MATH 2023 Fall 2021

Multivariable Calculus

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Chapter 15 **Vector Field**

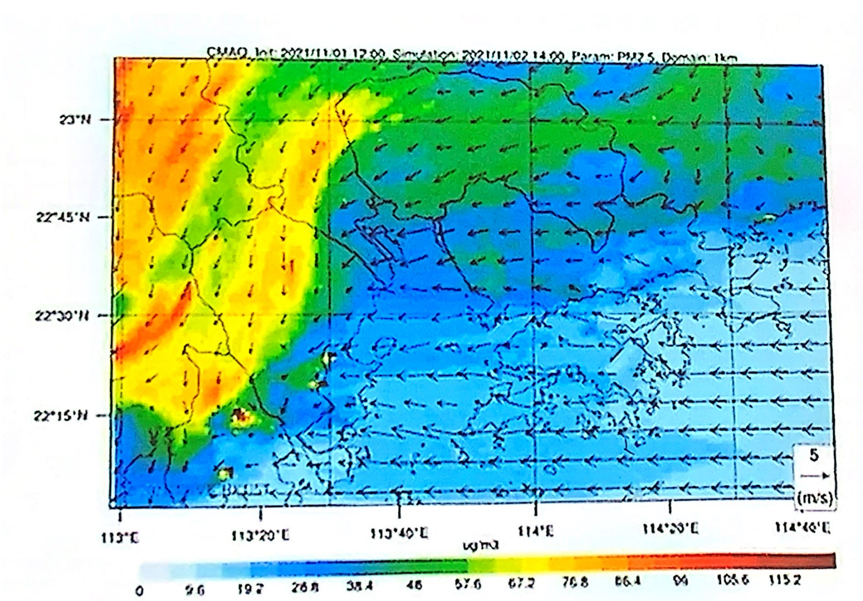
1 Intro. to Vector Field

So far, we have learned two kinds of functions involving vector:

- $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$: for each t, provides a *position* vector $< x(t), y(t), z(t) >$, so this is a (parametric) curve.
- $z = f(\mathbf{r}) = f(x_1, x_2, \cdots, x_n)$: for a given vector r, this gives a real number, so this is a function of *several variables.* This is also a **scalar field** since for any point $\mathbf{r}$ in **field**, it gives a scalar value.

Now we are looking at **vector-valued** function $\mathbf{F}$ of a vector $\mathbf{r}$, i.e., $\mathbf{F}(\mathbf{r})$. This is a **vector field**, which means for any point $\mathbf{r}$ in **field**, it gives a vector $\mathbf{F}(\mathbf{r})$.

You can consider a world map showing the *speed* and *direction* of wind.



You can see that in a 2D map(like above), if we put a vector on each point, the vector must have same dimension as the map, i.e., all vectors must also be 2D vectors.

$$\mathbf{F}(\mathbf{r}) = \begin{cases} (F_1(\mathbf{r}), F_2(\mathbf{r})) & \mathbf{r} = (x, y) & \color{blue}{2D} \\ (F_1(\mathbf{r}), F_2(\mathbf{r}), F_3(\mathbf{r})) & \mathbf{r} = (x, y, z) & \color{blue}{3D} \\ (F_1(\mathbf{r}), F_2(\mathbf{r}), \cdots, F_n(\mathbf{r})) & \mathbf{r} = (x_1, x_2, \cdots, x_n) & \color{blue}{nD} \end{cases}$$

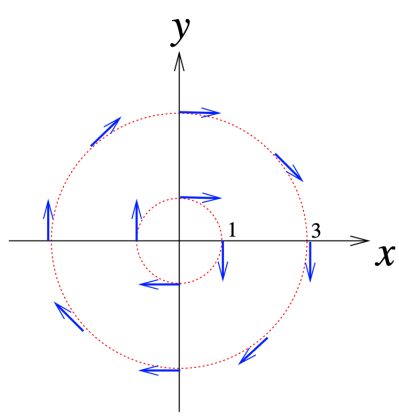
Summary: *dimension of* $\mathbf{F}$ *must be the same as* $\mathbf{r}$.

