MATH 2023 Fall 2021

Multivariable Calculus

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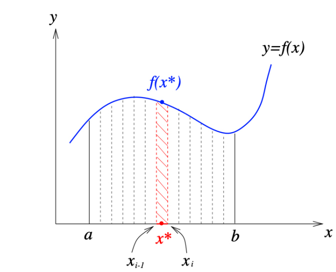
Chapter 14 **Multiple Integrations**

1 Double Integrals Over Rectangles

Recall that in single variable calculus, we divided a region into thin rectangles and use the **Riemann Sum** as integral.

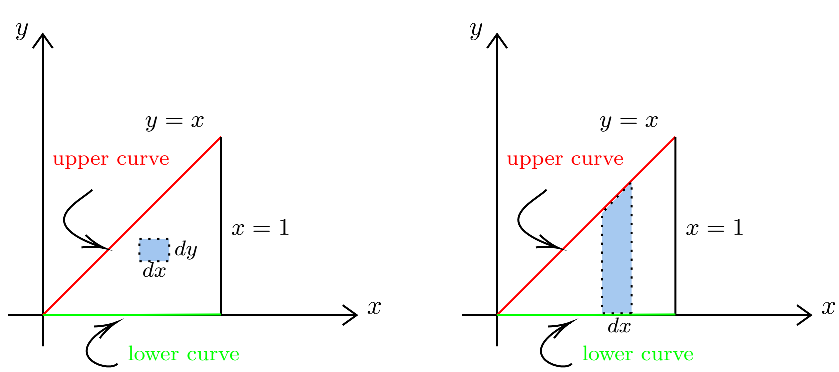


$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^\star)\delta x_i$$

Actually, instead of thin rectangles, we can use *small rectangles* to cover the area.

[Example.] Find the area bounded by $y = x$, $x = 1$ and $y = 0$.

[Solution.] (1) View the area as bounded by *upper curve* and *lower curve*.

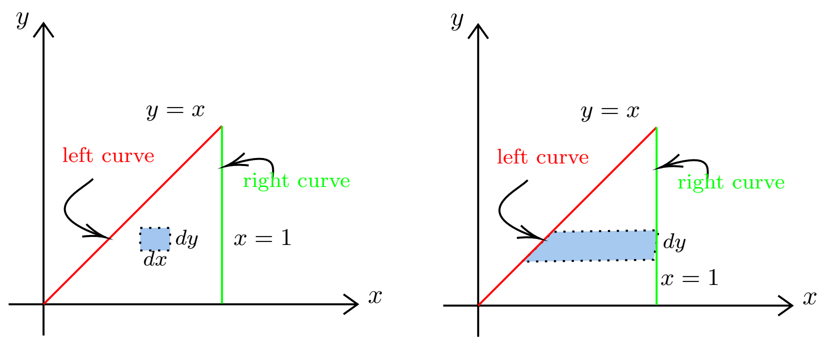


For the *small rectangles* with $dA = dxdy = dydx$, we first move it vertically, the lower bound is $y = 0$, and the upper bound is $y = x$, hence $\int_0^x dy$ is the shaded area in right image above.

Then we move the shaded area horizontally, the left bound is point $x = 0$, while the right bound is point $x = 1$, thus the total area is:

$$A = \int_0^1 \int_0^x dydx = \int_0^1 y\Big|_0^x dx = \int_0^1 xdx = \frac{1}{2}$$

(2) Alternatively, we can first move the rectangle horizontally, hitting *left curve* $x = y$ and *right curve* $x = 1$, hence the shaded area is $\int_y^1 dx$. note here we integral dx first, so when hitting the boundaries, we need to check x equals to what, i.e. $x = f(y)$. For example, here the two bounds are $x = y$ and $x = 1$.



Then we move the shaded area vertically, hitting lower bound $y = 0$(a point) and upper bound $y = 1$(a point), hence the total area is:

$$A = \int_0^1 \int_y^1 dxdy = \int_0^1 x\Big|_y^1 dy = \int_0^1 (1 - y)dy = \frac{1}{2}$$

2 Double Integrals Over General Regions

3 Double Integrals in Polar Coordinates

4 Change of Variables in Integrals

Recall that in single variable calculus, we often use a *substitution* to simplify an integral.

$$\int_a^b f(x)dx = \int_c^d f(g(u)) \cdot g'(u)\ du$$

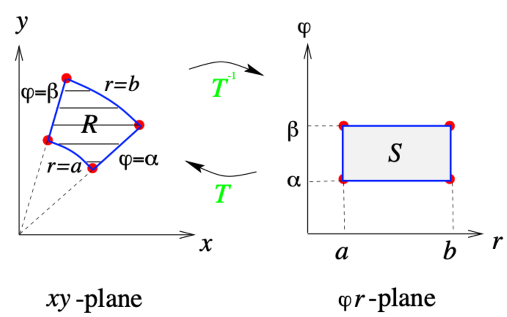
where $a = g(c)$ and $b = g(d)$. Notice that we can *view substitution as a kind of mapping*, and the change-of-variable process introduces *an additional factor* $g'(u)$ into the integrand.

This method can also be useful in multiple integrals. We have already seen one example: integration in *polar coordinate.*

$$\iint_R f(x,y)dA_{xy} = \iint_S f(r\cos\theta, r\sin\theta)rdrd\theta = \iint_S f(r\cos\theta, r\sin\theta)rdA_{r\theta}$$

In this example, *the additional factor* is r.

The *mapping* T is shown as below: we transform the region R into S, where S is an rectangle in θr-plane, which is easy to integrate.

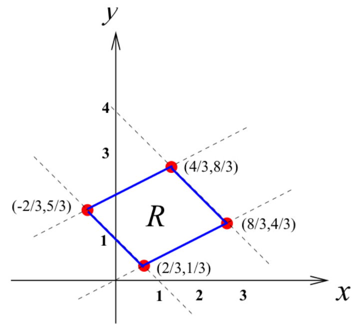


$$T(r,\theta) = (x,y) = (r\cos\theta, r\sin\theta)$$

[Example.] Find a change of variable.

.**1** Let R be the region bounded by the lines

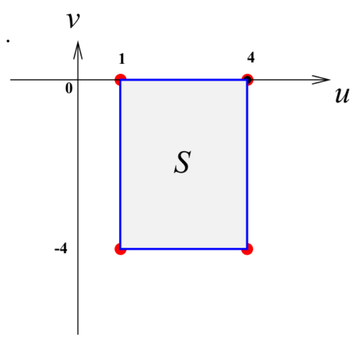
$$x - 2y = 0$$
$$x - 2y = -4$$
$$x + y = 4$$
$$x + y = 1$$



as shown. Find a transformation T from a region S to R such that S is a rectangular region (with sides parallel to the u- and v-axis).

