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On Manipulator Control by Exact Linearization

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Abstract—**Comments on the application to rigid link manipulators of geometric control theory, resolved acceleration control, operational space control, and nonlinear decoupling theory are given, and the essential unity of these techniques for externally linearizing and decoupling end effector dynamics is discussed. Exploiting the fact that the mass matrix of a rigid link manipulator is positive definite, and the fact that there is an independent input for each degree of freedom, it is shown that a necessary and sufficient condition for a locally externally linearizing and output decoupling feedback law to exist is that the end effector Jacobian matrix be nonsingular.**

I. Introduction

An "exactly linearizing" control makes a nonlinear system behave as if it has linear and decoupled dynamics. It has been known at least since the early 1970's [1]–[5] that exact linearization of manipulators in joint space is accomplished by the so-called inverse or computed torque technique. Efforts to accomplish decoupled linearization of end effector (EE) motions directly in task space began soon thereafter [6]–[14].

The work [6] is concerned with controlling the tip location of a three-link manipulator in the plane, and proceeds by the three explicit steps of: 1) decoupled linearization of tip behavior; 2) stabilization of the resulting tip dynamics; followed by 3) trajectory control of the now linearly behaving tip. Such clarity of approach will only be retrieved in the latter work of [19]–[22]. Reference [6] also presages future work in its dealing with the problems of manipulator redundancy and actuator saturation.

Resolved acceleration control (RAC) is developed in [7], [8]. RAC essentially extends the work of [6] to a six dof manipulator yielding linearized EE positional dynamics and "almost" linearized EE attitude dynamics (see Section III). This work did not make clear the three steps of [6] and consequently appears not to have been appreciated as a technique for exact linearization of EE motions. The fact that the attitude dynamics are not completely linearized also helped to obscure the appreciation of RAC as an exactly linearizing control technique.

References [9]–[11] apply nonlinear decoupling theory (NDT) to obtain decoupled linearization of a manipulator EE with simultaneous pole placement of the linearized EE dynamics. The simultaneous pole placement and linearization of EE dynamics is a blurring of the distinct steps 1) and 2) described above for the approach [6]. In [23] correspondences of NDT to RAC and the computed torque technique are discussed.

In [12]–[14], manipulator dynamics are expressed in the task space, or operational space of the EE. The resulting nonlinear end effector dynamics are then linearized by the computed torque method. Thus, the operational space control (OCS) of [12]–[14] can be viewed as a generalized computed torque technique. In [12] correspondences to RAC and the computed torque technique are noted.

Recently, geometric control theory (GCT) based techniques for exactly externally linearizing and decoupling general affine-in-the-input nonlinear systems have been developed [15]–[19]. References [15]–[19] extend the idea of feedback linearization for linearizing system state equations (the original problem considered by GCT) to include exact linearization of the input–output equations (hence, "external" linearization). These references give constructive sufficient conditions for local decoupled external linearization which produce the linearizing feedback law. Applications to manipulator control are given in [19]–[22], along with a clear control perspective which keeps the following steps distinct: 1) exactly linearize and decouple end effector dynamics to a canonical decoupled double integrator form, i.e., to Brunovsky canonical form (BCF); 2) effect a stabilizing loop (pole placement step); 3) perform feedforward precompensation to obtain nominal model following performance; 4) institute an error correcting feedback loop.

RAC, NDT, OCS, and GCT can be shown to give the same linearizing control law for exact external linearization and decoupling of EE motions [24]. (This equivalence is specific to the nonlinear systems considered here, viz. systems dynamically similar to rigid link manipulators. NDT and GCT apply to a much larger class than this, and so the equivalence to RAC and OSC holds for systems restricted to this class but not in general.) For example, the approaches discussed above are recovered within the GCT framework of [15]–[22] by making an appropriate choice of states and outputs and applying an output linearizing control. With q giving manipulator joint variables and y the EE position and orientation, the following associations can be made.

a) Computed torque method: state $x = (q, \dot{q})$, output q.

b) NDT, GCT, and (in a sense discussed in Section III) RAC: state $x = (q, \dot{q})$, output $y = h(q)$, where $h(\cdot)$ gives the manipulator forward kinematics.

c) OSC: state $x = (h(q), J\dot{q}) = (y, \dot{y})$, output $y = h(q)$, where J is the manipulator Jacobian matrix.

In this note these equivalences are discussed and a simple form for the linearizing control is given.

