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Efficient aeroacoustics optimization in the frequency domain using the adjoint method

E. Foglia[1], M. Daroukh[2], M. Buszyk[2], I. Salah el-Din[2], S. Moreau[3]

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05/05/2024

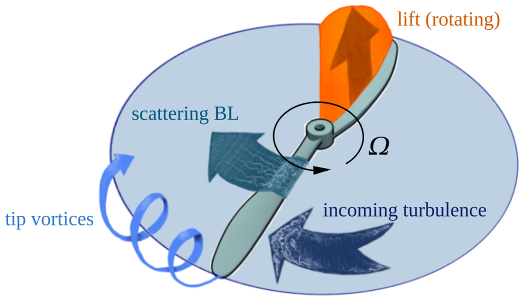
1982: the noise of a nearby elevated subway line in New York causes students to lose, on average, one year of education[a].

[a]Goldman 1982.

1982: the noise of a nearby elevated subway line in New York causes students to lose, on average, one year of education. **Now add drones**[a].

[a] Christian and Cabell 2017.

Noise production mechanisms in propellers are complex and, importantly, fundamentally linked to their aerodynamic performances.

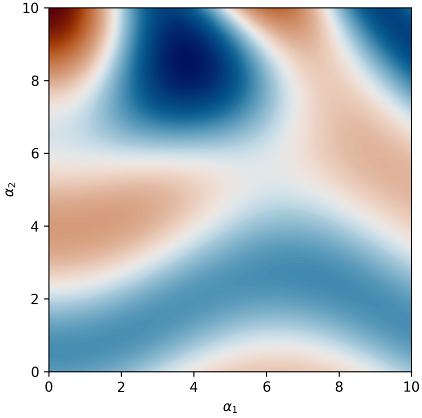


For this reason, simple models are usually insufficient to accurately predict both, and **full scale simulations are needed**.

Optimization as blind alpinism

Let α be you coordinates and $\mathcal{J}$ be the elevation.
Problem: find $\alpha^\star$ such that

$$\alpha^\star = \underset{\alpha}{\operatorname{argmin}}\, \mathcal{J}(\alpha)$$



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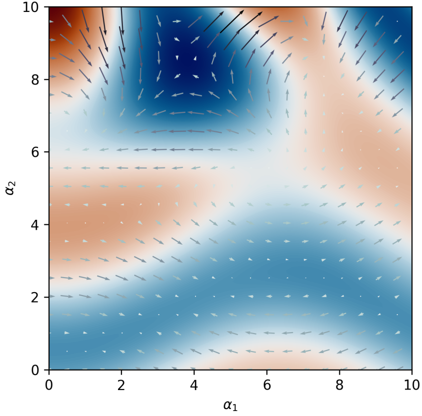

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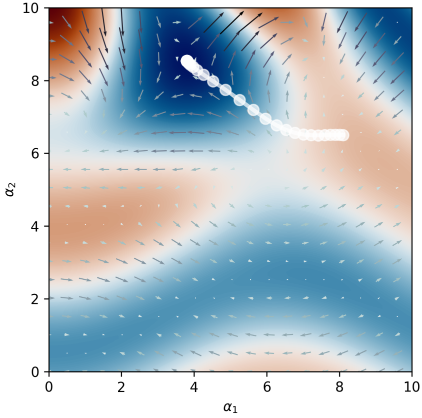
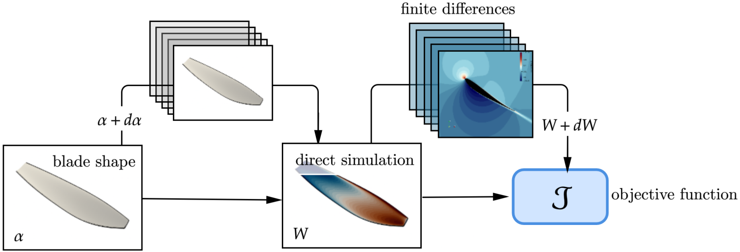


The need to go beyond finite differences

Efficient optimization requires the gradient of the objective function $\mathcal{J}$ with respect to the geometrical parameters α. There are mainly two ways to compute it: a perturbation method (finite differences) or the **adjoint method**.



