Introduction to Renormalization – QCD

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Abstract

We will show the renormalization procedures for Quantum Chromodynamics , and derive the renormalization group equation of strong coupling, α_s . As a non-Abelian gauge theory, the ghost fields have to be introduced. Because the contributions from the ghost fields and 3-gluon interaction, more Feynman diagrams of fermion self-energy, gauge boson self-energy and quark-quark-gluon interaction are needed to get $Z_{1,2,3}$. And the simple relation of $Z_1 = Z_2$ is not valid. We use the dimensional regularization to regularize the UV divergences in our calculation.

PACS numbers:

I. INTRODUCTION

In the previuos two lectures, we have introduced the procedures to regularize the divergences as calculating the one-loop Feynman diagrams of the Quantum Electrodynamics (QED). We also derived the running coupling which has a powerful prediction on the coupling strength at the different energy scales. In this lecture, we will use the same techniques to deal with another successful theory, Quantum Chromodynamics (QCD).

QCD is the theory describing the strong interaction of quarks and gluons at all energy scales. Unlike QED, QCD is a non-Abelian gauge theory described by $SU(3)$ gauge theory, the quanta, gluons, carry the color quantum numbers, and could interact with each other. This causes a significant results, asymptotic freedom, which means the coupling strength becomes weaker as energy scale becomes higher, we will see this features later as we derive the renormalization group equation. And also because of the asymptotic freedom, the interactions between quarks and gluons are so strong at low energy scale that the quarks will form bound states to be hadrons, either mesons ($\bar{q}q$ bound state) or baryons (qqq bound state). All the hadrons seen in Nature are color singlet objects, and no free quark is observed. Since we can not observe the colored object, it is interesting to know how the 3-color model be confirmed. The first suggestive experiment was $\pi \to \gamma\gamma$ decay, described by the Fig. 1 (a). Another experiment to confirm the number of colors was the ratio R, defined as $R \equiv \sigma(e^+e^- \to quarks)/\sigma(e^+e^- \to \mu^+\mu^-)$, shown in Fig. 1 (b). Since the coupling strength is strong at low energy, the perturbation calculations are not reliable anymore. The techniques to do the calculations at the low energy scale is Lattice QCD, we are not interested in this aspect in this notes. This note is organized as follows, we give an introduction of colors in Sec. II. The Lagrangian and renormalization of QCD are give in Sec. III. The Sec. IV is the summary of this notes. We also give the Feynman rules in Appendix. Some useful formulas can be found in the QED renormalization notes.

II. SU(3) AND COLORS

In the QCD, new quantum numbers are introduced, called colors. Colored Quark transforms as a fundamental representation under SU(3), Each quark carries one of the three colors which are red, blue and green. The gluon, quanta of QCD, transforms as an adjoint

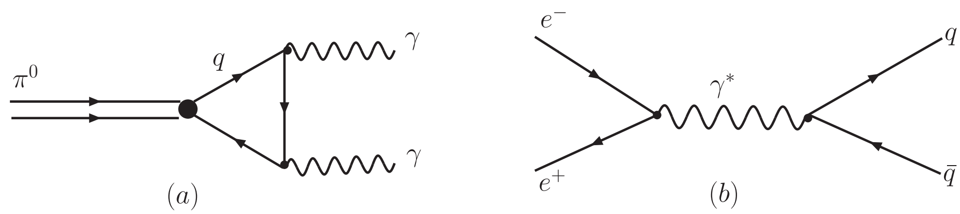


Figure 1: (a) π^0decays to $\gamma\gamma$, (b) e^+e^- annihilation to quarks.

representation under SU(3). The eight generators, t^a, of SU(3) should satisfy the Lie algebra,

$$[t^A, t^A] = if^{ABC}t^C, \ (A, B, C) = 1 \sim 8,$$

where f^{ABC} is the structure function. One of the representations for the generators is provided by Gell-Mann matrices, λ^a, which are hermitian and traceless,

$$t^A = \frac{1}{2}\lambda^A$$

$$\lambda^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ \lambda^2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ \lambda^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\lambda^4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \ \lambda^5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \ \lambda^6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$\lambda^7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \ \lambda^8 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

By convention, the normalization of the SU(N) matrices is chosen to be

$$Tr\{t^A t^B\} = \frac{1}{2}\delta^{AB}.$$

With this choice, we can have the color matrices obey the following relations which are needed in calculations,

$$\begin{aligned} \sum_A t^A_{ab} t^A_{bc} &= C_F \delta_{ac}\,,\, C_F = \frac{4}{3}; \\ Tr\{T^C T^D\} &= \sum_{A,B} f^{ABC} f^{ABD} = C_A \delta^{CD}\,,\ \text{where } (T^C)_{AB} \equiv -if^{CAB},\, C_A = 3. \end{aligned}$$

