
MATH 2023 Fall 2021

Multivariable Calculus

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Chapter 13 Application of Partial Derivatives

1 Extreme Values

Recall that in single variable calculus:

x_1 is a *relative maximum point*, if $f'(x_1) = 0$ and $f''(x_1) < 0$,

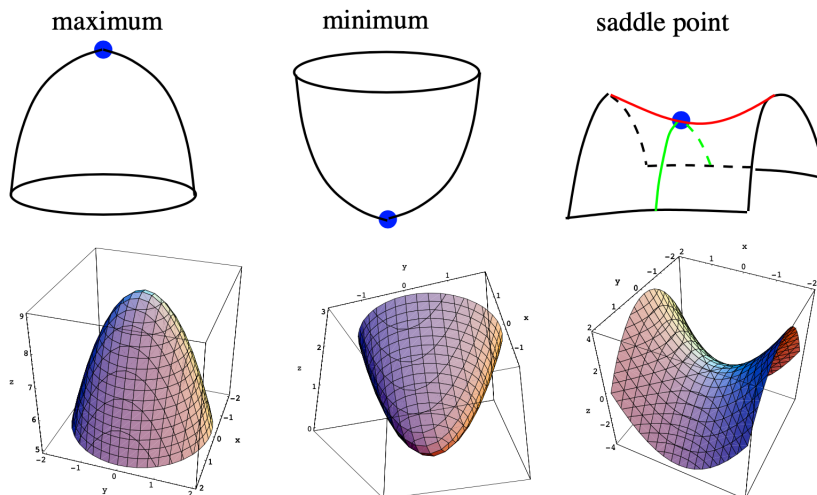
x_2 is a *relative minimum point*, if $f'(x_2) = 0$ and $f''(x_2) > 0$.

Similarly, in multi-variable calculus, the **critical point** is where

$$\nabla f(\mathbf{r}_0) = (f_{x_1}(\mathbf{r}_0), f_{x_2}(\mathbf{r}_0), \dots, f_{x_n}(\mathbf{r}_0)) = \mathbf{0}$$

And, if h has a **relative extremum** at a point \mathbf{r}_0 , then \mathbf{r}_0 is a **critical point**, and $\nabla f(\mathbf{r}_0) = \mathbf{0}$. However, if \mathbf{r}_0 is a critical point, we *cannot infer* that \mathbf{r}_0 is a relative extremum. The reason is similar in single variable calculus.

Different from single variable, a critical point which *is not a relative extremum* can be a **maximum point**, **minimum point**, or a **saddle point**.



However, to classify the critical points, we need the **second derivative test**, or **D-test**.

Second Derivative Test

Suppose $f(x, y)$ has a critical point at $\mathbf{r}_0 = (x_0, y_0)$ (i.e. $\nabla f(\mathbf{r}_0) = \mathbf{0}$) and the second partial derivative of $f(x, y)$ are continuous in a disk with center $\mathbf{r}_0 = (x_0, y_0)$. Let

$$D = \begin{vmatrix} f_{xx}(\mathbf{r}_0) & f_{xy}(\mathbf{r}_0) \\ f_{yx}(\mathbf{r}_0) & f_{yy}(\mathbf{r}_0) \end{vmatrix} = f_{xx}(\mathbf{r}_0)f_{yy}(\mathbf{r}_0) - f_{xy}^2(\mathbf{r}_0)$$

D	$f_{xx}(\mathbf{r}_0)$ or $f_{yy}(\mathbf{r}_0)$	nature of \mathbf{r}_0
> 0	> 0	relative minimum
> 0	< 0	relative maximum
< 0		saddle point
$= 0$		no conclusion can be drawn

I'd like to omit the proof of D-Test here.

This example shows basic use of D-Test.

[Example.] Find the relative minima and maxima of $f(x, y) = x^3 + y^3 - 3x - 12y + 20$.

$$f_x = 3x^2 - 3 \quad \text{and} \quad f_y = 3y^2 - 12$$

[Solution.] For critical points, $f_x = f_y = 0 \Rightarrow x = \pm 1, y = \pm 2$.

$\therefore (1, 2), (-1, 2), (1, -2), (-1, -2)$ are critical points.

To apply D-Test, compute: $f_{xx} = 6x, f_{yy} = 6y, f_{xy} = 0$, hence $D = f_{xx} \cdot f_{yy} - (f_{xy})^2 = 36xy$

Point	f_{xx}	f_{yy}	f_{xy}	D	Type
$(1, 2)$	6	12	0	72	min
$(-1, 2)$	-6	12	0	-72	saddle
$(1, -2)$	6	-12	0	-72	saddle
$(-1, -2)$	-6	-12	0	72	max

This example shows how to find extrema on a *closed* and *bounded* region.

