

Stochastic approximation with ‘controlled Markov’ noise

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Received 18 December 2004; received in revised form 14 May 2005; accepted 4 June 2005

Available online 18 July 2005

Abstract

Stochastic approximation algorithms with additional noise that can be modelled as a controlled Markov process are analyzed and shown to track the solutions of a differential inclusion defined in terms of the ergodic occupation measures associated with the controlled Markov process.

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Keywords: Stochastic approximation; Controlled Markov noise; Differential inclusions; Ergodic occupation measures; Invariant sets

1. Introduction

Stochastic approximation algorithms with ‘Markov’ noise have been extensively analyzed by means of their limiting o.d.e., see, e.g., [3]. The aim of this note is to extend these results to the case when the noise is ‘controlled Markov’, by relating it to a limiting differential inclusion defined in terms of the so-called ergodic occupation measures associated with the controlled Markov process. Stochastic approximation schemes whose limits are differential inclusions have been recently analyzed in [2].

The motivation for this study comes from the fact that many times the so-called noise process (which may not in fact be a physical noise—see the example below) is not Markov, but its lack of Markov property comes through its dependence on some possibly imperfectly known, possibly time-varying process which may be viewed as a ‘control’ for analysis purposes. Consider, for example, the case when the stochastic approximation scheme contributes the parameter estimation component of a self-tuning controller, whence the controlled state process would be what is called ‘controlled Markov noise’ here. The control cannot be a priori assumed

to be either Markov or stationary. Note that in this example the ‘control’ is in fact a physical control process, but this need not be so in general.

The next section derives some preliminary results, based on which the main result is established in Section 3.

2. Preliminaries

Specifically, we consider the iteration

$$x_{n+1} = x_n + a(n)[h(x_n, Y_n) + M_{n+1}], \quad (1)$$

where $\{Y_n\}$ is a random process taking values in a complete separable metric space S with dynamics we shall soon specify, and $h: \mathcal{R}^d \times S \rightarrow \mathcal{R}^d$ is jointly continuous in its arguments and Lipschitz in its first argument uniformly w.r.t. the second. $\{M_n\}$ is a martingale difference sequence w.r.t. the σ -fields $\mathcal{F}_n \triangleq \sigma(x_m, Y_m, M_m, m \leq n)$, $n \geq 0$, satisfying

$$E[\|M_{n+1}\|^2 | \mathcal{F}_n] \leq K_0(1 + \|x_n\|^2) \quad \forall n, \quad (2)$$

for some $K_0 > 0$. Stepsizes $\{a(n)\}$ satisfy the usual conditions:

$$\sum_n a(n) = \infty, \quad \sum_n a(n)^2 < \infty, \quad (3)$$

with the additional condition that they be eventually decreasing.

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¹ Research supported in part by Grant no. III.5(157)/99 from the Department of Science and Technology, Government of India.

We shall assume that $\{Y_n\}$ is an S -valued controlled Markov process controlled by two control processes: $\{x_n\}$ above and another random process $\{Z_n\}$ taking values in a compact metric space U . Thus

$$P(Y_{n+1} \in A | Y_n, Z_n, x_n, m \leq n) = \int_A p(dy | Y_n, Z_n, x_n), \quad n \geq 0,$$

for A Borel in S , where $(y, z, x) \in S \times U \times \mathcal{R}^d \rightarrow p(dw | y, z, x) \in \mathcal{P}(S)$ is a continuous map specifying the controlled transition probability kernel. (Here and in what follows, $\mathcal{P}(\cdots)$ will denote the space of probability measures on the complete separable metric space ‘ \cdots ’ with Prohorov topology—see, e.g., [5, Chapter 2]). We assume that the continuity in the x variable is uniform on compacts w.r.t. the other variables. We shall say that $\{Z_n\}$ is a stationary randomized control if for each n the conditional law of Z_n given $(Y_m, x_m, Z_{m-1}, m \leq n)$ is $\varphi(Y_n)$ for a fixed measurable map $\varphi : y \in S \rightarrow \varphi(y) = \varphi(y, dz) \in \mathcal{P}(U)$ independent of n . Thus, in particular, Z_n is conditionally independent of $(Y_m, x_m, Z_{m-1}, m \leq n)$ given Y_n for $n \geq 0$. By abuse of terminology, we identify the stationary randomized control above with the map $\varphi(\cdot)$.

If $x_n = x \forall n$ for a fixed deterministic $x \in \mathcal{R}^d$, then $\{Y_n\}$ will be a time-homogeneous Markov process under any stationary randomized control φ with the transition kernel

$$\bar{p}_{x,\varphi}(dw | y) = \int p(dw | y, z, x) \varphi(y, dz).$$

Suppose that this Markov process has a (possibly non-unique) invariant probability measure $\eta_{x,\varphi}(dy) \in \mathcal{P}(S)$. Correspondingly, we define the *ergodic occupation measure*

$$\Psi_{x,\varphi}(dy, dz) \triangleq \eta_{x,\varphi}(dy) \varphi(y, dz) \in \mathcal{P}(S \times U).$$

