PROBABILITIES OF COMPOUND EVENTS

7.DSP.C.8

CONTENTS

The types of documents contained in the unit are listed below. Throughout the unit, the documents are arranged by lesson.

LEARNING MAP INFORMATION	An overview of the standards, the learning map section, and the nodes addressed in this unit
TEACHER NOTES	A brief discussion describing the progression depicted in the learning map section with research-based recommendations for focusing instruction to foster student learning and an introduction to the unit's lessons
OVERVIEW OF INSTRUCTIONAL ACTIVITIES	A table highlighting the lesson goals and nodes addressed in each lesson of this unit
INSTRUCTIONAL ACTIVITY	A detailed walkthrough of the unit
INSTRUCTIONAL ACTIVITY STUDENT HANDOUT	A handout for the guided activity, intended to be paired with the Instructional Activity
INSTRUCTIONAL ACTIVITY SUPPLEMENT	A collection of materials or activities related to the Instructional Activity
STUDENT ACTIVITY	A work-alone activity for students
STUDENT ACTIVITY SOLUTION GUIDE	A solution guide for the work-alone activity with example errors, misconceptions, and links to the learning map section

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PROBABILITIES OF COMPOUND EVENTS

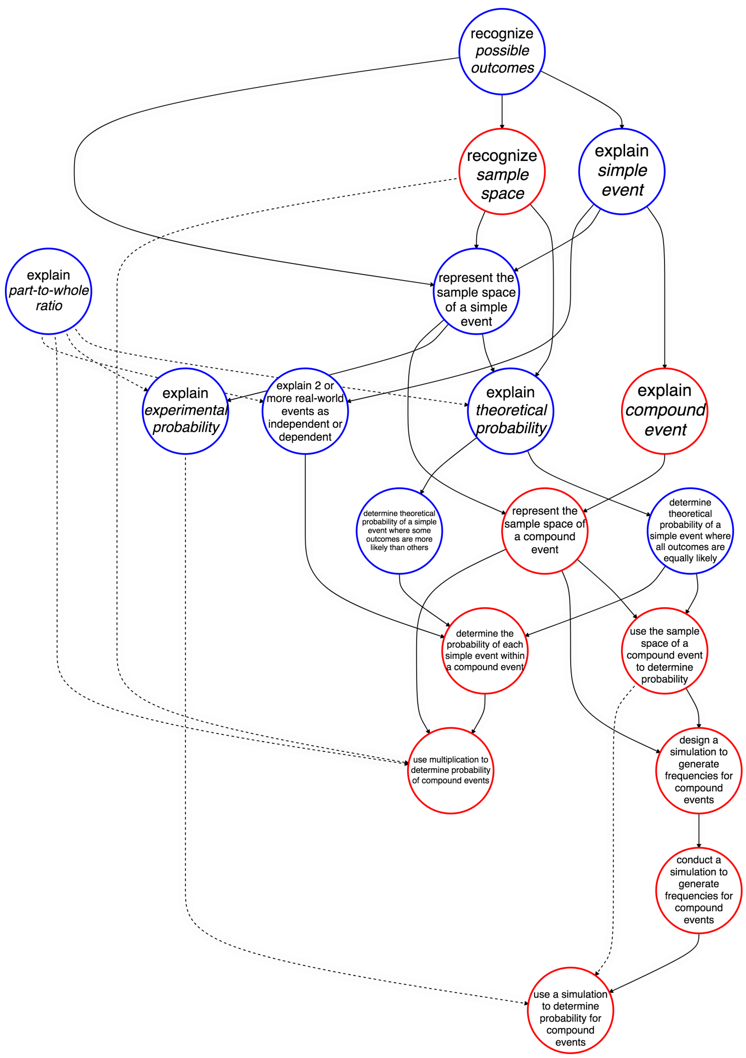
LEARNING MAP INFORMATION

STANDARDS

7.DSP.C.8 Find probabilities of compound events using organized lists, tables, tree diagrams, and simulation.

7.DSP.C.8.a Represent the sample space of a compound event.

7.DSP.C.8.b Design and use a simulation to generate frequencies for compound events.



**Learning map model of* 7.SP.8

Node Name	Node Description
CONDUCT A SIMULATION TO GENERATE FREQUENCIES FOR COMPOUND EVENTS	Conduct a simulation to generate frequencies for compound events. For example, to simulate an experiment where you are randomly choosing a male or female and then giving that person one of six conditions, you could flip a coin and roll a number cube as a simulation to determine gender and outcomes.
DESIGN A SIMULATION TO GENERATE FREQUENCIES FOR COMPOUND EVENTS	Design a simulation to generate frequencies for compound events. For example, to simulate an experiment where you are randomly choosing a male or female and then giving that person one of six conditions, you could flip a coin and roll a number cube as a simulation to determine gender and outcomes.
DETERMINE THE PROBABILITY OF EACH SIMPLE EVENT WITHIN A COMPOUND EVENT	Given two or more independent or dependent events, determine the probability of each simple event. Compound events could include selection problems with or without replacement and spinner problems where the result of the first spin dictates which spinner is used for the second spin.
DETERMINE THEORETICAL PROBABILITY OF A SIMPLE EVENT WHERE ALL OUTCOMES ARE EQUALLY LIKELY	Determine the probability of a simple event where all outcomes are equally likely. For example, drawing a green marble out of a bag where there are three blue marbles, three green marbles, and three red marbles.
DETERMINE THEORETICAL PROBABILITY OF A SIMPLE EVENT WHERE SOME OUTCOMES ARE MORE LIKELY THAN OTHERS	Determine the probability of a simple event where some outcomes are more likely than others. For example, drawing a green marble out of a bag where there are three blue marbles, four green marbles, and three red marbles.
EXPLAIN 2 OR MORE REAL-WORLD EVENTS AS INDEPENDENT OR DEPENDENT	Make known your understanding, formally or informally, of independent and dependent events in real-world examples. For example, the presence of clouds impacts the likelihood that it will rain, therefore these are dependent events. However, the day of the week does not impact the likelihood that it will rain, therefore these are independent events.
EXPLAIN *COMPOUND EVENT*	Make known your understanding that a compound event is composed of two or more simple events. For example, getting a heads and an even number when you toss a coin and roll a number cube.
EXPLAIN *EXPERIMENTAL PROBABILITY*	Make known your understanding that experimental probability is based on carrying out an experiment. Experimental probability is determined from the relative frequency (the ratio of the number of times the outcome occurred over the total number of trials).
EXPLAIN *PART-TO-WHOLE RATIO*	Make known your understanding that ratios can express a comparison between the number of parts to a given whole. These ratios are composed of two quantities measured in the same unit of measurement. For example, in the ratio of zebras to total animals in the zoo, the unit of measurement is one animal.
EXPLAIN *SIMPLE EVENT*	Make known your understanding that a simple event results in a single outcome. For example, drawing the ace of spades from a standard deck of cards.
EXPLAIN *THEORETICAL PROBABILITY*	Make known your understanding that theoretical probability is based on logical analysis of the outcomes, not through conducting an experiment. Theoretical probability equals the ratio of the number of favorable outcomes to the total number of outcomes.
RECOGNIZE *POSSIBLE OUTCOMES*	Identify or name possible outcomes of a given event.
RECOGNIZE *SAMPLE SPACE*	Identify or name all the possible outcomes of an event as the sample space.

REPRESENT THE SAMPLE SPACE OF A COMPOUND EVENT	Through writing or an appropriate assistive technology, represent the sample space of a compound event by organizing all possible outcomes using tables, tree diagrams, and organized lists.
REPRESENT THE SAMPLE SPACE OF A SIMPLE EVENT	Through writing or an appropriate assistive technology, list all possible outcomes of a simple event to represent the sample space of that simple event.
USE A SIMULATION TO DETERMINE PROBABILITY FOR COMPOUND EVENTS	Use a simulation to determine probability for compound events. For example, to simulate an experiment where you are randomly choosing a male or female and then giving that person one of six conditions, you could flip a coin and roll a number cube as a simulation to determine gender and outcomes.
USE MULTIPLICATION TO DETERMINE PROBABILITY OF COMPOUND EVENTS	Given two or more independent or dependent events, determine the joint probability of the events occurring by multiplying the probabilities of each event.
USE THE SAMPLE SPACE OF A COMPOUND EVENT TO DETERMINE PROBABILITY	Use the sample space of a compound event, where all of the simple events are independent and equally likely, to determine the probability of a specific outcome of the compound event. For example, flipping a coin and rolling a standard, six-sided number cube has 12 possible outcomes in its sample space (H1, T1, H2, T2, ...) and each outcome is equally likely, therefore the probability of each of the 12 possible outcomes is one-twelfth.

PROBABILITIES OF COMPOUND EVENTS

TEACHER NOTES

This unit includes the following documents:

- Learning Map Information
- Instructional Activity (three lessons)
- Instructional Activity Student Handout (for Lessons 2 and 3)
- Instructional Activity Supplement (for Lessons 1 and 2)
- Student Activity
- Student Activity Solution Guide

In this unit, students will make predictions about a variety of probabilistic scenarios in compound events, carry out experiments, compare predictions to experiment results, determine theoretical probabilities in compound events, and compare experimental and theoretical probabilities. Finally, students will design and conduct simulations to answer probability questions using what they have learned about experimental and theoretical probabilities in compound events.

RESEARCH

Students develop an understanding of chance behavior by discussing outcomes of a wide variety of probabilistic scenarios (Van de Walle, Bay-Williams, Lovin, & Karp, 2014). Students' understanding of probability and chance should be grounded in their experiences with data, which lay the foundation for making sense of theoretical probability. Teachers should provide ample opportunities for students to work with experimental probabilities before tying these or comparing them to theoretical probabilities, thereby maintaining the approach of creating a foundation of concrete activities to support more abstract ideas. To challenge existing beliefs about chance, it is important for students to work with actual data gathered through an experiment or sampling as they develop a more formal understanding of probability (Shaughnessy, 2003). Teachers are encouraged to use examples beyond flipping a coin and rolling a number cube to expose students to concepts of chance (Arbaugh, Scholten, & Essex, 2001). This will help acquaint students with a broad variety of realistic situations in which they will encounter chance in the real world. Additionally, hands-on games are a good way to engage students in probabilistic thinking (Wiest & Quinn, 1999). Such games give students opportunities to think about the consequences of different outcomes (e.g., "Where will my playing piece land if I roll a three? A five?").

There are four levels of development that have been identified as students progress in their probabilistic thinking (Jones, Thorton, Langrall, and Tarr (1999). A student operating at the first developmental level reasons subjectively and nonquantitatively (Jones et al., 1999). A student at this level might believe consecutive events are related when they are not (e.g., if a coin is flipped and lands on heads, a student operating at this level might believe tails is more likely to occur in the next flip), or believe that the person conducting an experiment can control the outcome (Jones et al., 1999). A student operating at the second developmental level is transitioning from subjective reasoning to naïve quantitative reasoning (Jones et al., 1999). A student operating at this level might make weak connections between sample space and probability before reverting to subjective reasoning (Jones et al., 1999). They may also believe a particular event will occur if it has been a long time since that event last occurred (Jones et al., 1999). A student reasoning at the third level uses more systematic strategies to list the outcomes of experiments and reasons quantitatively more consistently (Jones et al., 1999). Finally, a student reasoning at the fourth level regularly uses systematic strategies to generate outcomes of an experiment and determines experimental and theoretical probabilities (Jones et al., 1999). The following table summarizes the levels of probabilistic thinking (Jones et al., 1999).

LEVELS OF PROBABILISTIC THINKING

Adapted from Jones et al., 1999

Level 1	**Level 2**	**Level 3**	**Level 4**
Subjective, non quantitative reasoning; may believe the experimenter can control the outcome of an experiment	Transition from subjective reasoning to naïve quantitative reasoning; consistently identifies all outcomes for a one-stage experiment, but makes very weak connections between sample space and probability	More consistent use of quantitative reasoning to determine probabilities; systematic strategies used to list outcomes of one- and two-stage experiments	Accurately determines experimental and theoretical probabilities; systematic strategies to generate outcomes of an experiment

To help foster the progression through these levels of understanding, instruction should emphasize exploration rather than rules and definitions (Van de Walle et al., 2014). For example, given a compound event, students should make a predication about the probability of outcomes in that event, defend their prediction, then conduct an experiment or simulation and compare the results to the prediction (Van de Walle et al., 2014). Several researchers (e.g., Coffey & Richardson, 2005; Ely & Cohen, 2010; Shaughnessy, 2003; Van de Walle et al., 2014) agree that students need to make their own predictions about chance, test their predictions, then discuss and reconcile their findings compared to their predictions. Students should practice this same sequence of steps as they play probability games; they should first predict whether or not the game is fair, then play the game and compare the experimental and theoretical probabilities with their prediction (Norton, 2001). When students have conducted a probability experiment, they should discuss what would happen if additional trials were conducted (Edwards & Hensien, 2000). Only after students have predicted, experimented, and compared the results to the prediction is it appropriate for students to determine and connect the theoretical probability of a particular outcome to the results of the experiment (Lawrence, 1999; Wiest & Quinn, 1999).

After students have experience with probability scenarios, they can begin learning to design and conduct simulations and use the results to determine probabilities. Simulations are experiments that are designed to model the conditions of real events and are often conducted when experimenting with or manipulating the real situation is too dangerous, complex, or expensive (Van de Walle et al., 2014). It is important that this process is scaffolded for students, as they often struggle to select an appropriate device for simulations (Van de Walle et al., 2014). Using spinners as a convenient option because they can be created to fit the scenario, considering coins or chips for probabilities of one half, and utilizing number cubes when probabilities are compatible with six equally likely options, provides students with options to consider as they design their own simulations (Van de Walle et al., 2014). Random number generators should also be presented as an option because they can adapt to a variety of probability situations (Van de Walle et al., 2014).

Students should learn to use a variety of different representations to determine and list the sample space for an event. To promote a conceptual understanding of theoretical probability, a variety of representations can be used during instruction. For many students, creating a list of all possible outcomes and counting is intuitive and useful at first, but this approach has limitations when the number of events increases and when all events are not equally likely (Van de Walle et al., 2014). A matrix model is another option appropriate in cases where there are only two events. An area model is a conceptual approach that can be purposefully connected to students' knowledge of fractions; this model has been shown to help students understand experiments with two independent events with known probabilities (Van de Walle et al., 2014). The area model also helps students visualize how to combine probabilities in "and" and "or" situations (Van de Walle et al., 2014). The area model is limited in terms of the probabilistic situations it can model, but it leads to more symbolic approaches and helps to prepare students for and to explain tree diagrams (Van de Walle et al., 2014). After experimenting and working with probabilities, tree diagrams are generally easier for students to work with, and they can accommodate more situations with any number of events (Van de Walle et al., 2014).

AN EXAMPLE

Consider the sample space for the scenario of flipping a coin and rolling a number cube whose sides are 1, 1, 1, 2, 2, 3. Let H and T represent heads and tails on the coin respectively.

