MATH2023: Multivariable Calculus

2021 Fall Semester, HKUST

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This is the note of course **MATH2023:** *Multivariable Calculus* offered in 2021 Fall semester in HKUST by Professor Jimmy Chi Hung FUNG. The content is almost totally written by Ljm, with some images and examples chosen from Jimmy's lecture notes as well as his lectures.

I have written this note in a very short space of time and as a result it may contain many errors and inaccuracies, readers are welcome to email me at `enor2017@163.com` or create an issue on my GitHub repo `https://github.com/enor2017/CourseNotes/issues`.

Hope you enjoy this book!

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4.1 Extreme Values

Recall that in single variable calculus:

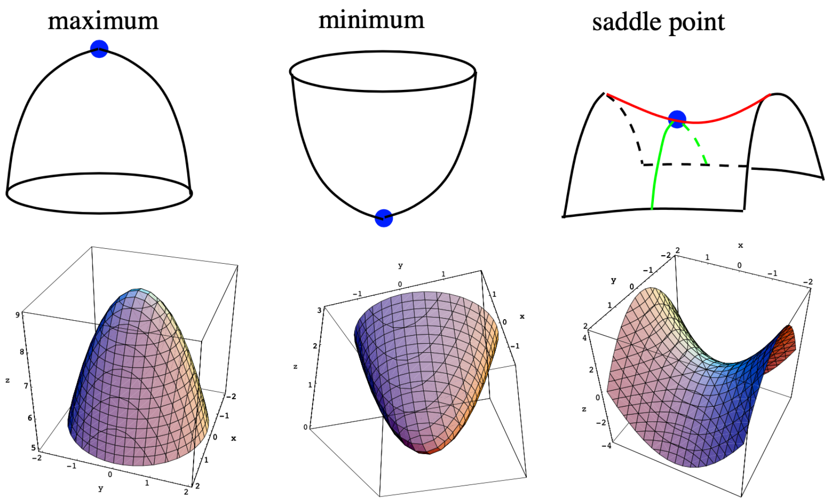
x_1 is a *relative* maximum point, if $f'(x_1) = 0$ and $f''(x_1) < 0$,
x_2 is a *relative* minimum point, if $f'(x_2) = 0$ and $f''(x_2) > 0$.

Similarly, in multi-variable calculus, the **critical point** is where

$$\boxed{\nabla f(\mathbf{r}_0) = (f_{x_1}(\mathbf{r}_0), f_{x_2}(\mathbf{r}_0), \cdots, f_{x_n}(\mathbf{r}_0)) = \mathbf{0}}$$

And, if h has a **relative extremum** at a point $\mathbf{r}_0$, then $\mathbf{r}_0$ is a **critical point**, and $\nabla f(\mathbf{r}_0) = \mathbf{0}$. However, if $\mathbf{r}_0$ is a critical point, we *cannot infer* that $\mathbf{r}_0$ is a relative extremum. The reason is similar in single variable calculus.

Different from single variable, a critical point which *is not a relative extremum* is a **saddle point**.



However, to classify the critical points, we need the **second derivative test**, or **D-test**.

Second Derivative Test

Suppose $f(x, y)$ has a critical point at $\mathbf{r}_0 = (x_0, y_0)$ (i.e. $\nabla f(\mathbf{r}_0) = \mathbf{0}$) and the second partial derivative of $f(x, y)$ are continuous in a disk with center $\mathbf{r}_0 = (x_0, y_0)$. Let

$$\boxed{D = \begin{vmatrix} f_{xx}(\mathbf{r}_0) & f_{xy}(\mathbf{r}_0) \\ f_{yx}(\mathbf{r}_0) & f_{yy}(\mathbf{r}_0) \end{vmatrix} = f_{xx}(\mathbf{r}_0) f_{yy}(\mathbf{r}_0) - f_{xy}^2(\mathbf{r}_0)}$$

D	$f_{xx}(\mathbf{r}_0)$ or $f_{yy}(\mathbf{r}_0)$	nature of $\mathbf{r}_0$
> 0	> 0	relative minimum
> 0	< 0	relative maximum
< 0		saddle point
$= 0$		no conclusion can be drawn

I'd like to omit the proof of D-Test here.

This example shows basic use of D-Test.

[Example.] Find the relative minima and maxima of $f(x, y) = x^3 + y^3 - 3x - 12y + 20$.

$$f_x = 3x^2 - 3 \quad \text{and} \quad f_y = 3y^2 - 12$$

[Solution.] For critical points, $f_x = f_y = 0 \quad \Rightarrow \quad x = \pm 1, y = \pm 2$.

$\therefore (1, 2), (-1, 2), (1, -2), (-1, -2)$ are critical points.

To apply D-Test, compute: $f_{xx} = 6x, f_{yy} = 6y, f_{xy} = 0$, hence $D = f_{xx} \cdot f_{yy} - (f_{xy})^2 = 36xy$

Point	f_{xx}	f_{yy}	f_{xy}	D	Type
$(1, 2)$	6	12	0	72	min
$(-1, 2)$	-6	12	0	-72	saddle
$(1, -2)$	6	-12	0	-72	saddle
$(-1, -2)$	-6	-12	0	72	max

This example shows how to find extrema on a *closed* and *bounded* region.

[Example.] Find the absolute extrema of the function

$$z = f(x, y) = xy - x - 3y$$

on the *closed* and *bounded* set R, where R is the triangular region with vertices $(0, 0), (0, 4)$ and $(5, 0)$.

[Solution.] $f_x = y - 1, f_y = x - 3, f_{xy} = f_{yx} = 1, f_{xx} = f_{yy} = 0, D = -1$

For critical points, $\nabla f = (f_x, f_y) = (0, 0) \Rightarrow x = 3, y = 1$.

This point is inside the domain. But we still need to find possible extreme points *on the boundary of domain.*

(1) Along OA :, $\mathbf{r}_0 = (0, 0), \mathbf{r}_1 = (5, 0)$, so the parametric representation of line OA is:

$$\mathbf{r}(t) = (1 - t)\mathbf{r}_0 + t\mathbf{r}_1, = (5t, 0),\ \ t \in [0, 1]$$

hence $z = f(\mathbf{r}(t)) = -5t,\ \ t \in [0, 1]$

So along OA, the possible extreme points are $(0, 0)$ and $(5, 0)$.

(2) Along OB :, similarly, $\mathbf{r}(t) = (0, 4t),\ \ t \in [0, 1],\ z = f(\mathbf{r}) = -12t,$

So along OB, the possible extreme points are $(0, 0)$ and $(0, 4)$.

(3) Along AB : $\mathbf{r}(t) = (5 - 5t, 4t),\ \ t \in [0, 1],\ z = -20t^2 + 13t - 5,\ \ t \in [0, 1]$

There is one critical point on AB, when $dz/dx = 0$, at $\left(\frac{27}{8}, \frac{13}{10}\right)$.

Then we compute the value of all possible extremum points,

(x, y)	$f(x, y)$
$(3, 1)$	-3
$\left(\frac{27}{8}, \frac{13}{10}\right)$	$-\frac{231}{80}$
$(0, 0)$	0
$(5, 0)$	-5
$(0, 4)$	-12

Therefore, absolute maximum value is 0 which occurs at $(0, 0)$, absolute minimum value is -12 which occurs at $(0, 4)$.

This example converts the problem to max/min problem.

[Example.] Find the points on the surface $z^2 = xy + 1$ that are closest to the origin.

[Solution.] $d^2 = (x-0)^2 + (y-0)^2 + (z-0)^2 = x^2 + y^2 + xy + 1 = f(x, y)$, only need to minimize this function.

This is a more comprehensive and tricker problem.

[Example.] Find absolute minimum and maximum value of $f(x,y) = 2x^3 + y^4$ on the set $D = \{(x,y)|x^2 + y^2 \leq 1\}$.

[Solution.] First, find the critical point of $f(x,y)$ on entire xy plane.

$f_x = 6x^2, f_y = 4y^3$, critical points: $f_x = f_y = 0 \Rightarrow x = y = 0$, and $f(0,0) = 0$.

Then, on the circle $x^2 + y^2 = 1$, eliminate y, we have:

$$g(x) = f(x,y) = x^4 + 2x^3 - 2x^2 + 1, \ -1 \leq x \leq 1$$
$$g'(x) = 4x^3 + 6x - 4x = 0$$

the equation has solutions: $(x,y) = (0, \pm 1), \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$, to check each of them:

$$f(0,\pm 1) = g(0) = 1, \ f\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right) = g\left(\frac{1}{2}\right) = \frac{13}{16}$$

Also, we need to check the *endpoints*, (since we are looking for min/max point of $g(x)$ on $x \in [-1,1]$)

$$g(1) = 2, \ g(-1) = -2$$

Therefore, the absolute minimum is $g(-1) = f(-1,0) = -2$, the absolute maximum is $g(1) = f(1,0) = 2$.

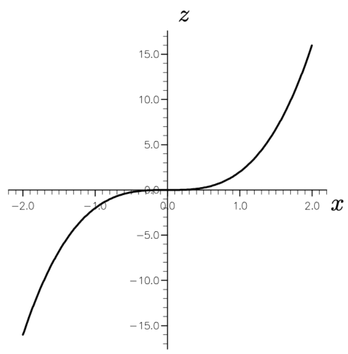
Notice that, if you are interested in the nature of the critical point at $(0,0)$, you may try D-test:

$$D = f_{xx}f_{yy} - f_{xy}^2 = 144xy^2 = 0 \quad \text{, at (0,0)}$$

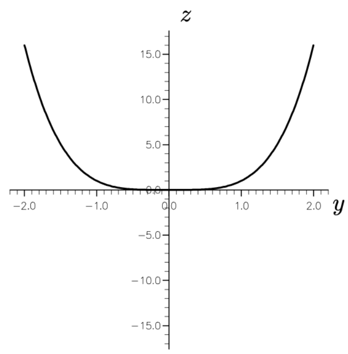
so the D-test fails, we have to use other methods to determine it:

In the plane $y = 0$, $f(x,0) = 2x^3$, and in the plane $x = 0$, $f(0,y) = y^4$.

In the plane $y = 0$, $f(x, 0) = 2x^3$



In the plane $x = 0$, $f(0, y) = y^4$



Thus, the critical point is a *saddle point.*

4.2 Lagrange multipliers

Motivation: sometimes we want to maximize/minimize $f(x, y)$ subject to $g(x, y) = k$.

How to find the maximum or minimum value?

1. Find all values of $\mathbf{r}$ and λ such that

$$\nabla f(\mathbf{r}) = \lambda \nabla g(\mathbf{r})$$

and

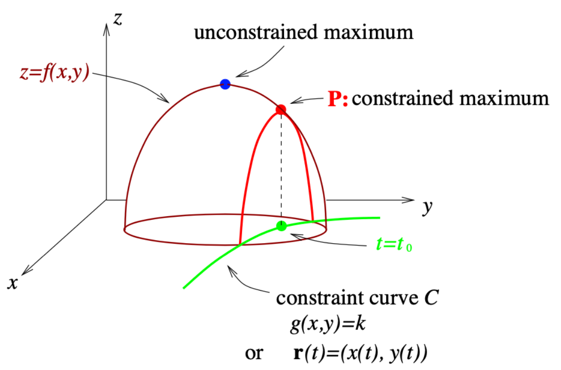
$$g(\mathbf{r}) = k$$

2. Evaluate f at all the points $\mathbf{r}$ that arise from step (1). The largest (smallest) of these values is the maximum (min) value of f.

Remark: Lagrange's method only finds critical points, it *does not tell* whether the function is maximized or minimized.

Proof of Lagrange's method:

Notice that, maximizing or minimizing a function $f(x_1, x_2, \cdots, x_n)$ subject to a constraint of $g(x_1, x_2, \cdots, x_n) = k$ is to restrict the point $(x_1, x_2, \cdots, x_n)$ to lie on the *level surface* S given by $g(x_1, x_2, \cdots, x_n) = k$. For example, if $n = 2$, maximize(or minimize) $z = f(x, y)$ subject to constraint curve $C : g(x, y) = k$ (shown in green) is to restrict the point (x, y) to lie on the red curve.



Suppose z has a maximum value at a point $\mathbf{P}$, and let C be the *constraint curve* with vector equation

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} \quad \text{be on} \quad g(x, y) = k.$$

Assume also that at the point $\mathbf{P}, t = t_0$.

Since on the constraint curve $C : z(t) = f(x(t), y(t))$,

and the point $\mathbf{P}$ should be a critical point. By using the chain rule,

$$\begin{aligned} \frac{dz}{dt} &= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} \\ &= \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) \cdot \left(\frac{dx}{dt}, \frac{dy}{dt}\right) \\ &= \nabla f \cdot \mathbf{r}'(t) \end{aligned}$$

Then at the point $\mathbf{P}$,

$$\left.\frac{dz}{dt}\right|_{t=t_0} = \nabla f\left(x\left(t_0\right), y\left(t_0\right)\right) \cdot \mathbf{r}'\left(t_0\right) = 0 \quad \text{(since } \mathbf{P} \text{ is critical point)}$$

Therefore,

$$\nabla f\left(x\left(t_0\right), y\left(t_0\right)\right) \perp \mathbf{r}'\left(t_0\right)$$

Moreover, since ∇g is normal vector to the level set,

$$\nabla g\left(x\left(t_0\right), y\left(t_0\right)\right) \perp \mathbf{r}'\left(t_0\right)$$

Therefore, $\nabla f \parallel \nabla g$, $\nabla f = \lambda \nabla g$.

The number λ is called a Lagrange multiplier.

This example shows how to use Lagrange's Method.

[Example.] Find the extreme values of $f(x,y) = x^2 - y^2$ subject to $x^2 + y^2 = 1$.

[Solution.] Find x, y and λ such that $\nabla f = \lambda \nabla g$, where $g = x^2 + y^2 = 1$(constant)

$$2x\mathbf{i} - 2y\mathbf{j} = \lambda(2x\mathbf{i} + 2y\mathbf{j})$$

, which gives

$$\begin{cases} 2x = & \lambda \cdot 2x \\ -2y = & \lambda \cdot 2y \end{cases} \quad \Rightarrow \quad \begin{cases} \lambda = 1 & \text{or} \quad x = 0 \\ \lambda = -1 & \text{or} \quad y = 0 \end{cases}$$

From $x^2 + y^2 = 1$, we have:

$$x = 0,\ y = \pm 1,\ \lambda = -1$$
$$y = 0,\ x = \pm 1,\ \lambda = 1$$

Therefore, f has possible extreme values at point $(0,1), (0,-1), (-1,0)$ and $(1,0)$. Evaluating f at these four points, we find that

$$f(0,1) = f(0,-1) = -1 \quad \text{(min)}$$
$$f(1,0) = f(-1,0) = 1 \quad \text{(max)}$$

Interpretation of λ

This will not be tested in exam.

Actually, λ has an interpretation which can be very useful.

Suppose M is the optimal value of $f(x, y)$ subject to the constraint $g(x, y) = c$.

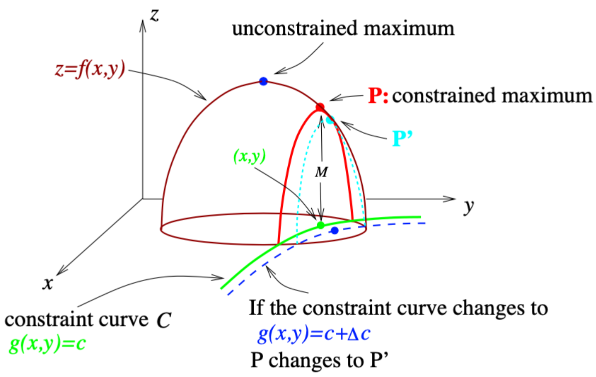
Then $f(x, y) = M$ for some ordered pair (x, y) that satisfies the three Lagrangian equations

$$\begin{aligned} f_x - \lambda g_x &= 0 \\ f_y - \lambda g_y &= 0 \\ g &= c \end{aligned}$$

Since $M = f(x, y)$

$$\begin{aligned} \frac{dM}{dc} &= \frac{\partial f}{\partial x}\frac{dx}{dc} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dc} \\ &= f_x\frac{dx}{dc} + f_y\frac{dy}{dc} \\ &= \lambda g_x\frac{dx}{dc} + \lambda g_y\frac{dy}{dc} \\ &= \lambda\left(g_x\frac{dx}{dc} + g_y\frac{dy}{dc}\right) \\ &= \lambda\frac{dg}{dc} = \lambda \end{aligned}$$

where dM/dc is evaluated at the optimal solution values. In other words, λ measures the *sensitivity* of the optimal value of f to change in c.



The example below shows the application of λ.

[Example.] Use Lagrangian multiplier to find the maximum and minimum values of the function $f(x, y) = 4x^3 + y^2$ subject to the constraint $2x^2 + y^2 = 1$.

If the constraint equation changes to $2x^2 + y^2 = 0.9$, estimate how this changes will affect the maximum and minimum values of f.

[Solution.] Let $g(x, y) = 2x^2 + y^2 = 1 = c$, then for $\nabla f = \lambda \nabla g$, we have $(12x^2, 2y) = \lambda(4x, 2y)$, i.e.,

$$\begin{aligned} 12x^2 &= \lambda 4x \\ 2y &= \lambda 2y \\ 2x^2 + y^2 &= 1 \end{aligned}$$

From the second equation, if $y \neq 0$, then $\lambda = 1$, substitute into the other two equations, we get $(x, y) = (0, \pm 1)$ or $\left(\frac{1}{3}, \pm\frac{\sqrt{7}}{3}\right)$

However, if $y = 0$, then from third equation, $x = \pm\frac{\sqrt{2}}{2}$, substitute to first eq, we get:

$$\begin{aligned} \lambda &= \frac{3\sqrt{2}}{2} \quad \text{when} \quad x = \frac{\sqrt{2}}{2} \\ \lambda &= -\frac{3\sqrt{2}}{2} \quad \text{when} \quad x = -\frac{\sqrt{2}}{2} \end{aligned}$$

λ	(x, y)	$f(x, y)$	nature
1	$(0, \pm 1)$	1	
1	$\left(\frac{1}{3}, \pm\frac{\sqrt{7}}{3}\right)$	$\frac{25}{27}$	
$\frac{3\sqrt{2}}{2}$	$\left(\frac{\sqrt{2}}{2}, 0\right)$	$\sqrt{2}$	max
$-\frac{3\sqrt{2}}{2}$	$\left(-\frac{\sqrt{2}}{2}, 0\right)$	$-\sqrt{2}$	min

Since $\frac{dM}{dc} = \lambda$, so $\Delta M \approx \lambda \Delta c$, in this case, $\Delta c = -0.1$, so at min point,

$$\Delta M = -\frac{3\sqrt{2}}{2} \cdot (-0.1) = \frac{3\sqrt{2}}{20} \quad \text{(increase)}$$

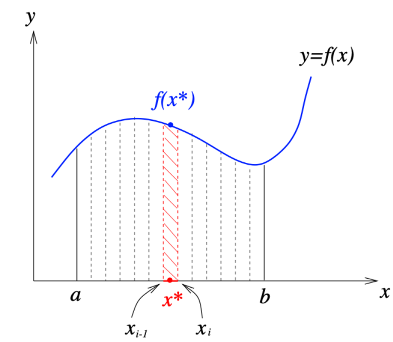
at max point,

$$\Delta M = \frac{3\sqrt{2}}{2} \cdot (-0.1) = -\frac{3\sqrt{2}}{20} \quad \text{(decrease)}$$

5. Chapter 14: Multiple Integrations

5.1 Double Integrals Over Rectangles

Recall that in single variable calculus, we divided a region into thin rectangles and use the **Riemann Sum** as integral.

