MATH 2023 Fall 2021

Multivariable Calculus

Written By: Ljm

Chapter 15 **Vector Field**

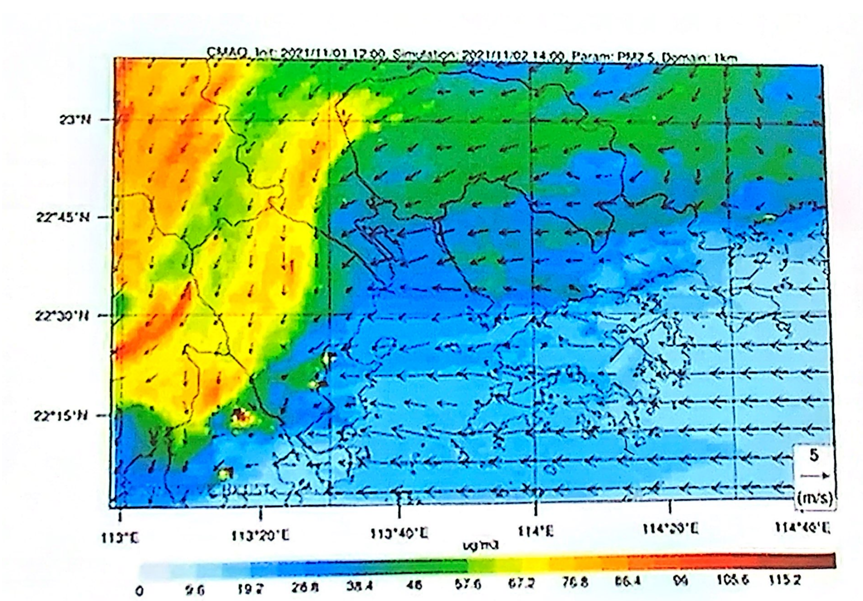
1 Intro. to Vector Field

So far, we have learned two kinds of functions involving vector:

- $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$: for each t, provides a *position* vector $< x(t), y(t), z(t) >$, so this is a (parametric) curve.
- $z = f(\mathbf{r}) = f(x_1, x_2, \cdots, x_n)$: for a given vector r, this gives a real number, so this is a function of *several variables*. This is also a **scalar field** since for any point $\mathbf{r}$ in **field**, it gives a scalar value.

Now we are looking at **vector-valued** function $\mathbf{F}$ of a vector $\mathbf{r}$, i.e., $\mathbf{F}(\mathbf{r})$. This is a **vector field**, which means for any point $\mathbf{r}$ in **field**, it gives a vector $\mathbf{F}(\mathbf{r})$.

You can consider a world map showing the *speed* and *direction* of wind.



You can see that in a 2D map(like above), if we put a vector on each point, the vector must have same dimension as the map, i.e., all vectors must also be 2D vectors.

$$\mathbf{F}(\mathbf{r}) = \begin{cases} (F_1(\mathbf{r}), F_2(\mathbf{r})) & \mathbf{r} = (x, y) & 2D \\ (F_1(\mathbf{r}), F_2(\mathbf{r}), F_3(\mathbf{r})) & \mathbf{r} = (x, y, z) & 3D \\ (F_1(\mathbf{r}), F_2(\mathbf{r}), \cdots, F_n(\mathbf{r})) & \mathbf{r} = (x_1, x_2, \cdots, x_n) & nD \end{cases}$$

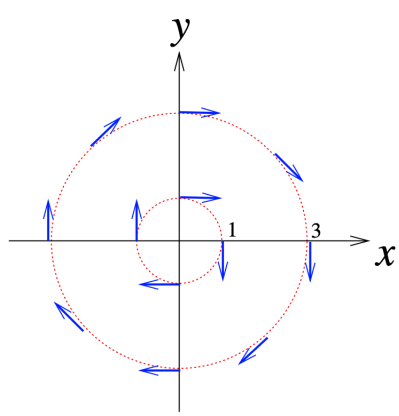
Summary: *dimension of* $\mathbf{F}$ *must be the same as* $\mathbf{r}$.

This is an example of vector field.

[Example.] Assume a vector field: $\mathbf{F}(x, y) = \dfrac{y\mathbf{i} - x\mathbf{j}}{\sqrt{x^2 + y^2}}$.

[Solution.] Notice that $||\mathbf{F}|| = \dfrac{y^2}{x^2 + y^2} + \dfrac{x^2}{x^2 + y^2} = 1$, all vectors $\mathbf{F}(x, y)$ are unit vectors. Moreover, let $\mathbf{r} = (x, y)$, then $\mathbf{r} \cdot \mathbf{F} = 0$, so $\mathbf{r} \perp \mathbf{F}$.

So all vectors are unit vectors tangent to circles centered at the origin with radius $\sqrt{x^2 + y^2}$.



2 Divergence and Curl

Recall that the **gradient operator** is a *vector operator*:

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \qquad \text{(a vector)}$$

If $\mathbf{F}(\mathbf{r}) = F_1(\mathbf{r})\mathbf{i} + F_2(\mathbf{r})\mathbf{j} + F_3(\mathbf{r})\mathbf{k}$, then we define:

- **divergence** of $\mathbf{F}$, written div $\mathbf{F}$:

$$\boxed{\operatorname{div}\mathbf{F} = \nabla\cdot\mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}}$$

- **curl** of $\mathbf{F}$, written curl $\mathbf{F}$:

$$\boxed{\operatorname{curl}\mathbf{F} = \nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right)\mathbf{i} - \left(\frac{\partial h}{\partial x} - \frac{\partial f}{\partial z}\right)\mathbf{j} + \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)\mathbf{k}}$$

This example shows basic computation of **divergence** and **curl**.

[Example.] Let $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $\mathbf{u} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$, where a, b and c are constants, show that

(a) $\nabla\cdot\mathbf{r} = 3$
(b) $\nabla\times\mathbf{r} = \mathbf{0}$
(c) $\nabla\cdot(\mathbf{u}\times\mathbf{r}) = 0$
(d) $\nabla\times(\mathbf{u}\times\mathbf{r}) = 2\mathbf{u}$.

