

Notes for fermionic 1PI and counter-terms

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March 1, 2025

1 First strategy

Here follow the calculation for the fermions 1PI at order $O(\frac{1}{k})$ (abelian):

$$-i\Sigma(q) = -i \int \frac{d^3p}{(2\pi)^3} \gamma^\mu \frac{2\pi}{k} \epsilon_{\mu\nu\alpha} \frac{p^\alpha}{p^2} \gamma^\nu \frac{(\not{p} + \not{q}) + m}{(p+q)^2 - m^2} . \quad (1)$$

We rewrite the **denominator** as follows:

$$\frac{1}{p^2} \frac{1}{(p+q)^2 - m^2} = \int dx \frac{1}{[x((p+q)^2 - m^2) + (1-x)p^2]^2} = \int_0^1 dx \frac{1}{[l^2 - \Delta]^2} ,$$

where $l = p + xq$ and $\Delta = q^2x(x-1) + xm^2$.

As for the **numerator**:

$$\begin{aligned} -i \frac{2\pi}{k} \gamma^\mu \gamma^\nu \epsilon_{\mu\nu\alpha} p^\alpha (\not{p} + \not{q} + m) &= -i \frac{2\pi}{k} (g^{\mu\nu} - i\epsilon^{\mu\nu\rho} \gamma_\rho) \epsilon_{\mu\nu\alpha} p^\alpha (\not{p} + \not{q} + m) \\ &= \frac{-2\pi}{k} (2\delta_\alpha^\rho) \gamma_\rho p^\alpha (\not{p} + \not{q} + m) = \frac{-4\pi}{k} \not{p}(\not{p} + \not{q} + m) . \end{aligned}$$

Which gives

$$-i\Sigma(q) = \frac{-4\pi}{k} \int_0^1 dx \int \frac{d^3p}{(2\pi)^3} \frac{1}{[l^2 - \Delta]^2} (\not{p}\not{p} + \not{p}\not{q} + \not{p}m) .$$

Changing to l :

$$\not{p}(\not{p} + \not{q} + m) = (l - xq)(l - xq + \not{q} + m) = l^2 - x\not{l}\not{q} + \not{l}\not{q} + \not{l}m - xq\not{l} - xq\not{q} - xqm + x^2q^2 .$$

Getting rid of linear contributions in l

$$-i\Sigma(q) = -\frac{4\pi}{k} \int_0^1 dx \int \frac{d^3l}{(2\pi)^3} \frac{l^2}{[l^2 - \Delta]^2} - \frac{4\pi}{k} \int_0^1 dx [q^2 x(x-1) - xm\not{q}] \int \frac{d^3l}{(2\pi)^3} \frac{1}{[l^2 - \Delta]^2} ,$$

and Wick rotating:

$$-i\Sigma(q) \stackrel{WR}{=} -i\frac{4\pi}{k} \int_0^1 dx \int \frac{d^3l}{(2\pi)^3} \frac{-l^2}{[l^2 + \Delta]^2} - i\frac{4\pi}{k} \int_0^1 dx [q^2 x(x-1) - xm\not{q}] \int \frac{d^3l}{(2\pi)^3} \frac{1}{[l^2 + \Delta]^2} .$$

NOTE: the first integral clearly shows UV divergences:

$$\int_a^\Lambda d^3l \frac{l^2}{[l^2 + \delta]^2} \sim^{a \rightarrow \infty} \int_a^\Lambda dl \frac{l^4}{l^4} \sim \Lambda .$$

Still, dimensional regulation is blind to power-law divergences, it only shows logarithmic ones.

Let us us dim-reg.

Recalling

$$\int \frac{d^d l}{(2\pi)^d} \frac{l^2}{[l^2 + \Delta]^2} = \frac{d}{2} \frac{\Gamma(1 - \frac{d}{2})}{(4\pi)^{d/2}} \frac{1}{\Delta^{1-\frac{d}{2}}} ; \quad \int \frac{d^d l}{(2\pi)^d} \frac{1}{[l^2 + \Delta]^2} = \frac{\Gamma[2 - \frac{d}{2}]}{(4\pi)^{d/2}} \frac{1}{\Delta^{2-\frac{d}{2}}}$$

No need to set $d = 3 - \epsilon$, since the Γ functions would show no pole for $\epsilon \rightarrow 0$.

