MATH2421 Final Cheat Sheet

Inclusion-Exclusion Principle:

$$P(A_1 \cup A_2 \cup \cdots \cup A_n) = \sum_{i=1}^{n} P(A_i) - \sum_{i \le i_1 < i_2 \le n} P(A_{i_1} A_{i_2}) + \cdots + (-1)^{r+1} \sum_{1 \le i_1 < \cdots < i_r \le n} P(A_{i_1} \cdots A_{i_r}) + \cdots + (-1)^{n+1} P(A_1 \cdots A_n)$$

General Multiplication Rule: $P(A_1A_2 \cdots A_n) = P(A_1)P(A_2|A_1)P(A_3|A_1A_2) \cdots P(A_n|A_1A_2 \cdots A_{n-1})$

Total Probability: $P(B) = P(B|A)P(A) + P(B|A^C)P(A^C)$

Bayes' formula: Events $A_1, \cdots, A_n$ partitions sample space, assume $P(A_i) > 0$ for $1 \le i \le n$. Let B be any event, then for any $1 \le i \le n$, we have $P(A_i|B) = \dfrac{P(B|A_i)P(A_i)}{P(B|A_1)P(A_1) + \cdots + P(B|A_n)P(A_n)}$

Probability mass function: $p_X(x) = \begin{cases} P(X = x) & \text{if } x = x_1, x_2, \cdots \\ 0 & \text{otherwise} \end{cases}$

Cumulative distribution function: $F_X(x) = P(X \le x)$ for $x \in \mathbb{R}$

Expected Value: $E(X) = \sum_x x p_X(x)$, $E[g(x)] = \sum_i g(x_i) p_X(x_i) = \sum_x g(x) p_X(x)$

Tail Sum Formula: $E(X) = \sum_{k=1}^{\infty} P(X \ge k) = \sum_{k=0}^{\infty} P(X > k)$

Variance: $\text{var}(X) = E(X - \mu)^2 = E(X^2) - [E(X)]^2$

Expected Value of Sum of RV: $E\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} E[X_i]$

Bernoulli random variable: $Be(p)$, $X = 1$ if success, 0 if failure.

$P(X = 1) = p, P(X = 0) = 1 - p, \qquad \mathbb{E}(X) = p, \text{var}(X) = p(1-p)$

Binomial random variable: $Bin(n, p)$, $X = \#$ of successes in n Bernoulli(p) trials.

For $0 \le k \le n$, $P(X = k) = \binom{n}{k} p^k q^{n-k} \qquad \mathbb{E}(X) = np, \text{var}(X) = np(1-p)$

Geometric random variable: $Geom(p)$, $X = \#$ of Bernoulli(p) trials required to obtain the first success.

For $k \ge 1$, $P(X = k) = pq^{k-1} \qquad \mathbb{E}(X) = \dfrac{1}{p}, \text{var}(X) = \dfrac{1-p}{p^2}.$

OR, $X' = \#$ of failures in Bernoulli(p) trials to obtain 1st success. $X = X' + 1$

For $k \ge 0$, $P(X' = k) = pq^k, \qquad \mathbb{E}(X') = \dfrac{1-p}{p}, \text{var}(X') = \dfrac{1-p}{p^2}$

Negative Binomial random variable: $NB(r, p)$, $X = \#$ of Bernoulli(p) trials required to obtain r success.

For $k \ge r$, $P(X = k) = \binom{k-1}{r-1} p^r q^{k-r}, \qquad \mathbb{E}(X) = \dfrac{r}{p}, \text{var}(X) = \dfrac{r(1-p)}{p^2}$

Note that $Geom(p) = NB(1, p)$, $\binom{k-1}{r-1} = (-1)^{r-1} \binom{-(k-r+1)}{r-1}$

Poisson Random Variable: $X \sim \text{Poisson}(\lambda) \qquad$ For $k \ge 0$, $P(X = k) = \dfrac{e^{-\lambda}\lambda^k}{k!}, \qquad \mathbb{E}(X) = \lambda, \text{var}(x) = \lambda$

Usually if $n > 20$ and $np < 15$, $\text{Bin}(n, p) \approx \text{Poisson}(np)$.

