Risolvere problemi di programmazione intera

Ricerca Operativa [035IN]

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DI TRIESTE

Enumeration. All feasible solutions are identified and the best one is picked up. It may not be practically viable. For instance, to solve the TSP in a complete graph with n nodes there are $(n-1)!$ feasible tours. Hence,

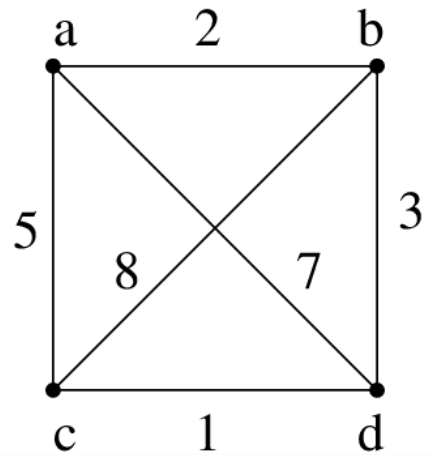
n	$n!$
10	3.6×10^6
100	9.33×10^{157}
1000	4.02×10^{2567}

Better ideas are needed.

- We are given a set on nodes $V = \{1, \ldots, n\}$ (e.g., cities) and a set of arcs $\mathcal{A}$.
- Arcs represent ordered pairs of cities between which direct travel is possible.
- For $(i, j) \in \mathcal{A}, c_{ij}$ is the direct travel time from city i to city j.
- The TSP aims at finding a tour, starting at city 1, that
  a) visits each other city exactly once and then returns to city 1
  b) takes the least total travel time

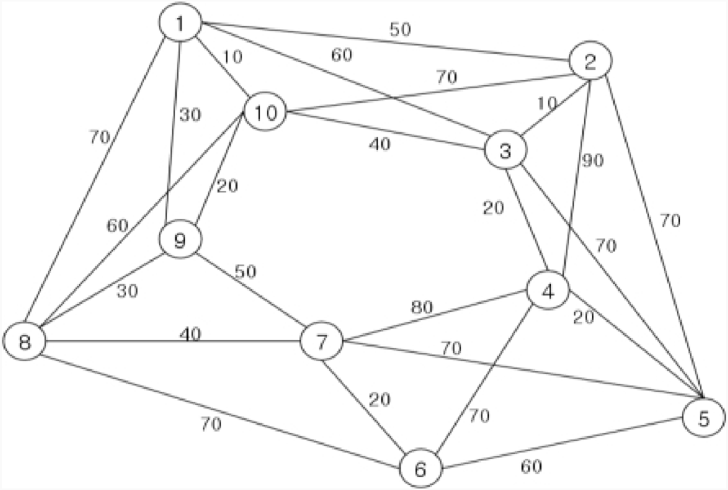
The Travelling Salesman Problem (TSP)

A tour that visits all nodes exactly once is called Hamiltonian tour. The TSP identifies the Hamiltonian tour of minimum cost.

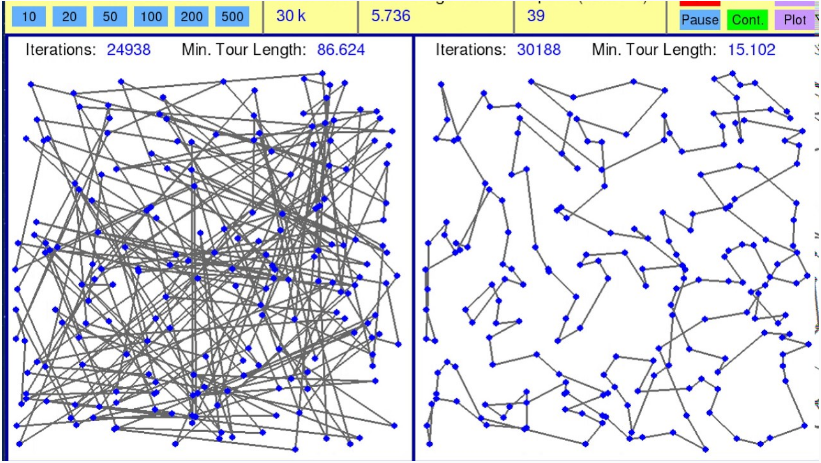


Three tours: A-B-D-C-A: 11; A-D-B-C-A: 23; A-D-C-B-A: 18.