LIST

S = {H1, H2, H3, T1, T2, T3}

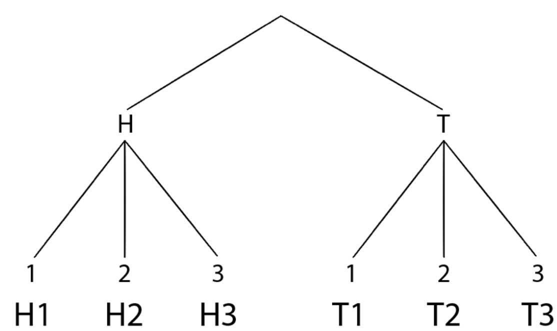
MATRIX

	1	**1**	**1**	**2**	**2**	**3**
H	H1	H1	H1	H2	H2	H3
T	T1	T1	T1	T2	T2	T3

AREA MODEL

	H	T
1	H1	T1
2	H2	T2
3	H3	T3

TREE DIAGRAM



Each of these representations helps students build the sample space for compound events in order to understand the possible outcomes. According to Shaughnessy (2003), students need more experience building sample spaces and listing all possible outcomes in probability experiments. This can be achieved through a variety of probability experiments and simulations. Before giving students formulas, algorithms, or computations, teachers should allow them to experiment with compound events and develop their own thinking and reasoning through visual representations (Ely & Cohen, 2010).

MISCONCEPTIONS

As students encounter various probability scenarios, terms such as "and", "or", "at least", and "no more than" can cause confusion for students (Van de Walle et al., 2014). The term "or" can be particularly difficult in probability, because the everyday meaning is not necessarily consistent with the mathematical meaning (Van de Walle et al., 2014). In mathematics, "or" includes both cases or events, while in everyday language "or" is interpreted as "strictly or" and is often used to mean only one out of two or more options (Van de Walle et al., 2014). Using lists, matrices, area models, and tree diagrams to develop and display the sample space and allow students to identify the cases which satisfy the problem condition helps students make sense of terms and operations used to determine probability in compound events (Van de Walle et al., 2014).

Another misconception students hold is a tendency to focus on absolute or actual size in a probability situation rather than the relative size of an outcome (Shaughnessy, 2003). For example, when comparing a scenario where two out of four objects are blue to a scenario where four out of eight objects are blue, rather than viewing the likelihood of selecting blue in both scenarios as equally likely, students may believe selecting blue in the scenario where four objects out of eight are blue to be more likely because there are more blue objects (Shaughnessy, 2003). In this situation, a student who believes more blue objects means that blue is more likely to be selected is applying additive reasoning – seeing "more than" instead of the ratio. The student who is able to think relatively and to understand that selecting blue in both scenarios is equally likely is able to use multiplicative reasoning, is able to accurately distinguish between additive and multiplicative structures, and has developed criteria to judge which structure is appropriate in a particular situation (Lamon, 1993).

Lastly, many students have intuitions about whether or not a game is fair, which tend to be more accurate with simple events than with compound events (Aspinwall & Shaw, 2000). Students need many experiences to overcome primitive, inaccurate intuitions, so designing tasks that require students to directly confront misconceptions helps students progress in their understanding (Aspinwall & Shaw, 2000; Ely & Cohen, 2010). One strategy is to introduce students to a game that is inherently unfair and, once students realize this is the case, ask students to adjust the game to make it fair (Coffey & Richardson, 2005). Another strategy is to use number cube games to confront misconceptions (Norton, 2001), as in rolling two number cubes and determining the difference has six possible outcomes (Norton, 2001). Students often believe that if one player wins when the difference is zero, one, or two, and the other player wins when the difference is three, four, or five, then the game is fair (Norton, 2001). However, even though each player has three ways of winning, they are not equally likely and the game is not fair (Norton, 2001). The following table displays all possible outcomes and their frequency for the difference of two number cubes.

POSSIBLE OUTCOMES FOR THE DIFFERENCE OF TWO NUMBER CUBES

		Result of Second Roll					
		1	**2**	**3**	**4**	**5**	**6**
Result of First Roll	**1**	0	1	2	3	4	5
	2	1	0	1	2	3	4
	3	2	1	0	1	2	3
	4	3	2	1	0	1	2
	5	4	3	2	1	0	1
	6	5	4	3	2	1	0

Additionally, tree diagrams can help students visually understand outcomes and their probabilities and begin to correct inaccurate beliefs they may have held (Aspinwall & Shaw, 2000).

LEARNING MAP INFORMATION

The learning map section for this sequence of activities begins with students' ability to recognize possible outcomes, explain part-to-whole ratios, and work with simple events and sample spaces. As students gain an understanding of the difference between theoretical and experimental probability and are able to determine the probabilities for simple events, they are ready to consider compound events. When working with compound events, students should be able to create and use the sample space to determine probabilities when the outcomes for each event are equally likely. This understanding builds the foundation for students to use multiplication to determine probabilities in compound events where the outcomes of each event are equally likely or where some outcomes are more likely to occur than others. Experiences with compound events prepare students to design and conduct their own simulations for a compound event and use the results to determine probabilities.

INSTRUCTIONAL ACTIVITIES

The activities in this unit are designed to expose students to a variety of models that can be used to determine and display sample spaces for compound events. Students will be presented with a variety of probabilistic scenarios for which they will make a prediction regarding the probability of a particular event. For each scenario, students will conduct the experiment, compare the results of the experiment to their prediction, then compute the theoretical probability for the event.

In Lesson 1, students will consider the possible outcomes related to having a single child, twins, triplets, etc., predict the likelihood of certain outcomes, and conduct an experiment. Students will then create the sample space for the event using organized lists and tree diagrams, reconsider their initial predictions, and determine the theoretical probability of various compound events. Finally, students will consider additional independent and dependent compound events, create the sample space, and determine probabilities.

In Lesson 2, students will participate in two different experiments after making predictions about the outcomes. First, students will roll two standard, six-sided number cubes and determine the difference in the values. One player wins if the difference is zero, one, or two, while the other player wins if the difference is three, four, or five. While this may appear to be a fair game, students will discover it is not fair as they experiment and create the sample space using a table. Students will then participate in an experiment with spinners and prizes, where they will establish through experimentation and theoretical probabilities which prize they should select to have the best chance of winning.

In Lesson 3, Students will use random number generators, coins, spinners, and/or number cubes to simulate a variety of compound events. Initial simulations will be guided in order to prepare students to design and conduct their own simulations. The results of each simulation will then be compared to the theoretical probability of the compound event.

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Probabilities of Compound Events

OVERVIEW OF INSTRUCTIONAL ACTIVITIES

Lesson	Learning Goal	Nodes Addressed
Lesson 1	Students will consider different chance scenarios, predict the likelihood of various events, conduct an experiment, create the sample space for the situation, reevaluate their initial predictions, determine the theoretical probability of compound events, and explain the difference between independent and dependent events.	▶ RECOGNIZE *SAMPLE SPACE* ▶ REPRESENT THE SAMPLE SPACE OF A SIMPLE EVENT ▶ EXPLAIN *COMPOUND EVENT* ▶ REPRESENT THE SAMPLE SPACE OF A COMPOUND EVENT ▶ EXPLAIN 2 OR MORE REAL-WORLD EVENTS AS INDEPENDENT OR DEPENDENT ▶ USE THE SAMPLE SPACE OF A COMPOUND EVENT TO DETERMINE PROBABILITY ▶ USE MULTIPLICATION TO DETERMINE PROBABILITY OF COMPOUND EVENTS ▶ DETERMINE THE PROBABILITY OF EACH SIMPLE EVENT WITHIN A COMPOUND EVENT
Lesson 2	Students will predict the result of a probabilistic situation, then conduct several trials in order to reconsider their prediction. After considering the data from an experiment, students will create the sample space and determine theoretical probabilities.	▶ EXPLAIN *COMPOUND EVENT* ▶ EXPLAIN 2 OR MORE REAL-WORLD EVENTS AS INDEPENDENT OR DEPENDENT ▶ RECOGNIZE SAMPLE SPACE ▶ REPRESENT THE SAMPLE SPACE OF A SIMPLE EVENT ▶ REPRESENT THE SAMPLE SPACE OF A COMPOUND EVENT ▶ USE THE SAMPLE SPACE OF A COMPOUND EVENT TO DETERMINE PROBABILITY ▶ DETERMINE THE PROBABILITY OF EACH SIMPLE EVENT WITHIN A COMPOUND EVENT ▶ USE MULTIPLICATION TO DETERMINE PROBABILITY OF COMPOUND EVENTS
Lesson 3	Students will learn how to design, conduct, and use simulations to estimate probabilities, then compare the simulation results to the theoretical probability of various events.	▶ DESIGN A SIMULATION TO GENERATE FREQUENCIES FOR COMPOUND EVENTS ▶ CONDUCT A SIMULATION TO GENERATE FREQUENCIES FOR COMPOUND EVENTS ▶ USE A SIMULATION TO DETERMINE PROBABILITY FOR A COMPOUND EVENT ▶ EXPLAIN 2 OR MORE REAL-WORLD EVENTS AS INDEPENDENT OR DEPENDENT ▶ DETERMINE THE PROBABILITY OF EACH SIMPLE EVENT WITHIN A COMPOUND EVENT ▶ USE MULTIPLICATION TO DETERMINE PROBABILITY OF COMPOUND EVENTS

PROBABILITIES OF COMPOUND EVENTS

INSTRUCTIONAL ACTIVITY

Lesson 1

LEARNING GOAL

Students will consider different chance scenarios, predict the likelihood of various events, conduct an experiment, create the sample space for the situation, reevaluate their initial predictions, determine the theoretical probability of compound events, and explain the difference between independent and dependent events.

PRIMARY ACTIVITY

Students will consider the possible outcomes related to having a single child, twins, triplets, etc., predict the likelihood of certain outcomes, and conduct an experiment. Students will then create the sample space for the event described using organized lists and tree diagrams, reconsider their initial predictions, and determine the theoretical probability of various compound events. Finally, students will consider additional independent and dependent compound events, create the sample space, and determine probabilities.

OTHER VOCABULARY

Students will need to know the meaning of the following terms:

- Event
- Simple event
- Compound event
- Dependent events
- Independent events
- Likelihood
- Outcome
- Probability
- Sample space
- Certain
- Possible
- Likely
- Unlikely
- Impossible

MATERIALS

- Coins or two-color counters/chips
- Paper
- Red, green, and blue counters or candies (optional)
- Instructional Activity Supplement (Recommend one copy for every one or two students.)

IMPLEMENTATION

This lesson begins with students considering the likelihood of having a boy or a girl. This simple event will lead into a discussion of compound events.

Pose the scenario of a couple having a baby to students.

Ask students what the gender of the baby could be. Students should state that it could be a boy or a girl.

Discuss with students that both genders are possible, and that for the purpose of the following activity, we are going to assume the chance of having a boy is the same as the chance of having a girl.

Provide students with either a coin or a two-colored counter and ask them to determine which side will represent a boy and which side will represent a girl.

Ask students what they believe the outcomes will be if they flip or spin the coin/counter 20 times to determine the gender of 20 hypothetical children. Students should estimate that approximately half the children will be boys and half the children will be girls, but that in 20 trials, it may not be exactly half and half.

Note that the *sample space*, or the possible outcomes, for this experiment is $S = \{B, G\}$, where B and G are abbreviations for boys and girls, respectively.

Require students to carry out the experiment, either individually or in pairs. Students should record their results in the "one baby" column on the Instructional Activity Supplement.

Once students have had an opportunity to conduct 10 trials, **ask** students to share their results and compare them to their predictions prior to the experiment.

Compile approximately 100 results and **discuss** that, as the number of trials increases, the results become closer to the theoretical probability, or what "should" happen.

Next, students will consider a compound event: having twins.

Ask students what could happen if a couple has twins. Students should indicate that they could have two boys, two girls, or a boy and a girl.

Ask students what they believe the outcomes will be if they use a coin or a counter to determine the gender combinations of 20 hypothetical sets of twins.

When making their predictions, students may not take into consideration that having a boy and a girl can occur in two different ways: a boy then a girl, or a girl then a boy. In this instance, they may predict that each outcome described will happen one third of the time. If students hold this misconception, do not correct them. Instead, **allow** them to carry out the experiment and discover why a boy and a girl will occur more frequently through experimentation and creating the sample space for the event.

> NOTE: Students likely have some intuitions about probability, some of which may be inaccurate. It is important to let students explore scenarios that may challenge these intuitions to help them grow in their understanding.

Require students to carry out 20 trials of the experiment, either individually or in pairs. Students should record their results in the order they occur in the "twins" column on the INSTRUCTIONAL ACTIVITY SUPPLEMENT, using the first coin flip as the "first" twin and the second coin flip as the "second" twin.

Once students have had an opportunity to conduct the trials, **ask** students to share their results and compare them to their predictions prior to the experiment. Students should notice that one boy and one girl occurs more frequently than two boys or two girls. If this is not apparent in a sample of 20 trials, **compile** approximately 100 results and remind students that, as the number of trials increases, the results become closer to the theoretical probability, or what "should" happen.

Ask students why one boy and one girl occurred more frequently than they may have predicted. Students should be able to explain that there were two outcomes where a boy and a girl could occur—having a boy then a girl, or a girl then a boy—whereas there was only one outcome where both children could be boys and only one outcome where both children could be girls.

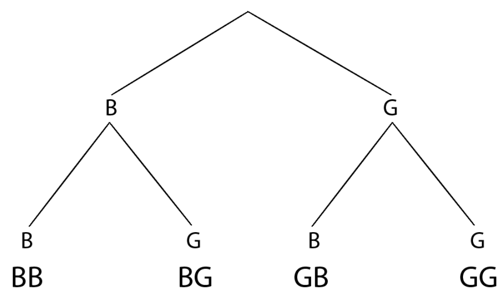
Ask students to identify the sample space for this experiment, using the previous experiment as a model. Students should identify the sample space for twins as $S = \{BB, BG, GB, GG\}$, where B and G are abbreviations for boys and girls, respectively.

Discuss that each of these four outcomes are equally likely to occur, therefore each can be expected to occur one fourth of the time. This means that having two boys would occur one fourth of the time, two girls would occur one fourth of the time, and one boy and one girl would occur one half of the time (one fourth plus one fourth).

Note that sometimes there aren't very many different possible outcomes and they are easy to list, whereas other times there are many possibilities which can be difficult to keep track of. For example, if there were nine kids, the number of possible outcomes increases significantly.

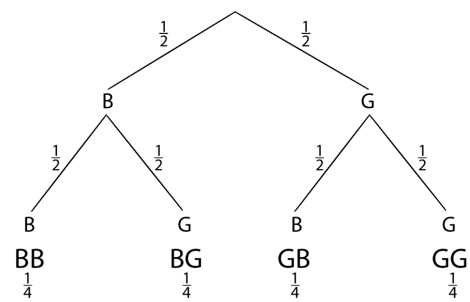
Model making a tree diagram, a tool that can be used to create the sample space for a compound event. Explain that the first branch represents the possibilities for the first twin and the second level of branches represents the possibilities for the second twin, based on the outcome of the first twin.

Following the outcomes along each branch produces the possible outcomes, which are written at the bottom of the diagram.



Remind students that the chance of having a boy is the same as the chance of having a girl, and therefore the probability is one half for each event.

Model including the probabilities in the tree diagram (along the branches) and the probabilities for each of the possible outcomes.