III. RENORMALIZATION OF QCD

In this section, we will proceed what we did in the QED and see how it becomes more complicated because of the self-interaction of gluons. First, we begin with the QCD Lagrangian,

$$\begin{aligned}\mathcal{L}_{QCD} &= -\frac{1}{4}F_{0\mu\nu}F^{0\mu\nu} + \bar{\Psi}_0(i\not{\partial} + m_0)\Psi_0 - \bar{c}_0^a\partial^2 c_0^a + g_0\bar{\Psi}_0\gamma_\mu t^a\Psi_0 A_0^{a\mu} - g_0 f_{abc}(\partial_\mu A_{0\nu}^a)A_0^{b\mu}A_0^{c\nu} \\ &\quad -g_0^2 f_{abc}f_{dec}A_{0\mu}^a A_{0\nu}^b A_0^{c\mu}A_0^{d\nu} - g_0 f_{abc}\bar{c}_0^a(\partial^\mu A_{0\mu}^b c_0^c), \qquad (1)\end{aligned}$$

where Ψ is the quark field, $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$, A is the gluon field and c is the ghost field. As we did in QED, the subscript "0" denotes the "bare" quantity. Now we introduce $Z's$ to define the normalized fields, therefore, we rewrite the Eq. (1) in terms of renormalized (without the subscript "0") quantities as

$$\begin{aligned}\mathcal{L}_{QCD} &= -\frac{1}{4}Z_3 F_{\mu\nu}F^{\mu\nu} + Z_2\bar{\Psi}(i\not{\partial} + m)\Psi - Z_c\bar{c}^a\partial^2 c^a + Z_1 g\bar{\Psi}\gamma_\mu t^a\Psi A^{a\mu} - Z_1^{3g} g f_{abc}(\partial_\mu A_\nu^a)A^{b\mu}A^{c\nu} \\ &\quad -Z_1^{4g} g^2 f_{abc}f_{dec}A_\mu^a A_\nu^b A^{c\mu}A^{d\nu} - Z_1^c g f_{abc}\bar{c}^a(\partial^\mu A_\mu^b c^c), \qquad (2) \\ &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\Psi}(i\not{\partial} + m)\Psi - \bar{c}^a\partial^2 c^a + g\bar{\Psi}\gamma_\mu t^a\Psi A^{a\mu} - g f_{abc}(\partial_\mu A_\nu^a)A^{b\mu}A^{c\nu} \\ &\quad -g^2 f_{abc}f_{dec}A_\mu^a A_\nu^b A^{c\mu}A^{d\nu} - g f_{abc}\bar{c}^a(\partial^\mu A_\mu^b c^c) + \mathcal{L}_{counter}.\end{aligned}$$

The Feyman rules are given in the Appendix.

By comparing Eqs. (1) and (2), we can easily have

$$\Psi_0 = \sqrt{Z_2}\Psi,\ A_0^{a\mu} = \sqrt{Z_3}A^{a\mu},\ c_0^a = \sqrt{Z_c}c^a \qquad (3)$$

and

$$Z_1 g = Z_2\sqrt{Z_3}g_0 \qquad (4)$$

$$Z_1^{3g} g = Z_3^{3/2} g_0 \qquad (5)$$

$$Z_1^{4g} g^2 = Z_3^2 g_0^2 \qquad (6)$$

$$Z_1^c g = Z_c\sqrt{Z_3}g_0\,. \qquad (7)$$

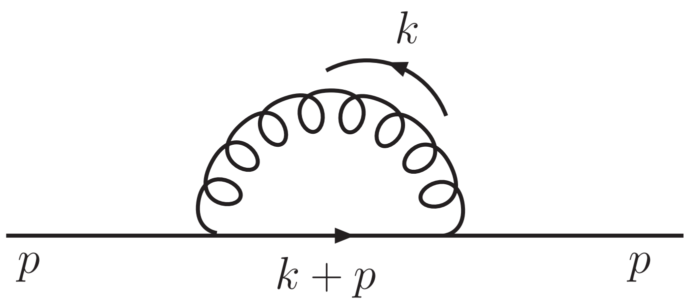


Figure 2: The quark self-energy Feynman diagram.

