Potential Flow Around a Regular Body in 2D

1 Settling the Problem

Suppose the fluids flows by given vector velocity field u. The velocity field can be written as the gradient $\nabla\phi$ of a velocity potential ϕ.

$$u = \nabla\phi \tag{1}$$

Let we assume that vector field u to be irrotational. It means $curl(u) = 0$. We can also write $\nabla \times u = 0$.

We consider the flow to be incompressible, we have $\nabla \cdot u = 0$. If we substitute (1), we have

$$\nabla \cdot u = 0 \tag{2}$$
$$\nabla \cdot \nabla\phi = 0 \tag{3}$$
$$\Delta\phi = 0 \tag{4}$$

From here we get:

$$\Delta\phi = 0 \tag{5}$$

Let consider fluid in an infinite domain. We have the velocity u_0 along x in one direction. This conditions can be written as

$$u = u_0\vec{e_x} \tag{6}$$

From (5) and (6), we have problem:

$$\Delta\phi = 0 \tag{7}$$
$$u = \nabla\phi \to u_0\vec{e_x} \quad \text{as } x \to \pm\infty \tag{8}$$

This problem has obvious solution i.e $\phi = u_0 x$.

Let we add a regular obstacle Ω_0. We consider $\partial\Omega_0$ to be C^1. It means the obstacle doesn't have angle nor discontinuity. On the border of the obstacle, the velocity is tangential.

$$u \cdot n = 0 \tag{9}$$

with n is the (exterior) normal vector of Ω_0. We substitute $u = \nabla\phi$ to (9), we get

$$\nabla\phi \cdot n = 0 \tag{10}$$

Equation (10) is called Homogeneous Neumann Boundary Condition. From equation (1), (5), (6) and (10), we obtain the problem:

$$\Delta\phi = 0 \quad \text{On: } \mathbb{R}^2 \setminus \overline{\Omega_0} \tag{11}$$
$$u = \nabla\phi \to u_0\vec{e_x} \quad \text{as } x \to \pm\infty \tag{12}$$
$$\nabla\phi \cdot n = 0 \quad \partial\Omega_0 = \Gamma_0 \tag{13}$$

We have to find the harmonic function that satisfies all the boundary conditions.

Without the obstacle, problem (7) has solution $u_0 x$. We can use it to find ϕ. Let we do the changing variable in order to get the solution of problem (11). Let $\psi = \phi - u_0 x$ and ϕ is the actual solution.

$$\psi = \phi - u_0 x \tag{14}$$
$$\Delta\psi = \Delta\phi - \Delta(u_0 x) \tag{15}$$
$$\Delta\psi = \Delta\phi \tag{16}$$
$$\nabla\psi = \nabla\phi - u_0 \vec{e_x} \tag{17}$$

when $x \to \pm\infty$ value of $\nabla\psi \to 0$. From here, we get

$$\nabla\psi \to 0 \quad \text{as } x \to \pm\infty \tag{18}$$

From the last condition in (11) we get

$$\nabla\psi \cdot n = \nabla(\phi - u_0 x) \cdot n \text{on } \Gamma_0 \tag{19}$$
$$= \nabla\phi \cdot n - \nabla(u_0 x) \cdot n \tag{20}$$
$$= 0 - u_0 \nabla x \cdot n \tag{21}$$
$$= -u_0 \vec{e_x} \cdot n \tag{22}$$
$$= -u_0 n_x \tag{23}$$

The problem after changing variable become

$$\Delta\psi = 0 \quad \text{in } \mathbb{R}^2 \setminus \overline{\Omega_0} \tag{24}$$
$$\nabla\psi \to 0 \quad \text{as } x \to \pm\infty \tag{25}$$
$$\nabla\psi \cdot n = -u_0 n_x \quad \text{on } \Gamma_0 \tag{26}$$

Once we solved the problem, if we want to retreive the velocity field, we simply compute:

$$u = \nabla\phi = \nabla(\psi + u_0 x) \tag{27}$$
$$= \nabla\psi + u_0 \vec{e_x} \tag{28}$$

