$$\begin{cases} x-2y+5z=7 \\ x-y+z=0 \end{cases} \Rightarrow C = \underbrace{\left(\begin{array}{ccc|c} 1 & -2 & 5 & 7 \\ 1 & -1 & 1 & 0 \end{array}\right)}_{A}$$

2x4

Tipo di una matrice: coppia di righe e colonne è spesso indicato $m \times n$.

L'insieme delle matrici $m \times n$ si indica con il simbolo $M_{m,n}(\mathbb{R})$

Oggi: matrice A le cui entrate sono quadrate: $A = (a_{ij})$

Esempio:

$$A = \begin{pmatrix} 1 & 0 \\ -7 & 9 \\ 10 & 4 \end{pmatrix} \qquad 3 \times 2 = 4$$

$\begin{pmatrix} 1 \\ 5 \\ 0 \end{pmatrix}$ Si chiama matrice colonna

la matrice $n \times 1$

$\begin{pmatrix} 0 \end{pmatrix}$ Si chiama matrice riga

la matrice $1 \times n$

$(4\ 3\ 1\ 5\ 0\ 2)$

Somma di matrici

Siano $A, B \in M_{m,n}(\mathbb{R})$

$+ : M_{m,n}(\mathbb{R}) \times M_{m,n}(\mathbb{R}) \to M_{m,n}(\mathbb{R})$

$A = (a_{ij}), \quad B = (b_{ij})$

$A + B := (a_{ij} + b_{ij})$

Esempio: $\begin{pmatrix} 3 & 0 \\ 1 & -1 \end{pmatrix} + \begin{pmatrix} 0 & 5 \\ 10 & 3 \end{pmatrix} = \begin{pmatrix} 3 & 5 \\ 11 & 2 \end{pmatrix}$

$\begin{pmatrix} 1 & 2 & 3 \\ 5 & 7 & 9 \end{pmatrix} + \begin{pmatrix} 1 & 2 & 3 \\ 5 & 7 & 9 \end{pmatrix} =$

Prodotto di una matrice per uno scalare

$\cdot : \mathbb{R} \times M_{m,n}(\mathbb{R}) \to M_{m,n}(\mathbb{R})$

$\lambda \cdot A = B := (\lambda a_{ij})$

$\lambda \in \mathbb{R}, \quad (a_{ij})$

$3 \begin{pmatrix} 1 & 5 \\ 0 & -3 \end{pmatrix} = \begin{pmatrix} 3 & 15 \\ 0 & -9 \end{pmatrix}$

Prodotto di matrici

Per definirlo ho bisogno di introdurre il prodotto scalare standard per due vettori di numeri reali:

$(x_1, \dots, x_n) \cdot (y_1, \dots, y_n) := x_1 y_1 + x_2 y_2 + \dots + x_n y_n$

Esempio: $(1, 0, -1) \cdot (5, 3, 2) = \sum_{i=1}^{3} x_i y_i$

$1 \cdot 4 + 0 \cdot 3 + (-1) \cdot 2 = 1 + 0 - 2 = -1$

$\cdot : M_{m,n}(\mathbb{R}) \times M_{n,p}(\mathbb{R}) \to M_{m,p}(\mathbb{R})$

$m \times n \quad n \times p$

$A \cdot B =$

$(a_{ij}) \quad (b_{ij})$

$\begin{pmatrix} r_1 \\ \vdots \\ r_m \end{pmatrix} \quad (c_1, \dots, c_p) = \begin{pmatrix} r_1 c_1 & r_1 c_2 & \dots & r_1 c_p \\ r_2 c_1 & r_2 c_2 & \dots & r_2 c_p \\ \vdots & & & \\ r_m c_1 & r_m c_2 & \dots & r_m c_p \end{pmatrix}$

$$\begin{array}{c|ccc} & c_1 & c_2 & c_3 \\ \hline r_1 & 1 & 0 & -1 \\ r_2 & 2 & -1 & 1 \end{array} \quad \begin{array}{c|ccc} & c_1 & c_2 & c_3 \\ \hline r_1 & 1 & 0 & -1 \\ r_2 & 2 & -1 & 1 \end{array} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 3 \end{pmatrix}$$

$m \times 3 \quad 3 \times 3 \qquad 2 \times 3$

$R_i \cdot C_j$

$$\begin{cases} x-y+z \\ x-y+0 \end{cases} \qquad C = \begin{pmatrix} 1 & -1 & 1 \\ 1 & -1 & 0 \end{pmatrix}$$

$A = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$

$X = \begin{pmatrix} x \\ y \end{pmatrix} \qquad B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\Updownarrow$?

