

# BLG 336E

## Analysis of Algorithms II

Lecture 6:

The Minimum Spanning Tree

Prim's Algorithm, Kruskal's Algorithm

# RECAP OF PREVIOUS LECTURES

## Stable Matching

- Gale-Shapley Algorithm

## Big-O Notation

- Asymptotically Tight Bounds
- Big Theta and Omega
- A Survey of runtimes

## Graphs

- Breadth First Search
- Depth First Search
- Testing Bi-partite
- Topological Ordering

## Greedy Algorithms

- Interval Scheduling
- Interval Partitioning
- Shortest Paths in a Graph(Dijkstra)

Week	Date	Topic
1	21-Feb	Introduction. Some representative problems
2	28-Feb	Stable Matching
3	7-Mar	Basics of algorithm analysis.
4	14-Mar	Graphs (Project 1 announced)
5	21-Mar	Greedy algorithms-I
6	28-Mar	Greedy algorithms-II
7	4-Apr	Divide and conquer (Project 2 announced)
8	11-Apr	Dynamic Programming I
9	18-Apr	Dynamic Programming II
10	25-Apr	Network Flow-I (Project 3 announced)
11	2-May	<b>Midterm</b>
12	9-May	Network Flow II
13	16-May	NP and computational intractability-I
14	23-May	NP and computational intractability-II

- Greedy algorithms for Minimum Spanning Tree.
- Agenda:
  1. What is a Minimum Spanning Tree?
  2. Short break to introduce some graph theory tools
  3. Prim's algorithm
  4. Kruskal's algorithm

## Minimum Spanning Trees

- What's the cheapest way to connect a graph

### Prim's Algorithm

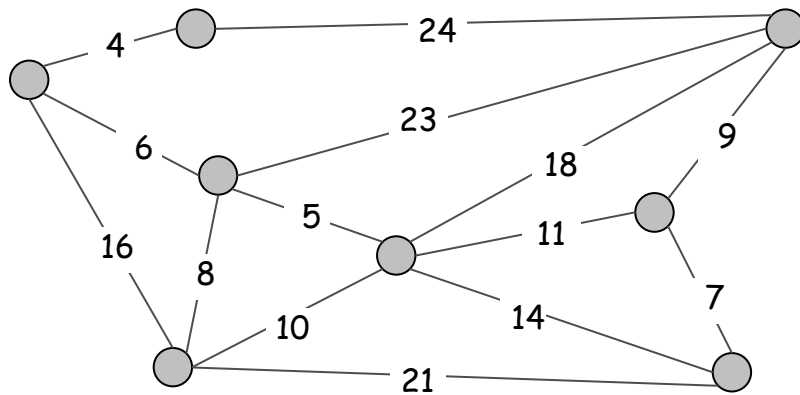
- simple and efficient algorithm for finding minimum spanning trees.

## 4.5 Minimum Spanning Tree

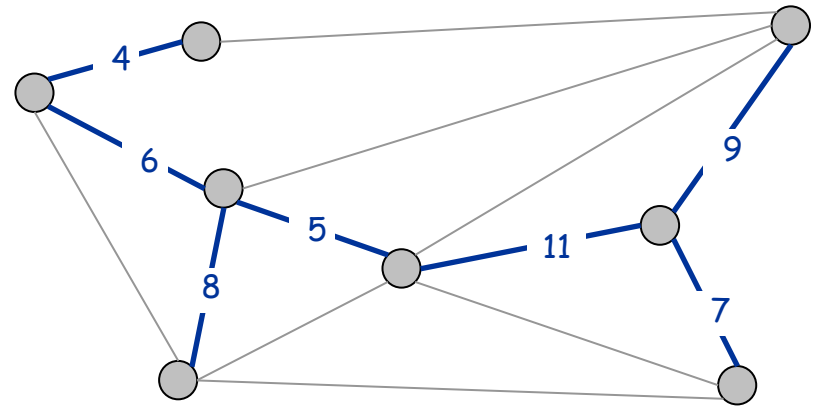
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# Minimum Spanning Tree

**Minimum spanning tree.** Given a connected graph  $G = (V, E)$  with real-valued edge weights  $c_e$ , an MST is a subset of the edges  $T \subseteq E$  such that  $T$  is a spanning tree whose sum of edge weights is minimized.



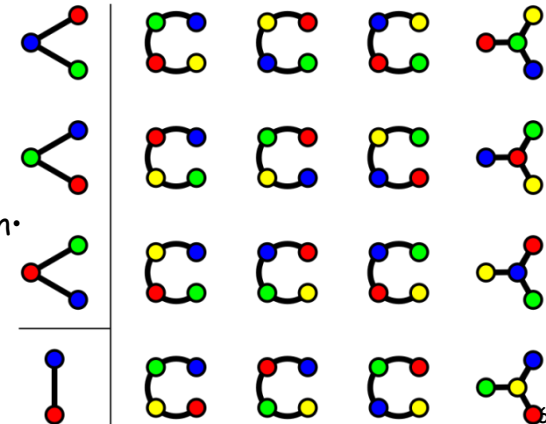
$G = (V, E)$



$T, \sum_{e \in T} c_e = 50$

**Cayley's Theorem.** There are  $n^{n-2}$  spanning trees of  $K_n$ .

↑  
can't solve by brute force  
#StayHome



# Greedy Algorithms

**Kruskal's algorithm.** Start with  $T = \emptyset$ . Consider edges in ascending order of cost. Insert edge  $e$  in  $T$  **unless doing so would create a cycle.**

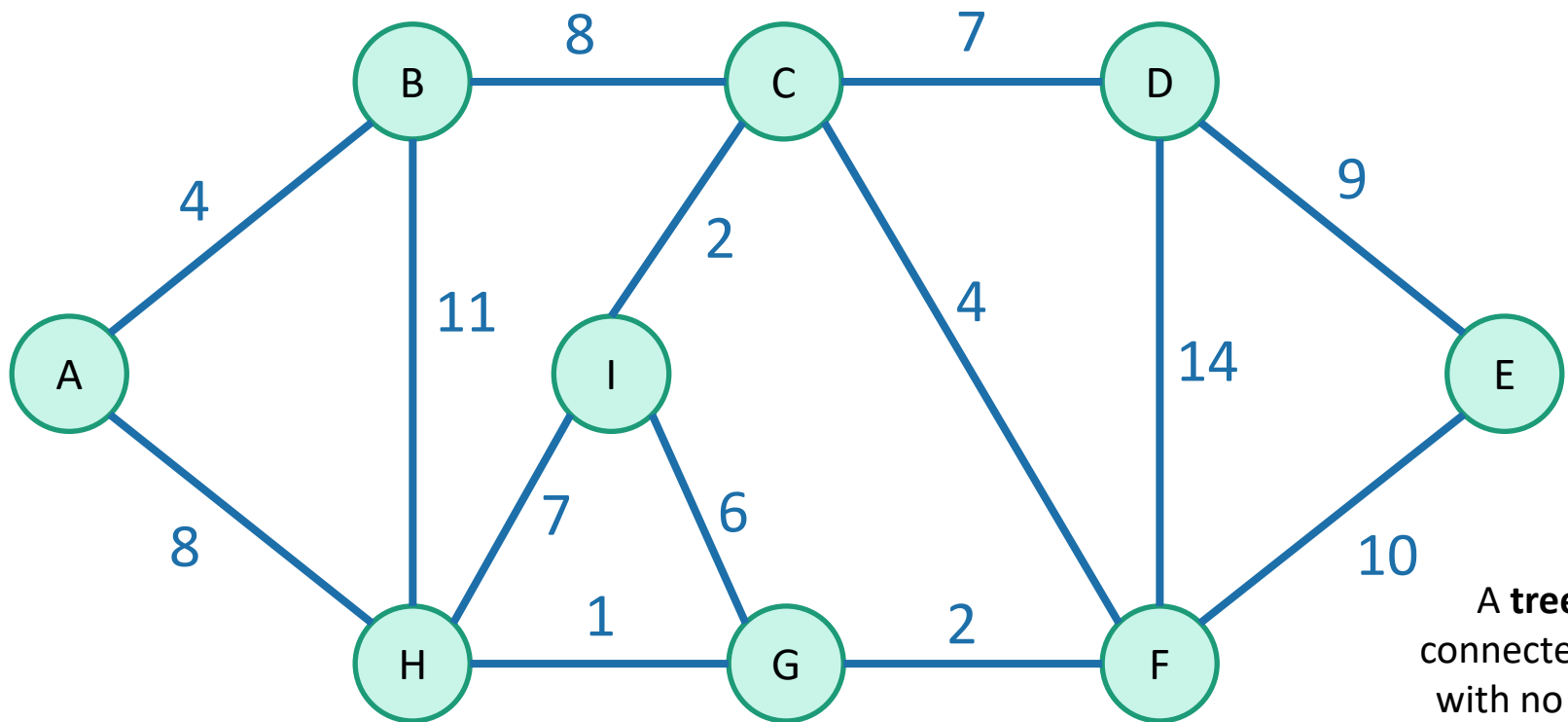
**Prim's algorithm.** Start with some root node  $s$  and greedily grow a tree  $T$  from  $s$  outward. At each step, **add the cheapest edge  $e$**  to  $T$  that has exactly one endpoint in  $T$ .

**Reverse-Delete algorithm.** Start with  $T = E$ . Consider edges in descending order of cost. Delete edge  $e$  from  $T$  unless doing so would disconnect  $T$ .

**Remark.** All three algorithms produce an MST.

# Minimum Spanning Tree

Say we have an undirected weighted graph



A **tree** is a connected graph with no cycles!

A **spanning tree** is a **tree** that connects all of the vertices.



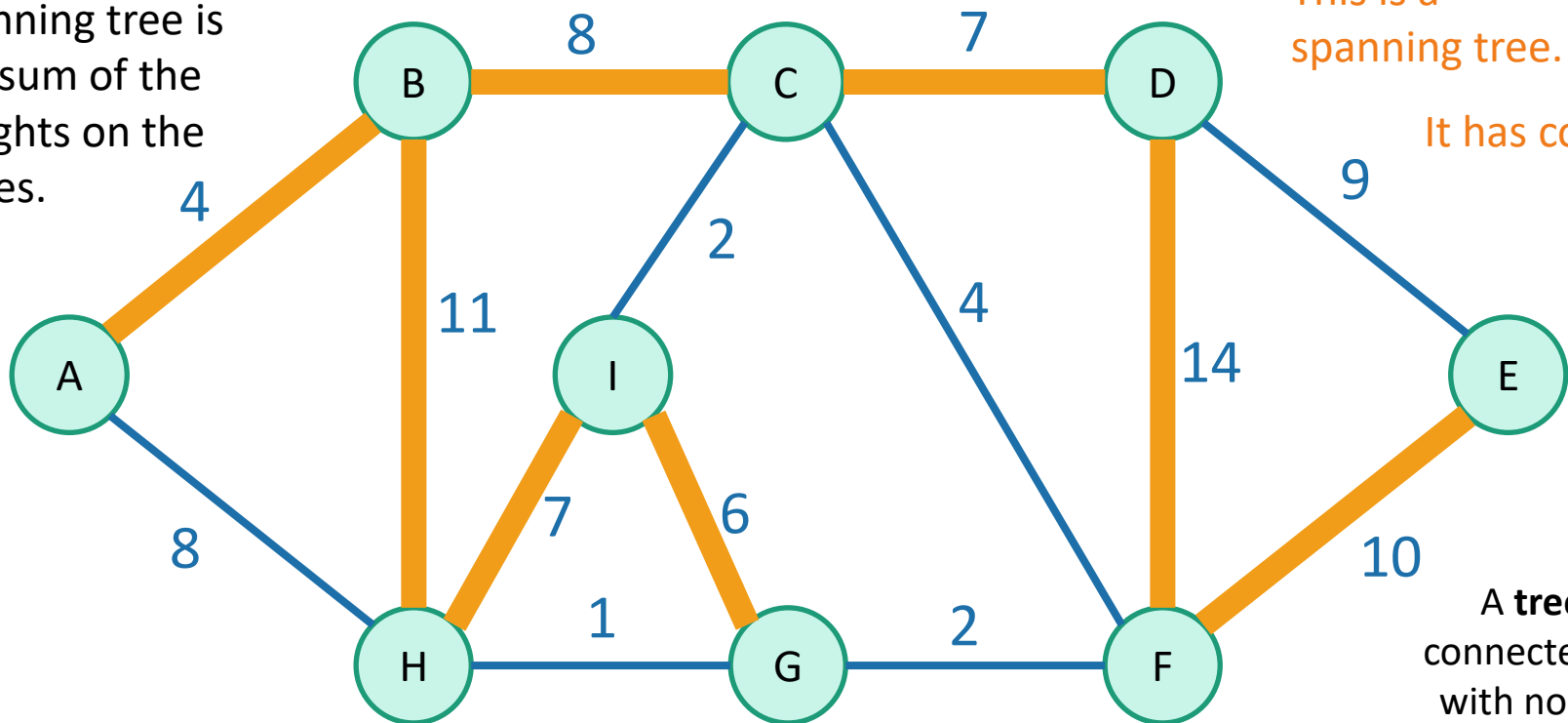
# Minimum Spanning Tree

Say we have an undirected weighted graph

The **cost** of a spanning tree is the sum of the weights on the edges.

This is a spanning tree.

It has cost 67

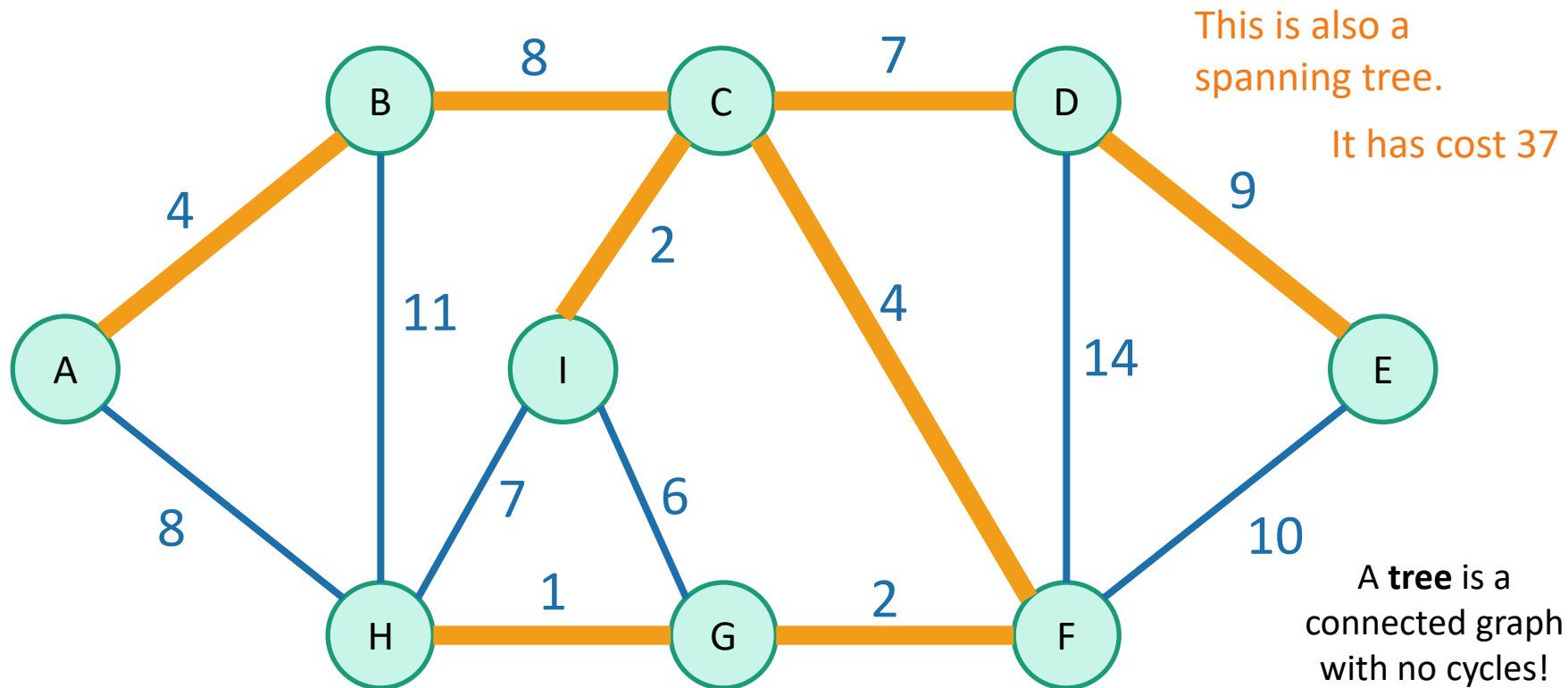


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# Minimum Spanning Tree

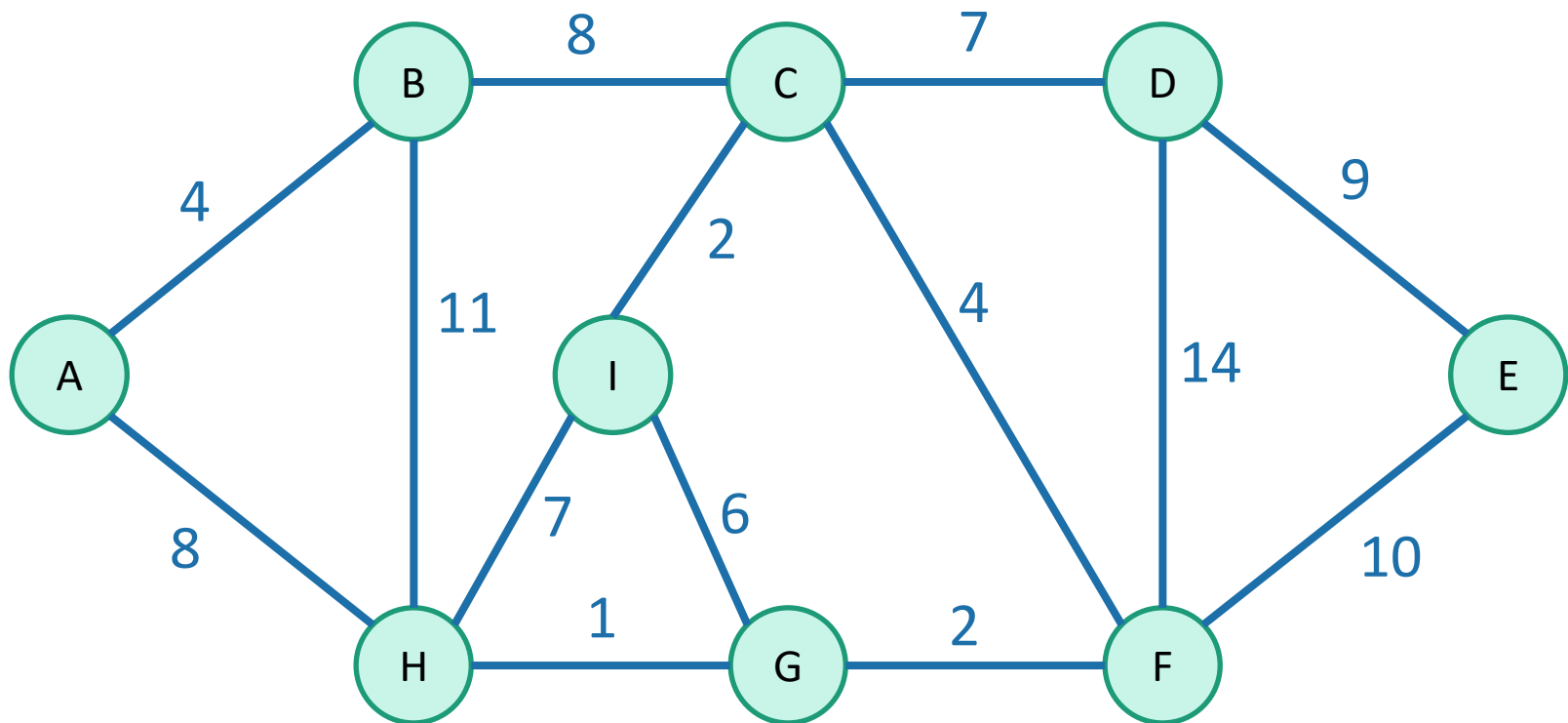
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# Minimum Spanning Tree

Say we have an undirected weighted graph



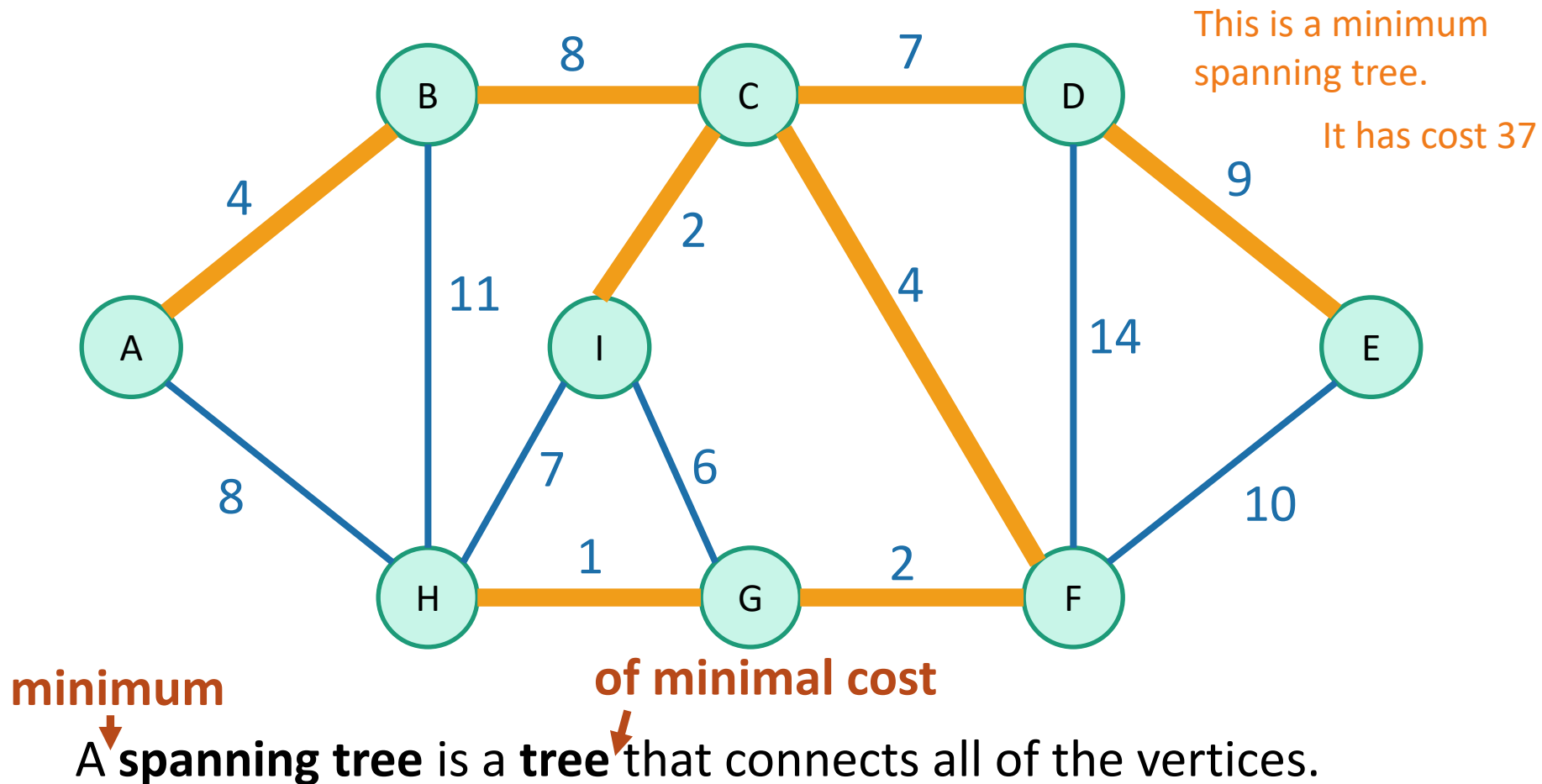
minimum

of minimal cost

A **spanning tree** is a **tree** that connects all of the vertices.

# Minimum Spanning Tree

Say we have an undirected weighted graph



# Why MSTs?

- Network design
  - Connecting cities with roads/electricity/telephone/...
- cluster analysis
  - eg, genetic distance
- image processing
  - eg, image segmentation
- Useful primitive
  - for other graph algs

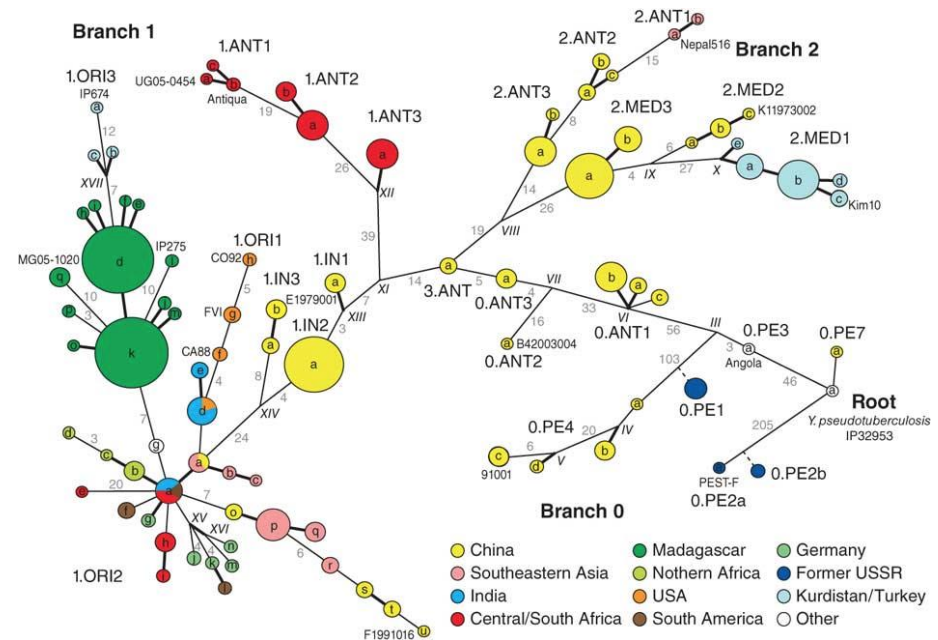
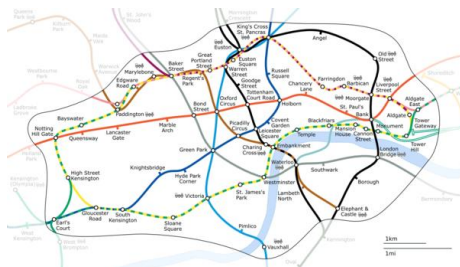


Figure 2: Fully parsimonious minimal spanning tree of 933 SNPs for 282 isolates of *Y. pestis* colored by location. Morelli et al. Nature genetics 2010

# How to find an MST?

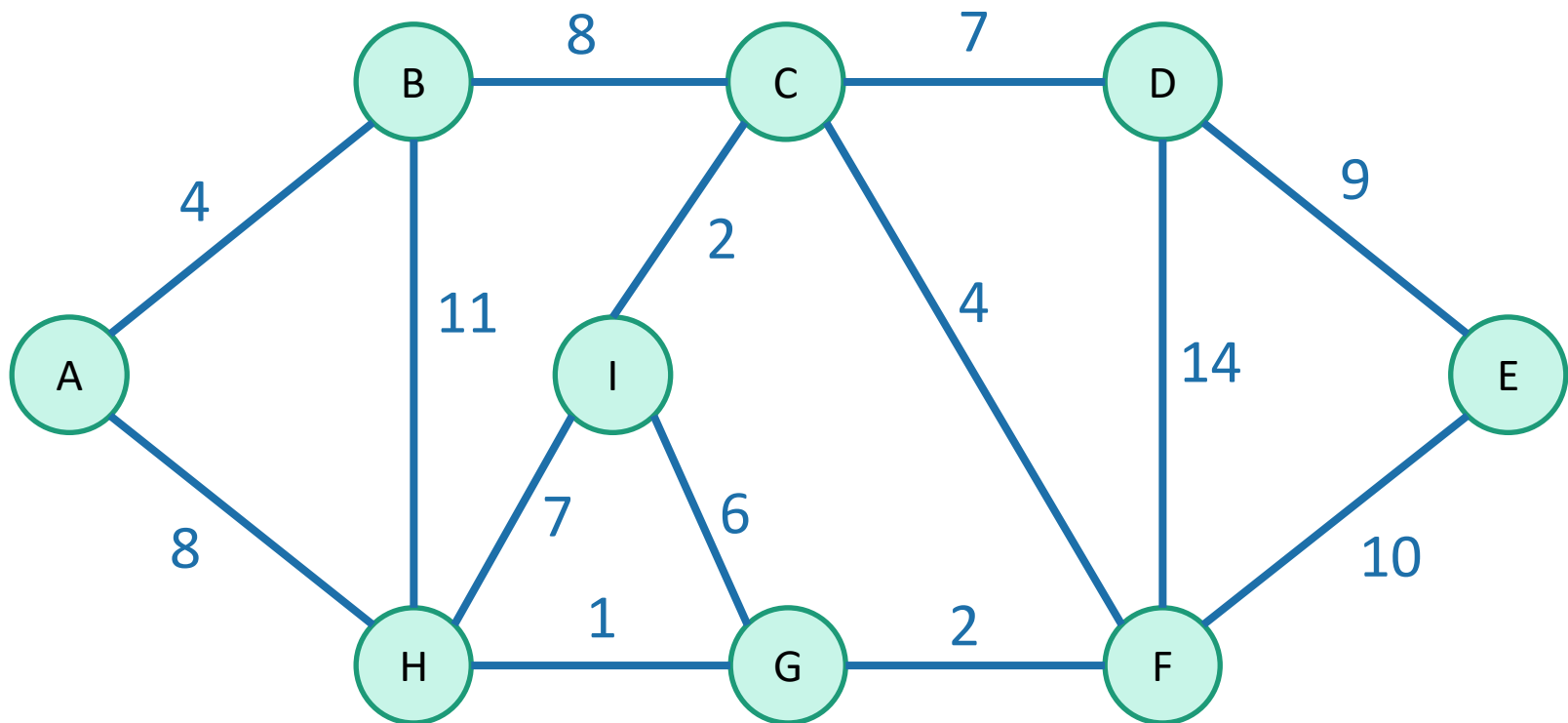
- Today we'll see two greedy algorithms.
- In order to prove that these greedy algorithms work, we'll need to show something like:

*Suppose that our choices so far  
haven't ruled out success.*

*Then the next greedy choice that we make  
also won't rule out success.*

- Here, **success** means finding an MST.

# Let's brainstorm some greedy algorithms!



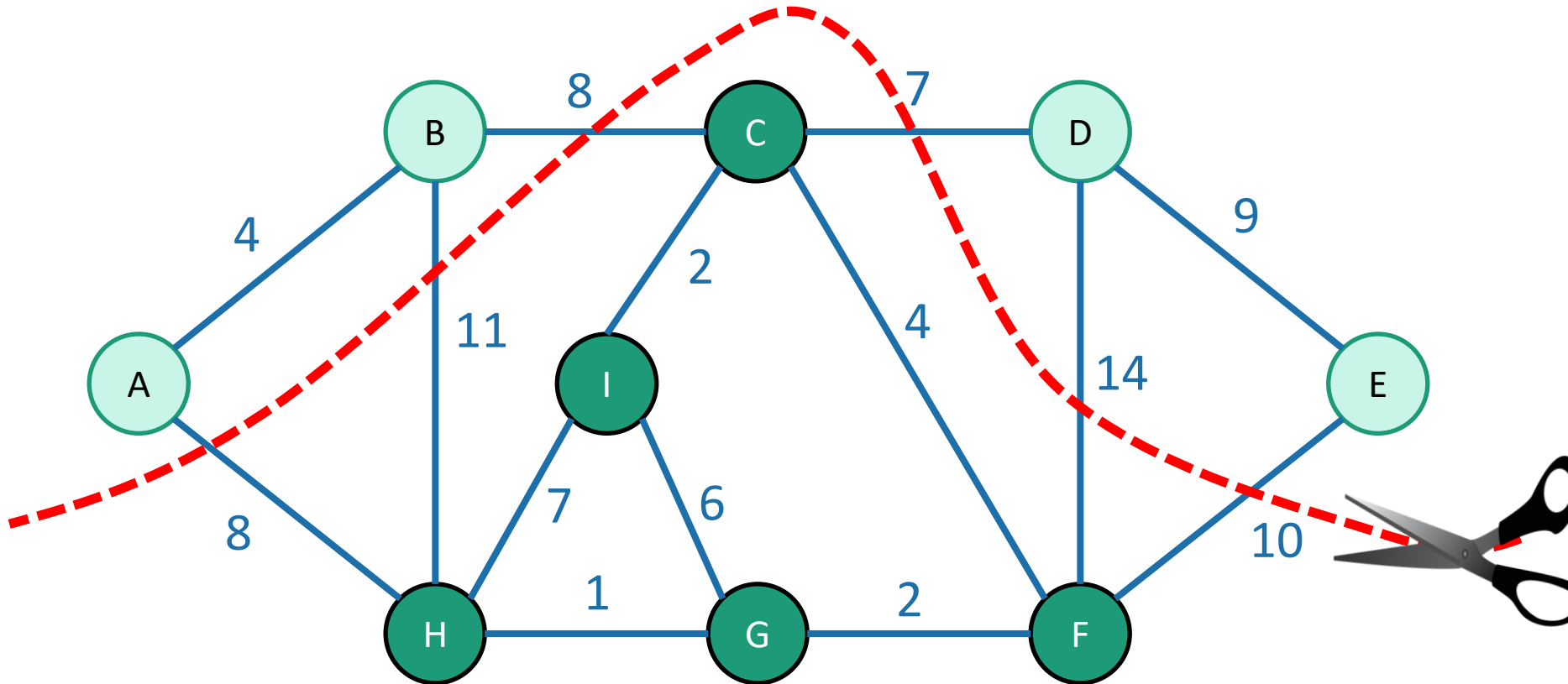
# Brief aside

for a discussion of cuts in graphs!



# Cuts in graphs

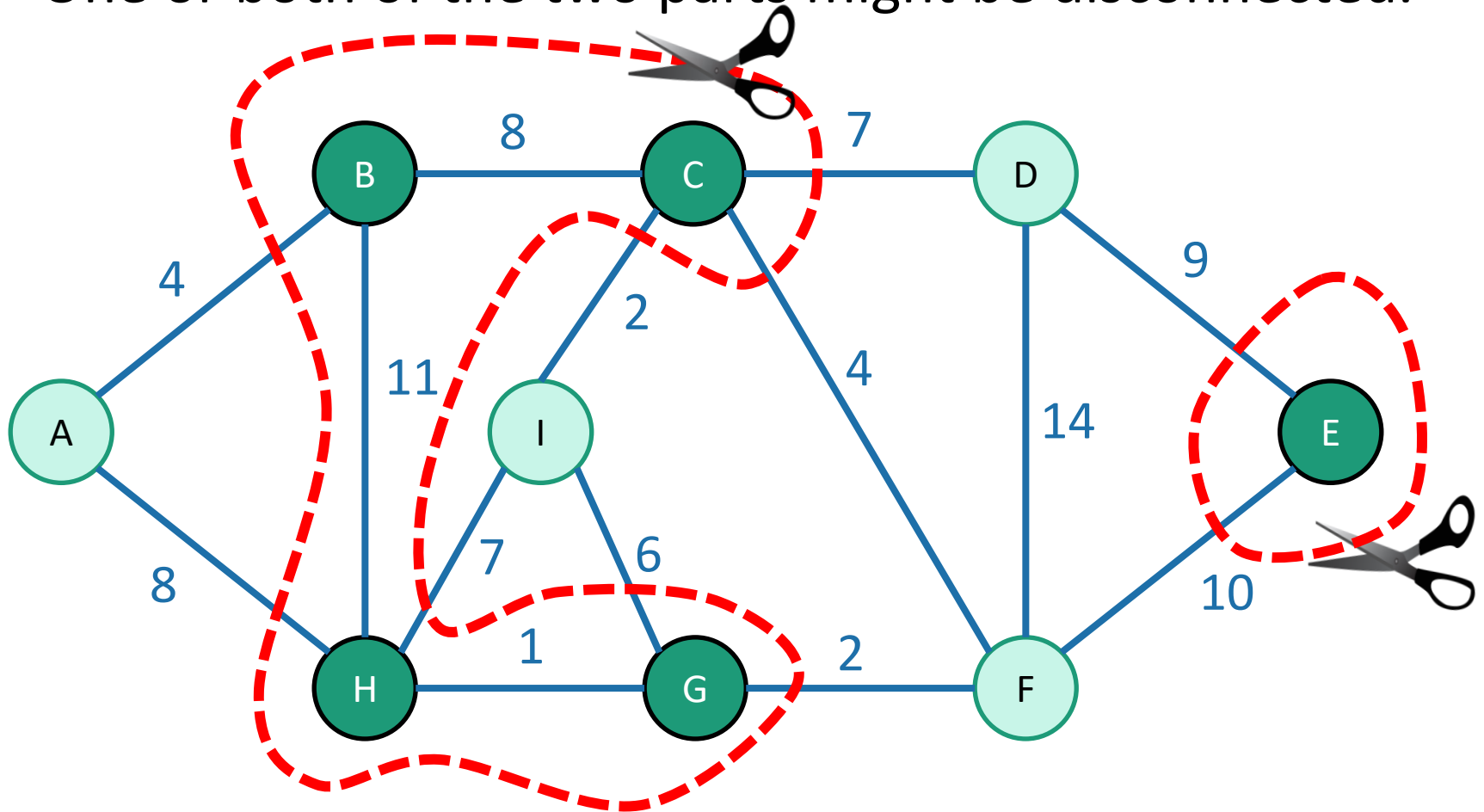
- A **cut** is a partition of the vertices into two parts:



This is the cut “{A,B,D,E} and {C,I,H,G,F}”

# Cuts in graphs

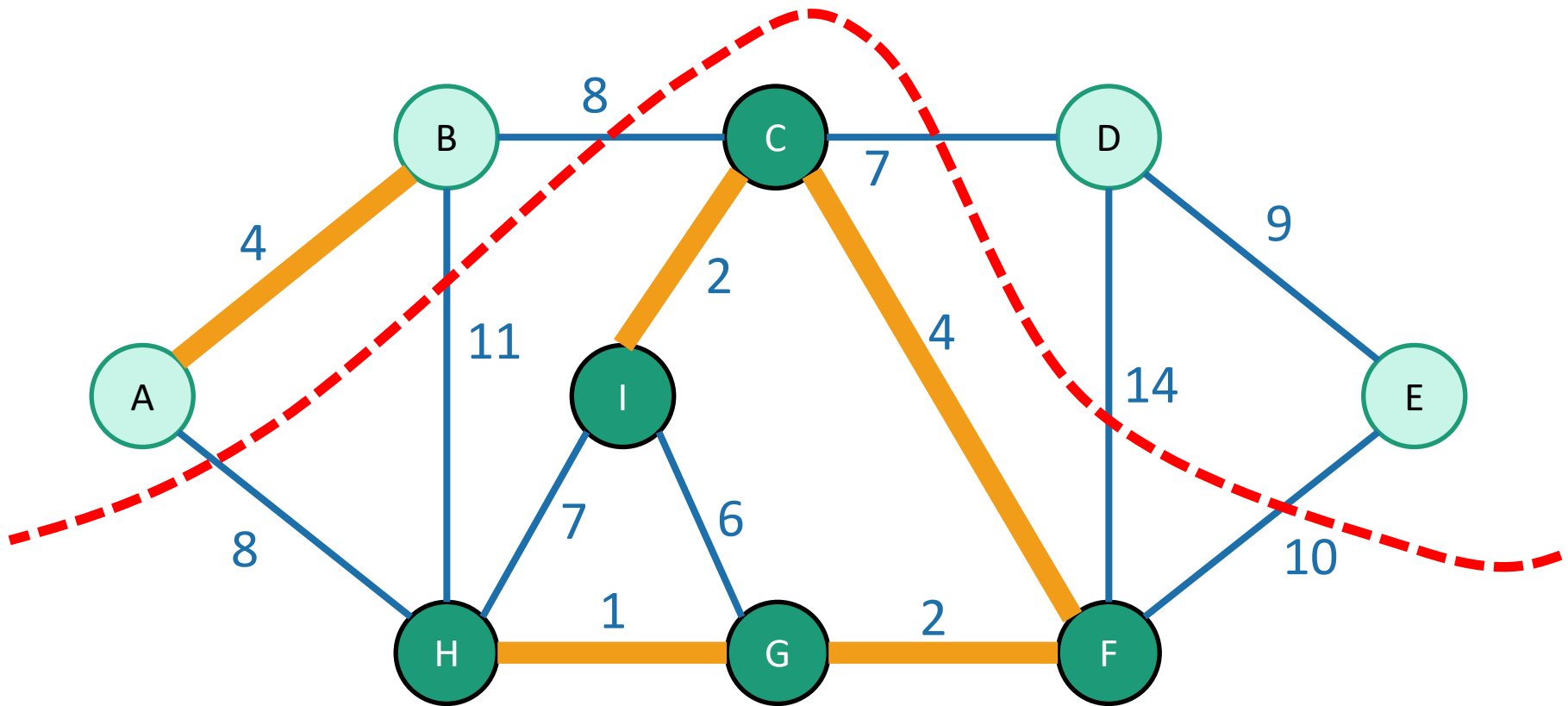
- One or both of the two parts might be disconnected.