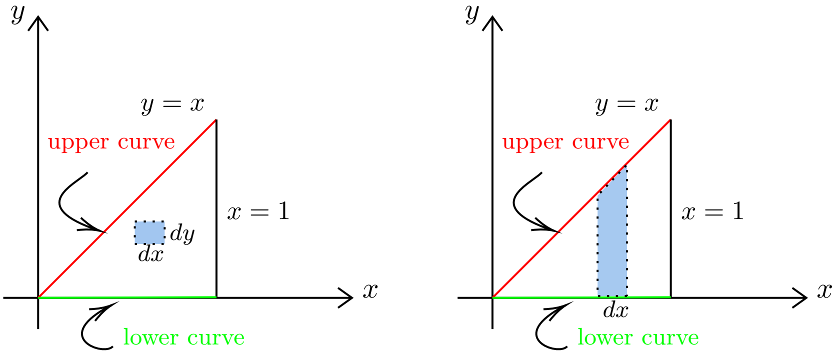


$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^\star)\delta x_i$$

Actually, instead of thin rectangles, we can use *small rectangles* to cover the area.

[Example.] Find the area bounded by $y = x$, $x = 1$ and $y = 0$.

[Solution.] (1) View the area as bounded by *upper curve* and *lower curve*.

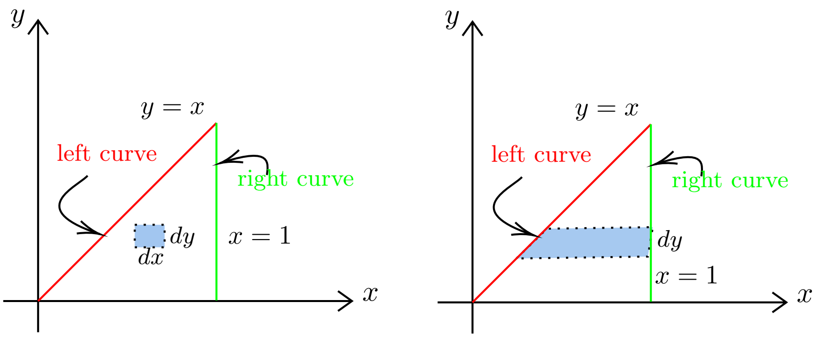


For the *small rectangles* with $dA = dxdy = dydx$, we first move it vertically, the lower bound is $y = 0$, and the upper bound is $y = x$, hence $\int_0^x dy$ is the shaded area in right image above.

Then we move the shaded area horizontally, the left bound is point $x = 0$, while the right bound is point $x = 1$, thus the total area is:

$$A = \int_0^1 \int_0^x dydx = \int_0^1 y\Big|_0^x dx = \int_0^1 xdx = \frac{1}{2}$$

(2) Alternatively, we can first move the rectangle horizontally, hitting *left curve* $x = y$ and *right curve* $x = 1$, hence the shaded area is $\int_y^1 dx$. note here we integral dx first, so when hitting the boundaries, we need to check x equals to what, i.e. $x = f(y)$. For example, here the two bounds are $x = y$ and $x = 1$.



Then we move the shaded area vertically, hitting lower bound $y = 0$(a point) and upper bound $y = 1$(a point), hence the total area is:

$$A = \int_0^1 \int_y^1 dxdy = \int_0^1 x\Big|_y^1 dy = \int_0^1 (1-y)dy = \frac{1}{2}$$

5.2 Double Integrals Over General Regions

5.3 Double Integrals in Polar Coordinates

5.4 Change of Variables in Integrals

Recall that in single variable calculus, we often use a *substitution* to simplify an integral.

$$\int_a^b f(x)dx = \int_c^d f(g(u)) \cdot g'(u)\ du$$

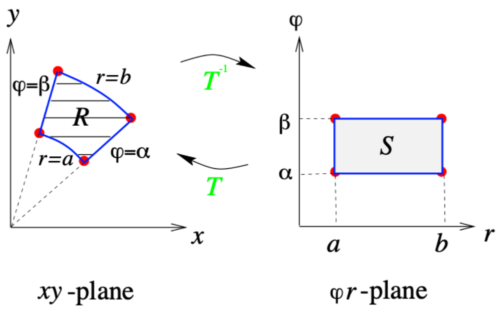
where $a = g(c)$ and $b = g(d)$. Notice that we can *view substitution as a kind of mapping*, and the change-of-variable process introduces *an additional factor* $g'(u)$ into the integrand.

This method can also be useful in multiple integrals. We have already seen one example: integration in *polar coordinate*.

$$\iint_R f(x,y)dA_{xy} = \iint_S f(r\cos\theta, r\sin\theta)rdrd\theta = \iint_S f(r\cos\theta, r\sin\theta)rdA_{r\theta}$$

In this example, *the additional factor* is r.

The *mapping* T is shown as below: we transform the region R into S, where S is an rectangle in θr-plane, which is easy to integrate.

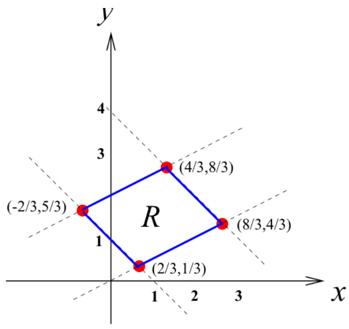


$$T(r,\theta) = (x,y) = (r\cos\theta, r\sin\theta)$$

[Example.] Find a change of variable.

.1 Let R be the region bounded by the lines

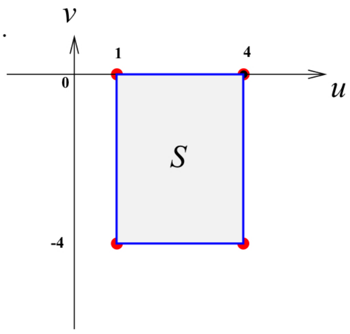
$$x - 2y = 0$$
$$x - 2y = -4$$
$$x + y = 4$$
$$x + y = 1$$



as shown. Find a transformation T from a region S to R such that S is a rectangular region (with sides parallel to the u- and v-axis).

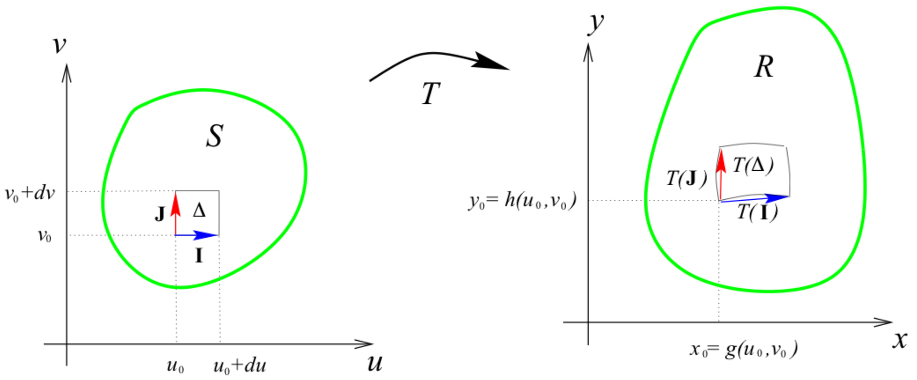
Let $u = x + y$, $v = x - 2y$, then $T(u, v) = (x, y) = \left(\frac{1}{3}(2u + v),\ \frac{1}{3}(u - v)\right)$.

$$v = 0$$
$$v = -4$$
$$u = 4$$
$$u = 1.$$



Note that the transformation T maps the vertices of the region S onto the vertices of the region R.

Now, to find $\iint_R f(x,y)dxdy$, if we make the change of variables $x = g(u,v), y = h(u,v)$, then *we are mapping things in uv-plane onto things in xy-plane.* For mapping function $T(u,v) = (g(u,v), h(u,v)) = (x,y)$, and assume the area S in uv-plane corresponds to region R in xy-plane, as shown below.



We still use the method that integrate all "small rectangles", Δ, as shown in uv-plane. Assume Δ locates at (u_0, v_0) and has area $dA = dudv$. Let

$\mathbf{I}$ be the vector from (u_0, v_0) to $(u_0 + du, v_0)$ and
$\mathbf{J}$ be the vector from (u_0, v_0) to $(u_0, v_0 + dv)$.

Then mapping T "takes" $\mathbf{I}$ to the vector $T(\mathbf{I})$ from $(g(u_0, v_0), h(u_0, v_0))$ to $(g(u_0 + du, v_0), h(u_0 + du, v_0))$. Notice the vector $T(\mathbf{I})$ is *not necessary a straight vector.* Now

$$\begin{aligned} T(\mathbf{I}) &= (g(u_0 + du, v_0) - g(u_0, v_0), h(u_0 + du, v_0) - h(u_0, v_0)) \\ &= \left(\frac{g(u_0 + du, v_0) - g(u_0, v_0)}{du}, \frac{h(u_0 + du, v_0) - h(u_0, v_0)}{du} \right) du \\ &= \left(\frac{\partial g}{\partial u}, \frac{\partial h}{\partial u} \right) du \quad \text{(for } du \to 0\text{)} \end{aligned}$$

Similarly, $T(\mathbf{J}) = \left(\frac{\partial g}{\partial v}, \frac{\partial h}{\partial v} \right) dv$.

Then the area of $T(\Delta)$ is

$$dxdy = \|T(\mathrm{I}) \cdot T(\mathrm{J})\| = \left| \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} dudv \right| = \left| \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} \right| dudv$$

which means, $dA_{xy} = |J|\, dA_{uv}$, where $|J|$ is the "additional factor" caused by this substitution, and it is called the **Jacobian** of mapping T, given by:

$$\boxed{T = \frac{\partial(x,y)}{\partial(u,v)} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}}$$

Therefore, the formula of *change of variable for two variables* is:

$$\boxed{\iint_{R=T(S)} f(x,y)dxdy = \iint_S f(T(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv}$$

while for *three variables*,

$$\boxed{\iiint_{R=T(S)} f(x,y,z)dxdydz = \iiint_S f(T(u,v,w)) \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| dudvdw}$$

This example shows substitution can be easy for some integrations.

[Example.] Find the area of $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

[Solution.] **Method 1:** directly integrate

$$\begin{aligned}
\frac{1}{4}A &= \int_0^a \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} dydx \\
&= \int_0^a b\sqrt{1-\frac{x^2}{a^2}}\ dx
\end{aligned}$$

We are familiar with substitution in single variable integration, let $x = a\sin\theta$, when $x = 0$, $\theta = 0$, and when $x = a$, $\theta = \dfrac{\pi}{2}$. Then,

$$\begin{aligned}
\frac{1}{4}A &= \int_0^{\frac{\pi}{2}} b(1-\sin^2\theta)^{\frac{1}{2}} a\cos\theta\ d\theta \\
&= ab\int_0^{\frac{\pi}{2}} \cos^2\theta\ d\theta \\
&= ab\int_0^{\frac{\pi}{2}} \frac{1}{2}(1-\cos 2\theta)\ d\theta \\
&= \frac{1}{4}ab\pi
\end{aligned}$$

Method 2: Mapping the ellipse to a disk.

Let $x = au,\ y = bv$, then $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ becomes $u^2 + v^2 = 1$.

The Jacobian of this mapping

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} a & 0 \\ 0 & b \end{vmatrix} = ab$$

Therefore, the area of ellipse

$$\iint_R dA_{xy} = \iint_S J\cdot dA_{xy} = \iint_S ab\ dA_{uv} = ab\pi$$

We observe that method 2 is much easier than method 1.

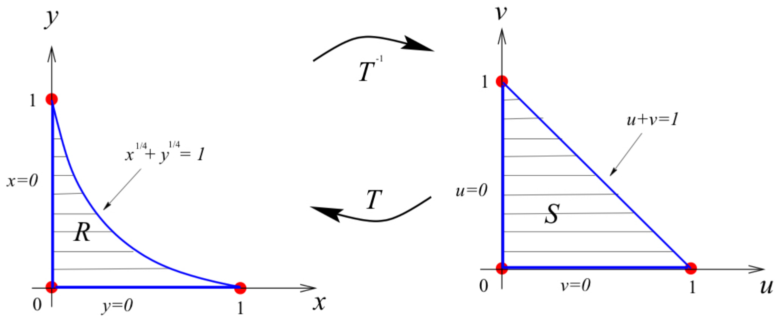
This example also uses Jacobian.

[Example.] Find the area bounded by $\sqrt[4]{x} + \sqrt[4]{y} = 1$ and the x and y axes.

[Solution.]

This integral would be tedious to evaluate directly because the region R is not 'simple'. So instead we find a suitable transformation of variables. Let

Let $u = \sqrt[4]{x}$, $v = \sqrt[4]{y}$, then $x = u^4$, $y = v^4$ and $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 4u^3 & 0 \\ 0 & 4v^3 \end{vmatrix} = 16u^3v^3$



$$\text{Area} = \iint\limits_R dxdy = \iint\limits_S \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv = \int_0^1 \int_0^{1-v} 16u^3v^3\, dudv = \frac{1}{70}$$

Sometimes, it's not easy to calculate $\dfrac{\partial(x,y)}{\partial(u,v)}$, since usually we substitute u and v as functions of x and y, so we always need to find the inverse function in order to calculate $\dfrac{\partial(x,y)}{\partial(u,v)}$. So we consider the relationship between $\dfrac{\partial(x,y)}{\partial(u,v)}$ and $\dfrac{\partial(u,v)}{\partial(x,y)}$.

Because of $\det A \det B = \det(AB)$,

$$
\begin{aligned}
\frac{\partial(x,y)}{\partial(u,v)} \cdot \frac{\partial(u,v)}{\partial(x,y)} &= \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \\
&= \begin{vmatrix} \frac{\partial x}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial x}{\partial v}\frac{\partial v}{\partial x} & \frac{\partial x}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial x}{\partial v}\frac{\partial v}{\partial y} \\ \frac{\partial y}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial y}{\partial v}\frac{\partial v}{\partial x} & \frac{\partial y}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial y}{\partial v}\frac{\partial v}{\partial y} \end{vmatrix} \\
&= \begin{vmatrix} \frac{\partial x}{\partial x} & \frac{\partial x}{\partial y} \\ \frac{\partial y}{\partial x} & \frac{\partial y}{\partial y} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1
\end{aligned}
$$

Therefore, if $x(u,v)$ and $y(u,v)$ have continuous first partial derivatives, and that

$$\frac{\partial(x,y)}{\partial(u,v)} \neq 0 \quad \text{at} \quad (u,v) \quad \text{(one-to-one map)}.$$

Then

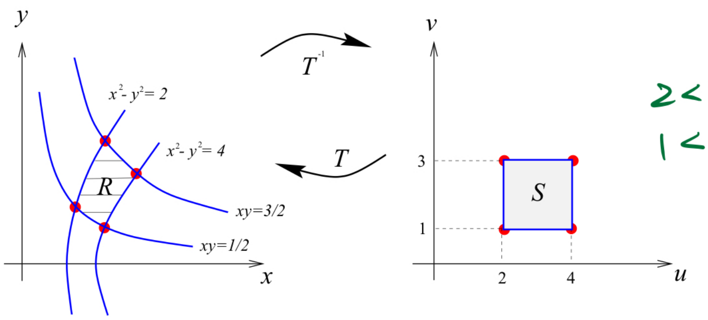
$$\boxed{\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{\dfrac{\partial(u,v)}{\partial(x,y)}}}$$

This example shows the situation when $\dfrac{\partial(x,y)}{\partial(u,v)}$ is difficult to compute.

[Example.] Find $\displaystyle\iint_A (x^2+y^2)\,dxdy$, where $A = \left[(x,y) \mid x,y>0, \quad 2 \leqslant x^2-y^2 \leqslant 4, \quad \frac{1}{2} \leqslant xy \leqslant \frac{3}{2}\right]$

[Solution.]

The change of the variables is motivated by the occurrence of the expressions x^2-y^2 and xy in the equations of the boundary.



$2<u<4$
$1<v<3$

Let $u=x^2-y^2$, $v=2xy$, then $(x^2+y^2)^2=(x^2-y^2)^2+4x^2y^2=u^2+v^2$ and

difficult to calculate

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{\dfrac{\partial(u,v)}{\partial(x,y)}} = \frac{1}{\begin{vmatrix} 2x & -2y \\ 2y & 2x \end{vmatrix}} = \frac{1}{4(x^2+y^2)} = \frac{1}{4\sqrt{u^2+v^2}}.$$

So $\displaystyle\iint_R (x^2+y^2)\,dxdy = \int_{v=1}^{3}\int_{u=2}^{4} \sqrt{u^2+v^2}\cdot\frac{1}{4\sqrt{u^2+v^2}}\,dudv$

Sometimes, though the given region is a relatively good one, but it's still difficult to directly integrate, maybe because the integrand is too complicated. See the below example:

[Example.]

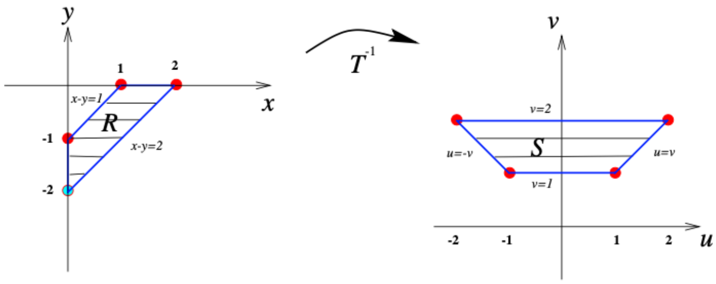
Evaluate the integral $\iint_R e^{(x+y)/(x-y)}\, dA$, where R is the trapezoidal region with vertices $(1,0)$, $(2,0)$, $(0,-2)$, and $(0,-1)$.

Since it is not easy to integrate $e^{(x+y)/(x-y)}$, we make a change of variables suggested by a form of the integrand. In particular, let

$$u = x+y, \qquad v = x-y.$$

These equations define a transformation T^{-1} from the xy-plane to the uv-plane.

