

BLG 336E

Analysis of Algorithms II

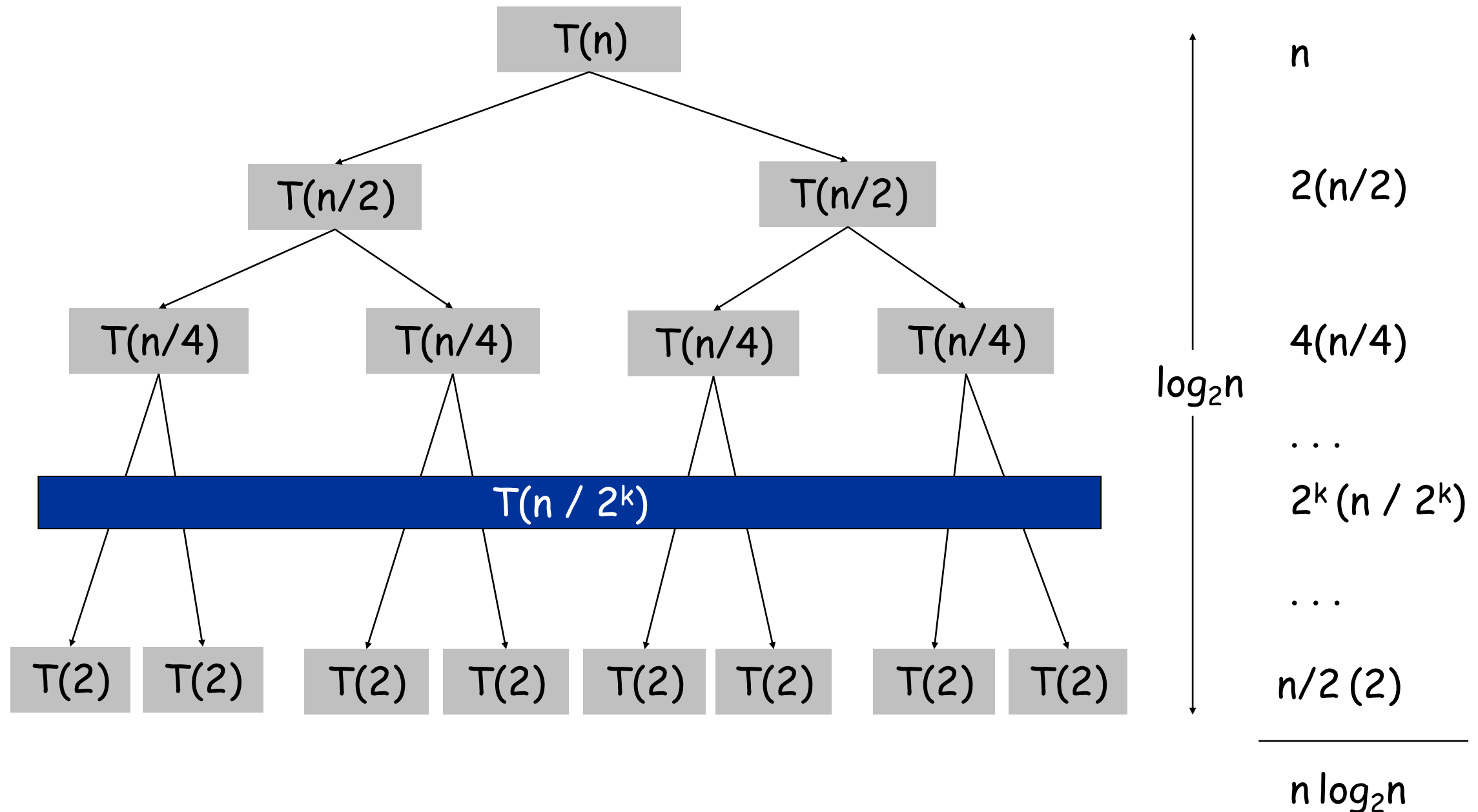
Lecture 8:

Dynamic Programming I

Weighted Interval Scheduling, Segmented Least Squares

Recap- Recursion Tree

$$T(n) = \begin{cases} 0 & \text{if } n = 1 \\ \underbrace{2T(n/2)}_{\text{sorting both halves}} + \underbrace{n}_{\text{merging}} & \text{otherwise} \end{cases}$$



Recap-Why Recurrences?

- The complexity of many interesting algorithms is easily expressed as a recurrence – especially divide and conquer algorithms
- The form of the algorithm often yields the form of the recurrence
- The complexity of recursive algorithms is readily expressed as a recurrence.

Recap-Why solve recurrences?

- To make it easier to compare the complexity of two algorithms
- To make it easier to compare the complexity of the algorithm to standard reference functions.

$$T(n) = a \cdot T\left(\frac{n}{b}\right) + O(n^d).$$

Recap-Master Theorem

$$T(n) = \begin{cases} O(n^d \log(n)) & \text{if } a = b^d \\ O(n^d) & \text{if } a < b^d \\ O(n^{\log_b(a)}) & \text{if } a > b^d \end{cases}$$

- Needlessly recursive integer mult.

- $T(n) = 4 T(n/2) + O(n)$
- $T(n) = O(n^2)$

$$\begin{aligned} a &= 4 \\ b &= 2 \\ d &= 1 \end{aligned}$$

$$a > b^d$$



- Karatsuba integer multiplication

- $T(n) = 3 T(n/2) + O(n)$
- $T(n) = O(n^{\log_2(3)} \approx n^{1.6})$

$$\begin{aligned} a &= 3 \\ b &= 2 \\ d &= 1 \end{aligned}$$

$$a > b^d$$



- MergeSort

- $T(n) = 2T(n/2) + O(n)$
- $T(n) = O(n \log(n))$

$$\begin{aligned} a &= 2 \\ b &= 2 \\ d &= 1 \end{aligned}$$

$$a = b^d$$



- That other one

- $T(n) = T(n/2) + O(n)$
- $T(n) = O(n)$

$$\begin{aligned} a &= 1 \\ b &= 2 \\ d &= 1 \end{aligned}$$

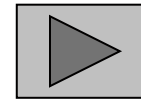
$$a < b^d$$



Recap-Counting Inversions: Combine

Combine: count blue-green inversions

- Assume each half is **sorted**.



See: 05demo-merge-

invert.ppt

- Count inversions where a_i and a_j are in different halves.
- Merge** two sorted halves into sorted whole

to maintain sorted invariant

3	7	10	14	18	19
---	---	----	----	----	----

2	11	16	17	23	25
6	3	2	2	0	0

13 blue-green inversions: $6 + 3 + 2 + 2 + 0 + 0$

Count: $O(n)$

2	3	7	10	11	14	16	17	18	19	23	25
---	---	---	----	----	----	----	----	----	----	----	----

Merge: $O(n)$

$$T(n) \leq 2T(n/2) + cn \Rightarrow T(n) = O(n \log n)$$

5.5 Integer Multiplication

This was kind of a big deal

$$XLIV \times XCVII = ?$$

$$\begin{array}{r} 44 \\ \times 97 \\ \hline \end{array}$$




Integer Multiplication

$$\begin{array}{r} 44 \\ \times 97 \\ \hline \end{array}$$

Integer Multiplication

n



233925720752752384623764283568364918374523856298
4562323582342395285623467235019130750135350013753
X

How fast is the grade-school
multiplication algorithm?

???

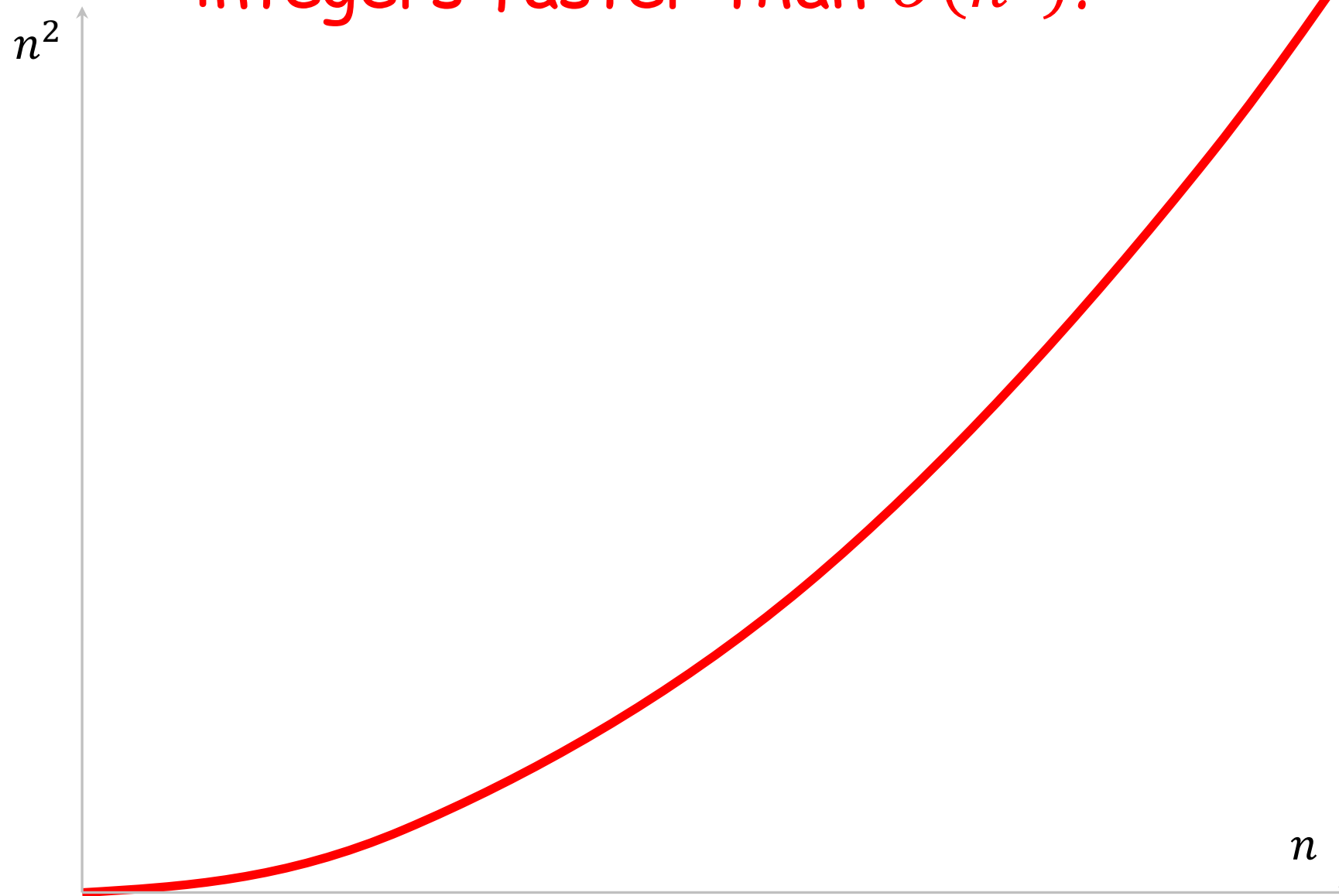
(How many one-digit operations?)

About n^2 one-digit operations

At most n^2 multiplications,
and then at most n^2 additions (for carries)
and then I have to add n different $2n$ -digit numbers...

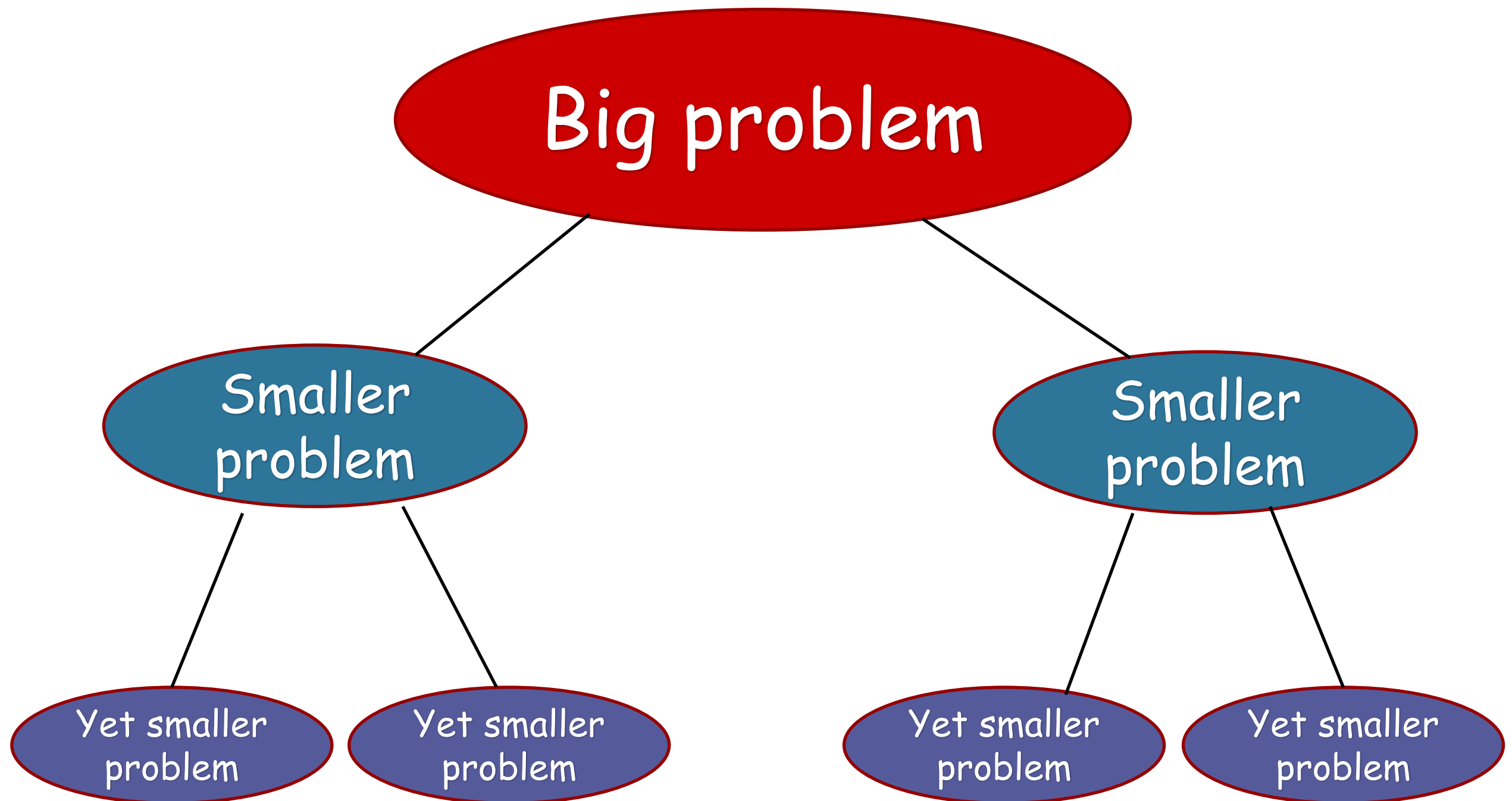
Can we do better?

Can we multiply n -digit
integers faster than $O(n^2)$?



Divide and conquer

Break problem up into smaller (easier)
sub-problems



Divide and conquer for multiplication

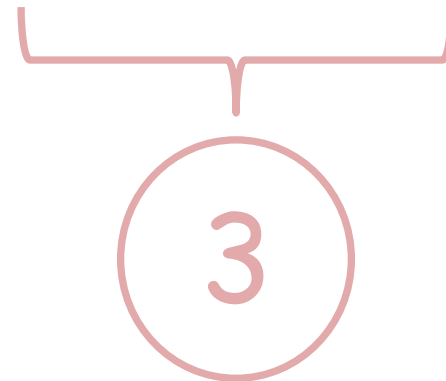
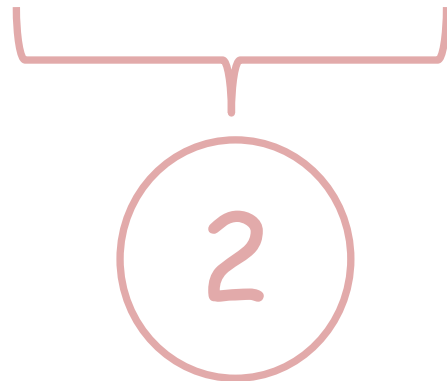
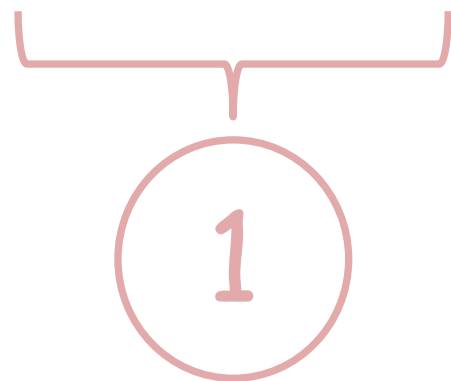
Break up an integer:

$$1234 = 12 \times 100 + 34$$

$$1234 \times 5678$$

$$= (12 \times 100 + 34) (56 \times 100 + 78)$$

$$= (12 \times 56) 10000 + (34 \times 56 + 12 \times 78) 100 + (34 \times 78)$$



One 4-digit multiply



Four 2-digit multiplies


More generally

Suppose n is even

Break up an n -digit integer:

$$[x_1 x_2 \cdots x_n] = [x_1 x_2 \cdots x_{n/2}] \times 10^{n/2} + [x_{n/2+1} x_{n/2+2} \cdots x_n]$$

$$\begin{aligned} x \times y &= (a \times 10^{n/2} + b)(c \times 10^{n/2} + d) \\ &= \underbrace{(a \times c)}_{\textcircled{1}} 10^n + \underbrace{(a \times d + c \times b)}_{\textcircled{2}} 10^{n/2} + \underbrace{(b \times d)}_{\textcircled{4}} \end{aligned}$$

One n -digit multiply  Four $(n/2)$ -digit multiplies

Divide and conquer algorithm

not very precisely...

