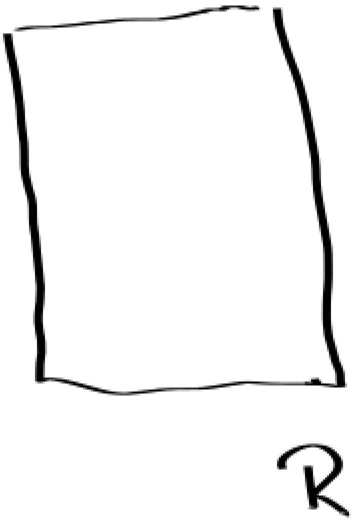
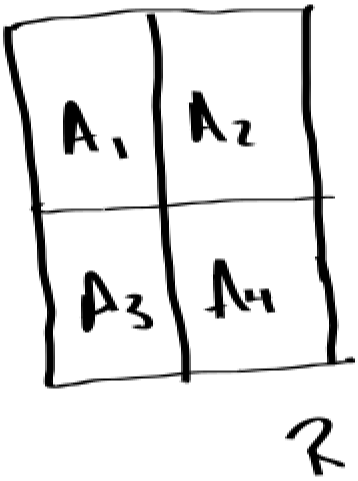
Proof of the Cauchy-Goursat Theorem for Rectangles.



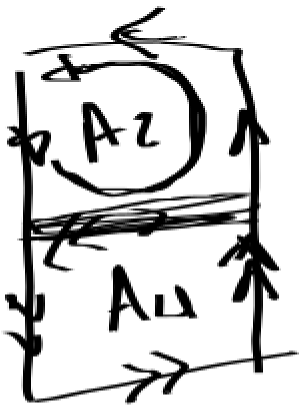
$$I(R) = \int_{\partial R} f(z)\,dz$$

Divide the rectangle R into four smaller ones: A_1, A_2, A_3, A_4



Note that

$$\underline{I}(R) = \underline{I}(A_1) + \underline{I}(A_2) + I$$



$\underline{I}(A_2)$

$\underline{I}(A_4)$

Note that inte in a common cancel.

- By triangle inequality

$$|I(R)| \leq \sum_{i=1}^{4} |I(A_i)|$$

Then at least on of the A_i's is such that

$$|I(A_i)| \geqslant \frac{I(R)}{4}$$

Re-name that rectangle A_i as R_1

- Repeat this procedure with R_1 then we can find R_2 such th

$$|I(R_2)| \geq \frac{|I(R_1)|}{4} \geqslant \frac{|I(R)|}{4^2}$$

- Then we have a collection of rectangles.

$$R \supset R_1 \supset R_2 \supset \cdots \supset R_n \supset \cdots$$

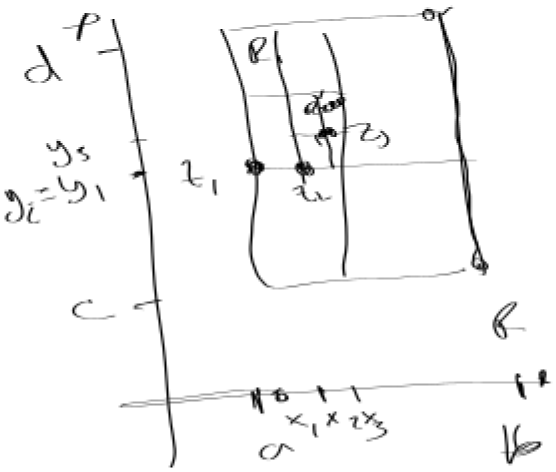
with the property that

$$|I(R_n)| \geq \frac{|I(R)|}{4^n} \quad \ldots \text{(eq. 1)}$$

- Consider the following sequence

$$z_n = x_n + i y_n$$

z_n is the lower-left corner of R_n.



The corresponding sequences
real numbers

$\{x_n\}, \{y_n\}$

are non-decreasing.

$x_n \le x_{n+1} \le b \quad n \in \mathbb{N}$

$y_n \le y_{n+1} \le d$

. For real numbers, monotone seqence
+ bounded

$\Downarrow$

limit exists.