Hypergeometric Random Variable: $H(n, N, m)$, a set of N balls, of which m are red and $N - m$ are blue. We choose n of these balls *without replacement*, $X = \#$ of red balls in sample.

For $0 \le x \le \min(m, n)$, $P(X = x) = \dfrac{\binom{m}{x}\binom{N-m}{n-x}}{\binom{N}{n}} \qquad \mathbb{E}(X) = \dfrac{nm}{N}, \text{var}(X) = \dfrac{nm}{N}\left[\dfrac{(n-1)(m-1)}{N-1} + 1 - \dfrac{nm}{N}\right]$

Expectation and Variance of Continuous RV:
$\mathbb{E}(X) = \int_{-\infty}^{\infty} x f_X(x)dx,\ \mathrm{var}(X) = \int_{-\infty}^{\infty} (x - \mathbb{E}(X))^2 f_X(x)dx = \mathbb{E}[(x-\mu_X)^2]$

Tail sum formula: $\mathbb{E}(X) = \int_0^{\infty} P(X > x)dx = \int_0^{\infty} P(X \geq x)dx$

Uniform Distribution: $X \sim U(a,b),\ f(x) = \frac{1}{b-a},\ a < x < b,\ \mathbb{E}(X) = \frac{a+b}{2},\ \mathrm{var}(x) = \frac{(b-a)^2}{12}$
$F_X(x) = 0,\ if\ x < a;\ \ \frac{x-a}{b-a},\ if\ a \leq x < b;\ \ 1,\ if\ b \leq x$

Normal distribution: $X \sim N(\mu, \sigma^2),\ f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}},\ -\infty < x < \infty,\ \mathbb{E}(X) = \mu, \mathrm{var}(X) = \sigma^2$

Standard normal distribution: $X \sim N(0,1),\ \phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}},\ \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{y^2}{2}} dy$
$P(a < Z < b) = \Phi(b) - \Phi(a),\ P(Z < b) = \Phi(b),\ \Phi(-x) = 1 - \Phi(x)$
$\frac{X-\mu}{\sigma} \sim N(0,1),\ \therefore Y \sim N(\mu, \sigma^2) \Rightarrow P(a < Y \leq b) = \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)$

Exponential distribution: $X \sim Exp(\lambda),\ f_X(x) = \lambda e^{-\lambda x},\ x \geq 0$, the c.d.f is $F_X(x) = 1 - e^{-\lambda x},\ x > 0$
$\mathbb{E}(X) = \frac{1}{\lambda},\ \mathrm{var}(X) = \frac{1}{\lambda^2}$

memoryless property of exp dist: $P(X > s+t | X > s) = P(X > t),\ s,t > 0$

Gamma distribution: $X \sim \Gamma(\alpha, \lambda),\ f_X(x) = \frac{\lambda e^{-\lambda x}(\lambda x)^{\alpha-1}}{\Gamma(\alpha)},\ x \geq 0,$ where $\lambda, \alpha > 0,\ \Gamma(\alpha) = \int_0^{\infty} e^{-y} y^{\alpha-1} dy$
$\mathbb{E}(X) = \frac{\alpha}{\lambda},\ \mathrm{var}(X) = \frac{\alpha}{\lambda^2},\ \Gamma(1) = 1,\ \Gamma(\alpha) = (\alpha-1)\Gamma(\alpha-1),\ \Gamma(n) = (n-1)!,\ \Gamma(1,\lambda) = \mathrm{Exp}(\lambda),\ \Gamma(1/2) = \sqrt{\pi}$