This is an example of vector field.

[Example.] Assume a vector field: $\mathbf{F}(x, y) = \dfrac{y\mathbf{i} - x\mathbf{j}}{\sqrt{x^2 + y^2}}$.

[Solution.] Notice that $||\mathbf{F}|| = \dfrac{y^2}{x^2 + y^2} + \dfrac{x^2}{x^2 + y^2} = 1$, all vectors $\mathbf{F}(x, y)$ are unit vectors. Moreover, let $\mathbf{r} = (x, y)$, then $\mathbf{r} \cdot \mathbf{F} = 0$, so $\mathbf{r} \perp \mathbf{F}$.

So all vectors are unit vectors tangent to circles centered at the origin with radius $\sqrt{x^2 + y^2}$.



2 Divergence and Curl

Recall that the **gradient operator** is a *vector operator*:

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \qquad \text{(a vector)}$$

If $\mathbf{F}(\mathbf{r}) = F_1(\mathbf{r})\mathbf{i} + F_2(\mathbf{r})\mathbf{j} + F_3(\mathbf{r})\mathbf{k}$, then we define:

- **divergence** of $\mathbf{F}$, written div $\mathbf{F}$:

$$\boxed{\operatorname{div}\mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}}$$

- **curl** of $\mathbf{F}$, written curl $\mathbf{F}$:

$$\boxed{\operatorname{curl}\mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right)\mathbf{i} - \left(\frac{\partial h}{\partial x} - \frac{\partial f}{\partial z}\right)\mathbf{j} + \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)\mathbf{k}}$$

This example shows basic computation of **divergence** and **curl**.

[Example.] Let $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $\mathbf{u} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$, where a, b and c are constants, show that

(a) $\nabla \cdot \mathbf{r} = 3$
(b) $\nabla \times \mathbf{r} = \mathbf{0}$
(c) $\nabla \cdot (\mathbf{u} \times \mathbf{r}) = 0$
(d) $\nabla \times (\mathbf{u} \times \mathbf{r}) = 2\mathbf{u}$.

[Solution.] (a) $\nabla \cdot \mathbf{r} = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right) \cdot (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 3$

(b) $\nabla \times \mathbf{r} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = \mathbf{0}$

(c) $\mathbf{u} \times \mathbf{r} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a & b & c \\ x & y & z \end{vmatrix} = (bz - cy)\mathbf{i} - (az - cx)\mathbf{j} + (ay - bx)\mathbf{k}$

$\therefore \nabla \cdot (\mathbf{u} \times \mathbf{r}) = \frac{\partial}{\partial x}(bz - cy) - \frac{\partial}{\partial y}(az - cx) + \frac{\partial}{\partial z}(ay - bx) = 0$

