Conformal symmetries in QCD

Work notes

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1 Background

1.1 The conformal group on Minkowski space

1.1.1 From infinitesimals to finite transformations

Conformal transformations (at most) rescale the metric with an overall, possibly position dependent factor. The key equation is *local*, it involves the differential of the coordinate transformation $x \mapsto y(x)$:

$$g_{\mu\nu}(x) \mapsto g_{\mu'\nu'}(x(y)) \frac{\partial y^{\mu'}}{\partial x^\mu} \frac{\partial y^{\nu'}}{\partial x^\nu} = \Omega^2(x) g_{\mu\nu}(x) \ . \tag{1.1.1}$$

$\Omega(x)$ is called the conformal factor of the transformation $x \mapsto y$.

If the original metric in x coordinates is diagonal, then so is the new one in y coordinates. In particular the tangents to the mesh lines, the vectors $[\frac{\partial y^\circ}{\partial x^\nu}]$ (the columns of the Jacobian) form a locally orthogonal basis whose norm has been rescaled via the conformal factor $\Omega(x)$. $\Omega(x)$ is nothing but the determinant of the Jacobian:

$$\frac{\partial y^\mu}{\partial x^\nu} = \Omega(x) R(x)^\mu{}_\nu \qquad \text{with} \qquad \Omega(x) := \det \frac{\partial y^\circ}{\partial x^\circ} \qquad \text{so that} \qquad \det([R(x)^\circ{}_\circ]) \stackrel{!}{=} 1 \ . \tag{1.1.2}$$

Such a factorization is possible for *any* coordinate transformation –even if they do not satisfy (1.1.1)– as long as $\Omega(x) \neq 0$ (in which case we are facing a problem with the geometry of the transformation that I have not seen a use of to date).

With Eq. (1.1.1) imposed, $R^\mu{}_\nu$ becomes an isometry of the metric – in our case an element in $\mathsf{SO}(1,3)$, or more generally $\mathsf{SO}(1,d-1)$.

Literature:

- A good first resource is Sec 4.1 in P. Di Francesco, P. Mathieu and D. Senechal, "Conformal Field Theory" [1] (Note that my notation is subtly different: $\Lambda(x) = \Omega^2(x)$.)
- Paul Ginsparg has a set of Les Houches lecture notes on arXiv that provide a lot of the "standard" applications of conformal field theory [2]. Tobias Osbourne has an interesting set of lecture notes on YouTube `https://www.youtube.com/watch?v=NGYX6gtObec`.
- Both Slava Rychkov's "EPFL Lectures on Conformal Field Theory in $D \geq 3$ Dimensions" [3] and Hugh Osborn's "Lectures on Conformal Field Theories (in more than two dimensions)" [4] `./../../BOOKS/Quantum Field Theory/Osborn H -- Lectures on conformal field theories.pdf`(local copy) also discuss how fields transform. Osborn's discussion, in particular, provides the best discussion I have seen to date of how to integrate back up from infinitesimal to finite transformations. Note that one cannot obtain transformations that lie outside the component of the identity element in this manner.
- Farnsworth, Luty, and Prilepina "Weyl versus Conformal Invariance in Quantum Field Theory" [5].

Eq. (1.1.1) is the perfect starting point to derive the infinitesimal versions of the conformal transformations in flat space. There the starting metric in Cartesian coordinates is denoted $\eta_{\mu\nu}$. Here we work with the standard Minkowki metric $[\eta_{\circ\circ}] = \text{diagonalmatrix}(1, -1, -1, \ldots)$.

To identify the full set of infinitesimal transformations, we introduce one parameter families of coordinate transformations $y_s(x)$ such that $y_0(x) = x$ and isolate the linear term as follows:

$$y_s(x) = x + s\epsilon(x) + \mathcal{O}(s^2) \tag{1.1.3}$$

Varying s the conformal factor $\Omega_s(x) := \Omega(y_s(x))$ will follow, but satisfy $\Omega_0(x) = 1$, since we start from the flat Minkowski metric. To leading order in s, we parametrize its behavior as

$$\Omega_s(x) = 1 + s\sigma(x) + \mathcal{O}(s^2) \tag{1.1.4}$$

and note that $\sigma(x)$ must be fully determined by $\epsilon(x)$. The equation that determines the relation emerges by isolating the terms linear in s on both sides of

$$\eta_{\mu'\nu'}\frac{\partial y_s^{\mu'}}{\partial x^\mu}\frac{\partial y_s^{\nu'}}{\partial x^\nu} = \Omega_s^2(x)\eta_{\mu\nu} \ . \tag{1.1.5}$$

To extract these, we apply $\frac{d}{ds}|_0$ to both sides to obtain the constraint equation

$$\eta_{\mu'\nu'}\frac{\partial \epsilon^{\mu'}}{\partial x^\mu}\eta^{\nu'}{}_\nu + \eta_{\mu'\nu'}\eta^{\mu'}{}_\mu\frac{\partial \epsilon^{\nu'}}{\partial x^\nu} = 2\sigma(x)\eta_{\mu\nu} \tag{1.1.6}$$

or more compactly

$$\partial_\mu\epsilon_\nu + \partial_\nu\epsilon_\mu = 2\sigma\eta_{\mu\nu} \tag{1.1.7}$$

Contracting the indices with $\eta^{\mu\nu}$ on both sides provides an equation for σ in terms of ϵ^μ:

$$\partial_\mu\epsilon^\mu = d\sigma \tag{1.1.8}$$

where $d = \eta^\mu{}_\mu$ is the dimension of space-time.

The equations look formidable and do not look easily amenable to solutions, but one can derive an equation for σ alone by considering

$$\begin{aligned}\partial_\rho\partial_\nu\epsilon_\mu &= \frac{1}{2}\Big(\partial_\rho(\partial_\mu\epsilon_\nu + \partial_\nu\epsilon_\mu) + \partial_\nu(\partial_\mu\epsilon_\rho + \partial_\rho\epsilon_\mu) - \partial_\mu(\partial_\rho\epsilon_\nu + \partial_\nu\epsilon_\rho)\Big)\\ &= \partial_\rho\sigma\eta_{\mu\nu} + \partial_\nu\sigma\eta_{\mu\rho} - \partial_\mu\sigma\eta_{\rho\nu}\end{aligned} \tag{1.1.9}$$

Acting with ∂^μ and using (1.1.8) to simplify the l.h.s. gives

$$\partial_\rho\partial_\nu d\sigma = 2\partial_\rho\partial_\nu\sigma - \partial^2\sigma\eta_{\rho\nu} \Leftrightarrow (d-2)\partial_\rho\partial_\nu\sigma = -\partial^2\sigma\eta_{\rho\nu} \tag{1.1.10}$$

Contracting with $\eta^{\rho\nu}$ yields $(d-1)\partial^2\sigma = 0$ which can be used in $(d-1)\times$ Eq. (1.1.10) to yield

$$(d-1)(d-2)\partial_\rho\partial_\nu\sigma = 0 \tag{1.1.11}$$

In $d = 1$ and $d = 2$ this provides no constraint, but for larger d we have $\partial_\rho\partial_\nu\sigma = 0$, i.e. σ is at most linear in x and can be parametrized as

$$\sigma(x) = \kappa + 2b.x \tag{1.1.12}$$

with a real scalar κ and a vector b, both independet of coordinates.

Now that the structure of σ is known, we can reinsert it into Eq. (1.1.9) to obtain

$$\partial_\rho\partial_\nu\epsilon_\mu(x) = +2b_\rho\eta_{\mu\nu} - 2b_\mu\eta_{\rho\nu} + 2b_\nu\eta_{\mu\rho} \tag{1.1.13a}$$

subject to the constraint

$$\partial_\mu\epsilon^\mu(x) = d(\kappa + 2b.x) \tag{1.1.13b}$$

Note that all of the r.h.s. of (1.1.13a) is x independent, so that ϵ can be at most quadratic in x. It must be of the form

$$\epsilon_\mu(x) = a_\mu + B_{\mu\alpha} x^\alpha + C_{\mu\alpha\beta} x^\alpha x^\beta \tag{1.1.14}$$

with x independent tensor coefficients a_μ, $B_{\mu\nu}$, and $C_{\mu\alpha\beta}$.

First note that only the part of C that is symmetric in the last two indices survives insertion into Eq. (1.1.14), so that any antisymmetric contribution will never show up in ϵ. We can thus restrict the ansatz to C that are symmetric in those last two indices: $C_{\mu\alpha\beta} = C_{\mu\beta\alpha}$.

Inserting this ansatz into $\partial_\mu\epsilon_\nu + \partial_\nu\epsilon_\mu = 2\sigma\eta_{\mu\nu}$ gives

$$B_{\nu\mu} + B_{\mu\nu} + 2C_{\nu\mu\alpha} x^\alpha + 2C_{\mu\nu\alpha} x^\alpha \stackrel{!}{=} 2(\kappa + 2b.x)\eta_{\mu\nu} \tag{1.1.15}$$

Since x is arbitrary, this splits into two independent equations:

$$B_{\nu\mu} + B_{\mu\nu} = 2\kappa\eta_{\mu\nu} \tag{1.1.16a}$$

$$C_{\nu\mu\alpha} x^\alpha + C_{\mu\nu\alpha} x^\alpha = 2b.x\eta_{\mu\nu} \tag{1.1.16b}$$

The first of these is all there is to constrain B. This clearly makes the statement that the symmetric part of B is fixed to equal $\kappa\eta_{\mu\nu}$, while the antisymmetric part remains fully unconstrained. We conclude

$$B_{\mu\nu} = \kappa\eta_{\mu\nu} + \omega_{\mu\nu} \qquad \text{with } \omega_{\mu\nu} = -\omega_{\nu\mu}\ . \tag{1.1.17}$$

The equation for C, Eq. (1.1.16b), is less direct and involves some symmetry constraints that are not explicitly evident.

We can isolate the relevant structures directly by returning to (1.1.9) and insert both the ansatz (1.1.14) on the l.h.s. and differentiate and our expression for σ on the right as we have previously done to obtain Eq. (1.1.13a):

$$\partial_\rho\partial_\nu\epsilon_\mu(x) = C_{\mu\alpha\beta}\left(\eta^\alpha{}_\nu\eta^\beta{}_\rho + \eta^\beta{}_\nu\eta^\alpha{}_\rho\right) = C_{\mu\nu\rho} + C_{\mu\rho\nu} \stackrel{!}{=} 2b_\rho\eta_{\mu\nu} - 2b_\mu\eta_{\rho\nu} + 2b_\nu\eta_{\mu\rho} \tag{1.1.18}$$

We obtain

$$C_{\mu(\nu\rho)} = b_\rho\eta_{\mu\nu} - b_\mu\eta_{\rho\nu} + b_\nu\eta_{\mu\rho} \tag{1.1.19}$$

Contracting this with $x^\nu x^\rho$ gives

$$C_{\mu\nu\rho} x^\nu x^\rho = 2x_\mu b.x - b_\mu x^2 \tag{1.1.20}$$

Now we have used all constraints and are left with

$$\frac{dy_s^\mu(x)}{ds}\Big|_{s=0} = \epsilon^\mu(x) = a^\mu - \omega^\mu{}_\nu x^\nu + \kappa x^\mu + (2x^\mu b.x - b^\mu x^2) \tag{1.1.21}$$

d special conformal transformations

1 rescaling

$\omega_{\mu\nu} = -\omega_{\nu\mu}$, $d(d-1)/2$ (Lorentz-)rotations

d translations

The special conformal transformations are the most complicated ingredient. They appear to induce a specific combination of rotations and rescalings that are not separated out in the formulation, both x-dependent.

1.1.2 Generators and $\mathsf{so}(2, d)$

If we assume for the moment that fields transform trivially, the infinitesimals lead to generators of the form

$$
\begin{aligned}
P_\mu &= -i\partial_\mu && (1.1.22a)\\
D &= -ix.\partial && (1.1.22b)\\
\tilde{X}_{\mu\nu} &= i(x_\mu\partial_\nu - x_\nu\partial_\mu) && (1.1.22c)\\
K_\mu &= -i(2x_\mu x.\partial - x^2\partial_\mu && (1.1.22d)
\end{aligned}
$$

These obey the commutation rules

$$
\begin{aligned}
[D, P_\mu] &= iP_\mu && (1.1.23a)\\
[D, K_\mu] &= -iK_\mu && (1.1.23b)\\
[K_\mu, P_\nu] &= 2i(\eta_{\mu\nu}D - \tilde{X}_{\mu\nu} && (1.1.23c)\\
[K_\rho, \tilde{X}_{\mu\nu}] &= && (1.1.23d)\\
[P_\rho, \tilde{X}_{\mu\nu}] &= && (1.1.23e)\\
[\tilde{X}_{\mu\nu}, \tilde{X}_{\rho\sigma}] &= i(\eta_{\nu\rho}\tilde{X}_{\mu\sigma} + \eta_{\mu\sigma}\tilde{X}_{\nu\rho} - \eta_{\mu\rho}\tilde{X}_{\nu\sigma} - \eta_{\nu\sigma}\tilde{X}_{\mu\rho} && (1.1.23f)
\end{aligned}
$$

Introducing

$$
\mathsf{J}_{\mu\nu} = \tilde{X}_{\mu\nu} \qquad \mathsf{J}_{d-1,d} = D \tag{1.1.24a}
$$

$$
\mathsf{J}_{d,\mu} = \frac{1}{2}(P_\mu - K_\mu) \qquad \mathsf{J}_{d-1,\mu} = \frac{1}{2}(P_\mu + K_\mu) \tag{1.1.24b}
$$

we end up with the canonical (Cartesian basis metric) commutation relations for the J_{ab} a

$$
[\mathsf{J}_{ab}, \mathsf{J}_{cd}] = i\big(\eta_{bc}\mathsf{J}_{ad} + \eta_{ad}\mathsf{J}_{bc} - \eta_{ac}\mathsf{J}_{bd} - \eta_{bd}\mathsf{J}_{ac}\big) \tag{1.1.25}
$$

where the latin indices run from 0 to d (i.e. from 0 to 5 in the physical case).

Please note for the moment that the sum and difference definitions in Eq. (1.1.24b). This looks very much like a transition from light cone coordinates to Cartesian coordinates. (The target behaves properly with the Cartesian metric!)

Since [1] prepend the two new coords and I have added them at the back I need to check signs and index assignment in this!

1.1.3 From infinitesimals to differentials

I have earlier alluded to a mix of rescalings and Lorentz-rotations inherent specifically in the special conformal transformations.

This mix of both rescalings and (Lorentz-)rotations emerges in an explicit and separable form if we consider what Eq. (1.1.21) implies for its differential, the analogous equation for the Jacobian. Simply differentiating with respect to x^ν we obtain:

$$
\frac{d}{ds}\Big|_0 \frac{\partial y_s^\mu(x)}{\partial x^\nu} = \frac{\partial \epsilon^\mu(x)}{\partial x^\nu} = \Big[\big(\kappa + 2b.x\big)\eta^\mu{}_\alpha - \big(\omega^\mu{}_\alpha + 2(b^\mu x_\alpha - x^\mu b_\alpha)\big)\Big]\frac{\partial y_s^\alpha(x)}{\partial x^\nu}\Big|_0 \tag{1.1.26}
$$

The correct way to interpret this equation away from $s = 0$ is

$$\frac{d}{ds}\frac{\partial y_s^\mu(x)}{\partial x^\nu} = \frac{\partial \epsilon^\mu(y_s)}{\partial x^\nu} = \Big[\big(\kappa + 2b.y_s\big)\eta^\mu{}_\alpha - \big(\omega^\mu{}_\alpha + 2(b^\mu y_{s\alpha} - y_s^\mu b_\alpha)\big)\Big]\frac{\partial y_s^\alpha(x)}{\partial x^\nu} \tag{1.1.27}$$

This is now a differential equation for $\frac{\partial y_s^\mu(x)}{\partial x^\nu}$ with the initial condition $\frac{\partial y_s^\alpha(x)}{\partial x^\nu}\Big|_0 = \eta^\mu{}_\nu$.

The equation contains a generator of scaling transformations that is present for both pure scaling transformations ($a = 0$, $b = 0$, $\kappa \neq 0$) and pure special conformal transformations ($a = 0$, $\kappa = 0$, $b \neq 0$) collectively represented by

$$\sigma_{b,\kappa}(y_s) := \kappa + 2b.y_s \tag{1.1.28}$$

and a generator of (Lorentz-)rotations

$$\hat\omega_b(x)^\mu{}_\nu := \omega^\mu{}_\nu + 2(b^\mu x_\nu - x^\mu b_\nu) \tag{1.1.29}$$

Using this notation, Eq. (1.1.27) becomes

$$\frac{d}{ds}\frac{\partial y_s^\mu(x)}{\partial x^\nu} = \left(\sigma_{b,\kappa}(y_s)\eta^\mu{}_\alpha - \hat\omega_b(y_s)^\mu{}_\alpha\right)\frac{\partial y_s^\mu(x)}{\partial x^\nu} \tag{1.1.30}$$

This structure is consistent with a factorized form of the solution as predicted in Eq. (1.1.2)

$$\frac{\partial y_s^\mu(x)}{\partial x^\nu} = \Omega_s(x) R_s(x)^\mu{}_\nu \tag{1.1.31}$$

Eq. (1.1.30) translates into a pair of decoupled differential equations for $\Omega_s(x)$ and $R_s(x)$ respectively with specific initial conditions:

$$\frac{d}{ds}\Omega_s(x) = \sigma_{b,\kappa}(y_s)\Omega_s(x) \qquad \Omega_0(x) = 1 \tag{1.1.32}$$

$$\frac{d}{ds}R_s(x)^\mu{}_\nu = -\hat\omega_{b,\kappa}(y_s)^\mu{}_\alpha R_s(x)^\alpha{}_\nu \qquad R_0(x)^\mu{}_\nu = \eta^\mu{}_\nu \tag{1.1.33}$$

The solutions satisfy the group property

$$\Omega_{s'+s}(x) = \Omega_{s'}(x_s)\Omega_s(x) \qquad [R_{s'+s}]^\mu{}_\nu(x) = R_{s'}(x_s)^\mu{}_\alpha R_s(x)^\alpha{}_\nu(x) \tag{1.1.34}$$

(verify this by inserting back!). and can be solved with formal solutions of the form

$$\Omega_s(x) = \exp\left\{\int_0^s ds' \sigma_{b,\kappa}(y_{s'})\right\} \tag{1.1.35}$$

$$[R_s(x)^\circ{}_\circ] = \mathsf{P}_s \exp\left\{-\int_0^s ds' [\hat\omega_b(y_s)^\circ{}_\circ]\right\} \tag{1.1.36}$$

These are immediately useful for pure Lorentz transformations (where $\hat\omega$ is a constant tensor and $\sigma = 0$) and rescalings (where $\hat\omega = 0$ and σ is a simple constant), but not for special conformal transformations, since for those our ingredients start to depend on y_s! Since we need to know both $\sigma_{b,\kappa}(y_s)$ and $\hat\omega_b(y_s)$, i.e. we need to know y_s in each step, but we cannot directly obtain it from these equations. For that we have to go back to Eq. (1.1.21) and promote it to

$$\frac{dy_s^\mu(x)}{ds} = \epsilon^\mu(y_s) = a^\mu - \omega^\mu{}_\nu y_s^\nu + \kappa y_s^\mu + (2y_s^\mu b.y_s - b^\mu y_s^2) \tag{1.1.37}$$

This is most easily solved separately for pure translations, pure (Lorentz-)rotations, pure scalings and pure special conformal transformations. The most complicated case of these is the special conformal transformations.

