MATH 2023 Fall 2021

Multivariable Calculus

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Chapter 16 **Vector Calculus**

1 The Divergence Theorem

Let G be a simple solid whose boundary surface S has *positive (outward)* orientation. When we find the **flux** of S in a *smooth* vector field $\mathbf{F}(x,y,z) = f(x,y,z)\mathbf{i} + g(x,y,z)\mathbf{j} + h(x,y,z)\mathbf{k}$, the **Divergence Theorem** gives:

$$\boxed{\oiint_S \mathbf{F}\cdot\widehat{\mathbf{n}}\,dS = \iiint_G \nabla\cdot\mathbf{F}\,dV}$$

where $\widehat{\mathbf{n}}$ is a unit normal vector pointing out of G.

In other words, *the total divergence within G equals the net flux emerging from G.*

Moreover, if the volume is very small, we can assume $\nabla\cdot\mathbf{F}$ is *constant* within the small volume, thus

$$\begin{aligned}\oiint_{\delta S} \mathbf{F}\cdot\widehat{\mathbf{n}}\,dS &= \nabla\cdot\mathbf{F}\iiint_G\,dV\\ &= \nabla\cdot\mathbf{F}\cdot\delta V\end{aligned}$$

Therefore,

$$\nabla\cdot\mathbf{F} = \lim_{\delta V\to 0}\frac{1}{\delta V}\oiint_{\delta S}\mathbf{F}\cdot\widehat{\mathbf{n}}\,dS$$

This is the definition of divergence given in lecture note Chapter 15.

Since exam will not cover the proof, I'd like to omit here.

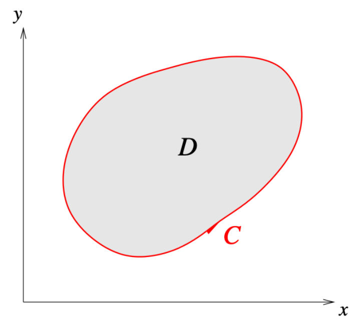
This example shows how Divergence Theorem simplifies the computation of flux.

2 Green's Theorem

2.1 Green's Theorem in Line Integral

In this part, we will go back to **line integral**, which we have done a lot.

Now consider doing line integral in a smooth simple **closed curve** C in the xy-plane, if $\mathbf{F}(\mathbf{r}) = P(\mathbf{r})\mathbf{i} + Q(\mathbf{r})\mathbf{j}$, then if we want to evaluate $\oint_C \mathbf{F} \cdot d\mathbf{r}$,



Counter-clockwise
– positive orientation

- If $\mathbf{F}$ is conservative, then line integral is 0, obviously.
- If $\mathbf{F}$ is not conservative, then **Green's Theorem** tells us

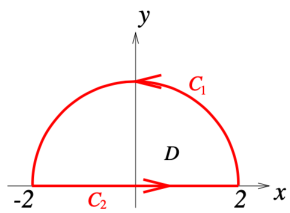
$$\boxed{\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_D (\nabla \times \mathbf{F}) \cdot \mathbf{k}\, dA}$$

Note k is the **normal** to xy-plane, or, normal to region D.

Since exam will not cover the proof, I'd like to omit here.

This example shows how Green's Theorem simplify computation.

[Example.] $\int_C xydx + 2x^2dy$, C consists of the segment from $(-2, 0)$ to $(2, 0)$ and top half of the circle $x^2 + y^2 = 4$.



[Solution.]

Method 1: use line integral:

$$\int_C xydx + 2x^2dy = \int_{C_1} xydx + 2x^2dy + \int_{C_2} xydx + 2x^2dy$$

Parametrize the two curves:

$$C_1 : \mathbf{r}(t) = (1-t)(-2, 0) + t(2, 0) = (4t - 2, 0) \quad 0 \leqslant t \leqslant 1$$
$$C_2 : \mathbf{r}(t) = (2\cos t, 2\sin t) \quad 0 \leqslant t \leqslant \pi$$

Then directly evaluate the two line integrals

$$\begin{aligned}
\int_{C_1} xydx + 2x^2dy &= \int_0^1 (4t-2) \cdot 0 \cdot 4dt + 2(4t-2)^2 \cdot (0) = 0 \\
\int_{C_2} xydx + 2x^2dy &= \int_0^\pi (2\cos t)(2\sin t)(-2\sin t)dt + 2(2\cos t)^2(2\cos t)dt \\
&= 8\int_0^\pi \left(-\cos t \sin^2 t + \cos^3 t\right) dt = 0
\end{aligned}$$

Thus $\int_C xydx + 2x^2dy = 0$.

