MATH370 HW3

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Feb 2022

Contents

1 Problem 2.3.1

Suppose that $x_0 \in (-1, 0)$ and $x_n \sqrt{x_{n-1} + 1} - 1$ for $n \in \mathbf{N}$.
Prove that $x_n \uparrow 0$ as $n \to \infty$.
What happens when $x_0 \in [-1, 0]$?

1.1 Proof

- If $x_n = \sqrt{x_{n-1} + 1} - 1 = (x_{n-1} + 1)^{0.5} - 1$ then $x_{n+1} = \sqrt{\sqrt{x_{n-1} + 1} - 1 + 1} - 1$
- $x_{n+1} = \sqrt{\sqrt{x_{n-1} + 1}} - 1 = (x_{n-1} + 1)^{0.25} - 1$
- We can see that$(x_{n-1} + 1)^{0.5} - 1 \neq (x_{n-1} + 1)^{0.25} - 1$
- Also note if we add 1 to each side $(x_{n-1} + 1)^{0.5} \neq (x_{n-1} + 1)^{0.25}$

- Note that while $x_n = (0, 1)$ then $(x_n)^{0.5} < (x_n)^{0.25}$ and since our actual $x_n = (-1, 0)$ and within the parentheses are are preforming $x_{n-1} + 1$, then we know on the interval $(-1, 0)$, $x_n > x_{n-1}$. Since this is strictly increasing on this interval and is bounded at 0 then as $n \to \infty$ $x_0 \to 0$. ■

1.2 What happens when $x_0 \in [-1, 0]$?

Then we have multiple possible convergences: $x_0 = -1$ or $x_0 = 0$.

2 Problem 2.3.2

Suppose that $0 \leq x_1 < 1$ and $x_{n+1} = 1 - \sqrt{1 - x_n}$ for $n \in \mathbf{N}$.

Prove that $x_n \downarrow 0$ and $\frac{x_{n+1}}{x_n} = \frac{1}{2}$ as $n \to \infty$.
What happens when $x_0 \in [-1, 0]$?

2.1 Proof for $x_n \downarrow 0$ as $n \to \infty$

- $x_{n+2} = 1 - \sqrt{1 - x_{n+1}}$
- $x_{n+2} = 1 + -((\sqrt{1 - x_n}))^{0.5} = 1 + -(1 - x_n)^{0.25}$
- Note that $1 + -(1 - x_n)^{0.25} \neq 1 + -(1 - x_n)^{0.5}$ except for when $x_1 = 0$.
- Also not that $1 + -(1 - x_n)^{0.25} \neq 1 + -(1 - x_n)^{0.5} \to (1 - x_n)^{0.25} \neq (1 - x_n)^{0.5}$ and since $1 - x_n$ where $0 \leq x_1 < 1$.
- Take not that when saying for $0 < y \leq 1$ then $y^{0.5} \leq y^{0.25}$.
- Since our case is a negative, thus $-(1 - x_n)^{0.25} < -(1 - x_n)^{0.5}$.
- Thus starting the sequence with an x_1 that is $0 \leq x_1 < 1$, then $x_n > n_{n+1}$ till the function gets to zero, which is our bound.
- Since we have a strictly decreasing sequence that is bounded by zero, under this condition $x_n \downarrow 0$.

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2.2 Proof for $\frac{x_{n+1}}{x_n} = \frac{1}{2}$ as $n \to \infty$

- So for $\frac{x_{n+1}}{x_n} = \frac{1-\sqrt{1-x_n}}{x_n}$
- $= \frac{1-(1-x_n)}{x_n(1-\sqrt{1-x_n})}$
- $lim_{n\to\infty} \frac{x_n}{x_n(1-\sqrt{1-x_n})} = \frac{1-1-1}{1(1+\sqrt{1-1})} = \frac{1}{2}$
- Actually this doesn't make entire sense to me.
- offfff

3 Problem 2.3.3

Suppose that $x_0 \geq 2$ and $x_n = 2 + \sqrt{x_{n-1} - 2}$ for $n \in \mathbf{N}$.

Use the Monotone Convergence Theorem to prove that either $x_n \to 2$ or $x_n \to 3$ as $n \to \infty$.