To arrive at $y_s(x)$ for a pure special conformal transformation Osborn [4] (in a footnote on p4) delivers the following: Parametrize

$$y_s = \alpha_s(x)x + \beta_s(x)b \tag{1.1.38}$$

with $\alpha(0) = 1$, $\beta(0) = 0$. Insert into Eq. (1.1.37) to find

$$\dot\alpha_s x^\mu + \dot\beta_s b^\mu = \dot y_s^\mu \overset{!}{=} 2b.y_s y_s^\mu - b^\mu y_s^2 = 2b.(\alpha_s x + \beta_s b)(\alpha_s x^\mu + \beta_s b^\mu) - b^\mu(\alpha_s x + \beta_s b)^2 \tag{1.1.39}$$

and isolate the coefficients of x^μ and b^μ on both sides:

$$\dot\alpha_s x^\mu + \dot\beta_s b^\mu = \big(2b.x\alpha^2 + 2b^2\alpha\beta\big)x^\mu + \big(2b.(\alpha_s x + \beta_s b)\beta - (\alpha_s x + \beta_s b)^2\big)b^\mu \tag{1.1.40}$$

which supposedly leads to

$$\dot\alpha = -2\alpha^2 b.x - 2\alpha\beta b^2 \qquad \dot\beta = \alpha^2 x^2 - \beta^2 b^2 \tag{1.1.41}$$

These are decoupled, giving

$$\frac{d}{ds}(\beta + f\alpha) = -b^2(\beta + f\alpha)^2 \qquad \text{for} \qquad f^2 b^2 = 2fb.x - x^2 \tag{1.1.42}$$

and hence integrated to

$$\beta + f\alpha = \frac{f}{1 + f^2 b^2 s} \tag{1.1.43}$$

Eliminate β leads to

$$\dot\alpha = 2(fb^2 - b.x)\alpha^2 - 2fb^2\frac{\alpha}{1 + f^2 b^2 s} \tag{1.1.44}$$

and can be integrated to give

$$\alpha = \frac{1}{1 + 2sb.x + s^2 b^2 x^2} \tag{1.1.45}$$

and used to recover

$$\beta = \alpha x^2 s \tag{1.1.46}$$

APPARENT CONFLICT WITH SCALE FACTOR in Di Francesco et al, but apparent agreement with Osborn?

The solution for y_s for special conformal transformation thus is

$$y_s^\mu = \frac{1}{1 - 2sb.x + s^2b^2x^2}(x^\mu - sx^2b^\mu) = \Omega_s^2(x)(x^\mu - sx^2b^\mu) = \Omega_s^2(x)\big(\eta^\mu{}_\nu - sb^\mu x_\nu\big)x^\nu \ . \tag{1.1.47}$$

Notice the Ω^2 on the r.h.s. in this expression. (This can be identified by evaluating Eq. (1.1.1) for the transformation given in Eq. (1.1.47).) The last version rewrites this as an x dependent linear operation on x.

Note that

$$y_s^2 = \Omega_s^4(x)(x - sx^2b)^2 = \Omega_s^4(x)(x^2 - sx^2b.x + s^2(x^2)^2b^2) = \Omega_s^2(x)x^2 \tag{1.1.48}$$

The Minkowski length of y_s is just a rescaled version of that of the original x.

This is equivalent to the observervation that $\Omega_s(x)(x + sx^2b)$ is a (Lorentz-)rotated version of x:

$$\big(\Omega_s(x)(x + sx^2b)\big)^2 = \Omega_s^2(x)(1 - 2sb.x + s^2b^2x^2)x^2 = x^2 \ . \tag{1.1.49}$$

This guarantees that there exists a (Lorentz-)rotation $\tilde{R}_s(x)$, such that

$$\Omega(x_s)(x^\mu - sx^2b^\mu) = \tilde{R}_s(x)^\mu{}_\nu x^\nu \tag{1.1.50}$$

so that even the full solution is of the form

$$y_s(x) = \Omega^2(x_s)(x^\mu - sx^2b^\mu) = \Omega_s(x)\tilde{R}_s(x)^\mu{}_\nu x^\nu \ . \tag{1.1.51}$$

The two rotations R_s and $\tilde{R}_s$ are not the same!

Let us work out how they relate (see `./Special-conformal-differential-vs-full.nb` for some mathematica checks)

These demonstrate that

$$\partial_\alpha y_s^\mu(x)\partial_\beta y_s^\nu(x)\eta^{\alpha\beta} = \frac{1}{(1 - 2b.x + b^2x^2)^2}\eta^{\mu\nu} = \Omega^4{}_s(x)\eta^{\mu\nu} \tag{1.1.52}$$

Note the power 4 that appears on Ω in this expression as opposed to the power 2 in Eq. (1.1.48). How does this come about on paper – maybe this is informative?

We can do all the calculations at $s = 1$ (absorb all factors of s into b)

We observe:

$$\partial_\nu y_s^\mu(x) = \frac{\partial\Omega^2(x)(x^\mu - x^2b^\mu)}{\partial x^\nu} = \Omega^2\Big[\frac{2(x^\mu - x^2b^\mu)(\partial_\nu\Omega)}{\Omega} + (\eta^\mu{}_\nu - 2b^\mu x_\nu)\Big] \tag{1.1.53}$$

and expect that, due to Eq. (1.1.56), the factor in square brackets is a (Lorentz-)rotation, the $R(x)$ we are looking for.

We isolate the η-term in this expression, regroup contributions and use

$$\partial_\nu\Omega = \Omega^3(b_\nu - b^2x_\nu) \tag{1.1.54}$$

to simplify the expression:

$$\partial_\nu y_s^\mu(x) = \Omega^2\eta^\mu{}_\nu + \Omega^2 2\Big[\Omega^2(x^\mu - x^2b^\mu)(b_\nu - b^2x_\nu) - b^\mu x_\nu\Big] \tag{1.1.55}$$

From this expresssion alone we conclude that

$$\partial_\alpha y^\mu_s(x)\partial_\beta y^\nu_s(x)\eta^{\alpha\beta} = \Omega^4{}_s(x)\eta^{\mu\nu} + f_1(b,x)x^\mu x^\nu + f_2(b,x)(b^\mu x^\nu + x^\mu b^\nu) + f_3(b,x)b^\mu b^\nu \tag{1.1.56}$$

so that the conformal factor must be Ω^4, all that remains to verify is that the three scalar functions $f_i(b,x)$ are indeed zero as is required for conformality.

We will need the following intermediate contractions:

$$(b - b^2x)^2 = b^2(1 - 2b.x + b^2x^2) = \Omega^{-2}b^2 \tag{1.1.57}$$

and thus

$$\begin{aligned}
&\Big[\Omega^2(x^\mu - x^2b^\mu)(b_\alpha - b^2x_\alpha) - b^\mu x_\alpha\Big]\Big[\Omega^2(x^\nu - x^2b^\nu)(b^\alpha - b^2x^\alpha) - b^\nu x^\alpha\Big] \\
= \quad & \Omega^2b^2(x^\mu - x^2b^\mu)(x^\nu - x^2b^\nu) \\
& - \Omega^2(x^\mu - x^2b^\mu)b^\nu(b.x - b^2x^2) - \Omega^2b^\mu(x^\nu - x^2b^\nu)(b.x - b^2x^2) \\
& + b^\mu b^\nu x^2 \\
= & \Omega^2b^2x^\mu x^\nu + \big(-\Omega^4b^2x^2(1 - 2b.x + b^2x^2) + \Omega^2(b^2x^2 - b.x)\big)\big(x^\mu b^\nu + b^\mu x^\nu\big) + x^2\Omega^2b^\mu b^\nu \\
= & \Omega^2b^2x^\mu x^\nu + \big(-\Omega^2b.x)\big)\big(x^\mu b^\nu + b^\mu x^\nu\big) + x^2\Omega^2b^\mu b^\nu
\end{aligned} \tag{1.1.58}$$

Since $b.x$ is both appearing explcitly and implicitly inside Ω^2, it may help to substitute

$$b.x = \frac{1}{2}\frac{\Omega^2(b^2x^2 + 1) - 1}{\Omega^2} \tag{1.1.59}$$

$$= \Omega^2b^2x^\mu x^\nu - \frac{1}{2}\big(\Omega^2(b^2x^2 + 1) - 1\big)\big(x^\mu b^\nu + b^\mu x^\nu\big) + x^2\Omega^2b^\mu b^\nu \tag{1.1.60}$$

This needs verifying – all but the $\eta^{\mu\nu}$ terms must cancel?:

$$\begin{aligned}
&\partial_\alpha y^\mu_s(x)\partial_\beta y^\nu_s(x)\eta^{\alpha\beta} - \Omega^4\eta^{\mu\nu} \\
&\quad = \Omega^4 2\Big[\Omega^2(x^\mu - x^2b^\mu)(b^\nu - b^2x^\nu) - b^\mu x^\nu\Big] \\
&\quad\quad + \Omega^4 2\Big[\Omega^2(x^\nu - x^2b^\nu)(b^\mu - b^2x^\mu) - b^\nu x^\mu\Big] \\
&\quad\quad + 4\Omega^4\Big[\Omega^2(x^\mu - x^2b^\mu)(b_\alpha - b^2x_\alpha) - b^\mu x_\alpha\Big]\Big[\Omega^2(x^\mu - x^2b^\mu)(b^\alpha - b^2x^\alpha) - b^\mu x^\alpha\Big] \\
&\quad = \Omega^4 2\Big[\Omega^2\big(2x^\mu x^\nu b^2 - 2x^2b^\mu b^\nu\big) + (b^\mu x^\nu + x^\mu b^\nu)(\Omega^2(b^2x^2 + 1) - 1)\big) \\
&\quad\quad + 2\big(-\Omega^2b^2x^\mu x^\nu - \Omega^2b.x\big(x^\mu b^\nu + b^\mu x^\nu\big) + x^2\Omega^2b^\mu b^\nu\big)\Big] \\
&\quad = \Omega^4 2\Big[0 \times x^\mu x^\nu + 0 \times \big(x^\mu b^\nu + b^\mu x^\nu\big) + 0 \times b^\mu b^\nu\Big] = 0
\end{aligned} \tag{1.1.61}$$

According to Eq. (1.1.31), we must be able to interpret this as a rotation of x times a rescaling with a single power of Ω_s:

QUESTIONABLE

Indeed, Thus, we identify

$$R_s(x)^\mu{}_\nu = \Omega(x_s)\big(\eta^\mu{}_\nu + sb^\mu x_\nu\big) = \mathsf{P}_s \exp\Big\{-\int_0^s ds'[\hat\omega_{b,\lambda}(x_s)^\circ{}_\circ]\Big\}^\mu{}_\nu \tag{1.1.62}$$

and

$$\Omega_s(x) = \frac{1}{\sqrt{1 - 2sb.x + s^2b^2x^2}} = \exp\Big\{\int_0^s ds' \sigma_{b,\lambda}(x_{s'})\Big\} \tag{1.1.63}$$

This exposes σ as

$$\sigma_b(x_s) = \frac{d}{ds}\ln\left(\frac{1}{\sqrt{1 - 2sb.x + s^2b^2x^2}}\right) = -\frac{b.x - sb^2x^2}{1 - 2sb.x + s^2b^2x^2} \tag{1.1.64}$$

albeit expressed in terms of x and s, instead of x_s.

1.1.4 Conformal transformations in Minkowski space – projectively from $\mathbb{R}^{2,d}$

The goal of this section is to introduce special conformal transformations in $\mathbb{R}^{1,d-1}$ Minkowski space as projective transformations that emerge from the defining representation of $\mathsf{SO}(2,d)$ acting on an enlarged $\mathbb{R}^{2,d}$ Minkowski space.

There are many ways to introduce conformal transformations in Minkowski space and I have collected a few versions:

- Callan [6] `./Callan_-_Introduction_to_Conformal_Invariance.pdf` sets things up for a linear representation on a signature $(+,-,-,-,-,+)$ space by introducing a point on a light ray in this space that is uniquely linked to a point in Minkowski space:

  $$X^a := [(x^\mu, \frac{1}{2}(1+x^2), \frac{1}{2}(1-x^2))]^a \tag{1.1.65}$$

  is a point on the light ray

  $$\big\{\lambda(x^\mu, \frac{1}{2}(1+x^2), \frac{1}{2}(1-x^2)) | x \in \mathbb{M}, \lambda \in \mathbb{R}\big\} \tag{1.1.66}$$

- Callan cites Dirac[7], which appears to be a little more systematic?

- JIM Wheeler (USU) `Wheeler-09Mar27zNotes.pdf` does an interesting variant: He introduces conformal transformations as transformations that map a lightcone centered on a point $a \in \mathbb{M}$ onto a lightcone centered on a a point $b \in \mathbb{M}$. He starts with the statement that a lightcone centered at a point a can be described by the equation

  $$0 = \lambda(x-a)^2 \tag{1.1.67}$$

  His parametrization seems to pick out a point (see `./Metric.nb`)

  $$(\lambda' a^\mu, \frac{\lambda'}{\sqrt{2}}(1 + \frac{1}{2}a^2), \frac{\lambda'}{\sqrt{2}}(1 - \frac{1}{2}a^2)) \tag{1.1.68}$$

  This matches with Eq. (1.1.65) by identifying $a^\mu = \sqrt{2}x^\mu$, $\lambda = \frac{\lambda'}{\sqrt{2}}$, although the two talk about different vectors.

- Slava Rychkov's "EPFL Lectures on Conformal Field Theory in $D \geq 3$ Dimensions" [3] makes a connection of this projective viewpoint with Weyl symmetries. See also Farnsworth, Luty, and Prilepina "Weyl versus Conformal Invariance in Quantum Field Theory" [5].

- A Mathematica notebook with checks is in `./Metric.nb`

- Schottenloher's *A mathematical introduction to conformal field theory* [8, ch 2] is what the title promises, modern and mathematical. Still only useful fragments.

- Fradkin[9] also looks like a reasonable text.

I will start from a slight variation of Wheeler's definition and add in what is missing in his very short description.

I will use $\mathbb{R}^{2,d}$ with signature $(+,-,\dots,-,+)$ with d minus signs and a "naive" embedding of Minkowski space in the first d components.

We can then define a section of the light cone in $\mathbb{R}^{2,d}$ from $\mathbb{R}^{1,d-1}$ via[1]

$$\mathbb{R}^{1,d-1} \to R^{2,d} \;, \qquad [x^\circ] \mapsto [\tilde{\mathsf{x}}^\star] = \begin{pmatrix} [x^\circ] \\ \frac{1}{\sqrt{2}}(\alpha + \frac{x^2}{2\alpha}) \\ \frac{1}{\sqrt{2}}(\alpha - \frac{x^2}{2\alpha}) \end{pmatrix} \tag{1.1.69}$$

where the structure of the last two component ensures that

$$\frac{\tilde{\mathsf{x}}^d + \tilde{\mathsf{x}}^{d+1}}{\sqrt{2}} = \alpha \tag{1.1.70}$$

takes a fixed value.

In standard terminology, one may think of this as a fixed value for a light cone component (in the d,d+1 subspace). To this end define "lightcone coordinates in the last two entries"

$$\mathsf{x}^\pm := \frac{\mathsf{x}^d \pm \mathsf{x}^{d+1}}{\sqrt{2}} \tag{1.1.71}$$

and identify the basis change isomorphism as the "orthogonal" matrix[2]

$$[\mathsf{W}^\star{}_\star] := \begin{pmatrix} \mathbb{1}_d & & \\ & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \in \mathrm{Lin}\left(\mathbb{R}^{2,d}\right) \tag{1.1.72}$$

The associated light cone metric then is

$$[\eta^\star{}_\star] = ([\mathsf{W}^\star{}_\star]^{-1})^t \begin{pmatrix} [\eta^\circ{}_\circ] & & \\ & -1 & \\ & & 1 \end{pmatrix} [\mathsf{W}^\star{}_\star]^{-1} = \begin{pmatrix} [\eta^\circ{}_\circ] & & \\ & & -1 \\ & -1 & \end{pmatrix} \tag{1.1.73}$$

The inner product in light cone coordinates reads

$$\mathsf{x} \star \mathsf{y} = \mathsf{x}.\mathsf{y} - \mathsf{x}^+\mathsf{y}^- - \mathsf{x}^-\mathsf{y}^+ \tag{1.1.74}$$

where the "$\star$" and "." notation is defined as

$$\mathsf{x} \star \mathsf{y} := \mathsf{x}_A \mathsf{y}^A \qquad \mathsf{x}.\mathsf{y} := \mathsf{x}_\mu \mathsf{y}^\mu \tag{1.1.75}$$

with capital latin indices summed from 0 to $d+1$ and greek indices summed from 0 to $d-1$.

---

[1] chosen such that it falls on the light cone in $\mathbb{R}^{2,d}$. Indeed with the given signature of the metric

$$\tilde{\mathsf{x}}^2 = x^2 - \frac{1}{\sqrt{2}}\left(\alpha + \frac{x^2}{2\alpha}\right)^2 + \frac{1}{\sqrt{2}}\left(\alpha - \frac{x^2}{2\alpha}\right)^2 = x^2 - \frac{1}{2}\left(\alpha + x^2 + \left(\frac{x^2}{2\alpha}\right)^2\right) + \frac{1}{2}\left(\alpha + x^2 - \left(\frac{x^2}{2\alpha}\right)^2\right) = 0$$

[2] "Orthogonal" as in "transpose equals inverse": $[\mathsf{W}^\star{}_\star]^t = [\mathsf{W}^\star{}_\star]^{-1}$.

In light cone coordinates[3]

$$[\mathsf{x}^\star] = \begin{pmatrix} \mathsf{x}^\circ \\ \mathsf{x}^- \\ \mathsf{x}^+ \end{pmatrix} \qquad \text{and} \qquad [\tilde{\mathsf{x}}^\star] = \begin{pmatrix} [x^\circ] \\ \frac{x^2}{2\alpha} \\ \alpha \end{pmatrix} \tag{1.1.76}$$

for a general element of $\mathbb{R}^{2,d}$ and an element in the section defined in Eq. (1.1.69) respectively.

The first goal is now to see in which sense we can invert the section Eq. (1.1.69) and what that means for the isometries of the inner product on $\mathbb{R}^{2,d}$. As a first step we will recast a geneal elemetn of $\mathbb{R}^{2,d}$ in a form that closely matches the form of $[\tilde{\mathsf{x}}^\star]$ and we will do so in light cone coordinates where the expressions are most compact.

Since x^2 features prominently the minus component ofthe expression for $[\tilde{\mathsf{x}}^\star]$, we use the invariance of the inner product to trade x^- for a combination of inner products and and x^+ factors via

$$\mathsf{x}^2 = \mathsf{x} \star \mathsf{x} = \mathsf{x.x} - 2\mathsf{x}^+\mathsf{x}^- \Rightarrow \mathsf{x}^- = \frac{\mathsf{x.x} - \mathsf{x}^2}{2\mathsf{x}^+} \qquad \text{provided that } \mathsf{x}^+ \neq 0. \tag{1.1.77}$$

We note immediately that for light cone elements (where $\mathsf{x}^2 = 0$) the numerator in the minus component is of the desired form.

