Tavole applicative

Corso di Controllo dei Robot

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January 17, 2019

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Delta robot

Delta robot

The Delta robot is a 3-DOF parallel kinematic machine developed by Reymond Clavel[1] in 1991. It mainly consists of three actuated kinematic chains linked at a common moving platform. Each chain is a serial connection of a revolute actuator, a rear-arm and a forearm (composed of two parallel rods forming a parallelogram). The rear-arms and the forearms are linked through ball-and-socket passive joints. The parallelogram structure of the forearms ensures that the moving platform stays always parallel to the fixed base. Figure 1 shows a schematic view of the Delta robot with its main elements highlighted.

[1] Reymond Clavel. *Conception d'un robot parallele rapide à 4 degres de liberté*. 1991.

Delta robot - Schematic view

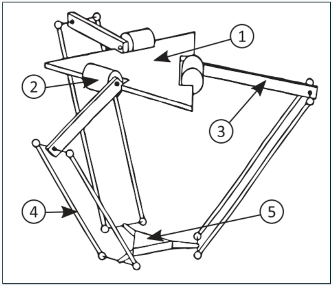


Figure 1: Schematic view of Delta robot

1. Fixed base-plate
2. Actuator
3. Rear-arm
4. Forearm
5. Moving platform

We consider a model with a ternary symmetric configuration with three kinematic chains disposed with a period of 120°.

Delta robot - Parameters

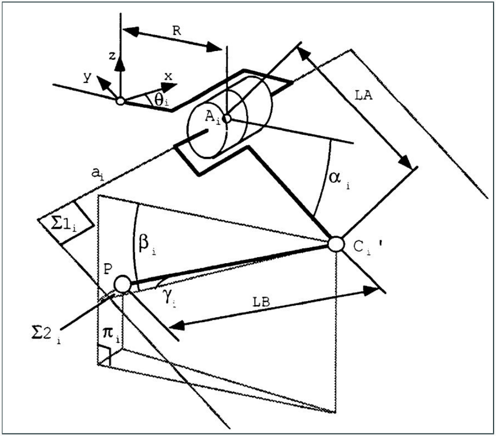


Figure 2: Delta robot length parameters and characteristic angles

Delta robot - Parameters

Parameter	Description	Value
l_A	Rear-arm length	$0.2m$
m_A	Rear-arm mass	$0.1Kg$
R	Base platform dimension	$0.126m$
l_B	Forearm length	$0.4m$
m_B	Forearm mass	$0.045Kg$
m_c	Elbow mass	$0.018Kg$
m_n	Moving platform mass	$0.1Kg$
I_{bi}	Rear-arm inertia	$Kg \times m^2$

Table 1: Delta robot geometric and dynamic parameters

Delta robot - Parameters

Analytical studies on the working volume of the Delta robot[2] showed that:

- A ratio $r = R/l_A < 0.63$ gives the most regular shape for the surface of the lower part of the working volume.
- If $r > 0.0484$ and $b = l_A/l_B > 1.75$ there is no singularity occurrence within the robot working volume.

Thus the parameters shown in table 1 have been chosen for the Delta model used in this project.

[2] L Rey and Reymond Clavel. "The Delta Parallel Robot". In: *Parallel Kinematic Machines. Advanced Manufacturing. Springer, London* (1999).

Delta robot - Reference system and state variables

The position of the End-effector

$$(x, y, z)^T$$

is described in a reference frame fixed to the base plate, as shown in figure 2.

The angles α_i of the actuated joints have been selected as state-variables to describe the robot dynamic:

$$q = (\alpha_1, \alpha_2, \alpha_3)^T$$

Since the moving platform is only translating we can study the model in figure 2 without loss of generality.
In this model the moving platform is reduced to an ideal point with a translation of the three kinematic chains.

