Rappr. param.

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = t_1 \underset{v_1}{\begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{pmatrix}} + t_2 \underset{v_2}{\begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{n2} \end{pmatrix}} + \dots + t_k \underset{v_k}{\begin{pmatrix} a_{1k} \\ a_{2k} \\ \vdots \\ a_{nk} \end{pmatrix}}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = s \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} + t \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \qquad \begin{cases} x = s \\ y = 2s \\ z = 4s + t \end{cases}$$

$$\boxed{U = \mathrm{Span}(v_1, \dots, v_k)}$$

Rappr. Cart.

$$AX = 0$$

$$\begin{cases} \_\_\_\_\_\_ = 0 \\ \_\_\_\_\_\_ = 0 \\ \\ \_\_\_\_\_\_ = 0 \end{cases}$$

Isomorfismi di spazi vettoriali

Spazio vettoriale $M_{2\times 2}(\mathbb{R})$

$\{(1 \ \pi)\} \in M \ (\mathbb{R})$

$$\left\{\begin{pmatrix}1&\pi\\0&7\end{pmatrix}\right\} \in M_{2\times 2}(\mathbb{R})$$

$$(\mathbb{R}, \mathbb{R}^4, +, \cdot)$$

$$(2.7, \sqrt{2}, 1, 0) \in \mathbb{R}^4$$

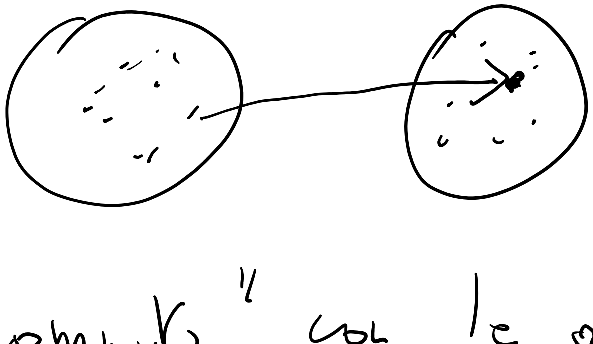
Consideriamo questa funzione

$$f: M_{2\times 2}(\mathbb{R}) \to \mathbb{R}^4$$

$$\boxed{f\left(\begin{pmatrix}a&b\\c&d\end{pmatrix}\right) := (a,b,c,d)}$$

$$f\left(\begin{pmatrix}1&\pi\\0&7\end{pmatrix}\right) = (1,\pi,0,7)$$

f è biunivoca (iniettiva e suriettiva)



f "commuta" con le operazioni di somma e prodotto per uno scalare.

$$(\mathbb{R}, V, +, \cdot) \qquad (\mathbb{R}, W, \oplus, \odot)$$

$$f(v_1+v_2) = f(v_1) \oplus f(v_2)$$

$$f(\lambda v) = \lambda \odot f(v)$$

$$\begin{tikzcd} v_1 \quad v_2 \arrow[d, "+"] & f(v_1) \quad f(v_2) \arrow[d, "\oplus"] \\ v \arrow[r, "f"] & f(v) \end{tikzcd}$$

$$(x+y)^2$$
$$\neq$$
$$x^2+y^2$$

$$f(\alpha v_1 + \beta v_2) = \alpha f(v_1) + \beta f(v_2)$$

$$\forall v_1, v_2 \in V \quad \forall \alpha, \beta \in \mathbb{R}$$

Si dice allora che f è un isomorfismo (di spazi vettoriali)

$$f\left(\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} + \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}\right) \overset{?}{=} f\left(\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}\right) + f\left(\begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}\right)$$

$f\left(\begin{pmatrix} c_1 & d_1 \end{pmatrix} + \begin{pmatrix} c_2 & d_2 \end{pmatrix}\right) \qquad f(\ldots) + f(\ldots)$

$$= f\left(\begin{pmatrix} a_1+a_2 & b_1+b_2 \\ c_1+c_2 & d_1+d_2 \end{pmatrix}\right) = (a_1+a_2,\, b_1+b_2,\, c_1+c_2,\, d_1+d_2)$$

$$(a_1, b_1, c_1, d_1) + (a_2, b_2, c_2, d_2) = (a_1+a_2,\, b_1+b_2,\, c_1+c_2,\, d_1+d_2)$$

$$f\left(\lambda \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) \overset{?}{=} \lambda \cdot f\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)$$

$$f\left(\begin{pmatrix} \lambda a & \lambda b \\ \lambda c & \lambda d \end{pmatrix}\right) = (\lambda a, \lambda b, \lambda c, \lambda d)$$

