Calculation sheet for Math Apptitude Test

Takumi Enomoto

1. Now $\frac{df}{dx} = \exp(-x) - x\exp(-x) = \exp(-x) - f$, so
$$\frac{d^2f}{dx^2} = -\exp(-x) - \frac{df}{dx} = -2\exp(-x) + f.$$

2. For $y = \ln|1-x|$, $\frac{df}{dx} = -\frac{1}{1-x}$. Hence $\frac{d^2f}{dx^2} = -\frac{1}{(1-x)^2}$ and
$$\frac{df}{dx} + x\frac{d^2f}{dx^2} = -\frac{1}{1-x} - \frac{x}{(1-x)^2} = -\frac{1}{(1-x)^2}.$$

3. Now $\frac{(x+h)^3-x^3}{h} = 3x^2 + 3xh + h^2$, so
$$\lim_{h\to 0}\frac{(x+h)^3-x^3}{h} = 3x^2.$$

4. If x is near 0 and $x \neq 0$, $\frac{2x+\sin x}{x(x-1)} = \frac{2+\frac{\sin x}{x}}{x-1}$. Using $\lim_{x\to 0}\frac{\sin x}{x} = 1$,
$$\lim_{x\to 0}\frac{2x+\sin x}{x(x-1)} = \lim_{x\to 0}\frac{2+\frac{\sin x}{x}}{x-1} = \frac{2+1}{0-1} = -3.$$

5. Using a variable transformation $x^2 = u$ in the domain $x \geq 0$, $2xdx = du$ and
$$I = \int_0^\infty x\exp(-x^2)dx = \frac{1}{2}\int_0^\infty \exp(-u)du = \frac{1}{2}.$$

6. Using $\frac{1}{x^2+3x+2} = \frac{1}{x+1} - \frac{1}{x+2}$,
$$\begin{aligned}\int_0^1 \frac{1}{x^2+3x+2}dx &= \int_0^1\left(\frac{1}{x+1} - \frac{1}{x+2}\right)dx \\ &= \Big[\ln(x+1) - \ln(x+2)\Big]_0^1 \\ &= (\ln 2 - \ln 3) - (\ln 1 - \ln 2) = -\ln\frac{3}{4}.\end{aligned}$$

7. The n-th order derivative of $f(x) = \exp x$ is $\exp x$, so the n-th order term in the Taylor series expansion of f about $x = -3$ is
$$\frac{e^{-3}}{n!}(x-(-3))^n.$$

8. The derivative of the function $\ln(1+x) = \frac{1}{1+x}$, so in some convergence radius,

$$\begin{aligned}
\ln(1+x) &= \int \frac{1}{1+x} dx = \int dx \sum_{n=0}^{\infty} (-1)^n x^n \\
&= \sum_{n=0}^{\infty} \int dx (-1)^n x^n \\
&= \sum_{n=0}^{\infty} \frac{1}{n+1} (-1)^n x^{n+1}.
\end{aligned}$$

9. k needs to satisfy $\int_{-\infty}^{\infty} p(x)dx = 1$. Now

$$\int_{-\infty}^{\infty} p(x)dx = \int_0^1 k(1-x^3)dx = k\left[x - \frac{1}{4}x^4\right]_0^1 = \frac{3}{4}k,$$

so $k = \frac{4}{3}$.

10. For given random variable X, the probablity $P(0 \le X \le \frac{1}{2})$ is

$$P(0 \le X \le \frac{1}{2}) = \int_0^{\frac{1}{2}} \frac{3}{4}(1-x^2)dx = \frac{3}{4}\left[x - \frac{1}{3}x^3\right]_0^{\frac{1}{2}} = \frac{11}{32}.$$

11. $E(X^2) = E((X-\mu)^2) + \mu^2 = \sigma^2 + \mu^2 = 2^2 + 4^2 = 20.$

12. The skew of the distribution with the mean μ and the standard derivation σ is $\frac{E((X-\mu)^3)}{\sigma^3}$. The uniform distribution is symmetric around its mean, so the integral $\int_0^1 (x - \frac{1}{2})^3 dx = 0$ and our skew is 0.