Then $\lim\limits_{n\to\infty} x_n = x_0$

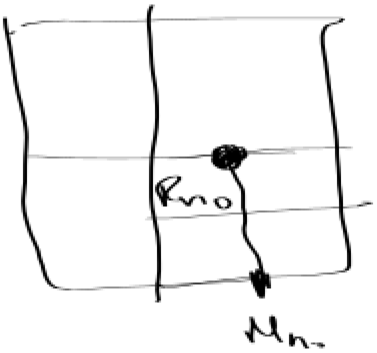
$\lim\limits_{n\to\infty} y_n = y_0$

Define $z_0 = x_0 + i y_0$

Claim: $z_0 \in R_n$, $n \in \mathbb{N}$.

Let $n_0 \in \mathbb{N}$ and choose the upper right corner of R_{n_0}

$$w_{n_0} = M_{n_0} + i N_{n_0}$$



M_{n_0} is an upper bound for $\{x_n\}$:

$$x_n \leq M_{n_0}$$

Even more:

$$x_n \leq x_0 \leq M_{n_0}$$

similarily with

$$y_n \leq y_0 \leq N_{n_0}$$

Since $\lim_{n \to \infty} |x_n - x_0| = 0$ $\quad z_0 = x_0 + y_0 \in R_{n_0}$

and $\lim_{n \to \infty} |y_n - y_0| = 0$

Since the choice of n_0 is arbitrary

Then $z_0 \in R_n$ for all $n \in \mathbb{N}$

(This is the only complex number with this property)

☒ f is holomorphic at z_0

$\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. if $\underline{\underline{|z - z_0| < \delta}}$

then

$$\left| \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \right| < \varepsilon$$

Choose $n \in \mathbb{N}$ s.t. $R_n \subset B_\delta(z_0)$

Note the following:

$$\int_{\partial R_n} f(z_0)\, dz = f(z_0) \int_{\partial R_n} dz$$

$$= 0$$

$$\int_{\partial R_n} f'(z_0)(z - z_0)\, dz =$$

$$= f'(z_0) \int_{\partial R_n} z\, dz - f'(z_0) z_0 \underbrace{\int_{\partial R_n} dz}_{=0}$$

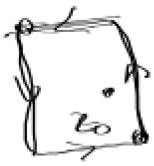
$$= 0$$

$$|I(R_n)| = \left| \int_{\partial R_n} f(z)\,dz \right|$$

$$= \left| \int_{\partial R_n} [f(z) - f(z_0) - f'(z_0)(z-z_0)]\,dz \right|$$

$$\leq \int_{\partial R_n} |f(z) - f(z_0) - f'(z_0)(z-z_0)|\,|dz|$$

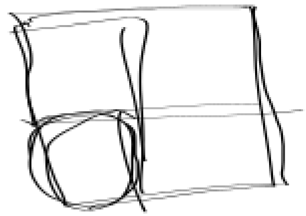
$$\leq \varepsilon \int_{\partial R_n} |z - z_0|\,|dz|$$



R_n

$$\leq \varepsilon D_n \int_{\partial R_n} |dz|$$

$$\leq \varepsilon D_n L_n \quad \text{--- (eq. 2)}$$



diagonal of R_n (D_n)

length of ∂R_n (L_n)

Note: $D_n = \frac{1}{2} D_{n-1} = \frac{1}{2^2} D_{n-2} = \cdots = \frac{D}{2^n}$

$L_n = \frac{1}{2} L_{n-1} = \frac{1}{2^2} L_{n-2} = \cdots = \frac{L}{2^n}$

Then from (eq. 1) and (eq. 2)

$$\frac{|I(R)|}{4^n} \leq |I(R_n)| \leq \frac{\varepsilon D L}{4^n}$$

$$\Rightarrow \quad |I(R)| \leq \varepsilon D L$$

since $\varepsilon > 0$ is arbitrary, then

$$|I(R)| = 0$$

$$\Rightarrow \boxed{I(R) = 0}$$

$\boxtimes$