This clearly satisfies

$$\begin{aligned} \int_S f(y) d\Psi_{x,\varphi}(dy, U) \\ = \int_S \int_U f(w) p(dw | y, z, x) d\Psi_{x,\varphi}(dy, dz), \end{aligned} \quad (4)$$

for bounded continuous $f : S \rightarrow \mathcal{R}$. Conversely, if some $\Psi \in \mathcal{P}(S \times U)$ satisfies the above for bounded $f \in C(S)$, then it must be of the form $\Psi_{x,\varphi}$ for some stationary randomized control φ . This is because we can always disintegrate Ψ as

$$\Psi(dy, dz) = \eta(dy) \varphi(y, dz)$$

with η, φ denoting resp. the marginal on S and the regular conditional law on U . Since $\varphi(\cdot)$ is a measurable map $S \rightarrow \mathcal{P}(U)$, it can be identified with a stationary randomized control. Eq. (4) then implies that η is an invariant probability measure under the ‘stationary randomized control’ φ .

We denote by $D(x)$ the set of all such ergodic occupation measures for the prescribed x . Since Eq. (4) is preserved

under convex combinations and convergence in $\mathcal{P}(S \times U)$, $D(x)$ is closed and convex. We also assume that it is compact. Once again, using the fact that (4) is preserved under convergence in $\mathcal{P}(S \times U)$, it follows that if $x(n) \rightarrow x$ in \mathcal{R}^d and $\Psi_n \rightarrow \Psi$ in $\mathcal{P}(S \times U)$ with $\Psi_n \in D(x(n)) \forall n$, then $\Psi \in D(x)$, implying upper semicontinuity of the set-valued map $x \rightarrow D(x)$.

Let $\delta_{(\cdot)}$ denote the Dirac measure at ‘ (\cdot) ’. We define a $\mathcal{P}(S \times U)$ -valued random process $\mu(t)$, $t \geq 0$, by

$$\mu(t) = \mu(t, dy) \triangleq \delta_{(Y_n, Z_n)}, \quad t \in [t(n), t(n+1)),$$

for $n \geq 0$. Also define for $t > s \geq 0$, $\mu_s^t \in \mathcal{P}(S \times U \times [s, t])$ by

$$\mu_s^t(A \times B) \triangleq \frac{1}{t-s} \int_B \mu(y, A) dy$$

for A, B Borel in $S \times U, [s, t]$ resp. Similar notation will be followed for other $\mathcal{P}(S \times U)$ -valued processes. Recall that S being a complete separable metric space, it can be homeomorphically embedded as a dense subset of a compact metric space \bar{S} . (See [5, Theorem 1.1.1, p. 2].) As any probability measure on $S \times U$ can be identified with a probability measure on $\bar{S} \times U$ that assigns zero probability to $(\bar{S} - S) \times U$, we may view $\mu(\cdot)$ as a random variable taking values in $\mathcal{U} \triangleq$ the space of measurable functions $v(\cdot) = v(\cdot, dy)$ from $[0, \infty)$ to $\mathcal{P}(\bar{S} \times U)$. This space is topologized with the coarsest topology that renders continuous maps $v(\cdot) \in \mathcal{U} \rightarrow \int_0^T g(t) \int f dv(t) dt \in \mathcal{R}$ for all $f \in C(\bar{S})$, $T > 0$ and $g \in L_2[0, T]$. (This topology, sometimes called the ‘stable’ topology, is quite common in the literature on existence of optimal controls (see, e.g., [4])). We shall assume that:

(*) For $f \in C(\bar{S})$, the function

$$(y, z, x) \in S \times U \times \mathcal{R}^d \rightarrow \int f(w) p(dw | y, z, x)$$

extends continuously to $\bar{S} \times U \times \mathcal{R}^d$.

Later on we see a specific instance of how this might come by, viz., in the Euclidean case. With a minor abuse of notation, we retain the original notation to denote this extension. Finally, we denote by $\mathcal{U}_0 \subset \mathcal{U}$ the subset $\{\mu(\cdot) \in \mathcal{U} : \int_{S \times U} \mu(t, dy) = 1 \forall t\}$ with the relative topology.

Lemma 2.1. \mathcal{U} is compact metrizable.

Proof. For $N \geq 1$, let $\{e_i^N(\cdot), i \geq 1\}$ denote a complete orthonormal basis for $L_2[0, N]$. Let $\{f_j\}$ be countable dense in the unit ball of $C(\bar{S})$. Then it is a convergence determining class for $\mathcal{P}(\bar{S})$. It can then be easily verified that

$$\begin{aligned} d(v_1(\cdot), v_2(\cdot)) &\triangleq \sum_{N \geq 1} \sum_{i \geq 1} \sum_{j \geq 1} 2^{-(N+i+j)} \\ &\times \left\| \int_0^N e_i^N(t) \int f_j dv_1(t) dt \right. \\ &\quad \left. - \int_0^N e_i^N(t) \int f_j dv_2(t) dt \right\| \wedge 1 \end{aligned}$$

