

Potential Flow Around a Regular Body in 2D

1 Setting of the Problem

Suppose the fluid flow is described by velocity vector field u . The velocity field can be written as the gradient of a velocity potential ϕ .

$$u = \nabla\phi \quad (1)$$

Let us assume that the vector field u is irrotational. It means $\text{curl}(u) = 0$. We can also write $\nabla \times u = 0$.

We consider the flow to be incompressible, we have $\nabla \cdot u = 0$. If we substitute (1), we have

$$\nabla \cdot u = 0 \quad (2)$$

$$\nabla \cdot \nabla\phi = 0 \quad (3)$$

$$\Delta\phi = 0 \quad (4)$$

From here we get:

$$\Delta\phi = 0 \quad (5)$$

Let us consider a fluid in an infinite domain. Far from the obstacle, we suppose that the velocity is uniform. Without loss of generality, we assume that its direction is \vec{e}_x . This condition can be written as

$$u = u_0\vec{e}_x \quad \text{as } |x| \rightarrow \infty, |y| \rightarrow \infty \quad (6)$$

From (5) and (6), we obtain the system:

$$\Delta\phi = 0 \quad (7)$$

$$u = \nabla\phi \rightarrow u_0\vec{e}_x \quad \text{as } x \rightarrow \pm\infty \quad (8)$$

Without any obstacle, this problem has the obvious solution i.e $\phi = u_0x$.

Let us add a regular obstacle Ω_0 . We consider $\partial\Omega_0$ to be C^1 . It means the obstacle doesn't have any angles nor discontinuities. On the border of the obstacle, the velocity is tangential.

$$u \cdot n = 0 \quad (9)$$

with n is the (exterior) normal vector of Ω_0 . We substitute $u = \nabla\phi$ to (9), we get

$$\nabla\phi \cdot n = 0 \quad (10)$$

Equation (10) is called an homogeneous Neumann boundary condition.

From equation (1), (5), (6) and (10), we obtain the problem:

$$\Delta\phi = 0 \quad \text{On: } \mathbb{R}^2 \setminus \overline{\Omega_0} = \Omega \quad (11)$$

$$u = \nabla\phi \rightarrow u_0\vec{e}_x \quad \text{as } x \rightarrow \pm\infty \quad (12)$$

$$\nabla\phi \cdot n = 0 \quad \partial\Omega_0 = \Gamma_0 \quad (13)$$

The problem is hence to find the harmonic function that satisfies all the boundary conditions.

Without the obstacle, problem (7) has solution u_0x . We can use it to find ϕ . Let us perform a change of variable in order to get the solution of problem (11). Let us define

$$\psi = \phi - u_0x \quad (14)$$

If we substitute (14) to the problem (11) we can obtain as follows:

$$\Delta\psi = \Delta\phi - \Delta(u_0x) \quad (15)$$

$$\Delta\psi = \Delta\phi \quad (16)$$

$$\nabla\psi = \nabla\phi - u_0\vec{e}_x \quad (17)$$

when $x \rightarrow \pm\infty$ value of $\nabla\psi \rightarrow 0$. From here, we get

$$\nabla\psi \rightarrow 0 \quad \text{as } x \rightarrow \pm\infty \quad (18)$$

From the last condition in (11) we get

$$\nabla\psi \cdot n = \nabla(\phi - u_0x) \cdot n \text{ on } \Gamma_0 \quad (19)$$

$$= \nabla\phi \cdot n - \nabla(u_0x) \cdot n \quad (20)$$

$$= 0 - u_0\nabla x \cdot n \quad (21)$$

$$= -u_0\vec{e}_x \cdot n \quad (22)$$

$$= -u_0n_x \quad (23)$$

The problem become, after changing variable:

$$\Delta\psi = 0 \quad \text{in } \Omega \quad (24)$$

$$\nabla\psi \rightarrow 0 \quad \text{as } x \rightarrow \pm\infty \quad (25)$$

$$\nabla\psi \cdot n = -u_0n_x \quad \text{on } \Gamma_0 \quad (26)$$

Once we solved the problem, if we want to retrieve the velocity field, we simply compute:

$$u = \nabla\phi = \nabla(\psi + u_0x) \quad (27)$$

$$= \nabla\psi + u_0\vec{e}_x \quad (28)$$

Remark 1. Uniqueness of solution problem (24) is not ensured. If ψ is a solution of (24) then $\tilde{\psi} = \psi + \text{const}$ is also a solution. This is not really a physical problem, because the only thing that matters to us is $\nabla\psi$.

$$\nabla\psi = \nabla\tilde{\psi} \quad (29)$$

From here we get the actual problem is

$$\Delta\psi = 0 \quad \text{in } \Omega \quad (30)$$

$$\nabla\psi \rightarrow 0 \quad \text{as } x \rightarrow \pm\infty? \quad (31)$$

$$\nabla\psi \cdot n = -u_0n_x \quad \text{on } \Gamma_0 \quad (32)$$

Remark 2. We didn't proof the existence and uniqueness solution of the (30) nor their regularity. We assume that the problem has all things we need. In order to obtain the uniqueness, we will have to set the constant?, which will be done by selecting a particular Green function.

