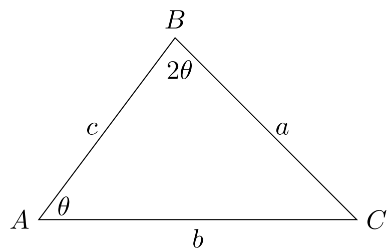
2024 AMC 12B Problem 22

Let $\triangle ABC$ be a triangle with integer side lengths and the property that $\angle B = 2\angle A$. What is the least possible perimeter of such a triangle?

(A) 13 **(B)** 14 **(C)** 15 **(D)** 16 **(E)** 17

Solution

Key Word Law of Sines, Law of Cosines, Trigonometric Identities, Property of Triangle



The fact that a, b and c are integers may be utilized.

According to the Law of Sines, the following equation is true.

$$\frac{b}{\sin 2\alpha} = \frac{a}{\sin \alpha}$$
$$\frac{b}{2\sin\alpha\cos\alpha} = \frac{a}{\sin\alpha}$$
$$\cos\alpha = \frac{b}{2a} \ (\because 0^\circ < \alpha < 90^\circ)$$

Moreover, using the Law of Cosines, an additional equation may be driven.

$$\cos\alpha = \frac{b^2 + c^2 - a^2}{2bc}$$

Substitute $\cos\alpha$ to $\cos\alpha = \frac{b}{2a}$.

$$\frac{b^2 + c^2 - a^2}{2bc} = \frac{b}{2a}$$
$$2a(b^2 + c^2 - a^2) = 2b^2c$$
$$2ab^2 + 2ac^2 - 2a^3 = 2b^2c$$
$$2b^2(a-c) + 2a(c^2 - a^2) = 0$$
$$2b^2(a-c) - 2a(a+c)(a-c) = 0$$
$$(a-c)(2b^2 - 2a(a+c)) = 0$$
$$(a-c)(2b^2 - 2a^2 - 2ac) = 0$$

Thus, $a - c = 0$ or $2b^2 - 2a^2 - 2ac = 0$ or both is true. However, when $a = c$, $\alpha = 45^\circ$, which leads to non-integer length for at least one of the sides.

$$\therefore 2b^2 - 2a^2 - 2ac = 0$$
$$b^2 = a^2 + ac$$
$$b^2 = a(a+c)$$

Because the least possible perimeter of such a triangle must be found, the values for a, b and c could be substituted.

b	a	c	$Validity$	
1	1	0	No	
2	1	3	No	$(\because 3 = 2 + 1)$
2	2	0	No	
3	1	8	No	$(\because 8 > 3 + 1)$
3	3	0	No	
4	1	15	No	$(\because 15 > 4 + 1)$
4	2	6	No	$(\because 6 = 4 + 2)$
4	4	0	No	
5	1	24	No	$(\because 24 > 5 + 1)$
6	1	35	No	$(\because 35 > 6 + 1)$
6	2	16	No	$(\because 16 > 6 + 2)$
6	3	9	No	$(\because 9 = 6 + 3)$
6	4	5	Yes	

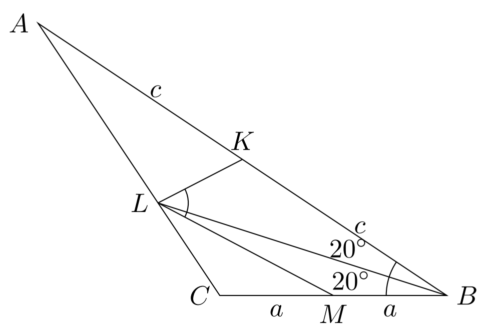
Therefore, the least perimeter is $6 + 4 + 5$, which is $\boxed{\text{(C) } 15}$. □

Problem

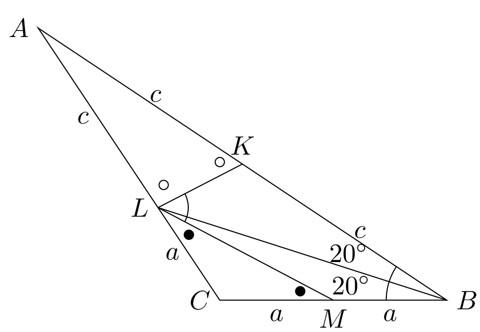
In $\triangle ABC$, $\angle B = 40°$ and $AB + BC = 2AC$. K and M are the mid-points of AB and BC respectively, while L is a point on AC such that BL bisects $\angle ABC$. Find $\angle KLM$.

