
MATH 2023 Fall 2021

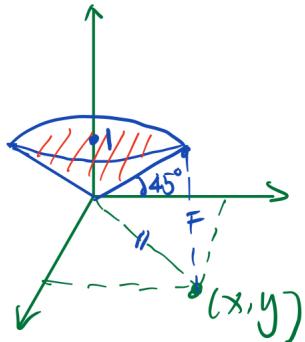
Multivariable Calculus

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Chapter 10 Vectors and Geometry in 3D

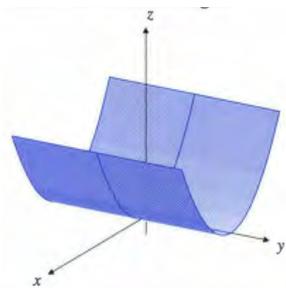
1 Coordinates in 3D

[Example.] $\{(x, y, z) \mid z \geq \sqrt{x^2 + y^2}\}$ is an ice-cream cone.



Remark: When one of the variables is missing from the equation, the equation represents a surface *parallel* to the axis of the missing variable.

[Example.] $z = x^2$, since y is missing, we draw a parabola and move it along the y axis.



2 Vectors

2.1 Vectors and Properties

In \mathbb{R}^3 , $\underline{r} = (a, b, c) = a\underline{i} + b\underline{j} + c\underline{k}$, $\|\underline{r}\| = \sqrt{a^2 + b^2 + c^2}$

Zero vector $\underline{0}$ has length zero and no specific direction.

Properties of vectors

- Equality: iff same **magnitude** and same **direction**. OR, their **components** equal
- Parallel: $\underline{r} = \lambda \underline{s}$
- Negative: $-\underline{r}$ has same magnitude but opposite direction.
- Addition, Subtraction, Scalar Multiplication: omit.

Unit Vector: $\hat{\underline{r}} = \frac{\underline{r}}{\|\underline{r}\|}$

Dot Product: $\underline{u} \cdot \underline{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$. **Remark:** $\underline{a} \cdot \underline{b} = \underline{a} \cdot \underline{c} \not\Rightarrow \underline{b} = \underline{c}$

Angle: $\underline{u} \cdot \underline{v} = \|\underline{u}\| \cdot \|\underline{v}\| \cos \theta$

Projections:

- Scalar projection: $s = \frac{\underline{u} \cdot \underline{v}}{\|\underline{v}\|} = \|\underline{u}\| \cos \theta$
- Vector projection of \underline{u} in the direction of \underline{v} : $\underline{u}_v = \frac{\underline{u} \cdot \underline{v}}{\|\underline{v}\|} \hat{\underline{v}} = \frac{\underline{u} \cdot \underline{v}}{\|\underline{v}\|^2} \underline{v}$

[Example.] Find the scalar and vector projections of vector $\underline{a} = \underline{i} - 2\underline{j} + \underline{k}$ on the vector $\underline{b} = 4\underline{i} - 4\underline{j} + 7\underline{k}$.

[Solution.]

$$s = \frac{\underline{a} \cdot \underline{b}}{\|\underline{b}\|} = \frac{4 + 8 + 7}{\sqrt{4^2 + 4^2 + 7^2}} = \frac{19}{9}$$

$$\underline{a}_b = s \cdot \frac{\underline{b}}{\|\underline{b}\|} = \frac{19}{81}(4\underline{i} - 4\underline{j} + 7\underline{k})$$

Position Vector: \underline{r} from the origin to the point (a, b, c) :

$$\underline{r} = a\underline{i} + b\underline{j} + c\underline{k}, \quad \|\underline{r}\| = \sqrt{a^2 + b^2 + c^2}$$

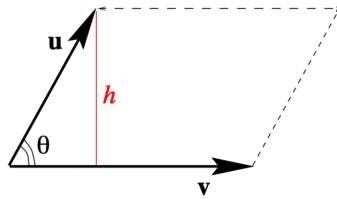
$$\hat{\underline{r}} = \frac{a\underline{i} + b\underline{j} + c\underline{k}}{\sqrt{a^2 + b^2 + c^2}}$$

Cross Product:

$$\begin{aligned}\underline{u} \times \underline{v} &= \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = (u_2 v_3 - u_3 v_2) \underline{i} - (u_1 v_3 - u_3 v_1) \underline{j} + (u_1 v_2 - u_2 v_1) \underline{k} \\ &= -\underline{v} \times \underline{u} \quad (\text{vector})\end{aligned}$$