This is the cut “{B,C,E,G,H} and {A,D,I,F}”

# Let $S$ be a set of edges in $G$

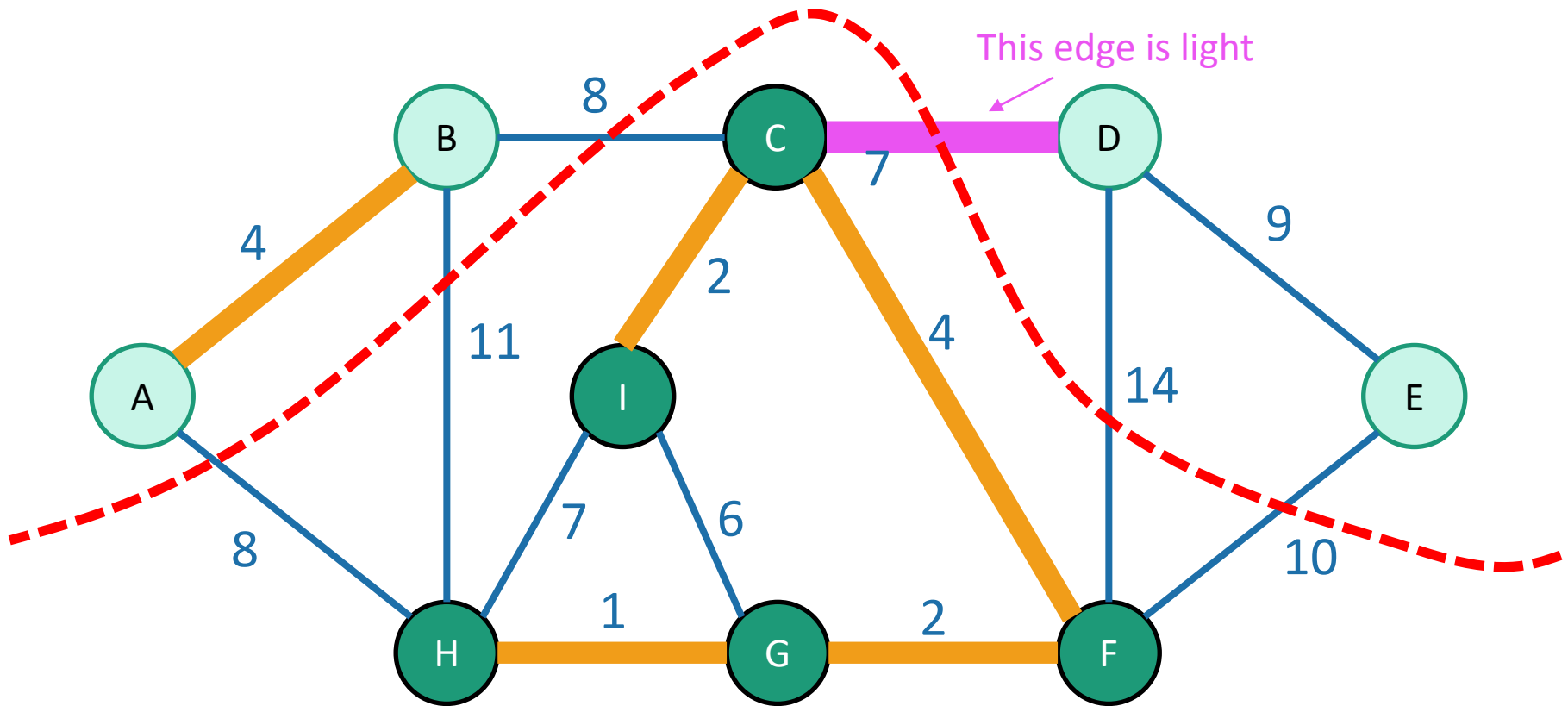
- We say a cut **respects**  $S$  if no edges in  $S$  cross the cut.
- An edge crossing a cut is called **light** if it has the smallest weight of any edge crossing the cut.



$S$  is the set of **thick orange** edges

# Let $S$ be a set of edges in $G$

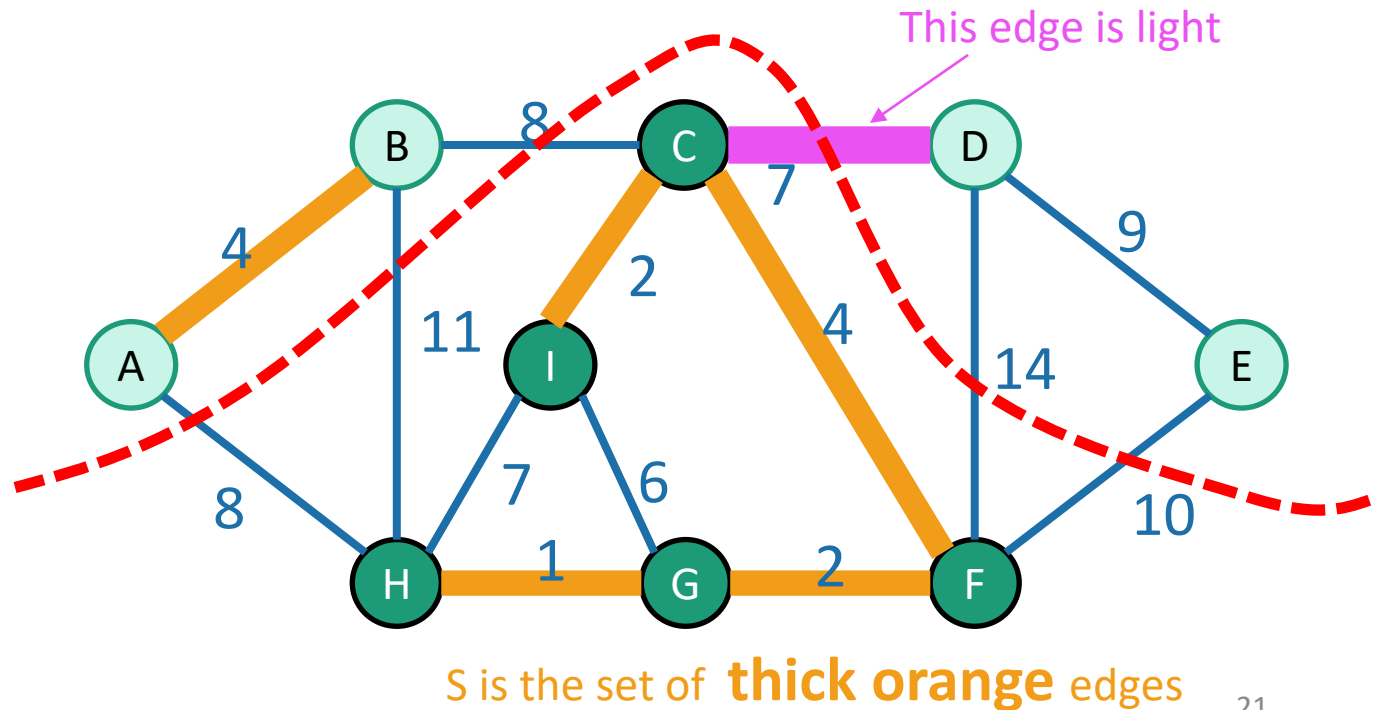
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# Lemma

- Let  $S$  be a set of edges, and consider a cut that respects  $S$ .
- Suppose there is an MST containing  $S$ .
- Let  $\{u,v\}$  be a light edge.
- Then there is an MST containing  $S \cup \{u,v\}$

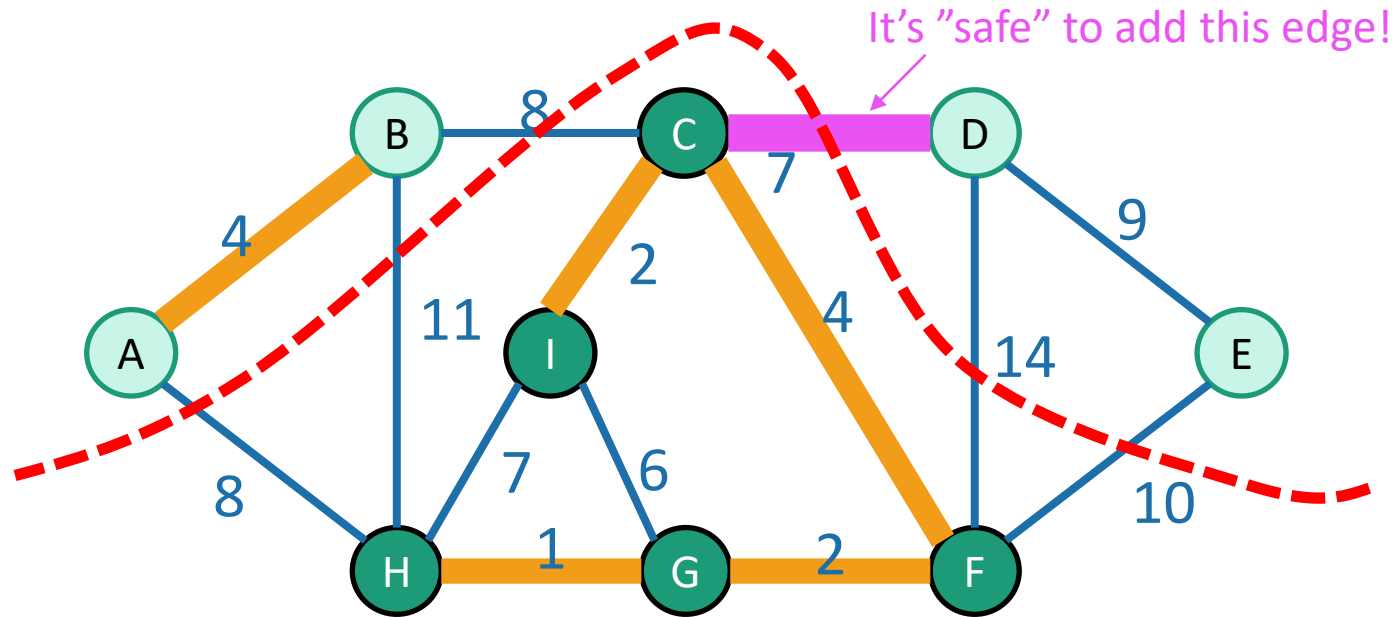


# Lemma

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Aka:

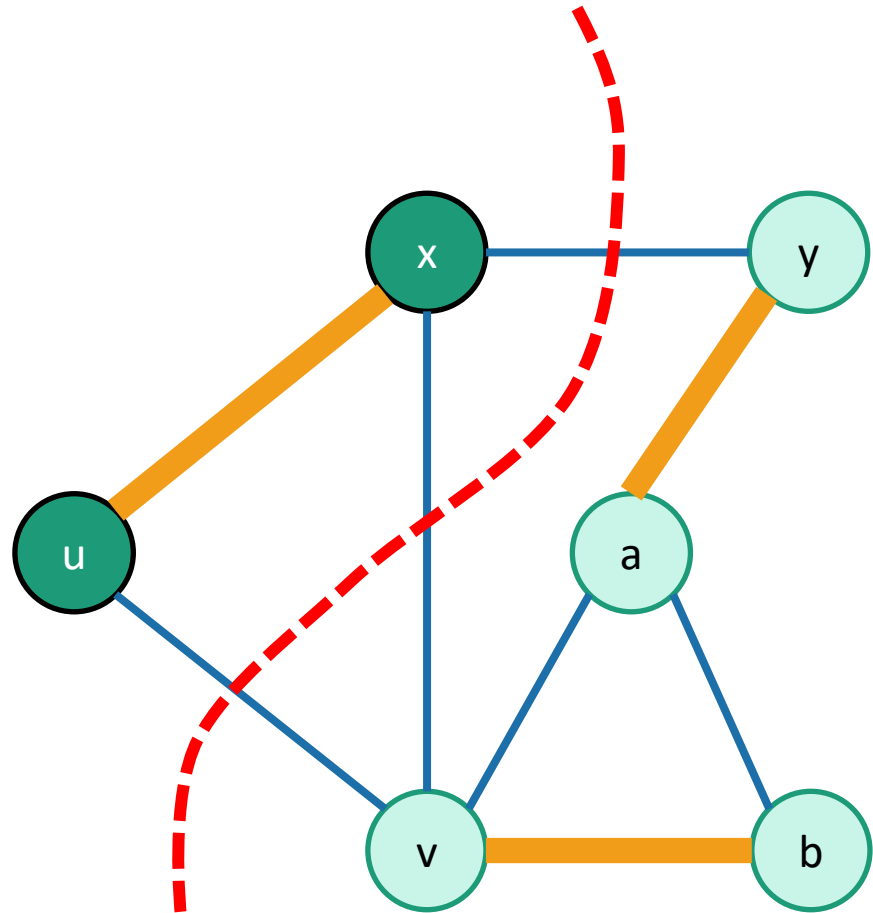
If we haven't ruled out the possibility of success so far, then adding a light edge still won't rule it out.



$S$  is the set of **thick orange** edges

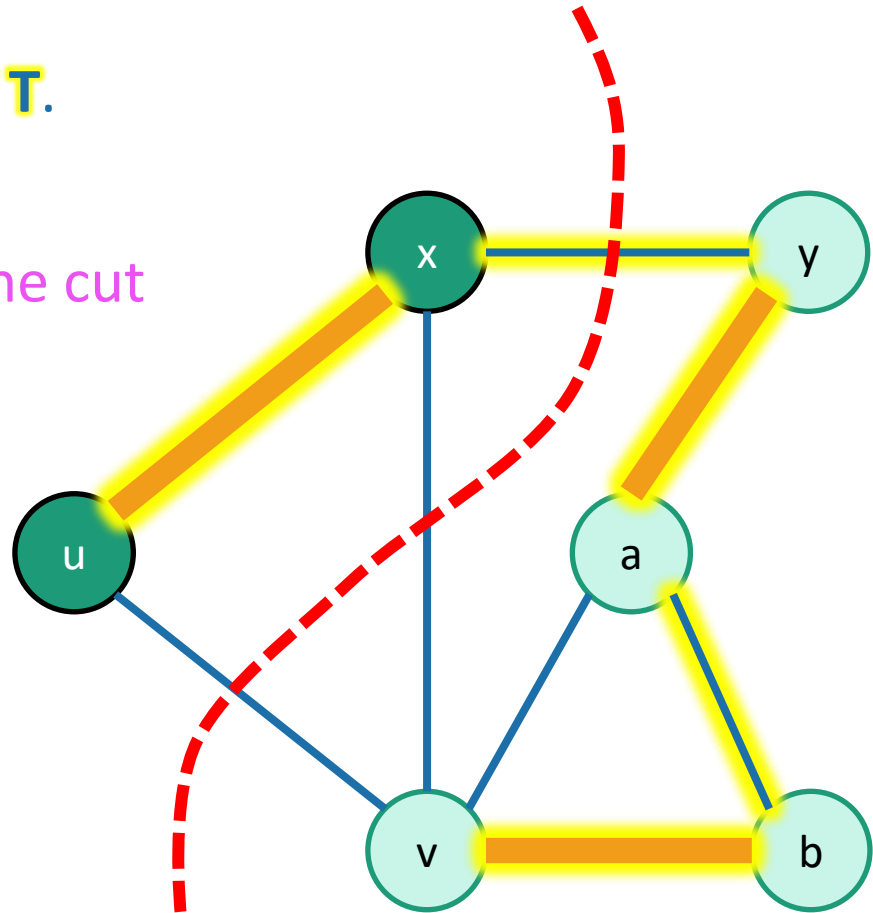
# Proof of Lemma

- Assume that we have:
  - a **cut** that respects **S**



# Proof of Lemma

- Assume that we have:
  - a **cut** that respects **S**
  - **S** is part of some **MST T**.
- Say that  $\{u, v\}$  is light.
  - lowest cost crossing the cut






# Proof of Lemma

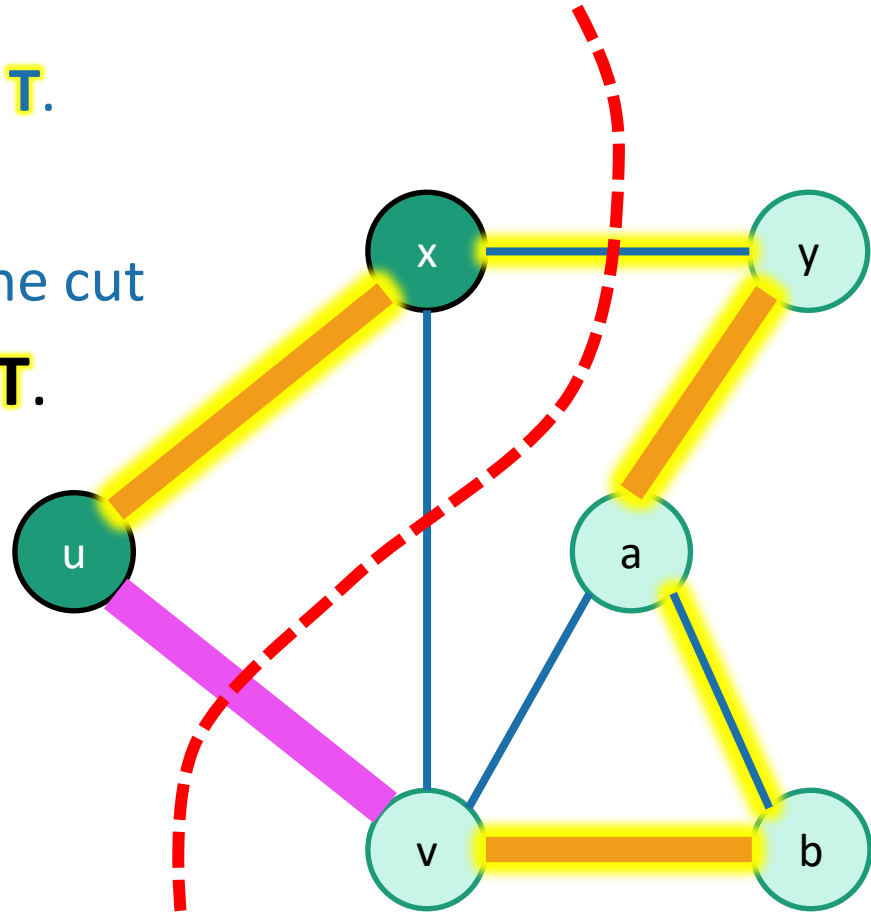
**Claim:** Adding any additional edge to a spanning tree will create a cycle.

**Proof:** Both endpoints are already in the tree and connected to each other.

- Assume that we have:
    - a **cut** that respects **S**
    - **S** is part of some **MST T**.
  - Say that **{u,v}** is light.
    - lowest cost crossing the cut
  - But say **{u,v}** is not in **T**.
    - So adding **{u,v}** to **T** will make a cycle.
- 
- otherwise  
done!

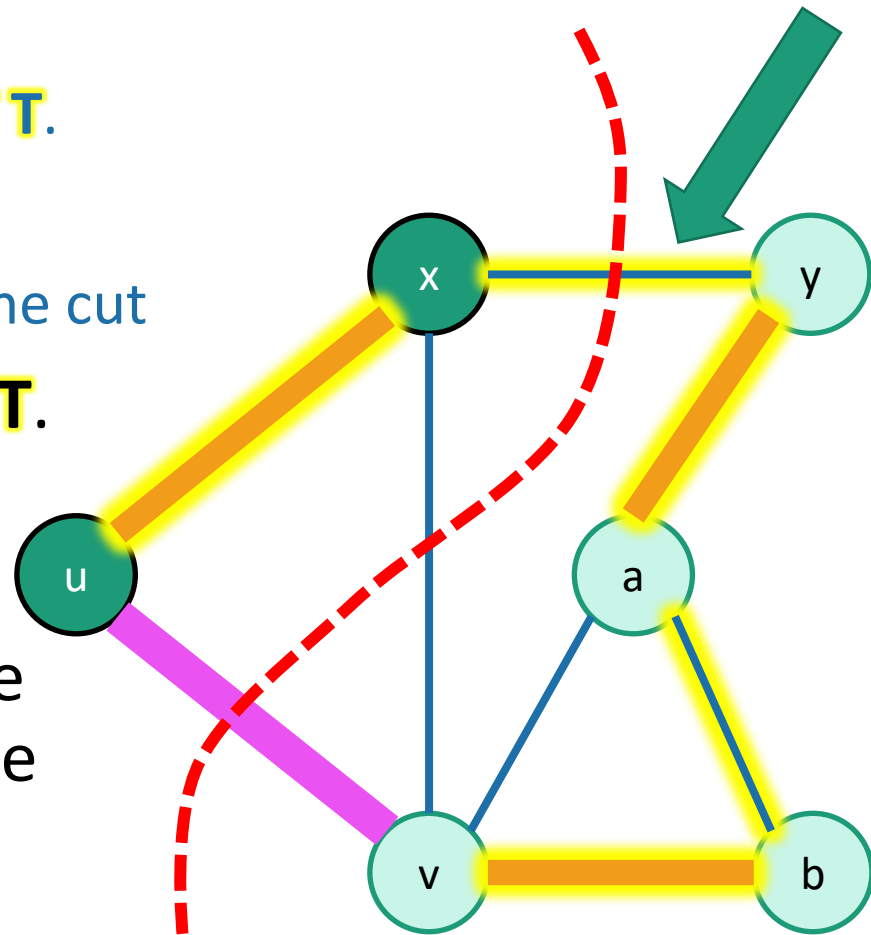
- B

Otherwise  
we're done!



# Proof of Lemma

- Assume that we have:
  - a **cut** that respects **S**
  - **S** is part of some **MST T**.
- Say that  $\{u,v\}$  is light.
  - lowest cost crossing the cut
- But say  $\{u,v\}$  is not in **T**.
  - So adding  $\{u,v\}$  to **T** will make a cycle.
- So there is at least one other edge in this cycle crossing the cut.
  - call it  $\{x,y\}$

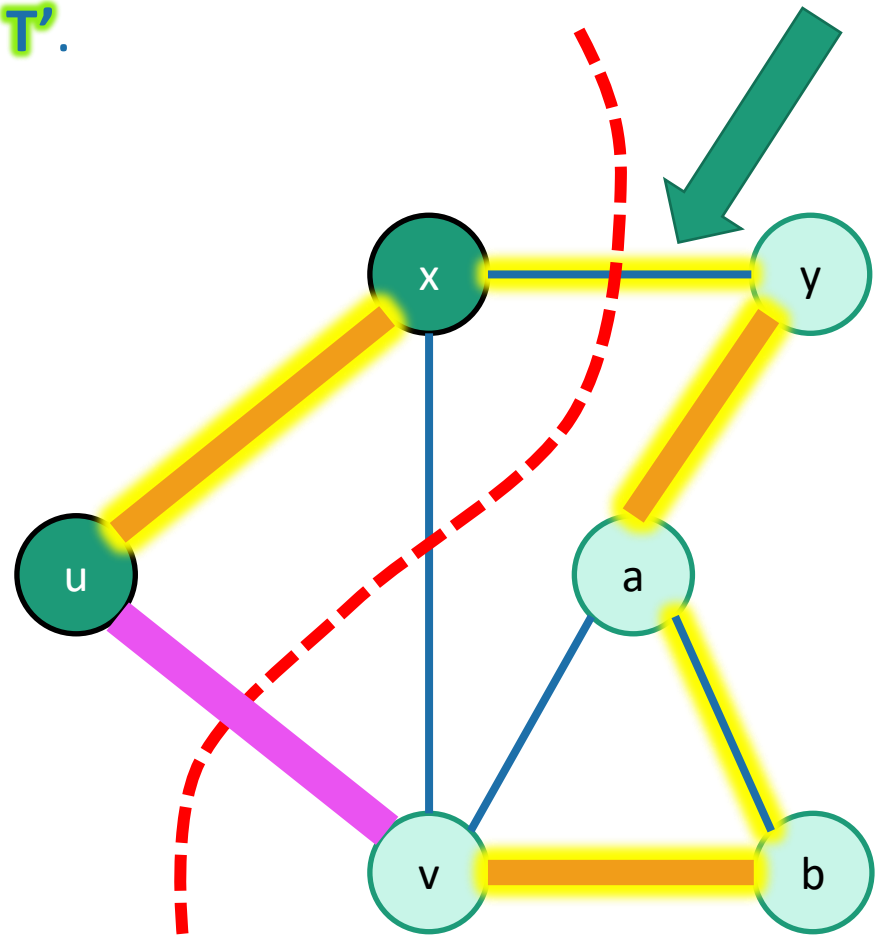


**Claim:** Adding any additional edge to a spanning tree will create a cycle.

**Proof:** Both endpoints are already in the tree and connected to each other.

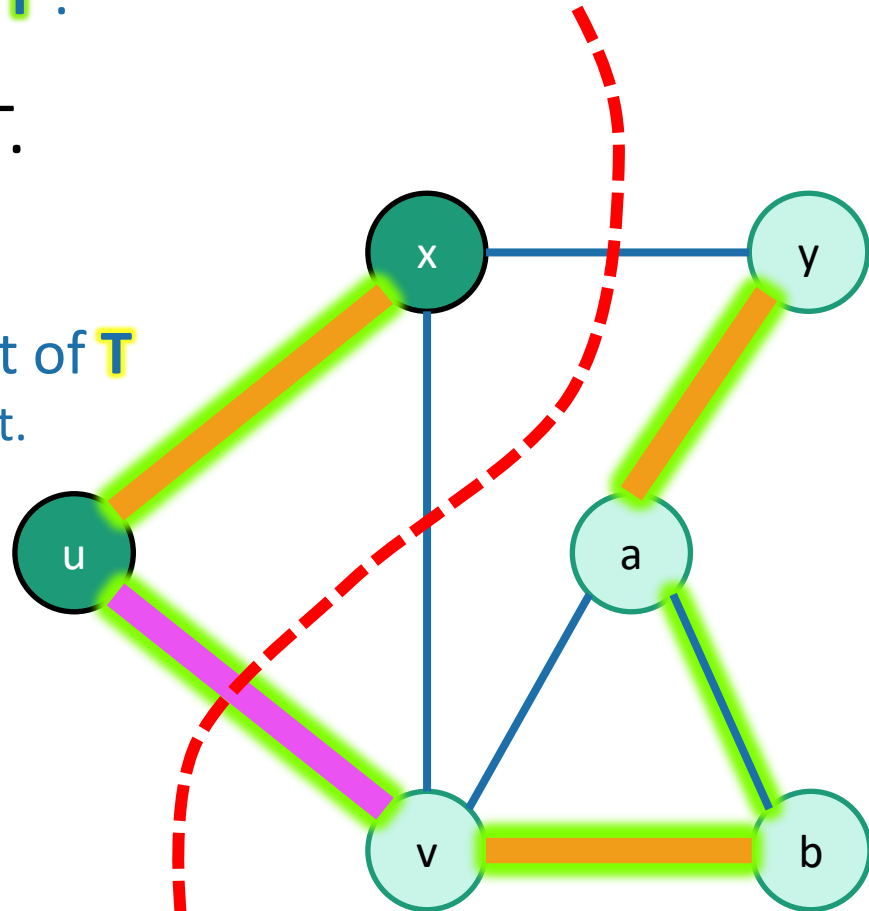
# Proof of Lemma ctd.

- Consider swapping  $\{u,v\}$  for  $\{x,y\}$  in **T**.
  - Call the resulting tree **T'**.



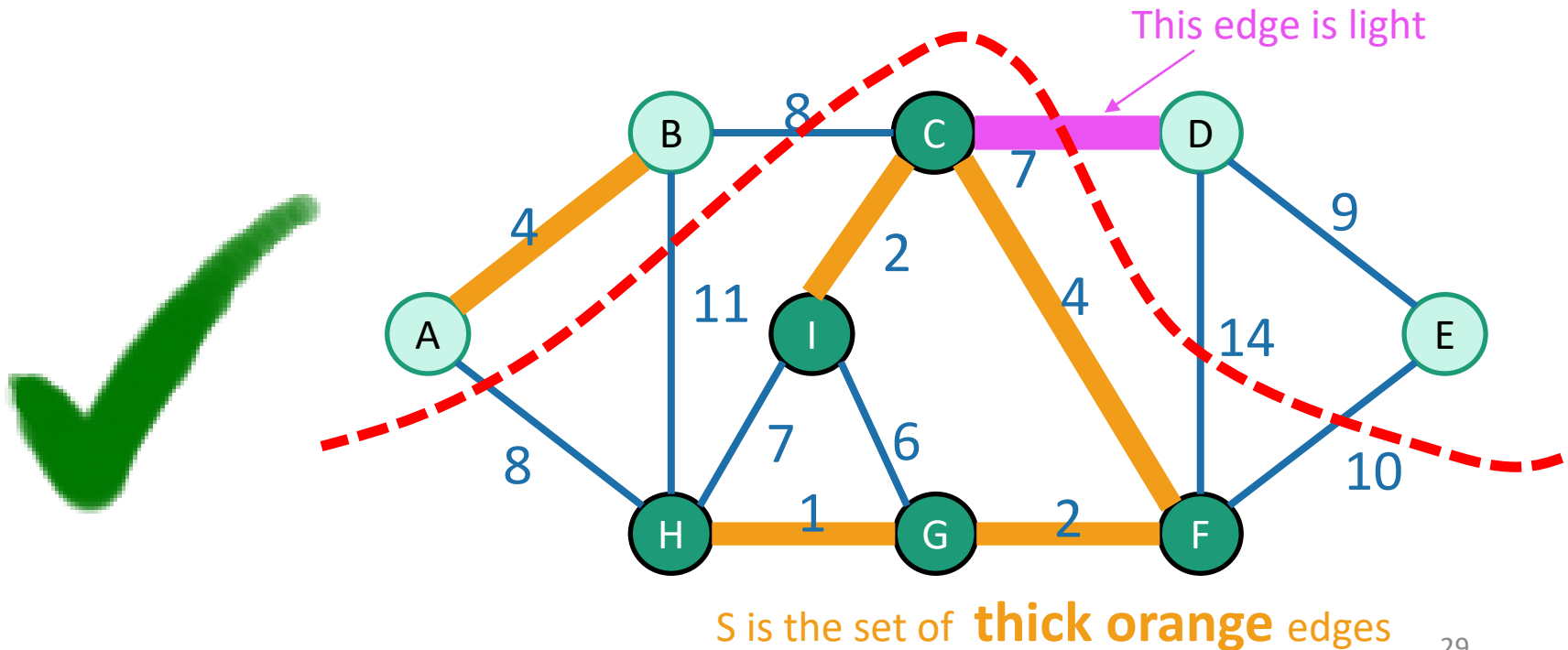
# Proof of Lemma ctd.

- Consider swapping  $\{u,v\}$  for  $\{x,y\}$  in  $\mathbf{T}$ .
  - Call the resulting tree  $\mathbf{T}'$ .
- **Claim:**  $\mathbf{T}'$  is still an MST.
  - It is still a tree:
    - we deleted  $\{x,y\}$
  - It has cost at most that of  $\mathbf{T}$ 
    - because  $\{u,v\}$  was light.
  - $\mathbf{T}$  had minimal cost.
  - So  $\mathbf{T}'$  does too.
- So  $\mathbf{T}'$  is an MST containing  $S$  and  $\{u,v\}$ .
  - This is what we wanted.



# Lemma

- Let  $S$  be a set of edges, and consider a cut that respects  $S$ .
- Suppose there is an MST containing  $S$ .
- Let  $\{u,v\}$  be a light edge.
- Then there is an MST containing  $S \cup \{u,v\}$



# End aside

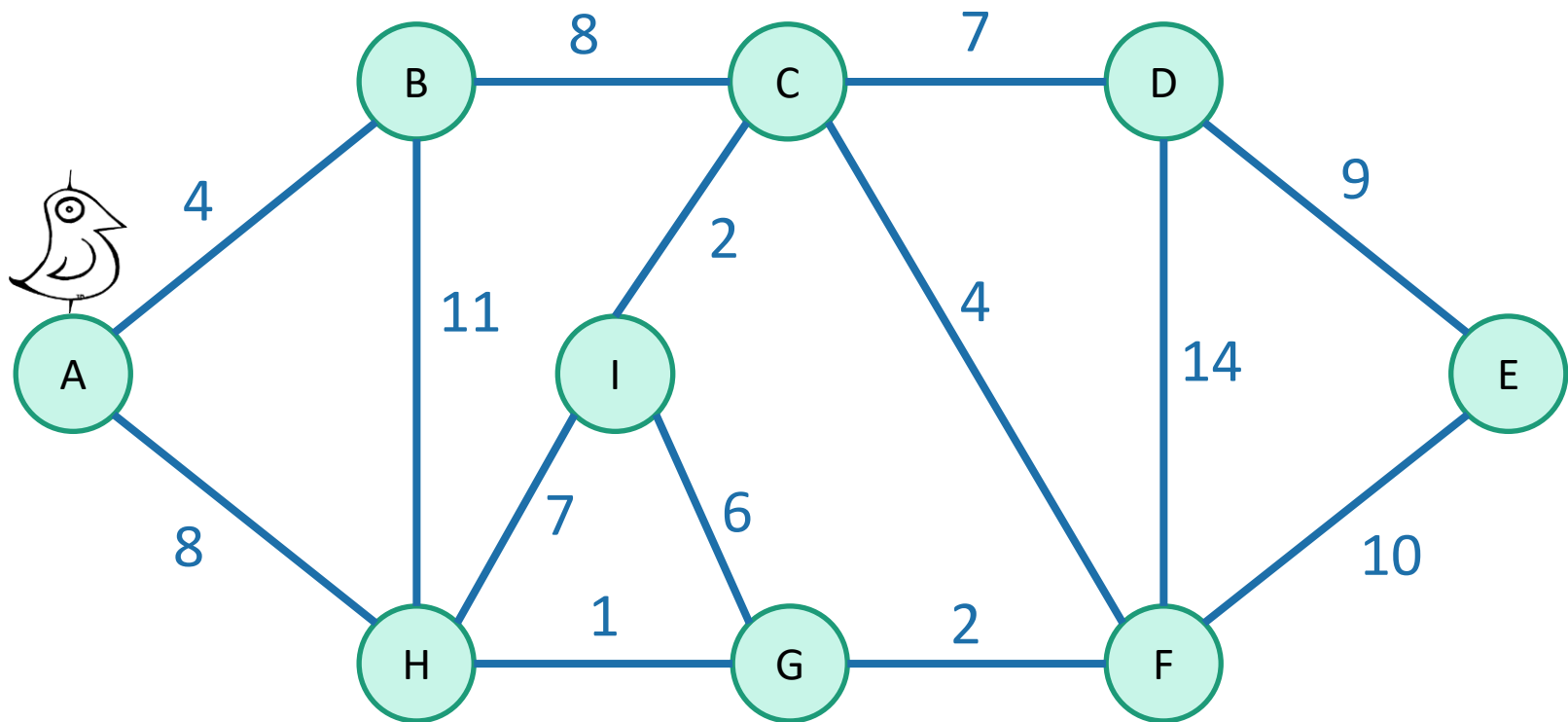
Back to MSTs!

# Back to MSTs

- How do we find one?
- Today we'll see **two greedy algorithms**.
- The strategy:
  - Make a **series of choices**, adding edges to the tree.
  - Show that each edge we add is **safe to add**:
    - we do not rule out the possibility of success
    - we will choose **light edges** crossing **cuts** and **use the Lemma**.
  - **Keep going** until we have an MST.

# Idea 1

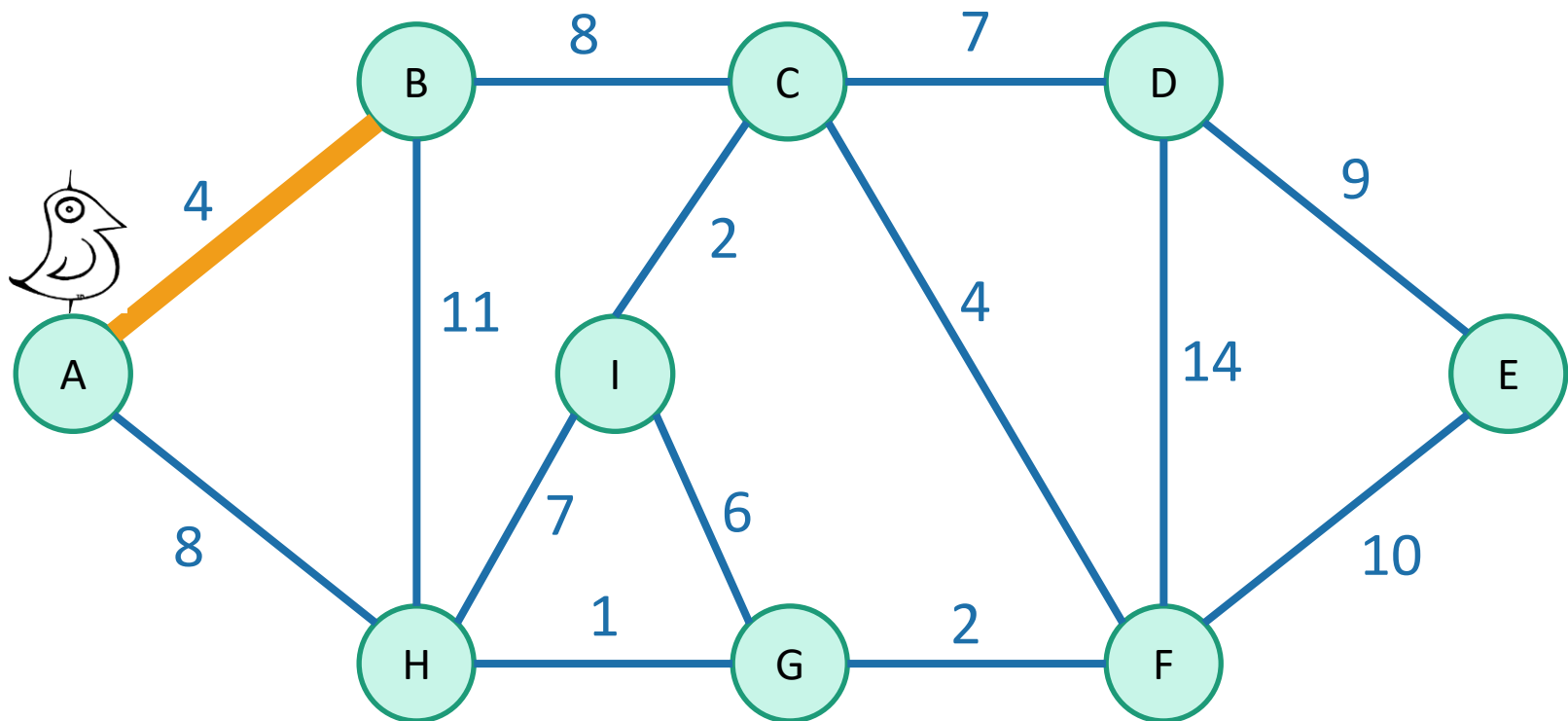
Start growing a tree, greedily add the shortest edge we can to grow the tree.