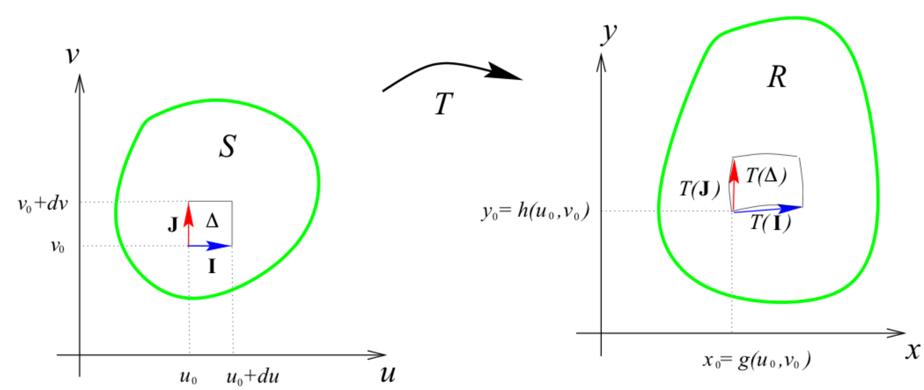
Let $u = x + y$, $v = x - 2y$, then $T(u, v) = (x, y) = \left(\frac{1}{3}(2u + v),\ \frac{1}{3}(u - v)\right)$.

$$v = 0$$
$$v = -4$$
$$u = 4$$
$$u = 1.$$



Note that the transformation T maps the vertices of the region S onto the vertices of the region R.

Now, to find $\iint_R f(x, y)dxdy$, if we make the change of variables $x = g(u, v), y = h(u, v)$, then *we are mapping things in uv-plane onto things in xy-plane*. For mapping function $T(u, v) = (g(u, v), h(u, v)) = (x, y)$, and assume the area S in uv-plane corresponds to region R in xy-plane, as shown below.



We still use the method that integrate all "small rectangles", Δ, as shown in uv-plane. Assume Δ locates at (u_0, v_0) and has area $dA = dudv$. Let

I be the vector from (u_0, v_0) to $(u_0 + du, v_0)$ and
J be the vector from (u_0, v_0) to $(u_0, v_0 + dv)$.

Then mapping T "takes" $\mathbf{I}$ to the vector $T(\mathbf{I})$ from $(g(u_0, v_0), h(u_0, v_0))$ to $(g(u_0 + du, v_0), h(u_0 + du, v_0))$. Notice the vector $T(\mathbf{I})$ is *not necessary a straight vector.* Now

$$\begin{aligned} T(\mathbf{I}) &= (g(u_0 + du, v_0) - g(u_0, v_0), h(u_0 + du, v_0) - h(u_0, v_0)) \\ &= \left(\frac{g(u_0 + du, v_0) - g(u_0, v_0)}{du}, \frac{h(u_0 + du, v_0) - h(u_0, v_0)}{du} \right) du \\ &= \left(\frac{\partial g}{\partial u}, \frac{\partial h}{\partial u} \right) du \quad \text{(for } du \to 0) \end{aligned}$$

Similarly, $T(\mathbf{J}) = \left(\frac{\partial g}{\partial v}, \frac{\partial h}{\partial v} \right) dv$.

Then the area of $T(\Delta)$ is

$$dxdy = \|T(\mathrm{I}) \cdot T(\mathrm{J})\| = \left| \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} dudv \right| = \left| \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u} \right| dudv$$

which means, $dA_{xy} = |J| \, dA_{uv}$, where $|J|$ is the "additional factor" caused by this substitution, and it is called the **Jacobian** of mapping T, given by:

$$\boxed{T = \frac{\partial(x, y)}{\partial(u, v)} = \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}}$$

Therefore, the formula of *change of variable for two variables* is:

$$\boxed{\iint_{R=T(S)} f(x, y) dxdy = \iint_S f(T(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| dudv}$$

while for *three variables*,

$$\boxed{\iiint_{R=T(S)} f(x, y, z) dxdydz = \iiint_S f(T(u, v, w)) \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| dudvdw}$$

This example shows substitution can be easy for some integrations.

[Example.] Find the area of $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

[Solution.] **Method 1:** directly integrate

$$\begin{aligned}\frac{1}{4}A &= \int_0^a \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} dydx \\ &= \int_0^a b\sqrt{1-\frac{x^2}{a^2}}\ dx\end{aligned}$$

We are familiar with substitution in single variable integration, let $x = a\sin\theta$, when $x = 0$, $\theta = 0$, and when $x = a$, $\theta = \dfrac{\pi}{2}$. Then,

$$\begin{aligned}\frac{1}{4}A &= \int_0^{\frac{\pi}{2}} b(1-\sin^2\theta)^{\frac{1}{2}} a\cos\theta\ d\theta \\ &= ab\int_0^{\frac{\pi}{2}} \cos^2\theta\ d\theta \\ &= ab\int_0^{\frac{\pi}{2}} \frac{1}{2}(1-\cos 2\theta)\ d\theta \\ &= \frac{1}{4}ab\pi\end{aligned}$$

Method 2: Mapping the ellipse to a disk.

Let $x = au,\ y = bv$, then $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ becomes $u^2 + v^2 = 1$.

The Jacobian of this mapping

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} a & 0 \\ 0 & b \end{vmatrix} = ab$$

Therefore, the area of ellipse

$$\iint_R dA_{xy} = \iint_S J \cdot dA_{xy} = \iint_S ab\ dA_{uv} = ab\pi$$

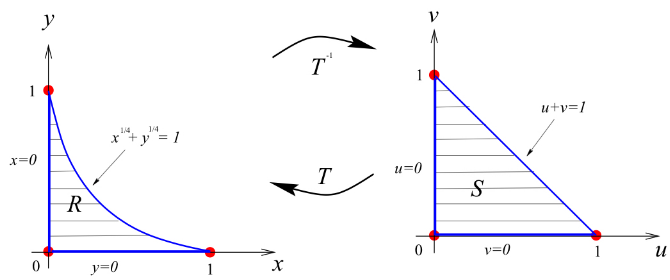
We observe that method 2 is much easier than method 1.

[**Example.**] Find the area bounded by $\sqrt[4]{x} + \sqrt[4]{y} = 1$ and the x and y axes.