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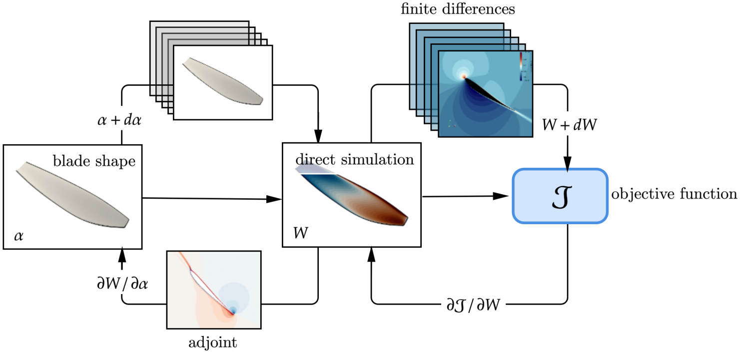

Definition: adjoint of a linear operator

[a] Given a linear operator $\boldsymbol{L} : V \to V$, where V is some vector space, and an inner product $\langle \cdot, \cdot \rangle$, the adjoint operator $\boldsymbol{L}^\dagger$ is the unique linear operator such that:

$$\langle \boldsymbol{w}^\dagger, \boldsymbol{L}\boldsymbol{w} \rangle = \langle \boldsymbol{L}^\dagger \boldsymbol{w}^\dagger, \boldsymbol{w} \rangle, \quad \forall \boldsymbol{w}, \boldsymbol{w}^\dagger$$

For the standard inner product in $\mathbb{R}^n$, $\langle \boldsymbol{u}, \boldsymbol{v} \rangle = \boldsymbol{u}^\top \boldsymbol{v}$, we have that $\boldsymbol{L}^\dagger = \boldsymbol{L}^\top$.

[a] Notice: this definition also works for operators in Hilbert spaces, with the correct boundary conditions on $\boldsymbol{w}^\dagger$

As the objective function $\mathcal{J}$ we can use the Source Power Level (PWL) as defined by Cantrell and Hart[1].

$$\mathcal{J} = PWL = 10\log_{10}\left(\int_{\mathcal{S}} \frac{p\bar{p}}{2\rho_0 c_0} F(\psi, M)d\mathcal{S}\right) - 10\log_{10}(W_{\text{ref}})$$

where $p = p(\boldsymbol{W}, \boldsymbol{X})$ is the (complex) Fourier coefficient of the acoustic pressure. The flow solution $\boldsymbol{W}$ and the mesh $\boldsymbol{X}$ are related by the discretized, steady-state Navier-Stokes equations as:

$$R(\boldsymbol{W}, \boldsymbol{X}) = 0$$

[1] Cantrell and Hart 1964.

By the chain rule we can write the gradient of p w.r.t. α_k as:

$$\frac{dp}{d\alpha_k} = \frac{\partial p}{\partial \boldsymbol{W}} \underbrace{\frac{d\boldsymbol{W}}{d\alpha_k}}_{\boldsymbol{w}_k} + \frac{\partial p}{\partial \boldsymbol{X}} \frac{d\boldsymbol{X}}{d\alpha_k}$$

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where the perturbation $\boldsymbol{w}_k$ solves the linearized system:

$$\frac{\partial R}{\partial \boldsymbol{W}} \boldsymbol{w}_k + \frac{\partial R}{\partial \boldsymbol{X}} \frac{d\boldsymbol{X}}{d\alpha_k} = 0$$

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$$\frac{\partial R}{\partial \boldsymbol{W}} \boldsymbol{w}_k + \frac{\partial R}{\partial \boldsymbol{X}} \frac{d\boldsymbol{X}}{d\alpha_k} = 0$$

By using a little algebra we can rewrite the gradient equation as:

$$\frac{dp}{d\alpha_k} = \left\langle \boldsymbol{w}^{\dagger}, -\frac{\partial R}{\partial \boldsymbol{X}}\frac{d\boldsymbol{X}}{d\alpha_k} \right\rangle + \frac{\partial p}{\partial \boldsymbol{X}}\frac{d\boldsymbol{X}}{d\alpha_k}$$

where $w^{\dagger}$ solves the adjoint system:

$$\left(\frac{\partial R}{\partial \boldsymbol{W}}\right)^{\top} \boldsymbol{w}^{\dagger} - \left(\frac{\partial p}{\partial \boldsymbol{W}}\right)^{\top} = 0$$

