30 октября.

Матан.

ИДЗ 1.

1. $f(x,y) = y + \cos\sqrt[3]{x^2+y^2}$

$\frac{\partial f}{\partial x} = -\sin\sqrt[3]{x^2+y^2}\,\frac{1}{3\sqrt{(x^2+y^2)^{\frac{2}{3}}}}\cdot 2x$

Не получается.

По опр.:

$\frac{\partial f}{\partial x} = \cancel{\quad}(x_0, y_0) = \lim\limits_{\Delta x\to 0}\frac{f(x_0+\Delta x, y_0) - f(x_0,y_0)}{\Delta x}$

$\frac{\partial f}{\partial x}(0,0) = \lim\limits_{\Delta x\to 0}\frac{f(\Delta x, 0) - f(0,0)}{\Delta x} =$

$= \lim\limits_{\Delta x\to 0}\frac{\cos\Delta x^{\frac{2}{3}} - 1}{\Delta x} = -\lim\limits_{\Delta x\to 0}\frac{\Delta x^{\frac{4}{3}}}{2\Delta x} =$

$= 0$

$\frac{\partial f}{\partial y}(0,0) = \lim\limits_{\Delta y\to 0}\frac{f(0,\Delta y) - f(0,0)}{\Delta y} =$

$= \lim\limits_{\Delta y\to 0}\frac{\Delta y + \cos\Delta y^{\frac{2}{3}} - 1}{\Delta y} = 1$

Проверим дифф-ть в (0,0):

$\Delta f(0,0) = f(\Delta x, \Delta y) - f(0,0) =$

$= \Delta y + \cos\sqrt[3]{\Delta x^2 + \Delta y^2} - 1 = \overset{=(0,1)}{A}\cdot h + o(\|h\|) =$

$= \cancel{\Delta x + \Delta y}\ \Delta y + o(\|h\|)$

$\cos\sqrt[3]{\Delta x^2 + \Delta y^2} - 1 = o(\|h\|)$

$$\lim_{(\Delta x, \Delta y)\to(0,0)} \frac{\cos\sqrt[3]{\Delta x^2 + \Delta y^2} - 1}{\sqrt{\Delta x^2 + \Delta y^2}}$$

$]\, t = \Delta x^2 + \Delta y^2 \qquad (\Delta x, \Delta y) \to (0,0) \Leftrightarrow t \to 0$

$$\lim_{t\to 0} \frac{\cos t^{\frac{1}{3}} - 1}{\sqrt{t}} = -\lim_{t\to 0} \frac{t^{\frac{2}{3}}}{2\sqrt{t}} = 0 \Rightarrow$$

$\Rightarrow f$ дифф-ма в т. $(0,0)$

$\cancel{df = A(dx, dy) = dy}$

$df(0,0) = (A\cdot(dx, dy))(0,0) = dy$

2.

$$\left(\frac{\partial z}{\partial y}\right)^2 \frac{\partial^2 z}{\partial x^2} - 2\frac{\partial z}{\partial x}\cdot\frac{\partial z}{\partial y}\cdot\frac{\partial^2 z}{\partial x \partial y} +$$

$$+\left(\frac{\partial z}{\partial x}\right)^2 \frac{\partial^2 z}{\partial y^2} = 0$$

$$z(x,y) \to x(y,z) \qquad \begin{cases} u = y \\ v = z \\ w = x \end{cases}$$

$$w(u,v)$$

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial u}\cdot 0 + \frac{\partial w}{\partial v}\cdot\frac{\partial z}{\partial x} = 1 \qquad \frac{\partial z}{\partial x} = \frac{1}{\frac{\partial w}{\partial v}}$$

$$\frac{\partial^2 w}{\partial x^2} = \left(\frac{\partial^2 w}{\partial v \partial u}\cdot\overset{=0}{\frac{\partial u}{\partial x}} + \frac{\partial^2 w}{\partial v^2}\frac{\partial v}{\partial x}\right)\cdot$$