3.1 Proof

- Note that also $x_{n+1} = (2 + \sqrt{(2 + \sqrt{x_{n-1} - 2}) - 2}) = (2 + \sqrt{(2 + \sqrt{x_{n-1} - 2}) - 2})$
- $= (2 + \sqrt{(\sqrt{x_{n-1} - 2})}) = (2 + \sqrt{(\sqrt{x_{n-1} - 2})}) = 2 + (x_{n-1} - 2)^{\frac{1}{4}} = 2 + \sqrt{x_{n-1} - 2}$.
- We can see that when $x_n = 2$ or$x_n = 3$, $2 + \sqrt{x_{n-1} - 2} = 2 + (x_{n-1} - 2)^{\frac{1}{4}}$, implying $x_{n-1} = x_n$ for those cases.........(1)
- Know for $y > 1$, $y^{0.25} < y^{0.5}$ and for $0 < y < 1$, $y^{0.25} > y^{0.5}$.
- This gives us insight that while $2 < x_0 < 3$ $x_n > x_{n-1}$, thus heading to 3 and the reverse is true when $x_0 > 3$. Then we have a case of $x_n < x_{n-1}$.
- Thus we can see from both sides x_{n-1} approaches 3.
- The exception to this is when $x_{n-1} = 2$ then$x_n = 2$
- Thus by Monotone Convergence Theorem, since when $3 > x_0 > 2$ is bounded above and at the same time is strictly increasing, it converges to a finite limit 3.
- $\blacksquare$

4 Problem 2.3.4

Suppose that $x_0 \in \mathbf{R}$ and $x_n = (1 + x_{n-1})/2$ for $n \in \mathbf{N}$.

Use the Monotone Convergence Theorem to prove that $x_n \to 1$ as $n \to \infty$.

4.1 Proof

- $x_n = (1 + x_{n-1})/2 = \frac{1}{2} + \frac{x_{n-1}}{2}$
- $x_{n+1} = (1 + (1 + x_{n-1})/2)/2 = \frac{1}{2} + (\frac{1}{2} + \frac{x_{n-1}}{2})/2 = \frac{1}{2} + \frac{1}{4} + \frac{x_{n-1}}{4}$
- Note $\frac{1}{2} + \frac{x_{n-1}}{2} \neq \frac{1}{2} + \frac{1}{4} + \frac{x_{n-1}}{4}$ except for when $x_n = 1$.
- In fact if we subtract $\frac{1}{2}$ from each side we get $\frac{x_{n-1}}{2} \neq \frac{1}{4} + \frac{x_{n-1}}{4}$.
- Note that for $x_0 < 1$ $\frac{x_{n-1}}{2} < \frac{1}{4} + \frac{x_{n-1}}{4}$
- Thus we can see when $x_n < 1$ then $x_{n+1} > x_n$
- but when we have $x_n > 1$ we can see that $x_{n+1} < x_n$.
- Therefore on the left side of 1 we have strictly increasing and on the right side of one we have strictly decreasing with a bound at 1, thus by Monotone Convergence Theorem we have $x_n \to 1$ as $n \to \infty$
- $\blacksquare$

5 Problem 2.3.7

Suppose that $E \subset \mathbf{R}$ is nonempty bounded set and that $\sup E \notin E$.

Prove that there exists a strictly increasing sequence $\{x_n\}$ that converges to $\sup E$ such that $x_n \in E$ for all $n \in N$.

5.1 Proof

- Honestly I have to look at this a whole lots more.
- Not even a 100 percent sure where to start with this.
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6 Problem 2.4.1

Prove that if $\{x_n\}$ is a sequence that satisfies

$$|x_n| \leq \frac{2n^2+3}{n^3+5n^2+3n+1}$$

6.1 Proof

- Note as $n \to \infty$ $\frac{2n^2+3}{n^3+5n^2+3n+1}\infty$ (the degree of the bottom function is larger). Then by squeeze theorem that $x_n \to 0$ as $n \to \infty$.
- Thus $\{x_n\}$ is Cauchy.

7 Problem 2.4.2

Suppose that $x_n \in \mathbf{Z}$ for $n \in \mathbf{N}$.

If $\{x_n\}$ is Cauchy, prove that x_n is eventually constant; that is, that there exist numbers $a \in \mathbf{Z}$ and $n \in \mathbf{N}$ such that $x_n = a$ for $n \geq N$.

7.1 Proof

- So knowing that $x_n \in \mathbf{Z}$ for $n \in \mathbf{N}$ and for this case Cauchy, then we know there is a $n, m \geq N$ such that $|x_n - x_m| < \epsilon$ for an $\epsilon > 0$ and $N \in \mathbf{N}$.
- Because $x_n \in \mathbf{Z}$ then the smallest $|x_n - x_m| = 0$ and the second possible smallest is $|x_n - x_m| = 1$.
- If $|x_n - x_m| \geq 1$ then this is not Cauchy.
- Thus for $|x_n - x_m| = 0$, x_n, x_m must equal the same. Therefore the sub sequence, $\{x_n\}$, starting at N for $n, m \geq N$, $x := a$.
- Thus our sequence $x_n := a$ or x_n converges to a.

8 Problem 2.4.7

1. Let E be a subset of $\mathbf{R}$. A point $a \in \mathbf{R}$ is called a cluster point of E if $E \cap (a - r, a + r)$ contains infinitely many points for every $r > 0$. Prove that a is a cluster point of E if and only if for each $r > 0$, $E \cap (a - r, a + r)$
$\{a\}$

2.

1. Honestly, again Not sure what to do here exactly.