Introduzione alla Programmazione Lineare

Ricerca Operativa [035IN]

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Linear programming problem

A linear programming (LP) problem is a optimisation problem such that

$$z = \max\{c(x) : x \in X \subseteq \mathbb{R}^n\}$$

or

$$z = \min\{c(x) : x \in X \subseteq \mathbb{R}^n\}$$

where

- the objective function $c(x) : \mathbb{R}^n \to \mathbb{R}$ is linear, i.e., $c(0) = 0$ and $c(\alpha x + \beta y) = \alpha c(x) + \beta c(y)$. Therefore $c(x) = cx$ where c is a vector in $\mathbb{R}^n$.
- the set X of feasible solutions is defined by linear constraints such as $h(x) = \gamma$ and/or $h(x) \leq \gamma$ and/or $h(x) \geq \gamma$, where $h(x) : \mathbb{R}^n \to \mathbb{R}$ is a linear function and γ is scalar in $\mathbb{R}$.

Linear programming problem

Formulation A LP problem can be formulated as

$$\max c^T x$$
$$Ax \leq b$$
$$x \geq 0$$

or

$$\begin{aligned}
\max Z = & c_1 x_1 + c_2 x_2 + \ldots + c_n x_n \\
& a_{11} x_1 + a_{12} x_2 + \ldots + a_{1n} x_n \leq b_1 \\
& a_{21} x_1 + a_{22} x_2 + \ldots + a_{2n} x_n \leq b_2 \\
& \vdots \\
& a_{m1} x_1 + a_{m2} x_2 + \ldots + a_{mn} x_n \leq b_m \\
& x_1, x_2, \ldots, x_n \geq 0
\end{aligned}$$

Notation

- m number of rows of matrix A
- n dimension of vector x and number of columns of matrix A
- c objective function vector
- A technology matrix
- b right-hand side vector (≥ 0 in the standard form)
- x decision variable vector
- $X = \{x : Ax \leq b, x \geq 0\}$, the set of feasible solutions

LP properties - Proportionality

The objective function and each constraint must be linear with respect to each decision variable; in other words, the measure of effectiveness and the use of the resource must be proportional to the level of each activity carried out individually, that is,

if $x_j \neq 0$ and $x_1 = x_2, \dots, = x_{j-1} = x_{j+1} =, \dots, = x_n = 0$, it has to be $Z = c_j x_j$ and $a_{ij} x_j \leq b_i$, $\forall i$, $\forall j$.

Therefore the following properties must always hold:

$$\frac{\partial Z}{\partial x_j} = c_j \ \forall j; \quad \frac{\partial b_i}{\partial x_j} \geq a_{ij} \ \forall i, \forall j,$$

that is, the marginal measure of effectiveness and the marginal use of each resource must be constant over the entire range of variation of the levels of each activity.

LP properties - Fixed charge

A case in which it is not possible to apply the LP is when we have a fixed charge. This happens whenever there is a preparation charge or a setup charge associated with an activity.

That is, if x is the level of a certain activity and the objective function is

$$f(x) = \begin{cases} 0 & x = 0 \\ K + cx & x > 0 \end{cases}$$

The property of proportionality is violated.

LP properties - Additivity

There must be no interactions between the various activities, that is, the total measure of effectiveness derived from the joint result of the activities must equal the sum of those quantities resulting from each activity carried out individually.

In practice, if $c_1x_1, c_2x_2, \ldots, c_nx_n$ is the measure of effectiveness for the activity $1, 2, \ldots, n$ conducted individually, it must be that $Z = c_1x_1 + c_2x_2 + \ldots, c_nx_n$ where Z is the total measure of effectiveness.

Similarly, for the total use of a resource.

The values of x_j are real numbers, i.e., $x \in \mathbb{R}^n$.