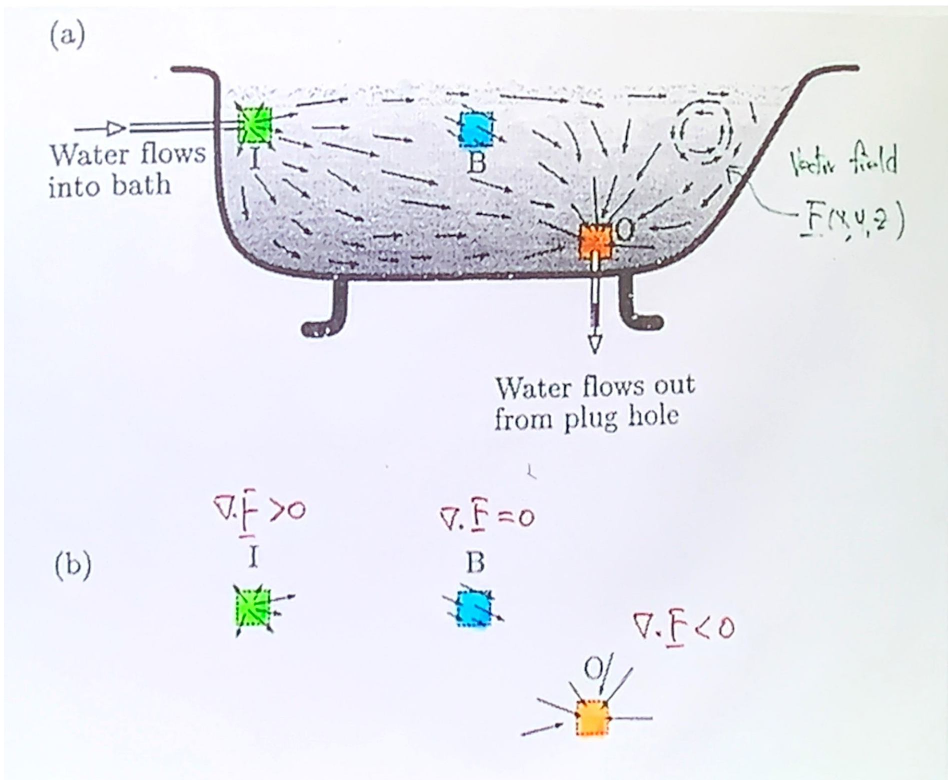
(d) $\nabla \times (\mathbf{u} \times \mathbf{r}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ bz - cy & -az + cx & ay - bx \end{vmatrix} = 2(a\mathbf{i} + b\mathbf{j} + c\mathbf{k}) = 2\mathbf{u}$

2.1 Interpretation of Divergence

Imagine water in a bath tank, if the **velocity** of water at any point of the tank is given by

$$\mathbf{u}(\mathbf{r}) = u_1(\mathbf{r})\mathbf{i} + u_2(\mathbf{r})\mathbf{j} + u_3(\mathbf{r})\mathbf{k}$$

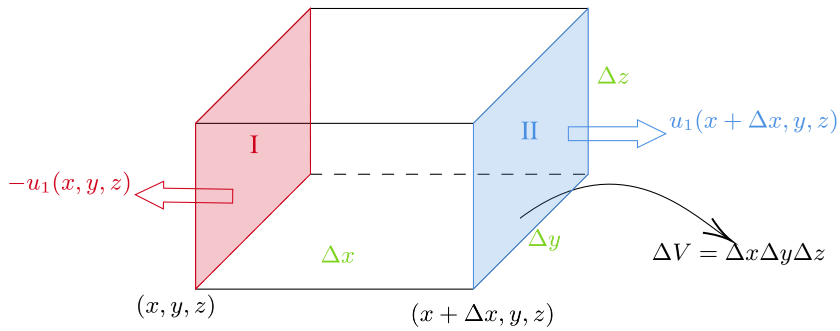
then **net outward flux per unit volume** is div $\mathbf{u} = \nabla \cdot \mathbf{u}$.



Moreover,

- If more water comes inside, then div $\mathbf{u} < 0$
- If more water comes outside, then div $\mathbf{u} > 0$
- If the amount of water comes inside equals to comes outside, then div $\mathbf{u} = 0$

This page proves the interpretation of divergence.



Imagine the box with volume $\Delta V = \Delta x \Delta y \Delta z$, firstly consider faces I and II, the total flux *out of* faces I and II, as shown above, is:

$$\begin{aligned}
&[u_1(x+\Delta x, y, z) - u_1(x, y, z)]\Delta y \Delta z \\
=&\frac{[u_1(x+\Delta x, y, z) - u_1(x, y, z)]}{\Delta x}\Delta x \Delta y \Delta z \\
=&\frac{\partial u_1}{\partial x}\Delta x \Delta y \Delta z, \qquad \text{(in the limit of } \Delta x \to 0)
\end{aligned}$$

Similarly, the two faces in the $y-$ and $z-$ direction contribute

$$\frac{\partial u_2}{\partial y}\Delta x \Delta y \Delta z, \quad \frac{\partial u_3}{\partial z}\Delta x \Delta y \Delta z$$

