A closer look to the United States presidential election

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January 2021

Abstract

One can imagine that the president of the biggest democracy in the world is elected by a popular vote where every citizen has the same weight. Yet it is not the case, maybe for historical reasons, the president of the United States is elected using an indirect election system. It has already happened, even in recent history, that a candidate manages to win without having the majority of the population on his side. In this paper we will try to analyse when this could happen and how to exploit this system in order to maximise the chance of victory.

1 Introduction

This past year has certainly been full of events, one of which is undoubtedly the 2020 United States presidential election held on Tuesday, November 3, where the democratic ticket Joe Biden defeated the republican incumbent president Donald Trump. Although Biden won with a margin of seven millions votes, there was uncertainty even three days after the official voting day. This is because the US presidential election uses a very particular and controversial [6] [10] indirect voting method.

This indirect voting method, known as electoral college [7], works by assigning to each state a prefix number of places in the electoral college based on their population size, then every state runs a popular vote to determine who to send in the electoral college. Almost all states follow a so called winner-tale-all-method, where they occupy all the places they are entitled to with supporters of the candidate who obtains the majority of the popular vote in that particular state.

This method has already been analysed in depth in the past [2] [5] and there are a lot of debates about its fairness. For this reason we want to analyse this system with a different approach, in particular we want to use linear programming and stochastic simulations in order to compare this indirect system with a simple popular vote, in particular we want to see how big is the impact that the distribution of the supporters of one candidate across the states can have. In other words we want to see how much significant is the difference between the direct and indirect vote when we change the population distribution.

In the first part we will detach ourselves from the US election and we will establish a similar theoretical model that we will use to run our simulations, the aim of which is to see if there is evidence that the distribution can influence the election. Then we will present a mixed integer linear program that takes into consideration the actual composition of our theoretical nation and suggests where the supporters of one candidate should move in order to maximize the probability of victory.

Lastly we will take into consideration all the US presidential election results of the past twenty years and we will see if our theoretical results are supported by the real world data. We will then summarize our results.

2 Main part

Before starting, we have to declare some notations, in particular we will take into consideration a theoretical nation of N individuals that can vote. We will partitioning this population into n states. We will use different methods to perform this subdivision. Then we have to declare the number k of representatives in the second round that will actually elect the president and how this representatives are allocated among the states. Let $p_1, ..., p_n$ be the number of seats for each states, it is obvious that we must have $p_i \approx \frac{|C_i|}{N}k$. Unfortunately the exact allocation where $p_i \in \mathbb{N}$ is not so straightforward. One of the most famous solutions to achieve this assignment problem is the Largest remainder method. The problem with this method is that under some circumstances we can encounter some paradoxes, such as the Alabama Paradox [3]. For this reason we will use the simpler Jefferson's method[1] implemented here [4] , where the only thing we have to do is to find a number $d \in \mathbb{R}$ such that:

$$\sum_{i=1}^{n} \left\lfloor \frac{|C_i|}{d} \right\rfloor = k$$

and then simply assign $p_i = \left\lfloor \frac{|C_i|}{d} \right\rfloor$.

2.1 Simulations

As already mentioned, the first thing we have done is to simulate an election to see if under some circumstances the indirect vote gives a different result from the popular vote. For this purpose we take into consideration a population of size $N = 50'005$ and we divide it equally into five categories based on the probability of each person to vote our party $(20\%, 40\%, 50\%, 60\%, 80\%)$. Clearly since the population is equally split the popular vote outcome is a Bernoulli distribution with probability 0.5. We also define $k = 401$ representatives in the second round and $n = 50$ states.

In the first simulation we assign every individual to a state with uniform probability By doing so we expect the outcome to be a Bernoulli variable with probability 0.5 as well. Indeed after running the simulation $10'000$ times we compute a statistical test [8] with null hypothesis $p = 0.5$. As expected we obtain a p-value of 0.865, for this reason we cannot claim that the indirect election system is different under the assumption that everyone chooses uniformly at random the state where to live.

The second thing we want to see is if the sizes of the states have an impact on our election. For this reason we perform again the same simulation as before, but this time we set that everyone chooses a state with probability normally distributed, where the states with index in the middle are more probable. Again we compute the p-value after $10'000$ simulations and obtain a value of 0.787 and again there is no evidence of differences between the two systems.

As the reader may have guessed if all the people follow the same distribution across the nation regardless of their political alignment we have that the two systems are statistically equivalent. For this reason we set up a last simulation where the entire population is again uniformed distributed except for the first category (the one that votes our party with probability of 20%) that has a 50% chance more to end up in one of the first half states. We run again the simulation and this time we found out that the other party won the elections 5841 times, this corresponds to a p-value of $5.579e-47$, clearly we can reject the null hypothesis of equal probability outcome and claim that the distribution could have a significant impact on the result. In particular, we can already speculate that one of the way to win the election is to concentrate their own supporters.

2.2 Linear programming and best distribution

As we have seen in the previous section the distribution of the people could be decisive for the final outcome. The goal of this section is to find out a distribution that leads to an almost deterministic victory for our part. Again we consider a model where the population is divided in five categories based on the probability of voting one party. This time we have that in each of this categories there are N_i people for $i \in \{1, 2, 3, 4, 5\}$, all the other things are equal to the previous section.

