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Dynamic modeling of SCARA robot based on Udwadia–Kalaba theory

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Abstract

With the aim of dynamic modeling of the SCARA robot subjected to space trajectory constraint to satisfy repetitive tasks with higher accuracy, a succinct and explicit equation of motion based on Udwadia–Kalaba theory is established. The trajectory constraint, which is regarded as the external constraints imposed on the system, is integrated into the dynamic modeling of the system dexterously. The explicit expression of constraint torques required to satisfy constraints and explicit dynamic equation of the system without Lagrange multiplier are obtained. However, constraint violation arises when the initial conditions are incompatible with the constraint equations. Baumgarte stabilization method is considered for constraint violation suppression. Simulations of the varying law of the generalized coordinate variables and the trajectories of the SCARA robot are performed to demonstrate the simplicity and accuracy of the method.

Keywords

SCARA robot, constraint, dynamic modeling, Udwadia–Kalaba theory, Baumgarte stabilization method

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Introduction

As a nonlinear dynamic system, the SCARA robot with four joints (e.g. three revolute and one prismatic) has better repeatability of robot movements and higher accuracy in their performance. The SCARA robot is required to follow the desired trajectory for many applications during the current stage of industrial development for productivity enhancement and quality improvement in manufactured products, such as pick and place, assembly, and packaging.

In fact, dynamic modeling of the SCARA robot to realize the desired trajectory is within the dynamic modeling of constrained multibody system. One of the key issues to address in multibody dynamics is to obtain explicit equations of motion for constrained dynamical systems.[1] Constrained dynamical systems are conventional model based on Lagrangian formulation in which Lagrangian multiplier is determined relied on problem-specific approaches.[2] When the number of the constraint equations is not equal to the number of generalized variables, the traditional Lagrangian formulation is invalidated. Moreover, there are many alternative approaches available to model constrained dynamical systems, which also need auxiliary variables. For example, Hamel's equations require Quasi Velocity as auxiliary term; Kane's[3] equations require a set of Pseudo-Generalized Speed as auxiliary terms; Gibbs–Appell formulation requires Pseudo-Velocity and Pseudo-Acceleration as auxiliary terms;[4] and Poincaré equations require a set of Pseudo-Velocity as auxiliary terms. It is worth noting that only the numerical solutions can be obtained by all of the aforementioned methods.

An explicit equation of motion for constrained systems, called the Udwadia–Kalaba equation, was proposed by Firdaus E. Udwadia and Robert E. Kalaba[5] in 1992, which opens up a new way of modeling

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complex multibody systems. This method provides a general structure for the explicit equations of motion for mechanical systems subjected to holonomic and nonholonomic equality constraints.[6,7] Moreover, it is not an issue for Udwadia–Kalaba equation based on the Gauss' principle of least constraint, and the optimal value of the generalized variables solution can still be obtained by the Moore–Penrose generalized inverse when the number of the constraint equations is not equal to the number of generalized variables. Udwadia and colleagues[8–11] also effectively dealt with the dynamics and control of nonlinear uncertain systems to benefit the basic research and further application of this method. It has been studied in some application fields, such as industrial robot,[12] parallel robot,[13] flexible multibody systems,[14] railway vehicles collision,[15] machine fish,[1] control of tethered satellites,[16] rigid multibody systems,[17] and mobile robots.[18]

Like many other robots, the SCARA robot is often required to achieve some specified space trajectories (i.e. trajectory constraint). Only a few scholars focus on the explicit dynamical equations for the SCARA robot subjected to constraints. In this article, a succinct and explicit equation of motion based on the Udwadia–Kalaba theory for the SCARA robot is established. However, constraint violation arises when the initial conditions are incompatible with the constraint equations. Due to the simplicity and easiness for computational implementation, Baumgarte stabilization method, proposed by Baumgarte[19] and then developed in numerous contributions, is probably the most popular and attractive technique to implement the control of constraint violation.[20] Therefore, Baumgarte stabilization method is considered for constraint violation suppression. The comparison of simulation results of the dynamic modeling shows that the derived equations can successfully address the dynamic modeling for the SCARA robot subjected to trajectory constraints to satisfy repetitive tasks with higher accuracy. Finally, conclusions of this work and future tasks are given.

Udwadia–Kalaba theory

The explicit equation of motion for a constrained dynamical system proposed by Udwadia and Kalaba[5] is derived in this section using a systematic general three-step procedure: (1) consideration of the so-called unconstrained system of equations obtained using Newtonian or Lagrangian mechanics, (2) description of the constraints required to model the given constrained system, and (3) expression of the constraint torques for system to satisfy the given constraints.

Unconstrained multibody dynamics

Considering an unconstrained multibody system characterized by n generalized coordinates $\mathbf{q} := [q_1\ q_2\ \cdots\ q_n]^T \in \mathbf{R}^n$, the dynamics of the system is described as

$$\mathbf{M}(\mathbf{q}, \mathrm{t})\ddot{\mathbf{q}} = \mathbf{Q}(\mathbf{q}, \dot{\mathbf{q}}, \mathrm{t}) \tag{1}$$

in which $\mathbf{M}(\mathbf{q}, t) \in \mathbf{R}^{n\times n}$ denotes a symmetric positive-definite mass matrix of the system related to $\mathbf{q}$, t; $\mathbf{Q}(\mathbf{q}, \dot{\mathbf{q}}, t) \in \mathbf{R}^n$ collects the normal and Coriolis inertial terms and the applied forces related to $\mathbf{q}$, $\dot{\mathbf{q}}$, t; $\mathbf{q} \in \mathbf{R}^n$ is the generalized coordinate n-vector; $\dot{\mathbf{q}} \in \mathbf{R}^n$ is the generalized velocity; and $\ddot{\mathbf{q}} \in \mathbf{R}^n$ is the generalized acceleration.

The generalized acceleration of the unconstrained dynamical system, defined as $\mathbf{a}(\mathbf{q}, \dot{\mathbf{q}}, t)$, is of the form

$$\mathbf{a}(\mathbf{q}, \dot{\mathbf{q}}, t) := \mathbf{M}^{-1}(\mathbf{q}, t)\mathbf{Q}(\mathbf{q}, \dot{\mathbf{q}}, t) \tag{2}$$

Constraint description

It is assumed that the system is subjected to p sufficiently smooth control requirements as constraints, which include all the usual varieties of holonomic and/or nonholonomic constraints and can be describes as[9]

$$\varphi_i(\mathbf{q}, \dot{\mathbf{q}}, t) = 0,\ i = 1, 2, \ldots, p \tag{3}$$

Equation (3) can be simplified as

$$\mathbf{\Phi} = [\varphi_1\ \varphi_2\ \cdots\ \varphi_p]^T = \mathbf{0} \tag{4}$$

Differentiating the usual constraint equation (4) with respect to time t once (for nonholonomic constraints) or twice (for holonomic constraints) yields a set of constraint equations in matrix form as[9]

$$\mathbf{A}(\mathbf{q}, \dot{\mathbf{q}}, t)\ddot{\mathbf{q}} = \mathbf{b}(\mathbf{q}, \dot{\mathbf{q}}, t) \tag{5}$$

in which $\mathbf{A}(\mathbf{q}, \dot{\mathbf{q}}, t) \in \mathbf{R}^{p\times n}$ denotes the constraint matrix, and $\mathbf{b}(\mathbf{q}, \dot{\mathbf{q}}, t) \in \mathbf{R}^p$ is a p-vector.

