MATH 2023 Fall 2021

Multivariable Calculus

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Chapter 16 **Vector Calculus**

1 The Divergence Theorem

Since exam will not cover the proof, I'd like to omit here.

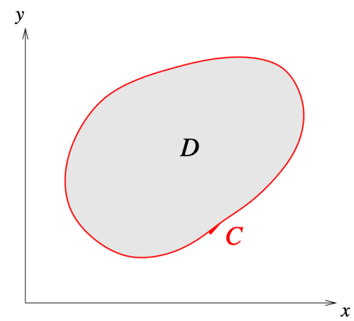
2 Green's Theorem

2.1 Green's Theorem in Line Integral

Recall what we have talked about in line integral:

1. Arc length,
2. Mass:

Now consider doing line integral in a smooth simple **closed curve** C in the xy-plane, if $\mathbf{F}(\mathbf{r}) = P(\mathbf{r})\mathbf{i} + Q(\mathbf{r})\mathbf{j}$, then if we want to evaluate $\oint_C \mathbf{F} \cdot d\mathbf{r}$,



Counter-clockwise
– positive orientation

- If $\mathbf{F}$ is conservative, then line integral is 0, obviously.
- If $\mathbf{F}$ is not conservative, then **Green's Theorem** tells us

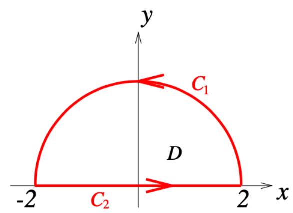
$$\boxed{\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_D (\nabla \times \mathbf{F}) \cdot \mathbf{k} \, dA}$$

Note k is the **normal** to xy-plane, or, normal to region D.

Since exam will not cover the proof, I'd like to omit here.

This example shows how Green's Theorem simplify computation.

[Example.] $\int_C xydx + 2x^2dy$, C consists of the segment from $(-2,0)$ to $(2,0)$ and top half of the circle $x^2+y^2=4$.



[Solution.]

Method 1: use line integral:

$$\int_C xydx + 2x^2dy = \int_{C_1} xydx + 2x^2dy + \int_{C_2} xydx + 2x^2dy$$

Parametrize the two curves:

$$C_1 : \mathbf{r}(t) = (1-t)(-2,0) + t(2,0) = (4t-2,0) \quad 0 \leqslant t \leqslant 1$$
$$C_2 : \mathbf{r}(t) = (2\cos t, 2\sin t) \quad 0 \leqslant t \leqslant \pi$$

Then directly evaluate the two line integrals

$$\int_{C_1} xydx + 2x^2dy = \int_0^1 (4t-2)\cdot 0 \cdot 4dt + 2(4t-2)^2 \cdot (0) = 0$$
$$\int_{C_2} xydx + 2x^2dy = \int_0^\pi (2\cos t)(2\sin t)(-2\sin t)dt + 2(2\cos t)^2(2\cos t)dt$$
$$= 8\int_0^\pi \left(-\cos t \sin^2 t + \cos^3 t\right) dt = 0$$

Thus $\int_C xydx + 2x^2dy = 0$.

Method 2: using Green's theorem:

$\mathbf{F} = (xy, 2x^2)$, hence $\nabla \times \mathbf{F} = (4x - x)\mathbf{k} = 3x\mathbf{k}$, then

$$\oint_C xydx + 2x^2dy = \iint_D 3xdA = \int_0^2 \int_0^\pi 3r\cos\theta \; rd\theta dr$$
$$= \int_0^2 3r^2 \sin\theta \Big|_0^\pi dr = 0$$

Actually, one may observe that $\iint_D 3xdA = 0$ directly, since $3x$ is a *odd* function in x, and the region D is *symmetric with respect to* y-axis.

2.2 Green's Theorem for computing Area

Recall that Green's Theorem states that:

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_D (\nabla \times \mathbf{F}) \cdot \mathbf{k} \, dA$$

Notice if $\mathbf{F}(\mathbf{r}) = P(\mathbf{r})\mathbf{i} + Q(\mathbf{r})\mathbf{j}$,

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C Pdx + Qdy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \iint_D (\nabla \times \mathbf{F}) \cdot \mathbf{k} \, dA$$

When $\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y} = 1$, then

$$\boxed{A = \iint_D dA = \oint_C Pdx + Qdy.}$$

For example, when $P = 0, Q = x$, or when $P = -y, Q = 0$, or when $P = -y/2, Q = x/2$,

$$A = \oint_C xdy = -\oint_C ydx = \frac{1}{2} \oint_C xdy - ydx$$

The two examples below shows how to use Green's Theorem to find area.