13. First we calculate $V = E((X-\mu)^2)$. Set $s = \frac{\sqrt{\sigma}}{\gamma}$, then

$$\begin{aligned}
V &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi s^2}} x \cdot x \exp\left(-\frac{(x-\mu)^2}{2s^2}\right) dx \\
&= \frac{1}{\sqrt{2\pi s^2}} \left\{ \left[x \cdot (-s^2) \exp\left(-\frac{x^2}{2s^2}\right) \right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} (-s^2) \exp\left(-\frac{x^2}{2s^2}\right) dx \right\} \\
&= \frac{1}{\sqrt{2\pi s^2}} \cdot s^2 \int_{-\infty}^{\infty} \exp\left(-\frac{x^2}{2s^2}\right) dx \\
&= \frac{1}{\sqrt{2\pi s^2}} \cdot s^2 \cdot \sqrt{2\pi s^2} = s^2.
\end{aligned}$$

Hence the standard derivation which we calculate is $s = \frac{\sqrt{\sigma}}{\gamma}$.

14. Here $f_x = 2x - 4y$, $f_{xx} = 2$, $f_y = -4x + 3y^2 + 4$, and $f_{yy} = 6y$, so $f_{xx} + f_{yy} = 2 + 6y$.

15. From given equation, $\frac{dy}{1+y^2} = \frac{dx}{1+x^2}$. Let $\theta_y = \tan^{-1} y$ and $\theta_x = \tan^{-1} x$, then an indefinite integral of $\frac{dy}{1+y^2}$ is θ_y and that of $\frac{dx}{1+x^2}$ is θ_x Hence, using some integral constant C_0, it holds $\theta_y = \theta_x + C_0$ and

$$\begin{aligned} y &= \tan(\theta_x + C_0) \\ &= \frac{\tan\theta_x + \tan C_0}{1 - \tan\theta_x \tan C_0} \\ &= \frac{x + C}{1 - Cx}, \end{aligned}$$

where $C = \tan C_0$.

16. From given equation, $\frac{dy}{1+y} = xdx$ and $\ln|1+y| = \frac{1}{2}x^2 + C$ by taking indefinite integration. Hence $y = A\exp\left(\frac{1}{2}x^2\right) - 1$, where $A = \pm\exp C$.

17. Let $u_0 = y$ and $u_1 = y'$, then given equation is equivalent to

$$\frac{d}{dx}\begin{pmatrix} u_1 \\ u_0 \end{pmatrix} = \begin{pmatrix} 4 & -13 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} u_1 \\ u_0 \end{pmatrix}$$

Hence the solution is

$$\begin{pmatrix} u_1 \\ u_0 \end{pmatrix} = \exp\left(x\begin{pmatrix} 4 & -13 \\ 1 & 0 \end{pmatrix}\right)\begin{pmatrix} a_1 \\ a_0 \end{pmatrix}$$

for some initial value $(u_1(0), u_0(0)) = (a_1, a_0)$. We need to calulate the right hand side explicitly. Set $A = \begin{pmatrix} 4 & -13 \\ 1 & 0 \end{pmatrix}$.

First we calculate the eigencvalue of the matrix A. Let E be the identity matrix. Solve $\det A - \lambda E = 0$, then $\lambda = 2 \pm 3\sqrt{-1}$.