In detail we get

$$\begin{pmatrix} [\mathsf{x}^\circ] \\ \mathsf{x}^- \\ \mathsf{x}^+ \end{pmatrix} = \begin{pmatrix} [\mathsf{x}^\circ] \\ \frac{\mathsf{x.x}-\mathsf{x}^2}{2\mathsf{x}^+} \\ \mathsf{x}^+ \end{pmatrix} = \frac{\mathsf{x}^+}{\alpha} \begin{pmatrix} \frac{[\mathsf{x}^\circ]}{\mathsf{x}^+/\alpha} \\ \frac{\mathsf{x.x}-\mathsf{x}^2}{2(\mathsf{x}^+)^2/\alpha} \\ \alpha \end{pmatrix} \tag{1.1.78}$$

for a general $\mathsf{x} \in \mathbb{R}^{2,d}$. A point on the light cone ($\mathsf{x}^2 = 0$) can be recast as

$$\left.\begin{pmatrix} [\mathsf{x}^\circ] \\ \mathsf{x}^- \\ \mathsf{x}^+ \end{pmatrix}\right|_{\mathsf{x}^2=0} = \frac{\mathsf{x}^+}{\alpha} \begin{pmatrix} \frac{[\mathsf{x}^\circ]}{\mathsf{x}^+/\alpha} \\ \frac{\mathsf{x.x}}{2(\mathsf{x}^+)^2/\alpha} \\ \alpha \end{pmatrix} \underset{x^\mu := \frac{\mathsf{x}^\mu}{\mathsf{x}^+/\alpha}}{=} \frac{\mathsf{x}^+}{\alpha} \begin{pmatrix} [x^\circ] \\ \frac{x^2}{2\alpha} \\ \alpha \end{pmatrix} \tag{1.1.79}$$

What this little calculation demonstrates is that every vector on the light cone in $\mathbb{R}^{2,d}$ lies on a light ray that contains a point that can be written in the form of $\tilde{\mathsf{x}}$ introduced in Eq. (1.1.69) and that the associated point in Minkowski space is simply

$$x^\mu := \frac{\mathsf{x}^\mu}{\mathsf{x}^+/\alpha} \tag{1.1.80}$$

Note that simultaneous rescalings of all components of x leave no trace in the associated x.

---

[3]

$$\begin{pmatrix} \mathbb{1}_d & & \\ & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} [x^\circ] \\ \frac{1}{\sqrt{2}}(\alpha + \frac{x^2}{2\alpha}) \\ \frac{1}{\sqrt{2}}(\alpha - \frac{x^2}{2\alpha}) \end{pmatrix} = \begin{pmatrix} [x^\circ] \\ \frac{x^2}{2\alpha} \\ \alpha \end{pmatrix} = \begin{pmatrix} [x^\circ] \\ \tilde{\mathsf{x}}^- \\ \tilde{\mathsf{x}}^+ \end{pmatrix}$$

What we get is a one to one correspondence of points in Minkowski space and multiplicative equivalence classes (rays) in $\mathbb{R}^{2,d}$ with the execption of $\mathsf{x}^+ = 0$. Using the standard notation for the equivalence classes from projective geometry,

$$\langle [\mathsf{x}^\circ] : \mathsf{x}^- : \mathsf{x}^+ \rangle := \left\{ \lambda \begin{pmatrix} [\mathsf{x}^\circ] \\ \mathsf{x}^- \\ \mathsf{x}^+ \end{pmatrix} | \lambda \in \mathbb{R} \right\} , \tag{1.1.81}$$

we can recast Eq. (1.1.78)

$$\langle [\mathsf{x}^\circ] : \mathsf{x}^- : \mathsf{x}^+ \rangle = \langle [\mathsf{x}^\circ] : \frac{\mathsf{x}.\mathsf{x} - \mathsf{x}^2}{2\mathsf{x}^+} : \mathsf{x}^+ \rangle = \langle \frac{[\mathsf{x}^\circ]}{\mathsf{x}^+/\alpha} : \frac{\mathsf{x}.\mathsf{x} - \mathsf{x}^2}{2(\mathsf{x}^+)^2/\alpha} : \alpha \rangle \tag{1.1.82}$$

while (1.1.79) for a point on the light cone yields

$$\langle [\mathsf{x}^\circ] : \mathsf{x}^- : \mathsf{x}^+ \rangle|_{\mathsf{x}^2=0} = \langle \frac{[\mathsf{x}^\circ]}{\mathsf{x}^+/\alpha} : \frac{\mathsf{x}.\mathsf{x}}{2(\mathsf{x}^+)^2/\alpha} : \alpha \rangle = \langle [x^\circ] : \frac{x^2}{2\alpha} : \alpha \rangle . \tag{1.1.83}$$

In this spirit it becomes evident that the (necessarily linear) isometries of $\mathbb{R}^{2,d}$, the defining linear representation of $\mathsf{SO}(2,d)$, induce a projective representation of this group on Minkowski space $\mathbb{R}^{1,d-1}$.

A general isometry of $\mathbb{R}^{2,d}$ will preserve $\mathsf{x}.\mathsf{y}$ for any pair of $\mathsf{x}, \mathsf{y} \in \mathbb{R}^{2,d}$ and $\mathsf{x}^2 = 0$ specifically, but will not respect (1.1.70), unless we restrict to $\mathsf{SO}(1, d-1)$ (general Lorentz rotations), which we find embedded into the top left corner:

$$\begin{aligned} &\mathsf{SO}(1, d-1) \hookrightarrow \mathsf{SO}(1, d, 1) \\ &\qquad [\Lambda^\circ{}_\circ] \mapsto [\tilde{\mathsf{\Lambda}}^\star{}_\star] := \begin{pmatrix} [\Lambda^\circ{}_\circ] & & \\ & 1 & \\ & & 1 \end{pmatrix} . \end{aligned} \tag{1.1.84}$$

This acts like a standard Lorentz transformation on what we have provisionally identified as the Minkowski space components via Eq. (1.1.80):

$$[x^\circ] = \frac{[\mathsf{x}^\circ]}{\mathsf{x}^+/\alpha} \mapsto \frac{[\Lambda^\circ{}_\circ][\mathsf{x}^\circ]}{\mathsf{x}^+/\alpha} = [\Lambda^\circ{}_\circ][x^\circ] , \tag{1.1.85}$$

since the the x^+ component remains unaffected altogether.

We are now primarily interested how the remaining group elements act and hope to identify scale transformations, translations, and special conformal transformations within them.

Since we know that the isometries of $\mathbb{R}^{2,d}$ are given by $\mathsf{SO}(1, d-1)$ in a linear representation we can explore the issue starting from its Lie algebra and reconstruct general group elements via matrix exponentiation.

The key observation about the generators is that their components with both indices up (or both down) are antisymmetric

$$\mathsf{X}^{\mu\nu} = -\mathsf{X}^{\nu\mu} \tag{1.1.86}$$

independent of the basis chosen to span the vector space we act on.[4]

[4]Under a basis change $\mathsf{A} \in \mathsf{GL}(\mathbb{R}^{2,d})$, $[\mathsf{X}^{\circ\circ}] \mapsto [\mathsf{X}'^{\circ\circ}] := (\mathsf{A}^{-1})^t [\mathsf{X}^{\circ\circ}] \mathsf{A}^{-1}$, so that $[\mathsf{X}^{\circ\circ}]^t = -[\mathsf{X}^{\circ\circ}]$ implies $[\mathsf{X}'^{\circ\circ}]^t = -[\mathsf{X}'^{\circ\circ}]$.

Let us now consider $\mathsf{so}(2,d)$ with $\mathsf{so}(1,d-1)$ acting on “Minkowski” space embedded into the top left corner.

In order to link up with the discussion of generators and commutation rules in Sec. 1.1.2 we consider the generators both in Cartesian and lightcone configurations.

We parametrize the most general element of $\mathsf{so}(2,d)$ in both Cartesian and $d, d+1$ lightcone coordinates as

$$[\tilde{\mathsf{X}}^{\star\star}] = \left(\begin{array}{c|c|c} [X^{\circ\circ}] & [v^\circ] & [u^\circ] \\ \hline [-v^\circ]^t & & \kappa \\ \hline [-u^\circ]^t & -\kappa & \end{array}\right) \tag{1.1.87}$$

for the Cartesian case and

$$[\mathsf{X}^{\star\star}] := ([\mathsf{W}^\star{}_\star]^{-1})^t[\tilde{\mathsf{X}}^{\star\star}][\mathsf{W}^\star{}_\star]^{-1} = \left(\begin{array}{c|c|c} [X^{\circ\circ}] & [w_-^\circ] & [w_+^\circ] \\ \hline [-w_-^\circ]^t & & \kappa \\ \hline [-w_+^\circ]^t & -\kappa & \end{array}\right) \qquad \text{where} \qquad w_\pm^\mu := \frac{v^\mu \pm u^\mu}{\sqrt{2}} \tag{1.1.88}$$

for the light cone case respectively. The second version is better adapted to the present purpose in the light of Eq. (1.1.70).

Let us just record the inverse transformation

$$v^\mu = \frac{w_+^\mu + w_-^\mu}{\sqrt{2}} \; ; \; u^\mu = \frac{w_+^\mu - w_-^\mu}{\sqrt{2}} \tag{1.1.89}$$

which will map onto (1.1.24b) later.

To obtain a group element we need to exponentiate $[\mathsf{X}^\star{}_\star]$, which is obtained from $[\mathsf{X}^{\star\star}]$ by lowering the the right index with the appropriate light cone metric Eq. (1.1.73). One obtains

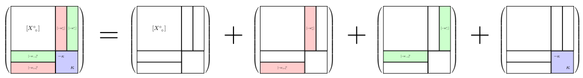
$$[\mathsf{X}^\star{}_\star] = \left(\begin{array}{c|c|c} [X^\circ{}_\circ] & [-w_+^\circ] & [-w_-^\circ] \\ \hline [-w_{-\circ}]^t & -\kappa & \\ \hline [-w_{+\circ}]^t & & \kappa \end{array}\right) \tag{1.1.90}$$

Note how the last two columns get swapped and multiplied by -1 and the remainder of the bottom two rows simply have an index lowered.

Now everything is ready for exponentiation and we will do so for the different types of transformation separately. To to so, we decompose $[\mathsf{X}^\star{}_\star]$ uniquely into a suitable set of independent components

$$[\mathsf{X}^\star{}_\star] = [X^\star{}_\star] + [iw_+.\mathsf{K}^\star{}_\star] + [iw_-.\mathsf{P}^\star{}_\star] + [i\kappa\mathsf{D}^\star{}_\star] \;, \tag{1.1.91}$$

or graphically

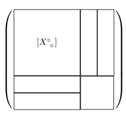


(1.1.92)

Note that the procedure uniquely defines a total of 9 matrices. There are four tensors, $\mathsf{P}^{\mu\star}{}_{\star}$, four tensors $\mathsf{K}^{\mu\star}{}_{\star}$, and one $\mathsf{D}^{\star}{}_{\star}$, all fully determined by linearity.

Now we exponentiate them *separately* and discuss their effect on elements of of both $\mathbb{R}^{2,d}$ and Minkowski space individually.

- **Lorentz rotations**,

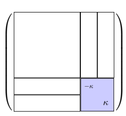
$$\exp[X^{\star}{}_{\star}][\mathsf{x}^{\star}] = \exp \left(\; \right)$$  $$\begin{pmatrix} [\mathsf{x}^{\circ}] \\ \mathsf{x}^{-} \\ \mathsf{x}^{+} \end{pmatrix} = \begin{pmatrix} \exp[\mathsf{X}^{\circ}{}_{\circ}] & & \\ & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} [\mathsf{x}^{\circ}] \\ \mathsf{x}^{-} \\ \mathsf{x}^{+} \end{pmatrix} = \begin{pmatrix} \exp[\mathsf{X}^{\circ}{}_{\circ}][\mathsf{x}^{\circ}] \\ \mathsf{x}^{-} \\ \mathsf{x}^{+} \end{pmatrix} , \tag{1.1.93}$$

  only affect $[\mathsf{x}^{\circ}]$ and leave the light cone components untouched. Restricting x to the light cone we see that they act as $\mathsf{SO}(1, d-1)$ elements on $x^{\mu} = \alpha \frac{\mathsf{x}^{\mu}}{\mathsf{x}^{+}}$:

$$[x^{\circ}] = \frac{[\mathsf{x}^{\circ}]}{\mathsf{x}^{+}/\alpha} \mapsto \frac{\exp[\mathsf{X}^{\circ}{}_{\circ}][\mathsf{x}^{\circ}]}{\mathsf{x}^{+}/\alpha} = \exp[\mathsf{X}^{\circ}{}_{\circ}][x^{\circ}] , \tag{1.1.94}$$

  irrespective of the value of α.

- **Scale transformations** originate from a proper boost in the $\mathsf{x}^{\pm}$ plane

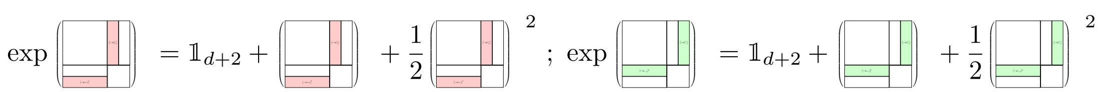
$$\exp[i\kappa \mathsf{D}^{\star}{}_{\star}][\mathsf{x}^{\star}] = \exp \left(\; \right)$$  $$\begin{pmatrix} [\mathsf{x}^{\circ}] \\ \mathsf{x}^{-} \\ \mathsf{x}^{+} \end{pmatrix} = \begin{pmatrix} [\eta^{\circ}{}_{\circ}] & & \\ & e^{-\kappa} & \\ & & e^{\kappa} \end{pmatrix} \begin{pmatrix} [\mathsf{x}^{\circ}] \\ \mathsf{x}^{-} \\ \mathsf{x}^{+} \end{pmatrix} = \begin{pmatrix} [\mathsf{x}^{\circ}] \\ e^{-\kappa}\mathsf{x}^{-} \\ e^{\kappa}\mathsf{x}^{+} \end{pmatrix} \tag{1.1.95}$$

  and leave $[\mathsf{x}^{\circ}]$ unchanged, but the associated Minkowski points are affected by the coordinate rescaling of Eq. (1.1.80)

$$[x^{\circ}] = \frac{[\mathsf{x}^{\circ}]}{\mathsf{x}^{+}/\alpha} \mapsto \frac{[\mathsf{x}^{\circ}]}{e^{-\kappa}\mathsf{x}^{+}/\alpha} = e^{\kappa}[x^{\circ}] , \tag{1.1.96}$$

  once more, the result is insensitive to the value of α.

The remaining the middle two contributions of Eq. (1.1.91) or (1.1.92) are nilpotent – all powers but the first two vanish identically. Thus the exponentials truncate:



(1.1.97)

The explicit expressions read

$$\exp[iw_{-}.\mathsf{P}^{\star}{}_{\star}] = \begin{pmatrix} [\eta^{\circ}{}_{\circ}] & & [w^{\circ}_{-}] \\ [w_{-\circ}] & 1 & \frac{w_{-}^{2}}{2} \\ & & 1 \end{pmatrix} \qquad \text{and} \qquad \exp[iw_{+}.\mathsf{K}^{\star}{}_{\star}] = \begin{pmatrix} [\eta^{\circ}{}_{\circ}] & [w^{\circ}_{+}] & \\ & 1 & \\ [w_{+\circ}] & \frac{w_{+}^{2}}{2} & 1 \end{pmatrix} \tag{1.1.98}$$

so that

$$\exp[iw_-.\mathsf{P}^\star{}_\star][\mathsf{x}^\star] = \begin{pmatrix} [\mathsf{x}^\circ] + [w_-^\circ]\mathsf{x}^+ \\ w_-.\mathsf{x} + \mathsf{x}^- + \frac{1}{2}w_-^2\mathsf{x}^+ \\ \mathsf{x}^+ \end{pmatrix} \qquad \text{and} \qquad \exp[iw_+.\mathsf{K}^\star{}_\star][\mathsf{x}^\star] = \begin{pmatrix} [\mathsf{x}^\circ] + [w_+^\circ]\mathsf{x}^- \\ \mathsf{x}^- \\ w_+.\mathsf{x} + \frac{1}{2}w_+^2\mathsf{x}^- + \mathsf{x}^+ \end{pmatrix} \tag{1.1.99}$$

Operating on light like vectors

$$\begin{pmatrix} [\mathsf{x}^\circ] \\ \mathsf{x}^- \\ \mathsf{x}^+ \end{pmatrix}\Bigg|_{\mathsf{x}^2=0} = \frac{\mathsf{x}^+}{\alpha}\begin{pmatrix} [\frac{\mathsf{x}^\circ}{\mathsf{x}^+/\alpha}] \\ \frac{\mathsf{x}.\mathsf{x}}{2(\mathsf{x}^+)^2/\alpha} \\ \alpha \end{pmatrix} \tag{1.1.100}$$

this results in translations and special conformal transformations on the Minkowski counterparts:

- **Translations on Minkowski space** are induced by $\exp[iw_-.\mathsf{P}^\star{}_\star]$ via

$$[x^\circ] = \frac{[\mathsf{x}^\circ]}{\mathsf{x}^+/\alpha} \mapsto \frac{[\mathsf{x}^\circ] + [w_-^\circ]\mathsf{x}^+}{\mathsf{x}^+/\alpha} = [x^\circ] + \alpha[w_-^\circ] =: [x^\circ] + [a^\circ] \tag{1.1.101}$$

  This is a is clearly a translation by $a = \alpha w_-$.

- **Special conformal transformations on Minkowski space** are induced by $\exp[iw_+.\mathsf{K}^\star{}_\star]$ via

$$\begin{aligned}[x^\circ] = \frac{[\mathsf{x}^\circ]}{\mathsf{x}^+/\alpha} \mapsto \frac{[\mathsf{x}^\circ] + [w_+^\circ]\mathsf{x}^-}{(w_+.\mathsf{x} + \frac{1}{2}w_+^2\mathsf{x}^- + \mathsf{x}^+)/\alpha} &= \frac{\frac{[\mathsf{x}^\circ]}{\mathsf{x}^+/\alpha} + [w_+^\circ]\frac{\mathsf{x}^-}{\mathsf{x}^+/\alpha}}{w_+.\frac{\mathsf{x}}{\mathsf{x}^+} + \frac{1}{2}w_+^2\frac{\mathsf{x}^-}{\mathsf{x}^+} + 1} = \frac{[x^\circ] + [w_+]\frac{x^2}{2}}{1 + w_+.x + \frac{1}{4}w_+^2 x^2} \\ &= \frac{[x^\circ] \pm [b^\circ]x^2}{1 \pm 2b.x + b^2x^2}\end{aligned} \tag{1.1.102}$$

  where, in the last line we have substituted $w_+ = \pm 2b$ (irrespective of α).

Theorem 1 – product form of conformal transformations:
Every conformal transformation can be decomposed into a product of

- Lorentz rotations
- translations
- scale transformations
- special conformal transformations

This needs proof! I would guess it helps to start from the above.

1.1.5 Conformal transformation and projective space

- Schottenloher
- Nice: Jadczyk[10] provides a discussion about finer points and criticizes some of the physics literature quite strongly.
  Original references of interest (Uhlmann [11]) but extends the explanations.