Remark 1. *Uniqueness of solution problem* (24) *is not ensured. If ψ is a solution of* (24) *then $\tilde{\psi} = \psi + const$ is also a solution. This is not really a physical problem, because the only thing that matters to us is $\nabla\psi$.*

$$\nabla\psi = \nabla\tilde{\psi} \tag{29}$$

From this remark, we have to select one of the solutions. We do it by adding a condition:

$$\psi \to 0 \quad \text{as} x \to \pm\infty, y \to \pm\infty \tag{30}$$

If ψ supposed regular, then

$$\psi \to 0 \quad \text{as } x \to \pm\infty \tag{31}$$
$$\Rightarrow \nabla\psi \to 0 \quad \text{as } x \to \pm\infty \tag{32}$$

From here we get the actual problem is

$$\Delta\psi = 0 \quad \text{in } \mathbb{R}^2 \setminus \overline{\Omega_0} \tag{33}$$
$$\psi \to 0 \quad \text{as } x \to \pm\infty? \tag{34}$$
$$\nabla\psi \cdot n = -u_0 n_x \quad \text{on } \Gamma_0 \tag{35}$$

Remark 2. *We didn't proof the existence and uniqueness solution of the* (33) *nor their regularity. We assume that the problem has all things we need.*

2 The Boundary Integral Equation

Let $G_{x',y'}(x,y)$ be the Green function of Laplacian problem 2D.

$$G_{x',y'}(x,y) = \frac{1}{2\pi}\ln(\sqrt{(x-x')^2+(y-y')^2}) \tag{36}$$

G has some properties:

$$\Delta G_{x',y'} = \delta_{x',y'} \tag{37}$$
$$G_{x',y'} \to 0 \quad \text{at} \quad (x,y)\to\infty \tag{38}$$

Let we see the problem:

$$\Delta\psi = 0 \quad \text{in } \mathbb{R}^2\setminus\overline{\Omega_0} \tag{39}$$
$$\psi \to 0 \quad \text{as } x\to\pm\infty \tag{40}$$
$$\nabla\psi\cdot n = -u_0 n_x \quad \text{on } \Gamma_0 \tag{41}$$

Let us multiply the first equation in (39) with $G_{x',y'}$ and integrate over $\mathbb{R}^2\setminus\overline{\Omega_0}$. By integrating by parts, we get:

$$\delta\psi = 0 \tag{42}$$
$$\Leftrightarrow \int_{\mathbb{R}^2\setminus\overline{\Omega_0}} \Delta\psi(x,y)G_{x',y'}(x,y)\,dxdy = 0 \tag{43}$$
$$\Leftrightarrow \int_{\Gamma_0}(\nabla\psi\cdot n)G_{x',y'}\,ds - \int_{\mathbb{R}^2\setminus\overline{\Omega_0}}\nabla\psi(x,y)\nabla G_{x',y'}(x,y)\,dxdy = 0 \tag{44}$$
$$\Leftrightarrow \int_{\Gamma_0}\Gamma_0(\nabla\psi\cdot n)G_{x',y'}\,ds - \left\{\int_{\Gamma_0}\psi\nabla G_{x',y'}(x,y)\cdot n\,ds1 - \int_{\mathbb{R}^2\setminus\overline{\Omega_0}}\Delta G_{x',y'}(x,y)\psi(x,y)\,dxdy\right\} = 0 \tag{45}$$

let we postpone the result of (45) until these few remarks.

Remark 3. *If $(x',y')\in\mathbb{R}^2\setminus\overline{\Omega_0}$, then we obtain the following results:*

1. *By using boundary condition*

$$\int_{\Gamma_0}\psi\nabla G_{x',y'}(x,y)\cdot n\,ds1 = \int_{\Gamma_0} -u_0 n_x G_{x',y'}\,ds \tag{46}$$
$$= -u_0\int_{\Gamma_0} n_x G_{x',y'}\,ds \tag{47}$$

2. *By definition of GreenFunction and dirac distribution*

$$\int_{\mathbb{R}^2\setminus\overline{\Omega_0}}\Delta G_{x',y'}(x,y)\psi(x,y)\,dxdy = \langle\delta_{x',y'},\psi\rangle_{D',D} \tag{48}$$
$$= \psi_{x',y'} \tag{49}$$

From these remarks, we can sum up the equation (45) when $(x',y')\in\mathbb{R}^2\setminus\overline{\Omega_0}$, not in boundary as

$$-u_0\int_{\Gamma_0} n_x G_{x',y'}(x,y)\,ds - \int_{\Gamma_0}\psi\nabla G_{x',y'}(x,y)\cdot\vec{n}\,ds + \psi_{x',y'} = 0 \tag{50}$$
$$\psi_{x',y'}(x,y) = \int_{\Gamma_0} n_x G_{x',y'}(x,y) + \psi\nabla G_{x',y'}(x,y)\cdot\vec{n}\,ds \tag{51}$$

We call equation (51) as Boundary Integral Formulation in $(x', y') \in \mathbb{R}^2 \setminus \overline{\Omega_0}$.