We could choose one relation among the Eqs. (4), (5), (6) and (7) to derive the running coupling g, or $\alpha_s(\equiv \frac{g^2}{4\pi})$. Here, we choose Eq. (4), since we already have the results for Z_1, Z_2 and Z_3 from QED. The extra works for QCD is that we need to take care about the colors and add in more diagrams. From Eq. (4), we have

$$g^2 = \frac{Z_2^2}{Z_1^2} Z_3 g_0^2 \,. \tag{8}$$

As we learn in QED case, g, $Z_{1,2,3}$ are scale dependent, therefore, we can take the derivative and get

$$\begin{aligned} \frac{dg^2}{d\mu^2} &= \left(\frac{2Z_2}{Z_1^2}\frac{dZ_2}{d\mu^2} Z_3 - \frac{2Z_2^2}{Z_1^3}\frac{dZ_1}{d\mu^2} Z_3 + \frac{Z_2^2}{Z_1^2}\frac{dZ_3}{d\mu^2} \right) g_0^2 \\ &= \left(\frac{2}{Z_2}\frac{dZ_2}{d\mu^2} - \frac{2}{Z_1}\frac{dZ_1}{d\mu^2} + \frac{1}{Z_3}\frac{dZ_3}{d\mu^2} \right) g^2 \end{aligned} \tag{9}$$

The Feynman diagram related to Z_2 is shown in Fig. 2. Compared to the diagram in QED, the only difference is that the color factor is needed. So the result is the result of QED multiplied by color factor, $t^A t^A$. From Eq. (26) in the QED notes, taking $n \to 4$ and $p^2 = -\mu^2$, we got the Z_2 in QCD is

$$Z_2 = 1 + \frac{\partial \sum_2}{\partial \not{p}} = 1 - \frac{\alpha_s}{4\pi} \frac{\pi^{n/2-2}\Gamma(2-n/2)}{(\mu^2)^{2-n/2}} C_F \tag{10}$$

where $\alpha_s \equiv g_s^2/(4\pi)$. The Feynman diagrams related to Z_1 are shown in Fig. 3. Form the vertex Feynman rule, we write the corrections of vertex from one loop diagrams as $-ig_s\Lambda^\mu t^A$. The Fig. 3 (a) can be easily got from the QED result, and multiply it by the factor factor, $t^B t^A t^B$, which we need to calculate.

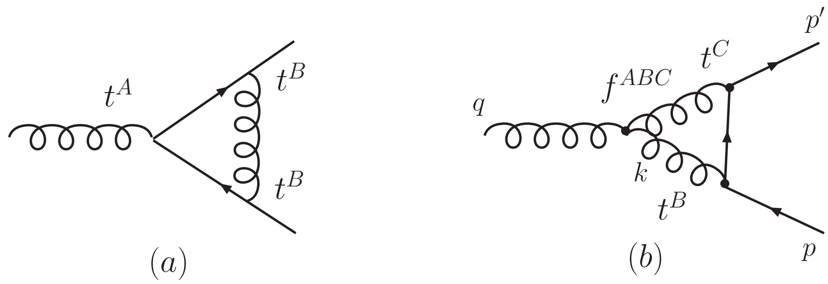


Figure 3: The Feynman diagrams of one-loop correction to vertex of quark-quark-gluon.

$$
\begin{aligned}
t^B t^A t^B &= t^B([t^A,\, t^B] + t^B t^A) \\
&= t^B(if^{ABC}t^C) + t^B t^B t^A \\
&= i\frac{f^{ABC}[t^B,\, t^C]}{2} + C_F t^A \\
&= if^{ABC}if^{BCD}t^D/2 + C_F t^A \\
&= -Tr\{T^A T^D\}\frac{t^D}{2} + C_F t^A \\
&= (\frac{-C_A}{2} + C_F)t^A
\end{aligned} \tag{11}
$$

Therefore,

$$
\begin{aligned}
\Lambda^\mu(\text{Fig.}3a) &= (\frac{-C_A}{2} + C_F)\Lambda^\mu(\text{QED},\, e \to g_s) \\
&= (\frac{-C_A}{2} + C_F)\frac{\alpha_s}{4\pi}\frac{\pi^{n/2-2}\Gamma(2-n/2)}{(\mu^2)^{2-n/2}}\,.
\end{aligned} \tag{12}
$$

Now we will work on the diagram Fig. 3 (b), $\Lambda'^\mu(p,\, p')$.

