Other topics in dynamics of mechanical systems (and not)

Most of topics treated in this section apply to mechanical systems, but their trivial extension to non-mechanical systems characterized by similar properties would apply as well.

Inverse problem of Lagrangian Mechanics

The **inverse problem of Lagrangian mechanics** is as follows: given a system of second-order ordinary differential equations

$$\ddot{u}^i = f^i\left(u^j, \dot{u}^j\right) \text{ for } 1 \leqslant i, j \leqslant n$$

that holds for times $0 \leq t \leq T$, does there exist a Lagrangian $L : [0, T] \times \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}$ for which these ordinary differential equations (E) are the Euler–Lagrange equations? In general, this problem is posed not on Euclidean space $\mathbf{R}^n$, but on an n-dimensional manifold M, and the Lagrangian is a function $L : [0, T] \times \mathrm{T}M \to \mathbf{R}$, where $\mathrm{T}M$ denotes the tangent bundle of M.

Jesse Douglas, in which he provided necessary and sufficient conditions for the problem to have a solution; these conditions are now known as the **Helmholtz conditions**

Passive dynamics

Another concept closely related to underactuation and dynamics is the one of Passive dynamics. Not always treated in traditional robotics textbooks.

Passive dynamics refers to the dynamical behavior of actuators, robots, or organisms when not drawing energy from a supply (e.g., batteries, fuel, ATP). Depending on the application, considering or altering the passive dynamics of a powered system can have drastic effects on performance, particularly energy economy, stability, and task bandwidth. Devices using no power source are considered "passive", and their behavior is fully described by their passive dynamics. In some fields of robotics (legged robotics in particular), design and more relaxed control of passive dynamics has become a complementary (or even alternative) approach to joint-positioning control methods developed through the 20th century. Additionally, the passive dynamics of animals have been of interest to biomechanists and integrative biologists, as these dynamics often underlie biological motions and couple with neuromechanical control. Particularly relevant fields for investigating and engineering passive dynamics include legged locomotion and manipulation.

Canonical motion equations of a robot

Applying above mentioned techniques for dynamic equations derivation, to mechanical system composed of rigid bodies, thus robots case is included, or more generally for any action of type

ones get a set of motion equations in the shape of (using joints variables vector q)

$$H(q)\ddot{q} + C(q, \dot{q})\dot{q} + G(q) = \tau$$

or alternatively using operational space formulation

$$\Lambda(x)\dot{v} + \mu(x, v) + \rho(q) = f$$

In last equations, x is a 6-D vector of operational-space coordinates giving the position and orientation of the robot's end-effector; v is the velocity of the end-effector; and f is the force exerted on the end-effector. x is typically a list of Cartesian coordinates, and Euler angles or quaternion components, and is related to v via a differential equation of the form

$$\dot{x} = E(x)v$$

μ and ρ are vectors of velocity-product and gravity terms, respectively.

These two are known as canonical equations of motion for robots. . More terms can be added to this equation, as required, to account for other dynamical effects (e.g., viscous friction). The effects of a force f exerted on the mechanism at the end-effector can be accounted for by adding the term $J^T f$ to the right side of first, where J is the Jacobian of the end-effector. . If the mechanism is a kinematic tree, then q contains every joint variable in the mechanism, otherwise it contains only an independent subset. The elements of q are generalized coordinates (but not all generalized coordinates correspond to in our interpretations). Likewise, the elements of $\dot{q}$, $\ddot{q}$, and are generalized velocities, accelerations, and forces.

Observe as these two forms are not all general case but heir choice is driven from a connection with physical meaning, related to joint space and task space they're associated to..

Transformations from joint space to task space

Connection between task space and joint space is obtained by following set of equations

- $v = J\dot{q}$
- $\dot{v} = J\ddot{q} + \dot{J}\dot{q}$
- $\tau = J^T f$
- $\Lambda = \left(JH^{-1}J^T\right)^{-1} \mu = \Lambda\left(JH^{-1}C\dot{q}.\, J\dot{q}\right)$
- $\rho = \Lambda J H^{-1}\tau_g$

These equations assume that m ≤ n (m is the dimension of operational-space coordinates), and that the Jacobian J has full rank.

System of equations order reduction

There are multiple different and ways to reduce one dynamic system oder, not all are equally effective.

One way is using simply a set of trivial auxiliary variables definition like using derivatives of generalized coordinates. Although in simpler cases this is equivalent to the introduction of conjugate momenta, assuming an unitary mass, or simply introducing a ratio over coordinate mass, this does not keep true in most complex cases where mass matrix structure plays a major role leading to more complex expressions.

Lyapunov equations

Introdcution of hamiltonian formalism lead in a quite natural way the introduction of a liapunov function helpful for system stability analysis

Robots mechanics and entropy

Since mechanics can be translated for many body systems into statistical themes, and statistical mechanics lead naturally to entropy concept, we can infer the existence of a concept of entropy valid for mechanical systems with many, or even a limited number of degree of freedom exploiting concepts like configuration space and occupied configurations. In a natural free motion one can infer configuration space is prone to be occupied uniformly, where the presence of hidden or unpreferred configurations space zones may indicate a posteriori some kind of discrimination on possible configurations workspace, as can happen for instance in presence of external forces/torques (performing work) like gravity, frictions, aerodynamic forces, or others, included control forces. This would be interesting direction of study also considering underactuation and for exploiting passive/free motion dynamics in design of operative space.

Geometric representation

The geometry of a robotic mechanism is conveniently defined by *attaching coordinate frames to each link*. While these frames could be located arbitrarily, it is advantageous both for consistency and computational efficiency to adhere to a *convention* for locating the frames on the links. Denavit and Hartenberg [2.26] introduced the foundational convention that has been adapted in a number of different ways, one of which is the convention introduced by Khalil and Dombre [2.27] used in many handbook. In all of its forms, the convention requires only four rather than six parameters to locate one coordinate frame relative to another. The Four parameters consist of two link parameters, the link length ai and the link twist ¸i, and two joint parameters, the joint offset di and the joint angle i. This parsimony is achieved through judicious placement of the coordinate frame origins and axes such that the $\hat{x}$ axis of one frame both intersects and is perpendicular to the $\hat{z}$ axis of the following coordinate frame. The convention is applicable to robotic mechanisms consisting of revolute and prismatic joints, so when multiple-degree-of-freedom joints are present, they are modeled as combinations of revolute and prismatic joints.

