RATIONAL VS. IRRATIONAL NUMBERS

8.NS.1,2

CONTENTS

The types of documents contained in the unit are listed below. Throughout the unit, the documents are arranged by lesson.

Learning Map Information	An overview of the standards, the learning map section, and the nodes addressed in this unit
Teacher Notes	A brief discussion describing the progression depicted in the learning map section with research-based recommendations for focusing instruction to foster student learning and an introduction to the unit's lessons
Overview of Instructional Activities	A table highlighting the lesson goals and nodes addressed in each lesson of this unit
Instructional Activity	A detailed walkthrough of the unit
Instructional Activity Supplement	A collection of materials or activities related to the Instructional Activity
Student Activity	A work-alone activity for students
Student Activity Solution Guide	A solution guide for the work-alone activity with example errors, misconceptions, and links to the learning map section

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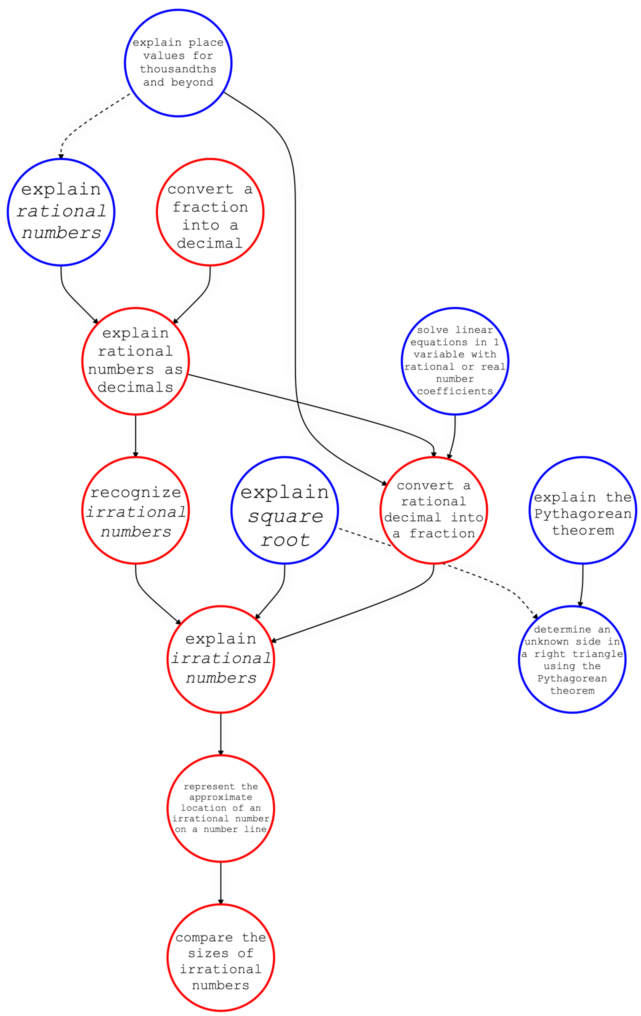
RATIONAL VS. IRRATIONAL NUMBERS

LEARNING MAP INFORMATION

STANDARDS

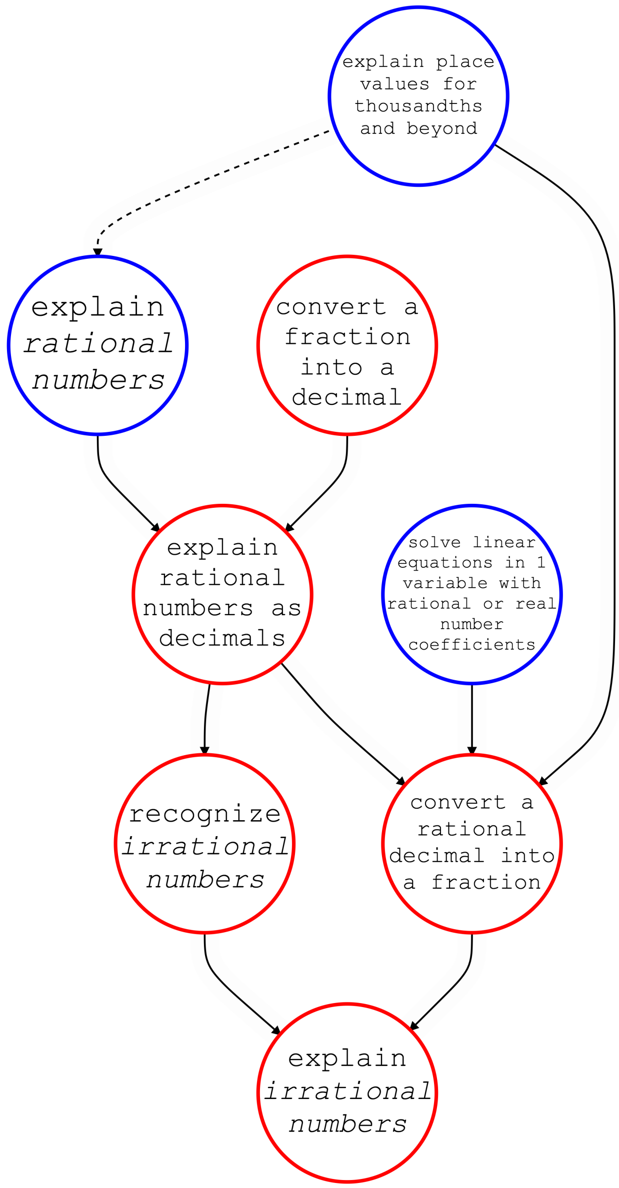
8.NS.1 Know that numbers that are not rational are called irrational. Understand informally that every number has a decimal expansion; for rational numbers show that the decimal expansion repeats eventually, and convert a decimal expansion which repeats eventually into a rational number.

8.NS.2 Use rational approximations of irrational numbers to compare the size of irrational numbers, locate them approximately on a number line diagram, and estimate the value of expressions (e.g., π^2). *For example, by truncating the decimal expansion of $\sqrt{2}$, show that $\sqrt{2}$ is between 1 and 2, then between 1.4 and 1.5, and explain how to continue on to get better approximations*.



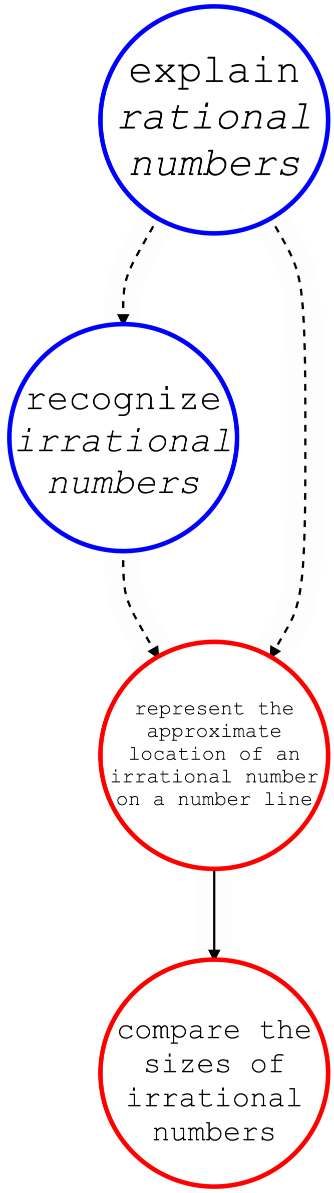
**Learning map model of* 8.NS.1,2

8.NS.1 Know that numbers that are not rational are called irrational. Understand informally that every number has a decimal expansion; for rational numbers show that the decimal expansion repeats eventually, and convert a decimal expansion which repeats eventually into a rational number.