To accomplish our task we will create a mixed integer linear programming that takes as an input the current distribution and calculate how the supporters of our party (the fifth category) should move across the nation in order to maximize the probability of victory. We will first present the linear program and then analyse it closely. The program takes as an input a matrix $X \in \mathbb{R}^{5\times n}$ where the entry i, j indicates how many people of type i live in state j and a vector p with the place of each state in the second round election. The variable z represents the best distribution of our supporters and the integer variable a represents if we wish to have the majority in one particular state.

$$
\begin{aligned}
&\text{maximize } m\\
&\text{variable } \; z, a \in \mathbb{R}^n, m \in \mathbb{R}\\
&\text{subject to}\\
&\qquad z \geq 0\\
&\qquad \sum_{i=1}^{n} y_i \leq N_5,\\
&\qquad a \in \{0,1\}^n\\
&\qquad \sum_{i=1}^{n} p_i a_i \geq \frac{k}{2}\\
&\qquad -0.3X_{1j} - 0.1X_{2j} + 0.1X_{1j} + 0.3z + H(1-a) \geq m \quad \forall j
\end{aligned}
$$

where H is a big constant. The goal of the program is to select a subset of states that are sufficient to win the election and then move the population so that the minimum probability of winning across this states is maximized. The first three constrains are clear, the fourth one ensures that, if we concentrate on a particular state ($a_i = 1$) then the probability of winning the election in that state should be greater than the variable m. If the state is not necessary for the victory ($a_i = 0$) the constrain is irrelevant since H is always bigger than any possible value of the other variables. Finally our goal is to maximize m, or in short we maximize the minimum probability of victory across a subset of states sufficient to win the election.

We recall, from the previous section, that one of the cases where we have a penalizing distribution is when the other party concentrates its supporters in the first half of the states. So, in order to test how effective is the program, we take this scenario and see what is the suggested distribution.

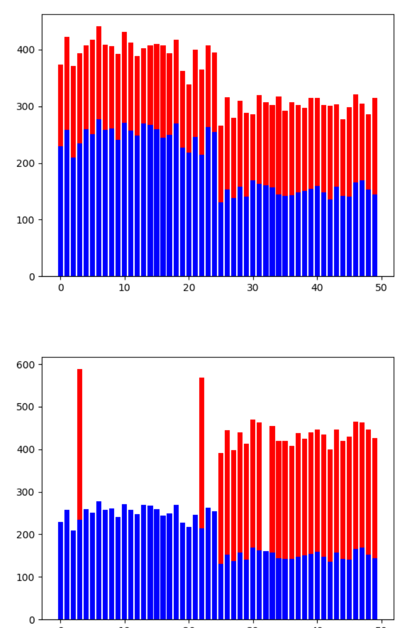

Figure 1: First the initial distribution, then the distribution suggested by the program

In figure 1 it is possible to see how the best solution is to concentrate our supporters in the other half without wasting any in the first half. After running some simulations with this scenario we notice that the simulated probability of victory for our party goes from 43.3% to 100%. Furthermore if we run again the same simulation with the difference that we have 25% less supporters, we find out that again with the best distribution it is possible to achieve 100% of victory.

As a minor remark, we point out that due to linear constrains we could not represent the probability of victory it self. For this reason we decided to minimize the expected difference of the supporter of each party. This should be proportional to the probability of winning.

2.3 Case study: United States

Now we want to conclude by analysing the US. We keep the five categories model and we compute the matrix $X \in \mathbb{R}^{5\times 51}$ (remember that the District of Columbia counts as a state during the presidential election) according with the results of the past five presidential elections.

After running some simulations we find out that with this model the Democratic party is able to win the election 98.8% of the times. For this reason we have decided to run our mixed integer linear program with the supporters of the Republican party. As already happened in the previous section, the results become almost deterministic with the victory of the Republican party. The reader can check in figure 2 how the program has changed the distribution of the electors.

Last thing we can do is to compute where the program has changed the distribution the most. The first five states are ['MN', 'WI', 'MI', 'VA', 'PA']. As one may expect these states are recurrent in the Swing State lists [9] during the past elections as our algorithm searches to maximize the probability of victory using the states that are already pendent.

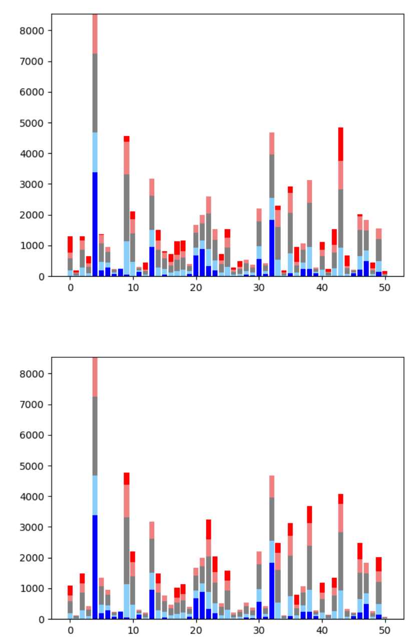

Figure 2: Population for each state divided in the five categories before and after running the program

3 Conclusion

As the reader may argue, moving the population among the states is not feasible in the short term and this research remains almost completely theoretical. Nevertheless, in the long term one could interpret the movement of the population as migratory flows or as the generation change. For this reason it could be useful to try to model, at least in part, the distribution of their own supporters according to one that could lead to a greater probability of victory.

4 Appendix 1

All the python programs and the data that we use in order to run the simulation, as well as an implementation of the mixed integer linear program could be found here:

https://github.com/enri8421/MLP

The seeds are fixed so that one can obtain the results presented in this paper.

References

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