2020 HKIMO Prelim Problem 20

Consider the Fibonacci sequence $1, 1, 2, 3, 5, 8, 13, \ldots$. What are the last three digits (from left to right) of the 2020^{th} term?

Solution

Key Word Implicit Form for Fibonacci Sequence, Binomial Expansion, Induction, Euler's Theorem, Property of Modular Arithmetic

The implicit form for Fibonacci Sequence is known to be $F_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$. In another words, the last three digits of

$$F_{2020} = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{2020} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{2020}$$

must be computed.

Binomial expansion could be utilized to effectively expand F_{2020}.

$$\begin{aligned}
\frac{1}{\sqrt{5}} \cdot \frac{1}{2^{2020}}\left(1+\sqrt{5}\right)^{2020} &= \frac{1}{\sqrt{5}} \cdot \frac{1}{2^{2020}}\left({}_{2020}C_0 \cdot 1^{2020} \cdot (\sqrt{5})^0 + {}_{2020}C_1 \cdot 1^{2019} \cdot (\sqrt{5})^1 + \cdots + {}_{2020}C_{2020} \cdot 1^0 \cdot (\sqrt{5})^{2020}\right) \\
\frac{1}{\sqrt{5}} \cdot \frac{1}{2^{2020}}\left(1-\sqrt{5}\right)^{2020} &= \frac{1}{\sqrt{5}} \cdot \frac{1}{2^{2020}}\left({}_{2020}C_0 \cdot 1^{2020} \cdot (\sqrt{5})^0 - {}_{2020}C_1 \cdot 1^{2019} \cdot (\sqrt{5})^1 + \cdots + {}_{2020}C_{2020} \cdot 1^0 \cdot (\sqrt{5})^{2020}\right) \\
F_{2020} &= \frac{1}{\sqrt{5}} \cdot \frac{1}{2^{2020}} \cdot 2\left({}_{2020}C_1 \cdot 1^{2019} \cdot (\sqrt{5})^1 + {}_{2020}C_3 \cdot 1^{2017} \cdot (\sqrt{5})^3 + \cdots + {}_{2020}C_{2019} \cdot 1^1 \cdot (\sqrt{5})^{2019}\right) \\
&= \frac{1}{2^{2019}}\left({}_{2020}C_1 \cdot (\sqrt{5})^0 + {}_{2020}C_3 \cdot (\sqrt{5})^2 + \cdots + {}_{2020}C_{2019} \cdot (\sqrt{5})^{2018}\right)
\end{aligned}$$

a represents the last three digits of the 2020^{th} term in the following equation.

$$F_{2020} \equiv a \pmod{1000}$$

Because $1000 = 2^3 \cdot 5^3$, the condition above may be split.

$$\begin{aligned}
F_{2020} &\equiv b \pmod{8} \\
F_{2020} &\equiv c \pmod{125}
\end{aligned}$$

Part I: $F_{2020} \equiv b \pmod{8}$

$$\frac{{}_{2020}C_1 \cdot (\sqrt{5})^0 + {}_{2020}C_3 \cdot (\sqrt{5})^2 + \cdots + {}_{2020}C_{2019} \cdot (\sqrt{5})^{2018}}{2^{2019}} \equiv b \pmod{8}$$

Lemma. *A repetition of remainders occurs when the Fibonacci sequence is divided by 8.*

Proof. First and foremost, replace 8 with n to manifest the situation.

$$1, 1, 2, 3, 5, n, n+5, 2n+5, 4n+2, 6n+7, 11n+1, 18n, \mathbf{19n+1, 37n+1, 56n+2, 74n+3}, \ldots$$

By induction, because the method to form the first 12 terms repeats, the remainders also retain repetition. □

b is equal to 3 because $2020 \equiv 4 \pmod{12}$.

Part II: $F_{2020} \equiv c \pmod{125}$

$$\frac{{}_{2020}C_1 \cdot (\sqrt{5})^0 + {}_{2020}C_3 \cdot (\sqrt{5})^2 + \cdots + {}_{2020}C_{2019} \cdot (\sqrt{5})^{2018}}{2^{2019}} \equiv c \pmod{125}$$

An impulse to remove the terms with $(\sqrt{5})^6$ or with a higher degree is created. However, 2^{2019} hinders the simplification. Therefore, the terms may be rewritten.

$${}_{2020}C_1 \cdot (\sqrt{5})^0 + {}_{2020}C_3 \cdot (\sqrt{5})^2 + \cdots + {}_{2020}C_{2019} \cdot (\sqrt{5})^{2018} \equiv 2^{2019}c \pmod{125}$$

Euler's theorem may be utilized for further simplification because $GCF(2, 125) = 1$.

$$\begin{aligned}
2^{\varphi(125)} &\equiv 1 \pmod{125} \\
2^{125\left(1-\frac{1}{5}\right)} &\equiv 1 \pmod{125} \\
2^{100} &\equiv 1 \pmod{125} \\
2^{2000} &\equiv 1 \pmod{125} \\
2^{2019} &\equiv 2^{19} \pmod{125} \\
2^{2019} &\equiv 24 \cdot 12 \pmod{125} \\
2^{2019} &\equiv 288 \pmod{125} \\
2^{2019} &\equiv 38 \pmod{125}
\end{aligned}$$

Therefore, the following relationships are true.

$$\begin{aligned}
{}_{2020}C_1 \cdot (\sqrt{5})^0 + {}_{2020}C_3 \cdot (\sqrt{5})^2 + \cdots + {}_{2020}C_{2019} \cdot (\sqrt{5})^{2018} &\equiv 38c \pmod{125} \\
{}_{2020}C_1 + {}_{2020}C_3 \cdot 5 + {}_{2020}C_5 \cdot 5^2 &\equiv 38c \pmod{125} \\
2020 + \frac{2020 \cdot 2019 \cdot 2018}{3 \cdot 2 \cdot 1} \cdot 5 + \frac{2020 \cdot 2019 \cdot \cdots \cdot 2015}{5!} \cdot 5^2 &\equiv 38c \pmod{125} \\
20 + 75 + 100 &\equiv 38c \pmod{125} \\
195 &\equiv 38c \pmod{125}
\end{aligned}$$

Therefore, $38c \equiv 195 \pmod{125}$. Moreover, because $GCF(38, 125) = 1$, $c \equiv \frac{195+125k}{38} \pmod{125}$. Therefore, $F_{2020} \equiv c \equiv 15 \pmod{125}$

Part III: Computing F_{2020}

$F_{2020} \equiv 3 \pmod{8}$ and $F_{2020} \equiv 15 \pmod{125}$ are true.

Let $F_{2020} = 8k + 3$.

$$\begin{aligned}
8k + 3 &\equiv 15 \pmod{125} \\
8k &\equiv 12 \pmod{125} \\
8k &\equiv 512 \pmod{125} \\
k &\equiv 64 \pmod{125} \ (\because GCF(8, 125) = 1)
\end{aligned}$$

Let $k = 125k' + 64$.
$F_{2020} = 1000k' + 67$, or the last three digits of the 2020^{th} term is $\boxed{515}$. $\square$