defines a metric on \mathcal{U} consistent with its topology. To show sequential compactness, take $\{v_n(\cdot)\} \subset \mathcal{U}$ and by a diagonal argument, pick a subsequence of $\{n\}$, denoted by $\{n\}$ again by abuse of terminology, such that for each j and N , $\int f_j dv_n(\cdot)|_{[0,N]} \rightarrow \alpha_j(\cdot)|_{[0,N]}$ weakly in $L_2[0, N]$ for some real valued measurable functions $\{\alpha_j(\cdot), j \geq 1\}$ on $[0, \infty)$ satisfying: $\alpha_j(\cdot)|_{[0,N]} \in L_2[0, N]$ for every $N \geq 1$. Fix N . Mimicking the proof of the Banach–Saks theorem [1, Theorem 1.8.4], let $n(1) = 1$ and pick $\{n(k)\}$ inductively to satisfy

$$\sum_{j=1}^{\infty} 2^{-j} \max_{1 \leq m < k} \left| \int_0^N \left(\int f_j dv_{n(k)}(t) - \alpha_j(t) \right) \times \left(\int f_j dv_{n(m)}(t) - \alpha_j(t) \right) dt \right| < \frac{1}{k},$$

which is possible because $\int f_j dv_n(\cdot)|_{[0,N]} \rightarrow \alpha_j(\cdot)|_{[0,N]}$ weakly in $L_2[0, N]$. Denote by $\|\cdot\|_2, \langle \cdot, \cdot \rangle_2$ the norm and inner product in $L_2[0, T]$. Then for $j \geq 1$,

$$\begin{aligned} & \left\| \frac{1}{m} \sum_{k=1}^m \int f_j dv_{n(k)}(\cdot) - \alpha_j(\cdot) \right\|_2^2 \\ & \leq \frac{1}{m^2} \left(2mN^2 + 2 \sum_{i=2}^m \sum_{\ell=1}^{i-1} \left\langle \int f_j dv_{n(i)}(\cdot) - \alpha_j(\cdot), \int f_j dv_{n(\ell)}(\cdot) - \alpha_j(\cdot) \right\rangle \right) \\ & \leq \frac{2N^2}{m^2} [m + 2^j(m-1)] \rightarrow 0, \end{aligned}$$

as $m \rightarrow \infty$. Thus

$$\frac{1}{m} \sum_{k=1}^m \int f_j dv_{n(k)}(\cdot) \rightarrow \alpha_j(\cdot)$$

strongly in $L_2[0, N]$ and hence a.e. along a subsequence $\{m(\ell)\}$ of $\{m\}$. Fix a $t \geq 0$ for which this is true. Let $v'(t)$ be a limit point in $\mathcal{P}(\bar{S} \times U)$ (which is a compact space by Prohorov's theorem—see [5, Chapter 2]), of the sequence

$$\left\{ \frac{1}{m(\ell)} \sum_{k=1}^{m(\ell)} v_{n(k)}(t), m \geq 1 \right\}.$$

Then $\alpha_j(t) = \int f_j dv'(t) \forall j$, implying that

$$[\alpha_1(t), \alpha_2(t), \dots] \in \left\{ \left[\int f_1 dv, \int f_2 dv, \dots \right] : v \in \mathcal{P}(\bar{S} \times U) \right\}$$

a.e. in $[0, N]$, where the ‘a.e.’ may be dropped by the choice of a suitable modification of the α_j 's. Then this is true for all t , because N was arbitrary. By a standard measurable selection theorem (see, e.g., [7]), it then follows that there

exists a $v^*(\cdot) \in \mathcal{U}$ such that $\alpha_j(t) = \int f_j dv^*(t) \forall t, j$. It follows that $d(v_n(\cdot), v^*(\cdot)) \rightarrow 0$. This completes the proof. \square

We assume the stability condition for $\{x_n\}$

$$\sup_n \|x_n\| < \infty \quad \text{a.s.} \quad (5)$$

$$E[\|x_n\|] < \infty \quad \forall n. \quad (6)$$

Eq. (6) is a mild condition. Eq. (5), i.e., the a.s. boundedness of iterates, usually has to be separately verified using a convenient stochastic Lyapunov type stability condition or otherwise. In addition, we shall need the following ‘stability’ condition for $\{Y_n\}$:

(\dagger) For any $t > 0$, the set $\{\mu_s^{s+t}, s \geq 0\}$ remains tight a.s.

By considering (\dagger) for t rational and then using the obvious continuity of the map $(s, t) \rightarrow \mu_s^t$, we may take a common set of zero probability outside which (\dagger) holds for all $t > 0$. A sufficient condition for (\dagger) when $S = \mathcal{R}^k$ will be discussed later.

Define time instants $t(0) = 0, t(n) = \sum_{m=0}^{n-1} a(m), n \geq 1$. By (3), $t(n) \uparrow \infty$. Let $I_n \triangleq [t(n), t(n+1)], n \geq 0$. Define a continuous, piecewise linear $\bar{x}(t), t \geq 0$, by $\bar{x}(t(n)) = x_n, n \geq 0$, with linear interpolation on each interval I_n . That is,

$$\bar{x}(t) = x_n + (x_{n+1} - x_n) \frac{t - t(n)}{t(n+1) - t(n)}, \quad t \in I_n.$$

Define $\tilde{h}(x, v) \triangleq \int h(x, y) v(dy, U)$ for $v \in \mathcal{P}(S \times U)$. For $\mu(\cdot)$ as above, consider the non-autonomous o.d.e.

$$\dot{x}(t) = \tilde{h}(x(t), \mu(t)). \quad (7)$$

Let $x^s(t), t \geq s$, denote the unique solution to (7) with $x^s(s) = \bar{x}(s)$, for $s \geq 0$. Likewise, let $x_s(t), t \leq s$, denote the unique solution to (7) ‘ending at s ’

$$\dot{x}_s(t) = \tilde{h}(x(t), \mu(t)), \quad t \leq s,$$

with $x_s(s) = \bar{x}(s), s \in \mathcal{R}$. Define also

$$\zeta_n = \sum_{m=0}^{n-1} a(m) M_{m+1}, \quad n \geq 1.$$

Then $(\zeta_n, \mathcal{F}_n), n \geq 1$, is a zero mean martingale.

