MATH 2023 Fall 2021
Multivariable Calculus

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Chapter 13 **Application of Partial Derivatives**

1 Extreme Values

Recall that in single variable calculus:

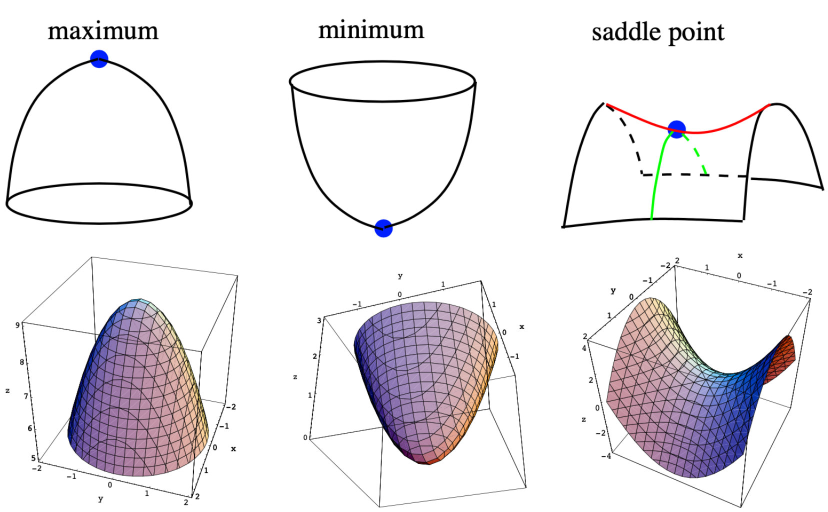
x_1 is a *relative* maximum point, if $f'(x_1) = 0$ and $f''(x_1) < 0$,
x_2 is a *relative* minimum point, if $f'(x_2) = 0$ and $f''(x_2) > 0$.

Similarly, in multi-variable calculus, the **critical point** is where

$$\boxed{\nabla f(\mathbf{r}_0) = (f_{x_1}(\mathbf{r}_0), f_{x_2}(\mathbf{r}_0), \cdots, f_{x_n}(\mathbf{r}_0)) = \mathbf{0}}$$

And, if h has a **relative extremum** at a point $\mathbf{r}_0$, then $\mathbf{r}_0$ is a **critical point**, and $\nabla f(\mathbf{r}_0) = \mathbf{0}$. However, if $\mathbf{r}_0$ is a critical point, we *cannot infer* that $\mathbf{r}_0$ is a relative extremum. The reason is similar in single variable calculus.

Different from single variable, a critical point which *is not a relative extremum* is a **saddle point**.



However, to classify the critical points, we need the **second derivative test**, or **D-test**.

Second Derivative Test

Suppose $f(x, y)$ has a critical point at $\mathbf{r}_0 = (x_0, y_0)$ (i.e. $\nabla f(\mathbf{r}_0) = \mathbf{0}$) and the second partial derivative of $f(x, y)$ are continuous in a disk with center $\mathbf{r}_0 = (x_0, y_0)$. Let

$$\boxed{D = \begin{vmatrix} f_{xx}(\mathbf{r}_0) & f_{xy}(\mathbf{r}_0) \\ f_{yx}(\mathbf{r}_0) & f_{yy}(\mathbf{r}_0) \end{vmatrix} = f_{xx}(\mathbf{r}_0) f_{yy}(\mathbf{r}_0) - f_{xy}^2(\mathbf{r}_0)}$$

D	$f_{xx}(\mathbf{r}_0)$ or $f_{yy}(\mathbf{r}_0)$	nature of $\mathbf{r}_0$
> 0	> 0	relative minimum
> 0	< 0	relative maximum
< 0		saddle point
$= 0$		no conclusion can be drawn

I'd like to omit the proof of D-Test here.

This example shows basic use of D-Test.

[Example.] Find the relative minima and maxima of $f(x, y) = x^3 + y^3 - 3x - 12y + 20$.

$$f_x = 3x^2 - 3 \quad \text{and} \quad f_y = 3y^2 - 12$$

[Solution.] For critical points, $f_x = f_y = 0 \quad \Rightarrow \quad x = \pm 1, y = \pm 2$.

$\therefore (1, 2), (-1, 2), (1, -2), (-1, -2)$ are critical points.

To apply D-Test, compute: $f_{xx} = 6x, f_{yy} = 6y, f_{xy} = 0$, hence $D = f_{xx} \cdot f_{yy} - (f_{xy})^2 = 36xy$

Point	f_{xx}	f_{yy}	f_{xy}	D	Type
$(1, 2)$	6	12	0	72	min
$(-1, 2)$	-6	12	0	-72	saddle
$(1, -2)$	6	-12	0	-72	saddle
$(-1, -2)$	-6	-12	0	72	max

This example shows how to find extrema on a *closed* and *bounded* region.