Delta robot - Direct kinematic

Direct kinematic is found following the method presented by Clavel in 1991.
Taking in mind the Delta robot representation of figure 2 one can simply find that C_i coordinates are given by the intersection of three circles of radius L_A belonging to the plane π_i and the sphere centred in P having radius L_B. Those conditions give a three equations system that can be solved to find the coordinates of the end-effector.

Delta robot - Direct kinematic

Coordinates of the point C_i in the base frame:

$$C_i = \begin{pmatrix} (R + L_A cos\alpha_i)cos\theta_i \\ (R + L_A cos\alpha_i)sin\theta_i \\ -L_A sin\alpha_i \end{pmatrix} \tag{1}$$

Equation of the sphere centred in P:

$$\left((R + L_A cos\alpha_i)\, cos\theta_i - x^2\right) + ((R + L_A cos\alpha_i)\, sin\theta_i - y)^2 + (L_A sin\alpha_i + z)^2 = L_B^2 \tag{2}$$

The system has two possible solutions. The one with negative z coordinate that belongs to the Delta robot workspace is selected.

Delta robot - Inverse kinematic

The inverse kinematic model let calculate the joint angles q_i as functions of the position of the end effector. The model here presented has been developed by Codourey[3] and has the advantage of removing the points of singularity contained in the model previously introduced by Clavel.
The rationale is still the intersection of a sphere and three circles but the computation is made for each angle in a frame centred in the centre of the $i-th$ joint and rotated with respect to the base frame of an angle θ_i.

[3] Alain Codourey. "Contribution à la commande des robots rapides et précis application au robot delta à entraînement direct". In: (1991), p. 188. DOI: 10.5075/epfl-thesis-922. URL: http://infoscience.epfl.ch/record/31400.

Delta robot - Dynamic model assumptions

- Ideal joints are considered.
- The rotational inertia of the forearm is neglected.
- The mass of each forearm is split up into two point-masses located at both ends of the forearm.

Delta robot - Dynamic model

We express the dynamic of the delta robot in classic matrix formulation:

$$\tau = M(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) \tag{3}$$

Where:

$$M(q) = \left(I_b + m_{nt} J^T J\right), \quad C(q,\dot{q}) = \left(J^T m_{nt} \dot{J}\right), \quad G(q) = -\Gamma_{Gb} - \Gamma_{Gn} \tag{4}$$

- I_b is the inertia matrix of the arms in joint space.
- m_{nt} is the total mass acting on the travelling plate.
- J is the Jacobian matrix.
- Γ_{Gn} is the gravity force acting on the moving platform.
- Γ_{Gb} is the gravity force acting on the rear-arms.

Delta robot - Working volume

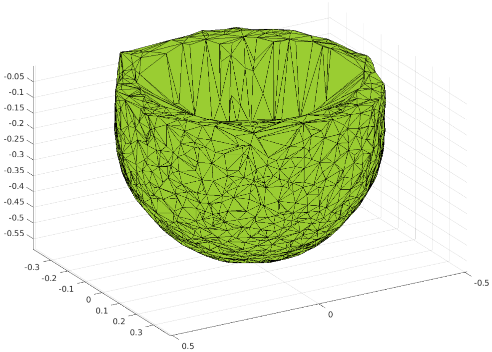


Figure 3: A convex hull of the workspace of the Delta robot

Delta robot - Working volume

In figure 3 a covex hull of the workspace of the Delta robot is reported. The surface has been generated as an $\alpha - \textit{shape}$[4] with $r_\alpha = 0.2$.
The geometric figure gives an analytical instrument to validate a sound reference trajectory generation for the Delta kinematic.

[4] H. Edelsbrunner, D. Kirkpatrick, and R. Seidel. "On the Shape of a Set of Points in the Plane". In: *IEEE Trans. Inf. Theor.* 29.4 (Sept. 2006), pp. 551–559. ISSN: 0018-9448. DOI: 10.1109/TIT.1983.1056714. URL: http://dx.doi.org/10.1109/TIT.1983.1056714.

