COMP2711 Homework4

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Question 1:

(a) Obviously $\forall x \geq 1, x^3 > x^3 \log_2 x$, so $n = 3$ is not enough. When $x \geq 1$, we can easily know $2x^2 + x^3 \log_2 x \geq 2x^4 + x^4 = 3x^4$. Therefore $n = 4$, with witnesses $C = 3, k = 1$.

(b) We know $\forall x \geq 1 (\log_2 x)^4 < x^5$, so $3x^5$ is dominant. When $x \geq 1$, $3x^5 + (\log_2 x)^4 \leq 3x^5 + x^5 = 4x^5$. Hence $n = 5$, with witnesses $C = 4, k = 1$.

(c) We know for all $x \geq 1$,

$$f(x) = \frac{x^4 + x^2 + 1}{x^4 + 1} = 1 + \frac{x^2}{x^4 + 1} < 1 + 1 = 2$$

Hence $n = 0$ as $f(x)$ is $O(1)$, with witnesses $C = 2, k = 1$.

(d) For all $x \geq 1$,

$$f(x) = \frac{x^3 + 5\log_2 x}{x^4 + 1} < \frac{x^3 + 5x^3}{x^4 + 1} < \frac{6x^3}{x^4 + 1} < \frac{6(x^4 + 1)}{x^4 + 1} = 6$$

Hence $n = 0$ as $f(x)$ is $O(1)$, with witnesses $C = 6, k = 1$.

Question 2:

Firstly, we denote the first player as Player A, second as Player B; denote (i, j) as the cell at the i-th row and j-th column.

Let $P(n)$: If now the board consists of only n cookies in the top row and n cookies in the left-most column, i.e., cookies fill all the cells from $(1, 1), (1, 2), \cdots, (1, n)$, and $(n, 1), (n, 2), \cdots, (n, n)$, and it's Player B's turn to pick cookies, then Player A has a winning strategy.

We want to prove $P(n)$ by strong induction.

Basis Step: It is obvious that $P(1)$ is true, since Player B is made to eat the poisoned cookie.

Inductive Step: For $k \geq 1$, we assume that $\forall i \leq k, P(i)$ is true, and we need to prove that $P(k + 1)$ is true. If Player B picks the cookie $(1, 1)$, then Player A wins. Otherwise, if Player B picks $(1, j)$ or $(j, 1)$ for some j with $2 \leq j \leq k + 1$, then Player A just picks $(j, 1)$ or $(1, j)$, correspondingly. After a round like this, the situation becomes $P(j - 1)$. From our assumption, we know $P(j - 1)$ is true, so that $P(k + 1)$ is true.

Therefore, by strong induction, we know $P(n)$ is true, so that Player A has a winning strategy.

Question 3:

Recursive Definition:

Basis Step: $(1, 1) \in S$.

Recursive Step: if $(a, b) \in S$, then $(a + 2, b) \in S, (a, b + 2) \in S, (a + 1, b + 1) \in S$.

Proof by Structural Induction:

Basis Step: Apparently, $(1, 1) \in S$ since $1 + 1 = 2$ is even.

Recursive Step: Assume $(a, b) \in S$, then we know $a+b$ is even. Thus, $(a+2, b), (a, b+2), (a+1, b + 1)$ all must in S, since $a + 2 + b = a + b + 2 = a + 1 + b + 1 = (a + b) + 2$ is even.

Therefore, according to the principle of structural induction, $a + b$ is even for all $(a, b) \in S$.

Question 4:

(a) Notice that $a^{2^k} = (a^{2^{k-1}})^2$.

procedure power (a : real number, n : positive integer)

if $n = 1$ **then return** a^2

else return $[\text{power}(a, n - 1)]^2$

(b) According to the algorithm in (a), in order to find a^{2^k}, we find $a^{2^{k-1}}$ instead since $a^{2^k} = (a^{2^{k-1}})^2$, and then we try to find $a^{2^{k-2}}, a^{2^{k-3}}, \cdots$ until we reach a^{2^0}, that is, $n = 1$. Thus, if we denote the the number of multiplications used as $T(n)$, then we will have:

$$T(n) = T(n - 1) + 1 \ (\forall n \geq 2)$$

$$T(1) = 1$$

(c) From (b), we have:

$$\begin{aligned} T(n) &= T(n-1) + 1 \\ T(n-1) &= T(n-2) + 1 \\ T(n-2) &= T(n-3) + 1 \\ &\vdots \\ T(2) &= T(1) + 1 \end{aligned}$$

Add up all equations above, we get:

$$\begin{aligned} T(n) + T(n-1) + \cdots + T(2) &= T(n-1) + T(n-2) + \cdots + T(2) + T(1) + (n-1) \\ T(n) &= T(1) + (n-1) \\ T(n) &= n \end{aligned}$$