[Example.] Find the absolute extrema of the function

$$z = f(x, y) = xy - x - 3y$$

on the *closed* and *bounded* set R , where R is the triangular region with vertices $(0, 0)$, $(0, 4)$ and $(5, 0)$.

[Solution.] $f_x = y - 1$, $f_y = x - 3$, $f_{xy} = f_{yx} = 1$, $f_{xx} = f_{yy} = 0$, $D = -1$

For critical points, $\nabla f = (f_x, f_y) = (0, 0) \Rightarrow x = 3, y = 1$.

This point is inside the domain. But we still need to find possible extreme points *on the boundary of domain*.

(1) Along OA :, $\mathbf{r}_0 = (0, 0)$, $\mathbf{r}_1 = (5, 0)$, so the parametric representation of line OA is:

$$\mathbf{r}(t) = (1 - t)\mathbf{r}_0 + t\mathbf{r}_1 = (5t, 0), \quad t \in [0, 1]$$

hence $z = f(\mathbf{r}(t)) = -5t$, $t \in [0, 1]$

So along OA , the possible extreme points are $(0, 0)$ and $(5, 0)$.

(2) Along OB :, similarly, $\mathbf{r}(t) = (0, 4t)$, $t \in [0, 1]$, $z = f(\mathbf{r}) = -12t$,

So along OB , the possible extreme points are $(0, 0)$ and $(0, 4)$.

(3) Along AB : $\mathbf{r}(t) = (5 - 5t, 4t)$, $t \in [0, 1]$, $z = -20t^2 + 13t - 5$, $t \in [0, 1]$

There is one critical point on AB , when $dz/dx = 0$, at $\left(\frac{27}{8}, \frac{13}{10}\right)$.

Then we compute the value of all possible extremum points,

(x, y)	$f(x, y)$
$(3, 1)$	-3
$\left(\frac{27}{8}, \frac{13}{10}\right)$	$-\frac{231}{80}$
$(0, 0)$	0
$(5, 0)$	-5
$(0, 4)$	-12

Therefore, absolute maximum value is 0 which occurs at $(0, 0)$, absolute minimum value is -12 which occurs at $(0, 4)$.

This example converts the problem to max/min problem.

[Example.] Find the points on the surface $z^2 = xy + 1$ that are closest to the origin.

[Solution.] $d^2 = (x - 0)^2 + (y - 0)^2 = x^2 + y^2 + xy + 1 = f(x, y)$, only need to minimize this function.

This is a more comprehensive and trickier problem.

[**Example.**] Find absolute minimum and maximum value of $f(x, y) = 2x^3 + y^4$ on the set $D = \{(x, y) | x^2 + y^2 \leq 1\}$.

[**Solution.**] First, find the critical point of $f(x, y)$ on entire xy plane.

$f_x = 6x^2, f_y = 4y^3$, critical points: $f_x = f_y = 0 \Rightarrow x = y = 0$, and $f(0, 0) = 0$.

Then, on the circle $x^2 + y^2 = 1$, eliminate y , we have:

$$\begin{aligned} g(x) &= f(x, y) = x^4 + 2x^3 - 2x^2 + 1, \quad -1 \leq x \leq 1 \\ g'(x) &= 4x^3 + 6x - 4x = 0 \end{aligned}$$

the equation has solutions: $(x, y) = (0, \pm 1), \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$, to check each of them:

$$f(0, \pm 1) = g(0) = 1, \quad f\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right) = g\left(\frac{1}{2}\right) = \frac{13}{16}$$

Also, we need to check the *endpoints*, (since we are looking for min/max point of $g(x)$ on $x \in [-1, 1]$)

$$g(1) = 2, \quad g(-1) = -2$$

Therefore, the absolute minimum is $g(-1) = f(-1, 0) = -2$, the absolute maximum is $g(1) = f(1, 0) = 2$.

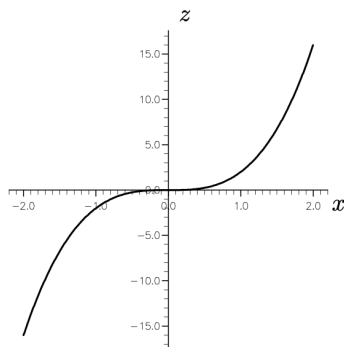
Notice that, if you are interested in the nature of the critical point at $(0, 0)$, you may try D -test:

$$D = f_{xx}f_{yy} - f_{xy}^2 = 144xy^2 = 0 \quad , \text{ at } (0, 0)$$

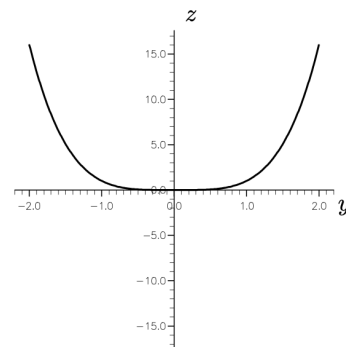
so the D -test fails, we have to use other methods to determine it:

In the plane $y = 0$, $f(x, 0) = 2x^3$, and in the plane $x = 0$, $f(0, y) = y^4$.