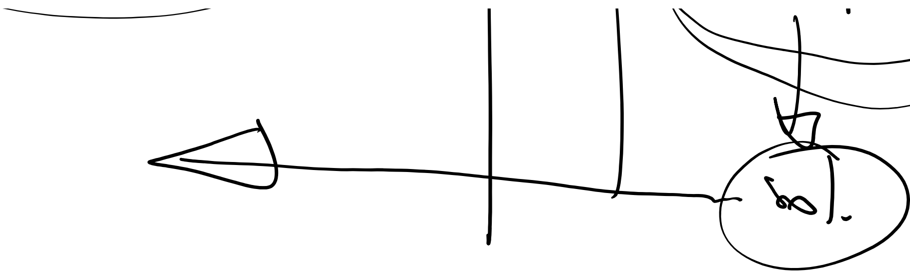
$$\lambda \cdot (a, b, c, d) = (\lambda a, \lambda b, \lambda c, \lambda d)$$

Quindi f è un isomorfismo da $M_{2\times 2}(\mathbb{R})$ a $\mathbb{R}^4$

numero di conigli $\longrightarrow$ c

" polli $\longrightarrow$ p

problema $\longrightarrow$ $\begin{cases} c+p = \ldots \\ 4c+2p = \ldots \end{cases}$



Def. Un omomorfismo (detto anche trasformazione lineare) tra due spazi vettoriali è un isomorfismo senza la richiesta di biunivocità.

NB: Fra gli omomorfismi $f: V \to W$ c'è n'è uno particolare: l'omomorfismo nullo:

$$f(v) = 0_W$$

Un esempio di una coppia di spazi vettoriali (V, W) tali che non ci sia nessun isomorfismo $f: V \to W$:

$$(\mathbb{R}^2, \mathbb{R}^3)$$

$\dim \mathbb{R}^2 = 2$ $\quad$ $\dim \mathbb{R}^3 = 3$

$(\ldots)$

$\dim \mathbb{R}^2 = 2 \qquad \dim \mathbb{R}^3 = 3$

C'è un isomorfismo "standard" che viene costruito quando è nota una base di V:
$\mathcal{B} = (v_1, \ldots, v_n)$

$$\Phi_{\mathcal{B}} : V \longrightarrow \mathbb{R}^n$$

Si definisce così:
sia $v \in V$. Sappiamo che v si può scrivere in modo unico come $x_1 v_1 + \ldots + x_n v_n$.

coord. di v risp. a $\mathcal{B}$.

$$v = x_1 v_1 + \ldots + x_n v_n$$

$$v = y_1 v_1 + \ldots + y_n v_n$$

$$\Downarrow$$

$$\underline{0} = \underline{(x_1 - y_1) v_1 + \ldots + (x_n - y_n) v_n}$$

$$\Downarrow$$

$x_1 = y_1$
$\vdots$
$x_n = y_n$

$$\Phi_{\mathcal{B}}(v) = \underline{(x_1, \ldots, x_n)}$$

Esempio: $M_{2\times 2}(\mathbb{R})$

$$B = \left(\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}_{E_{11}}, \underbrace{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}_{E_{12}}, \underbrace{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}}_{E_{21}}, \underbrace{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}}_{E_{22}} \right)$$

$$\Phi_B\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = (a, b, c, d)$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \underbrace{a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}_{\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}} + \underbrace{b\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}_{\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}} + \underbrace{c\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}}_{\begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix}} + \underbrace{d\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}}_{\begin{pmatrix} 0 & 0 \\ 0 & d \end{pmatrix}}$$

Prendiamo ora una base B' di $M_{2\times 2}(\mathbb{R})$ più "esotica".

$$B' = \left(\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right)$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = x\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + y\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} + z\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} + u\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} x+y+z+u & x+y+z \\ x+y & x \end{pmatrix}$$

$$\Updownarrow$$

$$\begin{cases} x+y+z+u = a \\ x+y+z = b \\ x+y = c \\ x = d \end{cases} \qquad \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & a \\ 1 & 1 & 1 & 0 & b \\ 1 & 1 & 0 & 0 & c \\ 1 & 0 & 0 & 0 & d \end{array}\right)$$

$$\begin{cases} x+y = c \\ x = d \end{cases} \qquad \left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & c \\ 1 & 0 & 0 & 0 & d \end{array}\right)$$

