COMP2711 Homework5

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Question 1:

(a) There are $5+11=16$ teachers and students in total. When they sit in a row, they provide 17 spaces(represent by $\circ$):

$$\circ 1 \circ 2 \circ 3 \circ \cdots \circ 15 \circ 16 \circ$$

Firstly, choose 4 spaces from those 17 spaces where guests can seat: $\binom{17}{4}$, then, guests can swap their positions and teachers and students can also swap their positions, this will be $4! \cdot 16!$. So the answer is:

$$\binom{17}{4} \cdot 4! \cdot 16!$$

(b) There are $5+11=16$ teachers and students in total. When they sit in a row, they provide 16 spaces. Firstly, choose 4 spaces from those 16 spaces where guests can seat: $\binom{16}{4}$, then, guests can swap their positions, which will be $4!$, and teachers and students can also swap their positions, but this time it will be $\frac{16!}{16}$, since arranged in circle. So the answer is:

$$\binom{16}{4} \cdot 4! \cdot \frac{16!}{16} = \binom{16}{4} \cdot 4! \cdot 15!$$

(c) There are $5+4=9$ teachers and guests in total. When they sit in a row, they provide 10 spaces. Since there are 11 students, it's impossible to let them sit in 10 spaces, i.e., $\binom{10}{11}=0$. Thus the answer is 0.

Question 2:

Note that the passcode follows non-decreasing order, so as long as we pick out 30 alphabets, there is only one possible order to arrange them. Since $2+7+1+1=11$ alphabets have already been chosen, we only need to pick $30-11=19$ alphabets from A $\sim$ Z.

If those 19 alphabets only contain one kind of letter,(e.g. all choose A/ all choose B, ...) then it's trivial that there are $\binom{26}{1}$ ways.

If those 19 alphabets contain two kinds of letter,(e.g. only contains A and B, ...) then we consider it in two steps: firstly choose 2 kinds of letter from 26 letters, $\binom{26}{2}$ ways, then we consider how many of them will we choose, respectively. This can be easily solved by thinking of putting 19 balls into 2 boxes, (where the boxes must be non-empty) so there are $\binom{18}{1}$ ways. Thus, there are $\binom{26}{2} \cdot \binom{18}{1}$ in total.

By the same method, if those 19 alphabets contain k different letters, there are $\binom{26}{k} \cdot \binom{18}{k-1}$ ways in total, where $1 \leq k \leq 19$.

To sum up, there are:

$$\sum_{k=1}^{19} \binom{26}{k} \cdot \binom{18}{k-1}$$

possible passwords.

Question 3:

Assume that there are $n-3$ apples and $n-2$ pears in total, and you would like to pick 10 fruits to eat. So we would like to calculate how many ways can you do that in two methods:

Method 1:

Since we just want to choose any 10 fruit in $2n-5$ fruites, so there are $\binom{2n-5}{10}$ ways.

Method 2:

We enumerate how many apples to choose: for example, we want to choose k apples and $10-k$ pears, out of $n-3$ apples and $n-2$ pears, so there are $\binom{n-3}{k} \cdot \binom{n-2}{10-k}$ ways. Note that $0 \leq k \leq 10$, so there're $\sum_{k=0}^{10} \binom{n-3}{k} \cdot \binom{n-2}{10-k}$ ways in total.

Therefore, we have:

$$\binom{2n-5}{10} = \sum_{k=0}^{10} \binom{n-3}{k} \cdot \binom{n-2}{10-k}$$

Question 4:

We try to simply consider one team as a whole(as one person), then there'll only be 19 people sitting in a row. Under this assumption, there are $\binom{10}{1} \cdot 19! \cdot 2!$ ways.

However, we find that some situations have been counted more than once. For example, if we denote 10 teams as $1, 2, \cdots 10$ and the two people inside one team as A, B, then the situation:

$$1_A 1_B 2_A 2_B 3_B 3_A 4_B 4_A \cdots 10_B 10_A$$

will be counted when we consider team 1 as a whole, and will also be counted when we consider team 2 as a whole, and also 3, 4, $\cdots$ 10 as a whole.

Thus, we need to use Inclusion-Exclusion Principle, and the final result will be:

$$\sum_{k=1}^{10} (-1)^{(k+1)} \cdot \binom{10}{k} \cdot (20-k)! \cdot (2!)^k$$