Design and optimization of lattice structures for aerospace applications

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List of Abbreviations

AD	Alternate Directions
BESO	Bi-directional ESO
CONLIN	CONvex LINearization
DOFs	Degrees Of Freedom
ESO	Evolutionary Structural Optimization
GA	Genetic Algorithm
GBMMA	Gradient Based MMA
GCMMA	Globally Convergent MMA
GGP	Generalized Geometry Projection
GP	Geometry Projection
HS	Hashin-Shtrikman
LP	Linear Programming
MIP	Mixed-Integer Programming
MMA	Method of Moving Asymptotes
MMC	Moving Morphable Components
NAND	Nested Analysis and Design
OC	Optimality Criteria
RVE	Representative Volume Element
SAND	Simultaneous Analysis and Design
SIMP	Solid Isotropic Material with Penalization Method
SLP	Sequential Linear Programming
SLSQP	Sequential Least Square Quadratic Programming
SQP	Sequential Quadratic Programming
TTO	Truss Topology Optimization

1 Literature review

This thesis focuses on the numerical optimization in the structural engineering domain. As such, one needs familiarity with existing optimization methods and contemporary engineering practices. The purpose of this chapter is to provide the reader with a non-exhaustive historical overview of structural optimization, particularly in the context of ultralight and modular structures. Additionally, we introduce crucial concepts and terminology that will be employed consistently throughout the document.

1.1 An introduction to structural optimization

Structural optimization is a multidisciplinary field within engineering that aims to systematically improve the performance–considering factors like mass, stiffness, and dynamic response–by optimizing their shape, material distribution, and overall design. Historically, structural optimization algorithms are categorized into three families: sizing, shape, and topology optimization. Sizing optimization concentrates on determining the optimal distribution of variables, where both the design and state variable domains are known *a priori* and remain constant during optimization. In contrast, shape optimization aims to discover the optimal shape of a predefined domain, treating the domain itself as a design variable allowing for flexibility in shaping the structure. Topology optimization goes further, involving the determination of features like the number, location, and shape of holes, as well as the connectivity of the structural domain. This approach offers a more comprehensive exploration of possibilities in structural design. A visual representation of the three families is provided in Fig. 1.1.

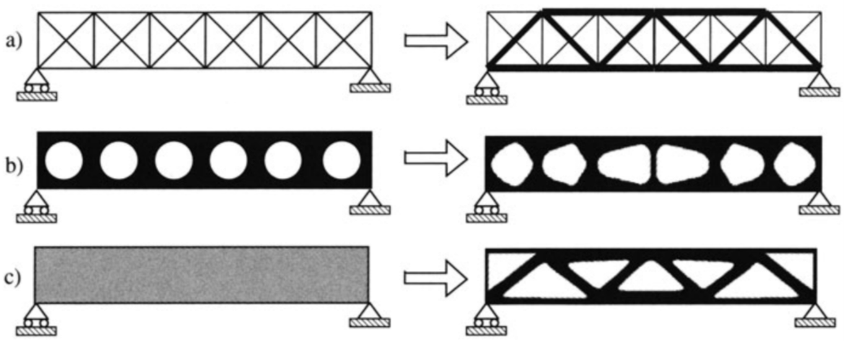


Figure 1.1: Visual representation of (a) size, (b) shape and (c) topology optimization [1].

Structural optimization involves using mathematical algorithms, computational models, and iterative analyses to explore and refine design solutions. For that reason, we introduce now the basic concepts and terminology behind numerical optimization. In numerical optimization, algorithms are employed to minimize or maximize a specific function by adjusting various design variables. The problem may or may not be subject to constraints. Formulating an optimization problem is a crucial step to prevent common conceptual errors, such as confusing constraints with objective functions. An incorrect problem formulation can lead to a failed solution or yield a mathematical optimum that lacks feasibility from an engineering perspective.

The most general formulation of an optimization problem is written as:

$$\begin{aligned} \min_{\boldsymbol{x}} \quad & f(\boldsymbol{x}) \\ \text{by varying} \quad & \boldsymbol{x} \in [l^-, l^+] \\ \text{s.t.} \quad & g_{\mathrm{e}}(\boldsymbol{x}) = 0 \\ & g_{\mathrm{i}}(\boldsymbol{x}) > 0, \end{aligned} \tag{1.1}$$

where $f(x)$ is the objective function to minimize, $\boldsymbol{x}$ is the vector of design variables bounded between l^- and l^+, and g_{e} and g_{i} represent the equality and inequality constraints, respectively.

Objective Function In numerical optimization, the objective function $f(\boldsymbol{x})$ represents the scalar that we aim to minimize. Should the goal be to maximize a function, one can achieve this by minimizing the opposite of that function, maintaining adherence to the convention. Common objective functions in structural design include the minimization of volume or structural compliance. The objective function can take the form of an explicit function or result from a highly complex computational procedure. The selection of the objective function is crucial to propose a design that is feasible from an engineering perspective, regardless of the precision of the optimization scheme employed.

Optimization problems are categorized in the literature based on how the objective function is with respect to design variables, whether linear, quadratic, or generally non-linear. It is possible to concurrently optimize multiple objective functions, but this usually results in a family of optimum designs with differing emphases on the various objectives called Pareto front. When possible, it is more straightforward to convert these diverse objectives into constraints [2].

2. Martins et al. (2021), 'Engineering Design Optimization'

Design Variables The design variables $\boldsymbol{x}$ are the parameters that the optimizer algorithm can change to minimize the objective function. Design variables could be continuous or discrete if only some distinct values are allowed (for example, only a certain size for a hole in a structural analysis). The optimization problem formulation allows for the lower and upper boundary for each design variable known in the literature as variable bounds.

Constraints The constraints are functions used to restrict the design variables in some way. They serve the purpose of preventing the algorithm from converging to a numerical minimum that is not feasible due to physical and engineering constraints. Similar to the objective function, constraint functions can take on linear, quadratic, or generally non-linear forms, and different algorithms must be applied accordingly.

Constraint functions can be further classified into two types: equality constraints (ge), which arise when the design variables are restricted to be equal to a fixed quantity, and inequality constraints (gi), which come into play when the design variables are required to be greater than or equal to a certain quantity.

1.1.1 Optimizers

The field of numerical optimizers is extensive. For that reason, our focus here will be specifically on algorithms employed in structural optimization. Various algorithm types have been applied to address structural optimization problems, predominantly categorized into three main families: optimality criteria, metaheuristic algorithms, and gradient-based strategies.

Optimality Criteria (OC) refer to mathematical conditions or rules used to assess and guide the modification of a design or structure to achieve the desired performance [3, 4]. In the context of topology optimization, OC are primarily applied in compliance minimization problems, as each element contributes independently to the overall compliance. Bendsøe used OC to seek the stiffest plate i.e. compliance minimization–that can be made of a given amount of material and, toghether with Kikuchi [5], they used OC to obtain optimal shape design of structural elements based on boundary variations without the use of remeshing. Later, Bendsøe and Sigmund [6, 7] introduced a heuristic update scheme for isotropic materials, while Allaire *et al.* [8] demonstrated the convergence proof for both isotropic and anisotropic materials using the Alternate Directions (AD) approach. In both methods mechanical analysis provides essential information for solving closed-form conditions, allowing iterative updates of variables until convergence is achieved.

