30 апреля.
Мат ан.
ДЗ № 5.

1. $\sum\limits_{n=1}^{+\infty} \frac{\operatorname{arctg}(n^2+n)}{3^n - n^2}$

$\frac{\operatorname{arctg}(n^2+n)}{3^n-n^2} \leq \frac{\pi}{2} \cdot \frac{1}{3^n - n^2} \sim \frac{\pi}{2} \cdot \frac{1}{3^n} \leq$

$\leq \frac{\pi}{2} \cdot \frac{1}{n^2}$ – сход.

Ответ: сх.

2. $\sum\limits_{n=1}^{\infty} \left(\cos \frac{1}{\sqrt{n}}\right)^{n^2}$

$\sqrt[n]{a_n} = \left(\cos \frac{1}{\sqrt{n}}\right)^n = \left(-\left(1 - \cos\frac{1}{\sqrt{n}}\right) + 1\right)^n =$

$= \left(\left(1 + \left(-\frac{1}{2n}\right)\right)^{-2n}\right)^{-\frac{1}{2}} = e^{-\frac{1}{2}} < 1 \Rightarrow$ сх.

по радик. Коши

Ответ: сх.

3. $\sum\limits_{n=1}^{+\infty} \frac{3^{2n}(2n)!}{n^n \, n!}$

$\frac{a_{n+1}}{a_n} = \frac{3^{2n+2} \cdot (2n+2)! \cdot n^n \cdot n!}{(n+1)^{n+1} (n+1)! \cdot 3^{2n} \cdot (2n)!} =$

$= \frac{9(2n+1)(2n+2) \cdot n^n}{(n+1)^2 (n+1)^n} \sim$

$$\sim \frac{9(2n+1)(2n+2)}{e(n+1)^2} \sim \frac{36}{e} > 1 \Rightarrow \text{расх}$$

по Даламберу

Ответ: расх.

4. $\sum\limits_{n=1}^{\infty} \frac{1}{(\ln n)^{\ln n}}$

$\sum\limits_{n=2}^{\infty} \frac{1}{(\ln n)^{\ln n}}$ $\quad \frac{1}{(\ln n)^{\ln n}}$ монотон

$$\int\limits_2^{\infty} \frac{1}{(\ln x)^{\ln x}} dx \overset{t=\ln x}{=} \int\limits_2^{\infty} \left(\frac{e}{t}\right)^t dt \leq$$

$$\left(\frac{e}{t}\right)^t \leq \left(\frac{1}{2}\right)^t$$

$$\leq \int\limits_2^{\infty} 2^{-t} dt = -\frac{1}{\ln 2} e^{-\ln 2 \cdot t} \Big|_2^{\infty} = \frac{1}{-\ln 2} -$$

$$-\frac{1}{\ln 2} 2^{-2} \Rightarrow \text{сх.}$$

Ответ: сх.

5. $\sum\limits_{n=1}^{\infty} (\sqrt[n]{n} - 1)$

$$a_n = e^{\frac{\ln n}{n}} - 1 \sim \frac{\ln n}{n}$$

$$\int\limits_1^{\infty} \frac{\ln x}{x} dx = \int\limits_1^{\infty} \ln x \, d\ln x = \frac{\ln^2 x}{2} \Big|_1^{\infty} -$$

– расх.

Ответ: расх.

6. $\sum\limits_{n=1}^{\infty} \frac{1\cdot 4\cdot \ldots \cdot (3n-2)\cdot 2\cdot 5 \cdot \ldots \cdot (3n+2)}{n!\,(n+1)!\cdot 9^n}$

$$\frac{a_n}{a_{n+1}} = \frac{(n+1)!\,(n+2)!\,9^{n+1}}{n!\,(n+1)!\,9^n\,(3n+1)(3n+5)} =$$

$$= \frac{9(n+1)(n+2)}{(3n+1)(3n+5)}$$

$$n\cdot\left(\frac{a_n}{a_{n+1}} - 1\right) = n\left(\frac{27n+18-18n-5}{9n^2+18n+5}\right) =$$

$$= \frac{9n^2+13n}{9n^2+18n+5}$$

$$\ln n\left(\frac{9n^2+13n}{9n^2+18n+5} - 1\right) = \ln n\left(\frac{-5n-5}{9n^2+18n+5}\right) \to$$

$\xrightarrow[n\to\infty]{} 0 < 1 \Rightarrow$ расх. по Бертрану

Ответ: расх.

