

Section 8.1: Analysis of Variance

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Outline

- ▶ Overview of Analysis of Variance
- ▶ The Basic ANOVA
- ▶ The ANOVA Model
- ▶ Checking Conditions

The Basic Idea of ANOVA

When we ask if a set of sample means gives evidence for **differences among the population means**, what matters is not how far apart the sample means are but how far apart they are **relative to the variability of individual observations**.

Baldi & Moore, 3rd ed., p.606

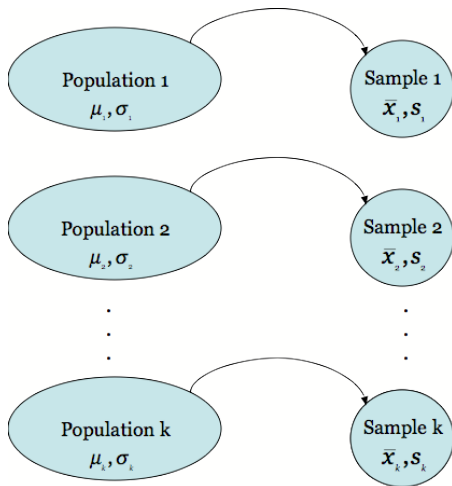
The Basic Idea of ANOVA

The basic idea is to compare measures of variability, both **between** the groups and **within** each group, as a way to assess how different the groups really are.

Lock5, p.540

Analysis of Variance Sampling Model

Draw **simple random samples** from k **independent** populations to compare population means μ_1, μ_2, \dots , and μ_k :



Condition: $\sigma_1 = \sigma_2 = \dots = \sigma_k = \sigma$

Conditions for Applying ANOVA

1. We have k independent SRSs, one from each of k populations.
2. Each of the k populations has a Normal distribution with an unknown mean μ_j .
3. All of the populations have the same standard deviation σ , whose value is unknown.

Case Study

Diet Restriction and Longevity

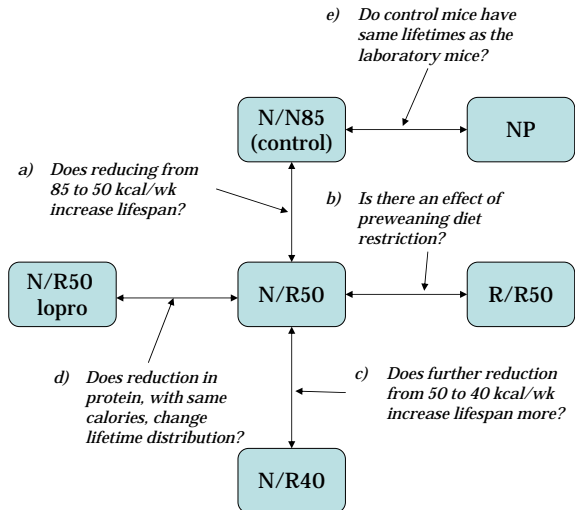
- ▶ Study of the effect of restricting caloric intake on life expectancy.
- ▶ Female mice randomly assigned to one of 6 treatment groups:
 1. **NP**: unlimited nonpurified standard diet for laboratory mice.
 2. **N/N85 (control group)**: fed normally both before and after weaning. Caloric intake controlled at 85 kcal/wk after weaning.
 3. **N/R50**: normal diet before weaning and reduced-calorie diet of 50 kcal/wk after weaning.
 4. **R/R50**: reduced-calorie diet of 50 kcal/wk before and after weaning.
 5. **N/R50 lopro**: normal diet before weaning, restricted diet of 50 kcal/wk after weaning with dietary protein content decreased with advancing age.
 6. **N/R40**: normal diet before weaning, severely reduced-calorie diet of 50 kcal/wk after weaning.
- ▶ Several questions of interest to be addressed.

Case Study

Diet Restriction and Longevity: Study Citation

Weindruch R, Walford RL, Fligiel S, Guthrie D. The retardation of aging in mice by dietary restriction: longevity, cancer, immunity and lifetime energy intake. *J Nutr*. 1986;116(4):641-654. doi:10.1093/jn/116.4.641

Planned Comparisons Among Groups in the Diet Restriction Study



Diet Restriction Study

Exploratory Plots

Use R to create plots to explore the possibility of differences among the means.

The Big Picture of ANOVA

- ▶ **Key question:** Is variability **across** populations greater than variability **within** populations
 - ▶ Variability **between** vs **within**
- ▶ **Hypothesis test:**
 - ▶ $H_0 : \mu_1 = \mu_2 = \dots = \mu_k$ versus
 - ▶ H_a : At least one $\mu_i \neq \mu_j$
- ▶ **Test statistic (F):** ratio of the variability **among** group (or treatment) means over the variability **within** samples.

$$F = \frac{\text{variation among the sample means}}{\text{variation among individuals in the same sample}}$$

- ▶ Large test statistic \Rightarrow evidence against H_0
- ▶ Results are summarized in the ANOVA table.