Hence net outward flux is:

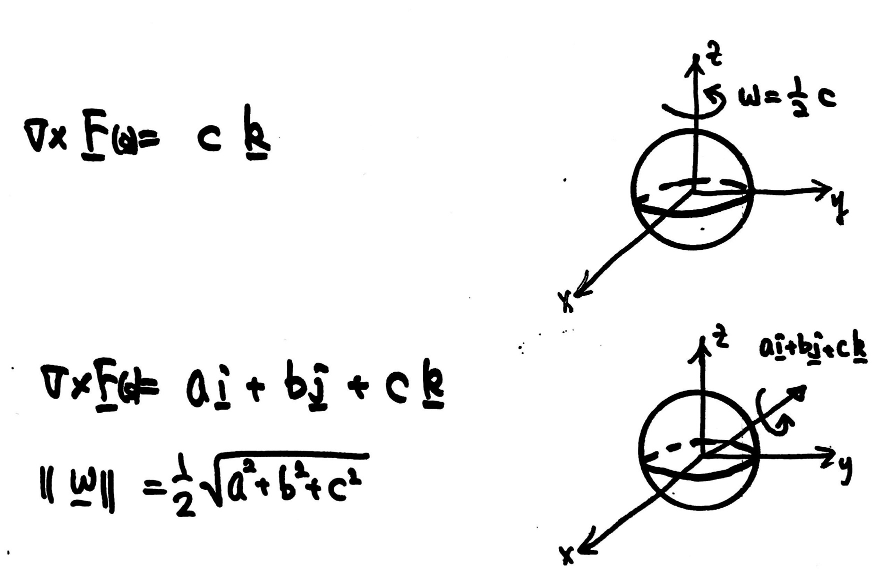
$$\left(\frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} + \frac{\partial u_3}{\partial z}\right) \cdot \Delta V$$

Therefore outward flux *per unit volume* is $\nabla \cdot \mathbf{u}$.

2.2 Interpretation of Curl

Curl is something related to rotation. Consider a small object flying in strong wind, where the speed and direction of wind can be treated as a vector field $\mathbf{F}$. If the object locates at position $\mathbf{r}$, then its rotation has some relation with curl $\mathbf{F}$.

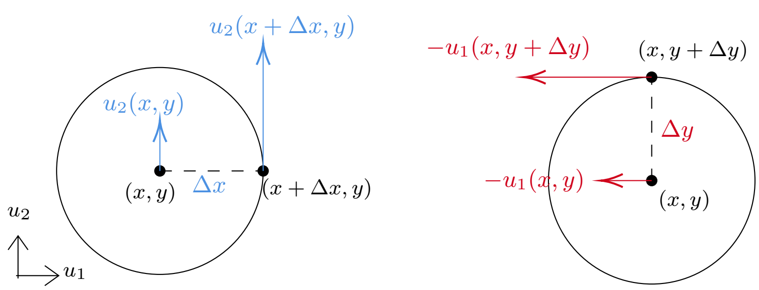
Actually, the object will rotate about the direction $\nabla \times \mathbf{F}(\mathbf{r})$(direction is determined by right-hand rule), and with angular speed $\omega = \frac{1}{2}||\nabla \times \mathbf{F}(\mathbf{r})||$.



The rest of this page prove the relation above.

Consider a disk in $xy-$plane, in y direction, the differential velocity *normal to* Δx is:

$$u_2(x + \Delta x) - u_2(x) = \frac{\partial u_2}{\partial x}\Delta x$$



Recall that $v = \omega r$, so the angular velocity is $\omega_1 = \dfrac{\partial u_2}{\partial x}$

Similarly, in the $y-$direction, (notice the negative sign)

$$-u_1(x, y + \Delta y) + u_1(x, y) = -\frac{\partial u_1}{\partial y}\Delta y, \qquad \omega_2 = -\frac{\partial u_1}{\partial y}$$

Thus the *averaged angular velocity* is: $\omega = \frac{1}{2}\left(\frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y}\right)$

The curl of this vector field is:

$$\operatorname{curl}\mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u_1 & u_2 & 0 \end{vmatrix} = \left(\frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y}\right)\mathbf{k} = 2\omega\mathbf{k}$$

Thus prove the result.