[**Solution.**]

This integral would be tedious to evaluate directly because the region R is not 'simple'. So instead we find a suitable transformation of variables. Let

Let $u = \sqrt[4]{x}$, $v = \sqrt[4]{y}$, then $x = u^4$, $y = v^4$ and $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 4u^3 & 0 \\ 0 & 4v^3 \end{vmatrix} = 16u^3v^3$



$$\text{Area} = \iint_R dxdy = \iint_S \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv = \int_0^1 \int_0^{1-v} 16u^3v^3\, dudv = \frac{1}{70}$$

Sometimes, it's not easy to calculate $\dfrac{\partial(x,y)}{\partial(u,v)}$, since usually we substitute u and v as functions of x and y, so we always need to find the inverse function in order to calculate $\dfrac{\partial(x,y)}{\partial(u,v)}$. So we consider the relationship between $\dfrac{\partial(x,y)}{\partial(u,v)}$ and $\dfrac{\partial(u,v)}{\partial(x,y)}$.

Because of $\det A \det B = \det(AB)$,

$$\begin{aligned}
\frac{\partial(x,y)}{\partial(u,v)} \cdot \frac{\partial(u,v)}{\partial(x,y)} &= \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \\
&= \begin{vmatrix} \frac{\partial x}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial x}{\partial v}\frac{\partial v}{\partial x} & \frac{\partial x}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial x}{\partial v}\frac{\partial v}{\partial y} \\ \frac{\partial y}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial y}{\partial v}\frac{\partial v}{\partial x} & \frac{\partial y}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial y}{\partial v}\frac{\partial v}{\partial y} \end{vmatrix} \\
&= \begin{vmatrix} \frac{\partial x}{\partial x} & \frac{\partial x}{\partial y} \\ \frac{\partial y}{\partial x} & \frac{\partial y}{\partial y} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1
\end{aligned}$$

Therefore, if $x(u,v)$ and $y(u,v)$ have continuous first partial derivatives, and that

$$\frac{\partial(x,y)}{\partial(u,v)} \neq 0 \quad \text{at} \quad (u,v) \quad \text{(one-to-one map)}.$$

Then

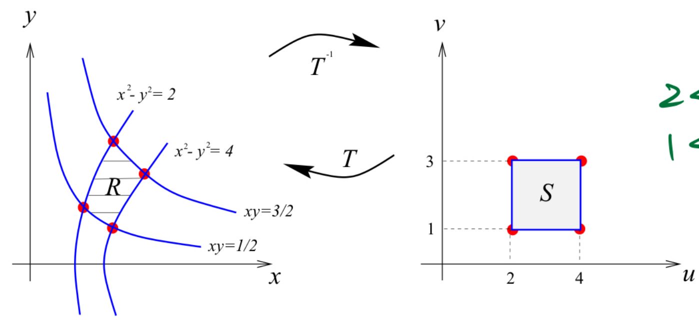
$$\boxed{\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{\dfrac{\partial(u,v)}{\partial(x,y)}}}$$

This example shows the situation when $\dfrac{\partial(x,y)}{\partial(u,v)}$ is difficult to compute.

[Example.] Find $\displaystyle\iint_A (x^2+y^2)\,dxdy$, where $A = \left[(x,y) \mid x, y > 0, \quad 2 \leqslant x^2 - y^2 \leqslant 4, \quad \tfrac{1}{2} \leqslant xy \leqslant \tfrac{3}{2}\right]$

[Solution.]

The change of the variables is motivated by the occurrence of the expressions $x^2 - y^2$ and xy in the equations of the boundary.



$2<u<4$
$1<v<3$

Let $u = x^2 - y^2$, $v = 2xy$, then $(x^2+y^2)^2 = (x^2-y^2)^2 + 4x^2y^2 = u^2 + v^2$ and

difficult to calculate

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{\dfrac{\partial(u,v)}{\partial(x,y)}} = \frac{1}{\begin{vmatrix} 2x & -2y \\ 2y & 2x \end{vmatrix}} = \frac{1}{4(x^2+y^2)} = \frac{1}{4\sqrt{u^2+v^2}}.$$

So $\displaystyle\iint_R (x^2+y^2)\,dxdy = \int_{v=1}^{3}\int_{u=2}^{4} \sqrt{u^2+v^2} \cdot \frac{1}{4\sqrt{u^2+v^2}}\,du\,dv$

Sometimes, though the given region is a relatively good one, but it's still difficult to directly integrate, maybe because the integrand is too complicated. See the below example:

[Example.]

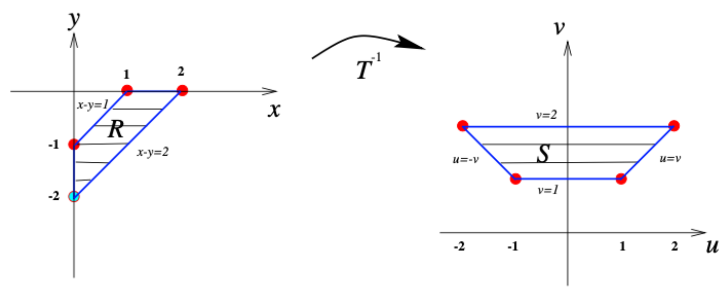
Evaluate the integral $\displaystyle\iint_R e^{(x+y)/(x-y)}\,dA$, where R is the trapezoidal region with vertices $(1,0)$, $(2,0)$, $(0,-2)$, and $(0,-1)$.

Since it is not easy to integrate $e^{(x+y)/(x-y)}$, we make a change of variables suggested by a form of the integrand. In particular, let

$$u = x + y, \qquad v = x - y.$$

These equations define a transformation T^{-1} from the xy-plane to the uv-plane.

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{\dfrac{\partial(u,v)}{\partial(x,y)}} = \frac{1}{\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}} = -\frac{1}{2}.$$



The sides of R lie on the lines

$$y = 0, \qquad x - y = 2, \qquad x = 0, \qquad x - y = 1$$

and the image lines in the uv-plane are

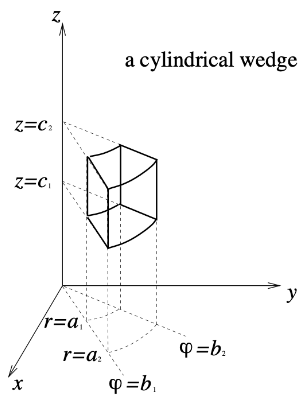
$$u = v, \qquad v = 2, \qquad u = -v, \qquad v = 1.$$