[Example.] Find the area of $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

[Solution.] Firstly parametrize the curve, let $x = a\cos\theta,\ y = b\sin\theta,\ 0 \le \theta \le 2\pi$, then

$$C:\ \mathbf{r}(\theta) = (a\cos\theta, b\sin\theta),\ 0 \le \theta \le 2\pi$$

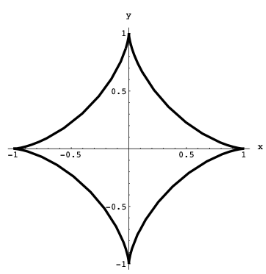
, If we choose $\mathbf{F}(\mathbf{r}) = P(\mathbf{r})\mathbf{i} + Q(\mathbf{r})\mathbf{j} = -\dfrac{y}{2}\mathbf{i} + \dfrac{x}{2}\mathbf{j}$, then we have

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \frac{1}{2} + \frac{1}{2} = 1$$

Hence,

$$\begin{aligned} D &= \frac{1}{2}\oint (xdy - ydx) \\ &= \frac{1}{2}\left[\int_0^{2\pi} a\cos\theta \cdot b\cos\theta\ d\theta + b\sin\theta \cdot a\sin\theta\ d\theta\right] \\ &= \frac{1}{2}ab \cdot \int_0^{2\pi} (\cos^2\theta + \sin^2\theta)\ d\theta = \pi ab \end{aligned}$$

[Example.] Find the area of the hypocycloid $x^{2/3} + y^{2/3} = a^{2/3}$.



[Solution.] Firstly parametrize the curve, let $x = a\cos^3\theta, y = a\sin^3\theta$, where $0 \leqslant \theta \leqslant 2\pi$,

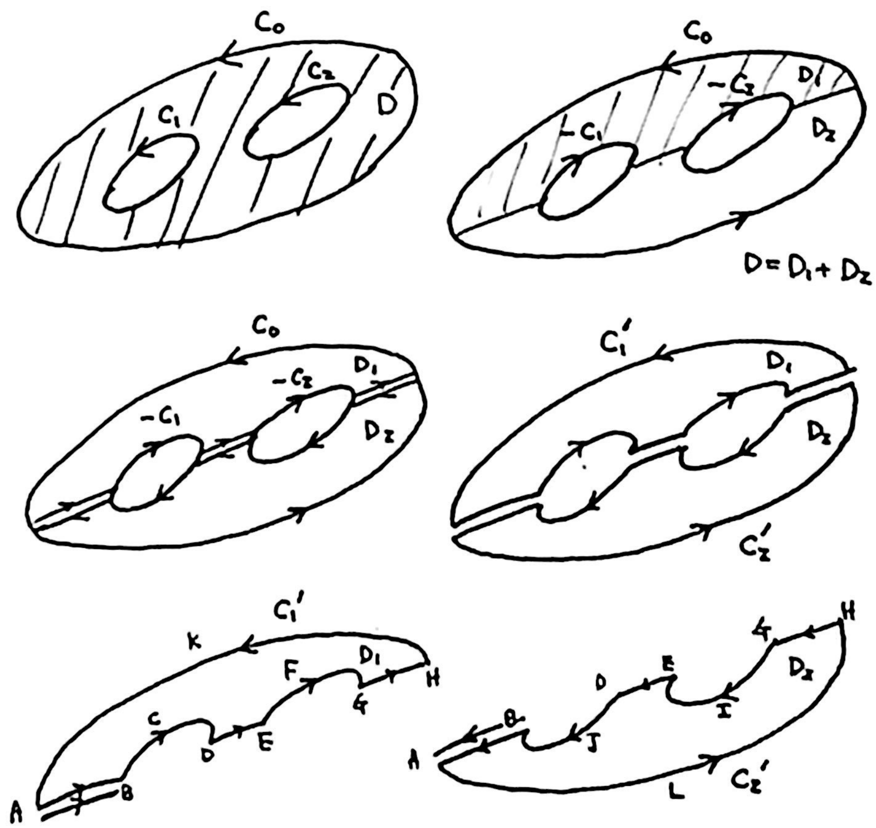
Again, use vector field $\mathbf{F}(\mathbf{r}) = P(\mathbf{r})\mathbf{i} + Q(\mathbf{r})\mathbf{j} = -\dfrac{y}{2}\mathbf{i} + \dfrac{x}{2}\mathbf{j}$,