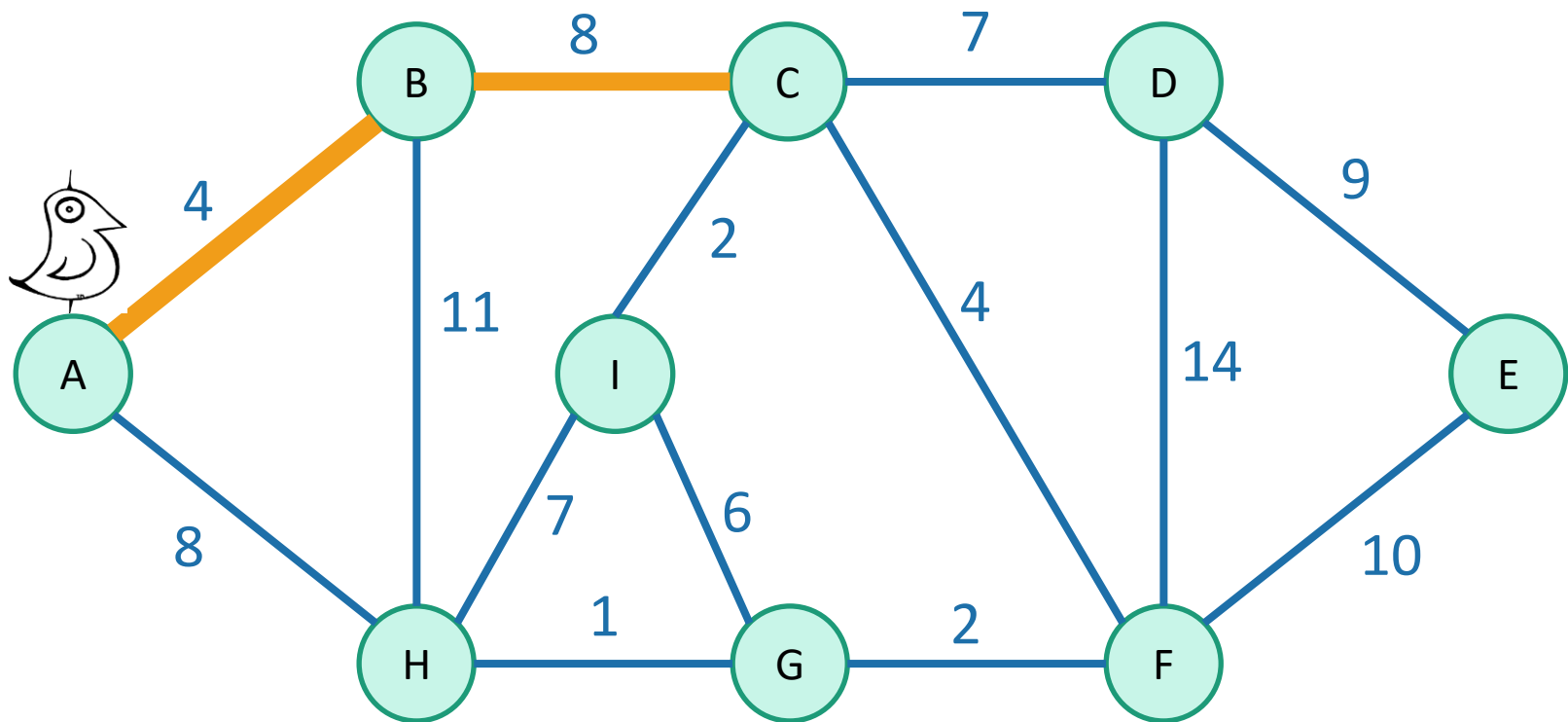
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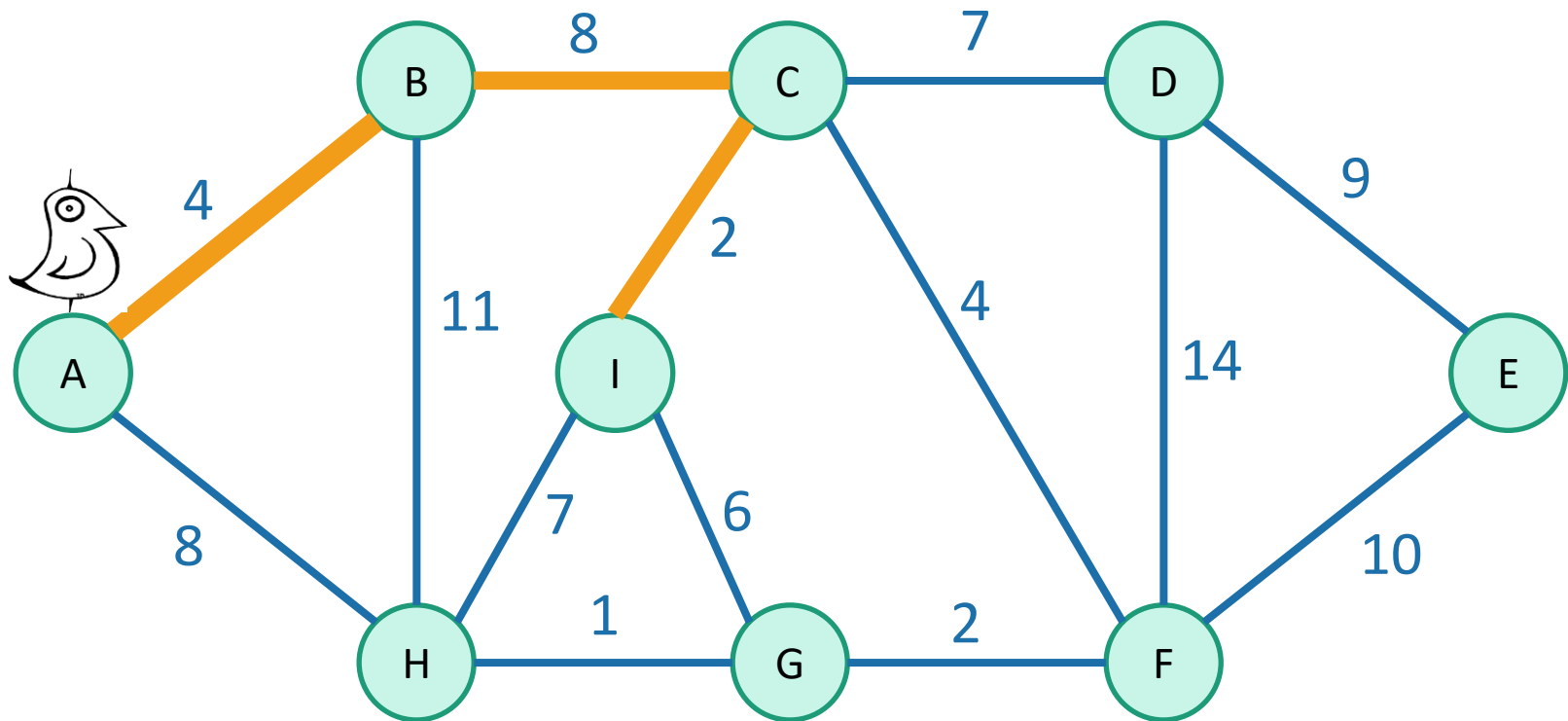
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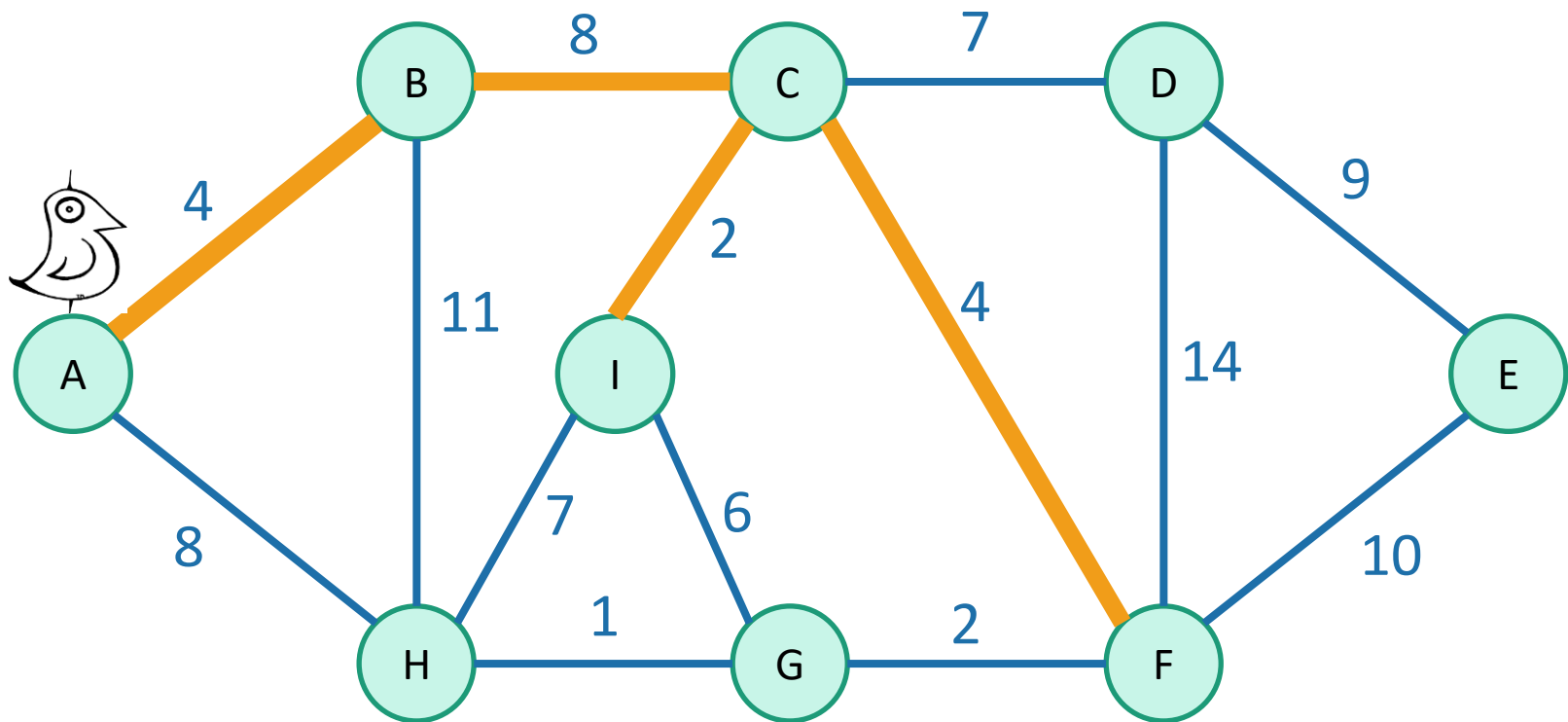
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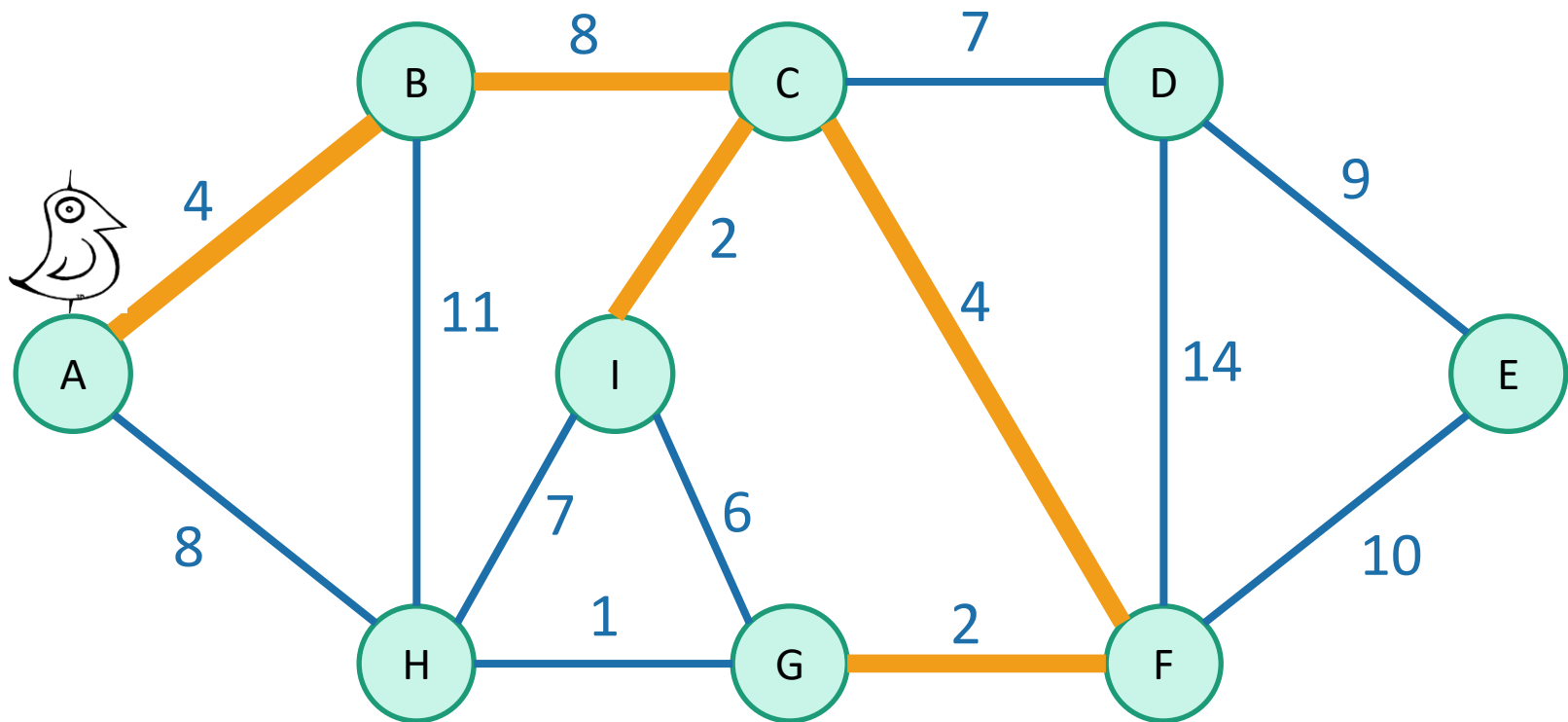
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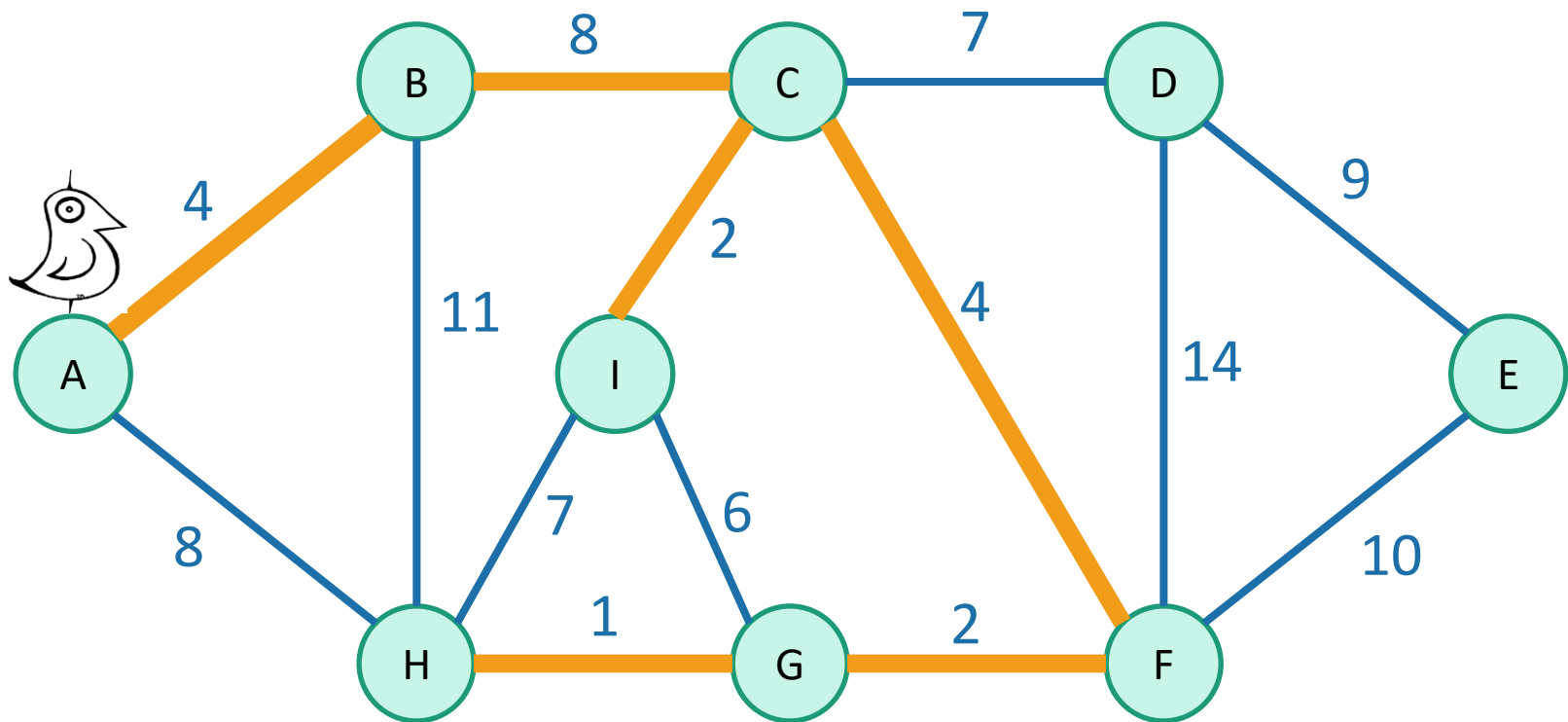
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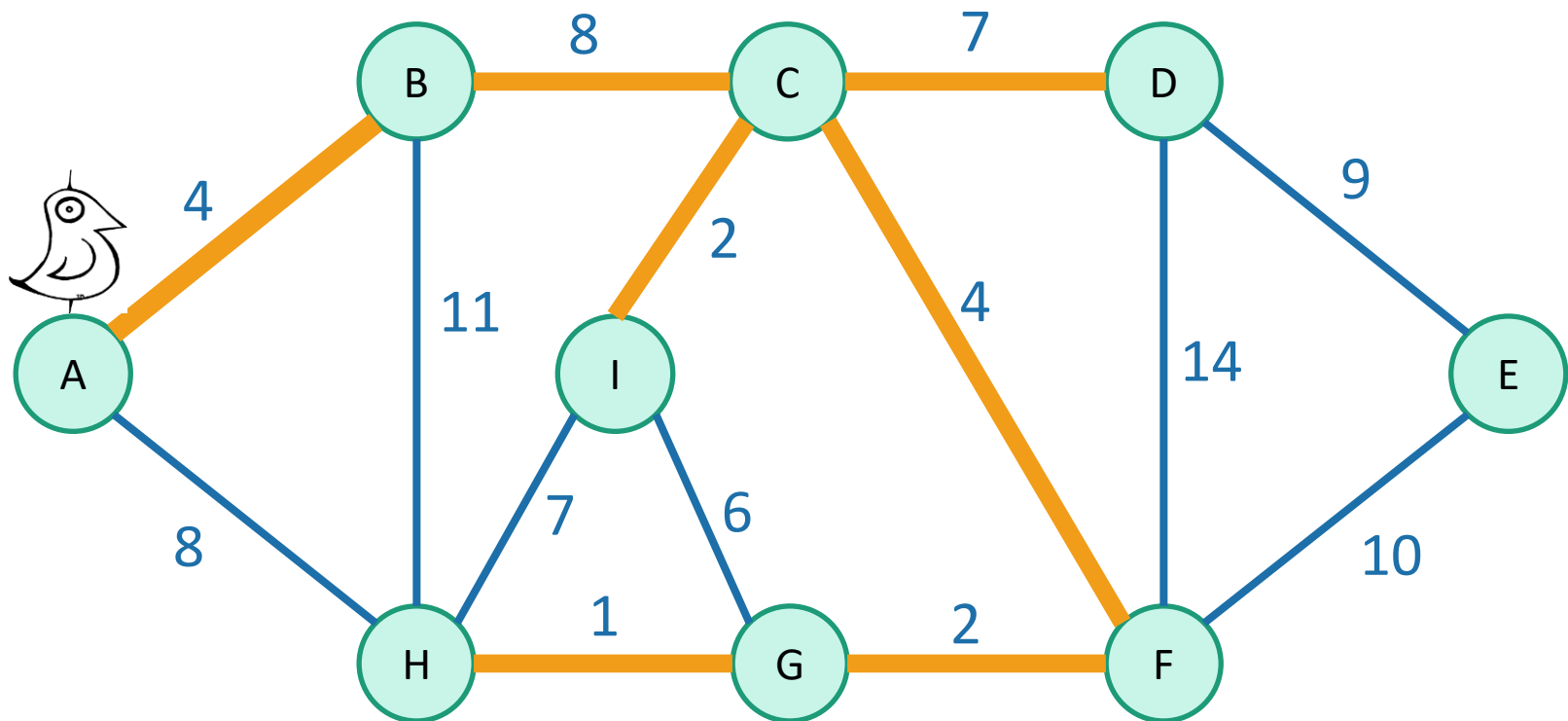
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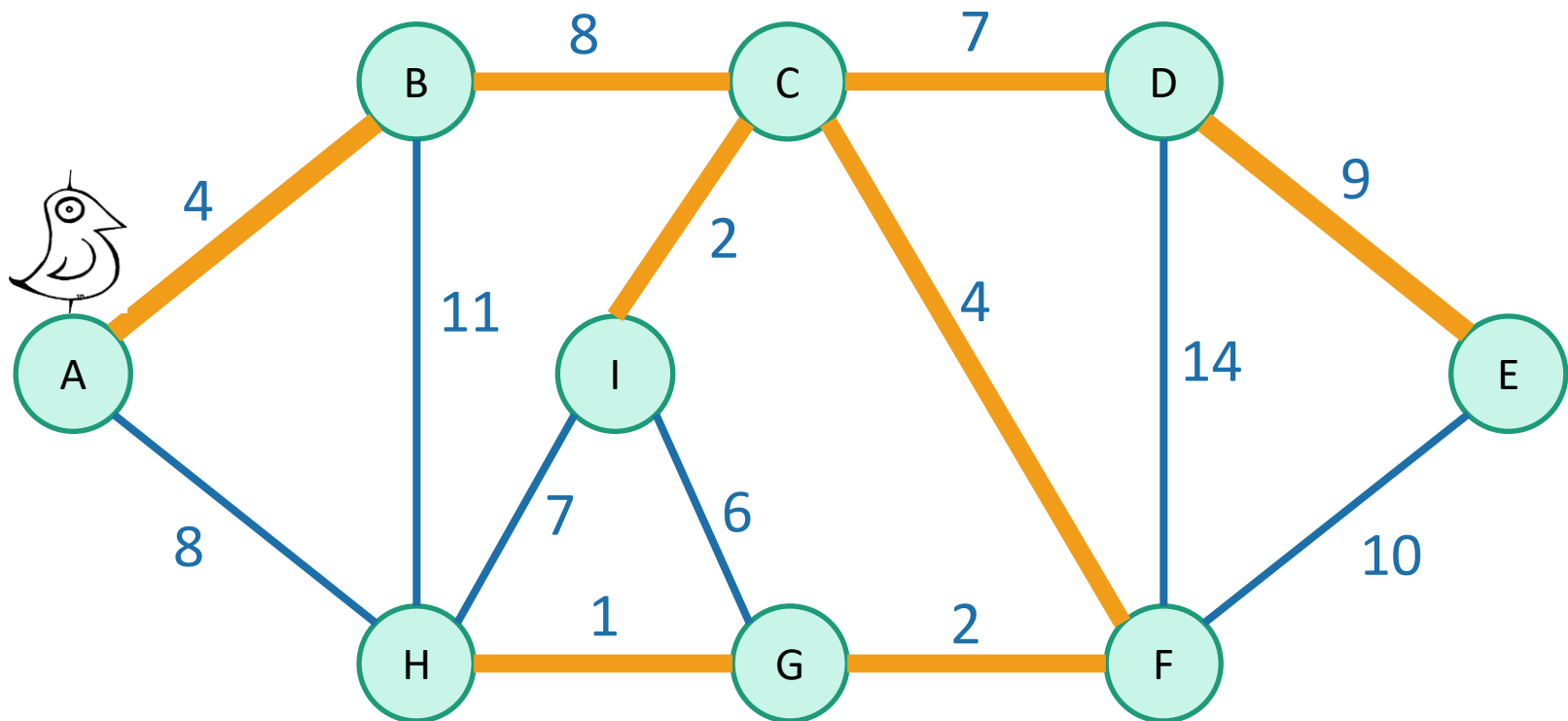
# Idea 1

Start growing a tree, greedily add the shortest edge we can to grow the tree.



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Start growing a tree, greedily add the shortest edge we can to grow the tree.





# We've discovered Prim's algorithm!

- `slowPrim( G = (V,E), starting vertex s )`:
  - Let  $(s,u)$  be the lightest edge coming out of  $s$ .
  - $MST = \{ (s,u) \}$
  - $verticesVisited = \{ s, u \}$
  - **while**  $|verticesVisited| < |V|$ :
    - find the lightest edge  $\{x,v\}$  in  $E$  so that:
      - $x$  is in  $verticesVisited$
      - $v$  is not in  $verticesVisited$
    - add  $\{x,v\}$  to  $MST$
    - add  $v$  to  $verticesVisited$
  - **return**  $MST$

*n iterations of this  
while loop.*

*Time at most  $m$  to  
go through all the  
edges and find the  
lightest.*

Naively, the running time is  $O(nm)$ :

- For each of  $n-1$  iterations of the while loop:
  - Go through all the edges.

# Two questions

## 1. Does it work?

- That is, does it actually return a MST?

## 2. How do we actually implement this?

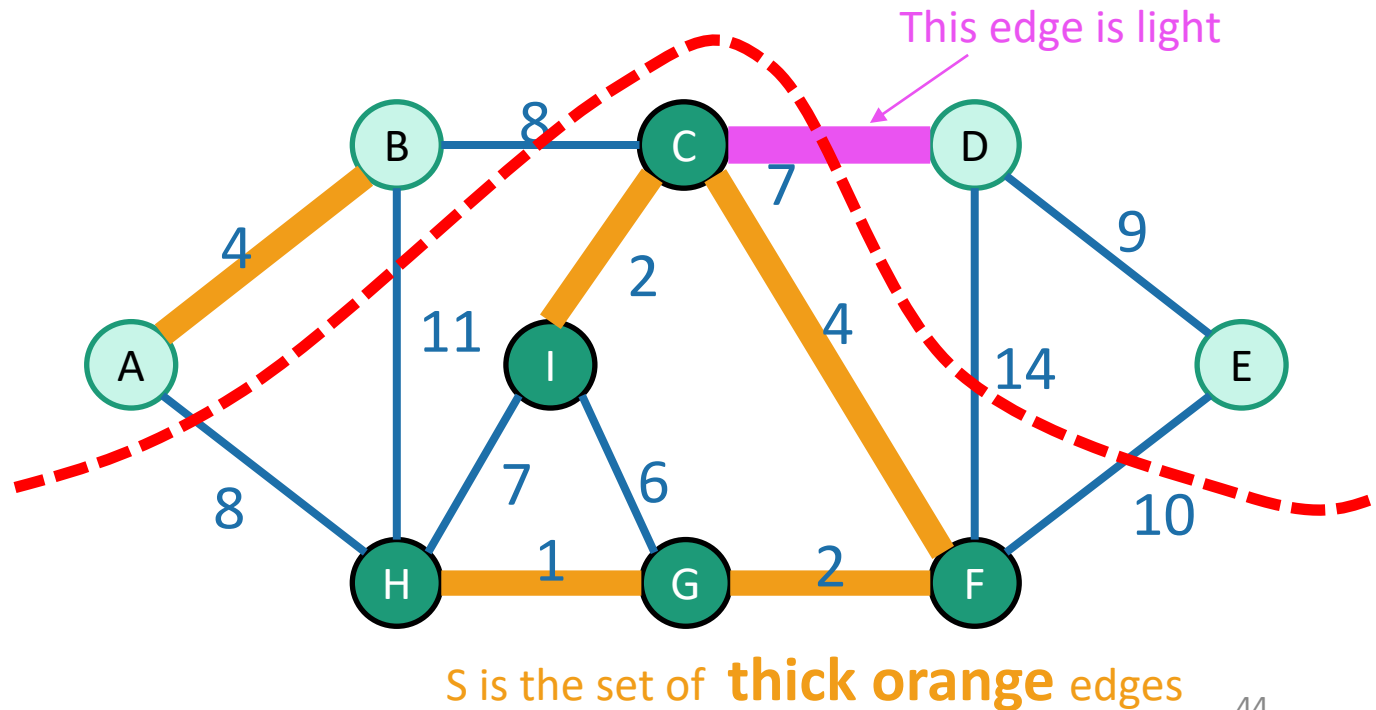
- the pseudocode above says “slowPrim”...

# Does it work?

- We need to show that our greedy choices **don't rule out success**.
- That is, at every step:
  - If there exists an MST that contains all of the edges  $S$  we have added so far...
  - ...then when we make our next choice  $\{u,v\}$ , there is still an MST containing  $S$  and  $\{u,v\}$ .
- Now it is time to use our lemma!

# Lemma

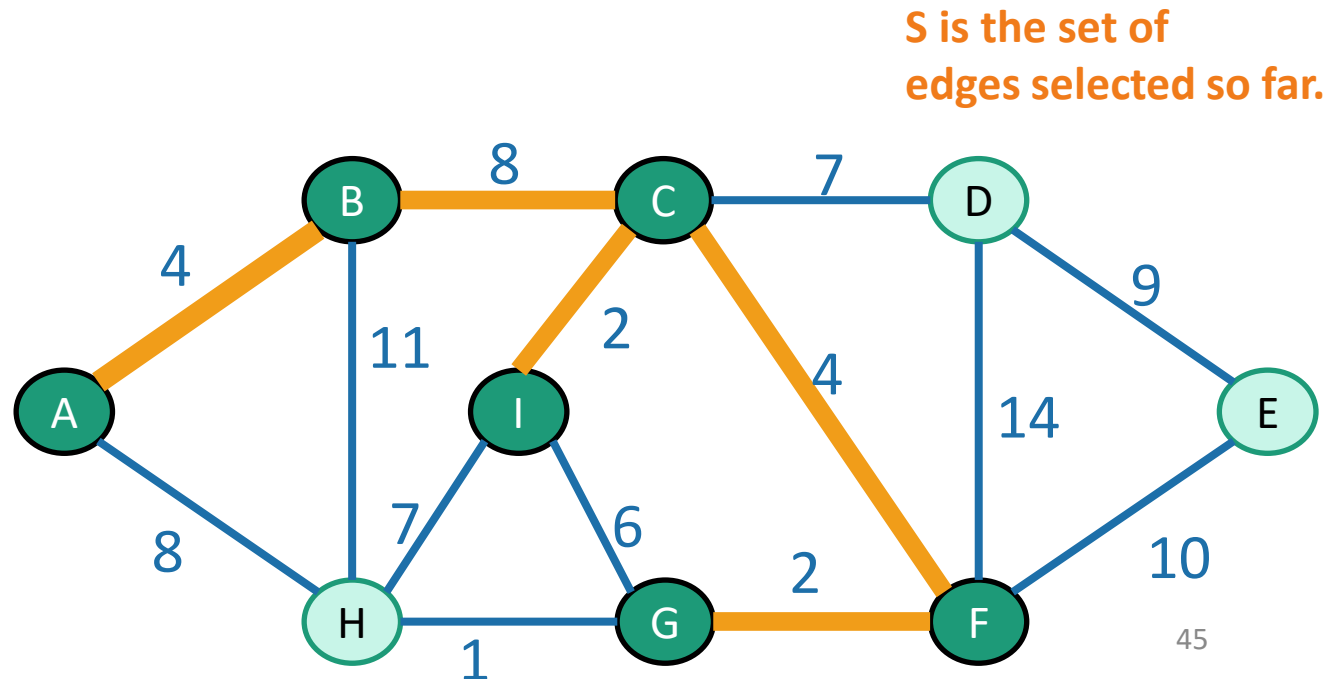
- Let  $S$  be a set of edges, and consider a cut that respects  $S$ .
- Suppose there is an MST containing  $S$ .
- Let  $\{u,v\}$  be a light edge.
- Then there is an MST containing  $S \cup \{u,v\}$



# Partway through Prim

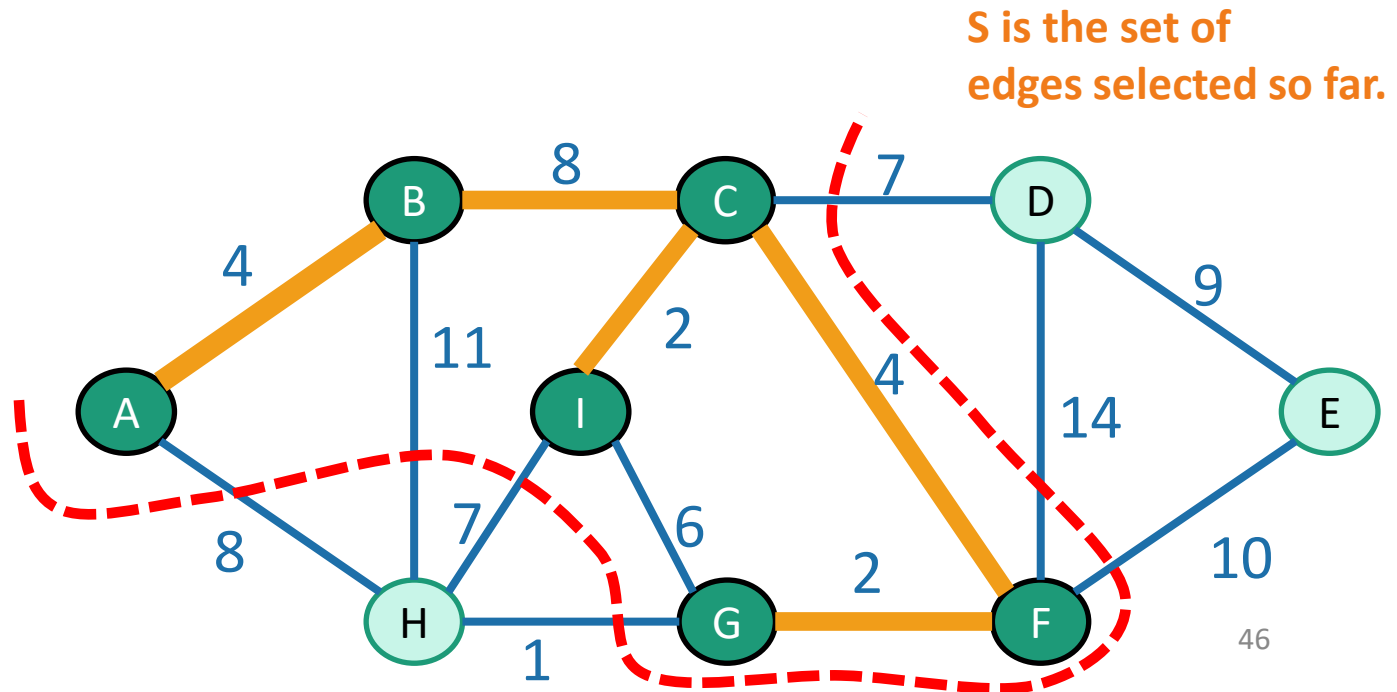
- Assume that our choices **S** so far don't rule out success
  - There is an MST extending them

How can we use our lemma to show that our next choice also does not rule out success?



# Partway through Prim

- Assume that our choices **S** so far don't rule out success
  - There is an MST extending them
- Consider the cut **{visited, unvisited}**
  - This cut respects S.

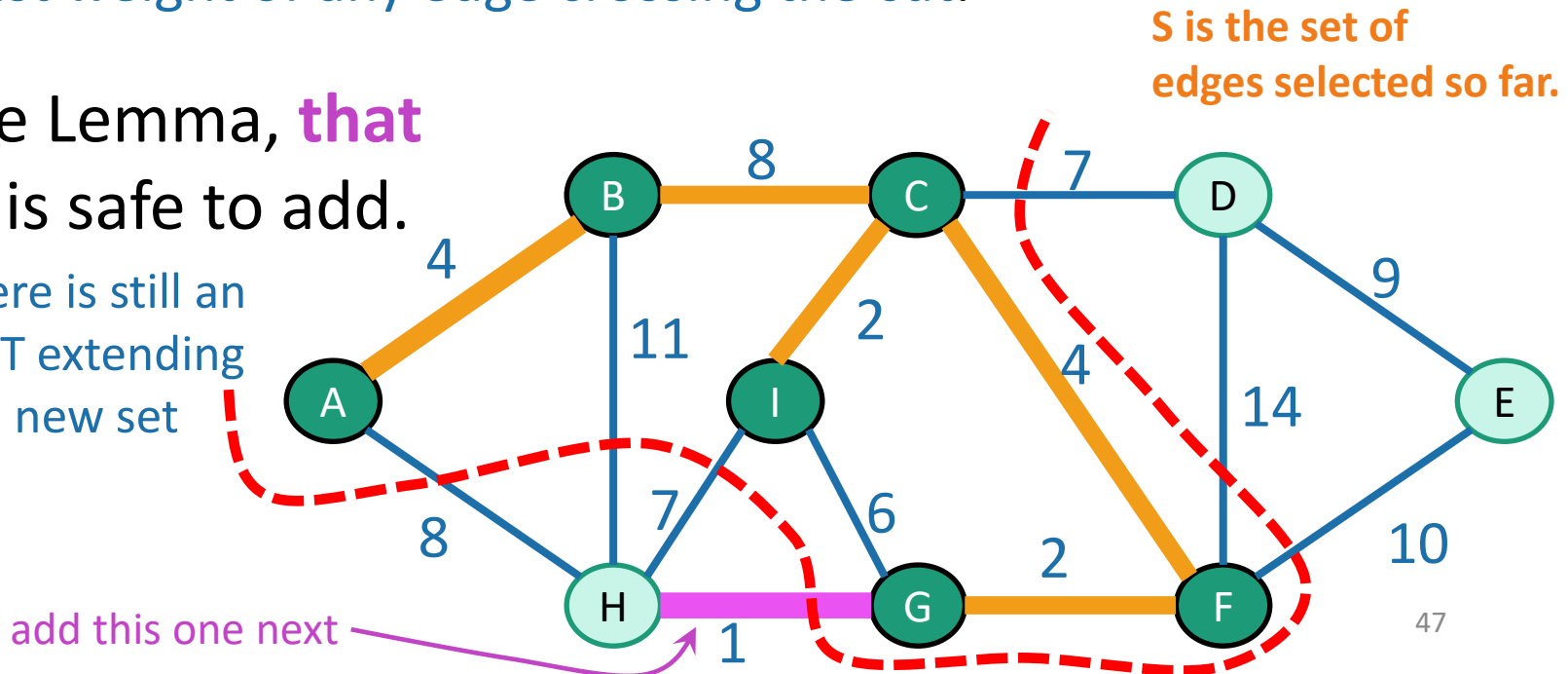


# Partway through Prim

- Assume that our choices **S** so far don't rule out success
  - There is an MST extending them
- Consider the cut **{visited, unvisited}**
  - This cut respects S.
- The edge we add next is a **light edge**.
  - Least weight of any edge crossing the cut.

- By the Lemma, **that edge** is safe to add.

- There is still an MST extending the new set



# Good news

- Our greedy choices **don't rule out success.**
- This is enough (along with an argument by induction) to guarantee correctness of Prim's algorithm.



# Formally(ish)

- Inductive hypothesis:
  - After adding the  $t$ 'th edge, there exists an MST with the edges added so far.
- Base case:
  - After adding the 0'th edge, there exists an MST with the edges added so far. **YEP.**
- Inductive step:
  - If the inductive hypothesis holds for  $t$  (aka, the choices so far are safe), then it holds for  $t+1$  (aka, the next edge we add is safe).
  - **That's what we just showed.**
- Conclusion:
  - After adding the  $n-1$ 'st edge, there exists an MST with the edges added so far.
  - At this point we have a spanning tree, so it better be minimal.

# Two questions

1. Does it work?

- That is, does it actually return a MST?

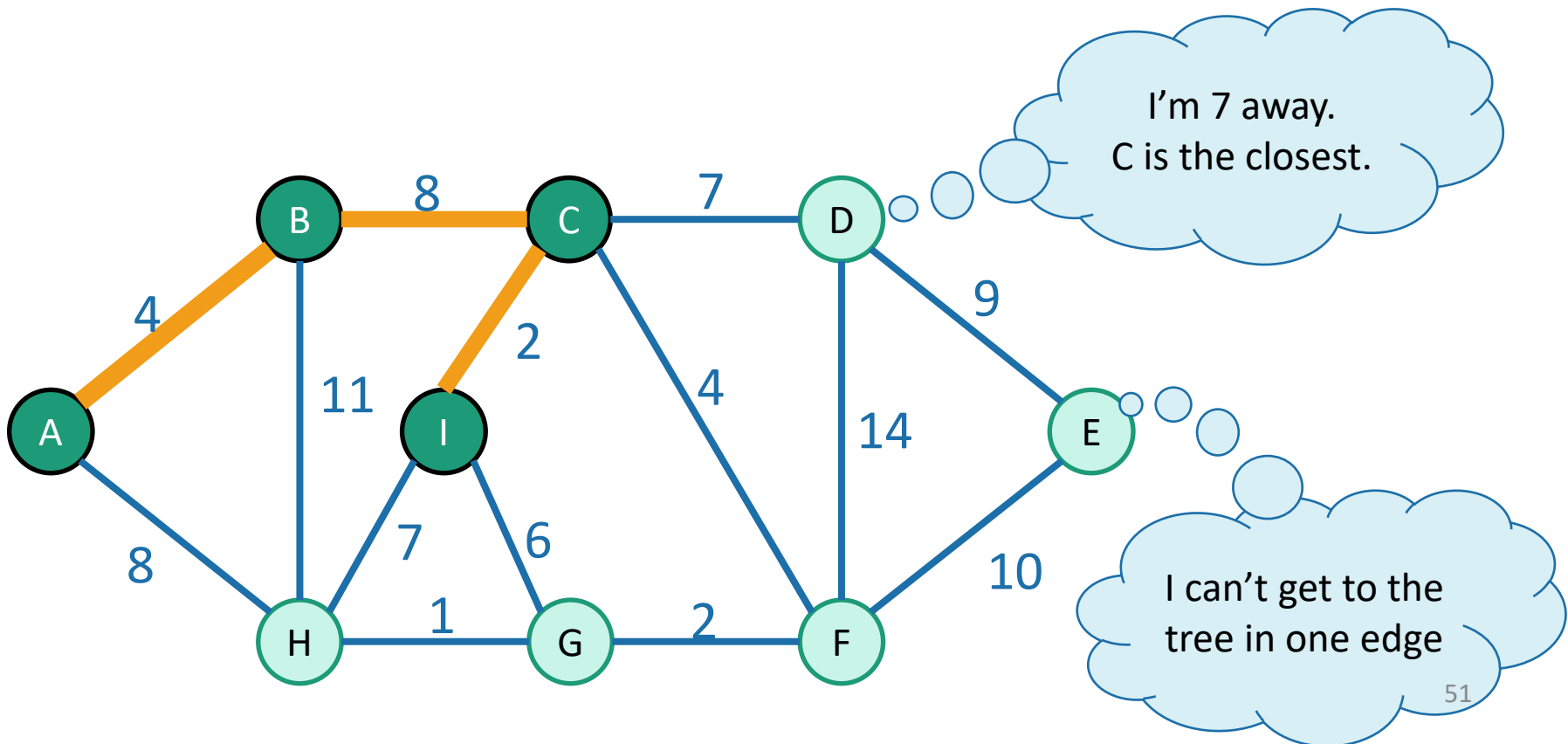
- **Yes!**

2. How do we actually implement this?

- the pseudocode above says “slowPrim”...

