MATH370 HW2

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Contents

1. Finish problem B and C
2. Note to self I don't think I did this problem totally correct. Looking back at what I did, I didn't use the defintion properly.
3. Oh shit have to do this one
4. f
5. Finish the proof, for zero.
6. Nothing so far

1 Introduction

2 Problem 2.2.0

Prove which of the following statements are true. Prove the true ones and provie counter examples to the false ones.

(a) if $x_n \to \infty$ and $y_n \to -\infty$, then $x_n + y_n \to 0$ as $n \to \infty$

(b) Prove that the sequence $\{n\}$ does not converge.

(c) Show that there exist unbounded sequences $x_n \neq y_n$ which satisfy the conclusion of part (a).

2.1 Problem 2.2.0.a

if $x_n \to \infty$ and $y_n \to -\infty$, then $x_n + y_n \to 0$ as $n \to \infty$

2.1.1 Proof

True. By

3 Problem 2.2.1

Prove each statement goes to zero

(a) $x_n = sin(log(n) + n^5 + e^{n^2})/n$

(b) $x_n = 2n/(n^2 + \pi)$

(c) $x_n = (\sqrt{2n} + 1)/(n + \sqrt{2})$

3.1 Problem 2.2.1.a

3.1.1 Proof A $x_n = sin(log(n) + n^5 + e^{n^2})/n$

Since for all n, $0 \leq sin(log(n) + n^5 + e^{n^2}) \leq 1$, x_n is dominated by the denominator as $n \to \infty$. We know $n \to \infty$ $\frac{1}{n}$ goes to zero. Thus by undeniable domination of the denominator, x_n goes to zero and n goes to ∞.

3.1.2 Proof B $x_n = 2n/(n^2 + \pi)$

Note $2n/(n^2 + \pi)$

- $\frac{2n}{(n^2+\pi)} = \frac{2n}{(n^2+\pi)} \frac{\frac{1}{n}}{\frac{1}{n}}$
- $\frac{\frac{2n}{n}}{(n^2+\pi)\frac{1}{n}} = \frac{2}{n+\frac{\pi}{n}} = 2\frac{1}{n+\frac{\pi}{n}}$
- Note $n + \frac{\pi}{n} \to \infty$ as $n \to \infty$.
- Thus as $\frac{2n}{n+\frac{\pi}{n}} \to 0$ as $n \to \infty$.

3.1.3 Proof C $(\sqrt{2n} + 1)/(n + \sqrt{2})$

- $(\sqrt{2n} + 1)/(n + \sqrt{2}) = \frac{\sqrt{2n}}{n+\sqrt{2}} + \frac{1}{n+\sqrt{2}}$
- $\frac{\sqrt{2}\sqrt{n}}{n+\sqrt{2}} \frac{\sqrt{n}}{\sqrt{n}} = \frac{\sqrt{2}n}{n\sqrt{n}+\sqrt{2}\sqrt{n}} = \frac{\sqrt{2}n}{n^{1.5}+\sqrt{2}n^{0.5}} = \frac{\sqrt{2}n}{n(n^{0.5}+\sqrt{2}n^{-0.5})} = \frac{\sqrt{2}}{(n^{0.5}+\sqrt{2}n^{-0.5})}$
- Then we would have $\frac{\sqrt{2}}{(n^{0.5}+\sqrt{2}n^{-0.5})} + \frac{1}{n+\sqrt{2}}$
- We can see that $lim_{n\to\infty}(n + \sqrt{2}) = \infty$
- Also $lim_{n\to\infty}(n^{0.5} + \sqrt{2}n^{-0.5}) = \infty$
- Then $lim_{n\to\infty}\frac{1}{(n+\sqrt{2})} = 0$ and $lim_{n\to\infty}\frac{1}{(n^{0.5}+\sqrt{2}n^{-0.5})} = 0$
- And then $lim_{n\to\infty}\frac{1}{(n+\sqrt{2})} + lim_{n\to\infty}\frac{1}{(n^{0.5}+\sqrt{2}n^{-0.5})} = lim_{n\to\infty}(\sqrt{2n} + 1)/(n + \sqrt{2}) = 0$
- $\blacksquare$

4 Problem 2.2.2

Use the definition 2.14 to prove that each of the foll lowing sequences diverges to $+\infty$ or to $-\infty$

(a) $x_n = n^2 - n$

(b) $x_n = n - 3n^2$

(c) $x_n = \frac{n^2+1}{n}$

4.1 Definition 2.14

Let $\{x_n\}$ be a sequence of real numbers. $\{x_n\}$is said to diverge to $+\infty$ if and only if for each $M \in \mathbf{R}$ there is an $N \in \mathbf{N}$ such that $n \geq N$ implies $x_n > M$

4.2 Problem 2.2.2.a

4.2.1 Proof A

- Note $x_n = n^2 - n = n(n-1)$.
- We can see that if $n = 1$ then $n(n-1) \to 0$, but for all $n > 1$, $n(n-1) > 0$.
- Choosing $M \geq 2$,we can always find a $\lfloor M \rfloor = N$ such that for all $n \geq N$, $\{x_n\} > M$.
- As for all $M \in \mathbf{R}$ where $2 \leq M$, then $M < \lfloor M \rfloor(\lfloor M \rfloor - 1) = N(N-1)$
- Thus $x_n = n^2 - n = n(n-1)$ diverge to $+\infty$.