(Assume n is a power of 2...)

x, y are n -digit numbers

Multiply(x, y):

- If $n=1$:

- Return xy

Base case: I've memorized my
1-digit multiplication tables...

- Write $x = a 10^{\frac{n}{2}} + b$

- Write $y = c 10^{\frac{n}{2}} + d$

a, b, c, d are
 $n/2$ -digit numbers

- Recursively compute ac, ad, bc, bd :

- $ac = \mathbf{Multiply}(a, c)$, etc..

- Add them up to get xy :

- xy = $ac 10^n + (ad + \underline{bc}) 10^{n/2} + \underline{bd}$

Make this pseudocode
more detailed! How
should we handle odd n ?
How should we implement
“multiplication by 10^n ”?

Think-Pair-Share

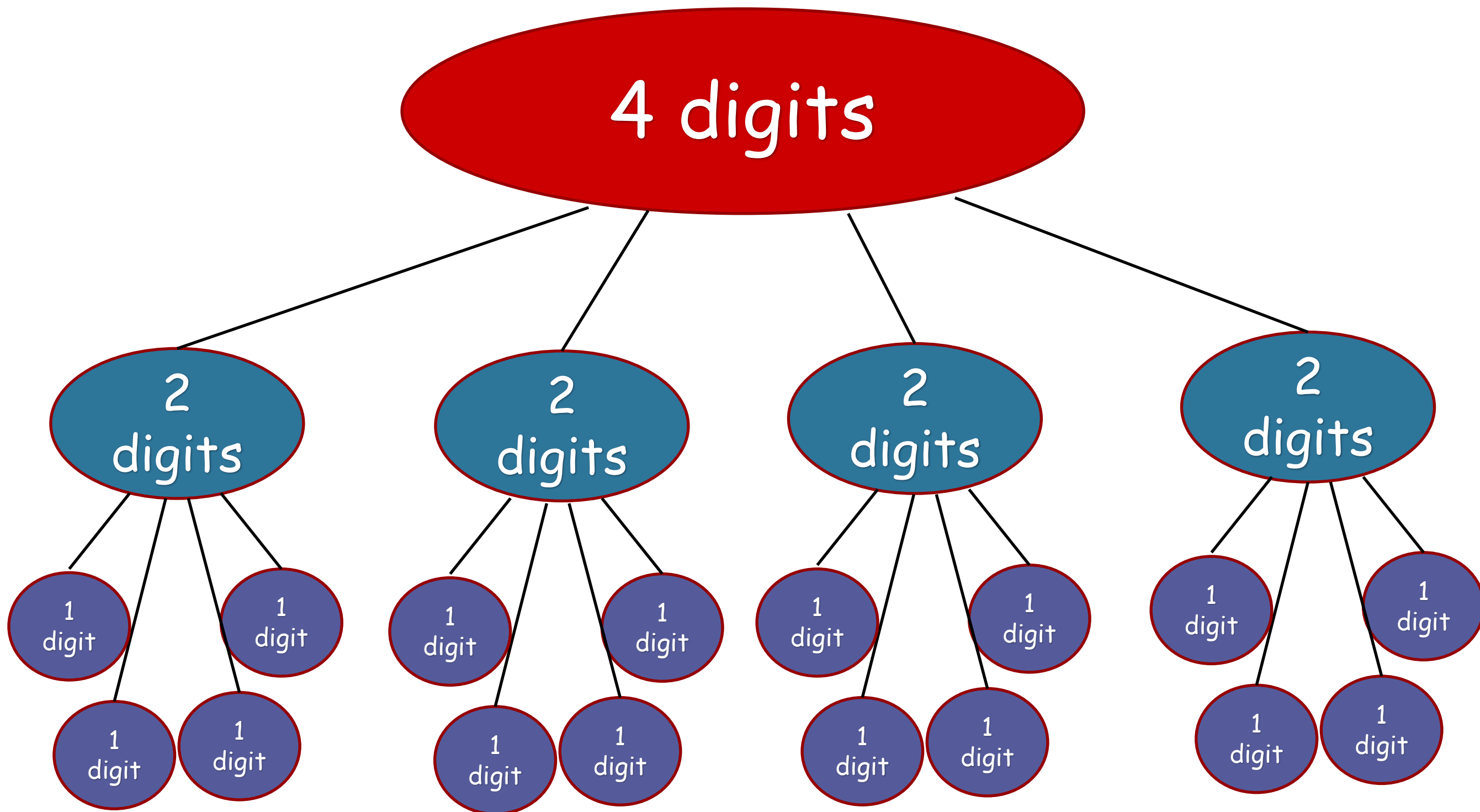
- We saw that this 4-digit multiplication problem broke up into four 2-digit multiplication problems

$$1234 \times 5678$$

- If you recurse on those 2-digit multiplication problems, how many 1-digit multiplications do you end up with total?

Recursion Tree

16 one-digit multiplies!



What is the running time?

- Better or worse than the grade school algorithm?
- How do we answer this question?
 1. Try it.
 2. Try to understand it analytically.

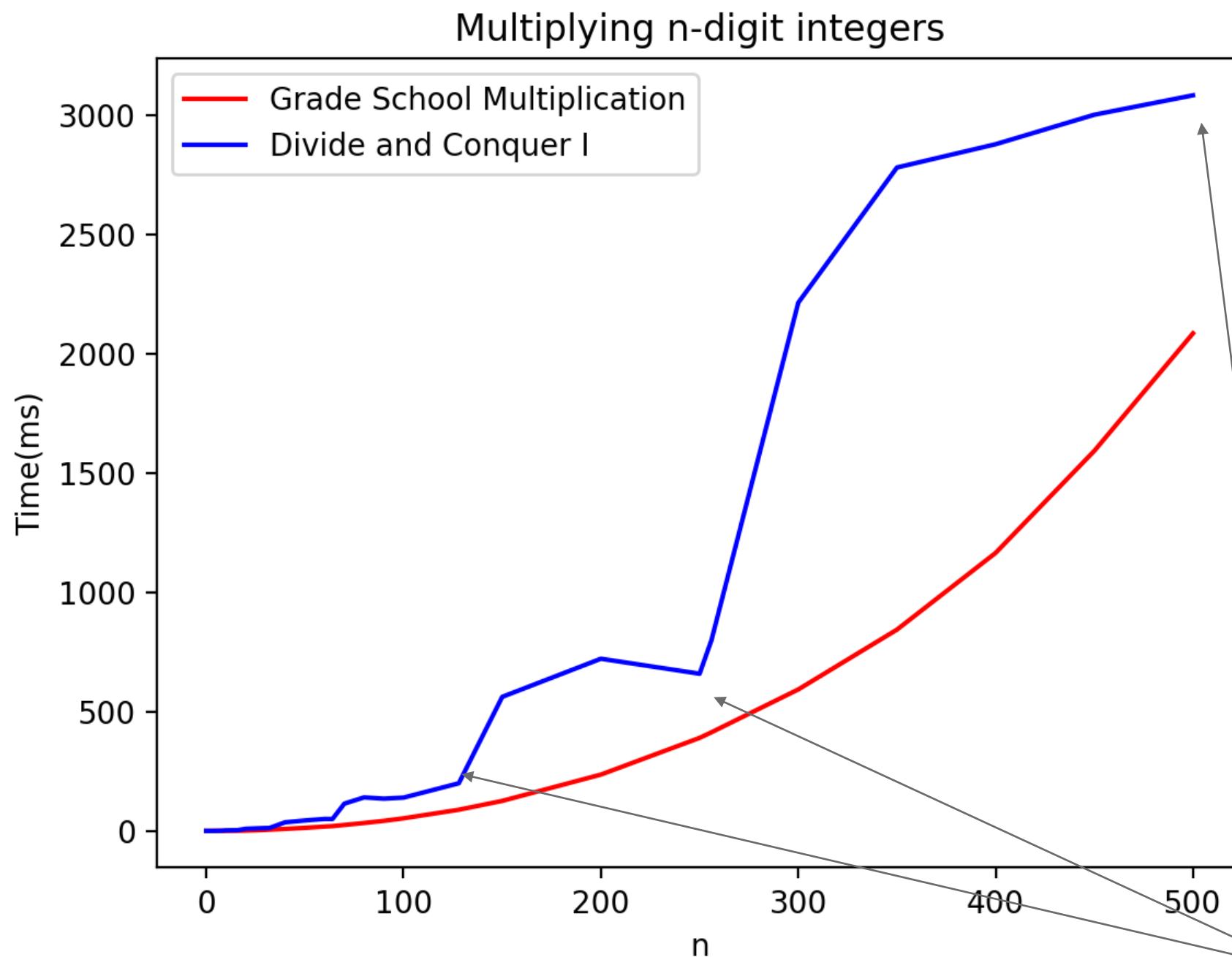
1. Try it.

Conjectures about running time?

Doesn't look too good
but hard to tell...

Maybe one implementation
is slicker than the other?

Maybe if we were to run it
to $n=10000$, things would
look different.



Something funny is happening at powers of 2... 19

2. Try to understand the running time analytically

- ~~. Proof by meta-reasoning:~~

~~It must be faster than the grade school algorithm, because we are learning it in an algorithms class.~~

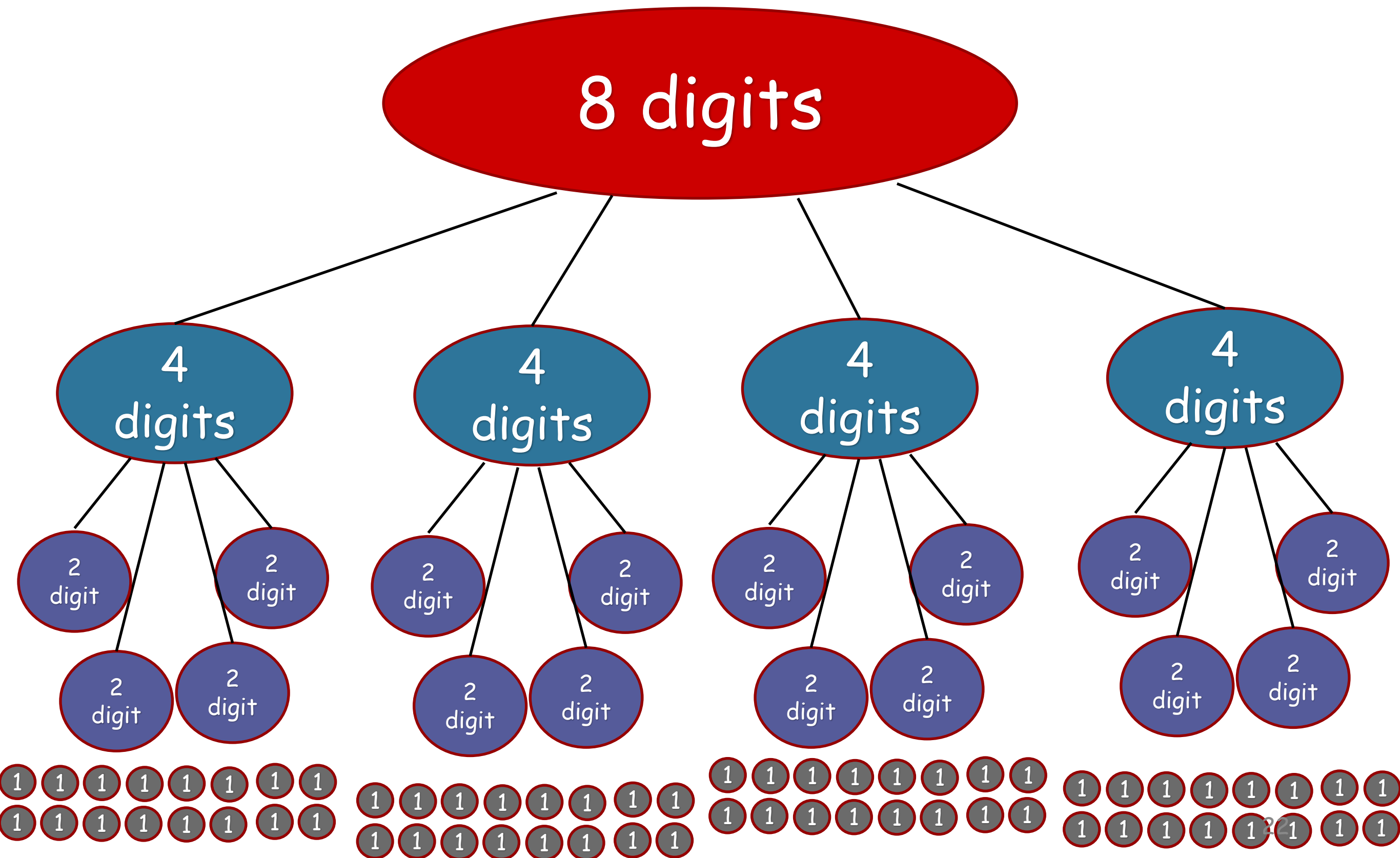
2. Try to understand the running time analytically

Think-Pair-Share:

- We saw that multiplying 4-digit numbers resulted in 16 one-digit multiplications.
- How about multiplying 8-digit numbers?
- What do you think about n -digit numbers?

Recursion Tree

64 one-digit multiplies!

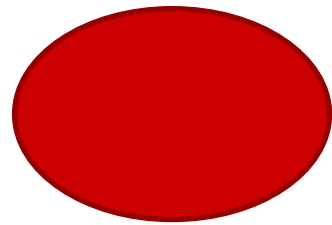


2. Try to understand the running time analytically

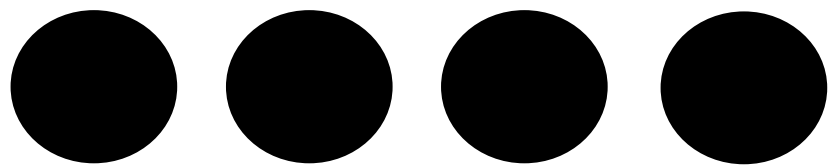
Claim:

The running time of this algorithm
is
AT LEAST n^2 operations.

There are n^2 1-digit problems

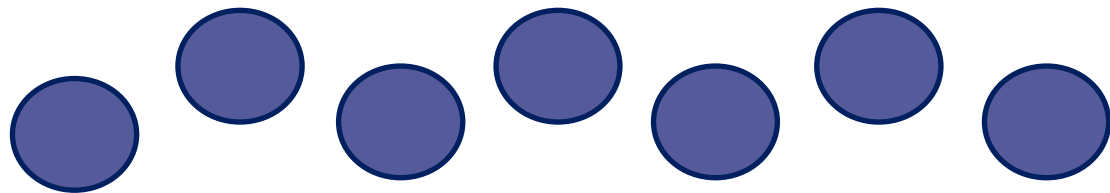


1 problem
of size n



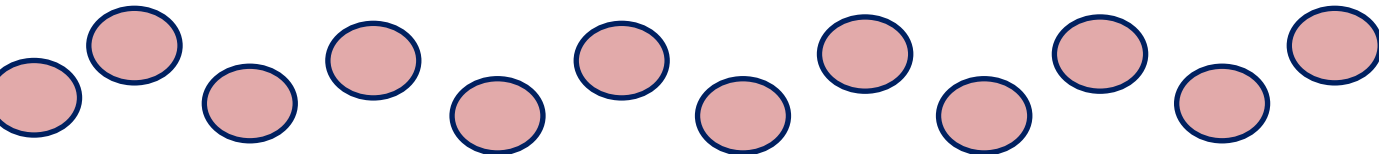
4 problems
of size $n/2$

...



4^t problems
of size $n/2^t$

...



