1 Rayleigh-Benard Convection

Rayleigh-Benard convection is described by the non-dimensionalized equations

$$
\begin{aligned}
&\boldsymbol{\nabla}\cdot\boldsymbol{u} = 0,\\
&\frac{1}{\mathrm{Pr}}\left(\partial_t\boldsymbol{u} + \boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{u}\right) = -\boldsymbol{\nabla}p + \mathrm{Ra}\theta\boldsymbol{e}_{\mathrm{z}} + \nabla^2\boldsymbol{u},\\
&\partial_t\theta + \boldsymbol{u}\cdot\boldsymbol{\nabla}\theta - u_z = \nabla^2\theta,
\end{aligned} \tag{1}
$$

with the boundary conditions

$$
u_z = \partial_z u_x = \partial_z u_y = \theta = 0 \qquad \text{at z} = 0, 1. \tag{2}
$$

We can rewrite this as a system of first order (in z) PDEs:

$$
\begin{aligned}
&\partial_x u + \partial_y v + \partial_z w = 0,\\
&\frac{1}{\mathrm{Pr}}\left(\partial_t u + u\partial_x u + v\partial_y u + w du\right) = -\partial_x p + \partial_x^2 u + \partial_y^2 u + \partial_z du,\\
&\frac{1}{\mathrm{Pr}}\left(\partial_t v + u\partial_x v + v\partial_y v + w dv\right) = -\partial_y p + \partial_x^2 v + \partial_y^2 v + \partial_z dv,\\
&\frac{1}{\mathrm{Pr}}\left(\partial_t w + u\partial_x w + v\partial_y w + w\partial_z w\right) = -\partial_z p + \mathrm{Ra}\theta +\\
&\qquad +\partial_x^2 w + \partial_y^2 w - (\partial_x du + \partial_y dv),\\
&\partial_t\theta + u\partial_x\theta + v\partial_y\theta + w d\theta - w = \partial_x^2\theta + \partial_y^2\theta + \partial_z d\theta,\\
&\partial_z u = du,\\
&\partial_z v = dv,\\
&\partial_z\theta = d\theta.
\end{aligned} \tag{3}
$$

Separating out the linear and nonlinear terms gives

$$
\begin{aligned}
&\partial_x u + \partial_y v + \partial_z w = 0, \\
&\frac{1}{\mathrm{Pr}}\partial_t u + \partial_x p - \partial_x^2 u - \partial_y^2 u - \partial_z du = -\frac{1}{\mathrm{Pr}}\left(u\partial_x u + v\partial_y u + wdu\right), \\
&\frac{1}{\mathrm{Pr}}\partial_t v + \partial_y p - \partial_x^2 v - \partial_y^2 v - \partial_z dv = -\frac{1}{\mathrm{Pr}}\left(u\partial_x v + v\partial_y v + wdv\right), \\
&\frac{1}{\mathrm{Pr}}\partial_t w + \partial_z p - \mathrm{Ra}\theta - \partial_x^2 w - \partial_y^2 w + \left(\partial_x du + \partial_y dv\right) \\
&\qquad\qquad = -\frac{1}{\mathrm{Pr}}\left(u\partial_x w + v\partial_y w + w\partial_z w\right), \\
&\partial_t \theta - w - \partial_x^2 \theta - \partial_y^2 \theta - \partial_z d\theta = -u\partial_x\theta - v\partial_y\theta - wd\theta, \\
&\partial_z u - du = 0, \\
&\partial_z v - dv = 0, \\
&\partial_z \theta - d\theta = 0.
\end{aligned} \tag{4}
$$