Constrained multibody dynamics

The intent needs to establish the explicit equations of motion with constraints. Additional generalized forces of constraints arise due to the presence of constraints. Accordingly, the actual explicit equation of motion of the constrained system can be assumed to take the form

$$\mathbf{M}(\mathbf{q}, t)\ddot{\mathbf{q}} = \mathbf{Q}(\mathbf{q}, \dot{\mathbf{q}}, t) + \mathbf{Q}^{\mathbf{c}}(\mathbf{q}, \dot{\mathbf{q}}, t) \tag{6}$$

in which $\mathbf{Q}^{\mathbf{c}}(\mathbf{q}, \dot{\mathbf{q}}, t) \in \mathbf{R}^n$ is the constraint forces imposed on the system.

Obtaining the constraint torques is a main goal in the modeling of constrained dynamical system. Udwadia and Kalaba[7] proposed that the constraint torque vector is explicitly given by

$$\mathbf{Q}^{\mathbf{c}} = \mathbf{M}^{1/2}\mathbf{B}^{+}\left(\mathbf{b} - \mathbf{A}\mathbf{M}^{-1}\mathbf{Q}\right) \tag{7}$$

in which $\mathbf{B}(\mathbf{q},t) := \mathbf{A}(\mathbf{q},\dot{\mathbf{q}},t)\mathbf{M}^{-1/2}(\mathbf{q},t)$, and the superscript " + " denotes the Moore–Penrose generalized inverse.

The explicit dynamical equation of the constrained system called as the Udwadia–Kalaba theory is denoted in the following general form[5,9]

$$\mathbf{M}\ddot{\mathbf{q}} = \mathbf{Q} + \mathbf{M}^{-1/2}\mathbf{B}^{+}\left(\mathbf{b} - \mathbf{A}\mathbf{M}^{-1}\mathbf{Q}\right) \tag{8}$$

Premultiplying both sides of equation (8) with $\mathbf{M}^{-1}$, the acceleration of the constrained system can be obtained

$$\ddot{\mathbf{q}} = \mathbf{M}^{-1}\mathbf{Q} + \mathbf{M}^{-1/2}\mathbf{B}^{+}\left(\mathbf{b} - \mathbf{B}\mathbf{M}^{-1/2}\mathbf{Q}\right) \tag{9}$$

The standard Lagrange multiplier formulation of the constrained dynamics is in the following form

$$\begin{bmatrix} \mathbf{M} & -\mathbf{A}^T \\ \mathbf{A} & \mathbf{0} \end{bmatrix}\begin{bmatrix} \ddot{\mathbf{q}} \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \mathbf{Q} \\ \mathbf{b} \end{bmatrix} \tag{10}$$

in which Lagrange multiplier $\boldsymbol{\lambda} = (\mathbf{A}\mathbf{M}^{-1}\mathbf{A}^T)^{-1}(\mathbf{b} - \mathbf{A}\mathbf{M}^{-1}\mathbf{Q})$.

Dynamic modeling of SCARA robot

Consider the SCARA robot illustrated in Figure 1. The generalized coordinates of the system are denoted as $\mathbf{q} = [q_1\ q_2\ q_3\ q_4]^T$.

Unconstrained dynamics

Result. The equations of motion of the unconstrained dynamical system are established first using the traditional Lagrange method

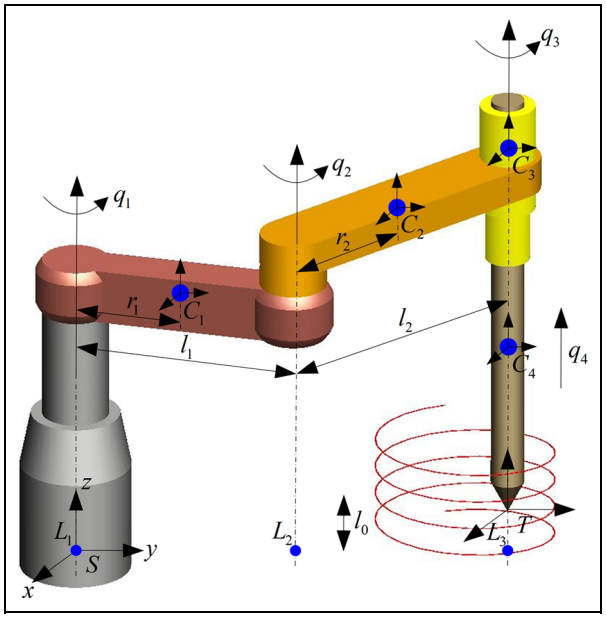

Figure 1. Sketch map of the SCARA robot manipulator subject to constraint.

$$\boldsymbol{\tau} = \mathbf{M}(\mathbf{q})\cdot\ddot{\mathbf{q}} + \mathbf{C}\cdot\dot{\mathbf{q}} + \mathbf{G}(\mathbf{q}) \tag{11}$$

in which $\mathbf{M}(\mathbf{q}) \in \mathbf{R}^{4\times4}$ is the mass matrix; $\mathbf{C}\cdot\dot{\mathbf{q}} \in \mathbf{R}^{4\times1}$ is the centrifugal force and Coriolis force matrix; $\mathbf{G}(\mathbf{q}) \in \mathbf{R}^{4\times1}$ is the gravitational force; $\mathbf{q},\dot{\mathbf{q}},\ddot{\mathbf{q}} \in \mathbf{R}^{4\times1}$ are the generalized position, velocity, and acceleration, respectively; $\boldsymbol{\tau} \in \mathbf{R}^{4\times1}$ is the joint torque vector; and[21]

$$M_{ij}(\mathbf{q}) = \sum_{l=\max(i,j)}^{n} \boldsymbol{\xi}_i^T \mathbf{A}_{li}^T \mathbb{M}_l' \mathbf{A}_{lj}\boldsymbol{\xi}_j \tag{12}$$

$$C_{ij}(\mathbf{q}) = \frac{1}{2}\sum_{k=1}^{n}\left(\frac{\partial M_{ij}}{\partial q_k} + \frac{\partial M_{ik}}{\partial q_j} - \frac{\partial M_{kj}}{\partial q_i}\right)\dot{q}_k \tag{13}$$

$$\mathbf{G}(\dot{\mathbf{q}},\mathbf{q}) = \frac{\partial \mathbf{V}}{\partial \mathbf{q}} \tag{14}$$

Proof. The coordinate frame relative to the base frame of the robot is represented as

$$g_{SC_i}(\mathbf{q}) = e^{\hat{\boldsymbol{\xi}}_1 q_1}\cdots e^{\hat{\boldsymbol{\xi}}_i q_i} g_{SC_i}(\mathbf{0}) \tag{15}$$

in which C_i is attached to the mass center of the ith link of the SCARA robot; S is the base frame of the robot; $g_{SC_i}(\mathbf{q}) = (\mathbf{R}_{SC_i}, \mathbf{t}_{SC_i}) \in SE(3)$ is the transformation between C_i and S, and $g_{SC_i}(\mathbf{0})$ is the transformation between C_i and S at $\mathbf{q} = \mathbf{0}$; $\hat{\boldsymbol{\xi}}_i$ is referred to a skew-symmetric matrix. For the revolute joint, the joint twists $\boldsymbol{\xi}_i$ is expressed as $\boldsymbol{\xi}_i = \begin{bmatrix} -\boldsymbol{\omega}_i \times \mathbf{L}_i \\ \boldsymbol{\omega}_i \end{bmatrix}$, in which $\boldsymbol{\omega}_i \in \mathbf{R}^3$ is a unit vector which specifies the direction of rotation; $\mathbf{L}_i \in \mathbf{R}^3$ is any point on the axis; for the prismatic joint, the twist $\boldsymbol{\xi}_i$ is expressed as $\boldsymbol{\xi}_i = \begin{bmatrix} \mathbf{v}_i \\ \mathbf{0} \end{bmatrix}$, in which $\mathbf{v}_i \in \mathbf{R}^3$ is a unit vector in the direction of translation.

