Exercise set 1

Advanced Course in Machine Learning

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Exercise 1

We want to evaluate the risk for the loss

$$\mathcal{L}(y, x, \alpha) = (y - \hat{y})^2 = (y - \alpha x)^2$$

Task a

The general definition of risk is the following:

$$\begin{aligned} R(\delta) &= \mathrm{E}_{p(y,x)}[\mathcal{L}(y, \delta(x))] \\ &= \int_{x,y} \mathcal{L}(y, \delta(x)) p(y, x) dx dy \end{aligned}$$

Where $\mathcal{L}(y, \delta(x))$ is the expected loss and $p(y, x)$ is the distribution that generates the data points. In this case, we know both:

$$\begin{gathered} \mathcal{L}(y, \delta(x)) = \mathcal{L}(y, x, \alpha) = (y - \hat{y})^2 = (y - \alpha x)^2 \\ p(x, y) = p(x) p(y|x) \end{gathered}$$

In the last equation, we know both $p(x)$ and $p(y|x)$:

- x is sampled uniformly in $[-3, 3]$, thus this is a simple continuous distribution. Therefore, its probability density function is

$$\mathbf{f_x}(x) = \begin{cases} \frac{1}{b-a} = \frac{1}{6} & for\ x\ \in [-3, 3] \\ 0 & otherwise \end{cases}$$

- $y|x$ is sampled uniformly in $[2x - \frac{1}{2}, 2x + \frac{1}{2}]$, this this is another simple continuous distribution. Therefore, its probability density function is

$$\mathbf{f_{y|x}}(y|x) = \begin{cases} \frac{1}{b-a} = 1 & for\ y\ \in [2x - \frac{1}{2}, 2x + \frac{1}{2}] \\ 0 & otherwise \end{cases}$$

Combining the two, we then the joint density function

$$\mathbf{f_{x,y}}(x, y) = \begin{cases} \frac{1}{6} & for\ x \in [-3, 3], y \in [2x - \frac{1}{2}, 2x + \frac{1}{2}] \\ 0 & otherwise \end{cases}$$

We can then compute the risk $R(\alpha)$:

$$\begin{aligned} R(q) &= \int_{x,y} \frac{1}{6} (y - \alpha x)^2 dx dy \\ &= \int_y \int_x \frac{1}{6} (y - \alpha x)^2 dx dy \\ &= \int_y \frac{-(y - \alpha x)^3}{18\alpha} dy \\ &= \frac{-(\alpha x - y)^4}{72\alpha} \end{aligned}$$

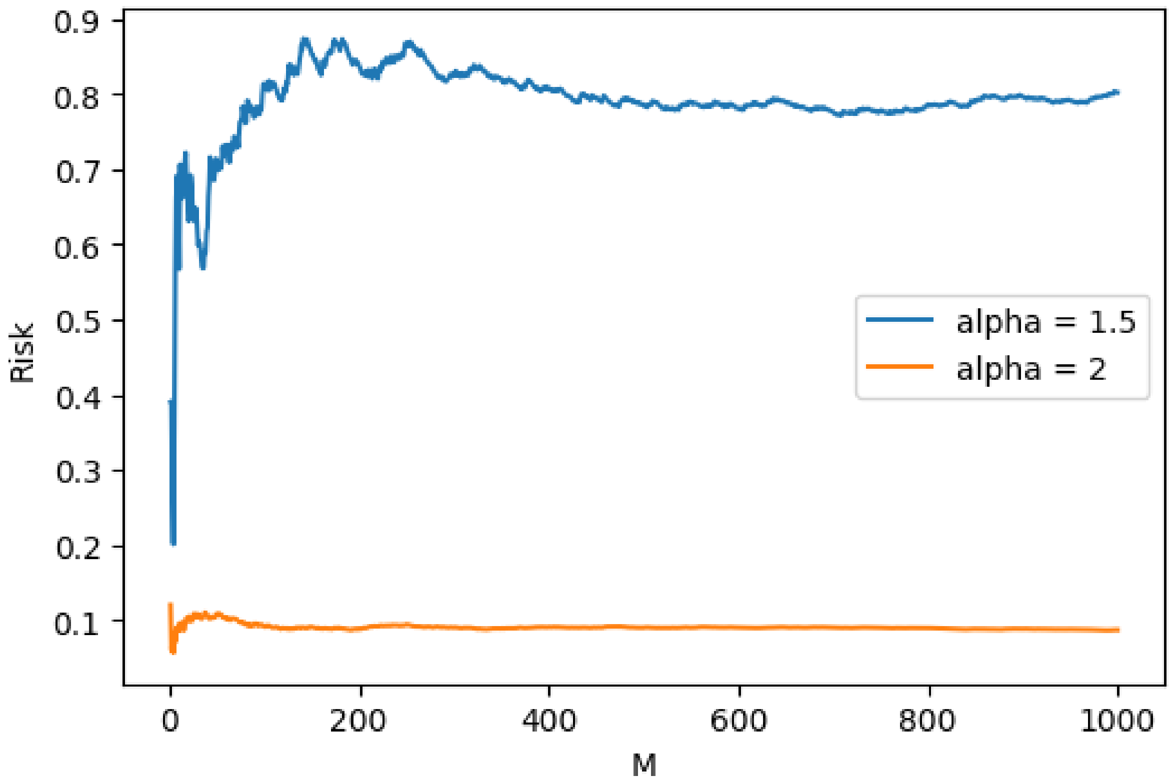
In order to find the alpha that gives the smalles risk, we set the last result to zero, calculating also the definite integral over y and x; after some math (which is not reported because both too long and trivial, since it's solving a fourth-degree equation), we find that the final result is $\alpha = 2$.

