

MATH 2023 Fall 2021

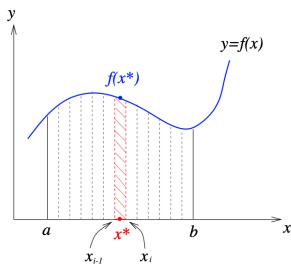
Multivariable Calculus

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Chapter 14 Multiple Integrations

1 Double Integrals Over Rectangles

Recall that in single variable calculus, we divided a region into thin rectangles and use the **Riemann Sum** as integral.

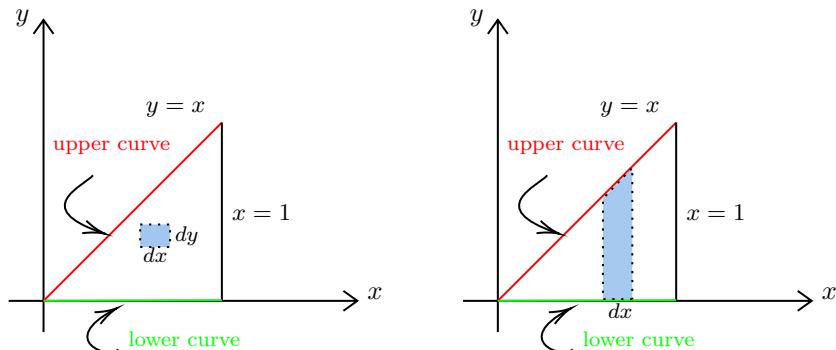


$$\int_a^b f(x)dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*)\delta x_i$$

Actually, instead of thin rectangles, we can use *small rectangles* to cover the area.

[Example.] Find the area bounded by $y = x$, $x = 1$ and $y = 0$.

[Solution.] (1) View the area as bounded by *upper curve* and *lower curve*.

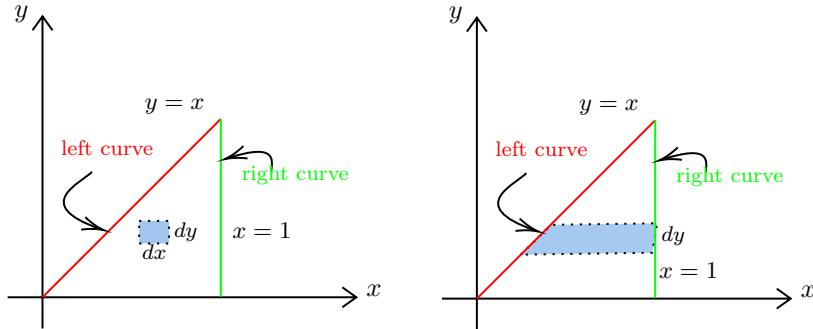


For the *small rectangles* with $dA = dxdy = dydx$, we first move it vertically, the lower bound is $y = 0$, and the upper bound is $y = x$, hence $\int_0^x dy$ is the shaded area in right image above.

Then we move the shaded area horizontally, the left bound is point $x = 0$, while the right bound is point $x = 1$, thus the total area is:

$$A = \int_0^1 \int_0^x dy dx = \int_0^1 y \Big|_0^x dx = \int_0^1 x dx = \frac{1}{2}$$

(2) Alternatively, we can first move the rectangle horizontally, hitting *left curve* $x = y$ and *right curve* $x = 1$, hence the shaded area is $\int_y^1 dx$. note here we integral dx first, so when hitting the boundaries, we need to check x equals to what, i.e. $x = f(y)$. For example, here the two bounds are $x = y$ and $x = 1$.



Then we move the shaded area vertically, hitting lower bound $y = 0$ (a point) and upper bound $y = 1$ (a point), hence the total area is:

$$A = \int_0^1 \int_y^1 dx dy = \int_0^1 x \Big|_y^1 dy = \int_0^1 (1 - y) dy = \frac{1}{2}$$

2 Double Integrals Over General Regions

3 Double Integrals in Polar Coordinates

4 Change of Variables in Integrals

Recall that in single variable calculus, we often use a *substitution* to simplify an integral.

$$\int_a^b f(x)dx = \int_c^d f(g(u)) \cdot g'(u) du$$

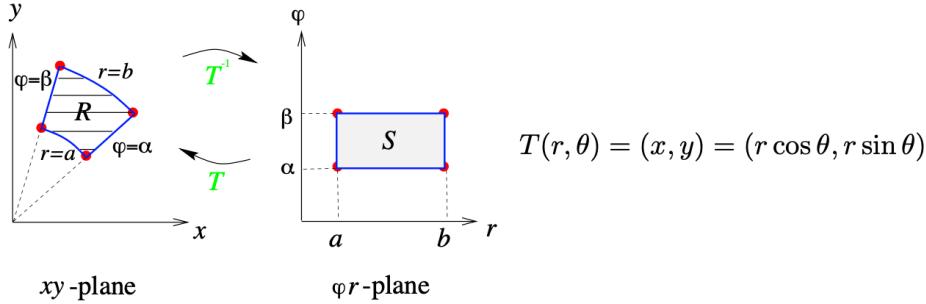
where $a = g(c)$ and $b = g(d)$. Notice that we can *view substitution as a kind of mapping*, and the change-of-variable process introduces *an additional factor* $g'(u)$ into the integrand.

This method can also be useful in multiple integrals. We have already seen one example: integration in *polar coordinate*.

$$\iint_R f(x, y)dA_{xy} = \iint_S f(r \cos \theta, r \sin \theta)r dr d\theta = \iint_S f(r \cos \theta, r \sin \theta)r dA_{r\theta}$$

In this example, *the additional factor* is r .

The *mapping T* is shown as below: we transform the region R into S , where S is an rectangle in θr -plane, which is easy to integrate.



[Example.] Find a change of variable.

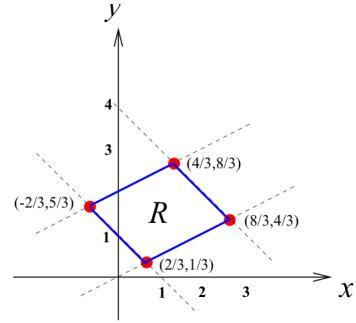
.1 Let R be the region bounded by the lines

$$x - 2y = 0$$

$$x - 2y = -4$$

$$x + y = 4$$

$$x + y = 1$$



as shown. Find a transformation T from a region S to R such that S is a rectangular region (with sides parallel to the u - and v -axis).

