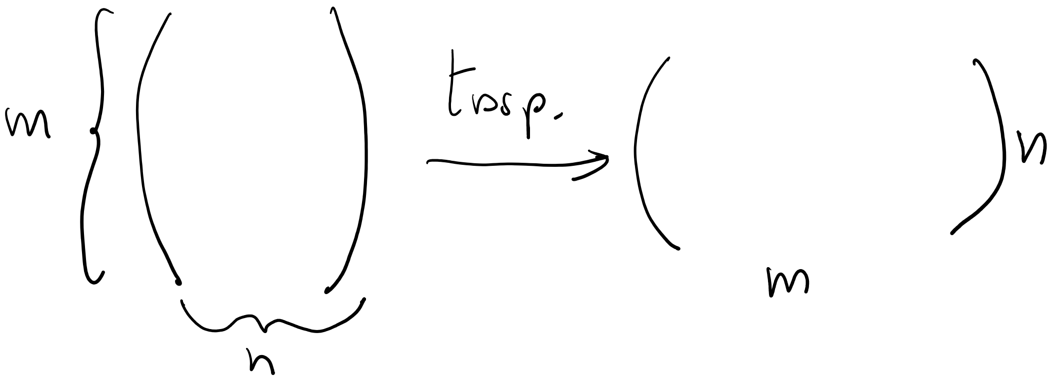
$${}^t(\underbrace{A}_{m\times h}\ \underbrace{B}_{h\times n}) = {}^t\underbrace{B}_{n\times h}\ {}^t\underbrace{A}_{h\times m}$$

$$\underbrace{m\times h \quad h\times n}_{m\times n} \qquad \underbrace{n\times h \quad h\times m}_{\boxed{n\times m}}$$

$$\boxed{n\times m}$$



$$(x+y)^2 = x^2+2xy+y^2$$

$${}^t(AB) \qquad\qquad {}^tB\ {}^tA$$

L'elem. di posto (i,j) in AB è $\sum_{r=1}^{h} a_{ir}b_{rj}$

L'elem di posto (i,j) in ${}^t(AB)$ è $\sum_{r=1}^{h} a_{jr}b_{ri}$

Scrivo come b'_{ij} l'elemento di posto (i,j) in ${}^tB$, cioè $b'_{ij} = b_{ji}$

Analogamente, scrivo come a'_{ij} l'elem. di posto (i,j) in ${}^tA$, cioè $a'_{ij} = a_{ji}$

posto (i,j) in ${}^tA$, cioè $\underline{a'_{ij} = a_{ji}}$

Allora l'elemento di posto $\underline{(i,j)}$ nella matrice ${}^tB\,{}^tA$ è

$$\underline{\sum_{r=1}^{h} b'_{ir}\, a'_{rj}}$$

Quindi l'elem. di posto $\underline{(i,j)}$ nella matrice ${}^tB\,{}^tA$ è

$$\sum_{r=1}^{h} b_{ri}\, a_{jr}$$

$$= \boxed{\sum_{r=1}^{h} a_{jr}\, b_{ri}}$$

$${}^t((AB)C) =$$

$$= {}^tC\; {}^t(AB)$$

$$= {}^tC\,({}^tB\,{}^tA)$$

$$= {}^t C\,({}^t B\,{}^t A)$$

$$ {}^t(ABC) = {}^t C\,{}^t B\,{}^t A$$

$$ {}^t(A+B) = {}^t A + {}^t B$$

$$ {}^t(\lambda A) = \lambda\,{}^t A$$

$$F : \mathbb{R}^3 \to \mathbb{R}^3 \qquad (\mathbb{R}, \mathbb{R}^3, +, \cdot)$$

$$F((x,y,z)) = {}^t\left(\underbrace{\underbrace{\begin{pmatrix}1 & 1 & 0\\ 1 & 2 & 1\\ 0 & 2 & 2\end{pmatrix}}_{3\times 3}\underbrace{\begin{pmatrix}x\\ y\\ z\end{pmatrix}}_{3\times 1}}_{3\times 1}\right)$$

Calcolare $\ker F$ $\operatorname{Im} F$

Calcolare $\ker F$, $\operatorname{Im} F$.

