2022 AMC 12B Problem 15

One of the following numbers is not divisible by any prime number less than 10. Which is it?

(A) $2^{606}-1$ **(B)** $2^{606}+1$ **(C)** $2^{607}-1$ **(D)** $2^{607}+1$ **(E)** $2^{607}+3^{607}$

Solution

Key Word Property of Modular Arithmetic, Euler's Theorem

Choice (A):

$$\begin{aligned}2^{606}-1 &\equiv (-1)^{606}-1 \pmod 3\\ &\equiv 0 \pmod 3\end{aligned}$$

Choice (B):

$$\begin{aligned}2^{\varphi(5)} &\equiv 1 \pmod 5\\ 2^4 &\equiv 1 \pmod 5\\ 2^{604} &\equiv 1 \pmod 5\\ 2^{606} &\equiv 4 \pmod 5\\ 2^{606}+1 &\equiv 0 \pmod 5\end{aligned}$$

Choice (C):

$$\begin{aligned}2^{\varphi(3)} &\equiv 1 \pmod 3\\ 2^2 &\equiv 1 \pmod 3\\ 2^{607} &\equiv 2 \pmod 3\\ 2^{607}-1 &\equiv 1 \pmod 3\end{aligned}$$

$$\begin{aligned}2^{\varphi(5)} &\equiv 1 \pmod 5\\ 2^4 &\equiv 1 \pmod 5\\ 2^{604} &\equiv 1 \pmod 5\\ 2^{607}-1 &\equiv 7 \pmod 5\end{aligned}$$

$$\begin{aligned}2^{\varphi(7)} &\equiv 1 \pmod 7\\ 2^6 &\equiv 1 \pmod 7\\ 2^{606} &\equiv 1 \pmod 7\\ 2^{607}-1 &\equiv 1 \pmod 7\end{aligned}$$

Therefore, $\boxed{\textbf{(C)}\ 2^{607}-1}$ is not divisible by any prime number less than 10. $\square$

2022 AMC 12B Problem 20

Let $P(x)$ be a polynomial with rational coefficients such that when $P(x)$ is divided by the polynomial x^2+x+1, the remainder is $x+2$, and when $P(x)$ is divided by the polynomial x^2+1, the remainder is $2x+1$. There is a unique polynomial of least degree with these two properties. What is the sum of the squares of the coefficients of that polynomial?

(A) 10 **(B)** 13 **(C)** 19 **(D)** 20 **(E)** 23

Solution

Key Word Property of Modular Arithmetic

First and foremost, the provided condition may be written.

$$\begin{aligned} P(x) &\equiv x+2 \pmod{x^2+x+1} \\ P(x) &\equiv 2x+1 \pmod{x^2+1} \end{aligned}$$

The relationship between the first and the second condition could be found.

$$\begin{aligned} (x^2+x+1)k+x+2 &\equiv 2x+1 \pmod{x^2+1} \\ (x^2+x+1)k &\equiv x-1 \pmod{x^2+1} \\ (x^2+x+1)k &\equiv x^2+x \pmod{x^2+1} \\ (x^2+1)k+xk &\equiv x^2+x \pmod{x^2+1} \\ xk &\equiv x^2+x \pmod{x^2+1} \\ k &\equiv x+1 \pmod{x^2+1} \end{aligned}$$

k could be rewritten as $k=(x^2+1)k'+x+1$. Therefore, $P(x)=(x^2+x+1)(x^2+1)k'+(x^2+x+1)(x+1)+x+2$. The degree of $P(x)$ is minimum when $k'=0$. Thereby, $P(x)=(x^2+x+1)(x+1)+x+2=x^3+2x^2+3x+3$. Thus, $1^1+2^2+3^2+3^2=\boxed{\textbf{E)}\ 23}$. $\square$

2022 AMC 12B Problem 23

Let $x_0, x_1, x_2, \ldots$ be a sequence of numbers, where each x_k is either 0 or 1. For each positive integer n, define

$$S_n = \sum_{k=0}^{n-1} x_k 2^k$$

Suppose $7S_n \equiv 1 \pmod{2^n}$ for all $n \geq 1$. What is the value of the sum

$$x_{2019} + 2x_{2020} + 4x_{2021} + 8x_{2022}?$$

(A) 6 **(B)** 7 **(C)** 12 **(D)** 14 **(E)** 15

Solution

Key Word Trial and Error, Euler's Theorem

Using the given conditions, it is evident that $x_{2019} + 2x_{2020} + 4x_{2021} + 8x_{2022} = \frac{S_{2023} - S_{2019}}{2^{2019}} = a$ (for simplicity). Moreover, because $7S_n \equiv 1 \pmod{2^n}$ is true, an impulse to multiply the numerator and denominator by 7 is created. Let $7S_{2023} = 2^{2023}k + 1$ and $7S_{2019} = 2^{2019}k' + 1$.

$$\begin{aligned} a = \frac{7S_{2023} - 7S_{2019}}{7 \cdot 2^{2019}} &= \frac{2^{2023}k + 1 - 2^{2019}k' - 1}{7 \cdot 2^{2019}} \\ &= \frac{2^4 k - k'}{7} \\ &= \frac{16k - k'}{7} \end{aligned}$$

Furthermore, because $0 \leq x_{2019} + 2x_{2020} + 4x_{2021} + 8x_{2022} \leq 15$, $0 \leq \frac{16k-k'}{7} \leq 15$. The order pairs (k, k') could be found through trial and error that satisfies conditions $7S_{2023} = 2^{2023}k + 1$ and $7S_{2019} = 2^{2019}k' + 1$. It is evident that $k \neq 0, 1, 2$ using Euler's Theorem.

k	k'	a	Validity
3	6	6	Yes $(\because 7 \mid 2^{2023} \cdot 3 + 1$ and $7 \mid 2^{2019} \cdot 6)$

Thereby, $x_{2019} + 2x_{2020} + 4x_{2021} + 8x_{2022} = \boxed{\textbf{(A) } 6}$. $\square$

Uploaded a new solution in AOPS!!