MATH 2023 Fall 2021

Multivariable Calculus

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Chapter 16 **Vector Calculus**

1 The Divergence Theorem

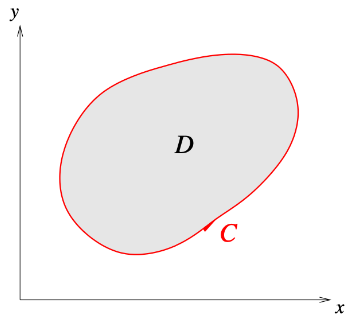
Since exam will not cover the proof, I'd like to omit here.

2 Green's Theorem

2.1 Green's Theorem in Line Integral

In this part, we will go back to **line integral**, which we have done a lot.

Now consider doing line integral in a smooth simple **closed curve** C in the xy-plane, if $\mathbf{F}(\mathbf{r}) = P(\mathbf{r})\mathbf{i} + Q(\mathbf{r})\mathbf{j}$, then if we want to evaluate $\oint_C \mathbf{F} \cdot d\mathbf{r}$,



Counter-clockwise
– positive orientation

- If $\mathbf{F}$ is conservative, then line integral is 0, obviously.
- If $\mathbf{F}$ is not conservative, then **Green's Theorem** tells us

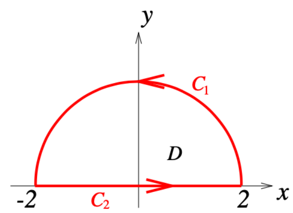
$$\boxed{\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_D (\nabla \times \mathbf{F}) \cdot \mathbf{k} \, dA}$$

Note k is the **normal** to xy-plane, or, normal to region D.

Since exam will not cover the proof, I'd like to omit here.

This example shows how Green's Theorem simplify computation.

[Example.] $\int_C xydx + 2x^2dy$, C consists of the segment from $(-2, 0)$ to $(2, 0)$ and top half of the circle $x^2 + y^2 = 4$.



[Solution.]

Method 1: use line integral:

$$\int_C xydx + 2x^2dy = \int_{C_1} xydx + 2x^2dy + \int_{C_2} xydx + 2x^2dy$$

Parametrize the two curves:

$$C_1 : \mathbf{r}(t) = (1-t)(-2,0) + t(2,0) = (4t-2, 0) \quad 0 \leqslant t \leqslant 1$$
$$C_2 : \mathbf{r}(t) = (2\cos t, 2\sin t) \quad 0 \leqslant t \leqslant \pi$$

Then directly evaluate the two line integrals

$$\begin{aligned}\int_{C_1} xydx + 2x^2dy &= \int_0^1 (4t-2)\cdot 0 \cdot 4dt + 2(4t-2)^2 \cdot (0) = 0 \\ \int_{C_2} xydx + 2x^2dy &= \int_0^\pi (2\cos t)(2\sin t)(-2\sin t)dt + 2(2\cos t)^2(2\cos t)dt \\ &= 8\int_0^\pi \left(-\cos t \sin^2 t + \cos^3 t\right) dt = 0\end{aligned}$$

Thus $\int_C xydx + 2x^2dy = 0$.

Method 2: using Green's theorem:

$\mathbf{F} = (xy, 2x^2)$, hence $\nabla \times \mathbf{F} = (4x - x)\mathbf{k} = 3x\mathbf{k}$, then

$$\begin{aligned}\oint_C xydx + 2x^2dy = \iint_D 3xdA &= \int_0^2 \int_0^\pi 3r\cos\theta \; rd\theta dr \\ &= \int_0^2 3r^2 \sin\theta \Big|_0^\pi dr = 0\end{aligned}$$

Actually, one may observe that $\iint_D 3xdA = 0$ directly, since $3x$ is a *odd* function in x, and the region D is *symmetric with respect to* y-axis.