Method 2: using Green's theorem:

$\mathbf{F} = (xy, 2x^2)$, hence $\nabla \times \mathbf{F} = (4x - x)\mathbf{k} = 3x\mathbf{k}$, then

$$\begin{aligned}
\oint_C xydx + 2x^2dy = \iint_D 3xdA &= \int_0^2 \int_0^\pi 3r\cos\theta \; rd\theta dr \\
&= \int_0^2 3r^2 \sin\theta \Big|_0^\pi dr = 0
\end{aligned}$$

Actually, one may observe that $\iint_D 3xdA = 0$ directly, since $3x$ is a *odd* function in x, and the region D is *symmetric with respect to* y-axis.

2.2 Green's Theorem for computing Area

Recall that Green's Theorem states that:

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_D (\nabla \times \mathbf{F}) \cdot \mathbf{k} \, dA$$

Notice if $\mathbf{F}(\mathbf{r}) = P(\mathbf{r})\mathbf{i} + Q(\mathbf{r})\mathbf{j}$,

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C Pdx + Qdy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \iint_D (\nabla \times \mathbf{F}) \cdot \mathbf{k} \, dA$$

When $\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y} = 1$, then

$$\boxed{A = \iint_D dA = \oint_C Pdx + Qdy.}$$

For example, when $P = 0, Q = x$, or when $P = -y, Q = 0$, or when $P = -y/2, Q = x/2$,

$$A = \oint_C xdy = -\oint_C ydx = \frac{1}{2} \oint_C xdy - ydx$$

The two examples below shows how to use Green's Theorem to find area.

[Example.] Find the area of $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

[Solution.] Firstly parametrize the curve, let $x = a\cos\theta,\ y = b\sin\theta,\ 0 \le \theta \le 2\pi$, then

$$C:\ \mathbf{r}(\theta) = (a\cos\theta, b\sin\theta),\ 0 \le \theta \le 2\pi$$

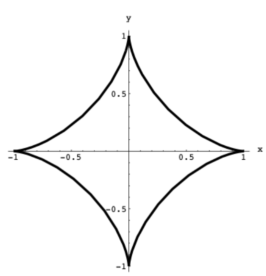
, If we choose $\mathbf{F}(\mathbf{r}) = P(\mathbf{r})\mathbf{i} + Q(\mathbf{r})\mathbf{j} = -\dfrac{y}{2}\mathbf{i} + \dfrac{x}{2}\mathbf{j}$, then we have

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \frac{1}{2} + \frac{1}{2} = 1$$

Hence,

$$\begin{aligned} D &= \frac{1}{2}\oint (xdy - ydx) \\ &= \frac{1}{2}\left[\int_0^{2\pi} a\cos\theta \cdot b\cos\theta\ d\theta + b\sin\theta \cdot a\sin\theta\ d\theta\right] \\ &= \frac{1}{2}ab \cdot \int_0^{2\pi} (\cos^2\theta + \sin^2\theta)\ d\theta = \pi ab \end{aligned}$$

[Example.] Find the area of the hypocycloid $x^{2/3} + y^{2/3} = a^{2/3}$.



[Solution.] Firstly parametrize the curve, let $x = a\cos^3\theta, y = a\sin^3\theta$, where $0 \leqslant \theta \leqslant 2\pi$,