Discuss that the probabilities for the possible outcomes are the product of the probabilities of the branches that compose that outcome. In this scenario, because having a boy and having a girl is equally likely each time and the first event is independent of the second event, the probabilities of each of the possible outcomes is the same.

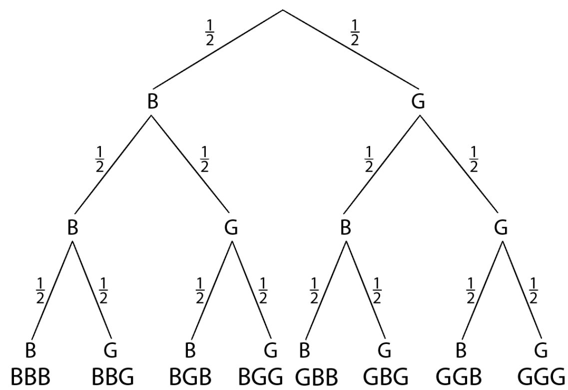
> NOTE: One misconception students may hold is that if the first child is a boy, it is more likely that the second child is a girl. However, it is important for students to understand that independent events are not impacted by what has occurred in the past. For example, just because someone has had five boys in a row, the sixth child has an equal chance of being a boy or a girl—it is not more likely that it will be a girl because the previous five children were boys.

Next, students will consider a case of triplets and quadruplets.

Ask students how likely it is that, in a set of triplets, all of the children will be boys. Based on their experience considering twins, students should anticipate that this is possible, but not very likely.

If desired, students can continue to simulate the genders in a set of triplets by tossing or spinning a coin or a counter. Students can record their results in the "triplets" column on the INSTRUCTIONAL ACTIVITY SUPPLEMENT.

Require students to draw a tree diagram to model a third child and include the probabilities along each branch. At the end of each branch, students should represent each of the possible outcomes (e.g., BBB, BGB, BGG). The following is an example of expected student work.



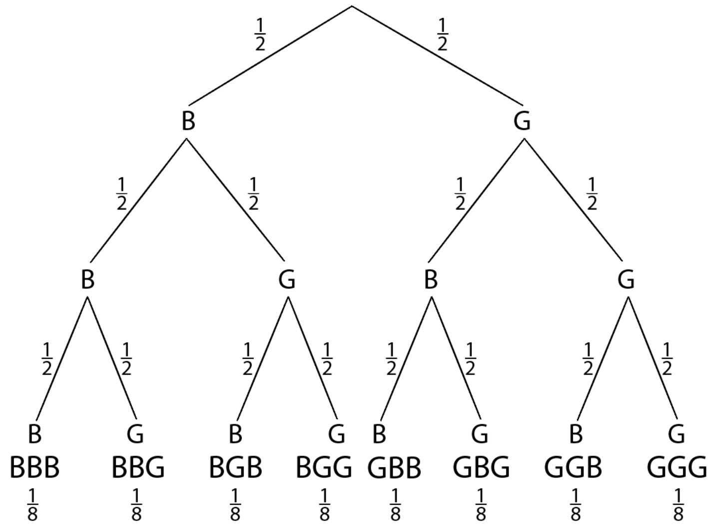
Ask students, based on the tree diagram, how likely it is that triplets would be all boys. Students should notice that only one of several possible outcomes is all boys. Therefore, while this is possible, it is not very likely to happen.

Ask students to determine the theoretical probability of having all boys by finding the product along the braches and using the sample space (because the events are independent and equally likely). Students should indicate that the probability of all boys is one out of eight or one eighth.

Require students to compare the theoretical probability of having all boys to the prediction they made prior to creating the tree diagram. **Ask** students to explain any differences in their predictions to the theoretical probability.

Relate the probability to the number of possible outcomes (eight) and that because each event, a boy or a girl, is equally likely throughout the situation, the possible outcomes are each equally likely. Additionally, the product of each of the branches is one eighth.

Require students to write the probabilities underneath each of the possible outcomes as shown in the following image and write the sample space (S = {BBB, BBG, BGB, BGG, GBB, GBG, GGB, GGG}) for this situation.



Ask students a variety of probability questions once the tree diagram is completed. The following are some examples of questions with answers. **Discuss** the meaning of words and phrases such as "or", "at least", and "no more than" as they arise.

- What is the probability of having all boys or all girls? ($\frac{2}{8}$ or $\frac{1}{4}$)
- What is the probability of having exactly two boys? ($\frac{3}{8}$)
- What is the probability of having at least two girls? ($\frac{4}{8}$ or $\frac{1}{2}$)
- What is the probability of having no more than one girl? ($\frac{4}{8}$ or $\frac{1}{2}$)
- What is the probability of having exactly two boys or exactly two girls? ($\frac{6}{8}$ or $\frac{3}{4}$)
- What is the probability of having at least 1 boy? ($\frac{7}{8}$)

Require students to draw a tree diagram to model a fourth child and include the probabilities along each branch. At the end of each branch, students should represent each of the possible outcomes (e.g., BBBB, BBGB, BGGG) and their probabilities ($\frac{1}{16}$).

Ask guiding questions as students work to ensure understanding.

GUIDING QUESTIONS

Elicit student thinking:

- What are some possible combinations of boys and girls in a family?
- Consider a family where the oldest child is a boy and the youngest child is a girl. Is this different than a family where the oldest child is a girl and the youngest child is a boy?
- Do you think the order of boys and girls is significant?
- If a family has a lot of kids, do you think it is possible for all of the kids to be boys? Explain.
- If a family has a lot of kids, do you think it is likely that all of the kids are girls? Explain.

Determine if the student can RECOGNIZE *SAMPLE SPACE*:

- If these are the possible outcomes of this experiment, as a group, what are they called?
- Can you describe what it means to determine the sample space for an event?

Determine if the student can REPRESENT THE SAMPLE SPACE OF A SIMPLE EVENT:

- What are the possible gender outcomes when someone has a (one) baby?
- What is the sample space for the event of having a (one) baby?

Determine if the student can EXPLAIN *COMPOUND EVENT*:

- When you consider the scenario of having one child, is this a simple event or a compound event? Explain.
- When you consider the scenario of having multiple children, is this a simple event or a compound event? Explain.
- If you were to roll a number cube and then flip a coin, is this a simple event or a compound event? Explain.
- Can you provide an example of a different compound event? What makes it a compound event?

Determine if the student can REPRESENT THE SAMPLE SPACE OF A COMPOUND EVENT:

- Can you tell me all the possible outcomes of this situation?
- How would you represent the possible outcomes of having quadruplets in a tree diagram?
- What information does this tree diagram provide? How can you use a tree diagram to create the sample space for this situation?

Determine if the student can EXPLAIN 2 OR MORE REAL-WORLD EVENTS AS INDEPENDENT OR DEPENDENT:

- If someone's first child is a boy, does this impact the likelihood of a boy or a girl if they have a second child? Why or why not?
- If someone has three boys, is it more likely that a fourth child would be a girl? Why or why not?

Determine if the student can USE THE SAMPLE SPACE OF A COMPOUND EVENT TO DETERMINE PROBABILITY:

- Which outcomes in the sample space represent having exactly two boys and two girls? (*E* = {BBGG, BGBG, BGGB, GBBG, GBGB, GGBB})
- Can you describe an event that is possible, but very unlikely to occur?
- Can you describe an event that will likely occur, but is not certain?
- What is the theoretical probability of having all boys? How can you determine this probability using the sample space?
- What is the theoretical probability of having exactly one girl? How can you determine this probability using the sample space?
- What is the theoretical probability of having at least one boy? How can you determine this probability using the sample space?
- What is the theoretical probability of having exactly two girls and two boys? How can you determine this probability using the sample space?
- How does the number of children impact the probability of having all girls or all boys? Explain your reasoning.

Determine if the student can USE MULTIPLICATION TO DETERMINE PROBABILITY OF COMPOUND EVENTS:

- What is the theoretical probability of having all girls? How can you determine this probability using multiplication?
- What is the theoretical probability of having exactly one boy? How can you determine this probability using multiplication (and addition)?
- What is the theoretical probability of having at least one girl? How can you determine this probability using multiplication (and addition)?
- What is the theoretical probability of having exactly two girls and two boys? How can you determine this probability using multiplication (and addition)?

Students should be required to explain their thinking and show their work as they consider the sample space for the genders of quadruplets. While it is recommended that students us a tree diagram in this lesson, it is possible that students could develop alternate methods for organizing lists or tables to represent the sample space. As long as students are able to work in an organized way and account for all possible outcomes, alternate strategies are acceptable.

Inform or **remind** students that the events considered so far—having a boy or a girl—are considered *independent events* because the outcome of one event does not impact the likelihood of future events. For example, whether you have a boy or girl as the first child, you still have the same chance of having a boy or girl as the second child.

Present students with the following scenario:

> Your friend is holding a bag of colored candies. There are five green candies, three blue candies, and two red candies. Your friend tells you he will give you the bag of candy if you draw two of the same color candies without replacing the first candy drawn. What is the probability you will draw two of the same colored candies?

Point out to students that after you select the first piece of candy, it will not go back in the bag before the next candy is selected.

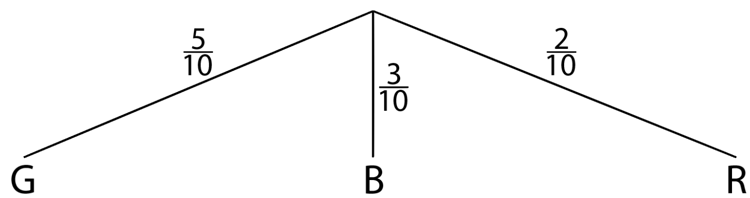
Ask students how many candies are in the bag as you select the first piece. Students should indicate there are ten candies before the first piece is drawn.

Ask students how many candies are in the bag as you select the second piece. Students should indicate there are nine candies in the bag when the second piece is drawn.

Ask students what color the candies in the bag are after the first piece of candy is drawn (before the second piece is drawn). Students should indicate that it depends what color was selected first.

Discuss that, in this situation, the second event depends on the outcome of the first, unlike in the scenario with having a boy or a girl. Because the second event depends on the outcome of the first, these are *dependent events*.

Draw or **show** the candies described (five green, three blue, and two red) to the class and begin a tree diagram with theoretical probabilities to model the situation, where G, B, and R represent drawing a green, blue, and red candy respectively.



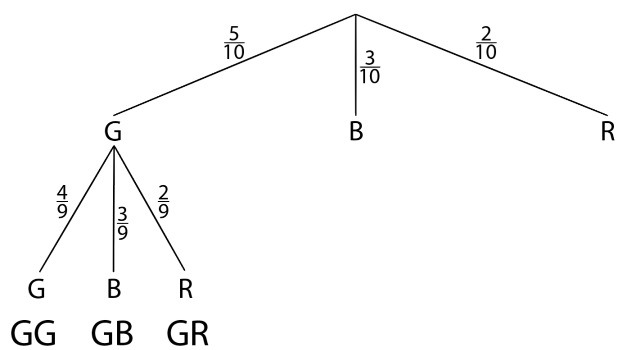
Model selecting a green candy on the first draw by removing one of the green candies.

Remind students that this candy will not be replaced before the second draw, so the candies they see are the ones that will be drawn from next.

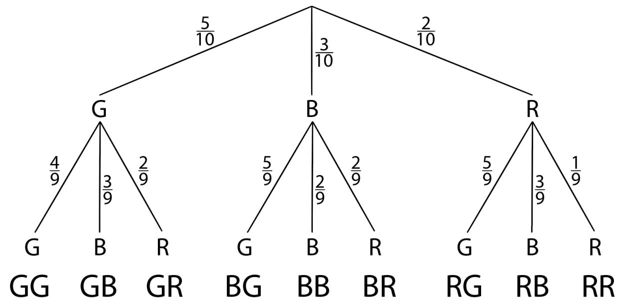
Ask students what colors are available to draw. They should indicate that green, blue, and red are all still possible.

Model this on the tree diagram, along with the probabilities of selecting each color in the second draw. **Note** that there are now only four green candies and nine candies total, so the theoretical probabilities will be different for the second draw than they were for the first.

Label each outcome at the end of each branch.



Repeat this process, modeling the selection of a blue candy on the first draw, then a red candy on the first draw. **Fill** in the tree diagram accordingly.



Ask students to first represent the sample space (S = {GG, GB, GR, BG, BB, BR, RG, RB, RR}), then identify the outcomes that result in keeping the bag of candy (E = {GG, BB, RR}).

Require students to determine the probability of drawing two greens, two blues, or two reds ($\frac{20}{90} + \frac{6}{90} + \frac{2}{90} = \frac{28}{90}$ or $\frac{14}{45}$).

GUIDING QUESTIONS

Elicit student thinking:

- Can you describe this situation in your own words?
- How are these events related?

Determine if the student can EXPLAIN 2 OR MORE REAL-WORLD EVENTS AS INDEPENDENT OR DEPENDENT:

- If you draw a marble and then put it back in the bag, are the probabilities of the second draw impacted?
- If you draw a marble and do not put it back in the bag, are the probabilities of the second draw impacted?

Determine if the student can DETERMINE THE PROBABILITY OF EACH SIMPLE EVENT WITHIN A COMPOUND EVENT:

- What are the theoretical probabilities for the first draw?
- If a blue candy is drawn first, what is the probability of drawing a second blue candy if the first blue candy is not replaced?
- If a green candy is drawn first, how does this impact the likelihood of drawing a blue or red candy in the second draw? Can you explain your thinking?
- Since the candy is not replaced after the first draw, what are the probabilities for the second draw if a green candy is drawn first? If a blue candy is drawn first? If a red candy is drawn first?

Determine if the student can USE MULTIPLICATION TO DETERMINE PROBABILITY OF COMPOUND EVENTS:

- How can you use multiplication to determine the probability of drawing a red candy then a blue candy?
- How can you use multiplication (and addition) to determine the probability of drawing a blue candy and a green candy in any order?

At the end of the activity, require students to create the sample space for the compound event of rolling a standard, six-sided number cube and flipping a coin. In addition to creating the sample space, require students to provide the probability of rolling a two and flipping tails ($\frac{1}{12}$), identify whether these are independent or dependent events, and explain how they know.

		Gender Results		
		One Baby	Twins	Triplets
Trial	1			
	2			
	3			
	4			
	5			
	6			
	7			
	8			
	9			
	10			
	11			
	12			
	13			
	14			
	15			
	16			
	17			
	18			
	19			
	20			

One Baby	B	G
Theoretical probability		
Experimental probability		

Twins	BB	BG	GB	GG
Theoretical probability				
Experimental probability				

Probabilities of Compound Events

INSTRUCTIONAL ACTIVITY

Lesson 2

LEARNING GOAL

Students will predict the result of a probabilistic situation, then conduct several trials in order to reconsider their prediction. After considering the data from an experiment, students will create the sample space and determine theoretical probabilities.

PRIMARY ACTIVITY

Students will participate in two different experiments after making predictions about the outcomes. First, students will roll two standard six-sided number cubes and determine the difference in the values. One player wins if the difference is zero, one, or two, while the other player wins if the difference is three, four, or five. While this may appear to be a fair game, students will discover it is not as they experiment and create the sample space using a table. Students will then participate in an experiment with spinners and prizes, where they will establish through experimentation and theoretical probabilities which prize they should select to have the best chance of winning.