PD with gravity compensation

Control equation:

$$\tau_{PD} = K_P e + K_D \dot{e} + G(q) \tag{5}$$

with

$$K_P = 1500\,, \qquad K_D = 60$$

Computed torque

Control equation:

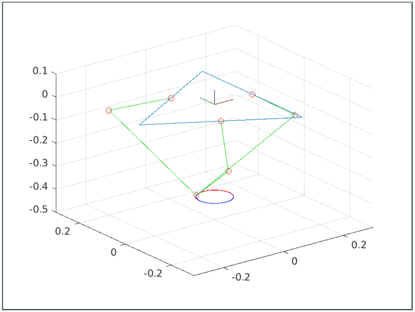
$$\tau_{CT} = M(q)\ddot{q}_d + C(q,\dot{q})\dot{q} + G(q) + K_p e + K_v \dot{e} \qquad (6)$$

with

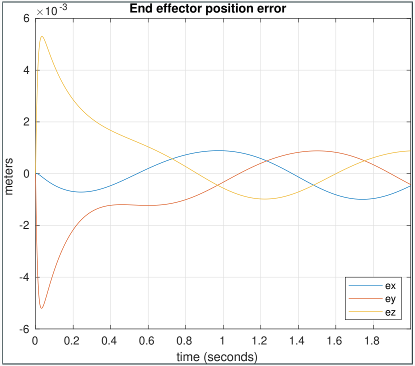
$$K_P = 500,\ K_D = 100$$

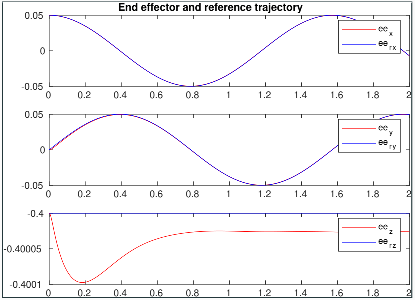
Computed torque

Descrizione traiettoria

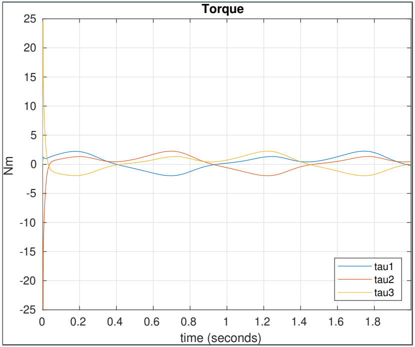


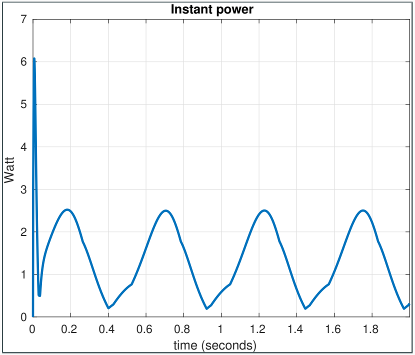
Computed torque





Computed torque





Backstepping

Control equation:

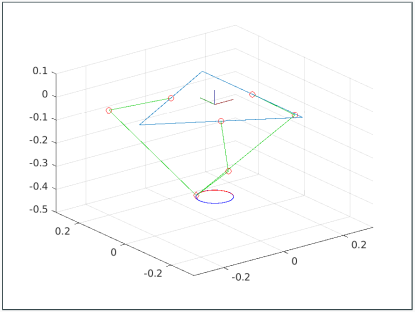
$$\tau_{BS} = M(q)\ddot{q}_r + C(q,\dot{q})\dot{q}_r + G(q) - K_d s + J^T e \qquad (7)$$

with

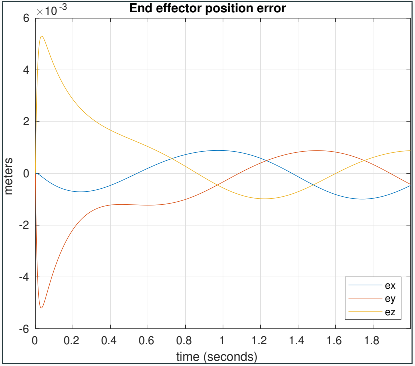
$$\ddot{q}_r = \ddot{q}_d - \Lambda\dot{e}, \quad \dot{q}_r = \dot{q}_d - \Lambda e, \quad s = \dot{q}_- \dot{q}_r, \quad K_d = 50, \quad \Lambda = 400$$