The body velocity of the mass center of the ith link is described as

$$\mathbf{V}_{SC_i}^b = \mathbf{J}(\mathbf{q})\dot{\mathbf{q}} \tag{16}$$

in which $\mathbf{J}(\mathbf{q})$ is the body Jacobian corresponding to $g_{SC_i}(\mathbf{q})$, and

$$\mathbf{J}(\mathbf{q}) = \begin{bmatrix} \boldsymbol{\xi}_1^{\dagger} & \cdots & \boldsymbol{\xi}_i^{\dagger} & \mathbf{0} & \cdots & \mathbf{0} \end{bmatrix} \tag{17}$$

in which $\boldsymbol{\xi}_j^{\dagger} = \mathrm{Ad}^{-1}_{(e^{\hat{\boldsymbol{\xi}}_j q_j}\cdots e^{\hat{\boldsymbol{\xi}}_i q_i}\mathbf{g}_{SC_i}(\mathbf{0}))}\boldsymbol{\xi}_j\ (j \le i)$ is the jth instantaneous joint twist relative to the ith link frame C_i.

The total kinetic energy of the robot has the form

$$\begin{aligned} T(\dot{\mathbf{q}},\mathbf{q}) &= \sum_{i=1}^{n} T_i(\dot{\mathbf{q}},\mathbf{q}) = \sum_{i=1}^{n}\frac{1}{2}\left(\mathbf{V}_{SC_i}^b\right)^T \mathbb{M}_i \mathbf{V}_{SC_i}^b \\ &= \sum_{i=1}^{n}\frac{1}{2}\dot{\mathbf{q}}^T\mathbf{J}_i^T(\mathbf{q})\mathbb{M}_i\mathbf{J}_i(\mathbf{q})\dot{\mathbf{q}} = :\frac{1}{2}\dot{\mathbf{q}}^T\mathbf{M}(\mathbf{q})\dot{\mathbf{q}} \end{aligned} \tag{18}$$

in which $T_i(\dot{\mathbf{q}}, \mathbf{q})$ and $\mathbb{M}_i$ are the kinetic energy and the generalized inertia matrix of the ith link, respectively.

The total potential energy has the form

$$V(\mathbf{q}) = \sum_{i=1}^{n} V_i(\mathbf{q}) = \sum_{i=1}^{n} m_i g h_i(\mathbf{q}) \tag{19}$$

in which $V_i(\mathbf{q})$ is the potential energy of the ith link; m_i is the mass of the ith link; g is the gravitational constant; and $h_i(\mathbf{q})$ is the height of the mass center of the ith link.

The Lagrange function can be obtained as

$$\begin{aligned} L(\dot{\mathbf{q}}, \mathbf{q}) &= \sum_{i=1}^{n} (T_i(\dot{\mathbf{q}}, \mathbf{q}) - V_i(\mathbf{q})) \\ &= \frac{1}{2}\dot{\mathbf{q}}^T \mathbf{M}(\mathbf{q})\dot{\mathbf{q}} - V(\mathbf{q}) = \frac{1}{2}\sum_{i,j=1}^{n} M_{ij}(\mathbf{q})\dot{q}_i\dot{q}_j - V(\mathbf{q}) \end{aligned} \tag{20}$$

So far, the equations of motion can be obtained by substituting into Lagrange's equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = F_i \tag{21}$$

in which F_i represents the generalized forces acting on the ith joint, and

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_i} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{j=1}^{n} M_{ij}\dot{q}_j\right) = \sum_{j=1}^{n}\left(M_{ij}\ddot{q}_j + \dot{M}_{ij}\dot{q}_j\right) \tag{22}$$

$$\frac{\partial L}{\partial q_i} = \frac{1}{2}\sum_{j,k=1}^{n}\frac{\partial M_{kj}}{\partial q_i}\dot{q}_k\dot{q}_j - \frac{\partial V}{\partial q_i} \tag{23}$$

1. The matrix $\mathbf{M}(\mathbf{q}) \in \mathbf{R}^{n\times n}$ is obtained using the following procedure.

In terms of the link Jacobians, $\mathbf{J}_i$, the manipulator inertia matrix is defined as

$$\mathbf{M}(\mathbf{q}) = \sum_{i=1}^{n} \mathbf{J}_i^T(\mathbf{q})\mathbb{M}_i\mathbf{J}_i(\mathbf{q}) \tag{24}$$

in which $\mathbf{J}_i(\mathbf{q}) = \mathrm{Ad}_{\mathbf{g}_{SC_i(\mathbf{0})}^{-1}}[\mathbf{A}_{i1}\boldsymbol{\xi}_1 \quad \cdots \quad \mathbf{A}_{ii}\boldsymbol{\xi}_i \ \mathbf{0} \quad \cdots \quad \mathbf{0}]$. $\mathrm{Ad}_{\mathbf{g}_{SC_i(\mathbf{0})}^{-1}}$ is referred to as the adjoint transformation associated with $\mathbf{g}_{SC_i(\mathbf{0})}^{-1}$. $\mathrm{Ad}_{\mathbf{g}_{SC_i}^{-1}}\mathbf{A}_{ij}\boldsymbol{\xi}_j$ represents the jth column of the body Jacobian for the ith link, in which $\mathbf{A}_{ij}$ is the adjoint transformation given by

$$\mathbf{A}_{ij} = \begin{cases} \mathrm{Ad}^{-1}_{(e^{\hat{\boldsymbol{\xi}}_{j+1}q_{j+1}}\cdots e^{\hat{\boldsymbol{\xi}}_i q_i})} & i > j \\ \mathbf{I} & i = j \\ \mathbf{0} & i < j \end{cases} \tag{25}$$

The inertia of the ith link reflected into the base frame of the manipulator, $\mathbb{M}_i'$, is given by

$$\mathbb{M}_i' = \mathrm{Ad}^T_{\mathbf{g}_{SC_i(\mathbf{0})}^{-1}}\mathbb{M}_i\mathrm{Ad}_{\mathbf{g}_{SC_i(\mathbf{0})}^{-1}} \tag{26}$$

Using the notation defined above, the inertia matrix for the manipulator can be obtained as[21]

$$M_{ij}(\mathbf{q}) = \sum_{l=\max(i,j)}^{n} \boldsymbol{\xi}_i^T\mathbf{A}_{li}^T\mathbb{M}_l'\mathbf{A}_{lj}\boldsymbol{\xi}_j \tag{27}$$

2. Substituting equations (22) and (23) into equation (21) and expanding $\dot{M}_{ij}$ in terms of partial derivatives, one can obtain

$$\begin{aligned} &\sum_{j=1}^{n} M_{ij}(\mathbf{q})\ddot{q}_j + \sum_{j,k=1}^{n}\left(\frac{\partial M_{ij}}{\partial q_k}\dot{q}_j\dot{q}_k - \frac{1}{2}\frac{\partial M_{kj}}{\partial q_i}\dot{q}_k\dot{q}_j\right) \\ &+ \frac{\partial V}{\partial q_i}(\mathbf{q}) = F_i, \quad i = 1, 2, \ldots, n \end{aligned} \tag{28}$$

Rearranging all the terms, equation (28) can be rewritten as

$$\begin{aligned} &\sum_{j=1}^{n} M_{ij}(\mathbf{q})\ddot{q}_j + \sum_{j,k=1}^{n}\Gamma_{ijk}\dot{q}_j\dot{q}_k + \frac{\partial V}{\partial \mathrm{q_i}}(\mathbf{q}) = F_i, \\ &i = 1, 2, \ldots, n \end{aligned} \tag{29}$$

in which $\Gamma_{ijk} = \frac{1}{2}\left(\frac{\partial M_{ij}(\mathbf{q})}{\partial q_k} + \frac{\partial M_{ik}(\mathbf{q})}{\partial q_j} - \frac{\partial M_{kj}(\mathbf{q})}{\partial q_i}\right)$.