OTHER VOCABULARY

Students will need to know the meaning of the following terms:

- Event
- Dependent event
- Independent event
- Simple event
- Compound event
- Outcome
- Probability
- Sample space

MATERIALS

- Standard six-sided number cubes
- Paperclips
- INSTRUCTIONAL ACTIVITY STUDENT HANDOUT
- INSTRUCTIONAL ACTIVITY SUPPLEMENT (Recommend one copy for every two students.)

IMPLEMENTATION

Students will first predict whether or not a game is fair, then play the game and create the sample space to evaluate the accuracy of their prediction.

Ask students what they believe it means for a game to be "fair". Students should have some intuitions about fair and unfair that they can contribute to the conversation.

After some discussion, **establish** that a fair game is a game where each player is equally likely to win.

Present the following game to students and **ask** them to consider whether or not they believe the game is fair.

> Player A and Player B each roll a standard six-sided number cube and subtract the values that are face up. If the difference is zero, one, or two, Player A wins. If the difference is three, four, or five, Player B wins.

Discuss students' predictions and why they believe the game is or is not fair. Do not come to a conclusion regarding whether or not the game is fair; students will discover this on their own as they play the game.

Require students to decide who is Player A and who is Player B and play the game at least 25 times, recording who wins each game on the first page of the INSTRUCTIONAL ACTIVITY SUPPLEMENT.

GUIDING QUESTIONS

Elicit student thinking:

- Would you rather be Player A or Player B? Why?
- Are you noticing any patterns as you play the game?

Determine if the student can EXPLAIN *COMPOUND EVENT*:

- Is rolling two number cubes a simple event or a compound event? How do you know?

Determine if the student can EXPLAIN 2 OR MORE REAL-WORLD EVENTS AS INDEPENDENT OR DEPENDENT:

- Does the result of the first roll impact the theoretical probabilities of the second roll? Explain your reasoning.
- Is rolling two number cubes an independent event or dependent events? How do you know?

After students have had a chance to play the game at least 25 times, **ask** the pairs whether Player A or Player B won more games overall. **Allow** several or all pairs to share who won more games.

As pairs report who won, it should be obvious to students that the game was not fair and that Player A had a greater chance of winning than Player B.

Ask students to informally explain why this may be the case. Students should be able to express that the numbers rolled were close together more frequently.

Provide the first page of the INSTRUCTIONAL ACTIVITY STUDENT HANDOUT to each student.

Explain that the chart shows all possible outcomes for the first roll and all possible outcomes for the second roll. The blank spaces are the result of subtracting the value of the first and second rolls.

Require students to fill in the chart representing the possible outcomes of rolling two standard, six-sided number cubes and subtracting their values. The following is an example of a completed chart.

		Result of Second Roll					
		1	**2**	**3**	**4**	**5**	**6**
Result of First Roll	**1**	0	1	2	3	4	5
	2	1	0	1	2	3	4
	3	2	1	0	1	2	3
	4	3	2	1	0	1	2
	5	4	3	2	1	0	1
	6	5	4	3	2	1	0

Ask students how the chart helps explain the results of game and whether or not the game is fair, then require them to answer Questions 1 – 3 on the INSTRUCTIONAL ACTIVITY STUDENT HANDOUT.

GUIDING QUESTIONS

Elicit student thinking:

- Is this game fair if Player A wins when the difference is zero, one, or two and Player B wins when the difference is three, four, or five? Explain.
- What is a fair game?
- How would you decide if a game is fair?

Determine if the student can RECOGNIZE *SAMPLE SPACE*:

- What are all possible outcomes of an event called?
- Can you describe what it means to determine the sample space for an event?

Determine if the student can REPRESENT THE SAMPLE SPACE OF A SIMPLE EVENT:

- What are the possible outcomes when you roll one number cube?
- What is the sample space for rolling a single number cube?

Determine if the student can EXPLAIN *COMPOUND EVENT*:

- When you consider the scenario of rolling one number cube, is this a simple event or a compound event? Explain.
- When you consider the scenario of rolling two or more number cubes, is this a simple event or a compound event? Explain.
- If I were to roll a number cube and then flip a coin, is this a simple event or a compound event? Explain.
- Can you provide an example of a compound event? What makes it a compound event?

Determine if the student can REPRESENT THE SAMPLE SPACE OF A COMPOUND EVENT:

- Can you tell me all the possible outcomes of rolling two number cubes and subtracting their values?
- How is the sample space represented in the table?
- Could you represent all the outcomes of this game in a tree diagram?

Determine if the student can USE THE SAMPLE SPACE OF A COMPOUND EVENT TO DETERMINE PROBABILITY:

- Which outcomes in the sample space represent Player A winning?
- Which outcomes in the sample space represent Player B winning?
- Can you describe an event that is possible but very unlikely to occur?
- Can you describe an event that will likely occur but is not certain?
- What is the theoretical probability Player A will win the original game?
- What is the theoretical probability Player B will win the original game?
- What is the theoretical probability of a difference of one?
- What is the theoretical probability of a difference of three or four?
- What is the theoretical probability of a difference of at least two?
- What is the theoretical probability of a difference less than five?
- [Refer to Question 3.] How do you know this game is fair? Use probabilities to explain your thinking.

Verify that students correctly found the theoretical probability of Player A winning to be $\frac{24}{36}$ or $\frac{2}{3}$ and the theoretical probability of Player B winning to be $\frac{12}{36}$ or $\frac{1}{3}$.

Ask students to share their responses to Question 3. If possible, select students who have used different approaches to make the game fair.

Be sure to **establish** during the course of the conversation that this is a compound event because it is composed of two simple events, and that the events are independent of each other because the result of one roll does not have any impact on the result of the next roll.

Next, students will consider a scenario with spinners and the likelihood of winning various prizes.

Provide pairs of students with the second page of the INSTRUCTIONAL ACTIVITY SUPPLEMENT.

Read and **discuss** the scenario together.

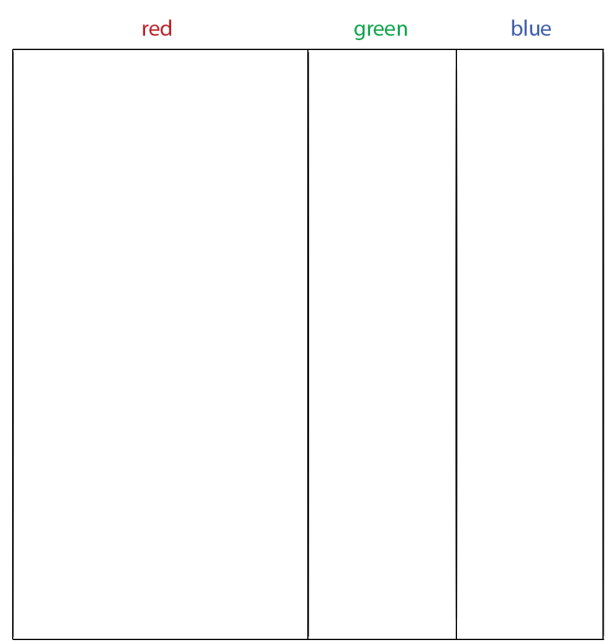
> A radio station is at a local store giving away prizes. They have set up a series of spinners to determine the prizes people will win, but there is a catch. You must correctly guess the prize before you spin the first spinner, then land on that prize in order to win. Which prize should you guess to have the best chance of winning?

Ask students which prize they think you are most likely to land on. While it is difficult to know the answer without creating a tree diagram, students can use probabilistic reasoning to support their predictions.

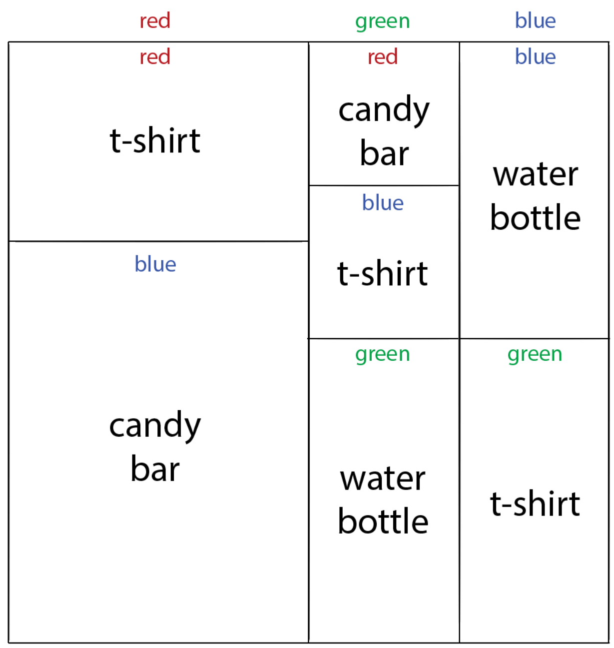
If time allows, students may benefit from considering an area model representing the spinner activity.

In addition to helping students visualize approximately how likely winning each prize is, area models incorporate fractional reasoning skills.

First, divide the space according to the probability of the first spinner.



Then, divide the outcomes of the first spin according to the probability of the second spinner, and label the prize won in each section.



Discuss in general what the area model shows about the probability of winning each prize, and **allow** students to revise their predictions based on the area model before conducting the experiment.

Pass out a paperclip and the third and fourth pages of the INSTRUCTIONAL ACTIVITY SUPPLEMENT to pairs of students.

The third page of the INSTRUCTIONAL ACTIVITY SUPPLEMENT contains the spinners necessary to play the game. Students should place a pencil inside the paperclip at the center of the spinner they are spinning in order to complete each spin.

The fourth page of the INSTRUCTIONAL ACTIVITY SUPPLEMENT contains a score card to keep track of how frequently each prize occurs.

Require students to play the game at least 25 times, recording who wins each game on the fourth page of the INSTRUCTIONAL ACTIVITY SUPPLEMENT.

Ask pairs of students to report how often the spinners resulted in each prize, and combine the data from several pairs so the class can begin to see trends in which prize occurs most often. Larger amounts of data should begin to show that the candy bar is the most likely prize, followed by the t-shirt, then the water bottle.

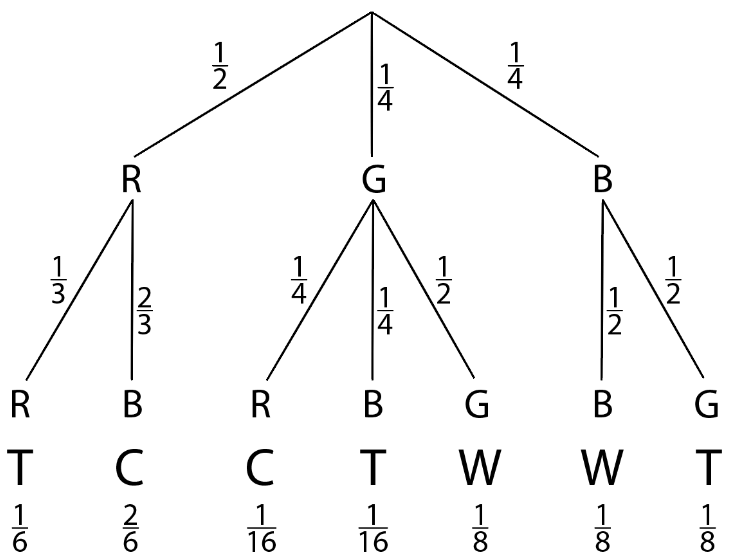
Compare the results of the experiment to students' predictions.

Discuss the trends students are seeing, then **require** students to create a tree diagram for the scenario to determine the theoretical probability of each prize occurring on the second page of the

INSTRUCTIONAL ACTIVITY STUDENT HANDOUT. Students should include the probabilities along each branch, then determine which prize is the result of the combination of spins and identify it at the end of each branch, along with its probability.

Remind students that the product of the probabilities on the branches is the probability of the outcome at the end of the branch.

The following is an example tree diagram with probabilities, where R, G, and B stand for red, green, and blue respectively, and C, T, and W stand for candy bar, t-shirt, and water bottle respectively.



Once students have created a tree diagram with probabilities, students should determine the probability of spinning and landing on each prize on the INSTRUCTIONAL ACTIVITY STUDENT HANDOUT (candy bar: $\frac{19}{48}$, t-shirt: $\frac{17}{48}$, water bottle: $\frac{12}{48}$, $\frac{2}{8}$, or $\frac{1}{4}$).

GUIDING QUESTIONS

Elicit student thinking:

- What would your strategy be for this game?
- Which prize seems most likely? Why?

Determine if the student can REPRESENT THE SAMPLE SPACE OF A SIMPLE EVENT:

- What are the possible outcomes for spinning this spinner?
- What is the sample space for this spinner?

Determine if the student can EXPLAIN *COMPOUND EVENT*:

- When you consider the scenario of spinning one spinner, is this a simple event or a compound event? Explain.
- When you consider the scenario of spinning more than one spinner, is this a simple event or a compound event? Explain.
- If you were to roll a number cube and then spin a spinner, is this a simple event or a compound event? Explain.
- Can you provide an example of a compound event? What makes it a compound event?

Determine if the student can REPRESENT THE SAMPLE SPACE OF A COMPOUND EVENT:

- How would you represent the possible outcomes of this game in a tree diagram?
- What information does this tree diagram provide?

Determine if the student can DETERMINE THE PROBABILITY OF EACH SIMPLE EVENT WITHIN A COMPOUND EVENT:

- How does the result of the first spin impact the next spin?
- What is the theoretical probability of spinning red on the first spinner? If you spin red first, what are the theoretical probabilities for the next spinner?
- What is the theoretical probability of spinning blue on the first spinner? If you spin blue first, what are the theoretical probabilities for the next spinner?
- What is the theoretical probability of spinning green on the first spinner? If you spin green first, what are the theoretical probabilities for the next spinner?

Determine if the student can USE MULTIPLICATION TO DETERMINE PROBABILITY OF COMPOUND EVENTS:

- What is the theoretical probability of winning a candy bar? How did you determine this value?
- What is the theoretical probability of winning a t-shirt? How did you determine this value?
- What is the theoretical probability of winning a water bottle? How did you determine this value?

At the end of the activity, teachers should require students to create a fair game with at least two events (e.g., rolling a number cube and spinning a spinner). Students should provide the sample space and theoretical probabilities to prove their game is fair.

Name________________________________

PROBABILITIES OF COMPOUND EVENTS

Lesson 2

WHAT'S THE DIFFERENCE?: POSSIBLE OUTCOMES

Use the following table to determine the difference for each combination of two rolls of a standard, six-sided number cube, then answer the questions below.

		Result of Second Roll					
		1	**2**	**3**	**4**	**5**	**6**
Result of First Roll	**1**						
	2						
	3						
	4						
	5						
	6						

1. If Player A wins when the difference is 0, 1, or 2, and Player B wins when the difference is 3, 4, or 5, is this a fair game? Explain your reasoning.

Name______________________________

2. What is the theoretical probability that Player A will win using these rules? What is the theoretical probability Player B will win using these rules?

3. Describe how the rules could be changed so that the game is fair. Provide theoretical probabilities to support your response.