**Learning map model of* 8.NS.1

8.NS.2 Use rational approximations of irrational numbers to compare the size of irrational numbers, locate them approximately on a number line diagram, and estimate the value of expressions (e.g., π^2). *For example, by truncating the decimal expansion of $\sqrt{2}$, show that $\sqrt{2}$ is between 1 and 2, then between 1.4 and 1.5, and explain how to continue on to get better approximations.*



**Learning map model of* 8.NS.2

Node Name	Node Description
COMPARE THE SIZES OF IRRATIONAL NUMBERS	Compare the sizes of irrational numbers using rational approximations of irrational numbers such as 3.14 for pi.
CONVERT A FRACTION INTO A DECIMAL	Convert a fraction into a decimal using long division. Decimals can be terminating or repeating.
CONVERT A RATIONAL DECIMAL INTO A FRACTION	Convert a rational number expressed as a decimal into fraction form.
DETERMINE AN UNKNOWN SIDE IN A RIGHT TRIANGLE USING THE PYTHAGOREAN THEOREM	Given a right triangle with two side lengths, apply the Pythagorean theorem to find the length of the missing side.
EXPLAIN *IRRATIONAL NUMBERS*	Make known your understanding that a real number that is not rational is irrational. Common examples are pi, non-repeating, non-terminating decimals, and square roots of non-perfect squares.
EXPLAIN PLACE VALUE FOR THOUSANDTHS AND BEYOND	Make known your understanding that the value of a digit is determined by its position in the number. For example, a digit in the thousandths place is worth that many thousandths, and a digit in the ten-thousandths place is worth that many ten-thousandths.
EXPLAIN *RATIONAL NUMBERS*	Make known your understanding that a rational number is a number that can be written as the quotient of two integers a/b, where b is not zero.
EXPLAIN RATIONAL NUMBERS AS DECIMALS	Make known your understanding that a rational number written as a decimal is either terminating or repeating. Non-terminating and non-repeating decimals are not rational numbers.
EXPLAIN *SQUARE ROOT*	Make known your understanding that the square root of a number c is the number b that, when squared, equals c.
EXPLAIN THE PYTHAGOREAN THEOREM	Make known your understanding that the Pythagorean theorem states for a right triangle with legs a and b and hypotenuse c, a^2 + b^2 = c^2.
RECOGNIZE *IRRATIONAL NUMBERS*	Identify or name nonrepeating, nonterminating decimals as irrational numbers.
REPRESENT THE APPROXIMATE LOCATION OF AN IRRATIONAL NUMBER ON A NUMBER LINE	Through writing or an appropriate assistive technology, represent the approximate location of an irrational number on a number line.
SOLVE LINEAR EQUATIONS IN 1 VARIABLE WITH RATIONAL OR REAL NUMBER COEFFICIENTS	Solve linear equations with rational or real number coefficients, including equations with solutions that require expanding expressions using the distributive property and collecting like terms.

ADDITIONAL NODES RELATED TO THIS UNIT OF INSTRUCTION

Node Name	Node Description	Related Node
EXPLAIN PLACE VALUE FOR HUNDREDTHS	Make known your understanding that the value of a digit is determined by its position in the number. A digit in the hundredths place is worth that many hundredths.	Prerequisite of EXPLAIN PLACE VALUE FOR THOUSANDTHS AND BEYOND

RATIONAL VS. IRRATIONAL NUMBERS

TEACHER NOTES

This unit includes the following documents:

- Learning Map Information
- Instructional Activity (two lessons)
- Instructional Activity Supplement (for Lessons 1 and 2)
- Student Activity
- Student Activity Solution Guide

In this unit, students will explore rational numbers and their decimal representations, then use the Pythagorean theorem to introduce the need for irrational numbers. Students will solve right triangles and position the irrational side on the number line in order to locate irrational numbers on the number line and compare their values to rational and irrational numbers.

RESEARCH

Authors support the notion that students need opportunities to develop an intuitive understanding of irrational numbers before working with their decimal approximations. Research has identified several misconceptions students hold related to the frequent use of decimal representations for irrational numbers. According to a study by Sirotic and Zazkis (2007), emphasizing the decimal representations of irrational numbers does not contribute to students' conceptual understanding. However, in practical situations, finite decimal approximations are often considered to be adequately precise when working with irrational numbers, contributing to students' conflicting understandings of irrational numbers (Sirotic & Zazkis, 2007). To improve student understanding of irrational numbers, it is important for students to be aware of the difference between the value of an irrational number and the value of its rational approximation (Sirotic & Zazkis, 2007).

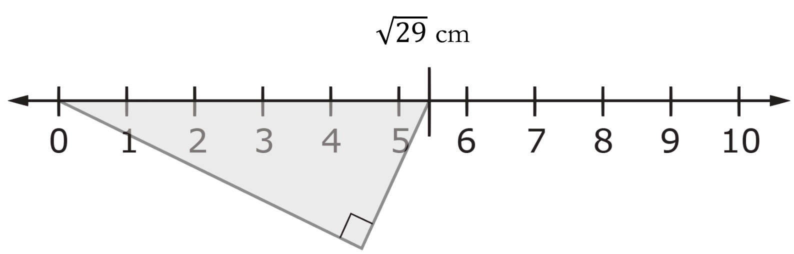
When considering rational and irrational numbers, some researchers have found the terms *rational* and *irrational* are obscure to many ninth and tenth grade students (Fischbein et al., 1995). For example, many students believed that $\sqrt{16}$ was not rational, and some students persisted with flawed understandings of irrational numbers, believing that irrational numbers are the same as negative numbers, non-terminating decimals, or non-whole numbers (Fischbein et al., 1995). In addition, many students are unaware of the difference or distinction between repeating (e.g., 0.416666666…) and non-repeating (e.g., 2.236067977499…) decimals (Fischbein et al., 1995). Emphasizing the difference between terminating (e.g., 3.14) and non-terminating (e.g., 3.141592653…) decimals is also important and may help students distinguish between an irrational number and its rational approximation.

To introduce the concept of irrational numbers, scholars suggest the use of the Pythagorean theorem. Historically, irrational numbers were discovered through the use of the Pythagorean theorem, where another type of number was needed to determine the length of the hypotenuse of a right triangle when its square was not a perfect square of a rational number. Therefore, it is appropriate to learn about the Pythagorean theorem before being introduced to irrational numbers, because it provides a motivating context for students that demands a new type of number (Popovic, 2015). Additionally, the Pythagorean theorem is accessible for many students and lends itself to explorations of irrational numbers on the number line, because irrational square roots can be represented in drawings of right triangles (Lewis, 2007; Sirotic & Zazkis, 2007). When working with students to determine how a five meter by one meter rectangle could be cut such that it could be made into a square table (using all the wood), Peled and Hershkovitz (1999) observed that most students were able to determine that, if the table exists, the side lengths must be $\sqrt{5}$ meters long. However, prior to considering irrational square roots as side lengths in right triangles, some students considered it impossible to measure a length $\sqrt{5}$ meters long because of the infinite decimal representation (Peled & Hershkovitz, 1999). The availability of a geometric representation of irrational square roots provided students with a more robust understanding and aided their conceptual development of irrational numbers; however, this geometric representation of irrational numbers was absent in many participants who had not received formal instruction to connect these concepts (Sirotic & Zazkis, 2007).

In addition to understanding irrational numbers as lengths in right triangles, a geometric representation can help students locate irrational numbers on the real number line (Sirotic & Zazkis, 2007). In one study, some students believed that, while a length of $\sqrt{5}$ was possible, $\sqrt{5}$ could not be located on the number line (Peled & Hershkovitz, 1999). By constructing a precise right triangle with an irrational side length determined using the Pythagorean theorem, students can construct a number line using the same unit of measure and orient the side with an irrational side length such that one endpoint is at zero and the other is on the positive side of the number line, demonstrating the location of the irrational number on the real number line. This provides a contrast for students who perceive the real number line as containing rational numbers only (Sirotic & Zazkis, 2007). Conceptualizing irrational numbers along with rational numbers is essential to students' understanding of the set of real numbers (Sirotic & Zazkis, 2007). Additionally, triangles and number lines allow teachers to challenge students to represent sums and differences of a rational and irrational number (e.g., $2 + \sqrt{29}$, $-3 - \sqrt{29}$) on the real number line (Coffey, 2001).

AN EXAMPLE

The following figure models using a right triangle with leg lengths of five centimeters and two centimeters to locate $\sqrt{29}$ centimeters, the length of the hypotenuse, on the number line.



LEARNING MAP INFORMATION

The learning map section for this sequence of activities begins with place value to the thousandths and beyond, which eventually connects to the ability to explain rational numbers (through comparisons of rational numbers, locating rational numbers on the number line, explaining situations in which opposites which make zero, and additive inverses). Explaining rational numbers then leads to considering the decimal representations of rational numbers. The Pythagorean theorem is included in this learning map section because, although it is not required, when combined with an understanding of rational numbers, the Pythagorean theorem contributes to students' readiness for understanding irrational numbers and how they complete the real number system, thereby setting them up to consider more advanced number system ideas such as continuity and infinity. Upon recognizing and explaining irrational numbers, students should be able to locate irrational numbers on a number line and compare the sizes of irrational numbers.

INSTRUCTIONAL ACTIVITIES

The activities in this unit are designed to promote student understanding of both rational and irrational numbers. The first lesson requires students to write fractions as decimals and look for patterns among the decimal representations, as well as convert rational decimals into fractions. The second lesson uses the Pythagorean theorem to introduce irrational numbers and locate them on a number line. Students then consider the decimal representation of irrational numbers and compare decimal representations of rational and irrational numbers. Finally, students compare irrational numbers to rational numbers and order irrational numbers.