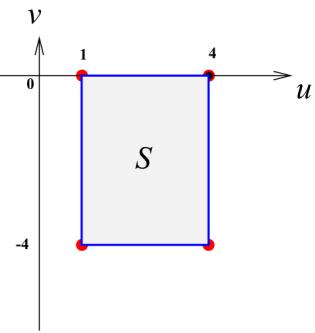
Let $u = x + y$, $v = x - 2y$, then $T(u, v) = (x, y) = \left(\frac{1}{3}(2u + v), \frac{1}{3}(u - v) \right)$.

$$v = 0$$

$$v = -4$$

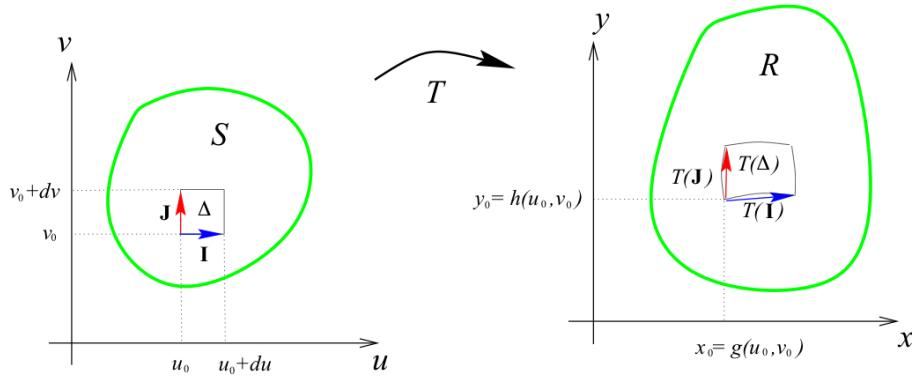
$$u = 4$$

$$u = 1.$$



Note that the transformation T maps the vertices of the region S onto the vertices of the region R .

Now, to find $\iint_R f(x, y) dxdy$, if we make the change of variables $x = g(u, v)$, $y = h(u, v)$, then we are mapping things in uv -plane onto things in xy -plane. For mapping function $T(u, v) = (g(u, v), h(u, v)) = (x, y)$, and assume the area S in uv -plane corresponds to region R in xy -plane, as shown below.



We still use the method that integrate all “small rectangles”, Δ , as shown in uv -plane. Assume Δ locates at (u_0, v_0) and has area $dA = dudv$. Let

I be the vector from (u_0, v_0) to $(u_0 + du, v_0)$ and

J be the vector from (u_0, v_0) to $(u_0, v_0 + dv)$.

Then mapping T “takes” \mathbf{I} to the vector $T(\mathbf{I})$ from $(g(u_0, v_0), h(u_0, v_0))$ to $(g(u_0 + du, v_0), h(u_0 + du, v_0))$. Notice the vector $T(\mathbf{I})$ is *not necessarily a straight vector*. Now

$$\begin{aligned} T(\mathbf{I}) &= (g(u_0 + du, v_0) - g(u_0, v_0), h(u_0 + du, v_0) - h(u_0, v_0)) \\ &= \left(\frac{g(u_0 + du, v_0) - g(u_0, v_0)}{du}, \frac{h(u_0 + du, v_0) - h(u_0, v_0)}{du} \right) du \\ &= \left(\frac{\partial g}{\partial u}, \frac{\partial h}{\partial u} \right) du \quad (\text{for } du \rightarrow 0) \end{aligned}$$

Similarly, $T(\mathbf{J}) = \left(\frac{\partial g}{\partial v}, \frac{\partial h}{\partial v} \right) dv$.

Then the area of $T(\Delta)$ is

$$dxdy = \|T(\mathbf{I}) \cdot T(\mathbf{J})\| = \left| \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} dudv \right| = \left| \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u} \right| dudv$$

which means, $dA_{xy} = |J| dA_{uv}$, where $|J|$ is the “additional factor” caused by this substitution, and it is called the **Jacobian** of mapping T , given by:

$$T = \frac{\partial(x, y)}{\partial(u, v)} = \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

Therefore, the formula of *change of variable for two variables* is:

$$\iint_{R=T(S)} f(x, y) dxdy = \iint_S f(T(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| dudv$$

while for *three variables*,

$$\iiint_{R=T(S)} f(x, y, z) dxdydz = \iiint_S f(T(u, v, w)) \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| dudvdw$$

This example shows substitution can be easy for some integrations.

[Example.] Find the area of $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

[Solution.] **Method 1:** directly integrate

$$\begin{aligned}\frac{1}{4}A &= \int_0^a \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} dy dx \\ &= \int_0^a b\sqrt{1-\frac{x^2}{a^2}} dx\end{aligned}$$

We are familiar with substitution in single variable integration, let $x = a \sin \theta$, when $x = 0$, $\theta = 0$, and when $x = a$, $\theta = \frac{\pi}{2}$. Then,

$$\begin{aligned}\frac{1}{4}A &= \int_0^{\frac{\pi}{2}} b(1 - \sin^2 \theta)^{\frac{1}{2}} a \cos \theta d\theta \\ &= ab \int_0^{\frac{\pi}{2}} \cos^2 \theta d\theta \\ &= ab \int_0^{\frac{\pi}{2}} \frac{1}{2}(1 - \cos 2\theta) d\theta \\ &= \frac{1}{4}ab\pi\end{aligned}$$

Method 2: Mapping the ellipse to a disk.

Let $x = au$, $y = bv$, then $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ becomes $u^2 + v^2 = 1$.

The Jacobian of this mapping

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} a & 0 \\ 0 & b \end{vmatrix} = ab$$

Therefore, the area of ellipse

$$\iint_R dA_{xy} = \iint_S J \cdot dA_{xy} = \iint_S ab dA_{uv} = ab\pi$$

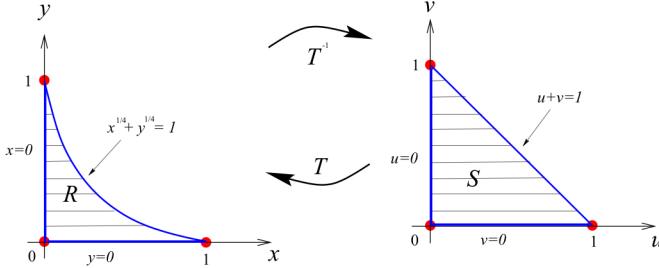
We observe that method 2 is much easier than method 1.

[Example.] Find the area bounded by $\sqrt[4]{x} + \sqrt[4]{y} = 1$ and the x and y axes.

[Solution.]

