

Design & Analysis of Algorithms

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Topic 2 Divide & Conquer

1 Intro: Binary Search

The main idea of Divide & Conquer is to solve a problem (such as of size n) by breaking it into one or more smaller (size less than n) problems. We use binary search example to illustrate that.

Problem: given an **sorted** array of length n , how to find the position of element x ; if x does not exist in the array, output nil.

Since the array is already sorted, it has a good property that: **for each item a_i , those who are larger than a_i must be on its right side, while smaller than a_i must be on its left side.** Hence we come up with an idea that we check the middle item mid first, then we will be able to know which direction to go: left or right, depending on the comparison of mid and x (the item we're looking for). If we go left, then the right half will be directly abandoned. Then we continue this process, check middle item each time, and abandon half items each time.

Algorithm 1: BinarySearch($a[], left, right, x$)

Data: $a[]$: the array given, x : the item to find

```
1 if left = right then
2   if  $a[left] = x$  then
3     return left
4   else
5     return nil
6   end
7 else
8    $mid = \lfloor (left + right) / 2 \rfloor$ 
9   if  $x \leq a[mid]$  then
10    BinarySearch( $a[], left, mid, x$ )
11  else
12    BinarySearch( $a[], mid + 1, right, x$ )
13  end
14 end
```

First call: BinarySearch($a[], 1, n, x$).

This algorithm is quite efficient, since each time we eliminate half of the array, with one additional comparison, until there is only one item left, when we will end the process.

Then let's analyse its time complexity. Let $T(n)$ be the number of comparisons needed for n elements, then we will have

$$T(n) = T(n/2) + 1, \quad T(1) = 1$$

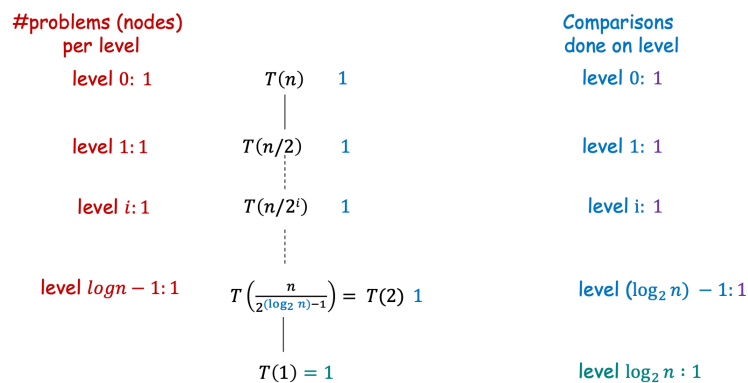
.

Solve this **recurrence**:

$$\begin{aligned} T(n) &= T(n/2) + 1 \\ &= [T(n/4) + 1] + 1 \\ &= T(n/4) + 2 \\ &= \dots \\ &= T(n/2^i) + i \end{aligned}$$

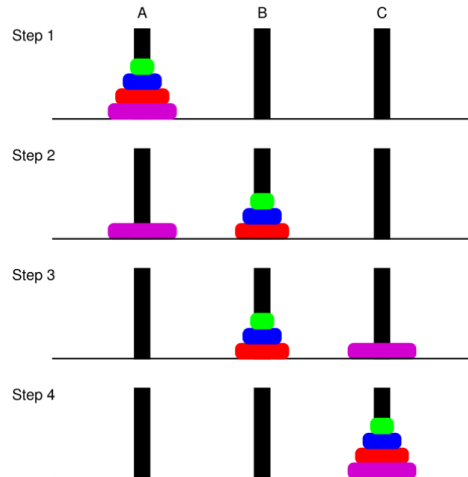
This process ends when reaching $T(1)$, i.e., $i = \log_2 n$, thus, $T(n) = T(1) + \log_2 n = \log_2 n + 1$.

We can also visualize this recurrence with recursion tree: (image from lecture note)



In each recursion step(level), we use 1 comparison(compare *mid* and *x*), then call recursion on a half of the original array. From the image above, we can easily notice there are total $1 + 1 + \dots + 1 = 1 + \log_2 n$ comparisons.