3. Prager et al. (1968), 'Problems of Optimal Structural Design'
4. Prager (1968), 'Optimality Criteria in Structural Design'

5. Bendsøe et al. (1988), 'Generating optimal topologies in structural design using a homogenization method'
6. Bendsøe (1995), 'Optimization of Structural Topology, Shape, and Material'
7. Sigmund (2001), 'A 99 line topology optimization code written in Matlab'
8. Allaire et al. (1996), 'The homogenization method for topology and shape optimization. Single and multiple loads case'

Metaheuristic (or gradient-free) algorithms offer a broader range of options compared to their gradient-based counterparts. While gradient-based algorithms typically conduct local searches, possess mathematical justification, and operate deterministically, metaheuristic algorithms are simpler and usually take much less developer time to use, and are perfect candidates for smaller problems. They find very diverse application cases and are useful when the design space is discrete, with multiple objective functions, or highly non-linear with many local minima (multimodal). The works authored by Conn *et al.* [9] and Audet and Hare [10] offer a more comprehensive exploration of gradient-free optimization algorithms. Evolutionary algorithms, a prominent category, simulate natural selection by retaining the fittest solutions in each generation while introducing mutations or cross-overs for improvement. A review on these optimization methods is given in the following reference [11]. These algorithms, also called Genetic Algorithm (GA), are employed for example by Balamurugan *et al.* [12] for compliance minimization, showcase versatility but face challenges with combinatorial considerations as the number of design variables increases [13]. Particle swarm and ant colony algorithms, inspired by nature, provide alternative strategies with randomness and new search directions. However, these non-gradient methods require regularization schemes for topology optimization, as outlined by Luh, Lin, and others [14, 15]. Targeting the resolution of Mixed-Integer Programming (MIP) problems, branch-and-bound algorithms divide the feasible set of the original problem into subsets through a process known as branching. These subsets are then further segmented to refine the partition of the feasible set. For each subset, lower bounds and optionally upper bounds on the objective function value are determined, a process referred to as bounding. Typically, the lower bounding problems are convex problems that can be efficiently solved to global optimality. Stolpe [16] addresses a volume minimization problem on a truss using a continuous branch-and-bound method, ensuring convergence to a globally optimal solution. The Evolutionary Structural Optimization (ESO) framework, initially a metaheuristic, removes less solicited elements iteratively [17, 18]. These methods offer freedom in optimization and improved convergence to local minima, especially in handling various optimization problems like buckling [19]. The ESO algorithm has been enhanced by the Bi-directional ESO (BESO) framework [20], which allows both removal and addition of elements.

Gradient-based algorithms in optimization leverage local information at a trial point to comprehend the shape of the local objective function in the neighborhood. This insight is crucial for determining the optimal direction to minimize the objective function. Typically, only the Jacobian (first derivative) is utilized, though more advanced algorithms incorporate the Hessian (second derivative). The computational demand for gradient calculation often constitutes the most resource-intensive aspect of the optimization loop. When constraints are present, solving the problem directly on the analytic response surface of the objective function becomes impractical. Consequently, the approach involves creating local approximations of the problem at the current design point using gradient information. These approx-

9. Conn et al. (2009), 'Introduction to Derivative-Free Optimization'

10. Audet et al. (2017), 'Derivative-Free and Blackbox Optimization'

11. Simon (2013), 'Evolutionary optimization algorithms'

12. Balamurugan et al. (2011), 'A two phase approach based on skeleton convergence and geometric variables for topology optimization using genetic algorithm'

13. Sigmund (2011), 'On the usefulness of non-gradient approaches in topology optimization'

14. Luh et al. (2009), 'Structural topology optimization using ant colony optimization algorithm'

15. Luh et al. (2011), 'A binary particle swarm optimization for continuum structural topology optimization'

16. Stolpe (2004), 'Global optimization of minimum weight truss topology problems with stress, displacement, and local buckling constraints using branch-and-bound'

17. Mattheck et al. (1990), 'A new method of structural shape optimization based on biological growth'

18. Xie et al. (1993), 'A simple evolutionary procedure for structural optimization'

19. Manickarajah et al. (1998), 'An evolutionary method for optimization of plate buckling resistance'

20. Young et al. (1999), '3D and multiple load case bi-directional evolutionary structural optimization (BESO)'

imations are designed so that specialized algorithms can efficiently solve them. The categorization of gradient-based algorithms is often based on how this local approximation is constructed.

The most used approximations in structural optimization includes among others Sequential Linear Programming (SLP), Sequential Quadratic Programming (SQP) and Sequential Least Square Quadratic Programming (SLSQP) [21], Method of Moving Asymptotes (MMA) [22] and its amelioration Globally Convergent MMA (GCMMA) [23] and Gradient Based MMA (GBMMA) [24], and CONvex LINearization (CONLIN) [25]. Specialized algorithms for solving the approximated problems are, among others, the primal-dual method and interior point method.

21. Kraft (1988), 'A software package for sequential quadratic programming'
22. Svanberg (1987), 'The method of moving asymptotes—a new method for structural optimization'
23. Svanberg (2002), 'A Class of Globally Convergent Optimization Methods Based on Conservative Convex Separable Approximations'
24. Bruyneel et al. (2002), 'A family of MMA approximations for structural optimization'
25. Fleury et al. (1986), 'Structural optimization'

An interior-point method is a numerical optimization algorithm used to solve constrained optimization problems. The key idea behind interior-point methods is to transform the constrained optimization problem into a sequence of unconstrained problems, allowing for efficient iterative solutions. The method introduces a barrier function that penalizes points outside the feasible region, effectively creating a "barrier" against leaving that region. This barrier function is incorporated into the objective function, and as the optimization progresses, it guides the search towards the interior of the feasible region. The term "interior point" originated from early methods that relied on interior penalty techniques, assuming the initial point was feasible. Nevertheless, contemporary interior-point methods as the open source IPOPT [26] are more versatile and can start from infeasible points. Rojas Labanda and Stolpe conducted a benchmark of various optimization algorithms and structural optimization formulations using a compliance minimization problem. Their findings highlight the efficacy of employing interior point algorithms such as IPOPT in topology optimization problems [27].

26. Wächter et al. (2006), 'On the implementation of an interior-point filter line-search algorithm for large-scale nonlinear programming'
27. Rojas Labanda et al. (2015), 'Benchmarking optimization solvers for structural topology optimization'

1.2 Ultra-lightweight structures optimization approaches

Two of the most frequently employed formulations for structural optimization are the minimization of volume while adhering to stress constraints and the minimization of compliance under volume constraints. Historically, the volume minimization formulation has been used in the first works of structural optimization of truss structures [28–30]. The problem was initially formulated in terms of member forces, ignoring the kinematic compatibility to obtain a Linear Programming (LP) problem. The formulation was modeled using the Simultaneous Analysis and Design (SAND) approach, in which the equations of nodal equilibrium are treated as equality constraints, and where both nodal displacements and the cross-sectional areas of truss members serve as design variables [31]. These methods are known in

28. Dorn et al. (1964), 'Automatic design of optimal structures'
29. Chan (1964), 'Optimum structural design and linear programming'
30. Hemp (1973), 'Optimum Structures'
31. Sankaranarayanan et al. (1994), 'Truss topology optimization with simultaneous analysis and design'

the literature as layout optimization or Truss Topology Optimization (TTO).

However, to attain greater design freedom, the structure optimization field later transitioned from truss structures to continuous discretization (also called density methods). While truss structures offered simplicity and ease of analysis, they imposed limitations on design due to their discrete member configurations and their inability to transmit moments, handle torsional effects, and represent complex structural elements such as plates or volumes. The continuum mesh offered instead more versatility [5, 32], and has since been used for multiple different applications, e.g. the design of optimized repetitive metamaterials [33–35], fluids optimization [36], modelization of self-weight of the structure [37], the simulation of advanced manufacturing constaints [38, 39], the design of compliant mechanism [40, 41], or the optimization for additive manufacturing [42]. The SAND approach is, however, incompatible with continuum meshes due to its excessive number of variables[1]. Given this limitation, a new approach was required to better handle the complexity of continuum meshes.

