MATH 2023   Fall 2021

Multivariable Calculus

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Chapter 11 **Vector Functions and Curves**

1 Vector Functions of One Variable

Vector-valued functions: the value of the functions is a vector. Used to represent **curves parametrically.**

$$\underline{r} = \underline{r}(t) = x(t)\underline{i} + y(t)\underline{j} + z(t)\underline{k}$$

As t varies, $\underline{r}$ traces a **space curve.** Such a curve is called a **parametric curve**.

2 Curves and Parametrizations

[**Example.**] The plane $x + y = 1$ intersects the paraboloid $z = x^2 + y^2$(recall this is rice bowl) in a parabola. Parametrize the whole parabola using $t = x$ as parameter.

[**Solution.**] Since $y = 1 - x$ and $z = x^2 + y^2 = 1 - 2t + 2t^2$, thus the required Parametrization is:

$$\underline{r}(t) = t\underline{i} + (1-t)\underline{j} + (1 - 2t + 2t^2)\underline{k}, \ -\infty < t < \infty$$

Parametrize the Curve of Intersection of Two Surfaces:

[**Example.**] Parametrize the curve of intersection of the plane $x+2y+4z = 4$ and the elliptic cylinder $x^2+4y^2 = 4$.

[**Solution.**] We begin with the equation $x^2 + 4y^2 = 4$, which is independent of z,

$$x = 2\cos t, \ y = \sin t, \quad (0 \leq t \leq 2\pi)$$

then we can solve the equation for z:

$$z = \frac{1}{4}(4 - x - 2y) = 1 - \frac{1}{2}(\cos t + \sin t)$$

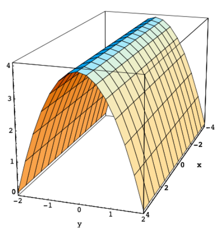
thus the intersection of the given two surfaces is:

$$\underline{r}(t) = 2\cos t\underline{i} + \sin t\underline{j} + \left(1 - \frac{\cos t + \sin t}{2}\right)\underline{k}, \quad (0 \le t \le 2\pi)$$

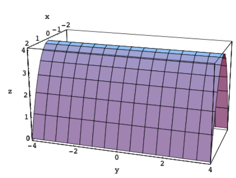
[Example.] Find the parametric equations of the curve of intersection of the surfaces $z = f(x,y) = a - x^2$ and $z = g(x,y) = a - y^2$.

[Solution.] Equal them to eliminate a, we get $y = \pm x$. Then let $x = t$, $y = \pm t$, $z = a - t^2$, so the equations will be:

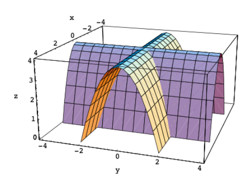
$$\underline{r}(t) = (t, t, a - t^2), \quad \text{or} \quad \underline{r}(t) = (t, -t, a - t^2)$$



$z = a - x^2$



$z = a - y^2$



3 Calculus of vector-valued functions

Limit:

$$\lim_{t\to a} \underline{r}(t) = \lim_{t\to a} x(t)\underline{i} + \lim_{t\to a} y(t)\underline{j} + \lim_{t\to a} z(t)\underline{k}$$

Continuous: at a if $\lim_{t\to a} \underline{r}(t) = \underline{r}(a)$.

Derivative:

$$\underline{r}'(t) = \lim_{h\to 0} \frac{\underline{r}(t+h) - \underline{r}(t)}{h} = x'(t)\underline{i} + y'(t)\underline{j} + z'(t)\underline{k}$$

Rules of differentiation:

- $(\underline{c})' = \underline{0}$
- addition and scalar multiplication: omit.
- $(f(t)\underline{r}(t))' = f'(t)\underline{r}(t) + f(t)\underline{r}'(t)$
- $(\underline{r_1}(t) \cdot \underline{r_2}(t))' = \underline{r_1}'(t) \cdot \underline{r_2}(t) + \underline{r_1}(t) \cdot \underline{r_2}'(t)$
- $(\underline{r_1}(t) \times \underline{r_2}(t))' = \underline{r_1}'(t) \times \underline{r_2}(t) + \underline{r_1}(t) \times \underline{r_2}'(t)$
- **chain rule:** $(\underline{r}(f(t)))' = \underline{r}'(f(t)) \cdot f'(t)$

[Example.] Find $\dfrac{d}{dt}||\underline{r}(t)||$.

[Solution.] Method 1:

$$\begin{aligned}
\underline{r}(t) &= (x(t), y(t), z(t)) \\
||\underline{r}(t)|| &= [x^2(t) + y^2(t) + z^2(t)]^{1/2} \\
\frac{d}{dt}||\underline{r}(t)|| &= \frac{1}{2} \cdot [x^2(t) + y^2(t) + z^2(t)]^{1/2} \cdot [2xx' + 2yy' + 2zz'] \\
&= \frac{1}{||\underline{r}||}(x, y, z) \cdot (x', y', z') = \frac{\underline{r} \cdot \underline{r}'}{||\underline{r}||}
\end{aligned}$$

Method 2: Notice $||\underline{r}||^2 = \underline{r} \cdot \underline{r}$,

$$\begin{aligned}
\frac{d}{dt}||\underline{r}||^2 &= \frac{d}{dt}(\underline{r} \cdot \underline{r}) \\
2||\underline{r}||\frac{d}{dt}||\underline{r}|| &= \underline{r}' \cdot \underline{r} + \underline{r} \cdot \underline{r}' \\
\frac{d}{dt}||\underline{r}|| &= \frac{\underline{r} \cdot \underline{r}'}{||\underline{r}||}
\end{aligned}$$

4 Higher order derivatives

- Position: $\underline{r}(t)$
- Velocity: $\frac{d}{dt}\underline{r}(t) = \underline{r}'(t)$
- Acceleration: $\frac{d}{dt}\underline{r}'(t) = \underline{r}''(t)$

5 Arc Length

$$\begin{aligned}
ds &= \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt \\
s &= \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt \\
&= \int_{t_1}^{t_2} ||\underline{r}'|| dt
\end{aligned}$$

[Example.] find the position of a point on the parametric curve

$$\underline{r}(t) = \cos^3 t\underline{i} + \sin^3 t\underline{j} + 2\underline{k}$$

that has arc length s unit from $\underline{r}(0) = (1, 0, 2)$.

[Solution.]

$$\int_0^{t_0} ||\underline{r}'(t)|| dt = s, \quad \underline{r}'(t) = -3\cos^2 t \sin t\underline{i} + 3\sin^2 t \cos t\underline{j}$$

$$s = \int_0^{t_0} (\cdots) dt = \frac{3}{2} \sin^2 t_0$$

$$t_0 = \sin^{-1} \sqrt{\frac{2s}{3}}$$