1.1.6 Conformal transformations in field theories

- Shapiro ./Shapiro-conform-2.pdf also shows (partially) how classical electrodynamics is conformally invariant. Codirla[12] does a bit more in this directions – still not quite what I need.
- Go, Kastrup and Mayer[13] gives a somewhat mathematical take on things that discusses local causality. They bring up the main irritants right away: Inversions $x^\mu \mapsto \frac{x^\mu}{x^2}$ are ill defined on the light cone. The conformal rescaling factors $\frac{1}{1-2c.x+c^2x^2}$ may be negative. Alternatively, look at this sign change as the sign of $(\frac{x}{x^2}-c)^2$, the lightcone for $\frac{x}{x^2}$ based at c. Again this happens where the rescaling factor is infinite.
- Bjørn Felsager[14] has a few sections that might help.
- Field theory texts on curved space times have partial discussions on the topic:
  - Birrell and Davies "Quantum Fields in Curved Space" [15] has an interesting page on conformal behavior of scalar fields providing Christoffels, and Riccis explicitly as well as the transformation behavior of the fields. Needs a rederivation.
  - Maxwell's equations in curved spacetime from wikipedia ((local copy)
  - Fewster[16]
  - Parker and Toms[17] provide a derivation for the scalar field results in Birrell and Davies in Sec. 2.2., *but they use infinitesimal transformations only.* It exposes the total derivative for the transformation law of the Lagrangian.
  - Think Penrose diagrams Halacek [18] Jadczyk [10]
  - Maxwell in curved space times is interesting in this context for two reasons:
    1. The cancellation of contributions and the resulting form invariance of the structure is a general case of what I am trying to see here.
    2. The finite transformations are doing something the GR setting does not do very often directly, it relates different points on the manifold. Maybe a good discussion of causality is useful in this context?

    References: Cabral & Lobo [19], Tsagas [20]
    Thorne and Macdonald [21]

1.1.6.1 Conformal transformation of geometric quantities in curved space time

Here are the claims from Birrell and Davies

- Claim 1:
$$\Gamma^\rho{}_{\mu\nu} \mapsto \Gamma^\rho{}_{\mu\nu} + \Omega^{-1}(\eta^\rho{}_\mu \Omega_{;\nu} + \eta^\rho{}_\nu \Omega_{;\mu} - g_{\mu\nu} g^{\rho\alpha}\Omega_{;\alpha}) \tag{1.1.103}$$
- Claim 2:
$$R^\nu{}_\mu \mapsto \bar{R}^\nu{}_\mu := \Omega^{-2} R^\nu{}_\mu - (d-2)\Omega^{-1}(\Omega^{-1})_{;\mu\rho} g^{\rho\nu} g^{\rho\sigma}\eta^\nu{}_\mu + (d-2)^{-1}\Omega^{-d}(\Omega^{d-2})_{;\rho\sigma} g^{\rho\sigma}\eta^\nu{}_\mu \tag{1.1.104}$$
- Claim 3:
$$R \mapsto \bar{R} := \Omega^{-2}R + 2(d-1)\Omega^{-3}\Omega_{;\mu\nu}g^{\mu\nu} + (d-1)(d-4)\Omega^{-4}\Omega_{;\mu}\Omega_{;nu}g^{\mu\nu} \tag{1.1.105}$$
- Claim 4:
$$(\Box + \frac{1}{4}\frac{d-2}{d-1}R)\phi \mapsto (\Box + \frac{1}{4}\frac{d-2}{d-1}\bar{R})\bar{\phi} = \Omega^{-\frac{d+2}{2}}(\Box + \frac{1}{4}\frac{d-2}{d-1}R)\phi \tag{1.1.106}$$

where

$$\Box\phi = g^{\mu\nu}\nabla_\mu\nabla_\nu\phi = \frac{1}{\sqrt{g}}\partial_\mu\sqrt{g}g^{\mu\nu}\partial_\nu\phi \qquad \bar{\phi} = \Omega^{-\frac{d-2}{2}}\phi \tag{1.1.107}$$

1.1.6.2 Scalar fields

1.1.7 Riemann spheres and beyond

- Complex analysis: automorphism groups on $\mathbb{C}$ and the Riemann sphere $\hat{\mathbb{C}} = \mathsf{CP}^1$ (notation varying) `./lec10.pdf` see also `./Kim.pdf`
  A very reasonable exposition can be found in Thomas Baird's `./../../BOOKS/Mathematical Tools/Baird Thomas -- projectivegeometrylecturenotes3.pdf`. This provides a clearer exposition of projective spaces, including the standard notion of coordinates that are easy to use, *projective coordinates*.
- Möbius transformations as a representation of $\mathsf{GL}(2,\mathbb{C})$. Very well executed in `./mobius-transformation.pdf`. Möbius transformations linked in with stereographic projections `./moebius.pdf`.
- Lectures on compact Riemann surfaces (B. Eynard) `./eynard_lectures.pdf`

1.1.8 Projective spaces and homogeneous coordinates

Definition 1 – projective space of a vector space:
Let V be a vector space. The projective space $P(V)$ is the set of one dimensional vector spaces.

If $V = \mathbb{F}^n$ we write $\mathbb{F}P^n := P(\mathbb{F}^{n+1})$

If V is a finite dimensional real vector space one may think of $P(V)$ as the space of directions. A few examples will help getting used to the idea:

- If V is one dimensional, the only one dimensional subspace is V itself, consequently $P(V)$ contains a single element we call a point in $P(V)$.
- If V is a two dimensional real vector space it is isomorphic to $\mathbb{R}^2$, equipped with its standard basis (Cartesian coordinates (x^1, x^2)). To characterize $P(\mathbb{R}^2)$ as a set we pursue two equivalent strategies
  1. Let $L \subset \mathbb{R}^2$ be a line parallel to the two-axis, say $L = \{x \in \mathbb{R}^2 | x^2 = 1\}$. With the exception of the two axis, each sub-vector-space of $\mathbb{R}^2$ (geometrically lines through the origin) intersects L in exactly one point, thus the elements $P(\mathbb{R}^2)$ are in one to one correspondence to $L \cup \{\infty\}$, where we have denoted the additional point representing the two-axis ∞ – the *point at infinity*.
  2. Equivalently we can observe that any line intersects a circle centered at the origin precisely twice in a pair of antipodal points. Topologically we can construct $P(\mathbb{R}^2)$ by taking a half circle and identifying the endpoints. This identifies $P(\mathbb{R}^2)$ with a circle.

  Any way you view it $P(\mathbb{R}^2)$ is a line with its two endpoints identified. We call it a projective line.
- Turning to $P(\mathbb{R}^2)$ we select the plane $x^3 = 1$ and note that every one dimensional subspace either intersects the plane at precisely one point or lies in the plane $x^3 = 0$, which we can treat like in the previous example. We get a bijection

$$P(\mathbb{R}^3) \cong \mathbb{R}^2 \cup P(\mathbb{R}^2) \cong \mathbb{R}^2 \cup \mathbb{R} \cup \{\infty\}$$

❯ By induction, we arrive at bijections

$$P(\mathbb{R}^{n+1}) \cong \mathbb{R}^n \cup \mathbb{R}^{n-1} \cup \ldots \cup \mathbb{R} \cup \mathbb{R}^0 \tag{1.1.108}$$

where we have denoted $\{\infty\}$ by $\mathbb{R}^0$.

The discussion above already hints at homogeneous coordinates, and with these we can construct an atlas any $P(\mathbb{F}^{n+1})$ that reveals them as n-dimensional manifolds. This motivates the notation $\mathbb{F}P^n$. We start the discussion by defining the line spanned by a vector $v \in V$

$$[v] := \{\lambda v | v \in V, \lambda \in \mathbb{F} \setminus \{0\} \tag{1.1.109}$$

so that $[\lambda v] = [v]$ for all nonzero $\lambda \in \mathbb{F}$. Choosing a basis we map this onto $\mathbb{F}^n$ ($n = \dim V$). We then write

$$[v] = [x^1 : \ldots : x^n] = [\lambda x^1 : \ldots : \lambda x^n] \tag{1.1.110}$$

Note that if we consider a subset $U_0 \subset P(V)$ with $x^i \neq 0$ for some i (this subset is well defined since $x^i \neq 0 \Leftrightarrow \lambda x^i \neq 0$ if $\lambda \neq 0$), we have

$$[x^1 : \ldots : x^i : \ldots : x^n] = [x^1/x^i : \ldots : 1 : \ldots : x^n/x^i] \tag{1.1.111}$$

Fixing the value of a nonzero component uniquely determines the representative and leaves us with an obvious bijection of points

$$U_0 \cong \mathbb{F}^{n-1}; \qquad [x^1/x^i : \ldots : 1 : \ldots : x^n] \mapsto (y^1, \ldots y^{n-1}) \tag{1.1.112}$$

Each choice of i provides us with a chart, the collection of these forms an atlas for the manifold $\mathbb{F}P^n$

❯ **Definition 2 – linear subspace of a projective space:**

A linear subspace of a projective space $P(V)$ is a subset $P(W) \subseteq P(V)$ consisting of all 1-dimensional subspaces of some vector subspace $W \subseteq V$

Projective spaces inherit some information from vector spaces, although clearly, information is lost in the process. For example, every linear map $A : V \to W$ induces a map from $P(V) \to P(W)$, provided it does not have a nontrivial kernel. Any kernel elements are mapped onto zero and thus do not define a 1-dimensional subspace of W, i.e. do not end up in $P(W)$. We must exclude these:

❯ **Definition 3 – induced projective maps:**

Any injective linear map $A : V \to W$ induces a map

$$\alpha : P(V) \to P(W); \qquad \alpha([v]) := [A(v)] \tag{1.1.113}$$

called the projective morphism or projective transformation induced by A.

This induction map is not not one to one – it is only unique up to rescaling:

> **Observation 1 – equality of induced projective maps:**
> Two injective linear maps $A : V \to W$ and $A' : V \to W$ determine the same projective morphism iff $A = \lambda A'$ for some nonzero $\lambda \in \mathbb{F}$.

This is obvious is one direction: Suppose $A = \lambda A'$, then $[A(v)] = [\lambda A'(v)] = [A'(v)]$ and we are done. The converse is the really interesting statement: To prove the statement, we must start from $[A(v)] = [A'(v)]$ for all $v \in V$. Then we choose a basis $v_1, \ldots, v_n$ for V. For all such basis elements we know that there exist nonzero scalars $\lambda_1, \ldots, \lambda_n$ such that $Av_i = \lambda_i A' v_i$. Another such constant exists for the sum of all basis elements: $A(v_1 + \ldots + v_n) = \lambda A'(v_1 + \ldots + v_n)$. Then

$$\sum_i \lambda A' v_i = \lambda A'(\sum_i v_i) = A(\sum_i v_i) = \sum_i A v_i = \sum_i \lambda_i A' v_i \qquad \Rightarrow \qquad 0 = \sum_i (\lambda - \lambda_i) A' v_i \tag{1.1.114}$$

Since A' is injective the $A'v_i$ are linearly independent and thus $\lambda_i = \lambda$ for all i.

Once we have chosen a basis for V and W and a specific pair of charts for the projective spaces (we pick the chart $[\boldsymbol{z} : z'] = [\boldsymbol{z}/z' : 1]$ for convenience of notation) we need to split the linear map in the from

$$\begin{pmatrix} \boldsymbol{z} \\ z' \end{pmatrix} \mapsto \begin{pmatrix} A & \boldsymbol{b} \\ \boldsymbol{c}^t & d \end{pmatrix} \begin{pmatrix} \boldsymbol{z} \\ z' \end{pmatrix} = \begin{pmatrix} A \cdot z + bz' \\ \boldsymbol{c}^t \cdot \boldsymbol{z} + dz' \end{pmatrix} \tag{1.1.115}$$

to expose the structure of the projected representation:

$$[\boldsymbol{z} : z'] = [\boldsymbol{z}/z' : 1] \mapsto [A\boldsymbol{z} + \boldsymbol{b}z' : \boldsymbol{c}^t \cdot \boldsymbol{z} + dz'] = [\frac{A\boldsymbol{z}/z' + \boldsymbol{b}}{\boldsymbol{c}^t \cdot \boldsymbol{z}/z' + d} : 1] \tag{1.1.116}$$

Again the case finite dimensional case with $V = W$ is of particular interest and leads to many instructive and often practically useful examples. In this case any injective linear map $A \in \mathrm{Lin}\,(V)$ is automatically surjective, hence invertible: $A \in \mathsf{GL}(V)$ and we can start to look at what the associated projective map looks

Projective transformations from a vector space to itself must of course also be induced by an injective linear map. For finite dimensional vector spaces we end up with a bijective linear map, which is thus invertible. These inverses produce

1.1.9 The Poincaré disk

This section provides background study material for Schwartz [22]

The Poincare disk is a projective chart for a fixed proper time hyperboloid in the forward lightcone in a $1+2$ dimensional space time. The chart is produced by a projection that copies the idea of the stereographic projection of a sphere.

To be specific we will project a hyperbola, say

$$x_0^2 - \boldsymbol{x}^2 = 1 \tag{1.1.117}$$

onto the unit disc using $(-1, \boldsymbol{0})$ as the projection center as shown in Fig. 1. If drawn correctly this seems to indicate that the geodesics are conic sections, their length between two points u and v should be related to the "Minkowski" overlap $u.v$

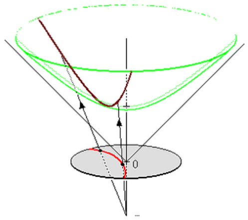


Fig. 1: *The Poincare disk obtained by hyperbolic projection. (Fig from Wikipedia)*

We have two mutually inverse maps, from the unit disk $D = \{\boldsymbol{x} \in \mathbb{R}^2 | \boldsymbol{x}^2 \leq 1\}$ into the "unit mass hyperbola" $H = \{y \in \mathbb{R}^3 | y^0 = \sqrt{\boldsymbol{y}_\perp^2 + 1}\}$

$$\boldsymbol{x} \mapsto \boldsymbol{y} = \frac{1}{1-\boldsymbol{x}^2}\begin{pmatrix} 1+\boldsymbol{x}^2 \\ 2x_1 \\ 2x_2 \end{pmatrix} \tag{1.1.118}$$

and its inverse

$$\boldsymbol{y} = \begin{pmatrix} y_0 \\ y_1 \\ y_2 \end{pmatrix} \mapsto \boldsymbol{x} = \frac{1}{1+y_0}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \tag{1.1.119}$$

By construction we have a $1, -1, -1$ signature metric in which

$$y.y = \frac{1}{(1-\boldsymbol{x}^2)^2}\left((1+\boldsymbol{x}^2)^2 - 4\boldsymbol{x}^2\right) = 1 \tag{1.1.120}$$

for every point on the hyperbola.

The geodesics on the hyperbola are simply the intersections of a plane fixed by the *origin* and two points on the hyperbola with the hyperbola: $(\alpha x + \beta y).(\alpha x + \beta y) \stackrel{!}{=} 1$ for $\boldsymbol{x}, \boldsymbol{y} \in H$

$$1 \stackrel{!}{=} \alpha^2 x.x + 2\alpha\beta x.y + \beta^2 y.y = \alpha^2 + 2\alpha\beta\frac{(1+\boldsymbol{x}^2)(1+\boldsymbol{y}^2) - 4\boldsymbol{x}\cdot\boldsymbol{y}}{(1-\boldsymbol{x}^2)(1-\boldsymbol{y})^2} + \beta^2 \tag{1.1.121}$$

To let $\mathsf{PSL}(2, \mathbb{R})$ act on the Poincare disk, we need maps that map the upper half plane onto the unit disk and back:

$$z \mapsto f(z) = \frac{z-i}{z+i} \tag{1.1.122}$$

takes the upper complex half plane onto the interior of the unit disk, the lower complex half plane onto its exterior. Its inverse can be written as

$$z \mapsto f^{-1}(z) = \frac{iz+i}{-z+1} \tag{1.1.123}$$

The composition of these are best derived via the underlying linear maps in $\mathsf{GL}(\mathbb{C}^2)$:

$$\begin{pmatrix} 1 & -i \\ 1 & +i \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} i & i \\ -1 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} a+d+i(b-c) & a-d-i(b+c) \\ a-d+i(b+c) & a+d-i(b-c) \end{pmatrix} =: \begin{pmatrix} \alpha & \beta \\ \beta^* & \alpha^* \end{pmatrix} \tag{1.1.124}$$

The result has a very regular structure and one notes that

$$\{\begin{pmatrix} \alpha & \beta \\ \beta^* & \alpha^* \end{pmatrix} | \alpha, \beta \in \mathbb{C}, |\alpha|^2 - |\beta|^2 = 1\}/\{\pm\mathbb{1}_2\} \tag{1.1.125}$$

provides a representation of $\mathsf{PSL}(2, \mathbb{C})$

1.1.10 Möbius transformations

Möbius Transformations Revealed is a short video by Douglas Arnold and Jonathan Rogness which depicts the beauty of Möbius transformations and shows how moving to a higher dimension reveals their essential unity. It was one of the winners in the 2007 Science and Engineering Visualization Challenge and was featured along with the other winning entries in the September 28, 2007 issue of journal Science.

Arnold also provides a short article describing the procedure at http://www.ima.umn.edu/~arnold/papers/moebius.pdf where the claim is made that every Möbius transformation can be written in terms of an element g of the Galilei Group $\mathsf{SO}(3) \times T_3$ (the semi direct product of rotations and translations in 3d space and a given version of the stereographic projection as $\mathsf{SP} \circ g \circ \mathsf{SP}^{-1}$

Introduction to Möbius transformations, the conformal transformations on the complex plane with all points at ∞ identified.

Introduce the projective complex plane

Thus we have arrived at

Definition 4 – Möbius transformations:
The map

$$f : \mathbb{C} \to \mathbb{C}; \qquad z \mapsto f(z) := \frac{az+b}{cz+d} \qquad a, b, c, d \in \mathbb{C} \text{ , with } ad - bc \neq 0 \tag{1.1.126}$$

is called a factional linear transformation or Möbius transformation

The condition $ad - bc \neq 0$ implies

- Neither the numerator nor the denominator can vanish identically
- a and c cannot both be zero (in which case f would be constant)
- b and d cannot both be zero (in which case f would be constant)
- The denominator cannot be a multiple of the numerator, since then $ad - bc = \alpha(ab - ba) = 0$. Thus numerator and denominator have no common factor and therefore f is a well defined, non-constant holomorphic function on the projective complex plane $\mathbb{P}$.

Let us now establish a group homomorphism between the standard group and the Möbius group $\mathsf{Aut}(\mathbb{P})$. Given the constraints in the definition we note that we can take any element in $\mathsf{GL}(2, \mathbb{C})$, i.e.

$$\mathsf{GL}(2, \mathbb{C}) = \left\{ M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{C}^{2\times 2} | \det(M) = ad - bc \neq 0 \right\} \tag{1.1.127}$$

and map it on a well defined Möbius transformation since its determinant condition mirrors that on the coefficients.

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto f_M(z) = \frac{az+b}{cz+d} \tag{1.1.128}$$

Note that this establishes that for every element in $\mathsf{Aut}(\mathbb{P})$ there is at least one element in $\mathsf{GL}(\mathbb{C}^2)$ in its preimage – the map is onto (or surjective). It is, however, not one to one (or injective) since M and λM for every $\lambda \in \mathbb{C} \setminus \{0\}$ map onto the same map:

$$f_{\lambda M}(z) = f_M(z) \tag{1.1.129}$$

Irrespective of this rather strong fibration, we observe that for $M, N \in \mathsf{GL}(2\mathbb{C})$

$$f_M \circ f_N = f_{MN} \tag{1.1.130}$$

We then get for free that every such map has the inverse $f_{M^{-1}}$, and in fact that these transformations form a group (closedness, associativity, unit element and inverses are all inherited from $\mathsf{GL}(2,\mathbb{C})$). Even the condition on the coefficients reduces to $0 \neq \det(MN) = \det(M)\det(N)$ and thus trivially holds for the composition $f_M \circ f_N$

We can use this to rescale our matrices M such that they satisfy $\det M = 1$. This takes us into the group

$$\mathsf{SL}(2,\mathbb{C}) := \left\{ M \in \mathbb{C}^{2\times 2} | \det(M) = ad - bc = 1 \right\} \tag{1.1.131}$$

but since the equation that we need to solve to find the rescaling factor for a given M is $\det(\lambda M) = \lambda^2 \det(M) \stackrel{!}{=} 1$ this rescaling operation is not unique. We always have the choice between $\pm\lambda$ to map any $M \in \mathsf{GL}(2,\mathbb{C})$ onto a suitable $\lambda M \in \mathsf{SL}(2,\mathbb{C})$. What we achieve is a two to one group homomorphism

$$\mathsf{SL}(2,\mathbb{C}) \to \mathsf{Aut}(\mathbb{C}P^1); M \mapsto f_M \text{ with } f_M(z) = \frac{M_{11}z + M_{12}}{M_{21}z + M_{22}} \tag{1.1.132}$$

whose kernel is the center of $\mathsf{SL}(2,\mathbb{C})$, the group $\{\pm\mathbb{1}_2\} \cong \mathbb{Z}_2$ and thus we see that

$$\mathsf{Aut}(\mathbb{C}P^1) \cong \mathsf{SL}(2,\mathbb{C})/\mathbb{Z}_2 =: \mathsf{PSL}(2,\mathbb{C}) \tag{1.1.133}$$

The last statement introduces the notion of a “projective special linear group”.