In density-based Nested Analysis and Design (NAND) approach, the nodal displacement (state) variables are eliminated from the optimization problem through a process where the structural equilibrium equation is solved every iteration instead of being used as a constraint of the optimization. This results in an independent nested phase where the state equation of structural equilibrium is solved separately from the optimization algorithm. This creates a dense coupling between displacement and material density variables, necessitating a computationally expensive sensitivity analysis within the nested algorithm, typically employing the adjoint method (more information about the adjoint method on the following resurces [2, 43]). Nevertheless, if the problem is reformulated as a compliance minimization with volume constraints, the problem is self-adjoint and the adjoint algorithm is no longer necessary to evaluate the gradient sensitivities [1], and this reduces considerably the computational times.

Both the TTO methods based on the ground structure and the density-based topology optimization approaches are good candidate for the optimization of ultra-light structures. We review here their main characteristics and numerical properties, starting from density-based approaches.

1.2.1 Density-based topology optimization

Let $\Omega \in \mathbb{R}^2$ be a rectangular domain in of dimensions X and Y, containing respectively N_x and N_y linear 4-nodes elements, for a total of $N_e = N_x N_y$ elements and M nodes (see Fig. 1.2). The objective of the optimization is the minimization of the compliance C of the structure, equivalent to finding the structure with the least possible nodal displacement with respect to a defined set of boundary conditions.

5. Bendsøe et al. (1988), 'Generating optimal topologies in structural design using a homogenization method'
32. Bendsøe (1989), 'Optimal shape design as a material distribution problem'
33. Sigmund (1994), 'Materials with prescribed constitutive parameters'
34. Zhang et al. (2006), 'Scale-related topology optimization of cellular materials and structures'
35. Collet et al. (2018), 'Topology optimization for microstructural design under stress constraints'
36. Borrvall et al. (2003), 'Topology optimization of fluids in Stokes flow'
37. Bruyneel et al. (2005), 'Note on topology optimization of continuum structures including self-weight'
38. Sigmund (2009), 'Manufacturing tolerant topology optimization'
39. Brackett et al. (2011), 'Topology Optimization for Additive Manufacturing'
40. Sigmund (1997), 'On the Design of Compliant Mechanisms Using Topology Optimization*'
41. Bruns et al. (2001), 'Topology optimization of non-linear elastic structures and compliant mechanisms'
42. Wang et al. (2020), 'Space-time topology optimization for additive manufacturing'

1: This proposition holds when referring to the end of the 1980s when computational power was scarce compared to what we have today.

2. Martins et al. (2021), 'Engineering Design Optimization'
43. Tortorelli et al. (1994), 'Design sensitivity analysis'
1. Bendsøe et al. (2004), 'Topology Optimization'

Compliance minimization formulation The Problem $\mathbb{T}_0$ is stated in terms of the design variables $\boldsymbol{\rho}$ as follows:

$$
\begin{aligned}
\min_{\boldsymbol{\rho}} \quad & C = \sum_i \boldsymbol{u}_{e,i}^T \boldsymbol{K}_{e,i} \boldsymbol{u}_{e,i} = \boldsymbol{f}^T \boldsymbol{u} && \forall i \in [0, \dots, N_\mathrm{e}] \\
\text{s.t.} \quad & \frac{\sum_i \left(\overline{\widetilde{\rho}}_i v_i \right) / V_0}{V_\mathrm{p}} - 1 \le 0 && \forall i \in [0, \dots, N_\mathrm{e}] \\
& \boldsymbol{K}\boldsymbol{u} = \boldsymbol{f} \\
& 0 \le \rho_i \le 1. && \forall i \in [0, \dots, N_\mathrm{e}]
\end{aligned} \qquad (\mathbb{T}_0)
$$

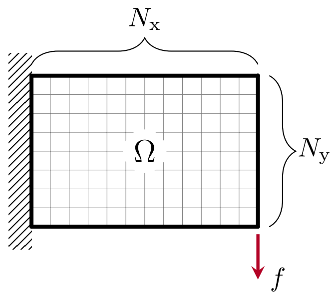


Figure 1.2: The domain Ω is discretized using $N_\mathrm{e} = N_\mathrm{x} N_\mathrm{y}$ continuous 4-nodes elements.

The design variables $\boldsymbol{\rho}$ are defined for every element of the structure as $\boldsymbol{\rho} = [\rho_1, \rho_2, \dots, \rho_{Ne}]^T$, with $\rho_i \in [0,1]$, $\forall i \in [0, \dots, N_\mathrm{e}]$. The physical densities $\overline{\widetilde{\boldsymbol{\rho}}}$ are related to the design variable $\boldsymbol{\rho}$ through density filtering and threshold projection [44], as explained later in the document. V_p is the prescribed volume fraction that acts as the constraint of the minimization problem, while v_i represents the area of the i-th element and V_0 is the total area of the domain Ω. $\boldsymbol{K}\boldsymbol{u} = \boldsymbol{f}$ is the state equation of the problem and defines the elastic response of the structure to an external nodal load $\boldsymbol{f} = [f_1, f_2, \dots, f_{2M}]^T$. The global stiffness matrix $\boldsymbol{K}$ is assembled from the element stiffness matrix $\boldsymbol{K}_{e,i}$ and $\boldsymbol{K}_{e,i} = E_i \boldsymbol{K}_{e,0}$ where $\boldsymbol{K}_{e,0}$ represents the element stiffness matrix relative to the chosen type of element (linear or quadratic) and $E_i(\overline{\widetilde{\rho}}_i)$ the Young's modulus of the i-th element.

44. Wang et al. (2011), 'On projection methods, convergence and robust formulations in topology optimization'

The material scheme used to interpolate between void and full material is the well-known Solid Isotropic Material with Penalization Method (SIMP) [32, 45] approach. It is governed by the equation:

32. Bendsøe (1989), 'Optimal shape design as a material distribution problem'

45. Bendsøe et al. (1999), 'Material interpolation schemes in topology optimization'

$$
E_i(\overline{\widetilde{\rho}}_i) = E_{\min} + \overline{\widetilde{\rho}}_i^{\,p} (E_0 - E_{\min}), \tag{1.2}
$$

where the parameter p penalizes the intermediate densities and pushes the result to a black-and-white result. E_0 is the Young's modulus of the dense material and $E_{\min}$ is a small value used to avoid the global stiffness matrix $\boldsymbol{K}$ from being singular when $\overline{\widetilde{\rho}}_i = 0$.

The SIMP exponent p is constrained to be greater than or equal to 1. From a physical perspective, the extreme case of $p = 1$ makes sense only in a two-dimensional optimization context, where it becomes equivalent to optimizing membrane thickness. When $p > 1$, the interpolation results in an equivalent homogenized stiffness tensor for intermediate densities, determined by the material-to-void ratio ρ. This mirrors microstructures conforming to the Hashin-Shtrikman (HS) conditions, which estimate the theoretical lower and upper bounds for the elastic modulus of a homogeneous, isotropic mixture of different materials based on their elastic modulus and volume fractions [46]. If the exponent p exceeds 3, Bendsøe [45] mathematically proves that the equivalent homogenized stiffness tensor adheres to the upper bound of the HS conditions (refer to Fig. 1.2). It is important to note

46. Hashin et al. (1963), 'A variational approach to the theory of the elastic behaviour of multiphase materials'

45. Bendsøe et al. (1999), 'Material interpolation schemes in topology optimization'

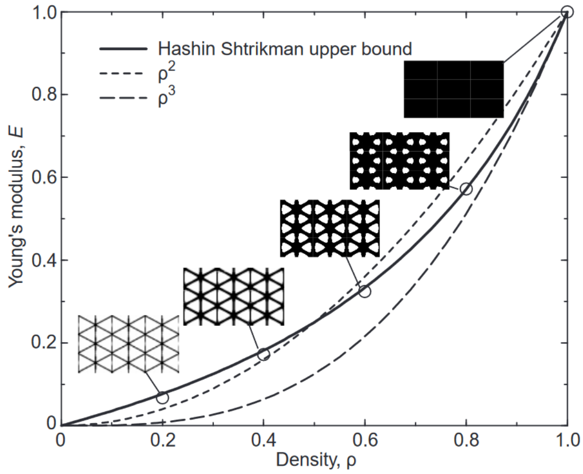


Figure 1.3: Comparison between SIMP model with the Hashin-Strikman upper bound, considering an isotropic material with a Poisson ratio of 1/3 mixed with void. The Hashin-Strikman upper bound is illustrated with microstructures approaching the specified bounds. [45].

that in the single-scale topology optimization context, deviating from the HS bounds for intermediate densities is allowed. The objective is to drive the density distribution towards a black-and-white result with minimal intermediate densities, without concerning ourselves with whether the equivalent homogenized stiffness tensor can be replicated by a real microstructure.