7. $\sum\limits_{n=1}^{\infty} \frac{\ln(1+n^{\alpha})}{\sqrt{1+n^{\alpha}}} \circledast \quad]\, t = \sqrt{1+n^{\alpha}}$

$\cancel{\alpha < 0 \Rightarrow n^{\alpha} \to 0}$

$\cancel{a_n \sim \frac{n^{\alpha}}{\sqrt{1+n^{\alpha}}} = \frac{n^{\frac{\alpha}{2}}}{\sqrt{\frac{1}{n^{\alpha}}+1}} \to 0}$

$\cancel{\circledast\ \frac{\ln t^2}{t} = \frac{2\ln t}{t}}$

$\cancel{\alpha = 0 \quad \frac{\ln 1}{\sqrt{2}} = 0 \quad \sum_{n=1}^{\infty} 0 \text{ сх.}}$

$\cancel{\alpha > 0: \; t \to \infty}$

$\cancel{\sum_{t=}^{\infty} \frac{2\ln t}{t} \text{ расх. (по № 5)}}$

$\cancel{\alpha < 0: \; t \to 1}$

~~ln~~

$$\int_1^{\infty} \frac{\ln(1+n^{\alpha})}{\sqrt{1+n^{\alpha}}} dn \underset{\alpha>0}{\overset{1}{=}} \int_1^{\infty} \frac{2\ln t}{t} d(t^2-1)^{\frac{1}{\alpha}} =$$

(II)

$$= \int_1^{\infty} \frac{2\ln t \, (t^2-1)^{\frac{1-\alpha}{\alpha}} 2t}{t} dt =$$

$$= \int_1^{\infty} 4\ln t \, (t^2-1)^{\frac{1}{\alpha}-1} dt$$

$$\ln t \, (t^2-1)^{\frac{1}{\alpha}-1} \sim \ln t \, \frac{t^{\frac{2}{\alpha}}}{t^2} = \ln t \, \frac{1}{t^{2-\frac{2}{\alpha}}}$$

$2 - \frac{2}{\alpha} \leq 1 \Rightarrow \alpha \leq 2$ расх.

$2 - \frac{2}{\alpha} > 1 \Rightarrow \alpha > 2$ сх.

$\alpha < 0: \; (=) \int_{\sqrt{2}}^{1} \frac{2\ln t}{t} d(t^2-1)^{\frac{1}{\alpha}}$ сх.

$\alpha = 0$: сх. <u>Ответ:</u> $0 < \alpha \leq 2$ расх

$\alpha \leq 0 \cup \alpha > 2$ сх.

8. $a_n = \left(1 - \sin\frac{\pi n^2}{2n^2+1}\right)^{\alpha}$

$\sin\frac{\pi n^2}{2n^2+1} = \cos\left(\frac{\pi}{2} - \frac{\pi n^2}{2n^2+1}\right) =$

$= \cos\frac{2\pi n^2 + \pi - 2\pi n^2}{4n^2+2} = \cos\frac{\pi}{4n^2+2}$

$\left(1 - \cos\frac{\pi}{4n^2+2}\right)^{\alpha} = \left(\frac{\pi^2}{2(4n^2+2)^2}\right)^{\alpha} \sim \left(\frac{\pi^2}{32 \cdot n^4}\right)^{\alpha}$

$4\alpha > 1 \Rightarrow \alpha > \frac{1}{4}$ сх.

$4\alpha \leq 1 \Rightarrow \alpha \leq \frac{1}{4}$ расх.