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{\dfrac{\partial(u,v)}{\partial(x,y)}} = \frac{1}{\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}} = -\frac{1}{2}.$$



The sides of R lie on the lines

$$y = 0, \qquad x-y = 2, \qquad x = 0, \qquad x-y = 1$$

and the image lines in the uv-plane are

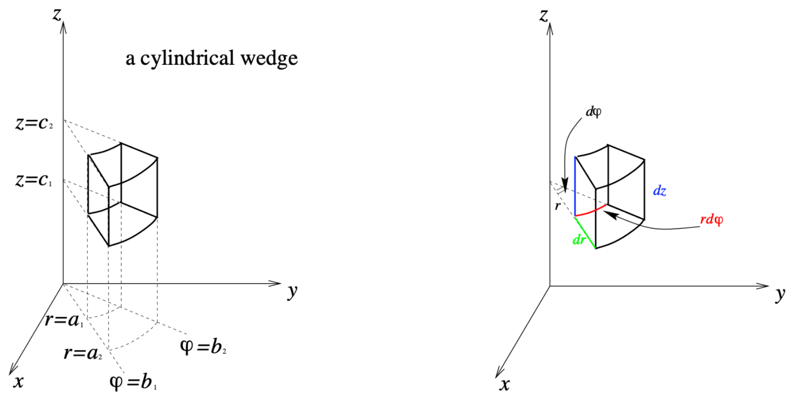
$$u = v, \qquad v = 2, \qquad u = -v, \qquad v = 1.$$

$$\begin{aligned} \therefore \quad \iint_R e^{(x+y)/(x-y)}\, dA &= \iint_S e^{u/v} \left| \frac{\partial(x,y)}{\partial(u,v)} \right| dudv \\ &= \int_1^2 \int_{-v}^{v} e^{u/v} \left(\frac{1}{2}\right) dudv \\ &= \end{aligned}$$

5.5 Triple Integrals in Cylindrical & Spherical Coordinates

Cylindrical coordinates are suited to problems *with axial symmetry*(the shape is around the z-axis)

The basic unit of cylindrical coordinate is shown below, and it's volume is $r\ drd\theta dz$



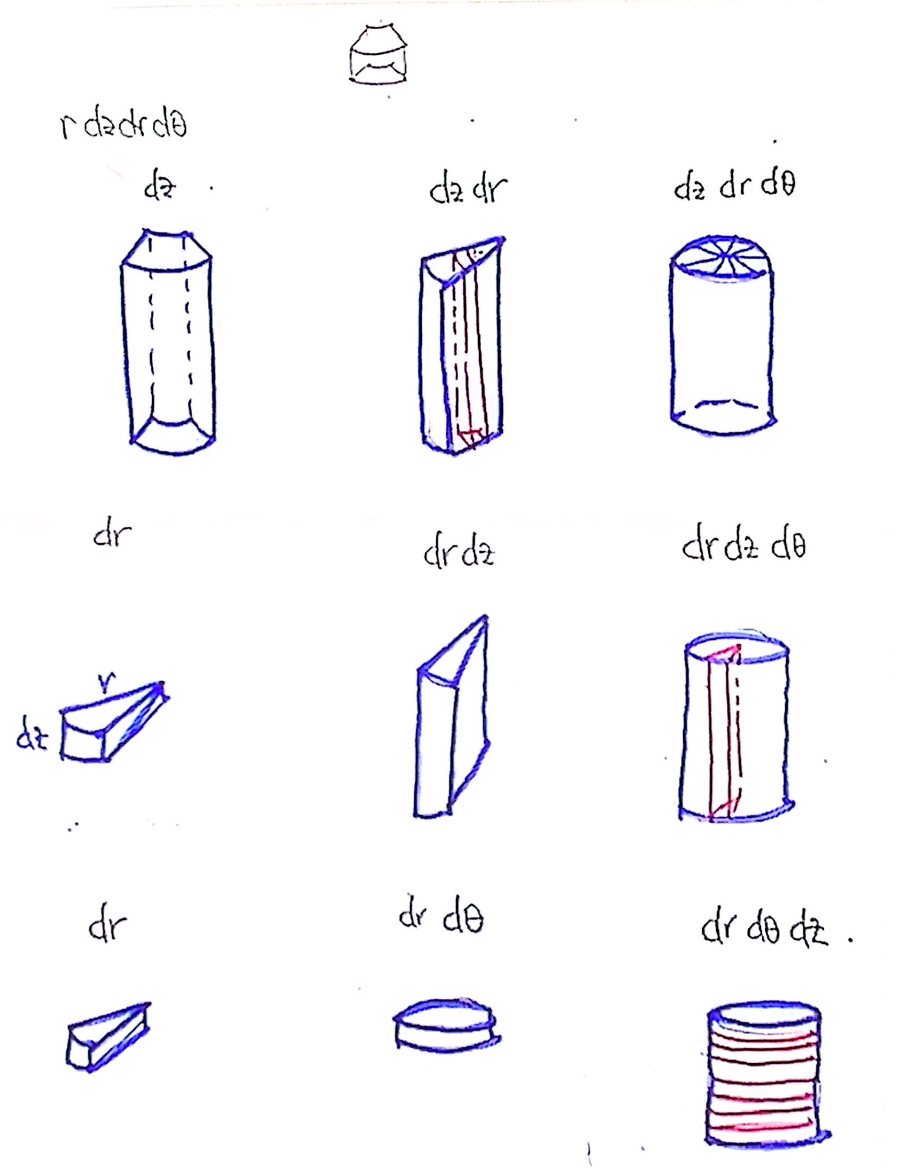
We can also find Jacobian in order to find the relation. For $x = r\cos\theta, y = r\sin\theta, z = z$, we have

$$\frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r$$

that is, $dxdydz \to r\ drd\theta dz$

$$\boxed{V_c = \iiint_V f dV = \iiint_{V(x,y,z)} f(x,y,z)dxdydz = \iiint_{V(r,\theta,z)} f(r,\theta,z)rdrd\theta dz}$$

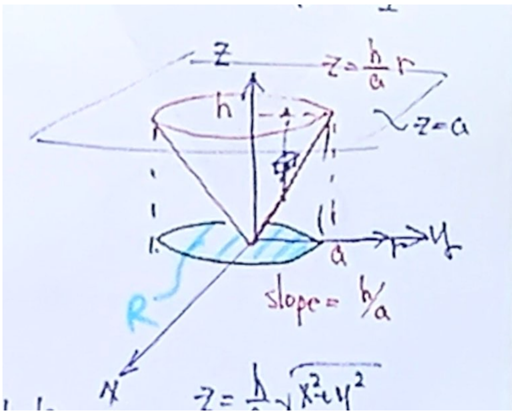
Moreover, same as before, we can freely change the *order of integration.* Below show three examples.



This example shows integration in cylindrical coordinates brings convenience sometimes.

[Example.] Find the volume of a circular cone with altitude h and a base of radius a.

[Solution.] To find the volume of the cone, we need to know the equation of its surface.



In $(x, y, z), z = \frac{h}{a}(x^2 + y^2)^{1/2}$. In $(r, \theta, z), z = \frac{h}{a}r$.

Method 1: Do the question by (x, y, z), and $dV = dzdydx$

$$V = 4\int_0^a \int_0^{\sqrt{a^2-x^2}} \int_{\frac{h}{a}(x^2+y^2)^{1/2}}^{h} dz\ dy\ dx$$

Method 2: Do the question by (r, θ, z), and $dV = r\ dzdrd\theta$

$$V = \int_0^{2\pi} \int_0^a \int_{\frac{h}{a}r}^{h} r\ dz\ dr\ d\theta = \frac{1}{3}\pi a^2 h$$

This example provides interpretation for triple integrals.

[Example.] Sketch the solid whose volume is given by the integral.

(a) $\displaystyle\int_0^{2\pi}\int_0^{2}\int_0^{4-r^2} r\ dzdrd\theta.$

(b) $\displaystyle\int_1^{3}\int_0^{\frac{\pi}{2}}\int_r^{3} r\ dzd\theta dr.$

[Solution.] Recall that for $\displaystyle\int_a^b\int_{g_1(x)}^{g_2(x)}\int_{\phi_1(x)}^{\phi_2(x)} dzdydx$, we can infer the volume of integration based on these six limits:

$z=\phi_1$ to $z=\phi_2$, (surfaces)
$y=g_1$ to $y=g_2$, (curves)
$x=a$ to $x=b$, (points)

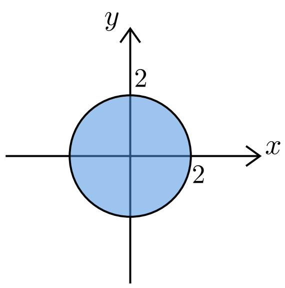
We can interpret the last two as "*area on the xy-plane*".

(a)

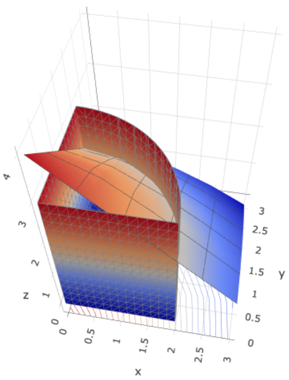
$z: z=0$ to $z=4-r^2$, (surfaces)

$$\left.\begin{array}{lll} r: r=0 & \text{to} & r=2 \\ \theta: \theta=0 & \text{to} & \theta=2\pi \end{array}\right\} \text{(area on } xy\text{-plane)}$$

The area on xy-plane is shown below left. We move the disk upward, hitting $z=4-r^2$, and the volume bounded is what we are looking for, as shown below right.



$r=0$ *to* $r=2$
$\theta=0$ *to* $\theta=2\pi$

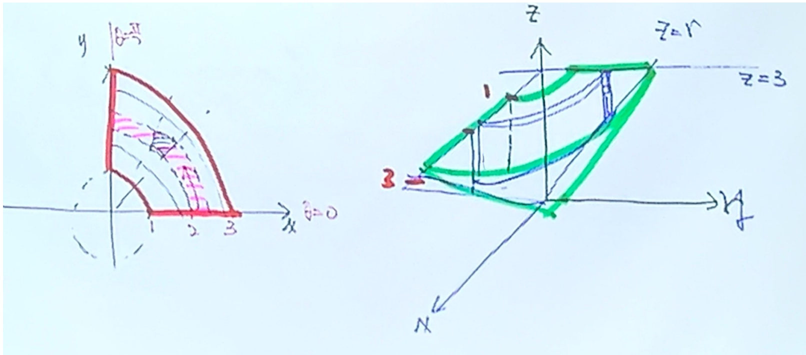


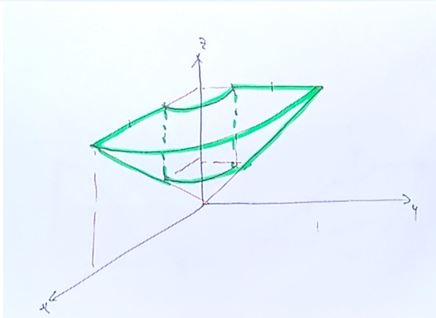
(b)

$z : z = r$ to $z = 3$, (surfaces)

$$\left.\begin{array}{llll} \theta : \theta = 0 & \text{to} & \theta = \frac{\pi}{2} \\ r : r = 1 & \text{to} & r = 3 \end{array}\right\} \text{(area on } xy\text{-plane)}$$

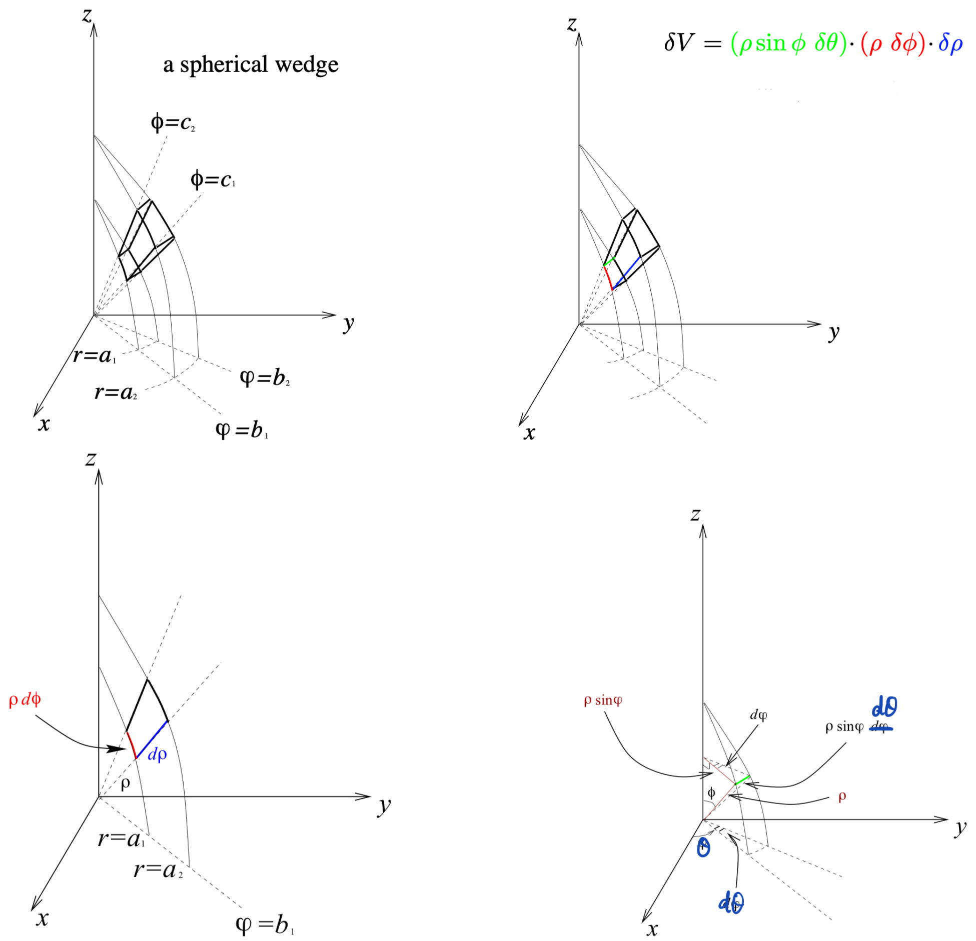
The procedure is similar to (a), we first draw the region on xy-plane, and move it from surface $z = r$ upwards to $z = 3$. The bounded region is what we're looking for.





Spherical coordinates are suited to problems *involving spherical symmetry*, and in particular, to regions bounded by *spheres* centered at the origin, *circular cones* with axes along the $z-$axis, *vertical planes* containing $z-$axis.

The basic unit of spherical coordinate is shown below, and it's volume is $dV = \rho^2 \sin\phi \; d\theta d\rho d\phi$



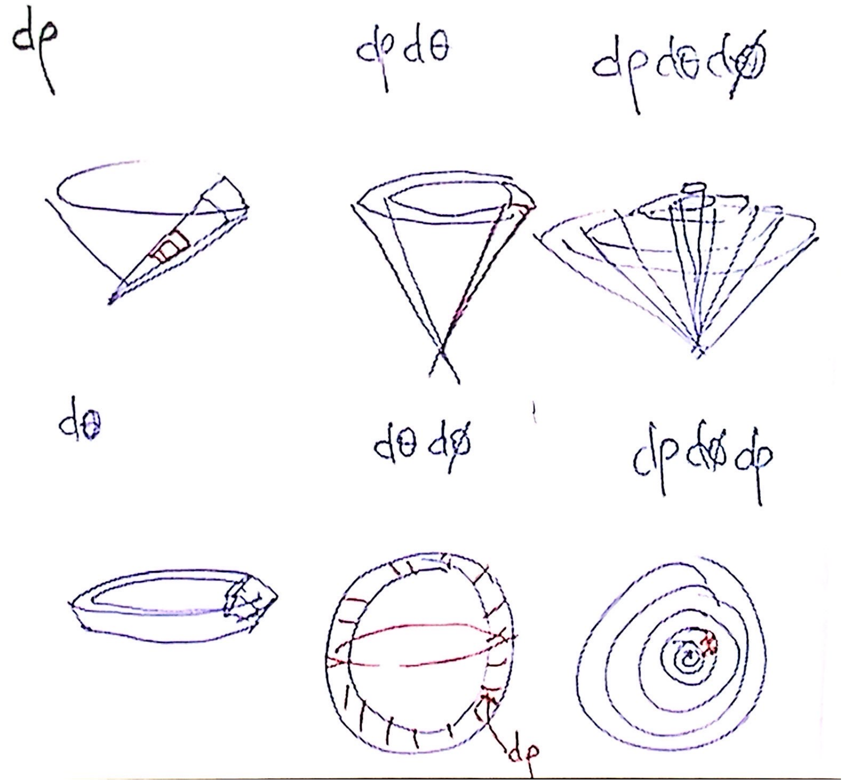
We can also find Jacobian in order to find the relation.

For spherical coordinates $x = \rho \sin\phi \cos\theta, y = \rho \sin\phi \sin\theta$ and $z = \rho \cos\phi$, then

$$\frac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)} = \begin{vmatrix} \sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \\ \cos\phi & -\rho\sin\phi & 0 \end{vmatrix} = \rho^2 \sin\phi$$

i.e. $dxdydz = \rho^2 \sin\phi d\rho d\theta d\phi$.

Again, different integration orders have different interpretations.


dρ
dρ dθ
dρ dθ dφ
dθ
dθ dφ
dρ dφ dρ
dρ

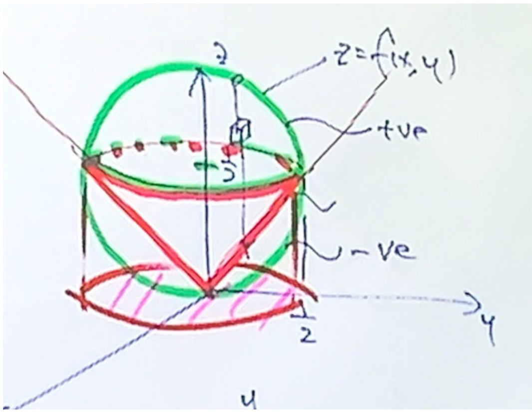
This example shows the choice of coordinate system can greatly affect the difficulty of computation of a multiple integral.

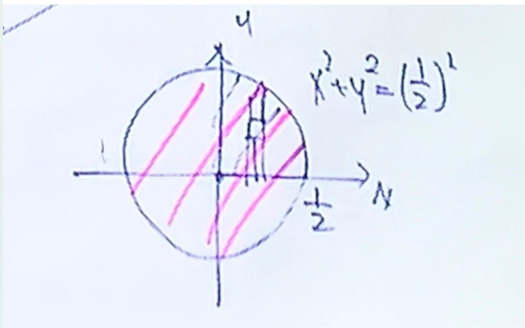
[Example.] Find the volume of the solid that lies above the cone $z = \sqrt{x^2 + y^2}$ and between the sphere $x^2 + y^2 + z^2 = z$.