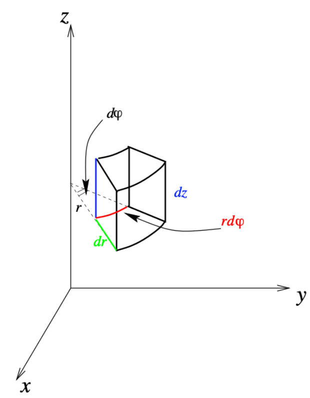
$$\begin{aligned} \therefore \quad \iint_R e^{(x+y)/(x-y)}\,dA &= \iint_S e^{u/v} \left| \frac{\partial(x,y)}{\partial(u,v)} \right| dudv \\ &= \int_1^2 \int_{-v}^{v} e^{u/v} \left(\frac{1}{2} \right) dudv \\ &= \end{aligned}$$

5 Triple Integrals in Cylindrical & Spherical Coordinates

Cylindrical coordinates are suited to problems *with axial symmetry*(the shape is around the z-axis)

The basic unit of cylindrical coordinate is shown below, and it's volume is $r\ drd\theta dz$





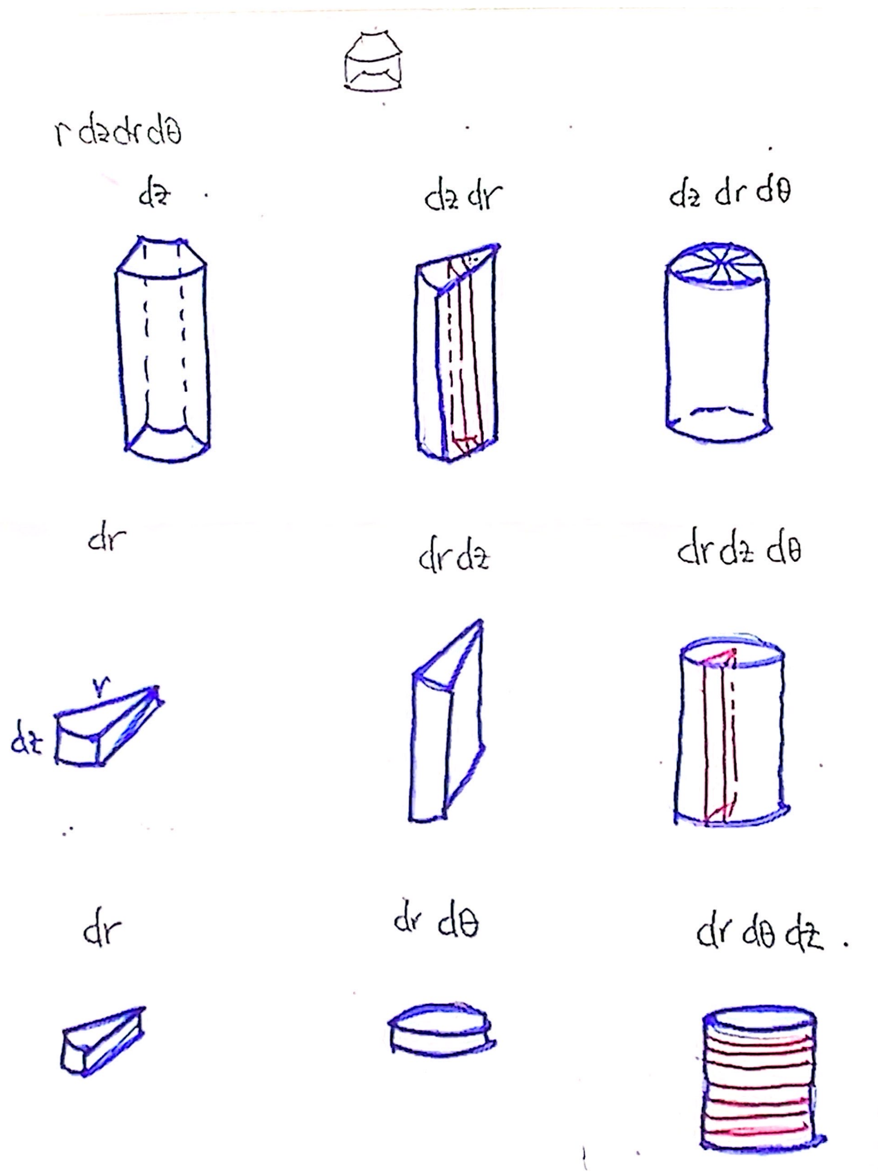
We can also find Jacobian in order to find the relation. For $x = r\cos\theta, y = r\sin\theta, z = z$, we have

$$\frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r$$

that is, $dxdydz \to r\ drd\theta dz$

$$\boxed{V_c = \iiint_V f dV = \iiint_{V(x,y,z)} f(x,y,z)dxdydz = \iiint_{V(r,\theta,z)} f(r,\theta,z) r dr d\theta dz}$$

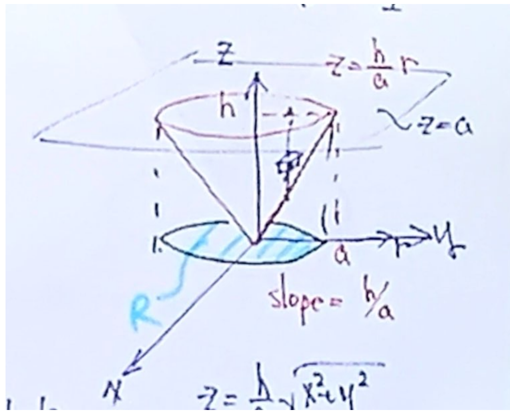
Moreover, same as before, we can freely change the *order of integration*. Below show three examples.



This example shows integration in cylindrical coordinates brings convenience sometimes.

[Example.] Find the volume of a circular cone with altitude h and a base of radius a.

[Solution.] To find the volume of the cone, we need to know the equation of its surface.



In (x, y, z), $z = \dfrac{h}{a}(x^2 + y^2)^{1/2}$. In (r, θ, z), $z = \dfrac{h}{a}r$.

Method 1: Do the question by (x, y, z), and $dV = dzdydx$

$$V = 4\int_0^a \int_0^{\sqrt{a^2-x^2}} \int_{\frac{h}{a}(x^2+y^2)^{1/2}}^{h} dz\ dy\ dx$$

Method 2: Do the question by (r, θ, z), and $dV = r\ dzdrd\theta$

$$V = \int_0^{2\pi} \int_0^a \int_{\frac{h}{a}r}^{h} r\ dz\ dr\ d\theta = \frac{1}{3}\pi a^2 h$$

This example provides interpretation for triple integrals.

[Example.] Sketch the solid whose volume is given by the integral.