Generalization of motion equations from geodetics on differential manifold motion

There's an alternative point of view that prove the generality of seen equations.
Starting from general relativity and geodetic principle, we get for a generic geodesic an equation respect proper time like, in tensorial formalism. Geodetic integral depend on metric interval and metrics tensor $g_{\mu\nu}$, can be put in parallel with kinetic energy for unforced systems, where inertia matrix play the role of metric tensor

$$\delta S = 0 = \delta \int g_{\mu\nu} dx^{\mu} dx^{\nu} = \delta \int g_{\mu\nu} \dot{x}^{\mu} \dot{x}^{\nu} d\tau$$

$$L = \dot{x}^{\mu} g_{\mu\nu} \dot{x}^{\nu} = \dot{q}^T M \dot{q}$$

$$\frac{\partial^2 x^{\mu}}{\partial s^2} + \Gamma_{ij}^{\ \ \mu} \frac{\partial x^i}{\partial s} \frac{\partial x^j}{\partial s} = 0$$

$$\frac{\partial^2 x^{\mu}}{\partial s^2} + \Gamma^{\mu}_{ij} \frac{\partial x^i}{\partial s} \frac{\partial x^j}{\partial s} = 0$$

where Γ is a pseudotensor (then do not transform in covariant way under a general transformation of coordinates), corresponding to Christoffel symbols of first type involved in geodesic equation, in covariant derivative and general relativity. In this equations we suppose to accept the passage from traditional newtonian assumption of absolute time to proper time, as most indicate and natural equivalent of traditional time notion measured in classic mechanics. Consequently speeds and accelerations appears like first and second derivatives of position coordinate over proper time. Adding formal passage of coordinate parametrization as function of generalized coordinates we get a substantially equivalent equation

$$\frac{\partial^2 x^{\mu}}{\partial q} \frac{\partial^2 q^k}{\partial s^2} + \Gamma_{ij}^{\ \ \mu} \frac{\partial x^i}{\partial q^h} \frac{\partial x^j}{\partial q^t} \dot{q}^h \dot{q}^t$$

$$\frac{\partial^2 x^{\mu}}{\partial q^k} \frac{\partial^2 q^k}{\partial s^2} + \Gamma^{\mu}_{ij} \frac{\partial x^i}{\partial q^h} \frac{\partial x^j}{\partial q^t} \dot{q}^h \dot{q}^t = 0$$

$$\frac{\partial q^k}{\partial s} = \dot{q}^k$$

$$\frac{\partial q^k}{\partial s} = \dot{q}^k$$

There's an additional observation to keep under reader attention, in this formulation there's no mass matrix appearing as factor for accelerations. This is due to essentially geometrical approach used into general relativity where trajectory is defined only from geometry underlying the system. Only indirectly forces of inertia (as well as other forces) appears in equations just modifying structure of geometry, accounting for mass induced accelerations, like supposed in general relativity

A further extension away from geodetic equations is obtained including additional *sources* or *forces* term too geodetic equations, that shall have same type of coordinate vector.

$$\frac{\partial^2 x^{\mu}}{\partial s^2} + \Gamma_{ij}^{\ \mu} \frac{\partial x^i}{\partial s} \frac{\partial x^j}{\partial s} = F$$

$$\frac{\partial^2 x^{\mu}}{\partial s^2} + \Gamma_{ij}^{\mu} \frac{\partial x^i}{\partial s} \frac{\partial x^j}{\partial s} = F^{\mu}$$

Beyond specific context of general relativity it is a proof of the fact that general motion equations assume a shape aforementioned in disregard of specific context, forces or other affecting factors.

This is rather easy to understand considering origins of Christofell symbols coming from *covariant derivatives* and *parallel transport*. They indeed account for total variation of momenta vectorial quantities along a curved path, so that it accounts for a modulus variation along directions coordinate (accelerations) and for a component due to trajectory curvature. Effective trajectory curvature is could be lead from constraints, or in a perfectly equivalent way, tue acting forces, since reactions arise to justify force balance in curved trajectories.

$$\frac{\partial v}{\partial s} + \Gamma v^i v^j$$

$$\frac{dv^k}{ds} = \frac{\partial v^k}{\partial s} + \Gamma^k_{ij} \frac{\partial v^i}{\partial s} \frac{\partial v^j}{\partial s} = \frac{\partial v^k}{\partial s} + \Gamma^k_{ij} \dot{v}^i \dot{v}^j$$

Part dependent on momenta (or velocities) vector are those we call usually centrifugal and coriolis terms, but may also assume a different meaning considering forces case, since describe the component of acting solicitations causing a variation in direction of motion.

Game theory and robot dynamics

Is possible to prove there's a connection between game theory and general systems dynamics, consequently to robot dynamics as well.
The proof is not straightforward, but an intuitive explanation is presented by common underlying principle of optimizations. Both dynamics and game theory derives from optimization principles, then is not hard to imagine a connection could be established between two study fields.
Indeed, more rigorously, there's proof that generic treatment of partial differential equations using characteristics method lead to Hamilton Jacobi equations, that are associated to hamiltonian system of equations describing system evolution over characteristic curve, or in other way, to time. Hamiltonian systems are in same way involved, under specific assumptions, used for modeling games problem.

Since we’re dealing with mechanical systems, there’s in the end a closure between game theory, system dynamics and mechanical systems dynamics, that could be interesting to be investigated in theretial field.

Consideration about applied methods in robotics field

So far we’ve just used Newton-Euler formulation, or Lagrange formulation. A remarkable place is deserved for Kane’s method suitable for generalization to quasi velocities.That’s classic literature stream approach. Other methods are known and minorly applied in literature as seen, like Maggi’s equations (usually not adopted because leading to DAE equations harder to solve even numerically) and because is possible to translate them in a constrained lagrange system; mostly ignored also hamiltonian approach although it could be easily be adopted and included, with some advantages.