[Solution.] The equation of sphere can be written as

$$x^2 + y^2 + \left(z - \frac{1}{2}\right)^2 = \left(\frac{1}{2}\right)^2 \Rightarrow \text{centre} \left(0, 0, \frac{1}{2}\right), \ \text{radius} = \frac{1}{2}$$

Method 1: Use Cartesian Coordinate:

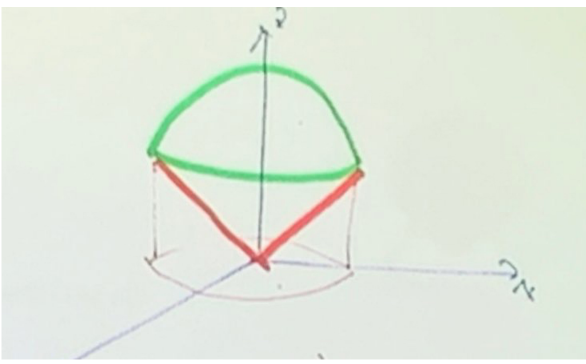


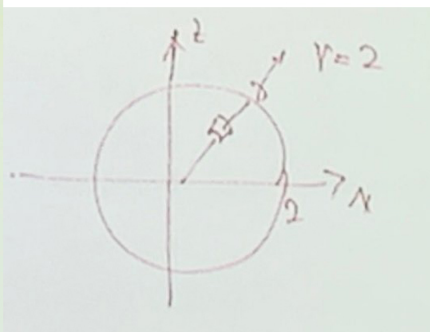


Cone: $z = \sqrt{x^2 + y^2}$
Sphere: $z = \frac{1}{2} + \sqrt{\frac{1}{4} - x^2 - y^2}$

$$V = 4 \int_0^{\frac{1}{2}} \int_0^{\sqrt{\frac{1}{4} - x^2}} \int_{\sqrt{x^2+y^2}}^{\frac{1}{2} + \sqrt{\frac{1}{4} - x^2 - y^2}} \, dzdydx$$

Method 2: Use Cylindrical Coordinate:



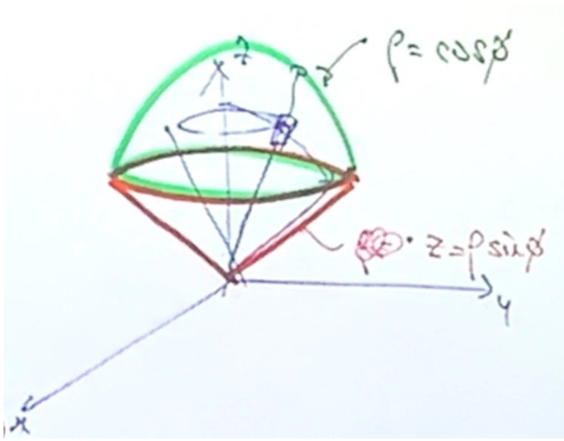


Cone: $z = \sqrt{x^2 + y^2} = r$

Sphere: $z = \frac{1}{2} + \sqrt{\frac{1}{4} - x^2 - y^2} = \frac{1}{2} + \sqrt{\frac{1}{4} - r^2}$

$$V = \int_0^{2\pi} \int_0^{2} \int_r^{\frac{1}{2}+\sqrt{\frac{1}{4}-r^2}} r \ dzdrd\theta$$

Method 3: Use Spherical Coordinate:



Cone: $z = \sqrt{x^2 + y^2} = \rho \sin \phi$
Sphere: $x^2 + y^2 + z^2 = z \Rightarrow \rho^2 = \rho \cos \phi \Rightarrow \rho = \cos \phi$, (since $\rho \neq 0$)

$$dV = \int_0^{\frac{\pi}{4}} \int_0^{2\pi} \int_0^{\cos \phi} \rho^2 \sin \phi \ d\rho d\theta d\phi$$

This example shows the situation when we need to integrate two parts.

[Example.] Find the volume of a cylinder with radius a and height h.

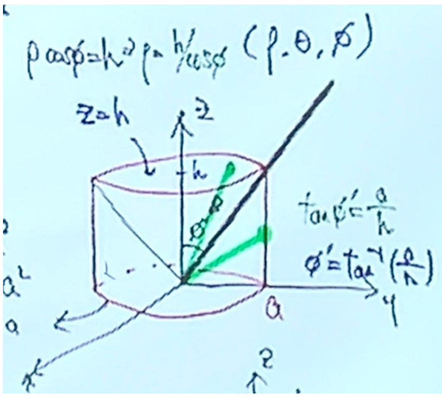
[Solution.] **Method 1:** Use Cylindrical Coordinate:

$$V = \int_0^{2\pi} \int_0^a \int_0^h r \; dzdrd\theta$$

This is really simple by cylindrical coordinate, but things become different when using spherical.

Method 2: Use Spherical Coordinate:

Notice that for different ϕ, the ρ is bounded by two different equations. So we cannot do it in one integration, instead, we need to separate these two situations.



For the above cone, the upper red surface is $z = h \Rightarrow \rho\cos\phi = h \Rightarrow \rho = h/\cos\phi$,

$$V_1 = \int_0^{\phi'} \int_0^{2\pi} \int_0^{\frac{h}{\cos\phi}} \rho^2 \sin\phi \; d\rho d\theta d\phi$$

For the bottom part, ρ moves along the green line, from 0 to $x^2 + y^2 = a^2 \Rightarrow \rho\sin\phi = a$,

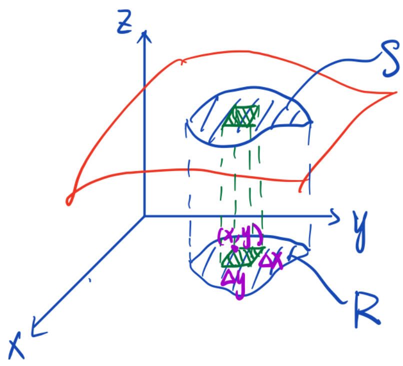
$$V_2 = \int_{\phi'}^{\frac{\pi}{2}} \int_0^{2\pi} \int_0^{\frac{a}{\sin\phi}} \rho^2 \sin\phi \; d\rho d\theta d\phi$$

For the bolded black line(the threshold), $\tan\phi' = \dfrac{a}{h} \Rightarrow \phi = \tan^{-1}\dfrac{a}{h}$.

$$V = V_1 + V_2$$

5.6 Surface Area

We now want to find the area of a surface. Finding an area on xy-plane is relatively easy, as we have discussed early this chapter, but things become much more complicated when we are focusing on an arbitrary surface. So, we think about *projecting the area onto xy-plane*.

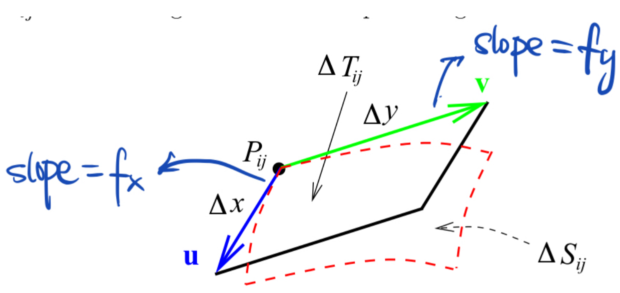


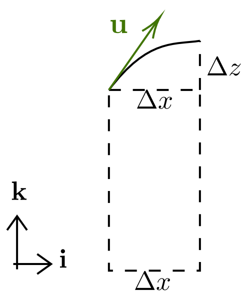
As shown above, we want to find the area of region S in a surface. The first thing to do is to project it onto xy-plane, resulting in a region R.

As usual, we use small rectangles to cover region R, as the green rectangle shown above, assume two sides are Δx and Δy, so the area of green rectangle is $\Delta A = \Delta x \Delta y$.

Then we project the rectangle up to the surface S, resulting in a "curved-parallelogram" surface, shown as red area in left-below image. To find this area, we know as long as Δx and Δy are small enough, the black parallelogram formed by Δx and Δy is a good approximation for that area. By the way, the area of parallelogram is $\mathbf{u} \times \mathbf{v}$.

How to represent $\mathbf{u}$ and $\mathbf{v}$? See the right-below image, the slope of vector $\mathbf{u}$ is $f_x = \dfrac{\Delta z}{\Delta x}$, so the width of $\mathbf{u}$ is Δx and the height of $\mathbf{u}$ is $\Delta z = \Delta x \cdot f_x$.(Notice this image is graphed vertically, i.e., in xz-plane)





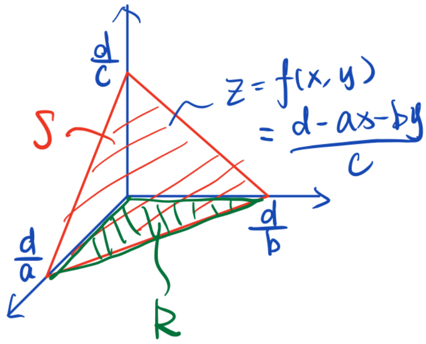
Therefore, $\mathbf{u} = \Delta x\mathbf{i} + 0\mathbf{j} + \Delta x \cdot f_x\mathbf{k}$, similarly, $\mathbf{v} = 0\mathbf{i} + \Delta y\mathbf{j} + \Delta y \cdot f_y\mathbf{k}$. Then

$$||\mathbf{u} \times \mathbf{v}|| = || - \Delta x \Delta y f_x \mathbf{i} - \Delta x \Delta y f_y \mathbf{j} + \Delta x \Delta y \mathbf{k}|| = \sqrt{(f_x)^2 + (f_y)^2 + 1} \cdot \Delta x \Delta y$$

When $\Delta x, \Delta y \to 0$, the area of S is given by:

$$\boxed{A = \iint_R \sqrt{(f_x)^2 + (f_y)^2 + 1}\, dA}$$

[Example.] Given a plane $ax + by + cz = d$, where $a, b, c, d > 0$. Find the area of the triangle bounded by the intersections of the plane and axes.(As the red shaded area shown)



[Solution.] The equation of surface $z = f(x, y)$ is given by $z = \dfrac{d - ax - by}{c}$.

To find the red area, we first *project it onto xy-plane*, resulting in green area R.

Thus the red area

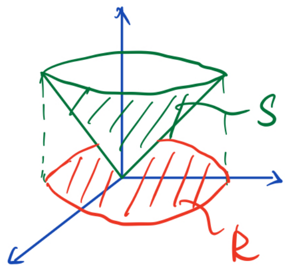
$$S = \iint_R \sqrt{1 + f_x^2 + f_y^2}\, dA_{xy}$$

We know that $f_x = -\dfrac{a}{c}$, $f_y = -\dfrac{b}{c}$, so $\sqrt{1 + f_x^2 + f_y^2} = \dfrac{\sqrt{a^2 + b^2 + c^2}}{c}$, then

$$\begin{aligned} S &= \iint_R \frac{\sqrt{a^2 + b^2 + c^2}}{c}\, dA_{xy} \\ &= \frac{\sqrt{a^2 + b^2 + c^2}}{c} \cdot (\text{area of R}) \\ &= \frac{\sqrt{a^2 + b^2 + c^2}}{c} \cdot \frac{1}{2} \cdot \frac{d}{a} \cdot \frac{d}{b} \\ &= \frac{d^2\sqrt{a^2 + b^2 + c^2}}{2abc} \end{aligned}$$

Notice the blue part is a constant.

[Example.] Find the surface area of the cone $z = \frac{h}{a}r$ (in cylindrical coordinate).



[Solution.] Project the cone onto xy-plane, resulting in red area R.

The surface is given by $z = f(x,y) = \frac{h}{a}\sqrt{x^2+y^2}$

Thus the green area

$$S = \iint_R \sqrt{1+f_x^2+f_y^2}\, dA_{xy}$$

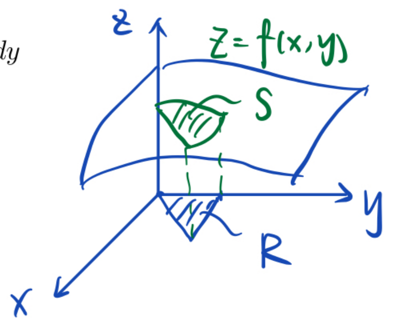
We know that $f_x = \frac{h}{a}\cdot\frac{x}{\sqrt{x^2+y^2}}$, $f_y = \frac{h}{a}\cdot\frac{y}{\sqrt{x^2+y^2}}$, so $\sqrt{1+f_x^2+f_y^2} = 1+\frac{h^2}{a^2}$, then

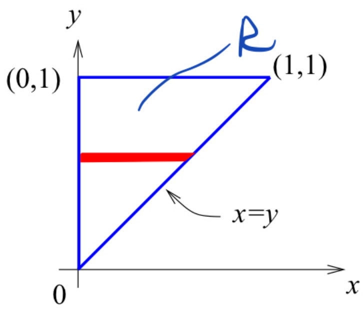
$$\begin{aligned}
S &= \iint_R \sqrt{a+\frac{h^2}{a^2}}\, dA_{xy} \\
&= \sqrt{a+\frac{h^2}{a^2}}\cdot \text{(Area of circle with radius a)} \\
&= \pi a \cdot \sqrt{a^2+h^2}
\end{aligned}$$

[Example.] Find the area of the surface $z = x + y^2$ that lies above the triangle with vertices (0,0), (1,1) and (0,1).

[Solution.]

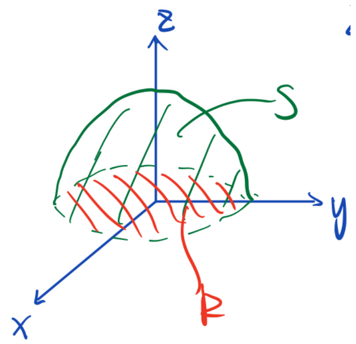
$$
\begin{aligned}
S &= \iint_R \sqrt{z_x^2 + z_y^2 + 1}\, dA = \int_0^1 \int_0^y \sqrt{1 + 4y^2 + 1}\, dx\, dy \\
&= \int_0^1 y\sqrt{2 + 4y^2}\, dy \\
&= \frac{2}{24}(2 + 4y^2)^{\frac{3}{2}}\Big|_0^1 \\
&= \frac{1}{6}(3\sqrt{6} - \sqrt{2}).
\end{aligned}
$$





This example uses substitution while evaluating integral.

[Example.] Find the surface of a sphere with radius a.



[Solution.] Again, project S onto xy-plane to get region R.

The equation of surface is given by $z = f(x, y) = \sqrt{a^2 - x^2 - y^2}$,

The green area:

$$S = \iint_R \sqrt{1 + f_x^2 + f_y^2}\ dA_{xy}$$

We know that $f_x = \dfrac{-x}{\sqrt{a^2 - x^2 - y^2}}$, $f_y = \dfrac{-y}{\sqrt{a^2 - x^2 - y^2}}$, so $\sqrt{1 + f_x^2 + f_y^2} = 1 + \dfrac{x^2 + y^2}{a^2 - x^2 - y^2}$, then

$$S = \iint_R \sqrt{1 + \frac{x^2 + y^2}{a^2 - x^2 - y^2}}\ dA_{xy}$$

Notice this integration is too difficult to calculate, so we consider using *polar coordinate* to substitute, let $r^2 = x^2 + y^2$, $dA = r\ drd\theta$, then

$$S = \iint_R \sqrt{1 + \frac{r^2}{a^2 - r^2}}\ r\ drd\theta = 2\pi a^2$$

6. Chapter 15: Vector Field

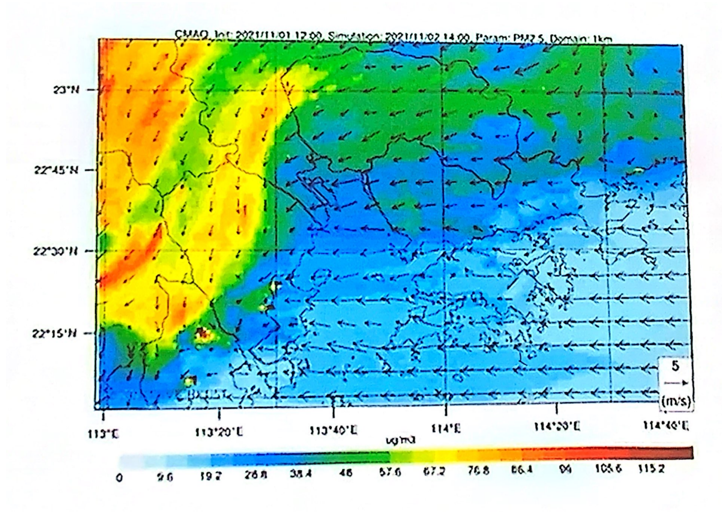
6.1 Intro. to Vector Field

So far, we have learned two kinds of functions involving vector:

- $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$: for each t, provides a *position* vector $< x(t), y(t), z(t) >$, so this is a (parametric) curve.
- $z = f(\mathbf{r}) = f(x_1, x_2, \cdots, x_n)$: for a given vector r, this gives a real number, so this is a function of *several variables*. This is also a **scalar field** since for any point $\mathbf{r}$ in **field**, it gives a scalar value.

Now we are looking at **vector-valued** function $\mathbf{F}$ of a vector $\mathbf{r}$, i.e., $\mathbf{F}(\mathbf{r})$. This is a **vector field**, which means for any point $\mathbf{r}$ in **field**, it gives a vector $\mathbf{F}(\mathbf{r})$.

You can consider a world map showing the *speed* and *direction* of wind.



You can see that in a 2D map(like above), if we put a vector on each point, the vector must have same dimension as the map, i.e., all vectors must also be 2D vectors.

$$\mathbf{F}(\mathbf{r}) = \begin{cases} (F_1(\mathbf{r}), F_2(\mathbf{r})) & \mathbf{r} = (x, y) & 2D \\ (F_1(\mathbf{r}), F_2(\mathbf{r}), F_3(\mathbf{r})) & \mathbf{r} = (x, y, z) & 3D \\ (F_1(\mathbf{r}), F_2(\mathbf{r}), \cdots, F_n(\mathbf{r})) & \mathbf{r} = (x_1, x_2, \cdots, x_n) & nD \end{cases}$$

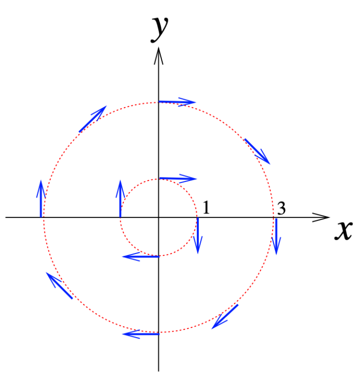
Summary: *dimension of* $\mathbf{F}$ *must be the same as* $\mathbf{r}$.

This is an example of vector field.

[Example.] Assume a vector field: $\mathbf{F}(x, y) = \dfrac{y\mathbf{i} - x\mathbf{j}}{\sqrt{x^2 + y^2}}$.

[Solution.] Notice that $||\mathbf{F}|| = \dfrac{y^2}{x^2 + y^2} + \dfrac{x^2}{x^2 + y^2} = 1$, all vectors $\mathbf{F}(x, y)$ are unit vectors. Moreover, let $\mathbf{r} = (x, y)$, then $\mathbf{r} \cdot \mathbf{F} = 0$, so $\mathbf{r} \perp \mathbf{F}$.

So all vectors are unit vectors tangent to circles centered at the origin with radius $\sqrt{x^2 + y^2}$.



6.2 Divergence and Curl

Recall that the **gradient operator** is a *vector operator*:

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \qquad \text{(a vector)}$$

If $\mathbf{F}(\mathbf{r}) = F_1(\mathbf{r})\mathbf{i} + F_2(\mathbf{r})\mathbf{j} + F_3(\mathbf{r})\mathbf{k}$, then we define:

- **divergence** of $\mathbf{F}$, written div $\mathbf{F}$:

$$\boxed{\operatorname{div}\mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}}$$

- **curl** of $\mathbf{F}$, written curl $\mathbf{F}$:

$$\boxed{\operatorname{curl}\mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right)\mathbf{i} - \left(\frac{\partial h}{\partial x} - \frac{\partial f}{\partial z}\right)\mathbf{j} + \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)\mathbf{k}}$$

This example shows basic computation of **divergence** and **curl**.

