Reduction of connections on vector bundles

Definition 1. *Let $\pi_P : P \to M$ be a principal G-bundle and $\pi_Q : Q \to M$ be a principal H-bundle. We say Q is a* reduction *of P if there is an embedding $\iota_Q : Q \to P$ and an injective homomorphism $\iota_H : H \to Q$ such that it holds that $\iota_Q(\eta h) = \iota_Q(\eta)\iota_H(h)$ for any $\eta \in Q$ and $h \in H$.*

Consider principal G-bundle $\pi_P : P \to M$, principal H-bundle $\pi_Q : Q \to M$, and associated bundle $P \times_\rho V$ for a representation $\rho : G \to GL(V)$. If there is an homomorphism $\iota_H : H \to G$, we can another associated bundle $Q \times_\sigma V$ for the restriction representation for $\sigma = \rho \circ \iota_H$ of H.

Proposition 1. *If Q is a reduction of P, then there is a natural isomorphism between $P \times_\rho V$ and $Q \times_\sigma V$.*

Proof. Consider a map $Q \times V \to P \times V$, $(\eta, v) \mapsto (\iota_Q(\eta), v)$. Then it holds that if $(\eta h, v) \sim (\eta, \sigma(h)v)$ then

$$\begin{aligned}
(\iota_Q(\eta h), v) &= (\iota_Q(\eta)\iota_H(h), v) \\
&\sim (\iota_Q(\eta), \rho(\iota_H(h))v) \\
&= (\iota_Q(\eta), \sigma(h)v).
\end{aligned}$$

Hence the map $\varphi : Q \times_\sigma V \to P \times_\rho V$, $\eta \times_\sigma v \mapsto \iota_Q(\eta) \times_\rho v$ is well-defined. We want to show that φ is a desired isomorphim.

We construct well-handled local trivializations of our bundles. First we consider Q and choose some local trivialization $\phi^Q : Q|_U \to U \times H$. Then we have a section $q : U \to Q|_U$ such that $\phi^Q \circ q(y) = (y, e)$. Using the section q, we can denote $\phi^Q(\eta) = (y, h)$ where $y = \pi_Q(\eta)$ and $\eta = q(y)h$.

We construct $\phi^P : P|_U \to U \times G$ which satisfies the following commutative diagram:

$$\begin{tikzcd}
P|_U \arrow[r, "\phi^P"] & U \times G \\
Q|_U \arrow[u, "\iota_Q"] \arrow[r, "\phi^Q"] & U \times H \arrow[u, "id\times\iota_H"'] \\
U \arrow[u, "q"] \arrow[ur, "id\times\{e\}"'] &
\end{tikzcd}$$

Let $p = \iota_Q \circ q$ and define $\phi^P : P|_U \to U \times G$ by $\phi^P(\xi) = (x, g)$ where $x = \pi_P(\xi)$ and $\xi = p(x)g$. Then it holds that for $\eta = q(y)h \in Q|_U$

$$\begin{aligned}
\iota_Q(\eta) &= \iota_Q(q(y)h) = \iota(q(y))\iota_H(h) \\
&= p(y)\iota_H(h) \\
\phi^P(\iota_Q(\eta)) &= (y, \iota_H(h)).
\end{aligned}$$

This shows the commutativity of upper rectangle. The commutativity of lower triangle follows by the definition of q. Note that $\phi^P \circ p(x) = (x, \iota_H(e)) = (x, e)$ for $x \in U$.

Next we construct local trivializations for E^P and E^Q. Define $\psi^Q(q(y) \times_\sigma v) = v$ and $\psi^P(p(x) \times_\sigma u) = u$. Then trivially it holds that the following commutative diagram:

$$\begin{tikzcd}
E^P|_U \arrow[r, "\psi^P"] & U \times V \\
E^Q|_U \arrow[u, "\varphi"] \arrow[r, "\psi^Q"] & U \times V \arrow[u, equal]
\end{tikzcd}$$

This implies that *varphi* is an isomorphism from E^Q to E^P.

$\square$

For a convinience in the following section, we re-state the exietence of well-handled local trivializations of our bundles:

Proposition 2. *For a local trivialization $\phi^Q : Q|_U \to U \times H$, difine*

- *a section $q : U \to Q|_U$ by $\phi^Q \circ q(y) = (y, e)$;*
- *a section $p : U \to P|_U$ by $p = \iota_Q \circ q$;*
- *a local trivialization $\phi^P : P|_U \to U \times G$ of P by $\phi^P(\xi) = (x, g)$ where $x = \pi_P(\xi)$ and $\xi = p(x)g$;*

- *a local trivialization* $\psi^Q : E^Q|_U \to U \times V$ *of* E^Q *by* $\psi^Q(q(y) \times_\sigma v) = v$*;*
- *a local trivialization* $\psi^P : E^P|_U \to U \times V$ *of* E^P *by* $\psi^P(p(x) \times_\sigma u) = u$*.*

Then it holds the following commutative diagrams:

$$\begin{array}{ccc} P|_U & \xrightarrow{\phi^P} & U \times G \\ {\scriptstyle \iota_Q}\uparrow & & \uparrow{\scriptstyle id \times \iota_H} \\ Q|_U & \xrightarrow{\phi^Q} & U \times H \end{array} \qquad \begin{array}{ccc} E^P|_U & \xrightarrow{\psi^P} & U \times V \\ {\scriptstyle \varphi}\uparrow & & \| \\ E^Q|_U & \xrightarrow{\psi^Q} & U \times V \end{array}$$

Definition 2. *Let* $\pi_P : P \to M$ *be a principal* G*-bundle and* $\pi_Q : Q \to M$ *be a principal* H*-bundle which is a reduction of* P*. We call a connection form* $\theta \in \Omega^1(P; \mathfrak{g})$ *is* reducible to Q *if* $(\ker\theta)_{\iota_Q(\eta)} \subset (\iota_Q)_{*\eta}(T_\eta Q)$ *for any* $\eta \in Q$*.*

Theorem 1. *Consider a (real or complex) Hermitian verctor bundle* (E, h)*, frame bandle* P *of* E*, and frame bandle* Q *of* (E, h)*. For a connection form* $\theta \in \Omega^1(P; \mathfrak{g})$ *the followings are equivalent:*

- *the induced connection* ∇^θ *on* E *is* h*-preserving;*
- θ *is reducible to* Q*.*