Tavole applicative

Corso di Controllo dei Robot

E. Puglisi   A. Ryals

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Ingegneria robotica e dell'automazione
Università di Pisa

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Delta robot

Delta robot

The Delta robot is a 3-DOF parallel kinematic machine developed by Reymond Clavel[1] in 1991. It mainly consists of three actuated kinematic chains linked at a common moving platform. Each chain is a serial connection of a revolute actuator, a rear-arm and a forearm (composed of two parallel rods forming a parallelogram). The rear-arms and the forearms are linked through ball-and-socket passive joints. The parallelogram structure of the forearms ensures that the moving platform stays always parallel to the fixed base. Figure 1 shows a schematic view of the Delta robot with its main elements highlighted.

[1] Reymond Clavel. *Conception d'un robot parallele rapide à 4 degres de liberté*. 1991.

Delta robot - Schematic view

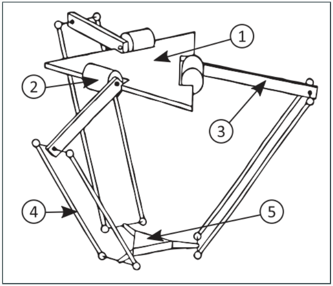


Figure 1: Schematic view of Delta robot

1. Fixed base-plate
2. Actuator
3. Rear-arm
4. Forearm
5. Moving platform

We consider a model with a ternary symmetric configuration with three kinematic chains disposed with a period of 120°.

Delta robot - Parameters

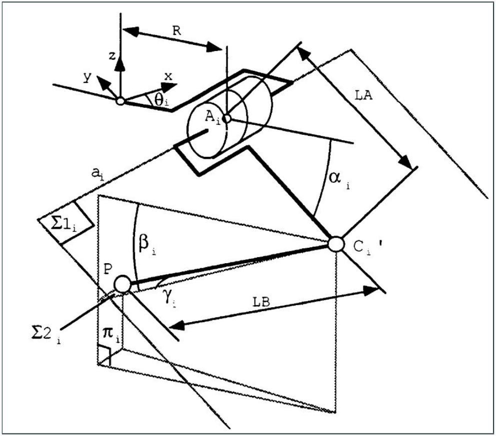


Figure 2: Delta robot length parameters and characteristic angles

Delta robot - Parameters

Parameter	Description	Value
l_A	Rear-arm length	$0.2m$
m_A	Rear-arm mass	$0.1Kg$
R	Base platform dimension	$0.126m$
l_B	Forearm length	$0.4m$
m_B	Forearm mass	$0.045Kg$
m_c	Elbow mass	$0.018Kg$
m_n	Moving platform mass	$0.1Kg$
I_{bi}	Rear-arm inertia	$Kg \times m^2$

Table 1: Delta robot geometric and dynamic parameters

Delta robot - Parameters

Analytical studies on the working volume of the Delta robot[2] showed that:

- A ratio $r = R/l_A < 0.63$ gives the most regular shape for the surface of the lower part of the working volume.
- If $r > 0.0484$ and $b = l_A/l_B > 1.75$ there is no singularity occurrence within the robot working volume.

Thus the parameters shown in table 1 have been chosen for the Delta model used in this project.

[2] L Rey and Reymond Clavel. "The Delta Parallel Robot". In: *Parallel Kinematic Machines. Advanced Manufacturing. Springer, London* (1999).

Delta robot - Reference system and state variables

The position of the End-effector

$$(x, y, z)^T$$

is described in a reference frame fixed to the base plate, as shown in figure 2.

The angles α_i of the actuated joints have been selected as state-variables to describe the robot dynamic:

$$q = (\alpha_1, \alpha_2, \alpha_3)^T$$

Delta robot - Direct kinematic

Since the moving platform is only translating we can study the model in figure 2 without loss of generality.
In this model the moving platform is reduced to an ideal point with a translation of the three kinematic chains.

Delta robot - Direct kinematic

Direct kinematic is found following the method presented by Clavel in 1991.
Taking in mind the Delta robot representation of figure 2 one can simply find that C_i coordinates are given by the intersection of three circles of radius L_A belonging to the plane π_i and the sphere centred in P having radius L_B. Those conditions give a three equations system that can be solved to find the coordinates of the end-effector.

Delta robot - Direct kinematic

Coordinates of the point C_i in the base frame:

$$C_i = \begin{pmatrix} (R + L_A cos\alpha_i)cos\theta_i \\ (R + L_A cos\alpha_i)sin\theta_i \\ -L_A sin\alpha_i \end{pmatrix} \tag{1}$$

Equation of the sphere centred in P:

$$\left((R + L_A cos\alpha_i)\, cos\theta_i - x^2\right) + ((R + L_A cos\alpha_i)\, sin\theta_i - y)^2 + (L_A sin\alpha_i + z)^2 = L_B^2 \tag{2}$$

The system has two possible solutions. The one with negative z coordinate that belongs to the Delta robot workspace is selected.

Delta robot - Inverse kinematic

The inverse kinematic model let calculate the joint angles q_i as functions of the position of the end effector. The model here presented has been developed by Codourey[3] and has the advantage of removing the points of singularity contained in the model previously introduced by Clavel.
The rationale is still the intersection of a sphere and three circles but the computation is made for each angle in a frame centred in the centre of the $i-th$ joint and rotated with respect to the base frame of an angle θ_i.

[3] Alain Codourey. "Contribution à la commande des robots rapides et précis application au robot delta à entraînement direct". In: (1991), p. 188. DOI: 10.5075/epfl-thesis-922. URL: http://infoscience.epfl.ch/record/31400.

Delta robot - Dynamic model assumptions

- Ideal joints are considered.
- The rotational inertia of the forearm is neglected.
- The mass of each forearm is split up into two point-masses located at both ends of the forearm.