[Example.] Let $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $\mathbf{u} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$, where a, b and c are constants, show that

(a) $\nabla \cdot \mathbf{r} = 3$
(b) $\nabla \times \mathbf{r} = \mathbf{0}$
(c) $\nabla \cdot (\mathbf{u} \times \mathbf{r}) = 0$
(d) $\nabla \times (\mathbf{u} \times \mathbf{r}) = 2\mathbf{u}$.

[Solution.] (a) $\nabla \cdot \mathbf{r} = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right) \cdot (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 3$

(b) $\nabla \times \mathbf{r} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = \mathbf{0}$

(c) $\mathbf{u} \times \mathbf{r} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a & b & c \\ x & y & z \end{vmatrix} = (bz - cy)\mathbf{i} - (az - cx)\mathbf{j} + (ay - bx)\mathbf{k}$

$$\therefore \nabla \cdot (\mathbf{u} \times \mathbf{r}) = \frac{\partial}{\partial x}(bz - cy) - \frac{\partial}{\partial y}(az - cx) + \frac{\partial}{\partial z}(ay - bx) = 0$$

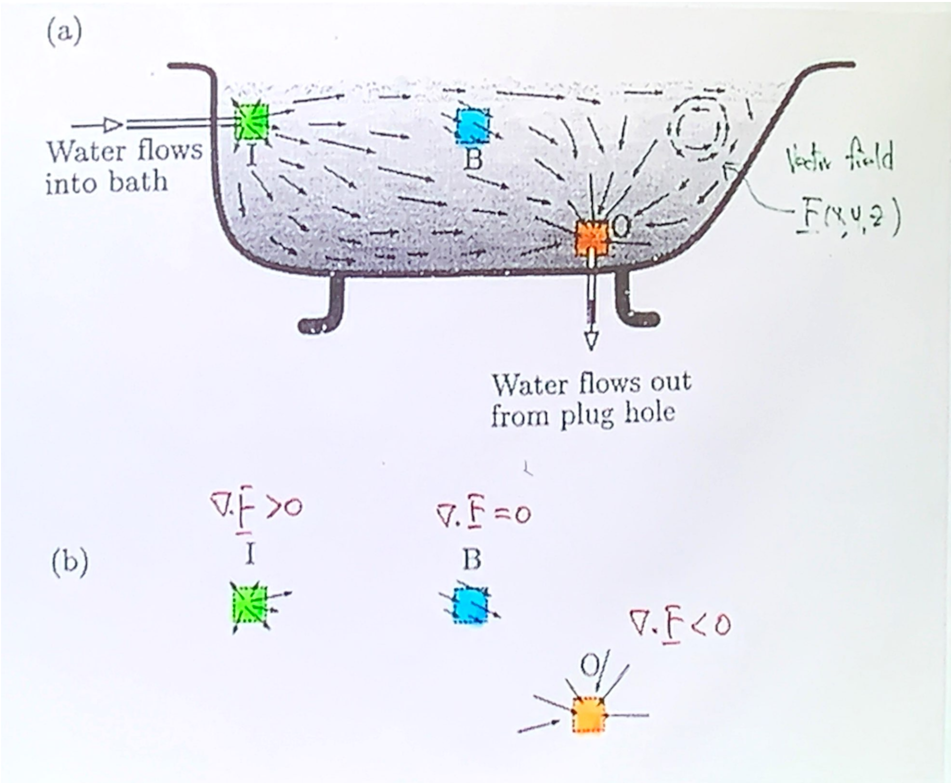
(d) $\nabla \times (\mathbf{u} \times \mathbf{r}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ bz - cy & -az + cx & ay - bx \end{vmatrix} = 2(a\mathbf{i} + b\mathbf{j} + c\mathbf{k}) = 2\mathbf{u}$

6.2.1 Interpretation of Divergence

Imagine water in a bath tank, if the **velocity** of water at any point of the tank is given by

$$\mathbf{u}(\mathbf{r}) = u_1(\mathbf{r})\mathbf{i} + u_2(\mathbf{r})\mathbf{j} + u_3(\mathbf{r})\mathbf{k}$$

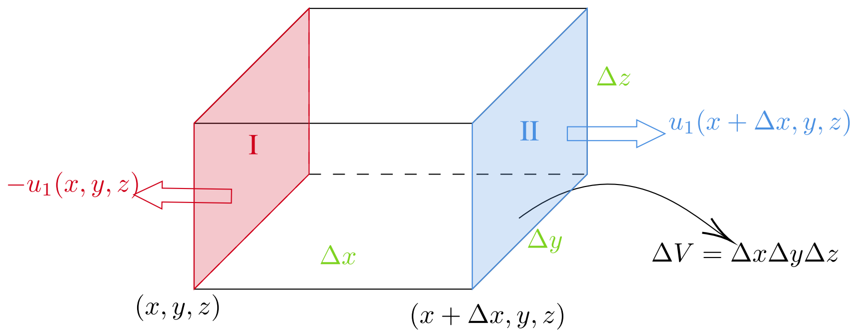
then **net outward flux per unit volume** is div $\mathbf{u} = \nabla \cdot \mathbf{u}$.



Moreover,

- If more water comes inside, then div $\mathbf{u} < 0$
- If more water comes outside, then div $\mathbf{u} > 0$
- If the amount of water comes inside equals to comes outside, then div $\mathbf{u} = 0$

This page proves the interpretation of divergence.



Imagine the box with volume $\Delta V = \Delta x \Delta y \Delta z$, firstly consider faces I and II, the total flux *out of* faces I and II, as shown above, is:

$$
\begin{aligned}
&[u_1(x+\Delta x, y, z) - u_1(x, y, z)]\Delta y \Delta z \\
=&\frac{[u_1(x+\Delta x, y, z) - u_1(x, y, z)]}{\Delta x}\Delta x \Delta y \Delta z \\
=&\frac{\partial u_1}{\partial x}\Delta x \Delta y \Delta z, \qquad \text{(in the limit of } \Delta x \to 0)
\end{aligned}
$$

Similarly, the two faces in the $y-$ and $z-$ direction contribute

$$
\frac{\partial u_2}{\partial y}\Delta x \Delta y \Delta z, \quad \frac{\partial u_3}{\partial z}\Delta x \Delta y \Delta z
$$

Hence net outward flux is:

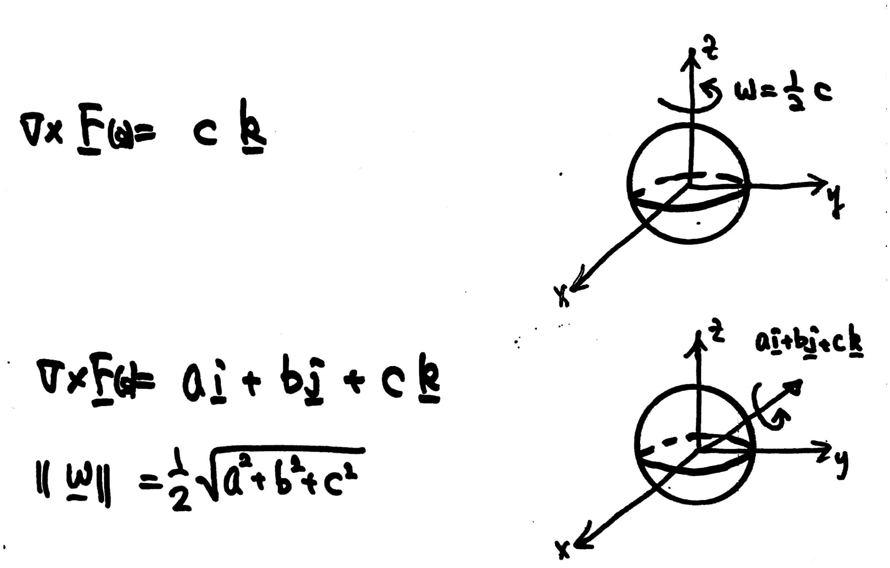
$$
\left(\frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} + \frac{\partial u_3}{\partial z}\right) \cdot \Delta V
$$

Therefore outward flux *per unit volume* is $\nabla \cdot \mathbf{u}$.

6.2.2 Interpretation of Curl

Curl is something related to rotation. Consider a small object flying in strong wind, where the speed and direction of wind can be treated as a vector field $\mathbf{F}$. If the object locates at position $\mathbf{r}$, then its rotation has some relation with curl $\mathbf{F}$.

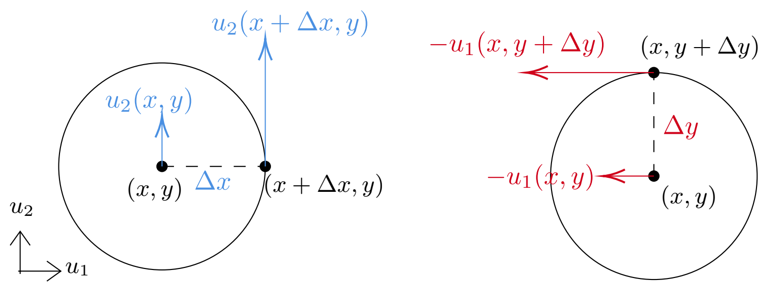
Actually, the object will rotate about the direction $\nabla \times \mathbf{F}(\mathbf{r})$(direction is determined by right-hand rule), and with angular speed $\omega = \frac{1}{2}||\nabla \times \mathbf{F}(\mathbf{r})||$.



The rest of this page prove the relation above.

Consider a disk in $xy-$plane, in y direction, the differential velocity *normal to* Δx is:

$$u_2(x + \Delta x) - u_2(x) = \frac{\partial u_2}{\partial x}\Delta x$$



Recall that $v = \omega r$, so the angular velocity is $\omega_1 = \dfrac{\partial u_2}{\partial x}$

Similarly, in the $y-$direction, (notice the negative sign)

$$-u_1(x, y + \Delta y) + u_1(x, y) = -\frac{\partial u_1}{\partial y}\Delta y, \qquad \omega_2 = -\frac{\partial u_1}{\partial y}$$

Thus the *averaged angular velocity* is: $\omega = \frac{1}{2}\left(\frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y}\right)$

The curl of this vector field is:

$$\operatorname{curl}\mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u_1 & u_2 & 0 \end{vmatrix} = \left(\frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y}\right)\mathbf{k} = 2\omega\mathbf{k}$$

Thus prove the result.

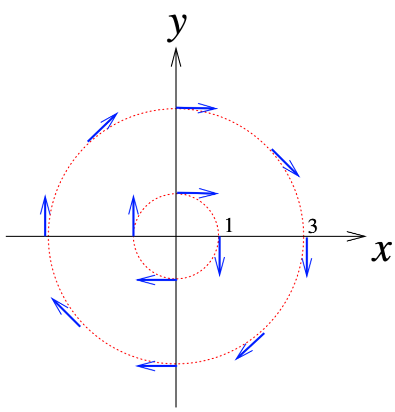
Below example is used to explain the meaning of curl, it's the same example in intro.

[Example.] Assume a vector field: $\mathbf{F}(x,y) = \dfrac{y\mathbf{i} - x\mathbf{j}}{\sqrt{x^2+y^2}}$.

[Solution.]

$$\begin{aligned} \operatorname{curl}\mathbf{F} = \nabla \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{y}{\sqrt{x^2+y^2}} & \frac{-x}{\sqrt{x^2+y^2}} & 0 \end{vmatrix} \\ &= \left[-\frac{\partial}{\partial x}\left(\frac{x}{\sqrt{x^2+y^2}}\right) - \frac{\partial}{\partial y}\left(\frac{y}{\sqrt{x^2+y^2}}\right)\right]\mathbf{k} \\ &= \mathbf{0} \end{aligned}$$

Consider a small object in the vector field, it doesn't rotate(since curl $\mathbf{F} = \mathbf{0}$ everywhere), it just move in circular, along the vector field.



This definition is optional.

Laplacian Operator

$$\begin{aligned}\nabla^2 = \nabla\cdot\nabla &= \left(\mathbf{i}\frac{\partial}{\partial x}+\mathbf{j}\frac{\partial}{\partial y}+\mathbf{k}\frac{\partial}{\partial z}\right)\cdot\left(\mathbf{i}\frac{\partial}{\partial x}+\mathbf{j}\frac{\partial}{\partial y}+\mathbf{k}\frac{\partial}{\partial z}\right)\\ &= \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\end{aligned}$$

∇^2 is a **scalar** differential operator. Note that

$$\begin{aligned}\nabla^2 f &= \frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}+\frac{\partial^2 f}{\partial z^2}\\ \nabla^2\mathbf{F} &= \nabla^2 F_1\mathbf{i}+\nabla^2 F_2\mathbf{j}+\nabla^2 F_3\mathbf{k}\end{aligned}$$

6.2.3 Vector differential identities

Let ϕ, ψ are scalar fields and $\mathbf{F}$ and $\mathbf{G}$ are vector fields, then

(a) $\nabla(\phi\psi) = \phi\nabla\psi + \psi\nabla\phi$

(b) $\nabla \cdot (\phi\mathbf{F}) = \nabla\phi \cdot \mathbf{F} + \phi(\nabla \cdot \mathbf{F})$

(c) $\nabla \times (\phi\mathbf{F}) = \nabla\phi \times \mathbf{F} + \phi(\nabla \times \mathbf{F})$

(d) $\nabla \cdot (\mathbf{F} \times \mathbf{G}) = (\nabla \times \mathbf{F}) \cdot \mathbf{G} - \mathbf{F} \cdot (\nabla \times \mathbf{G})$

(e) $\nabla \cdot (\nabla \times \mathbf{F}) = 0$

(f) $\nabla \times (\nabla\phi) = 0$

(g) $\nabla \times (\nabla \times \mathbf{F}) = \nabla(\nabla \cdot \mathbf{F}) - \nabla^2\mathbf{F}$

[Proof.] The basic idea is just calculate, there is no shortcuts.

(e)

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \mathbf{F}_1 & \mathbf{F}_2 & \mathbf{F}_3 \end{vmatrix} = \left(\frac{\partial \mathbf{F}_3}{\partial y} - \frac{\partial \mathbf{F}_2}{\partial z}\right)\mathbf{i} - \left(\frac{\partial \mathbf{F}_3}{\partial x} - \frac{\partial \mathbf{F}_1}{\partial z}\right)\mathbf{j} + \left(\frac{\partial \mathbf{F}_2}{\partial x} - \frac{\partial \mathbf{F}_1}{\partial y}\right)\mathbf{k}$$

$$\nabla \cdot (\nabla \times \mathbf{F}) = \frac{\partial^2 \mathbf{F}_3}{\partial x \partial y} - \frac{\partial^2 \mathbf{F}_2}{\partial x \partial z} - \frac{\partial^2 \mathbf{F}_3}{\partial y \partial x} + \frac{\partial^2 \mathbf{F}_1}{\partial y \partial z} + \frac{\partial^2 \mathbf{F}_2}{\partial z \partial x} - \frac{\partial^2 \mathbf{F}_1}{\partial z \partial y} = 0$$

(f) $\nabla\phi = \left(\dfrac{\partial \phi}{\partial x}, \dfrac{\partial \phi}{\partial y}, \dfrac{\partial \phi}{\partial z}\right)$

$$\nabla \times (\nabla\phi) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{\partial \phi}{\partial x} & \dfrac{\partial \phi}{\partial y} & \dfrac{\partial \phi}{\partial z} \end{vmatrix} = \left(\frac{\partial^2 \phi}{\partial y \partial z} - \frac{\partial^2 \phi}{\partial z \partial y}\right)\mathbf{i} - \left(\frac{\partial^2 \phi}{\partial x \partial z} - \frac{\partial^2 \phi}{\partial z \partial x}\right)\mathbf{j} + \left(\frac{\partial^2 \phi}{\partial x \partial y} - \frac{\partial^2 \phi}{\partial y \partial x}\right)\mathbf{k} = \mathbf{0}$$

Others: to be added.

6.3 Line Integral

Motivation: given a rope(parametrized space curve) $\mathbf{r}(t), a \leq t \leq b$, if the density at point (x, y, z) is given by function $\rho = f(x, y, z)$, we want to find the mass of this rope.

$$\boxed{\int_a^b f(x,y,z)\ ds = \int_C f(x,y,z)\ ds}$$

Recall when we computing arc length in Chapter 11, we knew that:

$$ds = ||\mathbf{r}'(t)||dt$$

Therefore, to calculate line integral $\int_C f(x,y,z)\ ds$, we only need to know:

1. $f(x,y,z)$
2. Region $C:\ \mathbf{r}(t) = (x(t), y(t), z(t)),\ a \leq t \leq b$

This example shows how to find line integral.

[Example.] Find $\int_C xy^4 ds, C$ is the right half of the circle $x^2 + y^2 = 16$.

[Solution.] In order to do this integral, we need the parametric form of the path C. Let $x = 4\cos t, y = 4\sin t$, the right half of the circle means $t \in [-\pi/2, \pi/2]$.

Hence the parametric equation of the curve C is $\mathbf{r}(t) = 4\cos t\mathbf{i} + 4\sin t\mathbf{j}$ with $t \in [-\pi/2, \pi/2]$, then

$$\mathbf{r}'(t) = -4\sin t\mathbf{i} + 4\cos t\mathbf{j} \quad \text{and} \quad \left\|\mathbf{r}'(t)\right\| = 4$$

Thus with $ds = ||\mathbf{r}'(t)||dt$,

$$\int_C xy^4 ds = \int_{-\pi/2}^{\pi/2} f(\mathbf{r}) \left\|\mathbf{r}'(t)\right\| dt = \int_{-\pi/2}^{\pi/2} \left[4^5 \cos t \sin^4 t\right] (4) dt = \frac{2 \cdot 4^6}{5}$$

Note that there are infinitely many ways to parametrize the curve C, for example, if we had parameterized C as

$$\mathbf{r}(t) = \sqrt{16 - t^2}\mathbf{i} + t\mathbf{j} \quad \text{where} \quad -4 \leqslant t \leqslant 4$$

Then

$$\mathbf{r}'(t) = -\frac{t}{\sqrt{16 - t^2}}\mathbf{i} + \mathbf{j}, \qquad \left\|\mathbf{r}'(t)\right\| = \sqrt{\frac{16}{16 - t^2}}$$

$$\int_C xy^4 ds = \int_C f(\mathbf{r}) \left\|\mathbf{r}'(t)\right\| dt = \int_{-4}^{4} \sqrt{16 - t^2} \times t^4 \times \sqrt{\frac{16}{16 - t^2}} dt = \frac{2 \cdot 4^6}{5}$$

Thus the line integral is *independent of parametrization* of the curve C.