This integral would be tedious to evaluate directly because the region R is not ‘simple’. So instead we find a suitable transformation of variables. Let

$$\text{Let } u = \sqrt[4]{x}, v = \sqrt[4]{y}, \text{ then } x = u^4, y = v^4 \text{ and } \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} 4u^3 & 0 \\ 0 & 4v^3 \end{vmatrix} = 16u^3v^3$$



$$\text{Area} = \iint_R dxdy = \iint_S \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du dv = \int_0^1 \int_0^{1-v} 16u^3v^3 du dv = \frac{1}{70}$$

Sometimes, it's not easy to calculate $\frac{\partial(x, y)}{\partial(u, v)}$, since usually we substitute u and v as functions of x and y , so we always need to find the inverse function in order to calculate $\frac{\partial(x, y)}{\partial(u, v)}$. So we consider the relationship between $\frac{\partial(x, y)}{\partial(u, v)}$ and $\frac{\partial(u, v)}{\partial(x, y)}$.

Because of $\det A \det B = \det(AB)$,

$$\begin{aligned} \frac{\partial(x, y)}{\partial(u, v)} \cdot \frac{\partial(u, v)}{\partial(x, y)} &= \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \\ &= \begin{vmatrix} \frac{\partial x}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial x}{\partial v} \frac{\partial v}{\partial x} & \frac{\partial x}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial x}{\partial v} \frac{\partial v}{\partial y} \\ \frac{\partial y}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial y}{\partial v} \frac{\partial v}{\partial x} & \frac{\partial y}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial y}{\partial v} \frac{\partial v}{\partial y} \end{vmatrix} \\ &= \begin{vmatrix} \frac{\partial x}{\partial x} & \frac{\partial x}{\partial y} \\ \frac{\partial y}{\partial x} & \frac{\partial y}{\partial y} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \end{aligned}$$

Therefore, if $x(u, v)$ and $y(u, v)$ have continuous first partial derivatives, and that

$$\frac{\partial(x, y)}{\partial(u, v)} \neq 0 \quad \text{at} \quad (u, v) \quad (\text{one-to-one map}).$$

Then

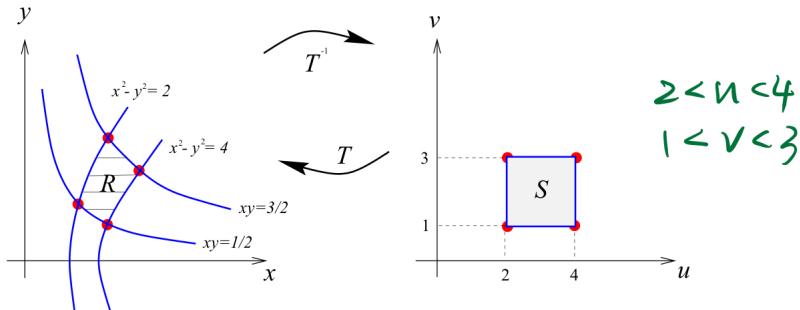
$$\boxed{\frac{\partial(x, y)}{\partial(u, v)} = \frac{1}{\frac{\partial(u, v)}{\partial(x, y)}}}$$

This example shows the situation when $\frac{\partial(x,y)}{\partial(u,v)}$ is difficult to compute.

[Example.] Find $\iint_A (x^2 + y^2) dx dy$, where $A = \{(x,y) \mid x, y > 0, \quad 2 \leq x^2 - y^2 \leq 4, \quad \frac{1}{2} \leq xy \leq \frac{3}{2}\}$

[Solution.]

The change of the variables is motivated by the occurrence of the expressions $x^2 - y^2$ and xy in the equations of the boundary.



Let $u = x^2 - y^2$, $v = 2xy$, then $(x^2 + y^2)^2 = (x^2 - y^2)^2 + 4x^2y^2 = u^2 + v^2$ and

difficult to calculate $\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{\frac{\partial(u,v)}{\partial(x,y)}} = \frac{1}{\begin{vmatrix} 2x & -2y \\ 2y & 2x \end{vmatrix}} = \frac{1}{4(x^2 + y^2)} = \frac{1}{4\sqrt{u^2 + v^2}}.$

So $\iint_R (x^2 + y^2) dx dy = \int_{v=1}^{3} \int_{u=2}^{4} \sqrt{u^2 + v^2} \cdot \frac{1}{4\sqrt{u^2 + v^2}} du dv$

Sometimes, though the given region is a relatively good one, but it's still difficult to directly integrate, maybe because the integrand is too complicated. See the below example:

[Example.]

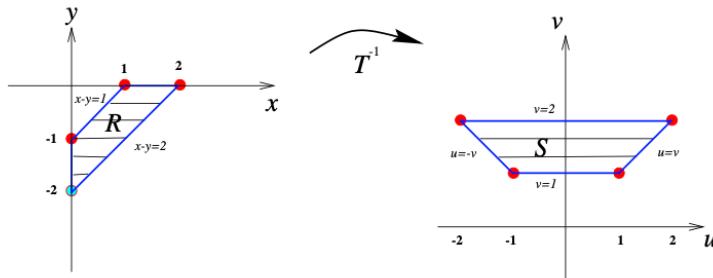
Evaluate the integral $\iint_R e^{(x+y)/(x-y)} dA$, where R is the trapezoidal region with vertices $(1, 0)$, $(2, 0)$, $(0, -2)$, and $(0, -1)$.

Since it is not easy to integrate $e^{(x+y)/(x-y)}$, we make a change of variables suggested by a form of the integrand. In particular, let

$$u = x + y, \quad v = x - y.$$

These equations define a transformation T^{-1} from the xy -plane to the uv -plane.

$$\frac{\partial(x, y)}{\partial(u, v)} = \frac{1}{\frac{\partial(u, v)}{\partial(x, y)}} = \frac{1}{\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}} = -\frac{1}{2}.$$



