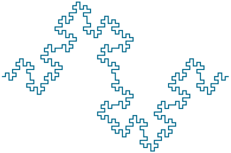
卡尔曼滤波

秦智



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内容

最优线性估计

一般模型

$$z = Hx + v$$

$$E(v) = 0, E(vv^T) = R$$

求 x 的最优估计值 $\hat{x}$

最优线性估计

考虑如何定义最优估计 $\hat{x}$

- $\hat{x}$ 应为 z, R 的函数
- 无偏

$$\widetilde{x} = x - \hat{x}$$
$$E(\widetilde{x}) = 0$$

- 误差的方差矩阵最小

$$E(\tilde{x}\tilde{x}^T) = min$$

- 最小二乘法

$$J = E[(z - H\hat{x})^T W(z - H\hat{x})] = min$$

- 线性

最优线性估计

- 最小二乘法 + 误差方差最小

$$J = E[(z - H\hat{x})^T W(z - H\hat{x})] = min$$

$$E(\tilde{x}\tilde{x}^T) = min$$

$$\Rightarrow \; W = R^{-1}$$

- 无偏 + 线性 + 误差方差最小

无偏且正交

二者一致
称为最优线性估计（加权最小二乘法）

加权最小二乘法

$$J = E[(z - H\hat{x})^T W (z - H\hat{x})] = min$$

$$\frac{\partial J}{\partial \hat{x}} = E[-2H^T W z + 2H^T W H \hat{x}] = 0$$

$$\frac{\partial^2 J}{\partial \hat{x}^2} = 2\left(H^T W H\right) > 0$$

$$\hat{x} = (H^T W H)^{-1} H^T W z$$

加权最小二乘法

$$\hat{x} = (H^T W H)^{-1} H^T W z$$

满足无偏，且线性

考虑怎样的 W 使误差矩阵最小

$$E(\tilde{x}\tilde{x}^T) = min$$

$$W = R^{-1}$$

递推最小二乘法

$$z_k = H_k x + v_k$$

$$\hat{x}_k = (H_k^T W_k H_k)^{-1} H_k^T W_k z_k$$

$$P_k = (H_k^T W_k H_k)^{-1}$$

$$P_{k+1} = P_k - P_k h_{k+1}^T (h_{k+1} P_k h_{k+1}^T + w_{k+1}^{-1})^{-1} h_{k+1} P_k$$

$$\hat{x}_{k+1} = \hat{x}_k + P_k h_{k+1}^T (h_{k+1} P_k h_{k+1}^T + w_{k+1}^{-1})^{-1} (\mathrm{z}_{k+1} - h_{k+1} \hat{x}_k)$$

最优估计的正交性（正交定理）

线性估计

$$\hat{X}(z_0, \cdots, z_r) = \left\{ \hat{x} | \hat{x} = \sum_{i=0}^{r} A_i z_i \right\}$$

最优线性估计，记为 $\hat{x}$ 满足

$$min\left\{ Tr\ Cov(x - \hat{x}, x - \hat{x}) : \hat{x} \in \hat{X}(z_0, \cdots, z_r) \right\}$$

正交性

$$Cov(x - \hat{x}, z_i) = 0$$

背景、模型及分类

离散卡尔曼滤波基本模型:

$$x(k+1) = \Phi(k+1,k)x(k) + \Gamma(k+1,k)w(k)$$
$$z(k) = H(k)x(k) + v(k)$$

$$z(0), z(1), \ldots, z(k) \longrightarrow \hat{x}(j/k)$$

分为三类问题

- 预测 $j > k$
- 滤波 $j = k$
- 平滑 $j < k$

非线性系统：拓展卡尔曼滤波

推导

两种自然的推导思路

- 卡尔曼最优滤波问题-最小二乘法推导
- 卡尔曼最优预测问题-正交定理推导

最优滤波问题-最小二乘法推导

最优预测问题-正交定理推导

$$x(k+1) = \Phi(k+1,k)x(k) + \Gamma(k+1,k)w(k)$$
$$z(k) = H(k)x(k) + v(k)$$

$$z(0), z(1), \ldots, z(k) \longrightarrow \hat{x}(k+1/k)$$

最优预测问题-正交定理推导

$$\hat{x}(k+1/k) =$$
$$\Phi(k+1,k)\hat{x}(k/k-1) + K(k)[z(k) - H(k)\hat{x}(k/k-1)]$$
$$K(k) =$$
$$\Phi(k+1,k)P(k/k-1)H^{\mathrm{T}}(k)\left[H(k)P(k/k-1)H^{\mathrm{T}}(k) + R(k)\right]^{-1}$$

$$P(k/k-1) := E\left[\tilde{x}(k/k-1)\tilde{x}^{\mathrm{T}}(k/k-1)\right]$$

$$P(k+1/k) = \Phi(k+1,k)P(k/k-1)\Phi^{T}(k+1,k)$$
$$-\Phi(k+1,k)P(k/k-1)H^{\mathrm{T}}(k)\left[H(k)P(k/k-1)H^{\mathrm{T}}(k) + R(k)\right]^{-1}$$
$$\cdot H(k)P(k/k-1)\Phi^{\mathrm{T}}(k+1,k) + \Gamma(k+1,k)Q(k)\Gamma^{\mathrm{T}}(k+1,k)$$

磁场下的带电粒子

平面上单个粒子在垂直平面的磁场下运动

$$t_k = kT \quad , \qquad t_0, t_1, \cdots, t_k$$

$$\vec{B} = B\hat{\vec{z}} \quad , \qquad \omega := \frac{qB}{m}$$

$$\begin{cases} \dot{\vec{v}} = \omega(-v_x\hat{\vec{y}} + v_y\hat{\vec{x}}) \\ \vec{x}(t_0) \\ \vec{v}(t_0) \end{cases}$$

$$X(k) = (x(t_k), y(t_k), v_x(t_k), v_y(t_k))^{\mathrm{T}}$$

磁场下的带电粒子

平面上单个粒子在垂直平面的磁场下运动

$$X(k+1) = \Phi(k)X(k) + \eta(k)$$
$$Z(k) = H(k)X(k) + \delta(k)$$

$$\Phi(k) = \begin{pmatrix} 1 & 0 & \frac{1}{\omega}\sin\theta & \frac{1}{\omega}(1-\cos\theta) \\ 0 & 1 & -\frac{1}{\omega}(1-\cos\theta) & \frac{1}{\omega}\sin\theta \\ 0 & 0 & \cos\theta & \sin\theta \\ 0 & 0 & -\sin\theta & \cos\theta \end{pmatrix}$$

$$\theta = \omega(t(k+1) - t(k))$$

$$H(k) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

磁场下的带电粒子

平面上单个带电粒子在垂直平面的磁场下运动

$$\Phi(k)=\begin{pmatrix}1 & 0 & \frac{1}{\omega}\sin\theta & \frac{1}{\omega}(1-\cos\theta)\\ 0 & 1 & -\frac{1}{\omega}(1-\cos\theta) & \frac{1}{\omega}\sin\theta\\ 0 & 0 & \cos\theta & \sin\theta\\ 0 & 0 & -\sin\theta & \cos\theta\end{pmatrix},\quad \theta=\omega(t(k+1)-t(k))$$

$$H(k)=\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\end{pmatrix}$$

$$R(k)=E(\delta(k)\delta(k)^T)=\begin{pmatrix}\sigma_\delta^2 & 0\\ 0 & \sigma_\delta^2\end{pmatrix},\qquad Q(k)=E(\eta(k)\eta(k)^T)=\begin{pmatrix}0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & \sigma_\eta^2 & 0\\ 0 & 0 & 0 & \sigma_\eta^2\end{pmatrix}$$

$$E(X(0))=(EX0_x, EX0_y, EX0_{vx}, EX0_{vy})^T$$

$$Cov(X(0),X(0))=\begin{pmatrix}\sigma_{X0\_x}^2 & 0 & 0 & 0\\ 0 & \sigma_{X0\_y}^2 & 0 & 0\\ 0 & 0 & \sigma_{X0\_vx}^2 & 0\\ 0 & 0 & 0 & \sigma_{X0\_vy}^2\end{pmatrix}$$