2.2 Green's Theorem for computing Area

Recall that Green's Theorem states that:

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_D (\nabla \times \mathbf{F}) \cdot \mathbf{k} \, dA$$

Notice if $\mathbf{F}(\mathbf{r}) = P(\mathbf{r})\mathbf{i} + Q(\mathbf{r})\mathbf{j}$,

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C Pdx + Qdy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \iint_D (\nabla \times \mathbf{F}) \cdot \mathbf{k} \, dA$$

When $\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y} = 1$, then

$$\boxed{A = \iint_D dA = \oint_C Pdx + Qdy.}$$

For example, when $P = 0, Q = x$, or when $P = -y, Q = 0$, or when $P = -y/2, Q = x/2$,

$$A = \oint_C xdy = -\oint_C ydx = \frac{1}{2} \oint_C xdy - ydx$$

The two examples below shows how to use Green's Theorem to find area.

[Example.] Find the area of $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

[Solution.] Firstly parametrize the curve, let $x = a\cos\theta,\ y = b\sin\theta,\ 0 \le \theta \le 2\pi$, then

$$C:\ \mathbf{r}(\theta) = (a\cos\theta, b\sin\theta),\ 0 \le \theta \le 2\pi$$

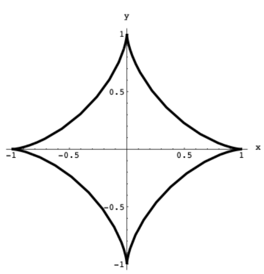
, If we choose $\mathbf{F}(\mathbf{r}) = P(\mathbf{r})\mathbf{i} + Q(\mathbf{r})\mathbf{j} = -\dfrac{y}{2}\mathbf{i} + \dfrac{x}{2}\mathbf{j}$, then we have

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \frac{1}{2} + \frac{1}{2} = 1$$

Hence,

$$\begin{aligned}
D &= \frac{1}{2}\oint (xdy - ydx) \\
&= \frac{1}{2}\left[\int_0^{2\pi} a\cos\theta \cdot b\cos\theta\ d\theta + b\sin\theta \cdot a\sin\theta\ d\theta\right] \\
&= \frac{1}{2}ab \cdot \int_0^{2\pi} (\cos^2\theta + \sin^2\theta)\ d\theta = \pi ab
\end{aligned}$$

[Example.] Find the area of the hypocycloid $x^{2/3} + y^{2/3} = a^{2/3}$.



[Solution.] Firstly parametrize the curve, let $x = a\cos^3\theta, y = a\sin^3\theta$, where $0 \leqslant \theta \leqslant 2\pi$,

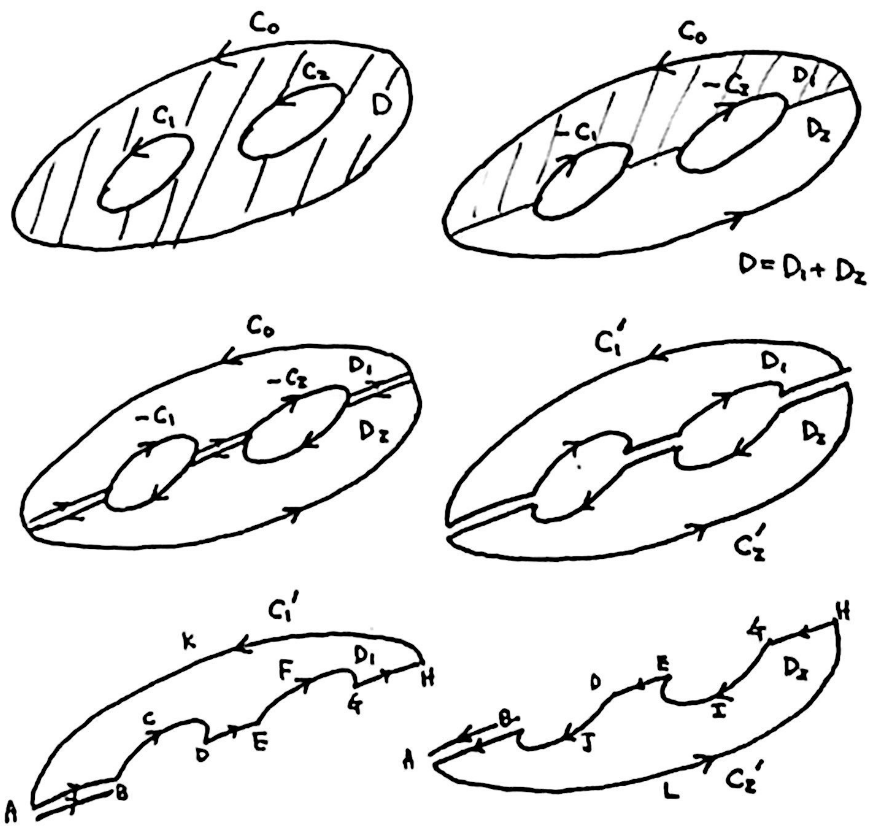
Again, use vector field $\mathbf{F}(\mathbf{r}) = P(\mathbf{r})\mathbf{i} + Q(\mathbf{r})\mathbf{j} = -\dfrac{y}{2}\mathbf{i} + \dfrac{x}{2}\mathbf{j}$,