(a) $\displaystyle\int_0^{2\pi}\int_0^{2}\int_0^{4-r^2} r\ dzdrd\theta.$

(b) $\displaystyle\int_1^{3}\int_0^{\frac{\pi}{2}}\int_r^{3} r\ dzd\theta dr.$

[Solution.] Recall that for $\displaystyle\int_a^b\int_{g_1(x)}^{g_2(x)}\int_{\phi_1(x)}^{\phi_2(x)} dzdydx$, we can infer the volume of integration based on these six limits:

$z=\phi_1$ to $z=\phi_2$, (surfaces)
$y=g_1$ to $y=g_2$, (curves)
$x=a$ to $x=b$, (points)

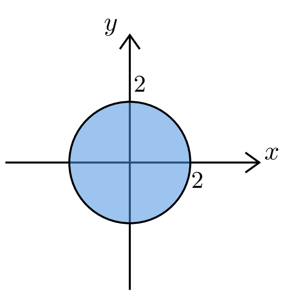
We can interpret the last two as "*area on the xy-plane*".

(a)

$z: z=0$ to $z=4-r^2$, (surfaces)

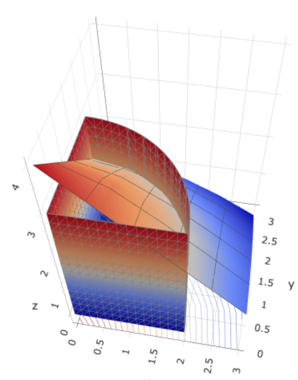
$$\left.\begin{array}{lll} r: r=0 & \text{to} & r=2 \\ \theta: \theta=0 & \text{to} & \theta=2\pi \end{array}\right\}\ \text{(area on } xy\text{-plane)}$$

The area on xy-plane is shown below left. We move the disk upward, hitting $z=4-r^2$, and the volume bounded is what we are looking for, as shown below right.



$r=0$ *to* $r=2$

$\theta=0$ *to* $\theta=2\pi$

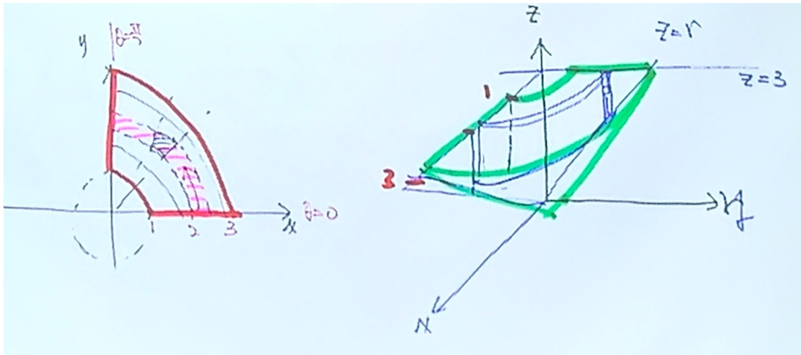


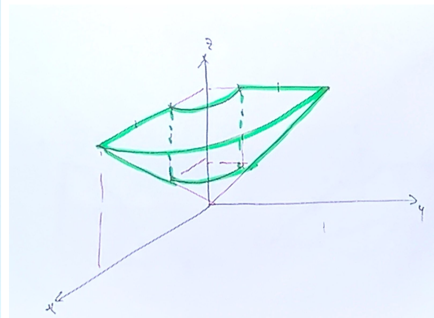
(b)

$z: z=r$ to $z=3$, (surfaces)

$$\left.\begin{array}{lll} \theta: \theta=0 & \text{to} & \theta=\frac{\pi}{2} \\ r: r=1 & \text{to} & r=3 \end{array}\right\}\ \text{(area on } xy\text{-plane)}$$

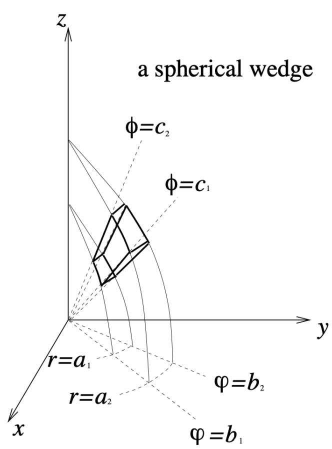
The procedure is similar to (a), we first draw the region on xy-plane, and move it from surface $z = r$ upwards to $z = 3$. The bounded region is what we're looking for.

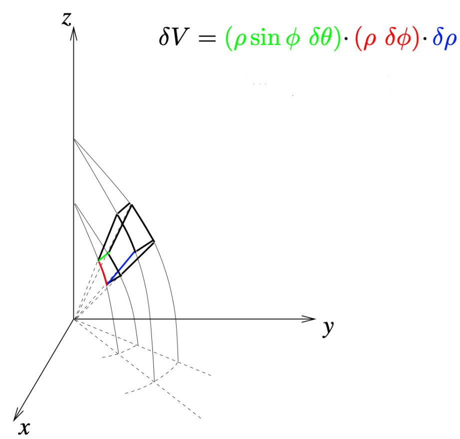


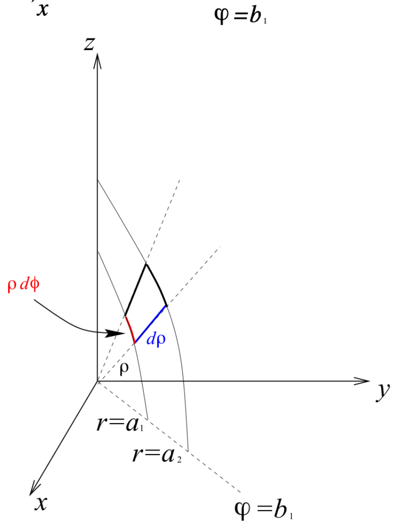


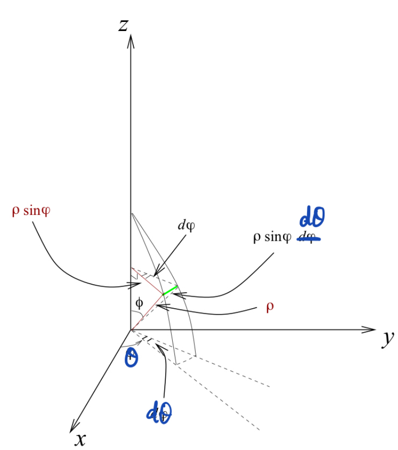
Spherical coordinates are suited to problems *involving spherical symmetry*, and in particular, to regions bounded by *spheres* centered at the origin, *circular cones* with axes along the $z-$axis, *vertical planes* containing $z-$axis.