Lemma 2.2. For any $T > 0$,

$$\lim_{s \rightarrow \infty} \sup_{t \in [s, s+T]} \|\bar{x}(t) - x^s(t)\| = 0, \quad \text{a.s.}$$

$$\lim_{s \rightarrow \infty} \sup_{t \in [s-T, s]} \|\bar{x}(t) - x_s(t)\| = 0, \quad \text{a.s.}$$

Proof. We shall only prove the first claim, as the arguments for proving the second claim are completely analogous.

Let $[t] \triangleq \max\{t(m) : t(m) \leq t\}$. Then by construction,

$$\bar{x}(t(n+m)) = \bar{x}(t(n)) + \sum_{k=0}^{m-1} a(n+k) \tilde{h}(\bar{x}(t(n+k)), \mu(t(n+k))) + \delta_{n,m}, \quad (8)$$

where $\delta_{n,m} \triangleq \zeta_{n+m} - \zeta_n$. Compare this with

$$\begin{aligned} x^{t(n)}(t(n+m)) &= \bar{x}(t(n)) + \int_{t(n)}^{t(n+m)} \tilde{h}(x^{t(n)}(t), \mu(t)) dt \\ &= \bar{x}(t(n)) + \sum_{k=0}^{m-1} a(n+k) \\ &\quad \times \tilde{h}(x^{t(n)}(t(n+k)), \mu(t(n+k))) \\ &\quad + \int_{t(n)}^{t(n+m)} (\tilde{h}(x^{t(n)}(y), \mu(y)) \\ &\quad - \tilde{h}(x^{t(n)}([y]), \mu([y]))) dy. \end{aligned} \quad (9)$$

Consider the integral on the right-hand side. In view of (5), $C_0 \triangleq \sup_n \|h(x_n)\| \vee \|x_n\| < \infty$ a.s. Then, by the Lipschitz property of h and (5), a straightforward application of the Gronwall inequality yields

$$\sup_{t \in [s, s+T]} \|x^s(t)\| \leq C_T < \infty$$

for a suitable $C_T > 0$ and hence

$$C \triangleq \sup_{t \in [s, s+T]} \|h(x^s(t))\| < \infty.$$

Thus,

$$\begin{aligned} &\left\| \int_{t(n)}^{t(n+m)} (\tilde{h}(x^{t(n)}(t), \mu(t)) \right. \\ &\quad \left. - \tilde{h}(x^{t(n)}([t]), \mu([t]))) dt \right\| \\ &\leq \int_{t(n)}^{t(n+m)} L \|x^{t(n)}(t) - x^{t(n)}([t])\| dt \\ &\leq CL \sum_{k=n}^{n+m} (t(k+1) - t(k))^2 \\ &\leq CL \sum_{k=n}^{\infty} a(k)^2 \xrightarrow{n \uparrow \infty} 0 \quad \text{a.s.,} \end{aligned} \quad (10)$$

where L is the Lipschitz constant of h in its first argument and the second inequality uses the fact

$$\begin{aligned} \|x^{t(n)}(t) - x^{t(n)}([t])\| &\leq \left\| \int_{[t]}^t \tilde{h}(x^{t(n)}(s), \mu(s)) ds \right\| \\ &\leq C(t - [t]) \\ &\leq Ca(m), \end{aligned}$$

for $t \in [t(m), t(m+1)]$. Also, in view of (2), (3), and (5), we have

$$\sum_n a(n)^2 E[\|M_{n+1}\|^2 | \mathcal{F}_n] < \infty, \quad \text{a.s.}$$

Thus by [5, Theorem 3.3.4, p. 53], ζ_n converges a.s. as $n \rightarrow \infty$. In particular,

$$\sup_{m \geq n} \|\delta_{n,m}\| \xrightarrow{n \uparrow \infty} 0, \quad \text{a.s.} \quad (11)$$

Subtracting (9) from (8) and taking norms, we have, in view of (10) and (11),

$$\begin{aligned} \|\bar{x}(t(n+m)) - x^{t(n)}(t(n+m))\| &\leq L \sum_{i=0}^{m-1} a(n+i) \|\bar{x}(t(n+i)) - x^{t(n)}(t(n+i))\| \\ &\quad + CL \sum_{k \geq n} a(k)^2 + \sup_{k \geq n} \|\delta_{n,k}\| \quad \text{a.s.} \end{aligned}$$

By the discrete Gronwall inequality, one then has

$$\begin{aligned} &\sup_{t \in [t(n), t(n)+T]} \|\bar{x}(t) - x^{t(n)}(t)\| \\ &\leq K_T \left(CL \sum_{k \geq n} a(k)^2 + \sup_{k \geq n} \|\delta_{n,k}\| \right) \quad \text{a.s.,} \end{aligned}$$

for a suitable $K_T > 0$. The claim follows from (10), (11) for the special case of $s \rightarrow \infty$ along $\{t(n)\}$. The general claim follows easily from this. \square

We shall also need the following lemma: let $\mu^n(\cdot) \rightarrow \mu^\infty(\cdot)$ in \mathcal{U}_0 .