TSP - maybe not too easy



TSP - it's difficult!!



TSP - Formulation

- Decision Variables

$$x_{ij} = \begin{cases} 1 & \text{if } j \text{ immediately follows } i \text{ on the tour} \\ 0 & \text{otherwise} \end{cases}$$

  Hence $x \in \{0,1\}^{|\mathcal{A}|}$

- Objective function

$$\min \sum_{(i,j)\in\mathcal{A}} c_{ij} x_{ij}$$

TSP - Constraint formulation

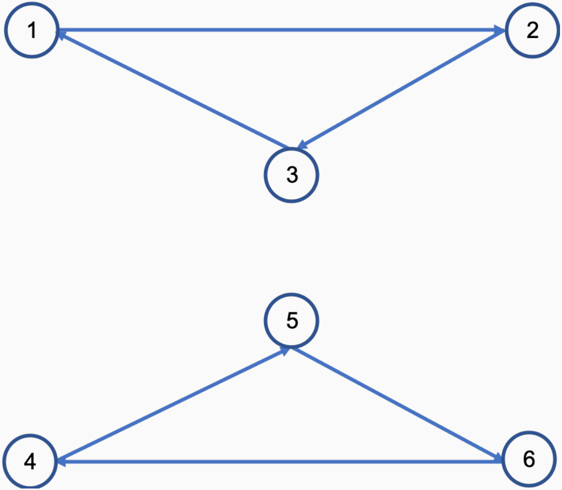
Each city is entered and left exactly once

$$\sum_{i:(i,j)\in\mathcal{A}} x_{ij} = 1 \text{ for } j \in V \tag{1}$$

$$\sum_{j:(i,j)\in\mathcal{A}} x_{ij} = 1 \text{ for } i \in V \tag{2}$$

However, constraints (1) and (2) are not sufficient to define tours since they are also satisfied by subtours.

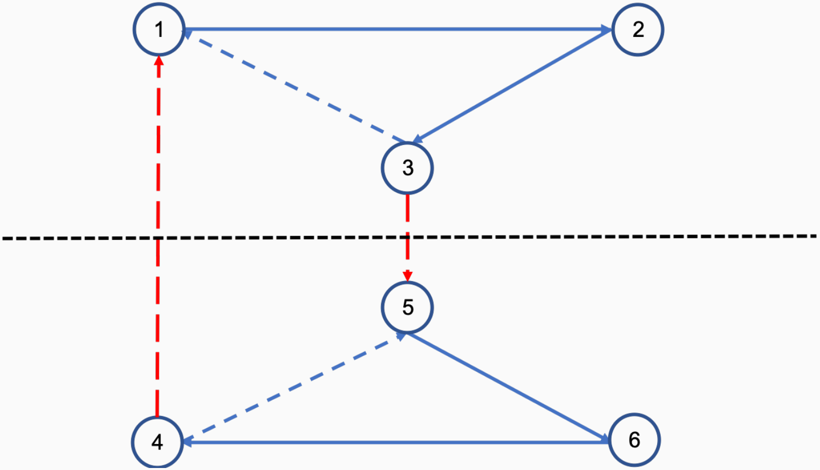
TSP - Subtours



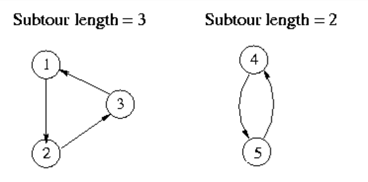
In any tour there must be an arc that goes from $\{1,2,3\}$ to $\{4,5,6\}$ and an arc that goes from $\{4,5,6\}$ to $\{1,2,3\}$. In general, for any $U \subset V$ with $2 \leq |U| \leq |V| - 2$, constraints

$$\sum_{\{(i,j)\in\mathcal{A}: i\in U, j\in V\setminus U\}} x_{ij} \geq 1 \tag{3}$$