[Solution.] (a) $\nabla\cdot\mathbf{r} = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)\cdot(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 3$

(b) $\nabla\times\mathbf{r} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = \mathbf{0}$

(c) $\mathbf{u}\times\mathbf{r} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a & b & c \\ x & y & z \end{vmatrix} = (bz - cy)\mathbf{i} - (az - cx)\mathbf{j} + (ay - bx)\mathbf{k}$

$\therefore \nabla\cdot(\mathbf{u}\times\mathbf{r}) = \frac{\partial}{\partial x}(bz - cy) - \frac{\partial}{\partial y}(az - cx) + \frac{\partial}{\partial z}(ay - bx) = 0$

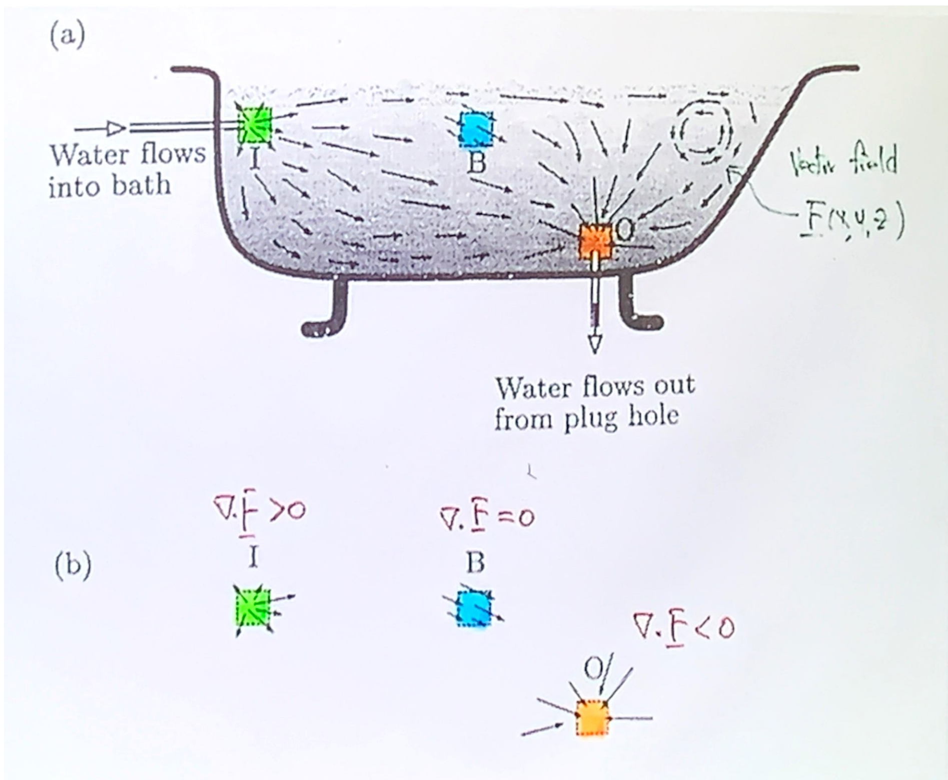
(d) $\nabla\times(\mathbf{u}\times\mathbf{r}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ bz - cy & -az + cx & ay - bx \end{vmatrix} = 2(a\mathbf{i} + b\mathbf{j} + c\mathbf{k}) = 2\mathbf{u}$

2.1 Interpretation of Divergence

Imagine water in a bath tank, if the **velocity** of water at any point of the tank is given by

$$\mathbf{u}(\mathbf{r}) = u_1(\mathbf{r})\mathbf{i} + u_2(\mathbf{r})\mathbf{j} + u_3(\mathbf{r})\mathbf{k}$$

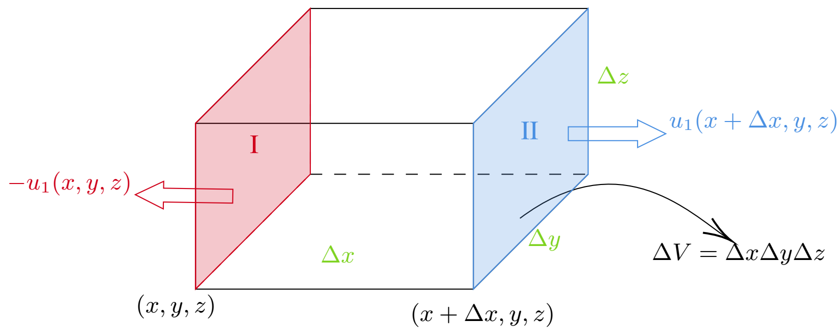
then **net outward flux per unit volume** is div $\mathbf{u} = \nabla \cdot \mathbf{u}$.



Moreover,

- If more water comes inside, then div $\mathbf{u} < 0$
- If more water comes outside, then div $\mathbf{u} > 0$
- If the amount of water comes inside equals to comes outside, then div $\mathbf{u} = 0$

This page proves the interpretation of divergence.



Imagine the box with volume $\Delta V = \Delta x \Delta y \Delta z$, firstly consider faces I and II, the total flux *out of* faces I and II, as shown above, is:

$$\begin{aligned}
&[u_1(x+\Delta x, y, z) - u_1(x, y, z)]\Delta y \Delta z \\
=&\frac{[u_1(x+\Delta x, y, z) - u_1(x, y, z)]}{\Delta x}\Delta x \Delta y \Delta z \\
=&\frac{\partial u_1}{\partial x}\Delta x \Delta y \Delta z, \qquad \text{(in the limit of } \Delta x \to 0)
\end{aligned}$$

Similarly, the two faces in the $y-$ and $z-$ direction contribute

$$\frac{\partial u_2}{\partial y}\Delta x \Delta y \Delta z, \quad \frac{\partial u_3}{\partial z}\Delta x \Delta y \Delta z$$

Hence net outward flux is:

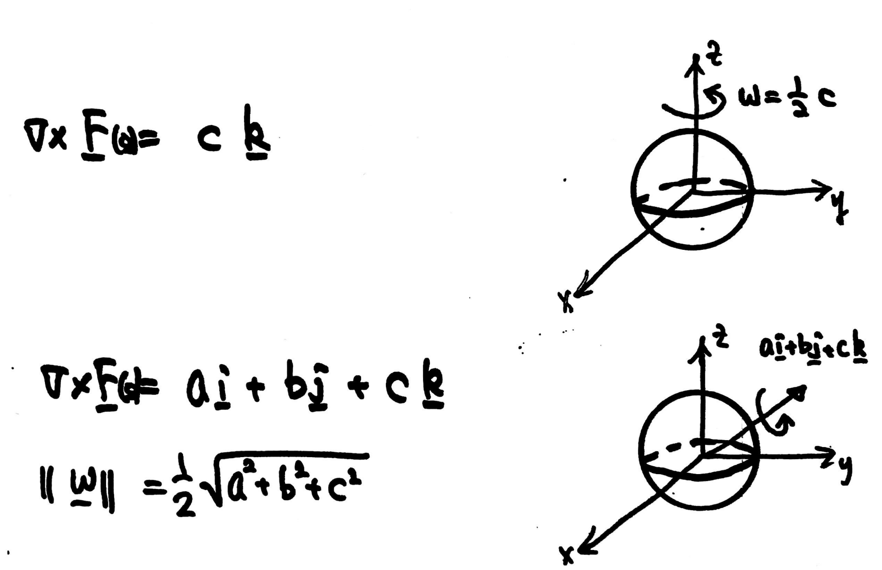
$$\left(\frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} + \frac{\partial u_3}{\partial z}\right) \cdot \Delta V$$

Therefore outward flux *per unit volume* is $\nabla \cdot \mathbf{u}$.

2.2 Interpretation of Curl

Curl is something related to rotation. Consider a small object flying in strong wind, where the speed and direction of wind can be treated as a vector field $\mathbf{F}$. If the object locates at position $\mathbf{r}$, then its rotation has some relation with curl $\mathbf{F}$.