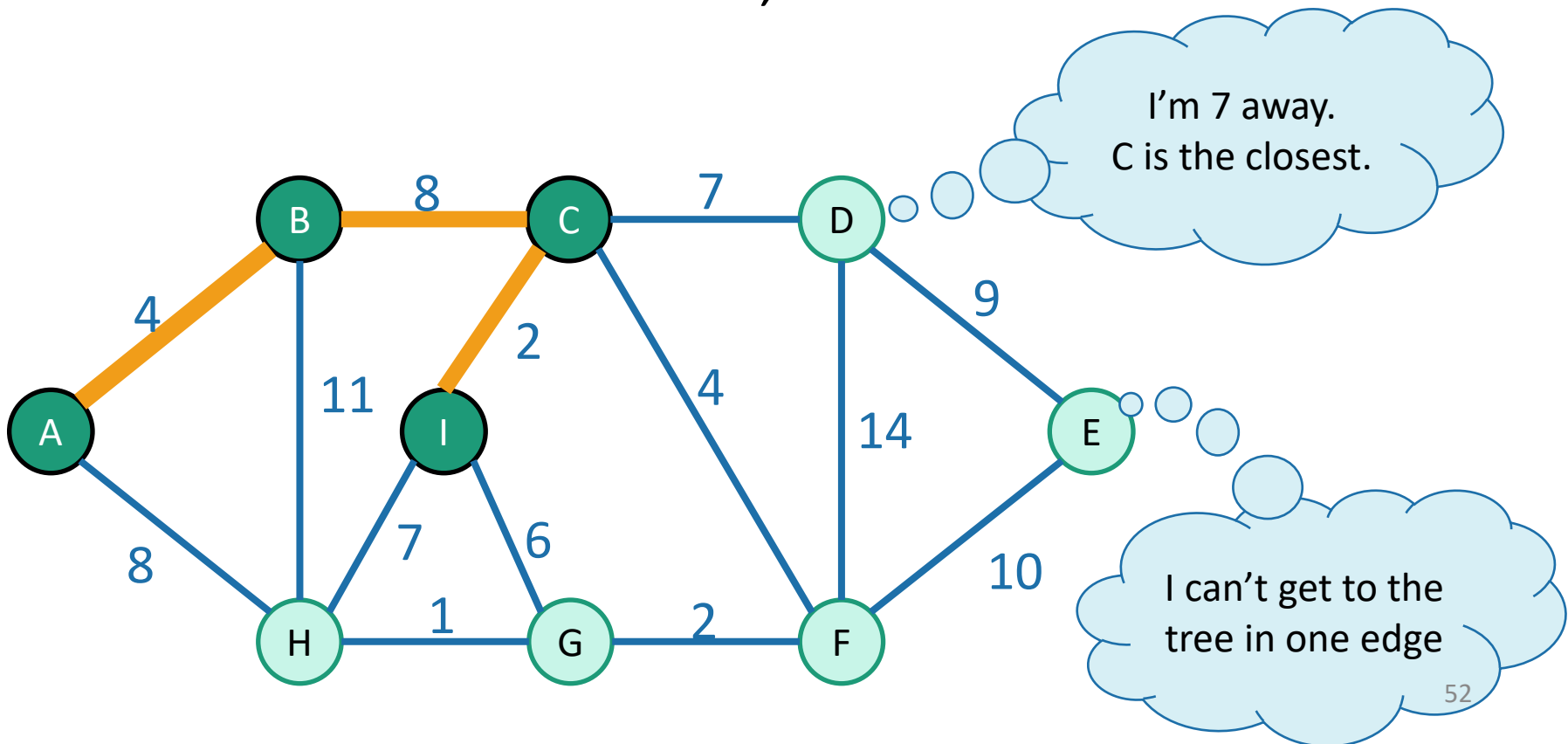
# How do we actually implement this?

- Each vertex keeps:
  - the **distance** from itself to the **growing spanning tree**
  - **how to get there.** if you can get there in one edge.



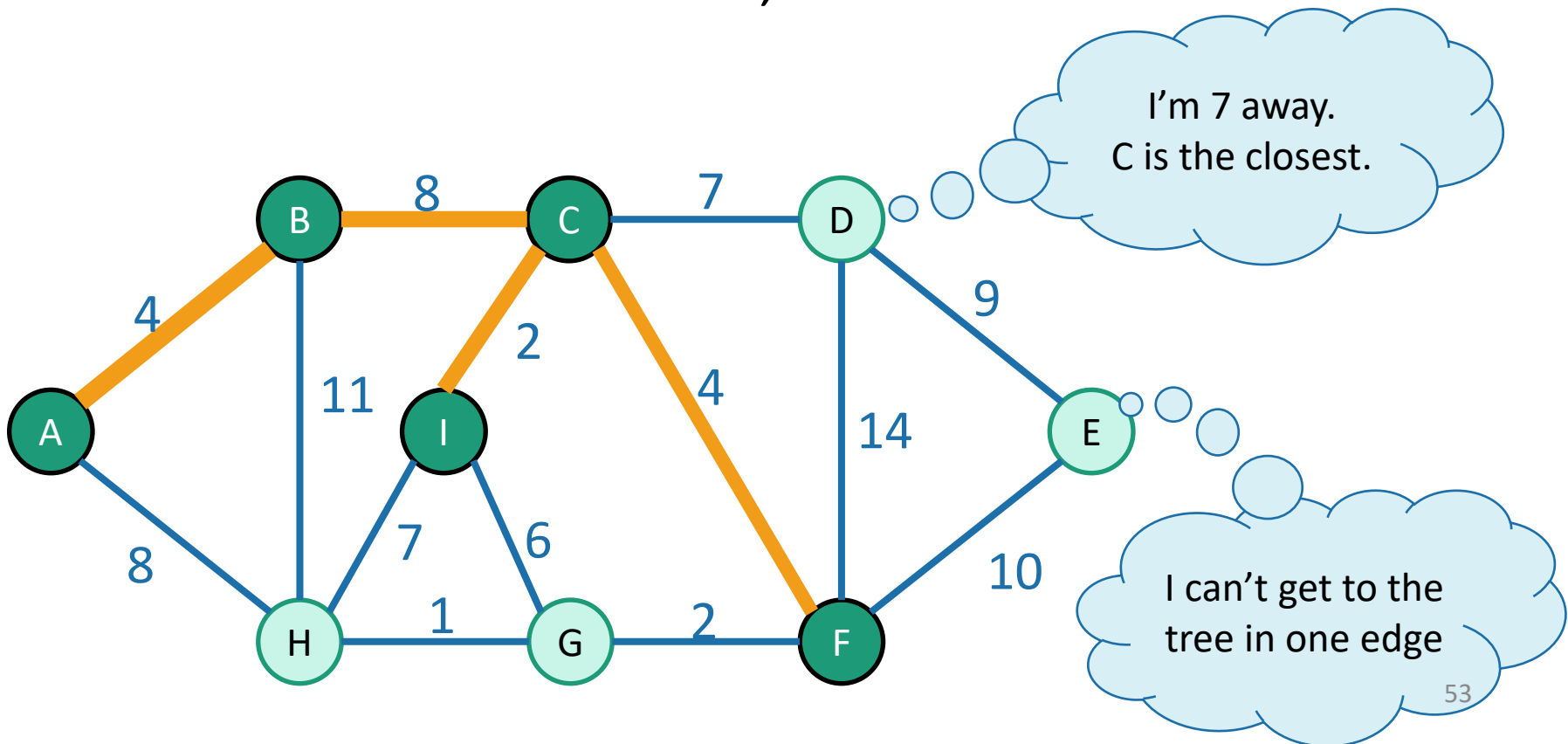
# How do we actually implement this?

- Each vertex keeps:
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- Choose the closest vertex, add it.



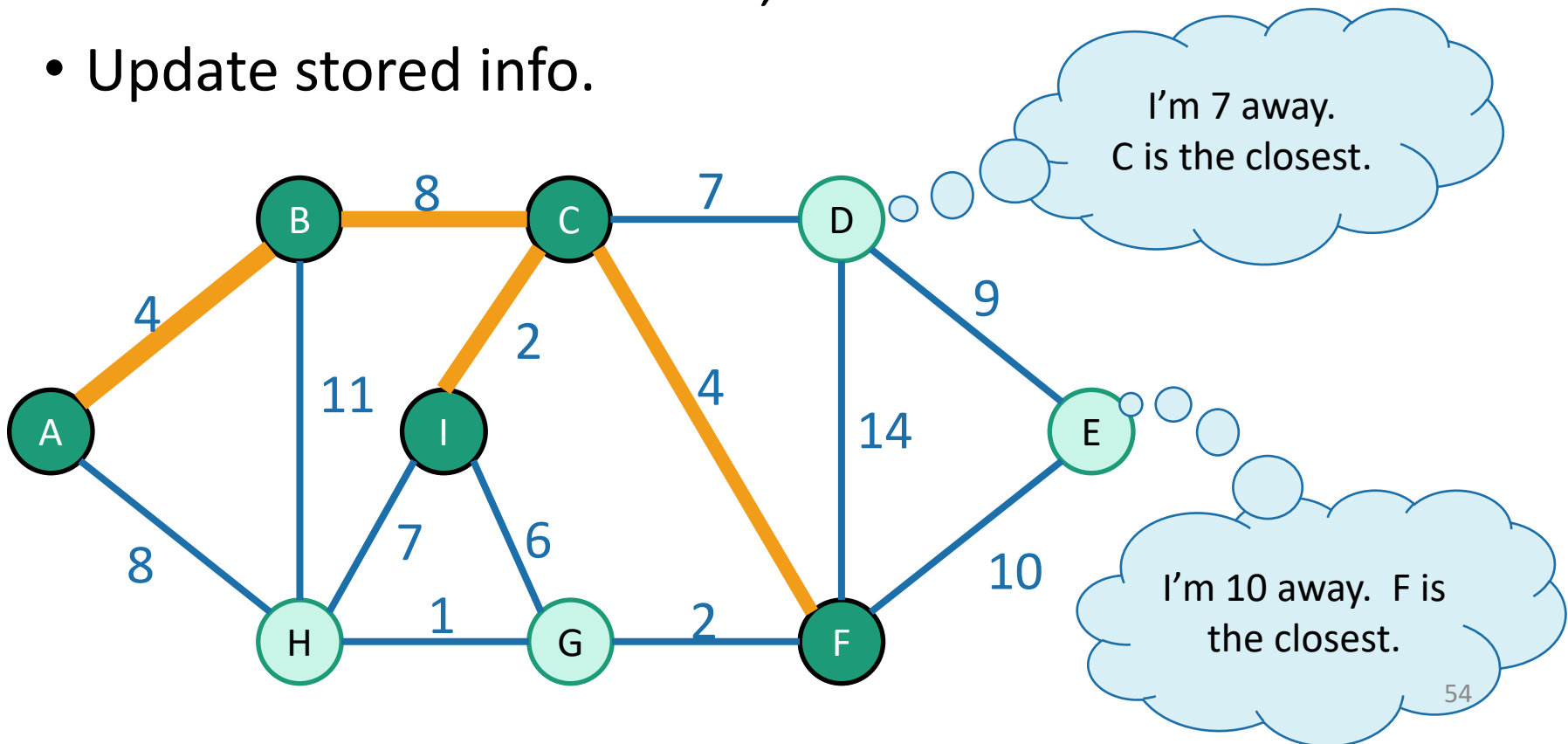
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# How do we actually implement this?

- Each vertex keeps:
  - the **distance** from itself to the **growing spanning tree**
  - **how to get there.** if you can get there in one edge.
- Choose the closest vertex, add it.
- Update stored info.



# Efficient implementation

Every vertex has a key and a parent

Until all the vertices are **reached**:



Can't reach x yet

x is "active"

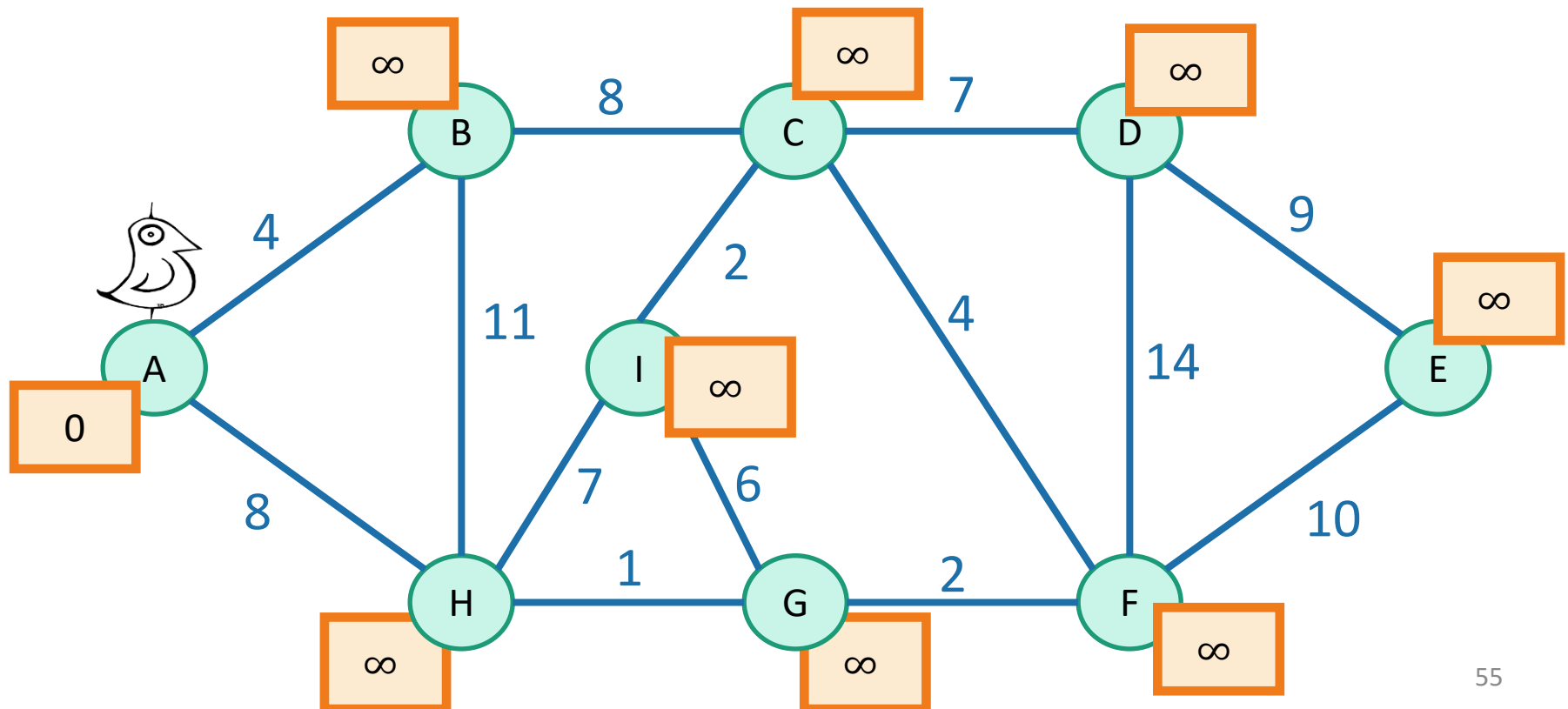
Can reach x



$k[x]$  is the distance of x from the growing tree



$p[b] = a$ , meaning that a was the vertex that  $k[b]$  comes from.



# Efficient implementation

Every vertex has a key and a parent

Until all the vertices are **reached**:

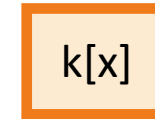
- Activate the **unreached** vertex  $u$  with the **smallest key**.



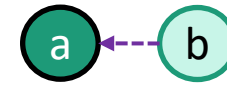
Can't reach  $x$  yet

$x$  is "active"

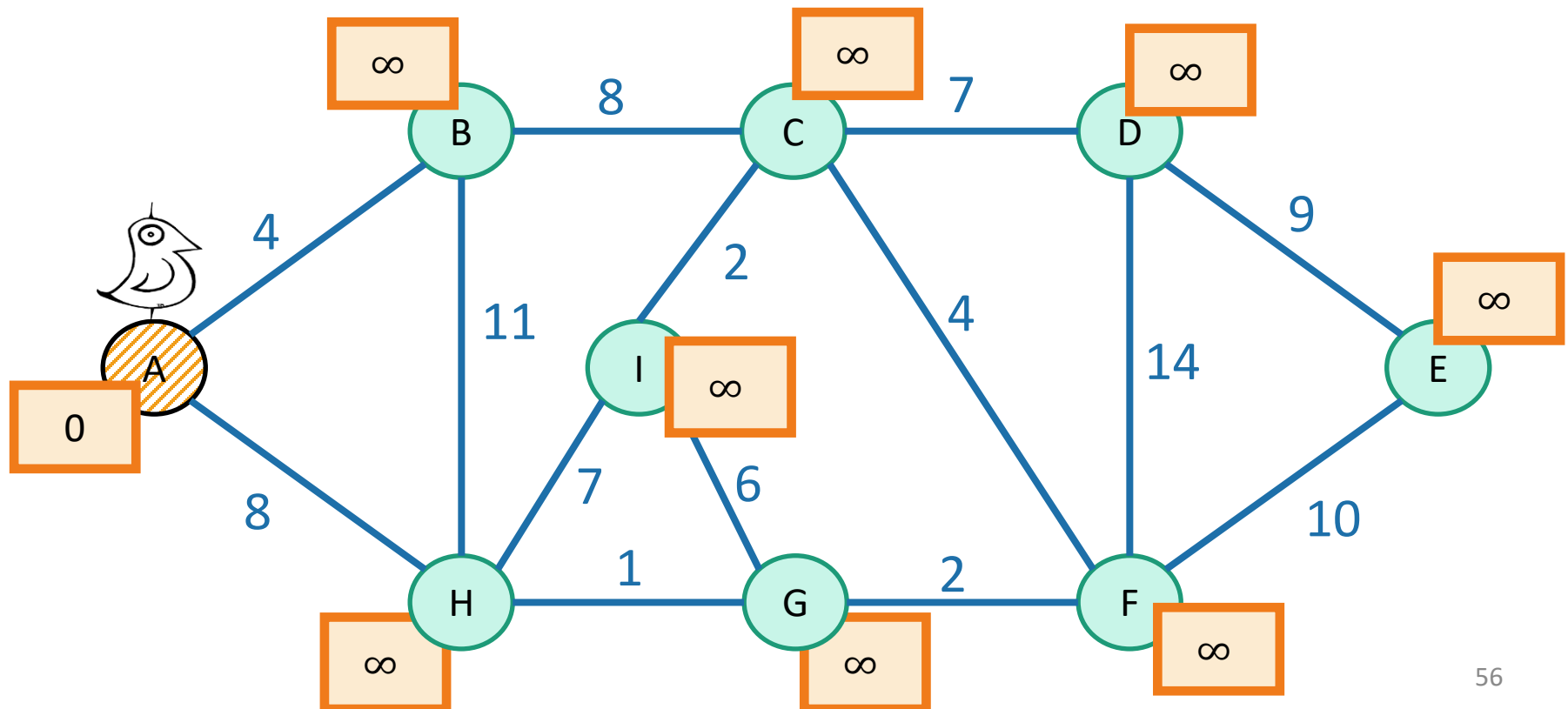
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# Efficient implementation

Every vertex has a key and a parent

Until all the vertices are **reached**:

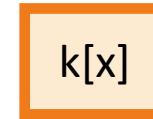
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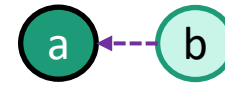
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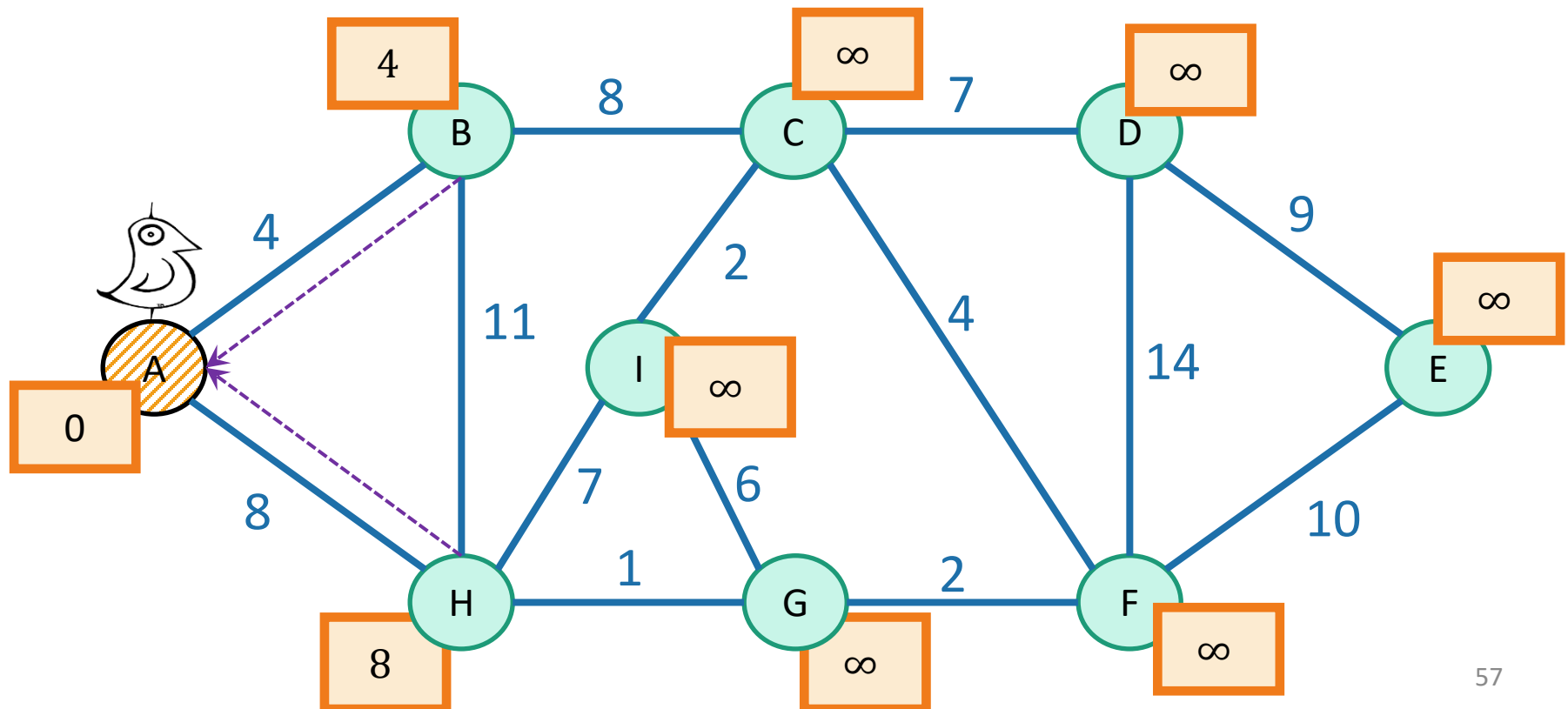
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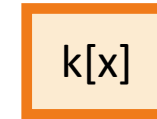
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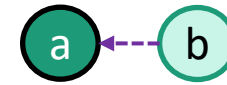
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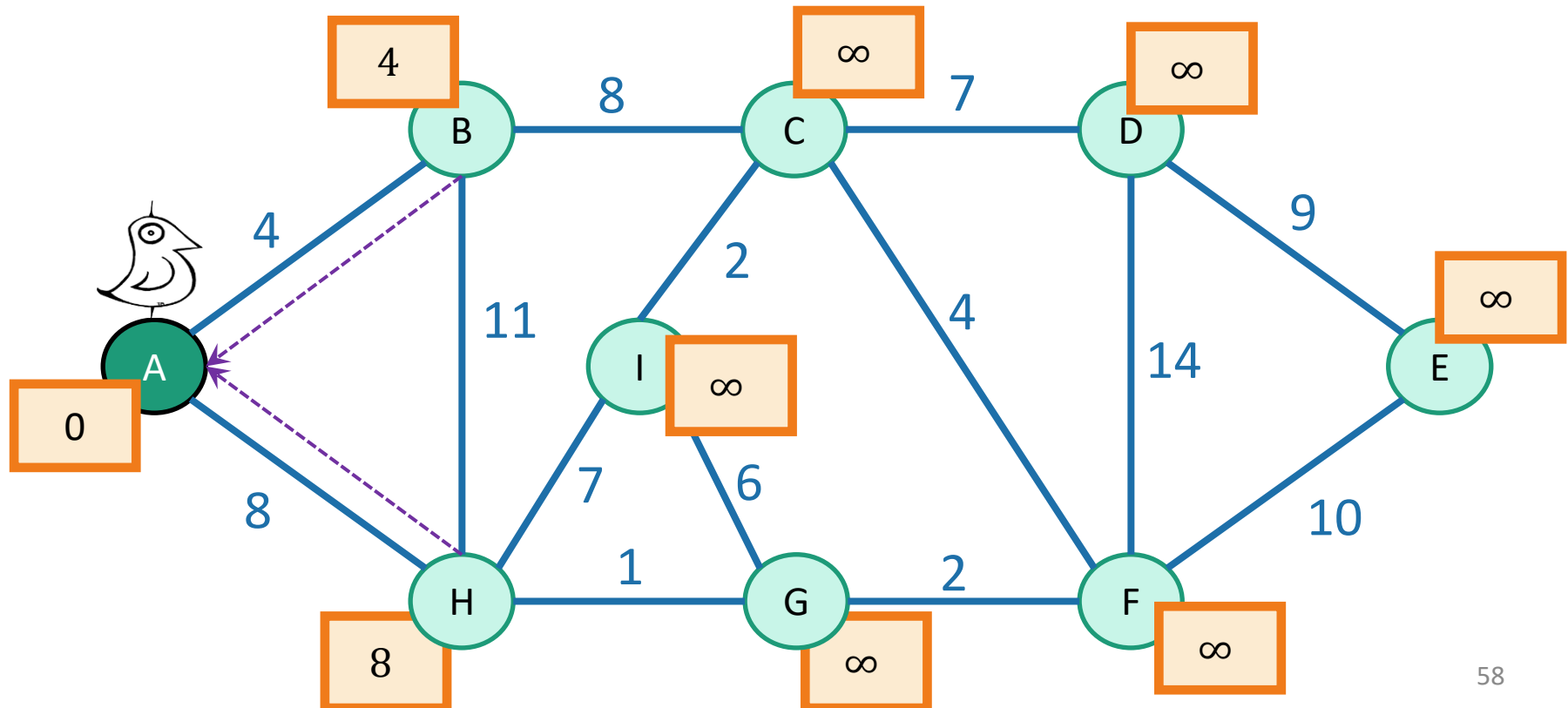
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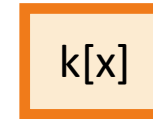
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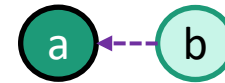
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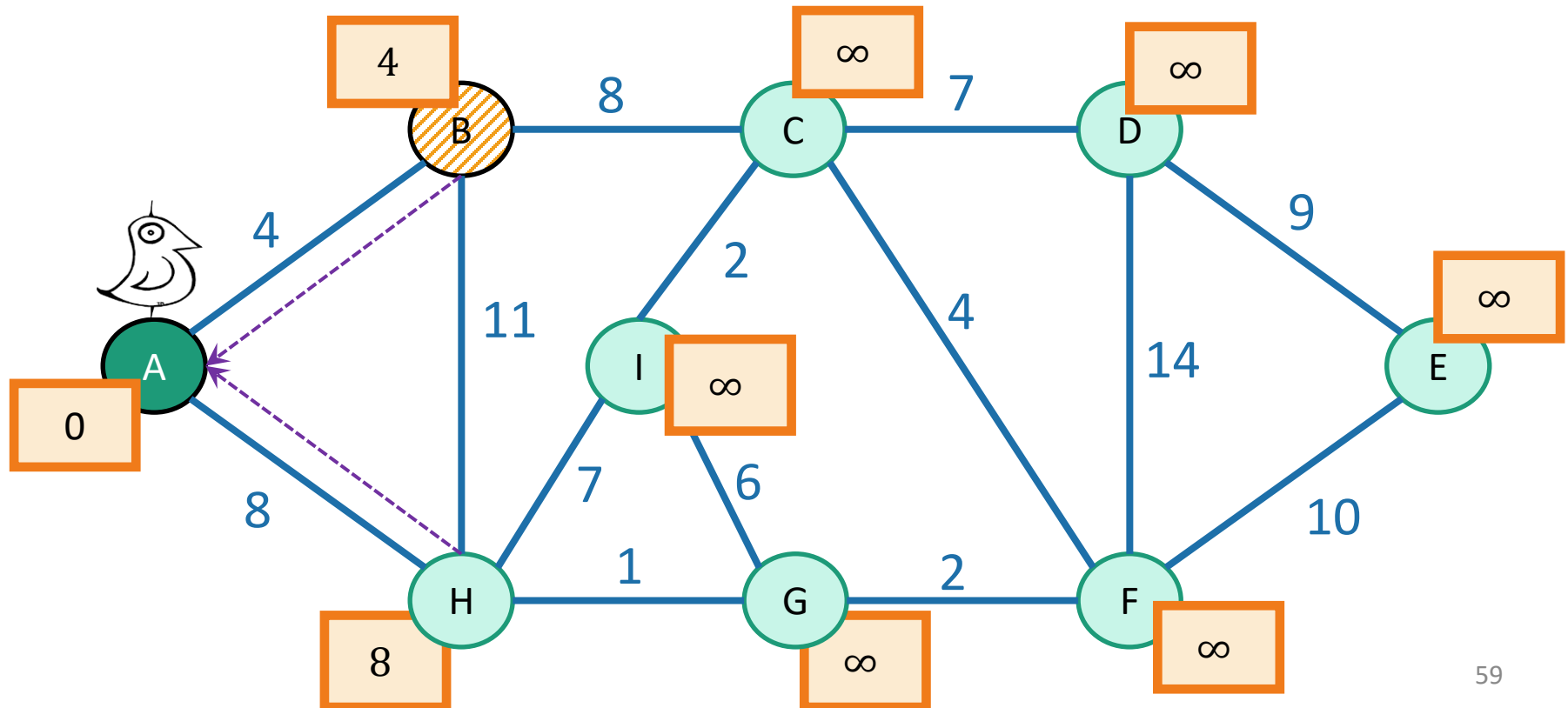
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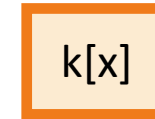
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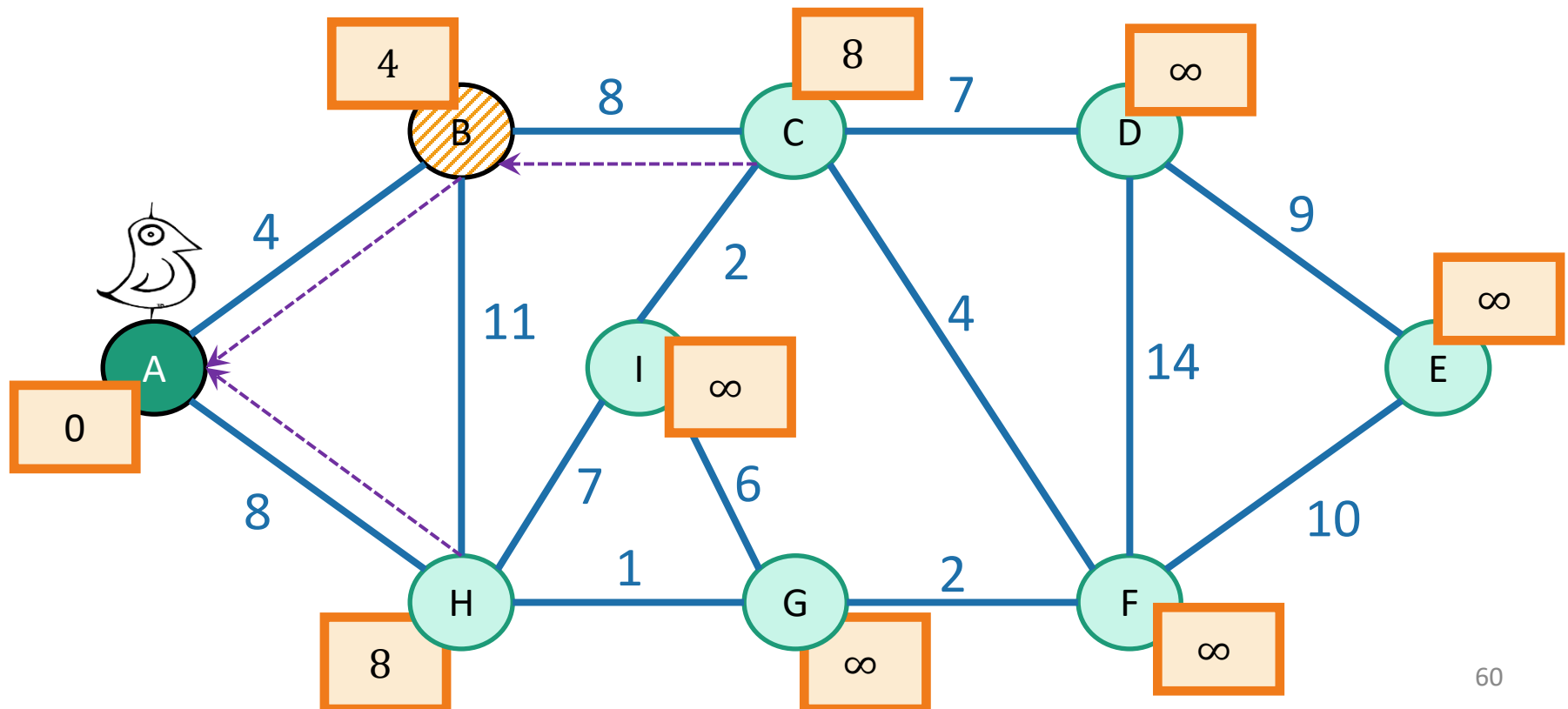
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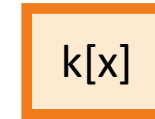
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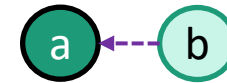
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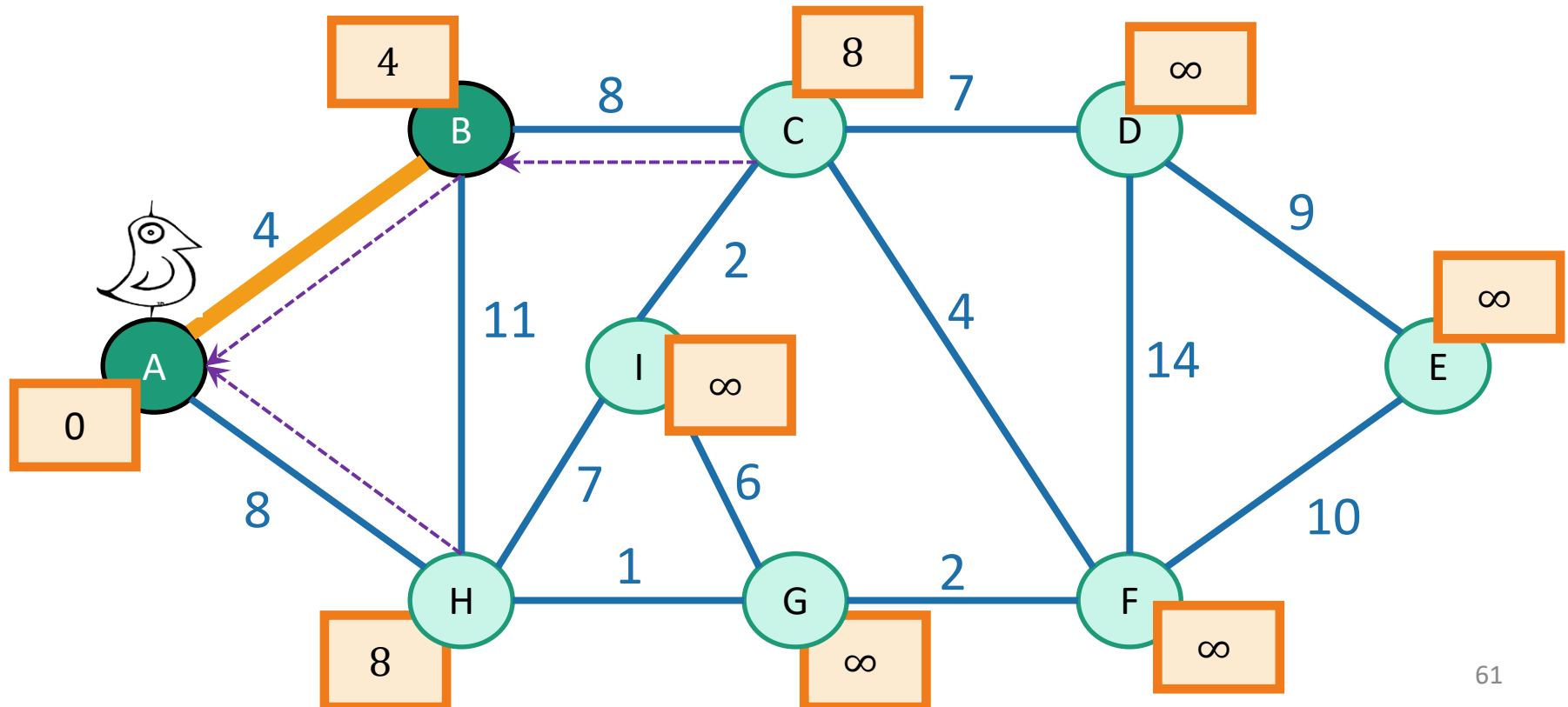
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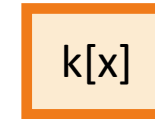
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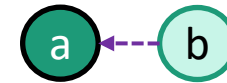
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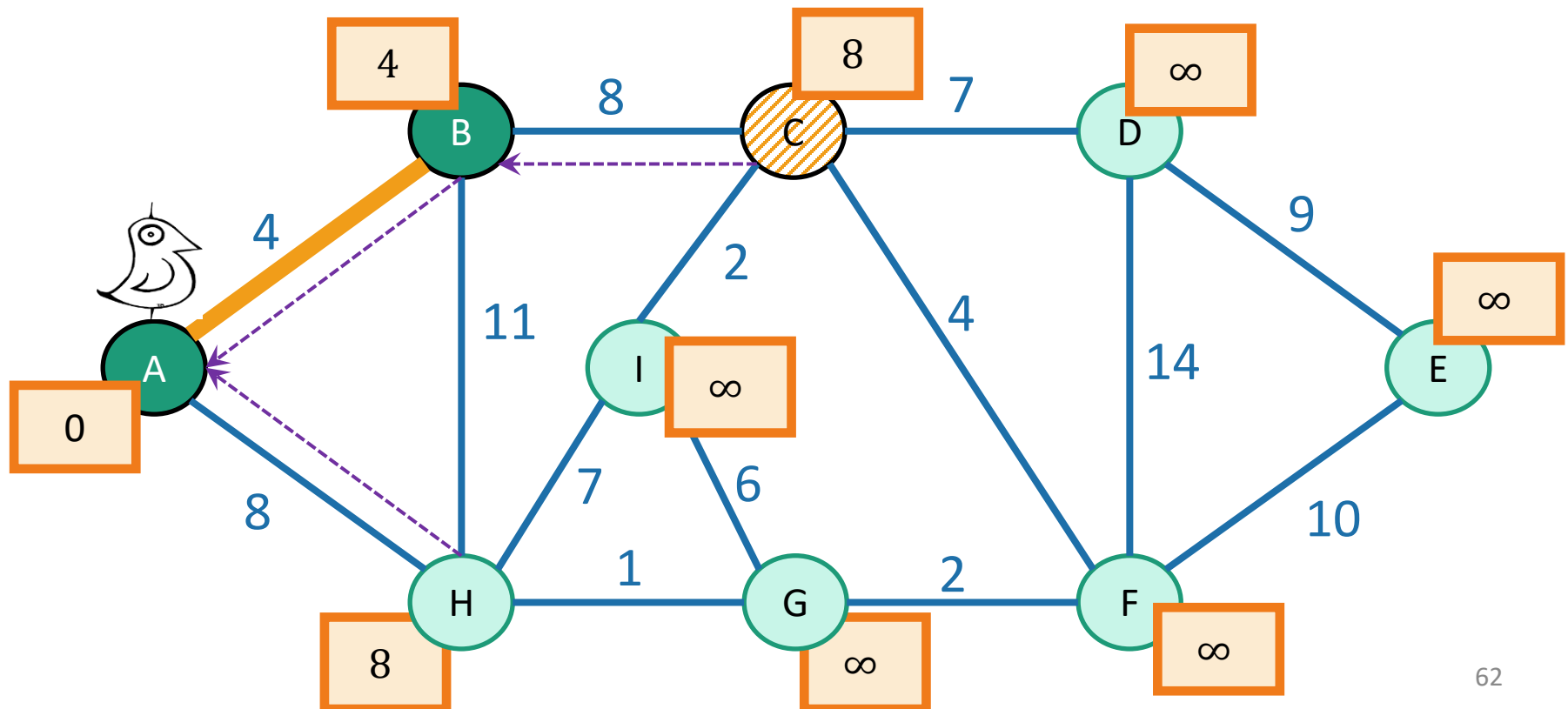
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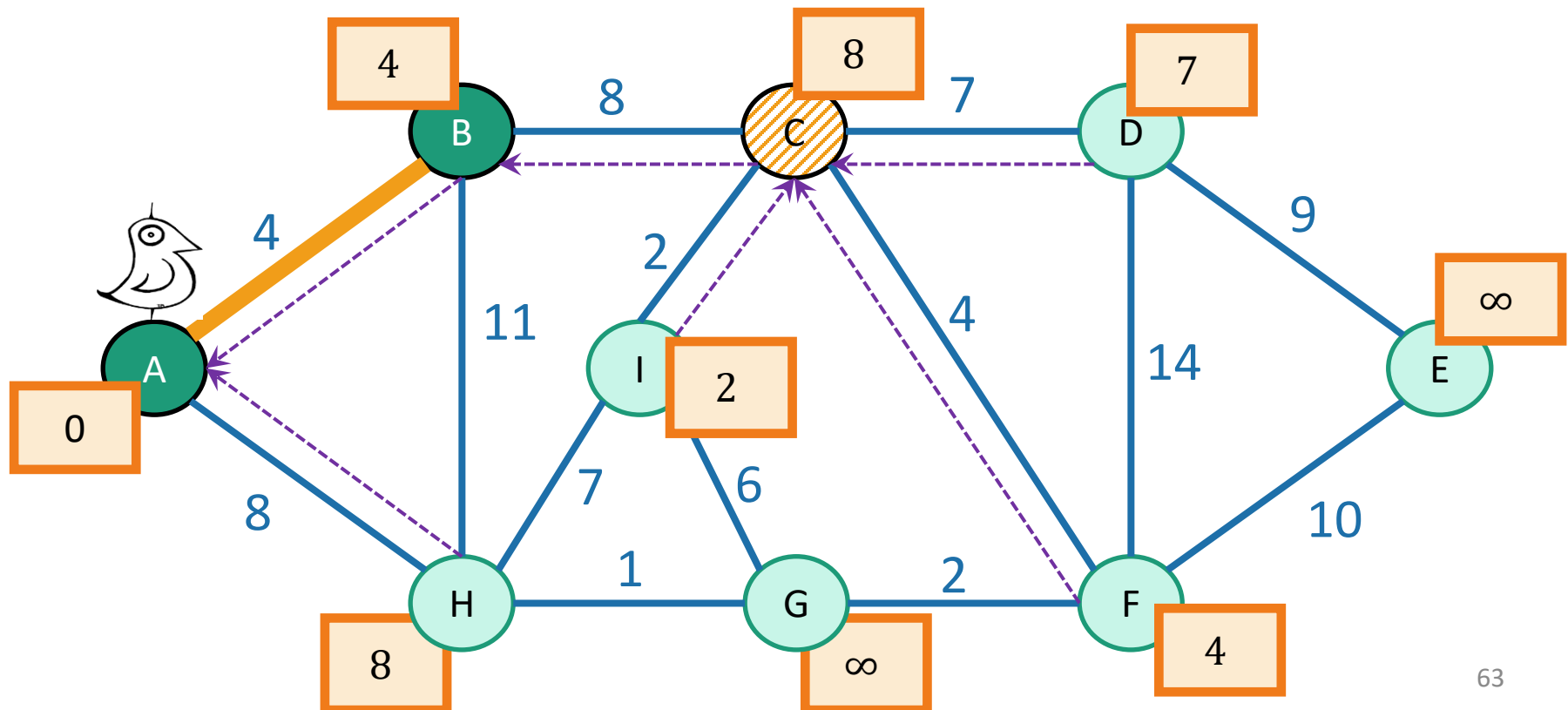
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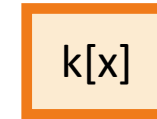
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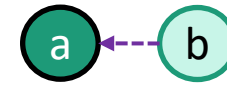
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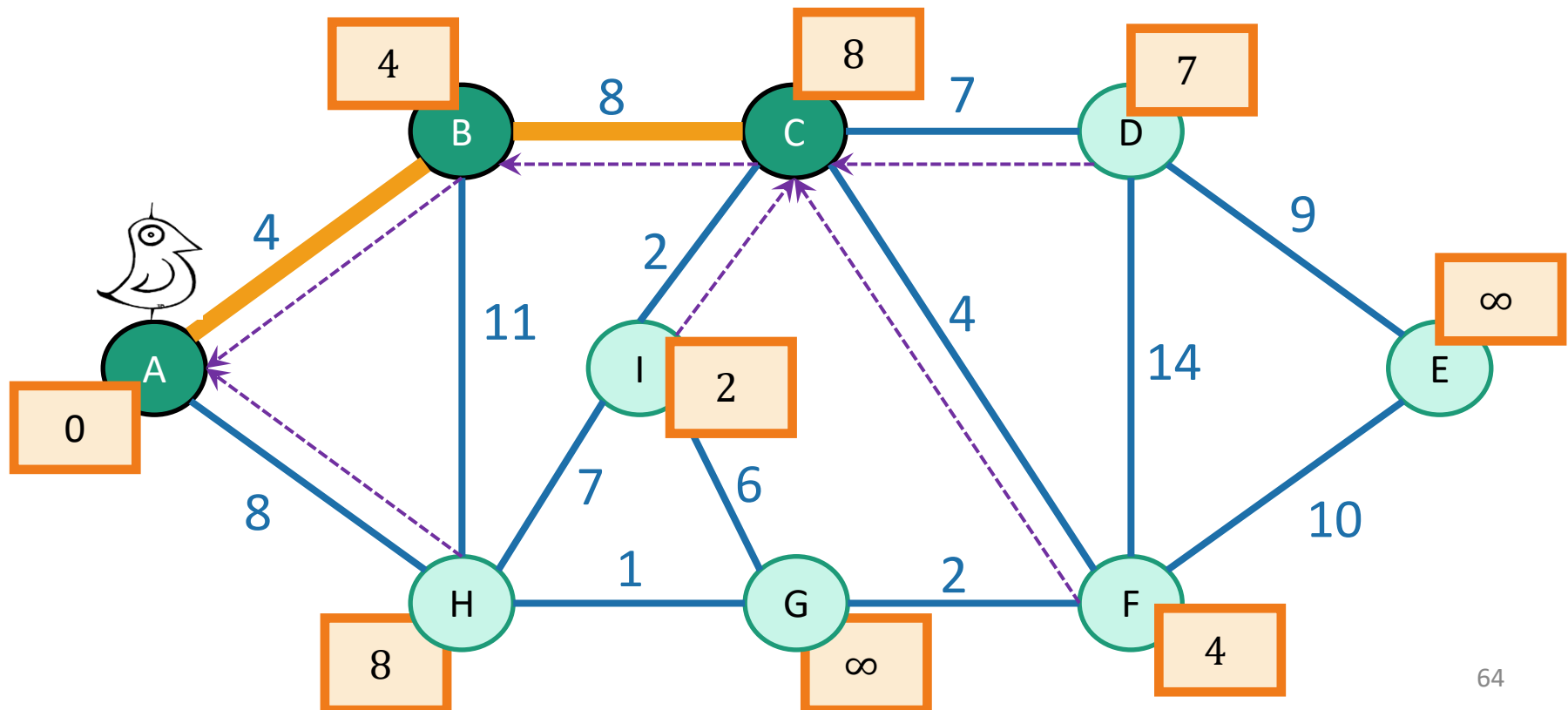
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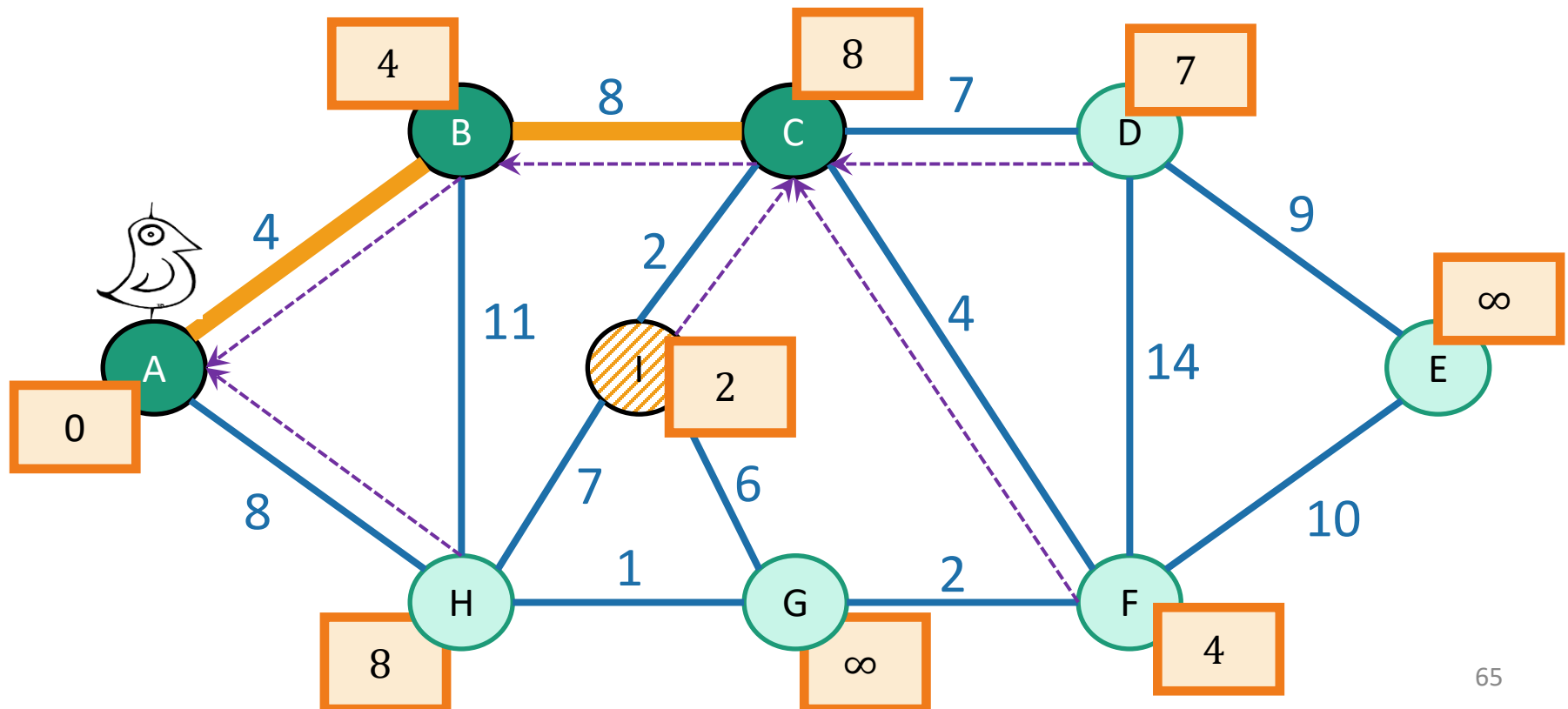
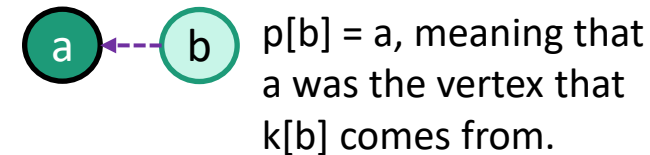
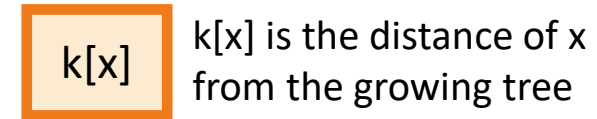
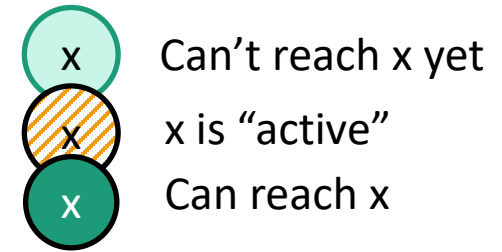