Perturbative approach

$$\frac{dp}{d\alpha_k} = \left\langle \left(\frac{\partial p}{\partial \boldsymbol{W}}\right)^\top, \boldsymbol{w}_k \right\rangle + \frac{\partial p}{\partial \boldsymbol{X}} \frac{d\boldsymbol{X}}{d\alpha_k}$$

$$\frac{\partial R}{\partial \boldsymbol{W}} \boldsymbol{w}_k + \frac{\partial R}{\partial \boldsymbol{X}} \frac{d\boldsymbol{X}}{d\alpha_k} = 0$$

Adjoint approach

$$\frac{dp}{d\alpha_k} = \left\langle \boldsymbol{w}^\dagger, -\frac{\partial R}{\partial \boldsymbol{X}} \frac{d\boldsymbol{X}}{d\alpha_k} \right\rangle + \frac{\partial p}{\partial \boldsymbol{X}} \frac{d\boldsymbol{X}}{d\alpha_k}$$

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The gradient: comparison

Perturbative approach

$$\frac{dp}{d\alpha_k} = \left\langle \left(\frac{\partial p}{\partial \boldsymbol{W}}\right)^\top, \boldsymbol{w}_k \right\rangle + \frac{\partial p}{\partial \boldsymbol{X}} \frac{d\boldsymbol{X}}{d\alpha_k}$$

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$$\left(\frac{\partial R}{\partial \boldsymbol{W}}\right)^\top \boldsymbol{w}^\dagger - \left(\frac{\partial p}{\partial \boldsymbol{W}}\right)^\top = 0$$

Acoustic pressure calculation

Apply the Ffowcs Williams and Hawkings[2] analogy on the surface of the blade (no volume terms!).

Thickness noise

$$Q = \rho_\infty \boldsymbol{u}^\Sigma \cdot \boldsymbol{n}$$

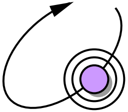
Monopolar nature, due to the accelaration of the fluid around the blade.

Loading noise

$$\boldsymbol{F} = p\boldsymbol{n} = \left[(\gamma - 1)\rho E - p_\infty\right] \boldsymbol{n}$$

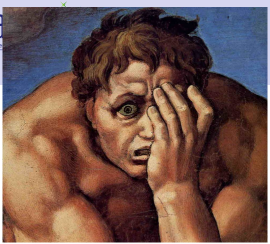
Dipolar nature, due to the circular motion of the forces on the blade. Viscous terms are neglected.

They both become efficient because of the **circular motion**



[2] Ffowcs Williams and Hawkings 1969.

Hanson's formula[3]



$$
\begin{aligned}
p_F^{(mB)}(\boldsymbol{x}) = -\frac{ImB^2\Omega e^{ImB\Omega\bar{R}^*_\beta/c_0}}{4\pi c_0 \bar{R}_\beta} e^{ImB\pi/2} \Bigg\{ \\
S_c \int_\Sigma J_{mB}(\kappa_0 R_0 S_s) e^{ImB\phi_0} e^{I\kappa_0 S_c y_1} F_a^{(0)}(\boldsymbol{y}) d\Sigma(\boldsymbol{y}) \\
+ \int_\Sigma J_{mB}(\kappa_0 R_0 S_s) \frac{e^{ImB\phi_0} e^{I\kappa_0 S_c y_1}}{M_{\text{rot}}} F_t^{(0)}(\boldsymbol{y}) d\Sigma(\boldsymbol{y}) \\
+ IS_s \int_\Sigma J'_{mB}(\kappa_0 R_0 S_s) e^{ImB\phi_0} e^{I\kappa_0 S_c y_1} F_r^{(0)}(\boldsymbol{y}) d\Sigma(\boldsymbol{y}) \Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
p_Q^{(mB)}(\boldsymbol{x}) = -\frac{ImB^2\Omega e^{-ImB\Omega\bar{R}^*_\beta/c_0}}{4\pi\beta^2 \bar{R}_\beta} \left(1 + M\frac{x_1}{\bar{R}_\beta}\right) e^{ImB\pi/2} \\
\times \int_\Sigma J_{mB}(\kappa_0 R_0 S_s) e^{I\kappa_0 S_c y_1} e^{ImB\phi_0} Q^{(0)}(\boldsymbol{y}) d\Sigma(\boldsymbol{y})
\end{aligned}
$$

