

VIII. Classification

- I) Logistic regression
 - I.1) The cross-entropy loss function
 - I.2) SoftMax regression
- 2) Applications
 - 2.1) Classifying data points in 2d
 - 2.2) Classifying signal/background with collider events

I)

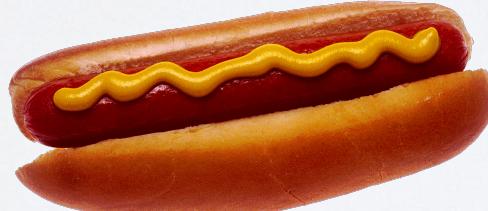
Logistic regression

- Discrete outcomes from the input data

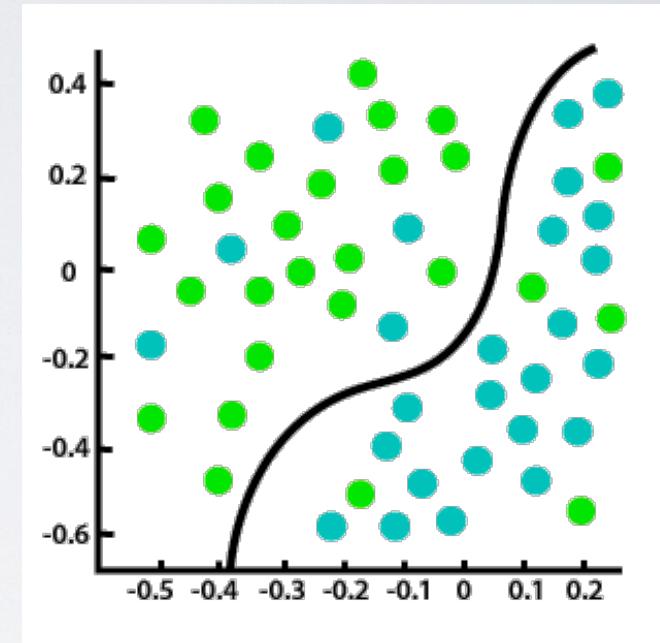


kitty or
croissant

hot dog



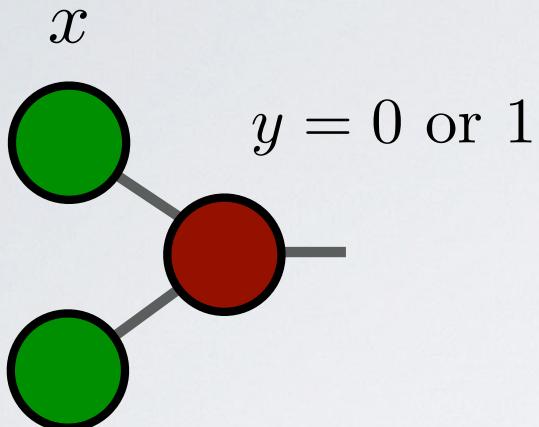
not hot dog



start with simple case:
output is 0 or 1

I.I)

Cross-entropy loss function



recall the Perceptron which
acted as a linear classifier

$$\sigma(\vec{x} \cdot \vec{w} + b) = \sigma(\vec{X} \cdot \vec{W})$$

theta/sign function

$$\sigma(z) = \Theta(z)$$

“hard” classifier

decide the answer is 0 or 1

sigmoid/logistic function

$$\sigma(z) = \frac{1}{1 + e^{-z}}$$

“soft” classifier

probability that the answer is 0 or 1

I.I)

Cross-entropy loss function

Probability that a data point $X=(l, x)$ belongs to a category $y=0, l$ for a model of weights W

$$P(y = 1|x, W) = \frac{1}{1 + e^{-X \cdot W}}$$

$$P(y = 0|x, W) = 1 - P(y = 1|x, W)$$

Boltzman distribution of a two state system

$$p_i = Q^{-1} e^{-E_i/(kT)} \quad p_1 = \frac{1}{1 + e^{-(E_1 - E_0)/(kT)}}$$

$$Q = e^{-E_0/(kT)} + e^{-E_1/(kT)} \quad p_0 = 1 - p_1$$

I.I)