Properties:

- $\underline{u} \times \underline{u} = \underline{0}$
- $\underline{a} \times (\underline{b} + \underline{c}) = \underline{a} \times \underline{b} + \underline{a} \times \underline{c}$
- $(\underline{u} \times \underline{v}) \cdot \underline{u} = 0, (\underline{u} \times \underline{v}) \cdot \underline{v} = 0$
- $\|\underline{u} \times \underline{v}\| = \|\underline{u}\| \cdot \|\underline{v}\| \sin \theta = \text{area of parallelogram.}$



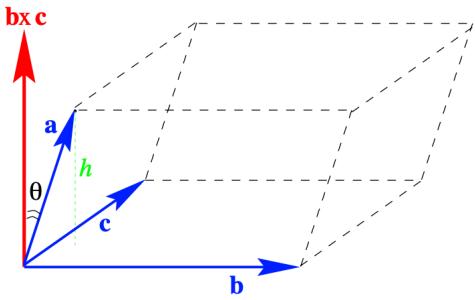
- $\|\underline{u} \times \underline{v}\|^2 = \|\underline{u}\|^2 \|\underline{v}\|^2 - (\underline{u} \cdot \underline{v})^2$
- $\underline{u} \times (\underline{v} \times \underline{w}) \neq (\underline{u} \times \underline{v}) \times \underline{w}$
- $\underline{a} \times \underline{b} = \underline{a} \times \underline{c} \not\Rightarrow \underline{b} = \underline{c}$

2.2 Triple Scalar Products

$$\underline{a} \cdot (\underline{b} \times \underline{c}) = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Geometric interpretation of $\underline{a} \cdot (\underline{b} \times \underline{c})$:

$$\begin{aligned}\underline{a} \cdot (\underline{b} \times \underline{c}) &= \|\underline{a}\| \|\underline{b} \times \underline{c}\| \cos \theta \\ &= (\|\underline{a}\| \cos \theta) \cdot \|\underline{b} \times \underline{c}\| \\ &= h \cdot \|\underline{b} \times \underline{c}\| \\ &= \text{volume of the parallelepiped}\end{aligned}$$

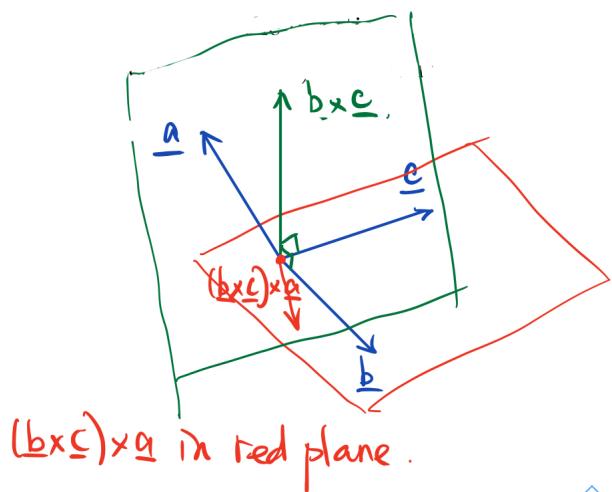


Another use of triple scalar product is for deciding whether three vectors are **in the same plane**. If so, then

$$(\underline{a} \times \underline{b}) \cdot \underline{c} = 0$$

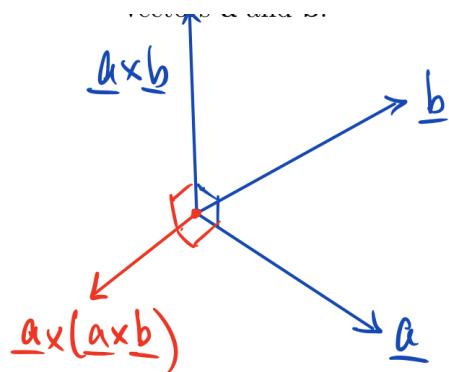
2.3 Triple Vector Products

$(\underline{a} \times \underline{b}) \times \underline{c}$ is in the plain containing \underline{a} and \underline{b} .



[Example.] Construct three mutually orthogonal vectors in space, making use of two non-parallel vectors \underline{a} and \underline{b} .