2 Example: Towers of Hanoi



In this example, we want to design an algorithm to move all n discs from peg A (start) to peg C (end), with the constraints: (1) move one disc at a time, and (2) cannot put larger disc on a smaller one. We are given another peg B (helper) where we can temporary storage our discs.

We still use the idea of **Divide & Conquer**, consider how we can turn a problem of n discs into a problem of $n - 1$? One possible solution is that, we can call recursion on upper $n - 1$ discs, i.e., move upper $n - 1$ discs to peg B (helper peg), then move the remaining (the biggest) disc to peg C (end peg), and finally move the $n - 1$ discs from peg B (helper) to peg C (end). The following pseudocode shows this idea.

Algorithm 2: MoveTower(n , $start$, $helper$, end)

Input: n : num of discs

```

1 if  $n = 1$  then
2   | move the only disc from  $start$  peg to  $end$  peg
3   | return
4 else
5   | // move first  $n - 1$  from  $start$  peg to  $helper$  peg
6   | // so this time ‘‘helper’’ peg will be the old  $end$  peg
7   | MoveTower( $n - 1$ ,  $start$ ,  $end$ ,  $helper$ )
8   | move the only disc from  $start$  peg to  $end$  peg
9   | // finally move first  $n - 1$  from  $helper$  peg to  $end$  peg
10  | // this time ‘‘helper’’ peg will be the old  $start$  peg
11  | MoveTower( $n - 1$ ,  $helper$ ,  $start$ ,  $end$ )
12 end
```

Now we would like to analyze the time complexity of this algorithm, in other words, how many **steps** are needed. Let $T(n)$ be the num of steps for n discs, each time, we first move $n - 1$ disks from $start$ to $helper$, costs $T(n - 1)$ steps; then we move the biggest disk to end peg, costs only 1 step; finally we move $n - 1$ disks from $helper$ to end , again costs $T(n - 1)$ steps. To sum up:

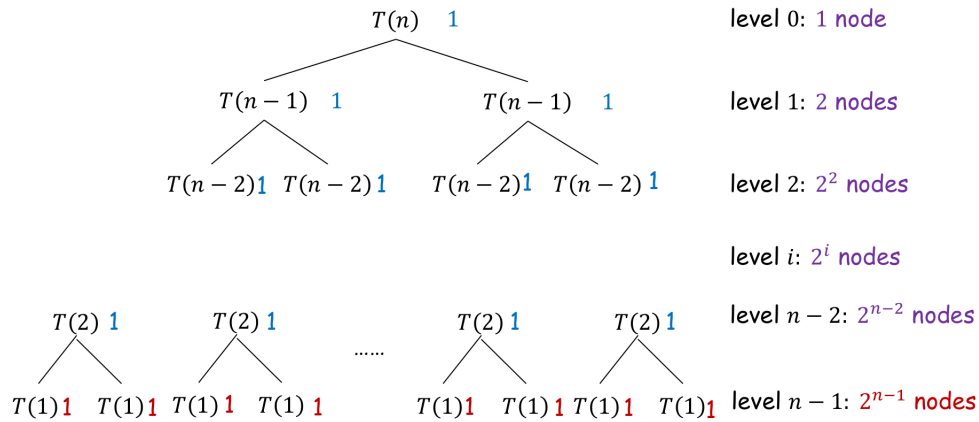
$$T(n) = 2T(n - 1) + 1$$

when $n > 1$, and $T(1) = 1$.

Now we solve the recurrence by the **expansion method**:

$$\begin{aligned}
 T(n) &= 2T(n-1) + 1 \\
 &= 2[2T(n-2) + 1] + 1 \\
 &= 4T(n-2) + 3 \\
 &= 4[2T(n-3) + 1] + 3 \\
 &= 8T(n-3) + 7 \\
 &= \dots \\
 &= 2^i T(n-i) + (2^i - 1) \\
 &= 2^{n-1} T(1) + (2^{n-1} - 1) \\
 &= 2^n - 1
 \end{aligned}$$

Or, with the recursion tree method:



There are, altogether, $1 + 2 + 2^2 + 2^3 + \dots + 2^{n-2} + 2^{n-1} = 2^n - 1$ nodes, and we are doing one work(step) each node, then the time complexity is again, $2^n - 1$.