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Rational vs. Irrational Numbers

OVERVIEW OF INSTRUCTIONAL ACTIVITIES

Lesson	Learning Goal	Nodes Addressed
Lesson 1	Students will discover that all rational numbers, when written in decimal form, either repeat or terminate. Students will then learn to write rational numbers represented in decimal form as fractions.	▶ CONVERT A FRACTION TO A DECIMAL ▶ EXPLAIN *RATIONAL NUMBERS* ▶ EXPLAIN RATIONAL NUMBERS AS DECIMALS ▶ EXPLAIN PLACE VALUES FOR THOUSANDTHS AND BEYOND ▶ CONVERT A RATIONAL DECIMAL INTO A FRACTION ▶ SOLVE LINEAR EQUATIONS IN 1 VARIABLE WITH RATIONAL OR REAL NUMBER COEFFICIENTS
Lesson 2	Students will locate irrational numbers on the number line, using right triangles as models. Students will then use the decimal expansion of irrational numbers to determine, with increasing precision, the rational numbers that the irrational value falls between.	▶ RECOGNIZE IRRATIONAL NUMBERS ▶ EXPLAIN *SQUARE ROOT* ▶ EXPLAIN *IRRATIONAL NUMBERS* ▶ REPRESENT THE APPROXIMATE LOCATION OF AN IRRATIONAL NUMBER ON A NUMBER LINE ▶ COMPARE THE SIZES OF IRRATIONAL NUMBERS

Rational vs. Irrational Numbers

INSTRUCTIONAL ACTIVITY

Lesson 1

LEARNING GOAL

Students will discover that all rational numbers, when written in decimal form, either repeat or terminate. Students will then learn to write rational numbers represented in decimal form as fractions.

PRIMARY ACTIVITY

Students will determine the decimal expansion of a variety of fractions and look for patterns among the decimals, eventually noting that all rational numbers either repeat or terminate in decimal form. Students will then learn to write terminating decimals as fractions and write repeating decimals as fractions.

OTHER VOCABULARY

Students will need to know the meaning of the following terms:

- Decimal
- Fraction
- Rational number
- Repeating decimal
- Terminating decimal

MATERIALS

- Scissors
- Calculators
- Instructional Activity Supplement A (Recommend one copy for every one to two students.)
- Instructional Activity Supplement B (Recommend one copy for every student.)

IMPLEMENTATION

Students will begin by considering various fractions and their equivalent decimal form, looking for patterns among the decimals.

Ask students if they remember how to convert a fraction to a decimal. Students should recall that in order to convert fractions to decimals, you can divide the numerator by the denominator, either using long division or a calculator.

Practice a few examples with students, including negative fractions and mixed numbers (e.g., $\frac{1}{4}, -\frac{10}{9}, 3\frac{2}{3}$).

Once students are comfortable changing fractions to decimals, **hand out** the INSTRUCTIONAL ACTIVITY SUPPLEMENT to individual students or pairs of students.

Require students to cut out each card, determine the decimal expansion for each fraction, and sort the cards based on patterns they notice among the decimal expansions.

GUIDING QUESTIONS

Elicit student thinking:

- What do you notice about the decimal expansions?
- What patterns do you see among the decimals? Can you see any other patterns?
- How could you group these cards?

Determine if the student can CONVERT A FRACTION INTO A DECIMAL:

- What process would you use to write this fraction as a decimal?
- When using long division to divide, how do you know when you can stop the long division process?
- Do the decimal and the fraction represent the same value? Why or why not?

Determine if the student can EXPLAIN *RATIONAL NUMBERS*:

- What is a rational number?
- What types of numbers do you think of when you hear the term "rational number"?
- Can you give me an example of a rational number? Explain how you chose this number.

Determine if the student can EXPLAIN RATIONAL NUMBERS AS DECIMALS:

- If each of these is a rational number, how would you describe rational numbers as decimals?
- What do you look for in a decimal to determine if it is a rational number?
- Can you describe what the decimal representation would look like for a value that is not rational?

Discuss the different ways students sorted the cards.

If students do not mention sorting cards based on whether the decimal terminated or repeated, **guide** the discussion to require students to consider the nature of the decimal.

By the end of the discussion, **ensure** it is clear to students that all rational decimals will either terminate or repeat.

After writing fractions as decimals, students will write terminating decimals as fractions.

Draw students' attention to the decimal expansion of $\frac{21}{2}$ (10.5) in order to **model** how to start with the terminating decimal and write the corresponding fraction using place value and simplification.

Ask students to read the decimal, properly utilizing their knowledge of place value (i.e., ten and five tenths).

Focus students' attention first on "ten", noting that it's a whole number.

Next, **focus** students' attention on "five tenths" and **ask** how they would represent five tenths as a fraction. Students should be able to indicate that this fraction has five as the numerator and ten as the denominator $\left(\frac{5}{10}\right)$.

Write ten and five tenths together as a mixed number $\left(10\frac{5}{10}\right)$, then **ask** students how they could simplify five tenths. Students should be able to indicate five tenths is equivalent to one half, therefore $10\frac{5}{10}$ is equivalent to $10\frac{1}{2}$.

Note that $10\frac{1}{2}$ is equivalent to $\frac{21}{2}$ and that either form is an acceptable fraction representation of the decimal.

Provide students with additional terminating decimals (the following problems along with the introductory example are provided in the INSTRUCTIONAL ACTIVITY SUPPLEMENT B), and **require** students to read the decimal and use the verbal representation to rewrite the decimal as a fraction. Examples could include the following:

- 0.3
- –2.75
- –0.125
- 7.46

GUIDING QUESTIONS

Elicit student thinking:

- What do you notice about this decimal?
- How does place value relate to writing decimals as fractions?
- Do you think any decimal can be written as a fraction using what you know about place value? Why or why not?

Determine if the student can EXPLAIN PLACE VALUES FOR THOUSANDTHS AND BEYOND:

- What is the place value of the last digit in this decimal?
- Using your knowledge of place value, how would you read this decimal?

Determine if the student can CONVERT A RATIONAL DECIMAL INTO A FRACTION:

- Can you read this decimal out loud?
- Based on how you would read this decimal, can you write an equivalent fraction?
- Are you able to simplify the fraction you wrote?

Finally, students will write repeating decimals as fractions.

Draw students' attention to the decimal expansion of $\frac{8}{3}$ $(2.\overline{6})$ in order to **model** how to rewrite a repeating decimal as a fraction.

Note that one digit (six) repeats.

Begin by assigning a variable, such as *n,* to the repeating decimal.

$$n = 2.666666\ldots$$

Because the decimal has one digit that repeats, **multiply** both sides of the equation by 10^1 or 10 and **write** the result above the original equation.

$$10n = 26.666666\ldots$$
$$n = 2.666666\ldots$$

Note that after the decimal point, the values in the first equation are identical to the values in the second equation. Therefore, subtracting the second equation from the first equation will result in a whole number.

$$10n = 26.666666\ldots$$
$$\underline{-(n = 2.666666\ldots)}$$
$$9n = 24$$

To solve for *n*, **divide** both sides of the equation by nine and **simplify** the fraction.

$$\frac{9n}{9} = \frac{24}{9}$$
$$n = \frac{8}{3}$$

Note that this simplified fraction is equivalent to the fraction they began with.

Require students to consider an example of a repeating decimal where more than one digit repeats. The following example models the procedure for writing the repeating decimal $-0.\overline{143}$ or –0.143143143… as a fraction.

Note that three digits (one, four, and three) repeat.

Begin by assigning a variable, such as *n,* to the repeating decimal.

$$n = -0.143143143\ldots$$

Because the decimal has three digits that repeat, **multiply** both sides of the equation by 10^3 or 1000 and **write** the result above the original equation.

$$\begin{aligned} 1000n &= -143.143143143\ldots \\ n &= -0.143143143\ldots \end{aligned}$$

Note that after the decimal point, the digits in the first equation are identical to the corresponding digits in the second equation. Therefore, subtracting the second equation from the first equation will result in a whole number.

$$\begin{aligned} 1000n &= -143.143143143\ldots \\ \underline{-(n} &\underline{= -0.143143143\ldots)} \\ 999n &= -143 \end{aligned}$$

To solve for *n*, **divide** both sides of the equation by 999 and **simplify** the fraction (in this case the fraction does not simplify).

$$\frac{999n}{999} = \frac{-143}{999}$$

$$n = \frac{-143}{999}$$

Require students to consider an example of a repeating decimal where a portion of the decimal does not repeat. The decimal used in the final example should have repeating digits following at least one nonrepeating digit. The following example models the procedure for writing the repeating decimal $0.3\overline{67}$ or 0.36767676767… as a fraction.

