MATH370 HW1

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1 Introduction

Defintion 2.1
A sequence of real numbers $\{x_n\}$ is said to converge to a real number $a \in \mathbf{R}$ if and only if for every $\epsilon > 0$ there is an $N \in \mathbf{N}$ (Which in general depend on ϵ) such that $\mathbf{N} \leq n$ implies $|x_n - a| < \epsilon$

2 Problem 2.1.1

Prove that the following limits exist

2.1 Problem 2.1.1.a

$2 - \frac{1}{n} \to 2$ as $n \to \infty$

2.1.1 Scratch work

Given $|(2 - \frac{1}{n}) - a| < \epsilon$
$|(2 - \frac{1}{n}) - 2| < \epsilon$
$|\frac{1}{n}| < \epsilon$

$\frac{1}{n} < \epsilon$
$\frac{1}{\epsilon} > n$
$|2 - \frac{1}{n}| - a < 2 < |2 + \frac{1}{n}| + a$

2.1.2 Formal Proof

Let $\epsilon > 0$
Note,$2 - \frac{1}{n} \to 2$ can be $\frac{1}{n} \to 0$ Choose a natural number N such that $N > \frac{1}{\epsilon}$

We now verify that this choice of N has the desired property.
Let $n > N$. Then, $n > \frac{1}{\epsilon}$ implies $\frac{1}{n} < \frac{1}{N} < \epsilon$ and hence $|2 - (\frac{1}{n}) - 2| < \epsilon$

■

2.2 Problem 2.1.1.b

$1 + \frac{\pi}{\sqrt{n}} \to 1$ as $n \to \infty$

2.2.1 Scratch work

$|1 + \frac{\pi}{\sqrt{n}} - 1| < \epsilon$ as $n \to \infty$

$|\frac{\pi}{\sqrt{n}}| < \epsilon$

$(\frac{\pi}{\sqrt{n}})^2 < \epsilon^2$
$\frac{\pi^2}{\sqrt{n}^2} < \epsilon^2$
$\frac{\pi^2}{\sqrt{n}^2} < \epsilon^2$
$\frac{1}{\sqrt{n}^2} < \frac{\epsilon^2}{\pi^2}$
$\sqrt{n}^2 < \frac{\pi^2}{\epsilon^2}$
$n > \frac{\pi^2}{\epsilon^2}$

2.2.2 Proof

Let $\epsilon > 0$
Note, $1 + \frac{\pi}{\sqrt{n}} \to 1$ can be $\frac{\pi}{\sqrt{n}} \to 0$ Using the ARCHMEDIAN PRINCIPLE, choose a natural number N such that $N > \frac{\pi^2}{\epsilon^2}$

We now verify that this choice of N has the desired property.
Let $n > N$. Then, $n > \frac{\pi^2}{\epsilon^2}$ implies $|\frac{\pi}{\sqrt{n}}| < |\frac{\pi}{\sqrt{N}}| < \epsilon$ and hence $|1 - \frac{\pi}{\sqrt{n}} - 1| < \epsilon$

■

3 Problem 2.1.2

For the following 2.1.2 problems:

1. the sequence of x_n converges to one.
2. Use definition 2.1 to prove the following limits exist.

3.1 Problem 2.1.2.a

$1 - 2x_n \to 3$ as $n \to \infty$

3.1.1 Work

So we want $|1 - 2x_n - 3| < \epsilon$
which is $|-2 - 2x_n| < \epsilon$
$2|-1 - x_n| < \epsilon$
$|-1 - x_n| > \frac{\epsilon}{2}$
$1 + x_n > \frac{\epsilon}{2}$
$x_n > \frac{\epsilon}{2} - 1$