The sides of R lie on the lines

$$y = 0, \quad x - y = 2, \quad x = 0, \quad x - y = 1$$

and the image lines in the uv -plane are

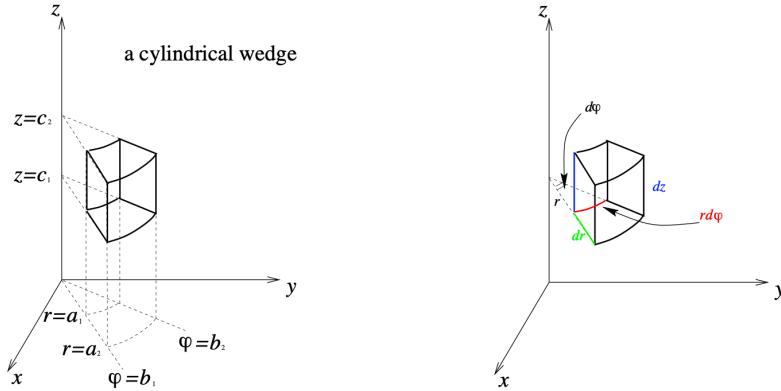
$$u = v, \quad v = 2, \quad u = -v, \quad v = 1.$$

$$\begin{aligned} \therefore \iint_R e^{(x+y)/(x-y)} dA &= \iint_S e^{u/v} \left| \frac{\partial(x, y)}{\partial(u, v)} \right| dudv \\ &= \int_1^2 \int_{-v}^v e^{u/v} \left(\frac{1}{2} \right) dudv \\ &= \end{aligned}$$

5 Triple Integrals in Cylindrical & Spherical Coordinates

Cylindrical coordinates are suited to problems *with axial symmetry*(the shape is around the z -axis)

The basic unit of cylindrical coordinate is shown below, and it's volume is $r drd\theta dz$



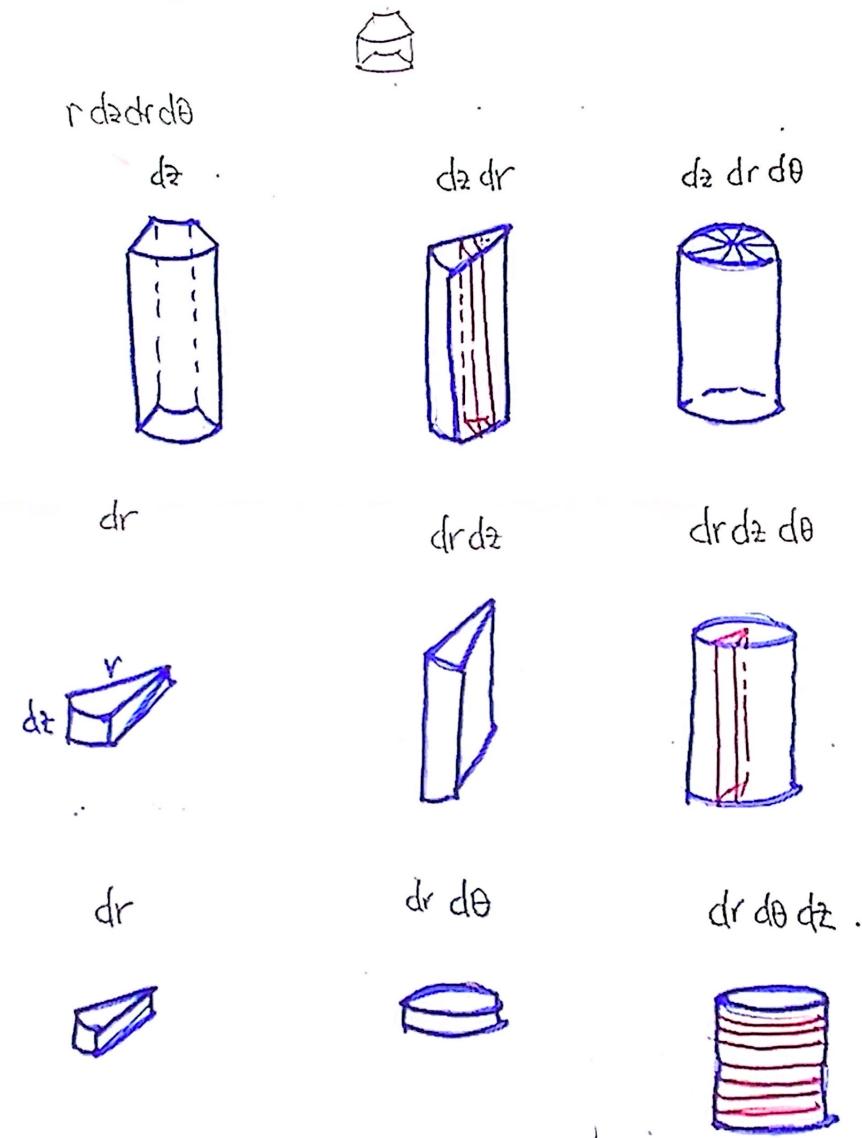
We can also find Jacobian in order to find the relation. For $x = r \cos \theta, y = r \sin \theta, z = z$, we have

$$\frac{\partial(x, y, z)}{\partial(r, \theta, z)} = \begin{vmatrix} \cos \theta & -r \sin \theta & 0 \\ \sin \theta & r \cos \theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r$$

that is, $dxdydz \rightarrow r drd\theta dz$

$$V_c = \iiint_V f dV = \iiint_{V(x,y,z)} f(x, y, z) dx dy dz = \iiint_{V(r,\theta,z)} f(r, \theta, z) r dr d\theta dz$$

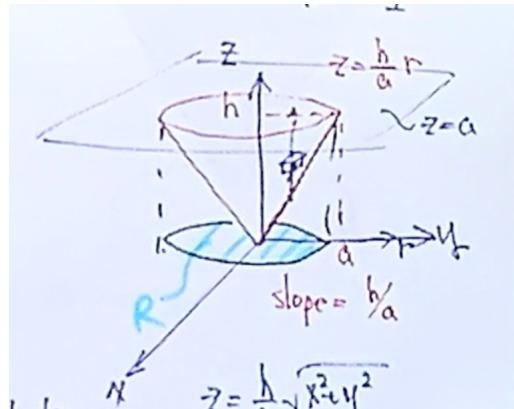
Moreover, same as before, we can freely change the *order of integration*. Below show three examples.



This example shows integration in cylindrical coordinates brings convenience sometimes.

[Example.] Find the volume of a circular cone with altitude h and a base of radius a .

[Solution.] To find the volume of the cone, we need to know the equation of its surface.



$$\text{In } (x, y, z), z = \frac{h}{a}(x^2 + y^2)^{1/2}. \quad \text{In } (r, \theta, z), z = \frac{h}{a}r.$$

Method 1: Do the question by (x, y, z) , and $dV = dz dy dx$

$$V = 4 \int_0^a \int_0^{\sqrt{a^2 - x^2}} \int_{\frac{h}{a}(x^2 + y^2)^{1/2}}^h dz dy dx$$

Method 2: Do the question by (r, θ, z) , and $dV = r dz dr d\theta$

$$V = \int_0^{2\pi} \int_0^a \int_{\frac{h}{a}r}^h r dz dr d\theta = \frac{1}{3}\pi a^2 h$$

This example provides interpretation for triple integrals.