Lemma 2.3. *If $x^n(\cdot)$, $n = 1, 2, \dots, \infty$, denote solutions to (7) corresponding to $\mu(\cdot) = \mu^n(\cdot)$ for $n = 1, 2, \dots, \infty$, and with $x^n(0) \rightarrow x^\infty(0)$, then $\lim_{n \rightarrow \infty} \sup_{t \in [s, s+T]} \|x^n(t) - x^\infty(t)\| \rightarrow 0$ for every $T > 0$.*

Proof. By our choice of the topology for \mathcal{U}_0 ,

$$\begin{aligned} &\int_0^t g(t) \int f d\mu^n(s) ds \\ &\quad - \int_0^t g(t) \int f d\mu^\infty(s) ds \rightarrow 0, \end{aligned}$$

for bounded continuous $g : [0, t] \rightarrow \mathcal{R}$, $f : \bar{S} \rightarrow \mathcal{R}$. Hence

$$\begin{aligned} &\int_0^t \int f(s, \cdot) d\mu^n(s) ds \\ &\quad - \int_0^t \int f(s, \cdot) d\mu^\infty(s) ds \rightarrow 0, \end{aligned}$$

for all bounded continuous $f : [0, t] \times \bar{S} \rightarrow \mathcal{R}$ of the form

$$f(s, w) = \sum_{m=1}^N a_m g_m(s) f_m(w),$$

for some $N \geq 1$, scalars a_i and bounded continuous real valued functions g_i, f_i on $[0, t]$, \bar{S} resp., for $1 \leq i \leq N$. By the Stone–Weierstrass theorem, such functions can uniformly approximate any $\bar{f} \in C([0, T] \times \bar{S})$. Thus, the above convergence holds true for all such \bar{f} , implying that $d\mu^n(s) ds \rightarrow d\mu^\infty(s) ds$ in $\mathcal{P}(\bar{S} \times [0, t])$ and hence in $\mathcal{P}(S \times [0, t])$. Thus, in particular

$$\left\| \int_0^t (\tilde{h}(x^\infty(s), \mu^n(s)) - \tilde{h}(x^\infty(s), \mu^\infty(s))) ds \right\| \rightarrow 0.$$

A straightforward application of the Arzela–Ascoli theorem implies that this convergence is in fact uniform for t in a compact set. Now for $t > 0$,

$$\begin{aligned} \|x^n(t) - x^\infty(t)\| &\leq \|x^n(0) - x^\infty(0)\| \\ &+ \int_0^t \|\tilde{h}(x^n(s), \mu^n(s)) - \tilde{h}(x^\infty(s), \mu^n(s))\| ds \\ &+ \left\| \int_0^t (\tilde{h}(x^\infty(s), \mu^n(s)) - \tilde{h}(x^\infty(s), \mu^\infty(s))) ds \right\|. \end{aligned}$$

In view of the foregoing, a straightforward application of the Gronwall inequality leads to the desired conclusion. \square

3. Main results

The key consequence of (\dagger) that we require is the following:

Lemma 3.1. *Almost surely, every limit point of $(\mu(s+\cdot), \bar{x}(s+\cdot))$ for $t > 0$ as $s \rightarrow \infty$ is of the form $(\tilde{\mu}(\cdot), \tilde{x}(\cdot))$, where $\tilde{\mu}(\cdot)$ satisfies: $\tilde{\mu}(t) \in D(\tilde{x}(t))$ and $\tilde{x}(\cdot)$ satisfies (7) with $\mu(\cdot) = \tilde{\mu}(\cdot)$.*

Proof. Let $\{f_i\}$ be a countable set of bounded continuous functions $S \times U \rightarrow \mathcal{R}$ that is a convergence determining class for $\mathcal{P}(S \times U)$. By replacing each f_i by $a_i f_i + b_i$ for suitable $a_i, b_i > 0$ we may suppose that $0 \leq f_i(\cdot) \leq 1$ for all i . For each i ,

$$\xi_n^i \triangleq \sum_{m=1}^{n-1} a(m) \left(f_i(Y_{m+1}) - \int f_i(w) p(dw|Y_m, Z_m, x_m) \right),$$

is a zero mean martingale with $\sup_n E[\|\xi_n^i\|^2] \leq \sum_n a(n)^2$. By the martingale convergence theorem, it converges a.s. Let $\tau(n, s) \triangleq \min\{m \geq n : t(m) \geq t(n) + s\}$ for $s \geq 0, n \geq 0$. Then as $n \rightarrow \infty$,

$$\sum_{m=n}^{\tau(n,t)} a(m) \left(f_i(Y_{m+1}) - \int f_i(w) p(dw|Y_m, Z_m, x_m) \right) \rightarrow 0, \quad \text{a.s.}$$

for $t > 0$. By our choice of $\{f_i\}$ and the fact that $\{a(n)\}$ are eventually non-increasing (this is the only time the latter property is used),

$$\sum_{m=n}^{\tau(n,s)} (a(m) - a(m+1)) f_i(Y_{m+1}) \rightarrow 0, \quad \text{a.s.}$$