Beta distribution: $X \sim \mathrm{Beta}(a,b),\ f(x) = \frac{1}{B(a,b)} x^{a-1}(1-x)^{b-1},\ 0 < x < 1$, where **beta function** $B(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}dx,\ \mathbb{E}(X) = \frac{a}{a+b},\ \mathrm{var}(X) = \frac{ab}{(a+b)^2(a+b+1)},\ B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$

Cauchy distribution: $X \sim \mathrm{Cauchy}(\theta),\ f(x) = \frac{1}{\pi} \frac{1}{1+(x-\theta)^2},\ -\infty < x < \infty,\ \mathbb{E}(X) = \infty,\ \mathrm{var}(X) = \infty$

De Moivre-Laplace Limit Thm: $X \sim \mathrm{Bin}(n,p)$, then for any $a < b$, $\mathrm{Bin}(n,p) \approx N(np, npq)$
$P\left(a < \frac{X-np}{\sqrt{npq}} \leq b\right) \approx \Phi(b) - \Phi(a)$

Continuity Correction: $X \sim \mathrm{Bin}(n,p),\ Z \sim N(0,1)$, then
$P(a \leq X \leq b) \approx P\left(\frac{a-0.5-np}{\sqrt{npq}} \leq Z \leq \frac{b+0.5-np}{\sqrt{npq}}\right),\ P(a < X < b) \approx P\left(\frac{a+0.5-np}{\sqrt{npq}} \leq Z \leq \frac{b-0.5-np}{\sqrt{npq}}\right)$

Dist of a func of a RV: For monotonic $Y = g(X),\ f_Y(y) = f_X\left(g^{-1}(y)\right)\left|\frac{d}{dy}g^{-1}(y)\right|$
If X is a RV with c.d.f F, then $F(X) \sim U(0,1)$.

Marginal distribution function: $F_X(x) = \lim_{y\to\infty} F_{X,Y}(x,y) = \lim_{y\to\infty} P(X \leq x, Y \leq y) = P(X \leq x)$. (c.d.f of X)

Marginal p.m.f: $p_X(x) = P(X = x) = \sum_{y\in\mathbb{R}} p_{X,Y}(x,y)$ **Marginal p.d.f:** $f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy$

Relation between p.d.f and c.d.f: $f_{X,Y}(x,y) = \frac{\partial^2}{\partial x \partial y} F_{X,Y}(x,y)$.

Independent: $p_{X,Y}(x,y) = p_X(x)p_Y(y),\ F_{X,Y}(x,y) = F_X(x)F_Y(y)$

Sum of Indep: $F_{X+Y}(x) = \int_{-\infty}^{\infty} F_X(x-t)f_Y(t)dt = \int_{-\infty}^{\infty} F_Y(x-t)f_X(t)dt,$
$f_{X+Y}(x) = \int_{-\infty}^{\infty} f_X(x-t)f_Y(t)dt = \int_{-\infty}^{\infty} f_Y(x-t)f_X(t)dt$

Some conclusions: $X_1, \cdots, X_n$ be n independent RV$\sim$ Exp(λ), then $\sum_{i=1}^{n} X_i \sim \text{Gamma}(n, \lambda)$.

$X_1, \cdots, X_n$ be n independent RV$\sim N(\mu_i, \sigma_i^2)$, then $\sum_{i=1}^{n} X_i \sim N\left(\sum_{i=1}^{n} \mu_i, \sum_{i=1}^{n} \sigma_i^2\right)$. (used to approx Binominal Dist.)

