4 \end{bmatrix} = +16 - 2(-4) + 2(4)$$

$$= -16 + 8 + 8$$

$$\begin{vmatrix} 2 & 2 & -1 \\ 4 & 0 & 2 \\ 0 & -4 & 4 \end{vmatrix} = 8 \times 2 - 32 + 16$$

$$= 0$$

Since the third order minor vanishes, therefore $\rho(A) \neq 3$

Consider a second order minor

$$\begin{vmatrix} 3 & 2 \\ 1 & 1 \end{vmatrix} = 3 - 2 = 1 \neq 0$$

There is a minor of order 2, which is not zero

$\therefore \rho(A) = 2$

(ii) $\begin{bmatrix} 1 & -3 & 1 & 2 \\ 0 & 1 & 2 & 3 \\ 3 & 4 & 1 & -2 \end{bmatrix}$

Sol^n let

$$B = \begin{bmatrix} 1 & -3 & 1 & 2 \\ 0 & 1 & 2 & 3 \\ 3 & 4 & 1 & -2 \end{bmatrix}$$

Order of B is 3×4

$\rho(A) \leq 3$

Consider the third order minor

$$\begin{vmatrix} 1 & -3 & 1 \\ 0 & 1 & 2 \\ 3 & 4 & 1 \end{vmatrix} = (1-8)+3(-0)+(-3)$$

$$= -7 - 18 - 3$$

$$= -28 \neq 0$$

Q.4. Find the eigen values and eigen vectors for the following matrices. Also determine algebraic multiplicity and geometric multiplicity of the matrices wherever possible.

(a) $\begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}$

Soln

$$A = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}$$

The characteristic eqn of matrix A is

$$|A - \lambda I| = \begin{vmatrix} 2-\lambda & 3 \\ 4 & 1-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)(1-\lambda) - 12 = 0$$

$$2 - 2\lambda - \lambda + \lambda^2 - 12 = 0$$

$$\lambda^2 - 3\lambda - 10 = 0$$

$$(\lambda - 5)(\lambda + 2) = 0$$

$$\lambda = 5 \text{ or } \lambda = -2$$

Eigen value is $(5, -2)$

(b) $\begin{bmatrix} 3 & 0 & 0 \\ 8 & 4 & 0 \\ 6 & 2 & 5 \end{bmatrix}$

Soln The characteristic eqn $|A - \lambda I| = 0$

$$\begin{vmatrix} 3-\lambda & 0 & 0 \\ 8 & 4-\lambda & 0 \\ 6 & 2 & 5 \end{vmatrix} = 0$$

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

$$S_1 = 12$$

$$S_2 = 20 + 15 + 12 = 47$$

$$S_3 = 60$$

$$\lambda^3 - 12\lambda^2 + 47\lambda - 60 = 0$$

$$(\lambda - 5)(\lambda^2 - 4\lambda - 3\lambda + 12) = 0$$

$$(\lambda - 5)(5 - 4)(5 - 3) = 0$$

Eigen Values are

$$\lambda = 3, 4, 5$$

A.M of $\lambda = 3$ is 1, $\lambda = 4$ is 1 and $\lambda = 5$ is 1.

$\lambda = 3$

$$\begin{bmatrix} 0 & 0 & 0 \\ 8 & 1 & 0 \\ 6 & 2 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 6 & 2 & 2 \\ 8 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad R_1 3$$

$$\begin{bmatrix} 1 & 1/3 & 1/3 \\ 80 & -5/3 & -8/3 \\ 0 & 0 & 0 \end{bmatrix} \quad R_1 \to \left(\frac{1}{10}\right) R_1, \; R_2 = R_2 - 8R_1$$

$$\begin{bmatrix} 1 & 1/3 & 1/3 \\ 0 & 1 & 8/5 \\ 0 & 0 & 0 \end{bmatrix} \quad R_2 \to \left(-\frac{3}{5}\right) R_2$$

$z = t$

$y + \frac{8}{5} z = 0 \qquad x = 3/5\, t$

$y = -\frac{8}{5} t$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3/5 \\ -8/5 \\ 1 \end{bmatrix} t$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -8 \\ 5 \end{bmatrix}$$

Eigen vector is for $\lambda = 3$ is $(3, -8, 5)\, t, t \in R$

$\lambda = 4$

$$\begin{bmatrix} -1 & 0 & 0 \\ 8 & 0 & 0 \\ 6 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{matrix} R_1 \to (-R_1) \\ R_2 \to R_2 + 8R_1 \\ R_3 \to R_3 + 6R_1 \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1/2 \\ 0 & 0 & 0 \end{bmatrix}$$