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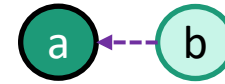
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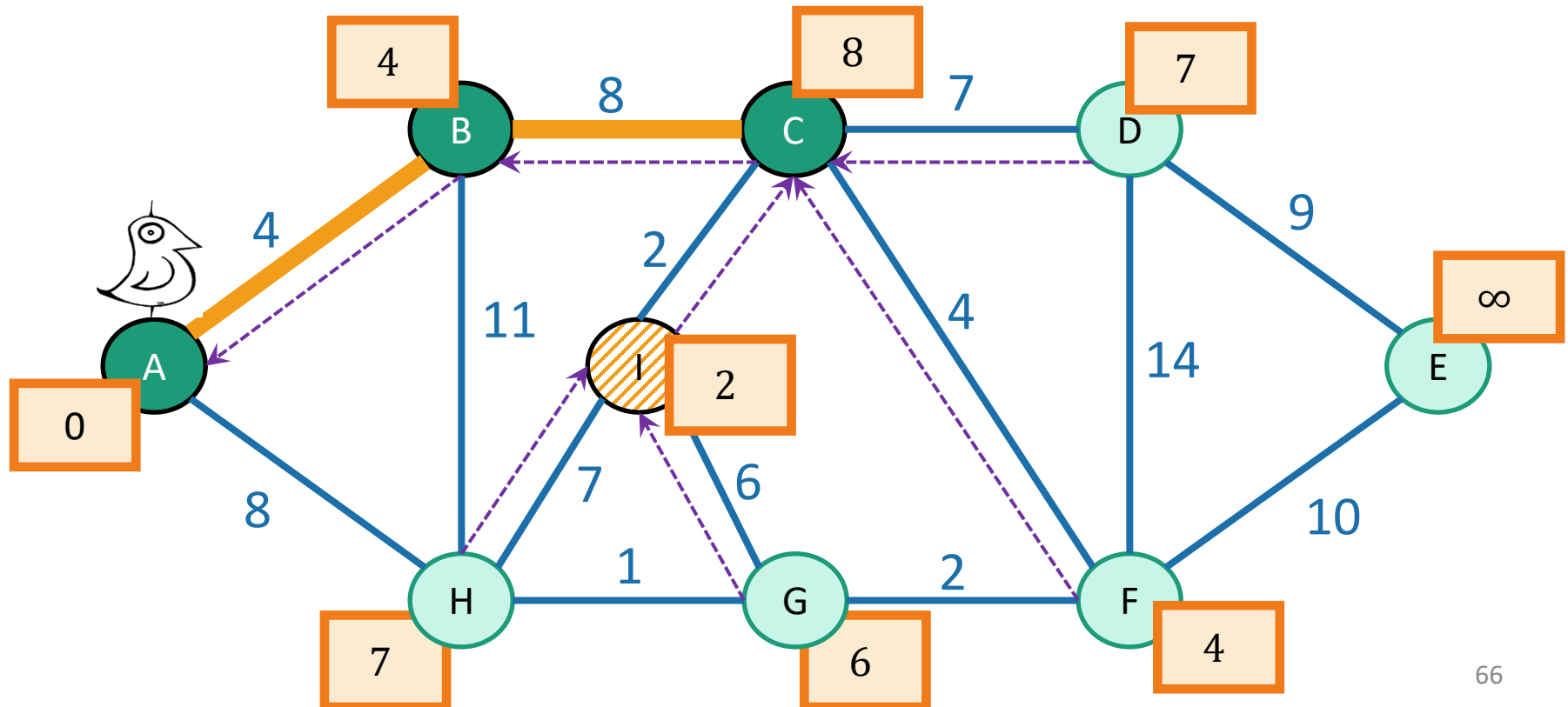
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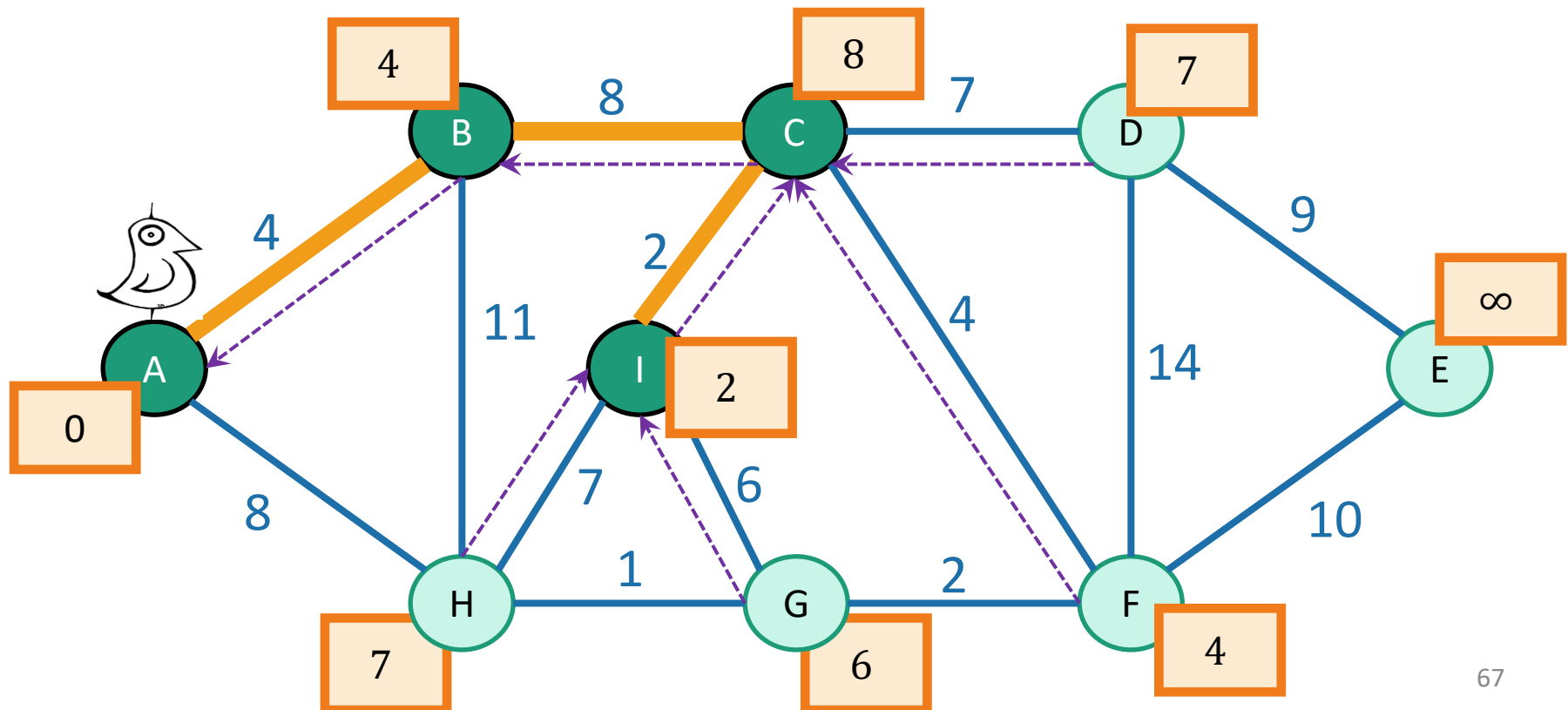
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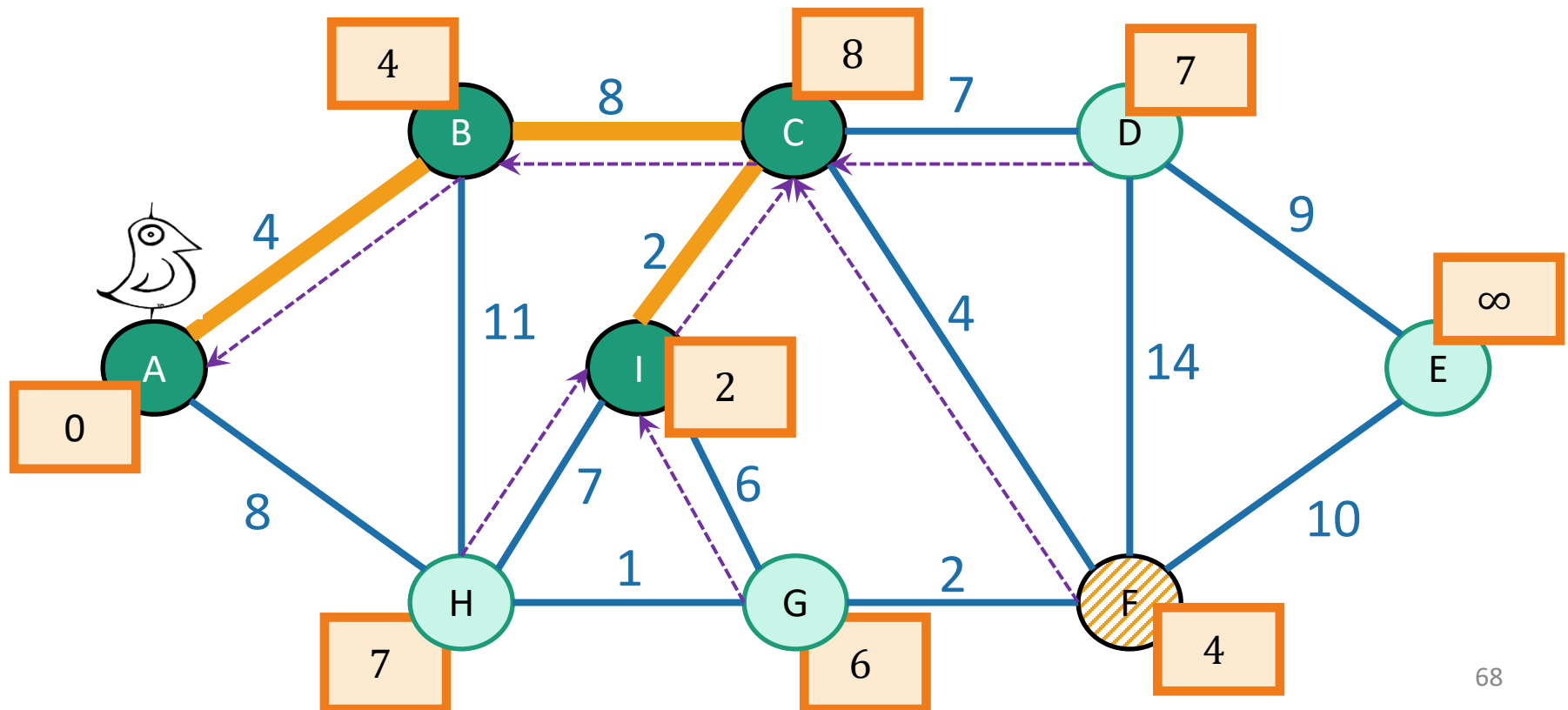
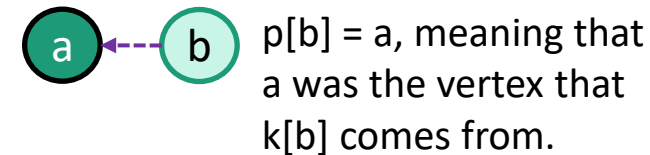
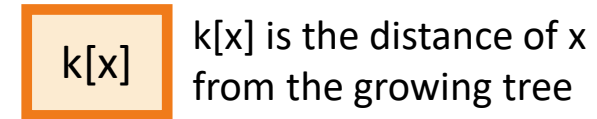
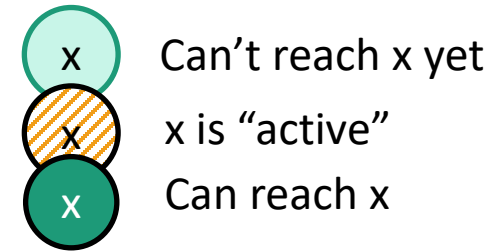


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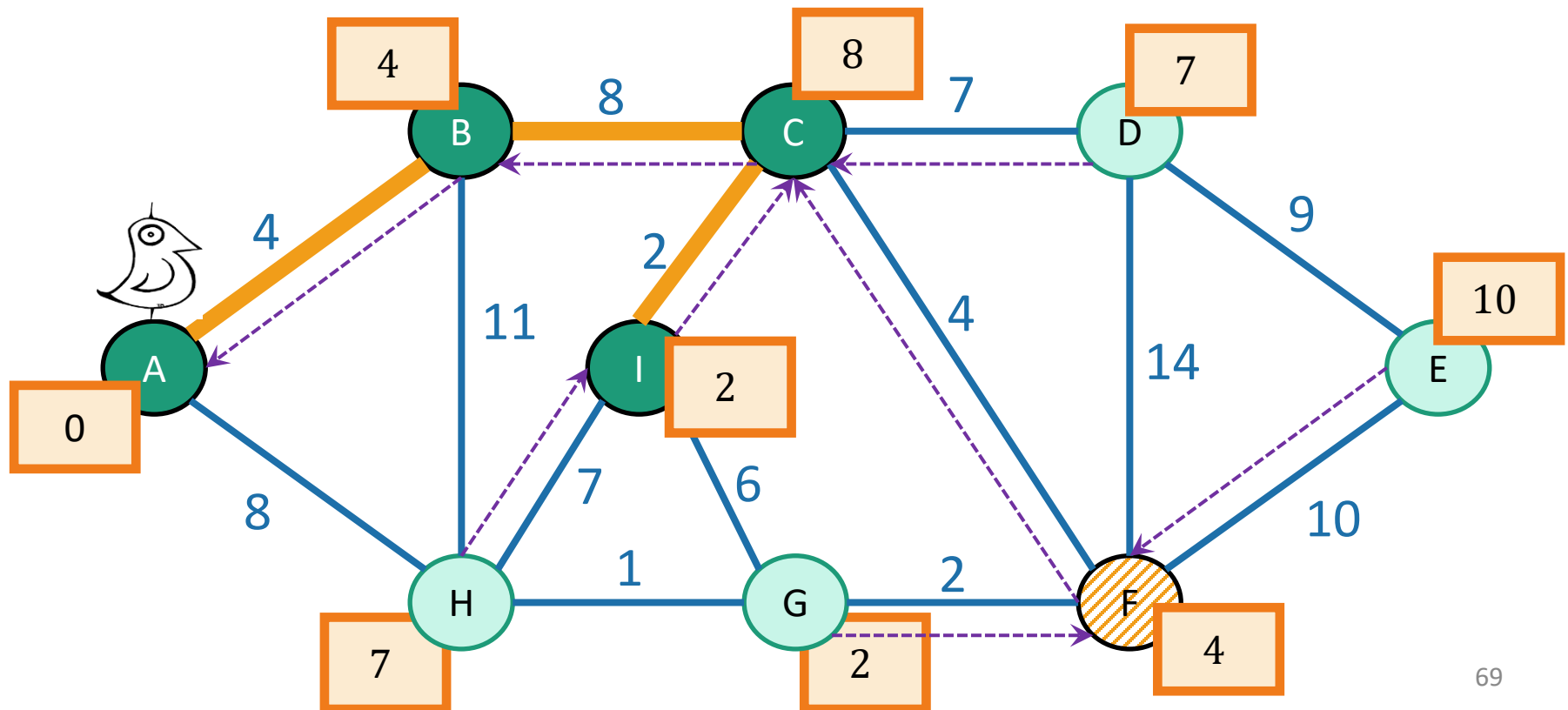
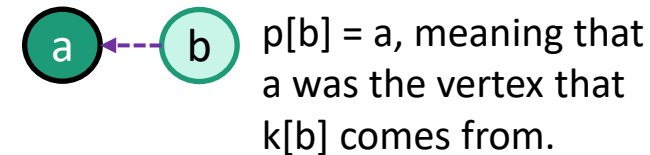
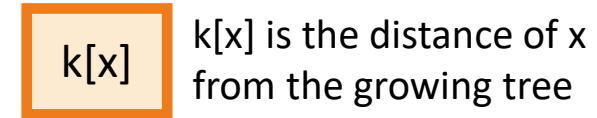
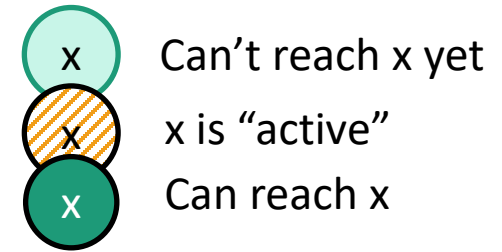


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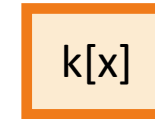
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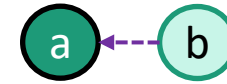
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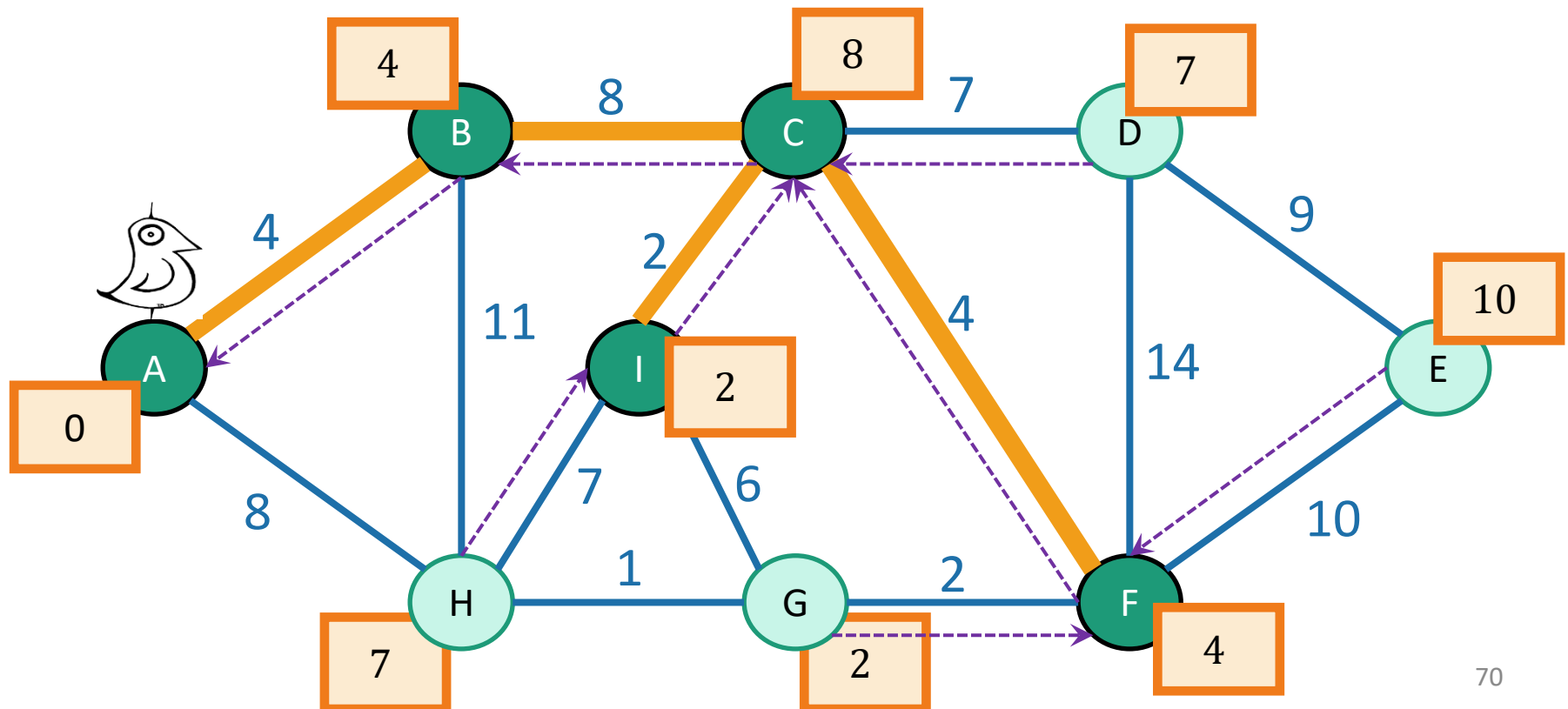
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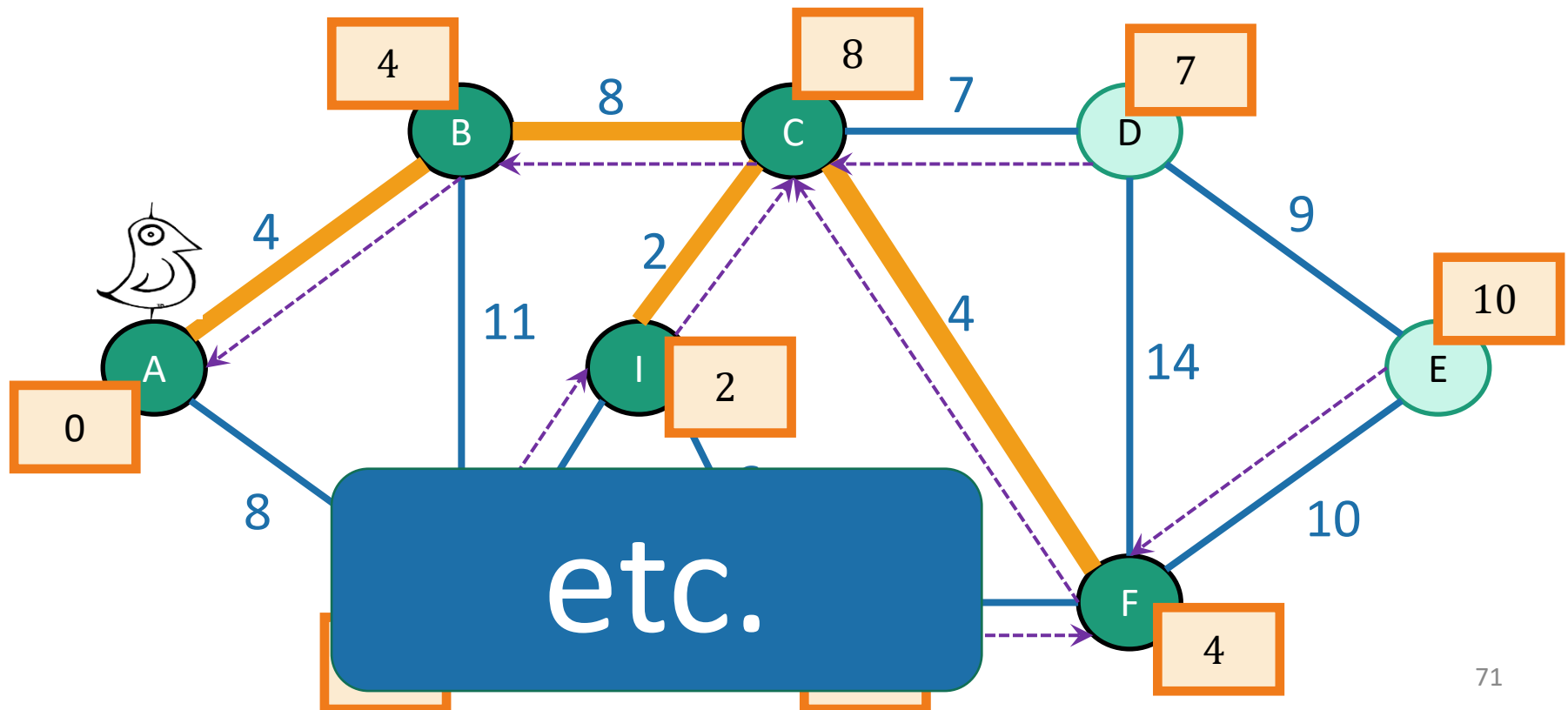
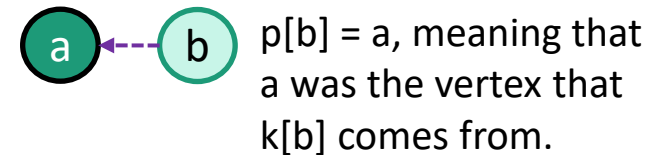
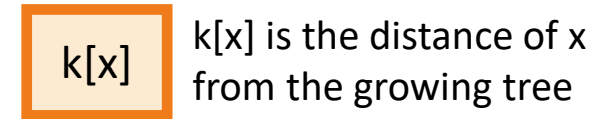
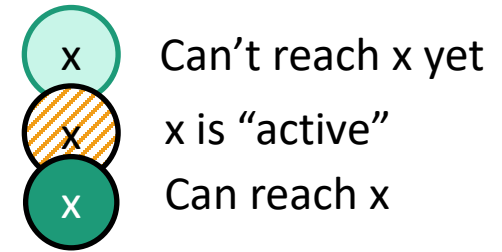


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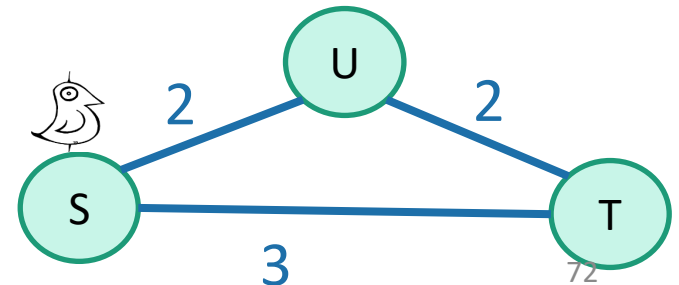
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# This should look pretty familiar

- Very similar to Dijkstra's algorithm!
- **Differences:**
  1. Keep track of  $p[v]$  in order to return a tree at the end
    - But Dijkstra's can do that too, that's not a big difference.
  2. Instead of  $d[v]$  which we update by
    - $d[v] = \min( d[v], d[u] + w(u,v) )$we keep  $k[v]$  which we update by
    - $k[v] = \min( k[v], w(u,v) )$
- To see the difference, consider:

Thing 2 is the big difference.

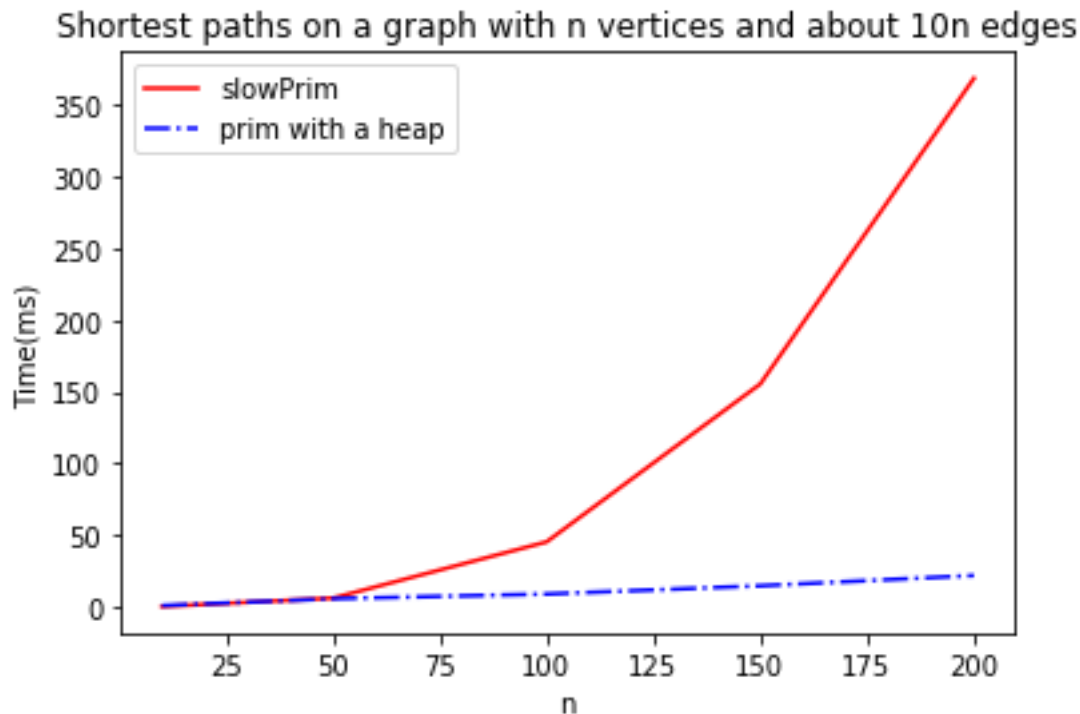




# One thing that is similar:

## Running time

- Exactly the same as Dijkstra:
  - $O(m \log(n))$  using a Red-Black tree as a priority queue.
  - $O(m + n \log(n))$  amortized time if we use a Fibonacci Heap.



# Two questions

1. Does it work?

- That is, does it actually return a MST?

- **Yes!**

2. How do we actually implement this?

- the pseudocode above says “slowPrim”...

