21 ноября.

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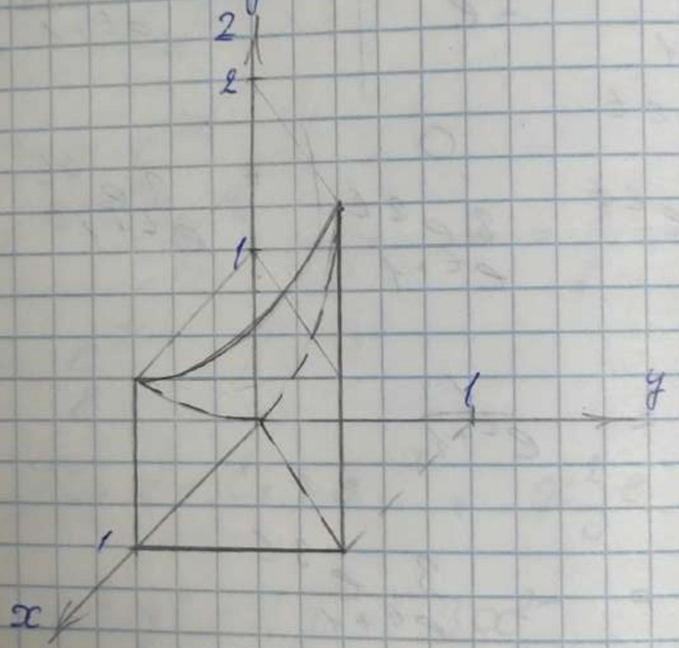
Дз № 7

1. Ω огр. $z = x^2 + y^2$ $y = x$ $x = 1$

$y = 0$ $z = 0$



$$\iiint\limits_{\Omega} f(x, y, z)\, dx\, dy\, dz =$$

$$= \int_0^1 dx \int_0^x dy \int_0^{x^2+y^2} f\, dz$$

1 способ:

2. $V_T = \iiint_T dx\,dy\,dz = \cancel{\int_{-R}^{R} dx \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} dy \int_{\sqrt{x^2+y^2}}^{H} dz} =$

$\cancel{= \int_{-R}^{R} dx \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} (H - \sqrt{x^2+y^2})\, dy}$

$= \int_0^H dz \iint_{D(z)} dx\,dz = \int_0^H dz\, S_{D(z)} = \int_0^H \frac{\pi R^2 z^2}{H^2}\, dz =$

$= \frac{\pi R^2}{H^2} \frac{z^3}{3} \Big|_0^H = \frac{\pi R^2 H^3}{3}$

$0 \le z \le H$, $(x,y) \in D(z):\ x^2 + y^2 \le \frac{R^2 z^2}{H^2}$

2 способ:

$(x,y) \in D:\ x^2 + y^2 \le R^2,\ \frac{H}{R}\sqrt{x^2+y^2} \le z \le H$

$V_T = \iint_D dx\,dy \int_{\frac{H}{R}\sqrt{x^2+y^2}}^{H} dz = \iint_D \left(H - \frac{H}{R}\sqrt{x^2+y^2}\right) dx\,dy =$

$= H \int_0^{2\pi} d\varphi \int_0^R \left(1 - \frac{r}{R}\right) r\, dr =$

$= 2\pi H \left(\frac{r^2}{2} - \frac{r^3}{3R}\right)\Big|_0^R = \frac{2\pi H R^2}{6} = \frac{\pi R^2 H}{3}$

3. $$\iiint\limits_{\Omega} f(\sqrt{x^2+y^2+z^2})\,dx\,dy\,dz$$

$$\Omega: \sqrt{x^2+y^2} \le z \le \sqrt{2-x^2-y^2}$$ Сектор ~~сферы~~ шара

Перейдём в сферические коор-ты.

$f(r)$

$$\int\limits_0^{\sqrt{2}} f(r)\,dr \int\limits_0^{2\pi} d\varphi \int\limits_0^{\frac{\pi}{4}} r^2 \sin\theta\, d\theta \ominus$$

$$r\sin\theta \le r\cos\theta \le \sqrt{2-r^2\sin^2\theta}$$

$$tg\,\theta \le 1$$

$$r^2\cos^2\theta \le 2 - r^2\sin^2\theta$$

$$r^2 \le 2$$

$$\ominus \int\limits_0^{\sqrt{2}} r^2 f(r)\, 2\pi \, -\cos\theta\Big|_0^{\frac{\pi}{4}} dr =$$

$$= +2\pi\left(-\frac{\sqrt{2}}{2}+1\right)\int\limits_0^{\sqrt{2}} r^2 f(r)\,dr$$

4. $\iiint\limits_{\Omega} \sqrt{x^2+y^2+z^2}\,dx\,dy\,dz \;=$

$\Omega = \{x^2+y^2+z^2 \le 2z\}$ ~~Сфера~~ Шар радиуса 1 с центром в (·) (0,0,1)

Перейдём в сферические коор-ты.