Task b and c

In order to compare the numerical approximation with the true value, I created the following code:

```
import numpy as np
from matplotlib import pyplot as plt

"""
Create artificial data, 1000 samples
    - x is sampled uniformly between -3 and 3
    - y is sampled uniformly between 2x-0.5 and 2x+0.5
"""
x = np.random.uniform(low=-3, high=3, size=1000)
y = []
for i in x:
    y.append(np.random.uniform(low=2*i-0.5, high=2*i+0.5, size=1)[0])

# R(alpha) with alpha=1.5
M = [i for i in range(1, 1001)]
alpha = 1.5
R = []

for i in M:
    r = 0
    for j in range(0, i):
        r += (y[j]-alpha*x[j])**2
    r/=i
    R.append(r)

# R(alpha) with alpha=2 (true value)
alpha = 2
R2 = []
for i in M:
    r = 0
    for j in range(0, i):
        r += (y[j]-alpha*x[j])**2
    r/=i
    R2.append(r)

# Plot the results
plt.plot(M, R, label='alpha = 1.5')
plt.plot(M, R2, label='alpha = 2')
plt.xlabel('M')
plt.ylabel('Risk')
plt.legend()
```

The result I got follows as a plot.



Exercise 2

We start with this optimization problem

$$\begin{aligned} &\min x^2 + y^2 \\ &\text{s.t. } 3x - y + 2 \leq 0 \end{aligned}$$

For convenience, we will call the function to optimize $f(x, y)$ and the inequality constraint $g(x, y)$.

Task a

The Lagrangian function for this problem is the following:

$$\begin{aligned} \Lambda(x, y, \lambda) &= f(x, y) + \lambda g(x, y) \\ &= x^2 + y^2 + \lambda(3x - y + 2) \end{aligned}$$

The constraints for x, y to be optimal are:

- There is $\lambda \geq 0$ such that $\nabla f = -\lambda \nabla g$;
- Either $\lambda = 0$ or $g(x, y) = 0$.

Task b

In order to find an optimal solution, we first compute the gradient of the lagrangian function calculating the partial derivatives with respect to x, y and λ. We then have that:

$$\nabla\Lambda = \begin{pmatrix} 2x + 3\lambda \\ 2y - \lambda \\ 3x - y + 2 \end{pmatrix}$$

We then solve the linear system $\nabla\Lambda = \mathbf{0}$. With trivial math we get:

$$x = -\tfrac{3}{5}$$

$$y = \tfrac{1}{5}$$

$$\lambda = \tfrac{2}{5}$$

Finally, we check if the results respect the KKT conditions reported in Task a. Since $\lambda \neq 0$, in order for the solution to be optimal we must find $g(x,y) = g\left(-\frac{3}{5}, \frac{1}{5}\right) = 0$; otherwise, the solution is not optimal. Let's then check:

$$g\left(-\frac{3}{5}, \frac{1}{5}\right) = 3\frac{-3}{5} - \frac{1}{5} + 2 = 0 \leq 0$$