Name________________________________

SPIN AND WIN: POSSIBLE OUTCOMES

In the following space, create a tree diagram to model the possible outcomes for the “Spin and Win” game. Include the theoretical probabilities on each branch and write the prize won at the bottom of each branch, along with its probability.

SPIN AND WIN: TREE DIAGRAM

Determine the theoretical probability of spinning and landing on each of the following prizes:

- Candy bar

- T-shirt

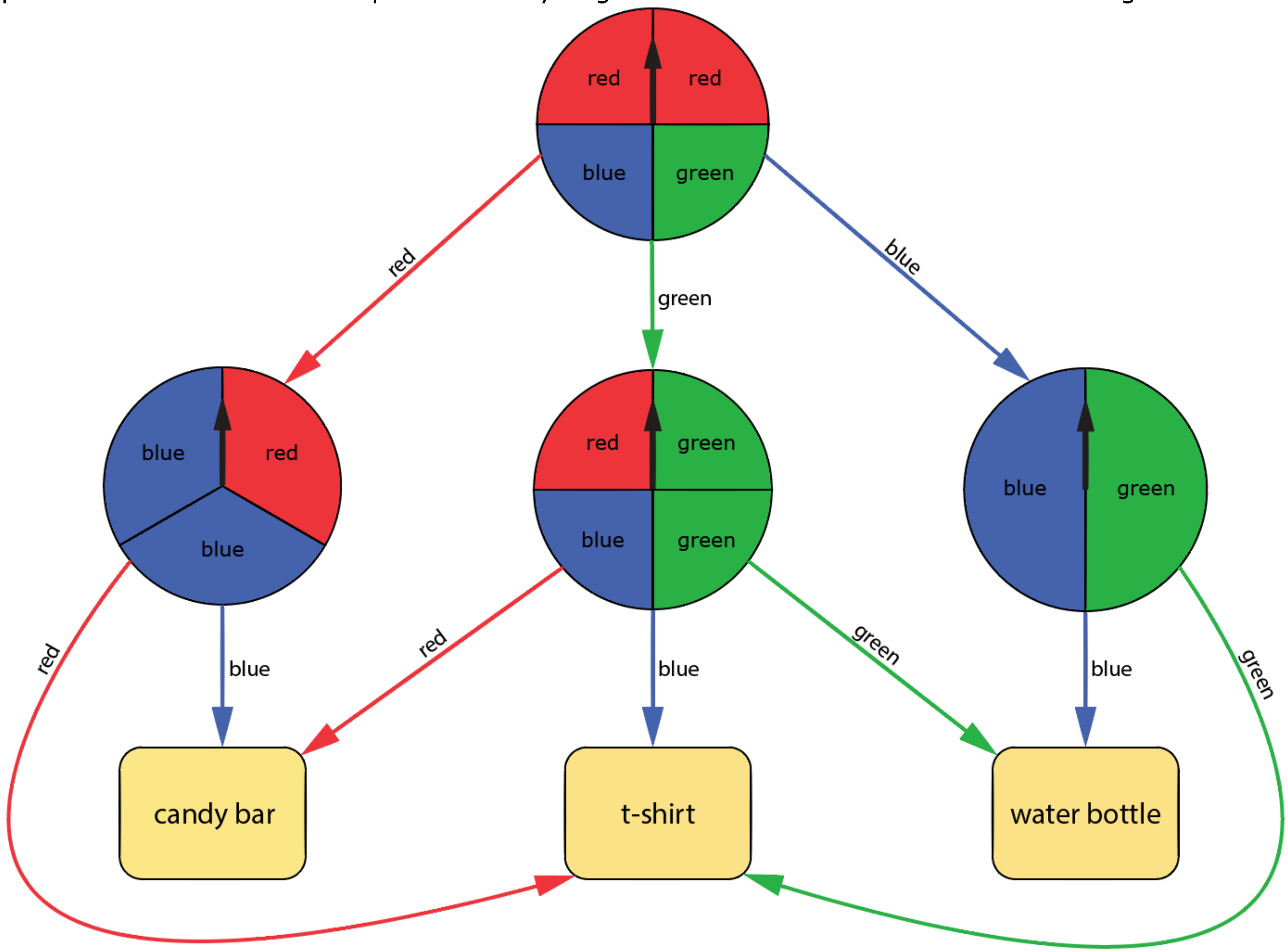
- Water bottle

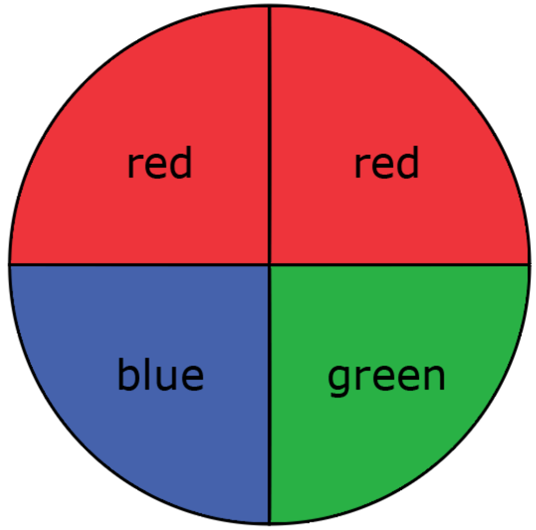
WHAT'S THE DIFFERENCE?: SCORE SHEET

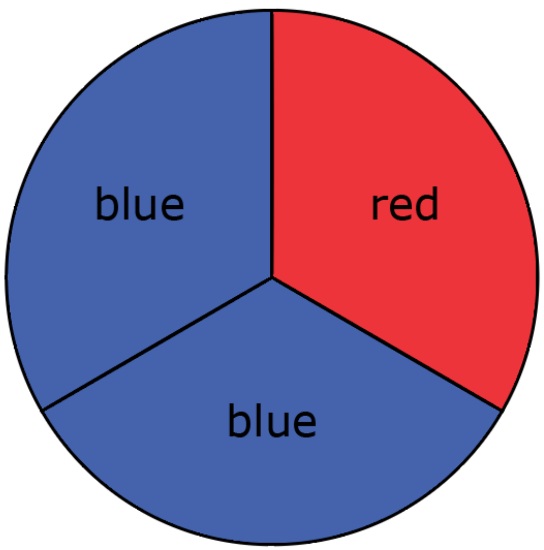
NUMBER OF WINS (TALLY)	
Player A (Difference of 0, 1, or 2)	**Player B** (Difference of 3, 4, or 5)

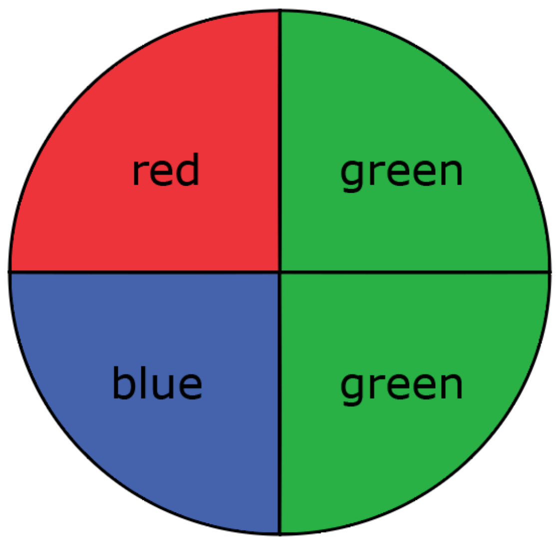
SPIN AND WIN: GAME BOARD

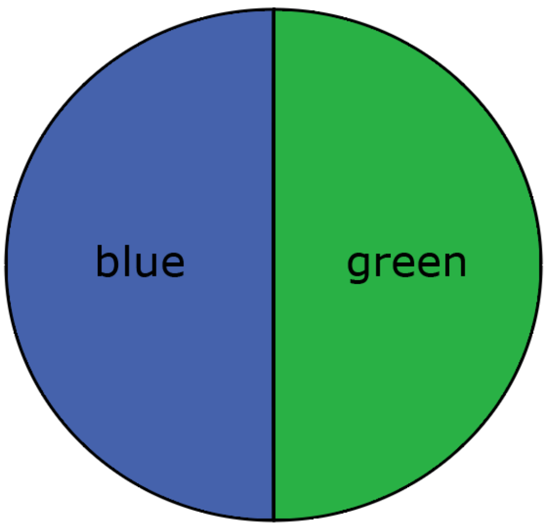
A radio station is at a local store giving away prizes. They have set up a series of spinners to determine the prizes people will win, but there is a catch. Before you spin the first spinner, you must correctly guess the prize, then land on that prize in order to win. Which prize should you guess to have the best chance of winning?




red
red
blue
green


blue
red
blue


red
green
blue
green


blue
green

PRIZE OUTCOMES (TALLY)		
candy bar	**t-shirt**	**water bottle**

Probabilities of Compound Events

INSTRUCTIONAL ACTIVITY

Lesson 3

LEARNING GOAL

Students will learn how to design, conduct, and use simulations to estimate probabilities, then compare the simulation results to the theoretical probability of various events.

PRIMARY ACTIVITY

Students will use random number generators, coins, spinners, and/or number cubes to simulate a variety of compound events. Initial simulations will be guided in order to prepare students to design and conduct their own simulation. The results of each simulation will then be compared to the theoretical probability of the compound event.

OTHER VOCABULARY

Students will need to know the meaning of the following terms:

- Compound event
- Experiment
- Outcome
- Random number
- Simulation
- Experimental probability
- Theoretical probability

MATERIALS

- Coins or two-color counters
- Standard, six-sided number cubes
- Variety of spinners or materials to create spinners (paper, compasses, straightedges, markers/crayons/colored pencils, paperclips)
- Internet access for teacher (optional for students)
- Instructional Activity Student Handout

IMPLEMENTATION

This lesson begins with a simulation using a random number generator, for which students will determine experimental and theoretical probabilities.

Ask the class a yes or no question for which you are reasonably certain some students will respond with “yes” and other students will respond with “no”. The question can be teacher generated or student generated. The example provided in this lesson will use the question, “Do you have a pet?”

Record each student’s response with his or her name so theoretical probabilities can be determined later and to ensure students respond consistently throughout the activity.

Generate a question about the data that has been collected. An example of a question which will be used in this lesson is, “What is the likelihood that all three randomly selected students have a pet?”

Assign each student a unique number, preferably consecutive numbers beginning with one.

Use a random number generator (e.g., www.random.org) or a calculator to generate random numbers to select the specified students to respond to the question.

NOTE: To generate random numbers on a TI-30X, press “PRB”, then arrow to the right and select “RANDI”. In parentheses, enter the range of values you want to generate separated by a comma (e.g., RANDI(1,30) will generate random numbers from one to 30).

In the example provided, the random number generator would provide a number, the teacher would ask that student whether they have a pet (or refer to the student’s original response), record the answer, then repeat two more times for two additional, randomly selected students.

After the necessary number of responses (three in this example), **determine** the answer to the question (whether all three randomly selected students have a pet in this example).

The following table is an example of how the data from the trials could be organized.

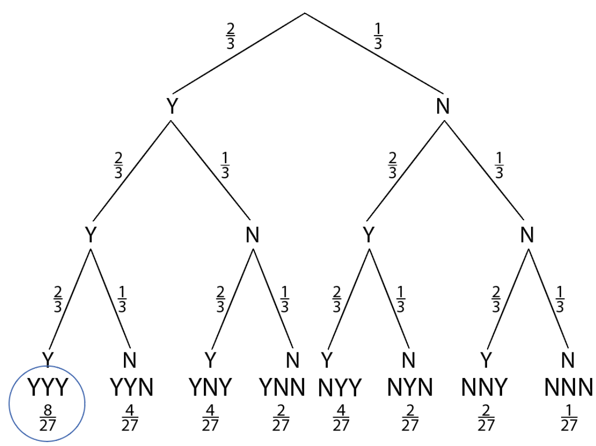
Random Numbers Generated	Has a pet?	All three students have a pet?
4, 9, 1	Y, N, Y	no
33, 12, 16	Y, Y, Y	yes
1, 8, 20	Y, Y, N	no
4, 5, 15	Y, N, Y	no
10, 3, 4	N, N, Y	no

It is recommended that at least 30 trials are conducted before determining the experimental probability. (Based on the trials in the experiment to this point, the experimental probability that all three randomly selected students have a pet is $\frac{1}{5}$.)

Once the experimental probability has been determined for the question, use the original class data to determine the theoretical probability that a randomly selected student responded to the question with "yes" and the probability that a randomly selected student responded to the question with "no". For example, if there were 24 students who responded "yes" and 12 students who responded "no", the theoretical probability of randomly selecting a student who answers "yes" is $\frac{24}{36}$ or $\frac{2}{3}$, and the theoretical probability of randomly selecting a student who answers "no" is $\frac{12}{36}$ or $\frac{1}{3}$.

Discuss whether the random selection of three students is an independent or dependent event. Because the random number generator can generate the same number twice, this is an independent event. Each student has an equal chance of being selected each time a random number is generated.

Create a tree diagram to determine the theoretical probability that all three randomly selected students have a pet. The following tree diagram is a model based on the hypothetical responses in this scenario, where Y and N represent yes and no responses respectively.



Compare the theoretical and experimental probabilities and **discuss** reasons why the experimental probability differs from the theoretical probability.

For more probability practice, **ask** additional questions about theoretical probabilities based on the tree diagram. For example, "What is the theoretical probability at least two randomly selected students have pets?"

GUIDING QUESTIONS

Elicit student thinking:

- What responses are possible if we ask one person this question?
- If we ask multiple people, how could each person answer?
- Does the first person's response impact the next person's response?

Determine if the student can DESIGN A SIMULATION TO GENERATE FREQUENCIES FOR COMPOUND EVENTS:

- How is the random number generator involved in this simulation?
- Why do we need a random number generator?
- How many people will be asked the question in each trial? Why?
- What will we look for in the responses?

Determine if the student can CONDUCT A SIMULATION TO GENERATE FREQUENCIES FOR COMPOUND EVENTS:

- What did we do to generate each response?
- How many people must be asked the question for each trial?
- What do the responses Y, Y, N (yes, yes, no) mean in this context?

Determine if the student can USE A SIMULATION TO DETERMINE PROBABILITY FOR COMPOUND EVENTS:

- Which outcome in the sample space represents that all three randomly selected students have a pet?
- Based on this simulation, what is the probability that all three randomly selected students have a pet?
- Based on this simulation, what is the probability that at least two randomly selected students do not have a pet?

Determine if the student can EXPLAIN 2 OR MORE REAL-WORLD EVENTS AS INDEPENDENT OR DEPENDENT:

- Does the first person's response impact the second person's response? Why or why not?
- If the first two people answer "yes", is the third person more likely to answer "no"? Why or why not?

Determine if the student can DETERMINE THE PROBABILITY OF EACH SIMPLE EVENT WITHIN A COMPOUND EVENT:

- Can you create a tree diagram including the theoretical probabilities for each event?

Determine if the student can USE MULTIPLICATION TO DETERMINE PROBABILITY OF COMPOUND EVENTS:

- What is the theoretical probability that all three randomly selected students have a pet?
- What is the theoretical probability that at least two randomly selected students do not have a pet?