[Example.] Find the absolute extrema of the function

$$z = f(x, y) = xy - x - 3y$$

on the *closed* and *bounded* set R, where R is the triangular region with vertices $(0,0), (0,4)$ and $(5,0)$.

[Solution.] $f_x = y - 1, f_y = x - 3, f_{xy} = f_{yx} = 1, f_{xx} = f_{yy} = 0, D = -1$

For critical points, $\nabla f = (f_x, f_y) = (0,0) \Rightarrow x = 3, y = 1$.

This point is inside the domain. But we still need to find possible extreme points *on the boundary of domain.*

(1) Along OA :, $\mathbf{r}_0 = (0,0), \mathbf{r}_1 = (5,0)$, so the parametric representation of line OA is:

$$\mathbf{r}(t) = (1-t)\mathbf{r}_0 + t\mathbf{r}_1, = (5t, 0),\ t \in [0,1]$$

hence $z = f(\mathbf{r}(t)) = -5t,\ t \in [0,1]$

So along OA, the possible extreme points are $(0,0)$ and $(5,0)$.

(2) Along OB :, similarly, $\mathbf{r}(t) = (0, 4t),\ t \in [0,1]$, $z = f(\mathbf{r}) = -12t$,

So along OB, the possible extreme points are $(0,0)$ and $(0,4)$.

(3) Along AB : $\mathbf{r}(t) = (5-5t, 4t),\ t \in [0,1]$, $z = -20t^2 + 13t - 5,\ t \in [0,1]$

There is one critical point on AB, when $dz/dx = 0$, at $\left(\frac{27}{8}, \frac{13}{10}\right)$.

Then we compute the value of all possible extremum points,

(x, y)	$f(x, y)$
$(3,1)$	-3
$\left(\frac{27}{8}, \frac{13}{10}\right)$	$-\frac{231}{80}$
$(0,0)$	0
$(5,0)$	-5
$(0,4)$	-12

Therefore, absolute maximum value is 0 which occurs at $(0,0)$, absolute minimum value is -12 which occurs at $(0,4)$.

This example converts the problem to max/min problem.

[Example.] Find the points on the surface $z^2 = xy + 1$ that are closest to the origin.

[Solution.] $d^2 = (x-0)^2 + (y-0)^2 + (z-0)^2 = x^2 + y^2 + xy + 1 = f(x, y)$, only need to minimize this function.

This is a more comprehensive and tricker problem.

[Example.] Find absolute minimum and maximum value of $f(x,y) = 2x^3 + y^4$ on the set $D = \{(x,y)|x^2+y^2 \le 1\}$.

[Solution.] First, find the critical point of $f(x,y)$ on entire xy plane.

$f_x = 6x^2, f_y = 4y^3$, critical points: $f_x = f_y = 0 \Rightarrow x = y = 0$, and $f(0,0) = 0$.

Then, on the circle $x^2 + y^2 = 1$, eliminate y, we have:

$$g(x) = f(x,y) = x^4 + 2x^3 - 2x^2 + 1, \ -1 \le x \le 1$$
$$g'(x) = 4x^3 + 6x - 4x = 0$$

the equation has solutions: $(x,y) = (0, \pm 1)$, $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$, to check each of them:

$$f(0,\pm 1) = g(0) = 1, \ f\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right) = g\left(\frac{1}{2}\right) = \frac{13}{16}$$

Also, we need to check the *endpoints*, (since we are looking for min/max point of $g(x)$ on $x \in [-1,1]$)

$$g(1) = 2, \ g(-1) = -2$$

Therefore, the absolute minimum is $g(-1) = f(-1,0) = -2$, the absolute maximum is $g(1) = f(1,0) = 2$.

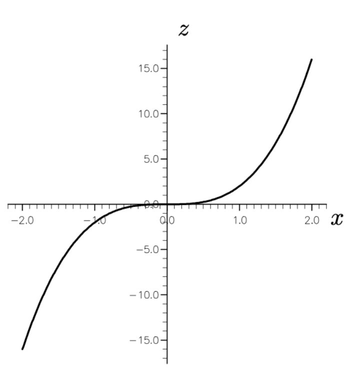
Notice that, if you are interested in the nature of the critical point at $(0,0)$, you may try D-test:

$$D = f_{xx}f_{yy} - f_{xy}^2 = 144xy^2 = 0 \quad \text{, at (0,0)}$$

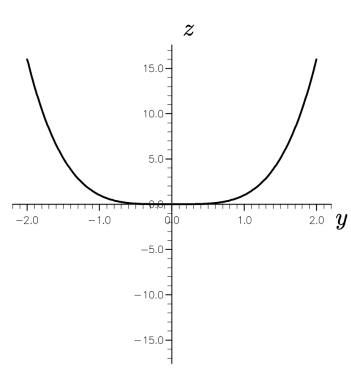
so the D-test fails, we have to use other methods to determine it:

In the plane $y = 0$, $f(x,0) = 2x^3$, and in the plane $x = 0$, $f(0,y) = y^4$.