3 Merge Sort

Now we again back to sorting, and we would like to introduce a new algorithm or sorting: Merge Sort. This is a typical example of divide & conquer, and its process is like: (1) we first divide array into two halves, (2) then we recursively sort each half, (which means we continuously divide it into halves, and then halves...) (3) finally **merge** two halves to get a whole.

The **merge** operation may confuse you most. Here it means combine two **sorted lists** into a whole sorted list. For example, given two sorted lists: $A = [2, 5, 7]$ and $B = [3, 4, 6, 10, 12]$, then after **merge** operation, we will get

$result = [2, 3, 4, 5, 6, 7, 10, 12]$. Since these two lists are sorted, we can do this process in $O(n)$ time, where n is the length of result list.(how many numbers in total) The basic idea is: we compare the first item of A and B , put the smaller one, say, $A[1]$, in the first position of result list, then we move on to the next item of A , but compare it still with the **first** item of B (since the first item of B has not yet inserted into result list), and again put the smaller one into result list, then continue move on. An example may help you understand the process:

- (1) Compare first items: $A = [2, 5, 7], B = [3, 4, 6, 10, 12]$, $2 < 3$, so $result = [2]$;
- (2) then compare 2nd in A and 1st in B , $A = [2, 5, 7], B = [3, 4, 6, 10, 12]$, $3 < 5$, so $result = [2, 3]$;
- (3) continue the process, similarly, $A = [2, 5, 7], B = [3, 4, 6, 10, 12]$, $4 < 5$, so $result = [2, 3, 4]$;
- (4) $A = [2, 5, 7], B = [3, 4, 6, 10, 12]$, $5 < 6$, so $result = [2, 3, 4, 5]$;
- (5) $A = [2, 5, 7], B = [3, 4, 6, 10, 12]$, $6 < 7$, so $result = [2, 3, 4, 5, 6]$;
- (6) $A = [2, 5, 7], B = [3, 4, 6, 10, 12]$, $7 < 10$, so $result = [2, 3, 4, 5, 6, 7]$;
- (7) Now, all items in A have already been inserted into result list so that no items can be compared with items in B . Then we simply add remaining items in B to result list, this will, obviously, ensure a sorted result list.(you may think of why) Hence $result = [2, 3, 4, 5, 6, 7, 10, 12]$

The pseudocode below shows the process: (below, append ∞ at the end of two lists can free us from considering the situation that one list is empty, like (7) above. Though different implementation, the idea is entirely the same)

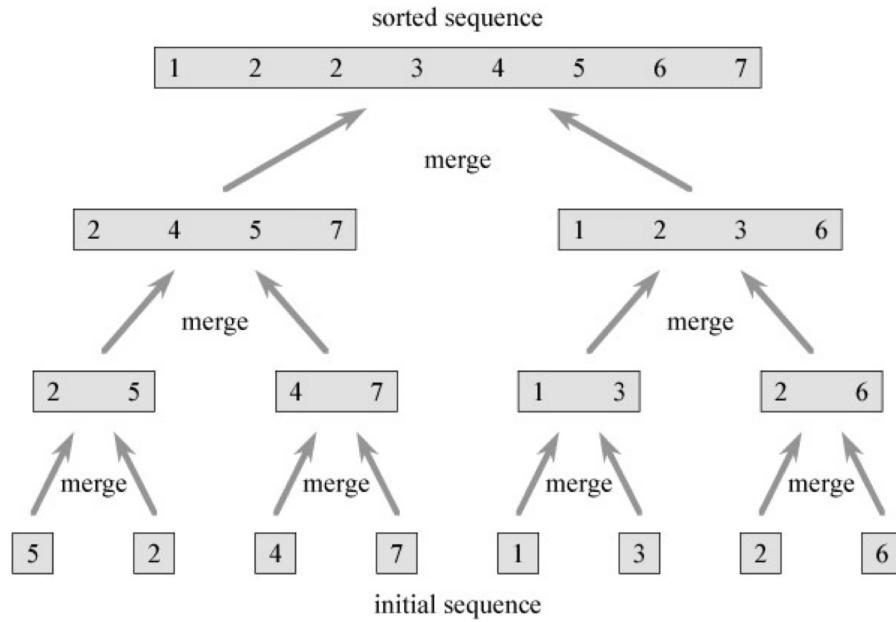
Algorithm 3: Merge(A , $left$, mid , $right$)

```

// merge two sorted list:  A[left...mid] and A[mid + 1...right]
1  L ← A[left...mid], R ← A[mid + 1...right]
2  append ∞ at the end of L and R      // see explanation above
3  i ← 1, j ← 1      // two pointers point at items in L and R
4  for k ← left to right do
    // always choose the smaller one to insert, and move on
5    if L[i] ≤ R[j] then
6      A[k] ← L[i]
7      i ← i + 1
8    else
9      A[k] ← R[j]
10     j ← j + 1
11  end
12 end