The matrix $\mathbf{C}(\dot{\mathbf{q}}, \mathbf{q})$ is considered to be defined in order to put equation (29) back into vector form[21]

$$C_{ij}(\dot{\mathbf{q}}, \mathbf{q}) = \sum_{k=1}^{n}\Gamma_{ijk}\dot{q}_K = \frac{1}{2}\sum_{k=1}^{n}\left(\frac{\partial M_{ij}}{\partial q_k} + \frac{\partial M_{ik}}{\partial q_j} - \frac{\partial M_{kj}}{\partial q_i}\right)\dot{q}_k \tag{30}$$

in which $\frac{\partial M_{ij}}{\partial q_k} = \sum_{l=\max(i,j)}^{n}\left([\mathbf{A}_{ki}\boldsymbol{\xi}_i, \boldsymbol{\xi}_k]^T\mathbf{A}_{lk}^T\mathbb{M}_l'\mathbf{A}_{lj}\boldsymbol{\xi}_j + \boldsymbol{\xi}_i^T\mathbf{A}_{li}^T\mathbb{M}_l'\mathbf{A}_{lk}\left[\mathbf{A}_{kj}\boldsymbol{\xi}_j, \boldsymbol{\xi}_k\right]\right)$.

3. Finally, the gravitational forces on the manipulator are written as[21]

$$\mathbf{G}(\dot{\mathbf{q}}, \mathbf{q}) = \frac{\partial \mathbf{V}}{\partial \mathbf{q}} \tag{31}$$

For the SCARA robot, considering $g_{ST} = (\mathbf{R}_{ST}, \mathbf{t}_{ST}) \in SE(3)$, the transformation between tool and base frames at $\mathbf{q} = \mathbf{0}$ is given by

$$g_{ST}(\mathbf{0}) = \begin{bmatrix} \mathbf{I} & \begin{pmatrix} 0 \\ l_1 + l_2 \\ l_0 \end{pmatrix} \\ \mathbf{0} & 1 \end{bmatrix} \tag{32}$$

$$e^{\hat{\xi}q} = \begin{bmatrix} e^{\hat{\omega}q} & \left(\mathbf{I} - e^{\hat{\omega}q}\right)(\boldsymbol{\omega} \times \mathbf{v}) + q\boldsymbol{\omega}\boldsymbol{\omega}^T\mathbf{v} \\ \mathbf{0} & 1 \end{bmatrix} \tag{37}$$

in which

$$\begin{aligned} e^{\hat{\omega}q} &= I + \hat{\omega}\mathrm{s}q + \hat{\omega}^2(1 - \mathrm{c}q) \\ &= \begin{bmatrix} \omega_1^2(1 - \mathrm{c}q) + \mathrm{c}q & \omega_1\omega_2(1 - \mathrm{c}q) - \omega_3\mathrm{s}q & \omega_1\omega_3(1 - \mathrm{c}q) + \omega_2\mathrm{s}q \\ \omega_1\omega_2(1 - \mathrm{c}q) + \omega_3\mathrm{s}q & \omega_2^2(1 - \mathrm{c}q) + \mathrm{c}q & \omega_2\omega_3(1 - \mathrm{c}q) - \omega_1\mathrm{s}q \\ \omega_1\omega_3(1 - \mathrm{c}q) - \omega_2\mathrm{s}q & \omega_2\omega_3(1 - \mathrm{c}q) + \omega_1\mathrm{s}q & \omega_3^2(1 - \mathrm{c}q) + \mathrm{c}q \end{bmatrix} \\ \mathrm{c}q &= \cos q, \quad \mathrm{s}q = \sin q \end{aligned}$$

For the three revolute joints, the direction of rotation is described as $\boldsymbol{\omega}_1 = \boldsymbol{\omega}_2 = \boldsymbol{\omega}_3 = [0 \quad 0 \quad 1]^{\mathrm{T}}$, and the points on the axis are described as $\mathbf{L}_1 = [0 \quad 0 \quad 1]^{\mathrm{T}}$, $\mathbf{L}_2 = [0 \quad l_1 \quad 0]^{\mathrm{T}}$, and $\mathbf{L}_3 = [0 \quad l_1 + l_2 \quad 0]^{\mathrm{T}}$. For the prismatic joint, the direction of translation is described as $\mathbf{v}_4 = [0 \quad 0 \quad 1]$. This yields twists

$$\begin{aligned} \boldsymbol{\xi}_1 &= [0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1]^{\mathrm{T}} \\ \boldsymbol{\xi}_2 &= [l_1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1]^{\mathrm{T}} \\ \boldsymbol{\xi}_3 &= [l_1 + l_2 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1]^{\mathrm{T}} \\ \boldsymbol{\xi}_4 &= [0 \quad 0 \quad 1 \quad 0 \quad 0 \quad 0]^{\mathrm{T}} \end{aligned} \tag{33}$$

The inertia matrix $\mathrm{M}(\mathbf{q}) \in \mathbf{R}^{4\times4}$ for the SCARA robot can be derived as[21]

$$\mathbf{M}(\mathbf{q}) = \begin{bmatrix} \alpha + \beta + 2\gamma\mathrm{c}_2 & \beta + \gamma\mathrm{c}_2 & \delta & 0 \\ \beta + \gamma\mathrm{c}_2 & \beta & \delta & 0 \\ \delta & \delta & \delta & 0 \\ 0 & 0 & 0 & m_4 \end{bmatrix} \tag{34}$$

in which $\alpha = I_{z1} + r_1^2m_1 + l_1^2m_2 + l_1^2m_3 + l_1^2m_4$, $\beta = I_{z2} + I_{z3} + I_{z4} + l_2^2m_3 + l_2^2m_4 + m_2r_2^2$, $\gamma = l_1l_2m_3 + l_1l_2m_4 + l_1m_2r_2$, and $\delta = I_{z3} + I_{z4}$. Note that $\mathrm{c}_i = \cos q_i$, $\mathrm{s}_i = \sin q_i$, $\mathrm{c}_{ij} = \cos(q_i + q_j)$, and $\mathrm{s}_{ij} = \sin(q_i + q_j)$.

The Coriolis matrix for the SCARA robot can be obtained as[21]

$$\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}}) = \begin{bmatrix} -\gamma\sin q_2\dot{q}_2 & -\gamma\sin q_2(\dot{q}_1 + \dot{q}_2) & 0 & 0 \\ \gamma\sin q_2\dot{q}_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{35}$$

The gravitational forces on the SCARA robot is expressed as[21]

$$\mathbf{G}(\mathbf{q}, \dot{\mathbf{q}}) = [0 \quad 0 \quad 0 \quad m_4g]^T \tag{36}$$