$$\begin{aligned}
A = \frac{1}{2}\oint_C xdy - ydx &= \frac{1}{2}\int_0^{2\pi} \left(a\cos^3\theta \times 3a\sin^2\theta\cos\theta d\theta + a\sin^3\theta \times 3a\cos^2\theta\sin\theta d\theta\right) \\
&= \frac{3}{2}a^2\int_0^{2\pi} \left(\cos^4\sin^2\theta + \sin^4\theta\cos^2\theta\right) d\theta \\
&= \frac{3}{2}a^2\int_0^{2\pi} \frac{1}{4}\sin^2 2\theta d\theta \\
&= \frac{3}{8}a^2\int_0^{2\pi} \left(\frac{1-\cos 4\theta}{2}\right) d\theta = \frac{3\pi}{8}a^2
\end{aligned}$$

2.3 General version of Green's Theorem

This part will not be covered in exam.

Recall that Green's Theorem only applies to *simple* and *closed* curve. However, it can be extended to apply to region with holes. We simply cut the region into some regions that without holes, for example:



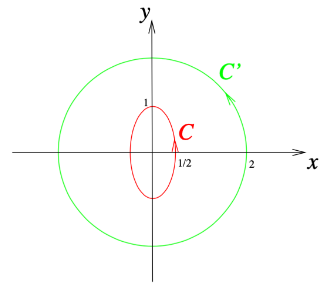
$$
\begin{aligned}
\iint_D &= \iint_{D_1} + \iint_{D_2} = \oint_{C_1'} + \oint_{C_2'} \\
&= \left(\int_{HKA} + \int_{AB} + \int_{BCD} + \int_{DE} + \int_{EFG} + \int_{GH} \right) + \left(\int_{ALH} + \int_{HG} + \int_{GIE} + \int_{ED} + \int_{DJB} + \int_{BA} \right) \\
&= \int_{C_0} - \int_{C_1} - \int_{C_2}
\end{aligned}
$$

Here is the example provided in lecture note.

Example $\displaystyle\oint_C \frac{-x^2y\,dx + x^3\,dy}{(x^2+y^2)^2}$, where C is the ellipse $4x^2+y^2=1$.

If C' is the circle $x^2+y^2=4$, then C is interior to C', and everywhere except at $(0,0)$. Note also that

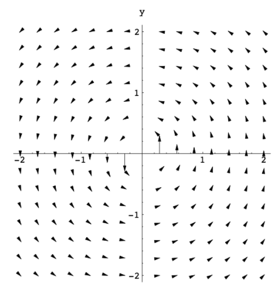
$$\frac{\partial}{\partial x}\left[\frac{x^3}{(x^2+y^2)^2}\right] = \frac{\partial}{\partial y}\left[\frac{-x^2y}{(x^2+y^2)^2}\right]$$



$$\therefore\ I = \oint_C \frac{-x^2y\,dx + x^2\,dy}{(x^2+y^2)^2} = \oint_{C'} \frac{-x^2y\,dx + x^3\,dy}{(x^2+y^2)^2}$$