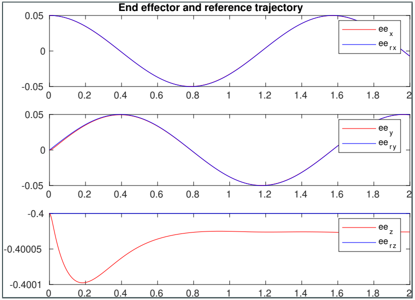
Backstepping

Descrizione traiettoria

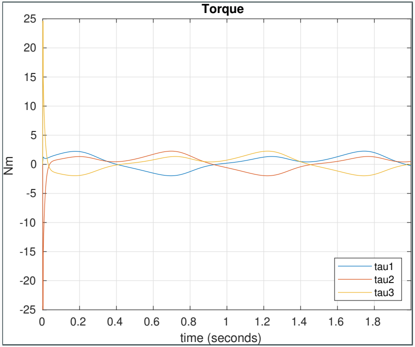


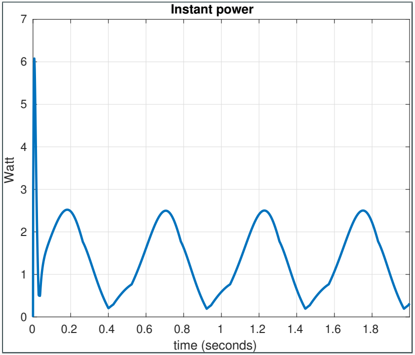
Backstepping





Backstepping





Adaptive backstepping

Ball and plate

Ball and plate

Ball and plate

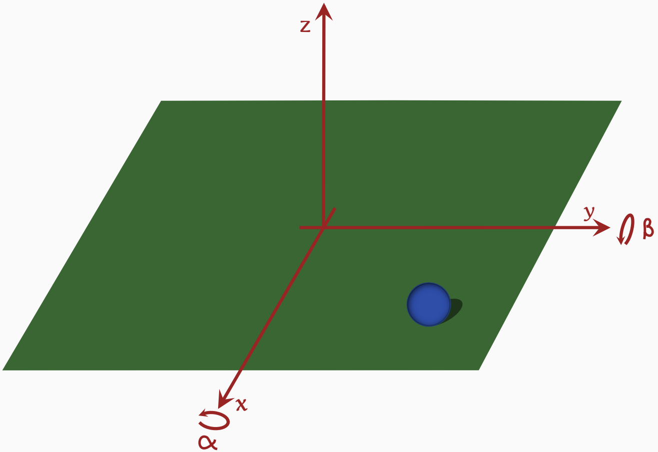


Figure 4: Coordinate frame of the ball and plate system

Ball and plate - Parameters

Parameter	Description	Value
m	Mass of the ball	$0.0109\,Kg$
r	Radius of the ball	$0.01\,m$
I_b	Ball inertia	$4.3563e^{-7}\,Kg \times m^2$
l_p	Plate side	$0.6\,m$
I_p	Plate inertia	$0.175\,Kg \times m^2$

Table 2: Ball and plate geometric and dynamic parameters

Ball and plate - Dynamic model

The general form of Euler-Lagrange for dynamic equations is used to describe the system:

$$\frac{d}{dt}\frac{\delta T}{\delta q_i} - \frac{\delta T}{\delta q_i} + \frac{\delta V}{\delta q_i} = Q_i \tag{8}$$

Where T is the kinetic energy, V is the potential energy, Q_i is the i-th generalized force and q_i id the i-th generalized coordinate. As generalized force we consider two torques acting on the plate ($Q_\alpha = \tau_\alpha, Q_\beta = \tau_\beta$). As generalized coordinates we select two ball position coordinates $[x, y]$ on the frame fixed to the plate and two plate inclination $[\alpha, \beta]$.