Delta robot - Dynamic model

We express the dynamic of the delta robot in classic matrix formulation:

$$\tau = M(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) \tag{3}$$

Where:

$$M(q) = \left(I_b + m_{nt}J^T J\right), \quad C(q,\dot{q}) = \left(J^T m_{nt} J\right), \quad G(q) = -\Gamma_{Gb} - \Gamma_{Gn}$$

- I_b is the inertia matrix of the arms in joint space.
- m_{nt} is the total mass acting on the travelling plate.
- J is the Jacobian matrix.
- Γ_{Gn} is the gravity force acting on the moving platform.
- Γ_{Gb} is the gravity force acting on the rear-arms.

Delta robot - Working volume

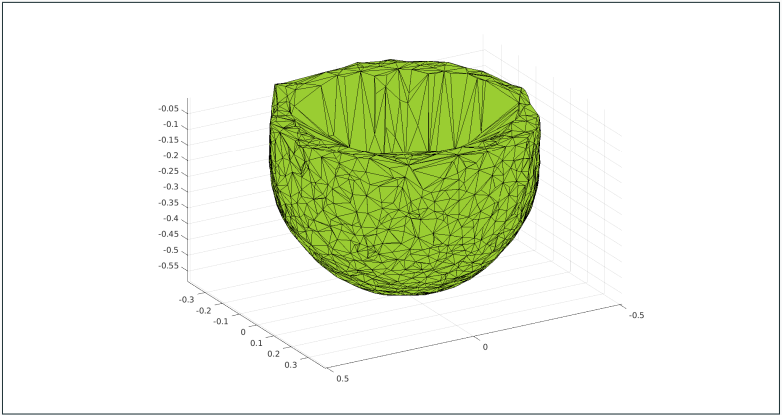


Figure 3: A convex hull of the workspace of the Delta robot

Delta robot - Working volume

In figure 3 a covex hull of the workspace of the Delta robot is reported. The surface has been generated as an $\alpha - \textit{shape}$[4] with $r_\alpha = 0.2$.
The geometric figure gives an analytical instrument to validate a sound reference trajectory generation for the Delta kinematic.

[4] H. Edelsbrunner, D. Kirkpatrick, and R. Seidel. "On the Shape of a Set of Points in the Plane". In: *IEEE Trans. Inf. Theor.* 29.4 (Sept. 2006), pp. 551–559. ISSN: 0018-9448. DOI: 10.1109/TIT.1983.1056714. URL: http://dx.doi.org/10.1109/TIT.1983.1056714.

PD with gravity compensation

Control equation:

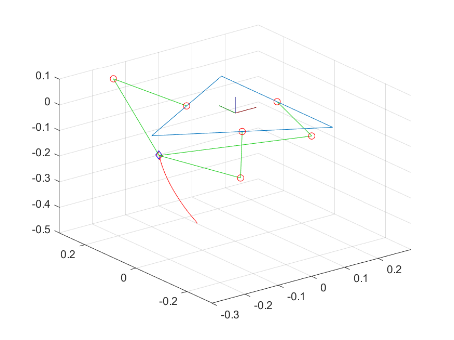
$$\tau_{PD} = K_P e + K_D \dot{e} + G(q) \tag{4}$$

with

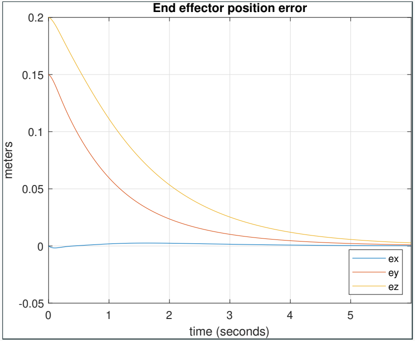
$$K_P = 1500\,, \qquad K_D = 60$$

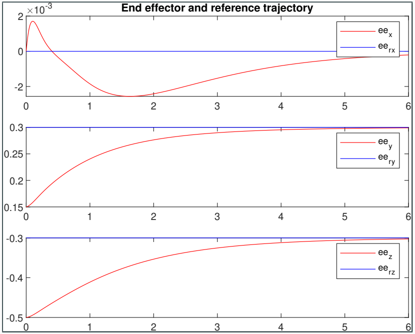
PD with gravity compensation

Point to point trajectory

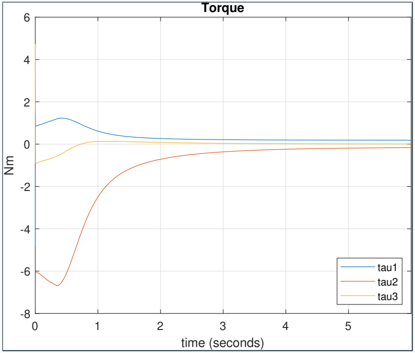


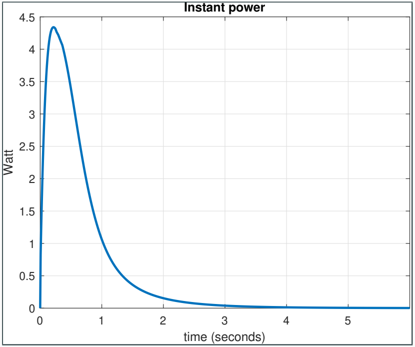
PD with gravity compensation





PD with gravity compensation





Computed torque

Control equation:

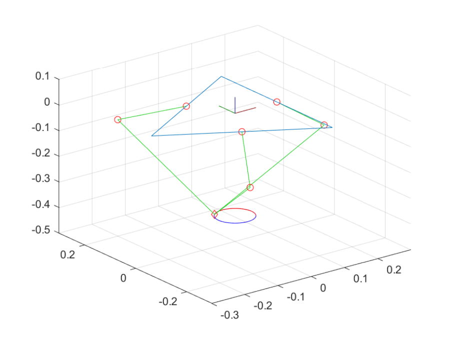
$$\tau_{CT} = M(q)\ddot{q}_d + C(q,\dot{q})\dot{q} + G(q) + K_p e + K_v \dot{e} \tag{5}$$

with

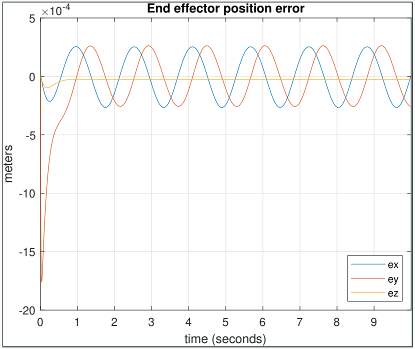
$$K_P = 500,\ K_D = 100$$

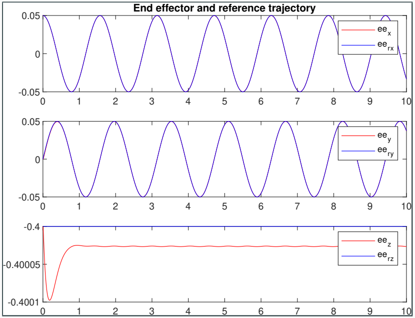
Computed torque

Circular trajectory

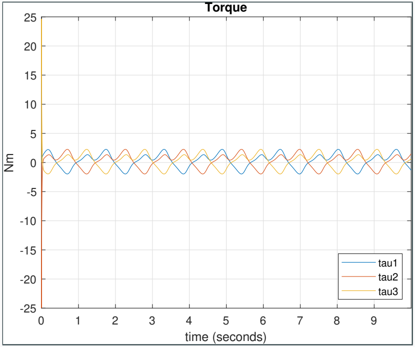


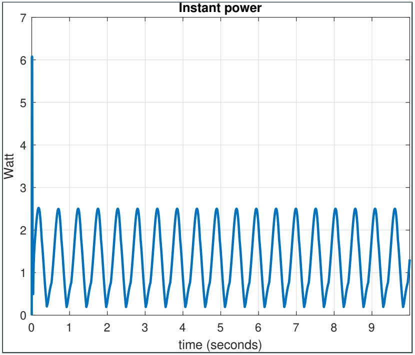
Computed torque





Computed torque





Backstepping

Control equation:

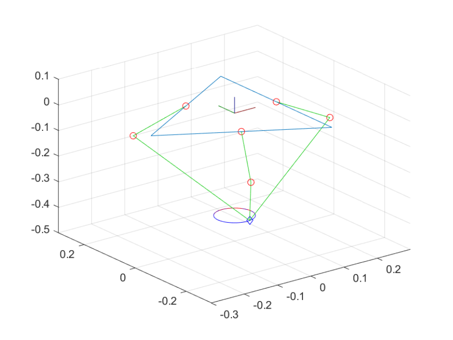
$$\tau_{BS} = M(q)\ddot{q}_r + C(q,\dot{q})\dot{q}_r + G(q) - K_d s + J^T e \tag{6}$$

with

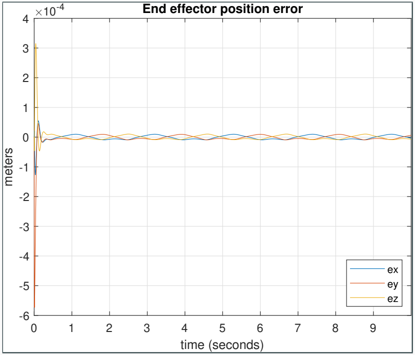
$$\ddot{q}_r = \ddot{q}_d - \Lambda\dot{e}, \quad \dot{q}_r = \dot{q}_d - \Lambda e, \quad s = \dot{q}_{-}\dot{q}_r, \quad K_d = 50, \quad \Lambda = 400$$

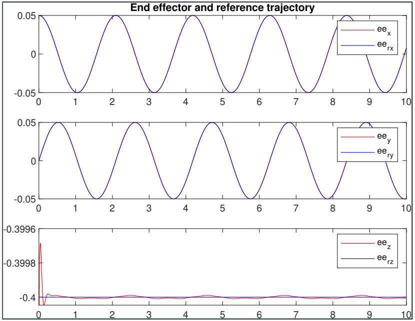
Backstepping

Circular trajectory

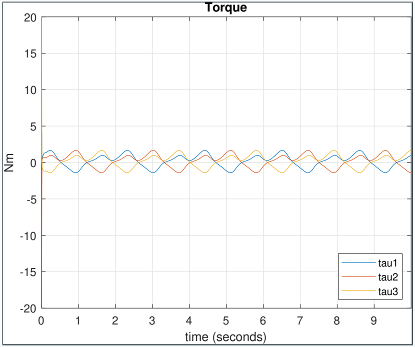


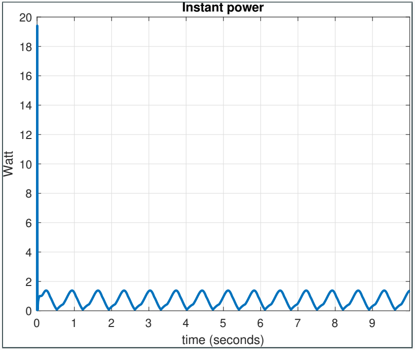
Backstepping





Backstepping





Adaptive backstepping

Dynamic parameters vector:

$$\pi = \begin{pmatrix} m_{nt} \\ I_{bi} \\ r_{Gb} \end{pmatrix} \begin{array}{ll} \rightarrow & \textit{Total moving platform mass} \\ \rightarrow & \textit{Inertia contribution for each upper arm} \\ \rightarrow & \textit{Upper arm center of mass} \end{array}$$