Cross-entropy loss function

Given a model, \vec{W} , the **likelihood** that it matches a data set $\mathcal{D} = \{\vec{x}_i, y_i\}$ is

$$P(\{\vec{x}_i, y_i\} | \vec{W}) = \prod_{i=1}^n [\sigma(\vec{X}_i \cdot \vec{W})]^{y_i} [1 - \sigma(\vec{X}_i \cdot \vec{W})]^{1-y_i}$$

where the loss function, \mathcal{L} , is $-\log$ (likelihood)

$$\mathcal{L}(\vec{W}) = - \sum_{i=1}^n y_i \log(\sigma(\vec{X}_i \cdot \vec{W})) + (1 - y_i) \log(1 - \sigma(\vec{X}_i \cdot \vec{W}))$$

This is called the cross-entropy

I.I)

Cross-entropy loss function

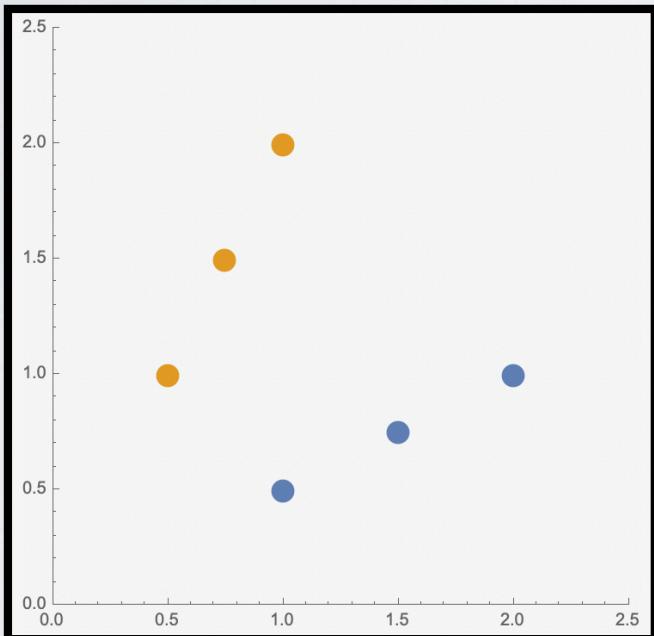
Minimising the cross entropy results in a set of optimal weights for our model

$$\frac{\partial}{\partial \vec{W}} \mathcal{L}(\vec{W}) = \sum_i \vec{X}_i \left(\sigma(\vec{X}_i \cdot \vec{W}) - y_i \right) = \sum_i \vec{X}_i (\hat{y}_i - y_i)$$

Training the model can be achieved through gradient descent methods

I.I)

Explicit example



from the data:

```
Data = {  
    {{2,1},0}, {{1,1/2},0}, {{3/2,3/4},0},  
    {{1,2},1}, {{1/2,1},1}, {{3/4,3/2},1}  
};
```

compute cross entropy loss function at values for the model parameters:

$$\begin{aligned}\mathcal{L}(&\{-1.5, 1, -1\}) \\ \mathcal{L}(&\{0, 1, -1\}) \\ \mathcal{L}(&\{-1.5, 1, 1\})\end{aligned}$$

implement the gradient of the loss function and find the minimum via a few iterations of gradient descent

$$\vec{\nabla} \mathcal{L} = 0$$

I.2)

SoftMax regression

The generalisation from a binary output (0,1) to a discrete list of values (0,1,2,...) (referred to as *labels*) requires generalising **Logistic Regression** to **SoftMax regression**

The generalisation is quite simple. Recall the Boltzman distribution of a multi-state system

$$p_I = Q^{-1} e^{-E_I/(kT)} \quad I = 1, \dots, M$$

$$Q = \sum_{I=1}^M e^{-E_I/(kT)} \quad \text{therefore} \quad \sum_{I=1}^M p_I = 1$$

