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1. $\begin{cases} \dot{x} = tx \\ \dot{y} = y^2 + t \end{cases}$ $\quad x(0) = 1 \quad y(0) = 0$

$$\begin{pmatrix} x \\ y \end{pmatrix}^{(0)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad f(t, x, y) = \begin{pmatrix} tx \\ y^2 + t \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix}^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \int_0^t \begin{pmatrix} t \\ t \end{pmatrix} dt = \begin{pmatrix} \frac{t^2}{2} + 1 \\ \frac{t^2}{2} \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix}^{(2)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \cancel{\int_0^x \begin{pmatrix} \frac{x^3}{2} + x & \frac{tx^2}{2} + t \\ x^4 & \frac{x^4}{48} + t \end{pmatrix} dt} =$$

$$\cancel{= \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} \frac{x^4}{4} + \frac{x^2}{2} \\ \frac{x^5}{24} + \frac{x^2}{2} \end{pmatrix} = \begin{pmatrix} \frac{x^4}{4} + \frac{x^2}{2} + 1 \\ \frac{x^5}{24} + \frac{x^2}{2} \end{pmatrix}}$$

$$+ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \int_0^t \begin{pmatrix} \frac{t^3}{2} + t \\ \frac{t^4}{4} + t \end{pmatrix} dt = \cancel{\frac{x^4}{8} + \frac{x^2}{2}} + 1$$

$$= \begin{pmatrix} \frac{t^4}{8} + \frac{t^2}{2} + 1 \\ \frac{t^5}{20} + \frac{t^2}{2} \end{pmatrix}$$

2. $\begin{cases} \dot{x} = \dfrac{x}{x-y} \\ \dot{y} = \dfrac{y}{x-y} \end{cases}$ $x > y > 0$

$\dot{x} - \dot{y} = 1$ $x = t + y + C$

$y = x - t - C$

$\dot{x} = \dfrac{x}{t+C}$ $\dfrac{dx}{x} = \dfrac{dt}{t+C}$

$\ln x = \ln(t+C) + C_2$

$\begin{cases} x = C_1(t+C) \\ y = C_1(t+C) - t - C \end{cases}$

3. $y''' - 5y'' + 6y' = x^2 - x$

$\lambda(\lambda^2 - 5\lambda + 6) = 0 \Leftrightarrow \lambda(\lambda - 3)(\lambda - 2) = 0$

$\lambda_1 = 0 \quad \lambda_2 = 2 \quad \lambda_3 = 3$

$y_0 = C_1 + C_2 e^{2x} + C_3 e^{3x}$

$\cancel{x^2} = x(ax^2 + bx + c) = y_1$

$y_1' = 3ax^2 + 2bx + c$

$y_1'' = 6ax + 2b$

$y_1''' = 6a$

$6a - 30ax - 10b + 18ax^2 + 12bx + 6c = x^2 - x$

$$\begin{cases} 18a = 1 \\ -30a + 12b = \cancel{0}\ -1 \\ 6a - 10b + 6c = 0 \end{cases} \Rightarrow \quad a = \frac{1}{18} \quad b = \frac{1}{18} \quad c = \frac{1}{27}$$

$y_1 = x\left(\frac{1}{18}\cancel{a}x^2 + \frac{1}{18}x + \frac{1}{27}\right)$

$y = y_0 + y_1 = C_1 + C_2 e^{2x} + C_3 e^{3x} + x\left(\frac{1}{18}x^2 + \frac{1}{18}x + \frac{1}{27}\right)$

4. $\begin{cases} \dot{x} = -3x - y + 5\cos t \\ \dot{y} = 2x \end{cases}$ $\qquad z(0) = \begin{pmatrix} \frac{3}{2} \\ 1 \end{pmatrix}$

$x' = \frac{y''}{2}$

$\frac{y''}{2} = -\frac{3y'}{2} - y + 5\cos t$

$y'' + 3y' + 2y = 10\cos t$

$\lambda^2 + 3\lambda + 2 = 0 \Leftrightarrow (\lambda + 2)(\lambda + 1) = 0$

$\lambda_1 = -1 \quad \lambda_2 = -2$

$y_0 = Ce^{-\cancel{x}t} + C_1 e^{-2\cancel{x}t}$

$y_1 = b\sin(\overset{t}{\cancel{x}}) + a\cos(\overset{t}{\cancel{x}})$

$y_1' = b\cos(\overset{t}{\cancel{x}}) - a\sin(\overset{t}{\cancel{x}})$

$y_1'' = -b\sin(\overset{t}{\cancel{x}}) - a\cos(\overset{t}{\cancel{x}})$

$(b - 3a)\sin t + (3b + a)\cos t = 10\cos t$

$\begin{cases} b - 3a = 0 \\ 3b + a = 10 \end{cases} \Rightarrow \begin{matrix} a = 1 \\ b = 3 \end{matrix}$

$y_1 = 3\sin t + \cos t$

$y = y_0 + y_1$

$\begin{cases} y = Ce^{-t} + C_1 e^{-2t} + 3\sin t + \cos t \\ x = \frac{y'}{2} = -\frac{\sin(t)}{2} + \frac{3\cos(t)}{2} - \frac{Ce^{-t}}{2} - C_1 e^{-2t} \end{cases}$

$y(0) = C + C_1 + 1 = 1$ $\qquad C = -2C_1$ $\qquad\Rightarrow\qquad C = 0$

$x(0) = \frac{3}{2} - \frac{C}{2} - C_1 = \frac{3}{2}$ $\qquad C_1 = 0$ $\qquad\qquad C_1 = 0$

5. $y'' + (2 - x^2)y = 0 \qquad y(0) = 0 \quad y'(0) = 1$

$$y = \sum_{k=0}^{\infty} a_k x^k, \quad y(0) = 0 \Rightarrow a_0 = 0$$

$$y' = \sum_{k=1}^{\infty} k a_k x^{k-1}, \quad y'(0) = 1 \Rightarrow a_1 = 1$$

$$y'' = \sum_{k=2}^{\infty} k(k-1) a_k x^{k-2}$$

$$y'' + (2 - x^2)y = 0 = \sum_{k=2}^{\infty} k(k-1) a_k x^{k-2} +$$

$$+ (2 - x^2) \sum_{k=0}^{\infty} a_k x^k$$

$$\sum_{k=0}^{\infty} (k+2)(k+1) a_{k+2} x^k + 2 \sum_{k=0}^{\infty} a_k x^k -$$

$$- \sum_{k=2}^{\infty} a_{k-2} x^k = 0$$

$$2a_2 + 6a_3 x + \sum_{k=2}^{\infty} (k+2)(k+1) a_{k+2} x^k + 2a_0 + 2a_1 x +$$

$$+ 2 \sum_{k=2}^{\infty} a_k x^k - \sum_{k=2}^{\infty} a_{k-2} x^k = 0$$

$$2a_0 + 2a_2 + (6a_3 + 2a_1)x + \sum_{k=2}^{\infty} ((k+2)(k+1) a_{k+2} + 2a_k -$$

$$- a_{k-2}) x^k = 0$$

$a_2 = 0$

$6a_3 + 2 = 0 \Rightarrow a_3 = -\frac{1}{3}$

$k=2: \ 12a_4 + 2a_2 - a_0 = 0$

$a_4 = 0$

$k=3: \ 20a_5 + 2a_3 - a_1 = 0$

$$a_5 = \frac{a_1 - 2a_3}{20} = \frac{\left(1 + \frac{2}{3}\right)}{20} = \frac{\frac{5}{3}}{20} = \frac{5}{60} = \frac{1}{12}$$