2 The Boundary Integral Equation

Let $G_{x',y'}(x, y)$ be the Green function of Laplacian problem 2D.

$$G_{x',y'}(x, y) = \frac{1}{2\pi} \ln(\sqrt{(x - x')^2 + (y - y')^2}) \quad (33)$$

G has some properties:

$$\Delta G_{x',y'} = \delta_{x',y'} \quad (34)$$

$$\nabla G_{x',y'} \rightarrow 0 \quad \text{at} \quad (x, y) \rightarrow \pm\infty \quad (35)$$

Let us examine the problem:

$$\Delta\psi = 0 \quad \text{in } \Omega \quad (36)$$

$$\psi \rightarrow 0 \quad \text{as } x \rightarrow \pm\infty \quad (37)$$

$$\nabla\psi \cdot n = -u_0 n_x \quad \text{on } \Gamma_0 \quad (38)$$

If we multiply the first equation in (36) with $G_{x',y'}$ and integrate over Ω . By integrating by parts, we get:

$$\Delta\psi = 0 \quad (39)$$

$$\Rightarrow \int_{\Omega} \Delta\psi G_{x',y'} = 0 \quad (40)$$

$$\Rightarrow \int_{\Gamma_0} (\nabla\psi \cdot n) G_{x',y'} - \int_{\Omega} \nabla\psi \nabla G_{x',y'} = 0 \quad (41)$$

$$\Rightarrow \int_{\Gamma_0} (\nabla\psi \cdot n) G_{x',y'} - \left\{ \int_{\Gamma_0} \psi \nabla G_{x',y'} \cdot n - \int_{\Omega} \Delta G_{x',y'} \psi \right\} = 0 \quad (42)$$

let us postpone the result of (42) until these few remarks.

Remark 3. If $(x', y') \in \Omega = \mathbb{R}^2 \setminus \overline{\Omega_0}$, then we obtain the following results:

1. By using boundary condition

$$\int_{\Gamma_0} \psi \nabla G_{x',y'}(x, y) \cdot n \, ds = \int_{\Gamma_0} -u_0 n_x G_{x',y'} \, ds \quad (43)$$

$$= -u_0 \int_{\Gamma_0} n_x G_{x',y'} \, ds \quad (44)$$

2. By definition of Green function and dirac distribution

$$\int_{\mathbb{R}^2 \setminus \overline{\Omega_0}} \Delta G_{x',y'}(x, y) \psi(x, y) \, dx dy = \langle \delta_{x',y'}, \psi \rangle_{D', D} \quad (45)$$

$$= \psi_{x',y'} \quad (46)$$

From these remarks, we can sum up the equation (42) when $(x', y') \in \mathbb{R}^2 \setminus \overline{\Omega_0}$, not in boundary as

$$-u_0 \int_{\Gamma_0} n_x G_{x',y'}(x, y) \, ds - \int_{\Gamma_0} \psi \nabla G_{x',y'}(x, y) \cdot \vec{n} \, ds + \psi_{x',y'} = 0 \quad (47)$$

$$\psi_{x',y'}(x, y) = - \int_{\Gamma_0} -u_0 n_x G_{x',y'}(x, y) - \psi \nabla G_{x',y'}(x, y) \cdot \vec{n} \, ds \quad (48)$$

We call equation (48) as Boundary Integral Formulation in $(x', y') \in \mathbb{R}^2 \setminus \overline{\Omega_0}$.

Next, We want to compute ψ on the boundary. It is because essentially, from (48), the value of ψ inside the domain, depend on the values ψ terms on the boundary. In order to obtain a closed integral equation, we want to evaluate $\psi(x', y')$ when $(x', y') \rightarrow \Gamma_0$. When we do this calculation, the problem appears. If $(x', y') \in \Gamma_0$, the singularities of G and ∇G are on the path of integration.

To overcome these singularities, let C_ε denote the sphere of radius ε with it's center in (x', y') , on the surface Γ_0 , and let Γ_0^ε denote the union of the Γ_0 and C_ε .

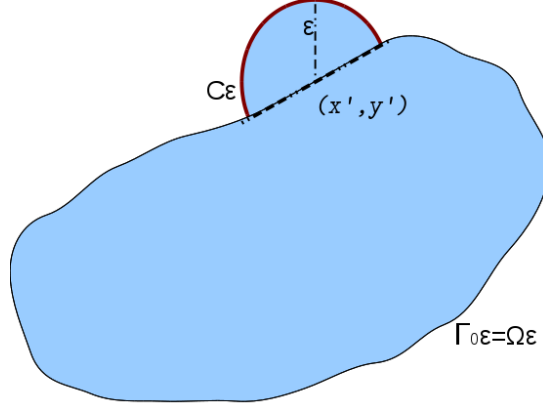


Figure 1: Domain to Avoid Singularity

Let us approximate the integral on Γ_0 with an integral on Γ_0^ε with $\varepsilon > 0$. From the Figure 1, as $\varepsilon \rightarrow 0$, we expect $\int_{\Gamma_0^\varepsilon = \Omega_\varepsilon} (\cdot) \rightarrow \int_{\Gamma_0} (\cdot)$. We know how to compute $\int_{\Gamma_0^\varepsilon = \Omega_\varepsilon} (\cdot)$, there are no singularities.

Let we calculate the Boundary Integral Formulation on the boundary. If $(x', y') \in \Gamma_0$, let we multiply equation (36) by $G_{x', y'}$ and integrate over Ω_ε .

$$\int_{\Omega_\varepsilon} \Delta \psi(x, y) G_{x', y'}(x, y) = 0 \quad (49)$$

$$\int_{\partial \Omega_\varepsilon} (G_{x', y'} \nabla \psi \cdot n) - \left\{ \int_{\partial \Omega_\varepsilon} \psi \nabla G_{x', y'}(x, y) \cdot n - \int_{\Omega_\varepsilon} \psi(x, y) \Delta G_{x', y'}(x, y) \right\} = 0 \quad (50)$$

$$- \int_{\partial \Omega_\varepsilon} G_{x', y'} u_0 n_x - \int_{\partial \Omega_\varepsilon} \psi \nabla G_{x', y'} \cdot n + 0 = 0 \quad (51)$$

$$\int_{\partial \Omega_\varepsilon} u_0 G_{x', y'} n_x + \psi \nabla G_{x', y'} \cdot n = 0 \quad (52)$$