磁场下的带电粒子

1. 首先得到 $\hat{X}(k+1/k)$——最优预测 (最优预测 $\to$ 最优滤波 $\to$ 最优光滑)

$$P(k/k-1) := E\left[\tilde{X}(k/k-1)\tilde{X}^{\mathrm{T}}(k/k-1)\right] \qquad (k \geq 0)$$

$$P(0/-1) := P(0) := Cov(E(0), E(0))$$

$$\begin{aligned}P(k+1/k) =& \Phi(k)P(k/k-1)\Phi^{T}(k)\\ &-\Phi(k)P(k/k-1)H^{\mathrm{T}}(k)\left[H(k)P(k/k-1)H^{\mathrm{T}}(k)+R(k)\right]^{-1}H(k)P(k/k-1)\Phi^{\mathrm{T}}(k)\\ &+Q(k)\end{aligned}$$

$$K_1(k) = \Phi(k)P(k/k-1)H^{\mathrm{T}}(k)\left[H(k)P(k/k-1)H^{\mathrm{T}}(k)+R(k)\right]^{-1}$$

$$\hat{X}(0/-1) := \hat{X}(0) = E(X(0))$$

$$\hat{X}(k+1/k) = \Phi(k)\hat{X}(k/k-1) + K_1(k)[Z(k) - H(k)\hat{X}(k/k-1)]$$

磁场下的带电粒子

2. 然后得到 $\hat{X}(k/k)$——最优滤波

$$\begin{aligned}\hat{X}(k/k) &= \hat{X}(k/k-1) + K_2(k)(z(k) - H(k)\hat{X}(k/k-1)) \\ K_2(k) &= P(k/k-1)H^{\mathrm{T}}(k)\left[H(k)P(k/k-1)H^{\mathrm{T}}(k) + R(k)\right]^{-1}\end{aligned}$$

$$\begin{aligned}P(k/k) &:= E\left[\tilde{X}(k/k)\tilde{X}^{\mathrm{T}}(k/k)\right] \qquad (k \geq 0) \\ P(k/k) &= P(k/k-1) - P(k/k-1)H^T(k)[H(k)P(k/k-1)H^T(k) + R(k)]^{-1}H(k)P(k/k-1) \\ P(0/0) &= P(0/-1) - P(0/-1)H^T(0)[H(0)P(0/-1)H^T(0) + R(0)]^{-1}H(0)P(0/-1)\end{aligned}$$

磁场下的带电粒子

3. 最后得到 $\hat{X}(k/N)$——最优光滑

$$
\begin{aligned}
\hat{X}(k/N) =& \hat{X}(k/k) + K_3(k)(\hat{X}(k+1/N) - \Phi(k)\hat{X}(k/k)) \\
K_3(k) =& P(k/k)\Phi^{\mathrm{T}}(k)P^{-1}(k+1/k)
\end{aligned}
$$

从 $X(N/N), K_3(N-1)$ 往前算。

磁场下的带电粒子-效果

https://github.com/enpg1qz/KalmanFilter_Example

Kalman.C :$\boldsymbol{X(k/k-1)}$(预测)，并给粒子实际运动时的磁场加了随机误差

Kalman_1DMeasure.C：只测量 $\boldsymbol{x}$ 坐标，进行对比

Kalman_filter.C： $\boldsymbol{X(k/N)}$(光滑)，与 $\boldsymbol{X(k/k-1)}$(预测) 对比

磁场下的带电粒子

效果图 1

磁场下的带电粒子

效果图 2

磁场下的带电粒子

效果图 3

接下来

- TPC 模拟
  - KalmanFilter：连续性问题、非线性问题
  - TPC 径迹重建文献
- CEE 模拟
- PPAC

Thank you!