Constraint description

The exponential of this twist is given by

The forward kinematic map of the SCARA robot is given by

$$\mathbf{g}_{ST}(\mathbf{q}) = e^{\hat{\xi}_1q_1}e^{\hat{\xi}_2q_2}e^{\hat{\xi}_3q_3}e^{\hat{\xi}_4q_4}g_{ST}(\mathbf{0}) = \begin{bmatrix} \mathbf{R}(\mathbf{q}) & \mathbf{p}(\mathbf{q}) \\ \mathbf{0} & 1 \end{bmatrix} \tag{38}$$

in which the individual exponentials have the form

$$e^{\hat{\xi}_1q_1} = \begin{bmatrix} \mathrm{c}_1 & -\mathrm{s}_1 & 0 & 0 \\ \mathrm{s}_1 & \mathrm{c}_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$e^{\hat{\xi}_2q_2} = \begin{bmatrix} \mathrm{c}_2 & -\mathrm{s}_2 & 0 & l_1\mathrm{s}_2 \\ \mathrm{s}_2 & \mathrm{c}_2 & 0 & l_1(1 - \mathrm{c}_2) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$e^{\hat{\xi}_3q_3} = \begin{bmatrix} \mathrm{c}_3 & -\mathrm{s}_3 & 0 & (l_1 + l_2)\mathrm{s}_3 \\ \mathrm{s}_3 & \mathrm{c}_3 & 0 & (l_1 + l_2)(1 - \mathrm{c}_3) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$e^{\hat{\xi}_4q_4} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & q_4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Expanding the terms in the product of exponentials formula yields

$$\begin{aligned} \mathbf{g}_{ST}(\mathbf{q}) &= \begin{bmatrix} \mathbf{R}(\mathbf{q}) & \mathbf{p}(\mathbf{q}) \\ \mathbf{0} & 1 \end{bmatrix} \\ &= \begin{bmatrix} \mathrm{c}_{123} & -\mathrm{s}_{123} & 0 & -l_1\mathrm{s}_1 - l_2\mathrm{s}_{12} \\ \mathrm{s}_{123} & \mathrm{c}_{123} & 0 & l_1\mathrm{c}_1 + l_2\mathrm{c}_{12} \\ 0 & 0 & 1 & l_0 + q_4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{aligned} \tag{39}$$

The end-effector of the SCARA robot moves along space trajectory to complete the given tasks. The desired space trajectory, which is regarded as constraints, is presented as follows.

The position constraints can be described as

$$\begin{aligned} x &= 0.05\sin(0.4\pi t) \\ y &= 0.35 + 0.05\cos(0.4\pi t) \\ z &= 0.02t \end{aligned} \tag{40}$$

The following elementary rotations about the x-, y-, and z-axis are defined as

$$\begin{aligned} R_x(\theta) &:= e^{\hat{\mathbf{x}}\theta} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \\ R_y(\theta) &:= e^{\hat{\mathbf{y}}\theta} = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \\ R_z(\theta) &:= e^{\hat{\mathbf{z}}\theta} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned} \tag{41}$$

ZYZ Euler angles are used to express the orientation of the end-effector of the SCARA robot. The equivalent rotation matrix is denoted as

$$\begin{aligned} {}^S_T R_{zyz}(\varphi,\psi,\gamma) &= R_z(\varphi)R_y(\psi)R_z(\gamma) = e^{\hat{z}\varphi}e^{\hat{y}\psi}e^{\hat{z}\gamma} \\ &= \begin{bmatrix} c\varphi c\psi c\gamma - s\varphi s\gamma & -c\varphi c\psi s\gamma - s\varphi c\gamma & c\varphi s\psi \\ s\varphi c\psi c\gamma + c\varphi s\gamma & -s\varphi c\psi s\gamma + c\varphi c\gamma & s\varphi s\psi \\ -s\psi c\gamma & s\psi s\gamma & c\psi \end{bmatrix} \end{aligned} \tag{42}$$

Conversely, equivalent ZYZ Euler angles can be solved by the rotation matrix, as shown below. If

$${}^S_T R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \begin{bmatrix} \mathrm{c}_{124} & -\mathrm{s}_{123} & 0 \\ \mathrm{s}_{123} & \mathrm{c}_{123} & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{43}$$

and $\sin\psi \neq 0$, then

$$\left.\begin{aligned} \varphi &= \mathrm{A}\tan 2(r_{23}, r_{13}) \\ \psi &= \mathrm{A}\tan 2\left(\sqrt{r_{31}^2 + r_{32}^2}, r_{33}\right) \\ \gamma &= \mathrm{A}\tan 2(r_{32}, -r_{31}) \end{aligned}\right\} \tag{44}$$

The solution of ψ is always in the range of $0 \le \psi \le 180^\circ$, although two solutions can be obtained from $\sin\psi = \sqrt{r_{31}^2 + r_{32}^2}$. The above solutions degenerate when $\psi = 0^\circ$ or 180°; at this point, only the sum or difference between φ and γ is obtained. $\varphi = 0^\circ$ is usually selected. If $\psi = 0^\circ$, then

$$\begin{cases} \alpha = 0^\circ \\ \beta = 0^\circ \\ \gamma = \mathrm{A}\tan 2(-r_{12}, r_{11}) \end{cases} \tag{45}$$

The orientation constraint equation combined with equation (43) can be derived as

$$q_1 + q_2 + q_3 = 0 \tag{46}$$

Now that there are the following four constraints in total

$$\begin{cases} -l_1\mathrm{s}_1 - l_2\mathrm{s}_{12} = 0.05\sin(0.4\pi t) \\ l_1\mathrm{c}_1 + l_2\mathrm{c}_{12} = 0.35 + 0.05\cos(0.4\pi t) \\ q_1 + q_2 + q_3 = 0 \\ q_4 + l_0 = 0.02t \end{cases} \tag{47}$$

In general, the constraint equations may be thought of as a set of p ($p = 4$) constraints of the form

$$\begin{aligned} \mathbf{\Phi}(t) &= \begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \varphi_4 \end{bmatrix} \\ &= \begin{bmatrix} -l_1\mathrm{s}_1 - l_2\mathrm{s}_{12} - 0.05\sin(0.4\pi t) \\ l_1\mathrm{c}_1 + l_2\mathrm{c}_{12} - 0.35 - 0.05\cos(0.4\pi t) \\ q_1 + q_2 + q_3 \\ q_4 + l_0 - 0.02t \end{bmatrix} = \mathbf{0} \end{aligned} \tag{48}$$

Differentiating equation (48) with respect to time t twice yields the second-order constraints

$$\mathbf{A}\ddot{\mathbf{q}} = \mathbf{b} \tag{49}$$

in which

$$\mathbf{A} = \begin{bmatrix} -l_1\mathrm{c}_1 - l_2\mathrm{c}_{12} & -l_2\mathrm{c}_{12} & 0 & 0 \\ -l_1\mathrm{s}_1 - l_2\mathrm{s}_{12} & -l_2\mathrm{s}_{12} & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{b} = \begin{bmatrix} -\dot{q}_1^2 l_1\mathrm{s}_1 - (\dot{q}_1 + \dot{q}_2)^2 l_2\mathrm{s}_{12} - 0.05 \times (0.4\pi)^2 \sin(0.4\pi t) \\ \dot{q}_1^2 l_1\mathrm{c}_1 + (\dot{q}_1 + \dot{q}_2)^2 l_2\mathrm{c}_{12} - 0.05 \times (0.4\pi)^2 \cos(0.4\pi t) \\ 0 \\ 0 \end{bmatrix}$$

Constraint torque description

Constraint torques for the system to meet the desired trajectory are explicitly determined by equation (7) in the form of

$$\mathbf{Q}^{\mathbf{c}} = \mathbf{M}^{1/2}(\mathbf{A}\mathbf{M}^{-1/2})^{+}(\mathbf{b} - \mathbf{A}\mathbf{M}^{-1}\mathbf{Q}) \tag{50}$$

