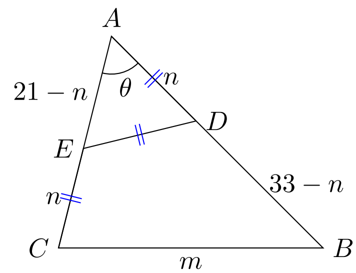
Problem

In $\triangle ABC$, D and E are points on AB and AC respectively. If $AB = 33$, $AC = 21$, $BC = m$ and $AD = DE = ED = n$ where m, n are integers, find the value of m.

Solution

Key Word Law of Cosines



Using angle θ and the Law of Cosines, two equations may be derived.

$$\begin{aligned}\cos\theta &= \frac{n^2 + (21-n)^2 - n^2}{2 \cdot n \cdot (21-n)} \\ &= \frac{21^2 + 33^2 - m^2}{2 \cdot 21 \cdot 33}\end{aligned}$$

Therefore,

$$\frac{21-n}{n} = \frac{21^2 + 33^2 - m^2}{2 \cdot 21 \cdot 33}.$$

By rearranging the equation, the relationship between n and m could be found, since n and m are positive integers.

$$\begin{aligned}21 \cdot 33(21-n) &= n(21^2 + 33^2 - m^2) \\ 21^2 \cdot 33 - 21 \cdot 33n &= n(21^2 + 33^2 - m^2) \\ n(21^2 + 33^2 + 21 \cdot 33 - m^2) &= 21^2 \cdot 33\end{aligned}$$

Using the property of triangle, the range for n and m could be narrowed.

Case I for m	**Case II for m**
$m < 33 \rightsquigarrow m + 21 > 33$	$m > 33 \rightsquigarrow 21 + 33 > m$
$\therefore 12 < m < 33$	$\therefore 33 < m < 54$

In another words, the range for m is $12 < m < 54$ because m could be 33. Inferring to the diagram, it is likely that the range for m is $12 < m < 33$.

Case I for n	**Case II for n**
$n < 21 - n \rightsquigarrow 2n > 21 - n$	$n > 21 - n \rightsquigarrow 21 - n + n > n$
$\therefore 7 < n < \frac{21}{2}$	$\therefore \frac{21}{2} < n < 21$

In another words, the range for n is $7 < n < 21$ because n is an integer.

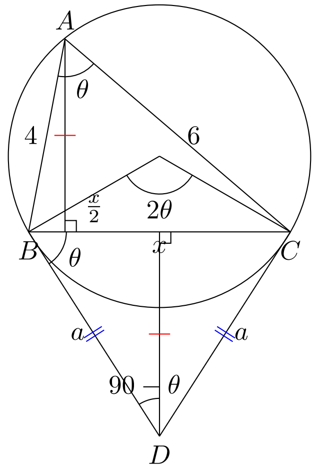
Before substituting every value in $n(21^2 + 33^2 + 21 \cdot 33 - m^2) = 21^2 \cdot 33$, it is evident that n must include $3, 7$ or 11. The only possible values for n is 9 and 11. However, 9 leads to non-integer value of m. Thus, $n = 11$ and $\boxed{m = 30}$. □

Problem

In $\triangle ABC$, $AB = 4$ and $AC = 6$. The two tangents to the circumcircle of $\triangle ABC$ at B and C intersect at D. If A and D are equidistant from the line BC, find the length of BC.

Solution I: The Typical Method

Key Word Law of Cosines, Triangle Area Formula



The diagram suggests that

$$\cos\theta = \frac{x}{2a} = \frac{4^2 + 6^2 - x^2}{2 \cdot 4 \cdot 6}.$$

Moreover, using the fact that $\triangle ABC = \triangle BCD$, a new equation could be written.

$$\frac{1}{2} \cdot 4 \cdot 6 \cdot \sin\theta = \frac{1}{2}a^2 \sin(180 - 2\theta).$$

Using trigonometric identities, the following expressions are true.