$\boxed{\ker F}$ Vogliamo trovare tutte le triple (x, y, z) t.c.

$$ {}^{t}\left(\underset{3\times 3}{\begin{pmatrix}1&1&0\\1&2&1\\0&2&2\end{pmatrix}} \underset{3\times 1}{\begin{pmatrix}x\\y\\z\end{pmatrix}} \right) = (0,0,0) $$

$$\Updownarrow$$

$$\boxed{\begin{pmatrix}1&1&0\\1&2&1\\0&2&2\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}}$$

Equazione del $\ker F$

$$ {}^{t}\begin{pmatrix}x\\y\\z\end{pmatrix} {}^{t}\begin{pmatrix}1&1&0\\1&2&1\\0&2&2\end{pmatrix} = \boxed{\underbrace{\underset{1\times 3}{(x\ \ y\ \ z)} \underset{3\times 3}{\begin{pmatrix}1&1&0\\1&2&2\\0&1&2\end{pmatrix}}}_{1\times 3}} $$

1×3

$$\begin{pmatrix}x+y\\x+2y+z\\2y+2z\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\begin{cases} x + y = 0 \\ x + 2y + z = 0 \\ 2y + 2z = 0 \end{cases}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 2 & 2 & 0 \end{array}\right)$$

$$\downarrow$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 2 & 2 & 0 \end{array}\right)$$

$$\downarrow$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\downarrow$$

matrice completa $\rightarrow$

$$\begin{array}{cccc} x & y & z & \\ \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \end{array}$$

compl. ridotta per righe

$$\begin{cases} x - z = 0 \\ y + z = 0 \\ \cancel{0 = 0} \end{cases}$$

$$\begin{cases} x = t \\ y = -t \\ z = t \end{cases} \qquad \begin{pmatrix} t \\ -t \\ t \end{pmatrix} = t \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

$$\ker F = \{(t,-t,t) : t \in \mathbb{R}\}$$

$$\ker F = \operatorname{Span}((1,-1,1))$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 2 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Prendiamo la base canonica di $\mathbb{R}^3$:

$$B = (\underset{e_1}{(1,0,0)}, \underset{e_2}{(0,1,0)}, \underset{e_3}{(0,0,1)})$$

$$F(e_1) = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 2 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$F(e_2) = \begin{pmatrix} 1 & \boxed{1} & 0 \\ 1 & \boxed{2} & 1 \\ 0 & \boxed{2} & 2 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$$

$$F(e_3) = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 2 & 2 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$$

Il prodotto di una matrice $A_{m \times n}$ per l'i-esimo vettore della base canonica di $\mathbb{R}^n$ (inteso come colonna) dà l'i-esima colonna di A.

$$\underbrace{\begin{pmatrix} 1 & 3 & 2 \\ 4 & 7 & 1 \\ 5 & -3 & 8 \\ 1 & 3 & 4 \end{pmatrix}_{4 \times 3} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}_{3 \times 1}}_{4 \times 1} = \begin{pmatrix} 3 \\ 7 \\ -3 \\ 3 \end{pmatrix}$$

Dobbiamo ora calcolare Im F.

$$F(x,y,z) = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 2 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\text{Im } F = \text{Span}\big((1\ 1\ 0), (1\ 2\ 2), (0\ 1\ 2)\big)$$

$$\operatorname{Im} F = \operatorname{Span}\left((1\ 1\ 0), (1\ 2\ 2), (0\ 1\ 2)\right)$$

$$v = x\, e_1 + y\, e_2 + z\, e_3 = (x, y, z)$$

$$F(v) = F(x\, e_1 + y\, e_2 + z\, e_3)$$

$$= x\, \underline{F(e_1)} + y\, \underline{F(e_2)} + z\, \underline{F(e_3)}$$

$$\operatorname{Im} F = \operatorname{Span}\left((1,1,0),(1,2,2),(0,1,2)\right)$$