Sum of Discrete RV: $X \sim \text{Poisson}(\lambda), Y \sim \text{Poisson}(\mu)$, then $X + Y \sim \text{Poisson}(\lambda + \mu)$
$X \sim \text{Bin}(n, p), Y \sim \text{Bin}(m, p)$, then $X + Y \sim \text{Bin}(n + m, p)$ $\quad X \sim \text{Geom}(p), Y \sim \text{Geom}(p)$, then $X + Y \sim NB(2, p)$

Conditional Dist.: (Discrete:) $p_{X|Y}(x|y) = \dfrac{p_{X,Y}(x, y)}{p_Y(y)}$, $F_{X|Y}(x|y) = P(X \leq x|Y = y)$

(Cont.:) $f_{X|Y}(x|y) = \dfrac{f_{X,Y}(x, y)}{f_Y(y)}$, $F_{X|Y}(x|y) = P(X \leq x|Y = y) = \int_{-\infty}^{x} f_{X|Y}(t|y)dt$

Joint p.d.f of Func of RV: $J(x, y) = \dfrac{\partial(u, v)}{\partial(x, y)} = \dfrac{\partial g}{\partial x}\dfrac{\partial h}{\partial y} - \dfrac{\partial g}{\partial y}\dfrac{\partial h}{\partial x}$, $f_{U,V}(u, v) = f_{X,Y}(x, y)|J(x, y)|^{-1}$

Expectation of Sum of RV: $\mathbb{E}[g(X, Y)] = \sum_y \sum_x g(x, y)p_{X,Y}(x, y)$, $\mathbb{E}[g(X, Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x, y)f_{X,Y}(x, y)dxdy$

Covariance: $\text{cov}(X, Y) = \mathbb{E}(X - \mu_X)(Y - \mu_Y)$, if $\text{cov}(X, Y) \neq 0$, then X, Y are correlated.

$\text{cov}(X, Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y)$, $\text{cov}\left(\sum_{i=1}^{n} a_i X_i, \sum_{j=1}^{m} b_j Y_j\right) = \sum_{i=1}^{n}\sum_{j=1}^{m} a_i b_j \text{cov}(X_i, Y_j)$,

$\text{var}\left(\sum_{k=1}^{n} X_k\right) = \sum_{k=1}^{n} \text{var}(X_k) + 2\sum_{1\leq i<j\leq n} \text{cov}(X_i, X_j)$, under **indep.**, $\text{var}\left(\sum_{k=1}^{n} X_k\right) = \sum_{k=1}^{n} \text{var}(X_k)$

Independent Case: X, Y independent, then $\mathbb{E}[g(X)h(Y)] = \mathbb{E}[g(X)]\mathbb{E}[h(Y)]$, $\text{cov}(X, Y) = 0$(reverse not true)

correlation coefficient: $\rho(X, Y) = \dfrac{\text{cov}(X, Y)}{\sqrt{\text{var}(X)\text{var}(Y)}}$

Conditional Expectation: $\mathbb{E}[X|Y = y] = \sum_x xp_{X|Y}(x|y) = \int_{-\infty}^{\infty} xf_{X|Y}(x|y)dx$, $\mathbb{E}\left[\sum_{k=1}^{n} X_k|Y = y\right] = \sum_{k=1}^{n} \mathbb{E}[X_k|Y = y]$

Expectation by Conditioning: $\mathbb{E}[X] = \mathbb{E}[\mathbb{E}(X|Y)] = \sum_y \mathbb{E}(X|Y = y)P(Y = y) = \int_{-\infty}^{\infty} \mathbb{E}(X|Y = y)f_Y(y)dy$

Probability by Conditioning: $P(A) = \sum_y P(A|Y = y)P(Y = y) = \int_{-\infty}^{\infty} P(A|Y = y)f_Y(y)dy$

conditional variance: $\text{var}(X|Y) = \mathbb{E}\big[(X - \mathbb{E}[X|Y])^2|Y\big]$, $\text{var}(X) = \mathbb{E}[\text{var}(X|Y)] + \text{var}(\mathbb{E}[X|Y])$
$\text{var}(Y|X) = \mathbb{E}(Y^2|X) - \mathbb{E}(Y|X)^2$

