

Given  $z = \cos 40^\circ + i \sin 40^\circ$ , express  $|z + 2z^2 + 3z^3 + \dots + 9z^9|^{-1}$  in the form of  $a \sin b$ .

$$40^\circ = \frac{40}{180\pi} \cdot \pi = \frac{2\pi}{9}$$

$$\begin{aligned} z &= \cos \frac{2\pi}{9} + i \sin \frac{2\pi}{9} \\ &= e^{i \frac{2\pi}{9}} \end{aligned}$$

$$\begin{aligned} z^9 &= 1 & z, z^2, \dots, z^9 \\ z^9 &= 1 \end{aligned}$$

$$\begin{aligned} z^9 - 1 &\approx 0 \\ (z-1)(z^8 + z^7 + \dots + 1) &\approx 0 \end{aligned}$$

$$|z| = 1$$

$$\begin{aligned} |1-z| &= \sqrt{(1-\cos \frac{2\pi}{9})^2 + \sin^2 \frac{2\pi}{9}} \\ &= \sqrt{2 - 2 \cos \frac{2\pi}{9}} \end{aligned}$$

$$\approx 2 \sqrt{1 - \cos \frac{2\pi}{9}}$$

$$\approx 2 \cdot \frac{\pi}{9}$$

$$\sum_{k=1}^9 k z^k$$

$$\begin{aligned} \hookrightarrow & \quad S = z + 2z^2 + \dots + 9z^9 \\ - & \quad \underline{Sz = z^2 + 2z^3 + \dots + 9z^{10}} \end{aligned}$$

$$\begin{aligned} S(1-z) &= z + z^2 + \dots + z^9 - 9z \\ \therefore S &= \frac{(z + z^2 + \dots + z^9) - 9z}{1-z} \end{aligned}$$

$$= \frac{-9z}{1-z}$$

$$\left| \frac{-9z}{1-z} \right|^{-1} = \left| \frac{1-z}{9z} \right|$$

$$= \frac{|1-z|}{|9z|}$$

$$= \frac{2 \cdot \frac{\pi}{9}}{9}$$

$$= \boxed{\frac{2}{81} \sin 20^\circ}$$

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