Note that the three does not repeat, but the alternating six and seven do.

Begin by assigning a variable, such as *n,* to the repeating decimal.

$$n = 0.36767676767\ldots$$

Because the decimal has two digits that repeat, **multiply** both sides of the equation by 10^2 or 100 and **write** the result above the original equation.

$$\begin{aligned} 100n &= 36.767676767\ldots \\ n &= 0.36767676767\ldots \end{aligned}$$

Note that after the decimal point, the first digits are different in the first and second equation, but after the first digit, all remaining digits after the decimal point in the first equation are identical to the values in the second equation. Therefore, subtracting the second equation from the first equation will result in a terminating decimal.

$$\begin{array}{r} 100n = 36.767676767\ldots \\ \underline{-(n = 0.36767676767\ldots)} \\ 99n = 36.4 \end{array}$$

To rewrite the equation so the value on the right side is a whole number, multiply both sides of the equation by 10.

$$10(99n) = 10(36.4)$$
$$990n = 364$$

To solve for n, **divide** both sides of the equation by 990 and **simplify** the fraction.

$$\frac{990n}{990} = \frac{364}{990}$$
$$n = \frac{182}{495}$$

Ask students how they know what power of ten they must multiply by both sides of the equation to ensure subtraction results in either a whole number or a terminating decimal. Students should notice that the power is based on the number of digits that repeat. For example, if one digit repeats, they must multiply by 10^1. If two digits repeat, they must multiply by 10^2. If three digits repeat, they must multiply by 10^3, and so forth.

Provide students with additional repeating decimals and **require** students to set up an equation to rewrite the repeating decimal as a fraction. Examples could include the following:

- $5.\overline{2}$
- $-0.\overline{67}$
- $0.\overline{122}$
- $1.1\overline{74}$

GUIDING QUESTIONS

Elicit student thinking:

- What do you notice about this decimal?
- Can you read this decimal using what you know about place value? Why or why not?

Determine if the student can CONVERT A RATIONAL DECIMAL INTO A FRACTION:

- How many digits repeat in this decimal?
- What should you multiply both sides of the equation by to ensure subtraction results in a whole number or a terminating decimal?
- Can you describe what you would do to write this repeating decimal as a fraction?
- How does subtraction help as you rewrite a repeating decimal as a fraction?

Determine if the student can SOLVE LINEAR EQUATIONS IN 1 VARIABLE WITH RATIONAL OR REAL NUMBER COEFFICIENTS:

- How would you solve this equation for the variable?
- What does the result represent?

Students should be required to show their work and explain their reasoning throughout the lesson in order to develop a better understanding of rational numbers and establish the similarities and differences between terminating and repeating decimals.

At the end of the activity, provide students with one terminating decimal and one repeating decimal, and require students to rewrite each decimal as a fraction.

Fraction: $\frac{11}{4}$ Decimal expansion: __________	Fraction: $\frac{-10}{2}$ Decimal expansion: __________	Fraction: $-\frac{1}{9}$ Decimal expansion: __________
Fraction: $\frac{8}{3}$ Decimal expansion: __________	Fraction: $\frac{11}{-6}$ Decimal expansion: __________	Fraction: $\frac{9}{8}$ Decimal expansion: __________
Fraction: $\frac{3}{10}$ Decimal expansion: __________	Fraction: $-\frac{5}{7}$ Decimal expansion: __________	Fraction: $\frac{15}{12}$ Decimal expansion: __________
Fraction: $\frac{21}{2}$ Decimal expansion: __________	Fraction: $4\ \frac{1}{11}$ Decimal expansion: __________	Fraction: $-6\ \frac{2}{25}$ Decimal expansion: __________
Fraction: $5\ \frac{4}{15}$ Decimal expansion: __________	Fraction: $1\ \frac{11}{100}$ Decimal expansion: __________	Fraction: $-3\ \frac{3}{5}$ Decimal expansion: __________

10.5

0.3

−2.75

−0.125

7.46

Rational vs. Irrational Numbers

INSTRUCTIONAL ACTIVITY

Lesson 2

LEARNING GOAL

Students will locate irrational numbers on the number line, using right triangles as models. Students will then use the decimal expansion of irrational numbers to determine, with increasing precision, the rational numbers that the irrational value falls between.

PRIMARY ACTIVITY

Students will use right triangles and the Pythagorean theorem to obtain a side length of a triangle with an irrational value in order to locate the irrational value on the real number line. Using the decimal expansion of the irrational value, students will discuss the difference between rational and irrational numbers and describe the rational numbers that the irrational value falls between. Students will then consider additional irrational values, their location on the number line, and approximations involving irrational numbers.

OTHER VOCABULARY

Students will need to know the meaning of the following terms:

- Rational
- Irrational
- Right triangle
- Leg (of a right triangle)
- Hypotenuse
- Pythagorean theorem

MATERIALS

- Calculators
- Rulers (with centimeter markings)
- Scissors
- Blank paper or notecards

- INSTRUCTIONAL ACTIVITY SUPPLEMENT (Recommend one copy for every student.)

IMPLEMENTATION

Prior to this lesson, students should be familiar with the Pythagorean theorem as a method to find the hypotenuse in a right triangle, given the lengths of the legs.

Review the Pythagorean theorem as a method for finding the length of one side of a right triangle when the other two sides are provided.

> NOTE: Although the Pythagorean theorem can be used to determine any side length in a right triangle when the other two sides are provided, this lesson only requires students to determine the length of the hypotenuse using the Pythagorean theorem.

Begin with a right triangle where all side lengths are whole numbers, as shown in Figure 1.

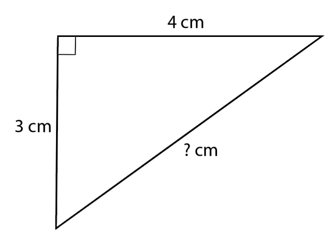


FIGURE 1

After establishing the length of the hypotenuse is $\sqrt{25}$ or five centimeters in the first example, **provide** students with a second example where the legs of the triangle are labeled and the hypotenuse is an irrational value. Students should not yet use a calculator to estimate the length of the hypotenuse. An example is provided in Figure 2.

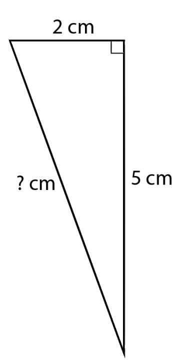


FIGURE 2

In computing the length of the hypotenuse, students should arrive at the value $\sqrt{29}$.

Ask students if they can, without touching a calculator, describe how long $\sqrt{29}$ centimeters is or where it would be located on the real number line.

Students may or may not be able to estimate this value, depending on their experience with square roots to this point. If students can provide an estimate, the following activity will either confirm or disprove their estimate. If students cannot provide an estimate, the following activity will help them do so in the future.

Next, students will recreate the triangle in Figure 2 in order to determine approximately how long the hypotenuse is and where the value is located on the number line.

Provide each student with a ruler, scissors, a piece of blank paper or a notecard, and the INSTRUCTIONAL ACTIVITY SUPPLEMENT.

Direct students to locate a corner of the paper or notecard and use that as their right angle in the right triangle, in order to ensure precision.

Require students to measure two centimeters from the corner/right angle with the ruler in one direction and five centimeters from the corner/right angle with the ruler in the other direction, marking each length.

Once students have the leg lengths marked, **require** students to use the ruler to connect the two points, without measuring, forming a right triangle.

Using scissors, students should cut along the hypotenuse, then mark the side lengths of the triangle "2 cm", "5 cm", and "$\sqrt{29}$ cm" accordingly.

On the INSTRUCTIONAL ACTIVITY SUPPLEMENT, **require** students to mark the first number line every centimeter, using the ruler, and label the number line such that, at the minimum, zero through six are represented.

Ask students to place one end of the hypotenuse at zero on the number line, with the hypotenuse lying on the number line such that the other end falls somewhere on the number line.

Require students to mark this point, as shown in Figure 3, located between 5 and 6 centimeters (slightly closer to 5), and label it $\sqrt{29}$ centimeters.

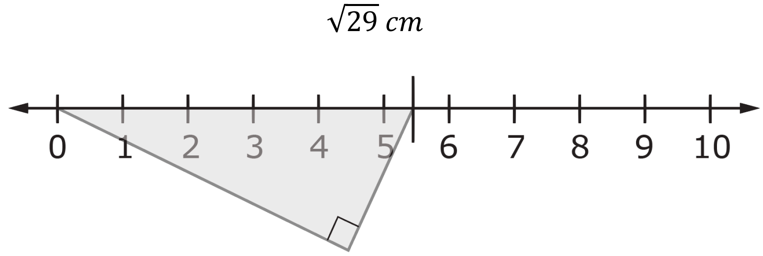


FIGURE 3

Discuss whether students' predictions regarding the value of $\sqrt{29}$ were correct.

