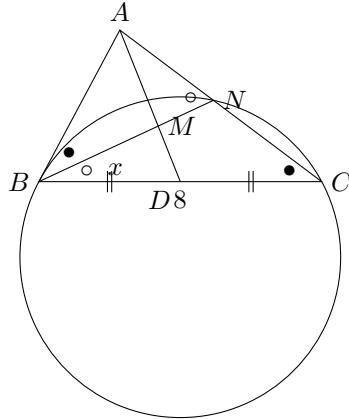


## Problem

In  $\triangle ABC$ ,  $D$  is the mid-point of  $BC$  and  $m$  is a point of  $AD$ . The extension of  $BM$  meets  $AC$  at  $N$ , and  $AB$  is tangent to the circumcircle of  $\triangle BCN$ . If  $BC = 8$  and  $BN = 6$ , find the length of  $BM$ .

### Solution

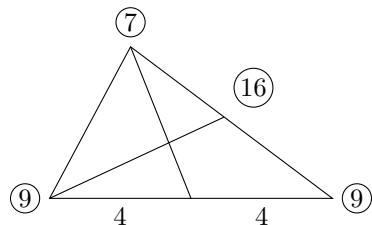


**Key Word** Mass Point, Menelaus's Theorem, Similar Triangle

<b>Property I</b> 	<b>Menelaus's Theorem</b> $\frac{c}{d} \cdot \frac{e}{f} \cdot \frac{a}{a+b} = 1$ $\frac{b}{a} \cdot \frac{g}{h} \cdot \frac{d}{d+c} = 1$
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### Solution I: Mass Point

Using Property I or Power of a Point Theorem, it is evident that  $AN : NC : AB = 3 : \frac{7}{3} : 4$ . Therefore, masses for each points could be assigned.



$$6 \cdot \frac{16}{16+9} = \boxed{\frac{96}{25}}$$

**Solution II: Menelaus's Theorem**

Menelaus's Theorem may be utilized to solve the problem.

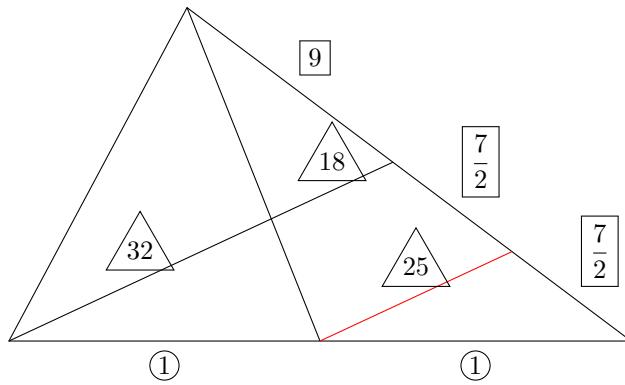
$$\frac{4}{4} \cdot \frac{x}{6-x} \cdot \frac{9}{16} = 1$$

Therefore,  $x$  could be calculated.

$$\begin{aligned} \frac{4}{4} \cdot \frac{x}{6-x} \cdot \frac{9}{16} &= 1 \\ 9x &= 16(6-x) \\ x &= \boxed{\frac{96}{25}} \end{aligned}$$

**Solution III: Similar Triangle**

A parallel line could be drawn to  $AC$  from  $D$ .



$$BM = 6 \cdot \frac{32}{32+18} = \boxed{\frac{96}{25}}$$

□

**Problem**

If  $a^3 - 3ab^2 = 11$  and  $b^3 - 3a^2b = 13$ , find the value of  $a^2 + b^2$ .