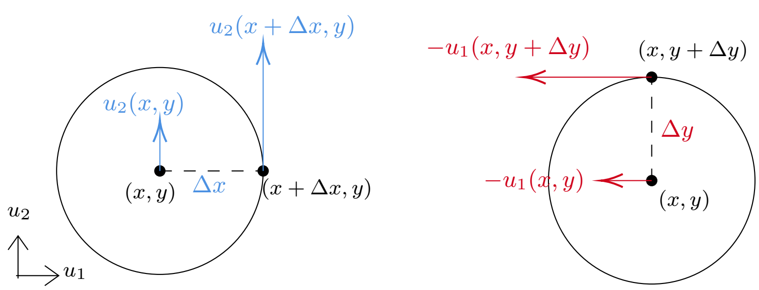
Actually, the object will rotate about the direction $\nabla \times \mathbf{F}(\mathbf{r})$(direction is determined by right-hand rule), and with angular speed $\omega = \frac{1}{2}||\nabla \times \mathbf{F}(\mathbf{r})||$.



The rest of this page prove the relation above.

Consider a disk in $xy-$plane, in y direction, the differential velocity *normal to* Δx is:

$$u_2(x+\Delta x) - u_2(x) = \frac{\partial u_2}{\partial x}\Delta x$$



Recall that $v = \omega r$, so the angular velocity is $\omega_1 = \dfrac{\partial u_2}{\partial x}$

Similarly, in the $y-$direction, (notice the negative sign)

$$-u_1(x, y+\Delta y) + u_1(x,y) = -\frac{\partial u_1}{\partial y}\Delta y, \qquad \omega_2 = -\frac{\partial u_1}{\partial y}$$

Thus the *averaged angular velocity* is: $\omega = \dfrac{1}{2}\left(\dfrac{\partial u_2}{\partial x} - \dfrac{\partial u_1}{\partial y}\right)$

The curl of this vector field is:

$$\operatorname{curl}\mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ u_1 & u_2 & 0 \end{vmatrix} = \left(\frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y}\right)\mathbf{k} = 2\omega\mathbf{k}$$

Thus prove the result.

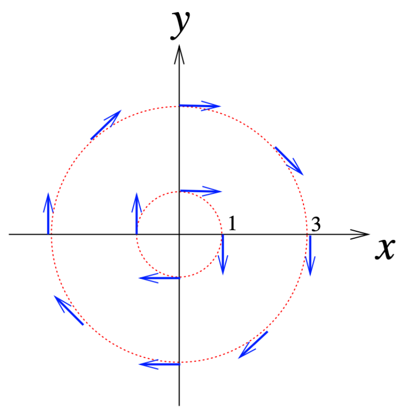
Below example is used to explain the meaning of curl, it's the same example in intro.

[Example.] Assume a vector field: $\mathbf{F}(x, y) = \dfrac{y\mathbf{i} - x\mathbf{j}}{\sqrt{x^2 + y^2}}$.

[Solution.]

$$\begin{aligned} \vec{\omega} = \nabla \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{y}{\sqrt{x^2+y^2}} & \dfrac{-x}{\sqrt{x^2+y^2}} & 0 \end{vmatrix} \\ &= \left[-\frac{\partial}{\partial x}\left(\frac{x}{\sqrt{x^2+y^2}}\right) - \frac{\partial}{\partial y}\left(\frac{y}{\sqrt{x^2+y^2}}\right)\right]\mathbf{k} \\ &= \mathbf{0} \end{aligned}$$

Consider a small object in the vector field, it doesn't rotate(since curl $\mathbf{F} = \mathbf{0}$ everywhere), it just move in circular, along the vector field.



This definition is optional.

Laplacian Operator

$$\begin{aligned}\nabla^2 = \nabla\cdot\nabla &= \left(\mathbf{i}\frac{\partial}{\partial x}+\mathbf{j}\frac{\partial}{\partial y}+\mathbf{k}\frac{\partial}{\partial z}\right)\cdot\left(\mathbf{i}\frac{\partial}{\partial x}+\mathbf{j}\frac{\partial}{\partial y}+\mathbf{k}\frac{\partial}{\partial z}\right)\\ &= \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\end{aligned}$$

∇^2 is a **scalar** differential operator. Note that

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}+\frac{\partial^2 f}{\partial z^2}$$

$$\nabla^2\mathbf{F} = \nabla^2 F_1\mathbf{i}+\nabla^2 F_2\mathbf{j}+\nabla^2 F_3\mathbf{k}$$

2.3 Vector differential identities

Let ϕ, ψ are scalar fields and $\mathbf{F}$ and $\mathbf{G}$ are vector fields, then

(a) $\nabla(\phi\psi) = \phi\nabla\psi + \psi\nabla\phi$

(b) $\nabla \cdot (\phi\mathbf{F}) = \nabla\phi \cdot \mathbf{F} + \phi(\nabla \cdot \mathbf{F})$

(c) $\nabla \times (\phi\mathbf{F}) = \nabla\phi \times \mathbf{F} + \phi(\nabla \times \mathbf{F})$

(d) $\nabla \cdot (\mathbf{F} \times \mathbf{G}) = (\nabla \times \mathbf{F}) \cdot \mathbf{G} - \mathbf{F} \cdot (\nabla \times \mathbf{G})$

(e) $\nabla \cdot (\nabla \times \mathbf{F}) = 0$

(f) $\nabla \times (\nabla\phi) = 0$

(g) $\nabla \times (\nabla \times \mathbf{F}) = \nabla(\nabla \cdot \mathbf{F}) - \nabla^2\mathbf{F}$

[Proof.] to be added.

3 Line Integral

Motivation: given a rope(parametrized space curve) $\mathbf{r}(t), a \leq t \leq b$, if the density at point (x, y, z) is given by function $\rho = f(x, y, z)$, we want to find the mass of this rope.

$$\int_a^b f(x, y, z)\ ds = \int_C f(x, y, z)\ ds$$

Recall when we computing arc length in Chapter 11, we knew that:

$$ds = ||\mathbf{r}'(t)||dt$$

Therefore, to calculate line integral $\int_C f(x, y, z)\ ds$, we only need to know:

1. $f(x, y, z)$
2. Region C : $\mathbf{r}(t) = (x(t), y(t), z(t)),\ a \leq t \leq b$

This example shows how to find line integral.

[Example.] Find $\int_C xy^4 ds, C$ is the right half of the circle $x^2 + y^2 = 16$.

[Solution.] In order to do this integral, we need the parametric form of the path C. Let $x = 4\cos t, t = 4\sin t$, the right half of the circle means $t \in [-\pi/2, \pi/2]$.