Ball and plate - Dynamic model

Kinetic energy of the ball:

$$T_b = \frac{1}{2}mv^2 + \frac{1}{2}I_b\omega^2 = \frac{1}{2}\left(m + \frac{I_b}{r^2}\right)\left(\dot{x}^2 + \dot{y}^2\right) \tag{9}$$

Kinetic energy of the plate:

$$T_p = \frac{1}{2}\left(I_b + I_p\right)\left(\dot{\alpha} + \dot{\beta}\right) + \frac{1}{2}m\left(\dot{\alpha}x + \dot{\beta}y\right)^2 \tag{10}$$

Potential energy:

$$V = mgh = mg(x\, sin\alpha + y\, sin\beta) \tag{11}$$

Ball and plate - Dynamic model

After some derivations we find the following non-linear system of equations:

$$
\begin{aligned}
\left(m + \frac{I_b}{r^2}\right)\ddot{x} - m\left(\dot{\alpha}\dot{\beta}y + \dot{\alpha}^2 x\right) + mg\,sin\alpha = 0 \\
\left(m + \frac{I_b}{r^2}\right)\ddot{y} - m\left(\dot{\alpha}\dot{\beta}x + \dot{\beta}^2 y\right) + mg\,sin\beta = 0 \\
\left(I_p + I_b + mx^2\right)\ddot{\alpha} + m\left(\ddot{\beta}xy + \dot{\beta}\left(\dot{x}y + x\dot{y}\right) + 2\dot{\alpha}\dot{x}x\right) + mgx\,cos\alpha = \tau_\alpha \\
\left(I_p + I_b + my^2\right)\ddot{\beta} + m\left(\ddot{\alpha}xy + \dot{\alpha}\left(\dot{x}y + x\dot{y}\right) + 2\dot{\beta}\dot{y}y\right) + mgy\,cos\beta = \tau_\beta
\end{aligned}
\tag{12}
$$

Ball and plate - Dynamic model

We express the dynamic in matrix form:

$$M(q) = \begin{pmatrix} \left(m + \frac{I_b}{r^2}\right) & 0 & 0 & 0 \\ 0 & \left(m + \frac{I_b}{r^2}\right) & 0 & 0 \\ 0 & 0 & \left(I_b + I_p + mx^2\right) & mxy \\ 0 & 0 & mxy & \left(I_b + I_p + my^2\right) \end{pmatrix}$$

$$C(q, \dot{q}) = \begin{pmatrix} 0 & 0 & -\dot{\alpha}x & -\dot{\alpha}y \\ 0 & 0 & -\dot{\beta}x & -\dot{\beta}y \\ 2\dot{\alpha}x & 0 & 0 & (\dot{x}y + x\dot{y}) \\ 0 & 2\dot{\beta}y & (\dot{x}y + x\dot{y}) & 0 \end{pmatrix}$$

$$G(q) = \begin{pmatrix} mg\ sin\alpha \\ mg\ sin\beta \\ mgx\ cos\alpha \\ mgx\ cos\beta \end{pmatrix}$$

Ball and plate - Dynamic model

Affine-in-control formulation:

$$\dot{x} = \begin{pmatrix} x_5 \\ x_6 \\ x_7 \\ x_8 \\ -B(q)^{-1}\left(C(q,\dot{q})\dot{q} + G(q)\right) \end{pmatrix} + \begin{pmatrix} 0_{4\times 2} \\ B(q)^{-1} \end{pmatrix} \begin{pmatrix} \tau_1 \\ \tau_2 \end{pmatrix} \tag{13}$$