In the plane $y = 0$, $f(x, 0) = 2x^3$



In the plane $x = 0$, $f(0, y) = y^4$



Thus, the critical point is a *saddle point*.

2 Lagrange multipliers

Motivation: sometimes we want to maximize/minimize $f(x, y)$ subject to $g(x, y) = k$.

How to find the maximum or minimum value?

1. Find all values of \mathbf{r} and λ such that

$$\nabla f(\mathbf{r}) = \lambda \nabla g(\mathbf{r})$$

and

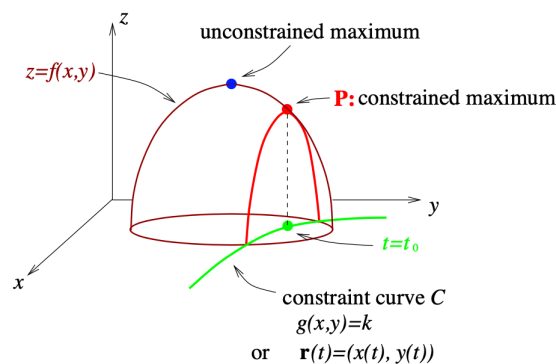
$$g(\mathbf{r}) = k$$

2. Evaluate f at all the points \mathbf{r} that arise from step (1). The largest (smallest) of these values is the maximum (min) value of f .

Remark: Lagrange's method only finds critical points, it *does not tell* whether the function is maximized or minimized.

Proof of Lagrange's method:

Notice that, maximizing or minimizing a function $f(x_1, x_2, \dots, x_n)$ subject to a constraint of $g(x_1, x_2, \dots, x_n) = k$ is to restrict the point (x_1, x_2, \dots, x_n) to lie on the *level surface* S given by $g(x_1, x_2, \dots, x_n) = k$. For example, if $n = 2$, maximize(or minimize) $z = f(x, y)$ subject to constraint curve $C : g(x, y) = k$ (shown in green) is to restrict the point (x, y) to lie on the red curve.



Suppose z has a maximum value at a point \mathbf{P} , and let C be the *constraint curve* with vector equation

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} \quad \text{be on} \quad g(x, y) = k.$$

Assume also that at the point \mathbf{P} , $t = t_0$.

Since on the constraint curve $C : z(t) = f(x(t), y(t))$,

and the point \mathbf{P} should be a critical point. By using the chain rule,

$$\begin{aligned}\frac{dz}{dt} &= \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} \\ &= \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right) \cdot \left(\frac{dx}{dt}, \frac{dy}{dt} \right) \\ &= \nabla f \cdot \mathbf{r}'(t)\end{aligned}$$

Then at the point \mathbf{P} ,

$$\left. \frac{dz}{dt} \right|_{t=t_0} = \nabla f(x(t_0), y(t_0)) \cdot \mathbf{r}'(t_0) = 0 \quad (\text{since } \mathbf{P} \text{ is critical point})$$

Therefore,

$$\nabla f(x(t_0), y(t_0)) \perp \mathbf{r}'(t_0)$$

Moreover, since ∇g is normal vector to the level set,

$$\nabla g(x(t_0), y(t_0)) \perp \mathbf{r}'(t_0)$$

Therefore, $\nabla f \parallel \nabla g$, $\nabla f = \lambda \nabla g$.

The number λ is called a Lagrange multiplier.

[This example shows how to use Lagrange's Method.](#)

[Example.] Find the extreme values of $f(x, y) = x^2 - y^2$ subject to $x^2 + y^2 = 1$.

[Solution.] Find x, y and λ such that $\nabla f = \lambda \nabla g$, where $g = x^2 + y^2 = 1$ (constant)

$$2x\mathbf{i} - 2y\mathbf{j} = \lambda(2x\mathbf{i} + 2y\mathbf{j})$$

, which gives

$$\begin{cases} 2x = \lambda \cdot 2x \\ -2y = \lambda \cdot 2y \end{cases} \Rightarrow \begin{cases} \lambda = 1 & \text{or } x = 0 \\ \lambda = -1 & \text{or } y = 0 \end{cases}$$

From $x^2 + y^2 = 1$, we have:

$$x = 0, y = \pm 1, \lambda = -1$$

$$y = 0, x = \pm 1, \lambda = 1$$

Therefore, f has possible extreme values at point $(0, 1)$, $(0, -1)$, $(-1, 0)$ and $(1, 0)$. Evaluating f at these four points, we find that

$$f(0, 1) = f(0, -1) = -1 \quad (\text{min})$$

$$f(1, 0) = f(-1, 0) = 1 \quad (\text{max})$$

Interpretation of λ

This will not be tested in exam.