- **Implement it basically the same way we’d implement Dijkstra!**

# What have we learned?

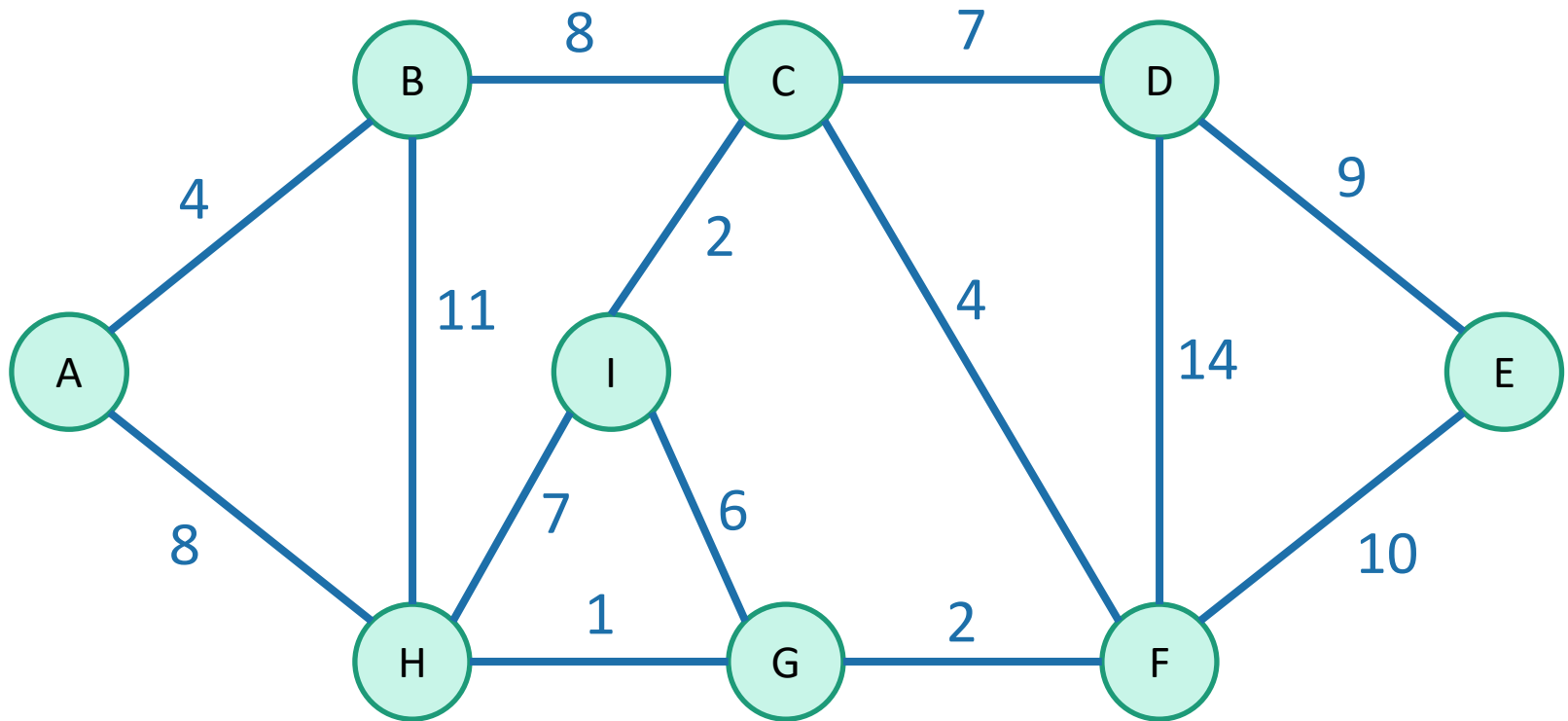
- Prim's algorithm greedily grows a tree
  - smells a lot like Dijkstra's algorithm
- It finds a Minimum Spanning Tree!
  - in time  **$O(m \log(n))$**  if we implement it with a Red-Black Tree.
  - In amortized time  **$O(m + n \log(n))$**  with a Fibonacci heap.
- To prove it worked, we followed the same recipe for greedy algorithms we saw last time.
  - Show that, at every step, we **don't rule out success**.

That's not the only greedy algorithm for MST!

# That's not the only greedy algorithm

what if we just always take the cheapest edge?

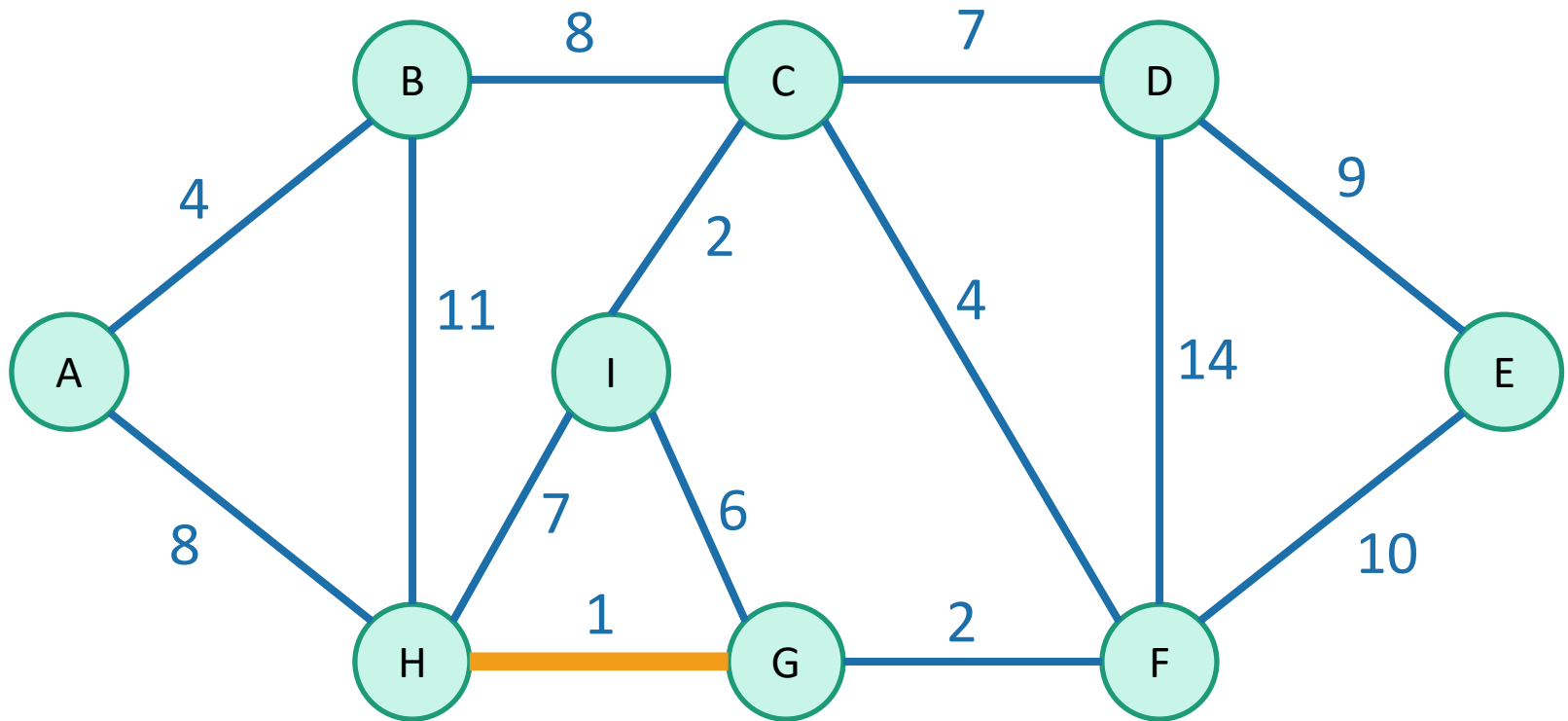
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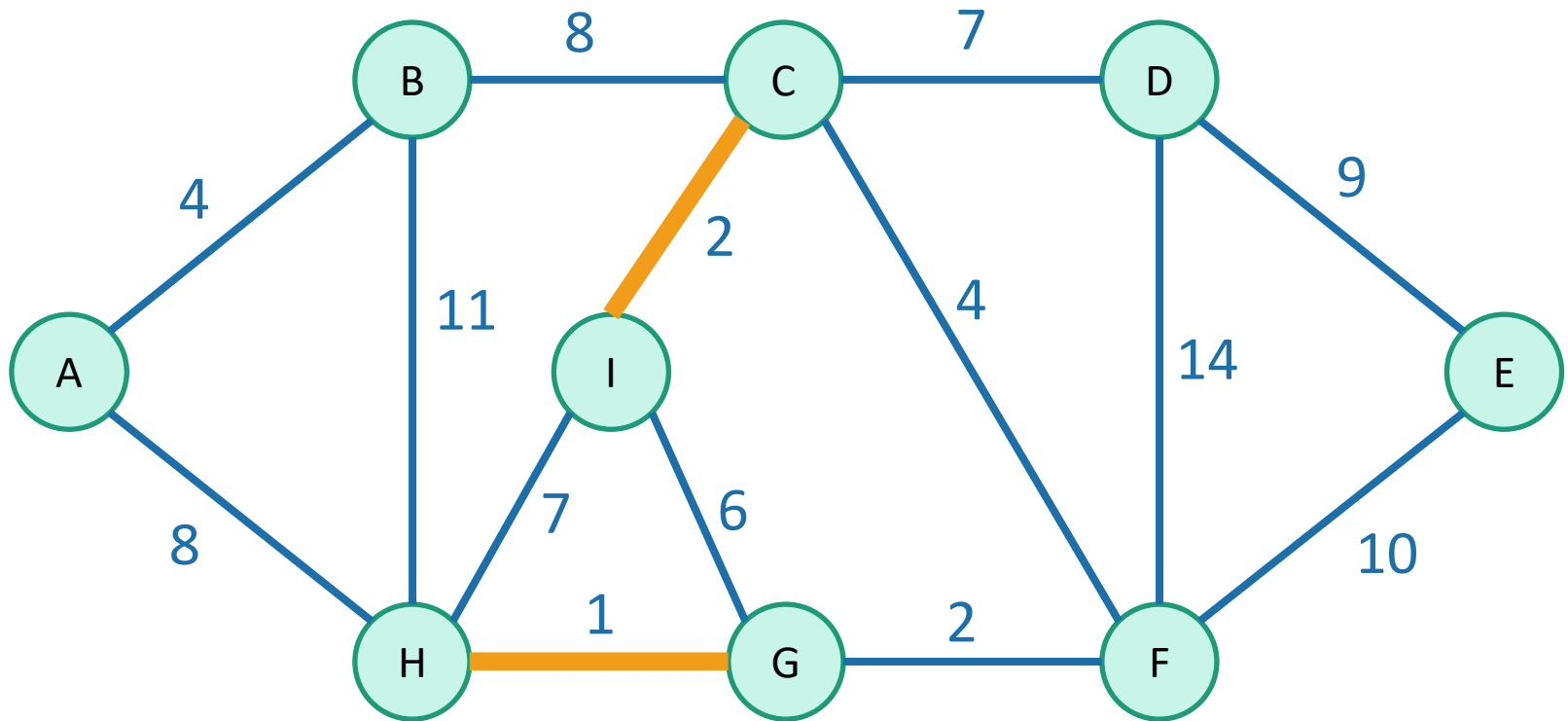
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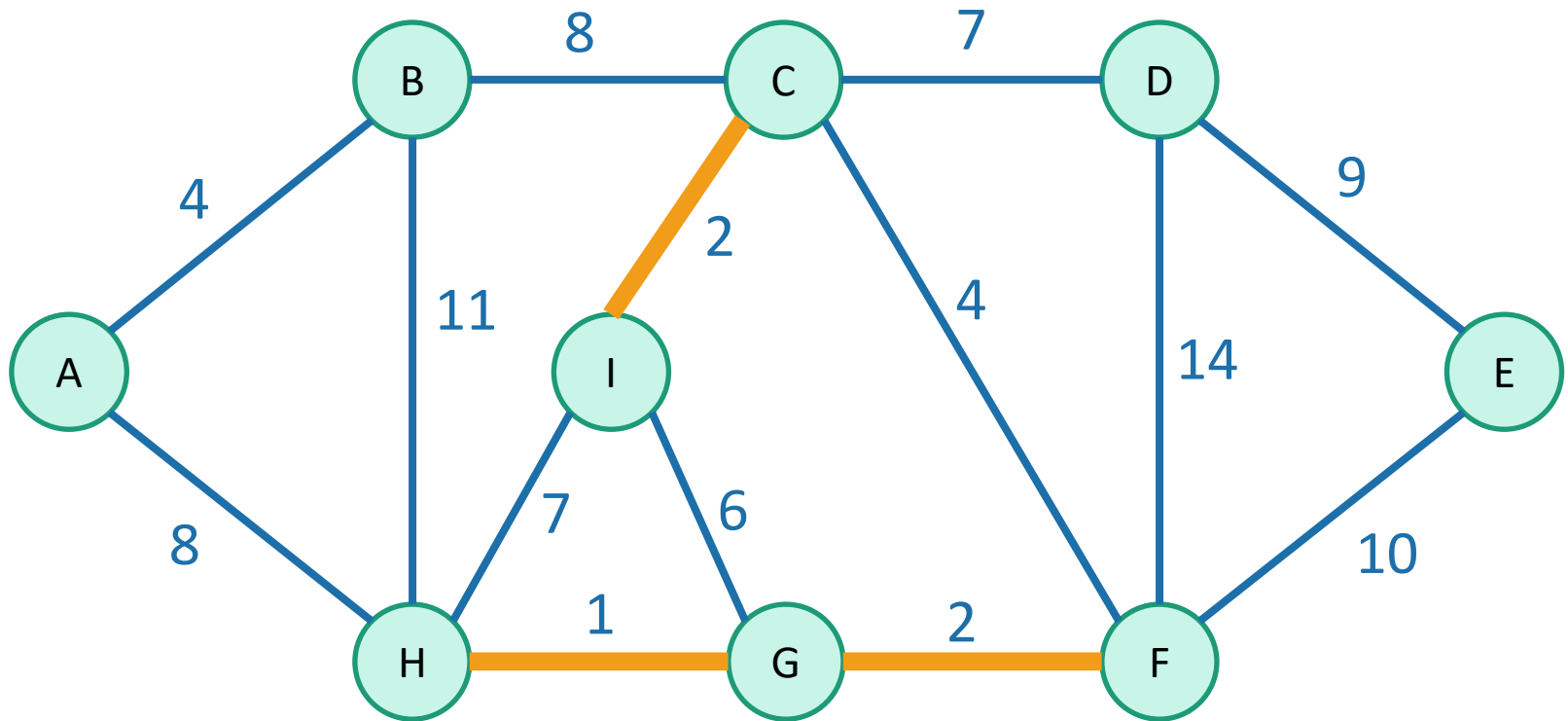
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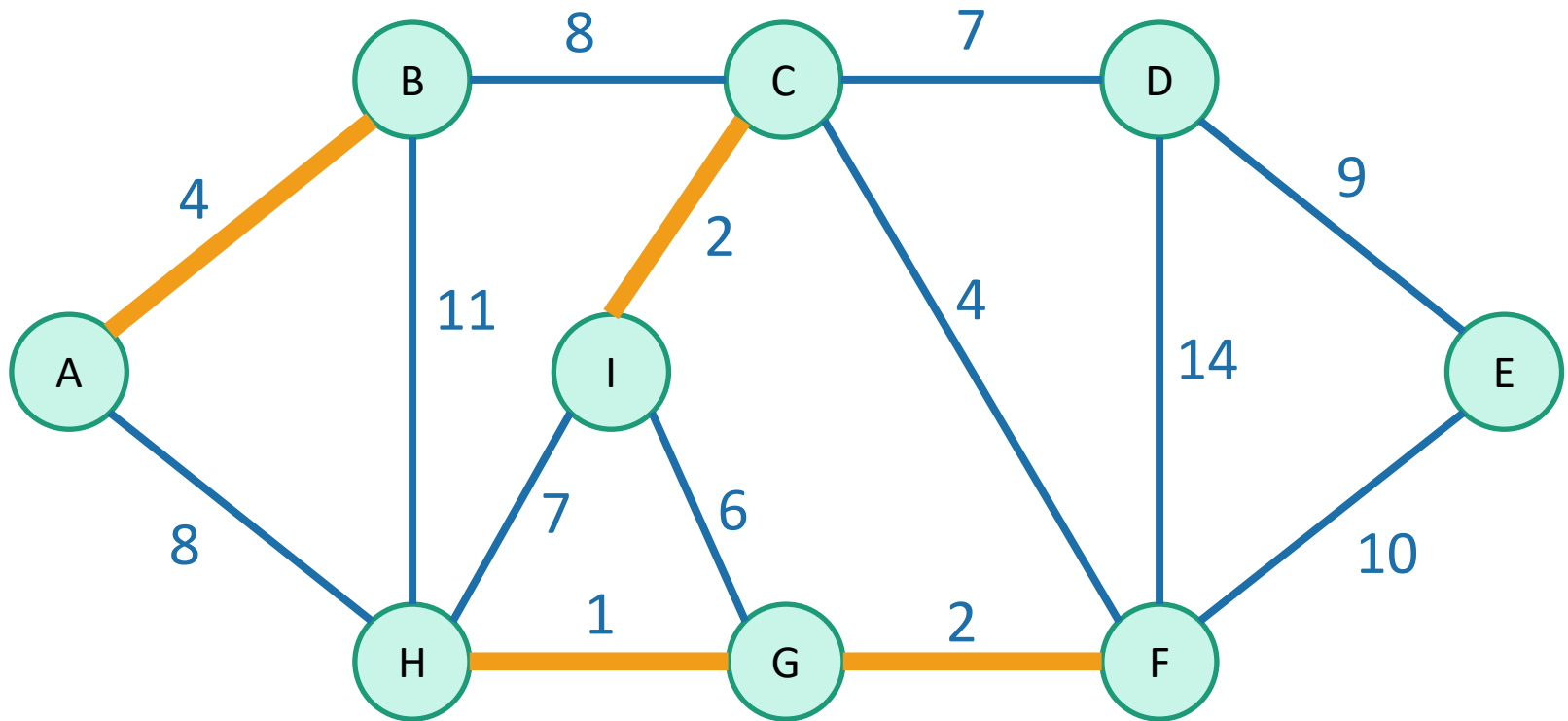




# That's not the only greedy algorithm

what if we just always take the cheapest edge?

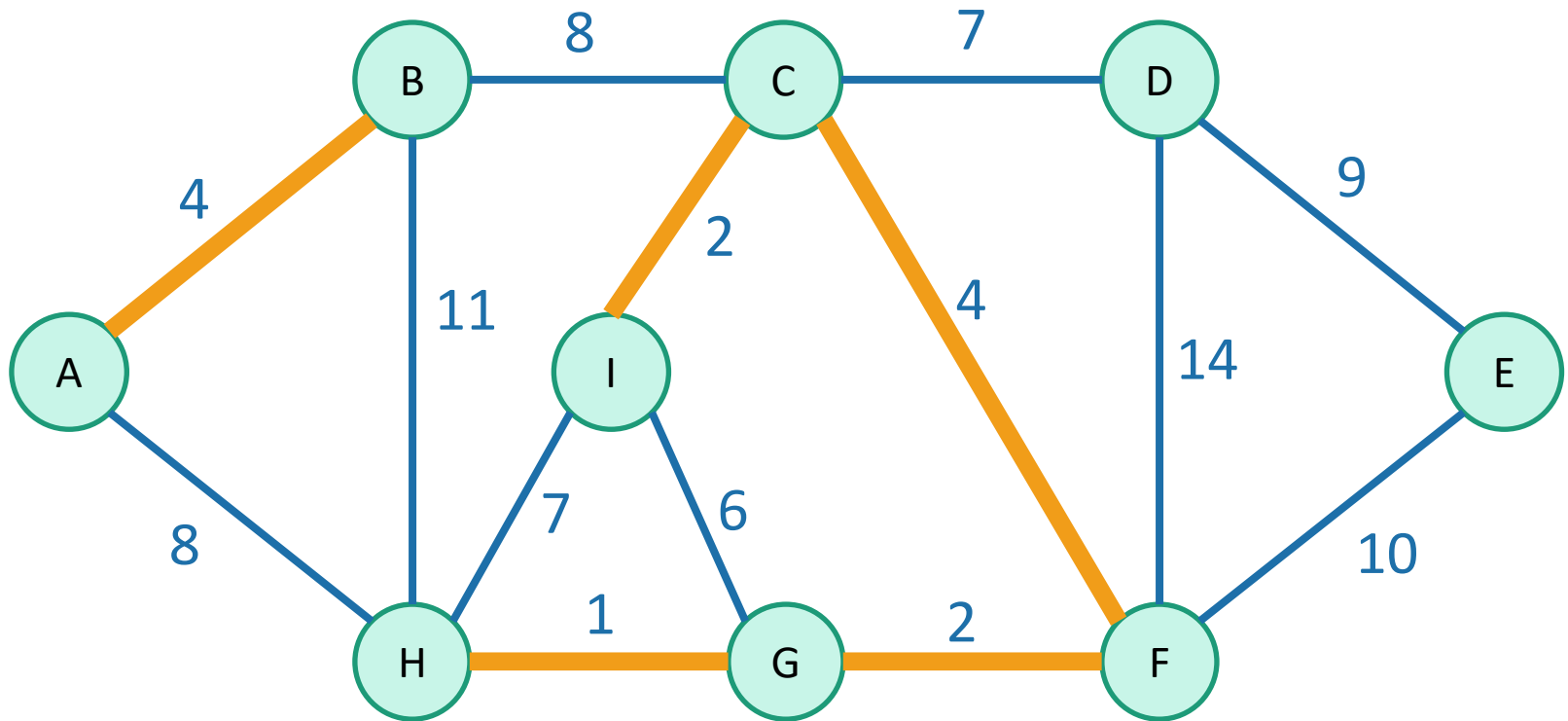
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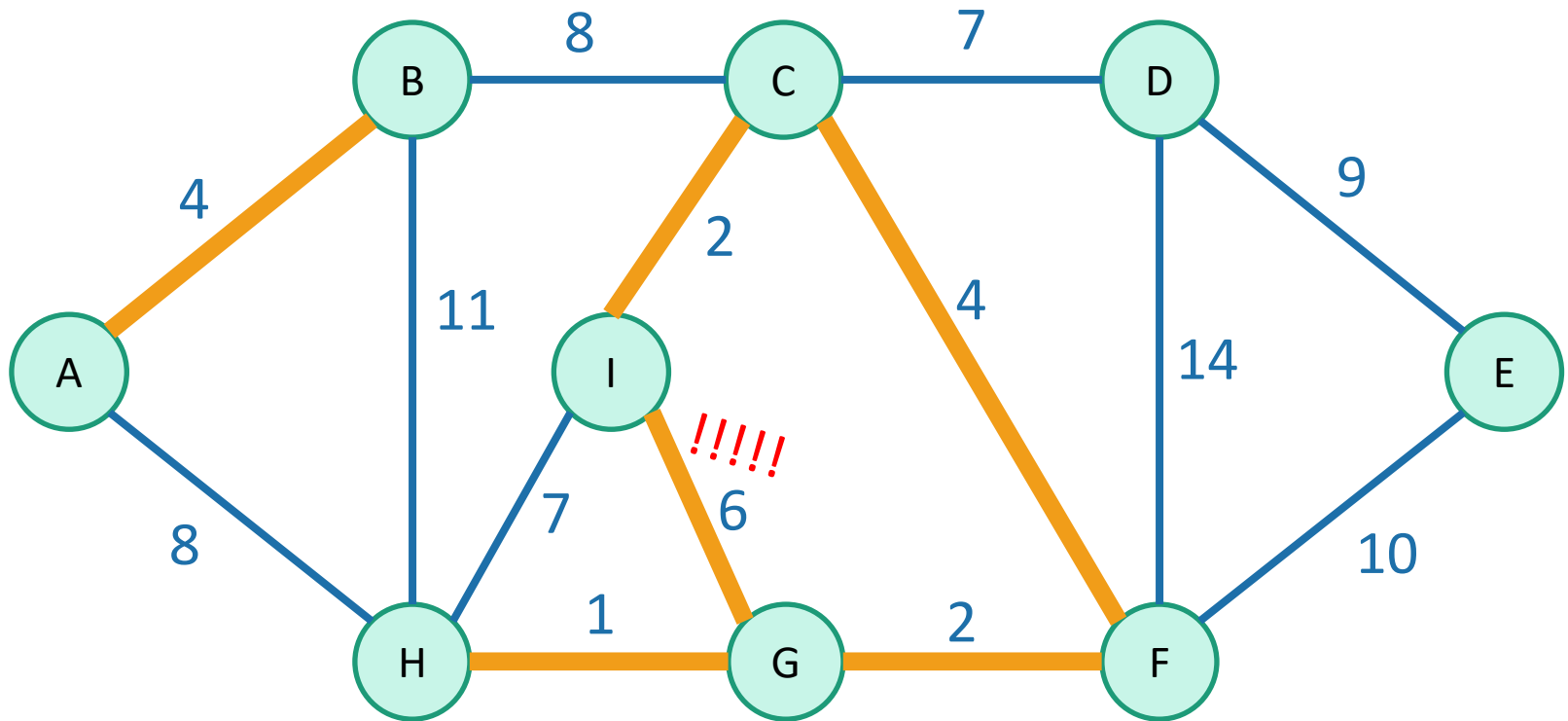
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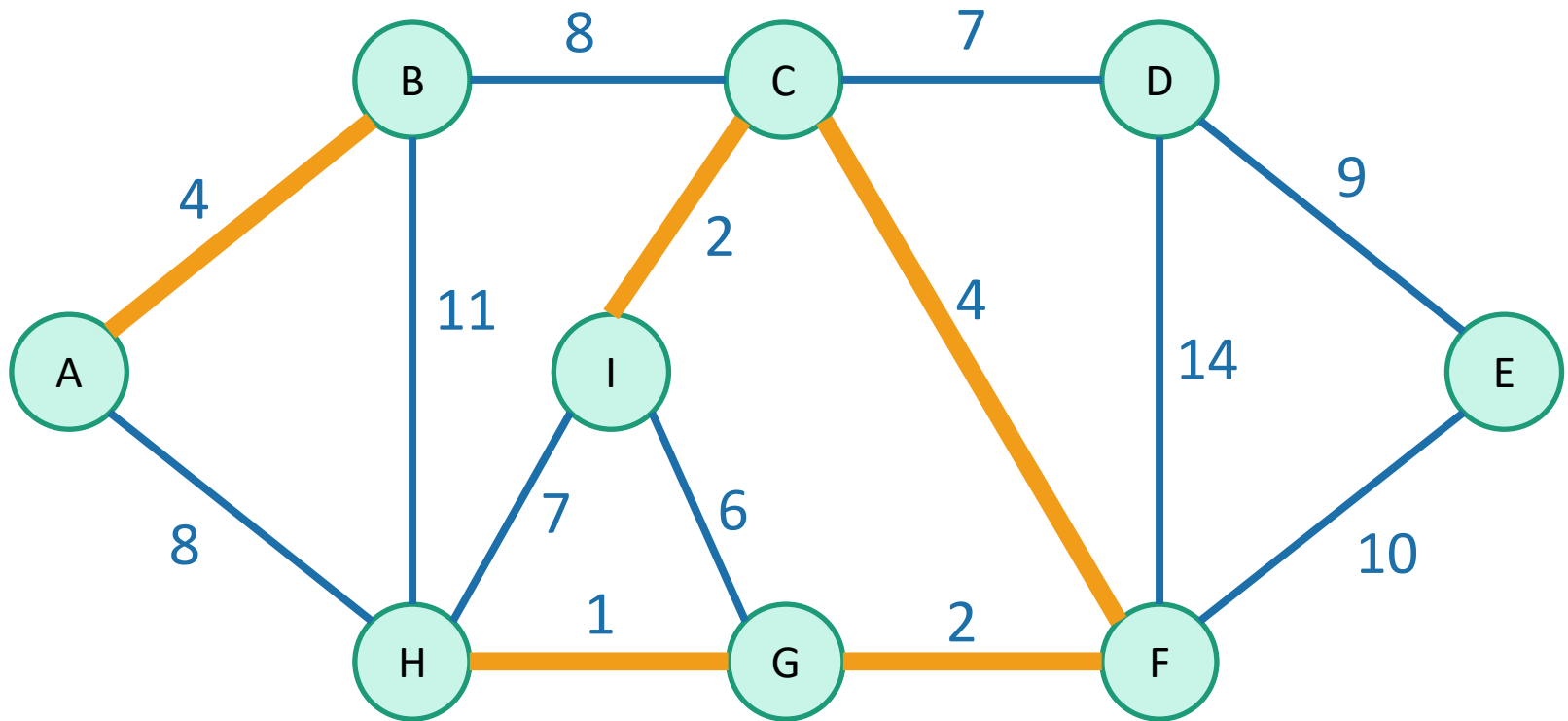
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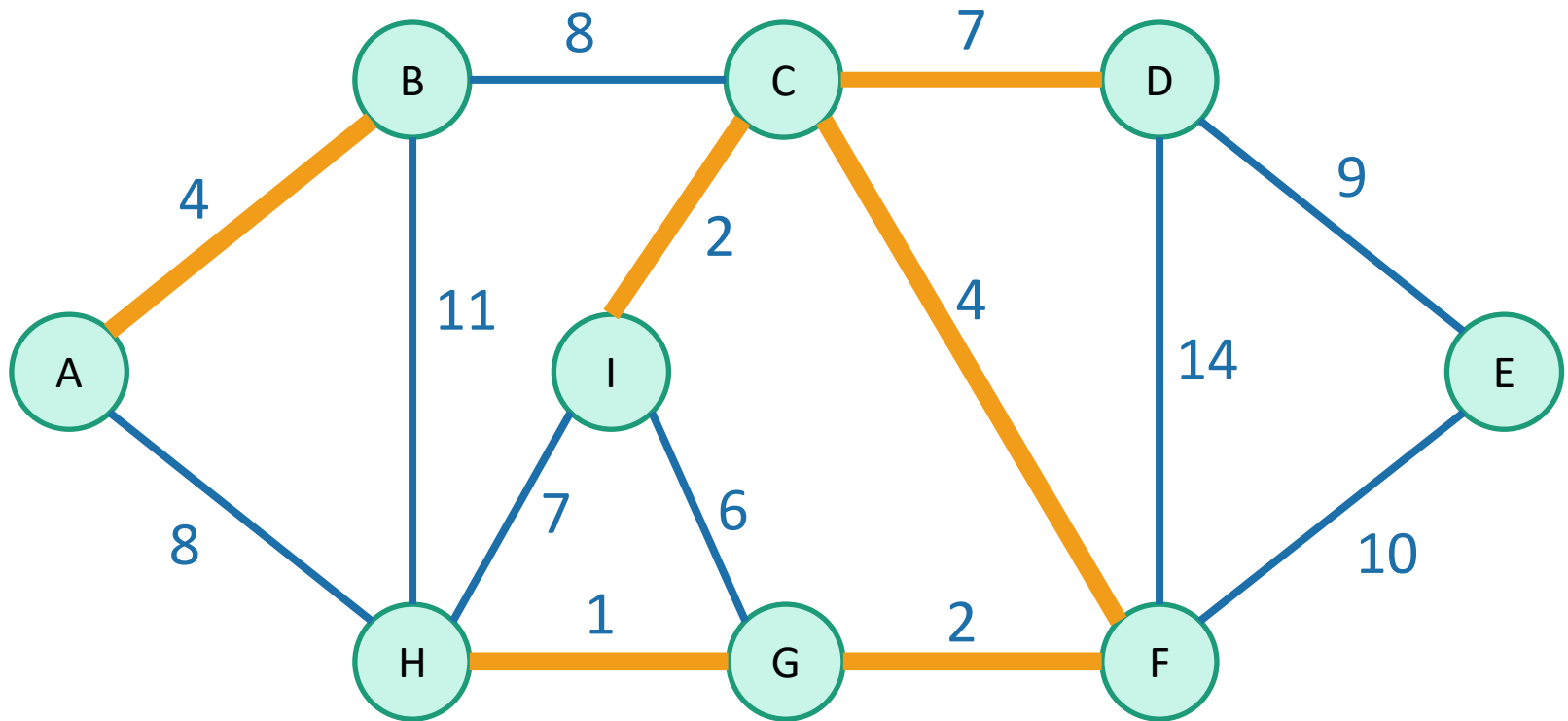
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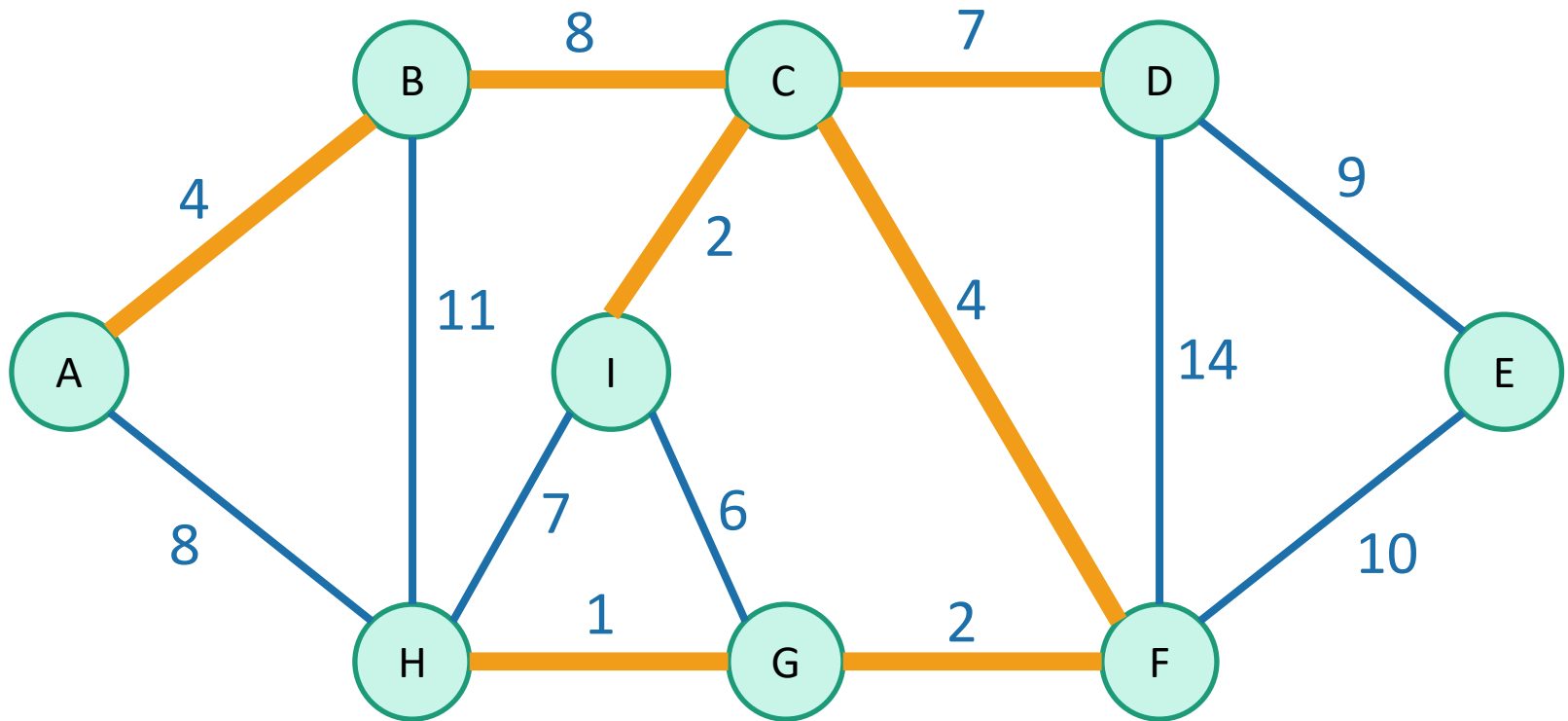
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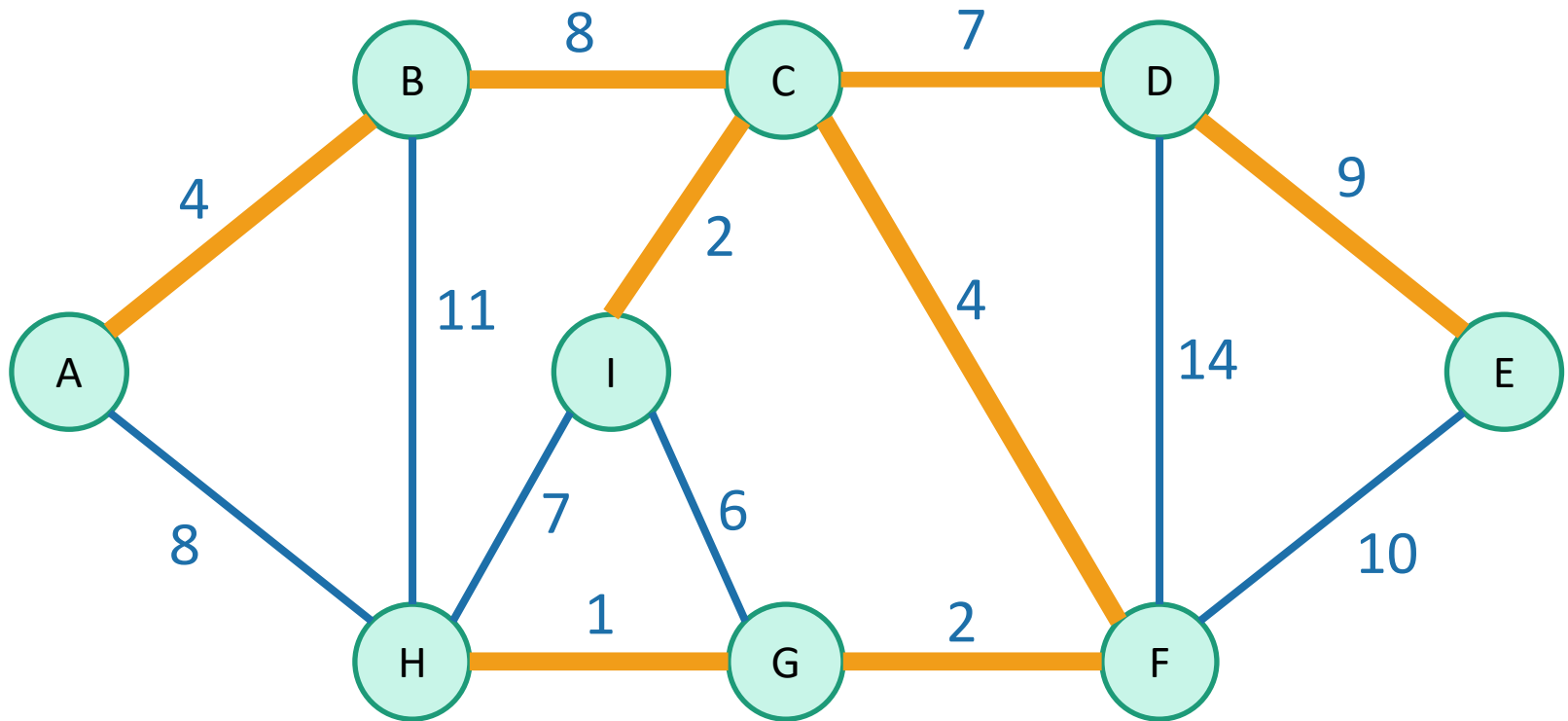
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# That's not the only greedy algorithm

what if we just always take the cheapest edge?  
whether or not it's connected to what we have so far?

That won't  
cause a cycle



# We've discovered Kruskal's algorithm!

- **slowKruskal**( $G = (V, E)$ ):
    - Sort the edges in  $E$  by non-decreasing weight.
    - $MST = \{\}$
    - **for**  $e$  in  $E$  (in sorted order):
      - **if** adding  $e$  to  $MST$  won't cause a cycle:
        - add  $e$  to  $MST$ .
    - **return**  $MST$
- m iterations through this loop*
- How do we check this?*

How **would** you  
figure out if added  $e$   
would make a cycle  
in this algorithm?

Naively, the running time is ???:

- For each of  $m$  iterations of the for loop:
  - Check if adding  $e$  would cause a cycle...



# Two questions

## 1. Does it work?

- That is, does it actually return a MST?

