MATH 2023 Fall 2021

Multivariable Calculus

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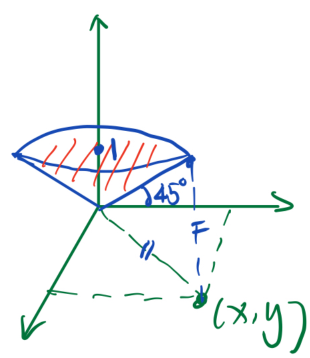
Chapter 10 **Vectors and Geometry in 3D**

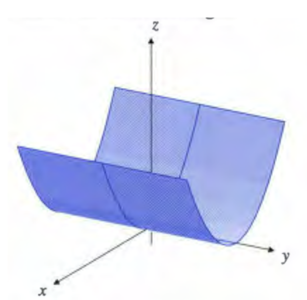
1 Coordinates in 3D

[Example.] $\{(x, y, z) \mid z \geq \sqrt{x^2 + y^2}\}$ is an ice-cream cone.



Remark: When one of the variables is missing from the equation, the equation represents a surface *parallel* to the axis of the missing variable.

[Example.] $z = x^2$, since y is missing, we draw a parabola and move it along the y axis.



2 Vectors

2.1 Vectors and Properties

In $\mathbb{R}^3$, $\underline{r} = (a, b, c) = a\underline{i} + b\underline{j} + c\underline{k}$, $||\underline{r}|| = \sqrt{a^2 + b^2 + c^2}$

Zero vector $\underline{0}$ has length zero and no specific direction.

Properties of vectors

- Equality: iff same **magnitude** and same **direction**. OR, their **components** equal
- Parallel: $\underline{r} = \lambda \underline{s}$
- Negative: $-\underline{r}$ has same magnitude but opposite direction.
- Addition, Subtraction, Scalar Multiplication: omit.

Unit Vector: $\hat{\underline{r}} = \dfrac{\underline{r}}{||\underline{r}||}$

Dot Product: $\underline{u} \cdot \underline{v} = u_1v_1 + u_2v_2 + u_3v_3$. **Remark:** $\underline{a} \cdot \underline{b} = \underline{a} \cdot \underline{c} \not\Rightarrow \underline{b} = \underline{c}$

Angle: $\underline{u} \cdot \underline{v} = ||\underline{u}|| \cdot ||\underline{v}|| \cos\theta$

Projections:

- Scalar projection: $s = \dfrac{\underline{u} \cdot \underline{v}}{||\underline{v}||} = ||\underline{u}|| \cos\theta$
- Vector projection of $\underline{u}$ in the direction of $\underline{v}$: $\underline{u}_{\underline{v}} = \dfrac{\underline{u} \cdot \underline{v}}{||\underline{v}||}\hat{\underline{v}} = \dfrac{\underline{u} \cdot \underline{v}}{||\underline{v}||^2}\underline{v}$

[Example.] Find the scalar and vector projections of vector $\underline{a} = \underline{i} - 2\underline{j} + \underline{k}$ on the vector $\underline{b} = 4\underline{i} - 4\underline{j} + 7\underline{k}$.

[Solution.]

$$s = \frac{\underline{a} \cdot \underline{b}}{||\underline{b}||} = \frac{4 + 8 + 7}{\sqrt{4^2 + 4^2 + 7^2}} = \frac{19}{9}$$

$$\underline{a}_{\underline{b}} = s \cdot \frac{\underline{b}}{||\underline{b}||} = \frac{19}{81}(4\underline{i} - 4\underline{j} + 7\underline{k})$$

Position Vector: $\underline{r}$ from the origin to the point (a, b, c):

$$\underline{r} = a\underline{i} + b\underline{j} + c\underline{k}, \qquad ||\underline{r}|| = \sqrt{a^2 + b^2 + c^2}$$

$$\hat{\underline{r}} = \frac{a\underline{i} + b\underline{j} + c\underline{k}}{\sqrt{a^2 + B^2 + c^2}}$$