$$
\text{Fig.}3(b) = -ig_s^3\frac{f^{ABC}t^C t^B}{(2\pi)^4}\int d^4k\frac{N^\mu}{D}, \tag{13}
$$

where

$$
N^\mu = \gamma_\lambda\, \not{k}\gamma_\nu(g^{\mu\nu}k^\lambda - 2g^{\nu\lambda}k^\mu + g^{\mu\lambda}k^\nu), \tag{14}
$$

$$
D = (k^2 + i\epsilon)(k^2 + 2p\cdot k + i\epsilon)(k^2 - 2q\cdot k + q^2 + i\epsilon). \tag{15}
$$

Here, we already drop the linear term in the N^μ, since the integral is zero for linear term. The color factor

$$f^{ABC}t^C t^B = \frac{1}{2}f^{ABC}[t^C,\, t^B]$$
$$= \frac{1}{2}f^{ABC}if^{CBD}t^D$$
$$= -\frac{i}{2}(T^A)_{BC}(T^D)_{CB}t^D$$
$$= -\frac{i}{2}C_A t^A. \tag{16}$$

The Λ(Fig.3b) then is

$$\Lambda(\text{Fig.}3b) = -i\frac{C_A}{2}g_s^2\frac{1}{(2\pi)^4}\int d^4k\frac{N^\mu}{D}. \tag{17}$$

By using the Feynman tricks to evaluate the denominator,

$$\frac{1}{D} = 2\int_0^1 dx\int_0^y dy\frac{1}{\{[k+px-(q+p)y]^2-\Delta\}^3}$$
$$= 2\int_0^1 dx\int_0^y dy\frac{1}{[\tilde{k}^2-\Delta]^3}, \tag{18}$$

where $\tilde{k} = k+px-(q+p)y$ and $\Delta = -q^2y(1-x)$. The numerator now is

$$N^\mu = 2k^2\gamma^\mu - 2\gamma^\nu \not{k}\gamma_\nu k^\mu$$
$$= 2(\tilde{k}^2+qxy)\gamma^\mu - 2(2-n)(\not{\tilde{k}} - x\not{p} + y\not{p}')(\tilde{k}-xp+yp')^\mu.$$

After taking $n\to 4$ and $p=p'$, i.e. $q\to 0$, for non-divergent term, and let $-q^2\to\mu^2$, we have

$$\Lambda(\text{Fig.}3b) = \frac{C_A}{2}g_s^2\frac{\pi^2}{(2\pi)^4}\int_0^1 dx\int_0^x dy\frac{6\Gamma(2-n/2)}{[-q^2y(1-x)+i\epsilon]^{2-n/2}}\gamma^\mu$$
$$= \frac{3C_A}{2}\frac{\alpha_s}{4\pi}\frac{\pi^{n/2-2}\Gamma(2-n/2)}{(\mu^2)^{2-n/2}}. \tag{19}$$

Combine Eq. (12) and Eq. (19), We shall have

$$Z_1 = 1-\frac{\alpha_s}{4\pi}\frac{\pi^{n/2-2}\Gamma(2-n/2)}{(\mu^2)^{2-n/2}}\left(\frac{-C_A}{2}+C_F+\frac{3C_A}{2}\right)$$
$$= 1-\frac{\alpha_s}{4\pi}\frac{\pi^{n/2-2}\Gamma(2-n/2)}{(\mu^2)^{2-n/2}}(C_F+C_A). \tag{20}$$

Now the only piece left to derive the running α_s is Z_3. The relevant Feynman diagrams are shown in Fig. 4.

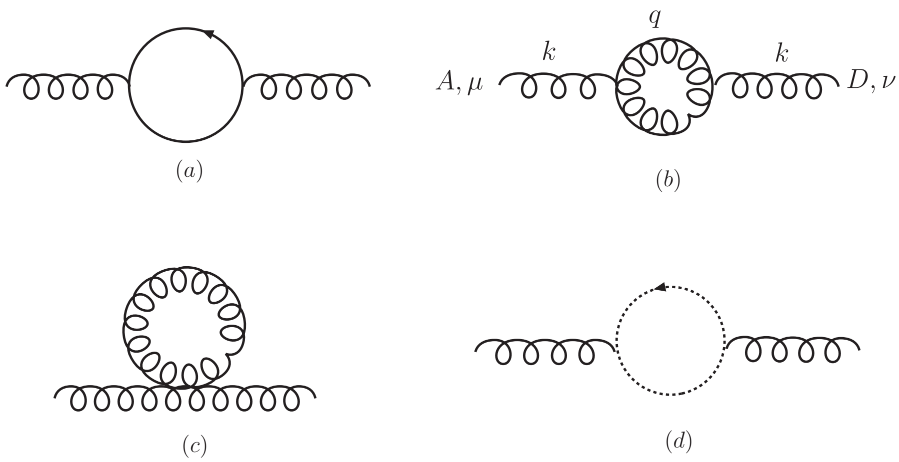


Figure 4: The one-loop contributions to the gluon self-energy.

