

Tavole applicative

Corso di Controllo dei Robot

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Table of contents

Delta robot

Direct kinematic

Inverse kinematic

Dynamic

Working volume

Control

PD with gravity compensation

Computed torque

Backstepping

Adaptive backstepping

Ball and plate

Dynamic

Structural properties

Controllability

Observability

Feedback linearization

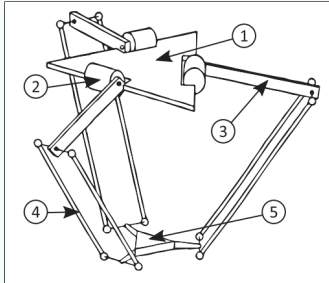
Control

Delta robot

The Delta robot is a 3-DOF parallel kinematic machine developed by Reymond Clavel¹ in 1991. It mainly consists of three actuated kinematic chains linked at a common moving platform. Each chain is a serial connection of a revolute actuator, a rear-arm and a forearm (composed of two parallel rods forming a parallelogram). The rear-arms and the forearms are linked through ball-and-socket passive joints. The parallelogram structure of the forearms ensures that the moving platform stays always parallel to the fixed base. Figure 1 shows a schematic view of the Delta robot with its main elements highlighted.

¹Reymond Clavel. *Conception d'un robot parallele rapide à 4 degres de liberté*. 1991.

Delta robot - Schematic view



1. Fixed base-plate
2. Actuator
3. Rear-arm
4. Forearm
5. Moving platform

Figure 1: Schematic view of Delta robot

We consider a model with a ternary symmetric configuration with three kinematic chains disposed with a period of 120° .

Delta robot - Parameters

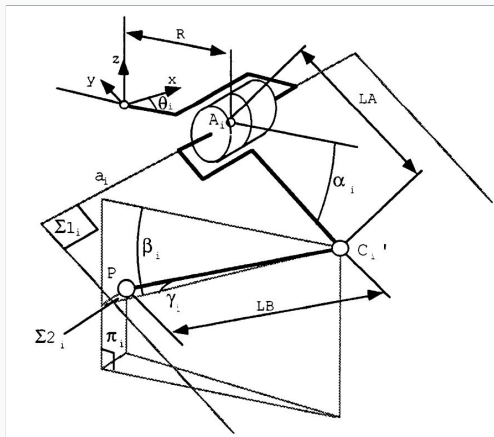


Figure 2: Delta robot length parameters and characteristic angles

Delta robot - Parameters

Parameter	Description	Value
l_A	Rear-arm length	$0.2m$
m_A	Rear-arm mass	$0.1Kg$
R	Base platform dimension	$0.126m$
l_B	Forearm length	$0.4m$
m_B	Forearm mass	$0.045Kg$
m_c	Elbow mass	$0.018Kg$
m_n	Moving platform mass	$0.1Kg$
I_{bi}	Rear-arm inertia	$Kg \times m^2$

Table 1: Delta robot geometric and dynamic parameters

Analytical studies on the working volume of the Delta robot² showed that:

- A ratio $r = R/l_A < 0.63$ gives the most regular shape for the surface of the lower part of the working volume.
- If $r > 0.0484$ and $b = l_A/l_B > 1.75$ there is no singularity occurrence within the robot working volume.

Thus the parameters shown in table 1 have been chosen for the Delta model used in this project.

²L Rey and Reymond Clavel. "The Delta Parallel Robot". In: *Parallel Kinematic Machines. Advanced Manufacturing*. Springer, London (1999).

Delta robot - Reference system and state variables

The position of the End-effector

$$(x, y, z)^T$$

is described in a reference frame fixed to the base plate, as shown in figure 2.

The angles α_i of the actuated joints have been selected as state-variables to describe the robot dynamic:

$$q = (\alpha_1, \alpha_2, \alpha_3)^T$$

Since the moving platform is only translating we can study the model in figure 2 without loss of generality.

In this model the moving platform is reduced to an ideal point with a translation of the three kinematic chains.

Direct kinematic is found following the method presented by Clavel in 1991.

Taking in mind the Delta robot representation of figure 2 one can simply find that C_i coordinates are given by the intersection of three circles of radius L_A belonging to the plane π_i and the sphere centred in P having radius L_B . Those conditions give a three equations system that can be solved to find the coordinates of the end-effector.

