Notes for fermionic 1PI and counter-terms

Enrico

March 1, 2025

1 First strategy

Here follow the calculation for the fermions 1PI at order $O(\frac{1}{k})$ (abelian):

$$-i\Sigma(\not{q}) = -i\int \frac{d^3p}{(2\pi)^3}\gamma^\mu \frac{2\pi}{k}\epsilon_{\mu\nu\alpha}\frac{p^\alpha}{p^2}\gamma^\nu \frac{(\not{p}+\not{q})+m}{(p+q)^2-m^2} \ . \tag{1}$$

We rewrite the **denominator** as follows:

$$\frac{1}{p^2}\frac{1}{(p+q)^2-m^2} = \int dx \frac{1}{[x((p+q)^2-m^2)+(1-x)p^2]^2} = \int_0^1 dx \frac{1}{[l^2-\Delta]^2} \ ,$$

where $l = p + xq$ and $\Delta = q^2x(x-1) + xm^2$.

As for the **numerator**:

$$\begin{aligned} -i\frac{2\pi}{k}\gamma^\mu\gamma^\nu\epsilon_{\mu\nu\alpha}p^\alpha(\not{p}+\not{q}+m) &= -i\frac{2\pi}{k}(g^{\mu\nu} - i\epsilon^{\mu\nu\rho}\gamma_\rho)\epsilon_{\mu\nu\alpha}p^\alpha(\not{p}+\not{q}+m) \\ &= \frac{-2\pi}{k}(2\delta^\rho_\alpha)\gamma_\rho p^\alpha(\not{p}+\not{q}+m) = \frac{-4\pi}{k}\not{p}(\not{p}+\not{q}+m) \ . \end{aligned}$$

Which gives

$$-i\Sigma(q) = \frac{-4\pi}{k}\int_0^1 dx \int \frac{d^3p}{(2\pi)^3}\frac{1}{[l^2-\Delta]^2}(\not{p}\not{p}+\not{p}\not{q}+\not{p}m) \ .$$

Changing to l:

$$\not{p}(\not{p}+\not{q}+m) = (\not{l}-x\not{q})(\not{l}-x\not{q}+\not{q}+m) = l^2 - x\not{l}\not{q} + \not{l}\not{q} + \not{l}m - x\not{q}\not{l} - x\not{q}\not{q} - x\not{q}m + x^2q^2 \ .$$

Getting rid of linear contributions in l

$$-i\Sigma(q) = -\frac{4\pi}{k}\int_0^1 dx \int \frac{d^3l}{(2\pi)^3}\frac{l^2}{[l^2-\Delta]^2} - \frac{4\pi}{k}\int_0^1 dx[q^2x(x-1) - xm\not{q}]\int \frac{d^3l}{(2\pi)^3}\frac{1}{[l^2-\Delta]^2} \ ,$$

and Wick rotating:

$$-i\Sigma(q) \overset{WR}{=} -i\frac{4\pi}{k}\int_0^1 dx \int \frac{d^3l}{(2\pi)^3}\frac{-l^2}{[l^2+\Delta]^2} - i\frac{4\pi}{k}\int_0^1 dx[q^2x(x-1) - xm\not{q}]\int \frac{d^3l}{(2\pi)^3}\frac{1}{[l^2+\Delta]^2} \ .$$

NOTE: the first integral clearly shows UV divergences:

$$\int_a^\Lambda d^3l \frac{l^2}{[l^2+\delta]^2} \sim^{a\to\infty} \int_a^\Lambda dl \frac{l^4}{l^4} \sim \Lambda \ .$$

Still, dimensional regulation is blind to power-law divergences, it only shows logarithmic ones.
Let us us dim-reg.
Recalling

$$\int \frac{d^dl}{(2\pi)^d}\frac{l^2}{[l^2+\Delta]^2} = \frac{d}{2}\frac{\Gamma(1-\frac{d}{2})}{(4\pi)^{d/2}}\frac{1}{\Delta^{1-\frac{d}{2}}} \ ; \ \int \frac{d^dl}{(2\pi)^3}\frac{1}{[l^2+\Delta]^2} = \frac{\Gamma[2-\frac{d}{2}]}{(4\pi)^{d/2}}\frac{1}{\Delta^{2-\frac{d}{2}}}$$