Next, We want to compute ψ on the boundary. It is because essentially, from (51), the value of ψ inside the domain, depend on the values ψ terms on the boundary. In order to obtain a closed integral equation, we want to evaluate $\psi(x', y')$ when $(x', y') \to \Gamma_0$. When we do this calculation, the problem appears. If $(x', y') \in \Gamma_0$, the singularities of G ang ∇G are on the path of integration.

To overcome these singularities, let us approximate the integral on Γ_0 with an integral on ${\Gamma_0}^{\varepsilon}$. With $\varepsilon > 0$.

picture here...

From the picture, as $\varepsilon \to 0$, we expect $\displaystyle\int_{{\Gamma_0}^{\varepsilon}=\Omega_\varepsilon} (\cdot) \to \int_{\Gamma_0} (\cdot)$. We know how to compute $\displaystyle\int_{{\Gamma_0}^{\varepsilon}=\Omega_\varepsilon} (\cdot)$, there are no singularities.

Let we calculate the Boundary Integral Formulation on the boundary. If $(x', y') \in \Gamma_0$, let we multiply equation (39) by $G_{x',y'}$ and integrate over Ω_ε.

$$\int_{\Omega_\varepsilon} \Delta\psi(x, y) G_{x',y'}(x, y) = 0 \tag{52}$$

$$\int_{\partial\Omega_\varepsilon} (G_{x',y'} \nabla\psi \cdot n) - \left\{ \int_{\partial\Omega_\varepsilon} \psi \nabla G_{x',y'}(x, y) \cdot n - \int_{\Omega_\varepsilon} \psi(x, y) \Delta G_{x',y'}(x, y) \right\} = 0 \tag{53}$$

$$-\int_{\partial\Omega_\varepsilon} G_{x',y'} u_0 n_x - \int_{\partial\Omega_\varepsilon} \psi \nabla G_{x',y'} \cdot n + 0 = 0 \tag{54}$$

$$\int_{\partial\Omega_\varepsilon} u_0 G_{x',y'} n_x + \psi \nabla G_{x',y'} \cdot n = 0 \tag{55}$$

Let us examine $\displaystyle\int_{\partial\Omega_\varepsilon} u_0 G_{x',y'} n_x + \psi \nabla G_{x',y'} \cdot n$ as $\varepsilon \to 0$

$$\int_{\partial\Omega_\varepsilon} u_0 G_{x',y'} n_x + \psi \nabla G_{x',y'} \cdot n = \int_{C_\varepsilon} u_0 G_{x',y'} n_x + \psi \nabla G_{x',y'} \cdot n + \int_{\partial\Omega_\varepsilon \setminus C_\varepsilon} u_0 G_{x',y'} n_x + \psi \nabla G_{x',y'} \cdot n \tag{56}$$

When $\varepsilon \to 0$, if $\displaystyle\int_{\partial\Omega_\varepsilon \setminus C_\varepsilon} u_0 G_{x',y'} n_x + \psi \nabla G_{x',y'} \cdot n$ has a limit, then it is called The cauchy Principal Value. We denote it $\displaystyle⨍_{\partial\Omega} u_0 G_{x',y'} n_x + \psi \nabla G_{x',y'} \cdot n$.

let us examine $\displaystyle\int_{C_\varepsilon} u_0 G_{x',y'} n_x + \psi \nabla G_{x',y'} \cdot n$ as $\varepsilon \to 0$.