Coordinates of the point C_i in the base frame:

$$C_i = \begin{pmatrix} (R + L_A \cos \alpha_i) \cos \theta_i \\ (R + L_A \cos \alpha_i) \sin \theta_i \\ -L_A \sin \alpha_i \end{pmatrix} \quad (1)$$

Equation of the sphere centred in P:

$$\left((R + L_A \cos \alpha_i) \cos \theta_i - x \right)^2 + \left((R + L_A \cos \alpha_i) \sin \theta_i - y \right)^2 + (L_A \sin \alpha_i + z)^2 = L_B^2 \quad (2)$$

The system has two possible solutions. The one with negative z coordinate that belongs to the Delta robot workspace is selected.

The inverse kinematic model let calculate the joint angles q_i as functions of the position of the end effector. The model here presented has been developed by Codourey³ and has the advantage of removing the points of singularity contained in the model previously introduced by Clavel.

The rationale is still the intersection of a sphere and three circles but the computation is made for each angle in a frame centred in the centre of the $i - th$ joint and rotated with respect to the base frame of an angle θ_i .

³Alain Codourey. "Contribution à la commande des robots rapides et précis application au robot delta à entraînement direct". In: (1991), p. 188. DOI: 10.5075/epfl-thesis-922. URL: <http://infoscience.epfl.ch/record/31400>.

Delta robot - Dynamic model assumptions

- Ideal joints are considered.
- The rotational inertia of the forearm is neglected.
- The mass of each forearm is split up into two point-masses located at both ends of the forearm.

Delta robot - Dynamic model

We express the dynamic of the delta robot in classic matrix formulation:

$$\tau = M(q)\ddot{q} + C(q, \dot{q})\dot{q} + G(q) \quad (3)$$

Where:

$$M(q) = (I_b + m_{nt}J^T J), \quad C(q, \dot{q}) = (J^T m_{nt}J), \quad G(q) = -\Gamma_{Gb} - \Gamma_{Gn} \quad (4)$$

- I_b is the inertia matrix of the arms in joint space.
- m_{nt} is the total mass acting on the travelling plate.
- J is the Jacobian matrix.
- Γ_{Gn} is the gravity force acting on the moving platform.
- Γ_{Gb} is the gravity force acting on the rear-arms.

Delta robot - Working volume

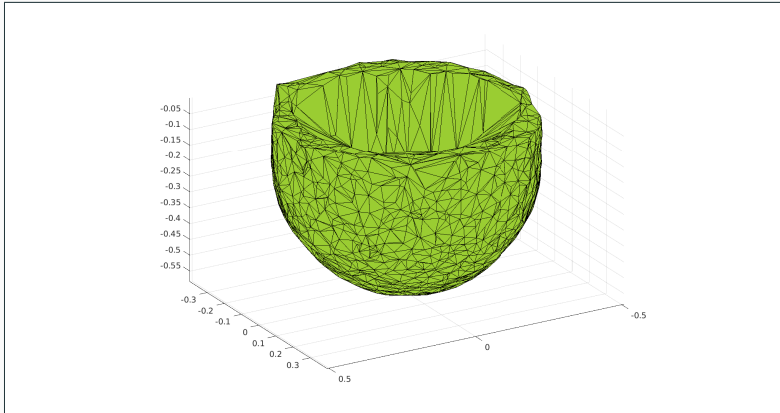


Figure 3: A convex hull of the workspace of the Delta robot

In figure 3 a convex hull of the workspace of the Delta robot is reported. The surface has been generated as an α - *shape*⁴ with $r_\alpha = 0.2$. The geometric figure gives an analytical instrument to validate a sound reference trajectory generation for the Delta kinematic.

⁴H. Edelsbrunner, D. Kirkpatrick, and R. Seidel. "On the Shape of a Set of Points in the Plane". In: *IEEE Trans. Inf. Theor.* 29.4 (Sept. 2006), pp. 551–559. ISSN: 0018-9448. DOI: 10.1109/TIT.1983.1056714. URL: <http://dx.doi.org/10.1109/TIT.1983.1056714>.

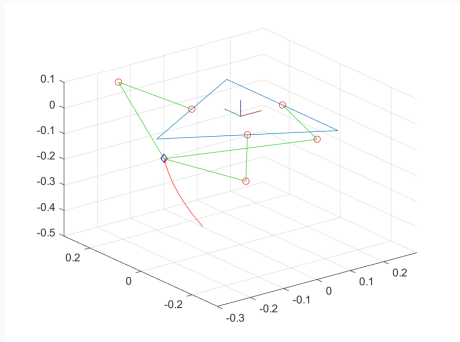
Control equation:

$$\tau_{PD} = K_P e + K_D \dot{e} + G(q) \quad (5)$$

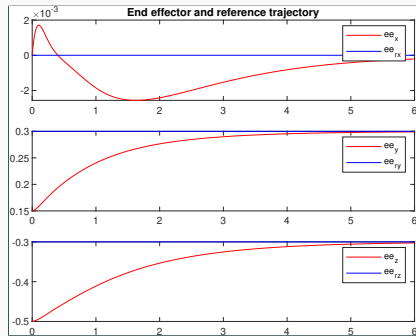
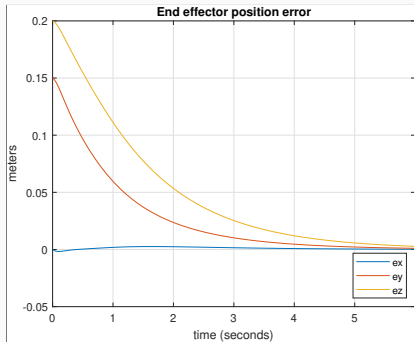
with

$$K_P = 1500, \quad K_D = 60$$

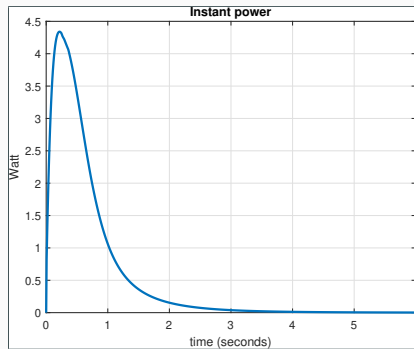
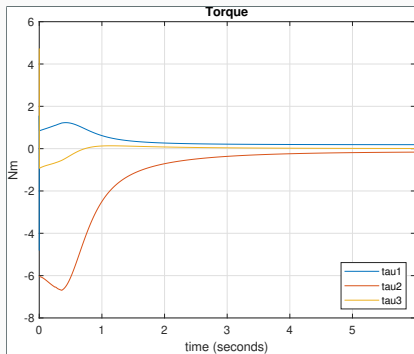
Descrizione traiettoria



PD with gravity compensation



PD with gravity compensation



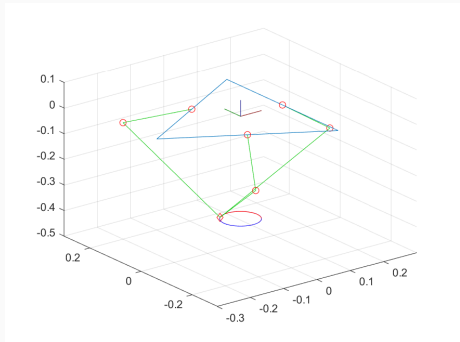
Control equation:

$$\tau_{CT} = M(q)\ddot{q}_d + C(q, \dot{q})\dot{q} + G(q) + K_p e + K_v \dot{e} \quad (6)$$

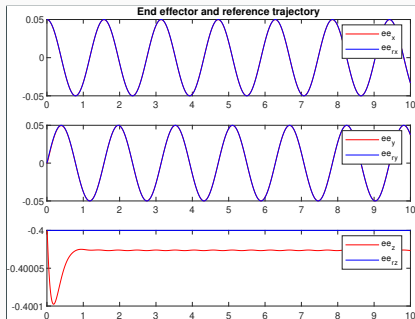
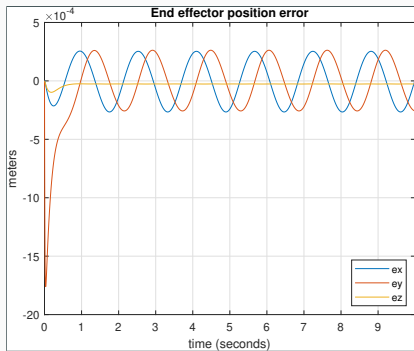
with

$$K_P = 500, K_D = 100$$

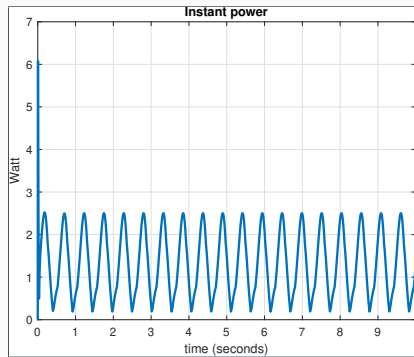
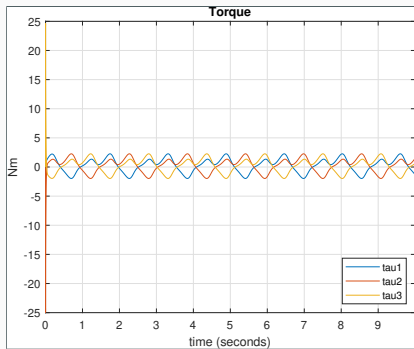
Descrizione traiettoria