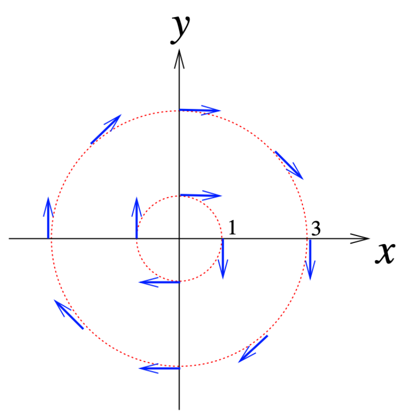
Below example is used to explain the meaning of curl, it's the same example in intro.

[Example.] Assume a vector field: $\mathbf{F}(x, y) = \frac{y\mathbf{i} - x\mathbf{j}}{\sqrt{x^2+y^2}}$.

[Solution.]

$$\begin{aligned} \vec{\omega} = \nabla \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{y}{\sqrt{x^2+y^2}} & \frac{-x}{\sqrt{x^2+y^2}} & 0 \end{vmatrix} \\ &= \left[-\frac{\partial}{\partial x}\left(\frac{x}{\sqrt{x^2+y^2}}\right) - \frac{\partial}{\partial y}\left(\frac{y}{\sqrt{x^2+y^2}}\right)\right]\mathbf{k} \\ &= \mathbf{0} \end{aligned}$$

Consider a small object in the vector field, it doesn't rotate(since curl $\mathbf{F} = \mathbf{0}$ everywhere), it just move in circular, along the vector field.



This definition is optional.

Laplacian Operator

$$\begin{aligned}\nabla^2 = \nabla \cdot \nabla &= \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right) \cdot \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right) \\ &= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\end{aligned}$$

∇^2 is a **scalar** differential operator. Note that

$$\begin{aligned}\nabla^2 f &= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \\ \nabla^2 \mathbf{F} &= \nabla^2 F_1 \mathbf{i} + \nabla^2 F_2 \mathbf{j} + \nabla^2 F_3 \mathbf{k}\end{aligned}$$

2.3 Vector differential identities

Let ϕ, ψ are scalar fields and $\mathbf{F}$ and $\mathbf{G}$ are vector fields, then

(a) $\nabla(\phi\psi) = \phi\nabla\psi + \psi\nabla\phi$

(b) $\nabla \cdot (\phi\mathbf{F}) = \nabla\phi \cdot \mathbf{F} + \phi(\nabla \cdot \mathbf{F})$

(c) $\nabla \times (\phi\mathbf{F}) = \nabla\phi \times \mathbf{F} + \phi(\nabla \times \mathbf{F})$

(d) $\nabla \cdot (\mathbf{F} \times \mathbf{G}) = (\nabla \times \mathbf{F}) \cdot \mathbf{G} - \mathbf{F} \cdot (\nabla \times \mathbf{G})$

(e) $\nabla \cdot (\nabla \times \mathbf{F}) = 0$

(f) $\nabla \times (\nabla\phi) = 0$

(g) $\nabla \times (\nabla \times \mathbf{F}) = \nabla(\nabla \cdot \mathbf{F}) - \nabla^2\mathbf{F}$

[Proof.]

3 Line Integral

Motivation: given a rope(parametrized space curve) $\mathbf{r}(t), a \leq t \leq b$, if the density at point (x, y, z) is given by function $\rho = f(x, y, z)$, we want to find the mass of this rope.

$$\int_a^b f(x,y,z)\ ds = \int_C f(x,y,z)\ ds$$

Recall when we computing arc length in Chapter 11, we knew that:

$$ds = ||\mathbf{r}'(t)||dt$$

Therefore, to calculate line integral $\int_C f(x, y, z)\ ds$, we only need to know:

1. $f(x, y, z)$
2. Region $C:\ \mathbf{r}(t) = (x(t), y(t), z(t)),\ a \leq t \leq b$

This example shows how to find line integral.