Emphasize that there are quantities around $\sqrt{29}$ students do know the exact value of, such as $\sqrt{25}$ (equivalent to five) and $\sqrt{36}$ (equivalent to six). **Note** that $\sqrt{29}$ falls between $\sqrt{25}$ and $\sqrt{36}$, therefore it makes sense that the location of $\sqrt{29}$ on the real number line is between 5 and 6.

Once students have located $\sqrt{29}$ on the number line using geometric reasoning rather than a calculator, allow them to type $\sqrt{29}$ into a calculator to view the decimal approximation (5.385164807134504…).

Note that although the digits reach the end of the calculator, they continue but cannot be represented on the screen forever.

If possible, **locate** two or more students whose calculators truncate the decimal at different digits to emphasize this point to students.

After establishing that the decimal does not end, **ask** students if they can find a repeating pattern in the digits (as they saw in LESSON 1 with rational numbers). Students should realize that they are unable to find a repeating pattern in the digits.

Tell students that when they come across decimals that continue infinitely without repeating, this indicates an *irrational number*. **Note** that irrational numbers can be placed on the number line with the rational values discussed in LESSON 1.

NOTE: MathisFun.com has an interactive number line which can be zoomed in and out to show the values between integers with increasing accuracy. This can provide a visual for students who struggle to make the connection between rational and irrational numbers and their placement on the number line. (https://www.mathsisfun.com/numbers/number-line-zoom.html is linked to the words "MathisFun.com")

Additionally, **note** that not all square roots are irrational values. For example $\sqrt{25}$ is exactly 5, a decimal that terminates, therefore it is a rational value.

Inform students that together, the rational and irrational numbers make up the entire set of *real numbers* on the real number line.

GUIDING QUESTIONS

Elicit student thinking:

- Can you describe the different types of numbers you know? What do these numbers have in common? How are they different?
- How can right triangles (or other shapes) help place numbers on the number line?

Determine if the student can RECOGNIZE *IRRATIONAL NUMBERS*:

- [Provide the student with a list of rational and irrational numbers.] Which values are irrational?
- [Point to an irrational number the student has been working with.] Is this number rational or irrational?

Determine if the student can EXPLAIN *SQUARE ROOT*:

- What do you consider as you try to simplify the square root of a number? What number are you looking for?
- What is the square root of 36? How did you determine this value? What was your thought process?

Determine if the student can EXPLAIN *IRRATIONAL NUMBERS*:

- What do you look at when deciding if a number is irrational?
- How can you tell whether this number is rational or irrational?

Determine if the student can REPRESENT THE APPROXIMATE LOCATION OF AN IRRATIONAL NUMBER ON A NUMBER LINE:

- [Point to an irrational number.] Does this number have a place on the number line? How do you know?
- How can you use a right triangle to help you place this number on a number line?
- Where would you locate this number on the number line? How did you determine this location?

Now that students have an understanding that $\sqrt{29}$ is irrational and the decimal approximation is 5.385164807134504…, students will approximate the rational numbers that $\sqrt{29}$ is between with increasing precision.

Beginning with whole numbers, **ask** students which two whole numbers $\sqrt{29}$ is located between. **Encourage** students to consider that $\sqrt{29}$ is greater than five but less than six.

Next, **ask** students to consider the tenths place. **Ask** students whether $\sqrt{29}$ is greater than or less than 5.3. Once students agree that $\sqrt{29}$ is greater than 5.3, **ask** students to provide the closest decimal, to the tenths place, $\sqrt{29}$ is less than. Students should identify that $\sqrt{29}$ is less than 5.4. Therefore, to the tenths place, $\sqrt{29}$ is between 5.3 and 5.4.

Continue this questioning to the hundredths place, thousandths place, and beyond if desired to help students continue to determine the rational values that the irrational value is between with increasing precision.

Next, repeat the previous reasoning with a new triangle.

Provide students with another example where the legs of the triangle are labeled and the hypotenuse is an irrational value. Students should not yet use a calculator to estimate the length of the hypotenuse. An example is provided in Figure 4.

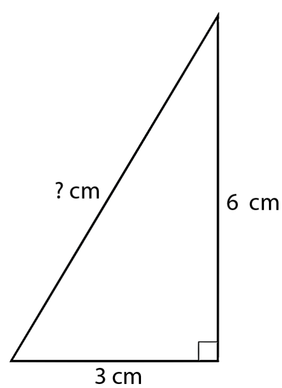


FIGURE 4

In computing the length of the hypotenuse, students should arrive at the value $\sqrt{45}$.

Ask students if they can, without touching a calculator, describe how long $\sqrt{45}$ centimeters is or where it would be located on the real number line.

Using the reasoning discussed in the previous example, students should be able to identify $\sqrt{45}$ centimeters as being more than six centimeters and less than seven centimeters, because $\sqrt{45}$ is located between $\sqrt{36}$ (equivalent to six) and $\sqrt{49}$ (equivalent to seven).

Next, students will recreate the triangle in Figure 4 in order to locate $\sqrt{45}$ on the number line.

Students will again need to use a ruler, scissors, a piece of blank paper or a notecard, and the INSTRUCTIONAL ACTIVITY SUPPLEMENT.

Direct students to locate an uncut corner of the paper or notecard and use that as their right angle in the right triangle, in order to ensure precision.

Require students to measure three centimeters from the corner/right angle with the ruler in one direction and six centimeters from the corner/right angle with the ruler in the other direction, marking each length.

Once students have the leg lengths marked, **require** students to use the ruler to connect the two points, without measuring, forming a right triangle.

Using scissors, students should cut along the hypotenuse, then mark the side lengths of the triangle "3 cm", "6 cm", and "$\sqrt{45}$ cm" accordingly.

On the INSTRUCTIONAL ACTIVITY SUPPLEMENT, **require** students to mark the second number line every centimeter, using the ruler, and label the number line such that, at the minimum, negative seven through positive seven are represented (after locating $\sqrt{45}$, students will use their understanding of symmetry and the number line to locate $-\sqrt{45}$).

Ask students to place one end of the hypotenuse at zero on the number line with the hypotenuse lying on the number line, such that the other end falls somewhere on the positive side of the number line.

Require students to mark this point, as shown in Figure 5, located between 6 and 7 centimeters (slightly closer to 7), and label it $\sqrt{45}$ centimeters.

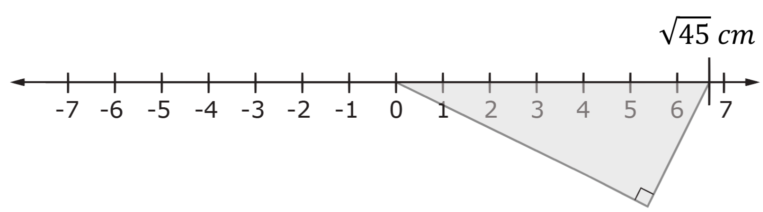


FIGURE 5

Discuss whether students' predictions regarding the location of $\sqrt{45}$ were correct.

Remind students that there are quantities around $\sqrt{45}$ students do know the exact value of, such as $\sqrt{36}$ (equivalent to six) and $\sqrt{49}$ (equivalent to seven). **Note** that $\sqrt{45}$ falls in between $\sqrt{36}$ and $\sqrt{49}$, therefore it makes sense that the location of $\sqrt{45}$ on the real number line is between 6 and 7. Students should be able to locate the two whole numbers that the square root is between without the use of a calculator. More precise approximations of the square root can be determined after finding the decimal expansion with a calculator.

Require students to answer questions without the use of a calculator, unless it is deemed necessary or as an accommodation.

Once students have located $\sqrt{45}$ on the number line using geometric reasoning, allow them to type $\sqrt{45}$ into a calculator to view the decimal approximation (6.708203932499369…).

Remind students that although the digits reach the end of the calculator, they continue but cannot be represented on the screen forever.

Ask students if they can find a repeating pattern in the digits (as they saw in LESSON 1 with rational numbers). Students should note that they are unable to find a repeating pattern in the digits and therefore should identify $\sqrt{45}$ as an irrational number.

Ask students how they could approximate the location of $-\sqrt{45}$ on the number line.