Linear in the parameters reformulation of the dynamic equations:

$$Y(q, \dot{q}, \ddot{q})\pi = \begin{pmatrix} J^T(\bar{g} + \dot{X}) & \ddot{q} & g \cos \bar{q} \end{pmatrix} \begin{pmatrix} m_{nt} \\ I_{bi} \\ r_{Gb} \end{pmatrix} = \tau$$

Adaptive backstepping

Estimation update law:

$$\dot{\hat{\pi}} = R^{-1} Y^T(q, \dot{q}, \ddot{q}) s \quad \textit{where } R = 30$$

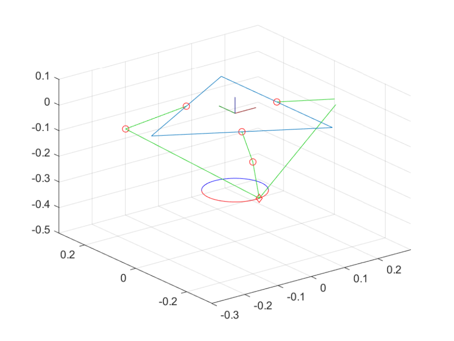
Control equation:

$$\tau_{AB} = Y(q, \dot{q}, \ddot{q}) \hat{\pi} - K_d s - e$$

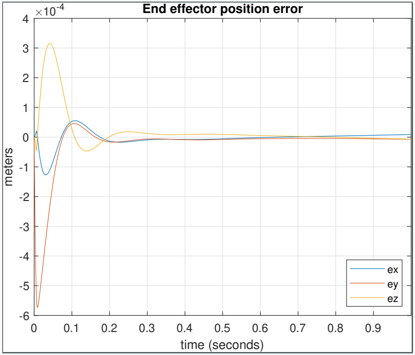
$$\ddot{q}_r = \ddot{q}_d - \Lambda \dot{e}, \quad \dot{q}_r = \dot{q}_d - \Lambda e, \quad s = \dot{q}_- \dot{q}_r, \quad K_d = 50, \quad \Lambda = 400$$

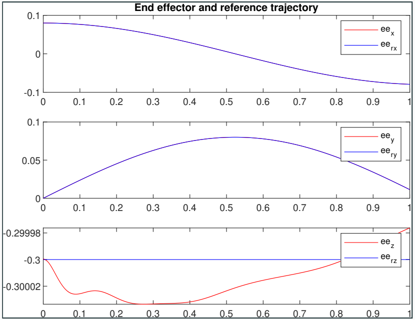
Adaptive backstepping

Circular trajectory

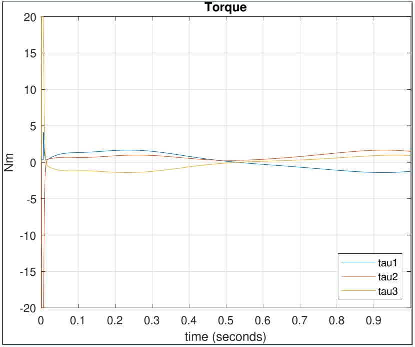


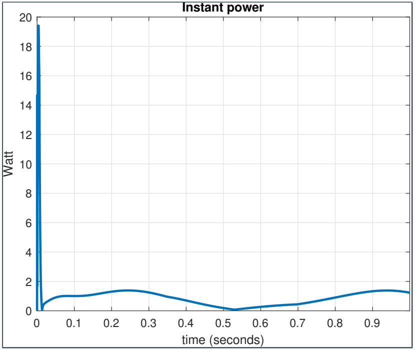
Adaptive backstepping





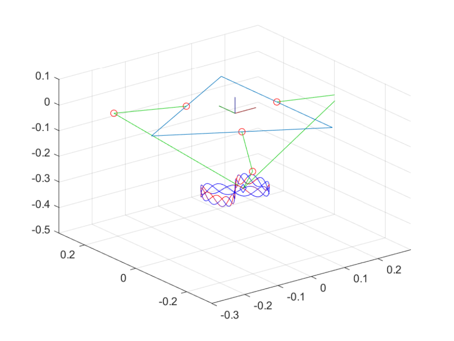
Adaptive backstepping



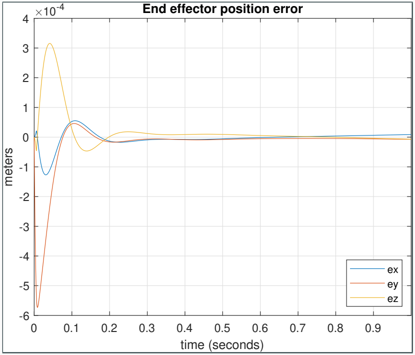


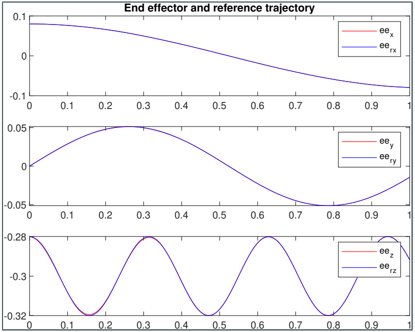
Adaptive backstepping

- ∞ trajectory with sinusoidal z

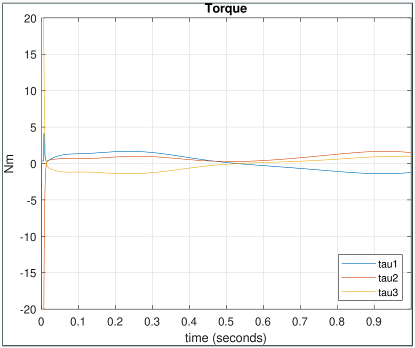


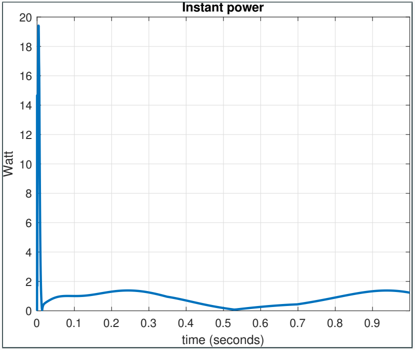
Adaptive backstepping





Adaptive backstepping





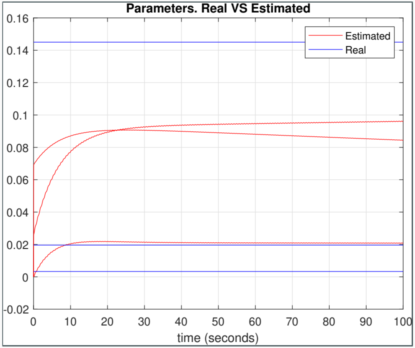
Adaptive backstepping - Parameters convergence

