Soft and Collinear Divergences

We may attempt to understand how to better construct an amplitude approximation by obtaining a better analytic understanding of the amplitude itself. We have already seen the compact structure for $e^+e^- \to q\bar{q}$ and $e^+e^- \to q\bar{q}g$:

$$\langle |A_{q\bar{q}}|^2 \rangle \sim \frac{S_{a1}^2 + S_{a2}^2}{S_{ab}^2}$$

$$\langle |A_{q\bar{q}g}|^2 \rangle \sim \frac{S_{a1}^2 + S_{a2}^2 + S_{b1}^2 + S_{b2}^2}{S_{ab} S_{13} S_{23}}$$

where

$$S_{ij} = (p_i + p_j)^2 \qquad i = 1, 3$$

[NB: $S_{ai} = (p_i - p_a)^2$ though we may like to adjust mom. conservation sum so $\sum p_i^\mu = 0$ and then all invariant are $S_{pq} = (p+q)^2$]

If we consider a pair of particles i and j separted with angle θ_{ij} [remember any pair of momenta will form a plane] and energies E_i and E_j then

$$2 p_i \cdot p_j = S_{ij} = 2 E_i E_j (1 - \cos\theta_{ij})$$

and so we see that

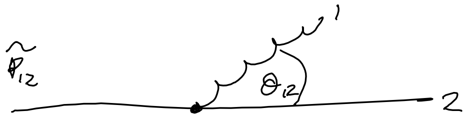
$s_{ij} \to 0$ if $\theta_{ij} \to 0$ [collinear]

$E_i, E_j \to 0$ [soft]

Since $p_i^2 = 0$, $p_i = E_i (1, \hat{n})$

normalised 3-vector.

hence we may dentoe $E_i \to 0$ as $p_i \to 0$. We also introduce the notation $i \parallel j$ to indicate the collinear limit $\theta_{ij} \to 0$.



Clearly $\langle |A|^2 \rangle \to \infty$ in the limits

$p_3 \to 0$, $1 \parallel 3$ and $2 \parallel 3$

but we can also determine a more detailed description as we approach the limit. For example let's write the collinear limit such that

$$A(\dots p_i\, p_j \dots) \xrightarrow{i \parallel j} A(\dots \tilde{p}_{ij} \dots)$$

where $\tilde{p}_{ij}^{\,2} = 0$. If the fraction of the momentum $\tilde{p}_{ij}$ coming from p_i in the limit is z_i then

$$p_i \xrightarrow{i \parallel j} z_i\, \tilde{p}_{ij}$$

$$p_j \xrightarrow{i \parallel j} z_j\, \tilde{p}_{ij} = (1 - z_i)\, \tilde{p}_{ij}$$

where $z_i + z_j = 1$ for $p_i + p_j \Rightarrow \tilde{p}_{ij}$.

We may therefore drop the index i and write $z_i \equiv z$. We may now consider the $e^+e^- \to q\bar{q}g$ limit:

$$\langle |A_{q\bar{q}g}|^2 \rangle = \frac{s_{a1}^2 + s_{a2}^2 + s_{b1}^2 + s_{b2}^2}{s_{ab}\, s_{13}\, s_{23}}$$

$$\xrightarrow{1 \parallel 3} \frac{z^2 s_{a\tilde{p}}^2 + s_{a2}^2 + z^2 s_{b\tilde{p}}^2 + s_{b2}^2}{s_{ab} \underbrace{s_{13}}_{\text{small}} (1-z) s_{2\tilde{p}}}$$

where mom. conservation after the limit is

$$p_a + p_b = \tilde{p}_{13} + p_2$$

This means that $s_{a\tilde{p}} = s_{b2}$ and

$s_{b\tilde{p}} = s_{a2}$ and $s_{2\tilde{p}} = s_{ab}$

$$\langle |A_{q\bar{q}g}|^2 \rangle = \frac{1+z^2}{(1-z)\, s_{13}} \times \left(\frac{s_{a2}^2 + s_{b2}^2}{s_{ab}^2} \right)$$

$$= \frac{1+z^2}{(1-z)\, s_{13}} \times \langle |A_{q\bar{q}}(a, b, \tilde{p}, 2)|^2 \rangle$$

so we have shown that the amplitude factorises in the limit with a well defined scaling law

$$\langle |A_{qqg}|^2 \rangle \xrightarrow{1\|3} \underbrace{P_{qg}(z)}_{\text{splitting function}} \frac{1}{S_{13}} \langle |A_{qq}|^2 \rangle$$