## 2. How do we actually implement this?

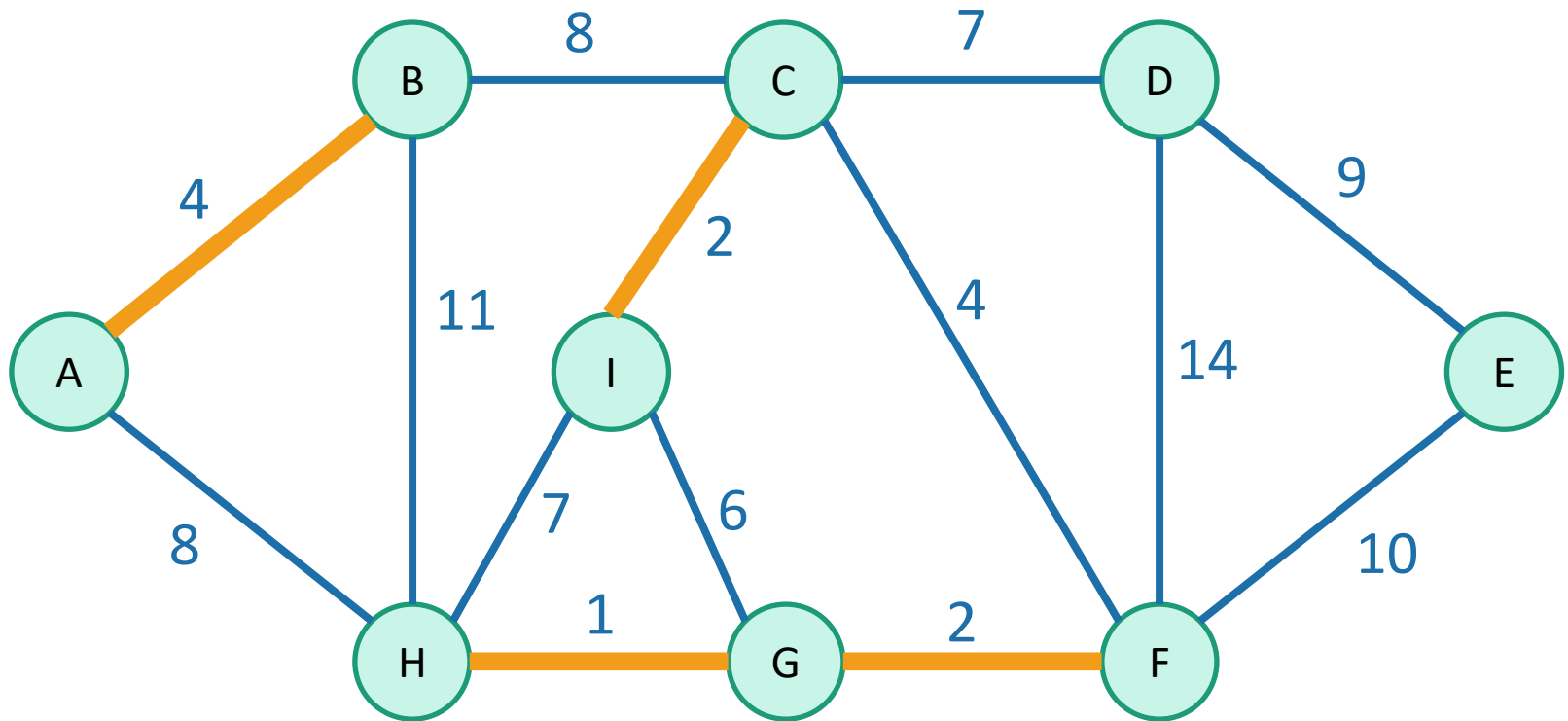
- the pseudocode above says “slowKruskal”...



Let's do this  
one first

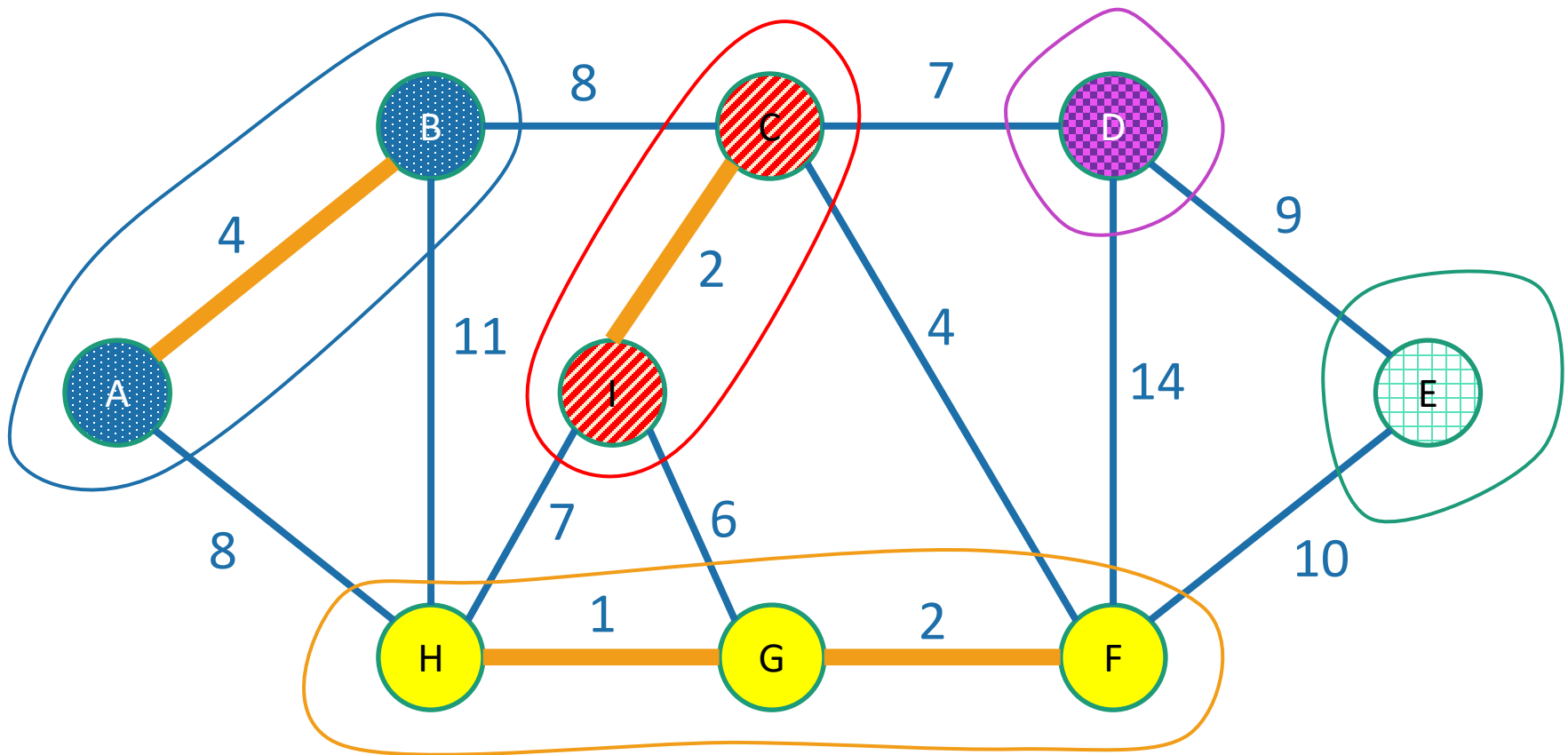
At each step of Kruskal's,  
we are maintaining a forest.

A **forest** is a  
collection of  
disjoint trees



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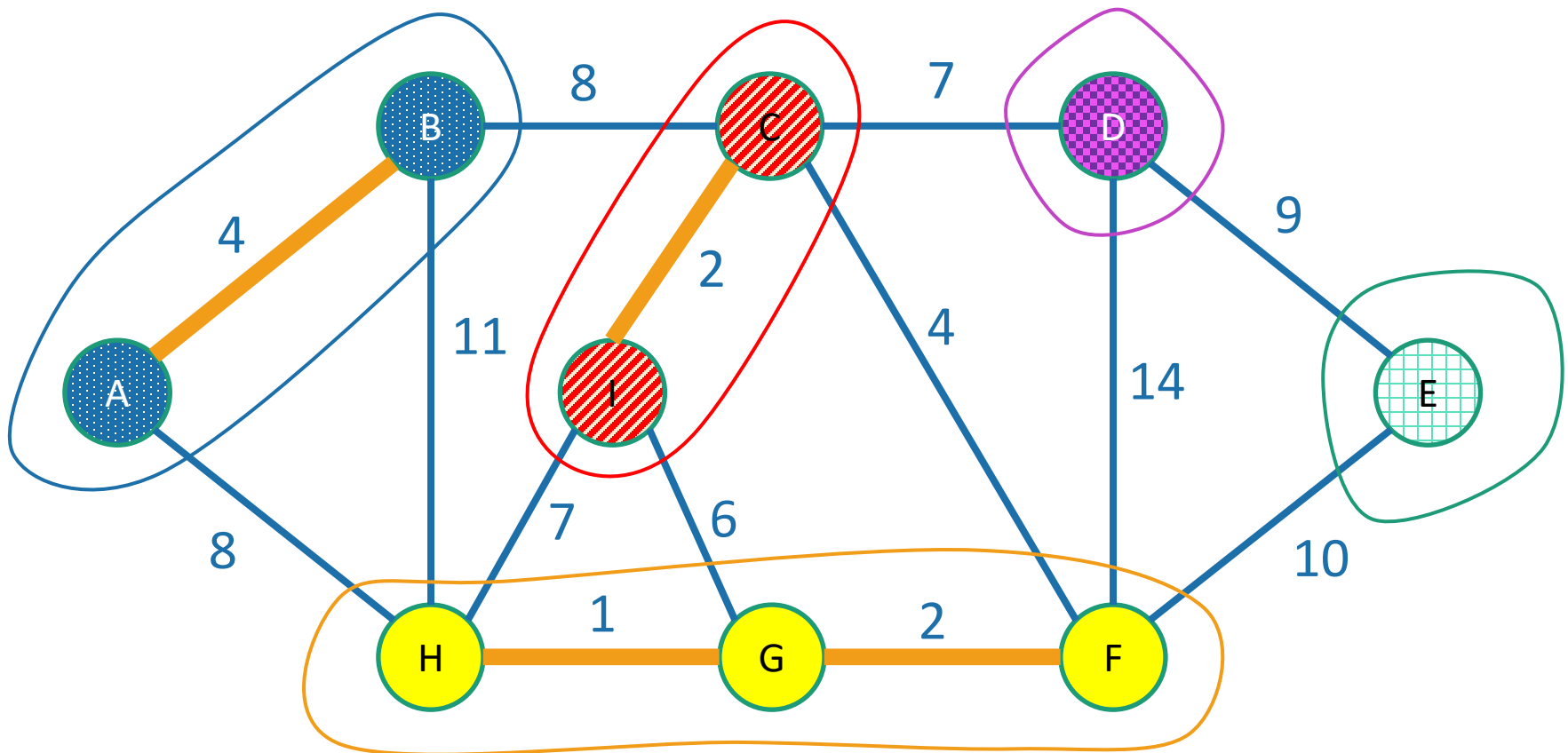
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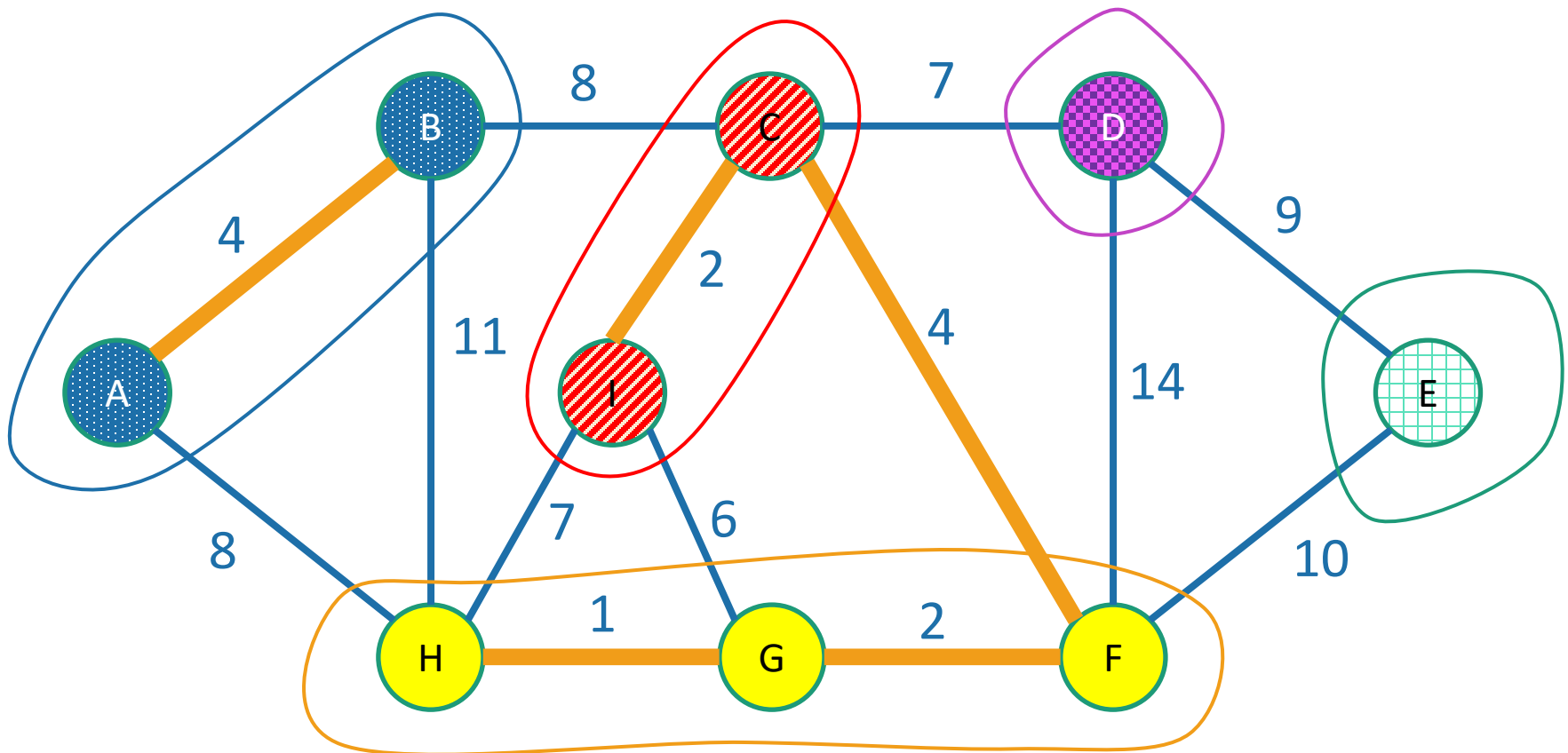
When we add an edge, we merge two trees:



At each step of Kruskal's,  
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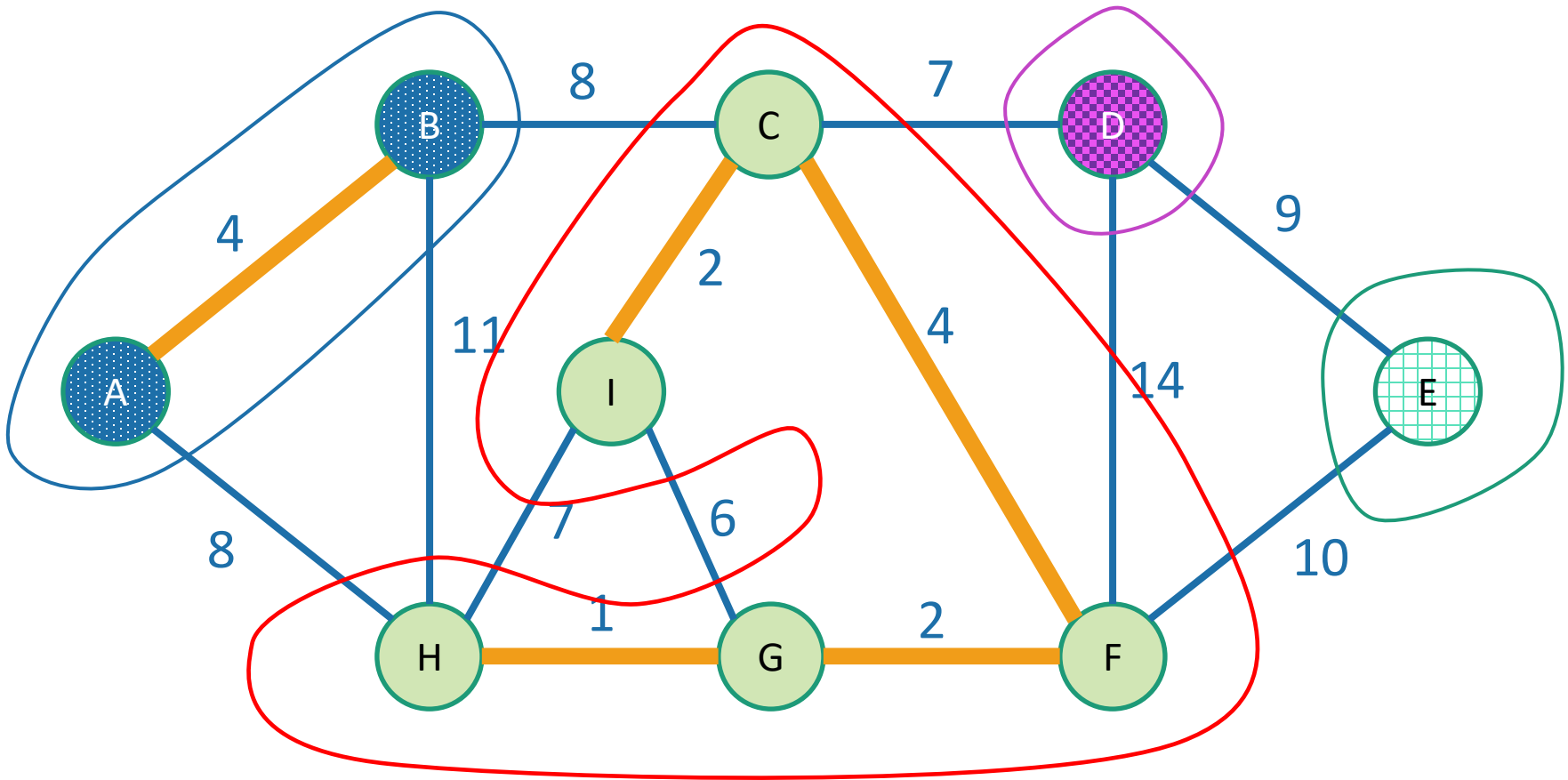
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At each step of Kruskal's,  
we are maintaining a **forest**.

A **forest** is a  
collection of  
disjoint trees

When we add an edge, we merge two trees:



We never add an edge within a tree since that would create a cycle.

# Keep the trees in a special data structure



“treehouse”?

# Union-find data structure

also called disjoint-set data structure

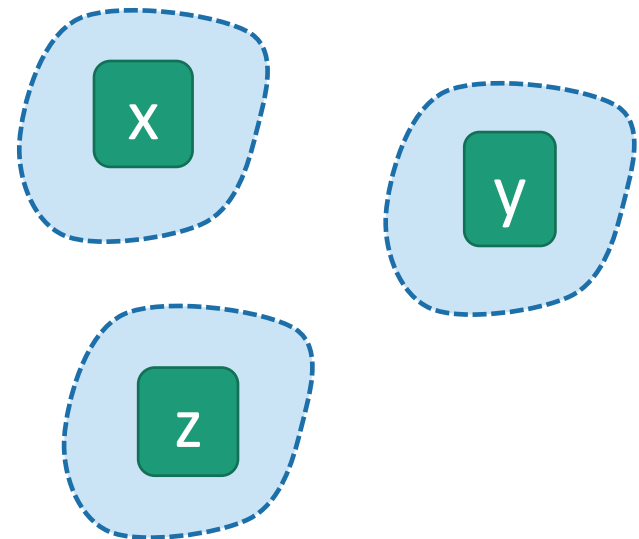
- Used for storing collections of sets
- Supports:
  - **makeSet(u)**: create a set {u}
  - **find(u)**: return the set that u is in
  - **union(u,v)**: merge the set that u is in with the set that v is in.

```
makeSet(x)
```

```
makeSet(y)
```

```
makeSet(z)
```

```
union(x, y)
```





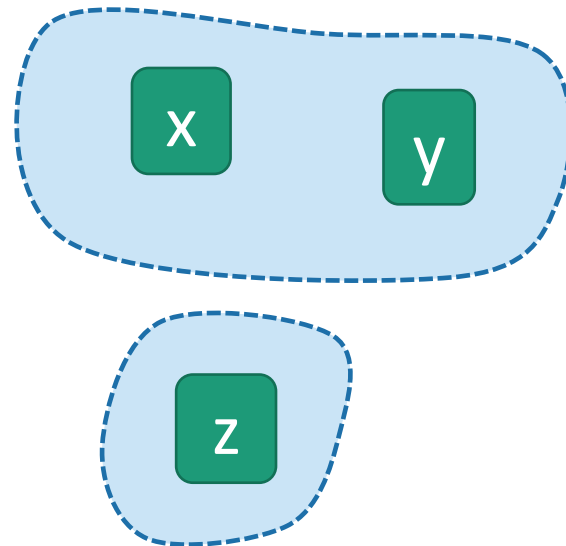
# Union-find data structure

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# Union-find data structure

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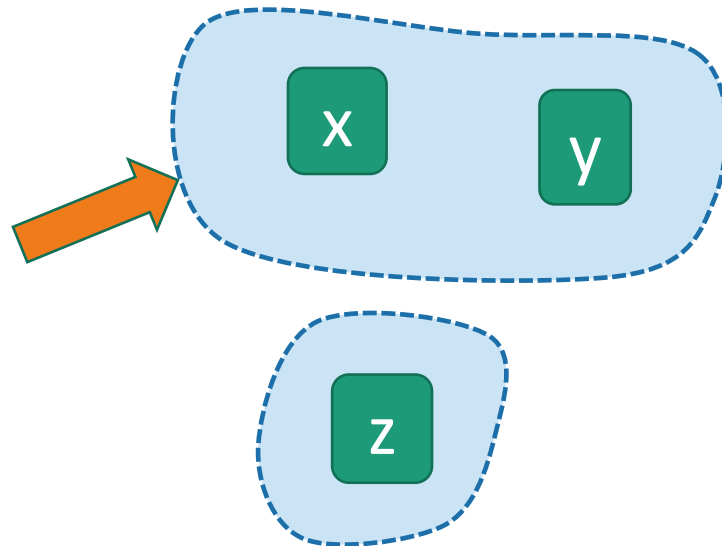
```
makeSet(x)
```

```
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```