```

After learning how **Merge** works, you now, hopefully, are able to understand how Merge Sort works, with the image below:



We break down array recursively, until one element left, and then merge from bottom to up. The complete pseudocode for Merge Sort is given below:

Algorithm 4: MergeSort(A , $left$, $right$)

```

1 if  $left = right$  then
2   | return
3 end
4  $mid \leftarrow \lfloor (left + right) / 2 \rfloor$ 
  // recursively divide array into two halves
5 MergeSort( $A$ ,  $left$ ,  $mid$ )
6 MergeSort( $A$ ,  $mid + 1$ ,  $right$ )
  // then merge from bottom to up
7 Merge( $A$ ,  $left$ ,  $mid$ ,  $right$ )

```

Firstly call **MergeSort**(A , 1, n) to sort array A .

As usual, we are interested in the running time of Merge Sort algorithm. Let $T(n)$ be the running time on an array of size n , it's not hard to find $T(n) \leq T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + O(n)$, when $n > 1$ and $T(1) = O(1)$.

Here we are actually able to simplify the equation. Firstly we can replace \leq with $=$, since we are interested in big- O upper bound of $T(n)$; and with the same reason, we can replace $O(n)$ with n , $O(1)$ with 1; finally, we can assume n is a power of 2 for the sake of simplicity but doesn't change the result at all, as $T(n) \leq T(n') \leq T(2n) = O(T(n))$ where n' is the smallest power of 2 such that $n' \geq n$.

Now we want to solve: $T(n) = 2T(n/2) + n$ for $n > 1$, and $T(1) = 1$.

$$\begin{aligned}
T(n) &= 2 \left(\frac{n}{2} \right) + n \\
&= 2 \left[2T \left(\frac{n}{4} \right) + \frac{n}{2} \right] + n = 2^2 \cdot T \left(\frac{n}{2^2} \right) + 2n \\
&= 2^2 \cdot \left[2T \left(\frac{n}{2^3} \right) + \frac{n}{2^2} \right] + 2n = 2^3 \cdot T \left(\frac{n}{2^3} \right) + 3n \\
&= \dots \\
&= 2^k \cdot T \left(\frac{n}{2^k} \right) + kn
\end{aligned}$$

We know the process ends with $\frac{n}{2^k} = 1$ i.e. $k = \log_2 n$, thus

$$\begin{aligned}
T(n) &= 2^{\log_2 n} T \left(\frac{n}{2^{\log_2 n}} \right) + n \cdot \log_2 n \\
&= n \log_2 n + n
\end{aligned}$$

In summary, merge sort runs in $O(n \log n)$ time. It is also worth pointing out that merge sort **always** runs in $O(n \log n)$ time, which means best case is the same as worst case, as you may think of it, the complexity of merge sort *does not depend on inputs*, it always break array down and then merge up.

4 Inversion Numbers

Given an array $A[1 \dots n]$, we say two elements $A[i]$ and $A[j]$ are **inverted** if $i < j$ but $A[i] > A[j]$, i.e., $A[i]$ appears before $A[j]$ but is larger than $A[j]$. The number of inverted pairs is called the **inversion number** of array A . Actually this is a useful measure, it provides us with an intuitive idea about how “sorted” an array is, larger inversion number implies a more unsorted array.