$$\begin{aligned}
\frac{1}{2} \cdot 4 \cdot 6 \cdot \sin\theta &= \frac{1}{2}a^2 \sin(180 - 2\theta) \\
\frac{1}{2} \cdot 4 \cdot 6 \cdot \sin\theta &= \frac{1}{2}a^2 \sin 2\theta \\
4 \cdot 6 \cdot \sin\theta &= a^2 \cdot 2\sin\theta\cos\theta \\
12 &= a^2 \cdot \cos\theta \ (\because 0^\circ < \theta < 90^\circ)
\end{aligned}$$

In another words,

$$\cos\theta = \frac{x}{2a} = \frac{4^2 + 6^2 - x^2}{2 \cdot 4 \cdot 6} = \frac{12}{a^2}.$$

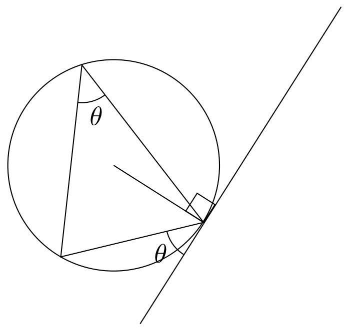
Because $\frac{x}{2a} = \frac{12}{a^2}$, $ax = 24$. From $\frac{4^2+6^2-x^2}{2\cdot4\cdot6} = \frac{52-x^2}{48} = \frac{12}{a^2}$, $52a^2 - a^2x^2 = 12 \cdot 48$. $a = \frac{24}{x}$ could be substituted.

$$\begin{aligned}
52 \cdot \frac{24^2}{x^2} - \frac{24^2}{x^2} \cdot x^2 &= 12 \cdot 48 \\
52 \cdot 24^2 - 24^2 \cdot x^2 &= 12 \cdot 48x^2 \\
(12 \cdot 48 + 24^2)x^2 &= 52 \cdot 24^2 \\
2x^2 &= 52 \\
x &= \boxed{\sqrt{26}}
\end{aligned}$$

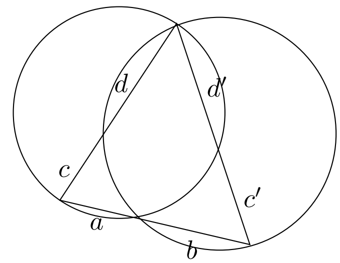
Solution II: The Smart Way

Key Properties

Property I

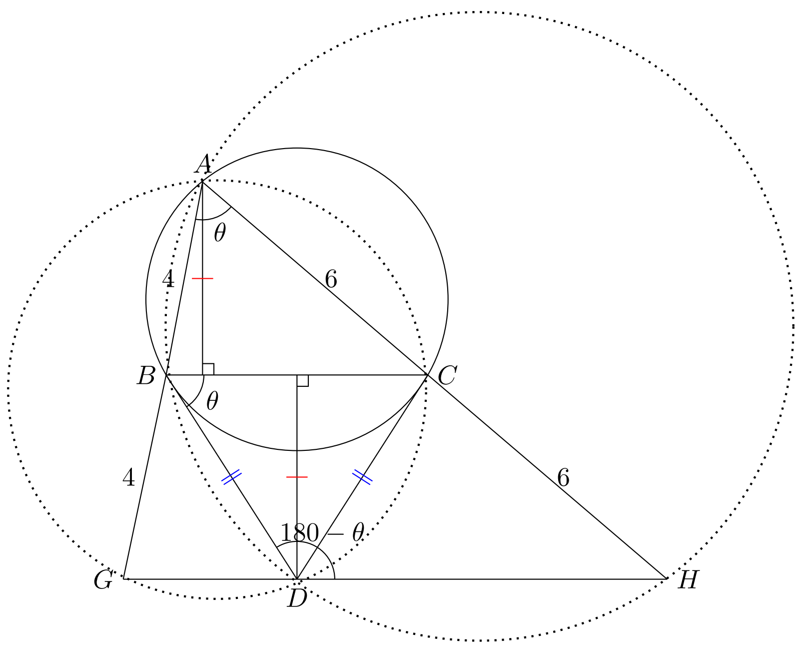


Property II



$$(a+b)^2 = c(c+d) + c'(c'+d')$$

The properties above may be utilized when the segments AB and AC are extended because $\square ABDH$ and $\square AGDC$ are cyclic.



