Matrici associate a trasf lineari.

$$v \overset{F}{\longmapsto} w \qquad\qquad x \longmapsto Ax$$

Sia $D : \mathbb{R}^{\leq 2}[x] \longrightarrow \mathbb{R}^{\leq 2}[x]$

$$D(2x^2 - x + 7) = 4x - 1$$

$$p \overset{D}{\longmapsto} \underset{p'}{D(p)} \qquad\qquad x \longmapsto Ax$$

Una base per $\mathbb{R}^{\leq 2}[x]$ è

$$B = \left(\underset{v_1}{1}, \underset{v_2}{x}, \underset{v_3}{x^2} \right)$$

$$p = 2x^2 - x + 7 = 2v_3 - 1v_2 + 7v_1 =$$

$$= 7v_1 - 1v_2 + 2v_3$$

$$p \underset{B}{\equiv} (7, -1, 2)$$

Dobbiamo ora costruire la matrice $A = M_{BB}(D)$ associata alla trasf. lin. D rispetto alle basi B, B.

$n = m = 3$

$$A = \begin{pmatrix} \boxed{\begin{matrix} 0 \\ 0 \\ 0 \end{matrix}} & \boxed{\begin{matrix} 1 \\ 0 \\ 0 \end{matrix}} & \boxed{\begin{matrix} 0 \\ 2 \\ 0 \end{matrix}} \end{pmatrix}$$

$D(v_1) = \mathbf{0} = 0v_1 + 0v_2 + 0v_3$

$D(v_2) = 1 = 1 \cdot 1 + 0x + 0x^2 = 1 \cdot v_1 + 0v_2 + 0v_3$

$D(v_3) = 2x = 0v_1 + 2v_2 + 0v_3$

$$1 \cdot v_1 + 0 v_2 + 0 v_3$$

$$\shortparallel$$

$$0 v_1 + 0 v_2 + 0 v_3$$

Prendiamo il polinomio

$$a + bx + cx^2 \in \mathbb{R}^{\leq 2}[x]$$

$$D(a + bx + cx^2) = b + 2cx$$

$\mathbb{R}^{\leq 2}[x]$ | $\mathbb{R}^3$

$$a + bx + cx^2 \longrightarrow (a, b, c)$$

$$D \downarrow \qquad\qquad A \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} b \\ 2c \\ 0 \end{pmatrix} \downarrow$$

$$b + 2cx \longrightarrow (b, 2c, 0)$$

$$\shortparallel$$

$$b \cdot 1 + 2c \cdot x + 0x^2$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} b \\ 2c \\ 0 \end{pmatrix}$$

$$2x^2 - x + 7 \underset{B}{\equiv} (7, -1, 2)$$

$$\shortparallel$$

$$p$$

$$\shortparallel$$

$$7 \cdot 1 + (-1)x + 2x^2$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 7 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \\ 0 \end{pmatrix}$$

$$D(2x^2 - x + 7) = (-1)\cdot 1 + 4x + 0x^2 = 4x - 1$$

$M_{3\times 3}(\mathbb{R})$ = spazio vettoriale delle matrici reali 3×3

$A(3)$ = spazio vettoriale delle matrici reali antisimmetriche 3×3

$$\begin{pmatrix} 1 & 7 & \boxed{5} \\ 7 & 0 & -10 \\ \boxed{5} & -10 & 5 \end{pmatrix} \in S(3)$$

$$\begin{pmatrix} 0 & 2 & 1 \\ -2 & 0 & 8 \\ -1 & -8 & 0 \end{pmatrix} \in A(3)$$

$$F : M_{3\times 3}(\mathbb{R}) \to A(3)$$

$$\boxed{F(M) = \underline{M} - \underline{{}^tM}}$$

$$F\begin{pmatrix} 1 & 0 & 1 \\ 0 & 4 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 4 & 5 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 3 \\ 0 & 4 & 7 \end{pmatrix} =$$

$$F\begin{pmatrix} 1 & 0 & 1 \\ 0 & 4 & 5 \\ 3 & 2 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 4 & 5 \\ 3 & 2 & -1 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 3 \\ 0 & 4 & 2 \\ 1 & 5 & -1 \end{pmatrix} =$$

$$= \begin{pmatrix} 0 & 0 & -2 \\ 0 & 0 & 3 \\ 2 & -3 & 0 \end{pmatrix} \in \underline{A(3)}$$

Nota: $M - {}^tM$ è <u>sempre</u> antisimmetrico!