On C', let $x = 2\cos\theta$, $y = 2\sin\theta$, where $0 \leqslant \theta \leqslant 2\pi$, then

$$\begin{aligned} I &= \int_0^{2\pi} \frac{-4\cos^2\theta\ 2\sin\theta(-2\sin\theta)\,d\theta + (2\cos\theta)^2\ 2\cos\theta\,d\theta}{16} \\ &= \int_0^{2\pi} \cos^2\theta\,d\theta = \int_0^{2\pi} \left(\frac{1+\cos 2\theta}{2}\right) d\theta = \pi. \end{aligned}$$



$$\mathbf{F}(\mathbf{r}) = \frac{-x^2y\,\mathbf{i} + x^3\,\mathbf{j}}{(x^2+y^2)^2}$$

3 Stokes' Theorem

Recall in Green's Theorem,

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_D (\nabla \times \mathbf{F}) \cdot \mathbf{k} dA$$

where $\mathbf{F}(\mathbf{r}) = (F_1, F_2)$, $C : \mathbf{r}(t) = (x(t), y(t))$, $a \le t \le b$

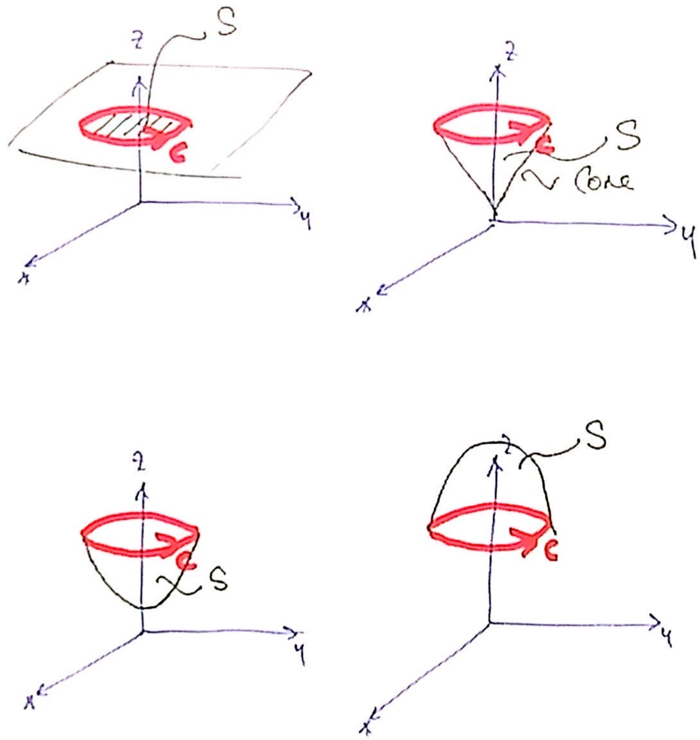
Now we want to extends this theorem into 3D space.

The **Stokes' Theorem** tells that if S is a *non-closed* surface, whose boundary consists of a closed smooth curve C with *positive orientation,* then

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \widehat{\mathbf{n}} \, dS$$

where $\mathbf{F}(\mathbf{r}) = (F_1, F_2, F_3)$, $C : \mathbf{r}(t) = \big(x(t), y(t), z(t)\big)$, $a \le t \le b$, and $\mathbf{r}(a) = \mathbf{r}(b)$ since the boundary is closed. $\widehat{\mathbf{n}}$ is unit normal vector of surface S.

However, you may have noticed that the theorem doesn't tell how to find S. When we evaluate a line integral on C, there are lots of surfaces S that can have boundary C.



This example gives a standard process for applying Stokes' Theorem and provides ideas of how to construct S.

[Example.] Find $\oint_C \mathbf{F}\cdot d\mathbf{r}$, where $\mathbf{F}(\mathbf{r}) = (y, x^2, y)$, $C:\ \mathbf{r}(t) = (\cos t, \sin t, 1),\ 0 \le t \le 2\pi$

[Solution.] Method 1: directly compute line integral.

$$\begin{aligned}\oint_C \mathbf{F}\cdot d\mathbf{r} &= \int_0^{2\pi} (\sin t, \cos^2 t, \sin t)\cdot(-\sin t, \cos t, 0)\ dt \\ &= \int_0^{2\pi} (-\sin^2 t + \cos^3 t)\ dt\end{aligned}$$

This is tedious.