[Example.] Find $\int_C xy^4 ds, C$ is the right half of the circle $x^2 + y^2 = 16$.

[Solution.] In order to do this integral, we need the parametric form of the path C. Let $x = 4\cos t, t = 4\sin t$, the right half of the circle means $t \in [-\pi/2, \pi/2]$.

Hence the parametric equation of the curve C is 4 $\mathbf{r}(t) = 4\cos t\mathbf{i} + 4\sin t\mathbf{j}$ with $t \in [-\pi/2, \pi/2]$, then

$$\mathbf{r}'(t) = -4\sin t\mathbf{i} + 4\cos t\mathbf{j} \quad \text{ and } \quad \|\mathbf{r}'(t)\| = 4$$

Thus with $ds = ||\mathbf{r}'(t)||dt$,

$$\int_C xy^4 ds = \int_{-\pi/2}^{\pi/2} f(\mathbf{r})\, \|\mathbf{r}'(t)\|\, dt = \int_{-\pi/2}^{\pi/2} \left[4^5 \cos t \sin^4 t\right](4)dt = \frac{2 \cdot 4^6}{5}$$

Note that there are infinitely many ways to parametrize the curve C, for example, if we had parameterized C as

$$\mathbf{r}(t) = \sqrt{16 - t^2}\mathbf{i} + t\mathbf{j} \quad \text{ where } \quad -4 \leqslant t \leqslant 4$$

Then

$$\mathbf{r}'(t) = -\frac{t}{\sqrt{16 - t^2}}\mathbf{i} + \mathbf{j}, \qquad \|\mathbf{r}'(t)\| = \sqrt{\frac{16}{16 - t^2}}$$

$$\int_C xy^4 ds = \int_C f(\mathbf{r})\, \|\mathbf{r}'(t)\|\, dt = \int_{-4}^{4} \sqrt{16 - t^2} \times t^4 \times \sqrt{\frac{16}{16 - t^2}} dt = \frac{2 \cdot 4^6}{5}$$

Thus the line integral is *independent of parametrization* of the curve C.

So far, we have been doing integration w.r.t. s, but we can also carry the integration w.r.t. x,

$$\int_C f(\mathbf{r}(t))\ dx$$

This example shows how to integrate w.r.t. x, y, z

[Example.] $f(\mathbf{r}(t)) = f(x, y, z) = xy + z$, $C : \mathbf{r}(t) = (x(t), y(t), z(t)) = (t^2, t^3, t), 0 \le t \le 1$

[Solution.] Since $dx = 2t\ dt,\ dy = 3t^2\ dt,\ dz = dt,$

$$\begin{aligned}
\int_C f(\mathbf{r}(t))\ dx &= \int_0^1 (t^2 \cdot t^3 + t)(2t)\ dt = \int_0^1 (2t^6 - 2t^2)\ dt = \cdots \\
\int_C f(\mathbf{r}(t))\ dy &= \int_0^1 (t^5 + t)(3t^2)\ dt = \cdots \\
\int_C f(\mathbf{r}(t))\ dz &= \int_0^1 (t^5 + t)\ dt = \cdots
\end{aligned}$$

But why are we doing this? Consider given $f(\mathbf{r}(t))$ and $C : \mathbf{r}(t)$, we integrate w.r.t. x, y, z, respectively:

$$\int_C f(\mathbf{r}(t))\ dx \qquad \int_C g(\mathbf{r}(t))\ dy \qquad \int_C h(\mathbf{r}(t))\ dz$$

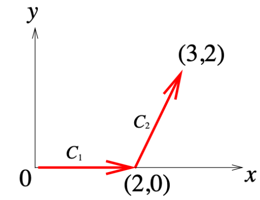
The summation of above three gives:

$$\begin{aligned}
&\int_C [f(\mathbf{r}(t))\ dx + g(\mathbf{r}(t))\ dy + h(\mathbf{r}(t))\ dz] \\
=&\int_C (f, g, h) \cdot (dx, dy, dz) \\
=&\boxed{\int_C \mathbf{F} \cdot d\mathbf{r}}
\end{aligned}$$

where $\mathbf{F}(\mathbf{r}) = (f(\mathbf{r}), g(\mathbf{r}), h(\mathbf{r}))$ is the given vector field.