$\Omega:\; r^2 \le 2r\cos\Theta$

$r \le 2\cos\Theta$

$$\cancel{= \int_0^2 r\,dr \int_0^{2\pi} d\varphi \int_0 r^2 \sin^2\Theta\, d\Theta}$$

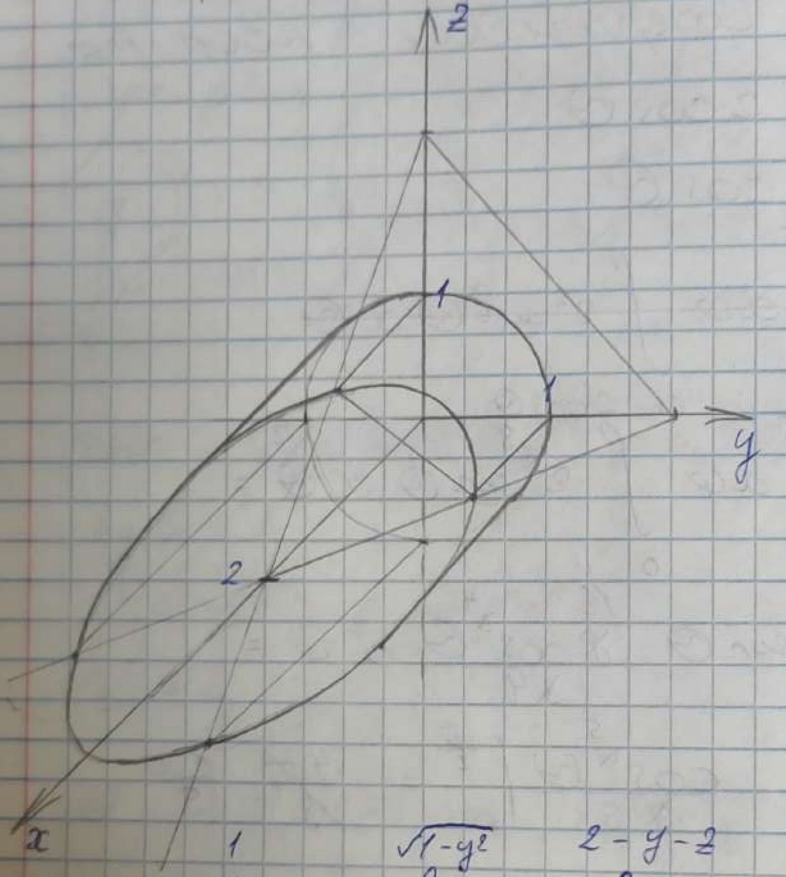
$$= \int_0^{\frac{\pi}{2}} d\Theta \int_0^{2\pi} d\varphi \int_0^{2\cos\Theta} r^{\cancel{2}\,3} \sin\Theta\, dr =$$

$$= 2\pi \int_0^{\frac{\pi}{2}} \sin\Theta\, \frac{\cancel{8}\,16\cos^{\cancel{3}\,4}\Theta}{\cancel{3}\,4}\, d\Theta =$$

$$= -\frac{\cancel{16}\,32\,\pi}{\cancel{3}\,4}\, \frac{\cos^{\cancel{4}\,5}\Theta}{\cancel{4}\,5}\Bigg|_0^{\frac{\pi}{2}} = \cancel{\frac{4\pi}{3}}\;\; \frac{8\pi}{5}$$

5. $\iiint\limits_{\Omega} (x+y+z)\,dx\,dy\,dz =$

Ω огр. $y^2+z^2=1 \quad x=0 \quad x+y+z=2$



$$= \int\limits_{\cancel{0}-1}^{1} dy \int\limits_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} dz \int\limits_{0}^{2-y-z} (x+y+z)\,dx =$$

$$= \int\limits_{-1}^{1} dy \int\limits_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \left(\frac{x^2}{2} + yx + zx\right)\Bigg|_{0}^{2-y-z} dz =$$

$$= \int_{-1}^{1} dy \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \left(-\frac{y^2}{2} - yz - \frac{z^2}{2} + 2\right) dz =$$

$$= \int_{-1}^{1} \left(-\frac{y^2 z}{2} - \frac{y z^2}{2} - \frac{z^3}{6} + 2z\right)\Bigg|_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} dy =$$

$$= \int_{-1}^{1} -y(1-y^2)\, dy = \cancel{\left(\frac{y^4}{4} - \frac{y^2}{2}\right)\Bigg|_{-1}^{1}} =$$

$$\cancel{= \frac{1}{4} - \frac{1}{2} + \frac{1}{4} - \frac{1}{2} = \frac{1}{2} - 1 = -\frac{1}{2}}$$

$$\neq \quad = \int_{-1}^{1} \left(-y^2\sqrt{1-y^2} - \frac{(\sqrt{1-y^2})^3}{\cancel{8}3} + 4\sqrt{1-y^2}\right) dy =$$

$$= \int_{-1}^{1} -\frac{\sqrt{1-y^2}}{3}(2y^2 - 11)\, dy =$$

$$= -\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos^2 u\,(2\sin^2 u - 11)}{3}\, du =$$

$$= \frac{7\arcsin(y)}{4} - \frac{y^3\sqrt{1-y^2}}{6} + \frac{23y\sqrt{1-y^2}}{12}\Bigg|_{-1}^{1} =$$

$$= \frac{7\pi}{4}$$

6.

$-R \leq x_4 \leq R$

$x_4 = const: \; x_1^2 + x_2^2 + x_3^2 \leq R^2 - x_4^2$

$$\int_{-R}^{R} dx_4 \iiint\limits_{D(R x_4^2)} dx_1\, dx_2\, dx_3 =$$

$V_{\text{шара радиуса } \sqrt{R^2 - x_4^2}} = \frac{4}{3}\pi (R^2 - x_4^2)^{\frac{3}{2}}$

$$= \frac{4}{3}\pi \int_{-R}^{R} (R^2 - x_4^2)^{\frac{3}{2}} dx_4 \ominus$$

$$\int_{-R}^{R} (R^2 - x^2)^{\frac{3}{2}} dx \underset{x = R\sin u}{=} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (R^2 - R^2 \sin^2 u)^{\frac{3}{2}} R\cos u\, du =$$

$$= \int R^4 \cos^4 u\, du = \frac{R^4 \cos^3(u) \sin u}{4} +$$

$$+ \frac{3R^4 \cos u \sin u}{8} + \frac{3R^4 u}{8} \Bigg|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{3\pi}{8} R^4$$

$$\ominus \frac{\pi^2}{2} R^4$$