MATH 2023 Fall 2021

Multivariable Calculus

Written By: Ljm

Chapter 13 **Application of Partial Derivatives**

1 Extreme Values

Recall that in single variable calculus:

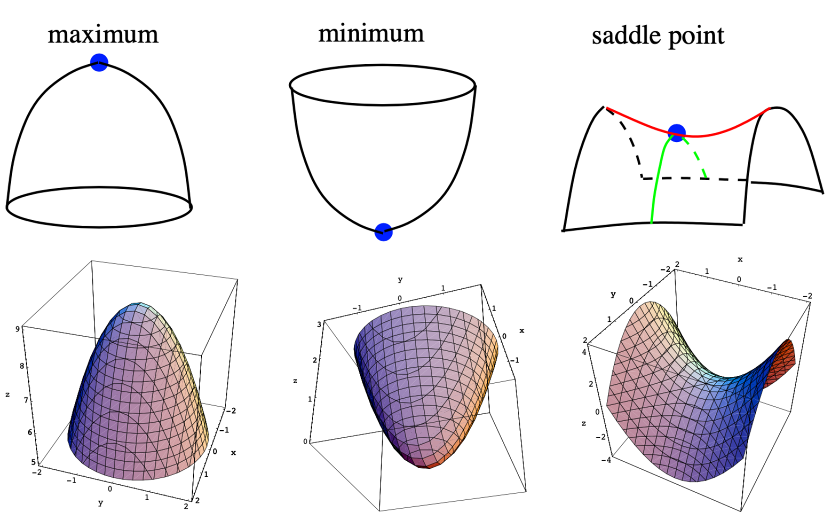
x_1 is a *relative* maximum point, if $f'(x_1) = 0$ and $f''(x_1) < 0$,
x_2 is a *relative* minimum point, if $f'(x_2) = 0$ and $f''(x_2) > 0$.

Similarly, in multi-variable calculus, the **critical point** is where

$$\boxed{\nabla f(\mathbf{r}_0) = (f_{x_1}(\mathbf{r}_0), f_{x_2}(\mathbf{r}_0), \cdots, f_{x_n}(\mathbf{r}_0)) = \mathbf{0}}$$

And, if h has a **relative extremum** at a point $\mathbf{r}_0$, then $\mathbf{r}_0$ is a **critical point**, and $\nabla f(\mathbf{r}_0) = \mathbf{0}$. However, if $\mathbf{r}_0$ is a critical point, we *cannot infer* that $\mathbf{r}_0$ is a relative extremum. The reason is similar in single variable calculus.

Different from single variable, a critical point which *is not a relative extremum* can be a **maximum point, minimum point,** or a **saddle point**.



However, to classify the critical points, we need the **second derivative test**, or **D-test**.

Second Derivative Test

Suppose $f(x, y)$ has a critical point at $\mathbf{r}_0 = (x_0, y_0)$ (i.e. $\nabla f(\mathbf{r}_0) = \mathbf{0}$) and the second partial derivative of $f(x, y)$ are continuous in a disk with center $\mathbf{r}_0 = (x_0, y_0)$. Let

$$\boxed{D = \begin{vmatrix} f_{xx}(\mathbf{r}_0) & f_{xy}(\mathbf{r}_0) \\ f_{yx}(\mathbf{r}_0) & f_{yy}(\mathbf{r}_0) \end{vmatrix} = f_{xx}(\mathbf{r}_0) f_{yy}(\mathbf{r}_0) - f_{xy}^2(\mathbf{r}_0)}$$

D	$f_{xx}(\mathbf{r}_0)$ or $f_{yy}(\mathbf{r}_0)$	nature of $\mathbf{r}_0$
> 0	> 0	relative minimum
> 0	< 0	relative maximum
< 0		saddle point
$= 0$		no conclusion can be drawn

I'd like to omit the proof of D-Test here.

This example shows basic use of D-Test.