Solution

Key Word Angle Bisector Theorem



According to the problem, $AC = a + c$. Moreover, using Angle Bisector Theorem, it is evident that $AL = (a + c) \cdot \frac{c}{a+c}$ and $LC = (a + c) \cdot \frac{a}{a+c}$.



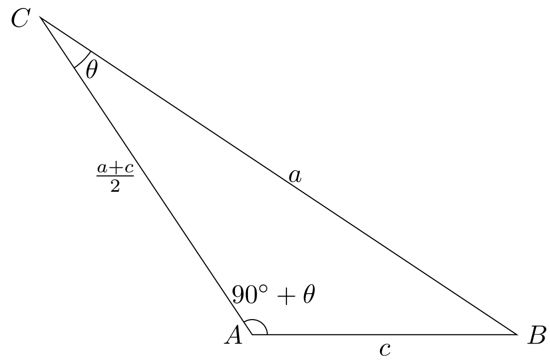
Using the property of triangle, $(180 - 2\circ) + (180 - 2\bullet) = 180 - 40$ is true. Thereby, $\circ + \bullet = 110$. In another words, $\boxed{\angle KLM = 70°}$. □

Problem

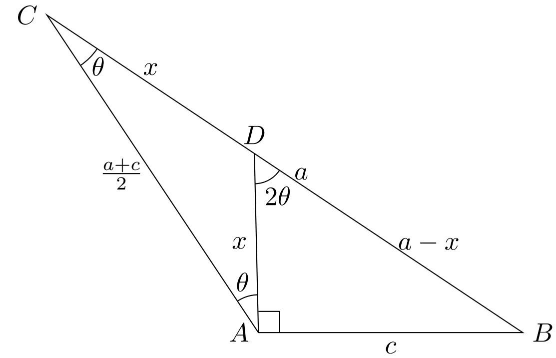
In $\triangle ABC$, $AB + BC = 2AC$ and $\angle A = \angle C + 90°$. Find $\cos B$.

Solution I

Key Word Law of Sines, Law of Cosines



A vertical line from A could be drawn.



$\cos(90° - 2\theta)$, or $\sin 2\theta$, is the value that is required to be computed. According to the diagram, it is evident that $\sin 2\theta = \frac{c}{a-x}$.

To compute the value of x, trigonometric identities may be utilized. $\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$ is true.

The Law of Sines may provide additional information.

$$\frac{\frac{a+c}{2}}{\sin(90-2\theta)} = \frac{c}{\sin\theta} = \frac{a}{\sin(90+\theta)}$$

In another words,

$$\begin{aligned} \frac{c}{\sin\theta} &= \frac{a}{\cos\theta} \\ \frac{c}{a} &= \frac{\sin\theta}{\cos\theta} \\ \tan\theta &= \frac{c}{a} \end{aligned}$$

Because $\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$, $\frac{c}{x} = \frac{2\cdot\frac{c}{a}}{1-(\frac{c}{a})^2}$.

$$\begin{aligned} a - x &= a - c\cdot\frac{1-(\frac{c}{a})^2}{2\cdot\frac{c}{a}} \\ &= a - (1 - \frac{c^2}{a^2})\cdot\frac{c}{2\cdot\frac{c}{a}} \\ &= a - (1 - \frac{c^2}{a^2})\cdot\frac{a}{2} \\ &= \frac{a}{2}\left(2 - \left(1 - \frac{c^2}{a^2}\right)\right) \\ &= \frac{a}{2}\left(1 + \frac{c^2}{a^2}\right) \\ &= \frac{a}{2}\left(\frac{a^2+c^2}{a^2}\right) \\ &= \frac{a^2+c^2}{2a} \end{aligned}$$

Therefore, the value of $\sin 2\theta = \frac{c}{\frac{a^2+c^2}{2a}} = \frac{2ac}{a^2+c^2}$.