[Example.] Sketch the solid whose volume is given by the integral.

$$(a) \int_0^{2\pi} \int_0^2 \int_0^{4-r^2} r \, dz \, dr \, d\theta.$$

$$(b) \int_1^3 \int_0^{\frac{\pi}{2}} \int_r^3 r \, dz \, d\theta \, dr.$$

[Solution.] Recall that for $\int_a^b \int_{g_1(x)}^{g_2(x)} \int_{\phi_1(x)}^{\phi_2(x)} dz \, dy \, dx$, we can infer the volume of integration based on these six limits:

$z = \phi_1$ to $z = \phi_2$, (surfaces)

$y = g_1$ to $y = g_2$, (curves)

$x = a$ to $x = b$, (points)

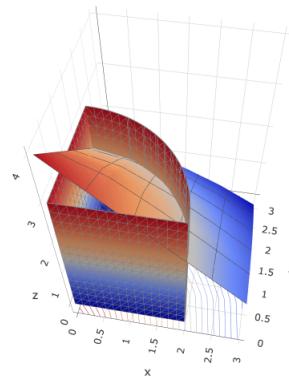
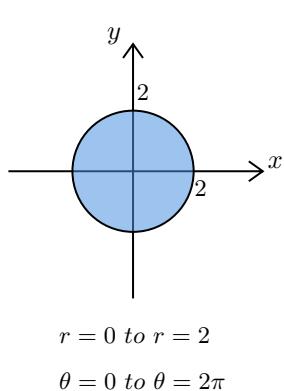
We can interpret the last two as “area on the xy -plane”.

(a)

$z : z = 0$ to $z = 4 - r^2$, (surfaces)

$$\left. \begin{array}{l} r : r = 0 \text{ to } r = 2 \\ \theta : \theta = 0 \text{ to } \theta = 2\pi \end{array} \right\} \text{(area on } xy\text{-plane)}$$

The area on xy -plane is shown below left. We move the disk upward, hitting $z = 4 - r^2$, and the volume bounded is what we are looking for, as shown below right.

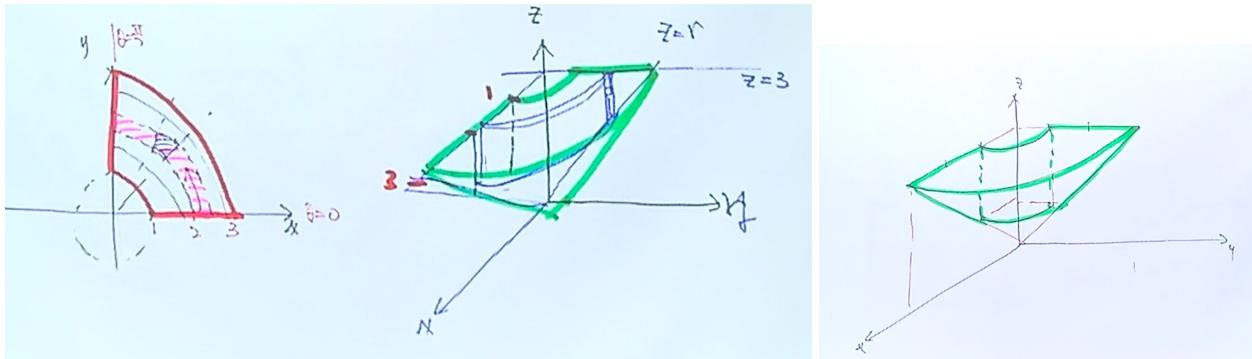


(b)

$z : z = r$ to $z = 3$, (surfaces)

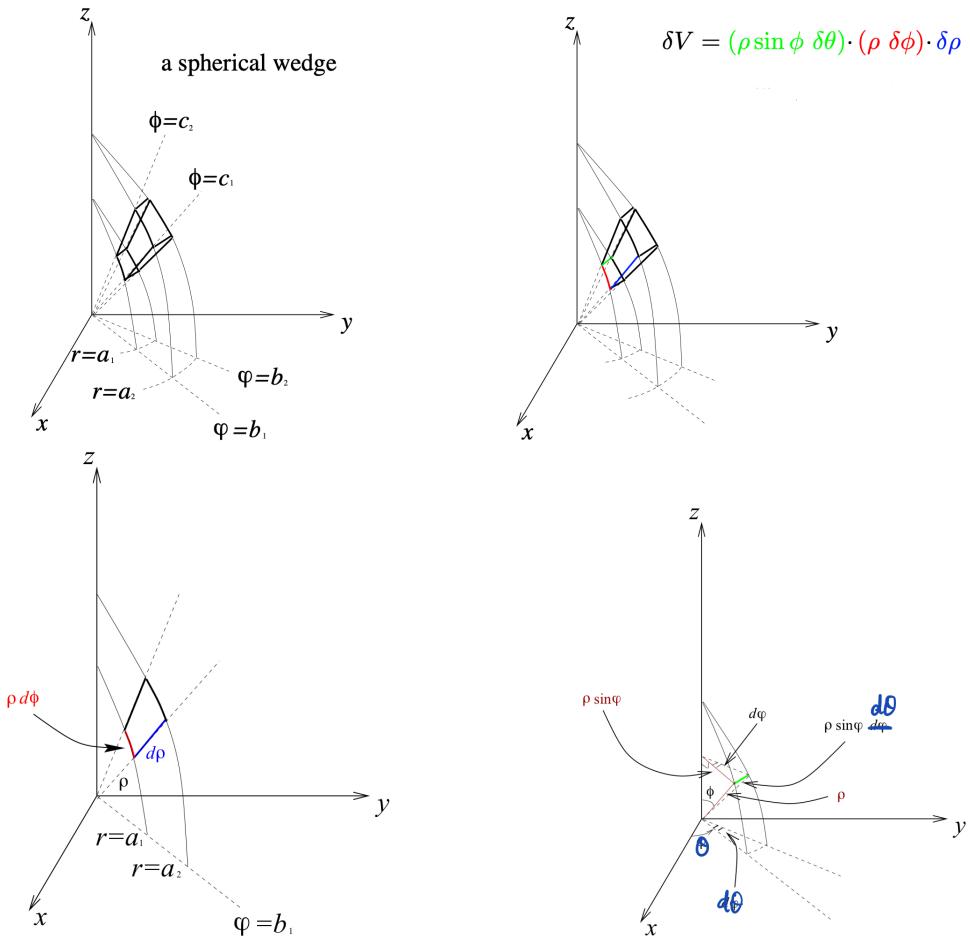
$$\left. \begin{array}{l} \theta : \theta = 0 \text{ to } \theta = \frac{\pi}{2} \\ r : r = 1 \text{ to } r = 3 \end{array} \right\} \text{(area on } xy\text{-plane)}$$

The procedure is similar to (a), we first draw the region on xy -plane, and move it from surface $z = r$ upwards to $z = 3$. The bounded region is what we're looking for.