Let us examine $\int_{\partial \Omega_\varepsilon} u_0 G_{x', y'} n_x + \psi \nabla G_{x', y'} \cdot n$ as $\varepsilon \rightarrow 0$

$$\int_{\partial \Omega_\varepsilon} u_0 G_{x', y'} n_x + \psi \nabla G_{x', y'} \cdot n = \int_{C_\varepsilon} u_0 G_{x', y'} n_x + \psi \nabla G_{x', y'} \cdot n + \int_{\partial \Omega_\varepsilon \setminus C_\varepsilon} u_0 G_{x', y'} n_x + \psi \nabla G_{x', y'} \cdot n \quad (53)$$

When $\varepsilon \rightarrow 0$, if $\int_{\partial \Omega_\varepsilon \setminus C_\varepsilon} u_0 G_{x', y'} n_x + \psi \nabla G_{x', y'} \cdot n$ has a limit, then it is called The cauchy

Principal Value. We denote it $\oint_{\partial \Omega} u_0 G_{x', y'} n_x + \psi \nabla G_{x', y'} \cdot n$.

let us examine $\int_{C_\varepsilon} u_0 G_{x',y'} n_x + \psi \nabla G_{x',y'} \cdot n$ as $\varepsilon \rightarrow 0$.

$$\int_{C_\varepsilon} u_0 G_{x',y'} n_x + \psi \nabla G_{x',y'} \cdot n = \int_{\theta^*}^{\theta^*+\pi} \left\{ \frac{1}{2\pi} \ln(\varepsilon) u_0 n_x + \psi \frac{1}{2\pi} \frac{1}{\varepsilon} \right\} \varepsilon d\theta \quad (54)$$

$$= \frac{1}{2\pi} \varepsilon \ln \varepsilon \int_{\theta^*}^{\theta^*+\pi} u_0 n_x d\theta + \int_{\theta^*}^{\theta^*+\pi} \frac{1}{2\pi} \psi d\theta \quad (55)$$

$$= \int_{\theta^*}^{\theta^*+\pi} \frac{1}{2\pi} \psi d\theta \quad (56)$$

$$= \frac{1}{2\pi} \psi \int_{\theta^*}^{\theta^*+\pi} d\theta \quad (57)$$

$$= \frac{1}{2\pi} \psi(x', y') \quad (58)$$

Result from (58) is called a “Residual Term”.

Now, we can compute (52) when $\varepsilon \rightarrow 0$ as follows:

$$\int_{\partial\Omega_\varepsilon} u_0 G_{x',y'} n_x + \psi \nabla G_{x',y'} \cdot n = 0 \quad (59)$$

$$\oint_{\partial\Omega} u_0 G_{x',y'} n_x + \psi \nabla G_{x',y'} \cdot n + \frac{1}{2\pi} \psi(x', y') = 0 \quad (60)$$

$$\frac{1}{2\pi} \psi(x', y') = - \oint_{\partial\Omega} u_0 G_{x',y'} n_x + \psi \nabla G_{x',y'} \cdot n \quad (61)$$

Next, we can calculate all these terms with numerical method.

3 Numerical Methods

3.1 Solving Boundary Integral Formulation on an Obstacle

We want to solve numerically Boundary Integral Formulation on the obstacle as we get in equation (61). First, we consider that on the boundary of obstacle, we have N partitions. In each partitions we have $\psi_i = \psi(x_i, y_i)$ and

$$e_i(x, y) = \begin{cases} 1 & \text{if } (x, y) \in S_i \\ 0 & \text{otherwise} \end{cases} = \chi_{S_i}$$

We obtain the value of ψ as

$$\psi(x, y) = \sum_{i=1}^N \psi_i e_i(x, y) \quad (62)$$

picture here...

For i from 1 to n we can write (61) as

$$\underbrace{\frac{1}{2\pi} \psi(x_i, y_i)}_1 + \underbrace{\oint_{\partial\Omega} \psi \nabla G_{x_i, y_i} \cdot n}_2 = - \underbrace{\oint_{\partial\Omega} G_{x_i, y_i} u_0 n_x}_3 \quad (63)$$

Let we see each items:

$$1. \quad \frac{1}{2\pi} \psi(x_i, y_i) = \frac{1}{2\pi} \psi_i$$

$$2. \oint_{\partial\Omega} \psi \nabla G_{x_i, y_i} \cdot n = \oint_{\partial\Omega} \left\{ \sum_{j=1}^N \psi_j e_j \right\} \nabla G_{x_i, y_i} \cdot n$$

$$\oint_{\partial\Omega} \psi \nabla G_{x_i, y_i} \cdot n = \oint_{\partial\Omega} \left\{ \sum_{j=1}^N \psi_j e_j \right\} \nabla G_{x_i, y_i} \cdot n \quad (64)$$

$$= \oint_{\partial\Omega} \left\{ \sum_{j=1}^N \psi_j \varepsilon_j \nabla G_{x_i, y_i} \cdot n \right\} \quad (65)$$

$$= \sum_{j=1}^N \left\{ \oint_{\partial\Omega} \psi_j e_j \nabla G_{x_i, y_i} \cdot n \right\} \quad (66)$$

$$= \sum_{j=1}^N \psi_j \left\{ \oint_{\partial\Omega} \nabla G_{x_i, y_i} \cdot n e_j \right\} \quad (67)$$

$$= \sum_{j=1}^N \psi_j \left\{ \oint_{S_j} \nabla G_{x_i, y_i} \cdot n \right\} \quad (68)$$

$$= \sum_{j=1}^N \psi_j m_{ij} = (M\psi)_i \quad (69)$$

with $M = (m_{ij})_{\substack{i=1 \dots N \\ j=1 \dots N}}$ and $\psi = (\psi_j)_{j=1, \dots, N}$.

$$3. - \oint_{\partial\Omega} G_{x_i, y_i} u_0 n_x$$

Since the domain is divided in line, we have $n = \begin{pmatrix} n_x \\ n_y \end{pmatrix}$ and $n_x = \sum_{i=1}^N n_{x_i} e_i$, $n_y = \sum_{i=1}^N n_{y_i} e_i$.