At this point after a presentation of different methods we can give a comparison of everyone’s advantages and disadvantages

Comparison between dynamic modeling methods

Lagrange	Kane	Hamilton	Newton	Newton-Euler	Maggi
Use of energy and potential		Use of total energy	Use of forces	Use of forces in body frame)	
Number of equations equal to system degree of freedom		Number of equations equal two times system degree of freedom	Number of equations equal to three times system degree of freedom	Number of equations equal to three times system degree of freedom	
Second order equations		First order equations			
Coordinate transformations are possible and allowed		Canonical transformations of four types are allowed			

Interpretation as connection

In mathematics, and specifically differential geometry, a **connection form** is a manner of organizing the data of a connection using the language of moving frames and differential forms.

In geometry, the notion of a **connection** makes precise the idea of transporting local geometric objects, such as tangent vectors or tensors in the tangent space, along a curve or family of curves in a *parallel* and consistent manner. There are various kinds of connections in modern geometry, depending on what sort of data one wants to transport. For instance, an affine connection, the most elementary type of connection, gives a means for parallel transport of tangent vectors on a manifold from one point to another along a curve. An affine connection is typically given in the form of a covariant derivative, which gives a means for taking directional derivatives of vector fields, measuring the deviation of a vector field from being parallel in a given direction.

Connections are of central importance in modern geometry in large part because they allow a comparison between the local geometry at one point and the local geometry at another point. Differential geometry embraces several variations on the connection theme, which fall into two major groups: the infinitesimal and the local theory. The local theory concerns itself primarily with notions of parallel transport and holonomy. The infinitesimal theory concerns itself with the differentiation of geometric data. Thus a covariant derivative is a way of specifying a derivative of a vector field along another vector field on a manifold.

Connections also lead to convenient formulations of *geometric invariants*, such as the curvature (see also curvature tensor and curvature form), and torsion tensor.

A connection form associates to each basis of a vector bundle a matrix of differential forms. The connection form is not tensorial because under a change of basis, the connection form transforms in a manner that involves the exterior derivative of the transition functions, in much the same way as the Christoffel symbols for the Levi-Civita connection. The main *tensorial* invariant of a connection form is its curvature form. In the presence of a solder form identifying the vector bundle with the tangent bundle, there is an additional invariant: the torsion form. In many cases, connection forms are considered on vector bundles with additional structure: that of a fiber bundle with a structure group.

In differential geometry and gauge theory, a **connection** is a device that defines a notion of parallel transport on the bundle; that is, a way to "connect" or identify fibers over nearby points. A **principal *G*-connection** on a principal G-bundle P over a smooth manifold *M* is a particular type of connection which is compatible with the action of the group *G*.

Indeed lagrangian mechanics have a close connection with differential geometry and manifolds, intended as variety where motion takes place. In addition parametrizations introduced into variety lead to different lie group representations and algebra, already remarked to exist in global coordinates (as special case).

Connections is a concept able to relating traditional notion of derivative with derivative in a smooth but not flat variety.

Covariant derivative

In mathematics, the **covariant derivative** is a way of specifying a derivative along tangent vectors of a manifold. Alternatively, the covariant derivative is a way of introducing and working with a connection on a manifold by means of a differential operator, to be contrasted with the approach given by a principal connection on the frame bundle – see affine connection. In the special case of a manifold isometrically embedded into a higher-dimensional Euclidean space, the covariant derivative can be viewed as the orthogonal projection of the Euclidean directional derivative onto the manifold's tangent space. In this case the Euclidean derivative is broken into two parts, the extrinsic normal component (dependent on the embedding) and the intrinsic covariant derivative component.

The name is motivated by the importance of changes of coordinate in physics: the covariant derivative transforms covariantly under a general coordinate transformation, that is, linearly via the Jacobian matrix of the transformation.

Gauge covariant derivative

In physics, the **gauge covariant derivative** is a means of expressing how fields vary from place to place, in a way that respects how the coordinate systems used to describe a physical phenomenon can themselves change from place to place.
If a physical theory is independent of the choice of local frames, the group of local frame changes, the gauge transformations, act on the fields in the theory while leaving unchanged the physical content of the theory. Ordinary differentiation of field components is not invariant under such gauge transformations, because they depend on the local frame. However, when gauge transformations act on fields and the gauge covariant derivative simultaneously, they preserve properties of theories that do not depend on frame choice and hence are valid descriptions of physics.

The first approach is to examine what is required for a generalization of the directional derivative to "behave well" under coordinate transitions. This is the tactic taken by the covariant derivative approach to connections: good behavior is equated with covariance. Here one considers a modification of the directional derivative by a certain linear operator, whose components are called the Christoffel symbols, which involves no derivatives on the vector field itself. The directional derivative $D_{\mathbf{u}}\mathbf{v}$ of the components of a vector $\mathbf{v}$ in a coordinate system φ in the direction $\mathbf{u}$ are replaced by a *covariant derivative*:

$$\nabla_u \mathbf{v} = D_u \mathbf{v} + \boldsymbol{\Gamma}\left(\right)\{\mathbf{u}, \mathbf{v}\}$$

where Γ depends on the coordinate system φ and is bilinear in $\mathbf{u}$ and $\mathbf{v}$. In particular, Γ does not involve any derivatives on $\mathbf{u}$ or $\mathbf{v}$. In this approach, Γ must transform in a prescribed manner when the coordinate system q is changed to a different coordinate system. This transformation is not tensorial, since it involves not only the *first derivative* of the coordinate transition, but also its *second derivative*.

Specifying the transformation law of Γ *is not sufficient to determine* Γ *uniquely*. Some other normalization conditions must be imposed, usually depending on the type of geometry under consideration.

The second approach is to use Lie groups to attempt to capture some vestige of symmetry on the space. This also most useful approach for physical perspective treatment. This is the approach of Cartan connections. The example above using rotations to specify the parallel transport of vectors on the sphere is very much in this vein.