$$\int_{C_\varepsilon} u_0 G_{x',y'} n_x + \psi \nabla G_{x',y'} \cdot n = \int_{\theta^*}^{\theta^*+\pi} \left\{ \frac{1}{2\pi} \ln(\varepsilon) u_0 n_x + \psi \frac{1}{2\pi} \frac{1}{\varepsilon} \right\} \varepsilon \, d\theta \tag{57}$$

$$= \frac{1}{2\pi} \varepsilon \ln \varepsilon \int_{\theta^*}^{\theta^*+\pi} u_0 n_x \, d\theta + \int_{\theta^*}^{\theta^*+\pi} \frac{1}{2\pi} \psi \, d\theta \tag{58}$$

$$= \int_{\theta^*}^{\theta^*+\pi} \frac{1}{2\pi} \psi \, d\theta \tag{59}$$

$$= \frac{1}{2\pi} \psi \int_{\theta^*}^{\theta^*+\pi} d\theta = \frac{1}{2\pi} \psi(x', y') \tag{60}$$

Result from (60) is called a "Residual Term".

Now, we can compute (55) when $\varepsilon \to 0$ as follows:

$$\int_{\partial\Omega_\varepsilon} u_0 G_{x',y'} n_x + \psi \nabla G_{x',y'} \cdot n = 0 \tag{61}$$

$$\fint_{\partial\Omega} u_0 G_{x',y'} n_x + \psi \nabla G_{x',y'} \cdot n + \frac{1}{2\pi}\psi(x',y') = 0 \tag{62}$$

$$\frac{1}{2\pi}\psi(x',y') = -\fint_{\partial\Omega} u_0 G_{x',y'} n_x + \psi \nabla G_{x',y'} \cdot n \tag{63}$$

Next, we can calculate all these terms with numerical method.

3 Numerical Methods

3.1 Solving Boundary Integral Formulation on an Obstacle

We want to solve numerically Boundary Integral Formulation on the ostacle as we get in equation (63). First, we consider that on the boundary of obstacle, we have N partitions. In each partitions we have

$$\psi(x,y) = \sum_{i=1}^{N} \psi_i e_i(x,y) \tag{64}$$

with $\psi_i = \psi(x_i, y_i)$ and

$$e_i(x,y) = \left\{ \begin{array}{ll} 1 & \text{if } (x,y) \in S_i \\ 0 & \text{otherwise} \end{array} \right. = \chi_{S_i}$$

.

picture here...

For i from 1 to n we can write (63) as

$$\underbrace{\frac{1}{2\pi}\psi(x_i,y_i)}_{1} + \underbrace{\fint_{\partial\Omega} \psi \nabla G_{x_i,y_i} \cdot n}_{2} = \underbrace{-\fint_{\partial\Omega} G_{x_i,y_i} u_0 n_x}_{3} \tag{65}$$

Let we see each items:

1. $\frac{1}{2\pi}\psi(x_i,y_i) = \frac{1}{2\pi}\psi_i$

2. $\displaystyle\fint_{\partial\Omega} \psi \nabla G_{x_i,y_i} \cdot n = \fint_{\partial\Omega} \left\{ \sum_{j=1}^{N} \psi_j \varepsilon_j \right\} \nabla G_{x_i,y_i} \cdot n$

$$\fint_{\partial\Omega} \psi \nabla G_{x_i,y_i} \cdot n = \fint_{\partial\Omega} \left\{ \sum_{j=1}^{N} \psi_j \varepsilon_j \right\} \nabla G_{x_i,y_i} \cdot n \tag{66}$$

$$= \fint_{\partial\Omega} \left\{ \sum_{j=1}^{N} \psi_j \varepsilon_j \nabla G_{x_i,y_i} \cdot n \right\} \tag{67}$$

$$= \sum_{j=1}^{N} \left\{ \fint_{\partial\Omega} \psi_j e_j \nabla G_{x_i,y_i} \cdot n \right\} \tag{68}$$

$$= \sum_{j=1}^{N} \psi_j \left\{ \fint_{\partial\Omega} \nabla G_{x_i,y_i} \cdot n e_j \right\} \tag{69}$$

$$= \sum_{j=1}^{N} \psi_j \left\{ \fint_{S_j} \nabla G_{x_i,y_i} \cdot n \right\} \tag{70}$$

$$= \sum_{j=1}^{N} \psi_j m_{ij} = (M\psi)_i \tag{71}$$

with

$$M = (m_{ij})_{\substack{i=1\cdots N \\ j=1\cdots N}}$$

and

$$\psi = (\psi_j)_{j=1,\cdots,N}.$$

3.2 Solving Boundary Integral Formulation in Domain