Next, students will help design a simulation to model having a child (boy or girl) with a particular eye color (brown, green, blue, or hazel).

Discuss first the possibilities when having a child (boy or girl) and ask students to think about how they could simulate this event in pairs. **Note** that for the purpose of the simulation, having a boy or a girl is equally likely.

Allow students time to consider the question and, ideally, come up with more than one way to simulate this event (e.g., coin toss, spinner, evens vs. odds on a number cube, random number generator, drawing items out of a bag, etc.).

Ask pairs of students for ideas and discuss how each idea presented would simulate the event.

Select one of the methods presented to use as a class. (This choice may be impacted by the materials available in your classroom.)

Next, **discuss** the eye color possibilities (brown, green, blue, and hazel). For the purpose of the simulation, provide the following hypothetical probabilities for eye color.

Brown: $\frac{1}{3}$ Green: $\frac{1}{3}$ Blue: $\frac{1}{6}$ Hazel: $\frac{1}{6}$

Ask students whether a coin toss would make sense for this simulation. Students should realize that a coin only has two sides, which is not adequate for this simulation.

Allow students time to consider the scenario and, ideally, come up with more than one way to simulate this event (e.g., spinner, numbers on a number cube, random number generator, drawing items out of a bag, etc.).

Ask pairs of students for ideas and discuss how each idea presented would simulate the event.

NOTE: It is important for students to assign the outcomes appropriately based on the theoretical probabilities provided. For example, on a spinner, one third would represent brown eyes, one third would represent green eyes, and the remaining third would be split in half such that one sixth of the spinner represents blue eyes and one sixth of the spinner represents hazel eyes. Alternatively, on a standard six-sided number cube, numbers one and two might represent brown eyes, three and four might represent green eyes, five might represent blue eyes, and six might represent hazel eyes.

Select one of the methods presented to use as a class. (This choice may be impacted by the materials available in your classroom.)

Require pairs of students to conduct the simulation at least 30 times and record their results in the INSTRUCTIONAL ACTIVITY STUDENT HANDOUT.

Once students have completed at least 30 simulations, **require** that they answer Questions 1 – 3 in the INSTRUCTIONAL ACTIVITY STUDENT HANDOUT using the data they collected.

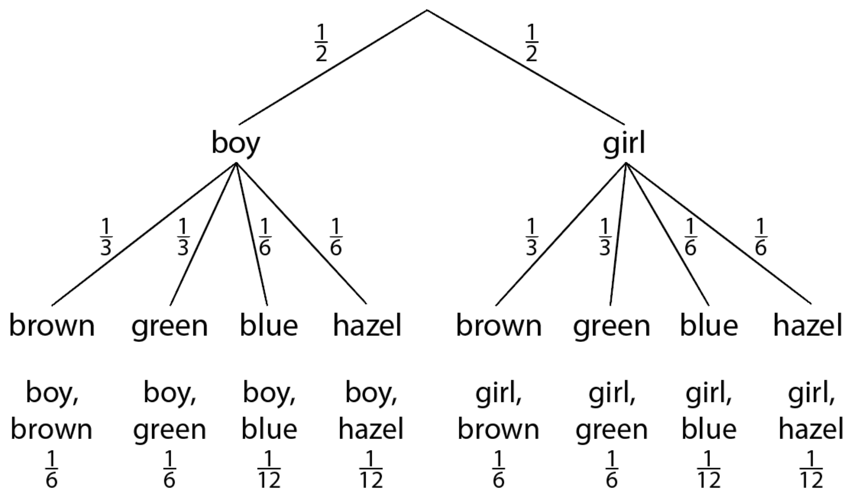
Discuss the experimental probabilities students arrived at in Questions 1 – 3 in the INSTRUCTIONAL ACTIVITY STUDENT HANDOUT.

Ask students whether the events in the simulation are independent or dependent. Students should come to the consensus that the events are independent, because the result of the first event does not impact the probabilities of the second (having a boy or girl does not make any of the eye colors more or less likely).

Note that different pairs will have different responses for each question based on the different outcomes of the simulations they conducted, though their method for determining the experimental probabilities should be consistent across pairs.

Next, **require** students to complete Questions 4 – 7 in the INSTRUCTIONAL ACTIVITY STUDENT HANDOUT. Question 4 reminds students of the theoretical probabilities of each event and asks them to create a tree diagram with probabilities to model the compound event. Questions 5 – 7 ask for the theoretical probabilities of the events students determined experimental probabilities for in Questions 1 – 3.

The following image is an example of a tree diagram for the compound event, similar to what students should create for Question 4 and correct probabilities for Questions 5 – 7 (explanations will vary).



- 5. $\frac{6}{12}$ or $\frac{1}{2}$
- 6. $\frac{3}{12}$ or $\frac{1}{4}$
- 7. $\frac{2}{6}$ or $\frac{1}{3}$

Ask guiding questions to check for student understanding as they work, then **review** students' responses as a class to ensure all students' understanding.

GUIDING QUESTIONS

Elicit student thinking:

- Can you describe this scenario in your own words?
- What could happen in this situation?
- Does it matter which event you consider first, gender or eye color? Why?

Determine if the student can DESIGN A SIMULATION TO GENERATE FREQUENCIES FOR COMPOUND EVENTS:

- What will you use to determine the gender of the baby in this simulation? Why did you choose this, and how do you know it accurately reflects the theoretical probabilities?
- What will you use to determine the eye color of the baby in this simulation? Why did you choose this and how do you know it accurately reflects the theoretical probabilities?

Determine if the student can CONDUCT A SIMULATION TO GENERATE FREQUENCIES FOR COMPOUND EVENTS:

- How did you carry out this simulation?
- What did you do to generate each event?
- What are you using to represent gender? What are you using to represent eye color?
- What does this result mean in the context of the problem?

Determine if the student can USE A SIMULATION TO DETERMINE PROBABILITY FOR COMPOUND EVENTS:

- Which outcome in the sample space represents a boy with blue eyes?
- Which outcomes in the sample space represent a girl with green or brown eyes?
- Based on this simulation, what is the probability of having a girl with any eye color?
- Based on this simulation, what is the probability of having a boy with blue or hazel eyes?
- Based on this simulation, what is the probability of having a child (girl or boy) with green eyes?

Determine if the student can DETERMINE THE PROBABILITY OF EACH SIMPLE EVENT WITHIN A COMPOUND EVENT:

- Can you create a tree diagram including the theoretical probabilities for each event?
- Does the order in which you create your tree diagram (gender then eye color versus eye color then gender) impact the probabilities for each simple event? Why or why not?

Determine if the student can USE MULTIPLICATION TO DETERMINE PROBABILITY OF COMPOUND EVENTS:

- What is the theoretical probability of having a girl with any eye color?
- What is the theoretical probability of having a boy with blue or hazel eyes?
- What is the theoretical probability of having a child (girl or boy) with green eyes?

Finally, students will design and conduct their own simulation for the scenario provided on Question 8 in the INSTRUCTIONAL ACTIVITY STUDENT HANDOUT.

> A cereal company puts one toy in each of their cereal boxes. There are three different toys and each toy is equally likely in each cereal box. What is the experimental probability that you get one of each toy after buying exactly three boxes of cereal?

Using any of the materials available in class, students should design and conduct their own simulation.

Require that students explain how they conducted the simulation on Question 9 in the INSTRUCTIONAL ACTIVITY STUDENT HANDOUT.

Ask the guiding questions provided to check student understanding as they work.

GUIDING QUESTIONS

Elicit student thinking:

- Can you describe the situation in your own words?
- What is the question asking?
- Does it matter what order you get the toys?

Determine if the student can DESIGN A SIMULATION TO GENERATE FREQUENCIES FOR COMPOUND EVENTS:

- How could you model this situation?
- How many outcomes do you need to consider? What can you use to represent these outcomes?
- What outcome(s) will represent the first toy? The second toy? The third toy?
- How do you make sure the theoretical probabilities are consistent from the problem situation to the simulation?

Determine if the student can CONDUCT A SIMULATION TO GENERATE FREQUENCIES FOR COMPOUND EVENTS:

- Which toy does this result represent?
- Why do you need three outcomes for each trial in this situation?
- Did you get all three toys in this trial? How do you know?

Determine if the student can USE A SIMULATION TO DETERMINE PROBABILITY FOR COMPOUND EVENTS:

- Which outcomes in the sample space represent getting one of each toy?
- Based on this simulation, what is the probability of getting one of each toy when you buy three boxes of cereal? How did you determine this probability?
- Based on this simulation, what is the probability of getting three of the same toys when you buy three boxes of cereal? How did you determine this probability?

At the end of the activity, require that students construct a tree diagram and determine the theoretical probability of getting one of each toy after buying exactly three boxes of cereal on Question 10 of the INSTRUCTIONAL ACTIVITY STUDENT HANDOUT.

Name________________________________

Probabilities of Compound Events

Lesson 3

TRIAL	OUTCOME (GENDER, EYE COLOR)
1	
2	
3	
4	
5	
6	
7	
8	
9	
10	
11	
12	
13	
14	
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16	
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18	
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20	
21	
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23	
24	
25	
26	
27	
28	
29	
30	

Name____________________________

1. What is the experimental probability of having a boy with any eye color? Explain how you arrived at your answer.

2. What is the experimental probability of having a girl with blue or green eyes? Explain how you arrived at your answer.

3. What is the experimental probability of having a child (girl or boy) with brown eyes? Explain how you arrived at your answer.

Name______________________________

4. Recall that for the purposes of this simulation, the theoretical probability of having a boy is $\frac{1}{2}$, the theoretical probability of having a girl is $\frac{1}{2}$, the theoretical probability of having a child with brown eyes is $\frac{1}{3}$, the theoretical probability of having a child with green eyes is $\frac{1}{3}$, the theoretical probability of having a child with blue eyes is $\frac{1}{6}$, and the theoretical probability of having a child with hazel eyes is $\frac{1}{6}$. Create a tree diagram with theoretical probabilities for the compound event of a child's gender and eye color.

5. What is the theoretical probability of having a boy with any eye color? Explain how you arrived at your answer.

Name____________________________

6. What is the theoretical probability of having a girl with blue or green eyes? Explain how you arrived at your answer.

7. What is the theoretical probability of having a child (girl or boy) with brown eyes? Explain how you arrived at your answer.

8. Design and conduct a simulation to model the following scenario and answer this question:

> A cereal company puts one toy in each of their cereal boxes. There are three different toys and each toy is equally likely in each cereal box. What is the experimental probability that you get one of each toy after buying exactly three boxes of cereal?

Name________________________________

TRIAL	OUTCOME (3 TOYS)
1	
2	
3	
4	
5	
6	
7	
8	
9	
10	
11	
12	
13	
14	
15	
16	
17	
18	
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23	
24	
25	
26	
27	
28	
29	
30	

Experimental probability of getting one of each toy if you buy exactly three boxes of cereal: ___________

Name____________________________

9. Describe how you conducted the simulation in Question 8. What materials did you use? How did you assign outcomes to specific events?

10. Create a tree diagram to determine the theoretical probability of getting one of each toy if you buy exactly three boxes of cereal.

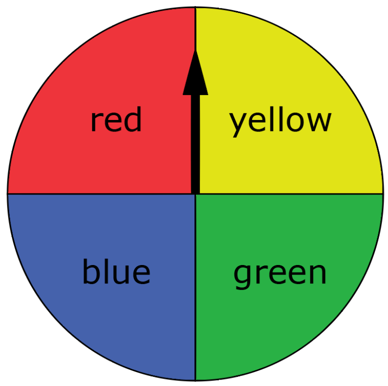
Theoretical probability of getting one of each toy if you buy exactly three boxes of cereal: __________

Name____

Probabilities of Compound Events

Lessons 1 – 3

1. Consider flipping a coin (with heads on one side, tails on the other) and spinning the following spinner.



1.a. Are these independent or dependent events? Explain your reasoning.

1.b. Create a tree diagram with theoretical probabilities along the branches to model flipping the coin then spinning the spinner.

Name________________________________

1.c. What is the sample space for this compound event?

1.d. What is the theoretical probability of flipping heads and spinning blue? Explain how you determined your answer.

1.e. What is the theoretical probability of flipping tails and spinning either yellow or green? Explain how you determined your answer.

2. Consider a bag with four blue candies, four red candies, and two purple candies, where one candy will be drawn and kept out of the bag, then a second candy will be drawn.

   2.a. Are these independent or dependent events? Explain your reasoning.

Name____

2.b. Create a tree diagram with theoretical probabilities along the branches to model selecting two candies from the bag if you <u>do not replace</u> the first candy that is drawn.

2.c. What is the sample space for this compound event?

2.d. Which outcome(s) in the sample space represents drawing two blue candies? What is the theoretical probability of drawing two blue candies? Explain how you determined your answer.

Name____

2.e. Which outcome(s) in the sample space represents drawing one red candy and one purple candy in any order? What is the theoretical probability of drawing one red candy and one purple candy in any order? Explain how you determined your answer.

3. You are going to play a game with your friend where you roll two standard, six-sided number cubes and determine the sum. Player A wins if the sum is 2, 3, 4, 10, 11, or 12. Player B wins if the sum is 5, 6, 7, 8, or 9. Is this a fair game? Justify your reasoning by showing your work and providing theoretical probabilities.

Name____

4. Your state recently reported that only two out of five people pass the written driving test on their first attempt. In the following questions, you will conduct a simulation and determine the likelihood that three randomly selected people who have taken the written driving test failed the test on their first attempt.

   4.a. Describe the design of a simulation that you can conduct to model the described scenario. Be sure to include what each outcome of the simulation represents in the context of the problem.

   4.b. Conduct 30 trials of the simulation and record the results of each trial in the following table.

TRIAL	OUTCOME	TRIAL	OUTCOME
1		16	
2		17	
3		18	
4		19	
5		20	
6		21	
7		22	
8		23	
9		24	
10		25	
11		26	
12		27	
13		28	
14		29	
15		30	

4.c. What outcome in the sample space represents that all three randomly selected people failed on their first attempt?

4.d. What is the experimental probability that all three randomly selected people who have taken the written test failed on their first attempt? Explain how you determined your answer.

4.e. Create a tree diagram with theoretical probabilities along the branches to model asking three people who have taken the written driving test whether they passed or failed on their first attempt.

4.f. What is the theoretical probability that all three randomly selected people who have taken the written test failed on their first attempt?

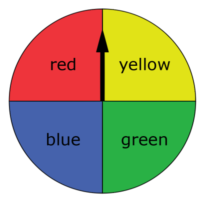
4.g. How does the theoretical probability compare to the experimental probability? Explain why you believe this is the case.

Probabilities of Compound Events

STUDENT ACTIVITY SOLUTION GUIDE

Lessons 1 – 3

1. Consider flipping a coin (with heads on one side, tails on the other) and spinning the following spinner.



1.a. Are these independent or dependent events? Explain your reasoning.