[3] Hanson and Parzych 1993.

$$p_F^{(mB)}(\boldsymbol{x}) = -\frac{ImB^2\Omega e^{ImB\Omega\bar{R}_\beta^*/c_0}}{4\pi c_0 \bar{R}_\beta} e^{ImB\pi/2} \Bigg\{ S_c \int_\Sigma J_{mB}(\kappa_0 R_0 S_s) e^{ImB\phi_0} e^{I\kappa_0 S_c y_1} F_a^{(0)}(\boldsymbol{y}) d\Sigma(\boldsymbol{y}) + \int_\Sigma J_{mB}(\kappa_0 R_0 S_s) \frac{e^{ImB\phi_0} e^{I\kappa_0 S_c y_1}}{M_{\mathrm{rot}}} F_t^{(0)}(\boldsymbol{y}) d\Sigma(\boldsymbol{y}) + IS_s \int_\Sigma J'_{mB}(\kappa_0 R_0 S_s) e^{ImB\phi_0} e^{I\kappa_0 S_c y_1} F_r^{(0)}(\boldsymbol{y}) d\Sigma(\boldsymbol{y}) \Bigg\}$$

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[3] Hanson and Parzych 1993.

Hold on: we are derivating...

$$\frac{\partial \mathcal{I}_{\lambda\mu,1}}{\partial Y_{\nu\xi,i}} = S_c \kappa_0 \left[S_s J_n'(\kappa_0 S_s R_{\lambda\mu}) \frac{\partial R_{\lambda\mu}}{\partial Y_{\nu\xi,i}} + I J_n(\kappa_0 S_s R_{\lambda\mu}) \left(\frac{c_0}{\Omega} \frac{\partial \phi_{\lambda\mu}}{\partial Y_{\nu\xi,i}} + S_c \delta_{i1} \frac{\partial C_{\lambda\mu 1}}{\partial Y_{\nu\xi 1}} \right) \right] e^{In\phi_{\lambda\mu}} e^{I\kappa_0 S_c C_{\lambda\mu 1}}$$

$$\frac{\partial \mathcal{I}_{\lambda\mu,2}}{\partial Y_{\nu\xi,i}} = I S_s \left(\frac{(\square)}{\Omega R_{\lambda\mu}/c_0} - (\star) \cdot \frac{c_0}{\Omega R_{\lambda\mu}^2} \frac{\partial R_{\lambda\mu}}{\partial Y_{\nu\xi,i}} \right)$$

$$\frac{\partial \mathcal{I}_{\lambda\mu,3}}{\partial Y_{\nu\xi,i}} = \kappa_0 \left[S_s J_n''(\kappa_0 S_s R_{\lambda\mu}) \frac{\partial R_{\lambda\mu}}{\partial Y_{\nu\xi,i}} + I J_n'(\kappa_0 S_s R_{\lambda\mu}) \left(\frac{c_0}{\Omega} \frac{\partial \phi_{\lambda\mu}}{\partial Y_{\nu\xi,i}} + S_c \delta_{i1} \frac{\partial C_{\lambda\mu 1}}{\partial Y_{\nu\xi 1}} \right) \right] e^{In\phi_{\lambda\mu}} e^{I\kappa_0 S_c C_{\lambda\mu 1}}$$

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$$\frac{\partial \mathcal{I}_{\lambda\mu,1}}{\partial Y_{\nu\xi,i}} = S_c \kappa_0 \Big[S_s J_n'(\kappa_0 S_s R_{\lambda\mu}) \frac{\partial R_{\lambda\mu}}{\partial Y_{\nu\xi,i}} + $$