Finally, the dynamic equation is obtained by equations (6), (7), and (50)

$$\mathbf{M}\ddot{\mathbf{q}} = \mathbf{Q} + \mathbf{Q}^{c} \tag{51}$$

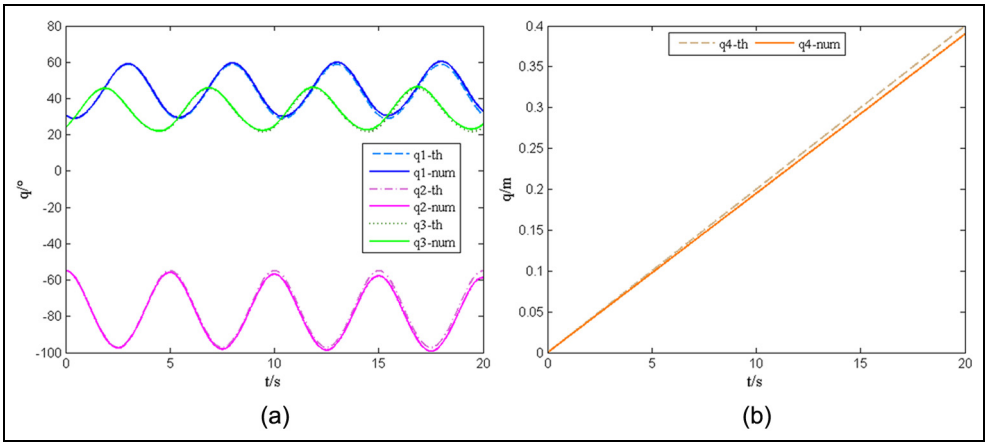

Figure 2. Comparisons of dynamic responses between numerical value and theoretical value: (a) q_1, q_2, q_3 and (b) q_4.

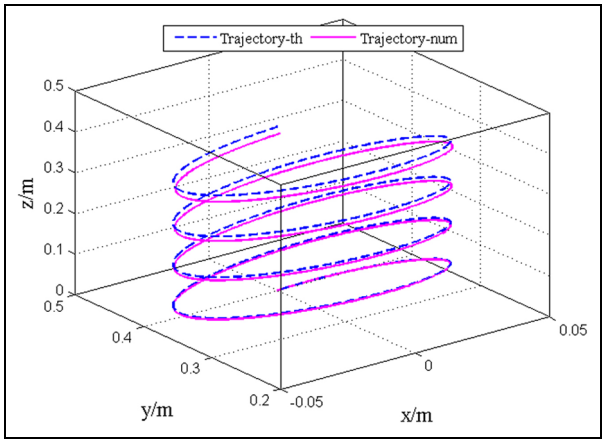

Figure 3. Comparison of the trajectories between numerical value and theoretical value.

Numerical simulations

The constraints must be satisfied at each instant of time during the maneuver including the initial time. In practice, however, it is usually quite difficult to meet these constraints at the initial time.[22] In other words, the initial conditions are incompatible with the constraint equations. It is assumed that the initial configuration for simulation are as follows: $q_1(0) = 30°$, $q_2(0) - 55°$, $q_3(0) = 24°$, and $q_4(0) = 0\,\text{m}$; $\dot{q}_1(0) = -0.157\,\text{rad/s}$, $\dot{q}_2(0) = 0.0001\,\text{rad/s}$, $\dot{q}_3(0) = 0.157\,\text{rad/s}$, and $\dot{q}_4(0) = 0.0195\,\text{m/s}$. The parameter values for simulation are as follows: $m_1 = 20\,\text{kg}$, $m_2 = m_3 = 15\,\text{kg}$, $m_4 = 0.5\,\text{kg}$, $l_0 = 0\,\text{m}$, $l_1 = 0.2\,\text{m}$, $l_2 = 0.25\,\text{m}$, $r_1 = 0.1\,\text{m}$, $r_2 = 0.125\,\text{m}$, $I_{z1} = 0.27\,\text{kg}\cdot\text{m}^2$, $I_{z2} = 0.31\,\text{kg}\cdot\text{m}^2$, $I_{z3} = 0.02\,\text{kg}\cdot\text{m}^2$, and $I_{z4} = 0.0001\,\text{kg}\cdot\text{m}^2$. The simulation results are shown in Figures 2–5.

Figure 2 represents the dynamic responses at each joint of the SCARA robot subjected to the space trajectory constraints, in which the solid curves and the dash curves represent the numerical value and the theoretical value, respectively. The numerical trajectories is slightly off the theoretical trajectory with time, as illustrated in Figure 3, in which the solid curve and the dash curve represent the numerical trajectory and the theoretical trajectory, respectively. However, the joint errors and

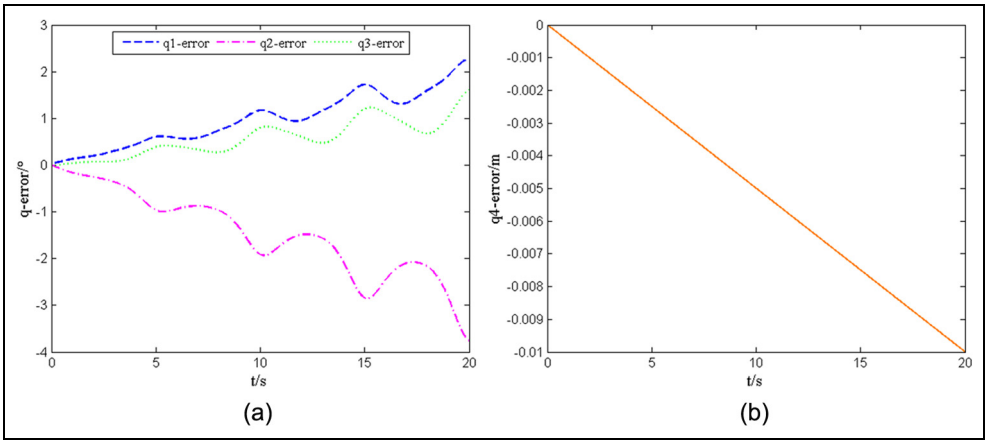

Figure 4. Sketch map of joint errors between numerical value and theoretical value: (a) q_1-error, q_2-error, q_3-error and (b) q_4-error.

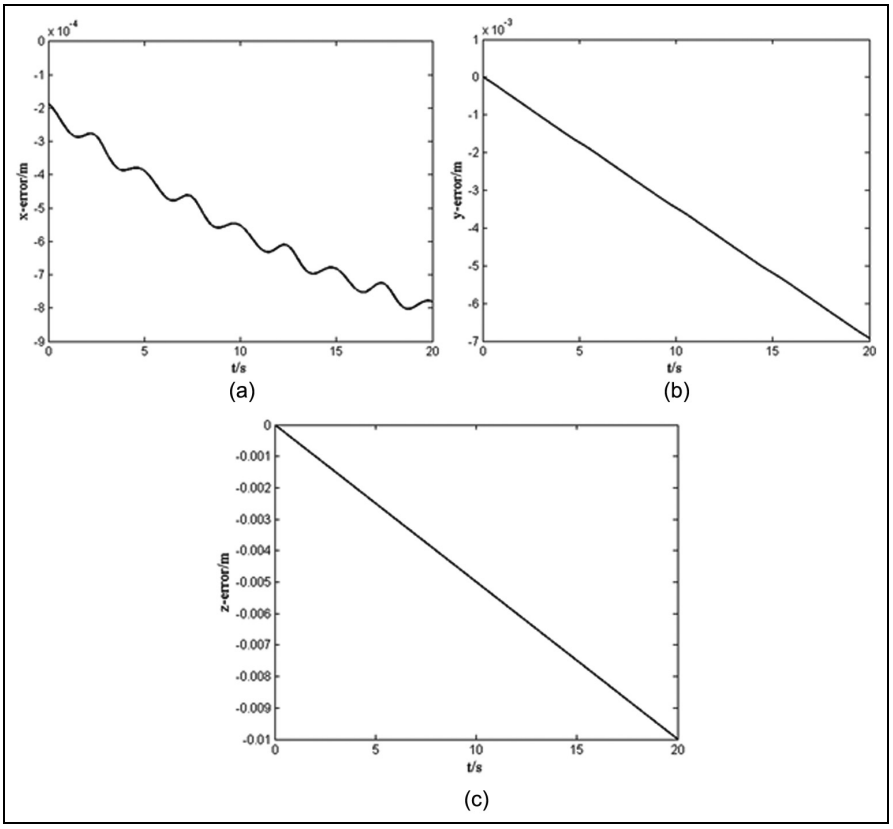

Figure 5. Displacement errors between numerical value and theoretical value: (a) *x*-error, (b) *y*-error, and (c) *z*-error.

displacement errors in the satisfaction of the constraints tend to increase in time and spoil reliability of the simulation results, as shown in Figures 4 and 5, respectively. The largest displacement errors correspond to about 8×10^{-4} m in *x*-axis, 7×10^{-3} m in *y*-axis, and 0.01 m in *z*-axis, which should not be neglected. Therefore, it is necessary to consider the numerical method for reducing the errors.