Guide students to use the symmetry in the number line to reflect the location of $\sqrt{45}$ across zero. Using the triangle, this can be modeled by placing one end of the hypotenuse at zero and the other end on the negative side of the number line, so that it falls between negative 6 and –7 but slightly closer to –7, as shown in Figure 6.

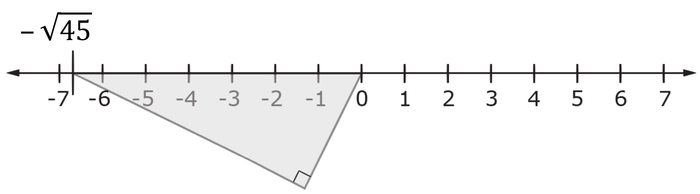


FIGURE 6

NOTE: To extend the concepts of irrational numbers, ask students to try modeling $-5 + \sqrt{45}$ (position the left side of the hypotenuse at –5 and observe the location of the the right side of the hypotenuse) or $6 - \sqrt{45}$ (position the right side of the hypotenuse at 6 and observe the location of the left side of the hypotenuse) on the number line and comparing these values to the result of computing the sum or difference in a calculator.

GUIDING QUESTIONS

Elicit student thinking:

- Can you describe the different types of numbers you know? What do these numbers have in common? How are they different?
- How can right triangles help place numbers on the number line?
- Are there other shapes that can help place numbers on the number line?

Determine if the student can RECOGNIZE *IRRATIONAL NUMBERS*:

- [Provide the student with a list of rational and irrational numbers.] Which values are irrational?
- [Point to an irrational number the student has been working with.] Is this number rational or irrational?

Determine if the student can EXPLAIN *SQUARE ROOT*:

- What do you consider as you try to simplify the square root of a number? What number are you looking for?
- What is the square root of 49? How did you determine this value? What was your thought process?

Determine if the student can EXPLAIN *IRRATIONAL NUMBERS*:

- What do you look at when deciding whether a number is irrational?
- How can you tell whether this number is rational or irrational?

Determine if the student can REPRESENT THE APPROXIMATE LOCATION OF AN IRRATIONAL NUMBER ON A NUMBER LINE:

- [Point to an irrational number.] Does this number have a place on the number line? How do you know?
- How can you use a right triangle to help you place this number on a number line?
- Where would you locate this number on the number line? How did you determine this location?

Next, require students to approximate the rational numbers that $\sqrt{45}$ is between with increasing precision.

Beginning with whole numbers, **ask** students which whole numbers $\sqrt{45}$ is between. Students should identify that $\sqrt{45}$ is greater than six but less than seven.

Next, **ask** students for two decimals, to the tenths place, that $\sqrt{45}$ is between. Students should identify that $\sqrt{45}$ is greater than 6.7 and less than 6.8.

Continue this questioning to the hundredths place, thousandths place, and beyond if desired to help students continue to determine the rational values that the irrational value is between with increasing precision.

Next, students will consider additional irrational values and compare irrational numbers.

Ask students to consider the value π (pi). Students should be familiar with pi from their experiences learning about area and circumference of circles in seventh grade.

Note that π is a symbol which has a value associated with it, and **ask** students if they know what this value is. It is likely students respond that pi is equal to 3.14 or $\frac{22}{7}$.

If students respond with either of these values, **ask** them if they are rational or irrational values. Students should indicate these values are rational; 3.14 is rational because the decimal terminates (and can be written as a fraction with integer values for the numerator and denominator), and $\frac{22}{7}$ is rational because it is a fraction (or a repeating decimal).

Require students to press the π button on their calculator and view the corresponding decimal.

Ask students if the decimal terminates (or ends). They should indicate that it does not. **Ask** students if the decimal repeats. They should indicate that it does not.

Ask students how they would classify a value whose decimal representation does not terminate or repeat. Students should come to the conclusion that pi is irrational.

Emphasize that the value of pi is not equal to the rational representations used in place of pi (e.g., 3.14 and $\frac{22}{7}$). Rather, the actual value of pi is greater than 3.14 and less than $\frac{22}{7}$.

Discuss how using rational approximations for pi rather than the actual value can affect the result of various computations involving the irrational number pi.

Using their knowledge of the value of pi, **require** students to estimate the values of π^2 and π^3.

Require students to place each value that has been discussed in the lesson ($\sqrt{29}$, $\sqrt{45}$, $-\sqrt{45}$, and π) on the remaining number line in the INSTRUCTIONAL ACTIVITY SUPPLEMENT, and to order the values from least to greatest using their locations on the number line.

GUIDING QUESTIONS

Elicit student thinking:

- What is the same about these values? What is different?
- Can you discuss how these values relate to the number line?

Determine if the student can REPRESENT THE APPROXIMATE LOCATION OF AN IRRATIONAL NUMBER ON A NUMBER LINE:

- [Point to an irrational number.] Does this number have a place on the number line? How do you know?
- Where would you locate this number on the number line? How did you determine this location?

Determine if the student can COMPARE THE SIZES OF IRRATIONAL NUMBERS:

- Which of these values is the greatest? Which of these values is the least?
- How can the number line help you order these values?
- How would you order these values from least to greatest?

Students should be required to compare or order additional irrational values, including the irrational value *e*. **Include** positive and negative irrational values.

NOTE: *e* is equal to approximately 2.718. It is the base of the natural log, and is the limit of $\left(1+\frac{1}{n}\right)^{n}$ as n approaches infinity, a formula related to compound interest. As this is likely the students' first introduction to the value *e*, it may be appropriate to provide a brief context for the number. However, at this level, it is acceptable for students to simply be introduced to *e* as another well-known irrational number and given its decimal approximation. The decimal approximation can also be found on a calculator.

At the end of the activity, **provide** students with a right triangle with two sides labeled and an irrational value for the hypotenuse. **Require** students to determine the length of the hypotenuse, then locate this value on the real number line.

Rational vs. Irrational Numbers

INSTRUCTIONAL ACTIVITY SUPPLEMENT

Lesson 2

Name____

Rational vs. Irrational Numbers

Lessons 1 & 2

1. Provide five **rational** numbers and explain how you know these numbers are rational. Provide a variety of answers. Provide more than one form or type of rational number.

2. Provide five **irrational** numbers and explain how you know these numbers are irrational.

Name______________________________________

3. Write the following decimal as a fraction in simplest form. Explain your reasoning.

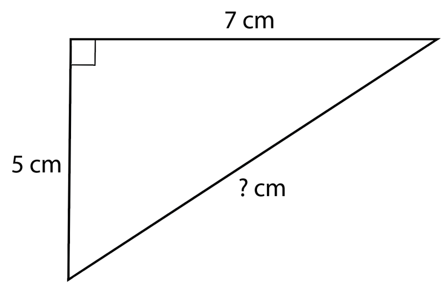
3.18

4. Write the following decimal as a fraction in simplest form. Show your work.

1.45454545…

Name____________________________________

5. Use the following right triangle to answer the questions provided.



5.a. Determine the length of the hypotenuse. Show your work.

5.b. Partition the number line provided such that each interval is one centimeter. Then cut out a triangle with the same dimensions as labeled in the triangle shown. Use the triangle to locate and label the length of the hypotenuse on the number line. Trace or glue the triangle you use to support your work.

Name______________________________________

6. Partition the number line provided such that each interval is one centimeter. Use a right triangle with leg lengths of eight centimeters and five centimeters to locate $-\sqrt{89}$ on the number line. Trace or glue the triangle you use to support your work.

7. Provide an irrational number located between 4.2 and 4.3 on the number line. Give your answer as a square root. Explain how you determined this number. You may use a calculator to help you.

Name____

8. Describe where $\sqrt{68}$ is located on a number line labeled with rational numbers to the hundredths place. Explain your reasoning.

9. Order the following values from least to greatest.

$$\sqrt{18},\ -\pi,\ 0,\ -\sqrt{23},\ \sqrt{11},\ e,\ -\sqrt{5}$$

Rational vs. Irrational Values

STUDENT ACTIVITY SOLUTION GUIDE

Lessons 1 & 2

1. Provide five **rational** numbers and explain how you know these numbers are rational. Provide a variety of answers. Provide more than one form or type of rational number.

CORRECT ANSWER

[Student responses will vary. Verify that students provide five values that fall into the following categories: fractions with integer values for the numerator and denominator (e.g., $\frac{-8}{3}$), decimals that terminate (e.g., 7.265), and/or decimals that repeat (e.g., 4.1212121212...).]

I know these numbers are rational because they can be or already are written as fractions with integer values for the numerator and denominator.

or

I know these numbers are rational because they are all fractions, decimals that terminate, or decimals that repeat.