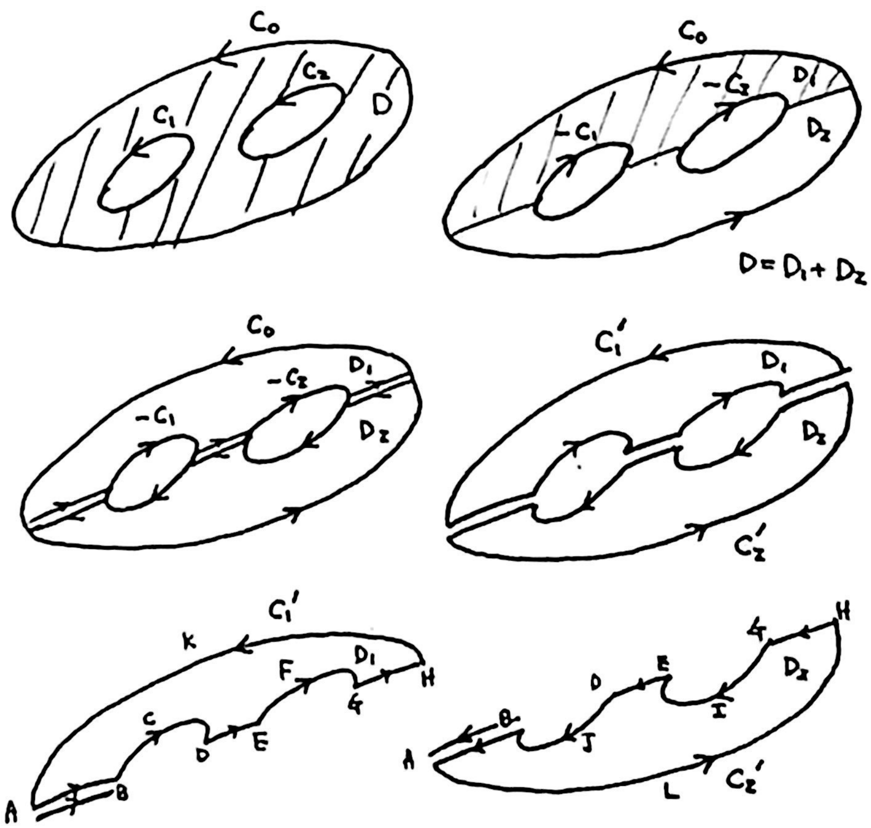
Again, use vector field $\mathbf{F}(\mathbf{r}) = P(\mathbf{r})\mathbf{i} + Q(\mathbf{r})\mathbf{j} = -\dfrac{y}{2}\mathbf{i} + \dfrac{x}{2}\mathbf{j}$,

$$\begin{aligned} A = \frac{1}{2}\oint_C xdy - ydx &= \frac{1}{2}\int_0^{2\pi} \left(a\cos^3\theta \times 3a\sin^2\theta\cos\theta d\theta + a\sin^3\theta \times 3a\cos^2\theta\sin\theta d\theta\right) \\ &= \frac{3}{2}a^2\int_0^{2\pi} \left(\cos^4\sin^2\theta + \sin^4\theta\cos^2\theta\right) d\theta \\ &= \frac{3}{2}a^2\int_0^{2\pi} \frac{1}{4}\sin^2 2\theta d\theta \\ &= \frac{3}{8}a^2\int_0^{2\pi} \left(\frac{1 - \cos 4\theta}{2}\right) d\theta = \frac{3\pi}{8}a^2 \end{aligned}$$

2.3 General version of Green's Theorem

This part will not be covered in exam.

Recall that Green's Theorem only applies to *simple* and *closed* curve. However, it can be extended to apply to region with holes. We simply cut the region into some regions that without holes, for example:

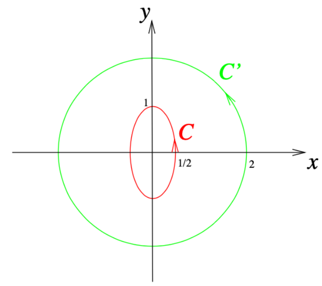


$$
\begin{aligned}
\iint_D &= \iint_{D_1} + \iint_{D_2} = \oint_{C_1'} + \oint_{C_2'} \\
&= \left(\int_{HKA} + \int_{AB} + \int_{BCD} + \int_{DE} + \int_{EFG} + \int_{GH}\right) + \left(\int_{ALH} + \int_{HG} + \int_{GIE} + \int_{ED} + \int_{DJB} + \int_{BA}\right) \\
&= \int_{C_0} - \int_{C_1} - \int_{C_2}
\end{aligned}
$$

Here is the example provided in lecture note.

Example $\displaystyle\oint_C \frac{-x^2y\,dx + x^3\,dy}{(x^2+y^2)^2}$, where C is the ellipse $4x^2+y^2=1$.