II. Dynamics of Finite-Dimensional Natural Systems

Many physical systems have finite-dimensional nonlinear dynamics of the form [25], [26]

$$M(q)\ddot{q} + C(q, \dot{q}) = \tau; \qquad \dot{q}, \ddot{q} \in R^n;$$
$$M(q) \in R^{n\times n}; \qquad M(q) = M^T(q) > 0, \ \forall q \tag{1}$$

where q evolves on a manifold of dimension n, $q \in \mathfrak{M}^n$. For example, $q \in R^n$ for a Cartesian manipulator, while $q \in T^n$ for a revolute manipulator. Typically (1) arises as a solution to the Lagrange equations $(d/dt)(\partial L/\partial \dot{q})L - (\partial L/\partial q) = Q^T$ where $L = T - U$, $T = 1/2\dot{q}^T M(q)\dot{q}$ is positive definite and autonomous, U is a conservative potential function, $Q = \tau + F$ are generalized forces, and F are dissipative or constraint forces. Manipulator dynamics can be obtained in this way and hence have the form (1). Such systems are known as natural systems [25], [26]. Not only is $M(q)$ positive definite for these systems, but $C(q, \dot{q})$ of (1) has terms which depend on $M(q)$ in a very special way [27]–[29]. In fact, natural systems are nongeneric in the class of all affine-in-the-input nonlinear systems [38], [39]. In addition to (1) describing the dynamics of a natural system, in (1) τ provides a direct independent input for each configuration degree of freedom—this is an additional assumption which we have made since our concern is with rigid link manipulators for which every joint can be actuated. Although we only exploit the fact that $M(q)$ is invertible for any q, and the fact that τ directly influences the system configuration degrees of freedom, it should be noted that the nongeneric structure of (1) has recently enabled important statements to be made on the existence of time optimal control laws [38]–[40], on the existence of globally stable control laws [27]–[33], on the existence of robust exponentially stable control laws [34], and on the existence of stable adaptive control laws [35]–[37] for the natural system (1). Recognizing the special properties of (1), it is not surprising that results yielding external linearizing behavior can be obtained much more easily than by application of NDT or GCT—theories which apply to the whole general class of smooth affine-in-the-input nonlinear systems.

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III. End Effector Kinematics and Control After Linearization

The system (1) is assumed to have a read-out map of either the form

$$y=h(q)\in R^{m},\ V=\dot y=J(q)\dot q\in R^{m},\ J(q)=\frac{\partial h}{\partial q}\in R^{m\times n}\tag{2}$$

or the more general form

$$y=h(q)\in \mathfrak{M}^{m},\ V=J_0(q)\dot q\in R^{m},\ J_0(q)\in R^{m\times n}\tag{3}$$

where in general $J_0 dq = V\,dt$ is not an exact differential [25], [26], $h(\cdot)$ is c^2 [44], [47] and defined on the manipulator configuration manifold $\mathfrak{M}^n$, $\mathfrak{M}^m$ is some m-dimensional output manifold, J or J_0 is c^1, and m and n can have different values. Often $h(\cdot)$ is smooth (i.e., c^∞) or even a diffeomorphism when the domain is suitably restricted. In subsequent discussion $V=\bar J\dot q$ will mean that $\bar J$ can be either J or J_0. Let the state of system (1) be $(q,\dot q)$. Then for $y=h(q)$, $V=\bar J(q)\dot q$ will be called the "velocity associated with the output y." Note that (2) is a special case of (3) where V, the velocity associated with y, is just $V=\dot y$ and $\mathfrak{M}^m = R^m$ giving $\bar J = J = \partial h/\partial q$. Also note that for the case (3), since h is c^2, it is still meaningful to talk about $\dot y = J\dot q$ and $J=\partial h/\partial q$, $J(q): T_q\mathfrak{M}^n \simeq R^n \to T_{h(q)}\mathfrak{M}^m$, but now $V\neq \dot y$ and $J\neq J_o$ is admitted as a possibility.