Spatial filtering and projection Multiple approaches have been developed to solve the problems linked to mesh discretization, such as mesh dependence or the checkerboard problem [47]. Filtering the sensitivity information of the optimization problem proved to be an effective approach to guarantee independence from mesh resolution [40, 48]. Another possibility is instead to directly filter the density field $\boldsymbol{\rho}$ using the 2D convolution operator [49]. The weight function w (or kernel) of the convolution is defined as:

47. Díaz et al. (1995), 'Checkerboard patterns in layout optimization'

40. Sigmund (1997), 'On the Design of Compliant Mechanisms Using Topology Optimization*'

48. Sigmund (1994), 'Design of Material Structures using Topology Optimization'

49. Sigmund (2007), 'Morphology-based black and white filters for topology optimization'

$$w(d_j) = R - d_j, \quad j \in \mathbb{N}_{i,R} \tag{1.3}$$

where $\mathbb{N}_{i,R}$ represent the set of elements lying within a circle of radius R centered on the i-th element and d_j is the distance of the j-th element to the center of the filter (see Fig. 1.4).

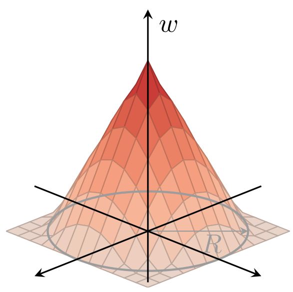


Figure 1.4: Kernel of the 2D convolution operator.

The filtered values of the design variable are calculated as:

$$\widetilde{\rho}_i = \frac{\sum_{j\in\mathbb{N}_{i,R}} w(d_j)v_j\rho_j}{\sum_{j\in\mathbb{N}_{i,R}} w(d_j)v_j}. \tag{1.4}$$

As the filtering phase produces a large number of gray elements, a smooth projection technique based on the *tanh* function is implemented [44]:

$$\bar{\widetilde{\rho}}_j = \frac{\tanh(\beta\eta) + \tanh(\beta(\widetilde{\rho}_j - \eta))}{\tanh(\beta\eta) + \tanh(\beta(1-\eta))}, \tag{1.5}$$

44. Wang et al. (2011), 'On projection methods, convergence and robust formulations in topology optimization'

where β is a parameter that defines the slope of this approximation function: the larger the value of β, the less intermediate elements are present in the structure topology. η is the threshold value of the projection.

In the domain of structural topology optimization, it is a widely adopted strategy to employ continuation methods. Introduced in the 90s [50, 51], they are used to converge towards more optimized structures. These methods solve a sequence of problems with increasing values of the SIMP material penalization parameter p. Many researchers such as Bendsøe and Sigmund [1] and Rozvany [52] consider, among others, continuation methods as a standard procedure in topology optimization. However, this approach comes at the expense of an increased number of iterations and, consequently, augmented computational time [53]. In an effort to mitigate this drawback, Rojas Labanda and Stolpe [54] have derived an automatic penalization scheme. This innovative scheme aims to reduce both the objective function value and the number of iterations, providing an improvement over the classical formulation with a fixed penalty parameter. While the literature is predominantly focused on the continuation scheme on the SIMP material penalization parameter p, it is worth noting that similar techniques could be employed for other optimization parameters e.g. the filter radius R or the projection parameter β.

50. Allaire et al. (1993), 'A Numerical Algorithm for Topology and Shape Optimization'
51. Allaire et al. (1993), 'Topology Optimization and Optimal Shape Design Using Homogenization'
1. Bendsøe et al. (2004), 'Topology Optimization'
52. Rozvany (2009), 'A critical review of established methods of structural topology optimization'
53. Petersson et al. (1998), 'Slope constrained topology optimization'
54. Rojas-Labanda et al. (2015), 'Automatic penalty continuation in structural topology optimization'

While topology optimization offers tremendous benefits in terms of weight reduction and structural efficiency, it is important to acknowledge the challenges associated with manufacturing such designs. The intricate and complex geometries generated through the optimization can pose difficulties in the fabrication process, often requiring advanced manufacturing techniques, specialized equipment, and specific constraints in the optimization [39, 55, 56]. Additionally, the computational time required for generating such optimized designs, particularly for low volume fractions typical of the aerospace domain, can be significant [57], impacting the overall efficiency of the design process. This remains true even with the use of adaptive meshes [58, 59]. Even if the freedom of the design space offered by continuum meshes is high, it is known that at very low volume fractions (e.g. ultralight structures), and, especially if buckling constraints and manufacturing considerations (e.g. minimum length scale), are taken into account, the optimal topology resemble to a truss-like structure [60]. As a result, a distinct branch of continuous topology optimization has emerged specifically tailored for optimizing truss-like structures, known as feature-Mapping topology optimization (also called topology optimization with explicitly defined components).

39. Brackett et al. (2011), 'Topology Optimization for Additive Manufacturing'
55. Zhou et al. (2002), 'Progress in Topology Optimization with Manufacturing Constraints'
56. Liu et al. (2018), 'Current and future trends in topology optimization for additive manufacturing'
57. Aage et al. (2017), 'Giga-voxel computational morphogenesis for structural design'
58. Salazar de Troya et al. (2018), 'Adaptive mesh refinement in stress-constrained topology optimization'
59. Zhang et al. (2020), 'Adaptive mesh refinement for topology optimization with discrete geometric components'
60. Sigmund et al. (2016), 'On the (non-)optimality of Michell structures'

1.2.2 Feature-Mapping topology optimization

Topology optimization methods using explicitly defined components have been developed to permit an easier interpretation of the solution, finding the optimal shape, size, and connectivity of compo-

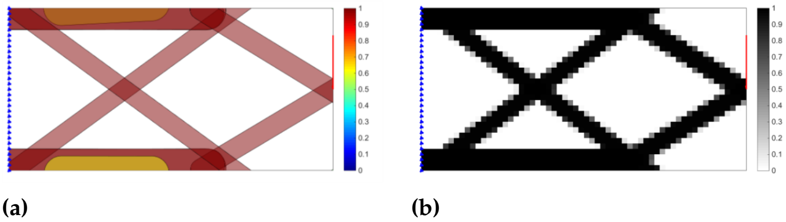


Figure 1.5: Component (a) and density (b) plot of a short cantilever beam optimized using the component-based topology optimization method GGP [66].

nents projected over a finite element continuum mesh. Two main feature-mapping methods applied to topology optimization have been developed [61], the Moving Morphable Components (MMC) approach [62, 63] and the Geometry Projection (GP) approach [64, 65], later combined in a unique methodology called Generalized Geometry Projection (GGP) [66]. Recently, the GP approach has been used to optimize light lattice structures, proving the effectiveness of the method to provide easy-to-interpret solutions [67]. Nevertheless, the optimization is still based on a density field projected on a continuum mesh, that needs to be refined to correctly discretize low volume fraction structures. Additionally, truss structure design naturally depends on constraints on maximum allowable stress and buckling which are all known for being difficult to implement on topology optimization using the Nested Analysis and Design (NAND) formulation. This is principally due to the singular optima (or topologies) phenomenon [68, 69] and the pseudo-modes of buckling of low-density elements [70].