The convergence of parameters π in an interval $[T, T+\Delta]$ is evaluated via the matrix:

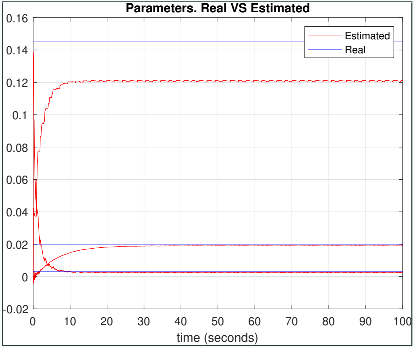
$$\Sigma = \int_{T}^{T+\Delta} Y^T(q, \dot{q}, \ddot{q}) Y(q, \dot{q}, \ddot{q}) dt$$

Convergence condition: $rank(\Sigma) = n = 3$

Adaptive backstepping - Parameters convergence



Circle trajectory



Infinity trajectory

In neither case the convergence condition is satisfied.

Ball and plate

Ball and plate

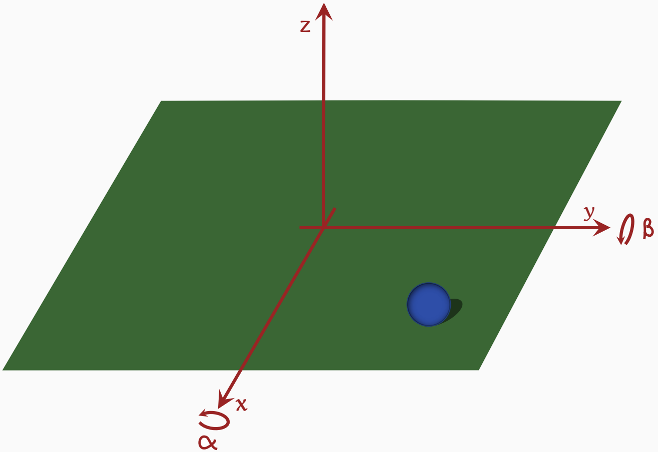


Figure 6: Coordinate frame of the ball and plate system

Ball and plate - Parameters

Parameter	Description	Value
m	Mass of the ball	$0.0109\,Kg$
r	Radius of the ball	$0.01\,m$
I_b	Ball inertia	$4.3563e^{-7}\,Kg \times m^2$
l_p	Plate side	$0.6\,m$
I_p	Plate inertia	$0.175\,Kg \times m^2$

Table 2: Ball and plate geometric and dynamic parameters

Ball and plate - Dynamic model

The general form of Euler-Lagrange for dynamic equations is used to describe the system:

$$\frac{d}{dt}\frac{\delta T}{\delta q_i} - \frac{\delta T}{\delta q_i} + \frac{\delta V}{\delta q_i} = Q_i \tag{7}$$

Where T is the kinetic energy, V is the potential energy, Q_i is the i-th generalized force and q_i id the i-th generalized coordinate. As generalized force we consider two torques acting on the plate ($Q_\alpha = \tau_\alpha, Q_\beta = \tau_\beta$). As generalized coordinates we select two ball position coordinates $[x, y]$ on the frame fixed to the plate and two plate inclination $[\alpha, \beta]$.

Ball and plate - Dynamic model

Kinetic energy of the ball:

$$T_b = \frac{1}{2}mv^2 + \frac{1}{2}I_b\omega^2 = \frac{1}{2}\left(m + \frac{I_b}{r^2}\right)\left(\dot{x}^2 + \dot{y}^2\right) \tag{8}$$

Kinetic energy of the plate:

$$T_p = \frac{1}{2}\left(I_b + I_p\right)\left(\dot{\alpha} + \dot{\beta}\right) + \frac{1}{2}m\left(\dot{\alpha}x + \dot{\beta}y\right)^2 \tag{9}$$

Potential energy:

$$V = mgh = mg(x\, sin\alpha + y\, sin\beta) \tag{10}$$

Ball and plate - Dynamic model

After some derivations we find the following non-linear system of equations:

$$
\begin{aligned}
\left(m+\frac{I_b}{r^2}\right)\ddot{x} - m\left(\dot{\alpha}\dot{\beta}y + \dot{\alpha}^2 x\right) + mg\,sin\alpha &= 0\\
\left(m+\frac{I_b}{r^2}\right)\ddot{y} - m\left(\dot{\alpha}\dot{\beta}x + \dot{\beta}^2 y\right) + mg\,sin\beta &= 0\\
\left(I_p + I_b + mx^2\right)\ddot{\alpha} + m\left(\ddot{\beta}xy + \dot{\beta}\left(\dot{x}y + x\dot{y}\right) + 2\dot{\alpha}\dot{x}x\right) + mgx\,cos\alpha &= \tau_\alpha\\
\left(I_p + I_b + my^2\right)\ddot{\beta} + m\left(\ddot{\alpha}xy + \dot{\alpha}\left(\dot{x}y + x\dot{y}\right) + 2\dot{\beta}\dot{y}y\right) + mgy\,cos\beta &= \tau_\beta
\end{aligned}
\tag{11}
$$