Hence the parametric equation of the curve C is 4 $\mathbf{r}(t) = 4\cos t\mathbf{i} + 4\sin t\mathbf{j}$ with $t \in [-\pi/2, \pi/2]$, then

$$\mathbf{r}'(t) = -4\sin t\mathbf{i} + 4\cos t\mathbf{j} \quad \text{and} \quad \|\mathbf{r}'(t)\| = 4$$

Thus with $ds = ||\mathbf{r}'(t)||dt$,

$$\int_C xy^4 ds = \int_{-\pi/2}^{\pi/2} f(\mathbf{r})\, \|\mathbf{r}'(t)\|\, dt = \int_{-\pi/2}^{\pi/2} \left[4^5 \cos t \sin^4 t\right](4)dt = \frac{2 \cdot 4^6}{5}$$

Note that there are infinitely many ways to parametrize the curve C, for example, if we had parameterized C as

$$\mathbf{r}(t) = \sqrt{16 - t^2}\mathbf{i} + t\mathbf{j} \quad \text{where} \quad -4 \leqslant t \leqslant 4$$

Then

$$\mathbf{r}'(t) = -\frac{t}{\sqrt{16 - t^2}}\mathbf{i} + \mathbf{j}, \qquad \|\mathbf{r}'(t)\| = \sqrt{\frac{16}{16 - t^2}}$$

$$\int_C xy^4 ds = \int_C f(\mathbf{r})\, \|\mathbf{r}'(t)\|\, dt = \int_{-4}^{4} \sqrt{16 - t^2} \times t^4 \times \sqrt{\frac{16}{16 - t^2}} dt = \frac{2 \cdot 4^6}{5}$$

Thus the line integral is *independent of parametrization* of the curve C.

So far, we have been doing integration w.r.t. s, but we can also carry the integration w.r.t. x,

$$\int_C f(\mathbf{r}(t))\ dx$$

This example shows how to integrate w.r.t. x, y, z

[Example.] $f(\mathbf{r}(t)) = f(x, y, z) = xy + z$, $C : \mathbf{r}(t) = (x(t), y(t), z(t)) = (t^2, t^3, t), 0 \leq t \leq 1$

[Solution.] Since $dx = 2t\ dt,\ dy = 3t^2\ dt,\ dz = dt,$

$$\begin{aligned}
\int_C f(\mathbf{r}(t))\ dx &= \int_0^1 (t^2 \cdot t^3 + t)(2t)\ dt = \int_0^1 (2t^6 - 2t^2)\ dt = \cdots \\
\int_C f(\mathbf{r}(t))\ dy &= \int_0^1 (t^5 + t)(3t^2)\ dt = \cdots \\
\int_C f(\mathbf{r}(t))\ dz &= \int_0^1 (t^5 + t)\ dt = \cdots
\end{aligned}$$

But why are we doing this? Consider given $f(\mathbf{r}(t))$ and $C : \mathbf{r}(t)$, we integrate w.r.t. x, y, z, respectively:

$$\int_C f(\mathbf{r}(t))\ dx \qquad \int_C g(\mathbf{r}(t))\ dy \qquad \int_C h(\mathbf{r}(t))\ dz$$

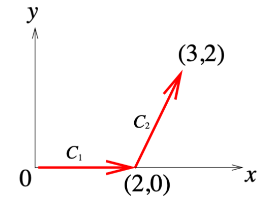
The summation of above three gives:

$$\begin{aligned}
&\int_C [f(\mathbf{r}(t))\ dx + g(\mathbf{r}(t))\ dy + h(\mathbf{r}(t))\ dz] \\
=&\int_C (f, g, h) \cdot (dx, dy, dz) \\
=&\boxed{\int_C \mathbf{F} \cdot d\mathbf{r}}
\end{aligned}$$

where $\mathbf{F}(\mathbf{r}) = (f(\mathbf{r}), g(\mathbf{r}), h(\mathbf{r}))$ is the given vector field.

The following two examples shows usage of integration w.r.t. x, y, z.

[Example.] $\int_C xydx + (x-y)dy, C$ consists of line segments from $(0,0)$ to $(2,0)$ and from $(2,0)$ to $(3,2)$.



[Solution.] We need the parametric function of C_1 and C_2, they are straight lines. Recall in Chapter 10 we know the parameterized curve of straight lines can be written as:

$$\mathbf{r}(t) = (1-t)\mathbf{r}_1 + t\mathbf{r}_2,\ 0 \le t \le 1$$

Hence

$$\begin{aligned} C1:\ \mathbf{r}(t) &= (1-t)(0,0) + t(2,0) = (2t, 0) = (x(t), y(t)),\ 0 \le t \le 1 \\ C2:\ \mathbf{r}(t) &= (1-t)(2,0) + t(3,2) = (2+t, 2t) = (x(t), y(t)),\ 0 \le t \le 1 \end{aligned}$$

Thus the integration is:

$$\begin{aligned} \int_C xydx + (x-y)dy &= \int_{C_1} xydx + (x-y)dy + \int_{C_2} xydx + (x-y)dy \\ &= \int_0^1 (2t)(0)2dt + (2t-0)(0) + \int_0^1 (2+t)(2t)dt + (2+t-2t)2dt \\ &= \int_0^1 (4t + 2t^2 + 2 + t - 2t)dt = \cdots \end{aligned}$$

Notice that: if $\mathbf{F}(\mathbf{r}) = (xy, x-y)$, and $\mathbf{r}_1 = (2t, 0), \mathbf{r}_2 = (2+t, 2t)$, the result above is exactly

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$$

Again, there are many ways to parameterized the curve, for example,

$C_1 : (x, 0),\ 0 \le x \le 2, \qquad C_2 : (x, 2x-4),\ 2 \le x \le 3.$

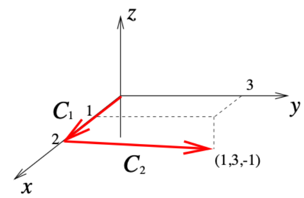
Then

$$\begin{aligned} \int_C xydx + (x-y)dy &= \int_{C_1} [xydx + (x-y)dy] + \int_{C_2} [xydx + (x-y)dy] \\ &= \int_0^2 0dx + \int_2^3 \left(2x^2 - 4x\right) dx + \int_2^3 (-x+4)2dx = \cdots \end{aligned}$$

[Example.]

$\mathrm{I} = \displaystyle\int_C yz\,dx + xz\,dy + xy\,dz$, C consists of line segments from $(0,0,0)$ to $(2,0,0)$, and from $(2,0,0)$ to $(1,3,-1)$.

[*Hint*: $\mathbf{r}(t) = (1-t)\mathbf{r}_0 + t\mathbf{r}_1$, where $0 \leqslant t \leqslant 1$.]



Let $C = C_1 + C_2$, where

C_1 : $(0,0,0)$ to $(2,0,0)$ $\Rightarrow$ $x = 2t$, $y = z = 0$, where $0 \leqslant t \leqslant 1$.

C_2 : $(2,0,0)$ to $(1,3,-1)$ $\Rightarrow$ $x = -t + 2$, $y = 3t$, $z = -t$, where $0 \leqslant t \leqslant 1$.

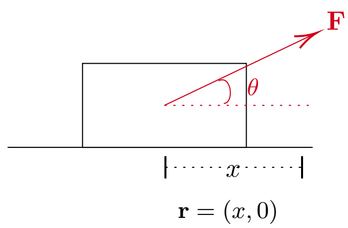
Then

$$\mathrm{I} = 0 + \int_0^1 \left[(3t^2) + 3(t^2 - 2t) - 3(2t - t^2)\right]\,dt =$$

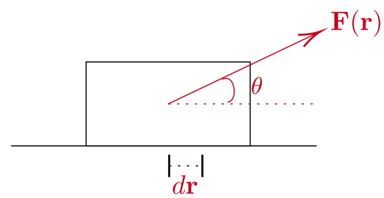
The final answer is -3.

