16 мая.

Дз № 7.

1. $f(x) = \frac{\sin x}{x^{\frac{2}{3}}}$

$f(x) \in L^2(\mathbb{R}, \lambda)$:

$$I = \int_{-\infty}^{+\infty} \frac{\sin^2 x}{x^{\frac{4}{3}}} dx = 2 \int_{0}^{+\infty} \frac{\sin^2 x}{x^{\frac{4}{3}}} dx =$$

$$= 2 \int_0^1 \frac{\sin^2 x}{x^{\frac{4}{3}}} dx + 2 \int_1^{\infty} \frac{\sin^2 x}{x^{\frac{4}{3}}} dx =$$

$$\underset{x \to 0}{\sim} \frac{1}{x^{-\frac{2}{3}}} \qquad \leq \frac{1}{x^{\frac{4}{3}}}$$

$\int_0^1 x^{\frac{2}{3}} dx$ сх.

$\int_1^{\infty} \frac{1}{x^{\frac{4}{3}}} dx = -3x^{-\frac{1}{3}}\Big|_1^{\infty}$ сх.

$\Rightarrow I$ сх.

$f(x) \notin L(\mathbb{R}, \lambda)$:

$$I = \int_{-\infty}^{+\infty} \left|\frac{\sin x}{x^{\frac{2}{3}}}\right| dx = 2 \int_0^{+\infty} \left|\frac{\sin x}{x^{\frac{2}{3}}}\right| dx \leq 2 \int_0^{+\infty} \frac{1}{x^{\frac{2}{3}}} dx =$$

$= 6x^{\frac{1}{3}}\Big|_0^{+\infty}$ расх.

$\Rightarrow I$ расх.

2. $$x_n(t) = \begin{cases} 1, & 0 \le t < \frac{1}{n} \\ t^{-\frac{1}{3}}, & \frac{1}{n} \le t < 1 \end{cases}$$

$x(t) = t^{-\frac{1}{3}}$ – предельная ф-я

$$(\| x_n(t) - x(t) \|_p)^p = \int_{[0,1]} (x_n(t) - x(t))^p d\lambda =$$

$$= \int_0^{\frac{1}{n}} (-1 + t^{-\frac{1}{3}})^p dt + \int_{\frac{1}{n}}^1 (t^{-\frac{1}{3}} - t^{-\frac{1}{3}})^p dt =$$

$$= \int_0^{\frac{1}{n}} (-1 + t^{-\frac{1}{3}})^p dt$$

$$< I = \int_0^1 (t^{-\frac{1}{3}} - 1)^p dt \sim \int_0^1 t^{-\frac{p}{3}} dt =$$

$$= t^{-\frac{p}{3}+1} \cdot \left(-\frac{p}{3}+1\right) \Big|_0^1 \quad \text{сх-ся при } p \le 3$$

$\Rightarrow I$ сх-ся при $p \le 3$

Тогда $\int_0^{\frac{1}{n}} (t^{-\frac{1}{3}} - 1)^p dt \xrightarrow[n \to \infty]{} 0$, как

хвост I вблизи особой точки 0.

$\Rightarrow x_n(t) \xrightarrow[n \to \infty]{} x\, t^{-\frac{1}{3}}$ при $1 \le p \le 3$

$\operatorname{esssup}_{[0,1]} |x_n(t) - x(t)| \neq 0 \Rightarrow$ при $p = +\infty$

сх-ти нет

3. $P_n(x) = \frac{1}{2^n n!} \frac{d^n (x^2-1)^n}{dx^n}$

а) $P_0(x) = 1$

$P_1(x) = \frac{1}{2} 2x = x$

$P_2(x) = \frac{1}{8} \frac{d}{dx} 2(x^2-1)\cdot 2x =$

$= \frac{1}{8} \cdot 4 (3x^2 - 1) = \frac{1}{2}(3x^2-1)$

$P_3(x) = \frac{1}{48} \frac{d^2}{dx^2} 3(x^2-1)^2 \cdot 2x =$

$= \frac{6}{48} \frac{d^2}{dx^2} \cancel{6}(x^5 - 2x^3 + x) =$

$= \frac{6}{48} \frac{d}{dx} (5x^4 - 6x^2 + 1) =$

$= \frac{6}{48} (20x^3 - 12x) = \frac{1}{2}(5x^3 - 3x)$

б) $\int_0^1 P_n(x) \cdot P_m(x)\, dx =$

$= \int_0^1 \frac{1}{2^{n+m} n!\, m!} ((x^2-1)^n)^{(n)}_x \cdot ((x^2-1)^m)^{(m)}_x\, dx$