3.1.2 Proof

Let > 0.
Then for $N \in \mathbf{N}$ with $N > \frac{}{2} - 1$
To verify that the choice of N is appropriate, let $n \in \mathbf{N}$ satisfy $n > N$.
Then, $n > N$ implies $n > \frac{}{2} - 1$, which is the same as saying $(n + 1)/2 <$. Finally this means

$$|1 - 2x_n - 3| <$$

Thus $|1 - 2x_n - 3| <$, meaning as x_n approaches 1, $1 - 3x_n$ approaches 3

3.2 Problem 2.1.2.b

$\frac{\pi x_n - 2}{x_n} \to \pi - 2$ as $n \to \infty$

3.2.1 2.1.2.b Thoughts, scratch work

- So we need to find the formula where this is always right for ϵ
- $\frac{\pi x_n - 2}{x_n} = \pi - \frac{2}{x_n}$
- $\pi - \frac{2}{x_n} \to \pi - 2$ and by subtracting π from each side.... $-\frac{2}{x_n} \to -2$
- $|-\frac{2}{x_n} - (-2)| < \epsilon$
- $|-\frac{2}{x_n} + 2| < \epsilon$
- $2|-\frac{1}{x_n} + 1| < \epsilon$

- $|-\frac{1}{x_n}+1|<\frac{\epsilon}{2}$
- $|-\frac{1}{x_n}+1|\leq|-\frac{1}{x_n}|+|1|<\frac{\epsilon}{2}$
- $\frac{1}{x_n}<\frac{\epsilon}{2}-1$
- $1<(\frac{\epsilon}{2}-1)x_n$
- $\frac{1}{(\frac{\epsilon}{2}-1)}>x_n$
- $\frac{1}{(\frac{\epsilon}{2}}-1)>x_n$
- $\frac{2}{\epsilon}-1>x_n$

3.2.2 Proof

- Let $\epsilon>0$ be an arbitrary positive number. Choose a natural number N satisfying $\frac{2}{\epsilon}-1>N$. We now verify that this choice of N has the desired property. Let $n\geq N$. Then, $n>\frac{2}{\epsilon}-1$ implies $2|-\frac{1}{x_n}+1|<\epsilon$, hence $|\pi-\frac{2}{x_n}-(\pi-2)|<\epsilon$.

4 Problem 2.1.4 *Bounded*

4.1 Problem 2.1.4.a

4.1.1 Initial thoughts

I am a little confused because it seems to me that they are asking to prove the definition of something is bounded or not? I guess this is a proof by contradiction?

4.1.2 Proof way1?

Suppose that bounded and $|x_n|$ is not $\leq C$ for all $n \in \mathbf{N}$, but that is a contradiction because $|x_n| \leq C$ for $C > 0$ must be by definition of being bounded.

4.1.3 Proof way2?

Choose a max from sequence so that $M = max(x_n)$ Then we see that $x_n \leq M\ \forall n \in \mathbf{N}$ and $M > 0$. By definition we have a bounded function.

Reflection on above, note for self, not grader I guess the above works. I am getting a little confused for $n\forall \in \mathbf{N}$. Note to self, really re-read the example and definition, I think there is a nuance I am missing/not understanding.

4.2 Problem 2.1.4.b

Suppose that $\{x_n\}$ *is bounded. Prove that* $\frac{x_n}{n^k} \to 0$*, as* $n \to \infty$ *for all* $\in \mathbf{N}$.

4.2.1 Problem 2.1.4.b thoughts

1. Knowing n goes to infinity and $\{x_n\}$ is bounded, we can use the definition bound so that $C = max(x_n)$ bounds $x_n\ \forall n \in \mathbf{N}$.
2. Since we know that the largest number is C, we could write $\frac{x_n}{n^k} \leq \frac{C}{n^k}$
3. From here we would use the definition of a limit of sequence.
4. The equation we would use in the proof to make that happen is $\frac{C}{n^k} < \epsilon$ and to solve for n, it would be $\sqrt[k]{\frac{C}{\epsilon}} < n$.
5. So let > 0 be an arbitrary positive number. Choose a natural number that satisfies $N > \sqrt[k]{\frac{C}{\epsilon}}$. We can see that choice of N has our desired property. Let $N \leq n$. Then, $\sqrt[k]{\frac{C}{\epsilon}} < n$ implies $\frac{C}{n^k} < \epsilon$ that implies that $\frac{1}{n^k} < \frac{\epsilon}{C}$ and hence $|\frac{x_n}{n^k} - 0| = |x_n||\frac{1}{n^k}| < C\frac{\epsilon}{C} < \epsilon$
6. ■

5 Problem 2.1.5 *Let sequence* b_n *be* ϵ *for sequence* x_n *converging to a*

let C be a fixed, positive constant. If $\{b_n\}$ is a sequence of non-negative numbers that converges to 0; and $\{x_n\}$ is a real sequence that satisfies $|x_n - a| \leq Cb_n$ for large n, prove that x_n converges to a.

