Exploring Advanced Calculus and Complex Analysis through Problem Solving

1. Evaluate the followings:

   i) $\int_0^\infty e^{-4x}\sqrt{x^5}\,dx$

   ii) $\int_0^\infty e^{\frac{x}{2}}x^5\,dx$

   iii) $\int_0^1 \frac{1}{\sqrt[3]{x^2-x^3}}\,dx$

   iv) $\int_0^1 \frac{x^2}{\sqrt{1-x^5}}\,dx$.

   v) $\int_0^{\pi/2} \cos^3\theta\,\sin^{\frac{5}{2}}\theta\;d\theta$

   vi) $\int_0^{\frac{\pi}{2}} 5\sin^{2.5}\theta\;d\theta$

2. Prove that: $\int_0^1 \left(\frac{1-x}{x}\right)^{\frac{1}{4}} dx = \frac{\pi}{2\sqrt{2}}$.

3. Prove that: $\int_0^1 \sqrt{1-x^4}\,dx = \frac{\left\{\Gamma\left(\frac{1}{4}\right)\right\}^2}{6\sqrt{2\pi}}$.

4. By using techniques involving the Gamma function find the exact value of
$$\int_0^\infty x^4\,e^{-\frac{1}{2}x^2}\,dx\,.$$
Give the answer in the form $k\sqrt{\frac{\pi}{2}}$, where k is an integer number.

5. By using techniques involving the Gamma function find the exact value of
$$\int_0^\infty x^6\,e^{-4x^2}\,dx\,.$$
Give the answer in the form $k\sqrt{\pi}$, where k is a rational constant.

6. Illustrate β-Γ function to calculate $\int_0^1 x^{\frac{3}{2}}\left(1-\sqrt{x}\right)^{\frac{1}{2}}dx$.

7. If $u = \cos^{-1}\left(\frac{xyz}{\sqrt{x^2+y^2+z^2}}\right)$ then show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = -\frac{2}{\tan u}$.

8. If $u = \sin^{-1}\left(\frac{x+y+z}{\sqrt{x}+\sqrt{y}+\sqrt{z}}\right)$ then show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = \frac{1}{2}\tan u$.

9. If $u = \cot^{-1}\left(\frac{x+y+z}{x^3+y^3+z^3}\right)$ then show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = \sin 2u$.

10. Verify Euler's theorem for the function $= \frac{y-\sqrt{xz}}{x^3-y^3+yz^2}$.

11. Verify Euler's theorem for the function $= \frac{x+y}{\sqrt{x^2+y^2+z^2}}$.

12. If $u = \sin(\sqrt{x} + \sqrt{y})$ then show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}(\sqrt{x} + \sqrt{y})\cos(\sqrt{x} + \sqrt{y})$.

13. Evaluate $\iint_R \frac{3}{2} e^{\frac{y}{\sqrt{x}}} dydx$, where R is the region of space bounded by $x=1, x=4, y=0$, $y=\sqrt{x}$.

14. Evaluate $\iint_R \left(\frac{x}{y}\right) dydx$, where R is the region of space bounded by $x=1, x=4, y=2$, $y=e^x$.

15. Evaluate $\iint_R (1+\theta)e^{-\theta} d\theta dr$, where R is the region of space bounded by $\theta=0$, $\theta=\ln r, r=1, r=2$.

16. Evaluate $\iint_R xye^{xy^2} dydx$, where R is the region of space bounded by $x=0, x=\ln 2$, $y=0, y=1$.

17. Evaluate $\iiint_R \cos\left(\frac{x}{y}\right) dzdxdy$, where R is the region of space bounded by $x=0$, $x=y^2$, $y=0, y=\frac{\pi}{2}$, $z=0, z=y$.

18. Evaluate $\iiint_R (xyz)\, dzdydx$, where R is the region of space bounded by $x=1, x=3$, $y=\frac{1}{x}, y=1$, $z=0, z=\sqrt{xy}$.

19. Evaluate $\iiint_R (xy^2z^3)\, dxdzdy$, where R is the region of space bounded by $x=z^2, x=y$, $y=0, y=2$, $z=\sqrt{y}, z=1$.

20. Find the modulus and argument of the following complex numbers:

i. $z = \frac{1+i}{1-i} - (1+2i)(2+2i) + \frac{3-i}{1+i}$

ii. $z = 2i(i-1) + \left(\overline{\sqrt{3}+i}\right)^3 + (1+i)\overline{(1-i)}$

iii. $z = \left(\frac{1}{1-2i} + \frac{3}{1+i}\right)\left(\frac{3+4i}{1-2i}\right)$

iv. $(2-i)^2 + \frac{7-4i}{2+i} - 8$

21. Construct the matrix form, polar form and exponential form of $z_1 + z_2$, where $z_1 = \frac{1+i}{1-i}$ and $z_2 = (1+i)^2$.

22. Construct the matrix form, polar form and exponential form of the complex number $z = \frac{2+6\sqrt{3}i}{5+\sqrt{3}i}$.

23. Construct the matrix form of $z_1 + z_2 z_3$, where $z_1 = (-1-2i)^4$, $z_2 = (7.5, 77^\circ)$ and $z_3 = (12.7, 153^\circ)$.

24. Construct the matrix form of $z_1 + z_2/z_3$, where $z_1 = (-3-5i)^5$, $z_2 = (5.7,\ 201^0\, 9'\, 23")$ and $z_3 = (21,\ 307^0\, 23')$.

25. Express $(3-5i)^7 + 7e^{5.2i} - 9.2\,(6.2,\ 272^0)$ in the matrix form.