Moment Generating Function: $M_X(t) = \mathbb{E}[e^{tX}] = \sum_X e^{tx}p_X(x) = \int_{-\infty}^{\infty} e^{tx}f_X(x)dx$, $\mathbb{E}(X^n) = M_X^{(n)}(0)$

If X, Y are independent, $M_{X+Y}(t) = M_X(t)M_Y(t)$

MGF for dist.: $X \sim \text{Be}(p)$, $M(t) = 1 - p + pe^t$, $\quad X \sim \text{Bin}(n, p), M(t) = (1 - p + pe^t)^n$

$X \sim \text{Geom}(p), M(t) = \dfrac{pe^t}{1 - (1 - p)e^t}$, $\quad X \sim \text{Poisson}(\lambda), M(t) = \exp(\lambda(e^t - 1))$

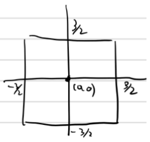
$X \sim U(\alpha, \beta), M(t) = \dfrac{e^{\beta t} - e^{\alpha t}}{(\beta - \alpha)t}$, $\quad X \sim \text{Exp}(\lambda), M(t) = \dfrac{\lambda}{\lambda - t}$ for $t < \lambda$, $\quad X \sim N(\mu, \sigma^2), M(t) = e^{\mu t + \frac{\sigma^2 t^2}{2}}$

Common Integrations: $\int xe^{ax}\, dx = \dfrac{1}{a^2}(ax - 1)e^{ax} + C$, $\int x^n e^{ax}\, dx = \dfrac{1}{a}x^n e^{ax} - \dfrac{n}{a}\int x^{n-1}e^{ax}\, dx$

The county hospital is located at the center of a square whose sides are 3 miles wide. If an accident occurs within this square, then the hospital sends out an ambulance. The road network is rectangular, so the travel distance from the hospital, whose coordinate are (0,0) to the point (x,y) is |x|+|y|. If an accident occurs at a point that is uniformly distributed in the square, find the expected travel distance of the ambulance.

Q2 Define (X,Y) as the coordinate of the accident

$$f_{XY}(x,y) = \begin{cases} \frac{1}{9} & |x|<\frac{3}{2} \quad |y|<\frac{3}{2} \\ 0 & \text{otherwise} \end{cases}$$



D—distance

$D = |X| + |Y|$

$\mathbb{E}D = \mathbb{E}|X| + \mathbb{E}|Y|$

$f_X(x) = \frac{1}{3} \quad |x|<\frac{3}{2} \qquad f_Y(y) = \frac{1}{3} \quad |y|<\frac{3}{2}$

$\Rightarrow \mathbb{E}|X| = \mathbb{E}|Y| = \frac{3}{4}$

$\Rightarrow \mathbb{E}D = \frac{3}{2}$

N people arrive separately to a professional dinner. Upon arrival, each person looks to see if he or she has any friends among those present. That person then sits either at the table of a friend or at an unoccupied table if none of those present is a friend. Assuming that each of the $\binom{N}{2}$ pairs of people is, independently, a pair of friends with probability p, find the expected number of occupied tables.

(Hint: Let X_i be 1 or 0, depending on whether the ith arrival sits at a previously occupied table.)

Q4 $X_i = \begin{cases} 1 & \text{i-th arrival sits at a previously occupied table} \\ 0 & \text{otherwise} \end{cases}$

$$\left.\begin{aligned} &\mathbb{P}(X_1=1)=1 \\ &\mathbb{P}(X_i=1)=(1-p)^{i-1} \quad i\geq 2 \end{aligned}\right\} \Rightarrow \mathbb{P}(X_i=1)=(1-p)^{i-1} \quad i\in[1,N]$$

$\mathbb{E}X_i = (1-p)^{i-1}$

X — # of occupied tables

$X = \sum_{i=1}^{N} X_i$

$\mathbb{E}X = \sum_{i=1}^{N} \mathbb{E}X_i = \sum_{i=1}^{N} (1-p)^{i-1} = \frac{1-(1-p)^N}{p}$

A total of n balls, numbered 1 through n, are put into n urns, also numbered 1 through n in such a way that ball i is equally likely to go into any of the urns 1,2,...,i.