The tensor of Fig. 4 (a) can be calculated easily by multiplying color factor to the QED result which we already have, i.e. Eq. (52) in lecture note II. The color factor is

$$Tr\{t^A t^A\} = \frac{1}{2}.$$

Therefore, the contribution from Fig. 4 (a) is

$$\begin{aligned}
\Pi_{(a)}(k^2 &= -\mu^2) = -\frac{1}{2}\frac{8\alpha_s}{4\pi}\frac{\pi^{n/2-2}\Gamma(2-n/2)}{(\mu^2)^{2-n/2}}\int_0^1 dx\frac{x(1-x)}{[x(1-x)]^{2-n/2}} \\
&= -\frac{1}{2}\frac{8\alpha_s}{4\pi}\frac{\pi^{n/2-2}\Gamma(2-n/2)}{(\mu^2)^{2-n/2}}\frac{\Gamma(2)\Gamma(2)}{\Gamma(4)} \\
&= -\frac{2}{3}\frac{\alpha_s}{4\pi}\frac{\pi^{n/2-2}\Gamma(2-n/2)}{(\mu^2)^{2-n/2}} \qquad (21)
\end{aligned}$$

The tensor of Fig. 4 (b) is

$$\begin{aligned}
\Pi^{\mu\nu}_{(b)}(k) &= -\frac{g^2 f^{ABC} f^{DCB}}{2(2\pi)^4 i}\int d^n q\frac{N^{\mu\nu}}{(q^2+i\epsilon)((q+k)^2+i\epsilon)} \\
&= \frac{g^2 C_A \delta^{AD}}{2(2\pi)^4 i}\int d^n q\frac{N^{\mu\nu}}{(q^2+i\epsilon)((q+k)^2+i\epsilon)}, \qquad (22)
\end{aligned}$$

where

$$\begin{aligned}
N^{\mu\nu} &= [(q+2k)^2+(q-k)^2]g^{\mu\nu} + n(2q+k)^\mu(2q+k)^\nu \\
&\quad +[-(q+2k)^\mu(2q+k)^\nu + (q-k)^\mu(q+2k)^\nu - (q-k)^\mu(2q+k)^\nu + \ \mu\leftrightarrow\nu]
\end{aligned}$$

Because $\int d^n q q^{-2}$ vanishes in dimensional regularization, any terms of the form

$$\int d^n q\frac{q^2}{q^2(q+k)^2}$$

should be consistently eliminated. The non-vanishing terms of $N^{\mu\nu}$ are

$$(2kq+5k^2)g^{\mu\nu}+n(2q+k)^\mu(2q+k)^\nu$$
$$+[-(q+2k)^\mu(2q+k)^\nu+(q-k)^\mu(q+2k)^\nu-(q-k)^\mu(2q+k)^\nu+\ \mu\leftrightarrow\nu].$$

Using Feynman trick, the denominator becomes

$$\int_0^1 dx\frac{1}{[(q+xk)^2+k^2x(1-x)]^2}\xrightarrow{q\to q-xk}\int_0^1 dx\frac{1}{[q^2-\Delta]^2},\tag{23}$$

where $\Delta=-k^2x(1-x)$, and the non-vanishing terms of $N^{\mu\nu}$ after symmetric integration are

$$4(n-\frac{3}{2})q^{\mu\nu}+[4k^2+(1-2x)k^2]g^{\mu\nu}+2[(\frac{n}{2}-1)(1-2x)^2-(1+x)(2-x)]k^\mu k^\nu.$$

After combining the denominators and using the useful integral

$$\int_0^1 dx(1-2x)F(x(1-x))=0,$$

the result for $\Pi^{\mu\nu}_{(b)}(k)$ is

$$\begin{aligned}\Pi^{\mu\nu}_{(b)}(k) &= \frac{g^2C_A\delta^{AD}\pi^{n/2}}{(2\pi)^4}\int_0^1\frac{dx}{(\Delta-i\epsilon)^{2-n/2}}\{[(n-\frac{3}{2})x(1-x)\Gamma(1-n/2)+2\Gamma(2-n/2)]k^2g^{\mu\nu}\\ &\quad -[(1-\frac{n}{2})(1-2x)^2+(1+x)(2-x)]\Gamma(2-n/2)k^\mu k^\nu.\end{aligned}\tag{24}$$