61. Wein et al. (2020), 'A review on feature-mapping methods for structural optimization'
62. Guo et al. (2014), 'Doing Topology Optimization Explicitly and Geometrically—A New Moving Morphable Components Based Framework'
63. Zhang et al. (2017), 'A new three-dimensional topology optimization method based on moving morphable components (MMCs)'
64. Norato et al. (2015), 'A geometry projection method for continuum-based topology optimization with discrete elements'
65. Zhang et al. (2016), 'A geometry projection method for the topology optimization of plate structures'
66. Coniglio et al. (2020), 'Generalized Geometry Projection'
67. Kazemi et al. (2020), 'Multi-material topology optimization of lattice structures using geometry projection'
68. Cheng et al. (1997), 'ε-relaxed approach in structural topology optimization'
69. Rozvany (2001), 'On design-dependent constraints and singular topologies'
70. Gao et al. (2015), 'Topology optimization of continuum structures under buckling constraints'
71. Gilbert et al. (2003), 'Layout optimization of large-scale pin-jointed frames'
72. Pedersen (1973), 'Optimal Joint Positions for Space Trusses'
73. Achtziger (2007), 'On simultaneous optimization of truss geometry and topology'
74. Descamps et al. (2013), 'A lower-bound formulation for the geometry and topology optimization of truss structures under multiple loading'

1.2.3 Truss topology optimization (TTO)

Truss Topology Optimization (TTO) focuses on optimizing the topology of the truss structure itself, instead of operating on a continuous mesh. It involves selecting the cross-sectional areas and the connectivity of a discrete and dense mesh called ground structure, aiming to minimize weight while satisfying structural constraints. The process is graphically presented in Fig. 1.6. Usually, the optimized structure is obtained as a subset of the initial ground structure, but multiple alternative approaches have been proposed since then, e.g. starting from a very coarse ground structure that is enriched during the analysis [71], or giving the nodes of a coarse ground structure the possibility to move, during [72–74], or after the optimization, simultaneously reducing the number of active members of the solution [75, 76]. Recently, a hybrid method based on the projection of explicitly defined components on a discrete ground structure has been proposed, easing the interpretation of the stiffening pattern of the optimized truss [77].

In the early works, the TTO problem was formulated in terms of member forces [28, 30] with plastic material modelization, ignoring the kinematic compatibility to obtain a LP problem. Formulated using the SAND approach, the equations of structural mechanics of the

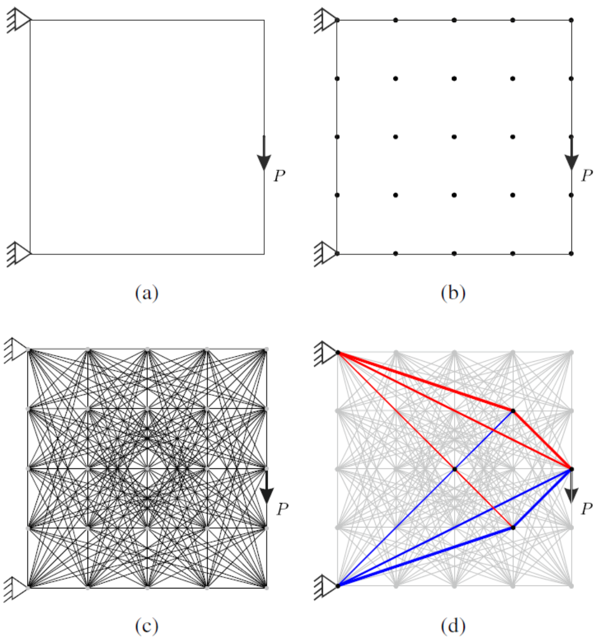


Figure 1.6: The TTO algorithm can be divided into 4 steps: (a) specification of the design space, loads, and boundary conditions; (b) discretization of the design space; (c) the ground structure is generated depending on the desired connectivity level; (d) resolution of the optimization problem and plot of the solution [78].

problem are imposed as constraints of the optimization and, contrary to NAND approaches, are not explicitly solved. Formulated that way, it is trivial to add maximum stress constraints compared to an equivalent NAND formulation. However, the SAND formulation with plastic material modelization only correctly predicts the mechanical behavior of statically determinate structures or mechanisms [79, 80]. Moreover, adding local buckling constraints to the optimization formulation is fundamental, as ultralight truss structures are often dominated by this mode of failure [60]. Multiple works in the field of truss structure optimization have focused on addressing these two crucial challenges [81–83].

75. He et al. (2015), 'Rationalization of trusses generated via layout optimization'
76. Lu et al. (2023), 'Reducing the number of different members in truss layout optimization'
77. Savine et al. (2021), 'A component-based method for the optimization of stiffener layout on large cylindrical rib-stiffened shell structures'
28. Dorn et al. (1964), 'Automatic design of optimal structures'
30. Hemp (1973), 'Optimum Structures'
79. Kirsch (1989), 'Optimal topologies of truss structures'
80. Rozvany et al. (1995), 'Layout Optimization of Structures'
60. Sigmund et al. (2016), 'On the (non-)optimality of Michell structures'
81. Kirsch (1980), 'Optimal design of trusses by approximate compatibility'
82. Cheng (1995), 'Some aspects of truss topology optimization'
83. Achtziger (1999), 'Local stability of trusses in the context of topology optimization Part I'
84. Maxwell (1870), 'I.—On Reciprocal Figures, Frames, and Diagrams of Forces'

Classical Michell structures The characteristics of these class of truss structures are described by some simple criteria that date to the end of the 19th and the beginning of the 20th century. When a structure is statically determinate — i.e. the structure is not a mechanism, and it is not over-constrained by the supports — the Maxwell theorem [84] states that:

$$\sum_{\forall i|q_i>0} \ell_i q_i + \sum_{\forall i|q_i<0} \ell_i q_i = \text{const.} \tag{1.6}$$

where ℓ_i and q_i represent the length and the axial force of the i-th member, respectively. The constant value at the right of Equation 1.6 depends on the nature of the boundary conditions and the material used. The Maxwell theorem dictates that any increment in compression forces must be counterbalanced by an equivalent increase in tension forces when the structure remains topologically unchanged. So for statically determinate structures the structure layout is not influenced by the ratio between σ_c and σ_t, Young's modulus E of the material, nor the force magnitude.

Starting from Maxwell's findings, Michell theorized two further criteria for optimal truss structures [85] valid when the maximum allowable stress is equal in tension and compression ($\sigma_t = \sigma_c$) and when the supports of the structure are statically determinate. The first one states that all the members of an optimal structure should present internal stress equal in magnitude to the maximum allowable value of the material – i.e. the structure is *fully stressed*. The second criterion asserts that the strain of all the members of the structure should be equal and there should be no other point having a strain higher than this value. As formulated, these two criteria are known as the Michell criteria. The second criterion was later generalized by Hemp [30] as:

85. Michell (1904), 'The limits of economy of material in frame-structures'

30. Hemp (1973), 'Optimum Structures'

$$-\frac{1}{\sigma_c} \leq \varepsilon \leq \frac{1}{\sigma_t}. \tag{1.7}$$

Compared to the second Michell criterion, Equation 1.7 permits to correct identification of the minimum volume structure even when different strength values for compression and tension and different support types are taken. These criteria are known as the Michell-Hemp criteria.