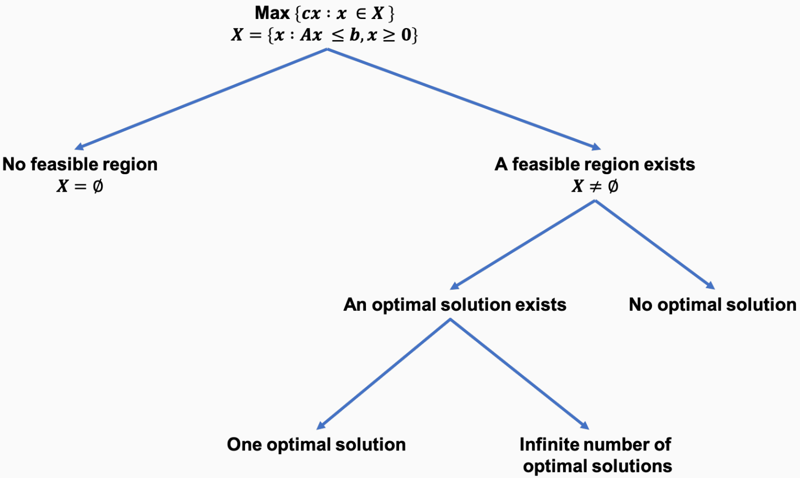
It should be noted that this does not always make sense, that is, sometimes the solution we are looking for must be integer. However, in that case, if the feasible region is not empty, we can always find a vector x that respects the constraints even without being integer.

We always exclude the case of complex values.

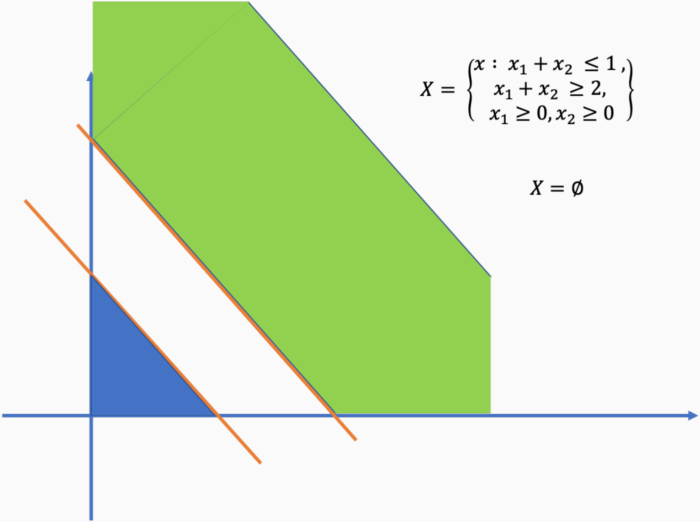
LP properties - Certainty

All the coefficients of the problem are constant real numbers known a priori. There is nothing stochastic, nothing random.

Possible outcomes of a LP problem



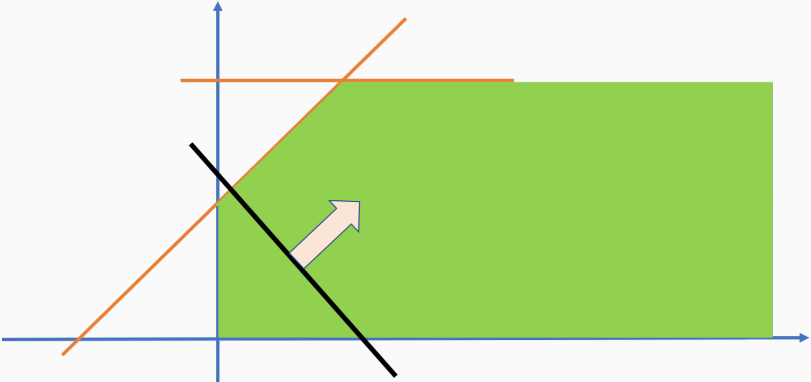
No feasible region



$$X = \left\{ \begin{array}{c} x : \; x_1 + x_2 \leq 1, \\ x_1 + x_2 \geq 2, \\ x_1 \geq 0, x_2 \geq 0 \end{array} \right\}$$