Notation

Key notation used in calculations for comparing variability
between groups and **within groups**:

k = number of groups

n_i = sample size for group i

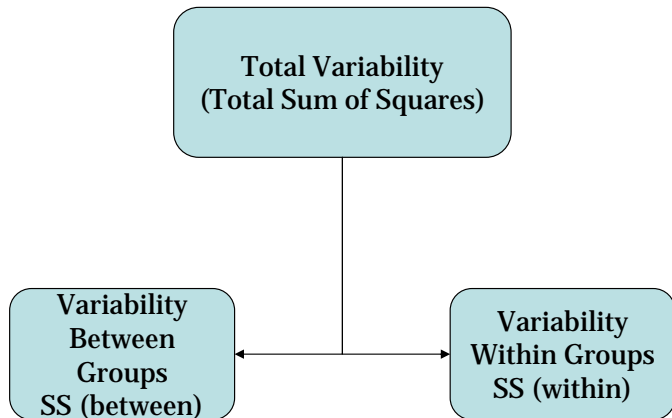
\bar{x}_i = sample mean for group i

s_i = standard deviation for group i

$n = \sum_{i=1}^k n_i$ = total sample size

$\bar{x} = \frac{\sum_{i=1}^k n_i \bar{x}_i}{n}$ = overall mean.

Apportioning Variability



Variability within Groups

Measure **variability within groups** by the sum of squared deviations from the group mean:

$$\text{SSE} = (n_1 - 1)s_1^2 + (n_2 - 1)s_2^2 + \cdots + (n_k - 1)s_k^2$$

Degrees of Freedom

- ▶ The **degrees of freedom within** samples is the sum of degrees of freedom for each sample.

$$\begin{aligned}\text{df}(\text{error}) &= (n_1 - 1) + (n_2 - 1) + \cdots + (n_k - 1) \\ &= n - k\end{aligned}$$

- ▶ This is also equal to the total sample size minus the number of groups.

Mean Square within Groups (Error)

In ANOVA, the **mean square** measures variability as the ratio of **sum of squares** to **degrees of freedom**.

$$\begin{aligned} \text{MSE} = \text{MS}(\text{error}) &= \frac{\text{SSE}}{\text{df}(\text{error})} \\ &= \frac{(n_1 - 1)s_1^2 + \cdots + (n_I - 1)s_k^2}{n - k} \end{aligned}$$

- The **pooled standard deviation** (s_p) estimates the common group-specific standard deviation:

$$s_p = \sqrt{\text{MS}(\text{error})}$$

Variability between Groups

Measure **variability between groups** by SS(between), a weighted sum of squared deviations of group means from the grand mean:

$$\text{SSG} = n_1(\bar{x}_1 - \bar{x})^2 + n_2(\bar{x}_2 - \bar{x})^2 + \cdots + n_k(\bar{x}_k - \bar{x})^2$$

The corresponding **degrees of freedom between groups** is one less than the number of groups:

$$\text{df}(\text{groups}) = k - 1$$

Mean Square between Groups

The mean square between groups, MSG or MS (treatment) is the **ratio** of the sum of squares over the degrees of freedom:

$$\begin{aligned}\text{MSG} &= \frac{\text{SSG}}{\text{df}(\text{groups})} \\ &= \frac{n_1(\bar{x}_1 - \bar{x})^2 + n_2(\bar{x}_2 - \bar{x})^2 + \cdots + n_k(\bar{x}_k - \bar{x})^2}{k - 1}\end{aligned}$$

The F Statistic

- ▶ The F statistic is the ratio of the mean square between to the mean square within.

$$F = \frac{MSG}{MSE}$$

- ▶ If the null hypothesis is true, F has an F distribution with $k - 1$ and $n - k$ degrees of freedom
- ▶ Use the StatKey F distribution web applet to find p -values when necessary or obtain them from R output for ANOVA using the **lm()** command we also used for regression.

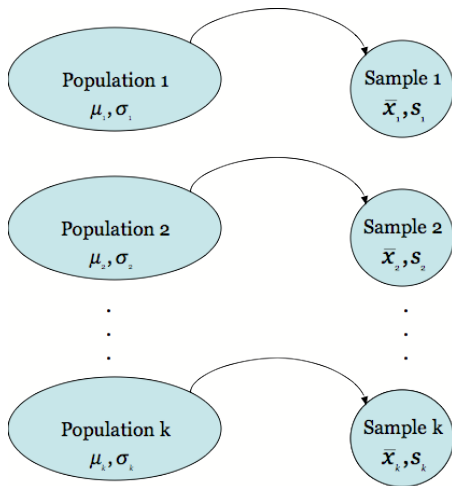
Diet Restriction Study

See [Moodle for R](#) commands and output.

Checking Conditions

Analysis of Variance Sampling Model

Draw samples from k **independent** populations to compare population means μ_1, μ_2, \dots , and μ_k :



Condition: $\sigma_1 = \sigma_2 = \dots = \sigma_k = \sigma$

Standard Conditions for ANOVA

- ▶ **Design conditions:**
 - ▶ **Random samples:** reasonable to consider observations a random sample from respective populations
 - ▶ **Independent samples:** the k samples are independent of each other
- ▶ **Population:**
 - ▶ **Normal:** Population distributions are normal (not crucial if n_i are large and similar)
 - ▶ **Equal standard deviations:**

$$\sigma_1 = \sigma_2 = \cdots = \sigma_k = \sigma$$

Rules of Thumb for Checking ANOVA Conditions

- ▶ Inference requires **random samples**.
- ▶ **Outliers** are always problematic! Plot your data.
- ▶ ANOVA methods are robust if group samples sizes are **similar and not too small**.
- ▶ Ratio of largest sample SD to smallest should not be much greater than **2**.
 - ▶ Biggest **problem** is if sample sizes are **unequal** and SD from a small sample is **much larger** than others.
- ▶ Normality is not critical if sample sizes n_i are **large** and approximately **equal**.