Method 2: Notice $\mathbf{r}(0) = \mathbf{r}(2\pi)$, so this is a *closed curve* in 3D, we can use **Stokes' Theorem**.

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_S (\nabla\times\mathbf{F})\cdot\widehat{\mathbf{n}}\ dS$$

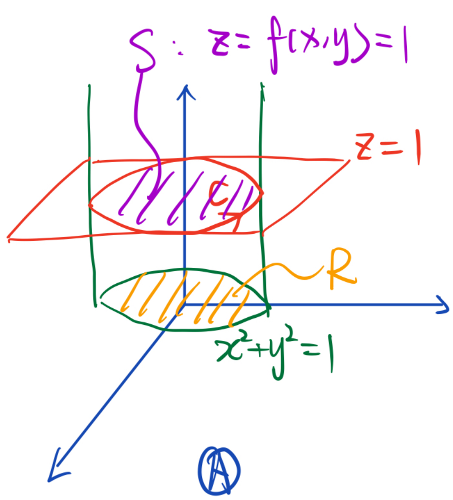
But S is not given, we need to find $S: z = f(x,y)$

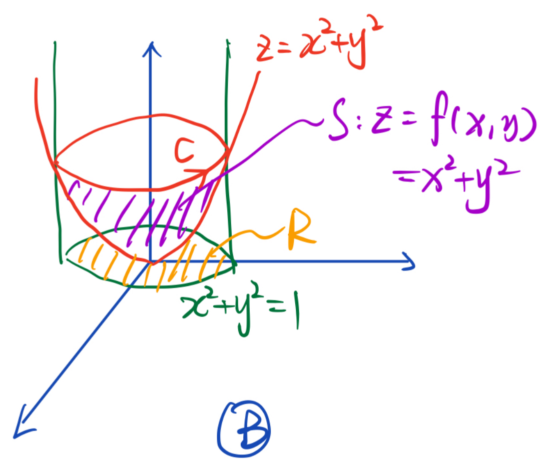
Idea: construct 2 surfaces whose *intersection* is the curve C.

From C: $\begin{cases} x(t) = \cos t \\ y(t) = \sin t \\ z(t) = 1 \end{cases}$, we can construct 2 surfaces by observing the relationship among x, y, z, for example,

$$\begin{cases} x^2+y^2 = 1 \\ z = 1 \end{cases} \qquad or \qquad \begin{cases} x^2+y^2=1 \\ z = x^2+y^2 \end{cases}$$

Their graphs are shown below:





We can see that for the first equation, the surface S is a circle, while for the second equation, the surface S is a "rice bowl". Either of them is ok for our calculation.

(1) Firstly, find the curl of vector field:

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y & x^2 & y \end{vmatrix} = \mathbf{i} + (2x-1)\mathbf{k}$$

(2) Next, find normal vector to the surface,

For (A), $z = f(x, y) = 1$, hence $\widehat{\mathbf{n}} = \mathbf{k}$.

For (B), let $G(x, y, z) = z - x^2 - y^2 = 0$(constant), this is a level set in 3D, hence

$$\mathbf{n} = \nabla G = (-2x, -2y, 1), \ \widehat{\mathbf{n}} = \frac{(-2x, -2y, 1)}{\sqrt{1 + 4x^2 + 4y^2}}$$

(3) Then, find surface integral, and thereby calculating the result:

For (A), $ds = \sqrt{1 + (f_x)^2 + (f_y)^2} dA = dA$, therefore,

$$\begin{aligned} \iint_S (\nabla \times F) \cdot \widehat{\mathbf{n}} \ dS &= \iint_R (1, 0, 2x-1) \cdot (0, 0, 1) \ dA \\ &= \iint_R (2x-1) \ dA \\ &= -\iint_R \ dA = -\pi \qquad \text{(}2x\ \textbf{is odd in}\ x\textbf{, and the region is symmetric w.r.t}\ y\text{)} \end{aligned}$$