$$\cdot\frac{\partial z}{\partial x} + \frac{\partial^2 z}{\partial x^2}\cdot\frac{\partial w}{\partial v} = 0$$

$$\frac{\partial^2 w}{\partial v^2}\left(\frac{\partial z}{\partial x}\right)^2 + \frac{\partial^2 z}{\partial x^2}\cdot\frac{\partial w}{\partial v} = 0$$

$$\frac{\partial^2 z}{\partial x^2} = -\frac{\frac{\partial^2 w}{\partial v^2}}{\left(\frac{\partial w}{\partial v}\right)^3}$$

$$\frac{\partial^2 w}{\partial x \partial y} = \left(\frac{\partial^2 w}{\partial v \partial u}\cdot\overset{1}{\frac{\partial u}{\partial y}} + \frac{\partial^2 w}{\partial v^2}\cdot\frac{\partial v}{\partial y}\right)\cdot\frac{\partial z}{\partial x} +$$

$$+\frac{\partial^2 z}{\partial x \partial y}\cdot\frac{\partial w}{\partial v} = 0$$

$$\frac{\partial w}{\partial y} = \frac{\partial w}{\partial u}\cdot 1 + \frac{\partial w}{\partial v}\cdot\frac{\partial z}{\partial y} = 0$$

$$\frac{\partial z}{\partial y} = -\frac{\frac{\partial w}{\partial u}}{\frac{\partial w}{\partial v}}$$

$$\frac{\partial^2 z}{\partial x\,\partial y} = -\left(\frac{\partial^2 w}{\partial v\,\partial u} + \cancel{\frac{\partial u}{\partial y}}\ \frac{\partial^2 w}{\partial v^2}\cdot\frac{\partial z}{\partial y}\right)\frac{\partial z}{\partial x} : \frac{\partial w}{\partial v} =$$

$$= \frac{-\left(\frac{\partial^2 w}{\partial v\,\partial u} + - \frac{\frac{\partial^2 w}{\partial v^2}\cdot\frac{\partial w}{\partial u}}{\frac{\partial w}{\partial v}}\right)}{\left(\frac{\partial w}{\partial v}\right)^2} =$$

$$= \frac{-\frac{\partial^2 w}{\partial v\,\partial u}\cdot\frac{\partial w}{\partial v} + \frac{\partial^2 w}{\partial v^2}\cdot\frac{\partial w}{\partial u}}{\left(\frac{\partial w}{\partial v}\right)^3}$$

$$\cancel{\frac{\partial^2 w}{\partial y^2} = \frac{\partial^2 w}{\partial u^2}\cdot\frac{\partial u}{\partial y} +}$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{-\left(\frac{\partial^2 w}{\partial u^2}\cdot\frac{\partial u}{\partial y} + \frac{\partial^2 w}{\partial u\,\partial v}\frac{\partial v}{\partial y}\right)\cdot\frac{\partial w}{\partial v} - \left(\frac{\partial^2 w}{\partial v\,\partial u}\cancel{+\partial^2}\cdot\frac{\partial u}{\partial y} + \frac{\partial^2 w}{\partial v^2}\cdot\frac{\partial v}{\partial y}\right)\frac{\partial w}{\partial u}}{\left(\frac{\partial w}{\partial v}\right)^2} =$$

$$= \frac{-\frac{\partial w}{\partial v}\left(\frac{\partial^2 w}{\partial u^2}\cdot\frac{\partial w}{\partial v} - \frac{\partial^2 w}{\partial u\,\partial v}\cdot\frac{\partial w}{\partial u}\right) - \left(\frac{\partial^2 w}{\partial v\,\partial u}\cdot\frac{\partial w}{\partial v} - \frac{\partial^2 w}{\partial v^2}\cdot\frac{\partial w}{\partial u}\right)\frac{\partial w}{\partial u}}{\left(\frac{\partial w}{\partial v}\right)^3}$$

$\frac{\partial^2 w}{\partial u\,\partial v}$, т.к. [illegible] дифф[illegible] 2

Подставим.