The behavior at these limits is fundamental in any gauge theory since infrared (IR) (soft or collinear) divergences must cancel in a physical observable. The splitting functions are therefore universal to any amplitude

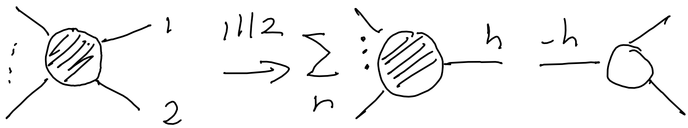
$P_{qg}(z) \equiv P_{q\to qg}(z)$ is

the probability that a quark radiates a collinear gluon. The soft limit also factorises but with a double singularity

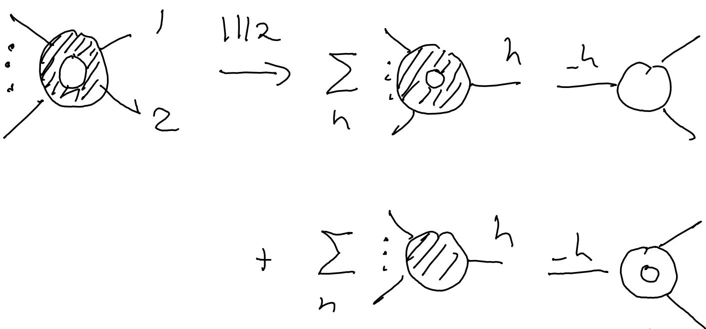
$$\langle |\mathcal{M}_{q\bar{q}g}|^2 \rangle \xrightarrow{p_3 \to 0} \underbrace{\frac{2 s_{ab}}{s_{13} s_{23}}}_{\text{Eikonal soft function}} \left(\frac{s_{1a}^2 + s_{1b}^2}{s_{ab}^2} \right)$$

These are the simplest factorisations for QCD amplitudes which quickly become more complicated:

At the amplitude level we have



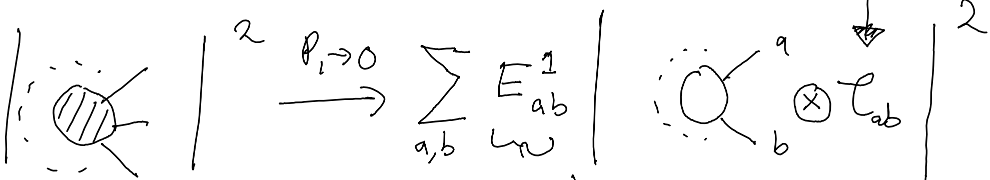
helicity ⇒ spin correlations



loop splitting amplitude

soft limits are in general colour correlated which makes a picture more difficult

colour space



eikonal

IR limits and Amplitude Neural Networks

The divergence of the amplitude in soft and collinear limits is clearly a problem for the neural network to learn.

1) we need a way to help point the network in the right direction

2) no single method to do this
   → please come up with a better one!

One option is to consider isolating particular regions of the phasespace so the NN doesn't have to learn many features at the same time.

We may even try to impose the factorisation into the network architecture — this always seemed like a complicated option since the factorisation is increasingly complicated for higher loops.

We can also look to another way of separating the (many) divergent regions employing a weighting factor (introduced by Frixione, Kunszt and Signer in 1995 – context of IR subtractions @ NLO)

$$\langle |A|^2 \rangle = \sum_{i,j} S_{ij} \langle |A|^2 \rangle$$

where $\sum_{i,j} S_{ij} = 1$ and

$$\mathcal{S}_{ij} \xrightarrow{i \parallel j} 1$$

$$\xrightarrow[\text{collinear limits}]{\text{all other}} \emptyset$$

$$\mathcal{S}_{ij} \xrightarrow{p_i \to 0} 1/2 \,, \quad S_{ij} \xrightarrow{p_j \to 0} 1/2$$

$$\xrightarrow[\text{soft limits}]{\text{all other}} \emptyset$$

a suitable definition is

$$\mathcal{S}_{ij} = \frac{1}{D} \frac{1}{S_{ij}}$$

where $D = \sum_{i,j} 1/S_{ij}$

we may now consider a NN for each FKS partition of $\langle |A|^2 \rangle$

$$\langle |A|^2 \rangle = \sum_{i,j} S_{ij} \langle |A|^2 \rangle$$

Does it do a better job?

Q: Possible improvements and tests

- passing FKS weights to loss function
- initial state singularities $\rightsquigarrow$ better cut configurations [discuss PDFs for $pp \to X$ rather than $e^+e^- \to X$]
- determine speed up for 1-loop $e^+e^- \to 3j$ @ 1-loop.