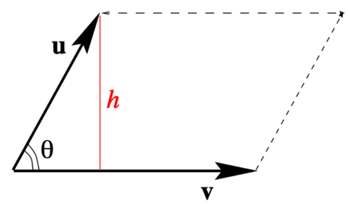
Cross Product:

$$\underline{u} \times \underline{v} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = (u_2v_3 - u_3v_2)\,\underline{i} - (u_1v_3 - u_3v_1)\,\underline{j} + (u_1v_2 - u_2v_1)\,\underline{k}$$

$$= -\underline{v} \times \underline{u} \quad \textbf{(vector)}$$

Properties:

- $\underline{u} \times \underline{u} = \underline{0}$
- $\underline{a} \times (\underline{b} + \underline{c}) = \underline{a} \times \underline{b} + \underline{a} \times \underline{c}$
- $(\underline{u} \times \underline{v}) \cdot \underline{u} = 0, (\underline{u} \times \underline{v}) \cdot \underline{v} = 0$
- $||\underline{u} \times \underline{v}|| = ||\underline{u}|| \cdot ||\underline{v}|| \sin\theta =$ area of parallelogram.



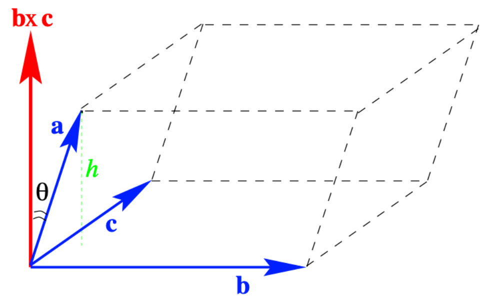
- $||\underline{u} \times \underline{v}||^2 = ||\underline{u}||^2||\underline{v}||^2 - (\underline{u} \cdot \underline{v})^2$
- $\underline{u} \times (\underline{v} \times \underline{w}) \neq (\underline{u} \times \underline{v}) \times \underline{w}$
- $\underline{a} \times \underline{b} = \underline{a} \times \underline{c} \not\Rightarrow \underline{b} = \underline{c}$

2.2 Triple Scalar Products

$$\underline{a} \cdot (\underline{b} \times \underline{c}) = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Geometric interpretation of $\underline{a} \cdot (\underline{b} \times \underline{c})$:

$$\begin{aligned} \underline{a} \cdot (\underline{b} \times \underline{c}) &= ||\underline{a}||\ ||\underline{b} \times \underline{c}|| \cos\theta \\ &= (||\underline{a}|| \cos\theta) \cdot ||\underline{b} \times \underline{c}|| \\ &= h \cdot ||\underline{b} \times \underline{c}|| \\ &= \textbf{volume of the parallelepiped} \end{aligned}$$

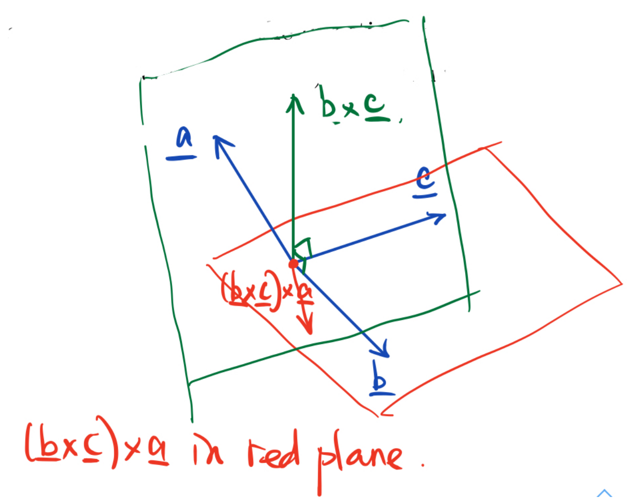


Another use of triple scalar product is for deciding whether three vectors are **in the same plane**. If so, then

