Relazione di Grassmann

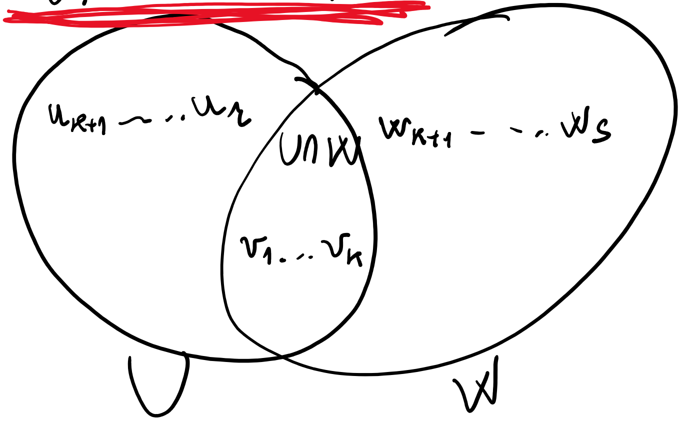
U, W sottospazi vettoriali di V (finitamente generato)

$$\underbrace{\dim U+W}_{r+s-k} = \underbrace{\dim U}_{r} + \underbrace{\dim W}_{s} - \underbrace{\dim U\cap W}_{-k}$$

Cenno della dimostrazione.

Prendiamo una base di $U \cap W$

$B_{U\cap W} = (v_1, \dots, v_k)$



Applicando due volte il teorema del completamento a una base, possiamo ottenere una base

$$\boxed{B_U = (v_1, \dots, v_k, u_{k+1}, \dots, u_r) \text{ di } U}$$

e una base

$$\boxed{B_W = (v_1, \dots, v_k, w_{k+1}, \dots, w_s) \text{ di } W.}$$

Affermo che

$B = (\dots$

$$B = (v_1, \dots, v_k, u_{k+1}, \dots, u_r, w_{k+1}, \dots, w_s)$$

è una base per $U+W$.

È chiaro che si tratta di un sistema di generatori per $U+W$:

$$\underset{\substack{\cap \\ U}}{u} + \underset{\substack{\cap \\ W}}{w} = \underbrace{\sum_i a_i v_i + \sum b_j u_j}_{u} + \underbrace{\sum c_i v_i + \sum d_h w_h}_{w}$$

$$= \sum (a_i + c_i) v_i + \sum b_j u_j + \sum d_h w_h$$

$$\sum a_i v_i + \sum b_j u_j + \cancel{\sum c_h w_h} = 0 \qquad c_h = 0$$

$$\Downarrow$$

$$\underbrace{\sum a_i v_i + \sum b_j u_j}_{\in U} = \underbrace{\boxed{-\sum c_h w_h}}_{\in W} \in U \cap W$$

$$-\sum c_h w_h = \sum x_i v_i$$

$$\Downarrow$$

$$\sum x_i v_i + \sum c_h w_h = 0$$

$$x_i = 0 \qquad c_h = 0$$

$$\Downarrow$$

$$\sum \underset{\substack{\| \\ 0}}{a_i}\, \underline{v_i} + \sum \underset{\substack{\| \\ 0}}{b_j}\, \underline{u_j} = 0$$

Avendo dimostrato che

$$B = (\underline{v_1, \dots, v_k}, \underline{u_{k+1}, \dots, u_r}, \underline{w_{k+1}, \dots, w_s})$$

è una base di $U+W$, si ha che

$$\dim U+W = k + (r-k) + (s-k)$$
$$= r + s - k$$

NOTA IMPORTANTE PER GLI ESERCIZI

1) Quando si deve lavorare con $U \cap W$ è bene usare le equazioni cartesiane,

$$U \cap W : \begin{cases} \boxed{\text{Eq. cartes. di } U} \\ \boxed{\text{Eq. cartes. di } W} \end{cases}$$

2) Quando si deve lavorare con

c) Quando si deve trovare con $U+W$ è bene usare le equazioni parametriche.