4 Line Integration in Vector Fields

We have talked so much about line integration above. However, this Chapter is called "Vector Field", so how does line integration relate to vector field? Consider a force pulling a box on the ground:



$$W = ||\mathbf{F}|| \cos\theta \cdot x$$
$$= \mathbf{F} \cdot \mathbf{r}$$

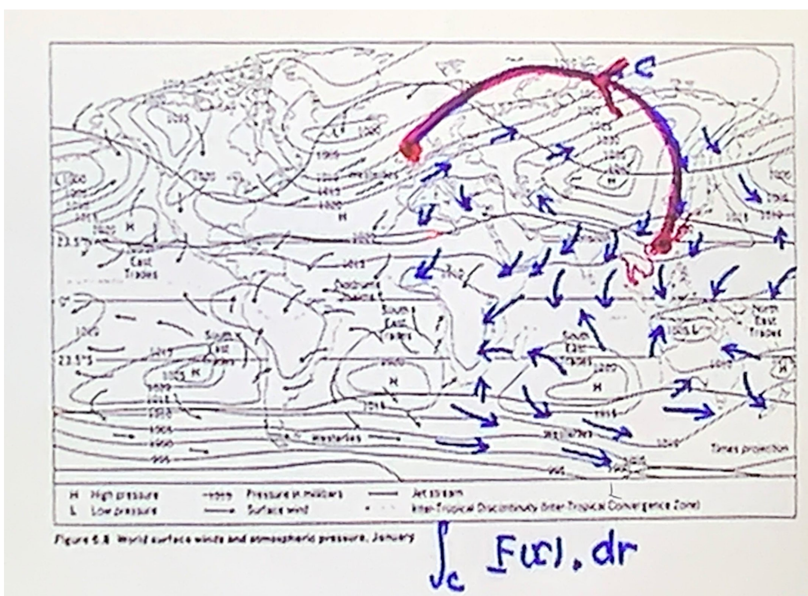


$$dW = \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r}$$
$$W = \int_a^b \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r}$$

On the left side, if the force is always the same, then total work done is $W = \mathbf{F} \cdot \mathbf{r}$, where $\mathbf{r}$ represents the direction and distance of moving. However, if the force is changing, i.e., $\mathbf{F}(\mathbf{r})$ varies for different $\mathbf{r}$, then the total work done is the integration given on the right.

You may have noticed the total work done is just the line integration we have discussed above.

Moreover, for a plane flying from one place to another, if we know the directions and speed of wind at any point on the map, the line integration on its path $\mathbf{r}$ is the total work done on plane, or, approximately how much oil is required.



The following two examples show that sometimes the line integral depends on path, while sometimes it does not.

[Example.] Find $\int_C \mathbf{F}\cdot d\mathbf{r}$, where $\mathbf{F}(x,y) = e^{x-1}\mathbf{i} + xy\mathbf{j}$ and C is given by

(a) $\mathbf{r}(t) = t^2\mathbf{i} + t^3\mathbf{j}; \quad 0 \leqslant t \leqslant 1.$
(b) $\mathbf{r}(t) = t\mathbf{i} + t\mathbf{j}; \quad 0 \leqslant t \leqslant 1.$

[Solution.]

(a)

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{r} &= \int_0^1 \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)dt = \int_0^1 \mathbf{F}\left(t^2,t^3\right)\cdot\mathbf{r}'(t)dt \\ &= \int_0^1 \left(e^{t^2-1}\mathbf{i} + t^5\mathbf{j}\right)\cdot\left(2t\mathbf{i} + 3t^2\mathbf{j}\right)dt \\ &= \int_0^1 \left(2te^{t^2-1} + 3t^7\right)dt = \left[e^{t^2-1} + \frac{3}{8}t^8\right]_0^1 = \frac{11}{8} - \frac{1}{e}\end{aligned}$$

(b)

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{r} &= \int_0^1 \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)dt = \int_0^1 \mathbf{F}(t,t)\cdot\mathbf{r}'(t)dt \\ &= \int_0^1 \left(e^{t-1}\mathbf{i} + t^2\mathbf{j}\right)\cdot(\mathbf{i}+\mathbf{j})dt \\ &= \int_0^1 \left(e^{t-1} + t^2\right)dt = \frac{4}{3} - \frac{1}{e}\end{aligned}$$

Note that the line integral depends on the path.

[Example.] Find $\int_C \mathbf{F}\cdot d\mathbf{r}$, where $\mathbf{F}(x,y) = y\mathbf{i} + x\mathbf{j}$ and C is given by

(a) $\mathbf{r}(t) = t\mathbf{i} + t\mathbf{j}; \quad 0 \leqslant t \leqslant 1.$ (b) $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j}; \quad 0 \leqslant t \leqslant 1$ (c) $\mathbf{r}(t) = t\mathbf{i} + t^3\mathbf{j}; \quad 0 \leqslant t \leqslant 1.$

[Solution.]

(a) $\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^1 (t\mathbf{i}+t\mathbf{j})\cdot(\mathbf{i}+\mathbf{j})dt = \int_0^1 2t dt = 1$

(b) $\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^1 \left(t^2\mathbf{i}+t\mathbf{j}\right)\cdot(\mathbf{i}+2\mathbf{j})dt = \int_0^1 3t^2 dt = 1$

(c) $\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^1 \left(t^3\mathbf{i}+t\mathbf{j}\right)\cdot\left(\mathbf{i}+3t^2\mathbf{j}\right)dt = \int_0^1 4t^3 dt = 1$

Note that this integral does not depend on path.

5 Conservative Vector Fields

So why different vector field cause different results in above examples?

Recall that if $\int_a^b f(x)dx = F(b) - F(a)$ (that the integral only depends on the *end points*) only if *anti-derivative* exists, otherwise, we cannot integrate it and hence cannot express it only with end points.

For example, $\int_1^2 e^x dx = e^2 - e^1$, but for $\int_1^2 e^{x^2} dx$, we cannot find it.

Similarly, if $\mathbf{F}(\mathbf{r}) = f(\mathbf{r})\mathbf{i} + g(\mathbf{r})\mathbf{j} + h(\mathbf{r})\mathbf{k}$ is the gradient of a function $\phi(\mathbf{r})$ on S, i.e.,

$$\mathbf{F}(\mathbf{r}) = \nabla\phi(\mathbf{r}) \quad \Rightarrow \quad f(\mathbf{r}) = \frac{\partial \phi}{\partial x}, \quad g(\mathbf{r}) = \frac{\partial \phi}{\partial y} \quad \text{and} \quad h(\mathbf{r}) = \frac{\partial \phi}{\partial z}$$

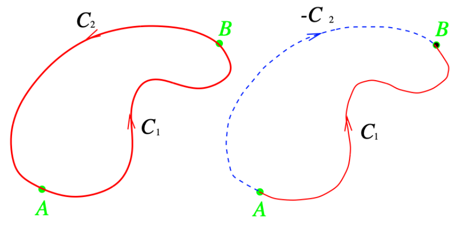
In that case, we say $\mathbf{F}(\mathbf{r})$ is a **conservative field** and ϕ is a *(scaler) potential function* of $\mathbf{F}$ on S.