**Solution**

**Key Word** DO NOT BE AFRAID TO INCREASE DEGREE

I am looking for  $a^2 + b^2$ . However, in the given equations  $a^3 - 3ab^2 = 11$  and  $b^3 - 3a^2b = 13$ , There are powers with odd numbers. What should I do? I think I want to square them to make the exponents even.

$$\begin{aligned} (a^3 - 3ab^2)^2 &= a^6 - 6a^4b^2 + 9a^2b^4 = 121 \\ (b^3 - 3a^2b)^2 &= b^6 - 6a^2b^4 + 9a^4b^2 = 169 \end{aligned}$$

An impulse to add the equations are created due to like terms.

$$a^6 - 6a^4b^2 + 9a^2b^4 + b^6 - 6a^2b^4 + 9a^4b^2 = a^6 + 3a^4b^2 + 3a^2b^4 + b^6 = (a^2 + b^2)^3 = 290$$

Therefore,  $a^2 + b^2 = \boxed{\sqrt[3]{290}}$

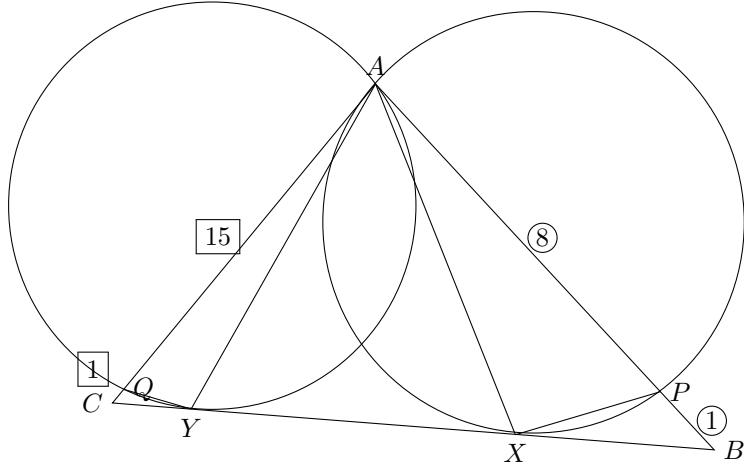
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## Problem

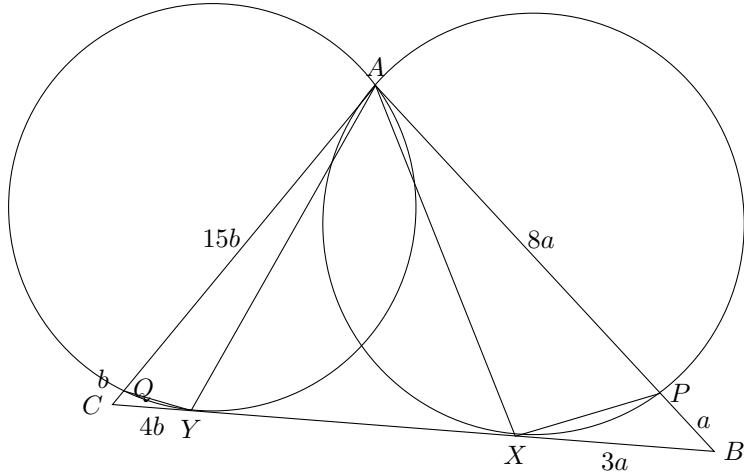
In  $\triangle ABC$ ,  $P$  and  $Q$  are points of  $AB$  and  $AC$  respectively such that  $AP : PB = 8 : 1$  and  $AQ : QC = 15 : 1$ .  $X$  and  $Y$  are points on  $BC$  such that the circumcircle of  $\triangle APX$  is tangent to both  $BC$  and  $CA$ , while the circumcircle of  $\triangle AQY$  is tangent to both  $AB$  and  $BC$ . Find  $\cos \angle BAC$ .

### Solution

**Key Word** Law of Cosines, Power of a Point Theorem



First and foremost, it is evident that  $\triangle BPX \sim \triangle BXA$  and  $\triangle CQY \sim \triangle CYA$ . Thereby, the diagram could utilize different variables.



Because the Law of Cosines utilize ratios, if the ratio between  $a$  and  $b$  are found,  $\cos \angle BAC$  could also be computed.

$AC = CX$  and  $AB = BY$  is true. Therefore,  $YX = 12b = 6a$ .

$$\cos \angle BAC = \frac{8^2 + 9^2 - 11^2}{2 \cdot 8 \cdot 9} = \boxed{\frac{1}{6}}$$

□