MATH 2421 Fall 2021

Probability

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Chapter 1 **Combinatorial Analysis**

1 Principle of Counting

Theorem. *(basic principle of counting) two experiments: one has m outcomes, the other has n outcomes, then together there are mn outcomes of the two experiments.*

Proof. prove by enumerating all (i, j) pairs of possible outcomes. ∎

Theorem. *(generalized BPOC) r experiments: each has $n_1, n_2 \cdots n_r$ outcomes, in total $n_1 \cdot n_2 \cdots n_r$ outcomes.*

Example: How many different 7-place license plates are possible, if first 3 are letters, final 4 are numbers, don't allow repetition?

Solution: $26 \times 25 \times 24 \times 10 \times 9 \times 8 \times 7 = 78,624,000$

2 Permutation

Theorem. *Permutation: different ordered arrangements. n distinct objects, total number of different arrangement is $n!$, with the convention $0! = 1$.*

Theorem. *n objects, of which n_1 are alike, n_2 are alike, $\cdots$, n_r are alike, there are*

$$\frac{n!}{n_1!n_2!\cdots n_r!}$$

different permutations.

Example: Ways to rearrange "Mississippi"?

Solution: $\dfrac{11!}{1!4!4!2!} = 34,650.$

Theorem. *For n people sitting in a circle: there are*

$$\frac{n!}{n} = (n-1)!$$

different arrangements.

Example: How many different ways can n different pearls string in a necklace?

Solution: $\dfrac{(n-1)!}{2}$, since the necklace can be flipped.(mirrored)

3 Combinations

Theorem. *n distinct objects, choose r to form a group:*

$$_nC_r = \binom{n}{r} = \frac{n!}{r!(n-r)!}$$

Some properties:

$$\binom{n}{r} = \binom{n}{n-r}, \; \binom{n}{0} = \binom{n}{n} = 1$$

And by convention, when $n \geq 0$ and $r < 0$ or $r > n$, $\binom{n}{r} = 0$.

Example: From a party of 3 from 20 people?

Solution: $\binom{20}{3} = 1140$

Example: What if A and B can't be chosen together?

Solution: (1) minus the situation that they are both chosen: $\binom{20}{3} - \binom{18}{1}\binom{2}{2} = 1122$, or (2) neither of them are chosen plus one of them chosen: $\binom{18}{3} + \binom{18}{2}\binom{2}{1} = 1122$.

Example: m antennas are defective and $n-m$ are functional. The system works if no two defectives are consecutive. How many different linear orderings?

Solution: Consider inserting m defective ones into the space of $n-m$ functional ones. $n-m$ antennas provide $n-m+1$ spaces, so we just choose m of them: $\binom{n-m+1}{m}$.

Theorem. *For $1 \leq r \leq n$,*

$$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$$

.

Proof. (Algebra proof)

$$RHS = \frac{(n-1)!}{(r-1)!(n-r)!} + \frac{(n-1)!}{r!(n-r-1)!} = \frac{(n-1)!r}{(r-1)!(n-r)!r} + \frac{(n-1)!(n-r)}{r!(n-r-1)!(n-r)} = \frac{(n-1)! \cdot [r+(n-r)]}{r!(n-r)!} = LHS$$

(Combinatorial proof)

LHS: number of choosing r balls from n balls. $\binom{n}{r}$

RHS: consider whether to choose the 1st ball:

- choose 1st ball, then choose $r-1$ from remaining $n-1$ balls
- don't choose 1st ball, then choose r from remaining $n-1$ balls

combine the two situations, $\binom{n-1}{r-1} + \binom{n-1}{r}$.

■

Theorem. *(Binomial Theorem) n be nonnegative integer, then*

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

Proof. (Combinatorial Proof) for $x^k y^{n-k}$, it means there are k brackets contribute x, while remaining $n-k$ brackets contribute y. ■

Example: How many subsets does a set of size n have?

Solution: There are $\binom{n}{k}$ subsets of size k, so add them together:

$$\sum_{k=0}^{n} \binom{n}{k} = (1+1)^n = 2^n.$$

Example: Proof: $\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0$.