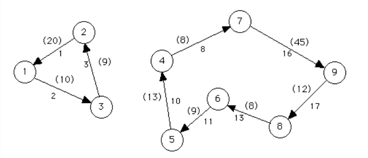
are satisfied by all tours, but every subtour violates at least one of them.


1
2
3
5
4
6

TSP - Subtours

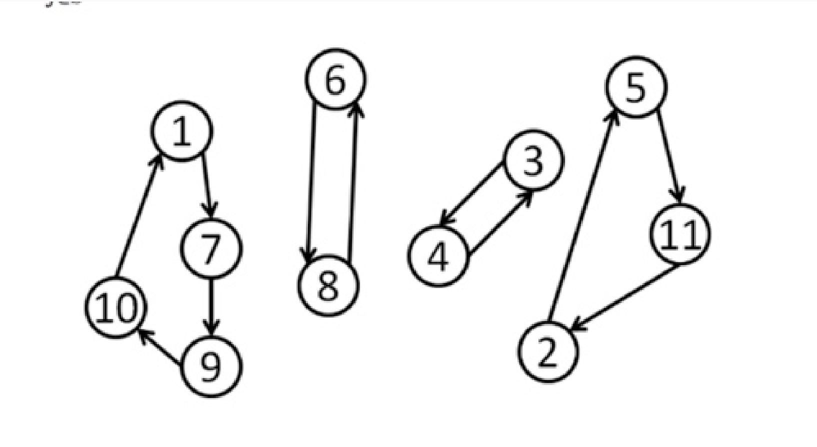


(a) 5 nodes

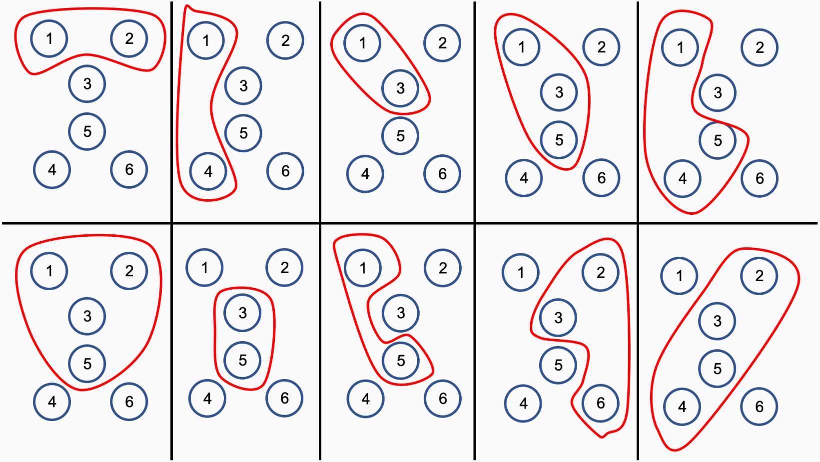


(b) 9 nodes

TSP - Subtours



TSP - Too many ways to choose U



TSP - Subtour elimination (ii)

An alternative way to eliminate subtours is to introduce constraints

$$\sum_{\{(i,j)\in\mathcal{A}: i\in U, j\in U\}} x_{ij} \leq |U| - 1 \ \forall U \subset V : 2 \leq |U| \leq |V| - 2 \tag{4}$$

But again we need a constraint for each $U \subset V$ such that $2 \leq |U| \leq |V| - 2$.