$\frac{n^2}{1}$ problems
of size 1

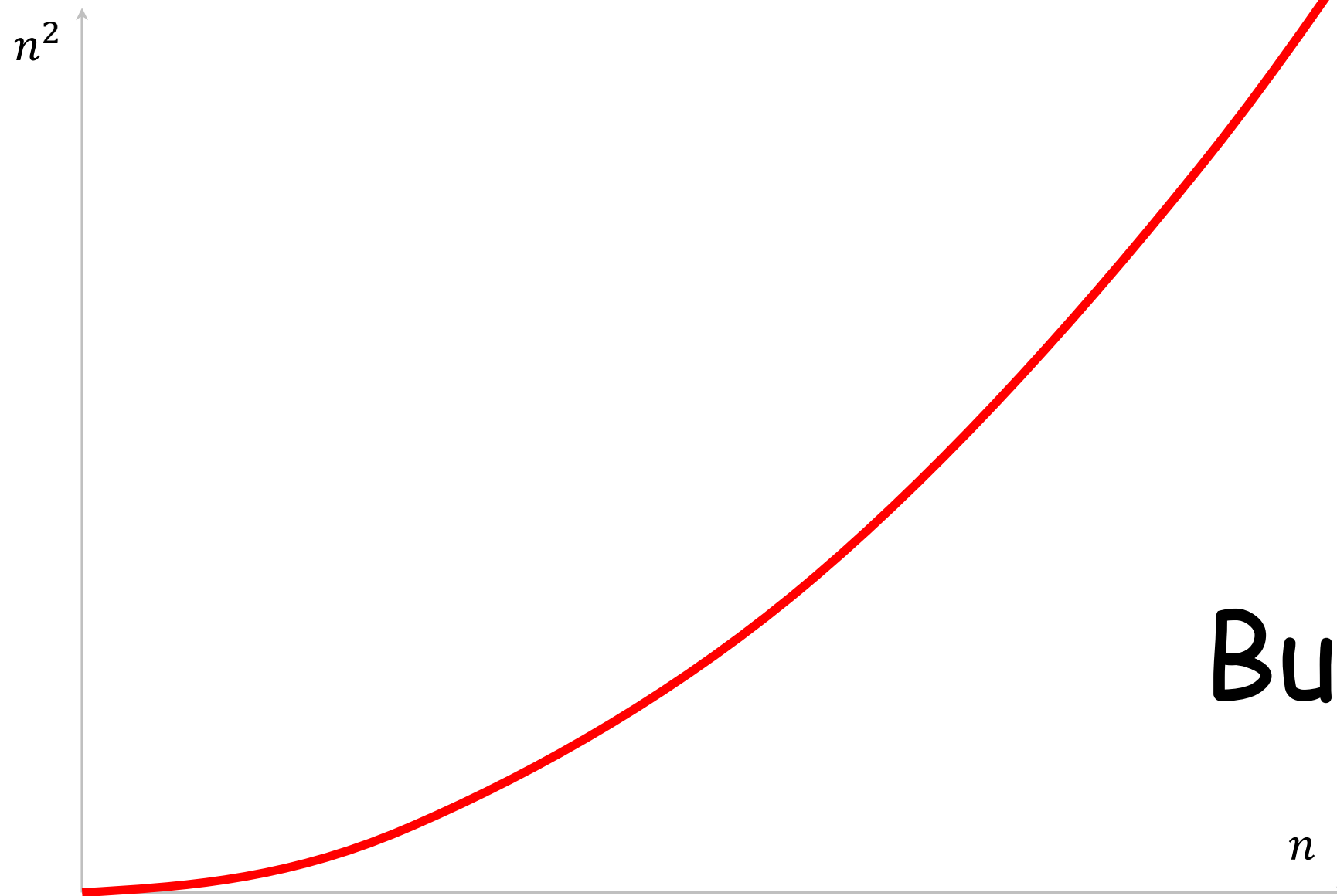
- If you cut n in half $\log_2(n)$ times, you get down to 1.
- So at level $t = \log_2(n)$ we get...

$$4^{\log_2 n} = n^{\log_2 4} = n^2$$

problems of size 1.

Note: this is just a cartoon - I'm not going to draw all 4^t circles!

That's a bit disappointing
All that work and still (at least) $O(n^2)$..



But wait!!

Divide and conquer **can** actually make progress

- Karatsuba figured out how to do this better!

$$\begin{aligned} xy &= (a \cdot 10^{n/2} + b)(c \cdot 10^{n/2} + d) \\ &= ac \cdot 10^n + (ad + bc)10^{n/2} + bd \end{aligned}$$

Need these three things



- If only we could recurse on three things instead of four...

Karatsuba integer multiplication

- Recursively compute these **THREE** things:

- ac
- bd
- $(a+b)(c+d)$

Subtract these off

get this

$$(a+b)(c+d) = ac + bd + bc + ad$$

- Assemble the product:

$$\begin{aligned} xy &= (a \cdot 10^{n/2} + b)(c \cdot 10^{n/2} + d) \\ &= ac \cdot 10^n + (ad + bc)10^{n/2} + bd \end{aligned}$$



How would this work?

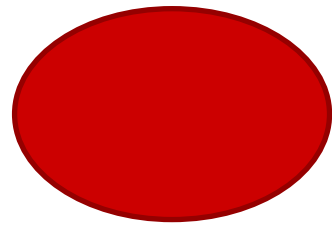
x, y are n-digit numbers

Multiply(x, y):

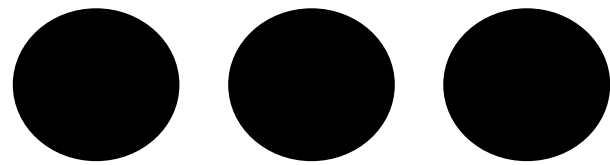
- **If** $n=1$:
 - **Return** xy
- Write $x = a 10^{\frac{n}{2}} + b$ and $y = c 10^{\frac{n}{2}} + d$
- $ac = \mathbf{Multiply}(a, c)$
- bd = **Multiply**(b, d)
- $z = \mathbf{Multiply}(\underline{a+b}, \underline{c+d})$
- xy = $ac 10^n + (z - ac - \underline{bd}) 10^{n/2} + \underline{bd}$
- **Return** xy

a, b, c, d are
 $n/2$ -digit numbers

What's the running time?

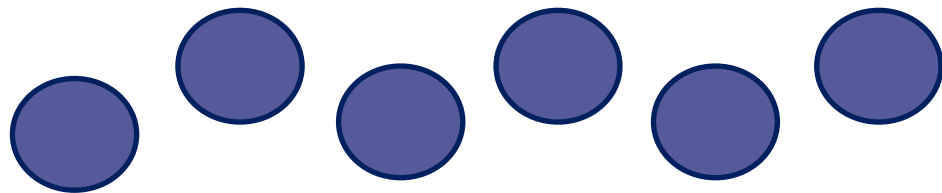


1 problem
of size n



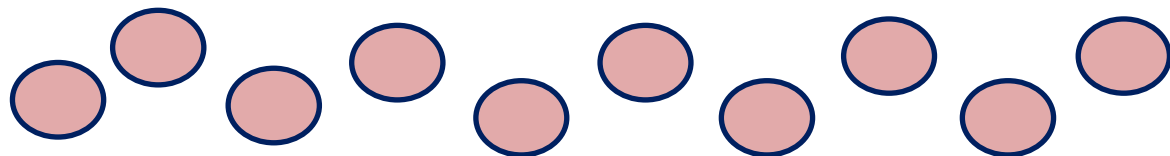
3 problems
of size $n/2$

...



3^2 problems
of size $n/2^2$

...



$n^{1.6}$ problems
of size 1

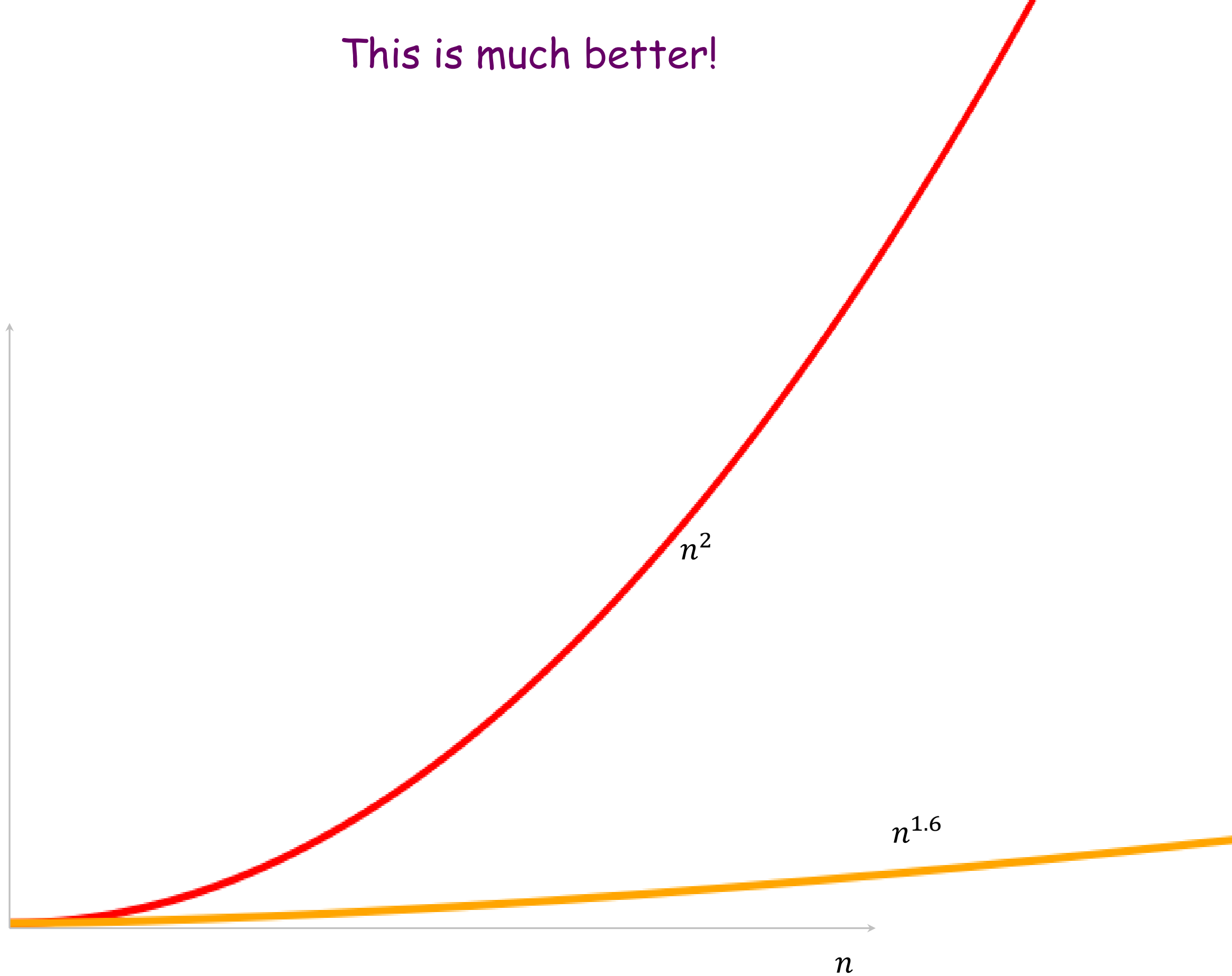
- If you cut n in half $\log_2(n)$ times, you get down to 1.
- So at level $t = \log_2(n)$ we get...

$$3^{\log_2 n} = n^{\log_2 3} \approx n^{1.6}$$

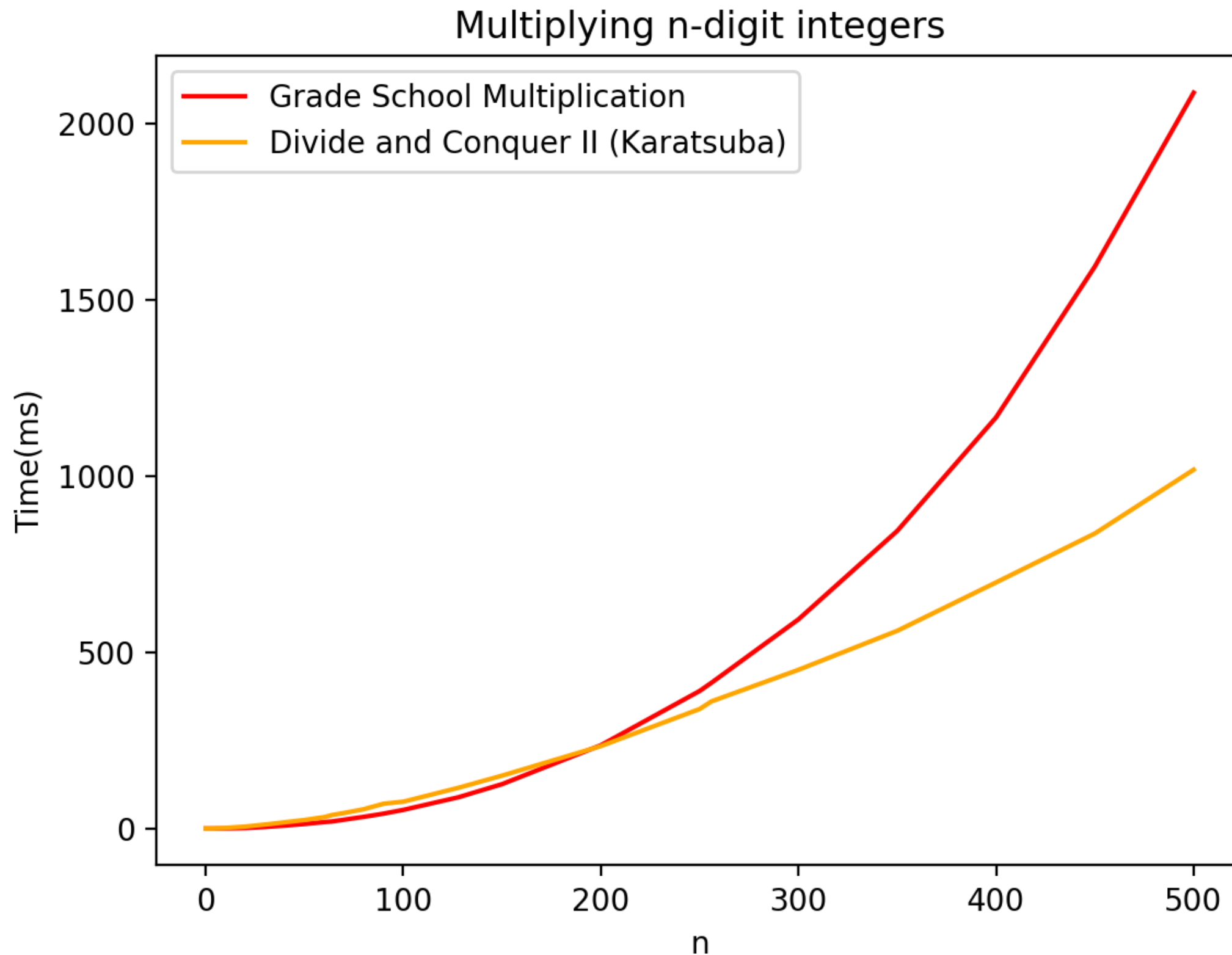
problems of size 1.

We aren't accounting for the work at the higher levels!
But we'll see later that this turns out to be okay.

This is much better!



We can even see it in real life!



Integer Arithmetic

Add. Given two n -digit integers a and b , compute $a + b$.

- $O(n)$ bit operations.

Multiply. Given two n -digit integers a and b , compute $a \times b$.

- Brute force solution: $\Theta(n^2)$ bit operations.

```
  12
  13
X-----
  36
 12
+-----
156
```

Decimal Multiplication

```
                1100
          1101
X-----
        1100
       0000
       1100
       1100
+-----
      1001100
```

Binary Multiplication

Divide-and-Conquer Multiplication: Warmup

To multiply two n -digit integers:

- Multiply four $\frac{1}{2}n$ -digit integers.
- Add two $\frac{1}{2}n$ -digit integers, and shift to obtain result.

$$\begin{aligned}x &= 2^{n/2} \times x_1 + x_0 \\y &= 2^{n/2} \times y_1 + y_0 \\xy &= \left(2^{n/2} \times x_1 + x_0\right) \left(2^{n/2} \times y_1 + y_0\right) = 2^n \times x_1 y_1 + 2^{n/2} \times (x_1 y_0 + x_0 y_1) + x_0 y_0\end{aligned}$$

$$T(n) = \underbrace{4T(n/2)}_{\text{recursive calls}} + \underbrace{cn}_{\text{add, shift}} \Rightarrow T(n) = \Theta(n^2)$$

etter
than brute force!