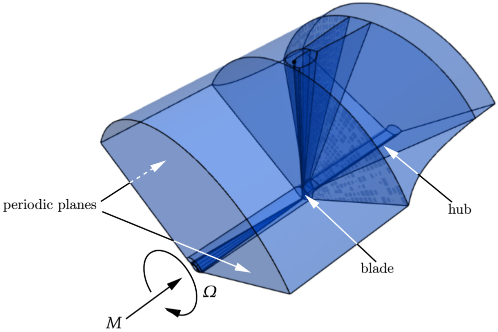
$$+ I J_n(\kappa_0 S_s R_{\lambda\mu}) \left(\frac{c_0}{\ldots} \frac{\partial \phi_{\lambda\mu}}{\ldots} \right. \ldots$$

$$+ I J_n'(\kappa_0 S_s R_{\lambda\mu}) \left(\frac{c_0}{\Omega} \frac{\partial \phi_{\lambda\mu}}{\partial Y_{\nu\xi,i}} + S_c \delta_{i1} \frac{\partial C_{\lambda\mu 1}}{\partial Y_{\nu\xi 1}} \right) \Big] e^{In\phi_{\lambda\mu}} e^{I\kappa_0 S_c C_{\lambda\mu 1}}$$

Why all this trouble?

Doing the derivation by hand is not a sterile exercise: **it allows to exchange mathematical complexity with programming and interpretability complexity!**

Numerical set up: HAD-1, isolated propeller



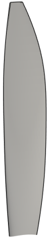


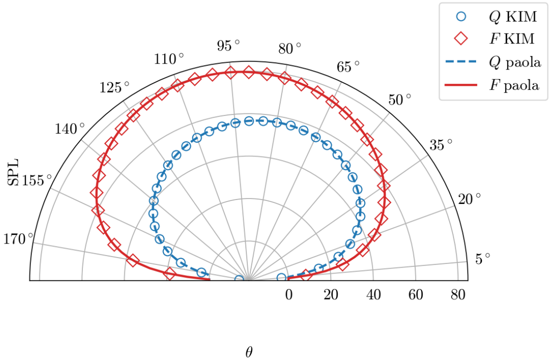
Figure: Blocking strategy (left) and blade surface (right)

Total number of cells ∼ 11 millions, number of cells on the blade surface ∼ 70 thousands. The RANS equations (Spalart Almaras) are solved using elsA.[4]

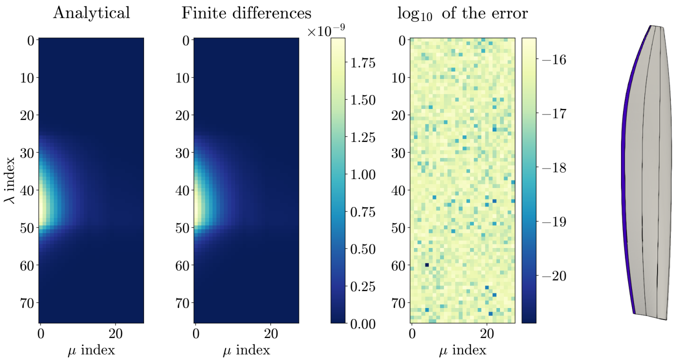
[4] Cambier, Heib, and Plot 2013.

The forward computation of the acoustic pressure is compared with the software KIM[a] on a semicircle of microphones at $125R_{\text{blade}}$ from the simulation centre.

[a] Prieur and Rahier 2001.



Validation: partial derivatives, $\partial p/\partial \boldsymbol{W}$



friendly reminder

$$\frac{dp}{d\alpha_k} = \left\langle \boldsymbol{w}^\dagger, -\frac{\partial R}{\partial \boldsymbol{X}}\frac{d\boldsymbol{X}}{d\alpha_k} \right\rangle + \frac{\partial p}{\partial \boldsymbol{X}}\frac{d\boldsymbol{X}}{d\alpha_k} \qquad \left(\frac{\partial R}{\partial \boldsymbol{W}}\right)^\top \boldsymbol{w}^\dagger - \left(\frac{\partial p}{\partial \boldsymbol{W}}\right)^\top = 0$$