$$- \oint_{\partial\Omega} G_{x_i, y_i} u_0 n_x = -u_0 \oint_{\partial\Omega} G_{x_i, y_i} n_x = -u_0 \oint_{\partial\Omega} G_{x_i, y_i} \sum_{j=1}^N n_{x_j} e_j \quad (70)$$

$$= -u_0 \oint_{\partial\Omega} \sum_{j=1}^N n_{x_j} e_j G_{x_i, y_i} \quad (71)$$

$$= -u_0 \sum_{j=1}^N \oint_{\partial\Omega} n_{x_j} e_j G_{x_i, y_i} \quad (72)$$

$$= -u_0 \sum_{j=1}^N n_{x_j} \oint_{\partial\Omega} G_{x_i, y_i} e_j \quad (73)$$

$$= -u_0 \sum_{j=1}^N n_{x_j} \int_{S_j} G_{x_i, y_i} \quad (74)$$

$$= -u_0 (PN_x)_i \quad (75)$$

We will compute matrices $P = (p_{i,j}) = \int_{S_j} G_{x_i, y_i}$ and $M = (m_{i,j}) = \oint_{S_j} \nabla G_{x_i, y_i} \cdot n$ which the calculating process divides in two cases.

1. $i \neq j$

Let we see the point (x_i, y_i) towards the other points in partition, such in figure 2.

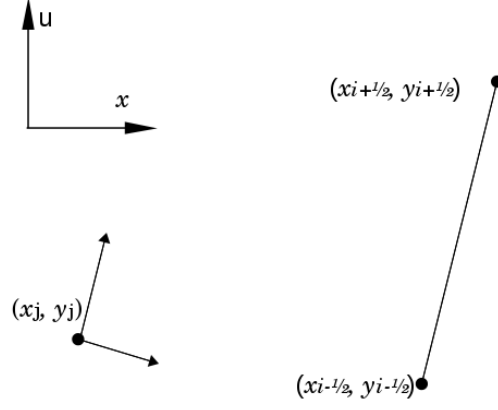


Figure 2: Points in obstacle

Since $\nabla G_{x_i, y_i}$ and G_{x_i, y_i} are radial functions, we can turn the axis in such a way more convenient to examine. In figure 3 we change the coordinate xu with $\eta\nu$. Let we say $H = x_i - x_{j-\frac{1}{2}}$ and θ is the angle that formed by r and ν .

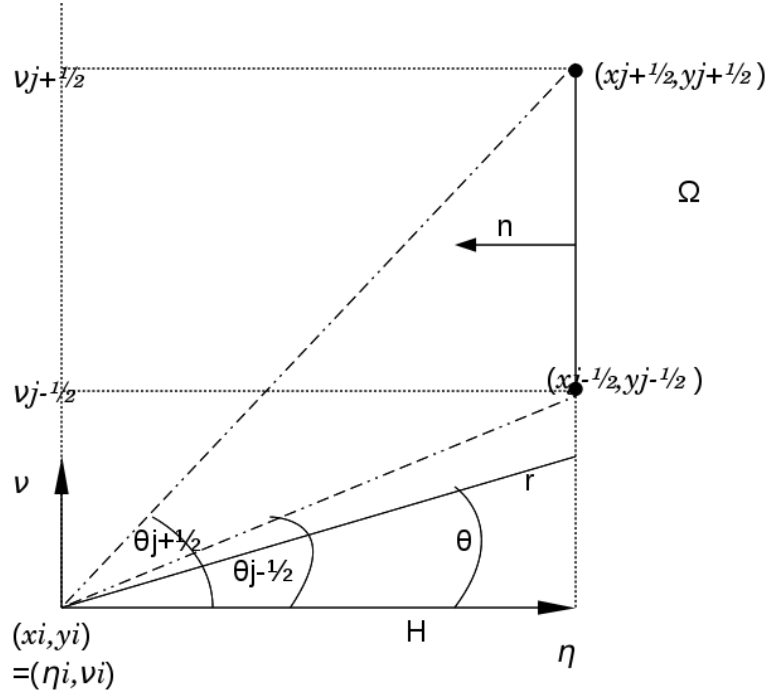


Figure 3: Change the axis and ordinat

In $\eta\nu$ coordinate, we can write Green function G as

$$G_{x_i, y_i}(\eta, \nu) = \frac{1}{2\pi} \ln \left(\sqrt{\eta^2 + \nu^2} \right) \quad (76)$$

we can also write η and ν in radial coordinates as follows:

$$\eta = r \cos \theta = H \quad (77)$$

$$\nu = r \sin \theta \quad (78)$$

$$\frac{\nu}{\eta} = \tan \theta \Leftrightarrow \nu = \eta \tan \theta \quad (79)$$

along the integration line, $\eta = H$.

$$\nu = H \tan \theta \Leftrightarrow d\nu = H \frac{1}{\cos^2 \theta} = H \sec^2 \theta d\theta \quad (80)$$

