MATH 2023 Fall 2021

Multivariable Calculus

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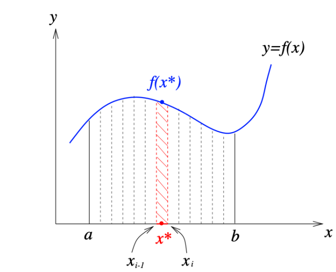
Chapter 14 **Multiple Integrations**

1 Double Integrals Over Rectangles

Recall that in single variable calculus, we divided a region into thin rectangles and use the **Riemann Sum** as integral.

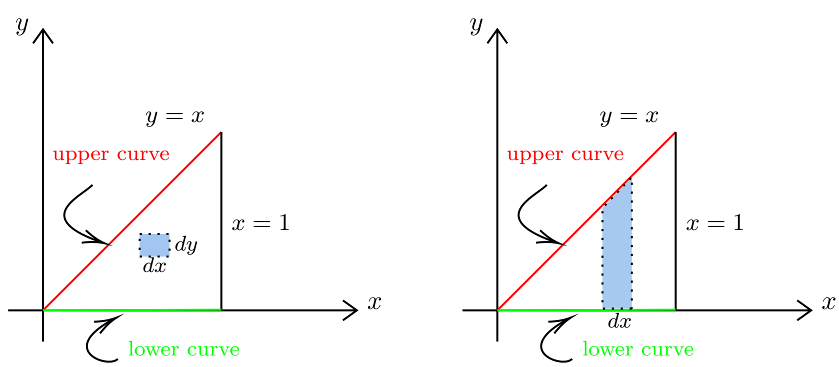


$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^\star)\delta x_i$$

Actually, instead of thin rectangles, we can use *small rectangles* to cover the area.

[Example.] Find the area bounded by $y = x$, $x = 1$ and $y = 0$.

[Solution.] (1) View the area as bounded by *upper curve* and *lower curve*.

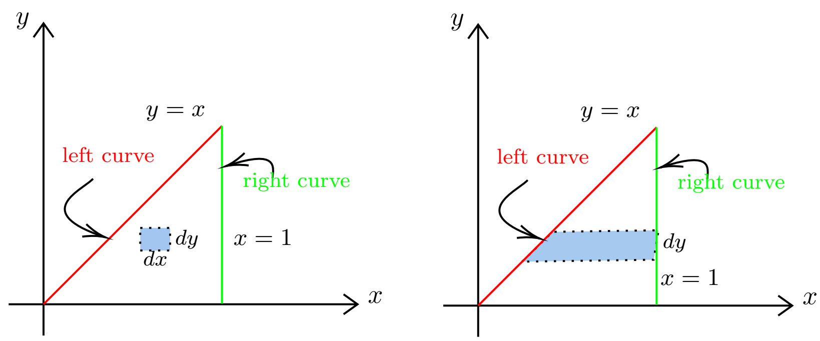


For the *small rectangles* with $dA = dxdy = dydx$, we first move it vertically, the lower bound is $y = 0$, and the upper bound is $y = x$, hence $\int_0^x dy$ is the shaded area in right image above.

Then we move the shaded area horizontally, the left bound is point $x = 0$, while the right bound is point $x = 1$, thus the total area is:

$$A = \int_0^1 \int_0^x dydx = \int_0^1 y\Big|_0^x dx = \int_0^1 xdx = \frac{1}{2}$$

(2) Alternatively, we can first move the rectangle horizontally, hitting *left curve* $x = y$ and *right curve* $x = 1$, hence the shaded area is $\int_y^1 dx$. note here we integral dx first, so when hitting the boundaries, we need to check x equals to what, i.e. $x = f(y)$. For example, here the two bounds are $x = y$ and $x = 1$.