The basic unit of spherical coordinate is shown below, and it's volume is $dV = \rho^2 \sin\phi \; d\theta d\rho d\phi$









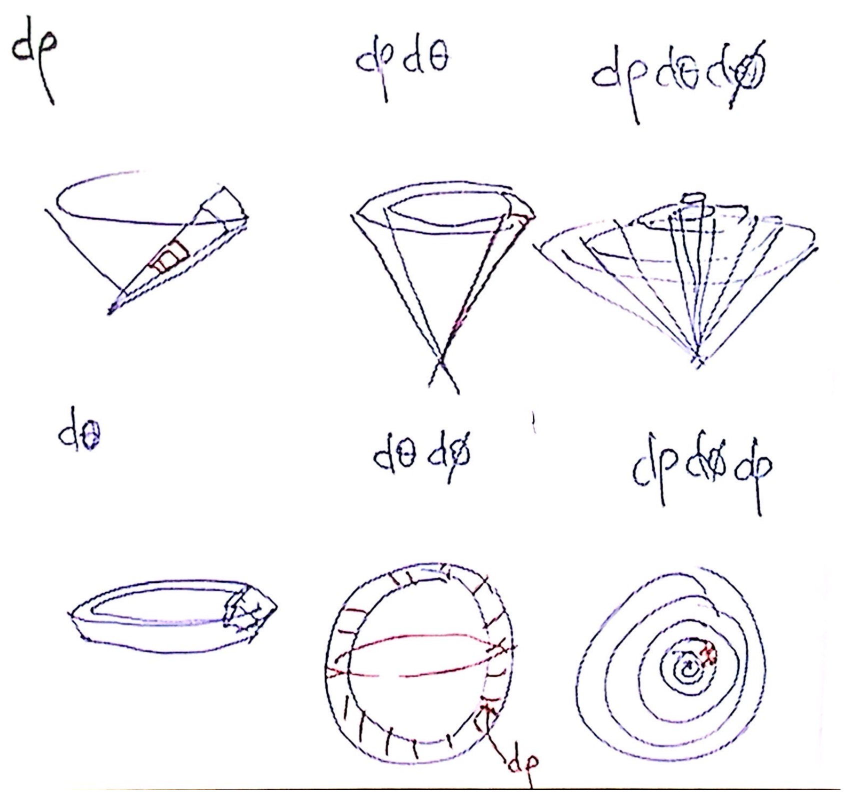
We can also find Jacobian in order to find the relation.

For spherical coordinates $x = \rho \sin\phi \cos\theta, y = \rho \sin\phi \sin\theta$ and $z = \rho \cos\phi$, then

$$\frac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)} = \begin{vmatrix} \sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \\ \cos\phi & -\rho\sin\phi & 0 \end{vmatrix} = \rho^2 \sin\phi$$

i.e. $dxdydz = \rho^2 \sin\phi d\rho d\theta d\phi$.

Again, different integration orders have different interpretations.



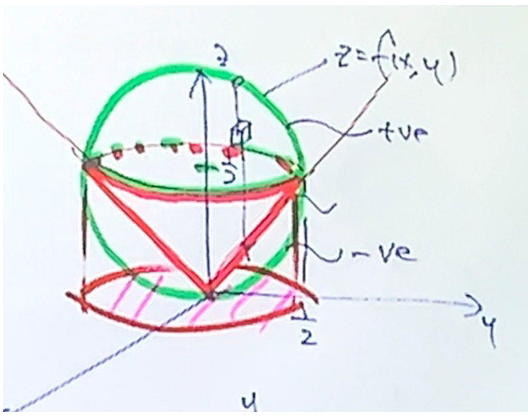
This example shows the choice of coordinate system can greatly affect the difficulty of computation of a multiple integral.

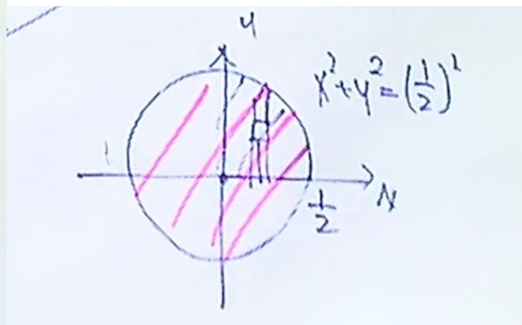
[Example.] Find the volume of the solid that lies above the cone $z = \sqrt{x^2+y^2}$ and between the sphere $x^2+y^2+z^2 = z$.

[Solution.] The equation of sphere can be written as

$$x^2 + y^2 + \left(z - \frac{1}{2}\right)^2 = \left(\frac{1}{2}\right)^2 \Rightarrow \text{centre} \left(0, 0, \frac{1}{2}\right), \text{ radius} = \frac{1}{2}$$

Method 1: Use Cartesian Coordinate:

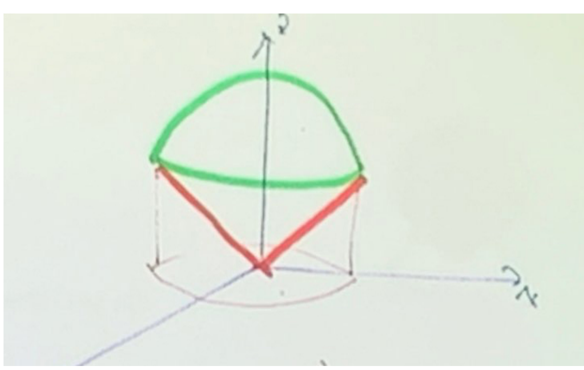


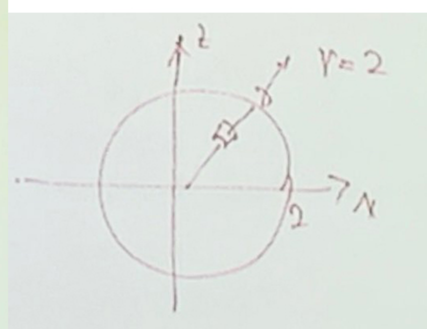


Cone: $z = \sqrt{x^2+y^2}$
Sphere: $z = \frac{1}{2} + \sqrt{\frac{1}{4} - x^2 - y^2}$

$$V = 4 \int_0^{\frac{1}{2}} \int_0^{\sqrt{\frac{1}{4}-x^2}} \int_{\sqrt{x^2+y^2}}^{\frac{1}{2}+\sqrt{\frac{1}{4}-x^2-y^2}} dzdydx$$

Method 2: Use Cylindrical Coordinate:

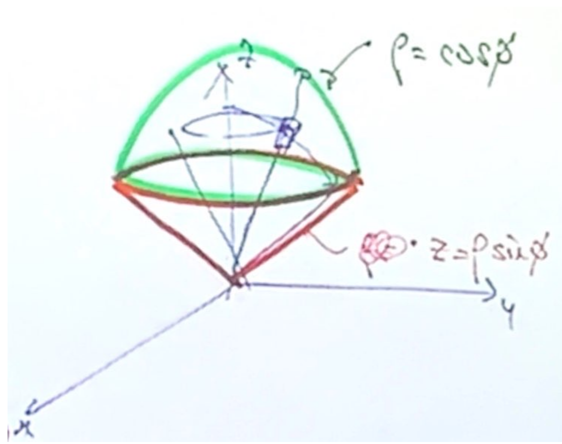




Cone: $z = \sqrt{x^2+y^2} = r$
Sphere: $z = \frac{1}{2} + \sqrt{\frac{1}{4} - x^2 - y^2} = \frac{1}{2} + \sqrt{\frac{1}{4} - r^2}$

$$V = \int_0^{2\pi} \int_0^2 \int_r^{\frac{1}{2}+\sqrt{\frac{1}{4}-r^2}} r \; dzdrd\theta$$

Method 3: Use Spherical Coordinate:



Cone: $z = \sqrt{x^2+y^2} = \rho \sin\phi$
Sphere: $x^2+y^2+z^2 = z \Rightarrow \rho^2 = \rho\cos\phi \Rightarrow \rho = \cos\phi$, (since $\rho \neq 0$)

$$dV = \int_0^{\frac{\pi}{4}} \int_0^{2\pi} \int_0^{\cos\phi} \rho^2 \sin\phi \; d\rho d\theta d\phi$$

This example shows the situation when we need to integrate two parts.

[Example.] Find the volume of a cylinder with radius a and height h.

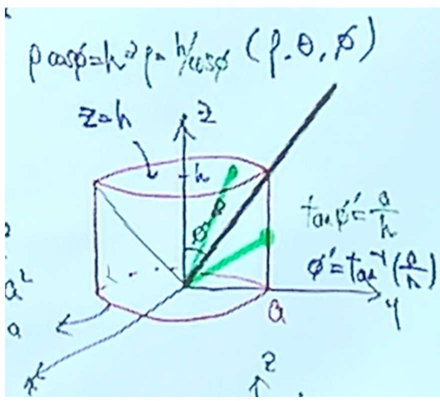
[Solution.] **Method 1:** Use Cylindrical Coordinate:

$$V = \int_0^{2\pi} \int_0^a \int_0^h r \; dzdrd\theta$$

This is really simple by cylindrical coordinate, but things become different when using spherical.

Method 2: Use Spherical Coordinate:

Notice that for different ϕ, the ρ is bounded by two different equations. So we cannot do it in one integration, instead, we need to separate these two situations.



For the above cone, the upper red surface is $z = h \Rightarrow \rho\cos\phi = h \Rightarrow \rho = h/\cos\phi$,

$$V_1 = \int_0^{\phi'} \int_0^{2\pi} \int_0^{\frac{h}{\cos\phi}} \rho^2 \sin\phi \; d\rho d\theta d\phi$$

For the bottom part, ρ moves along the green line, from 0 to $x^2 + y^2 = a^2 \Rightarrow \rho\sin\phi = a$,

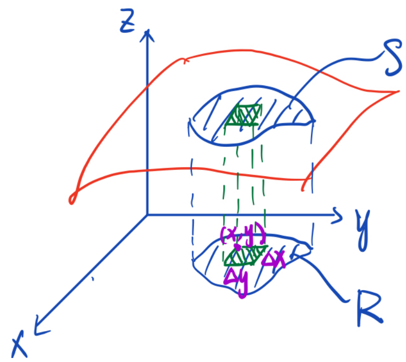
$$V_2 = \int_{\phi'}^{\frac{\pi}{2}} \int_0^{2\pi} \int_0^{\frac{a}{\sin\phi}} \rho^2 \sin\phi \; d\rho d\theta d\phi$$

For the bolded black line(the threshold), $\tan\phi' = \dfrac{a}{h} \Rightarrow \phi = \tan^{-1}\dfrac{a}{h}$.

$$V = V_1 + V_2$$

6 Surface Area

We now want to find the area of a surface. Finding an area on xy-plane is relatively easy, as we have discussed early this chapter, but things become much more complicated when we are focusing on an arbitrary surface. So, we think about *projecting the area onto xy-plane.*

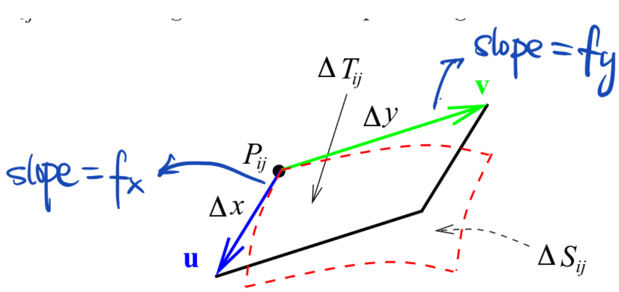


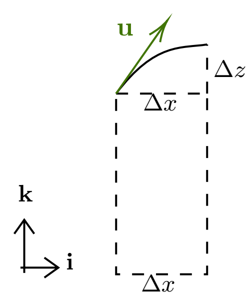
As shown above, we want to find the area of region S in a surface. The first thing to do is to project it onto xy-plane, resulting in a region R.

As usual, we use small rectangles to cover region R, as the green rectangle shown above, assume two sides are Δx and Δy, so the area of green rectangle is $\Delta A = \Delta x \Delta y$.

Then we project the rectangle up to the surface S, resulting in a "curved-parallelogram" surface, shown as red area in left-below image. To find this area, we know as long as Δx and Δy are small enough, the black parallelogram formed by Δx and Δy is a good approximation for that area. By the way, the area of parallelogram is $\mathbf{u} \times \mathbf{v}$.