$$-\frac{\cancel{\frac{\partial^2 w}{\partial v^2}\cdot\left(\frac{\partial w}{\partial u}\right)^2}^{\,1}}{\left(\frac{\partial w}{\partial v}\right)^5}+2\cdot\frac{\frac{\partial w}{\partial u}\left(-\cancel{\frac{\partial^2 w}{\partial v\,\partial u}\cdot\frac{\partial w}{\partial v}}^{\,2}+\right.}{\left(\frac{\partial w}{\partial v}\right)^5}$$

$$\frac{\left.+\cancel{\frac{\partial^2 w}{\partial v^2}\cdot\frac{\partial w}{\partial u}}\right)^{1}}{}-\frac{\frac{\partial w}{\partial v}\left(\frac{\partial^2 w}{\partial u^2}\cdot\frac{\partial w}{\partial v}-\right.}{}$$

$$\frac{\left.-\cancel{\frac{\partial^2 w}{\partial u\,\partial v}\cdot\frac{\partial w}{\partial u}}\right)^{2}-\left(\cancel{\frac{\partial^2 w}{\partial v\,\partial u}\cdot\frac{\partial w}{\partial v}}^{\,2}-\cancel{\frac{\partial^2 w}{\partial v^2}\cdot\frac{\partial w}{\partial u}}^{\,1}\right)\cdot}{\left(\frac{\partial w}{\partial v}\right)^5}$$

$$\frac{\cdot\frac{\partial w}{\partial u}}{}=0$$

$$\left(\frac{\partial w}{\partial v}\right)^2\frac{\partial^2 w}{\partial u^2}=0$$

$$\frac{\partial w}{\partial v}=0\Rightarrow x=f(y)\Rightarrow z=g(f(y),y)=$$

$$=p(y)\Rightarrow$$

$$\Rightarrow\frac{\partial z}{\partial x}=0=\frac{\partial^2 z}{\partial x^2}=\frac{\partial^2 z}{\partial x\,\partial y}$$

$$\frac{\partial^2 w}{\partial u^2}=0\Rightarrow\frac{\partial w}{\partial u}=C\Rightarrow w=Cu+C_1,\ \text{т.е.}$$

$$x=C\cdot y+C_1=f(y)$$

Аналогично пред.

Подставим. $0=0$.

3. $y=f(x)$ $y=(y_1,y_2)$ $x=(x_1,x_2)$

$$\begin{cases} F_1(x,y)=x_1 e^{y_1+y_2}+2y_1y_2-1 \\ F_2(x,y)=x_2 e^{y_1-y_2}-\dfrac{y_1}{y_2+1}-2x_1 \end{cases}$$

$M(1,2,0,0)$ - подходит $F(M)=0$

$F(x,y)=0$ $F=(F_1,F_2)$

$F\in C^1$ (очевидно)

$$F'_y=\begin{pmatrix} x_1e^{y_1+y_2}+2y_2 & x_1e^{y_1+y_2}+2y_1 \\ x_2e^{y_1-y_2}-\dfrac{1}{y_2+1} & -x_2e^{y_1-y_2}+\dfrac{y_1 \cancel{e_1\ y_2}}{(y_2^2+1)^2} \end{pmatrix}$$

$$F'_y\cancel{(1,2)=}(M)=\begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix}$$

$\det F'_y\neq 0$ в $\overset{\text{окр.}}{\vee}$ т. M $\Rightarrow$ F задаёт неявно диф-мое отображение $y=f(x)$ в окр. т. M
(очевидно, $F\in C^1$ в окр. т. M)

$y'=-(F'_y)^{-1}\cdot F'_x$

$$(F'_y(M))^{-1}=-\frac{1}{3}\cdot\begin{pmatrix} -2 & 1 \\ 1 & 1 \end{pmatrix}=\begin{pmatrix} \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} \end{pmatrix}$$

$$F'_x = \begin{pmatrix} e^{y_1+y_2} & 0 \\ \frac{e^{y_1-y_2}}{-2} & e^{y_1-y_2} \end{pmatrix}$$

$$F'_x(M) = \begin{pmatrix} 1 & 0 \\ -2 & \cancel{0}\,1 \end{pmatrix}$$

$$y'(M) = +\frac{1}{3}\begin{pmatrix} -4 & \cancel{0}\,1 \\ -1 & \cancel{0}\,1 \end{pmatrix} = \begin{pmatrix} -\frac{4}{3} & \cancel{0}\,+\frac{1}{3} \\ -\frac{1}{3} & \cancel{0}\,+\frac{1}{3} \end{pmatrix}$$