For rigid link manipulators moving in Euclidean three-space, typically $V=\operatorname{col}(\dot x,\omega)\in R^6$, where $x\in R^3$ gives the EE location, $\dot x$ the EE linear velocity, and $\omega\in R^3$ the EE angular rate of change. It is well known that ω is not the time derivative of any minimal (i.e., three-dimensional) representation of attitude, so that $V=\operatorname{col}(\dot x,\omega)=J_0(q)\dot q$ as in (3). In this case, we call $J_0(q)$ the "standard Jacobian" ([13] refers to J_0 as the "basic Jacobian"). It is also common to represent EE attitude by a proper orthogonal matrix $A\in SO(3)=\{A\,|\,A^TA=AA^T=I,\ \det A=+1\}\subset R^{3\times 3}$, where the columns of A determine EE fixed body axes in the usual way. It is well known that $\dot A=\tilde\omega A$ where $\tilde\omega v := \omega\times v$ for all $v\in R^3$. Thus, EE location and kinematics are often given by

$$y=(x,A)=h(q)\in R^3\times SO(3),\ V=\begin{pmatrix}\dot x\\ \omega\end{pmatrix}=J_0(q)q\in R^6$$
$$\dot A=\tilde\omega A,\ A\in SO(3),\ \tilde\omega^T=-\tilde\omega,\ J_0(q)\in R^{6\times n}\tag{4}$$

which should be compared to (3). Alternatively, we can take [cf. (2)]

$$y=\begin{pmatrix}x\\ \beta\end{pmatrix}=h(q)\in R^6,\ V=\dot y=\begin{pmatrix}\dot x\\ \dot\beta\end{pmatrix}=J(q)\dot q\in R^6,\ \beta\in\Omega\subset R^3.\tag{5}$$

$\beta\in\Omega\subset R^3$ is a minimal representation of EE attitude [i.e., of the rotation group $SO(3)$]. In general, $\beta=f(A)$ for some function $f(\cdot)$ which is many-to-one or undefined if the domain of $f(\cdot)$ on $SO(3)$ is not properly restricted. That is, because $SO(3)$ cannot be covered by a single coordinate chart, β is not valid for all possible EE orientations and there will be singularity of attitude representation unless we restrict EE attitude to some subregion of $SO(3)$ [25], [41], [42]. This restriction then forces β to be defined in the image of admissible attitudes, namely in some $\Omega\subset R^3$. (It may be true, however, that $\Omega=R^3$, as in the case of Euler–Rodriquez parameters where singularity of attitude representation corresponds to $\|\beta\|=\infty$ [42].) Typical β's are roll-pitch-yaw angles, axis/angle variables, Euler angles, Euler parameters, and Euler–Rodriquez parameters [25], [41]–[43]. The kinematical relationship between β and ω is given by

$$\dot\beta=\Pi(\beta)\omega\tag{6}$$

where $\Pi\in R^{3\times 3}$ will lose rank, i.e., become singular, precisely when β becomes a singular representation of EE attitude.

Note from (3)–(6) that $J=\operatorname{diag}(I,\Pi)J_0$. Generally, the standard Jacobian matrix J_0 will become singular only at a manipulator kinematic singularity, in which case J will also be singular. However, J can also be singular when $\beta=\beta(q)$ gives a singularity of EE attitude representation—this compounds the trajectory planning problem for EE motions, since now we must plan trajectories which avoid manipulator kinematic singularities and also ensure that $\beta(q)\in\Omega$.

Henceforth the system (1), (2) or (1), (3) will be said to be exactly externally linearized and decoupled if

$$\dot V=u\in R^6.\tag{7}$$

This is somewhat of an abuse of notation, as a consideration of the system (1), (4) shows; for $u=\operatorname{col}(u_1,u_2)$, $\dot V=u$ yields

$$\ddot x=u_1\in R^3,\ \dot\omega=u_2\in R^3,\ \dot A=\tilde\omega A.\tag{8}$$

Although EE positional dynamics are decoupled and linearized to $\ddot x=u_1$, attitude dynamics are nonlinear and given by $\dot\omega=u_2$, $\dot A=\tilde\omega A$. Equation (8) is precisely the sense in which RAC can be said to almost "exactly externally linearize and decouple" attitude dynamics, as was discussed in the Introduction. In the case of the system (1), (5), $\dot V=\operatorname{col}(u_1,u_2)$ gives

$$\ddot x=u_1\in R^3,\ \ddot\beta=u_2\in R^3,\ \beta\in\Omega\subset R^3\tag{9}$$

which can indeed be said to be exactly externally linearized and decoupled. Drawbacks to using (9) are that β must always be controlled to remain in Ω, trajectories involving β may be difficult to visualize, and the generalized force u_2, which drives β, may be nonintuitive. On the other hand, it is obvious how to obtain stable attitude tracking from (9). The advantage to using (8) is that ω and A are easily visualized entities, while u_2 is the ordinary torque that we are all familiar with. Fortunately, despite the nonlinear attitude dynamics, it is possible to use (8) to perform EE attitude tracking with asymptotically vanishing attitude error [7], [8].