```
makeSet(z)
```

```
union(x, y)
```

```
find(x)
```

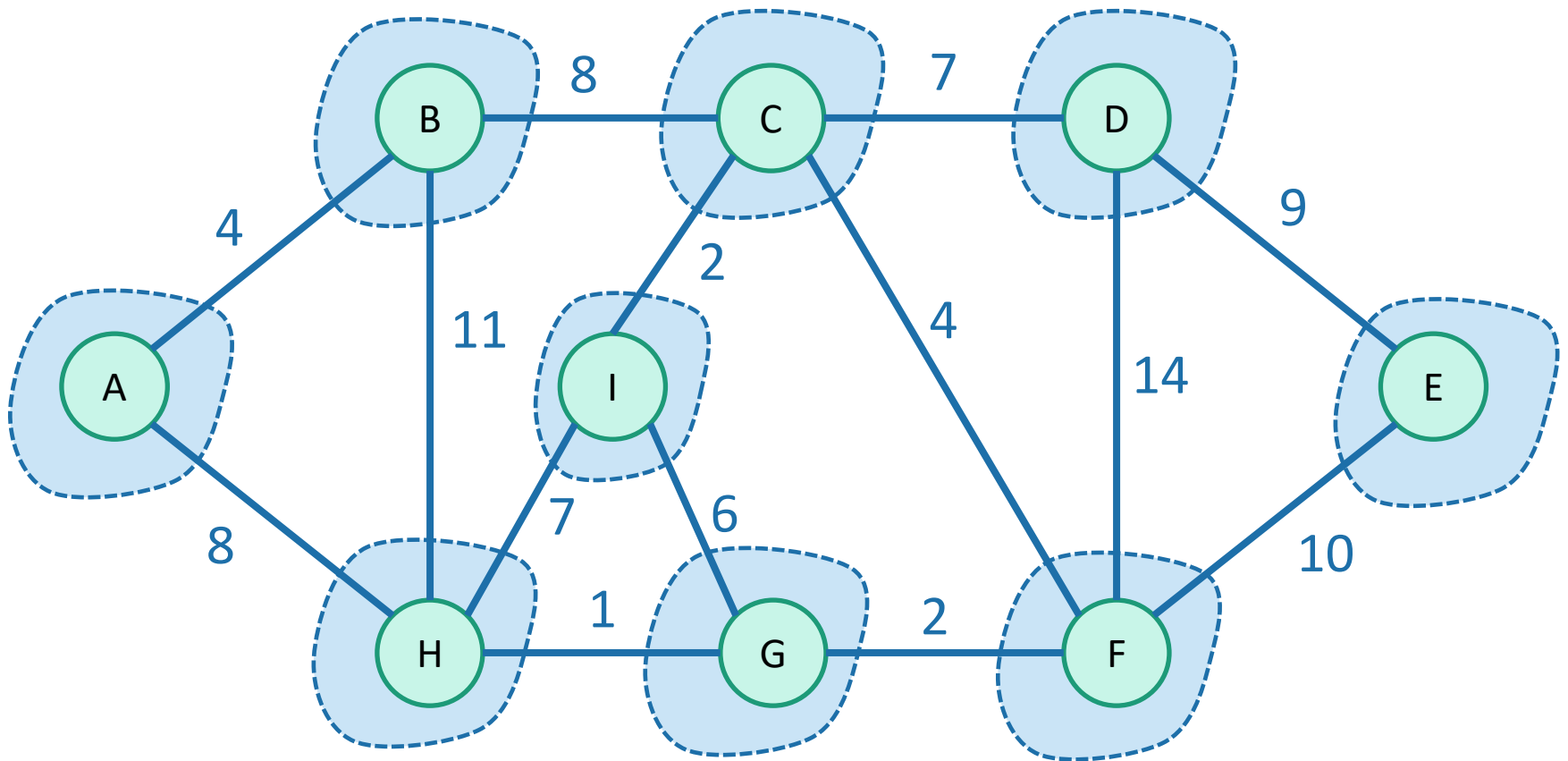


# Kruskal pseudo-code

- **kruskal**( $G = (V, E)$ ):
  - Sort  $E$  by weight in non-decreasing order
  - $MST = \{\}$  *// initialize an empty tree*
  - **for**  $v$  in  $V$ :
    - **makeSet**( $v$ ) *// put each vertex in its own tree in the forest*
  - **for**  $(u, v)$  in  $E$ : *// go through the edges in sorted order*
    - **if** **find**( $u$ )  $\neq$  **find**( $v$ ): *// if  $u$  and  $v$  are not in the same tree*
      - add  $(u, v)$  to  $MST$
      - **union**( $u, v$ ) *// merge  $u$ 's tree with  $v$ 's tree*
  - **return**  $MST$

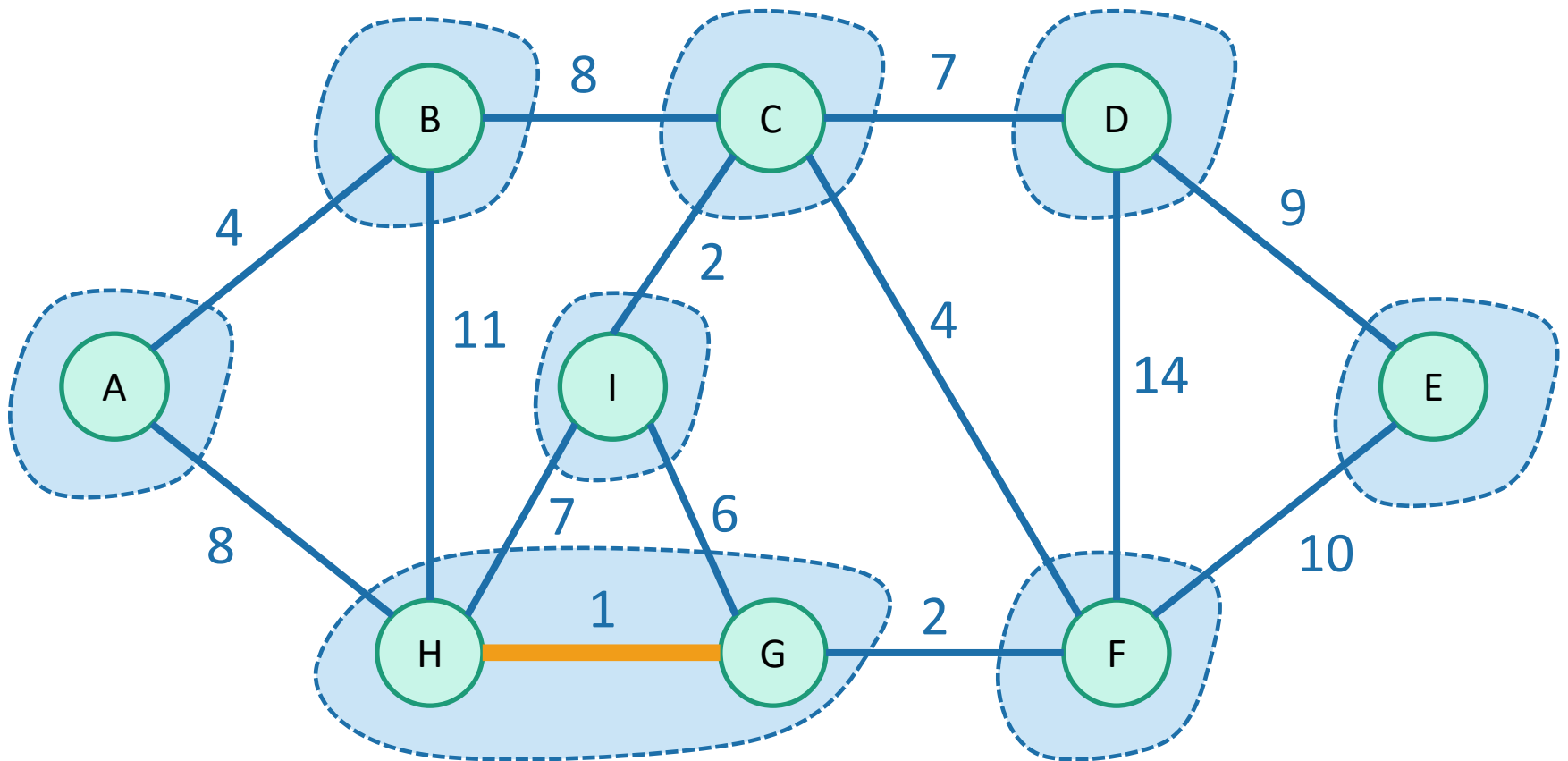
# Once more...

To start, every vertex is in its own tree.



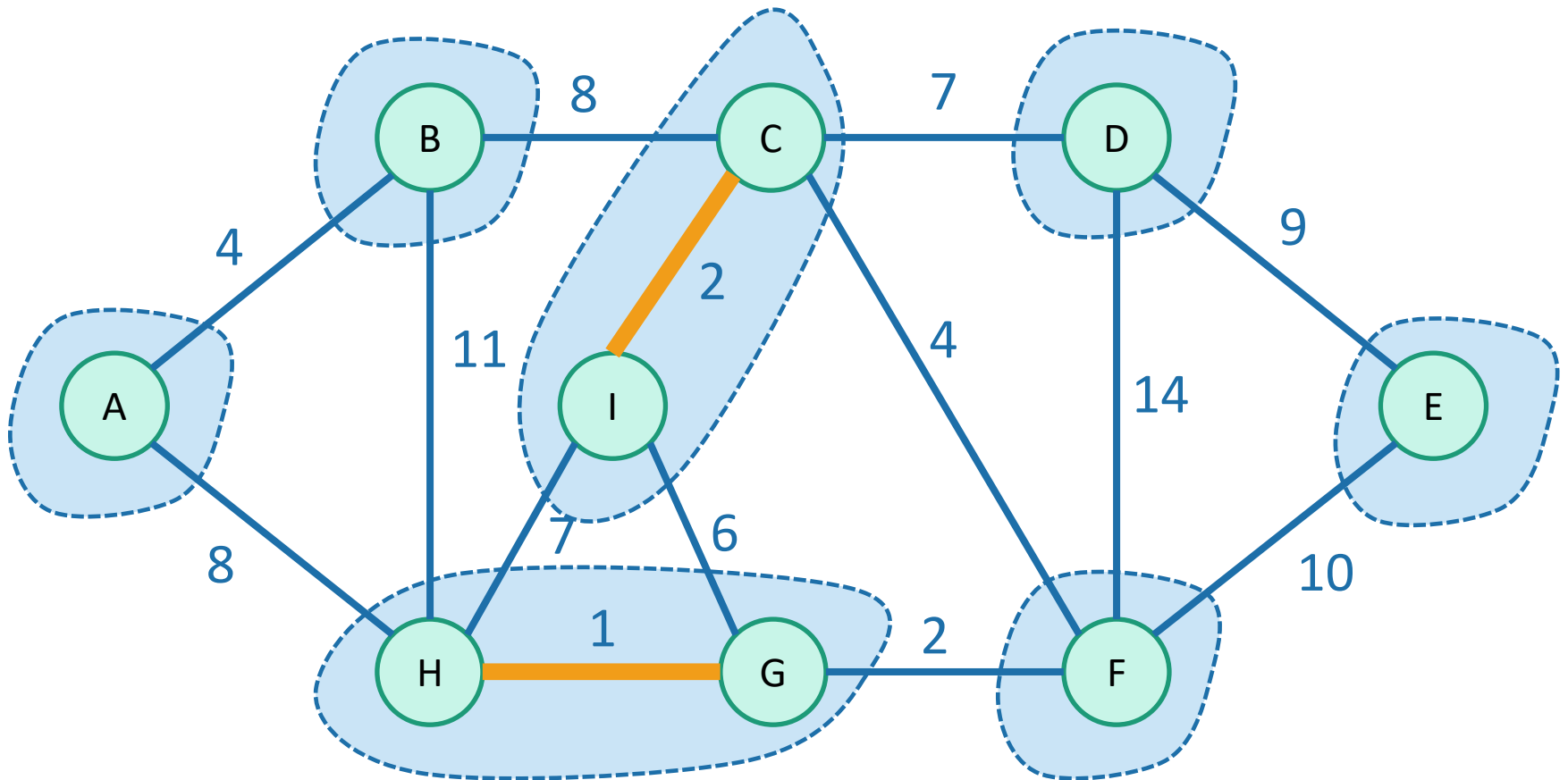
# Once more...

Then start merging.



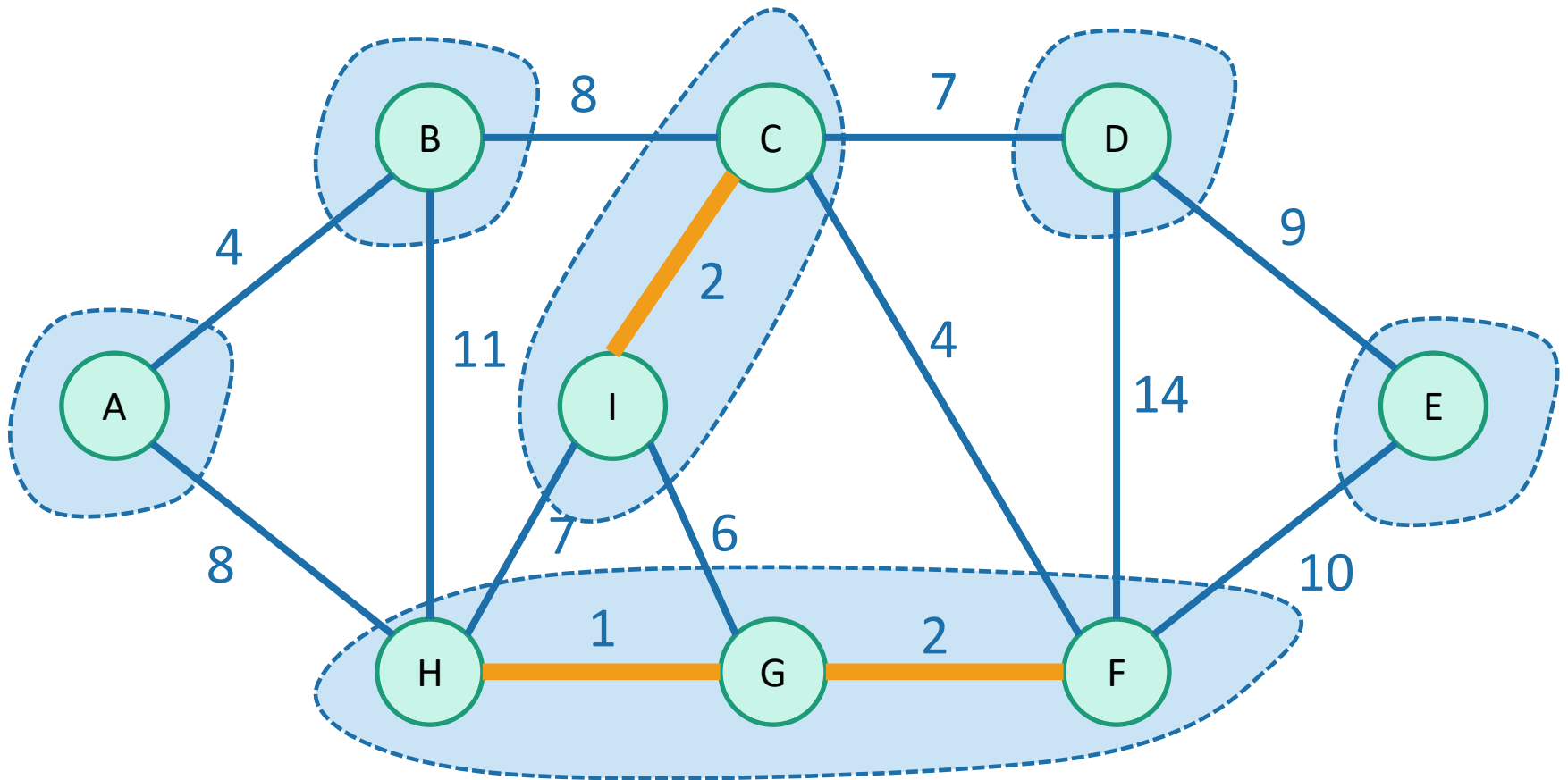
# Once more...

Then start merging.



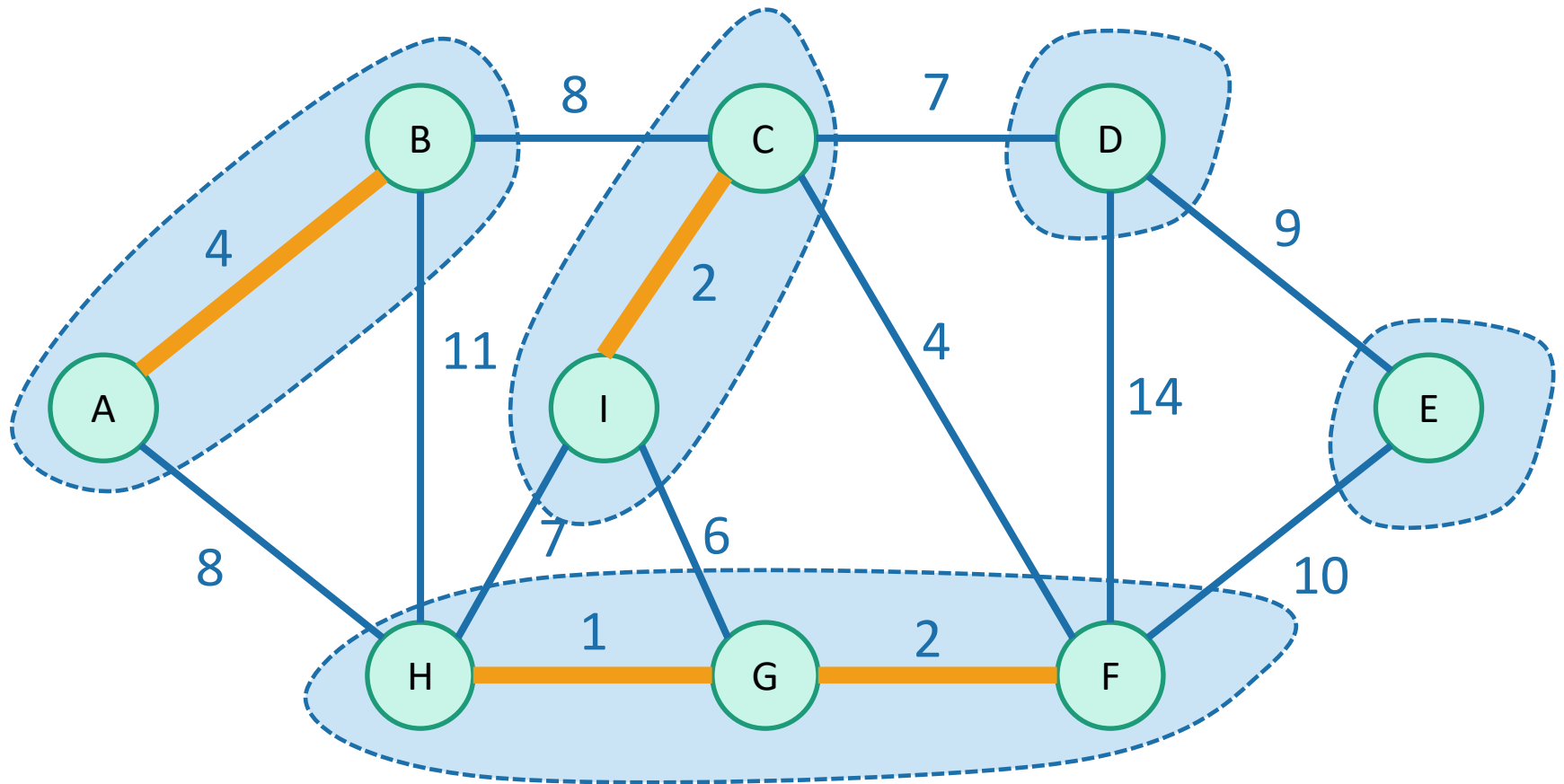
# Once more...

Then start merging.



# Once more...

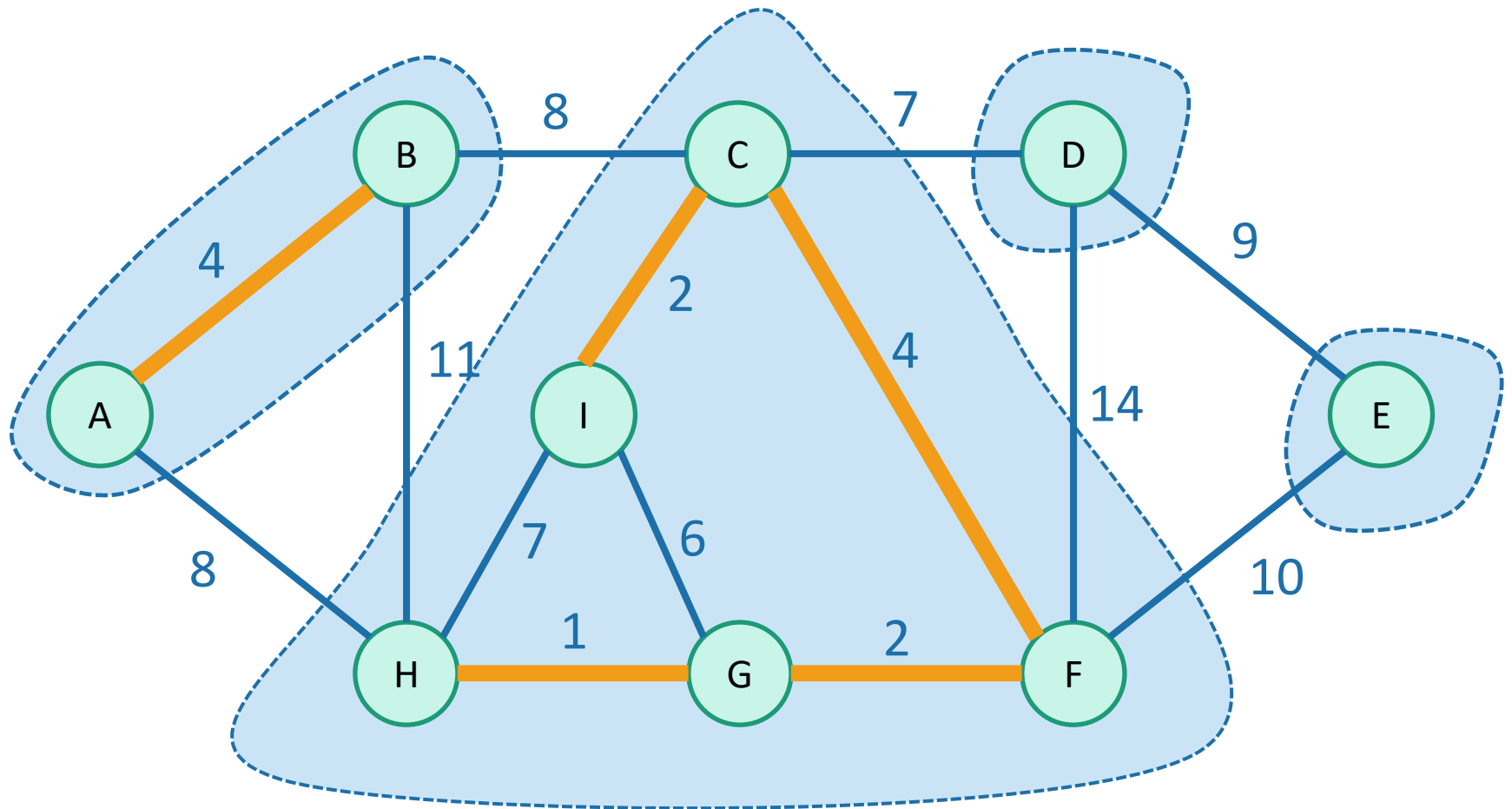
Then start merging.





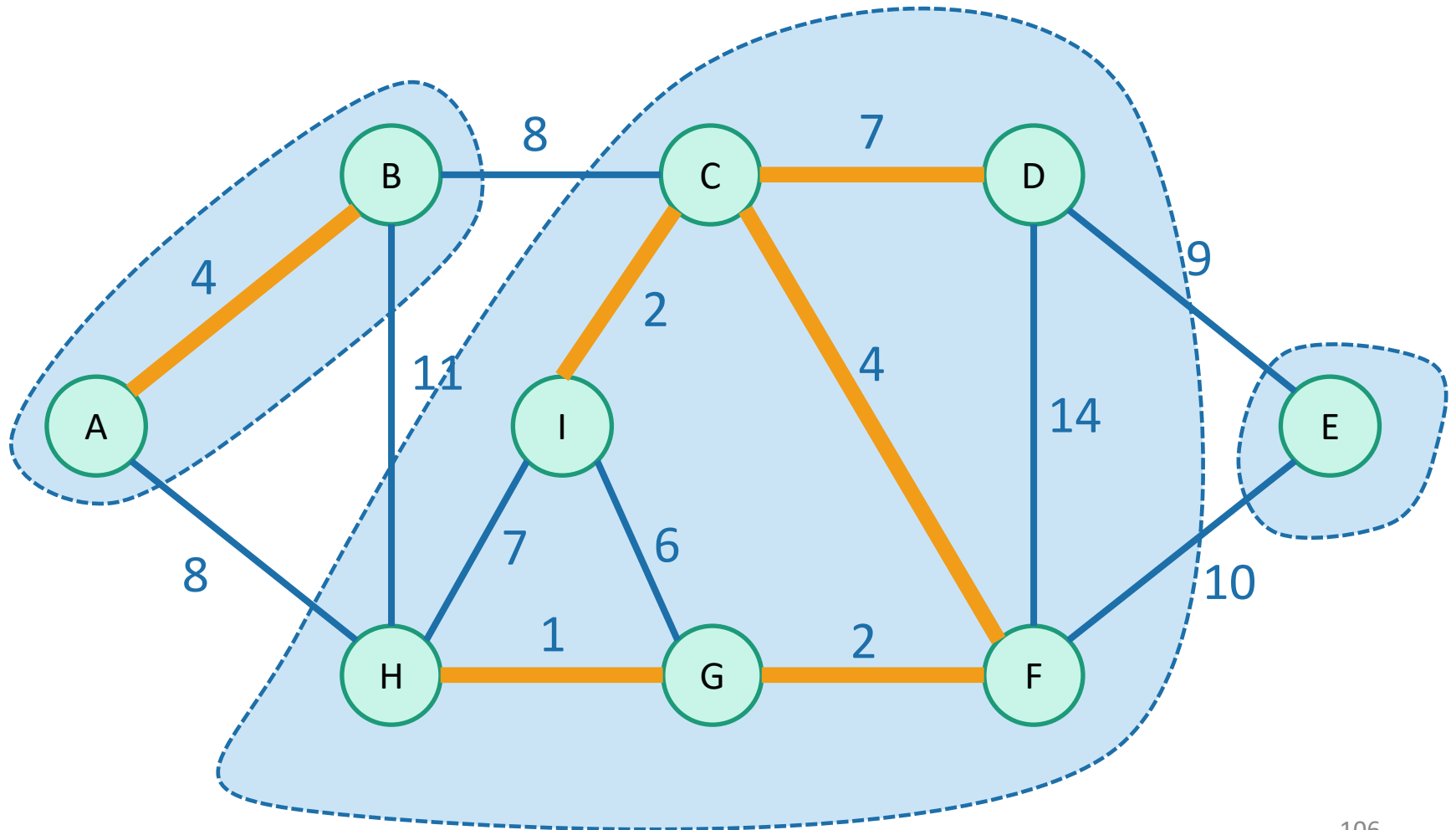
# Once more...

Then start merging.



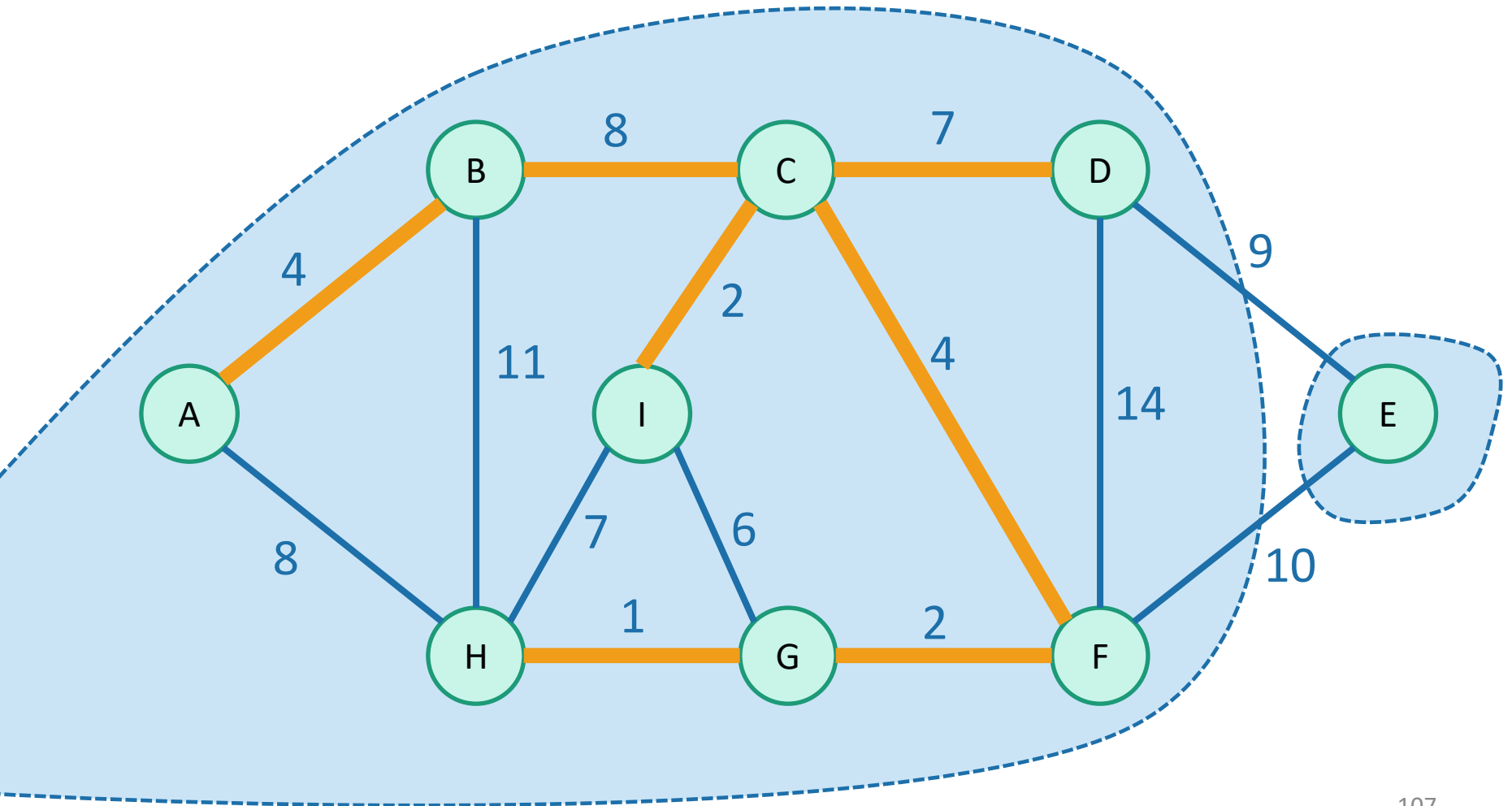
# Once more...

Then start merging.



# Once more...

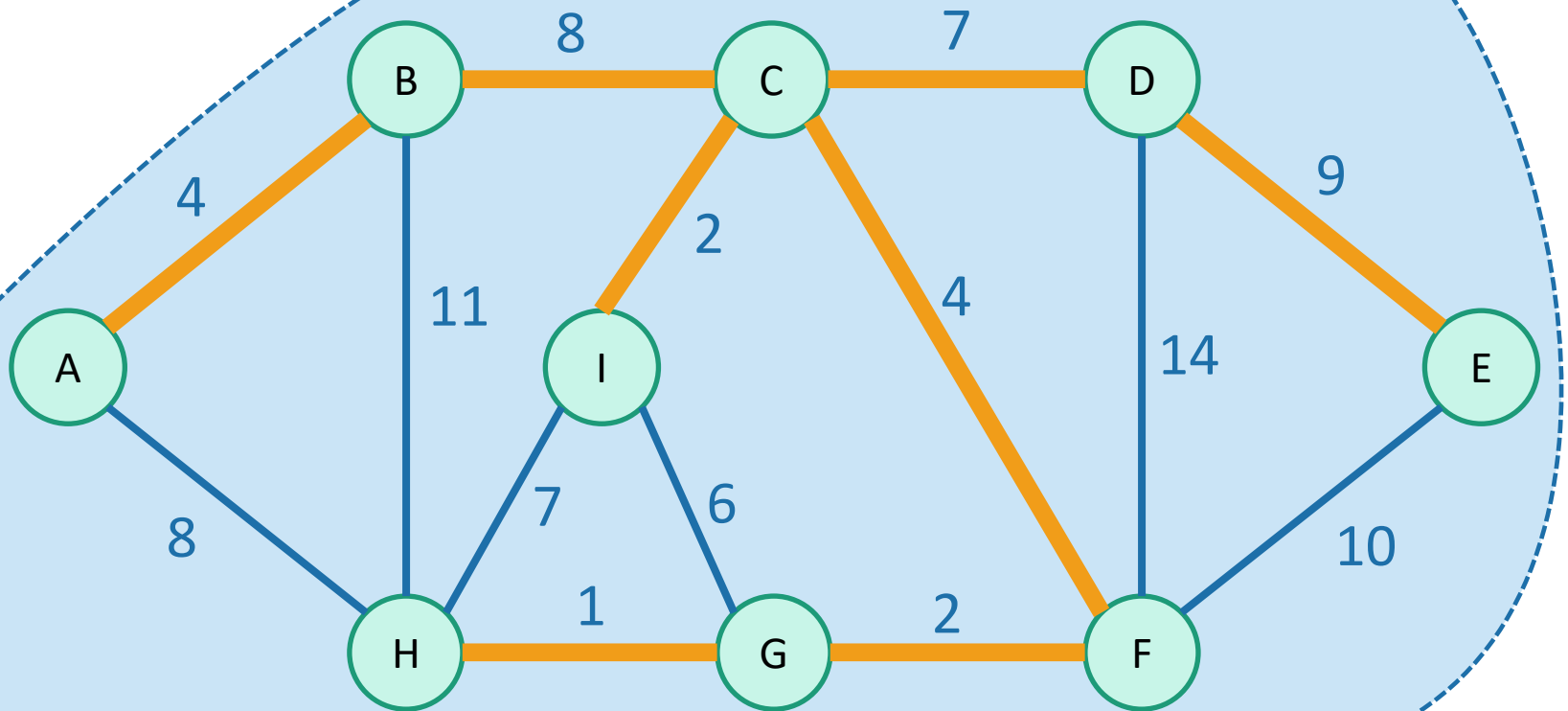
Then start merging.



Stop when we have one big tree!

# Once more...

Then start merging.



# Running time

- Sorting the edges takes  $O(m \log(n))$ 
  - In practice, if the weights are small integers we can use radixSort and take time  $O(m)$
- For the rest:
  - $n$  calls to **makeSet**
    - put each vertex in its own set
  - $2m$  calls to **find**
    - for each edge, **find** its endpoints
  - $n$  calls to **union**
    - we will never add more than  $n-1$  edges to the tree,
    - so we will never call **union** more than  $n-1$  times.
- Total running time:
  - Worst-case  $O(m \log(n))$ , just like Prim with an RBtree.
  - Closer to  $O(m)$  if you can do radixSort

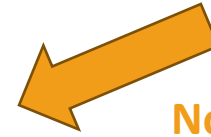
In practice, each of **makeSet**, **find**, and **union** run in constant time\*

\*technically, they run in *amortized time*  $O(\alpha(n))$ , where  $\alpha(n)$  is the *inverse Ackerman function*.  
 $\alpha(n) \leq 4$  provided that  $n$  is smaller than the number of atoms in the universe.

# Two questions

## 1. Does it work?

- That is, does it actually return a MST?



Now that we understand this “tree-merging” view, let’s do this one.

## 2. How do we actually implement this?

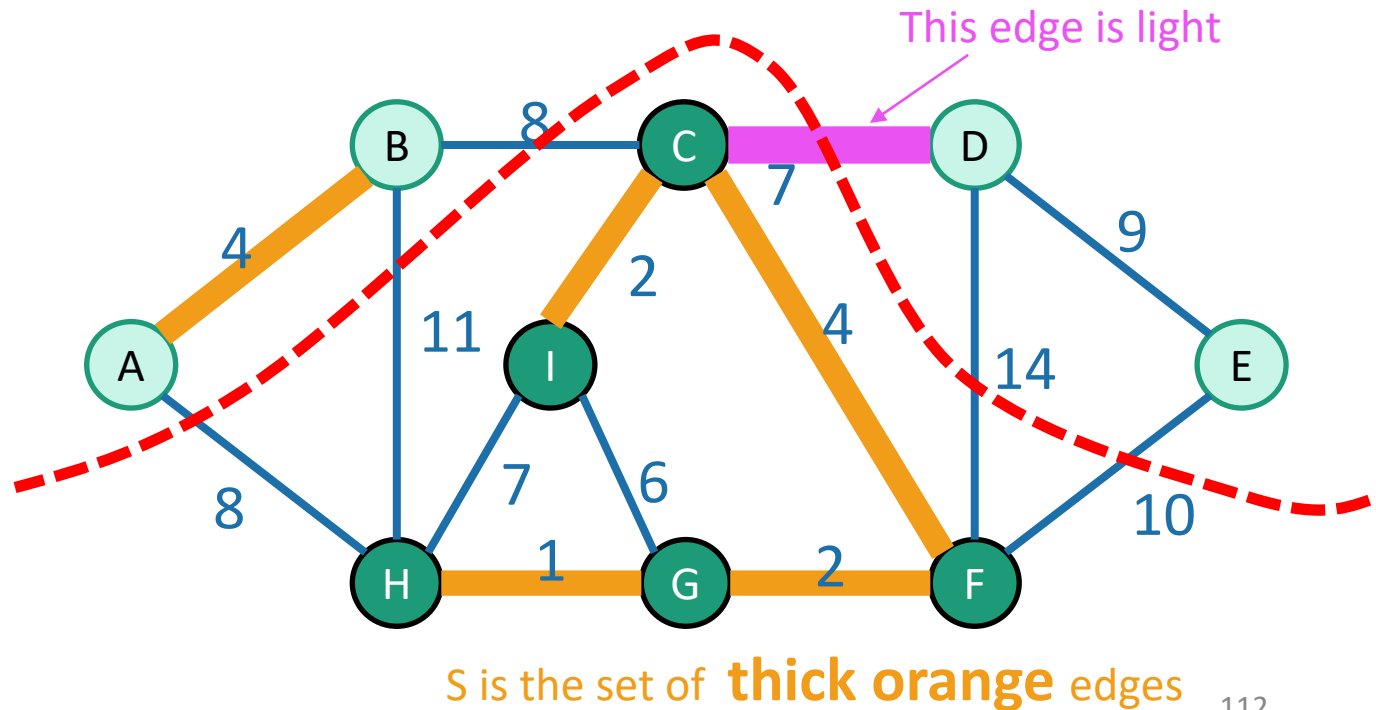
- the pseudocode above says “slowKruskal”...
  - **Worst-case running time  $O(m \log(n))$  using a union-find data structure.**

# Does it work?

- We need to show that our greedy choices **don't rule out success**.
- That is, at every step:
  - There exists an MST that contains all of the edges we have added so far.
- Now it is time to use our lemma!  
again!

# Lemma

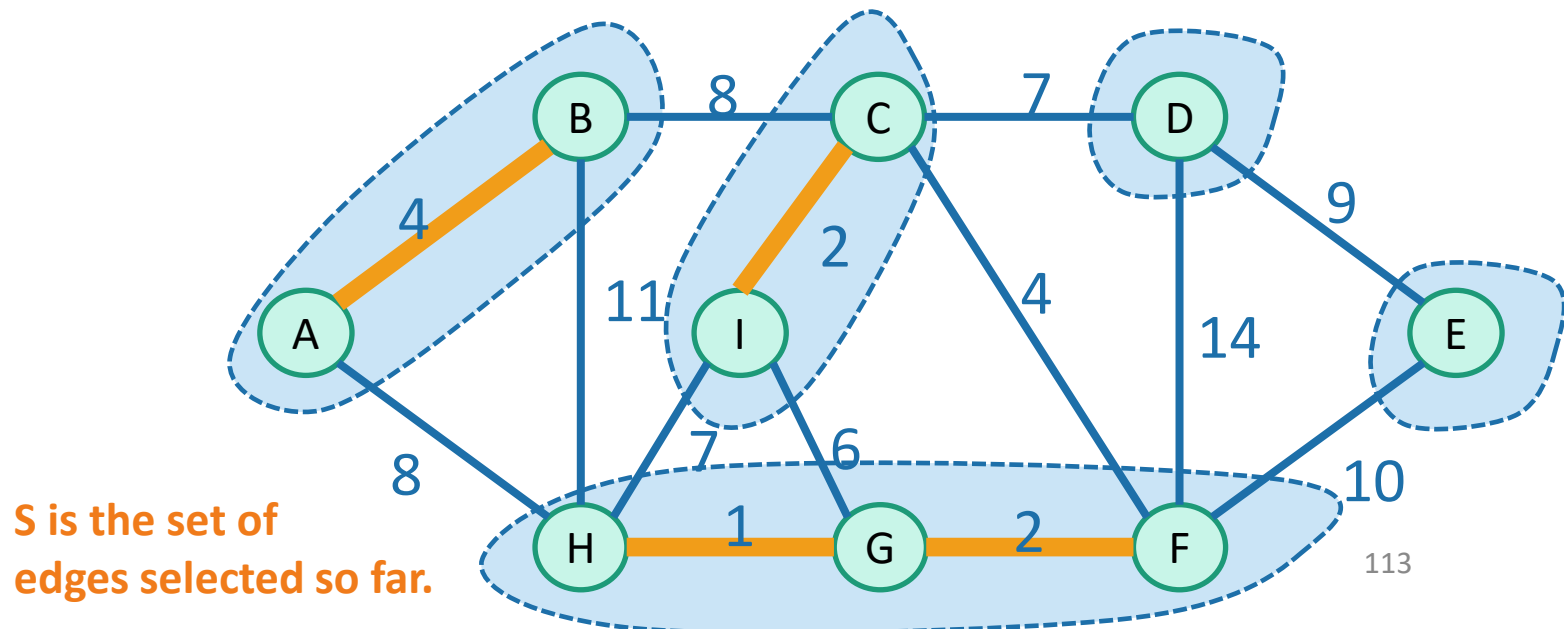
- Let  $S$  be a set of edges, and consider a cut that respects  $S$ .
- Suppose there is an MST containing  $S$ .
- Let  $\{u,v\}$  be a light edge.
- Then there is an MST containing  $S \cup \{u,v\}$





# Partway through Kruskal

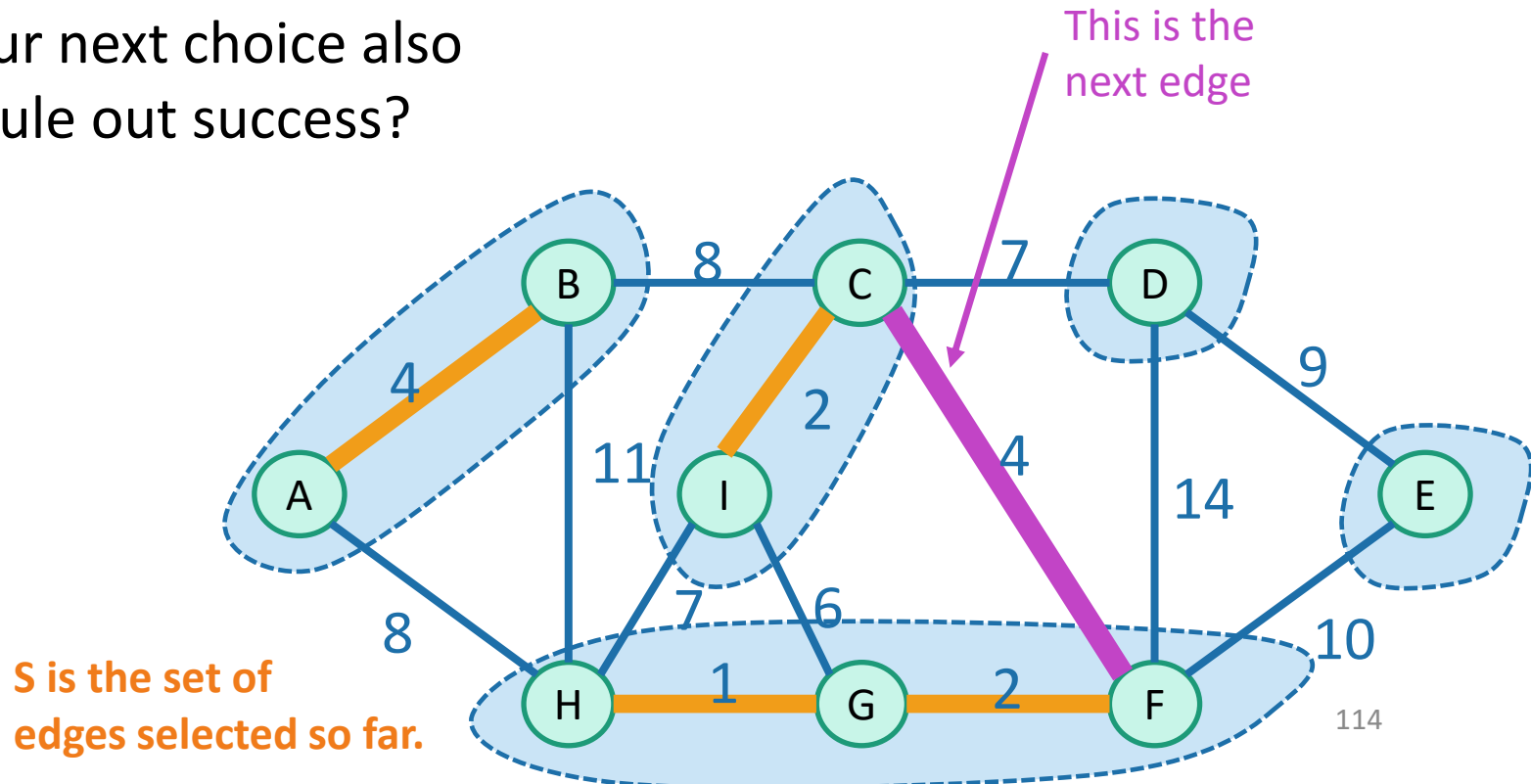
- Assume that our choices **S** so far don't rule out success.
  - There is an MST extending them
- The **next edge** we add will merge two trees, **T1**, **T2**



# Partway through Kruskal

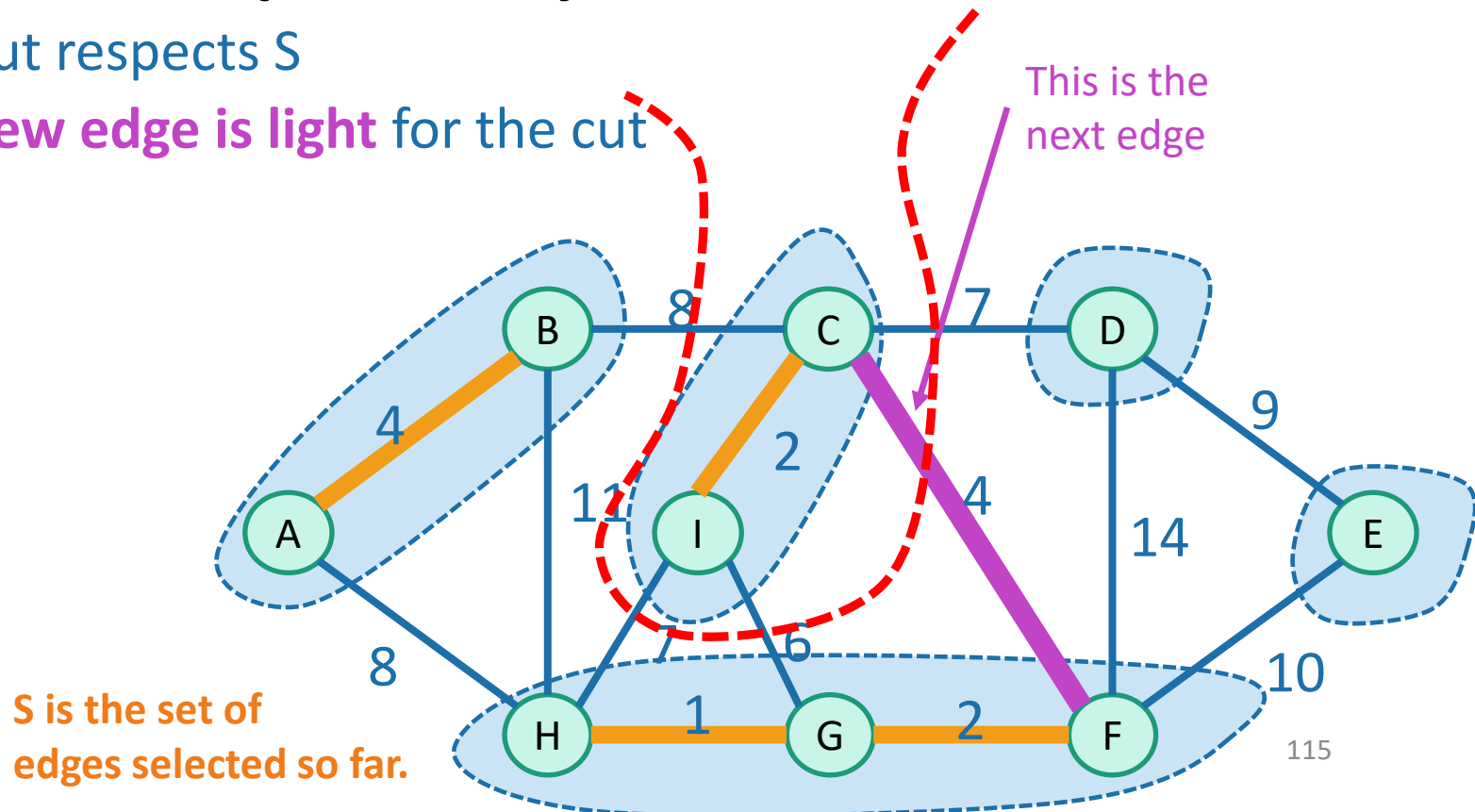
- Assume that our choices **S** so far don't rule out success.
  - There is an MST extending them
- The **next edge** we add will merge two trees, **T1**, **T2**

How can we use our lemma to show that our next choice also does not rule out success?



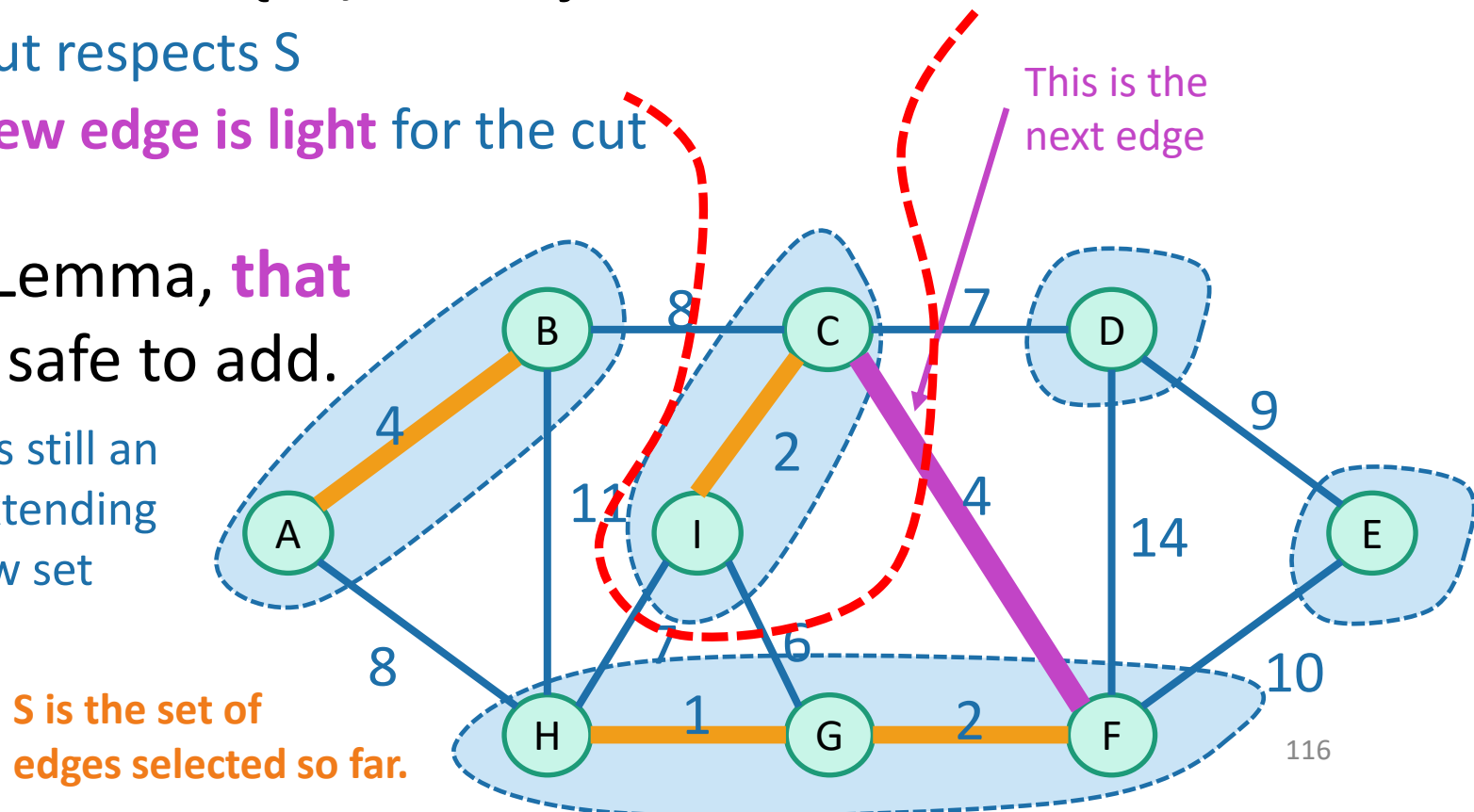
# Partway through Kruskal

- Assume that our choices **S** so far don't rule out success.
  - There is an MST extending them
- The **next edge** we add will merge two trees, **T1**, **T2**
- Consider the cut  $\{T1, V - T1\}$ .
  - This cut respects S
  - Our **new edge is light** for the cut



# Partway through Kruskal

- Assume that our choices **S** so far don't rule out success.
  - There is an MST extending them
- The **next edge** we add will merge two trees, **T1**, **T2**
- Consider the cut  $\{T1, V - T1\}$ .
  - This cut respects S
  - Our **new edge is light** for the cut
- By the Lemma, **that edge** is safe to add.
  - There is still an MST extending the new set



# Good news

- Our greedy choices **don't rule out success.**
- This is enough (along with an argument by induction) to guarantee correctness of Kruskal's algorithm.

# Two questions

1. Does it work?

- That is, does it actually return a MST?

- **Yes**

2. How do we actually implement this?

- the pseudocode above says “slowKruskal”...

- **Using a union-find data structure!**

# What have we learned?

- Kruskal's algorithm greedily grows a forest
- It finds a Minimum Spanning Tree in time  $O(m \log(n))$ 
  - if we implement it with a Union-Find data structure
  - if the edge weights are reasonably-sized integers and we ignore the inverse Ackerman function, basically  $O(m)$  in practice.
- To prove it worked, we followed the same recipe for greedy algorithms we saw last time.
  - Show that, at every step, we **don't rule out success**.

# Compare and contrast

- Prim:

- Grows a tree.
- Time  $O(m \log(n))$  with a red-black tree
- Time  $O(m + n \log(n))$  with a Fibonacci heap

Prim might be a better idea  
on dense graphs if you can't  
Sort edge weights

- Kruskal:

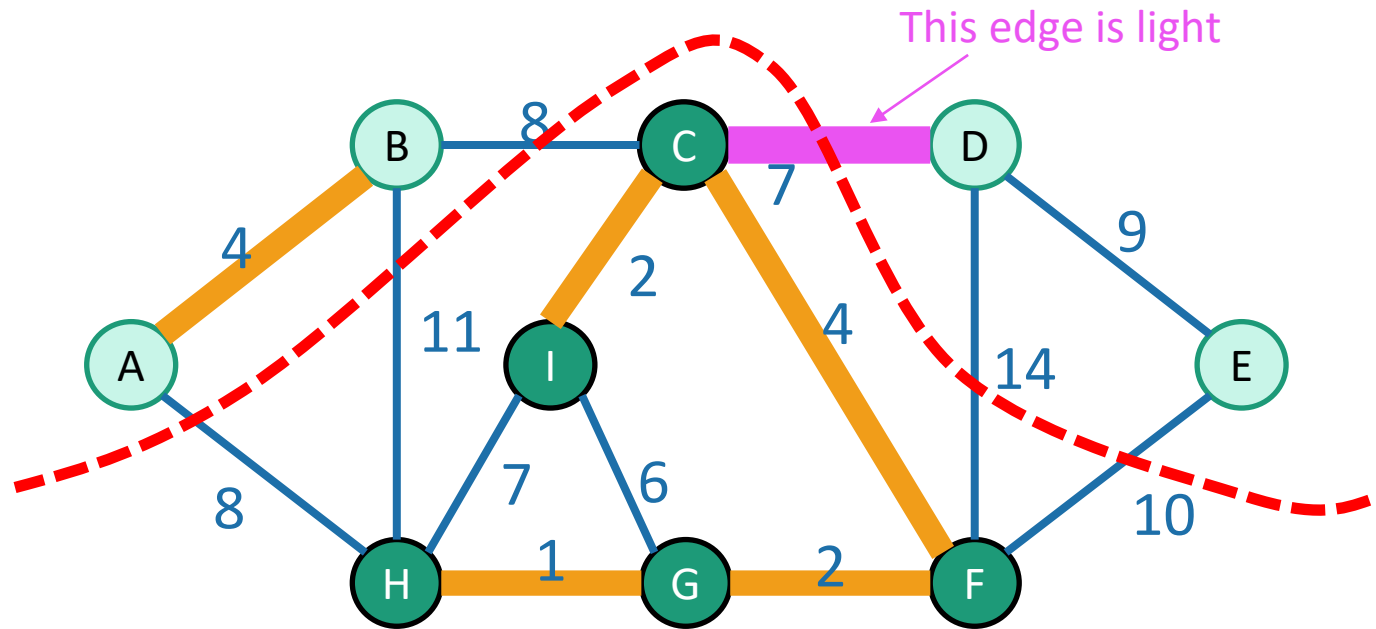
- Grows a forest.
- Time  $O(m \log(n))$  with a union-find data structure
- If you can do radixSort on the edge weights, morally  $O(m)$

Kruskal might be a better idea  
on sparse graphs if you can  
Sort edge weights



# Both Prim and Kruskal

- Greedy algorithms for MST.
- Similar reasoning:
  - Optimal substructure: subgraphs generated by cuts.
  - The way to make safe choices is to choose light edges crossing the cut.



S is the set of **thick orange** edges

# Can we do better?

State-of-the-art MST on connected undirected graphs

- Karger-Klein-Tarjan 1995:
  - $O(m)$  time randomized algorithm
- Chazelle 2000:
  - $O(m \cdot \alpha(n))$  time deterministic algorithm
- Pettie-Ramachandran 2002:
  - $O\left(\begin{array}{l} \text{The optimal number of comparisons} \\ N^*(n,m) \text{ you need to solve the} \\ \text{problem, whatever that is...} \end{array}\right)$  time deterministic algorithm

What is this number?

Do we need that silly  $\alpha(n)$ ?

Open questions!

# Recap

- Two algorithms for Minimum Spanning Tree
  - Prim's algorithm
  - Kruskal's algorithm
- Both are (more) examples of **greedy algorithms!**
  - Make a **series of choices.**
  - Show that at each step, your choice **does not rule out success.**
  - At the end of the day, you haven't ruled out success, so **you must be successful.**

## NEXT LECTURE

- Divide and Conquer
- Mergesort
- Counting Inversions
- Closest pair of points
- Karatsuba  
Multiplication

Week	Date	Topic
1	21-Feb	Introduction. Some representative problems
2	28-Feb	Stable Matching
3	7-Mar	Basics of algorithm analysis.
4	14-Mar	Graphs (Project 1 announced)
5	21-Mar	Greedy algorithms-I
6	28-Mar	Greedy algorithms-II
7	4-Apr	Divide and conquer (Project 2 announced)
8	11-Apr	Dynamic Programming I
9	18-Apr	Dynamic Programming II
10	25-Apr	Network Flow-I (Project 3 announced)
11	2-May	<b>Midterm</b>
12	9-May	Network Flow II
13	16-May	NP and computational intractability-I
14	23-May	NP and computational intractability-II