$(F'_y)^{-1}$:

$$\Delta = (x_1 e^{y_1+y_2} + 2y_2)\left(-x_2 e^{y_1-y_2} + \frac{y_1}{(y_2+1)^2}\right) - (x_1 e^{y_1+y_2} + 2y_1)\left(x_2 e^{y_1-y_2} - \frac{1}{y_2+1}\right)$$

$$(F'_y)^{-1} = \frac{1}{\Delta} \cdot \begin{pmatrix} -x_2 e^{y_1-y_2} + \frac{y_1}{(y_2+1)^2} & x_2 e^{y_1-y_2} - \frac{1}{y_2+1} \\ x_1 e^{y_1+y_2} + 2y_1 & x_1 e^{y_1+y_2} + 2y_2 \end{pmatrix}$$

$$y'_{1x_1} = -\frac{1}{\Delta} \cdot \left(-x_2 e^{2y_1} + \frac{e^{y_1+y_2}}{(y_2+1)^2} y_1 - 2x_2 e^{y_1-y_2} + \frac{2}{y_2+1}\right)$$

$$y'_{2x_2} = -\frac{1}{\Delta}\cdot\underbrace{\left(x_1 e^{2y_1} + 2y_2 e^{y_1-y_2}\right)}_{=p}$$

$$\Delta = ab - cd \qquad \frac{7}{9}$$

$$a'_{x_2} = e^{y_1+y_2}\, y'_{1x_2}\cdot y'_{2x_2}\, x_1 + 2y'_{2x_2}\Big|_M = +\frac{2}{3} + \frac{1}{9} = -\frac{\cancel{8}\ 7}{9}$$

$$b'_{x_2} = +e^{y_1-y_2}\cdot y'_{1x_2}\cdot y'_{2x_2}\cdot x_2 - 1\cdot e^{y_1-y_2} +$$

$$+\frac{y'_{1x_2}(y_2+1)^2 - 2(y_2+1)\,y'_{2x_2}\,y_1}{(y_2+1)^{\cancel{8}\,4}}\Big|_M = \frac{2}{9} - 1 +$$

$$+\frac{+\frac{1}{3}\cdot 1}{1} = -\frac{\cancel{10}\ 4}{9}$$

$$c'_{x_2} = e^{y_1+y_2}\, y'_{1x_2}\, y'_{2x_2}\, x_1 + 2y'_{1x_2}\Big|_M = \frac{1}{9} + \frac{2}{3} = +\frac{\cancel{5}\ 7}{9}$$

$$d'_{x_2} = e^{y_1-y_2} + e^{y_1-y_2}\, y'_{1x_2}(-y'_{2x_2})\,x_2 + \frac{y'_{2x_2}}{(y_2+1)^2}\Big|_M =$$

$$= 1 + \left(-\frac{2}{9}\right) + \frac{1}{3} = \frac{\cancel{4}}{\cancel{9}}\ \frac{10}{9}$$

$$a(M) = 1 \qquad b(M) = -2 \qquad c(M) = 1 \qquad d(M) = 1$$

$$\Delta(M) = -3 \qquad \Delta'_{x_2}(M) = a'_{x_2} b + b'_{x_2} a - (c'_{x_2} d +$$

$$+ d'_{x_2} c)\Big|_M = +\frac{\cancel{5}\ 7}{9}(-2) - \frac{\cancel{10}\ 4}{9}\cdot 1 - \left(+\frac{\cancel{5}\ 7}{9}\cdot 1 + \frac{\cancel{4}\ 10}{9}\cdot 1\right) = -\frac{35}{9}$$