Ball and plate - Dynamic model

We express the dynamic in matrix form:

$$M(q) = \begin{pmatrix} \left(m + \frac{I_b}{r^2}\right) & 0 & 0 & 0 \\ 0 & \left(m + \frac{I_b}{r^2}\right) & 0 & 0 \\ 0 & 0 & \left(I_b + I_p + mx^2\right) & mxy \\ 0 & 0 & mxy & \left(I_b + I_p + my^2\right) \end{pmatrix}$$

$$C(q, \dot{q}) = m \begin{pmatrix} 0 & 0 & -\dot{\alpha}x & -\dot{\alpha}y \\ 0 & 0 & -\dot{\beta}x & -\dot{\beta}y \\ 2\dot{\alpha}x & 0 & 0 & (\dot{x}y + x\dot{y}) \\ 0 & 2\dot{\beta}y & (\dot{x}y + x\dot{y}) & 0 \end{pmatrix}$$

$$G(q) = \begin{pmatrix} mg\ sin\alpha \\ mg\ sin\beta \\ mgx\ cos\alpha \\ mgx\ cos\beta \end{pmatrix}$$

Ball and plate - Dynamic model

Affine-in-control formulation:

$$\dot{x} = \begin{pmatrix} x_5 \\ x_6 \\ x_7 \\ x_8 \\ -B(q)^{-1}\left(C(q,\dot{q})\dot{q} + G(q)\right) \end{pmatrix} + \begin{pmatrix} 0_{4\times 2} \\ B(q)^{-1} \end{pmatrix} \begin{pmatrix} \tau_1 \\ \tau_2 \end{pmatrix} \tag{12}$$

Where

$$x = (x_b, y_b, \alpha, \beta, \dot{x}_b, \dot{y}_b, \dot{\alpha}, \dot{\beta})^T$$

Ball and plate - Change of coordinates

In order to simplify the analysis of the structural properties of the Ball and plate system, the following change of coordinates is adopted:

$$
\begin{aligned}
u_1 &= 2mx\dot{x}\dot{\alpha} - mgx\,cos\alpha - \left(I_p + I_b + mx^2\right)\ddot{\alpha} - m\dot{\beta}\left(\dot{x}y + \dot{y}x\right) - 2m\dot{\alpha}\dot{x}x \\
u_2 &= 2my\dot{y}\dot{\beta} - mgy\,cos\beta - \left(I_p + I_b + my^2\right)\ddot{\beta} - m\dot{\alpha}\left(\dot{x}y + \dot{y}x\right) - 2m\dot{\beta}\dot{y}y
\end{aligned}
$$

Ball and plate - Change of coordinates

We obtain the following system in affine form:

$$\dot{x} = \begin{pmatrix} x_5 \\ x_6 \\ x_7 \\ x_8 \\ \mathcal{E}(x_7 x_8 x_2 + x_7^2 x_1 - g\, sin x_3) \\ \mathcal{E}(x_7 x_8 x_1 + x_8^2 x_2 - g\, sin x_4) \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \qquad (13)$$

Where

$$\mathcal{E} = \frac{m r_b^2}{m r_b^2 + I_b}$$

Observability

Given the observation space $\mathcal{O}$ as the space containing all the repeated Lie-derivatives:

$$\mathcal{O} = \{h(\bar{x}),\ L_f h(\bar{x}), \dots, L_{g_i} L_f h(\bar{x}), \dots\}$$

The system results locally observable if $dim\,(d\mathcal{O}) = n$, where $d\mathcal{O}$ is the observability codistribution:

$$d\mathcal{O} = \left\{ \frac{\partial h(\bar{x})}{\partial x},\ \frac{\partial L_f h(\bar{x})}{\partial x}, \dots, \frac{\partial L_{g_i} L_f h(\bar{x})}{\partial x}, \dots \right\}$$

Observability

$$d\mathcal{O} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ \star & \star & \star & 0 & 0 & 0 & \star & \star \\ 0 & 0 & \star & 0 & 0 & 0 & \star & \star \\ 0 & 0 & \star & 0 & \star & \star & \star & \star \\ 0 & 0 & 0 & \star & \star & \star & \star & \star \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}_{48\times 8}$$

Where $\star$ elements represent non constant terms of $d\mathcal{O}(x)$ matrix.

Observability

In order to calculate $rank(d\mathcal{O})$ we perform the following columns and rows swapping:

$$\textit{columns } 5,6 \longleftrightarrow \textit{columns } 3,4$$

We obtain the following matrix:

$$d\tilde{\mathcal{O}} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \star & \star & 0 & 0 & \star & 0 & \star & \star \\ \star & \star & 0 & 0 & 0 & \star & \star & \star \\ 0 & 0 & \star & \star & \star & 0 & \star & \star \\ 0 & 0 & \star & \star & 0 & \star & \star & \star \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}_{48\times 8}$$

Observability

Studying the matrix obtained one can see that the first 4 rows are:

$$\begin{pmatrix} I & \emptyset \end{pmatrix}_{4\times 8}$$

It is thus sufficient to append 4 rows to this matrix to find a matrix completion of dimension $n = 8$.
We find a block matrix in the form:

$$d\tilde{\mathcal{O}} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

Given the determinant formula for block matrices:
$det(M) = det(A - BD^{-1}C)det(D)$

We can see that it is sufficient to study the submatrix D to conclude on eventual rank deficiency of $d\tilde{\mathcal{O}}$.