Then

$$\begin{aligned}
\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} &= \int_a^b \nabla\phi(\mathbf{r}(t)) \cdot \mathbf{r}'(t)dt \\
&= \int_a^b \left(\frac{\partial \phi}{\partial x}\frac{dx}{dt} + \frac{\partial \phi}{\partial y}\frac{dy}{dt} + \frac{\partial \phi}{\partial z}\frac{dz}{dt} \right) dt \\
&= \int_a^b \frac{d\phi}{dt} dt = \phi(x(b), y(b), z(b)) - \phi(x(a), y(a), z(a)) \\
&= \phi(\mathbf{r}(b)) - \phi(\mathbf{r}(a))
\end{aligned}$$

This depends only *on the endpoints* $\mathbf{r}(b)$ and $\mathbf{r}(a)$, not on the curve C.

This can also be expressed in circular integration, $\int_C \mathbf{F} \cdot d\mathbf{r}$ is *independent* of path in S if and only if $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ for *every closed path* C in S. Since if the integration depends only on end points, then integrate from a to b and then from b back to a will certainly gives result 0.



$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} - \int_{-C_2} \mathbf{F} \cdot d\mathbf{r} = 0$$

How to determine whether a vector field is *conservative* or not?

A continuously vector field $\mathbf{F}$ defined in a **simply-connected domain** S is conservative if and only if, it possesses *any one* of the following properties.

(i) It is the gradient of a scalar function, $\mathbf{F}(\mathbf{r}) = \nabla\phi(\mathbf{r})$.

(ii) Its line integral along any regular curve extending from a point P to a point Q is independent of the path.

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_{C_3} \mathbf{F} \cdot d\mathbf{r} = \cdots$$

(iii) Its line integral around any regular closed curved is zero, i.e. $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$.

(iv) $\nabla \times \mathbf{F} = \mathbf{0}$. (since $\nabla \times (\nabla \times \phi) = \mathbf{0}$)

Note that $(i) \Rightarrow (ii) \Rightarrow (iii)$, and it's trivial that $(iii) \Rightarrow (ii)$

For $(ii) \Rightarrow (i)$, this can be proved only if $\mathbf{F}$ us continuous on an *open connected region.* Proof is not required here.

For (iv), this is a **necessary** condition for the existence of a potential function ϕ, in other words, this is **not sufficient**, we also need to guarantee that the domain S must be **open and simply connected.**

What is **open and simply-connected** region?

Firstly, it must be *connected*, it cannot be divided into two parts that are separate. For example,

(1) $\mathbf{F}(\mathbf{r}) = \left(\frac{x}{x^2+y^2+z-1}, \frac{y}{x^2+y^2+z-1}, \frac{z^2}{x^2+y^2+z-1}\right)$ is defined on $\mathbb{R}^3 \backslash \{x^2+y^2+z=1\}$. This is not connected, because the region is divided into two parts by a "rice bowl".

(2) Also, $\mathbb{R}^3 \backslash \{x^2+y^2+z^2=1\}$ is not connected, because the region is divided into two parts by a ball shell.

Then, for *any curve* inside the region, if *at least one surface* inside the region with the curve *as boundary* does not have a hole on it, then the region is simply-connected. For example,

(1) $\mathbb{R}^2 \backslash \{(0,0)\}$ is not simply-connected. Consider any curve surround the origin, the only surface that with the curve as boundary is the "disk" bounded by the curve, and unfortunately it has a hole in the middle.

(2) $\mathbb{R}^3 \backslash \{(0,0,0)\}$ is simply-connected. For any curve inside the region, even if you choose a curve on xy-plane and contains $(0,0,0)$, we can easily find a surface bounded by the curve that does not contain the origin, for example a downside "rice bowl".

(3) $\mathbb{R}^3 \backslash \{x=y=0\}$ is not simply-connected. For example, if we choose a circle on $xy-$plane, like $x^2+y^2=1$, then *any* surface bounded by the curve must through $z-$axis, and since we've removed the $z-$axis, all those surfaces always have a hole on it.

(4) A donut is not simply-connected, we can choose any curve that surround the middle "hole".

This example shows how to find potential for a conservative vector field.

[Example.] Determine whether or not $\mathbf{F}(x,y) = (2x-3y)\mathbf{i} + (2y-3x)\mathbf{j}$ is a conservative vector field. If it is, find a function ϕ such that $\mathbf{F} = \nabla\phi$.

[Solution.] Notice that the curl of $\mathbf{F}$:

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial x} & 0 \\ 2x-3y & 2y-3x & 0 \end{vmatrix} = \mathbf{0}$$

And, $\mathbf{F}(x,y)$ is defined in $\mathbb{R}^2$, which is **open and simply-connected** domain. Therefore, $\mathbf{F}$ is conservative and $\mathbf{F} = \nabla\phi$ for some potential function ϕ.

To find f,

$$\frac{\partial \phi}{\partial x} = 2x - 3y \qquad (1)$$
$$\frac{\partial \phi}{\partial y} = 2y - 3x \qquad (2)$$

For (1), integrate w.r.t. x, $\phi(x,y) = x^2 - 3xy + g(y)$, differentiate it w.r.t. y, $\dfrac{\partial \phi}{\partial y} = 0 - 3x + g'(y)$.

Compare this equation with (2), we have $g'(y) = 2y \Rightarrow g(y) = y^2 + C$

Therefore

$$\phi(x,y) = x^2 - 3xy + y^2 + C.$$

This example shows curl $\mathbf{F} = \mathbf{0}$ is not sufficient for $\mathbf{F}$ to be conservative.

[Example.] Let $\mathbf{F}(x, y) = \dfrac{-y\mathbf{i} + x\mathbf{j}}{x^2 + y^2} = f(x, y)\mathbf{i} + g(x, y)\mathbf{j}$ (see also Ex. 1.2).

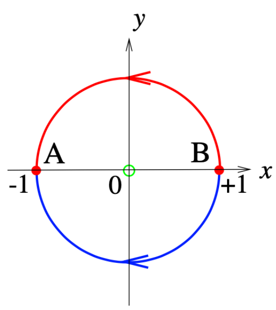
(a) Show that $\dfrac{\partial f}{\partial y} = \dfrac{\partial g}{\partial x}$.

(b) Show that $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r}$ is not independent of path.