What may surprise you is that inversion number has a close relation to insertion sort, and more concretely, **the number of swaps used by insertion sort is equals to inversion number**. We can prove it by induction:

Proof. Assuming the array has size n . Basic case $n = 2$ obviously holds.

Inductive step: assume correct for an array of size $n - 1$, i.e., the total number of swaps performed while insertion sorting $A[1 \dots n - 1]$ is equals to the inversion number of $A[1 \dots n - 1]$.

Let $x = A[n]$. Now, the remaining work by insertion sort is that we swap x with all items $A[j]$ such that $j < n$ and $A[j] > x$, notice that the number of those items is the same of inversions in which x **participates**. Therefore, adding these new inversions gives the full inversion number of $A[1 \dots n]$. \square

Now we will consider how to compute the inversion number of a given array with size n . One possible method is we

check all (i, j) pairs of given array, this requires $\binom{n}{2} = \Theta(n^2)$ running time. Another method uses the relation we proved above, running insertion sort and count the number of swaps we perform, but this also requires $\Theta(n^2)$ time since insertion sort requires $\Theta(n^2)$. How can we improve that? Come back to topic: divide and conquer!

Similar to previous problems, we divide array into two halves, and recursively count inversions in each half, but notice: we are missing something: we still need to count inversions where a_i and a_j are in different halves! We need to return the sum of those three quantities finally.

So the main problems is that, how we count the third quantity? Consider below situation, the two halves of array are: $[1, 5, 4, 8, 10, 2]$ and $[6, 9, 12, 11, 3, 7]$, how would you do that? You may count by hand, and knowing there are $5 - 3, 4 - 3, 8 - 6, \dots$ and in total 9 inversions with one item in 1st array and another in 2nd. But, it's really time consuming and totally a mess! We have no efficient algorithm to do this but to count one by one.

Fortunately, things will become much better if those two arrays are *sorted*. For example, $A = [3, 7, 10, 14, 18, 19]$ and $B = [2, 11, 16, 17, 23, 25]$. How will we do then? We can scan progressively through both lists, and for each item in B , we only need to find the smallest A item larger than it. In the lists above, for example, $A[1] = 3$ is larger than $B[1] = 2$, so all items in A form an inversion pair with $B[1]$; then we move to $B[2] = 11$, we try to find the smallest item in A larger than 11, so we move the pointer in A , $A[2] = 7 < 11$, $A[3] = 10 < 11$, until $A[4] = 14 > 11$, so each item in $A[4] \dots A[6]$ can form an inversion pair with $B[2]$. If we continue the process, we will finally get the inversion number formed between A and B , in $O(n)$ time. (Why is $O(n)$? Since we only iterate each item once during the whole process. You may find it quite similar to Merge operation in Merge Sort)

Algorithm 5: Count(A, l, mid, r)

```

// l means left, while r means right.
// notice here,  $L[1 \dots (mid - l + 1)]$  is corresponding to  $A[l \dots mid]$ , the subscript changes, try
// not be confused later.  $R$  also changes.
1  $L \leftarrow A[l \dots mid]$ ,  $R \leftarrow A[mid + 1 \dots r]$ 
2 (here assume)  $L, R$  already sorted
3  $i \leftarrow 1, j \leftarrow 1$  // two pointers for  $L$  and  $R$ 
4  $ans \leftarrow 0$  // total inversion number
// let  $i, j$  iterator over two arrays
5 while  $i \leq mid - l + 1$  and  $j \leq r - mid$  do
    // looking for smallest  $L$  item larger than  $R$ 
6   if  $L[i] \leq R[j]$  then
7      $i \leftarrow i + 1$ 
8   else
    // Found  $L[i] > R[j]!$ 
    // then  $L[i] \dots L[mid - l + 1]$  each can form an inversion pair with  $R[j]$ , remind here  $L$ 
    // subscript is diff from  $A$ , as stated above
    // so inversion num for  $R[j]$  is  $mid - l + 1 - i + 1 = mid - l - i + 2$ 
9      $inv \leftarrow (mid - l - i + 2)$ 
10     $ans \leftarrow ans + inv$ 
11     $j \leftarrow j + 1$ 
12  end
13 end
14 return  $ans$ 