$$\begin{aligned} A = \frac{1}{2}\oint_C xdy - ydx &= \frac{1}{2}\int_0^{2\pi} \left(a\cos^3\theta \times 3a\sin^2\theta\cos\theta d\theta + a\sin^3\theta \times 3a\cos^2\theta\sin\theta d\theta\right) \\ &= \frac{3}{2}a^2\int_0^{2\pi} \left(\cos^4\sin^2\theta + \sin^4\theta\cos^2\theta\right) d\theta \\ &= \frac{3}{2}a^2\int_0^{2\pi} \frac{1}{4}\sin^2 2\theta d\theta \\ &= \frac{3}{8}a^2\int_0^{2\pi} \left(\frac{1-\cos 4\theta}{2}\right) d\theta = \frac{3\pi}{8}a^2 \end{aligned}$$

2.3 General version of Green's Theorem

This part will not be covered in exam.

Recall that Green's Theorem only applies to *simple* and *closed* curve. However, it can be extended to apply to region with holes. We simply cut the region into some regions that without holes, for example:



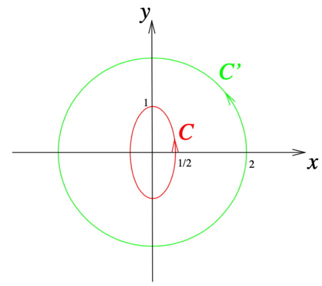
$$
\begin{aligned}
\iint_D &= \iint_{D_1} + \iint_{D_2} = \oint_{C_1'} + \oint_{C_2'} \\
&= \left(\int_{HKA} + \int_{AB} + \int_{BCD} + \int_{DE} + \int_{EFG} + \int_{GH}\right) + \left(\int_{ALH} + \int_{HG} + \int_{GIE} + \int_{ED} + \int_{DJB} + \int_{BA}\right) \\
&= \int_{C_0} - \int_{C_1} - \int_{C_2}
\end{aligned}
$$

Here is the example provided in lecture note.

Example $\displaystyle\oint_C \frac{-x^2y\,dx + x^3\,dy}{(x^2+y^2)^2}$, where C is the ellipse $4x^2+y^2=1$.

If C' is the circle $x^2+y^2=4$, then C is interior to C', and everywhere except at $(0,0)$. Note also that

$$\frac{\partial}{\partial x}\left[\frac{x^3}{(x^2+y^2)^2}\right] = \frac{\partial}{\partial y}\left[\frac{-x^2y}{(x^2+y^2)^2}\right]$$

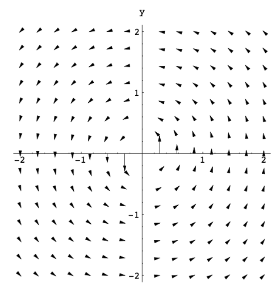


$$\therefore\ I = \oint_C \frac{-x^2y\,dx + x^2\,dy}{(x^2+y^2)^2} = \oint_{C'} \frac{-x^2y\,dx + x^3\,dy}{(x^2+y^2)^2}$$

On C', let $x = 2\cos\theta$, $y = 2\sin\theta$, where $0 \leqslant \theta \leqslant 2\pi$, then

$$I = \int_0^{2\pi} \frac{-4\cos^2\theta\ 2\sin\theta(-2\sin\theta)\,d\theta + (2\cos\theta)^2\ 2\cos\theta\,d\theta}{16}$$

$$= \int_0^{2\pi} \cos^2\theta\,d\theta = \int_0^{2\pi}\left(\frac{1+\cos 2\theta}{2}\right)\,d\theta = \pi.$$



$$\mathbf{F}(\mathbf{r}) = \frac{-x^2y\,\mathbf{i} + x^3\,\mathbf{j}}{(x^2+y^2)^2}$$

3 Stokes' Theorem

This is the end of Chapter 16, and the end of this course!