Thus,

$$\sum_{m=n}^{\tau(n,t)} a(m) \left(f_i(Y_m) - \int f_i(w) p(dw|Y_m, Z_m, x_m) \right) \rightarrow 0, \quad \text{a.s.}$$

Dividing by $\sum_{m=n}^{\tau(n,t)} a(m) \geq t$ and using $(*)$, (\dagger) and the uniform continuity of $p(dw|y, z, x)$ in x on compacts, we obtain

$$\int_{t(n)}^{t(n)+t} \left(f_i(y) - \int f_i(w) p(dw|y, z, \bar{x}(s)) \mu(s)(dy, dz) \right) ds \rightarrow 0, \quad \text{a.s.}$$

Fix a sample point in the probability one set on which the above holds for all i . Let $(\tilde{\mu}(\cdot), \tilde{x}(\cdot))$ be a limit point of $(\mu(s+\cdot), x^s(\cdot))$ in $\mathcal{U} \times C([0, \infty); \mathcal{R}^d)$ as $s \rightarrow \infty$. Then the above leads to

$$\int_0^t \int \left(f_i(y) - \int f_i(w) p(dw|y, z, \tilde{x}(s)) \right) \times \tilde{\mu}(s)(dy, dz) ds = 0 \quad \forall i. \quad (12)$$

By (\dagger) , $\tilde{\mu}_0^t(S \times U \times [0, t]) = 1 \quad \forall t$ and thus $\tilde{\mu}_s^t(S \times U \times [0, t]) = 1 \quad \forall t > s \geq 0$. By Lebesgue's theorem, $\tilde{\mu}(t)(S \times U) = 1$ for a.e. t . A similar application of Lebesgue's theorem in conjunction with (12) shows that

$$\int \left(f_i(y) - \int f_i(w) p(dw|y, z, \tilde{x}(t)) \right) \times \tilde{\mu}(t)(dy, dz) = 0 \quad \forall i,$$

for a.e. t . The qualification 'a.e. t ' here may be dropped throughout by choosing a suitable modification of $\tilde{\mu}(\cdot)$. By our choice of $\{f_i\}$, this leads to

$$\tilde{\mu}(dw, U) = \int p(dw|y, z, \tilde{x}(t)) \tilde{\mu}(t)(dy, dz).$$

The claim follows. \square

Consider the differential inclusion

$$\dot{x}(t) \in \hat{h}(x(t)), \quad (13)$$

where $\hat{h}(x) \triangleq \{\tilde{h}(x, v) : v \in D(x)\}$. Recall that a closed set $A \subset \mathcal{R}^d$ is said to be an invariant set (resp., a positively/negatively invariant set) for (13) if for $x \in A$ some trajectory $x(t)$, $-\infty < t < \infty$ (resp., $0 \leq t < \infty$ / $-\infty < t \leq 0$) of (13) with $x(0) = x \in A$ satisfies $x(t) \in A \quad \forall t \in \mathcal{R}$ (resp.,

$\forall t \geq 0, \forall t \leq 0$). It is said to be *chain transitive* in addition if for any $x, y \in A$ and any $T, \varepsilon > 0$, there exist $n \geq 1$ and points $x_0 = x, x_1, \dots, x_{n-1}, x_n = y$ such that the trajectory of (7) initiated at x_i meets with the ε -neighborhood of x_{i+1} at some $t \geq T$, for $0 \leq i < n$. (If we restrict to $y = x$ in the above, the set is said to be *chain recurrent*.)

Combining the above lemmas immediately leads to our main result.

Theorem 3.1. *Almost surely, $\{\bar{x}(s + \cdot), s \geq 0\}$ converge to a chain transitive invariant set of (13). In particular $\{x_n\}$ converge a.s. to such a set.*

Proof. Consider a sample point where (5) and the conclusions of Lemma 2.2 hold. Let A denote the set $\bigcap_{t \geq 0} \{\bar{x}(s) : s \geq t\}$. Since $\bar{x}(\cdot)$ is continuous and bounded, A will be nonempty compact and connected. If $x \in A$, there exist $s_n \uparrow \infty$ in $[0, \infty)$ such that $\bar{x}(s_n) \rightarrow x$. Then by the first part of the above lemma, it follows that as $n \uparrow \infty$, every limit point $(\tilde{x}(\cdot), \tilde{\mu}(\cdot))$ of $(\bar{x}(s_n + \cdot), \mu(s_n + \cdot))$ in $C([0, \infty); \mathcal{R}^d) \times \mathcal{U}$ satisfies: $\tilde{x}(\cdot)$ is a trajectory of (7) with $\mu(\cdot)$ replaced by $\tilde{\mu}(\cdot)$ and with $\tilde{x}(0) = x$. Thus $\bar{x}(s_n + t) \rightarrow \tilde{x}(t)$ along a subsequence for each $t > 0$, implying that $\tilde{x}(t) \in A$ as well. A similar argument works for $t < 0$, using the second part of Lemma 2.2. Thus, A is invariant under (13).