↑
assumes n is a power of 2

Karatsuba Multiplication

To multiply two n -digit integers:

- Add two $\frac{1}{2}n$ digit integers.
- Multiply **three** $\frac{1}{2}n$ -digit integers.
- Add, subtract, and shift $\frac{1}{2}n$ -digit integers to obtain result.

$$\begin{aligned}x &= 2^{n/2} \times x_1 + x_0 \\y &= 2^{n/2} \times y_1 + y_0 \\xy &= 2^n \times x_1 y_1 + 2^{n/2} \times (x_1 y_0 + x_0 y_1) + x_0 y_0 \\&= 2^n \times x_1 y_1 + 2^{n/2} \times ((x_1 + x_0)(y_1 + y_0) - x_1 y_1 - x_0 y_0) + x_0 y_0\end{aligned}$$

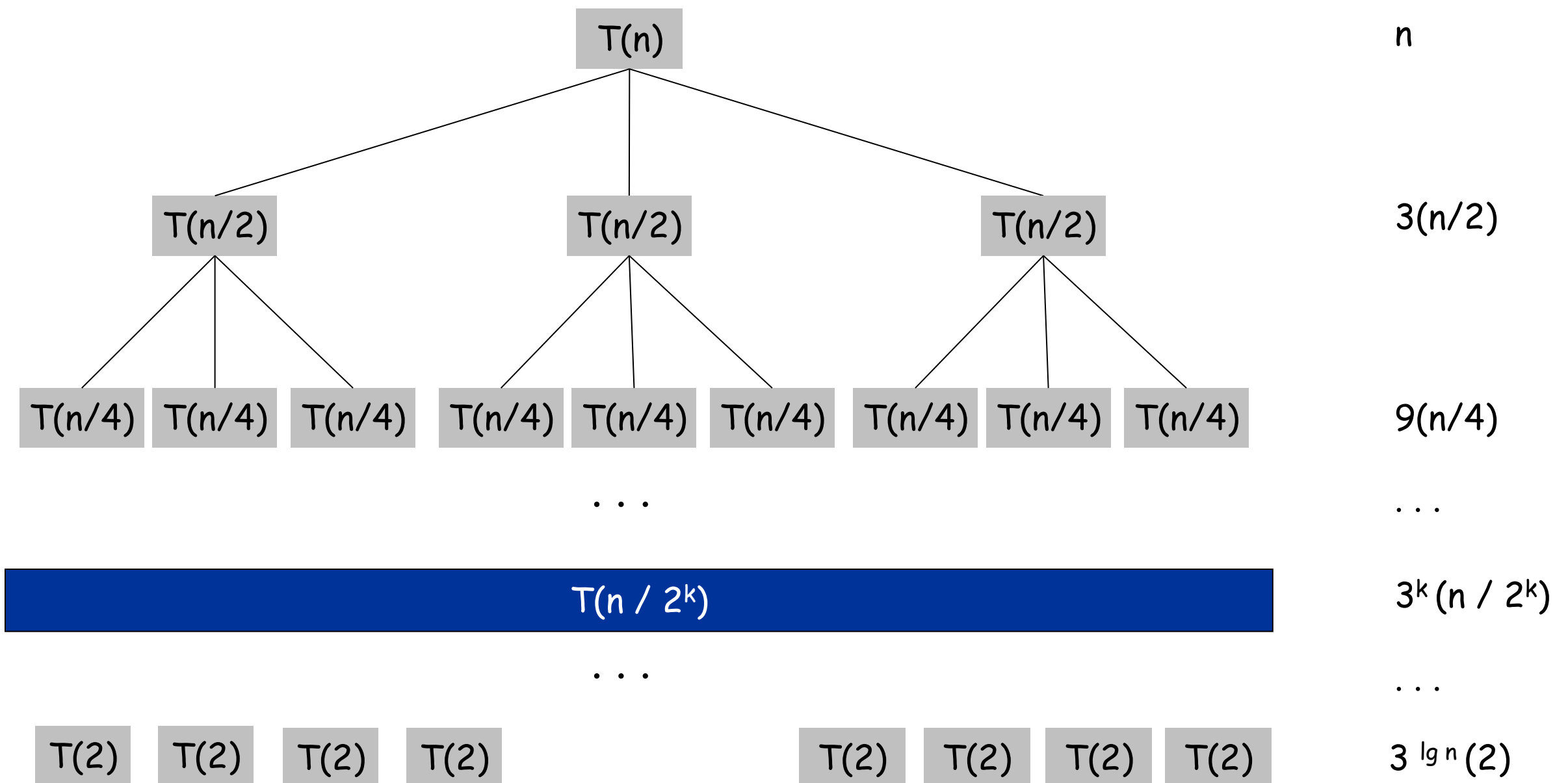
Theorem. [Karatsuba-Ofman, 1962] Can multiply two n -digit integers in $O(n^{1.585})$ bit operations.

$$T(n) = \underbrace{3T(n/2)}_{\text{recursive calls}} + \underbrace{cn}_{\text{add, subtract, shift}} \Rightarrow T(n) = O(n^{\log_2 3}) = O(n^{1.59})$$

Karatsuba: Recursion Tree

$$T(n) = \begin{cases} 0 & \text{if } n = 1 \\ 3T(n/2) + n & \text{otherwise} \end{cases}$$

$$T(n) = \sum_{k=0}^{\log_2 n} n \left(\frac{3}{2}\right)^k = \frac{\left(\frac{3}{2}\right)^{1+\log_2 n} - 1}{\frac{3}{2} - 1} = 3n^{\log_2 3} - 2$$



Matrix Multiplication

Matrix Multiplication

Matrix multiplication. Given two n -by- n matrices A and B , compute $C = AB$.

$$c_{ij} = \sum_{k=1}^n a_{ik} b_{kj}$$

$$\begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \times \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix}$$

Brute force. $\Theta(n^3)$ arithmetic operations.

Fundamental question. Can we improve upon brute force?

Matrix Multiplication: Warmup

Divide-and-conquer.

- Divide: partition A and B into $\frac{1}{2}n$ -by- $\frac{1}{2}n$ blocks.
- Conquer: multiply 8 $\frac{1}{2}n$ -by- $\frac{1}{2}n$ recursively.
- Combine: add appropriate products using 4 matrix additions.

$$\begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \cdot \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$$

$$\begin{aligned} C_{11} &= (A_{11} \cdot B_{11}) + (A_{12} \cdot B_{21}) \\ C_{12} &= (A_{11} \cdot B_{12}) + (A_{12} \cdot B_{22}) \\ C_{21} &= (A_{21} \cdot B_{11}) + (A_{22} \cdot B_{21}) \\ C_{22} &= (A_{21} \cdot B_{12}) + (A_{22} \cdot B_{22}) \end{aligned}$$

$$T(n) = \underbrace{8T(n/2)}_{\text{recursive calls}} + \underbrace{\Theta(n^2)}_{\text{add, form submatrices}} \Rightarrow T(n) = \Theta(n^3)$$

Matrix Multiplication: Key Idea

Key idea. multiply 2-by-2 block matrices with only **7** multiplications.

$$\begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$$

$$\begin{aligned} C_{11} &= P_5 + P_4 - P_2 + P_6 \\ C_{12} &= P_1 + P_2 \\ C_{21} &= P_3 + P_4 \\ C_{22} &= P_5 + P_1 - P_3 - P_7 \end{aligned}$$

$$\begin{aligned} P_1 &= (A_{11} - A_{12}) (B_{12} - B_{22}) \\ P_2 &= (A_{11} + A_{12}) B_{22} \\ P_3 &= (A_{21} + A_{22}) B_{11} \\ P_4 &= A_{22} (B_{21} - B_{11}) \\ P_5 &= (A_{11} + A_{22}) (B_{11} + B_{22}) \\ P_6 &= (A_{12} - A_{22}) (B_{21} + B_{22}) \\ P_7 &= (A_{11} - A_{21}) (B_{11} + B_{12}) \end{aligned}$$

- 7 multiplications.
- 18 = 10 + 8 additions (or subtractions).

Fast Matrix Multiplication

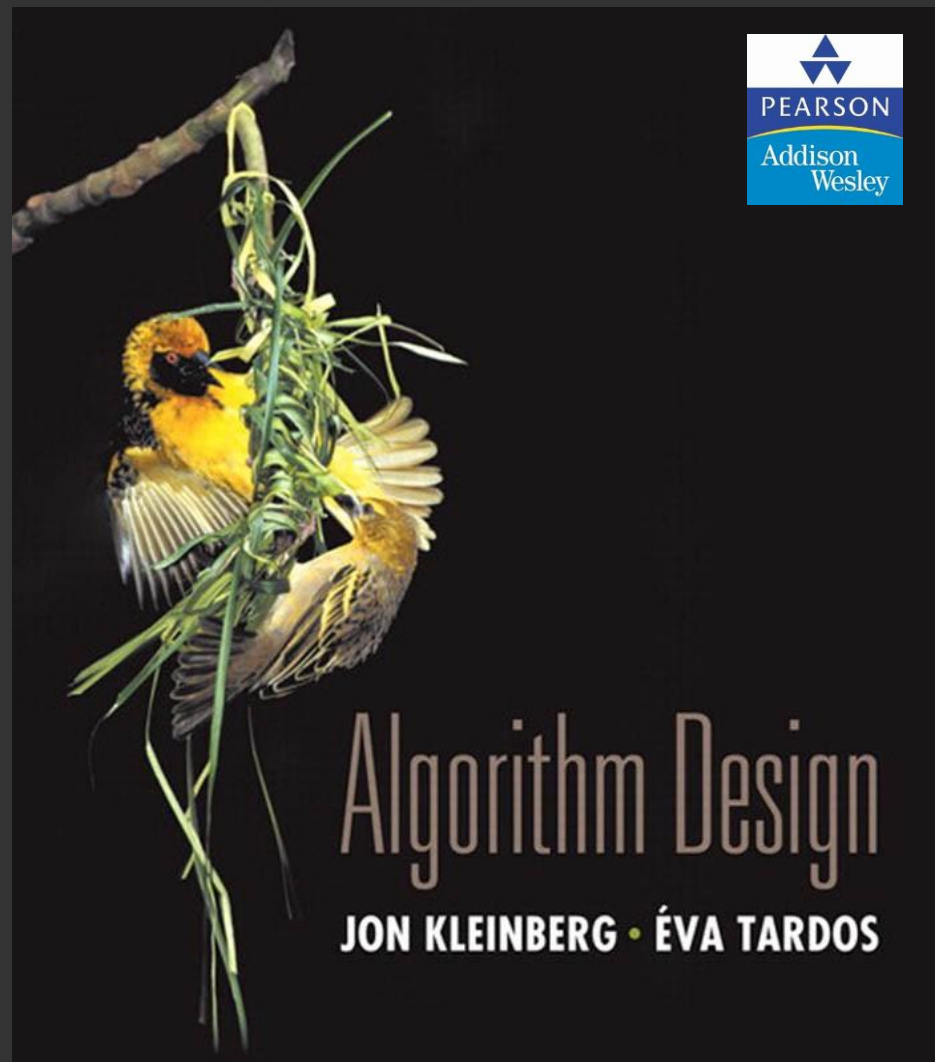
Fast matrix multiplication. (Strassen, 1969)

- Divide: partition A and B into $\frac{1}{2}n$ -by- $\frac{1}{2}n$ blocks.
- Compute: 14 $\frac{1}{2}n$ -by- $\frac{1}{2}n$ matrices via 10 matrix additions.
- Conquer: multiply 7 $\frac{1}{2}n$ -by- $\frac{1}{2}n$ matrices recursively.
- Combine: 7 products into 4 terms using 8 matrix additions.

Analysis.

- Assume n is a power of 2.
- $T(n)$ = # arithmetic operations.

$$T(n) = \underbrace{7T(n/2)}_{\text{recursive calls}} + \underbrace{\Theta(n^2)}_{\text{add, subtract}} \Rightarrow T(n) = \Theta(n^{\log_2 7}) = O(n^{2.81})$$



6. DYNAMIC PROGRAMMING I

- ▶ *weighted interval scheduling*
- ▶ *segmented least squares*
- ▶ *knapsack problem*
- ▶ *RNA secondary structure*

Lecture slides by Kevin Wayne Copyright © 2005 Pearson-

Addison Wesley

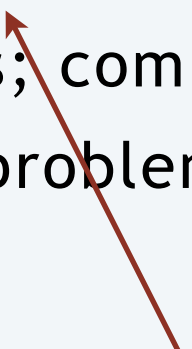
<http://www.cs.princeton.edu/~wayne/kleinberg-tardos>

Algorithmic paradigms

Greed. Process the input in some order, myopically making irrevocable decisions.

Divide-and-conquer. Break up a problem into **independent** subproblems; solve each subproblem; combine solutions to subproblems to form solution to original problem.

Dynamic programming. Break up a problem into a series of **overlapping** subproblems; combine solutions to smaller subproblems to form solution to large subproblem.



fancy name for
caching intermediate results
in a table for later reuse

Dynamic programming history

Bellman. Pioneered the systematic study of dynamic programming in 1950s.

Etymology.

- Dynamic programming = planning over time.
- Secretary of Defense had pathological fear of mathematical research.
- Bellman sought a “dynamic” adjective to avoid conflict.



THE THEORY OF DYNAMIC PROGRAMMING

RICHARD BELLMAN

1. **Introduction.** Before turning to a discussion of some representative problems which will permit us to exhibit various mathematical features of the theory, let us present a brief survey of the fundamental concepts, hopes, and aspirations of dynamic programming.

To begin with, the theory was created to treat the mathematical problems arising from the study of various multi-stage decision processes, which may roughly be described in the following way: We have a physical system whose state at any time t is determined by a set of quantities which we call state parameters, or state variables. At certain times, which may be prescribed in advance, or which may be determined by the process itself, we are called upon to make decisions which will affect the state of the system. These decisions are equivalent to transformations of the state variables, the choice of a decision being identical with the choice of a transformation. The outcome of the preceding decisions is to be used to guide the choice of future ones, with the purpose of the whole process that of maximizing some function of the parameters describing the final state.

Examples of processes fitting this loose description are furnished by virtually every phase of modern life, from the planning of industrial production lines to the scheduling of patients at a medical clinic; from the determination of long-term investment programs for universities to the determination of a replacement policy for machinery in factories; from the programming of training policies for skilled and unskilled labor to the choice of optimal purchasing and inventory policies for department stores and military establishments.