CORRECT ANSWER

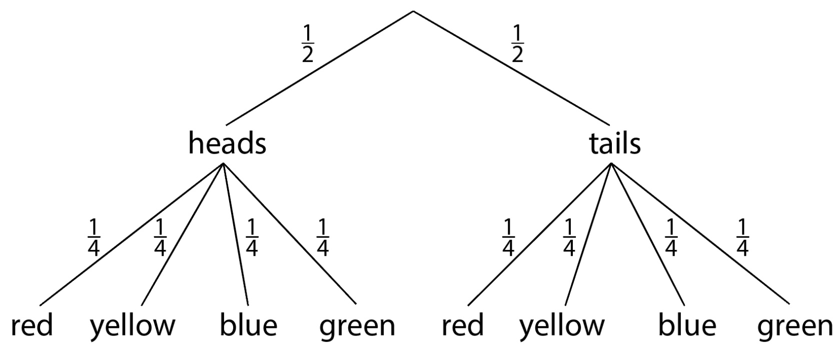
> These are independent events because the result of the coin flip does not impact the probabilities of the second event, spinning the spinner.

ERRORS, MISCONCEPTIONS, AND MISSING KNOWLEDGE

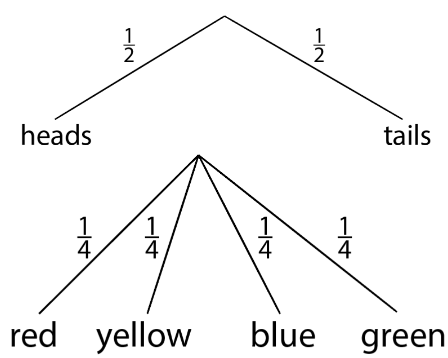
Example Error	Misconception	Missing Knowledge
These are dependent events because you flip the coin first, then spin the spinner.	does not consider whether probabilities of one event are affected by outcomes of the other, only identifies the events are related	EXPLAIN 2 OR MORE REAL-WORLD EVENTS AS INDEPENDENT OR DEPENDENT
These are independent events.	cannot explain what makes these independent events	EXPLAIN 2 OR MORE REAL-WORLD EVENTS AS INDEPENDENT OR DEPENDENT

1.b. Create a tree diagram with theoretical probabilities along the branches to model flipping the coin then spinning the spinner.

CORRECT ANSWER



ERRORS, MISCONCEPTIONS, AND MISSING KNOWLEDGE

Example Error	Misconception	Missing Knowledge
Incorrectly labels the probabilities along the branches. For example, labels all probabilities $\frac{1}{2}$, labels all probabilities $\frac{1}{4}$, or labels the probabilities of heads and tails $\frac{1}{4}$ and the probabiliites of red, yellow, blue, and green $\frac{1}{2}$.	does not consider that the first event and the second event have different theoretical probabilities or confuses the theoretical probabilities for the two events	DETERMINE THEORETICAL PROBABILITY OF A SIMPLE EVENT WHEN ALL OUTCOMES ARE EQUALLY LIKELY and DETERMINE THE PROBABILITY OF EACH SIMPLE EVENT WITHIN A COMPOUND EVENT
	represents the tree diagram with probabilities for each simple event separately, but does not represent how the events are related	EXPLAIN *COMPOUND EVENT*

1.c. What is the sample space for this compound event?

CORRECT ANSWER

> Using H and T for heads and tails respectively and R, Y, G, and B for red, yellow, green, and blue respectively, the sample space for this compound event is S = {HR, HY, HB, HG, TR, TY, TB, TG}.

ERRORS, MISCONCEPTIONS, AND MISSING KNOWLEDGE

Example Error	Misconception	Missing Knowledge
S = {H, T, R, Y, B, G} or S = {H, T} and S = {R, Y, B, G}	lists the outcomes of each simple event separately rather than the compound event	EXPLAIN *COMPOUND EVENT* and REPRESENT THE SAMPLE SPACE OF A COMPOUND EVENT
S = {R, Y, B, G, R, Y, B, G}	lists the outcomes of the second event from the bottom of the tree diagram without including the outcomes of the preceding event	EXPLAIN *COMPOUND EVENT* and REPRESENT THE SAMPLE SPACE OF A COMPOUND EVENT

1.d. What is the theoretical probability of flipping heads and spinning blue? Explain how you determined your answer.

CORRECT ANSWER

> The theoretical probability of flipping heads and spinning blue is $\frac{1}{8}$ because the theoretical probability of flipping heads is $\frac{1}{2}$, the theoretical probability of spinning blue is $\frac{1}{4}$, and the product of $\frac{1}{2}$ and $\frac{1}{4}$ is $\frac{1}{8}$.
>
> *OR*
>
> The theoretical probability of flipping heads and spinning blue is $\frac{1}{8}$ because each outcome in the sample space is equally likely. One out of eight outcomes is flipping heads and spinning blue, therefore the theoretical probability is $\frac{1}{8}$.

ERRORS, MISCONCEPTIONS, AND MISSING KNOWLEDGE

Example Error	Misconception	Missing Knowledge
$\frac{1}{2}$ and $\frac{1}{4}$	gives the probability of flipping heads, then gives the probability of spinning blue, but cannot determine the probability of the compound event	USE THE SAMPLE SPACE OF A COMPOUND EVENT TO DETERMINE PROBABILITY or USE MULTIPLICATION TO DETERMINE PROBABILITY OF COMPOUND EVENTS
$\frac{3}{4}$	adds the probabilities of the flipping heads and spinning blue when they should multiply	USE MULTIPLICATION TO DETERMINE PROBABILITY OF COMPOUND EVENTS

1.e. What is the theoretical probability of flipping tails and spinning either yellow or green? Explain how you determined your answer.

CORRECT ANSWER

The theoretical probability of flipping tails and spinning either yellow or green is $\frac{2}{8}$ or $\frac{1}{4}$, because the theoretical probability of flipping tails and spinning yellow is $\frac{1}{8}$, the theoretical probability flipping tails and spinning green is $\frac{1}{8}$, and the sum of $\frac{1}{8}$ and $\frac{1}{8}$ is $\frac{2}{8}$ or $\frac{1}{4}$.

OR

The theoretical probability of flipping heads and spinning blue is $\frac{2}{8}$ or $\frac{1}{4}$, because each outcome in the sample space is equally likely. Two out of eight outcomes are flipping tails and spinning yellow or flipping tails an spinning green, therefore the theoretical probability is $\frac{2}{8}$ or $\frac{1}{4}$.

ERRORS, MISCONCEPTIONS, AND MISSING KNOWLEDGE

Example Error	Misconception	Missing Knowledge
$\frac{1}{2}$ and $\frac{2}{4}$	gives the probability of flipping tails, then gives the probability of spinning yellow or green, but cannot determine the probability of the compound event	USE THE SAMPLE SPACE OF A COMPOUND EVENT TO DETERMINE PROBABILITY or USE MULTIPLICATION TO DETERMINE PROBABILITY OF COMPOUND EVENTS
$\frac{1}{64}$	multiplies $\frac{1}{8}$ and $\frac{1}{8}$ instead of adding to determine the probability of either event occurring	USE THE SAMPLE SPACE OF A COMPOUND EVENT TO DETERMINE PROBABILITY
$\frac{4}{4}$ or 1	adds the probabilities of the flipping tails and spinning yellow or green when they should multiply	USE MULTIPLICATION TO DETERMINE PROBABILITY OF COMPOUND EVENTS

2. Consider a bag with four blue candies, four red candies, and two purple candies, where one candy will be drawn and kept out of the bag, then a second candy will be drawn.

   2.a. Are these independent or dependent events? Explain your reasoning.

CORRECT ANSWER

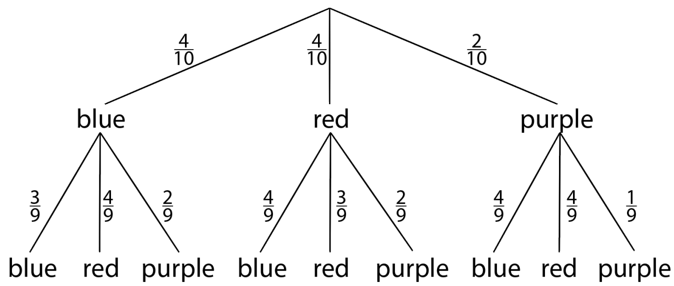
> These are dependent events because the result of the first draw impacts the probabilities of the second draw. For the first draw, there are 10 candies total, but for the second draw, there are only nine candies to choose from. Depending which color is drawn first, there will be one less of that color for the second draw.

ERRORS, MISCONCEPTIONS, AND MISSING KNOWLEDGE

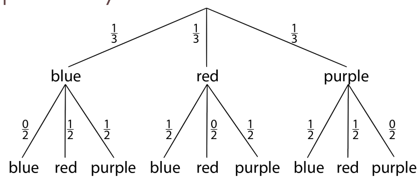
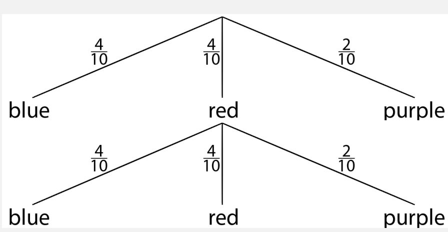
Example Error	Misconception	Missing Knowledge
These are independent events because you draw one candy first, then draw a differerent (separate) candy.	does not consider whether probabilities of one event are affected by outcomes of the other, only identifying that the events occur separately	EXPLAIN 2 OR MORE REAL-WORLD EVENTS AS INDEPENDENT OR DEPENDENT
These are dependent events.	cannot explain what makes these dependent events	EXPLAIN 2 OR MORE REAL-WORLD EVENTS AS INDEPENDENT OR DEPENDENT

2.b. Create a tree diagram with theoretical probabilities along the branches to model selecting two candies from the bag if you do not replace the first candy that is drawn.

CORRECT ANSWER



ERRORS, MISCONCEPTIONS, AND MISSING KNOWLEDGE

Example Error	Misconception	Missing Knowledge
Student incorrectly labels the probabilities along each branch $\frac{1}{3}$.	identifies the probability as one third because there are three possible outcomes for each draw; does not consider the number of each color candy in the bag and how that impacts the probability	DETERMINE THEORETICAL PROBABILITY OF A SIMPLE EVENT WHERE SOME OUTCOMES ARE MORE LIKELY THAN OTHERS and DETERMINE THE PROBABILITY OF EACH SIMPLE EVENT WITHIN A COMPOUND EVENT
Student incorrectly labels the probabilities of the first event in the tree diagram $\frac{1}{3}$, then uses conditional probabilistic reasoning when determining the probability of the second event. 	determines probabilities as though there is one of each color candy in the bag	DETERMINE THEORETICAL PROBABILITY OF A SIMPLE EVENT WHERE SOME OUTCOMES ARE MORE LIKELY THAN OTHERS
	represents the tree diagram for each simple event separately, but does not represent how the events are related or how the first event impacts the second	EXPLAIN 2 OR MORE REAL-WORLD EVENTS AS INDEPENDENT OR DEPENDENT, EXPLAIN *COMPOUND EVENT*, and REPRESENT THE SAMPLE SPACE OF A COMPOUND EVENT

2.c. What is the sample space for this compound event?

CORRECT ANSWER

Using B, R, and P for blue, red, and purple respectively, the sample space for this compound event is S = {BB, BR, BP, RB, RR, RP, PB, PR, PP}.

ERRORS, MISCONCEPTIONS, AND MISSING KNOWLEDGE

Example Error	Misconception	Missing Knowledge
S = {B, R, P, B, R, P} or S = {B, R, P} and S = {B, R, P}	lists the outcomes of each simple event rather than the compound event	EXPLAIN *COMPOUND EVENT* and REPRESENT THE SAMPLE SPACE OF A COMPOUND EVENT
S = {B, R, P, B, R, P, B, R, P}	lists the outcomes of the second event from the bottom of the tree diagram without including the outcomes of the preceding event	EXPLAIN *COMPOUND EVENT* and REPRESENT THE SAMPLE SPACE OF A COMPOUND EVENT

2.d. Which outcome(s) in the sample space represents drawing two blue candies? What is the theoretical probability of drawing two blue candies? Explain how you determined your answer.

CORRECT ANSWER

The outcome in the sample space that represents drawing two blue candies is E = {BB}.

The theoretical probability of drawing two blue candies is $\frac{12}{90}$ or $\frac{2}{15}$ because the theoretical probability of drawing the first blue candy is $\frac{4}{10}$ (or $\frac{2}{5}$), the theoretical probability of drawing a second blue candy is $\frac{3}{9}$ (or $\frac{1}{3}$), and the product of $\frac{4}{10}$ and $\frac{3}{9}$ is $\frac{12}{90}$ or $\frac{2}{15}$.

ERRORS, MISCONCEPTIONS, AND MISSING KNOWLEDGE

Example Error	Misconception	Missing Knowledge
The theoretical probability is $\frac{16}{100}$ or $\frac{4}{25}$.	does not account for the decrease in blue candies and total candies after the first blue candy is drawn	EXPLAIN 2 OR MORE REAL-WORLD EVENTS AS INDEPENDENT OR DEPENDENT and DETERMINE THE PROBABILITY OF EACH SIMPLE EVENT WITHIN A COMPOUND EVENT
$\frac{4}{10}$ and $\frac{3}{9}$	gives the probability of drawing a blue candy, then gives the probability of drawing another blue candy, but cannot determine the probability of the compound event	USE MULTIPLICATION TO DETERMINE PROBABILITY OF COMPOUND EVENTS
$\frac{66}{90}$ or $\frac{11}{15}$	adds the probabilities of drawing a blue then drawing a second blue when they should multiply	USE MULTIPLICATION TO DETERMINE PROBABILITY OF COMPOUND EVENTS

2.e. Which outcome(s) in the sample space represents drawing one red candy and one purple candy in any order? What is the theoretical probability of drawing one red candy and one purple candy in any order? Explain how you determined your answer.

CORRECT ANSWER

The outcomes in the sample space that represent drawing a red candy and a purple candy in any order are $E = \{RP, PR\}$.

The theoretical probability of drawing one red candy and one purple candy in any order is $\frac{16}{90}$ or $\frac{8}{45}$ because the theoretical probability of drawing a red candy then a purple candy is $\frac{4}{10} \cdot \frac{2}{9}$, which equals $\frac{8}{90}$ or $\frac{4}{45}$, and the theoretical probability of drawing a purple candy then a red candy is $\frac{2}{10} \cdot \frac{4}{9}$, which also equals $\frac{8}{90}$ or $\frac{4}{45}$. The sum of of the probabilities of these two outcomes is $\frac{16}{90}$ or $\frac{8}{45}$.