4.2.2 Proof B

$x_n = n - 3n^2$

- Note that $x_n = n - 3n^2 = n(1-3n)$.
- Using the same logic as above, we can see that for $n > 0$, $n(1-3n) < 0$.
- Also noting if we have an $M \in \mathbf{R}$ we can always use that M to create a N, where $N = \lfloor |M| \rfloor$ so that $n \geq N$ will imply $M > \lfloor |M| \rfloor(1 - 3\lfloor |M| \rfloor) = N(1-3N) \geq x_n$.
- Thus by definition 2.14 diverges.

4.2.3 Proof C

$$x_n = \frac{n^2+1}{n}$$

- Note $x_n = \frac{n^2+1}{n} = \frac{n^2}{n} + \frac{1}{n} = n + \frac{1}{n}$
- Since $n + \frac{1}{n} > n \geq N \geq M$, this implies for all $M \in \mathbf{R}$ there is an $N \in \mathbf{N}$ such that $n \geq N$ implies $x_n > M$.
- Thus, $\{x_n\}$ diverges.

5 Problem 2.2.3

Find the limit if it exists

(a) $x_n = \frac{2+3n-4n^2}{1-2n+3n^2}$

(b) $x_n = \frac{n^3+n-2}{2n^3+n-2}$

(c) $x_n = \sqrt{3n+2} - \sqrt{n}$

(d) $x_n = \frac{\sqrt{4n+1}-\sqrt{n-1}}{\sqrt{9n+1}-\sqrt{n+2}}$

5.1 (a) $\frac{2+3n-4n^2}{1-2n+3n^2}$

- $\frac{2+3n-4n^2}{1-2n+3n^2} = \frac{2+3n-4n^2}{1-2n+3n^2}\frac{\frac{1}{n^2}}{\frac{1}{n^2}}$
- $\frac{\frac{2}{n^2}+\frac{3}{n}-4}{\frac{1}{n^2}-\frac{2}{n}+3}$
- Since for any $y \in \mathbf{R}$, $lim_{n\to\infty}\frac{y}{n} = 0$
- $\frac{\frac{2}{n^2}+\frac{3}{n}-4}{\frac{1}{n^2}-\frac{2}{n}+3} = \frac{0+0-4}{0-0+3} = \frac{-4}{3}$

5.2 (b) $\frac{n^3+n-2}{2n^3+n-2}$

Same reasoning as above

- $\frac{n^3+n-2}{2n^3+n-2} = \frac{1+\frac{1}{n^2}-\frac{2}{n^3}}{2+\frac{1}{n^2}-\frac{2}{n^3}}$
- Note for for any $y \in \mathbf{R}$, $lim_{n\to\infty}\frac{y}{n^2} = 0$ and also for any $y \in \mathbf{R}$, $lim_{n\to\infty}\frac{y}{n^3} = 0$
- Thus, $\frac{1+0-0}{2+0-0} = \frac{1}{2}$

5.3 (c) $x_n = \sqrt{3n+2} - \sqrt{n}$

- $\sqrt{3n+2} - \sqrt{n}$
- We can see that $lim_{n\to\infty}\sqrt{n} = \infty$
- We can also see the same for $lim_{n\to\infty}\sqrt{3n+2} = \infty$
- We can see then that $lim_{n\to\infty}\sqrt{3n+2} - lim_{n\to\infty}\sqrt{n}$ will diverge.

5.4 (d) $x_n = \frac{\sqrt{4n+1}-\sqrt{n-1}}{\sqrt{9n+1}-\sqrt{n+2}}$

- $\frac{\sqrt{4n+1}-\sqrt{n-1}}{\sqrt{9n+1}-\sqrt{n+2}} = \frac{\sqrt{n}\sqrt{4+\frac{1}{n}}-\sqrt{n}\sqrt{1-\frac{1}{n}}}{\sqrt{9n+1}-\sqrt{n+2}} = \frac{\sqrt{n}\sqrt{4+\frac{1}{n}}-\sqrt{n}\sqrt{1-\frac{1}{n}}}{\sqrt{n}\sqrt{9+\frac{1}{n}}-\sqrt{n}\sqrt{1+\frac{2}{n}}}$
- $\frac{2-1}{3-1} = \frac{1}{2}$