Where

$$x = (x_b, y_b, \alpha, \beta, \dot{x}_b, \dot{y}_b, \dot{\alpha}, \dot{\beta})^T$$

Ball and plate - Change of coordinates

In order to simplify the analysis of the structural properties of the Ball and plate system, the following change of coordinates is adopted:

$$
\begin{aligned}
u_1 &= 2mx\dot{x}\dot{\alpha} - mgx\, cos\alpha - \left(I_p + I_b + mx^2\right)\ddot{\alpha} - m\dot{\beta}\left(\dot{x}y + \dot{y}x\right) - 2m\dot{\alpha}\dot{x}x \\
u_2 &= 2my\dot{y}\dot{\beta} - mgy\, cos\beta - \left(I_p + I_b + my^2\right)\ddot{\beta} - m\dot{\alpha}\left(\dot{x}y + \dot{y}x\right) - 2m\dot{\beta}\dot{y}y
\end{aligned}
$$

Ball and plate - Change of coordinates

We obtain the following system in affine form:

$$\dot{x} = \begin{pmatrix} x_5 \\ x_6 \\ x_7 \\ x_8 \\ \mathcal{E}(x_7 x_8 x_2 + x_3^2 x_1 - g\ sin x_3) \\ \mathcal{E}(x_7 x_8 x_1 + x_3^2 x_2 - g\ sin x_4) \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \tag{14}$$

Where $\mathcal{E} = \frac{mr_b^2}{mr_b^2 + I_b}$

Controllability

Chow theorem. If the accessibility distribution $\langle \Delta, \Delta_0 \rangle = n$ in x_0 then the system is said to be locally accessible in x_0.

Where $\Delta_0 = span\{g_1, g_2, ..., g_d\}$ and $\Delta = span\{f, g_1, g_2, ..., g_d\}$.

We build then the matrix $Q(x)$ as:

$$Q(x) = (g_1, g_2, ad_f g_1, ad_f g_2, \; ..., ad_f^{n-1} g_1, ad_f^{n-1} g_2) \tag{15}$$

And we evaluate its rank on the state space.

Controllability

$$Q(x) = \begin{pmatrix}
0 & 0 & \star & \star & \star & \star & \star & \star & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \star & \star & \star & \star & \star & \star \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \star & \star & \star & \star & \star & \star & \star & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \star & \star & \star & \star & \star & \star & \star \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

Where the $\star$ elements represent the non constant terms of the Q matrix.

Controllability

Performing row swapping, in order to calculate the matrix rank, we obtain the matrix $\tilde{Q}(x)$ as follows:

$$\tilde{Q}(x) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & \star & \star & 0 & \star & 0 & 0 & 0 & \dots \\ 0 & \star & \star & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & \star & 0 & 0 & \star & 0 & \dots \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & \dots \\ 0 & 0 & 0 & \star & 0 & \star & \star & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \dots \end{pmatrix}_{8\times 16} \tag{16}$$

Controllability

Rank condition. Evaluating the rank of matrix $\tilde{Q}(x_0)$ in the equilibria, the following cases are obtained:

- $x_4 \equiv \pi/2,\ x_8 \equiv 0 \implies \mathit{rank}(Q(x_0)) < 8.$
  Which represent the physical condition where $\beta = \pi/2$ with null angular velocity, the system is out of the range of interest.
- $x_3 \equiv \pi/2,\ x_7 \equiv 0 \implies \mathit{rank}(Q(x_0)) < 8.$
  Which represent the physical condition where $\alpha = \pi/2$ with null angular velocity; the same arguments as above hold.
- In all other cases we find $\mathit{rank}(Q(x_0)) = 8$ and the rank condition of controllability is satisfied.

