Potential Flow Around a Regular Body in 2D

1 Settling the Problem

Suppose the fluids flows by given vector velocity field u. The velocity field can be written as the gradient $\nabla\phi$ of a velocity potential ϕ.

$$u = \nabla\phi \tag{1}$$

Let we assume that vector field u to be irrotational. It means $curl(u) = 0$. We can also write $\nabla \times u = 0$.

We consider the flow to be incompressible, we have $\nabla \cdot u = 0$. If we substitute (1), we have

$$\begin{aligned}
&\nabla \cdot u = 0 \\
&\nabla \cdot \nabla\phi = 0 \\
&\Delta\phi = 0
\end{aligned}$$

From here we get:

$$\Delta\phi = 0 \tag{2}$$

Let consider fluid in an infinite domain. We have the velocity u_0 along x in one direction. This conditions can be written as

$$u = u_0\vec{e_x} \tag{3}$$

From (2) and (3), we have problem:

$$\begin{aligned}
\Delta\phi &= 0 \\
u = \nabla\phi &\to u_0\vec{e_x} \quad \text{as } x \to \pm\infty
\end{aligned} \tag{4}$$

This problem has obvious solution i.e $\phi = u_0 x$.

Let we add a regular obstacle Ω_0. We consider $\partial\Omega_0$ to be C^1. It means the obstacle doesn't have angle nor discontinuity. On the border of the obstacle, the velocity is tangential.

$$u \cdot n = 0 \tag{5}$$

with n is the (exterior) normal vector of Ω_0. We substitute $u = \nabla\Phi$ to (5), we get

$$\nabla\phi \cdot n = 0 \tag{6}$$

Equation (6) is called Homogeneous Neumann Boundary Condition. From equation (1), (2), (3) and (6), we obtain the problem:

$$\begin{aligned}
\Delta\phi = 0 &\quad \text{On: } \mathbb{R}^2 \setminus \overline{\Omega_0} \\
u = \nabla\phi \to u_0\vec{e_x} &\quad \text{as } x \to \pm\infty \\
\nabla\phi \cdot n = 0 &\quad \partial\Omega_0 = \Gamma_0
\end{aligned} \tag{7}$$

We have to find the harmonic function that satisfies all the boundary conditions.

Without the obstacle, problem (4) has solution $u_0 x$. We can use it to find ϕ. Let we do the changing variable in order to get the solution of problem (7). Let $\psi = \phi - u_0 x$ and ϕ is the actual solution.

$$
\begin{aligned}
\psi &= \phi - u_0 x \\
\Delta\psi &= \Delta\phi - \Delta(u_0 x) \\
\Delta\psi &= \Delta\phi \\
\nabla\psi &= \nabla\phi - u_0 \vec{e_x}
\end{aligned}
$$

when $x \to \pm\infty$ value of $\nabla\psi \to 0$. From here, we get

$$
\nabla\psi \to 0 \quad \text{as } x \to \pm\infty \tag{8}
$$

From the last condition in (7) we get

$$
\begin{aligned}
\nabla\psi \cdot n &= \nabla(\phi - u_0 x) \cdot n \text{on } \Gamma_0 \\
&= \nabla\phi \cdot n - \nabla(u_0 x) \cdot n \\
&= 0 - u_0 \nabla x \cdot n \\
&= -u_0 \vec{e_x} \cdot n \\
&= -u_0 n_x
\end{aligned}
$$

The problem after changing variable become

$$
\begin{aligned}
\Delta\psi &= 0 \quad \text{in } \mathbb{R}^2 \setminus \overline{\Omega_0} \\
\nabla\psi &\to 0 \quad \text{as } x \to \pm\infty \\
\nabla\psi \cdot n &= -u_0 n_x \quad \text{on } \Gamma_0
\end{aligned} \tag{9}
$$