Note that once (8) is obtained, it is easy to get (9) by use of the relationship (6). If we have $\dot\omega=u$, $\dot\beta=\Pi(\beta)\omega$, and $\beta\in\Omega$ so that $\Pi^{-1}(\beta)$ exists, then use of

$$\dot\omega=u,\ u=\Pi^{-1}(\beta)[\bar u-\dot\Pi(\beta)\omega]\tag{10}$$

gives

$$\ddot\beta=\Pi(\beta)\dot\omega+\dot\Pi(\beta)\omega=\bar u.\tag{11}$$

Therefore, having (8), we can perform attitude control directly on $\dot\omega=u_2$, $\dot A=\tilde\omega A$ or we can transform to $\ddot\beta=\bar u_2$ and then control.

IV. Comparison of GCT, NDT, and OSC

For brevity, we consider the nonredundant manipulator case, taking $n=6$ in (1), and we omit derivations. A more detailed discussion is given in [24]. Note that the system (1), (5) can be written as

$$\frac{d}{dt}\begin{pmatrix}q\\ \dot q\end{pmatrix}=\begin{pmatrix}\dot q\\ -M^{-1}C\end{pmatrix}+\begin{pmatrix}0\\ M^{-1}\end{pmatrix}\tau,\qquad y=h(q)\tag{12}$$

or, taking $Z=\operatorname{col}(q,\dot q)$,

$$\frac{d}{dt}Z=A(Z)+B(Z)\tau,\qquad y=H(Z)\tag{13}$$

where the definitions of A, B, and H are obvious. GCT asks: Does there exist i) a nonlinear feedback $\tau=Q(Z)+R(Z)u$; and ii) a nonlinear change of basis $x=X(Z)$ such that (12) is placed into BCF?

$$\frac{d}{dt}\begin{pmatrix}y\\ \dot y\end{pmatrix}=\begin{pmatrix}0 & I\\ 0 & 0\end{pmatrix}\begin{pmatrix}y\\ \dot y\end{pmatrix}+\begin{pmatrix}0\\ I\end{pmatrix}u\Leftrightarrow \ddot y=u.\tag{14}$$

The constructive sufficient conditions of [19]–[22] can be applied and give the linearizing and decoupling feedback law $\tau=-MJ^{-1}\partial J\dot q+C+MJ^{-1}u$, where $\partial J=[\Sigma_{k=1}^n(\partial J_{ik}/\partial q_j)\dot q_k]$. Although $\partial J\neq\dot J$, it is true that $\partial J\dot q=\dot J\dot q$ giving

$$\tau=-MJ^{-1}\dot J\dot q+MJ^{-1}u+C.\tag{15}$$

Note that J must be nonsingular for (15) to exist. This is consistent with the theory of [19]–[21] which provides sufficient conditions for *local* linearization. Note also that to implement (15), explicit expressions for M, J^{-1}, J, and C are required.

The NDT approach of [11] constructs the linearizing feedback in the following way. For the system (13) define

$$G(Z)=\frac{\partial}{\partial Z}\left(\left[\frac{\partial}{\partial Z}H(Z)\right]A(Z)\right),\ D^*(Z)=G(Z)B(Z),\quad H^*(Z)=G(Z)A(Z). \tag{16}$$

The use of

$$\tau=-D^{*-1}(Z)H^*(Z)+D^{*-1}(Z)u \tag{17}$$

will transform (12), (13) to $\ddot{y}=u$, i.e., to (14). It is straightforward to show that, for A, B, and H as in (12) and (13), (17) is precisely (15). Note that in (13) we take $Z=\text{col}(q,\dot{q})$ and *not* $Z=(q_1,\dot{q}_1,\cdots,q_n,\dot{q}_n)^T$. The latter choice for Z is taken in [11] and serves to obscure the final result—namely that (17) and (15) are equivalent.

Now consider the OSC approach of [12]–[14]. In this approach the EE coordinates y are viewed as generalized coordinates for the manipulator, and a change of basis $(q,\dot{q})\rightarrow(y,\dot{y})$ is made. In (1) let $C=b-g$ where b are the Coriolis forces and g the gravity forces. Restrict the domain of the system (1), (5) to ensure that $h(\cdot)$ is a bijection (and consequently $\det J(q)\neq 0$ on this restriction). This restriction means that, as for DGC and NDT, OSC gives a local result for external linearization. In [12]–[14], the effective EE dynamics are determined to be

$$\Lambda(y)\ddot{y}+c(y,\dot{y})=F,\ \tau=J^TF,\ \Lambda=J^{-T}MJ,\ c=U-P,$$
$$P=J^{-1}g,\ U=J^{-T}b-\Lambda\dot{J}\dot{q},\ q=h^{-1}(y),\ \dot{q}=J^{-1}\dot{y}. \tag{18}$$