The following two examples shows usage of integration w.r.t. x, y, z.

[Example.] $\int_C xydx + (x-y)dy, C$ consists of line segments from $(0,0)$ to $(2,0)$ and from $(2,0)$ to $(3,2)$.



[Solution.] We need the parametric function of C_1 and C_2, they are straight lines. Recall in Chapter 10 we know the parameterized curve of straight lines can be written as:

$$\mathbf{r}(t) = (1-t)\mathbf{r}_1 + t\mathbf{r}_2,\ 0 \le t \le 1$$

Hence

$$\begin{aligned} C1: \ \mathbf{r}(t) &= (1-t)(0,0) + t(2,0) = (2t, 0) = (x(t), y(t)),\ 0 \le t \le 1 \\ C2: \ \mathbf{r}(t) &= (1-t)(2,0) + t(3,2) = (2+t, 2t) = (x(t), y(t)),\ 0 \le t \le 1 \end{aligned}$$

Thus the integration is:

$$\begin{aligned} \int_C xydx + (x-y)dy &= \int_{C_1} xydx + (x-y)dy + \int_{C_2} xydx + (x-y)dy \\ &= \int_0^1 (2t)(0)2dt + (2t-0)(0) + \int_0^1 (2+t)(2t)dt + (2+t-2t)2dt \\ &= \int_0^1 (4t + 2t^2 + 2 + t - 2t)dt = \cdots \end{aligned}$$

Notice that: if $\mathbf{F}(\mathbf{r}) = (xy, x-y)$, and $\mathbf{r}_1 = (2t, 0), \mathbf{r}_2 = (2+t, 2t)$, the result above is exactly

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$$

Again, there are many ways to parameterized the curve, for example,

$C_1 : (x, 0),\ 0 \le x \le 2, \qquad C_2 : (x, 2x-4),\ 2 \le x \le 3.$

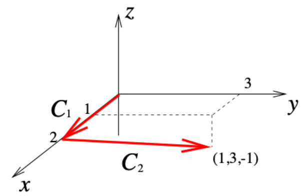
Then

$$\begin{aligned} \int_C xydx + (x-y)dy &= \int_{C_1} [xydx + (x-y)dy] + \int_{C_2} [xydx + (x-y)dy] \\ &= \int_0^2 0dx + \int_2^3 \left(2x^2 - 4x\right) dx + \int_2^3 (-x+4)2dx = \cdots \end{aligned}$$

[Example.]

$\text{I} = \displaystyle\int_C yz\,dx + xz\,dy + xy\,dz$, C consists of line segments from $(0,0,0)$ to $(2,0,0)$, and from $(2,0,0)$ to $(1,3,-1)$.

[*Hint*: $\mathbf{r}(t) = (1-t)\mathbf{r}_0 + t\mathbf{r}_1$, where $0 \leqslant t \leqslant 1$.]



Let $C = C_1 + C_2$, where

$C_1:$ $(0,0,0)$ to $(2,0,0)$ $\Rightarrow$ $x = 2t$, $y = z = 0$, where $0 \leqslant t \leqslant 1$.

$C_2:$ $(2,0,0)$ to $(1,3,-1)$ $\Rightarrow$ $x = -t+2$, $y = 3t$, $z = -t$, where $0 \leqslant t \leqslant 1$.

Then

$$\text{I} = 0 + \int_0^1 \left[(3t^2) + 3(t^2 - 2t) - 3(2t - t^2)\right]\,dt =$$

4 Line Integration in Vector Fields

We have talked so much about line integration above. However, this Chapter is called "Vector Field", so how does line integration relate to vector field?