[Solution.] (a)

$$\frac{\partial f}{\partial y} = \frac{y^2 - x^2}{(x^2 + y^2)^2} = \frac{\partial g}{\partial x}$$

(b) Notice $\mathbf{F}$ is defined on $\mathbb{R}^2 \backslash \{\mathbf{0}\}$, so it is not conservative. To show that it is dependent of path, we only need to given an counterexample. Consider on these two paths:



$C_1 : \quad x = \cos\theta, \quad y = \sin\theta$, with $\theta = 0$ to $\theta = \pi$
$C_2 : \quad x = \cos\theta, \quad y = \sin\theta$, with $\theta = 2\pi$ to $\theta = \pi$

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_0^\pi (-\sin\theta, \cos\theta) \cdot (-\sin\theta, \cos\theta) d\theta = \int_0^\pi d\theta = \pi$$
$$\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_{2\pi}^\pi d\theta = -\pi \neq \int_{C_1} \mathbf{F} \cdot d\mathbf{r}$$

Therefore, $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r}$ is not independent of path.

But if $\mathbf{F}$ is defined on $x > 0$, it will become a conservative field. ($x > 0$ is open and simply-connected.)

In summary, to find a line integral, you have following methods:

1. Carry out the integration directly:
$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_z^b g(t)dt \cdots$$
2. If $\nabla \times \mathbf{F} \neq \mathbf{0}$, nothing you can do, except for carrying the integration as above.
3. If $\nabla \times \mathbf{F} = \mathbf{0}$, there are two simpler ways:
   (a) Find ϕ such that $\mathbf{F} = \nabla\phi$, then
   $$\int_C \mathbf{F} \cdot d\mathbf{r} = \phi(\mathbf{r}(b)) - \phi(\mathbf{r}(a))$$
   (b) Carry the integration along a straight line path, i.e.
   $$\mathbf{r}(t) = (1-t)\mathbf{r}_1 + t\mathbf{r}_2,\ 0 \leq t \leq 1$$

This is a past final question in year 1994.

[Example.] For what values of b and c will

$$\mathbf{F}(x,y,z) = \left(y^2 + 2czx\right)\mathbf{i} + y(bx+cz)\mathbf{j} + \left(y^2 + cx^2\right)\mathbf{k}$$

have potential functions? For each pair of these values of b and c, find a potential function for $\mathbf{F}$.

[Solution.] For $\mathbf{F}(\mathbf{r})$ to have a potential function, iff $\nabla \times \mathbf{F}(\mathbf{r}) = \mathbf{0}$.

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{i} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y^2 + 2czx & y(bx+cz) & y^2 + cx^2 \end{vmatrix}$$

For the $\mathbf{i}$ component, $2y = yc \quad \Rightarrow \quad c = 2$

For the $\mathbf{j}$ component, $2cx = 2cx \quad \Rightarrow \quad c$ cannot be determined.

For the $\mathbf{k}$ component, $by = 2y \quad \Rightarrow \quad b = 2$

$\therefore b = 2$ and $c = 2$. In the case, we have

$$\mathbf{F}(\mathbf{r}) = \left(y^2 + 4zx\right)\mathbf{i} + y(2x+2z)\mathbf{j} + \left(y^2 + 2x^2\right)\mathbf{k} = \nabla\phi$$

Therefore

$$\frac{\partial \phi}{\partial x} = y^2 + 4zx \qquad (1)$$
$$\frac{\partial \phi}{\partial y} = 2xy + 2yz \qquad (2)$$
$$\frac{\partial \phi}{\partial z} = y^2 + 2x^2 \qquad (3)$$

From (1), we have

$$\phi(x,y,z) = xy^2 + 2x^2z + p(y,z) \qquad (4)$$
$$\phi_y(x,y,z) = 2xy + p_y(y,z) \qquad (5)$$

Comparing (2) and (5), we have $p_y(y,z) = 2yz \Rightarrow p(y,z) = y^2z + q(z)$. From (4), we have

$$\phi(x,y,z) = xy^2 + 2x^2z + y^2z + q(z) \qquad (6)$$
$$\phi_z(x,y,z) = 2x^2 + y^2 + q'(z) \qquad (7)$$

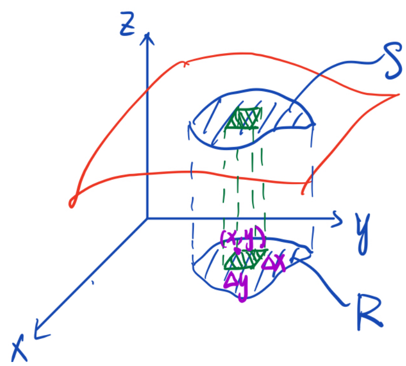
Comparing (7) and (3), we have $q'(z) = 0 \quad \Rightarrow \quad q(z) = k$ (constant).

$$\therefore \quad \phi(x,y,z) = xy^2 + 2x^2z + y^2z + k$$

6 Surface Integrals of Vector Fields

6.1 Surface Integrals

Recall in Chapter 14, we discussed about Surface Integrals:



The area of region S is given by:

$$\boxed{A = \iint_R \sqrt{(f_x)^2 + (f_y)^2 + 1}\; dA}$$

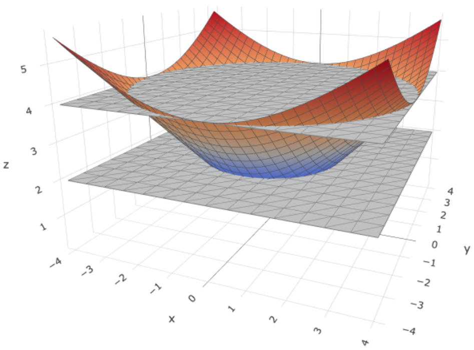
If we have a density function $\rho(x, y, z)$ defined on every point of the surface, then the mass of the surface is given by:

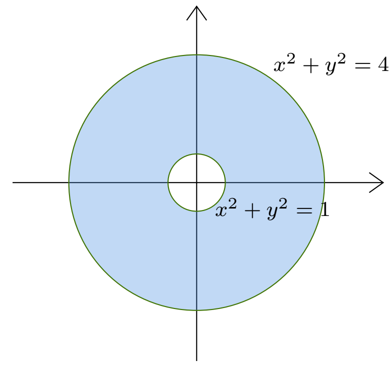
$$\boxed{M = \iint_R \rho(x, y, f(x, y))\sqrt{(f_x)^2 + (f_y)^2 + 1}\; dA}$$

This example shows basic computation of surface integration with density function.

[Example.] Find the mass of a thin funnel in the shape of a cone $z = \sqrt{x^2+y^2}, z \in [1,4]$, if its density function if $\rho(x,y,z) = 10 - z$.