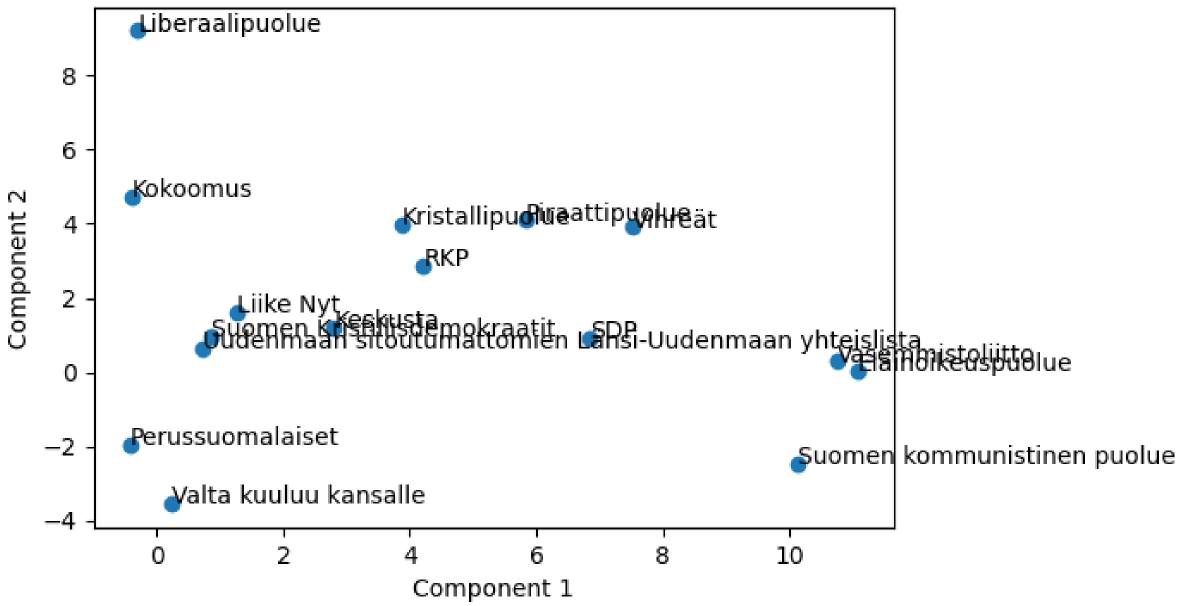
The constraints are respected, thus the solution is optimal.

Exercise 5

In order to apply PCA with 2 dimensions I used the code from Exercise 4. The additional code I used is the following:

```
# Data preparation
X = loadtxt('elec2022.txt', usecols=([i for i in range(1,56)]))
parties = loadtxt('elec2022.txt', usecols=(0))
 # (changed the file using commas for convenience)
parties_names = pd.read_csv('parties.txt', sep=',', header=None)

# Apply PCA using TMC code
x1, x2 = pca(X)

# Prepare output
df = pd.DataFrame()
df['party'] = parties
df['party'] = df['party'].map(parties_names.set_index(0)[1])
df['x1'] = x1
df['x2'] = x2
df = df.groupby(['party']).mean().reset_index()

# Plot the results
fig, ax = plt.subplots()
ax.scatter(df['x1'], df['x2'])
for i, txt in enumerate(df['party']):
    ax.annotate(txt, (df['x1'][i], df['x2'][i]))
plt.xlabel('Component 1')
plt.ylabel('Component 2')
fig.set_dpi(100)
```

The figure I created, with the party labels on each point, follows.


Liberaalipuolue
Kokoomus
Kristallipuolue
Piraattipuolue
Vihreät
RKP
Liike Nyt
Keskusta
SDP
Uudenmaan sitoutumattomien Länsi-Uudenmaan yhteislista
Perussuomalaiset
Suomen kommunistinen puolue
Valta kuuluu kansalle
8
6
4
2
0
−2
−4
0
2
4
6
8
10
Component 1
Component 2