6 2.2.5

6.1 Proof

6.1.1 For $X > 0$

- Let $\epsilon > 0$, then we must find an N such that $n \geq N$ implies $|\sqrt{x_n} - \sqrt{x}| < \epsilon$.
- Note that $|\sqrt{x_n} - \sqrt{x}| = |\sqrt{x_n} - \sqrt{x}|\frac{\sqrt{x_n}+\sqrt{x}}{\sqrt{x_n}+\sqrt{x}} = \frac{|x_n - x|}{\sqrt{x_n}+\sqrt{x}}$
- Since $\sqrt{x_n} + \sqrt{x} \geq \sqrt{x}$ and thus $\frac{1}{\sqrt{x_n}+\sqrt{x}} < \frac{1}{\sqrt{x}}$
- $\frac{|x_n - x|}{\sqrt{x_n}+\sqrt{x}} \leq \frac{|x_n - x|}{\sqrt{x}}$
- Because $x > 0$ and $x_n \to x$ then $x_n - x = 0$ and $n \to \infty$, thus there is a N such that $N \geq \epsilon\sqrt{x} + x$ such that for all $n \geq N$, $\frac{|x_n - x|}{\sqrt{x}} < \epsilon$

6.1.2 For $X = 0$

- We know that $x_n \to x$ as $n \to \infty$ and that $x = 0$.
- Let $\epsilon > 0$, then we must find an N such that $n \geq N$ so that $|\sqrt{x_n} - 0| = |\sqrt{x_n}| < \epsilon$.
- Note $|\sqrt{x_n}| < \epsilon \to x_n < \epsilon^2$.
- Using the inequality from section 1.2 $\to 0 \leq a \leq b \to 0 \leq a^2 \leq b^2 \to 0 \leq a^{0.5} \leq b^{0.5}$
- I don't entirely get the logic from here. I need to come back to this and understand this better

7 2.2.6

7.1 Thoughts

- hmmm I am going to have to think about this one.
- Note even sure what the question is stating entirely.
- So given a real number, there is a sequence of rational numbers that converge to the real number.
- I think there is squeeze theorom manipulation that could be used here, but why cannot we use definitions from section 1.2?

7.2 Proof

- Let a be any $\mathbf{R}$.
- Say for any n $n \in \mathbf{N}$ we have $a - \frac{1}{n} < a + \frac{1}{n}$.
- It should be noted that all terminating decimals are rational numbers that can be written as reduced fractions with denominators containing no prime number factors other than two or five.
- Thus we can have a sequence r_n(which is in $\mathbf{Q}$) that is between $a - \frac{1}{n}$ and $a + \frac{1}{n}$, meaning $a - \frac{1}{n} < r_n < a + \frac{1}{n}$.
- We would then have $|r_n - a| < \frac{1}{n}$.
- As $n \to \infty$, $\frac{1}{n} \to 0$.
- Then by squeeze theorem $r_n \to a$.

8 2.2.8

Find the limit if it exists

(a) Suppose that $0 \leq x_1 \leq 1$ and $x_{n+1} = 1 - \sqrt{1 - x_n}$ for $n \in \mathbf{N}$. If $x_n \to x$ as $n \to \infty$, then $x = 0$ or 1.

(b) Suppose that $x > 3$ and $x_{n+1} = 2 + \sqrt{x_n - 2}$ for $n \in \mathbf{N}$. If $x_n \to x$ as $n \to \infty$, then $x = 3$.

8.1 a

1. Given that $x_n \to x$ as $n \to \infty$.
2. Applying limits, we can get $lim(x_{n+1}) = lim(1) - lim(\sqrt{1 - x_n})$
3. $= 1 - \sqrt{lim(1) - lim(x_n)} = 1 - \sqrt{1 - x} = x$
4. Then $1 = x + \sqrt{1 - x}$
5. Which can only be true if $x = 1$ or 0.

8.2 b

1. Given that $x_n \to x$ as $n \to \infty$.
2. Applying limits, we can get $lim(x_{n+1}) = lim(2) + lim(\sqrt{x_n - 2}) = 2 + \sqrt{lim(x_n) - lim(2)} = 2 + \sqrt{x - 2} = x$
3. Then if $\sqrt{x - 2} = x - 2$
4. $1 = \frac{x-2}{\sqrt{x-2}}$
5. $1 = \sqrt{x - 2}$
6. $1 = x - 2$
7. $3 = x$

8.3 c

1. Given that $x_n \to x$ as $n \to \infty$.
2. Applying limits, we can get $lim(x_{n+1}) = lim(\sqrt{x_n + 2}) = \sqrt{lim(x_n) + lim(2)}$
3. $x = \sqrt{x + 2}$
4. $\frac{x}{\sqrt{x+2}} = 1$
5. $x = 2$ for $x_1 \geq 0$
6. if we squared both side we would get $x^2 = 2 + x = (x + 1)(x - 2)$
7. if $x_1 \geq -2$ then $x = -1$ and $x = 2$