The contribution from Fig. 4 (c) is

$$\begin{aligned}\Pi^{\mu\nu}_{(c)}(k) &= -\frac{g^2f^{BAC}f^{CDB}}{(2\pi)^4i}\int d^nq\frac{(q+k)^\mu q^\nu}{(q^2+i\epsilon)((q+k)^2+i\epsilon)}\\ &= \frac{g^2C_A\delta^{AD}}{(2\pi)^4i}\int_0^1 dx\int d^nq\frac{q^\mu q^\nu-x(1-x)k^\mu k^\nu}{(q^2-\Delta)^2}\\ &= -\frac{g^2C_A\delta^{AD}\pi^{n/2}}{(2\pi)^4}\\ &\quad \int_0^1\frac{dx}{(\Delta-i\epsilon)^{2-n/2}}x(1-x)[\frac{1}{2}\Gamma(1-n/2)k^2g^{\mu\nu}-\Gamma(2-n/2)k^\mu k^\nu].\end{aligned}\tag{25}$$

Adding Fig. 4 (b) and (c) we have

$$\begin{aligned}\Pi^{\mu\nu}_{(b)+(c)}(k) &= \frac{g^2C_A\delta^{AD}\pi^{n/2}}{(2\pi)^4}\int_0^1\frac{dx}{(\Delta-i\epsilon)^{2-n/2}}\{[(n-2)x(1-x)\Gamma(1-n/2)+2\Gamma(2-n/2)]k^2g^{\mu\nu}\\ &\quad -[2+(1-\frac{n}{2})(1-2x)^2]\Gamma(2-n/2)k^\mu k^\nu\}\\ &= \frac{g^2C_A\delta^{AD}\pi^{n/2}\Gamma(2-n/2)}{(2\pi)^4}\\ &\quad \int_0^1\frac{dx}{(\Delta-i\epsilon)^{2-n/2}}\{[2-2x(1-x)]k^2g^{\mu\nu}-[2+(1-\frac{n}{2})(1-2x)^2]k^\mu k^\nu\}.\end{aligned}\tag{26}$$

Now, we got to show that, the $k^2 g^{\mu\nu}$ term and $k^\mu k^\nu$ term are equal, then we can rewrite Eq. (26) as

$$\Pi^{\mu\nu}_{(b)+(c)}(k) = (k^2 g^{\mu\nu} - k^\mu k^\nu)\Pi(k).$$

We will start from the second term of the coefficient of $k^\mu k^\nu$, and show it is indeed equal to the second term of the coefficient of $k^2 g^{\mu\nu}$.

$$\begin{aligned}
(1-\frac{n}{2})\int_0^1 \frac{dx(1-2x)^2}{(\Delta - i\epsilon)^{2-n/2}} &= (1-\frac{n}{2})\int_0^1 \frac{dx(1-4x+4x^2)}{(\Delta - i\epsilon)^{2-n/2}} \\
&= (1-\frac{n}{2})\frac{1}{(-k^2)^{2-n/2}}\int_0^1 dx(x^{n/2-2} - 4x^{n/2-1} + 4x^{n/2})(1-x)^{n/2-2} \\
&= (1-\frac{n}{2})\frac{1}{(-k^2)^{2-n/2}}\{\frac{\Gamma(n/2-1)\Gamma(n/2-1)}{\Gamma(n-2)} \\
&\quad -4\frac{\Gamma(n/2)\Gamma(n/2-1)}{\Gamma(n-1)} + 4\frac{\Gamma(1+n/2)\Gamma(n/2-1)}{\Gamma(n)}\} \\
&= \frac{1}{(-k^2)^{2-n/2}}\frac{\Gamma(n/2)\Gamma(n/2)}{\Gamma(n)}[\frac{(n-1)(n-2)}{1-n/2} + 4(n-1) - 4\frac{n}{2}] \\
&= \frac{-2}{(-k^2)^{2-n/2}}\frac{\Gamma(n/2)\Gamma(n/2)}{\Gamma(n)} \\
&= -2\int_0^1 \frac{dx}{(\Delta - i\epsilon)^{2-n/2}} x(1-x).
\end{aligned}$$