Plastic material formulation The rigid-plastic formulation characterizes the material as entirely rigid up to the point of reaching the yield stress, denoted as σ_y, and subsequently assumes a constant stress level of σ_y once that threshold is exceeded. This formulation is a clear consequence of the application of the Michell-Hemp criteria and has thus been used in the very first work of TTO [28–30].

28. Dorn et al. (1964), 'Automatic design of optimal structures'
29. Chan (1964), 'Optimum structural design and linear programming'
30. Hemp (1973), 'Optimum Structures'

The ground structure approach The ground structure is a framework composed of various structural members that connect specified points or nodes in two- or three-dimensional space (see Fig. 1.7). These members can take the form of beams, columns, wires, or bars elements, depending on the specific structural requirements. In this thesis, we will deal with trusses, and so the chosen element is the bar. Since the nodes within the ground structure are considered pin-joints, all straight members exclusively face either tension or compression loads.

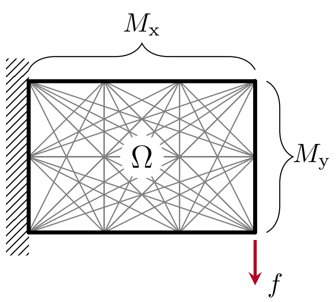


Figure 1.7: The domain Ω is discretized using a set of straight members connecting a set of nodes. This framework is known as the ground structure.

Depending on how the connectivity of the grid of nodes is, we can experience very different ground structures. In a fully connected ground structure, every node within the system is linked to every other node, resulting in a dense and redundant structural configuration. The number of bars N_{el} of a fully connected ground structure can be determined using the following formula:

$$N_{el} = \frac{M \cdot (M-1)}{2}, \tag{1.8}$$

where M represents the number of nodes of the structure.

In classic works, the ground structure is used as the start of the optimization, where the optimized structure is obtained as a subset of the initial ground structure, but multiple alternative approaches have been proposed since then, e.g. starting from a very coarse ground structure that is enriched during the analysis [71], or giving the nodes of a coarse ground structure the possibility to move, during [72–74], or after the optimization, simultaneously reducing the number of active members of the solution [75, 76].

71. Gilbert et al. (2003), 'Layout optimization of large-scale pin-jointed frames'
72. Pedersen (1973), 'Optimal Joint Positions for Space Trusses'
73. Achtziger (2007), 'On simultaneous optimization of truss geometry and topology'
74. Descamps et al. (2013), 'A lower-bound formulation for the geometry and topology optimization of truss structures under multiple loading'
75. He et al. (2015), 'Rationalization of trusses generated via layout optimization'
76. Lu et al. (2023), 'Reducing the number of different members in truss layout optimization'

Optimization formulation The volume minimization formulation with maximum stress constraints is stated in terms of members' cross-sectional areas $\boldsymbol{a}$ and member forces $\boldsymbol{q}$ as follows:

$$\begin{aligned} \min_{\boldsymbol{a},\boldsymbol{q}} \quad & V = \boldsymbol{\ell}^T \boldsymbol{a} && \text{(Volume minimization)} \\ \text{s.t.} \quad & \boldsymbol{B}\boldsymbol{q} = \boldsymbol{f} && (g_{\text{eq}}) \\ & -\sigma_c \boldsymbol{a} \le \boldsymbol{q} \le \sigma_t \boldsymbol{a} && (g_{\text{st,c}},\ g_{\text{st,t}}) \\ & \boldsymbol{a} \ge 0, \end{aligned} \qquad (\mathbb{P}_0)$$

where $\boldsymbol{B}$ is a $N_{\text{dof}} \times N_{\text{el}}$ matrix containing the direction cosines of the i-th member with respect to the i-th degree of freedom to calculate the nodal force equilibrium constraints $\boldsymbol{g}_{\text{eq}}$, and where N_{dof} is the number of Degrees Of Freedom (DOFs), equal to $2M$ or $3M$ for a two- or a three-dimensional load case, respectively. $\boldsymbol{q} = [q_1, q_2, \ldots, q_{N_{\text{el}}}]^T$ is the vector containing the internal member forces, with a positive sign when in tension, caused by the external load $\boldsymbol{f} = [f_1, f_2, \ldots, f_{N_{\text{dof}}}]^T$. The state variable $\boldsymbol{a} = [a_1, a_2, \ldots, a_{N_{\text{el}}}]^T$ represents the cross-sectional area of the N_{el} members of the structure. σ_c and σ_t are the compressive and tensile maximum allowable stresses of the material, respectively, used in the stress constraints $\boldsymbol{g}_{\text{st,c}}$ and $\boldsymbol{g}_{\text{st,t}}$. This formulation takes into account only the linear behavior of the structure and is equivalent to the original and well-studied member force formulation [1, 28].

1. Bendsøe et al. (2004), 'Topology Optimization'
28. Dorn et al. (1964), 'Automatic design of optimal structures'
85. Michell (1904), 'The limits of economy of material in frame-structures'
86. Parkes (1975), 'Joints in optimum frameworks'

The resolution of Problem $\mathbb{P}_0$ frequently produces complex structures made up of a multitude of small members that tend to the shapes of Michell structures (see Fig 1.8) [85]. While it is known that these structures are nearly optimal, one would want to limit the complexity of the resulting structure. Substituting $\boldsymbol{\ell}$ with $\tilde{\boldsymbol{\ell}} = [\ell_1 + s, \ell_2 + s, \ldots, \ell_{N\text{el}} + s]^T$ in the objective function of $\mathbb{P}_0$, one would penalize the appearance of small members [86]. $\tilde{\boldsymbol{\ell}}$ is called augmented member length and s the joint cost. This approach mimics the mesh-independency regularization filter of topology optimization, avoiding the inevitable apparition of structures with tiny features when a fine mesh is adopted.

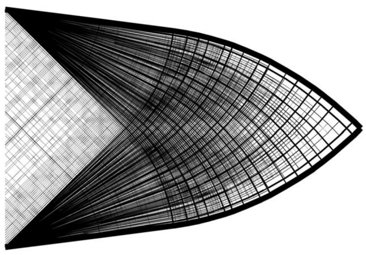
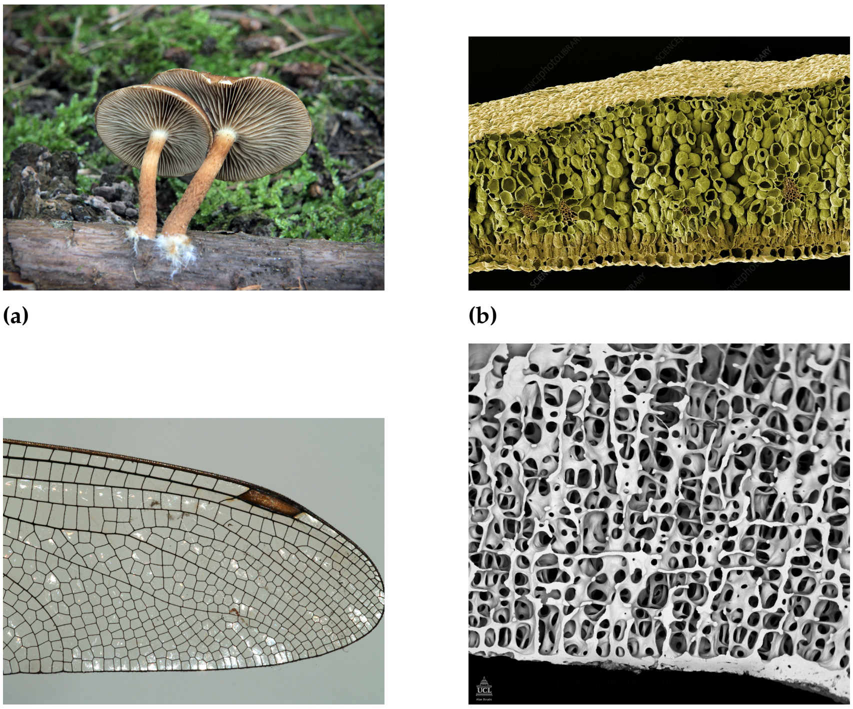


Figure 1.8: The optimal structures found by layout optimization tend at Michell-like structures, made up of a very large number of infinitesimal struts [71].



