A simple Hidden Markov Model

Alice and Bob are friends who live far apart. Each day, Bob either goes for a walk, goes to an amusement park, or plays video games all day. During their daily calls (aw besties) Bob tells Alice which activity he performed that day. The probability that Bob does each of the three activities is dependent on the weather. Alice does not know the exact weather conditions where Bob lives, but knows general trends.

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Weather **transition matrix** (A):

	Sun_{t+1}	Rain_{t+1}
Sun_t	.8	.2
Rain_t	.4	.6

Activities **emission matrix** (B):

	Walk	Park	Game
Sun	.6	.3	.1
Rain	.9	.05	.05

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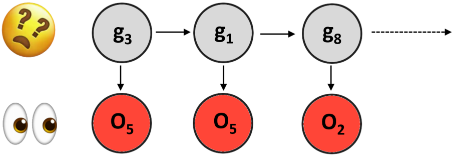
Hidden Markov Magic

A Likelihood Model for Sampled (Observed) Data: HMMs



General HMM:

- K hidden states, $\{g_1, ..., g_K\}$
- random variable $y_t \in \{g_i\}$
- M possible observations $\{O_1, ..., O_m\}$

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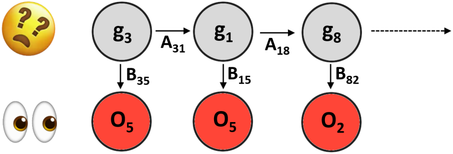

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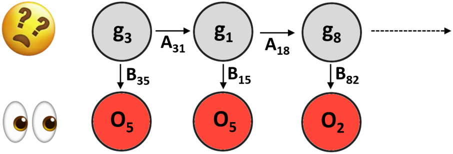

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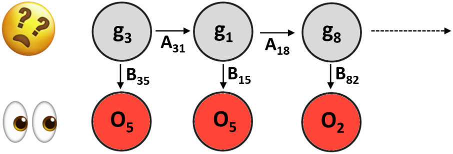

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- Discretized allele frequencies $\{g_i\}$
- random variable $y_t \in \{g_i\}$
- Sampled allele counts $\{O_j\}$
- $A =$ Gaussian transition probabilities
- $B_{ij} = \text{Bin}(O_j, g_i)$
- $\pi =$ uniform, delta function

Computing expected values in HMMs

Forward variable | Backward variable

$$\alpha_t(i) = \mathbb{P}(O_1, O_2, ...O_t, y_t = g_i),\ \beta_t(i) = \mathbb{P}(O_{t+1}, O_{t+2}, ...O_T | y_t = g_i)$$

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$$\downarrow$$

Posterior distribution of hidden state at time t:

$$\gamma_t(i) = \mathbb{P}(y_t = g_i | O) = \frac{\alpha_t(i)\beta_t(i)}{\mathbb{P}(\{O\})}$$

Joint posterior:

$$\xi_t(i,j) = \mathbb{P}(y_t = g_i, y_{t+1} = g_j | O) = \frac{\alpha_t(i) A_{ij} B_j(O_{t+1}) \beta_{t+1}(j)}{\mathbb{P}(\{O\})}$$

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We get likelihoods for free!

$$\mathbb{P}(\{O\}) = \sum_{i=1}^{K} \alpha_T(i)$$