$$\overbrace{{}^t(M - {}^tM)}^{{}^tN} = {}^tM - {}^t({}^tM) = {}^tM - M = -(-{}^tM + M)$$

$$= -\underbrace{(M - {}^tM)}_{N}$$

$${}^tN = -N \iff N \text{ è antisimmetrico!}$$

$$N = M - {}^tM$$

$$F(M_1 + M_2) = (M_1 + M_2) - {}^t(M_1 + M_2)$$

$$\overset{?}{=} \quad \text{sì}$$

$$F(M_1) + F(M_2) = (M_1 - {}^tM_1) + (M_2 - {}^tM_2)$$

$$\Rightarrow M_1 + M_2 - ({}^tM_1 + {}^tM_2) =$$

$$= (M_1 - {}^tM_1) + (M_2 - {}^tM_2)$$

$=$

$$= (M_1 - {}^tM_1) + (M_2 - {}^tM_2)$$

$$F(\lambda M) = \lambda M - {}^t(\lambda M) =$$

$$\| \text{sì} \quad = \lambda M - \lambda\, {}^tM$$

$$= \lambda(M - {}^tM)$$

$$\lambda F(M) = \lambda(M - {}^tM)$$

$$E_{23} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\underline{B_{M_{3\times 3}(\mathbb{R})}} = (E_{11}, E_{12}, E_{13}, E_{21}, E_{22}, E_{23}, E_{31}, E_{32}, E_{33})$$

$$B_{A(3)} = \left(\underbrace{\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{w_1}, \underbrace{\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}}_{w_2}, \underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}}_{w_3} \right)$$

NB.

$$\dim M_{m\times n}(\mathbb{R}) = m \times n$$

$$\dim \mathbb{R}^{\leq n}[x] = n+1$$

Esercizio : calcolare

$$A = M_{\mathcal{B}_{M_{3\times 3}(\mathbb{R})}\, \mathcal{B}_{A(3)}}(F)$$

$$A = \begin{pmatrix} \boxed{\begin{matrix}0\\0\\0\end{matrix}} & \boxed{\begin{matrix}1\\0\\0\end{matrix}} & \boxed{\begin{matrix}0\\1\\0\end{matrix}} & \boxed{\begin{matrix}-1\\0\\0\end{matrix}} & \boxed{\begin{matrix}0\\0\\0\end{matrix}} & \boxed{\begin{matrix}0\\0\\1\end{matrix}} & \boxed{\begin{matrix}0\\-1\\0\end{matrix}} & \boxed{\begin{matrix}0\\0\\-1\end{matrix}} & \boxed{\begin{matrix}0\\0\\0\end{matrix}} \end{pmatrix}$$

coord. di $F(E_{11})$ coord di $F(E_{12})$

$$F\Bigg(\underbrace{\begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix}}_{E_{11}}\Bigg) = \begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix} - \begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix} = \begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}$$

$$F\Bigg(\begin{pmatrix}0&1&0\\0&0&0\\0&0&0\end{pmatrix}\Bigg) = \begin{pmatrix}0&1&0\\0&0&0\\0&0&0\end{pmatrix} - \begin{pmatrix}0&0&0\\1&0&0\\0&0&0\end{pmatrix} = \begin{pmatrix}0&1&0\\-1&0&0\\0&0&0\end{pmatrix}$$

$$= 1w_1 + 0w_2 + 0w_3$$

$$F\Bigg(\underbrace{\begin{pmatrix}0&0&1\\0&0&0\\0&0&0\end{pmatrix}}_{\substack{\parallel\\ E_{13}}}\Bigg) = \begin{pmatrix}0&0&1\\0&0&0\\-1&0&0\end{pmatrix}$$

$$\begin{pmatrix}0&0&1\\0&0&0\\-1&0&0\end{pmatrix} = w_2 = 0w_1 + 1w_2 + 0w_3$$

$$F\Bigg(\begin{pmatrix}0&0&0\\1&0&0\\0&0&0\end{pmatrix}\Bigg) = \begin{pmatrix}0&0&0\\1&0&0\\0&0&0\end{pmatrix} - \begin{pmatrix}0&1&0\\0&0&0\\0&0&0\end{pmatrix} =$$

$$F\Bigg(\underbrace{\begin{pmatrix} \cdot & \cdot & \cdot \\ 0 & 0 & 0 \end{pmatrix}}_{E_{21}}\Bigg) - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} =$$