Quindi i vettori di Im F si possono TUTTI scrivere così:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = r \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + s \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} + t \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$$

$$\boxed{\begin{cases} x = r + s \\ y = r + 2s + t \\ z = \phantom{r +{}} 2s + 2t \end{cases}}$$

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 2 \\ 0 & 1 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{pmatrix}$$

$$\begin{pmatrix} \boxed{1 \quad 1 \quad 0} \\ \boxed{0 \quad 1 \quad 2} \\ 0 \quad 0 \quad 0 \end{pmatrix}$$

$$\operatorname{Im} F = \operatorname{Span}\left((1,1,0),(0,1,2)\right)$$

$$\begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\operatorname{Im} F = \operatorname{Span}\left(\underline{(1,0,-2),(0,1,2)}\right)$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = s\begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} + t\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$$

$$\boxed{\begin{cases} x = s \\ y = t \\ z = -2s + 2t \end{cases}}$$

$$\dot{z} = -2s + 2t$$

$\mathbb{R}^n$

$$\begin{cases} x_1 = s + t + h & = 1u \\ x_2 = 2s + 2t + 2h & = 2u \\ x_3 = 3s + 3t + 3h & = 3u \\ \vdots & \vdots \\ x_n = ns + nt + nh & = nu \end{cases}$$

$$s + t + h = u$$

Equazione dimensionale

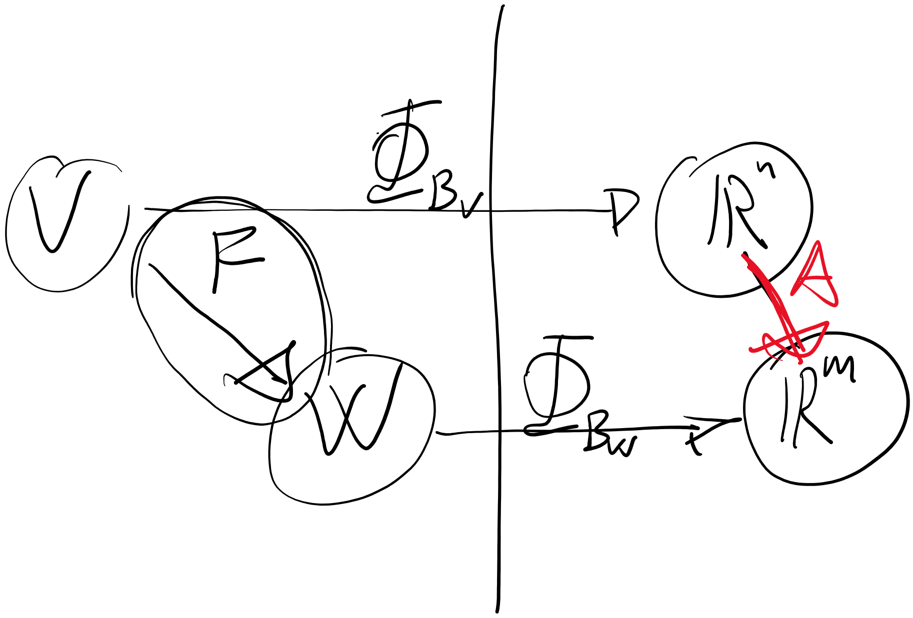
$$f : V \to W$$

$$\dim \ker f + \dim \operatorname{Im} f = \dim V$$

Es. $V = \mathbb{R}^{10}$

$$f : \mathbb{R}^{10} \to \mathbb{R}^7$$

$$\dim \ker f + \dim \operatorname{Im} f = 10$$



$$\begin{tikzcd}
V \arrow[r, "{B_V = (v_1, \dots, v_n)}"'] \arrow[d, "F"] & \mathbb{R}^n \arrow[d, "A"] \\
W \arrow[r, "{B_W = (w_1, \dots, w_m)}"'] & \mathbb{R}^m
\end{tikzcd}$$

$$A = ?$$

Ricetta per costruire la matrice A.