$$(\underline{a} \times \underline{b}) \cdot \underline{c} = 0$$

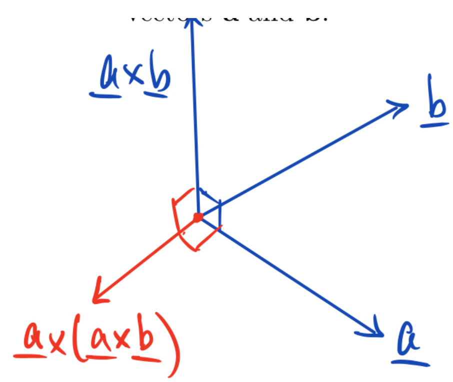
2.3 Triple Vector Products

$(\underline{a} \times \underline{b}) \times \underline{c}$ is in the plain containing $\underline{a}$ and $\underline{b}$.



[Example.] Construct three mutually orthogonal vectors in space, making use of two non-parallel vectors $\underline{a}$ and $\underline{b}$.

[Solution.] $\underline{a}$, $\underline{a} \times \underline{b}$, and $\underline{a} \times (\underline{a} \times \underline{b})$.



3 Planes and Lines

3.1 Planes

The equation of a plane: assume nonzero normal vector $\underline{n} = A\underline{i} + B\underline{j} + C\underline{k}$ and plane passes through $r_0 = (x_0, y_0, z_0)$,

$$\underline{n} \cdot (\underline{r} - \underline{r_0}) = 0$$

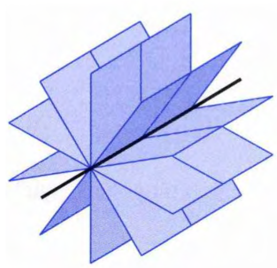
can also written in **standard form**:

$$Ax + By + Cz = Ax_0 + By_0 + Cz_0$$

A pencil of planes: a family of plans intersecting in a straight line. If two nonparallel planes have equations $A_1x + B_1y + C_1z = D1$ and $A_2x + B_2y + C_2z = D_2$, then

$$A_1x + B_1y + C_1z - D1 + \lambda(A_2x + B_2y + C_2z - D_2) = 0$$

represents a plane in the pencil, where $\lambda \in \mathbb{R}$.



[Example.] Find the equation of the plane through $P_0 = (-1, 4, 2)$ and containing the line of intersection of the planes $4x - y + z - 2 = 0$ and $2x + y - 2z - 3 = 0$.

[Solution.] Two methods:

(1) find three points.

Notice the two planes have infinity points intersected. We arbitrarily choose 2 of them, say $\underline{r_1} = (0, -7, -5), \underline{r_2} = (1, 3, 1)$. Another point is given in problem, $\underline{r_0} = (-1, 4, 2)$.

Then a normal vector can be $(\underline{r_1} - \underline{r_0}) \times (\underline{r_2} - \underline{r_0}) = (4, -13, 21)$

(2) find the pencil of planes.

The pencil of planes formed by the given two non-parallel planes is

$$4x - y + z - 2 + \lambda(2x + y - 2z - 3) = 0$$

Since the plane is inside the pencil and containing P_0, substitute $(-1, 4, 2)$ into the equation, we get $\lambda = -\frac{8}{5}$.

3.2 Lines

Vector parametric equation of straight line: with a point $\underline{r_0}$ and **direction vector** $\underline{v}$:

$$\underline{r}(t) = \underline{r_0} + t\underline{v}$$

[Example.] Find the vector equation for the line segment joins $P_1 = (2, 4, -1)$ and $P_2 = (5, 0, 7)$.