In both (3) and (4) the number of constraints is nearly $2^{|V|}$!!!

$$\frac{1}{2}\left[\binom{|V|}{2} + \binom{|V|}{3} + \cdots + \binom{|V|}{|V|-2}\right]$$

TSP - Formulation

$$\min \sum_{(i,j)\in\mathcal{A}} c_{ij}x_{ij}$$

$$\sum_{i:(i,j)\in\mathcal{A}} x_{ij} = 1 \text{ for } j \in V$$

$$\sum_{j:(i,j)\in\mathcal{A}} x_{ij} = 1 \text{ for } i \in V$$

$$\sum_{\{(i,j)\in\mathcal{A}:i\in U,j\in V\setminus U\}} x_{ij} \geq 1 \qquad \forall U \subset V : 2 \leq |U| \leq |V|-2$$

OR

$$\sum_{\{(i,j)\in\mathcal{A}:i\in U,j\in U\}} x_{ij} \leq |U|-1 \qquad \forall U \subset V : 2 \leq |U| \leq |V|-2$$

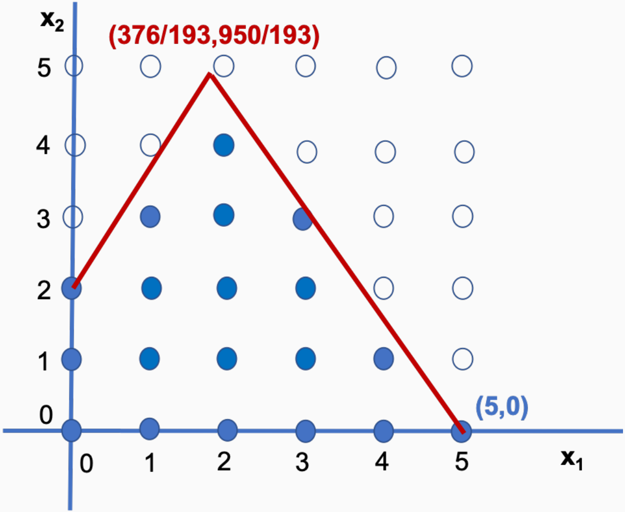
$$x \in \{0,1\}^{|\mathcal{A}|}$$

Disregarding variables' integrality constraints. Consider the following problem:

$$
\begin{aligned}
\max Z = & 1.00x_1 + 0.64x_2 \\
& 50x_1 + 31x_2 \leq 250 \\
& 3x_1 - 2x_2 \geq -4 \\
& x_1, x_2 \geq 0 \text{ and integer.}
\end{aligned}
$$

- The optimal integer solution is $(5, 0)$
- The optimal solution without considering variables' integrality constraints is $(376/193, 950/193) = (1.948, 4.922)$

How to solve an IP?



Disregarding variables' integrality constraints. Why not to round up and/or down the linear solution?

- The upper integer part $(\lceil 1.948 \rceil, \lceil 4.922 \rceil) = (2, 5)$ is NOT FEASIBLE (the first constraint is violated).
- The lower integer part $(\lfloor 1.948 \rfloor, \lfloor 4.922 \rfloor) = (1, 4)$ is NOT FEASIBLE (the second constraint is violated).
- The mixed choice $(\lfloor 1.948 \rfloor, \lceil 4.922 \rceil) = (1, 5)$ is NOT FEASIBLE (the second constraint is violated).
- The mixed choice $(\lceil 1.948 \rceil, \lfloor 4.922 \rfloor) = (2, 4)$ is feasible but NOT OPTIMAL: $Z(2, 4) = 4.56$, whereas $Z(5, 0) = 5$.

In addition, no rounding gives the values $(5, 0)$.

In conclusion, the linear solution appears to be useless to find the integer solution.

Optimality

Given an IP

$$z = \max\{c(x) : x \in X \subseteq \mathbb{Z}^n\}$$

how can we prove that a given point x^* is optimal?

We need to address this question because we saw that solution enumeration may not possible, whereas disregarding variables' integrality constraints may not provide useful information.

Therefore we need to find alternatives (i.e., algorithms) to solve an IP.

Bounds

The most common approach to solve IP problems is to find sequences of bounds until they are “close enough”.