If C' is the circle $x^2+y^2=4$, then C is interior to C', and everywhere except at $(0,0)$. Note also that



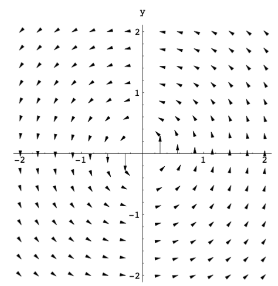
$$\frac{\partial}{\partial x}\left[\frac{x^3}{(x^2+y^2)^2}\right] = \frac{\partial}{\partial y}\left[\frac{-x^2y}{(x^2+y^2)^2}\right]$$

$$\therefore\ I = \oint_C \frac{-x^2y\,dx + x^2\,dy}{(x^2+y^2)^2} = \oint_{C'} \frac{-x^2y\,dx + x^3\,dy}{(x^2+y^2)^2}$$

On C', let $x = 2\cos\theta$, $y = 2\sin\theta$, where $0 \leqslant \theta \leqslant 2\pi$, then

$$I = \int_0^{2\pi} \frac{-4\cos^2\theta\ 2\sin\theta(-2\sin\theta)\,d\theta + (2\cos\theta)^2\ 2\cos\theta\,d\theta}{16}$$

$$= \int_0^{2\pi} \cos^2\theta\,d\theta = \int_0^{2\pi}\left(\frac{1+\cos 2\theta}{2}\right)d\theta = \pi.$$



$$\mathbf{F}(\mathbf{r}) = \frac{-x^2y\,\mathbf{i} + x^3\,\mathbf{j}}{(x^2+y^2)^2}$$

3 Stokes' Theorem

Recall in Green's Theorem,

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_D (\nabla \times \mathbf{F}) \cdot \mathbf{k} dA$$

where $\mathbf{F}(\mathbf{r}) = (F_1, F_2),\ C : \mathbf{r}(t) = (x(t), y(t)),\ a \leq t \leq b$

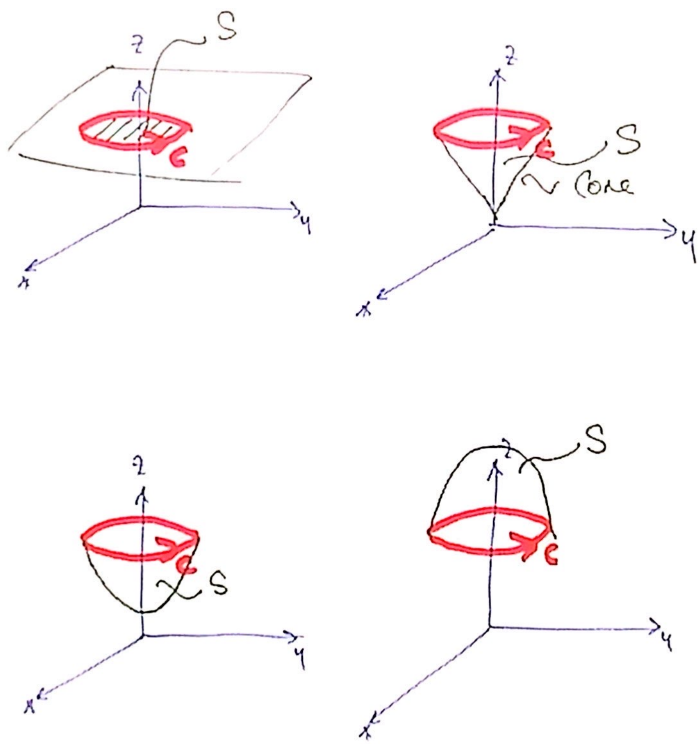
Now we want to extends this theorem into 3D space.

The **Stokes' Theorem** tells that if S is a *non-closed* surface, whose boundary consists of a closed smooth curve C with *positive orientation,* then

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \widehat{\mathbf{n}}\ dS$$

where $\mathbf{F}(\mathbf{r}) = (F_1, F_2, F_3),\ C : \mathbf{r}(t) = \big(x(t), y(t), z(t)\big),\ a \leq t \leq b$, and $\mathbf{r}(a) = \mathbf{r}(b)$ since the boundary is closed. $\widehat{\mathbf{n}}$ is unit normal vector of surface S.

However, you may have noticed that the theorem doesn't tell how to find S. When we evaluate a line integral on C, there are lots of surfaces S that can have boundary C.



This example gives a standard process for applying Stokes' Theorem and provides ideas of how to construct S.