Dynamic programming applications

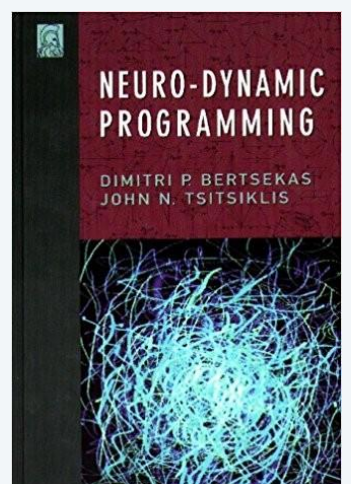
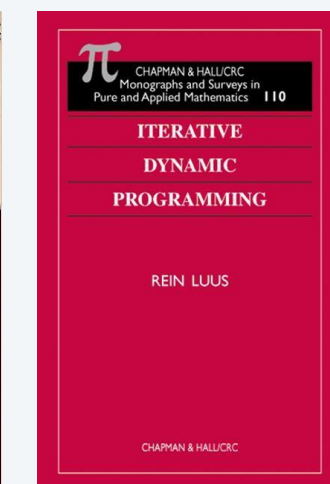
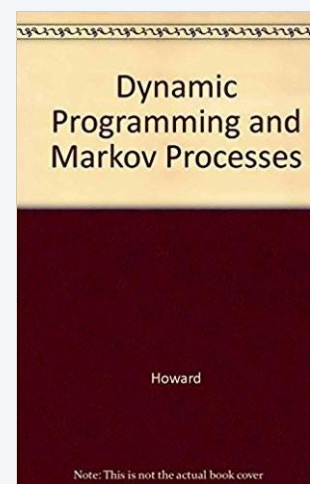
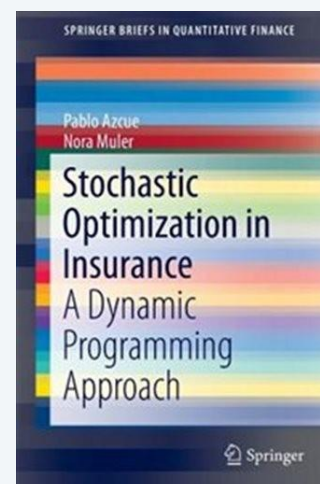
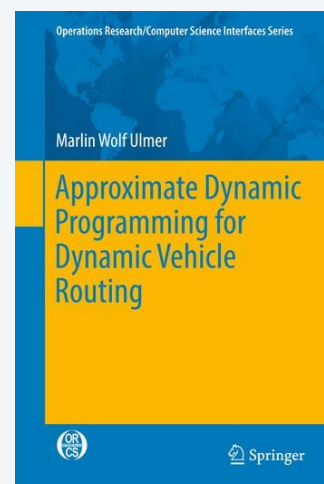
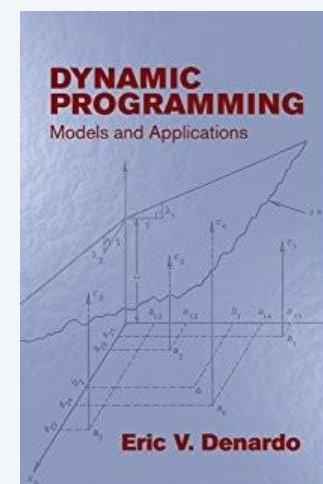
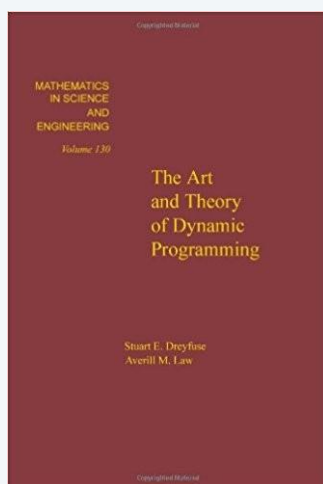
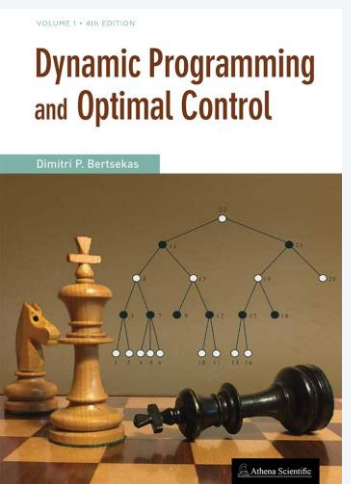
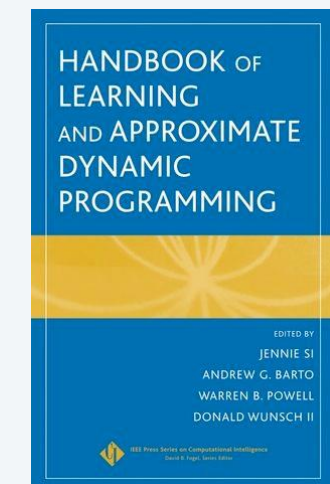
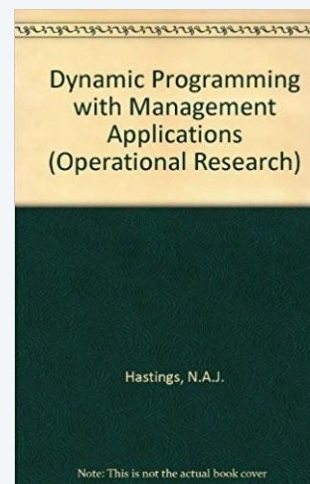
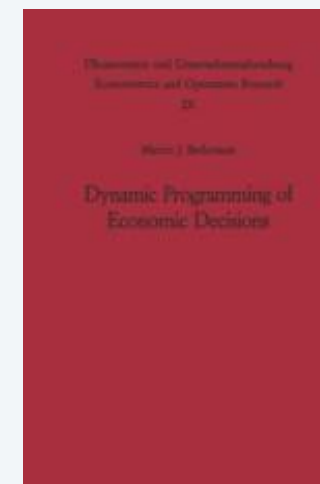
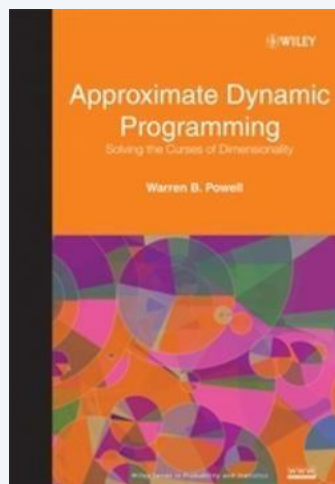
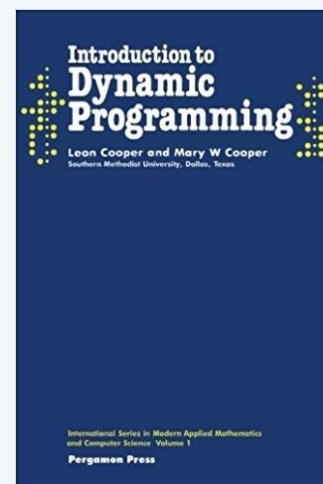
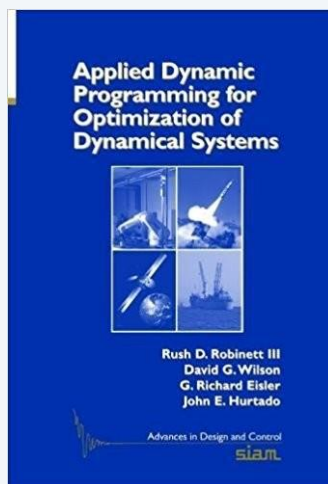
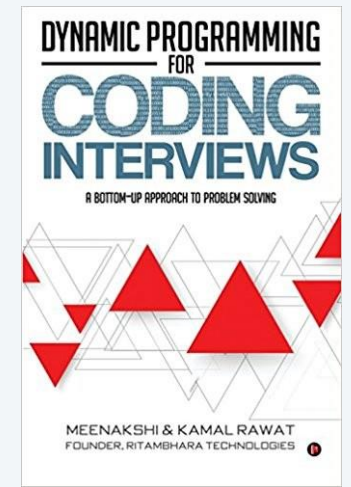
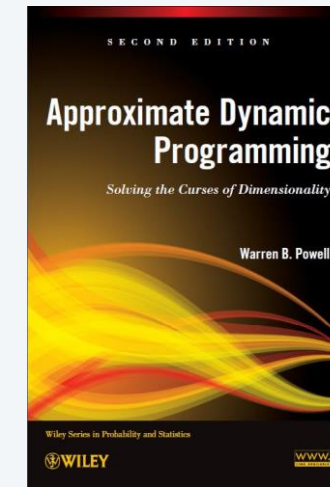
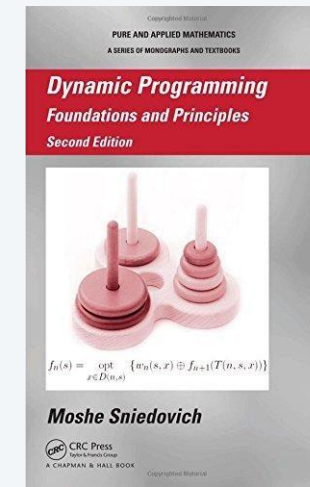
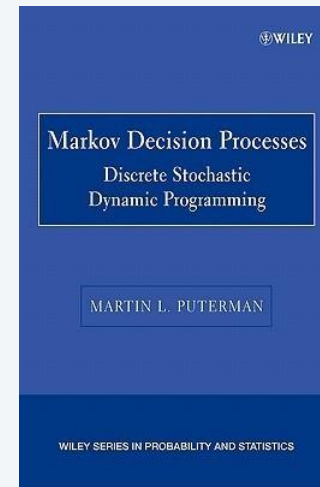
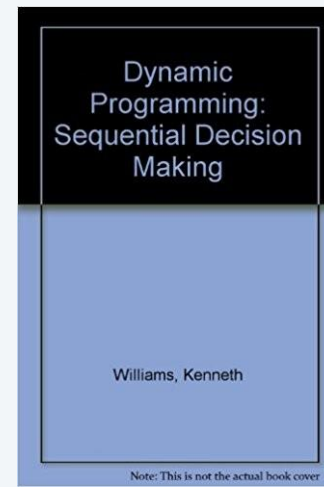
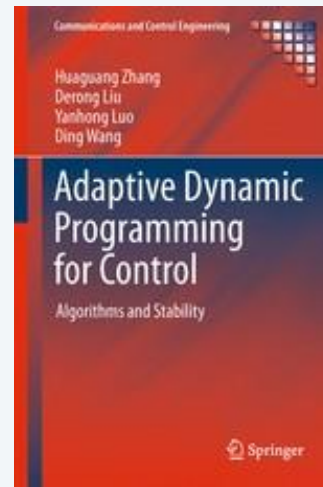
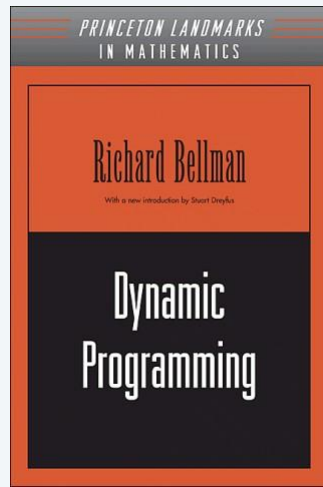
Application areas.

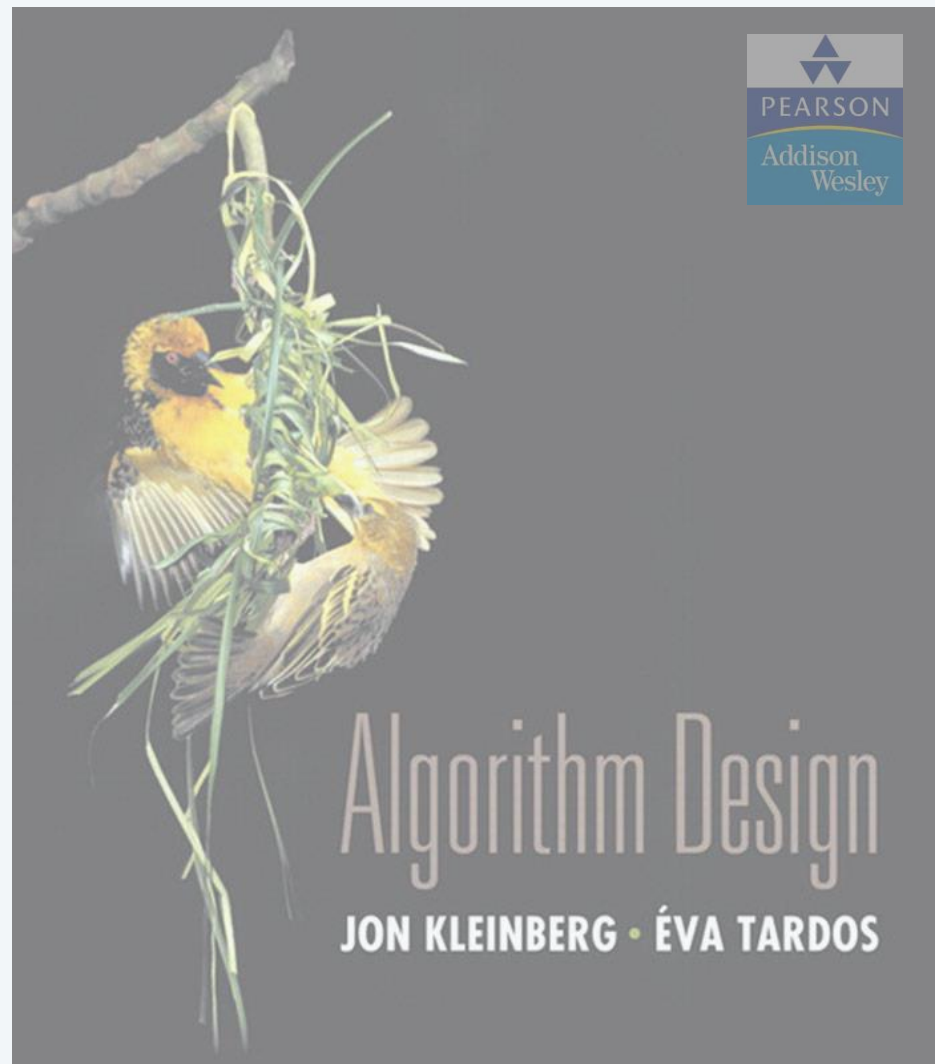
- Computer science: AI, compilers, systems, graphics, theory,
- Operations research.
- Information theory.
- Control theory.
- Bioinformatics.

Some famous dynamic programming algorithms.

- Avidan-Shamir for seam carving.
- Unix diff for comparing two files.
- Viterbi for hidden Markov models.
- De Boor for evaluating spline curves.
- Bellman-Ford-Moore for shortest path.
- Knuth-Plass for word wrapping text in T_1X .
- Cocke-Kasami-Younger for parsing context-free grammars.
- Needleman-Wunsch/Smith-Waterman for sequence alignment.

Dynamic programming books





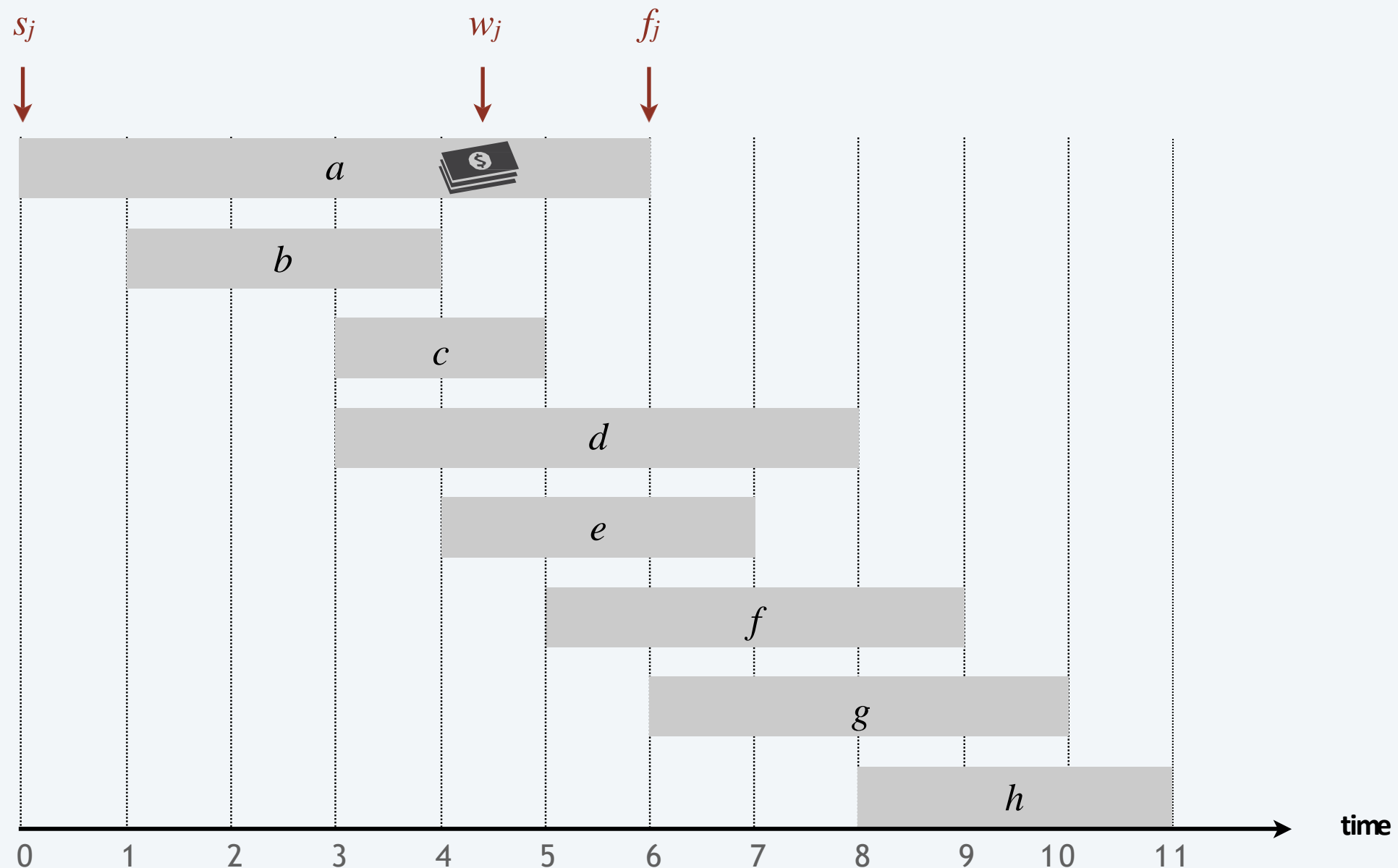
SECTIONS 6.1–6.2

6. DYNAMIC PROGRAMMING I

- ▶ *weighted interval scheduling*
- ▶ *segmented least squares*
- ▶ *knapsack problem*
- ▶ *RNA secondary structure*

Weighted interval scheduling

- Job j starts at s_j , finishes at f_j , and has weight $w_j > 0$.
- Two jobs are **compatible** if they don't overlap.
- Goal: find max-weight subset of mutually compatible jobs.