Observability

We select the following rows sets r_i of $d\tilde{\mathcal{O}}$:

- $r_1 = \{21, 22, 23, 6\} \implies det(D_1) = -2\,\mathcal{E}^4\,g^2\,{x_1}^2\,\cos x_4\,\sin x_3 \implies rank(d\tilde{\mathcal{O}}) < 8 \textit{ iff } x_1 \equiv 0 \;\vee\; x_3 \equiv 0 \;\vee\; x_4 \equiv \pi/2$
- $r_2 = \{37, 38, 5, 40\} \implies det(D_2) = -2\,\mathcal{E}^4\,g^2\,{x_2}^2\,\cos x_3\,\sin x_4 \implies rank(d\tilde{\mathcal{O}}) < 8 \textit{ iff } x_2 \equiv 0 \;\vee\; x_3 \equiv \pi/2 \;\vee\; x_4 \equiv 0$

Case $x_1 \equiv 0 \wedge x_2 \equiv 0$:

- $r_3 = \{5, 6, 23, 24\} \implies det(D_3) = 2\,\mathcal{E}^4\,g^2\,{x_5}^2\,\cos x_3\,\cos x_4 \implies rank(d\tilde{\mathcal{O}}) < 8 \textit{ iff } x_5 \equiv 0 \;\vee\; x_3 \equiv \pi/2 \;\vee\; x_4 \equiv \pi/2$
- $r_4 = \{5, 6, 39, 40\} \implies det(D_4) = 2\,\mathcal{E}^4\,g^2\,{x_6}^2\,\cos x_3\,\cos x_4 \implies rank(d\tilde{\mathcal{O}}) < 8 \textit{ iff } x_6 \equiv 0 \;\vee\; x_3 \equiv \pi/2 \;\vee\; x_4 \equiv \pi/2$

Observability

Case $x_3 \equiv 0 \;\wedge\; x_4 \equiv 0$:

- $r_5 = \{5, 6, 23, 24\} \implies det(D_5) = 2\,\mathcal{E}^4\, g^2\, {x_5}^2 \implies rank(d\tilde{\mathcal{O}}) < 8$ *iff* $x_5 \equiv 0$
- $r_6 = \{5, 6, 39, 40\} \implies det(D_6) = 2\,\mathcal{E}^4\, g^2\, {x_6}^2 \implies rank(d\tilde{\mathcal{O}}) < 8$ *iff* $x_6 \equiv 0$

Case $x_1 \equiv 0 \;\wedge\; x_2 \equiv 0 \;\wedge\; x_5 \equiv 0 \;\wedge\; x_6 \equiv 0$:

- $r_7 = \{5, 6, 7, 8\} \implies det(D_7) = \mathcal{E}^4\, g^4 \cos {x_3}^2 \cos {x_4}^2 \implies rank(d\tilde{\mathcal{O}}) < 8$ *iff* $x_3 \equiv \pi/2 \;\vee\; x_4 \equiv \pi/2$

Case $x_3 \equiv 0 \;\wedge\; x_4 \equiv 0 \;\wedge\; x_5 \equiv 0 \;\wedge\; x_6 \equiv 0$:

- $r_8 = \{5, 6, 7, 8\} \implies det(D_8) = \mathcal{E}^4\, g^4 \implies rank(d\tilde{\mathcal{O}}) < 8$ *iff* $x_3 \equiv \pi/2 \;\vee\; x_4 \equiv \pi/2$

Observability

Rank condition. Collecting the conditions highlighted above we can conclude for the global observability of the system since the matrix $rank(d\mathcal{O}) = n = 8$ everywhere with the exclusion of the subsets $S_1 = x_3 \equiv \pi/2$ and $S_2 = x_4 \equiv \pi/2$ that are out of the range of interest.

Controllability

Chow theorem. If the accessibility distribution $\langle \Delta, \Delta_0 \rangle = n$ in x_0 then the system is said to be locally accessible in x_0.

Where $\Delta_0 = span\{g_1, g_2, ..., g_d\}$ and $\Delta = span\{f, g_1, g_2, ..., g_d\}$.

We build then the matrix $Q(x)$ as:

$$Q(x) = (g_1, g_2, ad_f g_1, ad_f g_2,\ ..., ad_f^{n-1} g_1, ad_f^{n-1} g_2) \tag{14}$$

And we evaluate its rank on the state space.

Controllability matrix

$$
Q(x) = \begin{pmatrix}
0 & 0 & \star & \star & \star & \star & \star & \star & 0 & 0 & \star & \star & \star & \star & \star & \star \\
0 & 0 & \star & \star & \star & \star & \star & \star & 0 & 0 & \star & \star & \star & \star & \star & \star \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \star & \star & \star & \star & \star & \star & \star & 0 & \star & \star & \star & \star & \star & \star & \star \\
0 & \star & \star & \star & \star & \star & \star & \star & 0 & \star & \star & \star & \star & \star & \star & \star \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

Where $\star$ elements represent non constant terms of $Q(x)$ matrix.

Controllability matrix

In order to calculate $rank(Q(x))$ we perform the following columns and rows swapping:

$$column\ 2 \longleftrightarrow column\ 3$$

$$row\ 1 \longleftrightarrow row\ 7$$

$$column\ 10 \longleftrightarrow column\ 4$$

$$column\ 9 \longleftrightarrow column\ 8$$

$$row\ 2 \longleftrightarrow row\ 8$$

$$column\ 2 \longleftrightarrow column\ 8$$

Controllability matrix

We obtain the matrix $\tilde{Q}(x)$::

$$\tilde{Q}(x) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & \star & \star & \star & \star & \star & \star & \dots \\ 0 & 0 & \star & \star & \star & \star & \star & \star & \dots \\ 0 & 0 & 0 & 0 & \star & \star & \star & \star & \dots \\ 0 & 0 & 0 & 0 & \star & \star & \star & \star & \dots \end{pmatrix}_{8\times 16} \tag{15}$$

Controllability matrix

Once again we obtained a block submatrix in the form:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

with A trivial and $B = \emptyset$. However the equations appearing in the elements of the candidate submatrices D_i have a high degree of complexity and we are not able to conclude on a general result for accessibility.