(a) (b) (c) (d)

Figure 1.9: The natural evolution process frequently generates lattice materials and modular structures; (a) The spore-bearing gills of a Hypholoma fasciculare [91] (b) SEM image of a leaf microstructure [92] (c) cellular stucture of the wing of a dragonfly [93] (d) internal structure of a human bone [94].

1.3 Modular structures and lattice materials

Historically, material properties were modified by manipulating chemical composition, microstructure, and production processes [87]. Another avenue for enhancing material properties involves tailoring the spatial arrangement of solids and voids within the material. Referred to as architectured materials, this concept has gained significant traction in research, particularly with recent advancements in additive manufacturing. These materials, often observed in natural structures like bone microstructures or birds' beaks (refer to Fig. 1.9 for additional examples), have garnered interest due to the recognition that optimal structures exhibit stiffness across multiple scales [88, 89]. Additionally, Fleck noted [90] that one reason for structural hierarchy in engineering is to augment buckling strength. In fact, local buckling strength scales with the strut length l following l^{-2}, indicating that finer length scales contribute to higher buckling strength.

87. Schaedler et al. (2016), 'Architected Cellular Materials'

88. Kohn et al. (1986), 'Optimal design and relaxation of variational problems'

89. Allaire et al. (1999), 'On optimal microstructures for a plane shape optimization problem'

90. Fleck et al. (2010), 'Micro-architectured materials'

If we observe the Ashby material chart [95], where yield strength is plotted against density as shown in Fig. 1.10, it becomes evident that numerous empty spaces exist. Besides some unattainable areas delineated by the HS bounds [2], these empty spaces can be filled by lattices, extending the property space of actual materials.

95. Ashby (1999), 'Materials selection in mechanical design'

2: The HS bounds are the tightest bounds possible from the range of composite moduli for a two-phase isotropic mixture. In lattices, usually, the second material is void.

why modular, which are the advantages

define modules and subdomains

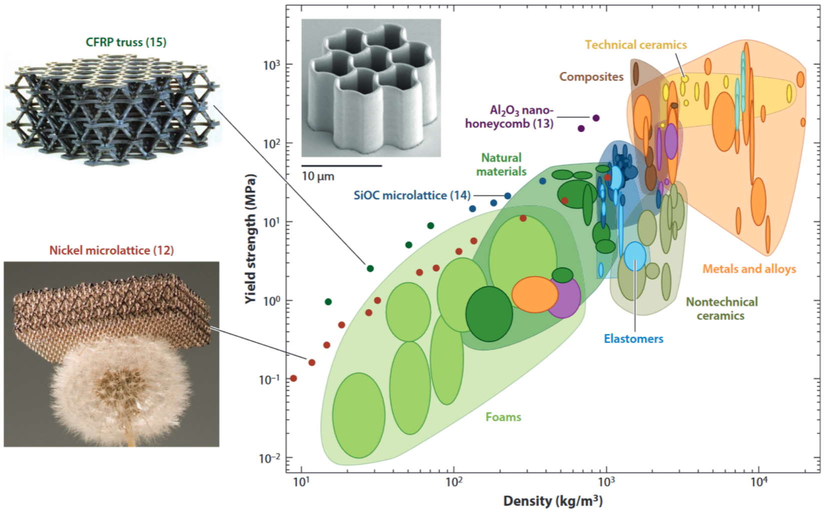


Figure 1.10: Density versus yield strength Ashby chart. Exploiting the architecture of the material as a variable to design new metamaterials, empty spaces of the graph can be filled (see dots) [87].

intro there are different type of approach that we could use, full scale and multiscale

intro multiscale.

numerical homogenization Multi-Scale Approaches A multi-scale approach is based on a microstructure in a representative elementary volume (REV). The macroscopic stiffness tensor of the REV [QH] is obtained by homogenization of the stiffness properties of its constituents. Several homogenization methods can be used such as strain energy methods [86] or asymptotic methods [87]. An example of asymptotic homogenization is given by Equation 2.11. chi0(ij) e are the test strains as presented in Figure 2.8 and chi(ij) e the corresponding response strains. The macroscopic stiffness properties are then used in the later process to set up the FEM analysis in topology optimization. Even when using a mixture of only isotropic material and void at the microscale, depending on the layout of the microstructure, an anisotropic macroscopic stiffness can be obtained

intro full scale

the problem on the number of subsections needed to correctly foresee the mechanical behaviour

Lattices can be classified as open- or closed-wall. Even if a closed-wall lattice would result in a more stiff structure, the interest of an open-cell is well described by Sigmund et al. [60]: "*we emphasize that the outcome of minimum compliance-type continuum topology optimization studies should always be of sheet type unless other constraints [...] that favour Michell-like structures have been explicitly stated*", where the possible constraints are:

60. Sigmund et al. (2016), 'On the (non-)optimality of Michell structures'

- **Structural and Microstructural stability**: The load for initiating buckling of a slender strut is much higher than that of a comparable plate with the same mass [96]. An open-cell structure,

96. Deshpande et al. (2001), 'Foam topology'

thus, is less prone to buckle compared to a closed-cell.

- **Porosity**: A porous open-wall cell permits a flow to pass through. This property enables the use of lattice as heat exchangers or to promote bone regrowth in biomedical scaffolds.
- **Manufacturing**: very thin walls are extremely difficult to manufacture. An open-cell design, while being far from easy to manufacture, is then preferable from a manufacturing standpoint.
- **Transparency**: The possibility to view through the structure can be beneficial for example for reparation and health monitoring.
- **Elegance/aesthetic**: According to Sigmund et al. [60] an open-cell design is a Michell-like structure, that are *inarguably beautiful* and *look elegant and efficient*.

In function of the size of the repeating cell, is it possible to distinguish a lattice material or a lattice structure, with the former having no scale separation between the cell and the macro-structure. Even if there is no clear threshold between the two, this definition will be very useful when we will talk about lattices optimization.

97. Ashby (2006), 'The properties of foams and lattices'

96. Deshpande et al. (2001), 'Foam topology'

96. Deshpande et al. (2001), 'Foam topology'

The relative density of the lattice is defined as:

$$\bar{\rho} = \frac{\rho_l}{\rho} \tag{1.9}$$

where ρ_l and ρ represent the density of the lattice and of the dense material, respectively [97].

We can define a stretching-dominated structure as a lattice whose constitutive struts faces only tension and compression loads. The nodal stiffness does not take part in the structural stiffness and the truss collapses by stretching of the struts. According to Desphande et al. [96] on a stretching-dominated truss *"freezing the joints of the triangulated structure has virtually no effect on the macroscopic stiffness or strength; although the struts bend, the frame is still stretching-dominated and the collapse load is dictated mainly by the axial strength of the struts.'*. An open-cell structure can thus be treated as a connected set of pin-jointed struts if it is stretching-dominated.