› **Observation 2 – elementary Möbius transformations:**

Every Möbius transformation can be decomposed into a composition of elementary transformations of the form

1. translations $z \mapsto z + \tilde{z}$, $\tilde{z} \in \mathbb{C}$
2. rescalings $z \mapsto \lambda z$, $0 \leq \lambda \in \mathbb{R}$
3. rotations $z \mapsto e^{I\theta} z$, $\theta \in [0, 2\pi[$
4. Möbius inversions $z \mapsto -1/z$ (note the $-$ sign and the comments in warning 1.1.)

To see this, consider $f(z) = \frac{az+b}{cz+d}$ and distinguish two cases: $c = 0$ and $c \neq 0$:

› $c = 0$:

$$f(z) = \frac{a}{d}z + \frac{b}{d} \tag{1.1.134a}$$

(scaling and rotation by a/d followed by translation by b/d)

- $c \neq 0$:
$$f(z) = \frac{ad-bc}{c^2}\left(-\frac{1}{z+d/c}\right) + \frac{a}{c} \tag{1.1.134b}$$
(translation by d/c followed by a an inversion, then a scaling and rotation by $(bc-ad)c^2$ followed by a translation by a/c.

Warning 1.1 – the complex inversion does not have $\det(M) = 1$:

Ironically the naive complex inversion $z \mapsto 1/z$ one is often offered in this context is not in $\mathsf{PSL}(2, \mathbb{C})$ since it corresponds to $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ which has determinant -1. This is easily fixed by taking $b = -1$ instead, i.e. the map

$$z \mapsto -\frac{1}{z} \qquad \text{i.e.} \qquad re^{i\phi} \mapsto -\frac{1}{r}e^{-i\phi} = \frac{1}{r}e^{-i(\phi-\pi)} \tag{1.1.135}$$

Note that this map preserves the upper and lower half planes

This still differs from the inversion on $\mathbb{R}^2$ which, in complex notation becomes $z \mapsto \frac{z}{|z|^2} = \frac{z}{zz^*} = \frac{1}{z^*}$, now by a reflection on the imaginary axis instead of a reflection on the real axis as the "naive" $z \mapsto 1/z$.

Observation 3 – $\mathsf{PSL}(2, \mathbb{R})$ preserves the upper and lower half planes:

We can naturally think of $\mathsf{PSL}(2, \mathbb{R})$ as a subgroup of $\mathsf{PSL}(2, \mathbb{C})$. As such it acts on the projective plane $P\mathbb{C}^1$.

Then, with all coefficients real, the transformations in Eqns. (1.1.134) consist only of compositions of translations parallel to the real axis, rescalings and Möbius inversions $z \mapsto -1/z$, each of which individually respect the half planes. [Beware of the minus sign in the latter, warning 1.1.]

Matthew Schwartz has a very interesting application of this in [22] "Non-global Logarithms at 3 Loops, 4 Loops, 5 Loops and Beyond"

1.1.11 The stereographic projection – beyond the unit sphere

`./stereographic.nb`

To set up a stereographic projection from a two sphere onto a plane we need to view both as hypersurfaces in $\mathbb{R}^3$. The standard setting embeds the sphere as a unit sphere at the origin and the plane as the x, y-plane through the origin.

To expose scaling features, I choose to use both an arbitrary radius for the sphere, setting

$$S^{(2)}(r) := \{\boldsymbol{w} \in \mathbb{R}^3 | \boldsymbol{w}^2 = r^2\} \tag{1.1.136}$$

and displace the plane by a vector $d\hat{\boldsymbol{z}}$:

$$P = \{\boldsymbol{y} \in \mathbb{R}^3 | \hat{\boldsymbol{z}} \cdot \boldsymbol{y} \stackrel{!}{=} d\} \tag{1.1.137}$$

The input for the projection, the map from the sphere onto the plane is the south pole at $\boldsymbol{s} = (0, 0, -r)^t$ and a point $\boldsymbol{w}$ on the sphere. This leads to a parametrization of the projection line in the form $\boldsymbol{s} + \alpha(\boldsymbol{w} - \boldsymbol{s})$ where α is a real parameter.

This line intersects P where

$$d = \hat{\boldsymbol{z}} \cdot (\boldsymbol{s} + \alpha(\boldsymbol{w} - \boldsymbol{s})) = -r + \alpha(w^3 + r) \Leftrightarrow \alpha = \frac{d+r}{w^3+r} \tag{1.1.138}$$

i.e. at a point

$$\boldsymbol{x} = \begin{pmatrix} \boldsymbol{x}_\perp \\ d \end{pmatrix} \text{ where } \boldsymbol{x}_\perp = \frac{d+r}{w^3+r} \begin{pmatrix} w^1 \\ w^2 \end{pmatrix} \tag{1.1.139}$$

This provides us with the stereographic projection from the two sphere $S^{(2)}(r)$ onto this plane, the map

$$SP : S^{(2)}(r) \subset \mathbb{R}^3 \to \mathbb{R}^2 \ , \qquad \boldsymbol{w} \mapsto \boldsymbol{x} = \frac{d+r}{w^3+r} \begin{pmatrix} w^1 \\ w^2 \end{pmatrix} \tag{1.1.140}$$

The inverse of this map uses the south pole and $\boldsymbol{x} \in P$ to parametrize the projection line as $\boldsymbol{s} + \beta(\boldsymbol{x} - \boldsymbol{s})$. The point on the sphere arises from solving

$$(\boldsymbol{s} + \beta(\boldsymbol{x} - \boldsymbol{s}))^2 = r^2 \tag{1.1.141}$$

for β. This quadratic equation has two solutions, the south pole at $\beta = 0$ and the projected point $\boldsymbol{w}$ at the other solution

$$\beta = \frac{2r(d+r)}{(d+r)^2 + \boldsymbol{x}_\perp^2} \qquad \text{with} \qquad \boldsymbol{w} = \frac{r}{(d+r)^2 + \boldsymbol{x}_\perp^2} \begin{pmatrix} 2(d+r)x^1 \\ 2(d+r)x^2 \\ (d+r)^2 - \boldsymbol{x}_\perp^2 \end{pmatrix} \ . \tag{1.1.142}$$

This expresses the inverse stereographic projection as

$$SP^{-1} : \mathbb{R}^2 \to S^{(2)}(r) \ , \qquad \boldsymbol{x} \mapsto \boldsymbol{w} = \frac{r}{1 + \boldsymbol{x}^2/(d+r)^2} \begin{pmatrix} 2x^1/(d+r) \\ 2x^2/(d+r) \\ 1 - \boldsymbol{x}^2/(d+r)^2 \end{pmatrix} \tag{1.1.143}$$

The form chosen in Eq. (1.1.143) exposes the scaling properties: The result maps onto the conventional expression if one measures the coordinates in the transverse plane in units of $d + r$ and scales $S^{(2)}(r)$ back to the unit sphere by dividing out r.

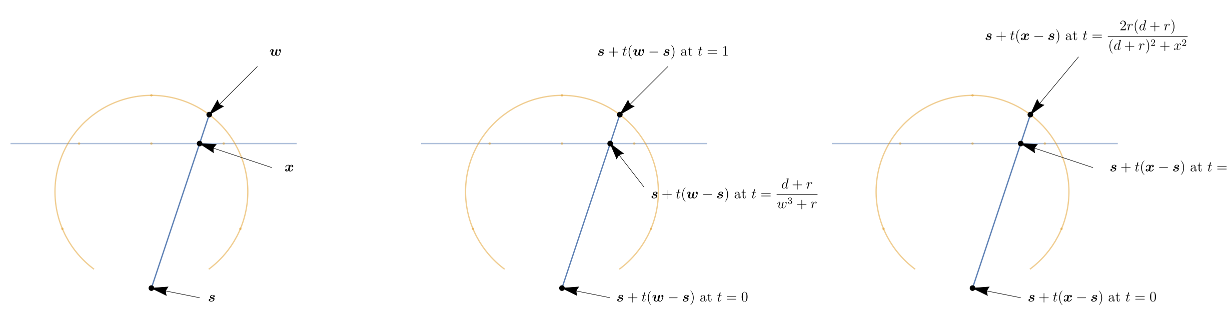


Fig. 2: *Sphere, plane and projection line for a stereographic projection from the south pole of* $S^{(2)}(r)$

To compare the embedding in Eq. (1.1.143) with spherical coordinates we need to untangle the equations

$$r \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix} \stackrel{!}{=} \frac{r}{1+\boldsymbol{x}^2/(d+r)^2} \begin{pmatrix} 2x^1/(d+r) \\ 2x^2/(d+r) \\ 1-\boldsymbol{x}^2/(d+r)^2 \end{pmatrix} \tag{1.1.144}$$

Purely geometrically we must have

$$\cos\phi = \frac{x^1}{|\boldsymbol{x}|} \qquad \sin\phi = \frac{x^2}{|\boldsymbol{x}|} \tag{1.1.145}$$

This leaves us with the identification

$$\sin\theta = \frac{2|\boldsymbol{x}/(d+r)|}{1+\boldsymbol{x}^2/(d+r)^2} \tag{1.1.146}$$

from the first two components. The third component announces

$$\cos\theta = \frac{1-\boldsymbol{x}^2/(d+r)^2}{1+\boldsymbol{x}^2/(d+r)^2} \qquad \Leftrightarrow \qquad \sin\theta = \sqrt{1-(\cos\theta)^2} = \frac{2|\boldsymbol{x}/(d+r)|}{1+\boldsymbol{x}^2/(d+r)^2} \tag{1.1.147}$$

All of this *is* consistent since $\sin(\theta)$ is positive on $[0,\pi]$.

Reminder 1.1 – circles and lines:

This is a reformatted quote from Wikipedia:

- **Behavior of the projection:** Circles on the sphere that do not pass through the point of projection are projected to circles on the plane. Circles on the sphere that do pass through the point of projection are projected to straight lines on the plane. These lines are sometimes thought of as circles through the point at infinity, or circles of infinite radius.
- **Behavior of the inverse projection:** All lines in the plane, when transformed to circles on the sphere by the inverse of stereographic projection, meet at the projection point. Parallel lines, which do not intersect in the plane, are transformed to circles tangent at projection point. Intersecting lines are transformed to circles that intersect transversally at two points in the sphere, one of which is the projection point.

Reminder 1.2 – great circles and their images:

Great circles are the geodesics on the sphere, they intersect the equator exactly twice. Their images under the stereographic project intersect the image of the equator exactly twice. (This is trivial.) The intersections lie on a line through the origin. (How to I show this?)

1.1.12 4-d conformal generalization

[See `./Vladimirov-Trans.nb` for algebra and plots.]

To set up notation we introduce two mutually dual lightlike vectors n and $\bar{n}$, i.e. vectors with $n^2 = \bar{n}^2 = 0$ and $n.\bar{n} = 1$ such that

$$x^\mu = x^+ \bar{n}^\mu + x^- n^\mu + x_\perp^\mu \tag{1.1.148}$$

This implies $x^- = n.x$, $x^+ = \bar{n}.x$.

To quote from [23]: "The rapidity divergences in many aspects resemble the UV divergences. The main difference is that the rapidity divergences are associated with the localization of gluons at the distant transverse plane $(-n)_\perp^\infty$, while the UV divergences are associated with the localization at a point. Let us build the conformal transformation which relates the plane $(-n)_\perp^\infty$ to a point (for simplicity we take the origin)."

1. translation by $\frac{\lambda-1}{2a}\bar{n}$ (In Alexei's notation $\{\frac{\lambda-1}{2a}, 0^-, 0_\perp\}$),
2. special conformal transformation along the light-cone direction n with the vector an ($\{0^+, a, 0_\perp\}$)
3. translation by $-\frac{1}{2a}\bar{n}$ ($\{-(2a)^{-1}, 0^-, 0_\perp\}$).

The resulting transformation reads

$$\mathcal{C}_{\bar{n}} : \{x^+, x^-, x_\perp\} \to \{\frac{-1}{2a}\frac{1}{\lambda + 2ax^+}, x^- + \frac{ax_\perp^2}{\lambda + 2ax^+}, \frac{x_\perp}{\lambda + 2ax^+}\}. \tag{1.1.149}$$

In the same manner we can build the transformation that relates the $(-\bar{n})_\perp^\infty$ to the origin,

$$\mathcal{C}_n : \{x^+, x^-, x_\perp\} \to \{x^+ + \frac{\bar{a}x_\perp^2}{\bar{\lambda} + 2\bar{a}x^-}, \frac{-1}{2\bar{a}}\frac{1}{\bar{\lambda} + 2\bar{a}x^-}, \frac{x_\perp}{\bar{\lambda} + 2\bar{a}x^-}\}. \tag{1.1.150}$$

The parameters a and λ are free real parameters.

Reminder 1.3 – special conformal transformations and units:

Special conformal transformations are obtaine by an inversion at the unit hyperola (unit sphere in Euclidean geometry), followed by a translation by a vector b followed by a second inversion at the unit hyperbola. The formula you find in a typical physics or mathematics text is

$$x^\mu \mapsto \frac{x^\mu}{x^2} \mapsto \frac{x^\mu}{x^2} + b^\mu \mapsto \frac{\frac{x^\mu}{x^2} + b^\mu}{(\frac{x^\nu}{x^2} + b^\nu)(\frac{x^\nu}{x^2} + b_\nu)} = \frac{x^\mu + x^2 b^\mu}{1 + 2b.x + b^2 x^2} \tag{1.1.151}$$

Clearly in this exposition, units are in a mess. If we want to view the inversion as a map from $\mathbb{M}$[length] into itself, we need to supply an inversion "radius" and write the fist step as

$$x^\mu \mapsto \frac{x^\mu}{x^2/R^2} \tag{1.1.152}$$

Without that we end up in $\mathbb{M}[\text{length}^{-1}]$. Only then does b carry the same units as x does. and we can interpret the chain of maps as

We end up with

$$x^\mu \mapsto \frac{x^\mu + x^2 b^\mu/R^2}{1 + 2x.b/R^2 + b^2/R^4 x^2} \tag{1.1.153}$$

and we see that we may equally well associate the inversion scale with b as b/R^2. This reinterprets inversion as a map $\mathbb{R}^n[\text{unit}] \to \mathbb{R}^n[\text{unit}^{-1}]$ and the translation as one on the latter space.

Reminder 1.4 – distance sqares and conformal factors:

We note that the norm squared of a vector simply rescales like

$$x^2 \mapsto \frac{(x+x^2b/R^2)^2}{(1+2x.b/R^2+b^2/R^4x^2)^2} = \frac{x^2+2x.b/R^2x^2+x^4b^2/R^4}{(1+2x.b/R^2+b^2/R^4x^2)^2} = \frac{x^2}{1+2x.b/R^2+b^2/R^4x^2} \tag{1.1.154}$$

The inner products of two vectors is more complicated

$$\begin{aligned} x_1.x_2 \mapsto & \frac{(x_1+x_1^2b/R^2).(x_2+x_2^2b/R^2)}{(1+2x_1.b/R^2+b^2/R^4x_1^2)(1+2x_2.b/R^2+b^2/R^4x_2^2)} \\ = & \frac{x_1.x_2+x_1^2b.x_2/R^2+x_2^2x_1.b/R^2+x_1^2x_2^2b^2/R^4}{(1+2x_1.b/R^2+b^2/R^4x_1^2)(1+2x_2.b/R^2+b^2/R^4x_2^2)} \end{aligned} \tag{1.1.155}$$

The distance squared of two vectors, however, simplifies again:

$$\begin{aligned} (x_1-x_2)^2 \mapsto & \left(\frac{x_1+x_1^2b/R^2}{1+2x_1.b/R^2+b^2/R^4x_1^2} - \frac{x_2+x_2^2b/R^2}{1+2x_2.b/R^2+b^2/R^4x_2^2}\right)^2 \\ = & \frac{x_1^2}{(1+2x_1.b/R^2+b^2/R^4x_1^2)} \\ & + 2\frac{(x_1+x_1^2b/R^2).(x_2+x_2^2b/R^2)}{(1+2x_1.b/R^2+b^2/R^4x_1^2)(1+2x_2.b/R^2+b^2/R^4x_2^2)} \\ & + \frac{x_2^2}{(1+2x_2.b/R^2+b^2/R^4x_2^2)} \\ = & \frac{(x_1-x_2)^2}{(1+2x_1.b/R^2+b^2/R^4x_1^2)(1+2x_2.b/R^2+b^2/R^4x_2^2)} \end{aligned} \tag{1.1.156}$$

with all b-dependent terms cancelling in the numerator after taking the common denominator.

The transformation

$$\mathcal{C}_{\bar{n}} = \begin{pmatrix} x^+ \\ \boldsymbol{x} \\ x^- \end{pmatrix} \mapsto \begin{pmatrix} -\frac{1}{2a}\frac{1}{\lambda+2ax^+} \\ \frac{\boldsymbol{x}}{\lambda+2ax^+} \\ x^- - \frac{a\boldsymbol{x}^2}{\lambda+2ax^+} \end{pmatrix} \xrightarrow{\lambda\to 0, a\to 1/\sqrt{2}} \begin{pmatrix} -\frac{1}{2x^+} \\ \frac{\boldsymbol{x}}{\sqrt{2}x^+} \\ x^- - \frac{\boldsymbol{x}^2}{2x^+} \end{pmatrix} \tag{1.1.157}$$

Note that this takes the subspace of vectors with fixed value of x^- onto a light cone based at nx^-, see Fig. 3.

$$x'^\mu = \left(-\frac{1}{2a}\bar{n}^\mu - a\boldsymbol{x}^2 n^\mu + x_\perp^\mu\right)\frac{1}{\lambda+2ax^+} + x^- n^\mu \tag{1.1.158}$$

This becomes obvious by noting that the (Minkowski) norm squared of the term in brackets vanishes and only the cross terms survives:

$$x^2 = 2x^+x^- - \boldsymbol{x}^2 \mapsto x'^2 = -\frac{x^-}{a(\lambda + 2ax^+)} \tag{1.1.159}$$

This is directly owed to the conformal nature of the transformation, which transforms the (light cone) metric like

$$\begin{pmatrix} & & 1 \\ & -\mathbb{1}_2 & \\ 1 & & \end{pmatrix} \mapsto \frac{1}{(\lambda + 2ax^+)^2} \begin{pmatrix} & & 1 \\ & -\mathbb{1}_2 & \\ 1 & & \end{pmatrix} \tag{1.1.160}$$

The difference squared becomes

$$\begin{aligned} (x-y)^2 \mapsto (x'-y')^2 &= 2(x'-y')^+(x'-y')^- - (\boldsymbol{x}'-\boldsymbol{y}')^2 \\ &= \frac{2(\lambda + 2ax^+) - (\lambda + 2ay^+))/2a(x-y)^- - (\boldsymbol{x}-\boldsymbol{y})^2}{(\lambda + 2ax^+)(\lambda + 2ay^+)} = \frac{(x-y)^2}{(\lambda + 2ax^+)(\lambda + 2ay^+)} \end{aligned} \tag{1.1.161}$$

This map covers the full light cone, it would appear it maps the hyperplanes $x^- =$ const. onto the cone based at x^-const. (i.e. at the point with coordinates $x^- =$ const., $x_\perp = 0$, $x^+ = 0$) in a one to one manner with its inverse given by Eq. (1.1.170). What is worth noting are the details of *how* the plane is wrapped around the cone.