Find the expected number of urns that are empty.

$X_i = \begin{cases} 1 & \text{ith urn is empty} \\ 0 & \text{not empty} \end{cases}$

$\mathbb{P}(X_i=1) = \frac{i-1}{i}\frac{i}{i+1}\cdots\frac{n-1}{n} = \frac{i-1}{n} \qquad \mathbb{E}X_i = \frac{i-1}{n}$

X — # of empty urns $\qquad X = \sum_{i=1}^{n} X_i$

$\mathbb{E}X = \sum_{i=1}^{n} \mathbb{E}X_i = \sum_{i=1}^{n} \frac{i-1}{n} = \frac{n-1}{2}$

Let X be the number of 1's and Y the number of 2's that occur in n rolls of a fair die.

Compute Cov(X,Y).

Q7 $P_{XY}(x,y) = \frac{\binom{n}{x}\binom{n-x}{y}4^{n-x-y}}{6^n} \qquad x+y\leq n \quad x,y\in[0,n]$

$4n+$

$P_X(x) = \sum_{y=0}^{n-x} \frac{\binom{n}{x}\binom{n-x}{y}4^{n-x-y}}{6^n} = \frac{\binom{n}{x}}{6^n}\cdot 5^{n-x}$

$P_Y(y) = \sum_{x=0}^{n-y} \frac{\binom{n}{y}\binom{n-y}{x}4^{n-x-y}}{6^n} = \frac{\binom{n}{y}}{6^n}\cdot 5^{n-y}$

$\operatorname{cov}(X,Y) = \sum_{x+y\leq n} xy\frac{\binom{n}{x}\binom{n-x}{y}4^{n-x-y}}{6^n} - \left(\sum_{x=0}^{n} x\binom{n}{x}\frac{5^{n-x}}{6^n}\right)^2$

$\sum_{x+y\leq n} xy\frac{\binom{n}{x}\binom{n-x}{y}4^{n-x-y}}{6^n} = \sum_{x=0}^{n}\frac{x}{6^n}\binom{n}{x}\sum_{y=0}^{n-x} y\binom{n-x}{y}4^{n-x-y} = \sum_{x=0}^{n}\frac{x}{6^n}\binom{n}{x}(n-x)5^{n-x-1} = \frac{n(n-1)}{6^2}$

$\sum_{x=0}^{n} x\binom{n}{x}\frac{5^{n-x}}{6^n} = \sum_{x=1}^{n} n\binom{n-1}{x-1}\frac{5^{n-1-(x-1)}}{6^n} = \frac{n}{6^n}6^{n-1} = \frac{n}{6}$

$\operatorname{cov}(X,Y) = \frac{n(n-1)}{36} - \frac{n^2}{36} = -\frac{n}{36}$

Let $X_1,\ X_2,\cdots,$ be independent with common mean μ and common variance σ^2. Set $Y_n = X_n + X_{n+1} + X_{n+2}$.

Find $Cov\left(Y_n,\ Y_{n+1}\right)$ and $Var\left(Y_n + Y_{n+1} + Y_{n+2}\right)$.