Recall that for the system (1), $M\ddot{q}+C=\tau$, the computed torque technique is to take $\tau=Mu+C$, yielding $M(\ddot{q}-u)=0\Rightarrow\ddot{q}=u$, since $M(q)>0$, $\forall q$. Similarly, in (18) $\Lambda(y)>0$ for every $y=h(q)$ where q is in the restricted domain. The choice of

$$F=\Lambda(y)u+c(y,\dot{y}),\qquad \tau=J^TF \tag{19}$$

in (18) then yields $\Lambda(y)(\ddot{y}-u)=0\Rightarrow\ddot{y}=u$. In this sense the work in [12]–[14] can be viewed as a generalized computed torque technique. From (18) and (19) it is straightforward to determine that τ of (19) is exactly τ of (15).

$JM^{-1}=D^*$ of (15), (16) and Λ^{-1} of (19) are the "decoupling matrices" [19], [20] for (13) and (18), respectively. The difficulty in feedback linearizing (13) is seen to arise from the decoupling matrix becoming singular, while the difficulty in feedback linearizing (19) is due to the change of basis $(q,\dot{q})\rightarrow(y,\dot{y})$ (since if y is a good generalized coordinate for the manipulator Λ^{-1} must be nonsingular). In either interpretation we see that the difficulty is due to the Jacobian matrix J becoming singular.

V. Derivation of a Feedback Law for Local Exact Decoupled External Linearization and Its Relationship to RAC and GCT

Recall that the system (1), (2) or (1), (3) is of the form

$$M(q)\ddot{q}+C(q,\dot{q})=\tau;\ q\in\mathfrak{M}^n;\ \dot{q},\ddot{q}\in R^n$$
$$y=h(q)\in\mathfrak{M}^m;\ h(\ )\text{ is }c^2;\ V=\bar{J}(q)\dot{q}\in R^m;\ \bar{J}(q)\text{ is }c^1;$$
$$M(q)\in R^{n\times n};\ M(q)=M(q)^T>0,\ \forall q\in\mathfrak{M}^n \tag{20}$$

where in general, it may be that $m\neq n$, $\mathfrak{M}^m\neq R^m$, $V\neq\dot{y}$, and $\bar{J}\neq J=\partial h/\partial q$. It is assumed that a necessary and sufficient condition for $h(q)$ to be onto some neighborhood of $y=h(q)$ in $\mathfrak{M}^m$ is that the mapping $\bar{J}(q)$ be onto R^m, i.e., we assume that $\bar{J}(q)$ is onto R^m if and only if $J(q)=\partial h(q)/\partial q$ is onto $T_{h(q)}\mathfrak{M}^m\simeq R^m$. This is a reasonable assumption; for example, when $\mathfrak{M}^m=R^m$, $V=\dot{y}=J(q)\dot{q}\in R^m$, and $\bar{J}=J=\partial h/\partial q$ this is trivially true. For the case $y=h(q)=(x,A)\in\mathfrak{M}^6=R^3\times SO(3)$ and $\bar{J}=J_0$ where $V=\text{col}(\dot{x},\omega)=J_0(q)\dot{q}$, the fact that $\dot{x}\in T_xR^3$ and $\dot{A}=\tilde{\omega}A\in T_ASO(3)$ means that for $J_0(q)$ onto, we can fill out a neighborhood of (x,A) and otherwise we cannot. (A general element of $T_ASO(3)$ is precisely of the form $\tilde{\omega}A$, $\tilde{\omega}\in R^{3\times 3}$ skew-symmetric, so that if $\omega=\omega(q)$ can be mapped onto R^3, $\tilde{\omega}A$ can be mapped onto $T_ASO(3)$ [44], [47].)