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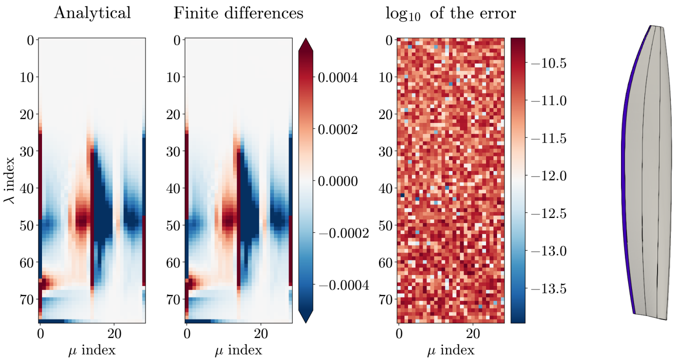

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$$\left(\frac{\partial R}{\partial \boldsymbol{W}}\right)^{\top} \boldsymbol{w}^{\dagger} - \left(\frac{\partial p}{\partial \boldsymbol{W}}\right)^{\top} = 0$$

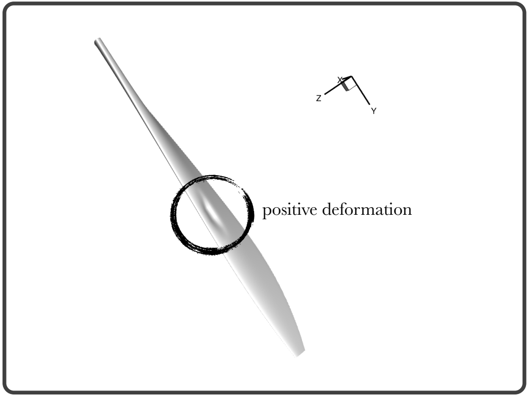
Analytical deformation:

$$X = X_0 + \Delta x = X_0 + \alpha e^{-(\boldsymbol{x}-\boldsymbol{x}_0)^2/2\sigma^2}$$

So that[a]:

$$\frac{dX}{d\alpha} = e^{-(\boldsymbol{x}-\boldsymbol{x}_0)^2/2\sigma^2}$$

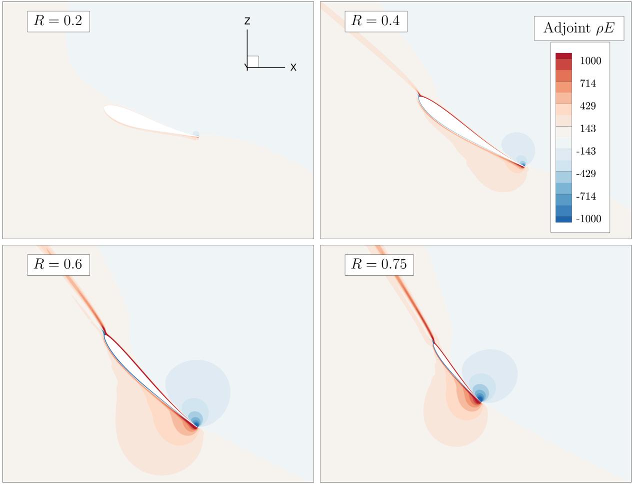
[a] clearly



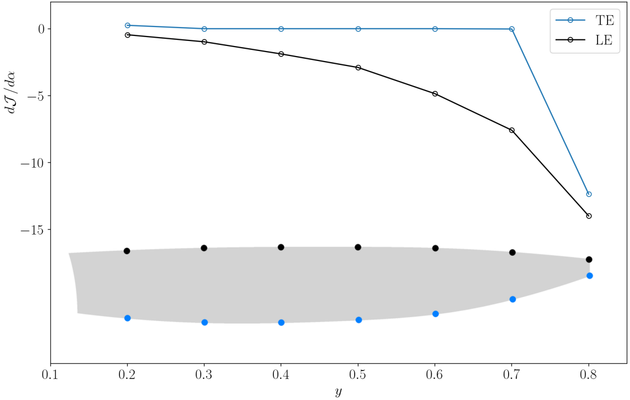
α [m]	$d\mathcal{J}/d\alpha$ finite-difference	$d\mathcal{J}/d\alpha$ adjoint	relative error
$\pm$0.001	-2.990	-2.841	**5.214 %**
$\pm$0.0001	-3.001	-2.840	**5.653 %**

5% error for a global effect ($\mathcal{J}$ is the integrated source power level!) for a local geometry modification? Not bad at all!

