MATH2421 Midterm Cheat Sheet

Permutation with objects alike: n objects of which n_1 are alike, n_2 are alike, $\cdots$, n_r alike: $\dfrac{n!}{n_1!n_2!\cdots n_r!}$

Circle arrangement: n people sitting in a circle, $(n-1)!$ permutations.

Properties of combinations: $\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$

Binomial Theorem: $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$

Multinomial Coefficients: n distinct items divided into r distinct groups of size $n_1, n_2, \cdots, n_r$, respectively, where $\sum n_i = n$, then there are $\binom{n}{n_1, n_2, \cdots, n_r} = \dfrac{n!}{n_1!n_2!\cdots n_r!}$ arrangements.

Multinomial Theorem: $(x_1 + x_2 + \cdots + x_r)^n = \sum_{(n_1,\cdots,n_r):n_1+\cdots+n_r=n} \binom{n}{n_1, n_2, \cdots, n_r} x_1^{n_1} x_2^{n_2} \cdots x_r^{n_r}$

Kolmogorov Axioms: Probability is a function satisfying:

1. For any event A, $0 \le P(A) \le 1$
2. Let S be the sample space, then $P(S) = 1$
3. For mutually exclusive events $A_1, A_2, \cdots$, there is $P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$

Inclusion-Exclusion Principle:

$$P(A_1 \cup A_2 \cup \cdots \cup A_n) = \sum_{i=1}^{n} P(A_i) - \sum_{i \le i_1 < i_2 \le n} P(A_{i_1} A_{i_2}) + \cdots + (-1)^{r+1} \sum_{1 \le i_1 < \cdots < i_r \le n} P(A_{i_1} \cdots A_{i_r}) + \cdots + (-1)^{n+1} P(A_1 \cdots A_n)$$

General Multiplication Rule: $P(A_1A_2\cdots A_n) = P(A_1)P(A_2|A_1)P(A_3|A_1A_2)\cdots P(A_n|A_1A_2\cdots A_{n-1})$

Total Probability: $P(B) = P(B|A)P(A) + P(B|A^C)P(A^C)$

Bayes' formula: Events $A_1, \cdots, A_n$ partitions sample space, assume $P(A_i) > 0$ for $1 \le i \le n$. Let B be any event, then for any $1 \le i \le n$, we have $P(A_i|B) = \dfrac{P(B|A_i)P(A_i)}{P(B|A_1)P(A_1) + \cdots + P(B|A_n)P(A_n)}$

Probability mass function: $p_X(x) = \begin{cases} P(X=x) & \text{if } x = x_1, x_2, \cdots \\ 0 & \text{otherwise} \end{cases}$

Cumulative distribution function: $F_X(x) = P(X \le x)$ for $x \in \mathbb{R}$

Expected Value: $E(X) = \sum_x x p_X(x)$, $E[g(x)] = \sum_i g(x_i) p_X(x_i) = \sum_x g(x) p_X(x)$

Tail Sum Formula: $E(X) = \sum_{k=1}^{\infty} P(X \ge k) = \sum_{k=0}^{\infty} P(X > k)$

Variance: $\mathrm{var}(X) = E(X-\mu)^2 = E(X^2) - [E(X)]^2$

Expected Value of Sum of RV: $E\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} E[X_i]$

Bernoulli random variable: $Be(p)$, $X = 1$ if success, 0 if failure.

$P(X = 1) = p, P(X = 0) = 1 - p, \qquad \mathbb{E}(X) = p, \text{var}(X) = p(1 - p)$

Binomial random variable: $Bin(n, p)$, $X = \#$ of successes in n Bernoulli(p) trials.

For $0 \leq k \leq n$, $P(X = k) = \binom{n}{k} p^k q^{n-k} \qquad \mathbb{E}(X) = np, \text{var}(X) = np(1 - p)$

Geometric random variable: $Geom(p)$, $X = \#$ of Bernoulli(p) trials required to obtain the first success.

For $k \geq 1$, $P(X = k) = pq^{k-1} \qquad \mathbb{E}(X) = \dfrac{1}{p}, \text{var}(X) = \dfrac{1-p}{p^2}.$

OR, $X' = \#$ of failures in Bernoulli(p) trials to obtain 1st success. $X = X' + 1$

For $k \geq 0$, $P(X' = k) = pq^k, \qquad \mathbb{E}(X') = \dfrac{1-p}{p}, \text{var}(X') = \dfrac{1-p}{p^2}$

Negative Binomial random variable: $NB(r, p)$, $X = \#$ of Bernoulli(p) trials required to obtain r success.

For $k \geq r$, $P(X = k) = \binom{k-1}{r-1} p^r q^{k-r}, \qquad \mathbb{E}(X) = \dfrac{r}{p}, \text{var}(X) = \dfrac{r(1-p)}{p^2}$

Note that $Geom(p) = NB(1, p)$, $\binom{k-1}{r-1} = (-1)^{r-1} \binom{-(k-r+1)}{r-1}$

Poisson Random Variable: $X \sim$ Poisson(λ) $\qquad$ For $k \geq 0$, $P(X = k) = \dfrac{e^{-\lambda}\lambda^k}{k!}, \qquad \mathbb{E}(X) = \lambda, \text{var}(x) = \lambda$

Usually if $n > 20$ and $np < 15$, Bin(n, p) $\approx$Poisson(np).

Hypergeometric Random Variable: $H(n, N, m)$, a set of N balls, of which m are red and $N - m$ are blue. We choose n of these balls *without replacement*, $X = \#$ of red balls in sample.

For $0 \leq x \leq \min(m, n)$, $P(X = x) = \dfrac{\binom{m}{x}\binom{N-m}{n-x}}{\binom{N}{n}} \qquad \mathbb{E}(X) = \dfrac{nm}{N}, \text{var}(X) = \dfrac{nm}{N}\left[\dfrac{(n-1)(m-1)}{N-1} + 1 - \dfrac{nm}{N}\right]$