For (B), $ds = \sqrt{1 + (f_x)^2 + (f_y)^2} dA = \sqrt{1 + 4x^2 + 4y^2} dA$, therefore,

$$\begin{aligned} \iint_S (\nabla \times F) \cdot \widehat{\mathbf{n}} \ dS &= \iint_R (1, 0, 2x-1) \cdot \frac{(-2x, -2y, 1)}{\sqrt{1 + 4x^2 + 4y^2}} \cdot \sqrt{1 + 4x^2 + 4y^2} dA \\ &= \iint_R (-2x + 2x - 1) \ dA \\ &= -\iint_R \ dA = -\pi \end{aligned}$$

[Example.] Evaluate $\int_C (y+\sin x)dx + \left(z^2+\cos y\right)dy + x^3 dz$, where $C:\ \mathbf{r}(t) = (\sin t, \cos t, \sin 2t), \quad 0 \leqslant t \leqslant 2\pi$

[Solution.] Note that C is a **closed space curve**, we can view it as circular integration on vector field:

$$\oint_C (y+\sin x)dx + (z^2+\cos y)dy + x^3 dz = \oint_C \mathbf{F}\cdot d\mathbf{r}$$

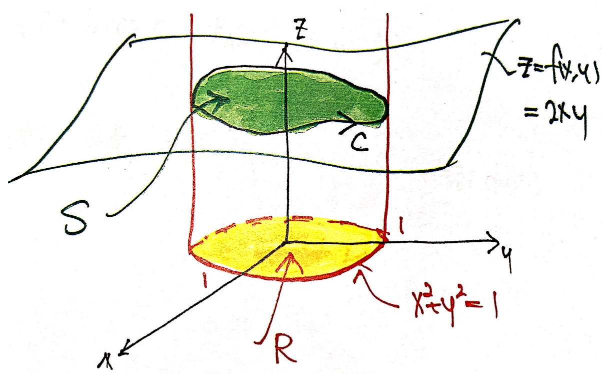
where $\mathbf{F}(x,y,z) = (y+\sin x, z^2+\cos y, x^3)$

Step 1: Find curl of vector field: $\nabla\times\mathbf{F} = \left(-2z, -3x^2, -1\right)$

Step 2: To apply Stokes' Theorem, we need to find a surface S,

$$\text{From } C\text{:} \begin{cases} x(t) = \sin t \\ y(t) = \cos t \\ z(t) = \sin 2t = 2\sin t\cos t \end{cases} \Rightarrow \begin{cases} x^2+y^2 = 1 \\ z = 1 \end{cases}$$

So C can be viewed as the intersection of two surfaces $x^2+y^2=1$ and $z = 2xy = f(x,y)$, and $z = 2xy$ is the S we need, while $R:\ x^2+y^2 = 1$ is the the projection of S onto xy-plane, which we will need in surface integral.



Step 3: find the normal to S:

$f(x,y,z) = z - 2xy = 0$(constant) is a level set in 3D, so

$$\mathbf{n} = \nabla f = (-2y, -2x, 1), \qquad \widehat{\mathbf{n}} = \frac{(-2y,-2x,1)}{\sqrt{1+4y^2+4x^2}}$$

Step 4: find surface integral: $dS = \sqrt{1+(f_x)^2+(f_y)^2}\ dA = \sqrt{1+4y^2+4x^2}\ dA$ Therefore,

$$\begin{aligned}\oint_C \mathbf{F}\cdot d\mathbf{r} &= \iint_S \nabla\times\mathbf{F}\cdot\widehat{\mathbf{n}}dS \\ &= \iint_R \left(-4xy, -3x^2, -1\right)\cdot\frac{(-2y,-2x,1)}{\sqrt{1+4y^2+4x^2}}\cdot\sqrt{1+4y^2+4x^2}\ dA \\ &= \iint_R \left(8xy^2+6x^3-1\right)dA \\ &= -\iint_R dA = -\pi \qquad \textit{(same trick again)}\end{aligned}$$

This is the end of Chapter 16, and the end of this course!