19 March 2025 11:06

Composizione di map lineari.

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[rr, bend left, "g\circ f"] & W \arrow[r, "g"] & U \end{tikzcd}$$

$\dim V = n$ $\qquad$ $\dim W = m$ $\qquad$ $\dim U = r$

B_V $\qquad$ B_W $\qquad$ B_U

$$v \underset{B_V}{\equiv} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \qquad f(v) \underset{B_W}{\equiv} \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix} \qquad g(f(v)) \underset{B_U}{\equiv} \begin{pmatrix} z_1 \\ \vdots \\ z_r \end{pmatrix}$$

$$\begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix} = M_{B_V B_W}(f) \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \qquad \begin{pmatrix} z_1 \\ \vdots \\ z_r \end{pmatrix} = M_{B_W B_U}(g) \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix}$$

$$\begin{pmatrix} z_1 \\ \vdots \\ z_r \end{pmatrix} = M_{B_W B_U}(g) \left(M_{B_V B_W}(f) \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \right)$$

$$\Downarrow$$

$$\begin{pmatrix} z_1 \\ \vdots \\ z_r \end{pmatrix} = \left(M_{B_W B_U}(g) \cdot M_{B_V B_W}(f) \right) \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

coord. del vettore $g(f(v))$

le coord. del vettore v

vettore $g(f(v))$
$\shortparallel$
$(g\circ f)(v)$

v

Sappiamo già che

$$\begin{pmatrix} z_1 \\ \vdots \\ z_r \end{pmatrix} = M_{B_V B_U}(g\circ f) \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

Esaminiamo il caso $\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

$$\begin{pmatrix} z_1 \\ \vdots \\ z_r \end{pmatrix} = M_{B_W B_U}(g)\, M_{B_V B_W}(f) \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$\shortparallel$

$$\begin{pmatrix} z_1 \\ \vdots \\ z_r \end{pmatrix} = M_{B_V B_U}(g\circ f) \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

coord. di $g(f(v))$

Prima colonna di A

A B

$\shortparallel$

$$\underset{2\times 3}{\underbrace{A}_{\;}} \quad A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}_{2\times 3} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}_{3\times 1} = \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix}_{2\times 1}$$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix}$$

$$\underset{B_V}{V} \xrightarrow{f} \underset{B_W}{W} \xrightarrow{g} \underset{B_U}{U}$$

$$M_{B_V B_W}(f) \qquad M_{B_W B_U}(g)$$

$$\underset{B_V}{V} \xrightarrow{g \circ f} \underset{B_U}{U}$$

$$M_{B_V B_U}(g \circ f)$$

$\check{B}_V$ $\qquad$ B_U

$$M_{B_V B_U}(g \circ f)$$

Esempio:

$$\mathbb{R}^n \xrightarrow{f} \mathbb{R}^m \xrightarrow{g} \mathbb{R}^s$$

Base canonica $\qquad$ Base canonica $\qquad$ Base canonica

$$\mathbb{R}^4 \xrightarrow{f} \mathbb{R}^2 \xrightarrow{g} \mathbb{R}^3$$

$$f(x_1, x_2, x_3, x_4) = (x_1 + x_2 + x_3,\ x_2 - x_4)$$

$$M_{\mathcal{E}^n \mathcal{E}^m}(f) = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}$$

$$f((\overset{x_1}{1}, \overset{x_2}{0}, \overset{x_3}{0}, \overset{x_4}{0})) = (1, 0)$$

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}$$

$$f((0, 1, 0, 0)) = (1, 1)$$

$$f((0, 0, 1, 0)) = (1, 0)$$

$$f((0, 0, 0, 1)) = (0, -1)$$

$$f((0,0,0,1)) = (0,-1)$$

$$g(y_1,y_2) = \boxed{(y_1+y_2,\ 0,\ -y_2)}$$

$$M_{\mathcal{E}^2\mathcal{E}^3}(g) = \begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 0 & -1 \end{pmatrix}$$

$$g((\overset{y_1}{1},\overset{y_2}{0})) = (1,0,0)$$

$$g((0,1)) = (1,0,-1)$$

$$\boxed{M_{\mathcal{E}^2\mathcal{E}^3}(g)\, M_{\mathcal{E}^4\mathcal{E}^2}(f)} =$$

$$\underset{3\times 2}{\begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 0 & -1 \end{pmatrix}} \underset{2\times 4}{\begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}} = \underset{3\times 4}{\boxed{\begin{pmatrix} 1 & 2 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}}}$$