In the plane $y = 0$, $f(x,0) = 2x^3$



In the plane $x = 0$, $f(0,y) = y^4$



Thus, the critical point is a *saddle point*.

2 Lagrange multipliers

Motivation: sometimes we want to maximize/minimize $f(x, y)$ subject to $g(x, y) = k$.

How to find the maximum or minimum value?

1. Find all values of $\mathbf{r}$ and λ such that

$$\nabla f(\mathbf{r}) = \lambda \nabla g(\mathbf{r})$$

   and

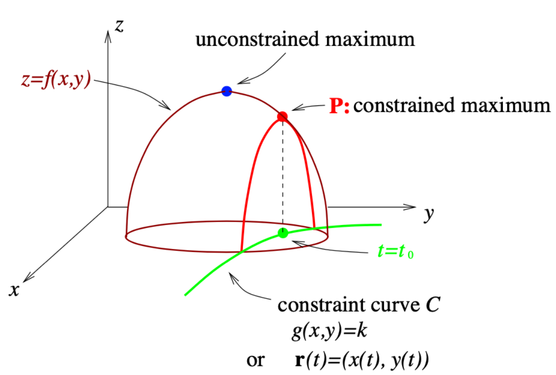
$$g(\mathbf{r}) = k$$

2. Evaluate f at all the points $\mathbf{r}$ that arise from step (1). The largest (smallest) of these values is the maximum (min) value of f.

Remark: Lagrange's method only finds critical points, it *does not tell* whether the function is maximized or minimized.

Proof of Lagrange's method:

Notice that, maximizing or minimizing a function $f(x_1, x_2, \cdots, x_n)$ subject to a constraint of $g(x_1, x_2, \cdots, x_n) = k$ is to restrict the point $(x_1, x_2, \cdots, x_n)$ to lie on the *level surface* S given by $g(x_1, x_2, \cdots, x_n) = k$. For example, if $n = 2$, maximize(or minimize) $z = f(x, y)$ subject to constraint curve $C : g(x, y) = k$ (shown in green) is to restrict the point (x, y) to lie on the red curve.



Suppose z has a maximum value at a point $\mathbf{P}$, and let C be the *constraint curve* with vector equation

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} \quad \text{be on} \quad g(x, y) = k.$$

Assume also that at the point $\mathbf{P}, t = t_0$.

Since on the constraint curve $C : z(t) = f(x(t), y(t))$, and the point $\mathbf{P}$ should be a critical point. By using the chain rule,

$$\begin{aligned}\frac{dz}{dt} &= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} \\ &= \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) \cdot \left(\frac{dx}{dt}, \frac{dy}{dt}\right) \\ &= \nabla f \cdot \mathbf{r}'(t)\end{aligned}$$

Then at the point $\mathbf{P}$,

$$\left.\frac{dz}{dt}\right|_{t=t_0} = \nabla f\left(x\left(t_0\right), y\left(t_0\right)\right) \cdot \mathbf{r}'\left(t_0\right) = 0 \quad \text{(since } \mathbf{P} \text{ is critical point)}$$

Therefore,

$$\nabla f\left(x\left(t_0\right), y\left(t_0\right)\right) \perp \mathbf{r}'\left(t_0\right)$$

Moreover, since ∇g is normal vector to the level set,

$$\nabla g\left(x\left(t_0\right), y\left(t_0\right)\right) \perp \mathbf{r}'\left(t_0\right)$$

Therefore, $\nabla f \parallel \nabla g$, $\nabla f = \lambda \nabla g$.

The number λ is called a Lagrange multiplier.

This example shows how to use Lagrange's Method.

[**Example.**] Find the extreme values of $f(x, y) = x^2 - y^2$ subject to $x^2 + y^2 = 1$.

[**Solution.**] Find x, y and λ such that $\nabla f = \lambda \nabla g$, where $g = x^2 + y^2 = 1$(constant)

$$2x\mathbf{i} - 2y\mathbf{j} = \lambda(2x\mathbf{i} + 2y\mathbf{j})$$

, which gives

$$\begin{cases} 2x = & \lambda \cdot 2x \\ -2y = & \lambda \cdot 2y \end{cases} \quad \Rightarrow \quad \begin{cases} \lambda = 1 & \text{or} \quad x = 0 \\ \lambda = -1 & \text{or} \quad y = 0 \end{cases}$$

From $x^2 + y^2 = 1$, we have:

$$\begin{aligned} x = 0,\ y = \pm 1,\ \lambda = -1 \\ y = 0,\ x = \pm 1,\ \lambda = 1 \end{aligned}$$

Therefore, f has possible extreme values at point $(0,1), (0,-1), (-1,0)$ and $(1,0)$. Evaluating f at these four points, we find that

$$\begin{aligned} f(0,1) = f(0,-1) = -1 \quad \text{(min)} \\ f(1,0) = f(-1,0) = 1 \quad \text{(max)} \end{aligned}$$

Interpretation of λ

This will not be tested in exam.