Error reducing

Constraint violation arises when the initial conditions are incompatible with the constraint equations. This makes the simulation results seriously different from real situations. Accordingly, it is necessary for the use of Baumgarte stabilization method to suppress constraint violation. In the standard formulation of the method, the (open-loop) second-order differential constraints, $\ddot{\mathbf{\Phi}} = \mathbf{0}$, are replaced with its stabilized (closed-loop) form[23]

$$\ddot{\mathbf{\Phi}} + \boldsymbol{\alpha}\dot{\mathbf{\Phi}} + \boldsymbol{\beta}\mathbf{\Phi} = \mathbf{0} \tag{52}$$

in which $\boldsymbol{\alpha} = diag\{\alpha_1, \alpha_2, \ldots, \alpha_p\}$ and $\boldsymbol{\beta} = diag\{\beta_1, \beta_2, \ldots, \beta_p\}$.

If α_i, $\beta_i > 0$, $\mathbf{\Phi}$ approaches to zero asymptotically. Thus, from equation (52), a more general constraint equation is obtained, which nonetheless retains the form of equation (49).[22] Thus, utilizing the Baumgarte stabilization method, the equations of motion for a system subjected to constraints are stated in the form

$$\begin{bmatrix} \mathbf{M} & -\mathbf{A}^T \\ \mathbf{A} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{q}} \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \mathbf{Q} \\ \mathbf{b} - \boldsymbol{\alpha}\dot{\mathbf{\Phi}} - \boldsymbol{\beta}\mathbf{\Phi} \end{bmatrix} \tag{53}$$

Then, equation (52) yields the following constraint equation

$$\begin{bmatrix} -l_1c_1 - l_2c_{12} & -l_2c_{12} & 0 & 0 \\ -l_1s_1 - l_2s_{12} & -l_2s_{12} & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \\ \ddot{q}_3 \\ \ddot{q}_4 \end{bmatrix} = \begin{bmatrix} -\dot{q}_1^2 l_1 s_1 - (\dot{q}_1 + \dot{q}_2)^2 l_2 s_{12} - 0.05 \times (0.4\pi)^2 \sin(0.4\pi t) \\ -\alpha_1(-\dot{q}_1 l_1 c_1 - (\dot{q}_1 + \dot{q}_2) l_2 c_{12} - 0.05 \times (0.4\pi) \cos(0.4\pi t)) \\ -\beta_1(-l_1 s_1 - l_2 s_{12} - 0.05 \sin(0.4\pi t)) \\ \dot{q}_1^2 l_1 c_1 + (\dot{q}_1 + \dot{q}_2)^2 l_2 c_{12} - 0.05 \times (0.4\pi)^2 \cos(0.4\pi t) \\ -\alpha_2(-\dot{q}_1 l_1 s_1 - (\dot{q}_1 + \dot{q}_2) l_2 s_{12} + 0.05 \times (0.4\pi) \sin(0.4\pi t)) \\ -\beta_2(l_1 c_1 + l_2 c_{12} - 0.35 - 0.05 \cos(0.4\pi t)) \\ -\alpha_3(\dot{q}_1 + \dot{q}_2 + \dot{q}_3) - \beta_3(q_1 + q_2 + q_3) \\ -\alpha_4(\dot{q}_4 - 0.02) - \beta_4(q_4 + l_0 - 0.02t) \end{bmatrix} \tag{54}$$

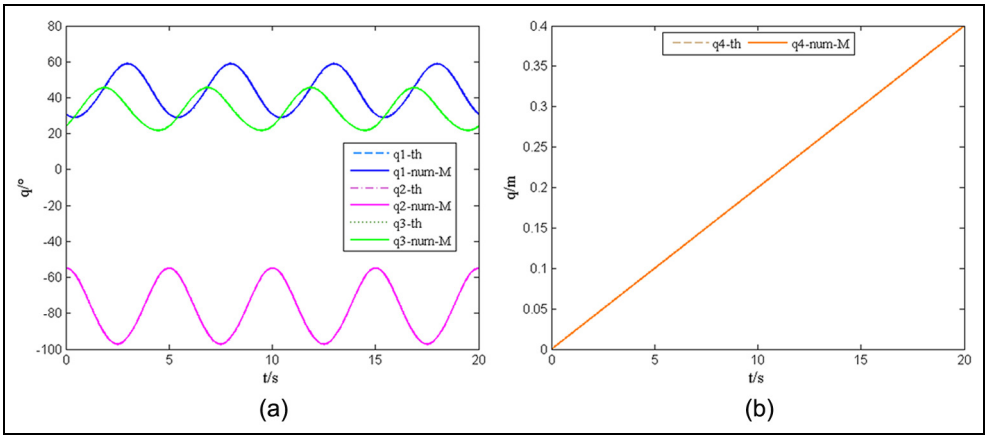

Figure 6. Comparisons of dynamic responses between modified numerical value and theoretical value: (a) q_1, q_2, q_3 and (b) q_4.

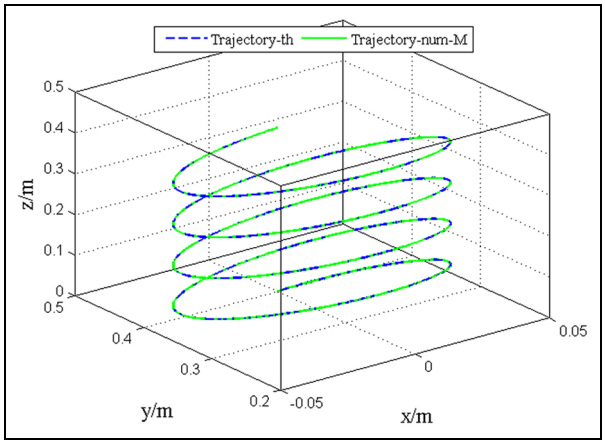

Figure 7. Comparison of the trajectories between modified numerical value and theoretical value.

Note that an improper choice of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ can lead to unacceptable results in the dynamic analysis of the multibody systems.[24] Baumgarte[19] highlighted that the suitable choice of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ is performed by numerical experiments. For simplicity, we choose $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0.5$ and $\beta_1 = \beta_2 = \beta_3 = \beta_4 = 200$. The simulation results are presented in Figures 6–10.

1. Figure 6 represents the dynamic response curves as a function of time t between modified numerical value and theoretical value, in which the solid curves and the dash curves represent the modified numerical value and the theoretical value, respectively.
2. The trajectories are almost coincident by comparing between modified numerical value and theoretical value, as illustrated in Figure 7, in which the solid curves and the dash curves represent the modified numerical value and the theoretical value, respectively. The overall effect of constraint trajectory obtained by modified numerical integration gets clear improvement.
3. Comparing with the qi-error between numerical value and theoretical value, the amplitude of qi-error between modified numerical value and theoretical value is substantially lowered, as illustrated in Figure 8. Obviously, the errors are within an acceptable range that the constraint relations in x-, y-, and z-axis are being maintained well as desired, as depicted in Figure 9.
4. Figure 10 represents the constraint torques to make the SCARA robot follow the space trajectory for precise application.