$$p'_{x_2} = \left(x_1 e^{2y_1} + 2y_2 e^{y_1-y_2}\right)'_{x_2} = x_1 e^{2y_1}\, 2\cdot y'_{1x_2} + 2y'_{2x_2} e^{y_1-y_2} +$$

$$+ 2y_2 e^{y_1-y_2}\cdot y'_{1x_2}(-y'_{2x_2})\Big|_M = +\frac{2}{3} + \frac{2}{3} = +\frac{4}{3}$$

$$p(M) = 1$$

$$y''_{2x_2x_2} = -\left.\frac{F'_{x_2}\Delta - \Delta'_{x_2}F}{\Delta^2}\right|_M =$$

$$= -\frac{+\frac{4}{3}\cdot(-3) + \frac{\cancel{3}5}{9}\cdot 1}{9} = -\frac{3\cancel{7}8}{81} + \frac{1}{81}$$

<u>Ответ:</u> $-\frac{3\cancel{7}^{5}}{81} + \frac{1}{81}$.

4. $f(x,y,z)=x^2+4y^2+4z^2$ при

$x+2y+3z=0$, $x^2+y^2+z^2=1$

$L(x,y,z,\lambda_1,\lambda_2)=x^2+4y^2+4z^2-$

$-\lambda_1(x+2y+3z)-\lambda_2(x^2+y^2+z^2-1)$

$\varphi'=\begin{pmatrix}1 & 2 & 3\\ 2x & 2y & 2z\end{pmatrix}$ $\quad 4z-6y\neq 0$

$rg\,\varphi'=2$ при $z\neq\frac{3}{2}y$

$L'_x=2x-\lambda_1-2\lambda_2x=0$

$L'_y=8y-2\lambda_1-2\lambda_2y=0$

$L'_z=8z-3\lambda_1-2\lambda_2z=0$

$x+2y+3z=0$

$x^2+y^2+z^2=1$

$x=\dfrac{\lambda_1}{2-2\lambda_2}$ $\quad y=\dfrac{\lambda_1}{4-\lambda_2}$ $\quad z=\dfrac{3\lambda_1}{8-2\lambda_2}$

$\dfrac{\lambda_1}{2-2\lambda_2}+\dfrac{2\lambda_1}{4-\lambda_2}+\dfrac{9\lambda_1}{8-2\lambda_2}=0$

$\lambda_1(4-\lambda_2)+2\lambda_1(2-2\lambda_2)+9\lambda_1(1-\lambda_2)=$
$=0$

$4\lambda_1-\lambda_1\lambda_2+4\lambda_1-4\lambda_1\lambda_2+9\lambda_1-$
$-9\lambda_1\lambda_2=0$

$17\lambda_1-14\lambda_1\lambda_2=0$

$$14\lambda_1\left(\frac{17}{14}-\lambda_2\right)=0 \Rightarrow \left[\begin{array}{l}\lambda_1=0 \\ \text{или} \\ \lambda_2=\frac{17}{14}\end{array}\right.$$

$$\cancel{2x-\frac{17}{7}x=0 \Rightarrow x=0}$$

$$\cancel{8y-\frac{17}{7}y=0 \Rightarrow}$$

~~$\lambda_1=0$: $x=0$ $y=0$ $z=0$ Не подходит~~.

$$\lambda_2=\frac{17}{14}:\quad x=\frac{\lambda_1}{2-\frac{17}{7}}=-\frac{7}{3}\lambda_1$$

$$y=\frac{\lambda_1}{4-\frac{17}{14}}=\frac{\cancel{39}}{\cancel{14}}\cancel{\lambda_1}\ \frac{14}{39}\lambda_1$$

$$z=\frac{3\lambda_1}{8-\frac{17}{7}}=\frac{7}{13}\lambda_1$$

$$\frac{49}{9}\lambda_1^2+\frac{196}{1521}\lambda_1^2+\frac{49}{169}\lambda_1^2=1$$