$$X = \emptyset$$

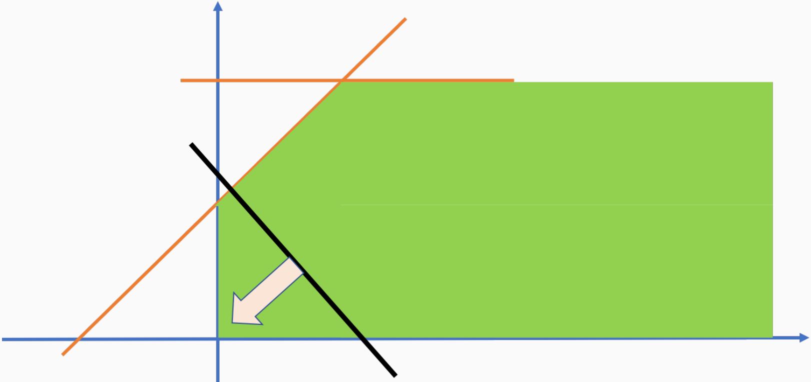
No optimal solution



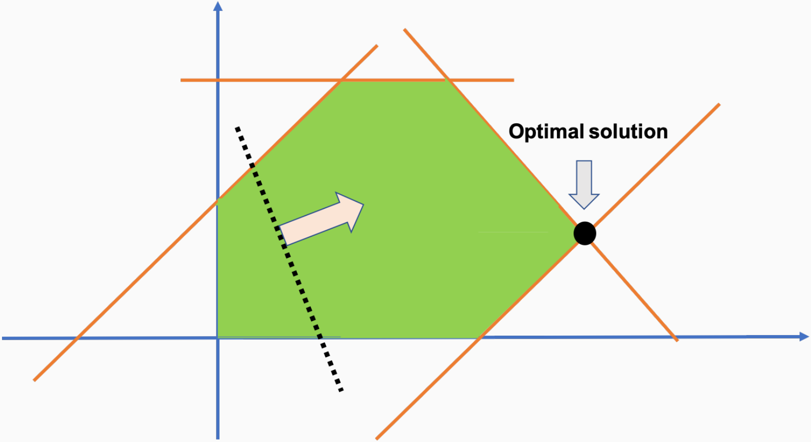
$\max x_1 + x_2$

$$X = \left\{ \begin{array}{c} x: \ -x_1 + x_2 \leq 2, \\ x_2 \leq 4, \\ x_1 \geq 0, x_2 \geq 0 \end{array} \right\}$$

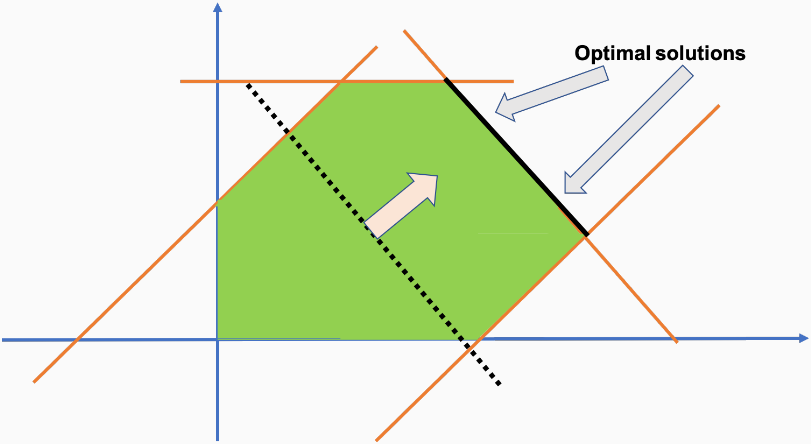
Optimal solution!



$$\min x_1 + x_2 \qquad X = \left\{ \begin{array}{c} x: \; -x_1 + x_2 \le 2, \\ x_2 \le 4, \\ x_1 \ge 0, x_2 \ge 0 \end{array} \right\}$$


Optimal solution

Infinite number of optimal solutions



Hyperplane

We call hyperplane the set $H = \{x \in \mathbb{R}^n : a^T x = b\}$, where $a \in \mathbb{R}^n$, $a \neq 0$, $b \in \mathbb{R}$. The regions $X^- = \{x \in \mathbb{R}^n : a^T x \leq b\}$ and $X^+ = \{x \in \mathbb{R}^n : a^T x \geq b\}$ are called half-spaces delimited by the supporting hyperplane H.