To visualize the map, note that the expression in Eq. (1.1.158) exposes that the transformation commutes with

- spatial rotations around the $\hat{z}$ axis $R(\alpha\hat{z})$ and with
- translations T_{sn} along n

so that

$$R(\alpha\hat{z})\mathcal{C}_{\bar{n}}R(\alpha\hat{z})^{-1} = \mathcal{C}_{\bar{n}} \qquad \text{and} \qquad T_{sn}\mathcal{C}_{\bar{n}}T_{sn}^{-1} = C_{\bar{n}} \tag{1.1.162}$$

This allows us to capture the salient features of the map by looking at its behavior at a hyperplane of fixed x^- and select a fixed one dimensional subspace within $x_\perp$ to reduce the dimensionality of the plots to 3d. This is done in Fig. 3.

The description of $\mathcal{C}_{\bar{n}}$ as a sequence of elementary conformal maps establishes that it is indeed a conformal map. It does not, however, highlight its geometric structure terribly clearly. A quick look at the final result hints at a more intuitive picture that separates out the two main geometric features of interest to us:

As a first step we invert the x^+ axis by mapping

$$x^+ \mapsto \frac{1}{\lambda + 2ax^+} \tag{1.1.163}$$

The complete 4-vector transforms as

$$x^\mu = \bar{n}^\mu x^+ + n^\mu x^- + x_\perp^\mu \mapsto \bar{n}^\mu \frac{1}{\lambda + 2ax^+} + n^\mu x^- + x_\perp^\mu =: \bar{n}^\mu \tilde{x}^+ + n^\mu \tilde{x}^- + \tilde{x}_\perp^\mu = \tilde{x}^\mu \tag{1.1.164}$$

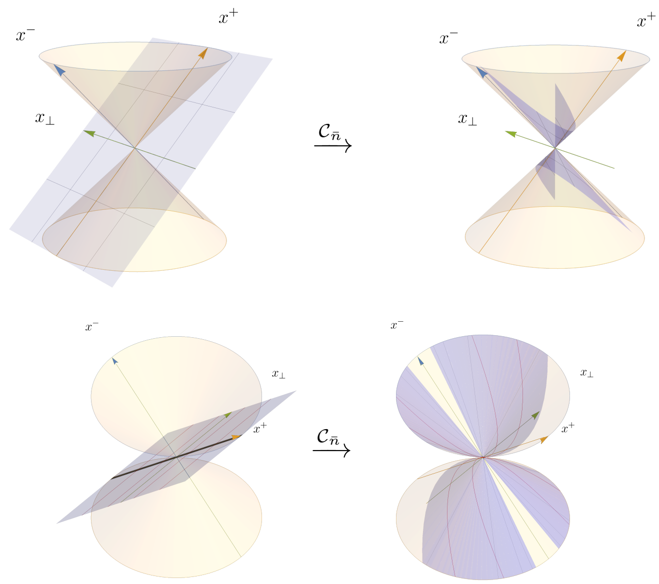


Fig. 3: *$\mathcal{C}_{\bar{n}}$ takes the subspace of vectors with fixed value of x^- onto a light cone based at nx^-. The images show the case $x^- = 0$. Domain and image are shaded in blue.*

The significance of this step is that the region just above λ is mapped to $+\infty$, the region just below to $-\infty$ and vice versa. The tangent plane in the figure is turned "inside out" and the split happens at λ. a simply provides the "inversion radius".

The length squared maps to

$$x^2 \mapsto (\tilde{x})^2 = 2\tilde{x}^+\tilde{x}^- - (\tilde{\boldsymbol{x}})^2 = 2\tilde{x}^+\tilde{x}^- - \tilde{\boldsymbol{x}}^2 = 2\frac{x^-}{\lambda + 2ax^+} - \boldsymbol{x}^2 \tag{1.1.165}$$

The second step then wraps the hyperplanes around the associated cones

$$\tilde{x}^\mu \mapsto \left(-\frac{1}{2a}\bar{n}^\mu - a\tilde{\boldsymbol{x}}^2 n^\mu + \tilde{x}^\mu_\perp\right)\tilde{x}^+ + \tilde{x}^- n^\mu \tag{1.1.166}$$

For a given x^-, a finite cylinder (a rectancle in the plots) $|\tilde{\boldsymbol{x}}| < |\tilde{\boldsymbol{x}}_{\max}|$, $|\tilde{x}^+| < |\tilde{x}^+_{\max}|$ is mapped onto the cone with "parabolale" in $|\boldsymbol{x}|$ traced out at each fixed value of $\tilde{x}^+$. The range restriction in $|\tilde{\boldsymbol{x}}|$ leads to a fixed spatial opening angle, solely dependent on $|\tilde{\boldsymbol{x}}_{\max}|$.

To calculate its value note that the spatial angle α between two points y_1 and y_2 on the image cone at 0 is given by

$$y_1 . y_2 = y_1^0 y_2^0 - \boldsymbol{y}_1 \cdot \boldsymbol{y}_2 = y_1^0 y_2^0 - |\boldsymbol{y}_1||\boldsymbol{y}_2|\cos\alpha = y_1^0 y_2^0(1 - \cos\alpha) \tag{1.1.167}$$

The opening angle is then contained in this expression if we insert

$$y_{1,2} \mapsto \left(-\frac{1}{2a}\bar{n}^\mu - a\tilde{\boldsymbol{x}}^2 n^\mu \pm \tilde{x}^\mu_\perp\right)\tilde{x}^+ = x^\mu_{(0)}\left(-\frac{1 + a\tilde{\boldsymbol{x}}^2}{2a}\frac{\tilde{x}^+}{\sqrt{2}}\right) + x^\mu_{(3)}\left(-\frac{1 - a\tilde{\boldsymbol{x}}^2}{2a}\frac{\tilde{x}^+}{\sqrt{2}}\right) \pm \tilde{x}^\mu_\perp \tag{1.1.168}$$

which leads to

The opening angle is given by

$$\cos\alpha = 1 - \frac{16a^2\boldsymbol{x}^2}{(1+2a^2\boldsymbol{x}^2)^2} \tag{1.1.169}$$

The inverse map then must take such cones onto hyperplanes at fixed x^-. To expose this we write it as

$$x^\mu = \left[(\frac{1}{2a}+y^+)\bar{n}^\mu - ay^2 n^\mu + y_\perp^\mu\right]\frac{1}{-2ay^+} = \left[\left(\frac{1}{2a}+y^+\right)\bar{n}^\mu + y_\perp^\mu\right]\frac{1}{-2ay^+} + \frac{y^2}{2y^+}n^\mu \tag{1.1.170}$$

and note that this maps the cone at the origin, $y^2 = 0$ onto the hyperplane tangent to it at $\bar{n}$. To coax out the more general structure not that a cone based at $n\alpha$ write y^2, which only features in the n term by splitting of a zero in the following form: $y^2 = 2y^+y^- - \boldsymbol{y}^2 = 2y^+(\Delta y^- + \frac{\boldsymbol{y}^2}{2y^+}) - \boldsymbol{y}^2 = 2y^+\Delta y^-$ so that

$$\begin{aligned} x^\mu &= \left[(\frac{1}{2a}+y^+)\bar{n}^\mu - ay^2 n + y_\perp^\mu\right]\frac{1}{-2ay^+} = \left[(\frac{1}{2a}+y^+)\bar{n}^\mu + y_\perp^\mu\right]\frac{1}{-2ay^+} + \frac{y^2}{2y^+}n^\mu \\ &= \left[\left(\frac{1}{2a}+y^+\right)\bar{n}^\mu + y_\perp^\mu\right]\frac{1}{-2ay^+} + \Delta y^- n^\mu \qquad \text{where} \qquad \Delta y^- = \frac{y^2}{2y^+} \end{aligned} \tag{1.1.171}$$

Here $y^2 = 2y^+y^- - \boldsymbol{y}^2$, the full Minkowski length, not just $\boldsymbol{x}^2$!

A note about conventions and minus signs: Vladimirov writes $x_\perp^2 = x_\perp.x_\perp = -\boldsymbol{x}\cdot\boldsymbol{x} = -\boldsymbol{x}^2$!

Correspondingly

$$\mathcal{C}_n = \begin{pmatrix} x^+ \\ \boldsymbol{x} \\ x^- \end{pmatrix} \mapsto \begin{pmatrix} x^+ - \frac{\bar{a}\boldsymbol{x}^2}{\bar{\lambda}+2\bar{a}x^-} \\ \frac{\boldsymbol{x}}{\bar{\lambda}+2\bar{a}x^-} \\ -\frac{1}{2\bar{a}}\frac{1}{\bar{\lambda}+2\bar{a}x^-} \end{pmatrix} \xrightarrow{\bar{\lambda}\to 0, a\to 1/\sqrt{2}} \begin{pmatrix} x^+ - \frac{\bar{a}\boldsymbol{x}^2}{\sqrt{2}x^-} \\ \frac{\boldsymbol{x}}{\sqrt{2}x^-} \\ -\frac{1}{2\bar{a}}\frac{1}{2x^-} \end{pmatrix} \tag{1.1.172}$$

Inner products of BK type vectors $x = x^+\bar{n} + x_\perp$ has contributions from transverse parts only and accordingly is purely spacelike:

$$(x-y)^2 = -(\boldsymbol{x}-\boldsymbol{y})^2 \tag{1.1.173}$$

which has contributions from both $x^2 = -\boldsymbol{x}^2$ and $y^2 = -\boldsymbol{y}^2$

Under the conformal transformation, both x' and y' become lightlike and we find

$$\begin{aligned} (x'-y')^2 = x'^2 - 2x'.y' + y'^2 = -2x'.y' &= -2\left(\frac{1}{2x^+}\frac{\boldsymbol{y}^2}{2y^+} + \frac{\boldsymbol{x}^2}{2x^+}\frac{1}{2y^+} - \frac{\boldsymbol{x}}{\sqrt{2}x^+}\cdot\frac{\boldsymbol{y}}{\sqrt{2}y^+}\right) \\ &= -\frac{1}{2x^+y^+}(\boldsymbol{x}-\boldsymbol{y})^2 \end{aligned} \tag{1.1.174}$$

Since the result is proportional to $x'.y' = y'^0x'^0(1-\cos\theta)$ this relates to coordinates on the "sphere of directions".

The Minkowski overlap of two such vectors becomes

$$\begin{aligned} x.y = |_{x^-=y^-=0} &= -\boldsymbol{x}\cdot\boldsymbol{y} \\ &\mapsto \frac{1}{2x^+}\frac{\boldsymbol{y}^2}{2y^+} + \frac{\boldsymbol{x}^2}{2x^+}\frac{1}{2y^+} - \frac{\boldsymbol{x}}{\sqrt{2}x^+}\cdot\frac{\boldsymbol{y}}{\sqrt{2}y^+} = \frac{1}{4x^+y^+}(\boldsymbol{x}-\boldsymbol{y})^2 \end{aligned} \tag{1.1.175}$$

2 Geometry of Wilson lines

Sources:

- Vladimirov: [23] and ./Vladimirov_2018_Edinburgh.pdf
- Hatta: [24]
- Hatta again, with slightly better discussion on stereographic[25]

3 QCD renormalization

Let me copy notation from [26]:

Modified quote from [26]:
To start with, let us recall some basic equations of QCD. The dimensionally regularized and renormalized Lagrangian is given by

$$\begin{aligned}\mathcal{L} &= Z_2\bar\psi i\,\partial\!\!\!/\psi + \bar Z_1\mu^\epsilon g\bar\psi \,B\!\!\!/^a t^a\psi \\ &- \frac{Z_3}{4}\left(F^a_{\mu\nu}\right)^2 - \frac{1}{2}\mu^\epsilon g Z_1\; f^{abc}F^a_{\mu\nu}B^b_\mu B^c_\nu - \frac{Z_4}{4}\mu^{2\epsilon}g^2\left(f^{abc}B^b_\mu B^c_\nu\right)^2 - \frac{1}{2\xi}\left(\partial_\mu B^a_\mu\right)^2 \\ &+ \widetilde{Z}_3\;\partial_\mu\bar\omega^a\partial_\mu\omega^a + \mu^\epsilon g\widetilde{Z}_1\; f^{abc}\partial_\mu\bar\omega^a B^b_\mu\omega^c,\end{aligned} \tag{3.0.1}$$

where $d = 4-2\epsilon$ is the space-time dimension and μ is a mass parameter with dimension one introduced in order to keep the coupling constant dimensionless. We introduce here the symbol $F^a_{\mu\nu}$ as a shorthand notation for the Abelian part of the full QCD field strength tensor $G^a_{\mu\nu} = F^a_{\mu\nu} + \mu^\epsilon gXf^{abc}B^b_\mu B^c_\nu$. The canonical dimension of the elementary fields are the following: $d^{\rm can}_\psi = \frac{3}{2}-\epsilon$ for fermions, $d^{\rm can}_G = 1-\epsilon$ for gluons, $d^{\rm can}_{\bar\omega} = d-2$ and $d^{\rm can}_\omega = 0$ for the anti-ghost and ghost fields, respectively. This Lagrangian is invariant under the following renormalized BRST-transformations:

$$\begin{aligned}&\delta^{\rm BRST}\psi = -i\mu^\epsilon g\widetilde{Z}_1\omega^a t^a\psi\delta\lambda, \quad \delta^{\rm BRST}B^a_\mu = \widetilde{Z}_3 D_\mu\omega^a\delta\lambda, \\ &\delta^{\rm BRST}\omega^a = \frac{1}{2}\mu^\epsilon g\widetilde{Z}_1 f^{abc}\omega^b\omega^c\delta\lambda, \quad \delta^{\rm BRST}\bar\omega^a = \frac{1}{\xi}\partial_\mu B^a_\mu\delta\lambda,\end{aligned} \tag{3.0.2}$$

where $\delta\lambda$ is a renormalized Grassman variable. The covariant derivative is defined as follows $D_\mu = \partial_\mu - i\mu^\epsilon gXT^aB^a_\mu$, where T^a is the generator in the fundamental ($T^a\phi_i = t^a_{ij}\phi_j$) or adjoint ($T^b\phi^a = if^{abc}\phi^c$) representations depending on the object it is acting on. The Ward-Takahashi identities imply the following relations between the renormalization constants $Z_1Z_3^{-\frac{3}{2}} = Z_4^{\frac{1}{2}}Z_3^{-1} = \widetilde{Z}_1\widetilde{Z}_3^{-1}Z_3^{-\frac{1}{2}} = \bar Z_1 Z_2^{-1}Z_3^{-\frac{1}{2}}$. In the MS scheme Z-factors are defined as Laurent series in ϵ: $Z(g,\epsilon) = 1 + Z^{[1]}(g,\epsilon)/\epsilon + Z^{[2]}(g,\epsilon)/\epsilon^2 + \ldots$. For our consequent discussion we take the renormalization constants Z_2, Z_3, $\widetilde{Z}_3$ and $X \equiv Z_1Z_3^{-1} = \widetilde{Z}_1\widetilde{Z}_3^{-1} = \bar Z_1 Z_2^{-1}$ as independent ones.
The renormalization group coefficients and the anomalous dimensions of the fields are

$$\beta_\epsilon(g) = \mu\frac{dg}{d\mu} = -\epsilon\, g + \beta(g), \quad \sigma = \mu\frac{d}{d\mu}\ln\xi = -2\gamma_G, \quad \gamma_\phi = \frac{1}{2}\mu\frac{d}{d\mu}\ln Z_\phi, \tag{3.0.3}$$

(ϕ runs over parton species entering the Lagrangian $\{\phi = \psi, G, \bar\omega\}$) where $Z_\psi = Z_2,\ Z_G = Z_3,\ Z_{\bar\omega} = \widetilde{Z}_3$. These coefficients are determined by the residues of the corresponding Z-factors [**?**, **?**] and in lowest order approximation, $\frac{\beta}{g} = \frac{\alpha_s}{4\pi}\beta_0 + \left(\frac{\alpha_s}{4\pi}\right)^2\beta_1 + \ldots$, $\gamma_\phi = \frac{\alpha_s}{2\pi}\gamma^{(0)}_\phi + \ldots$, they are given by

$$\beta_0 = \frac{4}{3}T_FN_f - \frac{11}{3}C_A, \quad \gamma^{(0)}_G(\xi) = \frac{2}{3}T_FN_f + \frac{C_A}{4}\left(\xi - \frac{13}{3}\right), \quad \gamma^{(0)}_\psi(\xi) = \frac{\xi}{2}C_F, \tag{3.0.4}$$

where the group theoretical factors are $C_A = 3$, $C_F = 4/3$, and $T_F = 1/2$ for $SU_c(3)$.

Comments on quote: The Z factors are introduced to related bare and regularized quantities

$$\psi_0 =: Z_2^{1/2}\psi \qquad A_0^\mu = Z_3^{1/2}A^\mu \tag{3.0.5}$$

The key logic in writing the above comes from dimensional counting, the fact that the regularized action S is dimensionless and that the regularized covariant derivative equals the bare covariant derivative.