Computed torque



Computed torque



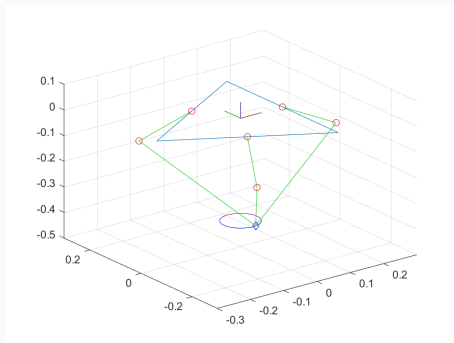
Control equation:

$$\tau_{BS} = M(q)\ddot{q}_r + C(q, \dot{q})\dot{q}_r + G(q) - K_d s + J^T e \quad (7)$$

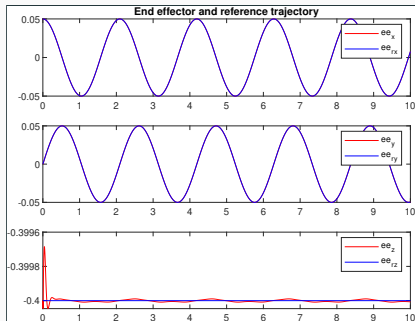
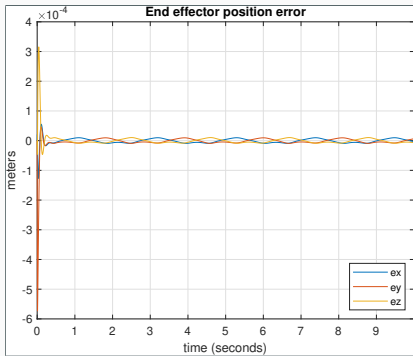
with

$$\ddot{q}_r = \ddot{q}_d - \Lambda \dot{e}, \quad \dot{q}_r = \dot{q}_d - \Lambda e, \quad s = \dot{q} - \dot{q}_r, \quad K_d = 50, \quad \Lambda = 400$$

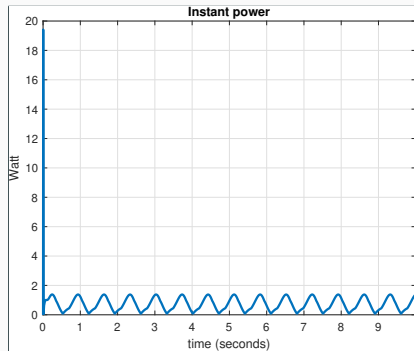
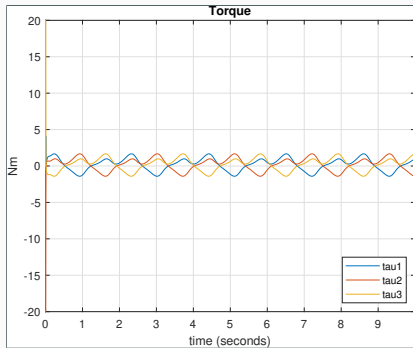
Descrizione traiettoria



Backstepping



Backstepping



Adaptive backstepping

Ball and plate

Ball and plate

Ball and plate

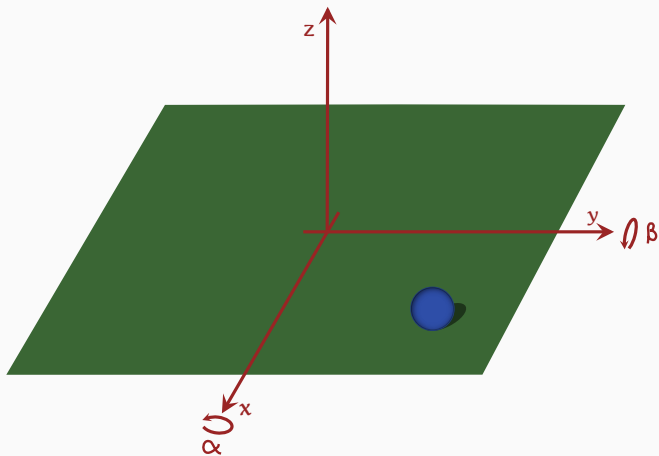


Figure 4: Coordinate frame of the ball and plate system

Ball and plate - Parameters

Parameter	Description	Value
m	Mass of the ball	0.0109 Kg
r	Radius of the ball	0.01 m
I_b	Ball inertia	$4.3563e^{-7} \text{ Kg} \times \text{m}^2$
l_p	Plate side	0.6 m
I_p	Plate inertia	$0.175 \text{ Kg} \times \text{m}^2$

