Vogliamo dimostrare che se

$A \in M_{m \times r}(\mathbb{R})$ $\quad A = (a_{ih})$

$B \in M_{r \times s}(\mathbb{R})$ $\quad B = (b_{hk})$

$C \in M_{s \times n}(\mathbb{R})$ $\quad C = (c_{kj})$

$$\underbrace{\underbrace{(AB)}_{m \times s}\underbrace{C}_{s \times n}}_{m \times n} = \underbrace{\underbrace{A}_{m \times r}\underbrace{(BC)}_{r \times n}}_{m \times n}$$

$D = AB \quad (d_{ik})$

$E = BC \quad (e_{hj})$

Devo verificare se l'elemento x_{ij} nella matrice $(AB)C$ coincide con l'elemento y_{ij} nella matrice $A(BC)$.

$$\boxed{z_{ij} = \sum_{r=1}^{r} a_{ir} b_{rj}}$$

$A = (a_{ir}) \qquad B = (b_{rj})$

colonna j-esima $\qquad z_{ij}$

riga i-esima

$$\underset{m \times r}{A}\begin{pmatrix} a_{i1} & a_{i2} & \cdots & a_{ir} \end{pmatrix} \underset{r \times n}{B}\begin{pmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{rj} \end{pmatrix} = \underset{AB}{\begin{pmatrix} \square \\ (i,j) \end{pmatrix}}$$

$$z_{ij} = a_{i1} b_{1j} + a_{i2} b_{2j} + \ldots + a_{ir} b_{rj}$$

$$= \sum_{k=1}^{r} a_{ik} b_{kj}$$

$a_{i1} b_{1j} + a_{i2} b_{2j} + \ldots + a_{ir} b_{rj}$

$(a_{ik})(b_{kj}) = (c_{ij})$

$$\left(a_{ik}\right)\left(b_{kj}\right) = \left(c_{ij}\right)$$

$\left(a_{ik}\right) = A$, $\left(b_{kj}\right) = B$, $c_{ij} = \sum_{k=1}^{r} a_{ik} b_{kj}$

$$\underbrace{(\underbrace{AB}_{D})C}_{m\times n} = \underbrace{A(\underbrace{BC}_{E})}_{m\times n}$$

$A B$: $m \times s$, $s \times n$ $\qquad$ $B C$: $m \times r$, $r \times n$

$D = AB \quad (d_{ik})$

$E = BC \quad (e_{hj})$

Devo verificare se l'elemento x_{ij} nella matrice $(AB)C$ coincide con l'elemento y_{ij} nella matrice $A(BC)$.

$$\boxed{d_{ik} = \sum_{h=1}^{r} a_{ih} b_{hk}}$$

$$x_{ij} = \sum_{k=1}^{s} d_{ik} c_{kj} =$$

$$= \sum_{k=1}^{s} \left(\sum_{h=1}^{r} a_{ih} b_{hk} \right) c_{kj}$$

$$\boxed{e_{hj} = \sum_{k=1}^{s} b_{hk} c_{kj}}$$

$$y_{ij} = \sum_{h=1}^{r} a_{ih} e_{hj}$$

$$= \sum_{h=1}^{r} a_{ih} \left(\sum_{k=1}^{s} b_{hk} c_{kj} \right)$$

$= ?$

$$x_{ij} = \sum_{k=1}^{s} \left(\sum_{h=1}^{r} a_{ih} b_{hk} \right) c_{kj}$$

$$x_{ij} = \sum_{k=1} \left(\sum_{h=1} a_{ih} b_{hk} \right) c_{kj}$$

$$= \sum_{k=1}^{s} \sum_{h=1}^{r} a_{ih} b_{hk} c_{kj}$$

$$y_{ij} = \sum_{h=1}^{r} a_{ih} \left(\sum_{k=1}^{s} b_{hk} c_{kj} \right)$$

$$= \sum_{h=1}^{r} \sum_{k=1}^{s} a_{ih} b_{hk} c_{kj}$$

$$\Downarrow$$

$$\forall (i,j) \quad x_{ij} = y_{ij}$$

cioè $(AB)C = A(BC)$

$$(AB)C = A(BC)$$

$$(A+B)+C = (A+B)+C$$

$$(A+B)C = AC + BC$$

$$C(A+B) = CA + CB$$

$$A + B = B + A$$

$$AI = IA = A$$

$AX = B$ dove $B = \mathbf{0} = \begin{pmatrix} 0 \\ 0 \\ \vdots \end{pmatrix}$

$$AX = B \quad \text{dove} \quad B = \mathbf{0} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$\begin{cases} x_1 + 2x_2 - 4x_3 = 0 \\ x_1 \qquad\quad + x_3 = 0 \end{cases} \iff \underbrace{\underset{2\times 3}{A}\ \underset{3\times 1}{X}}_{2\times 1} = \underset{2\times 1}{B} = \mathbf{0}$$

$$A = \begin{pmatrix} 1 & 2 & -4 \\ 1 & 0 & 1 \end{pmatrix} \quad 2\times 3$$

$$C = \begin{pmatrix} 1 & 2 & -4 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix} \quad 2\times 4$$

$$\underset{3\times 1}{X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}} \qquad \underset{2\times 1}{B = \begin{pmatrix} 0 \\ 0 \end{pmatrix}}$$