How to represent $\mathbf{u}$ and $\mathbf{v}$? See the right-below image, the slope of vector $\mathbf{u}$ is $f_x = \dfrac{\Delta z}{\Delta x}$, so the width of $\mathbf{u}$ is Δx and the height of $\mathbf{u}$ is $\Delta z = \Delta x \cdot f_x$.(Notice this image is graphed vertically, i.e., in xz-plane)





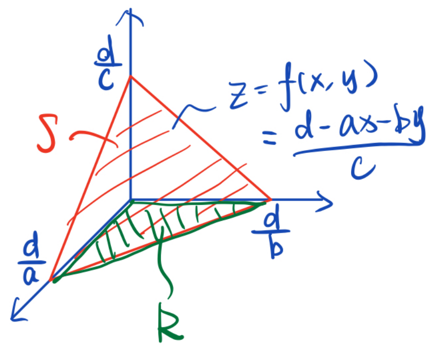
Therefore, $\mathbf{u} = \Delta x\mathbf{i} + 0\mathbf{j} + \Delta x \cdot f_x\mathbf{k}$, similarly, $\mathbf{v} = 0\mathbf{i} + \Delta y\mathbf{j} + \Delta y \cdot f_y\mathbf{k}$. Then

$$||\mathbf{u} \times \mathbf{v}|| = || - \Delta x \Delta y f_x \mathbf{i} - \Delta x \Delta y f_y \mathbf{j} + \Delta x \Delta y \mathbf{k}|| = \sqrt{(f_x)^2 + (f_y)^2 + 1} \cdot \Delta x \Delta y$$

When $\Delta x, \Delta y \to 0$, the area of S is given by:

$$\boxed{A = \iint_R \sqrt{(f_x)^2 + (f_y)^2 + 1}\; dA}$$

[Example.] Given a plane $ax + by + cz = d$, where $a, b, c, d > 0$. Find the area of the triangle bounded by the intersections of the plane and axes.(As the red shaded area shown)



[Solution.] The equation of surface $z = f(x, y)$ is given by $z = \dfrac{d - ax - by}{c}$.

To find the red area, we first *project it onto xy-plane*, resulting in green area R.

Thus the red area

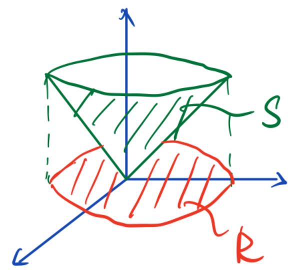
$$S = \iint_R \sqrt{1 + f_x^2 + f_y^2}\, dA_{xy}$$

We know that $f_x = -\dfrac{a}{c}$, $f_y = -\dfrac{b}{c}$, so $\sqrt{1 + f_x^2 + f_y^2} = \dfrac{\sqrt{a^2 + b^2 + c^2}}{c}$, then

$$\begin{aligned} S &= \iint_R \frac{\sqrt{a^2 + b^2 + c^2}}{c}\, dA_{xy} \\ &= \frac{\sqrt{a^2 + b^2 + c^2}}{c} \cdot (\text{area of R}) \\ &= \frac{\sqrt{a^2 + b^2 + c^2}}{c} \cdot \frac{1}{2} \cdot \frac{d}{a} \cdot \frac{d}{b} \\ &= \frac{d^2\sqrt{a^2 + b^2 + c^2}}{2abc} \end{aligned}$$

Notice the blue part is a constant.

[Example.] Find the surface area of the cone $z = \frac{h}{a} r$ (in cylindrical coordinate).



[Solution.] Project the cone onto xy-plane, resulting in red area R.

The surface is given by $z = f(x, y) = \frac{h}{a}\sqrt{x^2 + y^2}$

Thus the green area

$$S = \iint_R \sqrt{1 + f_x^2 + f_y^2}\, dA_{xy}$$

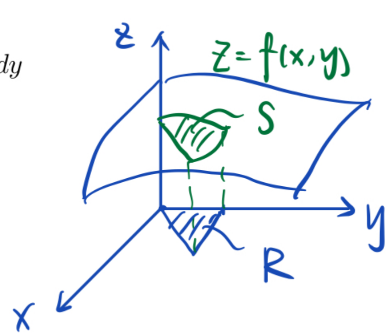
We know that $f_x = \frac{h}{a} \cdot \frac{x}{\sqrt{x^2+y^2}}$, $f_y = \frac{h}{a} \cdot \frac{y}{\sqrt{x^2+y^2}}$, so $\sqrt{1 + f_x^2 + f_y^2} = 1 + \frac{h^2}{a^2}$, then

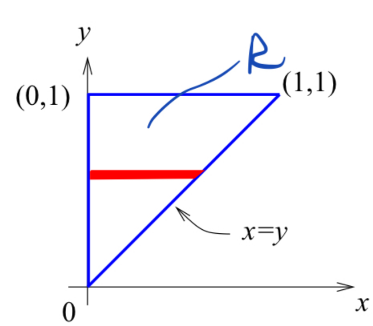
$$\begin{aligned} S &= \iint_R \sqrt{a + \frac{h^2}{a^2}}\, dA_{xy} \\ &= \sqrt{a + \frac{h^2}{a^2}} \cdot \text{(Area of circle with radius a)} \\ &= \pi a \cdot \sqrt{a^2 + h^2} \end{aligned}$$

[Example.] Find the area of the surface $z = x + y^2$ that lies above the triangle with vertices (0,0), (1,1) and (0,1).

[Solution.]

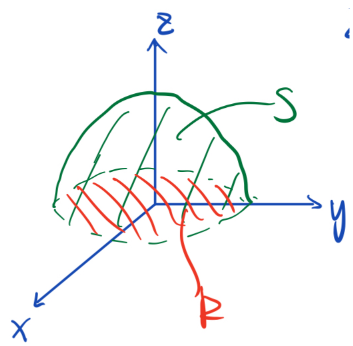
$$\begin{aligned} S &= \iint\limits_R \sqrt{z_x^2 + z_y^2 + 1}\, dA = \int_0^1 \int_0^y \sqrt{1 + 4y^2 + 1}\, dx\, dy \\ &= \int_0^1 y\sqrt{2 + 4y^2}\, dy \\ &= \frac{2}{24}(2 + 4y^2)^{\frac{3}{2}}\Big|_0^1 \\ &= \frac{1}{6}(3\sqrt{6} - \sqrt{2}). \end{aligned}$$





This example uses substitution while evaluating integral.

[Example.] Find the surface of a sphere with radius a.



[Solution.] Again, project S onto xy-plane to get region R.

The equation of surface is given by $z = f(x, y) = \sqrt{a^2 - x^2 - y^2}$,

The green area:

$$S = \iint_R \sqrt{1 + f_x^2 + f_y^2}\, dA_{xy}$$

We know that $f_x = \dfrac{-x}{\sqrt{a^2 - x^2 - y^2}}$, $f_y = \dfrac{-y}{\sqrt{a^2 - x^2 - y^2}}$, so $\sqrt{1 + f_x^2 + f_y^2} = 1 + \dfrac{x^2 + y^2}{a^2 - x^2 - y^2}$, then

$$S = \iint_R \sqrt{1 + \frac{x^2 + y^2}{a^2 - x^2 - y^2}}\, dA_{xy}$$

Notice this integration is too difficult to calculate, so we consider using *polar coordinate* to substitute, let $r^2 = x^2 + y^2$, $dA = r\ drd\theta$, then

$$S = \iint_R \sqrt{1 + \frac{r^2}{a^2 - r^2}}\ r\ drd\theta = 2\pi a^2$$