From these we get

1. Dimensions for the fields from their respective kinetic terms:
   a) We set $[\psi] = [\bar{\psi}]$ so that $[\psi] = [\bar{\psi}] = 2[\psi]$, note that $[\partial] = 1$ and read off that $2[\psi] + 1 = 4 - 2\epsilon$, i.e. $[\psi] = \frac{3-2\epsilon}{2}$.
   b) $2[A_\mu] + 2 = 4 - 2\epsilon$, i.e. $[A_\mu] = \frac{2-2\epsilon}{2}$
   c) as a consequence $[\bar{\psi}A_\mu\psi] - (4-2\epsilon) = [3 - 2\epsilon + \frac{2-2\epsilon}{2}] - (4-2\epsilon) = \epsilon$. For this reason we need a factor μ^ϵ in the covariant derivative to keep the coupling dimensionless
2. A relation between the Z to keep the covariant derivative

   $$Z_2\bar{\psi}i\,\not{\partial}\psi + \bar{Z}_1\mu^\epsilon g\bar{\psi}\not{B}^a t^a\psi \stackrel{!}{=} \bar{\psi}_0 i(\not{\partial} - ig_0\not{A}_0)\psi_0 \tag{3.0.6}$$

   This requires

   $$g_0 = \frac{\bar{Z}_1}{Z_2 Z_3^{1/2}}\mu^\epsilon g \tag{3.0.7}$$

   and leads to

   $$\partial_\mu - iX\mu^\epsilon gA_\mu \stackrel{!}{=} \partial - ig_0A_0 \tag{3.0.8}$$

   with $X = \frac{\bar{Z}_1}{Z_2}$ alone from the terms considered at this point.

dimensions for the fields

3.1 What we have discussed

We consider here the expectation value of a color singlet operator

$$\langle \mathrm{tr}\, U(\boldsymbol{x}) U^\dagger(\boldsymbol{y}) \rangle \propto \int \mathcal{D}\mathcal{A}^\mu \mathcal{D}\psi \mathcal{D}\overline{\psi}\, \mathrm{tr}\, U(\boldsymbol{x}) U^\dagger(\boldsymbol{y})\, e^{iS} \tag{3.1.1}$$

in the presence of a gluonic background field $\boldsymbol{A}^\mu$, where the quantum fluctuations $\mathcal{A}^\mu$ are integrated out. The action

$$S = \int d^D x\, \mathcal{L}(\psi, \overline{\psi}, \boldsymbol{A} + \mathcal{A}|x) \quad \text{with} \quad D = 4 - 2\epsilon \tag{3.1.2}$$

is dimensional regularized and the path ordered Wilson line might be defined as

$$U(\boldsymbol{x}) = \mathcal{P} \exp\left\{ ig \int_{-\infty}^{\infty} d\tau\, n_\mu A^\mu(x) \right\}, \quad x^\mu = \left\{ \tau(1+e^{-2\eta}), \boldsymbol{x}, \tau(1-e^{-2\eta}) \right\}. \tag{3.1.3}$$

An infinitesimal conformal transformation of the background field may be generated by the variation

$$\boldsymbol{G} = \int d^4 x\, \{G(x) \boldsymbol{A}^\mu(x)\} \frac{\delta}{\delta \boldsymbol{A}_\mu(x)} \tag{3.1.4}$$

and analogous transformations hold for the quantum fields

$$\mathcal{G} = \int d^4 x\, \{G(x) \mathcal{A}^\mu(x)\} \frac{\delta}{\delta \mathcal{A}_\mu(x)} + \cdots . \tag{3.1.5}$$

Here, $G(x)$ denotes any of the well-known differential operators

$$\hat{M}^x_{\alpha\beta} = x_\alpha \partial_\beta - x_\beta \partial_\alpha - \Sigma_{\alpha\beta}\,, \tag{3.1.6}$$

$$\hat{D}^x = l + x\cdot\partial\,, \tag{3.1.7}$$

$$\hat{K}^x_\alpha = 2x_\alpha(l + x\cdot\partial) - 2x^\beta \Sigma_{\alpha\beta} - x^2 \partial_\alpha\,. \tag{3.1.8}$$

DM: have to convince myself about the sense of this equation

We intentionally keep track of the variable affected with a supersript.

Hence, the variation of the path ordered Wilson line (3.1.3) reads

$$[\boldsymbol{G} + \mathcal{G}]\, U(\boldsymbol{A} + \mathcal{A}|\boldsymbol{x}) = \hat{G}(x) U(\boldsymbol{A} + \mathcal{A}|\boldsymbol{x}) \tag{3.1.9}$$

The (regularized) action is invariant under a general simultaneous Lorentz transformation of both background and fluctuation fields, hence

$$\boldsymbol{M}_{\alpha\beta} \int d^D x\, \mathcal{L}(\psi, \overline{\psi}, \boldsymbol{A} + \mathcal{A}|x) + \mathcal{M}_{\alpha\beta} \int d^D x\, \mathcal{L}(\psi, \overline{\psi}, \boldsymbol{A} + \mathcal{A}|x) = 0 \tag{3.1.10}$$

In the conformal case, however, the transformation is affected by the trace anomaly which we write as in [26]

$$\boldsymbol{D} \int d^D x\, \mathcal{L}(\psi, \overline{\psi}, \boldsymbol{A} + \mathcal{A}|x) + \mathcal{D} \int d^D x\, \mathcal{L}(\psi, \overline{\psi}, \boldsymbol{A} + \mathcal{A}|x) = \int d^D x\, \Delta(x) \tag{3.1.11}$$

and

$$\boldsymbol{K}_\alpha \int d^D x\, \mathcal{L}(\psi, \overline{\psi}, \boldsymbol{A} + \mathcal{A}|x) + \mathcal{K}_\alpha \int d^D x\, \mathcal{L}(\psi, \overline{\psi}, \boldsymbol{A} + \mathcal{A}|x) = \int d^D x\, 2x_\alpha\, \Delta(x)\,. \tag{3.1.12}$$

'Ward identities' can be derived by utilizing the parameterization invariance of the path integral w.r.t. integration over the quantum fields

$$\int \mathcal{D}\mathcal{A}^\mu \mathcal{D}\psi \mathcal{D}\overline{\psi}\ \left\{\mathcal{G}\mathrm{tr}\, U(\boldsymbol{x}) U^\dagger(\boldsymbol{y})\right\}\, e^{iS} + \int \mathcal{D}\mathcal{A}^\mu \mathcal{D}\psi \mathcal{D}\overline{\psi}\, \mathbf{tr}\, U(\boldsymbol{x}) U^\dagger(\boldsymbol{y})\ \{i\mathcal{G}S\}\ e^{iS} = 0\,. \tag{3.1.13}$$

Adding to that equation the variation of the background field

$$\boldsymbol{M}_{\alpha\beta}\langle \mathrm{tr}\, U(\boldsymbol{x}) U^\dagger(\boldsymbol{y})\rangle := \int \mathcal{D}\mathcal{A}^\mu \mathcal{D}\psi \mathcal{D}\overline{\psi}\ \left(\mathrm{tr}\left\{\boldsymbol{M}_{\alpha\beta} U(\boldsymbol{x}) U^\dagger(\boldsymbol{y})\right\} + \{i\boldsymbol{M}_{\alpha\beta} S\}\right)\, e^{iS} \tag{3.1.14}$$

and utilizing (3.1.9), we find

$$\left[\hat{M}^x_{\alpha\beta} + \hat{M}^y_{\alpha\beta}\right] \langle \mathrm{tr}\, U(\boldsymbol{x}) U^\dagger(\boldsymbol{y})\rangle = \boldsymbol{M}_{\alpha\beta}\langle \mathrm{tr}\, U(\boldsymbol{x}) U^\dagger(\boldsymbol{y})\rangle\,. \tag{3.1.15}$$

Analogously, we obtain for the dilation and special conformal Ward identity

$$\left[\hat{D}^x + \hat{D}^y\right] \langle \mathrm{tr}\, U(\boldsymbol{x}) U^\dagger(\boldsymbol{y})\rangle = \boldsymbol{D}\langle \mathrm{tr}\, U(\boldsymbol{x}) U^\dagger(\boldsymbol{y})\rangle - \langle \mathrm{tr}\, U(\boldsymbol{x}) U^\dagger(\boldsymbol{y})\ i \int d^D x\, \Delta(x)\rangle \tag{3.1.16}$$

and

$$\left[\hat{K}^x_\alpha + \hat{K}^y_\alpha\right] \langle \mathrm{tr}\, U(\boldsymbol{x}) U^\dagger(\boldsymbol{y})\rangle = \boldsymbol{K}_\alpha\langle \mathrm{tr}\, U(\boldsymbol{x}) U^\dagger(\boldsymbol{y})\rangle - \langle \mathrm{tr}\, U(\boldsymbol{x}) U^\dagger(\boldsymbol{y}) i \int d^D x\, 2x_\alpha\, \Delta(x)\rangle\,, \tag{3.1.17}$$

respectively.

An alternative way of expressing the content of these equations is to not that

$$\boldsymbol{M}_{\alpha\beta}\langle \mathrm{tr}(U_{\boldsymbol{x}} U^\dagger_{\boldsymbol{y}})\rangle = \langle (\boldsymbol{M}_{\alpha\beta} + \mathcal{M}_{\alpha\beta}) \mathrm{tr}(U_{\boldsymbol{x}} U^\dagger_{\boldsymbol{y}})\rangle \tag{3.1.18}$$

$$\boldsymbol{D}\langle \mathrm{tr}(U_{\boldsymbol{x}} U^\dagger_{\boldsymbol{y}}\rangle = \langle (\boldsymbol{D} + \mathcal{D} + i \int d^D x \Delta(x)) \mathrm{tr}(U_{\boldsymbol{x}} U^\dagger_{\boldsymbol{y}})\rangle \tag{3.1.19}$$

and similarly for special conformal transformations.

As our starting point we identify the generator of rapidity evolution with a boost of the background field in + direction

$$\frac{d}{dY} = \boldsymbol{M}^{+-} \tag{3.1.20}$$

3.2 Wilson lines and the conformal group

Link to cusp anomalous dimension

We start with Wilson lines that lie on or near the + lightcone direction in the presence of a background field that should be thought of as being created by fast moving color charges along the − lightcone direction.

In terms of lightcone vectors

$$n^\mu := \frac{1}{\sqrt{2}}(1,\mathbf{0},1)^\mu \qquad \bar{n}^\mu := \frac{1}{\sqrt{2}}(1,\mathbf{0},-1)^\mu \tag{3.2.1}$$

a boost in 3-direction is conveniently written in a spectral decomposition as

$$L(\eta \boldsymbol{e}_3)^\mu{}_\nu = g^\mu{}_\nu - \bar{n}^\mu n_\nu - n^\mu \bar{n}_\nu + e^{-\eta}\bar{n}^\nu n^\nu + e^\eta n^\nu \bar{n}^\nu \tag{3.2.2}$$

The directions that arise by boosting the time and 3-axes by η are conveniently written as

$$\begin{aligned} m^\mu &= \frac{1}{\sqrt{2}}(e^\eta n^\mu + e^{-\eta}\bar{n}^\mu) \quad \text{boosted time axis:} \quad m^\mu \xrightarrow{\eta\to 0} (1,\mathbf{0},0) \\ \bar{m}^\mu &= \frac{1}{\sqrt{2}}(e^\eta n^\mu - e^{-\eta}\bar{n}^\mu) \quad \text{boosted 3-axis:} \quad \bar{m}^\mu \xrightarrow{\eta\to 0} (0,\mathbf{0},1) \end{aligned} \tag{3.2.3}$$

where $m^2 = 1$, $\bar{m}^2 = -1$ and $m.\bar{m} = 0$. Conversely, the light cone directions can be expressed in terms of m and $\bar{m}$ as

$$n = \frac{e^{-\eta}}{\sqrt{2}}(m+\bar{m}) \qquad \bar{n} = \frac{e^{\eta}}{\sqrt{2}}(m-\bar{m}) \tag{3.2.4}$$

This affords us the component expansions of an arbitrary 4-vector

$$x^\mu = \bar{n}^\mu \underbrace{n.x}_{x^+} + n^\mu \underbrace{\bar{n}.x}_{x^-} + \boldsymbol{x}^\mu = m^\mu \underbrace{m.x}_{x^m} + \bar{m}^\mu \underbrace{(-\bar{m}.x)}_{x^{\bar{m}}} + \boldsymbol{x}^\mu \tag{3.2.5}$$

Note that all this implies

$$\begin{aligned} x^+ = n.x = \frac{e^{-\eta}}{\sqrt{2}}(m+\bar{m}).x \qquad & x^- = \bar{n}.x = \frac{e^{\eta}}{\sqrt{2}}(m-\bar{m}).x \\ x^m = m.x = \frac{1}{\sqrt{2}}(e^\eta x^+ + e^{-\eta}x^-) \qquad & -x^{\bar{m}} = \bar{m}.x = \frac{1}{\sqrt{2}}(e^\eta x^+ - e^{-\eta}x^-) \end{aligned} \tag{3.2.6}$$

so that we get

$$\partial^\mp = \frac{\partial}{\partial x^\pm} = \frac{\partial m.x}{\partial x^\pm}\frac{\partial}{\partial m.x} + \frac{\partial \bar{m}.x}{\partial x^\pm}\frac{\partial}{\partial \bar{m}.x} = \frac{e^{\pm\eta}}{\sqrt{2}}\left(\frac{\partial}{\partial m.x} \pm \frac{\partial}{\partial \bar{m}.x}\right) \tag{3.2.7}$$

$$\begin{aligned} x^+\partial^- + x^-\partial^+ &= x^+\frac{\partial}{\partial x^+} + x^-\frac{\partial}{\partial x^-} = \frac{1}{2}(m+\bar{m}).x(\frac{\partial}{\partial m.x} + \frac{\partial}{\partial \bar{m}.x}) + \frac{1}{2}(m-\bar{m}).x(\frac{\partial}{\partial m.x} - \frac{\partial}{\partial \bar{m}.x}) \\ &= m.x\frac{\partial}{\partial m.x} + \bar{m}.x\frac{\partial}{\partial \bar{m}.x} = x^m\frac{\partial}{\partial x^m} + x^{\bar{m}}\frac{\partial}{\partial x^{\bar{m}}} = x^m\partial^m - x^{\bar{m}}\partial^{\bar{m}} \end{aligned} \tag{3.2.8}$$

In the last step it is important to note that $\bar{m}$ is spacelike and that therefore $\partial^{\bar{m}} = -\frac{\partial}{\partial x^{\bar{m}}}$. The sign is indeed what we expect since the metric wants the corresponding sign in the spatial terms of $x_\mu \partial^\mu$.

$$
\begin{aligned}
x^+\partial^- - x^-\partial^+ = x^+\frac{\partial}{\partial x^+} - x^-\frac{\partial}{\partial x^-} &= \frac{1}{2}(m+\bar{m}).x(\frac{\partial}{\partial m.x} + \frac{\partial}{\partial \bar{m}.x}) - \frac{1}{2}(m-\bar{m}).x(\frac{\partial}{\partial m.x} - \frac{\partial}{\partial \bar{m}.x}) \\
&= \bar{m}.x\frac{\partial}{\partial m.x} + m.x\frac{\partial}{\partial \bar{m}.x} = -x^{\bar{m}}\frac{\partial}{\partial x^m} + x^m(-\frac{\partial}{\partial x^{\bar{m}}}) = -(x^{\bar{m}}\partial^m - x^m\partial^{\bar{m}}) \qquad (3.2.9)
\end{aligned}
$$

Now we are ready to write Wilson lines along the m-direction at $x^{\bar{m}} = 0$, i.e. along the path

$$
w^\mu(\tau) = \tau m^\mu + \boldsymbol{x} \qquad \text{with tangent} \qquad \frac{dw^\mu}{d\tau} = m^\mu \qquad (3.2.10)
$$

which only depends on the boost angle η, but neither on $\bar{m}$ nor the transverse components in $\boldsymbol{x}$. Note that $\tau = m.x = x^m$.

The latter depen resides in the gauge field via $A_\mu(w) = A_\mu(\tau m + \boldsymbol{x})$:

$$
U_{\boldsymbol{x},m} = P\exp\left[ig\mu^\epsilon \int_{-\infty}^{\infty} d\tau \frac{dw^\mu}{d\tau} A_\mu(w)\right] \qquad (3.2.11)
$$

where $A = \boldsymbol{A} + \mathcal{A}$.

An alternative regularization would be a momentum cutoff

$$
\begin{aligned}
U_x^\eta &= \mathrm{Pexp}\Big[ig\int_{-\infty}^{\infty} du\ p_1^\mu A_\mu^\sigma(up_1 + x_\perp)\Big] = \mathrm{Pexp}\Big[ig\int_{-\infty}^{\infty} dx^+\ \bar{n}^\mu A_\mu^\eta(x^+ n + \boldsymbol{x})\Big] \\
A_\mu^\eta(x) &= \int d^4k\ \theta(e^\eta - |\alpha_k|)e^{ik\cdot x}A_\mu(k) \qquad (3.2.12)
\end{aligned}
$$

where α is the Sudakov variable in p_1 direction ($p = \alpha p_1 + \beta p_2 + \boldsymbol{p}$).

Reminder 3.1 – Sudakov Variables:

(See [27] p18) Sudakov variables use light like vectors q' and p' to characterize the interaction plane in a two particle collision with incoming particles of momenta q and p:

$$
q = q' - xp' \qquad p = p' + M^2/sq' \approx p \qquad p'^2 = q'^2 = 0 \qquad s := 2p'.q' \approx 2p.q \qquad (3.2.13)
$$

which allows us to change variables to components along q' and p' as well as the transverse reminder. The relevant identities are:

$$
k^\mu = \alpha q'^\mu + \beta p'^\mu + \boldsymbol{k}^\mu \qquad k^2 = \alpha\beta s - \boldsymbol{k}^2 \qquad d^4k = \frac{s}{2}d\alpha d\beta d^2\boldsymbol{k}\ . \qquad (3.2.14)
$$

To identify with $+$ and $-$ components one must rescale q' and p' by appropriate fractions of $\sqrt{2}$. If we split symmetrically we get

$$
k^\mu = k^-\frac{q'^\mu}{\sqrt{2s}} + k^+\frac{p'^\mu}{\sqrt{2s}} + \boldsymbol{k}^\mu \qquad (3.2.15)
$$

Now we have for dilatations

$$
\begin{aligned}
(\boldsymbol{D}+\mathcal{D})U_{\boldsymbol{x},m} &= \int_{-\infty}^{\infty} d\tau U_{\boldsymbol{x},m;-\infty,\tau}\frac{dw^\mu}{d\tau}(\hat{D}_w A_\mu(w))U_{\boldsymbol{x},m;\tau,\infty} \\
&= \int_{-\infty}^{\infty} d\tau U_{\boldsymbol{x},m;-\infty,\tau}\frac{dw^\mu}{d\tau}(\underbrace{m.w\frac{\partial}{\partial m.w}}_{\tau\partial_\tau}+\underbrace{\bar{m}.w\frac{\partial}{\partial \bar{m}.w}}_{\to 0}+\boldsymbol{x}\cdot\partial_{\boldsymbol{x}}+\underbrace{l_A}_{1-\epsilon})A_\mu(w))U_{\boldsymbol{x},m;\tau,\infty} \\
&= \int_{-\infty}^{\infty} d\tau U_{\boldsymbol{x},m;-\infty,\tau}\frac{dw^\mu}{d\tau}(-1+\boldsymbol{x}\cdot\partial_{\boldsymbol{x}}+1-\epsilon)A_\mu(w))U_{\boldsymbol{x},m;\tau,\infty} \\
&= \left(\boldsymbol{x}\cdot\partial_{\boldsymbol{x}}-\epsilon\int d^D y A^a_\mu(y)\frac{\delta}{\delta A^a_\mu(y)}\right)U_{\boldsymbol{x},m} \qquad (3.2.16)
\end{aligned}
$$

Boosts see the regularization:

- For the off lightcone Wilson line we find

$$
\begin{aligned}
(\boldsymbol{M}^{+-}+\mathcal{M}^{+-})U_{\boldsymbol{x},m} &= \int_{-\infty}^{\infty} dx^m U_{\boldsymbol{x},m;-\infty,x^m} m^\mu\Big(-(x^{\bar{m}}\frac{\partial}{\partial x^m}+x^m\frac{\partial}{\partial x^{\bar{m}}})A_\mu(x^m m+x^{\bar{m}}\bar{m}+\boldsymbol{x})\Big)\Big|_{x^{\bar{m}}=0}U_{\boldsymbol{x},m;x^m,\infty} \\
&= -x^m\frac{\partial}{\partial x^{\bar{m}}}U_{\boldsymbol{x},m;\tau,\infty}-\int_{-\infty}^{\infty} d\tau U_{\boldsymbol{x},m;-\infty,\tau}\bar{m}^\mu A_\mu(w)U_{\boldsymbol{x},m;\tau,\infty} \qquad (3.2.17)
\end{aligned}
$$

  where I have used that

$$
\hat{M}^{+-}m = \frac{\partial}{\partial\eta}m = \frac{1}{\sqrt{2}}(e^\eta n - e^{-\eta}\bar{n}) = \bar{m} \qquad (3.2.18)
$$

  i.e. the geometry of the line, not the gauge field carries the regularization.