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & d \\ 1 & 1 & 0 & 0 & c \\ 1 & 1 & 1 & 0 & b \\ 1 & 1 & 1 & 1 & d \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & d \\ 0 & 1 & 0 & 0 & c-d \\ 0 & 1 & 1 & 0 & b-d \\ 0 & 1 & 1 & 1 & d-d \end{array}\right)$$

$$\to \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & d \\ 0 & 1 & 0 & 0 & c-d \\ 0 & 0 & 1 & 0 & b-c \\ 0 & 0 & 1 & 1 & a-c \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & d \\ 0 & 1 & 0 & 0 & c-d \\ 0 & 0 & 1 & 0 & b-c \\ 0 & 0 & 0 & 1 & a-b \end{array}\right)$$

$$\boxed{\begin{cases} x = d \\ y = c-d \\ z = b-c \\ u = a-b \end{cases}}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = d\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + (c-d)\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} + (b-c)\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} + (a-b)\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} \overset{a}{\overset{\shortparallel}{0}} & \overset{b}{\overset{\shortparallel}{0}} \\ \underset{c}{\underset{\shortparallel}{0}} & \underset{d}{\underset{\shortparallel}{0}} \end{pmatrix} = x\underline{\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}} + y\underline{\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}} + z\underline{\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}} + u\underline{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}$$

$$\begin{cases} x = 0 \\ y = 0 \\ z = 0 \\ u = 0 \end{cases}$$

$$\Phi_{B'}\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = (d, c-d, b-c, a-b)$$

Prendiamo lo spazio vettoriale $A(3)$

$$\left\{ \begin{pmatrix} 0 & x & y \\ \ldots \end{pmatrix} \right.$$

$$A(3)=\left\{\begin{pmatrix}0 & x & y\\ -x & 0 & z\\ -y & -z & 0\end{pmatrix} : x,y,z\in\mathbb{R}\right\}$$

$$\begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix}$$ non è ~~antisim~~ antisimmetrica!

E_{11}

$$\begin{pmatrix}0 & x & y\\ -x & 0 & z\\ -y & -z & 0\end{pmatrix}=\begin{pmatrix}0 & x & 0\\ -x & 0 & 0\\ 0 & 0 & 0\end{pmatrix}+\begin{pmatrix}0 & 0 & y\\ 0 & 0 & 0\\ -y & 0 & 0\end{pmatrix}+\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & z\\ 0 & -z & 0\end{pmatrix}$$

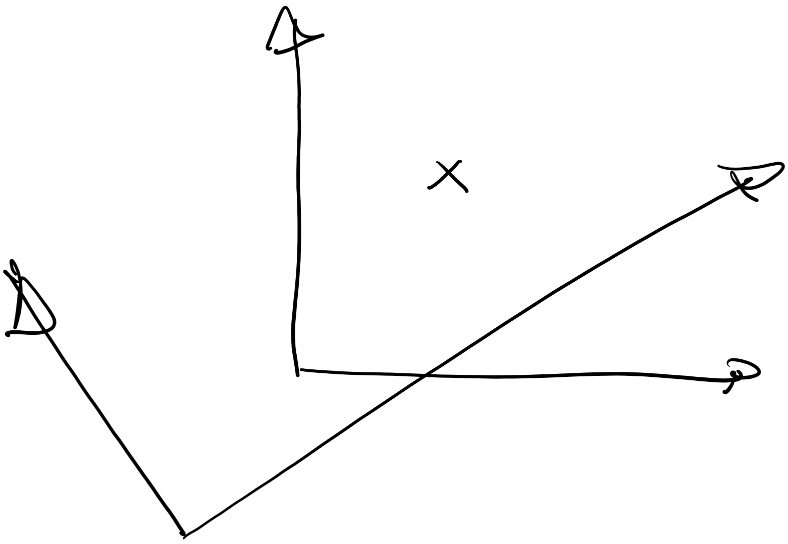
$$=\underline{x}\begin{pmatrix}0&1&0\\-1&0&0\\0&0&0\end{pmatrix}+\underline{y}\begin{pmatrix}0&0&1\\0&0&0\\-1&0&0\end{pmatrix}+\underline{z}\begin{pmatrix}0&0&0\\0&0&1\\0&-1&0\end{pmatrix}$$

$$B=\left(\underline{\begin{pmatrix}0&1&0\\-1&0&0\\0&0&0\end{pmatrix}},\underline{\begin{pmatrix}0&0&1\\0&0&0\\-1&0&0\end{pmatrix}},\underline{\begin{pmatrix}0&0&0\\0&0&1\\0&-1&0\end{pmatrix}}\right)$$ è

una base per lo s.v. $A(3)$.