c)

$$c_0(f) = \frac{(f, P_0)}{\|P_0\|^2}$$

$$(f, P_0) = \int_{\cancel{-\infty}\,0}^{\cancel{+\infty}\,1} \sin x \cdot 1\, dx = \cancel{0}-\cos x\Big|_0^1 = 1-\cos 1$$

$$\|P_0\|^2 = \int_{\cancel{-\infty}\,0}^{\cancel{+\infty}\,1} 1\, dx = 1$$

$$c_0(f) = \cancel{\cos 1}\ 1-\cos 1$$

$$c_1(f) = \frac{(f, P_1)}{\|P_1\|^2}$$

$$(f, P_1) = \int_0^1 x\sin x\, dx = -x\cos x\Big|_0^1 + \int_0^1 \cos x\, dx =$$

$$= -\cos \overset{1}{\cancel{x}} + \sin 1$$

$$\|P_1\|^2 = \int_0^1 x^{\cancel{1}\,2}\, dx = \cancel{3}\frac{x^3}{3}\Big|_0^1 = \frac{1}{3}$$

$$c_1(f) = -3\cos 1 + 3\sin 1$$

$$c_2(f) = \frac{(f, P_2)}{\|P_2\|^2}$$

$$(f, P_2) = \int_0^1 \frac{1}{2}(3x^2-1)\sin x\, dx =$$

$$= \frac{3}{2}\underbrace{\int_0^1 x^2 \sin x \, dx}_{\text{дважды по частям}} + \left(-\frac{1}{2}\right)\cdot \cos 1 =$$

$$= \overset{3}{\cancel{8}} \sin(1) + \overset{2}{\cancel{4}} \cos(1) - \frac{7}{2}$$

$$\|P_2\|^2 = \int_0^1 \frac{1}{4}(3x^2-1)^2 \, dx =$$

$$= \frac{1}{4}\int_0^1 (9x^4 - 6x^2 + 1)\, dx =$$

$$= \frac{1}{4}\left(\frac{9}{5}x^5 - 2x^3 + x\right)\Big|_0^1 = \frac{1}{5}$$

$$c_2(f) = 15\sin 1 + 10\cos 1 - \frac{35}{2}$$

в) $$\int_{\cancel{0}\,-1}^{1} P_n(x)\cdot P_m(x)\, dx =$$

$$= \frac{1}{2^{n+m}\, n!\, m!}\int_{-1}^{1} \left((x^2-1)^n\right)_x^{(n)} \cdot \left((x^2-1)^m\right)_x^{(m)} dx$$

Н.у.о. $m > n$.

$$I = \int_{-1}^{1} \left((x^2-1)^n\right)_x^{(n)} \cdot \left((x^2-1)^m\right)_x^{(m)} dx$$

① Док-м, что $I = 0$.

$$I = \left((x^2-1)^m\right)_x^{(m-1)} \left((x^2-1)^{\cancel{m}\,n}\right)_x^{(n)} \Big|_{-1}^{1} -$$

$$- \int_{-1}^{1} ((x^2-1)^m)_x^{(m-1)} ((x^2-1)^n)_x^{(n+1)} dx =$$

$$= - \int_{-1}^{1} ((x^2-1)^m)_x^{(m-1)} ((x^2-1)^n)_x^{(n+1)} dx \ominus$$

Т.к. $((x^2-1)^m)_x^{(m-1)} = 0$, ве при $x = \pm 1$, т.к. $x = \pm 1$ — корни $(x^2-1)^m$ (при дифф-и кратность корней умен. на 1)

$$\ominus (-1)^m \int_{-1}^{1} (x^2-1)^m \, \alpha ((x^2-1)^n)_x^{(m+n)} dx$$

$m > n \Rightarrow m+n > 2n \Rightarrow ((x^2-1)^n)_x^{(m+n)} = 0 \Rightarrow$

$\Rightarrow I = 0.$

d) $\tilde{f}(x) = C_0(f) P_0 + C_1(f) P_1 + C_2(f) P_2 =$

$= 1 - \cos 1 + (3\sin 1 - 3\cos 1)x +$

$+ (15 \sin 1 + 10 \cos 1 - \frac{35}{2}) \frac{1}{2} (3x^2 - 1)$

$$\|f(x) - \tilde{f}(x)\|^2 = \int_0^1 (\sin(x) - \tilde{f}(x))^2 dx =$$

$= \frac{1}{8}(-755 + 759 \sin(1) - 245 \sin(2) + 657 \cos(1) +$

$+ 33 \cos(2)) \Rightarrow \|f(x) - \tilde{f}(x)\| \approx 0,517732$