Table 2: Ball and plate geometric and dynamic parameters

Ball and plate - Dynamic model

The general form of Euler-Lagrange for dynamic equations is used to describe the system:

$$\frac{d}{dt} \frac{\delta T}{\delta \dot{q}_i} - \frac{\delta T}{\delta q_i} + \frac{\delta V}{\delta q_i} = Q_i \quad (8)$$

Where T is the kinetic energy, V is the potential energy, Q_i is the i -th generalized force and q_i is the i -th generalized coordinate. As generalized force we consider two torques acting on the plate ($Q_\alpha = \tau_\alpha$, $Q_\beta = \tau_\beta$). As generalized coordinates we select two ball position coordinates $[x, y]$ on the frame fixed to the plate and two plate inclination $[\alpha, \beta]$.

Ball and plate - Dynamic model

Kinetic energy of the ball:

$$T_b = \frac{1}{2}mv^2 + \frac{1}{2}I_b\omega^2 = \frac{1}{2} \left(m + \frac{I_b}{r^2} \right) (\dot{x}^2 + \dot{y}^2) \quad (9)$$

Kinetic energy of the plate:

$$T_p = \frac{1}{2} (I_b + I_p) (\dot{\alpha} + \dot{\beta})^2 + \frac{1}{2}m (\dot{\alpha}x + \dot{\beta}y)^2 \quad (10)$$

Potential energy:

$$V = mgh = mg(x \sin\alpha + y \sin\beta) \quad (11)$$

Ball and plate - Dynamic model

After some derivations we find the following non-linear system of equations:

$$\begin{aligned}\left(m + \frac{I_b}{r^2}\right) \ddot{x} - m \left(\dot{\alpha}\dot{\beta}y + \dot{\alpha}^2x\right) + mg \sin\alpha &= 0 \\ \left(m + \frac{I_b}{r^2}\right) \ddot{y} - m \left(\dot{\alpha}\dot{\beta}x + \dot{\beta}^2y\right) + mg \sin\beta &= 0 \\ (I_p + I_b + mx^2) \ddot{\alpha} + m \left(\ddot{\beta}xy + \dot{\beta}(\dot{x}y + x\dot{y}) + 2\dot{\alpha}\dot{x}x\right) + mgx \cos\alpha &= \tau_\alpha \\ (I_p + I_b + my^2) \ddot{\beta} + m \left(\ddot{\alpha}xy + \dot{\alpha}(\dot{x}y + x\dot{y}) + 2\dot{\beta}\dot{y}y\right) + mgy \cos\beta &= \tau_\beta\end{aligned}\tag{12}$$

Ball and plate - Dynamic model

We express the dynamic in matrix form:

$$M(q) = \begin{pmatrix} (m + \frac{l_b}{r^2}) & 0 & 0 & 0 \\ 0 & (m + \frac{l_b}{r^2}) & 0 & 0 \\ 0 & 0 & (l_b + l_p + mx^2) & mxy \\ 0 & 0 & mxy & (l_b + l_p + my^2) \end{pmatrix}$$
$$C(q, \dot{q}) = \begin{pmatrix} 0 & 0 & -\dot{\alpha}x & -\dot{\alpha}y \\ 0 & 0 & -\dot{\beta}x & -\dot{\beta}y \\ 2\dot{\alpha}x & 0 & 0 & (\dot{x}y + x\dot{y}) \\ 0 & 2\dot{\beta}y & (\dot{x}y + x\dot{y}) & 0 \end{pmatrix}$$
$$G(q) = \begin{pmatrix} mg \sin \alpha \\ mg \sin \beta \\ mgx \cos \alpha \\ mgx \cos \beta \end{pmatrix}$$

Affine-in-control formulation:

$$\dot{x} = \begin{pmatrix} x_5 \\ x_6 \\ x_7 \\ x_8 \\ -B(q)^{-1}(C(q, \dot{q})\dot{q} + G(q)) \end{pmatrix} + \begin{pmatrix} 0_{4 \times 2} \\ B(q)^{-1} \end{pmatrix} \begin{pmatrix} \tau_1 \\ \tau_2 \end{pmatrix} \quad (13)$$

Where

$$x = (x_b, y_b, \alpha, \beta, \dot{x}_b, \dot{y}_b, \dot{\alpha}, \dot{\beta})^T$$

Ball and plate - Change of coordinates

In order to simplify the analysis of the structural properties of the Ball and plate system, the following change of coordinates is adopted:

$$u_1 = 2mx\dot{\alpha} - mgx \cos\alpha - (I_p + I_b + mx^2) \ddot{\alpha} - m\dot{\beta} (\dot{x}y + \dot{y}x) - 2m\dot{\alpha}\dot{x}x$$

$$u_2 = 2my\dot{\beta} - mgy \cos\beta - (I_p + I_b + my^2) \ddot{\beta} - m\dot{\alpha} (\dot{x}y + \dot{y}x) - 2m\dot{\beta}\dot{y}y$$