Actually, λ has an interpretation which can be very useful.

Suppose M is the optimal value of $f(x, y)$ subject to the constraint $g(x, y) = c$.

Then $f(x, y) = M$ for some ordered pair (x, y) that satisfies the three Lagrangian equations

$$f_x - \lambda g_x = 0$$

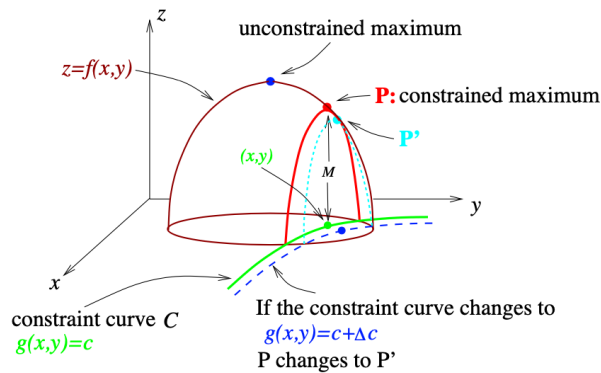
$$f_y - \lambda g_y = 0$$

$$g = c$$

Since $M = f(x, y)$

$$\begin{aligned} \frac{dM}{dc} &= \frac{\partial f}{\partial x} \frac{dx}{dc} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dc} \\ &= f_x \frac{dx}{dc} + f_y \frac{dy}{dc} \\ &= \lambda g_x \frac{dx}{dc} + \lambda g_y \frac{dy}{dc} \\ &= \lambda \left(g_x \frac{dx}{dc} + g_y \frac{dy}{dc} \right) \\ &= \lambda \frac{dg}{dc} = \lambda \end{aligned}$$

where dM/dc is evaluated at the optimal solution values. In other words, λ measures the *sensitivity* of the optimal value of f to change in c .



The example below shows the application of λ .

[**Example.**] Use Lagrangian multiplier to find the maximum and minimum values of the function $f(x, y) = 4x^3 + y^2$ subject to the constraint $2x^2 + y^2 = 1$.

If the constraint equation changes to $2x^2 + y^2 = 0.9$, estimate how this changes will affect the maximum and minimum values of f .

[**Solution.**] Let $g(x, y) = 2x^2 + y^2 = 1 = c$, then for $\nabla f = \lambda \nabla g$, we have $(12x^2, 2y) = \lambda(4x, 2y)$, i.e.,

$$\begin{aligned} 12x^2 &= \lambda 4x \\ 2y &= \lambda 2y \\ 2x^2 + y^2 &= 1 \end{aligned}$$

From the second equation, if $y \neq 0$, then $\lambda = 1$, substitute into the other two equations, we get $(x, y) = (0, \pm 1)$ or $\left(\frac{1}{3}, \pm \frac{\sqrt{7}}{3}\right)$

However, if $y = 0$, then from third equation, $x = \pm \frac{\sqrt{2}}{2}$, substitute to first eq, we get:

$$\begin{aligned} \lambda &= \frac{3\sqrt{2}}{2} \quad \text{when } x = \frac{\sqrt{2}}{2} \\ \lambda &= -\frac{3\sqrt{2}}{2} \quad \text{when } x = -\frac{\sqrt{2}}{2} \end{aligned}$$

λ	(x, y)	$f(x, y)$	nature
1	$(0, \pm 1)$	1	
1	$\left(\frac{1}{3}, \pm \frac{\sqrt{7}}{3}\right)$	$\frac{25}{27}$	
$\frac{3\sqrt{2}}{2}$	$\left(\frac{\sqrt{2}}{2}, 0\right)$	$\sqrt{2}$	max
$-\frac{3\sqrt{2}}{2}$	$\left(-\frac{\sqrt{2}}{2}, 0\right)$	$-\sqrt{2}$	min

Since $\frac{dM}{dc} = \lambda$, so $\Delta M \approx \lambda \Delta c$, in this case, $\Delta c = -0.1$, so at min point,

$$\Delta M = -\frac{3\sqrt{2}}{2} \cdot (-0.1) = \frac{3\sqrt{2}}{20} \quad (\text{increase})$$

at max point,

$$\Delta M = \frac{3\sqrt{2}}{2} \cdot (-0.1) = -\frac{3\sqrt{2}}{20} \quad (\text{decrease})$$

This is the end of Chapter 13.