No need to set $d = 3-\epsilon$, since the Γ functions would show no pole for $\epsilon \to 0$.
It gives finally:

$$\begin{aligned}
-i\Sigma(q) =& -i\frac{6}{4k}\int_0^1 dx\sqrt{q^2x(x-1)+xm^2} \\
& -i\frac{2}{4k}\left(\int_0^1 dx \frac{q^2x(x-1)}{\sqrt{q^2x(x-1)+xm^2}} - \int_0^1 dx\frac{xm\not{q}}{\sqrt{q^2x(x-1)+xm^2}}\right) \\
=& -\frac{i}{2k}\int_0^1 dx \frac{1}{\sqrt{q^2x(x-1)+xm^2}}\left[4q^2x(x-1)+3xm^2-xm\not{q}\right] \ .
\end{aligned}$$

From which we obtain the $\delta\mathbf{Z_m}$ **and** $\delta\mathbf{Z}_\psi$ **counter-terms** (ON-SHELL scheme) at order $O(\frac{1}{k})$:

$$\delta Z_m \cdot m = \Sigma(q=m) = \frac{1}{2k}\int_0^1 dx\frac{1}{mx}\left[4x^2m^2 - 4xm^2 + 3xm^2 - xm^2\right] = \frac{1}{2k}\int_0^1 dx\left[4xm - 2m\right] = 0 \ ,$$

$$\delta Z_\psi = \frac{d}{d\not{q}}\Sigma(\not{q})\Big|_{\not{q}=m} = \frac{1}{2k}\int_0^1 dx\left[\frac{2m^2x + 4m^2(x-1)x}{\sqrt{xm^2+m^2(x-1)x}}\right] = \frac{1}{2k}\int_0^1 dx(2m+4m(x-1)) = 0 \ .$$

2 Second strategy

Here follow the calculation for the fermions 1PI at order $O(\frac{1}{k})$

$$-i\Sigma(\not{q}) = -i\int\frac{d^3p}{(2\pi)^3}\gamma^\mu\frac{2\pi}{k}\epsilon_{\mu\nu\alpha}\frac{p^\alpha}{p^2}\gamma^\nu\frac{(\not{p}+\not{q})+m}{(p+q)^2-m^2}\,. \tag{2}$$

The **numerator** can be written as:

$$\begin{aligned}
-i\frac{2\pi}{k}\gamma^\mu\gamma^\nu\epsilon_{\mu\nu\alpha}p^\alpha(\not{p}+\not{q}+m) &= -i\frac{2\pi}{k}(g^{\mu\nu}-i\epsilon^{\mu\nu\rho}\gamma_\rho)\epsilon_{\mu\nu\alpha}p^\alpha(\not{p}+\not{q}+m)\\
&= \frac{-2\pi}{k}(2\delta^\rho_\alpha)\gamma_\rho p^\alpha(\not{p}+\not{q}+m) = \frac{-4\pi}{k}\not{p}(\not{p}+\not{q}+m)\\
&= -\frac{4\pi}{k}(p^2\mathbf{1}+\not{p}(\not{q}+m))\,.
\end{aligned}$$

Which gives:

$$\begin{aligned}
-i\Sigma(\not{q}) &= \frac{-4\pi}{k}\int\frac{d^3}{(2\pi)^3}\frac{p^2+\not{p}(\not{q}+m)}{p^2[(p+q)^2-m^2]}\\
&= -\frac{4\pi}{k}\left[\int\frac{d^3p}{(2\pi)^3}\frac{\mathbf{1}}{(p+q)^2-m^2}+\gamma_\mu\left(\int\frac{d^3p}{(2\pi)^3}\frac{p^\mu}{p^2[(p+q)^2-m^2]}\right)(\not{q}+m)\right]\,.
\end{aligned}$$

Focusing on the integral between partenthesis: in order the preserve the Lorentz structure, and since the only external momentum is q, it must be

$$\begin{aligned}
\int\frac{d^3p}{(2\pi)^3}\frac{p^\mu}{p^2[(p+q)^2-m^2]} &= A(q^2)\cdot q^\mu\,,\\
A(q^2) &= \frac{1}{q^2}\int\frac{d^3p}{(2\pi)^3}\frac{p\cdot q}{p^2[(p+q)^2-m^2]}\,.
\end{aligned}$$