Ball and plate - Change of coordinates

We obtain the following system in affine form:

$$\dot{x} = \begin{pmatrix} x_5 \\ x_6 \\ x_7 \\ x_8 \\ \mathcal{E}(x_7 x_8 x_2 + x_3^2 x_1 - g \sin x_3) \\ \mathcal{E}(x_7 x_8 x_1 + x_3^2 x_2 - g \sin x_4) \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \quad (14)$$

Where

$$\mathcal{E} = \frac{mr_b^2}{mr_b^2 + I_b}$$

Chow theorem. If the accessibility distribution $\langle \Delta, \Delta_0 \rangle = n$ in x_0 then the system is said to be locally accessible in x_0 .

Where $\Delta_0 = \text{span} \{g_1, g_2, \dots, g_d\}$ and $\Delta = \text{span} \{f, g_1, g_2, \dots, g_d\}$.

We build then the matrix $Q(x)$ as:

$$Q(x) = (g_1, g_2, \text{ad}_f g_1, \text{ad}_f g_2, \dots, \text{ad}_f^{n-1} g_1, \text{ad}_f^{n-1} g_2) \quad (15)$$

And we evaluate its rank on the state space.

$$Q(x) = \begin{pmatrix} 0 & 0 & \star & \star & \star & \star & \star & \star & 0 & 0 & \star & \star & \star & \star & \star & \star \\ 0 & 0 & \star & \star & \star & \star & \star & \star & 0 & 0 & \star & \star & \star & \star & \star & \star \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \star & \star & \star & \star & \star & \star & \star & 0 & \star & \star & \star & \star & \star & \star & \star \\ 0 & \star & \star & \star & \star & \star & \star & \star & 0 & \star & \star & \star & \star & \star & \star & \star \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Where the \star elements represent the non constant terms of $Q(x)$ matrix.

In order to calculate $\text{rank}(Q(x))$ we perform the following columns and rows swapping:

row 1 \longleftrightarrow row 7

column 2 \longleftrightarrow column 3

column 10 \longleftrightarrow column 4

column 9 \longleftrightarrow column 8

column 2 \longleftrightarrow column 12

column 2 \longleftrightarrow column 2

columns 5, 6, 7 \longleftrightarrow columns 10, 11, 12

We obtain the matrix $\tilde{Q}(x)$ as follows:

$$\tilde{Q}(x) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & \tilde{Q}_{22} & 0 & 0 & \star & \star & \star & 0 & \dots \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & \dots \\ 0 & \star & \star & \star & & & & 0 & \dots \\ 0 & \star & \star & \star & & & & 0 & \dots \\ 0 & \star & 0 & 0 & & & & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \dots \end{pmatrix}_{8 \times 16} \quad (16)$$

Where

$$\tilde{Q}_{22} = 2B_{x_5 x_7} + 4B_{x_6 x_8} + Bg \cos x_4$$

Rank condition. To evaluate the rank of $\tilde{Q}(x_0)$ is sufficient to look at rank deficiency conditions of \tilde{Q}_{22} and \mathfrak{J} . The following cases are obtained:

- $x_4 \equiv \pi/2 \wedge (x_8 \equiv 0 \vee x_6 \equiv 0) \wedge (x_7 \equiv 0 \vee x_5 \equiv 0) \implies \tilde{Q}_{22} \equiv 0 \implies \text{rank}(Q) < 8.$

Which represent the physical condition where $\beta = \pi/2$ or $\alpha = \pi/2$ with null angular velocity, the system is out of the range of interest.

- $(x_7 \equiv 0 \wedge x_8 \equiv 0 \wedge (x_3 \equiv \pi/2 \vee x_4 \equiv \pi/2)) \vee ((x_1 \equiv 0 \wedge x_2 \equiv 0 \vee x_5 \equiv 0 \vee x_6 \equiv 0) \wedge (x_3 \equiv \pi/2 \vee x_4 \equiv \pi/2)) \implies \text{rank}(\mathfrak{J}) < 3 \implies \text{rank}(Q) < 8.$

Which represent the physical condition where $\beta = \pi/2$ or $\alpha = \pi/2$ with null angular velocity; the same arguments as above hold.

- In all other cases we find $\text{rank}(Q(x_0)) \equiv 8$ and rank condition of controllability is satisfied.