Solution: Let $x=-1, y=1$, the problem $= [(-1)+1]^n = 0$.

4 Multinomial Coefficients

Example: n distinct items is to be divided into r distinct groups of size $n_1, n_2, \cdots, n_r$, where $\sum n_i = n$. How many different divisions are possible?

Solution: First choose n_1 from n items, then n_2 from remaining $n-n_1$ items, etc., we will get:

$$\begin{aligned}
&\binom{n}{n_1} \cdot \binom{n-n_1}{n_2} \cdots \binom{n-n_1-\cdots-n_{r-1}}{n_r} \\
&= \frac{n!}{n_1!(n-n_1)!} \cdot \frac{(n-n_1)!}{n_2!(n-n_1-n_2)!} \cdots \frac{(n-n_1-\cdots-n_{r-1})!}{n_r!(n-n_1-\cdots-n_r)!} \\
&= \frac{n!}{n_1! n_2! \cdots n_r!}
\end{aligned}$$

Notation: If $n_1 + n_2 + \cdots + n_r = n$, we define $\binom{n}{n_1, n_2, \cdots, n_r}$ by

$$\binom{n}{n_1, n_2, \cdots, n_r} = \frac{n!}{n_1! n_2! \cdots n_r!}$$

Example: 10 children are to be divided into A team and B team, of 5 each. Team A will play in one league and B in another. How many different divisions?

Solution: $\dfrac{10!}{5!5!} = 252$

Example: 10 children are to be divided into two teams of 5 each. How many different divisions?

Solution: Note that this example is different from the previous one, since now the two teams are not distinguishable, no A or B team. So the order of the two teams is irrelevant.

$$\frac{10!}{5!5!} \cdot \frac{1}{2!} = 126$$

Theorem. *(Multinomial Theorem)*

$$(x_1 + x_2 + \cdots + x_r)^n = \sum_{(n_1, \cdots, n_r): n_1 + \cdots + n_r = n} \binom{n}{n_1, n_2, \cdots, n_r} x_1^{n_1} x_2^{n_2} \cdots x_r^{n_r}$$

Proof. Basically the same idea as Binomial Theorem, there are n brackets in total, and we can choose $n_1, \cdots, n_r$ of them to be $x_1, \cdots, x_r$. ■

Example: Distinct positive integer vectors $(x_1, x_2, \cdots, x_r)$ satisfies $\sum x_i = n$?

Solution: Consider n balls, since positive required, we insert $r-1$ boards in *between* them(not both sides), then there will be $n-1$ spaces. The answer is $\binom{n-1}{r-1}$.

Example: Distinct non-negative integer vectors $(x_1, x_2, \cdots, x_r)$ satisfies $\sum x_i = n$?

Solution: Method 1: Let $y_i = x_i + 1$, then

$$x_1 + x_2 + \cdots + x_r = n$$

can be changed into:

$$(y_1 - 1) + (y_2 - 1) + \cdots + (y_r - 1) = n$$

that is,

$$y_1 + y_2 + \cdots + y_r = n + r$$

which is $\binom{n+r-1}{r-1}$.

Method 2: We mix n balls with $r-1$ boards, among those $n+r-1$ items, we choose $r-1$ items to be boards. After boards are chosen, they will automatically divide balls into r parts. This also gives $\binom{n+r-1}{r-1}$.

Example: 20 balls to put into 4 boxes. All balls must be put into boxes, but boxes can be empty. How many different divisions?

Solution: $n = 20, r = 4$, answer is $\binom{20+4-1}{4-1} = \binom{23}{3}$.

Example: 20 balls to put into 4 boxes. You can choose not put some balls into boxes, and boxes can be empty. How many different divisions?

Solution: Consider the balls that not put into boxes as another box. $n = 20, r = 5$, answer is $\binom{20+5-1}{5-1} = \binom{24}{4}$.

This is the end of lecture note. Last modified: Sep 6.