Actually, λ has an interpretation which can be very useful.

Suppose M is the optimal value of $f(x,y)$ subject to the constraint $g(x,y)=c$.

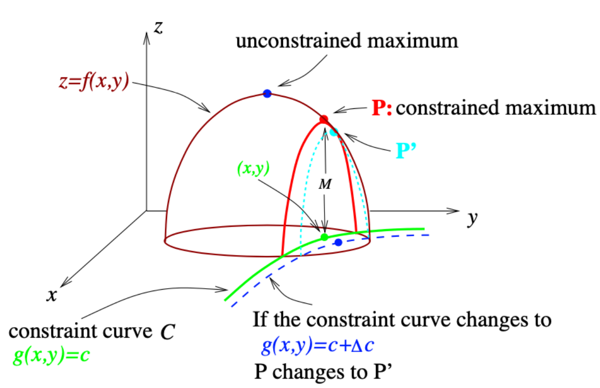
Then $f(x,y)=M$ for some ordered pair (x,y) that satisfies the three Lagrangian equations

$$\begin{aligned} f_x - \lambda g_x &= 0 \\ f_y - \lambda g_y &= 0 \\ g &= c \end{aligned}$$

Since $M = f(x,y)$

$$\begin{aligned} \frac{dM}{dc} &= \frac{\partial f}{\partial x}\frac{dx}{dc} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dc} \\ &= f_x\frac{dx}{dc} + f_y\frac{dy}{dc} \\ &= \lambda g_x\frac{dx}{dc} + \lambda g_y\frac{dy}{dc} \\ &= \lambda\left(g_x\frac{dx}{dc} + g_y\frac{dy}{dc}\right) \\ &= \lambda\frac{dg}{dc} = \lambda \end{aligned}$$

where dM/dc is evaluated at the optimal solution values. In other words, λ measures the *sensitivity* of the optimal value of f to change in c.



The example below shows the application of λ.

[Example.] Use Lagrangian multiplier to find the maximum and minimum values of the function $f(x, y) = 4x^3 + y^2$ subject to the constraint $2x^2 + y^2 = 1$.

If the constraint equation changes to $2x^2 + y^2 = 0.9$, estimate how this changes will affect the maximum and minimum values of f.

[Solution.] Let $g(x, y) = 2x^2 + y^2 = 1 = c$, then for $\nabla f = \lambda \nabla g$, we have $(12x^2, 2y) = \lambda(4x, 2y)$, i.e.,

$$\begin{aligned} 12x^2 &= \lambda 4x \\ 2y &= \lambda 2y \\ 2x^2 + y^2 &= 1 \end{aligned}$$

From the second equation, if $y \neq 0$, then $\lambda = 1$, substitute into the other two equations, we get $(x, y) = (0, \pm 1)$ or $\left(\frac{1}{3}, \pm\frac{\sqrt{7}}{3}\right)$

However, if $y = 0$, then from third equation, $x = \pm\frac{\sqrt{2}}{2}$, substitute to first eq, we get:

$$\begin{aligned} \lambda &= \frac{3\sqrt{2}}{2} \quad \text{when} \quad x = \frac{\sqrt{2}}{2} \\ \lambda &= -\frac{3\sqrt{2}}{2} \quad \text{when} \quad x = -\frac{\sqrt{2}}{2} \end{aligned}$$

λ	(x, y)	$f(x, y)$	nature
1	$(0, \pm 1)$	1	
1	$\left(\frac{1}{3}, \pm\frac{\sqrt{7}}{3}\right)$	$\frac{25}{27}$	
$\frac{3\sqrt{2}}{2}$	$\left(\frac{\sqrt{2}}{2}, 0\right)$	$\sqrt{2}$	max
$-\frac{3\sqrt{2}}{2}$	$\left(-\frac{\sqrt{2}}{2}, 0\right)$	$-\sqrt{2}$	min

Since $\frac{dM}{dc} = \lambda$, so $\Delta M \approx \lambda \Delta c$, in this case, $\Delta c = -0.1$, so at min point,

$$\Delta M = -\frac{3\sqrt{2}}{2} \cdot (-0.1) = \frac{3\sqrt{2}}{20} \quad \text{(increase)}$$

at max point,

$$\Delta M = \frac{3\sqrt{2}}{2} \cdot (-0.1) = -\frac{3\sqrt{2}}{20} \quad \text{(decrease)}$$

This is the end of Chapter 13.