[Solution.] direction vector $\underline{v} = (3, -4, 8)$, thus

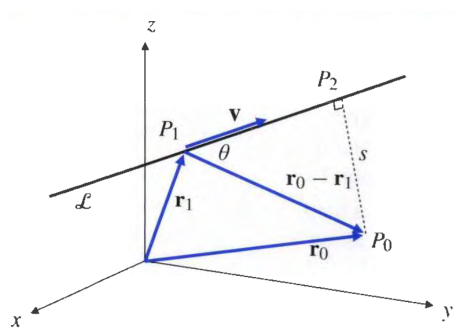
$$\underline{r}(t) = (2, 4, -1) + t \cdot (3, -4, 8), \ \ 0 \leq t \leq 1$$

[Example.] Find the intersection of line $\underline{r} = \underline{r_0} + t\underline{v}$ and plane $\underline{r} \cdot \underline{v} = d$, assume they are not parallel.

[Solution.] $(\underline{r_0} + t\underline{v}) \cdot \underline{n} = d$, solve for t.

3.3 Distances

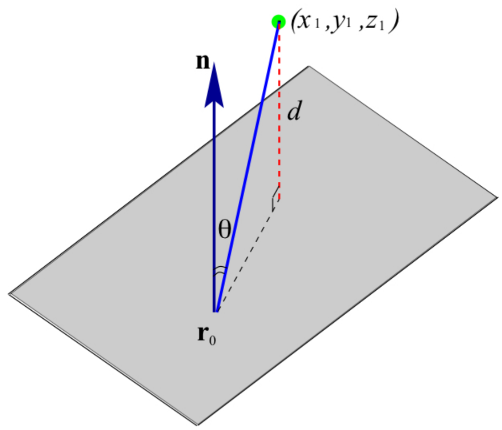
Point to Line:



$$\begin{aligned} d &= ||\underline{r_1} - \underline{r_0}|| \cdot \sin\theta \cdot 1 \\ &= ||\underline{r_1} - \underline{r_0}|| \cdot \sin\theta \cdot ||\underline{\hat{v}}|| \\ &= ||(\underline{r_1} - \underline{r_0}) \times \underline{\hat{v}}|| \end{aligned}$$

Point to Plane:

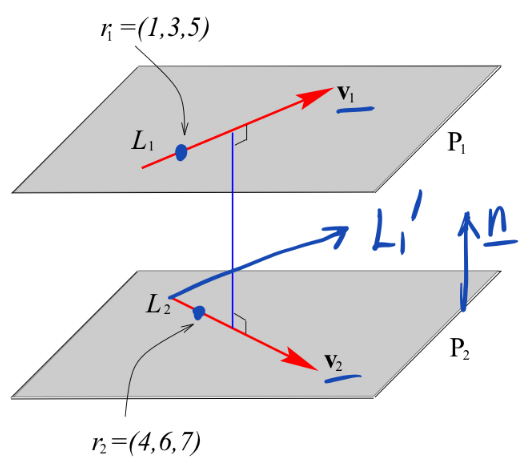
$$\begin{aligned} d &= ||\underline{r_1} - \underline{r_0}|| \cdot \cos\theta \cdot 1 \\ &= ||\underline{r_1} - \underline{r_0}|| \cdot \cos\theta \cdot ||\underline{\hat{n}}|| \\ &= |(\underline{r_1} - \underline{r_0}) \cdot \underline{\hat{n}}| \end{aligned}$$



Line to Plane(parallel): Take any point on the line $\rightarrow$ point to plane.

Parallel Lines: Same as point to line.

Skew Lines:



Make L_1 and L_1 inside the same plane, then normal vector of the plane is $\underline{n} = \underline{v_1} \times \underline{v_2}$.

Now it is the same as line to plane.

$$\begin{aligned} d &= |(\underline{r_1} - \underline{r_0}) \cdot \hat{\underline{n}}| \\ &= |(\underline{r_1} - \underline{r_0}) \cdot (\widehat{\underline{v_1} \times \underline{v_2}})| \end{aligned}$$

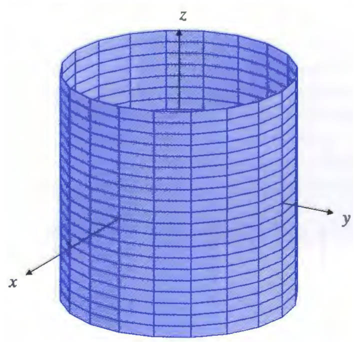
Plane to Plane(parallel): same as point to plane.