$\boxed{AX = B}$

$\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$2 \times 2 \quad 2 \times 1 \quad 2 \times 1$

$\begin{pmatrix} x-y \\ x-y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Leftrightarrow \begin{cases} x-y=1 \\ x-y=0 \end{cases}$

$\boxed{AX = B} \qquad 2x = 2$

$X = A^{-1}B \qquad \text{③} x = b \quad a \neq 0$

$x = \frac{b}{a}$

$$\cancel{\begin{pmatrix} 1 & 3 \\ 4 & 5 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 15 \\ 4 & 28 \end{pmatrix}}$$

$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

$K = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$

$\begin{cases} 2x-y=7 \\ x-y=1 \end{cases} \quad C = \begin{pmatrix} 2 & -1 & 7 \\ 1 & -1 & 1 \end{pmatrix}$

$\Updownarrow$

$AX = B$ dove $X = \begin{pmatrix} x \\ y \end{pmatrix}$

$\boxed{B = \begin{pmatrix} 7 \\ 1 \end{pmatrix}}$

$KA = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$KB = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 7 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ -6 \end{pmatrix}$

$(KA)X = K(AX) \Rightarrow KB$

$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} X = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 6 \\ -6 \end{pmatrix}$

$\Updownarrow$

$\begin{cases} x = 6 \\ y = -6 \end{cases}$

$AX = B$

$A^{-1} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$

$\cancel{K}\cancel{A}X = KB$

Notare che

$(A+B)+C = A+(B+C)$

$A+B = B+A$

$A+0 = A = 0+A$

$\begin{pmatrix} 0 & -1 \\ 0 & -2 \end{pmatrix}$

$A+B+0 = B+A$ per $B = -A = (-a_{ij})$

$\begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} + \begin{pmatrix} -1 & -1 \\ 0 & -2 \end{pmatrix}$

$(A+B) \cdot C = A \cdot C + B \cdot C$

$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix} \right) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix}$

$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix} \right) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 1 & 2 \end{pmatrix} =$

$= \begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix} + \begin{pmatrix} -1 & -1 \\ 0 & -2 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}$

$C(A+B) = CA + CB$

Il prodotto di matrici è associativo:

$(A \cdot B) \cdot C = A \cdot (B \cdot C)$

$a \cdot b = ba \qquad 2 \times 3 = 3 \times 2$

$AB \neq BA$

$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 2 & 5 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 7 \\ 6 & 19 \end{pmatrix}$

$\begin{pmatrix} 2 & 5 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 17 & 24 \\ 3 & 4 \end{pmatrix}$

$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \underbrace{\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}}_{A}$

$\boxed{IA = A} = AI$

Generalmente

$AB \neq BA$

$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$\underbrace{\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix}}_{A^{-1}} = \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{I}$

$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\quad \begin{cases} a_{11}a + a_{12}c = 1 \\ a_{11}b + a_{12}d = 0 \\ a_{21}a + a_{22}c = 0 \\ a_{21}b + a_{22}d = 1 \end{cases}$

$AX = B$

$\cancel{A^{-1}}AX = A^{-1}B$

$(A+B)\cancel{(A-B)} = \cancel{A^2} - B^2$

$A - B := A + (-B)$

$(A+B) \cdot (A-B) = A \cdot (A-B) + B(A-B) =$

$= A^2 - AB + BA - B^2$

$(A+B)^2 = A^2 + AB + BA + B^2$

Proviamo a dimostrare che

$\underbrace{(A \cdot B)}_{n \times p} \cdot C = \underbrace{A \cdot (B \cdot C)}_{m \times p}$

$A = (a_{ij}) \quad m \times n$

$B = (b_{ij}) \quad n \times s$

$C = (c_{ij}) \quad s \times n$

$AB = D = (d_{ij}) \quad m \times s$

$BC = E = (e_{ij}) \quad s \times n$

$R_i = (a_{i1}, \dots, a_{in})$

$C_t = \begin{pmatrix} b_{1t} \\ \vdots \\ b_{st} \end{pmatrix}$

$\boxed{a_{i1} \cdot b_{1t} + \dots + a_{is} b_{st}}$

Elemento di posto (i,t) nella matrice $A \cdot B$

$d_{it} = \boxed{\sum_{k=1}^{s} a_{ik} b_{kt}} \qquad \underbrace{\begin{pmatrix} R_1 \\ \vdots \\ R_m \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} C_1 & \dots & C_n \end{pmatrix}}_{B}$