I.2)

SoftMax regression

probability of being in a state I

$$P(y = I | \vec{x}, \{\vec{W}_J\}) = \frac{1}{1 + \sum_{J \neq I} e^{-\vec{X} \cdot (\vec{W}_J - \vec{W}_I)}}$$

which we may think of as a vector of probabilities

$$\vec{P}(\vec{x}, \{\vec{W}_J\}) = \begin{pmatrix} P(y = 1 | \vec{x}, \{\vec{W}_J\}) \\ P(y = 2 | \vec{x}, \{\vec{W}_J\}) \\ \vdots \\ P(y = M | \vec{x}, \{\vec{W}_J\}) \end{pmatrix}$$

1.2)

SoftMax regression

the data is then given as $\{\vec{x}, \vec{y}\}$

and so the loss function is

$$\mathcal{L}(\{\vec{W}_J\}) = - \sum_{i=1}^n \vec{y}_i \cdot \log(\vec{P}(\vec{x}_i, \{\vec{W}_J\})) + (\vec{1} - \vec{y}_i) \cdot \log(\vec{1} - \vec{P}(\vec{x}_i, \{\vec{W}_J\}))$$

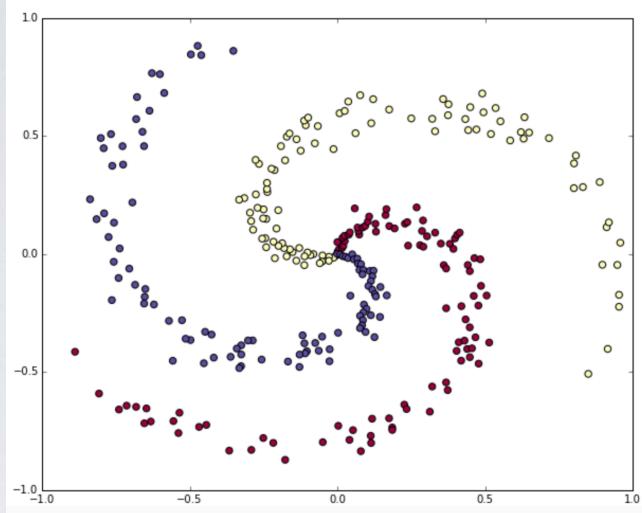


vector of 1s

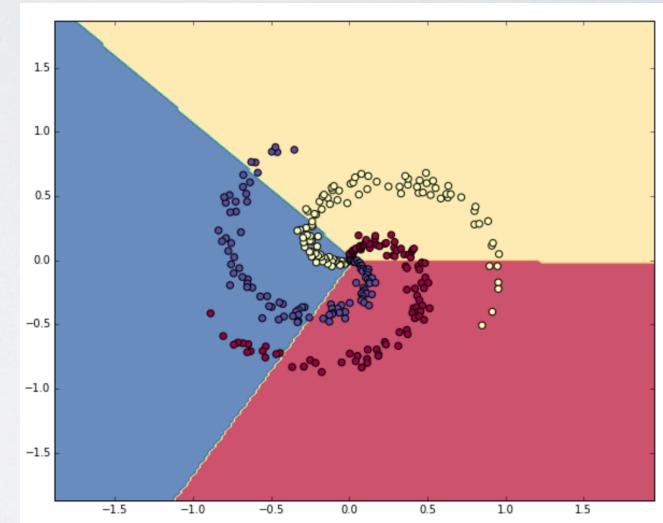
2.1)