Using property of similar triangle and Property II,

$$(2BC)^2 = 4(4+4) + 6(6+6) = 104$$

is true.

In another words, $\boxed{BC = \sqrt{26}}$.

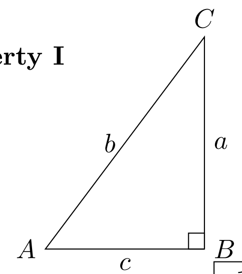
Problem

Let $\triangle ABC$, be an acute-angled triangle with incircle ω which is tangent to sides BC and CA at D and E, respectively. X is a point on the altitude from A to BC, and the circle ω' with diameter AX is tangent to ω. Denote by U and V, respectively, the points where CA and AB intersect ω' again. If $UV = 12$, $AX = 15$ and $AE = 24$, find the value of $BD \times DC$.

Solution

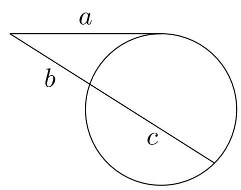
Key Properties

Property I



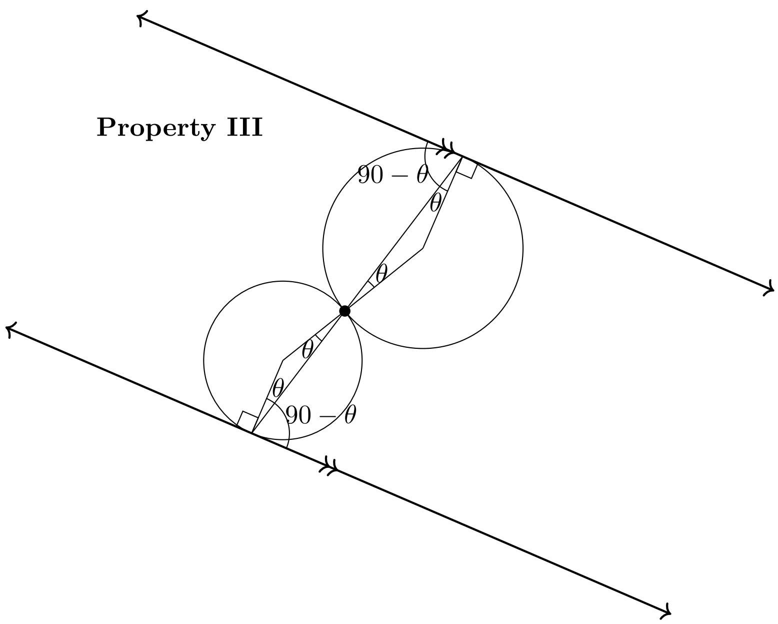
$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} = \boxed{\frac{1}{2r}}$$