Then we move the shaded area vertically, hitting lower bound $y = 0$(a point) and upper bound $y = 1$(a point), hence the total area is:

$$A = \int_0^1 \int_y^1 dxdy = \int_0^1 x\Big|_y^1 dy = \int_0^1 (1 - y)dy = \frac{1}{2}$$

2 Double Integrals Over General Regions

3 Double Integrals in Polar Coordinates

4 Change of Variables in Integrals

Recall that in single variable calculus, we often use a *substitution* to simplify an integral.

$$\int_a^b f(x)dx = \int_c^d f(g(u)) \cdot g'(u)\ du$$

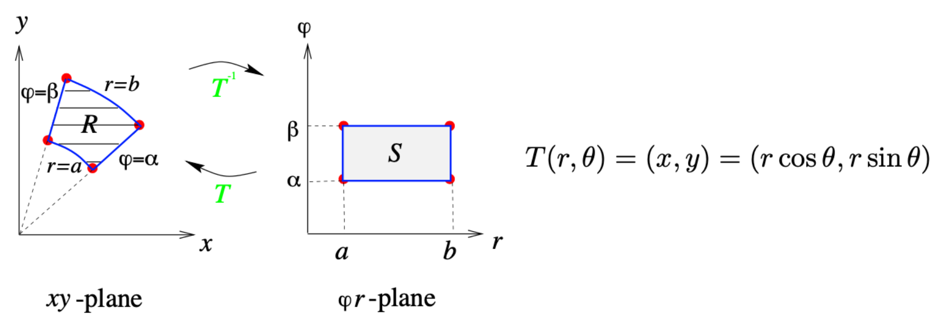
where $a = g(c)$ and $b = g(d)$. Notice that we can *view substitution as a kind of mapping*, and the change-of-variable process introduces *an additional factor* $g'(u)$ into the integrand.

This method can also be useful in multiple integrals. We have already seen one example: integration in *polar coordinate.*

$$\iint_R f(x,y)dA_{xy} = \iint_S f(r\cos\theta, r\sin\theta)rdrd\theta = \iint_S f(r\cos\theta, r\sin\theta)rdA_{r\theta}$$

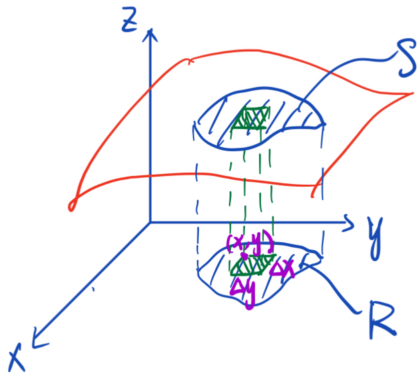
In this example, *the additional factor* is r.

The *mapping* T is shown as below: we transform the region R into S, where S is an rectangle in θr-plane, which is easy to integrate.



5 Surface Area

We now want to find the area of a surface. Finding an area on xy-plane is relatively easy, as we have discussed early this chapter, but things become much more complicated when we are focusing on an arbitrary surface. So, we think about *projecting the area onto xy-plane.*

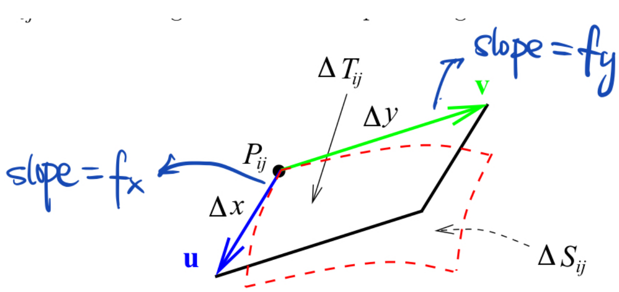


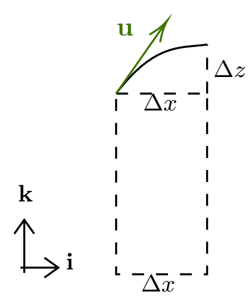
As shown above, we want to find the area of region S in a surface. The first thing to do is to project it onto xy-plane, resulting in a region R.

As usual, we use small rectangles to cover region R, as the green rectangle shown above, assume two sides are Δx and Δy, so the area of green rectangle is $\Delta A = \Delta x \Delta y$.