Given the observation space \mathcal{O} as the space containing all the repeated Lie-derivatives:

$$\mathcal{O} = \{h(\bar{x}), L_f h(\bar{x}), \dots, L_{g_i} L_f h(\bar{x}), \dots\}$$

The system results locally observable if $\dim(d\mathcal{O}) = n$, where $d\mathcal{O}$ is the observability codistribution:

$$d\mathcal{O} = \left\{ \frac{\partial h(\bar{x})}{\partial x}, \frac{\partial L_f h(\bar{x})}{\partial x}, \dots, \frac{\partial L_{g_i} L_f h(\bar{x})}{\partial x}, \dots \right\}$$

Observability

$$d\mathcal{O} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ d\mathcal{O}_{51} & d\mathcal{O}_{52} & d\mathcal{O}_{53} & 0 & 0 & 0 & d\mathcal{O}_{57} & d\mathcal{O}_{58} \\ 0 & 0 & d\mathcal{O}_{63} & 0 & 0 & 0 & d\mathcal{O}_{67} & d\mathcal{O}_{68} \\ 0 & 0 & d\mathcal{O}_{73} & 0 & d\mathcal{O}_{75} & d\mathcal{O}_{76} & d\mathcal{O}_{77} & d\mathcal{O}_{78} \\ 0 & 0 & 0 & d\mathcal{O}_{84} & d\mathcal{O}_{85} & d\mathcal{O}_{86} & d\mathcal{O}_{87} & d\mathcal{O}_{88} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}_{48 \times 8}$$

Observability

In order to calculate $\text{rank}(d\mathcal{O})$ we perform the following columns and rows swapping:

$$\text{columns } 5, 6 \longleftrightarrow \text{columns } 3, 4$$

We obtain the following matrix:

$$d\tilde{\mathcal{O}} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ d\tilde{\mathcal{O}}_{51} & d\tilde{\mathcal{O}}_{52} & 0 & 0 & d\tilde{\mathcal{O}}_{55} & 0 & d\tilde{\mathcal{O}}_{57} & d\tilde{\mathcal{O}}_{58} \\ d\tilde{\mathcal{O}}_{61} & d\tilde{\mathcal{O}}_{62} & 0 & 0 & 0 & d\tilde{\mathcal{O}}_{66} & d\tilde{\mathcal{O}}_{67} & d\tilde{\mathcal{O}}_{68} \\ 0 & 0 & d\tilde{\mathcal{O}}_{73} & d\tilde{\mathcal{O}}_{74} & d\tilde{\mathcal{O}}_{75} & 0 & d\tilde{\mathcal{O}}_{77} & d\tilde{\mathcal{O}}_{78} \\ 0 & 0 & d\tilde{\mathcal{O}}_{83} & d\tilde{\mathcal{O}}_{84} & 0 & d\tilde{\mathcal{O}}_{86} & d\tilde{\mathcal{O}}_{87} & d\tilde{\mathcal{O}}_{88} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}_{48 \times 8}$$

Where:

$$d\mathcal{O}_{51} = \mathcal{E}x_7^2$$

$$d\mathcal{O}_{57} = 2\mathcal{E}x_1x_7$$

$$d\mathcal{O}_{64} = -\mathcal{E}g \cos x_4$$

$$d\mathcal{O}_{73} = \mathcal{E}gx_7 \sin x_3$$

$$d\mathcal{O}_{77} = 2\mathcal{E}x_5x_7 - \mathcal{E}g \cos x_3$$

$$d\mathcal{O}_{86} = \mathcal{E}x_8^2$$

$$d\mathcal{O}_{53} = -\mathcal{E}g \cos x_3$$

$$d\mathcal{O}_{62} = \mathcal{E}x_8^2$$

$$d\mathcal{O}_{68} = 2\mathcal{E}x_2x_8$$

$$d\mathcal{O}_{75} = \mathcal{E}x_7^2$$

$$d\mathcal{O}_{84} = \mathcal{E}gx_8 \sin x_4$$

$$d\mathcal{O}_{88} = 2\mathcal{E}x_6x_8 - \mathcal{E}g \cos x_4$$

Rank condition. Evaluating the rank of the squared sub-matrix $d\mathcal{O}_{1-8}$ in the equilibria we can conclude for the global observability of the system since the matrix $rank(d\mathcal{O}) = n = 8$ everywhere.

Approximated Feedback linearization

Feedback linearization is only applicable to special cases of nonlinear systems that satisfy the constraints of controllability, involutivity and the existence of a relative degree equal to the dimension of the system or minimum phase property.