$$\Phi_B : A(3)\to\mathbb{R}^3$$

$$\underline{\Phi_B\left(\begin{pmatrix}0 & x & y\\ -x & 0 & z\\ -y & -z & 0\end{pmatrix}\right)=(x,y,z)}$$



Calcolare le coordinate del vettore $\begin{pmatrix}1 & 4\\ 0 & 0\end{pmatrix} \in M_{2\times 2}(\mathbb{R})$ rispetto alla base

$$\begin{pmatrix}1 & 2 & 1 & 0\\ 1 & 1 & 1 & 1\\ 1 & 0 & 0 & 1\\ 0 & 0 & 0 & 1\end{pmatrix} \longrightarrow \begin{pmatrix}1 & 0 & 0 & 1\\ 1 & 1 & 1 & 1\\ 1 & 2 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 2 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \rightarrow \boxed{\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}}$$

$$\rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \rightarrow \boxed{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}$$

$$B = \left(\begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right)$$

$$\begin{pmatrix} 1 & 4 \\ 0 & 0 \end{pmatrix} = x \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix} + y \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + z \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + u \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} x+y+z & 2x+y \\ x+y & y+z+u \end{pmatrix}$$

$$\begin{cases} x+y+z = 1 \\ 2x+y = 4 \\ x+y = 0 \\ y+z+u = 0 \end{cases}$$

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 1 \\ 2 & 1 & 0 & 0 & 4 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{array}\right)$$

$$\left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 2 & 1 & 0 & 0 & 4 \\ 0 & 1 & 1 & 1 & 0 \end{array}\right)$$

$$\left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & -1 & 0 & 0 & 4 \\ 0 & 1 & 1 & 1 & 0 \end{array}\right)$$

$$\left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{array}\right)$$

$$\left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 4 \end{array}\right)$$

$$\left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 3 \end{array}\right)$$

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 & -4 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 3 \end{array}\right) \Longrightarrow \begin{cases} x = 4 \\ y = -4 \\ z = 1 \\ u = 3 \end{cases}$$

$$\begin{pmatrix} 1 & 4 \\ 0 & 0 \end{pmatrix} \underset{B}{\equiv} (4, -4, 1, 3)$$

Prendiamo ora $V = \mathbb{R}^3[x]$

$$3x^3 - 2x^2 + 7$$

$$x^2 + \pi$$

$$x^2+\pi$$

$$S = \left(\boxed{x^3+x^2+x+1},\ x^2-x,\ x^3+3x^2-x+1,\ 2 \right)$$

S è una base per V ?

$$B = \left(\underset{v_1}{1}, \underset{v_2}{x}, \underset{v_3}{x^2}, \underset{v_4}{x^3} \right) \qquad x^3+x^2+x+1 = \underline{1v_1+1v_2+1v_3+1v_4}$$

$$\Phi_B\left(x^3+x^2+x+1\right) = (1,1,1,1)$$

$$\Phi_B\left(x^2-x\right) = \underline{(0,-1,1,0)}$$

$$\| $$

$$0\cdot 1+(-1)x+1x^2+0x^3$$

$$\Phi_B\left(x^3+3x^2-x+1\right) = (1,-1,3,1)$$

$$\Phi_B(2) = (2,0,0,0)$$

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & 1 & 0 \\ 1 & -1 & 3 & 1 \\ 2 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & -1 & 1 & 0 \\ 1 & -1 & 3 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 3 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 4 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1/2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 1/2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

card $\mathbb{Q}$ = card $\mathbb{N}$

$$\begin{array}{cccccccccccccc}
1 & \frac{1}{2} & \frac{1}{3} & \frac{2}{3} & \frac{1}{4} & \frac{3}{4} & \frac{1}{5} & \frac{2}{5} & \frac{3}{5} & \frac{4}{5} & \frac{1}{6} & \frac{5}{6} & \dots \\
2 & \frac{3}{2} & \frac{4}{3} & \frac{5}{3} & \dots \\
0 & -\frac{1}{2} & -\frac{2}{3} & -\frac{1}{3} & \dots \\
3 & \frac{5}{2} & \frac{7}{3} & \dots \\
-1 & -\frac{3}{2} & \dots
\end{array}$$