ERRORS, MISCONCEPTIONS, AND MISSING KNOWLEDGE

Example Error	Misconception	Missing Knowledge
Student provides irrational numbers rather than rational numbers.	confuses rational and irrational numbers	EXPLAIN *RATIONAL NUMBERS* and EXPLAIN *IRRATIONAL NUMBERS*
Student provides a combination of rational and irrational numbers.	does not understand rational numbers; may be confusing rational and real numbers	EXPLAIN *RATIONAL NUMBERS*
Student provides five rational numbers but cannot explain how they know they are rational.	identifies but cannot describe a rational number	EXPLAIN *RATIONAL NUMBERS*
NOTE: Students who provide only one or two representations of rational values may have an incomplete understanding of rational values and could benefit from revisiting the different representations of rational values.		

2. Provide five **irrational** numbers and explain how you know these numbers are irrational.

CORRECT ANSWER

[*Student responses will vary. Verify that students provide five numbers whose decimal representations do not terminate or repeat.*]

I know these numbers are irrational because they cannot be written as fractions with integer values for the numerator and denominator.

or

I know these numbers are irrational because their decimal representations do not terminate or repeat.

ERRORS, MISCONCEPTIONS, AND MISSING KNOWLEDGE

Example Error	Misconception	Missing Knowledge
Student provides square roots of perfect squares as examples of irrational numbers (e.g., $\sqrt{4}, \sqrt{25}$).	believes the square root of any value is irrational	RECOGNIZE *IRRATIONAL NUMBERS*
Student provides rational numbers rather than irrational values.	confuses rational and irrational numbers	EXPLAIN *RATIONAL NUMBERS* and EXPLAIN *IRRATIONAL NUMBERS*
Student provides five negative numbers.	believes negative numbers are irrational numbers	EXPLAIN *IRRATIONAL NUMBERS*
Student provides five decimals with infinite digits, including repeating decimals (e.g., 0.3333…).	believes if a decimal has an infinite number of digits it is an irrational number, regardless of whether or not it is a repeating decimal	EXPLAIN *IRRATIONAL NUMBERS*
Student provides a combination of five fractions, decimals, and/or negative values	believes numbers that are not whole numbers are irrational numbers	EXPLAIN *IRRATIONAL NUMBERS*
Student provides five irrational numbers but cannot explain how they know they are irrational.	identifies but cannot describe an irrational number	EXPLAIN *IRRATIONAL NUMBERS*
Student writes five decimals with no pattern but without an ellipse, thus creating a decimal that terminates and therefore a rational number.	does not show knowledge that an irrational number has an infinite number of digits	EXPLAIN IRRATIONAL NUMBERS

Example Error	Misconception	Missing Knowledge
Student writes a non-real number such as $\frac{4}{0}$.	confuses irrational numbers with nonreal numbers	EXPLAIN *IRRATIONAL* NUMBERS

3. Write the following decimal as a fraction in simplest form. Explain your reasoning.

3.18

CORRECT ANSWER

$3\frac{18}{100} = 3\frac{9}{50}$

This decimal is read "three and eighteen hundredths", therefore you can write the equivalent mixed number, $3\frac{18}{100}$, and simplify the fraction to $3\frac{9}{50}$.

[*NOTE: An improper fraction,* $\frac{159}{50}$*, is also an accurate response.*]

ERRORS, MISCONCEPTIONS, AND MISSING KNOWLEDGE

Example Error	Misconception	Missing Knowledge
$3\frac{18}{10} = 3\frac{9}{5}$	confuses or cannot distinguish the tenths and the hundredths place when reading the decimal	EXPLAIN PLACE VALUE FOR HUNDREDTHS
$3\frac{18}{1000} = 3\frac{9}{500}$	confuses or cannot distinguish the thousandths and the hundredths place when reading the decimal	EXPLAIN PLACE VALUE FOR HUNDREDTHS
$\frac{3.18}{1}$	knows numbers can be written with a denominator of one but does not understand how to use place value to write a terminating decimal as a fraction	CONVERT A RATIONAL DECIMAL INTO A FRACTION

4. Write the following decimal as a fraction in simplest form. Show your work.

1.45454545…

CORRECT ANSWER

Let $n = 1.45454545\ldots$

Multiply both sides of the equation by 100 (or 10^2) because there are two values in the repetend, resulting in

$100n = 145.45454545\ldots$

Subtract $n = 1.45454545\ldots$ from $100n = 145.45454545\ldots$ to create a terminating decimal and solve for n.

$$100n = 145.45454545\ldots$$
$$\underline{-(\,n = \quad 1.45454545\ldots)}$$
$$\frac{99n}{99} = \frac{144}{99}$$
$$n = \frac{144}{99}$$

$$n = \frac{48}{33}$$

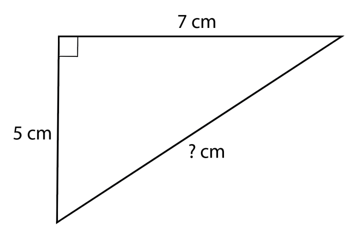
[*NOTE: A mixed number,* $1\frac{15}{33}$*, is also an accurate response.*]

ERRORS, MISCONCEPTIONS, AND MISSING KNOWLEDGE

Example Error	Misconception	Missing Knowledge
Student multiplies by 10 instead of 100 and believes the result of the subtraction is a rational number.	does not realize the digits to the right of the decimal must correspond to each other in order to create a rational number through subtraction	CONVERT A RATIONAL DECIMAL INTO A FRACTION
Student makes a mistake solving for the variable.	cannot solve a one step equation in the form $px = q$	SOLVE LINEAR EQUATIONS IN 1 VARIABLE WITH RATIONAL OR REAL NUMBER COEFFICIENTS

Example Error	Misconception	Missing Knowledge
Student does not know how to begin converting a decimal that repeats to a fraction.	does not understand the strategy of creating a decimal that is a factor of ten greater than the repeating decimal in order to use subtraction to obtain a rational number	CONVERT A RATIONAL DECIMAL INTO A FRACTION
Student does not show work and writes a fraction with a denominator of 9 or 99 or 999, etc.	does not know how to convert a rational decimal into a fraction but recalls that the examples in Lesson 1 often had denominators of nines and attempts to write a fraction similar to the examples	CONVERT A RATIONAL DECIMAL INTO A FRACTION

5. Use the following right triangle to answer the questions provided.



5.a. Determine the length of the hypotenuse. Show your work.

CORRECT ANSWER

$5^2 + 7^2 = c^2$

$25 + 49 = c^2$

$74 = c^2$

$\sqrt{74} = c$

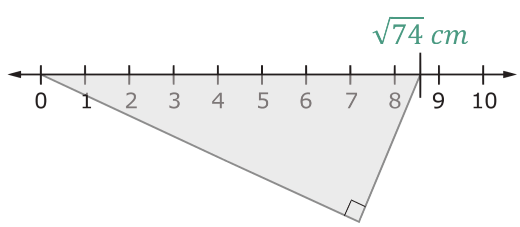
The length of the hypotenuse is $\sqrt{74}$ centimeters.

ERRORS, MISCONCEPTIONS, AND MISSING KNOWLEDGE

Example Error	Misconception	Missing Knowledge
The length of the hypotenuse is $\sqrt{12}$ centimeters.	forgets to square the lengths of the legs of the right triangle when using the Pythagorean theorem	DETERMINE AN UNKNOWN SIDE IN A RIGHT TRIANGLE USING THE PYTHAGOREAN THEOREM
The length of the hypotenuse is 74 centimeters.	gives the length of the hypotenuse squared rather than the length of the hypotenuse; forgets to take the square root of the sum of the leg lengths squared when using the Pythagorean theorem	DETERMINE AN UNKNOWN SIDE IN A RIGHT TRIANGLE USING THE PYTHAGOREAN THEOREM and EXPLAIN *SQUARE ROOT*

5.b. Partition the number line provided such that each interval is one centimeter. Then cut out a triangle with the same dimensions as labeled in the triangle shown. Use the triangle to locate and label the length of the hypotenuse on the number line. Trace or glue the triangle you use to support your work.