Let $\tilde{x}_1, \tilde{x}_2 \in A$ and $\varepsilon, T > 0$. Let $X(x, s, t)$ denote the solution to (7) starting at x at time s for $t \geq s$. Pick $\delta \in (0, \varepsilon/4)$ such that if $\|x - y\| < \delta$, then $\sup_{t \in [s, s+2T], s \geq 0} \|X(x, s, t) - X(y, s, t)\| < \varepsilon/4$, regardless of the choice of $\mu(\cdot)$. (This is possible by a standard Gronwall-based estimate based on the uniform Lipschitz condition on $\tilde{h}(\cdot, \mu)$.) Pick $n_0 \geq 1$ large enough so that for $n \geq n_0$, $\sup_{t \in [t(n), t(n)+2T]} \|\bar{x}(t) - x^{t(n)}(t)\| < \varepsilon/4$ and furthermore, $\bar{x}(t(n_0) + \cdot)$ is contained in the δ -neighborhood of A . (This is possible by Lemma 2.2.) Let $n_2 > n_1 > n_0$ be such that $\|\bar{x}(t(n_i)) - \tilde{x}_i\| < \delta$ for $i = 1, 2$. Set $s(1) = t(n_1)$ and pick $s(1) < s(2) < \dots < s_N = t(n_2)$ for some $N \geq 1$ such that $s(i) = t(k(i))$ for some $k(i)$ (thus $k(1) = n_1, k(N) = n_2$) and $T \leq s(n+1) - s(n) \leq 2T$ for $1 \leq n < N$. Let $y(1) = \tilde{x}_1, y(N) = \tilde{x}_2$ and pick $y(n), \in A, 1 < n < N$, such that $y(n)$ is in the δ -neighborhood of $\bar{x}(s(n))$ for each n in this range. For $1 \leq n < N$, let $y^n(s(n) + \cdot)$ denote a trajectory of (13) starting at $y(n)$ such that $\|y^n(s(n+1)) - x^{s(n)}(s(n+1))\| < \varepsilon/2$. This is possible for large n_0 . By our choice of n_0 , $\|x^{s(n)}(s(n+1)) - \bar{x}(s(n+1))\| < \varepsilon/4$. Finally, by our choice of $y(n+1)$, $\|y(n+1) - \bar{x}(s(n+1))\| < \delta < \varepsilon/4$. Hence $y^n(s(n+1))$ is in the ε -neighborhood of $y(n+1)$. This proves chain transitivity, completing the proof. \square

There are some important extensions of the foregoing that are immediate:

- When the set $\{\sup_n \|x_n\| < \infty\}$ has a positive probability not necessarily equal to one, we still have

$$\sum_n a(n) E[\|M_{n+1}\|^2 | \mathcal{F}_n] < \infty$$

a.s. on this set. [5, Theorem 3.3.4, p. 53], then tells us that ζ_n converges a.s. on this set. Thus by the same arguments as before (which are *pathwise*), Theorem 3.1 continues to hold ‘a.s. on the set $\{\sup_n \|x_n\| < \infty\}$ ’.

- While we took $\{a(n)\}$ to be deterministic in the foregoing, the above arguments would go through if $\{a(n)\}$ are random and bounded, satisfy (3), and the martingale difference property of $\{M_n\}$ along with (2) hold with \mathcal{F}_n redefined as $\sigma(x_m, M_m, a(m), m \leq n)$ for $n \geq 0$. In fact, the boundedness condition for random $\{a(n)\}$ could be relaxed by imposing appropriate moment conditions. There are applications, e.g., in system identification, when $\{a(n)\}$ are naturally random.
- The foregoing would go through even if we replaced (1) by

$$x_{n+1} = x_n + a(n)[h(x_n) + M_{n+1} + \varepsilon(n)], \quad n \geq 0,$$

where $\{\varepsilon(n)\}$ is a deterministic or random bounded sequence which is $o(n)$. This is because $\{\varepsilon(n)\}$ then contributes an additional error term in the proof of Lemma 2.2 which is also asymptotically negligible and hence does not affect the conclusions.

For special cases, more can be said, e.g., the following:

Corollary 3.1. *Suppose there is no additional control process $\{Z_n\}$ as above and for each $x \in \mathcal{R}^d$ and $x_n \equiv x \forall n$, $\{Y_n\}$ above is an ergodic Markov process with a unique invariant probability measure $v(x) = v(x, dy)$. Then (13) above may be replaced by the o.d.e.*

$$\dot{x}(t) = \tilde{h}(x(t), v(x(t))). \quad (14)$$

If $p(dw|y, x)$ denotes the transition kernel of this ergodic Markov process, then $v(x)$ is characterized by

$$\int \left(f(y) - \int f(w) p(dw|y, x) \right) v(x, dy) = 0$$

for bounded $f \in C(S)$. Since this equation is preserved under convergence in $\mathcal{P}(S)$, it follows that $x \rightarrow v(x)$ and therefore $x \rightarrow \tilde{h}(x, v(x))$ is a continuous map. This guarantees the existence of solutions to (14) by standard o.d.e. theory, though not their uniqueness. In general, the solution set for a fixed initial condition will be a non-empty compact subset of $C([0, \infty); \mathcal{R}^d)$. For uniqueness, we need $\tilde{h}(\cdot, v(\cdot))$ to be Lipschitz, which needs additional information about v and the transition kernel p .