Property II

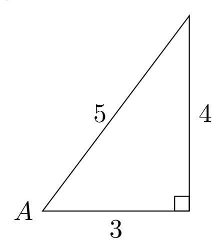


$$a^2 = b(b+c)$$

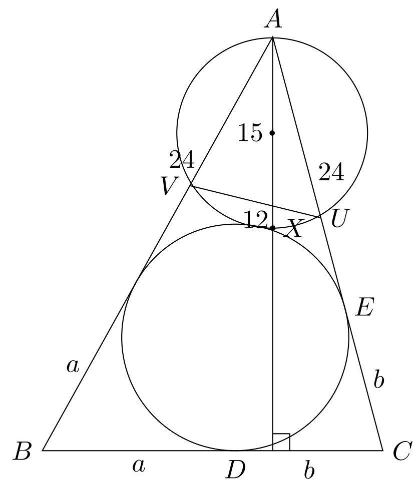
Property III



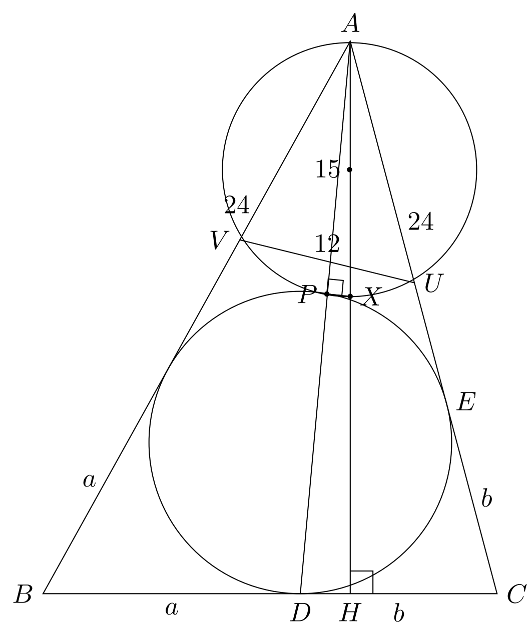
Trigger



$$\sin A = \frac{4}{5}$$

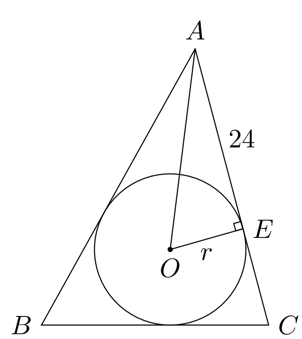


The properties may be utilized to further modify the diagram.

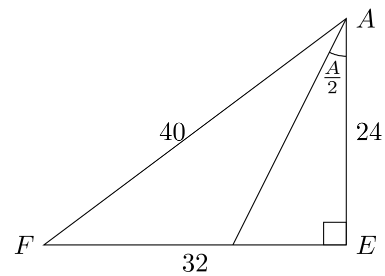


Using the property of similar triangle, it is evident that $\frac{AP}{AX} = \frac{AH}{AD}$. Using Power of a Point theorem, $AH = \frac{24^2}{15}$. Because AH is known, an impulse to use area formulas is created. Heron's Formula, area formula using the radius of inscribed circle, and the fundamental formula are the possible options.

From Property I, $\frac{\sin A}{12} = \frac{1}{15}$. In another words, $\sin A = \frac{4}{5}$.



From the extracted diagram, a new diagram may be form.



The Angle Bisector Theorem manifests that $r = 32 \cdot \frac{24}{40+24} = 12$.

The area formula involving the radius of inscribed circle and the fundamental formula provides the following equation.

$$\begin{aligned}
\frac{1}{2} \cdot 12 \cdot (2a + 2b + 48) &= \frac{1}{2} \cdot \frac{24^2}{15} \cdot (a+b) \\
(2a + 2b + 48) &= \frac{48}{15} \cdot (a+b) \\
(2a + 2b + 48) &= \frac{16}{5} \cdot (a+b) \\
10(a+b) + 240 &= 16(a+b) \\
a + b &= 40
\end{aligned}$$

The area could be represented in different form using the Heron's Formula.

$$\begin{aligned}
\text{Let } S =& \frac{24 + a + 24 + b + a + b}{2} = a + b + 24 \\
\text{Area} &= \sqrt{S(S - 24 - a)(S - 24 - b)(S - a - b)} \\
&= \sqrt{24ab(a + b + 24)}
\end{aligned}$$

$$\frac{1}{2} \cdot 12 \cdot (2a + 2b + 48) = \sqrt{24ab(a + b + 24)}$$

is true.

$$\begin{aligned}
\frac{1}{2} \cdot 12 \cdot (2a + 2b + 48) &= \sqrt{24ab(a + b + 24)} \\
6(128) &= \sqrt{24ab(64)} \\
6(128) &= 16\sqrt{6ab} \\
24 &= \sqrt{6ab} \\
ab &= 96
\end{aligned}$$

Therefore, $\boxed{BD \cdot DC = 96}$. $\square$