The Law of Cosines may provide supplementary information.

$$\cos\theta = \frac{a^2 + (\frac{a+c}{2})^2 - c^2}{2 \cdot a \cdot \frac{a+c}{2}}$$
$$\cos(90+\theta) = \frac{c^2 + (\frac{a+c}{2})^2 - a^2}{2 \cdot c \cdot \frac{a+c}{2}} = -\sin\theta$$

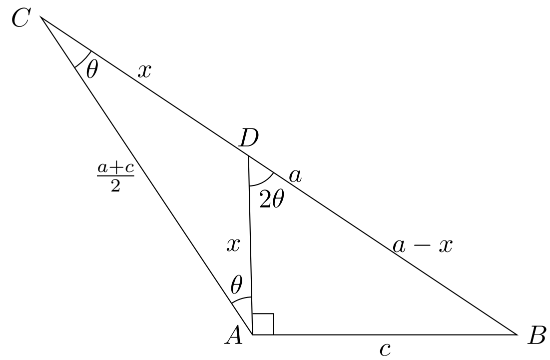
Using the fact $\tan\theta = \frac{c}{a}$ again, the following equations may be written.

$$\begin{aligned}
\frac{c}{a} &= \frac{\frac{a^2-(\frac{a+c}{2})^2-c^2}{2\cdot c\cdot \frac{a+c}{2}}}{\frac{a^2+(\frac{a+c}{2})^2-c^2}{2\cdot a\cdot \frac{a+c}{2}}} \\
&= \frac{\frac{a^2-(\frac{a+c}{2})^2-c^2}{c}}{\frac{a^2+(\frac{a+c}{2})^2-c^2}{a}} \\
\frac{c^2}{a^2} &= \frac{a^2-c^2-(\frac{a+c}{2})^2}{a^2-c^2+(\frac{a+c}{2})^2} \\
c^2\left(a^2-c^2+(\frac{a+c}{2})^2\right) &= a^2\left(a^2-c^2-(\frac{a+c}{2})^2\right) \\
a^2c^2 - c^4 + \frac{c^2}{4}(a+c)^2 &= a^4 - a^2c^2 - \frac{a^2}{4}(a+c)^2 \\
c^2(a^2-c^2) + \frac{c^2}{4}(a+c)^2 &= a^2(a^2-c^2) - \frac{a^2}{4}(a+c)^2 \\
c^2(a-c) + \frac{c^2}{4}(a+c) &= a^2(a-c) - \frac{a^2}{4}(a+c) \\
\frac{c^2}{4}(a+c) + \frac{a^2}{4}(a+c) &= a^2(a-c) - c^2(a-c) \\
(a+c)(\frac{c^2}{4}+\frac{a^2}{4}) &= (a-c)(a^2-c^2) \\
(\frac{c^2}{4}+\frac{a^2}{4}) &= (a-c)(a-c) \\
c^2+a^2 &= 4a^2 - 8ac + 4c^2 \\
3a^2+3c^2 &= 8ac \\
\frac{2ac}{a^2+c^2} &= \frac{3}{4}
\end{aligned}$$

$\therefore \sin 2\theta = \boxed{\frac{3}{4}}$.

Solution II

Key Word Trigonometric Identities, Law of Sines



From the diagram above, the Law of Sines could be used.

$$\frac{c}{\sin\theta} = \frac{a}{\sin(90+\theta)} = \frac{\frac{a+c}{2}}{\sin B} = \frac{a+c}{\sin\theta + \sin(90+\theta)}$$

In another words,

$$\begin{aligned} 2\sin B = \sin\theta + \sin(90+\theta) &= 2\sin\frac{\theta+(\theta+90)}{2}\cos\frac{\theta-(\theta+90)}{2} \\ &= 4\sin\frac{B}{2}\cos\frac{B}{2} \end{aligned}$$

Therefore,

$$\begin{aligned} 4\sin\frac{B}{2}\cos\frac{B}{2} &= 2\sin\frac{\theta+(\theta+90)}{2}\cos\frac{\theta-(\theta+90)}{2} \\ &= 2\sin\frac{2\theta+90}{2}\cos 45^\circ \\ &= \sqrt{2}\cos\frac{B}{2} \\ 4\sin\frac{B}{2} &= \sqrt{2} \\ \sin\frac{B}{2} &= \frac{\sqrt{2}}{4} \\ \sqrt{\frac{1-\cos B}{2}} &= \frac{\sqrt{2}}{4} \\ \frac{1-\cos B}{2} &= \frac{1}{8} \\ \therefore \boxed{\cos B = \frac{3}{4}} \end{aligned}$$