$$\underset{U}{\begin{cases} x_1 = \boxed{\quad} \\ \vdots \\ x_n = \boxed{\quad} \end{cases}} \qquad \underset{W}{\begin{cases} x_1 = \boxed{\quad} \\ \\ x_n = \boxed{\quad} \end{cases}} \Longrightarrow \underset{U+W}{\begin{cases} x_1 = \boxed{\quad} + \boxed{\quad} \\ \\ x_n = \boxed{\quad} + \boxed{\quad} \end{cases}}$$

$$\begin{pmatrix} x & y \\ z & u \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \lambda \\ 0 \end{pmatrix}$$

$$\| $$

$$\begin{pmatrix} x \\ z \end{pmatrix}$$

$$\Updownarrow$$

$$\begin{cases} x = \lambda \\ z = 0 \end{cases}$$

$$W = \left\{ \begin{pmatrix} x & y \\ z & u \end{pmatrix} : z = 0 \right\}$$

$$\begin{array}{cccc} x_1 & & x_3 & \\ \end{array}$$

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$x_1 + \underset{}{\overset{s}{\overset{\shortparallel}{x_4}}} = 0$$

$$x_3 = 0$$

$$\begin{cases} x_1 = -s \\ x_2 = r \\ x_3 = 0 \\ x_4 = s \end{cases}$$

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -s \\ r \\ 0 \\ s \end{pmatrix} = r \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -s & r \\ 0 & s \end{pmatrix}$$

$$B = \left(\underset{E_{11}}{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}, \underset{E_{12}}{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}, \underset{E_{21}}{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}}, \underset{E_{22}}{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}} \right)$$

$$U \cap W = \left\{ \begin{pmatrix} -s & r \\ 0 & s \end{pmatrix} : r, s \in \mathbb{R} \right\}$$

$U^\perp$ è l'insieme dei vettori ortogonali a TUTTI i vettori di U.

Cioè

$$U^\perp = \{ v \in V : \langle v, u \rangle = 0 \text{ per ogni } u \in U \}$$

Sapere che $\dim U^\perp = \underbrace{\dim V}_{= n} - \dim U$

$$\underbrace{\dim U}_{= 3} = \underbrace{\dim V}_{= 4} - \underbrace{\dim U^\perp}_{1}$$

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 2 & 3 \\ 2 & 0 & -1 & 0 \end{pmatrix}$$

$$\downarrow$$

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 4 \\ 0 & -2 & -3 & -2 \end{pmatrix}$$

$$\downarrow$$

$$r\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 3 & 6 \end{pmatrix} = 3$$

$$\begin{cases} x_1 = 2s & \Longrightarrow \quad s = \frac{1}{2}x_1 \\ x_2 = r+s \\ x_3 = 2r+s \\ x_4 = 3r+2s \end{cases}$$

$$\begin{cases} x_2 = r + \frac{1}{2}x_1 & \Longrightarrow \quad r = x_2 - \frac{1}{2}x_1 \\ x_3 = 2r + \frac{1}{2}x_1 \\ x_4 = 3r + x_1 \end{cases}$$

$$\begin{cases} x_3 = 2z + \frac{1}{2}x_1 \\ x_4 = 3z + x_1 \end{cases}$$

$$\begin{cases} x_3 = 2x_2 - x_1 + \frac{1}{2}x_1 = 2x_2 - \frac{1}{2}x_1 \\ x_4 = 3x_2 - \frac{3}{2}x_1 + x_1 = 3x_2 - \frac{1}{2}x_1 \end{cases}$$

$$\Updownarrow$$

$$\begin{cases} \frac{1}{2}x_1 - 2x_2 + x_3 = 0 \\ \frac{1}{2}x_1 - 3x_2 + x_4 = 0 \end{cases}$$

$$\Updownarrow$$

Eq. cartesiane di W
$$\begin{cases} x_1 - 4x_2 + 2x_3 = 0 \\ x_1 - 6x_2 + 2x_4 = 0 \end{cases}$$