Spherical coordinates are suited to problems *involving spherical symmetry*, and in particular, to regions bounded by *spheres* centered at the origin, *circular cones* with axes along the z -axis, *vertical planes* containing z -axis.

The basic unit of spherical coordinate is shown below, and it's volume is $dV = \rho^2 \sin \phi \, d\theta \, d\rho \, d\phi$



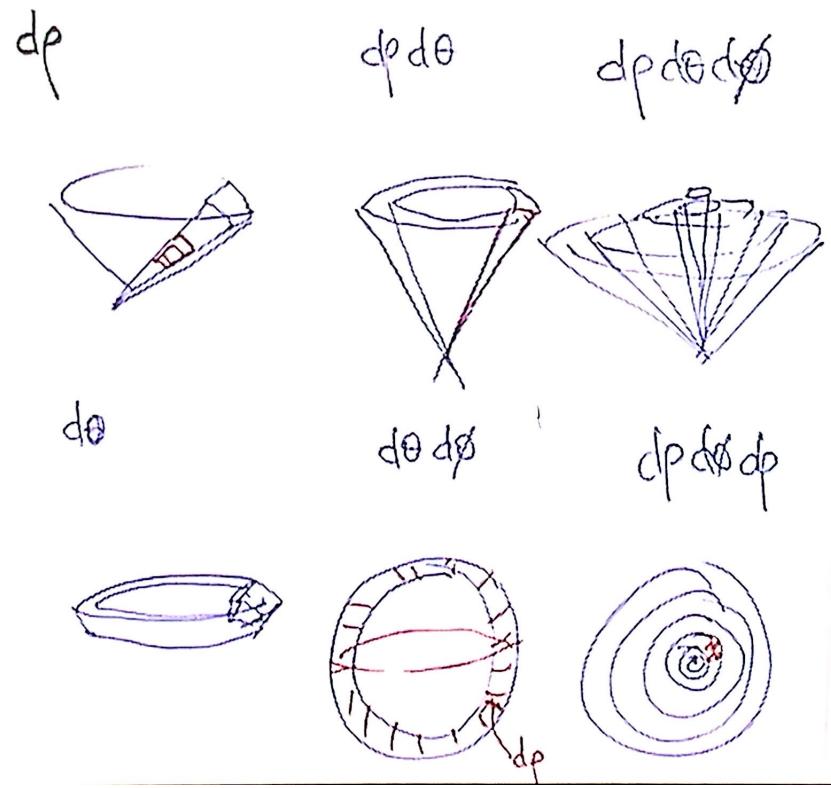
We can also find Jacobian in order to find the relation.

For spherical coordinates $x = \rho \sin \phi \cos \theta$, $y = \rho \sin \phi \sin \theta$ and $z = \rho \cos \phi$, then

$$\frac{\partial(x, y, z)}{\partial(\rho, \phi, \theta)} = \begin{vmatrix} \sin \phi \cos \theta & \rho \cos \phi \cos \theta & -\rho \sin \phi \sin \theta \\ \sin \phi \sin \theta & \rho \cos \phi \sin \theta & \rho \sin \phi \cos \theta \\ \cos \phi & -\rho \sin \phi & 0 \end{vmatrix} = \rho^2 \sin \phi$$

i.e. $dxdydz = \rho^2 \sin \phi d\rho d\theta d\phi$.

Again, different integration orders have different interpretations.



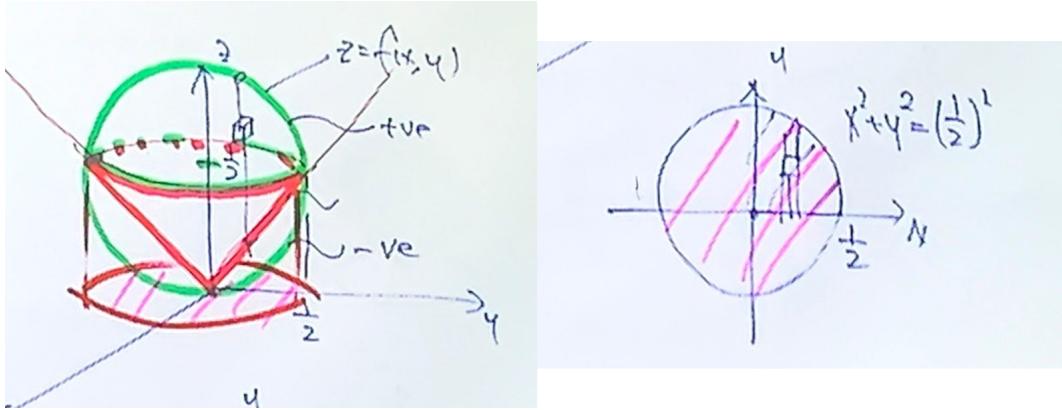
This example shows the choice of coordinate system can greatly affect the difficulty of computation of a multiple integral.

[Example.] Find the volume of the solid that lies above the cone $z = \sqrt{x^2 + y^2}$ and between the sphere $x^2 + y^2 + z^2 = z$.

[Solution.] The equation of sphere can be written as

$$x^2 + y^2 + \left(z - \frac{1}{2}\right)^2 = \left(\frac{1}{2}\right)^2 \Rightarrow \text{centre } \left(0, 0, \frac{1}{2}\right), \text{ radius} = \frac{1}{2}$$

Method 1: Use Cartesian Coordinate:

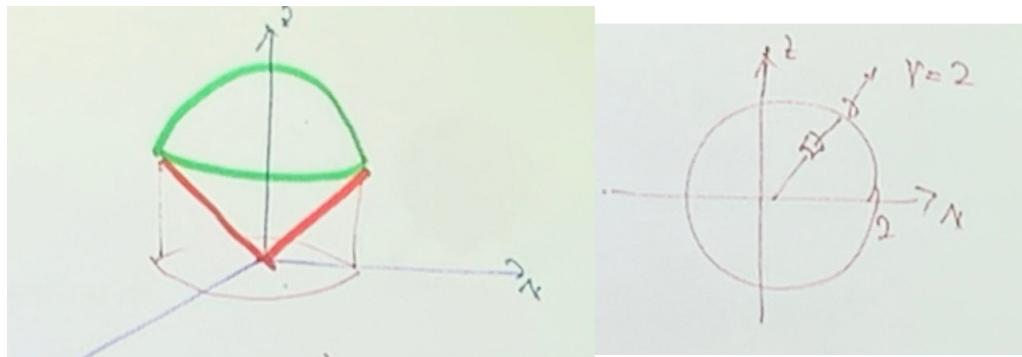


Cone: $z = \sqrt{x^2 + y^2}$

Sphere: $z = \frac{1}{2} + \sqrt{\frac{1}{4} - x^2 - y^2}$

$$V = 4 \int_0^{\frac{1}{2}} \int_0^{\sqrt{\frac{1}{4}-x^2}} \int_{\sqrt{x^2+y^2}}^{\frac{1}{2}+\sqrt{\frac{1}{4}-x^2-y^2}} dz dy dx$$

Method 2: Use Cylindrical Coordinate:

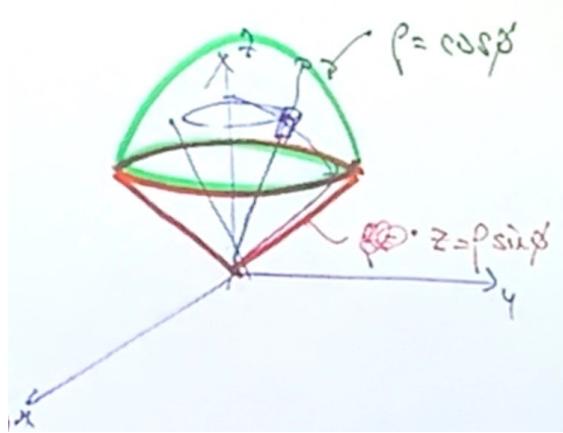


Cone: $z = \sqrt{x^2 + y^2} = r$

Sphere: $z = \frac{1}{2} + \sqrt{\frac{1}{4} - x^2 - y^2} = \frac{1}{2} + \sqrt{\frac{1}{4} - r^2}$

$$V = \int_0^{2\pi} \int_0^2 \int_r^{\frac{1}{2} + \sqrt{\frac{1}{4} - r^2}} r \, dz \, dr \, d\theta$$

Method 3: Use Spherical Coordinate:



Cone: $z = \sqrt{x^2 + y^2} = \rho \sin \phi$

Sphere: $x^2 + y^2 + z^2 = z \Rightarrow \rho^2 = \rho \cos \phi \Rightarrow \rho = \cos \phi$, (since $\rho \neq 0$)

$$dV = \int_0^{\frac{\pi}{4}} \int_0^{2\pi} \int_0^{\cos \phi} \rho^2 \sin \phi \, d\rho \, d\theta \, d\phi$$

This example shows the situation when we need to integrate two parts.

[**Example.**] Find the volume of a cylinder with radius a and height h .

[**Solution.**] **Method 1:** Use Cylindrical Coordinate:

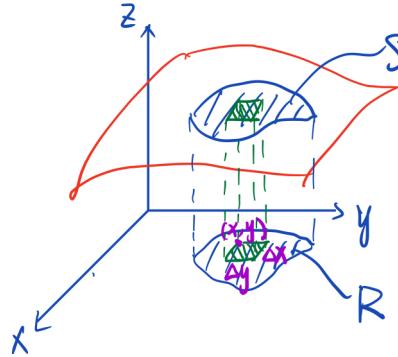
$$V = \int_0^{2\pi} \int_0^a \int_0^h r \, dz \, dr \, d\theta$$

This is really simple by cylindrical coordinate, but things become different when using spherical.

Method 2: Use Spherical Coordinate:

6 Surface Area

We now want to find the area of a surface. Finding an area on xy -plane is relatively easy, as we have discussed early this chapter, but things become much more complicated when we are focusing on an arbitrary surface. So, we think about *projecting the area onto xy -plane*.

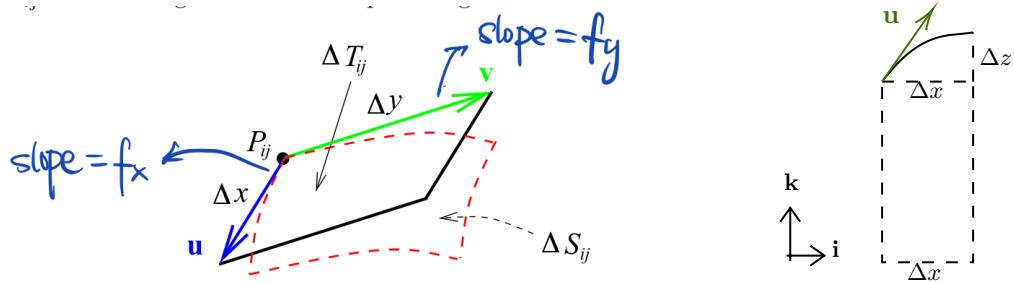


As shown above, we want to find the area of region S in a surface. The first thing to do is to project it onto xy -plane, resulting in a region R .

As usual, we use small rectangles to cover region R , as the green rectangle shown above, assume two sides are Δx and Δy , so the area of green rectangle is $\Delta A = \Delta x \Delta y$.

Then we project the rectangle up to the surface S , resulting in a “curved-parallelogram” surface, shown as red area in left-below image. To find this area, we know as long as Δx and Δy are small enough, the black parallelogram formed by Δx and Δy is a good approximation for that area. By the way, the area of parallelogram is $\mathbf{u} \times \mathbf{v}$.