CORRECT ANSWER



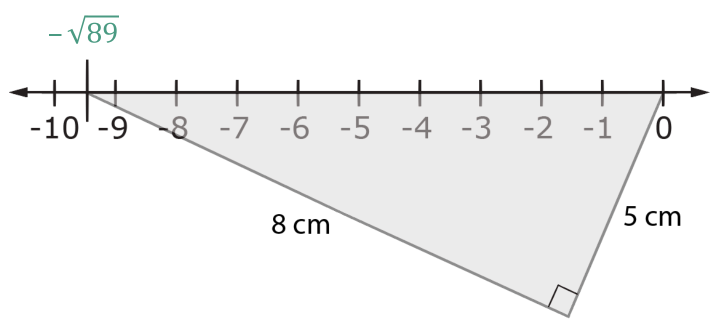
ERRORS, MISCONCEPTIONS, AND MISSING KNOWLEDGE

Example Error	Misconception	Missing Knowledge
Student inaccurately approximates the location of $\sqrt{74}$ without using a triangle.	cannot approximate an irrational value and does not understand how to use a geometric approach to locate an irrational number on the number line	REPRESENT THE APPROXIMATE LOCATION OF AN IRRATIONAL NUMBER ON A NUMBER LINE
Student inaccurately marks the intervals on the number line, then uses an accurately drawn triangle to locate $\sqrt{74}$.	does not understand the importance of the relationship between the length units in the triangle and precision when partitioning the number line	REPRESENT THE APPROXIMATE LOCATION OF AN IRRATIONAL NUMBER ON A NUMBER LINE

Example Error	Misconception	Missing Knowledge
Student represents $-\sqrt{74}$ on the number line.	does not distinguish the difference between $-\sqrt{74}$ and $\sqrt{74}$	REPRESENT THE APPROXIMATE LOCATION OF AN IRRATIONAL NUMBER ON A NUMBER LINE
Student lines up the initial vertex of the hypotenuse with 1 or a number other than 0.	does not recognize that the length of the hypotenuse must be shown with a starting value of zero	REPRESENT THE APPROXIMATE LOCATION OF AN IRRATIONAL NUMBER ON A NUMBER LINE

6. Use a right triangle with leg lengths of eight centimeters and five centimeters to locate $-\sqrt{89}$ on the number line. Trace or glue the triangle you use to support your work.

CORRECT ANSWER



ERRORS, MISCONCEPTIONS, AND MISSING KNOWLEDGE

Example Error	Misconception	Missing Knowledge
Student locates $\sqrt{89}$ on the number line.	does not understand how to locate a negative irrational number on the number line using a right triangle	REPRESENT THE APPROXIMATE LOCATION OF AN IRRATIONAL NUMBER ON A NUMBER LINE
Student incorrectly approximates the location of $-\sqrt{89}$ on the number line without using a right triangle.	does not understand how or why constructing a right triangle helps locate irrational square roots on the number line	REPRESENT THE APPROXIMATE LOCATION OF AN IRRATIONAL NUMBER ON A NUMBER LINE

Example Error	Misconception	Missing Knowledge
Student states $-\sqrt{89}$ cannot be located on the number line because it is a negative square root.	believes any square root with a negative sign associated with it is impossible to locate on a number line	REPRESENT THE APPROXIMATE LOCATION OF AN IRRATIONAL NUMBER ON A NUMBER LINE and EXPLAIN *IRRATIONAL NUMBERS*
Student lines up the initial vertex of the hypotenuse with 1 or a number other than 0.	does not recognize that the length of the hypotenuse must be shown with a starting value of zero	REPRESENT THE APPROXIMATE LOCATION OF AN IRRATIONAL NUMBER ON A NUMBER LINE
Student inaccurately marks the intervals on the number line, then uses an accurately drawn triangle to locate $-\sqrt{89}$.	does not understand the importance of the relationship between the length units in the triangle and precision when partitioning the number line	REPRESENT THE APPROXIMATE LOCATION OF AN IRRATIONAL NUMBER ON A NUMBER LINE
Student inaccurately marks the negative values on the number line, then uses an accurately drawn triangle to locate $-\sqrt{89}$.	does not understand how to label negative numbers on a number line	LOCATE NEGATIVE NUMBERS ON A NUMBER LINE

7. Provide an irrational number located between 4.2 and 4.3 on the number line. Give your answer as a square root. Explain how you determined this number. You may use a calculator to help you.

CORRECT ANSWER

[*Student responses may vary, but the most likely correct response is provided.*]

$\sqrt{18}$ is located between 4.2 and 4.3 on the number line.

I know $\sqrt{16} = 4$, and because 4.2 and 4.3 are a little greater than 4, I tried the square roots of numbers greater than 16. I found $\sqrt{17} \approx$ 4.1231056... which is still less than 4.2. Then I found $\sqrt{18} \approx 4.24264...$, which is greater than 4.2 and less than 4.3, and therefore would be located between 4.2 and 4.3 on the number line.

[*Another possible response.*]

$\sqrt{18}$ is located between 4.2 and 4.3 on the number line.

I found that 4.2^2 is 17.64 and 4.3^2 is 18.49. Since 18 is between 17.64 an 18.49, $\sqrt{18}$ must be between 4.2 and 4.3.

ERRORS, MISCONCEPTIONS, AND MISSING KNOWLEDGE

Example Error	Misconception	Missing Knowledge
$\sqrt{17}$ because $\sqrt{16}$ is equivalent to four and $\sqrt{17}$ is a little more than four, so it could be between 4.2 and 4.3 on the number line.	does not approximate the irrational number with enough precision to locate the value on a number line labeled to the tenths place	REPRESENT THE APPROXIMATE LOCATION OF AN IRRATIONAL NUMBER ON A NUMBER LINE
Between $\sqrt{4}$ and $\sqrt{5}$ because 4.2 and 4.3 are between four and five.	considers $\sqrt{4}$ and $\sqrt{5}$ as four and five, ignoring the meaning of the square root symbol, but associates the square root symbol with irrational numbers	RECOGNIZE *IRRATIONAL NUMBERS* and REPRESENT THE APPROXIMATE LOCATION OF AN IRRATIONAL NUMBER ON A NUMBER LINE
It is impossible to determine an irrational number located between 4.2 and 4.3 on the number line.	may know the value of the square root of a perfect square, but does not know how to use this reasoning to approximate values of irrational numbers	REPRESENT THE APPROXIMATE LOCATION OF AN IRRATIONAL NUMBER ON A NUMBER LINE

8. Describe where $\sqrt{68}$ is located on a number line labeled with rational numbers to the hundredths place. Explain your reasoning.

CORRECT ANSWER

Because the decimal representation of $\sqrt{68}$ is approximately 8.24621125..., $\sqrt{68}$ would be located between 8.24 and 8.25 on a number line labeled with rational numbers to the hundredths place.

ERRORS, MISCONCEPTIONS, AND MISSING KNOWLEDGE

Example Error	Misconception	Missing Knowledge
$\sqrt{68}$ is between eight and nine on the number line because 68 is between 64 and 81.	approximates the value of $\sqrt{68}$ to the whole number rather than the hundredths place	REPRESENT THE APPROXIMATE LOCATION OF AN IRRATIONAL NUMBER ON A NUMBER LINE

Example Error	Misconception	Missing Knowledge
It is impossible to determine the location of $\sqrt{68}$ on the number line because 68 is not a perfect square.	does not know how to use the decimal representation of an irrational number to describe the location of the number on a number line	REPRESENT THE APPROXIMATE LOCATION OF AN IRRATIONAL NUMBER ON A NUMBER LINE

9. Order the following values from least to greatest.

$$\sqrt{18}, -\pi, 0, -\sqrt{23}, \sqrt{11}, e, -\sqrt{5}$$

CORRECT ANSWER

Least → Greatest: $-\sqrt{23}, -\pi, -\sqrt{5}, 0, e, \sqrt{11}, \sqrt{18}$

ERRORS, MISCONCEPTIONS, AND MISSING KNOWLEDGE

Example Error	Misconception	Missing Knowledge
$e, -\pi, 0, -\sqrt{5}, \sqrt{11}, \sqrt{18}, -\sqrt{23}$ or $0, -\sqrt{5}, \sqrt{11}, \sqrt{18}, -\sqrt{23}, e, -\pi$	does not know that e and π are irrational numbers or does not know the value of e and π and only considers the value under the square root without considering if the square root is positive or negative	RECOGNIZE *IRRATIONAL NUMBERS* and COMPARE THE SIZES OF IRRATIONAL NUMBERS
Student misplaces e and $-\pi$ but orders the remaining values correctly.	does not know that e and π are irrational numbers or does not know the value of e and π	RECOGNIZE *IRRATIONAL NUMBERS*
$\sqrt{18}, \sqrt{11}, e, 0, -\sqrt{5}, -\pi, -\sqrt{23}$	orders the values from greatest to least	COMPARE THE SIZES OF IRRATIONAL NUMBERS
$-\pi, 0, -\sqrt{5}, e, \sqrt{11}, \sqrt{18}, -\sqrt{23}$	correctly orders $0, -\pi$, and e, but orders the square roots as though they are all positive; may believe you cannot have a negative associated with a square root	COMPARE THE SIZES OF IRRATIONAL NUMBERS