[Example.] Find the relative minima and maxima of $f(x, y) = x^3 + y^3 - 3x - 12y + 20$.

$$f_x = 3x^2 - 3 \quad \text{and} \quad f_y = 3y^2 - 12$$

[Solution.] For critical points, $f_x = f_y = 0 \quad \Rightarrow \quad x = \pm 1, y = \pm 2$.

$\therefore (1, 2), (-1, 2), (1, -2), (-1, -2)$ are critical points.

To apply D-Test, compute: $f_{xx} = 6x, f_{yy} = 6y, f_{xy} = 0$, hence $D = f_{xx} \cdot f_{yy} - (f_{xy})^2 = 36xy$

Point	f_{xx}	f_{yy}	f_{xy}	D	Type
$(1, 2)$	6	12	0	72	min
$(-1, 2)$	-6	12	0	-72	saddle
$(1, -2)$	6	-12	0	-72	saddle
$(-1, -2)$	-6	-12	0	72	max

This example shows how to find extrema on a *closed* and *bounded* region.

[Example.] Find the absolute extrema of the function

$$z = f(x, y) = xy - x - 3y$$

on the *closed* and *bounded* set R, where R is the triangular region with vertices $(0, 0), (0, 4)$ and $(5, 0)$.

[Solution.] $f_x = y - 1, f_y = x - 3, f_{xy} = f_{yx} = 1, f_{xx} = f_{yy} = 0, D = -1$

For critical points, $\nabla f = (f_x, f_y) = (0, 0) \Rightarrow x = 3, y = 1$.

This point is inside the domain. But we still need to find possible extreme points *on the boundary of domain.*

(1) Along OA :, $\mathbf{r}_0 = (0, 0), \mathbf{r}_1 = (5, 0)$, so the parametric representation of line OA is:

$$\mathbf{r}(t) = (1 - t)\mathbf{r}_0 + t\mathbf{r}_1, = (5t, 0),\ t \in [0, 1]$$

hence $z = f(\mathbf{r}(t)) = -5t,\ t \in [0, 1]$

So along OA, the possible extreme points are $(0, 0)$ and $(5, 0)$.

(2) Along OB :, similarly, $\mathbf{r}(t) = (0, 4t),\ t \in [0, 1]$, $z = f(\mathbf{r}) = -12t$,

So along OB, the possible extreme points are $(0, 0)$ and $(0, 4)$.

(3) Along $AB : \mathbf{r}(t) = (5 - 5t, 4t),\ t \in [0, 1]$, $z = -20t^2 + 13t - 5,\ t \in [0, 1]$

There is one critical point on AB, when $dz/dx = 0$, at $\left(\frac{27}{8}, \frac{13}{10}\right)$.

Then we compute the value of all possible extremum points,

(x, y)	$f(x, y)$
$(3, 1)$	-3
$\left(\frac{27}{8}, \frac{13}{10}\right)$	$-\frac{231}{80}$
$(0, 0)$	0
$(5, 0)$	-5
$(0, 4)$	-12

Therefore, absolute maximum value is 0 which occurs at $(0, 0)$, absolute minimum value is -12 which occurs at $(0, 4)$.

This example converts the problem to max/min problem.

[Example.] Find the points on the surface $z^2 = xy + 1$ that are closest to the origin.

[Solution.] $d^2 = (x - 0)^2 + (y - 0)^2 = x^2 + y^2 + xy + 1 = f(x, y)$, only need to minimize this function.

2 Lagrange multipliers

Motivation: sometimes we want to maximize/minimize $f(x,y)$ subject to $g(x,y)=k$.

How to find the maximum or minimum value?

1. Find all values of $\mathbf{r}$ and λ such that
$$\nabla f(\mathbf{r}) = \lambda \nabla g(\mathbf{r})$$
and
$$g(\mathbf{r}) = k$$
2. Evaluate f at all the points $\mathbf{r}$ that arise from step (1). The largest (smallest) of these values is the maximum (min) value of f.

Remark: Lagrange's method only finds critical points, it *does not tell* whether the function is maximized or minimized.