Dobbiamo ora calcolare la matrice

$$M_{\mathcal{E}^4\mathcal{E}^3}(g\circ f)$$

e vedere se si tratta della matrice appena calcolata.

$$g\circ f((x_1,x_2,x_3,x_4)) = g((x_1+x_2+x_3,\ x_2-x_4)) =$$

$$= (x_1+2x_2+x_3-x_4,\ 0,\ -x_2+x_4)$$

$$M_{\mathcal{E}^4\mathcal{E}^3}(g\circ f) = \begin{pmatrix} 1 & 2 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}$$

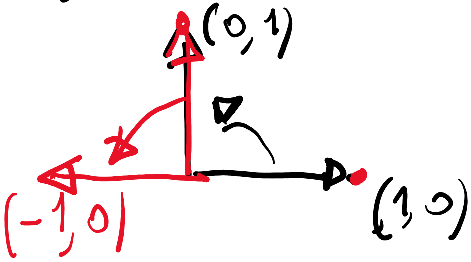
3 x 4

Come si possono rappresentare i movimenti rigidi del piano usando matrici.
che tengono fisso il punto (0,0)

Rotazione intorno a (0,0) di 90°.
antiorario

$r((1,0)) = (0,1)$

$r((0,1)) = (-1,0)$



$$M_{\mathcal{E}^2\mathcal{E}^2}(r) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$\underbrace{\boxed{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}}_{= M_{\mathcal{E}^2\mathcal{E}^2}(r^2)}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x \\ -y \end{pmatrix}$$

r^{-1} = rotazione di 90° in senso ORARIO

r^{-1} = rotazione di $90°$ in senso ORARIO

$r \circ r^{-1} = r^{-1} \circ r$ = identità

$$M_{\mathcal{E}^2\mathcal{E}^2}(r^{-1}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$r^{-1}((1,0)) = (0,-1)$$

$$r^{-1}((\quad))$$

$$\underbrace{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}_{M_{\mathcal{E}^2\mathcal{E}^2}(r^{-1})} \underbrace{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}_{\substack{\| \\ M_{\mathcal{E}^2\mathcal{E}^2}(r)}} = \underbrace{\underline{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}}_{\substack{\| \\ M_{\mathcal{E}^2\mathcal{E}^2}(\underset{\substack{\| \\ id}}{r^{-1} \circ r})}}$$

Definizione. Sia A una matrice $n \times n$. Se B è una matrice $n \times n$ tale che

$$AB = BA = I_n = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & \\ 0 & & & 1 \end{pmatrix}_{n \times n}$$

allora si dice che A ammette inversa

allora si dice che $\underset{n\times n}{A}$ ammette inversa
e B è l'inversa di A.
In simboli, si scrive $B = A^{-1}$.

$$\underset{A}{\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}} \underset{B}{\boxed{\begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Quindi $B = A^{-1}$

In generale A^{-1} non esiste.

Esempio:

$$\begin{pmatrix} 2 & 3 \\ 6 & 9 \end{pmatrix} \begin{pmatrix} x & y \\ z & t \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 2x+3z & 2y+3t \end{pmatrix} = \begin{pmatrix} 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 2x+3z & 2y+3t \\ 6x+9z & 6y+9t \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Updownarrow$$

$$\begin{cases} 2x+3z = 1 \\ \boxed{2y+3t = 0} \\ 6x+9z = 0 \\ \boxed{6y+9t = 1} \end{cases} \Longrightarrow 3(2y+3t) = 3\cdot 0$$

$$\Downarrow$$

$$6y+9t = 0$$

Un metodo per calcolare la matrice inversa (se esiste): il metodo delle matrici affiancate.