[Solution.] \underline{a} , $\underline{a} \times \underline{b}$, and $\underline{a} \times (\underline{a} \times \underline{b})$.



3 Planes and Lines

3.1 Planes

The equation of a plane: assume nonzero normal vector $\underline{n} = A\underline{i} + B\underline{j} + C\underline{k}$ and plane passes through $r_0 = (x_0, y_0, z_0)$,

$$\underline{n} \cdot (\underline{r} - \underline{r}_0) = 0$$

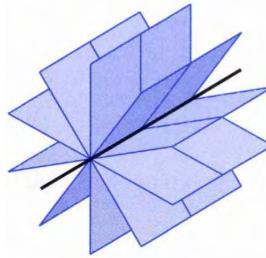
can also written in **standard form**:

$$Ax + By + Cz = Ax_0 + By_0 + Cz_0$$

A pencil of planes: a family of plans intersecting in a straight line. If two nonparallel planes have equations $A_1x + B_1y + C_1z = D_1$ and $A_2x + B_2y + C_2z = D_2$, then

$$A_1x + B_1y + C_1z - D_1 + \lambda(A_2x + B_2y + C_2z - D_2) = 0$$

represents a plane in the pencil, where $\lambda \in \mathbb{R}$.



[Example.] Find the equation of the plane through $P_0 = (-1, 4, 2)$ and containing the line of intersection of the planes $4x - y + z - 2 = 0$ and $2x + y - 2z - 3 = 0$.

[Solution.] Two methods:

(1) find three points.

Notice the two planes have infinity points intersected. We arbitrarily choose 2 of them, say $\underline{r}_1 = (0, -7, -5)$, $\underline{r}_2 = (1, 3, 1)$. Another point is given in problem, $\underline{r}_0 = (-1, 4, 2)$.

Then a normal vector can be $(\underline{r}_1 - \underline{r}_0) \times (\underline{r}_2 - \underline{r}_0) = (4, -13, 21)$

(2) find the pencil of planes.

The pencil of planes formed by the given two non-parallel planes is

$$4x - y + z - 2 + \lambda(2x + y - 2z - 3) = 0$$

Since the plane is inside the pencil and containing P_0 , substitute $(-1, 4, 2)$ into the equation, we get $\lambda = -\frac{8}{5}$.

3.2 Lines

Vector parametric equation of straight line: with a point \underline{r}_0 and **direction vector** \underline{v} :

$$\underline{r}(t) = \underline{r}_0 + t\underline{v}$$

[Example.] Find the vector equation for the line segment joins $P_1 = (2, 4, -1)$ and $P_2 = (5, 0, 7)$.

[Solution.] direction vector $\underline{v} = (3, -4, 8)$, thus

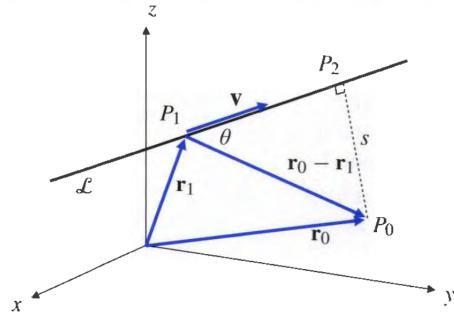
$$\underline{r}(t) = (2, 4, -1) + t \cdot (3, -4, 8), \quad 0 \leq t \leq 1$$

[Example.] Find the intersection of line $\underline{r} = \underline{r}_0 + t\underline{v}$ and plane $\underline{r} \cdot \underline{n} = d$, assume they are not parallel.

[Solution.] $(\underline{r}_0 + t\underline{v}) \cdot \underline{n} = d$, solve for t .

3.3 Distances

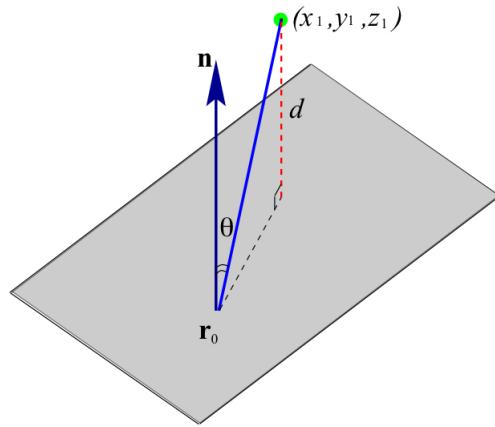
Point to Line:



$$\begin{aligned} d &= \|\underline{r}_1 - \underline{r}_0\| \cdot \sin \theta \cdot 1 \\ &= \|\underline{r}_1 - \underline{r}_0\| \cdot \sin \theta \cdot \|\hat{\underline{v}}\| \\ &= \|(\underline{r}_1 - \underline{r}_0) \times \hat{\underline{v}}\| \end{aligned}$$