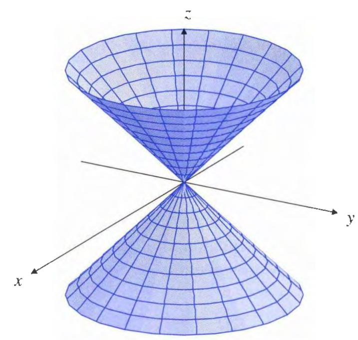
4 Quadric Surfaces

Spheres: $(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = a^2$.

Cylinders: $x^2 + y^2 = a^2$. (left below)

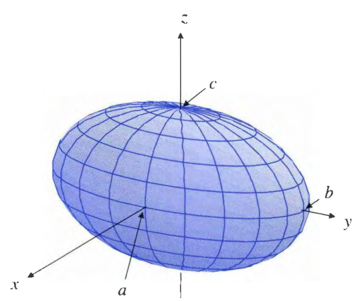
Cones: $z^2 = x^2 + y^2$. (right below)

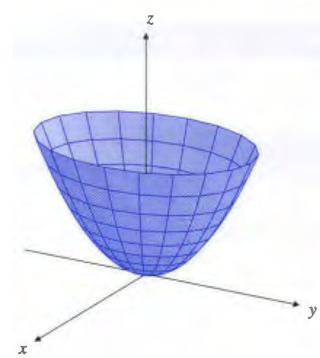




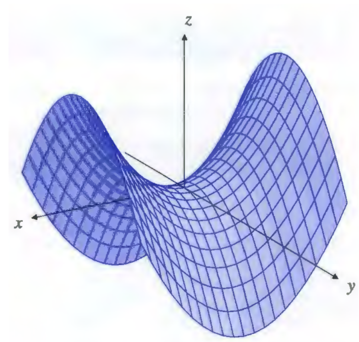
Ellipsoid: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$. (left below)

Elliptic paraboloid: $z = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2}$. (right below)





Hyperbolic paraboloid : $z = \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2}$.



Hyperboloids:

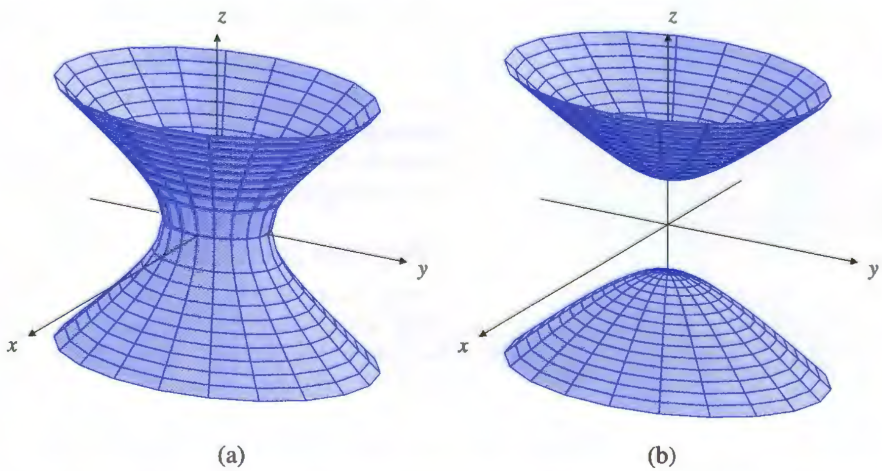


Figure 10.37

(a) The hyperboloid of one sheet $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$

(b) The hyperboloid of two sheets $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1$