Prendiamo l'insieme Sol S delle soluzioni del sistema lineare omogeneo S rappresentabile nella forma

$$AX = \mathbf{0}$$

Siano $X_1, X_2 \in \text{Sol } S$

$$X_1 = \begin{pmatrix} x_1^1 \\ \vdots \\ x_n^1 \end{pmatrix} \qquad X_2 = \begin{pmatrix} x_1^2 \\ \vdots \\ x_n^2 \end{pmatrix}$$

Allora anche $X_1 + X_2 \in \text{Sol } S$

perché

$$A(X_1 + X_2) =$$

$$= AX_1 + AX_2 =$$

$$= 0 + 0 = 0$$

Se α e β sono numeri reali

$\alpha X_1 + \beta X_2 \in \text{Sol } S$ → comb. lineare

$$\boxed{A(\alpha X_1 + \beta X_2)}$$

$$= A(\alpha X_1) + A(\beta X_2)$$

$$= \boxed{\alpha AX_1 + \beta AX_2} = \alpha 0 + \beta 0 = 0 + 0 = 0$$

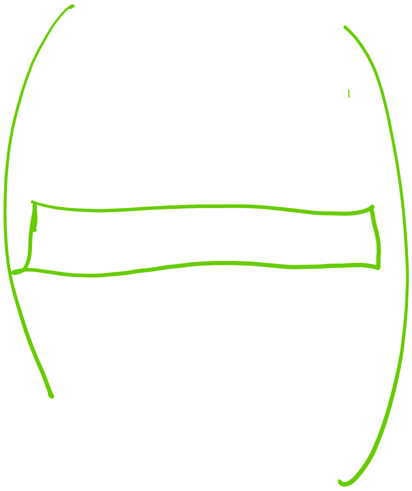
$A = (a_{ik})$ $\quad B = (b_{kj})$

$$\lambda B = \left(\underline{\lambda b_{rj}}\right)$$

$$\boxed{\lambda(AB) = A(\lambda B)}$$

$$\lambda\left(\sum_r a_{ir} b_{rj}\right) \qquad \sum_r a_{ir}\left(\lambda b_{rj}\right)$$

$$\lambda\left(\sum_r a_{ir} b_{rj}\right) = \sum_r \lambda a_{ir} b_{rj} = \sum_r a_{ir}\left(\lambda b_{rj}\right)$$



riga i-esima

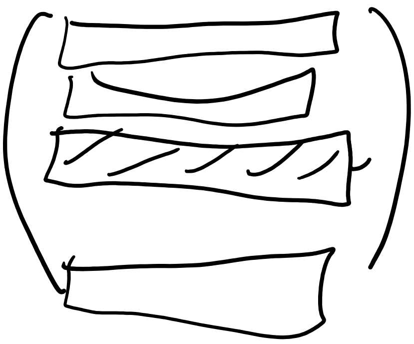
$$\sum_i \boxed{\sum_j t_{ij}} = \sum_j \sum_i t_{ij}$$

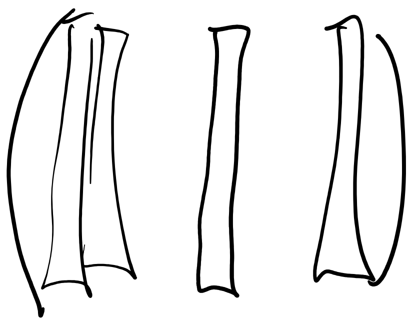
$$\begin{pmatrix} t_{ij} \end{pmatrix}$$

$r \times c$

somma dei termini nella riga i-esima

$$\sum_{i=1}^{r} \boxed{\sum_{j=1}^{c} t_{ij}} = \sum_{j=1}^{c} \boxed{\sum_{i=1}^{r} t_{ij}}$$





$$0$$

$$\begin{cases} x_1 + 2x_2 - 4x_3 = 0 \\ x_1 + x_3 = 0 \end{cases} \iff \underbrace{\underset{2\times 3}{A}\ \underset{3\times 1}{X}}_{2\times 1} = \underset{2\times 1}{B} = 0$$

$$A = \begin{pmatrix} 1 & 2 & -4 \\ 1 & 0 & 1 \end{pmatrix} \quad 2\times 3$$

$$C = \begin{pmatrix} 1 & 2 & -4 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix} \quad 2\times 4$$

$$\underset{3\times 1}{X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}} \qquad \underset{2\times 1}{B = \begin{pmatrix} 0 \\ 0 \end{pmatrix}}$$

$$\begin{pmatrix} 1 & 2 & -4 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 2 & -4 & 0 \end{pmatrix} \longrightarrow$$