Point to Plane:

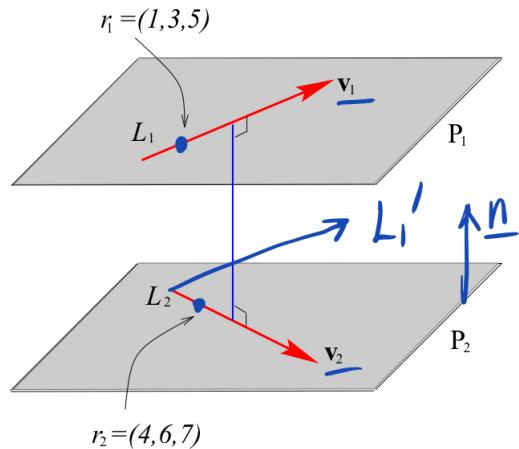
$$\begin{aligned} d &= \|\underline{r}_1 - \underline{r}_0\| \cdot \cos \theta \cdot 1 \\ &= \|\underline{r}_1 - \underline{r}_0\| \cdot \cos \theta \cdot \|\hat{\underline{n}}\| \\ &= |(\underline{r}_1 - \underline{r}_0) \cdot \hat{\underline{n}}| \end{aligned}$$



Line to Plane(parallel): Take any point on the line \rightarrow point to plane.

Parallel Lines: Same as point to line.

Skew Lines:



Make L_1 and L_1' inside the same plane, then normal vector of the plane is $\underline{n} = \underline{v}_1 \times \underline{v}_2$.

Now it is the same as line to plane.

$$\begin{aligned} d &= |(\underline{r}_1 - \underline{r}_0) \cdot \hat{\underline{n}}| \\ &= |(\underline{r}_1 - \underline{r}_0) \cdot (\widehat{\underline{v}_1 \times \underline{v}_2})| \end{aligned}$$

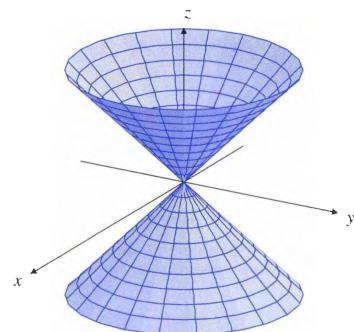
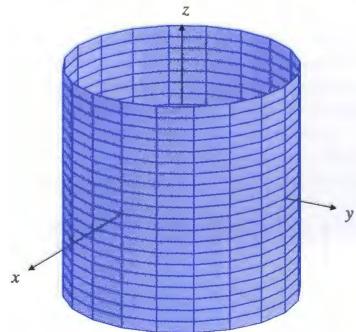
Plane to Plane(parallel): same as point to plane.

4 Quadric Surfaces

Spheres: $(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = a^2$.

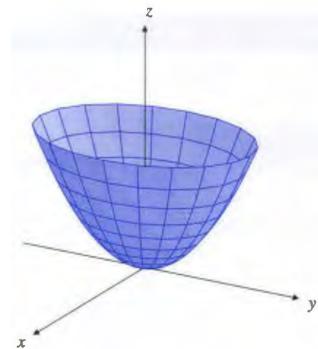
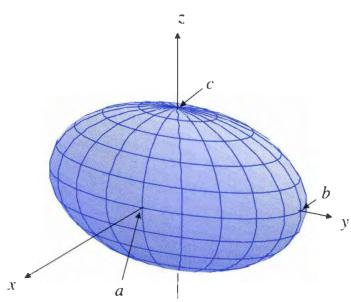
Cylinders: $x^2 + y^2 = a^2$. (left below)

Cones: $z^2 = x^2 + y^2$. (right below)

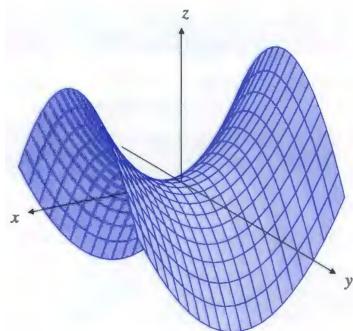


Ellipsoid: $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$. (left below)

Elliptic paraboloid: $z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$. (right below)



Hyperbolic paraboloid : $z = \frac{x^2}{a^2} - \frac{y^2}{b^2}$.