5.1 Proof?

- Note that since $\{b_n\}$ converges to 0 then therefore, $\forall \epsilon > 0$, thus there $\exists\ m \in \mathbf{N}$ such that $|b_n - 0| = |b_n| \leq 0$ whenever $m \leq n$.
- Since $|x_n - a| \leq Cb_n\ \forall n \in \mathbf{N}$, $|x_n - a| \leq \epsilon$ whenever $n \geq m$
- this implies that x_n converges to a. ■

6 Problem 2.1.6 *Constant Sequence Converge*

Let a be a fixed real number and define $x_n := a$ for $n \in \mathbf{N}$. Prove that the constant sequence converges.

6.1 thoughts

1. so intuitively, at first read, this seems like I should be bounding this from bottom and above.
2. so first I would show it is bounded above then show it was bounded below? Could I show that it is bounded in one swoop.
3. I think that would be an over complicating things. I could do this easily with just definition 2.1. As by the definition you can easily say that $|x_n - a| < \epsilon$

6.2 Proof

- Since $x_n := a$, then $|x_n - a| < \epsilon = 0$ for all $n \in \mathbf{N}$. Thus, by definition 2.1 x_n converges.

7 Problem 2.1.7

(a) Suppose that $\{x_n\}$ and $\{y_n\}$ converge to the same real number. Prove that $x_n - y_n \to 0$ and $n \to \infty$

(b) Prove that the sequence $\{n\}$ does not converge.

(c) Show that there exist unbounded sequences $x_n \neq y_n$ which satisfy the conclusion of part (a).

7.1 Part A

- Since we know $\{x_n\}$ and $\{y_n\}$ converge to the same number we can say:
- For all $n \geq N$ implies $|x_n - a| < \epsilon$.
- and for all $m \geq M$ implies $|y_m - a| < \epsilon$.
- Then by choosing the larger of n or m we can say(for example sake we will say n is larger)
- Then for all $n \geq N$ (and $n \geq M$) implies $|x_n - y_n| = 0 < \epsilon$.

