

# COMP2711 Homework4

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## Question 1:

(a) Obviously  $\forall x \geq 1, x^3 > x^3 \log_2 x$ , so  $n = 3$  is not enough. When  $x \geq 1$ , we can easily know  $2x^2 + x^3 \log_2 x \geq 2x^4 + x^4 = 3x^4$ . Therefore  $n = 4$ , with witnesses  $C = 3, k = 1$ .

(b) We know  $\forall x \geq 1, (\log_2 x)^4 < x^5$ , so  $3x^5$  is dominant. When  $x \geq 1, 3x^5 + (\log_2 x)^4 \leq 3x^5 + x^5 = 4x^5$ . Hence  $n = 5$ , with witnesses  $C = 4, k = 1$ .

(c) We know for all  $x \geq 1$ ,

$$f(x) = \frac{x^4 + x^2 + 1}{x^4 + 1} = 1 + \frac{x^2}{x^4 + 1} < 1 + 1 = 2$$

Hence  $n = 0$  as  $f(x)$  is  $O(1)$ , with witnesses  $C = 2, k = 1$ .

(d) For all  $x \geq 1$ ,

$$f(x) = \frac{x^3 + 5 \log_2 x}{x^4 + 1} < \frac{x^3 + 5x^3}{x^4 + 1} < \frac{6x^3}{x^4 + 1} < \frac{6(x^4 + 1)}{x^4 + 1} = 6$$

Hence  $n = 0$  as  $f(x)$  is  $O(1)$ , with witnesses  $C = 6, k = 1$ .

## Question 2:

Firstly, we denote the first player as Player A, second as Player B; denote  $(i, j)$  as the cell at the  $i$ -th row and  $j$ -th column.

Let  $P(n)$ : If now the board consists of only  $n$  cookies in the top row and  $n$  cookies in the left-most column, i.e., cookies fill all the cells from  $(1, 1), (1, 2), \dots, (1, n)$ , and  $(n, 1), (n, 2), \dots, (n, n)$ , and it's Player B's turn to pick cookies, then Player A has a winning strategy.

We want to prove  $P(n)$  by strong induction.

*Basis Step:* It is obvious that  $P(1)$  is true, since Player B is made to eat the poisoned cookie.

*Inductive Step:* For  $k \geq 1$ , we assume that  $\forall i \leq k, P(i)$  is true, and we need to prove that  $P(k+1)$  is true. If Player B picks the cookie  $(1, 1)$ , then Player A wins. Otherwise, if Player B picks  $(1, j)$  or  $(j, 1)$  for some  $j$  with  $2 \leq j \leq k+1$ , then Player A just picks  $(j, 1)$  or  $(1, j)$ , correspondingly. After a round like this, the situation becomes  $P(j-1)$ . From our assumption, we know  $P(j-1)$  is true, so that  $P(k+1)$  is true.

Therefore, by strong induction, we know  $P(n)$  is true, so that Player A has a winning strategy: only need to take  $(2, 2)$ , then the situation becomes  $P(n)$ .

**Question 3:****Recursive Definition:**

*Basis Step:*  $(1, 1) \in S$ .

*Recursive Step:* if  $(a, b) \in S$ , then  $(a + 2, b) \in S, (a, b + 2) \in S, (a + 1, b + 1) \in S$ .

**Proof by Structural Induction:**

*Basis Step:* Apparently,  $(1, 1) \in S$  since  $1 + 1 = 2$  is even.

*Recursive Step:* Assume  $(a, b) \in S$ , then we know  $a + b$  is even. Thus,  $(a + 2, b), (a, b + 2), (a + 1, b + 1)$  all must in  $S$ , since  $a + 2 + b = a + b + 2 = a + 1 + b + 1 = (a + b) + 2$  is even.

Therefore, according to the principle of structural induction,  $a + b$  is even for all  $(a, b) \in S$ .

**Question 4:**

(a) Notice that  $a^n = (a^{n/2})^2$ .

**procedure** power ( $a$  : real number,  $n$  : positive integer)

**if**  $n = 1$  **then return**  $a$

**else return** [power( $a, n/2$ )]<sup>2</sup>

(b) According to the algorithm in (a), in order to find  $a^n$ , we find  $a^{n/2}$  instead since  $a^n = (a^{n/2})^2$ , and then we try to find  $a^{n/4}, a^{n/8}, \dots$  until we reach  $a^1$ , that is, the end of our recursion. Thus, if we denote the the number of multiplications used as  $T(n)$ , then we will have:

$$T(n) = T\left(\frac{n}{2}\right) + 1 \quad (n \geq 2)$$

$$T(1) = 0$$

(c) Assume  $n = 2^k$  for some integer  $k \geq 0$ , then from (b), we have:

$$T(2^k) = T(2^{k-1}) + 1$$

$$T(2^{k-1}) = T(2^{k-2}) + 1$$

$$\vdots$$

$$T(2^1) = T(2^0) + 1$$

Add up all equations above, we get:

$$T(2^k) + T(2^{k-1}) + \dots + T(2^1) = T(2^{k-1}) + T(2^{k-2}) + \dots + T(2^1) + T(2^0) + (k)$$

$$T(2^k) = T(2^0) + (k)$$

$$T(n) = T(1) + \log_2 n = \log_2 n$$

Therefore, the number of multiplications is  $\log_2 n$ .