[Example.] Find $\oint_C \mathbf{F}\cdot d\mathbf{r}$, where $\mathbf{F}(\mathbf{r}) = (y, x^2, y)$, $C:\ \mathbf{r}(t) = (\cos t, \sin t, 1),\ 0 \le t \le 2\pi$

[Solution.] **Method 1:** directly compute line integral.

$$\begin{aligned}\oint_C \mathbf{F}\cdot d\mathbf{r} &= \int_0^{2\pi} (\sin t, \cos^2 t, \sin t)\cdot(-\sin t, \cos t, 0)\ dt\\ &= \int_0^{2\pi} (-\sin^2 t + \cos^3 t)\ dt\end{aligned}$$

This is tedious.

Method 2: Notice $\mathbf{r}(0) = \mathbf{r}(2\pi)$, so this is a *closed curve* in 3D, we can use **Stokes' Theorem**.

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_S (\nabla\times\mathbf{F})\cdot\widehat{\mathbf{n}}\ dS$$

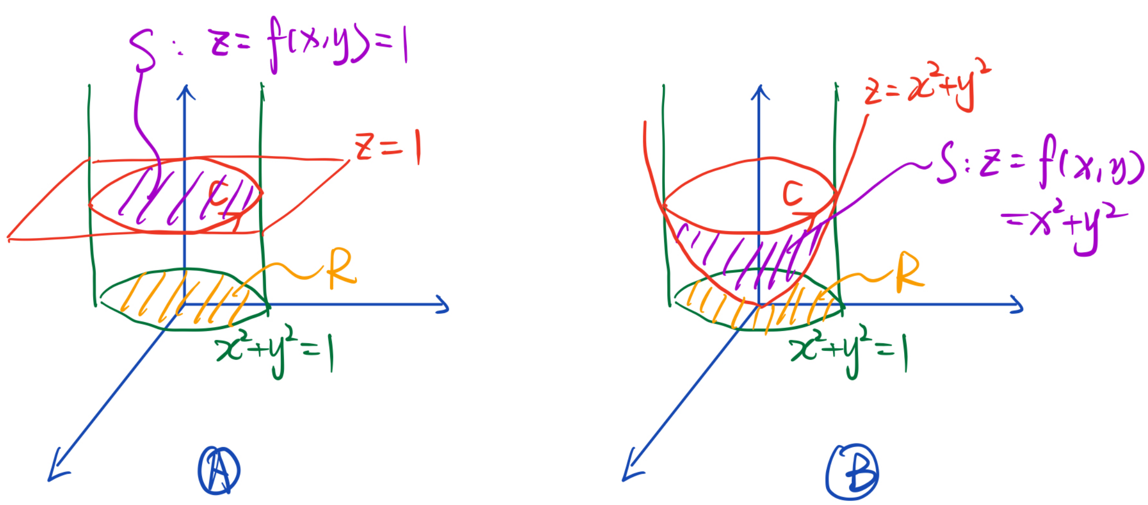
But S is not given, we need to find $S: z = f(x,y)$

Idea: construct 2 surfaces whose *intersection* is the curve C.

From C: $\begin{cases} x(t) = \cos t \\ y(t) = \sin t \\ z(t) = 1\end{cases}$, we can construct 2 surfaces by observing the relationship among x, y, z, for example,

$$\begin{cases} x^2+y^2 = 1 \\ z = 1\end{cases} \qquad or \qquad \begin{cases} x^2+y^2 = 1 \\ z = x^2+y^2\end{cases}$$

Their graphs are shown below:



We can see that for the first equation, the surface S is a circle, while for the second equation, the surface S is a "rice bowl". Either of them is ok for our calculation.

(1) Firstly, find the curl of vector field:

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y & x^2 & y \end{vmatrix} = \mathbf{i} + (2x-1)\mathbf{k}$$

(2) Next, find normal vector to the surface,

For (A), $z = f(x, y) = 1$, hence $\widehat{\mathbf{n}} = \mathbf{k}$.