The The contribution from Fig. 4 (d) is proportional to the integral

$$\int \frac{d^n q}{q^2},$$

which vanishes. Finally, we have

$$\begin{aligned}
\Pi^{\mu\nu}_{Gauge}(k) &= \Pi^{\mu\nu}_{(b)}(k) + \Pi^{\mu\nu}_{(c)}(k) + \Pi^{\mu\nu}_{(d)}(k) \\
&= (k^2 g^{\mu\nu} - k^\mu k^\nu)\Pi_{Gauge}(k),
\end{aligned}$$

where

$$\begin{aligned}
\Pi_{Gauge}(k) &= \frac{g^2 C_A \delta^{AD}\pi^{n/2}\Gamma(2-n/2)}{(2\pi)^4}\int_0^1 \frac{dx}{(\Delta - i\epsilon)^{2-n/2}}[2 - 2x(1-x)] \\
&= \frac{\alpha_s C_A \pi^{n/2-2}\Gamma(2-n/2)}{4\pi(-k^2)^{2-n/2}}\frac{5}{3} \\
&= \frac{5C_A}{3}\frac{\alpha_s \pi^{n/2-2}\Gamma(2-n/2)}{4\pi(\mu^2)^{2-n/2}},\ (k^2 \to -\mu^2) \qquad (27)
\end{aligned}$$

Combining Eq. (21) , (27) and considering n_f light fermions with masses lighter than scale μ, we have

$$
\begin{aligned}
Z_3 &= 1 + \Pi(k^2 = -\mu^2) \\
&= 1 + \frac{\alpha_s \pi^{n/2-2}\Gamma(2-n/2)}{4\pi(\mu^2)^{2-n/2}}(\frac{5C_A}{3} - \frac{2n_f}{3}).
\end{aligned} \tag{28}
$$

Let's recall the Z_1 and Z_2, which are

$$
\begin{aligned}
Z_1 &= 1 - \frac{\alpha_s}{4\pi}\frac{\pi^{n/2-2}\Gamma(2-n/2)}{(\mu^2)^{2-n/2}}(C_F + C_A), \\
Z_2 &= 1 - \frac{\alpha_s}{4\pi}\frac{\pi^{n/2-2}\Gamma(2-n/2)}{(\mu^2)^{2-n/2}}C_F.
\end{aligned}
$$

Therefore, the derivatives with respective to μ^2 are

$$
\frac{\partial Z_1}{\partial \mu^2} = \frac{\alpha_s}{4\pi\mu^2}\Gamma(3-n/2)(C_F + C_A) = \frac{\alpha_s}{4\pi\mu^2}(C_F + C_A), \tag{29}
$$

$$
\frac{\partial Z_2}{\partial \mu^2} = \frac{\alpha_s}{4\pi\mu^2}\Gamma(3-n/2)C_F = \frac{\alpha_s}{4\pi\mu^2}C_F, \tag{30}
$$

$$
\frac{\partial Z_3}{\partial \mu^2} = -\frac{\alpha_s}{4\pi\mu^2}\Gamma(3-n/2)(\frac{5C_A}{3} - \frac{2n_f}{3}) = -\frac{\alpha_s}{4\pi\mu^2}(\frac{5C_A}{3} - \frac{2n_f}{3}). \tag{31}
$$

We can see that the pole for $n \to 4$ are all eliminated. From the Eq. (9), the running of QCD coupling at one-loop is then

$$
\begin{aligned}
\frac{dg^2}{d\mu^2} &= \left(\frac{2}{Z_2}\frac{dZ_2}{d\mu^2} - \frac{2}{Z_1}\frac{dZ_1}{d\mu^2} + \frac{1}{Z_3}\frac{dZ_3}{d\mu^2}\right) g^2 \\
&= \left(2\frac{\alpha_s}{4\pi\mu^2}C_F - 2\frac{\alpha_s}{4\pi\mu^2}(C_F + C_A) - \frac{\alpha_s}{4\pi\mu^2}(\frac{5C_A}{3} - \frac{2n_f}{3})\right) g^2 \\
&= -\frac{\alpha_s}{4\pi\mu^2}(\frac{11}{3}C_A - \frac{2}{3}n_f)g^2 \\
\frac{dg^2}{d\mu^2} &= -\frac{\alpha_s}{12\pi\mu^2}(11C_A - 2n_f)g^2,
\end{aligned} \tag{32}
$$

, take $C_A = 3$ and $\alpha_s = \frac{g^2}{4\pi}$,

$$
\frac{d\alpha_s}{d\mu^2} = -\frac{\alpha_s^2}{12\pi\mu^2}(33 - 2n_f). \tag{33}
$$