$z = k, \; y = -\frac{1}{2} k, \; x = 0$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ -1/2 \\ 1 \end{bmatrix} k$$

Eigen vector for $\lambda = 4$ is $(0, -1, 2)\, k, k \in R$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}$$

$\lambda = 5$

$$= \begin{bmatrix} -2 & 0 & 0 \\ 8 & -1 & 0 \\ 6 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 8 & -1 & 0 \\ 6 & 2 & 0 \end{bmatrix} \quad R_1 \to \frac{1}{2} R_1$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 2 & 0 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - 8R_1 \\ R_3 \to R_3 - 6R_1 \end{matrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{matrix} (-1) R_2 \\ R_3 \to R_3 - 2R_2 \end{matrix}$$

$z = k, \; y = 0, \; x = 0$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

G.M of $\lambda = 3$ is 1 and $\lambda = 3$ is 1

$\lambda = 5$ is also 1

(c) $\begin{bmatrix} 2 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{bmatrix}$

Sol The characteristic eqn is $|A-\lambda I|=0$

$$\begin{vmatrix} 2-\lambda & 1 & 1 \\ 2 & 3-\lambda & 2 \\ 3 & 3 & 4-\lambda \end{vmatrix} = 0$$

$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

$S_1 = 9$

$S_2 = (12-6)+(8-3)+(6-2)$

$= 15$

$S_3 = 12-2-3 = 7$

$\lambda^3 - 9\lambda^2 + 15\lambda - 7 = 0$

$(\lambda-1)(\lambda^2 - 8\lambda + 7) = 0$

$(\lambda-1)(\lambda-7)(\lambda-1) = 0$

Eigen values

$\lambda = 1, 1, 7$

A.M of $\lambda = 1$ is 2 and $\lambda = 7$ is 1

Case-I $\lambda = 1$

$|A-\lambda I| = 0$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \end{matrix}$$

$z = k_1,\ y = k_2,\ x = -k_1 - k_2$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -k_1 - k_2 \\ k_2 \\ k_1 \end{bmatrix}$$

$$= k_1 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

Hence,

the Eigen vector of $\lambda = 1$ is

$k_1(-1,0,1) + k_2(-1,1,0) \ \ k_1, k_2 \in R$

GM of $\lambda = 1$ is 2

Case-II $\lambda = 7$

$$= \begin{bmatrix} -5 & 1 & 1 \\ 2 & -4 & 2 \\ 3 & 3 & -3 \end{bmatrix} = \begin{bmatrix} 1 & -1/5 & 1/5 \\ 2 & -4 & 2 \\ 3 & 3 & -3 \end{bmatrix} \quad R_1 \to \left(-\tfrac{1}{5}\right)R_1$$

$$= \begin{bmatrix} 1 & -1/5 & -1/5 \\ 2 & -4 & 2 \\ -1 & -1 & 1 \end{bmatrix} \quad R_3 \to (-1)R_3$$

$$= \begin{bmatrix} -5 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -1 \end{bmatrix} \quad R_2 \to \frac{1}{2} R_2, \quad R_3 \to R_3\left(\frac{1}{5}\right)$$

$$\begin{bmatrix} 1 & 1 & -1 \\ 1 & -2 & 1 \\ -5 & 1 & 1 \end{bmatrix} \quad R_{13}$$

$$\begin{bmatrix} 1 & 1 & -1 \\ 0 & -3 & 2 \\ 0 & 6 & -4 \end{bmatrix} \quad R_2 \to R_2 - R_1, \quad R_3 \to R_3 + 5R_1$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -2/3 \\ 0 & 0 & 0 \end{bmatrix} \quad R_2 \to \left(-\frac{1}{3}\right) R_2, \quad R_3 \to R_3 + 2R_2$$

$z = k$, $y = \frac{2}{3}k$, $x = \frac{1}{3}k$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1/3 \\ 2/3 \\ 1 \end{bmatrix} k$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

G.M of $\lambda = 1$ is 2 and $\lambda = 7$ is 1

d. $\begin{bmatrix} 2 & 2 & -1 \\ 5 & -1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$

Sol The characteristic eq^n is $|A - \lambda I| = 0$

$$\begin{vmatrix} 2-\lambda & 2 & -1 \\ 5-\lambda & -1-\lambda & 3 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0$$