1) Scrivi A e affiancala alla matrice I:

$$\left(A \;\middle|\; I \right)$$

$n \times 2n$

Esempio:

$$M = \left(\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right)$$

2) Fare operazioni riga su M fino a che a sinistra compare la matrice identica

a che a sinistra compare la matrice identica.

$$\left(\begin{array}{cc|cc} 1 & 0 & 1 & -1 \\ 1 & 1 & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{cc|cc} 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 2 \end{array}\right)$$

3) A questo punto a destra avrete A^{-1}.

$$\left(\begin{array}{cc|cc} 2 & 3 & 1 & 0 \\ 6 & 9 & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{cc|cc} 2 & 3 & 1 & 0 \\ 0 & 0 & -3 & 1 \end{array}\right)$$

$$\left(\begin{array}{cc|cc} 1 & \frac{3}{2} & 1 & 0 \\ 0 & 0 & -3 & 1 \end{array}\right)$$

Ora possiamo aggiungere al nostro calcolo matriciale l'uso della matrice inversa A^{-1} (quando esiste).

della matrice inversa A^{-1} (quando esiste).

$$(A+A^{-1})^2 = (A+A^{-1})(A+A^{-1})$$
$$= AA + AA^{-1} + A^{-1}A + A^{-1}A^{-1}$$
$$= A^2 + I + I + \underbrace{(A^{-1})^2}_{= A^{-2}}$$
$$= A^2 + 2I + A^{-2}$$

$$(AB)^{-1} = ?$$

Prop. Supponiamo che A e B siano invertibili (cioè che esistano A^{-1} e B^{-1}). Allora esiste $(AB)^{-1}$ e

$$(AB)^{-1} = \underline{B^{-1}A^{-1}}$$

Dim. $\underline{(AB)}\underline{(B^{-1}A^{-1})} =$

$$= A(BB^{-1})A^{-1}$$
$$= AIA^{-1}$$
$$= AA^{-1}$$

$$= A A^{-1}$$

$$= I$$

$$\underline{\left(B^{-1}A^{-1}\right)}\,\underline{\left(AB\right)}$$

$$= B^{-1}\underline{\left(A^{-1}A\right)}B$$

$$= \left(B^{-1} I\right) B$$

$$= B^{-1} B$$

$$= I$$

Tutto ciò dimostra la nostra proporzione.

$$\underset{n\times n}{A} = \left(a_{ij}\right) \qquad \underset{n\times n}{I} = \underline{\begin{pmatrix} 1 & 0 & \cdots & 0 \\ \vdots & \ddots & & \vdots \\ 0 & \cdots & & 1 \end{pmatrix}} = \left(\delta_{ij}\right)$$

Delta di Kronecker

$$\delta_{ij} = \begin{cases} \underline{1 \quad \text{se } i = j} \\ 0 \quad \text{se } i \neq j \end{cases}$$

$$(\delta_{ij}) = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & 0 \cdots & 0 \\ \vdots & & & \\ 0 & \cdots & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

$$A \quad I = A$$

$$I \quad A = A$$

$$(a_{ij})\,(\delta_{ij})$$

L'elemento di posto (i,j) nella matrice AI è $\sum_{r=1}^{n} a_{ir}\delta_{rj} = a_{ij}\delta_{jj} =$

$= a_{ij} \cdot 1$

$= a_{ij}$

Ciò vuol dire che

$$AI = A$$

La matrice

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & -1 \\ 0 & 1 & 1 \end{pmatrix}$$

è invertibile

Lo è se e solo se il metodo delle matrice appiarcate termina positivamente, cioè se e solo se la forma completamente ridotta di A è la matrice identica I, cioè se $r(A) = r(I) = 3$.

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 2 & -1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \end{array}\right.$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 2 & 1 & -1 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1/2 & -\frac{1}{2} & \frac{1}{2} \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{3}{2} & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & 0 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{array}\right)$$

$$\qquad \| \qquad\qquad\qquad \| $$
$$\qquad I \qquad\qquad\qquad A^{-1}$$

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & -1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} \frac{3}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$