7.2 Part B

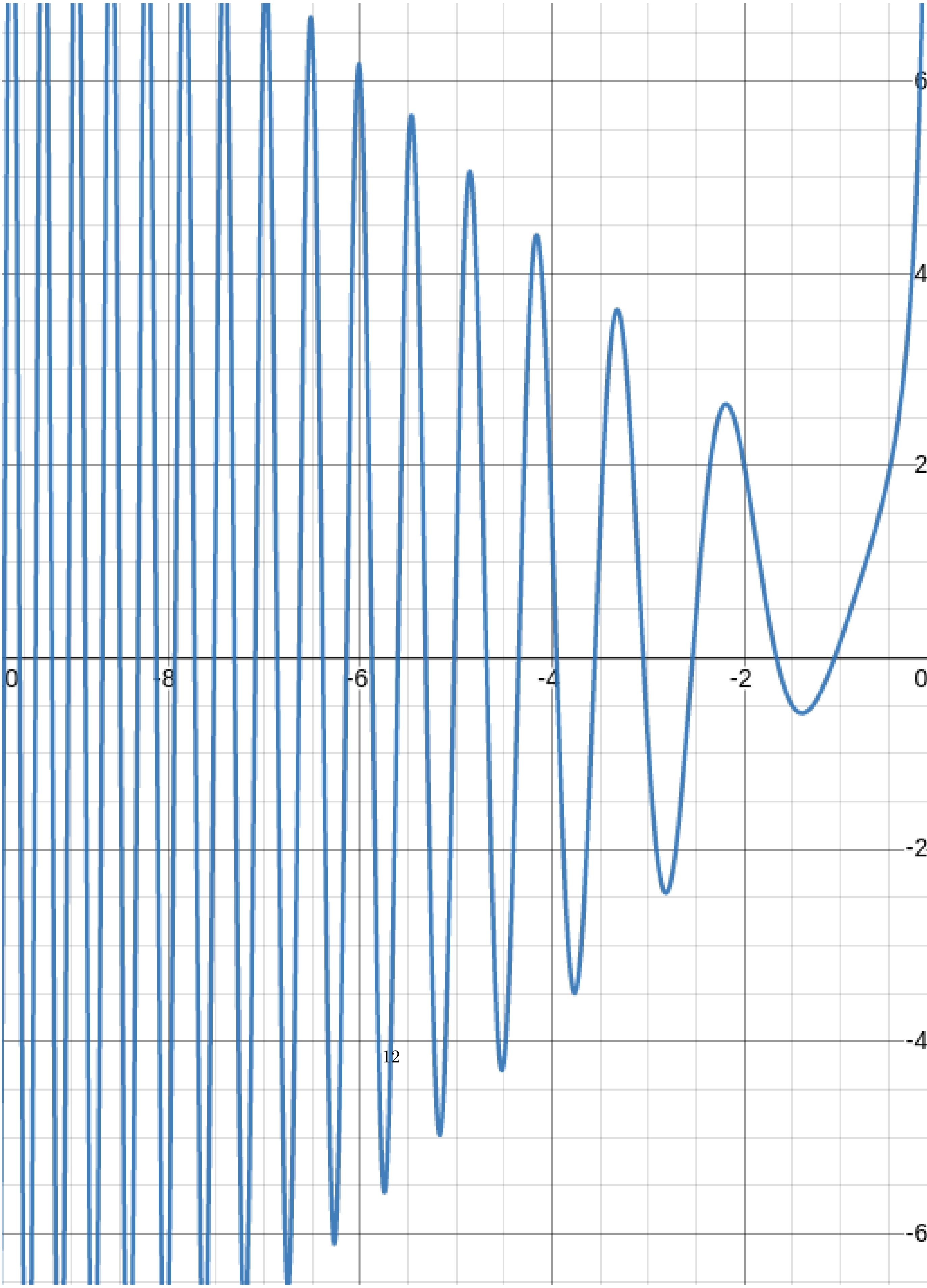
7.3 Thoughts

1. Doesn't this converge to infinity?
2. Okay for a curtain point, yes it doesn't converge.
3. So we can use the definition, say that not for every n there is an epsilon.
4. I assume there are a multiple ways to prove this. Some easier than others and some cleaner than others and some more related to what we are doing in this class.
5. My first thought is using the definition of bound to show it is not bounded or breaking definition 2.1 with a proof of contradiction.
6. $|x - a| < \epsilon$

7.4 Proof

1. To show this converges we must show the sequence will converge to a real number $a \in \mathbf{R}$ if and only if for every $\epsilon > 0$ there is an $N \in \mathbf{N}$ such that $\mathbf{N} \leq n$ implies $|x_n - a| < \epsilon$.
2. The problem with this is that the sequence $\{x\}$, as $|x_n - a| < \epsilon$ for an $n > N$, is that, $|x_n - a| < |x_{n+1} - a| < |x_{n+2} - a| < ...$ There is no choice of N, where $|x_n - a| < \epsilon$ for all $n > N$, as when n grows so does $|x_n - a|$. Thus $|x_n - a| < \epsilon$ will never be true, but rather $|x_n - a| \geq \epsilon$, which means our sequence diverges.

7.5 Part C



7.5.1 Thoughts

1. This problem is tricky.
2. The first time I did this problem I did something stupid, and basically used the definition of a convergent function for x_n and y_n without thinking that implies they are bounded.
3. I guess unbounded function would be something that goes to infinity. Or something semi-peculiar.
4. After a google search on examples of unbounded functions, that I like.

7.5.2 Proof

- Let sequence x_n be defined as $x_n sin(x_n^2)$ and sequence y_n be defined as $y_n sin(y_n^2) - \frac{1}{y_n}$.
- Obviously $x_n \neq y_n$ and $|x_n - y_n| = \frac{1}{y_n}$. Thus as y_n and y_n goes to infinity, $|x_n - y_n|$ goes to zero.