Lagrangian mechanics and symmetry

Galileo's transformation variance

A simple example of an interesting and very important symmetry: symmetry under uniform linear motion, known in classical mechanics as *Galileo's principle of relativity.* We will be surprised to learn that the classical action is not invariant under a Galilean transformation.
The action is not invariant under a Galilean transformation.
The two actions original and transformed) differs by a function that depends only on the coordinates of a given event so the mechanical laws are the same as determined by using any of them. Slightly more general considerations, but reasoning similar to that employed previously, demonstrate that the corresponding conservation law to Galilean transformation is related to the uniform motion of the center of mass.

Noether's theorem

Noether's theorem states that every continuous symmetry of the action of a physical system with conservative forces has a corresponding conservation law. This is the first of two theorems (see Noether's second theorem) published by mathematician Emmy Noether in 1918.[1] The action of a physical system is the integral over time of a Lagrangian function, from which the system's behavior can be determined by the principle of least action. This theorem only applies to continuous and smooth symmetries of physical space.

Noether's theorem is used in theoretical physics and the calculus of variations. It reveals the fundamental relation between the symmetries of a physical system and the conservation laws. It also made modern theoretical physicists much more focused on symmetries of physical systems. A generalization of the formulations on constants of motion in Lagrangian and Hamiltonian mechanics, it does not apply to systems that cannot be modeled with a Lagrangian alone. In particular, dissipative or actively forced systems with continuous symmetries need not have a corresponding conservation law. This lead to significative limitation of applicability of theoretical physics methods in applicative technical cases.

For N infinitesimal transformations leaving the action unchanged, having generators T_r and Q_r, Noether showed that the N quantities are conserved (constants of motion).

$$\left(\frac{\partial L}{\partial \dot{q}}\dot{q} - L\right)T_r - \frac{\partial L}{\partial \dot{q}}Q_r$$

Similarity of structure with hamiltonian formalism lead to reasons why it is preferable for treating general transformations and symmetries in systems dynamics.

A bunch of simple traditional study case includes:

- Time invariance: energy conservation
- Translational invariance: linear momenta conservation
- Rotational invariance: angular momenta conservation

consider a Lagrangian that does not depend on time $t \rightarrow t + \delta t$ leading to conserved quantity hamiltonian (this is de facto a legendre transformations)

$$H = \frac{\partial L}{\partial \dot{q}}\dot{q} - L = p\dot{q} - L$$

Consider a Lagrangian which does not depend on an ("ignorable") coordinate q_k; so it is invariant (symmetric) under changes $q_k \rightarrow q_k + \delta q_k$.$Q_k = 1$ Leading to momenta (conjugate) preservation:

$$P = p_k = \frac{\partial L}{\partial \dot{q}_k}$$

Lagrangian does not depend on the absolute orientation of the physical system in space, assume that the Lagrangian does not change under small rotations of an angle $\delta\theta$ about an axis **n**; such a rotation transforms the Cartesian coordinates by the equation

$$r \rightarrow r + \delta\theta n \times r$$

$$\frac{\partial L}{\partial \dot{q}}Q = \underbrace{p \cdot n \times r = n \cdot (r \times p)}_{\text{mixed product rule}} = n \cdot L$$

So far we talked of exact symmetries but we cannot exclude the existence of approximated or local symmetries that are broken on large, like after long time or for large displacement or rotations. Evenmore one can consider systems where symmetries sussist only on average like in periodic or quasi periodic systems. Applicability extend even more general local functional differentiable actions, including ones where the Lagrangian depends on higher derivatives of the fields.

Continuous and discrete and hybrid symmetries

Most of symmetries considered so fare are continuous. Other kind of symmetry exist that do not rely on continuous parameters and are not then treatable using Noether theorem. Even in case fo existing discrete symmetries existence of a conserved quantity (usually named charge in physics, leading to currents) is not assured.

General lagrangian formalism

Euler-Lagrange equations are expression of a more general variational principle extremization taking in this case the name of *action principle* (since involved integrand quantity L called generally lagrangian, is physically in units an action) that do not last only for mechanical systems but is of absolute generality in variational calculus, even for higher order of differentiation, as well valid for continuous systems and fields (infinite degree of freedom) and lagrangian depending on more than one free coordinate. For instance iit can be applied to continuum (involving then spatial coordinates and time) up to arbitrary differentiation order. General form of variant lagrange equations comes out quite easily from application iterated of integration by parts theorem applied to variational principle, leading to general expression (up to derivatives of order O, in one independent variable t and n dependent variables q indexed with h:

$$\delta \int L\left(q, \ldots, q^{(O)}\right) dt = 0$$

$$\frac{\partial L}{\partial q_h} - \sum_{k=1}^{O-1} (-1)^k \frac{d^k}{dt^k} \left(\frac{\partial L}{\partial q_h^{(k)}} \right) = 0$$

Extending to more free variables beyond time we get (using typical notation, with dependant variables vector f):

$$\sum_{j=0} \sum_{\mu_1 \leqslant \cdots \leqslant \mu_j} (-1)^j \partial_{\mu_1 \leqslant \cdots \leqslant \mu_j} \left(\frac{\partial L}{\partial f_{i,\mu_1 \ldots \mu_j}} \right) = 0$$

All other case are special case of.

Details can be found in Euler-Lagrange and in some variational calculus textbooks.

Collision and contact forces models

A primary relevance effect to account for in robotics modeling, but generally not considered at first step, regards the inclusion of interaction with external world and contact or impact with solid bodies or surfaces. One first effect regards collisions could occur between robot parts (especially when joints position limits are not imposed properly or considered) or in any case the presence of objects in robot workspace expose to potential contact or collision with, because expected to interact with (target), or undesired (like in case of improper interaction with environment) or in case of collaborative robotics. These objects are not necessarily fixed but can move too, and their nature could be heterogeneous (hard, soft, frail, resistant, movable, fixed, living or inanimate, living tissues and organs too). Especially for collaborative robots these considerations are of most importance since a direct interaction with environment entities is of primary relevance.

Unfortunately collision and contact modeling is numerically demanding, and could pose significative changes.

Safety implications are of primary importance and constitute an actual subject of research at applicative and academic level.