ERRORS, MISCONCEPTIONS, AND MISSING KNOWLEDGE

Example Error	Misconception	Missing Knowledge
Student provides either E = {RP} or E = {PR} as the sample space (but not both) and states the probability is $\frac{8}{90}$ or $\frac{4}{45}$.	only considers one outcome that results in a red candy and a purple candy rather than both outcomes	REPRESENT THE SAMPLE SPACE OF A COMPOUND EVENT
The theoretical probability is $\frac{16}{100}$ or $\frac{4}{25}$.	does not account for the decrease in total candies after the first candy is drawn	EXPLAIN 2 OR MORE REAL-WORLD EVENTS AS INDEPENDENT OR DEPENDENT and DETERMINE THE PROBABILITY OF EACH SIMPLE EVENT WITHIN A COMPOUND EVENT
The theoretical probability is $\frac{16}{2025}$ or $\frac{64}{8100}$.	multiplies $\frac{8}{90}$ and $\frac{8}{90}$ instead of adding to determine the probability of either event occurring	USE MULTIPLICATION TO DETERMINE PROBABILITY OF COMPOUND EVENTS
Red then purple: $\frac{4}{10}$ and $\frac{2}{9}$ Purple then red: $\frac{2}{10}$ and $\frac{4}{9}$	gives the probability of drawing a red candy then a purple candy as well as the probability of drawing a purple candy then a red candy, but cannot determine the probability of the compound event	USE MULTIPLICATION TO DETERMINE PROBABILITY OF COMPOUND EVENTS
$\frac{104}{90}$ or $\frac{52}{45}$	adds the probabilities of drawing a red candy then a purple candy and adds the probabilities of drawing a purple candy then a red candy when they should multiply	USE MULTIPLICATION TO DETERMINE PROBABILITY OF COMPOUND EVENTS

3. You are going to play a game with your friend where you roll two standard, six-sided number cubes and determine the sum. Player A wins if the sum is 2, 3, 4, 10, 11, or 12. Player B wins if the sum is 5, 6, 7, 8, or 9. Is this a fair game? Justify your reasoning by showing your work and providing theoretical probabilities.

CORRECT ANSWER

The following table is an example of how students may choose to show the possible outcomes for the sum of two standard six-sided number cubes. (Alternate methods are also acceptable.)

If Player A wins when a sum of 2, 3, 4, 10, 11, or 12 is rolled, then there are 12 out of 36 total possible outcomes which result in a Player A win. If Player B wins when a sum of 5, 6, 7, 8, or 9 is rolled, then there are 24 out of 36 total possible outcomes which result in a Player B win. Therefore, the theoretical probability of Player B winning $(\frac{24}{36}$ or $\frac{2}{3})$ is much greater than the theoretical probability of Player A winning $(\frac{12}{36}$ or $\frac{1}{3})$. Because one player has a better chance of winning than the other, this is not a fair game.

		Result of Second Roll					
		1	**2**	**3**	**4**	**5**	**6**
Result of First Roll	**1**	2	3	4	5	6	7
	2	3	4	5	6	7	8
	3	4	5	6	7	8	9
	4	5	6	7	8	9	10
	5	6	7	8	9	10	11
	6	7	8	9	10	11	12

ERRORS, MISCONCEPTIONS, AND MISSING KNOWLEDGE

Example Error	Misconception	Missing Knowledge
The game is not fair because Player A can win with more numbers than Player B.	only considers the number of possible outcomes, not how likely each outcome is to occur	USE THE SAMPLE SPACE OF A COMPOUND EVENT TO DETERMINE PROBABILITY
This game is not fair because Player A and Player B have different chances of winning.	may be referring to the fact that Player A and Player B win with different numbers; does not clearly describe why the game is not fair or provide theoretical probabilities to support their response	USE THE SAMPLE SPACE OF A COMPOUND EVENT TO DETERMINE PROBABILITY
The game is fair because there are 11 different sums and they are divided up as evenly as possible.	does not consider how likely each of the sums is to occur or use theoretical probabilities to support their response	USE THE SAMPLE SPACE OF A COMPOUND EVENT TO DETERMINE PROBABILITY

4. Your state recently reported that only two out of five people pass the written driving test on their first attempt. In the following questions, you will conduct a simulation and determine the likelihood that three randomly selected people who have taken the written driving test failed the test on their first attempt.

   4.a. Describe the design of a simulation that you can conduct to model the described scenario. Be sure to include what each outcome of the simulation represents in the context of the problem.

CORRECT ANSWER

Answers will vary but should include two out of five outcomes representing passing the written driving test on the first attempt and three of the five outcomes representing failing the written driving test on the first attempt.

A correct response may describe creating a spinner where two out of five equal size sections represent passing on the first attempt and three out of five equal size sections represent failing on the first attempt. Spinning the spinner three times represents asking three people whether they passed or failed on their first attempt.

Another correct response may describe placing two of one color of object in a bag to represent passing on the first attempt and three of another color of the same object in a bag to represent failing on the first attempt. Drawing an object, replacing it, drawing a second object, replacing it, and drawing a third object represents asking three people whether they passed or failed on their first attempt.

ERRORS, MISCONCEPTIONS, AND MISSING KNOWLEDGE

Example Error	Misconception	Missing Knowledge
The student describes a simulation with two equally possible outcomes, one which represents passing the test and the other which represents failing the test.	does not consider the theoretical probability of passing is $\frac{2}{5}$ and the theoretical probability of failing is $\frac{3}{5}$, meaning they are not equally likely and cannot be represented by events that are equally likely	DESIGN A SIMULATION TO GENERATE FREQUENCIES FOR COMPOUND EVENTS
Describes conducting one, two, or five events per trial rather than three independent events per trial.	does not recognize the simulation is modeling a compound event where each trial consists of three independent events	DESIGN A SIMULATION TO GENERATE FREQUENCIES FOR COMPOUND EVENTS

4.b. Conduct 30 trials of the simulation and record the results of each trial in the following table.

CORRECT ANSWER

Answers will vary but should include three results per outcome. An example is provided in the following table where P and F represent pass and fail respectively.

TRIAL	OUTCOME	TRIAL	OUTCOME
1	P F F	16	P F P
2	F P P	17	F F P
3	F P P	18	F F F
4	P P F	19	P F F
5	F F F	20	P P F
6	F F P	21	F P F
7	F P F	22	F F P
8	P P P	23	P F P
9	P F F	24	F F F
10	P P F	25	F F P
11	F P P	26	F F F
12	P P P	27	F F F
13	P F P	28	F P P
14	F P F	29	P F F
15	P F F	30	F P F

ERRORS, MISCONCEPTIONS, AND MISSING KNOWLEDGE

Example Error	Misconception	Missing Knowledge
Records a single "pass" or "fail" per trial.	does not recognize the simulation is modeling a compound event where each trial consists of three independent events	DESIGN A SIMULATION TO GENERATE FREQUENCIES FOR COMPOUND EVENTS and CONDUCT A SIMULATION TO GENERATE FREQUENCIES FOR COMPOUND EVENTS
Records two or five events ("pass" or "fail") per trial.	believes either two or five events should be conducted per trial because the question describes two out of five people passing on their first attempt	DESIGN A SIMULATION TO GENERATE FREQUENCIES FOR COMPOUND EVENTS and CONDUCT A SIMULATION TO GENERATE FREQUENCIES FOR COMPOUND EVENTS
Student fails to conduct the simulation after describing a simulation design.	cannot translate general, theoretical understanding of probabilistic situations and simulations to physically conducting a simulation	CONDUCT A SIMULATION TO GENERATE FREQUENCIES FOR COMPOUND EVENTS

4.c. What outcome in the sample space represents that all three randomly selected people failed on their first attempt?

CORRECT ANSWER

> E = {FFF} is the outcome that represents all three randomly selected people failed on their first attempt.

ERRORS, MISCONCEPTIONS, AND MISSING KNOWLEDGE

Example Error	Misconception	Missing Knowledge
A "fail" three trials in a row indicates that all three randomly selected people failed on their first attempt.	looks across trials containing a single event for three "fails" in a row rather than simulating all three independent events within a single trial, then repeating this process for the next trial	DESIGN A SIMULATION TO GENERATE FREQUENCIES FOR COMPOUND EVENTS and CONDUCT A SIMULATION TO GENERATE FREQUENCIES FOR COMPOUND EVENTS
Student indicates any outcome with at least one "fail" as an outcome in the sample space that represents all three randomly selected people failing on their first attempt.	does not attend to the fact that the question requires that all three randomly selected people failed their first attempt	USE A SIMULATION TO DETERMINE PROBABILITY FOR COMPOUND EVENTS

4.d. What is the experimental probability that all three randomly selected people who have taken the written test failed on their first attempt? Explain how you determined your answer.

CORRECT ANSWER

Answers will vary but should be based on the data gathered during the simulation. Students should explain that they counted the number of trials where all three events were "fail".

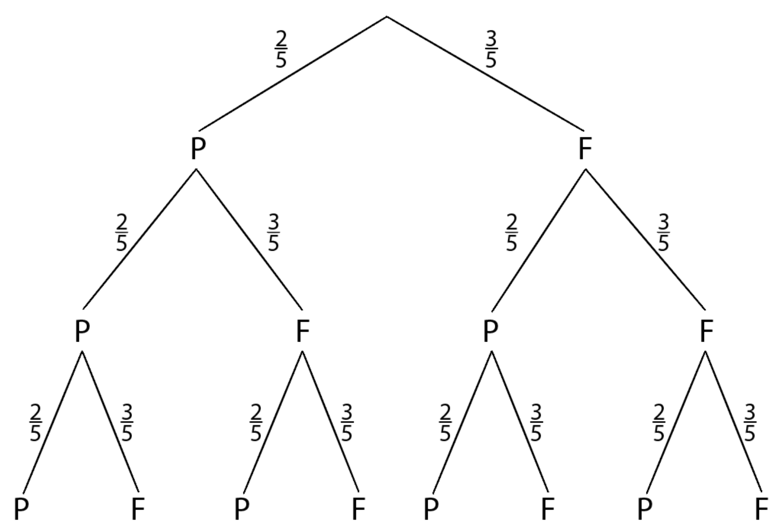
Based on the hypothetical data provided as the response in Question 4b, the experimental probability that all three randomly selected people who have taken the written test failed on their first attempt is $\frac{6}{30}$ or $\frac{1}{5}$.

ERRORS, MISCONCEPTIONS, AND MISSING KNOWLEDGE

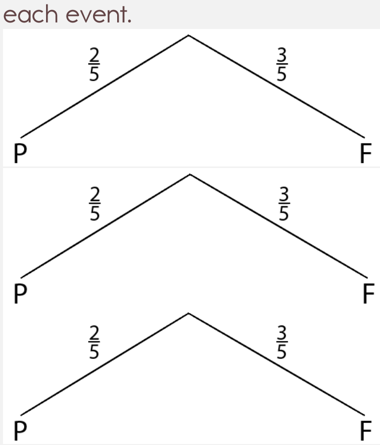
Example Error	Misconception	Missing Knowledge
Student miscounts the number of instances of F F F that occurred in the simulation.	correctly describes how to determine the experimental probability, but miscounts during the process	USE A SIMULATION TO DETERMINE PROBABILITY FOR COMPOUND EVENTS
Student records only one event per trial and counts how often F occurs three times in a row. The student may count occurences of more than three Fs in a row as multiple outcomes of three randomly selected people failing on their first attempt (e.g., F F F F might be counted as two different outcomes representing all three randomly selected people failling, because the student groups the first three Fs as one outcome and the last three Fs as a different outcome).	does not conduct the simulation such that each trial consists of three independent events, representing asking three randomly selected people who have taken the written driving test whether they passed or failed on their first attempt and/or double counting events as being part of two or more different trials	CONDUCT A SIMULATION TO GENERATE FREQUENCIES FOR COMPOUND EVENTS and USE A SIMULATION TO DETERMINE PROBABILITY FOR COMPOUND EVENTS

4.e. Create a tree diagram with theoretical probabilities along the branches to model asking three people who have taken the written driving test whether they passed or failed on their first attempt.

CORRECT ANSWER



ERRORS, MISCONCEPTIONS, AND MISSING KNOWLEDGE

Example Error	Misconception	Missing Knowledge
Student labels each probability as $\frac{1}{2}$.	does not take into consideration that the likelihood of passing and failing is not equal	DETERMINE THEORETICAL PROBABILITY OF A SIMPLE EVENT WHERE SOME OUTCOMES ARE MORE LIKELY THAN OTHERS and DETERMINE THE PROBABILITY OF EACH SIMPLE EVENT WITHIN A COMPOUND EVENT
Student labels each probability in the first event as $\frac{1}{2}$, then attempts to use conditional probabilistic reasoning in the second and third events.	inappropriately uses conditional probabilistic reasoning and determines probabilities as though there is an equal likelihood of passing and failing	EXPLAIN 2 OR MORE REAL-WORLD EVENTS AS INDEPENDENT OR DEPENDENT, DETERMINE THEORETICAL PROBABILITY OF A SIMPLE EVENT WHERE SOME OUTCOMES ARE MORE LIKELY THAN OTHERS, and DETERMINE THE PROBABILITY OF EACH SIMPLE EVENT WITHIN A COMPOUND EVENT
Student draws three separate, disconnected tree diagrams to represent each event. 	represents the tree diagram for each simple event separately, but does not represent how the events are related	EXPLAIN *COMPOUND EVENT*

4.f. What is the theoretical probability that all three randomly selected people who have taken the written test failed on their first attempt?

CORRECT ANSWER

> The theoretical probability that all three randomly selected people who have taken the written test failed on their first attempt is $\frac{27}{125}$.
> $\left(\frac{3}{5} \cdot \frac{3}{5} \cdot \frac{3}{5} = \frac{27}{125}\right)$

ERRORS, MISCONCEPTIONS, AND MISSING KNOWLEDGE

Example Error	Misconception	Missing Knowledge
The theoretical probability is $\frac{1}{8}$.	does not take into consideration that the likelihood of passing and failing are not the same; looks at the sample space and sees one out of eight outcomes is FFF	DETERMINE THEORETICAL PROBABILITY OF A SIMPLE EVENT WHERE SOME OUTCOMES ARE MORE LIKELY THAN OTHERS and USE MULTIPLICATION TO DETERMINE PROBABILITY OF COMPOUND EVENTS
$\frac{3}{5}, \frac{3}{5}, \frac{3}{5}$ or $\frac{3}{5}$	gives the probability for the simple event of selecting someone who failed the test on their first attempt, but cannot determine the probability of the compound event	USE MULTIPLICATION TO DETERMINE PROBABILITY OF COMPOUND EVENTS
$\frac{9}{5}$	adds the probabilities for the simple events of selecting someone who failed the test on their first attempt when they should multiply	USE MULTIPLICATION TO DETERMINE PROBABILITY OF COMPOUND EVENTS

4.g. How does the theoretical probability compare to the experimental probability? Explain why you believe this is the case.

CORRECT ANSWER

> *Answers will vary but should indicate that the experimental and theoretical probabilities are slightly different due to the small number of trials in the simulation.*

ERRORS, MISCONCEPTIONS, AND MISSING KNOWLEDGE

Example Error	Misconception	Missing Knowledge
The theoretical and experimental probabilities are different, but they should be the same.	does not understand that with a small number of trials, it is expected that experimental and theoretical probabilities will differ due to the element of chance in experiments, but that over an increased number of trials, they should become more similar	EXPLAIN *EXPERIMENTAL PROBABILITY* and EXPLAIN *THEORETICAL PROBABILITY*
The student acknowledges that the experimental and theoretical probabilities are different, but does not explain why.	does not understand that with a small number of trials, it is expected that the experimental and theoretical probabilities will differ due to the element of chance in experiments, but that over an increased number of trials, they should become more similar	EXPLAIN *EXPERIMENTAL PROBABILITY* and EXPLAIN *THEORETICAL PROBABILITY*