```

And, the whole algorithm for counting the inversion number will be:

Algorithm 6: Count-Inversion(A, l, r)

```
1 if  $l = r$  then
2   |   return 0
3 end
4  $mid \leftarrow \lfloor (l + r)/2 \rfloor$ 
5  $c_1 \leftarrow \text{Count-Inversion}(A, l, mid)$ 
6  $c_2 \leftarrow \text{Count-Inversion}(A, mid + 1, r)$ 
7 MergeSort( $A, l, mid$ )
8 MergeSort( $A, mid + 1, r$ )
9  $c_3 \leftarrow \text{Count}(A, l, mid, r)$ 
10 return  $c_1 + c_2 + c_3$ 
```

First call: Count-Inversion($A, 1, n$).

So far, you may think this is an excellent algorithm since we only use $O(n)$ in each recursion step. However, it isn't! Remember, we have assumed each half is already sorted, but in fact they are random. If we firstly run some sort algorithm, say, Merge Sort, and then do the counting above, the whole running time will be:

$$T(n) = 2T(n/2) + \Theta(n \log n + n) = 2T(n/2) + \Theta(n \log n)$$

One can show $T(n) = \Theta(n \log^2 n)$.

This is, to a certain degree, acceptable, compared to previous $\Theta(n^2)$, but we still want to improve that. We can easily notice the main problem lies in sorting, which uses $\Theta(n \log n)$ in each recursion step. How can we reduce, or even avoid this process?

This is indeed hard to think about, but we can combine the sorting process (more concretely, Merge sort) with the process which we count inversion pairs that form between the two halves. In other words, previously we only do counting between two halves, now we also perform Merge at the same time. What will this lead to? Consider from recursion bottom(1 item), to top, each time we Merge the two halves, as what we did in Merge Sort, and at the same time, count inversion pairs that cross the two halves. And since we Merge from bottom to top, the two halves will always be sorted.(this is exactly the same Merge in Merge Sort)

The paragraph above is still so abstract, at least for myself, perhaps it's better to look at how the algorithm implements.

Algorithm 7: Merge-and-Count(A, l, mid, r)

// same as previous algorithm, subscripts for L, R and A are different, remember this

```
1  $L \leftarrow A[l \dots mid], R \leftarrow A[mid + 1 \dots r]$ 
2 append  $\infty$  at the end of  $L$  and  $R$ 
3  $i \leftarrow 1, j \leftarrow 1$  // two iteration pointers for  $L$  and  $R$ 
4  $count \leftarrow 0$  // counter for inversion number
5 for  $k \leftarrow l$  to  $r$  do
6   if  $L[i] \leq R[j]$  then
7      $A[k] \leftarrow L[i]$ 
8      $i \leftarrow i + 1$ 
9   else
10     $A[k] \leftarrow R[j]$ 
11     $j \leftarrow j + 1$ 
12     $count \leftarrow count + (mid - l - i + 2)$ 
13   end
14 end
15 return  $count$ 
```

As you can find out above, apart from count inversion pairs between L and R , we merge them into a new array A , this is exactly what we did in merge sort, which maintain the “sorted” invariant. With the function above, the complete algorithm for finding inversion number for an array is displayed below.

Algorithm 8: Sort-and-Count(A, l, r)

```
1 if  $l = r$  then
2    $\text{return } 0$ 
3 end
4  $mid \leftarrow \lfloor (l + r) / 2 \rfloor$ 
5  $c_1 \leftarrow \text{Sort-and-Count}(A, l, mid)$ 
6  $c_2 \leftarrow \text{Sort-and-Count}(A, mid + 1, r)$ 
7  $c_3 \leftarrow \text{Merge-and-Count}(A, l, mid, r)$ 
8 return  $c_1 + c_2 + c_3$ 
```

First call: Sort-and-Count($A, 1, n$)

5 The Maximum Subarray Problem