Game theory and robot dynamics

Is possible to prove there's a connection between game theory and general systems dynamics, consequently to robot dynamics as well.
Indeed there's proof that generic treatment of partial differential equations using characteristics method lead to Hamilton Jacobi equations, that are associated to hamiltonian system of equations describing system evolution over characteristic curve, or in other way, to time.

Since we're dealing with mechanical systems, there's in the end a closure between game theory, system dynamics and mechanical systems dynamics, that could be interesting to be investigated in theretial field.

So far we've just used Newton-Euler formulation, or Lagrange formulation. That's classic literature stream approach, that's mostly ignoring hamiltonian approach although it could be easily be adopted and included, with some advantages.

Lagrange	**Hamilton**	**Newton**	**Newton-Euler**	
Use of energy and potential	Use of total energy	Use of forces	Use of forces in body frame)	
Number of equations equal to system degree of freedom	Number of equations equal two times system degree of freedom	Number of equations equal to three times system degree of freedom	Number of equations equal to three times system degree of freedom	
Second order equations	First order equations			
Coordinate transformations are possible and allowed	Canonical transformations of four types are allowed			

Consequently we can say that introduction of conjugate momenta could lead to a simpler formalism for robot dynamics representation, introducing in a mechanically and mathematically natural and justified way the introduction of a lower order set of equations, set of additional variables, and potential reduction of complexity like natural definition of motion constant and invariants, that in other formalisms would be harder to find and clearly depict.

Consequently hamilton equations are directly suitable for integration in their canonical form, where usually lagrange equations require some rearrangement. But on the other side using partial derivatives on less complex expression ease the understanding and structure of motion equations.
Indeed many of coriolis and centrifugal terms

Relation between momenta and mass matrix lead to

$$p = M\dot{q}$$
$$\dot{q} = M^{-1}p$$
$$\dot{p} = M\ddot{q} + \dot{M}\dot{q}$$
$$\dot{p} = M\ddot{q} + \frac{\partial M}{\partial q}\dot{q}\dot{q}$$

Ergo at least part of centrifugal and coriolis terms are included into momenta derivative in place of their explicit writing in term of generalized speeds. Indeed is easy to see how some relevant laws of motion like angular and linear momentum conservation, energy conservation and variable independence and consequent decoupling arise evidently in equations structure.

A second step is required to reconvert result into mor intuitive coordinates space, but is not such a hard task.

System of equations order reduction

There are multiple different and ways to reduce one dynamic system oder, not all are equally effective.

One way is using simply a set of trivial auxiliary variables definition like using derivatives of generalized coordinates. Although in simpler cases this is equivalent to the introduction of conjugate momenta, assuming an unitary mass, or simply introducing a ratio over coordinate mass, this does not keep true in most complex cases where mass matrix structure plays a major role leading to more complex expressions.

Lyapunov equations

Introdcution of hamiltonian formalism lead in a quite natural way the introduction of a liapunov function helpful for system stability analysis

Generalization of motion equations

There's an alternative point of view that prove the generality of seen equations.
Starting from general relativity and geodetic principle, we get for a generic geodesic an equation respect proper time like, in tensorial formalism

$$\frac{\partial^2 x^\mu}{\partial s^2} + \Gamma_{ij}^{\ \mu} \frac{\partial x^i}{\partial s} \frac{\partial x^j}{\partial s} = 0$$

where Γ is a pseudotensor (then do not transform in covariant way under a general transformation of coordinates). In this equations we suppose to accept the passage from traditional newtonian assumption of absolute time to proper time, as most indicate and natural equivalent of traditional time notion measured in classic mechanics. Consequently speeds and accelerations appears like first and second derivatives of position coordinate over proper time. Adding formal passage of coordinate parametrization as function of generalized coordinates we get a substantially equivalent equation

$$\frac{\partial^2 x^\mu}{\partial q} \frac{\partial^2 q^k}{\partial s^2} + \Gamma_{ij}^{\ \mu} \frac{\partial x^i}{\partial q^h} \frac{\partial x^j}{\partial q^t} \dot{q}^h \dot{q}^t$$

$$\frac{\partial q^k}{\partial s} = \dot{q}^k$$

There's an additional observation to keep under reader attention, in this formulation there's no mass matrix appearing as factor for accelerations. This is due to essentially geometrical approach used into general relativity where trajectory is defined only from geometry underlying the system. Only indirectly forces of inertia (as well as other forces) appears in equations just modifying structure of geometry, accounting for mass induced accelerations, like supposed in general relativity

A further extension of geodetic equations is obtained including additional sources and forces to this equations

$$\frac{\partial^2 x^\mu}{\partial s^2} + \Gamma_{ij}^{\ \mu} \frac{\partial x^i}{\partial s} \frac{\partial x^j}{\partial s} = F$$

Beyond specific context of general relativity it is a proof of the fact that general motion equations assume a shape aforementioned in disregard of specific context, forces or other affecting factors.

This is rather easy to understand considering origins of Christofell symbols coming from covariant derivatives and parallel transport. They indeed account for total variation of momenta vectorial quantities along a curved path, so that it accounts for a modulus variation along directions coordinate (accelerations) and for a component due to trajectory curvature. Effective trajectory curvature is could be lead from constraints, or in a perfectly equivalent way, tue acting forces, since reactions arise to justify force balance in curved trajectories.