A has distinct eigenvalues, so A is diagonalizable and $A = UDU^*$ for some unitary matrix and

$$D = \begin{pmatrix} 2+3\sqrt{-1} & 0 \\ 0 & 2-3\sqrt{-1} \end{pmatrix}.$$

Hence

$$\begin{aligned} \exp(xA) &= U\exp(xD)U^* \\ &= U\begin{pmatrix} e^{(2+3\sqrt{-1})x} & 0 \\ 0 & e^{(2-3\sqrt{-1})x} \end{pmatrix}U^* \\ &= U\begin{pmatrix} e^{2x}(\cos 3x + \sqrt{-1}\sin 3x) & 0 \\ 0 & e^{2x}(\cos 3x - \sqrt{-1}\sin 3x) \end{pmatrix}U^* \end{aligned}$$

and

$$\begin{pmatrix} u_1 \\ u_0 \end{pmatrix} = U\begin{pmatrix} e^{2x}(\cos 3x + \sqrt{-1}\sin 3x) & 0 \\ 0 & e^{2x}(\cos 3x - \sqrt{-1}\sin 3x) \end{pmatrix}U^*\begin{pmatrix} a_1 \\ a_0 \end{pmatrix}.$$

Now u_1 and u_0 take real values if the initial value is real, so taking real part after matrix multiplicaion

$$y = u_0 = e^{2x}(A\cos 3x + B\sin 3x).$$

for some constants A and B.

18. Consider $y = x^n u$, then

$$\begin{aligned} y' &= nx^{n-1}u + x^n u', \\ y'' &= n(n-1)x^{n-2}u + 2nx^{n-1}u' + x^n u'' \end{aligned}$$

and the ODE for u is

$$(n(n-1) - 4n + 6)\, x^n u + (2n-4)x^{n+1}u' + x^{n+2}u'' = 0.$$

We would like to choice n such that the coefficient of 0-th order in the ODE of u is 0. This coefficient is $n(n-1) - 4n + 6 = (n-2)(n-3)$, so we choice $n = 2$.

Then the ODE is

$$x^{n+2}u'' = 0$$

and the solution is $u = A + Bx$ for some constants A and B. Hence the solution y is

$$y = x^2 u = Ax^2 + Bx^3.$$

19. From $\begin{vmatrix} k & k \\ 8 & 4k \end{vmatrix} = 4k^2 - 8k = 4k(k-2)$, the solution is $k = 0, 2$.

20. The augumented matrix of given linear system is

$$\begin{pmatrix} 2 & 1 & -1 & 1 \\ 1 & 0 & -2 & -5 \\ 1 & -2 & 3 & 6 \end{pmatrix}.$$

Using elementary row operation,

$$\begin{aligned} \begin{pmatrix} 2 & 1 & -1 & 1 \\ 1 & 0 & -2 & -5 \\ 1 & -2 & 3 & 6 \end{pmatrix} &\to \begin{pmatrix} 0 & 1 & 3 & 11 \\ 1 & 0 & -2 & -5 \\ 0 & -2 & 5 & 11 \end{pmatrix} \\ &\to \begin{pmatrix} 0 & 1 & 3 & 11 \\ 1 & 0 & -2 & -5 \\ 0 & 0 & 11 & 33 \end{pmatrix} \\ &\to \begin{pmatrix} 0 & 1 & 3 & 11 \\ 1 & 0 & -2 & -5 \\ 0 & 0 & 1 & 3 \end{pmatrix}. \end{aligned}$$

This means that $z = 3, y = -3z + 11 = 2, x = 2z - 5 = 1$.

21. The determinant of given matrix is $20+0+(-9)-18-0-(-2)=-5$.

22. The inner product $\mathbf{u}\cdot\mathbf{v}=-6+0+2k=2k-6$, so the value of k such that $\mathbf{u}\cdot\mathbf{v}=0$ is $k=3$.

23. $|x|=\sqrt{2^2+(-3)^2}=\sqrt{13}$.

24. $f(\sin 2\theta,\cos 2\theta)=\sin^2 2\theta+\cos^2 2\theta=1$, so $\frac{df}{d\theta}=0$.

25. Solve $\begin{vmatrix} 2-\lambda & 2 \\ 1 & 3-\lambda \end{vmatrix}=(\lambda-1)(\lambda-4)=0$, then $\lambda=1,4$.

26. $\int_{-2}^{1}|x|dx=\int_0^2 xdx+\int_0^1 xdx=\frac{1}{2}(2^2+1)=\frac{5}{2}$.

End.