$\mathbb{R}^3 \qquad n = 3$

$$\det\begin{pmatrix} 2-\lambda & 1 & 1 \\ 1 & +2-\lambda & 1 \\ 0 & 0 & -\lambda \end{pmatrix} = \underbrace{(-1)^{3+3}}_{=1} \cdot (-\lambda)\det\begin{pmatrix} 2-\lambda & 1 \\ 1 & +2-\lambda \end{pmatrix} =$$

$(3,3)$

$$= -\lambda\left((2-\lambda)^2 - 1\right)$$

$$= \lambda\left(1 - (2-\lambda)^2\right)$$

$$= \lambda\,\underline{\left(1-(2-\lambda)^2\right)}$$

$$= \lambda\left(1-(2-\lambda)\right)\left(1+(2-\lambda)\right)$$

$$= \underset{0}{\lambda}\,(\underset{1}{\lambda-1})\,(\underset{3}{3-\lambda})$$

autovalori : 0, 1, 3

Esiste una base spettrale.

$\lambda = 1$

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & -3 & 1 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & -3 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$\downarrow$$

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$\downarrow$$

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{array}{c} \downarrow \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{array}$$

$$\begin{pmatrix} 2 & 1 & 1 \\ 1 & -2 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\det\begin{pmatrix} 2-\lambda & 1 & 1 \\ 1 & -2-\lambda & 1 \\ 0 & 0 & -\lambda \end{pmatrix} = -\lambda \det\begin{pmatrix} 2-\lambda & 1 \\ 1 & -2-\lambda \end{pmatrix} =$$

$$= -\lambda\left((2-\lambda)(-2-\lambda)-1\right)$$

$$= -\lambda\left((\lambda-2)(\lambda+2)-1\right)$$

$$= -\lambda\left(\lambda^2-4-1\right)$$

$$= -\lambda\left(\lambda^2-5\right)$$

$$= -\lambda\left(\lambda-\sqrt{5}\right)\left(\lambda+\sqrt{5}\right)$$

Autovalori: $0, \sqrt{5}, -\sqrt{5}$

Esiste una base spettrale.

Calcolo l'autospazio

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$U_{\sqrt{5}}$:

$$\begin{pmatrix} 2-\sqrt{5} & 1 & 1 \\ 1 & -2-\sqrt{5} & 1 \\ 0 & 0 & -\sqrt{5} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 2-\sqrt{5} & 1 & 0 \\ 1 & -2-\sqrt{5} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$(2-\sqrt{5})(-2-\sqrt{5}) =$$

$$= (\sqrt{5}-2)(\sqrt{5}+2) = 5-4 = 1$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & -2-\sqrt{5} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} x - (2+\sqrt{5})y = 0 \\ z = 0 \end{cases}$$

$$\begin{cases} x = (2+\sqrt{5})r \\ y = r \\ z = 0 \end{cases}$$

$$B_{U_{\sqrt{3}}} = ((2+\sqrt{5}, 1, 0))$$

$$\lambda^2 - 9\lambda + 22 \longrightarrow$$ radici complesse: $\frac{9 \pm \sqrt{-7}}{2}$

$$\Delta = (-9)^2 - 4 \cdot 1 \cdot 22 = 81 - 88 < 0$$

Non ci sono radici reali

autovalori in $\mathbb{C}$:

$\sqrt{-7} = \sqrt{-1 \cdot 7} = \sqrt{-1} \cdot \sqrt{7} = i \cdot \sqrt{7}$

$$4, \frac{9+\sqrt{-7}}{2}, \frac{9-\sqrt{-7}}{2}$$

$$4, \frac{9}{2} + i\frac{\sqrt{7}}{2}, \frac{9}{2} - i\frac{\sqrt{7}}{2}$$

$$\begin{pmatrix} 4 & 0 & 0 \\ 0 & \frac{9}{2}+i\frac{\sqrt{7}}{2} & 0 \\ 0 & 0 & \frac{9}{2}-i\frac{\sqrt{7}}{2} \end{pmatrix}$$