Phase Space Generation

Given an squared amplitude (averaged)

$$\langle |A|^2 \rangle : p_i^\mu \to \mathbb{R}$$

we would like to compute the integral over the <u>phase space</u> to find the (differential) cross-section.

$$\int d\underline{\Phi} \langle |A|^2 \rangle \propto \hat{\sigma}$$

$d\underline{\Phi}$ (also known as Lorentz Invariant Phase Space or LIPS) has the form

$$\boxed{d\underline{\Phi} = \left(\prod_{i=1}^{n_{final}} d^4 p_i \, \delta^{(+)}(p_i^2 - m_i^2) \right) \delta^{(4)}\left(Q - \sum_i p_i\right)}$$

where Q is the sum of the incoming momenta. Q^2 often written simply as s is the centre-of-mass energy. For each final state momenta we have an integral over <u>three</u> variables having applied the <u>on-shell</u> condition $\delta^{(+)}(p_i^2 - m_i^2)$ for +ve energy and mass m_i. In addition overall four-momentum conservation imposes four more constraints on the total number of parameters so we have

$$3n_{final} - 4$$

independent parameters in the phase space integral.

Since for $2 \to n_{final}$ scattering $n = n_{final} + 2$ we may also say the phase-space has $3n - 10$ parameters.

Isotropic phase space generation

For numerical integration we would like (at least we might imagine) to have an evenly distributed set of on-shell phase space points. We will only consider the case of $m_i = 0$ (massless particles).

We can use the RAMBO algorithm to do this [Kleiss, Stirling, Ellis (1986)]. What this does is to generate a "flat" phase space sampling which can be used in Monte-Carlo integration.

The algorithm can be described as follows:

1) generate random 4-component vectors from a uniform distribution

$$x_{\mu i} \quad , \quad \mu = 0, 3$$
$$, \quad i = 1, n_{final}$$
$$, \quad 0 < x_{\mu i} < 1$$

where $x_{\mu i}$ are given by a suitable random number generator, e.g.

$$\text{numpy.random.uniform}(\underbrace{0, 1}_{\text{range } 0,1}, \; 4)$$

$\uparrow$ 4-components

2) use $x_{\mu i}$ to construct isotropic momenta. For each i

$$c_i = 2x_{0i} - 1 \qquad \phi_i = 2\pi x_{1i}$$

$$\hat{p}_i^0 = -\log(x_{2i} x_{3i})$$

$$\hat{p}_i^1 = \hat{p}_i^0 \sqrt{1 - c_i^2} \cos(\phi_i)$$

$$\hat{p}_i^2 = \hat{p}_i^0 \sqrt{1 - c_i^2} \sin(\phi_i)$$

$$\hat{p}_i^3 = \hat{p}_i^0 \, c$$

so we now have $\hat{p}_i^\mu$ where $\hat{p}_i^2 = 0$.

3) Apply a <u>Lorentz boost</u> to $\hat{p}^\mu_i$ to put them in the centre-of-mass frame with total energy $S = Q^2$,

$$\hat{Q}^\mu = \sum_{i=1}^{n_{final}} \hat{p}^\mu_i$$

$$M = \sqrt{\hat{Q}^2}, \quad \vec{b} = -\vec{\hat{Q}}/M \qquad x = \frac{\sqrt{s}}{M}$$

$$\gamma = \frac{\hat{Q}^0}{M}, \quad a = \frac{1}{1+\gamma}$$

then,

$$p_i^0 = x\left(\gamma \hat{p}_i^0 + \vec{b}.\vec{\hat{p}}_i\right)$$

$$\vec{p}_i = x\left(\vec{\hat{p}}_i + \vec{b}\,\hat{p}_i^0 + a\,\vec{b}\,(\vec{b}\cdot\vec{\hat{p}}_i)\right)$$

where $\boxed{\sum_{i=1}^{n_{final}} \vec{p}_i = 0}$ and $\boxed{\sum_{i=1}^{n_{final}} p_i^0 = \sqrt{s}}$

Basic Monte Carlo Integration

The isotropic phase space generator maps randomly distributed values $0 < x_i < 1$ to on-shell phase space points. Importantly this means the Jacobian transformation on the integral is $\underline{1}$:

$$\int d\Phi_{n_{final}} = \int_0^1 \prod_{i=1}^{n_{final}} dx_i$$

and hence the total phase-space volume is equal to $\underline{1}$. The integration of a function (i.e. squared amplitude) is therefore

$$\sigma = \int_0^1 \prod dx_i \; f(\{p(x_i)\})$$

where $\{p(x_i)\}$ are momenta generated by RAMBO. The MC approximation is simply

$$\sigma = \frac{1}{N} \sum_{j=1}^{N} f(\{p(x_i^{(j)})\})$$

for a set of j random momenta.
Assuming in integral is well defined this should converge after some (potentially very large number of iterations).