So far, we have been doing integration w.r.t. s, but we can also carry the integration w.r.t. x,

$$\int_C f(\mathbf{r}(t))\ dx$$

This example shows how to integrate w.r.t. x, y, z

[Example.] $f(\mathbf{r}(t)) = f(x,y,z) = xy + z$, $C : \mathbf{r}(t) = (x(t), y(t), z(t)) = (t^2, t^3, t), 0 \le t \le 1$

[Solution.] Since $dx = 2t\ dt,\ dy = 3t^2\ dt,\ dz = dt$,

$$\begin{aligned}
&\int_C f(\mathbf{r}(t))\ dx = \int_0^1 (t^2 \cdot t^3 + t)(2t)\ dt = \int_0^1 (2t^6 - 2t^2)\ dt = \cdots \\
&\int_C f(\mathbf{r}(t))\ dy = \int_0^1 (t^5 + t)(3t^2)\ dt = \cdots \\
&\int_C f(\mathbf{r}(t))\ dz = \int_0^1 (t^5 + t)\ dt = \cdots
\end{aligned}$$

But why are we doing this? Consider given $f(\mathbf{r}(t))$ and $C : \mathbf{r}(t)$, we integrate w.r.t. x, y, z, respectively:

$$\int_C f(\mathbf{r}(t))\ dx \qquad \int_C g(\mathbf{r}(t))\ dy \qquad \int_C h(\mathbf{r}(t))\ dz$$

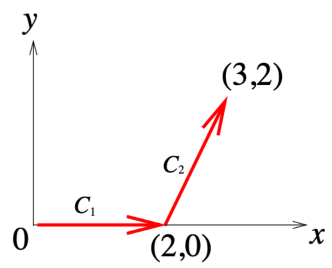
The summation of above three gives:

$$\begin{aligned}
&\int_C [f(\mathbf{r}(t))\ dx + g(\mathbf{r}(t))\ dy + h(\mathbf{r}(t))\ dz] \\
= &\int_C (f, g, h) \cdot (dx, dy, dz) \\
= &\boxed{\int_C \mathbf{F} \cdot d\mathbf{r}}
\end{aligned}$$

where $\mathbf{F}(\mathbf{r}) = (f(\mathbf{r}), g(\mathbf{r}), h(\mathbf{r}))$ is the given vector field.

The following two examples shows usage of integration w.r.t. x, y, z.

[Example.] $\int_C xydx + (x-y)dy, C$ consists of line segments from $(0,0)$ to $(2,0)$ and from $(2,0)$ to $(3,2)$.



[Solution.] We need the parametric function of C_1 and C_2, they are straight lines. Recall in Chapter 10 we know the parameterized curve of straight lines can be written as:

$$\mathbf{r}(t) = (1-t)\mathbf{r}_1 + t\mathbf{r}_2, \ 0 \le t \le 1$$

Hence

$$\begin{aligned}
C1: \ \mathbf{r}(t) &= (1-t)(0,0) + t(2,0) = (2t, 0) = (x(t), y(t)), \ 0 \le t \le 1 \\
C2: \ \mathbf{r}(t) &= (1-t)(2,0) + t(3,2) = (2+t, 2t) = (x(t), y(t)), \ 0 \le t \le 1
\end{aligned}$$

Thus the integration is:

$$\begin{aligned}
\int_C xydx + (x-y)dy &= \int_{C_1} xydx + (x-y)dy + \int_{C_2} xydx + (x-y)dy \\
&= \int_0^1 (2t)(0)2dt + (2t-0)(0) + \int_0^1 (2+t)(2t)dt + (2+t-2t)2dt \\
&= \int_0^1 (4t + 2t^2 + 2 + t - 2t)dt = \cdots
\end{aligned}$$

Notice that: if $\mathbf{F}(\mathbf{r}) = (xy, x-y)$, and $\mathbf{r}_1 = (2t, 0), \mathbf{r}_2 = (2+t, 2t)$, the result above is exactly

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$$

Again, there are many ways to parameterized the curve, for example,

$C_1 : (x, 0), \ 0 \le x \le 2, \qquad C_2 : (x, 2x-4), \ 2 \le x \le 3.$

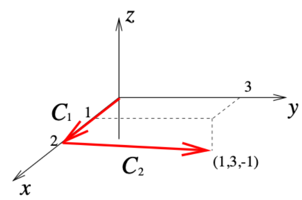
Then

$$\begin{aligned}
\int_C xydx + (x-y)dy &= \int_{C_1} [xydx + (x-y)dy] + \int_{C_2} [xydx + (x-y)dy] \\
&= \int_0^2 0dx + \int_2^3 \left(2x^2 - 4x\right) dx + \int_2^3 (-x+4)2dx = \cdots
\end{aligned}$$

[Example.]

$\mathrm{I} = \displaystyle\int_C yz\,dx + xz\,dy + xy\,dz$, C consists of line segments from $(0,0,0)$ to $(2,0,0)$, and from $(2,0,0)$ to $(1,3,-1)$.

[*Hint*: $\mathbf{r}(t) = (1-t)\mathbf{r}_0 + t\mathbf{r}_1$, where $0 \leqslant t \leqslant 1$.]



Let $C = C_1 + C_2$, where

C_1 : $(0,0,0)$ to $(2,0,0)$ $\Rightarrow$ $x = 2t$, $y = z = 0$, where $0 \leqslant t \leqslant 1$.

C_2 : $(2,0,0)$ to $(1,3,-1)$ $\Rightarrow$ $x = -t+2$, $y = 3t$, $z = -t$, where $0 \leqslant t \leqslant 1$.

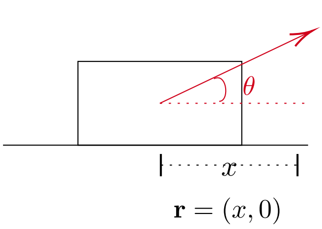
Then

$$\mathrm{I} = 0 + \int_0^1 \left[(3t^2) + 3(t^2 - 2t) - 3(2t - t^2)\right]\, dt =$$

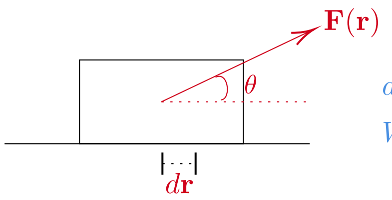
The final answer is -3.

6.4 Line Integration in Vector Fields

We have talked so much about line integration above. However, this Chapter is called "Vector Field", so how does line integration relate to vector field? Consider a force pulling a box on the ground:



$$W = ||\mathbf{F}|| \cos\theta \cdot x$$
$$= \mathbf{F} \cdot \mathbf{r}$$

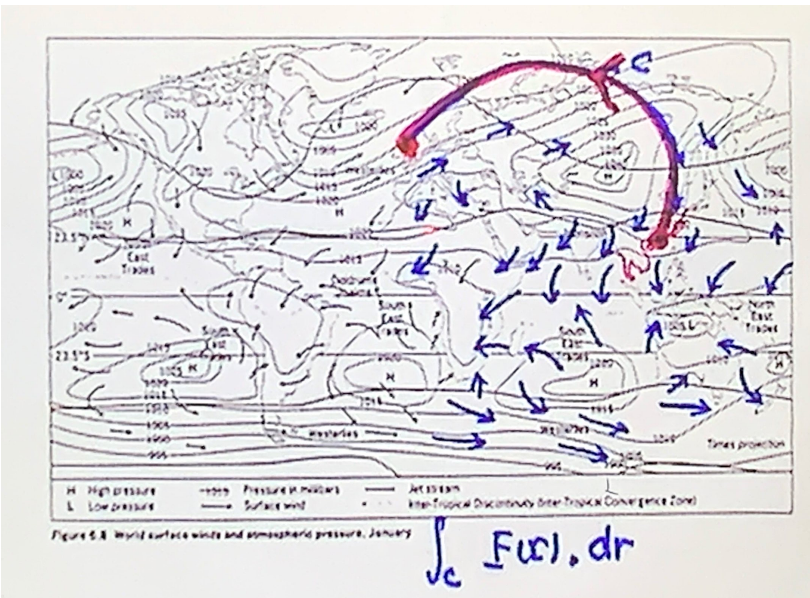


$$dW = \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r}$$
$$W = \int_a^b \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r}$$

On the left side, if the force is always the same, then total work done is $W = \mathbf{F} \cdot \mathbf{r}$, where $\mathbf{r}$ represents the direction and distance of moving. However, if the force is changing, i.e., $\mathbf{F}(\mathbf{r})$ varies for different $\mathbf{r}$, then the total work done is the integration given on the right.

You may have noticed the total work done is just the line integration we have discussed above.

Moreover, for a plane flying from one place to another, if we know the directions and speed of wind at any point on the map, the line integration on its path $\mathbf{r}$ is the total work done on plane, or, approximately how much oil is required.



The following two examples show that sometimes the line integral depends on path, while sometimes it does not.

[Example.] Find $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y) = e^{x-1}\mathbf{i} + xy\mathbf{j}$ and C is given by

(a) $\mathbf{r}(t) = t^2\mathbf{i} + t^3\mathbf{j}; \quad 0 \leqslant t \leqslant 1.$
(b) $\mathbf{r}(t) = t\mathbf{i} + t\mathbf{j}; \quad 0 \leqslant t \leqslant 1.$

[Solution.]

(a)

$$\begin{aligned}
\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^1 \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)dt = \int_0^1 \mathbf{F}\left(t^2, t^3\right) \cdot \mathbf{r}'(t)dt \\
&= \int_0^1 \left(e^{t^2-1}\mathbf{i} + t^5\mathbf{j}\right) \cdot \left(2t\mathbf{i} + 3t^2\mathbf{j}\right) dt \\
&= \int_0^1 \left(2te^{t^2-1} + 3t^7\right) dt = \left[e^{t^2-1} + \frac{3}{8}t^8\right]_0^1 = \frac{11}{8} - \frac{1}{e}
\end{aligned}$$

(b)

$$\begin{aligned}
\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^1 \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)dt = \int_0^1 \mathbf{F}(t, t) \cdot \mathbf{r}'(t)dt \\
&= \int_0^1 \left(e^{t-1}\mathbf{i} + t^2\mathbf{j}\right) \cdot (\mathbf{i} + \mathbf{j})dt \\
&= \int_0^1 \left(e^{t-1} + t^2\right) dt = \frac{4}{3} - \frac{1}{e}
\end{aligned}$$

Note that the line integral depends on the path.

[Example.] Find $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y) = y\mathbf{i} + x\mathbf{j}$ and C is given by

(a) $\mathbf{r}(t) = t\mathbf{i} + t\mathbf{j}; \quad 0 \leqslant t \leqslant 1.$ (b) $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j}; \quad 0 \leqslant t \leqslant 1$ (c) $\mathbf{r}(t) = t\mathbf{i} + t^3\mathbf{j}; \quad 0 \leqslant t \leqslant 1.$

[Solution.]

(a) $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 (t\mathbf{i} + t\mathbf{j}) \cdot (\mathbf{i} + \mathbf{j})dt = \int_0^1 2tdt = 1$

(b) $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 \left(t^2\mathbf{i} + t\mathbf{j}\right) \cdot (\mathbf{i} + 2\mathbf{j})dt = \int_0^1 3t^2dt = 1$

(c) $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 \left(t^3\mathbf{i} + t\mathbf{j}\right) \cdot \left(\mathbf{i} + 3t^2\mathbf{j}\right) dt = \int_0^1 4t^3dt = 1$

Note that this integral does not depend on path.

6.5 Conservative Vector Fields

So why different vector field cause different results in above examples?

Recall that if $\int_a^b f(x)dx = F(b) - F(a)$ (that the integral only depends on the *end points*) only if *anti-derivative* exists, otherwise, we cannot integrate it and hence cannot express it only with end points.

For example, $\int_1^2 e^x dx = e^2 - e^1$, but for $\int_1^2 e^{x^2} dx$, we cannot find it.

Similarly, if $\mathbf{F}(\mathbf{r}) = f(\mathbf{r})\mathbf{i} + g(\mathbf{r})\mathbf{j} + h(\mathbf{r})\mathbf{k}$ is the gradient of a function $\phi(\mathbf{r})$ on S, i.e.,

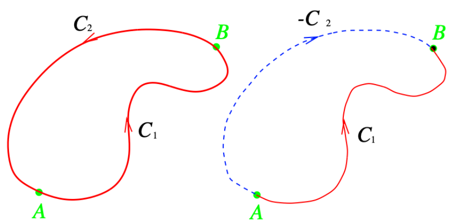
$$\mathbf{F}(\mathbf{r}) = \nabla\phi(\mathbf{r}) \quad \Rightarrow \quad f(\mathbf{r}) = \frac{\partial\phi}{\partial x}, \quad g(\mathbf{r}) = \frac{\partial\phi}{\partial y} \quad \text{and} \quad h(\mathbf{r}) = \frac{\partial\phi}{\partial z}$$

In that case, we say $\mathbf{F}(\mathbf{r})$ is a **conservative field** and ϕ is a *(scalar) potential function* of $\mathbf{F}$ on S. Then,

$$\begin{aligned}
\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} &= \int_a^b \nabla\phi(\mathbf{r}(t)) \cdot \mathbf{r}'(t)dt \\
&= \int_a^b \left(\frac{\partial\phi}{\partial x}\frac{dx}{dt} + \frac{\partial\phi}{\partial y}\frac{dy}{dt} + \frac{\partial\phi}{\partial z}\frac{dz}{dt} \right) dt \\
&= \int_a^b \frac{d\phi}{dt} dt = \phi(x(b), y(b), z(b)) - \phi(x(a), y(a), z(a)) \\
&= \phi(\mathbf{r}(b)) - \phi(\mathbf{r}(a))
\end{aligned}$$

This depends only *on the endpoints* $\mathbf{r}(b)$ and $\mathbf{r}(a)$, not on the curve C.

This can also be expressed in circular integration, $\int_C \mathbf{F} \cdot d\mathbf{r}$ is *independent* of path in S if and only if $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ for *every closed path* C in S. Since if the integration depends only on end points, then integrate from a to b and then from b back to a will certainly gives result 0.



$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} - \int_{-C_2} \mathbf{F} \cdot d\mathbf{r} = 0$$

How to determine whether a vector field is *conservative* or not?

A continuously vector field $\mathbf{F}$ defined in a **simply-connected domain** S is conservative if and only if, it possesses *any one* of the following properties.

(i) It is the gradient of a scalar function, $\mathbf{F}(\mathbf{r}) = \nabla\phi(\mathbf{r})$.

(ii) Its line integral along any regular curve extending from a point P to a point Q is independent of the path.

$$\int_{C_1} \mathbf{F}\cdot d\mathbf{r} = \int_{C_2} \mathbf{F}\cdot d\mathbf{r} = \int_{C_3} \mathbf{F}\cdot d\mathbf{r} = \cdots$$

(iii) Its line integral around any regular closed curved is zero, i.e. $\oint_C \mathbf{F}\cdot d\mathbf{r} = 0$.

(iv) $\nabla\times\mathbf{F} = \mathbf{0}$. (since $\nabla\times(\nabla\times\phi) = \mathbf{0}$)

Note that $(i) \Rightarrow (ii) \Rightarrow (iii)$, and it's trivial that $(iii) \Rightarrow (ii)$

For $(ii) \Rightarrow (i)$, this can be proved only if $\mathbf{F}$ us continuous on an *open connected region.* Proof is not required here.

For (iv), this is a **necessary** condition for the existence of a potential function ϕ, in other words, this is **not sufficient**, we also need to guarantee that the domain S must be **open and simply connected.**

What is **open and simply-connected** region?

Firstly, it must be *connected*, it cannot be divided into two parts that are separate. For example,

(1) $\mathbf{F}(\mathbf{r}) = \left(\dfrac{x}{x^2+y^2+z-1}, \dfrac{y}{x^2+y^2+z-1}, \dfrac{z^2}{x^2+y^2+z-1}\right)$ is defined on $\mathbb{R}^3\backslash\{x^2+y^2+z = 1\}$. This is not connected, because the region is divided into two parts by a "rice bowl".

(2) Also, $\mathbb{R}^3\backslash\{x^2+y^2+z^2 = 1\}$ is not connected, because the region is divided into two parts by a ball shell.

Then, for *any curve* inside the region, if *at least one surface* inside the region with the curve *as boundary* does not have a hole on it, then the region is simply-connected. For example,

(1) $\mathbb{R}^2\backslash\{(0,0)\}$ is not simply-connected. Consider any curve surround the origin, the only surface that with the curve as boundary is the "disk" bounded by the curve, and unfortunately it has a hole in the middle.

(2) $\mathbb{R}^3\backslash\{(0,0,0)\}$ is simply-connected. For any curve inside the region, even if you choose a curve

on xy-plane and contains $(0, 0, 0)$, we can easily find a surface bounded by the curve that does not contain the origin, for example a downside "rice bowl".

(3) $\mathbb{R}^3 \backslash \{x = y = 0\}$ is not simply-connected. For example, if we choose a circle on $xy-$plane, like $x^2 + y^2 = 1$, then *any* surface bounded by the curve must through $z-$axis, and since we've removed the $z-$axis, all those surfaces always have a hole on it.

(4) A donut is not simply-connected, we can choose any curve that surround the middle "hole".

This example shows how to find potential for a conservative vector field.

[Example.] Determine whether or not $\mathbf{F}(x, y) = (2x - 3y)\mathbf{i} + (2y - 3x)\mathbf{j}$ is a conservative vector field. If it is, find a function ϕ such that $\mathbf{F} = \nabla\phi$.

[Solution.] Notice that the curl of $\mathbf{F}$:

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial x} & 0 \\ 2x - 3y & 2y - 3x & 0 \end{vmatrix} = \mathbf{0}$$

And, $\mathbf{F}(x, y)$ is defined in $\mathbb{R}^2$, which is **open and simply-connected** domain. Therefore, $\mathbf{F}$ is conservative and $\mathbf{F} = \nabla\phi$ for some potential function ϕ.

To find f,

$$\frac{\partial \phi}{\partial x} = 2x - 3y \qquad (1)$$
$$\frac{\partial \phi}{\partial y} = 2y - 3x \qquad (2)$$

For (1), integrate w.r.t. x, $\phi(x, y) = x^2 - 3xy + g(y)$, differentiate it w.r.t. y, $\dfrac{\partial \phi}{\partial y} = 0 - 3x + g'(y)$.

Compare this equation with (2), we have $g'(y) = 2y \Rightarrow g(y) = y^2 + C$

Therefore

$$\phi(x, y) = x^2 - 3xy + y^2 + C.$$

This example shows curl $\mathbf{F} = \mathbf{0}$ is not sufficient for $\mathbf{F}$ to be conservative.

[Example.] Let $\mathbf{F}(x, y) = \dfrac{-y\mathbf{i} + x\mathbf{j}}{x^2 + y^2} = f(x, y)\mathbf{i} + g(x, y)\mathbf{j}$.

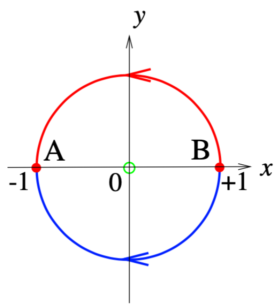
(a) Show that $\dfrac{\partial f}{\partial y} = \dfrac{\partial g}{\partial x}$.

(b) Show that $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r}$ is not independent of path.

[Solution.] (a)

$$\frac{\partial f}{\partial y} = \frac{y^2 - x^2}{(x^2 + y^2)^2} = \frac{\partial g}{\partial x}$$

(b) Notice $\mathbf{F}$ is defined on $\mathbb{R}^2\backslash\{\mathbf{0}\}$, so it is not conservative. To show that it is dependent of path, we only need to given an counterexample. Consider on these two paths:



$C_1: \quad x = \cos\theta, \quad y = \sin\theta$, with $\theta = 0$ to $\theta = \pi$

$C_2: \quad x = \cos\theta, \quad y = \sin\theta$, with $\theta = 2\pi$ to $\theta = \pi$

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_0^\pi (-\sin\theta, \cos\theta) \cdot (-\sin\theta, \cos\theta) d\theta = \int_0^\pi d\theta = \pi$$

$$\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_{2\pi}^\pi d\theta = -\pi \neq \int_{C_1} \mathbf{F} \cdot d\mathbf{r}$$

Therefore, $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r}$ is not independent of path.