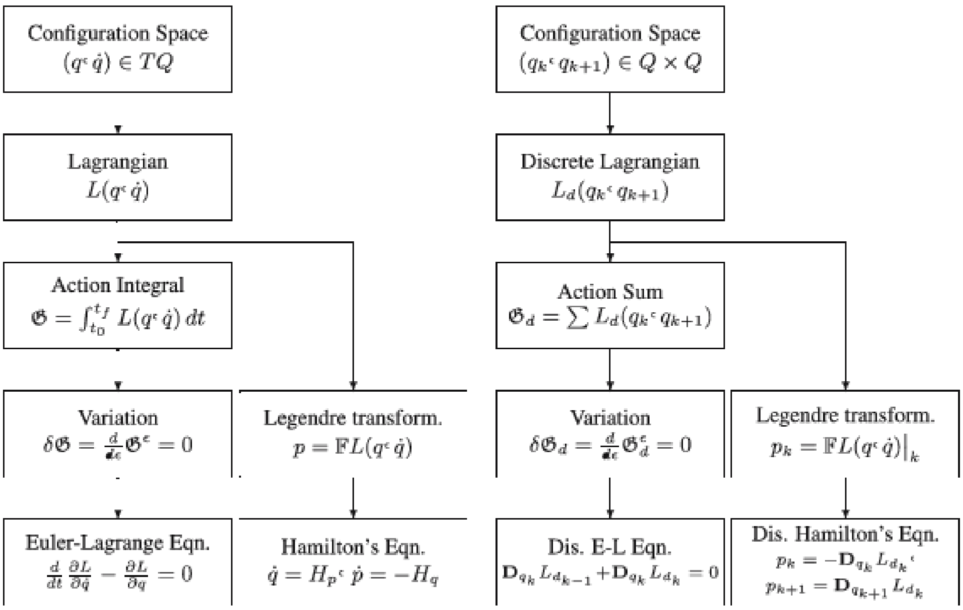
$\frac{\partial v}{\partial s} + \Gamma v^i v^j$

Part dependent on momenta (or velocities) vector are those we call usually centrifugal and coriolis terms, but may also assume a different meaning considering forces case, since descrivbe the component of acting solicitations causing a variation in direction of motion.

Recap of dynamics methods

Tabelle 4: Methoden der Dynamik

HUYGENS (konservativ)	$T + V = H$	... 1673
GIBBS & APPELL (hier: holonom)	$\left[\frac{\partial S}{\partial \ddot{\mathbf{z}}}\right]^T = \mathbf{Q}$	1879 & 1899
HAMILTON (hier: konservativ)	$\delta \int_{t_0}^{t_1} (T - V)\, dt = 0$	1834
HAMILTON (hier: konservativ, autonom)	$\dot{\mathbf{p}}^T = -\left[\frac{\partial H}{\partial \mathbf{z}}\right] \quad \dot{\mathbf{z}}^T = \left[\frac{\partial H}{\partial \mathbf{p}}\right]$	1834
LAGRANGE (hier: konservativ)	$\frac{d}{dt}\left[\frac{\partial T}{\partial \dot{\mathbf{z}}}\right] - \left[\frac{\partial T}{\partial \mathbf{z}}\right] + \left[\frac{\partial V}{\partial \mathbf{z}}\right] = 0$	1788
HAMEL-BOLTZMANN	$\left[\frac{d}{dt}\left[\frac{\partial T}{\partial \dot{\mathbf{s}}}\right] - \left[\frac{\partial T}{\partial \mathbf{s}}\right] - \mathbf{Q}^T\right]\delta\mathbf{s} + \frac{\partial T}{\partial \dot{\mathbf{s}}}\left[\frac{d\delta\mathbf{s} - \delta d\mathbf{s}}{dt}\right] = 0$	1904
	ZENTRAL – $\boxed{\frac{d}{dt}\left[\frac{\partial T}{\partial \dot{\mathbf{s}}}\delta\mathbf{s}\right] - \delta T - \delta W^e = 0}$ GLEICHUNG	
EULER (eliminierte Zwangskräfte)	$\sum_{i=1}^{p}\left\{\left[\frac{\partial \mathbf{v}_s}{\partial \dot{\mathbf{s}}}\right]^T[\dot{\mathbf{p}} + \tilde{\boldsymbol{\omega}}\mathbf{p} - \mathbf{f}^e] + \left[\frac{\partial \boldsymbol{\omega}}{\partial \dot{\mathbf{s}}}\right]^T[\dot{\mathbf{L}} + \tilde{\boldsymbol{\omega}}\mathbf{L} - \mathbf{l}^e]\right\}_i = 0$	1750/75

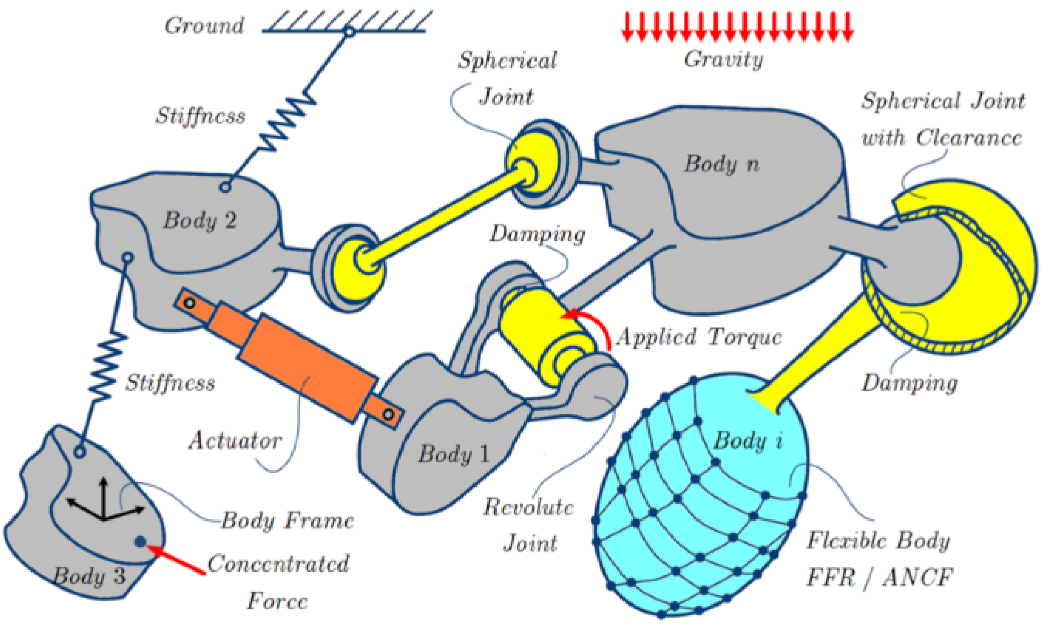


Articulated body algorithms ABA

Another algorithm developed by Featherstone is the articulated body algorithm.
This is a constraint propagation algorithm that performs forward dynamics propagation.