With using these values, we can get

$$p_{ij} = \oint_{S_j} G_{x_i, y_i} = \oint_{\nu_{j-\frac{1}{2}}}^{\nu_{j+\frac{1}{2}}} G_{x_i, y_i}(H, \nu) d\nu \quad (81)$$

$$= \int_{\theta_{j-\frac{1}{2}}}^{\theta_{j+\frac{1}{2}}} \frac{1}{2\pi} (\ln(H \sec \theta) H \sec^2 \theta) d\theta \quad (82)$$

$$= \frac{H}{2\pi} \int_{\theta_{j-\frac{1}{2}}}^{\theta_{j+\frac{1}{2}}} \ln(H \sec \theta) H \sec^2 \theta d\theta \quad (83)$$

$$= \frac{H}{2\pi} [\tan \theta \ln(H \sec \theta) + \theta - \tan \theta]_{\theta_{j-\frac{1}{2}}}^{\theta_{j+\frac{1}{2}}} \quad (84)$$

$$m_{ij} = \oint_{S_j} \nabla G_{x_i, y_i}(H, \nu) \cdot n \quad (85)$$

$$= \oint_{S_j} \frac{1}{2\pi} \left(\frac{\frac{2\eta}{2\sqrt{\eta^2+\nu^2}} \frac{1}{\sqrt{\eta^2+\nu^2}}}{\frac{2\nu}{2\sqrt{\eta^2+\nu^2}} \frac{1}{\sqrt{\eta^2+\nu^2}}} \right) \cdot n = \oint_{S_j} \frac{1}{2\pi} \frac{1}{\eta^2 + \nu^2} \begin{pmatrix} \nu \\ \eta \end{pmatrix} \cdot n d\nu \quad (86)$$

$$= \int_{\theta_{j-\frac{1}{2}}}^{\theta_{j+\frac{1}{2}}} \frac{1}{2\pi} \frac{1}{r^2} \begin{pmatrix} r \cos \theta \\ r \sin \theta \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 0 \end{pmatrix} H \sec^2 \theta d\theta \quad (87)$$

$$= \int_{\theta_{j-\frac{1}{2}}}^{\theta_{j+\frac{1}{2}}} \frac{-\cos \theta}{2\pi r} H \sec^2 \theta d\theta \quad (88)$$

$$= \int_{\theta_{j-\frac{1}{2}}}^{\theta_{j+\frac{1}{2}}} \frac{1}{2\pi H \sec^2 \theta} H \sec^2 \theta d\theta \quad (89)$$

$$= -\frac{1}{2\pi} \int_{\theta_{j-\frac{1}{2}}}^{\theta_{j+\frac{1}{2}}} d\theta \quad (90)$$

$$= -\frac{1}{2\pi} (\theta_{j+\frac{1}{2}} - \theta_{j-\frac{1}{2}}) \quad (91)$$

2. $i = j$

We can see figure 4 when $i = j$ in $\eta\nu$ coordinate.

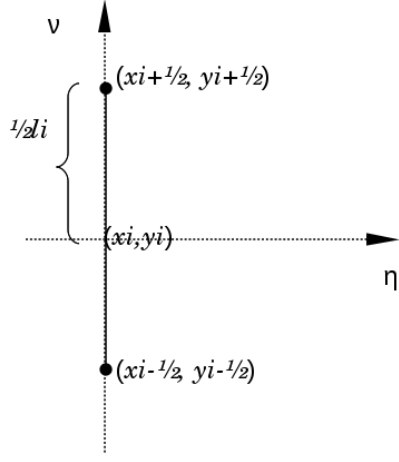


Figure 4: Position when i equal to j

$$P_{ij} = \oint_{S_i} G_{x_i, y_i} \quad (92)$$

$$= \lim_{\varepsilon \rightarrow 0} \int_{\nu_{i-\frac{1}{2}}}^{-\varepsilon} G_{x_i, y_i}(0, \nu) d\nu + \int_{\varepsilon}^{\nu_{i+\frac{1}{2}}} G_{x_i, y_i}(0, \nu) d\nu \quad (93)$$

$$= \lim_{\varepsilon \rightarrow 0} \int_{\nu_{i-\frac{1}{2}}}^{-\varepsilon} \frac{1}{2\pi} \ln(\sqrt{0 + \nu^2}) d\nu + \int_{\varepsilon}^{\nu_{i+\frac{1}{2}}} \frac{1}{2\pi} \ln(\sqrt{0 + \nu^2}) d\nu \quad (94)$$

$$= \lim_{\varepsilon \rightarrow 0} \int_{-\frac{l_i}{2}}^{-\varepsilon} \frac{1}{2\pi} \ln(|\nu|) d\nu + \int_{\varepsilon}^{\frac{l_i}{2}} \frac{1}{2\pi} \ln(|\nu|) d\nu \quad (95)$$

$$= \frac{2}{2\pi} \lim_{\varepsilon \rightarrow 0} \int_{\varepsilon}^{\frac{l_i}{2}} \frac{1}{2\pi} \ln(\nu) d\nu \quad (96)$$

$$= \frac{1}{\pi} \lim_{\varepsilon \rightarrow 0} (\nu \ln \nu - \nu)_{\varepsilon}^{\frac{l_i}{2}} \quad (97)$$

$$= \frac{1}{\pi} \lim_{\varepsilon \rightarrow 0} \left\{ \left(\frac{l_i}{2} \ln \left(\frac{l_i}{2} \right) - \frac{l_i}{2} \right) - (\varepsilon \ln \varepsilon - \varepsilon) \right\} \quad (98)$$

$$= \frac{1}{\pi} \left(\frac{l_i}{2} \ln \left(\frac{l_i}{2} \right) - \frac{l_i}{2} \right) \quad (99)$$