Classifying points in 2d

<https://cs231n.github.io/neural-networks-case-study/>

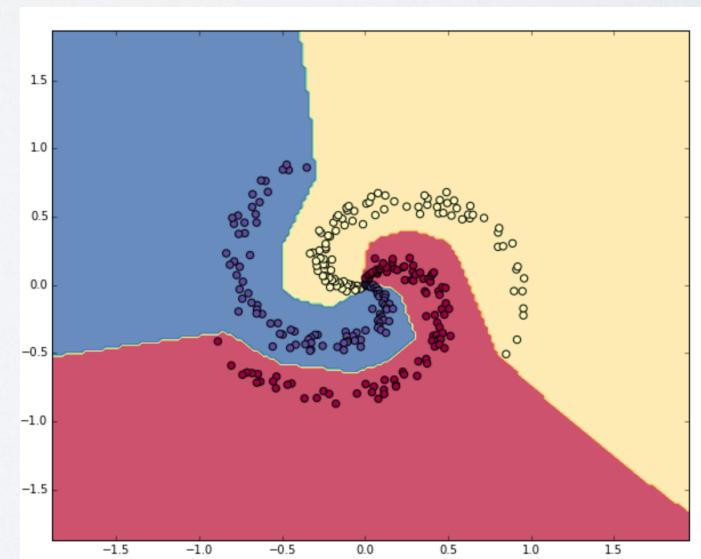


linear classifier



neural network
classifier

follow the tutorial to see
how to implement it



2.2)

Measuring performance: the ROC curve

ROC = Receiver Operating Characteristic

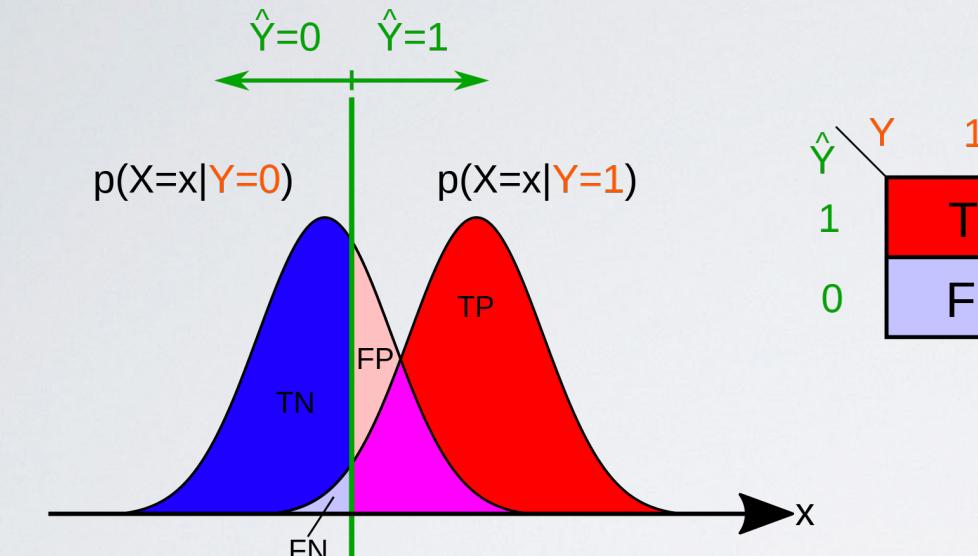
Once we've trained a classifier how do we interpret the inferred values?

- take a test: pass sample values to the trained model
- outcome is a probability between 0, 1 of the likelihood the test is positive. Let's say $>1/2$ means True and $<1/2$ is False

		actual result	
		True	False
test result	positive	True Positive	False Positive
	negative	False Negative	True Negative