Hyperboloids:

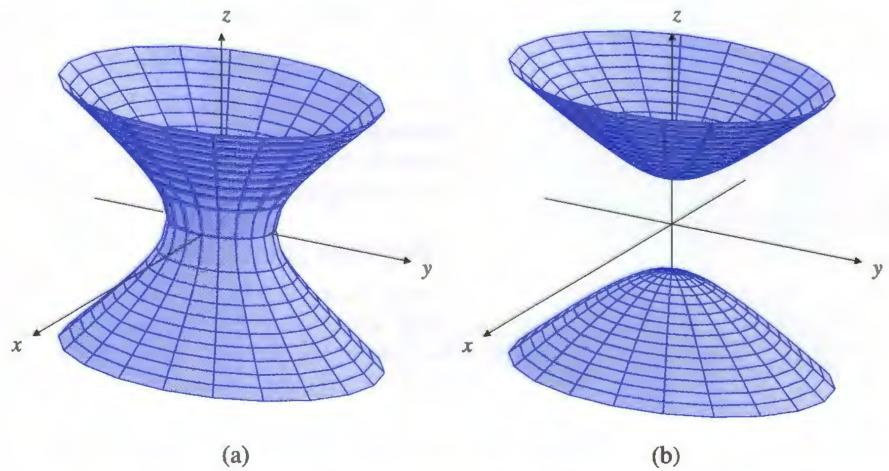
Figure 10.37

(a) The hyperboloid of one sheet

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

(b) The hyperboloid of two sheets

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1$$



5 Cylindrical and Spherical Coordinates

Cylindrical Coordinate: (r, θ, z)

$$x = r \cos \theta$$

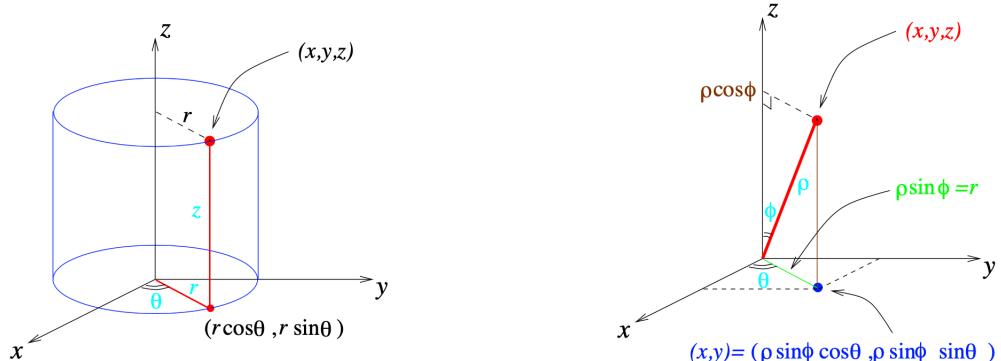
$$y = r \sin \theta$$

$$z = z$$

where $r \geq 0$, $0 \leq \theta < 2\pi$, and

$$r = \sqrt{x^2 + y^2}$$

$$\theta = \begin{cases} \tan^{-1}(y/x) & \text{if } x \geq 0 \\ \tan^{-1}(y/x) + \pi & \text{if } x < 0 \end{cases}$$



Spherical Coordinates: (ρ, θ, ϕ)

$$x = r \cos \theta = \rho \sin \phi \cos \theta$$

$$y = r \sin \theta = \rho \sin \phi \sin \theta$$

$$z = \rho \cos \phi$$

where $\rho \geq 0$, $0 \leq \theta < 2\pi$, $0 \leq \phi \leq \pi$, and

$$\begin{aligned}\rho &= \sqrt{x^2 + y^2 + z^2} \\ \theta &= \begin{cases} \tan^{-1}(y/x) & \text{if } x \geq 0 \\ \tan^{-1}(y/x) + \pi & \text{if } x < 0 \end{cases} \\ \phi &= \cos^{-1}(z/\rho)\end{aligned}$$