We conclude this section with a sufficient condition for (\dagger) when $S = \mathcal{R}^m$ for some $m \geq 1$. The condition, which is a variant of the well known Foster–Lyapunov criterion (see, e.g., [6, Chapters 11 and 12]), is that there exist a $V \in C(\mathcal{R}^m)$ such that $\lim_{\|x\| \rightarrow \infty} V(x) = \infty$ and furthermore,

$$\sup_n E[V(Y_n)^2] < \infty, \quad (15)$$

and for some compact $B \subset \mathcal{R}^m$ and scalar $\varepsilon_0 > 0$,

$$E[V(Y_{n+1})|\mathcal{F}_n] \leq V(Y_n) - \varepsilon_0 \quad (16)$$

a.s. on $\{Y_n \notin B\}$.

In the earlier framework, we now replace S by \mathcal{R}^m and \bar{S} by $\bar{\mathcal{R}}^m \triangleq$ the one point compactification of \mathcal{R}^m with the additional ‘point at infinity’ denoted simply by ‘ ∞ ’. Let $\{f_i\}$ denote a countable convergence determining class of functions for \mathcal{R}^m satisfying $\lim_{\|x\| \rightarrow \infty} |f_i(x)| = 0$ for all i . Thus, they extend continuously to $\bar{\mathcal{R}}^m$ with value zero at ∞ . Also, note that by the dominated convergence theorem, $\lim_{\|x\| \rightarrow \infty} \int f_i(w)p(dw|y, z, x) \rightarrow 0$ as well. We assume that this is uniformly in z, x , which verifies (*).

Lemma 3.2. *As $s \rightarrow \infty$, any limit point $(\mu^*(\cdot), x^*(\cdot))$ of $(\mu(s + \cdot), \bar{x}(s + \cdot))$ in $\mathcal{U} \times C([0, \infty); \mathcal{R}^d)$ is of the form*

$$\mu^*(t) = a(t)\tilde{\mu}(t) + (1 - a(t))\delta_\infty$$

for $t \geq 0$ and $a(\cdot)$ a measurable function $[0, \infty) \rightarrow [0, 1]$, where $\tilde{\mu}(t) \in D(x^*(t)) \forall t$.

Proof. For $\{f_i\}$ as above, argue as in the proof of Lemma 3.1 to conclude that

$$\begin{aligned} & \int \left(f_i(y) - \int f_i(w)p(dw|y, z, x^*(t)) \right) \\ & \times \mu^*(t)(dy, dz) = 0 \quad \forall i, \end{aligned}$$

for all t , a.s. Write $\mu^*(t) = a(t)\tilde{\mu}(t) + (1 - a(t))\delta_\infty$ with $a(\cdot) : [0, \infty) \rightarrow [0, 1]$ a measurable map, which is always possible: The decomposition is in fact unique for those t for which $a(t) < 1$ and $\tilde{\mu}(t)$ can be chosen arbitrarily when $a(t) = 1$. This ensures a measurable choice of $a(\cdot)$. Then when $a(t) > 0$ the above reduces to

$$\begin{aligned} & \int \left(f_i(y) - \int f_i(w)p(dw|y, z, x^*(t)) \right) \\ & \times \tilde{\mu}(t)(dy, dz) = 0 \quad \forall i, \end{aligned}$$

for all t . The claim follows. \square

Corollary 3.2. *(†) holds.*

Proof. Replacing f_i by V in the proof of above lemma and using (15) to justify the use of martingale convergence

theorem therein, we have

$$\begin{aligned} & \lim_{s \rightarrow \infty} \int_0^t \int \left(V(y) - \int V(w) \right. \\ & \times p(dw|y, z, \bar{x}(s + r))\mu(s + r)(dy, dz) \Big) dr = 0, \end{aligned}$$

a.s. Fix a sample point where this and the above lemma hold. Extend the map

$$\begin{aligned} & (y, z, x) \in S \times U \times \mathcal{R}^d \rightarrow V(y) \\ & - V(w)p(dw|y, z, x) \in \mathcal{R} \end{aligned}$$

to $\bar{S} \times U \times \mathcal{R}^d$ by setting it equal to $-\varepsilon_0$ when $x = \infty$. Then it is upper semicontinuous on $\bar{S} \times U \times \mathcal{R}^d$. Hence taking the above limit along an appropriate subsequence (denoted by, say, $\{s(n)\}$), we get

$$\begin{aligned} 0 & \leq -\varepsilon_0 \int_0^t (1 - a(s)) ds \\ & + \int_0^t a(s) \int \left(V(y) - \int V(w) \right. \\ & \times p(dw|y, z, x^*(s))\tilde{\mu}(s)(dy, dz) \Big) ds \\ & = -\varepsilon_0 \int_0^t (1 - a(s)) ds, \end{aligned}$$

by the preceding lemma. Thus $a(s) = 1$ a.e., where the ‘a.e.’ may be dropped by taking a suitable modification of $\mu^*(\cdot)$. This implies that the convergence of $\mu(s(n) + \cdot)$ to $\mu^*(\cdot)$ is in fact in \mathcal{U}_0 . This establishes (†). \square

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