

2020 HKIMO Prelim Problem 20

Consider the Fibonacci sequence 1, 1, 2, 3, 5, 8, 13, What are the last three digits (from left to right) of the 2020th term?

Solution

Key Word Implicit Form for Fibonacci Sequence, Binomial Expansion, Induction, Euler's Theorem, Property of Modular Arithmetic

The implicit form for Fibonacci Sequence is known to be $F_n = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2} \right)^n$. In another words, the last three digits of

$$F_{2020} = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^{2020} - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2} \right)^{2020}$$

must be computed.

Binomial expansion could be utilized to effectively expand F_{2020} .

$$\begin{aligned} \frac{1}{\sqrt{5}} \cdot \frac{1}{2^{2020}} \left(1 + \sqrt{5} \right)^{2020} &= \frac{1}{\sqrt{5}} \cdot \frac{1}{2^{2020}} \left({}_{2020}C_0 \cdot 1^{2020} \cdot (\sqrt{5})^0 + {}_{2020}C_1 \cdot 1^{2019} \cdot (\sqrt{5})^1 + \cdots + {}_{2020}C_{2020} \cdot 1^0 \cdot (\sqrt{5})^{2020} \right) \\ \frac{1}{\sqrt{5}} \cdot \frac{1}{2^{2020}} \left(1 - \sqrt{5} \right)^{2020} &= \frac{1}{\sqrt{5}} \cdot \frac{1}{2^{2020}} \left({}_{2020}C_0 \cdot 1^{2020} \cdot (\sqrt{5})^0 - {}_{2020}C_1 \cdot 1^{2019} \cdot (\sqrt{5})^1 + \cdots + {}_{2020}C_{2020} \cdot 1^0 \cdot (\sqrt{5})^{2020} \right) \\ F_{2020} &= \frac{1}{\sqrt{5}} \cdot \frac{1}{2^{2020}} \cdot 2 \left({}_{2020}C_1 \cdot 1^{2019} \cdot (\sqrt{5})^1 + {}_{2020}C_3 \cdot 1^{2017} \cdot (\sqrt{5})^3 + \cdots + {}_{2020}C_{2019} \cdot 1^1 \cdot (\sqrt{5})^{2019} \right) \\ &= \frac{1}{2^{2019}} \left({}_{2020}C_1 \cdot (\sqrt{5})^0 + {}_{2020}C_3 \cdot (\sqrt{5})^2 + \cdots + {}_{2020}C_{2019} \cdot (\sqrt{5})^{2018} \right) \end{aligned}$$

a represents the last three digits of the 2020th term in the following equation.

$$F_{2020} \equiv a \pmod{1000}$$

Because $1000 = 2^3 \cdot 5^3$, the condition above may be split.

$$\begin{aligned} F_{2020} &\equiv b \pmod{8} \\ F_{2020} &\equiv c \pmod{125} \end{aligned}$$

Part I: $F_{2020} \equiv b \pmod{8}$

$$\frac{{}_{2020}C_1 \cdot (\sqrt{5})^0 + {}_{2020}C_3 \cdot (\sqrt{5})^2 + \cdots + {}_{2020}C_{2019} \cdot (\sqrt{5})^{2018}}{2^{2019}} \equiv b \pmod{8}$$

Lemma. A repetition of remainders occurs when the Fibonacci sequence is divided by 8.

Proof. First and foremost, replace 8 with n to manifest the situation.

$$1, 1, 2, 3, 5, n, n+5, 2n+5, 4n+2, 6n+7, 11n+1, 18n, \mathbf{19n+1}, \mathbf{37n+1}, \mathbf{56n+2}, \mathbf{74n+3}, \dots$$

By induction, because the method to form the first 12 terms repeats, the remainders also retain repetition. \square

b is equal to 3 because $2020 \equiv 4 \pmod{12}$.

Part II: $F_{2020} \equiv c \pmod{125}$

$$\frac{_{2020}C_1 \cdot (\sqrt{5})^0 + {}_{2020}C_3 \cdot (\sqrt{5})^2 + \cdots + {}_{2020}C_{2019} \cdot (\sqrt{5})^{2018}}{2^{2019}} \equiv c \pmod{125}$$

An impulse to remove the terms with $(\sqrt{5})^6$ or with a higher degree is created. However, 2^{2019} hinders the simplification. Therefore, the terms may be rewritten.

$${}_{2020}C_1 \cdot (\sqrt{5})^0 + {}_{2020}C_3 \cdot (\sqrt{5})^2 + \cdots + {}_{2020}C_{2019} \cdot (\sqrt{5})^{2018} \equiv 2^{2019}c \pmod{125}$$

Euler's theorem may be utilized for further simplification because $GCF(2, 125) = 1$.

$$\begin{aligned} 2^{\varphi(125)} &\equiv 1 \pmod{125} \\ 2^{125(1-\frac{1}{5})} &\equiv 1 \pmod{125} \\ 2^{100} &\equiv 1 \pmod{125} \\ 2^{2000} &\equiv 1 \pmod{125} \\ 2^{2019} &\equiv 2^{19} \pmod{125} \\ 2^{2019} &\equiv 24 \cdot 12 \pmod{125} \\ 2^{2019} &\equiv 288 \pmod{125} \\ 2^{2019} &\equiv 38 \pmod{125} \end{aligned}$$

Therefore, the following relationships are true.

$$\begin{aligned} {}_{2020}C_1 \cdot (\sqrt{5})^0 + {}_{2020}C_3 \cdot (\sqrt{5})^2 + \cdots + {}_{2020}C_{2019} \cdot (\sqrt{5})^{2018} &\equiv 38c \pmod{125} \\ {}_{2020}C_1 + {}_{2020}C_3 \cdot 5 + {}_{2020}C_5 \cdot 5^2 &\equiv 38c \pmod{125} \\ 2020 + \frac{2020 \cdot 2019 \cdot 2018}{3 \cdot 2 \cdot 1} \cdot 5 + \frac{2020 \cdot 2019 \cdot \cdots \cdot 2015}{5!} \cdot 5^2 &\equiv 38c \pmod{125} \\ 20 + 75 + 100 &\equiv 38c \pmod{125} \\ 195 &\equiv 38c \pmod{125} \end{aligned}$$

Therefore, $38c \equiv 195 \pmod{125}$. Moreover, because $GCF(38, 125) = 1$, $c \equiv \frac{195+125k}{38} \pmod{125}$. Therefore, $F_{2020} \equiv c \equiv 15 \pmod{125}$

Part III: Computing F_{2020}

$F_{2020} \equiv 3 \pmod{8}$ and $F_{2020} \equiv 15 \pmod{125}$ are true.

Let $F_{2020} = 8k + 3$.

$$\begin{aligned} 8k + 3 &\equiv 15 \pmod{125} \\ 8k &\equiv 12 \pmod{125} \\ 8k &\equiv 512 \pmod{125} \\ k &\equiv 64 \pmod{125} \quad (\because GCF(8, 125) = 1) \end{aligned}$$

Let $k = 125k' + 64$.

$F_{2020} = 1000k' + 67$, or the last three digits of the 2020th term is $\boxed{515}$. □