$\lambda^3 - \lambda^2 S_1 + S_2 \lambda - S_3 = 0$

$S_1 = 2$

$S_2 = -1 + 2 - 2 - 10 = -11$

$S_3 = -2 - 10 = -12$

$\lambda^3 - 2\lambda^2 - 11\lambda + 12 = 0$

$(\lambda - 1)(\lambda^2 - \lambda - 12) = 0$

$(\lambda - 1)(\lambda - 4)(\lambda + 3) = 0$

Eigen values

$\lambda = 1, -3, 4$

G.M of $\lambda = 1$ is 1, $\lambda = -3$ is 1 and $\lambda = 4$ is 1

$\lambda = 1$

$$= \begin{bmatrix} 1 & 2 & -1 \\ 5 & -2 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & -1 \\ 0 & -12 & 8 \\ 0 & 0 & 0 \end{bmatrix} R_2 \to R_2 - 5R_1$$

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -2/3 \\ 0 & 0 & 0 \end{bmatrix} \left(-\frac{1}{12}\right) R_2$$

$z = k$, $y = \frac{2}{3}k$, $x = -\frac{1}{2}k$

Eigen vector of $\lambda = 1$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$$

$\lambda = 4$

$$= \begin{bmatrix} -2 & 2 & -1 \\ 5 & -5 & 3 \\ 0 & 0 & -3 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 2 & -1 \\ 5 & -5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{matrix} R_2 \to R_2 + R_3 \\ R_3 \to \left(-\frac{1}{3}\right) R_3 \end{matrix}$$

$$\begin{bmatrix} 5 & -5 & 0 \\ -2 & 2 & -1 \\ 0 & 0 & 1 \end{bmatrix} R_{12}$$

$$\begin{bmatrix} 1 & -1 & 0 \\ -2 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{matrix} \frac{1}{5} R_1 \\ R_2 \to R_2 + R_3 \end{matrix}$$

$$\begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \frac{1}{2} R_2$$

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} R_2 \to R_2 + R_1$$

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} R_{23}$$

$z = 0$, $y = k$, $x = k$

Eigen vector of $\lambda = 4$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

G.M of $\lambda = 1$ is 1, $\lambda = -3$ is 1 and $\lambda = 4$ is 1

Q10 Find a matrix that diagonalizes A, and determine $P^{-1}AP$ where.

$$A = \begin{bmatrix} 2 & 2 & -1 \\ 5 & -1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

SoM The characteristic eqn is $|A - \lambda I| = 0$

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

$$S_1 = 2$$

$$S_2 = -1 + 2 - 11 = -11$$

$$S_3 = -12$$

$$\lambda^3 - 2\lambda^2 - 11\lambda + 12 = 0$$

$$(\lambda - 1)(\lambda^2 - \lambda - 12) = 0$$

$$(\lambda - 1)(\lambda - 4)(\lambda - 3) = 0$$

$$\lambda = 1, -3, 4$$

Eigen Vectors are

$$\lambda = 1 \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} \quad \lambda = -3 \begin{bmatrix} -2 \\ 5 \\ 0 \end{bmatrix} \quad \lambda = 4 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Diagonalizes of A.

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$P = \begin{bmatrix} -1 & -2 & 1 \\ 2 & 5 & 1 \\ 3 & 0 & 0 \end{bmatrix}$$

$$P^{-1} = \begin{bmatrix} -1 & 2 & 3 \\ -2 & 5 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

$$P^{-1}AP = D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

Q11. Find characteristic polynomials and the inverse using Cayley-Hamilton theorem,

Q. $\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$

SoM Characteristic Matrix

$$\begin{bmatrix} 1-\lambda & 2 \\ 1 & 3-\lambda \end{bmatrix}$$

$$\lambda^2 - S_1\lambda + S_2 = 0$$

$$S_1 = 4$$

$$S_2 = 1$$

Characteristic Polynomials

$$\lambda^2 - 4\lambda + I = 0$$

By Cayley Hamilton theorem

$$A^2 - 4A + I = 0$$

multiplying by A^{-1}, we get,

$$A^{-1}(A^2 - 4A + I) = 0$$

$$A - 4I + A^{-1} = 0$$

$$A^{-1} = 4I - A$$

$$= \begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 3 & -2 \\ -1 & -1 \end{bmatrix}$$