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 2 & -5 & 0 \end{pmatrix} \longrightarrow \begin{array}{cccc} x_1 & x_2 & x_3 & \\ \hline \end{array}$$

$$\begin{array}{c} \\ \\ \end{array}\begin{array}{cccc} x_1 & x_2 & x_3 & \\ \end{array}$$

$$\begin{array}{cccc} x_1 & x_2 & x_3 & \end{array} \\ \begin{pmatrix} 1 & \boxed{0} & 1 & 0 \\ 0 & 1 & -\frac{5}{2} & 0 \end{pmatrix}$$

$$\begin{cases} x_1 = -t \\ x_2 = \frac{5}{2} t \\ x_3 = t \end{cases}$$

$$X_1 = \begin{pmatrix} -1 \\ \frac{5}{2} \\ 1 \end{pmatrix} \qquad X_2 = \begin{pmatrix} -2 \\ 5 \\ 2 \end{pmatrix}$$

$$X_1 + X_2 = \begin{pmatrix} -3 \\ \frac{15}{2} \\ 3 \end{pmatrix}$$

$$X_1 + X_2 = \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix}$$

Spazio vettoriale (reale)

$$(\mathbb{R}, V, +, \cdot)$$

insieme dei "vettori"

$+ : V \times V \to V$ $\qquad$ $\cancel{+(v_1, v_2)}$ $\quad v_1 + v_2$

$\cdot : \underline{\mathbb{R} \times V} \to V$ $\qquad$ $\cdot(\lambda, v)$ $\quad \lambda v$

Per ogni scelta dei vettori e degli scalari devono valere queste proprietà

$(V, +)$ è un gruppo abeliano (commutativo)

a) $(v_1 + v_2) + v_3 = v_1 + (v_2 + v_3)$ (la somma di vettori è associativa)

b) $\exists\, u \in V$ tale che

$$u + v = v + u = v$$

(u viene detto "vettore nullo" o "elemento neutro per +"). $\mathbf{0}$

c) $\forall\, v \in V \ \exists\, w \in V$ tale che

$$v + w = w + v = u$$

(w viene detto "opposto di v" …

d) …

$v + w = w + v = u$ ("w viene detto "opposto di v" e indicato col simbolo $-v$)

d) $v_1 + v_2 = v_2 + v_1$

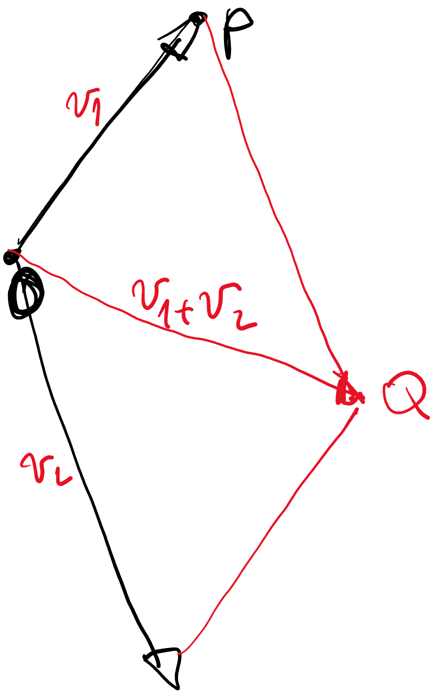
Inoltre

1) $\lambda(v_1 + v_2) = \lambda v_1 + \lambda v_2$

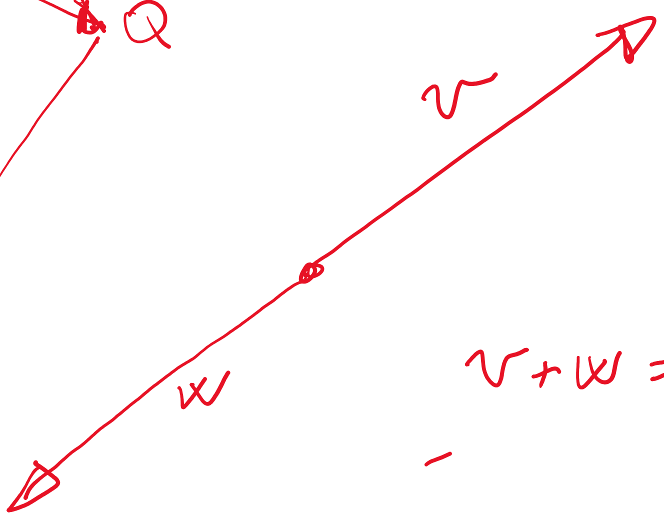
2) $(\alpha + \beta) v = \alpha v + \beta v$