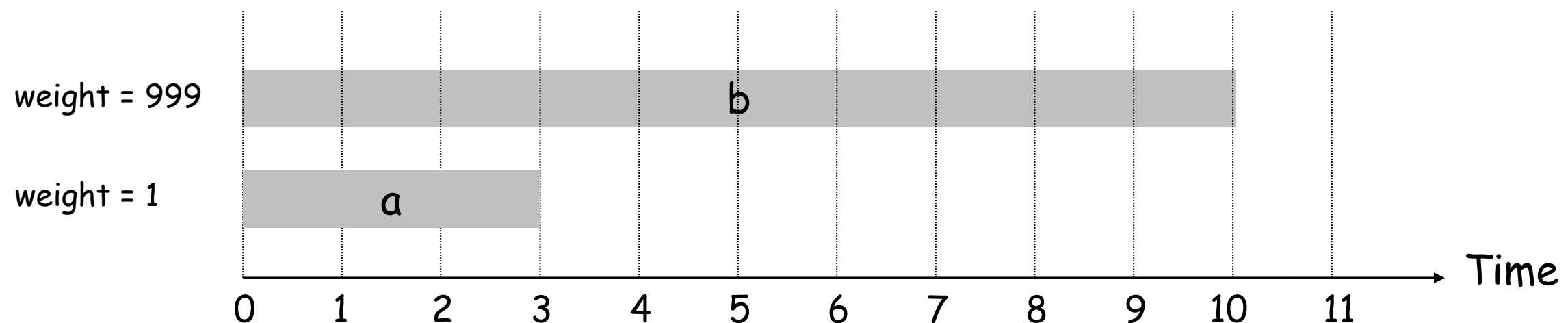


Unweighted Interval Scheduling Review

Recall. Greedy algorithm works if all weights are 1.

- Consider jobs in ascending order of finish time.
- Add job to subset if it is compatible with previously chosen jobs.

Observation. Greedy algorithm can fail spectacularly if arbitrary weights are allowed.

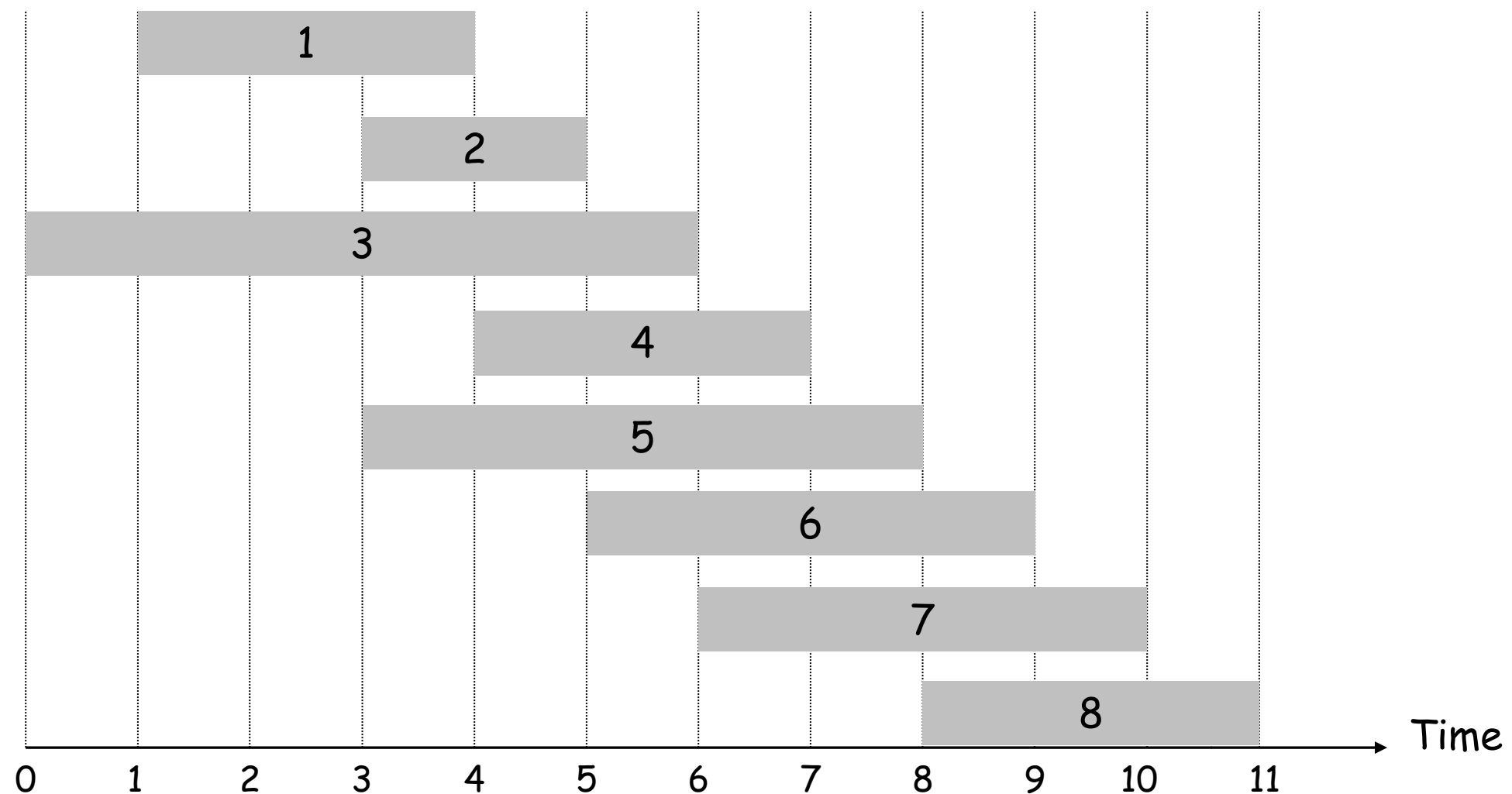


Weighted Interval Scheduling

Notation. Label jobs by finishing time: $f_1 \leq f_2 \leq \dots \leq f_n$.

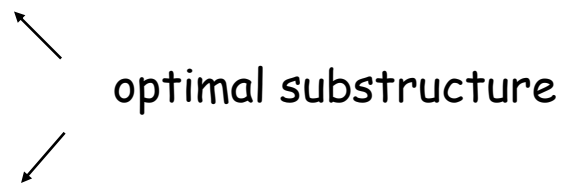
Def. $p(j)$ = largest index $i < j$ such that job i is compatible with j .

Ex: $p(8) = 5$, $p(7) = 3$, $p(2) = 0$.



Dynamic Programming: Binary Choice

Notation. $OPT(j)$ = value of optimal solution to the problem consisting of job requests $1, 2, \dots, j$.

- Case 1: OPT selects job j .
 - can't use incompatible jobs $\{ p(j) + 1, p(j) + 2, \dots, j - 1 \}$
 - must include optimal solution to problem consisting of remaining compatible jobs $1, 2, \dots, p(j)$
 - Case 2: OPT does not select job j .
 - must include optimal solution to problem consisting of remaining compatible jobs $1, 2, \dots, j-1$
- 

$$OPT(j) = \begin{cases} 0 & \text{if } j = 0 \\ \max \{ v_j + OPT(p(j)), OPT(j-1) \} & \text{otherwise} \end{cases}$$

Weighted Interval Scheduling: Brute Force

Brute force algorithm.

Input: $n, s_1, \dots, s_n, f_1, \dots, f_n, v_1, \dots, v_n$

Sort jobs by finish times so that $f_1 \leq f_2 \leq \dots \leq f_n$.

Compute $p(1), p(2), \dots, p(n)$

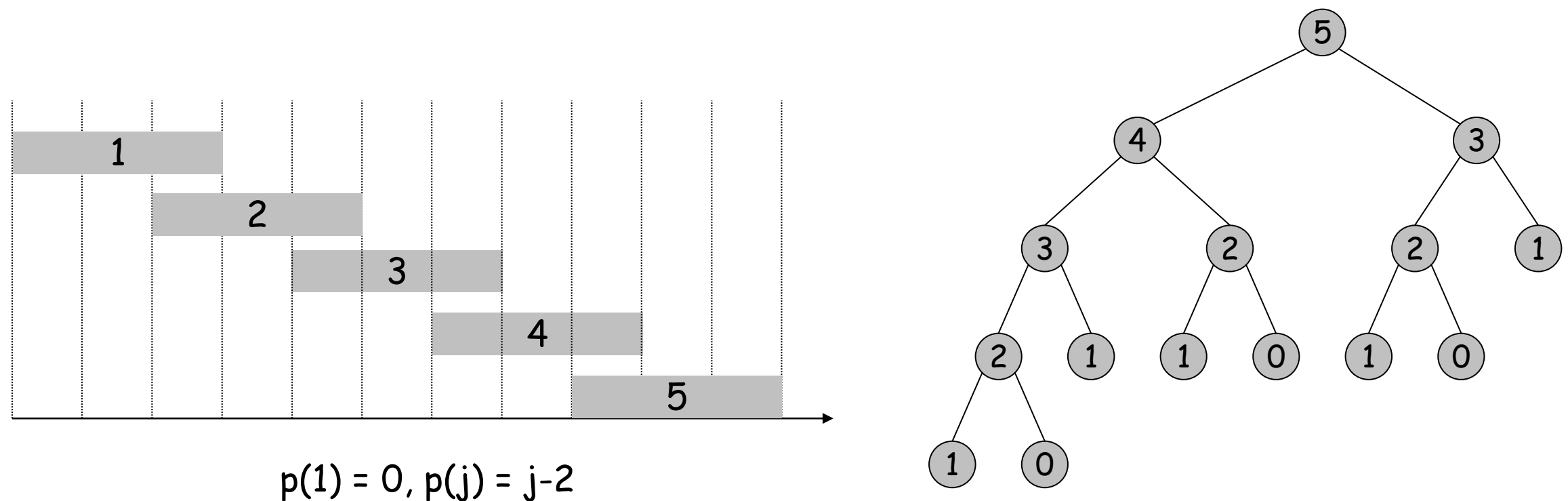
```
Compute-Opt(j) {  
    if (j = 0)  
        return 0  
    else  
        return max( $v_j + \text{Compute-Opt}(p(j))$ ,  $\text{Compute-Opt}(j-1)$ )  
}
```

Running time: $T(n) = T(p(n)) + T(n-1) + O(1) = \dots$

Weighted Interval Scheduling: Brute Force

Observation. Recursive algorithm fails spectacularly because of redundant sub-problems \Rightarrow exponential algorithms.

Ex. Number of recursive calls for family of "layered" instances grows like Fibonacci sequence.



Running time: $T(n) = T(p(n)) + T(n-1) + O(1) = T(n-2) + T(n-1) + O(1) = \Theta(\varphi^n)$
where $\varphi = 1.618$ is the golden ratio

Weighted Interval Scheduling: Memoization

Memoization. Store results of each sub-problem in a cache; lookup as needed.

```
Input:  $n, s_1, \dots, s_n, f_1, \dots, f_n, v_1, \dots, v_n$ 
```

```
Sort jobs by finish times so that  $f_1 \leq f_2 \leq \dots \leq f_n$ .
```

```
Compute  $p(1), p(2), \dots, p(n)$ 
```

```
for  $j = 1$  to  $n$ 
```

```
     $M[j] = \text{empty}$   $\leftarrow$  global array
```

```
 $M[j] = 0$ 
```

```
M-Compute-Opt( $j$ ) {
```

```
    if ( $M[j]$  is empty)
```

```
         $M[j] = \max(w_j + \text{M-Compute-Opt}(p(j)), \text{M-Compute-Opt}(j-1))$ 
```

```
    return  $M[j]$ 
```

```
}
```

Weighted Interval Scheduling: Running Time

Claim. Memoized version of algorithm takes $O(n \log n)$ time.

- Sort by finish time: $O(n \log n)$.
- Computing $p(\cdot)$: $O(n)$ after sorting by start time.
- $M\text{-Compute-Opt}(j)$: each invocation takes $O(1)$ time and either
 - (i) returns an existing value $M[j]$
 - (ii) fills in one new entry $M[j]$ and makes two recursive calls
- Progress measure $\Phi = \# \text{ nonempty entries of } M[\]$.
 - initially $\Phi = 0$, throughout $\Phi \leq n$.
 - (ii) increases Φ by 1 \Rightarrow at most $2n$ recursive calls, since there are only n entries to fill.
- Overall running time of $M\text{-Compute-Opt}(n)$ is $O(n)$. ▫

Remark. $O(n)$ if jobs are pre-sorted by start and finish times.

Automated Memoization

Automated memoization. Many functional programming languages (e.g., Lisp) have built-in support for memoization.

Q. Why not in imperative languages (e.g., Java)?

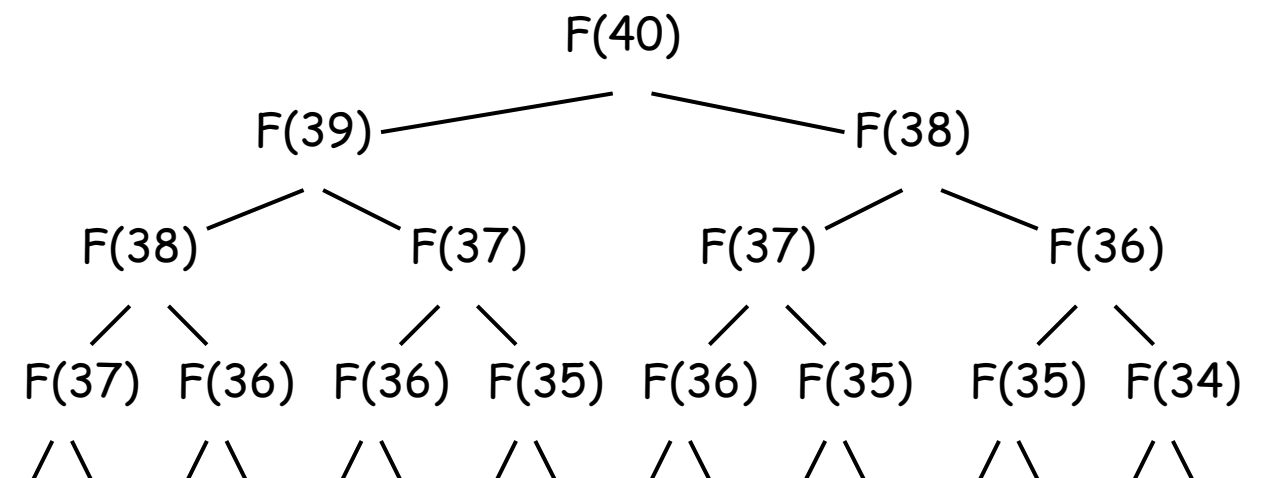
A. compiler's job is easier in pure functional languages since no side effects

```
(defun F (n)
  (if
    (<= n 1)
    n
    (+ (F (- n 1)) (F (- n 2))))))
```

Lisp (efficient)

```
static int F(int n) {
  if (n <= 1) return n;
  else return F(n-1) + F(n-2);
}
```

Java (exponential)



Weighted Interval Scheduling: Finding a Solution

Q. Dynamic programming algorithms computes optimal value. What if we want the solution itself?

A. Do some post-processing.

```
Run M-Compute-Opt(n)
Run Find-Solution(n)

Find-Solution(j) {
    if (j = 0)
        output nothing
    else if ( $v_j + M[p(j)] > M[j-1]$ )
        print j
        Find-Solution(p(j))
    else
        Find-Solution(j-1)
}
```

▫ # of recursive calls $\leq n \Rightarrow O(n)$.