As long as G is a radial function, ∇G is also radial (radial vector). We obtain

$$m_{ij} = \oint_{S_j} \nabla G_{x_i, y_i} \cdot n \quad (100)$$

$$= \lim_{\varepsilon \rightarrow 0} \int_{-\frac{l_i}{2}}^{-\varepsilon} \nabla G_{x_i, y_i} \cdot n(0, \nu) + \int_{\varepsilon}^{\nu_{i+\frac{1}{2}}} \nabla G_{x_i, y_i} \cdot n(0, \nu) \quad (101)$$

$$= \lim_{\varepsilon \rightarrow 0} \int_{-\frac{l_i}{2}}^{-\varepsilon} \frac{1}{2\pi} \frac{1}{\eta^2 + \nu^2} \begin{pmatrix} \nu \\ \eta \end{pmatrix} \begin{pmatrix} -1 \\ 0 \end{pmatrix} (0, \nu) + \int_{\varepsilon}^{\nu_{i+\frac{1}{2}}} \frac{1}{2\pi} \frac{1}{\eta^2 + \nu^2} \begin{pmatrix} \nu \\ \eta \end{pmatrix} \begin{pmatrix} -1 \\ 0 \end{pmatrix} (0, \nu) \quad (102)$$

$$= 0 \quad (103)$$

From the calculation above, we can write (63) as following equation system:

$$\left(\frac{1}{2} I + M \right) \psi = -u_0 P N_x \quad (104)$$

with

$$m_{ij} = \begin{cases} 0 & \text{if } i = j \\ -\frac{1}{2\pi}(\theta_{j+\frac{1}{2}} - \theta_{j-\frac{1}{2}}) & \text{otherwise} \end{cases}$$

and

$$p_{ij} = \begin{cases} \frac{1}{\pi} \left(\frac{l_i}{2} \ln \left(\frac{l_i}{2} \right) - \frac{l_i}{2} \right) & \text{if } i = j \\ \frac{H}{2\pi} [\tan \theta \ln(H \sec \theta) + \theta - \tan \theta]_{\theta_{j-\frac{1}{2}}}^{\theta_{j+\frac{1}{2}}} & \text{otherwise} \end{cases}$$

3.2 Solving Boundary Integral Formulation in Domain

We have to solve boundary integral formulation in domain $(x', y') \in \Omega$ as in equation (48).

$$\psi_{x',y'}(x, y) = - \int_{\Gamma_0} (-u_0 n_x) G_{x',y'}(x, y) + \int_{\Gamma_0} \psi \nabla G_{x',y'}(x, y) \cdot \vec{n} \quad (105)$$

If we call $-u_0 n_x$ and ψ respectively with Neum and Diri, we can write 105 as

$$\psi_{x',y'}(x, y) = - \int_{\Gamma_0} \text{Neum} G_{x',y'}(x, y) + \int_{\Gamma_0} \text{Diri} \nabla G_{x',y'}(x, y) \cdot \vec{n} \quad (106)$$

For solving Boundary Integral Formulation inside the domain, in order to get the potential flow, we wish to obtain $u = \nabla \psi$.

$$\nabla \psi = \begin{pmatrix} \partial_{x'} \psi \\ \partial_{y'} \psi \end{pmatrix} \quad (107)$$

First, let we compute $\partial_{x'} \psi$.

$$\partial_{x'} \psi(x', y') = \partial_{x'} \left(- \int_{\Gamma_0} \text{Neum} G_{x',y'}(x, y) + \int_{\Gamma_0} \text{Diri} \nabla G_{x',y'}(x, y) \cdot \vec{n} \right) \quad (108)$$

$$= \underbrace{- \int_{\Gamma_0} \text{Neum} (\partial_{x'} G_{x',y'}(x, y))}_1 + \underbrace{\int_{\Gamma_0} \text{Diri} (\partial_{x'} \nabla G_{x',y'}(x, y) \cdot \vec{n})}_2 \quad (109)$$

1. We have to find the value of $- \int_{\Gamma_0} \partial_{x'} G_{x',y'} \text{Neum}$

We use Green function as in equation 33. We obtain its derivative to x' as follow:

$$\partial_{x'} G_{x',y'}(x, y) = \frac{1}{2} \frac{x - x'}{(x - x')^2 + (y - y')^2} \quad (110)$$

We can see point (x', y') in coordinate $(x, y = u)$ in figure 5. we also can turn the axis and ordinat in such a convenient way as we see in figure 6.

in $\eta\nu$ coordinate, along the integration line, we get:

$$x - x' = \eta = r \cos \theta = H, r = H \sec \theta \quad (111)$$

$$y - y' = \nu = r \sin \theta \quad (112)$$

$$\frac{\eta}{\nu} = \tan \theta \quad (113)$$

$$\nu = \eta \tan \theta = H \tan \theta \quad (114)$$

$$d\nu = H \sec^2 \theta d\theta \quad (115)$$

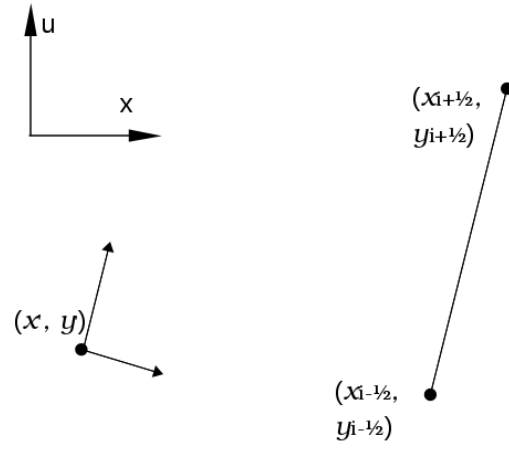


Figure 5: Points in Domain

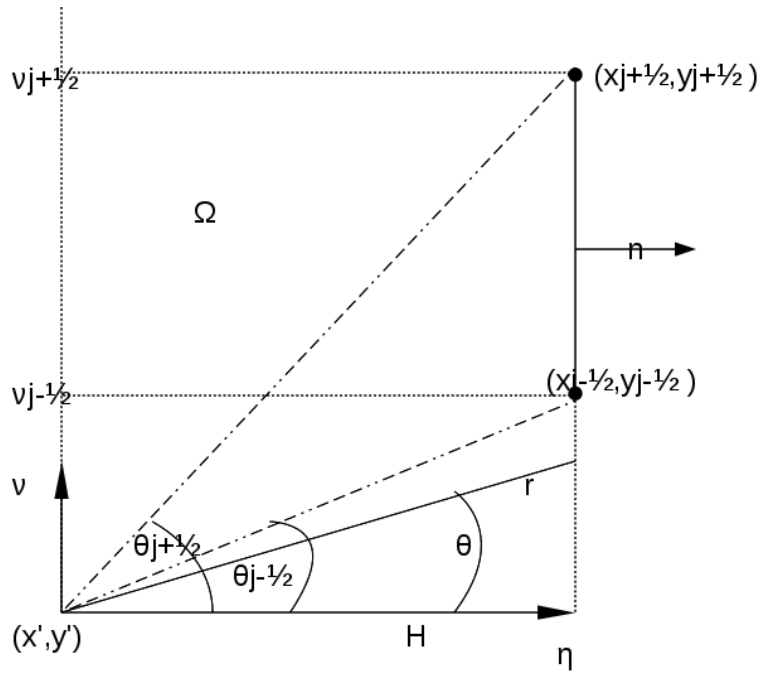


Figure 6: Change the axis and ordinat

Let us compute the integral

$$-\int_{\Gamma_0} \partial_{x'} G_{x',y'} \text{Neum} = -\sum_{i=1}^N \int_{S_i} \partial_{x'} G_{x',y'} \text{Neum} dS \quad (116)$$

$$= -\sum_{i=1}^N \text{Neum} \int_{\nu_{i-\frac{1}{2}}}^{\nu_{i+\frac{1}{2}}} \partial_{x'} G_{x',y'}(H, \nu) d\nu \quad (117)$$

$$= -\sum_{i=1}^N \frac{\text{Neum}}{2\pi} \int_{\theta_{i-\frac{1}{2}}}^{\theta_{i+\frac{1}{2}}} \frac{H}{(H \sec \theta)^2} H \sec^2 \theta d\theta \quad (118)$$

$$= -\sum_{i=1}^N \frac{\text{Neum}}{2\pi} \int_{\theta_{i-\frac{1}{2}}}^{\theta_{i+\frac{1}{2}}} d\theta \quad (119)$$

$$= -\sum_{i=1}^N \frac{\text{Neum}}{2\pi} \left(\theta_{i+\frac{1}{2}} - \theta_{i-\frac{1}{2}} \right) \quad (120)$$

2. We have to compute $\int_{\Gamma_0} \partial_{x'} \nabla G_{x',y'} \cdot n \text{diri}$

With using $\eta\nu$ coordinate, we can write

$$G_{x',y'}(\eta, \nu) = \frac{1}{2\pi} \ln \left(\sqrt{\eta^2 + \nu^2} \right) \quad (121)$$

From this equation and $n = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$, we get

$$\nabla G_{x',y'}(\eta, \nu) \cdot n = \frac{1}{2\pi r} \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix} \begin{pmatrix} -1 \\ 0 \end{pmatrix} \quad (122)$$

$$= -\frac{1}{2\pi H \sec^2 \theta} = \frac{\cos \theta}{2\pi H \sec \theta} \quad (123)$$

$$= \frac{\frac{x-x'}{r}}{2\pi r} = \frac{x-x'}{2\pi ((x-x')^2 + (y-y')^2)} \quad (124)$$

$$\partial_{x'} (\nabla G_{x',y'}(\eta, \nu) \cdot n) = \frac{1}{2\pi} \frac{(x-x')^2 - (y-y')^2}{((x-x')^2 + (y-y')^2)^2} \quad (125)$$

$$= \frac{H^2 - (r \sin \theta)^2}{(r^2)^2} \quad (126)$$

$$\int_{\Gamma_0} \partial_{x'} \nabla G_{x',y'} \cdot n \text{diri} = \sum_{i=1}^N \text{diri}_i \int_{\Gamma_0} \partial_{x'} \nabla G_{x',y'} \cdot n \quad (127)$$

$$= \sum_{i=1}^N \text{diri}_i \int_{\theta_{i-\frac{1}{2}}}^{\theta_{i+\frac{1}{2}}} \frac{1}{2\pi} \frac{H^2 - (r \sin \theta)^2}{(r^2)^2} d\theta \quad (128)$$

$$= \sum_{i=1}^N \text{diri}_i \int_{\theta_{i-\frac{1}{2}}}^{\theta_{i+\frac{1}{2}}} \frac{1}{2\pi} \frac{H^2 - H^2 \sec^2 \theta \sin^2 \theta}{H^4 \sec^4 \theta} d\theta \quad (129)$$

$$= \sum_{i=1}^N \text{diri}_i \int_{\theta_{i-\frac{1}{2}}}^{\theta_{i+\frac{1}{2}}} \frac{1}{2\pi} \frac{1 - \sec^2 \theta \sin^2 \theta}{H^2 \sec^4 \theta} d\theta \quad (130)$$

$$= \sum_{i=1}^N \text{diri}_i \frac{1}{2\pi H^2} \int_{\theta_{i-\frac{1}{2}}}^{\theta_{i+\frac{1}{2}}} \frac{1 - \sec^2 \theta \sin^2 \theta}{\sec^4 \theta} d\theta \quad (131)$$

$$= \sum_{i=1}^N \text{diri}_i \frac{1}{32\pi H^2} [4\theta + 4 \sin(2\theta) + \sin(4\theta)]_{\theta_{i-\frac{1}{2}}}^{\theta_{i+\frac{1}{2}}} \quad (132)$$

Form equation 120 and 132, we can conclude:

$$\partial_{x'} \psi(x', y') = - \sum_{i=1}^N \frac{\text{Neum}_i}{2\pi} \left(\theta_{i+\frac{1}{2}} - \theta_{i-\frac{1}{2}} \right) + \sum_{i=1}^N \text{diri}_i \frac{1}{32\pi H^2} [4\theta + 4 \sin(2\theta) + \sin(4\theta)]_{\theta_{i-\frac{1}{2}}}^{\theta_{i+\frac{1}{2}}} \quad (133)$$

Next, we will compute the value of $\partial_{y'} \psi(x', y')$.