Observability

Given the observation space $\mathcal{O}$ as the space containing all the repeated Lie-derivatives:

$$\mathcal{O} = \{h(\bar{x}),\ L_f h(\bar{x}), \dots, L_{g_i} L_f h(\bar{x}), \dots\}$$

The system results locally observable if $dim\,(d\mathcal{O}) = n$, where $d\mathcal{O}$ is the observability codistribution:

$$d\mathcal{O} = \left\{ \frac{\partial h(\bar{x})}{\partial x},\ \frac{\partial L_f h(\bar{x})}{\partial x}, \dots, \frac{\partial L_{g_i} L_f h(\bar{x})}{\partial x}, \dots \right\}$$

Observability

$$d\mathcal{O} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ d\mathcal{O}_{51} & 0 & d\mathcal{O}_{53} & 0 & 0 & 0 & d\mathcal{O}_{57} & 0 \\ 0 & d\mathcal{O}_{62} & 0 & d\mathcal{O}_{64} & 0 & 0 & 0 & d\mathcal{O}_{68} \\ 0 & 0 & d\mathcal{O}_{73} & 0 & d\mathcal{O}_{75} & 0 & d\mathcal{O}_{77} & 0 \\ 0 & 0 & 0 & d\mathcal{O}_{84} & 0 & d\mathcal{O}_{86} & 0 & d\mathcal{O}_{88} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}_{48\times 8}$$

Observability

Where:

$$d\mathcal{O}_{51} = \mathcal{E}x_7^2 \qquad d\mathcal{O}_{53} = -\mathcal{E}g\ cos x_3$$

$$d\mathcal{O}_{57} = 2\mathcal{E}x_1 x_7 \qquad d\mathcal{O}_{62} = \mathcal{E}x_8^2$$

$$d\mathcal{O}_{64} = -\mathcal{E}g\ cos x_4 \qquad d\mathcal{O}_{68} = 2\mathcal{E}x_2 x_8$$

$$d\mathcal{O}_{73} = \mathcal{E}g x_7\ sin x_3 \qquad d\mathcal{O}_{75} = \mathcal{E}x_7^2$$

$$d\mathcal{O}_{77} = 2\mathcal{E}x_5 x_7 - \mathcal{E}g\ cos x_3 \qquad d\mathcal{O}_{84} = \mathcal{E}g x_8\ sin x_4$$

$$d\mathcal{O}_{86} = \mathcal{E}x_8^2 \qquad d\mathcal{O}_{88} = 2\mathcal{E}x_6 x_8 - \mathcal{E}g\ cos x_4$$

Observability

Rank condition. Evaluating the rank of the squared sub-matrix $d\mathcal{O}_{1-8}$ in the equilibria we can conclude for the global observability of the system since the matrix $rank(d\mathcal{O}) = n = 8$ everywhere.

Approximated Feedback linearization

Feedback linearization is only applicable to special cases of nonlinear systems that satisfy the constraints of controllability, involutivity and the existence of a relative degree equal to the dimension of the system or minimum phase property.

Approximated Feedback linearization

The *Ball and plate* system described by the equations 14 fails to have full relative degree and does not fall under this class of systems. The Approximated Feedback Linearization (*AFL*) approach proposed by Ming et al.[5] is thus used to control the system. This method consists in a two-steps approximation: higher order coupling terms are neglected to reduce the system to two decoupled *Ball and beam* systems; then a second approximation is done to obtain an input-output feedback linearizable system.

[5] Ming Tzu Ho, Yusie Rizal, and Li Ming Chu. "Visual servoing tracking control of a ball and plate system: Design, implementation and experimental validation". In: *International Journal of Advanced Robotic Systems* 10 (2013). ISSN: 17298806. DOI: 10.5772/56525.