Since from Eqs. (10), (20) and (28), the $Z's = 1 + O(\alpha_s)$, i.e. $\partial Z/\partial\mu^2 \sim O(\alpha_s)$, therefore, in the second step of Eq. (32), we just keep the leading term which is at order $O(\alpha_s)$ inside the ( ). Solving Eq. (33), we have the relation of α_s at two different scales,

$$
\alpha_s(\mu_2^2) = \frac{\alpha_s(\mu_1^2)}{1 + \frac{\alpha_s(\mu_1^2)}{12\pi}(33 - 2n_f)\ln(\frac{\mu_2^2}{\mu_1^2})}. \tag{34}
$$

As we can see the asymptotic freedom that is, as energy scale becomes higher and higher the coupling α_s becomes weaker and weaker, because $(33 - 2n_f)$ is a positive value.

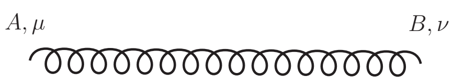
IV. CONCLUSION

In this notes, we have proceeded the renormalization of QCD. As a non-Abelian gauge theory, unlike QED, we needed to calculate more diagrams related to various renormalization Z factors, and take care of the color factors. The relations, $Z_1 = Z_2$, in QED case is no longer satisfied in QCD because the gluon will interact themselves and the ghost fields will also contribute. After carefully dealing with the poles for dimension $n \to 4$ in dimensional regularization, we see that all the singularities are eliminated in final results of α_s. And we can derive the β function and the running coupling $\alpha_s(\mu^2)$ to see the most important feature of QCD, the asymptotic freedom.

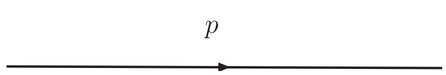
V. APPENDIX

Feynman rules of QCD:

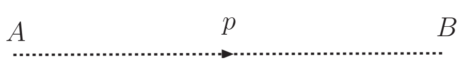
Propagators:



$-i\delta^{AB}[g^{\mu\nu} - (1-\lambda)\frac{p^\mu p^\nu}{p^2+i\epsilon}]/(p^2+i\epsilon)$, λ : Gauge choice.

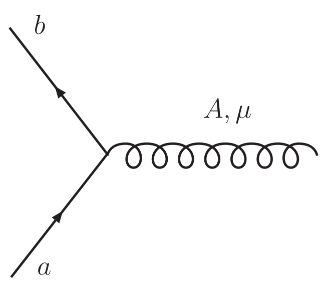


$\frac{i}{(\not p - m + i\epsilon)}$

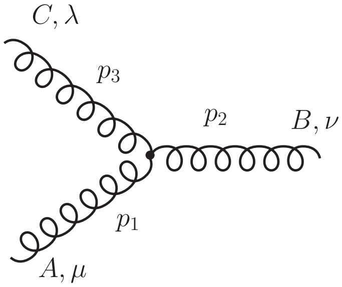


$\delta^{AB}\frac{i}{p^2+i\epsilon}$

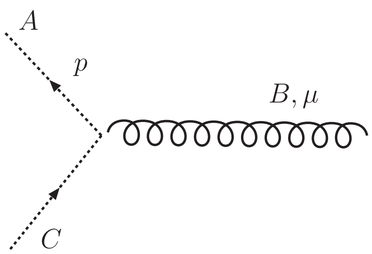
Vertex:



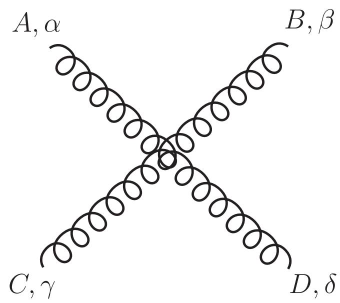
$-ig_s\gamma^\mu(t^A)_{ba}$



$$-g_s f^{ABC}\left[(p_1-p_2)^\lambda g^{\mu\nu}+(p_2-p_3)^\mu g^{\nu\lambda}+(p_3-p_1)^\nu g^{\lambda\mu}\right]$$



$$g_s f^{ABC} p^\mu$$



$$-ig_s^2 f^{XAC} f^{XBD}(g^{\alpha\beta}g^{\gamma\delta}-g^{\alpha\delta}g^{\beta\gamma})-ig_s^2 f^{XAD} f^{XBC}(g^{\alpha\beta}g^{\gamma\delta}-g^{\alpha\gamma}g^{\beta\delta})$$
$$-ig_s^2 f^{XAB} f^{XCD}(g^{\alpha\gamma}g^{\beta\delta}-g^{\alpha\delta}g^{\beta\gamma})$$