Q. $\begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$

Sol: The character eqn $|A - \lambda I| = 0$

$$\begin{bmatrix} -\lambda & 0 & -2 \\ 1 & 2-\lambda & 1 \\ 1 & 0 & 3-\lambda \end{bmatrix} = 0$$

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

$$S_1 = 5$$

$$S_2 = 6 + 2 = 8$$

$$S_3 = 4$$

Characteristic Polynomial

$$\lambda^3 - 5\lambda^2 + 6\lambda - 4 = 0$$

By Cayley Hamilton theorem

$$A^3 - 5A^2 + 6A - 4 = 0$$

multiply by A^{-1} then

$$A^{-1}(A^3 - 5A^2 + 6A - 4) = 0$$

$$A^2 - 5A + 6I - 4A^{-1} = 0$$

$$4A^{-1} = A^2 - 5A + 6I$$

$$A^{-1} = \frac{1}{4}(A^2 - 5A + 6I)$$

$$A^2 = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}\begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 0+0-2 & 0 & -6 \\ 2+1 & 4 & -2+2+3 \\ 3 & 0 & 9 \end{bmatrix} = \begin{bmatrix} -2 & 0 & -6 \\ 3 & 4 & 3 \\ 3 & 0 & 9 \end{bmatrix}$$

$$5A = \begin{bmatrix} 0 & 0 & -10 \\ 5 & 10 & 1 \\ 5 & 0 & 15 \end{bmatrix} \quad 6I = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

$$A^{-1} = \frac{1}{4}\left(A^2 - 5A + 6I\right)$$

$$= \frac{1}{4}\begin{bmatrix} -2 & 0 & -6 \\ 3 & 4 & 3 \\ 3 & 0 & 9 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 5 & 10 & 1 \\ 5 & 0 & 15 \end{bmatrix} + \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 4 & 0 & 4 \\ -2 & 0 & 2 \\ -2 & 0 & 0 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 1 & 0 & 1 \\ -1/2 & 0 & 1/2 \\ -1/2 & 0 & 0 \end{bmatrix}$$

(c) $\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

Sol^n The character eq^n is $|A - \lambda I| = 0$

$$\begin{vmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

$$S_1 = 6$$

$$S_2 = 3 + (3) + 3 = 9$$

$$S_3 = 6 + 1 - 1$$
$$= 6$$

Polynomial characteristic

$$\lambda^3 - 6\lambda^2 + 9\lambda - 6 = 0$$

By Cayley Hamilton theorem

$$A^3 - 6A^2 + 9A - 6 = 0$$

multiply by A^{-1} then, we get

$$A^{-1}(A^3 - 6A^2 + 9A - 6) = 0$$

$$A^2 - 6A + 9I - 6A^{-1} = 0$$

$$6A^{-1} = A^2 - 6A + 9I$$

$$A^{-1} = \frac{1}{6}\left(A^2 - 6A + 9I\right)$$

$$A^2 = A.A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 4+1+1 & -2-2-1 & 2+1+2 \\ -2-2-1 & 1+2+1 & -1-2-2 \\ 2+1+2 & -1-2-2 & 1+1+4 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}$$

$$A^{-1} = \frac{1}{6}\left(A^2 - 6A + 9I\right)$$

$$= \frac{1}{6}\begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 65 & -5 & 6 \end{bmatrix} - \begin{bmatrix} 12 & -6 & 6 \\ -6 & 12 & -6 \\ 6 & -6 & 12 \end{bmatrix} + \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

$$= \frac{1}{6}\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & -1 & 3 \end{bmatrix}$$

Qu. Reduce the quadratic form $3x_1^2 + 5x_2^2 + 3x_3^2 - 2x_2x_3 + 2x_3x_1 - 2x_1x_2$ into canonical form.

Sol^n

$$\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$$

The characteristic eq^n $|A - \lambda I| = 0$

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

$S_1 = 11$

$S_2 = 15 + 8 + 14$

$= 36$

$S_3 = 42 + 2 - 4$

$= 40$

Polynomial characteristic eq^n

$$\cancel{\lambda^3 - 11\lambda^2 + 36\lambda - 40 = 0}$$

$$\cancel{(\lambda - 3)(\lambda^2 - 8\lambda + 13) = 0}$$

$$\cancel{(\lambda - 3)(\lambda^2 - 8\lambda + 12 + 1) = 0}$$

$$\cancel{(\lambda - 3)(\lambda - 6)(\lambda - 2) + 1 = 0}$$

$$\lambda^3 - 11\lambda^2 + 36\lambda - 40 = 0$$