2022 AMC 12B Problem 17

How many 4×4 arrays whose entries are 0s and 1s are there such that the row sums (the sum of the entries in each row) are $1, 2, 3,$ and 4, in some order, and the column sums (the sum of the entries in each column) are also $1, 2, 3,$ and 4, in some order? For example, the array

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

satisfies the condition.

(A) 144 **(B)** 240 **(C)** 336 **(D)** 576 **(E)** 624

Solution

Key Word Counting Strategy

First and Foremost, each column and row could be numbered to find the relationship between each arrangement of columns and rows and the number of possible outcomes.

$$\begin{array}{cccc|c} ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? \\ \hline ? & ? & ? & ? & \end{array}$$

From the arrangement above, it is evident that when the 4 and 1 is chosen for some column and row, the possible cases for most 1 and 0 are chosen. The preceding diagram is the example of a case.

$$\begin{array}{cccc|c} 0 & 0 & 1 & 0 & 1 \\ 0 & ? & 1 & ? & ? \\ 0 & ? & 1 & ? & ? \\ 1 & 1 & 1 & 1 & 4 \\ \hline 1 & ? & 4 & ? & \end{array}$$

WLOG, only one possible case of arrangement exists for each arrangement of column and row.

Therefore, the number of possible arrangements is $4! \cdot 4! = \boxed{\textbf{D)}\ 576}$ $\square$

2022 AMC 12A Problem 22

Let c be a real number, and let z_1 and z_2 be the two complex numbers satisfying the equation $z^2 - cz + 10 = 0$. Points z_1, z_2, $\frac{1}{z_1}$, and $\frac{1}{z_2}$ are the vertices of (convex) quadrilateral $\mathcal{Q}$ in the complex plane. When the area of $\mathcal{Q}$ obtains its maximum possible value, c is closest to which of the following?

(A) 4.5 **(B)** 5 **(C)** 5.5 **(D)** 6 **(E)** 6.5

Solution

Key Word AM-GM Inequality, A Trick on Finding the Maximum Value

First and foremost, because z_n is a complex number, and the value of z could be found in terms of c, the given quadratic equation could be solved. WLOG, $z_1 = \frac{c+\sqrt{c^2-40}}{2}$ and $z_1 = \frac{c-\sqrt{c^2-40}}{2}$. If $c^2 - 40 \geq 0$, no quadrilateral will form. Thereby, it could be inferred that $c^2 - 40 < 0$.

Each complex number could be represented as a point.

$$\begin{gathered} z_1\left(\frac{c}{2}, \frac{\sqrt{40-c^2}}{2}\right) \\ z_2\left(\frac{c}{2}, \frac{-\sqrt{40-c^2}}{2}\right) \\ \frac{1}{z_1}\left(\frac{c}{20}, \frac{c-\sqrt{40-c^2}}{20}\right) \\ \frac{1}{z_2}\left(\frac{c}{20}, \frac{c+\sqrt{40-c^2}}{20}\right) \end{gathered}$$

Notice that vertices form a trapezoid. Therefore, the area of the trapezoid Q could be represented as

$$\frac{\left(\frac{c}{2}-\frac{c}{20}\right)\left\{\left(\frac{\sqrt{40-c^2}}{2}+\frac{\sqrt{40-c^2}}{2}\right)+\left(\frac{c+\sqrt{40-c^2}}{20}-\frac{c-\sqrt{40-c^2}}{20}\right)\right\}}{2}.$$

A RULE OF THUMB: Since we are finding the maximum value, any desired constant could be multiplied. Multiplying 80 and other constants to the equation above may provide a simplified form.

$$\begin{aligned} &(9c)(20\sqrt{40-c^2}+2\sqrt{40-c^2}) \\ \Rightarrow &9c(22\sqrt{40-c^2}) \\ \Rightarrow &c\sqrt{40-c^2} \end{aligned}$$

PLEASE DON'T USE OPTIMIZATION HERE!! What can we do? We can use AM-GM Inequality! According to the AM-GM Inequality,

$$40 - c^2 + c^2 \geq 2\sqrt{(40-c^2)\cdot c^2}.$$

$\sqrt{(40-c^2)\cdot c^2}$ reaches maximum when $40 - c^2 = c^2$. In another words, $c = \pm 2\sqrt{5}$. Using 2.2 for the approximation of $\sqrt{5}$, $\max(c) \approx 4.4$. Thus, the closest value in the answer choice is $\boxed{\textbf{(A)}\ 4.5}$. $\square$

Uploaded a new solution in AOPS!! Also, user name changed from "thinkingtree" to "MaPhyCom"!