Dynamic Programming Overview

Dynamic Programming = Recursion + Memoization

- 1 Formulate problem **recursively** in terms of solutions to **polynomially many** sub-problems
- 2 Solve sub-problems bottom-up, **storing optimal solutions**

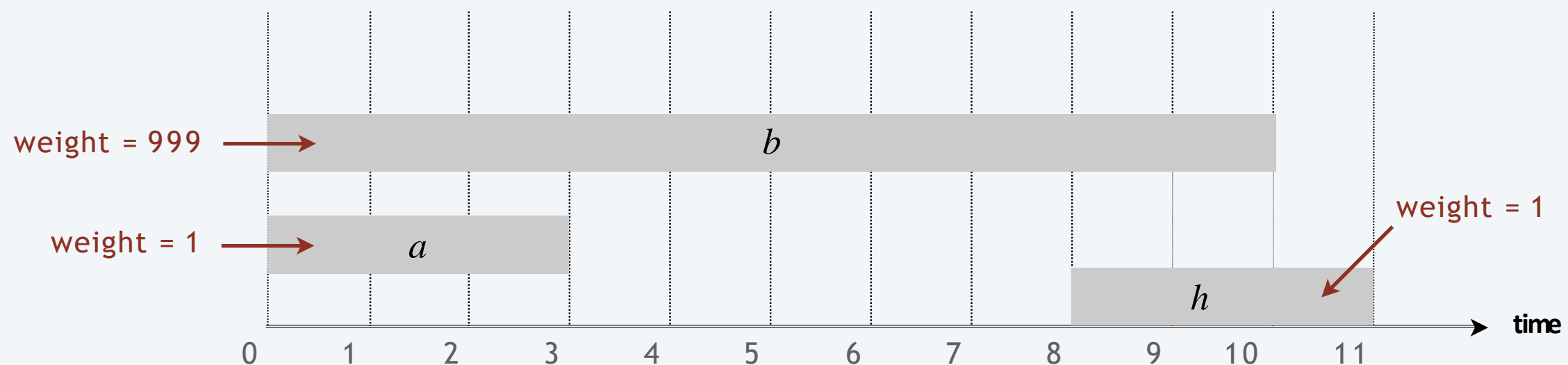
Earliest-finish-time first algorithm

Earliest finish-time first.

- Consider jobs in ascending order of finish time.
- Add job to subset if it is compatible with previously chosen jobs.

Recall. Greedy algorithm is correct if all weights are 1.

Observation. Greedy algorithm fails spectacularly for weighted version.



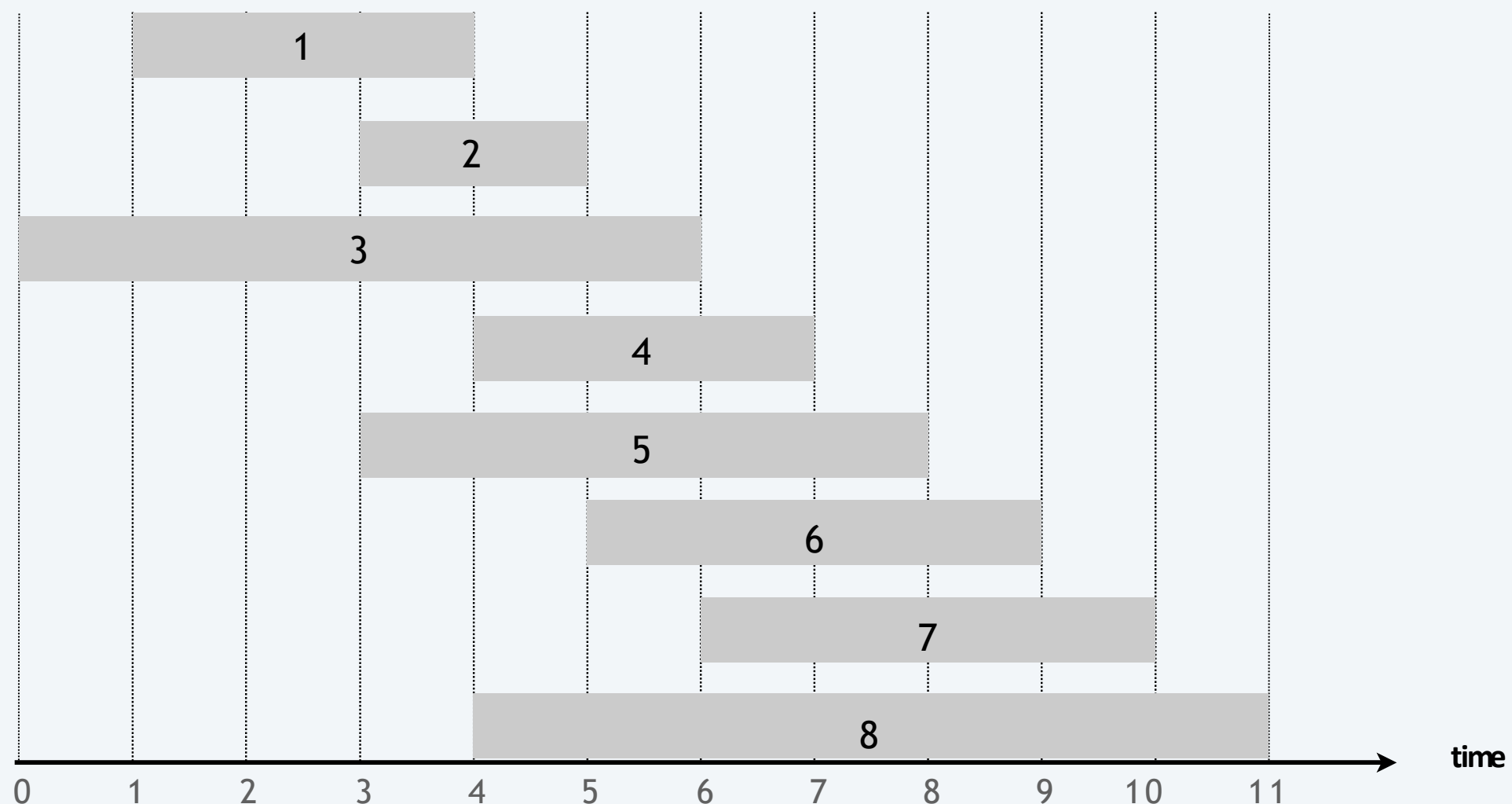
Weighted interval scheduling

Convention. Jobs are in ascending order of finish time: $f_1 \leq f_2 \leq \dots \leq f_n$.

Def. $p(j)$ = largest index $i < j$ such that job i is compatible with j .

Ex. $p(8) = 1, p(7) = 3, p(2) = 0$.

 i is leftmost interval that ends before j begins



Dynamic programming: binary choice

Def. $OPT(j)$ = max weight of any subset of mutually compatible jobs for subproblem consisting only of jobs $1, 2, \dots, j$.

Goal. $OPT(n)$ = max weight of any subset of mutually compatible jobs.

Case 1. $OPT(j)$ does not select job j .

- Must be an optimal solution to problem consisting of remaining jobs $1, 2, \dots, j - 1$.

Case 2. $OPT(j)$ selects job j .

- Collect profit w_j .
- Can't use incompatible jobs $\{ p(j) + 1, p(j) + 2, \dots, j - 1 \}$.
- Must include optimal solution to problem consisting of remaining compatible jobs $1, 2, \dots, p(j)$.

↖ ↗
optimal substructure property
(proof via exchange argument)

Bellman equation.
$$OPT(j) = \begin{cases} 0 & \text{if } j = 0 \\ \max \{ OPT(j - 1), w_j + OPT(p(j)) \} & \text{if } j > 0 \end{cases}$$

Weighted interval scheduling: brute force

BRUTE-FORCE ($n, s_1, \dots, s_n, f_1, \dots, f_n, w_1, \dots, w_n$)

Sort jobs by finish time and renumber so that $f_1 \leq f_2 \leq \dots \leq f_n$.

Compute $p[1], p[2], \dots, p[n]$ via binary search.

RETURN COMPUTE-OPT(n).

COMPUTE-OPT(j)

IF ($j = 0$)

RETURN 0.

ELSE

RETURN $\max \{ \text{COMPUTE-OPT}(j-1), w_j + \text{COMPUTE-OPT}(p[j]) \}$.



What is running time of **COMPUTE-OPT**(n) in the worst case?

A. $\Theta(n \log n)$

B. $\Theta(n^2)$

C. $\Theta(1.618^n)$

D. $\Theta(2^n)$



COMPUTE-OPT(j)

IF ($j = 0$)

RETURN 0.

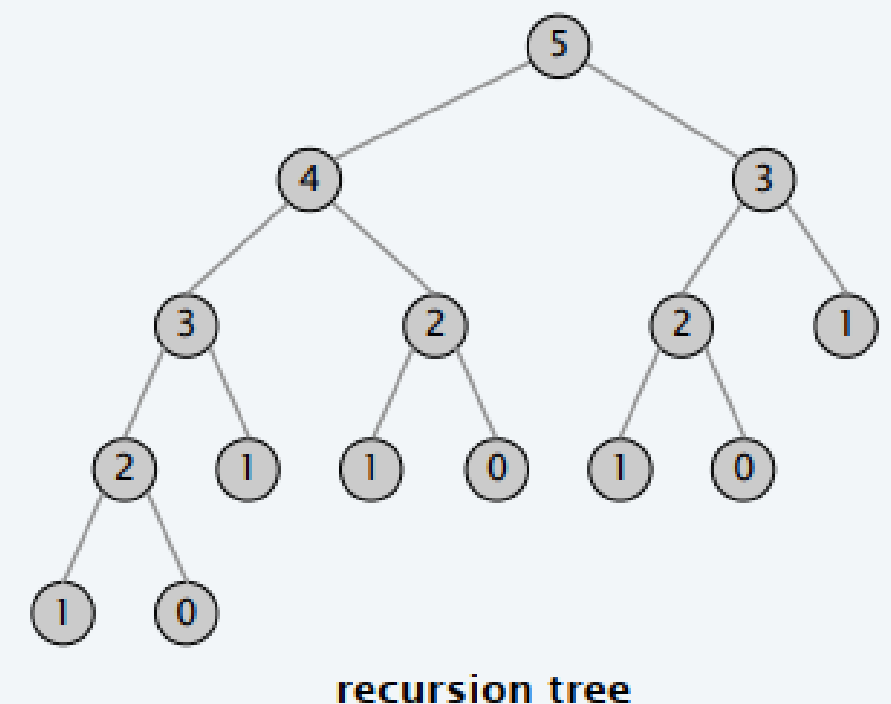
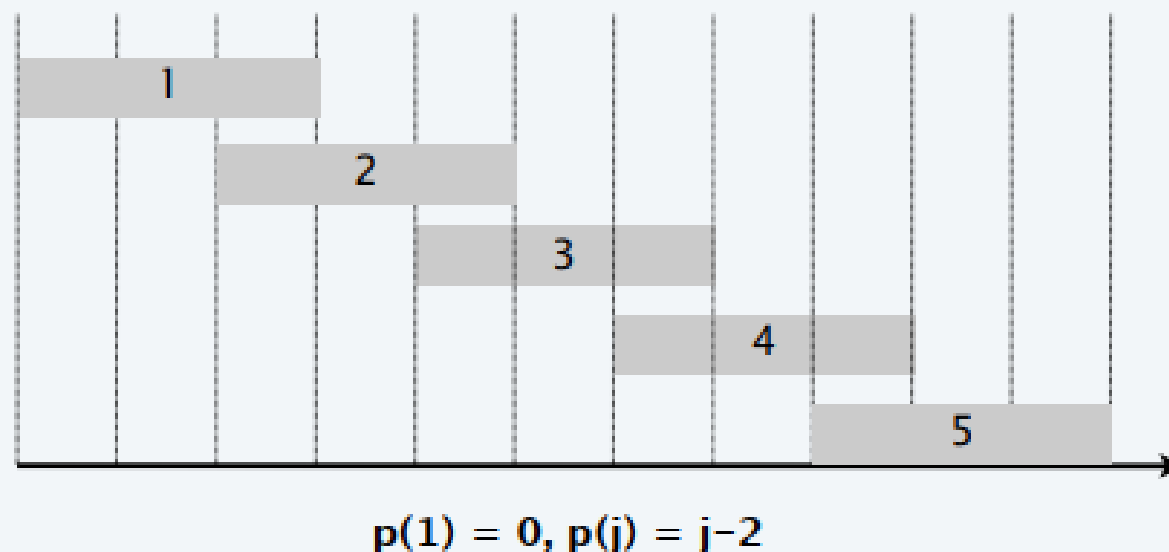
ELSE

RETURN $\max \{ \text{COMPUTE-OPT}(j-1), w_j + \text{COMPUTE-OPT}(p[j]) \}$.

Weighted interval scheduling: brute force

Observation. Recursive algorithm is spectacularly slow because of overlapping subproblems \Rightarrow exponential-time algorithm.

Ex. Number of recursive calls for family of “layered” instances grows like Fibonacci sequence.



Weighted interval scheduling: memoization

Top-down dynamic programming (memoization).

- Cache result of subproblem j in $M[j]$.
- Use $M[j]$ to avoid solving subproblem j more than once.

TOP-DOWN($n, s_1, \dots, s_n, f_1, \dots, f_n, w_1, \dots, w_n$)

Sort jobs by finish time and renumber so that $f_1 \leq f_2 \leq \dots \leq f_n$.

Compute $p[1], p[2], \dots, p[n]$ via binary search.

$M[0] \leftarrow 0$.  global array

RETURN M-COMPUTE-OPT(n).

M-COMPUTE-OPT(j)

IF ($M[j]$ is uninitialized)

$M[j] \leftarrow \max \{ \text{M-COMPUTE-OPT}(j-1), w_j + \text{M-COMPUTE-OPT}(p[j]) \}$.

RETURN $M[j]$.

Weighted interval scheduling: running time

Claim. Memoized version of algorithm takes $O(n \log n)$ time.

Pf.

- Sort by finish time: $O(n \log n)$ via mergesort.
- Compute $p[j]$ for each j : $O(n \log n)$ via binary search.
- M-COMPUTE-OPT(j): each invocation takes $O(1)$ time and either
 - (1) returns an initialized value $M[j]$
 - (2) initializes $M[j]$ and makes two recursive calls
- Progress measure $\Phi = \#$ initialized entries among $M[1..n]$.
 - initially $\Phi = 0$; throughout $\Phi \leq n$.
 - (2) increases Φ by 1 $\Rightarrow \leq 2n$ recursive calls.
- Overall running time of M-COMPUTE-OPT(n) is $O(n)$. ▪

Those who cannot remember the
past are condemned to repeat it.

- Dynamic Programming

Weighted interval scheduling: finding a solution

Q. DP algorithm computes optimal value. How to find optimal solution?

A. Make a second pass by calling FIND-SOLUTION(n).

FIND-SOLUTION(j)

IF ($j = 0$)

RETURN \emptyset .

ELSE IF ($w_j + M[p[j]] > M[j-1]$)

RETURN $\{j\} \cup \text{FIND-SOLUTION}(p[j])$.

ELSE

RETURN FIND-SOLUTION($j-1$).

$$M[j] = \max \{ M[j-1], w_j + M[p[j]] \}.$$

Analysis. # of recursive calls $\leq n \Rightarrow O(n)$.

Weighted interval scheduling: bottom-up dynamic programming

Bottom-up dynamic programming. Unwind recursion.

BOTTOM-UP($n, s_1, \dots, s_n, f_1, \dots, f_n, w_1, \dots, w_n$)

Sort jobs by finish time and renumber so that $f_1 \leq f_2 \leq \dots \leq f_n$.

Compute $p[1], p[2], \dots, p[n]$.

$M[0] \leftarrow 0$.

previously computed values

FOR $j = 1$ **TO** n

$M[j] \leftarrow \max \{ M[j-1], w_j + M[p[j]] \}.$

Running time. The bottom-up version takes $O(n \log n)$ time.

MAXIMUM SUBARRAY PROBLEM

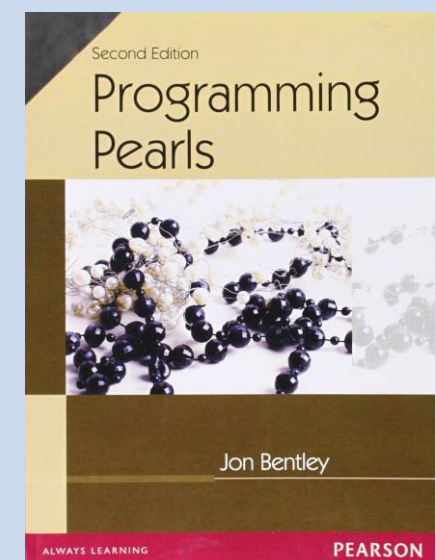


Goal. Given an array x of n integer (positive or negative), find a contiguous subarray whose sum is maximum.