[Solution.] The cone bounded by $z = 1$ and $z = 4$ and its projection onto $xy-$plane is shown below:





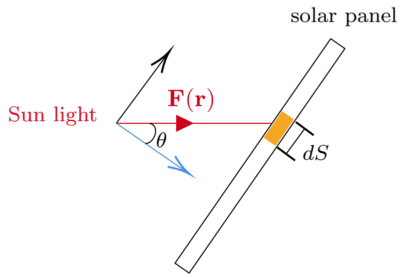
$z = f(x,y) = (x^2+y^2)^{\frac{1}{2}},\ f_x = \dfrac{x}{\sqrt{x^2+y^2}},\ f_y = \dfrac{y}{\sqrt{x^2+y^2}},$

hence $ds = \sqrt{1+(f_x)^2+(f_y)^2}dA = \sqrt{2}dA$,

$$\begin{aligned}\iint_S \rho(x,y,z)dS &= \iint_R \left(10 - \sqrt{x^2+y^2}\right)\sqrt{1+(f_x)^2+(f_y)^2}dA \\ &= \sqrt{2}\iint_R (10 - \sqrt{x^2+y^2})dA \\ &= \sqrt{2}\int_0^{2\pi}\int_1^4 (10-r)\ rdrd\theta = 108\sqrt{2}\pi\end{aligned}$$

6.2 Surface Integrals of vector fields: flux

Consider when sun light shines on a solar panel, only the energy that perpendicular to the panel will be received.(blue arrow below)



For a small area dS, the solar energy that it receives is:

$$
\begin{aligned}
&||\mathbf{F}(\mathbf{r})|| \cos\theta \cdot dS \\
=&||\mathbf{F}|| \cos\theta \cdot dS \cdot ||\widehat{\mathbf{n}}|| \\
=&(\mathbf{F} \cdot \widehat{\mathbf{n}}) dS
\end{aligned}
$$

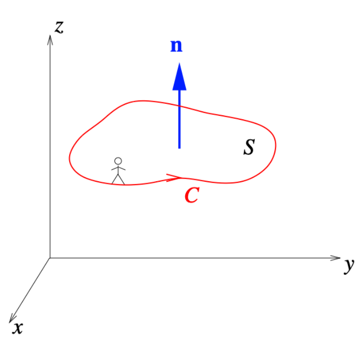
Thus the *total* solar energy on this panel is:

$$\boxed{\iint_S (\mathbf{F} \cdot \widehat{\mathbf{n}}) dS}$$

Suppose $\mathbf{F}(x, y, z)$ is a continuous vector field defined on a smooth, *oriented* surface S, the integral above is called the **flux** of $\mathbf{F}$ across S.

How to determine the **orientation** of the surface S, or, which $\mathbf{n}$ should we choose?

A non-closed surface S(i.e., the surface *not close a volume*) bounded by a *closed smooth curve* C, and the **positive orientation** around C means the surface will always be on your *left*, then your head pointing in the direction of $\mathbf{n}$.



This page summarize the steps to find flux.

(1) Find the normal vector to the surface $S : z = f(x, y)$.

Let $G(x, y, z) = z - f(x, y) = 0$ (constant), so this is a level set in 3D, hence

$$\begin{aligned}\mathbf{n} &= \nabla G = (G_x, G_y, G_z)\\ &= (-f_x, -f_y, 1)\\ \widehat{\mathbf{n}} &= \frac{(-f_x, -f_y, 1)}{\sqrt{1 + (f_x)^2 + (f_y)^2}}\end{aligned}$$

(2) Using $ds = \sqrt{1 + (f_x)^2 + (f_y)^2} dA$.

$$\begin{aligned}\iint_S (\mathbf{F} \cdot \widehat{\mathbf{n}}) dS &= \iint_R (F_1, F_2, F_3) \cdot \frac{(-f_x, -f_y, 1)}{\sqrt{1 + (f_x)^2 + (f_y)^2}} \cdot \sqrt{1 + (f_x)^2 + (f_y)^2} dA\\ &= \iint_R (F_1, F_2, F_3) \cdot (-f_x, -f_y, 1) dA\end{aligned}$$

This is an complete example of computing flux.

[Example.] Find the flux of $\mathbf{F} = y^3\mathbf{i} + z^2\mathbf{j} + x\mathbf{k}$ downward through the part of the surface $z = 4 - x^2 - y^2$ that lies above the plane $z = 2x + 1$.

[Solution.] First we need to find the curve of intersection between $z = 4 - x^2 - y^2$ and the plane $z = 2x + 1$,

$$\begin{aligned}4 - x^2 - y^2 &= 2x + 1\\ x^2 + 2x + 1 + y^2 &= 4\\ (x + 1)^2 + y^2 &= 2^2\end{aligned}$$

Next, find the normal to the surface S, let $G(x, y, z) = 4 - x^2 - y^2 - z = 0$ (constant), this is a level set in 3D, hence

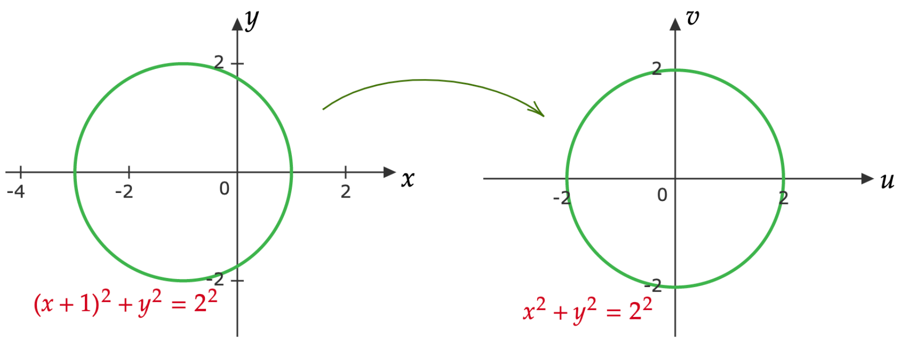
$$\begin{aligned}\mathbf{n} &= \nabla G = (G_x, G_y, G_z) = (-2x, -2y, -1)\\ \widehat{\mathbf{n}} &= \frac{(-2x, -2y, -1)}{\sqrt{1 + 4x^2 + 4y^2}}\end{aligned}$$

Then, using $ds = \sqrt{1 + (f_x)^2 + (f_y)^2} dA = \sqrt{1 + 4x^2 + 4y^2} dA$

$$\begin{aligned}\iint_S (\mathbf{F} \cdot \widehat{\mathbf{n}}) dS &= \iint_R \left(y^3, (4 - x^2 - y^2)^2, x\right) \cdot \frac{(-2x, -2y, -1)}{\sqrt{1 + 4x^2 + 4y^2}} \cdot \sqrt{1 + 4x^2 + 4y^2} dA\\ &= -\iint_R \left(2xy^3 + 2y(4 - x^2 - y^2)^2 - x\right) dA = -\iint_R x dA\end{aligned}$$

Notice the last step is because both $2xy^3$ and $2y(4-x^2-y^2)^2$ is *odd* in y, and the region R, $(x+1)^2+y^2=2^2$, is symmetric in y, hence the two items equals to 0, after integration.

However, it is still difficult to find $\iint_R x dA$ directly, since the area is not good, as you can see below. Then we consider using *substitution*, or *Jacobian*, to transform it, into a circle centered at origin.



Let $u=x+1, v=y$, then

$$\mathbf{J}=\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix}=\begin{vmatrix}1 & 0\\ 0 & 1\end{vmatrix}=1$$

$$-\iint_R x dA_{xy}=-\iint_{R_{uv}}(u-1)\mathbf{J}dA_{uv}=-\iint_{R_{uv}} u dA_{uv}+\iint_{R_{uv}} dA_{uv}=0+\iint_{R_{uv}} dA_{uv}=4\pi$$

This is the end of Chapter 15.