Enlai Yii
2064210

Homework_1

Exercise 2.4

(a)

```
1    function POW(x, n):
2         if n = 0:
3              return 1
4         return x * POW(x, n - 1)
```

(b)

With the pseudocode above, there is one comparison for the termination call (lines 2 and 3). And the recursive call (line 4) will be called, or cost, n - 1 times. So,

$+1 + (n - 1) = n - 1 + 1 = n = O(n)$

Thus, the runtime for the algorithm will be O(n).

(c)

```
function POW2(x, n):
     if n = 0:
          return 1
     if x % 2 = 0:
          return POW2(x, n/2) * POW2(x, n/2)
     return x * POW2(x, n/2) * POW2(x, n/2)
```

Exercise 2.10

```
function HW(n):
     if n = 0:
          return 0
     else if n = 1:
          return 1
     else:
          last_digit = n % 2
          if last_digit = 1:
               return 1 + HW(floor(n/2))
          else:
               return HW(floor(n/2))
```

Exercise 2.12

Ranking order of growth:

$$\frac{1}{n} \; < \; log^2(n) \; < \; \frac{n}{lg^4 n} \; < \; n^2 + n \; < \; 2^n \; < \; n!$$

Proving $\frac{1}{n} \; = \; O(log^2 n)$:

$$\lim_{n \to \infty} \frac{1/n}{lg^2 n} \; = \; \lim_{n \to \infty} \frac{1}{n \, lg^2 n} \; = \; 0$$

Proving $log^2(n) \; = \; O(\frac{n}{lg^4 n})$:

$$\lim_{n \to \infty} \frac{lg^2 n}{n/(lg^4 n)} \; = \; \lim_{n \to \infty} \frac{lg^6 n}{n} \; = \frac{1}{lg^6 2} \lim_{n \to \infty} \frac{ln^6 n}{n} \; = \frac{1}{lg^6 2} \lim_{n \to \infty} \frac{6ln^5 n * (1/n)}{1} \; =$$

. . . . . (continuing L' Hospital's Rule) . . . . . $= \frac{6!}{ln^6 2} \lim_{n \to \infty} \frac{1}{n} \; = \; 0$

Proving $\frac{n}{lg^4 n} \; = \; O(n^2 + n)$:

$$\lim_{n \to \infty} \frac{n/(lg^4 n)}{n^2 + n} \; = \; \lim_{n \to \infty} \frac{n}{n(n+1) \, lg^4 n} \; = \; \lim_{n \to \infty} \frac{1}{(n+1) lg^4 n} \; = \; 0$$

Proving $n^2 \; + \; n \; = \; O(2^n)$:

$$\lim_{n \to \infty} \frac{n^2 + n}{2^n} \; = \; \lim_{n \to \infty} \frac{2n+1}{2^n \, ln2} \; = \; \lim_{n \to \infty} \frac{2}{2^n \, ln^2 n} \; = \; 0$$

Proving $2^n \; = \; O(n!)$ using Induction:

P(n) : $2^n \; \leq \; n!$

<u>Base Case</u>:

P(0) : $2^0 \; = \; 0!$ is true.

<u>Induction Step</u>:

Assume P(n) is true. We want to show P(n + 1) is true.

$$2^{n+1} = 2^n * 2 \; \leq \; (n \, + \, 1)! \; = \; (n \, + \, 1)n!$$

So $2^{n+1} \; \leq \; (n \, + \, 1)!$, which means P(n + 1) is true.

Thus, 2^n is $O(n!)$.

Exercise 3.5

$n = 7x + 10y$, where $n \geq 54$

Proof By (Strong) Induction

Base Cases:

54 = 7 * 2 + 10 * 4
55 = 7 * 5 + 10 * 2
56 = 7 * 8 + 10 * 0
57 = 7 * 1 + 10 * 5
58 = 7 * 4 + 10 * 3
59 = 7 * 7 + 10 * 1
60 = 7 * 0 + 10 * 7
61 = 7 * 3 + 10 * 4 = 7 * (2 + 1) + 10 * 4
62 = 7 * 6 + 10 * 2 = 7 * (5 + 1) + 10 * 2
63 = 7 * 9 + 10 * 0 = 7 * (8 + 1) + 10 * 0

Inductive Step:

Assume that there are non-negative integers x and y for all k dollars that are greater than $54 will be true. We want to show that this is true for k + 1 dollars.

Using direct proof, we assume our base cases are true (54, 55, 56, …, 63). We can keep assuming that there will be k - m dollars will be true, where m is $54 \leq m \leq k$. So, we assume that k - 6 is true. Thus, there will be non-negative integers x and y for k - 6 and that will be greater than $54. Now, we can add one more $7 figurine, and we will have (k - 6) + 7 = k + 1 dollars with $7 and $10 figurines. 🔲

Exercise 3.8

<u>Rules</u>

1. If n is equal to 1, then Bob will win
2. If n is equal to 2, then Alice will win
3. If n is an even number, then Alice must remove k odd number of stones that is between $0 < k < n$ and $n \% k = 0$
4. If n is an odd number greater than 7, then Alice must remove k odd number of stones that is between $0 < k < n$ and $n \% k = 0$, where $k > 1$
5. Else, Alice must remove 1 stone (or $k = 1$) because it satisfies $0 < k < n$ and $n \% k = 0$
6. Repeat the rules until the game is over

<u>Proof By (Strong) Induction</u>

<u>Base Case</u>:
When $n = 1$, n is odd because $1 \% 2 = 1$, so we assume that Bob will win.
When $n = 2$, n is even because $2 \% 2 = 0$, so we assume that Alice will win.

<u>Inductive Step</u>:
Assume that the rules will determine whether Alice will win for x remaining stones. We want to show that the rules are true for $x + 1$ remaining stones.

Applying the rules for $x + 1$ stones, when x is an odd number greater than 7, there will be an odd number of stones greater than 1 removed, so the remaining stones will be even for Bob (or x(even)). With $x + 1$, the remaining stones for Bob will be odd, or x(even) + 1 = x(odd). So, the odd condition will be applied again and the remaining stones, x, will be even for Alice. If the remaining stones are even for Alice, then Alice will win the game.

Applying the rules for $x + 1$ stones, when x is an even number, there will be an odd number of stones that will be removed, so the remaining stones will be odd for Bob (or x(odd)). With $x + 1$, the remaining stones for Bob will be even, or x(odd) + 1 = x(even). So, the even condition will be applied and the remaining stones, x, will be odd for Alice. If x is greater than 7, then Alice will win the game because the remaining stones will become even after her turn. However, if x is less than or equal to 7, then Bob will win because the remaining stones will be odd since Alice can only remove 1 stone until the end of the game.

Thus, the rules are correct for $x + 1$ remaining stones.