General multibody problem formulations

Multibody Dynamics Robot dynamics can be regarded as a subset (or a specific application) of the broader discipline of multibody dynamics. Books on multibody dynamics include [3.3, 14, 35, 41–46]. Of course, multibody dynamics is, in turn, a subset of classical mechanics; and the mathematical foundations of the subject can be found in any good book on classical mechanics, such as [3.13].



Parallelizable procedures for multibody dynamics

In recent year an increasing attention has been attracted to the topic of parallelization. Indeed exploiting the fact that equations of motion could be assembled from sub components or even from individual bodies dynamics equations, end exploiting potential of parallelized computing, could lead to significative performances improvement in dealing with multibody systems.

In order to speed up the common dynamics computations, a number of algorithms have been developed for parallel and pipelined computers. For inverse dynamics, early work focused on speeding up the $O(n)$ RNEA on up to n processors [3.89, 90] while subsequent work resulted in $O\left(log_2 n\right)$ algorithms [3.91, 92]. For the $O\left(n^2\right)$ CRBA to compute the joint-space inertia matrix, early work resulted in $O\left(log_2 n\right)$ algorithms for n processors to compute the composite rigid-body inertias and diagonal elements of the matrix [3.93, 94]. Subsequent work resulted in $O\left(log_2 n\right)$ algorithms for $O\left(n^2\right)$ processors to compute the entire matrix [3.95, 96]. For forward dynamics, speedup was obtained for a multiple manipulator system on a parallel/pipelined supercomputer [3.97]. The first $O\left(log_2 n\right)$ algorithm for n processors was developed for an unbranched serial chain [3.98]. More recent work has focused on $O\left(log_2 n\right)$ algorithms for more complex structures [3.65, 99, 100].

One example of these algorithms is DCA: divide and conquer algorithm from Featherstone.

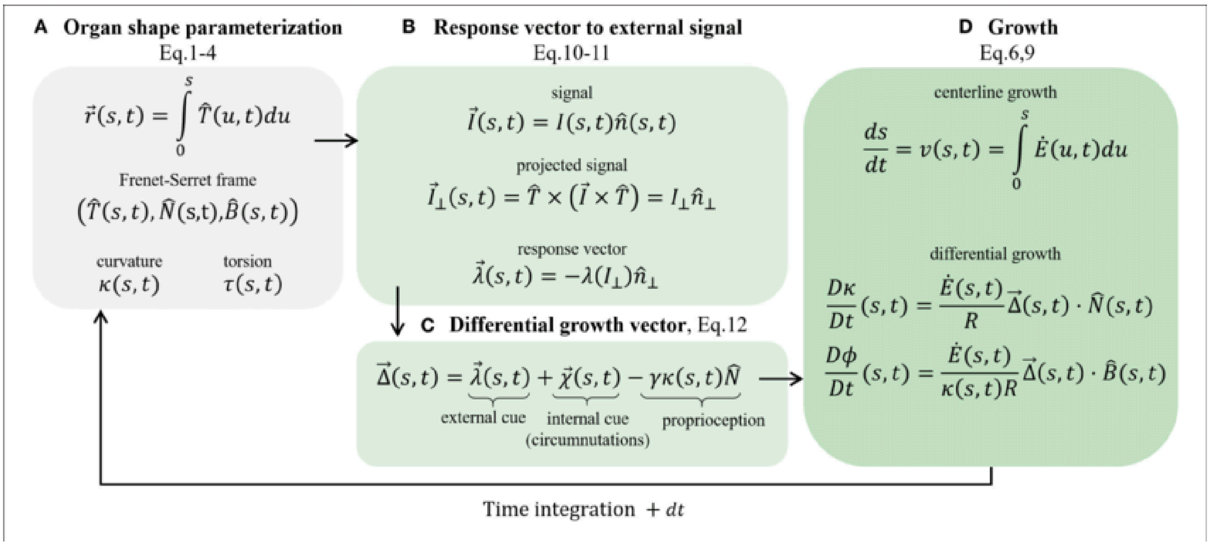
In this approach hinges are viewed as constraints between independent components of the system.
For a non parallel computing system advantages are rather limited.

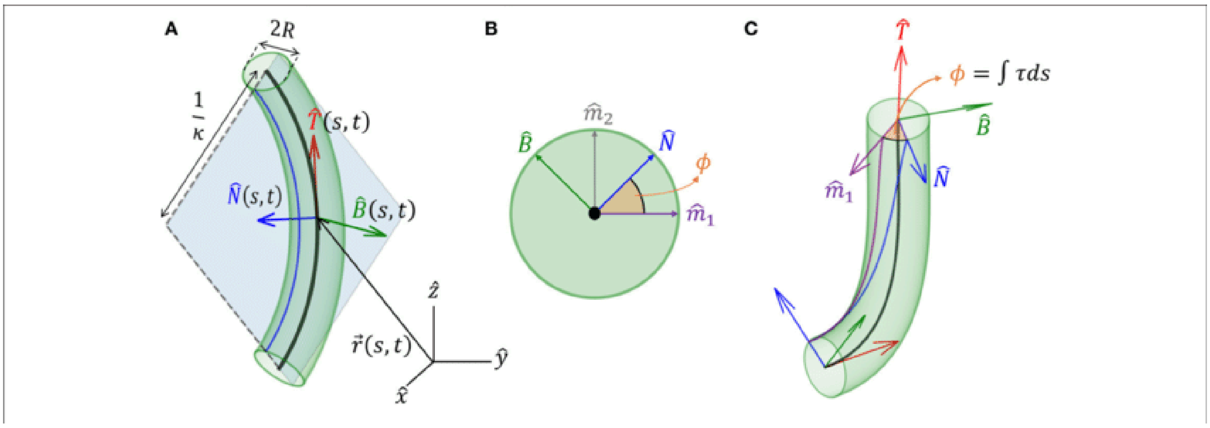
Topologically-Varying Systems

There are many robot mechanisms whose topology varies over time because of a change of contact conditions, especially with the environment. In legged vehicles, use of a compliant *ground-contact* model to compute the contact forces reduced the closed loop structure to a tree structure [3.101]. However, for cases in which the contacts are very stiff, *numerical integration problems* may result. In more recent work [3.40, 102] in which hard contact constraints are assumed, an efficient method was used to reduce the large number of coordinate variables from that which may be necessary in general-purpose motion analysis systems [3.43]. Also, they were able to automatically identify the variables as the structure varied and developed a method for computing the velocity boundary conditions after configuration changes [3.40, 102].