$$\begin{array}{cc} m \text{ righe} & \begin{pmatrix} \boxed{\begin{matrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{matrix}} & \boxed{\begin{matrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{matrix}} & \cdots & \boxed{\begin{matrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{matrix}} \end{pmatrix} \\ & n \text{ colonne} \end{array}$$

coordinate del vettore $\underline{F(v_1)}$ rispetto alla base B_W.

$$F(v_1) = a_{11} w_1 + a_{21} w_2 + \dots + a_{m1} w_m$$
$$F(v_2) = a_{12} w_1 + a_{22} w_2 + \dots + a_{m2} w_m$$
$$\vdots \qquad\qquad\qquad\qquad \vdots$$
$$F(v_n) = a_{1n} w_1 + a_{2n} w_2 + \dots + a_{mn} w_m$$

Teorema. Siano $(x_1, \dots, x_n)$ le coordinate del generico vettore $v \in V$ rispetto a B_V. Siano $(y_1, \dots, y_m)$ le coordinate del generico vettore $F(v) \in W$ rispetto a B_W.

Allora

$$\begin{pmatrix} \cancel{x}_1 \end{pmatrix} \qquad \begin{pmatrix} x_1 \end{pmatrix}$$

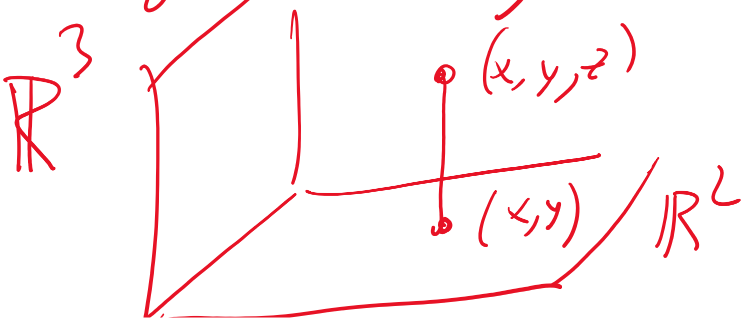
$$\underbrace{\begin{pmatrix} \cancel{x}_1 \\ \vdots \\ \cancel{x}_m \end{pmatrix}}_{F(v)} = A \underbrace{\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}}_{v}$$

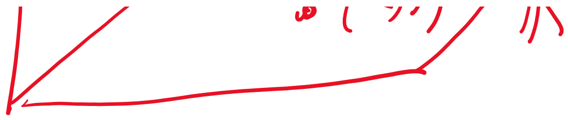
$$v \longmapsto F(v) = w$$

$$\underbrace{\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}}_{V} \qquad \underbrace{\begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix} = A \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}}_{W}$$

$$F : \overset{V}{\overset{\shortparallel}{\mathbb{R}^3}} \to \overset{W}{\overset{\shortparallel}{\mathbb{R}^2}}$$

$$F(x, y, z) = (x, y)$$





F è una map. lineare.

$$B_{\mathbb{R}^3} = \left((1,0,0), \underline{(0,1,0)}, \underline{(0,0,1)} \right)$$

$$B_{\mathbb{R}^2} = \left((1,0), (0,1) \right)$$

$$\begin{array}{c} A \\ \parallel \\ M_{B_V B_W}(F) \end{array} = \begin{pmatrix} \boxed{\begin{matrix} 1 \\ 0 \end{matrix}} & \boxed{\begin{matrix} 0 \\ 1 \end{matrix}} & \boxed{\begin{matrix} 0 \\ 0 \end{matrix}} \end{pmatrix}$$

coord. di $F(e_1)$

$\parallel$

$(1,0) = 1 e_1 + 0 e_2$

coord. di $F(e_2)$

$\parallel$

coord. di $F(e_3)$

$$\underline{F(\overset{v}{x,y,z}) = \overset{w}{(x,y)}}$$

$$\underbrace{\overset{A}{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}}}_{2\times 3} \underbrace{\begin{pmatrix} x \\ y \\ z \end{pmatrix}}_{3\times 1} = \begin{pmatrix} x \\ y \end{pmatrix}$$

2×1