Definition LEL: The system (20) can be locally exactly externally linearized and decoupled (LEL) over an open neighborhood $B^m(y')\subset\mathfrak{M}^m$ of $y'\in h(\mathfrak{M}^n)\subset\mathfrak{M}^m$ with the arm in the configuration $q'\in h^{-1}(y')$ if there is an open neighborhood of q', $B^n(q')\subset\mathfrak{M}^n$, such that $B^m(y')=h[B^n(q')]$ and if for any $u\in R^m$, $q\in B^n(q')$, and $\dot{q}\in T_q\mathfrak{M}^n\simeq R^n$ there exists a nonlinear feedback $\tau=F(q,\dot{q},u)$ such that V, the velocity associated with $y=h(q)\in B^m(y')$, obeys $\dot{V}=u$. ●

Note that for an EE to be LEL at y' it must be true that y' is in the range of $h(\cdot)$, i.e., y' must be a physically attainable EE position. Also, for a given EE location, $y\in h(\mathfrak{M}^n)$, a manipulator can physically be in only one of the possible configurations $h^{-1}(y')$. Thus, we can interpret $q'\in h^{-1}(y')$ to be the actual physical configuration of a manipulator. If the system (20) is not LEL at y' in the configuration $q'\in h^{-1}(y')$, it may be LEL at a different configuration $q\in h^{-1}(y')$.

Theorem LEL: A necessary and sufficient condition for (20) to be LEL at $y'\in h(\mathfrak{M}^n)$ in the configuration $q'\in h^{-1}(y')$ is that $\bar{J}(q')\in R^{m\times n}$ be onto, which is true iff $m\leq n$ and rank $\bar{J}(q')=m$. Furthermore, the locally exactly linearizing and decoupling feedback is given by

$$\tau=M(q)\xi+C(q,\dot{q}) \tag{21}$$

where ξ is any solution to

$$\bar{J}(q)\xi=-\dot{\bar{J}}(q)\dot{q}+u. \tag{22}$$

When $m=n$ this gives

$$\tau=-M(q)\bar{J}(q)^{-1}\dot{\bar{J}}(q)\dot{q}+M(q)\bar{J}(q)^{-1}u+C(q,\dot{q}). \tag{23}$$

●

Proof. Necessity: Suppose that $\dot{V}=\bar{J}(q')\ddot{q}'+\dot{\bar{J}}(q')\dot{q}'=u$ can be made to hold regardless of the values of $u\in R^m$ and $\dot{q}'$. This means that there must exist $\ddot{q}'\in R^n$ such that

$$\bar{J}(q')\ddot{q}'=-\dot{\bar{J}}(q')\dot{q}'+u. \tag{24}$$

If $\bar{J}(q')$ is not onto, then $Im\bar{J}(q')\subset R^m$ and $Im\bar{J}(q')\neq R^m$. Let u and $\dot{q}'$ be such that $-\dot{\bar{J}}(q')\dot{q}'+u\notin Im\bar{J}(q')$. Then there is no $\ddot{q}'$ for which (24) holds, yielding a contradiction. *Sufficiency:* By assumption $\bar{J}(q')$ is full rank and onto $\Leftrightarrow J(q')=\partial h(q')/\partial q$ is full rank and onto. Since $\bar{J}$ and J are c^1, there exist neighborhoods $B^m(y')$ and $B^n(q')$, $y'=h(q')$, such that $B^m(y')=h[B^n(q')]$ and such that $\bar{J}$ is full rank and onto when restricted to $B^n(q')$ [44]. Now consider any $q\in B^n(q')$ and its associated $y=h(q)\in B^m(y')$. Then, $V=\bar{J}(q)\dot{q}\Rightarrow$

$$\dot{V}=\bar{J}(q)\ddot{q}+\dot{\bar{J}}(q)\dot{q}. \tag{25}$$

Let ξ be any solution to (22). ξ is guaranteed to exist since $Im\bar{J}(q)=R^m$. Take τ to be (21), then

$$M\ddot{q}+C=\tau=M\xi+C\Rightarrow M(\ddot{q}-\xi)=0\Rightarrow\ddot{q}=\xi$$

which with (22) and (25) gives $\dot{V}=u$. ●

Comments:

1) Note that this result applies to all systems of the form (20), of which rigid link manipulators are a special case.

2) Note that with $y\in\mathfrak{M}^m$ and $\tau\in R^n$, the fact that we need $m\leq n$ can be interpreted to mean that there must be at least as many inputs as outputs.

3) When $\bar{J}=J=\partial h/\partial q$, $V=\dot{y}$, and $m=n$ we have that $\tau=-MJ^{-1}\dot{J}\dot{q}+MJ^{-1}u+C\Rightarrow\ddot{y}=u$ when $\det J\neq 0$. This is the same result provided by GCT, NDT, and OCS as seen in the last section [cf. (15)].

4) Note that in the proof we force $\ddot{q}=\xi$ precisely like $\ddot{q}=u$ is forced to happen in the computed torque method. In fact, for $y=q$ we have $J=I$ and $\dot{J}=0$ giving $\xi=u$. Thus, the exact linearizing control of (21), (22)

is seen to be a generalization of the computed torque method in a somewhat different, and perhaps more illuminating, way than OCS.