But if $\mathbf{F}$ is defined on $x > 0$, it will become a conservative field. ($x > 0$ is open and simply-connected.)

In summary, to find a line integral, you have following methods:

1. Carry out the integration directly:
$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b g(t)dt \cdots$$
2. If $\nabla \times \mathbf{F} \neq \mathbf{0}$, nothing you can do, except for carrying the integration as above.
3. If $\nabla \times \mathbf{F} = \mathbf{0}$, there are two simpler ways:
   (a) Find ϕ such that $\mathbf{F} = \nabla\phi$, then
   $$\int_C \mathbf{F} \cdot d\mathbf{r} = \phi(\mathbf{r}(b)) - \phi(\mathbf{r}(a))$$
   (b) Carry the integration along a straight line path, i.e.
   $$\mathbf{r}(t) = (1-t)\mathbf{r}_1 + t\mathbf{r}_2,\ 0 \leq t \leq 1$$

This is a past final question in year 1994.

[Example.] For what values of b and c will

$$\mathbf{F}(x,y,z) = \left(y^2 + 2czx\right)\mathbf{i} + y(bx+cz)\mathbf{j} + \left(y^2 + cx^2\right)\mathbf{k}$$

have potential functions? For each pair of these values of b and c, find a potential function for $\mathbf{F}$.

[Solution.] For $\mathbf{F}(\mathbf{r})$ to have a potential function, iff $\nabla \times \mathbf{F}(\mathbf{r}) = \mathbf{0}$.

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{i} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y^2 + 2czx & y(bx+cz) & y^2 + cx^2 \end{vmatrix}$$

For the $\mathbf{i}$ component, $2y = yc \quad \Rightarrow \quad c = 2$

For the $\mathbf{j}$ component, $2cx = 2cx \quad \Rightarrow \quad c$ cannot be determined.

For the $\mathbf{k}$ component, $by = 2y \quad \Rightarrow \quad b = 2$

$\therefore b = 2$ and $c = 2$. In the case, we have

$$\mathbf{F}(\mathbf{r}) = \left(y^2 + 4zx\right)\mathbf{i} + y(2x+2z)\mathbf{j} + \left(y^2 + 2x^2\right)\mathbf{k} = \nabla\phi$$

Therefore

$$\frac{\partial \phi}{\partial x} = y^2 + 4zx \qquad (1)$$
$$\frac{\partial \phi}{\partial y} = 2xy + 2yz \qquad (2)$$
$$\frac{\partial \phi}{\partial z} = y^2 + 2x^2 \qquad (3)$$

From (1), we have

$$\phi(x,y,z) = xy^2 + 2x^2z + p(y,z) \qquad (4)$$
$$\phi_y(x,y,z) = 2xy + p_y(y,z) \qquad (5)$$

Comparing (2) and (5), we have $p_y(y,z) = 2yz \Rightarrow p(y,z) = y^2z + q(z)$. From (4), we have

$$\phi(x,y,z) = xy^2 + 2x^2z + y^2z + q(z) \qquad (6)$$
$$\phi_z(x,y,z) = 2x^2 + y^2 + q'(z) \qquad (7)$$

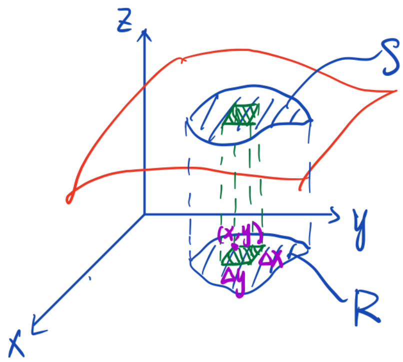
Comparing (7) and (3), we have $q'(z) = 0 \quad \Rightarrow \quad q(z) = k$ (constant).

$$\therefore \quad \phi(x,y,z) = xy^2 + 2x^2z + y^2z + k$$

6.6 Surface Integrals of Vector Fields

6.6.1 Surface Integrals

Recall in Chapter 14, we discussed about Surface Integrals:



The area of region S is given by:

$$\boxed{A = \iint_R \sqrt{(f_x)^2 + (f_y)^2 + 1}\ dA}$$

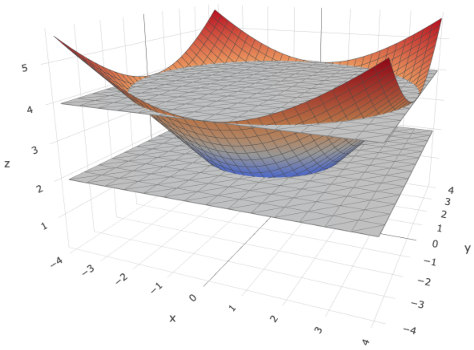
If we have a density function $\rho(x, y, z)$ defined on every point of the surface, then the mass of the surface is given by:

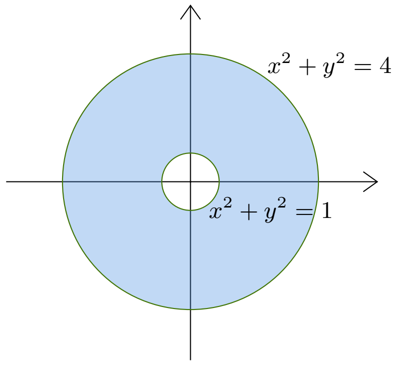
$$\boxed{M = \iint_R \rho(x, y, f(x, y))\sqrt{(f_x)^2 + (f_y)^2 + 1}\ dA}$$

This example shows basic computation of surface integration with density function.

[Example.] Find the mass of a thin funnel in the shape of a cone $z = \sqrt{x^2+y^2}, z \in [1,4]$, if its density function if $\rho(x,y,z) = 10 - z$.

[Solution.] The cone bounded by $z = 1$ and $z = 4$ and its projection onto $xy-$plane is shown below:





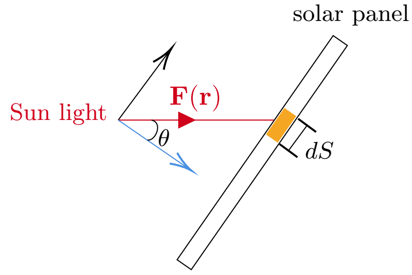
$z = f(x,y) = (x^2+y^2)^{\frac{1}{2}},\ f_x = \dfrac{x}{\sqrt{x^2+y^2}},\ f_y = \dfrac{y}{\sqrt{x^2+y^2}},$

hence $ds = \sqrt{1+(f_x)^2+(f_y)^2}dA = \sqrt{2}dA,$

$$\begin{aligned}
\iint_S \rho(x,y,z)dS &= \iint_R \left(10-\sqrt{x^2+y^2}\right)\sqrt{1+(f_x)^2+(f_y)^2}dA \\
&= \sqrt{2}\iint_R (10-\sqrt{x^2+y^2})dA \\
&= \sqrt{2}\int_0^{2\pi}\int_1^4 (10-r)\ rdrd\theta = 108\sqrt{2}\pi
\end{aligned}$$

6.6.2 Surface Integrals of vector fields: flux

Consider when sun light shines on a solar panel, only the energy that perpendicular to the panel will be received.(blue arrow below)



For a small area dS, the solar energy that it receives is:

$$\begin{aligned} &||\mathbf{F}(\mathbf{r})||\cos\theta \cdot dS \\ =&||\mathbf{F}||\cos\theta \cdot dS \cdot ||\widehat{\mathbf{n}}|| \\ =&(\mathbf{F}\cdot\widehat{\mathbf{n}})dS \end{aligned}$$

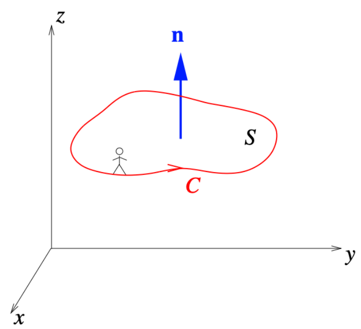
Thus the *total* solar energy on this panel is:

$$\boxed{\iint_S (\mathbf{F}\cdot\widehat{\mathbf{n}})dS}$$

Suppose $\mathbf{F}(x,y,z)$ is a continuous vector field defined on a smooth, *oriented* surface S, the integral above is called the **flux** of $\mathbf{F}$ across S.

How to determine the **orientation** of the surface S, or, which $\mathbf{n}$ should we choose?

A non-closed surface S(i.e., the surface *not close a volume*) bounded by a *closed smooth curve* C, and the **positive orientation** around C means the surface will always be on your *left*, then your head pointing in the direction of $\mathbf{n}$.



This page summarize the steps to find flux.

(1) Find the normal vector to the surface $S : z = f(x, y)$.

Let $G(x, y, z) = z - f(x, y) = 0$ (constant), so this is a level set in 3D, hence

$$\begin{aligned}\mathbf{n} &= \nabla G = (G_x, G_y, G_z)\\ &= (-f_x, -f_y, 1)\\ \widehat{\mathbf{n}} &= \frac{(-f_x, -f_y, 1)}{\sqrt{1 + (f_x)^2 + (f_y)^2}}\end{aligned}$$

(2) Using $ds = \sqrt{1 + (f_x)^2 + (f_y)^2} dA$.

$$\begin{aligned}\iint_S (\mathbf{F} \cdot \widehat{\mathbf{n}}) dS &= \iint_R (F_1, F_2, F_3) \cdot \frac{(-f_x, -f_y, 1)}{\sqrt{1 + (f_x)^2 + (f_y)^2}} \cdot \sqrt{1 + (f_x)^2 + (f_y)^2} dA\\ &= \iint_R (F_1, F_2, F_3) \cdot (-f_x, -f_y, 1) dA\end{aligned}$$

This is an complete example of computing flux.

[Example.] Find the flux of $\mathbf{F} = y^3\mathbf{i} + z^2\mathbf{j} + x\mathbf{k}$ downward through the part of the surface $z = 4 - x^2 - y^2$ that lies above the plane $z = 2x + 1$.

[Solution.] First we need to find the curve of intersection between $z = 4 - x^2 - y^2$ and the plane $z = 2x + 1$,

$$\begin{aligned}4 - x^2 - y^2 &= 2x + 1\\ x^2 + 2x + 1 + y^2 &= 4\\ (x + 1)^2 + y^2 &= 2^2\end{aligned}$$

Next, find the normal to the surface S, let $G(x, y, z) = 4 - x^2 - y^2 - z = 0$ (constant), this is a level set in 3D, hence

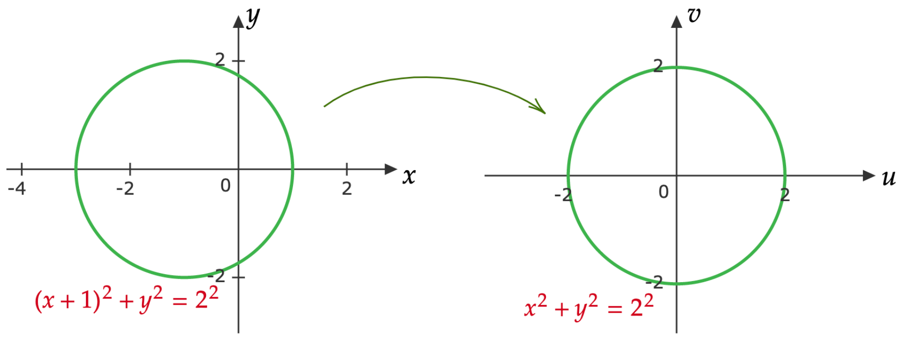
$$\begin{aligned}\mathbf{n} &= \nabla G = (G_x, G_y, G_z) = (-2x, -2y, -1)\\ \widehat{\mathbf{n}} &= \frac{(-2x, -2y, -1)}{\sqrt{1 + 4x^2 + 4y^2}}\end{aligned}$$

Then, using $ds = \sqrt{1+(f_x)^2+(f_y)^2}dA = \sqrt{1+4x^2+4y^2}dA$

$$\begin{aligned}\iint_S (\mathbf{F}\cdot\widehat{\mathbf{n}})dS &= \iint_R \left(y^3,(4-x^2-y^2)^2,x\right)\cdot\frac{(-2x,-2y,-1)}{\sqrt{1+4x^2+4y^2}}\cdot\sqrt{1+4x^2+4y^2}dA \\ &= -\iint_R \left(2xy^3+2y(4-x^2-y^2)^2-x\right)dA = -\iint_R xdA\end{aligned}$$

Notice the last step is because both $2xy^3$ and $2y(4-x^2-y^2)^2$ is *odd* in y, and the region R, $(x+1)^2+y^2=2^2$, is symmetric w.r.t. x, hence the two items equals to 0, after integration. (important!)

However, it is still difficult to find $\iint_R xdA$ directly, since the area is not good, as you can see below. Then we consider using *substitution*, or *Jacobian*, to transform it, into a circle centered at origin.



Let $u=x+1, v=y$, then

$$\mathbf{J}=\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix}\dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v}\\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v}\end{vmatrix}=\begin{vmatrix}1 & 0\\ 0 & 1\end{vmatrix}=1$$

$$-\iint_R xdA_{xy} = -\iint_{R_{uv}}(u-1)\mathbf{J}dA_{uv} = -\iint_{R_{uv}} udA_{uv}+\iint_{R_{uv}} dA_{uv} = 0+\iint_{R_{uv}} dA_{uv} = 4\pi$$

7. Chapter 16: Vector Calculus

7.1 The Divergence Theorem

Let G be a simple solid whose boundary surface S has *positive (outward)* orientation. When we find the **flux** of S in a *smooth* vector field $\mathbf{F}(x,y,z) = f(x,y,z)\mathbf{i} + g(x,y,z)\mathbf{j} + h(x,y,z)\mathbf{k}$, the **Divergence Theorem** gives:

$$\boxed{\oiint_S \mathbf{F} \cdot \widehat{\mathbf{n}}\, dS = \iiint_G \nabla \cdot \mathbf{F}\, dV}$$

where $\widehat{\mathbf{n}}$ is a unit normal vector pointing out of G.

In other words, *the total divergence within G equals the net flux emerging from G.*

Moreover, if the volume is very small, we can assume $\nabla \cdot \mathbf{F}$ is *constant* within the small volume, thus

$$\begin{aligned}\oiint_{\delta S} \mathbf{F} \cdot \widehat{\mathbf{n}}\, dS &= \nabla \cdot \mathbf{F} \iiint_G dV \\ &= \nabla \cdot \mathbf{F} \cdot \delta V\end{aligned}$$

Therefore,

$$\nabla \cdot \mathbf{F} = \lim_{\delta V \to 0} \frac{1}{\delta V} \oiint_{\delta S} \mathbf{F} \cdot \widehat{\mathbf{n}}\, dS$$

This is the definition of divergence given in lecture note Chapter 15.

Since exam will not cover the proof, I'd like to omit here. To be added. 2021 Dec 28th

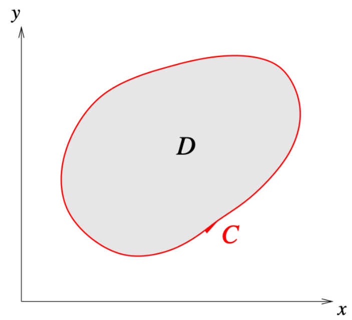
This example shows how Divergence Theorem simplifies the computation of flux.

7.2 Green's Theorem

7.2.1 Green's Theorem in Line Integral

In this part, we will go back to **line integral**, which we have done a lot.

Now consider doing line integral in a smooth simple **closed curve** C in the xy-plane, if $\mathbf{F}(\mathbf{r}) = P(\mathbf{r})\mathbf{i} + Q(\mathbf{r})\mathbf{j}$, then if we want to evaluate $\oint_C \mathbf{F} \cdot d\mathbf{r}$,



Counter-clockwise
– positive orientation

- If $\mathbf{F}$ is conservative, then line integral is 0, obviously.
- If $\mathbf{F}$ is not conservative, then **Green's Theorem** tells us

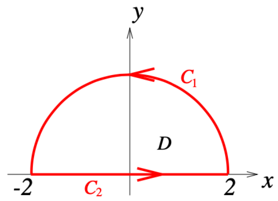
$$\boxed{\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_D (\nabla \times \mathbf{F}) \cdot \mathbf{k}\, dA}$$

Note k is the **normal** to xy-plane, or, normal to region D.

Since exam will not cover the proof, I'd like to omit here.

This example shows how Green's Theorem simplify computation.

[Example.] $\int_C xydx + 2x^2dy$, C consists of the segment from $(-2,0)$ to $(2,0)$ and top half of the circle $x^2+y^2=4$.



[Solution.]

Method 1: use line integral:

$$\int_C xydx + 2x^2dy = \int_{C_1} xydx + 2x^2dy + \int_{C_2} xydx + 2x^2dy$$

Parametrize the two curves:

$$C_1 : \mathbf{r}(t) = (1-t)(-2,0) + t(2,0) = (4t-2,0) \quad 0 \leqslant t \leqslant 1$$
$$C_2 : \mathbf{r}(t) = (2\cos t, 2\sin t) \quad 0 \leqslant t \leqslant \pi$$

Then directly evaluate the two line integrals

$$\int_{C_1} xydx + 2x^2dy = \int_0^1 (4t-2)\cdot 0 \cdot 4dt + 2(4t-2)^2 \cdot (0) = 0$$
$$\int_{C_2} xydx + 2x^2dy = \int_0^\pi (2\cos t)(2\sin t)(-2\sin t)dt + 2(2\cos t)^2(2\cos t)dt$$
$$= 8\int_0^\pi \left(-\cos t \sin^2 t + \cos^3 t\right) dt = 0$$

Thus $\int_C xydx + 2x^2dy = 0$.

Method 2: using Green's theorem:

$\mathbf{F} = (xy, 2x^2)$, hence $\nabla \times \mathbf{F} = (4x - x)\mathbf{k} = 3x\mathbf{k}$, then

$$\oint_C xydx + 2x^2dy = \iint_D 3xdA = \int_0^2 \int_0^\pi 3r\cos\theta \; rd\theta dr = \int_0^2 3r^2 \sin\theta\Big|_0^\pi dr = 0$$

Actually, one may observe that $\iint_D 3xdA = 0$ directly, since $3x$ is a *odd* function in x, and the region D is *symmetric with respect to* y-axis.

7.2.2 Green's Theorem for computing Area

Recall that Green's Theorem states that:

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_D (\nabla \times \mathbf{F}) \cdot \mathbf{k} \, dA$$

Notice if $\mathbf{F}(\mathbf{r}) = P(\mathbf{r})\mathbf{i} + Q(\mathbf{r})\mathbf{j}$,

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C Pdx + Qdy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \iint_D (\nabla \times \mathbf{F}) \cdot \mathbf{k} \, dA$$

When $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$, then

$$\boxed{A = \iint_D dA = \oint_C Pdx + Qdy.}$$

For example, when $P = 0, Q = x$, or when $P = -y, Q = 0$, or when $P = -y/2, Q = x/2$,

$$A = \oint_C xdy = -\oint_C ydx = \frac{1}{2} \oint_C xdy - ydx$$

The two examples below shows how to use Green's Theorem to find area.