Then we project the rectangle up to the surface S, resulting in a "curved-parallelogram" surface, shown as red area in left-below image. To find this area, we know as long as Δx and Δy are small enough, the black parallelogram formed by Δx and Δy is a good approximation for that area. By the way, the area of parallelogram is $\mathbf{u} \times \mathbf{v}$.

How to represent $\mathbf{u}$ and $\mathbf{v}$? See the right-below image, the slope of vector $\mathbf{u}$ is $f_x = \dfrac{\Delta z}{\Delta x}$, so the width of $\mathbf{u}$ is Δx and the height of $\mathbf{u}$ is $\Delta z = \Delta x \cdot f_x$.(Notice this image is graphed vertically, i.e., in xz-plane)





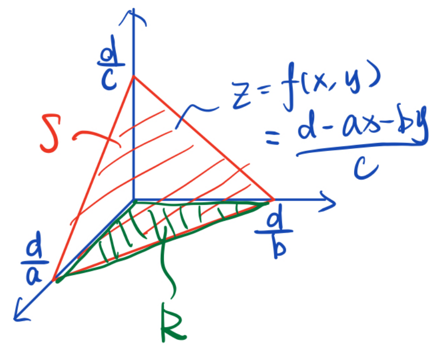
Therefore, $\mathbf{u} = \Delta x\mathbf{i} + 0\mathbf{j} + \Delta x \cdot f_x\mathbf{k}$, similarly, $\mathbf{v} = 0\mathbf{i} + \Delta y\mathbf{j} + \Delta y \cdot f_y\mathbf{k}$. Then

$$||\mathbf{u} \times \mathbf{v}|| = || - \Delta x\Delta y f_x\mathbf{i} - \Delta x\Delta y f_y\mathbf{j} + \Delta x\Delta y\mathbf{k}|| = \sqrt{(f_x)^2 + (f_y)^2 + 1} \cdot \Delta x\Delta y$$

When $\Delta x, \Delta y \to 0$, the area of S is given by:

$$A = \iint_R \sqrt{(f_x)^2 + (f_y)^2 + 1}\, dA$$

[Example.] Given a plane $ax + by + cz = d$, where $a, b, c, d > 0$. Find the area of the triangle bounded by the intersections of the plane and axes.(As the red shaded area shown)



[Solution.] The equation of surface $z = f(x, y)$ is given by $z = \dfrac{d - ax - by}{c}$.

To find the red area, we first *project it onto xy-plane*, resulting in green area R.

Thus the red area

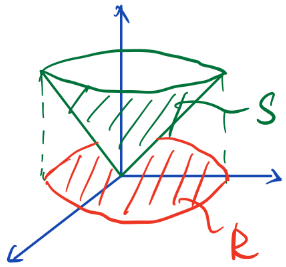
$$S = \iint_R \sqrt{1 + f_x^2 + f_y^2}\, dA_{xy}$$

We know that $f_x = -\dfrac{a}{c}$, $f_y = -\dfrac{b}{c}$, so $\sqrt{1 + f_x^2 + f_y^2} = \dfrac{\sqrt{a^2 + b^2 + c^2}}{c}$, then

$$\begin{aligned}
S &= \iint_R \frac{\sqrt{a^2 + b^2 + c^2}}{c}\, dA_{xy} \\
&= \frac{\sqrt{a^2 + b^2 + c^2}}{c} \cdot \text{(area of R)} \\
&= \frac{\sqrt{a^2 + b^2 + c^2}}{c} \cdot \frac{1}{2} \cdot \frac{d}{a} \cdot \frac{d}{b} \\
&= \frac{d^2 \sqrt{a^2 + b^2 + c^2}}{2abc}
\end{aligned}$$

Notice the blue part is a constant.

[Example.] Find the surface area of the cone $z = \dfrac{h}{a}r$ (in cylindrical coordinate).



[Solution.] Project the cone onto xy-plane, resulting in red area R.