$\operatorname{cov}(Y_n, Y_{n+1}) = 2\sigma^2 \quad \operatorname{cov}(Y_n, Y_n) = 3\sigma^2 \quad \operatorname{cov}(Y_n, Y_{n+2}) = \sigma^2$

$\operatorname{cov}(Y_n, Y_{n+j}) = 0 \quad j\geq 3$

$\operatorname{var}(Y_n + Y_{n+1} + Y_{n+2}) = \operatorname{var}(Y_n) + \operatorname{var}(Y_{n+1}) + \operatorname{var}(Y_{n+2}) + 2\operatorname{cov}(Y_n, Y_{n+1}) + 2\operatorname{cov}(Y_n, Y_{n+2}) + 2\operatorname{cov}(Y_{n+1}, Y_{n+2})$

$= 3\sigma^2\times 3 + 4\sigma^2 + 2\sigma^2 + 4\sigma^2$

$= 19\sigma^2$

There are two misshapen coins in a box; their probabilities for landing on heads when they are flipped are, respectively, 0.4 and 0.7. One of the coins is to be randomly chosen are flipped 10 times. Given that two of the first three flips landed on heads, what is the conditional expected number of heads in 10 flips?

X — # of heads in 10 flip

Y — # of heads in first three.

$\mathbb{P}(Y=2) = \frac{1}{2}\times 3\times(0.4)^2\times 0.6 + \frac{1}{2}\times 3\times(0.7)^2\times 0.3 = 0.3645$

$\mathbb{P}(X=i \mid Y=2) = \frac{\mathbb{P}(X=i\ Y=2)}{\mathbb{P}(Y=2)}$

$$\mathbb{P}(X=i, Y=2) = \begin{cases} 0 & i<2 \\ \frac{1}{2}\left[\binom{3}{2}\binom{7}{i-2}(0.4)^i(0.6)^{10-i} + \binom{3}{2}\binom{7}{i-2}(0.7)^i(0.3)^{10-i}\right] & 9\geq i\geq 2 \end{cases}$$

$\mathbb{E}[X \mid Y=2] = \frac{1}{\mathbb{P}(Y=2)}\sum_{i=2}^{9}\frac{i}{2}\left[\binom{3}{2}\binom{7}{i-2}0.4^i 0.6^{10-i} + \binom{3}{2}\binom{7}{i-2}0.7^i(0.3)^{10-i}\right]$

$= \frac{1}{0.3645}\left[0.6\times(0.4)^2\times\frac{3}{2}\times(7\times 0.4+2) + 0.3\times(0.7)^2\times\frac{3}{2}\times(7\times 0.7+2)\right]$

$= 6.07$

The joint density of X and Y is given by

$$f_{X,Y}(x,y) = \begin{cases} \frac{e^{-y}}{y}, & \text{if } 0<x<y, 0<y<\infty \\ 0, & \text{otherwise} \end{cases}$$

Compute $E(X^3|Y=y)$.

$f_Y(y) = \int_0^y \frac{e^{-y}}{y}dx$

$= e^{-y} \qquad 0<y<\infty$

$f_{X|Y}(x|y) = \frac{1}{y} \qquad 0<x<y \quad 0<y<\infty$

$\mathbb{E}(X^3|Y=y) = \int_0^y x^3\frac{1}{y}dx = \frac{1}{4}y^3$

A coin having probability p of coming up heads is continually flipped until both heads and tails have appeared. Find

(a) the expected number of flips;

(b) the probability that the last flip lands on heads.

(a) X — # of flips. $X\geq 2$

$\mathbb{P}(X=i) = (1-p)^{i-1}p + (1-p)p^{i-1}$

$\mathbb{E}X = \sum_{i=2}^{\infty} i\left((1-p)^{i-1}p + (1-p)p^{i-1}\right)$

$= p\sum_{i=2}^{\infty}\frac{-d(1-p)^i}{dp} + (1-p)\sum_{i=2}^{\infty}\frac{dp^i}{dp}$

$= -p\left(1-\frac{1}{p^2}\right) + (1-p)\frac{2p-p^2}{(1-p)^2}$

$= \frac{1}{p} + \frac{p}{1-p}$

(b) $\mathbb{P}(A) = \sum_{i=2}^{\infty}\mathbb{P}(A\cap\{X=i\})$

$= \sum_{i=2}^{\infty} p(1-p)^{i-1}$

$= p\frac{(1-p)}{p}$

$= 1-p$