$$\partial_{y'} \psi(x', y') = \partial_{y'} \left(- \int_{\Gamma_0} \text{Neum} G_{x',y'}(x, y) + \int_{\Gamma_0} \text{Diri} \nabla G_{x',y'}(x, y) \cdot \vec{n} \right) \quad (134)$$

$$= \underbrace{- \int_{\Gamma_0} \text{Neum} (\partial_{y'} G_{x',y'}(x, y))}_1 + \underbrace{\int_{\Gamma_0} \text{Diri} (\partial_{y'} \nabla G_{x',y'}(x, y) \cdot \vec{n})}_2 \quad (135)$$

1. We have to find the value of $-\int_{\Gamma_0} \partial_{y'} G_{x',y'} \text{Neum}$ With G in 33, we get

$$\partial_{y'} G_{x',y'} = \frac{y - y'}{2\pi ((x - x')^2 + (y - y')^2)}$$

Using this equation, we obtain:

$$-\int_{\Gamma_0} \partial_{y'} G_{x',y'} \text{Neum} = -\sum_{i=1}^N \text{Neum}_i \int_{\Gamma_0} \partial_{y'} G_{x',y'} \quad (136)$$

$$= -\sum_{i=1}^N \text{Neum}_i \int_{\Gamma_0} \frac{y-y'}{2\pi((x-x')^2+(y-y')^2)} \quad (137)$$

$$= -\sum_{i=1}^N \text{Neum}_i \int_{\theta_{i-\frac{1}{2}}}^{\theta_{i+\frac{1}{2}}} \frac{r \sin \theta}{2\pi r^2} d\theta \quad (138)$$

$$= -\sum_{i=1}^N \text{Neum}_i \int_{\theta_{i-\frac{1}{2}}}^{\theta_{i+\frac{1}{2}}} \frac{r \sin \theta}{2\pi H \sec \theta} d\theta \quad (139)$$

$$= -\sum_{i=1}^N \frac{\text{Neum}_i}{2\pi H} \int_{\theta_{i-\frac{1}{2}}}^{\theta_{i+\frac{1}{2}}} \sin \theta \cos \theta d\theta \quad (140)$$

$$= -\frac{1}{2\pi H} \sum_{i=1}^N \text{Neum}_i \left[-\frac{1}{2} \cos^2 \theta \right]_{\theta_{i-\frac{1}{2}}}^{\theta_{i+\frac{1}{2}}} \quad (141)$$

$$= \frac{1}{4\pi H} \sum_{i=1}^N \text{Neum}_i \left(\cos^2 \theta_{i+\frac{1}{2}} - \cos^2 \theta_{i-\frac{1}{2}} \right) \quad (142)$$

2. We have to compute $\int_{\Gamma_0} \partial_{y'} \nabla G_{x',y'} \cdot n \text{diri}$

We use equation 124 to compute $\int_{\Gamma_0} \partial_{y'} \nabla G_{x',y'} \cdot n \text{diri}$. First, we need $\partial_{y'} \nabla G \cdot n$.

$$\partial_{y'} \nabla G \cdot n = \frac{(x-x')(y-y')}{\pi((x-x')^2+(y-y')^2)^2} \quad (143)$$

$$\int_{\Gamma_0} \partial_{y'} \nabla G_{x',y'} \cdot n \text{diri} = \int_{\Gamma_0} \text{diri} \frac{(x-x')(y-y')}{\pi((x-x')^2 + (y-y')^2)^2} \quad (144)$$

$$= \frac{1}{\pi} \sum_{i=1}^N \text{diri}_i \int_{\theta_{i-\frac{1}{2}}}^{\theta_{i+\frac{1}{2}}} \frac{r \cos \theta r \sin \theta}{(r^2)^2} d\theta \quad (145)$$

$$= \frac{1}{\pi} \sum_{i=1}^N \text{diri}_i \int_{\theta_{i-\frac{1}{2}}}^{\theta_{i+\frac{1}{2}}} \frac{\cos \theta \sin \theta}{r^2} d\theta \quad (146)$$

$$= \frac{1}{\pi} \sum_{i=1}^N \text{diri}_i \int_{\theta_{i-\frac{1}{2}}}^{\theta_{i+\frac{1}{2}}} \frac{\cos \theta \sin \theta}{\left(\frac{H}{\cos \theta}\right)^2} d\theta \quad (147)$$

$$= \frac{1}{\pi H^2} \sum_{i=1}^N \text{diri}_i \int_{\theta_{i-\frac{1}{2}}}^{\theta_{i+\frac{1}{2}}} \cos^3 \theta \sin \theta d\theta \quad (148)$$

$$= -\frac{1}{4\pi H^2} \sum_{i=1}^N \text{diri}_i [\cos^4 \theta]_{\theta_{i-\frac{1}{2}}}^{\theta_{i+\frac{1}{2}}} \quad (149)$$

Form equation 142 and 149, we get:

$$\partial_{y'} \psi(x', y') = \frac{1}{4\pi H} \sum_{i=1}^N \text{Neum}_i \left(\cos^2 \theta_{i+\frac{1}{2}} - \cos^2 \theta_{i-\frac{1}{2}} \right) - \frac{1}{4\pi H^2} \sum_{i=1}^N \text{diri}_i [\cos^4 \theta]_{\theta_{i-\frac{1}{2}}}^{\theta_{i+\frac{1}{2}}} \quad (150)$$

$$= \frac{1}{4\pi H} \left(\sum_{i=1}^N \text{Neum}_i \left(\cos^2 \theta_{i+\frac{1}{2}} - \cos^2 \theta_{i-\frac{1}{2}} \right) - \frac{1}{H} \sum_{i=1}^N \text{diri}_i \left(\cos^4 \theta_{i+\frac{1}{2}} - \cos^4 \theta_{i-\frac{1}{2}} \right) \right) \quad (151)$$