The matrix version is:

$$
X = \begin{pmatrix} p \\ u \\ v \\ w \\ \theta \\ du \\ dv \\ d\theta \end{pmatrix}, \qquad M_0 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \mathrm{Pr}^{-1} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \mathrm{Pr}^{-1} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \mathrm{Pr}^{-1} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}. \tag{5}
$$

$$L_0 = \begin{pmatrix} 0 & \partial_x & \partial_y & 0 & 0 & 0 & 0 & 0 \\ \partial_x & -\partial_x^2 - \partial_y^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ \partial_y & 0 & -\partial_x^2 - \partial_y^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\partial_x^2 - \partial_y^2 & -\mathrm{Ra} & \partial_x & \partial_y & 0 \\ 0 & 0 & 0 & -1 & -\partial_x^2 - \partial_y^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}. \tag{6}$$

$$L_1 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}. \tag{7}$$

$$F(X) = \begin{pmatrix} 0 \\ -\frac{1}{\mathrm{Pr}}\left(u\partial_x u + v\partial_y u + wdu\right) \\ -\frac{1}{\mathrm{Pr}}\left(u\partial_x v + v\partial_y v + wdv\right) \\ -\frac{1}{\mathrm{Pr}}\left(u\partial_x w + v\partial_y w - w\partial_x u - w\partial_y v\right) \\ -u\partial_x\theta - v\partial_y\theta - wd\theta \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{8}$$

The boundary conditions are $w = du = dv = \theta = 0$ at $z = 0, 1$.

The 2D problem is easier:

$$X = \begin{pmatrix} p \\ u \\ w \\ \theta \\ du \\ d\theta \end{pmatrix}, \qquad M_0 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathrm{Pr}^{-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathrm{Pr}^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{9}$$

$$L_0 = \begin{pmatrix} 0 & \partial_x & 0 & 0 & 0 & 0 \\ \partial_x & -\partial_x^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\partial_x^2 & -\mathrm{Ra} & \partial_x & 0 \\ 0 & 0 & -1 & -\partial_x^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}. \tag{10}$$

$$L_1 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}. \tag{11}$$

$$F(X) = \begin{pmatrix} 0 \\ -\frac{1}{\mathrm{Pr}}\left(u\partial_x u + w du\right) \\ -\frac{1}{\mathrm{Pr}}\left(u\partial_x w - w\partial_x u\right) \\ -u\partial_x\theta - w d\theta \\ 0 \\ 0 \end{pmatrix}. \tag{12}$$

The boundary conditions are $w = du = \theta = 0$ at $z = 0, 1$.

2 Simplified System

Since we are having problems with the Rayleigh-Benard system, I propose that we consider the following simpler problem that has similar behavior:

$$\partial_t w - \mathrm{Ra}\theta - \nabla^2 w = 0, \tag{13}$$

$$\partial_t \theta - w - \nabla^2 \theta = 0. \tag{14}$$

The boundary conditions are $w = \theta = 0$ at $z = 0, 1$.

Assuming

$$w, \theta \sim \exp(-i\omega t + imx + inz), \tag{15}$$

with $\boldsymbol{k} = m\boldsymbol{e}_x + n\boldsymbol{e}_z$, we have

$$-i\omega w + k^2 w = \mathrm{Ra}\theta, \tag{16}$$

$$-i\omega\theta + k^2\theta = w, \tag{17}$$

so the dispersion relation is

$$(-i\omega + k^2)^2 = \mathrm{Ra}, \tag{18}$$

so

$$\omega = i\sqrt{\mathrm{Ra}} - ik^2. \tag{19}$$

We have instability when the imaginary part of ω is positive, so there is instability only when $k^2 < \sqrt{\mathrm{Ra}}$. In a box of size $(1, 1)$, the smallest possible wavenumber is $(0, 2\pi)$, so the critical Rayleigh number is $(2\pi)^4 \approx 1559$.

We can rewrite the system in terms of first order (in z) PDEs,

$$\partial_t w - \mathrm{Ra}\theta - \partial_x^2 w - \partial_z dw = 0, \tag{20}$$

$$\partial_t \theta - w - \partial_x^2 \theta - \partial_z d\theta = 0, \tag{21}$$

$$\partial_z w - dw = 0, \tag{22}$$

$$\partial_z \theta - d\theta = 0. \tag{23}$$

Writing this in matrix form, we have

$$X = \begin{pmatrix} w \\ \theta \\ dw \\ d\theta \end{pmatrix}, \qquad M_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{24}$$

$$L_0 = \begin{pmatrix} -\partial_x^2 & -\mathrm{Ra} & 0 & 0 \\ -1 & -\partial_x^2 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad L_1 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \tag{25}$$