Upper bound

If z is the optimal value of an IP problem, an upper bound is a value $\overline{z}$ such that $\overline{z} \geq z$.

Lower bound

If z is the optimal value of an IP problem, a lower bound is a value $\underline{z}$ such that $\underline{z} \leq z$.

Ideally, we would like to find $\overline{z}$ and $\underline{z}$ such that $\underline{z} = z = \overline{z}$.

Lower and upper bounds

From a practical point of view, any algorithm will look for a decreasing sequence of upper bounds

$$\overline{z_1} > \overline{z_2} > \ldots > \overline{z_s} \geq z$$

and an increasing sequence of lower bounds

$$\underline{z_1} < \underline{z_2} < \ldots < \underline{z_t} \leq z$$

and stops when

$$\overline{z_s} - \underline{z_t} \leq \epsilon$$

where ϵ is an appropriate non-negative value.

Lower bound

Every feasible solution $\hat{x} \in X$ provides a lower (or primal) bound $\underline{z} = c(\hat{x}) \leq z$.

For the problem

$$
\begin{aligned}
\max Z = & 1.00x_1 + 0.64x_2 \\
& 50x_1 + 31x_2 \leq 250 \\
& 3x_1 - 2x_2 \geq -4 \\
& x_1, x_2 \geq 0 \text{ and integer.}
\end{aligned}
$$

we saw, by rounding the optimal linear solution, that $\hat{x} = (2, 4)$ is a feasible solution such $\underline{z} = c(\hat{x}) = 4.56 \leq z$.

Findings upper bounds could be less obvious.

The most common idea is to replace a “difficult” IP problem by a simpler optimisation problem, whose optimal value is at least as large as z.

The simpler problem can be obtained by “relaxation”, i.e., by

- enlarging the set of feasible solutions so that one optimises over a larger set,
- replacing the $\max$ objective function by a function that has the same or a larger value everywhere.

Definition

A problem $(RP)\ z^R = \max\{f(x) : x \in T \subseteq \mathbb{R}^n\}$ is a relaxation of $(IP)\ z = \max\{c(x) : x \in X \subseteq \mathbb{Z}^n\}$ if:

(i) $X \subseteq T$, and

(ii) $f(x) \geq c(x)$ for all $x \in X$.

Proposition. If RP is a relaxation of IP, $z^R \geq z$

If x^* is an optimal solution of IP, $x^* \in X \subseteq T$ and $z = c(x^*) \leq f(x^*)$. As $x^* \in T$, $f(x^*)$ is a lower bound on z^R, and so $z \leq f(x^*) \leq z^R$. ∎

Hence, z^R is an upper bound!

Linear relaxation

Definition

For the integer program $\max\{cx : x \in X = P \cap \mathbb{Z}^n\}$ with formulation $P = \{x \in \mathbb{R}^n_+ : Ax \leq b\}$, the linear programming relaxation is the linear program $z^{LP} = \max\{cx : x \in P\}$.

Since $X = P \cap \mathbb{Z}^n \subseteq P$ and the objective function is unchanged, this is clearly a relaxation.

Linear relaxation - example 1

Original IP problem

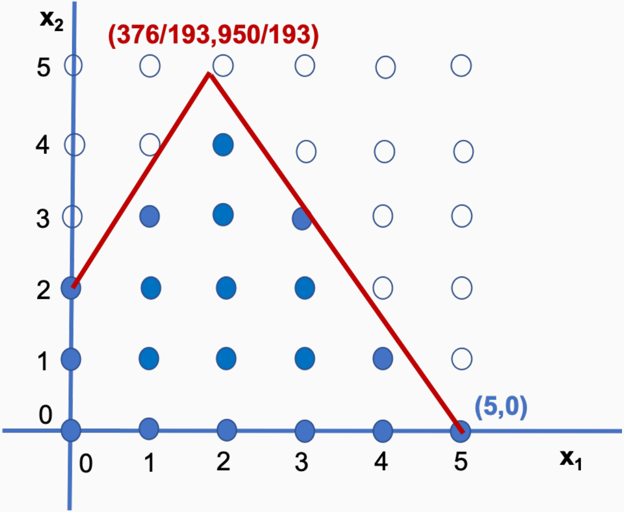
$$
\begin{aligned}
z = \max 1.00x_1 + 0.64x_2& \\
50x_1 + 31x_2 &\leq 250 \\
3x_1 - 2x_2 &\geq -4 \\
x_1, x_2 &\geq 0 \text{ and integer.}
\end{aligned}
$$