It gives finally:

$$\begin{aligned} -i\Sigma(q) &= -i\frac{6}{4k} \int_0^1 dx \sqrt{q^2 x(x-1) + xm^2} \\ &\quad - i\frac{2}{4k} \left(\int_0^1 dx \frac{q^2 x(x-1)}{\sqrt{q^2 x(x-1) + xm^2}} - \int_0^1 dx \frac{xm\not{q}}{\sqrt{q^2 x(x-1) + xm^2}} \right) \\ &= -\frac{i}{2k} \int_0^1 dx \frac{1}{\sqrt{q^2 x(x-1) + xm^2}} [4q^2 x(x-1) + 3xm^2 - xm\not{q}] . \end{aligned}$$

From which we obtain the $\delta\mathbf{Z}_m$ and $\delta\mathbf{Z}_\psi$ **counter-terms** (ON-SHELL scheme) at order $O(\frac{1}{k})$:

$$\delta Z_m \cdot m = \Sigma(q = m) = \frac{1}{2k} \int_0^1 dx \frac{1}{mx} [4x^2 m^2 - 4xm^2 + 3xm^2 - xm^2] = \frac{1}{2k} \int_0^1 dx [4xm - 2m] = 0 ,$$

$$\delta Z_\psi = \frac{d}{d\not{q}} \Sigma(\not{q}) \Big|_{\not{q}=m} = \frac{1}{2k} \int_0^1 dx \left[\frac{2m^2 x + 4m^2(x-1)x}{\sqrt{xm^2 + m^2(x-1)x}} \right] = \frac{1}{2k} \int_0^1 dx (2m + 4m(x-1)) = 0 .$$

2 Second strategy

Here follow the calculation for the fermions 1PI at order $O(\frac{1}{k})$

$$-i\Sigma(q) = -i \int \frac{d^3p}{(2\pi)^3} \gamma^\mu \frac{2\pi}{k} \epsilon_{\mu\nu\alpha} \frac{p^\alpha}{p^2} \gamma^\nu \frac{(\not{p} + \not{q}) + m}{(p+q)^2 - m^2} . \quad (2)$$

The **numerator** can be written as:

$$\begin{aligned} -i \frac{2\pi}{k} \gamma^\mu \gamma^\nu \epsilon_{\mu\nu\alpha} p^\alpha (\not{p} + \not{q} + m) &= -i \frac{2\pi}{k} (g^{\mu\nu} - i\epsilon^{\mu\nu\rho} \gamma_\rho) \epsilon_{\mu\nu\alpha} p^\alpha (\not{p} + \not{q} + m) \\ &= \frac{-2\pi}{k} (2\delta_\alpha^\rho) \gamma_\rho p^\alpha (\not{p} + \not{q} + m) = \frac{-4\pi}{k} \not{p} (\not{p} + \not{q} + m) \\ &= -\frac{4\pi}{k} (p^2 \mathbf{1} + \not{p}(\not{q} + m)) . \end{aligned}$$

Which gives:

$$\begin{aligned} -i\Sigma(q) &= \frac{-4\pi}{k} \int \frac{d^3p}{(2\pi)^3} \frac{p^2 + \not{p}(\not{q} + m)}{p^2[(p+q)^2 - m^2]} \\ &= -\frac{4\pi}{k} \left[\int \frac{d^3p}{(2\pi)^3} \frac{\mathbf{1}}{(p+q)^2 - m^2} + \gamma_\mu \left(\int \frac{d^3p}{(2\pi)^3} \frac{p^\mu}{p^2[(p+q)^2 - m^2]} \right) (\not{q} + m) \right] . \end{aligned}$$

Focusing on the integral between parenthesis: in order to preserve the Lorentz structure, and since the only external momentum is q , it must be

$$\begin{aligned} \int \frac{d^3p}{(2\pi)^3} \frac{p^\mu}{p^2[(p+q)^2 - m^2]} &= A(q^2) \cdot q^\mu , \\ A(q^2) &= \frac{1}{q^2} \int \frac{d^3p}{(2\pi)^3} \frac{p \cdot q}{p^2[(p+q)^2 - m^2]} . \end{aligned}$$