5) In addition to the invertibility of the manipulator mass matrix, a key reason that a relatively simple solution to the output linearizing problem can be found is that an independent input exists for every system configuration degree of freedom. In the case of manipulators with elastic joints, this condition may be violated, and it is not possible in general to feedback linearize using only static feedback of measured joint variables and rates. See [50], [51].

Consider the case of EE control given by the system (1), (4). Here $J = J_0$ where $V = \text{col}(\dot{x}, \omega) = J_0\dot{q}$. In this case, when $m = n$, (23) is

$$\tau = -MJ_0^{-1}\dot{J}_0\dot{q} + MJ_0^{-1}u + C. \tag{26}$$

When $\det J_0 \neq 0$, the use of (26) yields $\text{col}(\ddot{x}, \dot{\omega}) = \text{col}(u_1, u_2)$. This is precisely RAC [7], [8]. Theorem LEL can be interpreted as an extension of RAC to the redundant arm case which allows for the use of a minimal representation of EE attitude [24]. The more general case $m \leq n$ is given by

$$\tau = Mu + C, \quad J_0\xi = -\dot{J}_0\dot{q} + u. \tag{27}$$

By using the indirect form (27), τ can be obtained, after ξ has been found, by use of the Newton–Euler recursion [45]. Furthermore, ξ can be obtained recursively—either directly [46], or by first recursively obtaining J_0 and $\dot{J}_0$ and then solving for u by Gaussian elimination. The point to be drawn here is that (27) shows us how to perform exact external linearization without the need for an explicit manipulator model. After exactly linearizing to $\text{col}(\ddot{x}, \dot{\omega}) = \text{col}(u_1, u_2)$ one can perform EE tracking at this stage [7], [8], or one can continue to the form (11) by the use of (10).

When using (26) or (27), the only way that rank $J_0 < m$ can occur for $m \leq n$ is when the manipulator is at a mechanically singular configuration. Recall (Section IV) that in the case when a minimal representation of EE attitude is used, the resulting Jacobian matrix J will be rank deficient not just for a manipulator singularity, but at a configuration which leads to a singularity of attitude presentation. Thus, rank deficiency of J_0 is kinematically cleaner to understand. The necessity that rank $J_0 = m$ in order to use (26) or (27) allows two obvious, but important statements to be made: i) for a manipulator with a workspace boundary (ignoring joint stops), as in the case of a PUMA-type manipulator, exact linearization at the boundary is impossible; ii) for a nonredundant (6 dof) manipulator with workspace interior singularities, there cannot be exact linearization throughout the workspace interior. For a redundant manipulator with workspace interior singularities, it may be possible to avoid workspace interior configurations which cannot be exactly linearized by the use of self motions as described in [48], [49]. This is related to the multiplicity of solutions available for ξ in (27).

How does the control (23) fulfill the aim of GCT as stated in (12)–(14)? The nonlinear feedback (taking $V = \dot{y}$ and $J = \bar{J}$) $\tau = Q(Z) + R(Z)u = (C - MJ^{-1}\dot{J}\dot{q}) + (MJ^{-1})u$ applied to (12), (13) gives

$$\frac{d}{dt}\begin{pmatrix} q \\ \dot{q} \end{pmatrix} = \begin{pmatrix} 0 & I \\ 0 & -J^{-1}\dot{J} \end{pmatrix}\begin{pmatrix} q \\ \dot{q} \end{pmatrix} + \begin{pmatrix} 0 \\ J^{-1} \end{pmatrix} u. \tag{28}$$

Consider the local nonlinear change of basis given by

$$\begin{pmatrix} y \\ \dot{y} \end{pmatrix} = \begin{pmatrix} h(q) \\ J\dot{q} \end{pmatrix}; \begin{pmatrix} q \\ \dot{q} \end{pmatrix} = \begin{pmatrix} h^{-1}(y) \\ J^{-1}\dot{y} \end{pmatrix}.$$

The fact that $\dot{y} = J\dot{q}$ and $\ddot{y} = J\ddot{q} + \dot{J}\dot{q}$ gives

$$\frac{d}{dt}\begin{pmatrix} y \\ \dot{y} \end{pmatrix} = \begin{pmatrix} J & 0 \\ \dot{J} & J \end{pmatrix}\frac{d}{dt}\begin{pmatrix} q \\ \dot{q} \end{pmatrix}.$$