- To understand the momentum cutoff, one first needs to obtain the transformation properties of $A^\eta_\mu(x)$, given that the unregularized field $A_\mu(x)$ is a vector field. First we need the transformation property of $A_\mu(k)$:

$$
\begin{aligned}
i(\boldsymbol{M}^{\mu\nu}+\mathcal{M}^{\mu\nu})A^\alpha(k) := \int d^D x e^{ik.x} i(\boldsymbol{M}^{\mu\nu}+\mathcal{M}^{+-})A^\alpha(x) &= \int d^D x e^{ik.x}(x^\mu\partial^\nu - x^\nu\partial^\mu - \Sigma^{\mu\nu})A^\alpha(x) \\
&= (k^\nu\partial^\mu_k - k^\nu\partial^\mu_k - \Sigma^{\mu\nu})A^\alpha(k) \qquad (3.2.19)
\end{aligned}
$$

  so that

$$
\begin{aligned}
i(\boldsymbol{M}^{\mu\nu}+\mathcal{M}^{\mu\nu})A^{\eta\alpha}(x) &:= \int\frac{d^D k}{(2\pi)^D}e^{-ik.x}\theta(e^\eta-|\alpha|)(k^\nu\partial^\mu_k - k^\mu\partial^\nu_k - \Sigma^{\mu\nu})A^\alpha(k) \\
&= (x^\mu\partial^\nu - x^\nu\partial^\mu - \Sigma^{\mu\nu})A^{\eta\alpha}(x) \\
&\quad + \int\frac{d^D k}{(2\pi)^D}e^{-ik.x}\left[(k^\mu g^{\nu-}-k^\nu g^{\mu-})\frac{\partial}{\partial k^-}\theta(e^\eta - |k^-/\sqrt{2s}|)\right]A^\alpha(k)
\end{aligned}
\qquad (3.2.20)
$$

The cutoff is already adapted to $+-$ components and only there we find simplifications, only the second term in the second line survives

$$
\begin{aligned}
i(\boldsymbol{M}^{+-}+\mathcal{M}^{+-})A^{\eta\alpha}(x) &= (x^\mu\partial^\nu - x^\nu\partial^\mu - \Sigma^{\mu\nu})A^{\eta\alpha}(x) \\
&\quad + \int \frac{d^Dk}{(2\pi)^D} e^{-ik.x}\left[-k^-\frac{\partial}{\partial k^-}\theta(\sqrt{2}se^\eta - |k^-|)\right]A^\alpha(k) \\
&= (x^\mu\partial^\nu - x^\nu\partial^\mu - \Sigma^{\mu\nu})A^{\eta\alpha}(x) \\
&\quad + \int \frac{d^Dk}{(2\pi)^D} e^{-ik.x}\sqrt{2}se^\eta\left[\theta(k^-)\delta(\sqrt{2}se^\eta - k^-) + \theta(-k^-)\delta(\sqrt{2}se^\eta + k^-)\right]A^\alpha(k) \\
&= (x^\mu\partial^\nu - x^\nu\partial^\mu - \Sigma^{\mu\nu})A^{\eta\alpha}(x) \\
&\quad + \int \frac{d^Dk}{(2\pi)^D} e^{-ik.x}\sqrt{2}se^\eta\left[\frac{1}{\sqrt{2}se^\eta}\frac{\partial}{\partial\eta}\theta(\sqrt{2}se^\eta - |k^-|)\right]A^\alpha(k) \\
&= (x^\mu\partial^\nu - x^\nu\partial^\mu - \Sigma^{\mu\nu} + \frac{\partial}{\partial\eta})A^{\eta\alpha}(x)
\end{aligned}
\tag{3.2.21}
$$

The momentum space cutoff yields

$$
\begin{aligned}
(\boldsymbol{M}^{+-}+\mathcal{M}^{+-})U^\eta_{\boldsymbol{x}} &= \int\limits_{-\infty}^{\infty} dx^+ U_{\boldsymbol{x};-\infty,x^+}\Big((\underbrace{x^+\frac{\partial}{\partial x^+}}_{\to -1} - \underbrace{x^-}_{\to 0}\frac{\partial}{\partial x^-}\underbrace{-\Sigma^{+-}}_{\to 1} + \frac{\partial}{\partial\eta})A^-(x)^\eta\Big)\Big|_{x=x^+n+\boldsymbol{x}} U_{\boldsymbol{x};x^+,\infty} \\
&= \frac{\partial}{\partial\eta}U^\eta_{\boldsymbol{x}}
\end{aligned}
\tag{3.2.22}
$$

where $-\Sigma^{+-}A^\alpha = g^{+\alpha}A^- - g^{-\alpha}A^+ \xrightarrow{\alpha\to -} = 1A^- - 0A^+$

finish this

Two terms of a vanishing commuatator:

$$
\begin{aligned}
\frac{d}{dY}\boldsymbol{D}\langle \mathrm{tr} U_{\boldsymbol{x}}U^\dagger_{\boldsymbol{y}}\rangle &= \frac{d}{dY}\langle\left(\boldsymbol{x}\cdot\partial_{\boldsymbol{x}} + \boldsymbol{y}\cdot\partial_{\boldsymbol{y}} - \epsilon\int d^Dy A^a_\mu(y)\frac{\delta}{\delta A^a_\mu(y)} + i\int d^Dy\Delta(y)\right)\mathrm{tr}U_{\boldsymbol{x}}U^\dagger_{\boldsymbol{y}}\rangle \\
&= \left(\boldsymbol{x}\cdot\partial_{\boldsymbol{x}} + \boldsymbol{y}\cdot\partial_{\boldsymbol{y}}\right)\frac{d}{dY}\langle \mathrm{tr}U_{\boldsymbol{x}}U^\dagger_{\boldsymbol{y}}\rangle \\
&\quad + \frac{d}{dY}\langle\left(-\epsilon\int d^Dy A^a_\mu(y)\frac{\delta}{\delta A^a_\mu(y)} + i\int d^Dy\Delta(y)\right)\mathrm{tr}U_{\boldsymbol{x}}U^\dagger_{\boldsymbol{y}}\rangle \\
&= \left(\boldsymbol{x}\cdot\partial_{\boldsymbol{x}} + \boldsymbol{y}\cdot\partial_{\boldsymbol{y}}\right)\int d^{D-2}z K_{\boldsymbol{xzy}}\langle U^{ab}_{\boldsymbol{z}}\mathrm{tr}(t^aU_{\boldsymbol{x}}t^bU_{\boldsymbol{y}}) - C_f\mathrm{tr}U_{\boldsymbol{x}}U^\dagger_{\boldsymbol{y}}\rangle \\
&\quad + \frac{d}{dY}\langle\left(-\epsilon\int d^Dy A^a_\mu(y)\frac{\delta}{\delta A^a_\mu(y)} + i\int d^Dy\Delta(y)\right)\mathrm{tr}U_{\boldsymbol{x}}U^\dagger_{\boldsymbol{y}}\rangle \\
&= \int d^{D-2}z\ [\left(\boldsymbol{x}\cdot\partial_{\boldsymbol{x}} + \boldsymbol{y}\cdot\partial_{\boldsymbol{y}}\right), K_{\boldsymbol{xzy}}]\langle U^{ab}_{\boldsymbol{z}}\mathrm{tr}(t^aU_{\boldsymbol{x}}t^bU_{\boldsymbol{y}}) - C_f\mathrm{tr}U_{\boldsymbol{x}}U^\dagger_{\boldsymbol{y}}\rangle \\
&\quad + \int d^{D-2}z K_{\boldsymbol{xzy}}\left(\boldsymbol{x}\cdot\partial_{\boldsymbol{x}} + \boldsymbol{y}\cdot\partial_{\boldsymbol{y}}\right)\langle U^{ab}_{\boldsymbol{z}}\mathrm{tr}(t^aU_{\boldsymbol{x}}t^bU_{\boldsymbol{y}}) - C_f\mathrm{tr}U_{\boldsymbol{x}}U^\dagger_{\boldsymbol{y}}\rangle \\
&\quad + \frac{d}{dY}\langle\left(-\epsilon\int d^Dy A^a_\mu(y)\frac{\delta}{\delta A^a_\mu(y)} + i\int d^Dy\Delta(y)\right)\mathrm{tr}U_{\boldsymbol{x}}U^\dagger_{\boldsymbol{y}}\rangle
\end{aligned}
\tag{3.2.23}
$$

$$
\begin{aligned}
\boldsymbol{D}\frac{d}{dY}\langle \mathrm{tr} U_{\boldsymbol{x}} U^\dagger_{\boldsymbol{y}}\rangle &= \boldsymbol{D}\int d^{D-2}z K_{\boldsymbol{xzy}}\langle U^{ab}_{\boldsymbol{z}}\mathrm{tr}(t^a U_{\boldsymbol{x}} t^b U_{\boldsymbol{y}}) - C_f \mathrm{tr} U_{\boldsymbol{x}} U^\dagger_{\boldsymbol{y}}\rangle \\
&= \int d^{D-2}z K_{\boldsymbol{xzy}}\Big\langle \Big(\boldsymbol{x}\cdot\partial_{\boldsymbol{x}} + \boldsymbol{y}\cdot\partial_{\boldsymbol{y}} + \boldsymbol{z}\cdot\partial_{\boldsymbol{z}} - \epsilon\int d^D y A^a_\mu(y)\frac{\delta}{\delta A^a_\mu(y)} + i\int d^D y \Delta(y)\Big) \\
&\quad \big(U^{ab}_{\boldsymbol{z}}\mathrm{tr}(t^a U_{\boldsymbol{x}} t^b U_{\boldsymbol{y}}) - C_f \mathrm{tr} U_{\boldsymbol{x}} U^\dagger_{\boldsymbol{y}}\big)\Big\rangle \\
&= \int d^{D-2}z K_{\boldsymbol{xzy}}(\boldsymbol{x}\cdot\partial_{\boldsymbol{x}} + \boldsymbol{y}\cdot\partial_{\boldsymbol{y}} + \boldsymbol{z}\cdot\partial_{\boldsymbol{z}})\langle U^{ab}_{\boldsymbol{z}}\mathrm{tr}(t^a U_{\boldsymbol{x}} t^b U_{\boldsymbol{y}}) - C_f \mathrm{tr} U_{\boldsymbol{x}} U^\dagger_{\boldsymbol{y}}\rangle \\
&\quad + \int d^{D-2}z K_{\boldsymbol{xzy}}\Big\langle \Big(-\epsilon\int d^D y A^a_\mu(y)\frac{\delta}{\delta A^a_\mu(y)} + i\int d^D y \Delta(y)\Big) \\
&\qquad\qquad \big(U^{ab}_{\boldsymbol{z}}\mathrm{tr}(t^a U_{\boldsymbol{x}} t^b U_{\boldsymbol{y}}) - C_f \mathrm{tr} U_{\boldsymbol{x}} U^\dagger_{\boldsymbol{y}}\big)\Big\rangle \\
&= -\int d^{D-2}z [(\boldsymbol{z}\cdot\partial_{\boldsymbol{z}} + (D-2))K_{\boldsymbol{xzy}}]\langle U^{ab}_{\boldsymbol{z}}\mathrm{tr}(t^a U_{\boldsymbol{x}} t^b U_{\boldsymbol{y}}) - C_f \mathrm{tr} U_{\boldsymbol{x}} U^\dagger_{\boldsymbol{y}}\rangle \\
&\quad + \int d^{D-2}z K_{\boldsymbol{xzy}}(\boldsymbol{x}\cdot\partial_{\boldsymbol{x}} + \boldsymbol{y}\cdot\partial_{\boldsymbol{y}})\langle U^{ab}_{\boldsymbol{z}}\mathrm{tr}(t^a U_{\boldsymbol{x}} t^b U_{\boldsymbol{y}}) - C_f \mathrm{tr} U_{\boldsymbol{x}} U^\dagger_{\boldsymbol{y}}\rangle \\
&\quad + \int d^{D-2}z K_{\boldsymbol{xzy}}\Big\langle \Big(-\epsilon\int d^D y A^a_\mu(y)\frac{\delta}{\delta A^a_\mu(y)} + i\int d^D y \Delta(y)\Big) \\
&\qquad\qquad \big(U^{ab}_{\boldsymbol{z}}\mathrm{tr}(t^a U_{\boldsymbol{x}} t^b U_{\boldsymbol{y}}) - C_f \mathrm{tr} U_{\boldsymbol{x}} U^\dagger_{\boldsymbol{y}}\big)\Big\rangle
\end{aligned} \tag{3.2.24}
$$

In combination we find

$$
\begin{aligned}
0 =& [\boldsymbol{M}^{+-}, \boldsymbol{D}]\langle U_{\boldsymbol{x}} U^\dagger_{\boldsymbol{y}}\rangle = \int d^{D-2}z\ [(\boldsymbol{x}\cdot\partial_{\boldsymbol{x}} + \boldsymbol{y}\cdot\partial_{\boldsymbol{y}} + \boldsymbol{z}\cdot\partial_{\boldsymbol{z}} + (D-2))\,K_{\boldsymbol{xzy}}]\langle U^{ab}_{\boldsymbol{z}}\mathrm{tr}(t^a U_{\boldsymbol{x}} t^b U_{\boldsymbol{y}}) - C_f \mathrm{tr} U_{\boldsymbol{x}} U^\dagger_{\boldsymbol{y}}\rangle \\
&+ \boldsymbol{M}^{+-}\langle\left(-\epsilon\int d^D y A^a_\mu(y)\frac{\delta}{\delta A^a_\mu(y)} + i\int d^D y\Delta(y)\right)\mathrm{tr} U_{\boldsymbol{x}} U^\dagger_{\boldsymbol{y}}\rangle \\
&- \int d^{D-2}z K_{\boldsymbol{xzy}}\Big\langle \Big(-\epsilon\int d^D y A^a_\mu(y)\frac{\delta}{\delta A^a_\mu(y)} + i\int d^D y \Delta(y)\Big) \\
&\qquad \big(U^{ab}_{\boldsymbol{z}}\mathrm{tr}(t^a U_{\boldsymbol{x}} t^b U_{\boldsymbol{y}}) - C_f \mathrm{tr} U_{\boldsymbol{x}} U^\dagger_{\boldsymbol{y}}\big)\Big\rangle
\end{aligned} \tag{3.2.25}
$$

3.3 Operator redefinitions

Start from Ian and Giovanni's suggestionfor dipoles

fix factors!!!!

$$\begin{aligned}S^c_{\boldsymbol{xy}} :=& S_{\boldsymbol{xy}} + \alpha_s \# \int d^2z \mathcal{K}_{\boldsymbol{xzy}} \ln\Big(a^2 \mathcal{K}_{\boldsymbol{xzy}}\Big)(S_{\boldsymbol{xz}} S_{\boldsymbol{zy}} - S_{\boldsymbol{xy}}) \\ =& \big(1 + \mathsf{H}(a) + \mathcal{O}(\alpha_s^2)\big) S_{\boldsymbol{xy}}\end{aligned} \tag{3.3.1}$$

where

$$\mathsf{H}(a) := \alpha_s \# \int d^2z d^2u_1 d^2u_2 \mathcal{K}_{\boldsymbol{u_1 z u_2}} \ln\Big(a^2 \mathcal{K}_{\boldsymbol{u_1 z u_2}}\Big)\big(2U^{a_1 a_2}_{\boldsymbol{z}} i\boldsymbol{\nabla}^{a_1}_{u_1} i\bar{\boldsymbol{\nabla}}^{a_1}_{u_1} + \delta^{a_1 a_2}(i\boldsymbol{\nabla}^{a_1}_{u_1} i\boldsymbol{\nabla}^{a_2}_{u_2} + i\bar{\boldsymbol{\nabla}}^{a_1}_{u_1} i\bar{\boldsymbol{\nabla}}^{a_2}_{u_2})\big) \tag{3.3.2}$$

The latter form directly generalizes to operators other than the dipole. We also write the evolution equation as

$$\frac{d}{dY} S = -(H^{\text{LO}} + H^{\text{NLO}} + \mathcal{O}(\alpha_s^3)) S \tag{3.3.3}$$

Now we need to find the differential equation for the redefined object(s)

$$\begin{aligned}\frac{d}{dY} S^c =& \frac{d}{dY} S + \alpha_s \# \int d^2z \mathcal{K}_{\boldsymbol{xzy}} \ln\Big(a^2 \mathcal{K}_{\boldsymbol{xzy}}\Big) \frac{d}{dY}(S_{\boldsymbol{xz}} S_{\boldsymbol{zy}} - S_{\boldsymbol{xy}}) \\ =& \alpha_s \# \int d^2z \mathcal{K}_{\boldsymbol{xzy}}(S_{\boldsymbol{xz}} S_{\boldsymbol{zy}} - S_{\boldsymbol{xy}}) + (H^{\text{NLO}} + \mathcal{O}(\alpha_s^3)) S \\ &+ \alpha_s \# \int d^2z \mathcal{K}_{\boldsymbol{xzy}} \ln\Big(a^2 \mathcal{K}_{\boldsymbol{xzy}}\Big) \frac{d}{dY}(S_{\boldsymbol{xz}} S_{\boldsymbol{zy}} - S_{\boldsymbol{xy}}) \\ =& \alpha_s \# \int d^2z \mathcal{K}_{\boldsymbol{xzy}}(S_{\boldsymbol{xz}} S_{\boldsymbol{zy}} - S_{\boldsymbol{xy}}) + (H^{\text{NLO}} + \mathcal{O}(\alpha_s^3)) S \\ &+ \alpha_s \# \int d^2z \mathcal{K}_{\boldsymbol{xzy}} \ln\Big(a^2 \mathcal{K}_{\boldsymbol{xzy}}\Big) \frac{d}{dY}(S_{\boldsymbol{xz}} S_{\boldsymbol{zy}} - S_{\boldsymbol{xy}}) \\ =& \frac{d}{dY}\Big(S^c - \alpha_s \# \int d^2z \mathcal{K}_{\boldsymbol{xzy}} \ln\Big(a^2 \mathcal{K}_{\boldsymbol{xzy}}\Big)(S_{\boldsymbol{xz}} S_{\boldsymbol{zy}} - S_{\boldsymbol{xy}})\Big) \\ &+ \alpha_s \# \int d^2z \mathcal{K}_{\boldsymbol{xzy}} \ln\Big(a^2 \mathcal{K}_{\boldsymbol{xzy}}\Big) \frac{d}{dY}(S_{\boldsymbol{xz}} S_{\boldsymbol{zy}} - S_{\boldsymbol{xy}})\end{aligned} \tag{3.3.4}$$

The $\frac{d}{dY}$ terms on the right hand side need to be reexpressed in terms of S^c, which be behaves differently to order α_s.

- In the second term we may straightforwardly replace $S \mapsto S^c$, in the first term if we are to drop terms from $\mathcal{O}(\alpha_s^3)$. This yields

$$-H^{\text{LO}} H^a \tag{3.3.5}$$

- In the first term

3.4 Insights on the BMS side

- A large N_c discussion with quite interesting techniques – using the stereographics projection to use conformal symmetries in the "JIMWLK-plane" by Schwartz and [22]. This seems to talk both iterating the evolution equation and the emission in the amplitudes? (Soft part vs hard part)
- [28, 29] step beyond large N_c in an iterative manner. Soft or hard parts?

SCET: Thomas Becher's book [30].

Heinonen: reparametrization invariance with relation to Lorentz invariance [31]. This might help with what I am doing in JIMWLK in terms of scalings for the fields.

Timothy Cohen "As Scales Become Separated: Lectures on Effective Field Theory"[32] also has a modern discussion on reparametrization invariance.

Ian Stewart and Ira Rothstein[33] have a long article on SCET in the BFKL context. Maybe useful?

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