Substituting $p\cdot q = \frac{1}{2}[(p+q)^2-m^2-p^2-q^2+m^2]$:

$$A(q^2) = \frac{1}{2q^2}\int\frac{d^3}{(2\pi)^3}\left[\frac{1}{p^2}-\frac{1}{(p+q)^2-m^2}+\frac{m^2-q^2}{p^2[(p+q)^2-m^2]}\right]\,.$$

Placing it inside the total integral:

$$-i\Sigma(\not{q}) = -\frac{4\pi}{k}\left\{\int\frac{d^3p}{(2\pi)^3}\frac{\mathbf{1}}{(p+q)^2-m^2}+\frac{\not{q}(\not{q}+m)}{2q^2}\int\frac{d^3}{(2\pi)^3}\left[\frac{1}{p^2}-\frac{1}{(p+q)^2-m^2}+\frac{m^2-q^2}{p^2[(p+q)^2-m^2]}\right]\right\}\,.$$

Now, usingin dim-reg and recalling

$$\int\frac{d^dk}{(2\pi)^d}\frac{1}{(k^2+\Delta)^n} = \frac{1}{(4\pi)^{d/2}}\frac{\Gamma(n-d/2)}{\Gamma(n)}\left(\frac{1}{\Delta}\right)^{n-d/2}\,,$$

we can evaluate the different integrals inside $\Sigma(q)$. (WR : wick rotation, DR : dim-reg)

$$a.\ \int \frac{d^3p}{(2\pi)^3}\frac{1}{(p+q)^2-m^2} = \int \frac{d^3l}{(2\pi)^3}\frac{1}{l^2-m^2} \overset{WR}{=} -i\int \frac{d^3l}{(2\pi)^3}\frac{1}{l^2+m^2}$$

$$\overset{DR}{=} \frac{-i}{8\pi^{3/2}}(-2\sqrt{\pi})\cdot m = i\frac{m}{4\pi}$$

$$b.\ \int \frac{d^3p}{(2\pi)^3}\frac{1}{p^2} \overset{DR}{=} 0$$

$$c.\ \int \frac{d^3p}{(2\pi)^3}\frac{1}{p^2[(p+q)^2-m^2]} = \int_0^1 dx \int \frac{d^3k}{(2\pi)^3}\frac{1}{[k^2-\Delta]^2} \overset{WR}{=} i\int_0^1 dx\int \frac{d^3k}{(2\pi)^3}\frac{1}{[k^2+\Delta]^2}$$

$$\overset{DR}{=} i\frac{\sqrt{\pi}}{8\pi\sqrt{\pi}}\int_0^1 dx\frac{1}{(q^2x(x-1)+xm^2)^{\frac{1}{2}+\varepsilon}}$$

Where $l = p+q$; $k = p+xq$; $\Delta = q^2x(x-1)+xm^2$.
Note: in the first integral I subsituted $d = 3$ directly, whereas in the last one $d = 3-2\varepsilon$. The reason is to be found in the counter-term calculation at the end of the page.

This results in:

$$-i\Sigma(\not{q}) = -\frac{4\pi}{k}\left\{i\frac{m}{4\pi} + \frac{1}{2}\left(\mathbf{1}+\frac{\not{q}m}{q^2}\right)\left[-i\frac{m}{4\pi}+\frac{i}{8\pi}(m^2-q^2)\int_0^1 dx\frac{1}{(q^2x(x-1)+xm^2)^{\frac{1}{2}+\varepsilon}}\right]\right\}.$$

Finally, **the counter-terms** calculation in the ON-SHELL scheme:

$$\delta Z_m\cdot m = \Sigma(\not{p}=m) = 0\ ,$$

$$\delta Z_\psi = \frac{d}{d\not{q}}\Sigma(\not{q})|_{\not{q}=m} = \ldots = \frac{4\pi}{k}\left\{\frac{1}{8\pi}\frac{m^2}{q^2}+\frac{d}{d\not{q}}\left[\frac{1}{8\pi}(m^2-q^2)\int_0^1 dx\frac{1}{(q^2x(x-1)+xm^2)^{\frac{1}{2}+\varepsilon}}\right]\right\}\Bigg|_{\not{q}=m}$$

$$= \frac{1}{2k}-\frac{1}{k}\int_0^1 dx\,\frac{1}{x^{1+2\varepsilon}} = \frac{1}{2k}\left(1+\frac{1}{\varepsilon}\right)\ .$$