$$\lambda_1=\pm\frac{3\sqrt{13}}{7\sqrt{14}}\qquad A\left(-\sqrt{\frac{13}{14}},\ \frac{2}{13}\sqrt{\frac{13}{14}},\ \frac{3}{13}\sqrt{\frac{13}{14}}\right)$$

$$B\left(\sqrt{\frac{13}{14}},\ -\frac{2}{13}\sqrt{\frac{13}{14}},\ -\frac{3}{13}\sqrt{\frac{13}{14}}\right)$$

$$d^2L=(2-2\lambda_2)dx^2+(8-2\lambda_2)dy^2+(8-2\lambda_2)dz^2=$$

$$=-\frac{3}{7}dx^2+\frac{13}{7}dy^2+\frac{13}{7}dz^2$$

$N: \begin{cases} dx + 2dy + 3dz = 0 \\ 2x\,dx + 2y\,dy + 2z\,dz = 0 \end{cases}$

$(\cdot)A: \quad -2dx + \frac{4}{13}dy + \frac{6}{13}dz = 0$

$\cancel{\left(4 + \frac{4}{13}\right)dy} \quad -26dx + 4dy + 6dz = 0$

$-28dx + 3dz = 0 \Rightarrow dz = \frac{28}{3}dx$

$-29dx - 2dy = 0 \Rightarrow dy = -\frac{29}{2}dx$

$d^2L = -\frac{3}{7}dx^2 + \left(\frac{13}{7}\cdot\frac{29^2}{4} + \frac{13}{7}\cdot\frac{28^2}{4}\right)dx^2 > 0$

при $dx \neq \vec{0}$

$\Rightarrow$ усл. мин.

$(\cdot)B$: Аналогично.

$d^2L = \cancel{-\frac{3}{7}dx^2 +} \ldots \frac{169}{28}dz^2 > 0$ при $dz \neq 0 \Rightarrow$

$\Rightarrow$ усл. мин.

$\lambda_1 = 0$:

$(2 - 2\lambda_2)x = 0 \quad (8 - 2\lambda_2)y = 0 \quad (8 - 2\lambda_2)z = 0$

$\lambda_2 = 1$: не подходит, т.к. тогда $\begin{cases} x = 0 \\ x^2 = 1 \end{cases}$

$x = y = z = 0$ не подходит

$\lambda_2 = 4$: $\begin{cases} x = 0 \\ 2y + 3z = 0 \\ y^2 + z^2 = 1 \end{cases}$ $\quad y = -\frac{3}{2}z$

$\frac{9}{4}z^2 + z^2 = 1 \quad 13z^2 = 4$

N: $\begin{cases} dx + 2dy + 3dz = 0 \\ 6dy - 4dz = 0 \end{cases}$

$z = \pm\frac{2}{\sqrt{13}} \quad C\left(0, -\frac{3}{\sqrt{13}}, \frac{2}{\sqrt{13}}\right) (N_C = N_D)$ $\quad \begin{cases} dy = \frac{2}{3}dz \\ dx = \frac{13}{3}dz \end{cases}$

$D\left(0, \frac{3}{\sqrt{13}}, -\frac{2}{\sqrt{13}}\right)$

$d^2L = (2 - 2\lambda_2)dx^2 + (8 - 2\lambda_2)dy^2 + (8 - 2\lambda_2)dz^2 =$
$= -6\cancel{\lambda_2}\, dx^2 < 0$ при $dx \neq 0 \Rightarrow$ усл. макс.
в (·) C и (·) D.

(·)A:
$\begin{cases} -13\,dx + 2dy + 3dz = 0 \\ dx + 2dy + 3dz = 0 \end{cases}$

~~$14dx = 0 \Rightarrow dx = 0$~~

$14dx = 0 \Rightarrow dx = 0$

$dy = -\frac{3}{2}dz$

$d^2L = \frac{13}{7}\left(\frac{9}{4}dz^2 + dz^2\right) = \frac{169}{28}dz^2 > 0 \Rightarrow$ при $dz \neq 0$

$\Rightarrow$ усл. мин.