For (B), let $G(x, y, z) = z - x^2 - y^2 = 0$(constant), this is a level set in 3D, hence

$$\mathbf{n} = \nabla G = (-2x, -2y, 1),\ \widehat{\mathbf{n}} = \frac{(-2x, -2y, 1)}{\sqrt{1 + 4x^2 + 4y^2}}$$

(3) Then, find surface integral, and thereby calculating the result:

For (A), $ds = \sqrt{1 + (f_x)^2 + (f_y)^2}dA = dA$, therefore,

$$\begin{aligned} \iint_S (\nabla \times F) \cdot \widehat{\mathbf{n}}\ dS &= \iint_R (1, 0, 2x-1) \cdot (0, 0, 1)\ dA \\ &= \iint_R (2x-1)\ dA \\ &= -\iint_R\ dA = -\pi \qquad \textcolor{blue}{(2x \textbf{ is odd in } x, \ \ \textbf{and the region is symmetric w.r.t } y)} \end{aligned}$$

For (B), $ds = \sqrt{1 + (f_x)^2 + (f_y)^2}dA = \sqrt{1 + 4x^2 + 4y^2}dA$, therefore,

$$\begin{aligned} \iint_S (\nabla \times F) \cdot \widehat{\mathbf{n}}\ dS &= \iint_R (1, 0, 2x-1) \cdot \frac{(-2x, -2y, 1)}{\sqrt{1 + 4x^2 + 4y^2}} \cdot \sqrt{1 + 4x^2 + 4y^2}dA \\ &= \iint_R (-2x + 2x - 1)\ dA \\ &= -\iint_R\ dA = -\pi \end{aligned}$$

[Example.] Evaluate $\int_C (y+\sin x)dx + \left(z^2+\cos y\right)dy + x^3 dz$, where $C: \ \mathbf{r}(t) = (\sin t, \cos t, \sin 2t), \quad 0 \leqslant t \leqslant 2\pi$

[Solution.] Note that C is a **closed space curve**, we can view it as circular integration on vector field:

$$\oint_C (y+\sin x)dx + (z^2+\cos y)dy + x^3 dz = \oint_C \mathbf{F}\cdot d\mathbf{r}$$

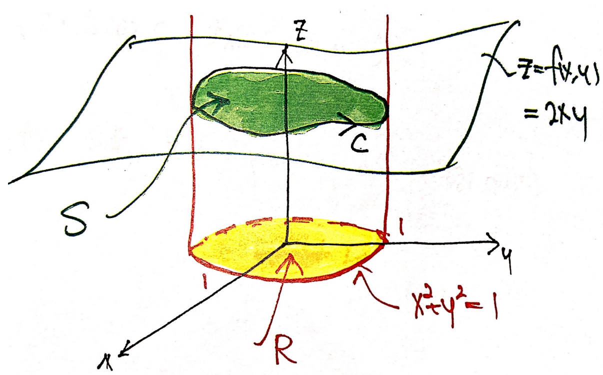
where $\mathbf{F}(x,y,z) = (y+\sin x, z^2+\cos y, x^3)$

Step 1: Find curl of vector field: $\nabla\times\mathbf{F} = \left(-2z, -3x^2, -1\right)$

Step 2: To apply Stokes' Theorem, we need to find a surface S,

From C: $\begin{cases} x(t) = \sin t \\ y(t) = \cos t \\ z(t) = \sin 2t = 2\sin t\cos t \end{cases} \Rightarrow \begin{cases} x^2+y^2 = 1 \\ z = 1 \end{cases}$

So C can be viewed as the intersection of two surfaces $x^2+y^2=1$ and $z = 2xy = f(x,y)$, and $z=2xy$ is the S we need, while $R: \ x^2+y^2=1$ is the the projection of S onto xy-plane, which we will need in surface integral.



Step 3: find the normal to S:

$f(x,y,z) = z - 2xy = 0$(constant) is a level set in 3D, so

$$\mathbf{n} = \nabla f = (-2y, -2x, 1), \qquad \widehat{\mathbf{n}} = \frac{(-2y,-2x,1)}{\sqrt{1+4y^2+4x^2}}$$

Step 4: find surface integral: $dS = \sqrt{1+(f_x)^2+(f_y)^2}\,dA = \sqrt{1+4y^2+4x^2}\,dA$ Therefore,

$$\begin{aligned}\oint_C \mathbf{F}\cdot d\mathbf{r} &= \iint_S \nabla\times\mathbf{F}\cdot\widehat{\mathbf{n}}dS \\ &= \iint_R \left(-4xy, -3x^2, -1\right)\cdot\frac{(-2y,-2x,1)}{\sqrt{1+4y^2+4x^2}}\cdot\sqrt{1+4y^2+4x^2}\,dA \\ &= \iint_R \left(8xy^2+6x^3-1\right)dA \\ &= -\iint_R dA = -\pi \qquad \textit{(same trick again)}\end{aligned}$$

This is the end of Chapter 16, and the end of this course!