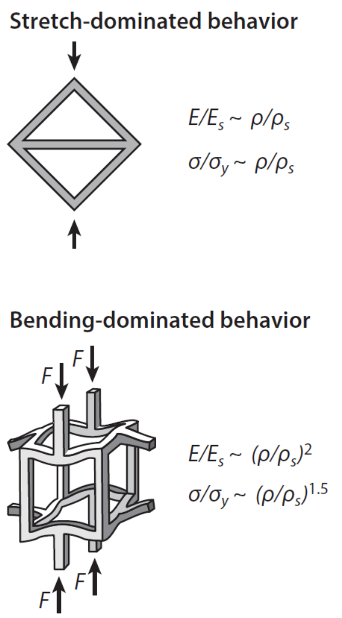


Figure 1.11: A stretch-dominated and a bending-dominated Representative Volume Element (RVE). Bending-dominated structure act as a mechanism if the joints cannot withstand moments. The scaling laws are different for the two structure families [87].

A stretching-dominated structure is typically 10 times more stiff and 3 times more strong than a bending-dominated structure at $\bar{\rho} = 0.1$ (see Fig. 1.11) [96]. However, in compression they present a softening post-yield response due to buckling of struts, making them non-interesting for energy absorption tasks. On the other side, bending-dominated lattices absorb more energy due to their plateau-like response.

Lattices structures and materials have various promising application as:

- Lattice structures and materials shows notable energy absorbing properties, even more if designed as bending-dominated [98–

Figure 1.12: [102].

100].

- As a direct consequence, lattice structures are set to be the possible new design scheme for aerodynamic structures. They show remarkable aeroelastic properties [101, 102].
- Lattices proved to be really interesting candidates to build scaffolds [103–105].
- Lattice materials present very good heat exchanger properties. This is due to their high surface-to-volume ratio and the turbulent mixing flow they create when a fluid pass through [106, 107].

Other interesting properties of lattice structures and materials are:

- Cellular structures can be designed to be reversibly assembled. This concept paves the way to rapid assembly and easy-to-repair structures. Various approach have been proposed, as using fasteners [102, 108, 109] or snap-fit joints [110]
- Lattice structures are naturally damage-tolerant structures [111, 112]. In a skin-lattice design, the skin is not load-bearing, and skin damage will not cause a progressive failure of the structure. Moreover, in case of rib damage, the damaged rib will be excluded from the load without any harm to the whole structure (see Fig. **??**).
- They open the way for an extensive use of robotization for the manufacture [113] and the assembly phases [114–117].

This field is now getting more and more interesting, especially in the aerospace field, as we can see by the recent applications by NASA project MADCAT [102] and Opgenoord PhD thesis [118].

manufacturing of modular structures and lattice materials

1.3.1 Modular structures and lattice materials optimization

1.3.2 Multi-scale structures optimization

98. Evans et al. (2010), 'Concepts for Enhanced Energy Absorption Using Hollow Micro-Lattices'
99. Schaedler et al. (2014), 'Designing Metallic Microlattices for Energy Absorber Applications'
100. Ozdemir et al. (2016), 'Energy absorption in lattice structures in dynamics'
101. Opgenoord et al. (2018), 'Aeroelastic Tailoring using Additively Manufactured Lattice Structures'
102. Cramer et al. (2019), 'Elastic shape morphing of ultralight structures by programmable assembly'
103. Hutmacher (2000), 'Scaffolds in tissue engineering bone and cartilage'
104. Mota et al. (2015), 'Additive manufacturing techniques for the production of tissue engineering constructs'
105. Nikolova et al. (2019), 'Recent advances in biomaterials for 3D scaffolds'
106. Lu et al. (1998), 'Heat transfer in open-cell metal foams'
107. Wadley et al. (2007), 'Thermal Applications of Cellular Lattice Structures'
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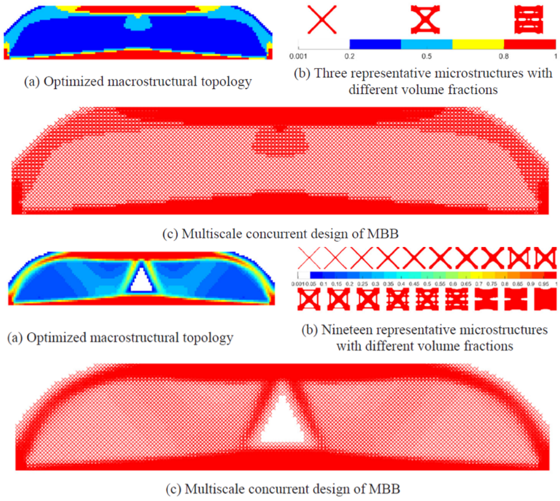
108. Cheung et al. (2013), 'Reversibly Assembled Cellular Composite Materials'
109. Jenett et al. (2017), 'Digital Morphing Wing: Active Wing Shaping Concept Using Composite Lattice-Based Cellular Structures'
110. Dong et al. (2015), 'Mechanical response of Ti–6Al–4V octet-truss lattice structures'



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1.3.3 Full-scale structures optimization

Although the aforementioned methods are able to design a structure consisting of a simple module repeated several times in the global domain, they suffer from a common limitation. Namely, the designs converge towards solutions with compromised structural performance (Huang and Xie 2008; Zhang and Sun 2006). The cause lies within the topological periodicity. The topology of the module is influenced most by the region with the highest compliance. The resulting module design is used at different locations in the structure, therefore not leading to the most optimal solution for these regions (Tugilimana et al. 2019). This shortcoming can be addressed by two approaches: (i) by defining additional module properties as design variables or (ii) by allowing more modules within the structure. Both approaches extend the solution space. The first approach, extending the solution space, was considered by allowing for rotation of a module. Allowing for rotations resulted in improved structural performance because rotation of the modules modifies the material distribution in the structure locally (Tugilimana et al. 2017). Also, in a 2D continuum setting, the one-to-one mapping of a module unit to the global domain is extended by by allowing the module unit to resize (Stromberg et al. 2011). [120]

111. Stolpe (2019), 'Fail-safe truss topology optimization'
112. Wu et al. (2021), 'Topology optimization of multi-scale structures'
113. Hunt et al. (2019), 'WrapToR composite truss structures'
114. Gershenfeld et al. (2015), 'Macrofabrication with Digital Materials'
115. Jenett et al. (2017), 'BILL-E: Robotic Platform for Locomotion and Manipulation of Lightweight Space Structures'
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102. Cramer et al. (2019), 'Elastic shape morphing of ultralight structures by programmable assembly'
118. Opgenoord (2018), 'Transonic Flutter Prediction and Aeroelastic Tailoring for Next-Generation Transport Aircraft'

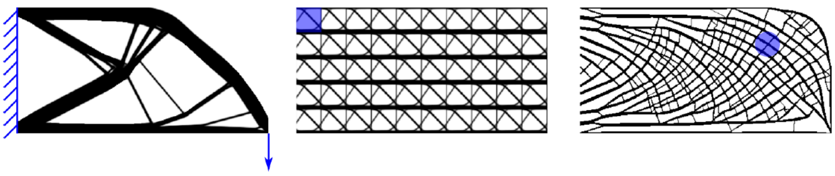

Figure 1.15: Three structures with the same volume are optimized for compliance minimization using three different methods: on the left, a classic mono-scale topology optimization algorithm. Middle: the variable linking method is used to enforce the pattern repetition on the structure. On the right an optimized structure with local volume constraints. The algorithms used to optimize the last two structures belong to the family of *full-scale* methods. [112].

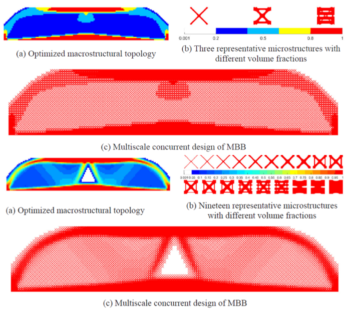

Figure 1.14: A multi-scale L-shape beam optimized by a multi-scale topology optimization algorithm [119]. Three different types of cells are used.

120. Bakker et al. (2021), 'Simultaneous optimization of topology and layout of modular stiffeners on shells and plates'

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