COMP2711 Homework2

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Question 1:

(a)

1. $\neg p$	premises
2. $\neg r$	premises
3. $\neg(p \vee q) \rightarrow r$	premises
4. $p \vee q \vee r$	3, equivalence
5. q	1,2,4 disjunctive syllogism

(b)

1. $\neg q$	premises
2. $p \rightarrow q$	premises
3. $\neg p \rightarrow (r \wedge s)$	premises
4. $\neg p$	1,2 modus tollens
5. $r \wedge s$	3,4 modus ponens
6. r	5 simplification

Question 2:

Let's begin with proving a lemma:

Lemma:

For any integer c, if c^2 is even, then c must be even.

Prove:

Use **prove by contraposition**, that is, we prove "For any integer c, if c is odd, then c^2 is odd." Since c is odd, we suppose $c = 2k + 1$, where $k \in \mathbb{N}$. Then

$$c^2 = (2k+1)^2 = (4k^2 + 4k + 1) = 2(2k^2 + 2k) + 1$$

is odd, thereby the lemma is proved.

Then we prove the original one: "For any integers a, b, c, if $a^2 + b^2 = c^2$, then a or b is even.", use **prove by contradiction:**

Suppose a and b are both odd, $a = 2m+1, b = 2n+1$, where $m, n \in \mathbb{N}$. Then

$$a^2 + b^2 = (2m+1)^2 + (2n+1)^2 = 2(2m^2 + 2m + 2n^2 + 2n + 1)$$

is even, hence c^2 is even.

According to **Lemma** we just proved, c must be even. Let $c = 2p, p \in \mathbb{N}$, then $a^2 + b^2 = c^2$ can be written as:

$$\begin{aligned} 2(2m^2 + 2m + 2n^2 + 2n + 1) &= 4p^2 \\ 4(m^2 + m + n^2 + n) + 1 &= 4p^2 \end{aligned}$$

However, it's obvious that left hand side is not divisible by 4, that is, the equation doesn't hold, contradiction found!

Therefore, the statement is proved.

Question 3:

(a) $f(x) = x + 1$. This function is injective since for every image $y \geq 1$, there exist a **unique** preimage $x = y - 1$; this function is not surjective since $y = 0$ doesn't have a preimage.

(b) $g(x) = \left\lfloor \dfrac{x}{2} \right\rfloor$. This function is not injective since $g(0) = g(1) = 0$; this function is surjective since for any y, you can find a preimage $x = 2y$

(c) $h(x) = x + (-1)^x$. Since identity function is forbidden, another straight forward idea is that we let $h(0) = 1, h(1) = 0, h(2) = 3, h(3) = 2, h(4) = 5, h(5) = 4, \cdots$, that is, making every pair of x "swap" their y. It's obvious that after this transformation, the function is still bijective, and there're many ways to express this function, such as the above one: $h(x) = x + (-1)^x$.

(d) $u(x) = 2 \cdot \left\lfloor \dfrac{x}{2} \right\rfloor$. This function is not injective since $u(0) = u(1) = 0$; this function is not surjective either since there doesn't exist such preimage x satisfying $u(x) = 1$ (as $\left\lfloor \frac{x}{2} \right\rfloor$ can never be $\frac{1}{2}$).

Question 4:

(a) Let $A = (0, 1], B = [1, 2)$, then $A \cap B = \{1\}$, which is a finite set.

(b) Let $A = (0, 1) \cup \mathbb{N}, B = (3, 4) \cup \mathbb{N}$, then $A \cap B = \mathbb{N}$, which is a countably infinite set.

(c) Let $A = (0, 3), B = (1, 4)$, then $A \cap B = (1, 3)$, which is an uncountably infinite set.