The surface is given by $z = f(x,y) = \dfrac{h}{a}\sqrt{x^2+y^2}$

Thus the green area

$$S = \iint_R \sqrt{1 + f_x^2 + f_y^2}\; dA_{xy}$$

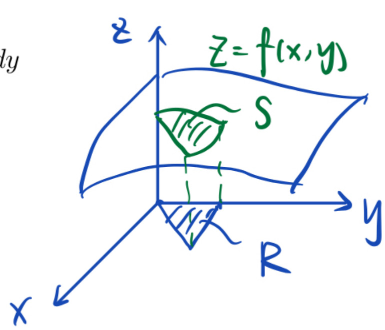
We know that $f_x = \dfrac{h}{a} \cdot \dfrac{x}{\sqrt{x^2+y^2}}$, $f_y = \dfrac{h}{a} \cdot \dfrac{y}{\sqrt{x^2+y^2}}$, so $\sqrt{1 + f_x^2 + f_y^2} = 1 + \dfrac{h^2}{a^2}$, then

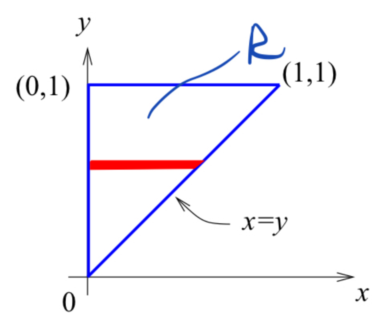
$$\begin{aligned} S &= \iint_R \sqrt{a + \frac{h^2}{a^2}}\; dA_{xy} \\ &= \sqrt{a + \frac{h^2}{a^2}} \cdot (\text{Area of circle with radius a}) \\ &= \pi a \cdot \sqrt{a^2 + h^2} \end{aligned}$$

[Example.] Find the area of the surface $z = x + y^2$ that lies above the triangle with vertices (0,0), (1,1) and (0,1).

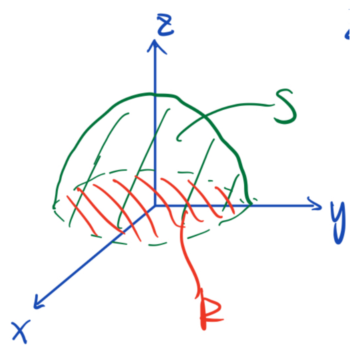
[Solution.]

$$\begin{aligned} S &= \iint\limits_R \sqrt{z_x^2 + z_y^2 + 1}\, dA = \int_0^1 \int_0^y \sqrt{1 + 4y^2 + 1}\, dx\, dy \\ &= \int_0^1 y\sqrt{2 + 4y^2}\, dy \\ &= \frac{2}{24}(2 + 4y^2)^{\frac{3}{2}} \Big|_0^1 \\ &= \frac{1}{6}(3\sqrt{6} - \sqrt{2}). \end{aligned}$$





[Example.] Find the surface of a sphere with radius a.



[Solution.] Again, project S onto xy-plane to get region R.

The equation of surface is given by $z = f(x, y) = \sqrt{a^2 - x^2 - y^2}$,

The green area:

$$S = \iint_R \sqrt{1 + f_x^2 + f_y^2}\, dA_{xy}$$

We know that $f_x = \dfrac{-x}{\sqrt{a^2 - x^2 - y^2}}$, $f_y = \dfrac{-y}{\sqrt{a^2 - x^2 - y^2}}$, so $\sqrt{1 + f_x^2 + f_y^2} = 1 + \dfrac{x^2 + y^2}{a^2 - x^2 - y^2}$, then

$$S = \iint_R \sqrt{1 + \frac{x^2 + y^2}{a^2 - x^2 - y^2}}\, dA_{xy}$$

Notice this integration is too difficult to calculate, so we consider using *polar coordinate* to substitute, let $r^2 = x^2 + y^2$, $dA = r\ dr d\theta$, then

$$S = \iint_R \sqrt{1 + \frac{r^2}{a^2 - r^2}}\ r\ dr d\theta = 2\pi a^2$$