Writing (28) as

$$\begin{pmatrix} J & 0 \\ \dot{J} & J \end{pmatrix}\frac{d}{dt}\begin{pmatrix} q \\ \dot{q} \end{pmatrix} = \begin{pmatrix} J & 0 \\ \dot{J} & J \end{pmatrix}\begin{pmatrix} 0 & I \\ 0 & -J^{-1}\dot{J} \end{pmatrix} \cdot \begin{pmatrix} h^{-1} & (y) \\ J^{-1} & \dot{y} \end{pmatrix} + \begin{pmatrix} J & 0 \\ \dot{J} & J \end{pmatrix}\begin{pmatrix} 0 \\ J^{-1} \end{pmatrix} u$$

we obtain the BCF

$$\frac{d}{dt}\begin{pmatrix} y \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix}\begin{pmatrix} y \\ \dot{y} \end{pmatrix} + \begin{pmatrix} 0 \\ I \end{pmatrix} u \Leftrightarrow \ddot{y} = u.$$

VI. Concluding Remarks

Recognizing the fundamental unity of RAC, GCT, OSC, and NDT [7]–[22] for exact linearization of manipulators, we can focus on their true differences—namely, differences in implementation detail and design philosophy. With the awareness that they all produce essentially the same linearizing feedback, we can ask why this particular feedback form is appropriate for manipulator-like systems.

OCS and RAC exploit the specific structure of such systems. Not surprisingly, the solutions arrived at, reflecting the philosophies and implementation perspectives of the researchers involved, are quite distinct in their flavor and presentation. Yet, since the properties specific to manipulator dynamics ultimately forced the solution, they are fundamentally the same. The important point here is that researchers consciously exploited the specific properties of a system of interest, but without pinpointing precisely what these properties were which made the system amenable to linearizing control.

GCT and NDT provide techniques for exactly linearizing general smooth affine-in-the-input dynamical systems. These techniques ignore any specific nongeneric structural properties that a system might have and as a consequence the solutions obtained are much less transparent than those of OCS or RAC. The strength of these approaches, particularly GCT, is that they can provide necessary and sufficient conditions for a system to be exactly linearizable and constructive sufficient conditions which produce the linearizing feedback when satisfied. Interestingly, when applied to the problem of manipulator exact linearization, the solutions obtained can be shown to be equivalent to those of RAC and OCS. Again, the structural properties of the system forced the solution. Certainly, once a solution is known to exist, it is reasonable to attempt to produce it from more physical arguments knowing now that the search is not fruitless. This leads to a reexamination of OCS and RAC.

The work of [17]–[22] stresses a perspective which serves to enable a clearer comparison between competing techniques for external linearization: Place the system in a standard linear canonical form before additional control efforts are made—this ensures that the process of linearizing the system is not mixed up with, and confused with, the process of stabilizing and controlling it. This perspective greatly aided the comparison of GCT, OCS, RAC, and NDT which resulted in [24]. In turn, this comparison focuses attention on the structural properties of manipulators.

Much current research makes it apparent that systems dynamically similar to rigid link manipulators have important structural properties which can be exploited to achieve results which are quite strong when compared to those available for general smooth affine in the inputs nonlinear systems [25]–[40]. Here we have seen that exploiting the nongeneric second-order form of system (1) which has an everywhere positive definite mass matrix, direct independent control of every configuration degree of freedom, and a c^2 locally onto readout map enables a simple form for the linearizing feedback.

One can also approach the problem of hybrid force/position control from the framework of the approaches discussed in this note. Unfortunately, lack of space precludes discussion here. For hybrid control from the OCS perspective see [52], for the RAC perspective see [53], and for approaches based on GCT see [54], [55].

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On Minimum-Fuel Control of Affine Nonlinear Systems

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Abstract—**The minimum-fuel control problem is investigated for a class of multiinput affine nonlinear systems whose associated Lie algebra is nilpotent. Interesting consequences of the maximum principle are deduced for such systems.**

I. Introduction

Optimal control theory [7] provides a systematic design method for modern control systems and thus plays an important role in linear control theory (more specifically, the linear quadratic regulator and linear quadratic Gaussian control theories). Roughly speaking, the success of optimal control theory in the context of linear systems is due to the ease of computation of the optimal control law. On the other hand, until now there has been a lack of systematic and reliable procedures for solving nonlinear optimal control problems. This is unfortunate, but some attempts have been made to resolve these difficulties; in [2] a Lie algebraic approach has been used to derive a set of quasi-linear partial differential equations which the optimal feedback law must satisfy.

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