Looking at the adjoint fields can give an idea of the most sensitive zones of the fluid.



The previous analysis is confirmed by placing 14 more Gaussian bumps along the leading and trailing edge, an operation that required only one adjoint solution.



Motivation

Excessive noise levels are a societal hazard. They need to be taken into account from the design phase.

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Adjoint and sensitivities

The adjoint method alleviates this difficulty, provided that one can linearize the solver and the objective function. **This issue has been the focus of this presentation in the context of rotor noise**.

Moving on: the bright future ahead

Step 1

Theory and validation are surely nice, but we need concrete results: this method is being currently applied to a **real case optimization**.

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Step 2

Hanson's formula is well suited for many instances of rotor noise, but fails for the biggest ones because of its inability to take volume sources (aka shocks) into account. We plan to move towards the **Ffowcs-Williams & Hawkings analogy**.

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Step 3: the game changer

Focusing on the frequency domain is not a whim: we wish to incorporate the adjoint framework into a **Time-Spectral Method** to be able to tackle unstationary behavior without the need for time dependent adjoint, which is very memory intensive.

Thank You!

Grazie per l'attenzione

Questions?

contact: `enrico.foglia@isae-supaero.fr`

To go deeper: paper n° PRVSN-08

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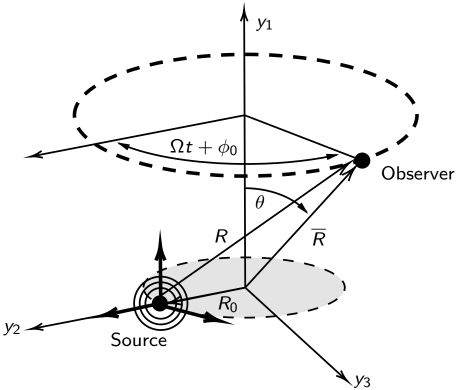
Even if the loading of the rotor is steady, it will still radiate sound in virtue of its circular motion. Let's set up the reference frame to study this condition.



Take again the problem of minimizing $\mathcal{J}(\boldsymbol{\alpha})$, subject to the constraint $R(\boldsymbol{W}, \boldsymbol{X}) = 0$. One can form the Lagrangian function:

$$\mathcal{L}(\boldsymbol{\alpha}, \boldsymbol{w}^\dagger) = \mathcal{J}(\boldsymbol{\alpha}) + \langle \boldsymbol{w}^\dagger, R(\mathbf{W}, \boldsymbol{X}) \rangle$$

The optimum is found for $\nabla \mathcal{L} = 0$. The partial derivative $\partial \mathcal{L} / \partial \boldsymbol{w}^\dagger$ only enforces the constraint. Interestingly the partial derivative w.r.t. the flow variables gives:

$$\left(\frac{\partial \mathcal{L}}{\partial \boldsymbol{W}}\right)^\top = \left(\frac{\partial \mathcal{J}}{\partial \boldsymbol{W}}\right)^\top + \left(\frac{\partial R}{\partial \boldsymbol{W}}\right)^\top \boldsymbol{w}^\dagger = 0$$

which is the adjoint equation. Finally the derivative w.r.t α_k reads:

$$\frac{d\mathcal{L}}{d\alpha_k} = \cancel{\left\langle \left(\frac{\partial \mathcal{J}}{\partial \boldsymbol{W}}\right)^\top + \left(\frac{\partial R}{\partial \boldsymbol{W}}\right)^\top \boldsymbol{w}^\dagger, \boldsymbol{w}_k \right\rangle} + \frac{\partial \mathcal{J}}{\partial \boldsymbol{X}} \frac{d\boldsymbol{X}}{d\alpha_k} + \left\langle \boldsymbol{w}^\dagger, \frac{\partial R}{\partial \boldsymbol{X}} \frac{d\boldsymbol{X}}{d\alpha_k} \right\rangle = \frac{d\mathcal{J}}{d\alpha_k}$$

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Backward waves

The adjoint of a wave equation $u_t + u_x = 0$, once discretized to get $\boldsymbol{L}\boldsymbol{u}^{n+1} = \boldsymbol{u}^n$ is still a travelling wave, but going backwards.