Approximated Feedback linearization

The *Ball and plate* system described by the equations 14 fails to have full relative degree and does not fall under this class of systems. The Approximated Feedback Linearization (AFL) approach proposed by Ming et al.⁵ is thus used to control the system. This method consists in a two-steps approximation: higher order coupling terms are neglected to reduce the system to two decoupled *Ball and beam* systems; then a second approximation is done to obtain an input-output feedback linearizable system.

⁵Ming Tzu Ho, Yusie Rizal, and Li Ming Chu. "Visual servoing tracking control of a ball and plate system: Design, implementation and experimental validation". In: *International Journal of Advanced Robotic Systems* 10 (2013). ISSN: 17298806. DOI: 10.5772/56525.

Approximated Feedback linearization

First approximation

$$\dot{x} = \begin{pmatrix} x_5 \\ x_6 \\ x_7 \\ x_8 \\ \mathcal{E}(\cancel{x_7 x_8 x_2} + x_3^2 x_1 - g \sin x_3) \\ \mathcal{E}(\cancel{x_7 x_8 x_1} + x_3^2 x_2 - g \sin x_4) \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \quad (17)$$

Assuming that the operating ranges of α and β are small, high order coupling terms are therefore small and neglected.

Approximated Feedback linearization

Second approximation

We start with the differentiation to find the Feedback linearization change of variables:

$$\xi_1 = h_1(x) = x_1$$

$$\dot{\xi}_1 = L_f h_1(x) = x_5$$

$$\dot{\xi}_2 = L_f^2 h_1(x) = \mathcal{E} x_1 x_7^2 - mg \sin x_3$$

$$\dot{\xi}_3 = L_f^3 h_1(x) + L_{g_1} L_f^2 h_1(x) = \mathcal{E} x_1 x_5 x_7^2 - x_7 mg \cos x_3 + \cancel{2\mathcal{E} m x_1 x_7 u_1}$$

The higher order term dependent from the input is discarded, we follow up differentiating to complete the feedback linearization:

$$\begin{aligned}\dot{\xi}_4 = L_f^4 h_1(x) + L_{g_1} L_f^3 h_1(x) + L_{g_2} L_f^3 h_1(x) = \\ \mathcal{E}^2 x_7^2 (x_1 x_7^2 - g \sin x_3) + \mathcal{E} g x_7^2 \sin x_3 + 2u_1 (\mathcal{E} x_5 x_7 - \mathcal{E} g \cos x_3)\end{aligned}$$

The same is done for input $h_2(x)$ to obtain $(\dot{\xi}_5, \dot{\xi}_6, \dot{\xi}_7, \dot{\xi}_8)$.

Approximated Feedback linearization

We collect the 4-th equations of the two chains in the following matrices:

$$\Gamma(x) = \begin{pmatrix} L_f^4 h_1(x) \\ L_f^4 h_2(x) \end{pmatrix} = \begin{pmatrix} \mathcal{E}^2 x_7^2 (x_1 x_7^2 - g \sin x_3) + \mathcal{E} g x_7^2 \sin x_3 \\ \mathcal{E}^2 x_8^2 (x_2 x_8^2 - g \sin x_4) + \mathcal{E} g x_8^2 \sin x_4 \end{pmatrix}$$

$$E(x) = \begin{pmatrix} L_{g_1} L_f^3 h_1(x) & L_{g_2} L_f^3 h_1(x) \\ L_{g_1} L_f^3 h_2(x) & L_{g_2} L_f^3 h_2(x) \end{pmatrix} = \begin{pmatrix} \mathcal{E} x_5 x_7 - \mathcal{E} g \cos x_3 & 0 \\ 0 & \mathcal{E} x_6 x_8 - \mathcal{E} g \cos x_4 \end{pmatrix}$$

Given the non singularity of matrix $E(x)$ we obtain the feedback linearizing control law:

$$U = -E^{-1}(x)\Gamma(x) + E^{-1}(x)\nu \quad (18)$$

Approximated Feedback Linearization

The approximate input-output feedback linearization for the system 17 is given by:

$$\begin{pmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \\ \dot{\xi}_3 \\ \dot{\xi}_4 \\ \dot{\xi}_5 \\ \dot{\xi}_6 \\ \dot{\xi}_7 \\ \dot{\xi}_8 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \\ \xi_5 \\ \xi_6 \\ \xi_7 \\ \xi_8 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} \quad (19)$$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \xi_1 \\ \xi_5 \end{pmatrix}$$

Full state feedback regulator

Given the full controllability and observability of the system, a feedback regulator is applied to the feedback linearized system to place the poles of the plant in the stable plane. The resulting gain matrix is:

$$K = \begin{pmatrix} 24 & 50 & 13 & 10 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 24 & 50 & 13 & 10 \end{pmatrix} \quad (20)$$