LP relaxation

$$
\begin{aligned}
z^{LP} = \max 1.00x_1 + 0.64x_2& \\
50x_1 + 31x_2 &\leq 250 \\
3x_1 - 2x_2 &\geq -4 \\
x_1, x_2 &\geq 0.
\end{aligned}
$$

- The optimal integer solution is $(5, 0)$ and $z = 5$
- The optimal solution of the linear relaxation is $(376/193, 950/193)$ and $z^{LP} = 984/193 = 5.098$.

Linear relaxation - example 1



Linear relaxation - example 1

For the IP problem

$$
\begin{aligned}
z = \max 1.00x_1 + 0.64x_2 & \\
50x_1 + 31x_2 &\leq 250 \\
3x_1 - 2x_2 &\geq -4 \\
x_1, x_2 &\geq 0 \text{ and integer.}
\end{aligned}
$$

We know that a lower bound is $\underline{z} = 4.560$ and an upper bound is $\overline{z} = 5.098$. The optimal value z therefore lies within

$$\underline{z} = 4.560 \leq z \leq 5.098 = \overline{z}.$$

In fact, $z = 5$. *The information we get by disregarding variables' integrality constraints can be very useful indeed.*

Consider the IP problem

$$
\begin{aligned}
z = \max 4x_1 - x_2 & \\
& 7x_1 - 2x_2 \leq 14 \\
& x_2 \leq 3 \\
& 2x_1 - 2x_2 \leq 3 \\
& x_1, x_2 \geq 0 \text{ and integer.}
\end{aligned}
$$

It is easy to see that $(2, 1)$ is a feasible solution, thus leading to the lower bound $\underline{z} = 7$. The optimal solution of the linear relaxation is $x^* = (20/7, 3)$ producing the upper bound $\overline{z} = 59/7 = 8.43$. Since all coefficients of the objective function are integer (i.e., $(4, -1)$) the optimal value z must to be integer as well. Hence, we can take as upper bound $\overline{z} = \lfloor 8.43 \rfloor = 8$. Therefore

$$\underline{z} = 7 \leq z \leq 8 = \overline{z}$$

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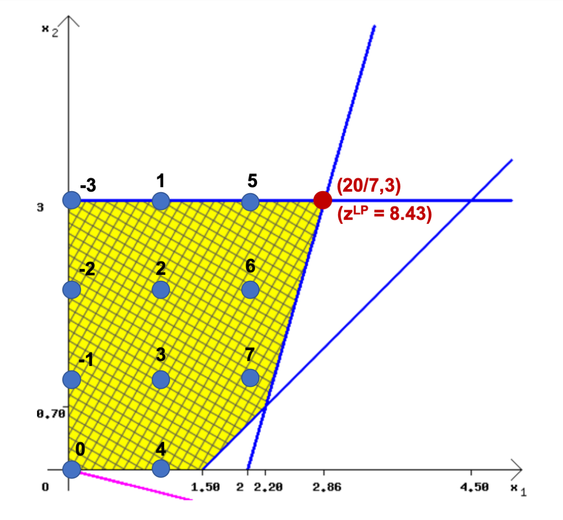

x_2
3
0.70
0
-3
1
5
-2
2
6
-1
3
7
0
4
(20/7,3)
(z^{LP} = 8.43)
1.50
2
2.20
2.86
4.50
x_1

Thank you for your attention