Polyhedron

We call (convex) polyhedron the intersection of a finite number of half-spaces and hyperplanes.

Polytope

We call polytope a limited polyhedron P, i.e., it exists a constant $M > 0$ such that $\|x\| \leq M$ for all $x \in P$

Key concepts

Lemma

The feasible region of a LP problem is a convex polyhedron

Vertices

The extreme points of a convex polyherdon are called vertices.

Key theorem

Theorem. The optimal solution is on a vertex

Given the LP problem

$$\max cx$$
$$Ax = b \ (b \geq 0)$$
$$x \geq 0$$

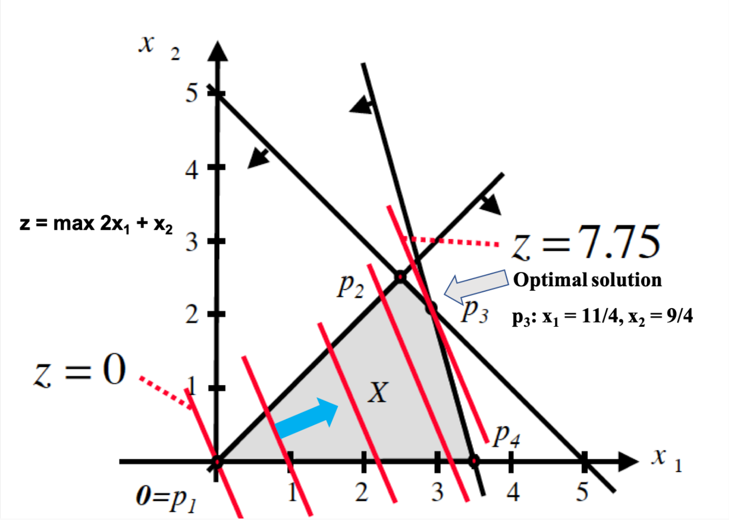
if $X \neq \emptyset$ and it has an optimal and finite solution, then it exists a vertex of X which is an optimal solution.

The proof relies on the fact that X is a polyhedron and hence a convex set.

LP problem - example

$$
\begin{aligned}
\max z = & 2x_1 + x_2 \\
& x_1 + x_2 \leq 5 \\
& -x_1 + x_2 \leq 0 \\
& 6x_1 + 2x_2 \leq 21 \\
& x_1, x_2 \geq 0
\end{aligned}
$$

The optimal solution is on a vertex



The optimal solution is on a vertex

Given a LP problem that has an optimal and finite solution, since there is certainly one on a vertex, we can think of limiting the search for the optimal solution to the set of vertices.

The problem therefore arises on how to identify (potentially all) the vertices starting from a representation of the polyhedron of the feasible solutions.

Finding all vertices - example

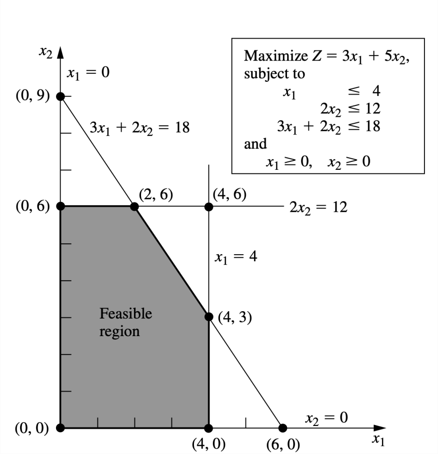


Figura 1: Constraint boundaries and corner-point solutions

Thank you for your attention