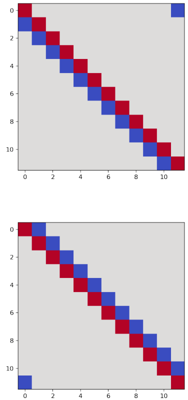

Figure: Top: matrix $\boldsymbol{L}$ for the forward wave equation (periodic BC); Bottom: adjoint $\boldsymbol{L}^\dagger$

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Self-adjoint operators

The heat operator $u_t - u_{xx} = 0$, once discretized, is invariant under the adjoint operation.

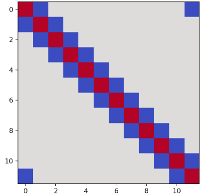

Figure: Heat operator $\boldsymbol{L}$ with periodic BC.

Simulation parameters

Parameters		
Name	Symbol	Value
Blade number	B	3
Blade height	R_{blade}	0.8
Flight Mach number	M	0.3
Rotation regime	Ω	2031 rpm
Static temperature	T_∞	288.15 K
Density	ρ_∞	1.2252 kg m^{-3}
Turbulent viscosity ratio	μ_t/μ	0.1

Table: Propeller cruise condition characteristics.

Parameters	
Name	Value
Inner Krylov space dimension	20
Tolerance inner GMRES	0.5
Outer Krylov space dimension	70
Tolerance outer GMRES	1×10^{-6}
Number of restarts	30
Preconditioner type	LU-RELAX
CFL	1×10^{3}
Deflated vectors	40

Table: GMRES parameters for the solution of the adjoint linear system.

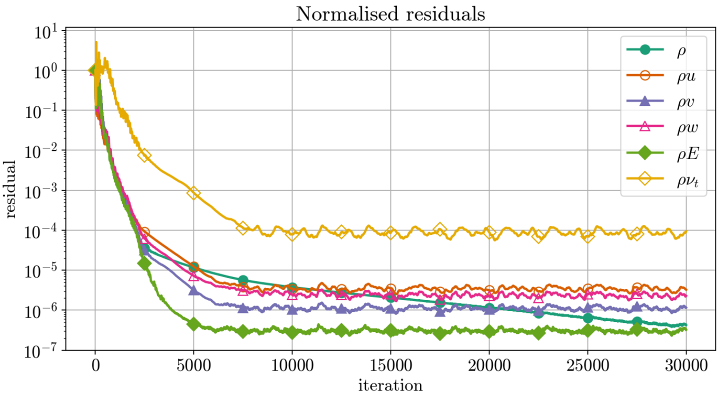

Figure: Residuals of the forward computation.

Convergence of the adjoint calculation

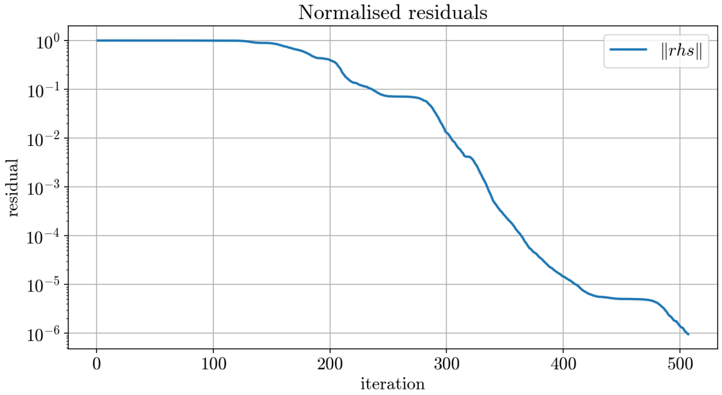

Figure: Residuals of the right-hand-side vector of the GMRES solver for the adjoint solution using the PWL as the objective function.