How to represent \mathbf{u} and \mathbf{v} ? See the right-below image, the slope of vector \mathbf{u} is $f_x = \frac{\Delta z}{\Delta x}$, so the width of \mathbf{u} is Δx and the height of \mathbf{u} is $\Delta z = \Delta x \cdot f_x$. (Notice this image is graphed vertically, i.e., in xz -plane)



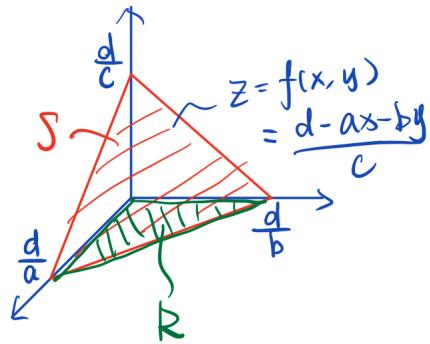
Therefore, $\mathbf{u} = \Delta x \mathbf{i} + 0 \mathbf{j} + \Delta x \cdot f_x \mathbf{k}$, similarly, $\mathbf{v} = 0 \mathbf{i} + \Delta y \mathbf{j} + \Delta y \cdot f_y \mathbf{k}$. Then

$$\|\mathbf{u} \times \mathbf{v}\| = \| -\Delta x \Delta y f_x \mathbf{i} - \Delta x \Delta y f_y \mathbf{j} + \Delta x \Delta y \mathbf{k} \| = \sqrt{(f_x)^2 + (f_y)^2 + 1} \cdot \Delta x \Delta y$$

When $\Delta x, \Delta y \rightarrow 0$, the area of S is given by:

$$A = \iint_R \sqrt{(f_x)^2 + (f_y)^2 + 1} \, dA$$

[Example.] Given a plane $ax + by + cz = d$, where $a, b, c, d > 0$. Find the area of the triangle bounded by the intersections of the plane and axes.(As the red shaded area shown)



[Solution.] The equation of surface $z = f(x, y)$ is given by $z = \frac{d - ax - by}{c}$.

To find the red area, we first *project it onto* xy -plane, resulting in green area R .

Thus the red area

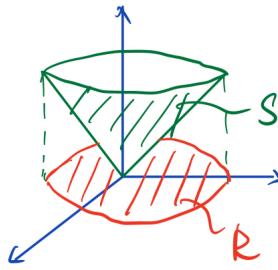
$$S = \iint_R \sqrt{1 + f_x^2 + f_y^2} dA_{xy}$$

We know that $f_x = -\frac{a}{c}$, $f_y = -\frac{b}{c}$, so $\sqrt{1 + f_x^2 + f_y^2} = \frac{\sqrt{a^2 + b^2 + c^2}}{c}$, then

$$\begin{aligned} S &= \iint_R \frac{\sqrt{a^2 + b^2 + c^2}}{c} dA_{xy} \\ &= \frac{\sqrt{a^2 + b^2 + c^2}}{c} \cdot (\text{area of } R) \\ &= \frac{\sqrt{a^2 + b^2 + c^2}}{c} \cdot \frac{1}{2} \cdot \frac{d}{a} \cdot \frac{d}{b} \\ &= \frac{d^2 \sqrt{a^2 + b^2 + c^2}}{2abc} \end{aligned}$$

Notice the blue part is a constant.

[Example.] Find the surface area of the cone $z = \frac{h}{a}r$ (in cylindrical coordinate).



[Solution.] Project the cone onto xy -plane, resulting in red area R .

The surface is given by $z = f(x, y) = \frac{h}{a} \sqrt{x^2 + y^2}$

Thus the green area

$$S = \iint_R \sqrt{1 + f_x^2 + f_y^2} dA_{xy}$$

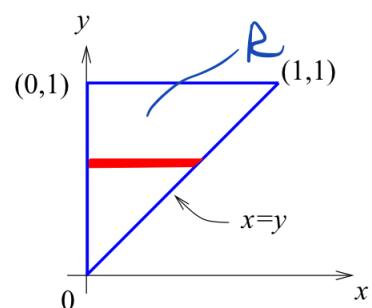
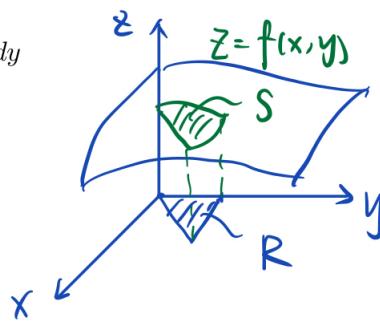
We know that $f_x = \frac{h}{a} \cdot \frac{x}{\sqrt{x^2 + y^2}}$, $f_y = \frac{h}{a} \cdot \frac{y}{\sqrt{x^2 + y^2}}$, so $\sqrt{1 + f_x^2 + f_y^2} = 1 + \frac{h^2}{a^2}$, then

$$\begin{aligned} S &= \iint_R \sqrt{a + \frac{h^2}{a^2}} dA_{xy} \\ &= \sqrt{a + \frac{h^2}{a^2}} \cdot (\text{Area of circle with radius } a) \\ &= \pi a \cdot \sqrt{a^2 + h^2} \end{aligned}$$

[Example.] Find the area of the surface $z = x + y^2$ that lies above the triangle with vertices $(0,0)$, $(1,1)$ and $(0,1)$.

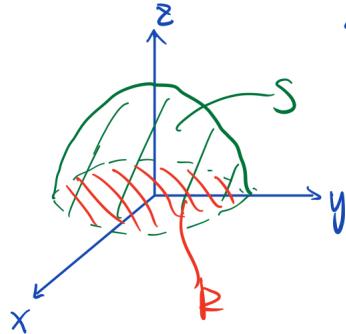
[Solution.]

$$\begin{aligned} S &= \iint_R \sqrt{z_x^2 + z_y^2 + 1} dA = \int_0^1 \int_0^y \sqrt{1 + 4y^2 + 1} dx dy \\ &= \int_0^1 y \sqrt{2 + 4y^2} dy \\ &= \frac{2}{24} (2 + 4y^2)^{\frac{3}{2}} \Big|_0^1 \\ &= \frac{1}{6} (3\sqrt{6} - \sqrt{2}). \end{aligned}$$



This example uses substitution while evaluating integral.

[Example.] Find the surface of a sphere with radius a .



[Solution.] Again, project S onto xy -plane to get region R .

The equation of surface is given by $z = f(x, y) = \sqrt{a^2 - x^2 - y^2}$,

The green area:

$$S = \iint_R \sqrt{1 + f_x^2 + f_y^2} dA_{xy}$$

We know that $f_x = \frac{-x}{\sqrt{a^2 - x^2 - y^2}}$, $f_y = \frac{-y}{\sqrt{a^2 - x^2 - y^2}}$, so $\sqrt{1 + f_x^2 + f_y^2} = 1 + \frac{x^2 + y^2}{a^2 - x^2 - y^2}$, then

$$S = \iint_R \sqrt{1 + \frac{x^2 + y^2}{a^2 - x^2 - y^2}} dA_{xy}$$

Notice this integration is too difficult to calculate, so we consider using *polar coordinate* to substitute, let $r^2 = x^2 + y^2$, $dA = r dr d\theta$, then

$$S = \iint_R \sqrt{1 + \frac{r^2}{a^2 - r^2}} r dr d\theta = 2\pi a^2$$