[Example.] Find the area of $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

[Solution.] Firstly parametrize the curve, let $x = a\cos\theta,\ y = b\sin\theta,\ 0 \le \theta \le 2\pi$, then

$$C:\ \mathbf{r}(\theta) = (a\cos\theta, b\sin\theta),\ 0 \le \theta \le 2\pi$$

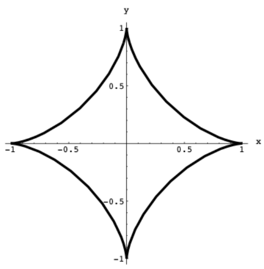
, If we choose $\mathbf{F}(\mathbf{r}) = P(\mathbf{r})\mathbf{i} + Q(\mathbf{r})\mathbf{j} = -\dfrac{y}{2}\mathbf{i} + \dfrac{x}{2}\mathbf{j}$, then we have

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \frac{1}{2} + \frac{1}{2} = 1$$

Hence,

$$\begin{aligned} D &= \frac{1}{2}\oint (xdy - ydx) \\ &= \frac{1}{2}\left[\int_0^{2\pi} a\cos\theta \cdot b\cos\theta\ d\theta + b\sin\theta \cdot a\sin\theta\ d\theta\right] \\ &= \frac{1}{2}ab \cdot \int_0^{2\pi} (\cos^2\theta + \sin^2\theta)\ d\theta = \pi ab \end{aligned}$$

[Example.] Find the area of the hypocycloid $x^{2/3} + y^{2/3} = a^{2/3}$.



[Solution.] Firstly parametrize the curve, let $x = a\cos^3\theta, y = a\sin^3\theta,\quad$ where $0 \leqslant \theta \leqslant 2\pi$,

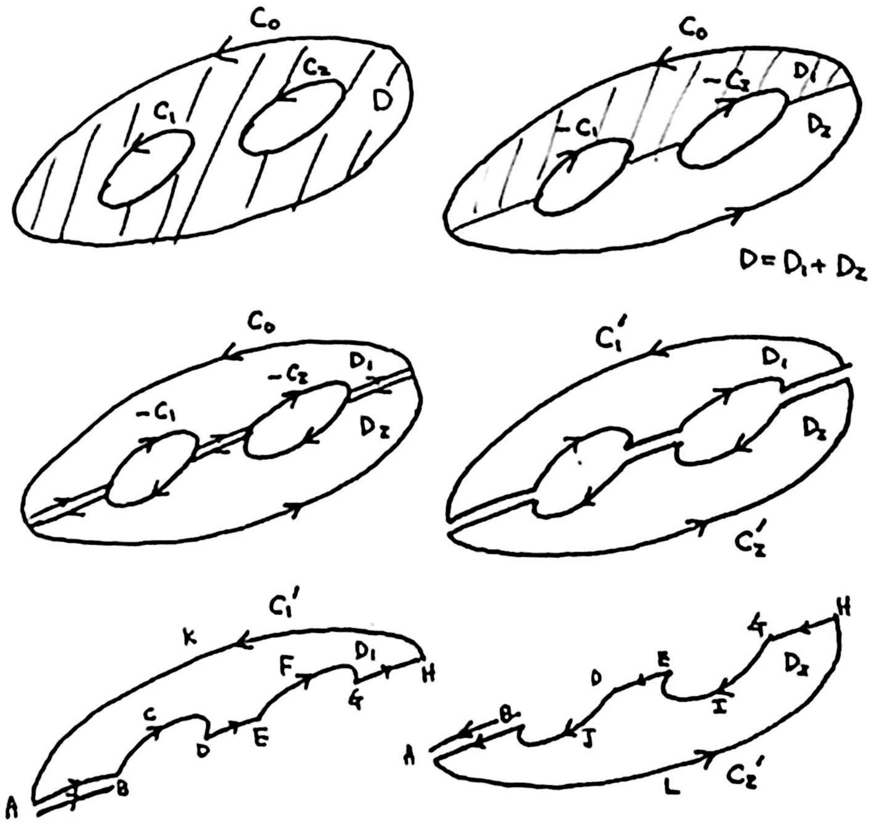
Again, use vector field $\mathbf{F}(\mathbf{r}) = P(\mathbf{r})\mathbf{i} + Q(\mathbf{r})\mathbf{j} = -\dfrac{y}{2}\mathbf{i} + \dfrac{x}{2}\mathbf{j}$,

$$\begin{aligned} A = \frac{1}{2}\oint_C xdy - ydx &= \frac{1}{2}\int_0^{2\pi} \left(a\cos^3\theta \times 3a\sin^2\theta\cos\theta d\theta + a\sin^3\theta \times 3a\cos^2\theta\sin\theta d\theta\right) \\ &= \frac{3}{2}a^2\int_0^{2\pi} \left(\cos^4\sin^2\theta + \sin^4\theta\cos^2\theta\right) d\theta \\ &= \frac{3}{2}a^2\int_0^{2\pi} \frac{1}{4}\sin^2 2\theta d\theta \\ &= \frac{3}{8}a^2\int_0^{2\pi} \left(\frac{1 - \cos 4\theta}{2}\right) d\theta = \frac{3\pi}{8}a^2 \end{aligned}$$

7.2.3 General version of Green's Theorem

This part will not be covered in exam.

Recall that Green's Theorem only applies to *simple* and *closed* curve. However, it can be extended to apply to region with holes. We simply cut the region into some regions that without holes, for example:

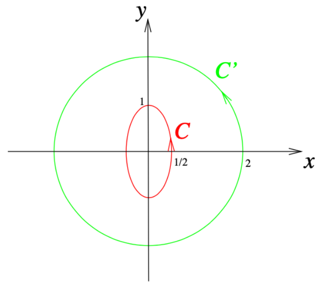


$$
\begin{aligned}
\iint_D &= \iint_{D_1} + \iint_{D_2} = \oint_{C_1'} + \oint_{C_2'} \\
&= \left(\int_{HKA} + \int_{AB} + \int_{BCD} + \int_{DE} + \int_{EFG} + \int_{GH} \right) + \left(\int_{ALH} + \int_{HG} + \int_{GIE} + \int_{ED} + \int_{DJB} + \int_{BA} \right) \\
&= \int_{C_0} - \int_{C_1} - \int_{C_2}
\end{aligned}
$$

Here is the example provided in lecture note.

Example $\displaystyle\oint_C \frac{-x^2y\,dx + x^3\,dy}{(x^2+y^2)^2}$, where C is the ellipse $4x^2+y^2=1$.

If C' is the circle $x^2+y^2=4$, then C is interior to C', and everywhere except at $(0,0)$. Note also that



$$\frac{\partial}{\partial x}\left[\frac{x^3}{(x^2+y^2)^2}\right] = \frac{\partial}{\partial y}\left[\frac{-x^2y}{(x^2+y^2)^2}\right]$$

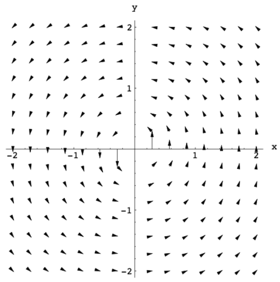
$$\therefore\ I = \oint_C \frac{-x^2y\,dx + x^2\,dy}{(x^2+y^2)^2} = \oint_{C'} \frac{-x^2y\,dx + x^3\,dy}{(x^2+y^2)^2}$$

On C', let $x = 2\cos\theta$, $y = 2\sin\theta$, where $0 \leqslant \theta \leqslant 2\pi$, then

$$I = \int_0^{2\pi} \frac{-4\cos^2\theta\ 2\sin\theta(-2\sin\theta)\,d\theta + (2\cos\theta)^2\ 2\cos\theta\,d\theta}{16}$$

$$= \int_0^{2\pi} \cos^2\theta\,d\theta = \int_0^{2\pi}\left(\frac{1+\cos 2\theta}{2}\right)d\theta = \pi.$$

$$\mathbf{F}(\mathbf{r}) = \frac{-x^2y\,\mathbf{i} + x^3\,\mathbf{j}}{(x^2+y^2)^2}$$



7.3 Stokes' Theorem

Recall in Green's Theorem,

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_D (\nabla \times \mathbf{F}) \cdot \mathbf{k} dA$$

where $\mathbf{F}(\mathbf{r}) = (F_1, F_2)$, $C : \mathbf{r}(t) = (x(t), y(t))$, $a \leq t \leq b$

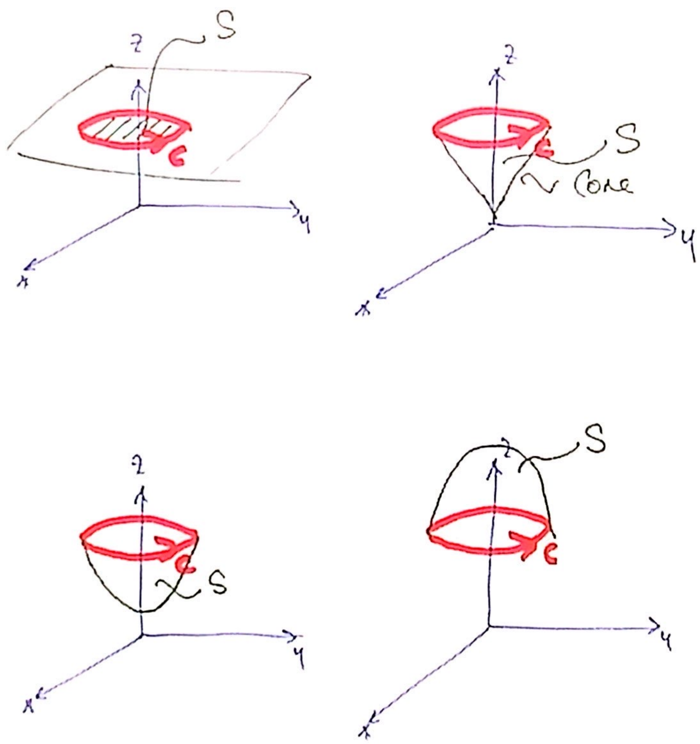
Now we want to extends this theorem into 3D space.

The **Stokes' Theorem** tells that if S is a *non-closed* surface, whose boundary consists of a closed smooth curve C with *positive orientation,* then

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \widehat{\mathbf{n}} \, dS$$

where $\mathbf{F}(\mathbf{r}) = (F_1, F_2, F_3)$, $C : \mathbf{r}(t) = \big(x(t), y(t), z(t)\big)$, $a \leq t \leq b$, and $\mathbf{r}(a) = \mathbf{r}(b)$ since the boundary is closed. $\widehat{\mathbf{n}}$ is unit normal vector of surface S.

However, you may have noticed that the theorem doesn't tell how to find S. When we evaluate a line integral on C, there are lots of surfaces S that can have boundary C.



This example gives a standard process for applying Stokes' Theorem and provides ideas of how to construct S.

[Example.] Find $\oint_C \mathbf{F}\cdot d\mathbf{r}$, where $\mathbf{F}(\mathbf{r}) = (y, x^2, y)$, $C:\ \mathbf{r}(t) = (\cos t, \sin t, 1),\ 0 \le t \le 2\pi$

[Solution.] Method 1: directly compute line integral.

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi} (\sin t, \cos^2 t, \sin t)\cdot(-\sin t, \cos t, 0)\ dt$$
$$= \int_0^{2\pi} (-\sin^2 t + \cos^3 t)\ dt$$

This is tedious.

Method 2: Notice $\mathbf{r}(0) = \mathbf{r}(2\pi)$, so this is a *closed curve* in 3D, we can use **Stokes' Theorem**.

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_S (\nabla\times\mathbf{F})\cdot\hat{\mathbf{n}}\ dS$$

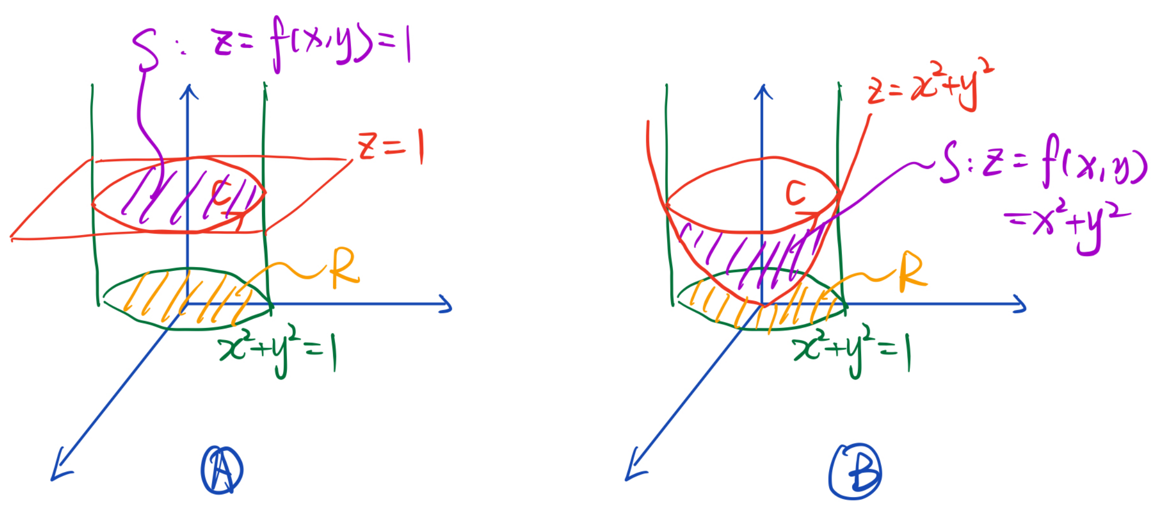
But S is not given, we need to find $S : z = f(x, y)$

Idea: construct 2 surfaces whose *intersection* is the curve C.

From C: $\begin{cases} x(t) = \cos t \\ y(t) = \sin t \\ z(t) = 1 \end{cases}$, we can construct 2 surfaces by observing the relationship among x, y, z, for example,

$$\begin{cases} x^2+y^2 = 1 \\ z = 1 \end{cases} \qquad \textit{or} \qquad \begin{cases} x^2+y^2 = 1 \\ z = x^2+y^2 \end{cases}$$

Their graphs are shown below:



We can see that for the first equation, the surface S is a circle, while for the second equation, the surface S is a "rice bowl". Either of them is ok for our calculation.

(1) Firstly, find the curl of vector field:

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y & x^2 & y \end{vmatrix} = \mathbf{i} + (2x-1)\mathbf{k}$$

(2) Next, find normal vector to the surface,

For (A), $z = f(x,y) = 1$, hence $\widehat{\mathbf{n}} = \mathbf{k}$.

For (B), let $G(x,y,z) = z - x^2 - y^2 = 0(\text{constant})$, this is a level set in 3D, hence

$$\mathbf{n} = \nabla G = (-2x, -2y, 1), \ \widehat{\mathbf{n}} = \frac{(-2x,-2y,1)}{\sqrt{1+4x^2+4y^2}}$$

(3) Then, find surface integral, and thereby calculating the result:

For (A), $ds = \sqrt{1+(f_x)^2+(f_y)^2}dA = dA$, therefore,

$$\begin{aligned} \iint_S (\nabla \times F) \cdot \widehat{\mathbf{n}} \ dS &= \iint_R (1,0,2x-1) \cdot (0,0,1) \ dA \\ &= \iint_R (2x-1) \ dA \\ &= -\iint_R \ dA = -\pi \qquad \textbf{(}2x \textbf{ is odd in } x\textbf{, and the region is symmetric w.r.t } y\textbf{)} \end{aligned}$$

For (B), $ds = \sqrt{1+(f_x)^2+(f_y)^2}dA = \sqrt{1+4x^2+4y^2}dA$, therefore,

$$\begin{aligned} \iint_S (\nabla \times F) \cdot \widehat{\mathbf{n}} \ dS &= \iint_R (1,0,2x-1) \cdot \frac{(-2x,-2y,1)}{\sqrt{1+4x^2+4y^2}} \cdot \sqrt{1+4x^2+4y^2}dA \\ &= \iint_R (-2x+2x-1) \ dA \\ &= -\iint_R \ dA = -\pi \end{aligned}$$

[Example.] Evaluate $\int_C (y+\sin x)dx+\left(z^2+\cos y\right)dy+x^3dz$, where $C: \mathbf{r}(t) = (\sin t, \cos t, \sin 2t), \quad 0 \leqslant t \leqslant 2\pi$

[Solution.] Note that C is a **closed space curve**, we can view it as circular integration on vector field:

$$\oint_C (y+\sin x)dx + (z^2+\cos y)dy + x^3dz = \oint_C \mathbf{F}\cdot d\mathbf{r}$$

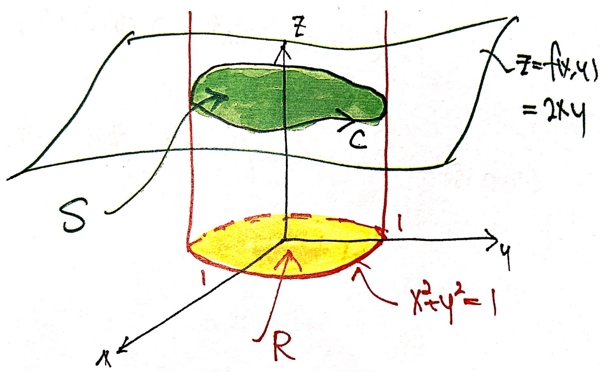
where $\mathbf{F}(x,y,z) = (y+\sin x, z^2+\cos y, x^3)$

Step 1: Find curl of vector field: $\nabla\times\mathbf{F} = \left(-2z, -3x^2, -1\right)$

Step 2: To apply Stokes' Theorem, we need to find a surface S,

From C: $\begin{cases} x(t) = \sin t \\ y(t) = \cos t \\ z(t) = \sin 2t = 2\sin t\cos t \end{cases} \Rightarrow \begin{cases} x^2+y^2 = 1 \\ z = 1 \end{cases}$

So C can be viewed as the intersection of two surfaces $x^2+y^2=1$ and $z = 2xy = f(x,y)$, and $z = 2xy$ is the S we need, while $R: \; x^2+y^2 = 1$ is the the projection of S onto xy-plane, which we will need in surface integral.



Step 3: find the normal to S:

$f(x,y,z) = z - 2xy = 0$(constant) is a level set in 3D, so

$$\mathbf{n} = \nabla f = (-2y, -2x, 1), \qquad \widehat{\mathbf{n}} = \frac{(-2y,-2x,1)}{\sqrt{1+4y^2+4x^2}}$$

Step 4: find surface integral: $dS = \sqrt{1+(f_x)^2+(f_y)^2}\; dA = \sqrt{1+4y^2+4x^2}\; dA$ Therefore,

$$
\begin{aligned}
\oint_C \mathbf{F}\cdot d\mathbf{r} &= \iint_S \nabla\times\mathbf{F}\cdot\widehat{\mathbf{n}}dS \\
&= \iint_R \left(-4xy, -3x^2, -1\right)\cdot\frac{(-2y,-2x,1)}{\sqrt{1+4y^2+4x^2}}\cdot\sqrt{1+4y^2+4x^2}\; dA \\
&= \iint_R \left(8xy^2+6x^3-1\right) dA \\
&= -\iint_R dA = -\pi \qquad \textit{(same trick again)}
\end{aligned}
$$