Approximated Feedback linearization

First approximation

$$\dot{x} = \begin{pmatrix} x_5 \\ x_6 \\ x_7 \\ x_8 \\ \mathcal{E}(\cancel{x_7 x_8 x_2} + x_3^2 x_1 - g\, sin x_3) \\ \mathcal{E}(\cancel{x_7 x_8 x_1} + x_3^2 x_2 - g\, sin x_4) \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \qquad (17)$$

Assuming that the operating ranges of α and β are small, high order coupling terms are therefore small and neglected.

Approximated Feedback linearization

Second approximation

We start with the differentiation to find the Feedback linearization change of variables:

$$\xi_1 = h_1(x) = x_1$$
$$\dot{\xi}_1 = L_f h_1(x) = x_5$$
$$\dot{\xi}_2 = L_f^2 h_1(x) = \mathcal{E} x_1 x_7^2 - mg\, sin x_3$$
$$\dot{\xi}_3 = L_f^3 h_1(x) + L_{g_1} L_f^2 h_1(x) = \mathcal{E} x_1 x_5 x_7^2 - x_7 mg\, cos x_3 + \cancel{2\mathcal{E} m x_1 x_7 u_1}$$

The higher order term dependent from the input is discarded, we follow up differentiating to complete the feedback linearization:

$$\dot{\xi}_4 = L_f^4 h_1(x) + L_{g_1} L_f^3 h_1(x) + L_{g_2} L_f^3 h_1(x) =$$
$$\mathcal{E}^2 x_7^2 (x_1 x_7^2 - g\, sin x_3) + \mathcal{E} g x_7^2\, sin x_3 + 2u_1(\mathcal{E} x_5 x_7 - \mathcal{E} g\, cos x_3)$$

The same is done for input $h_2(x)$ to obtain $\left(\dot{\xi}_5, \dot{\xi}_6, \dot{\xi}_7, \dot{\xi}_8\right)$.

Approximated Feedback linearization

We collect the 4-th equations of the two chains in the following matrices:

$$\Gamma(x) = \begin{pmatrix} L_f^4 h_1(x) \\ L_f^4 h_2(x) \end{pmatrix} = \begin{pmatrix} \mathcal{E}^2 x_7^2 (x_1 x_7^2 - g\ sinx_3) + \mathcal{E} g x_7^2\ sinx_3 \\ \mathcal{E}^2 x_8^2 (x_2 x_8^2 - g\ sinx_4) + \mathcal{E} g x_8^2\ sinx_x \end{pmatrix}$$

$$E(x) = \begin{pmatrix} L_{g_1} L_f^3 h_1(x) & L_{g_2} L_f^3 h_1(x) \\ L_{g_1} L_f^3 h_2(x) & L_{g_2} L_f^3 h_2(x) \end{pmatrix} = \begin{pmatrix} \mathcal{E} x_5 x_7 - \mathcal{E} g\ cosx_3 & 0 \\ 0 & \mathcal{E} x_6 x_8 - \mathcal{E} g\ cosx_4 \end{pmatrix}$$

Approximated Feedback Linearization

Given the non singularity of matrix $E(x)$ we obtain the feedback linearizing control law:

$$U = -E^{-1}(x)\Gamma(x) + E^{-1}(x)\nu \tag{18}$$

Approximated Feedback Linearization

The approximate input-output feedback linearization for the system 17 is given by:

$$\begin{pmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \\ \dot{\xi}_3 \\ \dot{\xi}_4 \\ \dot{\xi}_5 \\ \dot{\xi}_6 \\ \dot{\xi}_7 \\ \dot{\xi}_8 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \\ \xi_5 \\ \xi_6 \\ \xi_7 \\ \xi_8 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} \tag{19}$$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \xi_1 \\ \xi_5 \end{pmatrix}$$

Full state feedback regulator

Given the full controllability and observability of the system, a feedback regulator is applied to the feedback linearized system to place the poles of the plant in the stable plane. The resulting gain matrix is:

$$K = \begin{pmatrix} 24 & 50 & 13 & 10 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 24 & 50 & 13 & 10 \end{pmatrix} \tag{20}$$

PID Control