Growing robots

In addition to topologically varying robots we have also some studies about robots whose maybe topology do not change significatively, but size do.
IImplications are of interest because from different perspectives.





Alternative Representations

Spatial vectors are often used to express the equations of motion. There are various alternatives to the use of spatial vectors: other kinds of 6-D vector, 3-D vectors, 44 matrices, and the spatial operator algebra. All 6-D vector formalisms are ***similar***, but are **not** exactly the same. The main alternatives to spatial vectors are: screws [3.10–12], motors [3.47], Lie algebras [3.12, 48], and ad hoc notations. (An ad hoc notation is one in which 3-D vectors are grouped into pairs for the purpose of reducing the volume of algebra.) Three-dimensional vectors are the formalism used in most classical mechanics and multibody texts, and are also a precursor to 6-D vector and 44 matrix formalisms. 44 matrices are popular in robotics because they are very useful for kinematics. However, they are not so useful for dynamics. 44 matrix formulations of dynamics can be found in [3.37, 49, 50]. The spatial operator algebra was developed at the Jet Propulsion Laboratory (JPL) by Rodriguez, Jain, and others []. It uses 6N-dimensional vectors and 6N6N matrices, the latter regarded as linear operators. Examples of this notation can be found in [3.38, 51–53].

Vectorial formulation of the equations of motion that is usually called the Newtonian or *Newton–Euler* formulation. The main alternative is the *Lagrangian* formulation, in which the equations of motion are obtained via Lagrange's equation. Examples of the Lagrangian formulation can be found in [3.9, 10, 18, 54, 55]. *Kane's* method has also been applied in robotics [3.56, 57]. 3.7.4

Other approaches used in general dynamics, but with few applicative examples in literature are Gibbs-Appel’s method, Maggi equations, Hamilton equations,

Efficiency

Because of the need for real-time implementation, especially in control, the robotics community has focused on the problem of computational efficiency. For *inverse dynamics*, the $O(n)$ recursive Newton–Euler algorithm (RNEA) of Luh et al. [3.4] remains the most important algorithm. Further improvements to the algorithm are given in [3.58, 59]. For forward dynamics, the two algorithms presented in this chapter remain the most important for computational considerations: the $O(n)$ articulated-body algorithm (ABA) developed by Featherstone [3.1] and the $O(n^3)$ algorithm based on the composite-rigid-body algorithm (CRBA) of Walker and Orin [3.5]. Improvements were made in the ABA over the years [3.15, 17, 25] so that it was more efficient than the CRBA-based algorithm for decreasingly smaller values of n. However, more recent application of the CRBA to branched kinematic trees [3.26] and robotic systems with motion-controlled appendages [3.60] continue to show the viability of the CRBA approach. For the *joint-space inertia matrix*, the CRBA [3.5] is the most important algorithm. A number of improvements and modifications have been made over the years to increase its computational efficiency [3.15, 61–63]. For the *operational-space inertia matrix*, efficient $O(n)$ algorithms have been developed [3.28–30] and applied to increasingly complex systems [3.20, 31, 33]. The exploitation of branch-induced sparsity also results in an efficient algorithm [3.32].

The power model

From [Dynamic identification of robots with power model]

$$\dot{q}^T\Gamma = \dot{q}^T\left(\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q}\right) + \dot{q}^T\Gamma_f$$

$$\dot{q}^T\Gamma = \dot{q}^T\left(M\ddot{q} + C(q,\dot{q}) + G(q)\right) + \dot{q}^T\Gamma_f =$$

since gyroscopic and coriolis forces do not work

$$\dot{q}^T\Gamma = \dot{q}^T\left(M\ddot{q}\right) + \dot{q}^T\Gamma =$$

$$\dot{q}^T\Gamma = \left(\frac{dH(q,\dot{q})}{dt}\right) + \dot{q}^T\Gamma_f = \left(\frac{dh(q,\dot{q})}{dt}\right)\chi$$

$$\dot{q}^T\Gamma = \dot{q}^T\left(\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q}\right) + \dot{q}^T\Gamma_f$$

$$\dot{q}^T\Gamma = \dot{q}^T\left(M\ddot{q} + C\left(q\dot{;}q\right) + G\left(q\right)\right) + \dot{q}^T\Gamma_f$$

$$\dot{q}^T\Gamma = \dot{q}^T\left(M\ddot{q}\right) + \dot{q}^T\Gamma_f = \frac{dH\left(q,\dot{q}\right)}{dt} + \dot{q}^T\Gamma_f = \frac{dh\left(q,\dot{q}\right)}{dt}\chi$$

where friction effects have been included into model parameters.

$$\Gamma = D\left(q,\dot{q},\ddot{q}\right)\chi = \sum_{i=1}^{N_p} D_{:,i}\chi_i$$

where

$$D_{:,i}\left(q,\dot{q},\ddot{q}\right) = M_i\left(q\right)\ddot{q} + N_i\left(q,\dot{q}\right)$$
$$M_i = \frac{\partial M}{\partial X_i}$$
$$N_i = \frac{\partial N}{\partial X_i}$$

$$\begin{bmatrix} \Gamma_1 \\ \Gamma_2 \\ \vdots \\ \Gamma_{n-1} \\ \Gamma_n \end{bmatrix} = \begin{bmatrix} D^{1,1} & D^{1,2} & \dots & D^{1,n} \\ 0 & D^{2,2} & \dots & D^{2,n} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & D^{n-1,n-1} & D^{n-1,n} \\ 0 & \dots & 0 & D^{n,n} \end{bmatrix} \begin{bmatrix} X^1 \\ X^2 \\ \vdots \\ X^{n-1} \\ X^n \end{bmatrix}$$