12	5	-1	31	-61	59	26	-53	58	97	-93	-23	84	-15	6
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Applications. Computer vision, data mining, genomic sequence analysis, technical job interviews,



MAXIMUM RECTANGLE PROBLEM



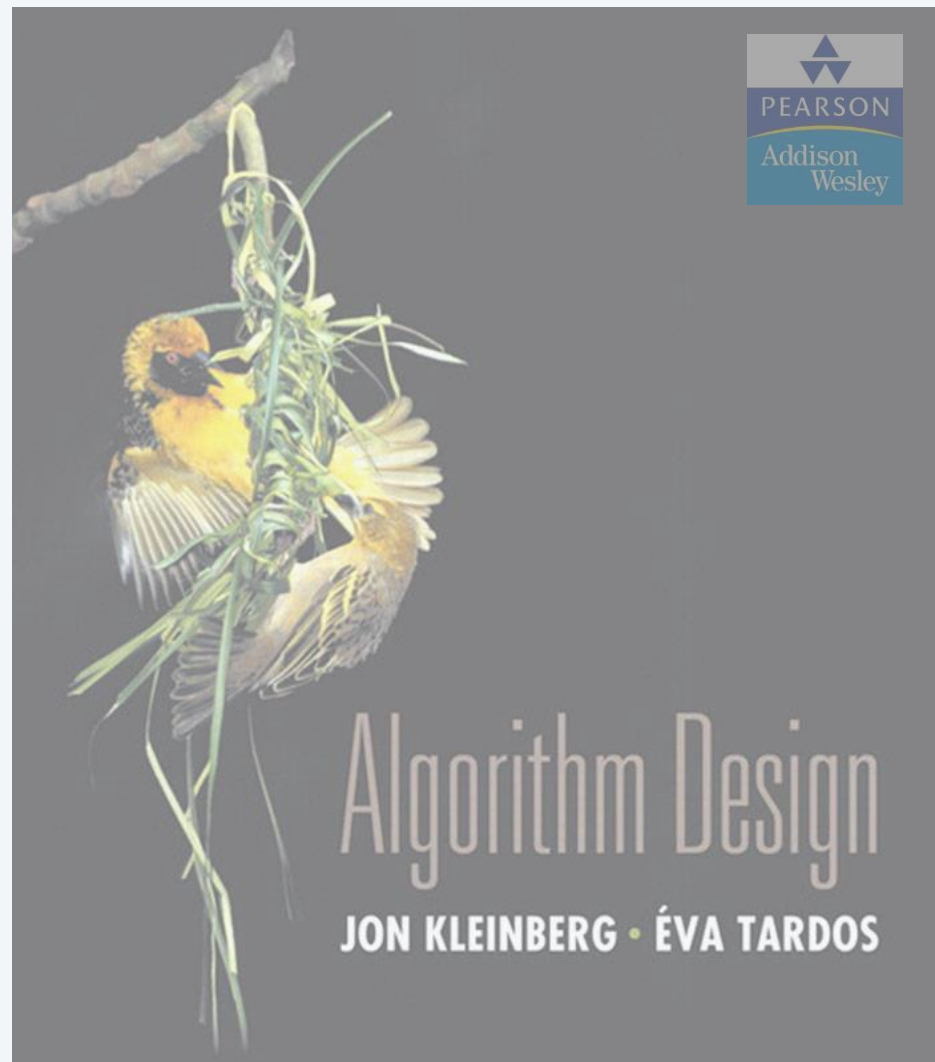
Goal. Given an n -by- n matrix A , find a rectangle whose sum is maximum.

$$A = \begin{bmatrix} -2 & 5 & 0 & -5 & -2 & 2 & -3 \\ 4 & -3 & -1 & 3 & 2 & 1 & -1 \\ -5 & 6 & 3 & -5 & -1 & -4 & -2 \\ -1 & -1 & 3 & -1 & 4 & 1 & 1 \\ 3 & -3 & 2 & 0 & 3 & -3 & -2 \\ -2 & 1 & -2 & 1 & 1 & 3 & -1 \\ 2 & -4 & 0 & 1 & 0 & -3 & -1 \end{bmatrix}$$

13

Applications. Databases, image processing, maximum likelihood estimation, technical job interviews, ...

6. DYNAMIC PROGRAMMING



SECTION 6.3

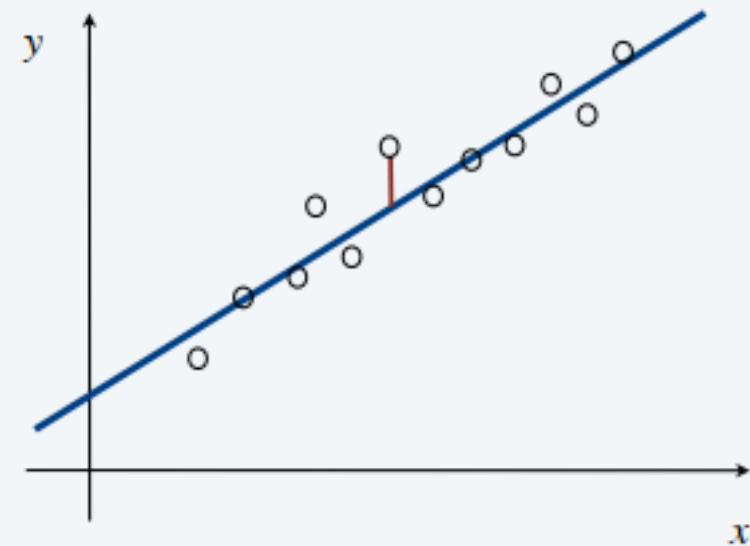
- ▶ *weighted interval scheduling*
- ▶ *segmented least squares*
- ▶ *knapsack problem*
- ▶ *RNA secondary structure*

Least squares

Least squares. Foundational problem in statistics.

- Given n points in the plane: $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$.
- Find a line $y = ax + b$ that minimizes the sum of the squared error:

$$SSE = \sum_{i=1}^n (y_i - ax_i - b)^2$$



Solution. Calculus \Rightarrow min error is achieved when

$$a = \frac{n \sum_i x_i y_i - (\sum_i x_i)(\sum_i y_i)}{n \sum_i x_i^2 - (\sum_i x_i)^2}, \quad b = \frac{\sum_i y_i - a \sum_i x_i}{n}$$

Segmented least squares

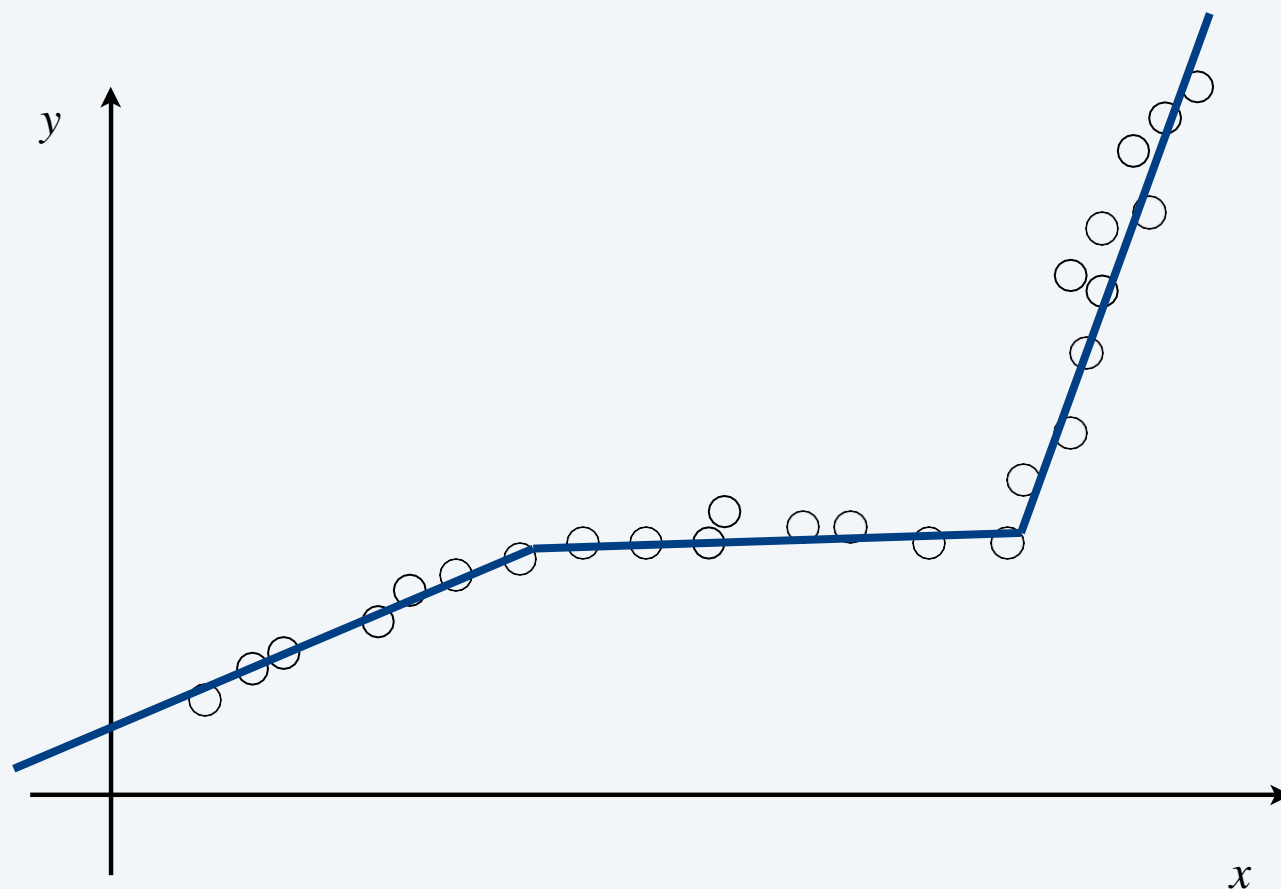
Segmented least squares.

- Points lie roughly on a sequence of several line segments.
- Given n points in the plane: $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$ with $x_1 < x_2 < \dots < x_n$, find a sequence of lines that minimizes $f(x)$.

Q. What is a reasonable choice for $f(x)$ to balance accuracy and parsimony?

↑
goodness of fit

↑
number of lines



2
5

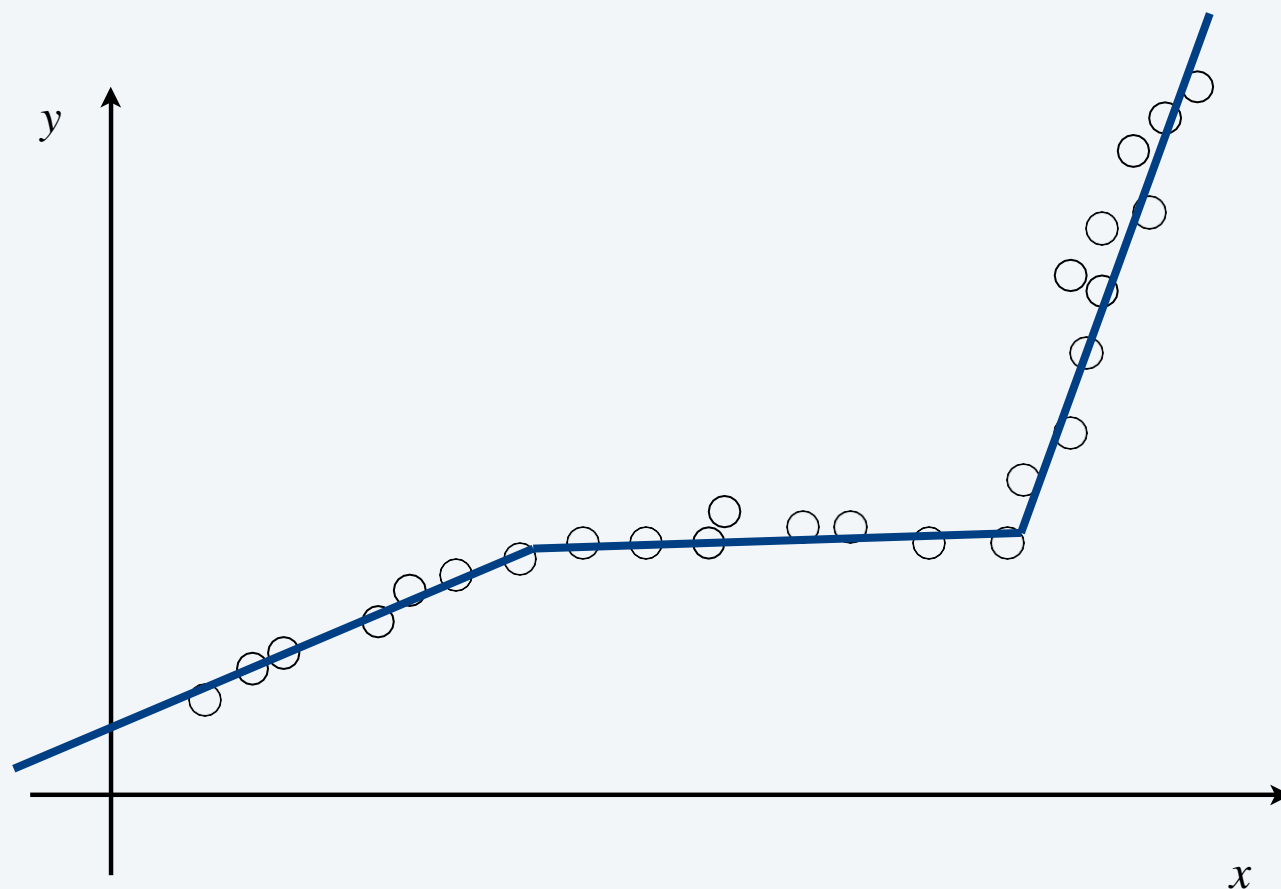
Segmented least squares

Segmented least squares.

- Points lie roughly on a sequence of several line segments.
- Given n points in the plane: $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$ with $x_1 < x_2 < \dots < x_n$, find a sequence of lines that minimizes $f(x)$.

Goal. Minimize $f(x) = E + c L$ for some constant $c > 0$, where

- E = sum of the sums of the squared errors in each segment.
- L = number of lines.




Dynamic programming: multiway choice

Notation.

- $OPT(j)$ = minimum cost for points p_1, p_2, \dots, p_j .
- e_{ij} = SSE for for points p_i, p_{i+1}, \dots, p_j .

To compute $OPT(j)$:

- Last segment uses points p_i, p_{i+1}, \dots, p_j for some $i \leq j$.
- $\text{Cost} = e_{ij} + c + OPT(i - 1).$  optimal substructure property (proof via exchange argument)

Bellman equation.

$$OPT(j) = \begin{cases} 0 & \text{if } j = 0 \\ \min_{1 \leq i \leq j} \{ e_{ij} + c + OPT(i - 1) \} & \text{if } j > 0 \end{cases}$$

Segmented least squares algorithm

SEGMENTED-LEAST-SQUARES(n, p_1, \dots, p_n, c)

FOR $j = 1$ TO n

FOR $i = 1$ TO j

Compute the SSE e_{ij} for the points p_i, p_{i+1}, \dots, p_j .

$M[0] \leftarrow 0$.

FOR $j = 1$ TO n

$M[j] \leftarrow \min_{1 \leq i \leq j} \{ e_{ij} + c + M[i-1] \}.$

previously computed value



$\frac{2}{8}$

RETURN $M[n]$.

Segmented least squares analysis

Theorem. [Bellman 1961] DP algorithm solves the segmented least squares problem in $O(n^3)$ time and $O(n^2)$ space.

Pf.

- Bottleneck = computing SSE e_{ij} for each i and j .

$$a_{ij} = \frac{n \sum_k x_k y_k - (\sum_k x_k)(\sum_k y_k)}{n \sum_k x_k^2 - (\sum_k x_k)^2}, \quad b_{ij} = \frac{\sum_k y_k - a_{ij} \sum_k x_k}{n}$$

- $O(n)$ to compute e_{ij} . ■

Remark. Can be improved to $O(n^2)$ time.

- For each i : precompute cumulative sums $\sum_{k=1}^i x_k$, $\sum_{k=1}^i y_k$, $\sum_{k=1}^i x_k^2$, $\sum_{k=1}^i x_k y_k$.
- Using cumulative sums, can compute e_{ij} in $O(1)$ time.

Dynamic programming summary

Outline.

← typically, only a polynomial
number of subproblems

- Define a collection of subproblems.
- Solution to original problem can be computed from subproblems.
- Natural ordering of subproblems from “smallest” to “largest” that enables determining a solution to a subproblem from solutions to smaller subproblems.

Techniques.

- Binary choice: weighted interval scheduling.
- Multiway choice: segmented least squares.
- Adding a new variable: knapsack problem.
- Intervals: RNA secondary structure.

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Top-down vs. bottom-up dynamic programming. Opinions differ.

NEXT LECTURE

- Dynamic Programming
- Segmented Least Squares
- Knapsack Problem

Week	Date	Topic
1	21-Feb	Introduction. Some representative problems
2	28-Feb	Stable Matching
3	7-Mar	Basics of algorithm analysis.
4	14-Mar	Graphs (Project 1 announced)
5	21-Mar	Greedy algorithms-I
6	28-Mar	Greedy algorithms-II
7	4-Apr	Divide and conquer (Project 2 announced)
8	11-Apr	Dynamic Programming I
9	18-Apr	Dynamic Programming II
10	25-Apr	Network Flow-I (Project 3 announced)
11	2-May	⁷⁹ Midterm
12	9-May	Network Flow II
13	16-May	NP and computational intractability-I
14	23-May	NP and computational intractability-II