$$= \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = -w_1$$

$$= (-1)w_1 + 0w_2 + 0w_3$$

$$F\Bigg(\underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{E_{22}}\Bigg) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$F\Bigg(\underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}}_{E_{23}}\Bigg) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} = w_3$$

$$\| $$

$$0w_1 + 0w_2 + 1w_3$$

$$F\Bigg(\underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}}_{E_{31}}\Bigg) = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} = -w_2$$

$$\| $$

$$0w_1 + (-1)w_2 + 0w_3$$

$$F\Bigg(\underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}}_{E_{32}}\Bigg) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} = -w_3$$

$$\| $$

$$0w_1 + 0w_2 + (-1)w_3$$

$$F\Bigg(\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \cdot & \cdot & \cdot \end{pmatrix}\Bigg) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \cdot & \cdot & \cdot \end{pmatrix}$$

$$F\left(\underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{= E_{33}}\right) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$M = \begin{pmatrix} 1 & 4 & 1 \\ 0 & 2 & -1 \\ -3 & 1 & 0 \end{pmatrix} = 1 \cdot E_{11} + 4 E_{12} + 1 E_{13} + 0 E_{21} + 2 E_{22} + (-1) E_{23} + (-3) E_{31} + 1 E_{32} + 0 E_{33}$$

$$F(M) = \begin{pmatrix} 1 & 4 & 1 \\ 0 & 2 & -1 \\ -3 & 1 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 & -3 \\ 4 & 2 & 1 \\ 1 & -1 & 0 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 4 & 4 \\ -4 & 0 & -2 \\ -4 & 2 & 0 \end{pmatrix}}_{N}$$

$$M \xrightarrow{F} N = F(M)$$

$$(1, 4, 1, 0, 2, -1, -3, 1, 0)$$

moltiplico per A (a sinistra) la colonna

$$(4, 4, -2)$$

$$W_1 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 4 & 4 \\ -4 & 0 & -2 \\ -4 & 2 & 0 \end{pmatrix} = 4 w_1 + 4 w_2 - 2 w_3$$

$$w_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

$$w_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

$$A = \begin{pmatrix} \boxed{\begin{matrix}0\\0\\0\end{matrix}} & \boxed{\begin{matrix}1\\0\\0\end{matrix}} & \boxed{\begin{matrix}0\\1\\0\end{matrix}} & \boxed{\begin{matrix}-1\\0\\0\end{matrix}} & \boxed{\begin{matrix}0\\0\\0\end{matrix}} & \boxed{\begin{matrix}0\\0\\1\end{matrix}} & \boxed{\begin{matrix}0\\-1\\0\end{matrix}} & \boxed{\begin{matrix}0\\0\\-1\end{matrix}} & \boxed{\begin{matrix}0\\0\\0\end{matrix}} \end{pmatrix}$$

$$\underline{(1,4,1,0,2,-1,-3,1,0)}$$

$$\begin{pmatrix} 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 4 \\ 1 \\ 0 \\ 2 \\ -1 \\ -3 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \\ -2 \end{pmatrix}$$

3×9 $\qquad$ 9×1 $\qquad$ 3×1

$$\underbrace{\begin{array}{c} \\ A \\ B \\ C \end{array} \begin{array}{c} \begin{matrix} A & B & C \end{matrix} \\ \begin{pmatrix} \frac{1}{3} & \frac{3}{5} & \frac{1}{15} \\ - & - & - \\ - & - & - \end{pmatrix} \end{array} \begin{pmatrix} 0.1 \\ 0.85 \\ 0.05 \end{pmatrix}} = \begin{pmatrix} \\ \\ \\ \end{pmatrix}$$

A	B	C
0.1	0.85	0.05

$$A \begin{pmatrix} p_1^0 \\ \vdots \\ p_n^0 \end{pmatrix} = \begin{pmatrix} p_1^1 \\ \vdots \\ p_n^1 \end{pmatrix}$$

$$A^2 \begin{pmatrix} p_1^0 \\ \vdots \\ p_n^0 \end{pmatrix} = \begin{pmatrix} p_1^2 \\ \vdots \\ p_n^2 \end{pmatrix}$$

$$\lim_{n \to \infty} A^n \begin{pmatrix} p_1^0 \\ \vdots \\ p_n^0 \end{pmatrix} = \begin{pmatrix} p_1^\infty \\ \vdots \\ p_n^\infty \end{pmatrix}$$

$$A \begin{pmatrix} q_1 \\ \vdots \\ q_n \end{pmatrix} = \begin{pmatrix} q_1 \\ \vdots \\ q_n \end{pmatrix}$$