Substituting $p \cdot q = \frac{1}{2}[(p+q)^2 - m^2 - p^2 - q^2 + m^2]$:

$$A(q^2) = \frac{1}{2q^2} \int \frac{d^3p}{(2\pi)^3} \left[\frac{1}{p^2} - \frac{1}{(p+q)^2 - m^2} + \frac{m^2 - q^2}{p^2[(p+q)^2 - m^2]} \right] .$$

Placing it inside the total integral:

$$-i\Sigma(q) = -\frac{4\pi}{k} \left\{ \int \frac{d^3p}{(2\pi)^3} \frac{\mathbf{1}}{(p+q)^2 - m^2} + \frac{\not{q}(\not{q} + m)}{2q^2} \int \frac{d^3p}{(2\pi)^3} \left[\frac{1}{p^2} - \frac{1}{(p+q)^2 - m^2} + \frac{m^2 - q^2}{p^2[(p+q)^2 - m^2]} \right] \right\} .$$

Now, using dim-reg and recalling

$$\int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2 + \Delta)^n} = \frac{1}{(4\pi)^{d/2}} \frac{\Gamma(n - d/2)}{\Gamma(n)} \left(\frac{1}{\Delta} \right)^{n-d/2} ,$$

we can evaluate the different integrals inside $\Sigma(q)$. (WR : wick rotation, DR : dim-reg)

$$a. \int \frac{d^3 p}{(2\pi)^3} \frac{1}{(p+q)^2 - m^2} = \int \frac{d^3 l}{(2\pi)^3} \frac{1}{l^2 - m^2} \stackrel{WR}{=} -i \int \frac{d^3 l}{(2\pi)^3} \frac{1}{l^2 + m^2} \\ \stackrel{DR}{=} \frac{-i}{8\pi^{3/2}} (-2\sqrt{\pi}) \cdot m = i \frac{m}{4\pi}$$

$$b. \int \frac{d^3 p}{(2\pi)^3} \frac{1}{p^2} \stackrel{DR}{=} 0$$

$$c. \int \frac{d^3 p}{(2\pi)^3} \frac{1}{p^2 [(p+q)^2 - m^2]} = \int_0^1 dx \int \frac{d^3 k}{(2\pi)^3} \frac{1}{[k^2 - \Delta]^2} \stackrel{WR}{=} i \int_0^1 dx \int \frac{d^3 k}{(2\pi)^3} \frac{1}{[k^2 + \Delta]^2} \\ \stackrel{DR}{=} i \frac{\sqrt{\pi}}{8\pi\sqrt{\pi}} \int_0^1 dx \frac{1}{(q^2 x(x-1) + xm^2)^{\frac{1}{2}+\varepsilon}}$$

Where $l = p + q$; $k = p + xq$; $\Delta = q^2 x(x-1) + xm^2$.

Note: in the first integral I subtituted $d = 3$ directly, whereas in the last one $d = 3 - 2\varepsilon$. The reason is to be found in the counter-term calculation at the end of the page.

This results in:

$$-i\Sigma(\not{q}) = -\frac{4\pi}{k} \left\{ i \frac{m}{4\pi} + \frac{1}{2} \left(1 + \frac{\not{q}m}{q^2} \right) \left[-i \frac{m}{4\pi} + \frac{i}{8\pi} (m^2 - q^2) \int_0^1 dx \frac{1}{(q^2 x(x-1) + xm^2)^{\frac{1}{2}+\varepsilon}} \right] \right\}.$$

Finally, **the counter-terms** calculation in the ON-SHELL scheme:

$$\delta Z_m \cdot m = \Sigma(\not{p} = m) = 0,$$

$$\delta Z_\psi = \frac{d}{d\not{q}} \Sigma(\not{q})|_{\not{q}=m} = \dots = \frac{4\pi}{k} \left\{ \frac{1}{8\pi} \frac{m^2}{q^2} + \frac{d}{d\not{q}} \left[\frac{1}{8\pi} (m^2 - q^2) \int_0^1 dx \frac{1}{(q^2 x(x-1) + xm^2)^{\frac{1}{2}+\varepsilon}} \right] \right\} \Big|_{\not{q}=m} \\ = \frac{1}{2k} - \frac{1}{k} \int_0^1 dx \frac{1}{x^{1+2\varepsilon}} = \frac{1}{2k} \left(1 + \frac{1}{\varepsilon} \right).$$