Controllability - particular case

Case $x_7 \equiv 0 \ \wedge \ x_8 \equiv 0$:

$c_1 = \{8, 10, 11, 12\} \implies det(D_1) = \mathcal{E}^4 \, g^4 \, \cos^2 x_3 \, \cos^2 x_4 \implies$
$rank(\tilde{Q}(x)) < 8 \; iff \; x_4 \equiv \pi/2$

In all the equilibria contained in this subspace we can conclude for accessibility since $x_4 \equiv \pi/2$ is out of the range of interest.

Approximated Feedback linearization

Feedback linearization is only applicable to special cases of nonlinear systems that satisfy the constraints of controllability, involutivity and the existence of a relative degree equal to the dimension of the system or minimum phase property.

Approximated Feedback linearization

Ball and plate system described by the equations 13 fails to have full relative degree and does not fall under this class of systems. The Approximated Feedback Linearization (*AFL*) approach proposed by Ming et al.[5] is thus used to control the system.

[5] Ming Tzu Ho, Yusie Rizal, and Li Ming Chu. "Visual servoing tracking control of a ball and plate system: Design, implementation and experimental validation". In: *International Journal of Advanced Robotic Systems* 10 (2013). ISSN: 17298806. DOI: 10.5772/56525.

Approximated Feedback linearization

This method consists in a two-steps approximation: higher order coupling terms are neglected to reduce the system to two decoupled *Ball and beam* systems; feedback linearization for those kind of systems have been studied by Sastry et al.[6] who introduced a second approximation in order to obtain an input-output feedback linearizable system.

[6] John Hauser, Sastry Shankar, and Petar Kokotovic. *Nonlinear Control VIa Approximate Input-Output Linearization: The Ball and Beam Example*. 1992.

Approximated Feedback linearization

First approximation

$$\dot{x} = \begin{pmatrix} x_5 \\ x_6 \\ x_7 \\ x_8 \\ \mathcal{E}(\cancel{x_7 x_8 x_2} + x_3^2 x_1 - g\, sin x_3) \\ \mathcal{E}(\cancel{x_7 x_8 x_1} + x_4^2 x_2 - g\, sin x_4) \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \tag{16}$$

Assuming that operating ranges of velocities $x_7 = \dot{\alpha}$ and $x_8 = \dot{\beta}$ are small, high order coupling terms are therefore small and neglected.

Approximated Feedback linearization

Second approximation

We start with the differentiation to find the Feedback linearization change of variables:

$$\xi_1 = h_1(x) = x_1$$
$$\dot{\xi}_1 = L_f h_1(x) = x_5$$
$$\dot{\xi}_2 = L_f^2 h_1(x) = \mathcal{E} x_1 x_7^2 - mg\, sinx_3$$
$$\dot{\xi}_3 = L_f^3 h_1(x) + L_{g_1} L_f^2 h_1(x) = \mathcal{E} x_1 x_5 x_7^2 - x_7 mg\, cosx_3 + \cancel{2\mathcal{E} m x_1 x_7 u_1}$$

The higher order term dependent from the input is discarded, we follow up differentiating to complete the feedback linearization:

$$\dot{\xi}_4 = L_f^4 h_1(x) + L_{g_1} L_f^3 h_1(x) + L_{g_2} L_f^3 h_1(x) =$$
$$\mathcal{E}^2 x_7^2 (x_1 x_7^2 - g\, sinx_3) + \mathcal{E} g x_7^2\, sinx_3 + 2u_1(\mathcal{E} x_5 x_7 - \mathcal{E} g\, cosx_3)$$

The same is done for input $h_2(x)$ to obtain $\left(\dot{\xi}_5, \dot{\xi}_6, \dot{\xi}_7, \dot{\xi}_8\right)$.

Approximated Feedback linearization

We collect the 4-th equations of the two chains in the following matrices:

$$\Gamma(x) = \begin{pmatrix} L_f^4 h_1(x) \\ L_f^4 h_2(x) \end{pmatrix} = \begin{pmatrix} \mathcal{E}^2 x_7^2 (x_1 x_7^2 - g\ sin x_3) + \mathcal{E} g x_7^2\ sin x_3 \\ \mathcal{E}^2 x_8^2 (x_2 x_8^2 - g\ sin x_4) + \mathcal{E} g x_8^2\ sin x_x \end{pmatrix}$$

$$E(x) = \begin{pmatrix} L_{g_1} L_f^3 h_1(x) & L_{g_2} L_f^3 h_1(x) \\ L_{g_1} L_f^3 h_2(x) & L_{g_2} L_f^3 h_2(x) \end{pmatrix} = \begin{pmatrix} \mathcal{E} x_5 x_7 - \mathcal{E} g\ cos x_3 & 0 \\ 0 & \mathcal{E} x_6 x_8 - \mathcal{E} g\ cos x_4 \end{pmatrix}$$

Approximated Feedback Linearization

Given the non singularity of matrix $E(x)$ we obtain the feedback linearizing control law:

$$U = -E^{-1}(x)\Gamma(x) + E^{-1}(x)\nu \tag{17}$$

Approximated Feedback Linearization

The approximate input-output feedback linearization for the system 16 is given by:

$$\begin{pmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \\ \dot{\xi}_3 \\ \dot{\xi}_4 \\ \dot{\xi}_5 \\ \dot{\xi}_6 \\ \dot{\xi}_7 \\ \dot{\xi}_8 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \\ \xi_5 \\ \xi_6 \\ \xi_7 \\ \xi_8 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} \tag{18}$$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \xi_1 \\ \xi_5 \end{pmatrix}$$

Full state feedback regulator

Given the full controllability and observability of the system, a feedback regulator is applied to the feedback linearized system to place the poles of the plant in the stable plane. The resulting gain matrix is:

$$K = \begin{pmatrix} 24 & 50 & 13 & 10 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 24 & 50 & 13 & 10 \end{pmatrix} \tag{19}$$

PID Control

PID Control