3) $\alpha(\beta v) = (\alpha\beta) v$

4) $1 v = v$

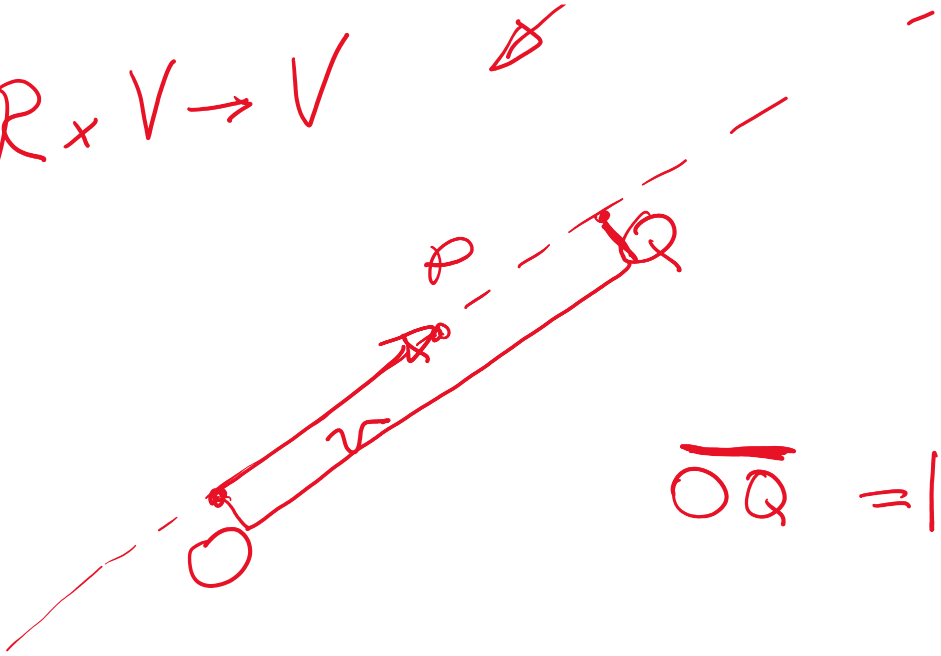


$+ : V \times V \to V$



$v + w = 0$

$\cdot : \mathbb{R} \times V \to V$



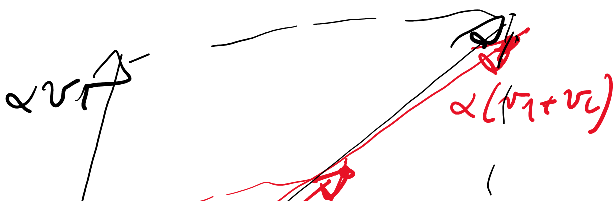
$\overline{OQ} = |\alpha| \, \overline{OP}$

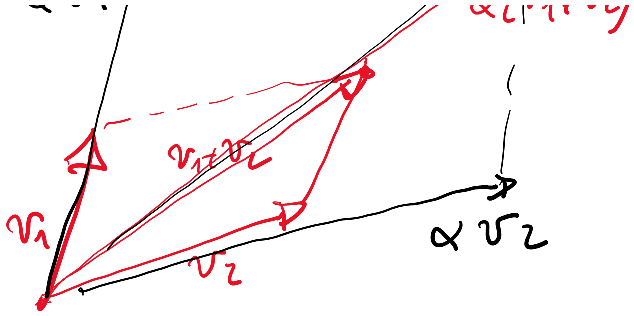
Dunque, detto V l'insieme dei "vettori del piano applicati a un punto fissato O", abbiamo che

$$(\mathbb{R}, V, +, \cdot)$$

è uno spazio vettoriale reale.

$$\alpha(v_1 + v_2) = \alpha v_1 + \alpha v_2$$





$V = \mathbb{R}[x]$

$2x^2 + \pi x$

x

$8x^{100} + 2x - 7$

$(\mathbb{R}, \mathbb{R}[x], \oplus, \odot)$ è uno spazio vettoriale reale

$v \quad x^2 + 3x + 7$

$w \quad -x^2 - 3x - 7$

$v + w = 0$

$$(\mathbb{R}, M_{m\times n}(\mathbb{R}), +, \cdot)$$

è uno spazio vettoriale reale