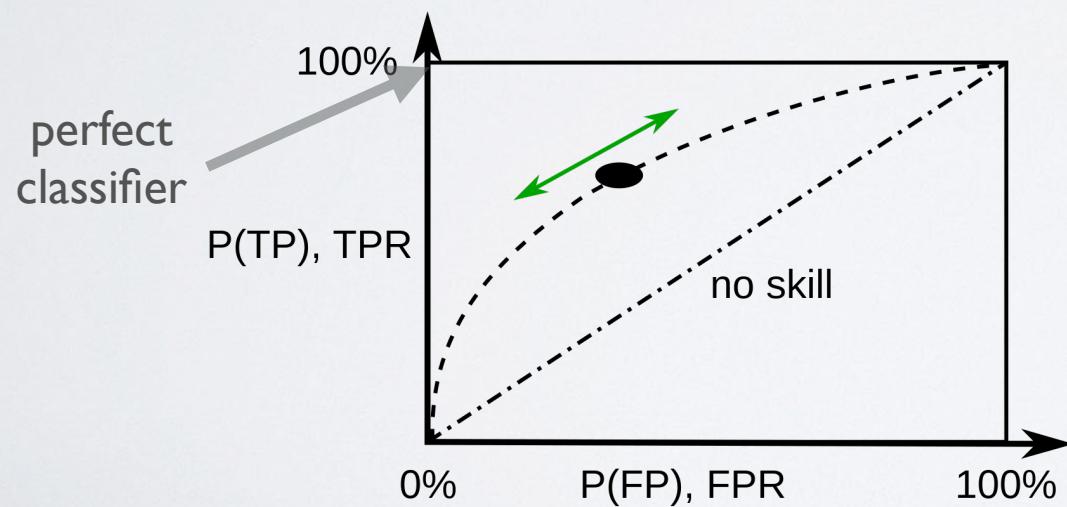
2.2)

Measuring performance: the ROC curve



	1	0
1	TP	FP
0	FN	TN

Y is the actual result



$P(X=x|Y=1)$ is the result from the model

$$P(X=x|Y=0) = 1 - P(X=x|Y=1)$$

$\hat{Y} = 1 \text{ if } P(X=x|Y=1) > \text{threshold},$
 $= 0 \text{ if } P(X=x|Y=1) < \text{threshold}$

(...figure from Wikipedia! sorry...)

2.2)

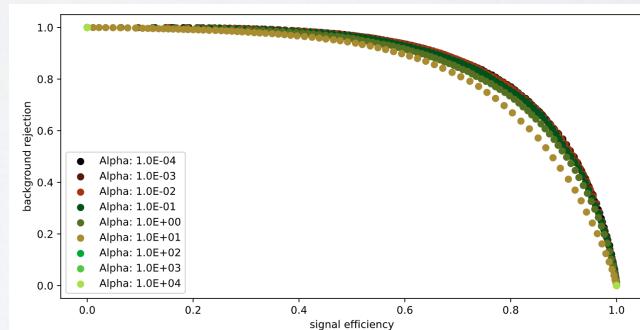
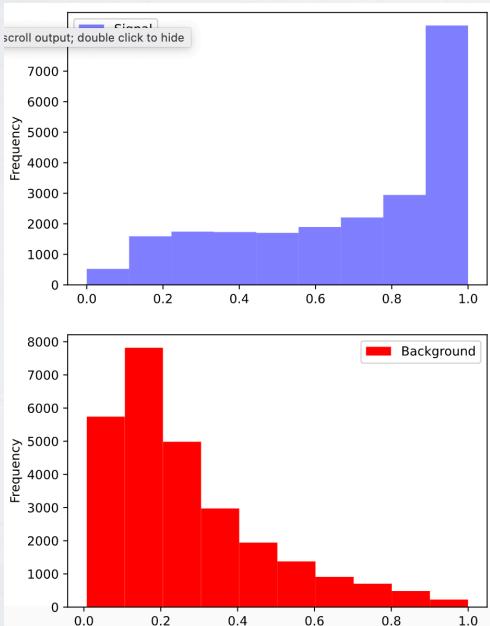
Discriminating signal from background

tutorial from: https://github.com/drckf/mlreview_notebooks/tree/master/jupyter_notebooks/notebooks

also in our repository with dataset: `LogisticRegression_SUSY.log`

- download a generic ‘SUSY’ dataset

MC generated samples based on theory models of Standard Model (SM = background) and supersymmetry (SUSY = signal)



NB: ROC as before but with $x \rightarrow 1 - x$