Conclusion

Aiming at dynamic modeling of the SCARA robot subjected to constraint, this article establishes the dynamic equation based on the Udwadia–Kalaba theory and reduces the errors using Baumgarte stabilization method. The comparison of numerical simulations confirms the ease of implementation and the high numerical accuracy of this methodology. The research process reveals some conclusions as follows:

1. The desired space trajectory, which is regarded as constraints imposed on the system, is integrated into the dynamic modeling of the system dexterously.
2. The explicit, closed-form expression for the constraint torques required to guarantee the end-effector of the SCARA robot to move along the desired space trajectory and the explicit dynamic equation of the system without Lagrange multiplier are obtained.

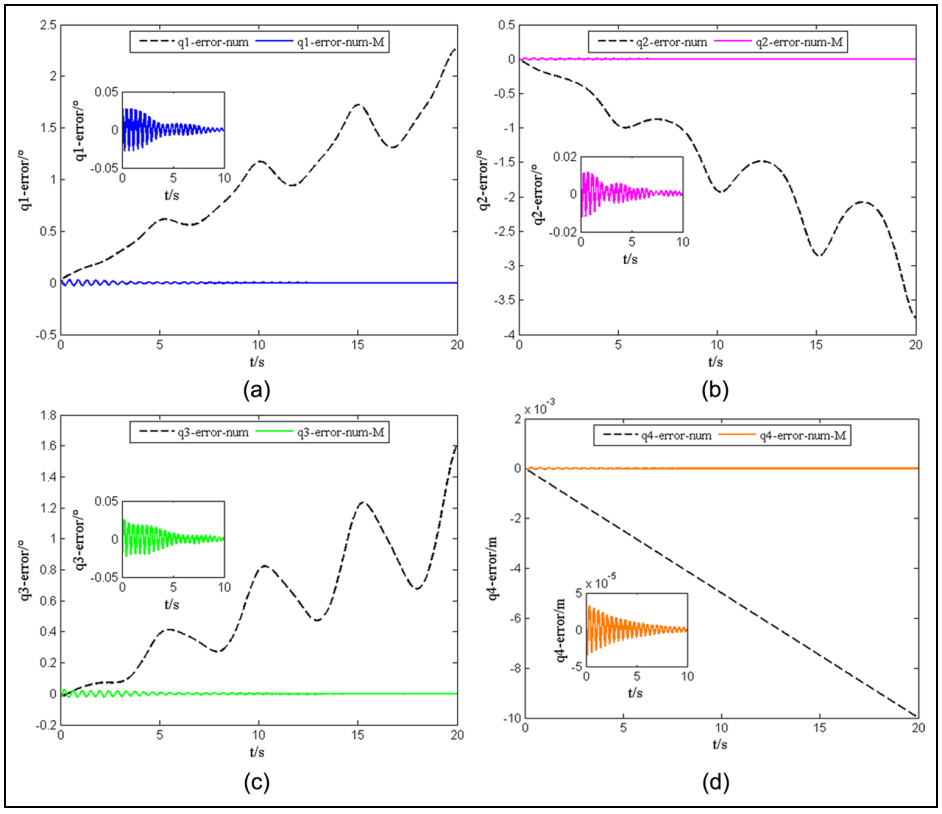

Figure 8. Comparisons of joint errors for-and-aft modification: (a) q_1-error, (b) q_2-error, (c) q_3-error, and (d) q_4-error.

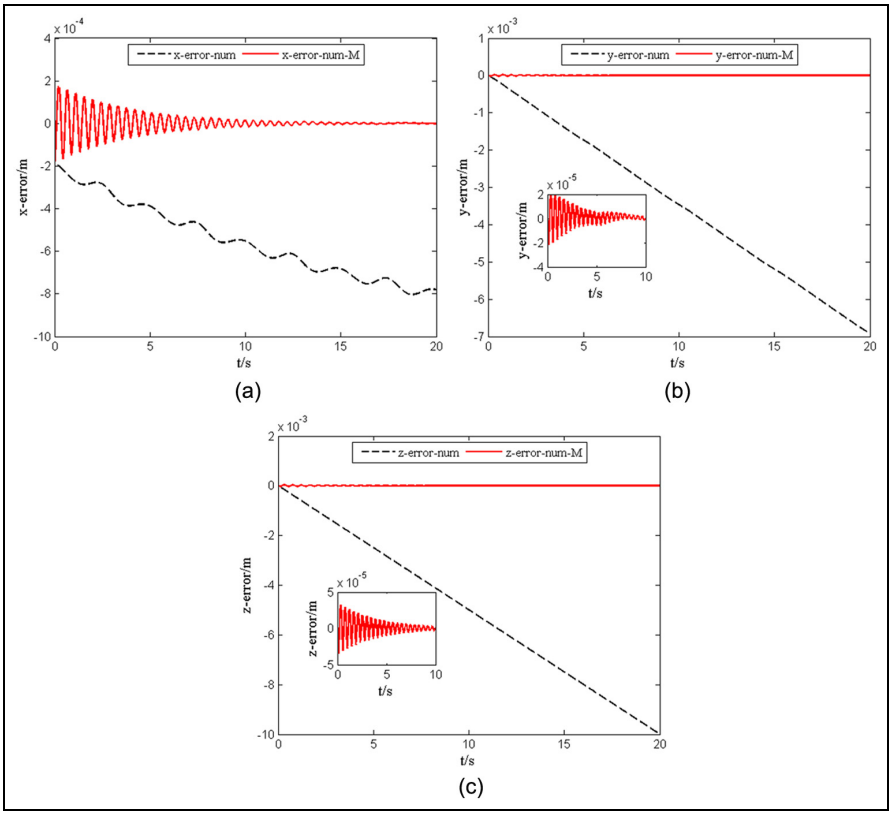

Figure 9. Comparison of displacement errors between for-and-aft modification: (a) x-error, (b) y-error, and (c) z-error.

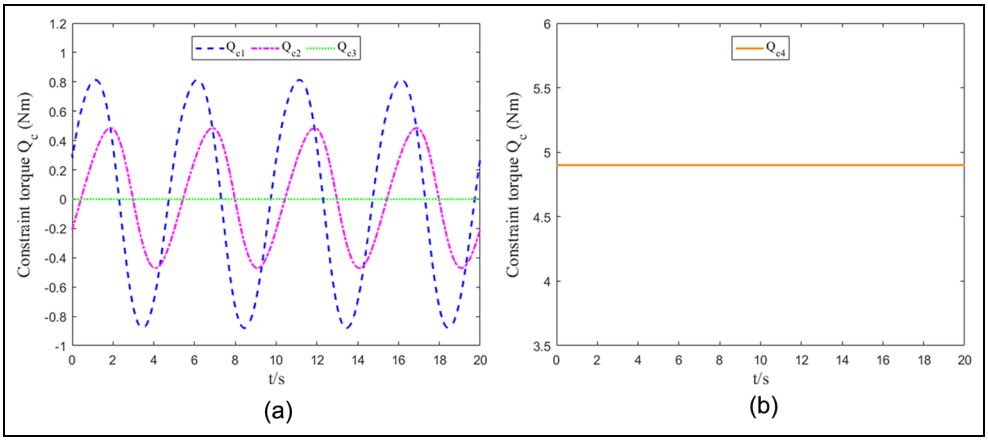

Figure 10. Constraint torques: (a) Qc1, Qc2, Qc3, and (b) Qc4.

3. Baumgarte stabilization method is used for constraint violation suppression to realize trajectory motion with high accuracy.

Declaration of conflicting interests

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